Poisson Representations for Spatial Population Models with Competition

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Abstract

In this thesis we want to extend the Kurtz-Rodrigues representation to spatial population models with competition. The Kurtz-Rodrigues representation consists of countably many particles forming together a so-called Poisson representation of the Dawson-Watanabe superprocess. Poisson representations are well-suited to combine the particle/individual perspective of the underlying branching particle systems with the measure-valued branching Markov processes obtained as high-density limits of the former making it possible to equip these limits with a genealogical structure. Competition refers to an additional death rate endured by the particles which depends on the current state of the population and can be interpreted as an increased mortality due to overcrowding.

Our approach is general enough to encompass two classes of spatial competition models which form respectively a generalization of the Bolker-Pacala models previously studied by Alison Etheridge and the SPDE equivalent of the logistic Feller-Diffusion investigated by Carl Mueller and Roger Tribe. In the spirit of Evans and Perkins we obtain our representations by cutting them out from the Kurtz-Rodrigues representations of the corresponding non-competitive counterparts. With this goal in our mind we develop an integration theory for the Kurtz-Rodrigues representation which is reminiscent of Perkins' stochastic calculus. In order to facilitate the development of our theory we introduce an ordered collection $(X_i, U_i)_{i=1}^{\infty}$ forming the particles of the Kurtz-Rodrigues representation and allowing us to utilize de Finetti's theorem to derive convergence and continuity results. Our integration theory provides us with a third coordinate $(Z_i)_{i=1}^{\infty}$ such that $(X_i, Z_i, U_i)_{i=1}^{\infty}$ has a well-defined high-density limit Ξ^{XZ}. The $(Z_i)_{i=1}^{\infty}$ are employed as death markers indicating which particles have prematurely died due to competition, while the remaining particles form the Poisson representation of the competitive model. The desired competitive model is obtained, when $(X_i, Z_i, U_i)_i^{\infty}$ and the corresponding measure-valued processes solve the associated "cut-out equation". Finally, we give a short outlook how such a representation can be used to study the extinction behavior of competitive models.

Our integration theory is a marriage between Perkins' stochastic calculus and the theory of Kurtz-Rodrigues. Combining both theories requires to work out some technical details, which will be done in the appendix. Last but not least, we discuss the Markov mapping theorem which is the foundation of the Kurtz-Rodrigues theory and we prove a version of this theorem general enough for our purpose.

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Contents

List of Symbols

An almost complete list of all frequently used symbols

Spaces of Functions, Measures and Etc.

$\mathrm{B}(E)$	Bounded Borel measurable functions on E	P. 17
$C([0,\infty),E)$	Continuous paths with values in E	P. 75
$C_b(E)$	Real-valued, bounded cont. functions on E	P. 17
$C_b^+(E)$	Elements of $C_b(E)$ with $f \geq 0$	P. 17
$C_b^2(\mathbb{R}^m)$	Twice cont. diff. $f \in C_b(\mathbb{R}^m)$ with bounded derivatives	D. 2.1.1
$C_c(\mathbb{R}^m)$	Elements of $C(\mathbb{R}^m)$ with compact support	P. 17
$C_b^{2,1}(\mathbb{R}^m \times \mathbb{R})$	Similar to $C_b^2(\mathbb{R}^m)$, but only one time	
	cont. differentiable in the last coord.	D. 4.2.3
$C_{lip}^+(E)$	Bounded, non-neg. Lipschitz-cont. functions	L. 2.6.4
$\mathbb{D}([0,\infty),E)$	Càdlàg paths with values in E	D. 2.4.1
$\mathbb{D}_t([0,\infty),E)$	Paths of $\mathbb{D}([0,\infty),E)$ stopped at time t	D. 2.4.1
$\widehat{\mathbb{D}}([0,\infty),E)$	State space of the path-valued process	D. 2.4.1
$\mathfrak{D}$	State space of $\mathbb{W}_i$, i.e. $\mathfrak{D} := \widehat{\mathbb{D}}([0,\infty),\mathbb{R}^{d+1})$	D. 2.4.7
E	Polish space	P. 17
E_X, E_Y	State spaces of X and Y in Appendix D.1	P. 201
$\mathfrak{G}(\mathbf{B},\dots)$	Test functions of $\mathbf{A}^o_{\mathbf{B}}$	D. 2.5.7
$\mathfrak{G}(\mathbf{B})$	Collection of all $\mathfrak{g}(\mathbf{B},\dots)$	D. 2.5.7
$\mathfrak{g}(\mathbf{B},\dots)$	Building blocks of $\mathfrak{G}(\mathbf{B},\dots)$ and $\mathbf{A}^o_{\mathbf{B}}$	D. B.2.7
$\mathfrak{g}(\mathbf{B})$	Collection of all $\mathfrak{g}(\mathbf{B},\dots)$	D. B.2.7
$\mathfrak{G}^Z(K,\dots)$	Test functions of $\mathbf{A}^o_{B_X,h}$	D. 4.2.7
$\mathfrak{g}^Z(\dots)$	Building blocks of $\mathfrak{G}^Z(K,\dots)$	D. 4.2.4
$\tilde{\mathfrak{g}}^Z(\dots)$	Building blocks of $\mathfrak{G}^Z(K,\dots)$	D. 4.2.4
$\mathfrak{g}^Z, \tilde{\mathfrak{g}}^Z$	Collection of all $\mathfrak{g}^Z(\dots)$ and $\tilde{\mathfrak{g}}^Z(\dots)$	D. 4.2.4
$\ell eb[0,\infty)$	Lebesgue measure over $[0,\infty)$	D. 177

$\mathcal{L}^1(\alpha)$	Space of integrable functions w.r.t. to the kernel α	D. D.1.1
$L^1(\mathbf{C}_\tau)$	Lebesgue space associated with the measure $\mathbf{C}_\tau$	D. 3.4.7
$\mathcal{L}^1(\mathbf{M})$	Space of integrands for our integration theory	D. 3.1.3
$\mathcal{L}^1_{loc}(\mathbf{M})$	Localized version of $\mathcal{L}^1(\mathbf{M})$	D. 3.1.4
$\mathcal{L}^{1,q}_{loc}(\mathbf{M})$	Subset of $\mathcal{L}^1_{loc}(\mathbf{M})$	D. 6.1.1
$\mathcal{L}^1(\mathbf{M}^\Xi)$	Space of integrands h used in Chapter 4	D. 4.1.2
$\mathcal{L}^1_{loc}(\mathbf{M}^\Xi)$	Localized version of $\mathcal{L}^1(\mathbf{M}^\Xi)$	D. 4.1.2
$\mathrm{Lip}_z(\dots)$	Special class of random Lipschitz functions	D. 5.1.1
$\mathbf{M}(E)$	Real-valued, Borel measurable functions on E	P. 17
$\mathcal{M}(E)$	Borel measures over E	P. 18
$\mathcal{M}_f(E)$	Finite measures over E	P. 18
$\mathcal{M}_1(E)$	Probability measures over E	P. 21
$\mathcal{N}(E)$	Integer-valued measures over E	P. 38
$\mathcal{N}_f(E)$	Finite int.-valued measures over E	P. 20
$\mathcal{N}_{lf}(E)$	Locally finite int.-valued measure over E	P. 38
$\overline{\mathcal{N}}(E\times[0,\infty))$	Int.-valued measures, loc. finite in the 2nd coord.	D. 1.1.2
$\mathcal{P}(\mathcal{F}^{\mathbf{\Xi},\mathbb{W}})$	Collection of predictable functions	D. 3.1.1
$\mathbf{S}(E)$	$\{(x_i,u_i)_{i=1}^\infty \in (E\times[0,\infty])^{\mathbb{N}};\ u_i \le u_{i+1}, i\in\mathbb{N}\}$	D. 2.0.2
$\mathbf{S}_{[0,\infty)}(E)$	$(x_i,u_i)_{i=1}^\infty \in \mathbf{S}(E)$ with $u_i<\infty$ for all $i\in\mathbb{N}$	D. 2.0.2
$\mathbf{S}_\infty(E)$	$(x_i,u_i)_{i=1}^\infty \in \mathbf{S}(E)$ with $u_i=\infty$ for some $i\in\mathbb{N}$	D. 2.0.2
$\mathcal{S}(\mathcal{F}^{\mathbf{\Xi},\mathbb{W}})$	Space of simple integrands in $\mathcal{L}^1(\mathbf{M})$	D. 3.1.5
$\bar{\mathcal{S}}(\mathcal{F}^{\mathbf{\Xi},\mathbb{W}})$	Equivalence classes of $\mathcal{S}(\mathcal{F}^{\mathbf{\Xi},\mathbb{W}})$ in $L^1(\mathbf{M})$	D. 3.1.5

Processes

L	The Lévy process based on $N\sim\mathbf{PPP}(\ell eb[0,\infty)^2)$	D. 2.3.7
$(L_i)_{i=1}^\infty$	Ordered system based on $(\tilde{L}_i)_{i=1}^\infty$	D. 2.3.8
$(\tilde{L}_i)_{i=1}^\infty$	Independent copies of L	D. 2.3.8

$(\mathfrak{L}_i)_{i=1}^{\infty}$	Historical processes associated with $(L_i)_{i=1}^{\infty}$	D. 2.3.8
$(\tilde{N}_i)_{i=1}^{\infty}$	Poisson point processes used to construct $(\tilde{L}_i)_{i=1}^{\infty}$	A. 2.1.2
$(\mathfrak{N}_i)_{i=1}^{\infty}$	Historical point processes	D. 2.3.6
$\mathcal{V}_{ji}$	Poisson point processes used to construct $(U_i)_{i=1}^{\infty}$	A. 2.1.2
P, Q	Auxiliary processes often defined during proofs	
$\widehat{\mathbf{Q}}$	De Finetti process of $(\widehat{X}_i)_{i=1}^{\infty}$	D. 7.1.3
$\mathbf{Q}^{\mathbb{W}}$	De Finetti process of $(\mathbb{W}_i)_{i=1}^{\infty}$	P. 2.6.5
$\mathbf{Q}^{\mathbb{W},m}$	Empirical measure of $(\mathbb{W})_{i=1}^{m}$	D. 2.5.2
$\tilde{\mathbf{Q}}^{\mathbb{W}}$	Preliminary versions of $\mathbf{Q}^{\mathbb{W}}$	D. 2.5.2
$\mathbf{Q}^{X}$	De Finetti process of $(X_i)_{i=1}^{\infty}$	R. 2.5.5
$\mathbf{Q}^{XZ}$	De Finetti process of $(X_i, Z_i)_{i=1}^{\infty}$.	D. 3.5.1
$\tilde{\mathbf{Q}}^{XZ,\tau}$	Preliminary versions of $\mathbf{Q}^{XZ,\tau}$ based on τ	D. 3.4.12
$\mathbf{Q}^{XZ,m}$	Empirical measure of $(X_i, Z_i)_{i=1}^{m}$	D. 3.5.1
$\varphi_t^{\boldsymbol{\Xi}}$	Density process of the Superbrownian motion	E. (7.15)
$(V_i)_{i=1}^{\infty}$	$U_1 = V_1$ and $V_{i+1} = U_{i+1} - U_i$, $i \in \mathbb{N}$	E. (2.2)
W	$W = (X, L)$ in Chap. 2, Lévy proc. in the App.	D. 2.4.4
$(W_i)_{i=1}^{\infty}$	Ordered system based on $(\tilde{W}_i)_{i=1}^{\infty}$	L. 2.4.8
$(\tilde{W}_i)_{i=1}^{\infty}$	Independent copies of W	L. 2.4.8
$(\mathfrak{W}_i)_{i=1}^{\infty}$	Historical processes of $(W_i)_{i=1}^{\infty}$	L. 2.4.8
$(\mathbb{W}_i)_{i=1}^{\infty}$	$\mathbb{W}_i = (t, \mathfrak{X}_i, \mathfrak{L}_i) = (t, \mathfrak{W}_i)$	L. 2.4.8
$\mathbb{W}$	Path-valued process of W	S. E.2
$(\widehat{\mathbf{W}}_i)_{i=1}^{\infty}$	$\widehat{\mathbf{W}}_i := (X_i(t), Z_i(t), U_i(t))$	D. 4.6.1
$(\hat{\mathbf{w}}_i)_{i=1}^{\infty}$	Elements of $\mathbb{R}^d \times \mathbb{R} \times [0, \infty]$	D. 4.6.1
$(U_i)_{i=1}^{\infty}$	Level system	D. 2.2.2
$(U_i^0)_{i=1}^{\infty}$	Initial values of $(U_i)_{i=1}^{\infty}$	A. 2.1.2
$\widehat{\boldsymbol{\xi}}$	Empirical process of $(\widehat{X}_i, \widehat{U}_i)_{i=1}^{\infty}$	D. 7.1.3
$\widehat{\boldsymbol{\Xi}}$	Intensity process of $(\widehat{X}_i, \widehat{U}_i)_{i=1}^{\infty}$	D. 7.1.3
$\boldsymbol{\xi}^{\mathbb{W}}$	Empirical process of $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$	D. 2.5.2
$\boldsymbol{\Xi}^{\mathbb{W}}$	Intensity process of $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$	P. 2.6.1

$\boldsymbol{\Xi}^{\mathbb{W},r}$	Point process of $(\mathbb{W}_i)_{i=1}^{\infty}$ with $U_i \leq r$	D. 2.5.2
$\boldsymbol{\xi}^{\mathbb{W},\geq r}$	Point process of $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$ with $U_i \geq r$	D. 2.5.2
$\tilde{\boldsymbol{\Xi}}^{\mathbb{W}}$	Preliminary versions of $\boldsymbol{\Xi}^{\mathbb{W}}$	D. 2.5.2
$\boldsymbol{\xi}^{X}$	Empirical process of $(X_i, U_i)_{i=1}^{\infty}$	R. 2.5.5
$\boldsymbol{\Xi}^{X}$	Intensity process of $(X_i, U_i)_{i=1}^{\infty}$	R. 2.5.5
$\boldsymbol{\xi}^{XZ}$	Empirical process of $(X_i, Z_i, U_i)_{i=1}^{\infty}$	D. 3.5.1
$\boldsymbol{\Xi}^{XZ}$	Intensity process of $(X_i, Z_i, U_i)_{i=1}^{\infty}$	D. 3.5.1
$\boldsymbol{\Xi}^{XZ,r}$	Point process of $(X_i, Z_i)_{i=1}^{\infty}$ with $U_i \leq r$	D. 3.5.1
$\boldsymbol{\xi}^{XZ,\geq r}$	Point process of $(X_i, Z_i, U_i)_{i=1}^{\infty}$ with $U_i \geq r$	D. 3.5.1
$(X_i^0)_{i=1}^{\infty}$	Initial values of $(X_i)_{i=1}^{\infty}$	A. 2.1.2
$\mathbb{X}$	Abstract Markov process	P. 1
X	Lévy process in $\mathbb{R}^d$ with second moment	A. 1.2.3
$(X_i)_{i=1}^{\infty}$	Ordered system based on $(\tilde{X}_i)_{i=1}^{\infty}$	D. 2.3.6
$(\tilde{X}_i)_{i=1}^{\infty}$	Independent copies of X	A. 2.1.2
$(\mathfrak{X}_i)_{i=1}^{\infty}$	Historical processes of $(X_i)_{i=1}^{\infty}$	D. 2.3.6
Y	Continuous modification of $\tilde{Y}$	C. 2.6.2
$\tilde{Y}$	Total mass of the intensity process, i.e. $\tilde{Y} := \boldsymbol{\Xi}^{\mathbb{W}}(\mathfrak{D})$	C. 2.6.2
Y^r	Number of particles with $U_i \leq r$, i.e. $Y^r := \boldsymbol{\Xi}^{\mathbb{W},r}(\mathfrak{D})$	L. 2.2.8
$(Z_i)_{i=1}^{\infty}$	Integrated processes based on an integrand $h \in \mathcal{L}^1_{loc}(\mathbf{M})$	D. 3.2.1 D. 3.4.4

Generators/Markov Kernels/Filtrations/Special Functions

$\mathbf{A}$	Generator of the process X in Appendix D.1	A. D.1
α	Markov kernel $\alpha : E_Y \to \mathcal{M}_1(E_X)$ in Appendix D.1	D. D.1.2
α^*, α_*	Pullback and pushforward associated with α	D. D.1.2
$\mathbf{A}_{\mathbf{B}}$	Gen. of the KR-rep. with spatial motion $\mathbf{B}$	D. B.2.9
$\mathbf{A}_{\mathbf{B}}^r$	Gen. of the KR-rep. with level cap r	D. B.5.2
$\mathbf{A}_{\mathbf{B}}^o$	Gen. of the ord. KR-rep. with spatial motion $\mathbf{B}$	D. 2.5.8

$\mathbf{A}^o_{B_X,h}$	Modified operator $\mathbf{A}^o_{\mathbf{B}}$	D. 4.2.1
$\mathbf{B}$	Generator of the abstract Markov process $\mathbb{X}$	P. 17
B_X	Weak generator of X	D. A.2.3
$B_{\mathbb{W}}$	Generator of $\mathbb{W}$	D. E.2.7
$\mathbf{C}$	Generator of the process Y in Appendix D.1	A. D.1
$\mathbf{C}_{\mathbf{B}}$	Generator of the DW-superprocess with spatial motion $\mathbf{B}$	D. B.1.4
$(B^\rho_W, B^{cov}_W, B^\eta_W)$	Characteristic triple of the Lévy process W	L. E.1.5
$(B^\rho_X, B^{cov}_X, B^\eta_X)$	Characteristic triple of the Lévy process X	A. 1.2.3
$\mathbf{D}_{\mathbf{B}}$	Branching particle system with spatial motion $\mathbf{B}$	D. B.37
$\overline{\mathcal{D}}(\mathbf{B})$	Collection of $\bar{g} = 1 - g$ with $g \in \mathcal{D}(\mathbf{B})$	D. B.4.2
ΔP_t	The jump given by $P_t - \lim_{s\uparrow t} P_s$	P. 48
Δ_L	Laplace operator	P. 19
$\mathbf{Exp}$	Markov kernel of the Exponential distribution	D. 2.0.1
F	Competition function	D. 1.2.1
$\mathcal{F}^\Phi$	σ-algebra containing the genealogical information	D. 2.3.4
$\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}$	Augmented version of the filtrations of $\mathbf{\Xi}^{\mathbb{W}}$	D. 2.5.2
$\mathcal{F}^{\mathbf{Q},Y,\mathbb{W}}$	Augmented version of the filtrations of $(\mathbf{Q}^{\mathbb{W}}, Y)$	D. 2.5.2
$\mathcal{F}^{\mathbf{\Xi},\mathbb{W},r}$	Augmented version of the filtrations of $\mathbf{\Xi}^{\mathbb{W},r}$	D. 2.5.2
$\mathcal{F}^{\mathbf{Q},\mathbb{W},m}$	Augmented version of the filtrations of $\mathbf{Q}^{\mathbb{W},m}$	D. 2.5.2
$\mathcal{F}^{\mathbf{\Xi},XZ}$	Augmented version of the filtration of $\mathbf{\Xi}^{XZ}$	D. 3.5.1
$\mathcal{F}^{\mathbf{\Xi},XZ,r}$	Augmented version of the filtration of $\mathbf{\Xi}^{XZ,r}$	D. 3.5.1
$\mathcal{F}^{\mathbf{Q},XZ,m}$	Augmented version of the filtration of $\mathbf{Q}^{XZ,m}$	D. 3.5.1
$\mathfrak{F}_{\varphi,f,\boldsymbol{g}}$	Test function of $B_{\mathbb{W}}$	D. E.2.4
γ	Projection $\gamma : E_X \to E_Y$	D. D.1.2
γ^*	Pullback associated with γ	D. D.1.2
$\gamma^{\mathbf{\Xi}}_E$	Map that generates $\mathbf{\Xi}^{\mathbb{W}}$	D. 1.18
$\gamma^{\mathbf{Q}}_E$	Map that generates $\mathbf{Q}^{\mathbb{W}}$	D. 2.42
$\gamma^{\mathbf{\Xi},r}_E$	Map that generates $\mathbf{\Xi}^{\mathbb{X},r}$	D. B.6.8
$\hat{g}^{\Delta,h}$	$\hat{g}^{\Delta,h}(\omega, x, z, p, s) := \hat{g}(x, z + h(\omega, x, p, s)) - \hat{g}(x, z)$	D. 5.2.1

G	Test function of $\mathbf{A}^o_{\mathbf{B}}$ or $\mathbf{A}^o_{B_X,h}$	D. 2.5.7 D. 4.2.1
h	Integrand, used to define $(Z_i)_{i=1}^{\infty}$, element of $\mathcal{L}^1_{loc}(\mathbf{M})$	E. (3.1)
$h[\cdot,\cdot]$	Integrand used to define the Cut-Out process	L. 7.1.2
H	Used to prove the exchangeability of $(X_i, Z_i, U_i)_{i=1}^{\infty}$	P. 3.3.2 P. 3.4.9 C. 3.4.10
$\overline{H}$	Extension of H	D.3.4.11
$\mathbf{Id}$	Identity map, i.e. $\mathbf{Id}(f) = f$ for all $f \in \mathbf{M}(E)$	L. A.2.6
L_f	Laplace functional defined on $\overline{\mathcal{N}}(E \times [0,\infty))$	D. B.2.5
$\hat{L}_{\hat{f}}$	Laplace functional defined on $\mathcal{M}(E)$	D. B.1.3
∇	The Nabla operator	P. 38
$\tilde{\mathfrak{P}}(E,\mathcal{F})$	Predictable σ-algebra of $\tilde{\Omega} \times E \times [0,\infty)$ based on $\mathcal{F}$	D. 7.1.1
P_t	Semigroup of the BSMP or of W depending on the context	S. A.2,E.1
P^x	Path law of the BSMP or of W depending on the context	S. A.2,E.1
$\mathfrak{P}_t$	Semigroup of the path-valued process	S. 2.4,E.2
$\mathfrak{P}^x$	Path law of the path-valued process	S. 2.4,E.2
$\mathbf{PPP}_E$	Distribution of a Poisson point process	D. 1.1.3
$\mathbf{PPP}^*_E$	Pull-back of functions associated with $\mathbf{PPP}_E$	L. B.3.7
$\pi_{\mathfrak{N}}$	Evaluation functions	D. 2.4.15
$\pi^f_{\mathfrak{X},\mathfrak{N}}$	Evaluation functions	L. 2.4.17
$\pi_{\mathfrak{L}}$	Evaluation functions	L. 2.4.14
$\pi_X, \pi_{\mathfrak{X}}$	Evaluation functions	L. 2.4.14
$\pi_{\mathfrak{X}}$	Evaluation functions	L. 2.4.14
Ψ	Maps $[h] \in \bar{\mathcal{S}}(\mathcal{F}^{\Xi,\mathbb{W}}) \subset L^1(\mathbf{M})$ to $[H] \in L^1(\mathbf{C}_\tau)$	L. 3.4.8
Ψ_τ	Maps $[h] \in L^1(\mathbf{M})$ to $[H] \in L^1(\mathbf{C}_\tau)$	P. 3.4.9
$(R_\lambda, \lambda > 0)$	Resolvent	D. A.2.5
$(\varrho_n, n \in \mathbb{N})$	Mollifier based on the heat kernel	E. (1.22)
$\varrho(x,\eta)$	Limit of $(\varrho_n, n \in \mathbb{N})$	E. (1.23)
$\mathbf{Uni}^r_E$	Markov kernel	D. 1.1.1

$\mathbf{Uni}_E^{r\,*}$	Pull-back of functions based on $\mathbf{Uni}_E^r$	D. B.6.5
Υ	Flow associated with the ODE $\dot{u} = au^2 - bu$	L. 2.2.5 L. 2.2.6

Miscellaneous

a	Branching rate, $a > 0$, fixed in Chap. 2 to 7	E. (1.1)
b	Drift, $b \in \mathbb{R}$, fixed in Chap. 2 to 7	E. (1.1)
BSMP	Borel strong Markov process	S. A.2
$\mathbf{BPS}(\dots)$	Branching particle system	S. B.4
$\mathbb{B}(E)$	Borel algebra of the Polish space E.	P. 22
b.p.	boundedly and pointwise convergence	E. (A.1)
$\mathbf{Comp}(\dots)$	Competition model	D. 1.2.1
$\mathbf{C}_\tau$	Measure over $C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D}$	D. 3.4.7
$\mathbf{Cut\text{-}Out}(\dots)$	The Cut-Out process	D. 7.1.3
$\mathbf{DW}(\dots)$	Dawson-Watanabe superprocess	D. B.1.1
$\mathfrak{L}(P\|Q)$	The distribution of P conditioned on Q.	E. (1.12)
$d_\Phi(i,j,t)$	The genealogical distance between i and j at time t	D. 2.3.2
$d_{\mathbb{D},E}$	Metric of the Sokorohod space $\mathbb{D}([0,\infty), E)$	R. 2.4.2
$d_{\widehat{\mathbb{D}},E}$	Metric of $\widehat{\mathbb{D}}([0,\infty), E)$	D. 2.4.1
$\mathbf{FE}$	Forward equation	D. D.1.6
G_i, G_i^{Bx}	Abbreviations used in Chapter 4	D. 4.6.1
$G_i^{\partial_u}, G_{j\downarrow i}$	Abbreviations used in Chapter 4	D. 4.6.1
$G_i^{\Delta h}$	Abbreviation used in Chapter 4	D. 4.6.1
$\mathbf{I}_0[h]$	$((X_i, Z_i, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ}, \mathbf{Q}^{XZ}) = \mathbb{I}_0[h]$	D. 3.5.1
$\mathbf{I}[h]$	Similar to $\mathbf{I}_0[h]$, but $\boldsymbol{\Xi}^{XZ}$ is replaced by its cont.mod.	D. 6.2.6
$\mathbf{KR}(\dots)$	Kurtz-Rodrigues representation	D. B.2.12
λ_b, λ_d	Birth and death rate of the Branching particle system.	P. 17
$\mathbf{MP}(A, \Theta_0)$	Martingale problem of A with initial dist. Θ_0	D. A.1.1

$\mathbf{M}$	Measure over $\Omega \times \mathbb{R}^d \times [0,\infty) \times [0,\infty)$	D. 3.1.2
$\mathbf{M}^{\Xi}$	Measure over $\Omega \times \mathbb{R}^d \times [0,\infty) \times [0,\infty)$	D. 4.1.1
Φ	Genealogical map	D. 2.3.1
$\mathfrak{P}(\mathcal{F}^{\mathbf{\Xi},\mathbb{W}})$	Predictable σ-algebra	D. 3.1.1
R.P.C.	Rogers-Pitman correspondence, $\alpha(y, \gamma^{-1}(y)) = 1$	P. 201
$span(V)$	Linear span of the set V	P. 58
$(\tau^Y_{\widehat{m}}, \widehat{m} \in \mathbb{N})$	$\tau^Y_{\widehat{m}} := \inf\{t \geq 0 : Y_t \geq \widehat{m}\}$	E. (4.7)
$\mathfrak{T}$	Collection of all finite $\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}$-stopping times	P. 84
$(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$	Loc. seq. of the semi-martingales from C. 4	E. (4.8)
$\mathcal{T}_{EX}$	Extinction time of $\mathbf{\Xi}^{\mathbb{W}}$	D. 2.2.3
τ_{EX}	Explosion time of the solution of $\dot{u} = au^2 - bu$	L. 2.2.5 L. 2.2.6
$\mathcal{T}^1_{\infty}$	Explosion time of the level process U_i	P. 2.2.2
$\|\|\| \cdot \|\|\|_{\mathbf{M}}$	L^1-norm based on the measure $\mathbf{M}$	D. 3.1.3
$\|\|\| \cdot \|\|\|_{\mathbf{C}_\tau}$	L^1-norm based on the measure $\mathbf{C}_\tau$	D. 3.4.7
$\mu \leq \nu$	For all $f \in C^+_b(E)$ holds $\mu(f) \leq \nu(f)$	D. 7.1.5

Chapter 1

Introduction

Particle models and their high-density limits play a big role in the mathematical description of biological populations. They are deployed to understand how evolutionary forces like mutation, selection and competition shape the genetic patterns observed in Nature. Due to their inherent stochastic component the study of these forces have been proven to be a fertile ground for probability theory, as it can been seen in the great variety of stochastic processes like the measure-valued branching Markov processes, the Fleming-Viot processes and the coalescent processes. While the mathematical theory of genetics is a rich branch of Mathematics, which is worthwhile to be studied for its own sake, many results are restricted to situations without interaction between the particles like competition or symbiotic branching. This gave rise to an increased research activity, especially in the field of genealogical descriptions. For example, Le, Pardoux and Wakolbinger extended the classical Ray-Knight theorem to the case of the logistic Feller diffusion, see [33]. Unfortunately it is not clear how this result can be modified to incorporate spatial models. Glöde considered genealogical trees in the context of autocatalytic branching processes, see [17], and Kielisch worked on the Kurtz-Rodrigues representation similar to us, but his focus was on symbiotic branching models, see [23]. A broad source of different genealogical constructions can be also found in the recent paper of Etheridge and Kurtz, see [13].
Before we present the competitive models we deem it necessary to discuss their non-competitive counterpart, the Dawson-Watanabe superprocess. We assume that E is a Polish space and we denote by $\mathbf{M}(E)$ the space of Borel measurable functions $f : E \to \mathbb{R}$, by $\mathrm{B}(E)$ the subset of bounded functions and by $C(E)$ the collection of continuous functions. We set $C_b(E) := C(E) \cap \mathrm{B}(E)$ and write $C_b^+(E)$ for the subset of non-negative functions, i.e. $f : E \to [0, \infty)$. When E is locally compact, then $C_c(E)$ contains the functions in $C(E)$ with compact support. The DW-superprocess is the high-density limit of time-continuous Galton-Watson processes with spatial motion, which we call Branching particle systems from now on. For the spatial motion we fix an abstract Markov process $\mathbb{X}$ with E as state space. The Branching particle system is a finite collection of particles that move through E like independent copies of $\mathbb{X}$. Each particle dies after an exponentially distributed time with rate $\lambda_d > 0$, but until its death it gives birth to new particles with rate $\lambda_b > 0$, which inherit at birth the current spatial position of their parent. There is no interaction between the living particles, each particle operates independently from the rest of the population.
Let us now choose two sequences $(\lambda_b^r, r > 0), (\lambda_d^r, r > 0)$ and two constants $a > 0, b \in \mathbb{R}$ with

$$a = \lim_{r\to\infty} \frac{\lambda_b^r + \lambda_d^r}{2r}, \qquad b = \lim_{r\to\infty} \lambda_b^r - \lambda_d^r. \tag{1.1}$$

We call a the **branching rate** and b the **drift**. Based on the work of Watanabe in [46], the corresponding Branching particle systems converge under an appropriate rescaling to a measure-valued branching Markov process $\mathbf{\Xi}^{\mathbb{X}}$, called the Dawson-Watanabe superprocess, which is a stochastic process with continuous paths in $\mathcal{M}_f(E)$, the space of finite measures over E. The Dawson-Watanabe superprocess is similar to the Brownian motion an universal limit, in the sense that a broad class of processes, not only the above described Branching particle systems, are converging to $\mathbf{\Xi}^{\mathbb{X}}$. Naturally this caught the attention of many mathematicians with the result that the literature about the Dawson-Watanabe superprocess is very broad and rich. A good introduction to the Dawson-Watanabe superprocess is given by Etheridge, see [12], and Perkins, see [40]. If one is interested in more general measure-valued branching Markov processes, we recommend the Saint Flour lecture notes of Dawson, see [9], and the book of Li, see [34].
During this thesis we use some of the deep insights of our precursors, most notability hereby are Perkins' stochastic calculus and the Kurtz-Rodrigues representation, but at the moment we will state only some basic facts needed for the later discussion of the competitive model. Let us write $\mu(\hat{g})$ for the integral $\int \hat{g}(x)\,\mu(dx)$, whenever the integral is well-defined for the function $\hat{g}$ and the measure μ. Denoting by $\mathbf{B}$ the generator of $\mathbb{X}$ and by $\mathcal{D}(\mathbf{B})$ its domain the DW-superprocess $\mathbf{\Xi}^{\mathbb{X}}$ is a Markov process characterized by the property that for all $\hat{g} \in \mathcal{D}(\mathbf{B})$ the process given by

$$M_{\hat{g}}(t) := \mathbf{\Xi}^{\mathbb{X}}_t(\hat{g}) - \mathbf{\Xi}^{\mathbb{X}}_0(\hat{g}) - \int_0^t \mathbf{\Xi}^{\mathbb{X}}_s(\mathbf{B}(\hat{g})) + b\mathbf{\Xi}^{\mathbb{X}}_s(\hat{g})\,ds, \quad t \geq 0, \tag{1.2}$$

is a continuous local $\mathcal{F}^{\mathbf{\Xi}}$-martingale with respect to the filtration of $\mathbf{\Xi}^{\mathbb{X}}$ with quadratic variation:

$$\langle M_{\hat{g}} \rangle_t = \int_0^t 2a\mathbf{\Xi}^{\mathbb{X}}_s(\hat{g}^2)\,ds, \quad t \geq 0. \tag{1.3}$$

This implies that the full-mass process $Y := \mathbf{\Xi}^{\mathbb{X}}(E)$ is a Feller diffusion with drift b and branching rate a, which can be obtained as a solution of the stochastic differential equation given by

$$dY_t = bY_t\,dt + \sqrt{2aY_t}\,d\hat{W}_t, \quad t \geq 0, \tag{1.4}$$

where $\hat{W}$ is a Brownian motion that lives on an extended probability space. If the spatial motion is given by a Brownian motion, then the corresponding Dawson-Watanabe superprocess, which is denoted by $\mathbf{\Xi}^X$, is called a Superbrownian motion (we write X instead of $\mathbb{X}$ for notational consistency with later chapters). If additionally $E = \mathbb{R}$, then $\mathbf{\Xi}^X$ is almost surely for all $t > 0$ absolutely continuous with respect to the Lebesgue measure and its density $\varphi^{\mathbf{\Xi}}$ is the unique solution of the stochastic partial differential equation given by

$$\varphi^{\mathbf{\Xi}}_t(x) = \frac{1}{2}\partial_x^2\varphi^{\mathbf{\Xi}}_t(x) + b\varphi^{\mathbf{\Xi}}_t(x) + \sqrt{2a\varphi^{\mathbf{\Xi}}_t(x)}d\mathcal{W}(t,x), \quad t > 0, x \in \mathbb{R}, \tag{1.5}$$

with $\mathcal{W}$ being white noise over $[0,\infty) \times \mathbb{R}$ (for more details see Section 7.4). This interesting connection to the theory of SPDE's was proven by Konno and Shiga in [27] and independently by Reimers in [41].
Our models with competition are often obtained from the non-competitive models by adding in the case of particle models a new death rate or in the case of high-density limit a new negative drift term. The logistic Feller diffusion $\widehat{Y}$ is the simplest high-density model with competition and is the solution of the stochastic differential equation given by

$$d\widehat{Y}_t = b\widehat{Y}_t\,dt - c\widehat{Y}_t^2\,dt + \sqrt{2a\widehat{Y}_t}\,d\hat{W}_t, \quad t \geq 0, \tag{1.6}$$

with $a > 0, b > 0$ and $c > 0$ (Later we will just assume $b \in \mathbb{R}$, but to present the competition models found in the literature, we have to assume $b > 0$). The difference to the Feller diffusion, see (1.4), is the negative quadratic drift $c\widehat{Y}_t^2$. This term arises, if we think of the logistic Feller diffusion as the high-density limit of a Branching particle system with no space and the death rate for each particle is raised by a term that is linear with respect to the population size. If we now look at the total number of particles, this turns into a negative quadratic term. A more advanced model with the spatial space being $E = \mathbb{R}$ has been considered by Müller and Tribe in a series of papers [36], [44] and [35]. They showed in [35] that the logistic counterpart of (1.5) given by

$$\widehat{\varphi}_t(x) = \frac{1}{6}\partial_x^2\widehat{\varphi}_t(x) + b\widehat{\varphi}_t(x) - \widehat{\varphi}_t^2(x) + \sqrt{\widehat{\varphi}_t(x)}\, d\mathcal{W}(t,x), \quad t > 0, x \in \mathbb{R}. \tag{1.7}$$

can be obtained as the rescaling limit of a sequence of long range voter models. The competition in (1.7) is local in the sense that only individuals at the same spatial position are in competition with each other. This is different in the Bolker-Pacala models introduced by Etheridge in [11], who was inspired by the work of Bolker and Pacala in [6] and [7]. If Δ_L denotes the Laplace operator in $\mathbb{R}^d$ and $\hat{\kappa} : [0,\infty) \to [0,\infty)$ is a bounded, continuous, decreasing function, then the Bolker Pacala model $\widehat{\mathbf{\Xi}}^{BP}$ can be characterized by saying that for each $\hat{g} \in \mathcal{D}(\Delta_L)$ the process

$$\begin{aligned} \hat{M}_t :=& \widehat{\mathbf{\Xi}}_t^{BP}(\hat{g}) - \widehat{\mathbf{\Xi}}_0^{BP}(\hat{g}) - \int_0^t \widehat{\mathbf{\Xi}}_s^{BP}(b\hat{g} + \frac{1}{2}\Delta_L(\hat{g}))ds \\ &+ \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c\hat{\kappa}(||x-y||)\hat{g}(x)\widehat{\mathbf{\Xi}}_s^{BP}(dy)\widehat{\mathbf{\Xi}}_s^{BP}(dx)ds, \quad t \geq 0, \end{aligned} \tag{1.8}$$

where $||x||^2 = \sum_{i=1}^d |x_i|^2$, is a continuous martingale with quadratic variation

$$\langle\, \widehat{\mathbf{\Xi}}^{BP} \rangle_t = \int_0^t 2a\widehat{\mathbf{\Xi}}^{BP}(\hat{g}^2)ds.$$

Competitive models are much harder to analyze than their non-competitive counterparts, because the negative, non-linear drift term added in the competitive models breaks the branching property, which tell us for example in the context of the Dawson-Watanabe superprocess, that the sum of two independent DW-superprocesses forms again a DW-superprocess. To express the branching property more formally, let us denote by "$*$" the convolution operator. If $(Q_t, t \geq 0)$ is the family of transition kernels corresponding to a measure-valued Markov process, then we can say that the process admits the branching property, if it holds

$$Q_t(\mu_1 + \mu_2, \cdot\,) = Q_t(\mu_1, \cdot\,) * Q_t(\mu_2, \cdot\,). \quad t \geq 0, \mu_1, \mu_2 \in \mathcal{M}_f(E).$$

The branching property of the Dawson-Watanabe superprocess is the result of the fact that the particles in the approximating particle systems have no interactions with each other. For more details on the branching property see Section 2.1 in [34]
A further tool in the theory of DW-superprocesses is the fact that expectation of the Laplace functional, $v_{\hat{g}}(t) := \mathbb{E}[\exp(\mathbf{\Xi}_t^{\mathbb{X}}(\text{-}\hat{g}))]$, satisfies a deterministic evolution equation. Luckily, in the case of the logistic Feller diffusions and the Mueller-Tribe SPDE, (1.7), this evolution equation can be replaced by a self-duality, because the competition is local. But the Bolker-Pacala models form non-local competitive models and hence they do not possess such a self-duality, which makes those even harder to analyze than the previous competitive models. We wish to build a Poisson representation for a class of competitive models that is broad enough to comprise the Müller-Tribe SPDE as well as the Bolker-Pacala models. We hope that those help to improve our understanding of these models. In Section 7.6, we sketch how such a representation could provide us with a new approach to study the extinction behavior of Bolker-Pacala models.

1.1 The Kurtz-Rodrigues Representation

In 2011 Kurtz and Rodrigues presented, see [32], a novel and unusual particle system with interesting properties, named now the Kurtz-Rodrigues representation. The KR-representation forms a Poisson representation for the Dawson-Watanabe superprocess and could be viewed as conceptual evolution of the previous lookdown-construction presented by Donnelly and Kurtz in [10] to which we refer as the Donnelly-Kurtz representation. There exists an interesting relationship between those two representation, which we explain in Remark 2.6.7. The Kurtz-Rodrigues representation is the basis for our Poisson representations, therefore we give here a more and less informal description of this model, more technical details can be found in the Appendix B.2.

As in the previous section, we fix an abstract Markov process $\mathbb{X}$ with state space E and generator $\mathbf{B}$, and we also fix three constants $a > 0, b \in \mathbb{R}$ and $r > 0$. Again a and b are named branching rate and drift, while r has the name "level cap". The Kurtz-Rodrigues representation $\boldsymbol{\xi}^{\mathbb{X},r}$ with level cap r is a stochastic process with values in $\mathcal{N}_f(E \times [0, r))$, the space of finite integer-valued measures over $E \times [0, r)$, whose atoms are interpreted as particles moving through the space $E \times [0, r)$. We call the first coordinate the "spatial position" and the second one the "level". The spatial positions behave as in the case of the previous discussed Branching particle systems, they evolve like independent copies of $\mathbb{X}$. The level follows the ordinary differential equation given by

$$\dot{u} = au^2 - bu. \tag{1.9}$$

As a consequence particles with a level below $\max\{b/a, 0\}$ are moving downwards, while the particles with level higher than $\max\{b/a, 0\}$ move upwards. For technical reasons we only consider level caps with $r \geq \max\{b/a, 0\}$. If a level hits r, the corresponding particle is considered dead and does not belong to $\boldsymbol{\xi}^{\mathbb{X},r}$ anymore. Until its demise each particle gives birth to new particles with rate $2a(r - u)$, where u is its current level. Each child is born at the spatial position of its parent and its level is uniformly distributed over (u, r) (recall u is the level of its parent).

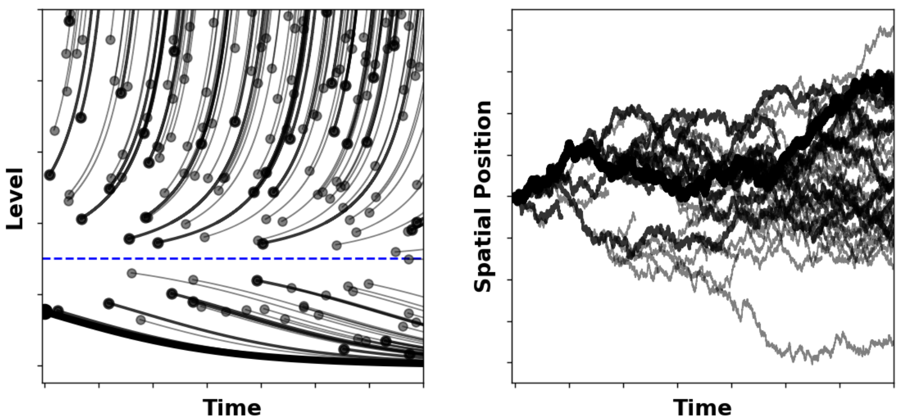

Figure 1.1: *The first three generations of a Kurtz-Rodrigues representation starting with one particle in generation one. The lines become thinner and more transparent with each generation. All particles have levels higher than the initial particle. The levels above b/a increase rapidly, levels below decrease.*

As in the case of the Dawson-Watanabe superprocess and the Bolker-Pacala models, we will give a characterization of the Kurtz-Representation as a martingale problem, but we will postpone many details to the appendix, see Section B.2. The martingale problem of the KR-representation is based on the Laplace functionals, hence, if $f : E \times [0, r) \to [0, \infty)$ is an element of $C^+(E \times [0, r))$, the set of non-negative continuous functions, the Laplace functional L_f is given by $\eta \mapsto \exp(-\eta(f))$, where $\eta \in \mathcal{N}_f(E \times [0, r))$ and $\eta(f) = \int f(x, u)\, \eta(x, u)$. If we write $g := \exp(-f)$, then L_f becomes

$$L_f(\boldsymbol{\xi}_t^{\mathbb{X},r}) = L_{\text{-}\log(g)}(\boldsymbol{\xi}_t^{\mathbb{X},r}) = \prod_{(x,u)\in \boldsymbol{\xi}_t^{\mathbb{X},r}} g(x, u),$$

the product on the right-hand side fits well with the interpretation that the atoms of $\boldsymbol{\xi}^{\mathbb{X},r}$ are particles. Now, we assume that $g \in \mathcal{D}(\mathbf{B})$ and $0 \leq g \leq 1$, the generator of $\boldsymbol{\xi}^{\mathbb{X},r}$ is informally given by

$$\begin{aligned}
\mathbf{A}^r_{\mathbf{B}}(L_{\text{-}\log(g)})(\eta) = {} & \exp(\text{-}\eta(\log(g))) \int_E \int_0^\infty \frac{\mathbf{B}(g)(x, u)}{g(x, u)}\ \eta(dx, du) \\
& + \exp(\text{-}\eta(\log(g))) \int_E \int_0^\infty \left(2a \int_u^\infty g(x, \tilde{u}) - 1\ d\tilde{u} \right) \eta(dx, du) \\
& + \exp(\text{-}\eta(\log(g))) \int_E \int_0^\infty \left[au^2 - bu \right] \frac{\partial_u g(x, u)}{g(x, u)}\ \eta(dx, du).
\end{aligned} \tag{1.10}$$

The dynamics of the levels in $\boldsymbol{\xi}^{\mathbb{X},r}$ may appear at first odd and unmotivated, but they are the central element of the Kurtz-Rodrigues representation. They have a very interesting property which is revealed, when we choose the right initial distribution for the levels. Therefore let us denote by $\mathcal{M}_1(\mathcal{N}_f(E \times [0, r)))$ the space of probability measures over $\mathcal{N}_f(E \times [0, r))$ and let us define:

Definition 1.1.1. *If E is a Polish space, the Markov kernel $\mathbf{Uni}^r_E : \mathcal{N}_f(E) \to \mathcal{M}_1(\mathcal{N}_f(E\times[0, r)))$ is defined by saying that $\mathbf{Uni}^r_E(\varrho)$ is for $\varrho = \sum_{x\in\varrho} \delta_x$ the distribution of the random measure $\eta := \sum_{x\in\varrho} \delta_{(x,U_x)}$, where $(U_x, x \in \varrho)$ is a collection of independent random variables uniformly distributed over $[0, r)$.*

Further, we write $\boldsymbol{\Xi}^{\mathbb{X},r}$ for the projection of $\boldsymbol{\xi}^{\mathbb{X},r}$ onto $\mathcal{M}_f(E)$, which means that

$$\boldsymbol{\Xi}^{\mathbb{X},r}_t := \sum_{(x,u)\in \boldsymbol{\xi}_t^{\mathbb{X},r}} \delta_x. \tag{1.11}$$

The dynamics of $\boldsymbol{\xi}^{\mathbb{X},r}$ show their special properties, if we choose as an initial distribution $\boldsymbol{\xi}_0^{\mathbb{X},r} \sim \mathbf{Uni}^r_E(\varrho)$ for some $\varrho \in \mathcal{N}_f(E)$. If so, the conditional distribution $\boldsymbol{\xi}_t^{\mathbb{X},r}$ based on the path of the projection $\boldsymbol{\Xi}^{\mathbb{X},r}$ up to time t is still $\mathbf{Uni}^r_E(\boldsymbol{\Xi}^{\mathbb{X},r}_t)$, i.e.

$$\mathfrak{L}(\boldsymbol{\xi}_t^{\mathbb{X},r} | \sigma(\boldsymbol{\Xi}^{\mathbb{X},r}_s, s \leq t)) = \mathbf{Uni}^r_E(\boldsymbol{\Xi}^{\mathbb{X},r}_t), \quad t \geq 0. \tag{1.12}$$

Less formally we can say that the path $(\boldsymbol{\Xi}^{\mathbb{X},r}_s, s \leq t)$ does not reveal any information about the levels. This effect becomes even more interesting by the fact that the projection $\boldsymbol{\Xi}^{\mathbb{X},r}$ is with respect to its own filtration a Branching particle system, where the particles move like $\mathbb{X}$, die with rate $ra - b$ and give birth to new particles with rate ra. Furthermore, the KR-representation is consistent, indeed if $\max\{b/a, 0\} \leq r_1 < r_2 < \infty$, and $\boldsymbol{\xi}^{\mathbb{X},r_2}$ is a KR-representation with level cap r_2, then its restriction $\boldsymbol{\xi}^{\mathbb{X},r_1}$ to $E \times [0, r_1)$, i.e.

$$\boldsymbol{\xi}_t^{\mathbb{X},r_1}(\Gamma) := \boldsymbol{\xi}_t^{\mathbb{X},r_2}(\Gamma \cap (E \times [0, r_1))), \quad \Gamma \in \mathbb{B}(E \times [0, \infty)), t \geq 0,$$

is again a KR-representation with level cap r_1 (with $\mathbb{B}(E\times[0,\infty))$ being the Borel algebra). This can be deduced from $\mathbf{A}^r_{\mathbf{B}}$ by observing that

$$\mathbf{A}^{r_1}_{\mathbf{B}}(L_{\text{-}\log(g)}) = \mathbf{A}^{r_2}_{\mathbf{B}}(L_{\text{-}\log(g)}),$$

whenever g has the property that $g(x,u)=1$ for $u \geq r_1$. We can make use of the consistency by defining a generator for a KR-representation $\boldsymbol{\xi}^{\mathbb{X}}$ with infinite level cap by setting

$$\mathbf{A}_{\mathbf{B}}(L_{\text{-}\log(g)}) = \lim_{r\to\infty} \mathbf{A}^r_{\mathbf{B}}(L_{\text{-}\log(g)})$$

for all g for which a $\tilde{r}>0$ exists with $g(x,u)=1$ for all $u\geq\tilde{r}$. The infinite KR-representation $\boldsymbol{\xi}^{\mathbb{X}}$ will be defined as a stochastic process with the following state space:

Definition 1.1.2. *If E is a Polish space, we denote by $\overline{\mathcal{N}}(E\times[0,\infty))$ the space of integer-valued measures ξ on $E\times[0,\infty)$ with the property: $\xi(E\times[0,r))<\infty, r>0$. We say that a sequence $(\xi_n)_{n=1}^\infty \subset \overline{\mathcal{N}}(E\times[0,\infty))$ is converging against an element $\xi\in\overline{\mathcal{N}}(E\times[0,\infty))$, when $\xi_n(g) \overset{n\to\infty}{\longrightarrow} \xi(g)$ for all $g\in C_b(E\times[0,\infty))$ with the property that there exists a $r\geq 0$ such that the support of g is contained in $E\times[0,r]$. We call this topology the mixed topology, because it is a mixture of the weak and the vague topology.*

The Kurtz-Rodrigues representation contains all the KR-representations with finite level cap due to the consistency property, which means, if we set

$$\boldsymbol{\xi}^{\mathbb{X},r}_t(\Gamma) := \boldsymbol{\xi}^{\mathbb{X}}_t(\Gamma\cap(E\times[0,r))), \quad \Gamma\in\mathbb{B}(E\times[0,\infty)), t\geq 0, r\geq\max\{b/a,0\}, \tag{1.13}$$

then we can obtain a sequence $(\boldsymbol{\xi}^{\mathbb{X},r}, r\geq\max\{b/a,0\})$ of KR-representations with finite level caps. Since the Kurtz-Rodrigues representations with infinite level cap is so important we will call it just the Kurtz-Rodrigues representation in the following chapters. The KR-representation with infinite level cap has an equivalent to (1.12).

Definition 1.1.3. *We define the Markov kernel $\mathbf{PPP}_E : \mathcal{M}_{lf}(E\times[0,\infty)) \to \mathcal{M}_1(\overline{\mathcal{N}}(E\times[0,\infty)))$ by saying that $\mathbf{PPP}_E(\overline{\mu})$ is the distribution of a Poisson point process with intensity measure $\overline{\mu}$, see Definition C.1.1.*

As in the case of a finite level cap we will define a projection. If $\boldsymbol{\Xi}^{\mathbb{X},r}$ is the projection of $\boldsymbol{\xi}^{\mathbb{X},r}$ on $\mathcal{M}_f(E)$ as in (1.11), then we define the projection of the KR-representation $\boldsymbol{\xi}^{\mathbb{X}}$ with infinite level cap on $\mathcal{M}_f(E)$ as

$$\boldsymbol{\Xi}^{\mathbb{X}}_t := \lim_{r\to\infty}\frac{1}{r}\boldsymbol{\Xi}^{\mathbb{X},r}_t = \lim_{r\to\infty}\frac{1}{r}\sum_{(x,u)\in\boldsymbol{\xi}^{\mathbb{X}}_t}\delta_x \mathbb{1}_{[0,r)}(u), \tag{1.14}$$

where the limit is taken in the weak-topology on $\mathcal{M}_f(E)$ and we map $\boldsymbol{\Xi}^{\mathbb{X}}$ to the nullmeasure $\mathbf{0}_E \notin \mathcal{M}_f(E)$, if the convergence does not hold. If we choose as initial distribution $\boldsymbol{\xi}^{\mathbb{X}}_0 \sim \mathbf{PPP}_E(\mu\otimes\ell eb[0,\infty))$ with $\mu\in\mathcal{M}_f(E)$ and $\ell eb[0,\infty)$ being the Lebesgue measure, then Theorem B.3.3 tells us that

$$\mathfrak{L}(\boldsymbol{\xi}^{\mathbb{X}}_t|\sigma(\boldsymbol{\Xi}^{\mathbb{X}}_s, s\leq t)) = \mathbf{PPP}_E(\boldsymbol{\Xi}^{\mathbb{X}}_t\otimes\ell eb[0,\infty)), \tag{1.15}$$

indeed the conditional distribution of $\boldsymbol{\xi}^{\mathbb{X}}_t$ based on the path of $\boldsymbol{\Xi}^{\mathbb{X}}$ up to time t, is the one of a Poisson point process with intensity measure $\boldsymbol{\Xi}^{\mathbb{X}}_t\otimes\ell eb[0,\infty)$. This means that the path $(\boldsymbol{\Xi}^{\mathbb{X}}_t, s\leq t)$ does not reveal any precise information about the levels. As a consequence the convergence in the definition of $\boldsymbol{\Xi}^{\mathbb{X}}$ holds for a fixed t almost surely, which follows immediately from (1.15). But most importantly, we can show that $\boldsymbol{\Xi}^{\mathbb{X}}$ is a Dawson-Watanabe superprocess with spatial motion $\mathbb{X}$, branching rate a and drift b (but we may have to replace $\boldsymbol{\Xi}^{\mathbb{X}}$ with a continuous modification).

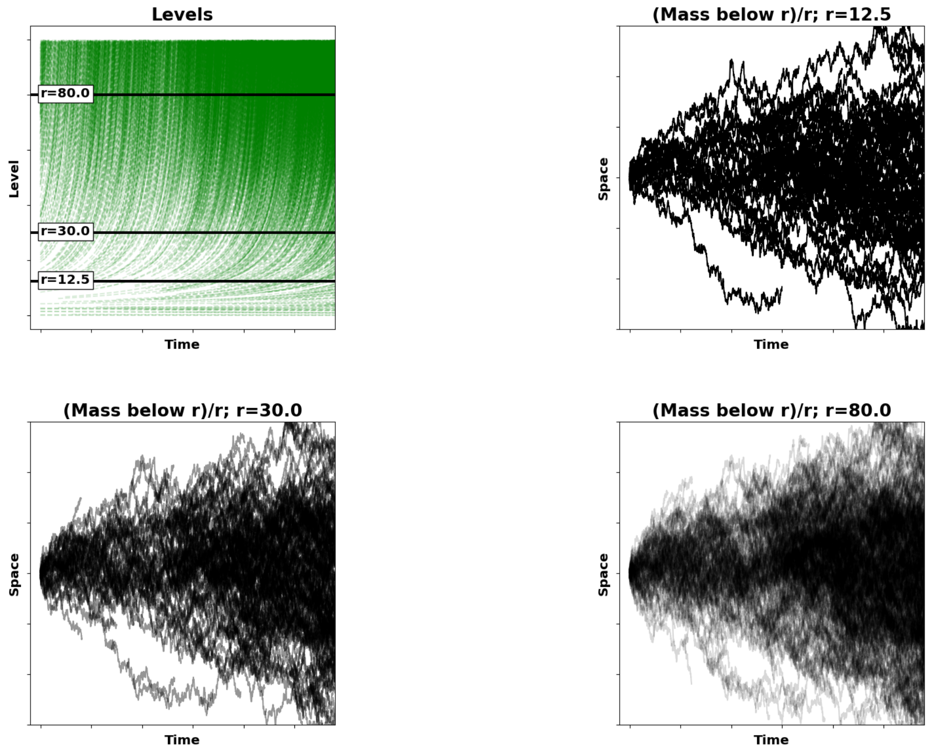

Figure 1.2: *In the upper left image we can find the realization of the levels of a KR-representation with infinite level cap, the remaining plots show the trajectory of* $\frac{1}{r}\Xi^{\mathbb{X},r}$ *defined as in* (1.13) *for* $r \in \{12.5, 30, 80\}$.

Since $\Xi^{\mathbb{X},r}$ was for each $r \geq \max\{b/a, 0\}$ a Branching particle system, the convergence (1.14) provides us with a variation of Watanabe's result, meaning that the DW-superprocess, $\Xi^{\mathbb{X}}$, is the rescaling limit of a sequence of Branching particle systems, $(\Xi^{\mathbb{X},r}, r \geq \max\{b/a, 0\})$. The consistency property of the KR-representation has the feature that we can interpret $\Xi^{\mathbb{X},r_1}$ as a subpopulation of $\Xi^{\mathbb{X},r_2}$, whenever $\max\{b/a, 0\} \leq r_1 < r_2 = \infty$ (this can not be achieved by combining Watanabe's convergence result with the Skorohod's representation theorem, see Theorem 1.8 in [14]).
The fact that $\Xi^{\mathbb{X}}$ is a DW-superprocess combined with (1.15) is the reason, why we call the Kurtz-Rodrigues representation a Poisson representation of the Dawson-Watanabe superprocess. The particles of $\boldsymbol{\xi}^{\mathbb{X}}$ die, when their level are converging to infinity, which happens in finite time due to the quadratic term in (1.9). Hereby we can observe that the parents always die after their children, which is an odd property, if we think of the particles in $\boldsymbol{\xi}^{\mathbb{X}}$ as individuals of a hypothetical population, therefore a more sensible interpretation is that the particles represent genealogies/families and the children of a particle are representations of subgenealogies/subfamilies. In this way it makes sense that the children, the subgenealogies, die before their parents, because the death of the main genealogy implies the death of all subgenealogies by definition. The level can be understood as an encoding of the longevity of the genealogy, particles with a lower level represent longer lasting genealogies.

The fact that the Kurtz-representation is a Poisson representation for the Dawson-Watanabe superprocess and also satisfies (1.12) may appear as a very unintuitive result, but it is a consequence of the so called Markov mapping theorem. Besides many technical assumptions the main ingredient for the Markov mapping theorem is the intertwining relationship between the generators of the two processes involved. Let us denote by $\mathbf{C_B}$ the generator of the DW-superprocess and by $\mathbf{PPP}_E^*$ the pull-back of function associated with $\mathbf{PPP}_E$, which is defined by mapping the function $F : \overline{\mathcal{N}}(E \times [0,\infty)) \to \mathbb{R}$ to the function $\hat{F} : \mathcal{M}_f(E) \to \mathbb{R}$ via

$$\hat{F}(\mu) := \int_{\overline{\mathcal{N}}(E\times[0,\infty))} F(\eta)\, \mathbf{PPP}_E(\mu \otimes \ell eb[0,\infty), d\eta), \quad \mu \in \mathcal{M}_f(E),$$

whenever the integral is well-defined for all $\mu \in \mathcal{M}_f(E)$. The operators $\mathbf{C_B}$ and $\mathbf{A_B}$ are intertwined in the following sense:

$$\mathbf{C_B} \circ \mathbf{PPP}_E^* = \mathbf{PPP}_E^* \circ \mathbf{A_B}$$

with "$\circ$" standing for the usual composition of operators. For more details see the appendices B.1, B.2 and B.3. For a proof of the Markov mapping theorem general enough to cover all our cases, see Appendix D.1.

The Kurtz-Rodrigues representation provides an intuitive understanding of the properties of the Dawson-Watanabe superprocess, which are hard to guess from looking at the martingale characterization given by (1.2) or when we think of the DW-superprocess as the high-density limit of the Branching particle systems. One of these properties which is particularly interesting for us is the decomposition of a supercritical DW-process ($b > 0$) into a subcritical DW-process ($b < 0$) with immigration generated by an immortal Branching particle system. This decomposition is obtained by dividing the particle population at the level boundary $u = b/a$. Particles with a level below will never die, they form a Branching particle system, which is a pure birth process with death rate $\lambda_d = 0$ and birth rate $\lambda_b = b$. The particles above b/a die fast and form a subcritical DW-process with drift $-b$, but the particles below b/a are generating constantly new particles above b/a resulting in an immigration term for the DW-superprocess. In Section 7.6 we are considering this decomposition for competing models. The above described decomposition is called the Evans-O'Connell backbone decomposition and was proved in the paper [15] from 1994. Back then, a deep understanding of the Dawson-Watanabe superprocess was necessary to realize this decomposition, but for us working with the Kurtz-Rodrigues representation it is a simple observation.

1.2 Main Theorem

In this section we present our main theorem, but for this we need some technical definitions. We start with the terms "competition model" and "Poisson representation".

Definition 1.2.1. *Let $\mathbf{B} : C_b(E) \supset \mathcal{D}(\mathbf{B}) \to C_b(E)$ be the generator of a well-posed martingale problem. Further we assume that $a > 0, b \in \mathbb{R}, \hat{\Theta}_0 \in \mathcal{M}_1(\mathcal{M}_f(E))$ and $F : E \times \mathcal{M}_f(E) \to [0, \infty]$ is a $\mathbb{B}(E \times \mathcal{M}_f(E))$-measurable function (for formal reasons we allow ∞). If $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and $\widehat{\boldsymbol{\Xi}} : \Omega \times [0, \infty) \to \mathcal{M}_f(E)$ is a $\mathcal{M}_f(E)$-valued process, then we write*

$$\widehat{\boldsymbol{\Xi}} \sim \mathbf{Comp}(\mathbf{B}, a, b, F, \hat{\Theta}_0),$$

when $\widehat{\boldsymbol{\Xi}}$ has the initial distribution $\hat{\Theta}_0$ and for all $\hat{g} \in \mathcal{D}(\mathbf{B})$ the process $M : \Omega \times [0, \infty) \to \mathbb{R}$ given by

$$\begin{aligned} \hat{M}(t) &:= \exp\Big(-\widehat{\boldsymbol{\Xi}}_t(\hat{g})\Big) - \exp\Big(-\widehat{\boldsymbol{\Xi}}_0(\hat{g})\Big) \\ &\quad - \int_0^t \Big[\widehat{\boldsymbol{\Xi}}_s(\mathbf{B}(\hat{g}) + b\hat{g}) - a\widehat{\boldsymbol{\Xi}}_s(\hat{g}^2) - \int_E \hat{g}(x) F(x, \widehat{\boldsymbol{\Xi}}_s)\widehat{\boldsymbol{\Xi}}_s(dx)\Big] \exp\big(-\widehat{\boldsymbol{\Xi}}_s(\hat{g})\big)\, ds, \quad t \geq 0, \end{aligned}$$

is a continuous local martingale with respect to the natural filtration of $\widehat{\boldsymbol{\Xi}}$.

If we apply Itô's formula, we can show that the above is equivalent to say that the process given by

$$\hat{M}(t) := \widehat{\boldsymbol{\Xi}}_t(\hat{g}) - \widehat{\boldsymbol{\Xi}}_0(\hat{g}) - \int_0^t \left[\widehat{\boldsymbol{\Xi}}_s(\mathbf{B}(\hat{g})) + b\widehat{\boldsymbol{\Xi}}_s(\hat{g}) - \int_E \hat{g}(x) F(x, \widehat{\boldsymbol{\Xi}}_s)\widehat{\boldsymbol{\Xi}}_s(dx)\right] ds, \tag{1.16}$$

is a continuous local martingale with respect to the natural filtration of $\widehat{\boldsymbol{\Xi}}$ with quadratic variation given by $\langle \widehat{\boldsymbol{\Xi}}(\hat{g}) \rangle_t = \int_0^t 2a\widehat{\boldsymbol{\Xi}}_s(\hat{g}^2) ds$.

Definition 1.2.2. *Let us assume that $\widehat{\boldsymbol{\xi}}$ is a $\overline{\mathcal{N}}(E \times [0, \infty))$-valued and $\widehat{\boldsymbol{\Xi}}$ is a $\mathcal{M}_f(E)$-valued process such that*

$$\mathfrak{L}(\widehat{\boldsymbol{\xi}}_t | \sigma(\widehat{\boldsymbol{\Xi}}_s, s \leq t)) = \mathbf{PPP}_E(\widehat{\boldsymbol{\Xi}}_t \otimes \ell eb[0, \infty)), \quad t \geq 0, \tag{1.17}$$

then we call $\widehat{\boldsymbol{\xi}}$ a Poisson representation of $\widehat{\boldsymbol{\Xi}}$. If the process $\widehat{\boldsymbol{\Xi}}$ satisfies

$$\widehat{\boldsymbol{\Xi}} \sim \mathbf{Comp}(\mathbf{B}, a, b, F, \hat{\Theta}_0),$$

then we call $\widehat{\boldsymbol{\xi}}$ a Poisson representation of $\mathbf{Comp}(\mathbf{B}, a, b, F, \hat{\Theta}_0)$.

Let us assume that E is a Polish space E. If $\boldsymbol{\xi}$ is a particle system given as a process with values in $\overline{\mathcal{N}}(E \times [0, \infty))$, then the "high-density limit" of $\boldsymbol{\xi}$, if it exists, is obtained by the map $\gamma_E^{\boldsymbol{\Xi}} : \overline{\mathcal{N}}(E \times [0, \infty)) \to \mathcal{M}_f(E)$ by

$$\gamma_E^{\boldsymbol{\Xi}}(\xi) := \begin{cases} \lim\limits_{r \to \infty} \frac{1}{r} \xi(\,\cdot \times [0, r]), & \text{if the limit exists in the weak top.} \\ \mathbf{0}_E, & \text{if the limit does not exist.} \end{cases} \tag{1.18}$$

In the case of a Poisson representation, we can conclude from (1.17) that

$$\widehat{\boldsymbol{\Xi}}_t = \gamma_E^{\boldsymbol{\Xi}}(\widehat{\boldsymbol{\xi}}_t) \text{ a.s.} \quad \forall t \geq 0.$$

There are two different classes of competitive models of our interest. We call these classes **(non-linear) Bolker Pacala models** and **(non-linear) singular interaction models**. These classes satisfy martingale problems which form a generalization of the martingale problems associated with the Bolker-Pacala models, see (1.8), and the Müller-Tribe SPDE, see (1.7). We call those non-linear, because the additional death rate experienced by each particle due to competition does not need to be linear with respect to $\widehat{\boldsymbol{\Xi}}$. Our competitive models will have particles, whose spatial motion is given by a Lévy process in $\mathbb{R}^d$ and, since $\mathbb{X}$ stands for an abstract Markov process, we switch our notation from $\mathbb{X}$ to X and $\mathbf{B}$ to B_X.

Assumptions 1.2.3. *We assume that X is a Lévy process on $\mathbb{R}^d$ with second moments, i.e. $\mathbb{E}[||X_t||^2] < \infty$ for all $t \geq 0$, when $X_0 = 0$ and with weak generator $B_X \subset C_b(\mathbb{R}^d) \times C_b(\mathbb{R}^d)$, see Definition A.2.3. We write $(B_X^\rho, B_X^{cov}, B_X^\eta)$ for the characteristic triple of X, where $B_X^\rho = (B_X^\rho(k))_{k=1}^d \in \mathbb{R}^d$, $B_X^{cov} = (B_X^{cov}(k,l))_{k,l=1}^d \in \mathbb{R}^{d\times d}$ being a symmetric, positive semidefinite matrix and $B_X^\eta \in \mathcal{M}(\mathbb{R}^d)$ being a Lévy measure, indeed $\int 1 \wedge ||x||^2 \, B^\eta(dx) < \infty$ and $B_X^\eta(\{0\}) = 0$. The assumption that X has second moments translates to*

$$\int_{\mathbb{R}^d} ||x||^2 \, B_X^\eta(dx) < \infty, \tag{1.19}$$

where $||x||^2 := \sum_{k=1}^d |x_{[k]}|^2$ for $x \in \mathbb{R}^d$, see Theorem 2.5.2. in [2].

We believe that most of the results can be extended to the case, where X is a more general Markov process in a locally compact space, but the above assumptions simplify many notations and the technically complexity of some proofs.

Definition 1.2.4 (Non-linear Bolker-Pacala Models)**.**
We say that the competitive model $\mathbf{Comp}(B_X, a, b, F, \hat{\Theta}_0)$ is a non-linear Bolker-Pacala model, if the competition function F can be written as

$$F(x,\mu) = \hat{F}(x, \pi(x,\mu)),$$

where $\hat{F} : \mathbb{R}^d \times [0,\infty) \to [0,\infty)$ and $\pi : \mathbb{R}^d \times \mathcal{M}_f(\mathbb{R}^d) \to [0,\infty)$ satisfy the following conditions:

1. *The function $\hat{F}$ is continuous and there exists for each $R > 0$ a constant $K_R > 0$ (which depends on R) such that the restriction $\hat{F}_R : \mathbb{R}^d \times [0,R] \to [0,\infty)$ of $\hat{F}$ on $\mathbb{R}^d \times [0,R]$ is bounded and the function $y \mapsto \hat{F}_R(x,y)$ is for each $x \in \mathbb{R}^d$ non-decreasing and Lipschitz continuous with Lipschitz constant K_R.*
2. *The function π is given by $\pi(x,\mu) := \int_{\mathbb{R}^d} \kappa(x,y)\,\mu(dy)$, $x \in \mathbb{R}^d, \mu \in \mathcal{M}_f(\mathbb{R}^d)$, where $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is given by $\kappa(x,y) = \hat{\kappa}(||x-y||)$ for a bounded, continuous decreasing function $\hat{\kappa} : [0,\infty) \to [0,\infty)$.*

Competition models with singular interactions will form a generalization of (1.7), they will only be well-defined if $\widehat{\boldsymbol{\Xi}}$ from Definition 1.2.1 is almost surely absolutely continuous with respect to the Lebesgue measure. For this reason we will only consider the space $E = \mathbb{R}$ and $B_X = \frac{1}{2}\partial_x^2$, indeed the spatial motion X is a Brownian motion. Let us repeat some basic details: A measure $\mu \in \mathcal{M}_f(\mathbb{R})$ is called absolutely continuous with respect to Lebesgue measure, if

$$\ell eb(\Gamma) = 0 \Rightarrow \mu(\Gamma) = 0, \quad \forall \Gamma \in \mathbb{B}(\mathbb{R}),$$

where ℓeb is the Lebesgue measure on $\mathbb{R}$. The theorem of Radon-Nikodỳm tells us that in this case, there exists a Borel measurable function $\varphi : \mathbb{R} \to [0,\infty)$ with the property:

$$\mu(\Gamma) = \int_\Gamma \varphi(x)\,dx, \quad \Gamma \in \mathbb{B}(\mathbb{R}), \tag{1.20}$$

or equivalently it holds for all non-negative Borel measurable functions $\hat{g} : \mathbb{R} \to [0, \infty)$

$$\mu(\hat{g}) = \int \hat{g}(x)\varphi(x)\,dx. \tag{1.21}$$

Of course the Lebesgue density φ is not unique, indeed changing the value of φ at countably many points will not change the correctness of (1.20) or (1.21), but φ is unique up to sets with Lebesgue measure 0. So for any two functions φ_1 and φ_2 satisfying (1.20) or (1.21) it holds $\ell eb(\tilde{\Gamma}) = 0$, where

$$\tilde{\Gamma} := \{x \in \mathbb{R} : \varphi_1(x) \neq \varphi_2(x)\}.$$

For our purposes it is necessary to express the density φ as a functional of μ. Therefore we define for $n \in \mathbb{N}$ the functional $\varrho_n : \mathbb{R} \times \mathcal{M}_f(\mathbb{R}) \to [0, \infty)$ by

$$\varrho_n(x, \eta) := \int_{\mathbb{R}} \frac{n}{\sqrt{2\pi}} \exp(n^2|x - y|^2/2)\ \eta(dy), \quad x \in \mathbb{R}, \tag{1.22}$$

and then we define $\varrho : \mathbb{R} \times \mathcal{M}_f(\mathbb{R}) \to [0, \infty]$ as

$$\varrho(x, \eta) := \liminf_{n\to\infty} \varrho_n(x, \eta). \tag{1.23}$$

If μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, then it holds

$$\ell eb\left(\{x \in \mathbb{R} : \varrho(x, \mu) = \infty\}\right) = 0$$

and $\widehat{\varphi} : \mathbb{R} \to [0, \infty]$ with $\widehat{\varphi}(x) = \varrho(x, \mu)$ is a Lebesgue density for μ, indeed (1.20) and (1.21) are satisfied by $\widehat{\varphi}$. In the case, where μ admits a continuous Lebesgue density φ with compact support, the convergence of $(\varrho_n(\cdot, \mu), n \in \mathbb{N})$ against $\varrho(\cdot, \mu)$ is even uniform, see the approximation theorem on Page 321 in [26].

Definition 1.2.5 (Non-linear Singular Interaction Models)**.**
We say that the competitive model $\mathbf{Comp}(B_X, a, b, F, \hat{\Theta}_0)$ *is a non-linear singular interaction models, if* $\mathbb{R}^d = \mathbb{R}$, $B_X = \frac{1}{2}\partial_x^2$ *is the generator of the one-dimensional Brownian motion and*

$$F(x, \mu) = \hat{F}(x, \varrho(x, \mu)), \tag{1.24}$$

where $\hat{F} : \mathbb{R} \times [0, \infty] \to [0, \infty)$ *satisfies the same conditions as in the case of the non-linear Bolker-Pacala Models,* $\varrho : \mathbb{R} \times \mathcal{M}_f(\mathbb{R}) \to [0, \infty]$ *is given by* (1.23) *and we use the convention that* $\hat{F}(x, \varrho(x, \mu)) = \lim_{y\to\infty} \hat{F}(x, y)$ *for the case, where* $\varrho(x, \mu) = \infty$*. Further we assume that the initial distribution* $\hat{\Theta}_0$ *has been chosen such that* $\widehat{\boldsymbol{\Xi}}_0$ *is absolutely continuous with respect to Lebesgue measure, more precisely we assume the existence of a random variable* $\widehat{\varphi}_0$ *with values in* $C_0(\mathbb{R})$*, the space of continuous compactly supported functions, such that*

$$\mathbb{P}\left[\widehat{\boldsymbol{\Xi}}_0(\hat{g}) = \int_{\mathbb{R}} \hat{g}(x)\widehat{\varphi}_0(x)\,ds,\ \forall \hat{g} \in C_b(\mathbb{R})\right] = 1.$$

To the best of our knowledge, it is not known, whether $\mathbf{Comp}(\mathbf{B}, a, b, F, \hat{\Theta}_0)$ is a well-posed martingale problem for a general non-linear Bolker-Pacala model or a general non-linear singular interaction model. But in the case of a (linear) Bolker-Pacala model the martingale problem (1.8) has indeed an unique solution, which can be proven with the Dawson-Girsanov transformation, see Theorem 7.9 in [12]. We get to know the Dawson-Girsanov transformation in Appendix B.7. A little bit more complicated is the case of the SPDE (1.7), the (linear) singular interaction model. In [44] Tribe considered a broad class of logistic SPDEs containing (1.7) and he showed in Theorem 2.2 of the same paper that these SPDEs have unique solutions, in the sense, that all solutions have the same law. But this does not necessarily imply the uniqueness of the corresponding martingale problems, because for this, one needs to show that each solution of the martingale problem implies a solution of the SPDE. Our main theorem is simply stated that we can construct Poisson representations for both classes of competitive models.

Theorem 1.2.6 (Main theorem)**.** *If* $\mathbf{Comp}(B_X, a, b, F, \hat{\Theta}_0)$ *is non-linear Bolker-Pacala model from Definition 1.2.4 or a non-linear singular interaction model from Definition 1.2.5 and X is a Lévy process with second moments, then there exists a Poisson representation for the competitive model* $\mathbf{Comp}(B_X, a, b, F, \hat{\Theta}_0)$.

Proof. See the end of Section 7.5. □

If $\mathbf{A}_{\mathbf{B},F}$ stands for the generator of our modified KR-representation, then $\mathbf{A}_{\mathbf{B},F}$ is given for $g \in \mathcal{D}(B_X)$ and $f := \text{-}\log(g)$ by

$$\begin{aligned}\mathbf{A}_{\mathbf{B},F}(L_f)(\eta) = {} & \exp(\text{-}\eta(f)) \int_{\mathbb{R}^d} \int_0^\infty \frac{B(g)(x,u)}{g(x,u)} \eta(dx, du) \\ & + \exp(\text{-}\eta(f)) \int_{\mathbb{R}^d} \int_0^\infty (au^2 - bu) \frac{\partial_u(g)(x,u)}{g(x,u)} \eta(dx, du) \\ & + \exp(\text{-}\eta(f)) \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} \int_u^\infty 2a[g(x,v) - 1]\, dv \eta(dx, du) \\ & + \exp(\text{-}\eta(f)) \int_{\mathbb{R}^d} \int_0^\infty F(x, \gamma^{\mathbf{\Xi}}_{\mathbb{R}^d}(\eta)) \frac{1 - g(x,u)}{g(x,u)} \eta(dx, du).\end{aligned}$$

The appearance of $\gamma^{\mathbf{\Xi}}_{\mathbb{R}^d}(\eta)$ in the last line makes the martingale problem of $\mathbf{A}_{\mathbf{B},F}$ very hard to work with. We can show that our Poisson representations form a solution of $\mathbf{MP}(\mathbf{A}_{\mathbf{B},F})$, but uniqueness of the martingale problem seems to be out of reach for the moment.

1.3 Summary

Our main idea, similar to an approach of Evans and Perkins used in a different context, is to "cut out" the Poisson representation $\widehat{\boldsymbol{\xi}}$ of the competitive model from a Kurtz-Rodrigues representation $\boldsymbol{\xi}^X$ of the Dawson-Watanabe process denoted by $\mathbf{\Xi}^X$. Based on Perkins' stochastic calculus we identify a random subset of particles forming the new process $\widehat{\boldsymbol{\xi}}$ such that its high-density limit $\widehat{\mathbf{\Xi}} := \gamma^{\mathbf{\Xi}}_{\mathbb{R}^d}(\boldsymbol{\xi})$ becomes a solution of the martingale problem $\mathbf{Comp}(\mathbf{B}, a, b, F, \hat{\Theta}_0)$, see Definition 1.2.1. This "cutting" should also not be a mere thinning of $\mathbf{\Xi}^X$, but it should also respect the genealogy of the particles.

We assume that $(X_i, U_i)_{i=1}^\infty$ is the collection of processes with values in $\mathbb{R}^d$ and $[0, \infty)$, representing the atoms of $\boldsymbol{\xi}^X$. Instead of atoms we will often say particles in the following, but note that (X_i, U_i) represents not necessarily the same particle for all time points, for more details see Chapter 2. The intuitive idea is to endow each particle $(X_i(t), U_i(t))$, $i \in \mathbb{N}$, with an additional label $Z_i(t)$ that has values in $\mathbb{R}$ and works as death marker, meaning that $Z_i(t) = 0$ indicates that the particle is still "alive" in $\widehat{\boldsymbol{\xi}}$ and $\boldsymbol{\xi}^X$, while we think of the particles as a "ghost particle" which has died due to competition in $\widehat{\boldsymbol{\xi}}$, when $Z_i(t) > 0$, but the particle still lives in $\boldsymbol{\xi}^X$, the original non-competitive KR-representation. These markers should be inherited by the children from their parents or else $\widehat{\boldsymbol{\xi}}$ will not have a sensible genealogy, because a "ghost particle" already dead in $\widehat{\boldsymbol{\xi}}$ could give birth to new particles in $\widehat{\boldsymbol{\xi}}$, which should be prevented.

Let us imagine for the moment that we have succeeded in defining $(Z_i)_{i=1}^{\infty}$ of particles $(X_i, U_i, Z_i)_{i=1}^{\infty}$ in a way that ensures the existence of the high-density/intensity process:

$$\boldsymbol{\Xi}_t^{XZ} = \lim_{r\to\infty} \frac{1}{r} \sum_{i=1}^{\infty} \delta_{(X_i(t), Z_i(t))} \mathbb{1}_{[0,r)}(U_i(t)) \quad t \geq 0. \tag{1.25}$$

The solution $\widehat{\boldsymbol{\Xi}} = (\widehat{\boldsymbol{\Xi}}_t, t \geq 0)$ of the martingale problem $\mathbf{Comp}(\mathbf{B}, a, b, F, \hat{\Theta}_0)$ would then be given by

$$\widehat{\boldsymbol{\Xi}}_t = \lim_{r\to\infty} \frac{1}{r} \sum_{i=1}^{\infty} \delta_{X_i(t)} \mathbb{1}_{[0,r)}(U_i(t)) \mathbb{1}_{\{0\}}(Z_i(t)), \quad t \geq 0, \tag{1.26}$$

or equivalently by

$$\widehat{\boldsymbol{\Xi}}_t(\Gamma) = \boldsymbol{\Xi}_t^{XZ}(\,\cdot\, \times \{0\}), \quad t \geq 0, \Gamma \in \mathbb{B}(\mathbb{R}^d).$$

But the process $\widehat{\boldsymbol{\xi}}$ is only a solution of the martingale problem $\mathbf{Comp}(\mathbf{B}, a, b, F, \hat{\Theta}_0)$, if we can arrange the dynamics of the label processes $(Z_i)_{i=1}^{\infty}$ in such a way that given $Z_i(t\text{-}) = 0$ at the moment t, the process Z_i will jump to $Z_i(t\text{-}) + 1$ in the time interval $[t, t + dt)$ with probability proportional to $dt \times F\big(X_i(t\text{-}), \hat{\boldsymbol{\Xi}}_t\big)$, independently for all $i \in \mathbb{N}$. Or in other words the process $\widehat{\boldsymbol{\xi}}$ has to be a solution of the martingale problem of the operator $\mathbf{A}_{\mathbf{B},F}$ defined at the end of Section 1.2.
Assuming that (1.26) exists, a naive approach for the construction of $(Z_i)_{i=1}^{\infty}$ is to assume that we have an independent family of Poisson processes $(\tilde{N}_i)_{i=1}^{\infty}$ on $[0, \infty) \times [0, \infty)$ with Lebesgue measure as intensity. We could use $(\tilde{N}_i)_{i=1}^{\infty}$ to define $(Z_i)_{i=1}^{\infty}$ by

$$Z_i(t) = \int_0^t \int_0^{\infty} \mathbb{1}_{[0, F(X_i(s), \hat{\boldsymbol{\Xi}}_s)]}(p)\, \tilde{N}_i(dp, ds), \quad t \geq 0, \tag{1.27}$$

(note that we can write $F(X_i(s), \hat{\boldsymbol{\Xi}}_s)$ instead of $F(X_i(s\text{-}), \hat{\boldsymbol{\Xi}}_{s\text{-}})$ in the integral because $\hat{\boldsymbol{\Xi}}$ will have continuous paths and X_i has no fixed points of discontinuity). While (1.27) is not completely wrong, the approach (1.27) is too simplistic, because (1.27) ignores the genealogical structure of the population. The death markers are not inherited from one generation to the next one; so if j represents a child of the particle represented by i born at time t, it must hold $Z_i(t\text{-}) = Z_j(t)$, which is not true for (1.27). Therefore let us introduce the genealogical map

$$\Phi : \Omega \times \mathbb{N} \times [0, \infty) \times [0, \infty) \to \mathbb{N},$$

where $\Phi(j, t, s) = i$ with $t > s$ and $i, j \in \mathbb{N}$ tells us that the particle with index i is at time s the ancestor of the particle with index j at time t (we set $\Phi(j, t, s) = j$ in the case $t \leq s$). With Φ in our hand we can define $(Z_i)_{i=1}^{\infty}$ for each $t \geq 0$ now by:

$$Z_i(t) = \sum_{k\in\mathbb{N}} \int_0^t \int_0^{\infty} \mathbb{1}_{\{\Phi(i,t,s)=k\}} \mathbb{1}_{[0, F(X_k(s), \widehat{\boldsymbol{\Xi}}_s)]}(p) \mathbb{1}_{[0, \mathcal{T}_{EX})}(s)\, \tilde{N}_k(dp, ds), \tag{1.28}$$

where we also added for technical convenience $\mathbb{1}_{[0,\mathcal{T}_{EX})}$ with $\mathcal{T}_{EX}$ being the extinction time of $\boldsymbol{\Xi}^X$, this addition of $\mathbb{1}_{[0,\mathcal{T}_{EX})}$ has the effect that everything stops evolving after the background population $\boldsymbol{\Xi}^X$ has died out. We discuss this in greater detail in Chapter 2 and Chapter 3. Once we have obtained $(Z_i)_{i=1}^{\infty}$ and assured the existence of the process $\boldsymbol{\xi}^{XZ}$ with "good properties" like continuity, we can verify via a semi-martingale decomposition for $\boldsymbol{\xi}^{XZ}$ and $\boldsymbol{\Xi}^{XZ}$, see Chapter 7, that $\widehat{\boldsymbol{\Xi}}$ from (1.26) is indeed a solution of the martingale problem $\mathbf{Comp}(\mathbf{B}, a, b, F, \hat{\Theta}_0)$.

While this program is so far in principle quite straightforward, a difficulty presents itself: There is a potential circularity in these definitions because the definition of the $(Z_i)_{i=1}^{\infty}$ as in (1.28) requires $\hat{\Xi}$ from (1.25), which in turn is a simple functional of Ξ^{XZ}, and for the definition of Ξ^{XZ} we need $(X_i, Z_i, U_i)_{i=1}^{\infty}$ which of course implies the necessity of defining $(Z_i)_{i=1}^{\infty}$ first. Hence, (1.25)–(1.28) forms in fact an implicit system of infinitely many equations and it is not a priori obvious why a solution should exist (nor why it is unique) and in fact even why it is possible to define the $(Z_i)_{i=1}^{\infty}$ in such a way that the family $(X_i, U_i, Z_i)_{i=1}^{\infty}$ guarantees the existence of the limits in (1.25) and (1.26) (recall that we simply pretended that Ξ^{XZ} and $\hat{\Xi}$ are well-defined, but of course this has to be established beforehand). The way forward is to divide the above task into two steps. We first develop in Chapter 3 to Chapter 6 an integration theory for the Kurtz-Rodrigues representation. Armed with our integration theory we are able to derive in our second step that the system of infinitely many equations implied by (1.25) and (1.28) has indeed a unique solution.

In our integration theory we allow the integrand

$$\mathbb{1}_{[0,F(X_i(s),\widehat{\Xi}_s)]}$$

in (1.28) to be replaced by a generic random integrand simply denoted by h and obeying certain conditions, notable a suitable notion of previsibility, see Definition 3.1.1. We prove for all $(X_i, Z_i, U_i)_{i=1}^{\infty}$ with $(Z_i)_{i=1}^{\infty}$ defined as in (1.28) with such a h that Ξ^{XZ} exists and admits a continuous modification, but also that

$$\boldsymbol{\xi}_t^{XZ} := \sum_{i=1}^{\infty} \delta_{(X_i(t),Z_i(t),U_i(t))}$$

forms a Poisson representation of Ξ^{XZ}. We also prove a convergence theorem, which discusses the convergence of $(\Xi^{XZ,n})_{n=1}^{\infty}$, if $(h_n)_{n=1}^{\infty}$ is converging to a different integrand $\tilde{h}$. This integration theory could be used in principle for other tasks, and the construction of our desired Poisson representations are "just" one application of this theory.

The proof that $\boldsymbol{\xi}^{XZ}$ is a Poisson representation of Ξ^{XZ} takes its inspiration from the Poisson mapping theorem which tells that if $\boldsymbol{\xi}^1$ is a Poisson point process over E_1 and $\Psi : E_1 \to E_2$ is a map between the spaces E_1 and E_2, then the image of $\boldsymbol{\xi}^2$ of $\boldsymbol{\xi}^1$ under Ψ, i.e.

$$\boldsymbol{\xi}^2(\Gamma) := \boldsymbol{\xi}^1(\Psi^{-1}(\Gamma)), \quad \Gamma \in \mathbb{B}(E_1),$$

($\boldsymbol{\xi}^2$ is the push forward of $\boldsymbol{\xi}^1$ under Ψ) is again a Poisson point process, this time over E_2. If μ_1 was the intensity measure of $\boldsymbol{\xi}^1$, then μ^2 given by $\mu^2(\Gamma) := \mu^1(\Psi^{-1}(\Gamma))$, $\Gamma \in \mathbb{B}(E_1)$, is the intensity measure of $\boldsymbol{\xi}^2$. But to make this approach work for $(\boldsymbol{\xi}^{XZ}, \Xi^{XZ})$, we need to express $(Z_i)_{i=1}^{\infty}$ as functionals of the historical processes $(\mathfrak{X}_i, \mathfrak{N}_i)_{i=1}^{\infty}$. While $X_i(t)$ is the current spatial position of the particle with index i at time t, $\mathfrak{X}_i(t_1, t_2)$ with $t_2 < t_1$ is the spatial position at time t_2 of the ancestor of the particle with index i at time t_1. Similarly we can define in an abuse of notation

$$\mathfrak{N}_i(t, dp, ds) = \sum_{k=1}^{\infty} \mathbb{1}_{\{\Phi(i,t,s)=k\}} \mathbb{1}_{[0,t]}(s) \tilde{N}_k(dp, ds), \quad t \geq 0,$$

which means that the measure $\mathfrak{N}_i(t)$ is identical with the measure $\mathbb{1}_{[0,t]}(s)\tilde{N}_k(dp, ds)$ on the set $\{(s,p); \Phi(i,t,s) = k\}$. The processes $(\mathfrak{X}_i, \mathfrak{N}_i)_{i=1}^{\infty}$ will get a rigorous definition in Section 2.3. Unfortunately things become again complicated. First, the theory of Kurtz-Rodrigues is formulated for time-homogeneous Markov processes, so we add a time coordinate to our processes, i.e. we set

$$\tilde{\mathbb{W}}_i(t) = (t, \mathfrak{X}_i(t), \mathfrak{N}_i(t)), \quad t \geq 0.$$

This leads to the notion of a path-valued process. Secondly, the path-valued process is not a Feller process, because its state space, a modified version of the Skorohod space, see Definition 2.4.1, is not locally compact (and can not easily transformed in locally compact space). Luckily, the class of Feller processes has a natural extension in form of the Borel strong Markov processes. We will discuss this class of Markov processes and how to combine the idea of a path-valued process with the theory of Kurtz-Rodrigues in the Sections A.2-B.6, E.2 and in Chapter 2. We can save a lot of work, when we encode the atoms of $(\tilde{N}_i)_{i=1}^{\infty}$ and $(\mathfrak{N}_i)_{i=1}^{\infty}$ by the jumps of a Lévy process. Indeed, if N is a Poisson point process with intensity measure $\ell eb[0,\infty) \otimes \ell eb[0,\infty)$, then

$$L(t) := \int_0^t \int_0^\infty e^{-p}\, N(dp, ds), \quad t \geq 0,\ i \in \mathbb{N},$$

is a pure-jump Lévy process with Lévy measure:

$$\nu(d\ell) = \mathbb{1}_{(0,1]}(\ell)\ell^{-1}\, d\ell.$$

Applying this encoding to $(\mathfrak{N}_i)_{i=1}^{\infty}$ will give us in Lemma 2.3.8 the historical processes $(\mathfrak{L}_i)_{i=1}^{\infty}$ and if we now set

$$\mathbb{W}_i(t) = (t, \mathfrak{X}_i(t), \mathfrak{L}_i(t)), \quad t \geq 0,\ i \in \mathbb{N},$$

then we have reduced everything to path-valued processes associated with Lévy processes, hence it is much easier to work with $(\mathbb{W}_i)_{i=1}^{\infty}$ instead of $(\tilde{\mathbb{W}}_i)_{i=1}^{\infty}$.
In Chapter 3 we introduce $\mathcal{L}^1_{loc}(\mathbf{M})$ as the right choice of integrands for our integration theory and show that for each $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ we can find a map $H_t,\ t \in [0,\infty)$, with

$$H_t((\boldsymbol{\Xi}^{\mathbb{W}}_s, s \leq t), \mathbb{W}_i(t)) = Z_i(t) \quad a.s., i \in \mathbb{N}.$$

This map or more precisely its extension

$$\overline{H}_t((\boldsymbol{\Xi}^{\mathbb{W}}_s, s \leq t), \mathbb{W}_i(t), U_i(t)) = (X_i(t), Z_i(t), U_i(t)) \quad a.s., i \in \mathbb{N},\ t \geq 0,$$

can be used to express $\boldsymbol{\xi}^{XZ}$ and $\boldsymbol{\Xi}^{XZ}$ as push forwards of $\boldsymbol{\xi}^{\mathbb{W}}$ and $\boldsymbol{\Xi}^{\mathbb{W}}$ under $\overline{H}$, then $\boldsymbol{\xi}^{XZ}$ will be a Poisson representation of $\boldsymbol{\Xi}^{XZ}$, because $\boldsymbol{\xi}^{\mathbb{W}}$ is one for $\boldsymbol{\Xi}^{\mathbb{W}}$. This is essentially the idea of Proposition 3.4.14 and Section 3.5, but we will alter our approach a little bit for technical convenience.
In Chapter 7 we can make the ideas alluded to in (1.25)–(1.28) precise and define the “cut-out” through “competitive deaths” triggered by a (for the moment externally given) measure-valued process $(\mathcal{V}_s)_{s\geq 0}$ as

$$\big((\widehat{X}_i, \widehat{U}_i)_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\Xi}}\big) = \textbf{Cut-Out}\big(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \mathcal{V})\big)$$

where

$$\widehat{\boldsymbol{\xi}}_t = \sum_{i=1}^{\infty} \delta_{(X_i(t), U_i(t))} \mathbb{1}_{\{0\}}(Z_i(t)) \quad \text{and} \quad \widehat{\boldsymbol{\Xi}}_t = \boldsymbol{\Xi}^{XZ}_t(\,\cdot\, \times \{0\}) \quad \text{for } t \geq 0,$$

with Z_i as in (1.28) but $\widehat{\boldsymbol{\Xi}}$ replaced with $\mathcal{V}$ for the moment (and $(\widehat{X}_i(t), \widehat{U}_i(t))_{i\in\mathbb{N}}$ is a re-numbering of $(X_i(t), U_i(t) : i \in \mathbb{N}, Z_i(t) = 0)$ which “closes the gaps”), see Definition 7.1.3. Now indeed, by the definition of Z_i in (1.28), a given particle $(\widehat{X}_i, \widehat{U}_i)$ currently “alive” in $\widehat{\boldsymbol{\xi}}$ will disappear through a “competitive death event” at a given time t at rate $F(X_i(t\text{-}), \mathcal{V}_t)$, i.e. it will stop to be counted for $\widehat{\boldsymbol{\xi}}$ even though its level $U_i(t)$ has not yet exploded.

Due the groundwork laid by our integration theory this gives a rigorous meaning to the infinite system of equations implied by (1.25) and (1.28), but we still have to ensure the existence of a self-consistent solution, indeed that we can plug in $\widehat{\boldsymbol{\Xi}}$ itself into (1.29) instead of the arbitrary process $\mathcal{V}$. This leads to the equation

$$\big((\widehat{X}_i, \widehat{U}_i)_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\Xi}}\big) = \textbf{Cut-Out}\big(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \widehat{\boldsymbol{\Xi}})\big), \tag{1.29}$$

which we call the Cut-Out equation in Section 7.2. We also present in Section 7.2 conditions for the competitive functions F which are sufficient to show that (1.29) admits a unique solution. These conditions, see Conditions 7.2.2, are general enough to cover the non-linear Bolker-Pacala models as well the non-linear singular interactive models.

There are four points for the competitive function F in the Conditions 7.2.2, these are too technical to explain in full detail here, but the first one requires F to be non-decreasing in the second coordinate, meaning that $F(x, \mu_1) \le F(x, \mu_2)$ for all $x \in \mathbb{R}^d$ and $\mu_1, \mu_2 \in \mathcal{M}_f(\mathbb{R}^d)$ with $\mu_1 \le \mu_2$, where "$\le$" refers to a "pointwise ordering", see Definition 7.1.5. We usa a Picard-type iteration scheme by defining recursively for all $n \in \mathbb{N}$:

$$\big((\widehat{X}_i^{\uparrow,n}, \widehat{U}_i^{\uparrow,n})_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}^{\uparrow,n}, \widehat{\boldsymbol{\Xi}}^{\uparrow,n}\big) = \textbf{Cut-Out}\big(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \widehat{\boldsymbol{\Xi}}^{\downarrow,n-1})\big), \quad n \in \mathbb{N}$$

with $\widehat{\boldsymbol{\Xi}}^{\downarrow,0} = \boldsymbol{\Xi}^X$ and

$$\big((\widehat{X}_i^{\downarrow,n}, \widehat{U}_i^{\downarrow,n})_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}^{\downarrow,n}, \widehat{\boldsymbol{\Xi}}^{\downarrow,n}\big) = \textbf{Cut-Out}\big(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \widehat{\boldsymbol{\Xi}}^{\uparrow,n})\big) \quad n \in \mathbb{N}.$$

By the non-decreasing nature of F we have:

$$\widehat{\boldsymbol{\Xi}}_t^{\uparrow,n} \le \widehat{\boldsymbol{\Xi}}_t^{\uparrow,n+1} \le \widehat{\boldsymbol{\Xi}}_t^{\downarrow,n+1} \le \widehat{\boldsymbol{\Xi}}_t^{\downarrow,n} \le \boldsymbol{\Xi}_t^X \quad \text{for all } t \ge 0 \text{ and } n \in \mathbb{N}, \tag{1.30}$$

where equality holds for all random measures, if $t = 0$. In particular $(\widehat{\boldsymbol{\Xi}}^{\uparrow,n})_{n=1}^{\infty}$ is increasing and $(\widehat{\boldsymbol{\Xi}}^{\downarrow,n})_{n=1}^{\infty}$ is decreasing, this suggests to define

$$\widehat{\boldsymbol{\Xi}}_t^{\uparrow} = \lim_{n\to\infty} \widehat{\boldsymbol{\Xi}}_t^{\uparrow,n} \le \widehat{\boldsymbol{\Xi}}_t^{\downarrow} = \lim_{n\to\infty} \widehat{\boldsymbol{\Xi}}_t^{\downarrow,n}, \quad t \ge 0, \tag{1.31}$$

where the limit holds in total variation norm. Ignoring some technical details, which are handled with the help of the second and third point of the Conditions 7.2.2, we obtain a solution of the pair of equations:

$$\begin{aligned} \big((\widehat{X}_i^{\uparrow}, \widehat{U}_i^{\uparrow})_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}^{\uparrow}, \widehat{\boldsymbol{\Xi}}^{\uparrow}\big) &= \textbf{Cut-Out}\big(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \widehat{\boldsymbol{\Xi}}^{\downarrow})\big), \\ \big((\widehat{X}_i^{\downarrow}, \widehat{U}_i^{\downarrow})_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}^{\downarrow}, \widehat{\boldsymbol{\Xi}}^{\downarrow}\big) &= \textbf{Cut-Out}\big(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \widehat{\boldsymbol{\Xi}}^{\uparrow})\big). \end{aligned} \tag{1.32}$$

By the construction of $\widehat{\boldsymbol{\Xi}}^{\uparrow}$ and $\widehat{\boldsymbol{\Xi}}^{\downarrow}$, we have $\widehat{\boldsymbol{\Xi}}_0^{\uparrow} = \widehat{\boldsymbol{\Xi}}_0^{\downarrow}$ and with the help of the fourth point in the Conditions 7.2.2, which can be roughly interpreted as a locally Lipschitz-condition for F, we can extend this equality to all time points, i.e.

$$\mathbb{P}[\widehat{\boldsymbol{\Xi}}_t^{\uparrow} = \widehat{\boldsymbol{\Xi}}_t^{\downarrow}, t \ge 0] = 1.$$

Setting $\widehat{\boldsymbol{\Xi}} := \widehat{\boldsymbol{\Xi}}^{\downarrow}$ and using $\widehat{\boldsymbol{\Xi}}^{\uparrow} = \widehat{\boldsymbol{\Xi}}^{\downarrow}$ in (1.32), we can see that $\widehat{\boldsymbol{\Xi}}$ is a solution of the Cut-Out equation (1.29).

In the last section of Chapter 7 we give an outlook how to apply our Poisson representation to study the extinction behavior of Bolker-Pacala models, this approach is based on Evans' and O'Connells backbone decomposition explained at the end of Section 1.1.

In [16] Evans and Perkins applied a cutting technique very similar to ours to study a system of two competing measure-valued processes which interact adversely with each other. But Evans and Perkins are working solely with high-density limit, they are not interested in Poisson representation, but this is not surprising, since Kurtz and Rodrigues' work is from the year 2011, while Evans and Perkins paper was published in 1998. But despite this, there is also a difference in the considered models. To explain this difference let us reduce Evans and Perkins' model to a situation, where there is no space, and the measure-valued processes become real-valued diffusions. In this scenario, the model of Evan and Perkins described in (v) in Section 1.5 in [16] reduces to a two-dimensional stochastic differential equation given by

$$\begin{aligned} dY_t^1 &= -cY_t^1 Y_t^2 dt + \sqrt{2aY_t^1} d\hat{W}_t^1, \\ dY_t^2 &= -cY_t^2 Y_t^1 dt + \sqrt{2aY_t^2} d\hat{W}_t^2 \end{aligned} \tag{1.33}$$

with $a > 0, c > 0$ and $\hat{W}^1, \hat{W}^2$ are two independent Brownian motions. Of course this is a complete injustice to the work of Evans and Perkins. But for us this reduction allows to point out the difference between our models and the ones of Evans and Perkins. Our models reduce in the same scenario to the logistic Feller diffusion:

$$d\widehat{Y}_t = b\widehat{Y}_t dt - c\widehat{Y}^2 dt + \sqrt{2a\widehat{Y}} d\hat{W}_t.$$

Our models are self-interacting, indeed the competition term has the form $c\widehat{Y}^2 dt$, while the populations in (Y^1, Y^2) are only affecting negatively the other one, but not themselves, because the competition terms have the form $cY_t^1 Y_t^2 dt$ in (1.33). This difference in the drifts leads to a difference in the proofs. Perkins and Evans construct similarly to us two sequences $(Y^{1,n}, n \in \mathbb{N})$ and $(Y^{2,n}, n \in \mathbb{N})$ via a Picard-type recursion with

$$\begin{aligned} Y^{1,1} &\leq Y^{1,2} \leq ... \leq Y^{1,n} \leq ...., \\ Y^{2,1} &\geq Y^{2,2} \geq ... \geq Y^{2,n} \geq .... \end{aligned}$$

and their solutions are obtained by setting $Y^i := \lim_{n\to\infty} Y^{i,n}$. But while we need to argue in an extra step, see Proposition 7.5.3, that the two limits $\widehat{\Xi}^{\uparrow}$ and $\widehat{\Xi}^{\downarrow}$ from (1.31) are in fact identical, this steps is unnecessary in the situation of Evans and Perkins, because Y^1 and Y^2 are already a solution of (1.33).

1.4 Outline

This thesis is organized in the following way: In Chapter 2 we construct and study a collection of processes $(X_i, U_i)_{i=1}^{\infty}$ and $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$, which we call the ordered Kurtz-Rodrigues representation. These processes belong to the particles of a Kurtz-Rodrigues representation and describe the current spatial position and level. We discuss the notion of path-valued processes and we study the behavior of $(X_i, U_i)_{i=1}^{\infty}$ and $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$ at the moment of extinction.
We develop our integration theory in Chapter 3 with the help of the ordered KR-representation, there we will introduce a third component $(Z_i)_{i=1}^{\infty}$, which depends on the chosen integrand h. The integrand must belong to the class $\mathcal{L}^1_{loc}(\mathbf{M})$ in order to ensure that $(Z_i)_{i=1}^{\infty}$ is well-defined and to maintain the exchangeability of $(X_i, Z_i, V_i)_{i=1}^{\infty}$ with $V_1 = U_1$ and $V_{i+1} = U_{i+1} - U_i$ for $i \in \mathbb{N}$. We show that $\boldsymbol{\xi}^{XZ}$ is a Poisson representation of $\boldsymbol{\Xi}^{XZ}$ with

$$\boldsymbol{\xi}^{XZ} := \sum_{i=1}^{\infty} \delta_{(X_i(t), Z_i(t), U_i(t))} \text{ and } \boldsymbol{\Xi}^{XZ} := \lim_{r\to\infty} \frac{1}{r} \sum_{i=1}^{\infty} \delta_{(X_i(t), Z_i(t))} \mathbb{1}_{[0,r)}(U_i(t)). \tag{1.34}$$

In Chapter 4 we derive several semi-martingale decompositions associated with the processes $(X_i, Z_i, U_i)_{i=1}^{\infty}$, $\boldsymbol{\xi}^{XZ}$ and $\boldsymbol{\Xi}^{XZ}$. These semi-martingale decomposition are used in Chapter 5 to prove a useful convergence theorem, see Proposition 5.2.3, which will be used in the proof of our main theorem in Chapter 7, but also to derive that $\boldsymbol{\Xi}^{XZ}$ admits a continuous modification in Chapter 6.
In the final Chapter 7, we start with introducing the Cut-Out process. We take a look at some of its properties and prove the Reversed order lemma, see Lemma 7.1.6. After this we start to study the Cut-Out equation

$$\big((\widehat{X}_i, \widehat{U}_i)_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\Xi}}\big) = \mathbf{Cut\text{-}Out}\big(\boldsymbol{\xi}^{\mathbb{W}}, F(\cdot, \widehat{\boldsymbol{\Xi}})\big),$$

and discuss conditions sufficient to ensure that this equation admits a unique solution. In Section 7.3 and 7.4 we show that these conditions are satisfied by the two classes of competitive models we are interested in, (non-linear) Bolker Pacala models and (non-linear) singular interaction models. Finally, we show in Section 7.5 that the Cut-Out equation has a unique solution from which we obtain our main theorem, Theorem 1.2.6. In the last section of Chapter 1.2 we sketch how one could apply our Poisson representation to study the extinction behavior of Bolker-Pacala models.
In the appendix we start with studying the notion of a martingale problem, see Appendix A.1, and the notion of a Borel strong Markov process, see Appendix A.2. Both notions play an important role in the Sections B.1-B.6, where we lay out the martingale characterization of the Dawson-Watanabe superprocess, the Kurtz-Rodrigues representation and the Branching particle system, further we use the Markov mapping theorem to prove the statements of Section 1.1. Throughout this thesis we make us of several properties associated with Poisson point process and its atoms, these properties are discussed in Section C.1 and C.2. Even more relevant is the Markov mapping theorem, which we present and prove in Section D.1. Since our spatial motions are given by Lévy processes, respectively by path-valued processes associated with such, we need to show that these admit a sufficiently nice generator, which suits the theory of Kurtz and Rodrigues, this happens in Sections E.1 and E.2.

Chapter 2

The Ordered Kurtz-Rodrigues Representation

Let us assume that $\mathbb{X}$ is an abstract Markov process with state space E and generator $\mathbf{B}$. In Section 1.1 we became acquainted with the Kurtz-Rodrigues representation $\boldsymbol{\xi}^{\mathbb{X}}$ as a stochastic processes with state space $\overline{\mathcal{N}}(E \times [0,\infty))$, recall Definition 1.1.2. We have characterized the KR-representation as the solution of the martingale problem of $\mathbf{A_B}$, the missing details can be found in Appendix B.2. Kurtz and Rodrigues proved the existence of the martingale problem $\mathbf{MP}(\mathbf{A_B})$ based on the abstract Markov mapping theorem, see Theorem A.15. in [32]. For our purpose we think it is convenient to a have collection of particles $(\mathbb{X}_i, U_i)_{i=1}^{\infty}$ such that their "empirical process"

$$\boldsymbol{\xi}_t^{\mathbb{X}} := \sum_{i=1}^{\infty} \delta_{(\mathbb{X}_i(t), U_i(t))}, \quad t \geq 0, \tag{2.1}$$

forms a Kurtz-Rodrigues representation. Of course one could start with the solution of $\mathbf{MP}(\mathbf{A_B})$ and claim the existence of $(\mathbb{X}_i, U_i)_{i=1}^{\infty}$, but a rigorous argumentation based on this approach requires to express everything as functionals of $\boldsymbol{\xi}^{\mathbb{X}}$, and all we know is that the functionals contained in the domain of $\mathbf{MP}(\mathbf{A_B})$ are martingales. Therefore we prefer the typical approach of giving an explicit construction of $(\mathbb{X}_i, U_i)_{i=1}^{\infty}$ and then we verify that $\boldsymbol{\xi}^{\mathbb{X}}$ is a solution of $\mathbf{MP}(\mathbf{A_B})$. The processes $(\mathbb{X}_i, U_i)_{i=1}^{\infty}$ will take values in $E \times [0,\infty]$, and the collection $(U_i)_{i=1}^{\infty}$ has the property that

$$\mathbb{P}\left[U_i(t) < U_{i+1}(t),\, \forall t \in [0,\infty)\right] = 1,$$

hence the name "ordered Kurtz-Rodrigues representation". We call $(\mathbb{X}_i)_{i=1}^{\infty}$ the spatial processes and $(U_i)_{i=1}^{\infty}$ the level processes. This ordering has advantages and disadvantages. One disadvantage is that the dynamics of $(\mathbb{X}_i, U_i)_{i=1}^{\infty}$ become more complicated, because the pair $(\mathbb{X}_i, U_i)$ for a fixed $i \in \mathbb{N}$ will not always represent the same particle/atom of $\boldsymbol{\xi}^{\mathbb{X}}$ for all time points, instead it will switch every time, when a new particle is born with an level below U_i. But this also means that the pair $(\mathbb{X}_1, U_1)$ will always represent the lowest particle, because all new particles have level higher than their parent. At the first glance this is not much, but note (1.15), i.e.

$$\mathfrak{L}(\,\boldsymbol{\xi}_t^{\mathbb{X}} \,|\, \sigma(\boldsymbol{\Xi}_s^{\mathbb{X}}, s \leq t)) = \mathbf{PPP}_E(\boldsymbol{\Xi}_t^{\mathbb{X}} \otimes \ell eb[0,\infty))$$

translates for the ordered system $(\mathbb{X}_i, U_i)_{i=1}^\infty$, see Proposition C.1.3, to

$$\mathfrak{L}((\mathbb{X}_i(t), V_i(t))_{i=1}^\infty | \sigma(\boldsymbol{\Xi}_s^{\mathbb{X}}, s \le t)) = \bigotimes_{i=1}^{\infty} \left(\mathbf{Q}_t^{\mathbb{X}} \otimes \mathbf{Exp}(Y_t)\right), \tag{2.2}$$

where $V_1 = U_1$, $V_i = U_i - U_{i-1}$ for $i \ge 2$, $Y_t = \boldsymbol{\Xi}^{\mathbb{X}}(E)$, $\mathbf{Q}_t^{\mathbb{X}} = \boldsymbol{\Xi}_t^{\mathbb{X}}/Y_t$ (on the set $\{Y_t := \boldsymbol{\Xi}_t^{\mathbb{X}}(E) > 0\}$, otherwise extra attention is required, see Proposition 2.6.5) and $\mathbf{Exp}$ is the Markov kernel given by:

Definition 2.0.1. *We define the map* $\boldsymbol{Exp} : [0, \infty] \to \mathcal{M}_1([0, \infty])$ *by saying that* $\boldsymbol{Exp}(\lambda)$ *is the exponential distribution with rate* λ *in the case of* $0 < \lambda < \infty$, $\boldsymbol{Exp}(0) = \delta_\infty$, *and* $\boldsymbol{Exp}(\infty) = \delta_0$.

Statement (2.2) will be often used by us to translate statements proven for the lowest pair $(\mathbb{X}_1, U_1)$, whose dynamic is undisturbed by the birth of new particles, to the whole population. We will now give characterization of the ordered system $(\mathbb{X}_i, U_i)_{i=1}^\infty$ as a martingale problem, but we postpone many details to Section 2.5. The state space of the ordered Kurtz-Rodrigues representation is given by:

Definition 2.0.2. *If* E *is a Polish space, then we denote by* $\mathbf{S}(E)$ *the subset of* $(E \times [0, \infty])^\infty$ *consisting of the elements* $(x_i, u_i)_{i=1}^\infty$ *with* $u_i \le u_{i+1}, i \in \mathbb{N}$. *We equip* $\mathbf{S}(E)$ *with the countable product of the topology of* $E \times [0, \infty]$, *which in turn is equipped with product topology of* E *and* $[0, \infty]$, *where the latter is considered to be the usual one-point compactification of* $[0, \infty)$ *(Alexandroff-Compactification, see Page 150 in [22]). Further we define* $\mathbf{S}_{[0,\infty)}(E)$ *as the subset of those elements* $(x_i, u_i)_{i=1}^\infty \in \mathbf{S}(E)$ *with* $u_i < \infty$ *for all* $i \in \mathbb{N}$ *and we set* $\mathbf{S}_\infty(E) := \mathbf{S}(E) \setminus \mathbf{S}_{[0,\infty)}(E)$.

Please note that if $(x_i, u_i)_{i=1}^\infty \in \mathbf{S}_{[0,\infty)}(E)$, then $\eta = \sum_{i=1}^\infty \delta_{(x_i, u_i)}$ is an element of $\overline{\mathcal{N}}(E \times [0, \infty))$, if and only if $\lim_{i \to \infty} u_i = \infty$. The generator of the ordered system $\mathbf{A}_{\mathbf{B}}^o$ will be defined on functions $G : \mathbf{S}(E) \to [0, 1]$ having the form

$$G((x_i, u_i)_{i=1}^\infty) = \prod_{i=1}^{\infty} g_i(x_i, u_i)$$

with $(g_i)_{i=1}^\infty$ being a suitable collection of functions. In Section 2.5 we will learn more about how to choose $(g_i)_{i=1}^\infty$, but for now it enough to know, that there should exist for each collection $(g_i)_{i=1}^\infty$ a level cap $r > 0$ such that $g_i(x_i, u_i) = 1$, if $u_i \ge r$, for all $i \in \mathbb{N}$. For G as above the function

$$\mathbf{A}_{\mathbf{B}}^o(G) : \mathbf{S}(E) \to \mathbb{R}$$

is given for $(x_i, u_i)_{i=1}^\infty \in \mathbf{S}_{[0,\infty)}(E)$, i.e. $u_i < \infty$ for all $i \in \mathbb{N}$, by

$$\begin{aligned}
&\mathbf{A}_{\mathbf{B}}^o(G)\big((x_i, u_i)_{i=1}^\infty\big) \\
&\quad = \prod_{l=1}^{\infty} g_l(x_l, u_l) \sum_{i=1}^{\infty} \frac{\mathbf{B}(g_i)(x_i, u_i)}{g_i(x_i, u_i)} &(2.3)\\
&\quad + \prod_{l=1}^{\infty} g_j(x_l, u_l) \sum_{i=1}^{\infty} (a u_i^2 - b u_i) \frac{\partial_u g_i(x_i, u_i)}{g_i(x_i, u_i)} &(2.4)\\
&\quad + \prod_{l=1}^{\infty} g_l(x_l, u_l) \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \int_{u_{i-1}}^{u_i} 2a \left(g_i(x_i, v) \prod_{m=j}^{\infty} \frac{g_{m+1}(x_m, u_m)}{g_m(x_m, u_m)} - 1 \right) dv &(2.5)
\end{aligned}$$

and for $(x_i, u_i)_{i=1}^\infty \in \mathbf{S}_\infty(E)$, i.e. $u_i = \infty$ for some $i \in \mathbb{N}$ (and hence $u_i = \infty$ for all $j \geq i$), by

$$\mathbf{A}^o_{\mathbf{B}}(G)\big((x_i, u_i)_{i=1}^\infty\big) = 0. \tag{2.6}$$

The first two lines of $\mathbf{A}^o_{\mathbf{B}}$ describing the spatial motion and the level dynamics are identical with the first two lines of $\mathbf{A}_{\mathbf{B}}$, see (1.10) or Definition B.2.9. This is not surprising, because when we assume that there is no new particle born with a level below U_i on the time interval $[t_1, t_2)$, then the collection $(\mathbb{X}_j, U_j)_{j=1}^i$ will evolve on $[t_1, t_2)$ like independent copies with $\mathbb{X}_j$ being a copy of the Markov process with generator $\mathbf{B}$ and U_j being the solution of the ODE $\dot{u} = au^2 - bu$.
But the Line (2.5) which is associated with the birth of new particles looks different as the corresponding line in $\mathbf{A}_{\mathbf{B}}$, this is due to the way how the birth of new particles is effecting the order of the particles. Let us fix an index i, then every particle with an index lower than i will give birth to a new particle with a level between $U_{i\text{-}1}$ and U_i independently with the rate $2a(U_i - U_{i\text{-}1})$. If this happens the new particle will inherit the spatial motion from its parent and its level will be uniformly chosen from the interval $[U_{i\text{-}1}, U_i)$. Further the pair $(\mathbb{X}_i, U_i)$ will now describe the spatial motion and the level of the new particle from now on until another new particle is born with an level below U_i. But the birth is also effecting all pairs $(\mathbb{X}_k, U_k)$ with $k > i$, because the particle having the k-th position in the ordering has suddenly the $k+1$-th position, hence its spatial motion and its level is now described by the pair $(\mathbb{X}_{k+1}, U_{k+1})$ and not by the pair $(\mathbb{X}_k, U_k)$ anymore, which results in a jump of $(\mathbb{X}_{k+1}, U_{k+1})$ to the value of $(\mathbb{X}_k, U_k)$.
Finally, (2.6) tells us that the whole system $(\mathbb{X}_i(t), U_i(t))_{i=1}^\infty$ stops evolving, when one of the level processes hits infinity. We will later see, that if a level process hits infinity, then all of the level processes $(U_i)_{i=1}^\infty$ hit infinity at the same time, and the level processes will hit infinity, if and only if $U_1 > \max\{b/a, 0\}$. Here it is important to distinguish between the processes $(X_i, U_i)_{i=1}^\infty$ and the particles of $\boldsymbol{\xi}^{\mathbb{X}}$. The particles of $\boldsymbol{\xi}^{\mathbb{X}}$ hit infinity at different time points, and this appears to be in contradiction to the statement that the $(U_i)_{i=1}^\infty$ hit infinity at the same time. This contradiction exists only apparently, because when the level of a particle in $\boldsymbol{\xi}^{\mathbb{X}}$ converges to infinity, then its index in $(\mathbb{X}_i(t), U_i(t))_{i=1}^\infty$ will also converge to infinity due to the new particles born with a lower level. This has the effect that there exists no index $j \in \mathbb{N}$ such that U_j converges to infinity. In Section 2.5 we will give more details about $\mathbf{A}^o_{\mathbf{B}}$ like the domain and we will formulate more rigorously a martingale problem for $(\mathbb{X}_i, U_i)_{i=1}^\infty$, see Definition 2.5.10. But in the end, the ordered KR-representation is just a tool for us, therefore we will not prove that the martingale problem $\mathbf{MP}(\mathbf{A}^o_{\mathbf{B}})$ is well-posed. On the contrary we will argue that the martingale problem may not even be well-defined, depending on the initial values of $(\mathbb{X}_i, U_i)_{i=1}^\infty$, for further explanations see Remark 2.4.11. But even when the ordered system $(\mathbb{X}_i, U_i)_{i=1}^\infty$ is a well-defined solution of $\mathbf{MP}(\mathbf{A}^o_{\mathbf{B}})$, the process $\boldsymbol{\xi}^{\mathbf{B}}$ from (2.1) is only an empirical KR-representation, if we chose $(U_i(0))_{i=1}^\infty$ such that $\lim_{i\to\infty} U_i(0) = \infty$. Otherwise there will be no newborn particles with levels above $U_\infty(t) = \lim_{i\to\infty} U_i(t)$, because by the definition of $\mathbf{A}^o_{\mathbf{B}}$ new particles are only born between existing ones. But these problems will not appear, if we choose an initial distribution such that:

$$\mathbb{P}\left[U_i(0) \stackrel{i\to\infty}{\longrightarrow} \infty\right] = 1 \quad \text{and} \quad \mathbb{P}\left[U_1(0) < U_2(0) < ...\right] = 1, \tag{2.7}$$

indeed the initial levels are all different from each other and form a sequence converging to infinity.

2.1 Ingredients

Here we state the building blocks for our explicit construction of the ordered Kurtz-Rodrigues representation. This explicit construction will help us to understand better the behavior of the processes describing the atoms of the Kurtz-Rodrigues representation, especially at the moment of extinction. We will restrict ourselves to Lévy processes for the spatial motion, which means that we switch from $\mathbb{X}$, denoting an abstract Markov process, to X, representing a Lévy process satisfying the Assumptions 1.2.3. The **weak** generator B_X of X can be defined as the collection of pairs $(g,\psi) \in C_b(\mathbb{R}^d) \times C_b(\mathbb{R}^d)$ with the property that the process

$$M(t) := g(X_t) - g(X_0) - \int_0^t \psi(X_s)\, ds \tag{2.8}$$

is a martingale with respect to the natural filtration of X, for more details see Definition A.2.3. If $(g,\psi_1),(g,\psi_2) \in B_X$, then $\psi_1 = \psi_2$, because if M_1 and M_2 are the martingales obtained by (2.8) for ψ_1 and ψ_2, then we can conclude that $M_3 := M_1 - M_2$ is a continuous martingale with finite variation and so M_3 is constant with $M_3(0) = 0$. So we can interpret B_X as a map defined on the subset $\mathcal{D}(B_X) \subset C_b(\mathbb{R}^d)$, where $g \in \mathcal{D}(B_X)$, if and only if there exists a $\psi \in C_b(\mathbb{R}^d)$ with $(g,\psi) \in B_X$, and given by $B_X(g) = \psi$. The domain $\mathcal{D}(B_X)$ contains $C_b^2(\mathbb{R}^d)$, see Lemma E.1.13.

Definition 2.1.1. *We define $C_b^2(\mathbb{R}^m)$ for $m \in \mathbb{N}$, as the set of twice continuous differentiable functions, which are bounded and all their derivatives are also bounded, i.e. if $\hat{g} \in C_b^2(\mathbb{R}^m)$, then*

$$||\hat{g}||_{\infty,2} := \sup_{x\in\mathbb{R}^m} |\hat{g}(x)| + \sup_{x\in\mathbb{R}^m} \sum_{i=1}^m |\partial_{x_i}\hat{g}(x)| + \sup_{x\in\mathbb{R}^m} \sum_{i,j=1}^m |\partial_{x_i x_j}\hat{g}(x)| < \infty.$$

By $C_b^{2,+}(\mathbb{R}^m)$ we denote the subset of non-negative functions.

If ∇ is the Nabla operator and if $g \in C_b^2(\mathbb{R}^d)$, then $B_X(g)$ can be written as:

$$\begin{aligned} B_X(g)(x) =& (B_X^\rho)^T \nabla g(x) + \frac{1}{2}\nabla^T B_X^{cov} \nabla g(x) \\ &+ \int_{\mathbb{R}^d} g(x+y) - g(x) - \mathbb{1}_{\{||y||\le 1\}} \left(y^T \nabla g(x)\right) B_X^\eta(dy),\ x \in \mathbb{R}^d, \end{aligned} \tag{2.9}$$

with $(B_X^\rho)^T \nabla g(x) = \sum_{k=1}^d B_X^\rho(k) \partial_{x_k} g(x)$ and $\nabla^T B_X^{cov} \nabla g(x) = \sum_{k,l=1}^d B_X^{cov}(k,l) \partial_{x_k}\partial_{x_l} g(x)$. It holds $B_X(g) \in C_b(\mathbb{R}^d)$.
Let us write $\mathcal{N}_{lf}(E)$ with E being a Polish space for the set of locally finite integer-valued measures over E. Our ordered system $(X_i,U_i)_{i=1}^\infty$ will be built from the following ingredients.

Assumptions 2.1.2. *We assume that the following random variables $(U_i^0, X_i^0; i \in \mathbb{N})$, $(\mathcal{V}_{ji}; i,j \in \mathbb{N}, i<j)$, $(\tilde{X}_i; i \in \mathbb{N})$ and $(\tilde{N}_i; i \in \mathbb{N})$ are defined on a common probability space $(\Omega,\mathcal{A},\mathbb{P})$ such that:*

1. *$U_i^0 : \Omega \to [0,\infty), X_i^0 : \Omega \to \mathbb{R}^d, i \in \mathbb{N}$, are random variables, $V_1^0 = U_1^0$ and $V_{i+1}^0 := U_{i+1}^0 - U_i^0$, $i \in \mathbb{N}$, such that*

$$\mathfrak{L}((X_i^0, V_i^0)_{i=1}^\infty | \mathbf{Q}_0^X, Y_0) \sim \bigotimes_{i=1}^\infty \left(\mathbf{Q}_0^X \otimes \boldsymbol{Exp}(Y_0)\right), \tag{2.10}$$

   where $V_1^0 = U_1^0$ and $V_{i+1}^0 := U_{i+1}^0 - U_i^0$, $i \in \mathbb{N}$, $\mathbf{Q}_0^X : \Omega \to \mathcal{M}_f(\mathbb{R}^d)$ is a random probability measure and $Y_0 : \Omega \to [0,\infty)$ is a random variable with $\mathbb{P}[Y_0 > 0] = 1$ and $\mathbb{E}[Y_0] < \infty$.

2. *$(\mathcal{V}_{ji}, 1 \le i < j < \infty)$ with $\mathcal{V}_{ji} : \Omega \to \mathcal{N}_{lf}([0,\infty)\times[0,\infty)), i < j < \infty$, consists of independent Poisson point process with intensity measure $2a\ell eb[0,\infty) \otimes \ell eb[0,\infty)$.*

3. *$(\tilde{X}_i)_{i=1}^{\infty}$ with $\tilde{X}_i : \Omega \times [0,\infty) \to \mathbb{R}^d, i \in \mathbb{N}$, is a collection of independent copies of X with $\tilde{X}_i(0) = 0$.*

4. *$(\tilde{N}_i)_{i=1}^{\infty}$ with $\tilde{N}_i : \Omega \to \mathcal{N}_{lf}([0,\infty) \times [0,\infty))$ consists of independent Poisson point process with intensity measure $\ell eb[0,\infty) \otimes \ell eb[0,\infty)$.*

The collections $((U_i^0, X_i^0)_{i=1}^{\infty}, \mathbf{Q}_0^X, Y_0), (\mathcal{V}_{ji}, 1 \le i < j < \infty), (\tilde{X}_i)_{i=1}^{\infty}$ and $(\tilde{N}_i)_{i=1}^{\infty}$ are independent from each other.

As consequence of (2.10) we know that (2.7) is satisfied. This will help to avoid problematic situations like in Remark 2.4.11.

2.2 The Level-System

We will continue with constructing $(U_i)_{i=1}^\infty$ using $(U_i^0)_{i=1}^\infty$ and $(\mathcal{V}_{ji}, 1 \le i < j < \infty)$. Note that in the following $U_i(t\text{-})$ stands for the left-limit of U_i at the time $t \in [0,\infty)$ and that j will always be the higher index of $j \in \mathbb{N}$ and $i \in \mathbb{N}$.

Definition 2.2.1 (Level System). *The collection of levels processes $(U_i, i \in \mathbb{N})$ with $U_i : \Omega \times [0,\infty) \to [0,\infty]$ is the solution of the infinite system of differential equation with jumps that is defined by setting for all $j \in \mathbb{N}$ and $t \in [0,\infty)$ by:*

$$U_j(t) := U_j^0 + \int_0^t \mathbb{1}_{[0,\infty)}(U_j(s\text{-})) \left[aU_j^2(s\text{-}) - bU_j(s\text{-})\right] ds \tag{2.11}$$

$$+ \sum_{i=1}^{j-1} \int_0^t \int_{U_{j\text{-}1}(s\text{-})}^{U_j(s\text{-})} (v - U_j(s\text{-}))\, \mathcal{V}_{ji}(dv, ds) \tag{2.12}$$

$$+ \sum_{i=2}^{j-1} \sum_{k=1}^{i-1} \int_0^t \int_{U_{i\text{-}1}(s\text{-})}^{U_i(s\text{-})} (U_{j\text{-}1}(s\text{-}) - U_j(s\text{-}))\, \mathcal{V}_{ik}(dv, ds), \tag{2.13}$$

where we interpret the inner integrals of (2.12) *and* (2.13) *as zero, if $(U_{j\text{-}1}(s\text{-}), U_j(s\text{-})) = (\infty,\infty)$ and $(U_{i\text{-}1}(s\text{-}), U_i(s\text{-})) = (\infty,\infty)$. Similarly the inner expression of integral in* (2.11) *becomes zero, if $U_j(s\text{-}) = \infty$.*

As we can see in (2.11) between two jumps U_j evolves like the solution of $\dot{u} = au^2 - bu$. From (2.12) and (2.13) we can conclude there are two types of jumps, (2.12) describes the jumps due the birth of a new particle with a level between $U_{j-1} - U_j$ and since $(U_i)_{i=1}^\infty$ are ordered, U_j will jump to the level of the new particle. The jumps of (2.13) are the result of new particles born with a level below of $U_{j\text{-}1}$.
A key observation is that the evolution of U_j depends only on the level processes that have a lower index, indeed on $U_1, U_2, ..., U_{j\text{-}1}$. So U_1 does not depend on any other process, U_2 only on U_1, U_3 only on U_1, U_2 and so on, hence the well-posedness of $(U_i)_{i=1}^\infty$ should follow from induction. But the solution of $\dot{u} = au^2 - bu$ will explode for all initial values with $u_0 > \max\{b/a, 0\}$, hence the level processes above this boundary will convergence against infinity. Although the jump parts (2.12) and (2.13) will in the case $j \ge 2$ always produce a jump, before U_j hits infinity, we have to exclude with argument that the frequency and the size of the jumps are too small to prevent that U_j reaches "∞", before $U_{j\text{-}1}$ does. This is a little bit problematic, because there exists no good answer, how U_j should behave, if it hits infinity at t, but $U_{j\text{-}1}(t) < \infty$. Due to (2.12), there should be an instantaneous jump, but what should be the new value of U_j after this jump? It can not be contained in $[U_{j\text{-}1}(t), U_{j\text{-}1}(t) + K]$, where K is an arbitrary large number, because then the jump could not have been instantaneous. To avoid this confusing problem, we have to carefully argue, why $U_{j\text{-}1}$ hits infinity at the same time as U_j. This argument will be provided by Lemma 2.2.7, which will allows us to prove that all of the processes $(U_i)_{i=1}^\infty$ hit infinity simultaneously.

Proposition 2.2.2. *If $(U_i^0, i \in \mathbb{N})$ and $(\mathcal{V}_{ji}; i,j \in \mathbb{N}, i < j)$ are like in Assumption 2.1.2, then it holds*

1. *There exists a set $\Omega_0 \subset \Omega$ with $\mathbb{P}[\Omega_0] = 1$ and a collection of processes $(U_i)_{i=1}^\infty$ with $U_i : \Omega \times [0,\infty) \to [0,\infty]$ such that $(U_i(\omega,\cdot))_{i=1}^\infty \in \mathbb{D}([0,\infty), [0,\infty]^\infty)$ is the unique solution of the system of equations from Definition 2.2.1 for every $\omega \in \Omega_0$.*

2. *Let us define for each $i \in \mathbb{N}$ the sequence of increasing stopping times $(\mathcal{T}_k^i; k \in \mathbb{N})$ by setting $\mathcal{T}_k^i := \inf\{t \ge 0 : U_i(t) > k\}$ for $k \in \mathbb{N}$. If we define $\mathcal{T}_\infty^i := \lim_{k\to\infty} \mathcal{T}_k^i$ as the explosion*

time of level process U_i, then it holds

$$\mathbb{P}\big[\mathcal{T}_\infty^i = \mathcal{T}_\infty^1 \text{ for all } i \in \mathbb{N}\big] = 1, \tag{2.14}$$

indeed all levels processes $(U_i)_{i=1}^\infty$ hit infinity at the same time.

Definition 2.2.3. *Due to* (2.14) *we define $\mathcal{T}_{EX} = \mathcal{T}_\infty^1$ and call $\mathcal{T}_{EX}$ the **extinction time**.*

Remark 2.2.4. *Although explosion time would be also fitting name in the current context, the name extinction time is based on the important Lemma 2.6.3.*

Before we prove Proposition 2.2.2 it is essential to understand the behavior of the ordinary differential equation $\dot{u} = au^2 - bu$. As previously mentioned the solution will explode, if the initial values u_0 is higher than $\max\{b/a, 0\}$ due to its quadratic term. But we are also interested in the integral of the solution due to its important role in the jump behavior of the processes $(U_i)_{i=1}^\infty$ given by (2.12) and (2.12).
Although the quantitative behavior is basically the same, the expressions describing the solution of $\dot{u} = au^2 - bu$ and its integral are different for the two cases $b \neq 0$ or $b = 0$, therefore we present each case in a separate lemma, Lemma 2.2.5 and Lemma 2.2.6, to avoid confusion.

Lemma 2.2.5. *For $a \in (0, \infty), b \in \mathbb{R} \setminus \{0\}$ the unique solution of the differential equation*

$$\dot{u}(t) = au^2(t) - bu(t), \quad t \geq 0,\, u(0) = u_0 \geq 0, \tag{2.15}$$

is given by $u(t) := \Upsilon(u_0, t)$, where the function $\Upsilon : [0, \infty) \times [0, \infty) \to [0, \infty]$ is defined by

$$\Upsilon(u_0, t) := \begin{cases} \frac{b}{a}\left[1 + e^{bt}(\frac{b}{au_0} - 1)\right]^{-1}, & t < \tau_{EX}(u_0), \\ \infty, & t \geq \tau_{EX}(u_0), \end{cases}$$

and the function $\tau_{EX} : [0, \infty) \to [0, \infty]$ is defined by

$$\tau_{EX}(u_0) := \begin{cases} -\frac{1}{b}\log\left(1 - \frac{b}{au_0}\right), & u_0 > \frac{b}{a}, \\ \infty, & u_0 \leq \frac{b}{a}. \end{cases}$$

The solution u is finite for $t < \tau_{EX}(u_0)$ and converges against infinity as t converges against $\tau_{EX}(u_0)$, when $u_0 > \max\{b/a, 0\}$ (therefore we call $\tau_{EX}(u_0)$ the explosion time). Further, if we integrate the solution, we obtain:

$$\int_0^t u(s)\, ds = \begin{cases} -\frac{1}{a}\left[\log\left(e^{-bt} + \frac{b}{au_0} - 1\right) - \log\left(\frac{b}{au_0}\right)\right], & t < \tau_{EX}(u_0) \\ \infty, & t \geq \tau_{EX}(u_0) \end{cases}$$

Hence the integral is also finite for all $t < \tau_{EX}(u_0)$, and converges against infinity, when t converges against $\tau_{EX}(u_0)$, when $u_0 > \max\{b/a, 0\}$.

Proof. Via differentiation we can verify that u defined as above is indeed a solution of the differential equation (2.15). If $\tilde{u}$ is a second solution, then we can show via *separation of variables* that $\tilde{u}$ is identical with u until explosion. That the solution converges to infinity, when $u_0 > \max\{b/a, 0\}$, follows from the fact that the denominator approaches 0 for $t \uparrow \tau_{EX}(u_0)$.
The integral can be calculated with the help of substitution and verified via differentiation. The convergence against infinity in the case $u_0 > \max\{b/a, 0\}$ follows from the fact that $e^{-bt} + \frac{b}{au_0} - 1$ converges against the value zero for $t \uparrow \tau_{EX}(u_0)$. □

Lemma 2.2.6. *In the case $b = 0$, the unique solution of the differential equation*

$$\dot{u}(t) = au^2(t), \quad t \geq 0,\, u(0) = u_0 \geq 0, \tag{2.16}$$

which we obtain from (2.15) *by setting $b = 0$, is given by $u(t) := \Upsilon(u_0, t)$, where the function $\Upsilon : [0, \infty) \times [0, \infty) \to [0, \infty]$ is defined by*

$$\Upsilon(u_0, t) := \begin{cases} \frac{u_0}{1-u_0at}, & t < \tau_{EX}(u_0), \\ \infty, & t \geq \tau_{EX}(u-0) \end{cases}$$

and $\tau_{EX} : [0, \infty) \to [0, \infty]$ is given by $\tau_{EX}(u_0) = \frac{1}{au_0}$. As in Lemma 2.15 *the solution u converges against infinity, when t converges against $\tau_{EX}(u_0)$, and, similar as before, when we integrate the solution, we obtain:*

$$\int_0^t u(s)\, ds = \begin{cases} -\frac{1}{a}\log(1 - u_0at), & t < \tau_{EX}(u_0), \\ \infty, & t \geq \tau_{EX}(u_0). \end{cases}$$

Hence, as in Lemma 2.15*, the integral of u is finite for $t < \tau_{EX}(u_0)$ and explodes at time $\tau_{EX}(u_0)$.*

Proof. The proof works the same as the one of Lemma 2.15. □

The key argument why all of the processes $(U_i)_{i=1}^{\infty}$ hit infinity simultaneously, is to show that U_j will not hit infinity at any time, as long as there exists a constant K with $U_i \leq K$ for $i \in \{1, ..., j\text{-}1\}$. This will achieved via a coupling argument with the process $\overline{U}$ from the following lemma.

Lemma 2.2.7. *For a given $n \in \mathbb{N}$ and a $r_1 \geq \max\{b/a, 0\}$ we assume that $\mathcal{V}$ is a Poisson point process with the intensity measure $2an\ell eb[r_1, \infty) \otimes \ell eb[0, \infty)$ defined on some probability space. When we define the stochastic process $\overline{U}$ as the solution of the following stochastic integral equation*

$$\overline{U}(t) := u_0 + \int_0^t a\overline{U}^2(s\text{-}) - b\overline{U}(s\text{-})\, ds + \int_0^t \int_{r_1}^{\overline{U}(s\text{-})} v - \overline{U}(s\text{-})\, \mathcal{V}^{r_1}(dv, ds)$$

with $\overline{U}(0) \in [r_1, \infty)$, then $\overline{U}$ is well-defined with càdlàg paths. Further $\overline{U}$ makes only finitely many jumps in finite time and if $\tau_k = \inf\{s > 0 : \overline{U}(s) \geq k\}$, then

$$\mathbb{P}\left[\lim_{k\to\infty} \tau_k = \infty\right] = 1. \tag{2.17}$$

The proof is essentially the Khasminskii's theorem adopted to the situation of Lemma 2.2.7 in several ways.

Proof. Recall the function τ_{EX} from Lemma 2.2.5 and 2.2.6, which tells us, when the solution of $\dot{u} = au^2 - bu$ will explode. Since the process $\overline{U}$ evolves like the differential equation $\dot{u} = au^2 - bu$ and it also holds $\overline{U}(0) \geq \max\{b/a, 0\}$, the process would hit infinity at the moment $\tau_{EX}(r_1)$, if it did not jump down before. This jump will happen almost surely, because the probability that $\overline{U}$ makes no jump up to time t is converging to 0, when t converges to $\tau_{EX}(r_1)$. To see this, let us recall the function Υ describing the flow of $\dot{u} = au^2 - bu$ from Lemma 2.2.5 ($b \neq 0$) and Lemma 2.2.6 ($b = 0$). The probability of no jump up to time t is given by

$$\begin{aligned} &\mathbb{E}\left[\exp\left(\int_0^t \int_{r_1}^{\overline{U}(s\text{-})} \mathbb{1}_{[0,\infty)^2}(v, s)\mathcal{V}^{r_1}(dv, ds)\right)\right] \\ &= \exp\left(-2an\int_0^t \left(\Upsilon(\overline{U}(0), s) - r_1\right)\, ds\right) \longrightarrow 1 \text{ for } t \to \tau_{EX}(\overline{U}(0)). \end{aligned}$$

If we define $(\hat{\tau}_k, k \in \mathbb{N})$ as the jump times of $\overline{U}$ with $\hat{\tau}_0 = 0$, then $\overline{U}(t)$ will be well-defined, if there exists a k such that $t \in [\hat{\tau}_k, \hat{\tau}_{k+1})$. But we need to verify that

$$\mathbb{P}\left[\lim_{k\to\infty} \hat{\tau}_k < \infty\right] = 0. \tag{2.18}$$

Since every jump of $\overline{U}$ corresponds to an atom of $\mathcal{V}$ and since $\mathcal{V}$ is a Poisson point process with locally finite intensity measure, (2.18) is equivalent to the statement that the integral of $\overline{U}$ does not explode and for this $\overline{U}$ needs to hit infinity in finite time. We will prove that this will not happen, i.e. the claim of (2.17) is true.
We start to note that $\overline{U}$ is a strong Markov process, at least until its explosion, and if we define the linear operator $\tilde{B} : C^1([r_1,\infty)) \to C([r_1,\infty))$ by setting for $f \in C^1([r_1,\infty))$:

$$\tilde{B}(f)(u) = (au^2 - bu)\partial_u f(u) + 2an\int_{r_1}^{u} f(v) - f(u)\,dv, \quad u \in [r_1,\infty),$$

then $\tilde{B}$ is the generator of $\overline{U}$ and we can conclude that the process M_k^f given by

$$M_k^f(t) := f(\overline{U}(t\wedge\tau_k)) - f(\overline{U}(0)) - \int_0^{t\wedge\tau_k} \tilde{B}(f)(\overline{U}(s))\,ds \tag{2.19}$$

is a local martingale. Note that M_k^f is even a proper martingale, because we stop $\overline{U}$ at τ_k. Our next step is to find a function f which is increasing, non-negative and it holds $\tilde{B}(f)(u) \le 0$ for all $u \in [r_1,\infty)$ (Note that $\overline{U}$ never leaves $[r_1,\infty)$). Our candidate for such a function is

$$g(u) = u^q \text{ with } q \in [0,\infty).$$

Applying $\tilde{B}$ to g gives us:

$$\begin{aligned}\tilde{B}(g)(u) &= aqu^{q+1} - bqu^q + 2an\left(\left[\frac{v^{q+1}}{q+1}\right]_{r_1}^{u} - (u-r_1)u^q\right)\\ &= a\left(q + \frac{2n}{q+1} - 2n\right)u^{q+1} + (2anr_1 - bq)\,u^q - \frac{2an}{q+1}r_1^{q+1}.\end{aligned}$$

If we choose $\hat{q} = \frac{2n-1}{2}$, then we have $\hat{q} + \frac{2n}{\hat{q}+1} - 2n < 0$, so we can find a $r_2 \ge r_1$ such that

$$\tilde{B}(g)(u) < 0, \;\; u \in [r_2,\infty).$$

We will now prove that $\overline{U}$ will always return to the interval $[r_1, r_2]$, so it can not hit infinity in finite time, which will imply the claim of the lemma. Let us define a new sequence of stopping times by $T_0 := \inf\{s > 0 : \overline{U}(s) > r_2\}$ and

$$T_k := \inf\{s > T_0 : \overline{U}(s) \le r_2 \text{ or } \overline{U}(s) = r_2 + k\}, \;\; k \in \mathbb{N}.$$

Note that T_k will be almost surely finite for all $k \in \mathbb{N}_0$, i.e. $\mathbb{P}[T_k = \infty] = 0$, because the only way that $\overline{U}$ does not hit the boundary k is by jumping infinitely often down, but every time $\overline{U}$ jumps, the probability that $\overline{U}$ jumps below r_2 is greater than $(r_2 - r_1)/(k - r_1)$. Since $\tilde{B}(g)(\overline{U}(s)) \le \infty$ for $s \in [T_0, T_k]$, $k \in \mathbb{N}$, we can apply the Optional sampling theorem and (2.19) to obtain for all $k \in \mathbb{N}$ that

$$\begin{aligned}\mathbb{E}[g(\overline{U}(T_0))] &= \mathbb{E}[g(\overline{U}(T_k))] - \mathbb{E}\left[\int_{T_0}^{T_k} \tilde{B}(g)(\overline{U}(s))\,ds\right]\\ &\ge \mathbb{E}[g(\overline{U}(T_k))] \ge \mathbb{P}[\overline{U}(T_k) = r_2 + k]g(r_2 + k),\end{aligned}$$

where we used that g is non-negative and that T_k is almost surely finite. By the definition of T_k, we know that $\overline{U}(T_k) \leq r_2$ implies that $T_k = T_{k+n}$ and $\overline{U}(T_{k+n}) \leq r_2$ for all $n \in \mathbb{N}$, therefore

$$\begin{aligned}\mathbb{P}[\exists k \in \mathbb{N} : \overline{U}(T_k) \leq r_2] &= \lim_{k\to\infty} \mathbb{P}[\overline{U}(T_k) = r_2] = 1 - \lim_{k\to\infty} \mathbb{P}[\overline{U}(T_k) = r_2 + k] \\ &\geq 1 - \lim_{k\to\infty} \mathbb{E}[g(\overline{U}(T_0))]/g(k) = 1.\end{aligned}$$

□

We are now going to prove that the level system, see Definition 2.2.1, is well-posed.

Proof of Proposition 2.2.2. Since $\mathcal{V}_{ji}$ is a Poisson point process with intensity measure

$$2a\ell eb[0,\infty) \otimes \ell eb[0,\infty),$$

we can conclude that

$$\mathbb{P}\left[\mathcal{V}_{ji}([0,K] \times [0,K]) < \infty,\, K \in \mathbb{N}, j \in \mathbb{N}, i \in \mathbb{N}\right] = 1. \tag{2.20}$$

We prove the claims of the Proposition 2.2.2 per induction over the level processes $(U_i, i \in \mathbb{N})$. The lowest level never jumps and just follows the differential equation $\dot{u} = au^2 - bu$, so we just set $U_1(t) := \Upsilon(U_i^0, t)$, where the function is taken from the Lemmas 2.2.5 and 2.2.6. Considering the second part of Proposition 2.2.2, there is nothing to show for $i = 1$.
Now let us assume that $U_1, U_2, ..., U_n : \Omega \times [0,\infty) \to [0,\infty]$ are well-defined for $n \in \mathbb{N}$ and it holds

$$\mathbb{P}\left[\mathcal{T}_\infty^i = \mathcal{T}_\infty^1, 1 \leq i \leq n\right] = 1. \tag{2.21}$$

The new process U_{n+1} follows the differential equation $\dot{u} = au^2 - bu$ until the first jump of the process P given by

$$P(t) := \sum_{j=2}^{n+1} \sum_{i=1}^{j-1} \int_0^t \int_{U_{j-1}}^{U_j(s-)} \mathbb{1}_{[0,\infty)\times[0,\infty)}(v,s) \mathcal{V}_{ji}(dv, ds).$$

The process P will jump before U_{n+1} hits infinity, since the integral of the solution of $\dot{u} = au^2 - bu$ explodes, before the solution hits infinity, see the Lemmas 2.2.5 and 2.2.6. After the first jump U_{n+1} evolves again like $\dot{u} = au^2 - bu$, until the second jump. Proceeding in this fashion we can conclude that U_{n+1} is well-defined until $\lim_{m\to\infty} \tau_m$, where $(\tau_m)_{m=1}^\infty$ are the jump times of P. Due to (2.20) the process P will perform only finitely many jumps until time t, if $\sup_{s\leq t} U_{n+1}(s) < \infty$, therefore $\lim_{m\to\infty} \tau_m < t$ implies that $\mathcal{T}_\infty^{n+1} < t$. We conclude that we need to show that $\mathcal{T}_\infty^{n+1} = \mathcal{T}_\infty^1$ almost surely.
Recall the stopping times $\mathcal{T}_k^i := \inf\{t \geq 0 : U_i(t) > k\}$ from Proposition 2.2.2. We are going to prove that $U_{n+1}(t)$ is well-defined and finite for all $t \leq \mathcal{T}_k^n$ and all k. It follows that $\mathcal{T}_\infty^{n+1} \geq \mathcal{T}_k^n$ for all $k \in \mathbb{N}$. But by definition it also holds $U_{n+1} \geq U_n$ (and so $\mathcal{T}_\infty^{n+1} \leq \mathcal{T}_\infty^n$), and hence we can conclude that $\mathcal{T}_\infty^{n+1} = \mathcal{T}_\infty^n = \mathcal{T}_\infty^1$. By setting $U_{n+1} = \infty$ for $t \geq \mathcal{T}_\infty^1$, it will follow that $U_{n+1}(t)$ is well-defined for all $t \in [0,\infty)$.
Let us fix an arbitrary $k_0 \in \mathbb{N}$ with $k \geq \max\{U_0^{n+1}, b/a\}$. We define the process $\overline{U} : \Omega \times [0,\infty) \to [0,\infty]$ as in the Lemma 2.2.7 with $\overline{U}(0) = \max\{U_0^{n+1}, k\}$ and where the Poisson point process $\mathcal{V}$ is defined by setting for each Borel set $\Gamma \subset [0,\infty) \times [0,\infty)$:

$$\mathcal{V}(\Gamma) := \sum_{j=2}^{n+1} \sum_{i=1}^{j-1} \mathcal{V}_{ji}(\Gamma \cap [k_0,\infty)^2).$$

We have chosen $\mathcal{V}$ and the initial value of $\overline{U}$ in such a way that it follows from the stochastic differential equation of $\overline{U}$, see Lemma 2.2.7, that $U_{n+1}(t) \leq \overline{U}(t)$ for $t \leq \mathcal{T}^n_{k_0}$. By Lemma 2.2.7 the process $\overline{U}$ does not explode in finite time, and hence U_{n+1} does neither at least until the stopping time $\mathcal{T}^n_{k_0}$. □

The next lemma tells us that there exists only finitely many particles with a level below r.

Lemma 2.2.8. *We define the process* $Y^r : \Omega \times [0, \infty) \to \mathbb{N}_0 \cup \{\infty\}$ *for* $r \geq \max\{b/a, 0\}$ *by setting*

$$Y^r(t) := \sum_{i=1}^{\infty} \mathbb{1}_{[0,r)}(U_i(t))$$

If the initial levels converge against infinity, i.e. (2.7) *holds true, then* $\mathbb{E}[Y^r(t)] < \infty, t \geq 0, r \geq 0$, *and* $\mathbb{P}[Y^r(t) < \infty, t \geq 0, r \geq 0] = 1$.

As a corollary of Proposition 2.6.6 we will later learn that Y^r is for $r \geq \max\{b/a, 0\}$ a time-continuous, binary branching Galton-Watson process with birth rate ra and death rate $ra - b$, but at the moment we have not proved that and for the proof of Proposition 2.6.6 we will need Lemma 2.2.8.

Proof. Note if we fix a level cap $r \in [\max\{b/a, 0\}, \infty)$, then we can couple Y^r with a pure birth process $\tilde{Y}^r$ with birth rate $2ra, Y^r(0) = \tilde{Y}^r$ and $Y^r \leq \tilde{Y}^r$. The claim follows easily from such a coupling by

$$\mathbb{E}[Y^r(t)] \leq \mathbb{E}[Y^r(t)] = \mathbb{E}[Y(0)]e^{2rat} < \infty.$$

In order to obtain such a coupling we begin by noting that that the number of births effecting Y^r up to time t, which we will denote by $\widehat{N}^r$, is given by

$$\widehat{N}^r(t) := \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \int_0^t \int_{U_{j\text{-}1}(s\text{-})}^{U_j(s\text{-})} \mathbb{1}_{[0,r)}(v) \mathcal{V}_{ij}(dv, ds) = \sum_{i=1}^{\infty} \int_0^t \int_{U_i(s\text{-})}^{\infty} \mathbb{1}_{[0,r)}(v) \mathcal{V}_i(dv, ds)$$

with $\mathcal{V}_i = \sum_{i=1}^{\infty} \mathcal{V}_{ij}$. So if we define $\tilde{Y}^r$ as the solution of the stochastic differential equation

$$\tilde{Y}^r(t) = \sum_{i=1}^{\infty} \int_0^t \int_0^r \mathbb{1}_{[0,\tilde{Y}^r(s\text{-})]}(i) \mathcal{V}_i(dv, ds)$$

with $\tilde{Y}^r(0) = Y^r(0)$, then $\tilde{Y}^r$ has the desired properties, because $Y^r \leq \widehat{N}^r \leq \tilde{Y}^r$. □

2.3 The Genealogy and the Historical Processes

Our previous steps provided not only well-defined level system $(U_i)_{i=1}^{\infty}$, but they also encode a genealogy between the particles. The genealogy is an interesting object in its own right, but for us it is especially important, because it is our key to define historical process in this section and the path-valued process in the next section. Therefore we define the genealogical map

$$\Phi : \Omega \times \mathbb{N} \times [0, \infty) \times [0, \infty) \to \mathbb{N},$$

where $\Phi(j, t, s) = i$ with $t > s$ and $i, j \in \mathbb{N}$ tells us that the particle with index i is at time s the ancestor of the particle with index j at time t (we set $\Phi(j, t, s) = j$ in the case $t \leq s$). The path $s \mapsto \Phi(j, t, s)$ is obtained by walking backwards in time, indeed we start with setting $\Phi(j, t, s) = j$ for all $s \in [t, \infty)$, then we will go backwards in time and $\Phi(j, t, \cdot)$ will jump at the moments, when a birth event occurs involving particles with an index lower or equal to current value of $\Phi(j, t, \cdot)$. Hereby $\Phi(j, t, \cdot)$ will remain constant between two such events and it will always jump down as we decrease s.

Definition 2.3.1. *We define the genealogical map Φ for $t < \mathcal{T}_{EX}$ as the solution of the following system*

$$\Phi(j,t,s) = j + \int_s^t \sum_{i=2}^{j} \sum_{k=1}^{i-1} \int_{U_{i\text{-}1}(\tilde{s}\text{-})}^{U_i(\tilde{s}\text{-})} (k-i) \mathbb{1}_{\{\Phi(j,t,\tilde{s}+)=i\}} \mathcal{V}_{ik}(dv, d\tilde{s}) \tag{2.22}$$

$$+ \int_s^t \sum_{i=2}^{j-1} \sum_{k=1}^{i-1} \int_{U_{i\text{-}1}(\tilde{s}\text{-})}^{U_i(\tilde{s}\text{-})} -\mathbb{1}_{\{\Phi(j,t,\tilde{s}+)>i\}} \mathcal{V}_{ik}(dv, d\tilde{s}), \tag{2.23}$$

where $\Phi(j, t, \tilde{s}+)$ stands for the right hand-side limit, i.e. $\Phi(j, t, \tilde{s}+) = \lim_{v \downarrow \tilde{s}} \Phi(j, t, v)$). Further we set

$$\Phi(j, t, s) = 1 \tag{2.24}$$

for all $j \in \mathbb{N}$, $t \geq \mathcal{T}_{EX}$ and $s \in [0, \infty)$.

The interpretation of the above equation is the following: (2.22) corresponds to the events, when the particle or its ancestor are involved in a birth event as a child, while (2.23) reflects the moments, where a particle is born with an index lower than the current index of our the particle or its ancestor. The justification of (2.24) is that "infinitesimally" before the moment of extinction all particles become closely related to the lowest particle, as we can see by Lemma 2.3.5, by (2.32) from Lemma 2.4.8 and by (2.51) from Proposition 2.6.5. We want to point out that Lemma 2.3.5 and (2.51) from Proposition 2.6.5 are the consequence of (2.22) and (2.23) and not of our choice in (2.24).

Definition 2.3.2. *We define the genealogical distance $d_\Phi : \Omega \times \mathbb{N} \times \mathbb{N} \times [0, \infty) \to [0, \infty]$ as*

$$d_\Phi(i, j, t) := \inf\{0 \leq s \leq t : \Phi(j, t, t-s) = \Phi(i, t, t-s)\},$$

so $d_\Phi(i, j, t)$ is the distance we have to go back in time to find the most recent common ancestor of the particles with index i and j at time t. We use the convention that $\inf\{\emptyset\} = \infty$, i.e. if $i, j \in \mathbb{N}$ are not related to each other at time t, then $d_\Phi(i, j, t) = \infty$.

Remark 2.3.3. *An alternative definition for the genealogical distance would be $\hat{d}_\Phi : \Omega \times (\mathbb{N} \times [0, \infty))^2 \to [0, \infty]$ with*

$$\hat{d}_\Phi((i, t_1), (j, t_2)) := t_1 + t_2 - 2 \inf\{0 \leq s \leq t_1 \wedge t_2 : \Phi(i, t_1, s) \neq \Phi(j, t_2, s)\}.$$

This definition has the advantage that we can compare two different individuals at different time points, but for us the more important information is d_Φ and not $\hat{d}_\Phi$.

The next lemma tells us that the genealogical distance is converging uniformly against zero, when t is approaching the extinction time.

Definition 2.3.4. *Let us assume that $\mathcal{F}^\Phi := \sigma(U_i(s), i \in \mathbb{N}, s < \infty) \vee \sigma(\mathcal{V}_{ji}, 1 \leq i < j < \infty)$ is the σ-algebra containing all the information about the levels and the genealogy.*

Lemma 2.3.5. *We can find for each $\epsilon > 0$ a $\mathcal{F}^\Phi$-measurable random variable $\iota_\Phi^\epsilon : \Omega \to [0, \infty)$ such that $d_\Phi(\omega, 1, i, t) \leq \epsilon$ for all $t \in [\iota_\Phi^\epsilon(\omega), \mathcal{T}_{EX})$ on the event $\{\mathcal{T}_{EX} < \infty\}$.*

Proof. Recall the function τ_{EX} from Lemma 2.2.5 (or from Lemma 2.2.6, if $b = 0$), where $\tau_{EX}(u_0)$ is the explosion time of the solution of the differential equation $\dot{u} = au^2 - bu$ with $u(0) = u_0$. Let us assume that $\mathcal{T}_{EX} < \infty$ and for simplicity that $\epsilon < \mathcal{T}_{EX}$. The first point implies that $U_1(0) > \max\{b/a, 0\}$ and so, with $T_1 := \mathcal{T}_{EX} - \epsilon$,

$$U_2(T_1) > U_1(T_1) > \max\{b/a, 0\}.$$

If we set $T_2 := T_1 + \tau_{EX}(U_2(T_1)) < T_1 + \tau_{EX}(U_1(T_1) = \mathcal{T}_{EX}$, then all particles alive after T_2 must be descendants of the lowest particle which are born after T_1, hence, if $t \in (T_2, \mathcal{T}_{EX}]$, then

$$d_\Phi(i, j, t) < t - T_1 < \mathcal{T}_{EX} - T_1 = \epsilon.$$

So we can choose $\iota_\Phi^\epsilon = T_2$. Note that $\mathcal{T}_{EX}, T_1$ and T_2 are $\mathcal{F}^\Phi$-measurable random variables, hence ι_Φ^ϵ is $\mathcal{F}^\Phi$-measurable. □

Now we use the genealogical map Φ and the ingredients $(\tilde{X}_i)_{i=1}^\infty$ and $(\tilde{N}_i)_{i=1}^\infty$ from Assumption 2.1.2 to build the spatial components $(X_i)_{i=1}^\infty$ together with historical processes $(\mathfrak{X}_i, \mathfrak{N}_i)_{i=1}^\infty$. While $X_i(t)$ is the spatial position at time t of the particle with index i at time t, $\mathfrak{X}_i(t_1, t_2)$ for $t_2 < t_1$ is the spatial position at time t_2 of the ancestor of the particle with the index i-th at time t. The map $\tilde{t} \mapsto \mathfrak{X}_i(t, \tilde{t})$, $0 \leq \tilde{t} \leq t$, is the spatial path from 0 to t of the particle's ancestor. The historical process $\mathfrak{N}_i$ has the same meaning as $\mathfrak{X}_i$, but the role of $\tilde{X}_i$ is replaced with $\tilde{N}_i$. Our notion of a historical process is closely related, but **not identical** with notion of a historical process found in the literature, for example as described in Section II.8 of [39].

Definition 2.3.6 (Historical Processes)**.** *Recall $(\tilde{X}_i)_{i=1}^\infty$ and $(\tilde{N}_i)_{i=1}^\infty$ from Assumption 2.1.2 and that $\mathcal{N}_{lf}([0, \infty) \times [0, \infty))$ is the space of locally finite integer-valued measures over $[0, \infty) \times [0, \infty)$. For all $i \in \mathbb{N}$ we define the processes $\mathfrak{X}_i : \Omega \times [0, \infty) \times [0, \infty) \to \mathbb{R}^d$ and $X_i : \Omega \times [0, \infty) \to \mathbb{R}^d$ by*

$$\begin{aligned} \mathfrak{X}_i(t_1, t_2) &:= X^0_{\Phi(i,t_1,0)} + \sum_{k=1}^\infty \int_0^{t_1 \wedge t_2} \mathbb{1}_{\{\Phi(i,t_1,s)=k\}} \mathbb{1}_{[0,\mathcal{T}_{EX})}(s) d\tilde{X}_i(s); \\ X_i(t) &:= \mathfrak{X}_i(t, t). \end{aligned}$$

When we are writing $\mathfrak{X}_i(t)$, then we are referring to the càdlàg path $\tilde{t} \mapsto \mathfrak{X}_i(t, \tilde{t})$. Further we define the process

$$\mathfrak{N}_i : \Omega \times [0, \infty) \to \mathcal{N}_{lf}([0, \infty) \times [0, \infty)),$$

where the random measure $\mathfrak{N}_i(t)$ has the property that for all $f \in C^+([0, \infty) \times [0, \infty))$:

$$\int_0^\infty \int_0^\infty f(p, s) \mathfrak{N}_i(t, dp, ds) := \sum_{k=1}^\infty \int_0^t \int_0^\infty f(p, s) \mathbb{1}_{\{\Phi(i,t,s)=k\}}(s) \mathbb{1}_{[0,\mathcal{T}_{EX})}(s) \tilde{N}_k(dp, ds).$$

We call $\mathfrak{X}_i$ and $\mathfrak{N}_i$ the ***historical processes****.*

In the next section we introduce the notion of a path-valued process, which works as a "wrapper" for the historical processes and allows an easy application of the Dawson-Watanabe superprocesses. In the appendix we work out the details needed to apply the Kurtz-Rodrigues construction to this "wrapper", but only for the case where the original process is a Lévy process. This will be sufficient for $(\mathfrak{X}_i)_{i=1}^{\infty}$, but not for $(\mathfrak{N}_i)_{i=1}^{\infty}$, therefore we reformulate $(\mathfrak{N}_i)_{i=1}^{\infty}$ as $(\mathfrak{L}_i)_{i=1}^{\infty}$ which are historical processes based on a Lévy process, but encode all the information about $(\mathfrak{N}_i)_{i=1}^{\infty}$, which will saves us a lot of work. If P is a process with càdlàg paths, then ΔP_t is the jump of P at time t, i.e. $\Delta P_t := P_t - P_{t\text{-}}$, where $P_{t\text{-}} = \lim_{s\uparrow t} P_s$.

Lemma 2.3.7. *1. If N is a Poisson point process with intensity measure $\ell eb[0,\infty)\otimes\ell eb[0,\infty)$, if we define the process L by setting*

$$L(t) = \int_0^t \int_0^\infty e^{-p} N(dp, ds), \tag{2.25}$$

then L is a pure-jump Lévy-process with triple $(0, 0, \mathbb{1}_{(0,1]}(l)\frac{1}{l}dl)$.

2. If we define the linear operator $B_L : C_c(\mathbb{R}) \to C(\mathbb{R})$ by setting for all $f \in C_c(\mathbb{R})$

$$B_L(f)(l) := \int_0^1 (f(l+\hat{l}) - f(l))\, \frac{1}{\hat{l}} d\hat{l},\ l \in \mathbb{R},$$

then the martingale problem of B_L is well-posed and L is its solution.

3. On the contrary, if $\hat{L}$ is a copy of L, then the random measure $\hat{N}$ given by

$$\hat{N} := \sum_{s\geq 0} \delta_{(s,\log(\text{-}\Delta\hat{L}(s)))},$$

where we sum over all jump times of $\hat{L}$, is a Poisson point process with intensity measure $\ell eb[0,\infty)\otimes\ell eb[0,\infty)$.

Proof. (1) Since N is Poisson point process with translation invariant intensity measure, the process L as the integral of a deterministic, time-invariant integrand with respect to N is a Lévy process. Obviously L is a pure jump process with positive jumps with a size in $(0,1)$, hence the Lévy triple of L has the form $(0, 0, B_L^\eta)$. Let $R(\delta)$ be the rate of jump of L, whose size is bigger than $\delta \in (0,1)$. Recalling the definition of L in (2.25) we can derive the following relationship between R and the Lévy measure B_L^η of L:

$$\int_0^\infty \mathbb{1}_{[0,\delta)}(y)\, B_L^\eta(dy) = R(\delta) = \int_0^\infty \mathbb{1}_{[0,\text{-}\log(\delta))}(y)\, dy.$$

A quick calculation gives us $R(\delta) = -\log(\delta)$. Taking the derivative gives us the above mentioned form of the Lévy-measure of L.
(2) This follows from Proposition E.1.10 from the appendix, where B_L takes the role of B_W.
(3) Since L and $\hat{L}$ are copies of each other and since $N = \sum_{s\geq 0}\delta_{(s,\text{-}\log(\Delta L(s)))}$, we can conclude from the fact that N is a Poisson point process with intensity measure $\ell eb[0,\infty)\otimes\ell eb[0,\infty)$, that the same is true for $\hat{N}$. □

This small lemma allows to encode the atoms of a Poisson point process by the jump times and jump sizes of a Lévy process. In the next lemma apply this insight to our situation.

Lemma 2.3.8. *Recall* $(\tilde{N}_i)_{i=1}^{\infty}$ *from Assumption 2.1.2. Let us define for all* $i \in \mathbb{N}$ *the processes:*

$$\begin{aligned}
&\tilde{L}_i : \Omega \times [0,\infty) \to \mathbb{R}, && \tilde{L}_i(t) := \int_0^t \int_0^\infty e^{-p} \tilde{N}_i(dp, ds) \\
&\mathfrak{L}_i : \Omega \times [0,\infty) \times [0,\infty) \to \mathbb{R}; && \mathfrak{L}_i(t_1,t_2) := \int_0^{t_1 \wedge t_2} \int_0^\infty e^{-p}\, \mathfrak{N}_i(t_1, dp, ds) \\
&L_i : \Omega \times [0,\infty) \to \mathbb{R}; && L_i(t) := \mathfrak{L}_i(t,t)
\end{aligned}$$

then $(\tilde{L}_i)_{i=1}^{\infty}$ *are independent copies of the Lévy process* L *from Lemma 2.3.7. Further it holds*

$$\mathfrak{L}_j(t_1,t_2) := \sum_{i=1}^{\infty} \int_0^{t_1 \wedge t_2} \mathbb{1}_{\{\Phi(j,t_1,s)=i\}} \mathbb{1}_{[0,\mathcal{T}_{EX})}(s) d\tilde{L}_i(s); \tag{2.26}$$

Proof. That $\tilde{L}_i$ is a copy of L follows from the Lemma 2.3.7 and $(\tilde{L}_i)_{i=1}^{\infty}$ form an independent sequence of processes, because $(\tilde{N}_i)_{i=1}^{\infty}$ does. For (2.26), we put the definition of $\mathfrak{N}_j$ in the definition of $\mathfrak{L}_j$ and then we use the definition of $(\tilde{L}_i)_{i=1}^{\infty}$, and obtain

$$\begin{aligned}
\mathfrak{L}_j(t_1,t_2) &= \sum_{i=1}^{\infty} \int_0^{t_1 \wedge t_2} \int_0^\infty \mathbb{1}_{\{\Phi(j,t_1,s)=i\}} e^{-p}\, \tilde{N}_i(dp, ds) \\
&= \sum_{i=1}^{\infty} \int_0^{t_1 \wedge t_2} \int_0^\infty \mathbb{1}_{\{\Phi(j,t_1,s)=i\}} \mathbb{1}_{[0,\mathcal{T}_{EX})}(s) d\tilde{L}_i(s).
\end{aligned}$$

□

2.4 Path-valued Processes

We begin this section by introducing the notion of path-valued process. Our presentation will be very close to the one of Perkins found in Chapter II.2 in [40]. The path-valued process transforms the historical process from the last section to a time-homogeneous Markov process and hence allows us to apply the results of the appendix to our processes in the following sections. We start with the state space of the path-valued process.

Definition 2.4.1. *Let* E *be a Polish space and* $\mathbb{D}([0,\infty), E)$ *be the space of càdlàg paths in* E. *We define the space* $\mathbb{D}_t([0,\infty), E)$ *for each* $t \geq 0$ *as the subset of* $\mathbb{D}([0,\infty), E)$ *consisting of the paths stopped at time* t, *i.e.*

$$\mathbb{D}_t([0,\infty), E) := \{x \in \mathbb{D}([0,\infty), E)\,;\ \boldsymbol{x}(s) = \boldsymbol{x}(t) \text{ for } s \geq t\}.$$

We define the space $\widehat{\mathbb{D}}([0,\infty), E)$ *as the union*

$$\widehat{\mathbb{D}}([0,\infty), E) := \cup_{t \geq 0} \big(\{t\} \times \mathbb{D}_t([0,\infty), E)\big).$$

Additionally we define a metric on $\widehat{\mathbb{D}}([0,\infty), E)$ *by*

$$d_{\widehat{\mathbb{D}},E}\big((t,\boldsymbol{x}),(s,\boldsymbol{y})\big) = d_{\mathbb{D},E}(\boldsymbol{x},\boldsymbol{y}) + |t-s|, \tag{2.27}$$

where $d_{\mathbb{D},E}$ *is a metrics on* $\mathbb{D}([0,\infty), E)$ *so that* $(\mathbb{D}([0,\infty), E), d_{\mathbb{D},E})$ *is a complete, separable metric space. We write* $\mathbb{B}(\hat{D}(E))$ *for the Borel algebra of the metric space* $(\widehat{\mathbb{D}}([0,\infty), E), d_{\widehat{\mathbb{D}},E})$.

Remark 2.4.2. *In order to simplify proofs we explicitly choose $d_{\mathbb{D},E}$ to be the metric on $\mathbb{D}([0,\infty),E)$ found in (5.2) on Page 117 in [14]. Let us define Λ as the collection of Lipschitz functions $\lambda:[0,\infty)\to[0,\infty)$ satisfying*

$$\gamma(\lambda) := \sup_{s>t\geq 0}\left|\log\left(\frac{\lambda(s)-\lambda(t)}{s-t}\right)\right| < \infty$$

For $\mathbf{x},\mathbf{y}\in\mathbb{D}([0,\infty),E)$ we define

$$d_{\mathbb{D},E}(\mathbf{x},\mathbf{y}) := \inf_{\lambda\in\Lambda}\{\gamma(\lambda)\vee\int_0^\infty e^{-u}\sup_{s\leq 0} d(\mathbf{x}(s\wedge u),\mathbf{y}(s\wedge u))\wedge 1\,du\}.$$

This metric $d_{\mathbb{D},E}$ has the property that

$$d_{\mathbb{D},E}(\mathbf{x},\mathbf{y}) \leq \sup_{s\geq 0} d(\mathbf{x}(s),\mathbf{y}(s)) \tag{2.28}$$

and note that $\sup_{s\geq 0} d(\mathbf{x}(s),\mathbf{y}(s)) = \sup_{s\in[0,t_1\vee t_2]} d(\mathbf{x}(s),\mathbf{y}(s)) < \infty$, if $\mathbf{x}\in\mathbb{D}_{t_1}([0,\infty),E)$ and $\mathbf{y}\in\mathbb{D}_{t_2}([0,\infty),E)$.

The additional time component has the function to ensure that the path-valued process is a time-homogeneous process.

Lemma 2.4.3. *The space $(\widehat{\mathbb{D}}([0,\infty),E),d_{\widehat{\mathbb{D}},E})$ is complete and separable. Indeed the space $\widehat{\mathbb{D}}([0,\infty),E)$ together with the topology implied by the metric $d_{\widehat{\mathbb{D}},E}$ is a Polish space.*

Proof. Please note that the metric space $(\mathbb{D}_t([0,\infty),E),d_{\widehat{\mathbb{D}},E})$ is homeomorphic to the complete separable metric space $(\mathbb{D}([0,t],E),d_t)$, where $\mathbb{D}([0,t],E)$ is the space of càdlàg paths from $[0,t]$ to E and d_t is a suitable metric, see the Theorem 12.2 in [4]. So we can choose the set $\Gamma := ((\mathfrak{w}_n^q,q),n\in\mathbb{N},q\in\mathbb{Q}\cap[0,\infty))$, where $(\mathfrak{w}_n^q,n\in\mathbb{N})\subset\mathbb{D}([0,q],E)$ is for a fixed $q\in\mathbb{Q}\cap[0,\infty)$ a countable and dense set in $(\mathbb{D}_q([0,\infty),E),d_{\widehat{\mathbb{D}},E})$ (note that this set is also dense in $(\mathbb{D}_s([0,\infty),E),d_{\mathbb{D},E})$, when $s\leq q$). The set Γ is countable and dense in $(\widehat{\mathbb{D}}([0,\infty),E),d_{\widehat{\mathbb{D}},E})$. $\square$

Definition 2.4.4. *If $W:\tilde{\Omega}\times[0,\infty)\to E$ is a process defined on some probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ with state space E, then we define the path-valued process*

$$\mathbb{W}:\tilde{\Omega}\times[0,\infty)\to\widehat{\mathbb{D}}([0,\infty),E)$$

associated with W by setting $\mathbb{W}(t)=(t,(W(t\wedge s),s\geq 0))$.

Remark 2.4.5. *In the Appendix E.2, see Corollary E.2.3, we we will learn that when the original process W has continuous paths, then the same is true for path-valued process $\mathbb{W}$.*

The path-valued process $\mathbb{W}$ associated with any stochastic process W is a Markov process, because the state $\mathbb{W}(t)$ contains all information about the path of $\mathbb{W}$ until time t. But the path-valued process is not a Feller process, because its state space is not locally compact. As a consequence many results associated with the theory of Feller processes are not directly applicable. It turns out that a good setting to study of path-valued processes is the so called class of Borel strong Markov process (BSMP) as defined the Saint-Flour Lectures by Edwin Perkins, see Chapter II.2 in [40]. The class of BSMP is very broad and contains processes like Lévy processes and superprocesses. It can be considered as a natural extension of the notion of a Feller process to general non-local compact Polish spaces like $\widehat{\mathbb{D}}([0,\infty),E)$. For more details about the Borel strong Markov processes, see Section A.2. Luckily the path-valued process associated with a BSMP is under mild conditions, which are satisfied by Lévy processes, again a BSMP.
In the next lemma we define the "product" of X and L, which we denote by W.

Lemma 2.4.6. *If we define W as $W = (X, L)$, where X is as in the Assumptions* 1.2.3 *and L is as in Lemma 2.3.7, where we assume that X and L are independent, then process W is a Lévy process in $\mathbb{R}^{d+1}$. The Lévy measure B_W^η of W satisfies $\int_{\mathbb{R}^d} ||w||^2 B_W^\eta(dw) < \infty$.*

Proof. The process W is a Lévy process, because X and L are Lévy processes and independent. Further by Assumption 1.2.3 it holds $\int_{\mathbb{R}^d} ||x||^2 B_X^\eta(dx) < \infty$, where B_X^η is the Lévy measure of X, and the Lévy measure of L is given by $dB_L^\eta(l) = \frac{1}{l}\mathbb{1}_{(0,1)}dl$, hence

$$\int_{\mathbb{R}^{d+1}} ||w||^2 B_W^\eta(dw) = \int_{\mathbb{R}^d} ||x||^2 B_X^\eta(dx) + \int_{\mathbb{R}} ||l||^2 B_L^\eta(dl) < \infty.$$

□

Definition 2.4.7. *Since we will work with path-valued processes of Lévy processes with state space $\mathbb{R}^{d+1}$, we introduce the notation:*

$$\mathfrak{D} := \widehat{\mathbb{D}}([0,\infty), \mathbb{R}^{d+1})$$

and we will often write an element $\mathfrak{w} \in \mathfrak{D}$ as a triple $(t, \mathbf{x}, \mathbf{l})$ with $\mathbf{x} \in \mathbb{D}([0,\infty), \mathbb{R}^d)$, $\mathbf{l} \in \mathbb{D}([0,\infty), \mathbb{R})$ and $t \in [0,\infty)$, hereby stands $\mathbf{x}$ for the path containing the first d coordinates and $\mathbf{l}$ for the path containing the $(d+1)$-th coordinate.

Lemma 2.4.8. *Let us define for each $i \in \mathbb{N}$:*

$$\begin{aligned}
&\tilde{W}_i : \Omega \times [0,\infty) \to \mathbb{R}^{d+1}, && \tilde{W}_i(t) := (\tilde{X}_i(t), \tilde{L}_i(t));\\
&W_i : \Omega \times [0,\infty) \to \mathbb{R}^{d+1}, && W_i(t) := (X_i(t), L_i(t));\\
&\mathfrak{W}_i : \Omega \times [0,\infty) \times [0,\infty) \to \mathbb{R}^{d+1}, && \mathfrak{W}_i(t_1,t_2) := (\mathfrak{X}_i(t_1,t_2), \mathfrak{L}_i(t_1,t_2));\\
&\mathbb{W}_i : \Omega \times [0,\infty) \to \mathfrak{D}, && \mathbb{W}_i(t) := (t, \mathfrak{W}_i(t,\cdot\,));
\end{aligned}$$

1. *The collection $(\tilde{W}_i)_{i=1}^\infty$ forms a collection of independent copies of the Lévy process W from Lemma 2.4.6. Further it holds for $t_1, t_2 \in [0,\infty)$:*

$$\mathfrak{W}_j(t_1,t_2) = \sum_{i=1}^{\infty} \int_0^{t_1 \wedge t_2} \mathbb{1}_{\{\Phi(j,t_1,s)=i\}} \mathbb{1}_{[0,\mathcal{T}_{EX})}(s) d\tilde{W}_i(s) \tag{2.29}$$

2. *The processes $(W_i)_{i=1}^\infty$ are the solution of an infinite system of stochastic differential equations given by*

$$\begin{aligned}
W_j(t) = \tilde{W}_j(0) &+ \int_0^t \mathbb{1}_{[0,\infty)}(U_j(s\text{-}))d\tilde{W}_j(s)\\
&+ \sum_{i=1}^{j-1} \int_0^t \int_{U_{j\text{-}1}(s\text{-})}^{U_j(s\text{-})} (W_i(s\text{-}) - W_j(s\text{-}))\, \mathcal{V}_{ji}(dv, ds)\\
&+ \sum_{i=2}^{j-1}\sum_{n=1}^{i-1} \int_0^t \int_{U_{i\text{-}1}(s\text{-})}^{U_i(s\text{-})} (W_{j\text{-}1}(s\text{-}) - W_j(s\text{-}))\, \mathcal{V}_{in}(dv, ds).
\end{aligned} \tag{2.30}$$

3. *On the event $\{\mathcal{T}_{EX} < \infty\}$ it holds for all $t \geq \mathcal{T}_{EX}$ and all $j \in \mathbb{N}$ that $\mathfrak{W}_j(t,s) = \mathfrak{W}_1(\mathcal{T}_{EX}\text{-}, s), s \geq 0$, and that*

$$\mathfrak{W}_1(\mathcal{T}_{EX}, s) = W_1(0) + \tilde{W}_1(s)\mathbb{1}_{[0,\mathcal{T}_{EX})}(s) + \tilde{W}_1(\mathcal{T}_{EX}\text{-})\mathbb{1}_{[\mathcal{T}_{EX},\infty)}(s), s \geq 0, \tag{2.31}$$

   and it holds for every $j \in \mathbb{N}$:

$$\mathbb{P}\left[\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}} \lim_{t\uparrow\mathcal{T}_{EX}} \sup_{s\geq 0} ||\mathfrak{W}_j(t,s) - \mathfrak{W}_1(\mathcal{T}_{EX}\text{-}, s)|| = 0\right] = 1. \tag{2.32}$$

Proof of Lemma 2.4.8. By definition, $(\tilde{W}_i)_{i=1}^{\infty}$ is identical with $(\tilde{X}_i, \tilde{L}_i)_{i=1}^{\infty}$ and it inherits its independency from the ones of $(\tilde{X}_i)_{i=1}^{\infty}$ and $(\tilde{L}_i)_{i=1}^{\infty}$. The identity in (2.29) follows directly from the similar identities for $\mathfrak{X}_j$ and $\mathfrak{L}_j$, see Lemma 2.3.8.
The second point, showing that $(W_i)_{i=1}^{\infty}$ is the solution of the infinite equation system (2.30), is more subtle. We start by defining for a fixed j the jump process $\widehat{N} : \Omega \times [0, \infty) \to \mathbb{N}_0 \cup \{\infty\}$ which jumps whenever a new particle is born with an index lower than j, i.e.

$$\widehat{N}(t) := \sum_{i=1}^{j-1} \sum_{n=i+1}^{j} \int_0^t \int_{U_{n\text{-}1}(s\text{-})}^{U_n(s\text{-})} \mathbb{1}_{[0,\infty)\times[0,\infty)}(v, s)\mathcal{V}_{ni}(dv, ds). \tag{2.33}$$

So when we define the stopping times $(\widehat{\tau}_k)_{k=0}^{\infty}$ as the jump times of $\widehat{N}$, then these are exactly the times in which the genealogy of $(\mathbb{W}_i, U_i)_{i=1}^{j}$ is changing. Indeed the genealogy remains the same between two jump times $\widehat{\tau}_k$ and $\widehat{\tau}_{k+1}$ in the sense that

$$\Phi(i, s_1, s) = \Phi(i, s_2, s), \quad s \geq 0,\ 1 \leq i \leq j, \tag{2.34}$$

whenever $\widehat{\tau}_k \leq s_1 \leq s_2 \leq \widehat{\tau}_k$. The identity (2.34) becomes crucial by showing that W_j is equal to the right-hand side of (2.30). Let us fix a time point t and let us divide the path of W_j up to t by

$$W_j(t) = W_j(t) - \sum_{\widehat{\tau}_k < t} \Delta W_j(\widehat{\tau}_k) + \sum_{\widehat{\tau}_k < t} \Delta W_j(\widehat{\tau}_k). \tag{2.35}$$

We can reformulate (2.35) by writing

$$W_j(t) - \sum_{\widehat{\tau}_k < t} \Delta W_j(\widehat{\tau}_k) = W_j(t) - W_j(\widehat{\tau}_K) + \sum_{\widehat{\tau}_k < K} W_j(\widehat{\tau}_{k+1}\text{-}) - W_j(\widehat{\tau}_k), \tag{2.36}$$

where K stands for the index of the last jump occurring before t, i.e. $K := \sup\{k; \widehat{\tau}_k < t\}$. The previous mentioned identity (2.34) tell us that for each $k \in \mathbb{N}$ and $1 \leq i \leq j$ we have that $W_i(\widehat{\tau}_k) := \mathfrak{W}_i(\widehat{\tau}_k, \widehat{\tau}_k)$ is identical with $\mathfrak{W}_i(\tilde{t}, \widehat{\tau}_k)$ as long as $\widehat{\tau}_k \leq \tilde{t} < \widehat{\tau}_{k+1}$. From (2.34) we can also conclude that $\Phi(i, s_1, s_2) = i$, whenever $\widehat{\tau}_k \leq s_1 \leq s_2 \leq \widehat{\tau}_{k+1}$, for all $1 \leq i \leq j$. Combining all these findings with $t \in [\widehat{\tau}_K, \widehat{\tau}_{K+1})$ we come to the conclusion that

$$\begin{aligned} W_j(t) - W_j(\widehat{\tau}_K) &= \mathfrak{W}_j(t, t) - \mathfrak{W}_j(t, \widehat{\tau}_K) = \sum_{i=1}^{\infty} \int_{\widehat{\tau}_K}^t \mathbb{1}_{\{\Phi(j,t,s)=i\}} \mathbb{1}_{[0,\infty)}(U_j(s\text{-}))\, d\tilde{W}_i(s) \\ &= \int_{\widehat{\tau}_K}^t \mathbb{1}_{[0,\infty)}(U_j(s\text{-}))\, d\tilde{W}_j(s). \end{aligned}$$

Based on the same thinking we can also derive for all stopping times $(\widehat{\tau}_k)_{k=0}^{\infty}$:

$$\begin{aligned} W_j(\widehat{\tau}_{k+1}\text{-}) - W_j(\widehat{\tau}_k) &= \lim_{\tilde{t} \uparrow \widehat{\tau}_{k+1}} W_j(\tilde{t}) - W_j(\widehat{\tau}_k) \\ &= \lim_{\tilde{t} \uparrow \widehat{\tau}_{k+1}} \int_{\widehat{\tau}_k}^{\tilde{t}} \mathbb{1}_{[0,\infty)}(U_j(s\text{-}))\, d\tilde{W}_j(s) = \int_{\widehat{\tau}_k}^{\widehat{\tau}_{k+1}} \mathbb{1}_{[0,\infty)}(U_j(s\text{-}))\, d\tilde{W}_j(s). \end{aligned}$$

All in all, (2.36) has been transformed into

$$\int_{\widehat{\tau}_K}^t \mathbb{1}_{[0,\infty)}(U_j(s\text{-}))\, d\tilde{W}_j(s) + \sum_{k<K} \int_{\widehat{\tau}_k}^{\widehat{\tau}_{k+1}} \mathbb{1}_{[0,\infty)}(U_j(s\text{-}))\, d\tilde{W}_j(s) = \int_0^t \mathbb{1}_{[0,\infty)}(U_j(s\text{-}))\, d\tilde{W}_j(s).$$

For the second and third line of (2.30), we need to rewrite the second sum in (2.35). This sum can be expressed as an integral with respect to $\widehat{N}$ by

$$\sum_{\widehat{\tau}_k < t} \Delta W_j(\widehat{\tau}_k) = \int_0^t \Delta W_j(\widehat{\tau}_k) d\widehat{N}_s$$

and by the definition of $\widehat{N}$, see (2.33), the integral is identical with

$$\sum_{i=1}^{j-1} \int_0^t \int_{U_{i\text{-}1}(s\text{-})}^{U_j(s\text{-})} (W_i(s\text{-}) - W_n(s\text{-})) \mathcal{V}_{ji}(dv, ds) + \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} \int_0^t \int_{U_{i\text{-}1}(s\text{-})}^{U_i(s\text{-})} (W_{n\text{-}1}(s\text{-}) - W_n(s\text{-})) \mathcal{V}_{ij}(dv, ds).$$

Altogether, we have seen that (2.35) is equivalent to the equation system (2.30), hence W_j is identical with the right-hand side of (2.30), or in other words $(W_i)_{i=1}^{\infty}$ is the solution of the infinite equation system (2.30).
We prove (2.32) by induction and we assume for the rest of the proof that we are working on the event $\{\mathcal{T}_{EX} < \infty\}$. The statement (2.32) is true for $i = 1$, because $\tilde{W}_1$ is a Lévy process and Lévy processes have no fixed jumps. Our induction hypothesis is now that (2.32) is true for $1 \leq i \leq j-1$ and as an induction step we prove (2.32) for a fixed $j \in \mathbb{N}$. For any $t \in [\widehat{\tau}_k, \widehat{\tau}_{k+1})$ we can decompose the path $s \mapsto \mathfrak{W}_j(t, s)$ into

$$\mathfrak{W}_j(t, s) = \mathfrak{W}_j(\widehat{\tau}_k, s), \quad s \leq \widehat{\tau}_k \tag{2.37}$$

and

$$\mathfrak{W}_j(t, s) = W_j(\widehat{\tau}_k) + \tilde{W}_j(t \wedge s) - \tilde{W}_j(\widehat{\tau}_k), \quad s \geq \tau_k. \tag{2.38}$$

Since this is true for all $t \in [\widehat{\tau}_k, \widehat{\tau}_{k+1})$, we can bound $\sup_{s\geq 0} ||\mathfrak{W}_j(t, s) - \mathfrak{W}_1(t, s)||$ for such a t by

$$\sup_{s\in[0,\widehat{\tau}_k)} ||\mathfrak{W}_j(\widehat{\tau}_k, s) - \mathfrak{W}_1(\widehat{\tau}_k, s)|| + ||W_j(\widehat{\tau}_k\text{-}) - W_1(\widehat{\tau}_k\text{-})|| \tag{2.39}$$

$$+ \sup_{s\in[\widehat{\tau}_k,\widehat{\tau}_{k+1})} ||\tilde{W}_j(s) - \tilde{W}_1(s) - \tilde{W}_j(\widehat{\tau}_k) + \tilde{W}_1(\widehat{\tau}_k)||. \tag{2.40}$$

So we can prove (2.32) for j, when we can show that (2.39) and (2.40) are converging to 0, when $k \to \infty$. According to the induction hypothesis, we know that it holds

$$\sup_{s\in[0,\widehat{\tau}_k)} ||\mathfrak{W}_i(\widehat{\tau}_k, s) - \mathfrak{W}_1(\widehat{\tau}_k, s)|| \overset{k\to\infty}{\longrightarrow} 0, \quad ||W_j(\widehat{\tau}_k\text{-}) - W_1(\widehat{\tau}_k\text{-})|||| \overset{k\to\infty}{\longrightarrow} 0,\ 1 \leq i \leq j\text{-}1.$$

Since $W_j(\widehat{\tau}_k)$ is an element of $\{W_1(\widehat{\tau}_k\text{-}), ..., W_{j\text{-}1}(\widehat{\tau}_k\text{-})\}$ and the path $\mathfrak{W}_j(\widehat{\tau}_k, \cdot)$ is an element of $\{\mathfrak{W}_1(\widehat{\tau}_k, \cdot), ..., \mathfrak{W}_{j\text{-}1}(\widehat{\tau}_k, \cdot)\}$, (2.39) is also converging to 0 for our fixed j, when k goes to infinity. For the remaining term (2.40) let us define

$$S_k^j(s) := \tilde{W}_1(\widehat{\tau}_k + s) - \tilde{W}_j(\widehat{\tau}_k + s) - \tilde{W}_j(\widehat{\tau}_k) + \tilde{W}_1(\widehat{\tau}_k),\ s \geq 0,\ k \in \mathbb{N}_0,$$

and also

$$\Gamma_k^{j,\epsilon} := \left\{ \sup_{s\in[0,\widehat{\tau}_{k+1}-\widehat{\tau}_k)} ||S_j^k(s)||_2^2 \geq \epsilon \right\}, k \in \mathbb{N}_0, \epsilon > 0.$$

The genealogical σ-algebra $\mathcal{F}^{\Phi}$ from Definition 2.3.4 contains the information about the genealogy, hence $(\tau_k)_{k=1}^{\infty}$ and $\mathcal{T}_{EX}$ are measurable with respect to $\mathcal{F}^{\Phi}$. But the processes $(\tilde{W}_i)_{i=1}^{\infty}$ are independent from $\mathcal{F}^\Phi$ and so S_k^j is for each k a Lévy process independent from $\mathcal{F}^\Phi$. If we apply Lemma E.1.12, then there exists a constant $K_S(\mathcal{T}_{EX}) > 0$ with

$$\mathbb{P}\left[\Gamma_k^{j,\epsilon}|\mathcal{F}^\Phi\right] \le \epsilon^{-2} K_S(\mathcal{T}_{EX})(\widehat{\tau}_{k+1} - \widehat{\tau}_k).$$

We can apply Lemma E.1.12, because S_k^j and $\tilde{W}$ have the same Lévy measure B_W^η and it holds $\int_{\mathbb{R}^{d+1}} ||w||_1 B_W^\eta(dw) < \infty$. We have:

$$\sum_{k=0}^{\infty} \mathbb{P}[\Gamma_k^{j,\epsilon}|\mathcal{F}^\Phi] \le \sum_{k=0}^{\infty} \epsilon^{-2} K_S(\mathcal{T}_{EX})(\widehat{\tau}_{k+1} - \widehat{\tau}_k) = \epsilon^{-2} K_S \mathcal{T}_{EX} < \infty.$$

Conditioned on $\mathcal{F}^\Phi$ it follows from Borel-Cantelli that only a finite number of the events $(\Gamma_k^{j,\epsilon})_{k=1}^\infty$ occurs for each $\epsilon > 0$ (on the event $\{\mathcal{T}_{EX} < \infty\}$), and so

$$\mathbb{P}\left[\mathcal{T}_{EX} < \infty, \limsup_{k\to\infty} \Gamma_k^{j,\epsilon}\right] = \mathbb{E}\left[\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}} \mathbb{E}\left[\limsup_{k\to\infty} \mathbb{1}_{\Gamma_k^{j,\epsilon}} |\mathcal{F}^\Phi\right]\right] = 0.$$

Since this is true for all $\epsilon > 0$, we can conclude that (2.40) must converge against 0 almost surely, if $k \to \infty$ (on the event $\{\mathcal{T}_{EX} < \infty\}$). Further, since $\sup_{s\ge 0} ||\mathfrak{W}_j(t,s) - \mathfrak{W}_1(t,s)||$ is for each $t \in [\widehat{\tau}_k, \widehat{\tau}_{k+1})$ bounded by (2.39) and (2.40), we have proven (2.32) for j, which completes our induction step. $\square$

Lemma 2.4.9. *Let us assume that $\overline{W}_1$ and $\overline{W}_2$ are two independent copies of W starting in zero. If we define the function $\vartheta : [0,\infty) \to [0,\infty)$ by*

$$\vartheta(t) := \mathbb{E}\left[\sup_{s\le t} |\overline{W}_1(s) - \overline{W}_2(s)|\right], \quad t \ge 0,$$

then ϑ is continuous with $\vartheta(0) = 0$.

Proof. Since $\overline{W}_i(0) = 0, i \in \{1,2\}$, we have $\vartheta(0) = 0$. The continuity of ϑ follows from the fact that $\tilde{\vartheta}(t) := \sup_{s\le t} |\overline{W}_1(s) - \overline{W}_2|$ is non-decreasing and càdlàg without fixed jumps. Applying the Lebesgue dominated convergence theorem together with $\mathbb{E}[\sup_{s\le t} ||W(s)||^2] < \infty$, see Lemma E.1.12, gives us now $\lim_{s\uparrow t} \vartheta(s) = \vartheta(t)$ and $\lim_{s\downarrow t} \vartheta(s) = \vartheta(t)$, hence ϑ is right- and left-continuous. $\square$

Remark 2.4.10. *Since $W_i = (X_i, L_i)$, we can express $(X_i)_{i=1}^\infty$ and respectively $(L_i)_{i=1}^\infty$ as the solution of the equation systems similar to (2.30), where we replace the role of $(\tilde{W}_i)_{i=1}^\infty$ with $(\tilde{X}_i)_{i=1}^\infty$, resp. with $(\tilde{L}_i)_{i=1}^\infty$. The derivation works in the same way. The convergence in (2.32) implies that*

$$\mathbb{P}\left[\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}} \lim_{t\uparrow\mathcal{T}_{EX}} \sup_{s\ge 0} ||\mathfrak{X}_j(t,s) - \mathfrak{X}_1(\mathcal{T}_{EX}\text{-}, s)|| = 0,\ i \in \mathbb{N}\right] = 1,$$
$$\mathbb{P}\left[\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}} \lim_{t\uparrow\mathcal{T}_{EX}} \sup_{s\ge 0} ||\mathfrak{L}_j(t,s) - \mathfrak{L}_1(\mathcal{T}_{EX}\text{-}, s)|| = 0,\ i \in \mathbb{N}\right] = 1,$$

where $\mathfrak{X}_1(\mathcal{T}_{EX}\text{-},\cdot)$ and respectively $\mathfrak{L}_1(\mathcal{T}_{EX}\text{-},\cdot)$ are defined similar as $\mathfrak{W}_1(\mathcal{T}_{EX}\text{-},\cdot)$ in (2.31) with $\tilde{W}_1$ replaced by $\tilde{X}_1$ or respectively $\tilde{L}_1$ depending on the process. Note further that (2.32) implies

$$\mathbb{P}\left[\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}} \lim_{t\to\mathcal{T}_{EX}} d_{\widehat{\mathbb{D}},E}(\mathbb{W}_i(t), \mathbb{W}_1(\mathcal{T}_{EX}\text{-})) = 0, i \in \mathbb{N}\right] = 1.$$

Remark 2.4.11. *It is worth mentioning that* $(\mathbb{W}_i)_{i=1}^{\infty}$ *is not well-defined, if we would allow* $\mathbb{P}[U_1(0) = U_2(0)] > 0$*, which is prevented by (2.7), because in this situation* U_1 *and* U_2 *are both not affected by births and explode simultaneously. This itself is not a problem, indeed* $(U_i)_{i=1}^{\infty}$ *would be still well-defined and the same would hold true for* $\mathbb{W}_1$ *and* $\mathbb{W}_2$*. But* $\mathbb{W}_2(t)$ *would not necessarily converge to* $\mathbb{W}_1(\mathcal{T}_{EX}\text{-})$ *and this causes the problem. Indeed, if we choose* $k := \max\{i \in \mathbb{N} : U_i(0) = U_1(0)\}$*, then* $\mathbb{W}_j(t)$ *for some* $j > k$ *would constantly switch between the values of* $\mathbb{W}_1, ..., \mathbb{W}_k$*, and since the latter are not converging against the same value,* $\mathbb{W}_j(\mathcal{T}_{EX}\text{-})$ *does not exist.*

Lemma 2.4.12. *Recall the* σ*-algebra* $\mathcal{F}^{\Phi}$ *from Lemma 2.3.5, which contains all information about the levels and the genealogy, then it holds for all* $i, j \in \mathbb{N}, t \geq 0$ *on the event* $\{d_{\Phi}(i,j,t) < \infty\} \in \mathcal{F}^{\Phi}$ *that*

$$\mathbb{E}\left[d_{\widehat{\mathbb{D}},E}\left(\mathbb{W}_i(t), \mathbb{W}_j(t)\right) | \mathcal{F}^{\Phi}\right] \leq \vartheta(d_{\Phi}(i,j,t)) \tag{2.41}$$

Remark 2.4.13. *We can also derive a similar inequality for the event* $\{d_{\Phi}(\tilde{i}, i, t) = \infty\}$*, indeed* $\tilde{i}$ *and* i *are not related, indeed on the event* $\{d_{\Phi}(\tilde{i}, i, t) = \infty\}$ *holds*

$$\mathbb{E}\left[d_{\widehat{\mathbb{D}},E}\left(\mathbb{W}_{\tilde{i}}(t), \mathbb{W}_i(t)\right) | \mathcal{F}^{\Phi}\right] \leq \mathbb{E}\left[|W_j(0) - W_i(0)|\right] + \vartheta(t).$$

The proof works identical as the one below.

Proof. Let us assume that the event $\{d_{\Phi}(i,j,t) < \infty\}$ is true, setting $t^* := t - d_{\Phi}(i,j,t) > 0$, we define $\overline{W}_i, \overline{W}_j : \Omega \times [0,\infty) \to \mathbb{R}^{d+j}$

$$\overline{W}_l(s) := \sum_{k=j}^{\infty} \int_{t^*}^{s+t^*} \mathbb{1}_{\{\Phi(l,t,\tilde{s})=k\}} d\tilde{W}_k(\tilde{s}), \quad s \in [0,\infty), l \in \{i,j\}.$$

Since $(\tilde{W}_k)_{k=1}^{\infty}$ are independent copies of the Lévy process W, which are also independent from $\mathcal{F}^{\Phi}$, and since $\Phi(i,t,s) \neq \Phi(j,t,s)$ for all $s \in [t^*, \infty)$, we can conclude that $(\overline{W}_i, \overline{W}_j)$ conditioned on $\mathcal{F}^{\Phi}$ are two independent copies of W starting in zero. By the expression (2.29) for $\mathfrak{W}_i(t,\cdot), \mathfrak{W}_j(t,\cdot)$ we have

$$\begin{aligned}\overline{W}_i(s) &= \mathfrak{W}_i(t, s+t^*) - \mathfrak{W}_i(t,t^*), \\ \overline{W}_j(s) &= \mathfrak{W}_j(t, s+t^*) - \mathfrak{W}_j(t,t^*), \quad s \in [0, t-t^*].\end{aligned}$$

Further since $\mathfrak{W}_i(t,s) = \mathfrak{W}_j(t,s)$ for $s \in [t^*, t]^c$ (which implies that $\mathfrak{W}_i(t,t^*) = \mathfrak{W}_j(t,t^*)$), it holds

$$d_{\widehat{\mathbb{D}},E}\left(\mathbb{W}_i(t), \mathbb{W}_j(t)\right) \leq \sup_{s \in [0,\infty)} |\mathfrak{W}_i(t,s) - \mathfrak{W}_j(t,s)| = \sup_{s \in [0,t-t^*]} |\overline{W}_i(s) - \overline{W}_j(s)|.$$

From the above inequality, the fact that $(\overline{W}_i, \overline{W}_j)$ conditioned on $\mathcal{F}^{\Phi}$ are independent and that $\{d_{\Phi}(i,j,t) < \infty\} \in \mathcal{F}^{\Phi}$ we can conclude that

$$\mathbb{E}\left[\mathbb{1}_{\{d_{\Phi}(i,j,t)<\infty\}} d_{\widehat{\mathbb{D}},E}\left(\mathbb{W}_j(t), \mathbb{W}_i(t)\right) | \mathcal{F}^{\Phi}\right] \leq \mathbb{1}_{\{d_{\Phi}(i,j,t)<\infty\}} \vartheta(d_{\Phi}(i,j,t)).$$

□

The main reasons for the introduction of the path-valued process $\mathbb{W}$ that is suits better in the theory of Kurtz and Rodrigues, but we still want to work with the historical processes. Therefore we want to express $\mathfrak{X}_i$ and $\mathfrak{Y}_i$ as functionals of $\mathbb{W}_i$. Before we proceed with the next section we want to discuss shortly how we can express the historical processes $(\mathfrak{X}_i, \mathfrak{Y}_i)_{i=1}^{\infty}$ as functionals of the path-valued process $(\mathbb{W})_{i=1}^{\infty}$. The case of $\mathfrak{X}_i$ and its relatives is straightforward:

Lemma 2.4.14. *We define the maps*

$$\begin{aligned}
&\pi_X\colon \mathfrak{D} \to \mathbb{R}^d, && (t,\mathbf{x},\mathbf{l}) \mapsto \mathbf{x}(t);\\
&\pi_{\mathfrak{X}}\colon \mathfrak{D}\times[0,\infty) \to \mathbb{R}^d, && ((t,\mathbf{x},\mathbf{l}),s) \mapsto \mathbf{x}(t\wedge s);\\
&\pi_{\mathfrak{L}}\colon \mathfrak{D}\times[0,\infty) \to \mathbb{R}, && ((t,\mathbf{x},\mathbf{l}),s) \mapsto \mathbf{l}(t\wedge s).
\end{aligned}$$

It holds for all $s,t\in[0,\infty)$ and all $i\in\mathbb{N}$ that:

$$\pi_X(\mathbb{W}_i(t)) = X_i(t); \qquad \pi_{\mathfrak{X}}(\mathbb{W}_i(t),s) = \mathfrak{X}_i(t,s); \qquad \pi_{\mathfrak{L}}(\mathbb{W}_i(t),s) = \mathfrak{L}_i(t,s).$$

Proof. All of this follows directly from the Definitions 2.3.6, 2.3.8 and 2.4.8. □

If we want to express $\mathfrak{N}_i$ as a functional of $\mathbb{W}_i$, the situation is a little bit more interesting.

Definition 2.4.15. *We define the map $\pi_{\mathfrak{N}} : \mathfrak{D}\times\mathbb{B}([0,\infty)\times[0,\infty)) \to [0,\infty]$ by setting for $\mathfrak{w} = (\mathbf{x},\mathbf{l},t)$ and every Borel set $\Gamma\in\mathbb{B}([0,\infty)\times[0,\infty))$:*

$$\pi_{\mathfrak{N}}(\mathfrak{w},\Gamma) = \lim_{\delta\downarrow 0} \sum_{s\le t,\Delta\mathbf{l}(s)>\delta} \mathbb{1}_\Gamma((-\log\Delta\mathbf{l}(s),s)),$$

where $\Delta\mathbf{l}(s)$ is the jump size of the path $\mathbf{l}$ at time s and the sum goes over all jumps with a size bigger than δ. Please note that the limit always exists in $[0,\infty]$, because the left-hand side forms an increasing sequence.

Lemma 2.4.16. *If $\Gamma\in\mathbb{B}([0,\infty)\times[0,\infty))$, then it holds $\pi_{\mathfrak{N}}(\mathbb{W}_i(t),\Gamma) = \mathfrak{N}_i(t,\Gamma)$.*

Proof. Recall that by definition of $\mathfrak{L}_i(t,\cdot)$ the path $s\mapsto\mathfrak{L}_i(t,s)$ has a jump of size $e^{-p}>0$ for some $p\ge 0$ at time t, if and only if $\mathfrak{N}(t)$ has an atom in (p,t), hence

$$\begin{aligned}
\pi_{\mathfrak{N}}(\mathbb{W}_i(t),\Gamma) &= \lim_{\delta\downarrow 0} \sum_{s\le t,\Delta\mathfrak{L}(t,s)>\delta} \mathbb{1}_\Gamma((-\log\Delta\mathfrak{L}(t,s),s))\\
&= \lim_{\delta\downarrow 0} \mathfrak{N}_i(t,\Gamma\cap\{(p,s)\in[0,\infty)^2; e^{-p}>\delta\}) = \mathfrak{N}_i(t,\Gamma).
\end{aligned}$$

□

Lemma 2.4.17. *For every measurable, non-negative map $f\in\mathbf{M}^+(\mathbb{R}^d\times[0,\infty)\times[0,\infty))$ we can find a measurable map $\pi^f_{\mathfrak{X},\mathfrak{N}} : \mathfrak{D}\to[0,\infty]$ such that*

$$\pi^f_{\mathfrak{X},\mathfrak{N}}(\mathbb{W}_i(t)) = \int_0^\infty\int_0^\infty f(\mathfrak{X}_i(t,s\text{-}),p,s)\,\mathfrak{N}_i(t,dp,ds) \quad t\in[0,\infty), i\in\mathbb{N}.$$

Proof. Recall that by definition of $\mathfrak{L}_i(t,\cdot)$ the path $s\mapsto\mathfrak{L}_i(t,s)$ has a jump of size $e^{-p}>0$ for some $p\ge 0$ at time t, if and only if $\mathfrak{N}(t)$ has an atom in (p,t), amongst other things this means that $\mathfrak{N}(t)$ has no atom at 0. Let us assume that $f(x,p,s) = \mathbb{1}_{\Gamma\times(p_1,p_2)\times[0,s_1)}(x,p,s)$, then the map $\pi^f_{\mathfrak{X},\mathfrak{N}}$ can be described as the point-wise limit of the sequence $(\pi^{f,k}_{\mathfrak{X},\mathfrak{N}}, k\in\mathbb{N})$, where $\pi^{f,k}_{\mathfrak{X},\mathfrak{N}} : \mathfrak{D}\to[0,\infty]$ is the measurable map given by

$$\pi^{f,k}_{\mathfrak{X},\mathfrak{N}}(t,\mathbf{x},\mathbf{l}) = \sum_{i=0}^{n-1} \mathbf{x}(t,s_1 i/n)\mathbb{1}_{[p_1+\frac{1}{n},p_2)}(\Delta^n(\mathbf{l}(t,(i+1)n^{-1}s_1) - \mathbf{l}(t,in^{-1}s_1)).$$

Consequently the statement is true for $f(x,p,s)$. Since the statement remains also true under multiplication and when we consider increasing sequences of functions, we can conclude by the monotone class theorem, see Theorem (2.2) from [42], that the statement is true for general $f\in\mathbf{M}^+(\mathbb{R}^d\times[0,\infty)\times[0,\infty))$. □

2.5 Connection to the Empirical KR-Rep. I

Here we show that our collection $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$ is a Kurtz-Rodrigues representation. We will focus on the more complicated case $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$ instead of $(X_i, U_i)_{i=1}^{\infty}$, but all the statement in this section are also true for $(X_i, U_i)_{i=1}^{\infty}$. We recall the state space of $\overline{\mathcal{N}}(E \times [0,\infty))$ of the Kurtz-Rodrigues representation with E being a Polish space, see Definition 1.1.2, and the map $\gamma_E^{\mathbf{\Xi}} : \overline{\mathcal{N}}(E \times [0,\infty)) \to \mathcal{M}_f(E)$ from (1.18).

Definition 2.5.1. *We define the map $\gamma_E^{\mathbf{Q}} : \mathbf{S}(E) \to \mathcal{M}_1(E)$ by*

$$\gamma_E^{\mathbf{Q}}\left((x_i, u_i)_{i=1}^{\infty}\right) := \begin{cases} \lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=1}^{n} \delta_{x_i}, & \text{if the limit exists in the weak topology,} \\ \delta_{\hat{x}}, & \text{if the limit does not exist in the weak topology,} \end{cases} \tag{2.42}$$

where $\hat{x} \in E$ is an arbitrary chosen element of E.

Definition 2.5.2. *Based on the previously constructed $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$ we define the processes*

$$\begin{aligned}
&\boldsymbol{\xi}^{\mathbb{W}} : \Omega \times [0,\infty) \to \overline{\mathcal{N}}(\mathfrak{D} \times [0,\infty)), && \boldsymbol{\xi}_t^{\mathbb{W}} := \sum_{i=1}^{\infty} \delta_{(\mathbb{W}_i(t), U_i(t))}; \\
&\tilde{\mathbf{\Xi}}^{\mathbb{W}} : \Omega \times [0,\infty) \to \mathcal{M}_f(\mathfrak{D}), && \tilde{\mathbf{\Xi}}_t^{\mathbb{W}} := \gamma_{\mathfrak{D}}^{\mathbf{\Xi}}(\boldsymbol{\xi}_t^{\mathbb{W}}); \\
&\tilde{\mathbf{Q}}^{\mathbb{W}} : \Omega \times [0,\infty) \to \mathcal{M}_1(\mathfrak{D}), && \tilde{\mathbf{Q}}_t^{\mathbb{W}} := \gamma_{\mathfrak{D}}^{\mathbf{Q}}((\mathbb{W}_i(t))_{i=1}^{\infty}); \\
&\tilde{Y} : \Omega \times [0,\infty) \to [0,\infty), && \tilde{Y}_t := \tilde{\mathbf{\Xi}}_t^{\mathbb{W}}(\mathfrak{D}).
\end{aligned}$$

Further we define the filtrations $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}, \mathcal{F}^{\mathbf{\Xi},\mathbb{W}}$ and $\mathcal{F}^{\mathbf{Q},Y,\mathbb{W}}$ as the complete, right-continuous versions of the natural filtrations of $\boldsymbol{\xi}^{\mathbb{W}}, \tilde{\mathbf{\Xi}}^{\mathbb{W}}$ and the pair $(\tilde{\mathbf{Q}}^{\mathbb{W}}, \tilde{Y})$. We are further defining for each $r \geq \frac{b}{a}$ and $m \in \mathbb{N}$ the processes

$$\begin{aligned}
&\mathbf{\Xi}^{\mathbb{W},r} : \Omega \times [0,\infty) \to \mathcal{N}_f(\mathfrak{D}), && \mathbf{\Xi}_t^{\mathbb{W},r} := \sum_{i=1}^{\infty} \delta_{\mathbb{W}_i(t)} \mathbb{1}_{[0,r)}(U_i(t)); \\
&\boldsymbol{\xi}^{\mathbb{W},\geq r} : \Omega \times [0,\infty) \to \overline{\mathcal{N}}(\mathfrak{D} \times [0,\infty)), && \boldsymbol{\xi}_t^{\mathbb{W},\geq r} := \sum_{i=1}^{\infty} \delta_{(\mathbb{W}_i(t), U_i(t))} \mathbb{1}_{[r,\infty)}(U_i(t)); \\
&\mathbf{Q}^{\mathbb{W},m} : \Omega \times [0,\infty) \to \mathcal{M}_1(\mathfrak{D}), && \mathbf{Q}_t^{\mathbb{W},m} := \sum_{i=1}^{m} \delta_{\mathbb{W}_i(t)}.
\end{aligned}$$

Further let us define the filtrations $\mathcal{F}^{\mathbf{\Xi},\mathbb{W},r}$ and $\mathcal{F}^{\mathbf{Q},\mathbb{W},m}$ as the right-continuous completion of the filtrations $\tilde{\mathcal{F}}^{\mathbf{\Xi},\mathbb{W},r}$ and $\tilde{\mathcal{F}}^{\mathbf{Q},\mathbb{W},m}$ given by $\tilde{\mathcal{F}}_t^{\mathbf{\Xi},\mathbb{W},r} = \sigma\big(\mathbf{\Xi}_s^{\mathbb{W},r}, \boldsymbol{\xi}_s^{\mathbb{W},\geq r}; s \leq t\big)$ and $\tilde{\mathcal{F}}_t^{\mathbf{Q},\mathbb{W},m} = \sigma\big(\mathbf{Q}_s^{\mathbb{W},m}, (\mathbb{W}_i(s), U_i(s))_{i=m}^{\infty}; s \leq t\big)$.

Remark 2.5.3. *We write $\tilde{\mathbf{\Xi}}^{\mathbb{W}}, \tilde{\mathbf{Q}}^{\mathbb{W}}$ and $\tilde{Y}$ instead of $\mathbf{\Xi}^{\mathbb{W}}, \mathbf{Q}^{\mathbb{W}}$ and Y to mark the preliminary status of these processes, it will be a consequence of the Kurtz-Rodrigues theory that the three processes admit continuous modifications, which will be denote by $\mathbf{\Xi}^{\mathbb{W}}, \mathbf{Q}^{\mathbb{W}}$ and Y.*

Remark 2.5.4. *If we recall the processes $(Y^r, r \geq \max\{b/a, 0\})$ from Lemma 2.2.8, with $Y_t^r := \sum_{i=1}^{\infty} \mathbb{1}_{[0,r)}(U_i(t))$, then it holds $Y_t^r = \mathbf{\Xi}_t^{\mathbb{W},r}(\mathfrak{D})$.*

Remark 2.5.5. *We could also define $\boldsymbol{\xi}^X, \mathbf{\Xi}^X, \mathbf{Q}^X$ and so forth, by replacing $(\mathbb{W}_i)_{i=1}^{\infty}$ and $\mathfrak{D}$ with $(X_i)_{i=1}^{\infty}$ and $\mathbb{R}^d$. The following statements in the rest of this chapter are in adapted form also true for $\boldsymbol{\xi}^X, \mathbf{\Xi}^X$ and $\mathbf{Q}^X$. The same holds for $\boldsymbol{\xi}^W, \mathbf{\Xi}^W, \mathbf{Q}^W$.*

We have to ask ourselves, are the processes defined above well-defined?

Lemma 2.5.6. *The processes from Definition 2.5.2 are well-defined.*

Proof. Since $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$ are well-defined, we can say the same of $\mathbf{Q}^{\mathbb{W},m}$ and $\mathbf{Q}^{\mathbb{W}}$. Considering $\boldsymbol{\xi}_t^{\mathbb{W}}, \tilde{\boldsymbol{\Xi}}_t^{\mathbb{W}}, \boldsymbol{\Xi}^{\mathbb{W},r}$ and $\boldsymbol{\xi}_t^{\mathbb{W},\geq r}$ all we need to show is that $\boldsymbol{\xi}_t^{\mathbb{W}}(\mathfrak{D} \times [0,r)) < \infty$ for all $t \geq 0$ almost surely. This follows from Lemma 2.2.8 which states $\mathbb{P}[Y_t^r < \infty, t \geq 0, r \geq 0] = 1$ and Remark 2.5.4. $\square$

The goal of the remaining part of this section is to show that $\boldsymbol{\xi}^{\mathbb{W}}$ is an empirical Kurtz-Rodrigues representation and we do this by showing that $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$ is an ordered KR-Rep. This in turn will be accomplished by showing $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$ satisfies the martingale problem associated with a linear operator $\mathbf{A}^o_{B_{\mathbb{W}}}$ which we will present below. This will be a very technical undertaking and those details are not essential for the understanding of the next section. A firm understanding of the empirical Kurtz-Rodrigues representation is recommended, so it may helpful to read the Appendix B.2 in parallel, where we also find the definition of many expressions used in this section.
We are going to define now the martingale problem of the ordered Kurtz-Rodrigues representation. Since we want to define it for the spatial motion X as well as $\mathbb{W}$, we work with the abstract spatial motion $\mathbb{X}$ and let $\mathbf{B}$ be once again its generator, which we use to define set of functions $\mathfrak{g}(\mathbf{B})$, which form the building block of the martingale problem. If V is a set contained in a vector space, we denote by $\mathit{span}(V)$ the linear span of V.

Definition 2.5.7. *Assume that E is a Polish space, $\mathbf{B} : \mathcal{D}(\mathbf{B}) \subset C_b(E) \to C_b(E)$ is a linear conservative operator and its domain $\mathcal{D}(\mathbf{B})$ is closed under multiplication. Fixing $K \in [0,\infty)$, $r > 0$, $m \in (0,1), n \in \mathbb{N} \cup \{\infty\}$ we define $\mathfrak{G}(\mathbf{B}, K, r, m, n) \subset C_b(\mathbf{S}(E))$ as the class of functions $G : \mathbf{S}(E) \to [0,1]$ with the form*

$$G\big((x_i, u_i)_{i=1}^{\infty}\big) = \prod_{i=1}^{\infty} g_i(x_i, u_i),$$

with $g_i \in \mathfrak{g}(\mathbf{B}, K, r, m)$, see Definition B.2.7, for $1 \leq i \leq n$ and $g_i = \mathbb{1}_{E \times [0,\infty)}$ for $i \geq n+1$, if $n \in \mathbb{N}$, and $g_i \in \mathfrak{g}(\mathbf{B}, K, r, m)$ for all $i \in \mathbb{N}$, if $n = \infty$. Further we define

$$\mathfrak{G}(\mathbf{B}) := \bigcup_{K>0, r>0, m\in(0,1), n\in\mathbb{N}} \mathfrak{G}(\mathbf{B}, K, r, m, n).$$

Definition 2.5.8. *Assume that $\mathbf{B} : \mathcal{D}(\mathbf{B}) \subset C_b(E) \to C_b(E)$ satisfies the Conditions B.2.2, $a > 0$ and $b \in \mathbb{R}$. For the parameters $(\mathbf{B}, a, b)$ we define the operator*

$$\mathbf{A}^o_{\mathbf{B}} : C(\mathbf{S}(E)) \supset \mathcal{D}(\mathbf{B}) \to C(\mathbf{S}(E))$$

in the following way: We set $\mathcal{D}(\mathbf{A}^o_{\mathbf{B}}) = \mathit{span}(\mathfrak{G}(\mathbf{B}))$ and we define $\mathbf{A}^o_{\mathbf{B}}(G) : \mathbf{S}(E) \to \mathbb{R}$ for $G := \prod_{i=1}^{\infty} g_i \in \mathfrak{G}(\mathbf{B})$ as the function given for $(x_i, u_i)_{i=1}^{\infty} \in \mathbf{S}(E) \cap (E \times [0,\infty))^{\infty}$, i.e. $u_i < \infty$ for $i \in \mathbb{N}$, by

$$\begin{aligned}
\mathbf{A}^o_{\mathbf{B}}(G)\big((x_i, u_i)_{i=1}^{\infty}\big) &= \prod_{l=1}^{\infty} g_l(x_l, u_l) \sum_{i=1}^{\infty} \frac{B(g_i)(x_i, u_i)}{g_i(x_i, u_i)} \\
&+ \prod_{l=1}^{\infty} g_l(x_l, u_l) \sum_{i=1}^{\infty} (a u_i^2 - b u_i) \frac{\partial_u g_i(x_i, u_i)}{g_i(x_i, u_i)} \\
&+ \prod_{l=1}^{\infty} g_l(x_l, u_l) \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \int_{u_{j-1}}^{u_j} 2a \left(g_i(x_i, v) \prod_{m=j}^{\infty} \frac{g_{m+1}(x_m, u_m)}{g_m(x_m, u_m)} - 1 \right) dv
\end{aligned}$$

and for $(x_i,u_i)_{i=1}^\infty \in \mathbf{S}(E)$ *with* $u_i = \infty$ *for some* $i \in \mathbb{N}$ *by* $\mathbf{A}^o_{\mathbf{B}}(G)\big((x_i,u_i)_{i=1}^\infty\big) = 0$.

Remark 2.5.9. *Note that the sums in the expression of* $\mathbf{A}^o_{\mathbf{B}}(G)$ *in the above definition are well-defined, because the sums consist of only finitely many terms. Indeed if* $\mathfrak{G}(\mathbf{B},K,r,m,n)$ *with* $n \in \mathbb{N}$, *then expressions inside the sums involving* g_i *with* $i > n$ *are zero. For this reason we have excluded the classes* $\mathfrak{G}(\mathbf{B},K,r,m,n)$ *with* $n = \infty$.

Definition 2.5.10. *We call the solution of the martingale problem* $\boldsymbol{MP}(\mathbf{A}^o_{\mathbf{B}})$ *an ordered Kurtz-Rodrigues representation.*

We will not prove here that the martingale problem $\mathbf{MP}(\mathbf{A}^o_{\mathbf{B}})$ has a unique solution. Since our main goal is to show that $\boldsymbol{\xi}^{\mathbb{W}}$ is an empirical KR-representation, we think of the martingale problem $\mathbf{MP}(\mathbf{A}^o_{\mathbf{B}})$ as an intermediate step, which we have to take but will not investigate more than necessary. Because it is sufficient for our purpose to show that $(\mathbb{W}_i,U_i)_{i=1}^\infty$ is a solution of $\mathbf{MP}(\mathbf{A}^o_{\mathbf{B}})$. This also makes it for us unnecessary to handle with an delicate problem, which is that the martingale problem $\mathbf{MP}(\mathbf{A}^o_{\mathbf{B}},(x_i,u_i)_{i=1}^\infty)$ is actually not well-defined for all $(x_i,u_i)_{i=1}^\infty \in \mathbf{S}(E)$. Indeed if $u_1 = u_2$, then the same problem at the moment of extinction occurs as described in Remark 2.4.11. Thankfully we can avoid such difficulties, since we assume that the initial levels satisfy the Assumption 2.10 and so all initial levels are different from each other. A further consequence of Assumption 2.10 is that the initial levels $(U_i(0))$ satisfy $U_i(0) \to \infty$ for $i \to \infty$. This is interesting, because if $G \in \mathfrak{G}(\mathbf{B},K,r_1,m,\infty)$ for some r_1, then $\mathbf{A}^o_{\mathbf{B}}(G)((x_i,u_i)_{i=1}^\infty)$ is still well-defined as long as the number of pairs (x_i,u_i) with a level below r_1 is finite, because if $u_i \geq r_1$, then the expression inside of the sums of $\mathbf{A}^o_{\mathbf{B}}(G)$ are equal to zero. Note that the latter is almost surely true for all time points for our level processes $(U_i)_{i=1}^\infty$ as a consequence of Lemma 2.2.8. This motivates the following extension of Definition 2.5.8.

Definition 2.5.11. *If* $G \in \mathfrak{G}(\mathbf{B},K,r,m,\infty)$, *then we define* $\mathbf{A}^o_{\mathbf{B}}(G) \in B(\mathbf{S}(E))$ *as in Definition 2.5.8, but with* $\mathbf{A}^o_{\mathbf{B}}(G)((x_i,u_i)_{i=1}^\infty) = 0$, *if the number of pairs* (x_i,u_i) *with a level below* r *is infinite. Note that this will almost surely never happen due to chosen initial distribution, see Assumption 2.10.*

After we have discussed the operator $\mathbf{A}^o_{\mathbf{B}}$, its test functions and the martingale problem $\mathbf{MP}(\mathbf{A}^o_{\mathbf{B}})$ we return to our $(\mathbb{W}_i,U_i)_{i=1}^\infty$. The state space $\mathfrak{D}$ of our path-valued process $\mathbb{W}$ takes the role of the Polish space E. Considering the linear operator $\mathbf{B}$ we will state the following proposition.

Proposition 2.5.12. *There exists a linear operator*

$$B_{\mathbb{W}} : C_b(\mathfrak{D}) \supset \mathcal{D}(B_{\mathbb{W}}) \to C_b(\mathfrak{D}),$$

such that the martingale problem $\boldsymbol{MP}(B_{\mathbb{W}},\delta_{(t_0,\mathfrak{w}_0)})$ *is well-posed for any* $(t_0,\mathfrak{w}_0) \in \mathfrak{D}$ *and the path-valued process* $\mathbb{W}$ *is the unique solution of this martingale problem. Further the domain* $\mathcal{D}(B_{\mathbb{W}})$ *and* $B_{\mathbb{W}}$ *satisfies the Conditions B.2.2 found in the Appendix B.1.*

Proof. See Definition (E.2.7) and Proposition E.2.16. □

In the appendix we also give an explicit form of $B_{\mathbb{W}}$. Since the explicit form $B_{\mathbb{W}}$ is very complicated and we make no use of the explicit form, we will not present it here and refer to the appendix. For us it is only important to know that such a operator $B_{\mathbb{W}}$ with the above properties exists.

Proposition 2.5.13. *Let us fix a $G \in \mathfrak{G}(B_{\mathbb{W}}, K, r, m, n)$, where $K \geq 0, r \geq \max\{b/a, 0\}$ and $n \in \mathbb{N}$ (i.e. $n \neq \infty$), then it holds for the ordered representation $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$ that the process $M : \Omega \times [0, \infty) \to \mathbb{R}$ given by*

$$M(t) = G\left((\mathbb{W}_i(t), U_i(t))_{i=1}^{\infty}\right) - G\left((\mathbb{W}_i(0), U_i(0))_{i=1}^{\infty}\right) - \int_0^t \mathbf{A}^o_{B_{\mathbb{W}}}(G)\left((\mathbb{W}_i(s), U_i(s))_{i=1}^{\infty}\right) ds$$

is a càdlàg martingale with respect to the filtration $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$.

Corollary 2.5.14. *The statements of Proposition 2.5.13 are also true, when we set $n = \infty$, i.e. for $G \in \mathfrak{G}(\mathbf{B}, K, r, m, \infty)$.*

Proof of Proposition 2.5.13. We introduce $\mathbf{W}_i = (\mathbb{W}_i, U_i), i \in \mathbb{N}$, and $\mathbf{W} = (\mathbb{W}_i, U_i)_{i=1}^{\infty}$ to save space and let us define the jump process $\widehat{N} : \Omega \times [0, \infty) \to \mathbb{N}_0$ by

$$\widehat{N}(t) := \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \int_0^t \int_{U_{j\text{-}1}(s\text{-})}^{U_j(s\text{-})} \mathbb{1}_{[0,r)}(U_1(s)) \mathcal{V}_{ji}(dv, ds),$$

hence $\widehat{N}$ counts the birth events affecting $(\mathbf{W})_{i=1}^{n}$, but stops to do so as soon as U_1 hits the barrier r. Let us write $(\widehat{\tau}_k)_{k=0}^{\infty}$ for the jump times of $\widehat{N}$ with the convention that $\widehat{\tau}_0 = 0$ and $\widehat{\tau}_k = \widehat{\tau}_{k\text{-}1}$, when there is no k-th jump, i.e. $k > \widehat{N}(t)$ for all $t \geq 0$. As an abbreviation we define $G_{j\downarrow i} : \mathfrak{D}^n \times [0, \infty] \to \mathbb{R}$ by setting for all $\mathbf{w} \in \mathfrak{D}^n \times [0, \infty]$:

$$G_{j\downarrow i}(\mathbf{w}, v) := \left[\prod_{k=1}^{j-1} g_k(\mathbf{w}_k, u_k)\right] (g_j(\mathbf{w}_i, v) - g_j(\mathbf{w}_j, v)) \left[\prod_{k=j}^{\infty} g_{k+1}(\mathbf{w}_k, u_k)\right].$$

Our next step is to decompose the path of $G(\mathbf{W})$ based on the times $(\widehat{\tau}_k)_{k=0}^{\infty}$.

$$\begin{aligned} G(\mathbf{W}(t)) - G(\mathbf{W}(0)) = G(\mathbf{W}(t)) - G(\mathbf{W}(\widehat{\tau}_{\widehat{N}(t)})) + \sum_{i=1}^{\widehat{N}(t)} G(\mathbf{W}(\widehat{\tau}_k\text{-})) - G(\mathbf{W}(\widehat{\tau}_{k\text{-}1})) \\ + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \int_0^t \int_{U_{j\text{-}1}(s\text{-})}^{U_j(s\text{-})} G_{j\downarrow i}(\mathbf{W}(s\text{-}), v) \mathcal{V}_{ji}(dv, ds). \end{aligned}$$

Let us set

$$P_k(t) := G(\mathbf{W}(t\text{-})) - G(\mathbf{W}(t \wedge \widehat{\tau}_{k\text{-}1})),$$

then we have almost surely the identity

$$G(\mathbf{W}(t)) - G(\mathbf{W}(0)) = \sum_{k=1}^{\infty} P_k(t \wedge \widehat{\tau}_k) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \int_0^t \int_{U_{j\text{-}1}(s\text{-})}^{U_j(s\text{-})} G_{j\downarrow i}(\mathbf{W}(s\text{-}), v) \mathcal{V}_{ji}(dv, ds),$$

because $P_k(t) = 0$ for $k > \widehat{N}(t)$ and $P_k(t) = G(\mathbf{W}(t)) - G(t \wedge \mathbf{W}(\widehat{\tau}_{k\text{-}1}))$ almost surely for $k = \widehat{N}(t)$, here we used also that $\mathbb{P}[\mathbf{W}(t\text{-}) = \mathbf{W}(t)] = 1$. When we stop the process $G(\mathbf{W})$ at the stopping time $\widehat{\tau}_{k_1}$ for a $k_1 \in \mathbb{N}$, then we have:

$$G(\mathbf{W}(t \wedge \widehat{\tau}_{k_1})) - G(\mathbf{W}(0)) = \sum_{k=1}^{k_1} P_k(t \wedge \widehat{\tau}_k) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \int_0^{t \wedge \widehat{\tau}_{k_1}} \int_{U_{j\text{-}1}(s\text{-})}^{U_j(s\text{-})} G_{j\downarrow i}(\mathbf{W}(s\text{-}), v) \mathcal{V}_{ji}(dv, ds).$$

With Lemma 2.4.8 in our mind we recall the set $(\tilde{W}_i)_{i=1}^{\infty}$ of independent copies of Lévy processes W used to build $(\mathbb{W}_i)_{i=1}^{\infty}$ and that $(\mathbb{W}_i)_{i=1}^{n}$ behaves between the stopping times $(\widehat{\tau}_k)_{k=0}^{\infty}$ like n independent copies of the path-valued process associated with W. Similarly $(U_i)_{i=1}^{n}$ evolves according to the differential equation $\dot{u} = au^2 - bu$ between two stopping times. In order to make use of Lemma 2.4.8 we want to construct for each $k \in \mathbb{N}$ a sequence $(\mathfrak{W}_i^k, U_i^k)_{i=1}^{\infty}$ such that the first n components $(\mathfrak{W}_i^k, U_i^k)_{i=1}^{n}$ are identical with $(\mathfrak{W}_i, U_i)_{i=1}^{n}$ on the time interval $[0, \widehat{\tau}_k)$, but while the original $(\mathfrak{W}_i, U_i)_{i=1}^{n}$ are affected by the birth events at time $(\widehat{\tau}_l, l \geq k)$ and the events after this one, the new processes $(\mathfrak{W}_i^k, U_i^k)_{i=1}^{n}$ continue their path by evolving like n independent copies of path-valued process and $(U_i^k)_{i=1}^{n}$ evolve according to the differential equation, this means that $(\mathfrak{W}_i^k, U_i^k)_{i=1}^{n}$ are not affected by birth events at $(\widehat{\tau}_l, l \geq k)$. We obtain $(\mathfrak{W}_i^k, U_i^k)_{i=1}^{\infty}$ by setting for $k \in \mathbb{N}_0$ and $1 \leq i \leq n$:

$$\mathfrak{W}_i^k(t,s) := \begin{cases} \mathfrak{W}_i(t,s), & t \leq \widehat{\tau}_{k\text{-}1}, s \geq 0 \\ \mathfrak{W}_i(\widehat{\tau}_{k\text{-}1}, s), & t > \widehat{\tau}_{k\text{-}1}, s \leq \widehat{\tau}_{k\text{-}1}, \\ \mathfrak{W}_i(\widehat{\tau}_{k\text{-}1}, \widehat{\tau}_{k\text{-}1}) + \tilde{W}_i(t \wedge s) - \tilde{W}_i(\widehat{\tau}_{k\text{-}1}), & t, s > \widehat{\tau}_{k\text{-}1}. \end{cases}$$

and $\mathfrak{W}_i^k = \mathfrak{W}_i$ for $k \geq n$. Further, we recall the function $\Upsilon : [0,\infty) \times [0,\infty) \to [0,\infty]$ from Lemma 2.15, if $b \neq 0$, or Lemma 2.16, if $b = 0$, which was defined by saying that $t \mapsto \Upsilon(u_0, t)$ is the solution of the differential equation $\dot{u} = au^2 - bu$ starting in u_0. We define for $k \in \mathbb{N}$ and $1 \leq i \leq n$ the processes:

$$U_i^k(t) := \begin{cases} U_i(t), & t \leq \widehat{\tau}_{k\text{-}1}, \\ \Upsilon(U_i(\widehat{\tau}_{k\text{-}1}), t - \widehat{\tau}_{k\text{-}1}), & t > \widehat{\tau}_{k\text{-}1}, \end{cases}$$

and set $U_i^k = U_i$ for $i > n$. If we now set

$$\mathbb{W}_i^k(t) = (t, \mathfrak{W}_i^k(t,\cdot)), \quad \mathbf{W}_i^k(t) = (\mathbb{W}_i^k(t), U_i^k(t)), \quad Q_k(t) = G(\mathbf{W}^k(t)) - G(\mathbf{W}^k(t \wedge \widehat{\tau}_{k\text{-}1})),$$

with $\mathbf{W}^k = (\mathbf{W}_i^k)_{i=1}^{\infty}$, then we obtain new processes with the property that for all $t \in [0, \widehat{\tau}_k)$ holds

$$\mathbf{W}_i^k(t) = \mathbf{W}_i(t), \quad G(\mathbf{W}^k(t)) = G(\mathbf{W}(t)), \quad Q_k(t) = P_k(t). \tag{2.43}$$

Further the processes $(\mathbf{W}_i^k)_{i=1}^{n} = (\mathfrak{W}_i^k, U_i^k)_{i=1}^{n}$ are not affected by the birth events at $(\widehat{\tau}_l, l \geq k)$, but instead evolve like n independent processes and hence show the desired behavior described before the definition of $\mathfrak{W}_i^k$. The new processes $(Q_k)_{k=1}^{\infty}$ can be used to write:

$$G(\mathbf{W}(t \wedge \widehat{\tau}_{k_1})) - G(\mathbf{W}(0)) = \tag{2.44}$$

$$\sum_{k=1}^{k_1} Q_k(t \wedge \widehat{\tau}_k) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \int_0^{\widehat{\tau}_{k_1}} \int_{U_{j\text{-}1}(s\text{-})}^{U_j(s\text{-})} G_{j\downarrow i}(\mathbf{W}(s\text{-}), v) \mathcal{V}_{ji}(dv, ds), \tag{2.45}$$

where we stopped at $\widehat{\tau}_{k_1}$ with $k_1 \in \mathbb{N}$. We wish to decompose the right-hand side into $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingales and predictable processes with finite variation. Following Lemma 2.4.8 we defining $A : \Omega \times [0,\infty) \times [0,\infty) \to \mathbb{R}$ and $(M_k)_{k=1}^{\infty}$ by setting

$$\begin{aligned} A_k(t_1,t_2) := &\int_{t_1}^{t_2} \prod_{l=1}^{\infty} g_l(\mathbb{W}_l^k(s\text{-}), U_l(s\text{-})) \sum_{i=1}^{\infty} \frac{B(g_i)(\mathbb{W}_i^k(s\text{-}), U_i(s\text{-}))}{g_i(\mathbb{W}_i^k(s), U_i(s\text{-}))} ds \\ &+ \int_{t_1}^{t_2} \prod_{l=1}^{\infty} g_j(\mathbb{W}_l^k(s\text{-}), U_l(s\text{-})) \sum_{i=1}^{\infty} (aU_i(s\text{-})^2 - bU_i(s\text{-})) \frac{\partial_u g_i(\mathbb{W}_i^k(s\text{-}), U_i(s\text{-}))}{g_i(\mathbb{W}_i^k(s\text{-}), U_i(s\text{-}))} ds. \\ M_k(t) := &\, Q_k(t) - A_k(\widehat{\tau}_{k\text{-}1}, t), \qquad k \in \mathbb{N}. \end{aligned}$$

We also define the process $A : \Omega \times [0,\infty) \times [0,\infty) \to \mathbb{R}$ in the same fashion as $(A_k, k \in \mathbb{N})$ but with $(\mathbf{W}_i^k)_{i=1}^{\infty}$ replaced with the original processes $(\mathbf{W}_i)_{i=1}^{\infty}$ and note that due to the identities in (2.43) we can conclude that

$$A_k(t_1,t_2) = A(t_1,t_2), \qquad \widehat{\tau}_k \le t_1 < t_2 \le \widehat{\tau}_{k+1}. \tag{2.46}$$

From Lemma 2.4.8 we know that M_k is a $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale. Considering the birth events, we define the two processes from which we can obtain an additional $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale by

$$\begin{aligned} M_{\mathcal{V}}(t) &:= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\int_0^t\int_{U_{j\text{-}1}(s\text{-})}^{U_j(s\text{-})} G_{j\downarrow i}(\mathbf{W}(s\text{-}),v)\,\overline{\mathcal{V}}_{ji}(dv,ds),\\ A_{\mathcal{V}}(t) &:= \sum_{j=2}^{m}\sum_{i=1}^{i-1}\int_0^t\int_{U_{j\text{-}1}(s\text{-})}^{U_j(s\text{-})} G_{j\downarrow i}(\mathbf{W}(s\text{-}),v)\,dvds, \end{aligned}$$

where $\overline{\mathcal{V}}_{ji}$ is the compensated version of the Poisson point process $\mathcal{V}_{ji}$. The processes $M_{\mathcal{V}}$ is a $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale, too. Using $Q_k = M_k + A_k$ and $A(\widehat{\tau}_{k\text{-}1}, t\wedge\widehat{\tau}_k) = A(0,t\wedge\widehat{\tau}_k)$ we can rewrite

$$\begin{aligned} \sum_{k=1}^{k_1} M_k(t\wedge\widehat{\tau}_k) + M_{\mathcal{V}}(t\wedge\widehat{\tau}_{k_1}) &= G(\mathbf{W}(t\wedge\widehat{\tau}_{k_1})) - G(\mathbf{W}(0)) - \sum_{k=1}^{k_1} A(0,t\wedge\widehat{\tau}_k) - A_{\mathcal{V}}(t\wedge\widehat{\tau}_{k_1})\\ &= G(\mathbf{W}(t\wedge\widehat{\tau}_{k_1})) - G(\mathbf{W}(0)) - A(0,t\wedge\widehat{\tau}_{k_1}) - A_{\mathcal{V}}(t\wedge\widehat{\tau}_{k_1}), \end{aligned}$$

where we used the identity (2.46) and the fact that $A(t_1,t_2)+A(t_2,t_3) = A(t_1,t_3)$ for $0 \le t_1 < t_2 < t_3 \le \infty$. As the sum of finitely many $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingales the left-hand side on the first line is also a $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale. If we let k_1 go to infinity, then $G(\mathbf{W}(t\wedge\widehat{\tau}_{k_1}))$ converges against $G(\mathbf{W}(t))$, because $\mathbf{W}$ has no fixed jumps and $\mathbb{P}[\widehat{\tau}_k = t, k\in\mathbb{N}] = 0$. In total we can observe the following convergences

$$\begin{aligned} G(\mathbf{W}(t\wedge\widehat{\tau}_{k_1})) &\overset{k_1\to\infty}{\longrightarrow} G(\mathbf{W}(t)), & M_{\mathcal{V}}(t\wedge\widehat{\tau}_{k_1}) &\overset{k_1\to\infty}{\longrightarrow} M_{\mathcal{V}}(t),\\ A(0,t\wedge\widehat{\tau}_k) &\overset{k_1\to\infty}{\longrightarrow} A(0,t), & A_{\mathcal{V}}(t\wedge\widehat{\tau}_k) &\overset{k_1\to\infty}{\longrightarrow} A_{\mathcal{V}}(t), \end{aligned}$$

where the convergence is almost surely. Since $G \in \mathfrak{G}(\mathbf{B},K,r,m,n)$, we can conclude that the above expressions are uniformly bounded in $L^1(\mathbb{P})$, hence the above convergence also holds true in $L^1(\mathbb{P})$ due to the Lebesgue dominated convergence theorem. In conclusion we have that

$$\sum_{k=1}^{k_1} M_k(t\wedge\widehat{\tau}_k) \overset{k_1\to\infty}{\longrightarrow} \sum_{k=1}^{\infty} M_k(t\wedge\widehat{\tau}_k)$$

also converges almost surely and in $L^1(\mathbb{P})$, which allows us to conclude that the infinity sum of the right-hand side is also a $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale. Therefore

$$\begin{aligned} \sum_{k=1}^{\infty} M_k(t\wedge\widehat{\tau}_k) + M_{\mathcal{V}}(t) &= G(\mathbf{W}(t)) - G(\mathbf{W}(0)) - A(0,t) - A_{\mathcal{V}}(t)\\ &= G(\mathbf{W}(t)) - G(\mathbf{W}(0)) - \int_0^t \mathbf{A}^o_{\mathbf{B}}(G)(\mathbf{W}(s\text{-}))\,ds \end{aligned}$$

is a $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale. □

Proof of Corollary 2.5.14. Note if $G \in \mathfrak{G}(B_{\mathbb{W}}, K, r, m, \infty)$ with $G = \prod_{j=1} g_j$, then $G^n := \prod_{j=1}^n g_j$ is an element of $\mathfrak{G}(B_{\mathbb{W}}, K, r, m, n)$. If we define A^n as we have defined A in Proposition 2.5.12 but G^n taking the role of G, then Proposition 2.5.12 tells us that

$$M^n(t) := G^n(\mathbf{W}(t)) - G^n(\mathbf{W}(0)) - \int_0^t A^n(s)\, ds$$

is a $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale. In order to prove the claim of Corollary 2.5.14 all we need to show is:

$$G^n(\mathbf{W}(t)) \overset{n\to\infty}{\longrightarrow} G(\mathbf{W}(t)), \quad \int_0^t A^n(s)\, ds \overset{n\to\infty}{\longrightarrow} \int_0^t A(s)\, ds, \quad \text{in } L^1(\mathbb{P}), t \geq 0,$$

because this implies $M^n(t) \to M(t)$ in $L^1(\mathbb{P})$ for each $t \geq 0$, which implies that M is a $\mathcal{F}^{\mathbb{W}}$-martingale. The convergence of $(G^n, n \in \mathbb{N})$ against G follows immediately, because G^n and G are products of $(g_j, j \in \mathbb{N})$ and g_j takes values in $[0,1]$, so G^n is decreasing against G, making $|G(\mathbf{W}(t)) - G^n(\mathbf{W}(t))|$ decreasing in n and so the convergence follows from Beppo-Levi. Considering the convergence of the integral part, we note that since $g_i \in \mathfrak{g}(B_{\mathbb{W}}, K, r, m)$ we have

$$|A^n(\mathbf{W}(t))| \leq \sum_{i=j}^{\infty} K \mathbb{1}_{[0,r)}(U_j(t)) \leq KY^r, \quad t \geq 0, n \in \mathbb{N} \cup \{\infty\}$$

with the convention that $A^\infty = A$. By the Lemma 2.2.8 we know that $\mathbb{E}[Y^r(t)] < \infty$, $t \geq 0$, and $\mathbb{P}[Y_s^r < \infty, s \in [0,\infty)] = 1$. Since $Y_t < \infty$ almost surely it follows $A^n(\mathbf{W}(t)) \to A(\mathbf{W}(t))$ almost surely and hence $\mathbb{E}[|A^n(t) - A(t)|] \to 0$, when n goes to infinity by Lebesgue's theorem. Because of $\int_0^t \mathbb{E}[|Y_t|]\, ds < \infty$, applying Lebesgue's theorem once more (and Fubini's) we get:

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}\left[\int_0^t |A^n(s) - A(s)|\, ds\right] &= \lim_{n\to\infty} \int_0^t \mathbb{E}\left[|A^n(s) - A(s)|\right]\, ds \\ &= \int_0^t \lim_{n\to\infty} \mathbb{E}\left[|A^n(s) - A(s)|\right]\, ds = 0. \end{aligned}$$

□

Corollary 2.5.15. *The process $\boldsymbol{\xi}^{\mathbb{W}}$ is an empirical Kurtz-Rodrigues representation, see Definition B.2.12.*

Proof. In order to show that $\boldsymbol{\xi}^{\mathbb{W}} \sim \mathbf{KR}(B_{\mathbb{W}}, a, b)$, we have to consider the operator $\mathbf{A_B}$ from Definition B.2.9 and we need to show that for all Laplace functionals $L_{\text{-}\log(g)} \in C_b(\overline{\mathcal{N}}(E \times [0,\infty)))$ with $L_{\text{-}\log(g)}(\eta) = \exp(-\eta(\log(g)))$ and with $g \in \mathfrak{g}(B_{\mathbb{W}}, K, r, m)$, see Definition B.2.7, it holds true that the process $M : \Omega \times [0,\infty) \to \mathbb{R}$ given by

$$M(t) := L_{\text{-}\log(g)}(\boldsymbol{\xi}_t^{\mathbb{W}}) - L_{\text{-}\log(g)}(\boldsymbol{\xi}_0^{\mathbb{W}}) - \int_0^t \mathbf{A_B}(L_{\text{-}\log(g)})(\boldsymbol{\xi}_s^{\mathbb{W}})\, ds$$

is a martingale. But when we choose $G \in \mathfrak{G}(B_{\mathbb{W}}, K, r, m, \infty)$ with $G = \prod_{j=1}^\infty g_j$, where $g_j = g$ with g being as above, then actually:

$$G((\mathbb{W}_i(t), U_i(t))) = \prod_{i=1}^{\infty} g(\mathbb{W}_i(t), U_i(t)) = L_{\text{-}\log(g)}(\boldsymbol{\xi}_t^{\mathbb{W}})$$

and similarly

$$G\left((\mathbb{W}_i(t), U_i(t))_{i=1}^\infty\right) = \mathbf{A_B}(L_{\text{-}\log(g)})(\boldsymbol{\xi}_s^{\mathbb{W}}),$$

hence M is a martingale by the Corollary 2.5.14. □

2.6 Connection to the Empirical KR-Rep. II

In Section 2.4 we constructed the ordered system $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$. In this section we show that (2.2) is true. In Section 2.5 we have proved that the empirical measure $\boldsymbol{\xi}^{\mathbb{W}}$ from Definition 2.5.2 is an empirical Kurtz-Rodrigues representation, i.e.

$$\boldsymbol{\xi}^{\mathbb{W}} \sim \mathbf{KR}(B_{\mathbb{W}}, a, b, \Theta_0),$$

see Definition B.2.12 from Appendix B.2 with $\Theta_0 \in \mathcal{M}_1(\overline{\mathcal{N}}(\mathfrak{D}))$ discussed below. We will now combine this result with the statements from Appendix B.3. Considering the initial distribution $\Theta_0 \in \mathcal{M}_1(\overline{\mathcal{N}}(\mathfrak{D}))$ we recall that by Assumption 2.10 we have:

$$\mathfrak{L}((X_i(0), V_i(0))_{i=1}^{\infty} | \mathbf{Q}_0^X, Y_0) = \bigotimes_{i=1}^{\infty} \big(\mathbf{Q}_0^X \otimes \mathbf{Exp}(Y_0)\big), \tag{2.47}$$

where $V_0 := U_0, V_i := U_i - U_{i-1}$ and where $\mathbf{Q}_0^X$ is some random measure over $\mathbb{R}^d$ and Y_0 is a $\mathbb{R}$-valued random variable with $Y > 0$ almost surely. In other words $(X_i(0), V_i(0))_{i=1}^{\infty}$ forms a vector of conditionally independent identically distributed random variables. Considering the path-valued processes $(\mathbb{W}_i, V_i)_{i=1}^{\infty}$, we know due to Definition 2.4.8 that

$$\mathbb{W}_i(0) = (0, \mathfrak{W}_i(0,\cdot)) = (0, \mathfrak{X}_i(0,\cdot), \mathfrak{L}_i(0,\cdot)),$$

with $\mathfrak{X}_i(0,\cdot)$ and $\mathfrak{L}_i(0,\cdot)$ being the constant paths given by $\mathfrak{X}_i(0,t) = X_i(0)$ and $\mathfrak{L}_i(0,t) = 0, i \in \mathbb{N}, t \geq 0$ defined in Definition 2.3.8. So from (2.47) it follows that

$$\begin{aligned} \mathfrak{L}((\mathbb{W}_i(0), V_i(0))_{i=1}^{\infty} | \mathbf{Q}_0^X, Y_0) &= \mathfrak{L}((\mathbb{W}_i(0), V_i(0))_{i=1}^{\infty} | \mathbf{Q}_0^{\mathbb{W}}, Y_0) \\ &= \bigotimes_{i=1}^{\infty} \big(\mathbf{Q}_0^{\mathbb{W}} \otimes \mathbf{Exp}(Y_0)\big), \end{aligned} \tag{2.48}$$

where $\mathbf{Q}_0^{\mathbb{W}}$ is a random measure over $\mathfrak{D}$, which can be obtained in two ways: The first one is by setting $\mathbf{Q} := \pi_*(\mathbf{Q}_0^X)$ with π_* being the push-forward of the map $\pi : \mathbb{R}^d \to \mathfrak{D}$ which maps $x \in \mathbb{R}^d$ to $\pi(x) = (0, \mathfrak{w})$, where $\mathfrak{w} = ((x,0), t \geq 0)$ is the constant path with value $(x,0) \in \mathbb{R}^{d+1}$. The other way is by $\mathbf{Q}_0^{\mathbb{W}} := \mathfrak{L}(\mathbb{W}_1 | \mathbf{Q}_0^X, Y_0)$. Either way this has the important implication that the initial distribution Θ_0 of the empirical process $\boldsymbol{\xi}^{\mathbb{W}}$ in (2.47) is a Poisson mixture, see Definition B.3.1. Together with the fact that $\boldsymbol{\xi}^{\mathbb{W}}$ is an empirical Kurtz-Rodrigues representation, we can apply Theorem B.3.3 the main result of Appendix B.2.

Proposition 2.6.1. *Under the Assumption 2.10 the process* $\tilde{\boldsymbol{\Xi}}^{\mathbb{W}}$ *from Definition 2.5.2 admits a continuous modification, which we denote by* $\boldsymbol{\Xi}^{\mathbb{W}}$. *Both processes are* $\boldsymbol{DW}(B_{\mathbb{W}}, a, b, \hat{\Theta}_0)$*-processes, where* $\hat{\Theta}_0 \in \mathcal{M}_1(\mathcal{M}_f(\mathfrak{D}))$ *is the distribution of* $Y_0\mathbf{Q}_0^{\mathbb{W}}$. *Further it holds for all finite* $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$*-stopping times* τ *that*

$$\mathfrak{L}(\boldsymbol{\xi}_\tau^{\mathbb{W}} | \mathcal{F}_\tau^{\boldsymbol{\Xi},\mathbb{W}}) = \mathbf{PPP}(\boldsymbol{\Xi}_\tau^{\mathbb{W}} \otimes \ell eb[0,\infty)) \tag{2.49}$$

and considering the processes $\boldsymbol{\Xi}^{\mathbb{W},r}$ *from Definition 2.5.2 we get:*

$$\frac{1}{r}\boldsymbol{\Xi}_\tau^{\mathbb{W},r} \overset{r\to\infty}{\longrightarrow} \boldsymbol{\Xi}_\tau^{\mathbb{W}} \quad a.s. \tag{2.50}$$

Note that it follows from (2.50) that it holds $\tilde{\boldsymbol{\Xi}}_0 = Y_0\mathbf{Q}_0^{\mathbb{W}}$.

Proof. By an application of Proposition C.1.3 it follows that the initial distribution Θ_0 of $\boldsymbol{\xi}^{\mathbb{W}}$ is a Poisson mixture and it holds:

$$\mathfrak{L}(\boldsymbol{\xi}_0^{\mathbb{W}}|\mathbf{Q}_0^{\mathbb{W}}, Y_0) = \mathbf{PPP}(\tilde{\boldsymbol{\Xi}}_0^{\mathbb{W}} \otimes \ell eb[0,\infty)).$$

Since the operator $B_{\mathbb{W}}$ satisfies the Conditions B.2.2, we can apply Theorem B.3.3 from which the above statements follow. □

Corollary 2.6.2. *The process $\tilde{Y}$ from Definition 2.5.2 admits a continuous modification, which is a Feller diffusion with drift b and branching rate a, meaning that it is a weak solution of the stochastic differential equation given by:*

$$dY_t = bY_t dt + \sqrt{2aY_t} d\hat{W}_t,$$

where $\hat{W}$ is a Brownian motion. It also holds $Y_t = \boldsymbol{\Xi}_t^{\mathbb{W}}(\mathfrak{D})$.

Proof. By defining Y by $Y_t = \boldsymbol{\Xi}_t^{\mathbb{W}}(\mathfrak{D})$, it follows from the fact that $\boldsymbol{\Xi}^{\mathbb{W}}$ is a continuous modification of $\tilde{\boldsymbol{\Xi}}^{\mathbb{W}}$ and that $\tilde{Y}_t = \tilde{\boldsymbol{\Xi}}_t^{\mathbb{W}}(\mathfrak{D})$, that Y is a continuous modification of $\tilde{Y}$. Since $\boldsymbol{\Xi}^{\mathbb{W}} \sim \mathbf{DW}(B_{\mathbb{W}}, a, b, \hat{\Theta}_0)$, it follows that Y is a Feller diffusion. □

For the next part let us recall the stopping time $\mathcal{T}_{EX}$ which was defined in Definition 2.2.3 as the time, when the lowest level U_1 hits infinity. We called $\mathcal{T}_{EX}$ the extinction time, the next lemma justifies this name.

Lemma 2.6.3. *Let us define the $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time $\tilde{\mathcal{T}}_{EX} = \inf\{t > 0 : Y_t = 0\}$, then it holds*

$$\mathbb{P}\left[\mathcal{T}_{EX} = \tilde{\mathcal{T}}_{EX}\right] = 1.$$

Proof. The identity on the event $\{\tilde{\mathcal{T}}_{EX} = \infty\} \cap \{\mathcal{T}_{EX} = \infty\}$ is clear. We begin with arguing why $\tilde{\mathcal{T}}_{EX} \leq \mathcal{T}_{EX}$, when $\mathcal{T}_{EX} < \infty$. By the definition of $\mathcal{T}_{EX}$ $U_i(t) = \infty, i \in \mathbb{N}$ for $t \geq \mathcal{T}_{EX}$. Consequently

$$\mathbb{P}\left[\mathbb{1}_{[\mathcal{T}_{EX},\infty)}(t)\boldsymbol{\xi}_t^{\mathbb{W},r}(\mathfrak{D}) = \mathbb{1}_{[\mathcal{T}_{EX},\infty)}(t) Y_t^r = 0, r > 0, t \geq \mathcal{T}_{EX}\right] = 1.$$

By the convergence (2.50) and the continuity of Y we can conclude that $Y_t = 0$ almost surely for all $t \geq \mathcal{T}_{EX}$, therefore $\tilde{\mathcal{T}}_{EX} \leq \mathcal{T}_{EX}$ almost surely. Next let us define

$$\mathcal{T}_k := \tilde{\mathcal{T}}_{EX} \wedge k.$$

Since $\mathcal{T}_k$ is finite, we have by (2.49) that

$$\boldsymbol{\xi}_{\mathcal{T}_k}^{\mathbb{W}} \mathbb{1}_{\{\mathcal{T}_k < k\}} = \mathbf{PPP}(\mathbf{0}_{\mathfrak{D}} \otimes \ell eb[0,\infty)) = \mathbf{0}_{\mathfrak{D}\times[0,\infty)},$$

therefore on the event $\{\mathcal{T}_k < k\}$ it holds $U_1(t) = \infty$ for all $t \geq \mathcal{T}_k$ almost surely and so $\mathcal{T}_{EX} \leq \mathcal{T}_k$. Since $\mathcal{T}_k = \tilde{\mathcal{T}}_{EX}$ on $\{\mathcal{T}_k < k\}$, we can see that $\mathcal{T}_{EX} \leq \tilde{\mathcal{T}}_{EX}$ on the event $\{\mathcal{T}_k < k\}$. Since this is true for all for all $k \in \mathbb{N}$, we can conclude that $\mathcal{T}_{EX} \leq \tilde{\mathcal{T}}_{EX}$ on the event $\{\tilde{\mathcal{T}}_{EX} < \infty\}$. In conclusion, if one of the stopping times $\tilde{\mathcal{T}}_{EX}$ and $\mathcal{T}_{EX}$ is finite, then the other one is smaller or equals to the first, hence this implies that the second one is also finite, from which we then can conclude that the first one is smaller or equal to the second one giving us the identity on the event $\{\tilde{\mathcal{T}}_{EX} < \infty\} \cup \{\mathcal{T}_{EX} < \infty\}$. □

From the identity between $\mathcal{T}_{EX}$ and $\tilde{\mathcal{T}}_{EX}$, we can conclude that $\mathcal{T}_{EX}$ is a $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time. This may appear surprising, because $\mathcal{T}_{EX}$ is defined as a hitting time of the process U_1, and U_1 is clearly not adapted to $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$, so one would expect $\mathcal{T}_{EX}$ to be "just" a $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-stopping time. We will see that $\mathbb{W}_1(\mathcal{T}_{EX}\text{-})$ (note that $\mathbb{W}_1(\mathcal{T}_{EX}\text{-}) = \mathbb{W}_1(\mathcal{T}_{EX})$) is measurable with respect to $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_{\mathcal{T}_{EX}}$. But before we prove that we introduce a technical lemma that will become very useful multiple times throughout this thesis.

Lemma 2.6.4. *Let (E, d_E) be a complete separable metric space and $C^+_{lip}(E)$ be the space of bounded, non-negative, d_E-Lipschitz-continuous functions. We can find a countable collection $(\hat{g})_{k=1}^\infty \subset C^+_{lip}(E)$ that is convergence determining for $\mathcal{M}_f(E)$.*

Proof. The set $C^+_{lip}(E)$ is closed under multiplication and addition. In order to see that $C^+_{lip}(E)$ strongly separates points, let us fix a point $\hat{x} \in E$ and a $\epsilon > 0$. If we define the function $\hat{g}$ by setting $\hat{g}(x) := d_E(x, \hat{x})$, then obviously $\inf\{|\hat{g}(x) - \hat{g}(\hat{x})| > 0 : x \in \Gamma^c_\epsilon(\hat{x})\}$, where $\Gamma_r(\hat{x})$ is the ball with radius r around $\hat{x}$. By the inverted triangle inequality it follows that $\hat{g}(x)$ is Lipschitz continuous. According to Lemma 2 from [5] there exists a countable collections of functions $(\hat{g})_{k=1}^\infty \subset C^+_{lip}(E)$ that is strongly separating points and closed under multiplication and addition. Since $\mathbb{1}_E \in C^+_{lip}(E)$, we can assume that $\hat{g}_1 = \mathbb{1}_E$. Since $(\hat{g})_{k=1}^\infty$ is an algebra, which is strongly separating points and contains $\mathbb{1}_E$, it follows by the Theorem 3.4.5.(b) from [14] that $(\hat{g})_{k=1}^\infty$ is convergences determining. □

The above observation is crucial for the next proposition, because it ensures that the process $\mathbf{Q}$ defined in the next proposition is $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$ adapted (which is important, because we want to condition on $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_\tau$ in (2.52)).

Proposition 2.6.5. *Let us define $\mathbf{Q}^{\mathbb{W}} : \Omega \times [0, \infty) \to \mathcal{M}_1(\mathfrak{D})$ with*

$$\mathbf{Q}^{\mathbb{W}}_t = \frac{\boldsymbol{\Xi}^{\mathbb{W}}_t}{Y_t} \mathbb{1}_{[0,\mathcal{T}_{EX})}(t) + \delta_{\mathbb{W}_1(\mathcal{T}_{EX}\text{-})} \mathbb{1}_{[\mathcal{T}_{EX},\infty)}(t),$$

then the process $\mathbf{Q}^{\mathbb{W}}$ is a modification of $\tilde{\mathbf{Q}}^{\mathbb{W}}$ that is continuous on $[0, \infty) \setminus \{\mathcal{T}_{EX}\}$ and with the property that

$$\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}} \mathbf{Q}^{\mathbb{W}}_{\mathcal{T}_{EX}-t}(\hat{g}) \overset{t\to 0}{\longrightarrow} \mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}} \hat{g}(\mathbb{W}_1(\mathcal{T}_{EX}\text{-})) \quad in\ L^1(\mathbb{P}) \tag{2.51}$$

for all $\hat{g} \in C^+_{lip}(E)$ (this implies $\mathbb{W}_1(\mathcal{T}_{EX}\text{-})$ is measurable with respect to $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_{\mathcal{T}_{EX}}$). Further for all finite $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping times τ it holds

$$\mathfrak{L}\big((\mathbb{W}_i(\tau), V_i(\tau))_{i=1}^\infty \big| \mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_\tau\big) = \bigotimes_{i=1}^{\infty} \big(\mathbf{Q}^{\mathbb{W}}_\tau \otimes \boldsymbol{Exp}(Y_\tau)\big). \tag{2.52}$$

and it holds

$$\frac{1}{m} \mathbf{Q}^{\mathbb{W},m}_\tau = \frac{1}{m} \sum_{i=1}^m \delta_{\mathbb{W}_i(\tau)} \overset{m\to\infty}{\Longrightarrow} \mathbf{Q}^{\mathbb{W}}_\tau \quad a.s., \tag{2.53}$$

where "$\Rightarrow$" for convergence in the weak topology of $\mathcal{M}_f(\mathfrak{D})$.

Proof. We prove the statement in the following order: First we prove (2.53) for finite stopping times with $\tau < \mathcal{T}_{EX}$. The reason for this restriction is that we do not know that

$$\sigma(\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}} \mathbb{W}_1(\mathcal{T}_{EX}\text{-})) \subset \mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_{\mathcal{T}_{EX}} \tag{2.54}$$

After this we derive (2.53) which also gives us that $\mathbf{Q}^{\mathbb{W}}$ is a modification of $\tilde{\mathbf{Q}}^{\mathbb{W}}$, then we prove (2.51). Since the latter implies (2.54) it follows immediately that (2.52) is true for general finite stopping times.
For $\tau < \mathcal{T}_{EX}$, let us define $\tilde{T}_k := \inf\{t \geq 0 : Y_t \leq \frac{1}{k}\}$ for $k \in \mathbb{N}$ and $\tilde{T}_0 = 0$, then we know from Proposition C.1.3 that (2.52) is true for $\tau \wedge \tilde{T}_k$ and $\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}_{\tau\wedge\tilde{T}_k}$. So it follows that

$$\begin{aligned}
\mathbb{1}_{[0,\mathcal{T}_{EX})}(\tau)\mathfrak{L}\big((\mathbb{W}_i(\tau),V_i(\tau))_{i=1}^{\infty}\big|\,\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}_{\tau}\big) &= \sum_{k=0}^{\infty}\mathbb{1}_{[\tilde{T}_k,\tilde{T}_{k+1})}(\tau)\mathfrak{L}\big((\mathbb{W}_i(\tau),V_i(\tau))_{i=1}^{\infty}\big|\,\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}_{\tau}\big)\\
&= \sum_{k=0}^{\infty}\mathbb{1}_{[\tilde{T}_k,\tilde{T}_{k+1})}(\tau)\mathfrak{L}\big((\mathbb{W}_i(\tau),V_i(\tau))_{i=1}^{\infty}\big|\,\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}_{\tau\wedge\tilde{T}_{k+1}}\big)\\
&= \sum_{k=0}^{\infty}\mathbb{1}_{[\tilde{T}_k,\tilde{T}_{k+1})}(\tau)\bigotimes_{i=1}^{\infty}\big(\mathbf{Q}^{\mathbb{W}}_{\tau\wedge\tilde{T}_{k+1}}\otimes\mathbf{Exp}(Y_{\tau\wedge\tilde{T}_{k+1}})\big)\\
&= \mathbb{1}_{[0,\mathcal{T}_{EX})}(\tau)\bigotimes_{i=1}^{\infty}\big(\mathbf{Q}^{\mathbb{W}}_{\tau}\otimes\mathbf{Exp}(Y_\tau)\big).
\end{aligned}$$

For $\tau \geq \mathcal{T}_{EX}$ it holds $(\mathbb{W}_i(\tau),U_i(\tau))_{i=1}^{\infty} = (\mathbb{W}_1(\mathcal{T}_{EX}),\infty)_{i=1}^{\infty}$, so (2.52) is true. Let us now consider a general $\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}$-stopping time τ. If we set again $\tau\wedge\tilde{T}_k$ with $\tilde{T}_k := \inf\{t \geq 0 : Y_t \leq \frac{1}{k}\}$, then we can conclude from the exchangeability of $(\mathbb{W}_i(\tau\wedge\tilde{T}_k))_{i=1}^{\infty}$, note that $\tau\wedge\tilde{T}_k \leq \mathcal{T}_{EX}$, it follows from Corollary C.1.7 that for all $k\in\mathbb{N}$ holds:

$$\lim_{m\to\infty}\mathbb{1}_{[0,\tilde{T}_k]}(\tau)\frac{1}{m}\mathbf{Q}^{\mathbb{W},m}_{\tau\wedge\tilde{T}_k} = \lim_{m\to\infty}\mathbb{1}_{[0,\tilde{T}_k]}(\tau)\frac{1}{m}\mathbf{Q}^{\mathbb{W},m}_{\tau} = \mathbb{1}_{[0,\tilde{T}_k]}(\tau)\mathbf{Q}^{\mathbb{W}}_{\tau}\ a.s.,$$

where the convergence holds in the weak topology of $\mathcal{M}_f(\mathfrak{D})$. This gives us

$$\mathbb{1}_{[0,\mathcal{T}_{EX})}(\tau)\mathbf{Q}^{\mathbb{W},m}_{\tau} \overset{m\to\infty}{\Longrightarrow} \mathbb{1}_{[0,\mathcal{T}_{EX})}(\tau)\mathbf{Q}^{\mathbb{W}}_{\tau}\ a.s.$$

On the event $\{\tau > \mathcal{T}_{EX}\}$, we have $\mathbb{W}_i(\tau) = \mathbb{W}_1(\mathcal{T}_{EX}\text{-})$ for all $i\in\mathbb{N}$, hence $\mathbf{Q}^m_\tau = \delta_{\mathbb{W}_1(\mathcal{T}_{EX}\text{-})}$ for all $m\in\mathbb{N}$, hence the convergence (2.53) holds true on the event $\{\tau > \mathcal{T}_{EX}\}$.
Considering (2.51) we fix $\hat{g}\in C^+_{lip}(E)$ and we recall the σ-algebra $\mathcal{F}^\Phi$ from Definition 2.3.4, which contains all the information about the genealogy. From (2.53) we can conclude

$$\begin{aligned}
\mathbb{E}\left[|\hat{g}(\mathbb{W}_1(t)) - \mathbf{Q}^{\mathbb{W}}_t(\hat{g})||\mathcal{F}^\Phi\right] &\leq \lim_{m\to\infty}\frac{1}{m}\mathbb{E}\left[|\hat{g}(\mathbb{W}_1(t)) - \mathbf{Q}^{m}_t(\hat{g})||\mathcal{F}^\Phi\right]\\
&\leq \lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\left[|\hat{g}(\mathbb{W}_1(t)) - \hat{g}(\mathbb{W}_i(t))||\mathcal{F}^\Phi\right]
\end{aligned}$$

If we apply that $\hat{g}$ is Lipschitz continuous with constant $K>0$ and then Lemma 2.4.9, we get

$$\mathbb{E}\left[|\hat{g}(\mathbb{W}_1(t)) - \hat{g}(\mathbb{W}_i(t))||\mathcal{F}^\Phi\right] \leq K\mathbb{E}\left[d_{\widehat{\mathbb{D}},E}\left(\mathbb{W}_1(t),\mathbb{W}_i(t)\right)|\mathcal{F}^\Phi\right] \leq K\vartheta(d_\Phi(1,i,t)).$$

By Lemma 2.3.5 we can find a $\mathcal{F}^\Phi$-measurable random variable ι^ϵ_Φ for a fixed $\epsilon>0$ such that $d_\Phi(\omega,1,i,t)\leq\epsilon$ for all $t\in[\iota^\epsilon_\Phi,\mathcal{T}_{EX})$ on the event $\{\mathcal{T}_{EX}<\infty\}$, and so it holds on the same event that

$$\mathbb{1}_{[\iota^\epsilon_\Phi,\mathcal{T}_{EX})}(t)\mathbb{E}\left[|\hat{g}(\mathbb{W}_1(t)) - \mathbf{Q}^{\mathbb{W}}_t(\hat{g})||\mathcal{F}^\Phi\right] \leq \mathbb{1}_{[\iota^\epsilon_\Phi,\mathcal{T}_{EX})}(t)K\vartheta(\epsilon). \tag{2.55}$$

Making use of the fact that $\lim_{t\to\mathcal{T}_{EX}}\mathbb{1}_{[\iota^\epsilon_\Phi,\mathcal{T}_{EX})}(t) = 1$ a.s. on $\{\mathcal{T}_{EX}<\infty\}$ we continue with

$$\begin{aligned}
&\lim_{t\to\mathcal{T}_{EX}}\mathbb{E}\left[|\hat{g}(\mathbb{W}_1(t)) - \mathbf{Q}^{\mathbb{W}}_t(\hat{g})||\mathcal{F}^\Phi\right] =\\
&\lim_{t\to\mathcal{T}_{EX}}\mathbb{1}_{[\iota^\epsilon_\Phi,\mathcal{T}_{EX})}(t)\mathbb{E}\left[|\hat{g}(\mathbb{W}_1(t)) - \mathbf{Q}^{\mathbb{W}}_t(\hat{g})||\mathcal{F}^\Phi\right] \leq K\vartheta(\epsilon).
\end{aligned}$$

Since this is true for all ϵ and since it holds $\vartheta(\epsilon) = 0$, it follows

$$\begin{aligned}
&\lim_{t\to 0} \mathbb{E}\left[\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}}|\hat{g}(\mathbb{W}_1(\mathcal{T}_{EX}-t)) - \mathbf{Q}^{\mathbb{W}}_{\mathcal{T}_{EX}-t}(\hat{g})|\right] \\
&= \mathbb{E}\left[\lim_{t\to 0} \mathbb{E}\left[\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}}|\hat{g}(\mathbb{W}_1(\mathcal{T}_{EX}-t)) - \mathbf{Q}^{\mathbb{W}}_{\mathcal{T}_{EX}-t}(\hat{g})||\mathcal{F}^{\Phi}\right]\right] = 0.
\end{aligned}$$

This gives us (2.51). We can conclude from (2.51) that $(\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}}\mathbf{Q}^{\mathbb{W}}_{\mathcal{T}_{EX}-t}(\hat{g}))$ is converging against $\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}}\hat{g}(\mathbb{W}_1(\mathcal{T}_{EX}\text{-}))$ in probability, therefore there exists a subsequence for which this convergence holds almost surely. From the latter we can conclude that

$$\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}}\hat{g}(\mathbb{W}_1(\mathcal{T}_{EX}\text{-}))$$

must be measurable with respect to $\mathcal{F}^{\Xi,\mathbb{W}}_{\mathcal{T}_{EX}}$, because the same holds true for $(\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}}\mathbf{Q}^{\mathbb{W}}_{\mathcal{T}_{EX}-t}(\hat{g}))$. If $(\hat{g})_{k=1}^{\infty}$ is the set from Lemma 2.6.4, then it holds $\sigma(\mathbb{1}_{\{\mathcal{T}_{EX}<\infty\}}\mathbb{W}_1(\mathcal{T}_{EX}\text{-})) = \sigma(\hat{g}_k(\mathbb{W}_1(\mathcal{T}_{EX}\text{-})), k \in \mathbb{N})$ and so (2.54) is true. □

These have been the most important consequences from the Kurtz-Rodrigues theory for our collection $(\mathbb{W}_i, U_i)_{i=1}^{\infty}$. These form the tools we will employ to construct Poisson representations for models with competition and we will begin in the next chapter by developing an integration theory. But in Chapter 6 we need the additional results that the particles with a level below r form a branching particle system, see Section B.4. Recall the Markov kernel $\mathbf{Uni}^r_E : \mathcal{N}_f(E) \to \mathcal{M}_1(\mathcal{N}_f(E \times [0,\infty)))$ from Definition 1.1.1 and the processes $\Xi^{\mathbb{W},r}, \Xi^{\mathbb{W},r}$ with from Definition 2.5.2, where we also introduced the filtration $\mathcal{F}^{\Xi,\mathbb{W},r}$ with $r \geq \max\{b/a, 0\}$.

Proposition 2.6.6. *The process $\Xi^{\mathbb{W},r}$ is for $r \geq \max\{b/a, 0\}$ a branching particle system, indeed $\Xi^{\mathbb{W},r} \sim \mathbf{D}(B_{\mathbb{W}}, ra, ra - b)$, and it holds for all $\mathcal{F}^{\Xi,\mathbb{W},r}$-stopping times τ that*

$$\mathfrak{L}(\boldsymbol{\xi}^{\mathbb{W},r}_{\tau}|\mathcal{F}^{\Xi,\mathbb{W},r}_{\tau}) = \mathbf{Uni}^r_{\mathfrak{D}}(\Xi^{\mathbb{W},r}_{\tau}). \tag{2.56}$$

Proof. Applying the Proposition B.6.2 gives us that $\Xi^{\mathcal{F}^{\Xi,\mathbb{W}},r} \sim \mathbf{D}(B_{\mathbb{W}}, ra, ra - b)$ and that

$$\mathfrak{L}(\boldsymbol{\xi}^{\mathbb{W},r}_{\tau}|\tilde{\mathcal{F}}^{\Xi,\mathbb{W},r}_{\tau}) = \mathbf{Uni}^r_{\mathfrak{D}}(\Xi^{\mathbb{W},r}_{\tau}),$$

where $\tilde{\mathcal{F}}^{\Xi,\mathbb{W},r}_t := \sigma(\Xi^{\mathbb{W},r}_s, s \leq t)$. The Lemma F.1.2 tells us now that for a fixed $t \geq 0$:

$$\mathfrak{L}(\boldsymbol{\xi}^{\mathbb{W},r}_t|\mathcal{F}^{\Xi,\mathbb{W},r}_t) = \mathfrak{L}(\boldsymbol{\xi}^{\mathbb{W},r}_t|\tilde{\mathcal{F}}^{\Xi,\mathbb{W},r}_t).$$

This in turn gives us (2.56) for a fixed $t \geq 0$ and by the Lemma D.1.15 we can extend (2.56) to an arbitrary finite $\mathcal{F}^{\Xi,\mathbb{W},r}$-stopping times. □

Remark 2.6.7. *We suspect that the ordered Kurtz-Rodrigues representation is very closely related to the Donnelly-Kurtz representation, indeed the latter is obtained from the former by forgetting the levels. To make this precise let us work again in the abstract setting, where $\mathbb{X}$ is an arbitrary Markov process. If $(\mathbb{X}_i, U_i)_{i=1}^{\infty}$ is an ordered KR-representation, then we assume that $(\mathbb{X}_i, V_i)_{i=1}^{\infty}$ is obtained as usual by setting $U_1 = V_1$ and $V_i = U_i - U_{i\text{-}1}$, $i \geq 2$, and $Y := \Xi^{\mathbb{X}}(E)$ is the full mass of the Dawson-Watanabe superprocess. When we now define the Markov kernel $\alpha : [0,\infty) \times E^{\infty} \to \mathcal{M}_1(E^{\infty} \times [0,\infty)^{\infty})$ by setting for each*

$$\alpha(y, (x_i)_{i=1}^{\infty}) = \bigotimes_{i=1}^{\infty} \delta_{x_i} \otimes \boldsymbol{Exp}(y), \quad ((x_i)_{i=1}^{\infty}, y) \in E^{\infty} \times [0,\infty).$$

Let us now assume that $\mathbf{A}^{DK}_{\mathbf{B}} \subset C_b([0,\infty) \times E^{\infty}) \times C([0,\infty) \times E^{\infty})$ is the generator of the Donnelly-Kurtz representation denoted as A in the second paragraph of Page 190 in [10] (the

Donnelly-Kurtz representation is constructed on Page 182). If $\tilde{\mathbf{A}}_{\mathbf{B}}^{o}$ *is the generator of* $(\mathbb{X}_i, V_i)_{i=1}^{\infty}$, *then we should be able to prove by a calculation the following intertwiner-relationship:*

$$\alpha^* \circ \tilde{\mathbf{A}}_{\mathbf{B}}^{o} = \mathbf{A}_{\mathbf{B}}^{DK} \circ \alpha^*.$$

By applying the extended Markov mapping theorem, see Theorem 3.6 in [28], it will follow that

$$\mathfrak{L}\left((\mathbb{X}_i(t), V_i(t))_{i=1}^{\infty} \middle| (Y_s, (\mathbb{X}_i(s))_{i=1}^{\infty}), s \leq t\right) = \alpha((Y_t, (\mathbb{X}_i(t))_{i=1}^{\infty})), \quad t \geq 0,$$

and that $(Y, (\mathbb{X}_i)_{i=1}^{\infty})$ *is with respect to its own filtration a Donnelly-Kurtz representation. Unfortunately we did not find the time to work out the details, but this result reflects the polar decomposition stating that the Dawson-Watanabe superprocess can be decomposed into a Feller-Diffusion and a Fleming-Viot process, see Section 4.3 and 4.4 in [12].*

Chapter 3

Integration

In this chapter we present the basics of our integration theory, in which we use the historical processes $(\mathfrak{X}_i, \mathfrak{N}_i)_{i=1}^{\infty}$ to define a third component $(Z_i)_{i=1}^{\infty}$, which we will employ later to cut out the Poisson representations for our competition models from the Kurtz-Rodrigues representation of the Dawson-Watanabe superprocess. Our integration theory can be described as the marriage of Perkins' stochastic calculus and the theory of Kurtz and Rodrigues. The motivation for the development of this integration theory in this chapter and the following is to provide us with a range of tools, which allow us to construct Poisson representations for the competition models without having to deal with subtle technical nuances.
The chapter has the following structure. In the first section we introduce the class of possible integrands and show that this class can be extended by localizing and approximated by more simple integrands. In the second and third sections we show that $(X_i(\tau), Z_i(\tau), V_i(\tau))_{i=1}^{\infty}$ are still conditionally independent, where τ is a $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time, when we base the integrated processes $(Z_i)_{i=1}^{\infty}$ on a simple integrand. We extend this statement for general integrands in the fourth section by approximation. Finally we use the fifth section to prove basic regularity properties of the processes $\boldsymbol{\xi}^{XZ} = (\boldsymbol{\xi}_t^{XZ}, t \geq 0)$ and $\mathbf{Q}^{XZ} = (\mathbf{Q}_t^{XZ}, t \geq 0)$ with $\boldsymbol{\xi}_t^{XZ} := \sum_{i=1}^{\infty} \delta_{(X_i(t), Z_i(t), U_i(t))}$ and $\mathbf{Q}^{XZ}(t)$ being the De Finetti measure of $(X_i(t), Z_i(t), U_i(t))_{i=1}^{\infty}$.

3.1 The Spaces of Integrands $\mathcal{L}^1(\mathbf{M})$ and $\mathcal{L}^1_{loc}(\mathbf{M})$

Here we define our class of integrands for our integration theory. Indeed we want to identify a class of functions

$$h : \Omega \times \mathbb{R}^d \times [0, \infty) \times [0, \infty) \to \mathbb{R}, \tag{3.1}$$

such that the corresponding integrated processes $Z_i : \Omega \times [0, \infty) \to \mathbb{R}$, $i \in \mathbb{N}$, defined by

$$Z_i(\omega, t) := \int_0^t \int_0^{\infty} h(\omega, \mathfrak{X}_i(\omega, t, s\text{-}), p, s)\, \mathfrak{N}_i(\omega, t, dp, ds), \quad (i, t) \in \mathbb{N} \times [0, \infty), \tag{3.2}$$

are well defined and satisfy additional properties which are crucial for our plans (Note that we will often omit ω in terms like (3.2), when it is not essential for the understanding to avoid unnecessarily long expressions). The first of these additional properties is that the collection $(X_i(\tau), Z_i(\tau), V_i(\tau))_{i=1}^{\infty}$ should be conditionally independent given $\mathcal{F}_{\tau}^{\boldsymbol{\Xi},\mathbb{W}}$, where τ is a finite $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time, as it was the case for $(W_i(\tau), V_i(\tau))$ (recall that $V_i = U_i - U_{i\text{-}1}, i \geq 2$, and $V_1 = U_1$). Consequently the randomness of the integrands h, indeed the dependency of h from Ω, should be adapted to the filtration $\mathcal{F}^{\Xi,\mathbb{W}}$. Further the function h should also be predictable with respect to the filtration $\mathcal{F}^{\Xi,\mathbb{W}}$, so that the integrated processes $(Z_i)_{i=1}^{\infty}$ become semi-martingales. Together this leads to the following definitions:

Definition 3.1.1. *We define the predictable σ-algebra $\mathfrak{P}(\mathcal{F}^{\Xi,\mathbb{W}})$ generated by the family of sets given by*

$$\begin{aligned}&\big\{\Gamma^1 \times \Gamma^2 \times \Gamma^3 \times \{0\};\ \Gamma^1 \in \mathcal{F}_0^{\Xi,\mathbb{W}}, \Gamma^2 \in \mathbb{B}(\mathbb{R}^d), \Gamma^3 \in \mathbb{B}([0,\infty))\big\}\\ \cup&\big\{\Gamma^1 \times \Gamma^2 \times \Gamma^3 \times (s_1, s_2];\ \Gamma^1 \in \mathcal{F}_{s_1}^{\Xi,\mathbb{W}}, \Gamma^2 \in \mathbb{B}(\mathbb{R}^d), \Gamma^3 \in \mathbb{B}([0,\infty)), 0 < s_1 < s_2 < \infty\big\}\end{aligned} \tag{3.3}$$

and we define the set of ***predictable functions*** $\mathcal{P}(\mathcal{F}^{\Xi,\mathbb{W}})$ *as the collection of functions*

$$h : \Omega \times \mathbb{R}^d \times [0,\infty) \times [0,\infty) \to \mathbb{R}, \tag{3.4}$$

which are measurable with respect to $\mathfrak{P}(\mathcal{F}^{\Xi,\mathbb{W}})$.

The class of predictable functions $\mathcal{P}(\mathcal{F}^{\Xi,\mathbb{W}})$ is too big. If we choose an arbitrary $h \in \mathcal{P}(\mathcal{F}^{\Xi,\mathbb{W}})$, then it is not clear, why the Z_i defined as in (3.2) should be well defined, since the integral does not need to exist. But the random variable $Z_i(t)$ will be well-defined, if we restrict ourselves to $h \in \mathcal{P}(\mathcal{F}^{\Xi,\mathbb{W}})$ with:

$$\mathbb{P}\bigg[\int_0^t \int_0^\infty |h(\mathfrak{X}_i(t, s\text{-}), p, s)| \mathfrak{N}_i(t, dp, ds) < \infty\bigg] = 1. \tag{3.5}$$

A sufficient condition for (3.5) would be to show that

$$\mathbb{E}\bigg[\int_0^t \int_0^\infty |h(\mathfrak{X}_i(t, s\text{-}), p, s)| \mathfrak{N}_i(t, dp, ds)\bigg] < \infty. \tag{3.6}$$

Based on this reasoning we will define the following measure $\mathbf{M}$ and the corresponding spaces $\mathcal{L}^1(\mathbf{M})$ and $L^1(\mathbf{M})$. Although it may be not directly obvious from the following definitions, the Lemma 3.4.3 shows, that if $h \in \mathcal{L}^1(\mathbf{M})$, then (3.6) will be true for all $t \geq 0$ and all $i \in \mathbb{N}$.

Definition 3.1.2. *We define the measures $\mathbf{M}$ on the measurable space*

$$\big(\Omega \times \mathbb{R}^d \times [0,\infty) \times [0,\infty), \mathfrak{P}(\mathcal{F}^{\Xi,\mathbb{W}})\big),$$

by setting for each non-negative, predictable function $h \in \mathcal{P}(\mathcal{F}^{\Xi,\mathbb{W}})$:

$$\mathbf{M}(h) := \int_0^\infty \int_0^\infty \mathbb{E}\left[h(X_1(s\text{-}), p, s) \mathbb{1}_{[0,\mathcal{T}_{EX})}(s)\right] dpds.$$

Note that the above integral $\mathbf{M}(h)$ is well-defined, because we assumed that h is non-negative.

Definition 3.1.3. *We define the space $\mathcal{L}^1(\mathbf{M}) \subset \mathcal{P}(\mathcal{F}^{\Xi,\mathbb{W}})$ consisting of the elements $h \in \mathcal{P}(\mathcal{F}^{\Xi,\mathbb{W}})$ with*

$$|||\, h \,|||_{\mathbf{M}} := \mathbf{M}(|h|) = \int_0^\infty \int_0^\infty \mathbb{E}\left[|h(X_1(s\text{-}), p, s) \mathbb{1}_{[0,\mathcal{T}_{EX})}(s)|\right] dpds < \infty. \tag{3.7}$$

By considering the equivalence classes in $\mathcal{L}^1(\mathbf{M})$ with respect to the semi-norm $|||\cdot|||_{\mathbf{M}}$, we obtain the Banach space $L^1(\mathbf{M})$.

A common and often quite useful tool in stochastic analysis are stopping times, we will employ these with great effect, for example we will extend our class $\mathcal{L}^1(\mathbf{M})$ of possible integrands. But these stopping times must be $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping times to guarantee that the integrated processes $(X_i, Z_i, V_i)_{i=1}^{\infty}$ remain exchangeable. Further the application of T should not destroy the predictability. But this is not a problem, because if T is a $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping time, then $\{T < t\}$ is measurable with respect to $\mathcal{F}_t^{\Xi,\mathbb{W}}$ (so T is also an optional time), since we assumed $\mathcal{F}^{\Xi,\mathbb{W}}$ to be the augmented version of the natural filtration of $\Xi^{\mathbb{W}}$, which implies that $\mathcal{F}^{\Xi,\mathbb{W}}$ is right-continuous. As a consequence the process $P = (\mathbb{1}_{[0,T]}(t), t \geq 0)$ is a $\mathcal{F}^{\Xi,\mathbb{W}}$-adapted process. Since P is left-continuous we can conclude that $h_T : \Omega \times \mathbb{R}^d \times [0,\infty) \times [0,\infty) \to \mathbb{R}$ given by $h_T(\omega, x, p, s) = \mathbb{1}_{[0,T(\omega)]}(t)$ is an element of $\mathcal{P}(\mathcal{F}^{\Xi,\mathbb{W}})$.

Definition 3.1.4. *We define $\mathcal{L}^1_{loc}(\mathbf{M})$ as the set of elements $h \in \mathcal{P}(\mathcal{F}^{\Xi,\mathbb{W}})$ for which we can find an increasing sequence $(T_n)_{n=1}^{\infty}$ of $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping times with $\mathbb{P}[T_n \overset{n\to\infty}{\longrightarrow} \infty] = 1$ and the property that $h_n := h\mathbb{1}_{[0,T_n]}$ is an element of $\mathcal{L}^1(\mathbf{M})$ for all $n \in \mathbb{N}$. We call such a sequence $(T_n)_{n=1}^{\infty}$ a $\mathcal{F}^{\Xi,\mathbb{W}}$-**localizing sequence** for h (since h and $\mathbb{1}_{[0,T_n]}$ are predictable, the same is true for $h\mathbb{1}_{[0,T_n]}$).*

A popular choice for localizing sequences $(T_n, n \in \mathbb{N})$ will be

$$T_n := \tau_n^Y := \inf\{s \geq 0 : Y_s \geq n\}$$

and $n \wedge \tau_n^Y$. Note that if $(T_n, n \in \mathbb{N})$ is a localizing sequence of $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ and $(\tilde{T}_n, n \in \mathbb{N})$ is second sequence of stopping times with $\mathbb{P}[\tilde{T}_n \leq T_n] = 1$, $n \in N$, and $\mathbb{P}[\tilde{T}_n \to \infty] = 1$, then $(\tilde{T}_n, n \in \mathbb{N})$ is also a localizing sequence of h.
A further common tool in stochastic analysis and far beyond is to introduce a class of simple functions, for which it is easy to establish certain properties, and then to extend these properties to a more general class of functions by approximation. In our case the more general class of functions is of course $\mathcal{L}^1_{loc}(\mathbf{M})$ and the class of simple functions is given in the definition below.

Definition 3.1.5. *We define the space of simple predictable integrands $\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}})$ as the elements h of $\mathcal{P}(\mathcal{F}^{\Xi,\mathbb{W}})$ with the form:*

$$h(\omega, x, p, s) = \mathbb{1}_{\Gamma^1}(\omega)\mathbb{1}_{\Gamma^2}(x)\mathbb{1}_{[p_1,p_2)}(p)\mathbb{1}_{(s_1,s_2]}(s), \tag{3.8}$$

*where $s_1 < s_2, \Gamma^1 \in \mathcal{F}_{s_1}^{\Xi,\mathbb{W}}$, $\Gamma^2 \in \mathbb{B}(\mathbb{R}^d)$ and $0 \leq p_1 < p_2 < \infty$. We denote **span**$(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$ as the linear span of $\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}})$.*

Remark 3.1.6. *If h_1 and h_2 are elements of **span**$(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$, then $h := |h_1 - h_2| \in$ **span**$(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$.*

Lemma 3.1.7. *The span **span**$(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$ is dense in $L^1(\mathbf{M})$ with respect to the norm $|||\cdot|||_{\mathbf{M}}$.*

Proof. We adapt the proof of Lemma 3.1.5 in [37] to our situation. **Step 1:** First let us assume that $h \in \mathcal{L}^1(\mathbf{M})$ is continuous with respect to x, p, s for all $\omega \in \Omega$, that h is bounded and that the support of $|h|$ is contained in $[m, m]^d \times [0, m] \times [0, m]$ for some fixed $m \in \mathbb{N}$. Let us set $[q] = \{1, 2, ..., q\} \subset \mathbb{N}$ for all $q \in \mathbb{N}$. When we define $(h_n)_{n=1}^{\infty} \in$ **span**$(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$ by setting for each $n \in \mathbb{N}$:

$$h_n(\omega, x, p, t) := \sum_{J \in [2^{n+1}-1]^d} \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^n-1} h(\omega, x_j, p_k, s_l)\mathbb{1}_{\Gamma_J}(x)\mathbb{1}_{[p_k,p_{k+1})}(p)\mathbb{1}_{(s_l,s_{l+1}]}(t),$$

with $\Gamma_J = \prod_{i=1}^d [j_i 2^{-n}m - m, (j_i + 1)2^{-n}m - m)$ for all $J = (j_1, j_2, ..., j_d) \in [2^{n+1}]$ and $p_i = s_i = i2^{-n}m$ for $0 \leq i \leq 2^n$, then $h_n \in$ **span**$(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$ and h_n converges against h pointwise due to the continuity of h. Applying Lebesgue dominated convergence theorem gives us now $\lim_{n\to\infty} |||\, h_n - h\, |||_{\mathbf{M}} = 0$. **Step 2:** Now we assume that $h \in \mathcal{L}^1(\mathbf{M})$ is bounded and the support of $|h|$ is still contained in $[m,m]^d \times [0,m] \times [0,m]$, but h is not necessarily continuous. Let us assume that $\phi : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ is a smooth function, whose support is contained in $[-1,1]^d \times [-1,1] \times [-1,1]$, and it holds $\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(x,p,s)\, dxdpds = 1$. We define a new sequence $(h_n)_{n=1}^{\infty} \subset \mathcal{L}^1(\mathbf{M})$ by setting:

$$h_n(\omega,x,p,t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}} h(\omega,\tilde{x},\tilde{p},\tilde{s}) n^{d+2} \phi\left(n(\tilde{x}-x), n(\tilde{p}-p), n(\tilde{s}-s)\right) d\tilde{x} d\tilde{p} d\tilde{s}$$

for all $(\omega,x,p,s) \in \mathbb{R}^d \times [0,\infty) \times [0,\infty)$, where we use the convention that $h(\omega,x,p,s) = 0$, if either $p < 0$ or $s < 0$. All members of the sequence $(h_n)_{n=1}^{\infty}$ are continuous, bounded and the support of $|h_n|$ is contained in $[m+1,m+1]^d \times [0,m+1] \times [0,m+1]$. **Step 3:** Finally let us assume that h is an arbitrary element of $\mathcal{L}^1(\mathbf{M})$ without any restrictions. If we choose $h_n := h\mathbb{1}_{|h|<n} \mathbb{1}_{[-n,n]^d \times [0,n] \times [0,n]}$ for each n, then h_n has the property of h from step 2. By Lebesgue dominated convergence theorem it follows again $\lim_{n\to\infty} |||\, h_n - h\, |||_{\mathbf{M}} = 0$. □

3.2 Integration for Simple Functions

We continue in this section by investigating the integration for our simple integrands. We discuss basic properties and prove one small but important lemma.

Definition 3.2.1. *For $h \in \mathtt{span}(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$ we define the integrated processes*

$$Z_i : \Omega \times [0,\infty) \to \mathbb{R}, \quad i \in \mathbb{N},$$

by setting for $(\omega,t) \in \Omega \times [0,\infty)$:

$$Z_i(\omega,t) := \int_0^t \int_0^\infty h(\omega, \mathfrak{X}_i(\omega,t,s\text{-}), p, s)\, \mathfrak{N}_i(\omega,t,dp,ds), \tag{3.9}$$

where the integral is defined as the Lebesgue integral with respect to the measure $\mathfrak{N}_i(\omega,t,dp,ds)$.

We want to understand the expression in (3.9) better. Let us assume that $h \in \mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}})$ with $h(\omega,x,p,s) = \mathbb{1}_{\Gamma^1}(\omega)\mathbb{1}_{\Gamma^2}(x)\mathbb{1}_{[p_1,p_2)}(p)\mathbb{1}_{(s_1,s_2]}(s)$, where $s_1 < s_2, \Gamma^1 \in \mathcal{F}^{\Xi,\mathbb{W}}_{s_1}, \Gamma^2 \in \mathbb{B}(\mathbb{R}^d)$ and $0 \le p_1 < p_2 < \infty$. If we fix a $(\omega,t) \in \Omega \times [0,\infty)$ and assume that the (deterministic) measure $\mathfrak{N}_i(\omega,t)$ is given by $\mathfrak{N}_i(\omega,t) = \sum_{k\in\mathbb{N}} \delta_{(\tilde{p}^i_k,\tilde{s}^i_k)}$ for a suitable collection $(\tilde{p}^i_k,\tilde{s}^i_k) \subset [0,\infty) \times [0,\infty)$, then $Z_i(\omega,t)$ can be written as

$$Z_i(\omega,t) := \mathbb{1}_{\Gamma^1}(\omega) \sum_{k\in\mathbb{N}} \mathbb{1}_{\Gamma^2}(\mathfrak{X}_i(\omega,t,\tilde{s}^i_k\text{-}))\mathbb{1}_{[p_1,p_2)\times(s_1,s_2]}(\tilde{p}^i_k,\tilde{s}^i_k). \tag{3.10}$$

In Definition 2.3.6 we build the historical processes $(\mathfrak{X}_i)_{i=1}^{\infty}$ and $(\mathfrak{N}_i)_{i=1}^{\infty}$ with the help of the genealogical map Φ, see Definition 2.3.1, and the Poisson point processes $(\tilde{N}_i)_{i=1}^{\infty}$ from the list of ingredients, see Assumption 2.1.2. So a further expression for Z_i would have been (note that we omit ω here):

$$Z_i(t) := \sum_{j=1}^{\infty} \int_0^t \int_0^\infty h(X_j(s\text{-}),p,s)\mathbb{1}_{\{\Phi(i,t,s)=j\}}\mathbb{1}_{[0,\mathcal{T}_{EX}]}(s)\, \tilde{N}_j(dp,ds). \tag{3.11}$$

Based on the appearance of the term $\mathbb{1}_{[0,\mathcal{T}_{EX}(\omega))}$ inside the above integral, we can conclude that Z_i will stop to evolve at the moment of extinction similar to $\mathbb{W}_i, \mathfrak{W}_i, W_i, \mathfrak{X}_i, X_i, \mathfrak{N}_i, \mathfrak{L}_i$ and L_i.

The expression (3.11) becomes much simpler in the case of $i = 1$, because of $\Phi(1, t, s) = 1$ for all $t, s \in [0, \infty)$ (recall the lowest particle is never affected by a birth event).

$$Z_1(t) = \int_0^t h(X_1(s\text{-}), p, s)\mathbb{1}_{[0,\mathcal{T}_{EX}]}(s)\tilde{N}_1(dp, ds). \tag{3.12}$$

This expression for Z_1 and together with the facts that $\tilde{N}_1$ is a Poisson point process with intensity measure $\ell eb[0,\infty) \otimes \ell eb[0,\infty)$ and that $h(\omega, X_1(\omega, s\text{-}), p, s)$ is predictable paves the way to the next lemma.

Lemma 3.2.2. *Let $h \in \mathtt{span}(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$ and Z_1 be the integrated process from (3.9), then we have for every finite $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping time τ:*

$$\mathbb{E}[Z_1(\tau)] = \int_0^\infty \int_0^\infty \mathbb{E}\big[h(X_1(s\text{-}), p, s)\mathbb{1}_{[0,\tau\wedge\mathcal{T}_{EX}]}(s)\big]dpds. \tag{3.13}$$

Remark 3.2.3. *It is not essential that τ is a stopping time with respect to $\mathcal{F}^{\Xi,\mathbb{W}}$, as long as it is a predictable stopping time with respect to a filtration $\tilde{\mathcal{F}}$ with the property that the compensated random measure $\overline{N}$ is a martingale measure with respect to $\tilde{\mathcal{F}}$.*

Proof. Due to the linearity of the integration it is sufficient to prove the claim for $h \in \mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}})$, hence we assume that h has the form found in (3.8), indeed

$$h(\omega, x, p, s) = \mathbb{1}_{\Gamma^1}(\omega)\mathbb{1}_{\Gamma^2}(x)\mathbb{1}_{[p_1,p_2)}(p)\mathbb{1}_{(s_1,s_2]}(s)$$

for $s_1 < s_2, \Gamma^1 \in \mathcal{F}^{\Xi,\mathbb{W}}_{s_1}$, $\Gamma^2 \in \mathbb{B}(\mathbb{R}^d)$ and $0 \leq p_1 < p_2 < \infty$. Based on Line (3.12), we can write

$$Z(\tau) = \int_{s_1}^{s_2} \int_{p_1}^{p_2} \mathbb{1}_{\Gamma_1}\mathbb{1}_{\Gamma_2}(X_1(s\text{-}))\mathbb{1}_{[0,\tau\wedge\mathcal{T}_{EX}]}(s)\mathbb{1}_{[p_1,p_2)\times(s_1,s_2]}(p, s)\tilde{N}_1(dp, ds).$$

If we define the process $P : \Omega\times[0, \infty) \to [0, 1]$ by $P(t) := \mathbb{1}_{\Gamma_1}\mathbb{1}_{\Gamma_2}(X_1(s\text{-}))\mathbb{1}_{[0,\tau\wedge\mathcal{T}_{EX}]}(s)$, then P is a left-continuous process and predictable with respect to the filtration $\tilde{\mathcal{F}}_t := \sigma(P(s), \tilde{N}_1(dp, ds), p \in [0, \infty), s \leq t)$. Therefore it is possible to approximate P pointwise in $\Omega \times [0, \infty)$ by the processes $P_n : \Omega \times [0, \infty) \to [0, 1], n \in \mathbb{N}$, given by

$$P_n(\omega, t) := P(\omega, 0) + \sum_{k=0}^{\infty}\sum_{l=0}^{2^n-1} P\left(\omega, k + \frac{l}{2^n}\right)\mathbb{1}_{(k+\frac{l}{2^n},k+\frac{l+1}{2^n}]}(t).$$

Since $\tilde{N}_1$ is a Poisson point process with intensity measure $\ell eb[0,\infty)\otimes\ell eb[0,\infty)$ and the restricted measure $\mathbb{1}_{[t,\infty)}\tilde{N}_1(dp, ds)$ is independent from $\tilde{\mathcal{F}}_t$, the expectation $\mathbb{E}[\int_{s_1}^{s_2}\int_{p_1}^{p_2} P^n(s)\,\tilde{N}_1(dp, ds)]$ is equal to

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{l=0}^{2^n-1}\mathbb{E}\left[P\left(k+\frac{l}{2^n}\right)\mathbb{E}\left[\tilde{N}_1\left([p_1,p_2]\times\left((k+\frac{l}{2^n})\wedge s_1, (k+\frac{l+1}{2^n})\wedge s_2\wedge t\right]\right)|\tilde{\mathcal{F}}_{k+\frac{l}{2^n}}\right]\right] \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{2^n-1}\mathbb{E}\left[P\left(k+\frac{l}{2^n}\right)\right][p_2-p_1][(k+\frac{l+1}{2^n})\wedge s_2\wedge t-(k+\frac{l}{2^n})\wedge s_1] \\
&= \int_0^\infty\int_0^\infty \mathbb{E}[P^n(s)]\mathbb{1}_{[p_1,p_2)\times(s_1,s_2]}(p,s)dpds = \int_0^\infty\int_0^\infty \mathbb{E}[P^n(s)\mathbb{1}_{[p_1,p_2)\times(s_1,s_2]}(p,s)]dpds.
\end{aligned}$$

We obtain the identity (3.13) by applying Lebesgue's theorem:

$$\mathbb{E}[Z_1(\tau)] = \mathbb{E}\left[\int_{s_1}^{s_2}\int_{p_1}^{p_2} P(s)\, \tilde{N}_1(dp, ds)\right] = \lim_{n\to\infty} \mathbb{E}\left[\int_{s_1}^{s_2}\int_{p_1}^{p_2} P_n(s)\, dpds\right]$$
$$= \int_0^\infty \int_0^\infty \lim_{n\to\infty} \mathbb{E}[P^n(s)\mathbb{1}_{[p_1,p_2)\times(s_1,s_2]}(p,s)]dpds$$
$$= \int_0^\infty \int_0^\infty \mathbb{E}[P(s)\mathbb{1}_{[p_1,p_2)\times(s_1,s_2]}(p,s)]dpds$$
$$= \int_0^\infty \int_0^\infty \mathbb{E}\big[h(X_1(s\text{-}), p, s)\mathbb{1}_{[0,\tau\wedge\mathcal{T}_{EX}]}(s)\big]dpds.$$

□

3.3 Poisson Mapping Theorem for Simple Functions

Our next step is motivated by the wish to establish the fact that $((X_i(\tau), Z_i(\tau), U_i(\tau))_{i=1}^\infty$, where τ is a finite $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time, are conditionally independent given $\mathcal{F}_\tau^{\boldsymbol{\Xi},\mathbb{W}}$. This will be achieved by using the fact that $((\mathfrak{W}_i(\tau), U_i(\tau))_{i=1}^\infty$ are conditionally independent given $\mathcal{F}_\tau^{\boldsymbol{\Xi},\mathbb{W}}$ and by the observation that the integrated process Z_i at time τ is a functional of $\mathbb{W}_i(\tau)$ and the path $(\boldsymbol{\Xi}_t^{\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}}, 0 \le t \le \tau)$. Indeed for a fixed $h \in \textit{span}(\mathcal{S}(\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}))$, we will show that there exists a suitable map H such that the integrated processes $(Z_i(\tau))_{i=1}^\infty$ can be expressed as

$$Z_i(\tau) = H^\tau((\boldsymbol{\Xi}_{s\wedge\tau}^{\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}}, s \ge 0), \mathbb{W}_i(\tau)).$$

It is important to notice that H^τ is independent from $i \in \mathbb{N}$, therefore we can use H to derive the conditional independence of $(X_i(\tau), Z_i(\tau), V_i(\tau))_{i=1}^\infty$ by the conditional independence of $(\mathbb{W}_i(\tau), V_i(\tau))_{i=1}^\infty$.
We are now formalizing and proving the above arguments. Therefore let us recall that $(\mathbb{W}_i, i \in \mathbb{N})$ are stochastic processes with state space $\mathfrak{D} = \widehat{\mathbb{D}}([0,\infty), \mathbb{R}^{d+1})$. and that $\boldsymbol{\Xi}^{\mathbb{W}}$ is a continuous process with state space $\mathcal{M}_f(\mathfrak{D})$. Since $\mathfrak{D}$ is Polish space, the same is also true for $\mathcal{M}_f(\mathfrak{D})$ and so we can understand $\boldsymbol{\Xi}^{\mathbb{W}} = (\boldsymbol{\Xi}_t^{\mathbb{W}}, t \ge 0)$ as a random variable in the space $C([0,\infty), \mathcal{M}_f(\mathfrak{D}))$, the space of continuous paths with values in $\mathcal{M}_f(\mathfrak{D})$. Let us denote by $\mathbb{B}(C([0,\infty), \mathcal{M}_f(\mathfrak{D})))$ the Borel σ-algebra of $C([0,\infty), \mathcal{M}_f(\mathfrak{D}))$ which is identical with the σ-algebra generated by the coordinate functions, because $\mathcal{M}_f(\mathfrak{D})$ is a Polish space, too. In order to formulate H as a functional of $\boldsymbol{\Xi}^{\mathbb{W}}$ and $\mathbb{W}_i$, we introduce the following definition.

Definition 3.3.1. *We write $\mathbf{M}(C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D})$ for the space of functions*

$$H : C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D} \to \mathbb{R},$$

which are measurable with respect to the Borel algebra $\mathbb{B}(C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D})$.

Lemma 3.3.2. *If $h \in \textit{span}(\mathcal{S}(\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}))$ is an element of the linear span of $\mathcal{S}(\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}})$ and $(Z_i)_{i=1}^\infty$ are the integrated processes defined as in* (3.9)*, then there exists a bounded*

$$H \in \mathbf{M}(C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D})$$

such that it holds for all $i \in \mathbb{N}$ and for all finite $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping times τ:

$$H\big((\boldsymbol{\Xi}_s^{\mathbb{W}}, s \ge 0), \mathbb{W}_i(\tau)\big) = H\big((\boldsymbol{\Xi}_{t\wedge\tau}^{\mathbb{W}}, t \ge 0), \mathbb{W}_i(\tau)\big) = Z_i(\tau) \text{ a.s.} \tag{3.14}$$

Remark 3.3.3. *Note that it holds $\mathbb{W}_i(\tau\text{-}) = \mathbb{W}_i(\tau)$ almost surely, as a consequence:*

$$H\big((\boldsymbol{\Xi}^{\mathbb{W}}_{t\wedge\tau}, t \geq 0), \mathbb{W}_i(\tau\text{-})\big) = Z_i(\tau) \text{ a.s.}$$

Note that $\mathbb{P}\left[\mathbb{W}_i(\tau\text{-}) \neq \mathbb{W}_i(\tau)\right] = 0$, because otherwise τ has a chance to coincide with a jump time of $\mathbb{W}_i$. This can not happen, because the jump times of $\mathbb{W}_i$ are inaccessible stopping times with respect to the filtration $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$, see Definition 2.5.2, while τ is a $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time, and since $\boldsymbol{\Xi}^{\mathbb{W}}$ is continuous, τ must be predictable with respect to $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}} \subset \mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$. Recall that the jumps performed by $\mathbb{W}$ are either results of the birth of new particles or due to the fact that $\mathbb{W}_i$ is constructed with the help of the Lévy processes $(\tilde{L}_i)_{i=1}^{\infty}$. New births correspond to atoms of the Poisson point processes $(\mathcal{V}_{ji}, 1 \leq i < j < \infty)$. Hence both jump-types are the result of Poisson point processes, making these inaccessible stopping times.

Proof of Lemma 3.3.2. By linearity it is sufficient to prove the claims for $h \in \mathcal{S}(\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}})$, hence we assume that h has the form given in (3.8), indeed

$$h(\omega, x, p, s) = \mathbb{1}_{\Gamma^1}(\omega)\mathbb{1}_{\Gamma^2}(x)\mathbb{1}_{[p_1,p_2)}(p)\mathbb{1}_{(s_1,s_2]}(s),$$

where $s_1 < s_2, \Gamma^1 \in \mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_{s_1},\ \Gamma^2 \in \mathbb{B}(\mathbb{R}^d)$ and $0 \leq p_1 < p_2 < \infty$. Putting h into the definition of $Z(\tau)$ in (3.9) we obtain:

$$Z_i(\tau) = \mathbb{1}_{\Gamma_1} \int_{s_1}^{s_2} \int_{p_1}^{p_2} \mathbb{1}_{\Gamma^2}(\mathfrak{X}_i(\tau, s\text{-}))\mathfrak{N}_i(\tau, dp, ds) = \mathbb{1}_{\Gamma_1}\pi^f_{\mathfrak{X},\mathfrak{N}}(\mathbb{W}_i(\tau)),$$

where $\pi^f_{\mathfrak{X},\mathfrak{N}} : \mathfrak{D} \to [0, \infty)$ is the function from Lemma 2.4.17 with

$$f(x, p, s) = \mathbb{1}_{\Gamma_2}(x)\mathbb{1}_{[p_1,p_2)\times(s_1,s_2]}(p, s).$$

It remains the term $\omega \to \mathbb{1}_{\Gamma_1}(\omega)$. Recall that $\Gamma_1 \in \mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_{s_1}$ and $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_{s_1}$ is by Definition 2.5.2 the completion of $\sigma\big(\boldsymbol{\Xi}^{\mathbb{W}}_s; s \leq s_1\big)$. Therefore the factorization lemma, see Corollary 1.97 in [24], ensures the existence of function $\pi_C : C([0, \infty), \mathcal{M}_f(\mathfrak{D})) \to \mathbb{R}$ with the property that $\pi_C((\boldsymbol{\Xi}^{\mathfrak{D}}_s(\omega), s \geq 0)) = \mathbb{1}_{\Gamma_1}(\omega)$. So we obtain H by setting for each $(\boldsymbol{\mu}, \mathbf{w}) \in C([0, \infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D}$:

$$H(\boldsymbol{\mu}, \mathbf{w}) := \pi_C(\boldsymbol{\mu})\pi^f_{\mathfrak{X},\mathfrak{N}}(\mathbf{w}).$$

Let us argue, why (3.14) is fulfilled for every finite stopping time τ and every $\omega \in \Omega$. Hereby we distinguish between the cases $\tau(\omega) \geq s_1$ and $\tau(\omega) < s_1$. For the first case, we note that $\Gamma_1 \in \mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_{s_1}$, therefore the value of the function π_C only depends on the path of $\boldsymbol{\Xi}^{\mathbb{W}}$ up to time s_1 and so for all ω with $\tau(\omega) \geq s_1$ we have

$$\pi_C((\boldsymbol{\Xi}^{\mathbb{W}}_{t\wedge\tau(\omega)}(\omega), t \geq 0)) = \pi_C((\boldsymbol{\Xi}^{\mathbb{W}}_{t\wedge s_1}(\omega), t \geq 0)) = \mathbb{1}_{\Gamma_1}(\omega), \text{ when } \tau(\omega) \geq s_1.$$

In conclusion (3.14) is true for $\tau(\omega) \geq s_1$. For the case $\tau(\omega) < s_1$, we note that all atoms of the deterministic measure $\mathfrak{N}_i(\omega, \tau(\omega))$ are contained in the set $[0, \infty) \times [0, \tau(\omega)]$ by the definition of $\mathfrak{N}_i$. Consequently for all $i \in \mathbb{N}$ we have $Z_i(\tau(\omega)) = 0$ and we have as well

$$\pi^f_{\mathfrak{X},\mathfrak{N}}(\mathbb{W}_i(\omega, \tau(\omega))) = \int_{s_1}^{s_2} \int_{p_1}^{p_2} \mathbb{1}_{\Gamma^2}(\mathfrak{X}_i(\omega, \tau(\omega), s\text{-}))\mathfrak{N}_i(\omega, \tau(\omega), dp, ds) = 0,$$

which implies that $H((\boldsymbol{\Xi}^{\mathbb{W}}_{t\wedge\tau(\omega)}(\omega), t \geq 0), \mathbb{W}_i(\omega, \tau(\omega))) = 0$. So in both cases, the statement (3.14) is true. $\square$

Due to Lemma 3.3.2 we could prove that $((X_i(\tau), Z_i(\tau), V_i(\tau))_{i=1}^{\infty}$, where τ is a finite $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping time, are conditionally independent given $\mathcal{F}_\tau^{\Xi,\mathbb{W}}$, as long as the integrand h is an element of the linear $span(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$. But we will not do it here, because we wish to prove it for all $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ simultaneously and for this we need to establish some further facts and extend the Lemma 3.3.2 to general integrands $h \in \mathcal{L}^1_{loc}(\mathbf{M})$. Hereby the Lemma 3.3.2 will be quite useful.

Corollary 3.3.4. *If $h \in span(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$ and $(Z_i, i \in \mathbb{N})$ are the integrated processes, then it holds for all finite $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping times τ that*

$$\begin{aligned}\mathbb{E}\big[Z_i(\tau)\big] &= \int_0^\infty \int_0^\infty \mathbb{E}\big[h(X_1(s), p, s)\mathbb{1}_{[0,\tau\wedge\mathcal{T}_{EX})}(s)\big]\, dpds \\ &= \mathbb{E}\bigg[\int_{\mathfrak{D}} \int_0^\infty \int_0^\infty h(\pi_{\mathfrak{X}}(\mathfrak{w}, s\text{-}), p, s)\, \pi_{\mathfrak{N}}(\mathfrak{w})(dp, ds)\mathbf{Q}_\tau^{\mathbb{W}}(\mathfrak{w})\bigg]\end{aligned}$$

It is especially true that

$$\begin{aligned}\mathbb{E}\big[|Z_i(\tau)|\big] &\le \int_0^\infty \int_0^\infty \mathbb{E}\big[|h(X_1(s), p, s)|\mathbb{1}_{[0,\tau\wedge\mathcal{T}_{EX})}(s)\big]\, dpds \\ &= |||\, h\mathbb{1}_{[0,\tau)}\, |||_{\mathbf{M}}.\end{aligned}$$

Proof. To establish the first equation we note that

$$\mathbb{E}[Z_i(\tau)] = \mathbb{E}\big[H\big((\Xi^{\mathbb{W}}_{t\wedge\tau}, t \ge 0), \mathbb{W}_i(\tau)\big)\big],$$

where H is the function from the Lemma 3.3.2. Since $(\mathbb{W}_i(\tau), i \in \mathbb{N})$ are conditionally independent and identically distributed given $\mathcal{F}_\tau^{\Xi,\mathbb{W}}$, we can conclude, that $(\Xi^{\mathbb{W}}, \mathbb{W}_i) \sim (\Xi^{\mathbb{W}}, \mathbb{W}_1)$. Together with the previous line we have

$$\mathbb{E}[Z_i(\tau)] = \mathbb{E}\big[H\big((\Xi^{\mathbb{W}}_{t\wedge\tau}, t \ge 0), \mathfrak{W}_1(\tau)\big)\big]$$

and with the Lemma 3.2.2 it follows that

$$\mathbb{E}[Z_i(\tau)] = \mathbb{E}[Z_1(\tau)] = \int_0^\infty \int_0^\infty \mathbb{E}\big[h(X_1(s\text{-}), p, s)\mathbb{1}_{[0,\tau\wedge\mathcal{T}_{EX})}(s)\big]\, dpds.$$

Considering the second equality, we assume again that h has the form

$$h(\omega, x, p, s) = \mathbb{1}_{\Gamma^1}(\omega)\mathbb{1}_{\Gamma^2}(x)\mathbb{1}_{[p_1,p_2)}(p)\mathbb{1}_{(s_1,s_2]}(s),$$

where $s_1 < s_2, \Gamma^1 \in \mathcal{F}_{s_1}^{\Xi,\mathbb{W}}$, $\Gamma^2 \in \mathbb{B}(\mathbb{R}^d)$ and $0 \le p_1 < p_2 < \infty$. Further we recall that $(\mathfrak{X}_i, \mathfrak{N}_i) = (\pi_{\mathfrak{X}}(\mathbb{W}_i), \pi_{\mathfrak{N}}(\mathbb{W}_i))$, where $\pi_{\mathfrak{X}}$ and $\pi_{\mathfrak{N}}$ are the functions found in Lemma 2.4.14 and Definition 2.4.15. Finally we remember that $\mathbb{W}_i(\tau)$ is distributed like $\mathbf{Q}^{\mathbb{W}}(\tau)$ given $\mathcal{F}_\tau^{\Xi,\mathbb{W}}$. All in all we can conclude

$$\begin{aligned}\mathbb{E}[Z_i(\tau)|\mathcal{F}_\tau^{\Xi,\mathbb{W}}] &= \mathbb{1}_{\Gamma_1}\mathbb{E}\left[\int_{s_1}^{s_2}\int_{p_1}^{p_2} \mathbb{1}_{\Gamma_2}(\mathfrak{X}_i(\tau, s\text{-}))\mathfrak{N}_i(\tau, p, s)\big|\mathcal{F}_\tau^{\Xi,\mathbb{W}}\big]\right] \\ &= \mathbb{1}_{\Gamma_1}\int_{\mathfrak{D}}\int_{s_1}^{s_2}\int_{p_1}^{p_2} \mathbb{1}_{\Gamma_2}(\pi_{\mathfrak{X}}(\mathfrak{w}, s\text{-}))\, \pi_{\mathfrak{N}}(\mathfrak{w})(dp, ds)\mathbf{Q}_\tau^{\mathbb{W}}(\mathfrak{w}) \\ &= \int_{\mathfrak{D}}\int_0^\infty\int_0^\infty h(\pi_{\mathfrak{X}}(\mathfrak{w}, s\text{-}), p, s)\, \pi_{\mathfrak{N}}(\mathfrak{w})(dp, ds)\mathbf{Q}_\tau^{\mathbb{W}}(\mathfrak{w}).\end{aligned}$$

For the inequality from the second part we note that

$$\begin{aligned}\mathbb{E}[|Z_i(\tau)|] &= \mathbb{E}\bigg[\bigg|\int_0^t\int_0^\infty h(\omega, \mathfrak{X}_i(\omega, t, s\text{-}), p, s)\, \mathfrak{N}_i(\omega, t, dp, ds)\bigg|\bigg] \\ &\le \mathbb{E}\bigg[\int_0^t\int_0^\infty |h(\omega, \mathfrak{X}_i(\omega, t, s\text{-}), p, s)|\, \mathfrak{N}_i(\omega, t, dp, ds)\bigg].\end{aligned}$$

Since $|h| \in \boldsymbol{span}(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$, we get from this and the first statement that

$$\mathbb{E}[|Z_i(\tau)|] = \int_0^\infty \int_0^\infty \mathbb{E}\big[|h(X_1(s), p, s)| \mathbb{1}_{[0,\tau\wedge\mathcal{T}_{EX})}(s)\big]\, dpds$$

which is by definition identical to $|||\, h\mathbb{1}_{[0,\tau)}\, |||_{\mathbf{M}}$. □

3.4 Integration for $\mathcal{L}^1(\mathbf{M})$ and $\mathcal{L}^1_{loc}(\mathbf{M})$

With the already established facts we are now ready to show that $\mathcal{L}^1(\mathbf{M})$ has been the right choice for our integrands. Indeed we we will show that if $h \in \mathcal{L}^1(\mathbf{M})$, then it holds for all finite $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping times τ:

$$\mathbb{E}\Big[\int_0^\tau \int_0^\infty |h(\mathfrak{X}_i(t, s\text{-}), p, s)| \mathfrak{N}_i(t, dp, ds)\Big] < \infty, \tag{3.15}$$

which will ensure that our integrated processes $(Z_i)_{i=1}^\infty$ associated with h are well defined. In order to elaborate on this more, we need to define additional measures.

Definition 3.4.1. *Let τ be a finite $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping time, then we associate with τ the following measures $\mathbf{M}_\tau$, $(\mathbf{M}^i_\tau)_{i=1}^\infty$ and $\hat{\mathbf{M}}_\tau$ which are defined on the measurable space*

$$\big(\Omega \times \mathbb{R}^d \times [0, \infty) \times [0, \infty), \mathfrak{P}(\mathcal{F}^{\Xi,\mathbb{W}})\big),$$

by setting for each non-negative, predictable functions $h \in \mathcal{P}(\mathcal{F}^{\Xi,\mathbb{W}})$:

$$\begin{aligned}
\mathbf{M}_\tau(h) &:= \int_0^\infty \int_0^\infty \mathbb{E}\left[h(X_1(s\text{-}), p, s)\mathbb{1}_{[0,\mathcal{T}_{EX}\wedge\tau)}(s)\right]\, dpds,\\
\mathbf{M}^i_\tau(h) &:= \mathbb{E}\Big[\int_0^\infty \int_0^\infty h(\mathfrak{X}_i(\tau, s\text{-}), p, s)\, \mathfrak{N}_i(\tau, dp, ds)\Big],\\
\hat{\mathbf{M}}_\tau(h) &:= \mathbb{E}\Big[\int_{\mathfrak{D}} \int_0^\infty \int_0^\infty h(\pi_{\mathfrak{X}}(\mathfrak{w}, s\text{-}), p, s)\, \pi_{\mathfrak{N}}(\mathfrak{w})(dp, ds)\mathbf{Q}^{\mathbb{W}}_\tau(\mathfrak{w})\Big].
\end{aligned}$$

Proposition 3.4.2. *Let τ be a finite $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping time, then the measures $\mathbf{M}_\tau$, $(\mathbf{M}^i_\tau)_{i=1}^\infty$ and $\hat{\mathbf{M}}_\tau$ are identical, indeed for all non-negative, predictable functions $h \in \mathcal{P}(\mathcal{F}^{\Xi,\mathbb{W}})$:*

$$\mathbf{M}_\tau(h) = \mathbf{M}^i_\tau(h) = \hat{\mathbf{M}}_\tau(h) = \mathbf{M}(h\mathbb{1}_{[0,\tau]}). \tag{3.16}$$

*Let us call this **the measure identity**.*

Proof. Note by Corollary 3.3.4 we have the identity 3.16 for all $h \in \boldsymbol{span}(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$. Now let us proceed by showing that the measures are σ-finite. Let us choose $\Gamma_n := \Omega \times \mathbb{R}^d \times [0, n] \times [0, n)$, then $h \in \mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}})$ and it holds

$$\mathbf{M}_\tau(h) = \mathbf{M}^i_\tau(h) = \hat{\mathbf{M}}_\tau(h) = \mathbf{M}(h\mathbb{1}_{[0,\tau]}) = n^2 < \infty.$$

In order to show the identity between the measures we compare the Definition of $\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}})$ with the collections of sets in (3.3) generating the predictable σ-algebra and conclude that $\sigma(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}})) = \mathcal{P}$. Since we already notice that the identity (3.16) is true for all $h \in \boldsymbol{span}(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$ by Corollary 3.3.4, we have proven the claim. □

Lemma 3.4.3. *Let $h \in \mathcal{L}^1(\mathbf{M})$, then it holds for all finite $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping times τ:*

$$\mathbb{E}\bigg[\int_0^\tau \int_0^\infty |h(\mathfrak{X}_i(\tau, s\text{-}), p, s)|\, \mathfrak{N}_i(\tau, dp, ds)\bigg] < \infty, \quad i \in \mathbb{N}. \tag{3.17}$$

Proof. By the definition of $\mathbf{M}^i_\tau$ it holds

$$\mathbb{E}\bigg[\int_0^\tau \int_0^\infty |h(\mathfrak{X}_i(\tau, s\text{-}), p, s)|\, \mathfrak{N}_i(\tau, dp, ds)\bigg] = \mathbf{M}^i_\tau(|h|)$$

and by Proposition 3.4.2 it follows

$$\mathbf{M}^i_\tau(|h|) = \mathbf{M}(|h|\mathbb{1}_{[0,\tau]}) \leq \mathbf{M}(|h|) = |||\, h\, |||_{\mathbf{M}} < \infty.$$

□

The measure identity from Proposition 3.4.2 and the Lemma 3.4.3 tell us that $\mathcal{L}^1(\mathbf{M})$ is the right space for our integration theory, because the integrated processes $(Z_i)_{i=1}^\infty$ will be well-defined due to (3.17).

Definition 3.4.4. *For a fixed $\mathcal{L}^1_{loc}(\mathbf{M})$, we define for each $i \in \mathbb{N}$ the integrated process Z_i : $\Omega \times [0, \infty) \to \mathbb{R}$ by setting for each $t \geq 0$:*

$$Z_i(\omega, t) = \int_0^t \int_0^\infty h(\omega, \mathfrak{X}_i(\omega, t, s\text{-}), p, s)\, \mathfrak{N}_i(\omega, t, dp, ds), \tag{3.18}$$

where we understand the integral for each $(t, \omega) \in \Omega \times [0, \infty)$ as a Lebesgue integral.

Before we proceed let us collect some properties of the integrated processes, which we will use constantly throughout this paper, in one lemma.

Lemma 3.4.5.

1. *The integrated processes $(Z_i)_{i=1}^\infty$ are well-defined for all $h \in \mathcal{L}^1_{loc}(\mathbf{M})$.*
2. *If $h \in \mathcal{L}^1(\mathbf{M})$, then it holds for $i \in \mathbb{N}$ and for every finite $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping time τ that*

$$\mathbb{E}[|Z_i(\tau)|] \leq |||\, h\mathbb{1}_{[0,\tau]}\, |||_{\mathbf{M}} \leq |||\, h\, |||_{\mathbf{M}}.$$

3. *Assume $h_1, h_2 \in \mathcal{L}^1(\mathbf{M})$ and that there exists a stopping time τ with $|||\, (h_1 - h_2)\mathbb{1}_{[0,\tau)}\, |||_{\mathbf{M}} = 0$. If $(Z^1_i)_{i=1}^\infty$ and $(Z^2_i)_{i=1}^\infty$ are the integrated processes corresponding to h_1 and h_2, then it holds*

$$\mathbb{P}\left[Z^1_i(s) = Z^2_i(s),\ s \in [0, \tau]\right] = 1. \tag{3.19}$$

4. *If $(Z^1_i)_{i=1}^\infty$ are the integrated processes for $h_1 \in \mathcal{L}^1_{loc}(\mathbf{M})$ and $(Z^2_i)_{i=1}^\infty$ are the integrated processes for $h_2 \in \mathcal{L}^1_{loc}(\mathbf{M})$ and $|||\, h_1 - h_2\, |||_{\mathbf{M}} = 0$, then Z^1_i and Z^2_i are indistinguishable from each other for all $i \in \mathbb{N}$.*
5. *If $(h_k, k \in \mathbb{N} \cup \{\infty\})$ is a sequence in $\mathcal{L}^1(\mathbf{M})$ with $|||\, h_\infty - h_k\, |||_{\mathbf{M}} \to 0$ for $k \to \infty$, and if $(Z^k_i)_{i=1}^\infty$ are the integrated processes for h_k, then it holds for $i \in \mathbb{N}$ and for every finite $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping time τ that*

$$\mathbb{E}\left[|Z^k_i(\tau) - Z^\infty_i(\tau)|\right] \overset{k\to\infty}{\longrightarrow} 0.$$

6. *If $(h_k, k \in \mathbb{N}\cup\{\infty\})$ is a sequence in $\mathcal{L}^1_{loc}(\mathbf{M})$ and there exists a localizing sequence $(T_n, n \in \mathbb{N})$ with $|||\,(h_\infty - h_k)\mathbb{1}_{[0,T_n]}\,|||_{\mathbf{M}} \to 0$ for $k \to \infty$, then it follows for all finite τ-stopping times and for all $i \in \mathbb{N}$ that $Z_i^k(\tau) \overset{k\to\infty}{\longrightarrow} Z_i^\infty(\tau)$ in probability. By considering a subsequence $(h_{k_j}, j \in \mathbb{N})$ of $(h_k, k \in \mathbb{N})$ it also holds $\mathbb{P}\Big[Z_i^{k_j}(\tau) \overset{j\to\infty}{\longrightarrow} Z_i^\infty(\tau),\ i \in \mathbb{N}\Big] = 1$.*

Proof. **1.** If $h \in \mathcal{L}^1(\mathbf{M})$, then $Z_i(t)$ is well-defined, because

$$\mathbb{P}\Bigg[\int_0^t \int_0^\infty |h(\mathfrak{X}_i(t,s\text{-}),p,s)|\mathfrak{N}_i(t,dp,ds) < \infty\Bigg] = 1 \tag{3.20}$$

by (3.17). If $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ and $(\tau_k)_{k=1}^\infty$ is a localizing sequence for h, $\mathbb{1}_{[0,\tau_k]}h \in \mathcal{L}^1(\mathbf{M})$, then (3.20) is satisfied, when we replace h by $\mathbb{1}_{[0,\tau_k]}h$, hence, if we define $\tilde{Z}_i^k$ like in (3.18) with h replaced by $\mathbb{1}_{[0,\tau_k]}h$, then $\tilde{Z}_i^k$ is well-defined. But $Z_i(t) = \tilde{Z}_i^k(t)$ on the event $\{\tau_k \le t\}$ and so $Z_i(t)$ is well-defined, if $t \le \tau_k$. Since $\tau_k \to \infty$, it follows that $\mathbb{P}[\cup_{k\in\mathbb{N}}\{\tau_k \le t\}] = 1$ and therefore $Z_i(t)$ is well-defined. **2.** The first inequality is the extension of the inequality of Corollary 3.3.4 to the case $\mathcal{L}^1_{loc}(\mathbf{M})$ and this extension can be proved with the same arguments as in the case of Corollary 3.3.4. The second inequality follows from the fact that $h\mathbb{1}_{[0,\tau]} \le h$. **3.** Note that it holds for all $i \in \mathbb{N}$ and all $t \in [0,\infty)$:

$$\begin{aligned}\mathbb{E}[|Z_i^1(t\wedge\tau) - Z_i^2(t\wedge\tau)|] &\le \mathbb{E}\left[\int_0^{t\wedge\tau}\int_0^\infty |h_1(\mathfrak{X}_i(t,s\text{-}),p,s) - h_2(\mathfrak{X}_i(t,s\text{-}),p,s)|\,\mathfrak{N}_i(t,dp,ds)\right]\\ &\le \int_0^t\int_0^\infty \mathbb{E}\left[|h_1(X_i(s\text{-}),p,s) - h_2(X_1(s\text{-}),p,s)|\right]\mathbb{1}_{[0,\tau]}(s)dpds\Big]\\ &\le |||\,(h_1 - h_2)\mathbb{1}_{[0,\tau)}\,|||_{\mathbf{M}}.\end{aligned}$$

Since the last expression is zero, we have $Z_i^1(t\wedge\tau) = Z_i^2(t\wedge\tau)$ a.s. and from this follows the claim, because Z_i^1 and Z_i^2 are càdlàg. **4.** This follows immediately from the previous point. **5.** As in Point **3** we have $\mathbb{E}\left[|Z_i^k(\tau) - Z_i^\infty(\tau)|\right] \le |||\,(h_k - h_\infty)\mathbb{1}_{[0,\tau)}\,|||_{\mathbf{M}}$ from which the claims follows directly. **6.** The convergence in $L^1(\mathbb{P})$ gives the convergence in probability, from which we can derive the a.s.-convergence for a subsequence. ☐

While the expression (3.18) for $(Z_i)_{i=1}^\infty$ will be very useful to prove the conditional independence of $((X_i(\tau), Z_i(\tau)))_{i=1}^\infty$ with τ being a finite $\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}$-stopping time, we also want to find another expression for $(Z_i)_{i=1}^\infty$, which describes the processes as the solution of a stochastic equation, similar as we did with $(W_i)_{i=1}^\infty, (X_i)_{i=1}^\infty$ and $(L_i)_{i=1}^\infty$, see (2.30) and Remark 2.4.10. This system of equations is more useful, when we want to do *calculations* with the $(Z_i)_{i=1}^\infty$, for example, when we derive the semi-martingale decompositions for our new processes.

Lemma 3.4.6. *For fixed $h \in \mathcal{L}^1_{loc}(\mathbf{M})$, the integrated processes $(Z_i)_{i=1}^\infty$ from definition 3.4.4 are the solution of the following system of stochastic equations, where it holds for all $i \in \mathbb{N}$ and $t < \mathcal{T}_{EX}$*

$$Z_j(t) = \int_0^t\int_0^\infty \mathbb{1}_{[0,\infty)}(U_j(s\text{-}))h(X_i(s\text{-}),p,s)\,\tilde{N}_j(dp,ds) \tag{3.21}$$

$$+\sum_{i=1}^{j-1}\int_0^t\int_{U_{j\text{-}1}(s\text{-})}^{U_j(s\text{-})} Z_i(s\text{-}) - Z_j(s\text{-})\mathcal{V}_{ji}(dv,ds) \tag{3.22}$$

$$+\sum_{i=2}^{j-1}\sum_{k=1}^{i-1}\int_0^t\int_{U_{i\text{-}1}(s\text{-})}^{U_i(s\text{-})} Z_{j\text{-}1}(s\text{-}) - Z_j(s\text{-})\mathcal{V}_{ik}(dv,ds). \tag{3.23}$$

and $Z_i(t) = Z_i(\mathcal{T}_{EX}\text{-})$ *for* $t \geq \mathcal{T}_{EX}$*, where* $(\tilde{N}_i)_{i=1}^{\infty}$ *and* $(\mathcal{V}_{ik}, i, k \in \mathbb{N})$ *are given by the ingredients list, see Assumption 2.1.2, and where* $(X_i)_{i=1}^{\infty}$ *are defined in Definition* (2.25).

Proof. The proof of this identity works in the same way as the proof of the identity (2.30), see the proof of Lemma 2.4.8. Recall the definition of the genealogical map Φ, see Definition 2.3.1. By the definition of $\mathfrak{N}_i$, see Definition 2.3.6, we have:

$$Z_i(t) = \sum_{j=1}^{\infty} \int_0^t \int_0^{\infty} h(X_j(s\text{-}), p, s) \mathbb{1}_{\{\Phi(i,t,s)=j\}}(s) \mathbb{1}_{[0,\mathcal{T}_{EX})}(s) \tilde{N}_j(dp, ds).$$

When there is no birth event effecting the particle i on the time interval $[s_1, s_2)$, this means $\Phi(i, t, s)$ is constant for $s_1 \leq s \leq t < s_2$, then

$$Z_i(t) - Z_i(s_1) = \int_{s_1}^t \int_0^{\infty} h(X_i(s\text{-}), p, s) \mathbb{1}_{[0,\mathcal{T}_{EX})}(s) \tilde{N}_i(dp, ds).$$

This gives (3.21). The jumps of (3.22) is the result of the birth events, where i is the child, and (3.23) comes from the birth events, where a particle is born below i. For more details, see the proof of Lemma 2.4.8. □

We wish now to extend the Lemma 3.3.2 to general $h \in \mathcal{L}^1_{loc}(\mathbf{M})$. We do so by approximating h by elements of $\mathit{span}(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$. This should allows us to obtain a function H for h as we did in Lemma 3.3.2 for the elements of $\mathit{span}(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$ by taking a limit in an appropriated space. We will now define this space. Naturally this space should be based on the space $\mathbf{M}(C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D})$ from Definition 3.3.1 which contains all measurable functions defined on $C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D}$ with values in $\mathbb{R}$. Recall also the measure valued process $\mathbf{Q}^{\mathbb{W}}$ from Proposition 2.6.5.

Definition 3.4.7. *For a fixed, finite* $\mathcal{F}^{\Xi,\mathbb{W}}$*-stopping time* τ*, we define the measure* $\mathbf{C}_\tau$ *over the measure space*

$$\big(C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D},\, \mathbb{B}\big(C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D}\big)\big)$$

by defining for each non-negative function $H \in \mathbf{M}(C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D})$ *the integral* $\mathbf{C}_\tau(H)$ *by setting*

$$\mathbf{C}_\tau(H) := \mathbb{E}\left[\int_{\mathfrak{D}} H(\Xi^{\mathbb{W}}, \mathfrak{w})\, \mathbf{Q}_\tau^{\mathbb{W}}(d\mathfrak{w})\right].$$

We define the space $\mathcal{L}^1(\mathbf{C}_\tau)$ *as the space of* $H \in \mathbf{M}(C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D})$ *with*

$$|||H|||_{\mathbf{C}_\tau} := \mathbf{C}_\tau(|H|) < \infty$$

and we denote by $L^1(\mathbf{C}_\tau)$ *the Banach space obtained by taking the equivalence classes with respect to the semi norm* $|||\cdot|||_{\mathbf{C}_\tau}$*.*

Lemma 3.4.8. *Let us denote* $\overline{\mathcal{S}}(\mathcal{F}^{\Xi,\mathbb{W}}) \subset L^1(\mathbf{M})$ *as the set of equivalence classes in* $\mathcal{L}^1(\mathbf{M})$ *associated with the elements of* $\mathit{span}(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$*. If we define the map*

$$\Psi : \overline{\mathcal{S}}(\mathcal{F}^{\Xi,\mathbb{W}}) \to L^1(\mathbf{C}_\tau),$$

by setting $\Psi([h])$ *to be the equivalence class* $[H]$ *of the function* H *in* $L^1(\mathbf{C}_\tau)$ *which is given by the Lemma 3.3.2 and satisfies* (3.14) *for* h*, then* Ψ *is a linear map with*

$$|||\Psi([h_1]) - \Psi([h_2])|||_{\mathbf{C}_\tau} \leq |||\, h_1 - h_2\, |||_{\mathbf{M}}, \quad h_1, h_2 \in \mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}). \tag{3.24}$$

Proof. For $h \in \mathit{span}(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$, let us define $(Z_i^h, i \in \mathbb{N})$ as the integrated processes. Let us fix two integrands h_1 and h_2 from $\mathit{span}(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$ and assume that $H_1, H_2 \in \mathbf{M}(C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D})$ are two functions satisfying (2.6.5) for h_1, h_2, i.e. $H_j(\Xi^{\mathbb{W}}_{\cdot\wedge\tau}, \mathbb{W}_i(\tau)) = Z_i^j(\tau)$ for $i \in \mathbb{N}$ and $j \in \{1,2\}$. By Proposition 2.6.5 the distribution of $\mathbb{W}_i(\tau)$ conditioned on $\mathcal{F}_\tau^{\Xi,\mathbb{W}}$ is the random measure $\mathbf{Q}_\tau^{\mathbb{W}}$. Since $\Xi^{\mathbb{W}}_{\cdot\wedge\tau}$ is $\mathcal{F}_\tau^{\Xi,\mathbb{W}}$ measurable, this gives us:

$$\begin{aligned}
|||H_1 - H_2|||_{\mathbf{C}_\tau} &= \mathbb{E}\left[\int_{\mathfrak{D}} |H_1(\Xi^{\mathbb{W}}_{\cdot\wedge\tau}, \mathfrak{w}) - H_2(\Xi^{\mathbb{W}}_{\cdot\wedge\tau}, \mathfrak{w})|\, \mathbf{Q}_\tau^{\mathbb{W}}(d\mathfrak{w})\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[|H_1(\Xi^{\mathbb{W}}_{\cdot\wedge\tau}, \mathbb{W}_i(\tau)) - H_2(\Xi^{\mathbb{W}}_{\cdot\wedge\tau}, \mathbb{W}_i(\tau))| \big| \mathcal{F}_\tau^{\Xi,\mathbb{W}}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[|Z_i^1(\tau) - Z_i^2(\tau)| \big| \mathcal{F}_\tau^{\Xi,\mathbb{W}}\right]\right] \\
&= \mathbb{E}[|Z_i^1(\tau) - Z_i^2(\tau)|].
\end{aligned}$$

The expression $\mathbb{E}[|Z_i^{h_1}(\tau) - Z_i^{h_2}(\tau)|]$ can now be bounded by:

$$\begin{aligned}
&\mathbb{E}[|Z_i^1(\tau) - Z_i^2(\tau)|] \\
&\leq \mathbb{E}\left[\int_0^t \int_0^\infty |h_1(\mathfrak{X}_i(t, s\text{-}), p, s) - h_2(\mathfrak{X}_i(t, s\text{-}), p, s)| \mathfrak{N}_i(t, dp, ds)\right] \\
&= \int_0^\infty \int_0^\infty \mathbb{E}\left[|h_1(X_1(s\text{-}), p, s) - h_2(X_1(s\text{-}), p, s)| \mathbb{1}_{[0,\tau\wedge\mathcal{T}_{EX}]}(s)\right] |dpds \\
&\leq |||\, h_1 - h_2 \,|||_{\mathbf{M}}.
\end{aligned}$$

In conclusion we get $|||H_1 - H_2|||_{\mathbf{C}_\tau} \leq |||\, h_1 - h_2 \,|||_{\mathbf{M}}$, which proves (3.24). This also makes Ψ is well defined, i.e. if it holds for the equivalence classes of h_1 and h_2 that $[h_1] = [h_2]$, then it also holds for equivalence classes of H_1 and H_2 that $[H_1] = [H_2]$, and so $\Psi([h_1]) = \Psi([h_2])$. □

Proposition 3.4.9. *The function Ψ from Lemma 3.4.8 can be for any fixed $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping time τ uniquely extended to a map*

$$\Psi_\tau : L^1(\mathbf{M}) \to L^1(\mathbf{C}_\tau)$$

such that for any $h \in \mathcal{L}^1(\mathbf{M})$ and any $H \in \Psi([h])$, it holds

$$H\big(\Xi^{\mathbb{W}}, \mathbb{W}_i(\tau)\big) = H\big((\Xi^{\mathbb{W}}_{t\wedge\tau}, t \geq 0), \mathbb{W}_i(\tau)\big) = Z_i(\tau) \text{ a.s.}, \tag{3.25}$$

where $(Z_i, i \in \mathbb{N})$ are the integrated processes for h, see Definition 3.1.2. Further it holds for all $h_1, h_2 \in \mathcal{L}^1(\mathbf{M})$ that

$$|||\Psi([h_1]) - \Psi([\tilde{h_2}])|||_{\mathbf{C}_\tau} \leq |||\, h_1 - h_2 \,|||_{\mathbf{M}}. \tag{3.26}$$

Proof. We recall that $\mathit{span}(\mathcal{S}(\mathcal{F}^{\Xi,\mathbb{W}}))$, or more precisely the set of equivalence classes $\overline{\mathcal{S}}$, is dense in $L^1(\mathbf{M})$ by the Proposition 3.1.7. We want to define Ψ_τ for $[h] \in L^1(\mathbf{M})$. Due to the density of $\overline{\mathcal{S}}$, there exists a approximating sequence $([h_n], n \in \mathbb{N}) \subset \overline{\mathcal{S}}$ with $|||\, |h - h_n| \,|||_{\mathbf{M}} \to 0$, when n goes to infinity. The latter implies that $([h_n], n \in \mathbb{N})$ is Cauchy sequence in $L^1(\mathbf{M})$ and by the inequality (3.24) it follows that $(\Psi([h_n]), n \in \mathbb{N})$ is a Cauchy sequence in $L^1(\mathbf{C}_\tau)$. By taking the limit in $L^1(\mathbf{C}_\tau)$, we can define $\Psi_\tau([h])$ by

$$\Psi_\tau([h]) = \lim_{n\to\infty} \Psi([h_n]).$$

By extending of Ψ_τ is this way to $L^1(\mathbf{M})$, the inequality (3.24) extends also to $L^1(\mathbf{M})$ by the continuity of the norms $|||\cdot|||_{\mathbf{C}_\tau}$ and $|||\cdot|||_{\mathbf{M}}$, this proves (3.26). In order to prove that (3.25) is still true, we need to show for all $i \in \mathbb{N}$ that $Z_i(\tau) = \tilde{Z}_i^\tau$ *a.s.*, where $(Z_i, i \in \mathbb{N})$ are the integrated processes from Definition 3.4.4 and $(\tilde{Z}_i^\tau, i \in \mathbb{N})$ are the random variables defined by $\tilde{Z}_i^\tau := H\big(\mathbf{\Xi}^{\mathbb{W}}, \mathbb{W}_i(\tau)\big)$. If $(Z_i^n, i \in \mathbb{N})$ are the integrated processes for $(h_n, n \in \mathbb{N})$, then $Z_i^n(\tau) \to Z_i(\tau)$ in $L^1(\mathbb{P})$ by Lemma 3.4.5, when n goes to infinity. If H_n is a member of the equivalence class $\Psi_\tau([h_n])$, then it holds by Lemma 3.3.2 for all $i, n \in \mathbb{N}$ that

$$Z_i^n(\tau) = H_n(\mathbf{\Xi}^{\mathbb{W}}, \mathbb{W}_i(\tau)) = H_n\big((\mathbf{\Xi}^{\mathbb{W}}_{t\wedge\tau}, t \geq 0), \mathbb{W}_i(\tau)\big) \text{ a.s.},$$

but at the same time it also holds due to the inequality (3.26) that

$$\begin{aligned}\mathbb{E}\left[|Z_i^n(\tau) - \tilde{Z}_i(\tau)|\right] &= \mathbb{E}\Big[\big|H_n(\mathbf{\Xi}^{\mathbb{W}}, \mathbb{W}_i(\tau)) - H\big(\mathbf{\Xi}^{\mathbb{W}}, \mathbb{W}_i(\tau)\big)\big|\Big] \\ &= |||\Psi([h_n]) - \Psi([\tilde{h}])|||_{\mathbf{C}_\tau} \leq |||\, h_n - h\,|||_{\mathbf{M}} \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

So $Z_i^n(\tau) \to \tilde{Z}_i^\tau$ in $L^1(\mathbb{P})$ for $n \to \infty$ as well, and since the limit in $L^1(\mathbb{P})$ must be unique, it follows that $Z_i^\tau = Z_i(\tau)$ almost surely. Analogously we can show that

$$Z_i(\tau) = H\big((\mathbf{\Xi}^{\mathbb{W}}_{t\wedge\tau}, t \geq 0), \mathbb{W}_i(\tau)\big) \text{ a.s.}$$

□

By localization we obtained the space $\mathcal{L}^1_{loc}(\mathbf{M})$ from $\mathcal{L}^1(\mathbf{M})$. We are now extending the Proposition to $\mathcal{L}^1_{loc}(\mathbf{M})$.

Corollary 3.4.10. *If $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ and $(Z_i(\tau))_{i=1}^\infty$ are the integrated processes for h as in Definition 3.4.4, then there exists a function*

$$H \in \mathbf{M}(C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D})$$

satisfying (3.25) *for $(Z_i(\tau))_{i=1}^\infty$.*

Proof. Since $h \in \mathcal{L}^1_{loc}(\mathbf{M})$, there exists a localizing sequence $(T_n, n \in \mathbb{N})$ of $\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}$-stopping times such that $\mathbb{P}\left[T_n \overset{n\to\infty}{\longrightarrow} \infty\right] = 1$ and it holds $h_n = h\mathbb{1}_{[0,T_n]} \in \mathcal{L}^1(\mathbf{M})$ ($(T_n, n \in \mathbb{N})$ is also increasing). If $(Z_i^n)_{i=1}^\infty$ are the integrated processes corresponding to $(h_n, n \in \mathbb{N})$, then according to the Point 3.19 of Lemma 3.4.5 it holds for all $n \in \mathbb{N}$:

$$\mathbb{P}\left[Z_i(T_n \wedge \tau) = Z_i^n(T_n \wedge \tau),\, n, i \in \mathbb{N}\right] = 1. \tag{3.27}$$

Further there exists for all $n \in \mathbb{N}$ a function $H_n \in \mathbf{M}(C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \times \mathfrak{D})$ with H_n being a member of the equivalence class $\Psi_{T_n\wedge\tau}(h_n)$, where $\Psi_{T_n\wedge\tau}$ is the map from Proposition 3.4.9, such that

$$Z_i(T_n \wedge \tau) = Z_i^n(T_n \wedge \tau) = H\big((\mathbf{\Xi}^{\mathbb{W}}_{t\wedge T_n\wedge\tau}, t \geq 0), \mathbb{W}_i(T_n \wedge \tau)\big) \text{ a.s.}$$

So we get with $T_0 = 0$:

$$\begin{aligned}Z_i(\tau) &= \sum_{n=1}^\infty \mathbb{1}_{(T_{n-1},T_n]}(\tau) Z_i(T_n \wedge \tau) = \sum_{n=1}^\infty \mathbb{1}_{(T_{n-1},T_n]}(\tau) Z_i^n(T_n \wedge \tau) \\ &= \sum_{n=1}^\infty \mathbb{1}_{(T_{n-1},T_n]}(\tau) H_n\big((\mathbf{\Xi}^{\mathbb{W}}_{t\wedge T_n\wedge\tau}, t \geq 0), \mathbb{W}_i(T_n \wedge \tau)\big) \\ &= \sum_{n=1}^\infty \mathbb{1}_{[T_{n-1},T_n]}(\tau) H_n\big((\mathbf{\Xi}^{\mathbb{W}}_{t\wedge\tau}, t \geq 0), \mathbb{W}_i(\tau)\big).\end{aligned}$$

Note that $\mathbb{1}_{[T_{n-1},T_n]}(\tau)$ is measurable with respect $\mathcal{F}_{T_n\wedge\tau} \subset \mathcal{F}_\tau$, so by the factorization Lemma for measurable functions, see Corollary 1.97 in [24], there exists a function $\pi_{C,n} : C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \to \mathbb{R}$ such that

$$\pi_{C,n}(\mathbf{\Xi}^{\mathbb{W}}, \mathbb{W}_i(\tau)) = \pi_{C,n}((\mathbf{\Xi}^{\mathbb{W}}_{t\wedge\tau}, t \geq 0), \mathbb{W}_i(\tau)) = \mathbb{1}_{[T_{n-1},T_n]}(\tau).$$

So if we set $H := \sum_{n=1}^{\infty} \pi_{C,n} H_n$, then H is the desired function. □

Definition 3.4.11. *We can extend the map H from Corollary 3.4.10 to a map* $\overline{H} : C([0,\infty), \mathcal{M}_f(\mathfrak{D})) \to \mathbb{R}^d \times \mathbb{R}$ *such that*

$$\overline{H}\big(\mathbf{\Xi}^{\mathbb{W}}, \mathbb{W}_i(\tau)\big) = \overline{H}\big((\mathbf{\Xi}^{\mathbb{W}}_{t\wedge\tau}, t \geq 0), \mathbb{W}_i(\tau)\big) = (X_i(\tau), Z_i(\tau)) \ a.s. \tag{3.28}$$

by simply setting $\overline{H} := (\pi_X, H)$.

We are finally able to prove that $(X_i(\tau), Z_i(\tau), V_i(\tau))_{i=1}^{\infty}$ is conditionally independent given $\mathcal{F}_\tau^{\mathbf{\Xi},\mathbb{W}}$. For technical reason we define for each $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ a collection of random measures $\{\tilde{\mathbf{Q}}^{XZ,\tau}, \tau \in \mathfrak{T}\}$, where $\mathfrak{T}$ is the collection of all finite $\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}$-stopping times. The introduction of this collection is an intermediate step introduced for the formulation of Proposition 3.4.14. Based on Proposition 3.4.14 we can show in the following section that there exists an $\mathcal{M}_1(\mathbb{R}^d \times \mathbb{R})$-valued process $\mathbf{Q}^{XZ}$ such that $\mathbf{Q}^{XZ}_\tau = \tilde{\mathbf{Q}}^{XZ,\tau}$.

Definition 3.4.12. *For a finite stopping time τ and a fixed $h \in \mathcal{L}$, we define the random measure*

$$\tilde{\mathbf{Q}}^{XZ,\tau} : \Omega \to \mathcal{M}_1(\mathbb{R}^d \times \mathbb{R})$$

by setting for all $f \in C_b(\mathbb{R}^d \times \mathbb{R})$:

$$\tilde{\mathbf{Q}}^{XZ,\tau}(f) := \int_{\mathfrak{D}} f\big(\overline{H}\big((\mathbf{\Xi}^{\mathbb{W}}_{t\wedge\tau}, t \geq 0), \mathfrak{w}\big)\big)\, \mathbf{Q}^{\mathbb{W}}_\tau(d\mathfrak{w}).$$

Remark 3.4.13. *Note that $\tilde{\mathbf{Q}}^{XZ,\tau}$ is measurable with respect to $\mathcal{F}_\tau^{\mathbf{\Xi},\mathbb{W}}$.*

Proposition 3.4.14. *Fix a $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ and let $(Z_i)_{i=1}^{\infty}$ be the integrated processes defined as in* (3.18). *Recall $Y_t = \mathbf{\Xi}^X_t(\mathbb{R}^d)$, $V_i = U_i - U_{i-1}$ for $i \geq 2$ and $V_1 = U_1$, then it holds for all finite stopping times τ that:*

$$\mathfrak{L}\left((X_i(\tau), Z_i(\tau), V_i(\tau))_{i=1}^{\infty} \big| \mathcal{F}_\tau^{\mathbf{\Xi},\mathbb{W}}\right) = \bigotimes_{i=1}^{\infty} \big(\tilde{\mathbf{Q}}^{XZ,\tau} \otimes \boldsymbol{Exp}(Y_\tau)\big), \tag{3.29}$$

where $\tilde{\mathbf{Q}}^{XZ,\tau}$ is the random probability measure from Definition 3.4.12.

Remark 3.4.15. *From* (3.29) *we can conclude*

$$\tilde{\mathbf{Q}}^{XZ,\tau} = \mathfrak{L}\big((X_i(\tau), Z_i(\tau))\big|\mathcal{F}_\tau^{\mathbf{\Xi},\mathbb{W}}\big). \tag{3.30}$$

Proof. From Proposition 2.6.5 we know that:

$$\mathfrak{L}\left((\mathbb{W}_i(\tau), V_i(\tau))_{i=1}^{\infty} \big| \mathcal{F}_\tau^{\mathbf{\Xi},\mathbb{W}}\right) = \bigotimes_{i=1}^{\infty} \big(\mathbf{Q}^{\mathbb{W},\tau} \otimes \mathbf{Exp}(Y_\tau)\big). \tag{3.31}$$

So if we assume that $f_1^{xz}, f_2^{xz}, ..., f_n^{xz} \in C_b(\mathbb{R}^d \times \mathbb{R})$ and $f_1^u, f_2^u, ..., f_n^u \in C_b([0,\infty))$, then we use the function $\overline{H}$ to write:

$$\mathbb{E}\Big[\prod_{i=1}^n f_i^{xz}(X_i(\tau), Z_i(\tau)) f_i^u(U_i(\tau))\Big| \mathcal{F}_\tau^{\mathbf{\Xi},\mathbb{W}}\Big]$$
$$= \mathbb{E}\Big[\prod_{i=1}^n f_i^{xz}\big(\overline{H}\big((\mathbf{\Xi}_{t\wedge\tau}^{\mathbb{W}}, t \geq 0), \mathbb{W}_i(\tau)\big)\big) f_i^u(U_i(\tau))\Big| \mathcal{F}_\tau^{\mathbf{\Xi},\mathbb{W}}\Big].$$

From (3.31) it follows that the above is equal to

$$\prod_{i=1}^n \int_{\mathfrak{D}} f_i^{xz}\big(\overline{H}\big((\mathbf{\Xi}_{t\wedge\tau}^{\mathbb{W}}, t \geq 0), \mathbb{W}_i(\tau)\big)\big) \mathbf{Q}_\tau^{\mathbb{W}}(d\mathfrak{w}) \cdot \mathbf{Exp}(Y_\tau)(f_i^u)$$

This proves (3.29). □

3.5 The Empirical, Intensity and de Finetti process of $(X_i, Z_i, U_i)_{i=1}^\infty$

In this section we are defining the important processes $\boldsymbol{\xi}^{XZ}$ and $\mathbf{Q}^{XZ}$. We are also proving that these processes have indeed the properties we are expecting. Recall the definition of the state space $\overline{\mathcal{N}}(\mathbb{R}^d \times \mathbb{R} \times [0,\infty))$, see Def.1.1.2, and the one of the functions $\gamma_{\mathbb{R}^d\times\mathbb{R}}^{\mathbf{\Xi}}$ and $\gamma_{\mathbb{R}^d\times\mathbb{R}}^{\mathbf{Q}}$, see (1.18) and (2.42). Compare the following definition with Definition 2.5.2.

Definition 3.5.1. *For $h \in \mathcal{L}_{loc}^1(\mathbf{M})$, let us define*

$$\big((X_i, Z_i, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ}, \mathbf{\Xi}^{XZ}, \mathbf{Q}^{XZ}\big) = \mathbb{I}_0\big[h\big], \tag{3.32}$$

where $Z_i : \Omega \times [0,\infty) \to \mathbb{R}$, $i \in \mathbb{N}$, is the integrated process defined in Definition 3.4.4, and the remaining processes are defined in the following way:

$$\boldsymbol{\xi}^{XZ} : \Omega \times [0,\infty) \to \overline{\mathcal{N}}(\mathbb{R}^d \times \mathbb{R} \times [0,\infty)); \quad \boldsymbol{\xi}_t^{XZ} := \sum_{i=1}^\infty \delta_{(X_i(t), Z_i(t), U_i(t))}, \tag{3.33}$$

$$\mathbf{\Xi}^{XZ} : \Omega \times [0,\infty) \to \mathcal{M}_f(\mathbb{R}^d \times \mathbb{R}); \quad \mathbf{\Xi}_t^{XZ} := \gamma_{\mathbb{R}^d\times\mathbb{R}}^{\mathbf{\Xi}}(\boldsymbol{\xi}_t^{XZ}), \tag{3.34}$$

$$\mathbf{Q}^{XZ} : \Omega \times [0,\infty) \to \mathcal{M}_1(\mathbb{R}^d \times \mathbb{R}); \quad \mathbf{Q}_t^{XZ} := \gamma_{\mathbb{R}^d\times\mathbb{R}}^{\mathbf{Q}}((X_i(t), Z_i(t))_{i=1}^\infty). \tag{3.35}$$

Further we define the filtrations $\mathcal{F}^{\mathbf{\Xi},XZ} := (\mathcal{F}_t^{\mathbf{\Xi},XZ}; t \geq 0)$ and $\mathcal{F}^{\mathbf{Q},XZ} := (\mathcal{F}_t^{\mathbf{Q},XZ}; t \geq 0)$ as the complete, right-continuous versions of the natural filtrations of $\mathbf{\Xi}^{XZ}$ and $\mathbf{Q}^{XZ}$. Further we are defining for each $r \geq \frac{b}{a}$ and $m \in \mathbb{N}$ the processes

$$\mathbf{\Xi}^{XZ,r} : \Omega \times [0,\infty) \to \mathcal{N}_f(\mathbb{R}^d \times \mathbb{R}), \quad \mathbf{\Xi}_t^{XZ,r} := \sum_{i=1}^\infty \delta_{(X_i(t), Z_i(t))} \mathbb{1}_{[0,r)}(U_i(t));$$

$$\boldsymbol{\xi}^{XZ,\geq r} : \Omega \times [0,\infty) \to \overline{\mathcal{N}}(\mathbb{R}^d \times \mathbb{R} \times [0,\infty)), \quad \boldsymbol{\xi}_t^{XZ,\geq r} := \sum_{i=1}^\infty \delta_{(X_i(t), Z_i(t), U_i(t))} \mathbb{1}_{[r,\infty)}(U_i(t));$$

$$\mathbf{Q}^{XZ,m} : \Omega \times [0,\infty) \to \mathcal{M}_1(\mathbb{R}^d \times \mathbb{R}), \quad \mathbf{Q}_t^{XZ,m} := \sum_{i=1}^m \delta_{(X_i(t), Z_i(t))}.$$

Further let us define the filtrations $\mathcal{F}^{\mathbf{\Xi},XZ,r}$ and $\mathcal{F}^{\mathbf{Q},XZ,m}$ as the right-continuous completion of the filtrations $\tilde{\mathcal{F}}^{\mathbf{\Xi},XZ,r}$ and $\tilde{\mathcal{F}}^{\mathbf{Q},XZ,m}$ given by $\tilde{\mathcal{F}}^{\mathbf{\Xi},XZ,r} = \sigma\big(\mathbf{\Xi}_s^{XZ,r}, \boldsymbol{\xi}_s^{XZ,\geq r}; s \leq t\big)$ and $\tilde{\mathcal{F}}^{\mathbf{Q},XZ,m} = \sigma\big(\mathbf{Q}_s^{XZ,m}, (X_i(s), Z_i(s), U_i(s))_{i=m}^\infty; s \leq t\big)$.

Remark 3.5.2. *We call $\boldsymbol{\xi}^{XZ}$ the empirical process, $\boldsymbol{\Xi}^{XZ}$ the intensity process and $\mathbf{Q}^{XZ}$ the De-Finetti measure of $(X_i, Z_i, i \in \mathbb{N})$. These names are based on the statements of Theorem 3.5.7.*

Using the functions $\gamma^{\boldsymbol{\Xi}}_{\mathbb{R}^d\times\mathbb{R}}$ to define the processes $\boldsymbol{\Xi}^{XZ}$ and $\mathbf{Q}^{XZ}$ as in the lines (3.34) and (3.35) has the advantage that for each $(\omega, t) \in \Omega \times [0, \infty)$ the values $\boldsymbol{\Xi}^{XZ}_t(\omega)$ and $\mathbf{Q}^{XZ}_t(\omega)$ are well defined and elements of $\mathcal{M}_f(\mathbb{R}^d \times \mathbb{R})$, respectively $\mathcal{M}_1(\mathbb{R}^d \times \mathbb{R})$, even for the elements $(\omega, t) \in \Omega \times [0, \infty)$, for which the sequences

$$\left(\frac{1}{r}\boldsymbol{\Xi}^{XZ,r}_t(\omega), r \geq \frac{b}{a}\right) \text{ and } \left(\frac{1}{n}\mathbf{Q}^{XZ,n}_t(\omega), n \in \mathbb{N}\right)$$

are not converging. Besides the fact that $\boldsymbol{\Xi}^{XZ}$ and $\mathbf{Q}^{XZ}$ are well-defined, we do not know any regularity properties of $\boldsymbol{\Xi}^{XZ}$ and $\mathbf{Q}^{XZ}$, yet. Note that $\boldsymbol{\Xi}^{XZ}$ is by definition adapted to $\mathcal{F}^{\boldsymbol{\Xi},XZ}$ and the same holds for $\mathbf{Q}^{XZ}$ and $\mathcal{F}^{\mathbf{Q},XZ}$. Additionally, we can not directly conclude that $\mathcal{F}^{\boldsymbol{\Xi},XZ}$ and $\mathcal{F}^{\mathbf{Q},XZ}$ contained in the filtration $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$.
Therefore we have two goals for the remaining part of this section. First, we want to prove that $\boldsymbol{\xi}^{XZ}$ is indeed a Poisson representation of $\boldsymbol{\Xi}^{XZ}$ and that $(X_i(\tau), Z_i(\tau), V_i(\tau))_{i=1}^{\infty}$ are for a finite $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time τ conditionally independent, identically distributed with distribution $\mathbf{Q}^{XZ}_\tau \otimes \mathbf{Exp}(Y(\tau))$ conditioned on $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_\tau$. The latter will follow immediately from Proposition 3.4.14, after we have proved for all finite $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping times that:

$$\mathbb{P}\left[\tilde{\mathbf{Q}}^{XZ,\tau} = \mathbf{Q}^{XZ}_\tau\right] = 1. \tag{3.36}$$

Our second goal is to show that the two processes $\boldsymbol{\Xi}^{XZ}$ and $\mathbf{Q}^{XZ}$ admit a $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-progressive modification. This also ensures the $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-adaptedness of processes like $\bar{A} : \Omega \times [0, \infty) \to \mathbb{R}$ with the form:

$$\bar{A} := \int_0^t \int_0^\infty \mathbb{1}_{[0,\tau)}(s)(\hat{f}(x, z + h(x, p, s)) - \hat{f}(x, z))\, dp\, \boldsymbol{\Xi}^{XZ}_s(dx, dz)ds, \tag{3.37}$$

where $h \in \mathcal{L}^1_{loc}(\mathbf{M})$, $\hat{f} \in C_b(\mathbb{R}^d \times \mathbb{R})$ and τ is a finite $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time. Later we will be able to prove that $\boldsymbol{\Xi}^{XZ}$ and $\mathbf{Q}^{XZ}$ have a continuous modifications, but for this we need that processes like $\bar{A}$ are $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-adapted.

Our first step in achieving our goals is to show that $\boldsymbol{\Xi}^{XZ}$ and $\mathbf{Q}^{XZ}$ are $\mathcal{F}^{\boldsymbol{\xi},XZ}$-progressive (recall that the filtration $\mathcal{F}^{\boldsymbol{\xi},XZ}$ contains the information of the individual particles and hence is much bigger than $\mathcal{F}^{\boldsymbol{\Xi},XZ}$). This will not only almost give us the existence of $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-progressive modification, but also ensures that the maps $\omega \mapsto \boldsymbol{\Xi}^{XZ}(\omega)_{\tau(\omega)}$ and $\omega \mapsto \mathbf{Q}^{XZ}(\omega)_{\tau(\omega)}$, where τ is a finite $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time, are actually random variables, because they are at least measurable with respect to the σ-algebra $\mathcal{F}^{\boldsymbol{\xi},XZ}_\tau$. The following additional processes are not only natural but also useful.

For the next lemma we recall that $\mathcal{M}_f(\mathbb{R}^d \times \mathbb{R})$ is equipped with the usual weak topology and $\overline{\mathcal{N}}(\mathbb{R}^d \times \mathbb{R} \times [0, \infty))$ with the mixed topology, see Def. 1.1.2.

Lemma 3.5.3. *The processes $\boldsymbol{\xi}^{XZ}$, $\boldsymbol{\Xi}^{XZ,r}$, $\boldsymbol{\xi}^{XZ,\geq r}$,$r \geq \frac{b}{a}$, and $\mathbf{Q}^{XZ,m}, m \in \mathbb{N}$, are well-defined and càdlàg in the topology of their state spaces.*

Proof. The processes $\boldsymbol{\xi}^{XZ}$, $\boldsymbol{\Xi}^{XZ,r}$ and $\boldsymbol{\xi}^{XZ,\geq r}$ are well defined, if it holds

$$\mathbb{P}[\boldsymbol{\xi}^{XZ}_t(\mathbb{R}^d \times \mathbb{R} \times [0, \tilde{r}]) < \infty;\ t \in [0, \infty), \tilde{r} \geq 0] = 1, \tag{3.38}$$

but this follows from the fact that $\boldsymbol{\xi}^{XZ}_t(\mathbb{R}^d \times \mathbb{R} \times [0, \tilde{r}]) = Y^r(t)$ and that $\mathbb{P}[Y^r(t) < \infty, t \geq 0, r \geq 0] = 1$, see Lemma 2.2.8. Since (X_i, Z_i, U_i) are càdlàg for all $i \in \mathbb{N}$, and due to line (3.38), it follows that $\boldsymbol{\xi}^{XZ}$, $\boldsymbol{\Xi}^{XZ,r}$ and $\boldsymbol{\xi}^{XZ,\geq r}$ are càdlàg. The process $\mathbf{Q}^{XZ,m}$ is also càdlàg for the same reason. □

Lemma 3.5.4. *The processes $\boldsymbol{\Xi}^{XZ}$ and $\mathbf{Q}^{XZ}$ are $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-progressive processes.*

Proof. The processes (X_i, Z_i, U_i) are $\mathcal{F}^{\boldsymbol{\xi},XZ}$-adapted processes for $i \in \mathbb{N}$, hence the same is true for $\boldsymbol{\Xi}^{XZ,r}$ and $\mathbf{Q}^{XZ,n}$ for all $r \geq \max\{b/a, 0\}$ and $n \in \mathbb{N}$. Since $\boldsymbol{\Xi}^{XZ,r}$ and $\mathbf{Q}^{XZ,n}$ have càdlàg paths, these processes are $\mathcal{F}^{\boldsymbol{\xi},XZ}$-progressive. From now on we will only consider the case of $\boldsymbol{\Xi}^{XZ}$, because the case of $\mathbf{Q}^{XZ}$ can be treated analogously. Let us now fix a $T \in [0,\infty)$. If we restrict $\boldsymbol{\Xi}^{XZ,r}$ and $\boldsymbol{\Xi}^{XZ}$ to maps from $\Omega \times [0,T]$ to $\mathcal{M}_f(\mathbb{R}^d \times \mathbb{R})$, then we know from the previous lines that $\boldsymbol{\Xi}^{XZ,r}, r \geq \max\{b/a, 0\}$ is $\mathcal{F}_T^{\boldsymbol{\xi},XZ} \times \mathbb{B}([0,T])$ measurable. Further since $\mathcal{M}_f(\mathbb{R}^d \times \mathbb{R})$ equipped with the weak topology, which we denote by τ^w, can be made to a complete metric space, if follows from the Lemma 1.10.(i) in [21] that the set given by

$$\Gamma := \left\{ (\omega, t) \in \Omega \times [0,T] : \left(r^{-1} \boldsymbol{\Xi}_t^{XZ,r}(\omega),\ r \geq \frac{b}{a} \right) \text{ converges in } \tau^w \right\}.$$

is an element of $\mathcal{F}_T^{\boldsymbol{\xi},XZ} \otimes \mathbb{B}([0,T])$. We are fixing a function $\hat{f} \in C_b^+(\mathbb{R}^d \times \mathbb{R})$. By the definition of the map $\gamma_{\mathbb{R}^d \times \mathbb{R}}^{\boldsymbol{\Xi}}$, it follows that the map $(\omega, t) \mapsto \boldsymbol{\Xi}_t^{XZ}(\omega)(\hat{f})$ is identical with the map

$$(\omega, t) \mapsto \mathbb{1}_\Gamma(\omega, t) \lim_{r \to \infty} \frac{1}{r} \boldsymbol{\Xi}_t^{XZ,r}(\omega)(\hat{f}). \tag{3.39}$$

Since $(\omega, t) \mapsto \boldsymbol{\Xi}_t^{XZ,r}(\omega)(\hat{f})$ is $\mathcal{F}_T^{\boldsymbol{\xi},XZ} \otimes \mathbb{B}([0,T])$-measurable, the map

$$(\omega, t) \mapsto \liminf_{r \to \infty} \frac{1}{r} \boldsymbol{\Xi}_t^{XZ,r}(\omega)(\hat{f})$$

is also $\mathcal{F}_T^{\boldsymbol{\xi},XZ} \otimes \mathbb{B}([0,T])$-measurable. Due to the Line (3.39) and the fact that $\Gamma \in \mathcal{F}_T^{\boldsymbol{\xi},XZ} \otimes \mathbb{B}([0,T])$, the map $(\omega, t) \mapsto \boldsymbol{\Xi}_t^{XZ}(\omega)(\hat{f})$ is $\mathcal{F}_T^{\boldsymbol{\xi},XZ} \otimes \mathbb{B}([0,T])$-measurable. Because this is true for all $\hat{f} \in C_b^+(\mathbb{R}^d \times \mathbb{R})$ and since there exists a countable separating class $(\hat{f}_k, k \in \mathbb{N}) \subset C_b^+(\mathbb{R}^d \times \mathbb{R})$ by Lemma 2.6.4, it follows that $\boldsymbol{\Xi}^{XZ}$ restricted on $\Omega \times [0,T]$ is a $\mathcal{F}_T^{\boldsymbol{\xi},XZ} \otimes \mathbb{B}([0,T])$-measurable map. Since T can be chosen arbitrarily, the process $\boldsymbol{\Xi}^{XZ}$ is $\mathcal{F}^{\boldsymbol{\xi},XZ}$-progressive. □

Lemma 3.5.5. *Let us denote by "$\Rightarrow$" the convergence in the weak topology. If τ is a finite $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time, then it holds*

$$\mathbb{P}\left[r^{-1} \boldsymbol{\Xi}_\tau^{XZ,r} \overset{r\to\infty}{\Longrightarrow} \boldsymbol{\Xi}_\tau^{XZ} \right] = 1, \quad \mathbb{P}\left[n^{-1} \mathbf{Q}_\tau^{XZ,n} \overset{n\to\infty}{\Longrightarrow} \mathbf{Q}_\tau^{XZ} \right] = 1 \tag{3.40}$$

and it holds

$$\mathbb{P}\left[\boldsymbol{\Xi}_\tau^{XZ} = Y_\tau \mathbf{Q}_\tau^{XZ} \right] = 1. \tag{3.41}$$

Further, if $\overline{H}$ is the map from Remark 3.4.10 satisfying (3.28) *for h and the processes $((X_i(\tau), Z_i(\tau)), i \in \mathbb{N})$, then it holds for a fixed $f \in C_b(\mathbb{R}^d \times \mathbb{R})$ almost surely:*

$$\begin{aligned} \boldsymbol{\Xi}_\tau^{XZ}(f) &= \lim_{r \to \infty} \frac{1}{r} \int_{\mathfrak{D}} f\left(\overline{H}\big((\boldsymbol{\Xi}_{t \wedge \tau}^{\mathbb{W}}, t \geq 0), \mathbf{w} \big) \right) \boldsymbol{\Xi}_\tau^{\mathbb{W},r}(d\mathbf{w}) \\ &= \int_{\mathfrak{D}} f\left(\overline{H}\big((\boldsymbol{\Xi}_{t \wedge \tau}^{\mathbb{W}}, t \geq 0), \mathbf{w} \big) \right) \boldsymbol{\Xi}_\tau^{\mathbb{W}}(d\mathbf{w}), \end{aligned} \tag{3.42}$$

and it also holds almost surely

$$\begin{aligned} \mathbf{Q}_\tau^{XZ}(f) &= \lim_{n \to \infty} \frac{1}{n} \int_{\mathfrak{D}} f\left(\overline{H}\big((\boldsymbol{\Xi}_{t \wedge \tau}^{\mathbb{W}}, t \geq 0), \mathbf{w} \big) \right) \mathbf{Q}_\tau^{\mathbb{W},n}(d\mathbf{w}) \\ &= \int_{\mathfrak{D}} f\left(\overline{H}\big((\boldsymbol{\Xi}_{t \wedge \tau}^{\mathbb{W}}, t \geq 0), \mathbf{w} \big) \right) \mathbf{Q}_\tau^{\mathbb{W}}(d\mathbf{w}) \end{aligned} \tag{3.43}$$

Proof. These statements follow from Corollary C.1.7 and Lemma C.1.6. □

Corollary 3.5.6. *It also true that $\boldsymbol{\Xi}^{XZ}_\tau$ and $\mathbf{Q}^{XZ}_\tau$ are measurable with respect to $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_\tau$ (this implies that $\boldsymbol{\Xi}^{XZ}_\tau$ and $\mathbf{Q}^{XZ}_\tau$ are $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-adapted).*

Proof. The adaptedness of $\boldsymbol{\Xi}^{XZ}$ and $\mathbf{Q}^{XZ}$ are a consequence of (3.42) and (3.43). □

Theorem 3.5.7. *Further it holds for all finite $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping times τ that*

$$\mathfrak{L}\big((X_i(t), Z_i(t), V_i(t))_{i=1}^{\infty}\big|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_\tau\big) = \bigotimes_{i=1}^{\infty}\big(\mathbf{Q}^{XZ}_\tau \otimes \boldsymbol{Exp}(Y_\tau)\big) \tag{3.44}$$

and that

$$\mathfrak{L}\big(\boldsymbol{\xi}^{XZ}_\tau\big|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_\tau\big) = \mathbf{PPP}(\boldsymbol{\Xi}^{XZ}_\tau \otimes \ell eb[0,\infty)). \tag{3.45}$$

If τ is a $\mathcal{F}^{\boldsymbol{\Xi},XZ}$-stopping time, then $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_\tau$ can be replaced by $\mathcal{F}^{\boldsymbol{\Xi},XZ}_\tau$ in (3.44) *and* (3.45).

We are ready to prove the existence of the $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-progressive modifications of $\boldsymbol{\Xi}^{XZ}$ and $\mathbf{Q}^{XZ}$. Unfortunately, if the stopping time τ has uncountably many different values and $\tilde{\boldsymbol{\Xi}}^{XZ}$ is the $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-progressive modification of $\boldsymbol{\Xi}^{XZ}$, then there is no reason why it should be true that

$$\mathbb{P}\left[\tilde{\boldsymbol{\Xi}}^{XZ}_\tau = \boldsymbol{\Xi}^{XZ}_\tau\right] = 1,$$

when the range of τ is uncountable. This is problematic in the sense, that the statements of Lemma 3.5.5 can only be extended to the $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-progressive modification $\tilde{\boldsymbol{\Xi}}^{XZ}$, when we consider $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping times with a countable range. Thankfully we do not need this extension, because we just use the progressive extension to prove that processes like $\overline{A}$ in (3.37) are $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-adapted. Still, we formulate the next theorem in such a way, that we can circumvent this problem. Later all this difficulties will vanish, after we have proved the existence of a continuous modification.

Proposition 3.5.8. *Assume that $\hat{\tau}$ is a $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time which is not necessarily finite, indeed we allow $\mathbb{P}[\hat{\tau} = \infty] > 0$, and that $\boldsymbol{\Xi}^{XZ}$ and $\mathbf{Q}^{XZ}$ are the processes defined in the lines* (3.34) *and* (3.35). *When we define the stopped processes $\boldsymbol{\Xi}^{XZ}_{\cdot\wedge\hat{\tau}}$ and $\mathbf{Q}^{XZ}_{\cdot\wedge\hat{\tau}}$ by setting*

$$(\omega, t) \mapsto \boldsymbol{\Xi}^{XZ}_{t\wedge\hat{\tau}(\omega)}(\omega), \qquad\qquad (\omega, t) \mapsto \mathbf{Q}^{XZ}_{t\wedge\hat{\tau}(\omega)}(\omega),$$

then the processes $\boldsymbol{\Xi}^{XZ}_{\cdot\wedge\hat{\tau}}$ and $\mathbf{Q}^{XZ}_{\cdot\wedge\hat{\tau}}$ admit $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-progressive modifications and the statements of Lemma 3.5.5 are also true for these $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-progressive modifications, when τ has a countable range.

Proof. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the probability space of our processes. Lemma 3.5.4 shows that the processes $\boldsymbol{\Xi}^{XZ}$ and $\mathbf{Q}^{XZ}$ are $\mathcal{F}^{\boldsymbol{\xi},XZ}$-progressive and so $\mathcal{A}$-measurable processes. Since $(\omega, t) \mapsto (\omega, t \wedge \hat{\tau}(\omega))$ is a $\mathcal{A} \otimes \mathbb{B}([0,\infty)) - \mathcal{A} \otimes \mathbb{B}([0,\infty))$ measurable map, we also can conclude that $\boldsymbol{\Xi}^{XZ}_{\cdot\wedge\tau}$ and $\mathbf{Q}^{XZ}_{\cdot\wedge\tau}$ are $\mathcal{A}$-measurable processes. By Lemma 3.5.5 we know that $\boldsymbol{\Xi}^{XZ}_{\cdot\wedge\hat{\tau}}$ and $\mathbf{Q}^{XZ}_{\cdot\wedge\hat{\tau}}$ are $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-adapted. It follows from Theorem 0.1 in [38] that $\boldsymbol{\Xi}^{XZ}_{\cdot\wedge\hat{\tau}}$ and $\mathbf{Q}^{XZ}_{\cdot\wedge\hat{\tau}}$ admit $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-progressive modifications. □

Chapter 4

Semi-Martingale-Decompositions

Let us assume that h is an element of $\mathcal{L}^1_{loc}(\mathbf{M})$ and let us set

$$\big((X_i, Z_i, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ}, \mathbf{Q}^{XZ}\big) = \mathbb{I}_0\big[h\big],$$

where we use $\mathbb{I}_0\big[h\big]$ as in Definition 3.5.1. Our goal of this chapter is to derive semi-martingale decompositions of real valued functionals belonging to the processes $(X_i, Z_i, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ}$ and $\boldsymbol{\Xi}^{XZ,r}$ with $r \geq \max\{b/a, 0\}$. Indeed, if $P : \Omega \times [0, \infty) \to \mathbb{R}$ is a process either given by $P := G((X_i, Z_i, U_i)_{i=1}^\infty), P := F(\boldsymbol{\xi}^{XZ})$, $P := \hat{F}(\boldsymbol{\Xi}^{XZ})$ or by $P := \hat{F}(\boldsymbol{\Xi}^{XZ,r})$, where G, F and $\hat{F}$ are functions with values in $\mathbb{R}$ (which will be specified in more detail below), then we wish to find two processes $M : \Omega \times [0, \infty) \to \mathbb{R}$ and $A : \Omega \times [0, \infty) \to \mathbb{R}$ with $M(0) = A(0) = 0$ such that

$$P(t) = P(0) + M(t) + A(t), \ t \geq 0, \tag{4.1}$$

where M is a local martingale with respect to some filtration $\tilde{\mathcal{F}}$ and A is a continuous (and hence predictable) process with finite variation adapted to the same filtration $\tilde{\mathcal{F}}$. The filtration $\tilde{\mathcal{F}}$ depends hereby on the underlying process. In total we have four cases:

Case I: The process P is given by $P = G((X_i, Z_i, U_i)_{i=1}^\infty)$ and the functional $G : \mathbf{S}(\mathbb{R}^d \times \mathbb{R}) \to [0, \infty)$, where $\mathbf{S}(\mathbb{R}^d \times \mathbb{R})$ is the ordered space from Definition 2.0.2, has the form

$$G(\mathbf{x}, \mathbf{z}, \mathbf{u}) = \prod_{i=1}^\infty g_i(x_i, z_i, u_i),$$

where $g_i : \mathbb{R}^d \times \mathbb{R} \times [0, \infty] \to [0, \infty)$ is for each $i \in \mathbb{N}$ an element of the class $\mathfrak{g}^Z$, which will be defined later in Definition 4.2.4 and has many similarities with the class $\mathfrak{g}(B)$ from Definition B.2.7. This case is divided in two subcases. The first case, Case **I.a** has the restriction that G only depends on a finite number of coordinates, indeed there exists a $n \in \mathbb{N}$ such that $g_i = \mathbb{1}_{\mathbb{R}^d \times \mathbb{R} \times [0,\infty)}$ for all $i \geq n$. The second case, Case **I.b**, allows G to depend on infinitely many coordinates, hence Case **I.a** is contained in **I.b** and it serves as an intermediate step. The process M from (4.1) is going to be a local martingale with respect to $\mathcal{F}^{\boldsymbol{\xi}, \mathbb{W}}$.

Case II: The process P is given by $P = F(\boldsymbol{\xi}^{XZ})$, where $F : \overline{\mathcal{N}}(\mathbb{R}^d \times \mathbb{R} \times [0, \infty)) \to [0, \infty)$ is a functional having one of the following two forms:

II.a: $F(\xi) := \xi(\tilde{g})$, **II.b:** $F(\xi) := \exp(-\xi(g))$.

The function $\tilde{g}$ is an element of $\tilde{\mathfrak{g}}^Z$, see Definition 4.2.4, and g is an element of $\mathfrak{g}^Z$. The process M from (4.1) is again a local martingale with respect to $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$.

__**Case III:**__ The process P is given by $P := \hat{F}(\boldsymbol{\Xi}^{XZ})$, where $\hat{F} : \mathcal{M}(\mathbb{R}^d) \to [0,\infty)$ is a function having one of the two forms:

III.a: $\hat{F}(\mu) := \mu(\hat{g})$, **III.b:** $\hat{F}(\mu) := \exp(-\mu(\hat{g}))$,

where $\hat{g}$ is an element of $C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$ in both cases. The process M from (4.1) will be a local martingale with respect to $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$ (later, we will write $\hat{M}$ instead of M).

__**Case IV:**__ The process P is given by $P := \boldsymbol{\Xi}^{XZ,r}(\hat{g})$ with $\hat{g} \in C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$. The process M will be a local martingale with respect to $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W},r}$ (and we will also write $\hat{M}$ instead of M).

This chapter is organized in the following way. In the first section we discuss the properties which must be satisfied by the integrand h such that the processes $\hat{A}$ and $\tilde{A}$ given by

$$\hat{A}(t) := \int_0^t \int_{\hat{E}} \int_0^\infty (\hat{g}(x, z + h(x,p,s)) - \hat{g}(x,z)) dp \boldsymbol{\Xi}_s^{XZ}(dx,dz)ds, \ t \geq 0, \tag{4.2}$$

$$\tilde{A}(t) := \int_0^t \int_{\overline{E}} \int_0^\infty (g(x, z + h(x,p,s), u) - g(x,z,u)) dp \boldsymbol{\xi}_{s\text{-}}^{XZ}(dx,dz)ds, \ t \geq 0, \tag{4.3}$$

are well-defined ($\hat{E} := \mathbb{R}^d \times \mathbb{R}$ and $\overline{E} := \mathbb{R}^d \times \mathbb{R} \times [0,\infty)$ and where $\hat{g}$ and g have been chosen as above). These kind of processes will appear in our semi-martingale decompositions. Fortunately it turns out that integrands from $\mathcal{L}^1_{loc}(\mathbf{M})$ are sufficient. In Section 3 we introduce the function classes $\tilde{\mathfrak{g}}^Z, \mathfrak{g}^Z$ and $\mathfrak{G}^Z$, which allow us to formulate the above cases in more detail. The proofs of the cases follow after the presentation of the cases. The main tool for the proof of Case I.a is the Itô formula for càdlàg semi-martingales and the Case I.b follows from Case I.a with the help of a martingale convergence theorem. The Case II.a is also derived via the Itô formula and the important Case II.b is actually just a special case of Case I.b. The derivation of the Case III.a and III.b is more interesting, in both cases we use the Poisson representation property of $(\boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ})$ and the conditional martingale lemma from Section D.2. The same lemma is used in the Case IV, but this time it in combination with the fact that $U_i(t), 1 \leq i \leq Y_t^r$, is uniformly distributed over $[0,r]$ conditioned on $\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W},r}$.
To motivate the derivation of the semi-martingale decompositions let us mention that these allow us to make efficient use of the tools of stochastic calculus like the Doob inequality, which we apply in the proof that $\boldsymbol{\Xi}^{XZ}$ admits a continuous modification in the weak topology. Further they play an important role in the derivation of our main theorem, the existence of a Poisson representation $\hat{\boldsymbol{\xi}}$ for a superprocess $\hat{\boldsymbol{\Xi}}$ with competition.

Remark 4.0.1. *Since A from* (4.1) *is predictable with respect to the corresponding filtration, the decomposition M and A is unique up to indistinguishability. Indeed let us assume $\tilde{M}$ and $\tilde{A}$ are processes with the same properties as M and A except that $\tilde{A}$ is predictable but not necessary continuous, then* (4.1) *tells us that $\tilde{M} - M = A - \tilde{A}$ and so $\dot{M} := \tilde{M} - M$ is a predictable local martingale with finite variation. But predictable martingales are continuous, see Proposition 22.16 in [21], and hence $\dot{M}$ must be constant with $\dot{M}(0) = 0$ by* (4.1). *Because of their uniqueness*

M and A are called the canonical decomposition and P a special semi-martingale in the terminology of Jean Jacod and Albert Shiryaev, see Definition 1.4.21.b) and 1.4.22. the term canonical decomposition should not be confused with the canonical representation of a semi-martingale, see Theorem 2.2.34 in [19]).

4.1 The Space $\mathcal{L}^1_{loc}(\mathbf{M}^\Xi)$

In the previous chapter we argued that the space $\mathcal{L}^1_{loc}(\mathbf{M})$ is a good choice for a space of possible integrands h to ensure that the corresponding integrated processes $(Z_i, i \in \mathbb{N})$ are well-defined. Let us therefore assume that

$$\big((X_i, Z_i, U_i)_{i=1}^{\infty}, \boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ}, \mathbf{Q}^{XZ}\big) = \mathbb{I}_0\big[h\big].$$

In the remaining part of this chapter (and this thesis) we will often encounter processes like $\hat{A}$ and $\tilde{A}$, see (4.2) and (4.3). Unfortunately the condition, that $h \in \mathcal{L}^1(\mathbf{M})$, is not sufficient to ensure that $\hat{A}$ or $\tilde{A}$ are well-defined or have finite first moments. So we will introduce a new class of integrand for this purpose.

Definition 4.1.1. *We define the measures $\mathbf{M}^\Xi$ on the measurable space*

$$\big(\Omega \times \mathbb{R}^d \times [0,\infty) \times [0,\infty), \mathfrak{P}(\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}})\big)\,,$$

by setting for each non-negative, predictable function $h \in \mathcal{P}$(Recall the predictable σ-algebra of $\mathfrak{P}$ and the class $\mathcal{P}$ of predictable integrands from Definition 3.1.1):

$$\mathbf{M}^\Xi(h){:=}\mathbb{E}\bigg[\int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} |h(x,p,s)|\, \boldsymbol{\Xi}^X_s(dx)dpds\bigg].$$

The measures $\mathbf{M}$ from Definition 3.1.2 and $\mathbf{M}^\Xi$ are closely related to each other, indeed we can derive from Proposition 3.5.5 that $\boldsymbol{\Xi}^X_s = Y_s \mathbf{Q}^X_s$.

Definition 4.1.2. *We define the space $\mathcal{L}^1(\mathbf{M}^\Xi)$ as the set of all predictable functions $h \in \mathcal{P}$ satisfying*

$$\mathbb{E}\bigg[\int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} |h(x,p,s)|\, \boldsymbol{\Xi}^X_s(dx)dpds\bigg] < \infty$$

and we define the space $\mathcal{L}^1_{loc}(\mathbf{M}^\Xi)$ as the set of predictable functions $h \in \mathcal{P}$ for which we can find an increasing sequence $(T_n)_{n=1}^\infty$ of $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping times with $\mathbb{P}[T_n \overset{n\to\infty}{\longrightarrow} \infty] = 1$ such that $\mathbb{1}_{[0,T_n]} h \in \mathcal{L}^1(\mathbf{M}^\Xi)$. for all $n \in \mathbb{N}$.

The next lemma, more precisely its corollary, will shows that in the previous chapter defined space $\mathcal{L}^1_{loc}(\mathbf{M})$ that is actually a subset of our new space $\mathcal{L}^1_{loc}(\mathbf{M}^\Xi)$.

Lemma 4.1.3. *If $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ with localizing sequence $(T_n)_{n=1}^\infty$ then it holds for all $n \in \mathbb{N}$ that*

$$\begin{aligned}\mathbb{E}\bigg[\int_0^{\tilde{T}_n} \int_{\mathbb{R}^d} \int_0^\infty |h(x,p,s)|\, dp \boldsymbol{\Xi}^X_s(dx)ds\bigg] \\ = \int_0^\infty \int_0^\infty \mathbb{E}\bigg[\int_{\mathbb{R}^d} \mathbb{1}_{[0,\tilde{T}_n)}(s)|h(x,p,s)|\boldsymbol{\Xi}^X_s(dx)\bigg]dpds < \infty,\end{aligned} \tag{4.4}$$

where $\tilde{T}_n := T_n \wedge \tau^Y_n$ and $\tau^Y_n := \inf\{s \geq 0 : Y_s \geq n\}$.

Before we prove Lemma 4.1.3, we state its most important consequence.

Corollary 4.1.4. *If h is an element of $\mathcal{L}^1_{loc}(\mathbf{M})$ with localizing sequence $(T_n)_{n=1}^\infty$, then h is also an element of $\mathcal{L}^1_{loc}(\mathbf{M}^\Xi)$ with localizing sequence $(T_n \wedge \tau_n^Y, n \in \mathbb{N})$.*

Remark 4.1.5. *So we have to keep in mind that $\mathcal{L}^1(\mathbf{M}) \subset \mathcal{L}^1_{loc}(\mathbf{M}) \subset \mathcal{L}^1_{loc}(\mathbf{M}^\Xi)$.*

Proof of Lemma 4.1.3. Considering the identity of the two integrals in (4.4), we note that $|h| \geq 0$, so it is allowed by Tonelli's theorem to switch the order of integration and it follows

$$\int_0^{\tilde{T}_n} \int_{\mathbb{R}^d} \int_0^\infty |h(x,p,s)|\, dp \mathbf{\Xi}_s^X(dx) ds = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{[0,\tilde{T}_n]}(s) |h(x,p,s)|\, \mathbf{\Xi}_s^X(dx) dp ds.$$

Taking the expectation and pulling the expectation inside the first two integrals, again by Tonelli, gives us (4.4). Finiteness follows by applying

$$\mathbb{P}\left[\mathbf{\Xi}_t^X = Y_t \mathbf{Q}_t^X,\, t \geq 0\right] = 1, \tag{4.5}$$

see (3.41) in Lemma 3.5.5 and recall that $\mathbf{\Xi}^X, Y$ and $\mathbf{Q}^X$ are continuous on $[0, \mathcal{T}_{EX})$ and both sides of the equality (4.5) are equal to 0 on $[\mathcal{T}_{EX}, \infty)$. It further holds for all $n \in \mathbb{N}$:

$$\mathbb{1}_{[0,\tilde{T}_n]}(t) Y_t \leq \mathbb{1}_{[0,\tilde{T}_n \wedge \mathcal{T}_{EX}]}(t) n, \quad t \in [0, \infty),$$

where $\mathcal{T}_{EX} := \inf\{s \geq 0 : Y_s\}$ is the extinction time. Applying both facts gives us

$$\mathbb{1}_{[0,\tilde{T}_n]}(t) \int_{\mathbb{R}^d} |h(x,p,t)|\, \mathbf{\Xi}_t^X(dx) \leq \mathbb{1}_{[0,\tilde{T}_n \wedge \mathcal{T}_{EX}]}(t) n \int_{\mathbb{R}^d} |h(x,p,t)|\, \mathbf{Q}_t^X(dx) \text{ a.s.}, \ t \geq 0. \tag{4.6}$$

Using that for each fixed $t \geq 0$ it is true that that $\mathbb{1}_{[0,\tilde{T}_n \wedge \mathcal{T}_{EX})}(t)$ is $\mathcal{F}_t^{\mathbf{\Xi},\mathbb{W}}$ measurable, that $X_1(t)$ conditioned on $\mathcal{F}_t^{\mathbf{\Xi},\mathbb{W}}$ has the distribution $\mathbf{Q}_t^X$ and that $X_1(t\text{-}) = X_1(t)$ almost surely gives us

$$\begin{aligned}\mathbb{E}\left[\int_{\mathbb{R}^d} \mathbb{1}_{[0,\tilde{T}_n \wedge \mathcal{T}_{EX}]}(t) |h(x,p,t)|\, \mathbf{Q}_t^X(dx)\right] &= \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_n \wedge \mathcal{T}_{EX}]}(t) \mathbb{E}\left[\int_{\mathbb{R}^d} |h(x,p,t)|\, \mathbf{Q}_t^X(dx) \Big| \mathcal{F}_t^{\mathbf{\Xi},\mathbb{W}}\right]\right] \\ &= \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_n \wedge \mathcal{T}_{EX}]}(t) \mathbb{E}\left[|h(X_1(t\text{-}),p,t)|\right]\right].\end{aligned}$$

Combining this result with (4.6) gives us now that

$$\begin{aligned}&\int_0^\infty \int_0^\infty \mathbb{E}\Big[\int_{\mathbb{R}^d} \mathbb{1}_{[0,\tilde{T}_n]}(s) |h(x,p,s)| \mathbf{\Xi}_s^X(dx)\Big] dp ds \\ &\qquad \leq \ n \int_0^\infty \int_0^\infty \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_n \wedge \mathcal{T}_{EX}]}(s) \mathbb{E}\left[|h(X_1(s\text{-}),p,s)|\right]\right] dp ds.\end{aligned}$$

The last expression is by the definition of the norm $|||\cdot|||_{\mathbf{M}}$ of the space $\mathcal{L}^1(\mathbf{M})$ identical with the expression $n|||\,\mathbb{1}_{[0,T_n)} h\,|||_{\mathbf{M}}$ and since $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ with localizing sequence $(T_n)_{n=1}^\infty$, it follows that $|||\,\mathbb{1}_{[0,T_n)} h\,|||_{\mathbf{M}}$ is finite, which in turn implies that integrals in (4.4) are finite. □

Proof of Corollary 4.1.4. This is a direct consequence of Lemma 4.1.3. □

4.2 Case I

For the rest of this chapter we **fix** an integrand $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ with a localizing sequence $(T_{\widehat{m}})_{\widehat{m}=1}^{\infty}$ and we also fix for the rest of this chapter

$$\big((X_i, Z_i, U_i)_{i=1}^{\infty}, \boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ}, \mathbf{Q}^{XZ}\big) = \mathbb{I}_0\big[h\big],$$

recall Definition 3.5.1. Further we define

$$\tau^Y_{\widehat{m}} := \inf\{t \geq 0 : Y_t \geq \widehat{m}\} \tag{4.7}$$

for each $\widehat{m} \in \mathbb{N}$, we can see by Corollary 4.1.4 that h is also an element of $\mathcal{L}^1_{loc}(\mathbf{M}^{\Xi})$ with localizing sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$, where

$$\tilde{T}_{\widehat{m}} := T_{\widehat{m}} \wedge \tau^Y_{\widehat{m}}, \quad \widehat{m} \in \mathbb{N}. \tag{4.8}$$

The sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$ remains also fixed for the rest of this chapter and will be a common localizing sequence of the semi-martingales presented in this chapter. We also could write $\tilde{T}^h_{\widehat{m}}$ instead of $\tilde{T}_{\widehat{m}}$ to make the dependence on h, which is given via $(T_{\widehat{m}}, \widehat{m} \in \mathbb{N})$, more explicit. The semi-martingale decompositions are based on the following formal definition of $\mathbf{A}^o_{B_X,h}(G)$. For this purpose let us recall the spaces $\mathbf{S}_{[0,\infty)}(\mathbb{R}^d \times \mathbb{R})$, $(x_i, u_i)_{i=1}^{\infty} \in \mathbf{S}(\mathbb{R}^d \times \mathbb{R})$ with $u_i < \infty$ for all $i \in \mathbb{N}$, and $\mathbf{S}_{\infty}(\mathbb{R}^d \times \mathbb{R})$, $(x_i, u_i)_{i=1}^{\infty} \in \mathbf{S}(\mathbb{R}^d \times \mathbb{R})$ with $u_i = \infty$ for some $i \in \mathbb{N}$, from Definition *D*. 2.0.2.

Definition 4.2.1. *Assume that $G : \mathbf{S}(\mathbb{R}^d \times \mathbb{R}) \to \mathbb{R}$ has the form*

$$G\left(\omega, (x_i, z_i, u_i)_{i=1}^{\infty}\right) = \prod_{i=1}^{\infty} g_i(x_i, z_i, u_i),$$

where $g_i : \mathbb{R}^d \times \mathbb{R} \times [0, \infty] \to [0, 1]$, $i \in \mathbb{N}$, then we define formally the function

$$\mathbf{A}^o_{B_X,h}(G) : \Omega \times \mathbf{S}(\mathbb{R}^d \times \mathbb{R}) \times [0, \infty) \to \mathbb{R}$$

for $\omega(x_i, z_i, u_i)_{i=1}^{\infty} \in \mathbf{S}_{[0,\infty)}(\mathbb{R}^d \times \mathbb{R}), t \in [0, \infty)$ by setting

$$\begin{aligned}
\mathbf{A}^o_{B_X,h}(G)\big(\,\omega\,, (x_i, z_i, u_i)_{i=1}^{\infty}, t\big) = &\\
&\prod_{l=1}^{\infty} g_l(x_l, z_l, u_l) \sum_{i=1}^{\infty} \frac{B_X(g_i)(x_i, z_i, u_i)}{g_i(x_i, z_i, u_i)}\\
&+ \prod_{l=1}^{\infty} g_l(x_l, z_l, u_l) \sum_{i=1}^{\infty} (a u_i^2 - b u_i) \frac{\partial_u g_i(x_i, z_i, u_i)}{g_i(x_i, z_i, u_i)}\\
&+ \prod_{l=1}^{\infty} g_l(x_l, z_l, u_l) \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \int_{u_{j\text{-}1}}^{u_j} 2a \bigg(g_j(x_i, z_i, v) \prod_{m=j}^{\infty} \frac{g_{m+1}(x_m, z_m, u_m)}{g_m(x_m, z_m, u_m)} - 1 \bigg) dv\\
&+ \prod_{l=1}^{\infty} g_l(x_l, z_l, u_l) \sum_{i=1}^{\infty} \int_0^{\infty} \left(\frac{g_i(x_i, z_i + h(\omega, x_i, p, t), u_i)}{g_i(x_i, z_i, u_i)} - 1 \right) dp,
\end{aligned}$$

and for $\omega \in \Omega, (x_i, z_i, u_i)_{i=1}^{\infty} \in \mathbf{S}_{\infty}(\mathbb{R}^d \times \mathbb{R}), t \in [0, \infty)$ by setting

$$\mathbf{A}^o_{B_X,h}(G)\big(\omega, (x_i, z_i, u_i)_{i=1}^{\infty}, t\big) = 0. \tag{4.9}$$

Remark 4.2.2. *In the following we will omit ω from $\mathbf{A}^o_{B_X,h}(G)(\omega,(x_i,z_i,u_i)_{i=1}^\infty,t)$ and write instead $\mathbf{A}^o_{B_X,h}(G)((x_i,z_i,u_i)_{i=1}^\infty,t)$, but please be aware that $\mathbf{A}^o_{B_X,h}(G)$ is a random function due to its dependence on h. Since h is predictable, the same is true for $\mathbf{A}^o_{B_X,h}(G)$.*

Comparing $\mathbf{A}^o_{B_X,h}(G)$ with the generator of the ordered Kurtz-Rodrigues representation $\mathbf{A}^o_{B_X}(G)$ from Definition 2.5.8, we can observe that the two expressions are almost identical except for the new line describing the behavior of the z-coordinate. In order to ensure that $\mathbf{A}^o_{B_X}(G)$ is a well-defined function, we will have to define a suitable class of test functions G, which we will denote by $\mathfrak{G}^Z$. The class $\mathfrak{G}^Z$ will have great similarities with $\mathfrak{G}(\mathbf{B})$ from Definition 2.5.7. But before we proceed, we need additional definitions.

Definition 4.2.3. *We define $C_b^{2,1}(\mathbb{R}^d\times\mathbb{R}\times[0,\infty))$ as the collection of continuous functions $\hat{g}$, which are twice continuously differentiable with regard to the (x,z)-coordinate and continuously differentiable with regard to the u-coordinate, and also satisfy*

$$||\hat{g}||_{\infty,2} := \sup_{(x,z,u)\in\mathbb{R}^d\times\mathbb{R}\times[0,\infty)}\Bigg(|g(x,z,u)| + \sum_{i=1}^m |\partial_{x_i} g(x,z,u)| + |\partial_z g(x,z,u)| + |\partial_u g(x,z,u)| \\ + \sum_{i,j=1}^m |\partial_{x_i x_j} g(x,z,u)| + |\partial_{zz} g(x,z,u)|\Bigg) < \infty.$$

We denote by $C_b^{2,1,+}(\mathbb{R}^d\times\mathbb{R}\times[0,\infty))$ the subset of non-negative functions.

Let us assume that $g\in C_b^{2,1}(\mathbb{R}^d\times\mathbb{R}\times[0,\infty))$ and let us define for each $(z,u)\in\mathbb{R}\times[0,\infty)$ the function $g^{zu}(x)=g(x,z,u)$, then $\tilde{g}^{zu}\in C_b^2(\mathbb{R}^d)$ and hence $\tilde{g}^{zu}\in\mathcal{D}(B_X)$, which in turn allows us to define $B_X(g)\in C(\mathbb{R}^d\times\mathbb{R}\times[0,\infty))$ as the function given by $B_X(g)(x,z,u)=B_X(g^{zu})(x)$.

Definition 4.2.4. *Fixing $K\in[0,\infty)$, $r>0$, $m\in(0,1)$ we define the two classes of test functions*

$$\tilde{\mathfrak{g}}^Z(K,r,m)\subset C_b^{2,1,+}(\mathbb{R}^d\times\mathbb{R}\times[0,\infty)),$$
$$\mathfrak{g}^Z(K,r,m)\subset C_b^{2,1,+}(\mathbb{R}^d\times\mathbb{R}\times[0,\infty)),$$

by saying that $\tilde{\mathfrak{g}}^Z(K,r,m)$ consists of nonnegative, continuous functions $\tilde{g}:\mathbb{R}^d\times\mathbb{R}\times[0,\infty)\to[0,1]$ satisfying:

1. *$|B_X(\tilde{g})|$, $|\partial_z\tilde{g}|$ and $|\partial_u\tilde{g}|$ are bounded by the constant K.*
2. *The support of the function $\tilde{g}$ is contained in $\mathbb{R}^d\times\mathbb{R}\times[0,r]$.*
3. *The image of $\tilde{g}$ is contained in $[0,m)$, i.e.*

$$0\le\tilde{g}(x,z,u)<m<1,\ (x,z,u)\in\mathbb{R}^d\times\mathbb{R}\times[0,\infty). \tag{4.10}$$

The second set $\mathfrak{g}^Z(K,r,m)$ consists of functions g with the form

$$g(x,z,u)=1-\tilde{g}(x,z,u), \tag{4.11}$$

where $\tilde{g}(x,z,u)\in\tilde{\mathfrak{g}}^Z(K,r,m)$. We also define

$$\tilde{\mathfrak{g}}^Z := \bigcup_{K>0,r>0,m\in(0,1)} \mathfrak{g}^Z(K,r,m) \text{ and } \mathfrak{g}^Z := \bigcup_{K>0,r>0,m\in(0,1)} \mathfrak{g}^Z(K,r,m).$$

Remark 4.2.5. *We will often use that, if $\tilde{g} \in \tilde{\mathfrak{g}}^Z$ and $g = 1 - \tilde{g} \in \mathfrak{g}^Z$, then it holds $B_X(\tilde{g}) = -B_X(g)$, $\partial_z \tilde{g} = -\partial_z g$ and $\partial_u \tilde{g} = -\partial_u g$.*

Lemma 4.2.6. *The sets $\tilde{\mathfrak{g}}^Z$ and $\mathfrak{g}^Z$ are closed under multiplication.*

Proof. Assume now that $g_1 = 1 - \tilde{g}_1 \in \mathfrak{g}^Z(K, r, m)$ and $g_2 = 1 - \tilde{g}_2 \in \mathfrak{g}^Z(K, r, m)$. In order to show that $g := g_1 g_2$ is again an element of $\mathfrak{g}^Z$, we need to show that

$$\tilde{g} = 1 - (1 - \tilde{g}_1)(1 - \tilde{g}_2) = 1 - \tilde{g}_1 - \tilde{g}_2 + \tilde{g}_1 \tilde{g}_2$$

is an element of $\tilde{\mathfrak{g}}^Z$. Since $\tilde{g}_1, \tilde{g}_2$ and $\tilde{g}_1 \tilde{g}_2$ are elements of $C_b^{2,1}(\mathbb{R}^d \times \mathbb{R} \times [0, \infty))$, the same is true for $\tilde{g}$ and hence there must exists a constant K with $|B_X(\tilde{g})| \leq K$ and $|\partial_z(\tilde{g})| \leq K$. For the condition (4.10) let us assume that (4.10) is satisfied by $\tilde{g}_1$ and $\tilde{g}_2$ for the constant $\hat{m} \in (0, 1)$ and note that we can write

$$\tilde{g} = 1 - (1 - \tilde{g}_1)(1 - \tilde{g}_2) \leq 1 - (1 - \hat{m})^2 =: m < 1.$$

From this we can conclude that $\tilde{g}$ satisfies (4.10) for the constant m. Further, if the support of $\tilde{g}_i$ is contained in $\mathbb{R}^d \times \mathbb{R} \times [0, r_i]$, $i \in \{1, 2\}$, then the support of $\tilde{g}$ is contained in $\mathbb{R}^d \times \mathbb{R} \times [0, \min\{r_1, r_2\})$. With a similar, less complex argument we can also see that $\tilde{\mathfrak{g}}^Z$ is closed under multiplication. ☐

If $\tilde{g} \in \tilde{\mathfrak{g}}^Z$ and $\hat{g} \in \mathfrak{g}^Z$, then these are functions with domain $\mathbb{R}^d \times \mathbb{R} \times [0, \infty)$ according to Definition 4.2.4. In Definition 4.2.7 we will interpret $\tilde{g}$ and $\hat{g}$ as functions with domain $\mathbb{R}^d \times \mathbb{R} \times [0, \infty]$, where $\tilde{g}(x, z, \infty) = 0$ and $g(x, z, \infty) = 1$ for all $(x, z) \in \mathbb{R}^d \times \mathbb{R}$. This is a continuous extension, because $\lim_{u \to \infty} \tilde{g}(x, z, u) = 0$ and $\lim_{u \to \infty} g(x, z, u) = 1$.

Definition 4.2.7. *For $K \in [0, \infty)$, $r > 0$, $m \in (0, 1), n \in \mathbb{N}_0 \cup \{\infty\}$, we define the set*

$$\mathfrak{G}^Z(K, r, m, n) \subset B(\mathbf{S}(\mathbb{R}^d \times \mathbb{R}))$$

as a collection of functions $G : \mathbf{S}(\mathbb{R}^d \times \mathbb{R}) \to \mathbb{R}$ with the form

$$G\left((x_i, z_i, u_i)_{i=1}^{\infty}\right) = \prod_{i=1}^{\infty} g_i(x_i, z_i, u_i), \tag{4.12}$$

with $g_i \in \mathfrak{g}^Z(K, r, m, n), 1 \leq i \leq n$, and $g_i = \mathbb{1}_{\mathbb{R}^d \times \mathbb{R} \times [0, \infty)}, n + 1 \leq i < \infty$, for $n \in \mathbb{N}$, $g_i = \mathbb{1}_{\mathbb{R}^d \times \mathbb{R} \times [0, \infty)}$ for $n = 0$ and $g_i \in \mathfrak{g}^Z(K, r, m, n), i \in \mathbb{N}$ for $n = \infty$.

Now we are ready to formulate the semi-martingale decompositions of the cases I.a and I.b. Since the proofs are quite long, we postpone them until the end of this chapter as we have mentioned it in the introduction of this chapter (we will do the same with the proofs for the Case II and Case III).

Proposition 4.2.8 (Case I.a)**.** *Assume that $G \in \mathfrak{G}^Z(K, r, m, n)$ with $K \in [0, \infty)$, $r > 0$, $m \in (0, 1)$ and finite $n \in \mathbb{N}$. If we define the process $A : \Omega \times [0, \infty) \to \mathbb{R}$ by setting for each $t \geq 0$:*

$$A(t) := \int_0^t \mathbf{A}^o_{B_X, h}(G)\big((X_i(s\text{-}), Z_i(s\text{-}), U_i(s\text{-}))_{i=1}^{\infty}, s\big)\, ds,$$

and the process $M : \Omega \times [0, \infty) \to \mathbb{R}$ by setting for each $t \geq 0$:

$$M(t) := G\left((X_i(t), Z_i(t), U_i(t))_{i=1}^{\infty}\right) - G\left((X_i(0), Z_i(0), U_i(0))_{i=1}^{\infty}\right) - A(t), \tag{4.13}$$

then A is a continuous $\mathcal{F}^{\boldsymbol{\xi}, \mathbb{W}}$-adapted process with finite variation and $E[|A(t \wedge \tilde{T}_{\widehat{m}})|] < \infty, t \in [0, \infty), \widehat{m} \in \mathbb{N}$, further the process M is a local $\mathcal{F}^{\boldsymbol{\xi}, \mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$ which admits a càdlàg modification.

Remark 4.2.9. *We recall that all evolution stops at the moment of extinction $\mathcal{T}_{EX}$, indeed*

$$\mathbb{P}[\widehat{\mathbf{W}}(\mathcal{T}_{EX}+t)=\widehat{\mathbf{W}}(\mathcal{T}_{EX}\text{-}),t\geq 0]=1,$$

where $\widehat{\mathbf{W}}=(X_i,Z_i,U_i)_{i=1}^{\infty}$, so one would guess that is necessary to stop the process A at $\mathcal{T}_{EX}$ to ensure that M is a martingale, but recall also that at the moment of extinction all levels $(U_i, i\in\mathbb{N})$ converge to infinity, which implies

$$\mathbb{P}[G(\widehat{\mathbf{W}}(t))\overset{t\to\mathcal{T}_{EX}}{\longrightarrow}0]=\mathbb{P}[\mathbf{A}^{o}_{B_X,h}(G)(\widehat{\mathbf{W}}(t\text{-}),t)\overset{t\to\mathcal{T}_{EX}}{\longrightarrow}0]=1.$$

and so we have the identity:

$$\int_0^t \mathbf{A}^{o}_{B_X,h}(G)(\widehat{\mathbf{W}}(s\text{-}),s)\,ds=\int_0^t \mathbf{A}^{o}_{B_X,h}(G)(\widehat{\mathbf{W}}(s\text{-}),s)\mathbb{1}_{[0,\mathcal{T}_{EX})}(s)\,ds.$$

So it is not necessary to stop the process A at time $\mathcal{T}_{EX}$.

Corollary 4.2.10 (Case I.b)**.** *The statements of Proposition 4.2.8 are also true for $G\in\mathfrak{G}^Z(K,r,m,n)$, when $n=\infty$.*

4.3 Case II

Recall that the Case II differs from the Case I in that we are considering in the Case II functionals of the process $\boldsymbol{\xi}^{XZ}$ and not of the process $\widehat{\mathbf{W}}=(X_i,Z_i,U_i)_{i=1}^{\infty}$. Please recall that the integrand h depends on ω, see also Remark 4.2.2.

Proposition 4.3.1 (Case II.a)**.** *Assume that $\tilde{g}\in\tilde{\mathfrak{g}}^Z$. When we define the stochastic process $A:\Omega\times[0,\infty)\to\mathbb{R}$ by setting for each $t\geq 0$:*

$$\begin{aligned}A(t):=&\int_0^t\int_{\overline{E}} B_X(\tilde{g})(x,z,u)\,\boldsymbol{\xi}^{XZ}_{s\text{-}}(dx,dz,du)\,ds\\&+\int_0^t\int_{\overline{E}}[au^2-bu]\partial_u(\tilde{g})(x,z,u)\,\boldsymbol{\xi}^{XZ}_{s\text{-}}(dx,dz,du)\,ds\\&+\int_0^t\int_{\overline{E}}2a\int_u^\infty\tilde{g}(x,z,v)\,dv\,\boldsymbol{\xi}^{XZ}_{s\text{-}}(dx,dz,du)\,ds\\&+\int_0^t\int_{\overline{E}}\int_0^\infty\tilde{g}(x,z+h(x,p,s),u)-\tilde{g}(x,z,u)\,dp\,\boldsymbol{\xi}^{XZ}_{s\text{-}}(dx,dz,du)\,ds,\end{aligned}\tag{4.14}$$

where $\overline{E}=\mathbb{R}^d\times\mathbb{R}\times[0,\infty)$, and we define the process $M:\Omega\times[0,\infty)\to\mathbb{R}$ by setting

$$M(t):=\boldsymbol{\xi}^{XZ}_t(\tilde{g})-\boldsymbol{\xi}^{XZ}_0(\tilde{g})-A(t),\ t\geq 0,\tag{4.15}$$

then A is a continuous $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-adapted process with finite variation and $E[|A(t\wedge\tilde{T}_{\widehat{m}})|]<\infty, t\in[0,\infty),\widehat{m}\in\mathbb{N}$, further the process M is a local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}},\widehat{m}\in\mathbb{N})$ which admits a càdlàg modification.

Proposition 4.3.2 (Case II.b)**.** *Assume that $g \in \mathfrak{g}^Z$ and $f := -\log(g)$. When we define the stochastic process $A : \Omega \times [0,\infty) \to \mathbb{R}$ by setting for each $t \geq 0$:*

$$
\begin{aligned}
A(t) := &\int_0^t \int_{\overline{E}} \exp(-\boldsymbol{\xi}_s^{XZ}(f)) \frac{B_X(g)(x,z,u)}{g(x,z,u)} \boldsymbol{\xi}_{s\text{-}}^{XZ}(dx,dz,du)ds \\
&+ \int_0^t \int_{\overline{E}} \exp(-\boldsymbol{\xi}_s^{XZ}(f))(au^2 - bu) \frac{\partial_u(g)(x,z,u)}{g(x,z,u)} \boldsymbol{\xi}_{s\text{-}}^{XZ}(dx,dz,du)ds \\
&+ \int_0^t \int_{\overline{E}} \exp(-\boldsymbol{\xi}_s^{XZ}(f)) \int_{\overline{E}} \int_u^\infty 2a[g(x,z,v) - 1]\, dv\, \boldsymbol{\xi}_{s\text{-}}^{XZ}(dx,dz,du)ds \\
&+ \int_0^t \int_{\overline{E}} \exp(-\boldsymbol{\xi}_s^{XZ}(f)) \int_0^\infty \frac{g(x,z+h(x,p,s),u) - g(x,z,u)}{g(x,z,u)}\, dp\, \boldsymbol{\xi}_{s\text{-}}^{XZ}(dx,dz,du)ds,
\end{aligned}
$$

where $\overline{E} = \mathbb{R}^d \times \mathbb{R} \times [0,\infty)$, and we define the process $M : \Omega \times [0,\infty) \to \mathbb{R}$ by setting

$$M(t) := \exp(-\boldsymbol{\xi}_t^{XZ}(f)) - \exp(-\boldsymbol{\xi}_0^{XZ}(f)) - A(t),\ t \geq 0,$$

then A is a continuous $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-adapted process with finite variation and $E[|A(t \wedge \tilde{T}_{\widehat{m}})|] < \infty, t \in [0,\infty), \widehat{m} \in \mathbb{N}$, further the process M is a local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}})_{\widehat{m}=1}^\infty$ which admits a càdlàg modification.

4.4 Case III

We are now presenting the semi-martingale decomposition of the functionals associated with the intensity process $\boldsymbol{\Xi}^{XZ}$. An important difference to the previous cases is that the processes obtained by the semi-martingale decompositions of Case III.a and Case III.b are adapted to the smaller filtration $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$, while the processes from the Cases I.a, I.b, II.a and II.b have been adapted to the bigger filtration $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$. By Definition 2.1.1 the space $C_b^2(\mathbb{R}^d \times \mathbb{R})$ consists of the functions $\hat{g} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ which are twice continuously differentiable and which have bounded derivatives. When we write $B_X(\hat{g})$, then we understand this expression as the application of the generator $B_X \subset C_b(\mathbb{R}^d) \times C_b(\mathbb{R}^d)$ to the function obtained from $\hat{g}$ by fixing the last coordinate. Further it is important to keep in mind that h as an element of $\mathcal{L}^1_{loc}(\mathbf{M})$ depends on Ω. But as usual we suppress the dependence of h on Ω and just write $h(x,p,s)$ instead of $h(\omega,x,p,s)$.

Proposition 4.4.1 (Case III.a)**.** *Assume that $\hat{g} \in C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$ and that the process $\hat{A} : \Omega \times [0,\infty) \to \mathbb{R}$ is given by*

$$
\begin{aligned}
\hat{A}(t) := &\int_0^t \int_{\hat{E}} B_X(\hat{g})(x,z) + b\hat{g}(x,z)\, \boldsymbol{\Xi}_{s\text{-}}^{XZ}(dx,dz)ds \\
&+ \int_0^t \int_{\hat{E}} \int_0^\infty \hat{g}(x,z+h(x,p,s)) - \hat{g}(x,z)\, dp\, \boldsymbol{\Xi}_{s\text{-}}^{XZ}(dx,dz)ds,\ t \geq 0,
\end{aligned}
$$

where $\hat{E} = \mathbb{R}^d \times \mathbb{R}$, and that the process $\hat{M} : \Omega \times [0,\infty) \to \mathbb{R}$ is given by

$$\hat{M}(t) := \boldsymbol{\Xi}_t^{XZ}(\hat{g}) - \boldsymbol{\Xi}_0^{XZ}(\hat{g}) - \hat{A}(t),\ t \geq 0, \tag{4.16}$$

then $\hat{A}$ is a continuous $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-adapted process with finite variation and $E[|\hat{A}(t \wedge \tilde{T}_{\widehat{m}})|] < \infty, t \in [0,\infty), \widehat{m} \in \mathbb{N}$, further the process $\hat{M}$ is a local $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$ which admits a càdlàg modification.

Proposition 4.4.2 (Case III.b). *Assume that $\hat{g} \in C_b^2(\mathbb{R}^d \times \mathbb{R})$ and that the process $\hat{A} : \Omega \times [0,\infty) \to \mathbb{R}$ is given by*

$$\begin{aligned}\hat{A}(t) := &\int_0^t \exp(-\Xi_s^{XZ}(\hat{g})) \int_{\hat{E}} \big[- B_X(\hat{g})(x,z) - b\hat{g}(x,z) + a\hat{g}^2(x,z)\big] \Xi_s^{XZ}(dx,dz)ds \\ &- \int_0^t \exp(-\Xi_s^{XZ}(\hat{g})) \int_{\hat{E}} \int_0^\infty \hat{g}(x, z + h(x,p,s)) - \hat{g}(x,z) dp \Xi_s^{XZ}(dx,dz)\, ds,\end{aligned} \tag{4.17}$$

where $\hat{E} = \mathbb{R}^d \times \mathbb{R}$, and that the process $\hat{M} : \Omega \times [0,\infty) \to \mathbb{R}$ is given by

$$\hat{M}(t) := \exp(-\Xi_t^{XZ}(\hat{g})) - \exp(-\Xi_0^{XZ}(\hat{g})) - \hat{A}(t),\ t \geq 0, \tag{4.18}$$

then $\hat{A}$ is a continuous $\mathcal{F}^{\Xi,\mathbb{W}}$-adapted process with finite variation and $E[|\hat{A}(t \wedge \tilde{T}_{\widehat{m}})|] < \infty, t \in [0,\infty), \widehat{m} \in \mathbb{N}$, further the process $\hat{M}$ is a local $\mathcal{F}^{\Xi,\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$ which admits a càdlàg modification.

4.5 Case IV

Recall the Definition 3.5.1, where we defined the process $\Xi^{XZ,r} : \Omega \times [0,\infty) \to \mathcal{M}_f(\mathbb{R}^d \times \mathbb{R})$ for $r \geq \max\{b/a, 0\}$ by setting

$$\Xi_t^{XZ,r} := \sum_{i=1}^\infty \delta_{(X_i(t),Z_i(t))} \mathbb{1}_{[0,r)}(U_i(t)).$$

Here we will present the semi-martingale decomposition of $\Xi^{XZ,r}(\hat{g})$, which will be used later in Section 6.1 to prove continuity of $\Xi^{XZ}(\hat{g})$. Like always we want to mention that h as an element of $\mathcal{L}^1_{loc}(\mathbf{M})$ depends on Ω and we suppress the dependency on Ω, so we just write $h(x,p,t)$ instead of $h(\omega,x,p,t)$.

Proposition 4.5.1. *Assume that $\hat{g} \in C_b^2(\mathbb{R}^d \times \mathbb{R})$ and that the process $\hat{A}^r : \Omega \times [0,\infty) \to \mathbb{R}$ is given by*

$$\begin{aligned}\hat{A}^r(t) = &\int_0^t \int_{\hat{E}} \big[B_X(\hat{g})(x,z) + b\hat{g}(x,z)\big]\, \Xi_{s\text{-}}^{XZ,r}(dx,dz)ds \\ &+ \int_0^t \int_{\hat{E}} \Big[\int_0^\infty \hat{g}(x, z + h(x,p,s)) - \hat{g}(x,z)\, dp\Big] \Xi_{s\text{-}}^{XZ,r}(dx,dz)ds,\ t \geq 0,\end{aligned}$$

where $\hat{E} = \mathbb{R}^d \times \mathbb{R}$, and that the process $M : \Omega \times [0,\infty) \to \mathbb{R}$ is given by

$$\hat{M}^r(t) := \Xi_t^{XZ,r}(\hat{g}) - \Xi_0^{XZ,r}(\hat{g}) - \hat{A}^r(t),$$

then $\hat{A}^r$ is a continuous $\mathcal{F}^{XZ,r}$-adapted process with finite variation and M is a càdlàg local $\mathcal{F}^{\Xi,\mathbb{W},r}$-martingale. If h has the localizing sequence $(T_n)_{n=1}^\infty$, then $\hat{M}$ is a local $\mathcal{F}^{\Xi,\mathbb{W},r}$-martingale with the localizing sequence $(T_n \wedge \tau_n^Y)_{n=1}^\infty$. If $h \in \mathcal{L}^1_{loc}(\mathbf{M}) \cap \mathcal{L}^1_{loc}(\mathbf{M}^\Xi)$, then M is a proper $\mathcal{F}^{\Xi,\mathbb{W},r}$-martingale.

Of course it is also possible to formulate an analogue of Case III.b for $\Xi^{XZ,r}$, but we do not need such a result in the following chapter. But we need an upper bound for $\langle \Xi^{XZ,r}(\hat{g})\rangle$, the compensator of the quadratic variation of $\Xi^{XZ,r}(\hat{g})$.

Proposition 4.5.2. *For $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ and $\hat{g} \in C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$, then $0 \leq t_1 < t_2 < \infty$:*

$$\begin{aligned}
\langle \boldsymbol{\Xi}^{XZ,r}(\hat{g}) \rangle_{t_2} - \langle \boldsymbol{\Xi}^{XZ,r}(\hat{g}) \rangle_{t_1} & \\
\leq \int_{t_1}^{t_2} \int_{\overline{E}} & \left[\nabla_x(\hat{g})(x,z)^T B_X^{cov} \nabla_x(\hat{g})(x,z) \right] \boldsymbol{\Xi}_{s\text{-}}^{XZ,r}(dx,dz)ds \\
+ \int_{t_1}^{t_2} \int_{\overline{E}} & \left[\int_{\mathbb{R}^d} \big(\hat{g}(x+y,z) - \hat{g}(x,z)\big)^2 B^{\eta}(dy) \right] \boldsymbol{\Xi}_{s\text{-}}^{XZ,r}(dx,dz)ds \\
+ \int_{t_1}^{t_2} \int_{\overline{E}} & \left[\int_0^{\infty} \big(\hat{g}(x,z+h(x,p,s)) - \hat{g}(x,z)\big)^2 dp \right] \boldsymbol{\Xi}_{s\text{-}}^{XZ,r}(dx,dz)ds \\
+ ||g||_{\infty}^2 \int_{t_1}^{t_2} & 2arY^r(s\text{-})ds,
\end{aligned} \tag{4.19}$$

where $\overline{E} = \mathbb{R}^d \times \mathbb{R}$ and $\nabla_x(\hat{g})(x,z) = (\partial_{x_i}\hat{g}(x,z))_{i=1}^d$ is the gradient of $\hat{g}$ with respect to the x-coordinate.

Remark 4.5.3. *Instead of an upper bound for $\langle \boldsymbol{\Xi}^{XZ,r}(\hat{g}) \rangle$ we can get the correct expression for $\langle \boldsymbol{\Xi}^{XZ,r}(\hat{g}) \rangle$, when we replace $||g||_{\infty}^2 \int_0^t 2arY^r(s\text{-})\,ds$ in the last line of (4.19) with*

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d \times \mathbb{R}} 2ar\hat{g}^2(x,z)\, \boldsymbol{\Xi}_s^{XZ,r}(dx,dz)ds.$$

But a rigorous derivation of this expression is far more difficult, and the upper bound given by (4.19) is sufficient for our purpose.

4.6 Proof of Case I

As we have seen the expression for the processes A and $\hat{A}$ with finite variation are quite long for all cases, therefore we introduce new notations to make them shorter.

Definition 4.6.1. *For $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ we define:*

1. *We define the processes $(\widehat{\mathbf{W}}_i)_{i=1}^{\infty}$ with $\widehat{\mathbf{W}}_i : \Omega \times [0,\infty) \to \mathbb{R}^d \times \mathbb{R} \times [0,\infty]$ for $i \in \mathbb{N}$ by*

$$\widehat{\mathbf{W}}_i(t) = (X_i(t), Z_i(t), U_i(t)),\ t \geq 0.$$

   Further, we define $\widehat{\mathbf{W}} : \Omega \times [0,\infty) \to \mathbf{S}(\mathbb{R}^d \times \mathbb{R})$ by setting $\widehat{\mathbf{W}} = (\widehat{\mathbf{W}}_i)_{i=1}^{\infty}$. We also write $\hat{\mathbf{w}} = (\hat{\mathbf{w}}_i)_{i=1}^{\infty}$ for the elements of $\mathbf{S}(\mathbb{R}^d \times \mathbb{R})$ with $\hat{\mathbf{w}}_i \in \mathbb{R}^d \times \mathbb{R} \times [0,\infty]$.

2. *For $j, i \in \mathbb{N}$ with $i < j, \hat{\mathbf{w}} \in \mathbf{S}(\mathbb{R}^d \times \mathbb{R}), p, v \in [0,\infty)$, we define $G_i(\hat{\mathbf{w}}) := \prod_{j \neq i} g_j(\hat{\mathbf{w}}_j)$ and*

$$\begin{aligned}
G_i^{B_X}(\hat{\mathbf{w}}) &:= G_i(\hat{\mathbf{w}}) B_X(g_i)(\hat{\mathbf{w}}_i), \\
G_i^{\partial_u}(\hat{\mathbf{w}}) &:= G_i(\hat{\mathbf{w}}) \left(au_i^2 - bu_i\right) \partial_u g_i(\hat{\mathbf{w}}_i), \\
G_{j\downarrow i}(\hat{\mathbf{w}}, v) &:= \left[\prod_{k=1}^{j-1} g_k(x_k, z_k, u_k) \right] g_j(x_i, z_i, v) \left[\prod_{k=j}^{\infty} g_{k+1}(x_k, z_k, u_k) \right], \\
G_i^{\Delta h}(\omega, \hat{\mathbf{w}}, p, t) &:= G_i(\hat{\mathbf{w}}) \left[g_i(x_i, z_i + h(\omega, x_i, p, t), u_i) - g_i(x_i, z_i, u_i) \right].
\end{aligned}$$

Remark 4.6.2. *Note that $G_i^{B_X}$ and $G_i^{\partial_u}$ corresponds to the evolution of the spatial motion and the level of the i-th particle. The expression $G_{j\downarrow i}$ corresponds to the event that the particle i gives birth to a new particle and this new particle obtains the index j (the j-th particle looks down on the particle i). Finally the expression $G_i^{\Delta h}$ corresponds to the jumps due to the z-coordinate. As usual the dependence of $G_i^{\Delta h}$ on Ω will be often suppressed, indeed we just write $G_i^{\Delta h}(\hat{\mathbf{w}}, p, t)$ instead of $G_i^{\Delta h}(\omega, \hat{\mathbf{w}}, p, t)$. This will be done to shorten the expressions, indeed we will often encounter lines with multiple expressions like $G_i^{\Delta h}(\widehat{\mathbf{W}}(t \wedge \tau), p, t)$, where τ is a stopping time, if we no not suppress the dependency on Ω, we get $G_i^{\Delta h}(\omega, \widehat{\mathbf{W}}(\omega, t \wedge \tau(\omega)), p, t)$. But sometimes ω will help the understanding, then we will not omit ω.*

Remark 4.6.3. *Using the new notation we can express $\mathbf{A}^o_{B_X,h}(G)(\hat{\mathbf{w}}, t)$, see Definition 4.2.1, as*

$$\sum_{i=1}^{\infty}\left[G_i^{B_X}(\hat{\mathbf{w}}) + G_i^{\partial_u}(\hat{\mathbf{w}}) + \sum_{j=i+1}^{\infty}\int_{u_{j\text{-}1}}^{u_j} G_{j\downarrow i}(\hat{\mathbf{w}}, v) - G(\hat{\mathbf{w}})dv + \int_0^{\infty} G_i^{\Delta h}(\hat{\mathbf{w}}, p, t)dp\right].$$

The proofs of the different semi-martingale decompositions can be roughly divided into two parts. The first part consists in showing that the processes involved have finite first moments and the second part in showing that the processes denoted by M and $\hat{M}$ are local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$- or $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-martingales. For the first part we will often make use of the following two lemmas.

Lemma 4.6.4. *Fix a quadruple $(K, R, m, n) \in [0, \infty) \times [0, \infty) \times (0, 1) \times (\mathbb{N}_0 \cup \{\infty\})$ and assume $G \in \mathfrak{G}^Z(K, r, m, n)$, then for all $\hat{\mathbf{w}} \in \mathbf{S}(\mathbb{R}^d \times \mathbb{R}), i, j \in \mathbb{N}$ with $j > i$ it holds:*

$$\left|G_i^{B_X}(\hat{\mathbf{w}})\right| \le K\mathbb{1}_{[0,r)}(u_i), \tag{4.20}$$

$$\left|G_i^{\partial_u}(\hat{\mathbf{w}})\right| \le \left(ar^2 + |b|r\right)K\mathbb{1}_{[0,r)}(u_i), \tag{4.21}$$

$$\int_{u_{j\text{-}1}}^{u_j}\left|G_{j\downarrow i}(\hat{\mathbf{w}}, v) - G(\hat{\mathbf{w}})\right|dv \le 2(r \wedge u_j - u_{j\text{-}1})\mathbb{1}_{[0,r)}(u_{j\text{-}1}), \tag{4.22}$$

$$\sum_{j=i+1}^{\infty}\int_{u_{j\text{-}1}}^{u_j}\left|G_{j\downarrow i}(\hat{\mathbf{w}}, v) - G(\hat{\mathbf{w}})\right|dv \le 2r\mathbb{1}_{[0,r)}(u_i). \tag{4.23}$$

Further, for all $\omega \in \Omega, \hat{\mathbf{w}} \in \mathbf{S}(\mathbb{R}^d \times \mathbb{R}), p \in [0, \infty), t \in [0, \infty)$ and $i \in \mathbb{N}$ holds

$$\int_0^{\infty} G_i^{\Delta h}(\omega, \hat{\mathbf{w}}(t), p, t)\,dp \le K\int_0^{\infty} h(\omega, x_i, p, t)\,dp\mathbb{1}_{[0,r)}(u_i), \tag{4.24}$$

here we did not suppressed the dependency on Ω, this inequality holds for all ω and not only almost surely.

Proof. Recall by the definition of $\mathfrak{G}^Z(K, r, m, n)$ that $G = \prod_{i=1}^{\infty} g_i$ with $g_i \in \mathfrak{g}^Z(K, r, m)$ for all $i \in \mathbb{N}$. So for all $i \in \mathbb{N}$ we have that $0 \le g_i \le 1$. As consequence it is also true for all $i \in \mathbb{N}$ that $0 \le G_i \le 1$. We start with the first two Inequalities (4.20) and (4.21) and note that it holds by Point 2 of the definition of $\mathfrak{g}^Z(K, r, m)$ that $g = g\mathbb{1}_{[0,r]}$. Together with Point 1 of the definition of $\mathfrak{g}^Z(K, r, m)$ it follows (4.20) and (4.21).
Considering the inequality (4.22), we need to prove that

$$\int_{u_{j\text{-}1}}^{u_j}\left|G_{j\downarrow i}(\hat{\mathbf{w}}, v) - G(\hat{\mathbf{w}})\right|dv = \int_{u_{j\text{-}1}}^{u_j \wedge r}\left|G_{j\downarrow i}(\hat{\mathbf{w}}, v) - G(\hat{\mathbf{w}})\right|dv. \tag{4.25}$$

From this, the right side of (4.22) follows immediately, because $|G_{j\downarrow i}|, |G| \le 1$ and obviously the right side of (4.25) is zero, when $u_{j\text{-}1} \ge r$ (so we can multiply (4.25) with the function $\mathbb{1}_{[0,r)}(u_{j\text{-}1})$ without changing its value). When $u_j \le r$, there is nothing to show. If we assume that $u_j > r$, then $u_k > r$ for $k \ge j$ and by Point 2 of 4.2.4 it follows $g_k(x_k, z_k, u_k) = 1$ and $g_{k+1}(x_k, z_k, u_k) = 1$. Therefore

$$\begin{aligned}\left|G_{j\downarrow i}(\hat{\mathbf{w}}, v) - G(\hat{\mathbf{w}})\right| &= \prod_{k=1}^{j-1} g_k(\hat{\mathbf{w}}_k) \left| g_j(x_i, z_i, v) \prod_{k=j}^{\infty} g_{k+1}(\hat{\mathbf{w}}_k) - g_j(\hat{\mathbf{w}}_j) \prod_{k=j}^{\infty} g_{k+1}(\hat{\mathbf{w}}_{k+1}) \right| \\ &= \prod_{k=1}^{j-1} g_k(\hat{\mathbf{w}}_k) \left| g_j(x_i, z_i, v) - 1 \right| \le \left| g_j(x_i, z_i, v) - 1 \right|.\end{aligned}$$

Since $|g_j(x_i, z_i, v) - 1| = 0$ for $v \ge r$ due to Point 2 of Definition 4.2.4, it holds $G_{j\downarrow i}(\hat{\mathbf{w}}, v) - G(\hat{\mathbf{w}}) = 0$ for $v > r$ and $u_j > r$. So we have also proved (4.25) for $u_j > r$, so the inequality (4.22) is true. The next inequality (4.23) follows from (4.22) and that $u_j \ge u_i, j \ge i$, by

$$\sum_{j=i+1}^{\infty} \int_{u_{j\text{-}1}}^{u_j} \left|G_{j\downarrow i}(\hat{\mathbf{w}}, v) - G(\hat{\mathbf{w}})\right| dv \le \sum_{j=i+1}^{\infty} 2(r \wedge u_j - u_{j\text{-}1}) \mathbb{1}_{[0,r)}(u_{j\text{-}1}) = 2(r - u_i) \mathbb{1}_{[0,r)}(u_i)$$

It follows (4.23). For (4.24), we recall that a function $g \in \mathfrak{g}^Z(K, r, m)$ is Lipschitz continuous in the z-coordinate with Lipschitz constant K due to Definition 4.2.4. So if $g_i \in \mathfrak{g}^Z(K, r, m)$, then with $|G_i(\hat{\mathbf{w}})| \le 1$ we get

$$\left|G_i^{\Delta h}(\hat{\mathbf{w}}, p, t)\right| = |G_i(\hat{\mathbf{w}})| \, |g_i(x_i, z_i + h(x_i, p, t), u_i) - g_i(x_i, z_i, u_i)| \le K |h(x_i, p, t)|. \quad (4.26)$$

Of course the above inequality is also true, when $g_i = \mathbb{1}_{\mathbb{R}^d \times \mathbb{R} \times [0,\infty)}$. Further it also holds

$$|g_i(x_i, z_i + h(x_i, p, t), u_i) - g_i(x_i, z_i, u_i)| = 0, \text{ if } u \ge r \quad (4.27)$$

either due to Point 2 of Definition 4.2.4 in the case $g_i \in \mathfrak{g}^Z(K, r, m)$ or because $g_i = \mathbb{1}_{\mathbb{R}^d \times \mathbb{R} \times [0,\infty)}$. We can combine (4.26) and (4.27) to get (4.24). □

Lemma 4.6.5. *If we define for $r \in [0, \infty)$ the processes*

$$\begin{aligned}P_1^r(t) &:= \int_0^t Y_s^r \, ds = \int_0^t Y_{s\text{-}}^r \, ds, \\ P_2^r(t) &:= \int_0^t (Y_s^r)^2 \, ds = \int_0^t (Y_{s\text{-}}^r)^2 \, ds, \\ P_3^r(t) &:= \int_0^t \int_0^{\infty} \sum_{i=1}^{Y_{s\text{-}}^r} |h(X_i(s\text{-}), p, s)| \, dpds = \int_0^t \int_0^{\infty} \sum_{i=1}^{Y_s^r} |h(X_i(s\text{-}), p, s)| \, dpds,\end{aligned}$$

then it holds $\mathbb{E}[P_1^r(t)] < \infty, \mathbb{E}[P_2^r(t)] < \infty$ and $\mathbb{E}[P_3^r(t \wedge \tilde{T}_{\widehat{m}})] < \infty$ for all $t \ge 0$ and $\widehat{m} \in \mathbb{N}$.

Proof. The different expressions for $P_1^r(t), P_2^r(t)$ and $P_3^r(t)$ follow from the fact that $Y_{t\text{-}}^r$ differs from Y_t^r, if and only if t is a jump time of Y^r, and since Y^r admits as a càdlàg process only countably many jumps, hence the identity follows from the fact that every countable sets has the Lebesgue measure zero.
We recall that the conditional distribution of Y^r conditioned on $\mathcal{F}_t^{\mathbf{\Xi}, \mathbb{W}}$ is a Poisson distribution with intensity rY_t, hence

$$\mathbb{E}\left[Y_t^r\right] = r\mathbb{E}[Y_t], \qquad \mathbb{E}\left[(Y_t^r)^2\right] = r^2 \mathbb{E}[Y_t^2] + r\mathbb{E}[Y_t]. \quad (4.28)$$

Since Y is a Feller diffusion with respect to the filtration $\mathcal{F}^{\Xi,\mathbb{W}}$ with drift b and branching rate a, the Itô-formula gives us for first two moments the following ODEs:

$$\frac{d}{dt}\mathbb{E}\left[Y_t\right] = b\mathbb{E}\left[Y_t\right], \quad \frac{d}{dt}\mathbb{E}\left[Y_t^2\right] = 2b\mathbb{E}\left[Y_t^2\right] + 2a\mathbb{E}\left[Y_t\right]. \tag{4.29}$$

which have the solutions:

$$\begin{aligned}\mathbb{E}\left[Y_t\right] &= \mathbb{E}[Y_0]e^{bt},\\ \mathbb{E}\left[(Y_t)^2\right] &= e^{2bt}\mathbb{E}[Y_0^2] + \frac{2a}{br^2}\mathbb{E}\left[Y_0\right]\left(e^{2bt} - e^{-bt}\right).\end{aligned}$$

We have together with (4.28)

$$\begin{aligned}\mathbb{E}\left[Y_t\right] &= r\mathbb{E}[Y_0]e^{bt},\\ \mathbb{E}\left[(Y_t)^2\right] &= r^2e^{2bt}\mathbb{E}[Y_0^2] + \frac{2ar^2}{b}\mathbb{E}\left[Y_0\right]\left(e^{2bt} - e^{-bt}\right) + r\mathbb{E}[Y_0]e^{bt}.\end{aligned} \tag{4.30}$$

This gives $\mathbb{E}[P_1^r(t)] < \infty$ and $\mathbb{E}[P_2^r(t)] < \infty$. For the next part we recall that it holds for all finite $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping times τ:

$$\mathfrak{L}\left(\boldsymbol{\xi}_\tau^X \middle| \mathcal{F}_\tau^{\Xi,\mathbb{W}}\right) = \mathbf{PPP}\left(\Xi_\tau^X \otimes \ell eb[0,\infty)\right). \tag{4.31}$$

Applying (4.31) to $\mathbb{E}[P_3^r(t \wedge \tilde{T}_{\widehat{m}})]$, using that $\mathbb{1}_{[0,\tilde{T}_{\widehat{m}})}$ measurable with respect to $\mathcal{F}_t^{\Xi,\mathbb{W}}$ gives us together with Tonelli's theorem which allows us to switch the order of integration:

$$\begin{aligned}\mathbb{E}&\left[\int_0^t \sum_{i=1}^{Y_s^r} \int_0^\infty |h(X_i(s\text{-}),p,s)|\,dpds\right] = \int_0^t\int_0^\infty \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_{\widehat{m}})}(s)\sum_{i=1}^\infty |h(X_i(s\text{-}),p,s)|\,\mathbb{1}_{[0,r)}(U_i(s))\right]dpds\\ &= \int_0^t\int_0^\infty \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_{\widehat{m}})}(s)\int_{\mathbb{R}^d\times[0,\infty)} |h(x,p,s)|\mathbb{1}_{[0,r)}(u)\boldsymbol{\xi}_s^X(dx,du)\right]dpds\\ &= \int_0^t\int_0^\infty \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_{\widehat{m}})}(s)\int_0^\infty\int_{\mathbb{R}^d} |h(x,p,s)|\mathbb{1}_{[0,r)}(u)\Xi_s^X(dx)du\right]dpds\\ &= r\int_0^t\int_0^\infty \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_{\widehat{m}})}(s)\int_{\mathbb{R}^d} |h(x,p,s)|\Xi_s^X(dx)\right]dpds,\end{aligned}$$

hereby the last expression is finite, because $\mathbb{1}_{[0,\tilde{T}_{\widehat{m}})}h$ is an element of $\mathcal{L}^1_{loc}(\mathbf{M}^\Xi)$. In conclusion $\mathbb{E}[P_3^r(t \wedge \tilde{T}_{\widehat{m}})] < \infty$. □

As a consequence of the two previous lemmas we get the following one which ensures that the processes of the Cases I.a, I.b, II.b and III.b. admit finite moments.

Lemma 4.6.6. *Let us fix a* $G \in \mathfrak{G}^Z(K,r,m,n)$ *for an arbitrary* $(K,r,m,n) \in [0,\infty) \times [0,\infty) \times (0,1) \times (\mathbb{N} \cup \{\infty\})$ *(note that we allow* $n = \infty$*). If we define the processes* $V_{B_X}^G, V_{\mathbf{b}}^G, V_h^G : \Omega \times [0,\infty) \to \mathbb{R}$ *by setting for each* $t \geq 0$*:*

$$\begin{aligned}V_{B_X}^G(t) &:= \sum_{i=1}^\infty G_i^{B_X}(\widehat{\mathbf{W}}(t\text{-})),\\ V_{\mathbf{b}}^G(t) &:= \sum_{i=1}^\infty G_i^{\partial_u}(\hat{\mathbf{w}})(\widehat{\mathbf{W}}(t\text{-})) - \sum_{i=1}^\infty\sum_{j=i+1}^\infty \int_{U_{j\text{-}1}(t\text{-})}^{U_j(t\text{-})} G_{j\downarrow i}\left(\widehat{\mathbf{W}}(t\text{-}),v\right) - G(\widehat{\mathbf{W}}(t\text{-}))\,dv,\\ V_h^G(t) &:= \sum_{i=1}^\infty \int_0^\infty G_i^{\Delta h}(\widehat{\mathbf{W}}(t\text{-}),p,t)\,dp,\end{aligned}$$

then it holds $\int_0^t |V_{B_X}^G(s)|ds \le KP_1^r(t)$, $\int_0^t |V_{\mathbf{b}}^G(s)|ds \le (ar^2{+}|b|r)KP_1^r(t){+}2rP_1^r(t)$ *and* $\int_0^t |V_h^G(s)|ds \le KP_3^r(t)$ *(it is important to note that these inequalities do not depend on n). This implies together with Lemma 4.6.5 that*

$$\int_0^t \mathbb{E}[|V_{B_X}^G(s)|]ds < \infty, \int_0^t \mathbb{E}[|V_{\mathbf{b}}^G(s)|]ds < \infty \text{ and } \int_0^t \mathbb{E}[\mathbb{1}_{[0,\tilde{T}_{\widehat{m}})}(s)|V_h^G(s)|]ds < \infty.$$

Remark 4.6.7. *Note that* $\mathbf{A}^o_{B_X,h}(G)(\widehat{\mathbf{W}}(t\text{-}),t) = V_{B_X}^G(t) + V_{\mathbf{b}}^G(t) + V_h^G(t)$.

Proof. The inequality for $|V_{B_X}^G(t)|$ follows from (4.20), the inequality for $|V_{\mathbf{b}}^G(t)|$ follows from (4.21) and (4.23) and the inequality for $|V_{B_X}^G(t)|$ from Inequality (4.24). □

We begin now to prove the correctness of the semi-martingale decomposition in the Case I.a. As mentioned before the proof splits basically into two parts, showing that the processes involved have finite first moments and that the process M defined in Proposition 4.2.8 is a local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$. Now we are proving the semi-martingale decomposition for the Case I.a. Our main tool is the Itô formula. If S is a m-dimensional semi-martingale and $f : \mathbb{R}^m \to \mathbb{R}$ is a twice continuous differentiable function, then the Itô formula, see Theorem 23.7 in [21], says

$$\begin{aligned} f(S(t)) = f(S(0)) + \sum_{i=1}^m \int_0^t \partial_i f(S(s\text{-}))\, dS(s) + \frac{1}{2}\sum_{i,j=1}^m \int \partial_{ij} f(S(s\text{-}))\, d[S_i,S_j]^c(s) \\ + \sum_{s<t}\Big(\Delta f(S(s)) - \sum_{i=1}^n \partial_i f(S(s\text{-}))\Delta S_i(s)\Big), \quad t \ge 0, \end{aligned}$$

where the sum in the second line is over of all jumps prior to time t (Note that a càdlàg paths has only countably many jumps) and where $[S_i,S_j]^c$ stands for the continuous part of the covariation, i.e. $[S_i,S_j]^c(t) = [S_i,S_j](t) - \sum_{s\le t}\Delta S_i(s)\Delta S_j(s)$.
We are going to combine the Itô formula with infinite systems of equations for $(U_i)_{i=1}^\infty$ and $(Z_i)_{i=1}^\infty$ presented in Definition 2.2.1 and Lemma 3.4.6. We will also formulate such a system for $(X_i)_{i=1}^\infty$. Let us recall the components $(\tilde{X}_i)_{i=1}^\infty$, see Assumptions 2.1.2, then we can express $X_j, j \in \mathbb{N}$, according to Remark 2.4.10 as the solution of:

$$\begin{aligned} X_j(t) = X_j^0(0) + &\int_0^t \mathbb{1}_{[0,\infty)}(U_j(s\text{-}))d\tilde{X}_j(s) \\ &+ \sum_{i=1}^{j-1}\int_0^t\int_{U_{j\text{-}1}(s\text{-})}^{U_j(s\text{-})} X_i(s\text{-}) - X_j(s\text{-})\mathcal{V}_{ji}(dv,ds) \\ &+ \sum_{i=2}^{j-1}\sum_{k=1}^{i-1}\int_0^t\int_{U_{i\text{-}1}(s\text{-})}^{U_i(s\text{-})} X_{j\text{-}1}(s\text{-}) - X_j(s\text{-})\mathcal{V}_{ik}(dv,ds). \end{aligned} \tag{4.32}$$

The derivation of this equation works in the same way as the derivation of the same type of equation system, see (2.30), for $(W_i)_{i=1}^\infty$, please see the proof of 2.4.8. The processes $(\tilde{X}_i)_{i=1}^\infty$ are independent copies of the Lévy process X with characteristic triple $(B_X^\rho, B_X^{cov}, B_X^\eta)$, see Assumption 1.2.3, and whose generator B_X has the form (2.9). The Lévy-Itô decomposition tells us that $\tilde{X}_i$ can be decomposed into a drift given by B_X^ρ, into a Brownian motion $\tilde{X}_i^c$ with covariation matrix E.1.5 and a Poisson point process $\tilde{X}_i^J$ with intensity measure $B_X^\eta \otimes \ell eb[0,\infty)$. With this new components the right-hand side of the first line, (4.32), of the infinite system becomes

$$\begin{aligned}\tilde{X}_j(0) + \int_0^t \mathbb{1}_{[0,\infty)}(U_j(s\text{-}))B_X^\rho\, ds + \int_0^t \mathbb{1}_{[0,\infty)}(U_j(s\text{-}))\, d\tilde{X}_j^c(s) \\ + \int_0^t \int_{\mathbb{R}^d} X_j(s\text{-}) + y - \mathbb{1}_{||y||\le 1}\, \tilde{X}_j^J(dy, ds).\end{aligned} \tag{4.33}$$

But before we also introduce another abbreviation: $g_i^{\Delta,X} : \mathbf{S}(\mathbb{R}^d \times \mathbb{R}) \times \mathbb{R}^d \to \mathbb{R}$, where $g_i^{\Delta,X}((x_i, z_i, u_i)_{i=1}^\infty, y)$ is given by

$$\big[g_i(x_i + y, z_i, u_i) - g_i(x_i, z_i, u_i) - y^T \nabla_X g(x_i, z_i, u_i)\big] G_i((x_i, z_i, u_i)_{i=1}^\infty) \tag{4.34}$$

for all $(x_i, z_i)_{i=1}^\infty \in \mathbf{S}(\mathbb{R}^d \times \mathbb{R}), y \in \mathbb{R}^d$.

Proof of Proposition 4.2.8. The first moment $\mathbb{E}[|A(t\wedge\tilde{T}_{\widehat{m}})|]$ is finite by Lemma 4.6.6 and Lemma 4.6.5. It remains to show that M is a local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$. Since $G \in \mathfrak{G}^Z(K, r, m, n)$ with $n < \infty$, the function G depends only on the first n coordinates, indeed G can be written as

$$G(\hat{\mathbf{w}}) = \prod_{i=1}^n g_i(\hat{\mathbf{w}}_i),$$

because $g_i = \mathbb{1}_{\mathbb{R}^d\times\mathbb{R}\times[0,\infty)}$ for $i > n$. We write ∇_x for the Nabla operator applied to the first d coordinates. Applying the Itô formula we can write the difference $G(\widehat{\mathbf{W}}(t)) - G(\widehat{\mathbf{W}}(0))$ for each $t \ge 0$ as:

$$\sum_{i=1}^n \int_0^t G_i(\widehat{\mathbf{W}}(s\text{-}))\nabla_x g_i(\widehat{\mathbf{W}}_i(s\text{-}))^T d\tilde{X}_i^c(s) + \frac{1}{2}\int_0^t \nabla_x^T B_X^{cov} \nabla_x g_i(\widehat{\mathbf{W}}_i(s\text{-}))ds \tag{4.35}$$

$$+ \sum_{i=1}^n \int_0^t \int_{\mathbb{R}^d} g_i^{\Delta,X}(\widehat{\mathbf{W}}(s\text{-}), y) G_i(\widehat{\mathbf{W}}(s\text{-}))\tilde{X}_i^J(dy, ds) \tag{4.36}$$

$$+ \sum_{i=1}^n \int_0^t G_i^{\partial_u}(\widehat{\mathbf{W}}(s\text{-}))\, ds \tag{4.37}$$

$$+ \sum_{i=1}^n \int_0^t \int_0^\infty G_i^{\Delta h}(\widehat{\mathbf{W}}(s\text{-}), p, s)\, \tilde{N}_i(dp, ds) \tag{4.38}$$

$$+ \sum_{i=1}^{n-1} \sum_{k=i+1}^n \int_0^t \int_{U_{k\text{-}1}(s\text{-})}^{U_k(s\text{-})} G_{k\downarrow i}(\widehat{\mathbf{W}}(s\text{-}), v) - G(\widehat{\mathbf{W}}(s\text{-}))\, d\mathcal{V}_{ki}(v, s). \tag{4.39}$$

To derive the desired semi-martingale decomposition of $G(\widehat{\mathbf{W}})$ we have to carefully rewrite (4.35)-(4.39). Let us write $P^{1,2,3}$ for the process consisting of the terms contained in (4.35)-(4.36) and $\overline{X}_i^J$ for the compensated version of $\tilde{X}_i^J$. The process $P^{1,2,3}$ transforms by adding and subtracting

0 into:

$$\begin{aligned}
P^{1,2,3}(t) &\pm \sum_{i=1}^{n}\int_0^t\int_{\mathbb{R}^d} g_i^{\Delta,X}(\widehat{\mathbf{W}}(s\text{-}),y)G_i(\widehat{\mathbf{W}}(s\text{-}))\,B_X^{\eta}(dy)ds \\
= &\quad \sum_{i=1}^{n}\int_0^t G_i(\widehat{\mathbf{W}}(s\text{-}))\nabla_x g_i(\widehat{\mathbf{W}}_i(s\text{-}))^T d\tilde{X}_i^c(s) + \int_0^t G_i^{B_X}(\widehat{\mathbf{W}}(s\text{-}))\,ds \\
&+ \sum_{i=1}^{n}\int_0^t\int_{\mathbb{R}^d} g_i^{\Delta,X}(\widehat{\mathbf{W}}(s\text{-}),y)G_i(\widehat{\mathbf{W}}(s\text{-}))\,\overline{X}_i^J(dy,ds) \\
= &\quad M^X(t) + \int_0^t G_i^{B_X}(\widehat{\mathbf{W}}(s\text{-}))\,ds,
\end{aligned}$$

where M^X consists of the two missing expressions and is a $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale. With $\overline{N}_i$ and $\overline{\mathcal{V}}_{ji}$ being the compensated versions of $\tilde{N}_i$ and $\overline{\mathcal{V}}_{ji}$ we define the processes:

$$\begin{aligned}
M^Z(t) &:= \sum_{i=1}^{n}\int_0^t\int_0^{\infty} G_i^{\Delta h}(\widehat{\mathbf{W}}(s\text{-}),p,s)\,\overline{N}(dp,ds) \\
M^{\mathcal{B}}(t) &:= \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\int_0^t\int_{U_{k\text{-}1}(s\text{-})}^{U_k(s\text{-})} G_{k\downarrow i}(\widehat{\mathbf{W}}(s\text{-}),v) - G(\widehat{\mathbf{W}}(s\text{-}))\,\overline{\mathcal{V}}_{ji}(dv,ds).
\end{aligned}$$

The processes M^X and $M^{\mathcal{B}}$ are $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingales and M^Z is a local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}},\widehat{m}\in\mathbb{N})$. We can rewrite (4.38) with M^Z into:

$$M^Z(t) + \sum_{i=1}^{n}\int_0^t\int_0^{\infty} G_i^{\Delta h}(\widehat{\mathbf{W}}(s\text{-}),p,s)\,dpds$$

and (4.39) with $M^{\mathcal{B}}$ into

$$M^{\mathcal{B}}(t) + 2a\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\int_0^t\int_{U_{k\text{-}1}(s\text{-})}^{U_k(s\text{-})} G_{k\downarrow i}(\widehat{\mathbf{W}}(s\text{-}),v) - G(\widehat{\mathbf{W}}(s\text{-}))\,dvds.$$

Let us recall the processes M and A from Proposition 4.2.8 and that

$$M(t) := G(\widehat{\mathbf{W}}(t)) - G(\widehat{\mathbf{W}}(0)) - A(t), \quad t \geq 0.$$

Our goal is to show that M is a local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingales with localizing sequence $(\tilde{T}_{\widehat{m}},\widehat{m}\in\mathbb{N})$ to prove our claim. Comparing the expression of A in Proposition 4.2.8 with our transformation of (4.35)-(4.36), we can see that

$$G(\widehat{\mathbf{W}}(t)) - G(\widehat{\mathbf{W}}(0)) - A(t) = M^X(t) + M^Z(t) + M^{\mathcal{B}}(t),$$

hence $M^X + M^Z + M^{\mathcal{B}} = M$. Therefore M is the sum of local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingales with localizing sequence $(\tilde{T}_{\widehat{m}},\widehat{m}\in\mathbb{N})$ and hence also a local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}},\widehat{m}\in\mathbb{N})$. The existence of a càdlàg modification follows from Doob's regularization theorem, see Theorem 6.27.(ii) in [21]. □

We will now derive the Case I.b from Case I.a.

Lemma 4.6.8. *Assume that $(\tilde{\Omega},\tilde{\mathcal{A}},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ is a stochastic base and that $(\tilde{M}_n)_{n=1}^{\infty}$ is a sequence of $\tilde{\mathcal{F}}$-martingales. Further assume that $\tilde{M}$ is a stochastic process with $\tilde{M}_n(t) \overset{n\to\infty}{\longrightarrow} \tilde{M}$ in $L^1(\tilde{\mathbb{P}})$ for all $t \geq 0$, then $\tilde{M}$ is also a $\tilde{\mathcal{F}}$-martingale.*

Proof. Since $\tilde{M}(t)$ is the $\mathrm{L}^1(\tilde{\mathbb{P}})$ limit of $(\tilde{M}_n(t))_{n=1}^{\infty}$, we can conclude that $\tilde{M}(t) \in \mathrm{L}^1(\tilde{\mathbb{P}})$ and that $\tilde{M}$ is adapted to $\tilde{\mathcal{F}}$. It remains to show that $\mathbb{E}[\tilde{M}(t)|\tilde{\mathcal{F}}_s] = \tilde{M}(s)$ for fixed $s < t$. For this we apply the triangle inequality, the martingale property and the Jensen inequality to obtain

$$\begin{aligned}\mathbb{E}\big[|\mathbb{E}[\tilde{M}(t)|\tilde{\mathcal{F}}_s] - \tilde{M}(s)|\big] &\leq \mathbb{E}\Big[\big|\mathbb{E}[\tilde{M}(t)|\tilde{\mathcal{F}}_s] - \mathbb{E}[\tilde{M}_n(t)|\tilde{\mathcal{F}}_s]\big|\Big] + \mathbb{E}\Big[\big|\mathbb{E}[\tilde{M}_n(t)|\tilde{\mathcal{F}}_s] - \tilde{M}(s)\big|\Big] \\ &\leq \mathbb{E}\Big[\mathbb{E}\big[|\tilde{M}(t) - \tilde{M}_n(t)|\big|\tilde{\mathcal{F}}_s\big]\Big] + \mathbb{E}\Big[\big|\tilde{M}_i(s) - \tilde{M}(s)\big|\Big] \\ &= \mathbb{E}\big[|\tilde{M}(t) - \tilde{M}_n(t)|\big] + \mathbb{E}\big[|\tilde{M}_i(s) - \tilde{M}(s)|\big] \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

□

Proof of Corollary 4.2.10. Since $G \in \mathfrak{G}^Z(K, r, m, \infty)$, there exists $g_i \in \mathfrak{g}^Z(K, r, m)$, $i \in \mathbb{N}$, such that $G = \prod_{i=1}^{\infty} g_i$. Now let us define $G_n \in \mathfrak{G}^Z(K, r, m, n), n \in \mathbb{N}$, by setting $G_n := \prod_{i=1}^{n} g_i$. For each $n \in \mathbb{N} \cup \{\infty\}$, we define the process $A_n : \Omega \times [0, \infty) \to \mathbb{R}$ by setting

$$A_n(t) := \int_0^t \mathbf{A}^o_{B_X,h}(G_n)(\widehat{\mathbf{W}}(s\text{-}), s)\, ds,\ t \geq 0,$$

and the process $M_n : \Omega \times [0, \infty) \to \mathbb{R}$ by setting

$$M_n(t) := G(\widehat{\mathbf{W}}(t)) - G(\widehat{\mathbf{W}}(0)) - \int_0^t \mathbf{A}^o_{B_X,h}(G_n)(\widehat{\mathbf{W}}(s\text{-}), s)\, ds,\ t \geq 0. \tag{4.40}$$

Now we make to observations: First, for finite n, i.e. $n \in \mathbb{N}$, the process M_n is a local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$ by Proposition 4.2.8, and second the processes A_∞ and M_∞ are identical with the processes A and M from our claim, see Proposition 4.2.10. In the following we want to apply the martingale convergence Lemma 4.6.8 to verify that M^∞ is also a $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$ which will proves our claim. By Lemma 4.6.5 it holds that for all $(\omega, t) \in \Omega \times [0, \infty)$

$$\begin{aligned}\int_0^t \mathbf{A}^o_{B_X,h}(G)(\widehat{\mathbf{W}}(\omega, s\text{-}), s) ds &= \int_0^t V^G_{B_X}(\omega, s) + V^G_{\mathbf{b}}(\omega, s) + V^G_h(\omega, s) ds \\ &\leq (ar^2 + |b|r + 1)KP^r_1(\omega, t) + 2rP^r_1(\omega, t) + KP^r_3(\omega, t).\end{aligned}$$

Since $E[|P^r_1(t \wedge \tilde{T}_{\widehat{m}})|] \leq E[|P^r_1(t)|] < \infty$ (P^r_1 is increasing) and $E[|P^r_3(t \wedge \tilde{T}_{\widehat{m}})|] < \infty$ by Lemma 4.6.5, it follows that sequence $(A_n(t \wedge \tilde{T}_{\widehat{m}}), n \in \mathbb{N} \cup \{\infty\})$ admits an integrable majorant. The same is true for $(G_n(\widehat{\mathbf{W}}(t \wedge \tilde{T}_{\widehat{m}}), n \in \mathbb{N} \cup \{\infty\})$, because $|G_n| \leq 1$ for all $n \in \mathbb{N} \cup \{\infty\}$. Further, from the definition of $(G_n, n \in \mathbb{N} \cup \{\infty\})$ and $\mathbf{A}^o_{B_X,h}(G_n)$, we can easily conclude that for all $(\omega, t) \in \Omega \times [0, \infty)$ holds

$$G_n(\widehat{\mathbf{W}}(\omega, t)) \overset{n\to\infty}{\longrightarrow} G_\infty(\widehat{\mathbf{W}}(\omega, t)) \text{ and}$$

$$\mathbf{A}^o_{B_X,h}(G_n)(\omega, t) \overset{n\to\infty}{\longrightarrow} \mathbf{A}^o_{B_X,h}(G_\infty)(\omega, t).$$

As a consequence it follows from Lebesgue dominated convergence theorem that $G_n(\widehat{\mathbf{W}}(0)) \overset{n\to\infty}{\longrightarrow} G_\infty(\widehat{\mathbf{W}}(0)), G_n(\widehat{\mathbf{W}}(t \wedge \tilde{T}_{\widehat{m}}) \overset{n\to\infty}{\longrightarrow} G_\infty(\widehat{\mathbf{W}}(t \wedge \tilde{T}_{\widehat{m}}))$ and $A_n(\widehat{\mathbf{W}}(t \wedge \tilde{T}_{\widehat{m}}) \overset{n\to\infty}{\longrightarrow} A_\infty(\widehat{\mathbf{W}}(t \wedge \tilde{T}_{\widehat{m}}))$ in $L^1(\mathbb{P})$. This and (4.40) gives us $M_n(t \wedge \tilde{T}_{\widehat{m}}) \overset{n\to\infty}{\longrightarrow} M_\infty(t \wedge \tilde{T}_{\widehat{m}}) = M(t \wedge \tilde{T}_{\widehat{m}})$. From Lemma 4.6.8 we conclude that M is a $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$. The existence of a càdlàg modification follows again from Theorem 6.27.(ii) in [21]. □

4.7 Proof of Case II

The following lemma will be used in the proof of the Case II.a and Case III.a and it is the analogue of Lemma 4.6.6.

Lemma 4.7.1. *Let us fix a* $\tilde{g} \in \tilde{\mathfrak{g}}^Z(K, r, m)$ *for an arbitrary* $(K, r, m) \in [0, \infty) \times [0, \infty) \times (0, 1)$. *It holds for all bounded* $\mathcal{F}^{\Xi,\mathbb{W}}$*-stopping times* τ*:*

$$\mathbb{E}\left[\boldsymbol{\xi}_\tau^{XZ}(\tilde{g})\right] \le \mathbb{E}\left[Y_\tau^r\right] = r\mathbb{E}\left[Y_\tau\right] < \infty. \tag{4.41}$$

Further, if we define for $n \in \mathbb{N} \cup \{\infty\}$ *the processes* $\hat{V}^n_{B_X}, \hat{V}^n_{\mathbf{b}}, \hat{V}^n_h : \Omega \times [0, \infty) \to [0, \infty)$ *by setting for each* $t \ge 0$*:*

$$\begin{aligned}
\hat{V}^n_{B_X}(t) &:= \sum_{i=1}^n B_X(\tilde{g})(\widehat{\mathbf{W}}_i(t\text{-})),\\
\hat{V}^n_{\mathbf{b}}(t) &:= \sum_{i=1}^n \left([aU_i(t\text{-})^2 - bU_i(t\text{-})]\partial_u(\tilde{g})(\widehat{\mathbf{W}}_i(t\text{-})) + 2a\int_u^\infty \tilde{g}(X_i(t\text{-}), Z_i(t\text{-}), v)\, dv \right),\\
\hat{V}^n_h(t) &:= \sum_{i=1}^n \int_0^\infty \tilde{g}(X_i(t\text{-}), Z_i(t\text{-}) + h(X_i(t\text{-}), p, t), U_i(t\text{-})) - \tilde{g}(\widehat{\mathbf{W}}_i(t\text{-}))\, dp\,,
\end{aligned}$$

then we get the following upper bounds

$$\begin{aligned}
\int_0^t |\hat{V}^n_{B_X}(s)| ds &\le KP^r_1(t),\\
\int_0^t |\hat{V}^n_{\mathbf{b}}(s)| ds &\le (ar^2 + |b|r)KP^r_1(t) + 2raP^r_1(t),\\
\int_0^t |\hat{V}^n_h(s)| ds &\le rKP^r_3(t),
\end{aligned}$$

(note that the upper bounds does not depend on n*), hence it holds* $\hat{V}^n_{B_X}(t) \le KP^r_3(t)$*. This implies together with Lemma 4.6.5 that for any* $\widehat{m} \in \mathbb{N}$

$$\int_0^t \mathbb{E}[|\hat{V}^n_{B_X}(s)|] ds < \infty, \int_0^t \mathbb{E}[|\hat{V}^n_{\mathbf{b}}(s)|] ds < \infty \text{ and } \int_0^t \mathbb{E}[\mathbb{1}_{[0,\tilde{T}_{\widehat{m}})}(s)|\hat{V}^n_h(s)|] ds < \infty.$$

Proof. The inequality in (4.41) follows from $|\tilde{g}| \le 1$ and $\text{supp}(g) \subset \mathbb{R}^d \times \mathbb{R} \times [0, r]$ by Definition Definition 4.2.4. The identity

$$\mathbb{E}[Y^r_\tau] = r\mathbb{E}[Y_\tau]$$

follows from the fact that Y_τ is conditioned on $\mathcal{F}^{\Xi,\mathbb{W}}_\tau$ a Poisson distributed random variable with intensity rY_τ. Since Y is under $\mathcal{F}^{\Xi,\mathbb{W}}$ a continuous time branching process with birth rate ra and death rate $ra - b$, we have

$$\mathbb{E}[Y_\tau] < \infty.$$

This proves the first claim. For the second and third claim we note that

$$B_X(\tilde{g})(\hat{\mathbf{w}}_i) \le K\mathbb{1}_{[0,r)}(u_i), [au_i^2 - bu_i]\partial_u(\tilde{g})(\hat{\mathbf{w}}_i) \le [a|r|^2 + |b|r]K\mathbb{1}_{[0,r)}$$

and that

$$|\tilde{g}(x, z + h(x,p,t), u) - \tilde{g}(x,z,u)| \le Kh(x,p,t)$$

by the same argumentation as in the proofs of (4.20), (4.21) and (4.24). Further $\tilde{g}(x,z,u) = 0$, if $u \ge r$ by Definition 4.2.4, hence

$$\int_{u_i}^{\infty} \tilde{g}(x,z,v)\,dv = \int_{u_i}^{r} \tilde{g}(x,z,v)\,dv \le r\mathbb{1}_{[0,r)}(u_i).$$

Applying these inequalities to $\hat{V}_{B_X}, \hat{V}_{\mathbf{b}}$ and $\hat{V}_h$ proves the second part of our claim. The third part follows directly from the second part and Lemma 4.6.5. □

Proof of Proposition 4.3.1. For $n \in \mathbb{N}$, we define the functions

$$S_n : \mathbf{S}(\mathbb{R}^d \times \mathbb{R}) \to \mathbb{R}, \quad S_n(\hat{\mathbf{w}}) := \sum_{i=1}^{n} g(\hat{\mathbf{w}}_i);$$
$$S_n^{\mathbf{b},i} : \mathbf{S}(\mathbb{R}^d \times \mathbb{R}) \times [0,\infty) \to \mathbb{R}, \quad S_n^{\mathbf{b},i}(\hat{\mathbf{w}}) := g(x_i, z_i, v) - g(x_n, z_n, u_n), \ i \in \mathbb{N}.$$

The function $\tilde{g}_i^{\Delta,X}$ is defined in the same way like $g_i^{\Delta,X}$ in (4.34) but with g replaced by $\tilde{g}$. With $(\tilde{X}_i^c, \tilde{X}_i^J)_{i=1}^{\infty}$ from (4.33) the Itô formula tells that $S_n(\widehat{\mathbf{W}}(t)) - S_n(\widehat{\mathbf{W}}(0))$, $t \ge 0$, is equal to

$$\sum_{i=1}^{n} \int_0^t \nabla_X \tilde{g}(\widehat{\mathbf{W}}_i(s\text{-}))\,d\tilde{X}_i^c(s) + \frac{1}{2}\int_0^t \nabla_X B^{cov} \nabla_X \tilde{g}(\widehat{\mathbf{W}}_i(s\text{-}))\,ds \tag{4.42}$$
$$+ \sum_{i=1}^{n} \int_0^t \int_{\mathbb{R}^d} \tilde{g}_i^{\Delta,X}(\widehat{\mathbf{W}}(s\text{-}), y)\,\tilde{X}_i^J(dy, ds) \tag{4.43}$$
$$+ \sum_{i=1}^{n} \int_0^t \left(aU_i^2(s\text{-}) - bU_i(s\text{-})\right) \partial_u \tilde{g}(\widehat{\mathbf{W}}_i(s\text{-}))\,ds \tag{4.44}$$
$$+ \sum_{i=1}^{\infty} \int_0^t \int_0^{\infty} \tilde{g}^{\Delta,h}(\widehat{\mathbf{W}}_i(s\text{-}), p, s)\,N_i(dp, ds) \tag{4.45}$$
$$+ \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} \int_0^t \int_{U_{k\text{-}1}(s\text{-})}^{U_k(s\text{-})} S_n^{\mathbf{b},i}(\widehat{\mathbf{W}}(s\text{-}), v)\,\mathcal{V}_{ki}(dv, ds) \tag{4.46}$$

We are dividing (4.42)-(4.46) into local martingales and continuous processes with finite variation, but before we need to add and subtract the compensators of $(\tilde{X}_i^J, \tilde{N}_i)_{i=1}^n$ and $(\mathcal{V}_{ki}, 1 \le i < k \le n)$. Writing $(\overline{X}_i^J, \overline{N}_i)_{i=1}^n$ and $(\overline{\mathcal{V}}_{ki}, 1 \le i < k \le n)$ for the compensated random measures, we introduce new processes by

$$M_n^X(t) := \sum_{i=1}^{n} \Bigg[\int_0^t \nabla_X \tilde{g}(\widehat{\mathbf{W}}_i(s\text{-}))\,dX_i^c(s) + \sum_{i=1}^{n} \int_0^t \int_{\mathbb{R}^d} \tilde{g}_i^{\Delta,X}(\widehat{\mathbf{W}}(s\text{-}), y)\,\overline{X}_i^J(dy, ds),$$
$$M_n^Z(t) := \sum_{i=1}^{n} \int_0^t \int_0^{\infty} \tilde{g}^{\Delta,h}(\widehat{\mathbf{W}}_i(s\text{-}), p, s)\,\overline{N}_i(dp, ds),$$
$$M_n^{\mathcal{B}}(t) := \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} \int_0^t \int_{U_{k\text{-}1}(s\text{-})}^{U_k(s\text{-})} S_n^{\mathbf{b},i}(\widehat{\mathbf{W}}(s\text{-}), v)\,\overline{\mathcal{V}}_{ki}(dv, ds),$$
$$A_n(t) := \sum_{i=1}^{n} \Bigg[\int_0^t B(\tilde{g})(\widehat{\mathbf{W}}_i(s\text{-})) + \left(aU_i^2(s\text{-}) - bU_i(s\text{-})\right) \partial_u \tilde{g}(\widehat{\mathbf{W}}_i(s\text{-}))\,ds$$
$$+ \int_0^t \int_0^{\infty} \tilde{g}^{\Delta,h}(\widehat{\mathbf{W}}_i(s\text{-}), p, s)\,dp + 2a \int_{U_i(s\text{-})}^{U_n(s\text{-})} S_n^{\mathbf{b},i}(\widehat{\mathbf{W}}(s\text{-}), v)\,dvds \Bigg],$$

where we used for the last line that

$$\sum_{k=i+1}^{n} \int_{U_{k\text{-}1}(s\text{-})}^{U_k(s\text{-})} S_n^{\mathbf{b},i}(\widehat{\mathbf{W}}(s\text{-}), v)\, dvds = \int_{U_i(s\text{-})}^{U_n(s\text{-})} S_n^{\mathbf{b},i}(\widehat{\mathbf{W}}(s\text{-}), v)\, dvds.$$

The processes M_n^X and $M_n^{\mathcal{B}}$ are $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingales, the process M_n^Z is a local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$ and A_n is a continuous process with finite variation. Comparing with (4.42)-(4.46) shows

$$S_n(\widehat{\mathbf{W}}(t)) - S_n(\widehat{\mathbf{W}}(0)) = M_n^X(t) + M_n^Z(t) + M_n^{\mathcal{B}}(t) + A_n(t),\ t \geq 0.$$

In order to prove Proposition 4.3.1 we need to show that S_n, A_n and $M_n := M_n^X + M_n^Z + M_n^{\mathcal{B}}$ are converging in $L^1(\mathbb{P})$ to $\boldsymbol{\xi}^{XZ}(\tilde{g}), A$ and M from (4.14) and (4.15). The processes $\boldsymbol{\xi}^{XZ}(\tilde{g}), A$ and M have the relation:

$$M(t) = \boldsymbol{\xi}_t^{XZ}(\tilde{g}) - \boldsymbol{\xi}_0^{XZ}(\tilde{g}) - A(t), \quad t \geq 0,$$

so it is sufficient to prove the $L^1(\mathbb{P})$ convergence for the right-hand side. This will be achieved with Lebesgue's theorem. By the definition of S_n:

$$S_n(t \wedge \tilde{T}_{\widehat{m}}) \overset{n\to\infty}{\longrightarrow} \boldsymbol{\xi}_{t\wedge\tilde{T}_{\widehat{m}}}^{XZ}(\tilde{g}) \quad a.s. \quad t \geq 0, \widehat{m} \in \mathbb{N}. \tag{4.47}$$

Further $(S_n(t\wedge\tilde{T}_{\widehat{m}}), n \in \mathbb{N})$ and $\boldsymbol{\xi}_{t\wedge\tilde{T}_{\widehat{m}}}^{XZ}(\tilde{g})$ are bounded by $KP_1^r(t\wedge\tilde{T}_{\widehat{m}})$, where P_1^r is taken from Lemma 4.7.1. Since

$$\mathbb{E}[P^r(t\wedge\tilde{T}_{\widehat{m}}] \leq \mathbb{E}[P^r(t)] < \infty, \tag{4.48}$$

Lebesgue's theorem extends the convergence from (4.47) to $L^1(\mathbb{P})$. The case of $(A_n)_{n=1}^{\infty}$ is more delicate. We first set $A_\infty = A$, so we can say that:

$$A_n(t) = \int_0^t \hat{V}_{B_X}^n(s) + \hat{V}_h^n(s) + \hat{V}_{\mathbf{b}}^n(s) + \hat{V}_n(s)\, ds, \quad t \geq 0, n \in \mathbb{N}\cup\{\infty\}, \tag{4.49}$$

where $\hat{V}_{B_X}^n, \hat{V}_{\mathbf{b}}^n, \hat{V}_h^n$ are from Lemma 4.7.1 and $\hat{V}_n$ is given by

$$\hat{V}_n(t) := 2a \sum_{i=1}^{n-1} \int_{U_i(s\text{-})}^{U_n(s\text{-})} \tilde{g}(\widehat{\mathbf{W}}_n(s\text{-}))\, dv,\ n \in \mathbb{N}, \quad \hat{V}_\infty(t) = 0, \quad t \geq 0.$$

The sequence $(\hat{V}_n(t))_{n=1}^{\infty}$ is converging to $\hat{V}_\infty = 0$ almost surely due to the combination of

$$U_i(t) \overset{i\to\infty}{\longrightarrow} \infty$$

and $\tilde{g}(x,z,u) = 0$ for all $u \geq r$. The fact that $\operatorname{supp}(\tilde{g}) \subset \mathbb{R}^d \times \mathbb{R} \times [0,r]$, implies that $\hat{V}_n(t) = 0$, when $U_n(t) \geq r$ and also that the i-th inner integral of $\hat{V}_n$ vanishes, when $U_i(t) \geq r$. If we combine both with $0 \leq \tilde{g} \leq 1$, we get

$$\begin{aligned}\hat{V}_n(t) &= 2a \sum_{i=1}^{n-1} \int_{U_i(s\text{-})}^{U_n(s\text{-})} \tilde{g}(\widehat{\mathbf{W}}_n(s\text{-}))\, dv \mathbb{1}_{[0,r)}(U_n(s\text{-}))\mathbb{1}_{[0,r)}(U_i(s\text{-})) \\ &\leq 2a \sum_{i=1}^{n-1} (U_n(s\text{-}) - U_i(s\text{-}))\, \mathbb{1}_{[0,r)}(U_n(s\text{-}))\mathbb{1}_{[0,r)}(U_i(s\text{-})) \\ &\leq 2ar \sum_{i=1}^{n-1} \mathbb{1}_{[0,r)}(U_i(s\text{-})) \leq 2arY_{s\text{-}}.\end{aligned}$$

This implies that $\int_0^t \hat{V}_n(s)\,ds \leq 2arP_1^r(t)$ for all $n \in \mathbb{N}$ and by (4.48) together with Lebesgue's theorem follows:

$$\int_0^{t\wedge\tilde{T}_{\widehat{m}}} \hat{V}_n(s)\,ds \overset{n\to\infty}{\longrightarrow} 0 \text{ in } L^1(\mathbb{P}). \tag{4.50}$$

Considering the remaining terms form (4.49), we recall also the process P_3^r from Lemma 4.6.5. With the help of Lemma 4.7.1 we can conclude that the integral

$$\int_0^t \hat{V}_{B_X}^n(s) + \hat{V}_h^n(s) + \hat{V}_{\mathbf{b}}^n(s)\,ds \tag{4.51}$$

is for all $t \geq 0$ and $\widehat{m} \in \mathbb{N} \cup \infty$ bounded by

$$(ar^2 + |b|r + 1)KP_1^r(t \wedge \tilde{T}_{\widehat{m}}) + 2aP_1^r(t \wedge \tilde{T}_{\widehat{m}}) + rKP_3^r(t \wedge \tilde{T}_{\widehat{m}}).$$

Since $\mathbb{E}[P_1^r(t \wedge \tilde{T}_{\widehat{m}})] < \infty$ and $\mathbb{E}[P_3^r(t \wedge \tilde{T}_{\widehat{m}})] < \infty$, we can conclude that

$$\int_0^{t\wedge\tilde{T}_{\widehat{m}}} \hat{V}_{B_X}^n(s) + \hat{V}_h^n(s) + \hat{V}_{\mathbf{b}}^n(s)\,ds \overset{n\to\infty}{\longrightarrow} A(t \wedge \tilde{T}_{\widehat{m}}), \quad L^1(\mathbb{P}).$$

It follows from Lemma 4.6.8 that M is a local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$. The existence of a càdlàg modification follows again by Theorem 6.27.(ii) in [21]. □

Now we turn to the Case II.b, which is contained in Case I.

Proof of Proposition 4.3.2. This will follow directly from Case I.b., indeed let us define the function $G : \mathbf{S}(\mathbb{R}^d \times \mathbb{R}) \to [0,1]$ by setting $G(\hat{\mathbf{w}}) = \prod_{i=1}^{\infty} g(\hat{\mathbf{w}}_i)$, then $G \in \mathfrak{G}_{2,\infty}^Z(K, m, r, \infty)$ and it holds

$$G(\widehat{\mathbf{W}}(t)) = \prod_{i=1}^{\infty} g(\widehat{\mathbf{W}}_i(t)) = \exp\left(\sum_{i=1}^{\infty} \log\left(g(\widehat{\mathbf{W}}_i(t))\right)\right) = \exp(-\boldsymbol{\xi}_t^{XZ}(f)),$$

where $f := -\log(g)$. From Corollary 4.2.10 we know that

$$M(t) = G(\widehat{\mathbf{W}}(t)) - G(\widehat{\mathbf{W}}(0)) - \int_0^t \mathbf{A}_{B_X,h}^o(G)(\widehat{\mathbf{W}}(s\text{-}), s)ds$$

is a $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale. Further $\mathbf{A}_{B_X,h}^o(G)$ becomes due to the special form of G:

$$\begin{aligned}
&\exp\left(-\sum_{i=1}^{\infty} f(\hat{\mathbf{w}}_i)\right) \sum_{i=1}^{\infty} \frac{B(g)(\hat{\mathbf{w}}_i)}{g(\hat{\mathbf{w}}_i)} \\
&+ \exp\left(-\sum_{i=1}^{\infty} f(\hat{\mathbf{w}}_i)\right) \sum_{i=1}^{\infty} (au_i^2 - bu_i) \frac{\partial_u g(\hat{\mathbf{w}}_i)}{g(\hat{\mathbf{w}}_i)} \\
&+ \exp\left(-\sum_{i=1}^{\infty} f(\hat{\mathbf{w}}_i)\right) \sum_{i=1}^{\infty} \sum_{l=1}^{i-1} \int_{u_{i\text{-}1}}^{u_i} 2a\left(g(x_l, z_l, v) - 1\right) dv \\
&+ \exp\left(-\sum_{i=1}^{\infty} f(\hat{\mathbf{w}}_i)\right) \sum_{i=1}^{\infty} \int_0^{\infty} \left(\frac{g(x_i, z_i + h(x_i, p, t), u_i)}{g(\hat{\mathbf{w}}_i)} - 1\right) dp.
\end{aligned}$$

Finally we just have to replace $\hat{\mathbf{w}}$ by $\widehat{\mathbf{W}}(s\text{-})$ and note that summing over all $i \in \mathbb{N}$ is the same as integrating with respect to $\boldsymbol{\xi}^{XZ}$. □

4.8 Proof of Case III

For the proof of Case III, it is essential to understand the connection between the test functions of the Case II and the ones of Case III. Let us assume that $\hat{g} \in C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$, then let us define $\tilde{g} \in \tilde{\mathfrak{g}}^Z$ by setting:

$$\tilde{g}(x,z,u) = \frac{1}{||\hat{g}||_\infty + \delta} \hat{g}(x,z) g_u(u),$$

where $g_u \in C^1([0,\infty))$ with $\operatorname{supp}(g_u) \subset [0,r]$ for some $r > 0$ and $\delta > 0$. Further let us define $g = 1 - \tilde{g} \in \mathfrak{g}^Z$ and $f := -\log(g)$. If we choose g_u such that $\int_0^\infty g_u(u)du = ||\hat{g}||_\infty + \delta$, then

$$\int_0^\infty \tilde{g}(x,z,u)\, du = \hat{g}(x,z).$$

If combine this with Proposition 3.5.7 and Lemma C.2.1, we obtain:

$$\mathbb{E}\left[\boldsymbol{\xi}_t^{XZ}(\tilde{g}) | \mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\right] = \boldsymbol{\Xi}_t^{XZ}(\hat{g}), \qquad \mathbb{E}\left[e^{-\boldsymbol{\xi}_t^{XZ}(f)} | \mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\right] = \mathbb{E}\left[\exp\left(-\boldsymbol{\Xi}_t^{XZ}(\hat{g})\right)\right].$$

These identities combined with the conditional martingale lemma, see Lemma D.2.1, are the core of the following proofs.

Proof of Proposition 4.4.1. Recall that $\tilde{g} \in \tilde{\mathfrak{g}}^Z(K,r,m)$. We define the processes $V_{B_X}, V_{\mathbf{b}}, V_h, A : \Omega \times [0,\infty) \to \mathbb{R}$ as

$$\begin{aligned}
V_{B_X}(\omega,t) &:= \int_{\overline{E}} B_X(g)(x,z,u)\, \boldsymbol{\xi}_{t\text{-}}^{XZ}(\omega,dx,dz,du),\\
V_{\mathbf{b}}(\omega,t) &:= \int_{\overline{E}} \left([au^2 - bu]\partial_u(g)(x,z,u) + 2a\int_u^\infty g(x,z,v)\, dv\right) \boldsymbol{\xi}_{t\text{-}}^{XZ}(\omega,dx,dz,du),\\
V_h(\omega,t) &:= \int_{\overline{E}} \int_0^\infty g(x,z+h(\omega,x,p,t),u) - g(x,z,u)\, dp\, \boldsymbol{\xi}_{t\text{-}}^{XZ}(\omega,dx,dz,du),\\
A(\omega,t) &:= \int_0^t V_{B_X}(\omega,s) + V_{\mathbf{b}}(\omega,s) + V_h(\omega,s) ds,
\end{aligned}$$

where $\overline{E} := \mathbb{R}^d \times \mathbb{R} \times [0,\infty)$. We will from now on suppress the dependence on Ω by omitting ω, when we write the above processes. Proposition 4.3.1 and Case II.a tells us that

$$M_t := \boldsymbol{\xi}_t^{XZ}(\tilde{g}) - \boldsymbol{\xi}_0^{XZ}(\tilde{g}) - A_t$$

is a local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$. Due to Lemma 4.7.1 we can apply the conditional martingale lemma D.2.1 to M and the filtration $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$. By doing so we obtain that

$$\overline{M}_t := \mathbb{E}\left[\boldsymbol{\xi}_t^{XZ}(g) | \mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\right] - \mathbb{E}\left[\boldsymbol{\xi}_0^{XZ}(g) | \mathcal{F}_0^{\boldsymbol{\Xi},\mathbb{W}}\right] - \int_0^t \mathbb{E}\left[V_{B_X}(s) + V_{\mathbf{b}}(s) + V_h(s) | \mathcal{F}_s^{\boldsymbol{\Xi},\mathbb{W}}\right] ds$$

is a local $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_{\widehat{m}}, \widehat{m} \in \mathbb{N})$. The idea of the rest of this proof is to show that for all $t \geq 0$ holds

$$\hat{A}(t) = \int_0^t \mathbb{E}\left[A_s | \mathcal{F}_s^{\boldsymbol{\Xi},\mathbb{W}}\right] ds,$$

where $\hat{A}$ is the process defined in the claim of this Proposition. This will be achieved by proving for all $t \geq 0$:

$$\mathbb{E}\left[\boldsymbol{\xi}_{t\text{-}}^{XZ}(g)|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\right] = \boldsymbol{\Xi}_t^{XZ}(\hat{g}), \tag{4.52}$$

$$\mathbb{E}\left[V_{B_X}(t)|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\right] = \int_{\hat{E}} B_X(\hat{g})(x,z)\,\boldsymbol{\Xi}_t^{XZ}(dx,dz), \tag{4.53}$$

$$\mathbb{E}\left[V_{\mathbf{b}}(t)|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\right] = \int_{\hat{E}} b\hat{g}(x,z)\,\boldsymbol{\Xi}_t^{XZ}(dx,dz), \tag{4.54}$$

$$\mathbb{E}\left[V_h(t)|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\right] = \int_{\hat{E}}\int_0^\infty \hat{g}(x,z+h(x,p,t)) - \hat{g}(x,z)\,dp\,\boldsymbol{\Xi}_t^{XZ}(dx,dz), \tag{4.55}$$

where $\hat{E} = \mathbb{R}^d \times \mathbb{R}$. The equalities will follow from Theorem 3.5.7, which tells us that for all $t \geq 0$ holds

$$\mathfrak{L}\big(\boldsymbol{\xi}_t^{XZ}\big|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\big) = \mathbf{PPP}\big(\boldsymbol{\Xi}_t^{XZ} \otimes \ell eb[0,\infty)\big). \tag{4.56}$$

and the two Lemmas C.2.1 and C.2.2. Indeed the Identity (C.7) from Lemma C.2.1 gives us

$$\mathbb{E}\big[\boldsymbol{\xi}_t^{XZ}(g)\big|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\big] = \int_0^\infty \int_{\hat{E}} g(x,z,u)\,\boldsymbol{\Xi}_t^{XZ}(dx,dz)\,du = \int_{\hat{E}} \hat{g}(x,z)\,\boldsymbol{\Xi}_t^{XZ}(dx,dz),$$

which proves Equation (4.52) (recall that $\hat{E} = \mathbb{R}^d \times \mathbb{R}$). Again applying Lemma C.2.1 and Lemma C.2.2 leads to

$$\begin{aligned}
\mathbb{E}\big[V_{B_X}(t)\big|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\big] &= \int_{\hat{E}}\int_0^\infty B_X(g)(x,z,u)\,du\,\boldsymbol{\Xi}_t^{XZ}(dx,dz),\\
&= \int_{\hat{E}} B_X(\hat{g})(x,z)\,\boldsymbol{\Xi}_t^{XZ}(dx,dz),\\
\mathbb{E}\big[V_h(t)\big|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\big] &= \int_{\hat{E}}\int_0^\infty g(x,z+h(x,p,t),u) - g(x,z,u)\,dp\,\boldsymbol{\Xi}_t^{XZ}(dx,dz)\,du\\
&= \int_{\hat{E}}\int_0^\infty \hat{g}(x,z+h(x,p,t)) - \hat{g}(x,z)\,dp\,\boldsymbol{\Xi}_t^{XZ}(dx,dz).
\end{aligned}$$

Similarly, when we first apply Identity (C.7) of Lemma C.2.1 and then the Identities (C.14) and (C.15) of Lemma C.2.2, we get:

$$\begin{aligned}
\mathbb{E}\big[V_{\mathbf{b}}(t)\big|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\big] &= \int_{\hat{E}}\int_0^\infty \left([au^2 - bu]\partial_u(g)(x,z,u) + 2a\int_u^\infty g(x,z,v)\,dv\right)du\,\boldsymbol{\Xi}_t^{XZ}(dx,dz)\\
&= \int_{\hat{E}} b\hat{g}(x,z)\,\boldsymbol{\Xi}_t^{XZ}(dx,dz).
\end{aligned}$$

The existence of a càdlàg modification of $\hat{M}$ follows from Theorem 6.27.(ii) in [21]. □

Proof of Proposition 4.4.2. We recall that $\hat{g} \in C_b^2(\mathbb{R}^d \times \mathbb{R}) \subset \mathcal{D}(B_X)$, that $\tilde{g} := 1 - g$ and $f := -\log(\tilde{g})$. As in the previous proof we start by define new stochastic processes, which we also call $V_{B_X}, V_{\mathbf{b}}$ and V_h as we have done in the proof of Proposition 4.4.1. Note that these new processes are very similar to their namesakes from the previous proof, but there are small differences. Nevertheless they play the same role as their namesakes in the following. The new processes are defined by

$$V_{B_X}(\omega,t) := \int_{\overline{E}} \exp(-\boldsymbol{\xi}^{XZ}_{t\text{-}}(\omega)(f)) \frac{B_X(\tilde g)(x,z,u)}{\tilde g(x,z,u)} \boldsymbol{\xi}^{XZ}_{t\text{-}}(\omega,dx,dz,du),$$

$$V_{\mathbf{b}}(\omega,t) := \int_{\overline{E}} \exp(-\boldsymbol{\xi}^{XZ}_{t\text{-}}(\omega)(f)) \Big[(au^2-bu)\frac{\partial_u \tilde g(x,z,u)}{\tilde g(x,z,u)} + \int_u^\infty 2a[\tilde g(x,z,v)-1]\,dv\Big] \boldsymbol{\xi}^{XZ}_{t\text{-}}(\omega,dx,dz,du),$$

$$V_h(\omega,t) := \int_{\overline{E}} \exp(-\boldsymbol{\xi}^{XZ}_{t\text{-}}(\omega)(f)) \int_0^\infty \frac{\tilde g(x,z+h(\omega,x,p,t),u)-\tilde g(x,z,u)}{\tilde g(x,z,u)}\,dp\, \boldsymbol{\xi}^{XZ}_{t\text{-}}(\omega,dx,dz,du),$$

where $\overline{E}=\mathbb{R}^d\times\mathbb{R}\times[0,\infty)$. As in the previous proof we will from now on omit ω. When we define the processes

$$A(t) = \int_0^t V_{B_X}(s)+V_{\mathbf{b}}(s)+V_h(s)\,ds,\ t\ge 0,$$
$$M(t) = \exp\left(-\boldsymbol{\xi}^{XZ}_t(f)\right) - \exp\left(-\boldsymbol{\xi}^{XZ}_0(f)\right) - A(t),\ t\ge 0,$$

then Proposition 4.3.2 tells us that M is a local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde T_{\widehat m},\widehat m\in\mathbb{N})$. For all $t\ge 0$ and $n\in\mathbb{N}$ holds $\mathbb{E}[\exp\left(-\boldsymbol{\xi}^{XZ}_{t\wedge\tilde T_{\widehat m}}(f)\right)]<\infty$, since $f\ge 0$, and $\mathbb{E}[|A(t\wedge\tilde T_{\widehat m})|]<\infty$, see Proposition 4.3.2. The conditional martingale lemma, see Appendix D.2, tells us that

$$\overline{A}(t) = \mathbb{E}[\overline{A}(t)|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_t] = \int_0^t \mathbb{E}\left[V_{B_X}(s)+V_{\mathbf{b}}(s)+V_h(s)|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_s\right]\,ds,\ t\ge 0,$$
$$\overline{M}(t) = \mathbb{E}[M_t|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_t] = \mathbb{E}\left[\exp\left(-\boldsymbol{\Xi}^{XZ}_t(\hat g)\right)|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_t\right] - \mathbb{E}\left[\exp\left(-\boldsymbol{\Xi}^{XZ}_0(\hat g)\right)|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_t\right] - \overline{A}(t),\ t\ge 0,$$

is a local $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde T_{\widehat m},\widehat m\in\mathbb{N})$. To prove our claim we need to show that $\hat A=\overline{A}$ and $\hat M=\overline{M}$, where $\hat A$ and $\hat M$ are the processes defined in (4.17) and (4.18). This will happen by showing:

$$\mathbb{E}\left[\exp(\boldsymbol{\xi}^{XZ}_t(\log(g)))|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_t\right] = \exp(-\boldsymbol{\Xi}^{XZ}_t(\hat g)),$$

and

$$\mathbb{E}\left[V_{B_X}(t)|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_t\right] = \hat L_{\hat g}(\boldsymbol{\Xi}^{XZ}_t)\int_{\hat E} B_X(\hat g)(x,z)\boldsymbol{\Xi}^{XZ}_t(dx,dz), \tag{4.57}$$

$$\mathbb{E}\left[V_{\mathbf{b}}(t)|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_t\right] = \hat L_{\hat g}(\boldsymbol{\Xi}^{XZ}_t)\int_{\hat E} b\hat g(x,z)\boldsymbol{\Xi}^{XZ}_t(dx,dz), \tag{4.58}$$

$$\mathbb{E}\left[V_h(t)|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_t\right] = \hat L_{\hat g}(\boldsymbol{\Xi}^{XZ}_t)\int_{\hat E}\int_0^\infty \hat g(x,z+h(x,p,t))-\hat g(x,z)dp\boldsymbol{\Xi}^{XZ}_t(dx,dz), \tag{4.59}$$

where $\hat E=\mathbb{R}^d\times\mathbb{R}$ and $\hat L_{\hat g}(\mu)=\exp(-\mu(\hat g))$. For this we use the fact that it holds for all $t\in[0,\infty)$:

$$\mathfrak{L}\left(\boldsymbol{\xi}^{XZ}_t\big|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_t\right) = \mathbf{PPP}\big(\boldsymbol{\Xi}^{XZ}_t\otimes \ell eb[0,\infty)\big).$$

It follows from (C.10) of Lemma C.2.1 that:

$$\mathbb{E}\left[\exp(-\boldsymbol{\xi}^{XZ}_t(f))\big|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_t\right] = \exp\left(\int_{\hat E}\int_0^\infty e^{-f(x,z,u)}-1\,du\,\boldsymbol{\Xi}^{XZ}_t(dx,dz)\right)$$

and combining this with $f = -\log(1-g)$, we can see that

$$\exp\left(\int_{\hat{E}}\int_0^\infty e^{-f(x,z,u)} - 1\,du\,\boldsymbol{\Xi}_t^{XZ}(dx,dz)\right) = \exp(-\boldsymbol{\Xi}_t^{XZ}(\hat{g})), \tag{4.60}$$

and so we have proven (4.57). In the same way it follows from (C.11) that $\mathbb{E}\big[V_{B_X}(t)\big|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\big]$ is equal to

$$\begin{aligned}&\exp\left(\int_{\hat{E}}\int_0^\infty e^{-f(x,z,u)} - 1\,du\,\boldsymbol{\Xi}_t^{XZ}(dx,dz)\right)\int_{\hat{E}}\int_0^\infty B_X(\tilde{g})(x,z,u)\,du\,\boldsymbol{\Xi}_t^{XZ}(dx,dz)\\ &= \exp(-\boldsymbol{\Xi}_t^{XZ}(\hat{g}))\int_{\hat{E}} B_X(\hat{g})(x,z)\,\boldsymbol{\Xi}_t^{XZ}(dx,dz).\end{aligned}$$

Further by the results of Lemma C.2.1 the expression $\mathbb{E}\big[V_h(t)\big|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\big]$ is equal to

$$\begin{aligned}\int_0^\infty\int_{\hat{E}}\int_0^\infty \tilde{g}(x,z+h(x,p,t),u) - \tilde{g}(x,z,u)\,du\,\boldsymbol{\Xi}_t^{XZ}(dx,dz)\,dp&\\ \cdot\exp\left(\int_{\hat{E}}\int_0^\infty e^{-f(x,z,u)} - 1\,du\,\boldsymbol{\Xi}_t^{XZ}(dx,dz)\right)&.\end{aligned}$$

Integrating with respect to u, using $\tilde{g} = 1-g$ and applying (4.60) turns the above into

$$-\int_{\hat{E}}\int_0^\infty \hat{g}(x,z+h(x,p,t)) - \hat{g}(x,z)\,dp\,\boldsymbol{\Xi}_t^{XZ}(dx,dz)\exp(-\boldsymbol{\Xi}_s^{XZ}(\hat{g})).$$

Finally, $\mathbb{E}\big[V_{\mathbf{b}}(t)\big|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\big]$ becomes:

$$\begin{aligned}\int_{\hat{E}}\int_0^\infty\left([au^2-bu]\partial_u\tilde{g}(x,z,u) + 2a\tilde{g}(x,z,u)\int_u^\infty \tilde{g}(x,z,v) - 1dv\right)\,du\,\boldsymbol{\Xi}_t^{XZ}(dx,dz)&\\ \times\exp\left(\int_{\hat{E}}\int_0^\infty e^{-f(x,z,u)} - 1\,du\boldsymbol{\Xi}_t^{XZ}(dx,dz)\right)&.\end{aligned}$$

Applying (C.13) and (C.15) from Lemma C.2.2 and (4.60) this transforms into

$$\mathbb{E}\big[V_{\mathbf{b}}(t)\big|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W}}\big] = \int_{\hat{E}}\big[\text{-}b\hat{g}(x,z) + a\hat{g}^2(x,z)\big]\,\boldsymbol{\Xi}_t^{XZ}(dx,dz)\exp(-\boldsymbol{\Xi}_t^{XZ}(\hat{g})).$$

So we have proved (4.57), (4.57), (4.58) and (4.59). The existence of a càdlàg modification of $\hat{M}$ follows from Theorem 6.27.(ii) in [21]. □

4.9 Proof of Case IV

Proof of Proposition 4.5.1. We assume that $\hat{g} \in C^{2,+}(\mathbb{R}^d\times\mathbb{R})$ and start by defining $\tilde{g} \in \tilde{\mathfrak{g}}^Z$, recall Definition 4.2.4, by setting $\tilde{g}(x,z,u) = \tilde{K}^{-1}\hat{g}(x,u)g_u(u)$, where $K > ||\hat{g}||_\infty$, $g_u : [0,\infty] \to [0,1]$ is continuously differentiable with support contained in $[0,r]$. If we set $\lambda_g := \int_0^r g_u(u)\,du$, then it holds

$$\int_0^r \tilde{g}(x,z,u)\,\frac{du}{r} = \frac{\lambda_g}{r\tilde{K}}\hat{g}(x,z). \tag{4.61}$$

Since $\tilde{g} \in \tilde{\mathfrak{g}}^Z$, we can apply Case II.a), see Proposition 4.3.1, to obtain a semi-martingale decomposition of $\boldsymbol{\xi}^{XZ}(\tilde{g})$, from which we can obtain a $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale by

$$M(t) := \boldsymbol{\xi}_t^{XZ}(\tilde{g}) - A(t),\ t \geq 0,$$

where A is the continuous process with finite variation (4.14). As in Case III we obtain the semi-martingale decomposition of Case IV by applying the conditional martingale lemma, see Lemma D.2.1, which tells us that

$$\tilde{M}^r(t) := \mathbb{E}\left[M_t|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W},r}\right] = \mathbb{E}\left[\boldsymbol{\xi}_t^{XZ,r}(\tilde{g})|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W},r}\right] - \mathbb{E}\left[A(t)|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W},r}\right] \tag{4.62}$$

is a $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W},r}$-martingale. We can conclude from Proposition 2.6.6 that

$$\mathfrak{L}(\boldsymbol{\xi}^{XZ,r}|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W},r}) = \mathbf{Uni}^r_{\mathbb{R}^d\times\mathbb{R}}(\boldsymbol{\Xi}_t^{XZ,r}), \tag{4.63}$$

where $\mathbf{Uni}^r_{\mathbb{R}^d\times\mathbb{R}}$ is the Markov kernel from Definition 1.1.1, we can conclude directly that

$$\mathbb{E}[\boldsymbol{\xi}_t^{XZ,r}(\tilde{g})|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W},r}] = \lambda_g r^{-1}\boldsymbol{\Xi}^{XZ,r}(\hat{g}),$$

where we used (4.61). The conditional martingale lemma D.2.1 together with (4.63) gives us:

$$\begin{aligned}
\mathbb{E}\left[A(t)|\mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W},r}\right] &= \int_0^t \int_{\overline{E}} \int_0^r B_X(\tilde{g})(x,z,u)\frac{du}{r}\,\boldsymbol{\Xi}_{s\text{-}}^{XZ}(dx,dz)\,ds \\
&+ \int_0^t \int_{\overline{E}} \int_0^r [au^2 - bu]\partial_u(\tilde{g})(x,z,u)\frac{du}{r}\boldsymbol{\Xi}_{s\text{-}}^{XZ}(dx,dz)\,ds \\
&+ \int_0^t \int_{\overline{E}} \int_0^r 2a\int_u^\infty \tilde{g}(x,z,v)\,dv\,\frac{du}{r}\boldsymbol{\Xi}_{s\text{-}}^{XZ}(dx,dz)\,ds \\
&+ \int_0^t \int_{\overline{E}} \int_0^\infty \int_0^r \tilde{g}(x,z+h(x,p,s),u) - \tilde{g}(x,z,u)\,\frac{du}{r}\,dp\,\boldsymbol{\Xi}_{s\text{-}}^{XZ}(dx,dz)\,ds,
\end{aligned} \tag{4.64}$$

Using the identity (4.61) the first and last line of (4.64) turn into:

$$\begin{aligned}
&\frac{\lambda_g}{r\tilde{K}}\int_0^t \int_{\overline{E}} B_X(\hat{g})(x,z)\boldsymbol{\Xi}_{s\text{-}}^{XZ}(dx,dz)ds \\
&+\frac{\lambda_g}{r\tilde{K}}\int_0^t \int_{\overline{E}} \int_0^\infty \hat{g}(x,z+h(x,p,s)) - \hat{g}(x,z)dp\boldsymbol{\Xi}_{s\text{-}}^{XZ}(dx,dz)ds
\end{aligned}$$

The two middle lines of (4.64) can be rewritten into

$$\int_0^t \int_{\overline{E}} \int_0^r [au^2\partial_u\tilde{g}(x,z,u) - 2a\int_u^\infty \tilde{g}(x,z,v)\,dv\,\frac{du}{r}\boldsymbol{\Xi}_{s\text{-}}^{XZ}(dx,dz)ds \tag{4.65}$$

$$- \int_0^t \int_{\overline{E}} \int_0^r bu\partial_u\tilde{g}(x,z,u)\frac{du}{r}\boldsymbol{\Xi}_{s\text{-}}^{XZ}(dx,dz)ds. \tag{4.66}$$

Using (C.14) from Lemma C.2.2 we can see that (4.65) turns into 0 and using (C.15) we can see that (4.66) turns into $b\lambda_g r^{-1}\hat{g}(x,z)$, again we applied Identity (4.61). Putting everything together we can conclude from (4.62) that

$$\frac{\lambda_g}{r\tilde{K}}\boldsymbol{\Xi}_t^{XZ,r}(\hat{g}) - \frac{\lambda_g}{r\tilde{K}}\hat{A}^r(t),$$

is a $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W},r}$-martingale, where $\hat{A}^r$ is the process with finite variation from the statement of the current proposition, see Proposition 4.5.1. By multiplying with $r\tilde{K}\lambda_g^{-1}$ and adding $\boldsymbol{\Xi}_0^{XZ,r}(\hat{g})$ we obtain the statement of Proposition 4.5.1. □

Proof of Proposition 4.5.2. Let us define $(\tau_k)_{k=0}^\infty$ by setting $\tau_0 = 0$ and $\tau_{k+1} = \inf\{s > \tau_k : |\Delta Y_s^r| > 0\}$, indeed the sequence $(\tau_k)_{k=0}^\infty$ corresponds to the times, where the number of particles with a level below r is changing either via birth or via death. It holds for all $t_2 \geq t_1 \geq 0$:

$$[\boldsymbol{\Xi}^{XZ,r}(\hat{g})]_{t_2} - [\boldsymbol{\Xi}^{XZ,r}(\hat{g})]_{t_1}$$
$$= \sum_{k=0}^{\infty} \mathbb{1}_{[t_1,t_2]}(\tau_k) \left(\Delta\boldsymbol{\Xi}_{\tau_k}^{XZ,r}(\hat{g})\right)^2 + \sum_{i=1}^{Y^r(\tau_k)} [\hat{g}(X_i, Z_i)]_{t_2\wedge\tau} - [\hat{g}(X_i, Z_i)]_{t_1\wedge\tau_k},$$

where $\Delta\boldsymbol{\Xi}_{\tau_k}^{XZ,r}(\hat{g})$ is the jump of $\boldsymbol{\Xi}^{XZ,r}(\hat{g})$ at time τ. Note that in the case of a death event it holds

$$\Delta\boldsymbol{\Xi}_{\tau_k}^{XZ,r}(\hat{g}) = -g(X_{Y^r(\tau_k\text{-})}(\tau_k\text{-}), Z_{Y^r(\tau_k\text{-})}(\tau_k\text{-}))$$

and in the case of a birth event, we assume that j is the index of the parent, it holds

$$\Delta\boldsymbol{\Xi}_{\tau_k}^{XZ,r}(\hat{g}) = g(X_j(\tau_k\text{-}), Z_j(\tau_k\text{-})).$$

This allows us to derive the following upper bound:

$$\sum_{k=0}^{\infty} \mathbb{1}_{[t_1,t_2]}(\tau_k) \left(\Delta\boldsymbol{\Xi}_{\tau_k}^{XZ,r}(\hat{g})\right)^2 \leq ||g||_\infty^2 \left([Y^r]_{t_2} - [Y^r]_{t_1}\right),$$

where we used that $\sum_{k=0}^\infty \mathbb{1}_{[t_1,t_2]}(\tau_k) = [Y^r]_t$. If $(N_i)_{i=1}^\infty$ are the Poisson point processes from Assumption 2.1.2, then we can write for $t \in [\tau_k, \tau_{k+1})$

$$Z_i(t) - Z_i(\tau_k) = \int_{\tau_k}^t \int_0^\infty h(X_i(s\text{-}), p, s) N_i(dp, ds)$$

The processes $X_i, 1 \leq i \leq Y_{\tau_k}^r$, behave like independent Lévy processes with characteristic triple $(B_X^\rho, B_X^{cov}, B_X^\eta)$ on the interval $[\tau_k, \tau_{k+1})$, indeed it holds $X_i(t) - X_i(\tau_k) = \tilde{X}_i(t) - \tilde{X}_i(\tau)$, where $(\tilde{X}_i)_{i=1}^\infty$ is the collection of independent Lévy processes from Assumption 2.1.2 with Lévy triple $(B_X^\rho, B_X^{cov}, B_X^\eta)$. According to the Lévy-Itô decomposition we can for each $\tilde{X}_i$ a $\mathbb{R}^d$-dimensional Brownian motion $\tilde{X}_i^c$ and a Poisson point process $\tilde{X}_i^J$ over $\mathbb{R}^d \times \mathbb{R} \times [0, \infty)$ such that:

$$\tilde{X}_i(t) = B_W^\rho t + (B_W^{cov})^{\frac{1}{2}} \tilde{X}_i^c(t) + \int_0^t \int_{\{||y||>1\}^c} y \, \tilde{X}_i^J(dy, ds) + \int_0^t \int_{\{||y||\leq 1\}} y \, [\tilde{X}_i^J - B_X^\eta](dy, ds).$$

Based on the Lévy-Itô decomposition of $\tilde{X}_i$ we can conclude that for each $k \in \mathbb{N}_0$ and $t \in [\tau_k, \tau_{k+1})$ holds

$$[\hat{g}(X_i, Z_i)]_t - [\hat{g}(X_i, Z_i)]_{\tau_k} =$$
$$\int_{\tau_k}^t \left[\nabla_x(\hat{g})(X_i(s), Z_i(s))^T B_X^{cov} \nabla_x(\hat{g})(X_i(s), Z_i(s))\right] ds$$
$$+ \int_{\tau_k}^t \int_{\mathbb{R}^d} \left(\hat{g}(X_i(s\text{-}), Z_i(s\text{-}) + y) - \hat{g}(X_i(s\text{-}), Z_i(s\text{-}))\right)^2 \tilde{X}_i^J(dp, ds)$$
$$+ \int_{\tau_k}^t \int_0^\infty \left(\hat{g}(X_i(s\text{-}), Z_i(s\text{-}) + h(X_i(s\text{-}), p, s)) - \hat{g}(X_i(s\text{-}), Z_i(s\text{-}))\right)^2 N_i(dp, ds),$$

If we define

$$P_t := ||\hat{g}||_\infty^2 [Y^r]_t + \sum_{i=1}^{Y^r(\tau_k)} [\hat{g}(X_i, Z_i)]_{t\wedge\tau} - [\hat{g}(X_i, Z_i)]_{\tau_k},$$

then

$$[\mathbf{\Xi}^{XZ,r}(\hat{g})]_{t_2} - [\mathbf{\Xi}^{XZ,r}(\hat{g})]_{t_1} \leq P_{t_2} - P_{t_1}$$

and further the compensator of P is given by the right-hand side of Inequality (4.19). This proves Inequality (4.19). $\square$

Chapter 5

Convergence Theorems

In this chapter we deal with the question, what happens if we have a sequence $(h_k)_{k=1}^{\infty} \subset \mathcal{L}^1_{loc}(\mathbf{M})$ that is converging to $h_\infty \in \mathcal{L}^1_{loc}(\mathbf{M})$ with respect to the norm $|||\cdot|||_{\mathbf{M}}$. If we define

$$\big((X_i, Z_i^k, U_i)_{i=1}^{\infty}, \boldsymbol{\xi}^{XZ,k}, \boldsymbol{\Xi}^{XZ,k}, \mathbf{Q}^{XZ,k}\big) = \mathbb{I}_0\big[h_k\big],\ k \in \mathbb{N}\cup\{\infty\},$$

then we would expect that $\boldsymbol{\Xi}^{XZ,k}$ is converging against $\boldsymbol{\Xi}^{XZ,\infty}$ in a suitable sense. We make this statement precise in this chapter, which is divided in two sections. In the first section we deal with the convergence for a fixed time point, in the second we extend this result to convergence, which holds uniformly over a compact time interval. The results of this chapter become useful in the rest of this thesis.

5.1 Convergence for a fixed time Point

We introduce the class $\mathrm{Lip}_z(\Omega\times\mathbb{R}^d\times\mathbb{R}, \mathcal{F}_\tau^{\boldsymbol{\Xi},\mathbb{W}})$, where τ is a $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time, because we want to handle two different sets of functions simultaneously. The first set is given by $C_b^2(\mathbb{R}^d\times\mathbb{R})$, see Definition 2.1.1, the space of twice continuous differentiable functions, which are bounded and have bounded derivatives. The second presented in Lemma 5.1.3 contains functions, which are random, but their randomness is measurable with respect to $\mathcal{F}_\tau^{\boldsymbol{\Xi},\mathbb{W}}$.

Definition 5.1.1. *Assume that τ is a finite $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time, we define $\mathbf{M}(\mathcal{F}_\tau^{\boldsymbol{\Xi},\mathbb{W}}\otimes\mathbb{B}(\mathbb{R}^d\times\mathbb{R}))$ as the collection of measurable functions $\hat{\mathbf{g}}:\Omega\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ that are measurable with respect to $\mathcal{F}_\tau^{\boldsymbol{\Xi},\mathbb{W}}\otimes\mathbb{B}(\mathbb{R}^d\times\mathbb{R})$. Further we denote by*

$$Lip_z(\Omega\times\mathbb{R}^d\times\mathbb{R}, \mathcal{F}_\tau^{\boldsymbol{\Xi},\mathbb{W}}) \subset \mathbf{M}(\mathcal{F}_\tau^{\boldsymbol{\Xi},\mathbb{W}}\otimes\mathbb{B}(\mathbb{R}^d\times\mathbb{R}))$$

the set of bounded $\hat{\mathbf{g}}$ for which a constant $K>0$ exists with

$$|\hat{\mathbf{g}}(\omega,x,z)-\hat{\mathbf{g}}(\omega,x,\tilde{z})| \le K|z-\tilde{z}| \quad \forall \omega\in\Omega, x\in\mathbb{R}^d, z,\tilde{z}\in\mathbb{R}.$$

Indeed the function $\hat{\mathbf{g}}$ is Lipschitz-continuous in the z-coordinate.

Lemma 5.1.2. *If τ is a finite $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time and $h\in\mathcal{L}^1_{loc}(\mathbf{M})$.*

$$\big((X_i, Z_i, U_i)_{i=1}^{\infty}, \boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ}, \mathbf{Q}^{XZ}\big) \ = \ \mathbb{I}_0\big[h_j\big],$$

then it holds for all $\hat{\mathbf{g}}\in\mathbf{M}(\mathcal{F}_\tau^{\boldsymbol{\Xi},\mathbb{W}}\otimes\mathbb{B}(\mathbb{R}^d\times\mathbb{R}))$ with $\boldsymbol{\Xi}_\tau^{XZ}(|\hat{\mathbf{g}}|)<\infty$ that

$$\frac{1}{r}\boldsymbol{\Xi}_\tau^{XZ,r}(\hat{\mathbf{g}}) = \frac{1}{r}\sum_{i=1}^{\infty}\hat{\mathbf{g}}(X_i(\tau),Z_i(\tau))\mathbb{1}_{[0,r)}(U_i(\tau)) \stackrel{r\to\infty}{\longrightarrow} \boldsymbol{\Xi}_\tau^{XZ}(\hat{\mathbf{g}}),$$

where the convergence holds almost surely and in $L^1(\mathbb{P})$.

Proof. Similar to Lemma C.1.5 we can show that $(\frac{1}{r}\boldsymbol{\Xi}^{XZ,r}_{\tau}(\hat{\mathbf{g}}), r \geq \max\{b/a, 0\})$ forms a backwards martingale with respect to the decreasing filtration $(\mathcal{F}^{\mathbb{W},r}, r \geq \max\{b/a, 0\})$. The convergence follows by the usual backwards martingale convergence. $\square$

Lemma 5.1.3. *Let us assume that τ is a finite $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-stopping time, $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ and $\hat{g} \in C^2_b(\mathbb{R}^d \times \mathbb{R})$. If we define $\hat{\mathbf{g}} : \Omega \times \mathbb{R}^d \times \mathbb{R}$ for fixed pair $(\hat{p}, \hat{s}) \in [0, \infty) \times [0, \infty)$ by setting:*

$$\hat{\mathbf{g}}(\omega, x, z) := \mathbb{1}_{[0,\tau(\omega)]}(\hat{s}) \left(\hat{g}(x, z + h(\omega, x, \hat{p}, \hat{s})) - \hat{g}(x, z)\right), \tag{5.1}$$

then the $\hat{\mathbf{g}}$ is an element of $Lip_z(\Omega \times \mathbb{R}^d \times \mathbb{R}, \mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_{\tau})$.

Functions of the form (5.1) will become important in the next section, where we apply the semi-martingale decomposition of Case III.a, see Proposition 4.4.1.

Proof. Since $\hat{g} \in C^2_b(\mathbb{R}^d \times \mathbb{R})$, it follows that $\hat{g}$ is Lipschitz-continuous with constant $||\nabla \hat{g}||_\infty$ recall Definition 2.1.1, hence we have:

$$\begin{aligned} |\hat{\mathbf{g}}(\omega, x, z_1) - \hat{\mathbf{g}}(\omega, x, z_2)| &\leq |\hat{g}(x, z_1 + h(\omega, x, \hat{p}, \hat{s})) - \hat{g}(x, z_2 + h(\omega, x, \hat{p}, \hat{s}))| \\ &\leq ||\nabla \hat{g}||_\infty |z_1 - z_2|. \end{aligned}$$

As an element of $\mathfrak{P}(\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}})$ the function h is measurable with respect to

$$\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}} \otimes \mathbb{B}(\mathbb{R}^d \times [0, \infty) \times [0, \infty)),$$

hence

$$(\omega, x, z, p, s) \mapsto \mathbb{1}_{[0,\tau(\omega)]}(\hat{s}) \left(\hat{g}(x, z + h(\omega, x, \hat{p}, \hat{s})) - \hat{g}(x, z)\right) \tag{5.2}$$

is $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}} \otimes \mathbb{B}(\mathbb{R}^d \times \mathbb{R}) \times [0, \infty) \times [0, \infty)$-measurable. Since $(\omega, x, z) \mapsto (\omega, x, \hat{s}, \tau(\omega) \wedge \hat{s})$ is $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_{\tau} \otimes \mathbb{B}(\mathbb{R}^d \times \mathbb{R})$-measurable (note that we can replace $\hat{s}$ in (5.2) by $\tau \wedge \hat{s}$). It follows that $\hat{\mathbf{g}}$ is in $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}} \otimes \mathbb{B}(\mathbb{R}^d \times \mathbb{R})$. $\square$

Proposition 5.1.4. *Let us assume that $(h_k, k \in \mathbb{N} \cup \{\infty\})$ are elements in $\mathcal{L}^1_{loc}(\mathbf{M})$ with common localizing sequence $(T_n, n \in \mathbb{N})$ in $\mathcal{L}^1_{loc}(\mathbf{M})$ and it holds additionally that*

$$|||\, \mathbb{1}_{[0,T_n]}(h_k - h_\infty)\, |||_{\mathbf{M}} \overset{k \to \infty}{\longrightarrow} 0,\ n \in \mathbb{N}.$$

If we define the processes $(\boldsymbol{\Xi}^{XZ,k}, k \in \mathbb{N})$ and $\boldsymbol{\Xi}^{XZ,\infty}$ by setting

$$\left((X_i, Z^k_i, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ,k}, \boldsymbol{\Xi}^{XZ,k}, \mathbf{Q}^{XZ,k}\right) = \mathbb{I}_0\left[h_k\right],$$

then it holds for all $\hat{\mathbf{g}} \in Lip(\Omega \times \mathbb{R}^d \times \mathbb{R}, \mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_{\tau})$ that

$$\mathbb{E}\left[\left|\boldsymbol{\Xi}^{XZ,k}_{t \wedge \tilde{T}_n}(\hat{\mathbf{g}}) - \boldsymbol{\Xi}^{XZ,\infty}_{t \wedge \tilde{T}_n}(\hat{\mathbf{g}})\right|\right] \overset{k \to \infty}{\longrightarrow} 0,\ n \in \mathbb{N}, t \geq 0,$$

where $\tilde{T}_n := T_n \wedge \tau^Y_n$ with $\tau^Y_n := \inf\{s \geq 0 : Y_s \geq n\}$.

Proof. If we define $(\hat{h}_k)_{k=1}^\infty$ with $\hat{h}_k := |h_k - h_\infty|$, then $|||\, \mathbb{1}_{[0,t \wedge \tilde{T}_n)} \hat{h}_k\, |||_{\mathbf{M}}$ converges against zero, when k goes to infinity. Let us define:

$$\left((X_i, \hat{Z}^k_i, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{X\hat{Z},k}, \boldsymbol{\Xi}^{X\hat{Z},k}, \mathbf{Q}^{X\hat{Z},k}\right) = \mathbb{I}_0[\hat{h}_k].$$

Our first step is to show that for all $k \in \mathbb{N}$ holds:

$$\mathbb{E}\left[\left|\mathbf{\Xi}^{XZ,k}_{t\wedge\tilde{T}_n}(\hat{\mathbf{g}}) - \mathbf{\Xi}^{XZ,\infty}_{t\wedge\tilde{T}_n}(\hat{\mathbf{g}})\right|\right] \leq K\mathbb{E}\left[\mathbf{\Xi}^{X\hat{Z},k}_{t\wedge\tilde{T}_n}(\hat{f})\right], \tag{5.3}$$

where $\hat{f} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is given by $f(x,z) = z$. Considering the integrated processes $(Z_i^k, k \in \mathbb{N}, i \in \mathbb{N})$ we have for every fixed $(i,k,t) \in \mathbb{N}^2 \times [0,\infty)$ the following inequality:

$$\begin{aligned}\left|Z_i^k(t) - Z_i^\infty(t)\right| &\leq \int_0^t \int_0^\infty |h_k(X_i(s\text{-}),p,s) - h_\infty(X_i(s\text{-}),p,s)|\, \mathfrak{N}_i(t,dp,ds) \\ &= \hat{Z}_i^k(t).\end{aligned} \tag{5.4}$$

From this inequality we can conclude that for all $r > 0$ and all $n \in \mathbb{N}$ holds:

$$\frac{1}{r}\left|\sum_{i=1}^\infty \hat{\mathbf{g}}(X_i(t\wedge\tilde{T}_n), Z_i^1(t\wedge\tilde{T}_n)) - \hat{\mathbf{g}}(X_i(t\wedge\tilde{T}_n), Z_i^2(t\wedge\tilde{T}_n))\right| \mathbb{1}_{[0,r)}(U_i(t\wedge\tilde{T}_n)) \tag{5.5}$$

$$\leq \frac{1}{r}\sum_{i=1}^\infty K\left|Z_i^1(t\wedge\tilde{T}_n) - Z_i^2(t\wedge\tilde{T}_n)\right| \mathbb{1}_{[0,r)}(U_i(t\wedge\tilde{T}_n))$$

$$\leq \frac{1}{r}\sum_{i=1}^\infty K\hat{Z}(t\wedge\tilde{T}_n)\mathbb{1}_{[0,r)}(U_i(t\wedge\tilde{T}_n)). \tag{5.6}$$

For (5.3) we need to show that (5.5) is converging against the left side of (5.3) and that (5.6) is converging against the right-hand side of (5.3) in $L^1(\mathbb{P})$, when r goes to infinity. In both cases this follow from Lemma 5.1.2, when we can prove the integrability condition. The latter is fullfilled for (5.5), because $\hat{\mathbf{g}}$ is bounded by a constant $\tilde{K}$ and it holds

$$\mathbb{E}\left[\left|\mathbf{\Xi}^{XZ,k}_{t\wedge\tilde{T}_n}(\hat{\mathbf{g}})\right|\right] \leq n\tilde{K}, \quad k \in \mathbb{N} \cup \{\infty\}.$$

The integrability condition is fullfilled for (5.6), because it holds for for all $n \in \mathbb{N}$:

$$\begin{aligned}\mathbb{E}\left[\mathbf{\Xi}^{X\hat{Z},k}_{t\wedge\tilde{T}_n}(\hat{f})\right] &= \mathbb{E}\left[Y_{t\wedge\tilde{T}_n}\mathbf{Q}^{X\hat{Z},k}_{t\wedge\tilde{T}_n}(\hat{f})\right] \leq n\mathbb{E}\left[\mathbf{Q}^{X\hat{Z},k}_{t\wedge\tilde{T}_n}(\hat{f})\right] \\ &\leq n\mathbb{E}\left[\hat{Z}^k(t\wedge\tilde{T}_n)\right] = n|||\,\mathbb{1}_{[0,t\wedge\tilde{T}_n]}\hat{h}_k\,|||_{\mathbf{M}} < \infty.\end{aligned} \tag{5.7}$$

It remains to argue that for all n and $t \geq 0$:

$$\mathbb{E}\left[\mathbf{\Xi}^{X\hat{Z},k}_{t\wedge\tilde{T}_n}(\hat{f})\right] \xrightarrow{k\to\infty} 0 \text{ in } L^1(\mathbb{P}). \tag{5.8}$$

But this follows from (5.7) and the fact that $|||\,\mathbb{1}_{[0,t\wedge\tilde{T}_n]}\hat{h}_k\,|||_{\mathbf{M}}$ converges against zero, if k goes to infinity. □

5.2 Uniform Convergence over a compact time Interval

We will extend the result of the previous section to uniform convergence over a compact time interval. For this we will not only make use of Proposition 5.1.4 from the last section, but we also apply the semi-martingale decomposition of Case III.a from Proposition 4.4.1.

Definition 5.2.1. *For the following proof we define for a $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ and an element $\hat{g} \in B(\mathbb{R}^d \times \mathbb{R})$ the function $\hat{g}^{\Delta,h} : \Omega \times \mathbb{R}^d \times \mathbb{R} \times [0,\infty) \times [0,\infty) \to \mathbb{R}$ by setting*

$$\hat{g}^{\Delta,h}(\omega,x,z,p,s) := \hat{g}(x, z + h(\omega,x,p,s)) - \hat{g}(x,z)$$

for all $\omega \in \Omega, x \in \mathbb{R}^d, z \in \mathbb{R}, p,s \in [0,\infty)$.

The next technical lemma is needed to apply Fubini's theorem during the proof of the main theorem of this section.

Lemma 5.2.2. *Assume that $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ and $\hat{g} \in C^2_b(\mathbb{R}^d \times \mathbb{R})$ and $\hat{g}^{\Delta,h}$ are as in Definition 5.2.1 and assume that $\tilde{T}_n := T_n \wedge \tau^Y_n$ for $n \in \mathbb{N}$, where $(T_n)_{n=1}^\infty$ is a localizing sequence of h in $\mathcal{L}^1_{loc}(\mathbf{M})$ and $\tau^Y_n := \inf\{s \geq 0 : Y_s \geq n\}$. If we define*

$$\big((X_i, Z_i, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ}, \mathbf{Q}^{XZ}\big) = \mathbb{I}_0\big[h\big],$$

then it holds that for all $n \in \mathbb{N}$ that

$$\int_0^\infty \int_0^\infty \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_n]} \boldsymbol{\Xi}^{XZ}_s(|\hat{g}^{\Delta,h}(\,\cdot\,, p, s)|)\right] dpds < \infty. \tag{5.9}$$

Further there exists a set $\Gamma \subset \Omega \times [0, \infty)$ of full measure such that it holds for all $(\omega, s) \in \Gamma$:

$$\int_{\hat{E}} \int_0^\infty \hat{g}^{\Delta,h}(\omega, x, z, p, s) dp \boldsymbol{\Xi}^{XZ}_s(\omega, dx, dz) = \int_0^\infty \boldsymbol{\Xi}^{XZ}_s(\omega, \hat{g}^{\Delta,h}(\omega, \cdot, \cdot, p, s)) dp, \tag{5.10}$$

where $\hat{E} = \mathbb{R}^d \times \mathbb{R}$ and where we switched the order of integration from $dp\boldsymbol{\Xi}^{XZ,k}(dx, dz)$ to $\boldsymbol{\Xi}^{XZ,k}(dx, dz)dp$.

Proof. Since $\hat{g} \in C^2_b(\mathbb{R}^d \times \mathbb{R})$, we know that $\hat{g}$ is Lipschitz-continuous with constant $||\nabla(\hat{g})||_\infty < \infty$, see Definition 2.1.1. From this we can conclude that

$$\begin{aligned}
&\mathbb{E}\bigg[\int_0^{\tilde{T}_n} \int_0^\infty \boldsymbol{\Xi}^{XZ}(|\hat{g}^{\Delta,h}(\,\cdot\,, p, s)|)\, dpds\bigg] \\
&\qquad \leq \ ||\nabla(\hat{g})||_\infty \mathbb{E}\bigg[\int_0^{\tilde{T}_n} \int_{\mathbb{R}^d} \int_0^\infty |h(x, p, s)| dp \boldsymbol{\Xi}^X_s(dx) ds\bigg]
\end{aligned}$$

and the second line is finite due to Lemma 4.1.3. Since the integral is finite for all $t \geq 0$ and all $n \in \mathbb{N}$, it is possible to conclude that there exists a set $\Gamma \subset \Omega \times [0, \infty)$ of full measure such that it holds for all $(\omega, s) \in \Gamma$:

$$\int_0^\infty |\boldsymbol{\Xi}^{XZ,k}(\hat{g}^{\Delta,h}(\,\cdot\,, p, s))|\, dp < \infty. \tag{5.11}$$

The identity of (5.10) for all $(\omega, s) \in \Gamma$ follows from Fubini. □

Proposition 5.2.3. *We assume that $(h_k)_{k=1}^\infty$ and h_∞ are elements of $\mathcal{L}^1_{loc}(\mathbf{M})$ with common localizing sequence $(T_n)_{n=1}^\infty$, and assume that $(h_k)_{k=1}^\infty$ converges against h_∞ locally in the sense that*

$$|||\, \mathbb{1}_{[0,T_n]}(h_k - h_\infty)\, |||_{\mathbf{M}} \overset{k\to\infty}{\longrightarrow} 0. \tag{5.12}$$

When we define the processes $(\boldsymbol{\Xi}^{XZ,k})_{k=1}^\infty$ and $\boldsymbol{\Xi}^{XZ,\infty}$ by setting

$$\big((X_i, Z^k_i, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ,k}, \boldsymbol{\Xi}^{XZ,k}, \mathbf{Q}^{XZ,k}\big) = \mathbb{I}_0\big[h_k\big],\ k \in \mathbb{N} \cup \{\infty\},$$

and assume that $\hat{g} \in C^{2,+}_b(\mathbb{R}^d \times \mathbb{R})$, then $\boldsymbol{\Xi}^{XZ,k}(\hat{g})$ converges locally in probability against $\boldsymbol{\Xi}^{XZ,\infty}(\hat{g})$, indeed for all $n, m \in \mathbb{N}$ and $\lambda > 0$ holds

$$\lim_{k\to\infty} \mathbb{P}\left[\sup_{s\leq m} \left|\boldsymbol{\Xi}^{XZ,k}_{s\wedge\tilde{T}_n}(\hat{g}) - \boldsymbol{\Xi}^{XZ,\infty}_{s\wedge\tilde{T}_n}(\hat{g})\right| \geq \lambda\right] = 0,$$

where $\tilde{T}_n := T_n \wedge \tau^Y_n$ with $\tau^Y_n := \inf\{t \geq 0 : Y_t \geq n\}$.

Proof. We fix $n \in \mathbb{N}$ and $m \in \mathbb{N}$ for the rest of the proof. If $\hat{f} \in C_b^2(\mathbb{R}^d \times \mathbb{R})$, then $\hat{f}$ is an element of $\mathrm{Lip}(\Omega \times \mathbb{R}^d \times \mathbb{R}, \mathcal{F}_\tau^{\mathbf{\Xi},\mathbb{W}})$, see Definition 5.1.1, so from Proposition 5.1.4 it follows

$$\mathbb{E}\left[\left|\mathbf{\Xi}_{t\wedge\tilde{T}_n}^{XZ,k}(\hat{f}) - \mathbf{\Xi}_{t\wedge\tilde{T}_n}^{XZ,\infty}(\hat{f})\right|\right] \stackrel{k\to\infty}{\longrightarrow} 0,\ t \geq 0. \tag{5.13}$$

We also know from Proposition 4.4.1 that we can find for each $k \in \mathbb{N} \cup \{\infty\}$ two processes $\hat{M}^k$ and $\hat{A}^k$ such that

$$\mathbf{\Xi}_t^{XZ,k}(\hat{g}) = \mathbf{\Xi}_0^{XZ,k}(\hat{g}) + \hat{M}_t^k + \hat{A}_t^k,\ \ t \in [0,\infty), \tag{5.14}$$

where $\hat{M}^k$ is a local $\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}$-martingale with localizing sequence $(\tilde{T}_n, n \in \mathbb{N})$ (note that the localizing sequence is for all $k \in \mathbb{N}$ the same) and $\hat{A}^k$ is a process with finite variation given by

$$\hat{A}_t^k = \int_0^t \left[\mathbf{\Xi}_s^{XZ,k}(B_X(\hat{g})) - b\mathbf{\Xi}_s^{XZ,k}(\hat{g}) + \int_0^\infty \mathbf{\Xi}_s^{XZ,k}(\hat{g}^{\Delta,h_k}(\,\cdot\,,\cdot\,,p,s))\,dp\right]ds, \tag{5.15}$$

where $\hat{g}^{\Delta,h_k}$ is given by Definition 5.2.1. Based on the above decomposition of $\mathbf{\Xi}_t^{XZ,k}(\hat{g})$, we can conclude that $\sup_{s\leq m}|\mathbf{\Xi}_{s\wedge\tilde{T}_n}^{XZ,k}(\hat{g}) - \mathbf{\Xi}_{s\wedge\tilde{T}_n}^{XZ,\infty}(\hat{g})|$ has the upper bound

$$\mathbf{U}^k(m) := \left|\mathbf{\Xi}_0^{XZ,k}(\hat{g}) - \mathbf{\Xi}_0^{XZ,\infty}(\hat{g})\right| + \sup_{s\leq m}\left|\hat{M}_{s\wedge\tilde{T}_n}^k - \hat{M}_{s\wedge\tilde{T}_n}^\infty\right| + \sup_{s\leq m}\left|\hat{A}_{s\wedge\tilde{T}_n}^k - \hat{A}_{s\wedge\tilde{T}_n}^\infty\right|. \tag{5.16}$$

This upper bound tells us that it is sufficient for our claim to show that $\mathbb{E}[\mathbf{U}^k(m)]$ is converging to 0 in probability, when k goes to infinity. Applying (5.13) to the first term of (5.16) we get that

$$\mathbb{E}\left[\left|\mathbf{\Xi}_0^{XZ,k}(\hat{g}) - \mathbf{\Xi}_0^{XZ,\infty}(\hat{g})\right|\right] \stackrel{k\to\infty}{\longrightarrow} 0, \tag{5.17}$$

which implies the convergence in probability. The remaining terms are more complicated. We begin by applying the Doob inequality, see Theorem 2.1.7 in [42], to the $\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}$-martingales $\hat{M}_{\cdot\wedge\tilde{T}_n}^k - \hat{M}_{\cdot\wedge\tilde{T}_n}^\infty$, which gives for all constant $\lambda > 0$:

$$\mathbb{P}\left[\sup_{s\leq m}\left|\hat{M}_{s\wedge\tilde{T}_n}^k - \hat{M}_{s\wedge\tilde{T}_n}^\infty\right| \geq \lambda\right] \leq \frac{1}{\lambda}\mathbb{E}\left[\left|\hat{M}_{m\wedge\tilde{T}_n}^k - \hat{M}_{m\wedge\tilde{T}_n}^\infty\right|\right]. \tag{5.18}$$

Coming back to the bound $\mathbf{U}^k(m)$ in (5.16), (5.17) and (5.18) tell us that the convergence of $\mathbf{U}^k(m)$ to 0 in probability will follow from:

$$\lim_{k\to\infty}\mathbb{E}\left[\left|\hat{M}_{m\wedge\tilde{T}_n}^k - \hat{M}_{m\wedge\tilde{T}_n}^\infty\right|\right] = 0, \qquad \lim_{k\to\infty}\mathbb{E}\left[\sup_{s\leq m}\left|\hat{A}_{s\wedge\tilde{T}_n}^k - \hat{A}_{s\wedge\tilde{T}_n}^\infty\right|\right] = 0. \tag{5.19}$$

By reordering the term in (5.14) we get that $\mathbb{E}[|\hat{M}_{m\wedge\tilde{T}_n}^k - \hat{M}_{m\wedge\tilde{T}_n}^\infty|]$ is bounded by

$$\mathbb{E}\left[\left|\mathbf{\Xi}_{m\wedge\tilde{T}_n}^{XZ,k}(\hat{g}) - \mathbf{\Xi}_{m\wedge\tilde{T}_n}^{XZ,\infty}(\hat{g})\right|\right] + \mathbb{E}\left[\left|\mathbf{\Xi}_0^{XZ,k}(\hat{g}) - \mathbf{\Xi}_0^{XZ,\infty}(\hat{g})\right|\right] + \mathbb{E}\left[\left|\hat{A}_{m\wedge\tilde{T}_n}^k - \hat{A}_{m\wedge\tilde{T}_n}^\infty\right|\right]. \tag{5.20}$$

Since the first two summands of (5.20) converge against zero by (5.13), we can prove the convergence of both terms in (5.19) by proving that

$$\mathbb{E}\left[\sup_{s\leq m}\left|\hat{A}_{s\wedge\tilde{T}_n}^k - \hat{A}_{s\wedge\tilde{T}_n}^\infty\right|\right] \stackrel{k\to\infty}{\longrightarrow} 0. \tag{5.21}$$

For this purpose we define for each $k \in \mathbb{N}$ the processes $\hat{A}^{1,k}$ and $\hat{A}^{2,k}$ by setting for $t \in [0, \infty)$:

$$\hat{A}_t^{1,k} := \int_0^t |\Xi_s^{XZ,k}(B_X(\hat{g})) - \Xi_s^{XZ,\infty}(B_X(\hat{g}))| + b|\Xi_s^{XZ,k}(\hat{g}) - \Xi_s^{XZ,\infty}(\hat{g})|\, ds, \tag{5.22}$$

$$\hat{A}_t^{2,k} := \int_0^t \int_0^\infty \big|\Xi_s^{XZ,k}(\hat{g}^{\Delta,h_k}(\,\cdot\,, p, s)) - \Xi_s^{XZ,\infty}(\hat{g}^{\Delta,h_\infty}(\,\cdot\,, p, s))\big|\, dpds. \tag{5.23}$$

The reason why we have defined these processes is that these are non-decreasing and it holds

$$\sup_{s \le m} \left| \hat{A}^k_{s \wedge \tilde{T}_n} - \hat{A}^\infty_{s \wedge \tilde{T}_n} \right| \le \hat{A}^{1,k}_{m \wedge \tilde{T}_n} + \hat{A}^{2,k}_{m \wedge \tilde{T}_n}, \tag{5.24}$$

so we just need to prove $\mathbb{E}\big[\hat{A}^{1,k}_{m \wedge \tilde{T}_n}\big]$ and $\mathbb{E}\big[\hat{A}^{2,k}_{m \wedge \tilde{T}_n}\big]$ converges against 0, when k goes to infinity to in order to prove (5.21). The theorem of Tonelli gives us

$$\mathbb{E}\big[\hat{A}^{1,k}_{m \wedge \tilde{T}_n}\big] = \int_0^t \mathbb{E}\big[|\Xi_s^{XZ,k}(B_X(\hat{g})) - \Xi_s^{XZ,\infty}(B_X(\hat{g}))| + b|\Xi_s^{XZ,k}(\hat{g}) - \Xi_s^{XZ,\infty}(\hat{g})|\big]\, ds.$$

The inner expression of the above integral converges for each s against 0, when k goes to infinity, due to (5.13). Further the inner expression is bounded by

$$2||B_X(\hat{g})||_\infty \mathbb{E}[Y_s] + |b|||\hat{g}||_\infty \mathbb{E}[Y_s]$$

and since $\int_0^t \mathbb{E}[Y_s]ds < \infty$, we can apply Lebesgue dominated convergence theorem to get that $\mathbb{E}\big[\hat{A}^{1,\infty}_m\big]$ converges against 0, when k goes to infinity.
Considering the remaining terms $\hat{A}^{2,k}$ we define the two new processes processes $\hat{A}^{3,k}$ and $\hat{A}^{4,k}$ by setting for each $t \ge 0$:

$$\hat{A}_t^{3,k} := \int_0^t \int_0^\infty \big|\Xi_s^{XZ,k}(\hat{g}^{\Delta,h_k}(\,\cdot\,,\,\cdot\,, p, s)) - \Xi_s^{XZ,k}(\hat{g}^{\Delta,h_\infty}(\,\cdot\,,\,\cdot\,, p, s))\big|\, dpds,$$
$$\hat{A}_t^{4,k} := \int_0^t \int_0^\infty \big|\Xi_s^{XZ,k}(\hat{g}^{\Delta,h_\infty}(\,\cdot\,,\,\cdot\,, p, s)) - \Xi_s^{XZ,\infty}(\hat{g}^{\Delta,h_\infty}(\,\cdot\,,\,\cdot\,, p, s))\big|\, dpds.$$

Due to the simple fact that it always holds

$$\begin{aligned}
&\big|\Xi_s^{XZ,k}(\hat{g}^{\Delta,h_k}(\,\cdot\,,\,\cdot\,, p, s)) - \Xi_s^{XZ,\infty}(\hat{g}^{\Delta,h_\infty}(\,\cdot\,,\,\cdot\,, p, s))\big| \\
&\le \big|\Xi_s^{XZ,k}(\hat{g}^{\Delta,h_k}(\,\cdot\,,\,\cdot\,, p, s)) - \Xi_s^{XZ,k}(\hat{g}^{\Delta,h_\infty}(\,\cdot\,,\,\cdot\,, p, s))\big| \\
&\quad + \big|\Xi_s^{XZ,k}(\hat{g}^{\Delta,h_\infty}(\,\cdot\,,\,\cdot\,, p, s)) - \Xi_s^{XZ,\infty}(\hat{g}^{\Delta,h_\infty}(\,\cdot\,,\,\cdot\,, p, s))\big|,
\end{aligned}$$

we can conclude that $\hat{A}^{2,k}(t) \le \hat{A}^{3,k}(t) + \hat{A}^{4,k}(t)$, so we will show that both terms convergence in $L^1(\mathbb{P})$ to 0. For the first term $\hat{A}^{3,k}$, we recall that

$$||\nabla(\hat{g})||_\infty := \sup_{(x,z) \in \mathbb{R}^d \times \mathbb{R}} |\nabla(\hat{g})(x, z)|$$

must be finite, because $\hat{g} \in C_b^2(\mathbb{R}^d \times \mathbb{R})$. Therefore $\hat{g}$ is Lipschitz-continuous with constant $||\nabla(\hat{g})||_\infty$, which gives us the upper bound

$$\begin{aligned}
\int_0^\infty |\hat{g}^{\Delta,h_k}(x, z, p, s) - \hat{g}^{\Delta,h_\infty}(x, z, p, s)|\, dp &\le \int_0^\infty |\hat{g}(x, z + h_k(x, p, s)) - \hat{g}(x, z + h_\infty(x, p, s))|\, dp \\
&\le ||\nabla(\hat{g})||_\infty \int_0^\infty |h_k(x, p, s) - h_\infty(x, p, s)|\, dp.
\end{aligned}$$

Hence for all $k \in N$ holds that

$$\mathbb{E}\big[\hat{A}^{3,k}_{m\wedge\tilde{T}_n}\big] \leq ||\nabla(\hat{g})||_\infty \mathbb{E}\bigg[\int_0^{m\wedge\tilde{T}_n} \int_{\overline{E}} \int_0^\infty |h_k(x,p,s) - h_\infty(x,p,s)|\, dp\, \boldsymbol{\Xi}^X_s(dx)\, ds\bigg],$$

where $\overline{E} = \mathbb{R}^d \times \mathbb{R}$. Since the expression inside of the integral does not contain an expression with the variable z, we could integrate with respect to the measure $dp\, \boldsymbol{\Xi}^X_s(dx)\, ds$ instead of $dp\, \boldsymbol{\Xi}^{XZ}_s(dx)\, ds$. We can rewrite the last expression to

$$\int_0^m \int_{\overline{E}} \int_0^\infty \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_n]}(s) Y_s |h_k(x,p,s) - h_\infty(x,p,s)| \mathbf{Q}^X_s(dx)\right] ds,$$

where we used $\boldsymbol{\Xi}^X_s = Y_s \mathbf{Q}^X_s$. Recalling the definition of the extinction time $\mathcal{T}_{EX} := \inf\{s \geq 0 : Y_s = 0\}$, the stopping time $\tau^Y_n := \inf\{s \geq 0 : Y_s \geq n\}$ and that $\tilde{T}_n := T_n \wedge \tau^Y_n$ that the above integral is bounded by

$$n|||\, \mathbb{1}_{[0,\tilde{T}_n]}(h_k - h_\infty)\, |||_{\mathbf{M}} \leq n|||\, \mathbb{1}_{[0,T_n)}(h_k - h_\infty)\, |||_{\mathbf{M}}.$$

This upper bound together with Assumption (5.12) allows us to conclude that

$$\mathbb{E}\big[\hat{A}^{3,k}_{m\wedge\tilde{T}_n}\big] \overset{k\to\infty}{\longrightarrow} 0. \tag{5.25}$$

To show the same statement is also true for $\mathbb{E}\big[\hat{A}^{4,k}_{t\wedge\tilde{T}_n}\big]$ we define the collection of functions $(\hat{\mathbf{w}}_k)_{k=1}^\infty, (\hat{\mathbf{o}}_k)_{k=1}^\infty, \hat{\mathbf{v}} : [0,\infty)^2 \to [0,\infty)$ by setting for all $p \geq 0, t \geq 0$ and $k \in \mathbb{N}$:

$$\hat{\mathbf{w}}_k(p,t) := \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_n]}(t) |\boldsymbol{\Xi}^{XZ,k}_t(\hat{g}^{\Delta,h_\infty}(\,\cdot\,,\,\cdot\,,p,t)) - \boldsymbol{\Xi}^{XZ,\infty}_t(\hat{g}^{\Delta,h_\infty}(\,\cdot\,,\,\cdot\,,p,t))|\right],$$

$$\hat{\mathbf{o}}_k(p,t) := \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_n]}(t) |\boldsymbol{\Xi}^{XZ,k}_t(\hat{g}^{\Delta,h_\infty}(\,\cdot\,,\,\cdot\,,p,t))|\right],$$

$$\hat{\mathbf{v}}(p,t) := ||\nabla(g)||_\infty \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_n]}(t) \int_{\mathbb{R}^d} |h_\infty(x,p,t)|\, \boldsymbol{\Xi}^X_t(dx)\right].$$

We have the following relationship between these functions, it holds for all $p \geq 0, t \geq 0, k \in \mathbb{N}$:

$$\hat{\mathbf{w}}_k(p,t) \leq \hat{\mathbf{o}}_k(p,t) + \hat{\mathbf{o}}_\infty(p,t) \leq 2\hat{\mathbf{v}}(p,t).$$

While the first inequality is obvious, the second one follows from the observation that $\hat{g}^{\Delta,h_\infty}(\,\cdot\,,p,s)(x,z,s) \leq ||\nabla(g)||_\infty |h(x,p,s)|$ for all k,x,z,p,s, which in turn implies for all $k \in \mathbb{N}\cup\{\infty\}$ and all $p \geq 0, t \geq 0$ that

$$\begin{aligned}\hat{\mathbf{o}}_k(p,t) &\leq ||\nabla(g)||_\infty \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_n]}(t) \int_{\mathbb{R}^{d+1}} |h(x,p,t)|\, \boldsymbol{\Xi}^{XZ,k}_t(dx,dz)\right] \\ &= ||\nabla(g)||_\infty \mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_n]}(t) \int_{\mathbb{R}^d} |h(x,p,t)|\, \boldsymbol{\Xi}^X_t(dx)\right] \\ &= \hat{\mathbf{v}}_k(p,t),\end{aligned}$$

where the first equality follows from the fact that the integrand h does not depend on z. The function $\hat{\mathbf{v}}$ is not only a majorant (up to a factor) for $(\hat{\mathbf{w}}_k)_{k=1}^\infty$ and $(\hat{\mathbf{o}}_k)_{k=1}^\infty$ but it is also integrable on $[0,\infty) \times [0,t]$ for a fixed $t \geq 0$, indeed it holds

$$\int_0^m \int_0^\infty \hat{\mathbf{v}}(p,s)\, dpds = ||\nabla(\hat{g})||_\infty \int_0^m \int_0^\infty \mathbb{E}\big[\mathbb{1}_{[0,\tilde{T}_n]}(t) |h_\infty(x,p,s)| \boldsymbol{\Xi}^X_s(dx)\big]\, dpds,$$

which is finite by the Lemma 4.1.3. Considering the functions $(\hat{\mathbf{w}}_k)_{k=1}^{\infty}$, we note that $\hat{\mathbf{g}}(\omega, x, z) = \hat{g}^{\Delta, h_\infty}(\omega, x, z, p, s)$ is by Lemma 5.1.3 an element of $\mathrm{Lip}_z(\Omega \times \mathbb{R}^d \times \mathbb{R}, \mathcal{F}_\tau^{\Xi, \mathbb{W}})$, see Definition 5.1.1. Since

$$\mathbb{E}\left[\mathbb{1}_{[0,\tilde{T}_n]}(t)\Xi_t^{XZ,k}(\hat{g}^{\Delta,h_\infty}(\,\cdot\,,\,\cdot\,,p,t))\right] < \infty$$

for all $t \in [0, \infty)$ and $k \in \mathbb{N}$, we can apply Proposition 5.1.4 to obtain:

$$\hat{\mathbf{w}}_k(p,t) \overset{k\to\infty}{\longrightarrow} 0, \quad \forall p \geq 0, t \geq 0.$$

Due to the pointwise convergence of $(\hat{\mathbf{w}}_k)_{k=1}^{\infty}$ to zero as mentioned above and due to the fact that $\hat{\mathbf{v}}$ forms an integrable majorant for the sequence $(\hat{\mathbf{w}}_k)_{k=1}^{\infty}$ we can apply Lebesgue's dominated convergence theorem and obtain:

$$\lim_{k\to\infty} \mathbb{E}\big[\hat{A}^{4,k}_{m\wedge\tilde{T}_n}\big] = \lim_{k\to\infty} \int_0^m \int_0^\infty \hat{\mathbf{w}}_k(p,s)\, dpds = \int_0^m \int_0^\infty \lim_{k\to\infty} \hat{\mathbf{w}}_k(p,s)\, dpds = 0,$$

where we have written $\mathbb{E}[\hat{A}^{4,k}_{m\wedge\tilde{T}_n}]$ as the integral $\int_0^m \int_0^\infty \hat{\mathbf{w}}_k(p,s)\, dpds$ by Tonelli's theorem. All in all we have proved that $\mathbb{E}[\hat{A}^{4,k}_{m\wedge\tilde{T}_n}]$ are converging against 0 for k going to infinity. Putting all together we can see that $\mathbb{E}[\hat{A}^{2,k}_{m\wedge\tilde{\tau}_n}]$ is converging against 0, so by (5.24). Hence (5.19) is true and we can conclude that the upper bound $\mathbf{U}_t^k$ from (5.16) is converging in probability to 0, which proves our claim. □

Corollary 5.2.4. *Assume that $(\Xi^{XZ,k}, k \in \mathbb{N} \cup \{\infty\})$ and $\hat{g}$ are as in Proposition 5.2.3, then there exists a subsequence $(\Xi^{XZ,\tilde{k}}(\hat{g}))_{\tilde{k}=1}^{\infty}$ of $(\Xi^{XZ,k}(\hat{g}))_{k=1}^{\infty}$ such that for all $m \in \mathbb{N}$ holds*

$$\sup_{s\leq m} |\Xi_s^{XZ,\tilde{k}}(\hat{g}) - \Xi_s^{XZ,\infty}(\hat{g})| \overset{\tilde{k}\to\infty}{\longrightarrow} 0 \ \ a.s.$$

Proof. From Proposition 5.2.3 we can conclude that there exists for each $n \in \mathbb{N}$ a subsequence $(\Xi^{XZ,\tilde{k}^n}(\hat{g}))_{\tilde{k}=1}^{\infty}$ of $(\Xi^{XZ,k}(\hat{g}))_{k=1}^{\infty}$ such that

$$\sup_{s\leq m} |\Xi_{s\wedge\tilde{T}_n}^{XZ,\tilde{k}^n}(\hat{g}) - \Xi_{s\wedge\tilde{T}_n}^{XZ,\infty}(\hat{g})| \overset{\tilde{k}^n\to\infty}{\longrightarrow} 0 \ \ a.s.,$$

where $(\tilde{T}_n)_{n=1}^{\infty}$ is defined as in Proposition 5.2.3. Since $(\tilde{T}_n)_{n=1}^{\infty}$ is increasing, we can choose the subsequences in such a way that these are contained in each other. We obtain the desired subsequence by a diagonal argument. □

Chapter 6

Continuity

Assume that h is an element of $\mathcal{L}^1_{loc}(\mathbf{M})$ and let us define as usual

$$\big((X_i, Z_i, U_i)_{i=1}^{\infty}, \boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ}\big) = \mathbb{I}_0\big[h\big]. \tag{6.1}$$

The goal of this chapter is to show that the intensity process $\boldsymbol{\Xi}^{XZ}$ admits a continuous modification for all possible integrands $h \in \mathcal{L}^1_{loc}(\mathbf{M})$. By continuous modification we mean a process $\hat{\boldsymbol{\Xi}}^{XZ} : \Omega \times [0, \infty) \to \mathcal{M}_f(\mathbb{R}^d \times \mathbb{R})$, which is continuous in the weak topology of $\mathcal{M}_f(\mathbb{R}^d \times \mathbb{R})$, and satisfies for all $t \geq 0$:

$$\mathbb{P}\big[\boldsymbol{\Xi}_t^{XZ}(\hat{g}) = \hat{\boldsymbol{\Xi}}_t^{XZ}(\hat{g}),\ g \in C_b^+(\mathbb{R}^d \times \mathbb{R})\big] = 1, \tag{6.2}$$

which is equivalent, since $C_b^+(\mathbb{R}^d \times \mathbb{R})$ is separating, to

$$\mathbb{P}\big[\boldsymbol{\Xi}_t^{XZ}(\Gamma) = \hat{\boldsymbol{\Xi}}_t^{XZ}(\Gamma),\ \Gamma \in \mathbb{B}(\mathbb{R}^d \times \mathbb{R})\big] = 1.$$

Further $\hat{\boldsymbol{\Xi}}^{XZ}$ should be adapted to the filtration $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$, but this follows immediately from the fact that $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$ is assumed to be complete and that $\boldsymbol{\Xi}^{XZ}$ is adapted to $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$.
This chapter is divided in two sections. In the first section we prove that the functional $\boldsymbol{\Xi}^{XZ}(\hat{g})$ with $\hat{g} \in C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$ being fixed has a continuous modification. We can use this result to derive rigorously an explicit expression for the quadratic variation of $\boldsymbol{\Xi}^{XZ}(\hat{g})$. In the second section we use this explicit expression for $\langle \boldsymbol{\Xi}^{XZ}(\hat{g}) \rangle$ to prove that $(\boldsymbol{\Xi}^{XZ}_{t \wedge \tau_k}, t \geq 0)$ forms for a suitable localizing sequence $(\tau_k)_{k=1}^{\infty}$ almost surely a tight family of measures. From the tightness we can derive that $\boldsymbol{\Xi}^{XZ}$ admits a continuous modification, which we denote by $\hat{\boldsymbol{\Xi}}^{XZ}$ in this chapter.
Since we will work in the following chapters entirely with the continuous modification $\hat{\boldsymbol{\Xi}}^{XZ}$ instead of the original process $\boldsymbol{\Xi}^{XZ}$, we will switch notation after this chapter and denote by $\boldsymbol{\Xi}^{XZ}$ the continuous modification. Further we replace $\mathbb{I}\big[h\big]$ with $\mathbb{I}_0\big[h\big]$ in definitions like (6.1) to mark that we are working with the continuous modification of the intensity process, see also Definition 6.2.6.

Besides its aesthetic value the fact that $\boldsymbol{\Xi}^{XZ}$ admits a continuous modification for every integrand $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ simplifies many arguments in the following chapters, e.g. we can see immediately that $\boldsymbol{\Xi}^{XZ}$ is predictable.

6.1 Continuity of the Functionals of the Intensity Process

The proof that $\Xi^{XZ}(\hat{g})$ admits a continuous modification, where $\hat{g} \in C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$, happens in two major steps. Before we consider a general integrand $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ in Proposition 6.1.8, we work as an intermediate step with the special case, where h is an element of the space $\mathcal{L}^{1,q}_{loc}(\mathbf{M})$ for some $q > 0$.

Definition 6.1.1. *We define $\mathcal{L}^{1,q}_{loc}(\mathbf{M}) \subset \mathcal{L}^1_{loc}(\mathbf{M})$ for a fixed $q > 0$ as the subset of those $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ for which holds $h(\omega, x, p, s) = 0$ for all $(\omega, x, p, s) \in \Omega \times \mathbb{R}^d \times [0, \infty) \times [0, \infty)$ with $p > q$.*

Obviously when we define $h^q := \mathbb{1}_{[0,q]}(p)h, q \in [0, \infty)$, for a fixed $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ with localizing sequence $(T_k, k \in \mathbb{N})$, then $h^q \in \mathcal{L}^{1,q}_{loc}(\mathbf{M})$ for each $q \in [0, \infty)$ and the sequence $(\mathbb{1}_{[0,T_k]}h^q, q \in [0, \infty))$ converges to $\mathbb{1}_{[0,T_k]}h$ in $\mathcal{L}^1(\mathbf{M})$ for each $k \in \mathbb{N}$.
The proof that $\Xi^{XZ}(\hat{g})$ admits a continuous modification, when $h \in \mathcal{L}^{1,q}_{loc}(\mathbf{M})$, is based on the following observation. If we define for each $r \geq \max\{b/a, 1\}$ the measure-valued process $\Xi^{XZ,r}$ as in Definition 2.5.2, then we should be able to prove that the sequence of processes

$$\left(\frac{1}{r}\Xi^{XZ,r}(\hat{g}), r \geq \max\{b/a, 1\}\right) \tag{6.3}$$

is converging in law against the process $\Xi^{XZ}(\hat{g})$ in the Skorohod space $\mathbb{D}([0, \infty), \mathbb{R})$ (we only consider values of r with $r \geq \max\{b/a, 1\}$ instead or $r \geq \max\{b/a, 0\}$, because we divide by r). This observation becomes useful, because the maximal jump size produced by $\frac{1}{r}\Xi^{XZ,r}(\hat{g})$ is bounded by $2r^{-1}||\hat{g}||_\infty$, which converges to 0, when r converges to infinity. The continuity of $\Xi^{XZ}(\hat{g})$ will follow from Lemma 6.1.2.

Lemma 6.1.2. *Fix $m \in \mathbb{N}$ and define the $J_m : \mathbb{D}([0, \infty), \mathbb{R}) \to \mathbb{R}$ by setting*

$$J_m(\mathbf{x}) = \sup_{t \in [0,m]} |\Delta \mathbf{x}(t)|$$

for all $\mathbf{x} \in \mathbb{D}([0, \infty), \mathbb{R})$. Further assume that $(\tilde{X}_n)_{n=1}^\infty$ and $\tilde{X}$ are processes with paths in $\mathbb{D}([0, \infty), \mathbb{R})$ and $\tilde{X}_n$ converges to $\tilde{X}$ in law on $\mathbb{D}([0, \infty), \mathbb{R})$. Then

$$\mathbb{P}[\tilde{X} \in C([0, \infty), \mathbb{R})] = 1,$$

if only if $J_m(\tilde{X}_n)$ converges in distribution against 0 for every m.

Proof. See Theorem 13.4 in [4] for the proof of the case, where $(\tilde{X}_n)_{n=1}^\infty, \tilde{X}$ are processes with paths in $\mathbb{D}([0, 1], \mathbb{R})$. With the help of the time change $\lambda(t) = t/m$ we can extend the result to the situation of $\mathbb{D}([0, m], \mathbb{R})$ and J_m. We can further extend the statement to $\mathbb{D}([0, \infty), \mathbb{R})$ by recalling that $\mathbb{P}[\tilde{X} \in C([0, \infty), \mathbb{R})] = 1$, if and only if $\mathbb{P}[\tilde{X}(\cdot \wedge m) \in C([0, m], \mathbb{R})] = 1$ for all $m \in \mathbb{N}$. □

Remark 6.1.3. *Instead of showing just convergence in path law, it may be possible to prove a stronger statement in the flavor of*

$$\mathbb{P}\left[\lim_{r \to \infty} \sup_{s \leq \tilde{T}_n} \left|\frac{1}{r}\Xi_s^{XZ,r}(\hat{g}) - \Xi^{XZ}(\hat{g})\right| = 0\right] = 1$$

by adapting the arguments of Lemma 3.4 and Lemma 3.5 from [10] to our situation. Unfortunately due to time constraints we can not follow this path.

Since it holds that $r^{-1}\Xi_t^{XZ,r}(\hat{g}) \to \Xi_t^{XZ}(\hat{g})$ almost surely for a fixed time $t \in [0,\infty)$, we just need to prove that (6.3) forms a tight family of processes in $\mathbb{D}([0,\infty),\mathbb{R})$ to show that (6.3) are converging in law against $\Xi^{XZ}(\hat{g})$ in the Skorohod space $\mathbb{D}([0,\infty),\mathbb{R})$.

Proposition 6.1.4 (Aldous's Tightness Criterion)**.** *Assume that $(\tilde{P}_n)_{n=1}^{\infty}$ is a sequence of stochastic processes. The sequence $(\tilde{P}_n)_{n=1}^{\infty}$ forms a tight sequence of processes in $\mathbb{D}([0,\infty),\mathbb{R})$, if and only if:*

1) For each $m \in \mathbb{N}$ holds

$$\lim_{\tilde{a}\to\infty} \limsup_{n\to\infty} \mathbb{P}\left[\sup_{s\le m} |\tilde{P}_s^n| \ge \tilde{a}\right] = 0. \tag{6.4}$$

2) For each $\epsilon > 0, \eta > 0$ and $m > 0$, there exist a δ_0 and an n_0 such that, if $\delta < \delta_0$ and $n \ge n_0$, and if θ is a discrete $\tilde{P}^n$-stopping time satisfying $\theta \le m$, then

$$\mathbb{P}\left[|\tilde{P}_{\theta+\delta}^n - \tilde{P}_\theta^n| \ge \epsilon\right] \le \eta. \tag{6.5}$$

Proof. See Theorem 16.10 in [4]. □

Lemma 6.1.5. *There exists a sequence of constants $(K_m)_{m=1}^{\infty}$ such that for all $m \in \mathbb{N}$ holds*

$$\mathbb{E}\left[\left(\sup_{s\le m} \frac{1}{r} Y_t^r\right)^2\right] \le K_m, \quad \forall r \ge \max\{b/a, 1\}.$$

Note that the constant does not depend on r.

Proof. Let us define the processes $(Q^r, r \ge \max\{b/a,1\})$ by setting $Q_t^r := (\sup_{s\le t} \frac{1}{r} Y_s^r)^2$ for $r \ge \max\{b/a,1\}$. We know that Y^r is a time-continuous Galton-Watson process with branching rate ra and drift b. This gives us the semi-martingale decomposition of $\frac{1}{r}Y^r$ by

$$\frac{1}{r} Y_t^0 = \frac{1}{r} Y_t^0 + \hat{M}_t^r + \int_0^t \frac{b}{r} Y_t^r \, ds,$$

where $\hat{M}^r$ is a pure jump-martingale with predictable quadratic variation

$$\langle \hat{M}^r \rangle_t = \frac{1}{r^2} \int_0^t 2arY_s^r \, ds.$$

If we define $\hat{M}_t^{r,*} := \sup_{s\le t} \hat{M}_s^r$, then we have

$$Q_t^r \le 3\left(\frac{1}{r} Y_0^r\right)^2 + 3\left(\hat{M}_t^{r,*}\right)^2 + 3\left(\int_0^t \frac{b}{r} Y_s^r ds\right)^2. \tag{6.6}$$

For $t \le m$ the Cauchy-Schwarz inequality gives us that

$$\left(\int_0^t \frac{b}{r} Y_s^r ds\right)^2 \le m \int_0^t \frac{b^2}{r^2} (Y_s^r)^2 ds \le m \int_0^t b^2 Q_s^r ds. \tag{6.7}$$

Further by the Burkholder-Davis-Gundy inequality there exists a constant c_2, which in independent from r, such that (we also apply $\mathbb{E}[Y_t^r] = \mathbb{E}[Y_0^r]e^{bt} = r\mathbb{E}[Y_0]e^{bt}$):

$$\begin{aligned} \mathbb{E}\left[\left(\hat{M}_t^{r,*}\right)^2\right] \le c_2 \mathbb{E}\left[\langle \hat{M}^r \rangle_t\right] &= \int_0^t \frac{2a}{r} \mathbb{E}\left[Y_s^r\right] ds \\ &= \int_0^t 2a\mathbb{E}\left[Y_0\right] e^{bs} ds = 2\frac{a}{b} \mathbb{E}\left[Y_0\right] (e^{bt} - 1). \end{aligned} \tag{6.8}$$

If we define the function $\hat{\mathbf{v}}^r(t) = \mathbb{E}[Q_t^r] = \mathbb{E}[(\sup_{s\le t} \frac{1}{r} Y_s^r)^2]$, we can combine (6.6), (6.7), (6.8) with $\hat{\mathbf{v}}^r(0) = \mathbb{E}[(\frac{1}{r}Y_0^r)^2] = \frac{1}{r}\mathbb{E}[Y_0] + \mathbb{E}[Y_0]^2$ to get:

$$\hat{\mathbf{v}}^r(t) \le 3\left(\mathbb{E}[Y_0] + \mathbb{E}[Y_0]^2 + \frac{2a}{|b|}\mathbb{E}\left[Y_0\right] e^{|b|m}\right) + 3m\int_0^t b^2 \hat{\mathbf{v}}_s ds, \quad t \in [0, m].$$

The Gronwall lemma tells us that $\hat{\mathbf{v}}^r$ is bounded on $[0, m]$ by:

$$\hat{\mathbf{v}}^r(t) \le Ae^{3m^2b^2} \text{ with } A := 3\left(\mathbb{E}[Y_0] + \mathbb{E}[Y_0]^2 + \frac{2a}{|b|}\mathbb{E}\left[Y_0\right] e^{|b|m}\right).$$

Hence we can choose $K_m := Ae^{4m^2b^2}$, which does not depend on r. □

Lemma 6.1.6. *Assume that $h \in \mathcal{L}^{1,q}_{loc}(\mathbf{M})$, $q > 0$, with localizing sequence $(T_n)_{n=1}^\infty$. Let us define*

$$\left((X_i, Z_i, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ}\right) = \mathbb{I}_0\left[h\right],$$

and assume that $\boldsymbol{\Xi}^{XZ,r}$ is given as in Definition 2.5.2. If we define for $\hat{g} \in C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$ and $r \ge \max\{b/a, 1\}$ the process $P^r : \Omega \times [0,\infty) \to [0,\infty)$ by

$$P_t^r := \frac{1}{r}\boldsymbol{\Xi}_t^{XZ,r}(\hat{g}), \qquad t \ge 0,$$

then $(P^r_{\cdot\wedge T_n}, r \ge \max\{b/a, 1\})$ is tight in $\mathbb{D}([0,\infty), \mathbb{R})$ for every fixed $n \in \mathbb{N}$.

Proof. We need to verify (6.4) and (6.5) of Aldous's criterion. We begin with (6.4) and note

$$\mathbb{P}\left[\left|\boldsymbol{\Xi}_t^{XZ,r}(\hat{g})\right| \le ||\hat{g}||_\infty Y_t^r,\, t \ge 0\right] = 1.$$

If (K_m) is the sequence of constants from Lemma 6.1.5, the Jensen inequality allows us to conclude that:

$$\mathbb{E}\left[\sup_{s\le m} \frac{1}{r}\boldsymbol{\Xi}_s^{XZ,r}(\hat{g})\right] \le ||\hat{g}||_\infty \sqrt{\mathbb{E}\left[\sup_{s\le m} \frac{1}{r} Y_s^r\right]^2} \tag{6.9}$$

$$\le ||\hat{g}||_\infty \sqrt{\mathbb{E}\left[\left(\sup_{s\le m} \frac{1}{r} Y_s^r\right)^2\right]} \le ||\hat{g}||_\infty \sqrt{K_m}. \tag{6.10}$$

Combining the above inequalities with the Markov inequality, it follows

$$\lim_{\tilde{a}\to\infty} \limsup_{r\to\infty} \mathbb{P}\left[\sup_{s\le m} |\tilde{P}_s^r| \ge \tilde{a}\right] = \lim_{\tilde{a}\to\infty} \frac{1}{\tilde{a}}\mathbb{E}\left[\sup_{s\le m} \frac{1}{r}||\hat{g}||_\infty Y_s^r\right] \le \lim_{\tilde{a}\to\infty} \frac{||\hat{g}||_\infty\sqrt{K_m}}{\tilde{a}} = 0.$$

For the second part, see (6.5), we will prove that for every $m \in \mathbb{N}$ there exists a continuous non-increasing function $F_m : [0,1) \to [0,\infty)$ with $F_m(0) = 0$ such that for all $r \ge \max\{b/a, 1\}$ and for all $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W},r}$-stopping times θ with $\theta \le m$ holds

$$\mathbb{E}\left[\left(P^r_{\theta+\delta} - P^r_\theta\right)^2\right] \le F_m(\delta), \quad \delta \in [0,1), \tag{6.11}$$

note that by showing the existence of F_m, we also show that $P^r_{\theta+\delta} - P^r_\theta$ admits a second moment. Further the existence of such a function, the second part of Aldous's criterion follows by the application of the Markov's inequality, indeed choose $\delta_0 > 0$ for a given $\epsilon > 0$ and $\eta > 0$ such that $F(\delta_0) \leq \epsilon^2\eta$, due to (6.11) and the fact that F_m is non-increasing, we have for all $\delta < \delta_0$:

$$\mathbb{P}\big[\big|P^r_{\theta+\delta} - P^r_\theta\big| \geq \epsilon\big] = \mathbb{P}\big[\big(P^r_{\theta+\delta} - P^r_\theta\big)^2 \geq \epsilon^2\big] \leq \frac{\mathbb{E}\big[\big(P^r_{\theta+\delta} - P^r_\theta\big)^2\big]}{\epsilon^2} \leq \frac{F_m(\delta_0)}{\epsilon^2} \leq \eta.$$

In order to prove the existence of F_m we note that we can use the semi-martingale decomposition of $\mathbf{\Xi}^{XZ,r}(\hat{g})$ from Proposition 4.5.1 to derive a semi-martingale decomposition for P^r (recall that $P^r := \frac{1}{r}\mathbf{\Xi}^{XZ,r}(\hat{g})$) by dividing the processes given by Proposition 4.5.1 by r, indeed we know that there exists

$$P^r_t = P^r_0 + \hat{M}^r_t + \hat{A}^r_t,$$

where $\hat{M}^r$ is a local $\mathcal{F}^{\mathbf{\Xi},\mathbb{W},r}$-martingale with localizing sequence $(\tilde{T}_n)_{n=1}^\infty$ and $\hat{A}^r$ being a continuous predictable processes with finite variance. Fixing a $m \in \mathbb{N}$, a $\mathcal{F}^{\mathbf{\Xi},\mathbb{W},r}$-stopping time $\theta \leq m$ a.s. and $\delta > 0$, we get:

$$\big(P^r_{\theta+\delta} - P^r_\theta\big)^2 \leq 2\left(\hat{M}^r_{\theta+\delta} - \hat{M}^r_\theta\right)^2 + 2\left(\hat{A}^r_{\theta+\delta} - \hat{A}^r_\theta\right)^2. \tag{6.12}$$

It is clear, that we need to find upper bounds for the two terms on the right side. The predictable quadratic variation of $\hat{M}^r$ is identical with the one of P^r (recall that $\hat{A}^r$ is predictable and has finite variation, hence quadratic variation is 0), and so we have

$$\langle\hat{M}^r\rangle_t = \langle P^r\rangle_t = \frac{1}{r^2}\langle\mathbf{\Xi}^{XZ,r}(\hat{g})\rangle_t.$$

From Lemma 4.5.2 we can now conclude that

$$\begin{aligned}\langle\hat{M}^r\rangle_{\theta+\delta} - \langle\hat{M}^r\rangle_\theta \leq & \int_\theta^{\theta+\delta} \frac{1}{r^2}\mathbf{\Xi}^{XZ,r}_s\left(\nabla_x(\hat{g})^T B^{cov}_X \nabla_x(\hat{g})\right) ds \\ & + \int_\theta^{\theta+\delta} \int_{\hat{E}} \int_{\mathbb{R}^d} \left(\hat{g}(x+y,z) - \hat{g}(x,z)\right)^2 B^\eta_X(dy)\, \frac{1}{r^2}\mathbf{\Xi}^{XZ,r}_s(dx,dz) \\ & + \int_\theta^{\theta+\delta} \int_{\hat{E}} \int_0^\infty \left(\hat{g}(x,z+h(x,p,s)) - \hat{g}(x,z)\right)^2 dp\, \frac{1}{r^2}\mathbf{\Xi}^{XZ,r}_s(dx,dz)ds \\ & + \int_\theta^{\theta+\delta} \frac{2a}{r} Y^r_{s\text{-}} ds,\end{aligned}$$

where $\hat{E} = \mathbb{R}^d \times \mathbb{R}$ and B^{cov}_X is the covariation matrix of the Brownian part and B^η_X is the Lévy-measure of the spatial motion, see Assumption 1.2.3. As a Lévy-measure B^η_X satisfies $\int ||y||^2 B^\eta_X(dy) < \infty$. Because of this and since $\hat{g} \in C^2_b(\mathbb{R}^d \times \mathbb{R})$ we have

$$\sup_{(x,z)\in\mathbb{R}^d\times\mathbb{R}} \left|\nabla_x(\hat{g})^T(x,z) B^{cov} \nabla_x(\hat{g})(x,z)\right| \leq K_{\hat{g}},$$

$$\sup_{(x,z)\in\mathbb{R}^d\times\mathbb{R}} \int_{\mathbb{R}^d} \left(\hat{g}(x+y,z) - \hat{g}(x,z)\right)^2 B^\eta(dy) \leq K_{\hat{g}},$$

when we define the constant $K_{\hat{g}}$ (using $(\hat{g}(x+y,z) - \hat{g}(x,z))^2 \leq ||D\hat{g}||_\infty ||y||^2$) by

$$K_{\hat{g}} := ||D\hat{g}||_\infty(1 + \int ||y||^2 B^\eta_X(dy)).$$

Next we recall that $h \in \mathcal{L}^{1,q}_{loc}(\mathbf{M})$, which means $h(x,p,s) = 0$, if $p \geq q$, and so

$$\int_0^\infty \left(\hat{g}(x,z+h(x,p,s)) - \hat{g}(x,z)\right)^2 dp \leq 4q||\hat{g}||^2_\infty.$$

Since the stopping time θ is smaller than m, $\langle \hat{M}^r \rangle_{\theta+\delta} - \langle \hat{M}^r \rangle_\theta$ can be bounded by

$$\int_\theta^{\theta+\delta} \left[2K_{\hat{g}} + 4q||\hat{g}||_\infty^2\right] \frac{1}{r^2} Y_s^r + \frac{2a}{r} Y_{s\text{-}}^r ds \leq \delta \left[2K_{\hat{g}} + 4q||\hat{g}||_\infty^2 + 2a\right] \sup_{s \leq m+\delta} \frac{1}{r} Y_s^r,$$

where we used that $\sup_{s \leq m+\delta} Y_s^r$ is almost surely identical with $\sup_{s \leq m+\delta} Y_{s\text{-}}^r$. As in (6.9) we can derive the upper bound

$$\frac{1}{r} \mathbb{E}\left[\sup_{s \leq m+\delta} Y_s^r \right] \leq \sqrt{K_{m+1}}, \quad r \geq \max\{b/a, 1\}.$$

From the above we can conclude that the local $\mathcal{F}^{\Xi,\mathbb{W},r}$-martingale

$$\hat{M}^r_{(\theta+\delta)\wedge t} - \hat{M}^r_{\theta \wedge t}, \quad t \geq 0,$$

is actually a proper martingale on the interval $[0, m+\delta]$ with second moments for all $r \geq \max\{b/a, 1\}$, because it holds for any stopping time θ which is smaller than m (and so $\theta + \delta \leq m + \delta$) that:

$$\mathbb{E}\left[\left(\hat{M}^r_{\theta+\delta} - \hat{M}^r_\theta \right)^2 \right] = \mathbb{E}\left[\langle \hat{M}^r \rangle_{(m+\delta)} - \langle \hat{M}^r \rangle_\theta \right] \leq \delta \left[2K_{\hat{g}} + 4q||\hat{g}||_\infty^2 + 2a\right] \sqrt{K_{m+1}}. \tag{6.13}$$

This completes the search for an upper bound for the first term of the right side of (6.12). Now we need to find an upper bound for the second term involving the finite variance process $\hat{A}^r$. Proposition 4.5.1 tells us that $\hat{A}^r$ is for fixed t given by (note that we have to divide by r for our current situation):

$$\hat{A}^r(t) = \int_0^t \frac{1}{r} \Xi_{s\text{-}}^{XZ,r} \left(B(\hat{g}) + b\hat{g}\right) + \int_{\hat{E}} \left[\int_0^q \hat{g}(x, z + h(x,p,s)) - \hat{g}(x,z)\, dp \right] \frac{1}{r} \Xi_{s\text{-}}^{XZ,r}(dx, dz) ds,$$

where $\hat{E} = \mathbb{R}^d \times \mathbb{R}$. When we define $K_{\hat{g}} := ||B_X(\hat{g}) + b\hat{g}||_\infty + 2q||\hat{g}||_\infty^2$, then $\hat{A}^r_{\theta+\delta} - \hat{A}_\theta$ is pointwise bounded by

$$\int_\theta^{\theta+\delta} K_{\hat{g}} \frac{1}{r} Y_s^r ds \leq \delta K_{\hat{g}} \sup_{s \leq m+\delta} \frac{1}{r} Y_s,$$

where $||\cdot||_\infty$ denotes the supremum. This upper bounds give us together with $\mathbb{E}[\sup_{s \leq m+\delta} \frac{1}{r^2}(Y_s^r)^2] \leq K_{m+1}$ and $r \geq \max\{b/a, 1\}$:

$$\mathbb{E}\left[\left(\hat{A}^r(\theta+\delta) - \hat{A}^r(\theta+\delta) \right)^2 \right] \leq \mathbb{E}\left[\left(\int_\theta^{\theta+\delta} K_{\hat{g}} \frac{1}{r} Y_s^r ds \right)^2 \right] \leq \delta^2 K_{\hat{g}}^2 K_{m+1}. \tag{6.14}$$

If now recall (6.12) and take on both sides the expectation, then we can conclude based on (6.13) and (6.14) that the desired function F_m is given by

$$F_m(\delta) = \delta \left[K_{\hat{g}} + 4q||\hat{g}||_\infty^2 + 2a\right] \sqrt{K_{m+1}} + \delta^2 K_{\hat{g}}^2 K_{m+1}.$$

□

Proposition 6.1.7. *Fix a* $q > 0$ *and assume that* $h \in \mathcal{L}^{1,q}_{loc}(\mathbf{M})$. *Let us define*

$$\left((X_i, Z_i, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ}\right) = \mathbb{I}_0[h],$$

then for every $\hat{g} \in C_b^2(\mathbb{R}^d \times \mathbb{R})$ *the process* $P : \Omega \times [0,\infty) \to [0,\infty)$ *given by* $P_t := \Xi_t^{XZ}(\hat{g})$ *admits a **continuous** modification adapted to* $\mathcal{F}^{\Xi,\mathbb{W}}$.

Proof. From the semi-martingale decomposition of P given by Proposition 4.4.1 we can conclude that there exists a càdlàg modification P^{cad} of P, indeed there exists a process P^{cad} with

$$\mathbb{P}[P^{cad} \in \mathbb{D}([0,\infty),\mathbb{R})] = 1 \text{ and } \mathbb{P}[P^{cad}_t = P_t] = 1, t \geq 0.$$

Assuming that P^r and $\mathbf{\Xi}^{XZ,r}$ are defined for $r > \max\{b/a, 1\}$ as in Lemma 6.1.6, the point *ii)* of Lemma 2.6.1 tells us that it holds for all $t \geq 0$ that

$$P^r_t := \frac{1}{r}\mathbf{\Xi}^{XZ,r}_t(\hat{g}) \overset{r\to\infty}{\longrightarrow} \mathbf{\Xi}^{XZ}_t(\hat{g}) = P_t = P^{cad}_t \, a.s.$$

So the finite dimensional distributions of $(P^r, r \geq \max\{b/a, 1\}$ converge against the one of P^{cad}, combined with the tightness in Lemma 6.1.6, this implies that $(P^r_{\cdot\wedge T_n}, r \geq \frac{b}{a} \vee 1)$ converges in law against the process $P^{cad}_{\cdot\wedge T_n}$ in the space $\mathbb{D}([0,\infty),\mathbb{R})$.
$\mathbf{\Xi}^{XZ,r}(\hat{g})$ only jumps, if it at least one of the processes

$$(\hat{g}(X_i, Z_i)\mathbb{1}_{[0,r]}(U_i))_{i=1}^{\infty}$$

makes a jump. Obviously the jump size of $\hat{g}(X_i, Z_i)\mathbb{1}_{[0,r]}(U_i)$ is bounded by $2||\hat{g}||_\infty$ for each $i \in \mathbb{N}$, hence a jump of $\mathbf{\Xi}^{XZ,r}(\hat{g})$ with a size bigger than $2||\hat{g}||_\infty$ is only possible, if two or more processes of $(\hat{g}(X_i, Z_i)\mathbb{1}_{[0,r]}(U_i))_{i=1}^{\infty}$ jump at the same moment. But that can only happen, when a new particle is born. Since two different particles are never born at the same time, the jump of $\mathbf{\Xi}^{XZ,r}(\hat{g})$ at the birth of a new particle is given by $\hat{g}(X_i(t\text{-}), Z_i(t\text{-}))$, where t is the moment of birth and i the index of the parent. Hence the jumps due to births are bounded by $||\hat{g}||_\infty$.
As a consequence the maximal jump height of the processes $\frac{1}{r}\mathbf{\Xi}^{XZ,r}(\hat{g})$ is for each r bounded by $2r^{-1}||\hat{g}||_\infty$, which converges against 0, when r goes to infinity. So by Lemma 6.1.2, it follows $\mathbb{P}[P^{cad}_{\cdot\wedge T_n} \in C([0,\infty),\mathbb{R})] = 1$. Since this true for all $n \in \mathbb{N}$ and $T_n \to \infty$, when n goes to infinity, it follows that $\mathbb{P}[P^{cad}_{\cdot\wedge T_n} \in C([0,\infty),\mathbb{R})] = 1$. Setting $\Gamma := \{P^{cad}_{\cdot\wedge T_n} \in C([0,\infty),\mathbb{R})\}$ we get the continuous modification P^{cont} of P by $P^{cont} = \mathbb{1}_\Gamma P$, which is adapted to $\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}$, since $\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}$ is complete. □

We are now going to extend the continuity of $\mathbf{\Xi}^{XZ}(\hat{g})$ to the case of a general $h \in \mathcal{L}^1_{loc}(\mathbf{M})$.

Proposition 6.1.8. *Assume that $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ and let us define*

$$\big((X_i, Z_i, U_i)_{i=1}^{\infty}, \boldsymbol{\xi}^{XZ}, \mathbf{\Xi}^{XZ}\big) = \mathbb{I}_0\big[h\big],$$

then for every $\hat{g} \in C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$ the process $P : \Omega \times [0,\infty) \to [0,\infty)$ given by $P_t := \mathbf{\Xi}^{XZ}_t(\hat{g})$ admits a ***continuous*** *modification.*

Proof. Assume that $(T_n)_{n=1}^{\infty}$ is a localizing sequence of h in $\mathcal{L}^1_{loc}(\mathbf{M})$. We define for each $q > 0$ the function $h^q \in \mathcal{L}^{1,q}_{loc}(\mathbf{M})$ by setting $h^q(\omega, x, p, s) = h(\omega, x, p, s)\mathbb{1}_{[0,q]}(p)$, then $(T_n)_{n=1}^{\infty}$ is not only a localizing sequence of h^q in $\mathcal{L}^1_{loc}(\mathbf{M})$, but it also holds for all $n \in \mathbb{N}$:

$$\begin{aligned}&\lim_{q\to\infty} |||\, \mathbb{1}_{[0,T_n]}(h - h^q)\, |||_{\mathbf{M}} \\ &= \int_0^t \int_q^\infty \mathbb{E}\left[\mathbb{1}_{[0,\mathcal{T}_{EX}\wedge T_n]} \lim_{q\to\infty} |(h(X_1(s), p, s) - h^q(X_1(s), p, s)|\right] dpds \\ &= 0,\end{aligned} \tag{6.15}$$

hereby we applied the convergence theorem of Beppo Levi. If we define for $q \geq 0$:

$$\big((X_i, Z^q_i, U_i)_{i=1}^{\infty}, \boldsymbol{\xi}^{XZ,q}, \mathbf{\Xi}^{XZ,q}\big) = \mathbb{I}_0\left[h^q\right],$$

then the previous Proposition 6.1.7 tells us that the process $P^q := \Xi^{XZ,q}(\hat{g})$ admits a continuous modification, which we also denote by P^q, and by Proposition 4.4.1 we know that P must also admit a càdlàg modification for which we also write P. Corollary 5.2.4 together with (6.15) tells us that there exists a sequence $(q_k)_{k=1}^{\infty}$ such that for all $m \in \mathbb{N}$ holds

$$\sup_{s \leq m} |P_s^{q_k} - P_s| \overset{k\to\infty}{\longrightarrow} 0 \text{ a.s.}$$

Since the space $C([0,m],\mathbb{R})$ is complete with respect to the supremum norm $||\cdot||$, it follows that there exists a set $\Gamma_m \subset \Omega$ with $\mathbb{P}[\Gamma_m] = 1$ with $t \mapsto P_t(\omega)$ is continuous up to time m for every $\omega \in \Gamma_m$. Setting $\Gamma := \cap_{m=1}^{\infty} \Gamma_m$ we have that $\mathbb{P}[\Gamma] = 1$ and $t \mapsto P_t(\omega)$ is continuous for all $t \geq 0$, then our continuous modification P^{cont} of P is given by $P^{cont} := \mathbb{1}_\Gamma P$. □

From the fact that $\Xi^{XZ}(\hat{g})$ is continuous for every $\hat{g} \in C_b^2(\mathbb{R}^d \times \mathbb{R})$, we can conclude that

$$[\Xi^{XZ}(\hat{g})] = \langle \Xi^{XZ}(\hat{g}) \rangle \ a.s. \tag{6.16}$$

The same is also true for $\exp(-\Xi^{XZ}(\hat{g}))$. We can now combine the semi-martingale decomposition of $\exp(-\Xi^{XZ}(2\hat{g}))$ and Itô's formula for continuous semi-martingales to derive an explicit expression for the quadratic variations $\langle \Xi^{XZ}(\hat{g}) \rangle$ and $\langle \exp(-\Xi^{XZ}(\hat{g})) \rangle$, which will become a quite useful tool in Proposition 6.2.1.

Proposition 6.1.9. *Assume that Ξ^{XZ} is like in Proposition 6.1.8 and $\hat{g} \in C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$, then $\Xi^{XZ}(\hat{g})$ and $\exp(-\Xi^{XZ}(\hat{g}))$ admit continuous modifications, which we also denote by $\Xi^{XZ}(\hat{g})$ and $\exp(-\Xi^{XZ}(\hat{g}))$, and their quadratic variations are given by:*

$$\langle \exp(-\Xi^{XZ}(\hat{g})) \rangle_t = \int_0^t 2a\, \Xi_s^{XZ}(\hat{g}^2) \exp(-2\, \Xi_s^{XZ}(\hat{g})) ds, \quad \forall t \geq [0,\infty), \tag{6.17}$$

$$\langle \Xi^{XZ}(\hat{g}) \rangle_t = \int_0^t 2a\, \Xi_s^{XZ}(\hat{g}^2) ds. \tag{6.18}$$

Proof. By Proposition 6.1.7 we can conclude that $\Xi^{XZ}(\hat{g})$ is continuous and hence the same is true for $\exp(-\Xi^{XZ}(\hat{g}))$ and $\exp(-\Xi^{XZ}(2\hat{g}))$.
Further by Proposition 4.4.2 there exist local $\mathcal{F}^{XZ}$-martingales $\hat{M}^1$ and $\hat{M}^2$ such that

$$\exp(-\Xi_t^{XZ}(\hat{g})) = \exp(-\Xi_0^{XZ}(\hat{g})) + \hat{M}_t^1 + \hat{A}_t^1, \tag{6.19}$$

$$\exp(-\Xi_t^{XZ}(2\hat{g})) = \exp(-\Xi_0^{XZ}(2\hat{g})) + \hat{M}_t^2 + \hat{A}_t^2. \tag{6.20}$$

with $\hat{A}^1$ and $\hat{A}^2$ being continuous predictable processes with finite variation given by

$$\hat{A}_t^1 = \int_0^t \Xi_s^{XZ}(a\hat{g}^2 - B(\hat{g}) - b\hat{g} + \hat{g}^{\Delta,h}) \exp(-\Xi_s^{XZ}(\hat{g})) ds \tag{6.21}$$

$$\hat{A}_t^2 = \int_0^t \Xi_s^{XZ}(4a\hat{g}^2 - 2B(\hat{g}) - 2b\hat{g} + 2\hat{g}^{\Delta,h}) \exp(-\Xi_s^{XZ}(2\hat{g})) ds \tag{6.22}$$

where $\hat{E} = \mathbb{R}^d \times \mathbb{R}$ and where we $\hat{g}^{\Delta,h}(x,z) = \int_0^\infty \hat{g}(x, z + h(x,z,p,s)) - \hat{g}(x,z)\, dp$ (note that here the meaning of $\hat{g}^{\Delta,h}$ is a **different** one than in Definition 5.2.1). Since $\exp(-\Xi^{XZ}(\hat{g}))$ is continuous, we can apply the Itô formula and obtain:

$$\langle \exp(-\Xi^{XZ}(\hat{g})) \rangle_t = \exp(-\Xi_t^{XZ}(2\hat{g})) - \exp(-\Xi_0^{XZ}(2\hat{g})) - \int_0^t 2 \exp(-\Xi_s^{XZ}(\hat{g}))\, dQ_s$$

with $Q = \exp(-\boldsymbol{\Xi}^{XZ}(\hat{g}))$. With the semi-martingale decompositions (6.19) and (6.20) we can rewrite the above expression for $\langle \exp(-\boldsymbol{\Xi}^{XZ}(\hat{g}))\rangle_t$ into:

$$\langle \exp(-\boldsymbol{\Xi}^{XZ}(\hat{g}))\rangle_t = \hat{M}_t^2 - \hat{M}_0^2 + \hat{A}_t^2 - \hat{A}_0^2 - \int_0^t 2\exp(-\boldsymbol{\Xi}_s^{XZ}(\hat{g}))d(\hat{M}_s^1 + \hat{A}_s^1).$$

Let us define the process $\hat{N} : \Omega \times [0,\infty) \to \mathbb{R}$ by

$$\begin{aligned}\hat{N}_t &:= \hat{M}_t^2 - \hat{M}_0^2 - \int_0^t 2\exp(-\boldsymbol{\Xi}_s^{XZ}(\hat{g}))d\hat{M}_s^1 \\ &= \langle \exp(-\boldsymbol{\Xi}^{XZ}(\hat{g}))\rangle_t - \hat{A}_t^2 + \hat{A}_0^2 + \int_0^t 2\exp(-\boldsymbol{\Xi}_s^{XZ}(\hat{g}))d\hat{A}_s^1.\end{aligned}$$

By the first line $\hat{N}$ must be a continuous local martingale, by the second line $\hat{N}$ has finite variation, so in conclusion: $\mathbb{P}[\hat{N}_t = 0, t \geq 0] = 1$. From the second line of $\hat{N}$ we get now:

$$\langle \exp(-\boldsymbol{\Xi}^{XZ}(\hat{g}))\rangle_t = \hat{A}_t^2 - \hat{A}_0^2 - \int_0^t 2\exp(-\boldsymbol{\Xi}_s^{XZ}(\hat{g}))d\hat{A}_s^1$$

Applying (6.21) to $d\hat{A}^1$ and (6.22) to $\hat{A}^2$ it remains

$$\langle \exp(-\boldsymbol{\Xi}^{XZ}(\hat{g}))\rangle_t = \int_0^t 2a\,\boldsymbol{\Xi}_s^{XZ}(\hat{g}^2)\exp(-2\,\boldsymbol{\Xi}_s^{XZ}(\hat{g}))ds.$$

This proves (6.17). Next we are calculating the quadratic variation of the continuous martingale $\boldsymbol{\Xi}^{XZ}(\hat{g})$. Since $\boldsymbol{\Xi}^{XZ}(\hat{g}) = -\log(Q)$ with $Q = \exp(-\boldsymbol{\Xi}^{XZ}(\hat{g}))$ we get by Itô's formula that

$$\boldsymbol{\Xi}_t^{XZ}(\hat{g}) = \boldsymbol{\Xi}_0^{XZ}(\hat{g}) + \int_0^t Q_s^{-1}\,dQ_s + \frac{1}{2}\int_0^t Q_s^{-2}\,d\langle Q\rangle_s.$$

From this we can conclude that the quadratic variation of $\boldsymbol{\Xi}^{XZ}(\hat{g})$ must be identical with the quadratic variation of $\int_0^t Q_s^{-1}\,dQ_s$, which is given by

$$\int_0^t (Q_s^{-1})^2\,d\langle Q\rangle_s = \int_0^t 2a\,\boldsymbol{\Xi}_s^{XZ}(\hat{g}^2)\,ds.$$

This proves (6.18). □

6.2 Continuity of the Intensity Process

We start with this section by proving the following local tightness result.

Proposition 6.2.1. *Let us assume that $h \in \mathcal{L}_{loc}^1(\mathbf{M})$ with localizing sequence $(T_n)_{n=1}^\infty$ and that*

$$\big((X_i, Z_i, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ}\big) = \mathbb{I}_0\big[h\big].$$

Let us define the stopping time $\tilde{T}_n := T_n \wedge \tau_n^Y$, where $\tau_n^Y := \inf\{s > 0 : Y_s \geq n\}$, and let us define the event $\Omega_{tight}^{n,m} \subset \Omega, n \in \mathbb{N}, m \in \mathbb{N}$, by

$$\Omega_{tight}^{n,m} := \{(\boldsymbol{\Xi}_{s\wedge\tilde{T}_n}^{XZ}, s \in [0,m] \cap \mathbb{Q}) \text{ is a tight family of measures}\}.$$

It holds $\mathbb{P}[\Omega_{tight}^{n,m}] = 1$ for all $n \in \mathbb{N}, m \in \mathbb{N}$.

This result will obtained by showing that for every $n \in \mathbb{N}$ and $m \in \mathbb{N}$ holds

$$\mathbb{E}\Big[\sup_{s\in\mathbb{Q}\cap[0,m\wedge\tilde{T}_n]}\big|\,\boldsymbol{\Xi}_s^{XZ}(\mathbb{1}_{\{||(x,z)||\geq\delta\}})\big|\Big] \overset{\delta\to\infty}{\longrightarrow} 0. \tag{6.23}$$

Unfortunately the function $\mathbb{1}_{\{||(x,z)||\geq\delta\}}$ is not smooth enough to be an element of $C_b^{2,+}(\mathbb{R}^d\times\mathbb{R})$, so we can not apply our previously obtained results directly to this function, but we can work around this problem by finding a suitable function $\hat{g}^\delta \in C_b^2(\mathbb{R}^d\times\mathbb{R})$, such that:

$$0 \leq \mathbb{1}_{\{||(x,z)||\geq\delta\}} \leq \mathbb{1}_{\mathbb{R}^d\times\mathbb{R}} - \hat{g}^\delta. \tag{6.24}$$

Lemma 6.2.2. *There exists a collection* $(\hat{g}^\delta, \delta > 1) \subset C_c^{2,+}(\mathbb{R}^d\times\mathbb{R})$ *with* $\hat{g}^\delta : \mathbb{R}^d\times\mathbb{R} \to [0,1]$ *and a constant* $\hat{K} > 0$ *such that*

i) $g^\delta(x,z) = 1$ *for* $||(x,z)|| \leq \delta - 1$ *and* $\hat{g}^\delta(x,z) = 0$ *for* $||(x,z)|| \geq \delta$,

ii) $B_X(\hat{g}^\delta)(x,z) \leq \hat{K}$ *for all* $(x,z) \in \mathbb{R}^d\times\mathbb{R}$,

iv) $B_X(\hat{g}^\delta)(x,z) \overset{\delta\to\infty}{\longrightarrow} 0$ *for all* $(x,z)\in\mathbb{R}^d\times\mathbb{R}$.

Proof. We take a smooth function $\phi : \mathbb{R}^d\times\mathbb{R} \to [0,1]$ whose support is contained in the ball $\{||(\tilde{x},\tilde{z})|| \leq 1\}$ and it holds $\int_{\mathbb{R}^d\times\mathbb{R}}\phi(\tilde{x})\,d\tilde{x} = 1$. By defining $\hat{g}^\delta$ as the convolution of ϕ and $\mathbb{1}_{\{|(x,z)|\geq\delta\}}$, it will have the desired properties. □

Proposition 6.2.3. *Let us assume the conditions of Lemma 6.2.1, then it holds for all* $m \in \mathbb{N}$ *and* $n \in \mathbb{N}$ *that*

$$\mathbb{E}\left[\sup_{s\in\mathbb{Q}\cap[0,m\wedge\tilde{T}_n]}\big|\,\boldsymbol{\Xi}_s^{XZ}(\mathbb{1}_{\{||(x,z)||\geq\delta\}})\big|\right] \overset{\delta\to\infty}{\longrightarrow} 0, \tag{6.25}$$

where $\tilde{T}_n := T_n \wedge \tau_n^Y$ *with* $\tau_n^Y := \inf\{s > 0 : Y_s \geq n\}$.

Proof of Proposition 6.2.3. Let us assume that $(g^\delta, \delta > 1)$ is taken from Lemma 6.2.2. When we define for $\delta > 1$ the function $\hat{\varphi}^\delta := \mathbb{1}_{\mathbb{R}^d\times\mathbb{R}} - \hat{g}^\delta$, then $\hat{\varphi}$ is an element of $C_b^{2,+}(\mathbb{R}^d\times\mathbb{R})$, so the process $\boldsymbol{\Xi}^{XZ}(\hat{\varphi})$ by Proposition 6.1.8 has continuous paths. Combining this with (6.24) gives us:

$$\sup_{s\in\mathbb{Q}\cap[0,m\wedge\tilde{T}_n]}\boldsymbol{\Xi}_s^{XZ}(\mathbb{1}_{\{||(x,z)||\geq\delta\}}) \;\leq\; \sup_{s\in[0,m\wedge\tilde{T}_n]}|\boldsymbol{\Xi}_s^{XZ}(\hat{\varphi}^\delta)| \;\; a.s. \quad \delta > 1, n\in\mathbb{N}. \tag{6.26}$$

By Proposition 4.4.1 we know that for $\delta > 1$ there exist a local $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-martingale $\hat{M}^\delta$ and a continuous process $\hat{A}^\delta$ with finite variation such that

$$\boldsymbol{\Xi}_t^{XZ}(\hat{\varphi}^\delta) = \boldsymbol{\Xi}_0^{XZ}(\hat{\varphi}^\delta) + \hat{M}^\delta(t) + \hat{A}^\delta(t), \;\; t \geq 0,$$

and by Proposition 6.1.9 we know that the quadratic variation $\langle\hat{M}^\delta\rangle$ is given by

$$\langle\hat{M}^\delta\rangle = \int_0^t \boldsymbol{\Xi}_s^{XZ}((\hat{\varphi}^\delta)^2)ds. \tag{6.27}$$

By Proposition 4.4.1 the processes $\hat{A}^\delta$,$\delta > 1$, is given by

$$\begin{aligned}\hat{A}^\delta(t) &= \int_0^t\left[\boldsymbol{\Xi}_s^{XZ}(B_X(\hat{\varphi}^\delta) + b\hat{\varphi}^\delta) + \int_0^\infty \boldsymbol{\Xi}_s^{XZ}(\hat{\varphi}^{\Delta,h,\delta}(\,\cdot\,,\,\cdot\,,p,s))dp\right]ds \\ &= \int_0^t\left[\boldsymbol{\Xi}_s^{XZ}(b\hat{\varphi}^\delta - B_X(\hat{g}^\delta)) - \int_0^\infty \boldsymbol{\Xi}_s^{XZ}(\hat{g}^{\Delta,h,\delta}(\,\cdot\,,\,\cdot\,,p,s))dp\right]ds,\end{aligned}$$

with

$$\hat\varphi^{\Delta,h,\delta}(x,z,p) = \hat\varphi^{\delta}(x,z+h(x,z,p)) - \hat\varphi^{\delta}(x,z),$$
$$\hat g^{\Delta,h,\delta}(x,z,p) = \hat g^{\delta}(x,z+h(x,z,p)) - \hat g^{\delta}(x,z)$$

(please note $B_X(\hat\varphi^\delta) = -B_X(\hat g^\delta)$ and $\hat\varphi^{\Delta,h,\delta} = -\hat g^{\Delta,h,\delta}$). Based on $\hat M^\delta$ and $\hat A^\delta$ we can define two new processes $\hat M^{\delta,*}$ and $\hat A^{\delta,*}$ by setting for $\hat M^{\delta,*}(t) := \sup_{s\le t} |\hat M^\delta_s|$ and

$$\hat A^{\delta,*}_t := \int_0^t \left[\Xi^{XZ}_s\big(|B_X(\hat g^\delta)| + b|\hat\varphi^\delta|\big) + \int_0^\infty \Xi^{XZ}_s(|\hat g^{\Delta,h,\delta}(\,\cdot\,,\,\cdot\,,p,s)|)\,dp\right] ds.$$

These new processes are upper bounds for $\hat M^\delta$ and $\hat A^\delta$, and if we combine this with the semi-martingale decompositions of $\Xi^{XZ}(\hat\varphi^\delta)$ we can derive from (6.26):

$$\sup_{s\in\mathbb{Q}\cap[0,m\wedge\tilde T_n]} \Xi^{XZ}_s(\mathbb{1}_{\{||(x,z)||\ge\delta\}}) \le |\Xi^{XZ}_0(\varphi^\delta)| + \hat M^{\delta,*}(m) + \hat A^{\delta,*}(m) \ a.s.$$

To prove our claim it is therefore sufficient to show that

$$\mathbb{E}[\,\Xi^{XZ}_0(|\varphi^\delta|)] + \mathbb{E}[\hat M^{\delta,*}(m\wedge\tilde T_n)] + \mathbb{E}\big[\hat A^{\delta,*}(m\wedge\tilde T_n)\big] \overset{\delta\to\infty}{\longrightarrow} 0. \tag{6.28}$$

We begin by proving that for all $(\omega,t)\in\Omega\times[0,\infty)$ holds

$$\Xi^{XZ}_t(\omega)(|B_X(\hat\varphi^\delta)|) \overset{\delta\to\infty}{\longrightarrow} 0,\ \ \Xi^{XZ}_t(\omega)(|\hat\varphi^\delta|) \overset{\delta\to\infty}{\longrightarrow} 0,\ \ \Xi^{XZ}_t(\omega)(|\hat\varphi^\delta|^2) \overset{\delta\to\infty}{\longrightarrow} 0. \tag{6.29}$$

This follows from Lebesgue's convergence theorem, since $\Xi^{XZ}_s(\omega)$ is by definition a finite measure for all $(t,\omega)\in\Omega\times[0,\infty)$ and we have for all $(x,z)\in\mathbb{R}^d\times\mathbb{R}$ that

$$|B_X(\hat g^\delta)(x,z)| \overset{\delta\to\infty}{\longrightarrow} 0, \quad |B_X(\hat g^\delta)(x,z)| \le \hat K,$$
$$|\hat\varphi^\delta(x,z)| \overset{\delta\to\infty}{\longrightarrow} 0, \qquad |\hat\varphi^\delta(x,z)| \quad \le 1,$$

where $\hat K$ is the constant from Lemma 6.2.2. Because $\Xi^{XZ}_0(\omega)(|\hat\varphi^\delta|) \le Y_0(\omega)$ for all $\omega\in\Omega$ and $\mathbb{E}[Y_0] < \infty$, we can conclude from (6.29) and a further application of Lebesgue's convergence theorem that

$$\lim_{\delta\to\infty} \mathbb{E}\big[\Xi^{XZ}_0(\hat\varphi^\delta)\big] = \mathbb{E}\big[\lim_{\delta\to\infty} \Xi^{XZ}_0(\hat\varphi^\delta)\big] = 0. \tag{6.30}$$

Considering the second term of (6.28), we combine the Jensen inequality and the Davis-Burkholder-Gundy inequality to get

$$\mathbb{E}[\hat M^{\delta,*}(m\wedge\tilde T_n)] \le \sqrt{\mathbb{E}[(\hat M^{\delta,*}(m\wedge\tilde T_n))^2]} \le 2\sqrt{\mathbb{E}[\langle\hat M^\delta\rangle_{m\wedge\tilde T_n}]}. \tag{6.31}$$

Since $\Xi^{XZ}_s(\omega)(|\hat g^\delta|^2) \le Y_s(\omega)$ for all $(\omega,s)\in\Omega\times[0,\infty)$ and

$$\mathbb{E}[\langle\hat M^\delta\rangle_{m\wedge\tilde T_n}] = \int_0^m \mathbb{E}\big[\mathbb{1}_{[0,\tilde T_n]}(s)\Xi^{XZ}_s(|\hat\varphi^\delta|^2)\big]\,ds \le \int_0^m \mathbb{E}[Y_s]\,ds = \mathbb{E}[Y_0]e^{bm} < \infty,$$

we can conclude from the expression for the quadratic variation $\langle\hat M^\delta\rangle$ in (6.27) that

$$\lim_{\delta\to\infty} \mathbb{E}[\langle\hat M^\delta\rangle_{m\wedge\tilde T_n}] = \int_0^m \mathbb{E}\left[\lim_{\delta\to\infty} \mathbb{1}_{[0,\tilde T_n]}(s)\Xi^{XZ}_s(|\hat\varphi^\delta|^2)\right] ds = 0$$

and from this and (6.31) follows

$$\mathbb{E}[\hat{M}^{\delta,*}(m \wedge \tilde{T}_n)] \overset{\delta\to\infty}{\longrightarrow} 0. \tag{6.32}$$

For the last term in (6.28) we split $\hat{A}^{\delta,*}$ into two different expressions, those are

$$\hat{A}^{\delta,*,1}_t := \int_0^t \Xi^{XZ}_s\big(|B_X(\hat{g}^\delta)| + b|\hat{\varphi}^\delta|\big)\,ds,\;\; \hat{A}^{\delta,*,2}_t := \int_0^t \int_0^\infty \Xi^{XZ}_s(|\hat{g}^{\Delta,h,\delta}(\,\cdot\,,p,s)|)\,dp\,ds.$$

For the first term we note that $\Xi^{XZ}_s(\omega)(|B_X(\hat{g}^\delta)|) \le \hat{K}Y_s(\omega)$ and $\Xi^{XZ}_s(\omega)(|\hat{g}^\delta|) \le Y_s(\omega)$ for all $(\omega,s) \in \Omega \times [0,\infty)$ and that

$$\mathbb{E}\bigg[\int_0^{m\wedge\tilde{T}_n} (\hat{K}+1)Y_s\,ds\bigg] = (\hat{K}+1)\int_0^m \mathbb{E}[\mathbb{1}_{[0,\tilde{T}_n]}Y_s]\,ds \le (\hat{K}+1)\mathbb{E}[Y_0]e^{bm} < \infty,$$

combining these with Lebesgue's convergence theorem implies

$$\lim_{\delta\to\infty} \mathbb{E}\big[\hat{A}^{\delta,*,1}_{m\wedge\tilde{T}_n}\big] = \int_0^m \mathbb{E}\big[\lim_{\delta\to\infty} \mathbb{1}_{[0,\tilde{T}_n]}(s)\Xi^{XZ}_s(B_X(\hat{g}^\delta) + b\hat{\varphi}^\delta)\big]\,ds = 0.$$

For $\hat{A}^{\delta,*,2}$, we note $|\hat{g}^{\Delta,h,\delta}(\omega,x,z,p,s)| \le \hat{K}h(\omega,x,p,s)$ for all $(\omega,x,z,p,s) \in \Omega\times\mathbb{R}^d\times\mathbb{R}\times[0,\infty)\times[0,\infty)$ and for all δ. Further by Lemma 4.1.3, it holds

$$\int_0^t\int_0^\infty \mathbb{E}\bigg[\mathbb{1}_{[0,\tilde{T}_n]}(s)\int |h(x,p,s)|\,\Xi^{XZ}_s(dx,dz)\bigg]dpds < \infty. \tag{6.33}$$

Hence the function $\hat{K}\,h$ is a majorant for the sequence $(|\hat{g}^{\Delta,h,\delta}|)_{\delta>1}$ which is integrable with respect to the measure $\mathbb{E}[\mathbb{1}_{[0,\tilde{T}_n]}(s)\Xi^{XZ}_s(dx,dz)]dpds$, which is a measure over $\Omega\times\mathbb{R}^d\times\mathbb{R}\times[0,\infty)\times[0,\infty)$. It also holds

$$|\hat{g}^{\Delta,h,\delta}|(\omega,x,z,p,s) \overset{\delta\to\infty}{\longrightarrow} 0, \quad (\omega,x,z,p,s)\in\Omega\times\mathbb{R}^d\times\mathbb{R}\times[0,\infty)\times[0,\infty).$$

So by Lebesgue's convergence theorem, it follows

$$\lim_{\delta\to\infty}\mathbb{E}[\hat{A}^{\delta,*,2}_{m\wedge\tilde{T}_n}] = \int_0^t\int_0^\infty \mathbb{E}\bigg[\mathbb{1}_{[0,\tilde{T}_n]}(s)\int_{\overline{E}} \lim_{\delta\to\infty}|\hat{g}^{\Delta,h,\delta}|\,\Xi^{XZ}_s(dx)\bigg]dpds = 0. \tag{6.34}$$

According to (6.33) and (6.33) give us together that

$$\mathbb{E}\big[\hat{A}^{\delta,*}_{m\wedge\tilde{T}_n}\big] \overset{\delta\to\infty}{\longrightarrow} 0. \tag{6.35}$$

Finally (6.28) follows from (6.35), (6.30) and (6.32). □

Now we can apply our results to prove the tightness statements of Proposition 6.2.1.

Proof of Proposition 6.2.1. For $k\in\mathbb{N}$, let us define $\mathcal{H}^{n,m}_k : \Omega \to [0,\infty)$ by

$$\mathcal{H}^{n,m}_k = \sup_{s\in\mathbb{Q}\cap[0,m\wedge\tilde{T}_n]} \big|\,\Xi^{XZ}_s(\mathbb{1}_{\{||(x,z)||\ge k\}})\big|,$$

then we have

$$\{\omega\in\Omega : \mathcal{H}^{n,m}_k(\omega) \overset{k\to\infty}{\longrightarrow} 0\} = \Omega^{n,m}_{tight}.$$

By (6.25) from Proposition 6.2.3 we know that $\mathcal{H}^{n,m}_k$ is converging against 0 in $L^1(\mathbb{P})$, when k goes to infinity, hence there exists an subsequence such that this convergence holds almost surely and since $(\mathcal{H}^{n,m}_k)$ is a decreasing sequence, this convergence can be extended to the whole sequence. □

Before we prove the main result of this chapter, we need a further small technical lemma.

Lemma 6.2.4. *There exists a countable set $(\hat{g}_k)_{k=1}^{\infty} \subset C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$ that is convergence determining.*

Proof. Note that $C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$ is closed under multiplication, contains $\mathbb{1}_{\mathbb{R}^d \times \mathbb{R}}$ and is strongly separating points, by Lemma 2 from [5] it follows that there exists a countable subset Γ of $C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$ with the same properties. If $\mu_1, \mu_2 \in \mathcal{M}_f(\mathbb{R}^d \times \mathbb{R})$ are two measures with $\mu_1(\hat{g}) = \mu_2(\hat{g})$ for all $\hat{g} \in \Gamma$, then it holds $\mu_1(\hat{g}) = \mu_2(\hat{g})$ for all $\hat{g} \in \mathit{span}(\Gamma)$, where $\mathit{span}(\Gamma)$ is the linear span of Γ. By Theorem 3.4.5 in cite [14] the set $\mathit{span}(\Gamma)$ is convergence determining and so we get that the same holds for Γ. □

Finally we are able to prove the main result of this chapter.

Theorem 6.2.5. *Let us assume that $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ and that*

$$\big((X_i, Z_i, U_i)_{i=1}^{\infty}, \boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ}\big) = \mathbb{I}_0\big[h\big],$$

then $\boldsymbol{\Xi}^{XZ}$ admits a continuous modification, indeed there exists a process

$$\hat{\boldsymbol{\Xi}}^{XZ} : \Omega \times [0, \infty) \to \mathcal{M}_f(\mathbb{R}^d \times \mathbb{R}),$$

which has continuous paths in $\mathcal{M}_f(\mathbb{R}^d \times \mathbb{R})$ with respect to the weak topology and it holds

$$\mathbb{P}\big[\boldsymbol{\Xi}_t^{XZ}(\hat{f}) = \hat{\boldsymbol{\Xi}}_t^{XZ}(\hat{f}) \text{ for all } f \in C_b^{+}(\mathbb{R}^d \times \mathbb{R})\big] = 1, \quad t \geq 0. \tag{6.36}$$

Proof of 6.2.5. For our purpose it is sufficient to prove that

$$(\boldsymbol{\Xi}^{XZ}_{t \wedge \tilde{T}_n}, t \in [0, m]) \tag{6.37}$$

is continuous for all fixed $n \in \mathbb{N}$ and $m \in \mathbb{N}$ (note that $\tilde{T}_n$ converges to infinity almost surely, when n goes to infinity). By Lemma 6.2.4 we know that there exists a countable, separating set $(\hat{g}_k)_{k=1}^{\infty} \subset C_b^{2,+}(\mathbb{R}^d \times \mathbb{R})$. By Proposition 6.1.8 the process $(\boldsymbol{\Xi}^{XZ}_{t \wedge \tilde{T}_n}(\hat{g}_k), t \in [0, m])$ admits a continuous modification for each $k \in \mathbb{N}$, which we denoted by P^k. Let us define

$$\Omega^k_{cont} := \Big\{ P^k_s = \boldsymbol{\Xi}^{XZ}_{s \wedge \tilde{T}_n}(\hat{g}^k), \ s \in \mathbb{Q} \cap [0, \infty) \Big\}.$$

Further let $\Omega^{n,m}_{tight} \subset \Omega$ be the event that $(\boldsymbol{\Xi}^{XZ}_{s \wedge \tilde{T}_n}, s \in [0, m] \cap \mathbb{Q})$ forms a tight family of measures in $\mathcal{M}_f(\mathbb{R}^d \times \mathbb{R})$. If we define Γ by

$$\Gamma := \bigcap_{k \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \Omega^{n,m}_{tight} \cap \Omega^k_{cont},$$

then we have $\mathbb{P}[\Gamma] = 1$. We know obtain the continuous modification of $(\boldsymbol{\Xi}^{XZ}_{t \wedge \tilde{T}_n, t \geq 0}$ by setting:

$$\hat{\boldsymbol{\Xi}}^{n,m}_t(\omega) := \mathbb{1}_{\{(\omega,t) \in \Gamma\}} \mu_{\omega,t},$$

where the measure $\mu_{\omega,t} \in \mathcal{M}_f(\mathbb{R}^d \times \mathbb{R})$ is obtained in the following way: We pick a sequence of time points $(t_i, i \in \mathbb{N}) \subset \mathbb{Q} \cap [0, m \wedge \tilde{T}_n(\omega)]$ with $t_i \to t$ for $i \to \infty$. Due to the tightness of $(\boldsymbol{\Xi}^{XZ}_{t_i}(\omega), i \in \mathbb{N})$, there exists a subsequence $(\tilde{t}_i, i \in \mathbb{N}) \subset (t_i, i \in \mathbb{N})$ such that $(\boldsymbol{\Xi}^{XZ}_{\tilde{t}_i}(\omega), i \in \mathbb{N})$ is converging in $\mathcal{M}_f(\mathbb{R}^d \times \mathbb{R})$. We define $\mu_{\omega,t}$ as the limit of this sequence. The measure $\mu_{\omega,t}$ is independent of the chosen sequence $(t_i, i \in \mathbb{N})$ and the chosen subsequence $(\tilde{t}_i, i \in \mathbb{N})$, because it always holds that

$$\mu_{\omega,t}(\hat{g}^k) = \lim_{i\to\infty} \Xi^{XZ}_{\tilde{t}_i}(\omega)(\hat{g}^k) = \lim_{i\to\infty} P^k_{\tilde{t}_i}(\omega) = P^k_t(\omega), \quad k \in \mathbb{N}, \tag{6.38}$$

and since $(\hat{g}_k)_{k=1}^\infty$ is separating, the measure $\mu_{\omega,t}$ is uniquely defined by $(P^k_t(\omega), k \in \mathbb{N})$. Further from the fact that $(\hat{g}_k)_{k=1}^\infty$ is even convergence determining, we can see that $\hat{\Xi}^{n,m}(\omega)$ is continuous. Indeed if $(t_i, i \in \mathbb{N}) \subset [0, m]$ is an arbitrary sequence converging to t, then it holds

$$\lim_{i\to\infty} \mu_{\omega,t_i}(\hat{g}^k) = \lim_{i\to\infty} P^k_{t_i}(\omega) = P^k_t(\omega) = \mu_{\omega,t}(\hat{g}^k), \quad k \in \mathbb{N},$$

and from the fact that $(\hat{g}_k)_{k=1}^\infty$ is convergence determining, it follows that $(\mu_{\omega,t_i}, i \in \mathbb{N})$ is converging against $\mu_{\omega,t}$ in the weak topology. Finally it follows from

$$\mathbb{P}[\hat{\Xi}^{n,m}_t(\hat{g}^k) = P^k_s = \Xi^{XZ}_{t\wedge\tilde{T}_n}(\hat{g}^k),\ k \in \mathbb{N}] = \mathbb{P}[\hat{\Xi}^{n,m}_t = \Xi^{XZ}_{t\wedge\tilde{T}_n}] = 1.$$

and the fact that $(\hat{g}_k)_{k=1}^\infty$ is separating that $\hat{\Xi}^{n,m}$ is modification of $\Xi^{XZ}_{t\wedge\tilde{T}_n}$. □

As we already mentioned at the beginning of this chapter, we will only work with the continuous modifications of the process Ξ^{XZ} from now on. We will mark this change by using the symbol $\mathbb{I}$ instead of $\mathbb{I}_0$. We summarize all of this with the next definition.

Definition 6.2.6. *For $h \in \mathcal{L}^1_{loc}(\mathbf{M})$ we define*

$$\big((X_i, Z_i, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ}, \Xi^{XZ}\big) = \mathbb{I}\big[h\big],$$

where $(X_i, Z_i, U_i)_{i=1}^\infty$ and $\boldsymbol{\xi}^{XZ}$ are identical with $(\tilde{X}_i, \tilde{Z}_i, \tilde{U}_i)_{i=1}^\infty$ and $\tilde{\boldsymbol{\xi}}^{XZ}$ from

$$\big((\tilde{X}_i, \tilde{Z}_i, \tilde{U}_i)_{i=1}^\infty, \tilde{\boldsymbol{\xi}}^{XZ}, \tilde{\Xi}^{XZ}\big) = \mathbb{I}_0\big[h\big],$$

but $\tilde{\Xi}^{XZ}$ is replaced by its continuous modification of Ξ^{XZ}.

Chapter 7

Poisson Representation of Competitive Models

7.1 The Cut-Out Process

We are going to apply our integration theory developed in the previous chapters to derive Poisson representations for the two different classes of competitive models presented in Section 1.2. We obtain our representation by cutting those out from $\boldsymbol{\xi}^X$, therefore we introduce the Cut-Out process in this section and discuss some basic properties. We will encounter different notions of predictability in this section.

Definition 7.1.1. *Let us assume that $\mathcal{F} := (\mathcal{F}_t, t \geq 0)$ is a filtration on some measure space $(\tilde{\Omega}, \tilde{\mathcal{A}})$, E and $\hat{E}$ are topological spaces, then we call $\mathcal{V} : \tilde{\Omega} \times E \times [0, \infty) \to \hat{E}$ a $\mathcal{F}$-predictable process, if $\mathcal{V}$ is measurable with respect to the σ-algebra given by*

$$\tilde{\mathfrak{P}}(E, \mathcal{F}) := \sigma\Big(\{\Gamma_1 \times \Gamma_2 \times \{0\}; \Gamma_1 \in \mathcal{F}_0, \Gamma_2 \in \mathbb{B}(E)\} \\ \vee \{\Gamma_1 \times \Gamma_2 \times (t_1, t_2]; \Gamma_1 \in \mathcal{F}_{t_1}, \Gamma_2 \in \mathbb{B}(E), 0 < t_1 < t_2 < \infty\}\Big).$$

If $E = \{0\}$, then we write $\tilde{\mathfrak{P}}(\mathcal{F})$, which is Borel isomorph with the traditional definition of predictable σ-algebra, see Definition 1.2.2.1 from [19].

This definition is an extension of the notion of predictability which we introduced in Section 3.1, i.e. if $h \in \mathcal{P}(\mathcal{F}^{\Xi,\mathbb{W}})$, then h is $\mathcal{F}^{\Xi,\mathbb{W}}$-predictable, and $\mathfrak{P}(\mathcal{F}^{\Xi,\mathbb{W}}) = \tilde{\mathfrak{P}}(E, \mathcal{F}^{\Xi,\mathbb{W}})$ with $E = \mathbb{R}^d \times [0, \infty)$. A sufficient criterion for $P : \tilde{\Omega} \times E \times [0, \infty) \to \hat{E}$ to be $\mathcal{F}$-predictable is that $\hat{E}$ is a metric space and P has the properties:

1. For each $T > 0$ the restriction on $\tilde{\Omega} \times E \times [0, T]$ is $\mathcal{F}_T \otimes \mathbb{B}(E) \otimes \mathbb{B}([0, \infty))$-measurable.
2. The map $t \mapsto P(\omega, x, t)$ is left-continuous for each $(\omega, x) \in \tilde{\Omega} \times E$.

This can can be seen by approximating P by the sequence $(P^n)_{n=1}^{\infty}$ pointwise, where $P^n(\omega, x, t) = P(\omega, x, n^{-1}\lfloor nt \rfloor)$, and the fact that the limit of measurable maps is again measurable, if the codomain $\hat{E}$ is a metric space, see Lemma 1.10 from [21]. Finally note that if $\mathcal{G}$ is the filtration on $\Omega \times E$ given by $\mathcal{G}_t := \mathcal{F}_t \otimes \mathbb{B}(E)$, $t \geq 0$, then we have the relations

$$\tilde{\mathfrak{P}}(E, \mathcal{F}) = \tilde{\mathfrak{P}}(\mathcal{G}) \simeq \tilde{\mathfrak{P}}(\mathcal{F}) \otimes \mathbb{B}(E),$$

where "$\simeq$" stands for the existence of a Borel isomorphism.

Lemma 7.1.2. *Assume that* $F : \mathbb{R}^d \times \mathcal{M}_f(\mathbb{R}^d) \to [0,\infty)$ *is measurable and the process* $\mathcal{V} : \Omega \times [0,\infty) \to \mathcal{M}_f(\mathbb{R}^d)$ *is* $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$*-predictable. If we define the map*

$$h[F,\mathcal{V}] : \Omega \times \mathbb{R}^d \times [0,\infty) \times [0,\infty) \to \{0,1\}$$

by setting $h[F,\mathcal{V}](\omega,x,p,s) = \mathbb{1}_{[0,F(x,\mathcal{V}_s(\omega))]}(p)$*, then* $h[F,\mathcal{V}]$ *is an element of* $\mathfrak{P}(\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}})$*, see Definition 3.1.1.*

Proof. We need to show that $h[F,\mathcal{V}]$ is $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-predictable. Since $\mathcal{V}$ is $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-predictable, the process $\overline{\mathcal{V}} : \Omega \times \mathbb{R}^d \times [0,\infty) \times [0,\infty) \to \mathcal{M}_f(\mathbb{R}^d) \times \mathbb{R}^d \times [0,\infty)$ given by $(\omega,x,p,t) \mapsto (\mathcal{V}_t(\omega),x,p,t)$ is also $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-predictable. Since the map $\overline{h} : \mathcal{M}_f(\mathbb{R}^d) \times \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ given by $(\mu,x,p) \mapsto \mathbb{1}_{[0,F(x,\mu)]}(p)$ is measurable, the claim follows from the fact that $h[F,\mathcal{V}] = \overline{h} \circ \overline{\mathcal{V}}$. □

Definition 7.1.3 (Cut-Out Process)**.** *Assume* F *and* $\mathcal{V}$ *are as in Lemma* 7.1.2 *and it holds additionally that* $h[F,\mathcal{V}] \in \mathcal{L}^1_{loc}(\mathbf{M})$*. In this case we define*

$$((\widehat{X}_i,\widehat{U}_i)_{i=1}^\infty, \widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\Xi}}) = \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,,\mathcal{V}))$$

by setting first

$$\big((X_i,Z_i,U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ}, \boldsymbol{\Xi}^{XZ}, \mathbf{Q}^{XZ}\big) = \mathbb{I}\big[h[F,\mathcal{V}]\big]$$

and then we define the processes $\widehat{\boldsymbol{\xi}} : \Omega \times [0,\infty) \to \overline{\mathcal{N}}(\mathbb{R}^d)$ *and* $\widehat{\boldsymbol{\Xi}} : \Omega \times [0,\infty) \to \mathcal{M}_f(\mathbb{R}^d)$ *as*

$$\widehat{\boldsymbol{\xi}}_t := \sum_{i=1}^\infty \delta_{(X_i(t),U_i(t))}\mathbb{1}_{\{0\}}(Z_i(t)), \qquad \widehat{\boldsymbol{\Xi}}_t := \boldsymbol{\Xi}^{XZ}(\ \cdot\ \times \{0\}\ )$$

and we define $(\widehat{X}_i(t),\widehat{U}_i(t))_{i=1}^\infty$ *for* $t \geq 0$ *as the ordered atoms of* $\widehat{\boldsymbol{\xi}}_t$*, where we set* $(\widehat{X}_i(t),\widehat{U}_i(t)) = (\dagger,\infty)$*, when* $t \geq \mathcal{T}_{EX}^{F,\mathcal{V}}$ *with* $\mathcal{T}_{EX}^{F,\mathcal{V}} = \inf\{t \geq 0 : \widehat{\boldsymbol{\xi}}_t(\mathbb{R}^d \times [0,\infty)) = 0\}$*.*

Lemma 7.1.4. *The process* $\widehat{\boldsymbol{\Xi}}$ *is continuous in the weak topology of* $\mathcal{M}_f(\mathbb{R}^d)$ *and if* τ *is a finite* $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$*-stopping time, then* $\widehat{\boldsymbol{\Xi}}_\tau = \gamma^{\boldsymbol{\Xi}}_{\mathbb{R}^d}(\widehat{\boldsymbol{\xi}}_\tau)$ *almost surely.*

Proof. We have proven that $\boldsymbol{\Xi}^{XZ}$ is continuous in the weak topology of $\mathcal{M}_f(\mathbb{R}^{d+1})$. If $f \in C_b(\mathbb{R}^d)$, then $f^{xz}(x,z) = f(x)\mathbb{1}_{[0,1]}(z)(1-z)$ is an element of $C_b(\mathbb{R}^{d+1})$. Consequently the process $(\boldsymbol{\Xi}^{XZ}_t(f^{xz}), t \geq 0)$ is continuous. Since $\widehat{\boldsymbol{\Xi}}_t(f) = \boldsymbol{\Xi}^{XZ}_t(f^{xz}), t \geq 0$, it follows that $(\widehat{\boldsymbol{\Xi}}_t(f), t \geq 0)$ is continuous, too. Because f has been arbitrary, it follows that $\widehat{\boldsymbol{\Xi}}$ is continuous in the weak topology. Similarly we know that:

$$\widehat{\boldsymbol{\Xi}}_\tau(f) = \boldsymbol{\Xi}^{XZ}_\tau(f^{xz}) = \lim_{r\to\infty}\frac{1}{r}\sum_{i=1}^\infty f^{xz}(X_i(\tau),Z_i(\tau))\mathbb{1}_{[0,r)}(U_i(\tau))$$
$$= \lim_{r\to\infty}\frac{1}{r}\sum_{i=1}^\infty f(X_i(t))\mathbb{1}_{\{0\}}(Z_i(\tau))\mathbb{1}_{[0,r)}(U_i(\tau)).$$

Since the equality of the first line holds for all possible $\overline{f} \in C_b^+(\mathbb{R}^{d+1})$, we have $\widehat{\boldsymbol{\Xi}}_\tau = \gamma^{\boldsymbol{\Xi}}_{\mathbb{R}^d}(\widehat{\boldsymbol{\xi}}_\tau)$. □

Definition 7.1.5.

1. *For $\mu, \nu \in \mathcal{M}_f(\mathbb{R}^d)$ we say $\mu \le \nu$ if and only if $\mu(f) \le \nu(f)$ for all $f \in C_b^+(\mathbb{R}^d)$.*
2. *Let $F : \mathbb{R}^d \times \mathcal{M}_f(\mathbb{R}^d) \to [0, \infty]$ be a measurable function. We say that F is non-decreasing in the $\mathcal{M}_f(\mathbb{R}^d)$-coordinate if and only if for all $\nu^+, \nu^- \in \mathcal{M}_f(\mathbb{R}^d)$ holds*
$$\nu^- \le \nu^+ \Rightarrow \forall x \in \mathbb{R}^d : F(x, \nu^-) \le F(x, \nu^+).$$
3. *If $\mathcal{V}^+, \mathcal{V}^- : \Omega \times [0, \infty) \to \mathcal{M}_f(\mathbb{R}^d)$ are $\mathcal{M}_f(\mathbb{R}^d)$-valued processes, we write $\mathcal{V}^- \preceq_{a.e.} \mathcal{V}^+$, if and only if $\mathcal{V}_t^- \le \mathcal{V}_t^+$ $\mathbb{P} \otimes \ell eb[0, \infty)$-almost everywhere.*

Lemma 7.1.6 (Reversed Order Lemma)**.** *Assume that $F : \mathbb{R}^d \times \mathcal{M}_f(\mathbb{R}^d) \to [0, \infty)$ is measurable, non-decreasing in the $\mathcal{M}_f(\mathbb{R}^d)$-coordinate and that $h[F, \cdot]$ is as in Lemma 7.1.2. When $h[F, \boldsymbol{\Xi}^X] \in \mathcal{L}^1_{loc}(\mathbf{M})$ with localizing sequence $(T_n, n \in \mathbb{N})$, $\mathcal{V}^+ : \Omega \times [0, \infty) \to \mathcal{M}_f(\mathbb{R}^d)$ and $\mathcal{V}^- : \Omega \times [0, \infty) \to \mathcal{M}_f(\mathbb{R}^d)$ are two $\mathcal{M}_f(\mathbb{R}^d)$-valued processes with*
$$\mathcal{V}^- \le \mathcal{V}^+ \le \boldsymbol{\Xi}^X \;\; \mathbb{P} \otimes \ell eb[0, \infty)\text{-a.e.},$$
then $h[F, \mathcal{V}^+] \in \mathcal{L}^1_{loc}(\mathbf{M})$ and $h[F, \mathcal{V}^+] \in \mathcal{L}^1_{loc}(\mathbf{M})$ with localizing sequence $(T_n, n \in \mathbb{N})$. Further, if we define
$$((\widehat{X}_i^+, \widehat{U}_i^+)_{i=1}^\infty, \widehat{\boldsymbol{\xi}}^+, \widehat{\boldsymbol{\Xi}}^+) = \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \mathcal{V}^+)) \tag{7.1}$$
$$((\widehat{X}_i^-, \widehat{U}_i^-)_{i=1}^\infty, \widehat{\boldsymbol{\xi}}^-, \widehat{\boldsymbol{\Xi}}^-) = \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \mathcal{V}^-)), \tag{7.2}$$
then it holds
$$\mathbb{P}\big[\, \forall t \ge 0 : 0 \le \widehat{\boldsymbol{\Xi}}_t^+ \le \widehat{\boldsymbol{\Xi}}_t^- \le \boldsymbol{\Xi}_t^X \big] = 1. \tag{7.3}$$

Proof. By (7.1) there exists a set $\Gamma \subset \Omega \times [0, \infty)$ with $\mathbb{P} \otimes \ell eb[0, \infty)(\Omega \times [0, \infty) \setminus \Gamma) = 0$, such that the relation (7.1) is true for all elements of Γ. Hence for $(\omega, s) \in \Gamma$ it is true, that for all $x \in \mathbb{R}^d$ holds $F(x, \mathcal{V}^-(\omega, s)) \le F(x, \mathcal{V}^+(\omega, s))$ and so
$$h^-(\omega, x, p, s) = \mathbb{1}_{[F(x, \mathcal{V}^-(\omega, s))]}(p) \le \mathbb{1}_{[F(x, \mathcal{V}^+(\omega, s))]}(p) = h^+(\omega, x, p, s)$$
for all $\omega \in \Omega, x \in \mathbb{R}^d, s \in [0, \infty)$ and $p \in \mathbb{R}$. So if we define the processes
$$\big((X_i, Z_i^+, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ,+}, \boldsymbol{\Xi}^{XZ,+}\big) = \mathbb{I}\big[h^+\big],$$
$$\big((X_i, Z_i^+, U_i)_{i=1}^\infty, \boldsymbol{\xi}^{XZ,-}, \boldsymbol{\Xi}^{XZ,-}\big) = \mathbb{I}\big[h^-\big],$$
then we can conclude from the last inequality, that for every $i \in \mathbb{N}$ and $t \ge 0$ holds almost surely
$$\begin{aligned} Z_i^-(t) &= \int_0^t \int_0^\infty h^-(\mathfrak{X}_i(t, s\text{-}), p, s)\; \mathfrak{N}_i(t, dp, ds) \\ &\le \int_0^t \int_0^\infty h^+(\mathfrak{X}_i(t, s\text{-}), p, s)\; \mathfrak{N}_i(t, dp, ds) = Z_i^+(t). \end{aligned}$$
Further we can assume that this inequality holds almost surely for all $t \ge 0$ and $i \in \mathbb{N}$ simultaneously, because Z_i^+ and Z_i^- are càdlàg. But if this inequality holds for all $i \in \mathbb{N}$, then it holds also, that for all $r \ge 0$ and $g \in C_b(\mathbb{R}^d)$:
$$\begin{aligned} \widehat{\boldsymbol{\Xi}}_t^- &\overset{a.s.}{=} \lim_{r \to \infty} \frac{1}{r} \sum_{i \in \mathbb{N}} \hat{g}(X_i(t)) \mathbb{1}_{\{0\} \times [0, r)}(Z_i^-(t), U_i(t)) \\ &\ge \lim_{r \to \infty} \frac{1}{r} \sum_{i \in \mathbb{N}} \hat{g}(X_i(t)) \mathbb{1}_{\{0\} \times [0, r)}(Z_i^+(t), U_i(t)) \overset{a.s.}{=} \widehat{\boldsymbol{\Xi}}_t^+. \end{aligned}$$

Since $\widehat{\boldsymbol{\Xi}}^-$ and $\widehat{\boldsymbol{\Xi}}^+$ are continuous in the weak topolgoy and since the relation "$\leq$" on $\mathcal{M}_f(\mathbb{R}^d)$ remains true under limits in the weak topology, we get that the above inequality between $\widehat{\boldsymbol{\Xi}}^-$ and $\widehat{\boldsymbol{\Xi}}^+$ is true in fact almost surely for all $t \geq 0$ simultaneously. $\square$

Proposition 7.1.7. *Let us assume that F and $\mathcal{V}$ as in Definition 7.1.3 are such that $h[F, \mathcal{V}] \in \mathcal{L}^1_{stop}(\mathbf{M})$ with localizing sequence $(T_n, n \in \mathbb{N})$. Further assume $\hat{g} \in \mathcal{D}(B)$. If we define*

$$((\widehat{X}_i, \widehat{U}_i)_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\Xi}}) = \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \mathcal{V})),$$

and if we define the process $M : \Omega \times [0, \infty) \to \mathbb{R}$ for each $t \geq 0$ by setting

$$M(t) := \widehat{\boldsymbol{\Xi}}_t(\hat{g}) - \widehat{\boldsymbol{\Xi}}_0(\hat{g}) - \int_0^t \int_{\mathbb{R}^d} \Big[B(\hat{g})(x) + b\hat{g}(x) - F(x, \mathcal{V}_s)\hat{g}(x) \Big] \widehat{\boldsymbol{\Xi}}_s(dx)ds, \tag{7.4}$$

then M is a local $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$-martingale with localizing sequence $\widehat{T}_n = T_n \wedge \tau_n^Y$ with

$$\tau_n^Y := \inf\{s \geq 0; Y_s \geq n\}.$$

Proof. If we assume that $\boldsymbol{\Xi}^{XZ}$ is given as in Definition 7.1.3, then this proposition is follows directly from Case III.a, see Proposition 4.4.1, because

$$\widehat{\boldsymbol{\Xi}}(\hat{g}) = \boldsymbol{\Xi}^{XZ}(\hat{g}^*),$$

where $\hat{g}^*(x, z) = \hat{g}(x)g_z(z)$ and g_z is a twice continuous differentiable with $g_z(0) = 1$ and $g_z(z) = 0$ for all $z \geq 1$. The Proposition 4.4.1 tells us that

$$\hat{M}_t = \boldsymbol{\Xi}_t^{XZ}(\hat{g}^*) - \boldsymbol{\Xi}_0^{XZ}(\hat{g}^*) - \hat{A}(t), \tag{7.5}$$

is a local martingale with

$$\begin{aligned} \hat{A}(t) := & \int_0^t \int_{\hat{E}} B(\hat{g}^*)(x, z) + b\hat{g}^*(x, z)\, \boldsymbol{\Xi}_{s\text{-}}^{XZ}(dx, dz)\, ds \\ & + \int_0^t \int_{\hat{E}} \int_0^{\infty} \hat{g}^*(x, z + \mathbb{1}_{[0, F(x, \mathcal{V}_s)]}(p)) - \hat{g}^*(x, z)\, dp\, \boldsymbol{\Xi}_{s\text{-}}^{XZ}(dx, dz)\, ds, \end{aligned}$$

with $\hat{E} = \mathbb{R}^d \times \mathbb{R}$ and for all $t \geq 0$. Using again $\boldsymbol{\Xi}^{XZ}(\hat{g}^*) = \widehat{\boldsymbol{\Xi}}(\hat{g})$, $t \geq 0$, we can see that (7.5) is identical with (7.4). $\square$

The next lemma will be used during our proof of existence.

Lemma 7.1.8. *Let $(\mu, n \in \mathbb{N}) \subset \mathcal{M}_f(\mathbb{R}^d)$ be a sequence of measures. Assume one of the following conditions is satisfied:*

1. *Assume $\mu_{n+1} \leq \mu_n$ for all $n \in \mathbb{N}$.*
2. *Assume $\mu_{n+1} \geq \mu_n$ for all $n \in \mathbb{N}$ and it exits a measure $\eta \in \mathcal{M}_f(\mathbb{R}^d)$ with $\mu_n \leq \eta$ for all $n \in \mathbb{N}$.*

Then there exits a measure $\mu \in \mathcal{M}_f(\mathbb{R}^d)$, such that for all Borel sets $\Gamma \in \mathbb{B}(\mathbb{R}^d)$ holds

$$\mu(\Gamma) = \lim_{n\to\infty} \mu_n(\Gamma). \tag{7.6}$$

This implies that $\mu_n \overset{n\to\infty}{\longrightarrow} \mu$ in the weak topology.

Remark 7.1.9. *As we will see, this convergence holds even in total variation.*

Proof. Let denote by $\mathcal{M}_f^S(\mathbb{R}^d)$ the space of signed measures over $\mathbb{R}^d$. The total variation norm is given for a signed measure $\tilde{\mu} \in \mathcal{M}_f^S(\mathbb{R}^d)$ by

$$|\tilde{\mu}| := \sup_{P \in \mathcal{P}} \sum_{\Gamma \in P} \tilde{\mu}(\Gamma),$$

where $\mathcal{P}$ is the set of finite partitions of $\mathbb{R}^d$. The space $(\mathcal{M}_f^S(\mathbb{R}^d), |\cdot|)$ forms a Banach space, see Page 22 in [47]. For a non-negative measure holds

$$|\tilde{\mu}| = \mu(\mathbb{R}^d).$$

If we consider the sequence $(\mu_n, n \in \mathbb{N})$, then it holds, that $\mu_k - \mu_n$ is an ordinary measure for $k \geq n$ due to $\mu_n \leq \mu_k$. It holds

$$|\mu_k - \mu_n| = \mu_k(\mathbb{R}^d) - \mu_n(\mathbb{R}^d)$$

for $k \geq n$. Since $(\mu_n(\mathbb{R}^d), n \in \mathbb{N})$ forms a Cauchy sequence in $[0, \infty)$, the same holds for $(\mu_n, \in \mathbb{N})$ in $(\mathcal{M}_f^S(\mathbb{R}^d), |\cdot|)$. Hence there exists a measure μ with the property (7.6). Since convergence in total variation implies convergence in the weak topology, the second part of the statement is also true.
The second case is analogous. This time it holds for $k \geq n$

$$|\mu_k - \mu_n| = \mu_n(\mathbb{R}^d) - \mu_k(\mathbb{R}^d)$$

and $(\mu_n(\mathbb{R}^d), n \in \mathbb{N})$ forms a Cauchy sequence in $[0, \eta(\mathbb{R}^d)]$. □

Definition 7.1.10. *In the first case of lemma* (7.1.8)*, we like to write* $\mu = \inf_{n\to\infty} \mu_n$ *and in the second we like to write* $\mu = \sup_{n\to\infty} \mu_n$.

7.2 The Cut-Out Equation: Conditions for Existence and Uniqueness

We obtain our Poisson representations for the two classes of competitive models from Section 1.2 by cutting these out from the Kurtz-Rodrigues representation of the Dawson-Watanabe superprocess. But the intensity process of the Cut-Out process will only be a solution of for the martingale problem $\mathbf{Comp}(\mathbf{B}, a, b, F, \hat{\Theta}_0)$, see Definition 1.2.1, of the competitive model, when the Cut-Out process satisfies the Cut-Out equation.

Definition 7.2.1 (Cut-Out Equation). *Assume $F, \mathcal{V}$ and $\widehat{\Xi}$ are as in Definition 7.1.3, i.e.*

$$((\widehat{X}_i, \widehat{U}_i)_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}, \widehat{\Xi}) = \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \mathcal{V})),$$

and it holds $\mathbb{P}[\forall t \geq 0 : \mathcal{V}_t = \widehat{\Xi}_t] = 1$, indeed $\mathcal{V}$ and $\widehat{\Xi}$ are indistinguishable, then we can write

$$((\widehat{X}_i, \widehat{U}_i,)_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}, \widehat{\Xi}) = \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \widehat{\Xi})). \tag{7.7}$$

and we say that $\widehat{\Xi}$ (or $\mathcal{V}$) is a solution of the ***Cut-Out-equation*** *for $(\boldsymbol{\xi}^{\mathbb{W}}, F)$. We say that the solution is unique, if two solutions $\widehat{\Xi}^1$ and $\widehat{\Xi}^2$ are indistinguishable.*

We are going to present that the Cut-Out equation has a solution, when F satisfies certain conditions described below. We show in Section 7.3 and 7.4 that these conditions are satisfied by the Bolker-Pacala models or the singular interaction models described in the Definitions 1.2.4 and 1.2.5. These two classes of competitive models have some similarities and some differences, and the following conditions for F have been chosen in such a way that we can treat both model classes in a unified way. In Section 7.5 we show that Cut-Out equation has a unique solution, if the following conditions are satisfied.

Conditions 7.2.2. *Let $F : \mathbb{R}^d \times \mathcal{M}_f(\mathbb{R}^d) \to [0, \infty)$ be a measurable function for which there exists a non-decreasing sequence $(T_k, k \in \mathbb{N})$ of $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping times with*

$$\mathbb{P}\left[T_k \to \infty \text{ for } k \to \infty\right] = 1$$

and a sequence $(K_k, k \in \mathbb{N}) \subset [0, \infty)$ of constants which have together the following properties:

1. *The function F is non-decreasing in the $\mathcal{M}_f(\mathbb{R}^d)$-coordinate.*
2. *It holds $\mathbb{1}_{[0,T_k]} h[F, \Xi^X] \in \mathcal{L}^1(\mathbf{M})$ for all $k \in \mathbb{N}$ (which is equivalent to saying that $h[F, \Xi^X] \in \mathcal{L}^1_{loc}(\mathbf{M})$ with localizing sequence $(T_k, k \in \mathbb{N})$).*
3. *For all sequences $(\mathcal{V}^n, n \in \mathbb{N})$ of $\mathcal{F}^{\Xi,\mathbb{W}}$-predictable $\mathcal{M}_f(\mathbb{R}^d)$-valued processes with $\mathcal{V}^n_t(\omega) \leq \Xi^X_t(\omega)$ for all $(\omega, t) \in \Omega \times [0, \infty), n \in \mathbb{N}$, which are either increasing or decreasing, indeed it either holds*

$$\mathcal{V}^n_t(\omega) \leq \mathcal{V}^{n+1}_t(\omega) \quad \forall \omega \in \Omega, t \in [0, \infty), n \in \mathbb{N}, \tag{7.8}$$

or it holds

$$\mathcal{V}^n_t(\omega) \geq \mathcal{V}^{n+1}_t(\omega), \quad \forall \omega \in \Omega, t \in [0, \infty), n \in \mathbb{N}, \tag{7.9}$$

then it holds for the process $\mathcal{V}^{\infty}(\omega, t) = \lim\limits_{n\to\infty} \mathcal{V}^n(\omega, t)$ that

$$|||\, \mathbb{1}_{[0,T_k]}(h[F, \mathcal{V}^n] - h[F, \mathcal{V}^{\infty}])\, |||_{\mathbf{M}} \overset{n\to\infty}{\longrightarrow} 0,\ k \in \mathbb{N}.$$

4. *For all pairs of $\mathcal{M}_f(\mathbb{R}^d)$-valued processes $\mathcal{V}^-, \mathcal{V}^+ : \Omega \times [0,\infty) \to \mathcal{M}_f(\mathbb{R}^d)$ with $\mathcal{V}^- \leq \mathcal{V}^+ \leq \Xi^X$ $\mathbb{P} \otimes \ell eb[0,\infty)$-a.e. holds*

$$\begin{aligned} &\mathbb{1}_{[0,T_k]}(s) \int_{\mathbb{R}^d} \left[F(x,\mathcal{V}_s^+) - F(x,\mathcal{V}_s^-)\right] \; \Xi_s^X(dx) \\ &\leq \mathbb{1}_{[0,T_k]}(s) K_k \left[\mathcal{V}_s^+(\mathbb{R}^d) - \mathcal{V}_s^-(\mathbb{R}^d)\right] \quad \mathbb{P} \otimes \ell eb[0,\infty)\text{-a.e.} \end{aligned} \tag{7.10}$$

In order to motivate these conditions, let us recall the steps described in the summary on Page 32. One key step was the construction of two sequences $(\widehat{\Xi}^{\uparrow,n})_{n=1}^\infty$ and $(\widehat{\Xi}^{\downarrow,n})_{n=1}^\infty$ of measure-valued processes, see Definition 1.31. The first point in the Conditions 7.2.2 is necessary to ensure that $(\widehat{\Xi}^{\uparrow,n})_{n=1}^\infty$ is increasing and that $(\widehat{\Xi}^{\downarrow,n})_{n=1}^\infty$ is decreasing. While the second point of the Conditions 7.2.2 is necessary for the application of our integration theory, the third point ensures that the limits $\widehat{\Xi}^\uparrow$ and $\widehat{\Xi}^\downarrow$ of our sequences, see (1.31), can be described as the solution of the two dimensional system (1.32). The last condition is necessary to show that $\widehat{\Xi}^\uparrow$ and $\widehat{\Xi}^\downarrow$ are identical.

7.3 Bolker-Pacala-Models and the C.O.-Eq.

In this section we verify that the non-linear Bolker-Pacala models from Definition 1.2.4 satisfy the Conditions 7.2.2. We recall that the competition function F has in the case of Bolker-Pacala models the form

$$F(x,\mu) = \hat{F}(x, \pi(x,\mu)), \tag{7.11}$$

where $\hat{F} : \mathbb{R}^d \times [0,\infty) \to [0,\infty)$ is a continuous function and

$$\pi(x,\mu) = \int_{\mathbb{R}^d} \hat{\kappa}(|x-y|)\, \mu(dy), \; x \in \mathbb{R}^d, \mu \in \mathcal{M}_f(\mathbb{R}^d) \tag{7.12}$$

with $\hat{\kappa} : [0,\infty) \to [0,\infty)$ being a bounded, continuous decreasing function.

Proposition 7.3.1. *If we assume that the competition function $F : \mathbb{R}^d \times \mathcal{M}_f(\mathbb{R}^d) \to [0,\infty)$ has the properties found in Definition 1.2.4, then we can pick a sequence of constants $(K_k)_{k=1}^\infty$ with*

$$\hat{F}(x,y) \leq K_k \text{ and } |\hat{F}(x,y_1) - \hat{F}(x,y_2)| \leq K_k|y_1 - y_2| \forall x \in \mathbb{R}, y_1, y_2 \in [0, k\hat{\kappa}(0)] \tag{7.13}$$

and we can define a sequence of $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping times given by

$$T_k := \tau_k^Y \wedge k \quad \text{with } \tau_k^Y := \inf\{t \geq 0 : Y_t \geq k\}$$

such that $(F, (K_k, \tau_k^Y)_{k=1}^\infty)$ satisfies the Conditions 7.2.2.

Proof. We start by noting that $\hat{F}$ is locally Lipschitz continuous by the first point of Definition 1.2.4 which gives us the existence of $(K_k)_{k=1}^\infty$ satisfying (7.13). We will prove every Condition of 7.2.2 step by step. The competition function is non-decreasing in the $\mathcal{M}_f(\mathbb{R})$- coordinate, because the function $\hat{F}$ from (7.11) (or Definition 1.2.4) is non-decreasing in the second coordinate and the same holds true for π from (7.12). Next we wish to argue that $h[F, \Xi^X]$ is an element of $\mathcal{L}^1_{loc}(\mathbf{M})$ with localizing sequence $(T_k, k \in \mathbb{N})$. Since

$$\pi(x,\mu) \leq \hat{\kappa}(0)\mu(\mathbb{R}^d) \quad \forall x \in \mathbb{R}, \mu \in \mathcal{M}_f(\mathbb{R}^d),$$

because $\hat{\kappa}$ is non-negative and assumed to be decreasing. Since $\Xi_t^X(\mathbb{R}^d) = Y_t \le k$ for all $t \le T_k$, it follows by 7.13, that

$$\begin{aligned}
|||\,\mathbb{1}_{[0,T_k]}h[F,\Xi^X]\,|||_{\mathbf{M}} &= \mathbb{E}\left[\int_0^\infty \int_0^\infty \mathbb{1}_{[0,T_k\wedge\mathcal{T}_{EX}]}(s)\mathbb{1}_{[0,F(X_1(s\text{-}),\Xi_s^X)]}(p)\,dp\,ds\right] \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{1}_{[0,T_k\wedge\mathcal{T}_{EX}]}(s)\hat{F}(X_1(s\text{-}),\hat{\kappa}(0)Y_s)\,ds\right] \le K_k T_k \wedge \mathcal{T}_{EX} \le K_k k
\end{aligned}$$

and hence $\mathbb{1}_{[0,T_k]}h[F,\Xi^X] \in \mathcal{L}^1(\mathbf{M})$, so $h[F,\Xi^X] \in \mathcal{L}^1_{loc}(\mathbf{M})$ with localizing sequence $(T_k, k\in\mathbb{N})$.
For the third condition, we assume that $(\mathcal{V}^n, n\in\mathbb{N})$ are $\mathcal{M}_f(\mathbb{R}^d)$-valued processes satisfying one of the order conditions given by (7.8) or (7.9). Note that both conditions imply that $(\mathcal{V}_t^n(\omega), n\in\mathbb{N})$ is a Cauchy sequence in the total variation norm for all $(\omega,t)\in\Omega\times[0,\infty)$. The pointwise limit is given by the process $\mathcal{V}^\infty(\omega,t)$, which must be $\mathcal{F}^{\Xi,\mathbb{W}}$-predictable. Since the total variation norm is stronger than the weak topology on $\mathcal{M}_f(\mathbb{R}^d)$, it follows for all $x\in\mathbb{R}^d$:

$$F(x,\mathcal{V}_t^n(\omega)) \overset{n\to\infty}{\longrightarrow} F(x,\mathcal{V}_t^\infty(\omega)), \quad \omega\in\Omega, t\in[0,\infty).$$

Further it holds due to (7.13) and $\mathcal{V}_t^n(\mathbb{R}^d) \le Y_t \le k$ for $t \le T_k$ that

$$\mathbb{1}_{[0,T_k\wedge\mathcal{T}_{EX}]}(t)F(X_1(s\text{-}),\mathcal{V}_t^n) \le \mathbb{1}_{[0,T_k\wedge\mathcal{T}_{EX}]}(t)K_k,\ n\in\mathbb{N}\cup\{\infty\}.$$

So we can apply Lebesgue dominated convergence theorem and obtain

$$\begin{aligned}
&|||\,\mathbb{1}_{[0,T_k)}(h[F,\mathcal{V}_t^n] - h[F,\mathcal{V}_t^\infty])\,|||_{\mathbf{M}} \\
&= \int_0^t \mathbb{E}\left[\mathbb{1}_{[0,T_k\wedge\mathcal{T}_{EX}]}(s)\left(F(X_1(s\text{-}),\mathcal{V}_s^n) - F(X_1(s\text{-}),\mathcal{V}_s^\infty)\right)\right]ds
\end{aligned}$$

is converging against 0, which proves the third point found in the Conditions 7.2.2.
For the fourth point we make use that $\hat{F}$ is locally Lipschitz continuous, see (7.13), and decreasing in the second coordinate, see Definition 1.2.4, combining these two properties of $\hat{F}$ results in

$$\hat{F}(x,y_1) - \hat{F}(x,y_2) \le K_k(y_1 - y_2)$$

for all $x\in\mathbb{R}^d, y_1,y_2\in[0,\hat{\kappa}(0)k]$ with $y_1 \ge y_2$. If we now assume that $\mathcal{V}^-$ and $\mathcal{V}^+$ are as in the fourth point of the Conditions 7.2.2, then due to $\mathcal{V}_t^- \le \mathcal{V}_t^+ \le \Xi_t^X$ for $t \le T_k$ we obtain

$$\begin{aligned}
&\mathbb{1}_{[0,T_k]}(s)\int_{\mathbb{R}^d}\left[F(x,\mathcal{V}_s^+) - F(x,\mathcal{V}_s^-)\right]\,\Xi_s^X(dx) \\
&\le K_k\mathbb{1}_{[0,T_k]}(s)\int_{\mathbb{R}^d}\left[\pi(x,\mathcal{V}_s^+) - \pi(x,\mathcal{V}_s^-)\right]\,\Xi_s^X(dx) \\
&\le K_k\mathbb{1}_{[0,T_k]}(s)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\kappa(x,y)\left[\mathcal{V}_s^+(dy) - \mathcal{V}_s^-(dy)\right]\,\Xi_s^X(dx)
\end{aligned}$$

and since $\kappa(x,y) = \hat{\kappa}(||x-y||) \ge 0, x,y\in\mathbb{R}^d, \mathcal{V}_s^+(dy) - \mathcal{V}_s^-(dy) \ge 0, y\in\mathbb{R}^d$ and the fact that $\kappa : [0,\infty)\to[0,\infty)$ is a non-decreasing function and that $Y_t = \Xi^X(\mathbb{R}^d) \le k$ for $t \le T_k$, we can bound the above by:

$$\begin{aligned}
&K_k\hat{\kappa}(0)\mathbb{1}_{[0,T_k]}(s)\int_{\mathbb{R}^d}\left[\mathcal{V}_s^+(\mathbb{R}^d) - \mathcal{V}_s^-(\mathbb{R}^d)\right]\,\Xi_s^X(dx) \\
&\le K_k\hat{\kappa}(0)\mathbb{1}_{[0,T_k]}(s)\Xi_s^X(\mathbb{R}^d)\left[\mathcal{V}_s^+(\mathbb{R}^d) - \mathcal{V}_s^-(\mathbb{R}^d)\right] \\
&\le K_k\hat{\kappa}(0)k\mathbb{1}_{[0,T_k]}(s)\left[\mathcal{V}_s^+(\mathbb{R}^d) - \mathcal{V}_s^-(\mathbb{R}^d)\right].
\end{aligned}$$

Therefore the fourth condition is also true. □

7.4 Singular Interaction Models and the C.O.-Eq.

In this section we prove that the competitive model with singular interactions satisfies the Conditions 7.2.2. By Definition 1.2.5 we assumed that for the spatial space $\mathbb{R}^d = \mathbb{R}$ and that the particles move like one-dimensional Brownian motion, hence the operator B_X from Definition 1.2.5 is the Laplace operator:

$$\Delta_L(f)(x) = \frac{1}{2}\partial_x^2 f(x), \quad x \in \mathbb{R}, f \in C^2(\mathbb{R}). \tag{7.14}$$

As a consequence the process $\mathbf{\Xi}^X$ is a Superbrownian motion with branching rate $a > 0$ and drift $b \in \mathbb{R}$, indeed $\mathbf{\Xi}^X \sim \mathbf{DW}(B_X, a, b)$. As we will see in Proposition 7.4.1, the random measure $\mathbf{\Xi}_t^X$ is almost surely for all time points $t > 0$ absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and the density $\varphi_t^{\mathbf{\Xi}}(\omega, dx) = \mathbf{\Xi}_t^X(dx)$ is the unique solution of

$$\varphi_t^{\mathbf{\Xi}}(x) = \frac{1}{2}\partial_x^2\varphi_t^{\mathbf{\Xi}}(x) + b\varphi_t^{\mathbf{\Xi}}(x) + \sqrt{2a\varphi_t^{\mathbf{\Xi}}(x)}d\mathcal{W}(t,x), \tag{7.15}$$

where $\mathcal{W}$ is white noise over $[0,\infty) \times \mathbb{R}$, see Chapter one in [45](it may be necessary to extend the probability space to find $\mathcal{W}$). Hereby (7.15) means that we are looking for function-valued process $\varphi^{\mathbf{\Xi}}$ such that for all $\hat{g} \in C_b^2(\mathbb{R})$ holds:

$$\int_{\mathbb{R}} \varphi_t^{\mathbf{\Xi}}(x)\hat{g}(x)\,dx = \mathbf{\Xi}_0^X(\hat{g}) + \int_0^t \int_{\mathbb{R}} \varphi_s^{\mathbf{\Xi}}(x)\left[\partial_x^2(\hat{g})(x) - b\hat{g}(x)\right] dxds \tag{7.16}$$

$$+ \int_0^t \int_{\mathbb{R}} \hat{g}(x)\sqrt{2a\varphi_s^{\mathbf{\Xi}}(x)}d\mathcal{W}(s,x), \tag{7.17}$$

where the second line stands for the stochastic integral with respect to the white noise $\mathcal{W}$, see the Section “Stochastic integration” in Chapter two in [45]. We collect this important properties of $\varphi^{\mathbf{\Xi}}$ in the next proposition. In Definition 1.2.5 we assumed that our competitive models start with a continuous Lebesgue density with compact support. Since our construction will imply that $\mathbf{\Xi}_0^X = \widehat{\mathbf{\Xi}}_0$, we can assume that the same is true for underlying KR-representation $\mathbf{\Xi}^X$. Considering our ingredients from Assumption 2.1.2, this means that the random probability measure $\mathbf{Q}_0^X$ must have a Lebesgue density with the same properties, the two densities will differ from each other by the factor Y_0.

Proposition 7.4.1. *Assume that there exists a random function $\varphi_0 : \Omega \to C_c^+(\mathbb{R})$ such that*

$$\mathbb{P}\left[\forall f \in C_b(\mathbb{R}) : \mathbf{\Xi}_0^X(\hat{g}) = \int_{\mathbb{R}} \hat{g}(x)\varphi_0(x)dx\right] = 1,$$

then there exists $C_c^+(\mathbb{R})$-valued process $\varphi^{\mathbf{\Xi}} : \Omega \times [0,\infty) \to C_c^+(\mathbb{R})$ with $\varphi_0^{\mathbf{\Xi}} = \varphi_0$ a.s. and such that

$$\mathbb{P}\left[\forall f \in C_b(\mathbb{R}), t \in [0,\infty) : \mathbf{\Xi}_t^X(f) = \int_{\mathbb{R}} f(x)\varphi_t^{\mathbf{\Xi}}(x)dx\right] = 1. \tag{7.18}$$

and for $\varrho : \mathbb{R} \times \mathcal{M}_f \to [0,\infty)$ from Definition 1.2.5 holds

$$\mathbb{P}\left[\varrho(x, \mathbf{\Xi}_t^X) = \varphi_t^{\mathbf{\Xi}}(x),\ \forall x \in \mathbb{R},\ \forall t > 0\right] = 1. \tag{7.19}$$

Proof. For this proof we assume that the drift b is zero in (7.15), because Perkins considers only this case, see Theorem III.4.2 in [40]. But this is not a problem for us, because the Dawson-Girsanov transformation allows us the extend the above statements from the case $b = 0$ to the

case $b \neq 0$, see Corollary B.7.8.
According to Theorem III.4.2.(a), there exists a continuous stochastic process

$$\tilde{\varphi}^{\boldsymbol{\Xi}} : \Omega \times (0, t) \to C_c(\mathbb{R}),$$

for which (7.18) is true, when we exclude $t = 0$. If we now define $\varphi_t^{\boldsymbol{\Xi}} := \tilde{\varphi}_t^{\boldsymbol{\Xi}}$ for $t > 0$ and $\varphi_0^{\boldsymbol{\Xi}} := \varphi_0$, then $\varphi^{\boldsymbol{\Xi}}$ satisfies (7.18). But for the proof of Corollary 7.4.2 we need that $(t, x) \mapsto \varphi_t^X(x)$ is jointly continuous on $[0, \infty) \times \mathbb{R}$. While this follows for $(0, \infty) \times \mathbb{R}$ from the fact that $\tilde{\varphi}^{\boldsymbol{\Xi}}$ has continuous paths in $C_c(\mathbb{R})$, the case $t = 0$ is problematic, and is often not considered. Unfortunately the arguments of Perkins used to prove the continuity of $\tilde{\varphi}^{\boldsymbol{\Xi}}$ can not be easily extended to include $t = 0$, therefore we present a workaround.
Let us define the space

$$C^+_{tem}(\mathbb{R}) := \{f \in C^+(\mathbb{R}) : ||f||_\lambda < \infty \ \forall \lambda > 0\} \text{ with } ||f||_\lambda = \sup_{x \in \mathbb{R}} |f(x)| e^{-\lambda |x|},$$

note that $C_c^+(\mathbb{R}) \subset C^+_{tem}(\mathbb{R})$ and that $C^+_{tem}(\mathbb{R})$ is a complete metric space equipped with $d(f, g) = \sum_{n \in \mathbb{N}} (1 \wedge ||f - g||_{1/n})$. Since (7.15) is a special case (up to a time scale) of the SPDE considered by Tribe in [44], it follows by Theorem 2.2 that (7.15) admits a solution $\hat{\varphi}^{\boldsymbol{\Xi}} : \tilde{\Omega} \times [0, \infty) \to C^+_{tem}(\mathbb{R})$ with $\hat{\varphi}_0^{\boldsymbol{\Xi}} \sim \tilde{\varphi}_0^{\boldsymbol{\Xi}}$ that is continuous with respect to the above metric d and that is unique in law on $C([0, \infty), C^+_{tem}(\mathbb{R}))$ (a compact overview about these results can be found on Pages 3 and 4 of [25]).
Perkins tells us now in [40] that every solution of (7.15) implies a Dawson-Watanabe superprocess, indeed if we take the solution of $\hat{\varphi}$ derived by Tribe and define $\hat{\boldsymbol{\Xi}}^X : \tilde{\Omega} \times [0, \infty) \to$ by setting $\hat{\boldsymbol{\Xi}}_t^X(\hat{g}) = \int_{\mathbb{R}} \hat{g}(x) \hat{\varphi}^{\boldsymbol{\Xi}}(x) dx$, then $\hat{\boldsymbol{\Xi}}^X \sim \mathbf{DW}(B_X, a, 0)$ according to Theorem III.4.2.(c). With the help of the function $\varrho : \mathcal{M}_f(\mathbb{R}) \to [0, \infty)$ from Definition 1.2.5 we have

$$\varrho(x, \hat{\boldsymbol{\Xi}}_t^X) = \hat{\varphi}_t^{\boldsymbol{\Xi}}(x) \text{ and } \varrho(x, \boldsymbol{\Xi}_t^X) = \varphi_t^{\boldsymbol{\Xi}}(x), \quad \forall x \in \mathbb{R}, t \in [0, \infty),$$

see the second part of the "approximation theorem" on Page 321 in [26]. Since $\hat{\boldsymbol{\Xi}}^X$ and $\boldsymbol{\Xi}^X$ have the same law, the density-processes $\hat{\varphi}^{\boldsymbol{\Xi}}$ and $\varphi^{\boldsymbol{\Xi}}$ must have same law on $C((0, \infty), C^+_{tem}(\mathbb{R}))$ (again $t = 0$ has been excluded). Due to the latter point and due to the continuity of $\hat{\varphi}^{\boldsymbol{\Xi}}$ on the whole of $[0, \infty)$, we can see that $(\varphi_{1/n}^{\boldsymbol{\Xi}})_{n=1}^\infty$ is almost surely a Cauchy-sequence in $C^+_{tem}(\mathbb{R})$. Since $C^+_{tem}(\mathbb{R})$ is complete, there exists a random variable $\hat{\varphi}_0$ which is the limit. Why should $\hat{\varphi}_0$ be identical with $\varphi_0^{\boldsymbol{\Xi}}$? Perkins states in Remark III.4.1. in [40] that

$$\mathbb{P}\left[\forall \hat{g} \in C_b(\mathbb{R}) : \lim_{t \to 0} \int_{\mathbb{R}} \hat{g}(x) \varphi_t^{\boldsymbol{\Xi}}(x) dx = \boldsymbol{\Xi}^X(\hat{g})\right] = 1.$$

Combining this with the convergence in $C^+_{tem}(\mathbb{R})$ we can conclude that:

$$\mathbb{P}\left[\forall \hat{g} \in C_c(\mathbb{R}) : \int_{\mathbb{R}} \hat{g}(x) \hat{\varphi}_0 dx = \lim_{t \to 0} \int_{\mathbb{R}} \hat{g}(x) \varphi_t^{\boldsymbol{\Xi}}(x) dx = \int_{\mathbb{R}} \hat{g}(x) \varphi_0^{\boldsymbol{\Xi}}(x) dx\right] = 1,$$

and so $\varphi_0^{\boldsymbol{\Xi}} = \hat{\varphi}_0^{\boldsymbol{\Xi}}$. Consequently $\varphi^{\boldsymbol{\Xi}}$ is continuous at $t = 0$ in $(C^+_{tem}(\mathbb{R}), d)$ and since $\varphi_0^{\boldsymbol{\Xi}} \in C_c^+(\mathbb{R})$ almost surely, this limit also holds in $(C_c^+(\mathbb{R}), || \cdot ||_\infty)$. □

Corollary 7.4.2. *If φ is the process given by Proposition 7.4.1 and if we define for $k \in \mathbb{N}$*

$$\tilde{T}_k(\omega) := \left\{ t > 0 : \sup_{s \leq t, x \in \mathbb{R}} \varphi_t^{\boldsymbol{\Xi}}(\omega, x) \geq k \right\}, \tag{7.20}$$

then $\tilde{T}_k$ is a $\mathcal{F}^{\boldsymbol{\Xi}, \mathbb{W}}$-stopping time and it holds $\tilde{T}_k \to \infty$ a.s. for $k \to \infty$.

Proof. According to Proposition 7.4.1 the Lebesgue density $\varphi^{\mathbf{\Xi}}$ is continuous in $C_c^+(\mathbb{R})$ with respect to the norm $||\cdot||_\infty$, hence it holds $\tilde{T}_k \to \infty$ a.s. for $k \to \infty$. Further by (7.19) and the joint continuity of $\varphi^{\mathbf{\Xi}}$ we have $\varphi_t^{\mathbf{\Xi}} = \varrho(x, \mathbf{\Xi}_t^X)$ almost surely and so

$$\{\tilde{T}_k \leq t\} = \{ \sup_{q \in \mathbb{Q}^+ \cap [0,t], p \in \mathbb{Q}} \varrho(p, \mathbf{\Xi}_q^X)\} \in \mathcal{F}_t^{\mathbf{\Xi},\mathbb{W}}.$$

□

We proceed by showing that the competition function F of the non-linear singular interaction models satisfies the Conditions 7.2.2.

Proposition 7.4.3. *Let us assume that the competition function $F : \mathbb{R} \times \mathcal{M}_f(\mathbb{R}) \to [0, \infty)$ is defined as in the definition of the non-linear singular interaction models, see Definition 1.2.5. If we define for all $k \in \mathbb{N}$*

$$T_k := \tilde{T}_k \wedge k, \tag{7.21}$$

where $(\tilde{T}_k, k \in \mathbb{N})$ is defined as in (7.20)*, and if we choose the constants $(K_k)_{k=1}^\infty$ such that*

$$\hat{F}(x, y) \leq K_k \text{ and } |\hat{F}(x, y_1) - \hat{F}(x, y_2)| \leq K_k |y_2 - y_1| \text{ for all } x \in \mathbb{R}, y_1, y_2 \in [0, k], \tag{7.22}$$

then the competition function F paired with $(T_k, K_k, k \in \mathbb{N})$ satisfies the Conditions 7.2.2.

Proof. We argue that the competition function defined as in Definition 1.2.5 is non-decreasing in the $\mathcal{M}_f(\mathbb{R})$ coordinate. Due to the form of F in (1.24) and because ϱ is non-decreasing in the $\mathcal{M}_f(\mathbb{R})$ coordinate and and $\hat{F}$ in non-decreasing in the second coordinate, the competition function F must be non-decreasing in the $\mathcal{M}_f(\mathbb{R})$ coordinate.
We will prove the remaining three conditions for a fixed stopping time T_k defined as in (7.21). Let $\varphi^{\mathbf{\Xi}} : \Omega \times [0, \infty) \times \mathbb{R} \to [0, \infty)$ be the Lebesgue density of the Superbrownian motion $\mathbf{\Xi}^X$ from Proposition 7.4.1, then by the definition of the stopping time T_k we know $\varphi^{\mathbf{\Xi}}(\omega, t, x) \leq k$ for all $x \in \mathbb{R}$, if $t \leq T_k(\omega)$, and this together with (7.22) allows us to conclude that

$$\begin{aligned}
\mathbb{1}_{[0,T_k \wedge \mathcal{T}_{EX}]}(t) F(X_1(t\text{-}), \mathbf{\Xi}_t^X) &= \mathbb{1}_{[0,T_k \wedge \mathcal{T}_{EX}]}(t) \hat{F}(X_1(t\text{-}), \varphi^{\mathbf{\Xi}}(t, X_1(t\text{-}))) \\
&\leq \mathbb{1}_{[0,T_k \wedge \mathcal{T}_{EX}]}(t) K_k.
\end{aligned}$$

And so it holds for all $k \in \mathbb{N}$ that

$$\begin{aligned}
|||\, \mathbb{1}_{[0,T_k)} h[F, \mathbf{\Xi}^X]\, |||_{\mathbf{M}} &= \mathbb{E}\left[\int_0^\infty \int_0^\infty \mathbb{1}_{[0,T_k \wedge \mathcal{T}_{EX}]}(s) \mathbb{1}_{[0,F(X_1(s\text{-}),\mathbf{\Xi}_s^X)]}(p)\, dp\, ds\right] \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{1}_{[0,T_k \wedge \mathcal{T}_{EX}]}(s) F(X_1(s\text{-}), \mathbf{\Xi}_s^X)\, ds\right] \\
&\leq K_k T_k \wedge \mathcal{T}_{EX} < K_k k.
\end{aligned}$$

This proves the second assumption in the Conditions 7.2.2.
Now, let us assume that $(\mathcal{V}^n)_{n=1}^\infty$ is the sequence of measure valued processes and $\mathcal{V}^\infty$ their pointwise limit from the third point of the Conditions 7.2.2. Due to Proposition 7.4.1 and the assumptions made about the sequence $(\mathcal{V}^n)_{n=1}^\infty$, there exists a set $\tilde{\Omega}$ with $\mathbb{P}[\tilde{\Omega}] = 1$ and with $\varphi^{\mathbf{\Xi}}(\omega, \cdot, \cdot)$ is a jointly continuous function for all $\omega \in \tilde{\Omega}$. holds that and that for all $(\omega, t) \in \Omega \times [0, \infty)$ holds that $(\mathcal{V}_t^n(\omega),)_{n=1}^\infty$ is either an increasing or decreasing sequence of measures bounded by $\mathbf{\Xi}_t^X(\omega)$ with respect to the order "$\leq$" defined in Definition 7.1.5. This fact allows us to perform

the following argumentation pointwise for all $(\omega, t) \in \Omega \times [0, \infty)$.
Due to the increasing or decreasing nature and the boundedness we can conclude that $(\mathcal{V}^n(\omega, t),)_{n=1}^{\infty}$ is a Cauchy sequence with respect to the total variation norm on $\mathcal{M}_f(\mathbb{R})$ and is converging against $\mathcal{V}^{\infty}(\omega, t)$. Further since $\mathbf{\Xi}_t^X(\omega)$ is absolutely continuous and $\mathcal{V}^n(\omega, t) \leq \mathbf{\Xi}_t^X(\omega)$ for all $n \in \mathbb{N}$, it follows that $\mathcal{V}^{\infty}(\omega, t) \leq \mathbf{\Xi}_t^X(\omega)$ and that $\mathcal{V}^n(\omega, t)$ is absolutely continuous with respect to the Lebesgue measure for all $n \in \mathbb{N} \cup \{\infty\}$. Let us define for $n \in \mathbb{N} \cup \{\infty\}$ the functions

$$d\mathcal{V}^n : \Omega \times [0, \infty) \times \mathbb{R} \to [0, \infty]; \qquad d\mathcal{V}^n(\omega, t, x) := \varrho(x, \mathcal{V}_t^n(\omega)),$$

then $(d\mathcal{V}^n(\omega, t, \cdot), n \in \mathbb{N} \cup \{\infty\})$ are Lebesgue densities for $(\mathcal{V}_t^n(\omega), n \in \mathbb{N} \cup \{\infty\})$. By the definition of T_k and $\tilde{T}_k$, see Definition 7.21 and (7.20), it holds, with $\varphi^{\mathbf{\Xi}}$ being the Lebesgue density of $\mathbf{\Xi}^X$ as before, that

$$\begin{aligned} \mathbb{1}_{[0, T_k(\omega) \wedge \mathcal{T}_{EX}(\omega))}(t) d\mathcal{V}^n(\omega, x, t) &\leq \mathbb{1}_{[0, T_k(\omega) \wedge \mathcal{T}_{EX}(\omega))}(t) d\varphi^{\mathbf{\Xi}}(\omega, t, x) \\ &\leq \mathbb{1}_{[0, T_k(\omega) \wedge \mathcal{T}_{EX}(\omega))}(t) k. \end{aligned} \tag{7.23}$$

So we can use the boundedness of the densities $(d\mathcal{V}^n, n \in \mathbb{N} \cup \{\infty\})$ to write

$$\begin{aligned} &||| \left(\mathbb{1}_{[0, T_k)}(h[F, \mathcal{V}^n] - h[F, \mathcal{V}^{\infty}]\right) |||_{\mathbf{M}} \\ &= \int_0^{\infty} \mathbb{E}\left[\int_0^{\infty} \mathbb{1}_{[0, T_k \wedge \mathcal{T}_{EX}]}(s) \left|\mathbb{1}_{[0, F(X_1(s\text{-}), \mathcal{V}_s^n))}(p) - \mathbb{1}_{[0, F(X_1(s\text{-}), \mathcal{V}_s^{\infty}))}(p)\right| dp\right] ds \\ &= \int_0^{\infty} \mathbb{E}\left[\mathbb{1}_{[0, T_k \wedge \mathcal{T}_{EX}]}(s) \left|F(X_1(s\text{-}), \mathcal{V}_s^n)) - F(X_1(s\text{-}), \mathcal{V}_s^{\infty}))\right|\right] ds \\ &= \int_0^{\infty} \mathbb{E}\left[\mathbb{1}_{[0, T_k \wedge \mathcal{T}_{EX}]}(s) \left|F(X_1(s\text{-}), \mathcal{V}_s^n)) - F(X_1(s\text{-}), \mathcal{V}_s^{\infty}))\right|\right] ds \\ &\leq \int_0^{\infty} K_k \mathbb{E}\left[\mathbb{1}_{[0, T_k \wedge \mathcal{T}_{EX}]}(s) \left|d\mathcal{V}^n(s, X_1(s)) - d\mathcal{V}^{\infty}(s, X_1(s))\right|\right] ds, \end{aligned} \tag{7.24}$$

note that we used in the last part that $\mathbb{P}[\forall s \geq 0 : X_1(s\text{-}) = X_1(s)] = 1$, because in the case of a Superbrownian motion the process X_1 is just a Brownian motion (stopped at time $\mathcal{T}_{EX}$). We are now applying that

$$\mathfrak{L}(X_1(s) | \mathcal{F}_t^{\mathbf{\Xi}, \mathbb{W}}) = \mathbf{Q}_t^X, \ t \geq 0.$$

and so (7.24) can be written into

$$\begin{aligned} &\int_0^{\infty} K_k \mathbb{E}\left[\mathbb{1}_{[0, T_k \wedge \mathcal{T}_{EX}]}(s) \mathbb{E}\left[\left|d\mathcal{V}^n(s, X_1(s)) - d\mathcal{V}^{\infty}(s, X_1(s))\right| \middle| \mathcal{F}_s^{\mathbf{\Xi}, \mathbb{W}}\right]\right] ds \\ &= \int_0^{\infty} K_k \mathbb{E}\left[\mathbb{1}_{[0, T_k \wedge \mathcal{T}_{EX}]}(s) \int_{\mathbb{R}} \left|d\mathcal{V}^n(s, x) - d\mathcal{V}^{\infty}(s, x)\right| \mathbf{Q}_s^X(dx)\right] ds. \end{aligned}$$

Fixing again a pair $(\omega, t) \in \tilde{\Gamma}^c$ we want to prove that

$$\mathbb{1}_{[0, T_k(\omega) \wedge \mathcal{T}_{EX}(\omega))}(t) \int_{\mathbb{R}} \left|d\mathcal{V}^n(\omega, t, x) - d\mathcal{V}^{\infty}(\omega, t, x)\right| \mathbf{Q}_t^X(\omega, dx) \overset{n \to \infty}{\longrightarrow} 0. \tag{7.25}$$

Since the measures $(\mathcal{V}_t^n(\omega),)_{n=1}^{\infty}$ converge against $\mathcal{V}_t^{\infty}(\omega)$ in total variation, it follows for the densities

$$\begin{aligned} &\int_{\mathbb{R}} \left|d\mathcal{V}^n(\omega, t, x) - d\mathcal{V}^{\infty}(\omega, t, x)\right| dx \\ &= \int_{d\mathcal{V}^n > d\mathcal{V}^{\infty}} d\mathcal{V}^n(\omega, t, x) - d\mathcal{V}^{\infty}(\omega, t, x)\, dx + \int_{d\mathcal{V}^n < d\mathcal{V}^{\infty}} d\mathcal{V}^{\infty}(\omega, t, x) - d\mathcal{V}^n(\omega, t, x)\, dx \\ &= \left(\mathcal{V}_t^n(\omega) - \mathcal{V}_t^{\infty}(\omega)\right)^+(\mathbb{R}) + \left(\mathcal{V}_t^n(\omega) - \mathcal{V}_t^{\infty}(\omega)\right)^-(\mathbb{R}) = \left|\mathcal{V}_t^n(\omega) - \mathcal{V}_t^{\infty}(\omega)\right|_{\mathbf{T.V.}} \overset{n \to \infty}{\longrightarrow} 0, \end{aligned}$$

where $(\mathcal{V}_t^n(\omega) - \mathcal{V}_t^\infty(\omega))^+$ and $(\mathcal{V}_t^n(\omega) - \mathcal{V}_t^\infty(\omega))^-$ are the positive, respectively the negative part of the signed measure $\mathcal{V}_t^n(\omega) - \mathcal{V}_t^\infty(\omega)$. So it holds

$$d\mathcal{V}^n(\omega, t) \overset{n\to\infty}{\longrightarrow} d\mathcal{V}^\infty(\omega, t) \text{ in } L^1(\ell eb(\mathbb{R})) \tag{7.26}$$

Note that $\mathbf{Q}_t^X = \mathbf{\Xi}_t^X / Y_t$ for $t < \mathcal{T}_{EX}$, and so $\mathbf{Q}_t^X$ is also absolutely continuous with respect to the Lebesgue measure for $t \leq \mathcal{T}_{EX}$ with density $\varphi^{\mathbf{Q},X}(t,x) = \varphi^{\mathbf{\Xi}}(t,x)/Y(t)$. Further for $t < T_k \wedge \mathcal{T}_{EX}$ it holds $\varphi^{\mathbf{Q},X}(t,x) \leq k/Y(t)$. With this upper bound it follows from (7.26) that

$$\begin{aligned}
&\mathbb{1}_{[0,T_k(\omega)\wedge\mathcal{T}_{EX}(\omega))}(t) \int_{\mathbb{R}} |d\mathcal{V}^n(\omega,t,x) - d\mathcal{V}^\infty(\omega,t,x)|\, \varphi^{\mathbf{Q},X}(t,dx) \\
&\leq \mathbb{1}_{[0,T_k(\omega)\wedge\mathcal{T}_{EX}(\omega))}(t) k/Y(t) \int_{\mathbb{R}} |d\mathcal{V}^n(\omega,t,x) - d\mathcal{V}^\infty(\omega,t,x)|\, dx \overset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

and this proves (7.25). Since

$$\mathbb{1}_{[0,T_k\wedge\mathcal{T}_{EX}]}(s) \int_{\mathbb{R}} |d\mathcal{V}^n(s,x) - d\mathcal{V}^\infty(s,x)|\, \mathbf{Q}_s^X(dx) \leq 2k(T_k \wedge \mathcal{T}_{EX}) \leq 2k^2,$$

it follows from (7.25) and the Lebesgue convergence theorem that

$$\begin{aligned}
&|||\,(\mathbb{1}_{[0,T_k]}(h[F,\mathcal{V}^n] - h[F,\mathcal{V}^\infty])\,|||_{\mathbf{M}} \\
&\leq \int_0^\infty K_k \mathbb{E}\left[\mathbb{1}_{[0,T_k\wedge\mathcal{T}_{EX}]}(s)\, |d\mathcal{V}^n(s, X_1(s)) - d\mathcal{V}^\infty(s, X_1(s))|\right] ds \\
&= \int_0^\infty K_k \mathbb{E}\left[\mathbb{1}_{[0,T_k(\omega)\wedge\mathcal{T}_{EX}(\omega)]}(t) \int_{\mathbb{R}} |d\mathcal{V}^n(\omega,s,x) - d\mathcal{V}^\infty(\omega,s,x)|\, \mathbf{Q}_s^X(dx)\right] ds \\
&\overset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

and this proves the third point of the Conditions 7.2.2.
For the last point of the Conditions 7.2.2 we fix again a $(\omega, t) \in \tilde{\Gamma}$ and note that as in the case of the measures $(\mathcal{V}_t^n(\omega))$ we can argue that $\mathcal{V}_t^+(\omega)$ and $\mathcal{V}_t^-(\omega)$ are absolute continuous, that

$$d\mathcal{V}^+(\omega,t,x) = \varrho(x, \mathcal{V}^+(\omega,t)); \quad d\mathcal{V}^-(\omega,t,x) = \varrho(x, \mathcal{V}^-(\omega,t));$$

are Lebesgue densities for $\mathcal{V}_t^+(\omega)$ and $\mathcal{V}_t^-(\omega)$ with the property that for all $k \in \mathbb{N}$ holds

$$d\mathcal{V}^-(\omega,t,x) \leq d\mathcal{V}^+(\omega,t,x) \leq \varphi^{\mathbf{\Xi}}(\omega,t,x) \leq k \text{ for } t \leq T_k.$$

For the fact that F is non-decreasing and Lipschitz-continuous on $[0,k]$ with constant K_k as in (7.22), we can conclude that it must hold for all $x \in \mathbb{R}, y^-, y^+ \in [0,k]$ with $y^- \leq y^+$ that

$$\varrho(x, y^+) - \varrho(x, y^-) \leq K_k(y^+ - y^-).$$

Therefore it holds

$$\begin{aligned}
&\mathbb{1}_{[0,T_k(\omega)\wedge\mathcal{T}_{EX}(\omega))}(t) \int_{\mathbb{R}} F(x, \mathcal{V}_t^+) - F(x, \mathcal{V}_t^-)\, \mathbf{\Xi}_s^X(dt) \\
&= \mathbb{1}_{[0,T_k(\omega)\wedge\mathcal{T}_{EX}(\omega))}(t) \int_{\mathbb{R}} \hat{F}(x, d\mathcal{V}^+(t,x)) - \hat{F}(x, d\mathcal{V}^-(t,x))\, \varphi^{\mathbf{\Xi}}(t,x)\, dx \\
&\leq \mathbb{1}_{[0,T_k(\omega)\wedge\mathcal{T}_{EX}(\omega))}(t) K_k \int_{\mathbb{R}} \left(d\mathcal{V}^+(t,x) - d\mathcal{V}^-(t,x)\right) k dx \\
&\leq \mathbb{1}_{[0,T_k(\omega)\wedge\mathcal{T}_{EX}(\omega))}(t) K_k k \left(\mathcal{V}_t^+(\mathbb{R}) - \mathcal{V}_t^-(\mathbb{R})\right)
\end{aligned}$$

as it was desired. This proves the fourth point of the Conditions 7.2.2. □

7.5 The Cut-Out Equation: Proof of Existence and Uniqueness

In order to prove our main theorem, see Theorem 1.2.6, we need to show that the Cut-Out equation

$$((\widehat{X}_i, \widehat{U}_i)_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}, \widehat{\Xi}) = \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \widehat{\Xi})) \tag{7.27}$$

from Section 7.2 has a unique solution, when the competitive function $F : \mathbb{R}^d \times \mathcal{M}_f(\mathbb{R}^d) \to [0, \infty]$ satisfies the Conditions 7.2.2. Since the Conditions 7.2.2 are true for non-linear Bolker-Pacala models and non-linear singular interactive models according to the sections 7.3 and 7.4, this will be sufficient to prove our main result, see Theorem 1.2.6, that these models admit a Poisson representation.

The proof that (7.27) has a solution consist of four steps, the first three steps prove the existence and the last step the uniqueness.

In our first step we use Lemma 7.1.6 to obtain two sequences $(\widehat{\Xi}^{\downarrow,n}, n \in \mathbb{N}_0)$ and $(\widehat{\Xi}^{\uparrow,n}, n \in \mathbb{N})$ of $\mathcal{F}^{\Xi,\mathbb{W}}$-predictable $\mathcal{M}_f(\mathbb{R}^d)$-valued processes, where the first one is decreasing and the second one is increasing with respect to the partial order $\leq$ of measures defined in Definition 7.1.5.

Proposition 7.5.1 (Step I). *Assume that the competitive function $F : \mathbb{R}^d \times \mathcal{M}_f(\mathbb{R}^d) \to [0, \infty)$ satisfies Conditions 7.2.2. In this case we can define two sequences $(\widehat{\Xi}^{\uparrow,n}, n \in \mathbb{N}_0)$ and $(\widehat{\Xi}^{\downarrow,n}, n \in \mathbb{N}_0)$ of $\mathcal{M}_f(\mathbb{R}^d)$-valued processes, which are continuous in the weak topology, with*

$$\widehat{\Xi}^{\uparrow,n} : \Omega \times [0, \infty) \to \mathcal{M}_f(\mathbb{R}^d), \quad \widehat{\Xi}^{\downarrow,n} : \Omega \times [0, \infty) \to \mathcal{M}_f(\mathbb{R}^d), \; n \in \mathbb{N}_0,$$

by beginning with $(\widehat{\Xi}^{\uparrow,0}, \widehat{\Xi}^{\downarrow,0}) = (0, \Xi^X)$ and then setting recursively for each $n \in \mathbb{N}$, first

$$((\widehat{X}_i^{\uparrow,n}, \widehat{U}_i^{\uparrow,n})_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}^{\uparrow,n}, \widehat{\Xi}^{\uparrow,n}) = \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \widehat{\Xi}^{\downarrow,n-1})) \tag{7.28}$$

and then

$$((\widehat{X}_i^{\downarrow,n}, \widehat{U}_i^{\downarrow,n})_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}^{\downarrow,n}, \widehat{\Xi}^{\downarrow,n}) = \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \widehat{\Xi}^{\uparrow,n})). \tag{7.29}$$

It holds for $(\widehat{\Xi}^{\downarrow,n}, n \in \mathbb{N}_0)$ and $(\widehat{\Xi}^{\uparrow,n}, n \in \mathbb{N}_0)$:

$$\mathbb{P}\big[\ \forall n, m \in \mathbb{N}_0, \forall t \geq 0 : \widehat{\Xi}_t^{\uparrow,n} \leq \widehat{\Xi}_t^{\downarrow,m}, \widehat{\Xi}_t^{\downarrow,n} \geq \widehat{\Xi}_t^{\downarrow,n+1}, \widehat{\Xi}_t^{\uparrow,n} \leq \widehat{\Xi}_t^{\uparrow,n+1}\ \big]{=}1. \tag{7.30}$$

Further there exist two $\mathcal{F}^{\Xi,\mathbb{W}}$-predictable processes $\widehat{\Xi}^{\downarrow,\infty}, \widehat{\Xi}^{\uparrow,\infty} : \Omega \times [0, \infty) \to \mathcal{M}_f(\mathbb{R}^d)$ with

$$\mathbb{P}\left[\ \forall t \geq 0 : \widehat{\Xi}_t^{\downarrow,n} \overset{n\to\infty}{\longrightarrow} \widehat{\Xi}_t^{\downarrow,\infty}, \widehat{\Xi}_t^{\uparrow,n} \overset{n\to\infty}{\longrightarrow} \widehat{\Xi}_t^{\uparrow,\infty} \text{ in total variation}\right] = 1.$$

Proof. We prove (7.30) and that the sequences $(\widehat{\Xi}^{\downarrow,n}, n \in \mathbb{N}_0)$ and $(\widehat{\Xi}^{\uparrow,n}, n \in \mathbb{N}_0)$ are well-defined and continuous by showing via induction that for all $N \in \mathbb{N}_0$ it is true that

$$h[F, \widehat{\Xi}^{\downarrow,n}], h[F, \widehat{\Xi}^{\uparrow,n}] \in \mathcal{L}^1_{loc}(\mathbf{M}), \; 0 \leq n \leq N, \tag{7.31}$$

with localizing sequence $(T_k, k \in \mathbb{N})$, where $(T_k, k \in \mathbb{N})$ is the sequence of $\mathcal{F}^{\Xi,\mathbb{W}}$-stopping times from the Conditions 7.2.2, and that

$$\mathbb{P}\big[\ \forall t \geq 0 : \widehat{\Xi}_t^{\uparrow,0} \leq \widehat{\Xi}_t^{\uparrow,1} \leq ... \leq \widehat{\Xi}_t^{\uparrow,N+1} \leq \widehat{\Xi}_t^{\downarrow,N+1} \leq ... \leq \widehat{\Xi}_t^{\downarrow,1} \leq \widehat{\Xi}_t^{\downarrow,0}\ \big]{=}1. \tag{7.32}$$

We start with $N = 0$. Recall that by the Conditions 7.2.2 we have for all $k \in \mathbb{N}$ that

$$\mathbb{1}_{[0,T_k)} h[F, \Xi^X] \in \mathcal{L}^1(\mathbf{M}) \tag{7.33}$$

and so $h[F, \Xi^X]$ is an element of $\mathcal{L}^1_{stop}(\mathbf{M})$ with localizing sequence $(T_k, k \in \mathbb{N})$. Since $\widehat{\Xi}^{\uparrow,0} = 0 \leq \Xi^X$ and $\widehat{\Xi}^{\downarrow,0} = \Xi^X$ and since the competitive function F is increasing in the $\mathcal{M}_f(\mathbb{R})$-coordinate, it follows from the reverse order Lemma 7.1.6 that

$$h[F, \widehat{\Xi}^{\uparrow,0}], h[F, \widehat{\Xi}^{\downarrow,0}] \in \mathcal{L}^1_{loc}(\mathbf{M}),$$

so if we define $\widehat{\Xi}^{\uparrow,1}$ and $\widehat{\Xi}^{\downarrow,1}$ as in (7.28) and (7.29), then those are well-defined measure-valued processes which are continuous with respect to the weak topology, see Lemma 7.1.4. The reverse order Lemma 7.1.6 tells us also that

$$\mathbb{P}\left[\ \forall t \geq 0 : \widehat{\Xi}^{\uparrow,0}_t = 0 \leq \widehat{\Xi}^{\uparrow,1}_t \leq \widehat{\Xi}^{\downarrow,1}_t \leq \Xi^X_t = \widehat{\Xi}^{\downarrow,0}_t\right] = 1.$$

Hence we have proven (7.31) and (7.32). The case $N > 1$ works analogously. Indeed let us assume we have proven (7.31) and (7.32) for $N \in \mathbb{N}$. In order to prove that in this case (7.31) and (7.32) are also true, when we replace N by $N + 1$, we need to show that

$$h[F, \widehat{\Xi}^{\downarrow,N+1}], h[F, \widehat{\Xi}^{\uparrow,N+1}] \in \mathcal{L}^1_{loc}(\mathbf{M}) \tag{7.34}$$

with localizing sequence $(T_k, k \in \mathbb{N})$ and that

$$\mathbb{P}\left[\forall t \geq 0 : \widehat{\Xi}^{\uparrow,N+1}_t \leq \widehat{\Xi}^{\uparrow,N+2}_t \leq \widehat{\Xi}^{\downarrow,N+2}_t \leq \widehat{\Xi}^{\downarrow,N+1}_t\right] = 1. \tag{7.35}$$

But since (7.32) is true for N, it holds

$$\mathbb{P}\left[\forall t \geq 0 : \widehat{\Xi}^{\uparrow,N+1}_t \leq \widehat{\Xi}^{\downarrow,N+1}_t \leq \Xi^X_t\right] = 1$$

and from this we can conclude (7.34) is true, due to the Reverse order lemma 7.1.6 and the fact that $h[F, \Xi^X] \in \mathcal{L}^1_{stop}(\mathbf{M})$ with localizing sequence $(T_k, k \in \mathbb{N})$. So we can define $\widehat{\Xi}^{\uparrow,N+2}$ and $\widehat{\Xi}^{\downarrow,N+2}$ as in (7.28) and (7.29), where in our current situation we have to set $n = N + 1$, and obtain well-defined $\mathcal{M}_f(\mathbb{R}^d)$-valued processes with continuous paths in $\mathcal{M}_f(\mathbb{R}^d)$ with respect to the weak topology. With the help of the reverse order Lemma 7.1.6 and the fact that

$$\mathbb{P}\left[\forall t \geq 0 : \widehat{\Xi}^{\uparrow,N}_t \leq \widehat{\Xi}^{\uparrow,N+1}_t \leq \widehat{\Xi}^{\downarrow,N+1}_t \leq \widehat{\Xi}^{\downarrow,N}_t\right] = 1,$$

which of course is a partial result of (7.32), we can conclude that it holds:

$$\mathbb{P}\left[\forall t \geq 0 : \widehat{\Xi}^{\downarrow,N+1}_t \geq \widehat{\Xi}^{\downarrow,N+2}_t \geq \widehat{\Xi}^{\uparrow,N+2}_t \geq \widehat{\Xi}^{\uparrow,N+1}_t\right] = 1$$

but this is just (7.35) in reversed order. Consequently the statements (7.31) and (7.32) also true for $N + 1$. So we can conclude by induction that these statements are true for all $N \in \mathbb{N}$, which proves that $(\widehat{\Xi}^{\downarrow,n}, n \in \mathbb{N}_0)$ and $(\widehat{\Xi}^{\uparrow,n}, n \in \mathbb{N}_0)$ consists of well-defined measure valued processes with paths that are continuous in the weak topology of $\mathcal{M}_f(\mathbb{R}^d)$ and that (7.30) is true.
Due to (7.30) is true for all $(\omega, t) \in \Omega \times [0, \infty)$ that

$$\widehat{\Xi}^{\uparrow,n}_t(\omega) \leq \widehat{\Xi}^{\uparrow,n+1}_t(\omega) \text{ and } \widehat{\Xi}^{\downarrow,n}_t(\omega) \geq \widehat{\Xi}^{\downarrow,n+1}_t(\omega)$$

This implies that $(\widehat{\Xi}^{\downarrow,n}_t(\omega), n \in \mathbb{N})$ is a converging sequence in total variation and due to the $\widehat{\Xi}^{\uparrow,n}_t(\omega) \leq \Xi^X_t(\omega)$ the same is true for the sequence $(\widehat{\Xi}^{\uparrow,n}_t(\omega), n \in \mathbb{N})$, see Lemma 7.1.8. Since the space $\mathcal{M}_f(\mathbb{R}^d)$ is a Banach space with respect to the total variation norm, we can define $\widehat{\Xi}^{\uparrow,\infty}_t(\omega)$ and $\widehat{\Xi}^{\downarrow,\infty}_t(\omega)$ as the limits of $(\widehat{\Xi}^{\downarrow,n}_t(\omega), n \in \mathbb{N})$ and $\widehat{\Xi}^{\uparrow,n}_t(\omega) \leq \Xi^X_t(\omega)$. As a consequence of the fact that $(\widehat{\Xi}^{\downarrow,n}, n \in \mathbb{N}_0)$ and $(\widehat{\Xi}^{\uparrow,n}, n \in \mathbb{N}_0)$ are continuous and adapted, they are $\mathcal{F}^{\Xi,\mathbb{W}}$-predictable, and therefore the same is also true for $\widehat{\Xi}^{\downarrow,\infty}$ and $\widehat{\Xi}^{\downarrow,\infty}$ which are given by

$$\widehat{\Xi}^{\downarrow,\infty}_t(\omega)(\hat{g}) := \lim_{n\to\infty} \widehat{\Xi}^{\downarrow,n}_t(\omega)(\hat{g}) \text{ and } \widehat{\Xi}^{\uparrow,\infty}_t(\omega)(\hat{g}) := \lim_{n\to\infty} \widehat{\Xi}^{\uparrow,n}_t(\omega)(\hat{g})$$

for each $(\omega, t) \in \Omega \times [0, \infty)$ and each $\hat{g} \in C_b(\mathbb{R}^d)$. □

Proposition 7.5.2 (Step II). *Assume that* $(\widehat{\Xi}^{\downarrow,n}, \widehat{\Xi}^{\uparrow,n}, n \in \mathbb{N}_0 \cup \{\infty\})$ *are the processes defined in Proposition 7.5.1. If we define with the help of* $\widehat{\Xi}^{\downarrow,\infty}$ *and* $\widehat{\Xi}^{\uparrow,\infty}$ *the processes*

$$\widehat{\Xi}^{\uparrow} : \Omega \times [0,\infty) \to \mathcal{M}_f(\mathbb{R}^d), \quad \widehat{\Xi}^{\downarrow} : \Omega \times [0,\infty) \to \mathcal{M}_f(\mathbb{R}^d)$$

by setting

$$((\widehat{X}_i^{\uparrow}, \widehat{U}_i^{\uparrow})_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}^{\uparrow}, \widehat{\Xi}^{\uparrow}) = \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\ \cdot\ , \widehat{\Xi}^{\downarrow,\infty})), \tag{7.36}$$
$$((\widehat{X}_i^{\downarrow}, \widehat{U}_i^{\downarrow})_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}^{\downarrow}, \widehat{\Xi}^{\downarrow}) = \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\ \cdot\ , \widehat{\Xi}^{\uparrow,\infty})), \tag{7.37}$$

then it is true, that the new processes $\widehat{\Xi}^{\uparrow}$ *and* $\widehat{\Xi}^{\downarrow}$ *are modifications of* $\widehat{\Xi}^{\uparrow,\infty}$ *and* $\widehat{\Xi}^{\downarrow,\infty}$*. Further, the vector* $(\widehat{\Xi}^{\uparrow}, \widehat{\Xi}^{\downarrow})$ *is the solution of the two-dimensional equation:*

$$\begin{aligned} ((\widehat{X}_i^{\uparrow}, \widehat{U}_i^{\uparrow})_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}^{\uparrow}, \widehat{\Xi}^{\uparrow}) &= \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\ \cdot\ , \widehat{\Xi}^{\downarrow})), \\ ((\widehat{X}_i^{\downarrow}, \widehat{U}_i^{\downarrow})_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}^{\downarrow}, \widehat{\Xi}^{\downarrow}) &= \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\ \cdot\ , \widehat{\Xi}^{\uparrow})). \end{aligned} \tag{7.38}$$

Further it holds

$$\mathbb{P}\big[\forall n, m \in \mathbb{N}, \forall t \geq 0: \ \widehat{\Xi}_t^{\uparrow,n} \leq \widehat{\Xi}_t^{\uparrow} \leq \widehat{\Xi}_t^{\downarrow} \leq \widehat{\Xi}_t^{\downarrow,m}\big] = 1. \tag{7.39}$$

Proof. Due to the fact that

$$\mathbb{P}\big[\ \forall n \in \mathbb{N}_0, \forall t \geq 0\ :\ \widehat{\Xi}_t^{\downarrow,n} \geq \widehat{\Xi}_t^{\downarrow,n+1}, \widehat{\Xi}_t^{\uparrow,n} \leq \widehat{\Xi}_t^{\uparrow,n+1}\ \big] = 1,$$

and due to

$$\mathbb{P}\left[\ \forall t \geq 0 : \widehat{\Xi}_t^{\downarrow,n} \overset{n\to\infty}{\longrightarrow} \widehat{\Xi}^{\downarrow,\infty}, \widehat{\Xi}_t^{\uparrow,n} \overset{n\to\infty}{\longrightarrow} \widehat{\Xi}^{\uparrow,\infty} \text{ in total variation}\right] = 1, \tag{7.40}$$

it holds according to the Point 3 of the Conditions 7.2.2 for all $k \in \mathbb{N}$ that

$$\begin{aligned} &|||\, \mathbb{1}_{[0,T_k)}(h[F, \widehat{\Xi}^{\uparrow,n}] - h[F, \widehat{\Xi}^{\uparrow,\infty}])\, |||_{\mathbf{M}} \overset{n\to\infty}{\longrightarrow} 0, \\ &|||\, \mathbb{1}_{[0,T_k)}(h[F, \widehat{\Xi}^{\downarrow,n}] - h[F, \widehat{\Xi}^{\downarrow,\infty}])\, |||_{\mathbf{M}} \overset{n\to\infty}{\longrightarrow} 0. \end{aligned} \tag{7.41}$$

Let us now define for $n \in \mathbb{N} \cup \{\infty\}$ (note that we set $\infty - 1 = \infty$):

$$\begin{aligned} \big((X_i, Z_i^{\uparrow,n}, U_i)_{i=1}^{\infty}, \boldsymbol{\xi}^{XZ,\uparrow,n}, \boldsymbol{\Xi}^{XZ,\uparrow,n}, \mathbf{Q}^{XZ,\uparrow,n}\big) &= \mathbb{I}\big[h[F, \boldsymbol{\Xi}^{\downarrow,n-1}]\big], \\ \big((X_i, Z_i^{\downarrow,n}, U_i)_{i=1}^{\infty}, \boldsymbol{\xi}^{XZ,\downarrow,n}, \boldsymbol{\Xi}^{XZ,\downarrow,n}, \mathbf{Q}^{XZ,\downarrow,n}\big) &= \mathbb{I}\big[h[F, \boldsymbol{\Xi}^{\uparrow,n}]\big]. \end{aligned}$$

Because of (7.41), Corollary 5.2.4 tells us that if we choose a $\hat{g}^x \in C_c^2(\mathbb{R}^d)$ and a $\hat{g}^z \in C^2(\mathbb{R})$ with $\hat{g}^z(0) = 1$ and $\hat{g}^z(z) = 0$ for all $z \geq 1$, then there exist two subsequences of $(\boldsymbol{\xi}^{XZ,\uparrow,n_m}, m \in \mathbb{N})$ and $(\boldsymbol{\xi}^{XZ,\downarrow,n_m}, m \in \mathbb{N})$ such that

$$\begin{aligned} &\sup_{s\leq t} |\boldsymbol{\Xi}_s^{XZ,\uparrow,n_m}(\hat{g}^{xz}) - \boldsymbol{\Xi}_s^{XZ,\uparrow,\infty}(\hat{g}^{xz})| \overset{m\to\infty}{\longrightarrow} 0 \ \ a.s., \, t \geq 0, \\ &\sup_{s\leq t} |\boldsymbol{\Xi}_s^{XZ,\downarrow,n_m}(\hat{g}^{xz}) - \boldsymbol{\Xi}_s^{XZ,\downarrow,n}(\hat{g}^{xz})| \overset{m\to\infty}{\longrightarrow} 0 \ \ a.s., \, t \geq 0, \end{aligned} \tag{7.42}$$

where $\hat{g}^{xz} = \hat{g}^x \hat{g}^z$. But by the definition of the cut-out process, see Definition 7.1.3, and the definition of $(\widehat{\Xi}^{\uparrow,n}, n \in \mathbb{N}_0)$ and $(\widehat{\Xi}^{\downarrow,n}, n \in \mathbb{N}_0)$ from Proposition 7.5.1, it holds

$$\boldsymbol{\Xi}_s^{XZ,\uparrow,n}(\hat{g}^{xz}) = \widehat{\Xi}^{\uparrow,n}(\hat{g}^x) \text{ and } \boldsymbol{\Xi}_s^{XZ,\downarrow,n}(\hat{g}^{xz}) = \widehat{\Xi}^{\downarrow,n}(\hat{g}^x), n \in \mathbb{N},$$

and

$$\Xi_s^{XZ,\uparrow,\infty}(\hat{g}^{xz}) = \widehat{\Xi}^{\uparrow}(\hat{g}^x) \text{ and } \Xi_s^{XZ,\downarrow,\infty}(\hat{g}^{xz}) = \widehat{\Xi}^{\downarrow}(\hat{g}^x).$$

So if we combine (7.42) with (7.40), then it holds for all $\hat{g}^x \in C_c^2(\mathbb{R}^d)$ and all $t \geq 0$ that

$$\widehat{\Xi}_t^{\uparrow,\infty}(\hat{g}^x) = \widehat{\Xi}_t^{\uparrow}(\hat{g}^x) \text{ and } \widehat{\Xi}_t^{\downarrow,\infty}(\hat{g}^x) = \widehat{\Xi}_t^{\downarrow}(\hat{g}^x) \ a.s.$$

Since there exists a countable, separating family $(\hat{g}_n^x, n \in \mathbb{N})$ in $C_c^2(\mathbb{R}^d)$, it holds that $\widehat{\Xi}_t^{\uparrow,\infty} = \widehat{\Xi}_t^{\uparrow}$ and $\widehat{\Xi}_t^{\downarrow,\infty} = \widehat{\Xi}_t^{\downarrow}$ almost surely. This also implies that

$$|||\, h[F, \widehat{\Xi}^{\uparrow}] - h[F, \widehat{\Xi}^{\uparrow,\infty}]\, |||_{\mathbf{M}} = 0, \ \ |||\, h[F, \widehat{\Xi}^{\downarrow}] - h[F, \widehat{\Xi}^{\downarrow,\infty}]\, |||_{\mathbf{M}} = 0. \tag{7.43}$$

So if we define $\overline{\Xi}^{\uparrow}$ and $\overline{\Xi}^{\downarrow}$ by setting

$$((\overline{X}_i^{\uparrow}, \overline{U}_i^{\uparrow})_{i=1}^{\infty}, \bar{\boldsymbol{\xi}}^{\uparrow}, \overline{\Xi}^{\uparrow}) = \textbf{Cut-Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \widehat{\Xi}^{\downarrow})),$$
$$((\overline{X}_i^{\downarrow}, \overline{U}_i^{\downarrow})_{i=1}^{\infty}, \bar{\boldsymbol{\xi}}^{\downarrow}, \overline{\Xi}^{\downarrow}) = \textbf{Cut-Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \widehat{\Xi}^{\uparrow})),$$

then (7.43) implies that $\widehat{\Xi}^{\uparrow}$ and $\overline{\Xi}^{\uparrow}$ are indistinguishable and $\widehat{\Xi}^{\downarrow}$ and $\overline{\Xi}^{\downarrow}$ are indistinguishable. Finally (7.39) follows from the fact that $\widehat{\Xi}^{\uparrow,\infty}$ and $\widehat{\Xi}^{\uparrow}$ are modifications and the fact that $\widehat{\Xi}^{\downarrow,\infty}$ and $\widehat{\Xi}^{\downarrow}$ are modifications combined with the fact that $\widehat{\Xi}^{\uparrow}, \widehat{\Xi}^{\downarrow}, (\widehat{\Xi}^{\uparrow,n}, n \in \mathbb{N})$ and $(\widehat{\Xi}^{\downarrow,n}, n \in \mathbb{N})$ are continuous in the weak topology. □

The third step is the most complicated one.

Proposition 7.5.3 (Step III)**.**
Assume that the two processes $\widehat{\Xi}^{\uparrow}$ and $\widehat{\Xi}^{\downarrow}$ are defined as in Proposition 7.5.2, then

$$\mathbb{P}\left[\forall t \geq 0 : \widehat{\Xi}_t^{\uparrow} = \widehat{\Xi}_t^{\downarrow}\right] = 1. \tag{7.44}$$

Proof. Fist let us argue, why it is sufficient to prove for a fixed $t \geq 0$, that

$$\mathbb{E}[\widehat{\Xi}_t^{\downarrow}(\mathbb{R}^d) - \widehat{\Xi}_t^{\uparrow}(\mathbb{R}^d)] = 0, \tag{7.45}$$

to obtain (7.44). Note, that we already know, that

$$\mathbb{P}\big[\forall t \geq 0 : \ \widehat{\Xi}_t^{\uparrow} \leq \widehat{\Xi}_t^{\downarrow}\big] = 1, \tag{7.46}$$

which implies amongst other things, that $\widehat{\Xi}_t^{\downarrow}(\mathbb{R}^d) - \widehat{\Xi}_t^{\uparrow}(\mathbb{R}^d) \geq 0$ almost surely. So by proving (7.45), we obtain

$$\widehat{\Xi}_t^{\downarrow}(\mathbb{R}^d) = \widehat{\Xi}_t^{\uparrow}(\mathbb{R}^d) \ a.s. \tag{7.47}$$

Since it holds almost surely for all Borel sets $\Gamma \in \mathbb{B}(\mathbb{R}^d)$, that

$$\widehat{\Xi}_t^{\downarrow}(\Gamma) - \widehat{\Xi}_t^{\uparrow}(\Gamma) \leq \widehat{\Xi}_t^{\downarrow}(\mathbb{R}^d) - \widehat{\Xi}_t^{\uparrow}(\mathbb{R}^d), \quad t \geq 0,$$

we can derive from (7.46) and (7.47) that

$$\widehat{\Xi}_t^{\downarrow} = \widehat{\Xi}_t^{\uparrow} \ a.s. \tag{7.48}$$

So by showing that (7.45) holds true, we have shown that the processes $\widehat{\Xi}^{\downarrow}$ and $\widehat{\Xi}^{\uparrow}$ are modifications of each other. But both are additionally continuous in the weak topology, hence (7.44)

follows from (7.48), which in turn followed from (7.45).
How do we prove (7.45)? We will make use of the Gronwall inequality and the semi-martingale decomposition of $\widehat{\mathbf{\Xi}}^{\uparrow}$ and $\widehat{\mathbf{\Xi}}^{\downarrow}$. Since $h[F, \widehat{\mathbf{\Xi}}^{\uparrow}], h[F, \widehat{\mathbf{\Xi}}^{\downarrow}] \in \mathcal{L}^1_{loc}(\mathbf{M})$ with localizing sequence $(T_k, k \in \mathbb{N})$ and due to (7.38), we can apply Proposition 7.1.7 to see that there exists local $\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}$-martingales $\hat{M}^{\uparrow}$ and $\hat{M}^{\downarrow}$ with localizing sequences

$$\widehat{\tau}_k := \tau_k^Y \wedge T_k, \quad \tau_k^Y := \{s > 0 : Y_s = \mathbf{\Xi}_s^X(\mathbb{R}^d) \geq k\}$$

such that it holds

$$\widehat{\mathbf{\Xi}}_t^{\downarrow}(\mathbb{R}^d) = \widehat{\mathbf{\Xi}}_0^{\downarrow}(\mathbb{R}^d) + \hat{M}^{\downarrow}(t) + \int_0^t b\, \widehat{\mathbf{\Xi}}_s^{\downarrow}(\mathbb{R}^d)\, ds - \int_0^t \int_{\mathbb{R}^d} F(x, \widehat{\mathbf{\Xi}}_s^{\uparrow})\, \widehat{\mathbf{\Xi}}_s^{\downarrow}(dx)\, ds \tag{7.49}$$

$$\widehat{\mathbf{\Xi}}_t^{\uparrow}(\mathbb{R}^d) = \widehat{\mathbf{\Xi}}_0^{\uparrow}(\mathbb{R}^d) + \hat{M}^{\uparrow}(t) + \int_0^t b\, \widehat{\mathbf{\Xi}}_s^{\uparrow}(\mathbb{R}^d)\, ds - \int_0^t \int_{\mathbb{R}^d} F(x, \widehat{\mathbf{\Xi}}_s^{\downarrow})\, \widehat{\mathbf{\Xi}}_s^{\uparrow}(dx)\, ds, \tag{7.50}$$

where $\hat{M}^{\downarrow}$ and $\hat{M}^{\uparrow}$ are local $\mathcal{F}^{\mathbf{\Xi},\mathbb{W}}$-martingales with $\hat{M}_0^{\downarrow} = 0$ and $\hat{M}_0^{\uparrow} = 0$ a.s. Let us define for each $k \in \mathbb{N}$ the function $v_k : [0, \infty) \to [0, \infty)$ by setting for each $t \geq 0$:

$$v_k(t) := \mathbb{E}\big[\mathbb{1}_{[0,\widehat{\tau}_k)}(t)\big(\widehat{\mathbf{\Xi}}_t^{\downarrow}(\mathbb{R}^d) - \widehat{\mathbf{\Xi}}_t^{\uparrow}(\mathbb{R}^d)\big)\big].$$

If we can show, that $v_k(t) = 0$ for all $k \geq 0$, then we can obtain our goal by applying the Lemma of Fatou, indeed

$$\begin{aligned}
\mathbb{E}[\widehat{\mathbf{\Xi}}_t^{\downarrow}(\mathbb{R}^d) - \widehat{\mathbf{\Xi}}_t^{\uparrow}(\mathbb{R}^d)] &= \mathbb{E}\left[\liminf_{k\to\infty} \mathbb{1}_{[0,\widehat{\tau}_k)}(t) \left(\widehat{\mathbf{\Xi}}_t^{\downarrow}(\mathbb{R}^d) - \widehat{\mathbf{\Xi}}_t^{\uparrow}(\mathbb{R}^d)\right)\right] \\
&\leq \liminf_{k\to\infty} \mathbb{E}\left[\mathbb{1}_{[0,\widehat{\tau}_k)}(t) \left(\widehat{\mathbf{\Xi}}_t^{\downarrow}(\mathbb{R}^d) - \widehat{\mathbf{\Xi}}_t^{\uparrow}(\mathbb{R}^d)\right)\right] \\
&= \lim_{k\to\infty} v_k(t) = 0.
\end{aligned}$$

We will show that $v_k(t) = 0$ for all $k \geq 0$ with the help of the Gronwall lemma, therefore we will prove that there exists a constant $\hat{K}_k \geq 0$, for which $v_k(t)$ can be estimated from above by

$$v_k(t) \leq \int_0^t \hat{K}_k v_k(s)\, ds. \tag{7.51}$$

Then the Gronwall lemma will tell us that v_k can be bounded by

$$v_k(t) \leq v_k(0) \exp\left(\hat{K}_k t\right).$$

But since

$$\widehat{\mathbf{\Xi}}_0^{\downarrow}(\mathbb{R}^d) = \widehat{\mathbf{\Xi}}_0^{\uparrow}(\mathbb{R}^d) = \widehat{\mathbf{\Xi}}^X(\mathbb{R}^d),$$

it follows that $v_k(t)$ is equal to zero for all time points $t \geq 0$. Consequently all we need to do is to show (7.51). Combining the semi-martingale decompositions of $\widehat{\mathbf{\Xi}}^{\downarrow}$ and $\widehat{\mathbf{\Xi}}^{\uparrow}$ from (7.49) and (7.50) with the fact that $\hat{M}^{\downarrow}(\,\cdot \wedge \widehat{\tau}_k)$ and $\hat{M}^{\uparrow}(\,\cdot \wedge \widehat{\tau}_k)$ are martingales with $\hat{M}^{\downarrow}(0) = 0$ and $\hat{M}^{\uparrow}(0) = 0$ gives us a first upper bound for v_k by

$$\begin{aligned}
v_k(t) &\leq \mathbb{E}\left[\widehat{\mathbf{\Xi}}_{t\wedge\widehat{\tau}_k}^{\downarrow}(\mathbb{R}^d) - \widehat{\mathbf{\Xi}}_{t\wedge\widehat{\tau}_k}^{\uparrow}(\mathbb{R}^d)\right] \\
&= \int_0^t b\mathbb{E}\big[\mathbb{1}_{[0,\widehat{\tau}_k]}(s)\big(\widehat{\mathbf{\Xi}}_s^{\downarrow}(\mathbb{R}^d) - \widehat{\mathbf{\Xi}}_s^{\uparrow}(\mathbb{R}^d)\big)\big]\, ds \\
&\quad + \int_0^t \mathbb{E}\Big[\mathbb{1}_{[0,\widehat{\tau}_k]}(s)\Big(\int_{\mathbb{R}^d} F(x, \widehat{\mathbf{\Xi}}_s^{\downarrow})\, \widehat{\mathbf{\Xi}}_s^{\uparrow}(dx) - \int_{\mathbb{R}^d} F(x, \widehat{\mathbf{\Xi}}_s^{\uparrow})\, \widehat{\mathbf{\Xi}}_s^{\downarrow}(dx)\Big)\Big]\, ds.
\end{aligned} \tag{7.52}$$

We are interested in finding a suitable upper bound for (7.52). By adding and subtracting the inner expression of (7.52) turns into:

$$\mathbb{1}_{[0,\widehat{\tau}_k)}(s)\Big(\int_{\mathbb{R}^d} F(x,\widehat{\Xi}^{\downarrow}_s)\,\widehat{\Xi}^{\uparrow}_s(dx) - \int_{\mathbb{R}^d} F(x,\widehat{\Xi}^{\uparrow}_s)\,\widehat{\Xi}^{\downarrow}_s(dx)\Big) \tag{7.53}$$

$$= \mathbb{1}_{[0,\widehat{\tau}_k)}(s)\Big(\int_{\mathbb{R}^d} F(x,\widehat{\Xi}^{\downarrow}_s)\,\widehat{\Xi}^{\uparrow}_s(dx) - \int_{\mathbb{R}^d} F(x,\widehat{\Xi}^{\uparrow}_s)\,\widehat{\Xi}^{\uparrow}_s(dx)\Big) \tag{7.54}$$

$$+ \mathbb{1}_{[0,\widehat{\tau}_k)}(s)\Big(\int_{\mathbb{R}^d} F(x,\widehat{\Xi}^{\uparrow}_s)\,\widehat{\Xi}^{\uparrow}_s(dx) - \int_{\mathbb{R}^d} F(x,\widehat{\Xi}^{\uparrow}_s)\,\widehat{\Xi}^{\downarrow}_s(dx)\Big). \tag{7.55}$$

The term (7.55) is negative. Indeed $F \geq 0$ and $\widehat{\Xi}^{\uparrow} \leq \widehat{\Xi}^{\downarrow}$, so

$$\int_{\mathbb{R}^d} F(x,\widehat{\Xi}^{\uparrow}_s)\,\Big(\widehat{\Xi}^{\uparrow}_s(dx) - \widehat{\Xi}^{\downarrow}_s(dx)\Big) \leq 0, \quad x \in \mathbb{R}^d, s \geq 0.$$

Therefore we can simply drop (7.55), since we are looking for an upper bound of (7.53). Considering (7.54) we note that due to $\widehat{\Xi}^{\uparrow}_s \leq \Xi^{\downarrow}_s$ for all $s \geq 0$ and the fact that F is non decreasing in the second coordinate, it follows that

$$F(x,\widehat{\Xi}^{\downarrow}_s) - F(x,\widehat{\Xi}^{\uparrow}_s) \geq 0, \quad s \in [0,\infty), x \in \mathbb{R}^d. \tag{7.56}$$

Since (7.56) is positive and $\widehat{\Xi}^{\downarrow}_s \leq \Xi^X_s$, we can bound (7.54) from above by

$$\begin{aligned} &\mathbb{1}_{[0,\widehat{\tau}_k)}(s)\int_{\mathbb{R}^d} F(x,\widehat{\Xi}^{\downarrow}_s) - F(x,\widehat{\Xi}^{\uparrow}_s)\widehat{\Xi}^{\uparrow}_s(dx) \leq \\ &\mathbb{1}_{[0,\widehat{\tau}_k)}(s)\left(\int_{\mathbb{R}^d} F(x,\widehat{\Xi}^{\downarrow}_s) - F(x,\widehat{\Xi}^{\uparrow}_s)\,\Xi^X_s(dx)\right). \end{aligned} \tag{7.57}$$

But this the situation of (7.10) from the Conditions 7.2.2, so (7.57) can be bounded from above again $\mathbb{P}\otimes \ell eb[0,\infty)$ almost everywhere by

$$\mathbb{1}_{[0,\widehat{\tau}_k)}(s)K_k\big[\widehat{\Xi}^{\downarrow}_s(\mathbb{R}^d) - \widehat{\Xi}^{\uparrow}_s(\mathbb{R}^d)\big].$$

So combining our results about (7.54) and (7.55) we get:

$$\begin{aligned} \mathbb{1}_{[0,\widehat{\tau}_k)}(s)\Big(&\int_{\mathbb{R}^d} F(x,\widehat{\Xi}^{\downarrow}_s)\,\widehat{\Xi}^{\uparrow}_s(dx) - \int_{\mathbb{R}^d} F(x,\widehat{\Xi}^{\uparrow}_s)\,\widehat{\Xi}^{\downarrow}_s(dx)\Big) \\ &\leq \mathbb{1}_{[0,\widehat{\tau}_k)}(s)K_k\big[\widehat{\Xi}^{\downarrow}_s(\mathbb{R}^d) - \widehat{\Xi}^{\uparrow}_s(\mathbb{R}^d)\big]\ \mathbb{P}\otimes \ell eb[0,\infty)\text{-}a.e. \end{aligned}$$

All in all, we can bound now the function v_k for all $t \geq 0$ by

$$v_k(t) \leq \int_0^t b\mathbb{E}\big[\mathbb{1}_{[0,\widehat{\tau}_k)}(s)\big(\widehat{\Xi}^{\downarrow}_s(\mathbb{R}^d) - \widehat{\Xi}^{\uparrow}_s(\mathbb{R}^d)\big)\big] + \int_0^t K_k\mathbb{E}\big[\mathbb{1}_{[0,\widehat{\tau}_k)}(s)\big(\widehat{\Xi}^{\downarrow}_s(\mathbb{R}^d) - \widehat{\Xi}^{\uparrow}_s(\mathbb{R}^d)\big)\big].$$

So the inequality required for the application of the Gronwall lemma takes the form:

$$v_k(t) \leq \int_0^t [|b| + K_k]v_k(s)\,ds.$$

Based on our previous thoughts this proves the claim. □

We combine all three previous steps in one theorem and add uniqueness.

Theorem 7.5.4. *If F satisfies the Conditions 7.2.2, then there exists a process $\widehat{\Xi} : \Omega \times [0, \infty) \to \mathcal{M}_f(\mathbb{R}^d)$ which solves the equation*

$$((\widehat{X}_i, \widehat{U}_i)_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}, \widehat{\Xi}) = \mathbf{Cut}\text{-}\mathbf{Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \widehat{\Xi})). \tag{7.58}$$

This solution is unique, in the sense, if there exists a second process $\overline{\Xi} : \Omega \times [0, \infty) \to \mathcal{M}_f(\mathbb{R}^d)$ being the solution of

$$((\overline{X}_i, \overline{U}_i)_{i=1}^{\infty}, \bar{\boldsymbol{\xi}}, \overline{\Xi}) = \mathbf{Cut}\text{-}\mathbf{Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \overline{\Xi})), \tag{7.59}$$

then $((\widehat{X}_i, \widehat{U}_i)_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}, \widehat{\Xi})$ and $((\overline{X}_i, \overline{U}_i)_{i=1}^{\infty}, \bar{\boldsymbol{\xi}}, \overline{\Xi})$ are indistinguishable from each other.

Proof. For the first part, if we set $\widehat{\Xi} = \widehat{\Xi}^{\downarrow}$, then this process satisfies the equation (7.58), because $(\widehat{\Xi}^{\downarrow}, \widehat{\Xi}^{\uparrow})$ form a solution of the equation system (7.38) and that $\widehat{\Xi}^{\downarrow}$ and $\widehat{\Xi}^{\uparrow}$ are indistinguishable from each other.
Assume now that (7.59) is a second solution to the competitive equation of F with driving signal $\boldsymbol{\xi}^{\mathbb{W}}$. It is sufficient to show that

$$\mathbb{P}\left[\widehat{\Xi}_t = \overline{\Xi}_t,\ t \geq 0\right] = 1. \tag{7.60}$$

We recall the sequences $(\widehat{\Xi}^{\uparrow,n}, n \in \mathbb{N}_0)$ and $(\widehat{\Xi}^{\downarrow,n}, n \in \mathbb{N}_0)$ from Proposition 7.5.1. Due to

$$\mathbb{P}\left[\forall t \geq 0 : 0 = \widehat{\Xi}_t^{\uparrow,0} \leq \overline{\Xi}_t \leq \widehat{\Xi}_t^{\downarrow,0} = \Xi_t^X\right] = 1.$$

If we now repeatedly apply (7.59) and the reversed order Lemma 7.1.6, we obtain:

$$\mathbb{P}\big[\ \forall n \in \mathbb{N},\ \forall t \geq 0 : \widehat{\Xi}_t^{\uparrow,n} \leq \overline{\Xi}_t \leq \widehat{\Xi}_t^{\downarrow,n}\big]. \tag{7.61}$$

But by Proposition 7.5.1, we know that $(\widehat{\Xi}^{\uparrow,n})_{n=0}^{\infty}$ and $(\widehat{\Xi}^{\downarrow,n})_{n=0}^{\infty}$ are both converging against $\widehat{\Xi}$ in total variation, indeed

$$\mathbb{P}\left[\ \forall t \geq 0 : \widehat{\Xi}_t^{\downarrow,n} \overset{n\to\infty}{\longrightarrow} \widehat{\Xi}_t^{\downarrow,\infty}, \widehat{\Xi}_t^{\uparrow,n} \overset{n\to\infty}{\longrightarrow} \widehat{\Xi}_t^{\uparrow,\infty} \text{ in t.v.}\right] = 1.$$

Therefore (7.60) follows from (7.61). From (7.60) follows also that

$$\mathbb{P}\left[((\widehat{X}_i(t), \widehat{U}_i(t))_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}_t) = ((\overline{X}_i(t), \overline{U}_i(t))_{i=1}^{\infty}, \bar{\boldsymbol{\xi}}_t)\right]. \tag{7.62}$$

Let us assume that $(\widehat{Z})_{i=1}^{\infty}$ and $(\overline{Z}_i)_{i=1}^{\infty}$ are the integrated processes used to construct the processes from (7.62) as it is done in Definition 7.1.3. Since

$$\widehat{Z}_i(t) = \int_0^t \int_0^{\infty} \mathbb{1}_{[F(\mathfrak{x}_i(t,s\text{-}), \widehat{\Xi}_s)]}(p)\, \mathfrak{N}_i(t, dp, ds),$$
$$\overline{Z}_i(t) = \int_0^t \int_0^{\infty} \mathbb{1}_{[F(\mathfrak{x}_i(t,s\text{-}), \overline{\Xi}_s)]}(p)\, \mathfrak{N}_i(t, dp, ds).$$

Because of (7.60), it holds $\widehat{Z}_i(t) = \hat{Z}_i(t)$ almost surely for all $i \in \mathbb{N}$ and $t \geq 0$ making $\widehat{Z}_i$ and $\widehat{Z}_i$ indistinguishable (both are càdlàg). We can now conclude that (7.62) is true based on Definition 7.1.3. □

We have now proved that the competitive equation admits a solution, and we assume for the rest of this section that $((\widehat{X}_i, \widehat{U}_i)_{i=1}^{\infty}, \widehat{\boldsymbol{\xi}}, \widehat{\Xi})$ is the solution of (7.58). All that remains to do is to show that $\widehat{\Xi}$ is a competitive model $\mathbf{Comp}(B_X, a, b, F, \Theta_0)$ and that $\widehat{\boldsymbol{\xi}}$ is a Poisson representation of $\widehat{\Xi}$. Both statements are direct consequences of our integration theory developed in Chapter 3 and in Chapter 4, therefore it is important to recall the definition of the Cut-out process $\mathbf{Cut}\text{-}\mathbf{Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \mathcal{V}))$, see Definition 7.1.3.

Definition 7.5.5. *We define the* $\mathcal{F}^{\widehat{\boldsymbol{\xi}}}$ *and* $\mathcal{F}^{\widehat{\boldsymbol{\Xi}}}$ *as the right-continuous completion of the natural filtrations of* $\widehat{\boldsymbol{\xi}}$ *and* $\widehat{\boldsymbol{\Xi}}$*.*

Lemma 7.5.6. *The process* $\widehat{\boldsymbol{\xi}}$ *is a Poisson representation of* $\widehat{\boldsymbol{\Xi}}$*. It holds for all finite* $\mathcal{F}^{\widehat{\boldsymbol{\Xi}}}$*-stopping times* τ *that*

$$\mathfrak{L}\left(\widehat{\boldsymbol{\Xi}}_\tau|\mathcal{F}^{\widehat{\boldsymbol{\Xi}}}_\tau\right) = \mathbf{PPP}_E(\widehat{\boldsymbol{\Xi}}_\tau \otimes \ell eb[0,\infty)). \tag{7.63}$$

Proof. Let us recall $((\widehat{X}_i,\widehat{U}_i)_{i=1}^\infty,\widehat{\boldsymbol{\xi}},\widehat{\boldsymbol{\Xi}}) = \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}},F(\,\cdot\,,\widehat{\boldsymbol{\Xi}}))$, indeed if $(\boldsymbol{\xi}^{XZ},\boldsymbol{\Xi}^{XZ})$ are given as in Definition 7.1.3, then $\widehat{\boldsymbol{\xi}} = \boldsymbol{\xi}^{XZ}(\cdot\times\{0\}\times\cdot)$ and $\widehat{\boldsymbol{\Xi}} = \boldsymbol{\Xi}^{XZ}(\,\cdot\,\times\{0\})$. Further by Theorem 3.5.7 we have that

$$\mathfrak{L}\big(\boldsymbol{\xi}^{XZ}_\tau\big|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_\tau\big) = \mathbf{PPP}(\boldsymbol{\Xi}^{XZ}_\tau\otimes \ell eb[0,\infty)).$$

Now let us fix $g^{xu}\in C_c^+(\mathbb{R}^d\times[0,\infty))$ and we set $g^{xzu} := g^{xu}g^z$ with $g^z\in C_c^+(\mathbb{R})$ with $g^z(0)=1$ and $g^z(z)=0$ for $|z|\geq 1$. Since $\mathcal{F}^{\widehat{\boldsymbol{\Xi}}}\subset\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}$, we can conclude

$$\begin{aligned}
&\mathbb{E}\left[\exp(-\widehat{\boldsymbol{\xi}}(g^{xu}))|\mathcal{F}^{\widehat{\boldsymbol{\Xi}}}_\tau\right]\\
&=\mathbb{E}\left[\mathbb{E}\left[\exp(-\boldsymbol{\xi}^{XZ}_\tau(g^{xzu}))|\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W}}_\tau\right]|\mathcal{F}^{\widehat{\boldsymbol{\Xi}}}_\tau\right]\\
&=\mathbb{E}\left[\exp\left(-\int_{\mathbb{R}^d\times\mathbb{R}}\int_0^\infty 1-\exp(g^{xzu}(x,z,u))\,du\boldsymbol{\Xi}^{XZ}_\tau(dx,dz)\right)|\mathcal{F}^{\widehat{\boldsymbol{\Xi}}}_\tau\right]\\
&=\exp\left(-\int_{\mathbb{R}^d}\int_0^\infty 1-\exp(g^{xu}(x,u))\,du\widehat{\boldsymbol{\Xi}}(dx)\right).
\end{aligned}$$

Since the class of Laplace functionals is separating for $\mathcal{M}_1(\overline{\mathcal{N}}(\mathbb{R}^d\times[0,\infty)))$ it follows 7.63 from Lemma C.2.1. □

Proposition 7.5.7. *The process* $\widehat{\boldsymbol{\Xi}}$ *solves the martingale problem described by* $\mathbf{Comp}(\widehat{B},a,b,F,\Theta_0)$*, see Definition 1.2.1.*

Proof. This proofs works similar to the proof of Proposition 7.1.7. Let us assume that $(\boldsymbol{\xi}^{XZ},\boldsymbol{\Xi}^{XZ})$ is given as in the proof of Lemma 7.5.6. Now for a fixed $\hat{g}\in C_b^+(\mathbb{R}^d)$ we set $\hat{g}^{xz}=\hat{g}\hat{g}^z$, where $\hat{g}^z\in C_b^2(\mathbb{R})$ with $\hat{g}^z(0)=1$ and $\hat{g}^z(z)=0$ with $|z|\geq 1$, Proposition 4.4.2 tells us that

$$\begin{aligned}
\hat{M}(t) :=&\exp(-\boldsymbol{\Xi}^{XZ}_t(\hat{g}^{xz}))-\exp(-\boldsymbol{\Xi}^{XZ}_0(\hat{g}^{xz}))-\hat{A}(t) &(7.64)\\
=&\exp(-\widehat{\boldsymbol{\Xi}}_t(\hat{g}))-\exp(-\widehat{\boldsymbol{\Xi}}_0(\hat{g}))-\hat{A}(t),\ t\geq 0, &(7.65)
\end{aligned}$$

is a local martingale, where the process $\hat{A}$ is given by (4.17) with $\hat{g}^{xz}$ taking the role of $\hat{g}$. The last line of Expression (4.17) takes in our case the form:

$$\int_0^t \exp(-\boldsymbol{\Xi}^{XZ}_s(\hat{g}^{xz}))\int_{\mathbb{R}^{d+1}}\int_0^\infty \hat{g}^{xz}(x,z+\mathbb{1}_{[0,F(x,\widehat{\boldsymbol{\Xi}}_s)]}(p))-\hat{g}^{xz}(x,z)dp\boldsymbol{\Xi}^{XZ}_s(dx,dz)ds$$

If we now apply the fact that $\boldsymbol{\Xi}^{XZ}_t(\hat{g}^{xz}))=\widehat{\boldsymbol{\Xi}}_t(\hat{g}))$, then the above turns into:

$$-\int_0^t\exp(-\widehat{\boldsymbol{\Xi}}_s(\hat{g}))\int_{\mathbb{R}^d}F(x,\widehat{\boldsymbol{\Xi}}_s)\hat{g}(x)\widehat{\boldsymbol{\Xi}}_s(dx)ds$$

So all in all, $\hat{A}(t)$ is identical to

$$\int_0^t\left[\widehat{\boldsymbol{\Xi}}_s(B_X(\hat{g})+b\hat{g})-a\widehat{\boldsymbol{\Xi}}_s(\hat{g}^2)-\int_{\mathbb{R}^d}\hat{g}(x)F(x,\widehat{\boldsymbol{\Xi}}_s)\widehat{\boldsymbol{\Xi}}_s(dx)\right]\exp\big(-\widehat{\boldsymbol{\Xi}}_s(\hat{g})\big)ds.$$

□

Proposition 7.5.7 together with Lemma 7.5.6 proves that $(\widehat{\boldsymbol{\xi}}, \widehat{\Xi})$ forms together a Poisson representation of $\mathbf{Comp}(B_X, a, b, F, \Theta_0)$. For completeness we state also the martingale problem satisfied by $\widehat{\boldsymbol{\xi}}$.

Proposition 7.5.8. *Let us assume that $g \in \mathfrak{g}(\mathbf{B})$, see Definition B.2.7, and that $f := -\log(g)$, then the process*

$$M(t) := \exp(-\widehat{\boldsymbol{\xi}}_t(f)) - \exp(-\widehat{\boldsymbol{\xi}}_0(f)) - A(t),\ t \geq 0,$$

is a local $\mathcal{F}^{\widehat{\boldsymbol{\xi}}}$-martingale, where the process A is given by (with $\overline{E} = \mathbb{R}^d \times \mathbb{R} \times [0,\infty)$)

$$\begin{aligned} A(t) := & \int_0^t \int_{\overline{E}} \exp(-\widehat{\boldsymbol{\xi}}_{s\text{-}}(f)) \frac{B(g)(x,u)}{g(x,u)} \widehat{\boldsymbol{\xi}}_{s\text{-}}(dx, du) ds \\ & + \int_0^t \int_{\overline{E}} \exp(-\widehat{\boldsymbol{\xi}}_{s\text{-}}(f))(au^2 - bu) \frac{\partial_u(g)(x,u)}{g(x,u)} \widehat{\boldsymbol{\xi}}_{s\text{-}}(dx, du) ds \\ & + \int_0^t \int_{\overline{E}} \exp(-\widehat{\boldsymbol{\xi}}_{s\text{-}}(f)) \int_{\overline{E}} \int_u^\infty 2a[g(x,v) - 1]\, dv \widehat{\boldsymbol{\xi}}_{s\text{-}}(dx, du) ds \\ & + \int_0^t \int_{\overline{E}} \exp(-\widehat{\boldsymbol{\xi}}_{s\text{-}}(f)) F(x, \widehat{\Xi}_s) \frac{1 - g(x,u)}{g(x,u)} \boldsymbol{\xi}^{XZ}_{s\text{-}}(dx, du) ds. \end{aligned}$$

Proof. From the definition of $\mathfrak{g}(\mathbf{B})$ we know that g must have the form:

$$g(x,u) = \prod_{j=1}^{l} \left[1 - g_j^x(x) g_j^u(u)\right] = 1 + \sum_{|J| \subset [l] \setminus \emptyset} (-1)^{|J|} g_J^x(x) g_J^u(u),$$

where $g_j^x : E \to [0,1)$ are elements of $\mathcal{D}(\mathbf{B})$, $g_j^u : [0,\infty) \to [0,1)$ are elements of $C^1([0,\infty))$ and $g_J^x := \prod_{j \in J} g_j^x$ and $g_J^u := \prod_{j \in J} g_j^u$ for $J \subset [l] \setminus \emptyset$ ($[l] := \{1, 2, ..., l\}$). Note that $\bar{g} := -g + 1 \in C_b^+(\mathbb{R}^d \times \mathbb{R})$. If we set $g^{xzu} := 1 - \bar{g} g^z$ with $g^z \in C_b^2(\mathbb{R})$ with $g^z(0) = 1$ and $g^z(z) = 0$ for $|z| \geq 1$, then it follows from the properties of $\mathfrak{g}(\mathbf{B})$, that $g^{xzu} \in\in \mathfrak{G}^Z$. Further it holds $\exp(-\widehat{\boldsymbol{\xi}}(f)) = \exp(-\boldsymbol{\xi}^{XZ}(f^{xzu}))$ with $f^{xzu} = -\log(g^{xzu})$. Our claim follows now from Proposition 4.3.2 in the same way how we derived Proposition 7.5.7 from Proposition 4.4.2. □

We wrap things up by explaining how the previous results prove the main theorem, see Theorem 1.2.6.

Proof of Theorem 1.2.6. According to the Sections 7.3 and 7.4 we know that the conditions 7.2.2 are satisfied by the non-linear Bolker-Pacala models and non-linear singular interactive models, hence if the competitive model $\mathbf{Comp}(B_X, a, b, F, \hat{\Theta}_0)$ belongs to one of the two classes, then the Cut-Out equation

$$((\widehat{X}_i, \widehat{U}_i)_{i=1}^\infty, \widehat{\boldsymbol{\xi}}, \widehat{\Xi}) = \mathbf{Cut\text{-}Out}(\boldsymbol{\xi}^{\mathbb{W}}, F(\,\cdot\,, \widehat{\Xi})) \tag{7.66}$$

has a solution. By Lemma 7.5.6 and Proposition 7.5.7 the pair $(\widehat{\boldsymbol{\xi}}, \widehat{\Xi})$ form a Poisson representation of $\mathbf{Comp}(B_X, a, b, F, \hat{\Theta}_0)$, recall Definition 1.2.2. □

7.6 Possible Application: Extinction

Even basic questions like "Does the population become extinct?" can be hard to answer for models with competition, where extinction is formally defined as

$$\mathbb{P}\left[\lim_{t \to \infty} \widehat{\Xi}_t(\mathbb{R}^d) = 0\right] = 1. \tag{7.67}$$

When $\widehat{\varphi}$ is the solution of the Mueller-Tribe SPDE, see (1.7), then Mueller and Tribe showed in Theorem 1 of [36], that there exists a constant $b_c > 0$ such that the population described by $\widehat{\varphi}$ will die out almost surely, when $b < b_c$ and the initial population φ_0 has compact support. If $b > b_c$, then the probability has a chance to survive. Other results about the extinction behavior have a similar taste. Hutzenthaler and Wakolbinger studied in [18] the stepping stone version of the logistic Feller diffusion, which is the solution of the following infinite system of stochastic differential equations:

$$
\begin{aligned}
d\widehat{\Xi}_t^{HW}(x) = \lambda \left(\sum_{y \in \mathbb{Z}^d} m(x,y) \widehat{\Xi}_t^{HW}(y) - \widehat{\Xi}_t^{HW}(x) \right) dt \\
+ b\widehat{\Xi}_t^{HW}(x) - c\left[\widehat{\Xi}_t^{HW}(x)\right]^2 + \sqrt{2a\widehat{\Xi}_t^{HW}(x)} dB_t(x), \quad x \in \mathbb{Z}^d,
\end{aligned}
\tag{7.68}
$$

where $(B(x), x \in \mathbb{Z}^d)$ is a collection of independent Brownian motions, $\lambda, a, b, c > 0$ and m is a irreducible normalized translation-invariant matrix, i.e. $\sum_{x \in \mathbb{Z}^d} m(0,x) = 1$ and $m(x,y) = m(0, x-y)$. While they studied even more general systems, their main result, Theorem 1, translates for (7.68) to the statement: For fixed parameters λ, m, a and c, the population $\widehat{\Xi}^{HW}$ dies out, if $b \leq b_c$, where b_c is given by

$$
\int_0^\infty \exp\left(b_c u - \frac{ac}{2}u^2\right) \lambda \exp(-\lambda u)\, du = 1.
$$

This statement can be found as Corollary 6 in [18], but it looks a little bit different there, because they use different parameters. Etheridge proved in the same paper [11], in which she introduced the Bolker-Pacala models (she calls models like (7.68) the stepping stone version of a Bolker-Pacala model), that for b big enough, the population will survive with a positive probability. The model of Mueller and Tribe, (1.7), and the model of Hutzenthaler and Wakolbinger, (7.68), are related to each other, in the sense that only particles at the same spatial location interact with each other. As a result of this local competition, there exists a Laplace self-duality, see Theorem 3 in [18] or Section 2 in [25], which is a valuable tool to investigate the extinction behavior. But the competition term in the Bolker-Pacala model $\widehat{\Xi}^{BP}$, see (1.8), is non-local, hence the Laplace self-duality does not hold and the models are less tractable. Still Etheridge could prove in [11] a result very similar to Mueller and Tribe. She showed, that for a fixed triple $(a, b, \hat{\kappa})$ there exists a constant $\hat{c}$ for which $\widehat{\Xi}^{BP}$ dies out, if $c > \hat{c}$ as long as the initial mass $\widehat{\Xi}_0^{BP}$ is finite. She also showed that for every fixed (a, c) we can find a $\widehat{b}$ such that $\widehat{\Xi}^{BP}$ dies out, when $b < \widehat{b}$, $\widehat{\Xi}_0^{BP}(\mathbb{R}^d) < \infty$ and $\sup_r r^{2-\delta}\hat{\kappa}(r) < \infty$ for some $\delta > 0$, see Theorem 1 in [11] (Etheridge could also prove extinction for the case where $\widehat{\Xi}^{BP}$ has infinite initial mass, but we will only discuss the finite case here). The argument of Etheridge depends on the compact support property, that means that, if $\widehat{\Xi}_0^{BP}$ has compact support, there will almost surely exists a compact set $\Gamma \subset \mathbb{R}^d$ with

$$
\cup_{s \leq t} \operatorname{supp}(\widehat{\Xi}_s^{BP}) \subset \Gamma.
$$

Unfortunately this is unlikely to hold, if the spatial motion of the particles has jumps (for a proof in the case of the corresponding superprocess without competition, see Theorem III.2.4. in [40]). We want to present a new approach which may allow to prove statements similar to the ones of Etheridge but with more general spatial motions. Let us assume that $((\widehat{X}_i, \widehat{U}_i,)_{i=1}^\infty, \widehat{\boldsymbol{\xi}}, \widehat{\Xi})$ are obtained as the solution of Cut-Out equation:

$$
((\widehat{X}_i, \widehat{U}_i,)_{i=1}^\infty, \widehat{\boldsymbol{\xi}}, \widehat{\Xi}) = \textbf{Cut-Out}(\boldsymbol{\xi}^{\mathbb{W}}, cF(\,\cdot\,, \widehat{\Xi})), \tag{7.69}
$$

see Definition 7.2.1. Since we are interested in the Bolker-Pacala case, see (1.8), we also assume that

$$F(x,\mu) := \int_{\mathbb{R}^d} \hat{\kappa}(||x-y||)\mu(dy), \quad x \in \mathbb{R}^d, \mu \in \mathcal{M}_f(\mathbb{R}^d).$$

By the previous sections (7.69) has a solution and $\widehat{\boldsymbol{\Xi}} \sim \mathbf{Comp}(B_X, a, b, cF, \Theta_0)$, which implies that $\widehat{\boldsymbol{\Xi}}$ is a Bolker-Pacala model. We are now going to divide the population $\widehat{\boldsymbol{\xi}}$ into two subpopulations. The first one consists of the particles with a levels lower than b/a, the remaining particles are contained in the second population. We also subtract b/a from the levels in the second population.

Definition 7.6.1. *We define the processes*

$$\begin{aligned}
&\widehat{\boldsymbol{\xi}}^1 : \Omega \times [0,\infty) \to \mathcal{N}_f(\mathbb{R}^d \times [0, b/a)), \quad && \widehat{\boldsymbol{\xi}}^1_t := \sum_{i=1}^{\infty} \delta_{(\widehat{X}_i(t), \widehat{U}_i(t))} \mathbb{1}_{[0,b/a)}(\widehat{U}_i(t));\\
&\widehat{\boldsymbol{\Xi}}^1 : \Omega \times [0,\infty) \to \mathcal{M}_f(\mathbb{R}^d), && \widehat{\boldsymbol{\Xi}}^1_t := \sum_{i=1}^{\infty} \delta_{\widehat{X}_i(t)} \mathbb{1}_{[0,b/a)}(\widehat{U}_i(t));\\
&\widehat{\boldsymbol{\xi}}^2 : \Omega \times [0,\infty) \to \overline{\mathcal{N}}(\mathbb{R}^d \times [0,\infty)), && \widehat{\boldsymbol{\xi}}^2_t := \sum_{i=1}^{\infty} \delta_{(\widehat{X}_i(t), \widehat{U}_i(t)-b/a)} \mathbb{1}_{[b/a,\infty)}(\widehat{U}_i(t));\\
&\widehat{\boldsymbol{\Xi}}^2 : \Omega \times [0,\infty) \to \mathcal{M}_f(\mathbb{R}^d), && \widehat{\boldsymbol{\Xi}}^2_t := \gamma^{\boldsymbol{\Xi}}_{\mathbb{R}^d}(\widehat{\boldsymbol{\xi}}^2_t).
\end{aligned}$$

This division of $\widehat{\boldsymbol{\xi}}$ into two processes $\widehat{\boldsymbol{\xi}}^1$ and $\widehat{\boldsymbol{\xi}}^2$ is based on the following observations. First, $\widehat{\boldsymbol{\Xi}}^2_t = \widehat{\boldsymbol{\Xi}}_t$ almost surely for $t \geq 0$, because:

$$\begin{aligned}
\widehat{\boldsymbol{\Xi}}^2_t &= \lim_{r\to\infty} \frac{1}{r} \sum_{i=1}^{\infty} \delta_{(\widehat{X}_i(t), \widehat{U}_i(t)-b/a)} \mathbb{1}_{[b/a,r)}(\widehat{U}_i(t))\\
&= \lim_{r\to\infty} \frac{1}{r} \sum_{i=1}^{\infty} \delta_{(\widehat{X}_i(t), \widehat{U}_i(t)-b/a)} \mathbb{1}_{[0,r)}(\widehat{U}_i(t) = \widehat{\boldsymbol{\Xi}}_t,
\end{aligned}$$

which in turn means $\gamma^{\boldsymbol{\Xi}}_{\mathbb{R}^d}(\widehat{\boldsymbol{\xi}}^2_t) = \widehat{\boldsymbol{\Xi}}_t$. Hence $\widehat{\boldsymbol{\Xi}}$ dies out, if and only if $\widehat{\boldsymbol{\xi}}^2$ dies out. Further, the particles in $\widehat{\boldsymbol{\xi}}^1$ give birth to new particles in $\widehat{\boldsymbol{\xi}}^2$, so $\widehat{\boldsymbol{\xi}}^2$ can not die out, as long as $\widehat{\boldsymbol{\xi}}^1$ did not die out, and we will see that $\widehat{\boldsymbol{\xi}}^2$ must die out, if $\widehat{\boldsymbol{\xi}}^1$ has died out. Finally, the differential equation describing the evolution of the levels, i.e.

$$\dot{u} = au^2 - bu,$$

is strictly negative for $u \in (0, b/a)$ and strictly positive for $u \in (b/a, \infty)$ (Note further that almost surely there will never be a particle with the level 0 or b/a), therefore the level of the particles in $\widehat{\boldsymbol{\xi}}^1$ will always remain below b/a and they can only die by competition. As a consequence $\widehat{\boldsymbol{\Xi}}$ can only become extinct, when $\widehat{\boldsymbol{\xi}}^1$ has died out before, which only happens, if $\widehat{\boldsymbol{\Xi}}^1$ does the same. We can also derive a quantitative description of the joint behavior of $(\widehat{\boldsymbol{\Xi}}^1, \widehat{\boldsymbol{\Xi}})$ in the form of a martingale characterization.

Proposition 7.6.2. *Recalling $\overline{\mathcal{D}}(B_X)$ from Definition B.4.2 let us assume that $\hat{g}_1 \in \overline{\mathcal{D}}(B_X), \hat{g}_2 \in \mathcal{D}(B_X)$, then the process $\hat{M}^{1,2}$ given by*

$$\hat{M}^{1,2}(t) := \exp(-\widehat{\boldsymbol{\Xi}}_t(\hat{g}_2)) \prod_{x\in\widehat{\boldsymbol{\Xi}}^1_t} \hat{g}_1(x) - \exp(-\widehat{\boldsymbol{\Xi}}_0(\hat{g}_2)) \prod_{x\in\widehat{\boldsymbol{\Xi}}^1_0} \hat{g}_1(x) - \hat{A}^{1,2}(t)$$

is a local martingale with respect to the natural filtration of $(\widehat{\mathbf{\Xi}}^1, \widehat{\mathbf{\Xi}})$ *(which we denote by* $\mathcal{F}^{\widehat{\mathbf{\Xi}}^1,\widehat{\mathbf{\Xi}}}$*), and where the process* $\hat{A}^{1,2}$ *is given for* $t \geq 0$ *by*

$$\int_0^t \Bigg(\exp(-\widehat{\mathbf{\Xi}}_s(\hat{g}_2)) \prod_{x\in\widehat{\mathbf{\Xi}}^1_{s\text{-}}} \hat{g}_1(x) \Bigg[-\widehat{\mathbf{\Xi}}_s(B_X(\hat{g}_2)) - b\widehat{\mathbf{\Xi}}_s(\hat{g}_2) + a\widehat{\mathbf{\Xi}}_s(\hat{g}_2^2)$$
$$+ c\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \hat{\kappa}(||x-y||)\hat{g}_2(x)\,\widehat{\mathbf{\Xi}}_s(dy)\widehat{\mathbf{\Xi}}_s(dx) - 2a\widehat{\mathbf{\Xi}}^1_{s\text{-}}(\hat{g}_2)$$
$$+ \int_{\mathbb{R}^d} \left(\frac{B_X(\hat{g}_1)(x)}{\hat{g}_1(x)} + b(\hat{g}_1(x)-1) + c\int_{\mathbb{R}^d} \hat{\kappa}(||x-y||)\widehat{\mathbf{\Xi}}_s(dy)(\hat{g}_1^{-1}(x)-1)\right)\widehat{\mathbf{\Xi}}^1_{s\text{-}}(dx)\Bigg]\Bigg) ds.$$

We give a sketch of the proof of this statement at the end of this section. Note that this is a competitive version of the Evans-O'Connell backbone decomposition, see [15]. If we compare this with the martingale problem described in Definition 1.2.1, then we realize that the process $\widehat{\mathbf{\Xi}}$ shows under the filtration $\mathcal{F}^{\widehat{\mathbf{\Xi}}^1,\widehat{\mathbf{\Xi}}}$ a different behavior than under its natural filtration. With the help of the Itô formula we can derive, that for each $\hat{g} \in \mathcal{D}(B_X)$

$$\hat{M}(t) := \widehat{\mathbf{\Xi}}_t(\hat{g}_2) - \widehat{\mathbf{\Xi}}_0(\hat{g}_2) - \hat{A}(t) \tag{7.70}$$

is a continuous local $\mathcal{F}^{\widehat{\mathbf{\Xi}}^1,\widehat{\mathbf{\Xi}}}$-martingale with $\langle \hat{M}\rangle_t = \int_0^t 2a\widehat{\mathbf{\Xi}}_s(\hat{g}_2^2)ds$ and $\hat{A}$ is a process with finite variance given by

$$\begin{aligned}\hat{A}(t) = &\int_0^t \widehat{\mathbf{\Xi}}_s(B_X(\hat{g}_2)) - b\widehat{\mathbf{\Xi}}_s(\hat{g}_2)ds \\ &- \int_0^t c\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \hat{\kappa}(||x-y||)\hat{g}_2(x)\,\widehat{\mathbf{\Xi}}_s(dy)\widehat{\mathbf{\Xi}}_s(dx) + 2a\widehat{\mathbf{\Xi}}^1_{s\text{-}}(\hat{g}_2)ds.\end{aligned}$$

Comparing this with the second line of (7.70), we notice that the linear part of the drift term has switched sign and is now equal to

$$-b\widehat{\mathbf{\Xi}}_s(\hat{g}_2) \tag{7.71}$$

instead of $b\widehat{\mathbf{\Xi}}_s(\hat{g}_2)$. We observe also an additional immigration term with the form $2a\widehat{\mathbf{\Xi}}^1_{s\text{-}}(\hat{g}_2)$. This immigration term is the result from the fact that the particles in $\widehat{\mathbf{\Xi}}^1$ are constantly giving birth to new particles in $\widehat{\boldsymbol{\xi}}^1$ which in turn produces new mass in $\widehat{\mathbf{\Xi}}$ which will keep the population $\widehat{\mathbf{\Xi}}$ alive as long as $\widehat{\mathbf{\Xi}}^1$ has not died out yet. But as soon as this has happened, the sign-switched drift term (7.71) ensure the extinction of $\widehat{\mathbf{\Xi}}$. This is on its own interesting. For our idea for the proof of extinction we start with $\tau_0 = 0$ and

$$\tau_{k+1} := \inf\{s > \tau_k : \widehat{\mathbf{\Xi}}^1_s(\mathbb{R}^d) \neq \widehat{\mathbf{\Xi}}^1_{\tau_k}(\mathbb{R}^d)\}, \quad k \in \mathbb{N}_0,$$

so $(\tau_k)_{k=1}^\infty$ are the moments, where the number of the particles in $\widehat{\mathbf{\Xi}}^1$ is changing. Next, we would like to show that we obtain by setting

$$\widehat{\mathbf{\Xi}}^1_{\tau_k}(\mathbb{R}^d) = N_k, \quad \forall k \geq \mathbb{N}_0,$$

a "random walk" $N = (N_k, k \in \mathbb{N}_0)$ which will almost surely hit 0. We proceed with defining for $(\varrho, \mu) \in \mathcal{N}_f(\mathbb{R}^d) \times \mathcal{M}_f(\mathbb{R}^d)$:

$$\mathbf{d}(\varrho,\mu) := \mathbb{P}\left[N_1 = N_0 - 1 \middle| (\widehat{\mathbf{\Xi}}^1_0, \widehat{\mathbf{\Xi}}_0) = (\varrho,\mu)\right],$$

which means that $\mathbf{d}(\varrho, \mu)$ is the probability that a particle dies in $\widehat{\Xi}^1$, before the first new particle is born in $\widehat{\Xi}^1$, (ϱ, μ) are hereby the initial values of $\widehat{\Xi}^1_0$ and $\widehat{\Xi}_0$. The key step would be to prove that

$$q := \inf\{\mathbf{d}(\varrho, \mu); \varrho \in \mathcal{N}_f(\mathbb{R}^d), \mu \in \mathcal{M}_f(\mathbb{R}^d)\} \geq 0.5. \tag{7.72}$$

Using a "pecking order" and the fact that the law of the Lévy process is invariant under translation, we should be able to argue that

$$q = \mathbf{d}(\delta_0, \mathbf{0}_{\mathbb{R}^d}),$$

where $\mathbf{0}_{\mathbb{R}^d}$ is the null measure over $\mathbb{R}^d$. So q is the probability that the one particle in $\widehat{\Xi}^1$ dies to the competition generated by its own children in $\widehat{\boldsymbol{\xi}}^2$, before it can give birth to a second particle in $\widehat{\boldsymbol{\xi}}^1$. Assuming that (7.72) is true, the strong Markov property of $(\widehat{\Xi}^1, \widehat{\Xi})$ will give that

$$\mathbb{P}\left[N_{k+1} = N_k - 1 | N_k = m\right] \geq q \geq 0.5,$$

and $\mathbb{P}[N_{k+1} = N_k + 1 | N_k = m] = 1 - q < 0.5$ for all $m \in \mathbb{N}$, hence N would hit 0 inevitably. Let us now shortly argue why (7.72) should intuitively be true. From Proposition 7.6.2 we can conclude that $\widehat{\Xi}^1$ is under the filtration $\mathcal{F}^{\widehat{\Xi}^1, \widehat{\Xi}}$ a Branching particle system with birth rate b and competitive death rate

$$c \int_{\mathbb{R}^d} \hat{\kappa}(||x - y||) \widehat{\Xi}_t(dy) \tag{7.73}$$

at the spatial position $x \in \mathbb{R}^d$ and the time $t \in [0, \infty)$. So if $\widehat{\Xi}$ increases, we observe more deaths in $\widehat{\Xi}^1$. The process $\widehat{\Xi}$ is under the filtration $\mathcal{F}^{\widehat{\Xi}^1, \widehat{\Xi}}$ a competitive model with negative drift b, competition rate (7.73) and an immigration term given by

$$2a\widehat{\Xi}^1_t. \tag{7.74}$$

So if we increase the branching rate a, then $\widehat{\Xi}$ grows faster due to the immigration, which increases the competition rate in $\widehat{\Xi}^1$, so increasing a should increase q from (7.72). Of course a bigger population $\widehat{\Xi}$ also leads to more competition in $\widehat{\Xi}$, but the competition and the immigration will balance each other out. So by fixing b and c, there should exists a critical value for a_c such that (7.72) is true for all $a > a_c$. This argument could generalize Etheridge's results, eliminating the need for the compact containment property. This approach may also work, when we consider more general branching mechanisms with infinite variance, which would extend Etheridge's result even more. In this case the process N would be a random walk with jump sizes in $\{-1\} \cup \mathbb{N}$, which would make it necessary to replace (7.72) with a statement in the flavor of

$$\inf\left\{\mathbb{E}\left[N_1 - N_0 \middle| (\widehat{\Xi}^1_0, \widehat{\Xi}_0) = (\varrho, \mu)\right], \varrho \in \mathcal{N}_f(\mathbb{R}^d), \mu \in \mathcal{M}_f(\mathbb{R}^d)\right\} < 0.$$

Proof of Proposition 7.6.2. Let us define $L_{g_1,g_2} : \mathcal{N}_f(\mathbb{R}^d \times [0, r)) \times \overline{\mathcal{N}}(\mathbb{R}^d \times [0, \infty)) \to [0, \infty)$ with $L_{g_1,g_2}(\varrho, \eta) := \prod_{x \in \varrho} g_1(x, u) \exp(-\eta(\log(g_2)))$ for $g_1 \in \mathfrak{g}(\mathbf{B}, b/a)$, see Definition B.11, $g_2 \in \mathfrak{g}(\mathbf{B})$. If we define the processes $V^1, V^2, V^3 : \Omega \times [0, \infty) \to \mathbb{R}$ by setting

$$\begin{aligned} V^k(t) := & L_{g_1,g_2}(\widehat{\boldsymbol{\xi}}^1_{t\text{-}}, \widehat{\boldsymbol{\xi}}^2_{t\text{-}}) \int_{\overline{E}} \left[\frac{B(g_k)(x, z, u)}{g_k(x, u)} + [au^2 + (-1)^k bu] \frac{\partial_u(g_k)(x, u)}{g_k(x, u)}\right. \\ & \left. + \int_{\overline{E}} \int_u^\infty 2a[g_k(x, v) - 1]\, dv + \int_{\mathbb{R}^d} c\hat{\kappa}(||x - y||) \frac{1 - g_k(x, u)}{g_k(x, u)} \widehat{\Xi}_t(dy)\right] \widehat{\boldsymbol{\xi}}^k_{t\text{-}}(dx, du), \quad k \in \{1, 2\} \\ V^3(t) := & L_{g_1,g_2}(\widehat{\boldsymbol{\xi}}^1_{t\text{-}}, \widehat{\boldsymbol{\xi}}^2_{t\text{-}}) \int_{\overline{E}} \int_0^\infty 2a[g_2(x, v) - 1]\, dv \widehat{\boldsymbol{\xi}}^1(dx, du), \end{aligned}$$

where $\overline{E} = \mathbb{R}^d \times [0,\infty)$, then we obtain a local $\mathcal{F}^{\boldsymbol{\xi},\mathbb{W}}$-martingale by setting

$$M^{1,2}(t) := L_{g_1,g_2}(\widehat{\boldsymbol{\xi}}^1_t, \widehat{\boldsymbol{\xi}}^2_t) - L_{g_1,g_2}(\widehat{\boldsymbol{\xi}}^1_0, \widehat{\boldsymbol{\xi}}^2_0) - \int_0^t V^1(s) + V^2(s) + V^3(s)\, ds. \tag{7.75}$$

This can be proven by extending the arguments used to prove Proposition 4.3.2. The process V^3 reflects the fact that $\widehat{\boldsymbol{\xi}}^1$ gives birth to new particles in $\widehat{\boldsymbol{\xi}}^2$. The difference between V^1 and V^2 with regard to the level dynamics is the results of the shift by $-b/a$ during the definition of a $\widehat{\boldsymbol{\xi}}^2$. This shift changes the ODE describing the dynamic of the levels, i.e. if $O = U - b/a$ and $\dot{U} = aU^2 - bU$, then $\dot{O} = aO^2 + bO$, because

$$\dot{O} = \dot{U} = a(O + b/a)^2 - b(O - b/a) = aO^2 + bO.$$

Recall the processes $\boldsymbol{\Xi}^{\mathbb{W},b/a}, \boldsymbol{\xi}^{\mathbb{W},b/a}, \boldsymbol{\xi}^{\mathbb{W},\geq b/a}$ from Definition 2.5.2, the state space $\mathfrak{D}$ of the path-valued process $\mathbb{W}$ and the Markov kernel $\mathbf{Uni}^r_E$ from Definition 1.1.1 with E being a Polish space and $r \in [0,\infty)$ a level cap. With the help of the Markov mapping theorem we are able to prove that

$$\mathfrak{L}((\boldsymbol{\xi}^{\mathbb{W},b/a}_t, \boldsymbol{\xi}^{\mathbb{W},\geq b/a})_t | \sigma(\boldsymbol{\Xi}^{\mathbb{W},b/a}_s, \boldsymbol{\Xi}^{\mathbb{W}}_s; s \leq t)) = \mathbf{Uni}^{\frac{b}{a}}_{\mathfrak{D}}(\boldsymbol{\Xi}^{\mathbb{W},b/a}_t) \otimes \mathbf{PPP}_{\mathfrak{D}}(\boldsymbol{\Xi}^{\mathbb{W}}_t \otimes \ell eb[0,\infty))$$

from this we can derive the same for $(\widehat{\boldsymbol{\xi}}^1, \widehat{\boldsymbol{\xi}}^2)$, i.e.

$$\mathfrak{L}((\widehat{\boldsymbol{\xi}}^1_t, \widehat{\boldsymbol{\xi}}^2_t) | \mathcal{F}^{\widehat{\boldsymbol{\Xi}}^1, \widehat{\boldsymbol{\Xi}}}_t) = \mathbf{Uni}^{\frac{b}{a}}_{\mathbb{R}^d}(\widehat{\boldsymbol{\Xi}}^1_t) \otimes \mathbf{PPP}_{\mathbb{R}^d}(\widehat{\boldsymbol{\Xi}}^2_t \otimes \ell eb[0,\infty)). \tag{7.76}$$

We can now prove our claim by applying (7.76) to (7.75) in combination with the conditional martingale lemma, see Lemma D.2.1, and the results of Lemma C.2.1 in a similar fashion how we proved Proposition 4.4.2 based on Proposition 4.3.2. □

Appendix A

Markov Process Theory

A.1 Martingale Problems

In this section we want to discuss the definition of a martingale problem and the path properties of its solution. The content of this section is based primarily on the the paper [3] by Abhay G. Bhatt and Rejeeva L. Karandikar, which provides a great overview about this topic and contains many examples. Our definition of a martingale problem is based on the Definition 2.1 found in this paper. We will introduced small changes, i.e. we use a slightly different notation and our processes are defined on the whole of $[0, \infty)$ and not only on a finite time interval of the form $[0, T]$ for some $T \in [0, \infty)$. The definition of Bhatt and Karandikar is a little bit different from the one given by Ethier and Kurtz in [14] with regard to some technical details. We shortly address these differences in Remark A.1.3 and explain why both definitions are basically equivalent, especially regarding to the question of uniqueness.

Definition A.1.1. *Assume that E is a Polish space, $(\Omega, \mathbb{F}, \mathbb{P})$ a probability space and $\mu \in \mathcal{M}_1(E)$ a probability measure. Considering a map*

$$B : \mathbf{M}(E) \supset \mathcal{D}(B) \to \mathbf{M}(E),$$

where $\mathbf{M}(E)$ is the space of Borel measurable functions $f : E \to \mathbb{R}$, we say that the stochastic process $X : \Omega \times [0, \infty) \to E$ is a solution to the (local) martingale problem $\boldsymbol{MP}(B, \mu)$ with respect to the filtration $\mathcal{F} := (\mathcal{F}_t, t \geq 0)$, when

1. *X is $\mathcal{F}$-progressively measurable,*
2. *X has the initial distribution μ, i.e. $X_0 \sim \mu$,*
3. *We have $\mathbb{E}\left[\int_0^t |B(f)(X_s)| ds\right] < \infty$ for all $f \in \mathcal{D}(B)$ and all $t \in [0, \infty)$,*
4. *The process $M_f : \Omega \times [0, \infty) \to \mathbb{R}$ given by*

$$M_f(t) := f(X_t) - f(X_0) - \int_0^t B(f)(X_s) ds, \quad t \in [0, \infty),$$

is a (local) $\mathcal{F}$-martingale for all $f \in \mathcal{D}(B)$.

If the filtration is not specified, then it is assumed that $\mathcal{F}$ is the natural filtration of X.

Definition A.1.2. *1. We say that X is a continuous, resp. càdlàg, solution of the (local) martingale problem, when X has almost surely continuous, resp. càdlàg, paths.*

2. *We say that the martingale problem $\boldsymbol{MP}(B, \mu)$ has a unique solution, if all solutions have the same finite dimensional distributions.*

3. *We say that the martingale problem $\boldsymbol{MP}(B, \mu)$ is well-posed, when there exists a solution and it is unique.*

4. *When we omit μ and just speak of the martingale problem $\boldsymbol{MP}(B)$, i.e. when we say that $\boldsymbol{MP}(B)$ is unique or well-posed, then we mean that $\boldsymbol{MP}(B, \mu)$ is well-posed for all possible initial distributions $\mu \in \mathcal{M}_1(E)$.*

5. *If we add the word "continuous" or "càdlàg" to the above definition, then we want to express that the martingale problem possesses the corresponding properties, when we only consider the set of continuous or càdlàg solutions.*

Of course every continuous solution is a càdlàg solution and every càdlàg solution is a progressively measurable solution. Hence the existence of a continuous solution implies the existence of a càdlàg solution, and the existence of càdlàg solution implies a progressively measurable solution. But the direction of implication is reversed, when we ask for uniqueness, indeed it is possible that $\mathbf{MP}(B, \mu)$ has a unique continuous solution, when we consider the only continuous processes, but it may happen that there exists more than one càdlàg or progressively measurable solution.

Remark A.1.3. *As already mentioned our definition differs from the one given by Ethier and Kurtz on Page 173 of [14] (or Page 224 for local martingale problems) in subtle nuances. First Kurtz and Ethier allow B to be multivalued, but we do not. Further a solution only has to be a $\mathcal{F}$-measurable processes, which is less restrictive than to ask for a $\mathcal{F}$-progressively measurable process. Additionally, if no filtration is specified, they use instead of the natural filtration the filtration $(\tilde{\mathcal{F}}_t, t \geq 0)$ with*

$$\tilde{\mathcal{F}}_t := \sigma(X_s, s \leq t) \vee \sigma\left(\int_0^t h(X(u))du;\ s \leq t,\ h \in B(E)\right).$$

We are now explaining why these differences are not substantial. If X is a $\mathcal{F}$-measurable process, then it admits a modification $\widehat{X}$ that is $\mathcal{F}$-progressively measurable, see Theorem 0.1 in [38]. Further, if X is progressive, it follows by Fubini's theorem that the process $P(t) := \int_0^t h(X(u))du$ is adapted to $\mathcal{F}_t := \sigma(X_s, s \leq t)$ and hence $\tilde{\mathcal{F}}_t = \mathcal{F}_t$. Since X and $\tilde{X}$ have the same finite dimensional distributions, uniqueness under all $\mathcal{F}$-progressive solutions implies uniqueness under all $\mathcal{F}$-measurable solutions.

A.2 Borel Strong Markov Processes

If one works with Markov processes whose state space is a locally compact Polish space, like $\mathbb{R}^d$, or it can be naturally embedded in such a space like $\mathcal{M}_f(\mathbb{R}^d)$ can be embedded in $\mathcal{M}_f(\overline{\mathbb{R}}^d)$, where $\overline{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$, then the notions of Feller-semigroup and Feller process offer a convenient framework to apply the theory of strong semigroups and the theory surrounding the theorem of Hille-Yosida as it is presented in Chapter 1 in [14]. But we encounter in Section 2.4 the idea of a path-valued process which takes values in a modified version of the Skorohod space $D(\mathbb{R}^m)$, the space of all càdlàg paths in $\mathbb{R}^m$, which is unfortunately not locally compact. So the theory of

Feller processes can not be applied so easily, but luckily Edwin Perkins provides us in Chapter II.2 of [40] with the notion of Borel strong Markov process which can be viewed as the natural extension of the notion Feller process to Polish spaces which are not locally compact. So it is possible to regain most theorems by replacing the notion of uniform convergence with a weaker convergence. We collect some of this results in this section.

Let (E, d) be a complete metric space with Borel algebra $\mathbb{B}(E)$. Further $B(E)$ is the space of bounded, measureable functions $f : E \to \mathbb{R}$ and $C_b(E)$ is the subspace of bounded, continuous functions. We write $\mathcal{M}_1(E)$ for the space of probability measures on E, which we equip with the weak topology. As usual $D(E)$ is the space of càdlàg paths in E with the topology implied by the usual Skorohod metric $d_{D(E)}$. Let $\mathbb{B}(D(E))$ be the Borel algebra of $(D(E), d_{D(E)})$, let $\pi_t : D(E) \to E$ for $t \geq 0$ be the coordinate map with $\pi_t(\mathbf{y}) = \mathbf{y}(t)$ for $\mathbf{y} \in D(E)$ and let $\mathcal{D} := (\mathcal{D}_t, t \geq 0)$ be the canonical right-continuous filtration on $D(E)$. We say that $(g_n)_{n=1}^{\infty} \subset \mathrm{B}(E)$ is converging boundedly and pointwise to $g \in \mathrm{B}(E)$, if $(||g_n||_\infty)$ is bounded and $\lim_{n\to\infty} g_n(x) = g(x)$ for all $x \in E$. We denote this by

$$g_n \xrightarrow{b.p.} g. \tag{A.1}$$

Our definition of a Borel strong Markov family is based on the notion of a Borel strong Markov process from Chapter II.2 of [40], more precisely it is a combination of the points (II.2.1), (II.2.2) and the Property (PC) from [40].

Definition A.2.1. *We call a family $(P^x, x \in E) \subset \mathcal{M}_1(D(E))$ of probability measures on $D(E)$ a Borel strong Markov family, if the following conditions are met:*

1. *The map $x \mapsto P^x$ is a measurable map from $(E, \mathbb{B}(E))$ to $(\mathcal{M}_1(D(E), \mathbb{B}(\mathcal{M}_1(D(E))))$, where $\mathbb{B}(\mathcal{M}_1(D(E)))$ is the Borel algebra generated by the weak topology on $\mathcal{M}_1(D(E))$.*

2. *The canonical process $(\pi_t, t \geq 0)$ on $(D(E), \mathbb{B}(D(E)), P^x)$ is for all $x \in E$ a strong $(\mathcal{D}_t, t \geq 0)$-Markov process whose semigroup $(P_t, t \geq 0)$, where $P_t(f)(x) = P^x(f(\pi_t))$, satisfies $P_t : C_b(E) \to C_b(E)$.*

*We call $(P^x, x \in E) \subset \mathcal{M}_1(D(E))$ a **continuous strong** Markov family, if the map $x \mapsto P^x$ is a continuous map between (E, d) and $(\mathcal{M}_1(E), \mathbb{B}(\mathcal{M}_1(E)))$.*

It will turn out that $t \mapsto P_t(f)(x)$ is continuous for $f \in C_b(E)$ and $x \in E$, while the right-continuity immediately follows from the requirement that the process is càdlàg, the left-continuity follows from the fact that we can show that every Borel strong Markov process is a Hunt process, see Lemma A.2.7. The difference to a Feller semigroup would be that $t \mapsto P_t(f)$ is continuous with respect to the $|| \cdot ||_\infty$-norm, but P would be also a map from $C_0(E)$ to $C_0(E)$, where $C_0(E)$ is the set of continuous functions that vanish at infinity.

Lemma A.2.2. *Let $(P_x)_{x\in E}$ be a Borel strong Markov family, then it is for every pair $(f, \psi) \in C_b(E) \times C_b(E)$ equivalent:*

1. *For every $x \in E$ and every process W with law P^x holds that $f(W_t) - f(W_0) - \int_0^t \psi(W_s)ds$ is a càdlàg martingale with respect to the natural filtration of W.*

2. *$\frac{P_t(f)-f}{t} \xrightarrow{b.p.} \psi$ for $t \to 0$.*

From the equivalence with the second point it follows that there exists at most only one ψ for each f such that $f(W_t) - f(W_0) - \int_0^t \psi(W_s)ds$ is a martingale.

Proof. See Proposition II.2.1 in [40]. □

Definition A.2.3. *We call the set $\mathbf{B}_F \subset C_b(E)\times C_b(E)$ given by the pairs $(f,\psi) \in C_b(E)\times C_b(E)$ satisfying one of the equivalent conditions stated in Lemma A.2.2 the **full weak generator**. We denote by dom($\mathbf{B}_F$) $\subset C_b(E)$ the collection of $f \in C_b(E)$ for which a $\psi \in C_b(E)$ exists such that $(f,\psi) \in \mathbf{B}_F$. Since $\mathbf{B}_F$ is single-valued we can interpret it as a map $\mathbf{B}_F : C_b(E) \supset \mathcal{D}(\mathbf{B}_F) \to C_b(E)$.*

Lemma A.2.4. *For all $f \in C_b(E)$ and all $t > 0$ holds $P_t(f) \in \mathcal{D}(\mathbf{B}_F)$. Further for all $f \in \mathcal{D}(\mathbf{B}_F)$ holds*

1. *$P_t(\mathbf{B}_F(f)) = \mathbf{B}_F(P_t(f))$ for all $t \geq 0$.*
2. *$P_t(f) - f = \int_0^t \mathbf{B}_F(P_s(f))ds$.*

Proof. See Proposition II.2.1 in [40] □

A further important tool is the resolvent.

Definition A.2.5. *For $\lambda > 0$ we define the λ-resolvent $R_\lambda : C_b(E) \to C_b(E)$ by*

$$R_\lambda(f) := \int_0^\infty e^{-\lambda s} P_s(f) ds.$$

Lemma A.2.6. *For all $f \in C_b(E)$ holds,*

1. *$R_\lambda(f) \in \mathcal{D}(\mathbf{B}_F)$.*
2. *$R_\lambda(f) \overset{b.p.}{\to} f$ for $\lambda \to 0$.*

*Let us denote by **Id** the identity on $C_b(E)$. For all $\lambda > 0$ holds*

3. *If $f \in C_b(E)$, then $(\mathbf{Id} - \lambda \mathbf{B}_F) R_\lambda f = f$.*
4. *If $f \in \mathcal{D}(\mathbf{B}_F)$, then $R_\lambda(\mathbf{Id} - \lambda \mathbf{B}_F) f = f$.*

Proof. See Proposition II.2.2. in [40] □

Lemma A.2.7. *Every Borel strong Markov process X is a Hunt process, indeed let us assume that $\mathcal{F}$ is the complete, right-continuous version of the natural filtration of X, then it holds*

$$\lim_{n\to\infty} X_{T_n} = X_T \quad a.s.$$

for all increasing sequences $(T_n)_{n=1}^\infty$ of $\mathcal{F}$-stopping times with $T_n \uparrow T$, where T is also a $\mathcal{F}$-stopping time.

Proof. This statement is the Exercise II.2.1. on Page 14 in [40]. Following the given hints, we define $Y := \lim_{n\to\infty} X_{T_n}$, note that $Y(\omega)$ is well-defined, because X is càdlàg. We are now going to show that $Y = X_T$ almost surely, for this it is sufficient to concentrate on bounded T, because for a general finite T, we can use the fact that $X_{T\wedge n} = X_T$ on $\{T \leq n\}$ and that $\mathbb{P}[T \leq n] \to 1$ for $n \to \infty$. The main step for this proof is to show that

$$\mathbb{E}[g(Y)f(X_T)] = \mathbb{E}[g(Y)f(Y)] \tag{A.2}$$

for all $g, f \in C_b(E)$. Based on this statement we can conclude that for any $h \in C_b(E)$ holds

$$\mathbb{E}[(h(X_T) - h(Y))^2] = \mathbb{E}[h^2(X_T)] - 2\mathbb{E}[h(X_T)h(Y)] + \mathbb{E}[h^2(Y)] = 0.$$

We prove (A.2) first for $h = R_\lambda(f)$ in the place of f, where $\lambda > 0$ and $f \in C_b(E)$. Since $R_\lambda(f)$ is converging against f boundedly and pointwise, we can apply Lebesgue theorem to derive (A.2). Since X is a strong Markov process, we have that

$$P_t(f)(X_T) = \mathbb{E}[f(X_{T+t})|X_T] = \mathbb{E}[f(X_{T+t})|\mathcal{F}_T].$$

When we combine this with definition of the resolvent, then it follows:

$$\mathbb{E}[g(Y)h(X_T)] = \mathbb{E}\left[g(Y)\int_0^\infty e^{-\lambda s}P_s(f)(X_T)ds\right] = \mathbb{E}\left[g(Y)\int_0^\infty e^{-\lambda s}\mathbb{E}[f(X_{T+s})|\mathcal{F}_T]ds\right]$$
$$= \mathbb{E}\left[\int_0^\infty e^{-\lambda s}\mathbb{E}[g(Y)f(X_{T+s})|\mathcal{F}_T]ds\right] = \mathbb{E}\left[\int_0^\infty e^{-\lambda s}g(Y)f(X_{T+s})ds\right]$$
$$= \mathbb{E}\left[g(Y)e^{\lambda T}\int_T^\infty e^{-\lambda s}f(X_s)ds\right]$$

Repeating those steps we can also show that

$$\mathbb{E}\left[g(X_{T_n})h(X_{T_n})\right] = \mathbb{E}\left[\int_0^\infty g(X_{T_n})e^{-\lambda s}f(X_{T_n+s})ds\right]$$
$$= \mathbb{E}\left[g(X_{T_n})e^{\lambda T_n}\int_{T_n}^\infty e^{-\lambda s}f(X_s)ds\right] \overset{n\to\infty}{\longrightarrow} \mathbb{E}\left[g(Y)e^{\lambda T}\int_T^\infty e^{-\lambda s}f(X_s)ds\right].$$

Since the limit of $\mathbb{E}[g(X_{T_n})h(X_{T_n})]$ is $\mathbb{E}[g(Y)h(Y)]$, this proves our claim. □

Appendix B

DW-Superprocesses and KR-Representations

B.1 Dawson-Watanabe Superprocess

In this section fills out the missing details from Chapter 1. We present two ways to characterize the Dawson-Watanabe superprocess as the solution of a martingale problem. The first one will be the classical way often found in the literature, while the second way is more convenient in the context of the Kurtz-Rodrigues representation and the Markov mapping theorem.
Let us assume that $\mathbf{B}_F$ is the full weak generator belonging to a Borel strong Markov family. Traditionally the Dawson-Watanabe superprocess is characterized as the solution of the following martingale problem see Section II.5 in Perkins' Saint Flour [40].

Definition B.1.1. *Let us assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and that $\Xi^{\mathbb{X}} : \Omega \times [0,\infty) \to \mathcal{M}_f(E)$ is a continuous $\mathcal{M}_f(E)$-valued process with natural filtration $\mathcal{F}^{\Xi} := (\mathcal{F}^{\Xi}_t, t \geq$ ß$)$ that satisfies: For every $\hat{g} \in \mathcal{D}(\mathbf{B}_F)$ the process given by*

$$M_{\hat{g}}(t) := \Xi^{\mathbb{X}}_t(\hat{g}) - \Xi^{\mathbb{X}}_0(\hat{g}) - \int_0^t \Xi^{\mathbb{X}}_s(\mathbf{B}_F(\hat{g})) + b\Xi^{\mathbb{X}}_s(\hat{g})\, ds \tag{B.1}$$

is a continuous local $\mathcal{F}^{\Xi}$-martingale with quadratic variation given by

$$\langle M_{\hat{g}} \rangle_t := \int_0^t 2a\Xi^{\mathbb{X}}_s(\hat{g}^2)\, ds. \tag{B.2}$$

*We call $\Xi^{\mathbb{X}}$ a **Dawson-Watanabe superprocess** with **spatial motion** given by $\mathbf{B}_F$, **branching rate** a and **drift** b. We often write $\Xi^{\mathbb{X}} \sim \boldsymbol{DW}(\mathbf{B}_F, a, b)$ or $\Xi^{\mathbb{X}} \sim \boldsymbol{DW}(\mathbf{B}_F, a, b, \hat{\Theta}_0)$ in the case of $\Xi^{\mathbb{X}}_0 \sim \hat{\Theta}_0$.*

Proposition B.1.2. *The local martingale problem from Definition B.1.1 is well-posed. Further, if $\mathbb{E}[\Xi^{\mathbb{X}}_0(E)] < \infty$, then the process $M_{\hat{g}}$ is a proper martingale for all $g \in \mathcal{D}(\mathbb{X})$. Further if $(\mathcal{P}_\mu, \mu \in \mathcal{M}_f(E))$, where $\mathcal{P}_\mu$ is the law of Ξ over $C([0,\infty), \mathcal{M}_f(E))$, then $(\mathcal{P}_\mu, \mu \in \mathcal{M}_f(E))$ forms a Borel strong Markov family.*

Proof. According to Theorem II.5.1.(a) from [40] the local martingale problem implied by Definition B.1.1 is well-posed for any initial distribution of $\Xi^{\mathbb{X}}$ and by Theorem II.5.1.(b) the solutions form together a Borel strong Markov process. Further by Proposition II.4.2. from [40] we also know that there exists a solution with $\mathbb{E}[\Xi_0^{\mathbb{X}}(E)] < \infty$ for which $M_{\hat{g}}$ is a proper martingale for all possible $\hat{g} \in \mathcal{D}(\mathbf{B}_F)$. □

While Definition B.1.1 establishes the connection to the standard literature about superprocesses, in the context of the Kurtz-Rodrigues representation and for the application of the Markov mapping theorem a different martingale problem for the Dawson-Watanabe superprocess based on the Laplace functionals is more convenient.

Definition B.1.3. *Let E be a Polish space and $\hat{f} \in C_b^+(E)$ be a continuous, non-negative function defined on E. We define the Laplace functional $\hat{L}_{\hat{f}} \in C_b^+(\mathcal{M}_f(E))$ by setting*

$$\hat{L}_{\hat{f}}(\mu) = \exp(-\mu(\hat{f})), \quad \mu \in \mathcal{M}_f(E).$$

Definition B.1.4. *Let us fix the parameters $(\mathbf{B}, a, b)$, where $\mathbf{B} : \mathcal{D}(\mathbf{B}) \subset C_b(E) \to C_b(E)$ is an operator satisfying the Conditions B.2.2, $a > 0$ and $b \in \mathbb{R}$. We define the map*

$$\mathbf{C_B} : C_b(\mathcal{M}_f(E)) \supset \mathcal{D}(\mathbf{C_B}) \to C(\mathcal{M}_f(E))$$

in the following way: We set $\mathcal{D}(\mathbf{C_B}) = \mathit{span}\{\hat{L}_{\hat{g}}; \hat{g} \in \hat{\mathcal{D}}(\mathbf{B})\}$ and we define $\mathbf{C_B}(\hat{L}_{\hat{g}})$ for $\hat{g} \in \mathcal{D}(\mathbf{B})$ by setting

$$\mathbf{C_B}(\hat{L}_{\hat{g}})(\mu) = -\big[\mu(\mathbf{B}(\hat{g})) + b\mu(\hat{g}) - a\mu(\hat{g}^2)\big] \exp\big(-\mu(\hat{g})\big), \quad \mu \in \mathcal{M}_f(E).$$

Proposition B.1.5. *The martingale problem $\boldsymbol{MP}(\mathbf{C_B})$ admits a $\Xi^{\mathbb{X}}$ solution with*

$$\mathbb{E}[\Xi_0^{\mathbb{X}}(E)] < \infty \text{ and } \int_0^t \mathbb{E}[\Xi_s^{\mathbb{X}}(E)]\, ds < \infty. \tag{B.3}$$

Proof. Every continuous solution of the martingale problem from Definition B.1.1 is also a solution of $\mathbf{MP}(\mathbf{C_B})$. Assume that $\Xi^{\mathbb{X}}$ is a solution of the martingale problem from Definition B.1.1. From (B.1), (B.2) and Itô's formula we can conclude that $\Xi^{\mathbb{X}}$ is a solution of $\mathbf{MP}(\mathbf{C_B})$. When we set $\hat{g} = \mathbb{1}_E$, then we can conclude from (B.1) that

$$\mathbb{E}[\Xi_t^{\mathbb{X}}(E)] = \mathbb{E}[\Xi_0^{\mathbb{X}}(E)] + \int_0^t b\mathbb{E}[\Xi_s^{\mathbb{X}}(E)]\, ds = \mathbb{E}[\Xi_0^{\mathbb{X}}(E)]e^{bt}.$$

This gives us (B.3). □

Remark B.1.6. *Note that we do not require that $\boldsymbol{MP}(\mathbf{C_B})$ is well-posed, although that could be derived easily with the help of the Log-Laplace equation, see Section 4.7 in [9] for situation of locally compact E or Page 43 in [39] for more general spaces E. With Log-Laplace equation we can derive that all solutions must have the same one-dimensional distribution, then the Theorem 4.4.2. from [14] tells us the that the martingale problem is well-posed. We do not need that $\boldsymbol{MP}(\mathbf{C_B})$ is well-posed, because the Markov mapping theorem, see Theorem D.1.13, just requires Proposition B.1.5. We obtain the well-posedness of $\boldsymbol{MP}(\mathbf{C_B})$ as a side effect of the Markov mapping theorem.*

B.2 Generator of the KR-Representation

Goal of this section is to characterize the Kurtz-Rodrigues representation as the solution of the well-posed martingale problem $\mathbf{MP}(\mathbf{A_B})$, hereby we fill out some of the gaps in Section 1.1. The operator $\mathbf{A_B}$ of the martingale problem, see Definition B.2.9 below, is based on a linear operator **B** characterizing the spatial motion of the particles. This operator **B** must satisfy certain conditions/restrictions, so we can apply the results of Section A.6 from [32] and the Markov mapping theorem. These conditions are very technical but for every reasonable Markov process it should be possible to find a suitable generator.

Definition B.2.1. *1. An operator* $\mathbf{A}$ *generates a Borel strong Markov family, if the martingale problem* $\boldsymbol{MP}(\mathbf{A})$ *is well-posed and every solution admits a càdlàg modification and if* $P_x \in \mathcal{M}_1(\mathbb{D}([0,\infty), E))$ *is the law belonging to the càdlàg modification of the solution* $\boldsymbol{MP}(\mathbf{A}, \delta_x)$*, then* $(P_x, x \in E)$ *forms a Borel strong Markov Family.*

2. If $\tilde{\mathbf{A}} : C(E) \supset \mathcal{D}(\tilde{\mathbf{A}}) \to C(E)$ *is a second linear operator, then we say that the martingale problem* $\boldsymbol{MP}(\mathbf{A})$ *is equivalent to* $\boldsymbol{MP}(\tilde{\mathbf{A}})$*, if every solution of* $\boldsymbol{MP}(\mathbf{A})$ *is a solution of* $\boldsymbol{MP}(\tilde{\mathbf{A}})$ *and vice versa.*

In the context of the Kurtz-Rodrigues theory we assume **B** to satisfy the following conditions:

Conditions B.2.2. *The operator* $\mathbf{B} : C_b(E) \supset \mathcal{D}(\mathbf{B}) \to C_b(E)$ *satisfies:*

1. *The martingale problem of* $\mathbf{B}$ *is well-posed.*
2. $\mathbf{B}$ *generates a Borel strong Markov family.*
3. $\mathcal{D}(\mathbf{B})$ *is separating.*
4. *The domain* $\mathcal{D}(\mathbf{B})$ *is closed under multiplication, linear combinations and contains* $\mathbb{1}_E$*, i.e.* $\mathcal{D}(\mathbf{B})$ *is an unital algebra.*
5. *There exists a countable set* $\Gamma \subset \mathbf{B}$*, such for every element* $(\hat{g}, \hat{h}) \in \mathbf{B}$*, the set* Γ *contains a sequence* $(\hat{g}_n, \hat{h}_n)_{n=1}^{\infty}$ *such that* $\hat{g}_n \overset{n\to\infty}{\longrightarrow} \hat{g}$ *and* $\hat{h}_n \overset{n\to\infty}{\longrightarrow} \hat{h}$ *uniformly for* $n \to \infty$*.*

The Conditions B.2.2 are inspired by the Conditions 3.1. in [32] but have been modified slightly to simply the proof of Lemma B.3.10.

We are now turning to the martingale characterization of the Kurtz-Rodrigues representation. Reflecting the situation of the Dawson-Watanabe superprocess, the KR-representation has the same three parameters $\mathbf{B}, a$ and b as before.

Lemma B.2.3. *We denote by* $\mathcal{N}_f(E)$ *the subspace of integer-valued measures in* $\mathcal{M}_f(E)$*. The space* $\mathcal{N}_f(E)$ *is a closed set under the weak topology on* $\mathcal{M}_f(E)$ *and hence also a Polish space.*

Proof. According to Lemma 9.1.V from [8] the space of boundedly finite integer-valued measures $\mathcal{N}_{lf}(E)$ is a closed set in the space of boundedly finite measures $\mathcal{M}_{lf}(E)$ in the weak-# topology, which is defined by saying that $(\mu_n)_{n=1}^{\infty}$ is converging against μ, if $\mu_n(f) \to \mu(f)$ for all $f \in C_b^+(E)$ which are vanishing outside of a bounded set. If $(\varrho_n)_{n=1}^{\infty} \subset \mathcal{N}_f(E) \subset \mathcal{N}_{lf}(E)$ are converging against $\varrho \in \mathcal{M}_f(E) \subset \mathcal{M}_{lf}(E)$ in the weak topology, then it also does in the weak-# topology, consequently $\varrho \in \mathcal{N}_{lf}(E)$, because the latter is closed and since it also holds $\varrho_n(E) \to \varrho(E)$, we can conclude that $\varrho(E)$ is finite and so $\varrho \in \mathcal{N}_f(E)$. □

Lemma B.2.4. *The space* $\overline{\mathcal{N}}(E \times [0,\infty))$ *together with mixed topology from Definition 1.1.2 is a Polish space.*

Proof. Note that $\mathcal{N}_f(E \times [n, n+1))$, the space of finite integer-valued measures over $E \times [n-1, n)$, is a Polish space with respect to the weak topology and so is $\prod_{n\in\mathbb{N}} \mathcal{N}_f(E \times [n-1, n))$ with regard to the product topology. Let us now define the map $\phi : \overline{\mathcal{N}}(E \times [0,\infty)) \to \prod_{n\in\mathbb{N}} \mathcal{N}_f(E \times [n-1, n))$ by setting $\phi(\eta) = (\eta_{|E\times[n-1,n)})_{n\in\mathbb{N}}$, then ϕ is a homeomorphism between these two spaces making $\overline{\mathcal{N}}(E \times [0,\infty))$ a Polish space. □

As in the case of the Dawson-Watanabe superprocess the Laplace functionals play an important role, but note that the domain is this time $\overline{\mathcal{N}}(E \times [0,\infty))$ and not $\mathcal{M}_f(E)$.

Definition B.2.5. *For $f \in C_b^+(E \times [0,\infty))$ we define the Laplace functional $L_f \in C_b^+(\overline{\mathcal{N}}(E \times [0,\infty)))$ by setting $L_f(\xi) = \exp(-\xi(f)),\ \xi \in \overline{\mathcal{N}}(E \times [0,\infty))$.*

Remark B.2.6. *Note, if there exists a $r > 0$ such that $supp(f) \subset E \times [0,r]$, then $L_f \in C_b^+(\overline{\mathcal{N}}(E \times [0,\infty))))$.*

Let us assume that $g \in C^+(E \times [0,\infty))$ is bounded away from 0 and bounded by 1, indeed we can find a value m_g such that $m_g \le g \le 1$, then $-\log(g)$ is an element of $C_b^+(E \times [0,\infty))$ and it holds

$$L_{-\log(g)}(\eta) = \prod_{(x,u)\in\eta} g(x,u). \tag{B.4}$$

If the KR-representation is characterized via the linear operator $\mathbf{A_B}$, then the domain of $\mathbf{A_B}$ will consist of Laplace functionals of the form (B.4). We are now going to specify the conditions for g. These conditions are based on the Conditions 3.1 in [32]. Let $\mathbf{B} : C_b(E) \supset \mathcal{D}(\mathbf{B}) \to C_b(E)$ be as in the context of Dawson-Watanabe superprocess an operator satisfying B.2.2. The building blocks of our test function are functions $g : E \times [0,\infty) \to (0,1]$ with the form

$$g(x,u) = \prod_{j=1}^{l} \left[1 - g_j^x(x) g_j^u(u)\right],\ (x,u) \in \mathbb{E} \times [0,\infty), \tag{B.5}$$

where $g_j^x : E \to [0,1)$ are elements of $\mathcal{D}(\mathbf{B})$ and $g_j^u : [0,\infty) \to [0,1)$ are elements of $C^1([0,\infty))$. The product form of g has the purpose to ensure that our test functions are closed under multiplication. When we multiply out g, we obtain

$$g(x,u) = 1 + \sum_{|J| \subset [l] \setminus \emptyset} (-1)^{|J|} g_J^x(x) g_J^u(u) \tag{B.6}$$

with $[l] := \{1,2,...,l\}, g_J^x := \prod_{j\in J} g_j^x$ and $g_J^u := \prod_{j\in J} g_j^u$ for $J \subset [l] \setminus \emptyset$. Since $\mathcal{D}(\mathbf{B})$ is closed under multiplication, we can apply $\mathbf{B}$ to g as a function of the x-variable and obtain

$$\mathbf{B}(g)(x,u) = \sum_{|J| \subset [l] \setminus \emptyset} (-1)^{|J|} \mathbf{B}(g_J^x)(x) g_J^u(u) \tag{B.7}$$

and obtain the partial derivative of g with respect to u by

$$\partial_u g(x,u) = \sum_{|J| \subset [l] \setminus \emptyset} (-1)^{|J|} g_J^x(x) \partial_u g_J^u(u). \tag{B.8}$$

Definition B.2.7. *Assume that E is a Polish space, that the linear operator $\mathbf{B} : \mathcal{D}(\mathbf{B}) \subset C_b(E) \to C_b(E)$ satisfies the Conditions B.2.2. Fixing $K \in [0,\infty),\ r > 0,\ m \in (0,1)$ we define the class of test functions $\mathfrak{g}(\mathbf{B}, K, r, m) \subset C_b^+(E \times [0,\infty))$ as the set of functions $g : E \times [0,\infty) \to (0,1]$ with*

$$g(x,u) = \prod_{j=1}^{l} \left(1 - g_j^x(x) g_j^u(u)\right),\ (x,u) \in \mathbb{E} \times [0,\infty), \tag{B.9}$$

where $g_j^x : E \to [0,1)$ and $g_j^u : [0,\infty) \to [0,1)$ for $1 \le j \le l \in \mathbb{N}$ and this collection of functions satisfies:

1. *It holds* $g_j^x \in \mathcal{D}(\mathbf{B})$ *for* $1 \leq j \leq n$ *and the function* $|\mathbf{B}(g)|$ *from* (B.7) *is bounded by the constant* K. *Similarly,* $g_j^u \in C^1([0,\infty))$ *for* $1 \leq j \leq n$ *and the function* $|\partial_u g|$ *from* (B.8) *is bounded by the constant* K.

2. *The support of the function* $g_j^u, 1 \leq j \leq l$, *is contained in the interval* $[0, r]$, *hence* $g(x,u) = 1$ *for* (x,u) *with* $u \geq r$.

3. *The image of* g *is contained in* $[m, 1]$, *i.e.*

$$0 < m \leq g(x,u) \leq 1, \; (x,u) \in E \times [0,\infty). \tag{B.10}$$

We also define for $r \in (0,\infty]$:

$$\mathfrak{g}(\mathbf{B}, r) := \bigcup_{K>0, \tilde{r}\in(0,r), m\in(0,1)} \mathfrak{g}(\mathbf{B}, K, \tilde{r}, m). \tag{B.11}$$

For $r = \infty$, *we prefer to write* $\mathfrak{g}(\mathbf{B})$ *instead of* $\mathfrak{g}(\mathbf{B}, \infty)$.

Remark B.2.8. *The set* $\mathfrak{g}(\mathbf{B})$ *is closed under multiplication and, if* $g \in \mathfrak{g}(\mathbf{B})$, *then the function* $f := -\log(g)$ *is an element of* $C_b^+(E)$.

The Kurtz-Rodritues representation is characterized as the solution of the martingale problem associated with the following operator.

Definition B.2.9. *Let* $\mathbf{B} : \mathcal{D}(\mathbf{B}) \subset C_b(E) \to C_b(E)$ *satisfy the Conditions B.2.2,* $a > 0$ *and* $b \in \mathbb{R}$. *For the parameters* $(\mathbf{B}, a, b)$ *we define the operator*

$$\mathbf{A_B} : C(\overline{\mathcal{N}}(E \times [0,\infty))) \supset \mathcal{D}(\mathbf{A_B}) \to C(\overline{\mathcal{N}}(E \times [0,\infty))),$$

where the domain $\mathcal{D}(\mathbf{A_B})$ *is given by the linear span of the Laplace functionals associated with the collection* $\mathfrak{g}(\mathbf{B})$, *indeed* $\mathcal{D}(\mathbf{A_B}) := \textit{span}\left\{L_{-\log(g)};\; g \in \mathfrak{g}(\mathbf{B})\right\}$, *and where the function* $\mathbf{A_B}(L_{-\log(g)})$ *is given for* $\eta \in \overline{\mathcal{N}}(E \times [0,\infty))$ *by*

$$\begin{aligned}\mathbf{A_B}(L_{-\log(g)})(\eta) &= L_{-\log(g)}(\eta) \int_E \int_0^\infty \frac{\mathbf{B}(g)(x,u)}{g(x,u)} \; \eta(dx,du) \\ &+ L_{-\log(g)}(\eta) \int_E \int_0^\infty \left(2a \int_u^\infty g(x,\tilde{u}) - 1 \; d\tilde{u}\right) \eta(dx,du) \\ &+ L_{-\log(g)}(\eta) \int_E \int_0^\infty \left[au^2 - bu\right] \frac{\partial_u g(x,u)}{g(x,u)} \; \eta(dx,du)\end{aligned}$$

with $\mathbf{B}(g)$ *being the application of the operator* $\mathbf{B}$ *to the function* $x \mapsto g(x,u)$, *where* $u \in [0,\infty)$ *is fixed (recall* (B.7)*).*

Remark B.2.10. *If* $\eta = \sum_{i=1}^\infty \delta_{(x_i,u_i)}$, *then* $\mathbf{A_B}(L_{-\log(g)})(\eta)$ *can be also expressed as:*

$$\begin{aligned}&\sum_{i=1}^\infty \prod_{i\neq j} g(x_j,u_j) \mathbf{B}(g)(x_i,u_i) \\ &+ \sum_{i=1}^\infty \prod_{i\neq j} g(x_j,u_j) 2a g(x_i,u_i) \int_{u_i}^\infty g(x_i,\tilde{u}) - 1 \, d\tilde{u} \\ &+ \sum_{i=1}^\infty \prod_{i\neq j} g(x_j,u_j) [a u_i^2 - b u_i] \partial_u g(x_i,u_i).\end{aligned}$$

Proposition B.2.11. *Let assume that* $\mathbf{B}, a, b$ *and* $\mathbf{A_B}$ *are as in Definition B.2.9, then the local martingale problem* $\boldsymbol{MP}(\mathbf{A_B})$ *is well-posed and admits a càdlàg solution.*

Proof. See Section 3.4. and Section A.6. in [32]. □

Definition B.2.12. *Let* $\mathbf{B}, a, b, \mathbf{A_B}$ *and* Θ_0 *as in Proposition B.2.11 and assume that* $(\Omega, \mathcal{A}, \mathbb{P})$ *is some probability space, we call a stochastic process*

$$\boldsymbol{\xi}^{\mathbb{X}} : \Omega \times [0, \infty) \to \overline{\mathcal{N}}(E \times [0, \infty)),$$

which is a càdlàg solution of $\boldsymbol{MP}(\mathbf{A_B}, \Theta_0)$ *a* ***Kurtz-Rodrigues representation*** *(or an* ***empirical Kurtz-Rodrigues representation****) with* ***spatial motion*** $\mathbf{B}$*,* ***branching rate*** $a > 0$ *and* ***drift*** $b \in \mathbb{R}$*. We write* $\boldsymbol{\xi}^{\mathbb{X}} \sim \boldsymbol{KR}(\mathbf{B}, a, b)$ *(or* $\boldsymbol{\xi}^{\mathbb{X}} \sim \boldsymbol{KR}(\mathbf{B}, a, b, \Theta_0)$*).*

B.3 Connection between DW-Superprocess and KR-Rep.

We are interested in the Kurtz-Rodrigues representation, because it is a Poisson representation of the Dawson-Watanabe superprocess. This is a result of the Markov mapping theorem and the right choice of the initial distribution, indeed the distribution of $\boldsymbol{\xi}_0^{\mathbb{X}}$ must be a mixture of Poisson point process distributions.

Definition B.3.1 (Poisson mixture)**.** *We say that* $\Theta_0 \in \mathcal{M}_1(\overline{\mathcal{N}}(E \times [0, \infty)))$ *is a* ***Poisson mixture*** *based on* $\hat{\Theta}_0 \in \mathcal{M}_1(E)$*, if*

$$\Theta_0 = \int_{\mathcal{M}_f(E)} \mathbf{PPP}_E(\mu \otimes \ell eb[0, \infty))\, \hat{\Theta}_0(d\mu), \tag{B.12}$$

and by this we mean that it holds for all $F \in C_b(\overline{\mathcal{N}}(E \times [0, \infty)))$*:*

$$\Theta_0(F) = \int_{\mathcal{M}_f(E)} \int_{\overline{\mathcal{N}}(E \times [0,\infty))} F(\boldsymbol{\xi})\, \mathbf{PPP}_E(\mu \otimes \ell eb[0, \infty), d\boldsymbol{\xi})\, \hat{\Theta}_0(d\mu).$$

In the special case, where $\hat{\Theta}_0 = \delta_\mu$ *for* $\mu \in \mathcal{M}_f(E)$*, we just write* $\Theta_0 = \mathbf{PPP}(\mu \otimes \ell eb[0, \infty))$*.*

Remark B.3.2. *The proof of Lemma B.3.7 implies that* $\mu \mapsto \mathbf{PPP}_E(\mu \otimes \ell eb[0, \infty))$ *is a continuous map, which makes it measurable and the above integrals well-defined.*

Theorem B.3.3. *Let us assume that* $\mathbf{B}, a, b$ *and* $\mathbf{A_B}$ *are given as in Definition B.2.9. Further let us assume* $\boldsymbol{\xi}^{\mathbb{X}}$ *is a Kurtz-Rodrigues representation with*

$$\boldsymbol{\xi}^{\mathbb{X}} \sim \boldsymbol{KR}(\mathbf{B}, a, b, \Theta_0)$$

and that the initial distribution $\Theta_0 \in \mathcal{M}_1(\overline{\mathcal{N}}(E \times [0, \infty)))$ *is a Poisson mixture based on* $\hat{\Theta}_0 \in \mathcal{M}_1(\mathcal{M}_f(E))$*, see Definition B.3.1. If we define the process* $\tilde{\boldsymbol{\Xi}}^{\mathbb{X}} : \Omega \times [0, \infty) \to \mathcal{M}_f(E)$ *by* $\tilde{\boldsymbol{\Xi}}^{\mathbb{X}} := \gamma_E^{\boldsymbol{\Xi}}(\boldsymbol{\xi}^{\mathbb{X}})$ *and assume that* $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{X}}$ *is the augmented filtration of the natural filtration of* $\tilde{\boldsymbol{\Xi}}^{\mathbb{X}}$*, then:*

1. *The process* $\tilde{\boldsymbol{\Xi}}^{\mathbb{X}}$ *admits a continuous modification* $\boldsymbol{\Xi}^{\mathbb{X}}$ *which is a Dawson-Watanabe superprocess with* $\boldsymbol{\Xi}^{\mathbb{X}} \sim (\mathbf{B}, a, b, \hat{\Theta}_0)$ *with respect to the filtration* $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{X}}$*.*

2. *For all finite* $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{X}}$*-stopping times* τ *it holds*

$$\mathfrak{L}(\boldsymbol{\xi}_\tau^{\mathbb{X}} | \mathcal{F}_\tau^{\boldsymbol{\Xi},\mathbb{X}}) = \mathbf{PPP}_E(\boldsymbol{\Xi}_\tau^{\mathbb{X}} \otimes \ell eb[0, \infty)).$$

3. *For $r \geq 0$ let us define $\boldsymbol{\Xi}^{\mathbb{X},r} : \Omega \times [0,\infty) \to \mathcal{M}_f(E)$ as $\boldsymbol{\Xi}^{\mathbb{X},r} = \boldsymbol{\xi}^{\mathbb{X}}(\,\cdot\, \times [0,r)))$, i.e. $\boldsymbol{\Xi}^{\mathbb{X},r}$ consists of the spatial positions of the atoms with a level below r, then it holds for every $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{X}}$-stopping time τ that*

$$\frac{1}{r}\boldsymbol{\Xi}^{\mathbb{X},r}_{\tau} \overset{r\to\infty}{\longrightarrow} \boldsymbol{\Xi}^{\mathbb{X}}_{\tau} \text{ a.s.},$$

where the convergence holds in the weak topology of $\mathcal{M}_f(E)$.

This theorem is an application of the Markov mapping theorem. For the latter we need to verify that $\mathbf{A_B}$ and the operator $\mathbf{C_B}$ from Definition B.1.4 which is associated with the Dawson-Watanabe superprocess satisfy an intertwiner relationship. This intertwiner relationship will be presented in Lemma B.3.9. We also need to show that $\mathbf{A_B}$ satisfies the Conditions D.1.8 from Appendix D.1.13. Since $\mathbf{A_B}(L_{\log(g)})$ is an unbounded function, we need a bounding function for $\mathbf{A_B}$ to apply the Markov mapping theorem.

Definition B.3.4. *Let us define $\psi : \overline{\mathcal{N}}(E \times [0,\infty)) \to [0,\infty)$ as the map given by*

$$\psi(\eta) := \int_{E\times[0,\infty)} e^{-u}\eta(dx,du) + 1 \tag{B.13}$$

and let us define $C_\psi(\overline{\mathcal{N}}(E \times [0,\infty)))$ as the subset of $C(\overline{\mathcal{N}}(E \times [0,\infty)))$ consisting of functions F for which we can find a constant K_F such that $|F| \leq K_F\psi$.

Lemma B.3.5. *The image of $\mathbf{A_B}$ is contained in $C_\psi(\overline{\mathcal{N}}(E \times [0,\infty)))$.*

Proof. Every function contained in the domain of $\mathbf{A_B}$ is a linear combination of functions with the form $L_{\text{-}\log(g)}$ with $L_{\text{-}\log(g)}(\eta) = \exp(\eta(\log(g)))$ and with $g \in \mathfrak{g}(\mathbf{B}, K, r, m) \subset C_b^+(E \times [0,\infty))$, where $K \in [0,\infty)$, $r > 0$, $m \in (0,1)$ are fixed. By the definition of $\mathfrak{g}(\mathbf{B}, K, r, m)$, see Definition B.2.7, and the one of $\mathbf{A_B}$, see Definition B.2.9, we can conclude that

$$|\mathbf{A_B}(L_{\text{-}\log(g)})(\eta)| \leq \left(1 + ar^2 + |b|r + 2ar)Ke^r(\int_{E\times[0,\infty)} e^{-u}\eta(dx,du) + 1\right).$$

□

The most important ingredient in the context of the Markov mapping theorem is the Markov kernel $\mathbf{PPP}_E$ and its relation with the projection γ_E.

Lemma B.3.6. *The Markov kernel $\mathbf{PPP}_E$ and the projection γ_E form together a **Rogers-Pitmann correspondence**, i.e. $\mathbf{PPP}_E(\mu, \gamma_E^{-1}(\mu)) = 1$.*

Proof. This follows from Corollary C.1.7. □

Let us recall that $\mathbf{M}(E)$ is the collection of measurable function $\hat{g} : E \to \mathbb{R}$.

Lemma B.3.7. *We define the pull-back*

$$\mathbf{PPP}_E^* : C_\psi(\overline{\mathcal{N}}(E \times [0,\infty))) \to \mathbf{M}(\mathcal{M}_f(E))$$

by setting for each $F \in C_\psi(\overline{\mathcal{N}}(E \times [0,\infty))$ and $\mu \in \mathcal{M}_f(E)$:

$$\mathbf{PPP}_E^*(F)(\mu) := \mathbb{E}\left[F(\boldsymbol{\xi})\right] \text{ with } \boldsymbol{\xi} \sim \mathbf{PPP}_E(\mu \otimes \ell eb[0,\infty)),$$

then $\mathbf{PPP}_E^*$ is well-defined on $C_\psi(\overline{\mathcal{N}}(E \times [0,\infty)))$ and continuous on $C_b(\overline{\mathcal{N}}(E \times [0,\infty)))$.

Proof. Let us write $\tilde{\psi}(x,u) = e^{-u}$, for $F \in C_\psi(\overline{\mathcal{N}}(E \times [0,\infty))$ it holds

$$|\mathbf{PPP}^*_E(F)(\mu)| = \mathbb{E}[|F(\boldsymbol{\xi})|] \leq K_F \mathbb{E}[\psi(\boldsymbol{\xi})] \leq K_F \mathbb{E}[\boldsymbol{\xi}(\tilde{\psi})] = K_F \mu(E),$$

so $\mathbf{PPP}^*_E(F)$ is well-defined. For $F \in C_b(\overline{\mathcal{N}}(E \times [0,\infty)))$ the continuity is a consequence of Lemma A.9 from [32] which we apply with $h_1 = \mathbb{1}_{E\times[0,\infty)}$. If we now approximate $\mathbf{PPP}^*_E(F)$ for a general $F \in C_\psi(\overline{\mathcal{N}}(E \times [0,\infty))$ by bounded functions, it follows $\mathbf{PPP}^*_E(F) \in \mathbf{M}(\overline{\mathcal{N}}(E \times [0,\infty)))$. ☐

Let us assume that $g \in \mathfrak{g}(\mathbf{B})$ and let us define $\hat{g} : E \to [0,\infty)$ by setting

$$\hat{g}(x) := \int_0^\infty 1 - g(x,u)\, du, \tag{B.14}$$

note that $\hat{g}$ is well-defined, because there exists a r, such that $1 - g(x,u) = 0$ for all (x,u) with $u \geq r$ (recall by Definition B.2.7 the support of the functions g^u_j is contained in a compact set and therefore the same is true for g^u_J from (B.18)). We can now make the following important observation: If $\tilde{\boldsymbol{\xi}} : \tilde{\Omega} \to \overline{\mathcal{N}}(E \times [0,\infty))$ is a Poisson point processes with an intensity measure $\mu \otimes \ell eb[0,\infty), \mu \in \mathcal{M}_f(E)$, from the identity (C.6) from Lemma C.2.1 it follows:

$$\mathbb{E}\left[L_{\text{-}\log(g)}(\tilde{\boldsymbol{\xi}})\right] = \exp\left(\int_E \int_0^\infty e^{\log(g)(x,u)} - 1\, du\, \mu(dx)\right) = \exp\left(-\mu(\hat{g})\right). \tag{B.15}$$

The immediate consequences of this observation are contained in the following lemma.

Lemma B.3.8. *If $g \in \mathfrak{g}(\mathbf{B})$, $\hat{g}$ is defined as in* (B.14) *and $\mathbf{PPP}^*_E$ is the pull-back from Lemma B.3.8, then it holds for the*

1. *The function $\hat{g}$ is an element of $\mathcal{D}(\mathbf{B})$.*
2. *For the functions $L_{\text{-}\log(g)}(\eta) = \exp(\eta(\log(g))), \eta \in \overline{\mathcal{N}}(E \times [0,\infty))$ and $\hat{L}_{\hat{g}} = \exp(-\mu(\hat{g})), \mu \in \mathcal{M}_f(E)$ holds*

   $$\mathbf{PPP}^*_E(L_{\text{-}\log(g)})(\mu) = \exp(-\mu(\hat{g})) = \hat{L}_{\hat{g}}(\mu), \quad \mu \in \mathcal{M}_f(E). \tag{B.16}$$

   Therefore $\mathbf{PPP}^*_E(L_{\text{-}\log(g)}) \in \mathcal{D}(\mathbf{C_B})$.
3. *Further for every $\hat{g} \in \hat{\mathcal{D}}(\mathbf{B})$, there exists a $g \in \mathfrak{g}(\mathbf{B})$, such that $\mathbf{PPP}^*_E(L_{\text{-}\log(g)}) = \hat{L}_{\hat{g}}$, which implies that*

   $$\mathbf{PPP}^*_E(\mathcal{D}(\mathbf{A_B})) = \mathcal{D}(\mathbf{C_B}). \tag{B.17}$$

Proof. Let us assume $g \in \mathfrak{g}(\mathbf{B}, K, r, m)$. We can write:

$$g(x,u) = \prod_{j=1}^{l} \left[1 - g^x_j(x) g^u_j(u)\right] = 1 + \sum_{|J| \subset [l] \setminus \emptyset} (-1)^{|J|} g^x_J(x) g^u_J(u), \tag{B.18}$$

where $[l] := \{1,2,...,l\}, g^x_J := \prod_{j\in J} g^x_j$ and $g^u_J := \prod_{j \in J} g^u_j$ for $J \subset [l] \setminus \emptyset$. Since $g^x_j \in \mathcal{D}(\mathbf{B})$ and since $\mathcal{D}(\mathbf{B})$ is closed under multiplication we can derive:

$$\mathbf{B}(\hat{g}) = \int_0^\infty \sum_{|J| \subset [l] \setminus \emptyset} (-1)^{|J|} \mathbf{B}(g^x_J)(x) g^u_J(u)\, du = -\int_0^\infty \mathbf{B}(g)(x,u)\, du, \tag{B.19}$$

where we could interchange $\mathbf{B}$ with the integral, because g_J^x does not depend on u, therefore $\hat{g} \in \mathcal{D}(\mathbf{B})$ and hence $\hat{L}_{\hat{g}} \in \mathcal{D}(\mathbf{C_B})$. If we consider the point process $\tilde{\boldsymbol{\xi}}$ appearing in (B.15), then $\tilde{\boldsymbol{\xi}} \sim \mathbf{PPP}_E(\mu)$, and so by (B.15) follows (B.16).
For (B.17) let us consider an arbitrary element $\hat{g}^*$ of $\mathcal{D}(\mathbf{B})$, then we choose

$$g = 1 - \frac{1}{||\hat{g}|| + 1} \hat{g}^* g_u \in \mathfrak{g}(\mathbf{B}),$$

where $g_u \in C^1([0, \infty))$ with $\operatorname{supp} g_u \subset [0, r]$ for some $r > 0$ has been chosen such that $\lambda := \int_0^r g_u(u) du = ||\hat{g}|| + 1$. If we define

$$\hat{g}(x) := \int_0^\infty 1 - g(x, u)\, du,$$

then $\hat{g}^* = \hat{g}$ and by (B.16) we have $\mathbf{PPP}_E^*(L_{-\log(g)}) = \hat{L}_{\hat{g}^*}$, which implies (B.17). ☐

Let us recall that $\mathbf{C_B}$ is the operator characterizing the Dawson-Watanabe superprocess and the operator $\mathbf{A_B}$ does the same for the Kurtz-Rodrigues representation. We are now ready to formulate and prove the intertwiner relationship between $\mathbf{A_B}$ and $\mathbf{C_B}$.

Lemma B.3.9. *If* $\mathbf{PPP}_E^* : C_b(\overline{\mathcal{N}}(E \times [0, \infty))) \to C_b(\overline{\mathcal{N}}(E))$ *is the pullback of the Markov kernel* $\mathbf{PPP}_E^*$ *from Lemma B.3.8, then it holds for all* $g \in \mathfrak{g}(\mathbf{B})$*:*

$$\mathbf{PPP}_E^* \circ \mathbf{A_B}(L_{-\log(g)})(\mu) = \mathbf{C_B} \circ \mathbf{PPP}_E^*(L_{-\log(g)})(\mu) \quad \mu \in \mathcal{M}_f(E). \tag{B.20}$$

Proof. Let us define $\hat{g}(x) := \int_0^\infty 1 - g(x, u)\, du, x \in E$, then by Lemma B.3.8 it holds $\hat{g} \in \mathcal{D}(\mathbf{B})$ and $\mathbf{PPP}_E^*(L_{-\log(g)}) = \hat{L}_{\hat{g}} \in \mathcal{D}(\mathbf{C_B})$. Therefore

$$\mathbf{C_B} \circ \mathbf{PPP}_E^*(L_{-\log(g)})(\mu) = \exp(-\mu(\hat{g})) \left[-\mu(\mathbf{B}(\hat{g})) - b\mu(\hat{g}) + a\mu(\hat{g}^2)\right], \quad \mu \in \mathcal{M}_f(E).$$

For a fixed μ and $g \in \mathfrak{g}(\mathbf{B})$ it holds by the definition of $\mathbf{PPP}_E^*$ that

$$\mathbf{PPP}_E^* \circ \mathbf{A_B}(L_{-\log(g)}) = \mathbb{E}\left[\mathbf{A_B}(L_{-\log(g)})(\boldsymbol{\xi})\right] \text{ with } \boldsymbol{\xi} \sim \mathbf{PPP}(\mu \otimes \ell eb[0, \infty)).$$

When we write out $\mathbb{E}\left[\mathbf{A_B}(L_{-\log(g)})(\boldsymbol{\xi})\right]$, we get:

$$\begin{aligned} \mathbb{E}\left[\mathbf{A_B}(L_{-\log(g)})(\boldsymbol{\xi})\right] &= \mathbb{E}\left[\exp(-\boldsymbol{\xi}(g)) \int_E \int_0^\infty \frac{\mathbf{B}(g)(x, u)}{g(x, u)}\, \boldsymbol{\xi}(dx, du)\right] &\text{(B.21)}\\ &+ \mathbb{E}\left[\exp(-\boldsymbol{\xi}(g)) \int_E \int_0^\infty \left(2a \int_u^\infty g(x, \tilde{u}) - 1\, d\tilde{u}\right) \boldsymbol{\xi}(dx, du)\right] &\text{(B.22)}\\ &+ \mathbb{E}\left[\exp(-\boldsymbol{\xi}(g)) \int_E \int_0^\infty \left[au^2 - bu\right] \frac{\partial_u g(x, u)}{g(x, u)}\, \boldsymbol{\xi}(dx, du)\right]. &\text{(B.23)} \end{aligned}$$

We can now apply (C.10) from Lemma C.2.1 to (B.21),(B.22) and (B.23) and obtain:

$$\begin{aligned} &\exp(-\mu(\hat{g})) \int_E \int_0^\infty \mathbf{B}(g)(x, u)\, du\, \mu(dx) &\text{(B.24)}\\ &+ \exp(-\mu(\hat{g})) \int_E \int_0^\infty g(x, u) \left(2a \int_u^\infty g(x, \tilde{u}) - 1\, d\tilde{u}\right) du\, \mu(dx) &\text{(B.25)}\\ &+ \exp(-\mu(\hat{g})) \int_E \int_0^\infty \left[au^2 - bu\right] \partial_u g(x, u)\, du\, \mu(dx) &\text{(B.26)} \end{aligned}$$

Recalling (B.19), we can see that the integral term in (B.24) is equal to

$$-\int_E \mathbf{B}(\hat{g})(x)\mu(dx). \tag{B.27}$$

By reordering the expressions of (B.25) and (B.26) to

$$\exp(-\mu(\hat{g}))\int_E\int_0^\infty u^2\partial_u g(x,u)+2\left(g(x,u)\int_u^R g(x,v)-1\,dv\right)du\mu(dx) \tag{B.28}$$

$$+\exp(-\mu(\hat{g}))\int_E\int_0^\infty u\,\partial_u g(x,u)\,du\mu(dx) \tag{B.29}$$

we can apply (C.13) and (C.15) from Lemma C.2.2 to (B.28) and (B.29) to obtain:

$$\exp(-\mu(\hat{g}))\left[a\int_E\left(\int_0^\infty 1-g(x,u)\,du\right)^2\mu(dx)-b\int_E\int_0^\infty 1-g(x,u)\,du\right].$$

Together we see that

$$\mathbf{PPP}_E^*\circ\mathbf{A_B}(L_{\text{-}\log(g)})(\mu)=\exp(-\mu(\hat{g}))\left[-\mu(\mathbf{B}(\hat{g}))-b\mu(\hat{g})+a\mu(\hat{g}^2)\right],$$

which is the expression of $\mathbf{C_B}\circ\mathbf{PPP}_E^*(L_{\text{-}\log(g)})(\mu)$. □

We are now proving that the generator $\mathbf{A_B}$ satisfies the Conditions D.1.8.

For the next lemma we need again the bounding function ψ form Definition B.3.4. Let us recall that a linear operator $\tilde{A}\subset C_b(E)\times C_b(E)$ is graph separable, when there exists a countable collection $(f_n,\tilde{A}(f_n))_{n=1}^\infty$ such that *tildeA* is contained in the b.p.-closure of linear span of $(f_n,\tilde{A}(f_n))_{n=1}^\infty$.

Lemma B.3.10. *If we define the operator $\tilde{\mathbf{A}}_\mathbf{B}$ as the linear span of $\{(L_{-\log(g)},\mathbf{A_B}(L_{-\log(g)})/\psi);g\in\mathfrak{g}(\mathbf{B})\}$, i.e. $\tilde{\mathbf{A}}_\mathbf{B}(L_{-\log(g)})(\eta)=\mathbf{A_B}(L_{-\log(g)})(\eta)/\psi(\eta)$, then $\tilde{\mathbf{A}}_\mathbf{B}$ is b.p.-graph separable.*

Proof. Let us recall that $\mathcal{D}(\mathbf{A_B})=\mathit{span}(\cup_{r>0}\{L_g;g\in\mathfrak{g}(\mathbf{B},r)\})$. By the Conditions B.2.2 there exists a countable collection $\Gamma_x\subset\mathcal{D}(\mathbf{B})$ such that we can find for each pair $(\hat{g},\mathbf{B}(\hat{g}))$ a sequence $(\hat{g}_n,n\in\mathbb{N})\subset\mathcal{D}(\mathbf{B})$ for which $(\hat{g}_n,\mathbf{B}(\hat{g}_n),n\in\mathbb{N})$ is converging uniformly to $(\hat{g},\mathbf{B}(\hat{g}))$. If Γ_u is a countable set being dense in $C_c^{1;+}([0,\infty))$ with respect to the norm $||g_u||_{\infty,1}:=||g_u||_\infty+||\partial_u g_u||_\infty$, then we define $\Gamma_{xu}:=\{g_xg_u,\ g_x\in\Gamma_x,g_u\in\Gamma_u\}$.

Let us assume that $L_{\text{-}\log(g)}\in\mathcal{D}(\mathbf{A_B})$ with $g(x,u)=\prod_{j=1}^l[1-g_j^x(x)g_j^u(u)]\in\mathfrak{g}(\mathbf{B},r)$, $(x,u)\in E\times[0,\infty)$. For each $1\le j\le k$, we can choose a sequence $(g_{j,n}^xg_{j,n}^u,n\in\mathbb{N})\in\Gamma_{xu}$ such that

$$(g_{j,n}^x,\mathbf{B}(g_{j,n}^x))\to(g_{j,n}^x,\mathbf{B}(g_{j,n}^x))\ \textit{b.p.}\quad g_{j,n}^u\to g_{j,n}^u\text{ w.r.t. }||\cdot||_{1,\infty}$$

and such that $\operatorname{supp}(g_{j,n}^u)\subset[0,r+1/n]$. If we define the function g_n by

$$g_n(x,u)=\prod_{j=1}^l[1-g_{j,n}^x(x)g_{j,n}^u(u)]\in\mathfrak{g}(\mathbf{B},r),$$

then it holds

$$\begin{gathered}g_n\to g\ ||\cdot||_\infty,\quad\mathbf{B}(g_n)\to\mathbf{B}(g)\ ||\cdot||_\infty,\quad g_{j,n}^u\to g_j^u\text{ w.r.t. }||\cdot||_{\infty,1}\\ \int_u^{r+1}g_n(\cdot,\tilde{u})-1\ d\tilde{u}\overset{n\to\infty}{\longrightarrow}\int_u^{r+1}g(\cdot,\tilde{u})-1\ d\tilde{u}\ ||\cdot||_\infty\end{gathered} \tag{B.30}$$

Since $\eta \in \overline{\mathcal{N}}(E \times [0,\infty))$, it holds $\eta(E \times [0,r]) = k \in \mathbb{N}$. We can find $(x_i, u_i) \in E \times [0,\infty)$, $1 \le i \le k$, such that $\eta = \sum_{i=1}^k \delta_{(x_i,u_i)}$. This allows us to write:

$$\mathbf{A_B}(L_{-\log(g_n)})(\eta) = \sum_{i=1}^{k} \prod_{j=1, j\neq i}^{k} g(x_j, u_j)\mathbf{B}(g_n)(x_i, u_i) \tag{B.31}$$

$$+ \sum_{i=1}^{k} \prod_{j=1}^{k} g_n(x_j, u_j) \left(2a \int_{u_i}^{r+1} g_n(x_i, \tilde{u}) - 1 \, d\tilde{u} \right) \tag{B.32}$$

$$+ \sum_{i=1}^{k} \prod_{j=1, j\neq i}^{k} g_n(x_j, u_j)[au_j^2 - bu_j]\partial_u g_n(x_i, u_i). \tag{B.33}$$

Since $\mathbf{A_B}(L_{-\log(g_n)})(\eta)$ consists only of finitely many terms, we can conclude from (B.30) that $\mathbf{A_B}(L_{-\log(g_n)})(\eta)$ is converging to $\mathbf{A_B}(L_{-\log(g)})(\eta)$ for n going to infinity. From the convergences in (B.30) and the fact that $1 \le g_n \le 1$ for all $n \in \mathbb{N}$, we can conclude that there exists a constant K such that:

$$|\mathbf{A_B}(L_{-\log(g_n)})| \le (1 + a(r+1)^2 + |b|(r+1) + 2a(r+1))Ke^{r+1} \int_{E\times[0,\infty)} e^{-u}\eta(dx, du)$$

Recall the function $\psi(\eta) = \int_{E\times[0,\infty)} e^{-u}\eta(dx, du)$ from Definition B.3.4. From the above upper bound it follows that $(||(\psi \vee 1)^{-1}\mathbf{A_B}(L_{-\log(g_n)})||_\infty, n \in \mathbb{N})$ is bounded. Therefor, if we define the countable set

$$\Gamma_{xu}^{lf} := \{L_{-\log(g)};\ g = \prod_{j=1}^{l}[1 - g_j^x g_j^u], l \in \mathbb{N}, (g_j^x, g_j^u) \in \Gamma_{xu}, 1 \le j \le l\},$$

then $\tilde{\mathbf{A}}_\mathbf{B}$ is contained in the *b.p.*-closure of

$$\{(L_{-\log(g)}, \mathbf{A_B}(L_{-\log(g)})/\psi), g \in \Gamma_{xu}^{lf}\}.$$

□

Lemma B.3.11. *The operator of $\mathbf{A_B}$ satisfies the Conditions D.1.8.*

Proof. The operator of $\mathbf{A_B}$ is conservative, because if we set $\hat{g}(x,u) = 1$, then $L_{-\log(\hat{g})} = \mathbb{1}_{\eta(E\times[0,\infty))} \in \mathcal{D}(\mathbf{A_B})$ and we can easily see that $\mathbf{A_B}(L_{-\log(\hat{g})}) = 0$.
The domain $\mathcal{D}(\mathbf{A_B})$ separates points, because if $\eta_1 \neq \eta_2$, then there exists $\hat{g} \in \mathcal{D}(\mathbf{B})$ and $r > 0$ with

$$\int_E \int_0^r \hat{g}(x)\eta_1(x,u) \neq \int_E \int_0^r \hat{g}(x)\eta_2(x,u),$$

because $\mathcal{D}(\mathbf{B})$ is separating for $\mathcal{M}_f(E)$ and $\eta_i(\cdot \times [0,r]) \in \mathcal{M}_f(E)$,$0 \le r < \infty, i \in \{1,2\}$. Choosing $g_u \in C^{1,+}([0,\infty))$ such that $g := 1 - \hat{g}g_u \in \mathfrak{g}(\mathbf{B}, r)$, then $L_{-\log(g)} \in \mathcal{D}(\mathbf{A_B})$ and it holds $L_{-\log(g)}(\eta_1) \neq L_{-\log(g)}(\eta_2)$. Further the domain is closed under multiplication, indeed assume that $L_{-\log(g_1)}, L_{-\log(g_2)} \in \mathcal{D}(\mathbf{A_B})$ with

$$g_i = \prod_{k=1}^{l_i}(1 - g_i^k), \quad l_i \in \mathbb{N}, g_i^k \in \mathfrak{g}(\mathbf{B}), 1 \le k \le l_i, i \in \{1,2\},$$

then $L_{-\log(g_1)}L_{-\log(g_2)} = L_{-\log(g_1)-\log(g_2)} = L_{-\log(g_1 g_2)}$. Since $g_1 g_2 \in \mathfrak{g}(\mathbf{B})$, it holds $L_{-\log(g_1 g_2)} \in \mathcal{D}(\mathbf{B})$.
By Lemma B.3.5 a bounding function for $\mathbf{A_B}$ is given by $\psi(\eta) := \int_{E\times[0,\infty)} e^{-u}\eta(dx, du)$.
The Point 4 follows from the fact that $\mathbf{MP}(\mathbf{A_B})$ is well-posed, and the Point 5 has been proven in Lemma B.3.10. □

We are now going to combine all the previous result with the Markov mapping theorem, see Theorem D.1.13, prove the main theorem of this section.

Proof of Theorem B.3.3. Recall that the bounding function in the context of the Kurtz-Rodrigues representation is given by $\psi(\eta) := \int_{E\times[0,\infty)} e^{-u}\eta(dx, du) + 1$, see Def. B.3.4, and hence

$$\mathbf{PPP}^*_E(\psi)(\mu) = \int_{E\times[0,\infty)} e^{-u}\mu(E)du + 1 = \mu(E) + 1, \tag{B.34}$$

see Lemma B.3.7 for the Definition of $\mathbf{PPP}^*_E$. Let us now assume that $\hat{\mathbf{\Xi}}^{\mathbb{X}} \sim \mathbf{DW}(\mathbf{B}, a, b, \hat{\Theta}_0)$ and that the initial distribution $\hat{\Theta}_0 \in \mathcal{M}_f(E)$ has been chosen in such a way that $\mathbb{E}[\hat{\mathbf{\Xi}}_0] < \infty$. From (B.1) we can conclude that

$$\mathbb{E}[\hat{\mathbf{\Xi}}^{\mathbb{X}}_t] = \mathbb{E}[\hat{\mathbf{\Xi}}_0] + \int_0^t b\mathbb{E}[\hat{\mathbf{\Xi}}^{\mathbb{X}}_s]ds = \mathbb{E}[\hat{\mathbf{\Xi}}^{\mathbb{X}}_0]e^{bt} < \infty,$$

hence $\int_0^t \mathbf{PPP}^*_E(\psi)(\tilde{\mathbf{\Xi}}^{\mathbb{X}}_s)ds < \infty$, $t \geq 0$. If $\boldsymbol{\xi}^{\mathbb{X}}$ is a càdlàg solution of $\mathbf{MP}(\mathbf{A_B}, \Theta_0)$ with $\Theta_0(d\eta) = \int \mathbf{PPP}(\mu, d\eta)\hat{\Theta}_0(d\mu)$, then we can conclude from Lemma B.3.11, Lemma B.3.9 and the Markov mapping theorem D.1.13 that the process $\tilde{\mathbf{\Xi}}^{\mathbb{X}} := \gamma^{\mathbf{\Xi}}_E(\boldsymbol{\xi}^{\mathbb{X}})$ has the same finite dimensional distributions as $\hat{\mathbf{\Xi}}^{\mathbb{X}}$. Since $\hat{\mathbf{\Xi}}^{\mathbb{X}}$ has continuous paths, we conclude by Lemma 2.24 in [21] that $\tilde{\mathbf{\Xi}}^{\mathbb{X}}$ admits a continuous modification $\mathbf{\Xi}^{\mathbb{X}}$. Further by the Markov mapping theorem we know that

$$\mathfrak{L}(\boldsymbol{\xi}^{\mathbb{X}}_t | \mathcal{F}^{\mathbf{\Xi},\mathbb{X}}_t) = \mathbf{PPP}(\mathbf{\Xi}_t \otimes \ell eb[0,\infty)), \tag{B.35}$$

where $\mathcal{F}^{\mathbf{\Xi},\mathbb{X}} := (\mathcal{F}^{\mathbf{\Xi},\mathbb{X}}_t, t \geq 0)$ is the completion of the natural filtration of $\mathbf{\Xi}^{\mathbb{X}}$. Since $\mathbf{\Xi}^{\mathbb{X}}$ is continuous and $\boldsymbol{\xi}^{\mathbb{X}}$ is càdlàg, we can extend (B.35) to arbitrary finite $\mathcal{F}^{\mathbf{\Xi},\mathbb{X}}$-stopping times by Lemma D.1.15. □

B.4 Branching Particle Systems

Recall the operator $\mathbf{B}$ from the Conditions B.2.2 that is describing a Borel strong Markov process with state space E. Let us assume that $\mathcal{I} := \cup_{k\in\mathbb{N}}\mathbb{N}^k$, then a Branching particle system consists of a population of particles indexed by $\mathcal{I}$. The number of particles is not constant, particles die and new particles are born. We denote the time of death by $\mathcal{D}_i$ and the time of birth by $\mathbf{b}_i$. Further each particles performs a spatial motion described by the processes $(\mathbb{X}_i, i \in \mathcal{I})$. The dynamics of the population is given by:

1. At $t = 0$ we start with a finite number of particles indexed by $1, 2, ..., k \in \mathbb{N}$.

2. Each living particles gives independently from the others birth to a new particle with rate $\lambda_b > 0$. If $i = (n_1, n_2, ..., n_k) \in \mathcal{I}$ is the index of the parent and if the newborn particle is the m-th child of i, then newborn particles has the index $j = (n_1, n_2, ..., n_k, m)$.

3. Each living particle dies with rate $\lambda_d > 0$.

4. If i and j are as above, then $\mathbb{X}_j$ evolves like an independent copy of the Markov process described by $\mathbf{B}$ starting at $\mathbf{b}_j$ at the position $\mathbb{X}_i(\mathbf{b}_j\text{-})$. For $t \notin [\mathbf{b}_j, \mathcal{D}_j)$ we set $\mathbb{X}_j(t) = \dagger$, where $\dagger$ is a point not contained in E.

A rigorous construction of such a particle system can be found in the Chapter 4.1 of [34]. If we write $\mathcal{I}(t) \subset \mathcal{I}$ for the collection of indicies of those particles which are alive at time t, then we can define the $\mathcal{N}_f(E)$-valued process $\boldsymbol{\Xi}^{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda} = (\lambda_d, \lambda_b)$, by setting

$$\boldsymbol{\Xi}_t^{\boldsymbol{\lambda}} := \sum_{i=1}^{\mathcal{I}(t)} \delta_{\mathbb{X}_i(t)}. \tag{B.36}$$

We like to write $\boldsymbol{\Xi}^{\mathbb{X},\boldsymbol{\lambda}} \sim \mathbf{BPS}(\mathbf{B}, \lambda_d, \lambda_b)$ or $\boldsymbol{\Xi}^{\mathbb{X},\boldsymbol{\lambda}} \sim \mathbf{BPS}(\mathbf{B}, \lambda_d, \lambda_b, \Theta_0)$, if $\boldsymbol{\Xi}_0^{\boldsymbol{\lambda}} \sim \Theta_0$. As in the case of the Dawson-Watanabe superprocess we wish to find an operator $\mathbf{D_B}$ such that $\boldsymbol{\Xi}^{\boldsymbol{\lambda}}$ is a solution of $\mathbf{MP}(\mathbf{D_B})$ and such that we can apply the Markov mapping theorem on $\mathbf{MP}(\mathbf{D_B})$. Again, as for Dawson-Watanabe superprocess, $\mathbf{D_B}$ is based on Laplace functionals.

Notation B.4.1. *As in the case of $\mathcal{M}_f(E)$ we write $\hat{L}_f$ for the Laplace functional given by $\hat{L}_f : \mathcal{N}_f(E) \to [0, \infty)$ with $\hat{L}_f(\varrho) = \exp(-\varrho(f))$, despite the fact that the domain is restricted to $\mathcal{N}_f(E)$ and is not $\mathcal{M}_f(E)$ (see Definition B.1.3).*

If $\varrho \in \mathcal{N}_f(E)$ is given by $\varrho = \sum_{i=1}^{\varrho(E)} \delta_{x_i}$, then

$$\hat{L}_{\text{-}\log(\bar{g})}(\varrho) = \prod_{i=1}^{\varrho(E)} \bar{g}(x_i)$$

The operator $\mathbf{D_B} : C_b^+(\mathcal{N}_f(E)) \supset \mathcal{D}(\mathbf{D_B}) \to C^+(\mathcal{N}_f(E))$, will be formally defined as

$$\mathbf{D_B}(\hat{L}_{\text{-}\log(\bar{g})})(\varrho) := \sum_{i=1}^{\varrho(E)} \prod_{i \neq j} \bar{g}(x_j) \left[\mathbf{B}(\bar{g})(x_i) + \lambda_d \left(1 - \bar{g}(x_i)\right) + \lambda_b(\bar{g}^2(x_i) - \bar{g}(x_i))\right]. \tag{B.37}$$

The expression of $\mathbf{D_B}(\hat{L}_{\text{-}\log(\bar{g})})$ is well-defined for all $\bar{g} \in \mathcal{D}(\mathbf{B})$, recall $\mathcal{D}(\mathbf{B}) \subset C_b^+(E)$, but for the Markov mapping theorem we need to modify $\mathcal{D}(\mathbf{B})$ in the following way:

Definition B.4.2. *We define $\overline{\mathcal{D}}(\mathbf{B})$ as the subset of $C_b^+(E)$ consisting of the elements $\bar{g}$ with the form $\bar{g} = 1 - \hat{g}$, where $\hat{g} \in \mathcal{D}(\mathbf{B})$ with $0 \le \hat{g} \le m_g$ for some constant $m_g \in (0, 1)$.*

Lemma B.4.3. *Recall the set $\mathfrak{g}(\mathbf{B}, r)$ from (B.11). If $T : \mathfrak{g}(\mathbf{B}, r) \to C_b^+(E)$ given by*

$$T(g)(x) := \frac{1}{r} \int_0^r g(x, u)\, du, \quad x \in E,$$

then $T(\mathfrak{g}(\mathbf{B}, r)) = \overline{\mathcal{D}}(\mathbf{B})$.

Proof. If $g \in \mathbf{B}, r)$, then it must have the form (B.5), which can be transformed to

$$g(x, u) = 1 + \sum_{|J| \subset [l] \setminus \emptyset} (-1)^{|J|} g_J^x(x) g_J^u(u)$$

with $[l] := \{1, 2, ..., l\}, g_J^x \in \mathcal{D}(\mathbf{B})$ and $g_J^u \in C^{1,+}([0, \infty))$ with $\operatorname{supp}(g_u^J) \subset [0, r]$. Hence

$$T(g)(x) = \sum_{|J| \subset [l] \setminus \emptyset} (-1)^{|J|} g_J^x(x) \frac{1}{r} \int_0^r g_J^u(u)\, du,$$

and so $T(g)$ is an linear combination of elements of $\mathcal{D}(\mathbf{B})$ and hence $g \in \mathcal{D}(\mathbf{B})$. Further since $m \leq g \leq 1$ for a constant $m > 0$ by the Definition of $\mathfrak{g}(\mathbf{B}, r)$, it holds $0 \leq T(g) \leq m$. In conclusion $T(g) \in \overline{\mathcal{D}}(\mathbf{B})$. For $\bar{g}^* \in \overline{\mathcal{D}}(\mathbf{B})$ with $0 \leq \bar{g}^* \leq m$ for a constant $m \in (0,1)$, we define $\hat{g} := 1 - \bar{g}^* \in \mathcal{D}(\mathbf{B})$. Let us now choose $g_u \in C^{1,+}([0,\infty))$ with support in $[0,r]$, with values in $[0,1]$ and $mr/\lambda < 1$, where $\lambda := \int_0^r g_u(u)du$. If we now set $g := 1 - (r/\lambda)\hat{g}g_u$, then $g \in \mathfrak{g}(\mathbf{B}, r)$ and it holds

$$T(g)(x) = \int_0^r 1 - \frac{r}{\lambda}\hat{g}(x)g_u(u)\frac{du}{r} = 1 - \hat{g}(x) = 1 - (1 - \bar{g}^*(x)) = \bar{g}^*(x).$$

□

Lemma B.4.4. *If $\overline{\mathbf{B}}$ is the restriction of $\mathbf{B}$ on $\overline{\mathcal{D}}(\mathbf{B})$, then the martingale problem of $\overline{\mathbf{B}}$ is well-posed. Further $\overline{\mathcal{D}}(\mathbf{B})$ is separating and closed under multiplication.*

Proof. The martingale problems $\mathbf{MP}(\overline{\mathbf{B}})$ and $\mathbf{MP}(\mathbf{B})$ are equivalent, because $\mathcal{D}(\overline{\mathbf{B}}) \subset \mathcal{D}(\mathbf{B})$ and $\mathcal{D}(\mathbf{B}) \subset \mathit{span}(\mathcal{D}(\overline{\mathbf{B}}))$. □

Definition B.4.5. *Let us fix the parameters $(\mathbf{B}, \lambda_d, \lambda_b)$, where $\mathbf{B} : \mathcal{D}(\mathbf{B}) \subset C_b(E) \to C_b(E)$ is a operator satisfying the Conditions B.2.2 and that $\lambda_d, \lambda_b \geq 0$. For the parameter $(\mathbf{B}, \lambda_d, \lambda_b)$, we set $\mathcal{D}(\mathbf{D_B})$ to be linear span of $\{\hat{L}_{-\log(\bar{g})}; \bar{g} \in \overline{\mathcal{D}}(\mathbf{B})\}$ with $\overline{\mathcal{D}}(\mathbf{B})$ as in Definition B.4.2 and we define $\mathbf{D_B}(\hat{L}_{-\log(\bar{g})})$ as in* (B.37).

Proposition B.4.6. *If $(\mathbf{B}, \lambda_d, \lambda_b)$ and $\mathbf{D_B}$ are as in Definition B.4.5, and the process $\boldsymbol{\Xi}^{\mathbb{X},\boldsymbol{\lambda}}$ is defined as in* (B.36) *with $\boldsymbol{\lambda} = (\lambda_d, \lambda_b)$ and $\mathbb{E}[\boldsymbol{\Xi}_0^{\mathbb{X},\boldsymbol{\lambda}}(E)] < \infty$, then $\boldsymbol{\Xi}^{\mathbb{X},\boldsymbol{\lambda}}$ is a solution of the martingale problem $\boldsymbol{MP}(\mathbf{D_B})$.*

Proof. Let us assume that $\mathcal{F}^{\boldsymbol{\Xi},\boldsymbol{\lambda}}$ is the natural filtration of $\boldsymbol{\Xi}^{\mathbb{X},\boldsymbol{\lambda}}$. If $\tau_0 = 0$ and $\tau_{k+1} := \inf\{t > \tau_k; \boldsymbol{\Xi}_t^{\mathbb{X},\boldsymbol{\lambda}}(E) \neq \boldsymbol{\Xi}_{\tau_k}^{\mathbb{X},\boldsymbol{\lambda}}(E)\}$ for $k \in \mathbb{N}$, then $\boldsymbol{\Xi}_t^{\mathbb{X},\boldsymbol{\lambda}} = \sum_{i\in\mathcal{I}(\tau_k)} \delta_{\mathbb{X}_i(s)}$ for $t \in [\tau_k, \tau_{k+1}]$ and, since $\mathbb{X}_i$ is an independent copy of the Markov process characterized by $\mathbf{B}$ on $[\tau_k, \tau_{k+1}]$ for $k \in \mathbb{N}_0$ and $i \in \mathcal{I}(\tau_k)$, we can conclude that the process $\overline{M}_k$ given by

$$\begin{aligned}\overline{M}_k(t) := &\prod_{i\in\mathcal{I}(\tau_k)} \bar{g}(\mathbb{X}_i(t \wedge \tau_k)) - \prod_{i\in\mathcal{I}(\tau_k)} \bar{g}(\mathbb{X}_i(\tau_k)) \\ &- \int_{\tau_k}^{\tau_k\wedge t} \sum_{i\in\mathcal{I}(\tau_k)} \prod_{j\in\mathcal{I}(\tau_k)\setminus\{i\}} \bar{g}(\mathbb{X}_j(s\text{-}))\mathbf{B}(\bar{g})(\mathbb{X}_i(s\text{-}))ds\end{aligned}$$

is a $\mathcal{F}^{\boldsymbol{\Xi},\boldsymbol{\lambda}}$-martingale. Further since every particle dies with rate $\lambda_d > 0$ and gives birth to a new particle with the same initial location with rate λ_b we can conclude that

$$\begin{aligned}\overline{M}_t^J := &\sum_{k\in\mathbb{N}} \mathbb{1}_{[0,t]}(\tau_k)\Delta\hat{L}_{-\log(\bar{g})}\left(\boldsymbol{\Xi}_{\tau_k}^{\mathbb{X},\boldsymbol{\lambda}}\right) \\ &- \int_0^t \sum_{i\in\mathcal{I}(s)} \prod_{j\in\mathcal{I}(s)\setminus\{i\}} \bar{g}(\mathbb{X}_i(s\text{-}))\left[\lambda_d\left(1 - \bar{g}(x_i)\right) + \lambda_b(\bar{g}^2(\mathbb{X}_i(s\text{-})) - \bar{g}(\mathbb{X}_i(s\text{-})))\right]ds\end{aligned}$$

is also a (local) martingale. Consequently we obtain a local $\mathcal{F}^{\boldsymbol{\Xi},\boldsymbol{\lambda}}$-martingale with localizing sequence $(\tau_k, k \in \mathbb{N})$ by

$$\overline{M}_t := \overline{M}_t^J + \sum_{k\in\mathbb{N}} \overline{M}_k(t).$$

We need to argue that $\overline{M}$ is a proper martingale. By the definition of the branching particle system we know that $Y^{\boldsymbol{\lambda}} := \boldsymbol{\Xi}^{\mathbb{X},\boldsymbol{\lambda}}(E)$ is time-continuous Galton-Watson process with death rate λ_d and birth rate λ_b, hence the sequence $(\overline{M}_{\cdot \wedge \tau_k})_{k=1}^{\infty}$ of martingales is bounded by the process

$$P_t := 2 \sum_{k \in \mathbb{N}} \mathbb{1}_{[0,t]}(\tau_k) + \int_0^t K Y_s^{\boldsymbol{\lambda}} \, ds,$$

where $K > 0$ is a suitable constant. Since Y is a Galton-Watson process, we have $\mathbb{E}[Y_t^{\boldsymbol{\lambda}}] = \mathbb{E}[Y_0^{\boldsymbol{\lambda}}] \exp([\lambda_b - \lambda_d] t)$ and so

$$\mathbb{E}[P_t] = \int_0^t (2 + K) \mathbb{E}[Y_0^{\boldsymbol{\lambda}}] e^{[\lambda_b - \lambda_d] s} ds < \infty.$$

The process $P = (P_t, t \geq 0)$ forms an integrable majorant for $(\overline{M}_{\cdot \wedge \tau_k})_{k=1}^{\infty}$, we can conclude that $\overline{M}$ is a martingale. □

Remark B.4.7. *Note that we do not prove that* $\boldsymbol{MP}(\mathbf{C_B})$ *is well-posed, this will follow as a side effect from the Markov Mapping theorem.*

B.5 Kurtz-Rodrigues Representation with finite Level-Cap

Here characterize the Kurtz-Rodrigues representation with finite level cap from Section 1.1 as a solution of a martingale problem, which will allow us to apply the Markov mapping theorem in order to prove (1.12). We recall from Section 1.1 that the state space of KR-representation with finite level cap is $\mathcal{N}_f(E \times [0,r))$, the space of finite integer-valued measures over $E \times [0,r)$. If $\boldsymbol{\xi}^{\mathbb{X}}$ is a KR-representation with infinite level cap and parameters $(\mathbf{B}, a, b)$, then the corresponding KR-representation $\boldsymbol{\xi}^{\mathbb{X},r}$ with **finite level cap** $r \geq \max\{b/a, 0\}$ is obtained by ignoring all particles with a level higher or equals to r, i.e.

$$\boldsymbol{\xi}_t^{\mathbb{X},r} := \boldsymbol{\xi}_t^{\mathbb{X}}(\,\cdot \cap (E \times [0,r))).$$

The martingale problem of $\boldsymbol{\xi}^{\mathbb{X},r}$ is identical with the one of $\boldsymbol{\xi}^{\mathbb{X}}$ except for the fact that $\boldsymbol{\xi}^{\mathbb{X},r}$ is a process with state space $\mathcal{N}_f(E \times [0,r))$ and not $\overline{\mathcal{N}}(E \times [0,\infty))$. We can interpret $\mathcal{N}_f(E \times [0,r))$ as subset of $\overline{\mathcal{N}}(E \times [0,\infty))$ consisting of those elements η which satisfy $\eta(E \times [r,\infty)) = 0$. The trace topology of the mixed topology of $\overline{\mathcal{N}}(E \times [0,\infty))$, see Definition 1.1.2, on $\mathcal{N}_f(E \times [0,r))$ coincides with the usual weak topology.

Notation B.5.1. *If* $L_f : \overline{\mathcal{N}}(E \times [0,\infty)) \to [0,\infty)$ *is the Laplace functional* $L_\eta = \exp -\eta(f)$ *for* $f \in C_b^+(E \times [0,\infty))$*, then we do not distinguish notationally between* L_f *itself and its restriction*

$$L_{f|\mathcal{N}_f(E \times [0,r))} : \mathcal{N}_f(E \times [0,r)) \to [0,\infty).$$

Definition B.5.2. *Let* $\mathbf{B} : \mathcal{D}(\mathbf{B}) \subset C_b(E) \to C_b(E)$ *satisfy the conditions B.2.2,* $a > 0, b \in \mathbb{R}$ *and* $\max\{b/a, 0\} \leq r < \infty$*. For the parameters* $(\mathbf{B}, a, b, r)$ *we define the operator*

$$\mathbf{A}_{\mathbf{B}}^r : C(\mathcal{N}_f(E \times [0,r))) \supset \mathcal{D}\left(\mathbf{A}_{\mathbf{B}}^r\right) \to C(\mathcal{N}_f(E \times [0,r))),$$

where the domain $\mathcal{D}\left(\mathbf{A}_{\mathbf{B}}^r\right)$ *is given by the linear span of the Laplace functionals associated with the collection* $\mathfrak{g}(\mathbf{B}, r)$*, see Definition B.11, indeed* $\mathcal{D}\left(\mathbf{A}_{\mathbf{B}}^r\right) := \mathit{span}\left\{L_{-\log(g)};\ g \in \mathfrak{g}(\mathbf{B}, r)\right\}$ *, and where the function* $\mathbf{A}_{\mathbf{B}}^r(L_{-\log(g)})$ *is given for* $\eta \in \overline{\mathcal{N}}_r(E \times [0,\infty))$ *by*

$$\mathbf{A}_{\mathbf{B}}^r(L_{-\log(g)})(\eta) = \mathbf{A}_{\mathbf{B}}(L_{-\log(g)})(\eta),$$

where $\mathbf{A}_{\mathbf{B}}(L_{-\log(g)})(\eta)$ *is defined as in Definition B.2.9.*

Proposition B.5.3. *If* $\mathbf{B}$ *is as in Definition B.5.2 and* $a > 0, b \in \mathbb{R}$ *and* $r \in [\max(b/a, 0), \infty)$, *then the martingale problem* $\boldsymbol{MP}(\mathbf{A}^r_{\mathbf{B}}, \Theta_0)$ *is well-posed for initial conditions satisfying:*

$$\int \eta(E \times [0, r))\, \Theta_0(d\eta) < \infty.$$

Proof. The same as in Proposition B.2.11. □

Definition B.5.4. *Let us assume that the parameters* $(\mathbf{B}, a, b, r)$ *and the operator* $\mathbf{A}^r_{\mathbf{B}}$ *are given as in Definition B.2.12.If* $(\Omega, \mathcal{A}, \mathbb{P})$ *is a probability space and the stochastic process*

$$\boldsymbol{\xi}^{\mathbb{X},r} : \Omega \times [0, \infty) \to \overline{\mathcal{N}}(E \times [0, \infty))$$

is a solution of the martingale problem $\boldsymbol{MP}(\mathbf{A}^r_{\mathbf{B}}, \Theta_0)$ *with* $\Theta_0 \in \mathcal{M}_1(\mathcal{N}_f(E \times [0, r)))$, *then we call* $\boldsymbol{\xi}^{\mathbb{X},r}$ *a Kurtz-Rodrigues representation with spatial motion* $\mathbf{B}$, *branching rate* $a > 0$, *drift* $b \in \mathbb{R}$, ***level cap*** r *and initial distribution* Θ_0.

B.6 Connection between BPS and KR-Rep. with finite Level-Cap

In Theorem B.3.3 we have seen that the KR-representation with infinite level cap is a Poisson representation for the Dawson-Watanabe superprocess, a similar relationship can be found between the Branching particle systems and the KR-representation with finite level cap. For Theorem B.3.3 we introduced the notion of a Poisson mixture, see Definition B.3.1, in the context of a finite level cap, this notion is replaced by a new kind of mixture.

Definition B.6.1. *Recall* $\mathbf{Uni}^r_E$ *from Definition 1.1.1. We say the distribution* Θ_0 *is a* r*-uniform mixture with* $r > 0$ *based on* $\overline{\Theta}_0 \in \mathcal{M}_1(\mathcal{M}_f(E))$, *if*

$$\Theta_0(F) = \int_{\mathcal{M}_f(E)} \mathbf{Uni}^r_E(\varrho)(F)\, \overline{\Theta}_0(d\varrho), \quad F \in C_b(\mathcal{N}_f(E \times [0, r))). \tag{B.38}$$

Theorem B.6.2. *Let us assume that* $\mathbf{B}, a, b$ *and* $\mathbf{A}_{\mathbf{B}}$ *are given as in Definition B.5.2. Further let us assume* $\boldsymbol{\xi}^{\mathbb{X}}$ *is a Kurtz-Rodrigues representation with finite level cap with*

$$\boldsymbol{\xi}^{\mathbb{X},r} \sim \boldsymbol{KR}(\mathbf{B}, a, b, r, \Theta_0)$$

and that the initial distribution $\Theta_0 \in \mathcal{M}_1(\mathcal{N}_f(E \times [0, r)))$ *is a* r*-uniform mixture, see* (B.38), *for a* $\overline{\Theta}_0 \in \mathcal{M}_1(\mathcal{N}_f(E))$. *If* $\boldsymbol{\Xi}^{\mathbb{X},r} : \Omega \times [0, \infty) \to \mathcal{M}_f(E)$ *is given by* $\boldsymbol{\Xi}^{\mathbb{X},r}_t := \gamma^{\boldsymbol{\Xi},r}_E(\boldsymbol{\xi}^{\mathbb{X}}_t)$ *and assume that* $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{X},r}$ *is the augmented filtration of the natural filtration of* $\tilde{\boldsymbol{\Xi}}^{\mathbb{X},r}$, *then:*

1. *The process* $\boldsymbol{\Xi}^{\mathbb{X},r}$ *is a Branching particle system, with respect to the filtration* $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{X},r}$ *and initial distribution* $\overline{\Theta}_0$, *i.e.* $\boldsymbol{\Xi}^{\mathbb{X},r} \sim \boldsymbol{BPS}(\mathbf{B}, ra, ra\text{-}b, \overline{\Theta}_0)$.
2. *For all finite* $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{X},r}$*-stopping times* τ, *it holds*

$$\mathfrak{L}(\boldsymbol{\xi}^{\mathbb{X},r}_\tau | \mathcal{F}^{\boldsymbol{\Xi},\mathbb{X},r}_\tau) = \mathbf{Uni}^r_E(\boldsymbol{\Xi}^{\mathbb{X},r}_\tau).$$

As in the case of Theorem B.3.3, the proof of Theorem B.6.2 is an application of the Markov mapping theorem and the most important ingredient is again an intertwiner relationship between the operators $\mathbf{D}_{\mathbf{B}}$ and $\mathbf{A}^r_{\mathbf{B}}$. We will proceed in the same way as in Section B.3.

Recall the bounding function $\psi : \overline{\mathcal{N}}(E \times [0, \infty)) \to [0, \infty)$ from Definition B.3.4 with $\psi(\eta) = 1 + \eta(\tilde{\psi})$ with $\tilde{\psi}(x, u) = 1 + e^{-u}$. The following definition is analogous to B.3.4.

Definition B.6.3. *We define $C_\psi(\mathcal{N}_f(E \times [0,r)))$ as the subset of $C(\mathcal{N}_f(E \times [0,r)))$ consisting of functions F for which we can find a constant K_F such that $|F| \leq K_F\psi$.*

Lemma B.6.4. *The image $\mathbf{Im}(\mathbf{A}^r_{\mathbf{B}})$ of the operator $\mathbf{A}^r_{\mathbf{B}}$ from Definition B.2.9 satisfies $\mathbf{Im}(\mathbf{A}^r_{\mathbf{B}}) \subset C_\psi(\mathcal{N}_f(E \times [0,r)))$.*

Proof. Since $\mathbf{A}^r_{\mathbf{B}} \subset \mathbf{A}_{\mathbf{B}}$, if we interpret both as subsets of $C_b(\overline{\mathcal{N}}(E\times[0,\infty)))\times C(\overline{\mathcal{N}}(E\times[0,\infty)))$, this follows from Lemma B.3.5. □

Definition B.6.5. *We define the pull-back $\mathbf{Uni}^{r\,*}_E : C_\psi(\mathcal{N}_f(E\times[0,r))) \to \mathbf{M}(\mathcal{N}_f(E))$ by setting*

$$\mathbf{Uni}^{r\,*}_E(F)(\varrho) = \mathbb{E}\left[F(\boldsymbol{\xi}^r)\right], \quad \boldsymbol{\xi}^r \sim \mathbf{Uni}^r_E(\varrho),$$

for all $F \in C_\psi(\mathcal{N}_f(E \times [0,r)))$ and $\varrho \in \mathcal{N}_f(E)$.

Lemma B.6.6. *The function $\mathbf{Uni}^{r\,*}_E(F)$ is continuous for $F \in C_b(\mathcal{N}_f(E\times[0,r)))$ and measurable for $F \in C_\psi(\mathcal{N}_f(E \times [0,r)))$. More specific, if $g \in \mathfrak{g}(\mathbf{B},r)$, then*

$$\mathbf{Uni}^{r\,*}_E(L_{\text{-}\log(g)})(\varrho) = \prod_{x\in\varrho}\int_0^r g(x,u)\,\frac{du}{r} = \exp\left(\int_E \log\left(\int_0^r g(x,u)\,\frac{du}{r}\right)\varrho(dx)\right) \quad \text{(B.39)}$$

for all $\varrho \in \mathcal{N}_f(E)$ and it holds

$$\mathbf{Uni}^{r\,*}_E(\mathcal{D}(\mathbf{A}^r_{\mathbf{B}})) = \mathcal{D}(\mathbf{D}_{\mathbf{B}}). \quad \text{(B.40)}$$

Remark B.6.7. *The statement $\mathbf{Uni}^{r\,*}_E(F)$ is continuous for all $F \in C_b(\mathcal{N}_f(E \times [0,r)))$ is equivalent to say that the kernel $\mathbf{Uni}^r_E$ is a continuous map with respect to the weak topologies on $\mathcal{N}_f(E)$ and $\mathcal{M}_1(\mathcal{N}_f(E \times [0,r)))$.*

Proof. If $\varrho \in \mathcal{N}_f(E)$, we can write η as $\sum_{i=1}^n \delta_{x_i} \in \mathcal{N}_f(E)$ for some $x_1, x_2, ..., x_n \in E$, where $n := \eta(E)$, then

$$\mathbf{Uni}^{r\,*}_E(L_{\text{-}log(g)})(\varrho) = \mathbb{E}\left[\prod_{i=1}^n g(x_i, U_i)\right],$$

where $U_1, ..., U_n$ are independent, uniformly over $[0,r]$ distributed random variables. This gives us (B.39). Now let us assume that $(\varrho_n, n \in \mathbb{N}) \subset \mathcal{M}_f(E)$ is converging against $\varrho \in \mathcal{M}_f(E)$, this implies that

$$\int_E \log\left(\int_0^r g(x,u)\,\frac{du}{r}\right)\varrho_n(dx) \overset{n\to\infty}{\longrightarrow} \int_E \log\left(\int_0^r g(x,u)\,\frac{du}{r}\right)\varrho(dx),$$

and so we have that $\mathbf{Uni}^{r\,*}_E(L_f)$ is a continuous function. Now we recall the Remark B.6.7 and note further that the continuity of the Markov kernel $\mathbf{Uni}^r_E$ is equivalent to the continuity of the transformed Laplace functionals $\mathbf{Uni}^{r\,*}_E(L_f)$ by Theorem 1.18 from [34]. This gives us the continuity of $\mathbf{Uni}^{r\,*}_E(F)$ for general $F \in C_b(\mathcal{N}_f(E \times [0,r)))$. The measurability of $\mathbf{Uni}^{r\,*}_E(F)$ for a general $F \in C_\psi(\mathcal{N}_f(E\times[0,r)))$ follows by the pointwise approximation with $F_n = \min\{F, n\} \in C_b(\mathcal{N}_f(E \times [0,r)))$.
From (B.39) and Lemma B.4.3, we can conclude that $\mathbf{Uni}^{r\,*}_E(L_{\text{-}\log(g)}) \in \mathcal{D}(\mathbf{D}_{\mathbf{B}})$. Finally, recall that $\mathcal{D}(\mathbf{D}_{\mathbf{B}}) = \mathit{span}\{\hat{L}_{\text{-}\log(\bar{g})}; \bar{g} \in \overline{\mathcal{D}}(\mathbf{B})\}$. If $\bar{g} \in \overline{\mathcal{D}}(\mathbf{B})$, then according to Lemma B.4.3, there there exists an element in $g \in \mathfrak{g}(\mathbf{B},r)$ with $\bar{g}(x) = \int_0^r g(x,u)du/r$, and so $\mathbf{Uni}^{r\,*}_E(L_{\text{-}\log(g)}) = \hat{L}_{\bar{g}}$. □

Definition B.6.8. *For a fixed $r \geq \max\{b/a, 0\}$ we define the map*

$$\gamma_E^{\mathbf{\Xi},r} : \mathcal{N}_f(E \times [0,r)) \to \mathcal{N}_f(E)$$

by setting $(\gamma_E^{\mathbf{\Xi},r})(\eta)$ to be the measure $\varrho \in \mathcal{M}_f(E)$ given by $\varrho(\Gamma) = \eta(\Gamma \times [0,r))$ for all Borel sets $\Gamma \in \mathbb{B}(E)$, i.e. ϱ is the projection of η on the space $\mathcal{M}_f(E)$.

Every $\eta \in \mathcal{N}_f(E \times [0,r))$ can be written as $\eta = \sum_{i=1}^n \delta_{(x_i,u_i)}$ with $n = \eta(E \times [0,r))$ and $x_1, x_2, ..., x_n \in E, u_1, u_2...u_n \in [0,r)$ and so $\gamma_E^{\mathbf{\Xi},r}(\eta)$ is given by

$$\gamma_E^{\mathbf{\Xi},r}(\eta) = \sum_{i=1}^n \delta_{x_i},$$

i.e. $\gamma_E^{\mathbf{\Xi},r}(\eta)$ just forgets the levels $u_1, u_2...u_n \in [0,r)$. From this point it is not hard to see, why the next lemma is true.

Lemma B.6.9. *For all $\varrho \in \mathcal{M}_f(E)$ holds true that*

$$\mathbf{Uni}_E^r(\varrho, (\gamma_E^{\mathbf{\Xi},r})^{-1}(\varrho)) = 1.$$

Proof. This follows from the previous lines. □

Proposition B.6.10. *Assume that $\mathbf{B} : \mathcal{D}(\mathbf{B}) \subset C_b(E) \to C_b(E)$ is a linear operator satisfying the Conditions B.2.2, that $a > 0, b \in \mathbb{R}$ and $r \in (\max\{b/a, 0\}, \infty)$ are fixed. Further let us assume that the operator*

$$\mathbf{D_B} : C_b^+(\mathcal{N}_f(E)) \supset \mathcal{D}(\mathbf{D_B}) \to C_b^+(\mathcal{N}(E)),$$

is defined as in Definition B.4.5 with the birth rate given by $\lambda_b := ra$ and the death rate given by $\lambda_d := ra - b$, and that the linear operator

$$\mathbf{A}_\mathbf{B}^r : C(\overline{\mathcal{N}}^r(E \times [0,\infty))) \supset \mathcal{D}(\mathbf{A}_\mathbf{B}^r) \to C(\overline{\mathcal{N}}^r(E \times [0,\infty)))$$

is defined as in the Definition B.2.9 with branching rate a, drift b and finite level cap r. Then it holds for all $L_{-\log(g)} \in \mathcal{D}(\mathbf{A}_\mathbf{B}^r)$ with $g \in \mathfrak{g}(\mathbf{B}, r)$, see (B.11)*:*

$$\mathbf{Uni}_E^{r\,*} \circ \mathbf{A}_\mathbf{B}^r(L_{-\log(g)}) = \mathbf{D_B} \circ \mathbf{Uni}_E^{r\,*}(L_{-\log(g)})$$

Proof. We know from (B.39) that

$$\mathbf{Uni}_E^{r\,*}(L_{-\log(g)})(\varrho) = \mathbb{E}[L_{\boldsymbol{\xi}}] = \mathbb{E}\left[\prod_{i=1}^n g(x_i, U_i)\right] = \prod_{i=1}^n \bar{g}^r(x_i)$$

with $\bar{g}^r(x) = \int_0^r g(x,u)du/r$. By Definition B.4.5, we have

$$\begin{aligned}&\mathbf{D_B} \circ \mathbf{Uni}_E^{r\,*}(L_{-\log(g)})(\varrho)\\ &= \sum_{i=1}^n \prod_{j=1, j\neq i}^n \bar{g}^r(x_j)\left[\sum_{x\in\varrho} \mathbf{B}(\bar{g}^r)(x_i) + (ra-b)\,(1-\bar{g}^r(x_i)) + ra(\bar{g}^r(x_i)-1)\bar{g}^r(x_i)\right].\end{aligned}$$

In order to see that $\mathbf{Uni}_E^{r\,*} \circ \mathbf{A}_\mathbf{B}^r(L_{-\log(g)})(\eta)$ is equal to the above expression, we assume that $\varrho = \sum_{i=1}^n \delta_{x_i} \in \mathcal{N}_f(E)$ for some $x_1, x_2, ..., x_n \in E$ and $n := \varrho(E)$, which is true for all $\varrho \in \mathcal{N}_f(E)$, then we can write

$$\mathbf{Uni}_E^{r\,*} \circ \mathbf{A}_\mathbf{B}^r(L_{-\log(g)})(\eta) = \mathbb{E}\left[\exp(-\boldsymbol{\xi}(\log(g)))\right], \tag{B.41}$$

with $\boldsymbol{\xi} := \sum_{i=1}^n \delta_{(x_i,U_i)}$, where $U_1, ..., U_n$ are independent random variables uniform distributed over $[0, r]$. From this and the Definition B.2.9, it follows:

$$
\begin{aligned}
\mathbb{E}\left[\mathbf{A_B}(L_{\text{-}\log(g)})(\boldsymbol{\xi})\right] &= \mathbb{E}\left[\sum_{i=1}^n \prod_{j=1}^n g(x_j, U_j) \int_0^\infty \frac{\mathbf{B}(g)(x_i, U_i)}{g(x_i, U_i)}\right] \\
&+ \mathbb{E}\left[\sum_{i=1}^n \prod_{j=1}^n g(x_j, U_j) \int_0^\infty \left(2a \int_{U_i}^\infty g(x_i, \tilde{u}) - 1 \, d\tilde{u}\right)\right] \\
&+ \mathbb{E}\left[\sum_{i=1}^n \prod_{j=1}^n g(x_j, U_j) \left[aU_i^2 - bU_i\right] \frac{\partial_u g(x_i, U_i)}{g(x_i, U_i)})\right] \\
&= \sum_{i=1}^n \prod_{j=1, j\neq i}^n \bar{g}^r(x_j) \left(E_1^i + E_2^i + E_3^i + E_4^i\right), \qquad \text{(B.42)}
\end{aligned}
$$

where E_1, E_2, E_3 and E_4 are given by

$$
\begin{aligned}
E_1^i &:= \int_0^r \mathbf{B}(g)(x_i, u) \frac{du}{r}, \qquad \text{(B.43)} \\
E_2^i &:= 2a \int_0^r g(x_i, u) \left(\int_u^\infty g(x_i, \tilde{u}) - 1 \, d\tilde{u}\right) \frac{du}{r}, \\
E_3^i &:= \int_0^r au^2 \partial_u g(x_i, u) du, \\
E_4^i &:= -\int_0^r bu \partial_u g(x_i, u) \frac{du}{r}.
\end{aligned}
$$

Applying the Lines (C.13) and (C.15) of Lemma C.2.2 wiht $R = r$ to $E_2 + E_3$ and E_4 gives us:

$$
\begin{aligned}
E_2^i + E_3^i &= 2r^{-1} a \left(\int_0^r 1 - g(x_i, u) \, du\right)^2 = r^{-1} a \left(r\bar{g}^r(x_i) - r\right)^2 \\
&= ra(1 - \bar{g}^r(x_i)) + ra(\bar{g}^r(x_i) - 1)\bar{g}^r(x_i), \\
E_4^i &= -\int_0^r b\tilde{g}(x_i, u) \frac{du}{r} = -b\bar{g}(x_i).
\end{aligned}
$$

If we combine (B.42) with the above and with (B.43) to can see that $\mathbf{Uni}_E^{r\,*} \circ \mathbf{A}_\mathbf{B}^r(L_{\text{-}\log(g)})(\eta)$ and $\mathbf{D_B} \circ \mathbf{Uni}_E^{r\,*}(L_{\text{-}\log(g)})(\varrho)$. □

Proof of Theorem B.6.2. We have proved in Lemma B.3.10 and in Lemma B.3.11 that $\mathbf{A_B}$ satisfies the Conditions D.1.8. Since $\mathbf{A}_\mathbf{B}^r$ is a restriction of $\mathbf{A_B}$ we can repeat the arguments to show that the same is true for $\mathbf{A}_\mathbf{B}^r$. Main difference to the situation of $\mathbf{A_B}$ comes into the form of the different Markov kernel used in the intertwiner relation. Recall the bounding function $\psi(\eta) = 1 + \int_E \int_0^\infty e^{-u} \eta(dxdu)$, then $\mathbf{Uni}_E^{r\,*} : \mathcal{M}_f(E) \to [0, \infty)$ is given by

$$
\mathbf{Uni}_E^{r\,*}(\psi)(\varrho) = 1 + \varrho(E).
$$

Let us now assume that $\boldsymbol{\Xi}^{\mathbb{X},r} \sim \mathbf{BPS}(\mathbf{B}, ra - b, ra)$, then $Y^r := \boldsymbol{\Xi}^{\mathbb{X},r}(E)$ is a time-continuous Galton-Watson process with death rate $ra - b$ and birth ra, hence $\mathbb{E}[Y_t^r] = \mathbb{E}[Y_0^r] \exp(bt)$ and so

$$
\int_0^t \mathbb{E}\left[\mathbf{Uni}_E^{r\,*}(\psi)(\boldsymbol{\Xi}_s^{\mathbb{X},r})\right] ds = \int_0^t \mathbb{E}[Y_s^r] ds = \frac{\exp(bt) - 1}{b} < \infty.
$$

The rest of Theorem B.6.2 follows now from intertwiner relation from Lemma B.6.10 and the Markov mapping theorem D.1.13. □

B.7 Dawson-Girsanov Transformation

We are going back to the Dawson-Watanabe superprocss from Section B.1. For the following let us assume that E is a locally compact space and $\mathbb{X}$ is a Feller process, indeed we assume that the semi-group $P_t : C_b(E) \to C_b(E)$ associated with the Borel strong Markov Family $(P_x, x \in E)$ from Conditions B.2.2 and given by $P_t(f)(x) = \mathbb{E}(f(\mathbb{X}(t)))$, where $\mathbb{X}$ has the law P_x, is restricted to the space $C_0(E)$ a Feller semi-group. We write $\mathbf{B}_F$ for the full generator $P_t : C_0(E) \to C_0(E)$. The following definition is based on Page 116 in [9].

Definition B.7.1. *Let us assume that $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is a filtered probability space with $\tilde{\mathcal{F}} := (\tilde{\mathcal{F}}_t \subset \tilde{\mathcal{A}}, t \geq 0)$ being a right-continuous filtration. We call $\tilde{\mathbf{M}} : \Omega \times \mathbb{B}(E) \times [0, \infty) \to \mathbb{R}$ a $\tilde{\mathcal{F}}$-martingale measure (or a L^2-$\tilde{\mathcal{F}}$-martingale measure), if*

1. *The process $\tilde{\mathbf{M}}(\Gamma) : \Omega \times [0, \infty) \to \mathbb{R}$ is a $L^2(\mathbb{P})$-integrable $\tilde{\mathcal{F}}$-martingale for every $\Gamma \in \mathbb{B}(E)$.*
2. *If $(\Gamma_n)_{n=1}^{\infty} \subset \mathbb{B}(E)$ is a disjoint collection of Borel sets and $\Gamma := \cup_{n=1}^{\infty} \Gamma_n$, then $\tilde{\mathbf{M}}_t(\Gamma)$ is the $L^2(\mathbb{P})$ limit of $(M_n)_{n=1}^{\infty}$, where $M_n(t) := \sum_{i=1}^{n} \tilde{\mathbf{M}}(\Gamma_i)_t$.*

We define the random set function $\boldsymbol{\eta} : \Omega \times \mathbb{B}(E) \times \mathbb{B}(E) \times [0, \infty) \to \mathbb{R}$ by setting $\tilde{\boldsymbol{\eta}}_t(\Gamma_1, \Gamma_2) := \langle \tilde{\mathbf{M}}(\Gamma_1), \tilde{\mathbf{M}}(\Gamma_2) \rangle_t$, indeed $\tilde{\boldsymbol{\eta}}_t(\Gamma_1, \Gamma_2)$ is the predictable covariation of $\tilde{\mathbf{M}}(\Gamma_1)$ and $\tilde{\mathbf{M}}(\Gamma_2)$ and so we call $\tilde{\boldsymbol{\eta}}$ the covariation functional of $\tilde{\mathbf{M}}$.

Definition B.7.2. *We call $\tilde{\mathbf{M}}$ a worthy $\tilde{\mathcal{F}}$-martingale measure, if there exists a random finite measure $\tilde{\boldsymbol{K}} : \Omega \to \mathcal{M}_f(E \times E \times [0, \infty))$ which is symmetric, positive definite, i.e.*

$$\int_0^{\infty} f(x, s) f(y, s) \tilde{\boldsymbol{K}}(dx, dy, ds) \geq 0$$

for all $f \in \mathbb{B}(E \times [0, \infty))$, and it holds

$$|\tilde{\boldsymbol{\eta}}_t(\Gamma_1, \Gamma_2)| \leq \tilde{\boldsymbol{K}}(\Gamma_1 \times \Gamma_2 \times [0, t]).$$

Further $P_t := \tilde{\boldsymbol{K}}(\Gamma_1 \times \Gamma_2 \times (0, t])$ is $\tilde{\mathcal{F}}$-predictable for all $\Gamma_1, \Gamma_2 \in \mathbb{B}(E)$, in the sense that P is measurable with respect to the σ-algebra given by $\sigma(\{\{0\} \times \Gamma; \Gamma \in \tilde{\mathcal{F}}_0\} \cup \{(s, t] \times \Gamma; 0 < s < t < \infty, \Gamma \in \tilde{\mathcal{F}}_s\})$.

Lemma B.7.3. *If $\tilde{\mathbf{M}}$ is a worthy $\tilde{\mathcal{F}}$-martingale measure with a dominating measure $\tilde{\boldsymbol{K}}$, then the covariation functional $\tilde{\boldsymbol{\eta}}$ can be extended to a proper random signed measure over $E \times E \times [0, \infty)$, which we will denote by the same symbol as the covariation functional.*

Proof. The proof of this statement can be found before Proposition 2.1 in [45]. □

For the following we assume that $\boldsymbol{\Xi}^{\mathbb{X}} \sim \mathbf{DW}(\mathbf{B}, a, b)$ with $\mathbb{E}[\boldsymbol{\Xi}_0^{\mathbb{X}}(E)] < \infty$ and that $\mathcal{F}^{\boldsymbol{\Xi}, \mathbb{X}}$ is usual augmented version of the natural filtration of $\boldsymbol{\Xi}^{\mathbb{X}}$.

Lemma B.7.4. *There exists a worthy $\mathcal{F}^{\boldsymbol{\Xi}, \mathbb{X}}$-martingale measure $\mathbf{M}$ such that for all $\hat{g} \in \mathcal{D}(\mathbf{B}_F)$ holds*

$$M_{\hat{g}}(t) = \int_0^t \int_E \hat{g}(x)\, \mathbf{M}(ds, dx), \quad \forall t \in [0, \infty), \tag{B.44}$$

where $M_{\hat{g}}$ is the martingale from (B.1) *and right side of* (B.44) *is a stochastic integral in the sense of Walsh, see the Chapter “Stochastic Integration” in [45]. Further the covariation measure $\boldsymbol{\eta}$ of $\mathbf{M}$ is given by*

$$\int_0^t \int_E \int_E f(x, s) g(y, s) \boldsymbol{\eta}(ds, dx, dy) = \int_0^t \int_E a f(x, s) g(x, s) \boldsymbol{\Xi}_s^{\mathbb{X}}(dx) \tag{B.45}$$

for all $f, g \in C_b(E \times [0, \infty))$ and $t \geq 0$.

Proof. See Theorem 7.25 in [34]. □

Lemma B.7.5. *Assume that $f : \Omega \times E \times [0, \infty) \to \mathbb{R}$ is a predictable map $\mathcal{F}^{\Xi,\mathbb{X}}$ in the sense of Definition 7.1.1 and we have*

$$\mathbb{E}\left[\int_0^t \Xi_s(f^2)\, ds\right] < \infty, \quad t \geq 0, \tag{B.46}$$

then the process $M_f : \Omega \times [0, \infty) \to \mathbb{R}$ given by

$$M_f(t) := \int_0^t \int_E f(x)\, \mathbf{M}(ds, dx)$$

is a continuous $\mathcal{F}^{\Xi,\mathbb{X}}$-martingale with $\mathbb{E}[M_f^2(t)] < \infty$ and $\langle M_f \rangle_t = \int_0^t a\Xi_s^{\mathbb{X}}(f^2)\, ds$, $t \geq 0$. If $f_1, f_2 : \Omega \times E \times [0, \infty) \to \mathbb{R}$ are $\mathcal{F}^{\Xi,\mathbb{X}}$-predictable and satisfy (B.46)*, then $\langle M_{f_1}, M_{f_2} \rangle_t = \int_0^t a\Xi_s^{\mathbb{X}}(f_1 f_2)\, ds$, $t \geq 0$.*

Proof. This is Perkins' Proposition II.5.4 in [40], where the fact that M_f is a continuous martingale is hidden in the statement that $M_f \in \mathcal{M}_{loc}$, which is the space of continuous local $\mathcal{F}^{\Xi,\mathbb{X}}$-martingales. □

Lemma B.7.6. *Let us assume that $\beta \in C_b(\mathcal{M}_f(E) \times E)$ is a measurable function. The process $\mathbf{Z} : \Omega \times [0, \infty) \to [0, \infty)$ given by*

$$\mathbf{Z}_t := \exp\left(\int_0^t \int_E \beta(\Xi_s^{\mathbb{X}}, x)\mathbf{M}(ds, dx) - \int_0^t \int_E \beta(\Xi_s^{\mathbb{X}}, x)^2 \Xi_s^{\mathbb{X}}(x) ds\right) \tag{B.47}$$

is a continuous $\mathcal{F}^{\Xi}$-martingale.

Proof. See Lemma 7.30 in [34] for the fact that $\mathbf{Z}$ is a $\mathcal{F}^{\Xi}$-martingale, the continuity follows from Lemma B.7.5. □

Proposition B.7.7. *If $\mathbf{Z}$ is defined as in Lemma B.7.6 and if we define the filtered probability space $(\Omega, \tilde{\mathcal{A}}, \mathbb{Q})$ with $\tilde{\mathcal{A}} := \cup_{t \geq 0} \mathcal{F}_t^{\Xi}$ and with*

$$\mathbb{Q}(\Gamma) := \mathbb{E}\left[\mathbb{1}_\Gamma \mathbf{Z}_t\right], \quad \Gamma \in \mathcal{F}_t,$$

then the process $\tilde{M}_{\hat{g}}$ given for $\hat{g} \in \mathcal{D}(\mathbf{B}_F)$ by

$$\begin{aligned}\tilde{M}_{\hat{g}}(t) := \Xi_t^{\mathbb{X}}(\hat{g}) - \Xi_0^{\mathbb{X}}(\hat{g}) &- \int_0^t \Xi_t^{\mathbb{X}}(\mathbf{B}(\hat{g})) + b\Xi^{\mathbb{X}}(\hat{g})\, ds \\ &- \int_0^t \int_E a\beta(\Xi_s^{\mathbb{X}}, x)\Xi_s^{\mathbb{X}}(dx) ds\end{aligned}$$

is a continuous $\mathcal{F}^{\Xi}$-martingale with quadratic variation $\langle \tilde{M}_{\hat{g}} \rangle = \langle M_{\hat{g}} \rangle$, where $\langle M_{\hat{g}} \rangle$ is the quadratic variation of the martingale $M_{\hat{g}}$ from Definition B.1.1.

Proof. Since $M_{\hat{g}}$ with

$$M_{\hat{g}}(t) := \Xi_t^{\mathbb{X}}(\hat{g}) - \Xi_0^{\mathbb{X}}(\hat{g}) - \int_0^t \Xi_t^{\mathbb{X}}(\mathbf{B}(\hat{g})) + b\Xi^{\mathbb{X}}(\hat{g})\, ds, \quad t \geq 0,$$

is a $\mathcal{F}^{\Xi}$-martingale with respect to the probability measure $\mathbb{P}$ by Definition B.1.1, Theorem 16.19 in [21] tells us that

$$\tilde{M_{\hat{g}}}(t) := M_{\hat{g}}(t) - \int_0^t \mathbf{Z}_s^{-1} d\langle \mathbf{Z}, M_{\hat{g}} \rangle_s, \quad t \geq 0, \tag{B.48}$$

is $\mathcal{F}^{\Xi}$-martingale with respect to the probability measure $\mathbb{Q}$. Based on Lemma B.7.5 we get a continuous martingale $M : \Omega \times [0, \infty) \to \mathbb{R}$ by

$$M_t = \int_0^t \int_E \beta(\Xi_s^{\mathbb{X}}, x)\, \mathbf{M}(ds, dx), \quad t \geq 0.$$

and an application of the Itô-formula shows that $\mathbf{Z}$ is the unique solution of the linear stochastic differential equation given by $d\mathbf{Z}_t = \mathbf{Z}_t dM_t$. Again by Lemma B.7.5:

$$\langle \mathbf{Z}, M_{\hat{g}} \rangle_t = \int_0^t \mathbf{Z}_s d\langle M, M_{\hat{g}} \rangle_s = \int_0^t \mathbf{Z}_s \int_E a\beta(\Xi_s^{\mathbb{X}}, x)\hat{g}(x)\Xi_s^{\mathbb{X}}(dx)ds.$$

Putting the above into (B.48) gives us our desired statement. □

The following corollary is very important for us, because it allows us to expand path properties of the Dawson-Watanabe superprocess from the critical case to the sub- and supercritical case.

Corollary B.7.8. *Let us assume that $b_1, b_2 \in \mathbb{R}$ are two different drifts. Further let us assume that Ξ^1, Ξ^2 are Dawson Watanabe superprocesses with $\Xi^1 \sim \boldsymbol{DW}(\mathbf{B}, a, b_1)$ and $\Xi^2 \sim \boldsymbol{DW}(\mathbf{B}, a, b_2)$ and it holds $\Xi_0^1 \sim \Xi_0^2$, i.e. Ξ^1 and Ξ^2 are have the same initial distribution. If we fix $T > 0$ and if we denote by $\mathbb{P}_i^T$ the probability measure on $C([0,T], \mathcal{M}_f(E))$ implied by the path law of Ξ^i, $i \in \{1,2\}$ restricted on the time interval $[0,T]$, then $\mathbb{P}_1^T$ and $\mathbb{P}_2^T$ are absolutely continuous with respect to each other. Especially for all Borel sets $\Gamma \in \mathbb{B}(C([0,T], \mathcal{M}_f(E)))$ holds*

$$\mathbb{P}_1^T(\Gamma) = 1 \quad \Leftrightarrow \quad \mathbb{P}_2^T(\Gamma) = 1.$$

Proof. We apply Proposition $B.7.7$ to the function $\beta(\mu, x) = (b_2 - b_1)/a$ to transform the measure $\mathbb{P}_1^T$ to $\mathbb{P}_2^T$, i.e. if $\mathbf{Z}$ is the process B.47 and if we define the new probability measure on $C([0,T], \mathcal{M}_f(E))$ by setting for all

$$\mathbb{Q}(\Gamma) = \mathbb{P}_1^T(\mathbf{Z}_T \mathbb{1}_\Gamma), \quad \forall \Gamma \in \mathbb{B}(C([0,T], \mathcal{M}_f(E))),$$

then the process Ξ^1 becomes under the new law $\mathbb{Q}$ a solution of the martingale problem associated with $\mathbf{DW}(\mathbf{B}, a, b_2)$, see Definition B.1.1, and since the martingale problem of $\mathbf{DW}(\mathbf{B}, a, b_2)$ is well-posed, $\mathbb{Q}$ must be identical with $\mathbb{P}_2^T$. Therefore $\mathbb{P}_2^T$ must be absolutely continuous with respect to $\mathbb{P}_1^T$. By switching the role of Ξ^1 and Ξ^2 we obtain that $\mathbb{P}_1^T$ is absolutely continuous with respect to $\mathbb{P}_2^T$. □

Appendix C

Poisson and Cox Processes

C.1 Cox Processes and Conditional Independence

In this section we discuss the relationship between the notion of conditional independence and the one of a Cox process. First we define a Poisson point process and a Cox process. Let us assume that $\tilde{E}$ is a Polish space.

Definition C.1.1. *If $\boldsymbol{\xi}$ is a random measure over $\tilde{E}$ and $\mu \in \mathcal{M}(\tilde{E})$ is a measure over μ, then we say that $\boldsymbol{\xi}$ is a Poisson point process with intensity measure μ, i.e. $\boldsymbol{\xi} \sim \mathbf{PPP}(\mu)$, when the random variable $\boldsymbol{\xi}(\Gamma)$ is Poisson distributed with intensity $\mu(\Gamma)$ for all $\Gamma \in \mathbb{B}(\tilde{E})$ and when $\boldsymbol{\xi}(\Gamma_1), ..., \boldsymbol{\xi}(\Gamma_k)$ are independent for all finite collections of pairwise disjunct sets $\Gamma_1, ..., \Gamma_k \in \mathbb{B}(\tilde{E})$.*

Definition C.1.2. *If $\boldsymbol{\xi}$ and $\tilde{\boldsymbol{\Xi}}$ are random measures with*

$$\mathfrak{L}(\boldsymbol{\xi}|\tilde{\boldsymbol{\Xi}}) = \mathbf{PPP}(\boldsymbol{\Xi}),$$

i.e. conditioned on $\tilde{\boldsymbol{\Xi}}$ the random measure $\boldsymbol{\xi}$ is a Poisson point process with intensity measure $\tilde{\boldsymbol{\Xi}}$, then $\boldsymbol{\xi}$ is a Cox process directed by $\tilde{\boldsymbol{\Xi}}$, see Page 16 in [20].

In the special case where $\tilde{E} = E \times [0, \infty)$ with E being a Polish space and $\tilde{\boldsymbol{\Xi}} = \boldsymbol{\Xi} \otimes \ell eb[0, \infty)$ with $\boldsymbol{\Xi}$ being a random measure over E, there exists an interesting relationship

Proposition C.1.3. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathcal{F} \subset \mathcal{G} \subset \mathcal{A}$ be two sub-σ-algebras of $\mathcal{A}$. Further let us assume that $\boldsymbol{\xi} : \Omega \to \overline{\mathcal{N}}(E \times [0, \infty))$ is a $\mathcal{G}$-measurable random measures over $E \times [0, \infty)$. Then the following two statements are equivalent:*

1. *There exists a $\mathcal{F}$-measurable random measure $\tilde{\boldsymbol{\Xi}} : \Omega \to \mathcal{M}_f(E)$ with $\mathbb{P}[\boldsymbol{\Xi}(E) > 0] = 1$ and such that*

$$\mathfrak{L}(\boldsymbol{\xi}|\mathcal{F}) = \mathbf{PPP}_E(\boldsymbol{\Xi} \otimes \ell eb[0, \infty)).$$

2. *There exists two sequences $X_i : \Omega \to E$, $i \in \mathbb{N}$, and $V_i : \Omega \to [0, \infty]$, $i \in \mathbb{N}$, of $\mathcal{G}$-measurable random variables with $\boldsymbol{\xi} := \sum_{i=1}^{\infty} \delta_{(X_i, U_i)}$, where $U_i = \sum_{j=1}^{i} V_j$. Further there exists a $\mathcal{F}$-measurable random measure $Q : \Omega \to \mathcal{M}_1(E)$ and $\mathcal{F}$-measurable $Y : \Omega \to [0, \infty)$ with*

$$\mathfrak{L}((X_i, V_i)_{i=1}^{\infty}|\mathcal{F}) = \bigotimes_{i=1}^{\infty} \big(\mathbf{Q} \otimes \boldsymbol{Exp}(Y)\big). \tag{C.1}$$

If one and hence both statements are true, we have $Y := \Xi(E)$ and $\mathbf{Q} = \Xi/Y$ on $\{Y > 0\}$. If $\mathbb{P}[Y > 0] = 1$, then $(X_i, V_i)_{i=1}^{\infty}$ and $\mathbf{Q}$ are unique.

Proof. "1. $\Rightarrow$ 2." We start by setting $Y := \Xi(E)$, hence Y is $\mathcal{F}$-measurable. On the set $\{Y = 0\}$ we pick an arbitrary element x_0 and set $X_i := x_0, V_i := \infty, i \in \mathbb{N}$, and $\mathbf{Q} := \delta_{x_0}$. We continue by defining the process $P : \Omega \times [0, \infty) \to \mathcal{M}_f(E)$ by setting $P_t := \boldsymbol{\xi}(\cdot \times [0, t))$ for $t \geq 0$. We write U_i for the i-th jump time of P and $X_i := \Delta \boldsymbol{\xi}_t$ for the i-th jump. Since P is a process with values in the Polish space $\mathcal{M}_f(E)$, its jump times and jumps are adapted to the natural filtration of P, hence (X_i, U_i) are measurable with respect to $\mathcal{G} \supset \sigma(\boldsymbol{\xi}) = \sigma(P(s), s \geq 0)$. On the set $\{Y > 0\} \in \mathcal{F}$ and conditioned on $\mathcal{F}$ the process P is a càdlàg compound Poisson point process with rate Y, whose jumps are independent, identically distributed with distribution $\mathbf{Q} := \Xi/Y$.
"2. $\Rightarrow$ 1." Let us consider the Laplace functional $L_f : \overline{\mathcal{N}}(E \times [0, \infty)) \to [0, \infty)$ with $L_f(\eta) := \exp(-\eta(f))$, where $f \in C_b(E \times [0, \infty))$ with $f(x, u) = 0$, if $u \geq 0$. The collection of this kind of Laplace functionals form a separating class for $\mathcal{M}_1(\overline{\mathcal{N}}(E \times [0, \infty)))$. Let us assume that $\tilde{\boldsymbol{\xi}}$ is a Poisson point process with intensity measure $\mu \otimes \ell eb[0, \infty)$, where $\mu \in \mathcal{M}_f(E)$. By the Lemma C.2.1 we know that

$$\mathbb{E}\left[L_f(\tilde{\boldsymbol{\xi}})\right] = \exp\left(\int_E \int_0^{\infty} e^{-f(x,u)} - 1 \, du \mu(dx)\right). \tag{C.2}$$

Let us consider $\boldsymbol{\xi}$. We set $Y^r := \boldsymbol{\xi}(E \times [0, r))$. Since $\boldsymbol{\xi} = \sum_{i=1}^{\infty} \delta_{(X_i, U_i)}$, if follows from C.1 that Y^r is Poisson distributed with rate rY conditioned on $\mathcal{F}$. Let us assume that $\varrho : \Omega \times \mathbb{N} \to \mathbb{N}$ is on $\{Y^r = n\}$ a random uniformly distributed permutation of $[n] = \{1, ..., n\}$ with $\varrho(k) = k$ for $k > n$. If we define $(\tilde{X}_i, \tilde{U}_i)_{i=1}^{\infty} := (X_{\varrho(i)}, U_{\varrho(i)})_{i=1}^{\infty}$, then $\boldsymbol{\xi} = \sum_{i=1}^{\infty} \delta_{(\tilde{X}_i, \tilde{U}_i)}$ and

$$\mathfrak{L}((\tilde{X}_i, \tilde{U}_i)_{i=1}^{Y^r} | \mathcal{G}) = \bigotimes_{i=1}^{Y^r} (\mathbf{Q} \otimes \mathcal{U}_{[0,r)}),$$

where $\mathcal{U}_{[0,r)}$ is the uniform distribution over $[0, r]$. From this we can conclude with $g := e^{-f}$:

$$\begin{aligned}
\mathbb{E}\left[L_f(\boldsymbol{\xi}) | \mathcal{F}\right] &= \sum_{k=0}^{\infty} \left(\int_E \int_0^r g(x, u) \, \frac{du}{r} \mathbf{Q}(dx)\right)^k \frac{r^k Y^k}{k!} e^{-krY} \\
&= \exp\left(\int_E \int_0^r g(x, u) - 1 \, du \mathbf{Q}(dx) Y\right) \\
&= \exp\left(\int_E \int_0^r e^{f(x,u)} - 1 \, du \Xi(dx)\right).
\end{aligned}$$

Comparing the above with (C.2) proves our claim. □

For the rest of this section we assume that $(\Omega, \mathcal{A}, \mathbb{P})$, $\mathcal{F}$, $\mathcal{G}$, $\boldsymbol{\xi}$, Ξ, $\mathbf{Q}$, Y, $(X_i)_{i=1}^{\infty}, (U_i)_{i=1}^{\infty}$ and $(V_i)_{i=1}^{\infty}$ are given as in Proposition C.1.3 and that the one and hence both statements from Proposition C.1.3 are true.

Definition C.1.4. *We are defining for all $r>0$ and $m\in\mathbb{N}$ the additional processes:*

$$\begin{aligned}
&\boldsymbol{\xi}^r:\Omega\to\overline{\mathcal{N}}(E\times[0,\infty)), &&\boldsymbol{\xi}^r:=\sum_{i=1}^{\infty}\delta_{(X_i,U_i)}\mathbb{1}_{[0,r)}(U_i);\\
&\boldsymbol{\xi}^{\geq r}:\Omega\to\overline{\mathcal{N}}(E\times[0,\infty)), &&\boldsymbol{\xi}^{\geq r}:=\sum_{i=1}^{\infty}\delta_{(X_i,U_i)}\mathbb{1}_{[r,\infty)}(U_i);\\
&\boldsymbol{\Xi}^r:\Omega\to\mathcal{N}_f(E), &&\boldsymbol{\Xi}^r:=\sum_{i=1}^{\infty}\delta_{X_i}\mathbb{1}_{[0,r)}(U_i);\\
&Y^r:\Omega\to\mathbb{N}_0, &&Y^r:=\boldsymbol{\Xi}^r(E);\\
&\mathbf{Q}^m:\Omega\to\mathcal{M}_1(E), &&\mathbf{Q}^m:=\sum_{i=1}^{m}\delta_{X_i};
\end{aligned}$$

For each $r>0$ and each $m\in\mathbb{N}$, we define the ***decreasing*** *filtrations $\mathcal{F}^{\boldsymbol{\Xi}}$ and $\mathcal{F}^{\mathbf{Q}}$ as the* ***decreasing*** *filtrations $\tilde{\mathcal{F}}^{\boldsymbol{\Xi}}=(\tilde{\mathcal{F}}^{\boldsymbol{\Xi},r},r>0)$ and $\tilde{\mathcal{F}}^{\mathbf{Q}}=(\tilde{\mathcal{F}}^{\mathbf{Q},m},m\in\mathbb{N})$ with $\tilde{\mathcal{F}}^{\boldsymbol{\Xi},r}:=\sigma(\boldsymbol{\Xi}^r,\boldsymbol{\xi}^{\geq r})$ and $\tilde{\mathcal{F}}^{\mathbf{Q},m}:=\sigma(\mathbf{Q}^m,(X_i,U_i)_{i=m+1}^{\infty})$.*

Lemma C.1.5. *Let $\hat{\mathbf{g}}:\Omega\times E\to\mathbb{R}$ be a $\mathcal{F}\times\mathbb{B}(E)$-measurable function ($\hat{\mathbf{g}}$ is random, but the randomness is $\mathcal{F}$-measurable).*

1. *If $\mathbb{E}[|\hat{\mathbf{g}}(X_1)|]<\infty$, then $(\frac{1}{m}\mathbf{Q}^m(\hat{\mathbf{g}}),m\in\mathbb{N})$ is a $\mathcal{F}^{\mathbf{Q}}$-backwards martingale.*
2. *If additionally $\mathbb{E}[Y|\hat{\mathbf{g}}(X_1)|]<\infty$, then $(\frac{1}{r}\boldsymbol{\Xi}^r(\hat{\mathbf{g}}),r\geq 0)$ is a $\mathcal{F}^{\boldsymbol{\Xi}}$-backwards martingale.*

Proof. The integration conditions are needed to ensure that the conditional expectations are well-defined in the usual sense and note that we have the following identities

$$\mathbb{E}[Y|\hat{\mathbf{g}}(X_1)|]=\mathbb{E}[Y\mathbf{Q}(|\hat{\mathbf{g}}|)]=\mathbb{E}[\boldsymbol{\Xi}(|\hat{\mathbf{g}}|)].$$

For a general $m\in\mathbb{N}$ we write $\mathcal{S}(m)$ for the set of permutations of the set $[m]:=\{1,2,...,m\}$. Let us fix $m_1,m_2\in\mathbb{N}$ with $m_1<m_2$, then it holds true that

$$\frac{1}{m_1}\mathbb{E}\left[\mathbf{Q}^{m_1}(\hat{\mathbf{g}})\middle|\mathcal{F}^{\mathbf{Q},m_2}\right]=\frac{1}{m_2}\sum_{i=1}^{m_2}\sum_{\phi\in\mathcal{S}(m_2)}\mathbb{1}_{\{\phi(i)\leq m_1\}}\frac{\hat{\mathbf{g}}(X_{\phi(i)})}{m_2!}=\sum_{i=1}^{m_2}\frac{\hat{\mathbf{g}}(X_{\phi(i)})}{m_2}=\frac{1}{m_2}\mathbf{Q}^{m_2}(\hat{\mathbf{g}}),$$

where the second equality follows from the fact that there exists $m_1(m_2-1)!$ permutations in $\mathcal{S}(m_2)$ with $\phi(i)\leq m_1$. This proves the first claim. For the second claim we fix $0<r_1<r_2<\infty$. In the following we make use of the fact that the σ-algebra $\tilde{\mathcal{F}}^{\boldsymbol{\Xi},r_2}$ knows the random set $A:=\{X_1,...,X_{Y^r}\}$, but it does not know the levels $U_1,...,U_{Y^r}$ or the ordering of the values in A. Using this fact gives us:

$$\frac{1}{r_1}\mathbb{E}\left[\boldsymbol{\Xi}^{r_1}(\hat{\mathbf{g}})\middle|\tilde{\mathcal{F}}^{\boldsymbol{\Xi},r_2}\right]=\frac{1}{r_1}\sum_{i=1}^{Y_2^r}\hat{\mathbf{g}}(X_i)\int_0^{r_2}\mathbb{1}_{[0,r_1)}(u)\frac{du}{r_2}=\frac{1}{r_2}\boldsymbol{\Xi}^{r_2}(\hat{\mathbf{g}}).$$

This proves the second claim. □

Lemma C.1.6. *For $\mathcal{J}=\mathbb{N}$ or $\mathcal{J}=(0,\infty)$, assume that $(\Phi_j,j\in\mathcal{J})$ is a collection of random measures on a Polish space E and that $\mathcal{H}=(\mathcal{H}_j,j\in\mathcal{J})$ is a decreasing filtration. If we obtain for* *every $f \in C_b(E)$ by $(\Phi_j(f), j \in \mathcal{J})$ a $\mathcal{H}$-backwards martingale, which is either right-continuous or left-continuous in the weak topology of $\mathcal{M}_f(E)$ for the case $\mathcal{J} = (0, \infty)$, in the sense that*

$$\mathbb{E}[\Phi_{j_2}(f)|\mathcal{H}_{j_2}] = \Phi_{j_2}(f) \quad \text{for} \quad j_1 < j_2,$$

then there exists a finite random measure Φ such that

$$\mathbb{P}\left[\Phi_j \overset{j\to\infty}{\Longrightarrow} \Phi\right] = 1,$$

where "$\Rightarrow$" stands for the convergence in the weak topology of $\mathcal{M}_f(E)$. If $\mathcal{H}_\infty = \bigcap_{j\in\mathcal{J}} \mathcal{H}_j$, then the limit Φ is given by

$$\Phi(f) = \mathbb{E}\left[\Phi_j(f)\middle|\mathcal{H}_\infty\right], \quad \forall j \in \mathcal{I}, f \in B(E). \tag{C.3}$$

The above lemma is based on the ideas found in Lemma 7.14.(b) in David J. Aldous' Saint Flour lecture notes from the year 1983, see [1]. We extended the proof of Aldous in such a way that is not necessary to assume that $\mathcal{J} = \mathbb{N}$ or that $(\Phi_j, j \in \mathcal{J})$ is a sequence of random probability measures.

Proof. A proof for the case, where the index set is $\mathbb{N}$ can be found as the proof of Lemma 7.14.(b) in [1]. We are considering the situation where $\mathcal{J} = (0, \infty)$ and $(\Phi_j(f), j \in \mathcal{J})$ is left-continuous for every $f \in C_b(E)$, but the right-continuous case works identically. Note that the case of $\mathcal{J} = \mathbb{N}$ can be considered as a special case, indeed if $(\tilde{\Phi}_n, n \in \mathbb{N})$ satisfies the conditions of Lemma C.1.6, then the same holds true for the sequence given by $\Phi_j := \tilde{\Phi}_{\lfloor j \rfloor}$ for $j \in [0, \infty)$ and the limits of are identical. We will begin by proving that $(\Phi_j, j \in \mathcal{J})$ form almost surely a tight family of measures. Let us define the measure $\overline{\Phi}$ by defining for every $f \in C_b^+(E)$ the integral $\overline{\Phi}(f)$ by

$$\overline{\Phi}(f) := \mathbb{E}\big[\Phi_j(f)\big].$$

Due to the martingale property $\overline{\Phi}(f)$ does not depend on the choice of j. Further it holds that $\overline{\Phi}$ is finite, because $(\Phi_j(\mathbb{1}_E) = \Phi_j(E), j \in \mathcal{J})$ is a backwards martingale and so by definition of a backwards martingale it is implied that $\overline{\Phi}(E) = \mathbb{E}[\Phi_j(E)] < \infty$. Since E is polish, the measure $\overline{\Phi}$ must be tight, and so there exists a sequence of compact sets $(\Gamma_n, n \in \mathbb{N})$ such that $\overline{\Phi}(\Gamma_n^c) < 2^{-2n}$. By Doob's maximal inequality and the left-continuity (or the right-continuity) we get

$$\mathbb{P}\left[\sup_{j\in\mathcal{J}} \Phi_j(\Gamma_n^c) > 2^{-n}\right] \le 2^n \overline{\Phi}(\Gamma_n^c) \le 2^{-n}.$$

Setting $A_n := \{\sup_{j\in\mathcal{J}} \Phi(\Gamma_n^c) > 2^{-n}\}$ it follows from the Lemma of Borel-Cantelli that

$$\mathbb{P}[(\Phi_j, j \in \mathcal{J}) \text{ is tight}] = 1 - \mathbb{P}\left[\limsup_{n\to\infty} A_n\right] = 1. \tag{C.4}$$

Now let us assume that $(f_k, k \in \mathbb{N}) \subset C_b^+(E)$ is the convergence determining set from Lemma 2.6.4. Since $(\Phi_j(f_k), j \in \mathcal{J})$ is a backwards martingale for all $k \in \mathbb{N}$, we get that

$$\mathbb{P}\left[\lim_{j\to\infty} \Phi_j(f_k) \text{ is converging for all } k \in \mathbb{N}\right] = 1. \tag{C.5}$$

Let us define

$$\tilde{\Omega} := \left\{\lim_{j\to\infty} \Phi_j(f_k) \text{ is converging for all } k \in \mathbb{N}\} \cap \{(\Phi_j, j \in \mathcal{J}) \text{ is tight}\right\},$$

then $\mathbb{P}[\tilde{\Omega}] = 1$. Fixing $\omega \in \tilde{\Omega}$ we know that $(\Phi_j(\omega), j \in \mathcal{J})$ has a converging subsequence $(\Phi_{j_m}, m \in \mathbb{N})$ which is converging in the weak topology against some measure $\mu_\omega \in \mathcal{M}_f(E)$. Let now assume that whole sequence $(\Phi_j(\omega), j \in \mathcal{J})$ is not converging against $\mu_\omega \in \mathcal{M}_f(E)$. Since $(f_k, k \in \mathbb{N})$ is converging determining, there must exists a $\hat{k}$ and a second subsequence $(\Phi_{j_i}(\omega), i \in \mathbb{N})$ such that

$$\lim_{i\to\infty} \Phi_{j_i}(\omega)(f_{\hat{k}}) \neq \mu_\omega(f_{\tilde{k}}) = \lim_{m\to\infty} \Phi_{j_m}(\omega)(f_{\hat{k}}).$$

But this would be a contradiction to the fact that

$$\omega \in \left\{ \lim_{j\to\infty} \Phi_j(f_k) \text{ is converging for all } k \in \mathbb{N} \right\},$$

hence $(\Phi_j(\omega), j \in \mathcal{J})$ must convergence in the weak topology against μ_ω. We obtain Φ by setting $\Phi(\omega) = \mu_\omega$ for $\omega \in \tilde{\Omega}$ and $\Phi(\omega) = \nu$ for $\omega \notin \tilde{\Omega}$, where ν is some fixed measure over E. The expression for Φ in (C.3) follows once again from the fact that $(\Phi_j(f), j \in \mathcal{J})$ is a backwards martingale. □

Corollary C.1.7. *It holds that* $(\frac{1}{r}\mathbf{\Xi}^r, r > 0)$ *is converging almost surely in the weak topology against* $\mathbf{\Xi}$ *and* $(\frac{1}{m}\mathbf{Q}^m, m \in \mathbb{N})$ *is converging almost surely in the weak topology against* $\mathbf{Q}$.

Proof. The weak convergence for $(\frac{1}{m}\mathbf{Q}^m, m \in \mathbb{N})$ follows immediately from Lemma C.1.5.1 and Lemma C.1.6. The same would also hold for $(r^{-1}\mathbf{\Xi}^r, r \geq 0)$ with C.1.5.2, if $\mathbb{E}[\mathbf{\Xi}(g)] = \mathbb{E}[Y|g(X_1)|]$ holds for all $g \in C_b(E)$. Otherwise we first set $\Phi_r^k := \mathbb{1}_{\{|Y|<k\}}\mathbf{\Xi}^r$ for all $r > 0$ and $k \in \mathbb{N}$. Applying the Lemma C.1.5.2 to the function $\hat{g} := \mathbb{1}_{\{|Y|<k\}} f$, where $f \in C_b^+(E)$, we can conclude from $\mathbb{E}[\tilde{Y}\hat{g}(X_1)] \leq k||f||_\infty$ that the sequence $(\Phi_r^k(f), r > 0)$ forms a backwards martingale. Since this true for all $f \in C_b^+(E)$, we can apply Lemma C.1.6 to the sequence $(\Phi_r(f), r > 0)$ and obtain:

$$\mathbb{P}\left[\mathbb{1}_{\{|Y|<k\}}\frac{1}{r}\mathbf{\Xi}^r \overset{r\to\infty}{\Longrightarrow} \mathbb{1}_{\{|Y|<k\}}\mathbf{\Xi}\right] = 1,$$

where $\Phi_r^k = \mathbb{1}_{\{|Y|<k\}}\mathbf{\Xi}^r$, $\Phi^k = \mathbb{1}_{\{|Y|<k\}}$ and "$\Rightarrow$" stands for the weak convergence in $\mathcal{M}_f(E)$. Since this is true for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \mathbb{P}[|Y| < k] = 1$, it follows

$$\mathbb{P}\left[\frac{1}{r}\mathbf{\Xi}^r \overset{r\to\infty}{\Longrightarrow} \mathbf{\Xi}\right] = 1.$$

□

C.2 Poisson Integration

The following lemma can be found as Lemma A.3 in [32].

Lemma C.2.1. *Let* E *be a Polish space,* ν *be a* σ*-finite Borel measure over* E *and* $\boldsymbol{\xi}$ *be a Poisson random measure on* E *with intensity measure* ν*. If* $f \in \mathcal{L}^1(\nu)$*, then it holds*

$$\mathbb{E}\left[\exp\left(\int_E f(x)\,\boldsymbol{\xi}(dx)\right)\right] = \exp\left(\int_E e^{f(x)} - 1\,dv(x)\right), \tag{C.6}$$

$$\mathbb{E}\left[\int_E f(x)\,\boldsymbol{\xi}(dx)\right] = \int_E f(x)\,\nu(dx), \tag{C.7}$$

$$\mathbf{Var}\left[\int_E f(x)\,\boldsymbol{\xi}(dx)\right] = \int_E f^2(x)\,\nu(dx). \tag{C.8}$$

Assume for next part that $(X_i)_{i=1}^{\infty}$ *are* E*-valued random variables such that* $\boldsymbol{\xi} = \sum_{i=1}^{\infty} \delta_{X_i}$. *If* $g \geq 0$ *and* $\log(g) \in \mathcal{L}^1(\nu)$, *then*

$$\mathbb{E}\Big[\prod_{i=1}^{\infty} g(X_i)\Big] = \exp\Big(\int_E g(x) - 1\,\nu(dx)\Big). \tag{C.9}$$

If $gh, g-1 \in \mathcal{L}^1(\nu)$, *then*

$$\begin{aligned}\mathbb{E}\Big[\sum_{j=1}^{\infty} h(X_j) \prod_{i=1}^{\infty} g(X_i)\Big] &= \mathbb{E}\left[\boldsymbol{\xi}(h) e^{\boldsymbol{\xi}(\log(g))}\right] \\ &= \int_E h(x) g(x)\,\nu(dz) e^{\int_E g(x)-1\,\nu(dx)},\end{aligned} \tag{C.10}$$

$$\begin{aligned}\mathbb{E}\Big[\sum_{i\neq j}^{\infty} h(X_i)h(X_j) \prod_{k=1}^{\infty} g(X_k)\Big] &= \mathbb{E}\left[\left(\boldsymbol{\xi}(h)^2 - \boldsymbol{\xi}(h^2)\right) e^{\boldsymbol{\xi}(\log(g))}\right] \\ &= \Big(\int_E h(x)g(x)\,\nu(dx)\Big)^2 e^{\int_E g(x)-1\,\nu(dx)}.\end{aligned} \tag{C.11}$$

Proof. See the proof of Lemma A.3. in [32]. □

Lemma C.2.2. *Let us assume that* E *is a Polish space and that* $g : E \times [0,\infty) \to \mathbb{R}$ *is a bounded continuously functions with the additional properties: 1.The image of* g *is contained in* $[0,1]$, *i.e.* $0 \leq g(x,u) \leq 1$ *for all* $(x,u) \in E \times [0,\infty)$, *2.For each* $x \in E$, *the function* $g_x : [0,\infty) \to \mathbb{R}$ *given by* $g_x(u) = g(x,u)$ *is continuous differentiable 3.When we define* $\tilde{g} \in C_b(E \times [0,\infty))$ *by setting* $\tilde{g}(x,u) := 1 - g(x,u)$ *for each* $(x,u) \in \mathbb{E} \times [0,\infty)$, *then its support is contained in* $E \times [0,r]$. *If these properties are satisfied, then:*

1. *For all* $R \in [r,\infty]$, *it holds that* $\hat{g}^R \in C_b(E)$ *with* $\hat{g}^R : E \to \mathbb{R}$ *given by*

$$\hat{g}^R(x) = \int_0^R 1 - g(x,u)\,du,\ \ x \in E. \tag{C.12}$$

Note that the values of $\hat{g}^R$ *do not depend on the chosen* R *as long as* $R \in [r,\infty]$.

2. *It it holds for each* $R \in [r,\infty]$:

$$\int_0^R u^2 \partial_u g(x,u) + 2\left(g(x,u)\int_u^R g(x,v) - 1\,dv\right) du = \left(\int_0^R 1 - g(x,u)\,du\right)^2, \tag{C.13}$$

$$\int_0^R u^2 \partial_u \tilde{g}(x,u) + 2\left(\int_u^R \tilde{g}(x,v)\,dv\right) du = 0, \tag{C.14}$$

$$\begin{aligned}\int_0^R u\,\partial_u g(x,u)\,du &= -\int_0^R u\,\partial_u \tilde{g}(x,u)\,du \\ &= \int_0^R \tilde{g}(x,u)\,du = \int_0^R 1 - g(x,u)\,du.\end{aligned} \tag{C.15}$$

Further, if we define $J_1(R,x), J_2(R,x)$ *and* $J_3(R,x)$ *for* $R \in [r,\infty]$ *as the left-hand side (or as the right-hand side) of* (C.13), (C.14) *and* (C.15), *then those values do not depend on the value of* $R \in [r,\infty]$, *indeed*

$$J_i(R,x) = J_i(\tilde{R},x) \text{ for all } R,\tilde{R} \in [r,\infty], x \in E \text{ and } i \in \{1,2,3\}. \tag{C.16}$$

Further we have $J_1(R,\cdot), J_2(R,\cdot), J_3(R,\cdot) \in C_b(E)$.

Proof. Since $0 \leq \tilde{g} = 1 - g \leq \mathbb{1}_{E\times[0,r]}$, we can derive that $\hat{g}^R \in C_b(E)$ by Lebesgue's dominated convergence theorem. The values of $\hat{g}^R$ do not depend on R, because the support of $1 - g$ is contained in $E \times [0, r]$. We continue with arguing why (C.16) is true and why $J_1(R, \cdot), J_2(R, \cdot), J_3(R, \cdot) \in C_b(E)$, but this follows from the fact that expressions on the right side of (C.13), (C.14) and (C.15) are continuous functionals of g^R from (C.12), which does not depend on the chosen R and is an element of $C_b(E)$.
We continue with proving the identities found in the above lines. Due to (C.16) we can assume that R is finite.
Starting with (C.13) we multiply out the brackets of the right-hand side and this gives us:

$$\int_0^R \int_0^R 1 - g(x,u) - g(x,v) - g(x,u)g(x,v)\,dvdu$$
$$= R^2 - 2R \int_0^R g(x,u)du + 2\int_0^R \int_u^R g(x,u)g(x,v)dudv. \tag{C.17}$$

Considering the left-hand side of (C.13), we apply partial integration to the first term and obtain

$$\left[u^2 g(x,u)\right]_0^R - \int_0^R 2ug(x,u)\,du + \int_0^R 2g(x,u) \int_u^R g(x,v) - 1\,dvdu.$$

It holds $\left[au^2 g(x,u)\right]_0^R = R^2$, because $g(x,R) = 1$, since $R \geq r$, so by multiplying out the brackets of the remaining terms transforms the expression into:

$$R^2 - 2\int_0^R ug(x,u)\,du - 2\int_0^R \int_u^R g(x,u)\,dvdu + 2\int_0^R \int_u^R g(x,u)g(x,v)\,dvdu. \tag{C.18}$$

Because $\int_0^R \int_u^R g(x,u)\,dvdu = \int_0^R (R-u)g(x,u)\,dvdu$, the two middle terms (C.18) add up to $2R\int_0^R g(x,u)\,du$ and so (C.17) as well as (C.18) are identical.
For the Identity (C.14) we apply partial integration to first term of the left-hand side and obtain

$$[u^2\tilde{g}(x,u)]_0^R - \int_0^R 2u\tilde{g}(x,u) + 2\int_u^R \tilde{g}(x,v)\,dvdu = [u^2\tilde{g}(x,u)]_0^R.$$

The remain term $[u^2\tilde{g}(x,u)]_0^R$ also vanishes, because of $\tilde{g}(x,R) = 0$, since $R \geq r$.
The first and last identity of (C.15) follow from the definition of $\tilde{g}$. For the interesting middle one, we apply partial integration to $-\int_0^R u\,\partial_u\tilde{g}(x,u)$ and we obtain:

$$-\left[u\partial_u\tilde{g}(x,u)\right]_0^R + \int_0^R \tilde{g}(x,u)\,du.$$

Again, $[u\partial_u\tilde{g}(x,u)]_0^R = 0$, because of $\tilde{g}(x,R) = 0$, since $R \geq r$. $\square$

Appendix D

Technical Results

D.1 Markov Mapping Theorem

The importance of the Markov mapping theorem for this thesis can not be overstated, therefore we want to present a proof. Our proof will be based on more assumptions than the version proved by Kurtz used in [30].
Let us consider two Polish spaces E_X and E_Y called the big space and the small space. Further, let us assume that $X : \Omega \times [0, \infty) \to E_X$ is a time homogeneous Markov process with state space E_X and that $\gamma : E_X \to E_Y$ is a measurable function. The Markov mapping theorem gives an answer to the question under which conditions the process $Y := \gamma(X)$ is again a Markov process and it also gives an description of the conditional distribution of X_t based on the path $(Y_s, 0 \leq s \leq t)$.
We begin with semigroup version which was presented in 1981 by L.C.G. Rogers and J.W. Pitman in [43], and which is to the best of our knowledge the first formulation of the Markov mapping theorem. The presentation of Rogers and Pitman results makes the introduction of new notation necessary (Recall that $\mathbf{M}(E)$ stands for the class of measurable and $\mathrm{B}(E)$ for the subclass of bounded, measurable functions).

Definition D.1.1. *If $\alpha : E_Y \to \mathcal{M}_1(E_X)$ is a Markov kernel, we denote by $\mathcal{L}(\alpha)$ the subset of $\mathbf{M}(E_X)$ given by*

$$\mathcal{L}^1(\alpha) := \left\{ g \in \mathbf{M}(E_X) : \int_{E_X} |g(x)| \alpha(y, dx) < \infty \text{ for all } y \in E_Y \right\}.$$

Definition D.1.2. *Let us assume that E_Y and E_X are Polish spaces, that $\alpha : E_Y \to \mathcal{M}_1(E_X)$ is a Markov kernel and that $\gamma : E_X \to E_Y$ is a measurable function.*

1. *We call the Markov kernel α and the function γ in a* ***Rogers-Pitman correspondence, R.P.C.*** *if it holds $\alpha(y, \gamma^{-1}(y)) = 1$ for all $y \in E_Y$, which makes γ a surjective map.*

2. *We denote by $\alpha^* : \mathcal{L}^1(\alpha) \to \mathbf{M}(E_Y)$ the pullback of functions which is given for $g \in C_b^+(E_X)$ by $\hat{g}(y) = \int_{E_Y} g(x) \alpha(y, dx)$.*

3. *We denote by $\alpha_* : \mathcal{M}_1(E_Y) \to \mathcal{M}_1(E_X)$ the push forward of measures, where the measure $\alpha_*(\theta) = \hat{\Theta} \in \mathcal{M}_1(E_X)$ is defined for $\Theta \in \mathcal{M}_1(E_Y)$ by setting $\hat{\Theta}(g) = \Theta(\alpha^*(g))$ for all $g \in C_b^+(E_X)$.*

4. We denote by $\gamma^* : \mathbf{M}(E_Y) \to \mathbf{M}(E_X)$ the pullback of functions given by $\gamma^*(\hat{g}) := \hat{g} \circ \gamma$ for all $\hat{g} \in \mathbf{M}(E_Y)$.

5. We say α is a continuous Markov kernel, if α is a continuous map with respect to the topology of E_Y and the weak topology on $\mathcal{M}_1(E_X)$. Note, that this is equivalent to say that $\alpha^*(C_b(E_X)) \subset C_b(E_Y)$.

Note that $\gamma^*(\mathrm{B}(E_Y)) \subset \mathcal{L}^1(\alpha)$ for any γ and any α. Using the pullback α^* and the pushforward γ^* the Rogers-Pitman correspondence can be formulated as

$$\alpha^* \circ \gamma_* = \mathbf{Id}_Y,$$

where $\mathbf{Id}_Y$ stands for the identity on the space $\mathrm{B}(E_Y)$. So being in a R.P.C. is equivalent to the statement that the pushforward γ^* is the right inverse of α^* on $\mathrm{B}(E_Y)$.

Theorem D.1.3 (Markov-Mapping, Semigroup-Version, Rogers-Pitman 1981)**.** *Let us assume that X is a Markov process with semigroup $(P_t, t \geq 0)$ with $P_t : B(E_X) \to B(E_X), t \geq 0$, and that α and γ are like in the Definition D.1.2. Further, let us define the process Y by $Y := \gamma(X)$ and the linear operator $Q_t : B(E_Y) \to B(E_Y), t \geq 0$, by setting $Q_t := \alpha^* \circ P_t \circ \gamma^*$. Under the conditions that*

1. The Markov kernel α and the function γ are in a Rogers-Pitman correspondence.

*2. The Markov kernel α is an **Intertwiner** for P and Q on $B(E_X)$, that means*

$$\alpha^* \circ P_t(g) = Q_t \circ \alpha^*(g), \quad t \geq 0,$$

for all $g \in B(E_X)$.

Then Q is a Markov semigroup and, under the additional assumption that $X_0 \sim \alpha_*(\mu)$ for some probability measure $\mu \in \mathcal{M}(E_Y)$, the process Y is a Markov process with the semigroup Q and initial distribution μ, i.e. $Y_0 \sim \mu$. Further, it holds for all $t \geq 0$:

$$\mathfrak{L}(X_t | \mathcal{F}_t^Y) = \alpha(Y_t), \tag{D.1}$$

where $\mathcal{F}^Y$ is the natural filtration of Y.

Proof. Using Condition 1 and the semigroup property of P we obtain for all $t_1, t_2 \in [0, \infty)$:

$$Q_{t_1} \circ Q_{t_2} = \alpha^* \circ P_{t_1} \circ \gamma^* \circ \alpha^* \circ P_{t_2} \circ \gamma^* = \alpha^* \circ P_{t_1+t_2} \circ \gamma^* = Q_{t_1+t_2}.$$

We will now prove (D.1) by showing for all $g \in \mathrm{B}(E_X)$, $0 = t_0 < t_2 < ... < t_n = t$ and $\hat{f}_0, \hat{f}_1, ..., \hat{f}_n \in \mathrm{B}(E_Y)$ that the expectation $\mathbb{E}\left[g(X_t) \prod_{i=0}^n \hat{f}_i(Y_{t_i})\right]$ is equal to

$$\int_{E_Y} \cdots \int_{E_Y} \left[\alpha^*(g)(y_n) \prod_{i=0}^n \hat{f}(y_i)\right] Q_{\delta t_n}(y_{n\text{-}1}, dy_n) \ldots Q_{\delta t_1}(y_0, dy_1) \mu(dy_0), \tag{D.2}$$

where $\delta t_k = t_k - t_{k-1}$. From the Markov property of X and the definition of Y as $\gamma(X)$ we know that $\mathbb{E}\left[g(X_t) \prod_{i=1}^n \hat{f}_i(Y_{t_i})\right]$ is equal to

$$\int_{E_Y} \int_{E_X} \cdots \int_{E_X} \left[g(x_n) \prod_{i=0}^n \hat{f}(\gamma(x_i))\right] P_{\delta t_n}(x_{n\text{-}1}, dx_n) \ldots P_{\delta t_1}(x_0, dx_1) \alpha_*(y, dx_0) \mu(dy). \tag{D.3}$$

The idea of proving the equality between (D.2) and (D.3) is to repeatedly apply the intertwiner relationship $\alpha^* \circ P_t = Q_t \circ \alpha^*$ and the fact that we can drag out $\hat{f}_i$ due to the R.P.C. To make this simple and nice idea more transparent let us define the linear operators $\tilde{Q}_i := \hat{f}_{i\text{-}1} Q_{\delta t_i}$ and $\tilde{P}_i := \gamma^*(\hat{f}_{i\text{-}1}) P_{\delta t_i}$ for $1 \leq i \leq n$. Then we get for an arbitrary $h \in \mathrm{B}(E_X), \leq i \leq n$ and $y \in E_Y$ by first applying the R.P.C., second equality, and then the Intertwiner relation, third equality, that

$$\begin{aligned}\alpha^* \circ \tilde{P}_i(h)(y) &= \int_{E_X} \int_{E_X} \hat{f}_{i\text{-}1}(\gamma(x_{i\text{-}1})) h(x_i) P_{\delta t_i}(x_{i\text{-}1}, dx_i) \alpha(y, dx_{i\text{-}1}) \\ &= \hat{f}_{i\text{-}1}(y) \big(\alpha^* \circ P_{\delta t_i}(h)(y)\big) = \hat{f}_{i\text{-}1}(y) \big(Q_{\delta t_i} \circ \alpha^*(h)(y)\big) = \tilde{Q} \circ \alpha^*(h)(y).\end{aligned}$$

In short we get the modified intertwiner relation:

$$\alpha^* \circ \tilde{P}_i = \tilde{Q}_i \circ \alpha^*, \quad 0 \leq i \leq n. \tag{D.4}$$

Now we are using the new operators to express (D.3) as

$$\mu(\alpha^* \circ \tilde{P}_1 \circ \ldots\ldots \circ \tilde{P}_n(\gamma^*(\hat{f}_n) g)).$$

Applying consecutively the modified intertwiner relation (D.4) we get

$$\mu(\tilde{Q}_1 \circ \ldots\ldots \circ \tilde{Q}_n(\alpha^*(\gamma^*(\hat{f}_n) g)))$$

and since $\alpha^*(\gamma^*(\hat{f}_n) g)(y) = \hat{f}_n(y) \alpha^*(g)(y)$ for all $y \in E_Y$ due to the R.P.C., this is equal to

$$\mu(\tilde{Q}_1 \circ \ldots\ldots \circ \tilde{Q}_n(\hat{f}_n \alpha^*(g))).$$

The above is now equal to (D.2) and so we have proved that the equality of (D.2) and (D.3). Since the selection of $g, t_0, t_1, ..., t_n, \hat{f}_1, ..., \hat{f}_n$ has been arbitrary it follows (D.1). From this the Markov property of Y follows immediately, because for an arbitrary $\hat{f} \in \mathrm{B}(E_Y)$ and $0 \leq s < t < \infty$ it follows by the Markov property of X, third equality, that

$$\mathbb{E}\left[\hat{f}(Y_t) | \mathcal{F}_s^Y\right] = \mathbb{E}\left[\mathbb{E}\left[\hat{f}(\gamma(X_s)) | \mathcal{F}_s^X\right] | \mathcal{F}_t^Y\right] = \mathbb{E}\left[\mathbb{E}\left[\hat{f}(\gamma(X_t)) | X_s\right] | \mathcal{F}_t^Y\right].$$

Using the equality $\mathbb{E}[\hat{f}(\gamma(X_t)) | X_s] = P_{t\text{-}s}(\hat{f} \circ \gamma)(X_s)$, we conclude that

$$\mathbb{E}\left[\hat{f}(Y_t) | \mathcal{F}_s^Y\right] = \mathbb{E}[P_{t\text{-}s}(\hat{f} \circ \gamma)(X_s) | \mathcal{F}_s^Y].$$

If we now apply $\mathfrak{L}(X_t | \mathcal{F}_t^Y) = \alpha(Y_t)$, we can write

$$\mathbb{E}\left[P_{t\text{-}s}(\hat{f} \circ \gamma)(X_s) | \mathcal{F}_s^Y\right] = \int_{E_Y} P_{t\text{-}s}(\hat{f})(\gamma(x)) \alpha(Y_s, dx) = \alpha^* \circ P_{t\text{-}s} \circ \gamma^*(\hat{f})(Y_s),$$

where we used $\gamma^*(\hat{f}) = \hat{f} \circ \gamma$ in the rightmost expression. If we now apply the intertwiner relation and that $\alpha^* \circ \gamma_*(\hat{f}) = \hat{f}$ due to the Rogers-Pitman Correspondence, we can finally see that

$$\mathbb{E}\left[\hat{f}(Y_t) | \mathcal{F}_s^Y\right] = \alpha^* \circ P_{t\text{-}s} \circ \gamma^*(\hat{f})(Y_s) = Q_{t\text{-}s} \circ \alpha^* \circ \gamma^*(\hat{f})(Y_s) = Q_{t\text{-}s}(\hat{f})(Y_s),$$

hence Y is a Markov process with semigroup Q. □

Proposition D.1.4 (Markov-Mapping, Generator Version)**.** *Let us assume that α and γ are given as in Definition D.1.2 and that α is a continuous Markov kernel. Further let $\mathcal{P}_X = (\mathcal{P}_X^x, x \in E_X) \subset \mathcal{M}_1(\mathbb{D}([0,\infty), E_X))$ and $\mathcal{Q}_Y = (\mathcal{Q}_Y^y, y \in E_Y) \subset \mathcal{M}_1(\mathbb{D}([0,\infty), E_Y))$ be two Borel strong Markov families. Let $(P_t, t \geq 0)$ and $(Q_t, t \geq 0)$ be the two semigroups and suppose that $\boldsymbol{A} : C_b(E_X) \supset \mathcal{D}(\boldsymbol{A}) \to C_b(E_X)$ and $\boldsymbol{C} : C_b(E_Y) \supset \mathcal{D}(\boldsymbol{C}) \to C_b(E_Y)$ are the corresponding weak generators and $\mathbf{U}_X^\lambda$ and $\mathbf{U}_Y^\lambda$, $\lambda > 0$, are the λ-resolvents. If α and γ are in a Rogers-Pitman correspondence, then the following statements are equivalent:*

1. *It holds $\alpha^* \circ P_t(g) = Q_t \circ \alpha^*(g)$ for all $g \in C_b(E_X)$ and $t \geq 0$.*
2. *It holds $\alpha^*(\mathcal{D}(\boldsymbol{A})) \subset \mathcal{D}(\boldsymbol{C})$ and $\alpha^* \circ \boldsymbol{A}(f) = \boldsymbol{C} \circ \alpha^*(f)$ for all $f \in \mathcal{D}(\boldsymbol{A})$.*
3. *It holds $\mathbf{U}_Y^\lambda \circ \alpha^*(g) = \alpha^* \circ \mathbf{U}_X^\lambda(g)$ for all $g \in C_b(E_X)$ and $\lambda \in (0,\infty)$.*

Remark D.1.5. *Please note that the continuity of α is only needed to ensure that $\alpha^*(g)$ is continuous for $g \in \mathcal{D}(\boldsymbol{A})$, which is required because the domain $\mathcal{D}(\boldsymbol{C})$ must be a subset of $C_b(E_Y)$ by the definition of the weak generator of a Borel strong Markov process.*

Proof. $1 \Rightarrow 2$: Fixing $g \in \mathcal{D}(\mathbf{A})$ we get for all $t > 0$:

$$\frac{Q_t \circ \alpha^*(g)(y) - \alpha^*(g)(y)}{t} = \frac{\alpha^* \circ P_t(g)(y) - \alpha^*(g)(y)}{t} = \int_{E_X} \frac{P_t(g)(x) - g(x)}{t} \alpha(y, dx).$$

Since $g \in \mathcal{D}(\mathbf{A})$ it holds $b.p.\text{-}\lim_{t\to\infty}(P_t(g) - g)/t = \mathbf{A}(g)$, that means $(P_t(g) - g)/t$ converges pointwise against $\mathbf{A}(g)$ and it exists a upper bound $K > 0$ with $\sup_t ||(P_t(g) - g)/t||_\infty + ||\mathbf{A}(g)||_\infty < K$. So it follows by Lebesgue and the above that:

$$\int_{E_X} \frac{P_t(g)(x) - g(x)}{t} \alpha(y, dx) \overset{t\to\infty}{\longrightarrow} \int_{E_X} \mathbf{A}(g)(x)\alpha(y, dx)$$

and since the inner expression of the left integral is bounded by the constant K, we can conclude that $b.p.\text{-}\lim_{t\to\infty}(Q_t \circ \alpha * (g) - \alpha^*(g))/t = \alpha^* \circ \mathbf{A}(g)$. By the definition of the weak generator $\mathbf{C}$ it follows that $\alpha^*(g) \in \mathcal{D}(\mathbf{C})$ and that

$$\mathbf{C} \circ \alpha^*(g) = \mathbf{C}(\alpha^*(g)) = b.p.\text{-}\lim_{t\to\infty}(Q_t \circ \alpha^*(g) - \alpha^*(g))/t = \alpha^* \circ \mathbf{A}(g).$$

$2) \Rightarrow 3)$: We get immediately that $\alpha^* \circ (\lambda - \mathbf{A}) = (\lambda - \mathbf{C}) \circ \alpha^*$ for any $f \in \mathcal{D}(\mathbf{A})$. Using this together with $\mathbf{U}_X^\lambda(C_b(E_X)) \subset \mathcal{D}(\mathbf{A}), \mathbf{U}_Y^\lambda \circ (\lambda - \mathbf{C})(\hat{f}) = \hat{f}$ for $\hat{f} \in \mathcal{D}(\mathbf{C})$ and $\mathbf{U}_X^\lambda \circ (\lambda - \mathbf{A})(g) = g$ for $g \in C_b(E_X)$, it follows

$$\begin{aligned}\alpha^* \circ \mathbf{U}_X^\lambda(g) &= \mathbf{U}_Y^\lambda \circ (\lambda - \mathbf{C}) \circ \alpha^* \circ \mathbf{U}_X^\lambda(g) = \mathbf{U}_Y^\lambda \circ \alpha^* \circ (\lambda - \mathbf{A}) \circ \mathbf{U}_X^\lambda(g)\\ &= \mathbf{U}_Y^\lambda \circ \alpha^*(g)\end{aligned}$$

for all $g \in C_b(E_X)$. $3) \Rightarrow 1)$: We get for $\lambda > 0$ and $g \in C_b(E_X)$ that

$$\int_0^\infty e^{-\lambda s} \alpha^* \circ P_s(g)\, ds = \alpha^* \circ \mathbf{U}_X^\lambda(g) = \mathbf{U}_Y^\lambda \circ \alpha^*(g) = \int_0^\infty e^{-\lambda s} \alpha^* \circ Q_s(g)\, ds.$$

Since this true for all $\lambda \in [0,\infty)$, it follows $\alpha^* \circ P_t(g) = Q_t \circ \alpha^*(g)$ for all $t \geq 0$. □

A huge drawback of this formulation of Proposition D.1.4 is that it is necessary to prove the intertwiner relationship for all elements contained in the **full** weak generator of X. This is hard to do in general, therefore it would be convenient, if we could show that it sufficient to prove the intertwiner relationship for a linear operator $\mathbf{A}$ which is just a "core" for X, meaning that the martingale problem of $\mathbf{MP}(\mathbf{A})$ is well-posed. By employing advanced technical terminology this is possible. We start by introducing the important concept of a forward equation and their solutions.

Definition D.1.6. *If $\boldsymbol{A} : C(E_X) \supset \mathcal{D}(\boldsymbol{A}) \to C(E_X)$ is a linear operator and $\mu \in \mathcal{M}_1(E_X)$, then we call a probability measure valued map $\nu : [0,\infty) \to \mathcal{M}_1(E_X)$ a solution of the forward equation of $\boldsymbol{FE}(\boldsymbol{A},\mu)$, if for all $f \in \mathcal{D}(\boldsymbol{A})$ and $t \geq 0$ holds $\int_0^t \nu_s(|\boldsymbol{A}(f)|)ds < \infty$ and*

$$\nu_t(f) = \nu_0(f) + \int_0^t \nu_s(\boldsymbol{A}(f))ds.$$

*We say that the solution of the forward equation $\boldsymbol{FE}(\boldsymbol{A},\mu)$ is **unique**, if there exists at most one solution, and **well-posed**, if there exists exactly one solution.*

Another important concept is the bounding function.

Definition D.1.7. *If $\boldsymbol{A} : C(E_X) \supset \mathcal{D}(\boldsymbol{A}) \to C(E_X)$ is a linear operator, then we call $\psi : E_X \to [1,\infty)$ a bounding function for $\boldsymbol{A}$, if ψ is continuous and we can find for each $f \in \mathcal{D}(\boldsymbol{A})$ a constant c_f with $|\boldsymbol{A}(f)| \leq c_f\psi$.*

Obviously if the process X is a solution of the martingale problem $\mathbf{MP}(\mathbf{A},\mu)$ and there exists a bounding function ψ for $\mathbf{A}$ with the property that

$$\int_0^t \mathbb{E}[\psi(X_s)]ds < \infty \quad \text{for all } t \geq 0,$$

then $\nu^X : [0,\infty) \to \mathcal{M}_1(E_X)$ defined by setting $\nu_t^X(g) = \mathbb{E}[g(X_t)]$ for all $g \in C_b(E_X)$ and all $t \geq 0$ is a solution of the forward equation of $\mathbf{FE}(\mathbf{A},\nu_0^X)$.
If the solution of $\mathbf{FE}(\mathbf{A},\nu_0)$ is unique, then the one-dimensional distribution of the martingale problem $\mathbf{MP}(\mathbf{A},\nu_0)$ is also unique and we can conclude that the same holds for the solution of $\mathbf{MP}(\mathbf{A},\nu_0)$ by Proposition 4.4.2 from [14]. The reverse need not be true in general, see Page 6 of [31].
But under mild conditions on the operator $\mathbf{A}$ it is possible to prove that every solution ν of $\mathbf{FE}(\mathbf{A},\nu_0)$ implies a solution X with $X_t \sim \nu_t$ for all $t \geq 0$, which extends the uniqueness from $\mathbf{MP}(\mathbf{A},\nu_0)$ to $\mathbf{FE}(\mathbf{A},\nu_0)$.

Conditions D.1.8.

1. *$\boldsymbol{A} : C_b(E_X) \subset \mathcal{D}(\boldsymbol{A}) \to C(E_X)$ is a conservative operator, i.e. $\mathbb{1}_{E_X} \in \mathcal{D}(\boldsymbol{A})$ and $\boldsymbol{A}(\mathbb{1}_{E_X}) = 0$.*
2. *The domain $\mathcal{D}(\boldsymbol{A})$ is closed under multiplication and separates points, i.e. for each $x, \tilde{x} \in E_X$ there exists a $f \in \mathcal{D}(\boldsymbol{A})$ such that $f(x) \neq f(\tilde{x})$.*
3. *There exists a bounding function $\psi : E_X \to [1,\infty)$ for $\boldsymbol{A}$.*
4. *The linear operator $\boldsymbol{A}_0$ is dissipative and there exists a sequence of functions $\mu_n : E_X \to \mathcal{M}_1(E_X)$ and $\lambda_n : E_X \to [0,\infty)$ such that for each $f \in \mathcal{D}(\boldsymbol{A})$ holds*

$$\boldsymbol{A}(g)(x) = \lim_{n\to\infty} \lambda_n(x) \int_{E_X} (f(y) - f(x))\mu_n(x,dy)$$

   for each $x \in E_X$. Note that we can replace $\boldsymbol{A}(g)$ with $\boldsymbol{A}_0(g)$ by replacing λ_n by λ_n/ψ.
5. *The linear operator $\boldsymbol{A}_0 := \{(f, (\psi \vee 1)^{-1}\boldsymbol{A}(f)), f \in \mathcal{D}(\boldsymbol{A})\}$ is graph separable, indeed there exists a countable set $(f_i, i \in \mathbb{N})$ such that $\boldsymbol{A}_0$ is contained in the b.p.-closure of the linear span of $((f_i, \boldsymbol{A}_0(f_i)), i \in \mathbb{N})$ in $B(E_X) \times B(E_X)$.*

Remark D.1.9. *These conditions are mild in the sense that almost all operators whose martingale problem is well-posed will also satisfy these conditions.*

Remark D.1.10. *The above conditions are based on the Conditions 2.1 formulated in [28], but we could combine the points i) and ii) of [28] to our Point 1, because we are only considering linear operators whose image is contained in $C(E_X)$. We have also changed the order of the different points a little bit.*

The next lemma gives us the existence of a solution of $\mathbf{MP}(\mathbf{A},\mu)$ as long as a solution of $\mathbf{FE}(\mathbf{A},\mu)$ exists.

Lemma D.1.11. *Let us assume that $\boldsymbol{A} : C(E_X) \supset \mathcal{D}(\boldsymbol{A}) \to C(E_X)$ is a linear operator satisfying the Conditions D.1.8. If there exists a measurable map $\nu : [0,\infty) \to \mathcal{M}_1(E_X)$ such that ν is a solution of the forward equation $\boldsymbol{FE}(\boldsymbol{A},\nu_0)$ and it holds*

$$\int_0^t \nu_s(\psi)ds < \infty \quad \text{for all } t > 0,$$

where ψ is the bounding function for $\boldsymbol{A}$ from the second point of the Conditions D.1.8, then there exists a solution X of the martingale problem $\boldsymbol{MP}(\boldsymbol{A},\nu_0)$ with $X_t \sim \nu_t$ for all $t \geq 0$.

Proof. Here, we combined the first part of Definition 2.7 with the first part of the Lemma 2.8, both from [28]. The above statement is proved by the Lemma 2.8. Since the proof is based on several other highly non-trivial results from [29], we will not present it here. □

Lemma D.1.12. *Assume that $\boldsymbol{C} : C(E_Y) \supset \mathcal{D}(\boldsymbol{C}) \to C(E_Y)$ is a linear operator and $\tilde{\psi} : E_Y \to \mathbb{R}$ is a measurable function. Further let us assume that for all $\mu \in \mathcal{M}_1(E)$ holds that, if Y_1 and Y_2 are two solutions of $\boldsymbol{MP}(\boldsymbol{C},\mu)$ with*

$$\int_0^t \mathbb{E}[Y_i(s)]\, ds < \infty, \quad t \geq 0, i \in \{1,2\}, \tag{D.5}$$

then Y_1 and Y_2 have the same one-dimensional distribution, i.e. $\mathbb{E}[f(Y_1)] = \mathbb{E}[f(Y_2)]$ for all $f \in C_b(E)$. Under this condition the solutions Y_1 and Y_2 have the same finite dimensional distributions and are Markov processes.

Proof. Note that this statement is almost identical with Theorem 4.4.2.(a) and 4.4.2.(b) from [14] except that we are only considering solutions satisfying (D.5). The argumentation in the proof of Theorem 4.4.2.(a) and 4.4.2.(b) remains valid, if we only consider solutions satisfying (D.5). □

Theorem D.1.13 (Markov Mapping theorem)**.** *Let us assume that $\boldsymbol{A} : C(E_X) \supset \mathcal{D}(\boldsymbol{A}) \to C(E_X)$ and $\boldsymbol{C} : C(E_Y) \supset \mathcal{D}(\boldsymbol{C}) \to C(E_Y)$ are linear operators, that α and γ are given as in Definition D.1.2. We are further assuming that*

1. *$\boldsymbol{A}$ satisfies the Conditions D.1.8. Further the local martingale problem $\boldsymbol{MP}(\boldsymbol{A})$ is well-posed and admits a càdlàg solution.*

2. *For every $y \in E_Y$, the martingale problem $\boldsymbol{MP}(\boldsymbol{C},\hat{\Theta}_0^1)$, $\Theta_0^1 \in \mathcal{M}_1(E_Y)$, admits a solution and every solution $\tilde{Y}$ of $\boldsymbol{MP}(\boldsymbol{C},\hat{\Theta}_0^1)$ satisfies*

   $$\int_0^t \mathbb{E}[\alpha^*(\psi)(\tilde{Y}_s)]\, ds < \infty, \quad t \geq 0,$$

   where ψ is the bounding functions for $\boldsymbol{A}$ from the Conditions D.1.8.

3. α and γ are in a Rogers-Pitman correspondence, i.e. $\alpha(y, \gamma^{-1}(y)) = 1$ for all $y \in E_Y$ and it holds $\alpha^*(\mathcal{D}(\boldsymbol{A})) \subset \mathcal{D}(\boldsymbol{C})$, $\alpha^*(C_b(E_X))$ is separating and $\alpha^* \circ \boldsymbol{A}(f) = \boldsymbol{C} \circ \alpha^*(f)$ for all $f \in \mathcal{D}(\boldsymbol{A})$.

If X is a solution of $\boldsymbol{MP}(\boldsymbol{A}, \Theta_0)$, where $\Theta_0(dx) = \int_{E_Y} \alpha(y, dx)\hat{\Theta}_0^2(dy)$, $\hat{\Theta}_0^2 \in \mathcal{M}_1(E_Y)$, and $Y := \gamma(X)$, then Y is a solution of $\boldsymbol{MP}(\boldsymbol{C}, \hat{\Theta}_0)$ and it holds

$$\mathfrak{L}(X_t|\mathcal{F}_t^Y) = \alpha(Y_t).$$

Further the martingale problem $\boldsymbol{MP}(\boldsymbol{C})$ is well-posed and $y \mapsto \mathcal{Q}_Y^y$ is measurable, where $\mathcal{Q}_Y^y \in \mathcal{M}_1(E_Y^{[0,\infty)})$ is the distribution of the solution of $\boldsymbol{MP}(\boldsymbol{C}, \delta_y)$.

Proof. If $\mathcal{P}_X^x, x \in E_X$, is the law of the solution $\mathbf{MP}(\mathbf{A}, \delta_x)$, then due to the fact that $\mathbf{A}_0$ from the Conditions D.1.8 is graph-separable and the Theorem 4.4.6 in [14], it follows that the map $x \mapsto \mathcal{P}_X^x$ is measurable. If we combine this fact with the fact that $\mathbf{MP}(\mathbf{A})$ is well-posed, the Theorem 4.4.2 c) tells us that every solution is a strong Markov process. Let us assume that $(P_t, t \geq 0)$ is the semigroup associated with $(\mathcal{P}_X^x, x \in E_X)$ and note that the function $P_t(g)$ is measurable for all $g \in \mathrm{B}(E)$ and $t \geq 0$.
Let us define the collection of maps $(Q_t, t \geq 0)$ by setting

$$Q(f)(\tilde{y}) := \mathbb{E}_{\tilde{y}}[f(\tilde{Y}(t))], \quad f \in C_b(E_Y), t \geq 0, \tilde{y} \in E_Y,$$

where $\tilde{Y}$ is the solution of $\mathbf{MP}(\mathbf{C}, \delta_{\tilde{y}})$. Now let us assume that $y \in E_Y$ and Y be a solution of the martingale problem $\mathbf{MP}(\mathbf{C}, \delta_y)$. Since $\alpha^*(f) \in \mathcal{D}(\mathbf{C})$ for $f \in \mathcal{D}(\mathbf{A})$, the process

$$M_f(t) = \alpha^*(f)(Y_t) - \alpha^*(f)(Y_0) - \int_0^t \mathbf{C} \circ \alpha^*(f)(Y_s)ds$$

is a martingale and hence it holds for all $t \geq 0$:

$$\mathbb{E}_y\left[\alpha^*(f)(Y_t)\right] = \mathbb{E}_y\left[\alpha^*(f)(Y_0)\right] + \int_0^t \mathbb{E}_y\left[\mathbf{C} \circ \alpha^*(f)(Y_s)\right] ds,$$

where we have written $\mathbb{E}_y$ for the expectation with respect to $\mathbb{P}_y$. Using $\alpha^* \circ \mathbf{A}(f) = \mathbf{C} \circ \alpha^*(f)$ this turns into

$$\mathbb{E}_y\left[\alpha^*(f)(Y_t)\right] = \mathbb{E}_y\left[\alpha^*(f)(Y_0)\right] + \int_0^t \mathbb{E}_y\left[\alpha^* \circ \mathbf{A}(f)(Y_s)\right] ds.$$

Note that this is true for all $f \in \mathcal{D}(\mathbf{A})$ If we use the fact that $\alpha^*(C_b(E_X))$ is separating to define the measure-valued map $\mu^y : [0, \infty) \to \mathcal{M}_1(E_X)$ by setting

$$\mu_t^y(g) = \mathbb{E}_y[\alpha^*(g)(Y_t)], \quad g \in C_b(E_X),$$

then μ^y is a solution for the forward equation $\mathbf{FE}(\mathbf{A}, \alpha(y))$. According to Lemma D.1.8 there exists a solution X to the local martingale problem $\mathbf{MP}(\mathbf{A}, \alpha(y))$ which is a proper martingale with

$$X_t \sim \mu_t^y. \tag{D.6}$$

On the other hand the martingale problem of $\mathbf{MP}(\mathbf{A}, \alpha(y))$ is well-posed with the unique solution given by $\int_{E_X} \mathcal{P}_X^x(\cdot)\alpha(y, dx) \in \mathcal{M}_1(\mathbb{D}([0, \infty), E))$. Therefore, if we define $\tilde{\nu}_t^y \in \mathcal{M}_1(E_X)$ by setting $\tilde{\nu}_t^y(g) = \alpha^* \circ P_t(g)(y)$, $g \in C_b(E_X), t \geq 0$, then it must also holds

$$X_t \sim \tilde{\nu}_t^y. \tag{D.7}$$

We can conclude from the identities (D.6) and (D.7) that

$$Q_t \circ \alpha(g)(y) = \alpha_* \circ P_t(g)(y), \quad g \in C_b(E_X), t \geq 0. \tag{D.8}$$

Since y has been arbitrary it follows that $\alpha^* \circ P_t = Q_t \circ \alpha^*$ for all $t \geq 0$. From this we can also conclude that Q_t is measurable, because $P_t(g)$ for all $g \in \mathcal{D}(\mathbf{A})$ is measurable, $\alpha^*(\mathcal{D}(\mathbf{A}))$ separable for $\mathcal{M}_f(E_Y)$, and $\mathbf{A}$ is graph separable. Let us now fix $\hat{\Theta}_0 \in \mathcal{M}_1(E_Y)$. If X is a solution of $\mathbf{MP}(\mathbf{A}, \alpha(y))$, $Y = \gamma(X)$ and

$$\tilde{Q}_t := \alpha^* \circ P_t \circ \gamma_*,$$

then $\tilde{Q}_t = Q_t$,$t \geq 0$, by (D.8), where we used $\alpha^* \circ \gamma_* = \mathbf{Id}_Y$ and that $\alpha^*(C_b(E_X))$ is separating. Hence by the Rogers-Pitman version of the Markov mapping theorem, see Proposition $D.1.3$, Y is a Markov process with Markov semigroup $(Q_t, t \geq 0)$, and it also holds

$$\mathfrak{L}(X_t | \mathcal{F}_t^Y) = \alpha(Y_t). \tag{D.9}$$

If $\tilde{Y}$ is another solution of $\mathbf{MP}(\mathbf{C}, \hat{\Theta}_0)$ with $\int_0^t \mathbb{E}[\alpha^*(\psi)(Y_s)]ds < \infty$,$t \geq 0$, then we can repeat the above argumentation to conclude that for all $g \in \mathcal{D}(\mathbf{A})$

$$\mathbb{E}[\alpha^*(g)(\tilde{Y}_t)] = \int_{E_Y} \alpha_* \circ P_t(g)(y)\hat{\Theta}_0(dy) = \int_{E_Y} Q_t \circ \alpha(g)(y)\hat{\Theta}_0(dy).$$

Since $\alpha^*(\mathcal{D}(\mathbf{A}))$ is separating, it follows that $\tilde{Y}$ and Y have the one-dimensional distribution and by Lemma D.1.12 they have the same finite dimensional distributions, hence $\mathbf{MP}(\mathbf{C}, \hat{\Theta}_0)$ is well-posed in regard to the solutions with $\int_0^t \mathbb{E}[\alpha^*(\psi)(Y_s)]ds < \infty$,$t \geq 0$. Since Y is a Markov process with semigroup $(Q_t, t \geq 0)$ and since Q_t is measurable, it follows that the map $y \mapsto \mathcal{Q}_Y^y$ is measurable. □

Remark D.1.14. *Comparing our formulation of the Markov mapping theorem with the one found in [30], we observe that the proof of our version is shorter. This is due to the fact that we assumed that the martingale problem* $\boldsymbol{MP}(\boldsymbol{A})$ *is well-posed, implying that* X *is a Markov process, which in turn allows to apply the Rogers-Pitman version of the Markov mapping theorem, see Theorem D.1.3. Kurtz in [30] does not assume that there exists a solution to the martingale problem* $\boldsymbol{MP}(\boldsymbol{A})$ *beforehand, instead he constructs a solution based on the assumption that* $\boldsymbol{MP}(\boldsymbol{C})$ *admits a solution satisfying* $\int_0^t \mathbb{E}[\alpha^*(\psi)(\tilde{Y}_s)]\, ds < \infty$*. Leaving out some technical details this is together with a weaker version of the Assumptions D.1.8 enough for him to show that it is possible to construct a pair* (X, Y)*, where* Y *is solution of* $\boldsymbol{MP}(\boldsymbol{C})$*,* X *is a solution of* $\boldsymbol{MP}(\boldsymbol{A})$ *and we have*

$$\mathfrak{L}(X_t | Y_s, 0 \leq s \leq t) = \alpha(Y_t).$$

He needs not to assume that X *or* Y *are Markov processes or that their martingale problem is well-posed. But if one applies this Markov mapping theorem to a process* $\tilde{X}$ *which is a solution of* $\boldsymbol{MP}(\boldsymbol{A})$*, but* $\boldsymbol{MP}(\boldsymbol{A})$ *is not well-posed, then it is hard to justify why Kurtz' process* X *has the same law as* $\tilde{X}$*.*

In Remark (vii) on Page 575 of the paper [43] Rogers and Pitman mentioned that under suitable topological assumptions one can derive a strong version of (D.1), in the sense that the fixed time point t is replaced by a $\mathcal{F}^Y$-stopping time τ, i.e. it holds

$$\mathfrak{L}(X_\tau | \mathcal{F}_\tau^Y) = \alpha(Y_\tau)$$

They hinted that X being a Feller process and that α and γ being continuous should suffice, but they neither formulate a precise statement nor a proof. Here, we formulate a version which is sufficient for us. As a short reminder: We call a filtration $\mathcal{F}$ right-continuous, if it holds $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $s \in [0, \infty)$, where $\mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$. We call a stopping time τ finite, if $\mathbb{P}[\tau < \infty] = 1$.

Lemma D.1.15. *Let $\alpha : E_Y \to \mathcal{M}_1(E_X)$ be a Markov kernel. Let us assume that X is a process with state space E_X and Y is a process with state space E_Y, both defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $\mathcal{F}$ be a filtration such that Y is adapted to $\mathcal{F}$ and $\hat{\mathcal{F}}$ be its completion with respect to $\mathbb{P}$. Let us assume that $\mathfrak{L}(X_t|\mathcal{F}_t) = \alpha(Y_t)$ for all fixed $t \in [0, \infty)$, then it holds for all finite $\hat{\mathcal{F}}$-stopping times τ with countably many values:*

$$\mathfrak{L}(X_\tau|\hat{\mathcal{F}}_\tau) = \alpha(Y_\tau). \tag{D.10}$$

If $X, Y, \hat{\mathcal{F}}$ are right-continuous and α is a continuous Markov kernel, then (D.10) *is true for all finite $\hat{\mathcal{F}}$-stopping times τ.*

Remark D.1.16. *It is important to keep in mind that* (D.10) *will in general not be true for stopping times τ that are adapted to the filtration of X or any bigger filtration.*

Proof of Lemma D.1.15. Since $\hat{\mathcal{F}}$ differs from $\mathcal{F}$ only by $\mathbb{P}$-nullsets, it also holds $\mathfrak{L}(X_t|\hat{\mathcal{F}}^Y_t) = \alpha(Y_t)$ for all $t \in [0, \infty)$. In order to show that $\mathfrak{L}(X_\tau|\hat{\mathcal{F}}_\tau) = \alpha(Y_\tau)$ for a fixed finite $\hat{\mathcal{F}}$-stopping time τ, we need to prove for all $f \in C_b(E_X)$ that for the conditional expectations holds:

$$\mathbb{E}[f(X_\tau)|\hat{\mathcal{F}}_\tau] = \alpha(Y_\tau)(f) := \int_{E_X} f(x)\alpha(Y_\tau, dx)$$

We fix $f \in C_b(E_X)$ for the rest of this proof and begin by assuming that τ has countable many values, i.e. there exits $(t_i, i \in \mathbb{N}) \supset [0, \infty)$ and $(\Gamma_i, i \in \mathbb{N}) \subset \hat{\mathcal{F}}$ with $\Gamma_i \in \hat{\mathcal{F}}_{t_i}, i \in \mathbb{N}$ and $\Gamma_i \cap \Gamma_j \neq \emptyset$ such that $\tau = \sum_{i=1}^\infty t_i \mathbb{1}_{\Gamma_i}$. Using the dominated convergence theorem for conditional expectations, see theorem 8.14.(viii) in [24], we get

$$\mathbb{E}[f(X_\tau)|\hat{\mathcal{F}}_\tau] = \sum_{i=1}^\infty \mathbb{E}[f(X_{t_i})\mathbb{1}_{\Gamma_i}|\hat{\mathcal{F}}_\tau] = \sum_{i=1}^\infty \mathbb{E}[f(X_{t_i})|\hat{\mathcal{F}}_{t_i}]\mathbb{1}_{\Gamma_i},$$

where we used for the second equality that $\mathbb{E}[f(X_{t_i})\mathbb{1}_{\Gamma_i}|\mathcal{F}_\tau] = \mathbb{E}[f(X_{t_i})\mathbb{1}_{\Gamma_i}|\mathcal{F}_{t_i}]$ almost surely by the local property of the conditional expectation, see Lemma 6.2 in [21], and Lemma 7.1.(ii) in [21]. Hence we can conclude

$$\mathbb{E}[f(X_\tau)|\hat{\mathcal{F}}_\tau] = \sum_{i=1}^\infty \alpha(Y_{t_i})(f)\mathbb{1}_{\Gamma_i} = \alpha(Y_\tau)(f).$$

Now let us assume that $\hat{\mathcal{F}}$ is right-continuous, α is continuous and that τ is an arbitrary finite $\hat{\mathcal{F}}$-stopping time. By Lemma 7.4 in [21] there exists a sequence of countable valued stopping times $(\tau_k, k \in \mathbb{N})$ such that $\tau_k \downarrow \tau$. Using the triangle inequality and Jensen inequality we get for each $k \in \mathbb{N}$:

$$\begin{aligned}\mathbb{E}\left[\left|\mathbb{E}\left[f(X_\tau)|\hat{\mathcal{F}}_\tau\right] - \alpha(Y_\tau)(f)\right|\right] \leq\ & \mathbb{E}\left[\left|\mathbb{E}\left[f(X_\tau)|\hat{\mathcal{F}}_\tau\right] - \mathbb{E}\left[f(X_\tau)|\hat{\mathcal{F}}_{\tau_k}\right]\right|\right] \\ &+ \mathbb{E}\left[\mathbb{E}\left[|f(X_\tau) - f(X_{\tau_k})|\,|\hat{\mathcal{F}}_{\tau_k}\right]\right] \\ &+ \mathbb{E}\left[\left|\mathbb{E}\left[f(X_{\tau_k})|\hat{\mathcal{F}}_{\tau_k}\right] - \alpha(Y_\tau)(f)\right|\right].\end{aligned}$$

We will prove that the three expression on the right side converge against 0 for $k \to \infty$. In the first line we use that $\hat{\mathcal{F}}$ is right-continuous. Since $\tau_k \geq \tau$, it holds $\hat{\mathcal{F}}_{\tau_k} \supset \hat{\mathcal{F}}_\tau$, hence $(\mathbb{E}[f(X_\tau)|\hat{\mathcal{F}}_{\tau_k}], k \in \mathbb{N})$ forms a uniform integrable backwards martingale, hence $\mathbb{E}[f(X_\tau)|\hat{\mathcal{F}}_{\tau_k}] \to \mathbb{E}[f(X_\tau)| \cap_{k\in\mathbb{N}} \hat{\mathcal{F}}_{\tau_k}] = \mathbb{E}[f(X_\tau)|\hat{\mathcal{F}}_\tau]$ in $L^1(\mathbb{P})$, since $\hat{\mathcal{F}}$ is right-continuous. In the second line we use that X is càdlàg. Indeed we have

$$\mathbb{E}\left[\mathbb{E}\left[|f(X_\tau) - f(X_{\tau_k})|\,|\hat{\mathcal{F}}_{\tau_k}\right]\right] = \mathbb{E}[|f(X_\tau) - f(X_{\tau_k})|] \overset{k\to\infty}{\to} 0,$$

because X is càdlàg and $f \in C_b(E_X)$. In the third line we use that Y is càdlàg. We recall that $\mathbb{E}[f(X_{\tau_k})|\hat{\mathcal{F}}_{\tau_k}] = \alpha(Y_{\tau_k})$, because τ_k is discrete. So by the right-continuity of Y, the continuity of the Markov kernel α and the fact that $\alpha^*(f) \in C_b(E_Y)$, we have that $\alpha(Y_{\tau_k})(f) \to \alpha(Y_\tau)(f)$ in $L^1(\mathbb{P})$. All in all, it follows that

$$\mathbb{E}\left[\left|\mathbb{E}\left[f(X_\tau)|\hat{\mathcal{F}}_\tau\right] - \alpha(Y_\tau)(f)\right|\right] = 0$$

and so $\mathbb{E}\left[f(X_\tau)|\hat{\mathcal{F}}_\tau\right] = \alpha(Y_\tau)$ almost surely. □

D.2 Conditional Martingale Lemma

In this section we prove the conditional martingale lemma which is used to derive the Cases III.a, III.b and IV from the Cases II.a and II.b, in Chapter 4. This lemma can be found as Lemma A.13 in [32], but we added some additional conditions to make the processes well-defined.

Lemma D.2.1 (Conditional Martingale Lemma)**.** *Let us assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space, $\mathcal{F} := (\mathcal{F}_t, t \geq 0)$ is a filtration contained in $\mathcal{A}$ and that $P, V : \Omega \times [0,\infty) \to \mathbb{R}$ are $\mathcal{F}$-adapted càdlàg processes with*

$$\mathbb{E}[|P(t)|] + \int_0^t \mathbb{E}[|V(s)|]ds < \infty,\ \mathbb{E}[|V(t)|] < \infty, \quad t \geq 0, \tag{D.11}$$

and for which

$$M(t) := P(t) - \int_0^t V(s)\,ds, \quad t \geq 0,$$

is a $\mathcal{F}$-martingale. If $\mathcal{G} := (\mathcal{G}_t, t \geq 0)$ is a second filtration with $\mathcal{G}_t \subset \mathcal{F}_t$, $t \geq 0$, and there exists a $\mathcal{G}$-progressively measurable process $Q : \Omega \times [0,\infty) \to \mathbb{R}$ with $Q(t) = \mathbb{E}[V(t)|\mathcal{G}_t]$, then

$$\hat{M}(t) := \mathbb{E}[P(t)|\mathcal{G}_t] - \int_0^t Q(s)\,ds, \quad t \geq 0,$$

is a $\mathcal{G}$-martingale.

Remark D.2.2. *The formulation of Lemma D.2.1 found in [32] makes besides of $\mathbb{E}[|P(t)|] + \int_0^t \mathbb{E}[|V(s)|]ds < \infty$, $t \geq 0$, and the condition that M is a martingale, no further assumptions about P and V. Since the proof in [32] does not explain, why $(\omega, t) \mapsto \mathbb{E}[V(t)|\mathcal{G}_t](\omega)$ should be $\mathcal{G}$-progressively measurable, which appears to us to be necessary to ensure that $\hat{M}$ is $\mathcal{G}$-adapted process and also not explains, why $\mathbb{E}[V(t)|\mathcal{G}_t]$ should be well-defined for all $t \geq 0$, we added some reasonable extra conditions to the processes involved in Lemma D.2.1 to ensure that everything is well-defined.*

Proof of Lemma D.2.1. We have $\mathbb{E}[\hat{M}(t)] \in L^1(\mathbb{P})$ due to $\mathbb{E}[|P(t)|] + \int_0^t \mathbb{E}[|V(s)|]ds < \infty$. Since Q is a $\mathcal{F}$-progressively measurable process, it follows from Fubini's theorem that $t \mapsto \int_0^t Q(s)ds$ is $\mathcal{G}$-adapted, making $\hat{M}$ a $\mathcal{G}$-adapted process. Let us assume that $\epsilon > 0$ and $\Gamma \in \mathcal{G}_t \subset \mathcal{F}_t$, then it holds

$$\begin{aligned}
\mathbb{E}[(\hat{M}(t+\epsilon) - \hat{M}(t))\mathbb{1}_\Gamma] &= \mathbb{E}\left[\left(\mathbb{E}[P(t+\epsilon)|\mathcal{G}_{t+\epsilon}] - \mathbb{E}[P(t)|\mathcal{G}_t] - \int_t^{t+\epsilon} Q(s)\,ds\right)\mathbb{1}_\Gamma\right] \\
&= \mathbb{E}[\mathbb{E}[P(t+\epsilon)\mathbb{1}_\Gamma|\mathcal{G}_{t+\epsilon}]] - \mathbb{E}[\mathbb{E}[P(t)\mathbb{1}_\Gamma|\mathcal{G}_t]] - \int_t^{t+\epsilon} \mathbb{E}[\mathbb{E}[V(s)\mathbb{1}_\Gamma|\mathcal{G}_s]]\,ds \\
&= \mathbb{E}[P(t+\epsilon)\mathbb{1}_\Gamma] - \mathbb{E}[P(t)\mathbb{1}_\Gamma] - \int_t^{t+\epsilon} \mathbb{E}[V(s)\mathbb{1}_\Gamma]\,ds \\
&= \mathbb{E}[\mathbb{E}[P(t+\epsilon)\mathbb{1}_\Gamma|\mathcal{F}_t]] - \mathbb{E}[\mathbb{E}[P(t)\mathbb{1}_\Gamma|\mathcal{F}_t]] - \int_t^{t+\epsilon} \mathbb{E}[\mathbb{E}[V(s)\mathbb{1}_\Gamma|\mathcal{F}_t]]\,ds \\
&= \mathbb{E}\left[\mathbb{E}\left[P(t+\epsilon) - P(t) - \int_t^{t+\epsilon} V(s)\,ds\middle|\mathcal{F}_t\right]\mathbb{1}_\Gamma\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[(M(t+\epsilon) - M(t))|\mathcal{F}_t\right]\mathbb{1}_\Gamma\right] = 0.
\end{aligned}$$

Consequently $\mathbb{E}[\hat{M}(t+\epsilon)|\mathcal{G}_t] = \mathbb{E}[\hat{M}_t|\mathcal{G}_t] = \hat{M}_t$ and so we know that $\hat{M}$ is a $\mathcal{G}$-martingale. □

Corollary D.2.3. *Let us assume that the processes $P, V, Q, M, \hat{M}$ and the filtrations $\mathcal{F}, \mathcal{G}$ are defined as in D.2.1, but instead of the integrable conditions* (D.11) *we have*

$$\mathbb{E}[|P(t \wedge T_n)|] + \int_0^t \mathbb{E}[|V(s \wedge T_n)|]ds < \infty, \mathbb{E}[|V(t \wedge T_n)|] < \infty, \quad t \geq 0,\ n \in \mathbb{N},$$

where $(T_n, n \in \mathbb{N})$ is a localizing sequence of $\mathcal{G}$-stopping times, i.e. $T_n \to \infty$, when n goes to infinity, and $T_n \leq T_{n+1}$ for all $n \in \mathbb{N}$. Further M is "just" a local $\mathcal{F}$-martingale with localizing sequence $(T_n, n \in \mathbb{N})$. Then $\hat{M}$ is a local $\mathcal{G}$-martingale with localizing sequence $(T_n, n \in \mathbb{N})$.

Proof. We just apply D.2.1 to the stopped processes $P(\cdot \wedge T_n), V(\cdot \wedge T_n), Q(\cdot \wedge T_n), M(\cdot \wedge T_n)$ and $\hat{M}(\cdot \wedge T_n)$. □

Appendix E

Path-valued Processes

E.1 Lévy Processes

The spatial motion of our particles are either Lévy processes or the path-valued processes associated with a Lévy process. The application of the Markov mapping theorem requires that we can characterize the spatial motion processes as the solution of martingale problem associated with a linear operator satisfying some regularity conditions. We will do this for the path-valued process in the next section, the goal of the current sections is to present facts about Lévy processes, which will become useful in the next section and other places. We begin with the definition of a Lévy process.

Definition E.1.1. *Let us assume that $(\Omega, \mathbb{F}, \mathbb{P})$ is a probability space and $W : \Omega \times [0, \infty) \to \mathbb{R}^m$ is a stochastic process. We call W a Lévy process starting in $x \in \mathbb{R}^m$, when*

1. $\mathbb{P}[W_0 = x] = 1$.
2. *W has independent and stationary increments.*
3. *W is continuous in probability, indeed for all $a > 0$ and for all $t \geq 0$ holds that*

$$\lim_{s \to t} \mathbb{P}[|W(t) - W(s)| > a] = 0.$$

This definition is taken from the book *Lévy Processes and Stochastic Calculus* by David Applebaum, see the Section 1.3 in [2], but we took the freedom and allowed the Lévy process to start from a different value than the origin. We start with a regularization lemma.

Lemma E.1.2. *Every Lévy process admits a càdlàg modification.*

Proof. The existence of the càdlàg modification follows from the Theorem 13.1 in [21] and the continuity in probability. □

Our next step is the introduction of the Lévy-Itô-decomposition, which tells us that every Lévy process can be written as the sum of a drift, a Brownian and Poisson component.

Definition E.1.3. *We say that $(B^\rho_W, B^{cov}_W, B^\eta_W)$ is a characteristic triple (we also call it a Lévy triple), when $B^\rho_W = (B^\rho_W(k))_{k=1}^m \in \mathbb{R}^m$, $B^{cov}_W = (B^{cov}_W(k,l))_{k,l=1}^m \in \mathbb{R}^{m \times m}$ is a symmetric, positive semidefinite matrix and $B^\eta_W \in \mathcal{M}(\mathbb{R}^m)$ is a Lévy measure, indeed B^η_W satisfies $\int_{\mathbb{R}^m} 1 \wedge ||x||^2 \, B^\eta_W(dx) < \infty$.*

Remark E.1.4. *Sometimes the condition* $\int_{\mathbb{R}^m} 1 \wedge ||x||^2 \, B_W^\eta(dx) < \infty$ *is stated as*

$$\int_{\mathbb{R}^m} ||x||^2(1+||x||^2)^{-1} \, B_W^\eta(x),$$

which is equivalent, because $||x||^2(1+||x||^2)^{-1} \leq 1 \wedge ||x||^2 \leq 2||x||^2(1+||x||^2)^{-1}$. *Both variation can be often found in the literature.*

Proposition E.1.5 (Lévy-Itô decomposition)**.** *If W is a Lévy process and $\widehat{W}$ is its càdlàg modification, then there exists a Lévy-triple* $(B_W^\rho, B_W^{cov}, B_W^\eta)$, *a m-dimensional Brownian motion W^c and a Poisson Point process W^J over $\mathbb{R}^m \times [0,\infty)$ with intensity measure $B_W^\eta \otimes \ell eb[0,\infty)$, such that W^c and W^J are independent, adapted to the filtration of $\widehat{W}$ (hence they are adapted to completion of the filtration of W) and it holds almost surely for all $t \geq 0$ simultaneously*

$$\widehat{W}(t) = B_W^\rho t + (B_W^{cov})^{\frac{1}{2}} W^c(t) + \int_0^t \int_{\{||y||>1\}^c} y \, W^J(dy,ds) + \int_0^t \int_{\{||y||\leq 1\}} y \, \overline{W}^J(dy,ds) \tag{E.1}$$

where $(B_W^{cov})^{\frac{1}{2}} \in \mathbb{R}^{m\times m}$ *is the unique positive, semidefinite matrix with* $(B_W^{cov})^{\frac{1}{2}}(B_W^{cov})^{\frac{1}{2}} = B_W^{cov}$ *and* $\overline{W}^J$ *is the compensated Poisson process* W^J *with* $\overline{W}^J(dy,ds) = W^J(dy,ds) - B_W^\eta(dy,ds)$.

Proof. This Lévy-Itô decomposition is proven in Theorem 2.4.16 in [2] and by Corollary 2.4.21. □

From the Lévy-Itô we can conclude that the finite dimensional distribution of a Lévy process is uniquely specified by the characteristic triple. Further, since every summand on the right-hand side of (E.1) is a Lévy process by itself and since the class of Lévy processes is closed under addition, we can conclude there exists for every characteristic triple a Lévy process.

Definition E.1.6. *We say that W is a Lévy process with characteristic triple* $(B_W^\rho, B_W^{cov}, B_W^\eta)$, *when* $(B_W^\rho, B_W^{cov}, B_W^\eta)$ *is the triple from the Lévy-Itô-decomposition of W.*

Proposition E.1.7.

1. *Let us assume that W is a Lévy process with Lévy triple* $(B_W^\rho, B_W^{cov}, B_W^\eta)$ *starting in $0 \in \mathbb{R}^m$. If we define $P_t : C_0(\mathbb{R}^m) \to C_0(\mathbb{R}^m)$ for $t \geq 0$ by setting for each $f \in C_0(\mathbb{R}^m)$*

$$P_t(f)(x) := \mathbb{E}\left[f(x+W(t))\right],$$

   then $(P_t, t \geq 0)$ is the Markov semigroup of W and $(P_t, t\geq 0)$ forms a Feller semigroup.

2. *If we define $(P^x, x \in \mathbb{R}^m) \subset \mathcal{M}_1(\mathbb{D}([0,\infty),\mathbb{R}^m))$ by setting P^x for $x \in \mathbb{R}^m$ to be the law of W^x, which is a copy of W starting in $x \in \mathbb{R}^m$, then $(P^x, x \in \mathbb{R}^m) \subset \mathcal{M}_1(\mathbb{D}([0,\infty),\mathbb{R}^m))$ forms a continuous strong Markov family.*

Proof. A Lévy process is a Feller process by Theorem 3.1.9 from [2] and by the definition of a Feller process the corresponding semigroup $(P_t, t \geq 0)$ is a Feller semigroup, see Page 150 in [2]. Now we prove that $(P^x, x \in \mathbb{R}^m)$ is a continuous strong Markov family. If $W_x := x + W_0$, where W is a copy of W starting in 0, then $W_x \sim P_x$, and that if $(x_n, n \in \mathbb{N})$ is converging against x, then $||W_{x_n} - W_x||_\infty \to 0$ almost surely. □

Proposition E.1.8. *Let us define $B_W : C_b\mathbb{R}^m \subset \mathcal{D}(B) \to C_0(\mathbb{R}^m)$ as the weak generator of $(P^x, x \in \mathbb{R}^m)$, see Definition A.2.3, then $dom(B_W) \subset \mathtt{span}(C_c^2(\mathbb{R}^m) \cup \{\mathbb{1}_{\mathbb{R}^m}\})$ and if holds $g \in C_b(\mathbb{R}^m)$ by*

$$\begin{aligned} B_W(g)(x) = (B_W^\rho)^T \nabla g(x) + \nabla^T B_W^{cov} \nabla g(x) \\ + \int_{\mathbb{R}^m} g(x+y) - g(x) - \mathbb{1}_{\{||y||\leq 1\}}\big(y^T \nabla g(x)\big) B_W^\eta(dy), \quad x \in \mathbb{R}^d, \end{aligned}$$

with $(B_W^\rho)^T \nabla g(x) = \sum_{k=1}^m B_W^\rho(k)\partial_{x_k} g(x)$ *and* $\nabla^T B_W^{cov} \nabla g(x) = \sum_{k,l=1}^m B_W^{cov}(k,l)\partial^2_{x_k x_l} g(x)$. *If* W_0 *is a random variable with* $W_0 \sim \mu$, *where* $\mu \in \mathcal{M}_1(\mathbb{R}^m)$, *then the process* W_μ *given by* $W_\mu(t) = W_0 + W(t)$ *is a solution of the martingale problem* $\boldsymbol{MP}(B, \mu)$.

Proof. This follows from Proposition 4.1.7. in [14] and Theorem 3.3.3 from [2]. □

Remark E.1.9. *In the book [14] the authors Ethier and Kurtz use* B_W *given by*

$$\begin{aligned} B_W(g)(x) =& (\tilde{B}_W^\rho)^T \nabla g(x) + \nabla^T B_W^{cov} \nabla g(x) \\ &+ \int_{\mathbb{R}^d} g(x+y) - g(x) - \frac{y^T \nabla g(x)}{1 + ||y||^2} B_W^\eta(dy), \quad x \in \mathbb{R}^m. \end{aligned}$$

The differences between (E.2) *and* (E.2) *are the drift constant and the compensator of the Poisson part. Both expression are identical, because it holds*

$$B_W^\rho - \tilde{B}_W^\rho = \int_{\mathbb{R}^d} \left(\frac{y}{1 + ||y||^2} - \mathbb{1}_{\{||y|| \leq 1\}} y \right) B_W^\eta(dy).$$

We think that (E.2) *is aesthetically more pleasant, but* (E.2) *one makes the application of Taylors approximation theorem to the Poisson part more straightforward, as we do in the proof of Lemma E.1.11.*

Proposition E.1.10. *Let us denote by* $\overline{B}_W$ *the restriction of* B_W *on* $\boldsymbol{span}(C_c^2(\mathbb{R}^m) \cup \{\mathbb{1}_{\mathbb{R}^m}\})$, *i.e.* $\overline{B}_W := \{(f, B_W(f));\ f \in \boldsymbol{span}(C_c^2(\mathbb{R}^m) \cup \{\mathbb{1}_{\mathbb{R}^m}\})\}$. *The martingale problem of the operator* B_W *is well-posed, indeed if* W *is a solution of the martingale problem of* B_W, *then* W *is a Lévy-process with Levy-triple* $(B_W^\rho, B_W^{cov}, B_W^\eta)$ *and admits càdlàg modification.*

Proof. If we define the set $B_{W|C_c^\infty(\mathbb{R}^m)} := \{(f, B_W(f));\ f \in C_c^\infty(\mathbb{R}^m)\}$, then the martingale problem $\mathbf{MP}(B_{W|C_c^\infty(\mathbb{R}^m)}, \mu)$ is well-posed for any probability measure $\mu \in \mathcal{M}_1(E)$ according to the Theorem 8.3.3 from [14]. By Proposition E.1.8 we know that the solution is given by the Lévy process W, if $W \sim \mu$.
Now, let us assume that $\overline{W}$ is an arbitrary solution of the martingale problem $\mathbf{MP}(\overline{B}_W, \mu)$. Then W is also a solution of $\mathbf{MP}(B_{W|C_c^\infty(\mathbb{R}^m)}, \mu)$, because $B_{W|C_c^\infty(\mathbb{R}^m)} \subset \overline{B}_W$. By the uniqueness of $\mathbf{MP}(B_{W|C_c^\infty(\mathbb{R}^m)}, \mu)$ it follows that $\overline{W}$ and W have the same finite dimensional distributions. Hence $\overline{W}$ is a Lévy process with triple $(B_W^\rho, B_W^{cov}, B_W^\eta)$ and has a càdlàg modification by Lemma E.1.2. □

An important prerequisite for the Markov mapping theorem is that the operator of the martingale problem is graph separable. This conditions is mild, in sense that many possible operators/generators satisfy this condition, the proof that the conditions is satisfied is also not very difficult in our case, but very technical.

Lemma E.1.11. *The restriction* $\overline{B}_W$ *of* B_W *on* $\boldsymbol{span}(C_c^2(\mathbb{R}^m) \cup \{\mathbb{1}_{\mathbb{R}^m}\})$ *is strong graph separable, indeed there exists a countable collection* $\tilde{B} \subset \overline{B}_W$ *(here we interpret* $\overline{B}_W$ *as a subset of* $B(\mathbb{R}^m) \times B(\mathbb{R}^m)$*) with the property that for any element* $(f, g) \in \overline{B}_W$ *we can find a sequence* $(f_n, g_n)_{n=1}^\infty$ *contained in* $\tilde{B}$ *such that* $f_n \to f$ *and* $g_n \to g$ *uniformly.*

Proof. We pick a countable family $\tilde{C}^2(\mathbb{R}^m)$ that is dense in $\boldsymbol{span}(C_c^2(\mathbb{R}^m) \cup \{\mathbb{1}_{\mathbb{R}^m}\})$ with respect to the norm

$$||f||_{\infty,2} := ||f||_\infty + \sum_{i=1}^m ||\partial_{x_i} f||_\infty + \sum_{i,j=1}^m ||\partial_{x_i x_j} f||_\infty. \tag{E.2}$$

We claim, that if we set $\tilde{B} := \{(f, B_W(f)); f \in \tilde{C}^2(\mathbb{R}^m)\}$, then the set $\tilde{B}$ has the desired properties.
Let us fix an arbitrary pair $(f, B_W(f)) \in \overline{B}_W$. Since $\tilde{C}^2(\mathbb{R}^m)$ is dense with respect to the norm $||\cdot||_{\infty,2}$, there must exists a sequence $(f_n)_{n=1}^{\infty}$ in $\tilde{C}^2(\mathbb{R}^m)$ such that $||f - f_n||_{\infty,2} \to 0$. It remains to show that $B_W(f_n) \to B_W(f)$ uniformly. During this proof we will write $f_\infty = f$. It holds that $B_W(f)(x) - B_W(f)(x), x \in \mathbb{R}^m$ is equal to:

$$(B^\rho)^T \nabla f(x) + \nabla^T B^{cov} \nabla f(x) - (B^\rho)^T \nabla f_n(x) - \nabla^T B^{cov} \nabla f_n(x) \tag{E.3}$$

$$+ \int_{\{||y||>1\}} f_n^\Delta(x+y) - f_n^\Delta(x) B_W^\eta(dy), \tag{E.4}$$

$$+ \int_{\{||y||\leq 1\}} f_n^\Delta(x+y) - f_n^\Delta(x) - y^T \nabla (f_n^\Delta(x) B_W^\eta(dy), \tag{E.5}$$

where $f_n^\Delta := f - f_n$. The uniform convergence for the drift and Brownian part, see (E.3), follows immediately from the convergence with respect to $||\cdot||_{\infty,2}$. For (E.4) we note that B_W^η is a Lévy measure, i.e. $B_W(\mathbb{R}^m \setminus \{||y|| \leq 1\}) < \infty$, and so:

$$\sup_{x\in\mathbb{R}^m} \left| \int_{\{||y||>1\}} f_n^\Delta(x+y) - f_n^\Delta(x) B_W^\eta(dy) \right| \leq 2B_W(\{||y|| > 1\}) ||f_n^\Delta||_{\infty,2} \overset{n\to\infty}{\longrightarrow} 0.$$

For (E.5), let us define $R^n : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ by

$$R^n(\tilde{x}, \tilde{y}) = \frac{1}{2} \sum_{i,j} \partial_{x_i} \partial_{x_j} f_n^\Delta(\tilde{x}, \tilde{y})(\tilde{x}_i - \tilde{y}_i)(\tilde{x}_j - \tilde{y}_j).$$

By applying the multidimensional Taylor formula, see “Taylorformel mit Rest” on Page 65 in [26], it holds for all $\tilde{x}, \tilde{y} \in \mathbb{R}^m$:

$$f_n^\Delta(x+y) - f_n^\Delta(x) - y^T \nabla f_n^\Delta(x) = R^n(x, x + t^n(x,y)y),$$

where $t^n : \mathbb{R}^m \times \mathbb{R}^m \to [0,1]$ is the function implicit implied by the mean value theorem. From $||f_n^\Delta||_{\infty,2} \to 0$, we can conclude that $||R^n|| \to 0$, therefore the integral (E.5) converges uniformly against 0, because $\int_{\{||y||\leq 1\}} ||y||^2 B_W^\eta(dy) < \infty$ and of

$$\begin{aligned}
&\int_{\{||y||\leq 1\}} f_n^\Delta(x+y) - f_n^\Delta(x) - y^T \nabla f_n^\Delta(x) B_W^\eta(dy) \\
&= \int_{\{||y||\leq 1\}} R^n(x, x + t^n(x,y)y) ||y||^2 B_W^\eta(dy) \\
&\leq ||R^n||_\infty \int_{\{||y||\leq 1\}} ||y||^2 B_W^\eta(dy).
\end{aligned}$$

□

Lemma E.1.12. *Let us assume W is a Lévy process with Lévy triple $(B_W^\rho, B_W^{cov}, B_W^\eta)$ and $W(0) = 0$. If $\int_{\mathbb{R}^m} ||w||_2^2 B_W^\eta(dw) < \infty$, where $||w||^2 = \sum_{k=1}^d |w_k|^2, w \in \mathbb{R}^m$, then there exists a constant K_1 such that*

$$\mathbb{E}[[W_i, W_j]_t] \leq K_1 t.$$

Further there exists an increasing function $K_2 : [0,\infty) \to [0,\infty)$ such that $\mathbb{E}[\sup_{s\leq t} ||W_s||_2^2] \leq K(T)t$ for all $0 \leq t \leq T < \infty$. The latter especially implies

$$\mathbb{P}\left[\sup_{s\leq t} ||W|| \geq \epsilon\right] \leq \frac{1}{\epsilon^2} K(T)t. \tag{E.6}$$

Proof. Let us define $\nu := \int_{\mathbb{R}^d} ||x||^2 B_W^\eta(dx) < \infty$, then we can derive from the Lévy-Itô decomposition that

$$\begin{aligned}\mathbb{E}\left[|[W_i, W_j]_t|\right] &\leq |B_W^{cov}(i,j)|t + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} |x_i x_j| B_W^\eta(dx)\right] \\ &\leq (\sum_{k,l=1}^m |B_W^{cov}(i,j)| + \nu)t < \infty.\end{aligned}$$

Hence K_1 is given by $|B_W^{cov}(i,j)| + \nu$. The Lévy-Itô decompositions give us also the semimartingale decomposition $W_i = M_i + A_i$ which is given by

$$M_i(t) := \sum_{i=1}^m (B_W^{cov})^{\frac{1}{2}}(i,j) W_i^c(t) + \int_0^t \int_{\mathbb{R}^d} x_i \overline{W}^J(dx, ds), \quad A_i(t) := [B_W^\rho(i) + \mu_i]t,$$

where $\mu_i := \int_{||w|| \geq 1} |x_i| B_W^\eta(dx) \leq \nu < \infty$ and $\overline{W}^J$ is the compensated process. Combining Jensen's inequality and Doob's L^2-inequality gives for $\mathbb{E}[\sup_{s\leq t} ||W_s||^2]$ the upper bound:

$$\mathbb{E}\left[\sup_{s\leq t} 2||M_s||^2 + 2||A_s||^2\right] \leq \sum_{i=1}^m 2[B_W^\rho(i) + \mu_i]^2 t^2 + 2\mathbb{E}\left[[M_i]_t\right] \leq K_W(T)t,$$

where $K(T) := 2T\sum_{i=1}^m [B_W^\rho(k) + \nu]^2 + 2\sum_{i=1}^m K_1$. Applying Markov inequality gives us (E.6). □

Lemma E.1.13. *The space $C_b^2(\mathbb{R}^m) \subset \mathcal{D}(B_W)$, see Definition 2.1.1, is contained in the domain of the weak generator of W.*

Proof. Let us assume that $\hat{g} \in C_b^{2,+}(\mathbb{R}^m)$ and that $(P_t, t \geq 0)$, where $P_t : C_b(\mathbb{R}^m) \to C_b(\mathbb{R}^m)$ for $t \geq 0$, is the Markov semigroup of W. If $W(0) = 0$ and $W_x = W + x$, $x \in \mathbb{R}^m$, then $P_t(\hat{g}) = \mathbb{E}[\hat{g}(W_x(t))]$. Let us define the function $\hat{\mathbf{w}} : (0,\infty) \times \mathbb{R}^m \to \mathbb{R}$ by

$$\hat{\mathbf{w}}_t(x) := \frac{P_t(\hat{g})(x) - \hat{g}(x)}{t}, \quad x \in \mathbb{R}^m, t > 0.$$

We need to show that $t \mapsto \hat{\mathbf{w}}_t(x)$ is converging to $B_W(\hat{g})(x)$ for each $x \in \mathbb{R}^m$, when $t \to 0$, and that $(||\hat{\mathbf{w}}_t||_\infty, t \in [0,1])$ is bounded. We recall (W^c, W^J) from the Lévy-Itô decomposition, see Lemma E.1.5. The Itô-formula for Lévy processes, see Theorem 4.4.7 in [2], tells us

$$\begin{aligned}&\hat{g}(W_x(t)) - \hat{g}(x) = \int_0^t (B_W^\rho)^T \nabla \hat{g}(W_x(s\text{-}))ds \\ &+ \sum_{k,l=1}^m \int_0^t \partial_{x_k} \hat{g}(W_x(s\text{-}))(B_W^{cov})^{\frac{1}{2}}(k,l) dW_{[l]}^c(s) + \frac{1}{2}\int_0^t \nabla^T B_W^{cov} \nabla g(W_x(s\text{-}))ds \quad \text{(E.7)} \\ &+ \int_0^t \int_{\mathbb{R}^m} \hat{g}(W_x(s\text{-}) + y) - \hat{g}(W_x(s\text{-})) - \mathbb{1}_{\{||y|| \leq 1\}} \left(y^T \nabla g(W_x(s\text{-}))\right) B_W^\eta(dy)ds \\ &+ \int_0^t \int_{\mathbb{R}^m} \hat{g}(W_x(s\text{-}) + y) - \hat{g}(W_x(s\text{-})) - \mathbb{1}_{\{||y|| \leq 1\}} \left(y^T \nabla g(W_x(s\text{-}))\right) \overline{W}^J(dy, ds). \quad \text{(E.8)}\end{aligned}$$

By applying the multidimensional Taylor formula, see "Taylorformel mit Rest" on Page 65 in [26], it holds

$$\hat{g}(W_x(s\text{-}) + y) - \hat{g}(W_x(s\text{-})) - \mathbb{1}_{\{||y|| \leq 1\}} \left(y^T \nabla g(W_x(s\text{-}))\right) \leq ||y||^2 ||\hat{g}||_{2,\infty}, \quad y \in \mathbb{R}^m,$$

where $||\hat{g}||_{2,\infty}$ is the norm of $\hat{g}$ in $C_b^2(\mathbb{R}^m) \subset \mathcal{D}(B_W)$, see Definition 2.1.1. For a more detailed derivation of this bound, please see the proof of Lemma E.1.11. Due to this bound, the fact that $\hat{g}$ is bounded and the fact that $B_W^\eta(dy)$ is a Lévy measure, it follows that the integral in (E.8) forms a martingale. Similar, since $\partial_{x_k}\hat{g}, 1 \leq k \leq m$, is bounded, the Itô-integrals in (E.7) are also martingales. Taking the expectation it remains:

$$
\begin{aligned}
&\mathbb{E}\left[\hat{g}(W_x(t)) - \hat{g}(x)\right] = \\
&\int_0^t \mathbb{E}\left[(B_W^\rho)^T \nabla \hat{g}(W_x(s\text{-}))\right] ds + \frac{1}{2}\int_0^t \mathbb{E}\left[\nabla^T B_W^{cov} \nabla g(W_x(s\text{-}))\right] ds \\
&+ \int_0^t \int_{\mathbb{R}^m} \mathbb{E}\left[\hat{g}(W_x(s\text{-}) + y) - \hat{g}(W_x(s\text{-})) - \mathbb{1}_{\{||y|| \leq 1\}}\left(y^T \nabla g(W_x(s\text{-}))\right)\right] B_W^\eta(dy) ds
\end{aligned}
$$

The right hand side of the above can be bounded by tK, where

$$
K := \left(||B_W^\rho||_2^{\frac{1}{2}} ||\hat{g}||_{2,\infty} + \int_{||y|| \leq 1} ||y||^2 ||\hat{g}||_{2,\infty} B_W^\eta(dy) + 2||\hat{g}||_{2,\infty} B_W^\eta(\{||x|| > 1\}) \right).
$$

Hence $\sup_{t\in[0,1]} ||\hat{\mathbf{w}}_t||_\infty$ is finite. Taking the derivative with respect to t shows the convergence of the map $t \mapsto \hat{\mathbf{w}}_t(x)$ to $B_W(\hat{g})(x)$ for $t \to 0$. □

E.2 The Generator of a Path-valued Process

In his lectures notes Perkins also discusses the characterization of the path-valued process via the help of a well-posed martingale problem corresponding to linear operator $\mathfrak{B}$w. Unfortunately, to the best of our knowledge no explicit expression of $B_{\mathbb{W}}$ is given in the literature, which is problematic, because without an explicit expression, it is difficult to check, that this linear operator satisfies the conditions of the theory of the Kurtz-Rodrigues representation. Therefore we present an explicit version $B_{\mathbb{W}}$ and the proof of the following proposition. We will not give more details on the operator and the proof of Proposition 2.5.12, because both are very complicated.

As we have already discussed in the previous Section E.1 it is necessary for us to characterize the spatial processes of our particles as the solution of a martingale problem. Therefore in this section we will characterize the path-valued process $\mathfrak{W}$ of a Lévy process W as the unique solution of a **well-posed** martingale problem. This martingale problem should not only be well-posed, but the linear operator $B_{\mathbb{W}}$ associated with the martingale problem should also satisfy the **Conditions 3.1** found in the Paper [32]. This is important for us, because it allows us to apply the Markov mapping theorem to the Kurtz-Rodrigues representation with spatial motion defined by $B_{\mathbb{W}}$. While there exists in the literature possible candidates for $B_{\mathbb{W}}$, which characterize $\mathfrak{W}$ via a well-posed martingale problem, these candidates usually do not satisfy the Conditions 3.1. Either their domain is not closed under multiplication, or their domain and their image is not contained in the set of continuous functions.

Before we discuss our choice for $B_{\mathbb{W}}$ in details, let us repeat some properties of the state space of $\mathfrak{W}$, which play crucial role, when one wishes to define the path-valued process of a process with càdlàg paths via a martingale problem.

Therefore let us recall that, if W is a Lévy process in $\mathbb{R}^m$, then the state space of the path-valued process $\mathfrak{W}$ has the form

$$
\widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m) := \bigcup_{t\geq 0} \left(\{t\} \times \mathbb{D}_t([0,\infty),\mathbb{R}^m)\right),
$$

where $\mathbb{D}_t([0,\infty),\mathbb{R}^m)$ is the collection of càdlàg paths which are constant from the time point t onwards (i.e. $\mathfrak{w} \in \mathbb{D}_t([0,\infty),\mathbb{R}^m)$, if and only if $\mathfrak{w}(\tilde{t}) = \mathfrak{w}(t)$ for all $\tilde{t} \geq t$). Further the topology of $\widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m)$ is the one implied by a metric with the form

$$d_{\widehat{\mathbb{D}},E}\big((t,\mathfrak{w}),(s,\tilde{\mathfrak{w}})\big) = d_{\mathbb{D},E}(\mathfrak{w},\tilde{\mathfrak{w}}) + |t-s| \tag{E.9}$$

with $d_{\mathbb{D},E}$ being a metric generating the Skorohod J-1-topology on $\mathbb{D}([0,\infty),\mathbb{R}^m)$, the space of càdlàg paths.

Remark E.2.1. *In this section we will assume that the Skorohod metric $d_{\mathbb{D},E}$ in* (E.9) *is the one found in Chapter* 3 *of [4]. There $d_{\mathbb{D},E}(\mathfrak{w},\tilde{\mathfrak{w}}) := \sum_{n=1}^{\infty} \frac{1}{2^n} d_{\mathbb{D},E,2^n}(\mathfrak{w},\tilde{\mathfrak{w}})$, where for a fixed $T > 0$:*

$$d_{\mathbb{D},E,T}(\mathfrak{w},\tilde{\mathfrak{w}}) := \inf \left\{ \sup_{t\in[0,T]} |\phi(t) - t| + \sup_{t\in[0,T]} ||\mathfrak{w}(\phi(t)) - \tilde{\mathfrak{w}}(t)|| \right\},$$

where the infimum is taken over all time changes. Note that this metric does not make $\mathbb{D}([0,\infty),\mathbb{R}^m)$ a complete metric space, for completeness one needs to use the more complicated metric defined on the Page 117 in [14]. But when we are dealing with questions of continuity it is easier to work with the above metric.

Let us fix an element $(t_0,\mathfrak{w}_0) \in \widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m)$. We want to construct a path-valued process starting in $W(0) = \mathfrak{w}_0(t_0)$, therefore let us assume that $(\Omega,\mathbb{F},\mathbb{P})$ is suitable probability space and that $W : \Omega \times [0,\infty) \to \mathbb{R}^m$ is a Lévy process corresponding to the operator B_W starting in $\mathfrak{w}_0(t_0)$, i.e $W(0) = \mathfrak{w}_0(t_0)$. The associated path-valued process $\mathbb{W} : \Omega \times [0,\infty) \to \widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m)$ is obtained by setting $\mathbb{W}(t) := (t_0 + t, \mathfrak{W}(t))$, where $\mathfrak{W} : \Omega \times [0,\infty) \to \mathbb{D}([0,\infty),\mathbb{R}^m)$ is the process whose value for fixed time $t \in [0,\infty)$ is the path $s \to \mathfrak{W}(t,s)$ is given by

$$\mathfrak{W}(t,s) := \begin{cases} W(t \wedge (s-t_0)), & s > t_0, \\ \mathfrak{w}_0(s), & s \leq t_0. \end{cases}$$

For the rest of this section we also fix the following notation (which will be useful during the formulation of the following results), we write:

$$\mathfrak{P}_{(t_0,\mathfrak{w}_0)} \in \mathcal{M}_1\left(\widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m)^{[0,\infty)}\right), \quad (t_0,\mathfrak{w}_0) \in \widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m), \tag{E.10}$$

for the distribution of the path-valued process $\mathbb{W}$ with initial state $(t_0,\mathfrak{w}_0)$, and we write $(\mathfrak{P}_t, t \geq 0)$ for the semigroup defined by

$$\mathfrak{P}_t(\mathfrak{F})(t,\mathfrak{w}) = \mathbb{E}[\mathfrak{F}(\mathbb{W}_t)|\mathbb{W}(0) = (t_0,\mathfrak{w}_0)] \tag{E.11}$$

for $(t,\mathfrak{w}) \in \widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m)$, $\mathfrak{F} \in C_b(\widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m))$. One difficulty in defining a linear operator characterizing the law of $\mathbb{W}$ is that the evaluation of a càdlàg path at a time point, i.e. any map of the form $\mathfrak{w} \mapsto \mathfrak{w}(t_1)$ for some $t_1 \in [0,\infty)$, is not a continuous functional in the Skorohod topology. Consequently a functional of the form $(t,\mathfrak{w}) \mapsto \mathfrak{w}(t_1)$ for some $t_1 \in [0,\infty)$ is not continuous in the topology of $\widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m)$. But the specific topology of $\widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m)$ allows one exception, and this is the map

$$(t,\mathfrak{w}) \mapsto \mathfrak{w}(t).$$

This exception is crucial for us and we prove its continuity in the next lemma.

Lemma E.2.2. *The evaluation map $\pi : \widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m) \to \mathbb{R}^m$ given by $\pi(t,\mathfrak{w}) := \mathfrak{w}(t)$ is continuous in the topology generated by the metric $d_{\widehat{\mathbb{D}},E}$.*

Proof. We begin with the observation that for all $0 \leq s < T < \infty$ holds $\mathbb{D}_s([0,\infty),\mathbb{R}^m) \subset \mathbb{D}_T([0,\infty),\mathbb{R}^m)$ and so

$$\pi(s,\tilde{\mathfrak{w}}) = \tilde{\mathfrak{w}}(T) \tag{E.12}$$

for all $(s,\tilde{\mathfrak{w}}) \in \widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m)$ with $s \leq T$. Let us now assume that $(t_1,\mathfrak{w}_1),(t_2,\mathfrak{w}_2) \in \widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m)$. When we choose a time point T such that $T \geq t_1$ and $T \geq t_2$, then it follows from (E.12) that

$$|\pi(t_1,\mathfrak{w}_1) - \pi(t_2,\mathfrak{w}_2)| = |\mathfrak{w}_1(T_1) - \mathfrak{w}_2(T_2)| \tag{E.13}$$

for all $T_1, T_2 \geq T$. We obtain:

$$\begin{aligned}|\pi(t_1,\mathfrak{w}_1) - \pi(t_2,\mathfrak{w}_2)| &\leq \inf_{\tilde{T}\geq T} |\mathfrak{w}_1(\tilde{T}) - \mathfrak{w}_2(T)| + \inf_{\tilde{T}\geq T} |\mathfrak{w}_1(T) - \mathfrak{w}_2(\tilde{T})| \\ &\leq 2 \inf_{\phi} \sup_{s\geq 0} |\mathfrak{w}_1(\phi(s)) - \mathfrak{w}_2(s)|,\end{aligned}$$

where the infimum in the above line is taken over all time changes $\phi : [0,\infty) \to [0,\infty)$. It follows

$$|\pi(t_1,\mathfrak{w}_1) - \pi(t_2,\mathfrak{w}_2)| \leq 2d_{\mathbb{D},E}(\mathfrak{w}_1,\mathfrak{w}_2) \leq 2d_{\widehat{\mathbb{D}},E}\big((t_1,\mathfrak{w}_1),(t_2,\mathfrak{w}_2)\big),$$

when $d_{\mathbb{D},E}$ is given as in the Remark E.2.1. □

Corollary E.2.3. *The path-valued process $\mathfrak{W}$ has càdlàg paths or continuous paths with respect to the metric $d_{\widehat{\mathbb{D}},E}$ from* (E.9)*, if the original Lévy process has càdlàg paths or continuous paths with respect to standard euclidean norm in $\mathbb{R}^m$.*

Proof. This follows directly from (E.9) and Lemma E.2.2. □

Definition E.2.4. *Let us define for $\varphi \in C_c^1([0,\infty)) \cup \{\mathbb{1}_{[0,\infty)}\}$, $f \in \textbf{span}(C_c^2(\mathbb{R}^m) \cup \{\mathbb{1}_{\mathbb{R}^m}\})$ and $\boldsymbol{g} = (g_1,...,g_k), g_i \in C_c(\mathbb{R}^{m+1}), 1 \leq i \leq k$, the function $\mathfrak{F}_{\varphi,f,\boldsymbol{g}} : \widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m) \to \mathbb{R}$ by setting*

$$\mathfrak{F}_{\varphi,f,\boldsymbol{g}}(t,\mathfrak{w}) := \varphi(t) f(\mathfrak{w}(t)) \prod_{i=1}^{k} \int_0^t g_i(\mathfrak{w}(s),s)\, ds \tag{E.14}$$

for all $(t,\mathfrak{w}) \in \widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m)$.

Remark E.2.5. *When we omit $\varphi, f, \boldsymbol{g}$ from the substring of $\mathfrak{F}_{\varphi,f,\boldsymbol{g}}$, then we interpret this as if the corresponding factor has been set to one, e.g.*

$$\mathfrak{F}_{\varphi,f}(t,\mathfrak{w}) = \varphi(t)f(\mathfrak{w}(t)), \quad \mathfrak{F}_{\varphi}(t,\mathfrak{w}) = \varphi(t), \quad \mathfrak{F}_{f,\boldsymbol{g}}(t,\mathfrak{w}) = f(\mathfrak{w}(t)) \prod_{i=1}^{k} \int_0^t g_i(\mathfrak{w}(s),s)\, ds$$

and $\mathfrak{F}(t,\mathfrak{w}) = \mathbb{1}_{\widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m)}$. This list of examples is not exhaustive, but it should convey the idea.

Lemma E.2.6. *The test functions defined in Definition E.2.4, which includes the functions from Remark E.2.5, are continuous.*

Proof. Let φ, f, g be defined as in the Definition E.2.4. The statement should be clear for test functions of the form $\mathfrak{F}_{\varphi}(t\mathfrak{w}) = \varphi(t)$ and $\mathfrak{F}_f(t,\mathfrak{w}) = f(\mathfrak{w}(t))$ due to the definition of the metric in (E.9) and the Lemma E.2.2. Since continuity is maintained under multiplication it remains to argue that test functions with the form $\mathfrak{F}_g(t,\mathfrak{w}) = \int_0^t g(\mathfrak{w}(s))ds$ are continuous. Let us assume $(t_n,\mathfrak{w}_n)_{n=1}^{\infty}$ is a sequence in $\widehat{\mathbb{D}}([0,\infty),\mathbb{R}^m)$ converging against $(t,\mathfrak{w})$. Since this implies

$\lim_{n\to\infty} t_n = t$, there exists an upper bound for $(t_n)_{n=1}^{\infty}$ and t which we denote by $T \in [0,\infty)$, which gives us in turn the upper bound:

$$|\mathfrak{F}_g(t_n, \mathfrak{w}_n) - \mathfrak{F}_g(t_n, \mathfrak{w}_n)| \le \int_0^T |g(\mathbb{W}_n(s), s) - g(\mathbb{W}(s), s)|\, ds.$$

Since g is bounded and continuous, the right-hand side of the above inequality converges to zero due to Lebesgue dominated convergence theorem. □

Definition E.2.7. *We denote by $B_{\mathbb{W}}$ the linear operator*

$$B_{\mathbb{W}} : C_b(\widehat{\mathbb{D}}([0,\infty), \mathbb{R}^m)) \supset \mathcal{D}(B_{\mathbb{W}}) \to C_b(\widehat{\mathbb{D}}([0,\infty), \mathbb{R}^m)),$$

whose domain $\mathcal{D}(B_{\mathbb{W}})$ is the linear span of $\bigcup_{k=0}^{\infty} \mathbf{\Gamma}_k$, where

$$\mathbf{\Gamma}_k := \Big\{ \mathfrak{F}, \mathfrak{F}_\varphi, \mathfrak{F}_f, \mathfrak{F}_{\boldsymbol{g}}, \mathfrak{F}_{\varphi,f}, \mathfrak{F}_{\varphi,\boldsymbol{g}}, \mathfrak{F}_{f,\boldsymbol{g}} \mathfrak{F}_{\varphi,f,\boldsymbol{g}};\ \varphi \in C_c^1([0,\infty)), f \in \boldsymbol{span}(C_c^2(\mathbb{R}^m) \cup \{\mathbb{1}_{\mathbb{R}^m}\}),$$
$$\boldsymbol{g} = (g_1, ..., g_k), g_i \in C_c(\mathbb{R}^{m+1}), 1 \le i \le k \Big\}, k \in \mathbb{N}_0,$$

and whose value for $\mathfrak{F}_{\varphi,f,\boldsymbol{g}} \in \mathbf{\Gamma}_k$ is the function $B_{\mathbb{W}}(\mathfrak{F}_{\varphi,f,\boldsymbol{g}}) : \widehat{\mathbb{D}}([0,\infty), \mathbb{R}^m) \to \mathbb{R}$ given by

$$\begin{aligned} B_{\mathbb{W}}(\mathfrak{F}_{\varphi,f,\boldsymbol{g}})(t, \mathfrak{w}) := {}& \dot{\varphi}(t) f(\mathfrak{w}(t)) \prod_{i=1}^{k} \int_0^t g_i(\mathfrak{w}(s), s)\, ds \\ &+ \varphi(t) B_W(f)(\mathfrak{w}(t)) \prod_{i=1}^{k} \int_0^t g_i(\mathfrak{w}(s), s)\, ds \\ &+ \varphi(t) f(\mathfrak{w}(t)) \sum_{j=1}^{k} g_j(\mathfrak{w}(t), t) \prod_{i=1, i\neq j}^{k} \int_0^t g_i(\mathfrak{w}(s), s)\, ds. \end{aligned}$$

Remark E.2.8. *Analogously to Remark E.2.5, when we apply the operator $B_{\mathbb{W}}$ to a test function, where φ, f or $\boldsymbol{g}$ is missing in the substring, e.g. $\mathfrak{F}_{\varphi,f}$ or $\mathfrak{F}_{f,\boldsymbol{g}}$, then we set the corresponding part in the expression of B_W to zero, e.g.*

$$\begin{aligned} \mathfrak{F}_{\varphi,f}(t, \mathfrak{w}) &:= \dot{\varphi}(t) f(\mathfrak{w}(t)) + \varphi(t) B_W(f)(\mathfrak{w}(t)) \\ B_{\mathbb{W}}(\mathfrak{F}_{f,\boldsymbol{g}})(t, \mathfrak{w}) &:= B_W(f)(\mathfrak{w}(t)) \prod_{i=1}^{k} \int_0^t g_i(\mathfrak{w}(s), s)\, ds \\ &\quad + f(\mathfrak{w}(t)) \sum_{j=1}^{k} g_j(\mathfrak{w}(t), t) \prod_{i=1, i\neq j}^{k} \int_0^t g_i(\mathfrak{w}(s), s)\, ds. \end{aligned}$$

Note that this is consistent with Remark E.2.5.

Proposition E.2.9. *Let $(\mathfrak{P}_t, t \ge 0)$ be the semi-group of the path-valued process corresponding to the Lévy process characterized by the martingale problem $\boldsymbol{MP}(B_W)$ and $\mathfrak{F}_{\varphi,f,\boldsymbol{g}} \in \mathcal{D}(B_{\mathbb{W}})$, indeed see* (E.11)*, then it holds*

$$\frac{\mathfrak{P}_t(\mathfrak{F}_{\varphi,f,\boldsymbol{g}}) - \mathfrak{F}_{\varphi,f,\boldsymbol{g}}}{t} \xrightarrow{t\to 0} B_{\mathbb{W}}\left(\mathfrak{F}_{\varphi,f,\boldsymbol{g}}\right) \ b.p. \tag{E.15}$$

Further, let $\mathbb{W}$ be a process whose law is that of the path-valued process starting $(t_0, \mathfrak{w}_0) \in \widehat{\mathbb{D}}([0,\infty), \mathbb{R}^m)$, then $\mathbb{W}$ is a solution of the martingale problem $\boldsymbol{MP}(B_{\mathbb{W}}, \delta_{(t_0,\mathfrak{w}_0)})$.

Proof. If W is a Lévy process with generator B_W and $\mathbb{W}$ is the path-valued process constructed for the initial state $(t_0, \mathbf{x}_0)$ as above, then we can prove (E.15) by showing that:

$$\frac{1}{t}\left(\mathbb{E}\left[\mathfrak{F}_{\varphi,f,\boldsymbol{g}}(\mathbb{W}(t))\right] - \mathfrak{F}_{\varphi,f,\boldsymbol{g}}(t_0, \mathfrak{w}_0)\right) \stackrel{t\to 0}{\longrightarrow} B_{\mathbb{W}}\left(\mathfrak{F}_{\varphi,f,\boldsymbol{g}}\right)(t_0, \mathfrak{w}_0) \tag{E.16}$$

for an arbitrary chosen. We also need to show that the left side of (E.16) is bounded uniformly in t and $(t_0, \mathbf{x}_0)$. We know that the statements are true for the case $\mathfrak{F}_{\varphi,f}$ (and its special cases $\mathfrak{F}_{\varphi}$ and $\mathfrak{F}_f$), because

$$\mathfrak{F}_{\varphi,f}(\mathbb{W}(t)) = \varphi(t) f(W(t))$$

and W is a solution of the martingale problem of B_W. Therefore it is sufficient for us, when we only consider test functions with the form

$$\mathfrak{F}_{f,\boldsymbol{g}}(t, \mathfrak{w}) = f(\mathfrak{w}(t)) \prod_{i=1}^{k} \int_0^t g_i(\mathfrak{w}(s_i), s_i)\, ds_i.$$

Considering the remaining cases $\mathfrak{F}_{\boldsymbol{g}}(t, \mathfrak{w}) = \prod_{i=1}^{k} \int_0^t g_i(\mathfrak{w}(s_i), s_i)\, ds_i$ and

$$\mathfrak{F}_{\varphi,f,\boldsymbol{g}}(t, \mathfrak{w}) = \varphi(t) f(\mathfrak{w}(t)) \prod_{i=1}^{k} \int_0^t g_i(\mathfrak{w}(s_i), s_i)\, ds_i = \varphi(t) \mathfrak{F}_{f,\boldsymbol{g}}(t, \mathfrak{w})$$

we can say that $\mathfrak{F}_{\boldsymbol{g}}$ is just a special case of $\mathfrak{F}_{f,\boldsymbol{g}}$ with $f = \mathbb{1}_{\mathbb{R}^m}$ and $\mathfrak{F}_{\varphi,f,\boldsymbol{g}}$ will follow immediately from the case of $\mathfrak{F}_{f,\boldsymbol{g}}$, since the time component of $\mathbb{W}$ is just the deterministic process given by $(\omega, t) \to t_0 + t$. For the rest of this proof we choose a fixed constant $K \geq 1$ with the property that $|f|, |B_W(f)|, |g_1|, ..., |g_k| \leq K$ and we proceed with the decomposition:

$$\begin{aligned}
&\mathbb{E}\left[\mathfrak{F}_{f,\boldsymbol{g}}(\mathbb{W}(t))\right] - \mathfrak{F}_{f,\boldsymbol{g}}(\mathbb{W}(0)) = \mathbb{E}\left[f(W(t))\mathfrak{F}_{\boldsymbol{g}}(\mathbb{W}(t))\right] - f(W(0))\mathfrak{F}_{\boldsymbol{g}}(\mathbb{W}(0)) \\
&= \mathbb{E}\left[f(W(t))\left(\mathfrak{F}_{\boldsymbol{g}}(\mathbb{W}(t)) - \mathfrak{F}_{\boldsymbol{g}}(\mathbb{W}(0))\right)\right] + \mathbb{E}\left[(f(W(t)) - f((\mathbb{W}(0)))\mathfrak{F}_{\boldsymbol{g}}(\mathbb{W}(0))\right].
\end{aligned} \tag{E.17}$$

Considering the second term in the above, we note that $\mathfrak{F}_{\boldsymbol{g}}(\mathbb{W}(0)) = \mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0)$ is deterministic, so we can pull it out from the expectation. Combining this with the fact that W is a solution of the martingale problem of B_W, we obtain for all $t \geq 0$ the upper bound

$$\frac{1}{t}\left|\mathbb{E}\left[(f(W(t)) - f((t_0, \mathfrak{w}_0))\mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0)\right]\right| \leq \frac{1}{t}\int_0^t \mathbb{E}\left[|B_W(f)(W(s))|\right]\, ds \cdot |\mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0)|.$$

Since $|B_W(f)|$ is bounded by the constant K and $\mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0)$ is bounded by $K(T+1)^k$ for all initial states $(t_0, \mathfrak{w}_0)$, when we choose T such that the support of g_i is for all $i \in \{1, ..., k\}$ contained in the set $\mathbb{R}^m \times [0, T]$, we can conclude that

$$\frac{1}{t}\left|\mathbb{E}\left[(f(W(t)) - f((t_0, \mathfrak{w}_0))\mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0)\right]\right|$$

is bounded by $K^2(T+1)^k$ for every $t \geq 0$ and every initial state $(t_0, \mathfrak{w}_0)$. That W is a solution of the martingale problem of B_W allows us also to conclude that

$$\begin{aligned}
\frac{1}{t}\mathbb{E}\left[(f(W(t)) - f((t_0, \mathfrak{w}_0))\mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0)\right] &\stackrel{t\to 0}{\longrightarrow} B_W(f)(W(0))\mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0) \\
&= B_W(f)(\mathfrak{w}_0(t_0))\mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0).
\end{aligned}$$

This gives the b.p. convergence of the second term of (E.17). Considering the first term of (E.17) we start with the observation that $\mathfrak{F}_{\boldsymbol{g}}(\mathbb{W}(t)) - \mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0)$ can be written as

$$\begin{aligned}\mathfrak{F}_{\boldsymbol{g}}(\mathbb{W}(t)) - \mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0) &= \prod_{i=1}^{k} \mathfrak{F}_{g_i}(\mathbb{W}(t)) - \prod_{i=1}^{k} \mathfrak{F}_{g_i}(\mathbb{W}(0)) \\ &= \sum_{i=1}^{k} \left[\prod_{j=1}^{i-1} \mathfrak{F}_{g_j}(\mathbb{W}(t)) \right] \left(\mathfrak{F}_{g_i}(\mathbb{W}(t)) - \mathfrak{F}_{g_i}(\mathbb{W}(0)) \right) \left[\prod_{j=i+1}^{k} \mathfrak{F}_{g_j}(\mathbb{W}(0)) \right]. \end{aligned} \tag{E.18}$$

with $\mathfrak{F}_{g_i}(t, \mathfrak{w}) = \int_0^t g_i(\mathfrak{w}(s), s)\, ds$. But that is not all, because we can also write

$$\mathfrak{F}_{g_i}(\mathbb{W}(t)) - \mathfrak{F}_{g_i}(\mathbb{W}(0)) = \int_0^t g_i(W(s), s)\, ds. \tag{E.19}$$

Again choosing $T > 0$ such that the support of g_i is for all $i \in \{1, ..., k\}$ contained in the set $\mathbb{R}^m \times [0, T]$ and using the fact that $|f|, |g_1|, ..., |g_k| \leq K$, we can observe that it holds

$$\begin{aligned}\frac{1}{t} \left| f(W(t)) \left(\mathfrak{F}_{\boldsymbol{g}}(\mathbb{W}(t)) - \mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0) \right) \right| &\leq \frac{K^k}{t} \sum_{i=1}^{k-1} \left| \mathfrak{F}_{g_i}(\mathbb{W}(t)) - \mathfrak{F}_{g_i}(\mathbb{W}(0)) \right| \\ &\leq \frac{K^k}{t} (k-1) t K = K^{k+1}(k-1).\end{aligned}$$

Note that this bound is true, independent from the values of $\mathbb{W}$ on $[0, T]$ and the chosen initial state $(t_0, \mathfrak{w}_0)$. This uniform bound allows us to apply Lebesgue dominated convergence theorem to the first term of (E.17) and so we obtain:

$$\lim_{t \to \infty} \frac{1}{t} \mathbb{E} \left[f(W(t)) \left(\mathfrak{F}_{\boldsymbol{g}}(\mathbb{W}(t)) - \mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0) \right) \right] = \mathbb{E} \left[\lim_{t \to \infty} f(W(t)) \frac{\mathfrak{F}_{\boldsymbol{g}}(\mathbb{W}(t)) - \mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0)}{t} \right]$$

and by (E.18) and (E.19) we can see that

$$\lim_{t \to 0} \frac{\mathfrak{F}_{\boldsymbol{g}}(\mathbb{W}(t)) - \mathfrak{F}_{\boldsymbol{g}}(t_0, \mathfrak{w}_0)}{t} = \sum_{i=1}^{k} g_i(W(t_0), t_0) \prod_{j \neq i}^{k} \int_0^t g_j(\mathfrak{W}(t, s), s)\, ds.$$

This gives us the b.p. convergence of the first term in (E.17). □

The next lemma tells us under which conditions the dynamic of the functional $f(X)$, where X is the solution of martingale problem associated with an operator $\mathbf{A}$ and $f \in \mathcal{D}(\mathbf{A})$, can be described by a (random) differential equation, meaning that the evolution of $f(X)$ contains no Brownian part, no Jump part, only drift. This lemma will become quite useful for the proof that the martingale problem of $B_{\mathbb{W}}$ is well-posed.

Lemma E.2.10. *Let us assume that E is a Polish space, $\mathbf{B} : B(E) \supset \mathcal{D}(\mathbf{B}) \to B(E)$ is a linear operator and the process X is a solution of the martingale problem of $\mathbf{B}$. When $f \in B(E)$ has the property that $f, f^2 \in \mathcal{D}(\mathbf{B})$ and $\mathbf{B}(f^2) - 2fB(f) = 0$, then it holds*

$$f(X_t) - f(X_0) = \int_0^t \mathbf{B}(f)\, ds \quad a.s. \quad \forall t \in [0, \infty).$$

Proof. See Lemma 2.1. from [3]. □

We will also need the following small technical lemma.

Lemma E.2.11. *Let us assume that $\mathfrak{w}_1$ and $\mathfrak{w}_2$ are two càdlàg paths in $\mathbb{R}^m$. Further let us assume that there exist two time points $t_1 \leq t_2$ such that the equality*

$$\int_{t_1}^{t_2} g_n(\mathfrak{w}_1(s), s)ds = \int_{t_1}^{t_2} g_n(\mathfrak{w}_2(s), s)ds \tag{E.20}$$

is true for a countable family $(g_n, n \in \mathbb{N}) \subset C_c(\mathbb{R}^{m+1})$ of functions that is dense in $C_c(\mathbb{R}^{m+1})$ with respect to the uniform topology, then it holds $\mathfrak{w}_1(s) = \mathfrak{w}_2(s)$ for all $s \in [t_1, t_2)$.

Proof. Since $(g_n, n \in \mathbb{N}) \subset C_c(\mathbb{R}^{m+1})$ is dense in $C_c(\mathbb{R}^{m+1})$ with respect to the uniform topology, we can conclude that the equality of (E.20) is true for any element of $C_c(\mathbb{R}^{m+1})$. Now let us assume that there exists a t_3 with $t_1 \leq t_3 < t_2$ such that $\mathfrak{w}_1(t_3) = \mathfrak{w}_2(t_3)$. In this case we choose a function $g_w \in C_c(\mathbb{R}^m)$ with the property that $g_w(\mathfrak{w}_1(t_3)) < g_w(\mathfrak{w}_2(t_3))$. Since $s \mapsto g_w(\mathfrak{w}_i(s))$ is right continuous for $i \in \{1, 2\}$, there must exists a $\delta > 0$ and an ϵ such that $g_w(\mathfrak{w}_1(s)) < \epsilon < g_w(\mathfrak{w}_2(s))$ for all $s \in [t_3, t_3 + \epsilon)$. Now let us choose a non-negative function $g_t \in C_c([0, \infty))$ with $\operatorname{supp}(g_t) \subset [t_3, t_3 + \epsilon)$ and $\int_{t_1}^{t_2} g_t(s)ds = 1$. It follows

$$\int_{t_1}^{t_2} g_w(\mathfrak{w}_1(s))g_t(s)ds < \epsilon < \int_{t_1}^{t_2} g_w(\mathfrak{w}_1(s))g_t(s)ds.$$

But since the product $g_w g_t$ is also an element of $C_c(\mathbb{R}^{m+1})$, this produces a contradiction. $\square$

Proposition E.2.12. *The martingale problem of $B_{\mathbb{W}}$ is well-posed, indeed let us assume $\mathbb{M}$ is an progressive process defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ which is a solution of the martingale problem $\boldsymbol{MP}(B_{\mathbb{W}}, (t_0, \mathfrak{w}_0))$ and with the law*

$$\mathfrak{P}_{\mathbb{M}} \in \mathcal{M}_1(\widehat{\mathbb{D}}([0, \infty), \mathbb{R}^m)^{\infty}),$$

then $\mathfrak{P}_{\mathbb{M}}$ is identical with $\mathfrak{P}_{(t_0, \mathfrak{w}_0)}$ from (E.10)*, which is the law of the path-valued process associated with the Lévy process with generator B_W. Further $\mathbb{M}$ admits a càdlàg modification.*

Proof. Let us assume that $\mathbb{M}$ is a solution of the martingale problem $\mathbf{MP}(B_{\mathbb{W}}, \delta_{\mathbb{M}_0})$, where $\mathbb{M}_0 := (t_0, \mathbb{M}_0) \in \widehat{\mathbb{D}}\mathbb{R}^m$ is an arbitrary initial state. Further we denote by

$$\mathfrak{P}_{\mathbb{M}} \in \mathcal{M}_1 \left(\widehat{\mathbb{D}}([0, \infty), \mathbb{R}^m)^{[0, \infty)} \right)$$

the law of $\mathbb{M}$.
Let us define the processes $\mathfrak{T} : \Omega \times [0, \infty) \to [0, \infty)$ and $\mathfrak{M} : \Omega \times [0, \infty) \to \mathbb{D}([0, \infty), \mathbb{R}^m)$ as the unique processes given by

$$\mathbb{M}(t) = (\mathfrak{T}(t), \mathfrak{M}(t)).$$

Further we define the peak process $M : \Omega \times [0, \infty) \to \mathbb{R}^m$ by setting $M(t) := \mathfrak{M}(t, \mathfrak{T}(t))$. In order to prove that the law $\mathfrak{P}_{\mathbb{M}}$ is indeed the law of the path-valued of the Lévy process associated with the generator B_W and to prove that $\mathbb{M}$ admits a càdlàg modification, we need to show that M is a solution of the martingale problem for B_W and that M admits a càdlàg modification. Further if $\widehat{M}$ is the càdlàg modification, then we need to show that

$$\mathbb{P}\left[\forall s \in [t_0, t_0 + t] : \mathfrak{M}(t, s) = \widehat{M}(s) \right] = 1 \tag{E.21}$$

and also that $\mathfrak{T} = t_0 + t$ almost surely for all $t \in [0, \infty)$.
We start with M. By considering test function with the form $\mathfrak{F}_f(t, \mathfrak{w}) = f(\mathfrak{w}(t))$ we can conclude that

$$f(M(t)) - f(M(0)) - \int_0^t B_W(f)(M(s))ds$$
$$= \mathfrak{F}_f(\mathbb{M}(t)) - \mathfrak{F}_f(\mathbb{M}(0)) - \int_0^t B_{\mathbb{W}}(\mathfrak{F}_f)(\mathbb{M}(s))ds$$

is a martingale. Hence M is a solution of the martingale problem of B_W starting in the origin, indeed by Proposition E.1.10 the process M is a Lévy process and admits a càdlàg modification denoted by $\widehat{M}$.
We continue with $\mathfrak{T}$. Applying $B_{\mathbb{W}}$ to the class of test functions with the form $\mathfrak{F}(t, \mathfrak{w}) = \varphi(t)$ with $\varphi \in C^1([0, \infty))$ we obtain

$$B_{\mathbb{W}}(\mathfrak{F}_\varphi^2)(t, \mathfrak{w}) - 2\mathfrak{F}_\varphi(t, \mathfrak{w})B_{\mathbb{W}}(\mathfrak{F}_\varphi)(t, \mathfrak{w}) = 0.$$

for all $(t, \mathfrak{w}) \in \widehat{\mathbb{D}}([0, \infty), \mathbb{R}^m)$. According to Lemma E.2.10 we can conclude that

$$\varphi(\mathfrak{T}(t)) - \varphi(\mathfrak{T}(0)) = \int_0^t \dot{\varphi}((s))ds \quad a.s.,$$

therefore $\mathfrak{T}$ admits the deterministic process $\widehat{\mathfrak{T}}(t) = t_0 + t$ as modification.
Next we prove (E.21). By the definition of $\widehat{M}$ and M we already know that

$$\mathbb{P}[\mathfrak{M}(t, t) = M(t) = \widehat{M}(t)] = 1$$

is true for any fixed $t \geq 0$. To obtain the same result for the entire path up to time t, it is sufficient according to Lemma E.20 to show that

$$\int_{t_0}^{t_0+t} g(\mathfrak{M}(t, s), s)ds = \int_0^t g(\widehat{M}(s), s)ds \ a.s. \tag{E.22}$$

for an arbitrary fixed function $g \in C_c(\mathbb{R}^{m+1})$. Using the function $\mathfrak{F}_g(t, \mathfrak{w}) = \int_0^t g(\mathfrak{w}(s), s)\, ds$ we can rewrite the left-hand side in (E.22) by

$$\int_{t_0}^{t_0+t} g(\mathfrak{M}(t, s), s)ds = \int_0^{t_0+t} g(\mathfrak{M}(t, s), s)ds - \int_0^{t_0} g(\mathfrak{M}(0, s), s)ds = \mathfrak{F}_g(\mathbb{M}(t)) - \mathfrak{F}_g(\mathbb{M}(0)),$$

where we used that $\mathfrak{M}(0, s) = \mathfrak{M}(t, s)$ for all $s \leq t_0$. Combining Lemma E.2.10 with $\mathfrak{F}_g, \mathfrak{F}_g^2 \in \mathcal{D}(B_{\mathbb{W}})$ and that $B_{\mathbb{W}}(\mathfrak{F}_g^2) = 2\mathfrak{F}_g B_{\mathbb{W}}(\mathfrak{F}_g)$ we can obtain the identity:

$$\mathfrak{F}_g(\mathbb{M}(t)) - \mathfrak{F}_g(\mathbb{M}(0)) = \int_0^t B_{\mathbb{W}}(\mathfrak{F}_g)(\mathbb{M}(s))ds, \quad a.s.,$$

and using $B_{\mathbb{W}}(\mathfrak{F}_g)(\mathbb{M}(s)) = g(M(s), s)$ gives us the identity:

$$\mathfrak{F}_g(\mathbb{M}(t)) - \mathfrak{F}_g(\mathbb{M}(0)) = \int_0^t g(M(s), s)ds, \quad a.s.$$

Combining the previous steps we get.

$$\int_{t_0}^{t_0+t} g(\mathfrak{M}(t, s), s)ds = \int_0^t B_{\mathbb{W}}(\mathfrak{F}_g)(\mathbb{M}(s))ds$$

Since $\int_0^t \mathbb{E}[|g(\mathfrak{T}(s), M(s)) - g(\widehat{\mathfrak{T}}(s), \widehat{M}(s))|]\,ds = 0$, we can replace M by its càdlàg modification $\widehat{M}$ and we obtain the desired equality in (E.22). Choosing a countable collection $(g_n, n \in \mathbb{N}) \subset C_c(\mathbb{R}^{m+1})$ that is dense in $C_c(\mathbb{R}^{m+1})$ with respect to the uniform topology, we obtain (E.21) by the Lemma E.2.11. In our last step we define the process $\widehat{\mathfrak{M}} : \Omega \times [0,\infty) \to \cup_{t\geq 0}\mathbb{D}([0,\infty), \mathbb{R}^m)$ by setting for all t

$$\widehat{\mathfrak{M}}(t,s) := \begin{cases} \widehat{M}(t \wedge (s - t_0)); & s > t_0 \\ \mathfrak{w}(s); & s \leq t_0. \end{cases}$$

From (E.21) we can conclude that the process $\widehat{\mathbb{M}} = (\widehat{\mathfrak{T}}, \widehat{\mathfrak{M}})$ is a càdlàg modification of $\mathbb{M} = (\mathfrak{T}, \mathfrak{M})$. Further since $\widehat{M}$ is a copy of the Lévy process implied by B_W, the law $\widehat{\mathbb{M}}$ must be $\mathfrak{P}$. Because $\mathbb{M}$ and $\widehat{\mathbb{M}}$ have the same law, it must hold $\mathfrak{P}_{\mathbb{M}} = \mathfrak{P}$. □

Proposition E.2.13. *The operator $B_{\mathbb{W}}$ generates a Borel strong Markov family, indeed if we define for each $(t_0, \mathfrak{w}_0) \in \widehat{\mathbb{D}}([0,\infty), \mathbb{R}^m)$*

$$\mathfrak{P}_{(t_0,\mathfrak{w}_0)} \in \mathcal{M}_1\left(\mathbb{D}\left([0,\infty), \widehat{\mathbb{D}}([0,\infty), \mathbb{R}^m)\right)\right)$$

as the path law of the càdlàg version of the unique solution of $\boldsymbol{MP}(B_{\mathbb{W}}, \delta_{(t_0,\mathfrak{w}_0)})$, then this forms a Borel strong Markov family.

Remark E.2.14. *In* (E.10) *we defined $\mathfrak{P}_{(t_0,\mathfrak{w}_0)}$ as an element of $\mathcal{M}_1\left(\widehat{\mathbb{D}}([0,\infty), \mathbb{R}^m)^{[0,\infty)}\right)$. But in Proposition E.2.12 we proved that every process whose finite dimensional distributions are given by the probability measure from* (E.10) *also admits a càdlàg modification, so the difference between these two formulations is purely formal.*

Proof. Since the Martingale problem $\mathbf{MP}(B_{\mathbb{W}})$ is well-posed and every solution admits a càdlàg modification, and we need to argue, why $(\mathfrak{P}_{(t_0,\mathfrak{w}_0)}, (t_0, \mathfrak{w}_0) \in \widehat{\mathbb{D}}([0,\infty), \mathbb{R}^m))$ forms a Borel strong Markov family. Therefore let us assume that W is the underlying Lévy process and that $(P_x, x \in \mathbb{R}^m)$ forms the family of paths laws on $\mathbb{D}([0,\infty), \mathbb{R}^m)$ associated with W. By Lemma E.1.7 $(P_x, x \in \mathbb{R}^m)$ forms a continuous strong Markov family. Our claim follows now from Proposition II.2.5. in [40]. □

Proposition E.2.15.

1. *The domain $\mathcal{D}(B_{\mathbb{W}})$ is separating.*
2. *There exists a countable subset $\tilde{B} \subset B_{\mathbb{W}}$ such that for every $(\mathfrak{F}, B_{\mathbb{W}}(\mathfrak{F}))$ the set $\tilde{B}$ contains a sequence $(\mathfrak{F}_n, B_{\mathbb{W}}(\mathfrak{F}_n), n \in \mathbb{N})$ such that $\mathfrak{F}_n \to \mathfrak{F}$ and $B_{\mathbb{W}}(\mathfrak{F}_n) \to B_{\mathbb{W}}(\mathfrak{F})$ uniformly.*

Proof. 1. By definition $\mathcal{D}(B_{\mathbb{W}})$ forms an algebra, so in order to prove that $\mathcal{D}(B_{\mathbb{W}})$ is separating, we only need to show that $\mathcal{D}(B_{\mathbb{W}})$ is point separating, see Theorem 4.4.5 in [14]. Therefore let us assume that $(t_1, \mathfrak{w}_1)$ and $(t_2, \mathfrak{w}_2)$ are two different elements of $\widehat{\mathbb{D}}([0,\infty), \mathbb{R}^m)$. We have to show, that there exists a test function $\tilde{\mathfrak{F}} \in \mathcal{D}(B_{\mathbb{W}})$ such that $\tilde{\mathfrak{F}}(t_1, \mathfrak{w}_1) \neq \tilde{\mathfrak{F}}(t_2, \mathfrak{w}_2)$.
We begin with the case that $t_1 \neq t_2$. Since $C^1([0,\infty))$ is point separating, we can choose a $\varphi \in C^1([0,\infty))$ with $\varphi(t_1) \neq \varphi(t_2)$. Then it is sufficient to use the function $\mathfrak{F}_\varphi(t, \mathfrak{w}) = \varphi(t)$.
If $t_1 = t_2$, but $\mathfrak{w}(t_1) \neq \mathfrak{w}(t_2)$, we proceed similarly. Since $\mathit{span}(C_c^2(\mathbb{R}^m) \cup \{\mathbb{1}_{\mathbb{R}^m}\})$ is also separating, we choose an element f with $f(\mathfrak{w}(t_1)) \neq f(\mathfrak{w}(t_1))$ and then we choose $\mathfrak{F}_f(t, \mathfrak{w}) = f(\mathfrak{w}(t))$.
If $t_1 = t_2$ and there exists a $\hat{s} \in [0, t_1)$ such that $\mathfrak{w}_1(\hat{s}) \neq \mathfrak{w}_2(\hat{s})$, then there must exists a $g \in C_c(\mathbb{R}^{m+1})$ such that it holds for $\mathfrak{F}_g(t, \mathfrak{w}) = \int_0^t g(\mathfrak{w}(s), s)ds$ with $\mathfrak{F}_g(t_1, \mathfrak{w}_1) \neq \mathfrak{F}_g(t_2, \mathfrak{w}_2)$, or else we could conclude from Lemma E.2.11 that $\mathfrak{w}_1(s) = \mathfrak{w}_2(s)$ for all $s \in [0, t_1)$.

2. The domain $\mathcal{D}(B_{\mathbb{W}})$ of $B_{\mathbb{W}}$ is defined as the linear span of $\cup_{n\in\mathbb{N}_0}\mathbf{\Gamma}_n$, see Def. E.2.7, hence it is sufficient, when we can prove the statement for $B_{\mathbb{W}}$ restricted to $\mathbf{\Gamma}_n$.
For this purpose let us fix a set $\tilde{C}([0,\infty))$ that is dense in $\boldsymbol{span}(C_c^1([0,\infty)) \cup \{\mathbb{1}_{[0,\infty)}\})$ with respect to the norm

$$||\varphi||_{\infty,1} = ||\varphi||_\infty + ||\dot{\varphi}||_\infty.$$

Hence if $\mathfrak{F}_\varphi$ is a test function in $\mathbf{\Gamma}_n$ with the form $\mathfrak{F}_\varphi(t,\mathfrak{w}) = \varphi(t)$ for an element φ in $\boldsymbol{span}(C_c^1([0,\infty)) \cup \{\mathbb{1}_{[0,\infty)}\})$, note that $B_{\mathbb{W}}(\mathfrak{F}_\varphi)(t,\mathfrak{w}) = \dot{\varphi}(t)$, then we can choose a sequence $(\varphi_n)_{n=1}^\infty$ in $\tilde{C}([0,\infty))$ converging against φ in the norm $||\cdot||_{\infty,1}$, and we can define the sequence $(\mathfrak{F}_{\varphi_n})$ with $\mathfrak{F}_{\varphi_n}(t,\mathfrak{w}) = \varphi_n(t)$, then it follows that $(\mathfrak{F}_{\varphi_n}, n\in\mathbb{N})$ and $(B_{\mathbb{W}}(\mathfrak{F}_{\varphi_n}), n\in\mathbb{N})$ are converging bounded and pointwise against $(\mathfrak{F}_\varphi, B_{\mathbb{W}}(\mathfrak{F}_\varphi))$.
Next let us consider the countable set $\tilde{C}^2(\mathbb{R}^m)$ from the proof of Lemma E.1.11 that is dense in $\boldsymbol{span}(C_c^2(\mathbb{R}^m) \cup \{\mathbb{1}_{\mathbb{R}^m}\})$ with respect to the norm $||\cdot||_{\infty,2}$ from (E.2). If the test function $\mathfrak{F}_f$ is given by $\mathfrak{F}_f(t,\mathfrak{w}) = f(\mathfrak{w}(t))$ for a $f \in \boldsymbol{span}(C_c^2(\mathbb{R}^m) \cup \{\mathbb{1}_{\mathbb{R}^m}\})$, note that implies $B_{\mathbb{W}}(\mathfrak{F}_f)(t,\mathfrak{w}) = B_W(f(t))$, then we can choose according to Lemma E.1.11 a sequence in $(f_n, n\in\mathbb{N})$ such that $((f_n, B_W(f_n), n\in\mathbb{N})$ are converging against $(f, B_W(f))$ uniformly. Consequently the same holds for $(\mathfrak{F}_{f_n}, B_{\mathbb{W}}(f_n), n\in\mathbb{N})$ and $(\mathfrak{F}_f, B_{\mathbb{W}}(f))$.
For the next part we consider a countable set countable set $\tilde{\Gamma}_c$ that is dense in $C_c(\mathbb{R}^m+1)$ with respect to the supremum norm $||\cdot||_\infty$. If $\mathfrak{F}_{\boldsymbol{g}}$ is a test function with the form

$$\mathfrak{F}_{\boldsymbol{g}}(t,\mathfrak{w}) = \prod_{i=1}^k \int_0^t g_i(\mathfrak{w}(s),s)\,ds$$

for $\boldsymbol{g} = (g_1,...,g_k), g_i \in \boldsymbol{span}(C_c(\mathbb{R}^{m+1}) \cup \{\mathbb{1}_{\mathbb{R}^{m+1}}\}), 1\le i\le k$, which implies that

$$B_{\mathbb{W}}(\mathfrak{F})(t,\mathfrak{w}) := \sum_{j=1}^k g_j(\mathfrak{w}(t),t) \prod_{i=1,i\neq j}^k \int_0^t g_i(\mathfrak{w}(s),s)ds,$$

then we can choose $\boldsymbol{g}_n = (g_1^n,...,g_k^n)$, $g_i^n \in \tilde{\Gamma}_c, 1\le i\le k.$ with $(g_i^n, n\in\mathbb{N})$ is converging uniformly against g_i for $1\le i\le k$. If we define the test functions $\mathfrak{F}_{\boldsymbol{g}_n}$ by setting

$$\mathfrak{F}_{\boldsymbol{g}_n}(t,\mathfrak{w}) = \prod_{i=1}^k \int_0^t g_i^n(\mathfrak{w}(s),s)ds,$$

then $(\mathfrak{F}_{\boldsymbol{g}_n}, B_{\mathbb{W}}(\mathfrak{F}_{\boldsymbol{g}_n}), n\in\mathbb{N})$ is converging against $(\mathfrak{F}_{\boldsymbol{g}}, B_{\mathbb{W}}(\mathfrak{F}_{\boldsymbol{g}}))$ uniformly.
Finally let us define the set $\tilde{\mathbf{\Gamma}}_k$ as

$$\tilde{\mathbf{\Gamma}}_k := \Big\{\mathfrak{F}, \mathfrak{F}_\varphi, \mathfrak{F}_f, \mathfrak{F}_{\boldsymbol{g}}, \mathfrak{F}_{\varphi,f}, \mathfrak{F}_{\varphi,\boldsymbol{g}}, \mathfrak{F}_{f,\boldsymbol{g}}\mathfrak{F}_{\varphi,f,\boldsymbol{g}};\ \varphi\in\tilde{C}([0,\infty)), f\in\tilde{C}^2(\mathbb{R}^m),$$
$$\boldsymbol{g} = (g_1,...,g_k), g_i\in\tilde{\Gamma}_c, 1\le i\le k\Big\}$$

If $\mathfrak{F}_{\varphi,f,\boldsymbol{g}}$ is now an element of Γ_n with the general form

$$\mathfrak{F}_{\varphi,f,\boldsymbol{g}}(t,\mathfrak{w}) = \varphi(t)f(\mathfrak{w}(t))\prod_{i=1}^k \int_0^t g_i(\mathfrak{w}(s),s)ds.$$

then we can write $\mathfrak{F}_{\varphi,f,\boldsymbol{g}} = \mathfrak{F}_\varphi\mathfrak{F}_f\mathfrak{F}_{\boldsymbol{g}}$ and

$$B_{\mathbb{W}}(\mathfrak{F}_{\varphi,f,\boldsymbol{g}}) = B_{\mathbb{W}}(\mathfrak{F}_\varphi)\mathfrak{F}_f\mathfrak{F}_{\boldsymbol{g}} + \mathfrak{F}_\varphi B_{\mathbb{W}}(\mathfrak{F}_f)\mathfrak{F}_{\boldsymbol{g}} + \mathfrak{F}_\varphi\mathfrak{F}_f B_{\mathbb{W}}(\mathfrak{F}_{\boldsymbol{g}}).$$

Since the uniform convergence is maintained under linear combination and multiplication, let us choose $\mathfrak{F}_{\varphi_n}, \mathfrak{F}_{f_n}$ and $\mathfrak{F}_{(}\boldsymbol{g}_n)$ as in the upper paragraphs, and let us define $\mathfrak{F}_n := \mathfrak{F}_{\varphi_n}\mathfrak{F}_{f_n}\mathfrak{F}_{(}\boldsymbol{g}_n)$ and it follows that $(\mathfrak{F}_{\varphi_n}, B_{\mathbb{W}}(\mathfrak{F}_{\varphi_n}))$ is uniformly against $(\mathfrak{F}_{\varphi,f,\boldsymbol{g}}, B_{\mathbb{W}}(\mathfrak{F}_{\varphi,f,\boldsymbol{g}}))$. □

Proposition E.2.16. *There exists an operator $\hat{B}_{\mathbb{W}}$ that satisfies the Conditions B.2.2 for path-valued process of a Lévy process.*

Proof. This is a combination of Proposition E.2.12 and E.2.15. □

Appendix F

Auxiliary Results

F.1 Second Construction

In this section we are going to prove Lemma F.1.2, which is applied in the proof Proposition 2.6.6, which is needed for the Case IV in Chapter 4, which in turn is essential for the proof of the continuity of Ξ^{XZ} in Chapter 5. Therefore let us fix a $r \geq \max\{b/a, 0\}$ and let us assume that $\boldsymbol{\xi}^{\mathbb{W},r}, \Xi^{\mathbb{W},r}$ and Y^r are the processes obtained in Lemma 2.2.8 and Definition 2.5.2. We will now construct a new $\overline{\mathcal{N}}(E \times [0,\infty))$-valued process $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$ such that

$$\tilde{\boldsymbol{\xi}}^{\mathbb{W}} = \boldsymbol{\xi}^{\mathbb{W},r} + \tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}.$$

The process $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$ will be divided into three components, indeed we define three $\overline{\mathcal{N}}(E \times [0,\infty))$-valued process $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,1}$,$\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,2}$,$\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,3}$ with

$$\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r} = \tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,1} + \tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,2} + \tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,3}.$$

We start with the most complex one, $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,1}$, which represents the population descended from the initial particles with a level above r. This means that the process $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,1}$ will a be also Kurtz-Rodrigues representation similar to $\boldsymbol{\xi}^{\mathbb{W}}$, but all particles will have a level above r. We can construct $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,1}$ in the same fashion as we did with $\boldsymbol{\xi}^{\mathbb{W}}$. Therefore let us assume that

$$(U_i^{0,2}, X_i^{0,2})_{i=1}^{\infty}, (\mathcal{V}_{ji}^{k,2}, 1 \leq i < j < \infty), (\tilde{X}_i^2)_{i=1}^{\infty} \text{ and } (N_i^2)_{i=1}^{\infty} \tag{F.1}$$

are independent copies of

$$(U_i^0, X_i^0)_{i=1}^{\infty}, (\mathcal{V}_{ji}^k, 1 \leq i < j < \infty), (\tilde{X}_i)_{i=1}^{\infty} \text{ and } (N_i)_{i=1}^{\infty}.$$

Since all particles in $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,1}$ should have a level higher than r, we modify $(U_i^{0,2})_{i=1}^{\infty}$ by setting $\hat{U}_i^{0,2} := U_i^{0,2}$, $i \in \mathbb{N}$. We can now repeat the steps of Chapter 2 for the construction of $\boldsymbol{\xi}^{\mathbb{W}}$ to obtain $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,1}$, but we use $(\hat{U}_i^{0,2})_{i=1}^{\infty}$ and (F.1) instead of (F.2).

The next process $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,2}$ consists of the particles born by the particles inside of $\boldsymbol{\xi}^{\mathbb{W},r}$ but with a level higher than r. For this purpose let us assume that $(\mathcal{E}_k, k \in \mathbb{N}_0)$ is a collection of independent Poisson counting processes with rate 1, $(S_{k,n}; k,n \in \mathbb{N}), (U_{k,n}; k,n \in \mathbb{N})$ are two sets of independent on $(0,1)$ uniformly distributed random variables and $(\tilde{W}_{k,n}; k,n \in \mathbb{N})$ consists of independent Lévy processes in $\mathbb{R}^d$, which start at 0 and are copies of W, see Lemma 2.4.6. We also need the following lemma.

Lemma F.1.1. *Let us assume E is a Polish space, then there exists a measurable map $\phi_E : \mathcal{M}_1(E) \times [0,1] \to E$ such that for all over $[0,1]$ uniformly distributed random variable S and all $\mu \in \mathcal{M}_1(E)$ holds that $\tilde{S} := \phi(\mu, S)$ is a μ-distributed random variable on E.*

Proof. This follows directly from Lemma 2.22 in [21]. $\square$

We are now defining a collection $(W_{k,n}, U_{k,n})$ of processes that will form the atoms of $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,2}$. Next we define the "initial values" and birth times of the particles. Our first step is to set:

$$\mathcal{N}_k(t) := \mathcal{E}_k\left(\int_0^t Y^r(s)\,ds\right).$$

The birth times will be the jumps times $(\tilde{\tau}_{k,n}; n, n \in \mathbb{N}_0)$ of $(\mathcal{E}_k, k \in \mathbb{N}_0)$, where $\tilde{\tau}_{k,n}$ is the n-th jumping time of $\mathcal{E}_k$ with the convention that $\tilde{\tau}_{k,0} = 0$ and we do allow the value infinity, because there is the possibility that Y^r dies out. Indeed let us set $\mathcal{T}^r_{EX} := \inf\{t > 0 : Y^r(t) = 0\}$, which is extinction time, and let us define the process $\tilde{\mathbf{Q}}^{\mathbb{W},r}$ by setting

$$\tilde{\mathbf{Q}}^{\mathbb{W},r}_t := \mathbb{1}_{[0,\mathcal{T}^r_{EX})}(t)\frac{1}{Y^r_t}\tilde{\boldsymbol{\Xi}}^{\mathbb{X},r}_t + \mathbb{1}_{[0,\mathcal{T}^r_{EX})}(t)\delta_\dagger,$$

where $\dagger$ is a point not contained in $\mathfrak{D}$. After the extinction time $\mathcal{T}^r_{EX}$ no $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r,2}$ will

We are now defining 'initial values"

$$(\hat{W}_{k,n}, U_{k,n})\phi_{\mathfrak{D}}\left(\frac{\boldsymbol{\Xi}^{\mathbb{W},r}_{\tilde{\tau}_{k,n}\text{-}}}{Y^r(\tilde{\tau}_{k,n}\text{-})}, S_{k,n}\right).$$

Since $(U_i^{0,2}, X_i^{0,2})_{i=1}^\infty$ is a copy of $(U_i^0, X_i^0)_{i=1}^\infty$ with

Let us assume that the level system $(U_i)_{i=1}^\infty$, the genealogy Φ and the system of path-valued processes $(\mathbb{W}_i)_{i=1}^\infty$ have been constructed as in Definition 2.2.1, 2.3.1 and as in Lemma 2.4.8. We are now fixing $r \geq \max\{b/a, 0\}$. If $\boldsymbol{\Xi}^{\mathbb{W},r}$ and Y^r are the processes obtained in Lemma 2.2.8 and Definition 2.5.2 from $(\mathbb{W}_i, U_i)_{i=1}^\infty$. Let us define the filtration $\tilde{\mathcal{F}}^{\mathbb{W},r} := \sigma(\boldsymbol{\Xi}^{\mathbb{W},r})$ and the sequence of $(\tau_k)_{k=0}^\infty$ of $\tilde{\mathcal{F}}^{\mathbb{W},r}$-stopping times by setting $\tau_0 = 0$ and $\tau_{k+1} = \inf\{t > \tau_k : \Delta Y^r = -1\}$, $k \in \mathbb{N}_0$. The stopping time τ_k is the moment, where we observe the k-th death in the population $\boldsymbol{\Xi}^{\mathbb{W},r}$ meaning the particle with the highest level below r before τ_k is hitting the barrier r. Let us fix a $r \geq \max\{b/a, 0\}$. In the Definition 2.5.2 we defined the processes $\boldsymbol{\xi}^{\mathbb{W},r}, \boldsymbol{\xi}^{\mathbb{W},\geq r}$ and $\boldsymbol{\Xi}^{\mathbb{W},r}$, but also the filtration $\mathcal{F}^{\boldsymbol{\Xi},\mathbb{W},r}$, which is the right-continuous completion of $\sigma(\boldsymbol{\Xi}^{\mathbb{W},r}, \boldsymbol{\xi}^{\mathbb{W},\geq r})$. If we set $\tilde{\mathcal{F}}^{\boldsymbol{\Xi},\mathbb{W},r} := (\tilde{\mathcal{F}}^{\boldsymbol{\Xi},\mathbb{W},r}_t, t \geq 0)$ with $\tilde{\mathcal{F}}^{\boldsymbol{\Xi},\mathbb{W},r}_t := \sigma(\boldsymbol{\Xi}^{\mathbb{W},r}_s, s \leq t)$, then the difference between $\tilde{\mathcal{F}}^{\boldsymbol{\Xi},\mathbb{W},r}$ is the information contained in the path of $\boldsymbol{\xi}^{\mathbb{W},\geq r}$. The purpose of this section is to prove the following statement.

Lemma F.1.2. *It holds for all $t \geq 0$:*

$$\mathfrak{L}(\boldsymbol{\xi}^{\mathbb{W},r}_t | \mathcal{F}^{\boldsymbol{\Xi},\mathbb{W},r}_t) = \mathfrak{L}(\boldsymbol{\xi}^{\mathbb{W},r}_t | \tilde{\mathcal{F}}^{\boldsymbol{\Xi},\mathbb{W},r}_t).$$

The intuitive interpretation of the Lemma F.1.2 is that the path $(\boldsymbol{\xi}^{\mathbb{W},\geq r}_s, s \leq t)$ does not contain any new information about $\boldsymbol{\xi}^{\mathbb{W},r}_t$, if we already know the path $(\boldsymbol{\Xi}^{\mathbb{W},r}_s, s \leq t)$. This should not be surprising, because $\boldsymbol{\xi}^{\mathbb{W},r}_t$ is a functional of $\boldsymbol{\Xi}^{\mathbb{W},r}_t$ and the path of $\boldsymbol{\Xi}^{\mathbb{W},r}$ is unaffected by the path of $\boldsymbol{\xi}^{\mathbb{W},\geq r}$ and the way how the path of $\boldsymbol{\xi}^{\mathbb{W},\geq r}$ is affected by $\boldsymbol{\Xi}^{\mathbb{W},r}$ does not reveal any new information about the levels of the particles contained in $\boldsymbol{\xi}^{\mathbb{W},r}_t$. We will now transform this intuition in a proof based on the observation that we can think about $(\boldsymbol{\xi}^{\mathbb{W},\geq r}_s, s \leq t)$ as a functional of $(\boldsymbol{\Xi}^{\mathbb{W},r}_s, s \leq t)$ and some additional randomness that is independent from the path of $\boldsymbol{\xi}^{\mathbb{W},r}$ and $\boldsymbol{\Xi}^{\mathbb{W},r}$. Based on this observation the Lemma F.1.2 follows from the next lemma.

Lemma F.1.3. *Let us assume that P and Q are two independent random variables with values in the Polish spaces E_P and E_Q. Let us assume that $\phi : E_P \to \overline{E}_P$ and $\psi : \overline{E}_P \times E_Q \to \overline{E}_Q$ are measurable maps, where $\overline{E}_P$ and $\overline{E}_Q$ are also Polish spaces. Let us define the random variables $\overline{P} := \phi(P)$ and $\overline{Q} := \psi(Q, \overline{P})$, then it holds for the conditional distribution:*

$$\mathfrak{L}(\, P \,|\, \sigma(\overline{P})) = \mathfrak{L}(\, P \,|\, \sigma(\overline{Q}, \overline{P})).$$

Proof. Since $\overline{P}$ and Q are independent, it follows from $\sigma(\overline{Q}, \overline{P}) \subset \sigma(Q, \overline{P})$ that for all bounded measurable functions $f : \overline{E}_P \to \mathbb{R}$ holds

$$\mathbb{E}[f(P)|\sigma(\overline{Q}, \overline{P})] = \mathbb{E}[\mathbb{E}[f(P)|\sigma(Q, \overline{P})]|\sigma(\overline{Q}, \overline{P})] = \mathbb{E}[\mathbb{E}[f(P)|\sigma(\overline{P})]|\sigma(\overline{Q}, \overline{P})] = \mathbb{E}[f(P)|\sigma(\overline{P})].$$

□

While it is obvious that $\boldsymbol{\Xi}^{\mathbb{W},r}$, $\boldsymbol{\xi}^{\mathbb{W},r}$ and $\boldsymbol{\xi}^{\mathbb{W},\geq r}$ should take the role of $\overline{P}, P$ and $\overline{Q}$, it is difficult to identify Q. Therefore we will make a detour, instead to work with $\boldsymbol{\xi}^{\mathbb{W},\geq r}$ we will construct a new process $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$ which is not only a copy of $\boldsymbol{\xi}^{\mathbb{W},\geq r}$ but also $(\boldsymbol{\xi}^{\mathbb{W},r}, \tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r})$ has the same joint distribution as $(\boldsymbol{\xi}^{\mathbb{W},r}, \boldsymbol{\xi}^{\mathbb{W},\geq r})$. For this it is sufficient to construct $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$ in such a way that the $\overline{\mathcal{N}}(E \times [0,\infty))$-valued process given by

$$\tilde{\boldsymbol{\xi}}^{\mathbb{W}} = \boldsymbol{\xi}^{\mathbb{W},r} + \tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}.$$

is again a Kurtz-Rodrigues representation.

Remark F.1.4. *While $\boldsymbol{\xi}^{\mathbb{W},r}$ is given and unaffected by $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$, the other way around this not true, $\boldsymbol{\xi}^{\mathbb{W},r}$ effects $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$ in two ways. First, the particles in $\boldsymbol{\xi}^{\mathbb{W},r}$ give birth to new particles with levels higher than r, hence to particles belonging to $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$, Second, if a particle in $\boldsymbol{\xi}^{\mathbb{W},r}$ dies, meaning its level hits r, the particles lives on in $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$, indeed such particles in $\boldsymbol{\xi}^{\mathbb{W},r}$ continue their "life" by immigrating to $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$.*

Recall the state space $\mathfrak{D} = \widehat{\mathbb{D}}([0,\infty), \mathbb{R}^{d+1})$ of our path-valued processes $(\mathbb{W}_i)_{i=1}^{\infty}$, see Definition 2.4.7 and Lemma 2.4.8. As in Chapter 2 we build a collection $(\mathbb{W}_i^r, U_i^r)_{i=1}^{\infty}$ of processes with values in $\mathfrak{D} \times [0,\infty)$ and these will form the atoms of $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$. For the construct of $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$ we will make us of the following list components which is roughly based on our Ingredients list 2.1.2.

Assumptions F.1.5.

1. *$(\tilde{S}_i, i \in \mathbb{N})$ and $(S_i^k; i,k \in \mathbb{N}_0)$ are a collections of independent random variables uniformly distributed over $(0,1)$.*
2. *$(\mathcal{V}_{ji}^r, 1 \leq i < j < \infty)$ are a collection of independent Poisson point processes over $[0,\infty) \times [0,\infty)$ with intensity measure $2a\ell eb[0,\infty) \otimes \ell eb[0,\infty)$.*
3. *$(\mathcal{E}_j^{k,r}, j \in \mathbb{N})$ is a collection of independent Poisson point processes over $[0,\infty) \times [0,\infty)$ with intensity measure $2a\ell eb[0,\infty) \otimes \ell eb[0,\infty)$.*

We assume further that the different components are independent from each other and also independent from components from the Assumption 2.1.2, which implies the independence from $\boldsymbol{\xi}^{\mathbb{W},r}$, since $\boldsymbol{\xi}^{\mathbb{W},r}$ is built from the latter.

Our first step is to use the $(X_i^{0,r}, U_i^{0,r})_{i=1}^{\infty}$ is a collection random variables with values in $[0,\infty) \times \mathbb{R}^d$, such that if we set $\tilde{\boldsymbol{\xi}}^{X,\geq r} := \sum_{i=1}^{\infty} \delta_{(X_i^{0,2}, U_i^{0,2})}$, then

$$\mathfrak{L}(\tilde{\boldsymbol{\xi}}^{X,\geq r}|\mathbf{Q}_0^X, Y_0) = \mathbf{PPP}(\boldsymbol{\Xi}_0^X \otimes \ell eb[r,\infty)),$$

recall that $\boldsymbol{\Xi}_0^X = Y_0 \mathbf{Q}_0^X$.

Lemma F.1.6. *There exists a map*

$$\rho : \mathcal{M}_1(\mathbb{R}^d) \times [0,\infty) \times [0,1] \to \mathbb{R}^d \times [0,\infty)$$

such that, if $(\mu, y) \in \mathcal{M}_1(\mathbb{R}^d) \times [0,\infty)$ and S is a $[0,1]$-uniformly distributed random variable, then $(X,V) := \rho(\mu, y, S)$ is a random variable with distribution $\mu \otimes \boldsymbol{Exp}(y)$.

Proof. The existence of ρ follows, when we apply the Lemma 2.22 from [21] to the Markov kernel $\alpha : \mathcal{M}_1(\mathbb{R}^d) \times [0,\infty) \to \mathcal{M}_1(\mathbb{R}^d \times [0,\infty))$ given by $\alpha(\mu, y) = \mu \otimes \mathbf{Exp}(y)$. □

With ρ in our hand we first set $(X_i^{0,r}, V_i^{0,r}) := \rho(\mathbf{Q}_0^X, Y_0, \tilde{S}_i)$, $i \in \mathbb{N}$, then $U_1^{0,r} = r + V_1^{0,r}$ and $U_i^{0,r} = r + \sum_{j=1}^{i} V_j^{0,r}$.

The dynamics of $(U_i^r)_{i=1}^\infty$ are very similar to one of $(U_i)_{i=1}^\infty$, see Definition 2.2.1, but there are the additional Lines (F.4), (F.6) and (F.7) due to the observation made in Remark F.1.4.

Definition F.1.7 (Level System II)**.** *The levels processes $(U_i^r, i \in \mathbb{N})$ with $U_i^r : \Omega \times [0,\infty) \to [0,\infty]$ are the solution of an infinite system of differential equation with jumps which is given for all $j \in \mathbb{N}$ and $t \in [0,\infty)$ by:*

$$U_j^r(t) := U_j^r + \int_0^t \mathbb{1}_{[0,\infty)}(U_j^r(s\text{-})) \left[a\left(U_j^r(s\text{-})\right)^2 - bU_j^r(s\text{-})\right] ds \tag{F.2}$$

$$+ \sum_{i=1}^{j-1} \int_0^t \int_{U_{j\text{-}1}^r(s\text{-})}^{U_j^r(s\text{-})} v - U_j^r(s\text{-}) \mathcal{V}_{ji}^r(dv, ds) \tag{F.3}$$

$$+ \sum_{i=2}^{j-1} \sum_{k=1}^{\infty} \int_0^t \int_{U_{j\text{-}1}^r(s\text{-})}^{U_j^r(s\text{-})} [v - U_j^r(s\text{-})] \mathbb{1}_{\{k \le Y_{s\text{-}}^r\}} \mathcal{E}_i^{k,r}(dv, ds) \tag{F.4}$$

$$+ \sum_{i=2}^{j-1} \sum_{k=1}^{i-1} \int_0^t \int_{U_{i\text{-}1}^r(s\text{-})}^{U_i^r(s\text{-})} U_{j\text{-}1}^r(s\text{-}) - U_j^r(s\text{-}) \mathcal{V}_{ik}^r(dv, ds) \tag{F.5}$$

$$+ \sum_{i=2}^{j-1} \sum_{k=1}^{\infty} \int_0^t \int_{U_{i\text{-}1}^r(s\text{-})}^{U_i^r(s\text{-})} [U_{j\text{-}1}^r(s\text{-}) - U_j^r(s\text{-})] \mathbb{1}_{\{k \le Y_{s\text{-}}^r\}} \mathcal{E}_i^{k,r}(dv, ds) \tag{F.6}$$

$$+ \mathbb{1}_{\{j=1\}} \int_0^t [r - U_j^r(s\text{-})] \mathbb{1}_{\{\Delta Y_s^r = -1\}} dY_s^r. \tag{F.7}$$

Again as in Definition 2.2.1, we interpret the inner integrals of (F.3) (F.4) *and* (F.5), (F.6) *as zero, if $(U_{j\text{-}1}^r(s\text{-}), U_j^r(s\text{-})) = (\infty,\infty)$ and $(U_{i\text{-}1}^r(s\text{-}), U_i^r(s\text{-})) = (\infty,\infty)$. Similar the inner expression of integral in* (2.11) *becomes zero, if $U_j^r(s\text{-}) = \infty$.*

The Lines (F.2), (F.3) and (F.5) have their counterpart in Definition 2.2.1, where we defined the dynamics of $(U_i)_{i=1}^\infty$. The new Lines (F.4) and (F.6) describe the effect that new particles are born into $\tilde{\boldsymbol{\xi}}^{\mathbb{W}, \ge r}$ by $\boldsymbol{\xi}^{\mathbb{W},r}$. The last Line (F.7) only effects the process U_1^r and it described the jumps of U_1^r due to the deaths in $\boldsymbol{\xi}^{\mathbb{W},r}$ which occur, when the level of a particle in $\boldsymbol{\xi}^{\mathbb{W},r}$ hits the value r. If this happens, then the dying particle becomes the lowest particle in $\boldsymbol{\xi}^{\mathbb{W},r}$.
We are now going to define the spatial processes $(\mathbb{W}_i^r)_{i=1}^\infty$, hereby our procedure will differ greatly from the one of Chapter 2, becaue we will make use of the following lemma.

Lemma F.1.8. *Recall that $(\mathfrak{P}_{(t_0,\mathfrak{w}_0)}, (t_0, \mathfrak{w}_0) \in \mathfrak{D})$ is the Borel strong Markov family of the path-valued process $\mathbb{W}$ associated with the Lévy process W. There exists a measurable maps such that*

$$\Psi : \mathcal{M}_1(\mathfrak{D}) \times [0,1] \to \mathbb{D}([0,\infty), \mathfrak{D}))$$

such that, if $S \sim U_{(0,1)}$ and $\Theta \in \mathcal{M}_1(\mathfrak{D})$, then $\Psi := \phi(\mu, S)$ is a process with law $\int_{\mathfrak{D}} \mathfrak{P}_{(t_0,\mathfrak{w}_0)} \mu(d(t_0, \mathfrak{w}_0))$.

Proof. Let us define the Markov kernel $\alpha : \mathcal{M}_1(\mathfrak{D}) \to \mathcal{M}_1(\mathbb{D}([0,\infty), \widehat{\mathbb{D}}([0,\infty), \mathbb{R}^{d+1}))))$ by setting $\alpha(\mu) := \int_{\mathfrak{D}} \mathfrak{P}_{(t_0,\mathfrak{w}_0)}\mu(d(t_0,\mathfrak{w}_0))$, then we can apply Lemma 2.22 from [21] to obtain Ψ. $\square$

We start by setting $\mathbb{W}_i^{r,0} = (0, \mathfrak{X}_i^r(0), \mathfrak{L}_i^r(0))$, where $(\mathfrak{X}_i^r(0), \mathfrak{L}_i^r(0))$ is the constant path in $\mathbb{D}([0,\infty), \mathbb{R}^{d+1})$ with $(\mathfrak{X}_i^r(0,s), \mathfrak{L}_i^r(0,s)) = (X_i^{0,r}, 0)$.

We begin with the process $\mathbb{W}_1^r$, therefore let us define the stopping times $(\tau_k^1)_{k=0}^\infty$ as the jump times of the level process U_1^r (but $\tau_0^i = 0$), indeed $(\tau_k^1)_{k=0}^\infty$ represents the moments where a new particle becomes the particle with the lowest level in $\boldsymbol{\xi}^{\mathbb{W},\geq r}$. Let us also define

$$J_1 := \{s > 0 : \Delta Y_s^r = -1\} \subset (\tau_k^1)_{k=0}^\infty$$

as the collections of those moments in which a particle in $\boldsymbol{\xi}^{\mathbb{W},r}$ dies. If $\boldsymbol{\xi}^{\mathbb{W},r}$ goes instinct in finite time, i.e. $\mathcal{T}_{EX}^r < \infty$ with

$$\mathcal{T}_{EX}^r := \{s \geq 0 : Y_s^r = 0\},$$

then J_1 is finite and there exists $\hat{k} \in \mathbb{N}$ such that $\tau_{\hat{k}}^1 := \max J_1 = \mathcal{T}_{EX}^r$. Further, if $\mathcal{T}_{EX}^r < \infty$, then $\tau_{\hat{k}}^1 = \mathcal{T}_{EX}^r$ is the last time that U_1^r and so $\tau_k^1 = \infty$ for $k > \hat{k}$. If $\tau_k^1 \notin J_1$ and $\tau_k^1 < \mathcal{T}_{EX}^r$, then τ_k^1 is a moment in which a particle inside of $\boldsymbol{\xi}^{\mathbb{W},r}$ gives birth to a particle with a level between r and U_1^r. If $\tau_k^1 \in J_1$, then

$$\Delta \boldsymbol{\xi}_{\tau_k^1}^{\mathbb{W},r} = \delta_{\mathbb{W}_{Y^r(\tau_k^1\text{-})}(\tau_k^1\text{-})},$$

(recall that $(\mathbb{W}_i, U_i)_{i=1}^\infty$ are processes forming $\boldsymbol{\xi}^{\mathbb{W}}$ and $\boldsymbol{\xi}^{\mathbb{W},r}$). For our next step we need the following map.

Lemma F.1.9. *Recall that $\mathfrak{D} = \widehat{\mathbb{D}}([0,\infty), \mathbb{R}^{d+1})$ and that $(\mathfrak{P}_{(t_0,\mathfrak{w}_0)}, (t_0,\mathfrak{w}_0) \in \mathfrak{D})$ is the Borel strong Markov family of the path-valued process $\mathbb{W}$ associated with the Lévy process W. There exists a measurable maps such that*

$$\Psi : \mathcal{M}_1(\mathfrak{D}) \times [0,1] \to \mathbb{D}([0,\infty), \mathfrak{D}))$$

such that, if $S \sim U_{(0,1)}$ and $\Theta \in \mathcal{M}_1(\mathfrak{D})$, then $P := \phi(\mu, S)$ is a process with law $\int_{\mathfrak{D}} \mathfrak{P}_{(t_0,\mathfrak{w}_0)}\mu(d(t_0,\mathfrak{w}_0))$.

Proof. Let us define the Markov kernel $\alpha : \mathcal{M}_1(\mathfrak{D}) \to \mathcal{M}_1(\mathbb{D}([0,\infty), \widehat{\mathbb{D}}([0,\infty), \mathbb{R}^{d+1}))))$ by setting $\alpha(\mu) := \int_{\mathfrak{D}} \mathfrak{P}_{(t_0,\mathfrak{w}_0)}\mu(d(t_0,\mathfrak{w}_0))$, then we can apply Lemma 2.22 from [21] to obtain Ψ. $\square$

For the construct of $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$ we will make us of the following list components which is roughly based on our Ingredients list 2.1.2. Based on this observations we define

$$\mathbb{W}_1(t) := \mathbb{W}_1^k(t - \tau_k^1), \quad \text{if } t \in [\tau_k^1, \tau_{k+1}^1),$$

where the collection of $\mathfrak{D}$-valued processes $(\mathbb{W}_1^k, k \in \mathbb{N}_0)$ is given by $\mathbb{W}_1^0 = \Psi(\delta_{\mathbb{W}_1^{r,0}}, S_1^0)$ and for $k \in \mathbb{N}$ by

$$\mathbb{W}_1^k := \begin{cases} \Psi(\Delta \boldsymbol{\xi}_{\tau_k^1}^{\mathbb{W},r}, S_1^k), & \text{if } \tau_k^1 \in J_1, \\ \Psi(\tilde{\mathbf{Q}}_{\tau_k^1}^{\mathbb{W},r}, S_1^k), & \text{if } \tau_k^1 \notin J_1 \text{ and } \tau_k^1 < \infty. \end{cases}$$

For purely formal reason we set $\mathbb{W}_1^k = (\tilde{t}, \tilde{\mathfrak{w}})$, where $(\tilde{t}, \tilde{\mathfrak{w}})$ is an arbitrary point in $\mathfrak{D}$, for $k \in \mathbb{N}$ with $\tau_k^1 = \infty$.

For the process $\mathbb{W}_i^r$ with $i > 1$, we assume that $\mathbb{W}_1^r, ..., \mathbb{W}_{i\text{-}1}^r$ have been already defined, and therefore we can define $\tilde{\mathbf{Q}}^{\mathbb{W},r,i\text{-}1} := \frac{1}{i\text{-}1}\sum_{j=1}^{i-1}\delta_{\mathbb{W}_j^r}$. We write $(\tau_k^2)_{k=0}^{\infty}$ for the jump times of U_i^r. Let us define $J_i := \{s > 0 : U_i^r(s) = U_{i\text{-}1}^r(s\text{-})\}$, then $\tau_k \notin J_i$ is one of the moments, where a new particle with a level between U_i^r and $U_{i\text{-}1}^r$ is born. Otherwise, if $\tau_k \in J_i$, then a new particle is born with a level below $U_{i\text{-}1}^r$, and the particle with index i takes over the identity of the particle with index $i-1$. Based on this observations we define

$$\mathbb{W}_i(t) := \mathbb{W}_i^k(t - \tau_k^i), \quad \text{if } t \in [\tau_k^i, \tau_{k+1}^i),$$

where the collection of $\mathfrak{D}$-valued processes $(\mathbb{W}_i^k, k \in \mathbb{N}_0)$ is given by $\mathbb{W}_i^0 = \Psi(\delta_{\mathbb{W}_i^{r,0}}, S_i^0)$ and for $k \in \mathbb{N}$ by

$$\mathbb{W}_i^k := \begin{cases} \Psi(\delta_{\mathbb{W}_{i\text{-}1}(\tau_k^i\text{-})}, S_i^k), & \text{if } \tau_k^i \in J_i \\ \Psi(\tilde{\mathbf{Q}}^{\mathbb{W},r,i}_{\tau_k^i}, S_i^k), & \text{if } \tau_k^i \notin J_i \text{ and } \tau_k^i < \infty. \end{cases}$$

Proposition F.1.10. *If we define the $\overline{\mathcal{N}}(\mathfrak{D} \times [0,\infty))$-valued processes $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$ and $\tilde{\boldsymbol{\xi}}^{\mathbb{W}}$ by*

$$\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r} = \sum_{i=1}^{\infty} \delta_{\mathbb{W}_i^r(t), U_i^r(t)}$$

and $\tilde{\boldsymbol{\xi}}^{\mathbb{W}} = \boldsymbol{\xi}^{\mathbb{W},r} + \tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$, then $\tilde{\boldsymbol{\xi}}^{\mathbb{W}}$ is a KR-representation, indeed a solution of the martingale problem $\boldsymbol{MP}(\mathbf{A_B})$.

Proof. For the proof we can repeat the arguments in Section 2.5 which we used to prove Corollary 2.5.15. □

Proof of Lemma F.1.2. Let us fix a $t \geq 0$. We start to apply the Lemma F.1.3 to show that

$$\mathfrak{L}(\boldsymbol{\xi}_t^{\mathbb{W},r} | \mathcal{F}_t^{\boldsymbol{\Xi},\mathbb{W},r}) = \mathfrak{L}(\boldsymbol{\xi}_t^{\mathbb{W},r} | \tilde{\mathcal{F}}_t^{\boldsymbol{\Xi},\mathbb{W},r}). \tag{F.8}$$

For the application of Lemma F.1.2 we set

$$P := (\boldsymbol{\xi}_s^{\mathbb{W},r}, s \leq t), \quad \overline{P} := (\boldsymbol{\Xi}_s^{\mathbb{W},r}, s \leq t) \quad \overline{Q} := (\tilde{\boldsymbol{\xi}}_s^{\mathbb{W},\geq r}, s \leq t).$$

As the random variable Q we choose the components from Assumption F.1.5. Since the path $(\boldsymbol{\Xi}_s^{\mathbb{W},r}, s \leq t)$ is measurable with respect to the path $(\boldsymbol{\xi}_s^{\mathbb{W},r}, s \leq t)$, and since the path $(\tilde{\boldsymbol{\xi}}_s^{\mathbb{W},\geq r}, s \leq t)$ is measurable with respect to $(\boldsymbol{\Xi}_s^{\mathbb{W},r}, s \leq t)$ and the components from Assumption F.1.5 we can conclude by the factoring corollary, see Corollary 1.97 from [24], that there must exists measurable functions ϕ and ψ with $\overline{P} = \phi(P)$ and $\overline{Q} = \psi(Q, \overline{P})$. Lemma F.1.3 tells us now that (F.8) is true, because the components from Assumption F.1.5 are independent from $(\boldsymbol{\xi}_s^{\mathbb{W},r}, s \leq t)$. Further the process $\tilde{\boldsymbol{\xi}}^{\mathbb{W}} := \boldsymbol{\xi}^{\mathbb{W},r} + \tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$ from Proposition F.1.10 is a KR-representation. The same is true for

$$\boldsymbol{\xi}^{\mathbb{W}} := \boldsymbol{\xi}^{\mathbb{W},r} + \boldsymbol{\xi}^{\mathbb{W},\geq r}.$$

Further $\tilde{\boldsymbol{\xi}}^{\mathbb{W},\geq r}$ and $\boldsymbol{\xi}^{\mathbb{W},\geq r}$ have the same initial distribution. Since the martingale problem $\mathbf{MP}(\mathbf{A_B})$, see Definition B.2.7, is well-posed, we can conclude that $(\boldsymbol{\xi}_s^{\mathbb{W},r}, \tilde{\boldsymbol{\xi}}_s^{\mathbb{W},\geq r}, s \leq t)$ has the same joint distribution as $(\boldsymbol{\xi}_s^{\mathbb{W},r}, \boldsymbol{\xi}_s^{\mathbb{W},\geq r}, s \leq t)$. Then the Lemma F.1.2 follows from (F.8). □

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