A History of Configurations from Möbius to Coxeter

Dissertation zur Erlangung des Grades 'Doktor der Naturwissenschaften' am Fachbereich Physik, Mathematik und Informatik der Johannes Gutenberg-Universität in Mainz

Author: Sima Faghihi

Datum: 10/2/2021

Contents

Introduction

Configurations were first explicitly defined as special types of geometric objects by Theodor Reye in 1876. Their "prehistory", however, is far older and surprisingly rich. Important examples of configurations were discovered throughout the course of the nineteenth century, and two of the most famous examples date back to the seventeenth century (the configurations of Pascal and Desargues). These refer to special geometric incidence structures, finite sets of points, lines, and planes. Some configurations are trivial – three lines in the plane that form a triangle (a $(3,2)$ configuration) or four planes that form a tetrahedron (a $(4,3)$. These correspond to figures in general position, whereas other configurations reflect very subtle geometric properties associated with figures in special position. Pascal's theorem, for example, starts with six points on a conic, which leads to three more points that lie on a line. The special case of Pascal's theorem in which the conic degenerates into two lines with three points on each line is often called Pappus's Theorem. This leads to a $(9,3)$ configuration of points and lines in the real projective plane.

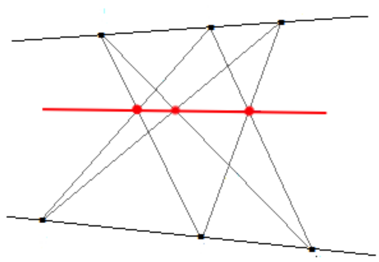


Figure 0.1 A $(9,3)$ configuration derived from Pappus's Theorem

While many special studies have been devoted to configurations, this topic has never been studied historically in a systematic way.[1] One reason for this is that the topic fell out of favor after around 1920 when abstract trends began to dominate modern research. This does not mean that configurations were entirely forgotten, however. Rather the original concept was transformed from a geometrical idea into a more algebraic one, just as projective geometry itself became a more abstract discipline. Viewed in this more modern setting, configurations appear as a special topic within combinatorics and group theory. In the hands of several mathematicians, starting around 1930, configurations gradually became

[1]Configurations appear, for example, in several specialized studies on graph theory by Harald Gropp; these do not, however, deal with the larger transformation in geometrical research that took place during the period covered here.

part of graph theory. In this thesis, I describe how this transformational process gradually unfolded. At the same time, I show how the history of configurations was strongly connected with parallel developments in group theory and combinatorics.

For the purposes of this study, it will be convenient to treat the history of configurations as unfolding in three periods, following a scheme used by David Hilbert to characterize three phases in the history of invariant theory: 1) naive, 2) formal and 3) critical. During the first, covering roughly the five decades that began with the work of A.F. Möbius and ended with Theodor Reye's first explication of the phenomenon, geometers became increasingly aware of the importance of special configurations of points, lines, and planes, especially in the context of a new finding: the Reye configuration. Thus, Ludwig Otto Hesse highlighted the importance of the configuration of nine inflection points that lie in fours on twelve lines in $P^2(\mathbb{C})$ as a key structure for studying nonsingular cubic curves. In the 1860s, Ludwig Schläfli classified cubic surfaces by making use of a double-six of lines, which comprise twelve of the 27 lines that lie on nonsingular cubic surfaces. Around the same time, E.E. Kummer uncovered the special (16_6) configuration formed by the singular points and planes of Kummer surfaces, quartics with the maximal number of singularities. All of these examples were taken up by Camille Jordan in his *Traité des substitutions et des équations algébriques* (1870). Jordan showed how their symmetries could be exploited in order to reduce the corresponding algebraic equations to simpler types. Thus, the interplay between Galois theory and configurations was already well documented even before Reye introduced the general concept of a geometrical configuration. This background in algebraic geometry, discussed in Chapter 3, greatly enriched the theory once Reye called attention to the general case.

The second period thus begins in 1876 when Theodor Reye first alluded to the general phenomenon of configurations in the second edition of his *Geometrie der Lage*. In the following passage, he described the Desargues configuration (10_3) in these words:

> Die den Satz erläuternde Figur verdient Beachtung als Repräsentant einer Gattung von merkwürdigen, durch eine gewisse Regelmässigkeit ausgezeichneten Configurationen. Sie besteht aus 10 Punkten und 10 Geraden; auf jeder der Geraden liegen drei von den 10 Punkten, und durch jeden dieser Punkte gehen drei von den 10 Geraden. [Reye 1876, p. 5]

Six years later, Reye introduced the notation n_k for a configuration in the real projective plane (or space) consisting of n points and lines (or planes), k of which are incident [Reye 1882, p. 94]. Soon after the publication of this article in the first volume of *Acta Mathematica*, a burst of interest in this topic took place among the geometers of this era.

The 1880s thus witnessed the first systematic studies of geometric configurations in the plane or space with various new discoveries by S. Kantor, V. Martinetti, A. Schönflies, C. Segre, G. Veronese, E. Steinitz and others.[2]

In 1888 Jan de Vries introduced the symbol (p_γ, q_π) to denote a configuration with p points and q lines such that γ lines pass through each point and π points lie on each line, where these numbers satisfy the property that $p \cdot \gamma = q \cdot \pi$ [Steinitz 1910a, p. 482][Vries 1888, p. 63]. Later, this symbol was generalized to describe configurations in spaces of higher dimension. This was a natural development once geometers no longer took $\mathbb{R}^2$ and $\mathbb{R}^3$ (or their extensions to the projective plane and space) as the primary settings for geometrical objects. Once higher-dimensional geometries were widely accepted, the classical cases simply became examples of an $\mathbb{R}^n$ or $P^n(\mathbb{R})$. E.H. Moore formulated a definition for general geometric configurations in $\mathbb{R}^n$ [Moore 1896, p. 265] by associating a square matrix to a given configuration. This matrix-notation for configurations was later adopted by mathematicians such as Oswald Veblen and F.W. Levi [Veblen and Young 1910, p. 38], [Levi 1929] (see Section 1.2). Other writers used simple numbering schemes to indicate the incidence relations in a configuration (see Section 2.2.2). By exhibiting such a scheme, they could establish the existence of a specific type of configuration, but only in an abstract sense.

As will be seen in Chapter 2, most of the more important configurations for geometry were already known before the advent of the second period, an assessment supported by the discussion in [Hilbert and Cohn-Vossen 1932]. Yet even during that second period, which saw the first systematic studies of abstract configurations, most investigators were primarily interested in cases corresponding to real figures in the plane or 3-space; Ernst Steinitz called these schemes realizable configurations. In his dissertation from 1894 on projective configurations, Steinitz investigated the question of when an abstract $(n,3)$ configuration can be realized in the plane. In certain respects, Steinitz can be thought of as a transitional figure between the second and third periods of the present study. His research survey [Steinitz 1910a] summarized many of the main results of the second period, but in doing so he also emphasized the close connections between configurations and the symmetry groups associated with them. In fact, he pointed to the fertility of starting with a known finite group that acts on a given space as a standard way to derive important configurations. Steinitz began his own mathematical research with a topic on

[2] Much of this literature was summarized in 1910 by Ernst Steinitz in [Steinitz 1910a], an article for the *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*. This monumental multi-volume work was intended as a survey of nearly all mathematical knowledge available at the time. Its numerous volumes, published from 1898 to 1935, contained articles by eminent scholars dealing with all major branches of mathematics.

realizable configurations, but he also contributed a number of important results to group theory as well as to the theory of polyhedra. Still, his best-known and most influential work was the article “Algebraische Theorie der Körper”, which also appeared in 1910. Its impact can easily be traced through the early algebraic works of Emmy Noether.

Band 3-1-1 of the German Encyclopedia was published over the course of the period 1906 to 1910. The nine articles contained in it provide an overview of geometric topics, few of which reflect the spirit of axiomatization that would later dominate research on foundations of geometry. Steinitz’s survey was entitled “Configurations in Projective Geometry”, entirely consistent with the established position of projective geometry during this period. By the end of the 1920s, however, abstract algebra and topology had emerged not only as leading research fields but also as symbols for an entirely new trend in mathematics. Mathematicians turned away from studies focused on concrete geometrical objects in favor of general theories based on rigorously defined abstract concepts. Although configurations never quite disappeared during this third phase, they were now mainly studied from this more modern perspective.

Rather than following the chronological course of events, I have chosen to structure this study so that the reader first gains an idea of the modern theory, drawing especially on Coxeter’s expository paper [Coxeter 1950].[3] This paper gives a very good overview of many of the most important configurations studied in the past, but discussed from the point of view of graph theory and the corresponding automorphism groups. Later in life, Coxeter became a celebrated mathematician, famous for bringing classical geometry back to life (see [Roberts 2006]). Coxeter’s book *Regular Polytopes* (1948) was also a highly successful monograph. More recently, many new investigations on the subject of configurations have taken place, as illustrated by [Gruenbaum 2009] and [Artebani and Dolgachev 2009]. The present study, however, only goes up to the middle of the twentieth century.

The first chapter begins with a brief discussion of developments that preceded the main periods of interest for this study. Two of the most important configurations in plane geometry derive from famous incidence theorems that were first presented by Girard Desargues and Blaise Pascal in the seventeenth century. The works of these two pioneering figures aimed at reconstructing the older theory of conic sections by means of projective principles. It was not until the nineteenth century, however, that these ideas caught the attention of leading geometers, a number of whom found a series of new plane configurations derived in connection with the classical theorem of Pascal. If one

[3]The three other most important works for this study are [Steinitz 1910a], [Levi 1929], and [Hilbert and Cohn-Vossen 1932].

joins six arbitrary ordered points on a conic to form a hexagon, then the three pairs of opposite sides meet in three points that lie on a straight line, called the Pascal line. (In the case of a degenerate conic, the corresponding configuration is a special type of 9_3, as noted above.) In 1828, Jakob Steiner directed the attention of geometers to the complete figure obtained by taking all possible orderings of six fixed points on a conic. This leads to 60 different hexagons, and thus 60 different Pascal lines altogether. These pass in threes through 60 points, forming a 60_3 configuration known as a Pascal Hexagram (or Hexagrammum Mysticum). Numerous additional configurations were found over the decades that followed, and all of these were summarized in [Ladd-Franklin 1879].

Chapter 1 also touches on modern studies of configurations beginning around 1930, when they were viewed as part of another branch of mathematics, namely graph theory. During the second half of the twentieth century graph theory became a standard course taught in mathematics faculties, one reason being its many connections with topics in group theory, geometry, knot theory, etc. Graph theory also has many applications in science and engineering. Since many types of problems can be represented by graphs, this field came to serve as a common language utilized by mathematicians, chemists, electrical engineers, and social scientists. One of the earliest examples of this was proposed by Arthur Cayley in 1874 when he related his work on trees to contemporaneous studies of chemical compositions. Nowadays investigations on the subject of configurations rely heavily on their graphical presentations. Various graph-theoretic concepts and methods can also be used to study configurations more closely.

A leading mathematician from the third period was Friedrich Wilhelm Levi, who explicitly connected configurations to graph theory. Levi was also the author of the first monograph devoted to configurations. His *Geometrische Konfigurationen* [Levi 1929] was written as a textbook for mathematics teachers and advanced students, and it reflects this new modern spirit by showing how configurations can be studied by drawing on combinatorics and group theory. At the same time, in this book only a few traces can be found pointing to the geometric background that originally motivated Reye and others. Instead, it focuses on those configurations that became part of a more abstract combinatorial theory, leaving aside the question of whether a configuration can be realized in a given space.

Levi's book has just six chapters, the first on group theory and the second on combinatorial topology. These are intended as brief introductions to these topics, whereas Chapter 3 is somewhat longer and discusses the simplest projective configurations (embedded in the plane or 3-space). Chapter 4 concerns polyhedral configurations, starting with Desargues' theorem. These are special types that arise by taking plane sections of a polyhedral figure. Chapter 5 is entirely devoted to the Pascal figure and its various elaborations. Chapter 6 then passes from configurations to a discussion of regular polytopes and their groups, etc. All this is very much in the spirit of Coxeter's work from the mid-century.

Levi later taught a lecture course in India, published as *Finite geometrical systems* [Levi 1942], in which he invented a bipartite graph that gives a faithful representation of the incidence structure for the points and lines (planes) of a configuration. This so-called Levi graph thus provides a one-to-one correspondence with a given configuration. Levi's textbook provides a very detailed account of this construction, which was later popularized by H.S.M. Coxeter, who coined the terminology "Levi graph" in his article [Coxeter 1950].

A different type of graph construction was introduced by Karl Menger and also popularized by Coxeter in [Coxeter 1948]. Menger graphs, however, do not determine a configuration uniquely, unlike Levi graphs, which are essentially isomorphic with their configurations. Using the latter, for example, one can study the symmetries of configurations, thereby linking group theory and combinatorics. Coxeter's writings emphasized this linkage between geometrical configurations, graph theory, and group theory, whereas the latter two topics are missing in the book by Hilbert and Cohn-Vossen. In terms of popularization, Coxeter is the most important figure for the third stage in history of configurations. His publications were among the few remarkable works on this topic that appeared in the mid-twentieth century during the era of Bourbaki.

The second chapter focuses on classical geometric configurations, that is finite incidence structures that arise either in the projective plane or 3-space. Many of the most important examples were presented in *Anschauliche Geometrie* [Hilbert and Cohn-Vossen 1932], which did much to popularize this topic. This book was based on a lecture course with the same title taught by David Hilbert no fewer than four times in Göttingen during the 1920s. This might seem at first very surprising given that Hilbert staked his mathematical reputation on axiomatic methods, whereas his colleague Felix Klein famously promoted ideas based on physical intuition and geometrical drawings and models. These aspects of mathematical understanding were, however, by no means foreign to Hilbert, who enjoyed immense success teaching this course. In the hands of Stephan Cohn-Vossen, their book became a classic that continues to be read even today. Since [Hilbert and Cohn-Vossen 1932] was intended merely as an introduction to topics in geometry, the presentations are highly informal, nothing like the systematic style found in most mathematical textbooks. The goal was merely to awaken interest in various geometrical topics. Hilbert and Cohn-Vossen also neglected to give references or even the names of those mathematicians who made important contributions to these developments. Their book thus contains almost no historical information relating to how configurations arose, when they were studied, and by whom. Nevertheless, it serves as a useful guide for this study, and I will refer to it throughout Chapter 2. The third chapter of [Hilbert and Cohn-Vossen 1932], entitled "Projective Configurations", presents many examples and offers a very readable overview of this topic. The authors mainly describe important configurations in the real plane or 3-space, omitting those in complex spaces, but also discussing abstract configurations that cannot be embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$. They also make no references to group theory, except for some brief remarks about the automorphism groups of configurations. In this regard, Ernst Steinitz's encyclopedia article [Steinitz 1910a] proves to be very useful as a guide to this aspect of the theory, as well as for its overview of the literature. Using it, I have in many cases been able to give precise references to works that Hilbert drew upon. Hilbert and Cohn-Vossen's book can be seen in this respect as a successful popularization of certain results described by Steinitz. His survey, on the other hand, represents the most important source for understanding how the theory developed beginning with Reye's work, which first launched the subject.

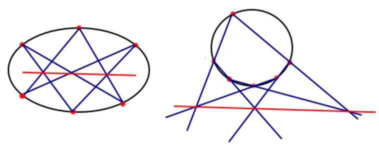


Figure 0.2 Figures illustrating Pascal's Theorem

The third chapter describes classical configurations that arose during the first period in the context of algebraic geometry. This topic was only mentioned in passing by Hilbert and Cohn-Vossen in their book, since they wanted to avoid the use of algebra and analysis whenever possible. One can discuss many properties of conic sections and quadric surfaces without algebra, of course, but these geometrical objects are essentially devoid of interesting configurations (the only exception being those derived from Pascal's Theorem). Cubic and quartic curves and surfaces, on the other hand, offered geometers a whole array of new possibilities. Several of these will be mentioned, but the main focus will fall on two important special cases, namely, the Hessian configuration and the Schläfli double-six configuration. As mentioned above, the Hessian configuration arises naturally from the 9 inflection points common to a 1-parameter family of nonsingular cubic curves in the complex projective plane. These 9 points lie in threes on 12 lines that pass through them in fours to form a $(9_4, 12_3)$ configuration. Schläfli's double-six consists of 12 lines and 30 points, which lie in groups of five on the lines forming a $(30_2, 12_5)$ configuration.

The fourth chapter discusses groups associated with some of the more significant configurations. Since configurations have rich symmetry properties, they make for interesting objects of study from the perspective of group theory. The investigation and classification of finite groups was a major new field of research at the end of the nineteenth century, which naturally led to questions about those groups that act on special objects in the plane or 3-space (e.g. the Platonic solids). The group of a geometric configuration refers to the totality of collineations that leave the configuration invariant. An important example comes from the Hesse configuration arising from the nine inflection points of plane cubic curves. The collineation group which acts on Hesse's configuration (generally on a Hessian pencil) was discovered by Camille Jordan in 1878. In this chapter we mainly study the Hessian group of 216 collineations which acts on the Hessian pencil while keeping the Hessian configuration invariant. Among the groups associated with configurations, the Hessian group was frequently studied by mathematicians. In this chapter, we will investigate the group and its subgroups as presented in three historical sources [Newson 1901], [Winger 1925], [Shaub and Schoonmaker 1931]. It should be noted that an abstract configuration can also be studied by way of its automorphism group, which acts by permuting objects of the same type (points, lines or planes) while leaving the incidence structure of the configuration as a whole invariant. In the case of the Hessian configuration these groups are isomorphic (see Brieskorn and Knoerrer 1986, pp. 297–298).

The earliest studies connecting group theory and configurations can be found in the works of algebraists and geometers from the 1890s, e.g. Arthur Schönflies and Eliakim Hastings Moore. A lesser know figure illustrating this rising interest in group-theoretic aspects of configurations was Julius Feder, a student of Theodor Reye at the University of Strassburg. His study [Feder 1895] examined the group and subgroups associated with the famous Reye configuration of 12 points and 16 lines in 3-space. Reye, a well-known synthetic geometer who first identified configurations as geometric objects, had not originally considered this approach. The fact that he supervised this dissertation thus shows how quickly this group-theoretic approach had spread. Ten years later, William Burnside discovered a new configuration associated with the Valentiner group of 360 plane collineations in [Burnside 1905]. This is a $(45_8, 120_3)$ consisting of 45 points and 120 lines, which Burnside showed was closely related to the Hessian configuration. The strong interest in the properties of finite collineation groups during this second period carried over into the third, by which time research in group theory completely overshadowed the earlier interest in geometric configurations. With the exception of a a few papers written by Coxeter, barely any traces of this topic can be found after 1950, despite the fact that Hilbert considered it an open field of major importance.

The fifth chapter summarizes the main findings in the previous chapters. It also notes certain parallels between the history of configurations and two other fields of research that were prominent during the nineteenth century: invariant theory and the theory of groups. The study closes with appendices that present brief biographical information about three of the mathematicians who worked on configurations. The final appendix (section 6.1.3) on Seligman Kantor presents part of an interesting letter that Kantor wrote to Luigi Cremona, a document that sheds light on the early history of configurations.

1 Graph Theory and Configurations

1.1 Historical Roots of Graph Theory

The general topic of combinatorics has never received much attention in the historical literature, though the history of graph theory has been studied to some extent. Graph theory, a special field within combinatorics, traditionally traces its history back to the famous Königsberg bridge problem, which was solved by Leonhard Euler in 1736.[1]

The old city of Königsberg was set on both sides of the Pregel River, which contained two islands that were spanned by seven bridges. The people of Königsberg entertained themselves by trying to find a way to cross each of these seven bridges just once, a difficult problem. In fact it was impossible for them to find such a route, as Euler showed in 1735. By treating this problem abstractly as a graph, he could easily prove that such a path does not exist.

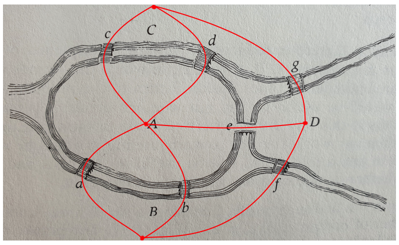


Figure 1.1 A graph based on the seven bridges of Königsberg [Biggs, Lloyd, and Wilson 1986, p. 3]

Euler's article hardly launched a new theory, and thus should be seen in the context of recreational mathematics, as practiced by amateurs for many centuries. Thus, the study of graphs was largely motivated by special problems like "the knight's tour", which was treated by the French mathematician Alexandre Theophile Vandermonde in 1771. This question has a long history, its aim being to find a sequence of moves for a knight on the chessboard so that the figure falls on each square just once and finishes on the same square on which it began. Vandermonde used this problem to illustrate his ideas on analysis situs, the field of inquiry initiated by Leibniz. "The knight's tour" can be regarded as a special case of the general problem of finding a circuit on a graph which passes through each vertex just once. In this case, the graph has sixty-four vertices, one for each square of the chessboard, and two vertices are joined by an edge whenever a

[1]The following brief discussion on the historical roots of graph theory in this section is based on [Biggs, Lloyd, and Wilson 1986].

knight can legally move from one to the other. Sixty-four edges thus form a circuit if they pass through all 64 vertices.

Another problem which played an important role in the development of graph theory was "the four-colour problem". This states that any map in a plane can be filled in using no more than four-colours, where no two adjacent countries can have the same colour. This problem was first posed by Francis Guthrie in 1852; its first written reference appeared in that same year in a letter sent to Sir William Rowan Hamilton by Augustus De Morgan, Professor of mathematics at University College London. In his letter, De Morgan mentioned that he was asked about the problem by one of his students. Many failed attempts to prove this followed, including incorrect proofs submitted by Cayley, Kempe, and others, though no one doubted that the conjecture was true. Finally, after more than a century, the problem was solved by Kenneth Appel and Wolfgang Haken in 1976 by means of a computer program. This was the first ever computer-generated mathematical proof and it led to considerable controvery at the time.

In the meantime, older results were reinterpreted in the language of graph theory, for example Euler's polyhedral formula. In 1752 Euler wrote two papers on the polyhedral formula and proved that $v-e+f=2$ (v is the number of vertices, e is the number of edges and f is the number of faces). By using a suitable projection into a plane, this formula remains valid for the corresponding polyhedral graph [Biggs, Lloyd, and Wilson 1986, p. 78]. In 1916, Ernst Steinitz proved that such polyhedral graphs are precisely the 3-connected planar graphs. (A connected graph G is said to be k-connected if it has more than k vertices and remains connected whenever fewer than k vertices are removed.)

Another central result for planar graphs was published by the Polish mathematician Kazimierz Kuratowski in 1930 [Kuratowski 1930]. This paper actually addressed a question concerning Peano continua, i.e. compact, connected subsets of a metric space given by the continuous image of a closed interval.[2] Previous investigators had found necessary conditions for a Peano continuum to be spatial, i.e. not embeddable in a plane. Kuratowski identified two simple spatial Peano continua P_1, P_2 and proved that any non-planar Peano continuum must contain a subset homeomorphic either to P_1 or to P_2. The topological properties of these two objects are easily seen to be equivalent to two standard graphs: $K_{3,3}$, the bipartite graph with 3 plus 3 vertices, and K_5, the complete graph with 5 vertices.

[2] Biggs, Lloyd, and Wilson noted that Kuratowski's theorem was not conceived as pertaining to graph theory: "The original paper was written from the standpoint of analytic topology. Kuratowski dealt with a structure rather more general than a graph, called a 'continu Péanien'; we lose nothing by translating this phrase by the word 'graph', provided that we remember that it signifies a topological realization of a graph as a system of points and lines" [Biggs, Lloyd, and Wilson 1986].

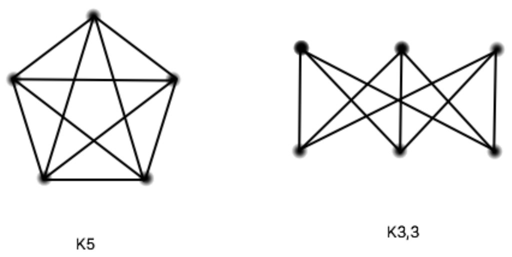


Figure 1.2 The two obstructions to planarity

Eventually, Kuratowski's theorem came to be interpreted as a theorem that answered the question: when can a graph be embedded in the plane, i.e., when can it be drawn in such a way that its paths only intersect at the vertices. This, then, can always be done so long as the graph does not contain a subgraph of type $K_{3,3}$ or type K_5; in other words, these two graphs are the only obstructions to planarity.[3,4] These results and other

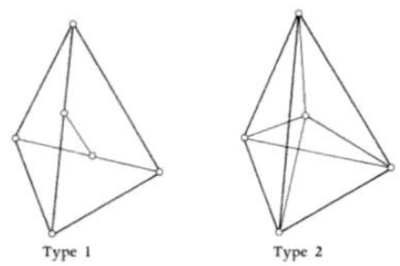


Figure 1.3 Kuratowski's figures: Type 1 is a $K_{3,3}$ and Type 2 a K_5 [Biggs, Lloyd, and Wilson 1986, p. 145].

similar ones mark only the beginnings of modern graph theory. They can be considered as the first traces of a combinatorial science, but for a long time they were seen either as mathematical curiosities [Levi 1942, p. 1] or as belonging to some other mathematical discipline, e.g. topology, as was the case with Kuratowski's theorem. In this sense, the historical development of the subject of graph theory differs sharply from most other branches of mathematics. The first textbook on graph theory, Dénes König's *Theorie der endlichen und unendlichen Graphen*, was only published in 1936, which happened to be the two-hundredth anniversary of Euler's article on the Königsberg bridges problem. As we shall see, the theory of configurations was much older, and in fact by 1900 it was already quite far more developed.[5]

[3] In the same year, Karl Menger proved a more restrictive result for planar graphs in [Menger 1930].

[4] $K_{3,3}$ as a graph can not be embedded in the plane, since its edges cross other than its vertices. But $K_{3,3}$ can be a planar configuration with symbol $(6_3, 9_2)$.

[5] As further evidence of how graph theory emerged much later, consider what König wrote about the early study by Julius Petersen [Petersen 1891]: "Diese Abhandlung von Petersen, an der auch Sylvester beteiligt

1.2 Configurations associated with Pascal's Theorem

Incidence theorems are today familiar from elementary synthetic geometry, particularly in textbooks that provide detailed accounts of the properties of triangles. This interest, however, arose quite late and was probably strongly influenced by the revival of synthetic geometry in the nineteenth century, which was led by French and German mathematicians. The British educational system remained steeped in a curriculum dominated by Euclid's *Elements*, a tradition that emphasized theorems involving areas, volumes, and shapes of figures, but only rarely the special positions of points and lines. Nevertheless, it is true that the far less familiar works of Pappus of Alexandria contain several such results, including the famous incidence relation that arises by taking three points on two lines and joining these in pairs to form an extended hexagon. Some fifteen hundred years later, geometers came to realize that Pappus's Theorem was merely a special case of the more famous Theorem of Pascal in which the six points lie on a conic curve [Coxeter 1974, p. 85].

Conic sections, of course, also have a long history, though the beginnings are quite obscure. The Greek tradition had already reached a high degree of refinement by around 200 B.C. when Apollonius of Perga wrote the eight books of his *Conica*. Not all of these eight survived in Latin translations, only the first four; books 5, 6, and 7 survived in Arabic, and these were published in English translation by Gerald Toomer in 1990. Leading European mathematicians of the 16th and 17th centuries took a deep interest in these works, which were edited by Isaac Barrow and Edmond Halley, among others. Alongside this classical tradition, Girard Desargues (1591–1661) developed a novel new approach based on viewing conic curves as projections of circles. Few of Desargues' contemporaries found his work intelligible, however, with one notable exception, Blaise Pascal (1623–1662).

In 1639, the 16-year-old Pascal sketched an outline for an entirely new theory of conics based on Desargues' principles, which he published the next year in a small pamphlet entitled *Essay pour les coniques*. Lemma 1 in this text states a result that can be understood as equivalent to what today is called Pascal's Theorem, familiar in nearly all books on the history of geometry. Pascal afterward spent several years writing a major treatise on conics, a work that continued to circulate after his death. One of those who studied it was Leibniz, who expressed admiration for Pascal's work, but unfortunately this text was lost not long afterward. In a letter to one of Pascal's heirs from 30 August, 1676, Leibniz

ist, ist sicherlich eine der bedeutendsten Arbeiten über Graphentheorie, scheint aber mehr als 25 Jahre lang fast gänzlich unbeachtet geblieben zu sein" [König 1936, p. 156].

discussed the content of this treatise in some detail. This letter was published in 1779 in the fifth volume of *Oeuvres de Blaise Pascal* edited by C. Bossut. Had this manuscript survived it would have very likely altered the course of projective geometry, which only became fashionable starting in the 1820s.

Pascal's theorem concerns six points in the plane that lie in a special position, namely on a conic (since five points in general position determine a unique conic). If these six arbitrary points are joined in any order to form a "hexagon", then the three pairs of opposite sides will meet in three points that lie on a straight line, often called the Pascal line. There are many different ways to prove this fundamental theorem, one being to start with a proof for the circle and then continuously deform the entire figure by passing from the circle to a conic by projection and section. This was likely the method of proof used by Pascal [Veblen and Young 1910, p. 111]. Since this is a general theorem concerning any six points on a conic, the inscribed hexagon need not be a convex figure and its sides may, of course, cross one another, as shown below, where the red lines are Pascal lines.

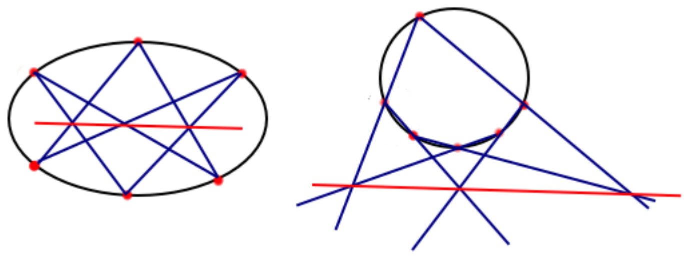


Figure 1.4 Two figures illustrating Pascal's Theorem

Although the Theorem of Pascal became very famous, the circumstances mentioned make it rather evident that its fame came much later. This also helps to account for why it was not until 1806 that Charles Julien Brianchon found the well-known dual theorem that bears his name. Brianchon's Theorem states that if a hexagon is circumscribed about a conic section, then the lines connecting its opposite vertices meet in a point [Coxeter 1974, p. 83]. The principle of duality would soon become a favorite motif for Joseph Diez Gergonne (1771–1859), who began editing the *Annales de mathématiques pures et appliquées* in 1810. Note that in Brianchon's theorem the conic must be conceived as a locus of lines rather than as a locus of points in accordance with duality.

Within the theory of plane configurations, one has the important special case known as Pappus's Theorem. Instead of taking 6 points on a conic section, one has triples of points on two lines, which form a degenerate conic. This leads to 3 more points on a line, so altogether an incidence structure with 9 lines and 9 points that form a $(9,3)$ configuration,

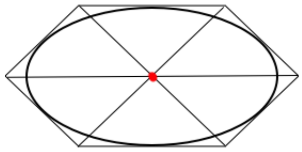


Figure 1.5 Brianchon's Theoremplanarity

one of three such possible cases. The famous Theorem of Desargues leads to a $(10,3)$ configuration, starting from two triangles in the plane that lie in perspective. This figure thus involves 7 points and 9 lines, and by extending corresponding sides one gets 3 additional points of intersection. Desargues' Theorem then asserts that these 3 points lie on a line, producing a $(10,3)$ configuration. Like the Pappus theorem, the theorem of Desargues in the plane begins with points that lie in special position.

The spatial version of Desargues' Theorem, on the other hand, follows by taking any 5 planes in general position, much like 4 planes determine a tetrahedron, which forms a $(4_3,6_2,4_3)$ configuration. Taking 3 of the 5 planes determines a vertex point P, while the fourth and fifth planes intersect the first three in two triangles whose vertices lie in perspective with respect to P. The corresponding sides of the two triangles are coplanar, so extending these we get 3 more points, thus in all 1 + 6 + 3 = 10. The last 3 points are collinear because the lie on the intersection of the fourth and fifth planes. This shows that the Desargues configuration is an immediate consequence of the incidence structure for five planes in projective 3-space.

By the 1820s, the theorems of Pascal and Desargues had become standard results in the new synthetic geometry championed by Jean-Victor Poncelet (1788–1867) and many others. In 1828, Jakob Steiner directed the attention of geometers to the complete figure obtained by taking six points on a conic in different orders. Six given points on a conic section can thus form a hexagon in 60 different ways, and thus overall there are 60 Pascal lines. From this time forward, several leading geometers took a great deal of interest in exploring various additional incidence structures related to these 60 lines. Among those who made contributions to this problem were Julius Plücker, Arthur Cayley, Thomas Kirkman and George Salmon. Through the works of these scholars, a maze of special configurations developed, a situation that Christine Ladd (1847–1930) encountered in the late 1870s when she was studying for her doctorate at Johns Hopkins University.

In an article on "The Pascal Hexagram" [Ladd-Franklin 1879], she proposed a new notation for the various other lines and points connected with the Pascal lines. She began her article by describing the properties of the various points and lines, while developing a few additional properties of the figure. An earlier description of these appeared in the fifth edition of Salmon's *Conic Sections*, to which she referred. A summary of her results can be given in the language of configurations:

Steiner proved that the 60 Pascal lines are concurrent by triples that meet in 20 points. This fact can be interpreted as a configuration of points and lines with the symbol $(20_3, 60_1)$.

Kirkman found 60 new points – beyond the 20 Steiner points – in which triples of the 60 Pascal lines also meet, thereby forming a (60_3) configuration.

Plücker proved that 20 Steiner points lie in fours on 15 lines, three through each point. This leads to a configuration with the symbol $(20_3, 15_4)$.

Cayley and Salmon discovered at the same time that the 60 Kirkman points lie in threes on 20 lines. Salmon then showed that these 20 lines meet in threes in 15 points each lying on 4 lines. These two findings can be expressed as $(60_1, 20_3)$ and $(15_4, 20_3)$ configurations.

Christine Ladd summarized all of these results schematically using the following simple figure (here the capital letters represent points and the lower-case letters are lines) [Ladd-Franklin 1879, p. 4]:

I: the Salmon points
H: the Kirkman points
G: the Steiner points
g: the Cayley-Salmon lines
h: the Pascal lines
i: the Plücker lines

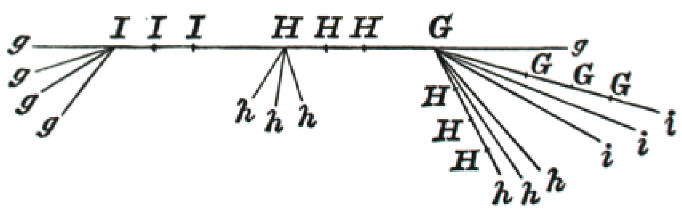


Figure 1.6

This particular bundle of configurations is in many ways typical for the first period of our study. Quite often what one finds in studies of algebraic curves and surfaces from this era is that finite incidence structures play an important role. These configurations, once found, often led to others that are closely related. It should be kept in mind, however, that up until the 1880s geometers only viewed these as special cases. The general phenomenon itself might have been noticed, but Reye was the first to address the phenomenon of configurations as a topic worthy of systematic investigation.

1.3 Traces of Combinatorics in the Definition of Configurations

Yet only a few years after Theodor Reye presented his original definition of a geometrical configuration, the concept already changed in the hands of several mathematicians. One can also easily follow some of the trends toward treating configurations as combinatorial schemes after ca. 1880.[6] In 1887, Vittorio Martinetti first considered configurations from a combinatorial point of view. He found a recursive method for constructing all (n_3) configurations, if all $(n-1)_3$ configurations are known. Using this method, he was able to construct all 31 configurations of type (11_3) [Steinitz 1910a, p. 486], [Gropp 2004, p. 83]. To construct this recursive method, Martinetti did not care at all about geometrical properties. Furthermore, the language is changed. Instead of points and lines he used the words “numbers" (numeri) and “columns"(colonne) [Gropp 2004, p. 83]. This shows that he really was one of the first to study configurations from an abstract standpoint.

So, Martinetti did not address the question of whether these 31 cases of type (11_3) could be realized in the plane, but in 1895 R. Daublebsky von Sterneck drew diagrams of all 31, which suggested they were. This appeared in a paper devoted to constructing all 12_3 configurations, in which Daublebsky von Sterneck claimed there were exactly 228 cases. In 1991, however, H. Gropp found that this list was not quite correct: there exists an additional more configuration 12_3. The fact that this error went undetected for nearly a century suggests the relative lack of interest in configurations during most of the twentieth century [Gropp 1997b, p. 139]. With the advent of modern computer technology, though, this long dormant field of research blossomed again.

E.H. Moore, who began his career working in algebraic geometry, switched to group theory by the 1890s, and by the turn of the century had moved into axiomatics and foundations of analysis [Parshall and Rowe 1994, p. 372]. As Acting Head (until 1896 when he became Head) of the Mathematics Department at the newly founded University of Chicago, Moore spearheaded the trend toward research in abstract theories that became

[6] The definition of configurations, presented as an incidence structure in the beginning with the symbols (n_k) and (p_γ, q_π), was also influenced by combinatorics, as will be seen below.

fairly dominant in the United States. Moore was an ambitious man, and since he alone among the Chicago mathematicians was American – his colleagues Oskar Bolza and Heinrich Maschke were from Germany – he had perhaps the most to prove [Parshall and Rowe 1994, p. 373].

Moore was also deeply influenced by the general trend toward abstraction, including Hilbert's work on the foundations of geometry from around the turn of the century. In his *Grundlagen der Geometrie* (1899), Hilbert set forth five groups of axioms, each providing a new layer of structures and theorems, without however making any direct claims about the ontological status of the objects under discussion. Points, lines, planes, etc. were, as always before, the fundamental objects, but Hilbert made very clear, these were merely names. The axioms described the properties they satisfied – that, for example, two distinct points determine a line – but the words point and line had no further meaning or importance than this. As Hilbert once famously said, one could just as well speak of tables, chairs, and beer mugs so long as these objects fulfilled certain axioms.

E.H. Moore already took an important step in this direction of abstraction when he generalized the notion of a geometrical incidence configuration to objects in higher dimensional spaces. In 1896, Moore wrote three articles under the general heading "Tactical Memoranda", in which he introduced the term tactical configuration for the first time. Here he treated the "incidence" relations between certain sets of objects by using matrices. Moore formulated the notion of a general geometric configuration in a flat space of n dimensions by using matrix-notation. His idea of a tactical configuration of rank n generalized the notion of a configuration of points and lines (planes) to a configuration of n different sets, which can be described as follows:

We have n sets of objects (letters), such that each set contains a certain number of objects (letters), i.e, the cardinality of the first set is a_1 , the cardinality of the second set is a_2 , and in general, the cardinality of the i^{th} set is a_i.

We denote every object of each set by the symbols λ_{ij} , $i = 1,2,\cdots,n$ and $j = 1,2,\cdots,a_i$, where the first suffix shows the set to which the object belongs, and the second picks out the object in that set. For example, λ_{25} is the fifth object in the second set.

Then he defined a certain incidence relation between objects of different sets, so that one can say that λ_{ij} is or is not incident with $\lambda_{i'j'}$ $(i \neq i')$.

Moore tabulated these incidence relations using a table of incidences, in which one line might be:

$\lambda_{i_1 j_1}[\lambda_{i'_1 j'_1}, \lambda_{i'_2 j'_2}, \cdots, \lambda_{i'_k j'_k}, \cdots]$, which reads: $\lambda_{i_1 j_1}$ is incident with $\lambda_{i'_k j'_k}$, $(k = 1,2,\cdots)$ [Moore 1896, p. 265]

For example, we can apply this definition to the Pascal configuration (9_3), a planar configuration that will be discussed in 2.1.1.3. In this case, we have two sets of points and lines. As will be described later, Hilbert and Cohn-Vossen gave the table of incidences for this configuration:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	1	1	2	2	3	3	4	5
2	4	6	4	7	6	5	6	7
3	5	7	8	9	8	9	9	8

The first line of the above table shows the 9 lines of the configuration, denoted by the numbers in parentheses. The 9 points of the configuration are distributed by threes in the nine columns so that the points in one column have incidences with the line which is placed in the first row of the same column . Using the definition presented by E.H. Moore, we can place the nine lines of the configuration into the first set and the nine points of the configuration into the second set. Thus we can translate the incidence relations of the above table into Moore's formula, as follows:

The first column of the table coincides with: $\lambda_{11}[\lambda_{21}, \lambda_{22}, \lambda_{23}]$, which means that line number 1 is incident with the points 1, 2, and 3.

The second column of the table coincides with: $\lambda_{12}[\lambda_{21}, \lambda_{24}, \lambda_{25}]$, which means that line number 2 is incident with the points 1, 4, and 5.

The third column of the table coincides with: $\lambda_{13}[\lambda_{21}, \lambda_{26}, \lambda_{27}]$, which means that line number 3 is incident with the points 1, 6, and 7, and likewise, we can write the other columns of the table into the formula as above.

E. H. Moore indicated in [Moore 1895] that geometrical tactical configurations are examples of such linear configurations of rank n, but he expressed the view that the term "tactical configuration" should be understood as merely a name. The terminology "tactical configurations" was later used in Robert D. Carmichael's book *Introduction to the Theory of Groups of Finite Order*. Its last chapter, entitled "Tactical Configurations," contains a brief introduction to these and the groups characterized by them [Carmichael 1937]. Brieskorn and Knörrer also called the "Hessian configuration" a tactical configurations and referred to Carmichael's book as well as Steinitz's article [Steinitz 1910a] in [Brieskorn and Knoerrer 1986, p. 296]. This terminology also appears in an article by Leonard Eugene

Dickson entitled, "The Group of a Tactical Configuration" [Dickson 1905]. Dickson was one of Moore's first students, earning his Ph.D. in 1896 [Parshall and Rowe 1994, p. 379].

Following Moore's publications, others made similar use of matrix notation for describing configurations, as for example in *Projective Geometry*, Vol. I by Oswald Veblen[7] and John Wesley Young [Veblen and Young 1910, p. 38]. In their book, the authors described configurations by using square matrices, as in this example:

$$
\begin{array}{l|ccc|}
 & \begin{matrix}1\\ \text{point}\end{matrix} & \begin{matrix}2\\ \text{line}\end{matrix} & \begin{matrix}3\\ \text{plane}\end{matrix} \\
\hline
\mathbf{1\ point} & a_{11} & a_{12} & a_{13} \\
\mathbf{2\ line} & a_{21} & a_{22} & a_{23} \\
\mathbf{3\ plane} & a_{31} & a_{32} & a_{33} \\
\hline
\end{array}
$$

Friedrich W. Levi also used matrix notation in his book [Levi 1929, p. 3]. He called such an incidence matrix an "Inzidenztafel" for representing a planar configuration. For a configuration with p points and g lines, Levi let the rows of the matrix correspond to the points and the columns to the lines of the configuration. If the point ν is incident with the line μ, then there will be an X mark in the entry for the ν-th row and μ-th column, otherwise there will be only a point. One can obtain such an incidence matrix algebraically in the following way:

Let V be a matrix with p rows and 3 columns; in the ν-th row we have $x_1^\nu, x_2^\nu, x_3^\nu$, which give the coordinates of the point ν. Likewise, let U be a matrix with 3 rows and p columns; in the μ-th column we have $u_1^\mu, u_2^\mu, u_3^\mu$, which are the coordinates of the line μ. The matrix $J = V.U$ then has the entries $a_{\nu\mu} = x_1^\nu u_1^\mu + x_2^\nu u_2^\mu + x_3^\nu u_3^\mu$ in the ν-th row and μ-th column. If $a_{\nu\mu} = 0$, then the point and line are incident, so we place the mark X in the entry for the ν-th row and μ-th column of the incidence matrix. If, on the other hand, $a_{\nu\mu} \neq 0$ we place a point in the corresponding entry. The incidence matrix for a configuration (p_ν, g_μ) has the property that in every row there are ν incidence marks X and in every column there are μ incidence marks X; the total number of X marks is then $p\nu = g\mu$.

[7] Veblen was a student at the University of Chicago, where he completed a doctoral dissertation under Moore's direction. This presented an axiom system for geometry that differed from Hilbert's, which was based on the (undefined) notions of point, line, and plane; Veblen's system was based rather on the notions of point and order, and he proved that his axioms were both complete as well as independent [Parshall and Rowe 1994, p. 384].

After describing the concept of incidence matrix, F. Levi represents an example for it in his book, which could be found in Fig. 1.7 [Levi 1929, p. 3]:

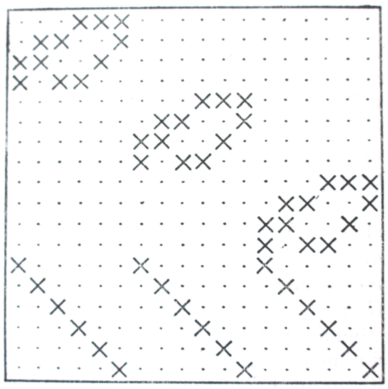


Figure 1.7

Levi noted that an incidence matrix characterizes uniquely the equivalence class of a configuration; for example, each of the different types of 10_3 configurations corresponds to a unique incidence matrix. Like Moore's approach, Levi's matrix representation applies to any abstract configuration. The designation of the objects in an incidence matrix as points and lines has no geometrical significance. A renumbering of the points (lines) of the configuration then corresponds to a permutation of rows (columns) of the incidence matrix, and vice versa. So the configuration remains invariant under permutations of the rows (columns) of the corresponding incidence matrix.

1.4 Representing Configurations with Graphs

Moore's work on tactical configuration suggests that the direction of research on the theory of configurations was undergoing change. This was a natural development as algebraic methods came to dominate classical synthetic geometry, reducing the importance of $\mathbb{R}^2$ and $\mathbb{R}^3$ (or their extensions to the projective plane and space) as the primary settings for geometrical objects. Once higher-dimensional geometries become accepted in geometry, these spaces just became examples of an $\mathbb{R}^n$. One could still ask whether a given abstract configuration can be embedded in some $\mathbb{R}^n$, but this question no doubt had rather little appeal. The general answer was obviously yes, so the only point would be to determine the smallest dimension possible. Kuratowski's theorem on nonplanar graphs (see introduction) represents a good example of older topological interests, and it may be significant that this result appeared just as modern graph theory was emerging.

One of the earliest studies explicitly devoted to graph theory was published in 1891 by Julius Petersen (Die Theorie der regulären graphen [Petersen 1891]). A regular graph has the property that each of its vertices has the same degree. For example, the vertices of a cube form a regular graph of degree $d = 3$. Note that it can also be viewed as a $(8_3, 12_2)$ configuration, a special case of Steinitz's theorem (see below). Since n, k configurations involving points and lines can be viewed as regular graphs, the results Peterson proved could have been brought to bear in studies of these configurations, which from the standpoint of modern graph theory can be regarded as regular bipartite graphs. This interplay, however, appears not to have taken place until much later. Only in retrospect did these types of connections appear to be obvious. The graph-theorist Harald Gropp has identified examples of this kind, for example in the work of Ernst Steinitz. In his dissertation from 1894 on projective configurations, Steinitz obtained the following result for an n_p configuration: "In dem Schema einer jeden Cf. n_p lassen sich die Elemente innerhalb der Colonnen in der Weise anordnen, dass jede Horizontalreihe jedes der Elemente $1, 2, \cdots n$ einmal (und also auch nur einmal) enthält." As Gropp points out, this result is equivalent to a theorem of Dénes König, which asserts that every regular bipartite graph has a 1-factor [Gropp 1997a, p. 160]. In fact, if such a graph has degree k, then it breaks up into k 1-factors. König proved this in 1916, presumably without any knowledge of Steinitz's earlier result.

During the second period of research on configurations, the question of realizability was central. Seen from a mathematical standpoint, Kuratowski's theorem from 1930 has a direct bearing on this question, but it had little historical importance for configurations since it came too late to influence these investigations. Graph theorists took up the question of when a graph can be embedded in a surface. A natural question to ask in this context was whether one can always determine the surface of minimum genus. This suggests that the issue of geometric realization for abstract configurations can be seen as related to the problem of determining whether a graph can be embedded in a geometrical object, e.g., a surface of minimum genus.

Steinitz did not address this problem directly, but he did undertake related work on polyhedra, some of which was published posthumously by Hans Rademacher in E. Steinitz, *Vorlesungen über die Theorie der Polyeder unter Einschluss der Elemente der Topologie*. One of Steinitz's fundamental results concerns the skeletons of convex polyhedra. By omitting the face structure, any polyhedron gives rise to a graph, called its skeleton, with corresponding vertices and edges. In 1916, Steinitz proved that the skeletons of convex polyhedra are precisely the 3-connected planar graphs.[8] (A connected graph G is said to be k-connected if it has more than k vertices and remains connected whenever fewer than k vertices are removed.) It is easy to see that the skeleton of every convex polyhedron is planar and 3-connected using Schlegel diagrams to project the skeleton onto a plane (see [Hilbert and Cohn-Vossen 1932, pp. 127–135]). One places a light source near one face of the polyhedron and a plane on the opposite side. The shadows of the polyhedral edges will be projected as lines that form a planar graph. The converse direction – showing that a 3-connected planar graph always be embedded in a convex polyhedron – is more difficult to prove.

Steinitz's theorem relates the connectivity of planar graphs to their realizability as skeletons of convex polyhedra. This theorem can be used as a general criterion for embedding an (n_3) configuration in the plane, i.e., for deciding whether an abstract (n_3) is realizable as a plane configuration. Steinitz showed already in 1894 that in cases when this cannot be done, one can still find a plane embedding for all but one line of the configuration. This earlier result can be interpreted in the language of bipartite graphs rather than abstract configurations (which can obviously be understood as equivalent to a special class of bipartite graphs, as will be described in the next section).

[8] This theorem was first published in 1922 in E. Steinitz, Polyeder und Raumeinteilungen, Encyclopädie der mathematischen Wissenschaften, Band 3.

1.4.1 Levi Graphs

Harold Scott MacDonald Coxeter (1907–2003), known as Donald to friends and colleagues, was a British-Canadian geometer. He received many awards and honors throughout his life and is regarded as one of the most important geometers of the 20th century. He was best known for his work on regular polytopes and higher-dimensional geometries, but also as a champion of classical geometry in a period when there was a growing tendency to approach geometry via algebra. He also did fundamental work in other areas of mathematics, in particular group theory and graph theory.

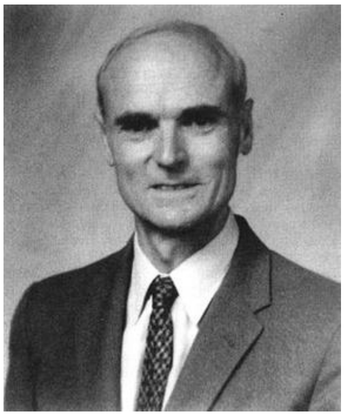


Figure 1.8 Harold Scott MacDonald Coxeter (Wikipedia)

Coxeter was perhaps the leading popularizer of configurations after Hilbert and Cohn-Vossen. In 1950, he wrote an article on "Self-dual configurations and regular graphs", in which he called attention to *Levi graphs*, which were barely known at this time. These graphs were first introduced in [Levi 1942, p. 5] as a tool for describing configurations. Coxeter summarized Levi's description of them as follows:

> ... "we represent the points and lines (or planes) of the configuration by dots of two colors, say "red nodes" and "blue nodes," with the rule that two nodes differently colored are joined whenever the corresponding elements of the configurations are incident. (Two nodes of the same color are never joined.) [Coxeter 1950, p. 413].

In modern terms, a Levi graph is simply a bipartite graph in which the vertices of one part correspond to the points of a given configuration, whereas the vertices of the other part correspond to the lines (planes) of the same configuration. Two vertices of this Levi graph are then connected by an edge if and only if the corresponding point and line (plane) are incident. Coxeter did not use the terminology of bipartite graphs; he simply followed Levi in defining this graph by associating the points and lines (or planes) of a configuration with two batches of red and blue dots. Graph theory was a young branch of mathematics in the year 1942 when Levi's book was published, and the concept of "bipartite graph" was not so established among mathematicians. We should also keep in mind the circumstances in which he wrote it, namely, as a displaced German scholar working in India during the Second World War. Levi mentioned in the foreword of his book that he had no ready access to books and journals in India at that time.

The following is based on Coxeter's presentation of some simple examples of Levi graphs, in which a configuration with symbol (m_c, n_d) is represented by m red nodes and n blue nodes. Each red node is joined with c blue nodes and each blue node to d red nodes, so there are $cm = dn$ edges altogether. A very simple example is given by a triangle – the plane configuration (3_2) – its corresponding Levi graph can be represented by a hexagon with alternating red and blue vertices. Another example is the complete

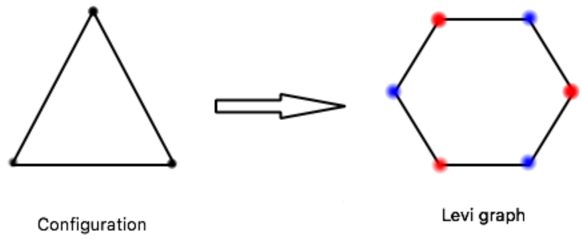


Figure 1.9

quadrangle $(4_3, 6_2)$: the Levi graph can be represented by a tetrahedron with a red node at each vertex and a blue node at the midpoint of each edge. A simple example in three

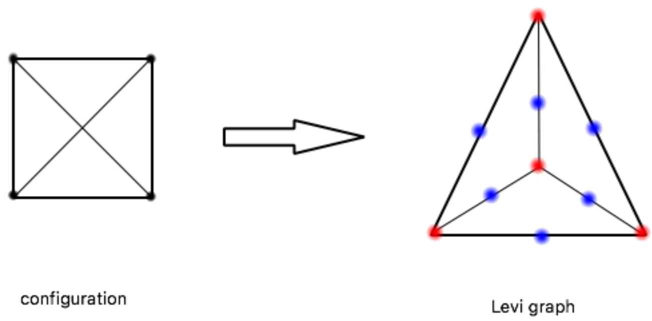


Figure 1.10

dimensions is the Levi graph for the tetrahedron 4_3, which consists of the vertices and edges of a cube, where the vertices are alternately colored red and blue. Another instance in three dimensions comes from the Möbius configuration 8_4 (see Section 2.1.1.2), for which the Levi graph can be represented by the vertices and edges of a four-dimensional hypercube.

In 1846, Cayley remarked that by taking five points in general position in projective 3-space, one obtains ten points and ten lines that form a Desarguesian configuration 10_3. Cayley labeled the ten points:

$$12, 23, 34, 45, 15, 13, 24, 35, 14, 25$$

and the ten lines:

$$345, 145, 125, 123, 234, 245, 135, 124, 235, 134,$$

utilizing the rule that the point ij is incident with the line ijk.

Coxeter illustrated this incidence structure with the projective diagram shown in Fig. 1.11. This is identical with Desargues' configuration 10_3, which can be represented by the Levi graph in Fig. 1.12. [Coxeter 1950, pp. 434,435], [Cayley 1846, p. 214]

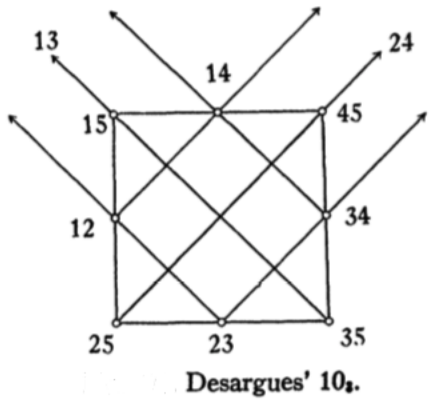


Figure 1.11 [Coxeter 1950, p. 435]

This configuration can be regarded in ten ways as a pair of triangles in perspective; for example, the two triangles with vertices $14, 24, 34$ and $15, 25, 35$ are in perspective.[9]

Coxeter's representations of Levi graphs were constructed by making full use of symmetries that make it easy to visualize their structure. This can, of course, be done in various ways. Fig. 1.13 shows a second representation of the Levi graph for Desargues' configuration, which is identical to the graph shown in Fig. 1.12.

Coxeter expressed the correlation between configurations and Levi graphs by saying that: "... every configuration can be represented in a unique manner by a "Levi graph"

[9]Another example: two triangles with vertices $14, 45, 34$ and $12, 25, 23$ are in perspective with 24 as the center of perspectivity; the sides of these two triangles meet two by two in the three points: 35, 15 and 13.

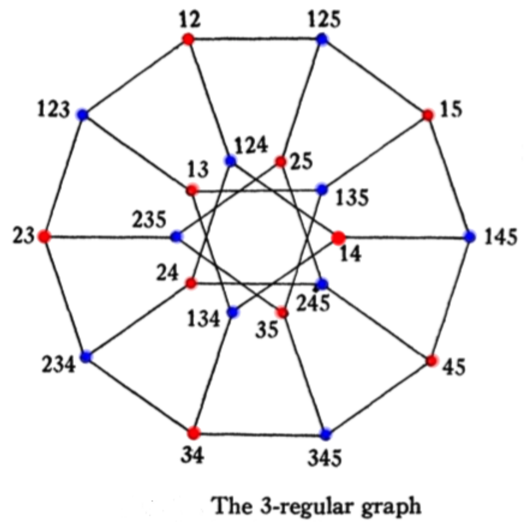


Figure 1.12 The Levi graph of the Desargues configuration. [Coxeter 1950, p. 435]

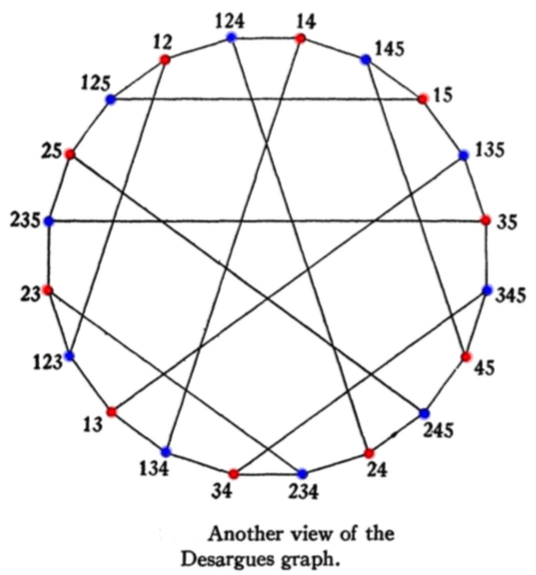


Figure 1.13 A second representation of the same Levi graph. [Coxeter 1950, p. 435]

[Coxeter 1950, p. 418]. In this way, its easy to see how the theory of configurations can be considered as a part of graph theory. In fact, a configuration and its Levi graph are isomorphic: the automorphism group of the configuration (including reciprocities as well as symmetries) is identical with the group of the graph. In his article, Coxeter calculated the order of this group for an (n_3) configuration with an s-regular Levi graph $(s > 1)$, which is just $2^s 3n$. These results suggest the modern connections linking configurations with graph theory and group theory. Connections with finite geometries will also be discussed in Section 1.4.2. This interplay of ideas was illustrated schematically in [Pisanski and Servatius 2004, p. 3], as reproduced here:

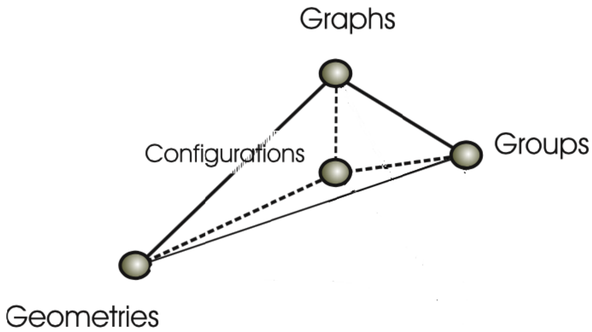


Figure 1.14 Configurations in their Modern Mathematical Context

As already mentioned, Friedrich Wilhelm Levi (1888–1966) was the first author of a textbook on configurations, Levi 1929. He taught at the University of Leipzig until 1935, when he was dismissed by the Nazi government because of his Jewish ancestry. In January 1936, he took up the position of Hardinge Professor of Higher Mathematics at the University of Calcutta, remaining until 1948 (for further information on his biography, see the appendix in chapter 6).

Figure 1.15 F.W. Levi (Professorenkatalog der Universität Leipzig)

In India, Levi taught a 6-part course on finite geometrical systems in 1940 – published as a book with the same title [Levi 1942]. According to its foreword, Levi originally intended to publish these lectures in a revised form, which would have included additional results from other literature that was not available to him in Calcutta. He changed his mind, however, after receiving urgent requests from students of mathematics as well as from researchers in statistics that he not delay publication.

In these six lectures, Levi makes use of finite and infinite geometric methods; algebra also plays an important role, but most of all combinatorial techniques, which appear in all six parts of the book. Throughout, Levi studies finite geometrical systems and tries to show their applications to practical problems in the "Design of Experiments".[10] Thus, in the first lecture he regarded "regular finite graphs" as the simplest finite geometrical systems, and he related these to "block design" based on a set of abstract objects called "varieties". Suppose there are v different varieties and each one is taken r times, then one gets vr marks. These marks are then distributed into b subsets called "blocks". Each block is composed of k marks of different varieties, so obviously $vr{=}bk$. In his third lecture, Levi represented a block system in the most obvious manner, namely, by a tactical configuration (v_r, b_k) that represents the varieties by points and the blocks by straight lines or by planes [Levi 1942, p. 18]. Finally, Levi connects configurations (or a block system) with even graphs (bipartite graphs) as discussed above.[11]

1.4.2 Menger Graphs

Another type of graph that was attached to configurations was introduced by the Austrian-American mathematician Karl Menger (1902–1985). He proposed representing a configuration of points and lines by a graph for which the vertices correspond to points of the configuration and the edges correspond to two points of the configuration that lie on one of its lines. Thus, for example, in the case of an n_3 configuration, the Menger graph contains n vertices and $3n$ edges without common interior points. Karl Menger introduced this construct in a course on projective geometry that he taught at Notre Dame University in 1945, but he seems never to have published on this topic. The term Menger graph was introduced for the first time by Coxeter in his article "Configurations and Maps" [Coxeter 1948] and then in "Self-dual configurations and regular graphs" [Coxeter 1950]. In [Coxeter 1948], he elaborated on Menger's idea of representing a configuration of points and lines by a graph, and then embedding this graph into a surface of minimal genus.

The first publication on this topic was a Ph.D. thesis written by Sister Mary Petronia van Straten under the Menger's supervision in 1949. This dissertation, entitled "The topology of the configurations of Desargues and Pappus", applied Menger's construct to three

[10] Through the work of R. A. Fisher and his associates this came to form an important part of modern statistical theory. [Levi 1942, p. 1]

[11] The terminology of even graphs (paare Graphen) was used by Dénes König in his textbook from 1936 *Theorie der endlichen und unendlichen Graphen*, one of the few sources cited by Levi.

special cases: the 7_3 configuration arising from Fano's finite projective geometry with seven points and seven lines, the Pappus configuration 9_3, and the Desargues configuration 10_3. Van Straten also investigated the question of planarity of the Menger graphs for certain configurations. Her results will be described briefly below.

As noted earlier, various types of planar graphs have interesting geometrical properties. For example, polyhedral graphs satisfy Euler's formula $v-e+f=2$ if one counts the unbounded exterior as a single face. By Kuratowski's theorem, for a graph to be non-planar it must contain a subgraph of the form either K_5 (the complete graph on five vertices) or $K_{3,3}$ (complete bipartite graph on with 3 plus 3 vertices).[12] Van Straten showed that the Menger graph of the Pappus configuration is non-planar, but it can be embedded in a torus. She showed further that the Menger graph of Desargues configuration is also non-planar, whereas this graph cannot be embedded in a torus.

As noted in the preceding section, a Levi graph carries the same structural information as the original configuration, but this is not the case with Menger graphs, which cannot be used to determine a configuration uniquely. In fact, Coxeter gave several examples of distinct configurations that lead to identical Menger graphs:

1. The complete heptagon and Fano's plane;
2. The complete enneagon (nonagon) and the $(9_4, 12_3)$ configuration that arises from the nine inflection points of a general cubic curve in the complex projective plane (see chapter 3 for details);[13]
3. Möbius' configuration 8_4 in both real and complex 3-space;
4. The complete pentagram and the configuration of Desargues (an example introduced by Sister Petronia van Straten in her Ph.D. thesis).

In closing this chapter, I briefly explain examples 1, 2, and 4.

1) After presenting the definition of the Menger graphs in her article, Sister Mary Petronia van Straten introduced the first example to illustrate how two different configurations can lead to the same Menger graph. She pointed out, further, that whereas the complete heptagon K_7 can be embedded in the torus it cannot be topologically embedded in the real projective plane [Straten 1949, p. 5]. The complete heptagon has 7 points and 21 lines. Generally each complete polygon K_n is a configuration with symbol $(n_{n-1}, (n(n-1)/2)_2)$ and by Kuratowski's Theorem these are nonplanar graphs for $n \geq 5$. Note also that the

[12] More precisely, a finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$. The same result was proved independently by Orrin Frink and P. A. Smith in the *Transactions of the American Mathematical Society* a few months after Kuratowski's publication [Biggs, Lloyd, and Wilson 1986, pp. 147–148].

[13] Here Coxeter referred to *Finite Groups* by Miller, Blichfeldt and Dickson.

Menger graph for such a configuration is simply K_n. So the Menger graph of K_7, viewed as a $(7_6, 21_2)$ configuration, is K_7, and it suffices to show this for the Fano plane, which is a (7_3) configuration. A schematic picture of the Fano plane looks like this:

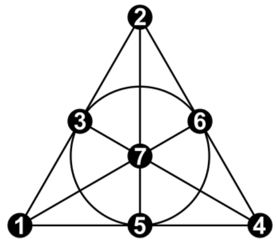


Figure 1.16

To draw its Menger graph, we first choose seven vertices, labeled 1 to 7, corresponding to the seven points of the configuration, as shown in Figure 1.17. As can be seen from Figure 1.16, point 1 is connected with all 6 other points of the configuration, so we draw 6 edges joining vertex 1 with the other vertices. Since the same property holds for all other points of the Fano configuration, the resulting Menger graph is obviously a K_7.

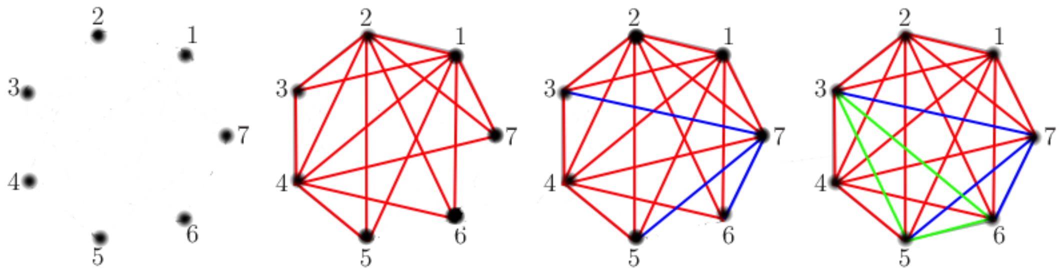


Figure 1.17

2) For the second example, the argument is identical. We merely need to note, referring to the schematic figure below illustrating the incidence structure for a Hessian configuration, that it shares the same key property with K_9, the complete nonagon representing a $(9_8, 36_2)$ configuration. Whereas the 8 lines that pass through each vertex in the latter obviously connect that point with all the others, in the Hessian configuration the same holds true for the 4 lines that pass through each of its points. Hence, both configurations have the same Menger graph, which is isomorphic with K_9.

To draw the Menger graph of the Hessian configuration we first associate nine vertices with the nine points of the configuration. Starting with point 1, we draw edges connecting it with the other 8 points, including the 6 and 8, which are drawn in red in Figure 1.19. Since the Hessian configuration is fully symmetric, we repeat this construction for the

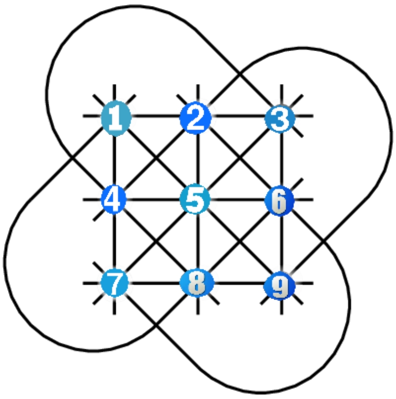


Figure 1.18

remaining points in order. The last edge drawn will connect the points 8 and 9, producing a complete nonagon.

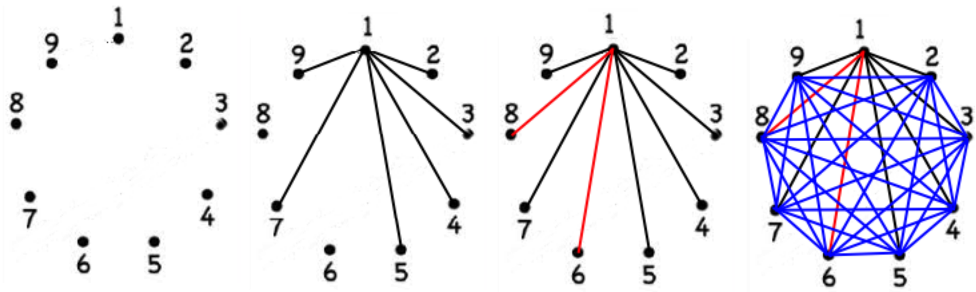


Figure 1.19

4) The fourth example is also easy to show. As can be seen from a picture of the complete pentagram, its five lines intersect in 10 points, which form a $(5_4, 10_2)$ configuration. If we take any of these 10 points it will be connected to 6 other points in the configuration.

Desargues' configuration is a $(10, 3)$ that can be represented in a number of ways. As noted above in Section 1.4.1, Cayley showed that one can construct a Desarguesian figure by taking any five points in general position in 3-space. These determine 10 lines and 10 planes, which when intersected by an arbitrary plane yield a Desarguesian configuration. Its points can be represented by the 10 unordered pairs of numbers ij and its lines by triples ijk, where $i, j, k \in \{1, 2, 3, 4, 5\}$ and ij is incident with ijk, thus whenever its entries appear in the latter triple. Coxeter gave the schematic in Fig. 1.11 as a representation of Desargues' configuration [Coxeter 1950,

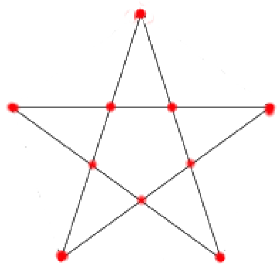


Figure 1.20

p. 435]. From this figure, we see that it shares the same property with the $(5_4, 10_2)$ configuration, namely that each of its 10 points lie on lines connecting 6 other points in the configuration. So both have the same Menger graph.

2 Geometric Configurations and Abstract Incidence Structures

2.1 Popularization of Configurations

In 1932, David Hilbert (1862-1943) and Stefan Cohn-Vossen (1902-1936)[1] published their co-authored book *Anschauliche Geometrie* [Hilbert and Cohn-Vossen 1932]. This was based on a course first taught by Hilbert in 1920-21 at the University of Göttingen. Hilbert designed this course in an effort to popularize mathematics for young men returning to the University after the war, choosing a subject which could be presented in a lively and intuitive style [Reid 1996, p. 154]. An Ausarbeitung of Hilbert's original course was prepared by W. Rosemann and can be found at the Mathematics Institute of the University of Göttingen (see Fig. 2.3).

Figure 2.1 David Hilbert (Wikipedia)

Stefan Cohn-Vossen reworked the original lecture notes, added some details in a few places and arranged for its publication in book form. This remarkable book is richly illustrated with over 300 pictures that include photos of models from the Göttingen collection.

The published text contains a good deal of densely packed information, presented with a minimum of technicalities. The authors explain mathematical ideas clearly, elegantly, and above all, with penetrating insight. In a few short strides, Hilbert and Cohn-Vossen carry the reader from the elements to the very heart of a geometrical topic. This classic text remains in print today, one of the few mathematics books still widely read nearly a century after its publication.

[1] Cohn-Vossen died tragically at the age of 34 from pneumonia as a Jewish refugee in Moscow [Siegmund–Schultze 1998, p. 133].

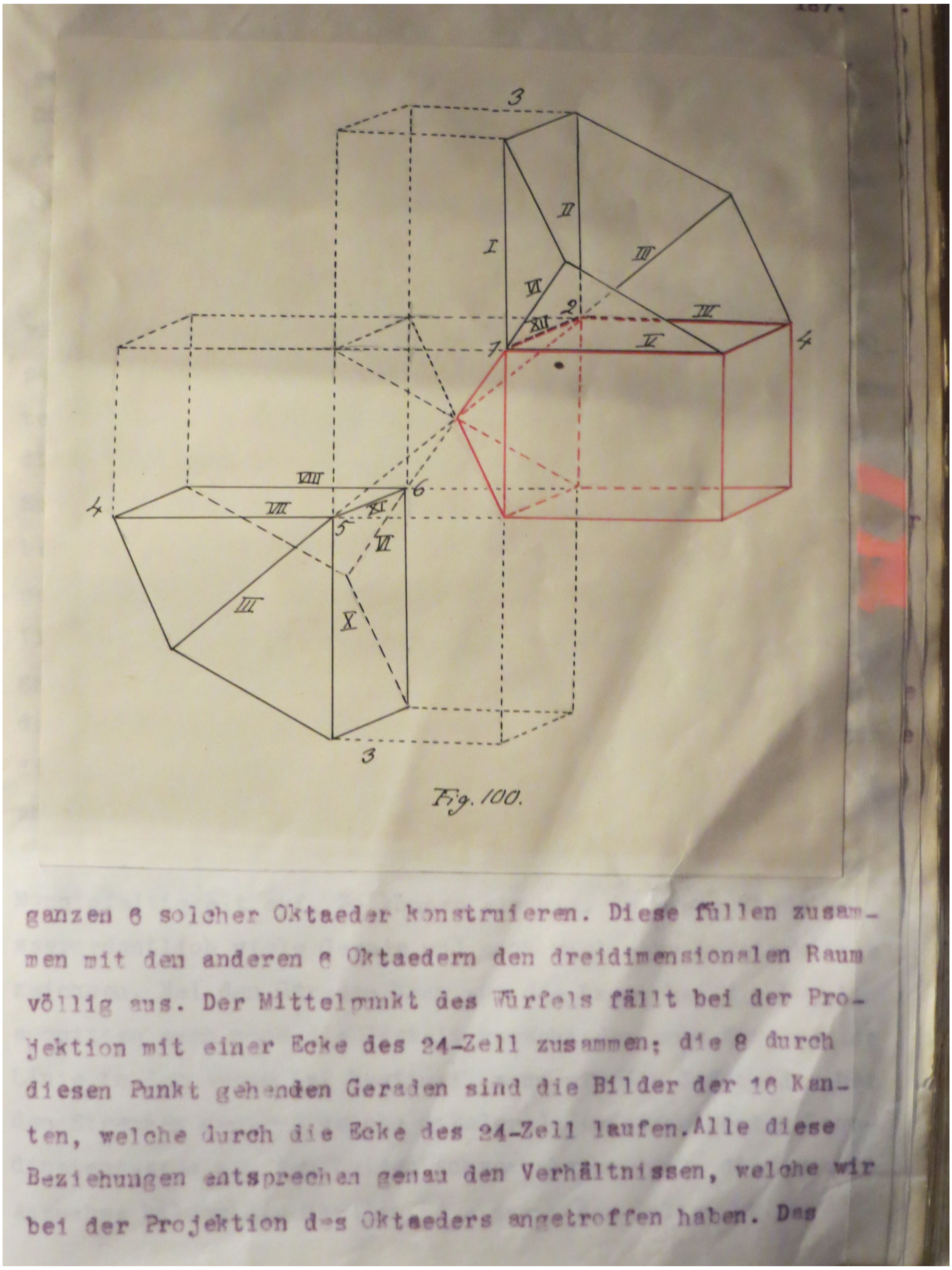
Figure 2.2 Stefan Cohn-Vossen (Wikipedia)

This book was translated in 1952 by Peter Nemenyi (1927-2002),[2] who studied with Max Dehn at Black Mountain College before taking up graduate studies at Princeton University. Nemenyi was presumably also responsible for the aptly chosen title of the English translation: *Geometry and the Imagination*. Reviewing this translation in 1953, Coxeter wrote that the original text "has been a classic for twenty years. Its breadth of outlook is reminiscent of Klein's Elementary Mathematics from an Advanced Standpoint. Although it deals with elementary topics, it reaches the fringe of our knowledge in many directions. ... this excellent translation by Dr. Nemenyi may help to restore interest in geometry, a subject that seems lately to have lost favor in America" [Coxeter 1953, p. 117].

The book's contents are easily summarized since each of its six chapters is devoted to a special topic in geometry. The first (The Simplest Curves and Surfaces) mainly concerns the conic sections and quadric surfaces, including the construction of points on a central quadric by means of a loop of thread strung tightly against two fixed focal conics, an ellipse and a hyperbola, that lie in orthogonal planes. The second chapter (Regular Systems of Points) offers an original introduction to point lattices and quadratic forms, crystallographic groups, and regular polyhedra. The drawings of polyhedra are especially striking because of the skillful use of perspective where edges nearer to the eye of the reader appear thicker than those that are farther away, as illustrated in Fig. 2.4.

The third chapter (Projective Configurations) will be discussed in detail below. The authors note that this topic had long been a thriving area of research, though by 1932 it was largely forgotten and they hoped to revitalize it again. This chapter discusses planar configurations of type n_3 – those consisting of n points and n lines such that 3 points lie on every line and 3 lines pass through each point – for $7 \leq n \leq 10$. The cases $n = 9$ and $n = 10$ include the figures that derive from the theorems of Pappus and Desargues, both of which play important roles in Hilbert's *Grundlagen der Geometrie* [Hilbert 1899]. The authors also describe two important spatial configurations: Reye's configuration with 12 points, 16 lines and 12 planes and Schläfli's double-six, which leads to a construction of the 27 lines on a general cubic surface.

[2] In Wikipedia, the translation is incorrectly attributed to Paul Nemenyi (1895-1952), Peter's father.



ganzen 6 solcher Oktaeder konstruieren. Diese füllen zusammen mit den anderen 8 Oktaedern den dreidimensionalen Raum völlig aus. Der Mittelpunkt des Würfels fällt bei der Projektion mit einer Ecke des 24-Zell zusammen; die 8 durch diesen Punkt gehenden Geraden sind die Bilder der 16 Kanten, welche durch die Ecke des 24-Zell laufen. Alle diese Beziehungen entsprechen genau den Verhältnissen, welche wir bei der Projektion des Oktaeders angetroffen haben. Das

Figure 2.3 A page from Hilbert's original lecture course "Anschauliche Geometrie" prepared by Wilhelm Rosemann.

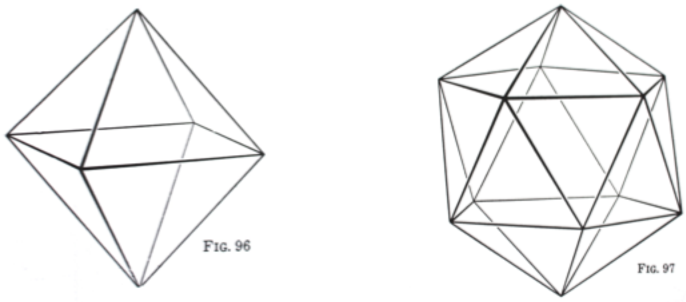


Figure 2.4 Two Platonic Solids

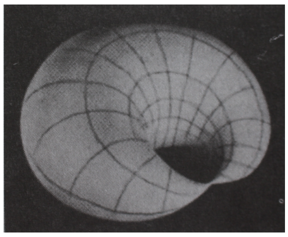


Figure 2.5 Model of a Dupin Cyclide

The fourth chapter (Differential Geometry) begins with a discussion of curvature and its ramifications, then moves to minimal surfaces, before entering a lengthier account of Euclidean and non-Euclidean geometry. An intuitive understanding of surface curvature is followed by a photo showing Klein's model of the Apollo Belvedere with parabolic curves marked on the face and neck. Elliptic and hyperbolic geometry are illustrated by models of surfaces of constant positive, resp. negative curvature. The role of stereographic projection in hyperbolic geometry also receives attention. One remarkable section in this chapter is entitled "eleven properties of the sphere." Hilbert used this rubric to describe other geometrical objects similar to the sphere in the sense that they share with it one or more of its eleven properties. A final section gives a brief account of the then recent progress on Plateau's problem due to work by Jesse Douglas and Tibor Radó.

The fifth chapter (Kinematics) is much shorter and begins with a discussion of Peaucellier's inversor. This planar linkage transforms rotational motion into motion along a straight line. This leads to a brief survey of special plane curves generated kinematically either by a movable point or a movable tangent line. The chapter ends with a short section on motions of rigid bodies in space. Chapter six (Topology) begins with polyhedra and the theorem of Euler, followed by a classification of surfaces based on the Euler-Poincaré characteristic, i.e. $V-E+F=3-h$. This leads to some examples of non-orientable surfaces, beginning with the heptahedron, a surface with $h=2$ that can be deformed into Steiner's famous Roman surface. A table then summarizes the five different topological types of surfaces (with and without boundary) that can arise by identifying points on the opposite edges of a square. The three without boundary – the torus, Klein bottle, and projective plane – are discussed at greater length. In the case of the projective plane, its representation as Boy's surface is illustrated by photos of a wire model in the Göttingen collection. The final section deals with the 4-colour problem – the problem of colouring a map using only four colours so that no two adjacent regions have the same colour[3] – is discussed in relation to the related problem of differentiating neighboring regions on a map by number. The latter problem had been solved for the plane, sphere (both require 4) as well as the projective plane (6) and torus (7). This final section, far more than the chapter on configurations, pointed to a rich field of problems for future research in graph theory.

In the remainder of the present chapter, we will follow the discussion of configurations in *Anschauliche Geometrie* fairly closely, while supplementing this with references to the primary literature the authors drew upon, though usually without attribution. It is worth noting that the original title of chapter three was merely "Konfigurationen", but it appears as "Projective Configurations" in the English translation, which also conforms with the title of Steinitz's report [Steinitz 1910a], a major source for [Hilbert and Cohn-Vossen 1932]. In their introduction to third chapter, the authors called attention to the importance of this subject by writing: "Es sei erwähnt, daß eine Zeitlang die Konfigurationen als das wichtigste Gebiet der ganzen Geometrie angesehen wurden." They also included a footnote in which they referred to F.W. Levi's book *Geometrische Konfigurationen* [Levi 1929] for a comprehensive treatment of the subject. This indicates that Cohn-Vossen was well aware that this field of research had taken on a far more abstract character over the course of the 1920s.

2.1.1 Planar Configurations

After introducing the definition of configurations in 1876, Reye published an article in the first volume of *Acta Mathematica*,[4] entitled "Das Problem der Configurationen."[5]. In this article he introduced the definition of planar configurations as follows: "Eine Configuration n_i in der Ebene besteht aus n Punkten und n Geraden in solcher Lage, dass jede der n Geraden i von den n Punkten enthält und durch jeden der n Punkte i von den

[3] The 4-colour problem was still only a conjecture in 1932, but it was proved in 1976 by Kenneth Appel and Wolfgang Haken, a famous result since this was the first major theorem to be proved by means of a computer.

[4] Reye's standing as a mathematician is illustrated by the fact that Gösta Mittag-Leffler, the founder of the journal, interrupted his honeymoon so that he could personally speak to Reye and ask for his support for the new journal by submitting a paper for publication [O'Connor and Robertson 2012].

[5] He mentioned in a footnote that he had already written on configurations in volume 86 of the *Crelle-Borchardt* journal and in his *Synthetische Geometrie der Kugeln, mit einer Einleitung in die analytische Geometrie der Kugelsysteme*, page 54.

n Geraden gehen" [Reye 1882, p. 94] This notion of a finite geometrical configuration soon appeared too narrow and was therefore extended to include other cases. Thus, a (p_λ, l_π) "plane configuration" consists of p points and l lines in a plane such that λ lines pass through each point and π points lie on each line. The four numbers p, l, λ, π must furthermore satisfy the condition $p\lambda = l\pi$. This notation was introduced by J. de Vries in [Vries 1888, p. 63] [Steinitz 1910a, p. 482]. The simplest planar configurations are geometrically trivial: thus, $(1_1, 1_1)$ corresponds to a single point on a straight line, whereas $(3_2, 3_2)$ represents a triangle. Extending this notion to spatial configurations, the elementary figures are points, straight lines, and planes that satisfy corresponding incidence relations. In *Konfigurationen der projektiven Geometrie* [Steinitz 1910a], the following notation was introduced for configurations involving points, lines, and planes:

$$(A_b^{\mathfrak{c}}, B_a^{\gamma}, C_{\mathfrak{a}}^{\beta})$$

which means A points lying on b lines and on $\mathfrak{c}$ planes, with B lines each with a points and lying in γ planes, and C planes, each containing $\mathfrak{a}$ points and β lines.

One of the first to investigate configurations systematically was the largely forgotten Prague mathematician Seligman Kantor, who studied under Luigi Cremona in Rome. On Christmas day, 25 December 1882, he summarized some of his findings in a letter to Cremona (see the transcription in the Appendix).

In fact, in this letter Kantor claimed he was the first to undertake systematic investigations on configurations: "Überhaupt möchte ich Gewicht auf den Umstand legen, dass, soviel ich weiss, vor mir niemand (selbst nicht der divinatorische Sylvester ungeachtet seiner Bemerkung im Am. Jorn. of Math. 1880) von einem "Problem der Configurationen in diesem präzisen Sinne gesprochen hat." In 1881, Kantor found that there is just one type of (7_3) and (8_3) configuration, but three different (9_3) configurations (the Pascal configuration and 2 others were discussed by Hilbert and Cohn-Vossen), and ten different (10_3) configurations (one being the Desargues configuration).

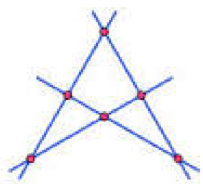


Figure 2.6 A Complete Quadrilateral

He also showed how to construct at least one configuration (m, n) for allowable m and n.

The symbol $(6_2, 4_3)$ corresponds to a complete quadrilateral (see Figure 2.6), whereas the inverse symbol $(4_3, 6_2)$ represents its dual figure, a complete quadrangle. Both play a central role in plane projective geometry. The configuration $(6_2, 4_3)$ and its dual $(4_3, 6_2)$

can both be embedded in the real projective plane.

For configurations in which the number of points equals the number of lines, i.e. $p = l$, hence $\lambda = \pi$, the symbol is written p_λ. Here the cases $\lambda = 1$ and $\lambda = 2$ are insignificant: for $\lambda = 1$, the configuration (p_1) consists of p lines with a single point on each; the case $\lambda = 2$ or (p_2) corresponds to planar polygons with p vertices and p sides. The case $\lambda = 3$, on the other hand, leads to many interesting configurations. Here p must be at least seven, since three lines pass through every point and three points lie on every line (see Figure 2.7). In this section we examine the properties of planar configurations with the symbol (p_3) for $7 \leq p \leq 10$.

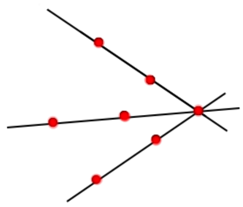


Figure 2.7

2.1.1.1 The Configuration (7_3)

As noted above, there is only one (7_3) configuration, familiar as a finite geometric configuration of points and lines with coordinates from the field $\mathbb{Z}_2$. This incidence structure of seven points and seven lines is often called the Fano plane after the Italian mathematician Gino Fano, who derived it in 1891 in connection with his axiomatic studies on projective geometries.[6] Its points can be written in homogeneous coordinates as follows: $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, $P_3 = (0,0,1)$, $P_4 = (0,1,1)$, $P_5 = (1,0,1)$, $P_6 = (1,1,0)$, $P_7 = (1,1,1)$ [Pisanski and Servatius 2013, p. 5]. Its seven lines are then: $x = 0$, $y = 0$, $z = 0$, $x+y = 0$, $x+z = 0$, $y+z = 0$, $x+y+z = 0$. We can easily check the incidences between these seven points and seven lines, for example, points $(1,0,0)$ are on these three lines: $y = 0$, $z = 0$, $y+z = 0$ and the line $z = 0$ contains these three points: $(1,1,0)$, $(1,0,0)$, $(0,1,0)$. Six of these seven lines can be drawn in the

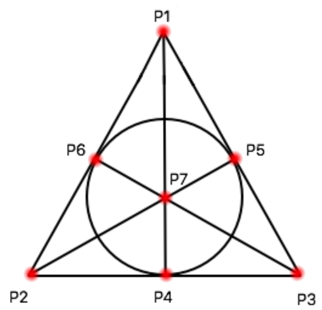


Figure 2.8 The seven points and seven lines of Fano plane

[6] The Fano plane is often mentioned as a minimal model for a finite projective plane. The (7_3) configuration was found earlier, in 1888, by A. Schönflies as well as by H. Schroeter [Gruenbaum 2009, p. 61].

real projective plane, but not all seven. The line $x+y+z=0$ passes through the points P_4, P_5, P_6, which lie on the circle shown in Figure 2.8, where the three sides of the triangle are given by the lines $x=0, y=0, z=0$.

In the complete quadrangle given by P_1, P_2, P_3, P_7 and the 6 lines joining them, the three diagonal points P_4, P_5, P_6 fail to lie on a line. Were they collinear, then the seven points and seven lines would form a Fano subplane. In 1956, Andrew M. Gleason proved that if every complete quadrangle in a finite projective plane extends to a Fano subplane (that is, has collinear diagonal points) then the plane must be Desarguesian.

Hilbert and Cohn-Vossen demonstrated a standard method for finding configurations with the symbol (p_3) [Hilbert and Cohn-Vossen 1932, p. 98]. This method, however, only establishes the existence of an abstract incidence configuration, hence it provides a necessary but not sufficient condition for the existence of a geometrical configuration. Here we illustrate the general procedure by taking the configuration (7_3) and following [Hilbert and Cohn-Vossen 1932, p. 98].

We label the p points 1 through p and the p lines (1) through (p). Then we set up a rectangular array with $p\lambda$ entries in which the λ points incident with any given line are arranged in a column; there will then be p columns corresponding to the p lines. For the case of the configuration (7_3) this leads to the following format:

$$\begin{array}{cc} & \overbrace{\begin{array}{ccccccc}(1)&(2)&(3)&(4)&(5)&(6)&(7)\end{array}}^{p} \\ \lambda & \left\{\begin{array}{ccccccc}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{array}\right. \end{array}$$

Three conditions must then be satisfied when filling the entries in this array:

1) The numbers in any column must all be different to guarantee that there are three distinct points on every line.

2) Two different columns cannot have two numbers in common (otherwise the corresponding lines for these columns coincide).

3) Every number must occur exactly three times in the entire array, since three lines pass through every point.

It should be noted that the order of the columns does not affect the incidence scheme. These may be freely interchanged, since this merely corresponds to a renumbering of the lines, which, just as with a renumbering of the points, does not alter the configuration. These three conditions are clearly necessary for the geometrical realization of a p_λ configuration, but they are not sufficient because the geometric realization of a table also depends on some geometric or algebraic properties that cannot be expressed in terms of a purely combinatorial scheme. As we have seen with the case of (7_3), which turns out to have a unique representation, this does not lead to a planar configuration.

To construct this unique (7_3) scheme, we denote the points on the first line by 1, 2, and 3. Since two more lines pass through point 1 and cannot contain the points 2 or 3, let us label the points of the second line by 1, 4 and 5, and those of the third by 1, 6 and 7. So part of the array is now filled:

In the matrices given below, there should be a top line with the 7 lines (1), . . ., (7).

$$\begin{array}{ccccccc} 1 & 1 & 1 & . & . & . & . \\ 2 & 4 & 6 & . & . & . & . \\ 3 & 5 & 7 & . & . & . & . \end{array}$$

Since each of the numbers 2 and 3 must appear twice in the remaining columns, but not in the same column, we may place them accordingly:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 4 & 6 & . & . & . & . \\ 3 & 5 & 7 & . & . & . & . \end{array}$$

Now the remaining eight places must be filled with 4, 5, 6, 7, and each of these numbers must appear exactly twice. So, by placing 4 in two different columns and doing the same with 5, we obtain, for example:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 4 & 6 & 4 & 5 & 4 & 5 \\ 3 & 5 & 7 & . & . & . & . \end{array}$$

We can then use 6 and 7 to fill the remaining four slots to complete the table, for example in this form:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 4 & 6 & 4 & 5 & 4 & 5 \\ 3 & 5 & 7 & 6 & 7 & 7 & 6 \end{array}$$

As noted above, the existence of this table does not imply that a planar configuration (7_3) actually exists. Using analytic geometry, it can be shown, in fact, that these incidence conditions lead to an incomplete system of linear equations. The following intuitive synthetic argument also shows the impossibility of embedding a (7_3) configuration in a plane. Following the numbering in the array, draw straight lines (1) and (2) intersecting in point 1 and choose arbitrary points 2, 3 and 4, 5 on the lines (1) and (2), respectively. Then draw lines (4) and (7), which join the pairs of points 2, 4 and 3, 5, respectively, and intersect in point 6. Similarly, the pairs of points 2, 5 and 3, 4 determine the lines (5) and (6), and these intersect in point 7. Now all seven points are determined, but the three points 1, 6, and 7 are also required to be collinear, which clearly cannot be the case. Therefore, the configuration (7_3) cannot be embedded in the plane.

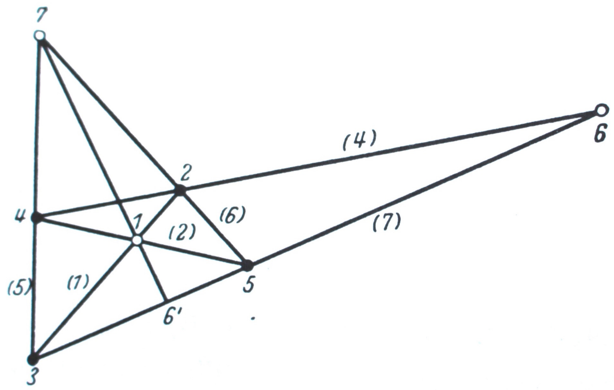


Figure 2.9 [Hilbert and Cohn-Vossen 1932, p. 100]

2.1.1.2 The Configuration (8_3)

Using the same method introduced above leads to an array for the configuration (8_3), which like that for (7_3) is essentially unique:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	1	1	2	2	3	3	4
2	4	6	3	7	4	5	5
5	8	7	6	8	7	8	6

The corresponding schematic diagram can be drawn as follows:

(8_3) is a self-dual configuration and may be interpreted as consisting of two quadrilaterals (1234 and 5678) each of which is inscribed in as well as circumscribed about the other. Clearly, such a configuration cannot be embedded in the real plane. Using analytic geometry, the incidence conditions in this case give rise to a system of equations whose only solutions are complex (unlike the configuration (7_3) for which the system of equations has no solution at all). Thus, although the configuration (8_3) cannot be embedded in the real plane, it can be realized in the complex plane [Hilbert and Cohn-Vossen 1932, p. 101]. The historical origins of this result will now be briefly explained.

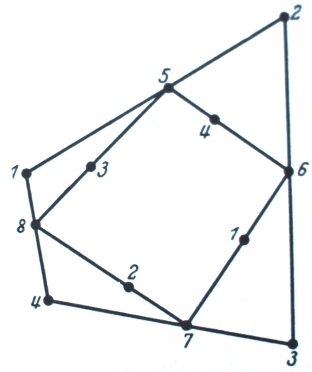


Figure 2.10 [Hilbert and Cohn-Vossen 1932, p. 101]

In 1828, the German mathematician August Ferdinand Möbius (1790–1868) asked whether there exists a pair of p-sided polygons with the property that the vertices of one lie on the lines through the edges of the other, and vice versa. He gave a combinatorial solution to this problem in the case $p = 4$, corresponding to the abstract scheme given above for an (8_3) configuration.

He also proved the impossibility of finding a pair of real quadrangles, each inscribed in the other. Möbius did, however, show that a corresponding (8_4) configuration can be realized in 3-space by means of mutually in- and circumscribed tetrahedra (see 2.3.2). He also remarked that: "Auf Vielecke von mehreren Seiten habe ich die Untersuchung nicht ausgedehnt" [Moebius 1828, p. 446], [Coxeter 1950]. It was not until 1881, thus during the early second phase in the history of configurations, that this problem was taken up again. One year later, Seligmann Kantor showed that the problem of finding a pair of mutually inscribed p-gons can be solved in the complex projective plane [Kantor 1882c] [Coxeter 1950]. Kantor described his results in a letter to Luigi Cremona, part of which is transcribed in the appendix of this dissertation. We here briefly describe his method of proof.

Figure 2.11 August Ferdinand Möbius (Wikipedia)

Assuming p is even, we denote the vertices of the first p-gon by $1,3,5,\dots 2p-1$ and require that these points lie on the respective sides 02, 24, $\dots 2(p-2)2(p-1)$ of a second p-gon with vertices $0,2,4,\dots 2(p-1)$.

The vertices of this second p-gon must then lie on the sides 13, 35, $\dots (2p-3)(2p-1)$ of the first. To realize this combinatorial scheme in the complex projective plane, Kantor considered a tentative position for the point 1 on the side 02. He then determines further points on the remaining sides by means of a chain of perspectivities; this leads eventually to a point $1'$ on the original side 02. The problem will then be solved if the point $1'$ coincides with 1. Kantor then notes that the resulting projectivity is hyperbolic,[7] which means there will be two solutions [Kantor 1882c, pp. 916-917][Kantor 1882a, p. 1291][Coxeter 1950, p. 428]. In the case of quadrangles ($p = 4$), we have the above picture. Möbius proved that in this simplest case the projectivity is necessarily elliptic; hence the configuration 8_3 cannot be constructed in the real plane. [Coxeter 1950, p. 428].[8]

Coxeter gave an embedding of an 8_3 [Coxeter 1950, p. 428], using the following complex projective coordinates involving the cube roots of unity ($\omega = (-1+3^{1/2}i)/2$ and $\omega^2 = (-1-3^{1/2}i)/2$):

$1:(1,0,0)$, $2:(0,0,1)$, $3:(\omega,-1,1)$, $4:(-1,0,1)$,
$5:(-1,\omega^2,1)$, $6:(1,\omega,0)$, $7:(0,1,0)$, $8:(0,-1,1)$.

Coxeter called this plane configuration a Möbius-Kantor (8_3), in honor of August Ferdinand Möbius and Seligmann Kantor.

A Levi graph of the Möbius-Kantor configuration is shown in Figure 2.12. The vertices of one color represent the points of the configuration, while the vertices of the other represent its lines. Since the configuration and its dual are equivalent, the choice of color for points and lines is interchangeable.

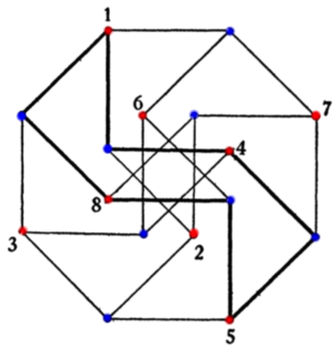


Figure 2.12 [Coxeter 1950, p. 430]

Although an (8_3) cannot be embedded in the real plane, this configuration is not without geometrical interest. In fact, it can be extended to the configuration $(9_4, 12_3)$, which plays an important role in the theory of third-order nonsingular plane curves (see chapter 3). A simple construction for this extension can be obtained by adding the four diagonals of the two quadrilaterals, which can be shown to meet in a common point. This then yields 9 points and 12 lines which form a $(9_4, 12_3)$ configuration. If we omit any one of the points in the configuration $(9_4, 12_3)$ and the lines passing through it – for example, the point 9 and the four lines (9), (10), (11), (12) – the remaining figure is an (8_3) configuration.

Nonsingular cubic curves have nine points of inflection, at most three of which can be

[7] A hyperbolic projectivity has two fixed points.
[8] An elliptic projectivity has no fixed points.

real. Every straight line connecting two of these inflection points must pass through a third. (By Bézout's theorem, a straight line cannot meet a third-order curve in more than three points.) We thus have $p = 9, \pi = 3$, but also $\lambda = 4$ because if we select any one point, the remaining eight will be collinear with it in pairs; thus each point is incident with four straight lines. From the formula $l = p\lambda/\pi$ we have $l = 12$. The table for this configuration, which is essentially unique[9], is as follows:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
1	1	1	2	2	3	3	4	1	2	5	6
2	4	6	3	7	4	5	5	3	4	7	8
5	8	7	6	8	7	8	6	9	9	9	9

2.1.1.3 The (9_3) Configurations

The case $p = 9$ gives rise to three essentially different configurations, all of which can be realized in the real plane. Seligmann Kantor obtained these three different (9_3) configurations for the first time, and Theodor Reye also mentioned this result in his influential article [Reye 1882, p. 94]. The most important of these three is known as the Pascal configuration, which is related to the Pascal-Brianchon theorem (see the discussion in chapter 3). Hilbert and Cohn-Vossen called this the most important configuration in all of geometry. It actually depends on a special case of Pascal's theorem[10] known as Pappus's theorem, which arises when the given conic curve degenerates into two straight lines.[11]

We shall denote the Pascal configuration with the symbol $(9_3)_1$ and use the symbols $(9_3)_2$ and $(9_3)_3$ for the other two configurations of type (9_3). A table for the incidence structure of the Pascal configuration $(9_3)_1$ appears as follows:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	1	1	2	2	3	3	4	5
2	4	6	4	7	6	5	6	7
3	5	7	8	9	8	9	9	8

[9] It is the only possible table for configuration $(9_4, 12_3)$.

[10] Pascal's theorem was formulated by Blaise Pascal in 1639 when he was 16 years old. It was published the following year as a broadside entitled "Essai pour les coniques. par B. P."

[11] Pappus of Alexandria was one of the last important Greek mathematicians in Antiquity; he lived around 290–350 A. D. His theorem, one of the oldest in projective geometry, asserts that if the six vertices of a hexagon lie alternately on two lines, then the three pairs of opposite sides meet in three collinear points [Coxeter 1969, p. 38]. Since the two given lines may be interpreted as a degenerate conic, the theorem of Pappus is a special case of Pascal's theorem.

To construct this planar configuration, we begin with two arbitrary points, say 8 and 9, and draw the corresponding lines that pass through them, thus (4), (6) and (9) passing through 8, and (5), (7) and (8) through 9. The intersection of these six lines produces six new points that we label 2, 3, 4, 5, 6 and 7. These six points then fix the positions of the remaining lines (1), (2) and (3). According to the table, the line (1) passes through 2 and 3, and the line (2) through 4 and 5. Their point of intersection is then labeled 1. The line (3), which is determined by 6 and 7, must then pass through 1. This condition is automatically satisfied due to Brianchon's theorem, despite the arbitrary choice of the points 8 and 9 as well as for the three straight lines through each of these points.

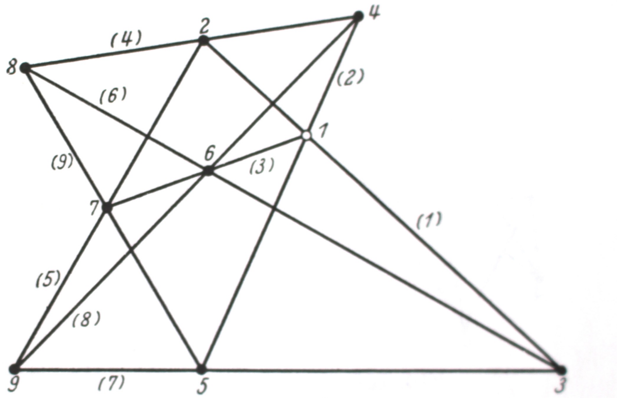


Figure 2.13 [Hilbert and Cohn-Vossen 1932, p. 103]

A similar procedure can be used to construct planar figures corresponding to the other two $(9)_3$ configurations. If we follow this construction step by step, we find that the last incidence condition is no longer satisfied automatically. This suggests why $(9_3)_2$ and $(9_3)_3$ are not of such fundamental importance as $(9_3)_1$, since they do not express a general theorem in projective geometry. Hilbert already underscored the fundamental importance of closure theorems, like the theorem of Pappus, in his *Grundlagen der Geometrie* [Hilbert 1899]. During the period when geometers began to search systematically for new configurations, they turned up a great number of new examples, but few had the same significance for other branches of mathematics as those already found during the period before Reye opened this field of investigation.

Hilbert and Cohn-Vossen avoided using tables in their discussion of the $(9_3)_2$ and $(9_3)_3$ configurations. Instead, they used two different methods to differentiate between these two configurations and $(9_3)_1$. The first method was due to Seligmann Kantor in 1882, who introduced the concept of "Restfiguren" ("remainder figures") in [Kantor 1882c][Gruenbaum 2009, p. 30]. He used this technique to classify the 10 different configurations with the symbol $(10)_3$, as will be described in the next section.

The second method used by Hilbert and Cohn-Vossen involved the notion of regularity. Regular configurations were introduced by Arthur Schönflies in his article entitled "Ueber die regelmässigen Configurationen n_3" [Schoenflies 1887] ([Steinitz 1910a, p. 488]). This concept relates to the automorphism group of a configuration, which will be discussed in chapter 4. An automorphism of a configuration is a mapping of the configuration into itself that permutes points and lines in such a way as to preserve the incidence structure. A configuration is called regular if its automorphism group acts transitively, i.e. if it contains enough transformations so that any point of the configuration can be mapped to any other. This definition applies to abstract configurations, independent of any embedding. So for the study of the automorphisms of a configuration it suffices to consider any abstract scheme. Thus, one can easily see that $(9_3)_1$ is regular just by exchanging columns in its table .

We now turn to a discussion of the three distinct (9_3) configurations using Kantor's method of "Restfiguren". The following figures give plane realizations of the $(9_3)_2$ and $(9_3)_3$ configurations:

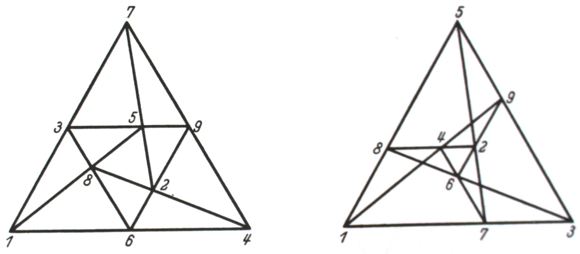


Figure 2.14 [Hilbert and Cohn-Vossen 1932, p. 107]

In any configuration (p_3), each point is connected with exactly six other points by lines of the configuration. Thus, in the case $p = 9$ every point is *not* connected with two other points of the configuration. For example, in the Pascal configuration $(9_3)_1$, the points 8 and 9 are not connected with 1. Furthermore, there is no line connecting 8 with 9, so 1, 8 and 9 form a triangle of unconnected points. Similarly 2, 5, 6 and 3, 4, 7 form such triangles. These three triangles form the Restfiguren for the Pascal configuration. Kantor's procedure amounts to connecting the three unconnected points for each point of the configuration, which for the points 1, 8 and 9 simply produces the lines of the triangle, as with the other two cases. Kantor notes that the three triangles that arise from this procedure happen to lie in 3-fold perspective, and the centers of perspectivity for any two are just the vertices of the third triangle. He also points out that the same is true for the Restfigur viewed as 9 lines (3 Dreiseite). In this case, the three lines of any one Dreiseit will be the 3 perspectivity axes for the other two [Kantor 1882c, p. 926], [Rosanes 1870].

Using Restfiguren, Kantor was able to construct a complementary figure for any given configuration. The points of the complementary figure are the same, but one connects any two points in the complementary figure if they are not already connected in the given configuration.

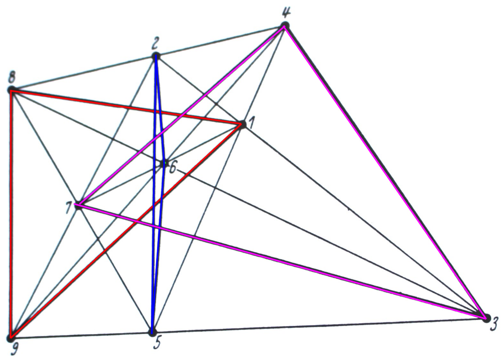


Figure 2.15 [Hilbert and Cohn-Vossen 1932, p. 108]

Kantor then showed that the remainder figure for a $(9_3)_2$ is a nonagon, which is simultaneously inscribed and circumscribed in a cubic curve, whereas in the case $(9_3)_3$ one obtains a hexagon and a triangle. [Kantor 1882c, p. 925] This shows that the three configurations (9_3) do not merely differ in the positions of their points but are essentially different. Furthermore, a $(9_3)_3$ cannot be a regular configuration because an automorphism maps points of the hexagon only to other points of hexagon and never to points of the triangle.

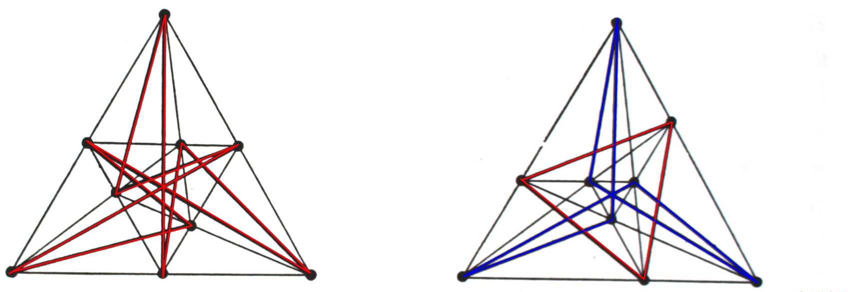


Figure 2.16 [Hilbert and Cohn-Vossen 1932, p. 108]

2.1.1.4 The (10_3) Configurations

Ten points and ten lines in the plane constitute a (10_3) configuration if three points lie on each line and three lines pass through each point. In this case, however, there are ten different configurations (10_3), only one of which is not realizable in the real plane. One of these is particularly important because its close relationship with Desargues' theorem[12] and is therefore called Desargues' configuration. We will discuss Desargues' theorem in space in the next section; the planar configuration can then be obtained by projection. Below is a picture illustrating Desargues' configuration in the plane.

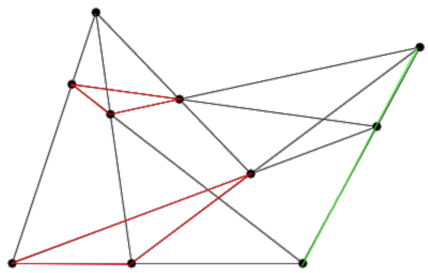


Figure 2.17

Hilbert and Cohn-Vossen wrote that there are nine other (10_3) configurations, but they only stated this result ([Hilbert and Cohn-Vossen 1932, p. 127]) without further discussion. They presented no tables or other methods for differentiating between these cases, as they had for the three cases of (9_3) configurations. Already fifty years earlier, Kantor gave a complete analysis of all ten (10_3) configurations; his work marks the beginning of systematic research on n_3 configurations [Kantor 1882a, p. 1299] [Steinitz 1910a, p. 486]. Reye also mentioned Kantor's findings in [Reye 1882, p. 94].

Using his method of Restfiguren, Kantor first identified three possibilities for the connectivity of the three points not connected with a given point:

1. The three points lie on a line, which Kantor denoted by the symbol "$\cdot - \cdot - \cdot$",
2. The three points are not collinear and only one point is connected with the other two, denoted by "$\wedge$",
3. The three points are not collinear, but all three points are connected, denoted by "$\triangle$".

Kantor then identified the "remainder figures" associated with each of the ten points of a configuration. He furthermore claimed that two configurations will be identical if and only if their associated remainder figures are identical. Clearly this is a necessary condition, and it happens to be correct in the case of (10_3) configurations, but not in general. In 1887 Vittorio Martinetti refuted this claim by exhibiting two different configurations (11_3) with

[12] This result was proved by Girard Desargues, though he never published this theorem. His friend and pupil Abraham Bosse mentioned it in an appendix of his book *Maniére universelle de M. Desargues pour practiquer la perspective* (*Universal Method of M. Desargues for using Perspective*), which was published in 1648.

the same remainder figures [Steinitz 1910a, p. 487].

Kantor's analysis led to the following 10 different cases for the remainder figures of (10_3) configurations:

(A) 10 $\triangle$	(B) 10 $\cdot - \cdot - \cdot$	(C) 6 $\cdot - \cdot - \cdot$ 4 $\triangle$	(D) 2 $\cdot - \cdot - \cdot$ 6 $\wedge$ 2 $\triangle$	(E) 9 $\wedge$ 1 $\triangle$
(F) 9 $\wedge$ 1 $\cdot - \cdot - \cdot$	(G) 4 $\cdot - \cdot - \cdot$ 6 $\wedge$	(H) 1 $\cdot - \cdot - \cdot$ 3 $\wedge$ 6 $\triangle$	(J) 3 $\wedge$ 7 $\triangle$	(K) 4 $\triangle$ 6 $\wedge$

These ten cases correspond to the following pictures of configurations (10_3) based on those produced by Kantor in [Kantor 1882a, p. 1315]:

If we examine tables associated with these 10 different configurations, we see that one of them contains the table for the unique configuration (7_3). As already noted, the (7_3) cannot be realized even in the complex plane because its system of equations is incompatible. This configuration appears as *C* among Kantor's figures, but this does not correspond to a geometric realization since some of its lines are slightly curved [Gruenbaum 2009, p. 75]. This error in Kantor's work was first pointed out by Heinrich Schroeter in 1889 [Gropp 1997b, p. 142]. All the other nine (10_3) configurations can, however, be embedded in the real plane. Desargues' configuration is distinguished from the other nine by the fact that for it alone the final incidence condition for its construction will automatically be satisfied. In 1990 B. Sturmfels and N. White proved that all 31 11_3 and all 229 12_3 configurations can be realized in the plane, whereas this was still an open question in the 1990s for the three next cases 13_3, 14_3, and 15_3 [Gropp 1997b].

2.1.2 Spatial Configurations

As pointed out in the introduction, the concept of a configuration was also introduced for incidence structures in three-dimensional space. For points and planes, a spatial configuration requires that every point be incident with the same number of planes and every plane to contain the same number of points. Apart from configurations of points and planes, one can also consider spatial configurations consisting of points and straight lines. These two points of view are often considered together when they apply to the same figure.

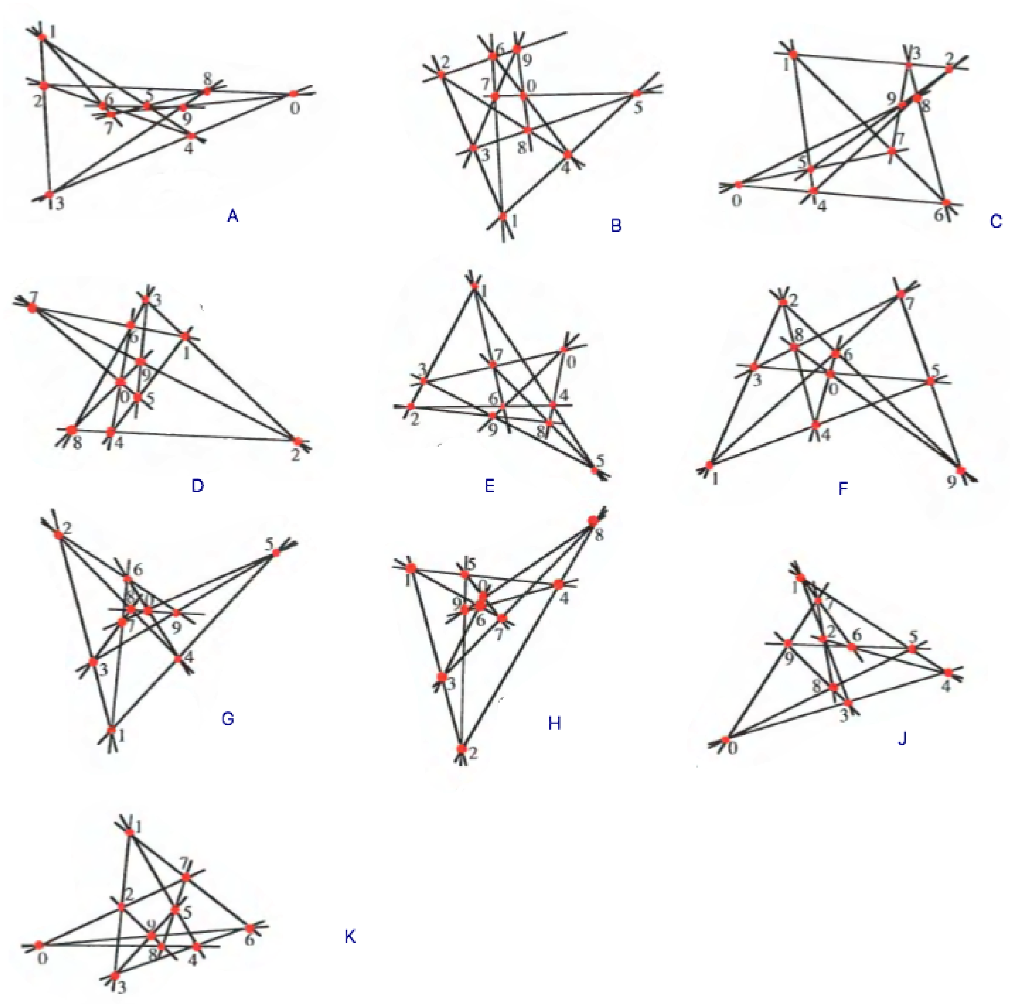


Figure 2.18 The reproduced pictures were obtained from [Gruenbaum 2009, p. 74]

In analogy with the cases just considered, the number of points and planes may well be equal. When this occurs, then every point of a configuration is incident with n planes, and every plane of the configuration must also contain n points. We denote such a configuration by the symbol (p_n). Here we take n to be at least 4, since $n < 4$ leads only to trivial cases in three-dimensional space. Since a (p_4) configuration does not exist for $p \leq 7$, the simplest case in space is the (8_4) configuration first studied by Möbius. After this, however, the number of configurations rapidly proliferate, making it next to impossible to gain an overview of all of them. For example, there are 26 (9_4) configurations, all of which can be realized geometrically. Hilbert and Cohn-Vossen only examined two spatial configurations in greater detail: Reye's configuration and Schläfli's double-six. We take these up here as well, but begin first with Desargues' spatial configuration.

2.1.2.1 Desargues' Configuration

Hilbert and Cohn-Vossen called Desargues' theorem the simplest and at the same time the most important theorem in three-dimensional projective geometry. Its significance led to Hilbert's pioneering work on so-called Desarguesian geometries in [Hilbert 1899]. In [Hilbert and Cohn-Vossen 1932] the theorem is given as follows:

Two non-coplanar triangles ABC and $A'B'C'$ lie in perspective with respect to the point O, the common point of intersection of the lines connecting corresponding vertices. Then the three pairs of lines of the corresponding sides intersect in three points R, S, and T that lie on a line (see Figure 2.19).

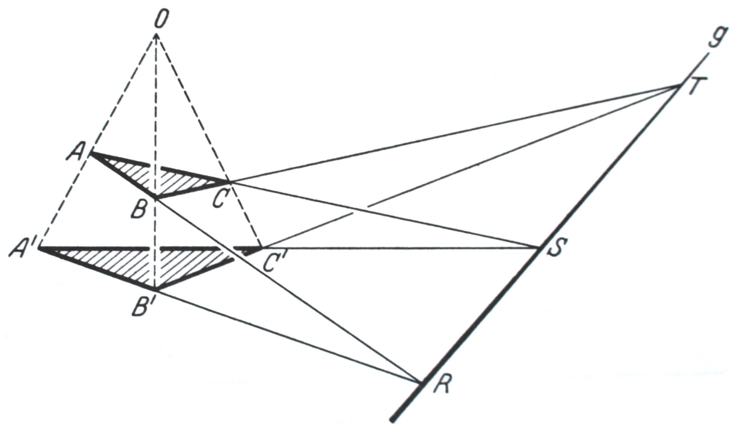


Figure 2.19 [Hilbert and Cohn-Vossen 1932, p. 121]

A special case of particular importance occurs when the two triangles are coplanar. If the plane happens to be embedded in space, then Hilbert and Cohn-Vossen show how the planar figures can be lifted to a spatial figure like Figure 2.20 of the spatial theorem. This means that every planar Desarguesian figure can be derived via projection from a three-dimensional Desarguesian figure, from which it is easy to see that the Desarguesian configuration (10_3) is embeddable in the plane.

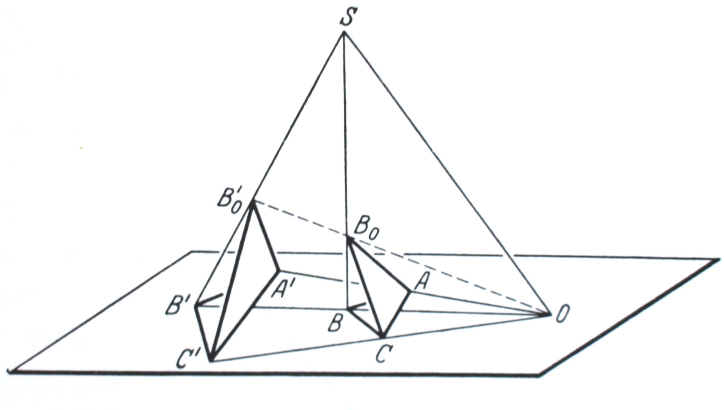


Figure 2.20 [Hilbert and Cohn-Vossen 1932, p. 122]

This configuration shares with Pascal's configuration the property that the last incidence condition is automatically satisfied when the figure is constructed step by step from its table. Moreover, Desargues' configuration, like the Pascal configuration, is self-dual because it represents both Desargues' theorem and its converse. The theorem of Brianchon is the dual of Pascal's theorem, but both lead to the same configuration when the conic degenerates to two lines, the case of Pappus's theorem.

Hilbert and Cohn-Vossen showed how the Desargues configuration can be represented by a pair of mutually inscribed and circumscribed pentagons. Very likely they took this from [Kantor 1882a, p. 1291]. In this article, Kantor labeled the vertices of one pentagon I, II, III, IV, V and the other 1, 2, 3, 4, 5. The mutually inscribed and circumscribed pentagons can then be positioned in three different ways, as Kantor showed using the following tables:

case 1	I II	contains	3	and	1 2	contains	II
	II III		4		2 3		III
	III IV		5		3 4		IV
	IV V		1		4 5		V
	V I		2		5 1		I

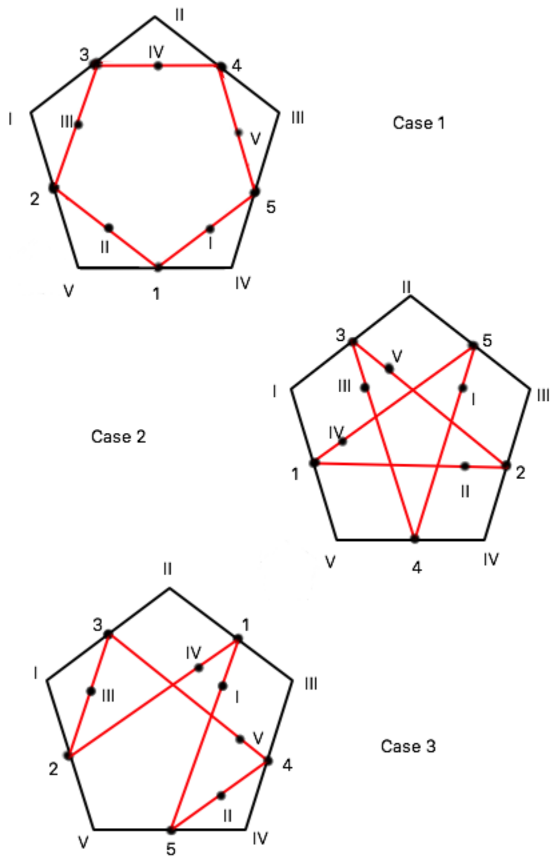


Figure 2.21

case 2	I II	contains	3	and	1 2	contains	II
	II III		5		2 3		V
	III IV		2		3 4		III
	IV V		4		4 5		I
	V I		1		5 1		IV

case 3	I II	contains	3	and	1 2	contains	IV
	II III		1		2 3		III
	III IV		4		3 4		V
	IV V		5		4 5		II
	V I		2		5 1		I

For each of these case, one can draw corresponding figures, which look like this:

Kantor concluded that only cases 1 and 2 lead to configurations. According to Coxeter, case 1 gives rise to a “non-Desarguesian” 10_3 configuration [Coxeter 1950, p. 438]. Hilbert and Cohn-Vossen only consider case 2, which they associate with Desargues’ configuration. They begin by looking for pentagons within the spatial configuration such that no three consecutive vertices are collinear. These vertices will form five points in general position in space, i.e. no four can be coplanar and no three collinear. The vertices of a second pentagon must then lie on the edges of the first pentagon and these along with their connecting edges must belong to the configuration.

2.1.2.2 Möbius' Configuration

The Möbius configuration with symbol (8_4) consists of two mutually inscribed and circumscribed tetrahedra. Thus, each vertex of one tetrahedron lies on a face plane of the other, and vice versa. Möbius published his findings on this arrangement of tetrahedra in [Moebius 1828]. There he also proved that if two tetrahedra have the property that seven of their vertices lie on corresponding face planes of the other tetrahedron, then the eighth vertex also lies on the plane of its corresponding face, thus forming a configuration of this type. Thus, like the configurations of Pascal and Desargues, the incidence structure underlying the Möbius configuration expresses a geometric theorem.

Hilbert and Cohn-Vossen stated that there are altogether five configurations (8_4), information they likely gained from the survey article by Ernst Steinitz [Steinitz 1910a]. Steinitz actually described this result more precisely, citing research by P. Muth, G. Bauer, and V. Martinetti, who showed there are five (8_4) configurations with the property that at most two planes have two points in common, and dually that at most two points are common to two planes [Steinitz 1910a, p. 493] (This condition means that every three points are non-collinear and dually, that no three planes have a line in common.) Steinitz remarked, however, that there are also 10 other (8_4) configurations which do not satisfy this condition; furthermore, all fifteen of these configurations are realizable in real three-dimensional space [Steinitz 1910a, p. 494]. He observed further that the Kummer configuration is the only (16,6) configuration that satisfies this same condition.

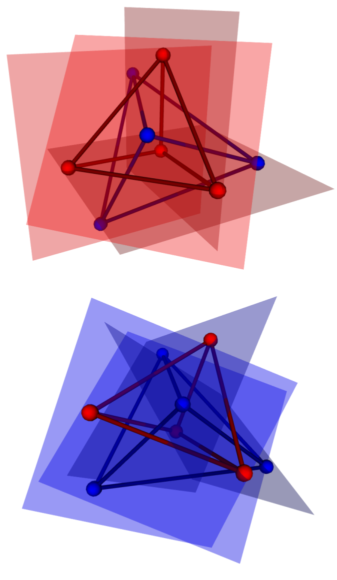


Figure 2.22 Möbius configuration (Wikipedia)

Coxeter presented the Levi graph of the Möbius configuration, which is a four-dimensional hypercube, as shown in Figure 2. 23. [Coxeter 1950, p. 415]

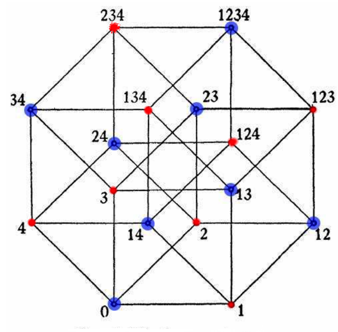


Figure 2.23 [Coxeter 1950, p. 415]

2.1.2.3 Reye's Configuration

Reye's configuration consists of twelve points and sixteen lines, forming a planar configuration with the symbol $(12_4, 16_3)$. It also can be realized in projective space as a configuration of points and planes with the symbol (12_6).

This incidence structure was actually first discovered even before Möbius began to contemplate mutually inscribed tetrahedra. The famous French geometer Jean-Victor Poncelet already noticed it in connection with the centers of similitude of four spheres [Poncelet 1822][Steinitz 1910a, p. 497].[13] The first deeper investigation began much later, however, with works by Cyparissos Stephanos, Giuseppe Veronese, and of course, Theodor Reye [Stephanos 1879], [Reye 1882].

Figure 2.24 Theodor Reye (Wikipedia)

After mentioning the concept of configurations in the second edition of his *Geometrie der Lage*, Reye introduced it explicitly by definition in [Reye 1882], where he discussed the configuration of 12 points and 16 lines on the very first page of his article: "There are 12 centers of similitude for four spheres with three points on each of 16 lines and 6 points on each of 12 planes, and 3 of the 12 planes pass through each of 16 lines and 6 of the 12 planes pass through each of the 12 points" [Reye 1882, p. 93]. One month later, Reye wrote another article on the same subject with the title "Die Hexaëder und die Octaëder Configurationen $(12_6, 16_3)$"; this article was also published in the first volume of Acta Mathematica.

Reye's configuration embodies a theorem of projective geometry, hence the last inci-

[13] In his pioneering *Traité des propriétés projectives des figures*, Poncelet investigated the differences between metric and incidence properties [Wussing 1984, p. 27].

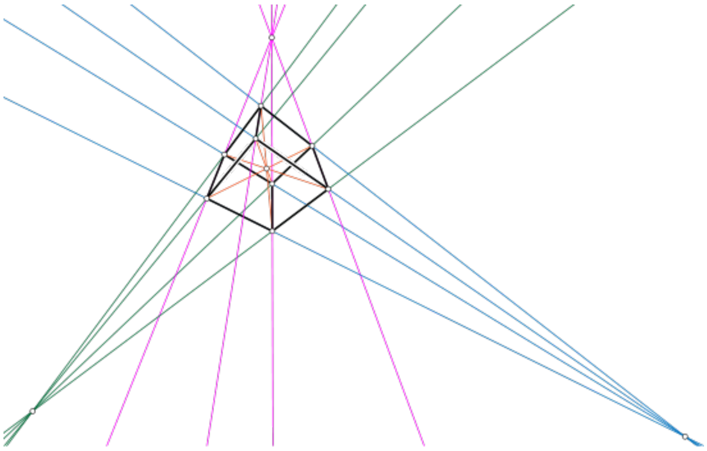


Figure 2.25 The Reye Configuration (Wikipedia)

dence involved in its construction follows as a result. Hilbert and Cohn-Vossen arranged the points in a special symmetrical order to facilitate the visualization of the configuration, as can be seen in the Fig. 2.26.

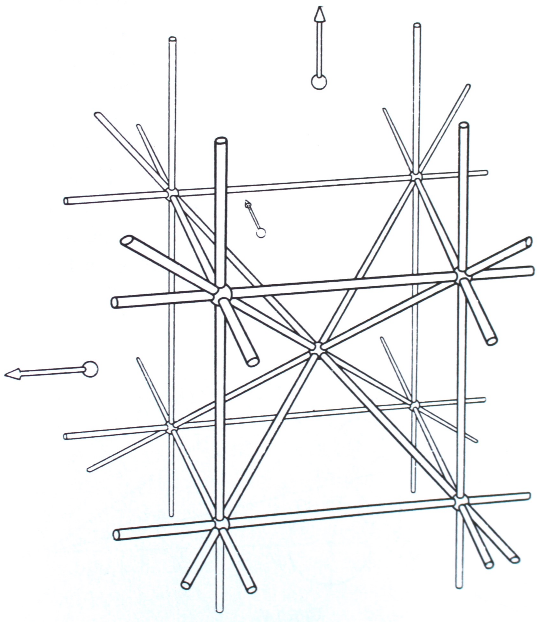


Figure 2.26 [Hilbert and Cohn-Vossen 1932, p. 135]

The twelve points of the configuration appear here as the eight vertices of a cube together with its center point and the three ideal points at infinity where the parallel edges of the cube meet. Its twelve planes consist of the six faces of the cube and the six diagonal planes passing through opposite pairs of edges. There are thus six configuration points in each plane: four vertices and two ideal points on each of the planes containing a face of the cube, and four vertices, the center of the cube, and an ideal point on each of the diagonal planes. Six planes pass through each point: the six diagonal planes pass through the center of the cube, three face planes and three diagonal planes pass through each vertex, and four face planes and two diagonal planes through each of the ideal points. These twelve points and lines thus form a configuration with the symbol (12_6).

The figure can also be viewed as a configuration of points and lines with the same 12 points, but with 16 lines: the twelve edges and the four diagonals of the cube. Three points of the configuration lie on each of these lines: two vertices and one ideal point on each edge of the cube; two vertices and the center on each of its diagonals. Furthermore, there are four lines through each point : three edges and one diagonal through each vertex of the cube; four diagonals through the center of the cube; and four edges through each ideal point. Therefore, the points and lines of Reye's configuration form a configuration of type $(12_4, 16_3)$.

Hilbert and Cohn-Vossen also discussed Reye's configuration in relation to the system of centers of similitude of four spheres, following Poncelet's initial observation. [14] The centers of similitude of two circles or spheres divide the line of centers in the ratio of their radii. In the case of two circles that lie outside one another, these two points are determined by the internal and external tangents to the circles.

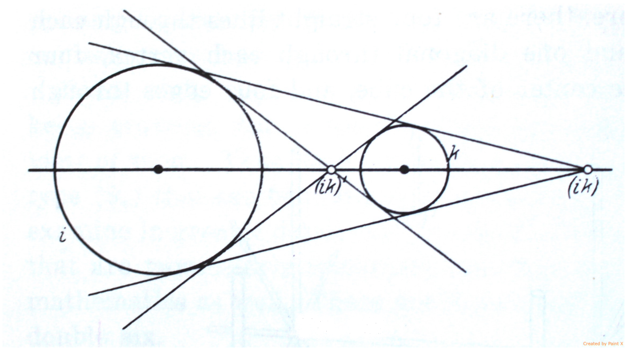


Figure 2.27 [Hilbert and Cohn-Vossen 1932, p. 136]

For three circles or spheres with centers: 1, 2, and 3 there are three internal and three external centers of similitude, making six in all. These six points lie on four straight lines called the axes of similitude. According to a theorem of Monge, the centers of similitude and the axes of similitude form the six points and four lines of a complete quadrilateral, which is a plane projective configuration.

[14] It was Poncelet who introduced the fundamental distinction between "projective" and "nonprojective" properties of figures, that is, between properties that are always preserved by central projections and properties that are usually destroyed by such projections. [Wussing 1984, p. 27]

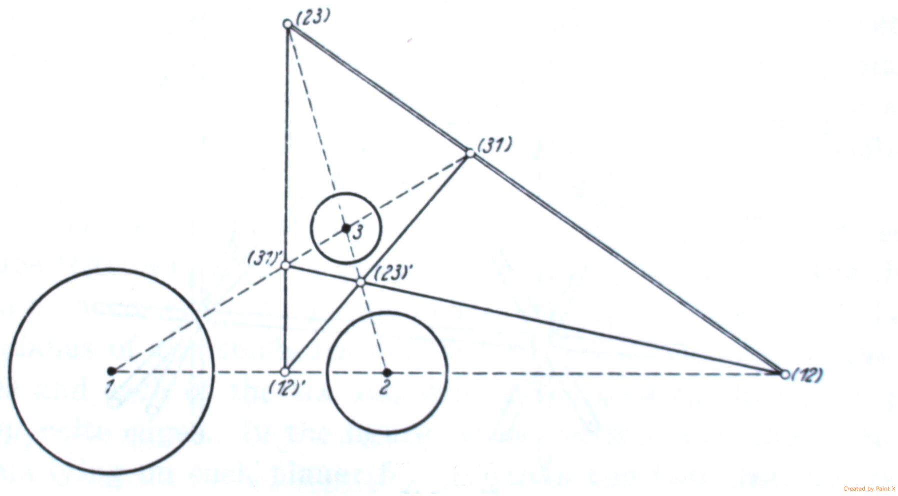


Figure 2.28 [Hilbert and Cohn-Vossen 1932, p. 136]

The analogous spatial configuration arises from four spheres whose centers 1, 2, 3, 4 are not coplanar and no three of which lie on a straight line. Six different pairs can be chosen from the spheres, each of which gives rise to an external and an internal center of similitude; therefore, there are twelve centers of similitude in all. Furthermore, there are 16 axes of similitude, since there are four different ways to select three spheres out of four, and each set of three spheres gives rise to four different axes of similitude. Thus, altogether the centers and axes of similitude constitute a projective configuration of type $(12_4, 16_3)$, shown in Fig. 2.29.

To see that it is identical with Reye's configuration, we need to identify twelve corresponding planes that constitute a projective spatial configuration of type (12_6). These are: 1. the four planes containing the centers of three of the four spheres; 2. one plane containing all six external centers of similitude; 3. the four planes containing three external and three internal centers of similitude; 4. three planes containing the following points: (12, 34) , (13, 24) , (14, 23).

So Fig. 2.29 is projectively equivalent to Fig. 2.26 if the three points $(12), (12)'$, and (34) are moved to infinity in mutually perpendicular directions so that they assume the positions of the three ideal points. The eight points (13), (14), (23), (24), $(13)'$, $(14)'$, $(23)'$, and $(24)'$ then become the vertices of a cube, and $(34)'$ becomes its center. The Fig. 2.29 can represent a plane configuration of type $(12_4, 16_3)$ formed by the centers and axes of similitude of four coplanar circles. Reye's configuration of points and planes is both regular and self-dual.

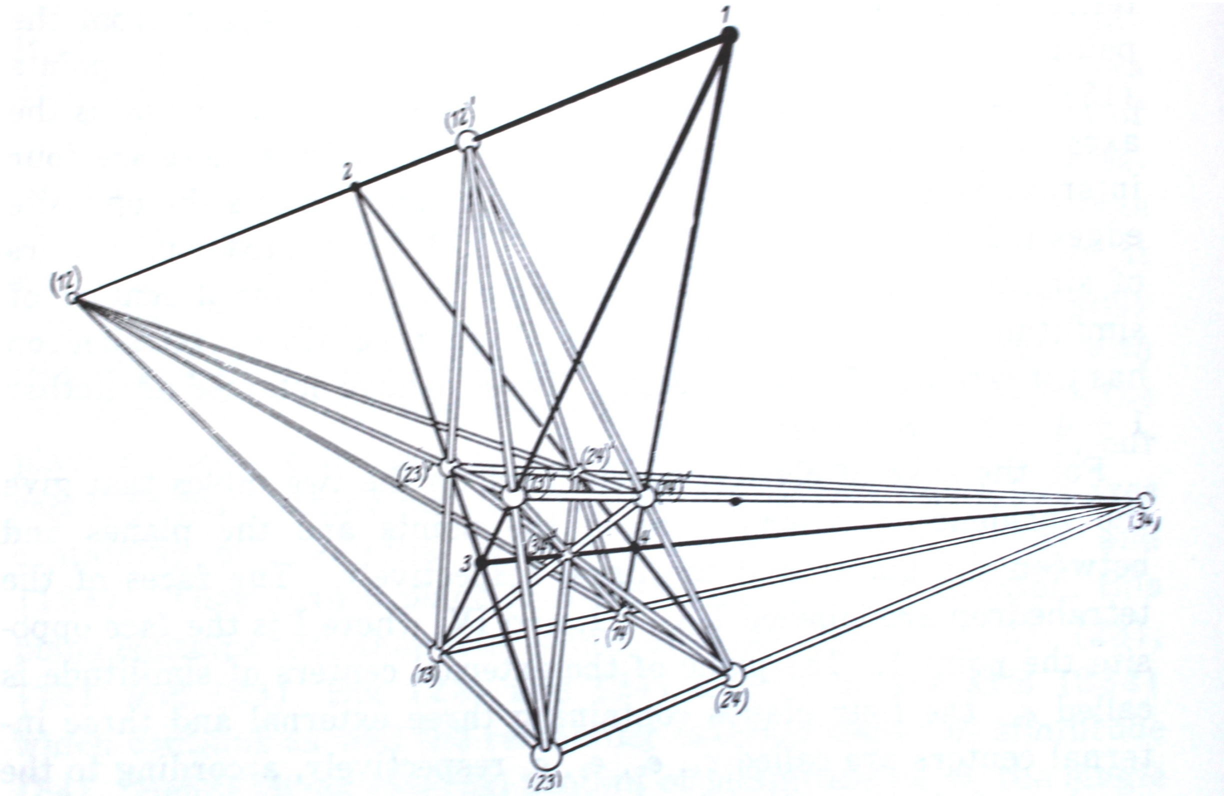


Figure 2.29 [Hilbert and Cohn-Vossen 1932, p. 140]

An important connection also exists between Reye's configuration and the tetrahedra that form a so-called "desmic system". The latter structure was investigated for the first time in 1879 by Cyparissos Stephanos, who related this notion to the 12 points and 16 lines of a Reye configuration. A desmic system is formed by a set of three tetrahedra in projective space, positioned so that for any pair each edge of one cuts a pair of opposite edges of the other. In such a system, every line passing through two vertices of two different tetrahedra also passes through a vertex of the third tetrahedron. The 12 vertices and 16 lines formed in this way constitute the points and lines of a Reye configuration. Figure 2.30 shows two desmic tetrahedra. The third tetrahedron is not shown, since it has one vertex at the center and the other three lie on the plane at infinity.

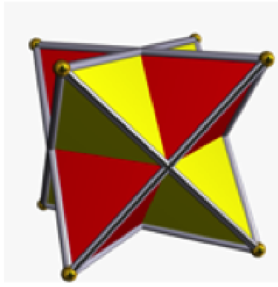


Figure 2.30 A Desmic System (Wikipedia)

The following is a picture of two desmic tetrahedra from the article of Stephanos.

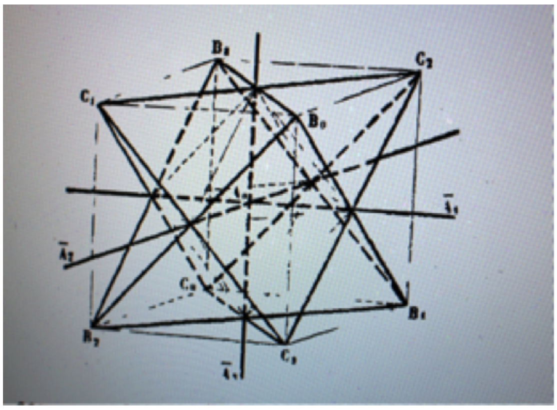


Figure 2.31

3 Configurations in Algebraic Geometry

Three particularly important configurations are discussed in this chapter: the Hessian configuration, Schläfli's double-six, and the Kummer configuration. The researches on these three configurations were undertaken, respectively, by Otto Hesse (1844), Ludwig Schläfli (1858) and Ernst Kummer (1866). They thus belong to the pre-history of configurations and arose from studies of cubic curves, cubic surfaces, and quartic surfaces, which mark the beginning of algebraic geometry studied mainly from the standpoint of projective geometry.

The discovery of the 27 lines on nonsingular cubic surfaces has long been understood as one of the key breakthroughs that help launch classical algebraic geometry in the mid-19th century. Before this time, geometers had focused a good deal of attention of quadric surfaces, which Euler had classified algebraically a century earlier. Quadrics were relatively easy to visualize, and in some cases they could be handled as ruled surfaces. In no case, however, did they lead to a finite configuration of points and lines.

Special classes of quartic surfaces were also investigated intensely during the 1860s and 1870s. Since a general classification of quartics, like the one Schläfli undertook for cubic surfaces, would have led to hopeless complications, most studies focused on objects with several singularities. Whereas an irreducible cubic surface can have at most 4 singular points, a quartic can have as many as 16. The first quartic surface to be studied in considerable detail was the Fresnel wave surface, which happens to be connected with a famous problem in optics: the phenomenon of double refraction. This arises when a light beam entering a crystal is refracted in two different directions, so that it splits into two beams in the course of passing through the glass.

In the 1840s Arthur Cayley took up the study of certain quartic surfaces he called tetrahedroids [Hudson 1990, pp. 89-94]. These have the property that the four planes of a tetrahedron meet the surface in two conics that pass through four singularities of the surface. The Fresnel surface corresponds to a special case in which each pair of conics contains a circle, and where these pairs of curves only intersect in real points in one of the four planes. The other twelve singular points are imaginary, so there are sixteen altogether. Analytic geometers allowed for the possibility that points, lines, and planes may have imaginary coordinates, though these imaginary elements were usually sharply distinguished from those that are real.

Eventually it became clear that from a projective viewpoint the Fresnel surface belonged to a special class of quartics known today as Kummer surfaces. Among all quartics, the Kummer surfaces are distinguished by the property of having sixteen isolated singular points. These 16 singular points occupy a very special position in 3-space: they lie in groups of six in each of the 16 singular planes, or tropes. Each of these 16 tropes is a tangent plane to the surface that touches it along a conic section containing 6 of the 16 singularities. These sets of six points thus also lie in a special position, since only five coplanar points determine a conic. In fact, the singular points and planes of a Kummer surface form a symmetric $(16,6)$ configuration since six singular planes pass through each of the singular points of the surface, which is self-dual.

Jordan introduced a useful notation for this incidence structure by using two square matrices. The points are labeled (A_{ij}) and the planes (α_{ij}), $i,j=1,\ldots,4$. A point and a plane are then incident when exactly one of the two indices agrees, thus they appear either in the same row or the same column, but not both. Geometrically, one can then see that two corresponding rows or columns form a tetrahedron where each pair of entries represents a vertex and the opposite plane. A pair of rows and columns is then a pair of tetrahedra that form a Möbius $(8,4)$ configuration [Steinitz 1910a, p. 501]. A very rich interplay of various configurations arises when Kummer surfaces are studied within the context of line geometry, a theory first developed by Felix Klein around 1870. Klein showed that a Kummer surface is the associated singularity surface for a 1-parameter family of quadratic line complexes, a relationship that bears a striking resemblance analytically to the theory of concentric surfaces in 3-space (see [Hudson 1990] for details).

An older configuration, already mentioned in chapter 2, was uncovered by Otto Hesse in 1844 in connection with cubic curves. These were the first truly new objects to emerge in algebraic geometry, although Descartes had sketched a tentative theory that pointed toward special classes of higher-order curves. Mainly, though, he and Fermat had determined that the theory of second-order curves coincides with the conic sections. Before discussing Hesse's approach to cubic curves, we briefly describe some of the earlier work devoted to the theory of cubic curves.

3.1 Early discoveries on Cubic Curves

The English mathematician, astronomer, and physicist Isaac Newton (1642-1727) was the first to undertake a systematic study of real cubic curves, which he classified in an appendix to his *Opticks* [Dolgachev 2012, p. 160]. He classified these curves in 72 different types, although it turned out that he had overlooked six others that belong to this scheme. These distinctions were made algebraically, but otherwise Newton's analysis was largely synthetic and conceptually his approach was in the spirit of Apollonius's account of the properties of conic sections.

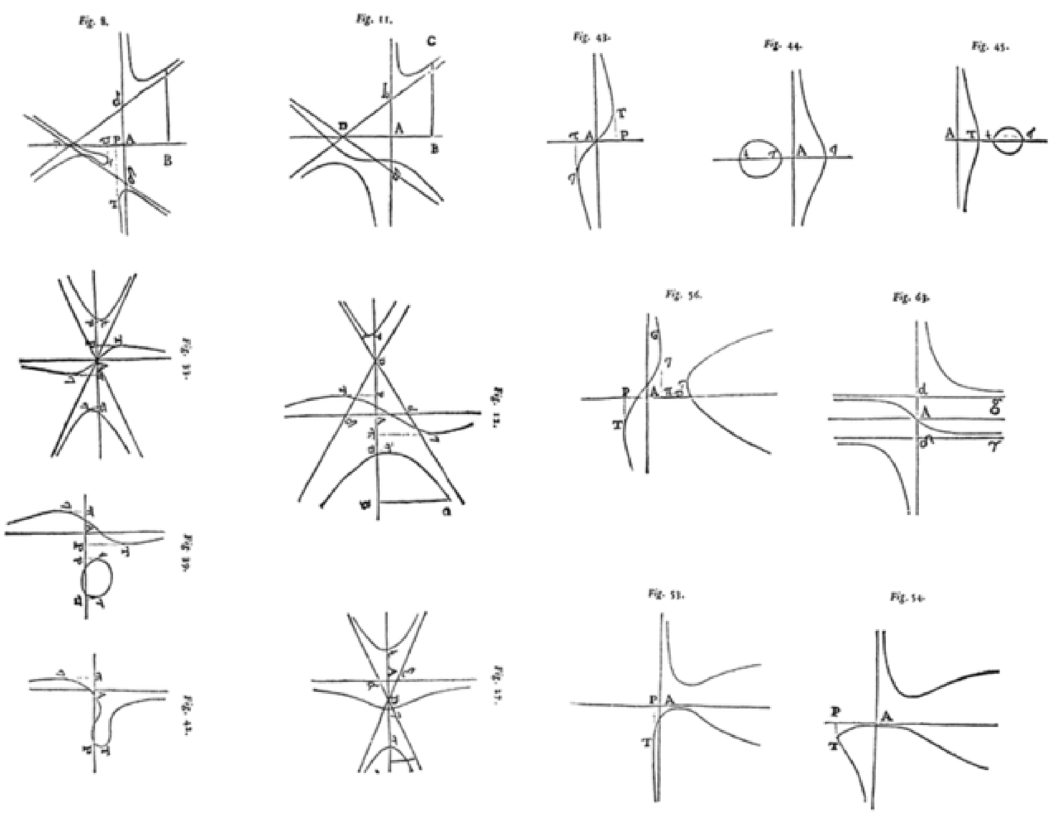


Figure 3.1 Fifteen of the 72 types of cubics classified by Newton ([Brieskorn and Knoerrer 1986, pp. 284–285])

Like many later investigators, Newton was especially interested in showing how these different types of cubics behaved at infinity, as can be seen from the asymptotes in several of the pictures above. If we imagine these as projective curves where the branches along asymptotes eventually meet, then these pictures show that a real cubic may have one or two connected components. Several pictures also show inflection points on the curves, and indeed there will always be at least one. Somewhat later, the French mathematician

Jean Paul de Gua de Malves (1713–1785) and the Scottish mathematician Colin Maclaurin (1698–1746) made a notable discovery, namely that when a line connects two inflection points on a cubic curve it will also pass through a third inflection point [Pascal 1910, p. 384] [Kohn and Loria 1908, p. 475]. Maclaurin showed, furthermore, that a cubic could have at most three real inflection points and these, therefore, always lie on a line.

In the nineteenth century, the German mathematician Julius Plücker (1801–1868) presented a detailed classification of cubic curves in his *System der analytischen Geometrie* (1835). In this work, he also gave an elegant proof of Pascal's theorem that implicitly illuminated the connection between families of cubic curves and the Pascal configuration. Plücker's proof made use of an abbreviated notation in which a single letter was used to write the equation for a line as $p = 0$. He then considered two triples of lines p, q, r and p', q', r' that determine nine points of intersection. If six of these lie on a conic C_2, then Pascal's theorem asserts that the remaining three points – say $P = p \cap p', Q = q \cap q', R = r \cap r'$ – fall on a line. Plücker next considers the 1-parameter pencil of cubic curves C_μ given by $pqr - \mu p'q'r' = 0$, and notes that all of them pass through the nine points of intersection. The pencil C_μ fills the entire plane, so choosing some arbitrary seventh point on C_2 there will be a C_3 in C_μ that meets C_2 in all seven points. By Bezout's theorem, this can only happen when $C_2 \subset C_3$, which implies that C_3 decomposes into a conic and a line. This proves that the three points P, Q, R must be collinear. Seen from the standpoint of configurations, Plücker's proof shows that every Pascal configuration 9_3 determines a pencil of cubics that contains this configuration as a degenerate case. An analogous situation occurs with regard to the Hessian inflection point configuration, which will be taken up below.

Plücker was a pioneering figure in the realm of curve theory. One of his many achievements was to determine the number of inflection points of an algebraic curve depending on the number and type of its singularities. When a curve of degree n has no singularities, then it has $3n(n-2)$ inflection points. Thus, a nonsingular cubic has 9 inflection points, and Plücker also proved that 3 will be real and the other 6 imaginary [Kohn and Loria 1908, p. 475]. This seemed to confirm the results obtained in the eighteenth century by Colin MacLaurin [Artebani and Dolgachev 2009, p. 235]. Arthur Cayley regarded Plücker's results on inflection points of plane algebraic curves as the greatest discovery in the entire history of geometry [Wussing 1984, p. 167]. This assessment seems quite odd today, however, especially since Otto Hesse soon afterward found a far better method for handling this problem based on what today is called the Hessian curve associated with a given algebraic curve.

The basic idea in any case is to find the inflection points of a given curve C by intersecting C with another associated curve C'. This idea goes back to de Gua's work from 1740 and was then carried out by Plücker in 1835. Plücker's C', however, was not "covariant", i.e. its definition depended on the choice of coordinate system [Brieskorn and Knoerrer 1986, p. 288]. In 1844, Hesse found a covariant curve, the Hessian curve $H(C)$, and he showed that the inflection points of a given curve C in the complex projective plane are precisely those cut out by $H(C)$, thus $C \cap H(C)$. If C is defined by the homogeneous equation $f(x_0,x_1,x_2) = 0$, then the equation of $H(C)$ of C is given by

$$\det\left(\frac{\partial^2 \mathrm{f}}{\partial \mathrm{x_i} \partial \mathrm{x_j}}\right) = 0.$$

For a curve C_m of order m, $H(C_m)$ has order $3(m-2)$, so by Bézout's theorem C_m can have at most $3(m-2)$ inflection points. In particular, the Hessian curve of a cubic is again a cubic, so in this case the maximum number of inflection points will be $C \cap H(C) = 3 \times 3 = 9$. There are, indeed, exactly 9 if and only if C_m is a nonsingular cubic. Moreover, these 9 points cannot lie in general position since only 8 points suffice to determine a cubic curve. The three real inflection points lie on a line, and since the entire figure is symmetric, the 9 inflection points lie in triples on 12 lines, the Hessian inflection-point configuration $(9_4, 12_3)$ for nonsingular cubic curves. This configuration has been named after Otto Hesse, who was the first to study its properties in [Hesse 1844a] and [Hesse 1844b].

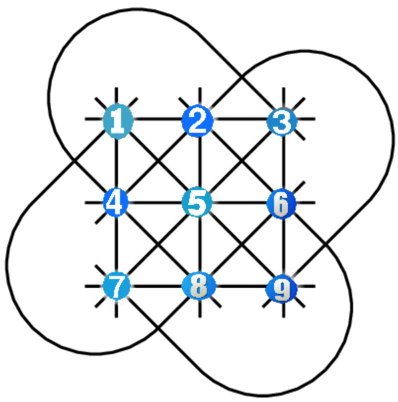


Figure 3.2

3.1.1 The Hessian Configuration

The Hessian configuration thus arose from Otto Hesse's investigations of cubic curves. Hilbert and Cohn-Vossen did not describe the properties of this configuration, except to mention how one can obtain its table by extending the table of the configuration (8_3). We gave this table earlier as well as a schematic presentation for an abstract configuration $(9_4, 12_3)$ in the figure 3.2.

From Hesse's work, we see that this configuration can be realized geometrically in the complex projective plane. The equation of any non-singular plane cubic curve can be written in Hessian normal form, using (x, y, z) as homogeneous coordinates:[1]

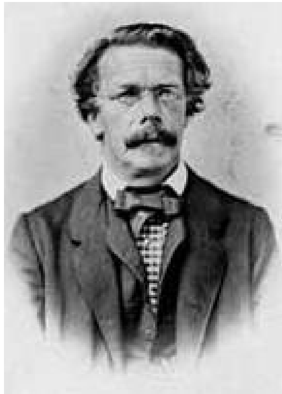


Figure 3.3 Otto Hesse (Wikipedia)

$$x^3 + y^3 + z^3 + \lambda xyz = 0,$$

where λ is a complex parameter and $\lambda^3 + 27 \neq 0$.

The pencil of cubics C_λ [Weber 1896, pp. 399-404] was classically called a "syzygetic pencil" of cubic curves. The term "syzygy" originates from astronomy, where it refers to three planets that are aligned [Artebani and Dolgachev 2009, p. 235].[2] It was commonly employed in invariant theory to express an algebraic relation between covariant expressions, which is the case here, as will be seen below.

To find the inflection points for this pencil, we calculate the Hessian determinant D_λ for the Hessian curve $H(C_\lambda)$:

$$D_\lambda = \begin{vmatrix} 6x & \lambda z & \lambda y \\ \lambda z & 6y & \lambda x \\ \lambda y & \lambda x & 6z \end{vmatrix} = -6\lambda^2(x^3 + y^3 + z^3) + (216 + 2\lambda^3)xyz = \alpha(x^3 + y^3 + z^3) + \beta xyz.$$

This shows that the Hessian curve $H(C_\lambda)$ given by $\alpha(x^3 + y^3 + z^3) + \beta xyz = 0$ belongs to the pencil of cubics C_λ (set $\lambda = \frac{\beta}{\alpha}$), which accounts for why this is a syzygetic pencil. Since $C_\lambda \cap H(C_\lambda)$ are the 9 inflection points, it follows that every curve in the pencil shares them. The non-singular curves of this system are precisely the cubics for which $\lambda^3 + 27 \neq 0$. For $\alpha = 0$ one obtains $xyz = 0$, which gives the three lines $x = 0$, $y = 0$ and $z = 0$ of the coordinate triangle. The three roots of $\lambda^3 + 27 = 0$ are $\lambda = -3, -3\omega$

[1] The following presentation is based on [Brieskorn and Knoerrer 1986, pp. 293–298].

[2] This terminology is attributed to Luigi Cremona in [Artebani and Dolgachev 2009, p. 235] and to Arthur Cayley in [Shaub and Schoonmaker 1931, p. 389].

or $-3\omega^2$, where ω and ω^2 are the imaginary cube roots of unity. In each of these three cases, the cubic decomposes into three lines corresponding to the linear factors in the equations for the C_λ with:

$\lambda = -3$: $(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z) = 0$

$\lambda = -3\omega$ gives: $(x+y+\omega z)(x+\omega y+z)(x+\omega^2 y+\omega^2 z) = 0$

$\lambda = -3\omega^2$ gives: $(x+y+\omega^2 z)(x+\omega y+\omega z)(x+\omega^2 y+z) = 0$

These form the 12 lines of the Hessian configuration, 4 of which are real and the other 8 imaginary. Since any two curves of the pencil meet in the 9 inflection points, an easy way to calculate their coordinates is to let $\lambda = 0, \infty$ and find the intersection of these two curves:

$$\begin{cases} xyz = 0 & \text{if} \quad \alpha = 0 \\ x^3+y^3+z^3 & \text{if} \quad \beta = 0 \end{cases}$$

The coordinates of the nine points of intersection are then:

$$\begin{array}{ccc} (0, 1, \text{-}1) & (1, 0, \text{-}1) & (\text{-}1, 1, 0) \\ (0,\omega^2,-1) & (\omega,0,-1) & (-1,\omega,0) \\ (0,\omega,-1) & (\omega^2,0,-1) & (-1,\omega^2,0) \end{array}$$

Since the coordinates of these nine points are independent of λ, these are the inflection points for any curve in the Hessian pencil. The 12 lines of the configuration fall into 4 sets of 3 lines forming 4 triangles. Hesse discovered these four triangles and proved that each contains all 9 inflection points. The existence of these four triangles reduces the degree of the equation for these 9 points, as Jordan showed in his Traité [Lé 2015, p. 321]. Felix Klein emphasized the importance of this finding in his obituary for Hesse:

> Hesse seized on the problem of the algebraic determination of the nine inflection points. Because one can sort out four triangles with twelve lines on which the points lie three by three, the solution of the equation of the ninth degree depends on an equation of the fourth degree. It was a first remarkable example that geometry gave to the theory of equations .... (Klein 1875)

In two papers published in Crelle in 1849, Hesse elaborated on this configuration by studying an associated dual figure determined by the 9 harmonic lines, each of which arises from an inflection point on the nonsingular cubic curve. The result is a $(12_3, 9_4)$ configuration obtained as follows. If $p \in C_3$ is an inflection point, then its associated harmonic line h_p is given by the locus of points that lie harmonically with respect to the three points $p, p', p'' \subset C_3 \cap l_p$ for any line l_p that passes through p. Maclaurin had shown that this harmonic locus is a line when p is a real inflection point, and Plücker then studied the figure formed by all 9 harmonic lines. The 12 harmonic points that complete the configuration are related to the four inflectional triangles in the $(9_4, 12_3)$ configuration. Each inflection point p lies on 4 lines, which form one of the sides of these four triangles, and its opposite vertex is a harmonic point. There are thus 12 harmonic points in all, and 4 of these lie on each of the 9 lines h_p.

A drawing of the Hessian pencil can be found in [Grove 1906, p. 15], a doctoral dissertation submitted to the John Hopkins University [Artebani and Dolgachev 2009, p. 236]. To obtain Figure 3.4, one transforms the pencil by putting $x_1 = x+y+1$, $x_2 = -(x-1)$, $x_3 = -(y-1)$. Under this transformation, equation $x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3 = 0$ of the pencil becomes:

$$xy(x+y) + \frac{2(1-m)}{1+2m}(x^2 + xy + y^2) + 1 = 0,$$

which is of the form:

$$xy(x+y) + \mu(x^2 + xy + y^2) + 1 = 0,$$

where $\mu = \frac{2(1-m)}{1+2m}$. In Fig. 3.4, the inflectional lines are drawn solid and the harmonic polars as dashed lines. Several cubics with special parameters are shown: $\mu = 0, 2$, $\mu = -(1-\sqrt{3}), -(1+\sqrt{3})$, $\mu = 1/3, -2$ and etc. [Grove 1906, p. 15].

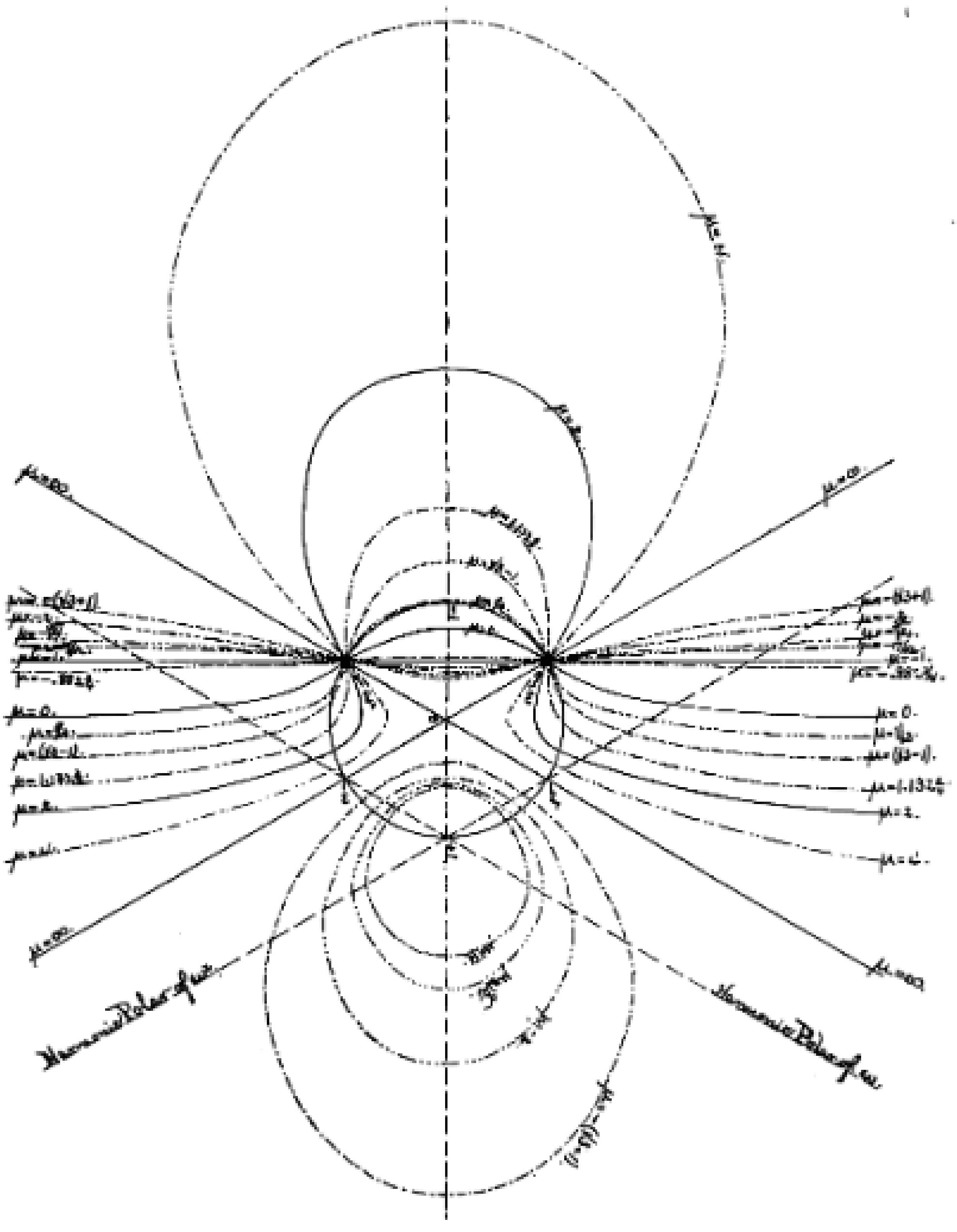


FIG. 1.—The Syzygetic Pencil of Cubics.

Figure 3.4 [Grove 1906, p. 15]

3.2 Schläfli's Double-Six

Algebraic geometry in 3-space gained new impulse after the discovery that the theory of cubic surfaces was intimately connected with the configuration of the 27 lines that lie on these surfaces. Ludwig Schläfli discovered this configuration in 1858 in connection with his studies on the classification of cubic surfaces. Ten years earlier, Arthur Cayley and George Salmon showed that exactly 27 lines lie on a nonsingular cubic surface, and in some cases all 27 can be real. Schläfli afterward came to the realization that all 27 of these can be constructed starting with the 12 lines formed by a so-called double-six, which forms a subconfiguration $(30_2, 12_5)$ of the larger one, a $(135_2, 27_{10})$ spatial configuration. Both were found, of course, well before the concept of a configuration was formulated by Reye in 1882. Visually cubics, on the other hand, presented a major challenge for geometers, beginning with the problem of building a spatial model for Schläfli's double-six, let alone the entire $(135_2, 27_{10})$ configuration. In 1871 Christian Wiener succeeded in building a model of a cubic surface showing all 27 lines, and soon thereafter Adolf Weiler produced a model for the diagonal surface shown in Fig. 3.7.

The Schläfli double-six was the final configuration investigated in the third chapter of [Hilbert and Cohn-Vossen 1932]. A double-six can be represented by the following scheme.

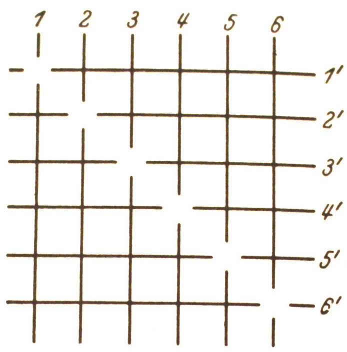


Figure 3.5 Schema for the Schläfli double-six ([Hilbert and Cohn-Vossen 1932, p. 146])

From this, we see there are 30 points of intersection, so the 12 lines form a regular configuration of points and lines with the symbol $(30_2, 12_5)$. A spatial representation of the double-six can be constructed by choosing one of the lines of each set of six to correspond with a face of a cube. In this way, all the lines lie in the planes formed by a cube. This can also be illustrated by the two visible desmic tetrahedra shown in Figure 2.31, for which the pairs of intersecting edges lie in the six faces of a cube.

Schläfli's double-six thus consists of 30 points and 12 lines, the latter divided into two sets of six skew lines. Hilbert and Cohn-Vossen label these $1, 2, \ldots, 6, 1', 2', \ldots, 6'$ and describe Schläfli's method of construction roughly as follows. First, one takes an arbitrary line 1 and three mutually skew lines $2', 3', 4'$ that intersect it. The lines in space that meet these three form a second-degree ruled surface. There then exists a line $5'$ skew to lines $2', 3', 4'$ that intersects 1 but does not lie on this quadric surface. A second line in space, labeled 6, also meets the mutually skew lines $2', 3', 4', 5'$ but not 1. Then one can find a line $6'$ that intersects 1 but none of the four lines $2', 3', 4', 5'$. Now one can repeat the above argument, noticing for example that besides 1 a second line, denoted 5, meets the four skew lines $2', 3', 4', 6'$. Similarly, one obtains the lines $2, 3, 4$. Finally, there will be a second line other than $6'$ that intersects the lines $2, 3, 4, 5$. What is more, this final line $1'$ will also intersect with 6, thereby closing the configuration.

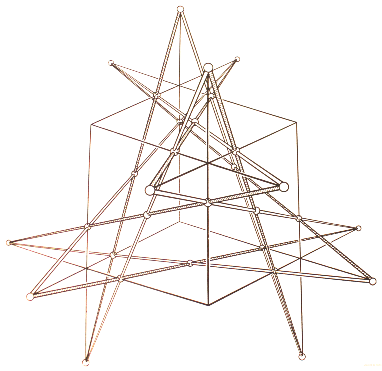


Figure 3.6 Spatial configuration of a Schläfli double-six ([Hilbert and Cohn-Vossen 1932, p. 147])

This final assertion depends on the fact that the non-intersecting lines in this double-six construction are in general position, and that all 12 lines lie on an irreducible cubic surface. Since a general cubic surface depends on 19 parameters, one can pass such a surface through 19 suitably chosen points in space. Following [Hilbert and Cohn-Vossen 1932, p. 163], we proceed by selecting four points on the line 1 and three points on each of the lines $2'$ to $6'$ that intersect line 1. These 19 points determine a third-order surface F_3. Since the line 1 has four points in common with F_3, the entire line must lie on the surface. But this means that F_3 must also have at least four points in common with each of the lines $2'$ to $6'$, so these lines lie on the surface as well. From this it follows that F_3 also contains the lines 2 through 6 , since each of these intersects four lines that lie on the surface. Finally F_3 contains $1'$ for the same reason. This shows that one can easily find a third-order surface that contains a given double-six configuration. Using this fact, Hilbert and Cohn-Vossen can easily show that the line $1'$ must meet line 6. For if it

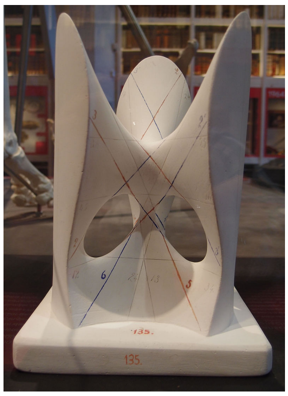


Figure 3.7 Model of Clebsch surface (Collection of Mathematical Models and Instruments, Göttingen University)

did not, then there would be a second line g, other than $5'$, that intersects lines $2,3,4,6$. This would mean that g lies on F_3, but also that the four lines $g,1',5',6'$ all intersect with the three skew lines $2,3,4$. They would therefore belong to a ruled quadric that lies on F_3, which contradicts that F_3 was assumed to be an irreducible cubic. So the line g must be identical with line $1'$ and the latter must intersect lines $2,3,4,5,6$.

It is now easy to add several more lines to the configuration which also lie on F_3. Fifteen additional lines can be obtained by intersecting planes spanned by two intersecting lines of the double-six. For example, we write (12) to denote the intersection line of the planes spanned by 1 and $2'$ and by $1'$ and 2. The line (12) lies on F_3 because it meets the four lines $1,1',2,\ 2'$, all of which lie on F_3. Since fifteen different pairs can be chosen from the numbers 1 to 6, there are fifteen lines altogether that, like (12), lie on F_3. Thus, there will be $12+15=27$ lines on a general cubic surface F_3, and these form an enlarged configuration that contains the $(30_2,12_5)$ configuration of Schläfli.

It can then be shown that any two lines, such as (12) and (34), whose symbols have no number in common will have a point of intersection. Thus, (12) meets (34) as well as (35), (36), (45), (46), and (56). Since (12) also intersects $1,1',2$ and $2'$, it meets ten lines in the enlarged configuration. This applies to all fifteen additional lines as well as the lines of the double-six itself. For example, the line 1 intersects the five lines $2'$ through $6'$ and also the five lines $(12),(13),(14),(15),(16)$. Thus, there are ten points on each of the twenty-seven lines and two lines passing through each point, so the enlarged configuration has the symbol $(135_2,27_{10})$. The configuration is regular and 72 different double-sixes can be found within it.

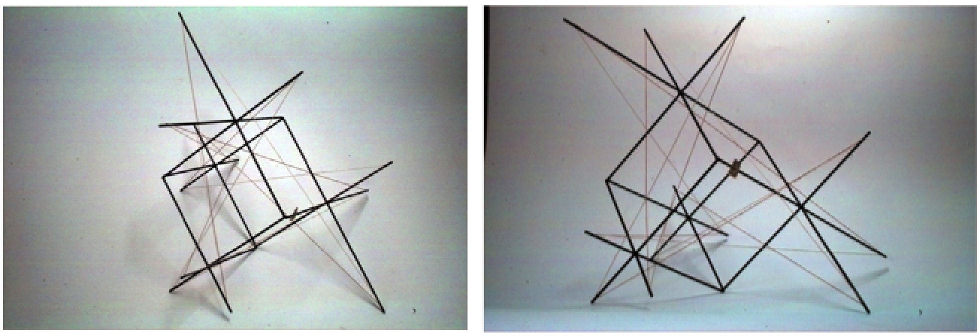


Figure 3.8 Model of Schläfli's double-six (Collection of Mathematical Models and Instruments, Göttingen University)

These last two examples were the most complex cases presented by Hilbert and Cohn-Vossen. Both fall in the first period of our study, which preceded the systematic investigations that began in the 1880s. One could discuss many other interesting examples from this period – the $(16,6)$ configuration associated with Kummer surfaces being one of the more famous – but this only goes to show that the prehistory of configurations, the period from Möbius to Reye, was full of important discoveries that have largely been ignored by historians of mathematics. The present study only scratches the surface of this very large topic. [Hilbert and Cohn-Vossen 1932] summarized some of the more famous results, but since this was intended as an elementary text they focused on real geometrical configurations. As noted above, all 27 lines on a nonsingular cubic surface *can* be real, but in fact Schläfli's whole approach to classifying cubics depend on knowing how many of the 27 lines were real.

4 Group theory and Configurations

4.1 Historical Roots of Group theory

The historical roots of group theory can be found in three different mathematical domains: number theory, theory of algebraic equations, and geometry [Wussing 1984, p. 255]. The concept itself was essentially implicit in the first field, but became explicit in the second, whereas in geometry the notion of continuous groups developed in direct analogy with the concept of finite groups in the theory of equations. The early geometrical investigations of Lie and Klein in the early 1870s form the third principal development, whereas Gauss's theory of binary quadratic forms launched the first. The works of Lagrange, Cauchy, Abel, and Galois on permutation groups and their applications to the theory of equations represent the mainstream. [Wussing 1984, pp. 167-229], [Rowe 1989, p. 211]

In 1801, Carl Friedrich Gauss extended Leonhard Euler's work on modular arithmetic by introducing concepts related to abelian groups. Gauss developed the theory of binary quadratic forms $ax^2 + 2bxy + cy^2$, where a, b, c are integers by studying their behavior under modular transformations. This led to a partitioning of forms into classes and an implicit group operation defined by composition on these classes [Wussing 1984, pp. 55-61] [O'Connor and Robertson 1996]. The beginnings of permutation group theory can be traced back to a paper from 1770 by Lagrange on the theory of cubic and quartic equations. Twenty years later, Ruffini drew on Lagrange's work in an attempt to show that the general quintic equation is unsolvable by radicals. Ruffini's work introduced groups of permutations. In a paper from 1802, he showed that the group of permutations associated with an irreducible equation is transitive [Wussing 1984, pp. 80-84] [O'Connor and Robertson 1996]. Cauchy played a major role in developing the theory of permutation groups. His first paper on this subject was published in 1815, in which he studied the permutations of roots of equations. Thirty years later, in 1844, he published a fundamental work on the theory of permutations in its own right that led to a subsequent study by Arthur Cayley in 1849. Cayley's interest in abstract group was so far ahead of the times. He already published two papers on this topic in 1854, and he returned to it in 1878 with four more papers containing many new results. In one of them, he proved the fundamental theorem that every finite group can be represented as a group of permutations [O'Connor and Robertson 1996].

Evariste Galois was the first to recognize that the theory of algebraic equations is reflected in the structure of the group of permutations of its roots. He was also the first to use the word *groupe* in its modern sense, but his early death in 1832 and the novelty of his ideas led to a long and slow reception period. Camille Jordan took up the theory in the 1860s, when he defined the concept of isomorphism of permutation groups and proved an early version of the Jordan-Hölder theorem. Jordan also classified groups of motions in his *Mémoire sur les groupes de mouvements* from 1868-1869, in which he investigated motions obtained by composing translations and rotations, leaving aside reflections and deformations [Wussing 1984, p. 196]. These studies were perhaps the earliest to investigate groups of geometrical transformations [Van der Waerden 1985, p. 141].

From the 1860s through the 1880s, the classical theory of invariants was an important topic in algebra. The invariant-theoretic approach to analytic projective geometry was also important as a first step in the transition to the group-theoretic classification of geometries. In his Erlangen Program from 1872, Felix Klein generalized the invariant-theoretic approach to other geometries, which led to the study of various geometries based on their transformation groups [Wussing 1984, p. 178]. In the early 1870s, the joint efforts of Lie and Klein elaborated the concept of a transformation group. In his Erlanger programm, Felix Klein has described how the different branches of geometry can be regarded as the study of those properties of suitable spaces which are preserved under appropriate groups of transformations [Birkhoff and Mac Lane 1961, p. 125]

Klein also exploited the group-theoretic viewpoint to develop a new theory of equations in the context of geometric Galois theory. In his *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, Klein showed how the symmetry groups of the Platonic solids lead to finite groups of linear fractional transformations in a single complex variable. This approach to linear groups in two or more variables was later exploited to study the symmetry groups of geometric configurations [Steinitz 1910a]. In his Vorlesungen über das Ikosaeder, Klein wrote:

> ... What we are concerned with in the sequel is not really the various figures but rather the rotations and reflections or, briefly, the elementary geometric operations that bring them into coincidence with themselves. The figures are, for us, only the means of orientation that enables us to survey the totality of

certain rotations or other transformations... . [Wussing 1984, pp. 202-203]

4.2 Groups associated with Configurations

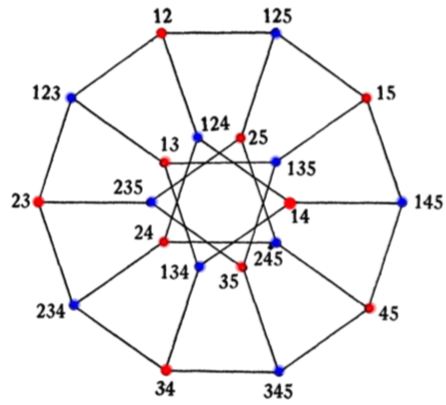
Geometric configurations have rich symmetric properties, which make them interesting objects from the perspective of group theory. A symmetry of a configuration leads to an isometry that maps the configuration into itself thereby preserving incidence relations [Gruenbaum 2009, pp. 32-33]. All symmetries of a configuration form a group, namely a group of symmetries of the configuration. The term symmetry is sometimes also applied to abstract configurations, in which case it refers to a permutation of the elements of the configuration that preserves its incidence relations. Here one distinguishes symmetries that map points to points and lines to lines from duality mappings that exchange the type of element. For purposes of clarity, we will refer to these mappings on abstract configurations as automorphisms, reserving the term symmetry for isometries that preserve the incidence relations of a geometric configuration (a configuration embedded in a geometrical space).

From the well-known classification of isometries of the Euclidean plane it follows that the symmetry group of a geometric configuration is either a cyclic group C_n or a dihedral group D_n, where n is a positive integer. The group C_n consists of rotations about a center. The group D_n consists of the same rotations as its subgroup C_n, together with n reflections. [Gruenbaum 2009, p. 34] For example, the dihedral group for a regular polygon with n points and sides, which is a planar configuration (n_2), has $2n$ different symmetries: n rotational symmetries and n reflection symmetries.

Three non-abelian rotation groups are associated with the Platonic solids: a G_{12} with the tetraehdron; a G_{24} with the cube and octahedron; and a G_{60} with the icosahedron and dodecahedron. As with the dihedral groups, the full symmetry groups arise by adjoining reflections and lead to groups that are twice as large. For example, for the cube, the number of these isometries is 48, 24 rotations and 24 reflections. The rotations are: 9 rotations of the cube about the 3 axes connecting midpoints of opposite faces, 6 rotations about the 6 axes connecting midpoints of opposite sides, and 8 rotations about the 4 diagonals of the cube. Together with the identity, this gives: 9+6+8+1 = 24 rotations of the cube. [Birkhoff and Mac Lane 1961, p. 124]

Coxeter considered abstract configurations from a combinatorial point of view in his article [Coxeter 1950] and their associated automorphism groups. This group is the totality of substitutions that leave the configuration invariant by permuting the objects of every same set (of points, lines or planes) such that no incidence is lost and no new incidence added.

In a configuration (p_λ, l_π) if $p = l$, and consequently $\lambda = \pi$, the configuration is self-dual and its symbol (p_λ, p_λ) is abbreviated to (p_λ), as was mentioned in 2.1.1 . In this case the group of the configuration may include not only symmetries which permute the points among themselves but also reciprocities which interchange points and lines (or planes) in accordance with the principle of duality [Coxeter 1950, p. 413].



The 3-regular graph

Figure 4.1 [Coxeter 1950, p. 435]

The complete graph K_n is an $n_{n-1}, n(n-1)/2_2$ configuration, which cannot be realized in the plane for $n \geq 5$ by Kuratowski's Theorem. Its automorphism group is S_n since every pair are vertices is joined by a unique line. The group of a graph are those permutations of nodes which leave them joined as before. As was mentioned in 1.4.1, Coxeter noted that the group of a configuration is identical with the group of its Levi graph. To compute the order of the group, he calculated the order of the associated group of its Levi graph. For an (n_3) configuration with an s-regular Levi graph ($s > 1$), the order of its group is $2^s 3n$ [Coxeter 1950, p. 418]. (A graph is said to be s-regular if its s-arcs[1] are all alike while its $s+1$-arcs are not all alike, that is, if the group is transitive on the s-arcs but not on the $s+1$-arcs.) As we saw in section 1.4.1, the Levi graph for the Desargues configuration (10_3) can be represented as in Figure 4.1, which is obviously a $3-$regular graph. So, using the formula $2^s 3n$, with $n = 10$ and $s = 3$, the order of its group is $2^3.3.10 = 240$. Therefore, the associated group of Desargues configuration is also of order 240 [Coxeter 1950, p. 436].

4.2.1 Early Researches on Groups associated with Configurations

After the appearance of Jordan's Traité in 1870, research on groups underwent a fundamental change of character. Before 1870, only two kinds of groups were considered, namely groups of substitutions (or permutations) and groups of geometrical transformations. After 1870, investigations began on the structure of groups independent of their representation

[1] An ordered sequence of s distinct branches, consecutively adjacent form a continuous path which is called an s-arc.

by permutations or transformations [Van der Waerden 1985, p. 137]. Within the context of projective geometry, Klein and Lie studied various groups of linear transformations that left a figure or family of figures invariant. For geometric configurations, one could investigate the full invariance group of collineations, whereas for abstract configurations or their graphs the associated automorphism groups are substitution groups.

A few years after introducing geometrical configurations by Reye, the original concept was changed by the Italian mathematician Vittorio Martinetti in 1887. He first considered configurations from a combinatorial point of view. Then he developed a recursive method for the construction of all configuration n_3, if all $(n-1)_3$ configurations are known and applied it to construct all 31 abstract configurations 11_3. He also computed the groups of all these 31 configurations. [Steinitz 1910a, p. 486], [Gropp 1990a, pp. 264-265], [Gropp 2004, p. 83]

The Göttingen geometer Arthur Schönflies made several important contributions to the study of groups associated with abstract configurations. His earliest work concerned geometry and kinematics, but he later turned to set theory and crystallography. Schönflies is perhaps best known for his classification of crystallographic space groups, a problem Klein suggested to him in the late 1880s. By 1891 Schönflies had found the complete list of 230 such groups, published one year later in *Krystallsysteme und Krystallstruktur*. This classic study introduced the Schönflies notation, which is still used today to describe crystallographic point groups. [O'Connor and Robertson 2010a][2]

Figure 4.2 Arthur Moritz Schoenflies (Wikipedia)

In the same year Martinetti's work appeared, Schönflies published his paper Über die regelmässigen Configurationen n_3 [Schoenflies 1887] in which he studied the automorphism groups of these types of configurations. He restricted this investigation, however, to groups that act transitively on the given configuration,[3] which means the group contains enough transformations so that every point of the configuration can be transformed into every other point. Generally the automorphism group of a configuration n_3 will not be transitive.

[2]The classification of the crystallographic space groups was done independently by E. S. Fedorov. Schönflies corresponded with Fedorov and they corrected some minor errors in both classifications before publishing their classification. [O'Connor and Robertson 2010a]

[3]The action of a group G on a set X (here by a set we mean a configuration) is called transitive if for each pair of points (resp. lines or planes) x, y in X there exists a g in G such that g(x) = y.

Schönflies defined regular configurations as follows: “If the group of a configuration is transitive, then that configuration behaves the same with regards to every point, and should be therefore called regular.” The problem of finding all regular n_3 configurations was first attacked in this article, although most of the configurations discussed in earlier previous works (such as S. Kantor’s articles) were regular.[4] Schönflies computed the automorphism groups of the 8_3 and 9_3 configurations, which are of order 48 and 108, respectively.

4.3 Examples of Groups associated with Geometric Configurations

Since most of the familiar or more interesting examples of configurations came from geometry, the more common practice was to study their properties by way of their collineation groups. Ernst Steinitz studied the finite collineation groups of configurations in the last three sections of his survey article. This discussed examples, including Reye’s group, Klein’s group, the Hessian group, Valentiner’s group, etc., citing the rich literature from the preceding period [Steinitz 1910a, pp. 505-516]. Theodor Reye, as a traditional synthetic geometer, made no reference to group theory when he introduced the configuration that bears his name. In the 1890s, however, his student at the University of Strassburg, Julius Feder, investigated the Reye configuration $(12_6, 16_3)$ and its associated collineation group of order 2304 [Feder 1895], [Steinitz 1910a, p. 499].

4.3.1 Klein’s group G_{168}

In 1875, Klein began to investigate the connections between the icosahedron and the modular equation of degree 5. This led him to determine a classification of all binary linear groups by means of isometries of regular polygons and polyhedra (i.e., the cyclic, dihedral, tetrahedral, octahedral, and icosahedral groups) [Gray 1986, pp. 83-87], [Brechenmacher 2011, p. 343]. In 1878 Klein began to publish a series of papers generalizing his earlier work to equations and transformations of higher degree than 5. These works form an important stage in the development of Galois theory, including his important paper “Über die Transformation siebenter Ordnung der elliptischen functionen” [Gray 1986].

In this paper, Klein described a group of 168 linear substitutions $\omega' = \frac{\alpha\omega+\beta}{\gamma\omega+\delta}$ which

[4] Hilbert and Cohn-Vossen mentioned that the configurations 7_3, 8_3, the Hessian, Pascal, and Reye configurations, and Schläfli’s double-six are regular [Hilbert and Cohn-Vossen 1932, p. 108].

permute the roots of the modular equation. As Galois had known, the group of the modular equation of order 7 consists of 168 linear substitutions $\omega' = \frac{\alpha\omega+\beta}{\gamma\omega+\delta}$ with coefficients in the integers reduced mod 7. Klein noted that G_{168} is a simple group and he determined all its subgroups [Gray 1986, p. 227]. He then exploited this group-theoretic information to study the properties of a particular fourth-degree curve that came to be known as the Klein quartic.

4.3.2 Hessian group G_{216}

Among the collineation groups arising directly from configurations, mathematicians discussed the Hessian group G_{216} quite frequently. As discussed earlier in 3.1, the Hessian configuration $(9_4, 12_3)$ can be realized geometrically in the complex projective plane. As the following sections will show, interest in this group and its connections with the geometry of cubic curves continued well into the twentieth century.

4.3.2.1 Jordan's Analysis of the Hessian group

Camille Jordan (1838–1922) formulated and expounded the permutation-theoretic group concept in his *Traité des substitutions et des équations algébriques*[Wussing 1984, pp. 141-144][Jordan 1870]. Among other objectives, he aimed to show how the theory of substitutions could be used to deduce various geometric connections between special configurations of points and lines [Van der Waerden 1985, pp. 126-131][Brechenmacher 2011, p. 342]. The third chapter 3 of Book 3 of the Traité is devoted to geometrical applications of Galois theory, beginning with the "Equation de M. Hesse". In this section, Jordan observed that the configuration of inflection points on a cubic coincides with the points and lines in the affine plane over the field of 3 elements [Jordan 1870, p. 302]. He denoted the nine inflection points of a plane cubic curve by the symbol (xy), obtaining the following scheme:

$$\begin{array}{ccc} (0\,0) & (\,0\,1) & (0\,2) \\ (1\,0) & (\,1\,1) & (1\,2) \\ (2\,0) & (\,2\,1) & (2\,2) \end{array}$$

The twelve lines then correspond to formal products $(xy)(x'y')(x''y'')$, where each line contains three inflection points. These satisfy the relations $x+x'+x'' = y+y'+y'' \equiv 0$ (mod 3).

In 1878, Jordan investigated the finite linear groups in both the binary and the ternary domains. He deduced six types of groups; the list appears in matrix form in [Gray 1986, pp. 377, 378]. It is noteworthy that he did not find the simple group G_{168} first found and studied by Felix Klein in 1879. A special case was subgroup of order 216 generated by mI, A, B, sDE, and tED, where:

$$A= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & \Theta^2 \end{pmatrix} \qquad B= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$D= \begin{pmatrix} j & 0 & 0 \\ 0 & j\Theta^2 & 0 \\ 0 & 0 & j \end{pmatrix} \qquad E= a \begin{pmatrix} 1 & 1 & \Theta \\ 1 & \Theta & 1 \\ 0 & \Theta^2 & \Theta^2 \end{pmatrix}$$

$t^2 = s^2$ or ms^2, $\Theta^3 = 1$, $j^3 = \Theta$, $a^3 = \frac{1}{3(1-\Theta^2)}$.

Jordan identified this collineation group G_{216} as the Hessian group in honor of Otto Hesse. [Jordan 1878, p. 209] Not long afterward it appeared in the works of several other mathematicians. For example, the Hessian group was taken up in Alexander Witting's Göttingen. dissertation on hyperelliptic functions [Witting 1887]. It also plays a role in an important paper by Heinrich Maschke on quaternary forms [Maschke 1889]. Machke showed that the Hessian group is a subgroup of the quaternary collineation group of order 25920. In [Newson 1901], H.B. Newson studied the geometric properties of this group and its sub-groups, while referring to Maschke's paper as the standard reference on the subject. William Burnside, an English mathematician, developed the Hessian configuration very elegantly by pure geometry while taking up the relation of this configuration to the Valentiner group. The existence of this group of collineations G_{360} was established by the Danish mathematician, Herman Valentiner, in (*Die endelige Transformations-Gruppen Theorie, 1889*) [Burnside 1905].

The Hessian group was also treated in Heinrich Weber's *Lehrbuch der Algebra*, Vol.II. , pp. 400-410 [Weber 1896]. This book was distinguished by its excellent presentation of the material and was reprinted a number of times [Wussing 1984, p. 251].[5]

[5] Weber exerted considerable influence on group-theoretic terminology and developed a unified system of concepts for various areas of algebra. For example, the German term "Normalteiler" (normal subgroup) goes back to him [Wussing 1984, p. 251].

4.3.2.2 The Hessian Group in the Twentieth Century

Another article which is concerned this subject was written by H. C. Shaub and Hazel E. Schoonmaker in 1931 [Shaub and Schoonmaker 1931]. The authors mention in the footnote that the treatment of the paper was suggested by Professor Virgil Snyder in a recent course in cubic curves at Cornell University.[6] Virgil Snyder was one of the earliest and most influential algebraic geometers in America. He received a doctorate from the University of Göttingen under Felix Klein in 1895. The authors also referred to an article by R.M. Winger entitled "The ternary Hesse group and its invariants". [Winger 1925]

The discussion of the Hessian group that follows will draw on three of above articles, namely: [Shaub and Schoonmaker 1931], [Winger 1925], [Newson 1901]. These are especially useful because they provide more detailed accounts of the geometry than earlier studies, thereby revealing the action of the Hessian group and its subgroups on the Hessian configuration.[7] More precisely, Newson and Schoonmaker-Shaub emphasized the geometric properties of the Hessian group, whereas the main purpose of Winger's paper was to give "a comprehensive and detailed analysis, - such e. g. as Heinrich Weber, in [Weber 1896, pp. 400–410], gives of Klein G_{168}, assuming a minimum of technical group theory" [Winger 1925, pp. 60-61].

As noted earlier, the Hessian group of 216 collineations in the plane is intimately related to the projective geometry of a pencil of cubics that pass through the same nine points of inflection. Using homogeneous coordinates, the equation of any non-singular plane cubic can be written in the standard form:

$$x^3+y^3+z^3+\lambda xyz=0$$

Taking λ as a variable parameter, this equation represents ∞^1 cubics whose nine points of inflection are [Shaub and Schoonmaker 1931, p. 389]:

$$\begin{array}{ccc} (0, 1, -1) & (1, 0, -1) & (-1, 1, 0) \\ (0,\omega^2,-1) & (\omega,0,-1) & (-1,\omega,0) \\ (0,\omega,-1) & (\omega^2,0,-1) & (-1,\omega^2,0) \end{array}$$

Furthermore, there are twelve inflectional lines, each of which contains three of these

[6] Hazel Schoonmaker remained at Cornell as resident doctor for a year after she received her doctorate, taking two mathematics courses and preparing typed notes for a course she had taken from Virgil Snyder on plane cubic curves. Those notes are now in the archives at Cornell.

[7] Brieskorn and Knörrer also studied the Hessian group in their book, [Brieskorn and Knoerrer 1986, pp. 296-300]

inflection points, whereas four inflectional lines pass through each point. The equation of these twelve lines are:

(1) $xyz = 0$

(2) $(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z) = 0$

(3)$(x+y+\omega z)(x+\omega y+z)(x+\omega^2 y+\omega^2 z) = 0$

(4) $(x+y+\omega^2 z)(x+\omega y+\omega z)(x+\omega^2 y+z) = 0$

Each of these degenerate cubics consists of three lines which form a triangle, the first being the triangle of reference. These are the inflectional triangles of the pencil of cubics, each of which intersects the pencil of cubics in the same nine points. For brevity we number the nine points in the following square array:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}$$

This accords with the following diagram showing the four inflectional triangles, which appears in [Shaub and Schoonmaker 1931, p. 389].

Each inflectional triangle contains all nine points of inflection and the 12 edges of these fours triangles constitute the 12 lines of the Hessian configuration. As can be seen from the diagram, three inflection points lie on each line (edge) and four lines pass through each point. Any two of these triangles are in six-fold perspective for the vertices of the other two triangles; and any three are perspective from some vertex of the fourth. The existence of these four triangles was proved by Hesse, who showed that the twelve lines, counting the inflection points three by three, formed four triangles, each containing all nine inflection points. [Lé 2015, p. 321] As pointed out in the previous chapter, a harmonic polar line corresponds to each inflection point, yielding nine harmonic lines. To find the equations of these nine harmonic lines, we take the equation of the pencil $f = x^3 + y^3 + z^3 + \lambda xyz = 0$ and calculate its polar conic:

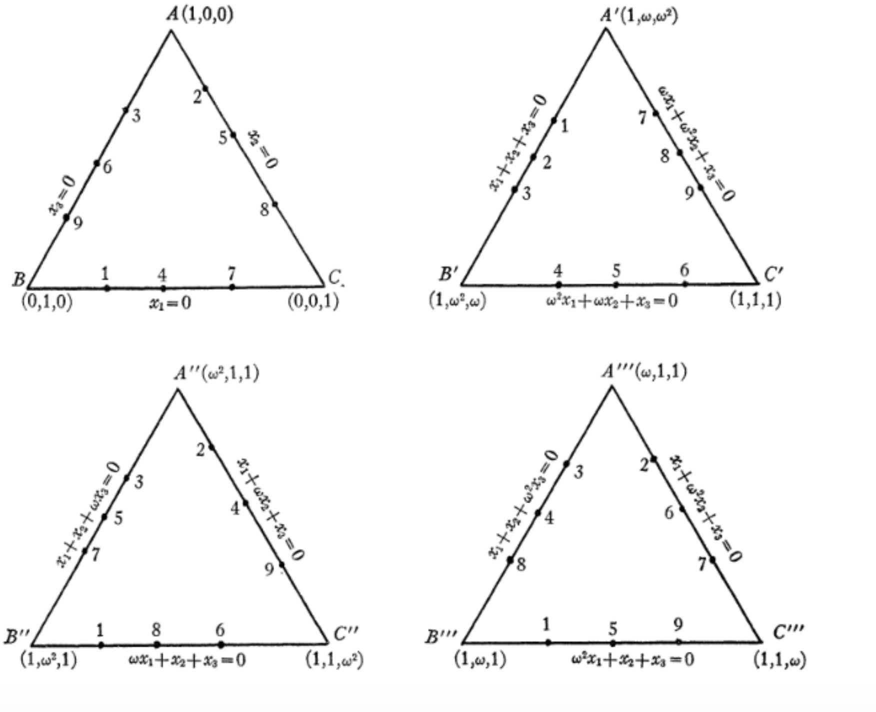


Figure 4.3

$$f' = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' = 0$$

.

For an inflection point (x', y', z'), this polar conic will degenerate into two lines, and for the inflection point $1 = (0,1,-1)$ it reduces to $(3y+3z-\lambda x)(y-z) = 0$, where $y-z=0$ is the harmonic line of $1 = (0,1,-1)$ [Shaub and Schoonmaker 1931, p. 391]. The following is a list of the nine harmonic lines: [Newson 1901, p. 15]

$$\begin{array}{lll} \text{y-z=0}, & \omega^2 y - z = 0, & \omega y - z = 0 \\ \text{z-x=0}, & \omega^2 z - x = 0, & \omega z - x = 0 \\ \text{x-y=0}, & \omega^2 x - y = 0, & \omega x - y = 0 \end{array}$$

In figure 4.3, any pair out of the four such triangles are fully perspective with centers of perspectivity located at the vertices of the other two. Taking the first two triangles shown in fig. 4.3 with vertices: $(1,0,0),(0,1,0),(0,0,1)$ and $(1,1,1)$, $(1,\omega,\omega^2)$, $(1,\omega^2,\omega)$, their corresponding vertices lie in pairs on nine lines, such that three lines pass through each vertex. These nine lines meet in six other points, which constitute the vertices of the other two triangles in fig. 4.3, thereby forming a $(12_3, 9_4)$ configuration. The nine lines, in fact, are precisely the nine harmonic lines given above, and the configuration

$(12_3, 9_4)$ is the dual of the $(9_4, 12_3)$ Hessian configuration [Winger 1925, p. 68]. The four triangles as a whole are invariant under the group G_{216}. Following Winger, we shall call the 9 inflection points and 9 harmonic lines A and α respectively, and the 12 vertices and 12 sides of the triangles B and β. The 9 points A and 9 lines α are projectively related to the four triangles, so they are invariant under any transformation that leaves the triangles invariant, hence under the entire group G_{216}.

The subgroups of the Hessian group have been investigated by several authors. Winger, following Witting, presented G_{216} with five generators S, T, R, V, W satisfying several relations, among them that $S^3 = T^3 = R^2 = V^4 = W^3 = 1$. The generators S, T together generate an abelian G_9, which combined with the reflexion R yields the important subgroup G_{18}. This is the only normal subgroup of the Hessian group that carries each curve C_λ of the Hessian pencil into itself [Brieskorn and Knoerrer 1986, p. 298]. This fact was long known, as was pointed out by Newson, who referred to the 1876 edition of Clebsch-Lindemann, *Vorlesungen über Geometrie, I*, p. 512: "It is a well known fact in the theory of plane cubics that every non-singular cubic C can be projectively transformed into itself in eighteen different ways," and that these eighteen collineations form a group G_{18}. Since G_{18} is a normal subgroup of G_{216}, we can form the quotient group G_{216}/G_{18}, which is isomorphic with the tetrahedral group T_{12} [Winger 1925, p. 64]. The subgroup structure of T_{12} can then be used to examine the subgroups of G_{216}.

For example, T_{12} contains nine subgroups: the identity, three reflexions, four cyclic G_3's and a G_4. These correspond, respectively, to nine subgroups of G_{216}: the normal subgroup G_{18}, three conjugate G_{36}'s, four conjugate G_{54}'s, and a G_{72}. Winger refers to these as the principal subgroups of the Hessian group. He then proceeds to consider the action of its cyclic subgroups, distinguishing between the two types of collineations that can arise. The first has a single fixed triangle, the second will be a homology, which has a fixed center and axis. Taking again the abelian G_9, it contains four cyclic G_3's generated by S, T, ST, ST^2. Each of these can be enlarged in three ways by combining with reflexions, so that altogether twelve dihedral subgroups of order six will lie in G_{18}. A cyclic G_6 will contain two elements of period six, two homologies of period three, and one reflexion, plus the identity. Altogether, G_{216} contains 36 cyclic G_6's that have 72 elements of order 6, 24 homologies of order 3 (each appearing three times), and 9 reflexions (each appearing four times) [Winger 1925, p. 67].

The collineations in a G_6 are thus generated by a homology of order three and a reflexion, where these mappings are related to the Hessian configuration and its dual. Thus, the center and axis of the reflexion are given by one of the nine A points with its corresponding harmonic line α; the center and axis of the homology are determined by one of the twelve vertices B and its opposite side β. Furthermore, the inflection point A lies on β, the axis of the homology, and the vertex B lies on α, the axis of the reflexion, as shown in Fig. 4.4 [Winger 1925, p. 71].

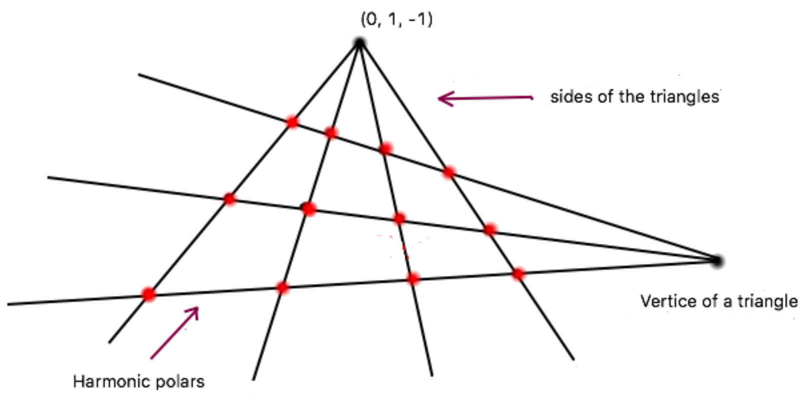


Figure 4.4

The lines α and β intersect in a D point, which will be a third fixed point that determines the invariant triangle for this G_6. The action of this cyclic subgroup can now be easily described. Take any point P in the plane and draw lines through it and the two centers A and B. Applying the homology twice will map P to two other points of the line through B, and applying the reflection maps P to a point P' on the line through A. The images of P' under the homology are then determined by drawing the line joining it with B. This produces a closed set of six points, which reveals the action of G_6 on an arbitrary point P.

The schematic picture shown in Fig. 4.4 applies to all 36 cyclic G_6's contained in the Hessian group. Geometrically, these collineations depend on the nine inflection points A with their corresponding harmonic lines α, the twelve vertices B of the four inflexional triangles with their opposite sides β, and the 36 D points, which are the points of intersection of the lines α with the lines β. These and many more similar findings can be found in Winger's paper, which was first presented at a meeting of the San Francisco section of the American Mathematical Society in June 1925. The fact that this discussion of the Hessian group appeared long after it was first studied by Camille Jordan may seem surprising, but it also suggests that such an analysis was by this time no longer at the forefront of research interests.

5 Conclusion

The theory of configurations went through three stages of varied duration, which followed a pattern Hilbert described as the naive, formal, and critical phases in the development of a theory. Hilbert introduced these terms in describing the history of invariant theory up to 1893. For the history of configurations, these phases reflect the different contexts in which they appeared, first in geometry, then in studies based on combinatorics and finite groups, and finally in connection with graph theory. In this study, they appear in reverse order in Chapters 1 to 3, whereas Chapter 4 considers the larger context suggested by the history of group theory, which only emerged as an abstract theory based on axioms around 1900.

Theodor Reye was the first to identify various geometrical configurations as illustrations of a general concept. He already pointed to such a notion of configurations in connection with Desargues' Theorem in the second edition of *Geometrie der Lage* [Reye 1876]. He then called attention to the problem of configurations in his short paper [Reye 1882], published in the very first volume of *Acta Mathematica*. Soon thereafter, the formal phase of research began with a large number of publications devoted to a systematic study of various special types. During the naive period, various important configurations were uncovered in algebraic geometry. Three of the most important examples – the Hessian configuration, Schläfli's double-six, and the Kummer configuration – are discussed in Chapter 3. Detailed analysis of the Hessian configuration from the standpoint of group theory only took place in the early twentieth century; this is discussed in detail in Chapter 4.

Soon after Reye introduced the concept of configurations, a number of investigators began to approach this topic systematically by using combinatorial ideas. They mostly studied the different types of configurations n_3 for $7 \leq n \leq 12$. Many of the early investigations up until 1887, including those by S. Kantor, E. Steinitz and others, focused on those configurations that can be embedded in the real projective plane, thus the primary context of interest was geometry, just as for Reye. A new orientation toward the study of configurations arose, however, in the work of Martinetti, Schönflies, and E. H. Moore. In 1887, Martinetti developed a recursive method for constructing all configurations n_3, and he applied this to construct all 31 configurations of type 11_3. Unlike Kantor, who drew diagrams for all ten types of 10_3 configurations – one of which was incorrect as only nine of these can be realized in the plane – Martinetti only dealt with the combinatorial possibilities. In 1895, R. Daublebsky von Sterneck drew plane diagrams for all 31 cases [Gropp 1997b], though a modern proof that all 31 are actually realizable in the plane only appeared much later.

In the seventies and eighties of the nineteenth century the group concept gradually took on significance in geometry, number theory, and in various areas of analysis [Wussing 1984]. In his Erlangen program form 1872, Felix Klein signaled the transition from the narrower invariant-theoretic concept associated with analytic projective geometry to an approach based on other transformation groups, which had not yet been investigated from a systematic point of view. Klein also related this program to finite groups, thereby developing a geometric approach to Galois theory that led to much new research starting in the 1880s. Since configurations have rich symmetry properties, several mathematicians studied them from the point of view of group theory. Geometers considered various finite collineation groups that left a figure invariant. A famous case relates to finite groups of linear fractional transformations of a single complex variable, which Klein related by stereographic projection to rotation groups of Platonic solids as well as regular polyons. This approach to geometric configurations based on finite linear groups was discussed in Ernst Steinitz's survey article for the *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* [Steinitz 1910a].

The study of configurations as abstract objects also opened the possibility of considering their automorphism groups, thus permutations of the vertices that leave the configuration as a whole invariant. This approach appeared in works by A. Schönflies, better known for his work on crystallographic groups. In [Schoenflies 1887] he studied the automorphism groups of abstract configurations, singly out those whose groups were transitive. This was probably the first time that the notion of transitivity entered the theory of transformation groups in geometry. Thus, by the end of the century the study of configurations reflected how this part of geometry was reshaped by combinatorial methods and group theory, which gradually led to investigations of configurations as abstract objects. This shift in methodology plays no part in Ernst Steinitz's survey, published in 1910, but one can easily recognize how groups came to play a major part in the later studies he described.

By the time Steinitz wrote his report, this topic in geometry no longer attracted very many researchers, no doubt partly because classifying the number of cases for configurations of type n_3 or other types was simply too complicated. During the 1920s, David Hilbert sought to revive interest in configurations through his lecture course on "Anschauliche Geometrie", which he taught four different times in Göttingen (WS 1920/21, SS 1923, SS 1925, SS 1928). The original Ausarbeitung from the first course served as the basis for the book prepared by Stephan Cohn-Vossen and published under the same title. In chapter 3, devoted to projective configurations, they took the traditional standpoint that also dominates in Steinitz's survey. Using combinatorial arguments, they discuss various simpler n_3 configurations and then show that some of these cannot be embedded in a real plane or 3-space. They also make no references to group theory, except for some brief remarks about the automorphism groups of configurations.

The transition to graph theory and combinatorics in the late 1920s was promoted by F.W. Levi's *Geometrische Konfigurationen* [Levi 1929], the first textbook devoted to configurations. Levi wrote this book when he was teaching at the University of Leipzig, but after being forced out there in 1935 he took a position at the University of Calcutta in 1936. He later taught a lecture course on finite geometrical systems in India, in which he connected configurations with graph theory by identifying the points and lines with the vertices of a regular bipartite graph. He published these new ideas in [Levi 1942], though so-called Levi graphs were only first popularized by H.S.M. Coxeter in his article [Coxeter 1950]. By this time, interest in abstract configurations overshadowed the earlier one, which had focused on geometric configurations that can be embedded in a projective space. This naturally transformed research on configurations into a topic within graph theory and geometric groups, i.e., the finite automorphism groups associated with these special graphs. During this time, Coxeter continued to keep the classical tradition in view, for example with his article "Desargues configurations and their collineation groups" [Coxeter 1975]. A revival of these topics began in the 1880s as part of wider developments in discrete mathematics, but these lie beyond the chronological bounds of the present study.

6 Appendix

6.1 Biographies

6.1.1 Friedrich Wilhelm Levi

Friedrich Wilhelm Daniel Levi (1888-1966) was born in Mulhouse. He received his Ph.D. in 1911 under Heinrich Martin Weber at the University of Strasbourg with his dissertation entitled "Integral domains and fields of third order" (Integritätsbereiche und Körper dritten Grades). Levi habilitated at the University of Leipzig in 1919 with the Habilitationsschrift "Countable abelian groups" (Abelsche Gruppen mit abzählbaren Elementen). He taught mathematics in Leipzig until 1935, though he never held a permanent position there. [1] The main areas of his research were combinatorial topology and group theory. Levi wrote his book *Geometrische Konfiguration*, the first textbook on the subject of configurations, in 1929.

In 1935 Levi was removed from the Leipzig faculty by the Nazi government because of his Jewish ancestry. That same year, Richard Courant received an offer from the University of Calcutta; Courant declined and recommended Max Dehn instead. Dehn, who had been dismissed from his professorship in Frankfurt in 1935, expressed his interest, but wrote to Courant in June 1935: "It is very doubtful whether the board there will be interested to hire an old German geometer, given that Indian interests lean towards the opposite direction" [Siegmund–Schultze 1998, p. 123].[2] In fact, Dehn was not appointed in Calcutta and he remained without a position; in 1940 he escaped from Norway via

[1] His position was: nichtplanmäßiger außerordentlicher Professor für Mathematik an der Mathematisch-Naturwissenschaftlichen Abteilung der Philosophischen Fakultät der Universität Leipzig.
[2] Dehn was apparently alluding to the strong Indian tradition in number theory.

Sweden and the Trans-Siberian Railway to the United States, where he taught up until his death in 1952. In the meantime, the position in Calcutta went to a younger geometer: in January 1936, Friedrich Levi was appointed Hardinge Professor of Higher Mathematics, a chair he held until 1948. During this time, his mother and a sister remained in Germany and were murdered in the Holocaust.

At least in the beginning, Levi experienced considerable difficulties adapting to India, as becomes clear from a letter he wrote to Richard von Mises in 1937: “The Bengali are very intelligent and lethargic people, they usually have an admirable memory. Real penetration of the subject is replaced by memorizing” [Siegmund–Schultze 1998, p. 123]. After the war, Levi helped to build the Tata Institute for fundamental research in Bombay. In spite of being renowned in India, he hoped to return to Europe, in part because he found living in India very hard. He suffered from health problems caused by the climatic conditions. He also was unhappy about his low salary as well as the unstable political situation. Furthermore, Levi disliked being cut off from European mathematicians and missed the mathematical culture in Germany. In India he had almost no access to German mathematical literature.

Levi retired from Calcutta University in 1948 at the age of 60. In 1950, he traveled to Europe and lectured in England, the Netherlands, and Germany, where he also visited Oberwolfach. Hermann Ludwig Schmid wanted to help Levi gain a permanent position in Germany, and in 1952 he was appointed full professor at the recently founded Free University in West Berlin. Friedrich Wilhelm Levi was one of three Jewish emigrant mathematicians who returned to permanent positions at a German university after the war. The other two were Hans Hamburger, who returned to Cologne in 1953, and Reinhold Baer, who accepted a professorship in Frankfurt in 1956. Levi taught at the Free University until 1956 when he accepted a professorship in Freiburg, where he spent the remainder of his life [Remmert 2015, pp. 33–37].

6.1.2 Ernst Steinitz

Ernst Steinitz came from a large Jewish family in Upper Silesia. He was born in Laurahütte (today part of Siemianowice in Poland) on 13 June 1871. As a youngster he showed exceptional talent for music as well as remarkable mathematical abilities. He studied piano for 13 years and maintained great interest in music throughout his life. After graduating from a classical Gymnasium in 1890, Steinitz studied mathematics at the University of Breslau. He received his doctorate there in 1894. His dissertation, written under the supervision of Jacob Rosanes, was entitled "Über die Construktion der Configurationen n_3." In it he showed that every abstract n_3 configuration can be realized in the Euclidean plane with the possible exception of a single line, which can then be drawn as a conic curve.

In 1897 Steinitz was appointed Privatdozent at the Technische Hochschule Berlin-Charlottenburg after submitting his habilitation thesis. In Berlin he soon came in contact with Issai Schur and Edmund Landau, who were both studying for their doctorates there. Steinitz joined the Deutsche Mathematiker-Vereinigung in 1897, the year he first met Otto Toeplitz. The two became close friends and often made music together. On two occasions, David Hilbert wrote in support of Steinitz for professorships. The first time was in 1909 when Steinitz was being considered for an extraordinary professorship at the University of Würzburg. Hilbert considered him as a strong candidate for that position and wrote:

> Steinitz is no longer a very young experienced lecturer of great versatility and who has worked on numbers theory, set theory, polyhedron geometry and analysis situs; he has recently been on the recommended list almost everywhere but has not been appointed due to adverse circumstances. From a personal point of view he is, without doubt, extremely likable as well as very modest and agreeable. [O'Connor and Robertson 2010b]

Steinitz was not appointed at the University of Würzburg. The following year he returned to Breslau, where he gained a titular professorship at the Technical College. In 1911 he married his cousin Martha, who was also a talented pianist. Ernst and Martha Steinitz had one child, their son Erhard, born on 6 August 1912. Steinitz spent ten years in Breslau before he finally gained a regular professorial appointment at the Christian-Albrechts University in Kiel in 1920. Hilbert again wrote in support of him:

> My first suggestion would be Steinitz, who deserves it and is up and coming. You will get on well with him. From a purely scientific point of view I consider him to be the most successful researcher among the persons mentioned. [O'Connor and Robertson 2010b]

Following Steinitz, Hilbert listed Felix Hausdorff and Ludwig Bieberbach. The second chair in mathematics at Kiel also became vacant around the same time and was filled by Steinitz's friend Otto Toeplitz. At Kiel Steinitz taught a variety of courses including algebra, number theory, complex analysis, topology, geometry, vector analysis and mechanics.

He also organized a research seminar there with Otto Toeplitz and Helmut Hasse. Ernst Steinitz died from heart problems in 1928. After the Nazis came to power in 1933, his wife and son emigrated to Palestine. Sadly, however, Martha Steinitz chose to return to Breslau; she became a victim of Nazism and died in 1942 [Johnsen n.d.].

6.1.3 Seligmann Kantor

Seligmann Kantor was born on 6 December 1857 in Soborten near Teplitz (now Teplice in the Czech Republic). Kantor studied mathematics and physics at the Technische Hochschule in Vienna, but also in Rome with Luigi Cremona as well as in Strassburg and Paris. In 1881 he habilitated at the Deutsche Technische Hochschule (DTH) in Prague and became a Privatdozent there two years later, remaining until 1888. He was once considered for a professorship in Vienna, but the prevailing anti-Semitic atmosphere ended his chances for an appointment there. During that time, Seligmann Kantor was the victim of a street attack, which led the faculty to conclude he was an inappropriate candidate for a professorship. Shortly afterward, Kantor left academic life. He died on 21 March 1903 in Soborten [Gropp 2004, p. 81], [Kowalewski 1950, pp. 249-251], [Bečvářová 2016, pp. 372-383].

Although little is known about the life of Seligmann Kantor, there do exist a few letters that he wrote to Luigi Cremona after returning from Rome. One of these letters [Kantor 1882b], written on 25 December 1882, is of interest for this study. In this letter, Kantor mentioned that he attended Cremona's lecture courses for two semesters, which he called the most useful and most beautiful time in his life. He also summarized his recent work on configurations. A transcription of this letter from Seligmann Kantor to Luigi Cremona appears below.

Ober-Krclni Prag, den 25. Dec. 1882

Hochgeehrter Herr!

Zwar weiss ich nicht, ob Sie sich noch des jungen Mannes erinnern, der vor einigen Jahren das Glück hatte, in zwei Semestern zu Rom an Ihren Vorlesungen teilzunehmen und viel weniger bin ich sicher, ob Sie sich der ihm vor der Abreise gegebenen Erlaubnis erinnern, Ihnen, über geometrische Dinge zu schreiben oder Ihr Urteil über seine Arbeiten und deren Veröffentlichung anzurufen. Aber da ich meinerseits die von mir damals in Rom verlebten Tage zu den nützlichsten und den schönsten meines Lebens zählen muss, so mögen Sie es nur dieser frisch in mir lebenden und unvergänglichen Erinnerung zu Gute halten, wenn Sie es mich wagen sehen, mit diesem Schreiben von jener Erlaubnis Gebrauch zu machen. Fürs Erste seien mir einige Worte zu meinen beifolgend übersandten Abhandlungen gestattet. Die Verhältnisse gestatten mir leider bisher keineswegs, meine Kräfte dem Gegenstande mit jener Uneingeschränktheit und Continuität zu widmen, die er beansprucht. Möchten Sie aber, hochgeehrter Herr, vielleicht dennoch aus den Arbeiten ersehen, dass ich bestrebt war, der Berechtigung jenes überaus schmeichelhaften Urteiles, das Sie seinerzeit über mich zu fällen die gütige Nachsicht hatten, wenigstens um einen Schritt näher zu kommen und ich wäre glücklich, wenn Sie finden wollten, dass er mir in gewissem Grade gelungen sei! Was die beiden Abhandlungen über Configurationen betrifft, die auf einem gerade in letzter Zeit von mehreren Seiten gestreiften Untersuchungsgebiete stehen, so habe ich bereits in einer früheren Abhandlung, von der ich seinerzeit Mitteilung zu machen mir erlaubte, zum ersten Male den Satz dargethan, wie man wenigstens eine Configuration (m,n) für beliebiges m und n construiren könne. Überhaupt möchte ich Gewicht auf den Umstand legen, dass, soviel ich weiss, vor mir niemand (selbst nicht der divinatorische Sylvester ungeachtet seiner Bemerkung im Am. Jorn. of Math. 1880) von einem “Problem der Configurationen" in diesem präzisen Sinne gesprochen hat. Man hat nur da und dort gelegentlich analytischer oder geometrischer Untersuchungen die Existenz besonders eigenthümlicher ober äusserst specieller (gewöhnlich mit den Einheitswurzeln zusammenhängender) Configurationen entdeckt und eine oder die andere genauer untersucht. In den beiden vorliegenden Arbeiten versuche ich schrittweise zu einer vollständigen Aufstellung der Configurationen vorzudringen und so eine systematische Behandlung des Problemes anzubahnen und behandele $(3,3)_8$, $(3,3)_9$ und $(3,3)_{10}$. Dass ich die Anwendung auf Curven 3.Ordnung in der 1. Abhandlung nicht habe von der Hand weisen wollen, wird man im Hinblicke auf die neue Theorie der bekannten “Inflexionstripel", die sich dabei ergab, entschuldigen. In der 2. Abhandlung dürften in den letzten 5 Paragraphen auch einige das ganze Problem umfassende allgemeine Gedanken niedergelegt sein. Es sei mir gestattet, namentlich auf die gewissermassen “darwinistische" successive Entstehung aller Configurationen derselben Anzahlen aus einer einzigen und auf die Einteilungen aufmerksam zu machen. Aus der 3. Abhandlung will ich, abgesehen von dem eigentlichen Gegenstande, der überhaupt neu sein mag, einige Details hervorheben, welche Anknüpfungspunkte für abzweigende Untersuchungsarten vorstellen und durch die ich eben die Tragweite der Arbeit erweisen möchte. Namentlich: Die Verallgemeinerung eines Cayleyschen Ortes auf p.6 und die daraus gezogenen Consequenzen. Eine endliche Gruppe linearer Substitutionen im ternären Gebiete p.8 und eine zahlentheoretische Formel. Die Anwendung auf die bekannte, von mir mit (B) bezeichnete Configuration $(3,3)_{10}$ zweier perspektiver Dreiecke, wofür ich auch gleich auf eine demnächst in den "Mathematischen Annalen" erscheinende Note (Auszug aus einem Schreiben an Herrn Brill in München) verweisen möchte. Die Anwendung auf eine gewisse Classe von Curven (u. analog Flächen) mit zerfallender Hessischer Determinante [3]) und das dabei auftauchende Polarisierungsproblem p.11, welches ich übrigens auch nunmehr allgemein gelöst habe. [Was die auf p.10 gemachte Anmerkung hinsichtlich der cyclischen Configurationen betrifft, so darf ich gerade hier vielleicht auf ein Gespräch mit Herrn Prof. Battaglini hinweisen, dem gegenüber ich in einer seiner Vorlesungen von der Existenz solcher Punktgruppen sprach, welche (wie ich damals sagte) die Einheitswurzeln darstellen.] Der Satz c) auf p.13 und die Resultate auf p.15 eröffnen einen geraden Weg zum Studium der (1,1) Correspondenzen auf C_3 (p=1), das bisher immer noch sehr vague ist, wie man z. B. am Salmons higher pl[ane] c[urves] ersieht. Wenn es angenehm sein sollte, so würde ich mir demnächst einige Mitteilungen über meine diesbezüglichen Resultate erlauben. Aus dem Abschnitte V will ich das Büschel von Curven 6. Ordnung mit 62 Doppelpunkten und 6 Berührungspunkten herausgreifen. Indem ich Abschnitt I und II von B) nicht zerreissen will, sei es gestattet, auf covarianten Verwandtschaftsgebüsche p.34 und 35 und das genaue Studium einer F_4 , die eigentlich schon in Ihrer Theorie der Oberflächen enthalten ist (cf. p.32 Anmerkung), die Aufmerksamkeit zu lenken. Die auf p.38 unten aufgezeigte neue Eigenschaft der Kreisen einer F_3 war mir von grossem Interesse.

Bevor ich zu dem letzten und Hauptpunkte dieses Schreibens übergehe, habe ich ein kleines Anliegen vorzubringen. Analog der beifolgenden Arbeit über dil.(?) Transformationen habe ich nämlich eine, Theorie der linearen Systeme quadratischer Transformationen schon seit Langem durchgeführt und wünsche sie nur noch in einigen Punkten zu completiren, bevor ich sie der Öffentlichkeit übergebe. Es wäre mir dann angenehm, sie sei es in den Annali di Matematica, sei es in den Atti dell Accademia dei Lincei zu veröffentlichen, nur müsste ich, wenn ich sie in italienischer Sprache abfasse, sicher sein, dass sie in Rom vor dem Drucke von einem verständigen Leser in Rücksicht auf sprachliche Unzukömmlichkeiten durchgesehen, eventuell corrigirt werde. Könnten Sie vielleicht, hochgeehrter Herr, in dieser Hinsicht durch Ihre wertvolle Vermittlung behilflich zu meiner Beruhigung sein? Was die Arbeit selbst betrifft, so erliegt sie zuvor Ihrem Urteile. Ich komme schliesslich zu der etwas heiklen Angelegenheit, in der ich Ihren gütigen Rat anspreche. Wie ich nämlich (erst im October d.J.) aus dem "Giornale"

[3] Die tetraedral-symmetrischen des Herrn de la Gournerie

des Herrn Battaglini ersehen habe, sind die periodischen rationalen Transformationen von der Akademie zu Neapel zu einem Gegenstande der Preisfrage für 1882 gemacht worden. Nun habe ich mich, wie Ihnen vielleicht sogar bekannt sein wird, in Arbeiten dass den Sitzungsberichten der Wiener Akademie, namentlich in einer über successive quadratische Transformationen mit hiehergehörigen Untersuchungen beschäftigt, unter Anderem eine zwar einfache, aber das ganze Problem (auch für rationale Transformationen) beherrschende Bedingung angegeben. Ausserdem besitze ich eine Reihe im Jahre 1880 in Paris begonnener, seither weitergeführter Untersuchungen über den Gegenstand mit mehreren sehr schönen und interessanten Resultaten. Nichtsdestoweniger kann ich bei der Eingeschränktheit meiner Zeit und wol auch bei der Eingeschränktheit meiner Kräfte nicht hoffen, das Problem in einer Art zu erledigen, von der ich glauben dürfte, dass sie die Akademie befriedigen könne. Darum habe ich die Absicht, das, was ich darüber besitze, ehestens an einem geeigneten Orte zu publiciren, um mir etwa anderweitig einlaufenden Arbeiten gegenüber jedenfalls die Priorität zu wahren. Es frägt sich mir nun, ob ich mit diesen gedruckten Arbeiten die Akademie zu Neapel wenigstens zu einer Erwähnung derselben veranlassen könnte. Und da ich voraussetze, dass Sie hochgeehrter Herr, mit den Usancen der italienischen Akademien vertraut sind, so glaube ich, mich hiermit jedenfalls an die competenteste Stelle für einen Rathschlag in dieser Sache gewendet zu haben. Ich darf wol aber um so berechtiger um Discretion in der Angelegenheit bitten, da ich mich vielleicht doch noch entschliessen könnte, eine Zusammenstellung meiner Resultate dem Urteile jener geehrten Akademie zu unterbreiten. Ich wäre hoch erfreut, wenn Ihre Musse, hochgeehrter Herr, eine Beantwortung meiner Desiderata in wenigen Zeilen zulassen würde; sollte ich aber mit meiner letzten Bitte oder in Rücksicht auf Ihre so in Anspruch genommene Thätigkeit meine Befugnis überschritten haben, so ignoriren Sie gütigst meine Bitte, ohne mir dieselbe übel zu nehmen. Genehmigen Euer Hochwohlgeboren die Versicherung meiner besonderen Hochachtung und aufrichtigen Verehrung, mit der ich zeichne ergebenst

S. Kantor, Privatdozent
Adresse: Prag, deutsche technische Hochschule.

Illustrations

Fig. 1.8: https://en.wikipedia.org/wiki/Harold_Scott_MacDonald_Coxeter#/media/File:Coxeter.jpg

Fig. 1.15: https://research.uni-leipzig.de/catalogus-professorum-lipsiensiu leipzig/Levi_263/

Fig. 1.16: https://en.wikipedia.org/wiki/Fano_plane#/media/File:Fano_plane_nimbers.svg

Fig. 1.18: https://en.wikipedia.org/wiki/Hesse_configuration#/media/File:Hesse_configuration.svg

Fig. 2.1: https://en.wikipedia.org/wiki/David_Hilbert#/media/File:Hilbert.jpg
Fig. 2.2: https://en.wikipedia.org/wiki/Stefan_Cohn-Vossen#/media/File:StephanCohnVossen_MFO12644.jpg
Fig. 2.3: https://lamington.wordpress.com/about/

Fig. 2.11: https://en.wikipedia.org/wiki/August_Ferdinand_MÃűbius#/media/File:August_Ferdinand_MÃűbius.jpg

Fig. 2.17: https://mathworld.wolfram.com/DesarguesConfiguration.html

Fig. 2.22: https://en.wikipedia.org/wiki/MÃűbius_configuration#/media/File:Mobius_configuration.png

Fig. 2.24: https://en.wikipedia.org/wiki/Theodor_Reye#/media/File:Theodor_Reye.jpeg

Fig. 2.25: https://en.wikipedia.org/wiki/Reye_configuration#/media/File:Reye_configuration.svg

Fig. 2.30: https://en.wikipedia.org/wiki/Desmic_system#/media/File:Compound_of_two_tetrahedra.png

Fig. 3.3: https://en.wikipedia.org/wiki/Otto_Hesse#/media/File:Ludwig_Otto_Hesse.jpg

Fig. 3.7: https://en.wikipedia.org/wiki/Clebsch_surface#/media/File:Modell_der_DiagonalflÃd'che_von_Clebsch_-Schilling_VII,_1_-_44-.jpg

Fig. 3.8: http://modellsammlung.uni-goettingen.de/index.php?lang=en&r=2&sr=37&m=401#modelimages-1,http://modellsammlung.uni-goettingen.de/index.php?lang=en&r=2&sr=37&m=401#modelimages-2

Fig. 4.2: https://en.wikipedia.org/wiki/Arthur_Moritz_Schoenflies#/media/File:Arthur_Schonflies.jpg

Bibliography

Artebani, Michela and Igor V. Dolgachev (2009). 'The Hesse pencil of plane cubic curves'. In: *European Mathematical Society* 55 (3/4), pp. 235–273.

Bečvářová, Martina (2016). *Matematika na Německé univerzitě v Praze v letech 1882–1945*. Univerzita Karlova v Praze Nakladatelstvi Karolinum.

Biggs, N. L., E. K. Lloyd, and R. J. Wilson (1986). *Graph Theory 1736–1936*. Oxford University Press.

Birkhoff, Garrett and Saunders Mac Lane (1961). *A Survey of Modern Algebra*. The MacMillan Company.

Blichfeldt, Hans Frederik (1917). *Finite collineation groups, with an introduction to the theory of groups of operators and substitution groups*. The University of Chicago Press.

Brechenmacher, Frédéric (2011). 'Self-Portraits with Évariste Galois (and the shadow of Camille Jordan)'. In: *Revue d'histoire des mathématiques* 17, pp. 273–371.

Brieskorn, Egbert and Horst Knoerrer (1986). *Plane Algebraic Curves*. Birkhaeuser Basel.

Burnside, William (1905). 'On the Hessian configuration and its connection with the group of 360 plane collineations'. In: *Proceedings of the London Mathematical Society* 4 (2), pp. 54–71.

Carmichael, Robert Daniel (1937). *Introduction to the Theory of Groups of finite order*. Boston/New York: Ginn company.

Cayley, Arthur (1846). 'Sur quelques théorèmes de la ǵeométrie de position'. In: *Journal für die reine und angewandte mathematik(Crelle)* 31, pp. 213–226.

– (1864). 'On the Notions and Boundaries of Algebra'. In: *Collected Mathematical Papers* 5 (347).

Coxeter, Harold Scott MacDonald (1948). 'Configurations and maps'. In: *Reports of a Mathematical Colloquium* 8 (2), pp. 18–38.

– (1950). 'Self–dual configurations and regular graphs'. In: *Bulletin of the American Mathematical Society* 56, pp. 413–455.

– (1953). 'Review: Intuitive Geometry'. In: *The Scientific Monthly* 76.2, pp. 117–118.

– (1969). *Introduction to Geometry*. Wiley.

– (1974). *Projective Geometry*. 2nd ed. University of Toronto Press.

Coxeter, Harold Scott MacDonald (1975). 'Desargues configurations and their collineation groups'. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 78.2, pp. 227–246.

– (1977). 'The Pappus configuration and the self-inscribed octagon. I, II, III.' In: *Indagationes Mathematicae* 39, pages.

– (1983). 'My graph'. In: *Proceedings of the London Mathematical Society* 3.46, pp. 117–136.

Dickson, Leonard E. (1905). 'The group of a Tactical Configuration'. In: *Bulletin of the American Mathematical Society* 11.4, pp. 177–179.

Dolgachev, Igor V. (2002). 'Abstract configurations in algebraic geometry'. In: *Proc. Fano Conference, Torino*.

– (2012). *Classical Algebraic Geometry: A Modern View*. Cambridge University Press.

Fano, Gino (1892). 'Sui postulati fondamentali della geometria proiettiva in uno spazio lineare a un numero qualunque di dimensioni.' In: *Giornale di Matematiche* 30, pp. 106–132.

Feder, Julius (1895). 'Die Configuration $(12_6, 16_3)$ und die zugehoerige Gruppe von 2304 Collineationen und Correlationen.' In: *Mathematische Annalen* 47, pp. 385–417.

Gray, Jeremy (1986). *Linear differential equations and group theory from Riemann to Poincaré*. Vol. 2. Princeton: Princeton University Press.

Green, Judy and Jeanne LaDuke (2009). *Pioneering Women in American Mathematics: The Pre–1940 PhD's*. Vol. 34. History of Mathematics. American Mathematical Society/London Mathematical Society.

Gropp, Harald (1990a). 'On the existence and nonexistence of configurations n_k'. In: *journal of combinatorics information and system sciences* 15, pp. 34–48.

– (1990b). 'On the history of configurations'. In: *Symposium On Structures in Mathematical Theories, A. Diez, J. Echeverria, and A. Ibarra*, pp. 263–268.

– (1993). 'Configurations and graphs'. In: *Discrete Mathematics* 111, pp. 269–276.

– (1997a). 'Configurations and graphs II'. In: *Discrete Mathematics* 164, pp. 155–163.

– (1997b). 'Configurations and their realizations'. In: *Discrete Mathematics* 174, pp. 137–151.

– (2004). 'Configurations between geometry and combinatorics'. In: *Discrete Applied Mathematics* 138, pp. 79–88.

Grove, C. (1906). 'I. The syzygetic pencil of cubics with a new geometrical development of its Hesse Group, (G_{216}); II. The complete Pappus hexagon'. In: *Dissertation, Baltimore, Md.,* Gruenbaum, Branko (2009). *Configurations of Points and Lines*. American Mathematical Society.

Hesse, Otto (1844a). 'Ueber die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variabeln'. In: *Journal für die reine und angewandte mathematik(Crelle)* 28, pp. 68–96.

– (1844b). 'Ueber die Wendepunkte der Curven dritter Ordnung'. In: *Journal für die reine und angewandte mathematik(Crelle)* 28, pp. 97–102.

Hilbert, David (1899). *Grundlagen der Geometrie*. Springer.

Hilbert, David and Stephan Cohn-Vossen (1932). *Anschauliche Geometrie*. Springer.

Hudson, R. W. H. (1990). *Kummer's Quartic Surface*. Cambridge University Press.

Johnsen, Karsten. 'Famous scholars from Kiel: Ernst Steinitz'. In: *https://www.uni-kiel.de/grosse-forscher/index.php?nid=steinitzlang=epr=1*.

Jordan, Camille (1870). *Traite des substitutions et des equations algebriques*. Gauthier–Villars.

– (1877). 'Mémoire sur les équations différentielles linéaires á intégrate algébrique'. In: *Journal für die reine und angewandte mathematik(Crelle)* 84, pp. 89–215.

– (1878). 'Mémoire sur les équations différentielles linéaires à intégrale algébrique'. In: *Journal für die reine und angewandte mathematik(Crelle)* 84, pp. 89–215.

Kantor, Seligmann (1882a). 'Die Configurationen $(3,3)_{10}$'. In: *Sitzungsberichte der Mathematisch-Naturwissenschaftlichen Classe der Kaiserlichen Akademie der Wissenschaften, Wien* 84 (1), pp. 1291–1315.

– (1882b). 'Letter to Luigi Cremona'. In: *Kantor's letter deposited in the Luigi Cremona's fond Legato Itala Cremona Cozzolino of the Mazzini Institute of Genoa*.

– (1882c). 'Ueber die Configurationen (3, 3) mit den Indices 8, 9 und ihren Zusammenhang mit den Curven dritter Ordnung'. In: *Sitzungsberichte der Mathematisch–Naturwissenschaftlichen Classe der Kaiserlichen Akademie der Wissenschaften, Wien* 84 (1), pp. 915–932.

Klein, Felix (1871). 'Ueber eine geometrische Repräsentation der Resolventen algebraischer Gleichungen'. In: *Mathematische Annalen* 4.

– (1884). *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom 5ten Grade*. Teubner(Leipzig).

Kohn, G. and Gino Loria (1908). 'Spezielle ebene algebraische Kurven'. In: *Encyklopaedie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* 3, pp. 461–634.

König, Dénes (1936). *Theorie der endlichen und unendlichen Graphen*. Nachdr. d. Ausg. Leipzig.

Kowalewski, Gerhard (1950). *Bestand und Wandel – Meine Lebenserinnerungen, zugleich ein Beitrag zur neueren Geschichte der Mathematik*. Oldenbourg – Muenchen.

Kuratowski, Kazimierz (1930). 'Sur le problème de courbes gauche en Topologie'. In: *Fundamenta Mathematicae* 15, pp. 271–282.

Ladd-Franklin, Christine (1879). 'The Pascal Hexagram'. In: *American Journal of Mathematics* 2, pp. 1–12.

Lé, Francois (2015). '"Geometrical equations": Forgotten premises of Felix Klein's Erlanger Programm'. In: *Historia Mathematica* 24, pp. 315–342.

Levi, Friedrich Wilhelm (1929). *Geometrische Konfigurationen*. Verlag von S. Hirzel, Leipzig.

– (1942). *Finite geometrical systems*. University of Calcutta.

Maschke, Heinrich (1889). 'Aufstellung des vollen Formensystems einer quaternären Gruppe von 51840 linearen Substitutionen'. In: *Mathematische Annalen* 33 (3), pp. 317–344.

Menger, Karl (1930). 'Ueber plaettbare Dreiergraphen und Potenzen nichtplaettbarer Graphen'. In: *Anzeiger der Akademie der Wissenschaften in Wien* 67, pp. 85–86.

Miller, George Abram, Hans Frederik Blichfeldt, and Leonard Eugene Dickson (1916). *Theory and applications of finite groups*. John Wiley Sons.

Moebius, August Ferdinand (1828). 'Kann von zwei dreiseitigen Pyramiden eine jede in Bezug auf die andere um- und eingeschrieben zugleich heissen?' In: *Journal für die reine und angewandte mathematik(Crelle)* 3, pp. 281–286.

Moore, Eliakim Hastings (1895). 'Concerning Jordan's Linear Groups'. In: *American Mathematical Society* 2.2, pp. 33–43.

– (1896). 'Tactical Memoranda I–III'. In: *American Journal of Mathematics* 18.3, pp. 264–290.

Newson, Henry Byron (1901). 'On the group of 216 collineations in the plane'. In: *Kansas University Quarterly* 10, pp. 13–32.

O'Connor, J. J. and E. F. Robertson (1996). 'The development of group theory'. In: *https://mathshistory.st-andrews.ac.uk/HistTopics/Development$_g$roup$_t$heory/*.

– (2010a). 'Arthur Moritz Schönflies'. In: *https://mathshistory.st-andrews.ac.uk/ Biographies/Schonflies/*.

– (2010b). 'Ernst Steinitz'. In: *http://mathshistory.st-andrews.ac.uk/ Biographies/Steinitz.html*.

– (2012). 'Karl Theodor Reye'. In: *https://mathshistory.st-andrews.ac.uk/Biographies/Reye/*.

Parshall, Karen Hunger and David E. Rowe (1994). *The Emergence of the American Mathematical Research Community 1876 – 1900: J.J. Sylvetser, Felix Klein, and E.*

H. Moore. Vol. 8. History of Mathematics. American Mathematical Society/London Mathematical Society.

Pascal, Ernesto (1910). *Repertorium der Hoeheren Mathematik*. Vol. 2. Teubner, Leipzig.

Petersen, Julius (1891). 'Die Theorie der regulären graphs'. In: *Acta Mathematica* 15, pp. 193–220.

Pisanski, Tomaz and Brigitte Servatius (2004). *Configurations from a Graphical Viewpoint*. Preprint.

– (2013). *Configurations from a Graphical Viewpoint*. Birkhaeuser.

Poncelet, Jean-Victor (1822). *Traite des proprietes projectives des figures*. Paris Gauthier-Villars.

Reid, Constance (1996). *Hilbert*. Springer.

Remmert, Volker R. (2015). 'Forms of Remigration: Émigré Jewish Mathematicians and Germany in the Immediate Postwar Period'. In: *The Mathematical Intelligencer* 37.1, pp. 30–40.

Reye, Theodor (1876). *Geometrie der Lage I*. 2nd ed. Verlag von S. Hirzel, Leipzig.

– (1882). 'Das Problem der Configurationen'. In: *Acta mathematica* 1, pp. 93–96.

Roberts, Siobhan (2006). *King of Infinite Space: Donald Coxeter, the Man Who Saved Geometry*. House of Anansi Press.

Rosanes, J. (1870). 'Ueber Dreiecke in perspectivischer Lage'. In: *Mathematische Annalen* 2 (4), pp. 549–552.

Rossiter, Margaret W. (1982). *Women Scientists in America. Baltimore*. The Johns Hopkins University Press.

Rowe, David E. (1989). 'Klein, Lie, and the Geometric Background of the Erlangen Program'. In: *Rowe, David E. and McCleary, John (eds.), The History of Modern Mathematics: Ideas and their Reception* 1, pp. 209–273.

Rowe, David E. and John McCleary (1989). *The History of Modern Mathematics: Ideas and their reception*. Vol. 1. Academic Press.

Schoenflies, Arthur Moritz (1887). 'Ueber die regelmaessigen Configurationen n^3'. In: *Mathematische Annalen* 31, pp. 43–69.

Schroeter, Heinrich (1888). 'Ueber lineare Konstruktionen zur Herstellung der Konfigurationen n_3'. In: *Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, pp. 193–236.

– (1891). 'Die Hessesche Configuration $(12_4, 16_3$'. In: *Journal für die reine und angewandte mathematik(Crelle)* 108, pp. 269–312.

Shaub, H. C. and H. E. Schoonmaker (1931). 'The Hessian configuration and its relation to the group of order 216'. In: *American Mathematical Monthly* 38, pp. 388–393.

Siegmund–Schultze, Reinhard (1998). *Mathematicians Fleeing from Nazi Germany: Individual Fates and Global Impact*. Princeton University Press.

Steinitz, Ernst (1897). 'Ueber die Unmoeglichkeit, gewisse Configurationen n_3 in einem geschlossenen Zuge zu durchlaufen.' In: *Monatshefte für Mathematik und Physik* 8, pp. 293–296.

– (1910a). 'Konfigurationen der projektiven Geometrie'. In: *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* 3, pp. 481–516.

– (1910b). 'Ueber Konfigurationen'. In: *Archiv Math. Phys., 3rd Ser.* 16, pp. 289–313.

Stephanos, Cyparissos (1879). 'Sur les systémes desmiques de trois tétraedres'. In: *Bulletin des sciences mathématiques et astronomiques,Sér. 2* 3.1, pp. 424–456.

Sterneck, R. Daublebsky von (1894). 'Die Configurationen (11_3).' In: *Monatshefte Math. Phys.* 5, pages.

– (1895). 'Die Configurationen (12_3).' In: *Monatshefte Math. Phys.* 6, pages.

Straten, Sister Mary Petronia van (1949). 'The topology of the configurations of Desargues and Pappus'. In: *Reports of a Mathematical Colloquium* 2.8, pp. 3–17.

Van der Waerden, Bartel L. (1985). *A History of Algebra*. Springer.

Veblen, Oswald and J.W. Young (1910). *Projective Geometry*. Vol. 1. Projective Geometry. Ginn.

Vries, Jan de (1888). 'Ueber gewisse ebene Configurationen'. In: *Acta Mathematica* 12, pp. 63–81.

Weber, Heinrich (1896). *Lehrbuch der Algebra*. Vol. 2. Braunschweig : Friedrich Vieweg und Sohn.

Winger, R. M. (1925). 'The Ternary Hesse Group and its Invariants'. In: *University of Washington Publications in Mathematics* 1, pp. 60–80.

Witting, Alexander (1887). 'Ueber Jacobi'sche Functionen k-ter Ordnung zweier Variabler'. In: *Mathematische Annalen* 29, pp. 157–170.

Wussing, Hans (1984). *The Genesis to the History of the Abstract Group Concept, A Contribution to the History of the Origin of Abstract Group Theory*. The MIT Press.

Summary (German)

In dieser Dissertation wird die Geschichte der geometrischen Konfigurationen und deren Verbindung zu zeitgenössischen Entwicklungen in der Gruppentheorie und Kombinatorik untersucht. Konfigurationen wurden erstmals 1876 von Theodore Reye als eine allgemeine geometrische Struktur definiert, obwohl viele wichtige Konfigurationen schon lange zuvor entdeckt worden waren. Zwei klassische Fälle – die Pascal Konfiguration (9_3) und die Desargues Konfiguration (10_3) – gehen beispielsweise auf Sätze zurück, die im 17. Jahrhundert bewiesen wurden. In der vorliegenden Studie werden zunächst Arbeiten ab den 1820er Jahren untersucht, die entstanden, nachdem A.F. Möbius eine wichtige räumliche Konfiguration beschrieb, die durch zwei jeweils gegenseitig ein – und umbeschriebene Tetraeder im Raum entsteht. In dieser naiven Phase der Forschung über Konfigurationen , die ungefähr bis 1880 vorhielt, findet sich lediglich ein Interesse an geometrischen Beispielen. Die zweite oder formelle Phase ist durch eine zunehmende Untersuchung der abstrakten Konfigurationen gekennzeichnet, obwohl die vorherrschenden Fragestellungen aus der Geometrie und Gruppentheorie stammten. Die ersten systematischen Studien erschienen in den 1880er Jahren und damit kurz nachdem Reye auf das grundlegende Phänomen der Konfigurationen aufmerksam gemacht hatte. Hier wurden die Konfigurationen unter abstrakteren Gesichtspunkten untersucht, was dazu führte, dass sie deutlich später als spezielle bipartite Graphen verstanden wurden. Zwischen 1880 und 1930 verstand man Konfigurationen jedoch vorrangig als Inzidenzstrukturen von Punkten, Geraden und Ebenen in einem reellen oder komplexen projektiven zwei – oder dreidimensionalen Raum verstanden.

Zwei der ersten Mathematiker, die sich mit dem Thema beschäftigten – S. Kantor und A. Schönflies – konzentrierten sich in ihrer Forschung auf n_3 – Konfigurationen, speziell auf diejenigen, die durch gegenseitig eingeschriebene Polygone dargestellt werden können. Schönflies gehörte auch zu den ersten, die sich mit der Gruppe der Automorphismen von regulären Konfigurationen beschäftigten, insbesondere in Fällen, in denen diese Gruppe transitiv ist. Konfigurationen verfügen über viele symmetrische Eigenschaften, was sie zu interessanten Objekten für die Gruppentheorie, insbesondere die Theorie endlicher kollinearer Gruppen, machte. In Nachfolge von C. Jordan und F. Klein begannen sich

Mathematiker mit Konfigurationen zu beschäftigen, die in Zusammenhang mit diesen Gruppen und ihren Untergruppen stehen. Ein wichtiges Beispiel ist die Gruppe G_{216}, die auf der Hesse – Konfiguration operiert und in Verbindung mit den neun Wendepunkten von ebenen kubischen Gleichungen steht. Um 1900 hatte sich gezeigt, dass Kurven und Oberflächen, die in der algebraischen Geometrie untersucht wurden, viele überraschende Beispiele von endlichen geometrischen Konfigurationen enthielten. Wilhelm Levi veröffentlichte 1929 *Geometrische Konfigurationen*, das erste Lehrbuch über Konfigurationen, das mit einer Schwerpunktsetzung auf Gruppen, Kombinatorik und Graphentheorie einen moderneren Zugang zu dem Thema bietet. Zur gleichen Zeit präsentierte David Hilbert in seiner beliebten Vorlesung Anschauliche Geometrie den klassischen Zugang zu Konfigurationen. Ausserhalb Göttingens wurde dieser Kurs in weiten Kreisen durch das gleichnamige Buch bekannt, dass Hilbert 1932 gemeinsam mit Stefan Cohn-Vossen veröffentlichte. Das darin enthaltene dritte Kapitel enthält viele wichtige Beispiele und bietet einen sehr lesbaren Überblick über das Thema.

Die dritte, kritische und abschliessende Phase war von Untersuchungen von Konfigurationen unter graphentheoretischen Aspekten bestimmt. F.W. Levi bot in den 1940er Jahren eine sechsteilige Vorlesungsreihe über finite geometrische Systeme im indischen Kalkutta an. Die Veröffentlichung dieses Kurses im Jahr 1942 war sein zweiter wichtiger Beitrag zur Theorie der Konfigurationen. Levi zeigte hier, wie Konfigurationen von Punkten und Geraden (Ebenen) durch spezielle bipartite Graphen dargestellt werden können. Nach dem 2. Weltkrieg wurde diese Form der Darstellung unter dem Namen “Levi – Graph" vom bedeutenden Geometer H.S.M. Coxeter berühmt gemacht/verbreitet/bekannt gemacht. Coxeter besass ein tiefes Verständnis der klassischen Geometrie und konnte den Konfigurationen daher eine weitere Funktion zuweisen, indem er ihre Verbindungen zu algebraischer Geometrie, Gruppentheorie und Kombinatorik aufzeigte.

Summary (English)

In this dissertation, I investigate the history of geometrical configurations and their connections with developments in group theory and combinatorics. Configurations were first defined as a general type of geometric structure by Theodore Reye in 1876, although many important examples of configurations had been discovered well before he introduced this concept. Two classical examples – the Pascal configuration (9_3) and Desargues' configuration (10_3) – derive from theorems that date back to the seventeenth century. The present study begins in the 1820s, when A. F. Möbius discovered an important spatial configuration formed by two mutually inscribed and circumscribed tetrahedra in space. The naive stage[4] in the study of configurations, which lasted until around 1880, merely reflected an interest in examples from geometry.

The second period or formal stage introduced investigations of abstract configurations, although the dominant interests stemmed from geometry and group theory. The first systematic studies began in the 1880s, thus soon after Reye called attention to the general phenomenon of configurations. These investigations took a more abstract point of view that would later lead to an interpretation of configurations as special types of bipartite graphs. During the period 1880–1930, however, configurations were primarily understood as incidence structures involving points, lines, and planes in a real or complex projective space of two or three dimensions.

Two early investigators, S. Kantor and A. Schönflies, concentrated on n_3 configurations, especially those that could be represented by mutually inscribed polygons. Schönflies was also one of the first to consider the automorphism groups of regular configurations, especially those cases with transitive automorphism groups. Configurations have rich symmetry properties, which made them interesting objects of study from the perspective of group theory, in particular the theory of finite collineation groups. Beginning with the work of C. Jordan and F. Klein, mathematicians began to consider configurations associated with these groups and their subgroups. An important example is the group G_{216} which acts on the Hessian configuration, a famous example connected with the

[4] David Hilbert wrote in 1893 that a mathematical theory usually goes through three periods: 1) naive, 2) formal and 3) critical.

nine inflection points of plane cubic curves. By 1900 it became clear that curves and surfaces studied in algebraic geometry contained many surprising examples of finite geometrical configurations. In 1929 Friedrich Wilhelm Levi published *Geometrische Konfigurationen*, the first and only textbook on configurations. This text reflects more modern interests in groups, combinatorics, and graph theory. Around this time, David Hilbert was promoting interest in the classical approach to configurations in his popular lecture course on Anschauliche Geometrie. Outside G öttingen this course gained a wide audience through the book by the same title, published by Hilbert and Stefan Cohn-Vossen in 1932. Its third chapter, entitled "Projective Configurations", presents many examples and a very readable overview of this topic.

Finally, the third critical stage was devoted to researches on configurations from the point of view of graph theory. In the 1940s, F.W. Levi later offered a lecture course on finite geometrical systems in Calcutta, India. He published this 6-part course of lectures in 1942, his second important contribution to the theory of configurations. Levi here invented a representation of configurations of points and lines (planes) by means of a special type of bipartite graph. After World War II, this construction was popularized by the distinguished geometer H.S.M. Coxeter, who coined the term Levi graph for it. Coxeter had a very deep knowledge of classical geometry, and in his hands, configurations took on new life via their connections with algebraic geometry, group theory, and combinatorics.

Eidesstattliche Erklärung

Hiermit versichere ich gemäss 12 Abs. 3e der Promotionsordnung des Fachbereichs 08, Physik, Mathematik und Informatik der Johannes Gutenberg-Universität Mainz vom 02.12.2013:

Ich habe die jetzt als Dissertation vorgelegte Arbeit selbständig verfasst. Es wurden ausschliesslich die angegebenen Quellen und Hilfsmittel verwendet. Von der Ordnung zur Sicherung guter wissenschaftlicher Praxis in Forschung und Lehre und vom Verfahren zum Umgang mit wissenschaftlichem Fehlverhalten habe ich Kenntnis genommen. Ich habe oder hatte die jetzt als Dissertation vorgelegte Arbeit nicht schon als Prüfungsarbeit für eine andere Prüfung eingereicht. Ich hatte weder die jetzt als Dissertation vorgelegte Arbeit noch Teile davon an einer anderen Stelle als Dissertation eingereicht.