# Log Toroidal Families

## Dissertation

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## Summary

We study families of logarithmic varieties with mild singularities, the log toroidal families. They generalize and unify various classes of spaces with controlled singularities, including toroidal varieties, toroidal embeddings, semistable degenerations, log smooth morphisms, and toric log Calabi–Yau spaces. Starting from Kato's toroidal characterization of log smoothness and Gross–Siebert's local models for the singularities of a toric log Calabi–Yau space, we construct elementary log toroidal families from combinatorial data as étale local models for the singularities which we allow in a log toroidal family. We study the reflexive de Rham complex $W^\bullet_{X/S}$ of a log toroidal family and prove the Hodge–de Rham degeneration for proper log toroidal families over a log point $S = \mathrm{Spec}\ (Q \to k)$. This in particular settles a conjecture of Danilov on the cohomology of toroidal pairs $(X, D)$. This thesis is an expanded version of the article [22], where we prove the Hodge–de Rham degeneration and apply it to obtain a smoothing of a normal crossing space as well as a toroidal crossing space.

## Zusammenfassung

Wir untersuchen Familien logarithmischer Varietäten mit leichten Singularitäten, die log-toroidalen Familien. Diese verallgemeinern und vereinheitlichen verschiedene Begriffe eines Raumes mit kontrollierten Singularitäten, unter anderem toroidale Varietäten, toroidale Einbettungen, halbstabile Entartungen, logarithmisch glatte Morphismen und torische log-Calabi-Yau-Räume. Ausgehend von Katos toroidaler Charakterisierung der logarithmischen Glattheit und den lokalen Modellen des Gross-Siebert-Programms für die Singularitäten torischer log-Calabi-Yau-Räume konstruieren wir elementare log-toroidale Familien aus kombinatorischen Daten, welche als lokale Modelle für die Singularitäten dienen, die wir in log-toroidalen Familien zulassen. Wir untersuchen den reflexiven de-Rham-Komplex $W^\bullet_{X/S}$ einer log-toroidalen Familie und beweisen die Entartung der Hodge-de-Rham-Spektralfolge an $E_1$ für eigentliche log-toroidale Familien über einem logarithmischen Punkt $S = \mathrm{Spec}\ (Q \to k)$. Insbesondere ist damit eine Vermutung Danilovs über die Kohomologie toroidaler Paare $(X, D)$ gezeigt. Diese Dissertation ist eine erweiterte Version der Arbeit [22], in der wir die Entartung der Spektralfolge zeigen und als Anwendung die Glättung vieler Räume mit normalen oder toroidalen Kreuzungen erhalten.

**Foreword**

This is the final version of my PhD thesis. It has undergone minor revision since I submitted the first version in March 2020. Thanks to the reporter [redacted] kind comments, many typographical and grammatical and even some mathematical errors have been removed.

Simon Felten, February 2021

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# 1 Introduction

In this PhD Thesis we develop the concepts of *generically log smooth family* and of *log toroidal family* as tools to study degenerations of varieties. The concepts are generalizations of log smooth morphisms and deeply rooted both in log geometry and in the Gross–Siebert approach to mirror symmetry. This thesis is an expanded version of parts of the paper [22] on smoothing toroidal crossing spaces. There we introduce log toroidal families for the first time, prove the degeneration of a Hodge–de Rham spectral sequence, and apply it to obtain smoothings of toroidal crossing spaces. Some of the ideas are already contained in my Master thesis [20] and developed here further. We will indicate throughout the text what is taken from earlier works. Whereas [22] heads to the application—the smoothing of toroidal crossing spaces and in particular normal crossing spaces—here we delve into the details of the basic theory with a special focus on cohomology and spectral sequences as well as connections to related concepts. In the main part, we assume the reader to be acquainted with log geometry, but we drop this assumption in the Introduction to make it accessible to a wider audience.

### Cohomology of Varieties

Given a compact complex Kähler manifold $X$, one of the key results in complex geometry is the Hodge decomposition

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X,\Omega^p_X),$$

which relates cohomology of the topological space $X$ to sheaf cohomology of the Kähler differential forms $\Omega^p_X$. The quasi-isomorphism $\mathbb{C} \simeq \Omega^\bullet_X$ shows that the former is isomorphic to the hypercohomology $\mathbb{H}^k(X,\Omega^\bullet_X)$, so the Hodge decomposition implies that the Hodge–de Rham spectral sequence

$$E_1^{p,q} := H^q(X,\Omega^p_X) \Rightarrow \mathbb{H}^{p+q}(X,\Omega^\bullet_X) \cong H^{p+q}(X,\mathbb{C})$$

associated to the Hodge filtration on $\Omega^\bullet_X$ degenerates at $E_1$. More generally, an open—i.e., non-compact—Kähler manifold $X^\circ$ can be embedded into a compact Kähler manifold $X$ as an open subset such that the complement $D = X \setminus X^\circ \subset X$ is a (reduced) divisor with normal crossings. After choosing local coordinates $z_1,...,z_n$ on $X$ such that $D = \{z_1 \cdot ... \cdot z_r = 0\}$ for $r \leq n$, we find that the sheaf $\Omega^1_X(\log D)$ of differential forms with log poles in $D$ is locally free on generators $\frac{dz_1}{z_1},...,\frac{dz_r}{z_r},dz_{r+1},...,dz_n$. There is a canonical isomorphism $H^k(X^\circ,\mathbb{C}) \cong \mathbb{H}^k(X,\Omega^\bullet_X(\log D))$, allowing the study of the cohomology of $X^\circ$ via the spectral sequence

$$E_1^{p,q} := H^q(X,\Omega^p_X(\log D)) \Rightarrow \mathbb{H}^{p+q}(X,\Omega^\bullet_X(\log D)),$$

which degenerates at $E_1$ as well.

The algebro-geometric analog of a compact Kähler manifold is a smooth projective scheme $X/\mathbb{C}$, and a reduced divisor $D \subset X$ is *normal crossing* if it is isomorphic to

$$\{z_1 \cdot ... \cdot z_r\} \subset \mathbb{A}^n$$

locally in the étale topology. One of the key ingredients in the above results is the easy local structure of the embedding $D \subset X$; it is generalized by toric geometry—we say $(X,D)$ is a *toric pair* if $X$ is a toric variety and $D \subset X$ is a union of reduced toric prime divisors, e.g. $D = \emptyset$. Following Danilov's foundational work [14] in toric geometry, $(X,D)$ is a *toroidal pair* if it is locally isomorphic to a toric pair $(Y,E)$ in the étale topology (where we might choose different pairs at different points). If we can choose $E$ to be the full toric boundary, i.e., the union of all toric Weil divisors, then $X \setminus D \subset X$ is a toroidal embedding in the sense of [56]. In any case, there is an open $j : U \subset X$ with $\mathrm{codim}(X \setminus U, X) \geq 2$ such that $U$ is smooth and $D|_U \subset U$ is a smooth divisor, in particular it has normal crossings. It allows Danilov to define the de Rham complex as the direct image

$$\Omega^\bullet_X(\log D) := j_*\Omega^\bullet_U(\log D|_U)$$

from the normal crossing pair $(U, D|_U)$. As a consequence of the theory we develop in this thesis, we obtain:

**Theorem 1.1.** *Let $(X, D)$ be a toroidal pair with $X$ proper. Then the Hodge–de Rham spectral sequence*

$$E_1^{p,q} := H^q(X, \Omega_X^p(\log D)) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D))$$

*degenerates at $E_1$.*

This has been conjectured by Danilov in [14, §15]. The case $D = \varnothing$ has been settled by Danilov himself in [15], and the case where $X \setminus D \subset X$ is a toroidal embedding follows from earlier results in logarithmic geometry as we shall see below. Related results are also contained in [72].

### Hodge Numbers in Mirror Symmetry

For $X$ as above, the numbers $h^{p,q}(X) := \dim_{\mathbb{C}} H^q(X, \Omega_X^p)$ are called the Hodge numbers of $X$. Mirror symmetry in its most basic form of numerical mirror symmetry predicts the existence of pairs $(X, \check{X})$ of Calabi–Yau manifolds of dimension $n$ such that $h^{p,q}(X) = h^{n-p,q}(\check{X})$. If $f : \mathcal{X} \to S$ is a family of compact Kähler manifolds, then the relative Hodge–de Rham spectral sequence

$$E_1^{p,q} := R^q f_* \Omega_{\mathcal{X}/S}^p \Rightarrow R^{p+q}\Omega_{\mathcal{X}/S}^\bullet$$

degenerates at $E_1$ and the sheaves $E_1^{p,q}$ are locally free (of constant rank), a fact that—in the algebraic setting—was first proven by Deligne in [16]. Thus, the Hodge numbers $h^{p,q}(\mathcal{X}_s)$ of the fibers $\mathcal{X}_s = f^{-1}(s)$ are constant in the family. If $X', \check{X}'$ are deformations of $X, \check{X}$ and $(X, \check{X})$ is a (numerical) mirror pair, then $(X', \check{X}')$ is one as well.

### Degenerations

A degeneration is a flat family $f : \mathcal{X} \to \Delta$ of complex spaces over the unit disk

$$\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$$

which is smooth over the punctured disk $\Delta^* = \Delta \setminus \{0\}$. We shall be especially interested in the behavior of Hodge numbers in a degeneration.

The simplest type of degenerations is a semistable degeneration, i.e., a degeneration which is locally of the form

$$\mathbb{C}^n \to \mathbb{C}, \quad (z_1, ..., z_n) \to z_1 \cdot ... \cdot z_r,$$

for $r \leq n$. If $\pi : \mathcal{Y} \to \Delta$ is an algebraic degeneration (obtained by analytification of a map of varieties), then by the Semistable Reduction Theorem of [56], there is a map $\Delta \to \Delta, z \mapsto z^k$, and a diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\phi"] \arrow[dr, "f"] & \tilde{\mathcal{Y}} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d, "\pi"] \\ & \Delta \arrow[r, "z \mapsto z^k"] & \Delta \end{tikzcd}$$

with Cartesian square such that $\phi : \mathcal{X} \to \tilde{\mathcal{Y}}$ is a blow-up isomorphic over $\Delta^*$ and $f : \mathcal{X} \to \Delta$ is semistable. Thus in some sense, every degeneration has a semistable model, explaining the relevance of semistable degenerations.

A generalization of Deligne's above mentioned degeneration result to semistable families $f : \mathcal{X} \to \Delta$ was obtained by Steenbrink in [71]. Namely, the variation of Hodge structures on $\Delta^*$ obtained from the smooth family $\mathcal{X}^* \to \Delta^*$ cannot be extended to $\Delta$ as a variation of Hodge structures, but one gets an abstract limiting mixed Hodge structure in $0 \in \Delta$ by a purely Hodge-theoretic analysis started in [70]. In order to obtain a geometric interpretation of the limiting mixed Hodge structure, Steenbrink studies the relative differential forms $\Omega^1_{\mathcal{X}/\Delta}(\log \mathcal{X}_0)$ with log poles in $\mathcal{X}_0 = f^{-1}(0)$ and finds on the cohomology of

$$\Omega^\bullet_{\mathcal{X}/\Delta}(\log \mathcal{X}_0) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_{\mathcal{X}_0}$$

a mixed Hodge structure. In the course of this, he proves that the Hodge–de Rham spectral sequence

$$E_1^{p,q} := R^q f_* \Omega^p_{\mathcal{X}/\Delta}(\log \mathcal{X}_0) \Rightarrow R^{p+q} f_* \Omega^\bullet_{\mathcal{X}/\Delta}(\log \mathcal{X}_0)$$

degenerates at $E_1$. If we consider

$$h^{p,q}(\mathcal{X}_0) := \dim_\mathbb{C} H^q(\mathcal{X}_0, \Omega^p_{\mathcal{X}/\Delta}(\log \mathcal{X}_0) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_{\mathcal{X}_0})$$

the Hodge number of $\mathcal{X}_0$, it remains constant in the degeneration by [71, Thm. 2.8].

### Log Geometry

Logarithmic geometry studies schemes $(X, \mathcal{O}_X)$ endowed with a sheaf of monoids $\mathcal{M}_X$ and a monoid homomorphism $\alpha : \mathcal{M}_X \to (\mathcal{O}_X, \cdot)$ such that $\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism. The morphism $\alpha$ is called a log structure and $(X, \mathcal{O}_X, \mathcal{M}_X, \alpha)$ is called a log scheme. Every scheme $X$ carries the trivial log structure $\mathcal{M}_X = \mathcal{O}_X^*$. A pair $(X, D)$ consisting of a variety $X$ and a Weil divisor $D \subset X$ gives rise to a log scheme by taking the divisorial log structure associated to $D \subset X$, i.e.,

$$\mathcal{M}_X(V) = \{f \in \mathcal{O}_X(V) \mid f|_{V \setminus D} \in \mathcal{O}_X^*(V \setminus D)\}$$

is the sheaf of functions invertible on $X \setminus D$. Given a morphism $f : X \to S$ of log schemes, a log derivation with values in a coherent sheaf $\mathcal{E}$ is a pair $(D, \Delta)$ where $D : \mathcal{O}_X \to \mathcal{E}$ is a relative derivation and $\Delta : \mathcal{M}_X \to \mathcal{E}$ is a monoid homomorphism satisfying some relation. As in the classical setting, there is a universal log derivation $(d, \delta) : (\mathcal{O}_X, \mathcal{M}_X) \to \Omega^1_{X/S}$ with the analogous universal property. The sheaf $\Omega^1_{X/S}$ is called the sheaf of log differential forms. If $(X, D)$ is a normal crossing pair, then $\Omega^1_{(X,D)/k} \cong \Omega^1_X(\log D)$. This justifies the name *log structure* and explains the relevance of differential forms with log poles from a more conceptual perspective.

The prototypical example $A_P$ of a log scheme is constructed from a sharp toric monoid $P$, i.e., $P$ is the intersection of a rational polyhedral cone with the lattice and $0 \in P$ is its only invertible. We obtain $A_P$ by endowing the affine toric variety Spec $\mathbb{C}[P]$ with the divisorial log structure defined by the union $D_P$ of all toric divisors. Homomorphisms $\theta : Q \to P$ of monoids induce homomorphisms $A_\theta : A_P \to A_Q$ of log schemes since $A_\theta^{-1}(D_Q) \subset D_P$.

Translating Grothendieck's geometric-functorial characterization of smoothness, i.e., the infinitesimal lifting criterion, to log geometry yields the notion of *log smoothness*. Under mild assumptions on $\theta : Q \to P$, e.g. injective and $\theta^{gp}$ has a torsion-free cokernel, the induced morphism $A_\theta : A_P \to A_Q$ is log smooth. Conversely, by Kato's toroidal characterization of log smoothness in [50], every log smooth morphism $f : X \to S$ is locally in the étale topology a base change of some composition

$$\mathbb{A}^r \times A_P \xrightarrow{\pi} A_P \xrightarrow{A_\theta} A_Q,$$

where $\mathbb{A}^r \times A_P$ carries the divisorial log structure induced by the divisor $\mathbb{A}^r \times D_P = \pi^{-1}(D_P)$. Given a geometric point $\bar{x} \in X$, this can be done in such a way that $\bar{x}$ maps to $(0,0) \in \mathbb{A}^r \times A_P$. Note that $\mathbb{A}^r \times A_P$ is *not* the same as $A_{\mathbb{N}^r \oplus P}$ because the latter carries a different log structure, which is induced by the divisor $D_{\mathbb{N}^r \oplus P}$. If $f : X \to S$ is log smooth, then $\Omega^1_{X/S}$ is locally free. If a pair $(X, D)$ is normal crossing or if $X \setminus D \subset X$ is a toroidal embedding, then the structure morphism $(X, D) \to \text{Spec } k$ (the latter with the trivial log structure) is log smooth. This explains why $\Omega^1_X(\log D)$ is locally free in these situations. Similarly, if $f : \mathcal{X} \to \Delta$ is a semistable degeneration, then $f : (\mathcal{X}, \mathcal{X}_0) \to (\Delta, 0)$ is log smooth, and we have $\Omega^1_{(\mathcal{X},\mathcal{X}_0)/(\Delta,0)} \cong \Omega^1_{\mathcal{X}/\Delta}(\log \mathcal{X}_0)$. Steenbrink's degeneration result for semistable families $f : \mathcal{X} \to \Delta$ is a special case of the more general degeneration of the Hodge–de Rham spectral sequence for log smooth morphisms, which is proven by Illusie–Kato–Nakayama in [47, Cor. 7.2]. Illusie–Kato–Nakayama gives—under additional technical hypotheses—a natural generalization of Deligne's degeneration result in the smooth case to the logarithmic setting:

**Theorem 1.2** ([47]). *Let $S^\dagger/\mathbb{Q}$ and $f : X^\dagger \to S^\dagger$ be a proper, exact, and log smooth morphism of fs log schemes. Then the Hodge–de Rham spectral sequence*

$$E_1^{p,q} := R^q f_* \Omega^p_{X^\dagger/S^\dagger} \Rightarrow R^{p+q} f_* \Omega^\bullet_{X^\dagger/S^\dagger}$$

*degenerates at $E_1$, the sheaves $E_1^{pq}$ are locally free, and their formation commutes with base change.*

The theorem implies the constancy of log Hodge numbers in log smooth degenerations. E.g. the hypotheses are satisfied for semistable degenerations. Moreover, if $(X, D)$ is a pair such that $X \setminus D \subset X$ is a toroidal embedding, then it implies Danilov's conjecture in that special case.

### Toric Degenerations

Whereas more traditional approaches to mirror symmetry relate a variety $X$ to its mirror $\check{X}$, the Gross–Siebert approach in [29, 30, 31, 32] relates a toric degeneration $\mathcal{X} \to \mathcal{S}$ over the base $\mathcal{S} = \operatorname{Spec} k[\![t]\!]$ to its mirror degeneration $\check{\mathcal{X}} \to \mathcal{S}$. Roughly speaking, this is a proper flat morphism $\mathcal{X} \to \mathcal{S}$ where the generic fiber $\mathcal{X}_\eta$ corresponds to the variety $X$ and the central fiber $\mathcal{X}_0$ is obtained by gluing toric varieties along toric divisors. A toric degeneration becomes a log morphism $(\mathcal{X}, \mathcal{X}_0) \to (\mathcal{S}, 0)$ using the corresponding divisorial log structures. As a matter of definition, it is log smooth outside a closed subset $Z \subset \mathcal{X}$ of relative codimension $\geq 2$. The central fiber $\mathcal{X}_0$ is considered a combinatorial model of $\mathcal{X}_\eta$ and mirror symmetry becomes essentially a question of finding invariants that are shared by $\mathcal{X}_\eta$ and $\mathcal{X}_0$, dualizing the combinatorial data and studying these invariants under the duality. A precise definition of a toric degeneration is given in [29].

**Example 1.3.** This is the standard example of a toric degeneration, given e.g. in the introduction of [30]. We consider the degeneration of a smooth quartic surface

$$E = \{X^4 + Y^4 + Z^4 + W^4 = 0\} \subset \mathbb{P}^3$$

into an arrangement of four planes $D := \{XYZW = 0\} \subset \mathbb{P}^3$. The pencil defined by $D, E$ has the total space

$$\mathcal{X} = \{T_0(X^4 + Y^4 + Z^4 + W^4) - T_1 XYZW = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3,$$

where $X, Y, Z, W$ are homogeneous coordinates of $\mathbb{P}^3$, and $T_0, T_1$ are homogeneous coordinates of $\mathbb{P}^1$. The space $\mathcal{X}$ is the blow-up of $\mathbb{P}^3$ in $D \cap E$ via the second projection, and the first projection defines a flat projective family $\varphi : \mathcal{X} \to \mathbb{P}^1$. We denote by $S \subset \mathbb{P}^1$ some neighborhood of $0 = [0:1]$ such that $D = \varphi^{-1}(0)$ is the only singular fiber of the restricted family, which we denote by $f : X \to S$. The singular fiber $D$ consists of four copies of $\mathbb{P}^2$ intersecting in a union $L$ of six lines $\mathbb{P}^1$ like the faces of a tetrahedron, exhibiting $D$ as four toric varieties $\mathbb{P}^2$ glued along toric divisors. It is depicted in Figure 1.1. The set

$$Z := L \cap \{X^4 + Y^4 + Z^4 + W^4 = 0\} \subset \mathbb{P}^3$$

consists of 24 points, four on each line. It is the singular locus of the total space $X$, i.e., $U = X \setminus Z$ is regular. The divisor $D|_U \subset U$ is a simple normal crossing divisor whereas $D \subset X$ is not a normal crossing divisor since $X$ is not regular in the points $z \in Z$. The log morphism $(X, D) \to (S, 0)$ is log smooth outside $Z$ because there it is a semistable family, but not log smooth in $Z$. We obtain a toric degeneration by base change to a formal neighborhood of $0 \in S$.

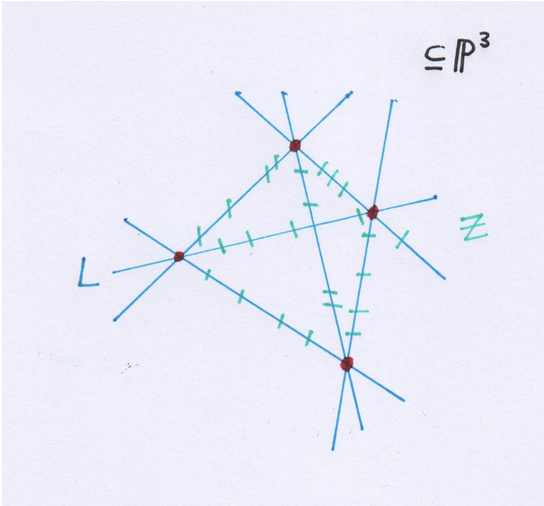


Figure 1.1: The central fiber $D$ consists of four copies of $\mathbb{P}^2$ intersecting in a union $L$ of six copies of $\mathbb{P}^1$. On each $\mathbb{P}^1$ there are 4 singularities of $X$. This picture is also contained in my Master Thesis.

The most important problem in the Gross–Siebert approach has been, until its solution in [32], the *reconstruction problem.* This is the question if, given a toric log Calabi–Yau space $\mathcal{X}_0$—i.e., a potential central fiber of a toric degeneration—there is a toric degeneration with this central fiber. Namely, given a toric degeneration $\mathcal{X} \to \mathcal{S}$, there is a candidate $\check{\mathcal{X}}_0$ for the mirror central fiber, but a priori no toric degeneration $\check{\mathcal{X}} \to \mathcal{S}$. Whereas the solution in [32] goes another way, the original attempt in [31] has been to mimic the deformation theory of log smooth morphisms and obtain the toric degeneration via a Bogomolov–Tian–Todorov result. In this original approach, Gross–Siebert addresses the problem of controlling the log singularities in toric degenerations by introducing local models for the singularities in very much the same way as the morphisms $A_\theta : A_P \to A_Q$ are local models for log smooth morphisms by Kato's toroidal characterization.

**Example 1.4.** Let $Q = \mathbb{N}$ and let $P \subset \mathbb{N}^3$ be the submonoid generated by

$$T := (1,0,0),\ X := (1,0,1),\ Y := (1,1,0),\ W := (1,1,1)$$

which is depicted in Figure 1.2. The associated monoid ring is $A_{st} := \mathbb{C}[x,y,t,w]/(xy - tw)$. We set $\theta(1) := T$ and write

$$f : X_{st} := \mathrm{Spec}\ A_{st} \to \mathbb{A}^1$$

for the geometric morphism. It is log smooth if we use the full toric boundary to define the log structure, but if we use $\{t = 0\}$ on both spaces, it is log smooth only away from the origin, cf. [31, Ex. 1.11] or the author's Master thesis for a detailed analysis. It turns out that étale locally around the 24 singularities, the family of Example 1.3 is isomorphic as a log morphism to the above $f : X_{st} \to \mathbb{A}^1$, cf. also Example 4.30 below. Thus to study the singularities of the degeneration, it is sufficient to study the singularities of $f : X_{st} \to \mathbb{A}^1$. The subscript is due to the fact that we consider $f : X_{st} \to \mathbb{A}^1$ the standard example of this type of structure.

The local models yield the notion of a *divisorial deformation* of a toric log Calabi–Yau space $\mathcal{X}_0$; these deformations correspond to infinitesimal thickenings of $\mathcal{X}_0$ induced by a toric degeneration $\mathcal{X} \to \mathcal{S}$. Among other partial results toward the reconstruction theorem, Gross–Siebert proves in [31, Thm. 4.1] the degeneration at $E_1$ of the Hodge–de Rham spectral sequence associated to a divisorial deformation $f_A : \mathcal{X}_A \to \mathrm{Spec}\ A$ over an Artinian ring $A$. When we write $j : \mathcal{X}_A \smallsetminus Z \subset \mathcal{X}_A$ for the inclusion, this spectral sequence is constructed

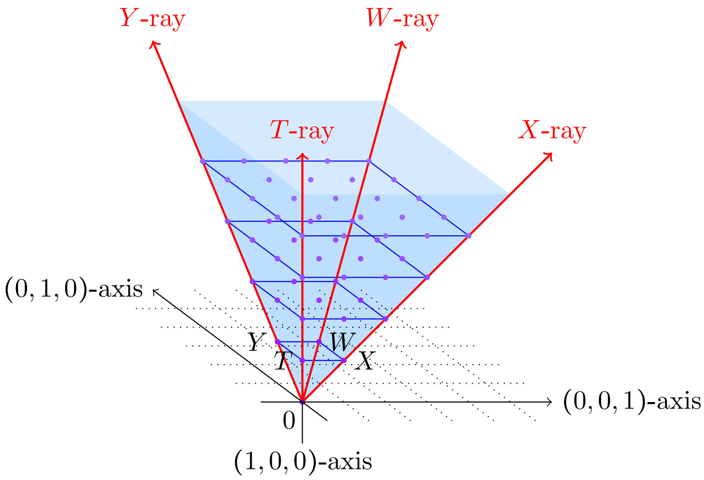


Figure 1.2: The four rays generated by the vectors span the cone in blue. The monoid $P$ consists of the marked points, which are the integral points within the cone. The $(1,0,0)$-axis goes upwards. The picture is also included in my Master Thesis.

from the direct image de Rham complex $j_*\Omega^\bullet_{(\mathcal{X}_A\setminus Z)/A}$, similarly to the de Rham complex of toroidal pairs $(X,D)$. This original approach has been completed recently in [22]—partially as a consequence of the theory developed in this PhD thesis. Beyond the reconstruction theorem, the degeneration shows that $h^{p,q}(\mathcal{X}_0) = h^{p,q}(\mathcal{X}_\eta)$, i.e., log Hodge numbers of $\mathcal{X}_0$ coincide with Hodge numbers of $\mathcal{X}_\eta$. Since in many cases $h^{p,q}(\mathcal{X}_0) = h^{n-p,q}(\check{\mathcal{X}}_0)$, this proves a form of numerical mirror symmetry, see [31, 69] for precise statements.

### Generically Log Smooth Families

The example of toric degenerations shows that many interesting geometric degenerations are not log smooth, but can nonetheless be studied with tools of logarithmic geometry. In this thesis, we develop a systematic theory out of Gross–Siebert's idea to control log singularities with local models obtained from toric morphisms. This gives rise to the notions of *generically log smooth family* and of *log toroidal family*.

Roughly speaking, a generically log smooth family is a flat morphism $f : X \to S$ of schemes together with a distinguished open $j : U \subset X$, with log structures on $U$ and $S$, and with the structure of a log morphism on $f|_U$ such that $f|_U$ is log smooth and saturated. There need not even a log structure on $Z := X \setminus U$ be defined. This is the essential structure in the degeneration of the smooth quartic in Example 1.3. The precise definition of a generically log smooth family is Definition 2.2.

To define Hodge numbers of the fibers of a generically log smooth family $f : X \to S$, we need to define its de Rham complex. Our strategy—taken from Gross–Siebert—can be motivated most easily by the situation for normal varieties. Namely, if $Y$ is a normal variety, then the regular locus $j : U := Y_{reg} \subset Y$ forms an open subset whose complement $Y_{sing}$ has codimension $\geq 2$. Reflexive differentials forms

$$\Omega^{[p]}_Y := j_*\Omega^p_U = (\Omega^p_Y)^{**}$$

have turned out to be better behaved than Kähler differentials, so we just copy the definition to generically log smooth families and define the (log) de Rham complex $W^\bullet_{X/S} := j_*\Omega^\bullet_{U/S}$. The good properties of $\Omega^{[p]}_Y$ (in particular coherence) crucially depend on the fact that $j_*\mathcal{O}_U = \mathcal{O}_Y$. To imitate this construction in the relative setting, we require that, for a generically log smooth family, $Z$ has codimension $\geq 2$ in every fiber, and that $f : X \to S$ is a Cohen–Macaulay morphism, such that we have indeed $j_*\mathcal{O}_U = \mathcal{O}_X$. With this definition, we expect log Hodge numbers to be constant under suitable hypotheses. As in the case of smooth families, we need a spectral sequence to study this question. The Hodge filtration on $W^\bullet_{X/S}$ is given by $F^pW^m_{X/S} = W^m_{X/S}$ if $m \geq p$ and $F^pW^m_{X/S} = 0$ otherwise. It induces a filtration $F^\bullet\mathbb{H}^n$ on $\mathbb{H}^n(X/S) \coloneqq R^{p+q}f_*W^\bullet_{X/S}$ as the images of $R^{p+q}f_*F^pW^\bullet_{X/S}$. Moreover, it gives rise to the *Hodge–de Rham spectral sequence*

$$E(X/S): \qquad E_1^{pq} = R^qf_*W^p_{X/S} \Rightarrow R^{p+q}f_*W^\bullet_{X/S}$$

with abutment

$$E_\infty^{pq} = F^p\mathbb{H}^{p+q}(X/S)/F^{p+1}\mathbb{H}^{p+q}(X/S)$$

the subquotients of $F^\bullet\mathbb{H}^{p+q}(X/S)$.

**Philosophy 1.5.** *Let $f : X \to S$ be a reasonable generically log smooth family. Then the Hodge–de Rham spectral sequence $E(X/S)$ should degenerate at $E_1$. The sheaves $R^qf_*W^p_{X/S}$ and $R^nf_*W^\bullet_{X/S}$ should be locally free, and their formation should commute with base change.*

A proof of this statement in some special cases is the heart of this thesis, see the discussion below. Log toroidal families as explained below in characteristic 0 are certainly reasonable in the sense of the philosophy, but we will not determine its meaning beyond.

### Log Toroidal Families

It is difficult to approach Philosophy 1.5 in full generality since in a generically log smooth family $f : X \to S$ we have very few control over the log singularities in $Z = X \setminus U$. To illustrate just one point, we consider a Cartesian diagram

$$\begin{tikzcd} Y \arrow[r, "c"] \arrow[d, "g"] & X \arrow[d, "f"] \\ T \arrow[r, "b"] & S \end{tikzcd}$$

of generically log smooth families. In view of Philosophy 1.5, we expect the canonical homomorphism $c^*W^p_{X/S} \to W^p_{Y/T}$ to be an isomorphism. In fact this is a key step in those cases where we will prove the statement of Philosophy 1.5, but a priori we only know it to be an isomorphism on $c^{-1}(U)$. To prove the isomorphism, it is sufficient to show that $c^*W^p_{X/S}$ is reflexive, which is a local property. Thus, in order to approach Philosophy 1.5, we need to control the local structure of $f : X \to S$ around points $z \in Z$. We define a log toroidal family to be a generically log smooth family whose log singularities are controlled by a local model in the same way as the local structure of log smooth morphisms is controlled by $A_P \to A_Q$.

Let us introduce the local models. Given an injective (and saturated) homomorphism $\theta : Q \to P$ of sharp toric monoids, we turn the geometric family

$$A_\theta : A_P = \operatorname{Spec} \mathbb{Z}[P] \to \operatorname{Spec} \mathbb{Z}[Q] = A_Q$$

into a generically log smooth family by endowing $A_P$ with the divisorial log structure associated to a toric divisor $D \supset A_\theta^{-1}(D_Q)$. When we choose $D = D_P$, this yields the local models of log smooth morphisms. In the Gross–Siebert program, every local model has $Q = \mathbb{N}$ and $D = A_\theta^{-1}(0)$. The family $f : X_{st} \to \mathbb{A}^1$ in Example 1.4 has this form. Since we consider the local models as building blocks of log toroidal families, we call them *elementary log toroidal families.*

**Definition 1.6** (provisional)**.** A *log toroidal family* $f : X \to S$ is a generically log smooth family which is étale locally of the form $(\operatorname{Spec} \mathbb{Z}[P], D) \times_{A_Q} S \to S$ for some map $S \to A_Q$ and some generically log smooth family $(\operatorname{Spec} \mathbb{Z}[P], D) \to A_Q$ with a toric divisor $D \subset \operatorname{Spec} \mathbb{Z}[P]$ as above.

This is not the precise definition, which is rather intricate and contained in Section 4. The degeneration of the smooth quartic in Example 1.3 is a log toroidal family. If $(X, D)$

is a toroidal pair, then endowing $X$ with the divisorial log structure yields a log toroidal family over $S = \mathrm{Spec}\ \mathbb{C}$ (with the trivial log structure) as well. The name "log toroidal" is in analogy with toroidal varieties, which are étale locally isomorphic to toric varieties, but accounts for the fact that we have introduced a log structure.

After studying the de Rham complex of elementary log toroidal families in Section 6, one of our key results is that the formation of $W^\bullet_{X/S}$ commutes with base change.

**Theorem 1.7.** *Let $k$ be a field, and let $S/k$ be an fs log scheme defined over $k$. Let $f : X \to S$ be a log toroidal family, and let $g : Y \to T$ be the base change along a strict morphism $b : T \to S$. Then the canonical map $c^*W^p_{X/S} \to W^p_{Y/T}$ is an isomorphism.*

In general, the formation of $W^\bullet_{X/S}$ does not commute with base change even for elementary log toroidal families, see Example 6.8.

### The Hodge–de Rham Spectral Sequence

Having the base change Theorem 1.7, which allows us to specify the meaning of "reasonable", we upgrade Philosophy 1.5 to a conjecture.

**Conjecture 1.8.** *Let $k \supset \mathbb{Q}$ be a field, let $S/k$ be an fs log scheme over it, and let $f : X \to S$ be a proper log toroidal family of relative dimension $d$. Then $E(X/S)$ degenerates at $E_1$. The sheaves $R^q f_* W^p_{X/S}$ and $R^n f_* W^\bullet_{X/S}$ are locally free, and their formation commutes with base change.*

The main result of this thesis is that Conjecture 1.8 holds in several important cases. Let $S = \mathrm{Spec}\ (Q \to k)$ be the log structure on the point induced by the inclusion $S \to A_Q$ of the origin. We say $f : X \to S$ is a log toroidal family *with respect to* $S \to A_Q$ if we can always choose this particular map in Definition 1.6, i.e., if the family is locally isomorphic to base changes of $(\mathrm{Spec}\ \mathbb{Z}[P], D) \to A_Q$ along this map $S \to A_Q$.

**Theorem 1.9.** *Let $S = \mathrm{Spec}\ (Q \to k)$ for a sharp toric monoid $Q$ and a field $k \supset \mathbb{Q}$, and let $f : X \to S$ be a proper log toroidal family of relative dimension $d$ with respect to $S \to A_Q$. Then $E(X/S)$ degenerates at $E_1$.*

We prove this result by an adaptation of the method of Deligne–Illusie in [17]. The degeneration comes down to a dimension count. In positive characteristic, this can be achieved by a decomposition in the derived category induced by the Cartier isomorphism. Then we compare the characteristic-0 case and the positive-characteristic case via a spreading out of $f : X \to S$ to a base of finite type over $\mathbb{Z}$. We construct the spreading out in Section 4.3 and the Cartier isomorphism and the decomposition in the log toroidal setting in Section 7. The second partial result is Conjecture 1.8 over one-dimensional infinitesimal bases.

**Theorem 1.10.** *Let $S = S_m := \mathrm{Spec}\ (\mathbb{N} \overset{1 \mapsto t}{\to} \mathbb{C}[t]/(t^{m+1}))$ and let $f : X \to S$ be a proper log toroidal family of relative dimension $d$ with respect to $S \to A_{\mathbb{N}}$. Then:*

1. *$R^q f_* W^p_{X/S}$ is a free $\mathbb{C}[t]/(t^{m+1})$-module whose formation commutes with base change.*
2. *The spectral sequence $R^q f_* W^p_{X/S} \Rightarrow R^{p+q} f_* W^\bullet_{X/S}$ degenerates at $E_1$.*

We prove this with an idea of Steenbrink in [71] in the guise of Gross–Siebert's adaptation in [31]. The key point is Lemma 8.4, which implies that a certain map of complexes is a quasi-isomorphism. The proof of this statement in [31] has a gap because the differential in the complex is not $\mathcal{O}_X$-linear. From Theorem 1.10, we deduce the statement of Conjecture 1.8 in some more cases, e.g. some particular higher-dimensional bases. Note that Kawamata–Namikawa states—using Steenbrink's idea as well—in [55, Lemma 4.1] a degeneration result over more general (Artinian) bases. However, their proof fails for some bases, so we restrict to the above mentioned ones.

### Relatively Log Smooth Families

Example 1.3 shows us that at least some interesting degenerations are not log smooth. We introduce in this thesis the concept of log toroidal family to study them, but this is not the only possible approach. Nakayama–Ogus introduce in [64] the concept of relatively (log) smooth morphism. Essentially, a morphism

$$f : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$$

of log schemes is relatively log smooth if there is a second log structure $\mathcal{H}$ on $X$ and an embedding $\mathcal{M}_X \subset \mathcal{H}$ such that $(X, \mathcal{H}) \to (S, \mathcal{M}_S)$ is log smooth.

**Example 1.11.** The family $f : X_{st} \to \mathbb{A}^1$ of Example 1.4 is relatively log smooth, where $\mathcal{M}_X$ is the divisorial log structure of $\{t = 0\}$ and $\mathcal{H}$ is the divisorial log structure of the full toric boundary.

In Section 5, we introduce a variant of this concept, the *relatively log smooth family.* It turns out that this variant is closely related to log toroidal families in the sense that, given a relatively log smooth family $f : X \to S$, there is an open subset $U \subset X$ such that the generically log smooth family obtained by restricting the log structure of $X$ to $U$ is log toroidal. Conversely, though there are log toroidal families that do not arise from relatively log smooth ones, many log toroidal families are in fact relatively log smooth (after extension of the log structure to the whole of $X$).

### Outlook

The smoothing application in our article [22] heavily depends on the deformation theory of log toroidal families over Spec $(\mathbb{N} \to \mathbb{C})$. There we used the intricate approach of Gross–Siebert to divisorial deformations of toric log Calabi–Yau spaces developed in [31]. This approach certainly does not fit all log toroidal families: For example it is an open problem, if in general, log toroidal deformations are locally unique. A systematic deformation theory of log toroidal families is thus subject to future studies. Once local uniqueness is established, we expect that log toroidal deformations are controlled by a $\mathbb{C}[\![Q]\!]$-linear predifferential graded Lie algebra in the sense of [21], generalizing our recent adaptation of the method of Chan–Leung–Ma of [12] in the smoothing result.

Classically, for a smooth family $f : X \to S$, there is a variation of Hodge structures on $R^k f_* \Omega^\bullet_{X/S}$. Hodge structures have been generalized to log Hodge structures, see e.g. [53, 24], but the problem of constructing log Hodge structures is notoriously difficult and only achieved in a few special cases, see [52, 23, 24]. It would be interesting to construct a variation of log Hodge structures on $R^k f_* W^\bullet_{X/S}$ for a log toroidal family $f : X \to S$.

Given a log analytic space $X$ its Kato–Nakayama space $X_{log}$ is a topological realization of $X$ in the sense that it translates log geometric features of $X$ into topological features of $X_{log}$. Since in a log toroidal family $f : X \to S$ the space $X$ is a log scheme only on $U$, a priori it has no Kato–Nakayama space. However, Nakayama–Ogus study in [64] the topological realization for relatively log smooth morphisms, which turns out to be well-behaved. It might be possible to adapt the construction to the log toroidal setting. This is interesting for us because the construction of log Hodge structures heavily depends on the topological realization. Another direction is to study the relationship between the fibers of $f_{log} : X_{log} \to S_{log}$ and the topology of a smoothing. For first results in this direction, see Ogus' book [67].

Another approach to the log Hodge structures is via resolution of the log singularities. Given a log toroidal family $f : X \to S$, we search for a proper map $h : \tilde{X} \to X$ of log schemes which is an isomorphism on $U \subset X$ and has the property that $f \circ h : \tilde{X} \to S$ is log smooth. We then expect that

$$Rh_* W^\bullet_{\tilde{X}/S} = W^\bullet_{X/S},$$

so we may consider a log Hodge structure of $f \circ h : \tilde{X} \to S$ as a log Hodge structure of $f : X \to S$. In the spirit of Voevodsky's $h$-topology of [76] and the study of differential forms in the $h$-topology by Huber–Jörder in [43], it might even suffice to have the resolution only locally on $X$. Moreover, in case $Rh_*W^\bullet_{\tilde{X}/S} \neq W^\bullet_{X/S}$, we might consider $Rh_*W^\bullet_{\tilde{X}/S}$ the better differential forms—cf. also the recent preprint [68] for the case of positive characteristic.

### Comparison with [22]

Let us indicate how this thesis relates to our paper [22]. Section 2 on generically log smooth families is an expansion of §2 in [22]. Subsections 2.1 and 2.2 containing the definition and basic properties of generically log smooth families and their de Rham complex are essentially the content in [22]. Only [22, Prop. 2.8] on the dualizing sheaf and the results on analytification have been moved to their own Subsections 2.3 respective 2.7. Subsection 2.4 is essentially contained in my paper [21] on the use of Gerstenhaber algebras in log smooth deformation theory, the remaining subsections are new. Section 3 on elementary log toroidal families is in its core identical to §3 in [22]. Besides some additional remarks, it has been extended by a study of the horizontal locus, the non-strict base change, and many examples. Section 4 is an expansion of §4 in [22]. A study of the possibility to find local models such that any given one point corresponds to the origin in the local model has been added as Subsection 4.2. §9 in [22] on spreading out has been moved to Section 4 as Subsection 4.3. Furthermore, examples have been added. Section 5 on relatively log smooth families is completely new. Section 6 on differential forms on log toroidal families consists of the essentially unchanged content of §7 and §8 of [22]. In Subsection 6.4 on the local analytic theory, a proof has been added which is omitted in [22]. Section 7 on the theory in positive characteristic contains the essentially unchanged contents of §10 and §11 in [22]. Section 8 on the Hodge–de Rham degeneration contains—besides the essentially unchanged content of §12 in [22]—a slight generalization to higher-dimensional infinitesimal and formal base spaces. All sections from [22] which I have included in this thesis as well as §5 are my own contribution to [22]. The remaining sections §§1, 6, 13 are my coauthors' Helge Ruddat and Matej Filip's contribution.

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*Addendum in the final version:* I thank the reporter [redacted] for valuable mathematical, grammatical, and typographical suggestions to be included in this final version.

# 2 Generically Log Smooth Families

A *generically log smooth family* $f : X \to S$ is a generalization of a saturated log smooth morphism in the sense that it needs to be log smooth and saturated only on some open $U \subset X$. This notion is a technical framework for our study of log toroidal families below. Log structures in this section are assumed to be in the étale topology.

## 2.1 Definition and Basic Properties

This section follows closely §2 in our paper [22]. If $f : X \to S$ is a finite type morphism of Noetherian schemes, we say a Zariski open $U \subset X$ satisfies the *codimension condition* (CC) if the relative codimension of $Z := X \setminus U$ is $\geq 2$, i.e., for every point $s \in S$ with $X_s, U_s$ the fibers over the residue field $\kappa(s)$, we have

$$\mathrm{codim}(X_s \setminus U_s, X_s) \geq 2. \tag{CC}$$

*Remark* 2.1. Since $\dim(\mathcal{O}_{X_s,z}) = \mathrm{codim}(\overline{\{z\}}, X_s)$, we have $\mathrm{codim}(X_s \setminus U_s, X_s) \geq 2$ if and only if $\dim(\mathcal{O}_{X_s,z}) \geq 2$ for all $z \in X_s \setminus U_s$. In particular, if $g : X' \to X$ is surjective and étale, then $U \subset X$ satisfies (CC) if and only if $g^{-1}(U) \subset X'$ satisfies (CC).

Recall that a Cohen–Macaulay morphism is a flat morphism with Cohen–Macaulay fibers.

**Definition 2.2.** A *generically log smooth family* $f : X \to S$ consists of:

- a finite type Cohen–Macaulay morphism $f : X \to S$ of Noetherian schemes,
- a Zariski open $j : U \subset X$ satisfying (CC), and
- a saturated and log smooth morphism $f : (U, \mathcal{M}_U) \to (S, \mathcal{M}_S)$ of fine saturated log schemes.

To the complement $Z := X \setminus U$, we refer as the *log singular locus* even though $f$ might extend log smoothly to it. We say two generically log smooth families $f, f' : X \to S$ with the same underlying morphism of schemes are *equivalent* if there is some $\tilde{U} \subset U \cap U'$ satisfying (CC) with $\mathcal{M}_U|_{\tilde{U}} \cong \mathcal{M}'_{U'}|_{\tilde{U}}$ compatibly with all data.

The notion of equivalence is due to the fact that we do not care about the precise $U$. However, for technical simplicity we assume some $U$ fixed. The name *log singular locus* is in analogy with [30].

**Example 2.3.** Let $f : X \to S$ be a log smooth and saturated morphism of Noetherian fine saturated log schemes. Then $f$ is flat by [50, 4.5] and has Cohen–Macaulay fibers by [73, II.4.1]. We see that $f : X \to S$ gives a generically log smooth family for $U = X$.

*Remark* 2.4. Not every log smooth morphism is saturated, e.g. see [48, Rem. 9.1] for a log smooth morphism that is not even integral.

**Example 2.5.** Let $f : X \to S$ be the degeneration of the smooth quartic surface from Example 1.3. Endow $S$ with the divisorial log structure defined by $0 \in S$ and $X$ with the divisorial log structure defined by $D \subset X$. Then with $U = X \setminus Z$, we have a generically log smooth family $f : X \to S$.

If $T \to S$ is a morphism of Noetherian fine saturated log schemes, then the base change $f_T : X_T \to T$ as a generically log smooth family is defined in the obvious way: We take the fiber product $f_T : X_T \to T$ of underlying schemes, and we form the fiber product $U_T \to T$ in the category of *all* log schemes. This has the effect that the underlying scheme of the log fiber product $U_T$ is indeed the fiber product of underlying schemes, so $U_T \subset X_T$ is a Zariski open subset which satisfies (CC). The log scheme $U_T$ is fine and saturated because $f : U \to S$ is saturated.

*Remark* 2.6. If $f : U \to S$ is not saturated, the fiber product $U_T$ in the category of *all* log schemes might not be saturated. When we take the fiber product $U_T$ in the category of *fine saturated* log schemes instead, the map $U_T \to X_T$ might not be an open immersion, cf. [67, III, Cor. 2.1.6]. Thus we require $f : U \to S$ to be saturated.

It is well-known that on a Cohen–Macaulay scheme $X$, for a closed subset $Z \subset X$ of codimension $\geq 2$, every regular function $g \in \mathcal{O}_X(X \setminus Z)$ can be extended to $X$. The codimension condition (CC) is what we need for a relative version. This relative version is crucial for all structures that we will construct as a direct image from $U$ to be well-behaved.

**Lemma 2.7.** *Let $f : X \to S$ be a Cohen–Macaulay morphism of Noetherian schemes, and let $j : U \subset X$ satisfy* (CC). *Then $j_*\mathcal{O}_U \cong \mathcal{O}_X$.*

*Proof.* This is a special case of [41, 3.5]. Note that our (CC) is a stronger assumption than the condition on the codimension in [41, 3.5]. □

Let $f : X \to S$ be a generically log smooth family. Using the language of EGA in [35, Def. 5.9.9], we call a sheaf $\mathcal{F}$ *$Z$-closed* (resp. *$Z$-pure*) if the natural map $\mathcal{F} \to j_*(\mathcal{F}|_U)$ is an isomorphism (resp. injective). Most notably, two $Z$-closed sheaves that agree on $U$ are entirely equal. Furthermore, every reflexive sheaf is $Z$-closed.

*Remark* 2.8. Since $j_*\mathcal{O}_U = \mathcal{O}_X$, the direct image $j_*\mathcal{M}_U \to j_*\mathcal{O}_U = \mathcal{O}_X$ yields a log structure, which is compatible with $S$, so every generically log smooth family is canonically a log morphism $X \to S$. However, we do not know if this is independent of the choice of $U$, and if this is compatible with base change. Thus we consider the log structure only defined on $U$.

**Example 2.9.** Toric degenerations as defined in [30, 29] are generically log smooth families. Indeed, let $k$ be an algebraically closed field, let $R$ be a $k$-algebra which is a discrete valuation ring with residue field $k$ (e.g. $R = k[\![t]\!]$), and let $f : \mathcal{X} \to \mathcal{S} = \operatorname{Spec} R$ be a toric degeneration (with $\mathcal{X}$ a scheme). Since $\mathcal{X}$ is Gorenstein and $f$ is flat, $f$ is not only a Cohen–Macaulay, but even a Gorenstein morphism. Setting $U = \mathcal{X} \setminus Z$ the complement of the log singular locus $Z$, we find that the open $U$ satisfies (CC). We endow $\mathcal{X}$ with the divisorial log structure defined by $\mathcal{X}_0 \subset \mathcal{X}$ as in [30, Prop. 4.6]. Since the local models $f_{\bar{x}} : Y_{\bar{x}} \to \mathbb{A}^1$ for neighborhoods $U_{\bar{x}}$ of points $\bar{x} \in U$ are log smooth and saturated, so is $f : U \to \mathcal{S}$. There is a small technical issue in showing that the (divisorial) log structure on $U_{\bar{x}}$ is indeed the inverse image of the (divisorial) log structure on $Y_{\bar{x}}$. We expect that this can be settled by a careful analysis using Deligne–Faltings log structures as in [67, III, 1.7].

## 2.2 The de Rham Complex $W^\bullet_{X/S}$

This section follows [22, §2]. We define a de Rham complex $W^\bullet_{X/S}$ for every generically log smooth family $f : X \to S$ as explained in the Introduction. Since we assumed the log structure to be defined only on $U \subset X$, we cannot just take the usual log de Rham complex. Moreover, when we take the global log structure of Remark 2.8, its de Rham complex might be badly behaved:

**Example 2.10.** Consider the family $f : X_{st} \to \mathbb{A}^1$ of Example 1.4 with the divisorial log structure defined by $t = 0$ on source and target. It is a generically log smooth family. The sheaf $\Omega^1_{X_{st}/\mathbb{A}^1}$ of log differential forms is *not* a coherent sheaf at the origin, see [31, Ex. 1.11] or my Master Thesis. In particular the divisorial log structure on $X_{st}$ is not coherent.

Instead, we follow the philosophy of Zariski–Steenbrink–Danilov and define the de Rham complex as the direct image from $U$.

**Definition 2.11.** For a generically log smooth family $f : X \to S$, the *de Rham complex* is defined as $W^\bullet_{X/S} := j_*\Omega^\bullet_{U/S}$, where $\Omega^\bullet_{U/S}$ denotes the log de Rham complex of $f : U \to S$. The $\mathcal{O}_X$-module of degree *$m$ log polyvector fields* is $\Theta^m_{X/S} := j_* \bigwedge^m \mathcal{D}er_{U/S}(\mathcal{O}_U)$.

*Remark* 2.12. The use of $j_*\Omega^1_U$ as differential forms has a long history. Danilov uses them in [14] as differential forms on toric varieties, cf. Example 2.16 below. Steenbrink uses them for orbifolds in [71]. In [2] Ambro calls them Zariski–Steenbrink differentials, and in [8], Blickle calls $j_*\Omega^\bullet_U$ the Zariski–de Rham complex. It seems that the name *Zariski differentials* for $j_*\Omega^1_U$ was first introduced by Knighten in [57] in his study of differential forms on quotients of varieties. The name refers to Zariski's book [78] on algebraic surfaces, where the idea is more or less implicit. Reflexive differential forms are also intensely studied on normal varieties in the context of the Minimal Model Program, cf. [27, 26] and related work. The famous Lipman–Zariski conjecture, tracing back to [60], states that a normal complex space with locally free tangent sheaf $\Theta^1_X$ is smooth. This is of course equivalent to $j_*\Omega^1_U$ being locally free. An early reference studying reflexive sheaves (essentially for their own sake) is [40]. Here $Z$-closed sheaves are called *normal in the sense of Barth*, referring to [7]. Our inspiration to use reflexive sheaves comes from the Gross–Siebert program, where they are used e.g. in [31] to study toric log Calabi–Yau spaces and their deformations.

**Lemma 2.13.** *The $\mathcal{O}_X$-modules $W^m_{X/S}$ and $\Theta^m_{X/S}$ are coherent and reflexive, and they depend only on the equivalence class of $f : X \to S$.*

*Proof.* Let $\tilde{U} \subset U$ also satisfy (CC). Lemma 2.7 shows that $j_*\Omega^\bullet_{\tilde{U}/S} = j_*\Omega^\bullet_{U/S}$ since $\Omega^m_{U/S}$ is finite locally free, so $W^\bullet_{X/S}$ depends only on the equivalence class of $f$. It is clear that it is quasi-coherent. For any sheaf $\mathcal{G}$ on $U$, the direct image $j_*\mathcal{G}$ is $Z$-closed, so in particular $W^m_{X/S}$ is $Z$-closed. Writing $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ for the dual, Lemma 2.7 shows that $\mathcal{F}^\vee$ is $Z$-closed for all $\mathcal{F}$. In particular $(W^m_{X/S})^{\vee\vee}$ is a $Z$-closed sheaf and coincides with $W^m_{X/S}$ on $U$, hence $(W^m_{X/S})^{\vee\vee} = W^m_{X/S}$, and $W^m_{X/S}$ is reflexive. By the extension theorem [33, 9.4.8], there is a coherent sheaf $\mathcal{G}$ that restricts to $W^m_{X/S}$ on $U$. Now $\mathcal{G}^{\vee\vee} = W^m_{X/S}$ since both are $Z$-closed and agree on $U$; hence $W^m_{X/S}$ is also coherent. The argument for $\Theta^m_{X/S}$ is similar. ☐

**Lemma 2.14.** *We have $W^m_{X/S} = \mathcal{H}om(\Theta^m_{X/S}, \mathcal{O}_X)$ and $\Theta^m_{X/S} = \mathcal{H}om(W^m_{X/S}, \mathcal{O}_X)$.*

*Proof.* The statement is clear on $U$, where all sheaves are locally free. The statement follows since all sheaves are $Z$-closed. ☐

*Remark* 2.15. As we will see in Lemma 4.9, there is a dense open subset $U_{str} \subset U$ where $f : U \to S$ is strict. Thus, if $f$ has relative dimension $d$, then rk $W^1_{X/S} = d$. This shows that $W^1_{X/S}$ indeed has expected rank $d$. In full generality this fails, for there are non-strict log morphisms where the rank of $\Omega^1_{X/S}$ is *not* the relative dimension of underlying schemes, e.g. Spec $(\mathbb{N} \to \mathbb{C}) \to$ Spec $(0 \to \mathbb{C})$ (which is ideally log smooth once endowed with the appropriate sheaves of ideals).

**Example 2.16.** Toric pairs give generically log smooth families. Let $X$/Spec $R$ be a toric variety over a Noetherian base ring $R$. The fibers over points in Spec $R$ are normal (and Cohen–Macaulay), so there is a regular open $U \subset X$ whose complement has relative codimension $\geq 2$ over Spec $R$. Now $U \to$ Spec $R$ is log smooth and saturated for the trivial log structure on Spec $R$ and any divisorial log structure on $X$ coming from a toric divisor $D$ on $X$. Hence $X \to$ Spec $R$ is a generically log smooth family. The differential forms $W^\bullet_{X/S}$ coincide with what is called *reflexive* or *Danilov* or *Zariski–Steenbrink differentials* with log poles in $D$, i.e., they coincide with the complex

$$\Omega^\bullet_X(\log D)$$

which we considered in the Introduction. If $D$ is taken to be empty, then we recover the differential forms considered by Danilov in [14, §4].

## 2.3 The Log Canonical Bundle $W^d_{X/S}$

A Cohen–Macaulay morphism $f : X \to S$ admits a relative dualizing *sheaf* $\omega_{X/S}$. The morphism $f$ is Gorenstein if and only if $\omega_{X/S}$ is a line bundle (cf. [5, 0C08]), and we say it is Calabi–Yau if $\omega_{X/S} \cong \mathcal{O}_X$. Recall that for a smooth morphism $f : X \to S$ of relative dimension $d$ of schemes, the relative dualizing sheaf is $\omega_{X/S} \cong \Omega^d_{X/S}$. In analogy, we define:

**Definition 2.17.** Let $f : X \to S$ be generically log smooth of relative dimension $d$. Then $f : X \to S$ is

- *log Gorenstein* if $W^d_{X/S}$ is a line bundle.
- *log Calabi–Yau* if $W^d_{X/S} \cong \mathcal{O}_X$.

*Remark* 2.18. By Remark 2.15, the sheaf $W^d_{X/S}$ is reflexive of rank 1, so our definition makes sense.

The (classical) relative dualizing sheaf $\omega_{X/S}$ is closely related to $W^d_{X/S}$ as well. To make the relationship precise, let us recall the *horizontal locus* $H_{X/S} \subset X$, which was introduced by Tsuji in [75].

**Construction 2.19.** We construct the horizontal locus. For $\theta : Q \to P$ a homomorphism of monoids, a prime $\mathfrak{p} \subset P$ is *horizontal* with respect to $\theta$ if $\theta(Q) \subset P \setminus \mathfrak{p}$. Following [75, §2], this gives rise to an ideal

$$I_{P/Q} := \{p \in P \mid p \in \mathfrak{p} \text{ for all horizontal } \mathfrak{p} \subset P \text{ with } \mathrm{ht}(\mathfrak{p}) = 1\} \subset P,$$

and, for a log smooth morphism $f : X \to S$ of fs log schemes, to a sheaf of ideals $\mathcal{I}_f \subset \mathcal{M}_X$ defined by

$$\mathcal{I}_f(U) := \{m \in \mathcal{M}_X(U) \mid \forall \bar{x} \in U : m_{\bar{x}} \in I_{\mathcal{M}_{X,\bar{x}}/\mathcal{M}_{S,f(\bar{x})}}\} .$$

The induced ideal $\mathcal{J}_f := \mathcal{I}_f\mathcal{O}_X \subset \mathcal{O}_X$ is quasi-coherent and defines a closed subscheme $H_{X/S} \subset X$, which is flat over $S$. We call it the *horizontal locus*, or, in case it is a divisor, the *horizontal divisor* of $f : X \to S$. If $f : X \to S$ is *vertical*, i.e., the induced morphisms on the ghost stalks are vertical in the sense of [67, I, Def. 4.3.1], then $\mathcal{I}_f = \mathcal{M}_X$ and $\mathcal{J}_f = \mathcal{O}_X$, so $H_{X/S} = \varnothing$.

*Remark* 2.20. If $\pi : P \to \bar{P} = P/P^*$ is the projection, then $I_{P/Q} = \pi^{-1}(I_{\bar{P}/\bar{Q}})$. Thus, if $b : T \to S$ is *strict* and $g : Y \to T$ the fiber product, then $\mathcal{I}_g$ is generated by the image of $c^{-1}\mathcal{I}_f \to \mathcal{M}_Y$. In particular, the ideal $\mathcal{J}_g$ is generated by the image of $c^{-1}\mathcal{J}_f \to \mathcal{O}_Y$, and thus $H_{Y/T} = H_{X/S} \times_S T$. We do not know if this still holds for more general $b : T \to S$.

Given a generically log smooth family $f : X \to S$ of relative dimension $d$, we apply the above theory to the log smooth morphism $f : U \to S$. We then obtain an ideal sheaf $\mathcal{J}_f \subset \mathcal{O}_U$ and a closed subset $H_{X/S} \subset U$, which is flat over $S$. We define $\mathcal{J}_f W^m_{X/S} := j_*(\mathcal{J}_f \Omega^m_{U/S})$. The following result is also contained in our paper [22] and essentially goes back to Tsuji's work [75].

**Proposition 2.21.** *Let $f : X \to S$ be a generically log smooth family of relative dimension $d$ over the standard log point $S = \mathrm{Spec}\ (\mathbb{N} \to k)$ (where $1 \mapsto 0$). Assume $X$ is Gorenstein. Then $\omega_{X/S} \cong \mathcal{J}_f W^d_{X/S}$. In particular, if $f : U \to S$ is vertical, then $\omega_{X/S} \cong W^d_{X/S}$.*

*Proof.* The dualizing sheaf $\omega_{X/S}$ is locally free by the Gorenstein assumption. On $U$, the isomorphism is [75, Theorem 2.21, (ii)], and since both sheaves are $Z$-closed, the statement follows. □

**Corollary 2.22.** *Let $f : X \to S = \mathrm{Spec}\ (\mathbb{N} \to k)$ be generically log smooth and vertical. If $X$ is Gorenstein, then $f : X \to S$ is log Gorenstein, and if $X$ is Calabi–Yau, then $f : X \to S$ is log Calabi–Yau.*

*Proof.* Recall that $X$ Gorenstein means $\omega_{X/S}$ is a line bundle, and $X$ Calabi–Yau means $\omega_{X/S} \cong \mathcal{O}_X$. □

## 2.4 The Gerstenhaber Algebra $G^\bullet_{X/S}$

Gerstenhaber algebras carry a structure both of graded commutative and of graded Lie algebra. To our knowledge, they first appeared in Gerstenhaber's famous work [25] on the cohomology of an associative algebra. Here, we endow the graded commutative algebra of polyvector fields with a variant of the Schouten–Nijenhuis bracket and thus obtain a structure of Gerstenhaber algebra on it. For log smooth morphisms, this is also done in my paper [21], where we use the Gerstenhaber algebra of polyvector fields to study deformation theory. It follows ideas of Chan–Leung–Ma in [12], where an abstract deformation theory based on Gerstenhaber algebras is developed. Since this theory is also important for the smoothing application in our paper [22], we follow here the conventions of [12]. For a generically log smooth family $f : X \to S$, we set

$$G^\bullet_{X/S} \coloneqq j_* \bigwedge^{-\bullet} \Theta^1_{U/S}$$

concentrated in negative degrees $-d \le \bullet \le 0$, i.e., we have $G^{-m}_{X/S} = \Theta^m_{X/S}$. We follow the grading convention of Chan–Leung–Ma in [12]. We use the new symbol $G^\bullet_{X/S}$ in analogy with [12] and to distinguish it from $\Theta^\bullet_{X/S}$. The construction of the Gerstenhaber algebra structure should be viewed as a preliminary to construct the Batalin–Vilkovisky structure below in the log Calabi–Yau case. This in turn is important for the deformation-theoretic results of [12].

**Definition 2.23.** A *Gerstenhaber algebra* on $X$ is a graded abelian sheaf $\mathcal{G}^\bullet$ together with two bilinear operations

$$- \wedge - : \mathcal{G}^p \times \mathcal{G}^q \to \mathcal{G}^{p+q} \quad \text{and} \quad [-,-] : \mathcal{G}^p \times \mathcal{G}^q \to \mathcal{G}^{p+q+1}$$

satisfying the relations

- $x \wedge (y \wedge z) = (x \wedge y) \wedge z \quad \text{and} \quad x \wedge y = (-1)^{|x||y|}(y \wedge x)$,
- $[x, y \wedge z] = [x,y] \wedge z + (-1)^{(|x|+1)|y|} y \wedge [x,z]$,
- $[x,y] = -(-1)^{(|x|+1)(|y|+1)}[y,x]$,
- $[x,[y,z]] = [[x,y],z] + (-1)^{(|x|+1)(|y|+1)}[y,[x,z]]$,

where $|x|$ is the degree of a homogeneous element $x$. Moreover, $(\mathcal{G}^\bullet, \wedge)$ is unital, i.e., there is $1 \in \mathcal{G}^0$ such that $1 \wedge x = x$.

A *morphism of Gerstenhaber algebras* is a graded map $\phi^\bullet : \mathcal{G}^\bullet_1 \to \mathcal{G}^\bullet_2$ that is compatible with $\wedge$ and $[-,-]$ and such that $\phi^0(1) = 1$.

*Remark* 2.24. Often Gerstenhaber algebras are defined with $[-,-]$ having degree $-1$ instead of $+1$. In this case, the polyvector fields $\bigwedge^p T^1_X$ on a smooth manifold $X$ form a Gerstenhaber algebra with their natural grading. However, as explained above, we follow [12] for the grading, so we have functions in degree 0, vector fields in degree $-1$ etc.

*Remark* 2.25. Note that $(\mathcal{G}^\bullet, \wedge)$ is a graded commutative algebra and $(\mathcal{G}^\bullet[-1], [-,-])$ is a graded Lie algebra. By the *odd Poisson identity* (which is the second identity above) $[x,-]$ acts as a derivation of degree $|x|+1$.

*Remark* 2.26. The odd Poisson identity is symmetric. Indeed, by skew-symmetry it is equivalent to

$$[x \wedge y, z] = x \wedge [y,z] + (-1)^{(|z|+1)|y|}[x,z] \wedge y \ .$$

On differential forms and vector fields, we have the evaluation pairing

$$\langle -,- \rangle : W^1_{X/S} \times G^{-1}_{X/S} \to \mathcal{O}_X$$

induced by the homomorphism $h_\theta : \Omega^1_{X/S} \to \mathcal{O}_X$ associated to a derivation $\theta \in \Theta^1_{X/S}$, and the Lie bracket

$$[-,-] : G^{-1}_{X/S} \times G^{-1}_{X/S} \to G^{-1}_{X/S}$$

as e.g. defined in [67, V, Prop. 2.1.2]. It can be extended to a bracket on the whole of $G^\bullet_{X/S}$, namely the Schouten–Nijenhuis bracket.

**Lemma 2.27.** *Let $f : X \to S$ be a morphism of finite type of coherent Noetherian log schemes. Then there is a unique $f^{-1}(\mathcal{O}_S)$-bilinear bracket*

$$[-,-]_{sn} : \Theta^p_{X/S} \times \Theta^q_{X/S} \to \Theta^{p+q-1}_{X/S}$$

*called the* Schouten–Nijenhuis bracket *such that*

- $[x,y]_{sn} = -(-1)^{(|x|+1)(|y|+1)}[y,x]_{sn}$ *(skew-symmetry),*
- $[x, y \wedge z]_{sn} = [x,y]_{sn} \wedge z + (-1)^{(|x|+1)|y|} y \wedge [x,z]_{sn}$ *(odd Poisson identity),*
- $[g,h]_{sn} = 0$ *for* $g, h \in \mathcal{O}_X$,
- $[\theta, g]_{sn} = \langle dg, \theta\rangle$ *for* $\theta \in \Theta^1_{X/S}, g \in \mathcal{O}_X$,
- $[\theta, \xi]_{sn} = [\theta, \xi]$ *for* $\theta, \xi \in \Theta^1_{X/S}$ *(Lie bracket).*

*Proof.* For existence, we define a bracket via the formulas $[g,h]_{sn} = 0$ and

$$[\theta_0 \wedge \ldots \wedge \theta_n, h]_{sn} := \sum_{i=0}^{n} (-1)^{n-i} \langle dh, \theta_i\rangle \theta_0 \wedge \ldots \wedge \hat{\theta}_i \wedge \ldots \wedge \theta_n,$$

$$[\theta_0 \wedge \ldots \wedge \theta_n, \xi_0 \wedge \ldots \wedge \xi_m]_{sn} := \sum_{i=0}^{n}\sum_{j=0}^{m} (-1)^{i+j} [\theta_i, \xi_j] \wedge \theta_0 \wedge \ldots \wedge \hat{\theta}_i \wedge \ldots \wedge \theta_n \wedge \xi_0 \wedge \ldots \wedge \hat{\xi}_j \wedge \ldots \wedge \xi_m$$

for $g, h \in \mathcal{O}_X, \theta_i, \xi_i \in \Theta^1_{X/S}$. These formulas are taken from [61]. Uniqueness holds by the odd Poisson identity since $\Theta^p_{X/S}$ is generated by elements $\theta_1 \wedge \ldots \wedge \theta_p$ with $\theta_i \in \Theta^1_{X/S}$. □

*Remark* 2.28. The Schouten–Nijenhuis bracket satisfies the Jacobi identity

$$[x, [y,z]_{sn}]_{sn} = [[x,y]_{sn}, z]_{sn} + (-1)^{(|x|+1)(|y|+1)}[y, [x,z]_{sn}]_{sn};$$

this shows that $\Theta^\bullet_{X/S}[1]$ is a graded Lie algebra, cf. [62, Lemma VII.13].

The Schouten–Nijenhuis bracket $[-,-]_{sn}$ induces two Gerstenhaber algebra structures on $G^\bullet_{X/S}$:

**Definition 2.29.** On $G^\bullet_{X/S}$ we define a bracket $[-,-]_{sn} = j_*[-,-]_{sn}$ as the direct image of the Schouten–Nijenhuis bracket on $\Theta^\bullet_{U/S}$, and a second bracket $[-,-]_g := (-1)\cdot[-,-]_{sn}$. Both turn $G^\bullet_{X/S}$ into a Gerstenhaber algebra.

We consider $[-,-]_g$ the natural bracket for the Gerstenhaber algebra structure, again following [12].

## 2.5 The Batalin–Vilkovisky Module $W^\bullet_{X/S}$

The de Rham complex $W^\bullet_{X/S}$ forms some sort of module over the Gerstenhaber algebra $(G^\bullet_{X/S}, \wedge, [-,-]_g)$, a so-called *Batalin–Vilkovisky module*. Then the pair $(G^\bullet_{X/S}, W^\bullet_{X/S})$ of algebra and module is considered a *differential calculus* for the generically log smooth family $f : X \to S$, cf. [58] in the abstract context or any textbook on differential geometry for the original context of these structures. To construct the module structure essentially means to define a contraction of polyvector fields with differential forms. This contraction map also plays an important role in the deformation-theoretic results of [12]. More abstractly such a contraction is used in Iacono's abstract Bogomolov–Tian–Todorov theorem of [45] which can be used to prove homotopy abelianity of dglas. Beyond the scope of this thesis we expect this to be relevant for the deformation theory of log toroidal families.

Roughly following [58], but adjusting for the signs in [12], if $\mathcal{G}^\bullet$ is a Gerstenhaber algebra, then by a *Gerstenhaber module* we mean a $\mathbb{Z}$-graded sheaf $\mathcal{M}^\bullet$ of abelian groups with two bilinear maps

$$\lrcorner : \mathcal{G}^p \times \mathcal{M}^q \to \mathcal{M}^{p+q} \quad \text{and} \quad \mathcal{L}_-(-) : \mathcal{G}^p \times \mathcal{M}^q \to \mathcal{M}^{p+q-1}$$

such that the following identities hold:

- $1 \lrcorner m = m \quad \text{and} \quad (x \wedge y) \lrcorner m = x \lrcorner \; y \lrcorner m$,
- $\mathcal{L}_{[x,y]}(m) = \mathcal{L}_x(\mathcal{L}_y(m)) - (-1)^{(|x|+1)(|y|+1)}\mathcal{L}_y(\mathcal{L}_x(m))$,

- $x \lrcorner \mathcal{L}_y(m) = (-1)^{|y|+1}([x,y] \lrcorner m) + (-1)^{|x|(|y|+1)}\mathcal{L}_y(x \lrcorner m)$.

The last condition is called the *mixed Leibniz rule*. We call a Gerstenhaber module a *Batalin–Vilkovisky module* over the Gerstenhaber algebra $\mathcal{G}^\bullet$ if we have a linear differential $d : \mathcal{M}^p \to \mathcal{M}^{p+1}$ with $d^2 = 0$ and such that the so-called *Lie–Rinehart homotopy formula*

$$(-1)^{|x|}\mathcal{L}_x(m) = d(x \lrcorner m) - (-1)^{|x|}(x \lrcorner dm) \tag{1}$$

holds.

*Remark* 2.30. Our version of the Lie–Rinehart homotopy formula as well as the so-called mixed Leibniz identity in our definition of a Gerstenhaber module differ from [58] by signs.

*Remark* 2.31. If $(\mathcal{M}^\bullet, d)$ is a complex and $\lrcorner : \mathcal{G}^p \times \mathcal{M}^q \to \mathcal{M}^{p+q}$ a $(\mathcal{G}^\bullet, \wedge)$-module structure, then a candidate for $\mathcal{L}$ can be defined via the Lie–Rinehart formula (1). It turns $\mathcal{M}^\bullet$ into a Batalin–Vilkovisky module if and only if

$$\{\mathcal{L}_x, y \lrcorner\}(m) := \mathcal{L}_x(y \lrcorner m) - (-1)^{(|x|+1)|y|}(y \lrcorner \mathcal{L}_x(m)) = [x,y] \lrcorner m$$

holds. Indeed, this is equivalent to the mixed Leibniz identity, and the remaining identity follows since the graded commutator $\{-,-\}$ of operators is a Lie bracket, i.e., we find $\{d, \mathcal{L}_x\} = 0$ from (1) and thus $\mathcal{L}_{[x,y]} = \{\mathcal{L}_x, \mathcal{L}_y\}$ again using (1).

**Lemma 2.32.** *Let $f : X \to S$ be a morphism of finite type of coherent Noetherian log schemes. Then there is a unique family of homomorphisms of sheaves of abelian groups*

$$\lrcorner_k : \Theta^1_{X/S} \times \Omega^k_{X/S} \to \Omega^{k-1}_{X/S}$$

*satisfying $\theta \lrcorner \omega = \langle \omega, \theta \rangle$ for $\omega \in \Omega^1_{X/S}$, and*

$$\theta \lrcorner (\omega \wedge \eta) = (\theta \lrcorner \omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge (\theta \lrcorner \eta) \ .$$

*This map is called the* contraction map.

*Proof.* Uniqueness is obvious. For $\theta \in \Theta^1_{X/S}$, the map $\theta \lrcorner$ is induced by

$$(\Omega^1_{X/S})^{\otimes k} \to \Omega^{k-1}_{X/S}, \quad \omega_1 \otimes \ldots \otimes \omega_k \mapsto \sum_{i=1}^k (-1)^{i+1}\langle \omega_i, \theta \rangle \omega_1 \wedge \ldots \wedge \widehat{\omega_i} \wedge \ldots \wedge \omega_k,$$

which maps $\omega_1 \otimes \ldots \otimes \omega_k$ to 0 as soon as $\omega_i = \omega_j$ for $i \neq j$. ☐

*Remark* 2.33. The contraction $\lrcorner_k$ is $\mathcal{O}_X$-linear and satisfies the two relations

$$\begin{aligned} \theta \lrcorner (\theta' \lrcorner \omega) &= -[\theta' \lrcorner (\theta \lrcorner \omega)], \\ [\theta, \theta'] \lrcorner \omega &= [\theta \lrcorner d(\theta' \lrcorner \omega)] + [\theta \lrcorner (\theta' \lrcorner d\omega)] \\ &\quad - [d(\theta' \lrcorner \theta \lrcorner \omega)] - [\theta' \lrcorner d(\theta \lrcorner \omega)]. \end{aligned}$$

An elegant way to see that the second relation holds is by considering the right hand side a map $\mathcal{P}_{\theta,\theta'} : \Omega^\bullet_{X/S} \to \Omega^{\bullet-1}_{X/S}$. It satisfies the bilinearity assumption of the lemma, and for $\omega$ a closed 1-form, we easily see $\mathcal{P}_{\theta,\theta'}(\omega) = h_{[\theta,\theta']}(\omega)$. Because $\Omega^1_{X/S}$ is generated by closed forms, the uniqueness statement (applied to $[\theta, \theta']$) yields the relation.

Following e.g. [62], we can iteratively apply the contraction to obtain a contraction

$$\lrcorner : \Theta^q_{X/S} \times \Omega^p_{X/S} \to \Omega^{p-q}_{X/S}, \quad (\theta_q \wedge \ldots \wedge \theta_1) \lrcorner \omega = \theta_q \lrcorner \ldots \lrcorner \theta_1 \lrcorner \omega,$$

of polyvector fields for $q \leq p$. For $p = q$, it induces a pairing

$$\langle -, - \rangle : \Omega^p_{X/S} \times \Theta^p_{X/S} \to \mathcal{O}_X, \quad \langle \omega, \theta \rangle = \theta \lrcorner \omega,$$

given by the explicit formula

$$\langle \omega_p \wedge \ldots \wedge \omega_1, \theta_1 \wedge \ldots \wedge \theta_p \rangle = \sum_{\sigma \in S_p} (-1)^\sigma \prod_{i=1}^p \langle \omega_i, \theta_{\sigma(i)} \rangle;$$

the pairing is perfect if $\Omega^1_{X/S}$ is locally free.

*Remark* 2.34. Be aware of the indexing of inputs in $\langle -,- \rangle$. The pairing $\langle -,- \rangle$ does *not* correspond to plugging in $p$ vector fields in a $p$-form, but differs from that by a sign $(-1)^{\frac{(p-1)p}{2}}$.

**Definition 2.35.** Let $f : X \to S$ be a generically log smooth family. Then the *contraction*

$$\lrcorner \; : G^{-q}_{X/S} \times W^p_{X/S} \to W^{p-q}_{X/S}$$

is defined as the direct image $j_* \lrcorner$ of the contraction on $U$. For $\theta \in G^{-q}_{X/S}$, the *Lie derivative* is defined by the Lie–Rinehart homotopy formula

$$(-1)^{|\theta|}\mathcal{L}_\theta(\omega) = d(\theta \lrcorner \omega) - (-1)^{|\theta|}(\theta \lrcorner d\omega)$$

as an operator $\mathcal{L}_\theta : W^p_{X/S} \to W^{p-q+1}_{X/S}$.

We show now that $W^\bullet_{X/S}$ together with the contraction and Lie derivative is a Batalin–Vilkovisky module over $G^\bullet_{X/S}$. We have $1 \lrcorner \omega = \omega$ and $\theta \lrcorner \theta' \lrcorner \omega = (\theta \wedge \theta') \lrcorner \omega$ by definition, and moreover

$$\mathcal{L}_{\theta\wedge\theta'}(\omega) = (-1)^{|\theta'|}\mathcal{L}_\theta(\theta' \lrcorner \omega) + \theta \lrcorner \mathcal{L}_{\theta'}(\omega) \tag{2}$$

and $\{d, \mathcal{L}_\theta\} = 0$ by direct computation, where $\{-,-\}$ is the graded commutator of operators.

**Lemma 2.36.** *The identity* $\{\mathcal{L}_\theta, \theta' \lrcorner\}(\omega) = -[\theta,\theta']_{sn} \lrcorner \omega = [\theta,\theta']_g \lrcorner \omega$ *holds, where*

$$\{\mathcal{L}_\theta, \theta' \lrcorner\}(\omega) := \mathcal{L}_\theta(\theta' \lrcorner \omega) - (-1)^{(|\theta|+1)|\theta'|}(\theta' \lrcorner \mathcal{L}_\theta(\omega))$$

*is the graded commutator. In particular, the complex* $W^\bullet_{X/S}$ *is a Batalin–Vilkovisky module over* $G^\bullet_{X/S}$ *with bracket* $[-,-]_g$.

*Proof.* The identity is obvious for $|\theta| = |\theta'| = 0$. For $\theta \in G^{-1}_{X/S}$ and a function $f \in G^0_{X/S}$, we have

$$\{\mathcal{L}_\theta, f \lrcorner\}(\omega) = -df \wedge (\theta \lrcorner \omega) - (\theta \lrcorner (df \wedge \omega)) = -(\theta \lrcorner df) \wedge \omega$$

by Lemma 2.32, and similarly $\{\mathcal{L}_f, \theta \lrcorner\}(\omega) = (\theta \lrcorner df)\wedge\omega = -[f,\theta]_{sn} \lrcorner \omega$ since $\mathcal{L}_f(\omega) = df\wedge\omega$. For $\theta, \theta' \in G^{-1}_{X/S}$, we have

$$\{\mathcal{L}_\theta, \theta' \lrcorner\}(\omega) = -[\theta,\theta']_{sn} \lrcorner \omega$$

by Remark 2.33. Now, if the identity that we want to show holds for $\theta, \theta', \xi, \xi'$, then also

$$\begin{aligned}
[\theta\wedge\theta', \xi] \lrcorner \omega = {} & \theta \lrcorner \mathcal{L}_{\theta'}(\xi \lrcorner \omega) - (-1)^{|\xi|(|\theta'|+1)}(\theta \lrcorner \; \xi \lrcorner \mathcal{L}_{\theta'}(\omega)) \\
& + (-1)^{(|\xi|+1)|\theta'|}\mathcal{L}_\theta(\xi \lrcorner \; \theta' \lrcorner \omega) + (-1)^{|\xi|(|\theta|+|\theta'|+1)+|\theta'|}\xi \lrcorner \mathcal{L}_\theta(\theta' \lrcorner \omega) \\
= {} & \{\mathcal{L}_{\theta\wedge\theta'}, \xi \lrcorner\}(\omega)
\end{aligned}$$

by (2), and

$$\begin{aligned}
[\theta, \xi\wedge\xi'] \lrcorner \omega &= \mathcal{L}_\theta(\xi \lrcorner \xi' \lrcorner \omega) - (-1)^{(|\xi|+|\xi'|)(|\theta|+1)}(\xi \lrcorner \xi' \lrcorner \mathcal{L}_\theta(\omega)) \\
&= \{\mathcal{L}_\theta, \xi\wedge\xi' \lrcorner\}(\omega),
\end{aligned}$$

which prove the identity by induction since on $U$, the sheaf $G^{-p}_{X/S}$ is locally generated by sections that are products of vector fields. For the corollary, use Remark 2.31. $\square$

## 2.6 Log Calabi–Yau Families

In case $f : X \to S$ is log Calabi–Yau, the Gerstenhaber algebra $G^\bullet_{X/S}$ of polyvector fields can be enhanced to a *Batalin–Vilkovisky algebra.* These algebras form an important tool in deformation theory since they often allow to prove a deformation functor unobstructed, see e.g. [6, Lemma 2.1]. In particular, they are used in [12] to prove a logarithmic unobstructedness result. For us, a Batalin–Vilkovisky algebra is a Gerstenhaber algebra $\mathcal{G}^\bullet$ together with an operator $\Delta : \mathcal{G}^p \to \mathcal{G}^{p+1}$ such that $\Delta(1) = 0$ and $\Delta^2 = 0$, and such that

$$(-1)^{|x|}[x,y] = \Delta(x\wedge y) - \Delta(x)\wedge y - (-1)^{|x|}x\wedge\Delta(y)$$

holds.

*Remark* 2.37. Again, we follow [12]. In the terminology of [45], what we defined is a Batalin–Vilkovisky algebra of degree $-1$ (however, we do not have a differential, so it is not a *differential* Batalin–Vilkovisky algebra). Instead of introducing the Lie bracket on its own, one can equivalently require $\Delta$ to be a second order differential operator, i.e., it satisfies an explicit identity which can be found e.g. in [45], and then *define* the Lie bracket via the above formula.

If $\Omega \in W^d_{X/S}$ is a volume form, then we obtain an isomorphism

$$\eta = \eta_\Omega : G^{-p}_{X/S} \cong W^{d-p}_{X/S}, \quad \theta \mapsto (\theta \lrcorner \Omega),$$

which depends on $\Omega$.

**Definition 2.38.** For $f : X \to S$ a log Calabi–Yau generically log smooth family and $\Omega \in W^d_{X/S}$ a volume form, the *Batalin–Vilkovisky operator*

$$\Delta := \Delta_\Omega : G^{-p}_{X/S} \to G^{-p+1}_{X/S}$$

is given by $\Delta = \eta^{-1} \circ d \circ \eta$, where $d : W^{d-p}_{X/S} \to W^{d-p+1}_{X/S}$ is the de Rham differential.

*Remark* 2.39. In general, the operator $\Delta_\Omega$ depends on $\Omega$. Indeed, the formula

$$\langle \Omega, \xi \wedge \theta \rangle = \xi \lrcorner \theta \lrcorner \Omega = \langle \eta(\theta), \xi \rangle$$

implies $\Delta_{g\Omega}(\theta) = \Delta_\Omega(\theta) + g^{-1} \cdot \eta^{-1}(dg \wedge \eta_\Omega(\theta))$ for a function $g \in H^0(X, \mathcal{O}^*_X)$. In particular, if $f_*\mathcal{O}_X = \mathcal{O}_S$, then $\Delta$ is independent of $\Omega$.

*Remark* 2.40. The isomorphism $\eta$ gives an isomorphism

$$(\mathcal{G}^\bullet_{X/S}[-d], (-1)^d \Delta) \cong (W^\bullet_{X/S}, d)$$

of complexes. Here we need the factor $(-1)^d$ to account for the factor introduced by the shift functor.

Before we proceed, we express $\Delta$ at $\bar{x} \in U$ in an explicit basis. Let $\omega_1, ..., \omega_d \in (W^1_{X/S})_{\bar{x}}$ be a basis with $d\omega_i = 0$, and let $\theta_1, ..., \theta_d \in (\Theta^1_{X/S})_{\bar{x}}$ be the dual basis, i.e., $\langle \omega_i, \theta_j \rangle = \delta_{ij}$ for the perfect pairing. For $I = \{i_1, ..., i_p\}$ with $i_1 < ... < i_p$, we set $\omega_I = \omega_{i_1} \wedge ... \wedge \omega_{i_p}$ and $\theta_I = \theta_{i_1} \wedge ... \wedge \theta_{i_p}$ to denote the canonical induced bases of the exterior powers, which moreover satisfy

$$\langle \omega_I, \theta_J \rangle = (-1)^{\frac{(p-1)p}{2}} \delta_{IJ},$$

where $p = |I|$. There is a sign function $\varepsilon(I, J)$ depending on two disjoint indices defined by $\omega_I \wedge \omega_J = \varepsilon(I, J) \cdot \omega_{I \cup J}$, which satisfies the identity

$$\varepsilon(I, J) \cdot \varepsilon(I \cup J, K) = \varepsilon(I, J \cup K) \cdot \varepsilon(J, K).$$

For $\Phi \in \mathcal{O}_{X,\bar{x}}$ the function such that the volume form is $\Omega = \Phi \cdot \omega_1 \wedge ... \wedge \omega_d$, we have

$$\eta(\theta_I) = (-1)^{\frac{(p-1)p}{2}} \cdot \varepsilon(I, I^c) \cdot \Phi \cdot \omega_{I^c},$$

where $I^c$ is the complement of $I$ in $\{1, ..., d\}$ and $p = |I|$. The equation $\eta\Delta(g\theta_I) = d\eta(g\theta_I)$ is satisfied by

$$\Delta(g\theta_I) = (-1)^{p+1} \sum_{i \in I} \varepsilon(I \setminus \{i\}, \{i\}) \cdot \Phi^{-1} \cdot \langle d(g\Phi), \theta_i \rangle \cdot \theta_{I \setminus \{i\}},$$

which thus gives $\Delta(g\theta_I)$.

*Remark* 2.41. Our sign $(-1)^{\frac{(p-1)p}{2}} \varepsilon(I, I^c)$ in the formula for $\eta$ is precisely the sign given explicitly in [59, p. 11]. Moreover, since sorting $\theta_I \wedge \theta_{I^c}$ requires $\sum_{k=1}^p (i_k - i_{k-1} - 1)(p - k + 1)$ transpositions (once we set $i_0 = 0$), we have

$$\varepsilon(I, I^c) = (-1)^{(\sum_k i_k) - \frac{p(p+1)}{2}};$$

this shows that our sign is equal to $(-1)^{(\sum i_k) - p}$, the sign given explicitly in [44, 6.1].

**Lemma 2.42** (Tian–Todorov Lemma). *We have*

$$-(-1)^{|x|}[x,y]_{sn} = \Delta(x \wedge y) - \Delta(x) \wedge y - (-1)^{|x|} x \wedge \Delta(y).$$

*In particular, the algebra* $(\mathcal{G}^\bullet_{X/S}, \wedge, [-,-]_g, \Delta)$ *is a Batalin–Vilkovisky algebra.*

*Proof.* It suffices to show the identity at points $\bar{x} \in U$. Denoting the right hand side by $G(x,y)$, for functions $g, h \in \mathcal{O}_{X,\bar{x}}$ we have $G(g,h) = 0 = -[g,h]_{sn}$. Moreover,

$$G(g\theta_I, h) = (-1)^{p+1} g \sum_{i \in I} \varepsilon(I \smallsetminus \{i\}, \{i\}) \langle dh, \theta_i \rangle \theta_{I \smallsetminus \{i\}} = -(-1)^p [g\theta_I, h]_{sn},$$

where $p = |I|$, and for $I \cap J = \varnothing$, we have

$$\begin{aligned} G(g\theta_I, h\theta_J) &= (-1)^{p+q+1} \sum_{i \in I} \varepsilon(I \cup J \smallsetminus \{i\}, \{i\}) \cdot \varepsilon(I,J) \cdot \langle g dh, \theta_i \rangle \theta_{I \cup J \smallsetminus \{i\}} \\ &\quad + (-1)^{p+q+1} \sum_{j \in J} \varepsilon(I \cup J \smallsetminus \{j\}, \{j\}) \cdot \varepsilon(I,J) \cdot \langle h dg, \theta_j \rangle \theta_{I \cup J \smallsetminus \{j\}} \\ &= -(-1)^p [g\theta_I, h\theta_J]_{sn}, \end{aligned}$$

where $q = |J|$, and where we use $[\theta_i, \theta_j] = 0$, which follows from [67, V, Prop. 2.1.2] (though the bracket there in equation (2.1.2.) uses another sign convention). In case $I \cap J = \{m\}$, we find

$$\begin{aligned} &G(g\theta_I, h\theta_J) \\ &= -(-1)^{p+q+1} \cdot \varepsilon(J \smallsetminus \{m\}, \{m\}) \cdot \varepsilon(I, J \smallsetminus \{m\}) \cdot \langle g dh - h dg, \theta_m \rangle \cdot \theta_{I \cup J} \\ &= -(-1)^p [g\theta_I, h\theta_J]_{sn}, \end{aligned}$$

and for $|I \cap J| \geq 2$, we have $G(g\theta_I, h\theta_J) = 0$ as well as $[g\theta_I, h\theta_J]_{sn} = 0$. □

*Remark* 2.43. The most elegant approach would be to check the conditions of Lemma 2.27, but the odd Poisson identity is hard.

## 2.7 Analytification

Let $f : X \to S$ be a generically log smooth family with $S/\mathbb{C}$ of finite type. Then by the general GAGA results in [37, Exposé XII], the analytification $f^{an} : X^{an} \to S^{an}$ is a flat morphism of complex analytic spaces with Cohen–Macaulay fibers. According to [67, V, 1.1], we obtain also a log smooth and saturated morphism $f^{an} : U^{an} \to S^{an}$ of fs log analytic spaces. If $S = \operatorname{Spec} A$ for a local Artinian $\mathbb{C}$-algebra $A$, then $S = S^{an}$ as locally ringed spaces, and moreover $S$ is itself Cohen–Macaulay, so $X$ and $X^{an}$ are Cohen–Macaulay.

**Lemma 2.44.** *If $X$ is a Cohen–Macaulay complex space of pure dimension $d$ and $Z \subset X$ is a closed analytic subset of codimension $\geq 2$, then setting $U := X \smallsetminus Z$, we have $j_*\mathcal{O}_U = \mathcal{O}_X$.*

*Proof.* It is sufficient to prove $\mathcal{H}^i_Z(\mathcal{O}_X) = 0$ for $i = 0, 1$. When we write

$$S_m := S_m(\mathcal{O}_X) := \{x \in X \mid \operatorname{depth} \mathcal{O}_{X,x} \leq m\},$$

by [9, Thm. 3.6], it is sufficient to prove $\dim(Z \cap S_{k+2}) \leq k$ for all $k \geq 0$. This holds since depth $\mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x} = d$. □

*Remark* 2.45. We do not know if there is a relative version as in the algebraic case.

The analytic log de Rham complex $\Omega^\bullet_{U^{an}/S^{an}}$ on $U^{an}$ gives rise to the de Rham complex $W^{\bullet,an}_{X/S} := (j^{an})_* \Omega^\bullet_{U^{an}/S^{an}}$. The reflexive coherent analytic sheaf $(W^m_{X/S})^{an}$ is $Z^{an}$-closed by the lemma, so it is isomorphic to $W^{m,an}_{X/S}$, which is thus coherent. If $f : X \to S$ is proper, then there is an isomorphism

$$R^q(f^{an})_* W^{p,an}_{X/S} \cong R^q f_* W^p_{X/S}, \tag{3}$$

where we do not need to analytify the right hand side since $S = S^{an}$.

**Lemma 2.46.** *If $f : X \to S$ is proper, we have an isomorphism*

$$R^m(f^{an})_* W^{\bullet,an}_{X/S} \cong R^m f_* W^\bullet_{X/S}$$

*of hypercohomologies.*

*Proof.* Let $W^\bullet := W^{\bullet,Zar}_{X/S}$ be the de Rham complex in the Zariski topology, and let $W^\bullet \to L^{\bullet\bullet}$ be an injective Cartan–Eilenberg resolution. After writing $\varepsilon : X^{an} \to X$ for the comparison map, functoriality yields a map $f_*(L^{\bullet\bullet}) \to f_*\varepsilon_*\varepsilon^{-1}(L^{\bullet\bullet}) = f^{an}_*(\varepsilon^{-1}L^{\bullet\bullet})$ of double complexes. Because $\varepsilon^{-1}$ is exact, the discussion in [34, p. 33] yields, given an injective Cartan–Eilenberg resolution $W^{\bullet,an}_{X/S} \to L'^{\bullet\bullet}$, a map $\varepsilon^{-1}L^{\bullet\bullet} \to L'^{\bullet\bullet}$ of resolutions. In total, we obtain an induced map $f_*(L^{\bullet\bullet}) \to f^{an}_*(L'^{\bullet\bullet})$ of double complexes, which gives a morphism of associated (Hodge–de Rham) spectral sequences. On the $E_1$ page, this is the comparison map (3), which is an isomorphism. Thus, we obtain an isomorphism of abutments. The comparison between the étale and the Zariski hypercohomology is analogous. □

*Remark* 2.47. One cannot construct a comparison map $L^{\bullet\bullet} \to \varepsilon_* L'^{\bullet\bullet}$ directly for the latter might not be a Cartan–Eilenberg resolution (since $\varepsilon_*$ might not be exact).

# 3 Elementary Log Toroidal Families

In this section, we study elementary log toroidal families $f : A_{P,\mathcal{F}} \to A_Q$ constructed from elementary toroidal data $(Q \subset P, \mathcal{F})$. After specifying an appropriate open $U_{P/Q} \subset A_{P,\mathcal{F}}$, they are examples of generically log smooth families and serve as models for the singularities that we allow in a log toroidal family. Unlike in the introduction, here for a sharp toric monoid $Q$ we write $A_Q := \mathrm{Spec}\ (Q \to \mathbb{Z}[Q])$ for the associated log scheme, i.e., we are working over $\mathbb{Z}$. The notation $A_Q$ is taken from Ogus' book [66, 67]. Elementary log toroidal families are generalizations of the log morphism $A_P \to A_Q$ in the sense that we allow another log structure on Spec $\mathbb{Z}[P]$ which is specified by the set $\mathcal{F}$ of toric prime divisors. We denote Spec $\mathbb{Z}[P]$ with this log structure by $A_{P,\mathcal{F}}$.

*Remark* 3.1. Since this is important for the construction of $f : A_{P,\mathcal{F}} \to A_Q$, we briefly recall the geometry of Spec $\mathbb{Z}[P]$. For a face $F \subset P$, there is a closed embedding

$$V_F := \mathrm{Spec}\ \mathbb{Z}[F] \subset \mathrm{Spec}\ \mathbb{Z}[P]$$

defined by $z^p \mapsto 0$ for $p \notin F$. If $F \subset P$ is a facet, then $V_F$ is called a *toric divisor*, and the union of all toric divisors is denoted by $D_P$. For a face $F \subset P$, we also have the localization $P_F$ of $P$ in $F$, which equals the submonoid in $P^{gp}$ generated by $P$ and $-F$. It gives an open embedding

$$U_F := \mathrm{Spec}\ \mathbb{Z}[P_F] \subset \mathrm{Spec}\ \mathbb{Z}[P]$$

via the inclusion $P \subset P_F$ in $P^{gp}$. The two constructions are related by the set-theoretic formula

$$\mathrm{Spec}\ \mathbb{Z}[P] \setminus U_F = \bigcup_{F \notin K} V_K,$$

where the union is over faces $K$ such that $F \notin K$.

## 3.1 Definition and Basic Properties

**Definition 3.2.** An *elementary (log) toroidal datum* $(Q \subset P, \mathcal{F})$ (ETD for short) consists of a saturated injection $Q \to P$ of sharp toric monoids and a set $\mathcal{F}$ of facets of $P$ containing all facets that do not contain $Q$. Set

$$\mathcal{F}_{\min} := \underbrace{\{F \subset P \text{ a facet} \,|\, Q \notin F\}}_{\text{vertical facets}},$$

so $\mathcal{F}_{\min} \subset \mathcal{F} \subset \mathcal{F}_{\max}$, where $\mathcal{F}_{\max}$ is the set of all facets.

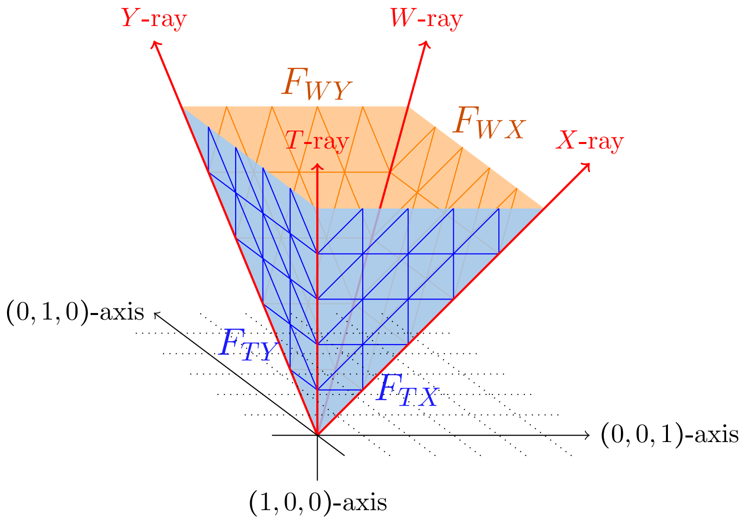


Figure 3.1: We see the four facets of the monoid $P$ in Example 3.6. We have the two faces $F_{TX} = \langle T, X \rangle, F_{TY} = \langle T, Y \rangle$ containing $Q$ in blue, and the two maximal essential faces $F_{WX} = \langle X, W \rangle, F_{WY} = \langle Y, W \rangle$ in orange. The elements of the facets are the lattice points on the intersection of the lines of the grid. We also see $E = F_{WX} \cup F_{WY}$ in orange. This picture is also contained in my Master Thesis.

*Remark* 3.3. The homomorphism $Q \to P$ being saturated is equivalent to $P$ being a free $Q$-set whose canonical basis is a union of faces of $P$, cf. [67, I, Cor. 4.6.11, Thm. 4.8.14, Cor. 1.4.3]. Below, we denote the union of the faces giving the basis by $E$.

We interpret an ETD $(Q \subset P, \mathcal{F})$ geometrically as the morphism

$$\underline{f} : \mathrm{Spec}\ \mathbb{Z}[P] \to \mathrm{Spec}\ \mathbb{Z}[Q]$$

of schemes. We have

$$\underline{f}^{-1}(D_Q) = \bigcup_{F \in \mathcal{F}_{\min}} V_F \subset \mathrm{Spec}\ \mathbb{Z}[P],$$

whereas for a facet $F \notin \mathcal{F}_{\min}$ the composition $V_F \subset \mathrm{Spec}\ \mathbb{Z}[P] \to \mathrm{Spec}\ \mathbb{Z}[Q]$ is flat and surjective. Thus we call the divisors $V_F$ *vertical* if $F \in \mathcal{F}_{\min}$, and we call them *horizontal* if $F \notin \mathcal{F}_{\min}$. Endowing $\mathrm{Spec}\ \mathbb{Z}[Q]$ with the divisorial log structure from $D_Q$ and $\mathrm{Spec}\ \mathbb{Z}[P]$ with the divisorial log structure from $\bigcup_{F \in \mathcal{F}} V_F$, we obtain a log morphism $f : A_{P,\mathcal{F}} \to A_Q$, which we call an *elementary log toroidal family.* We work here with Zariski log structures, which however coincide with the direct image of the corresponding étale log structures by [67, III, Prop. 1.6.5].

*Remark* 3.4. We denote the (reduced) union of horizontal divisors in $\mathcal{F}$ by

$$D^h_{\mathcal{F}} = \bigcup_{F \in \mathcal{F} \setminus \mathcal{F}_{\min}} V_F \subset \mathrm{Spec}\ \mathbb{Z}[P]\ .$$

We will see below in Lemma 3.21 that this gives indeed the horizontal locus (in the sense of Construction 2.19) of $f : A_{P,\mathcal{F}} \to A_Q$ on the log smooth locus $U_{P/Q}$ of $f$. Besides, note that $D^h_{\mathcal{F}}$ is defined by the ideal $\bigcap_{F \in \mathcal{F} \setminus \mathcal{F}_{\min}} \mathbb{Z}[P \setminus F]$.

**Example 3.5.** Let $P$ be a sharp toric monoid. Then $(0 \subset P, \mathcal{F}_{\max})$ is an ETD. Its elementary log toroidal family is the affine toric variety $A_P$ with its standard log structure given by the full toric boundary.

**Example 3.6.** Consider the map $\theta : \mathbb{N} \to P$ of Example 1.4 and let

$$\mathcal{F} = \{\langle X, W\rangle, \langle Y, W\rangle\}$$

where $\langle\cdot\rangle$ denotes the generated face. Then $\mathcal{F}$ equals $\mathcal{F}_{\min}$ and corresponds to the orange faces in Figure 3.1. We obtain an ETD $(Q \subset P, \mathcal{F})$ whose geometric realization is the family $f : X_{st} \to \mathbb{A}^1$ with the divisorial log structure defined by $t = 0$ on source and target.

**Example 3.7.** If $(Q \subset P, \mathcal{F})$ is an ETD and $r \geq 0$, then we obtain another ETD

$$(Q \times \{0\} \subset P \times \mathbb{N}^r, \mathcal{F}'),$$

where $\mathcal{F}' = \{F \times \mathbb{N}^r \,|\, F \in \mathcal{F}\}$. The canonical morphism $A_{P\times\mathbb{N}^r,\mathcal{F}'} \to A_{P,\mathcal{F}}$ is smooth (on underlying schemes) and strict on the log smooth locus $U_{P/Q} \times \mathbb{A}^r$ (defined below).

Following Remark 3.3 above, we denote the union of faces of $P$ that gives the generating set of the free $Q$-action by $E$. We call a face $F$ of $P$ contained in $E$ an *essential face*. Every $p \in P$ has a unique decomposition $p = e + q$ with $e \in E, q \in Q$, hence

$$E \times Q \to P, \quad (e, q) \mapsto e + q, \tag{4}$$

is bijective ([67, I, Thm. 4.8.14], cf. [49, Lemma 1.1]). Furthermore, $E = P \smallsetminus (Q^+ + P)$ where $Q^+ = Q \smallsetminus 0$ is the maximal ideal. Moreover, projecting $E$ to $P^{gp}/Q^{gp}$ is injective and its image $\bar{P} \subset P^{gp}/Q^{gp}$ is a monoid since it is also the image of $P$. Note that $\bar{P}^{gp} = P^{gp}/Q^{gp}$. A choice of splitting $P^{gp} \cong \bar{P}^{gp} \oplus Q^{gp}$ yields a unique map of sets $\varphi : \bar{P} \to Q^{gp}$ so that $\mathrm{id} \times \varphi : \bar{P} \to \bar{P} \oplus Q^{gp}$ is a section of the projection $P \to \bar{P}$ with the property that its image is $E$, so

$$P = \{(\bar{p}, q) \in \bar{P} \oplus Q^{gp} \,|\, \exists \tilde{q} \in Q : q = \varphi(\bar{p}) + \tilde{q}\}. \tag{5}$$

**Lemma 3.8.** *The morphism $\underline{f} : \operatorname{Spec} \mathbb{Z}[P] \to \operatorname{Spec} \mathbb{Z}[Q]$ induced by the injection $Q \subset P$ of monoids is a Cohen–Macaulay morphism of relative dimension $d = \operatorname{rk}(P^{gp}/Q^{gp})$.*

*Proof.* Since $P$ is free as a $Q$-set (generated by $E$), $\operatorname{Spec} \mathbb{Z}[P]$ is flat over $\operatorname{Spec} \mathbb{Z}[Q]$. By [35, Cor. 6.3.5] the total space of a faithfully flat morphism of Noetherian schemes is Cohen–Macaulay if and only if the base and all fibers are. By Hoechster's theorem (see [42]) the fibers of $\operatorname{Spec} \mathbb{Z}[P] \to \operatorname{Spec} \mathbb{Z}$ are Cohen–Macaulay, hence $\operatorname{Spec} \mathbb{Z}[P]$ and $\operatorname{Spec} \mathbb{Z}[Q]$ are Cohen–Macaulay. Now flatness of $\underline{f}$ implies that it is Cohen–Macaulay. □

The composition

$$A_P = A_{P,\mathcal{F}_{\max}} \to A_{P,\mathcal{F}} \xrightarrow{f} A_Q$$

is saturated since $Q \to P$ is saturated. Because $A_P \to A_{P,\mathcal{F}}$ is given by embedding one log structure as a sheaf of faces into another, it is exact, so by [67, I, Prop. 4.8.5(2)], the morphism $f$ is saturated.

*Remark* 3.9. Indeed, $A_P \to A_Q$ is integral resp. saturated if and only if $Q \to P$ is. This follows from the fact that the ghost stalks of $A_P, A_Q$ are quotients of $P, Q$ by faces.

We next want to define an open set $U = U_{P/Q}$ in the domain of $\underline{f}$ that satisfies (CC). Below, $f$ will turn out to be log smooth on $U$. We will actually define its complement, and for this we need a good understanding of the faces of $P$.

**Lemma 3.10.** *Let $F \subseteq P$ be a face. Set $\bar{F} := F \cap E$, $Q' := Q \cap F$, then*

$$F = \bar{F} + Q' := \{\bar{f} + q' | \bar{f} \in \bar{F}, q' \in Q'\}.$$

*Since $E$ is a union of faces of $P$, so is $\bar{F}$. Note also that $Q'$ is a face of $Q$.*

*Proof.* By the decomposition (4), any element in $F$ has the form $\bar{f} + q$ with $\bar{f} \in E, q \in Q$. Since $F$ is a face, $\bar{f}, q$ are both in $F$, hence $F \subseteq \bar{F} + Q'$. The reverse inclusion is clear. □

Consider the set of *bad faces* of $P$ defined as

$$\mathcal{B} = \left\{ \bar{F} + Q' \;\middle|\; \begin{array}{c} \bar{F} \text{ is a union of essential faces of rank at most } d-2 \\ Q' \text{ is a face of } Q,\ \bar{F}+Q' \text{ is a face of } P \end{array} \right\}.$$

As explained in Remark 3.1 there is a 1-1 correspondence between faces $F \subset P$ and torus orbit closures $V_F \subset \operatorname{Spec} \mathbb{Z}[P]$. Similarly, for $Q'$ a face of $Q$, we have a torus orbit closure $V_{Q'} \subset \operatorname{Spec} \mathbb{Z}[Q]$.

**Lemma 3.11.** *Given $\bar{F} + Q' \in \mathcal{B}$, we find that $V_{\bar{F}+Q'}$ is flat over $V_{Q'} \subset \operatorname{Spec} \mathbb{Z}[Q]$. Furthermore, if $X$ is a fiber of $\underline{f}$, then $\operatorname{codim}(X \cap V_{\bar{F}+Q'}, X) \geq 2$.*

*Proof.* Since $\bar{F} + Q'$ is free as a $Q'$-set, $\mathbb{Z}[\bar{F} + Q']$ is a free $\mathbb{Z}[Q']$-module, so the flatness statement follows. The origin 0 given by the prime ideal $(z^q | q \in Q^+)$ is contained in $V_{Q'}$, let $X_0$ be the fiber over it. It suffices to check the codimension condition for this particular fiber. But note that $X_0 \cap V_{\bar{F}+Q'} = \bigcup_{F \subset \bar{F}} V_F$, where the union runs over faces $F$ of $P$ contained in $\bar{F}$, and we have $\dim V_F \leq d-2$ by the assumption on $\bar{F}$. □

Set

$$U := U_{P/Q} := \operatorname{Spec} \mathbb{Z}[P] \setminus \left( \bigcup_{B \in \mathcal{B}} V_B \right). \tag{6}$$

As explained in Remark 3.1, for every face $F \subset P$, we have an open subset $U_F \subset \operatorname{Spec} \mathbb{Z}[P]$.

**Lemma 3.12.** *We find $U = \bigcup_F U_F$ where the union is over the essential faces $F$ of rank $d-1$.*

*Proof.* Since $U$ is a union of torus orbits, it suffices to check that any torus orbit contained in $U$ is contained in some $U_F$ for $F$ essential of rank $d-1$. Every torus orbit is given by $O_G := \operatorname{Spec} \mathbb{Z}[G^{gp}]$ for $G$ a face of $P$. Assume $O_G \subseteq U$. We use Lemma 3.10 to write $G = \bar{G} + Q'$. If $\operatorname{rk} \bar{G} \leq d-2$, then $G \in \mathcal{B}$, so $O_G \nsubseteq U$. Hence, $\dim \bar{G} \geq d-1$ and $\bar{G}$ contains some essential face $F$ of rank $d-1$. Then $F$ is also contained in $G$, and thus $O_G$ is contained in $U_F$. Conversely, since $O_F$ is not in any $V_B$, the assertion follows. □

**Lemma 3.13** (Theorem 3.5 in [50] or Theorem 4.1 in [48])**.** *If $\mathcal{F} = \mathcal{F}_{\max}$, then $f$ is log smooth.*

**Proposition 3.14.** *The map $f : A_{P,\mathcal{F}} \to A_Q$ is a generically log smooth family when we use $U = U_{P/Q}$ as the specified dense open of log smoothness.*

*Proof.* The assertion is clear if $d = 0$—which is equivalent to $P = Q$—, so assume $d > 0$. Given Lemma 3.8 and the saturatedness, it remains to verify that $U$ satisfies (CC) and that $f$ is log smooth on $U$. Note that Lemma 3.11 implies that $U$ satisfies (CC) since the complement of $U$ is the union of closed sets each of which has codimension at least two in each fiber.

To see that $f$ is log smooth on $U$, by Lemma 3.12, it suffices to check that $f$ is log smooth on $U_F$ for $F$ essential of rank $d-1$. Let $F$ be such a face. Set $\bar{P}_F := P_F / F^{gp}$ and note that the projection of $Q$ to $\bar{P}_F$ is injective because $F^{gp} \cap Q = \{0\}$. There is an isomorphism $P_F \cong F^{gp} \times \bar{P}_F$ commuting with the injection of $Q$ that is $\{0\} \times Q$ on the right.

The log structure on $U_F$ is a divisorial log structure given by a set of divisors each of which pulls back from $\operatorname{Spec} \mathbb{Z}[\bar{P}_F]$, so we may consider the corresponding divisorial log structure on $\operatorname{Spec} \mathbb{Z}[\bar{P}_F]$ to upgrade this to a log scheme $\bar{U}_F$. We have a factorization $U_F \to \bar{U}_F \to A_Q$ with the first map a smooth projection from a product, that is therefore strict, hence log smooth. It thus suffices to show that $\bar{U}_F \to A_Q$ is log smooth. Note that $\bar{U}_F \to A_Q$ is the log morphism of an ETD with $d = 1$. The following lemma finishes the proof. □

**Lemma 3.15.** *Assume that $f : A_{P,\mathcal{F}} \to A_Q$ has one-dimensional fibers, i.e., $d = 1$. Then $f$ is log smooth.*

*Proof.* We are done by Lemma 3.13 if $\mathcal{F} = \mathcal{F}_{\max}$, and this always holds if $Q$ meets the interior of $P$. So assume $Q$ is contained in a proper face of $P$, then by Lemma 3.10 it is in fact a facet of $P$, and then $\bar{P} = \mathbb{N}$, and consequently $P = \mathbb{N} \times Q$. A facet of $P$ that is not $Q$ is in $\mathcal{F}_{\min} = \{\mathbb{N} \times F \,|\, F \text{ is a facet of Q}\}$. Hence $\mathcal{F} \subsetneq \mathcal{F}_{\max}$ implies $\mathcal{F} = \mathcal{F}_{\min}$, and thus $f$ is strict. Since $\underline{f}$ is smooth, we find that $f$ is log smooth. □

*Remark* 3.16. The third situation of Figure 3.2 below is an example of this situation.

**Corollary 3.17.** *It is possible to find open subsets $U_1$ and $U_2$ so that $U_{P/Q} = U_1 \cup U_2$ and $A_P|_{U_1} = A_{P,\mathcal{F}}|_{U_1}$ and $f : U_2 \subset A_{P,\mathcal{F}} \to A_Q$ is strict and smooth.*

*Proof.* Let $\mathcal{E}_1$ be the set of essential faces of rank $d-1$ such that, when applying the proof of Lemma 3.15 to $\bar{U}_F \to A_Q$ from the proof of the proposition, we are in the case $\mathcal{F} = \mathcal{F}_{max}$, and let $\mathcal{E}_2$ be the set of faces where we are in case $\mathcal{F} = \mathcal{F}_{min}$. Then for $F \in \mathcal{E}_1$, we have $A_P|_{U_F} = A_{P,\mathcal{F}}|_{U_F}$, and for $F \in \mathcal{E}_2$, the morphism $U_F \to A_Q$ is strict and smooth. Now we define $U_1 = \bigcup_{F\in\mathcal{E}_1} U_F$ and $U_2 = \bigcup_{F\in\mathcal{E}_2} U_F$. □

*Remark* 3.18. If $F \subset P$ is a face, we can easily determine whether $A_P|_{U_F} = A_{P,\mathcal{F}}|_{U_F}$ or not: Equality holds if and only if for every facet $H \subset P$ with $F \subset H$, we have $H \in \mathcal{F}$. Indeed, the facets with $F \subset H$ correspond to the toric divisors intersecting $U_F$. Moreover, we need to test only facets with $Q \subset H$, too. In case $F$ is an essential face of rank $d-1$, the generated face $\langle F, Q\rangle$ is at least a facet by a dimension argument, so we need to test at most one facet: Thus $A_P|_{U_F} = A_{P,\mathcal{F}}|_{U_F}$ for $\langle F, Q\rangle = P$ or $\langle F, Q\rangle \in \mathcal{F}$, and $U_F \to A_Q$ is strict and smooth otherwise.

*Remark* 3.19. Every elementary log toroidal family $f : A_{P,\mathcal{F}} \to A_Q$ is *generically strict*, by which we mean that there is an open $U_{str} \subset U$, dense in every fiber, on which $f$ is strict. Indeed, for an essential face $F$ of rank $d$, i.e., a maximal essential face, we have $\langle F, Q\rangle = P$, so $A_{P,\mathcal{F}}|_{U_F} = A_P|_{U_F}$. Moreover, the map $U_F \to A_Q$ is the projection from the product $A_Q \times A_{P_F^{gp}}$, so it is strict and smooth. Using Remark 3.1 and the proof of Lemma 3.11, we find that the complement of $\bigcup_{F\in\mathcal{E}} U_F$ has dimension $\leq d-1$ in every fiber where the union is over the essential faces of rank $d$.

*Remark* 3.20. Let $R$ be a regular ring. Then base change along $\mathrm{Spec}\, R \to \mathrm{Spec}\, \mathbb{Z}$ gives a generically log smooth family $A_{P,\mathcal{F}} \times R \to A_Q \times R$. Since $R$ is regular, $A_Q \times R$ is log regular and hence carries a compactifying log structure by [51, Thm. 11.6]. We see that it is the divisorial log structure induced by $D_Q \times R$. Similarly, $U_{P/Q} \times R$ is log regular, so it carries the divisorial log structure induced by $\bigcup_{F\in\mathcal{F}} \mathrm{Spec}\, R[F]$. This means that the construction of $f : A_{P,\mathcal{F}} \to A_Q$ can be carried out with any regular domain $R$ instead of $\mathbb{Z}$, yielding a generically log smooth family $A_{P,\mathcal{F}} \times R \to A_Q \times R$. We do not know (and do not care) if the two log structures (from base change and from the direct divisor construction) also coincide outside $U_{P/Q} \times R$. This holds in the relatively log smooth case studied in Section 5 since both the base change to $R$ and the log structure defined by divisors over $R$ are relatively coherent with the same relative chart.

### The Horizontal Locus

We use the decomposition $U = U_1 \cup U_2$ of Corollary 3.17 to determine the horizontal locus $H_{P,\mathcal{F}} \subset U$ of $f : U \to S$ in the sense of Construction 2.19.

**Lemma 3.21.** *The horizontal locus $H_{P,\mathcal{F}} \subset U_{P/Q}$ is*

$$H_{P,\mathcal{F}} = U_{P/Q} \cap \bigcup_{F\in\mathcal{F}\setminus\mathcal{F}_{\min}} \mathrm{Spec}\, \mathbb{Z}[F] = U_{P/Q} \cap D^h_{\mathcal{F}} \,.$$

*Proof.* Any horizontal prime $\mathfrak{p} \subset P$ is of the form $\mathfrak{p} = P \setminus F$ for a facet $F \subset P$ with $Q \subset F$ (and vice versa), so for $\mathcal{F} = \mathcal{F}_{\max}$, Tsuji's result [74, Cor. 2.6] gives that $H := H_{P,\mathcal{F}_{\max}}$ is defined by the ideal $\bigcap_{F\in\mathcal{F}_{max}\setminus\mathcal{F}_{\min}} \mathbb{Z}[P \setminus F]$. More generally, we have $H_{P,\mathcal{F}} \cap U_1 = H \cap U_1$ and $H_{P,\mathcal{F}} \cap U_2 = \varnothing$ (because $U_2 \to A_Q$ is strict, hence vertical) as an equality of defining ideals. Since also $D^h_{\mathcal{F}} \cap U_1 = U_1 \cap H$ and $D^h_{\mathcal{F}} \cap U_2 = \varnothing$, the claim follows. □

*Remark* 3.22. The morphism $A_P \to A_Q$ is vertical if and only if $Q \to P$ is vertical. Thus for $\mathcal{F} = \mathcal{F}_{\max}$, the map $f : A_{P,\mathcal{F}} \to A_Q$ is vertical if and only if $Q$ is not contained in any proper face. Furthermore, the morphism $A_P \to A_Q$ is vertical on $U_G$ for an essential face $G$ of rank $d-1$ if and only if $\langle G, Q\rangle = P$. Thus in view of Remark 3.18, the family $f : A_{P,\mathcal{F}} \to A_Q$ is vertical (on $U_{P/Q}$) if and only if for all essential faces $G$ of rank $d-1$, we have $\langle G, Q\rangle \notin \mathcal{F}$. In particular, for $\mathcal{F} = \mathcal{F}_{\min}$, the family $f : A_{P,\mathcal{F}} \to A_Q$ is vertical. This is of course what we expect from a notion of verticality since $A_{P,\mathcal{F}_{\min}}$ has the divisorial log structure given by $f^{-1}(D_Q)$. Conversely, if $f : A_{P,\mathcal{F}} \to A_Q$ is vertical on $U_{P/Q}$, then $\mathcal{F} = \mathcal{F}_{\min}$. Indeed, if $H \in \mathcal{F} \setminus \mathcal{F}_{\min}$, then $H \cap E = \bigcup_i G_i$ decomposes into essential faces $G_i$ of rank $d-1$, and $\langle G_i, Q\rangle = H$ for each of them.

### The Base Change of Elementary Log Toroidal Families

If $(Q \subset P, \mathcal{F})$ is an ETD and $\beta : Q \to Q'$ a homomorphism to a sharp toric monoid, then the fiber product

$$g : A_{P,\mathcal{F}} \times_{A_Q} A_{Q'} \to A_{Q'}$$

is a generically log smooth family. If it is the elementary log toroidal family of *some* ETD $(Q' \subset P', \mathcal{F}')$, then $P' = Q' \oplus_Q P$ is the pushout of monoids.

**Example 3.23.** For $(Q \to P) = (\mathbb{N} \to \mathbb{N}^2,\ 1 \mapsto (1,1))$ and $\beta : \mathbb{N} \to 0$, we have $P' = \mathbb{Z}$. Thus, in general, $P'$ is *not* sharp.

Therefore, in general, $g$ is *not* the elementary log toroidal family of any ETD. Nevertheless, we can get very closely: The pushout $P' := Q' \oplus_Q P$ is a fine and saturated monoid since $Q \to P$ is saturated. In view of (5), writing $\varphi' : \bar{P} \to Q^{gp} \to Q'^{gp}$, we make an explicit ansatz

$$P' = \{(\bar{p}, q') \in \bar{P} \oplus Q'^{gp} \mid \exists \tilde{q} \in Q' : q' = \varphi'(\bar{p}) + \tilde{q}\}.$$

This ansatz has the universal property of the pushout. In particular, the monoid $P'$ is toric, and $Q' \to P'$ is injective. We find a canonical basis $E'$ of $P'$ as a $Q'$-set, which is in bijection with $E$ via the canonical map $\gamma : P \to P'$. Moreover, mapping $F' \mapsto \gamma^{-1}(F')$ is a bijection between the facets $F' \subset P'$ containing $Q'$ and the facets $F \subset P$ containing $Q$. Indeed, via projection both are in bijection with the facets of $\bar{P} = \mathrm{im}(P \to P^{gp}/Q^{gp})$. We define a set of facets

$$\mathcal{F}' := \{F' \subset P' \text{ a facet} \mid Q' \notin F' \text{ or } \gamma^{-1}(F') \in \mathcal{F}\}$$

of $P'$ and a divisor $D' = \bigcup_{F' \in \mathcal{F}'} \mathrm{Spec}\ \mathbb{Z}[F']$. This gives rise to a log morphism which we denote $f' : A_{P',\mathcal{F}'} \to A_{Q'}$ by abuse of notation. In view of Lemma 3.12, we define an open $U' = \bigcup U_{G'} \subset A_{P',\mathcal{F}'}$, where the union is over the essential rank $d-1$ faces $G' \subset E'$. We decompose

$$U' = U'_1 \cup U'_2,$$

where, in analogy with Corollary 3.17, $U'_1 = \bigcup_{F'} U_{F'}$ is the union over those essential faces $F'$ of rank $d-1$ which have the property that, for every facet $F' \subset H'$, we have $H' \in \mathcal{F}'$, and where $U'_2$ is the union over the remaining faces.

*Remark* 3.24. We get a log morphism $c : A_{P',\mathcal{F}'} \to A_{P,\mathcal{F}}$. Indeed, for $D = \bigcup_{F \in \mathcal{F}} \mathrm{Spec}\ \mathbb{Z}[F]$ and $D'_{\mathrm{vert}} = f'^{-1}(D_{Q'})$ the union of the vertical divisors, we get $D' = c^{-1}(D) \cup D'_{\mathrm{vert}}$ as an equality of subsets, so $c^{-1}(D) \subset D'$.

**Proposition 3.25.** *We have $U' = c^{-1}(U)$, and with these opens,*

$$\begin{tikzcd} A_{P',\mathcal{F}'} \arrow[r, "c"] \arrow[d, "f'"] & A_{P,\mathcal{F}} \arrow[d, "f"] \\ A_{Q'} \arrow[r, "b"] & A_Q \end{tikzcd}$$

*is a Cartesian diagram of generically log smooth families. There is an ETD $(Q' \subset P'', \mathcal{F}'')$ and a strict open immersion $k : A_{P',\mathcal{F}'} \subset A_{P'',\mathcal{F}''}$ compatible with the morphism to the base such that $U' = k^{-1}(U_{P''/Q'})$.*

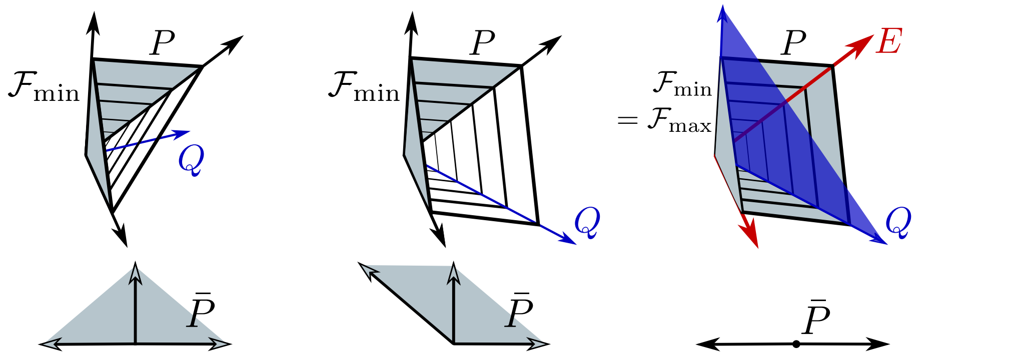


Figure 3.2: Three examples of a saturated injection $Q \subset P$ and the projection $\bar{P}$, the outer two are log smooth, the middle one gives Example 2.10. This picture is also included in [22].

*Proof.* We choose a splitting of $P' \to P'/P'^*$ over $Q'$, which then induces an isomorphism $P' \cong P'/P'^* \oplus P'^*$, and a submonoid $\mathbb{N}^r \subset P'^*$ such that $(\mathbb{N}^r)^{gp} = P'^*$. Then the monoid $P'' := P'/P'^* \oplus \mathbb{N}^r \subset P'$ and the collection

$$\mathcal{F}'' := \{F'/P'^* \oplus \mathbb{N}^r | F' \in \mathcal{F}'\}$$

of faces form an ETD $(Q' \subset P'', \mathcal{F}'')$; the strict open immersion $k$ is given by the localization in the face $L := 0 \oplus \mathbb{N}^r \subset P''$. Intersection with $P'' \subset P'$ induces a bijection between the facets of $P'$ and the facets of $P''$ containing $L$, under which $\mathcal{F}'$ corresponds to $\mathcal{F}''$.

We find $k^{-1}(U_{P''/Q'}) = U'$. Indeed, for an essential face $G \subset E''$ of rank $d-1$ with $L \subset G$, the localization $L^{-1}G \subset E'$ is an essential face of rank $d-1$ in $P'$, and every such face is of this form. Now $k^{-1}(U_G) = U_{L^{-1}G}$, and for $G$ with $L \notin G$, we can find an essential face $L \subset \tilde{G} \subset E''$ of rank $d-1$ with $k^{-1}(U_G) \subset k^{-1}(U_{\tilde{G}})$. Next, applying the criterion of Remark 3.18 to essential faces $G \supset L$, we find that $A_{P',\mathcal{F}'}|_{U_1'} = A_{P'}|_{U_1'}$ and that $U_2' \to A_{Q'}$ is strict and smooth.

Turning to the morphism $c$, if $F'$ is a rank $d-1$ essential face, then, in view of Lemma 3.10, $\gamma^{-1}(F') = \bar{F} + \ker(\beta)$ and $\bar{F}$ is a union of rank $d-1$ essential faces of $P$. If $G$ is any one of them, then $c^{-1}(U_G) = U_{F'}$. We say $G$ is of *first type*, and we write $\tilde{U} = \bigcup_G U_G$, where the union is over faces of first type. In the other event, if $F$ is an essential face of $P$ of rank $d-1$ and $\gamma(F) \subset E'$ is *not* contained in a rank $d-1$ essential face of $P'$, then it hits the interior of a rank $d$ essential face $G'$ of $P'$, and then for any facet $F'$ of $G'$, $c^{-1}(U_F) \subset U_{F'}$. This shows $U' = c^{-1}(\tilde{U}) = c^{-1}(U)$. Looking only at faces of first type, we decompose $\tilde{U} = \tilde{U}_1 \cup \tilde{U}_2$ according to the criterion of Remark 3.18 and find $U_1' = c^{-1}(\tilde{U}_1)$ and $U_2' = c^{-1}(\tilde{U}_2)$. Because the diagram is Cartesian in the category of log schemes both on $\tilde{U}_1$ and on $\tilde{U}_2$, it is a Cartesian diagram of generically log smooth families. □

*Remark* 3.26. We have $c^{-1}(D^h_{\mathcal{F}}) = k^{-1}(D^h_{\mathcal{F}''})$ as closed subschemes. After denoting the horizontal locus of $f' : A_{P',\mathcal{F}'} \to A_{Q'}$ by $H'$, this means $H' = A_{Q'} \times_{A_Q} H_{P,\mathcal{F}}$.

## 3.2 Examples

Figure 3.2 shows some examples of saturated injections $Q \subset P$. Below, we give some more concrete examples of ETDs.

**Example 3.27.** The saturated injection $\mathbb{N} \to \mathbb{N}^2, 1 \mapsto (1,1)$, satisfies $\mathcal{F}_{\min} = \mathcal{F}_{\max}$ and gives an ETD $(\mathbb{N} \subset \mathbb{N}^2, \mathcal{F})$. Geometrically, this is

$$f : \operatorname{Spec} \mathbb{Z}[x,y] \to \operatorname{Spec} \mathbb{Z}[t], \;\; f^*t = xy$$

with the divisorial log structure defined by $\{t = 0\}$ respective $\{xy = 0\}$. Applying the construction from Example 3.7 with $r = 1$ to it, we obtain a log smooth and saturated morphism

$$f : (\mathbb{A}^3, \{xy = 0\}) \to (\mathbb{A}^1, \{0\}),$$

which is associated to the ETD with homomorphism $\mathbb{N} \to \mathbb{N}^3, 1 \mapsto (1,1,0)$, and set of facets $\mathcal{F}_{\min} = \{\mathbb{N} \times 0 \times \mathbb{N}, 0 \times \mathbb{N} \times \mathbb{N}\}$.

**Non-Example 3.28.** The map $\mathbb{N} \to \mathbb{N}^r, 1 \mapsto (a_1, ..., a_r)$, is saturated if and only if $0 \le a_i \le 1$ for all $i$. Thus, if $a_i \ge 2$ for some $i$, then $(\mathbb{N} \subset \mathbb{N}^r, \mathcal{F})$ is *not* an ETD. Indeed, e.g.

$$f : \mathbb{A}^2 \to \mathbb{A}^1, \quad (x,y) \mapsto x^2 y,$$

has a non-reduced fiber $f^{-1}(0) \cong \operatorname{Spec} k[x,y]/(x^2 y)$, which is not possible for an elementary log toroidal family.

**Example 3.29.** Generalizing Example 3.6, the family $\{tw - z_1 \cdot ... \cdot z_k = 0\} \to \mathbb{A}^1_t$ has the structure of an elementary log toroidal family. Let $k \ge 1$ and let $P^k_{pnc} \subset \mathbb{N}^{k+1}$ be the submonoid spanned by

$$T = e_0, \quad W = (k-1) \cdot e_0 + e_1 + ... + e_k, \quad Z_i = e_0 + e_i \quad (1 \le i \le k),$$

where $e_0, ..., e_k$ is the standard basis of $\mathbb{N}^{k+1}$. Since

$$P^k_{pnc} = \left\{ p \in \mathbb{Z}^{k+1} \mid \forall 1 \le i \le k : \langle e_i, p \rangle \ge 0, \left\langle e_0 - \sum_{j \ne 0,i} e_j, p \right\rangle \ge 0 \right\},$$

it is a sharp toric monoid. The homomorphism $\theta : \mathbb{N} \to P^k_{pnc}, 1 \mapsto T$, is locally exact by the exactness criterion [67, I, Prop. 2.1.16(5)]. Because the complement of the ideal $J = T + P^k_{pnc}$ is

$$E^k_{pnc} = \bigcup_{i=1}^k \left\{ p \in P^k_{pnc} \mid \left\langle e_0 - \sum_{j \ne 0,i} e_j, p \right\rangle = 0 \right\} =: \bigcup_{i=1}^k F_i^{ver},$$

it is a radical ideal, and $\theta : \mathbb{N} \to P^k_{pnc}$ is saturated by [67, I, Thm. 4.8.14(5)]. Every facet of $P^k_{pnc}$ is $h^{-1}(0)$ for a function $h : P^k_{pnc} \to \mathbb{N}$ that evaluates 0 on at least $k$ elements of the set $\{T, W, Z_1, ..., Z_k\}$, so the facets of $P^k_{pnc}$ are $F_i^{ver}$ defined by the formula above and $F_i^{hor} = \{p \in P^k_{pnc} \mid \langle e_i, p \rangle = 0\}$. Thus, setting

$$\mathcal{F}_{\min} := \{F_1^{ver}, ..., F_k^{ver}\},$$

we obtain an ETD $(\mathbb{N} \subset P^k_{pnc}, \mathcal{F}_{\min})$. Geometrically, we have a log morphism

$$X^k_{pnc} = \operatorname{Spec} \mathbb{Z}[t, w, z_1, ..., z_k]/(tw - z_1 \cdot ... \cdot z_k) \to \mathbb{A}^1,$$

where both schemes carry the divisorial log structure defined by $\{t = 0\}$. The subscript 'pnc' is short for 'pencil of normal crossing divisors', a name by which we emphasize the role of $P^k_{pnc}$ in showing that certain pencils of normal crossing divisors are log toroidal families, see Example 4.30 below.

**Example 3.30** (ETDs of Gross–Siebert type)**.** As explained in the Introduction, the local models that are used by Gross–Siebert to control the singularities of toric log Calabi–Yau spaces have been a major inspiration for elementary log toroidal families. In fact, the constructions in [31, Constr. 2.1] of the local models yield ETDs. For a lattice $M'$ with dual lattice $N'$ and a convex lattice polytope $\tau \subset M'_{\mathbb{R}}$ with $\dim \tau = \dim M'_{\mathbb{R}}$, we get a cone

$$C'(\tau) := \{(rm, r) \mid r \ge 0, m \in \tau\} \subset M'_{\mathbb{R}} \oplus \mathbb{R}$$

and a (dual) sharp toric monoid $P' := C'(\tau)^\vee \cap (N' \oplus \mathbb{Z})$. The projection $M' \oplus \mathbb{Z} \to \mathbb{Z}$ induces an element $\rho' \in P'$, which gives a saturated injection $\mathbb{N} \to P', 1 \to \rho'$, and thus for $q \ge 0$ a saturated injection $\mathbb{N} \to P' \oplus \mathbb{N}^q$. We turn it into an ETD by choosing $\mathcal{F} = \mathcal{F}_{\min}$. With $\Delta_0 := \tau$ and $q$ (Newton) polytopes $\Delta_1, ..., \Delta_q \subset M'_{\mathbb{R}}$, we define functions

$$\check{\psi}_i(n) := -\inf\{\langle n, m\rangle \mid m \in \Delta_i\}$$

on $N'$ for $0 \le i \le q$. They give a sharp toric monoid

$$P := \{n + \sum_{i=0}^{q} a_i e_i^* \mid \forall 0 \le i \le q : a_i \ge \check{\psi}_i(n)\} \subset N' \oplus \mathbb{Z}^{q+1},$$

where $e_0^*, ..., e_q^* \in \mathbb{Z}^{q+1}$ is the standard basis, and the saturated injection is $\mathbb{N} \to P, 1 \mapsto e_0^*$. We turn it into an ETD by choosing $\mathcal{F} = \mathcal{F}_{\min}$. If we set $\Delta_1 = ... = \Delta_q = \{0\}$, then $\check{\psi}_1 = ... = \check{\psi}_q = 0$, and we recover $P = P' \oplus \mathbb{N}^q$ from above. We say an ETD $(\mathbb{N} \subset P, \mathcal{F}_{\min})$ that arises from this construction is of *Gross–Siebert type*. Indeed, all local models that are relevant to [31] have this form. Setting

$$K := \langle \bigcup_{i=0}^{q} \Delta_i \times \{e_i\}\rangle \subset M'_{\mathbb{R}} \oplus \mathbb{R}^{q+1}$$

the cone generated by the set, we have $P = K^\vee \cap (N' \oplus \mathbb{Z}^{q+1})$ and Spec $\mathbb{Z}[P]$ is Gorenstein.

# 4 Log Toroidal Families

A *log toroidal family* $f : X \to S$ is a generically log smooth family whose singularities in $z \in Z$ are controlled by elementary log toroidal families. Whereas in the context of the Gross–Siebert program, the local models happen to control a log singularity that arises from another construction—see [31, Thm. 2.6]—we turn this property into a definition. This is analogous to the situation of toroidal varieties: A variety is *toroidal* if it is étale locally isomorphic to a toric variety. As explained in the Introduction, controlling the local structure of the singularities in $z \in Z$ provides powerful tools for a log toroidal family that we do not have for a generically log smooth family. Namely, to study the global cohomology of $W^\bullet_{X/S}$, it is insufficient to know only what happens on $U \subset X$. We will see this in the forthcoming sections when we study e.g. differential forms and the Cartier isomorphism.

## 4.1 Definition and Basic Properties

We give our precise definition of a log toroidal family. It is slightly technical and intricate. We start with the definition of a local model.

**Definition 4.1** (Local Models)**.** Let $f : X \to S$ be a generically log smooth family. For a geometric point $\bar{s} \in S$, a *base chart* at $\bar{s}$ is a strict étale morphism $(\tilde{S}, \bar{s}) \to (S, \bar{s})$ and a map $a : \tilde{S} \to A_Q$ given by a chart $Q \to \mathcal{M}_{\tilde{S}}$ of the log structure which is neat at $\bar{s}$. For a geometric point $\bar{x} \in X$ and a base chart at $f(\bar{x})$, a *local model* at $\bar{x}$ is a diagram

$$\begin{tikzcd}
 & (V, g^{-1}(U)) \arrow[dl, "g"'] \arrow[dd] \arrow[dr, dotted, "h"] & & \\
(X, U) \arrow[dd, "f"'] & & (L, U_L) \arrow[dl] \arrow[dr, "c"] & \\
 & \tilde{S} \arrow[dl] \arrow[dr, "a"] & & (A_{P,\mathcal{F}}, U_{P/Q}) \arrow[dl] \\
S & & A_Q, &
\end{tikzcd} \tag{LM}$$

where $g : V \to X$ is an étale neighborhood of $\bar{x}$ (of underlying schemes) and the bottom right diagonal map is given by an ETD $(Q \subset P, \mathcal{F})$. The solid arrows are morphisms of schemes and log morphisms on the specified opens, whereas $h : V \to L$ is an étale morphism only of underlying schemes. The bottom right diamond is Cartesian, in particular $U_L = c^{-1}(U_{P/Q})$. Moreover, we have an open $\tilde{U} \subset V$ satisfying (CC), such that $\tilde{U} \subset g^{-1}(U) \cap h^{-1}(U_L)$ and there is an isomorphism $g^*\mathcal{M}_X \cong h^*\mathcal{M}_L$ of the two log structures on $\tilde{U}$ such that the composed maps to $\tilde{S}$ coincide. Finally, we have $c \circ h(\bar{x}) = 0 \in A_{P,\mathcal{F}}$.

For a base chart $S \leftarrow \tilde{S} \to A_Q$ and a Zariski open $W \subset X$, a *local model* is a diagram (LM) as above with $g(V) = W$ (and no requirement on some $\bar{x} \in X$).

**Definition 4.2** (Log Toroidal)**.** A *log toroidal family* is a generically log smooth family $f : X \to S$ such that there is a Zariski cover $X = \bigcup_i W_i$ and local models for the $W_i$ (over various base charts $S \leftarrow \tilde{S}_i \to A_{Q_i}$).

If $S \cong \mathrm{Spec}\ (Q \to B)$, then $\tilde{S} = S$ and $a : S \to A_Q$ given by the chart $Q \to B$ is a base chart. We say $f : X \to S$ is *log toroidal with respect to* $a : S \to A_Q$, if we can choose the local models over this base chart.

**Example 4.3.** Every elementary log toroidal family $f : A_{P,\mathcal{F}} \to A_Q$ is log toroidal.

**Example 4.4.** Toric varieties $X$ are log toroidal families over a trivial base. Indeed, let $\underline{X}$ be a toric variety over $S = \mathrm{Spec}\ \mathbb{Z}$ and $D \subset X$ a reduced toric divisor. Let $X$ be the log space $\underline{X}$ with divisorial log structure defined by $D$ and let $S$ have the trivial log structure, then $X \to S$ is a log toroidal family since it is locally in the Zariski topology Example 4.3 with $Q = 0$.

We give much more examples below in Section 4.4. We recommend to browse through the examples before reading on. In particular, we will prove that saturated log smooth morphisms are log toroidal families. This is essentially a consequence of Kato's toroidal characterization of log smoothness.

*Remark* 4.5. We use $\tilde{U}$ since we do not care about the precise chosen log smooth locus $U \subset X$. In this way, e.g. $f : A_P \to A_Q$ with $U = A_P$ instead of $U = U_{P/Q}$ can be considered a log toroidal family. However, this definition means we cannot easily understand the local structure of constructions that do depend on the precise $U$ just by considerations in the local models. This applies in particular to the horizontal locus $H_{X/S} \subset U$ as well as to the deformation theory of log toroidal families, which we do not consider in this thesis.

*Remark* 4.6. There are three natural options to define a log toroidal family: First, we could require a local model *at* every point $\bar{x} \in X$, i.e., we require $(c \circ h)(\bar{x}) = 0 \in A_{P,\mathcal{F}}$. Secondly, we could require finitely many points $\bar{x}_i \in X$ and local models at them such that $X = \bigcup_i g_i(V_i)$. For the definition above we decided to use the third option employing local models for Zariski opens $W \subset X$. This gives us (compared to the second option) more flexibility since we do not need to give a point mapping to $0 \in A_{P,\mathcal{F}}$. With our definition, the family $f : A_{P,\mathcal{F}} \setminus \{0\} \to A_Q$ is obviously log toroidal, and this flexibility is also crucial for the proof of Proposition 4.7 below. However, we prove in Proposition 4.15 that, over an algebraically closed base field $k$, there is no difference between the three options. I.e., in this situation, we actually have local models controlling the local geometry around every point (which is the first option).

### Base Change

We study under which conditions the notion of log toroidal family is stable under base change. If $f : X \to S$ is a log toroidal family, and $b : S' \to S$ is *strict*, then the fiber product $f' : X' \to S'$ of generically log smooth families is easily seen to be a log toroidal family: The map $\tilde{T} = T \times_S \tilde{S} \to A_Q$ is a base chart, and the local models carry over to $f' : X' \to S'$. If $b : S' \to S$ is *not* strict, the situation is more involved:

**Proposition 4.7.** *Let $f : X \to S$ be a log toroidal family, and let $b : S' \to S$ be a morphism from a Noetherian fs log scheme $S'$. Assume that for every geometric point $\bar{s}' \in S'$, the order of the torsion subgroup of $\mathcal{M}^{gp}_{S'/S,\bar{s}'}$ is invertible in $k(\bar{s}')$. Then the fiber product $f' : X' = X \times_S S' \to S'$ is a log toroidal family.*

*Proof.* The family $f' : X' \to S'$ is generically log smooth with $U' = U \times_S S'$. For a geometric point $\bar{s}' \in S'$, there is some base chart on $S$ with $\bar{s}' \to \tilde{S}_i$, so we can find some étale neighborhood $\tilde{S}' \to S'$ of $\bar{s}'$ fitting into a chart for $S' \to S$, i.e., into a commutative diagram

$$\begin{tikzcd} \tilde{S}' \arrow[r] \arrow[d] & \tilde{S}_i \arrow[d] \\ A_{Q'} \arrow[r] & A_{Q_i}, \end{tikzcd}$$

where $\tilde{S}' \to A_{Q'}$ is an exact chart at $\bar{s}'$. Because the order of the torsion subgroup of $\mathcal{M}^{gp}_{S'/S,\bar{s}'}$ is invertible in $k(\bar{s}')$, we can assume that $Q' \to \mathcal{M}_{\tilde{S}'}$ is neat at $\bar{s}'$ by [67, III, Thm. 1.2.7]. In particular, $Q'$ is a sharp toric monoid, and we obtain local models via base change along $A_{Q'} \to A_{Q_i}$ using Proposition 3.25. □

*Remark* 4.8. The order of the torsion subgroup of $\mathcal{M}^{gp}_{S'/S,\bar{s}'}$ being invertible in $k(\bar{s}')$ is a very technical condition. We need it here to ensure the existence of a chart for $S' \to S$ which is neat—not only exact—at $\bar{s}'$. We expect that a stronger version of Proposition 3.25 would render the assumption superfluous. However, the condition is virtually always satisfied, e.g. when $S' \to S$ is strict or (more generally) saturated or if $S/\mathbb{Q}$.

### Generic Strictness

As we have seen in Remark 3.19, every elementary log toroidal family is generically strict. This easily generalizes to log toroidal families and even generically log smooth families.

**Lemma 4.9.** *Let $f : X \to S$ be a log toroidal family. Then there is an open $U_{str} \subset U$ dense in every fiber on which $f$ is strict. The same holds for generically log smooth families.*

*Proof.* In a local model (LM), denote the strict locus of Remark 3.19 by $W \subset U_{P/Q}$, and take the union over all $g((c \circ h)^{-1}(W) \cap \tilde{U}) \subset X$ for all local models. For generically log smooth families, note that the saturated log smooth morphism $f : U \to S$ is a log toroidal family by Example 4.25 below. □

**Corollary 4.10.** *Let $f : X \to S$ be a generically log smooth family. Then the forgetful map $\Theta^1_{X/S} \to \Theta^1_{\underline{X}/\underline{S}}$ forgetting the log part of the derivation is injective.*

*Proof.* Since $\mathcal{O}_X \to j_*\mathcal{O}_{U_{str}}$ is injective, this follows from [31, Prop. 1.3]. □

**Corollary 4.11.** *Let $f : X \to S$ be a generically log smooth family of relative dimension $d$. Then* rk $\Omega^1_{U/S} = d$.

*Remark* 4.12. In general, if $f : X \to S$ is a flat finite type morphism of Noetherian schemes with reduced fibers and $j : U \to X$ is an open subset which is dense in every fiber, then $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective. For lack of reference, we briefly indicate the proof: It suffices to assume $S = \mathrm{Spec}\ R$ and $X = \mathrm{Spec}\ A$ affine. First, let $R = k$ be a field. Then prime avoidance implies that there is a non-zero divisor $a \in A$ such that $X_a := \{a \neq 0\} \subset U$ is a principal dense open subset. In particular, the localization map $A \to A_a$ is injective; hence, the restriction $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective as well. Next, let $R$ be a local Artinian ring with residue field $R/\mathfrak{m} = k$. We find a non-zero divisor $a_0 \in A/\mathfrak{m}A$, which defines a principal dense open $(X \times_R k)_{a_0} \subset U \times_R k$, and lift it to $a \in A$. This $a$ is a non-zero divisor as well; namely, the base change from $R$ to $k$ of the multiplication map $\mu_a : A \to A$ is injective and has flat cokernel, so this holds for $\mu_A$ as well. Thus, the localization map $A \to A_a$ is injective. From here, we easily generalize to complete local Noetherian, local Noetherian, and finally arbitrary Noetherian bases $S = \mathrm{Spec}\ R$.

### Vertical Families

We study under which conditions a log toroidal family is vertical (in the sense of [67]). Verticality is closely related to the condition $\mathcal{F} = \mathcal{F}_{\min}$. For a saturated log smooth morphism $f : X \to S$ and a point $\bar{x} \in X$ at which $f$ is not vertical, consider a local model at $\bar{x}$ as in

Example 4.25 with ETD $(Q \subset P, \mathcal{F})$. Since $f$ is not vertical at $\bar{x}$, we have $\mathcal{F} \neq \mathcal{F}_{\min}$, so $H_{P,\mathcal{F}} \subset A_{P,\mathcal{F}}$ is a non-empty divisor flat over $A_Q$. In particular, the horizontal locus

$$H_{V/\tilde{S}} = h^{-1}(c^{-1}(H_{P,\mathcal{F}})) \subset V$$

is pure of codimension 1 in every fiber and non-empty, and the same holds for $H_{X/S} \subset X$. If $U \subset X$ satisfies (CC), then $f : U \to S$ is *not* vertical, for $H_{X/S} \cap U \neq \varnothing$. This shows that a generically log smooth family $f : X \to S$ is vertical (on $U$) if and only if it is vertical on some $\tilde{U} \subset U$ satisfying (CC).

**Lemma 4.13.** *A log toroidal family $f : X \to S$ is vertical if there is a Zariski covering $X = \bigcup_i W_i$ and a local model for every $W_i$ with $\mathcal{F} = \mathcal{F}_{\min}$. Conversely, if $f : X \to S$ is vertical, then for every local model at* some *$\bar{x} \in X$ (i.e., such that $0 \in \mathrm{Im}(c \circ h)$), we have $\mathcal{F} = \mathcal{F}_{\min}$.*

*Proof.* For an ETD $(Q \subset P, \mathcal{F}_{\min})$, the map $A_{P,\mathcal{F}_{\min}} \to A_Q$ is vertical on $U_{P/Q}$, so if we have such a local model for $W_i$, then $f : U \to S$ is vertical on $g(\tilde{U}) \subset W_i$. Since their union satisfies (CC), we find $f : U \to S$ vertical. Conversely, for a local model at $\bar{x}$ with $\mathcal{F} \neq \mathcal{F}_{\min}$, the horizontal locus $H_{P,\mathcal{F}}$ is a non-empty divisor. In particular, we have $h^{-1}(c^{-1}(H_{P,\mathcal{F}}))$ a non-empty divisor, contradicting $H_{U/S} = \varnothing$ by $f : U \to S$ being vertical. $\square$

*Remark* 4.14. A vertical log toroidal family can have local models not satisfying $\mathcal{F} = \mathcal{F}_{\min}$ if there is no point $\bar{x} \in X$ mapping to $0 \in A_{P,\mathcal{F}}$. Consider e.g. $(0 \subset \mathbb{N}, \{\mathbb{N}\})$ and $X = A_{\mathbb{N}} \setminus \{0\}$, which is log-trivial—hence vertical—over the log-trivial point.

## 4.2 Local Models at Points

As we have seen in Remark 4.6, a priori there is a difference between local models for open subsets $W \subset X$ and local models at points $\bar{x}$. The first one is easier to check, the second one is more convenient to study the local structure of log toroidal families. In this section, we show that, given a log toroidal family $f : X \to S$ over an algebraically closed field $k$ (which has local models for opens $W_i \subset X$ by definition), there exist local models at points $\bar{x} \in X$. This is important for the relative degeneration in Theorem 8.2 since it depends on a local computation that we carry out on $0 \in A_{P,\mathcal{F}}$ of a local model. The key ideas of the result seem to be standard. In particular, the situation is similar for toroidal varieties—Danilov defines them in [14] via a local isomorphism to $0 \in \mathrm{Spec}\ \mathbb{C}[P]$, and [14, Rem. 13.2] just says that toric varieties are toroidal.

**Proposition 4.15.** *Let $k$ be an algebraically closed field, let $S/k$, let $f : X \to S$ be a log toroidal family, and let $\bar{x} \in X$ be a $k$-valued point. Then there is a base chart $S \leftarrow \tilde{S} \to A_Q$ at $f(\bar{x})$ and a local model at $\bar{x}$.*

*Proof.* The claim follows from a local construction: Let $(Q \subset P, \mathcal{F})$ be an ETD, and let $\bar{x} \in A_{P,\mathcal{F}} \times k$ be a $k$-valued point with defining ideal $\mathfrak{p} \subset k[P]$. We construct a diagram starting on the left with the base change along $\mathrm{Spec}\ k \to \mathrm{Spec}\ \mathbb{Z}$ of $f : A_{P,\mathcal{F}} \to A_Q$:

$$\begin{tikzcd}
A_{P,\mathcal{F}} \times k \arrow[d, "f"] & U_F \times k \arrow[l, hook'] \arrow[r, "\gamma"] \arrow[d] \arrow[rrd, dashed] \arrow[dr, phantom, "(S)"] & A_{\bar{P},\bar{\mathcal{F}}} \times \mathbb{G}_m^r \times k \arrow[r, hook, "i"] \arrow[d, "A_\sigma"] \arrow[dr, "\phi"] & A_{\bar{P}\times\mathbb{N}^r,\bar{\mathcal{F}}\times\mathbb{N}^r} \times k \arrow[d] \arrow[dl, dashed] \\
A_Q \times k & U_G \times k \arrow[l, hook'] \arrow[r, "\beta"] & A_{\bar{Q}} \times k & A_{\bar{P},\bar{\mathcal{F}}} \times k \arrow[l]
\end{tikzcd}$$

For the face $F := \{p \in P \mid z^p \notin \mathfrak{p}\}$, we find $\bar{x} \in U_F \times k = \mathrm{Spec}\ k[P_F] \subset A_{P,\mathcal{F}} \times k$, and setting $G := F \cap Q \subset Q$, we have $f(U_F \times k) \subset U_G \times k = \mathrm{Spec}\ k[Q_G]$. The natural projections $Q_G \to \bar{Q} := Q_G/G^{gp}$ and $P_F \to \bar{P} := P_F/F^{gp}$ admit splittings $\sigma' : \bar{Q} \to Q_G$ and $\sigma : \bar{P} \to P_F$ which fit into a commutative diagram with the maps $Q_G \to P_F$ and $\bar{Q} \to \bar{P}$. Indeed, the quotient $\bar{P}^{gp}/\bar{Q}^{gp}$ is torsion free, so $\bar{P}^{gp} \cong \bar{Q}^{gp} \oplus \bar{P}^{gp}/\bar{Q}^{gp}$. Thus we can choose a splitting $\bar{Q}^{gp} \to Q_G^{gp}$ and a compatible splitting $\bar{P}^{gp} \to P_F^{gp}$. Since $Q_G \to \bar{Q}$ and $P_F \to \bar{P}$ are exact, we obtain $\sigma', \sigma$ by restriction. Since $Q_G \cong G^{gp} \oplus \bar{Q}$, the map $\beta : U_G \to A_{\bar{Q}}$ induced by $\sigma' : \bar{Q} \to Q_G$ is strict and smooth. With

$$\bar{\mathcal{F}} := \{K/F^{gp} \mid K \in \mathcal{F}, F \subset K\},$$

we obtain an ETD $(\bar{Q} \subset \bar{P}, \bar{\mathcal{F}})$ and thus a morphism $A_{\bar{P},\bar{\mathcal{F}}} \times k \to A_{\bar{Q}} \times k$. Because $P_F \cong F^{gp} \oplus \bar{P}$, the map $A_\sigma : U_F \to A_{\bar{P},\bar{\mathcal{F}}}$ is smooth (on underlying schemes)—indeed, it is the projection $U_F \cong \mathrm{Spec}\ \mathbb{Z}[\bar{P}] \times \mathrm{Spec}\ \mathbb{Z}[F^{gp}] \to \mathrm{Spec}\ \mathbb{Z}[\bar{P}]$. Moreover, since

$$A_\sigma^{-1}(\bar{D}) := A_\sigma^{-1}\Big(\bigcup_{K \in \bar{\mathcal{F}}} \mathrm{Spec}\ \mathbb{Z}[K]\Big) = U_F \cap \left(\bigcup_{K \in \mathcal{F}} \mathrm{Spec}\ \mathbb{Z}[K]\right),$$

the map $A_\sigma$ is a log morphism. It is strict on $A_\sigma^{-1}(U_{\bar{P}/\bar{Q}})$ since there, the inverse image log structure from $A_{\bar{P},\bar{\mathcal{F}}}$ is log regular, hence divisorial, and they have the same log-trivial locus. Choosing a splitting $F^{gp}/G^{gp} \to F^{gp}$ of the projection, we obtain a factorization

$$U_F \xrightarrow{\gamma} A_{\bar{P},\bar{\mathcal{F}}} \times \mathbb{G}_m^r \xrightarrow{\phi} A_{\bar{P},\bar{\mathcal{F}}}$$

with $r = \mathrm{rk}(F^{gp}/G^{gp})$, where we endow $A_{\bar{P},\bar{\mathcal{F}}} \times \mathbb{G}_m^r$ with the divisorial log structure defined by $\phi^{-1}(\bar{D})$. With the opens $\phi^{-1}(U_{\bar{P}/\bar{Q}})$ and $A_\sigma^{-1}(U_{\bar{P}/\bar{Q}})$, the square $(S)$ in the middle of the above diagram becomes a Cartesian square of generically log smooth families.

By definition of $F$, we have $A_\sigma(\bar{x}) = 0 \in A_{\bar{P},\bar{\mathcal{F}}}$. Let $\bar{y} \in \mathbb{G}_m^r \times k \subset \mathbb{A}^r \times k$ be the image of $\gamma(\bar{x}) \in A_{\bar{P},\bar{\mathcal{F}}} \times \mathbb{G}_m^r \times k$ under the projection to $\mathbb{G}_m^r$. Because $k$ is algebraically closed, there is a (unique) translation morphism $\lambda : \mathbb{A}^r \times k \to \mathbb{A}^r \times k$ with $\lambda(\bar{y}) = 0$, which gives rise to an open immersion $i : A_{\bar{P},\bar{\mathcal{F}}} \times \mathbb{G}_m^r \times k \to A_{\bar{P},\bar{\mathcal{F}}} \times \mathbb{A}^r \times k$ with $i(\gamma(\bar{x})) = 0$. The latter space is $A_{\bar{P}\times\mathbb{N}^r,\bar{\mathcal{F}}\times\mathbb{N}^r} \times k$, where we use the construction of Example 3.7. ☐

**Corollary 4.16.** *Let $k$ be an algebraically closed field, let $A$ be an Artinian $k[\![Q]\!]$-algebra, and let $S = \mathrm{Spec}\ (Q \to A)$. Let $f : X \to S$ be a log toroidal family with respect to $S \to A_Q$, and let $\bar{x} \in X$ be a $k$-valued point. Then there is a local model at $\bar{x}$ with base chart $S \to A_Q$.*

*Proof.* We keep the notation from the above proof. Since $S$ has only one (set-theoretic) point, we find $G = 0$, so $\bar{Q} = Q$. Thus the construction does not change the base chart. ☐

*Remark* 4.17. For a log toroidal family $f : X \to S$ with respect to $S \to A_Q$, where $S$ is *not* a punctual scheme, there is no analog of the Corollary—some points might not map to $0 \in A_Q$, so we need to change the base chart.

## 4.3 Spreading Out

To spread out a $\mathbb{Q}$-variety $X$ means to find a finitely generated subring $B \subset \mathbb{Q}$ and a $B$-scheme $X_B$ such that $X \cong X_B \times_B \mathbb{Q}$. The scheme $X_B$ can be considered an integral model of $X$. This allows to reduce problems about $X$ to positive characteristic; it is proven for schemes in EGA, see [36]. For log smooth morphisms, spreading out was done by Tsuji in [74]. Here we prove the analogous result for log toroidal families, another step toward the Degeneration Theorem 8.1.

We fix a sharp toric monoid $Q$, a field $k \supset \mathbb{Q}$ and set $S = \mathrm{Spec}\ (Q \to k)$ where the map $Q \to k$ is $q \mapsto 0$ except $0 \mapsto 1$. We choose distinct subrings $B_\lambda \subseteq k$ for all $\lambda$ in some index set $\Lambda$ so that any two $B_{\lambda_1}, B_{\lambda_2}$ are both contained in a third $B_\lambda$. We say $\lambda_1 \leq \lambda_2$ if $B_{\lambda_1} \subseteq B_{\lambda_2}$. Furthermore, we require $\varinjlim_\lambda B_\lambda = k$ and that each $B_\lambda$ is of finite type over $\mathbb{Z}$. We get log schemes $S_\lambda = \mathrm{Spec}\ (Q \to B_\lambda)$ each with a strict map $S \to S_\lambda$, and in fact we have $S = \varprojlim_\lambda S_\lambda$.

**Proposition 4.18.** *Let $f : X \to S$ be a log toroidal family of relative dimension $d$. Then there is $\lambda \in \Lambda$ and a log toroidal family $f_\lambda : X_\lambda \to S_\lambda$ so that $f$ is obtained by base change from $f_\lambda$, i.e., there is a Cartesian square*

$$\begin{array}{ccc} X & \longrightarrow & X_\lambda \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_\lambda} \\ S & \longrightarrow & S_\lambda \end{array}$$

*of generically log smooth families. If $f$ is separated and/or proper, we can assume $f_\lambda$ to be so, too.*

*Proof.* By classical spreading out—see [36, Thm. 8.8.2 (ii)], [36, Thm. 8.10.5] and [36, Thm. 11.2.6 (ii)]—we can find a $\lambda \in \Lambda$ and a morphism $f_\lambda : X_\lambda \to S_\lambda$ that is finitely presented and flat, and an isomorphism $S \times_{S_\lambda} X_\lambda \cong X$ over $S$. If $f : X \to S$ is separated respective proper, we can choose $f_\lambda$ moreover separated respective proper. Using [36, Corollaire 12.1.7(iii)] and [36, Thm. 8.10.5], we can choose $\lambda$ such that $f_\lambda$ is a Cohen–Macaulay morphism. Since these decompose disjointly over the relative codimension, again by increasing $\lambda$ if needed, we may assume that $f_\lambda$ has relative dimension $d$.

We next spread out $U$ such that $U_\lambda \subset X_\lambda$ satisfies (CC). We do this by spreading out its complement $Z$. Indeed, by [5, 05M5, Lemma 31.16.1], we can increase $\lambda$ so that every fiber of $Z_\lambda \to S_\lambda$ has dimension $\leq d-2$ and then define $U_\lambda := X_\lambda \setminus Z_\lambda$.

Now a straightforward generalization of the method employed in [74, 4.11.1] yields that, for appropriate $\lambda$, we can find a log structure on $U_\lambda$ and upgrade $f_\lambda$ to a log morphism such that $U_\lambda$ is fs and $f_\lambda$ is log smooth and saturated. While Tsuji uses absolute charts to construct the log structure, we choose relative charts $A_{P_i} \to A_Q$ with saturated injections $Q \subset P_i$.

Finally—again by possibly increasing $\lambda$—we show that the family $f_\lambda : X_\lambda \to S_\lambda$ is log toroidal. We fix a finite covering $\{V_i \to X\}$ with local models $(Q \subset P_i, \mathcal{F}_i)$ as in Definition 4.2, and for each of them, we construct a diagram

$$\begin{tikzcd}
 & V_i \arrow[rr] \arrow[dl, "g"'] \arrow[dd, dotted, "h"] & & V_{i,\lambda} \arrow[dl, "g_\lambda"'] \arrow[dd, dotted, "h_\lambda"] & & \\
X \arrow[rr, "p_\lambda"'] \arrow[dd, "f"] & & X_\lambda \arrow[dd, "f_\lambda"] & & & \\
 & L_i \arrow[rr] \arrow[dl, "r"'] & & L_{i,\lambda} \arrow[rr] \arrow[dl, "r_\lambda"'] & & A_{P_i,\mathcal{F}_i} \arrow[dl] \\
S \arrow[rr, "q_\lambda"'] & & S_\lambda \arrow[rr, "a_\lambda"'] & & A_Q &
\end{tikzcd}$$

Namely, we first spread out $V_i \to S$ to $V_{i,\lambda} \to S_\lambda$. Then $L_{i,\lambda}$ is defined by base change, and we construct the étale morphisms of schemes $g_\lambda : V_{i,\lambda} \to X_\lambda$ and $h_\lambda : V_{i,\lambda} \to L_{i,\lambda}$ also by spreading out. We can assume that $X_\lambda$ is covered by $\{V_{i,\lambda} \to X_\lambda\}$ and that $\tilde{U}_i \subset V_i$ spreads out to an open $\tilde{U}_{i,\lambda} \subset V_{i,\lambda}$ satisfying (CC). We get two log structures $(g_\lambda)^*_{log}\mathcal{M}_{X_\lambda}$ and $(h_\lambda)^*_{log}\mathcal{M}_{L_{i,\lambda}}$ on $\tilde{U}_{i,\lambda}$, which we identify by [74, 4.11.3]. By the same Lemma, the two morphisms $(g \circ f)^*_{log}\mathcal{M}_{S_\lambda} \to \mathcal{M}_{\tilde{U}_{i,\lambda}}$ coming from $f_\lambda \circ g_\lambda$ respective $r_\lambda \circ h_\lambda$ coincide. Since $\{V_i \to X\}$ is a finite covering, we can find $\lambda$ that admits the above construction for all $V_i$ simultaneously. □

*Remark* 4.19. The proof shows that, if $f : X \to S$ is only a generically log smooth family, then we can find a generically log smooth spread out. If the generically log smooth family is vertical, then the spread out is vertical. Indeed, in this case the local models on $U$ as constructed in Example 4.25 are vertical. Similarly, if $f : X \to S$ is relatively log smooth, then we can find a relatively log smooth spread out.

## 4.4 Examples

We illustrate the scope where log toroidal families occur. This includes toroidal varieties which are *not* log smooth as well as toroidal embeddings and the saturated log smooth morphisms. Most interestingly, many degenerations are log toroidal families.

### Toroidal Varieties

Toroidal varieties $X$ and toroidal pairs $(X, D)$ (see e.g. [14, §15] for a definition, cf. also the Introduction) are log toroidal families over a trivial base. Indeed, fix a normal domain $R$, and let $\underline{X}$ be a normal variety over Spec $R$ and $D \subset X$ a Weil divisor such that $(\underline{X}, D)$ is étale locally (over Spec $R$) isomorphic to a pair $(Y, E)$, where $Y$ is an affine toric variety and $E$ a toric divisor—not necessarily the entire toric boundary! Then $X \to \operatorname{Spec} R$ is a log toroidal family; here, $X$ is $\underline{X}$ with the divisorial log structure given by $D$ (in the étale topology), and Spec $R$ has the trivial log structure.

**Example 4.20.** A toroidal curve is smooth. Indeed, every toroidal curve is normal hence smooth.

**Example 4.21.** Toroidal surfaces are given by sharp toric monoids of rank 2; the latter are completely classified. By taking the dual of [13, Prop. 10.1.1], they are all given by $\operatorname{Cone}(e_1, de_2 + ke_1) \cap \mathbb{Z}^2$ for parameters $d, k$ satisfying $d > 0, 0 \leq k < d$, and $\gcd(d, k) = 1$, where $e_1, e_2$ is the standard basis of $\mathbb{Z}^2$. All of them are cyclic quotient singularities. In particular, every toroidal surface is an orbifold.

Taking $k = d - 1$, we obtain the *du Val singularities* of type $A_k$ defined by

$$\{xy - z^{k+1} = 0\} \subset \mathbb{A}^3,$$

cf. [13, Ex. 10.1.5]. The du Val singularities of types $D_k, E_6, E_7, E_8$ are *not* toroidal since they are non-abelian quotient singularities.

*Remark* 4.22. Du Val singularities are precisely the ADE-singularities as given e.g. in the list [28, p. 145]. They are isolated surface singularities, and they are precisely the simple hypersurface singularities in dimension 2. Since surface singularities of types $D_k, E_6, E_7, E_8$ are not toroidal, we can give explicit examples of non-toroidal surfaces immediately—the surfaces $\{w^2 + x^3 + y^4 = 0\}$ and $\{w^2 + x^3 + y^5 = 0\}$ are not toroidal.

**Example 4.23.** In [65], Namikawa studies deformations of (complex) Calabi–Yau threefolds $X$ with terminal singularities. Such a threefold $X$ is a toroidal variety if and only if it has only $A_1$-rational double points as singularities. In particular, none of the Kleinian singularities in [65, Def. 5.4] except $A_1$ can appear if $X$ is toroidal.

*Proof.* Since $X$ is Cohen–Macaulay (because it is toroidal) and the canonical divisor $K_X$ is Cartier, $X$ is Gorenstein. Thus, by the classification of three-dimensional terminal toric singularities ([13, Thm. 11.4.21]), every singularity is given by $xy - tw = 0$, hence an $A_1$-rational double point. ☐

*Remark* 4.24. The cone defining the monoid from Example 3.6, i.e., the variety $\{xy - tw = 0\}$, is not simplicial, so a toroidal variety is not an orbifold at $A_1$-rational double points. In particular, there are toroidal threefolds which are not orbifolds.

### Saturated Log Smooth Morphisms

**Example 4.25.** If $f : X \to S$ is a saturated log smooth morphism, it is a log toroidal family with $U = X$. Indeed, there is a local model $A_{\tilde{P},\mathcal{F}} \to A_Q$ at every geometric point $\bar{x} \to X$. It is everywhere log smooth, not only on $U_{\tilde{P}/Q}$, and the map $V \to A_{\tilde{P},\mathcal{F}}$ is everywhere strict.

*Proof.* Let $S \to A_Q$ be a chart which is neat at $f(\bar{x})$ (possibly after shrinking $S$). Then in a neighborhood $V$ of $\bar{x}$, we obtain the diagram

$$\begin{tikzcd}
V \arrow[r] \arrow[rr, bend left] \arrow[d] & A_P \arrow[d] & A_{P \times \mathbb{N}^r, \mathcal{F}'} \arrow[l] \arrow[dl] \\
S \arrow[r] & A_Q &
\end{tikzcd}$$

by [67, IV, Thm. 3.3.3], where $V \to A_P$ is a chart which is neat at $\bar{x}$. In particular, we have a saturated injection $Q \to P$ of sharp toric monoids (injectivity follows from the fact that integral local maps are exact and hence $s$-injective). The map $A_{P \times \mathbb{N}^r, \mathcal{F}'} \to A_P$ obtained from Example 3.7 applied to $(Q \subset P, \mathcal{F}_{\max})$ is the map constructed in [67, IV, Thm. 3.3.3]. Since $V \to A_{P \times \mathbb{N}^r, \mathcal{F}'} \times_{A_Q} S$ is strict étale, we have a local model with ETD $(Q \subset P \times \mathbb{N}^r, \mathcal{F}')$. ☐

*Remark* 4.26. This is *not* a trivial consequence of Kato's toroidal characterization of log smoothness in [50, Thm. 3.5]. For, this yields a *smooth* map $X \to A_P \times_{A_Q} S$, but not an *étale* map.

**Example 4.27.** Semistable degenerations are saturated log smooth morphisms, hence log toroidal families.

### Toroidal Embeddings

Let us work over an algebraically closed field $k \supset \mathbb{Q}$. A *toroidal embedding* $U \subset X$, which has been originally defined in [56], gives rise to log smooth and saturated log schemes over Spec $k$ by endowing $X$ with the divisorial log structure defined by $X \setminus U$. As observed e.g. in [2], the definition is equivalent to the existence of an étale roof

$$(X, x) \xleftarrow{u} (X', x') \xrightarrow{v} (A_P, p)$$

for every closed point $x \in X$, where $A_P$ is an affine toric variety, $p \in A_P$ is a point, and we have $u^{-1}(U) = v^{-1}(A_P^*)$. A *toroidal morphism* $f : (X, U_X) \to (S, U_S)$ of toroidal embeddings has been defined e.g. in [1]. Using [4, Cor. 2.6], we can find étale roofs as above, a toric morphism $A_P \to A_Q$, and the middle map such that we have a commutative diagram

$$\begin{tikzcd}
(X, x) \arrow[d] & (X', x') \arrow[l] \arrow[r] \arrow[d] & (A_P, p) \arrow[d] \\
(S, f(x)) & (S', s') \arrow[l] \arrow[r] & (A_Q, q).
\end{tikzcd}$$

Since $f$ is dominant and $\hat{\mathcal{O}}_{S,f(x)}$ is a domain, the induced map $\hat{\mathcal{O}}_{S,f(x)} \to \hat{\mathcal{O}}_{X,x}$ is injective by [38, I. Cor. 3.9.8]. We find $k[Q] \to k[P]$ injective, so the kernel and the torsion part of the cokernel of $Q^{gp} \to P^{gp}$ are finite groups of order invertible in $k$. This proves $f : X \to S$ log smooth, a fact that has been also observed e.g. in [18] with a slightly different definition. On the other hand, not every toroidal morphism is saturated, e.g. $k[t] \to k[t], t \mapsto t^3$. However, weakly semistable families in the sense of Abramovich–Karu as defined in [1] are. They have been introduced to generalize semistable reduction to higher dimensional bases. To us, they are just another example illustrating the wide range where log toroidal families occur. We recall the definition for convenience of the reader.

**Definition 4.28** ([1])**.** A *weakly semistable family* is a flat morphism $f : X \to S$ of projective varieties with connected fibers together with structures of toroidal embeddings $U_S \subset S$ and $U_X \subset X$ such that $f^{-1}(U_S) = U_X$ and $f$ is toroidal, equidimensional, and has reduced fibers. Moreover, $S$ is nonsingular. The family $f : X \to S$ is *semistable* if additionally $X$ is smooth.

Note that this definition of semistability is more general than the classical one—we do not assume $S$ to be one-dimensional.

**Example 4.29.** Weakly semistable families are log smooth and saturated.

*Proof.* The log smooth map $A_P \to A_Q$ is log flat. Since $X' \to S'$ is flat and $S'$ is log regular, the result [67, IV, Thm. 4.3.5] yields $X' \to S'$ integral. It has reduced fibers, so it is saturated by [67, IV, Thm. 4.3.6]. □

### Degenerations

Recall that we consider the theory of log toroidal families a tool to study degenerations of varieties. We give here some evidence that in fact many degenerations carry the structure of a log toroidal family. The following construction generalizes the degeneration of the smooth quartic surface in Example 1.3.

**Example 4.30.** Pencils of normal crossing divisors are log toroidal families. Let $k = \bar{k}$ be an algebraically closed field, let $D = \{F = 0\} \subset \mathbb{P}^n_k$ be a normal crossing divisor of degree $d$, let $E = \{G = 0\} \subset \mathbb{P}^n_k$ be a smooth hypersurface of degree $d$, and assume that $D \cup E = \{FG = 0\}$ is a normal crossing divisor. Defining

$$Y := \{TF - SG = 0\} \subset \mathbb{P}^1_k \times \mathbb{P}^n_k,$$

where $[S : T]$ are the variables of $\mathbb{P}^1_k$, we obtain a morphism $\mathbf{f} : Y \to \mathbb{P}^1_k$ via projection. Then there is a neighborhood $S \subset \mathbb{P}^1_k$ of $0 = [0 : 1]$ and an open $U \subset X := \mathbf{f}^{-1}(S)$ such that $f : X \to S$ with the divisorial log structures defined by $f^{-1}(0)$ and $0$ is a proper log toroidal family.

*Proof.* For a $k$-valued point $\bar{x} \in \mathbb{P}^n_k$, let $H = H_{\bar{x}}$ be a homogeneous polynomial of degree $d$ such that $\bar{x} \in \{H \neq 0\}$. Since $D \cup E$ is normal crossing, we can find étale morphisms

$$\mathbb{P}^n_k \xleftarrow{\gamma} V_{\bar{x}} \xrightarrow{\eta} \mathbb{A}^n_k$$

such that $\gamma^*(F/H \cdot G/H) = \eta^*(z_1 \cdot ... \cdot z_k)$ for some $0 \leq k \leq n$ (where the empty product is 1). Because $z_i$ is a prime in the étale local ring $(\mathbb{A}^n_k)_{\bar{0}}$, we can choose $V_{\bar{x}}, \gamma, \eta$ such that $\gamma^*(F/H) = \eta^*(z_1 \cdot ... \cdot z_\ell)$ and $\gamma^*(G/H) = \eta^*(z_{\ell+1} \cdot ... \cdot z_k)$. In particular, by using [19, Prop.-IV-25], the second projection $\pi : Y \to \mathbb{P}^n_k$ can be identified with the blow-up of $\mathbb{P}^n_k$ in $D \cap E$. In particular, the scheme $Y$ is integral and $\mathbf{f} : Y \to \mathbb{P}^1_k$ is flat because $Y$ dominates the smooth curve $\mathbb{P}^1_k$.

Because $\mathbf{f} : Y \to \mathbb{P}^1_k$ is proper and has a non-singular fiber $E = \mathbf{f}^{-1}(\infty)$, there is an open neighborhood $S \ni 0$ such that $\mathbf{f}^{-1}(0) = D$ is the only singular fiber over $S$. We take this $S$ as the base of $f : X \to S$. Denoting $D_2 \subset D$ the double locus of the normal crossing divisor $D$, we set $U = X \setminus (D_2 \cap E)$ under the identification $D = f^{-1}(0)$. The scheme $\pi^{-1}(\mathbb{P}^n_k \setminus (D_2 \cap E))$ is étale locally the blow-up of $\mathbb{A}^{n-2} \subset \mathbb{A}^n$, so it is a regular scheme; this shows that $U$ is a regular scheme. Thus $f : U \to S$ is a semistable degeneration and hence saturated and log smooth with the given log structures. Around a point $\bar{x} \in X \setminus U$, on $\mathbb{A}^1_k \times V_{\bar{x}}$ the scheme $X$ is locally given by the equation

$$\gamma^*(F/H) - s \cdot \gamma^*(G/H) = \eta^*(z_1 \cdot ... \cdot z_{k-1}) - s\eta^*(z_k) = 0,$$

where $\gamma^*(G/H) = \eta^*(z_k)$, i.e., $\ell = k - 1$ because $E$ is a smooth hypersurface. Thus a local model is given by the ETD $(\mathbb{N} \subset P^{k-1}_{pnc}, \mathcal{F}_{\min})$ from Example 3.29. ☐

**Example 4.31.** Setting $D = \{xyz = 0\}$ and $E = \{x^3 + y^3 + z^3 = 0\}$ we obtain the *Hesse pencil*

$$\lambda(x^3 + y^3 + z^3) + \mu xyz = 0$$

of cubic curves, cf. [3]. Since it is a log toroidal family with fiber dimension 1, it is actually log smooth.

**Example 4.32.** Setting $D = \{x_0x_1x_2x_3 = 0\} \subset \mathbb{P}^3$ the union of the four coordinate planes and $E = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\} \subset \mathbb{P}^3$ the Fermat quartic, we obtain the degeneration of quartic surfaces of Example 1.3. For

$$D = \{x_0x_1x_2x_3x_4 = 0\} \subset \mathbb{P}^4 \quad \text{and} \quad E = \{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \subset \mathbb{P}^4,$$

we obtain essentially the famous family of quintic threefolds, which is often studied in mirror symmetry, e.g. in [10, 63].

**Question 4.33.** The construction in Example 4.30 can be easily generalized as follows: Let $P/k$ be a scheme, let $\mathcal{L}$ be a line bundle on $P$, and let $F, G \in H^0(P, \mathcal{L})$ be sections defining hypersurfaces $D, E$. Let $\mathbb{P}^1 = \operatorname{Proj} k[S, T]$, and let $p : P \times \mathbb{P}^1 \to P, q : P \times \mathbb{P}^1 \to \mathbb{P}^1$ be the two projections. Then

$$T \otimes F - S \otimes G \in H^0(P \times \mathbb{P}^1, p^*\mathcal{L} \otimes q^*\mathcal{O}(1))$$

defines a hypersurface $Y \subset P \times \mathbb{P}^1$ and a family $\mathbf{f} : Y \to \mathbb{P}^1$ via projection. What are the conditions on $P, \mathcal{L}, F, G$ to make $\mathbf{f}$ a log toroidal family?

*Remark* 4.34. When we take $P = \mathbb{P}_\Delta$ for a reflexive polytope $\Delta$ and $\mathcal{L} = \mathcal{O}_P(1)$, this construction is closely related to the construction of toric degenerations out of the data of Batyrev duality, see [29]. Here one takes $F$ corresponding to $0 \in \Delta$ such that $D$ is the toric boundary divisor in $P = \mathbb{P}_\Delta$, and $G$ to be general. The result is a (in good cases) smooth variety $E$, which degenerates to a union of toric varieties.

**Example 4.35.** Many structures of the Gross–Siebert program are log toroidal families. If $(B, \mathcal{P})$ is a positive and simple integral affine manifold with singularities and a polyhedral decomposition, and if $s$ are lifted open gluing data, then the toric log Calabi–Yau space $X_0^\dagger(B, \mathcal{P}, s)$ is a log toroidal family over Spec $(\mathbb{N} \to \mathbb{C})$ by [31, Thm. 2.6]. Indeed, the local models are the ETDs of Example 3.30. More generally, by Ruddat's generalization in [69, Prop. 2.8], log Calabi–Yau spaces of *complete intersection type* are log toroidal over Spec $(\mathbb{N} \to \mathbb{C})$. Divisorial deformations of $X_0^\dagger(B, \mathcal{P}, s)$ are log toroidal families essentially by definition.

# 5 Relatively Log Smooth Families

We introduce in this section a variant of the concept of relatively (log) smooth family. Relatively log smooth families take into account the incoherencies occuring in "nature" (as in the family $f : X_{st} \to \mathbb{A}^1$ in Example 1.4, see the Introduction) by weakening the coherence assumption of the log structure on $X$ to *relative coherence*. This notion is discussed in Ogus' book [67], but has been used earlier in Nakayama–Ogus work [64] and, relying on this, by Gross–Siebert in [31]. We review it here and give some new elementary results which are important for our further study. Relatively (log) smooth families $f : X(\mathcal{F}) \to Y$ have been introduced by Nakayama-Ogus in [64] in the context of studying the Betti realizations (also called Kato–Nakayama spaces) of log smooth morphisms. It is shown there ([64, Thm. 3.7]) that smooth as well as relatively smooth families are submersive on the level of Betti realizations $X_{log}(\mathcal{F}) \to Y_{log}$. In particular, if $f : X(\mathcal{F}) \to Y$ is proper, then $f_{log}$ is a topological fiber bundle with nice fibers. We give a slightly different definition of relatively log smooth family and show that the concept is intimately related to log toroidal families. Indeed, Proposition 5.13 below shows that every relatively log smooth family can be endowed with an open $U \subset X$ which turns it into a log toroidal family.

### Relative Coherence

Relative coherence generalizes the notion of coherence of a log structure. Let $(X, \mathcal{H})$ be a fine log scheme, and let $\beta : P \to \mathcal{H}$ be a chart. For a face $G \subset P$, we have a sheaf $\mathcal{G}_G$ of faces in $\mathcal{H}$ given by
$$\mathcal{G}_G(U) = \langle \beta(G) \rangle \subset \mathcal{H}(U)$$
for an open $U \subset X$, where $\langle \beta(G) \rangle$ is the face in $\mathcal{H}(U)$ generated by $\beta(G)$. The sheaf $\mathcal{G}_G$ is itself a log structure. A sheaf of faces $\mathcal{M} \subset \mathcal{H}$ is *relatively coherent* if (étale) locally we can find charts and faces such that $\mathcal{M} = \mathcal{G}_G$. Globalizing this, we call a log structure $\mathcal{M}$ on a scheme $X$ *relatively coherent*—without reference to a global $\mathcal{H}$—if (étale) locally on $X$, the log structure is isomorphic to some $\mathcal{G}_G$ as above. In this case, the chart $\beta : P \to \mathcal{H}$ together with $G \subset P$ is called a *relative chart*.

*Remark* 5.1. The construction of $\mathcal{G}_G$ works as well for the Zariski and the étale topology. If $\mathcal{G}_G \subset \mathcal{H}$ is in the étale topology and $G \subset P \to \mathcal{H}$ is a relative chart, then the chart also defines Zariski log structures $\tilde{\mathcal{G}}_G \subset \tilde{\mathcal{H}}$ whose associated étale log structures are $\mathcal{G}_G \subset \mathcal{H}$. Thus, if convenient, étale locally we can assume a relatively coherent log structure to be defined in the Zariski topology.

*Remark* 5.2. If $\mathcal{G}_G \subset \mathcal{H}$ is defined by the relative chart $G \subset P \xrightarrow{\beta} \mathcal{H}$, then $(\mathcal{G}_G)_{\bar{x}} \subset \mathcal{H}_{\bar{x}}$ is the face generated by $\beta(G)$, and the same holds for the ghost sheaves.

*Remark* 5.3. By definition, $\mathcal{M} \subset \mathcal{H}$ is relatively coherent if $\mathcal{M} = \mathcal{G}_G$ for *some* chart $\beta : P \to \mathcal{H}$. By [67, II, Prop. 2.6.5], if $\beta : P \to \mathcal{H}$ is *any* chart around a point $x \in X$ (with $\mathcal{H}$ fixed), there is a face $G \subset P$ such that $G \subset P \to \mathcal{H}$ is a relative chart for $\mathcal{M}$ in some neighborhood of $x$. We can choose $G = \beta^{-1}(\mathcal{M}_x)$.

For a fine log scheme $(X, \mathcal{H})$ and a (fine) chart $\beta : P \to \mathcal{H}$, there is equivalently a strict morphism $b : (X, \mathcal{H}) \to A_P$. If $\mathcal{M} \subset \mathcal{H}$ is relatively coherent with relative chart $G \subset P \to \mathcal{H}$, then it fits into a commutative diagram

$$\begin{tikzcd} (X,\mathcal{H}) \arrow[r] \arrow[d, "b"'] & (X,\mathcal{M}) \arrow[r] \arrow[d, "b_G"'] & (X,\mathcal{O}_X^*) \arrow[d] \\ A_P \arrow[r] & A_{P,G} \arrow[r] & \mathrm{Spec}\ \mathbb{Z}[P] \end{tikzcd} \tag{RC}$$

because $(b^\flat)^{-1}(b_*\mathcal{M}) \subset \mathcal{M}_P$ is a sheaf of faces in the log structure of $A_P$ which contains the image of $G \subset P \to \mathcal{M}_P$.

**Lemma 5.4.** *The diagram* (RC) *is Cartesian. In particular, the map* $b_G : (X, \mathcal{M}) \to A_{P,G}$ *is strict. Conversely, if we have a Cartesian diagram* (RC) *with* $P$ *fine, then* $\mathcal{M} \subset \mathcal{H}$ *is relatively coherent.*

*Proof.* Denoting the log structure on $A_{P,G}$ by $\mathcal{M}_{P,G}$, we have log structures

$$\mathcal{T} = b_{log}^*\mathcal{M}_{P,G} \to \mathcal{M} \to \mathcal{H}$$

on $X$. Because $\overline{\mathcal{T}} \subset \overline{\mathcal{H}}$ is a sheaf of faces, also $\mathcal{T} \subset \mathcal{H}$ is a sheaf of faces. Since $\mathcal{T}$ contains the image of $\beta(G)$, we have $\mathcal{M} \subset \mathcal{T}$, so $\mathcal{T} = \mathcal{M}$; this shows that $b_G$ is strict and the squares are Cartesian. The converse is obvious: There is a relatively coherent log structure $\mathcal{G}_G \subset \mathcal{H}$ defined by $G \subset P \to \mathcal{H}$, which satisfies $\mathcal{G}_G \cong b_{log}^*\mathcal{M}_{P,G} \cong \mathcal{M}$. ☐

**Corollary 5.5.** *Let* $g : Y \to X$ *be a strict morphism of log schemes, and assume* $X$ *relatively coherent. Then* $Y$ *is relatively coherent.*

Let us relate relatively coherent log structures to ETDs. Therefore, let $P$ be a sharp toric monoid, and let $G \subset P$ be a face. Then $X := A_P = \mathrm{Spec}\ (P \to \mathbb{Z}[P])$ is a fine log scheme, and $\beta : P \to \mathcal{M}_X$ is a chart. We denote $X$ with the relatively coherent log structure $\mathcal{G}_G$ by $A_{P,G}$.

**Lemma 5.6.** *Let* $G \subset P$ *be a face in a sharp toric monoid* $P$*, and let*

$$\mathcal{F}(G) := \{F \subset P \text{ a facet } \mid G \notin F\} \ .$$

*Then* $A_{P,G}$ *carries the divisorial log structure defined by* $\bigcup_{F\in\mathcal{F}(G)} \mathrm{Spec}\ \mathbb{Z}[F]$.

*Proof.* Let $\mathcal{H}$ be the log structure on $A_P$ (which is divisorial for the union of all toric divisors) with a chart $\beta : P \to \mathcal{H}$, and let $\mathcal{M} \subset \mathcal{H}$ be the divisorial log structure defined by $D = \bigcup_{F\in\mathcal{F}(G)} \mathrm{Spec}\ \mathbb{Z}[F]$, which is a sheaf of faces. For a (geometric) point $\bar{x} \in A_P$, setting $K_{\bar{x}} = \beta^{-1}(\mathcal{O}_{\bar{x}}^*) \subset P$, we have

$$\{z^p \neq 0\} \cap U \supseteq \mathrm{Spec}\ \mathbb{Z}[P_G] \cap U$$

in some neighborhood $U \ni \bar{x}$ if and only if $p \in \langle G, K_{\bar{x}}\rangle$. Because $\mathrm{Spec}\ \mathbb{Z}[P] \setminus D = \mathrm{Spec}\ \mathbb{Z}[P_G]$, this shows $\beta(p)_{\bar{x}} \in \mathcal{M}_{\bar{x}}$ if and only if $p \in \langle G, K_{\bar{x}}\rangle$. In particular, for $g \in G$, we have

$$\{z^g \neq 0\} \supseteq \mathrm{Spec}\ \mathbb{Z}[P_G];$$

this implies $\beta(g) \in \mathcal{M}$, so $\mathcal{G}_G \subset \mathcal{M}$. Now assume $m \in \overline{\mathcal{M}}_{\bar{x}}$ for the ghost sheaf. Because $P/K_{\bar{x}} \cong \overline{\mathcal{H}}_{\bar{x}}$ via $\beta$, we can find $p \in P$ with $\overline{\beta(p)}_{\bar{x}} = m$. Then $p \in \langle G, K_{\bar{x}}\rangle$ and therefore $m \in \overline{\mathcal{G}_G}_{\bar{x}}$, so $\mathcal{G}_G = \mathcal{M}$. ☐

**Example 5.7.** Let $Q \subset G \subset P$ be sharp toric monoids such that $G \subset P$ is a face and $Q \subset P$ is saturated. Then we get an ETD $(Q \subset P, \mathcal{F}(G))$. The lemma shows $A_{P,\mathcal{F}(G)} = A_{P,G}$ everywhere, not only on $U_{P/Q}$, so $A_{P,\mathcal{F}(G)}$ is relatively coherent. Because $\mathcal{F}_{\max} = \mathcal{F}(P)$ and $\mathcal{F}_{\min} = \mathcal{F}(\langle Q\rangle)$, where $\langle Q\rangle \subset P$ is the face generated by $Q$, these two elementary log toroidal families carry always a relatively coherent log structure.

**Relatively Log Smooth Families**

**Definition 5.8.** A *relatively log smooth family* is a morphism $f : X \to S$ of log schemes such that:

- $\underline{f} : \underline{X} \to \underline{S}$ is a finite type morphism of Noetherian schemes.
- $S$ is fine and saturated.
- There is an étale covering $\{V_i\}$ of $X$ such that we can find an embedding $\mathcal{M}_X|_{V_i} \subset \mathcal{H}_i$ as a relatively coherent sheaf of faces in a fine and saturated log structure $\mathcal{H}_i$ such that $(V_i, \mathcal{H}_i) \to S$ is log smooth and saturated.

**Example 5.9.** Let $Q \subset G \subset P$ be as in Example 5.7 with associated ETD $(Q \subset P, \mathcal{F}(G))$. Then $f : A_{P,\mathcal{F}(G)} \to A_Q$ is a relatively log smooth family. Since we have $\mathcal{F}_{\max} = \mathcal{F}(P)$ and $\mathcal{F}_{\min} = \mathcal{F}(\langle Q \rangle)$, these two elementary log toroidal families are always relatively log smooth families.

*Remark* 5.10. Our definition differs from the one given by Nakayama–Ogus in [64, Def. 3.6]. Namely, they do not require any saturatedness condition, and we do not require any condition on the quotients $\mathcal{H}_i/\mathcal{M}_X$ (which however has in [64] mainly the purpose of making the fibers in the Betti realization topological *manifolds* instead of topological spaces with toric singularities). Moreover, [64] is working in the category of log analytic spaces whereas we are using schemes.

Unlike generically log smooth families, relatively log smooth families are actual morphisms of log schemes. However, we will see below that we can find an open $U \subset X$ which makes them generically log smooth (and even log toroidal). Relatively log smooth families are stable under *strict* base change, i.e., for $f : X \to S$ a relatively log smooth family and $b : T \to S$ a strict morphism with $T$ Noetherian, the log morphism $g : X \times_S T \to T$ is a relatively log smooth family. In fact, Lemma 5.4 shows the log structure on $X \times_S T$ relatively coherent. We do not know if the notion of relatively log smooth family is stable under *non-strict* base change.

**Lemma 5.11.** *Let $f : X \to S$ be a relatively log smooth family. Then $\underline{f} : \underline{X} \to \underline{S}$ is a Cohen–Macaulay morphism with reduced fibers. The log morphism $f : X \to S$ is saturated, exact, and s-injective.*

*Proof.* The statement about $\underline{f}$ follows from [67, IV, Thm. 4.3.5, Thm. 4.3.6] applied to $(V_i, \mathcal{H}_i) \to S$. Since $\mathcal{M} \subset \mathcal{H}_i$ is exact, the statement about $f$ follows from the respective properties of $(V_i, \mathcal{H}_i) \to S$. □

If $(Q \subset P, \mathcal{F})$ is an ETD and $\mathcal{M}_\mathcal{F}, \mathcal{M}_P$ are the log structures of $A_{P,\mathcal{F}}, A_P$, then the log structure $\mathcal{M}_\mathcal{F} \subset \mathcal{M}_P$ is relatively coherent if and only if $\mathcal{F} = \mathcal{F}(G)$ for some face $G \subset P$ such that $Q \subset G$. Indeed, if it is relatively coherent, then by Remark 5.3 there is some face $G \subset P$ with $\mathcal{M}_\mathcal{F} = \mathcal{G}_G$ in a neighborhood of $0 \in A_{P,\mathcal{F}}$, so $\mathcal{M}_\mathcal{F} = \mathcal{M}_{\mathcal{F}(G)}$. We conclude $\mathcal{F} = \mathcal{F}(G)$. Since $G = \bigcap_{F \in \mathcal{F}_{\max} \setminus \mathcal{F}(G)} F$, we find $Q \subset G$.

**Non-Example 5.12.** There is an ETD $(Q \subset P, \mathcal{F})$ such that $\mathcal{M}_\mathcal{F} \subset \mathcal{M}_P$ is *not* relatively coherent. Namely, take the monoid $P$ from Example 3.6, the inclusion $0 \subset P$, and the two opposite facets $\mathcal{F} = \{\langle X, W \rangle, \langle Y, T \rangle\}$. Then there is no face $G \subset P$ with $\mathcal{F} = \mathcal{F}(G)$. We conjecture that $A_{P,\mathcal{F}}$ is not relatively coherent at all, but strictly speaking, we have no proof because $\mathcal{M}_\mathcal{F}$ might be relatively coherent in another coherent log structure.

A relatively log smooth family $f : X \to S$ admits local models like a log toroidal family. To construct it, let $\bar{x} \in X$, let $\bar{s} = f(\bar{x})$, and let $S \leftarrow \tilde{S} \to A_Q$ be a chart which is neat at $\bar{s}$. Choosing some $(V_i, \mathcal{H}_i)$ such that $\bar{x} \in V_i$ and applying [67, IV, Thm. 3.3.3] to the log smooth morphism $(V_i, \mathcal{H}_i) \to S$, we find a neighborhood $V \ni \bar{x}$ and the dashed quadrilateral in the diagram

$$
\begin{tikzcd}
 & L_P \times \mathbb{A}^r \arrow[rr] \arrow[dr] \arrow[dd] & & A_P \times \mathbb{A}^r \arrow[dr] \arrow[dd] & \\
(V,\mathcal{H}) \arrow[ur] \arrow[rrrr, dashed, bend left=10] \arrow[dd] \arrow[dddr, dashed] & & L_P \arrow[rr] \arrow[dd] & & A_P \arrow[dd] \arrow[dddl, dashed] \\
 & L_{P,G} \times \mathbb{A}^r \arrow[rr] \arrow[dr] \arrow[dd] & & A_{P,G} \times \mathbb{A}^r \arrow[dr] \arrow[dd] & \\
(V,\mathcal{M}) \arrow[ur] \arrow[rrrr, dotted, bend left=10] \arrow[dr] & & L_{P,G} \arrow[rr] \arrow[dl] & & A_{P,G} \arrow[dl] \\
 & \tilde{S} \arrow[rr, dashed] & & A_Q &
\end{tikzcd}
$$

such that $(V,\mathcal{H}) \to A_P$ is a neat chart at $\bar{x}$. By Remark 5.3 and Lemma 5.4, we can find a face $G \subset P$ and a strict map $(V,\mathcal{M}) \to A_{P,G}$ fitting into the diagram (after shrinking $V$). The projections $A_P \times \mathbb{A}^r \to A_P$ and $A_{P,G} \times \mathbb{A}^r \to A_{P,G}$ are strict. By [67, IV, Thm. 3.3.3], we have a morphism $V \to \mathbb{A}^r$ inducing the maps $(V,\mathcal{H}) \to A_P \times \mathbb{A}^r$ and $(V,\mathcal{M}) \to A_{P,G} \times \mathbb{A}^r$. The spaces $L_P, L_{P,G}$ are defined as the fiber product along $\tilde{S} \to A_Q$, and by [67, IV, Thm. 3.3.3], the morphism $(V,\mathcal{H}) \to L_P \times \mathbb{A}^r$ is strict and étale, so $(V,\mathcal{M}) \to L_{P,G} \times \mathbb{A}^r$ is strict and étale. Since the charts are neat, the homomorphism $Q \to P$ of sharp toric monoids is injective and saturated. The morphism $A_{P,G} \times \mathbb{A}^r \to A_Q$ is associated to the ETD $(Q \subset P \times \mathbb{N}^r, \mathcal{F}(G) \times \mathbb{N}^r)$ from Example 3.7. Therefore we have a local model at $\bar{x}$ in the sense of Definition 4.2.

**Proposition 5.13.** *Let $f : X \to S$ be a relatively log smooth family. Then there is an open $U \subset X$ satisfying* (CC) *such that $f : X \to S$ is a log toroidal family.*

*Proof.* By the above construction, for every geometric point $\bar{x} \in X$, we get an open $U_{\bar{x}} \subset V$ on which $f : X \to S$ is log smooth as the preimage of $U_{P/Q} \times \mathbb{A}^r \subset A_{P,G} \times \mathbb{A}^r$. After writing $g_{\bar{x}} : V \to X$ for the étale maps and defining

$$U := \bigcup_{\bar{x} \in X} g_{\bar{x}}(U_{\bar{x}}),$$

this $U \subset X$ is a Zariski open satisfying (CC) since the property (CC) is local in the étale topology. By Remark 5.11, the family $f : X \to S$ is generically log smooth. Since we already have constructed local models, it is a log toroidal family. ☐

**Question 5.14.** Is it possible to reconstruct the relatively log smooth family from the log toroidal family? I.e., given a log toroidal family $f : X \to S$, is there at most one possibility to extend the log structure from $U$ to $X$ such that it is relatively log smooth? What are conditions for a log toroidal family to be induced from a relatively log smooth one? By Lemma 4.13 and Example 5.9, if $f : X \to S$ is vertical, then it comes locally from a relatively log smooth family. Are these extensions compatible?

From a technical perspective, relatively log smooth families are more satisfactory than log toroidal families because there is no ambiguity in the log structure on $Z$. Sometimes, it is slightly easier to check that a family is relatively log smooth than log toroidal because we do not need to care about $U$.

**Example 5.15.** Let $f : X \to \mathbb{A}^1$ be a toroidal morphism, i.e., étale locally on $X$ given by $A_P \to \mathbb{A}^1$ (for saturated injections $\mathbb{N} \subset P$). Endowing $X$ with the divisorial log structure defined by $f^{-1}(0)$ and $\mathbb{A}^1$ with the divisorial log structure defined by $0 \in \mathbb{A}^1$ turns $f : X \to \mathbb{A}^1$ into a relatively log smooth family. In particular, the pencils of normal crossing divisors in Example 4.30 are relatively log smooth families. Concretely, we see that the degeneration of smooth quartics in Example 1.3 is relatively log smooth.

# 6 Differential Forms of Log Toroidal Families

We study the de Rham complex $W^\bullet_{X/S}$ of a log toroidal family $f : X \to S$ and prove two base change results, namely Theorems 6.15 and 6.16 below. These results depend on a careful analysis of differential forms on elementary log toroidal families $f : A_{P,\mathcal{F}} \to A_Q$. In fact, the base change property defined in Definition 6.14 is local in the étale topology, and some elementary log toroidal families have it by explicit computation. To this end, we start with an explicit computation of $W^\bullet_{X/S}$ on elementary log toroidal families.

## 6.1 Differential Forms on Elementary Log Toroidal Families

We fix a principal ideal domain $R$ as a base ring, e.g. $R = \mathbb{Z}$ or $R = \mathbb{C}$. Given an ETD $(Q \subset P, \mathcal{F})$, the construction of $f : A_{P,\mathcal{F}} \to A_Q$ carries through when replacing $\mathbb{Z}$ by $R$ as explained in Remark 3.20 because $R$ is regular. We compute the differential forms $W^\bullet_f$ of this elementary log toroidal family, using the following elementary lemma.

**Lemma 6.1.** *Let $n, m \geq 0$ and $G_1, ..., G_r \subset R^n$ be submodules each of which is a direct summand, then the natural map $\bigwedge^m_R(\bigcap_i G_i) \to \bigcap_i \bigwedge^m_R G_i$ is an isomorphism.*

*Proof.* The case of a *field* $R$ is given by Danilov in [14]. For the general case, compare to the situation over $K = \mathrm{Quot}(R)$ and use the summand intersection property, cf. [77]: Let $R$ be a principal ideal domain and let $H, H' \subset R^s$ be direct summands. Then $H \cap H' \subset R^s$ is a direct summand. □

First consider the absolute case, i.e., an ETD $(Q \subset P, \mathcal{F})$ with $Q = 0$, and denote by $f : A_{P,\mathcal{F}} \to \mathrm{Spec}\ R$ the associated log toroidal family. One checks that $U = U_{P/0}$ from (6) is simply the complement of codimension two strata. Recall from Example 2.16 that

$$W^m := W^m_{A_{P,\mathcal{F}}/\mathrm{Spec}\ R}$$

are just the Danilov differentials with log poles in the divisor given by the facets in $\mathcal{F}$. Danilov already computed these in [14, Proposition 15.5] over a field, but because of Lemma 6.1, the same calculation works over $R$, and we obtain the following result.

**Proposition 6.2** (absolute case). *We have a grading $\Gamma(A_P, W^m) = \bigoplus_{p\in P}(W^m)_p$ with*

$$(W^m)_p = \bigwedge_R^m \left( \bigcap_{\substack{F\in\mathcal{F}_{\max}\setminus\mathcal{F} \\ p\in F}} F^{gp} \otimes_{\mathbb{Z}} R \right),$$

*where the intersection is $P^{gp} \otimes_{\mathbb{Z}} R$ if the index set is empty.*

*Remark* 6.3. See Remark 6.7 below for an explanation how this relates to derivations and what the differential is.

Let us next assume we have a general ETD $(Q \subset P, \mathcal{F})$, and let again $f$ denote the associated log toroidal family and $W^m_f := W^m_{A_{P,\mathcal{F}}/A_Q}$ the differentials. Note that, since $\mathcal{F}$ contains all vertical facets, every facet in $\mathcal{F}_{\max} \setminus \mathcal{F}$ contains $Q$. We obtain the following generalization.

**Proposition 6.4** (general case). *We have a grading $\Gamma(A_P, W^m_f) = \bigoplus_{p\in P}(W^m_f)_p$ with*

$$(W^m_f)_p = \bigwedge_R^m \left( \left( \bigcap_{\substack{F\in\mathcal{F}_{\max}\setminus\mathcal{F} \\ p\in F}} F^{gp} \otimes_{\mathbb{Z}} R \right) \Big/ (Q^{gp} \otimes_{\mathbb{Z}} R) \right),$$

*where the intersection is $P^{gp} \otimes_{\mathbb{Z}} R$ if the index set is empty. Since $Q^{gp} \subset P^{gp}$ splits, we can equivalently take the quotient before the intersection.*

*Proof.* We can compose $f$ with the projection to Spec $R$ to relate the current situation to that of Proposition 6.2. The open set $U^{\mathrm{abs}}$ in the absolute case is the complement of $Z^{\mathrm{abs}}$, the union of all codimension two strata. Hence, $U^{\mathrm{abs}}$ is covered by $U_F$ where $F$ runs over the facets of $P$. On the other hand, the open set $U$ for $f$ as given in (6) has a cover $U_F$ where $F$ runs over the essential faces of rank $d-1$ by Lemma 3.12. Obviously, $U^{\mathrm{abs}} \subseteq U$. Note that, since $W^m_f$ is locally free on $U$ and $\mathcal{O}_U$ is $Z^{\mathrm{abs}}$-closed, we find that $W^m_f$ is not only $Z$-closed, but also $Z^{\mathrm{abs}}$-closed. Consider the commutative diagram of solid arrows

$$\begin{tikzcd} 0 \arrow[r] & f^*\Omega_{A_Q/\mathrm{Spec}\ R} \arrow[r, "\iota"] \arrow[d, equal] & W^1_{A_{P,\mathcal{F}}/\mathrm{Spec}\ R} \arrow[r] \arrow[d] & W^1_f \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & f^*\Omega_{A_Q/\mathrm{Spec}\ R} \arrow[r] & W^1_{A_P/\mathrm{Spec}\ R} \arrow[r] \arrow[ul, dotted] & W^1_{A_P/A_Q} \arrow[r] & 0, \end{tikzcd} \tag{7}$$

where the top row is obtained by pushing it forward from $U^{\mathrm{abs}}$. The bottom sequence is obtained from tensoring the sequence $0 \to Q^{gp} \to P^{gp} \to P^{gp}/Q^{gp} \to 0$ with $\mathcal{O}_{A_P}$, in particular, it is exact and splits. Hence the dotted diagonal arrow exists and commutes with the other maps. Therefore, $\mathrm{coker}(\iota)$ is a direct summand of $W^1_{A_{P,\mathcal{F}}/\mathrm{Spec}\ R}$, in particular $Z^{\mathrm{abs}}$-closed. Moreover, $\mathrm{coker}(\iota) \to W^1_f$ is an isomorphism on $U^{\mathrm{abs}}$, and since both sheaves are $Z^{\mathrm{abs}}$-closed, we have $\mathrm{coker}(\iota) = W^1_f$, and thus the top row is exact and splits.

Let $\langle f^*\Omega_{A_Q/\mathrm{Spec}\ R}\rangle$ denote the homogeneous ideal in the sheaf of exterior algebras $W^\bullet_{A_{P,\mathcal{F}}/\mathrm{Spec}\ R}$ generated by $f^*\Omega_{A_Q/\mathrm{Spec}\ R}$. The split exactness above gives the split exactness of the following sequence

$$0 \to \langle f^*\Omega_{A_Q/\mathrm{Spec}\ R}\rangle_m \to W^m_{A_{P,\mathcal{F}}/\mathrm{Spec}\ R} \to W^m_f \to 0.$$

Since $A_P$ is affine and $\langle f^*\Omega_{A_Q/\mathrm{Spec}\ R}\rangle$ coherent, applying $\Gamma(A_P,\cdot)$ to this sequence yields another exact sequence, which already gives that $\Gamma(A_P, W^m_f)$ is $P$-graded. We have

$$\Gamma(A_P, f^*\Omega_{A_Q/\mathrm{Spec}\ R}) = Q^{gp} \otimes_{\mathbb{Z}} R[P]\ .$$

Set

$$\mathbf{F}_p := \left( \bigcap_{\substack{F \in \mathcal{F}_{\max} \smallsetminus \mathcal{F} \\ p \in F}} F^{gp} \otimes_{\mathbb{Z}} R \right),$$

and let $\langle Q^{gp} \otimes R\rangle \subseteq \bigwedge^\bullet_R \mathbf{F}_p$ be the homogeneous ideal generated by $Q^{gp} \otimes R$. One computes $\Gamma(A_P, \langle f^*\Omega_{A_Q/\mathrm{Spec}\ R}\rangle_m)_p = \langle Q^{gp} \otimes R\rangle_m$. Using Proposition 6.2, in degree $p \in P$, we obtain the exact sequence

$$0 \to \langle Q^{gp} \otimes R\rangle_m \to \bigwedge^m_R \mathbf{F}_p \to (W^m_f)_p \to 0.$$

Using a splitting of the injection $(Q^{gp} \otimes R) \subseteq \mathbf{F}_p$ and comparing lead to the assertion. □

**Example 6.5.** For the elementary log toroidal family $f : X_{st} \to \mathbb{A}^1$ of Example 1.4, Proposition 6.4 yields the following: The monoid $P$ is divided into four regions

$$P_0 := P \smallsetminus (F_{TX} \cup F_{TY}), \quad P_X := F_{TX} \smallsetminus Q, \quad P_Y := F_{TY} \smallsetminus Q, \quad P_{XY} := Q$$

on which the graded piece $(W^1_f)_p$ is constant. Here $F_{TX}$ and $F_{TY}$ denote facets of $P$ as in Figure 3.1 at Example 3.6. Using the isomorphism $P^{gp} \otimes k / Q^{gp} \otimes k \cong k \oplus k$, we get:

$$\begin{aligned} (W^1_f)_p &= (P^{gp} \otimes k)/(Q^{gp} \otimes k) = k \oplus k && \text{for } p \in P_0, \\ (W^1_f)_p &= (F^{gp}_{TX} \otimes k)/(Q^{gp} \otimes k) = 0 \oplus k && \text{for } p \in P_X, \\ (W^1_f)_p &= (F^{gp}_{TY} \otimes k)/(Q^{gp} \otimes k) = k \oplus 0 && \text{for } p \in P_Y, \\ (W^1_f)_p &= (Q^{gp} \otimes k)/(Q^{gp} \otimes k) = 0 \oplus 0 && \text{for } p \in P_{XY}. \end{aligned}$$

This is visualized in Figure 6.1. For the result, cf. also my Master Thesis.

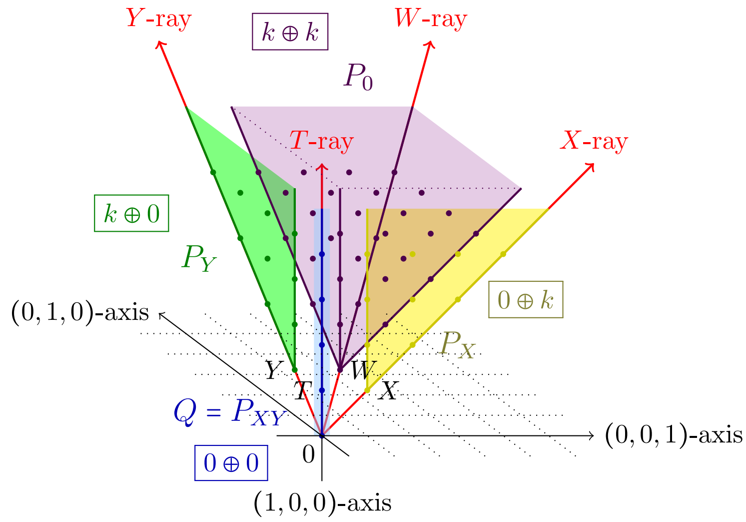


Figure 6.1: We see the four regions $P_{XY}, P_X, P_Y, P_0$ of $P$ of Example 6.5 with the respective graded pieces $(W_f^1)_p$. This figure is also contained in my Master Thesis.

**Corollary 6.6.** *For all $m$, the module $W_f^m$ is flat over $A_Q$.*

*Proof.* By Proposition 6.4, we find that $\Gamma(A_P, W_f^m)$ is a free $R[Q]$-module. □

*Remark* 6.7. We explain the meaning of our description of $W_f^1$. We have the open subscheme $A_P^* = \mathrm{Spec}\ R[P^{gp}] \subset A_{P,\mathcal{F}}$, which is log trivial. The $R$-linear derivations on it are given by

$$\mathrm{Der}(A_P^*) = \bigoplus_{p \in P^{gp}} z^p \cdot \mathrm{Hom}(P^{gp}, R),$$

acting as $(z^p \phi)(z^q) = \phi(q) z^{p+q} \in R[P^{gp}]$. The $R[Q]$-linear derivations are those derivations with $\phi(q) = 0$ for $q \in Q$, so they are given by $\bigoplus_{p \in P^{gp}} z^p \cdot \mathrm{Hom}(P^{gp}/Q^{gp}, R)$. The restriction map

$$\Gamma(A_{P,\mathcal{F}}, \Theta_f^1) \to \bigoplus_{p \in P^{gp}} z^p \cdot \mathrm{Hom}(P^{gp}/Q^{gp}, R)$$

is injective and thus identifies $\Theta_f^1$ with some submodule. The differentials $W_f^1$ are the dual of $\Theta_f^1$ and are thus identified with a submodule of the module $\bigoplus_{p \in P^{gp}} z^p \cdot (P^{gp}/Q^{gp}) \otimes R$ via the pairing

$$\bigoplus_{p \in P^{gp}} z^p \cdot \mathrm{Hom}(P^{gp}/Q^{gp}, R) \times \bigoplus_{p \in P^{gp}} z^p \cdot (P^{gp}/Q^{gp}) \otimes R \to R[P^{gp}]$$

given by $(z^p \phi, z^q r) \mapsto \phi(r) z^{p+q}$. Indeed, it is possible but very tedious to compute $W_f^1$ along these lines, cf. my Master Thesis. Within this framework, one can compute the differential of the de Rham complex: We find first $d(z^p) = z^p \cdot [p]$ for functions $z^p \in W_f^0$ and then $d(z^p \cdot n) = z^p \cdot [p] \wedge n$ for $z^p \cdot n \in W_f^m$.

## 6.2 Base Change for Elementary Log Toroidal Families

We study under which conditions the formation of $W_{X/S}^\bullet$ commutes with base change, for $f : X \to S$ an elementary log toroidal family $A_{P,\mathcal{F}} \to A_Q$. Thus, let $\mathcal{T}$ be a Noetherian ring and $T = \mathrm{Spec}\ \mathcal{T} \to \mathrm{Spec}\ R[Q]$ be any map. Denote by $\sigma$ the composition $Q \to R[Q] \to \mathcal{T}$, which makes $T$ a coherent log scheme. Define $Y$ by the fiber diagram of log toroidal families

$$\begin{tikzcd} Y \arrow[r, "c"] \arrow[d] & A_{P,\mathcal{F}} \arrow[d, "f"] \\ T \arrow[r] & A_Q. \end{tikzcd} \tag{8}$$

Concretely, we study when the natural map $c^*W^m_f \to W^m_{Y/T}$ is an isomorphism. This holds if $f$ is log smooth since then $W^m_f = \Omega^m_f$ are the ordinary log differentials, which satisfy this isomorphism property by their universal property. In particular, $c^*W^m_f \to W^m_{Y/T}$ is always an isomorphism on the open set $V := c^{-1}(U)$. The following example shows that it is not an isomorphism in general. For a subset $I \subset P$, let $\langle I\rangle$ be the smallest face of $P$ containing $I$.

**Example 6.8.** Let $P$ be the submonoid of $\mathbb{Z}^2$ generated by $(1,0),(1,1),(1,2)$, and let $Q = 0$. The monoid $P$ has two facets $H_1 = \langle(1,0)\rangle$ and $H_2 = \langle(1,2)\rangle$, and setting $\mathcal{F} = \varnothing$ yields an ETD. Let $f : A_{P,\mathcal{F}} \to A_Q = \operatorname{Spec}\mathbb{Z}$ be the corresponding map. Now set $\mathcal{T} = \mathbb{Z}/2\mathbb{Z}$, inducing the natural map $T = \operatorname{Spec}\mathcal{T} \to \operatorname{Spec}\mathbb{Z}$ and a fiber diagram as above. One checks that $c^*W^1_f \to W^1_{Y/T}$ is *not* an isomorphism by computing both terms via Proposition 6.2. It suffices to check the degree $p = 0$, indeed, $(W^1_f)_0 = H_1^{gp} \cap H_2^{gp} = 0$ but

$$(W^1_{Y/T})_0 = (H_1^{gp} \otimes \mathbb{Z}/2\mathbb{Z}) \cap (H_2^{gp} \otimes \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}\cdot(1,0) \subset (\mathbb{Z}/2\mathbb{Z})^2.$$

Hence, $((W^1_f)\otimes_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z})_0 = 0$, but $(W^1_{Y/T})_0 \neq 0$.

The example teaches that base change is related to the (non-)commuting of intersection and tensor product. The following lemma (that is an elementary exercise in Tor groups) will help us. We say a ring $\mathcal{T}$ is of *characteristic* $\geq p_0$ if, for all of its residue fields $\kappa_{\mathfrak{p}}$, it holds that $\operatorname{char}\kappa_{\mathfrak{p}} \geq p_0$ or $\operatorname{char}\kappa_{\mathfrak{p}} = 0$.

**Lemma 6.9.** *Let $G$ be a finitely generated $\mathbb{Z}$-module and $H, H' \subset G$ be two submodules. Then there is $p_0 \geq 0$ such that, for every ring $\mathcal{T}$ of characteristic $\geq p_0$, we have*

$$(H \cap H')\otimes\mathcal{T} = (H\otimes\mathcal{T}) \cap (H'\otimes\mathcal{T}),$$

*and each term here is a submodule of $G\otimes\mathcal{T}$.*

In the general situation, observe that we have $\Gamma(Y,\mathcal{O}_Y) = \bigoplus_{e\in E} z^e\cdot\mathcal{T}$ with multiplication

$$z^{e_1}\cdot z^{e_2} = z^e\cdot\sigma(q) \quad \text{whenever} \quad e_1 + e_2 = e + q$$

with $e \in E, q \in Q$ under the canonical decomposion from (4). Similarly, Proposition 6.4 gives

$$\Gamma(Y, c^*W^m_f) = \bigoplus_{e\in E} z^e\cdot((W^m_f)_e \otimes_R \mathcal{T}). \tag{9}$$

**Lemma 6.10.** *Recall $V = c^{-1}(U)$. Equivalent are*

1. *the map $c^*W^m_f \to W^m_{Y/T}$ is an isomorphism,*
2. *$c^*W^m_f$ is reflexive,*
3. *the restriction map $\rho : \Gamma(Y, c^*W^m_f) \to \Gamma(V, c^*W^m_f)$ is surjective.*

*Proof. (1)⇒(2)*: $W^m_{Y/T}$ is reflexive; *(2)⇒(3)*: $c^*W^m_f$ is $(Y\setminus V)$-closed; *(3)⇒(1)*: Consider the commutative square

$$\begin{tikzcd} \Gamma(Y, c^*W^m_f) \arrow[r] \arrow[d, "\rho"] & \Gamma(Y, W^m_{Y/T}) \arrow[d] \\ \Gamma(V, c^*W^m_f) \arrow[r] & \Gamma(V, W^m_{Y/T}), \end{tikzcd}$$

where the right vertical map is an isomorphism because $W^m_{Y/T}$ is reflexive by Lemma 2.13. The bottom horizontal map is an isomorphism by what we said just before Example 6.8. Now *(1)* holds if the top horizontal map is an isomorphism, which follows from *(3)* if $\rho$ is additionally injective. This injectivity is a general fact that we prove next. Recall that $A_{P,\mathcal{F}_{\max}} = A_P$ and we have a map $A_P \to A_{P,\mathcal{F}}$ that gives us another commutative square

$$\begin{tikzcd} \Gamma(Y, c^*W^m_f) \arrow[r] \arrow[d, "\rho"] & \Gamma(Y, c^*W^m_{A_P/A_Q}) \arrow[d] \\ \Gamma(V, c^*W^m_f) \arrow[r] & \Gamma(V, c^*W^m_{A_P/A_Q}). \end{tikzcd} \tag{10}$$

Since $A_P \to A_Q$ is log smooth and $W^m_{A_P/A_Q} = \Omega^m_{A_P/A_Q}$ a free sheaf, the right vertical map is an isomorphism. We get that $\rho$ is injective if the top horizontal map is injective. The latter can be computed from Proposition 6.4. Indeed, this follows from (9) since for every $e \in E$, the cokernel of $(W^m_f)_e \to (W^m_{A_P/A_Q})_e$ is a free $R$-module. $\square$

We next provide a useful criterion for the surjectivity of $\rho$. Let $\mathcal{E}$ be the set of essential faces of $P$ of rank $d-1$. By Lemma 3.12, $U$ is covered by $\{U_F | F \in \mathcal{E}\}$. Set $V_F = c^{-1}(U_F)$, so these cover $V$. For each $F \in \mathcal{E}$, choose $e_F \in F$ in the relative interior, i.e., $\langle e_F \rangle = F$.

**Theorem 6.11.** *Write $M_p := (W^m_f)_p$ for short, and assume that, for every subset $\mathcal{E}' \subset \mathcal{E}$ and every $e \in E$, the natural map*

$$\left(\bigcap_{F\in\mathcal{E}'} M_{e+e_F}\right) \otimes_R \mathcal{T} \to \bigcap_{F\in\mathcal{E}'} (M_{e+e_F} \otimes_R \mathcal{T})$$

*is an isomorphism. Then $\rho$ is surjective.*

*Proof.* We write $M = \Gamma(A_P, W^m_f)$, $N = \Gamma(A_P, W^m_{A_P/A_Q})$ and $N_p$ for the degree $p$ part of $N$. By Proposition 6.4, $M_p$ and $N_p$ only depend on $\langle p \rangle$. We are going to use that, for $p_1, p_2 \in P$, it holds

$$\langle p_1 + p_2 \rangle = \langle \langle p_1 \rangle \cup \langle p_2 \rangle \rangle. \tag{11}$$

We have a natural injection $M \subseteq N$ by Proposition 6.4. Given $\mu \in \Gamma(V, c^*W^m_f)$, we want to show it has a preimage under $\rho$. We do have a unique preimage $\nu$ under the right vertical map of (10), so in $N \otimes_{R[Q]} \mathcal{T}$ and we are going to show that this preimage lies in $M \otimes_{R[Q]} \mathcal{T}$. Say $\nu = \sum_e z^e \cdot n_e$ with $n_e \in N_e \otimes \mathcal{T}$ is such that $\nu|_V = \mu$. In particular, $\nu|_{V_F} = \mu|_{V_F}$ for all $F \in \mathcal{E}$. There is some large $a \geq 1$ so that, for each $F \in \mathcal{E}$, there are $m_{F,e} \in M_e \otimes \mathcal{T}$ such that

$$\mu|_{V_F} = z^{-ae_F} \sum_e z^e \cdot m_{F,e},$$

and therefore $\nu|_{V_F} = \mu|_{V_F}$ implies

$$z^{ae_F} \sum_e z^e \cdot n_e \;\in\; \bigoplus_{e\in E} z^e \cdot (M_e \otimes_R \mathcal{T}) \;\subset\; \bigoplus_{e\in E} z^e \cdot (N_e \otimes_R \mathcal{T}).$$

If $e + ae_F = \tilde{e} + q$ is the decomposition $P = E \times Q$, then $n_e \cdot \sigma(q) \in M_{\tilde{e}} \otimes_R \mathcal{T}$. By (11),

$$e + ae_F \in E \iff \langle e + e_F \rangle \subset E \iff e + e_F \in E$$

and if this holds, then $\sigma(q) = 1$, so setting

$$\mathcal{E}_e := \{F \in \mathcal{E} \mid e + e_F \in E\},$$

we obtain $n_e \in \bigcap_{F\in\mathcal{E}_e} (M_{e+ae_F} \otimes_R \mathcal{T})$ and $M_{e+ae_F} = M_{e+e_F}$. Note that $\mathcal{E}_e$ does not depend on the chosen $e_F$. Using the assumption, we get

$$n_e \in \bigcap_{F\in\mathcal{E}_e} (M_{e+e_F} \otimes_R \mathcal{T}) = \left(\bigcap_{F\in\mathcal{E}_e} M_{e+e_F}\right) \otimes_R \mathcal{T}.$$

For the next step, define $\mathcal{F}_e = \{H \in \mathcal{F}_{\max} \setminus \mathcal{F} \mid \exists F \in \mathcal{E}_e : e + e_F \in H\}$. We use Lemma 6.1 to compute

$$\bigcap_{F \in \mathcal{E}_e} M_{e+e_F} = \bigwedge_R^m \left( \bigcap_{H \in \mathcal{F}_e} \frac{H^{gp} \otimes_{\mathbb{Z}} R}{Q^{gp} \otimes_{\mathbb{Z}} R} \right).$$

We finally claim that $\mathcal{F}_e = \{H \in \mathcal{F}_{\max} \setminus \mathcal{F} \mid e \in H\}$; indeed, given an $H$ in the latter, we just need to exhibit an $F \in \mathcal{E}$ that is also contained in $H$ with $\langle e, F \rangle \subset E$, which can be done since $H \cap E$ is a union of faces in $\mathcal{E}$. Thus, $n_e \in M_e \otimes_R \mathcal{T}$, so indeed $\nu \in M \otimes_{R[Q]} \mathcal{T}$, and we are done. □

**Corollary 6.12.** *Let $(Q \subset P, \mathcal{F})$ be an ETD, $\mathcal{T}$ a Noetherian ring and $T = \operatorname{Spec} \mathcal{T} \to A_Q$ a strict morphism of log schemes. Then $c^* W_f^m$ is reflexive and $c^* W_f^m \to W_{Y/T}^m$ an isomorphism provided that the composition*

$$R \to R[Q] \to \mathcal{T}$$

*is flat, e.g. when $R$ is a field.*

As Example 6.8 shows, the conditions of Lemma 6.11 are not always satisfied in case $R = \mathbb{Z}$. However, we do get close:

**Corollary 6.13.** *Let $(Q \subset P, \mathcal{F})$ be an ETD, and assume $f : A_{P,\mathcal{F}} \to A_Q$ to be defined over $R = \mathbb{Z}$. Then there is a $p_0 = p_0(Q \subset P, \mathcal{F})$ such that, for $T = \operatorname{Spec} \mathcal{T} \to A_Q$ with a Noetherian ring $\mathcal{T}$ of characteristic $\geq p_0$, the sheaf $c^* W_f^m$ is reflexive, and $c^* W_f^m \to W_{Y/T}^m$ is an isomorphism.*

*Proof.* Apply Lemma 6.9 recursively and use that the modules $M_p \subset \bigwedge_{\mathbb{Z}}^m (P^{gp}/Q^{gp})$ are free direct summands and that the set of situations to consider for the assumption of Theorem 6.11 is finite. □

## 6.3 Global Base Change

We globalize the base change results that we obtained for elementary log toroidal families. Since $W_{X/S}^\bullet$ is not always compatible with base change by Example 6.8, we give a name to the situation in which it is.

**Definition 6.14** (BC)**.** A generically log smooth family $f : X \to S$ has the *base change property* if, for every strict morphism $T \to S$ of Noetherian fs log schemes, every $m \in \mathbb{Z}$ and $c$ the map given by the Cartesian diagram

$$\begin{tikzcd} Y \arrow[r, "c"] \arrow[d, "g"'] & X \arrow[d, "f"'] \\ T \arrow[r, "b"] & S, \end{tikzcd} \tag{BC}$$

the sheaf $c^* W_{X/S}^m$ is reflexive, or equivalently, the natural map $c^* W_{X/S}^m \to W_{Y/T}^m$ is an isomorphism.

**Theorem 6.15** (Base Change over Fields)**.** *Let $f : X \to S$ be a log toroidal family defined over a field $k$, then $f$ has the base change property.*

*Proof.* Reflexivity is étale local and $S$ is defined over $k$, so we can assume $S, T$ affine and that we have a diagram

$$\begin{tikzcd} Y \arrow[r, "c"] \arrow[d] & X \arrow[d, "h"] & \\ L_T \arrow[r] \arrow[d] & L \arrow[r, "d"] \arrow[d] & A_{P,\mathcal{F}} \times k \arrow[d, "f"] \\ T \arrow[r, "b"] & S \arrow[r] & A_Q \times k \end{tikzcd}$$

with Cartesian squares and $h$ étale. Now Corollary 6.12 shows $W^m_{X/S} \cong (d \circ h)^* W^m_f$ and $W^m_{Y/T} \cong (d \circ h \circ c)^* W^m_f$, so we conclude $c^* W^m_{X/S} \cong W^m_{Y/T}$. □

**Theorem 6.16** (Generic Base Change)**.** *Let $f : X \to S$ be a log toroidal family. Then there is a finite set of prime numbers $p_1, ..., p_N \in \mathbb{Z}$ such that if $f^\circ : X^\circ \to S^\circ$ is obtained from $f$ by inverting $p_1, ..., p_N$ (i.e., base change to* Spec $\mathbb{Z}_{p_1 \dots p_N}$*), then $f^\circ$ has the base change property.*

*Proof.* This follows analogously from the local statement Corollary 6.13. Since there is a finite cover by local models, we need to invert only finitely many primes. □

*Remark* 6.17. In case $f : X \to S$ is log Gorenstein, we have locally $W^d_{X/S} \cong \mathcal{O}_X$. Thus, locally $\Theta^m_{X/S} \cong W^{d-m}_{X/S}$ and therefore $c^* \Theta^m_{X/S}$ is reflexive as well.

An application of the above theorems is the following lemma, which is crucial for the degeneration of the Hodge–de Rham spectral sequence.

**Lemma 6.18** (cf. Prop. 6.6 in [46])**.** *Let $f : X \to S$ be a proper log toroidal family with $S$ affine, and let $b : T \to S$ with $T$ affine. Assume $c^* W^m_{X/S} = W^m_{Y/T}$ holds for all $m$. Then we have isomorphisms*

$$Lb^* Rf_* W^p_{X/S} \to Rg_* W^p_{Y/T}, \tag{12}$$

$$Lb^* Rf_* W^\bullet_{X/S} \to Rg_* W^\bullet_{Y/T} \tag{13}$$

*in $D^b(T)$. If, for fixed $p$, all $R^q f_* W^p_{X/S}$ are locally free of constant rank, then* (12) *induces an isomorphism*

$$b^* R^q f_* W^p_{X/S} \xrightarrow{\cong} R^q g_* W^p_{Y/T}.$$

*If, for all $n$, the sheaf $R^n f_* W^\bullet_{X/S}$ is locally free of constant rank, then* (13) *induces an isomorphism*

$$b^* R^n f_* W^\bullet_{X/S} \xrightarrow{\cong} R^n g_* W^\bullet_{X/S}.$$

*Proof.* Since we know the flatness of $W^m_{X/S}$ over $S$, which is Corollary 6.6, the proof becomes identical to that in [46, Prop. 6.6]. □

### The Tangent Sheaf $\Theta^1_{X/S}$

We briefly indicate the behavior of the tangent sheaf $\Theta^1_{X/S}$ and of the Gerstenhaber algebra $G^\bullet_{X/S}$ under base change. Let $f : X \to S$ be a generically log smooth family with the base change property, and consider a base change diagram (BC). We obtain an $\mathcal{O}_X$-module homomorphism

$$T_c : \Theta^1_{X/S} \to c_* \Theta^1_{Y/T}$$

by pulling back homomorphisms $\varphi : W^1_{X/S} \to \mathcal{O}_X$ since $c^* W^1_{X/S} \cong W^1_{Y/T}$. This means $T_c(\varphi)$ is the unique $\mathcal{O}_Y$-module homomorphism which renders the diagram

$$\begin{array}{ccc} W^1_{X/S} & \xrightarrow{\varphi} & \mathcal{O}_X \\ \downarrow & & \downarrow \\ c_* W^1_{Y/T} & \xrightarrow{c_* T_c(\varphi)} & c_* \mathcal{O}_Y \end{array}$$

commutative. Using notation from Section 2.4 we find:

**Lemma 6.19.** *Assume $f : X \to S$ has the base change property. Then, for $\theta, \xi \in \Theta^1_{X/S}$ and $\omega \in W^1_{X/S}$, we have $[T_c(\theta), T_c(\xi)] = T_c[\theta, \xi]$ and $\langle c^* \omega, T_c(\theta) \rangle = c^* \langle \omega, \theta \rangle$. In particular, we have an induced morphism $G^\bullet_{X/S} \to c_* G^\bullet_{Y/T}$ of Gerstenhaber algebras.*

*Proof.* Use the unicity of $T_c(\phi)$ for the first identity and the commutative diagram for the second identity. The last statement follows from the construction of the Gerstenhaber structure, especially Lemma 2.27. $\square$

*Remark* 6.20. One might wonder to what extent the Gerstenhaber algebra is functorial (like the de Rham complex). Unlike differential forms, for morphisms $c : Y \to X, f : X \to S$, there is *no* (natural) morphism $\Theta^1_{X/S} \to c_*\Theta^1_{Y/S}$ in general. Thus the Gerstenhaber algebra is less functorial than the de Rham complex.

In the log Gorenstein case, i.e., when $W^d_{X/S}$ is a line bundle, Remark 6.17 yields:

**Corollary 6.21.** *Let $f : X \to S$ be a log Gorenstein generically log smooth family with the base change property. Then we have an isomorphism $c^*\Theta^m_{X/S} \cong \Theta^m_{Y/T}$.*

**Question 6.22.** Is the log Gorenstein assumption necessary?

## 6.4 Absolute Differential Forms and the Analytic Theory

For a log toroidal family $Y \to T$, where $T$ is defined over a field $k$, *absolute* differential forms are defined as $W^\bullet_Y := j_*\Omega^\bullet_{V/k}$. Here $j : V \subset Y$ is the specified open of log smoothness. Absolute differential forms over $T = \mathrm{Spec}\ \mathbb{C}[t]/t^{k+1}$ play an important role in Steenbrink's method to deduce the relative degeneration from the absolute one. We employ this method in Section 8 to prove Theorem 8.2. Here, we study absolute differential forms both algebraically and analytically on families $Y \to T$ which are base changes of $A_{P,\mathcal{F}} \to A_Q$ for an ETD $(Q \subset P, \mathcal{F})$. More precisely, consider a monoid ideal $K \subset Q$, let $(K) \subset k[Q]$ denote the corresponding monomial ideal of $k[Q]$, and set $\mathcal{T} = k[Q]/(K)$. The map $T = \mathrm{Spec}\ \mathcal{T} \to k[Q]$ is the natural one, and $Y \to T$ is defined by (8) as before. For the analytic part, we will assume $k = \mathbb{C}$.

#### Absolute Differential Forms on $Y \to T$

Setting $E_K := P \smallsetminus (P + K)$, we generalize the union of essential faces $E$, indeed we have $E = E_{Q\smallsetminus\{0\}}$. Combining Proposition 6.4 with Corollary 6.12 (for $R = k$) gives the following result, which also generalizes [31, Corollary 1.13].

**Corollary 6.23.** *We have*

$$\Gamma(Y, W^m_{Y/T}) \cong \bigoplus_{e \in E_K} z^e \cdot \bigwedge^m \left( \bigcap_{H \in \mathcal{F}_{\max} \smallsetminus \mathcal{F} : e \in H} (H^{gp} \otimes k)/(Q^{gp} \otimes k) \right)$$

*with differential $d(z^e \cdot n) = z^e \cdot [e] \wedge n$.*

With $c : Y \to A_{P,\mathcal{F}}$ the notation from before, we apply $c^*$ to the split exact sequence given by the top row of (7) and obtain another split exact sequence. The left term is free and $c^*W^m_f$ is reflexive by Corollary 6.12. Hence $c^*W^m_{A_{P,\mathcal{F}}/k}$ is reflexive. With $V = c^{-1}(U)$, we find the natural surjection $c^*\Omega^\bullet_{U/k} \to \Omega^\bullet_{V/k}$ to be an isomorphism (e.g. by local freeness of both). We thus have $c^*W^m_{A_{P,\mathcal{F}}/k} \cong W^m_Y$. Plugging this into Proposition 6.2 yields the following result.

**Corollary 6.24.** *We have*

$$\Gamma(Y, W^m_Y) \cong \bigoplus_{e \in E_K} z^e \cdot \bigwedge^m \left( \bigcap_{H \in \mathcal{F}_{\max} \smallsetminus \mathcal{F} : e \in H} H^{gp} \otimes k \right)$$

*with differential $d(z^e \cdot n) = z^e \cdot e \wedge n$.*

*Remark* 6.25. This description of differential forms is inspired by [31, Prop. 1.12], which is essentially the same result in less generality.

**Local Analytic Theory**

We keep the setup and notation from before (with $k = \mathbb{C}$), so $(Q \subset P, \mathcal{F})$ is an ETD and $K \subset Q$ a monoid ideal. We additionally assume that $Q \setminus K$ is finite, so $\mathcal{T} = \mathbb{C}[Q]/(K)$ is an Artinian local ring. For $P^+ = P \setminus \{0\}$, let $\mathbb{C}[\![P]\!]$ be the completion of $\mathbb{C}[P]$ in $(P^+)$. It is the complete local ring of $A_P$ at 0. Ogus determines the analytic local ring inside $\mathbb{C}[\![P]\!]$ as follows:

**Lemma 6.26** (V, Prop. 1.1.3 in [67]). *For every local homomorphism $h : P \to \mathbb{N}$, i.e., we have $h^{-1}(0) = \{0\}$ and we may view $h$ as a grading, it holds*

$$\mathcal{O}_{A_P^{an},0} = \left\{ \sum_{p \in P} \alpha_p z^p \;\middle|\; \alpha_p \in \mathbb{C},\ \sup_{p \in P^+} \left\{ \frac{\log |\alpha_p|}{h(p)} \right\} < \infty \right\} \subset \mathbb{C}[\![P]\!].$$

We have $\Gamma(Y, \mathcal{O}_Y) \cong \mathbb{C}[E_K] := \bigoplus_{e \in E_K} \mathbb{C} \cdot z^e$ with $z^e \cdot z^{e'} = z^{e+e'}$ if $e + e' \in E_K$ and $z^e \cdot z^{e'} = 0$ otherwise. By [39, Cor. 3.2] and Lemma 6.26, the complete local ring at the origin in $Y^{an}$ is

$$\hat{\mathcal{O}}_{Y,0} \cong (\mathbb{C}[Q]/(K)) \otimes_{\mathbb{C}[\![Q]\!]} \mathbb{C}[\![P]\!] \cong \left\{ \sum_{e \in E_K} \alpha_e z^e \right\} =: \mathbb{C}[\![E_K]\!].$$

Lemma 6.26 together with Krull's intersection theorem and the surjectivity of the map $\mathcal{O}_{A_P^{an},0} \to \mathcal{O}_{Y^{an},0}$ yields

$$\mathcal{O}_{Y^{an},0} = \left\{ \sum_{e \in E_K} \alpha_e z^e \in \mathbb{C}[\![E_K]\!] \;\middle|\; \sup_{e \in E_K \setminus 0} \left\{ \frac{\log |\alpha_e|}{h(e)} \right\} < \infty \right\}. \tag{14}$$

The next lemma describes the analytic stalks of $\mathcal{O}_Y$-modules of a particular form. Note that the differential forms $W^m_{Y/T}$ and $W^m_Y$ are of this form by Corollaries 6.23 and 6.24.

**Lemma 6.27.** *Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional $\mathbb{C}$-vector space with a Hermitian inner product. For every $e \in E_K$, let $V_e \subset V$ be subvector spaces so that*

$$\tilde{V} := \bigoplus_{e \in E_K} z^e \cdot V_e \subset V[E_K]$$

*is a $\mathbb{C}[E_K]$-module. Assume moreover that $V_e \subset V$ depends only on the set*

$$F(e) := \{H \subset P \text{ a facet} \mid Q \subset H, e \in H\}.$$

*Set $V[\![E_K]\!] := \prod_{e \in E_K} z^e \cdot V_e$, and let $\mathcal{V}^{an} := \tilde{V} \otimes_{\mathbb{C}[E_K]} \mathcal{O}_{Y^{an}}$ be the coherent analytic sheaf associated to $\tilde{V}$. We find its stalk at the origin to be*

$$\mathcal{V}^{an}_0 \cong \left\{ \sum_{e \in E_K} z^e \cdot v_e \in V[\![E_K]\!] \;\middle|\; \sup_{e \in E_K \setminus 0} \left\{ \frac{\log \|v_e\|}{h(e)} \right\} < \infty \right\},$$

*where $\|\cdot\|$ denotes the absolute value on $V$ induced by $\langle \cdot, \cdot \rangle$.*

*Proof.* For brevity, we denote the right hand side of the claim by $M$. By definition, we have $\mathcal{V}^{an}_0 = \tilde{V} \otimes_{\mathbb{C}[E_K]} \mathcal{O}_{Y^{an},0}$. Denoting the completed stalk by $\hat{V}_0$, we get a sequence of injections $\tilde{V}^{an}_0 \to \hat{V}_0 \to V[\![E_K]\!]$ and find the image of $\mathcal{V}^{an}_0 \to V[\![E_K]\!]$ to be in $M$ by a direct computation.

For every set $\mathcal{G}$ of facets of $P$ containing $Q$, we set

$$S_{\mathcal{G}} := \{e \in P \mid F(e) = \mathcal{G}\} \subset \bigcap_{F \in \mathcal{G}} F =: F_{\mathcal{G}},$$

so $F_{\mathcal{G}} \subset P$ is a face, and $S_{\mathcal{G}} \subset F_{\mathcal{G}}$ is an ideal. There are $s^1_{\mathcal{G}}, ..., s^k_{\mathcal{G}} \in S_{\mathcal{G}}$ with $S_{\mathcal{G}} = \bigcup_{i=1}^k (s^i_{\mathcal{G}} + F_{\mathcal{G}})$, and we can find subsets $T^i_{\mathcal{G}} \subset s^i_{\mathcal{G}} + F_{\mathcal{G}}$ such that $S_{\mathcal{G}} = \coprod_{i=1}^k T^i_{\mathcal{G}}$ disjointly. Given an element $v = \sum z^e v_e \in M$, we want to show it in the image of $\mathcal{V}^{an}_0 \to V[\![E_K]\!]$. We decompose $v$ as

$$v = \sum_{\mathcal{G}} \sum_i \sum_{e \in E_K \cap T^i_{\mathcal{G}}} z^e v_e.$$

Choosing one orthonormal basis $v_{\mathcal{G}}^1, ..., v_{\mathcal{G}}^\ell$ of all $V_e$ with $e \in E_K \cap S_{\mathcal{G}}$, we expand $v_e = \sum_j \alpha_e^j v_{\mathcal{G}}^j$. Without loss of generality, we assume $s_{\mathcal{G}}^i \in E_K$ for otherwise $E_K \cap T_{\mathcal{G}}^i = \varnothing$. Setting $\delta_{ip} = 1$, if $s_{\mathcal{G}}^i + p \in E_K \cap T_{\mathcal{G}}^i$ and $\delta_{ip} = 0$ otherwise, we write

$$\sum_{e \in E_K \cap T_{\mathcal{G}}^i} z^e v_e = \sum_{p \in F_{\mathcal{G}}} z^{s_{\mathcal{G}}^i + p} \delta_{ip} \sum_{j=1}^{\ell} \alpha_{s_{\mathcal{G}}^i + p}^j v_{\mathcal{G}}^j = \sum_{j=1}^{\ell} \left( \sum_{p \in F_{\mathcal{G}}} z^p \delta_{ip} \alpha_{s_{\mathcal{G}}^i + p}^j \right) \cdot z^{s_{\mathcal{G}}^i} v_{\mathcal{G}}^j.$$

Defining $f_{\mathcal{G},i,j} := \sum_{p \in F_{\mathcal{G}}} z^p \delta_{ip} \alpha_{s_{\mathcal{G}}^i + p}^j \in \mathbb{C}[\![E_K]\!]$, we have

$$\sup_{p \in E_K \setminus 0} \left\{ \frac{\log |\alpha_{s_{\mathcal{G}}^i + p}^j|}{h(p)} \right\} < \infty,$$

hence $f_{\mathcal{G},i,j} \in \mathcal{O}_{Y^{an},0}$. Namely $|\alpha_{s_{\mathcal{G}}^i + p}^j| \le \|v_{s_{\mathcal{G}}^i + p}\|$ since the basis is orthonormal, and furthermore $\frac{1}{h(p)} \le \frac{1}{h(s_{\mathcal{G}}^i + p)} \cdot (1 + h(s_{\mathcal{G}}^i))$, so the set is bounded. Hence $\mathcal{V}_0^{an} \xrightarrow{\cong} M \subset V[\![E_K]\!]$. ☐

# 7 Differential Forms in Characteristic $p > 0$

We adapt the positive characteristic part of Deligne–Illusie's approach to the Hodge–de Rham spectral sequence in [17] to our setting. To this end, we introduce the Cartier isomorphism for log toroidal families. It is a key ingredient for the Decomposition Theorem 7.11 below, which lies at the heart of the $E_1$-degeneration of the Hodge–de Rham spectral sequence and is an analog of [17, Thm. 2.1].

## 7.1 The Cartier Isomorphism

In this section, we define the *Cartier homomorphism* for a generically log smooth family $f : X \to S$ in characteristic $p > 0$. We then prove that it is an isomorphism if $f$ is log toroidal. Similar to [8], we first study the situation on $U$ and then examine its extension to all of $X$. Let $F_S : S \to S$ be the absolute log Frobenius on the base, i.e., given by taking $p$-th power in $\mathcal{M}_S$ and $\mathcal{O}_S$ respectively. Similarly, $F_X : X \to X$ is the absolute log Frobenius on $X$ (with log part defined on $U$). We define $f' : X' \to S$ and the relative Frobenius $F$ by the Cartesian square

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[rr, bend left, "F_X"] \arrow[dr, "f"'] & X' \arrow[r, "s"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ & S \arrow[r, "F_S"] & S. \end{tikzcd}$$

Set $U' := s^{-1}(U)$ and $Z' = X' \setminus U'$.

**Theorem 7.1** ([50]). *We have a canonical (Cartier) isomorphism of $\mathcal{O}_{U'}$-modules*

$$C_U^{-1} : \Omega_{U'/S}^m \to \mathcal{H}^m(F_* \Omega_{U/S}^\bullet),$$

*which is compatible with $\wedge$ and satisfies $C^{-1}(a) = F^*(a)$ for $a \in \mathcal{O}_{X'}$ and $C^{-1}(\mathrm{dlog}(s^* q)) = \mathrm{dlog}(q)$ for $q \in \mathcal{M}_U$.*

*Proof.* This is [50, Thm. 4.12(1)] once we identify $U'' = U'$: Kato considers the factorization $U \xrightarrow{g} U'' \xrightarrow{h} (U')^{\mathrm{int}} \xrightarrow{i} U'$ of $F|_U$, where $i$ is the integralization of $U'$ and $g \circ h$ is the unique factorization of this weakly purely inseparable morphism, where $h$ is étale and $g$ purely inseparable, using [50, Prop. 4.10(2)]. Now $i$ is an isomorphism because $f$ is integral. By [67, III, Cor. 2.5.4], since $f : U \to S$ is saturated, the relative Frobenius $F : U \to U'$ is exact. The uniqueness of the factorization $g \circ h$ implies that $h$ is an isomorphism. ☐

Since $W^m_{X'/S}$ is $Z'$-closed, pushing forward the inverse of $C_U^{-1}$ to $X'$, we obtain a homomorphism

$$C : \mathcal{H}^m(F_* W^\bullet_{X/S}) \to W^m_{X'/S},$$

which is an isomorphism on $U'$. We obtain the following lemma.

**Lemma 7.2.** *The map $C$ is an isomorphism if and only if $\mathcal{H}^m(F_* W^\bullet_{X/S})$ is $Z'$-closed.*

**Definition 7.3.** We say that a generically log smooth family $f : X \to S$ in positive characteristic has the *Cartier isomorphism property* if $C$ is an isomorphism for all $m \geq 0$.

By Theorem 7.1, the cohomology sheaf $\mathcal{H}^m(F_* W^\bullet_{X/S})$ is locally free on $U'$, hence it is $Z'$-closed if and only if it is reflexive. Reflexivity can be checked étale locally.

**Lemma 7.4.** *Let $(Q \subset P, \mathcal{F})$ be an ETD, let $b : T \to A_Q$ be strict with $\underline{T}$ = Spec $\mathcal{T}$, and consider the Cartesian diagram*

$$\begin{tikzcd} Y \arrow[r, "c"] \arrow[d, "g"'] & A_{P,\mathcal{F}} \arrow[d, "f"] \\ T \arrow[r, "b"] & A_Q. \end{tikzcd}$$

*Then $\mathcal{H}^m(F_* W^\bullet_{Y/T})$ is reflexive.*

**Corollary 7.5.** *Every log toroidal family $f : X \to S$ over $\mathbb{F}_p$ has the Cartier isomorphism property.*

*Proof of Lemma 7.4.* Set $V := c^{-1}(U_P)$ and let $Y', V'$ be the base changes by the absolute Frobenius $F_T$. Let $F : Y \to Y'$ be the relative Frobenius. Inspired by the Frobenius decomposition [17, Thm. 2.1], we construct a homomorphism $\phi^\bullet : \bigoplus_m W^m_{Y'/T}[-m] \to F_* W^\bullet_{Y/T}$ of *complexes of $\mathcal{O}_{Y'}$-modules*, which induces an isomorphism in cohomology. Since the left hand side has zero differentials, the assertion then follows from the reflexivity of $W^m_{Y'/T}$ given by Lemma 2.13.

Similar to Section 6.2, we find explicitly that $R' := \Gamma(Y', \mathcal{O}_{Y'}) = \bigoplus_{e \in E} z^e \cdot \mathcal{T}$ with

$$z^{e_1} \cdot z^{e_2} = z^e \cdot \sigma(q)^p \quad \text{whenever} \quad e_1 + e_2 = e + q$$

with $e \in E, q \in Q$. We have $s^*(z^e \cdot t) = z^e \cdot t^p$ and $F^*(z^e \cdot t) = z^{p \cdot e} \cdot t$. After writing $W^m_e := (W^m_f)_e \otimes_{\mathbb{F}_p} \mathcal{T}$, the module $\Gamma(Y', W^m_{Y'/T})$ is given by the $\mathcal{T}$-module $\bigoplus_{e \in E} z^e \cdot W^m_e$, on which $R'$ acts as

$$(z^{e_1} \cdot t_1) \cdot [z^{e_2} \cdot (w \otimes t_2)] = z^e \cdot (w \otimes \sigma(q)^p t_1 t_2) \quad \text{whenever} \quad e_1 + e_2 = e + q$$

with $e \in E, q \in Q$. Similarly, $\Gamma(Y', F_* W^m_{Y/T})$ is given by the same $\mathcal{T}$-module, however now $R'$ acting via $F^*$ as

$$(z^{e_1} \cdot t_1) \cdot [z^{e_2} \cdot (w \otimes t_2)] = z^e \cdot (w \otimes \sigma(q) t_1 t_2) \quad \text{whenever} \quad p \cdot e_1 + e_2 = e + q.$$

Note the subtle difference. The differential on $F_* W^\bullet_{Y/T}$ is given by

$$d(z^e \cdot (w \otimes t)) = z^e \cdot ([e] \wedge w \otimes t).$$

We define

$$\phi^\bullet : \bigoplus_m W^m_{Y'/T}[-m] \to F_* W^\bullet_{Y/T}, \quad z^e \cdot (w \otimes t) \mapsto z^{p \cdot e} \cdot (w \otimes t)$$

and claim that $\mathcal{H}^m(\phi^\bullet)$ is an isomorphism. Indeed, first note that $\phi^\bullet$ itself is injective. Then set $E_p = \{p \cdot e | e \in E\}$. We have $\mathrm{im}(\phi^m) = \bigoplus_{e \in E_p} z^e \cdot W^m_e$ because $W^m_e = W^m_{e/p}$ for $e \in E_p$ by Prop. 6.4. Denoting the coboundaries of $F_* W^m_{Y/T}$ by $B^m$, we have $\mathrm{im}(\phi^m) \cap B^m = 0$ since $0 = [e] \in W^1_e$ for $e \in E_p$ because $e = pe'$ and $p$ is zero in $\mathcal{T}$. This readily gives that $\mathcal{H}^m(\phi^\bullet)$ is injective. For surjectivity, if $e \notin E_p$, observe that $[e] \neq 0$, so if $w \in W^m_e$, then $[e] \wedge w = 0$ if and only if there is some $w' \in W^{m-1}_e$ with $[e] \wedge w' = w$. ☐

*Remark* 7.6. We believe that $\mathcal{H}^m(\phi^\bullet)$ is the log Cartier isomorphism on $V'$, but we did not prove it and do not need it.

## 7.2 The Decomposition of $F_* W^\bullet_{X_0/S_0}$

We prove Theorem 7.11 below, which is a log version of the Decomposition Theorem [17, Thm. 2.1] in the setting of generically log smooth families. Though rather technical, it is the key ingredient in the proof of the Degeneration Theorem 8.1. The assumption for $f : X \to S$ to be saturated on the log smooth locus allows a simpler approach than [50, Thm. 4.12].

Our setting is as follows: Let $k$ be a perfect field with char $k = p$, and let $Q$ be a sharp toric monoid. We set

$$S_0 = \mathrm{Spec}\ (Q \to k) \quad \text{and} \quad S = \mathrm{Spec}\ (Q \to W_2(k))$$

where in both cases $Q \ni q \mapsto 0$ except $0 \mapsto 1$. Here $W_2(k)$ is the ring of second Witt vectors, which is flat over $\mathbb{Z}/p^2\mathbb{Z}$ since $k$ is perfect. The Frobenius endomorphism on $k$ becomes an endomorphism $F_0$ of $S_0$ via $Q \ni q \mapsto pq$. Similarly, its lift to $W_2(k)$ defined via $(a_1, a_2) \mapsto (a_1^p, a_2^p)$ becomes[1] an endomorphism $F_S$ of $S$ that restricts to $F_0$ on $S_0$. Let $f : X \to S$ be a generically log smooth family and let $f_0 : X_0 \to S_0$ be its restriction to $S_0$. We consider the commutative diagram of generically log smooth families

$$\begin{tikzcd}
 & X_0 \arrow[rr, "F"] \arrow[dl, "i"'] \arrow[ddrr, bend left, "f_0"'] & & X'_0 \arrow[rr, "s"] \arrow[dl, "i'"'] \arrow[dd, "f'_0"] & & X_0 \arrow[dl] \arrow[dd, "f_0"] & & \\
X \arrow[rr, dotted, "G"] \arrow[ddrr, bend right, "f"'] & & X' \arrow[rr] \arrow[dd, "f'"] & & X \arrow[dd, "f"] & & & \\
 & & & S_0 \arrow[rr, "F_0"] \arrow[dl] & & S_0 \arrow[rr] \arrow[dl] & & \mathrm{Spec}\ \mathbb{F}_p \arrow[dl] \\
 & & S \arrow[rr, "F_S"] & & S \arrow[rr] & & \mathrm{Spec}\ \mathbb{Z}/p^2\mathbb{Z}, &
\end{tikzcd}$$

where $X'_0, X'$ are defined by requiring the front and back square to be Cartesian and $F$ is the relative Frobenius, i.e., $F$ is induced by the back square's Cartesianness using the Frobenius endomorphisms on $X_0$ and $S_0$. Since $X$ does not have a Frobenius, we do *not* easily obtain the dotted arrow $G$ in a similar way, and in general, it does not exist globally. We call a locally defined morphism $G$ that fits into the diagram a *local Frobenius lifting*. Because the (Zariski or étale) topologies are identified along $F$ and $i$, we can define Frobenius liftings simply at the level of sheaves:

**Definition 7.7.** Let $Y' \to X'$ be an étale open. Then a Frobenius lifting $G : Y \to Y'$ on $Y'$ consists of a ring homomorphism $G^* : \mathcal{O}_{Y'} \to G_*\mathcal{O}_Y$ yielding a morphism of schemes and a monoid homomorphism $G^* : \mathcal{M}_{Y'}|_{V'} \to G_*\mathcal{M}_Y|_{V'}$ defined on some $V' \subset Y'$ satisfying $(CC)$, yielding a log morphism. Two Frobenius liftings are considered equal if they are equal on some smaller (Zariski) open satisfying $(CC)$. The Frobenius liftings form an étale sheaf of sets $\mathcal{F}rob(X, X')$.

*Remark* 7.8. We need the flexibility of $V'$ in the definition of $\mathcal{F}rob(X, X')$ to construct Frobenius liftings from local models as they occur for log toroidal families. We will see below that we could have as well required the log part to be defined on $Y' \cap U'$, see the proof of Proposition 7.10.

Let $j : U' \subset X'$ denote the pullback of $U \subset X$ and $Z' = X' \setminus U'$. By Lemma 2.7, $\mathcal{F}rob(X, X') = j_*\mathcal{F}rob(X, X')|_{U'}$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf defining $X_0 \subset X$. Because $X$ is flat over $\mathbb{Z}/p^2\mathbb{Z}$ (in particular because $k$ is perfect), we have $\mathcal{I} = p \cdot \mathcal{O}_X \cong \mathcal{O}_{X_0}$. Using $\mathcal{I}^2 = 0$, one checks that $F_*\mathcal{I}$ is an $\mathcal{O}_{X'}$-module. Considering derivations on $U'$ with values in $F_*\mathcal{I}$, we obtain a sheaf of groups

$$\mathcal{G} := j_*\mathcal{D}er_{U'/S}(F_*\mathcal{I}) = j_*\mathcal{H}om(\Omega^1_{U'/S}, F_*\mathcal{I}),$$

which agrees with $\mathcal{H}om(W^1_{X'/S}, F_*\mathcal{I})$ because $F_*\mathcal{I}$ is $Z'$-closed by Lemma 2.7.

[1] Warning: This is *not* the $p$th power map on $W_2(k)$ and thus depends on the chosen chart.

**Lemma 7.9.** *The restriction $\mathcal{F}rob(X,X')|_{U'}$ is a $\mathcal{G}|_{U'}$-torsor, so $\mathcal{F}rob(X,X')$ is a $\mathcal{G}$-pseudo-torsor.*

*Proof.* Let $\mathcal{D}$ be the sheaf of sets on $U'$ given by étale local deformations of the diagram

$$\begin{tikzcd} U_0 \arrow[r, "i' \circ F"] \arrow[d, "i"'] & U' \arrow[d, "f'"] \\ U \arrow[r, "f"] & S \end{tikzcd}$$

in the sense of [67, IV, Def. 2.2.1], i.e., $\mathcal{D}$ is the sheaf of morphisms $U \to U'$ making the diagram commute. The sheaf $\mathcal{D}$ is a $\mathcal{G}|_{U'}$-pseudo-torsor by [67, IV, Thm. 2.2.2], and because $f' : U' \to S$ is smooth, it is a torsor. Because $\Omega^1_{U'/S}$ is locally free, $\mathcal{D}$ is locally isomorphic to $(F_*\mathcal{I})^{\oplus d}$. By Lemma 2.7, $\mathcal{D}$ is $\tilde{Z}$-closed for every $\tilde{Z} \subset X'$ satisfying $\operatorname{codim}(\tilde{Z}, X') \geq 2$. By this property, the obvious homomorphism $\mathcal{D} \to \mathcal{F}rob(X,X')|_{U'}$ is an isomorphism of sheaves of sets making $\mathcal{F}rob(X,X')|_{U'}$ a $\mathcal{G}|_{U'}$-torsor. □

**Proposition 7.10.** *Let $Y' \to X'$ be an étale open and $G : Y \to Y'$ a local Frobenius lifting. Then there is a canonical homomorphism of complexes*

$$\phi_G : W^1_{Y'_0/S_0}[-1] \to F_* W^\bullet_{Y_0/S_0}$$

*inducing the Cartier isomorphism in first cohomology on $U'_0 \cap Y'_0$. If $h \in \mathcal{G}(Y')$, then $\phi_G$ and $\phi_{h\cdot G}$ are related by*

$$\phi_{h\cdot G} = \phi_G + (F_* d) \circ \tilde{h},$$

*where $\tilde{h} : W^1_{Y'_0/S_0} \to F_*\mathcal{I} \cong F_* W^0_{Y_0/S_0}$ is the induced homomorphism.*

*Proof.* We choose $V' = U' \cap Y'$ for the representative of $G$. The straightforward log version of the construction of [46, Prop. 3.8] yields a homomorphism $\Omega^1_{V'_0/S_0} \to F_*\Omega^1_{V_0/S_0}$, and this has also been used implicitly by Kato in [50, Thm. 4.12]. Applying $j_*$ yields $(\phi_G)^1$, and we define the other $(\phi_G)^m$ to be 0. The resulting $\phi_G$ does not depend on $V'$ since the involved sheaves are $\tilde{Z}$-closed for every $\tilde{Z} \subset Y'_0$ satisfying $\operatorname{codim}(\tilde{Z}, Y'_0) \geq 2$, so $\phi_G$ is well-defined. The construction yields that $\mathcal{H}^1(\phi_G)$ is the Cartier isomorphism of Theorem 7.1 on $V'_0 = U'_0 \cap Y'_0$. The second statement is similar to [46, Lemma 5.4,(5.4.1)] except that we use the more elegant language of torsors (as already remarked in [17, Rem.2.2 (iii)]) which renders the analog of [46, Lemma 5.4,(5.4.2)] trivial. □

**Theorem 7.11** (Decomposition Theorem)**.** *Let $f : X \to S$ be a generically log smooth family, assume that $f_0 : X_0 \to S_0$ has the Cartier isomorphism property (Def. 7.3), and assume that $\mathcal{F}rob(X,X')$ is a $\mathcal{G}$-torsor. Then we have a quasi-isomorphism*

$$\bigoplus_{m<p} W^m_{X'_0/S_0}[-m] \to \tau_{<p} F_* W^\bullet_{X_0/S_0}$$

*in $D^b(X'_0)$, where $\tau_{<p}$ means the truncation of a complex.*

*Proof.* Because $\mathcal{F}rob(X,X')$ is a torsor, we can find an étale cover $\mathcal{Y} = \{Y'_\alpha\}$ of $X'$ such that we have a local Frobenius lifting $G_\alpha : Y_\alpha \to Y'_\alpha$. We obtain an induced cover $\mathcal{Y}_0$ of $X'_0$. On the log smooth locus $U'_0 \subset X'_0$, we can apply an argument as implicitly used in [50, Thm. 4.12]: using Proposition 7.10, the gluing method of Step **B** in the proof of [46, Thm. 5.1] yields a homomorphism

$$\varphi : \Omega^1_{U'_0/S_0}[-1] \to \check{\mathcal{C}}^\bullet(\mathcal{Y}_0 \cap U'_0, F_*\Omega^\bullet_{U_0/S_0}) =: \check{\mathcal{C}}^\bullet_U$$

of complexes of sheaves, where $\check{\mathcal{C}}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet)$ refers to the total sheaf Čech complex for a cover $\mathfrak{U}$ and a complex of sheaves $\mathcal{F}^\bullet$. We also have the natural quasi-isomorphism

$$\psi : F_* W^\bullet_{X_0/S_0} \to \check{\mathcal{C}}^\bullet(\mathcal{Y}_0, F_* W^\bullet_{X_0/S_0}).$$

When we use $\psi$ and that the question is local, Proposition 7.10 gives that $\varphi$ induces the Cartier isomorphism on $U_0'$ for $\mathcal{H}^1$. Now let $0 \le m < p$. With the antisymmetrization map

$$a_m : \Omega^m_{U_0'/S_0}[-m] \to (\Omega^1_{U_0'/S_0}[-1])^{\otimes m}$$

defined by $a_m(\omega_1 \wedge ... \wedge \omega_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) \omega_{\sigma(1)} \otimes ... \otimes \omega_{\sigma(m)}$, we obtain a morphism

$$\varphi^m : \Omega^m_{U_0'/S_0}[-m] \xrightarrow{a_m} (\Omega^1_{U_0'/S_0}[-1])^{\otimes m} \xrightarrow{\varphi^{\otimes m}} (\check{\mathcal{C}}^\bullet_U)^{\otimes m} \to \check{\mathcal{C}}^\bullet_U,$$

where the last map is induced by the wedge product on $F_* \Omega^\bullet_{U_0/S_0}$. Note that the various $\varphi^m$ are compatible with the wedge product of $\Omega^\bullet_{U_0'/S_0}$ and of the cohomology of $F_* \Omega^\bullet_{U_0/S_0}$, hence $\varphi^m$ induces the Cartier isomorphism in cohomology. Taking the sum, we obtain a quasi-isomorphism

$$\varphi^\bullet : \bigoplus_{m<p} \Omega^m_{U_0'/S_0}[-m] \to \tau_{<p} \check{\mathcal{C}}^\bullet_U.$$

Since $j_* \check{\mathcal{C}}^\bullet_U = \check{\mathcal{C}}^\bullet(\mathcal{Y}_0, F_* W^\bullet_{X_0/S_0})$, we obtain the desired homomorphism in $D^b(X_0')$ as the composition $\psi^{-1} \circ j_* \varphi^\bullet$. It is a quasi-isomorphism because $f_0 : X_0 \to S_0$ has the Cartier isomorphism property by assumption. ☐

*Remark* 7.12. Assuming a lifting of $f_0 : X_0 \to S_0$, this is a generalization of [17, Thm. 2.1]. The results [17, Cor. 3.7] and [46, Thm. 5.1] assume a lifting of $f_0' : X_0' \to S_0$ instead. These results do not generalize well to the generically log smooth setting.

We like to apply this theorem to the case of a log toroidal family. It remains only to show that $\mathcal{F}rob(X, X')$ is a torsor:

**Proposition 7.13.** *In the above situation, assume $f : X \to S$ is a log toroidal family with respect to $S \to A_Q$. Then $\mathcal{F}rob(X, X')$ is a $\mathcal{G}$-torsor, i.e., Frobenius liftings exist locally.*

*Proof.* Let $(Q \subset P, \mathcal{F})$ be an ETD from a local model of $f : X \to S$, and consider the diagram

$$\begin{tikzcd} L \arrow[r, dotted, "F"] \arrow[d] & L \arrow[r, "c"] \arrow[d] & A_{P,\mathcal{F}} \arrow[d] \\ S \arrow[r, "F_S"] & S \arrow[r, "a"] & A_Q. \end{tikzcd}$$

For the local existence of a Frobenius lifting, it suffices to show that there is a scheme morphism $F : L \to L$ and a log morphism on $c^{-1}(U_P)$ such that the diagram commutes and the induced map $F \times_S S_0$ on $L_0 = L \times_S S_0$ is the absolute Frobenius. Then $F$ plays the role of an absolute Frobenius on $L$, and its induced relative Frobenius gives rise to a local Frobenius lifting on $X'$ via the local model. The scheme $\underline{L}$ is affine with $\mathcal{O}(L) = \bigoplus_{e \in E} z^e \cdot W_2(k)$, allowing us to define $F : \underline{L} \to \underline{L}$ via $F^*(z^e \cdot w) := z^{pe} \cdot F_S^*(w)$. Note the maps

$$M := \mathrm{Spec}\ (P \to \mathcal{O}(L)) \to L \to \mathrm{Spec}\ (Q \to \mathcal{O}(L)) =: N,$$

and by using $W_i := c^{-1}(U_i)$ with the notation of Corollary 3.17, observe that $M|_{W_1} = L|_{W_1}$ and $L|_{W_2} = N|_{W_2}$. On $N$ and $M$, we get morphisms $F_N : N \to N$ and $F_M : M \to M$ by mapping $q \mapsto p \cdot q$ on the monoids and using $F^*$ on the rings. They are compatible with each other and with the maps to $S$, and moreover, $F_N \times_S S_0$ and $F_M \times_S S_0$ are the absolute Frobenii on $N_0, M_0$. We define partially $F|_{W_1} := F_M|_{W_1}$ and $F|_{W_2} := F_N|_{W_2}$. Because $N|_{W_1 \cap W_2} = L|_{W_1 \cap W_2} = M|_{W_1 \cap W_2}$, these definitions agree on $W_1 \cap W_2$, and we obtain a log morphism defined on $c^{-1}(U_P) = W_1 \cup W_2$. This gives the desired map. ☐

# 8 The Hodge–de Rham Spectral Sequence

In this section, we prove the degeneration of the Hodge–de Rham spectral sequence for log toroidal families.

### The Absolute Case

We start with the absolute case over $S = \mathrm{Spec}\ (Q \to k)$.

**Theorem 8.1.** *Let $S = \mathrm{Spec}\ (Q \to k)$ for a sharp toric monoid $Q$ and a field $k \supset \mathbb{Q}$, and let $f : X \to S$ be a proper log toroidal family of relative dimension $d$ with respect to $S \to A_Q$. Then $E(X/S)$ degenerates at $E_1$.*

*Proof.* The proof is analogous to [17, 46]. Because $E_\infty^{pq}$ is a subquotient of $E_1^{pq}$, it suffices to show

$$\sum_{p+q=n} \dim R^q f_* W^p_{X/S} = \dim R^n f_* W^\bullet_{X/S}. \tag{15}$$

We spread out $f : X \to S$ to a proper log toroidal family $f_\lambda : X_\lambda \to S_\lambda$ of relative dimension $d$ with respect to $S_\lambda \to A_Q$ using Proposition 4.18. Shrinking $S_\lambda$, we can assume $R^q f_{\lambda *} W^p_{X_\lambda/S_\lambda}$ and $R^n f_{\lambda *} W^\bullet_{X_\lambda/S_\lambda}$ locally free of constant rank and that $S_\lambda \to \mathrm{Spec}\ \mathbb{Z}$ is smooth. By Theorem 6.16, we can assume furthermore that $W^m_{X_\lambda/S_\lambda}$ commutes with every base change, and inverting all primes that are smaller than $d$, we can assume $\mathrm{char}\ \kappa(s) > d$ for every residue field of a closed point $s \in S_\lambda$ (which is necessarily a finite field by [46, Prop. 6.4]).

For a closed point $\kappa = \kappa(s) \to S_\lambda$, the geometric-functorial characterization of smoothness provides a splitting

$$T_0 = \mathrm{Spec}\ \kappa \to T = \mathrm{Spec}\ W_2(\kappa) \to S_\lambda,$$

which induces by strict base change a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X_\lambda \arrow[d, "f_\lambda"'] & Y \arrow[l] \arrow[d, "g"'] & Y_0 \arrow[l] \arrow[d, "g_0"'] \\ S \arrow[r] & S_\lambda & T \arrow[l] & T_0 \arrow[l] \end{tikzcd}$$

of proper log toroidal families. By Lemma 6.18, it suffices to prove the equality (15) for $g_0 : Y_0 \to T_0$. Since $g : Y \to T$ is log toroidal with respect to $T \to A_Q$, we have a decomposition

$$\bigoplus W^m_{Y_0'/T_0}[-m] \cong (F_0)_* W^\bullet_{Y_0/T_0}$$

in $D^b(Y_0')$ by Theorem 7.11 (where our notation is analogous to Section 7.2). Indeed, $\mathrm{char}\ \kappa = p > d$. Since $F_0 : Y_0 \to Y_0'$ is a homeomorphism, we have

$$R^n g_{0*} W^\bullet_{Y_0/T_0} = R^n g'_{0*} F_{0*} W^\bullet_{Y_0/T_0} \cong \bigoplus_{m=0}^{d} R^n g'_{0*} W^m_{Y_0'/T_0}[-m].$$

The field $\kappa$ is perfect, so we have by flat base change

$$R^n g'_{0*} W^m_{Y_0'/T_0}[-m] = R^{n-m} g'_{0*} s^* W^m_{Y_0/T_0} = F_\kappa^* R^{n-m} g_{0*} W^m_{Y_0/T_0},$$

where $F_\kappa : T_0 \to T_0$ is the absolute Frobenius, and $s : Y_0 \to Y_0$ is its base change. □

### The Relative Case over a One-Dimensional Base

In this case, we need analytic methods, so we restrict to $k = \mathbb{C}$. Let $A_m := \mathbb{C}[t]/(t^{m+1})$ and $S_m := \mathrm{Spec}\ (\mathbb{N} \xrightarrow{1 \mapsto t} A_m)$.

**Theorem 8.2.** *Let $S = S_m$, and let $f : X \to S$ be a proper log toroidal family of relative dimension $d$ with respect to $S \to A_\mathbb{N}$. Then:*

1. *$R^q f_* W^p_{X/S}$ is a free $A_m$-module whose formation commutes with base change.*

2. *The spectral sequence $R^q f_* W^p_{X/S} \Rightarrow R^{p+q} f_* W^\bullet_{X/S}$ degenerates at $E_1$.*

*Proof.* By Corollary 6.12, the formation of $W^p_{X/S}$ commutes with base change, which is an ingredient for the classical base change theorem, e.g. [16, §3], [54, Theorem (8.0)]. It thereby suffices to show the surjectivity of

$$\mathbb{H}^k(X, W^\bullet_{X/S}) \to \mathbb{H}^k(X_0, W^\bullet_{X_0/S_0}).$$

We prove this with the idea of [71, Section (2.6)], cf. [55, Lemma 4.1] and [31, Thm. 4.1]. We define a complex

$$\mathcal{L}^\bullet := W^{\bullet,an}_X[u] = \bigoplus_{s=0}^{\infty} W^{\bullet,an}_X \cdot u^s, \qquad d(\alpha_s u^s) = d\alpha_s \cdot u^s + s\delta(\rho) \wedge \alpha_s \cdot u^{s-1},$$

of analytic sheaves, where $\rho = f^*(1) \in \mathcal{M}_{X^{an}}$ and $\delta : \mathcal{M}_{X^{an}} \to W^{1,an}_X$ is the log part of the universal derivation. Here $W^{\bullet,an}_X$ denotes absolute differentials as in Corollary 6.24. Projection to the $u^0$-summand composed with $W^{\bullet,an}_X \to W^{\bullet,an}_{X/S}$ yields a map $\mathcal{L}^\bullet \to W^{\bullet,an}_{X/S}$, whose composition with $W^{\bullet,an}_{X/S} \to W^{\bullet,an}_{X_0/S_0}$ fits into an exact sequence

$$0 \to \mathcal{K}^\bullet \to \mathcal{L}^\bullet \xrightarrow{\phi^\bullet} W^{\bullet,an}_{X_0/S_0} \to 0$$

of complexes, which defines $\mathcal{K}^\bullet$. By Corollary 4.16, we find for every $\mathbb{C}$-valued point $\bar{x} \in X$ a local model at $\bar{x}$, so we may use Corollaries 6.23, 6.24 and Lemma 6.27 to have a local description of this sequence. Lemma 8.4 below verifies that $\mathcal{K}^\bullet$ is acyclic for all ETDs over $\mathbb{N}$, so $\phi^\bullet$ is a quasi-isomorphism and the theorem follows by the discussion in Section 2.7. ☐

*Remark* 8.3. There are problems with similar theorems in earlier works: The generalization from a one-dimensional base to higher dimensions in [55, p. 404] is flawed, which then also affects [31, Theorem 4.1]. In addition, there is a gap in the proof of [31, Theorem 4.1] related to the fact that the de Rham differential of $\Omega^\bullet_{X/S}$ is not $\mathcal{O}_X$-linear. Since our result encompasses the one-parameter base case of [31, Theorem 4.1], Theorem 8.2 closes the latter gap.

**Lemma 8.4.** *Let $(\mathbb{N} \subset P, \mathcal{F})$ be an ETD, and let $f : X \to S = S_m$ be the base change of $A_{P,\mathcal{F}} \to A_{\mathbb{N}}$ along $S_m \to A_{\mathbb{N}}$. With $0 \in A_{P,\mathcal{F}}$ denoting the origin, we have $\mathcal{H}^k(\mathcal{K}^\bullet)_0 = 0$ for all $k$.*

*Proof.* We choose Hermitian inner products on the two vector spaces $L := P^{gp} \otimes \mathbb{C}$ and $W := (P^{gp} \otimes \mathbb{C})/(\mathbb{N}^{gp} \otimes \mathbb{C})$. With $K = (m+1) + \mathbb{N} \subset \mathbb{N}$, we recall $E_K$ from §6.4. For $e \in E_K$, we define

$$L_e := \bigcap_{H \in \mathcal{F}_{\max} \smallsetminus \mathcal{F} : e \in H} H^{gp} \otimes \mathbb{C} \qquad \text{and} \qquad W_e := \bigcap_{H \in \mathcal{F}_{\max} \smallsetminus \mathcal{F} : e \in H} (H^{gp} \otimes \mathbb{C})/(\mathbb{N}^{gp} \otimes \mathbb{C}).$$

By Lemma 6.27, elements of $\mathcal{L}^k_0$ are formal sums

$$(\ell_{e,s}) := \sum_{s=0}^{N} \sum_{e \in E_K} u^s z^e \ell_{e,s}\ , \quad \ell_{e,s} \in \bigwedge^k L_e\ , \quad \sup_{\substack{e \in E_K \smallsetminus 0 \\ 1 \le s \le N}} \{\log \|\ell_{e,s}\| / h(e)\} < \infty,$$

and elements of $W^{k,an}_{X_0/S_0,0}$ are formal sums

$$(w_e) := \sum_{e \in E} z^e \cdot w_e, \quad w_e \in \bigwedge^k W_e\ , \quad \sup_{e \in E \smallsetminus 0} \{\log \|w_e\| / h(e)\} < \infty.$$

Note that $(\ell_{e,s})$ is summed over $E_K$, whereas $(w_e)$ is summed over $E$. We denote the kernel of $\pi : \bigwedge^k L_e \to \bigwedge^k W_e$ by $K^k_e$ and observe $\phi((\ell_{e,s})) = (\pi(\ell_{e,0}))$, so $(\ell_{e,s}) \in \mathcal{K}^k_0$ if and only if $\ell_{e,0} \in K^k_e$ for all $e \in E$. With $\bar{\rho} := 1 \otimes 1 \in \mathbb{N}^{gp} \otimes \mathbb{C}$ we have $\delta(\rho) = z^0 \cdot \bar{\rho} \in W^1_X$ and thus

$$d((\ell_{e,s})) = (e \wedge \ell_{e,s} + (s+1)\bar{\rho} \wedge \ell_{e,s+1}). \tag{16}$$

Let $(\ell_{e,s}) \in \mathcal{K}^0_0$ and assume $d((\ell_{e,s})) = 0$. Since $\ell_{e,s} \in \mathbb{C}$, for $e \neq 0$ by descending induction in $s$ starting from $\ell_{e,N}$, we find $\ell_{e,s} = 0$. We have $\ell_{0,0} = 0$, and ascending induction yields $\ell_{0,s} = 0$. Thus $\mathcal{H}^0(\mathcal{K}^\bullet)_0 = 0$.

Next, let $(\ell_{e,s}) \in \mathcal{K}^{k+1}_0$ for $k \ge 0$ with $d((\ell_{e,s})) = 0$. Starting with $e = 0$, we construct $(\tau_{e,s}) \in \mathcal{K}^k_0$ with $d((\tau_{e,s})) = (\ell_{e,s})$ using the following claim.

**Claim 8.5.** *Let $(L, \langle\cdot,\cdot\rangle)$ be a $\mathbb{C}$-vector space of finite dimension with a Hermitian inner product. Let $0 \neq p \in L$ and $k \geq 0$, and assume $\ell \in \bigwedge^{k+1} L$ with $p \wedge \ell = 0$. Then there is a $\tilde{\ell} \in \bigwedge^k L$ with $p \wedge \tilde{\ell} = \ell$ and $\|p\| \cdot \|\tilde{\ell}\| = \|\ell\|$.*

*Proof.* Let $\ell_1 := \frac{p}{\|p\|}, \ell_2, ..., \ell_n$ be an orthonormal basis of $L$, and $\{\ell_{i_1...i_k}\}$ the induced basis of $\bigwedge^k L$. If $\ell = \sum \alpha_{i_1...i_{k+1}} \ell_{i_1...i_{k+1}}$ satisfies the assumption, then $\tilde{\ell} = \frac{1}{\|p\|} \sum \alpha_{1 i_2...i_{k+1}} \ell_{i_2...i_{k+1}}$ is a solution. □

We set $\tau_{0,0} = 0$. Writing out (16) for $e = 0$ yields

$$d\big(\ell_{0,0} + \ell_{0,1}u + \ell_{0,2}u^2 + ...\big) = \bar{\rho} \wedge \ell_{0,1} \;+\; 2\bar{\rho} \wedge \ell_{0,2}u \;+\; 3\bar{\rho} \wedge \ell_{0,3}u^2 \;+\; ...$$

and therefore $\bar{\rho} \wedge \ell_{0,i} = 0$ for $i > 0$. Since $\ell_{0,0} \in K_0^0$, we also have $\bar{\rho} \wedge \ell_{0,0} = 0$. By Claim 8.5, there is $\tau_{0,s+1} \in \bigwedge^k L_0$ with $\bar{\rho} \wedge \tau_{0,s+1} = \ell_{0,s}$, and we are done with the case $e = 0$. For $e \neq 0$, we need to care about convergence. Without loss of generality, $N \geq 1$. Since $e \wedge \ell_{e,N} = 0$, we can find by Claim 8.5 an element $\tau_{e,N} \in \bigwedge^k L_e$ with $e \wedge \tau_{e,N} = \ell_{e,N}$ and $\|\tau_{e,N}\| \cdot \|e\| = \|\ell_{e,N}\|$. For $s \geq 1$, we construct $\tau_{e,s} \in \bigwedge^k L_e$ by descending induction. Because of $e \wedge (\ell_{e,s} - (s+1)\bar{\rho} \wedge \tau_{e,s+1}) = 0$, there is $\tau_{e,s}$ with $e \wedge \tau_{e,s} = \ell_{e,s} - (s+1)\bar{\rho} \wedge \tau_{e,s+1}$ and

$$\|\tau_{e,s}\| \cdot \|e\| = \|\ell_{e,s} - (s+1)\bar{\rho} \wedge \tau_{e,s+1}\|. \tag{17}$$

For $e \notin E$, we go one step further and construct $\tau_{e,0} \in \bigwedge^k L_e$ with the same method, but for $e \in E$, the construction of $\tau_{e,0} \in K_e^k$ is more intricate. We need another claim:

**Claim 8.6.** *Let $(L, \langle\cdot,\cdot\rangle)$ be a $\mathbb{C}$-vector space of finite dimension with a Hermitian inner product. Let $0 \neq V, Y \subset L$ be subspaces with $V \cap Y = 0$. Then there is a constant $\gamma > 0$ with the following property: For every subspace $H$ with $V \subset H \subset L$ and $k \geq 0$, let $K_H^k$ be the kernel of $\bigwedge^k H \to \bigwedge^k (H/V)$. Then for every $0 \neq p \in Y \cap H$ and every $\ell \in K_H^{k+1}$ with $p \wedge \ell = 0$, there is a $\tilde{\ell} \in K_H^k$ with $p \wedge \tilde{\ell} = \ell$ and $\gamma \cdot \|p\| \cdot \|\tilde{\ell}\| \leq \|\ell\|$.*

*Proof.* Let $p = (p_1, p_2)$ be the decomposition of $p$ under $L = V \oplus V^\perp$, so $\|p\|^2 = \|p_1\|^2 + \|p_2\|^2$. Since $V \cap Y = 0$, we have for $\gamma^2 := \inf_{0 \neq p \in Y} \|p_2\|^2/\|p\|^2$ that $0 < \gamma \leq 1$. Let $\ell_0 := \frac{p_2}{\|p_2\|}, \ell_1, \ell_2, ...$ be an orthonormal basis of $H$; then $\bar{\ell}_0 = \frac{p}{\|p\|}$, $\bar{\ell}_i := \ell_i$ for $i > 0$, is an ordinary basis of $H$. For $\ell = \sum \alpha_{i_0...i_k} \bar{\ell}_{i_0...i_k} \in K_H^{k+1}$ with $p \wedge \ell = 0$, we define $\tilde{\ell} := \frac{1}{\|p\|} \sum \alpha_{0 i_1...i_k} \bar{\ell}_{i_1...i_k} \in K_H^k$ to have $p \wedge \tilde{\ell} = \ell$. We also find

$$\|\ell\|^2 = \left\| \sum \alpha_{0i_1...i_k} \frac{p}{\|p\|} \wedge \ell_{i_1...i_k} \right\|^2 \geq \left\| \sum \alpha_{0i_1...i_k} \frac{p_2}{\|p\|} \wedge \ell_{i_1...i_k} \right\|^2 \geq \gamma^2 \cdot \|p\|^2 \cdot \|\tilde{\ell}\|^2.$$

□

We apply Claim 8.6 to $L = P^{gp} \otimes \mathbb{C}$. Let $F_e \subset P$ be the face generated by $e$ and $Y = F_e^{gp} \otimes \mathbb{C}$. Let $V = \mathbb{N}^{gp} \otimes \mathbb{C}$ and $H = L_e$, so $K_H^k = K_e^k$. Then $e \wedge (\ell_{e,0} - \bar{\rho} \wedge \tau_{e,1}) = 0$, so we find $\tau_{e,0} \in K_e^k$ with $e \wedge \tau_{e,0} = \ell_{e,0} - \bar{\rho} \wedge \tau_{e,1}$ and

$$\gamma \cdot \|\tau_{e,0}\| \cdot \|e\| \leq \|\ell_{e,0} - \bar{\rho} \wedge \tau_{e,1}\|. \tag{18}$$

The factor $\gamma$ depends on $Y$, but there are only finitely many faces generated by elements $e \in E$, so we take for $\gamma$ the minimum over them and furthermore $\gamma < 1$. Applying the triangle inequality to the right hand side of (18) and using induction and (17) yields

$$\|\tau_{e,s}\| \leq \frac{1}{\gamma} \cdot \frac{1}{\|e\|} \sum_{k=s}^{N} \left( \frac{\|\bar{\rho}\|}{\|e\|} \right)^{k-s} \cdot \frac{k!}{s!} \cdot \|\ell_{e,k}\|$$

for all $e \neq 0$. Because $\inf_{e \neq 0}\{\|e\|\} > 0$, there is a bound $M > 1$ independent of $e$ such that $\|\tau_{e,s}\| \leq M \cdot \max_k\{\|\ell_{e,k}\|\}$, which proves

$$\sup_{e \in E_K \setminus 0} \{\log\|\tau_{e,s}\|/h(e)\} < \infty$$

and thus $(\tau_{e,s}) \in \mathcal{K}_0^k$. By construction, $d((\tau_{e,s})) = (\ell_{e,s})$, so $\mathcal{H}^k(\mathcal{K}^\bullet)_0 = 0$. □

Similar to Deligne's classical work [16], the degeneration over infinitesimal bases extends to families over $S = \mathrm{Spec}\ (\mathbb{N} \to \mathbb{C}[\![t]\!])$. Since [16] explains only very roughly how it works, we do it here in more detail.

**Corollary 8.7.** *Let $S = \mathrm{Spec}\ (\mathbb{N} \to \mathbb{C}[\![t]\!])$, and let $f : X \to S$ be a proper log toroidal family of relative dimension $d$ with respect to $S \to A_{\mathbb{N}}$. Then the Hodge–de Rham spectral sequence $E(X/S)$ degenerates at $E_1$, and $R^q f_* W^p_{X/S}$ is locally free.*

*Proof.* Let $S_m$ be as above and define proper log toroidal families $f_m : X_m \to S_m$ by base change. Let $b_m : S_m \to S$ and $c_m : X_m \to X$ as well as $b_{mn} : S_m \to S_n$ and $c_{mn} : X_m \to X_n$ be the closed immersions. Choose injective Cartan–Eilenberg resolutions $W^\bullet_{X_m/S_m} \to L^{\bullet\bullet}_m$ and $W^\bullet_{X/S} \to L^{\bullet\bullet}$ (in the category of e.g. $f^{-1}\mathcal{O}_S$ modules). Similar to the proof of Lemma 2.46, we find maps

$$L^{\bullet\bullet} \to c_{n*}L^{\bullet\bullet}_n \to c_{m*}L^{\bullet\bullet}_m$$

of resolutions that are unique up to homotopy (in particular, these maps are compatible with composition only up to homotopy). Applying $f_*$, taking the spectral sequence of a double complex, using $f_* c_{m*} = b_{m*} g_*$ and the exactness of $b_{m*}$ we find maps of spectral sequences

$$E(X/S) \to b_{n*}E(X_n/S_n) \to b_{m*}E(X_m/S_m)$$

of $\mathcal{O}_S$-modules. A priori, these maps are not unique, but [11, XV. Prop. 6.1] shows them unique from $E_2$ on. However, on $E_1$ the maps are the canonical maps

$$R^q f_* W^p_{X/S} \to b_{m*} R^q f_{m*} W^p_{X_m/S_m}$$

since the double complexes are Cartan–Eilenberg resolutions (hence the rows are injective resolutions of the pieces). Therefore, we have maps of spectral sequences that are compatible with composition. The spectral sequences $b_{m*}E(X_m/S_m)$ degenerate at $E_1$, so their limit $\varprojlim_k b_{m*}E(X_m/S_m)$ is a spectral sequence (though inverse limits are not exact) which moreover degenerates at $E_1$. The induced map

$$E(X/S) \to \varprojlim_k b_{m*}E(X_m/S_m)$$

is an isomorphism since it is on $E_1$ by [34, III. Thm. 4.1.5]. (Note that $R^q f_* W^p_{X/S}$ is already a complete module.) Thus $E(X/S)$ degenerates at $E_1$. The modules $R^q f_{m*} W^p_{X_m/S_m}$ are flat, and the maps between them are surjective, so their limit $R^q f_* W^p_{X/S}$ is flat (hence locally free) by [5, 0912, Lemma 15.27.4]. □

*Remark* 8.8. The spectral sequence $E(X/S)$ also degenerates for proper log toroidal families over $S = \mathrm{Spec}\ (\mathbb{N} \to \mathbb{C}[t]_{(t)})$. Namely, denoting by $\hat{X} \to \hat{S}$ the base change along the completion $\mathbb{C}[t]_{(t)} \to \mathbb{C}[\![t]\!]$, the natural map $E(X/S) \to E(\hat{X}/\hat{S})$ is injective (by the Krull intersection theorem). The cohomologies $R^q f_* W^p_{X/S}$ are locally free since $\hat{S} \to S$ is faithfully flat.

### The Relative Case over a Higher-Dimensional Base

The case of a higher-dimensional base can be reduced to the one-dimensional case by an idea of Chan–Leung–Ma in [12]. Let $Q$ be a sharp toric monoid with maximal ideal $Q^+ = Q \setminus \{0\}$, and set $A_m = \mathbb{C}[Q]/(Q^+)^{m+1}$.

**Lemma 8.9.** *Let $(Q^+)^n \subset I \subset J \subset \mathbb{C}[Q]$ be monomial ideals with $\dim_{\mathbb{C}}(J/I) = 1$. Let $h : Q \to \mathbb{N}$ be a homomorphism such that $I = h^{-1}(z^{k+1})$ and $J = h^{-1}(z^k)$. Moreover, let $S' = \mathrm{Spec}\ (Q \to \mathbb{C}[Q]/I)$, let $f' : X' \to S'$ be a proper log toroidal family with respect to $S' \to A_Q$, let $S = \mathrm{Spec}\ (Q \to \mathbb{C}[Q]/J)$, and let $f : X \to S$ be the base change along $S \to S'$. Then*

$$\mathbb{H}^i(X', W^\bullet_{X'/S'}) \to \mathbb{H}^i(X, W^\bullet_{X/S})$$

*is surjective.*

*Proof.* Let
$$T = \mathrm{Spec}\ (\mathbb{N} \to \mathbb{C}[z]/z^k), \quad T' = \mathrm{Spec}\ (\mathbb{N} \to \mathbb{C}[z]/z^{k+1}),$$
and let $g : Y \to T, g' : Y' \to T'$ be the log toroidal families induced by base change along $A_{\mathbb{N}} \to A_Q$. Note that all four log toroidal families now have the same underlying topology. We obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & W^\bullet_{X_0/S_0} \otimes J/I \arrow[r, "\iota_X"] \arrow[d, "\cong"] & W^\bullet_{X'/S'} \arrow[r, "\pi_X"] \arrow[d] & W^\bullet_{X/S} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W^\bullet_{Y_0/T_0} \otimes (z^k)/(z^{k+1}) \arrow[r] & W^\bullet_{Y'/T'} \arrow[r, "\pi_Y"] & W^\bullet_{Y/T} \arrow[r] & 0
\end{tikzcd}$$

of complexes where $\pi_X$ and $\pi_Y$ are surjective due to Theorem 6.15. The left vertical map is an isomorphism. Namely, $T_0 \to S_0$ is the identity on underlying schemes, so $W^\bullet_{X_0/S_0} \cong W^\bullet_{Y_0/T_0}$, and $J/I \cong (z^k)/(z^{k+1})$ since $\dim_{\mathbb{C}}(J/I) = 1$. Theorem 8.2 shows $\mathbb{H}^\bullet(Y', \pi_Y)$ surjective, so $\mathbb{H}^\bullet(X', \iota_X)$ is injective, and therefore $\mathbb{H}^\bullet(X', \pi_X)$ is surjective. □

*Remark* 8.10. Note that we do not use any base change results along non-strict morphisms (besides $T_0 \to S_0$).

**Lemma 8.11.** *Let $I = (Q^+)^n$. Then there is a finite sequence of monomial ideals*
$$Q^+ = I_0 \supset I_1 \supset \ldots \supset I$$
*such that $\dim_{\mathbb{C}} I_k/I_{k+1} = 1$, and such that, for every $k$, there is a homomorphism $h_k : Q \to \mathbb{N}$ with $I_k = h_k^{-1}(z^i)$ and $I_{k+1} = h_k^{-1}(z^{i+1})$.*

*Proof.* This follows straightforwardly from the discussion just below [12, Lemma 4.16]. □

**Theorem 8.12.** *Let $S = S_m := \mathrm{Spec}\ (Q \to A_m)$, and let $f : X \to S$ be a proper log toroidal family with respect to $S \to A_Q$. Then:*

1. *$R^q f_* W^p_{X/S}$ is a free $A_m$-module whose formation commutes with base change.*
2. *The spectral sequence $R^q f_* W^p_{X/S} \Rightarrow R^{p+q} f_* W^\bullet_{X/S}$ degenerates at $E_1$.*

**Corollary 8.13.** *Let $S = \mathrm{Spec}\ (Q \to \mathbb{C}[\![Q]\!])$ and let $f : X \to S$ be a proper log toroidal family of relative dimension $d$ with respect to $S \to A_Q$. Then the Hodge–de Rham spectral sequence $E(X/S)$ degenerates at $E_1$, and $R^q f_* W^p_{X/S}$ is locally free.*

*Proof.* The proof is literally the same as of Corollary 8.7. □

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## Education

[redacted]

## Scientific Works

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- Simon Felten: *Log Smooth Deformation Theory via Gerstenhaber Algebras*, manuscripta math. (2020). https://doi.org/10.1007/s00229-020-01255-6
- Simon Felten, Matej Filip, Helge Ruddat: *Smoothing Toroidal Crossing Spaces*, arXiv: 1908.11235.
- Simon Felten: *Smoothing Normal Crossing Spaces*, in Oberwolfach Report 27/2019
- Simon Felten: *Good Differential Forms for a Family of Schemes*, Master Thesis, [redacted]
- Simon Felten, Stefan Müller-Stach: *A diophantine equation for sums of consecutive like powers*, EMS, volume 70, 2015, pp. 117-124.

## Conferences, Workshops, Seminars

[redacted]

## Talks

| | |
|---|---|
| 2021 | *Log Toroidal Families*, PhD Defense, online |
| 2020 | *Aus dem Leben eines Mathematikers*, Science Slam, Mainz |
| 2020 | *Kabeljau und Calabi-Yau: Wie ordnet der Geometer sein Aquarium?*, Berlin |
| 2020 | *Rational Homotopy Theory*, Mainz |
| 2019 | *Smoothing Normal Crossing Spaces*, Freiburg |
| 2019 | *Potential Density of Abelian Varieties*, Mainz |
| 2019 | *Archimedes and the Regular Heptagon*, Thessaloniki |
| 2019 | *The Quartet Theory of Human Emotions*, San Giovanni |
| 2019 | *Colimits in $\infty$-Categories*, Mainz |
| 2019 | *Smoothing Normal Crossing Spaces*, MFO |
| 2019 | *Algebro-Geometric Codes*, Bad Tölz |
| 2019 | *Foundations of Quantum Computing*, Mainz |
| 2018 | *Mirror Symmetry of Families*, Nürnberg |
| 2018 | *Eutrophication*, Neubeuern |
| 2018 | *The Hodge–to–deRham Degeneration for Toroidal Families*, Hamburg |

## Awards and Grants

[redacted]

## Honorary and Scientific Service

- [redacted]
- [redacted]
- [redacted]
- [redacted]
- [redacted]
- [redacted]

## Further Experience and Qualification

[redacted]

## Hobbies

- [redacted]
- [redacted]

# 11 Previously Published Parts

The reader finds below the article "Smoothing Toroidal Crossing Spaces", which contains most of the original material in this thesis. It has been uploaded to the arXiv in August 2019 and is up to now (February 2021) not accepted in a journal. We append it here since the University's regulations require to add all previous publications of results of the thesis.

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# SMOOTHING TOROIDAL CROSSING SPACES

SIMON FELTEN, MATEJ FILIP, HELGE RUDDAT

Abstract. We prove the existence of a smoothing for a toroidal crossing space under mild assumptions. By linking log structures with infinitesimal deformations, the result receives a very compact form for normal crossing spaces. The main approach is to study log structures that are incoherent on a subspace of codimension two and prove a Hodge to de Rham degeneration theorem for such log spaces. We show that new developments of Bogomolov-Tian-Todorov theory can be applied to obtain smoothings. The theory relates to recent work in mirror symmetry and the construction of Frobenius manifold structures. It has potential applications to the classification of Fano fourfolds.

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## 1. Introduction

For two smooth components $Y_1, Y_2$ meeting in a smooth divisor $D$ a folkloristic statement says that a necessary condition for $X = Y_1 \cup Y_2$ to have a smoothing is that the two normal bundles are dual to each other, i.e. $\mathcal{N}_{D/Y_1} \otimes \mathcal{N}_{D/Y_2} \cong \mathcal{O}_D$. This statement is actually incorrect. It is true only with the further requirement that the total space of the smoothing be itself smooth. Conceptually, $\mathcal{N}_{D/Y_1} \otimes \mathcal{N}_{D/Y_2} \cong$

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$\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X) =: \mathcal{T}^1_X$ and Friedman famously coined the notion of *d-semistability* which is saying $\mathcal{T}^1_X \cong \mathcal{O}_D$ [12]. We are going to generalize the situation by only requiring $\mathcal{T}^1_X$ to be *generated by global sections* (and beyond). For a choice of $s \in \Gamma(D, \mathcal{T}^1_X)$, the total space of the smoothing will then be of the local form $xy = tf$ where $t$ is the deformation parameter, $Y_1 = V(x), Y_2 = V(y)$ and $f$ represents $s$ in a local trivialization of $\mathcal{T}^1_X$. The total space of the smoothing has singularities precisely along $s = 0$. The local form $xy = tf$ has been found to be abundant in mirror symmetry applications [7, 14, 15, 16, 6, 13, 1, 21].

We work more generally with a *normal crossing space*, that is a connected variety $X$ over $\mathbb{C}$ étale locally of the form $z_1 \cdot \ldots \cdot z_k = 0$ for varying $k \leq \dim X + 1$. We call a flat map $\mathcal{X} \to \mathbb{D}$ for $\mathbb{D}$ a holomorphic disk a *smoothing* of $X$ if the central fiber is isomorphic to $X$ and the general fiber is smooth. If a smoothing exists, then we call $X$ *smoothable*. We say that a normal crossing space has *effective anti-canonical class* if the dual of its dualizing sheaf $\omega_X$ can be represented by a reduced divisor $E$ that meets the strata of $X$ transversely, that is, étale locally along $E$, $X$ is equivalent to $E \times \mathbb{A}^1$. We prove the following theorem.

**Theorem 1.1.** *Let $X$ be a proper normal crossing space with effective anti-canonical class. If $\mathcal{T}^1_X$ is generated by global sections and $X_{\mathrm{sing}}$ is projective, then $X$ is smoothable.*

The only purpose of the projectivity condition is to apply Bertini's theorem to have available a "nice" section of the line bundle $\mathcal{T}^1_X$ on $X_{\mathrm{sing}}$. Both the projectivity assumption as well as the global generatedness assumption on $\mathcal{T}^1_X$ can thus be removed if there exists a *schön* section of $\mathcal{T}^1_X$, that is a section whose vanishing locus $Z$ is reduced and $X_{\mathrm{sing}}$ is locally along $Z$ equivalent to $Z \times \mathbb{A}^1$. We also prove a more general theorem for toroidal crossing spaces that we give down below (Theorem 1.7). Theorem 1.1 provides a lot more flexibility than existing smoothing results, notably Friedman's [12] for surfaces, Kawamata-Namikawa's [31] for d-semistable Calabi-Yaus and Gross-Siebert's [16] allowing a singular total space but with much stronger requirements on $X$.

**Example 1.2.** The union $X$ of $d$ hyperplanes in general position in $\mathbb{P}^n$ is smoothable to a degree $d$ hypersurface but none of the existing results is able to predict the smoothability of $X$ abstractly. Indeed, the total space of the smoothing is singular since it requires blowing up the base locus of the smoothing pencil. On the other hand, $\mathcal{T}^1_X$ is generated by global sections. Theorem 1.1 predicts the smoothability if $d \leq n + 1$.

**Example 1.3.** The simplest type of normal crossing space is one with two smoothly intersecting components: let $Y$ be a smooth Fano manifold with $-K_Y$ very ample, let $D$ be a smooth section of $-K_Y$ and $X$ be the normal crossing space obtained by identifying two copies of $Y$ along $D$. Then $\mathcal{T}^1_X \cong \mathcal{N}^{\otimes 2}_{D/Y}$ is generated by global sections and $X$ is Calabi-Yau, so Theorem 1.1 provides a smoothing of $X$. For Fano threefolds $Y$ that are complete intersections in products of weighted projective spaces the smoothing gives Calabi-Yau threefolds of Euler numbers $-106, -122, -138, -156, -128, -156, -176, -256, -260, -296$. While double intersection situations can be birationally modified to be tractible by the smoothing result in [31], this is no longer true for triple (and higher) intersection situations [32] but Theorem 1.1 provides smoothings.

Theorem 1.1 considerably facilitates the construction of Calabi-Yau and Fano manifolds. Our work generalizes the Gross-Siebert program towards allowing non-toric components in the central fiber as well as more flexibility in the local structure, cf. Example 1.8. While we work with toroidal local models, Alessio Corti communicated to us he also finds non-toric local models to be relevant when smoothing singular Fanos for the classification of Fano fourfolds by means of their Landau-Ginzburg mirrors, cf. [8]. If push-forward of differentials from the log smooth locus can be verified to commute with base change for such local models, then we would be able to treat the models considered by these other authors by the methods of this paper and obtain smoothings for the relevant degenerate Fanos.

Furthermore, our results enable the construction of versal Calabi-Yau families and conjecturally a logarithmic Frobenius manifold structure in a formal neighborhood of the extended moduli space, see [3], [25, Theorem 1.3]. Since the smoothing deformations are given by Maurer-Cartan solutions in the Gerstenhaber algebra of (log) polyvector fields §13.1, we expect the combination of our work with [25] to make maximal degenerations amenable to deformation quantization and homological mirror symmetry.

1.1. **Method of Proof.** The first step towards proving Theorem 1.1 is to furnish $X$ with a log structure, an idea already found in [31, 16]. We build a connection between these two works. A sheaf of sets $\mathcal{LS}_X$ on $X$ classifying log smooth structures locally on $X$ over the standard log point $S$ has been defined and studied in [14]. We show in §5 there is a canonical map $\mathcal{LS}_X \to \mathcal{T}^1_X$ with the property that a section $s \in \Gamma(X_{\mathrm{sing}}, \mathcal{T}^1_X)$ yields a log smooth structure on $U := X \setminus V(s)$, i.e. we obtain a log smooth morphisms $U \to S$. Note that the complement $Z := V(s)$ has codimension two in $X$. Using Bertini's theorem with the projectivity of $X_{\mathrm{sing}}$, we can assume that $Z$ is schön as defined above.

In the fashion of Zariski-Steenbrink-Danilov, we consider the differential forms $W^k_{X/S} := j_*\Omega^k_{U/S}$ for $j : U \hookrightarrow X$ the inclusion. A key ingredient for the smoothing of $X$ is the knowledge that the Hodge to de Rham spectral sequence for $W^\bullet_{X/S}$ degenerates at $E_1$. This requires close control over $W^k_{X/S}$ along $Z$ which we gain by using [15, 35] to obtain a particular type of *elementary log toroidal* local models for the log structure near $Z$. For the proof of the Hodge to de Rham degeneration, we adapt the one by Deligne-Illusie [11]: spreading out to finite characteristic and using the Cartier isomorphism. The hardest technical part is to show that the sheaves $W^\bullet_{X/S}$ commute with base change because $j_*$ and $\otimes$ don't commute in general. We show this holds if the characteristic of the base field is large enough by explicit computation in the elementary log toroidal local models. (Base change may fail for low characteristics by Example 7.5.) We settle a conjecture by Danilov [9, 15.9] along the way (Theorem 1.4 below).

To show the unobstructedness of log deformations of $X$, we use recent advancements of the Bogomolov-Tian-Todorov theory motivated by the study of mirror symmetry, starting with [30] and [3] which got cultivated to work in algebraic geometry by [23]. All these works however produce equisingular deformations (because they are intended for deforming smooth spaces). The crucial difference to our setup is that while we prescribe local deformations by the log structure, these are not locally trivial deformations. Most recently, this difficulty in the theory has been addressed in [25] which adapts perfectly to our situation to produce a formal deformation in the prescribed local models, see §13. We found the framework of Gerstenhaber algebras to be the most effective to think about the theory which governs our way of parsing [25] in §13.1. At this point, the assumption about effectiveness of $\omega_X^{-1}$ enters the proof, so that one obtains an isomorphism of $W^\bullet_{X/S}(\log E)$ with the Gerstenhaber algebra of log polyvector fields $\mathrm{PV}^\bullet$ and has the Batalin-Vilkovisky operator $\Delta$ available by transporting the de Rham differential to $\mathrm{PV}^\bullet$ which is used in §13.2.

To improve the resulting formal smoothing to an analytic smoothing, we use the Grauert-Douady space and Artin approximation as already done in [21].

1.2. **Toroidal Pairs and Danilov's Conjecture.** A *toroidal pair* $(X, D)$ is a variety $X$ over a field $\Bbbk$ of characteristic zero with Weil divisor $D \subset X$ such that $X$ is étale locally equivalent to an affine toric variety with $D$ identified with a reduced toric divisor (not necessarily the entire toric boundary). Danilov defined the sheaf of differentials $\tilde\Omega^p_X(\log D)$ as the push-forward of the usual Kähler differentials with log poles $\Omega^p_X(\log D)$ from the locus where $X$ is regular.

**Theorem 1.4** (Danilov's conjecture)**.** *Given a proper toroidal pair $(X, D)$, the Hodge to de Rham spectral sequence*

$$E_1^{p,q} = H^q(X, \tilde\Omega^p_X(\log D)) \Rightarrow \mathbb{H}^{p+q}(X, \tilde\Omega^\bullet_X(\log D))$$

*degenerates at $E_1$.*

Special cases of this theorem were known before: when $X$ has at worst orbifold singularities [38], for $D = \emptyset$ [4] and for $D$ locally the entire toric boundary [40, 24]. We believe that our methods can be extended to prove generalizations of the Akizuki-Nakano-Kodaira vanishing theorem.

1.3. **Toroidal Crossing Spaces, their Log Structures and Orbifold Smoothings.** If $V = \operatorname{Spec} \Bbbk[P]$ is an affine toric variety given by some toric monoid $P$, consider the map of sheaves $a : \underline{P} \to \mathcal{O}_V, p \mapsto z^p$ with $\underline{P}$ denoting the constant sheaf. We obtain a sheaf of monoids $\mathcal{P}_V = \underline{P}/a^{-1}(\mathcal{O}_V^\times)$. Now $V$ is Gorenstein if and only if the toric boundary $D$ in $V$ is a Cartier divisor, hence given as the zero locus of a monomial $\mathbb{1} \in P$.

**Definition 1.5** (Siebert, Schröer [37])**.** A *toroidal crossing space* is an algebraic space $X$ over $\Bbbk$ together with a sheaf of monoids $\mathcal{P}$ with global section $\mathbb{1} \in \Gamma(X, \mathcal{P})$ such that for every point $x \in X$, étale locally at $x$, $X$ permits a smooth map to the toric boundary $D_x$ in $V_x = \operatorname{Spec} \Bbbk[\mathcal{P}_x]$ so that $\mathcal{P}$ is isomorphic to the pullback of $\mathcal{P}_{V_x}$ and mapping $\mathbb{1}_x$ to the monomial in $\mathcal{P}_x$ whose divisor is $D_x$.

A toroidal crossing space $X$ is automatically Gorenstein, we denote its dualizing line bundle by $\omega_X$. The boundary divisor in a Gorenstein toric variety is naturally a toroidal crossing space. General hyperplane sections of projective toroidal crossing spaces are again naturally toroidal crossing spaces.

**Lemma 1.6.** *A normal crossing space is naturally a toroidal crossing space by setting* $\mathcal{P}_x := \mathbb{N}^k$ *and* $\mathbb{1}_x = (1, 1, ..., 1) \in \mathbb{N}^k$ *whenever* $X$ *is locally at* $x$ *given by* $z_1 \cdot ... \cdot z_k = 0$. *(This isn't the only way to turn a normal crossing space into a toroidal crossing one but we will always refer to this one.)*

The class of toroidal crossing spaces is closed under forming products (but not so the class of normal crossing spaces). The sheaf $\mathcal{P}$ provides what Gross and Siebert call a *ghost structure* for $X$ ([14, Definition 3.16]), an ingredient to define the sheaf $\mathcal{LS}_X$ ([14, Definition 3.19]) whose sections are in bijection with log structures on $X$ together with a log smooth map to the standard log point $S$. By [14], $\mathcal{LS}_X$ embeds into the coherent sheaf $\bigoplus_C j_{\tilde{C},*}\mathcal{N}_{\tilde{C}}$ where the sum is over the irreducible components $C$ of $X_{\text{sing}}$, $j_C : \tilde{C} \to C \to X$ the composition of normalization and closed embedding and $\mathcal{N}_{\tilde{C}}$ is a line bundle on $\tilde{C}$. The sheaf $\mathcal{LS}_X$ often doesn't have global sections. It suffices however to give a section $s$ of $\mathcal{LS}_X$ on a dense open set $U$ that contains the generic points of the minimal strata of $X$ so that each component $s_C \in \Gamma(U \cap C, \mathcal{N}_C)$ of $s$ extends to a section of $\mathcal{N}_C$ on all of $C$ by acquiring simple zeros. The zeros define a reduced Cartier divisor $Z_{\tilde{C}}$ for each $\tilde{C}$. Set $Z = \bigcup_C j_C(Z_{\tilde{C}}) \subset X$. The construction of local models along $Z$ in [15] was generalized in [35]: locally the coherent log structure given by $s$ on $U$, extends to an incoherent log structure on $X$ that is still given by certain toric local models, namely from a divisor in an affine toric variety that is not the entire toric boundary, e.g. like in the definition of toroidal pair above. A section $s$ of $\mathcal{LS}_X$ on a dense open set $U$ will be called *simple* if it extends to $X$ by simple zeros and the resulting $Z_C$ satisfy the simpleness criterion in §6. Our most general smoothing result is the following.

**Theorem 1.7.** *Let* $X$ *be a proper toroidal crossing space with a simple section* $s$ *of* $\mathcal{LS}_X$ *on a dense open set* $U$. *Assume that* $\omega_X^{-1}$ *permits a section whose divisor of* *zeroes $E$ meets all strata of $X$ and $Z$ transversely (e.g. when $\omega_X^{-1} \cong \mathcal{O}_X$, $E = \emptyset$), then $X$ is smoothable to an orbifold with terminal singularities.*

There is a precise derivation of the types of singularities of the orbifold smoothing from knowing $\mathcal{P}$ and $Z$, e.g. for a normal crossing space there will be no singularities in the smoothing and thus combined with the Bertini argument and linking $\mathcal{LS}_X$ with $\mathcal{T}^1_X$, we find that Theorem 1.7 implies Theorem 1.1 (see Proposition 6.12). The definition of c.i.t. in [35] permitted more general singularities and in fact we can still obtain a log toroidal morphism from such a section but don't have the uniqueness of local deformations (Theorem 6.13) which is an ingredient for the smoothing method.

**Example 1.8.** Following [14], let $(B, \mathscr{P}, \varphi)$ be a closed oriented tropical manifold with singular locus combinatorially c.i.t. then the associated space $X_0(B, \mathcal{P}, s)$ with its vanilla gluing data and log structure satisfies the assumptions of Theorem 1.7 for $E = \emptyset$. Smoothings for such spaces had been constructed in [16] under the stronger assumption of local rigidity (e.g. the quintic threefold degeneration in $\mathbb{P}^4$ is not locally rigid but c.i.t.).

1.4. **The Hodge-to-de-Rham Spectral Sequence.** We refer to [28, 26, 14, 34] for basic notions of log geometry. Let $f : X \to S$ be a log toroidal family as defined in Definition 4.1 below. A toroidal pair $(X, D)$ yields an example by giving $X$ the divisorial log structure from $D$ and making $S$ the log-trivial point. The families $X$ over the standard log point featured in Theorem 1.7 give further examples. Also, a saturated relatively log smooth morphism $f : X \to S$ in the sense of [33] is an example. The complex $W^\bullet_{X/S}$ (see page 3) gives rise to a spectral sequence

$$E(X/S) : E_1^{pq} = R^q f_* W^p_{X/S} \Rightarrow R^{p+q} f_* W^\bullet_{X/S}.$$

Let $Q$ be a sharp toric monoid and $\Bbbk$ be a field of characteristic zero. We prove the following theorems.

**Theorem 1.9.** *Let $S = \mathrm{Spec}(Q \to \Bbbk)$ and $f : X \to S$ be a proper log toroidal family (with respect to $S \to A_Q$). Then $E(X/S)$ degenerates at $E_1$.*

Theorem 1.9 implies Theorem 1.4 since $W^p_{X/S} = \tilde{\Omega}^p_X(\log D)$ whenever $f$ comes from a toroidal pair. We conjecture the statement of Theorem 1.9 to hold also for an arbitrary coherent base $S$ over a field of characteristic zero. To prove Theorem 1.9, we adapt the proof of the degeneration in [11] as follows: since $f$ is proper, it suffices to verify

$$(*) \qquad \sum_{p+q=n} \dim\, R^q f_* W^p_{X/S} = \dim\, R^n f_* W^\bullet_{X/S}.$$

In §9, we show that $f : X \to S$ spreads out to a log toroidal family $\phi : \mathfrak{X} \to \mathcal{S} = \mathrm{Spec}(Q \to B)$ where $\mathbb{Z} \subset B \subset \Bbbk$ is a subring such that $B/\mathbb{Z}$ is of finite type. Spreading out of log smooth morphisms over a log-trivial base has been done before in [40, 4.11.1] but $f$ is only generically log smooth. Then for suitable fields $k \supset \mathbb{F}_p$, with $W_2(k)$ denoting the ring of second Witt vectors, we obtain by base change a diagram with Cartesian squares

$$\text{(SO)} \qquad \begin{array}{ccccccc} X & \longrightarrow & \mathfrak{X} & \longleftarrow & \mathfrak{X}_W & \longleftarrow & \mathfrak{X}_k \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_W} & & \downarrow{\scriptstyle \phi_k} \\ S & \longrightarrow & \mathcal{S} & \longleftarrow & \operatorname{Spec} W_2(k) & \longleftarrow & \operatorname{Spec} k. \end{array}$$

In §8 we investigate the behavior of $W^\bullet$ under base change which leads to equalities like $\dim_{\Bbbk} R^q f_* W^p_{X/S} = \dim_k R^q(\phi_k)_* W^p_{\mathfrak{X}_k/k}$, i.e. it suffices to show $(*)$ for $\phi_k : \mathfrak{X}_k \to \operatorname{Spec} k$. In §10 we construct the Cartier isomorphism for log toroidal families in positive characteristic which we then apply in §11 to obtain the Frobenius decomposition of $F_* W^\bullet_{\mathfrak{X}_k/k}$ where $F$ is the relative Frobenius. Finally, in §12, we put the pieces together and prove Theorem 1.9.

We prove a modest but important generalization of Theorem 1.9 to the relative case using Katz's method that first appeared in [38]. This requires a detailed understanding of the analytification of the absolute differentials $W_X^{\bullet,an}$ with respect to base change as given in §7.2 and §12.1.

**Theorem 1.10.** *Let $S = S_m := \operatorname{Spec}(\mathbb{N} \overset{1\mapsto t}{\to} \mathbb{C}[t]/(t^{m+1}))$ and let $f : X \to S$ be a proper log toroidal family with respect to $S \to A_{\mathbb{N}}$. Then:*

1. *$R^q f_* W^p_{X/S}$ is a free $\mathbb{C}[t]/(t^{m+1})$-module whose formation commutes with base change.*
2. *The spectral sequence $R^q f_* W^p_{X/S} \Rightarrow R^{p+q} f_* W^\bullet_{X/S}$ degenerates at $E_1$.*

There are problems with similar theorems in earlier works: the generalization from a one-dimensional base to higher dimensions in [31, p. 404] is flawed which then also affects [15, Theorem 4.1]. In addition, there is a gap in the proof of [15, Theorem 4.1] related to the fact that the de Rham differential of $\Omega^\bullet_{X/S}$ isn't $\mathcal{O}_X$-linear. Since our result encompasses the one-parameter base case of [15, Theorem 4.1], Theorem 1.10 closes the latter gap.

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**Conventions.** We use $\underline{X}$ to refer to the underlying scheme of a log scheme $X$. Given a map $P \to A$ from a monoid $P$ into the multiplicative monoid of a ring $A$, we refer to the associated log scheme by $\operatorname{Spec}\big(P \to A\big)$.

## 2. Generically Log Smooth Families

A *log toroidal family* will be a generalization of a saturated log smooth morphism. We first introduce the weaker notion of a *generically log smooth family* that already enjoys some useful properties. Log structures in this section are assumed to be in the étale topology. If $f : X \to S$ is a finite type morphism of Noetherian schemes, we say a Zariski open $U \subset X$ satisfies the *codimension condition* (CC) if the relative codimension of $Z := X \setminus U$ is $\geq 2$, i.e. for every point $s \in S$ with $X_s, U_s$ the fibers,

$$\text{codim}(X_s \setminus U_s, X_s) \geq 2. \tag{CC}$$

Recall that a Cohen-Macaulay morphism is a flat morphism with Cohen-Macaulay fibers.

**Definition 2.1.** A *generically log smooth family* consists of:

- a finite type Cohen-Macaulay morphism $f : X \to S$ of Noetherian schemes,
- a Zariski open $j : U \subset X$ satisfying (CC),
- a saturated and log smooth morphism $f : (U, \mathcal{M}_U) \to (S, \mathcal{M}_S)$ of fine saturated log schemes.

The complement $Z := X \setminus U$ we refer to as the *log singular locus* even though $f$ might extend log smoothly to it. We say two generically log smooth families $f, f' : X \to S$ with the same underlying morphism of schemes are *equivalent*, if there is some $\tilde{U} \subset U \cap U'$ satisfying (CC) with $\mathcal{M}_U|_{\tilde{U}} \cong \mathcal{M}'_{U'}|_{\tilde{U}}$ compatibly with all data.

If $T \to S$ is a morphism of fine saturated log schemes, then the base change $X_T \to T$ as a generically log smooth family is defined in the obvious way, taking fiber products in the category of *all* log schemes. Note that we need $f : U \to S$ saturated to ensure that $U_T$ is again a fine saturated log scheme. The notion of equivalence is due to the fact that we don't care about the precise $U$. However, for technical simplicity we assume some $U$ fixed. The name *log singular locus* is in analogy with [14].

**Definition 2.2.** For a generically log smooth family $f : X \to S$, the *de Rham complex* is defined as $W^\bullet_{X/S} := j_*\Omega^\bullet_{U/S}$ where $\Omega^\bullet_{U/S}$ denotes the log de Rham complex. We also define the $\mathcal{O}_X$-module of degree $m$ log polyvector fields $\Theta^m_{X/S} := j_* \bigwedge^m \mathcal{D}er_{U/S}(\mathcal{O}_U)$.

**Lemma 2.3.** *Let $f : X \to S$ be a Cohen-Macaulay morphism of Noetherian schemes, and let $j : U \subset X$ satisfy* (CC). *Then $j_*\mathcal{O}_U \cong \mathcal{O}_X$.*

*Proof.* This is a special case of [22, 3.5]. Note that our (CC) is a stronger assumption than the condition on the codimension in [22, 3.5]. □

Let $X \to S$ be a generically log smooth family. Using the language of [18, Def. 5.9.9], a sheaf $\mathcal{F}$ we call *$Z$-closed* if the natural map $\mathcal{F} \to j_*(\mathcal{F}|_U)$ is an isomorphism.

Most notably, two $Z$-closed sheaves that agree on $U$ are entirely equal. By their definition, $W^m_{X/S}$ as well as $\Theta^m_{X/S}$ are $Z$-closed. Furthermore, every reflexive sheaf is $Z$-closed.

**Lemma 2.4.** *The $\mathcal{O}_X$-modules $W^m_{X/S}$ and $\Theta^m_{X/S}$ are coherent and reflexive and these depend only on the equivalence class of $f : X \to S$.*

*Proof.* Let $\tilde{U} \subset U$ also satisfy (CC). We have by Lemma 2.3 that $j_*\Omega^\bullet_{\tilde{U}/S} = j_*\Omega^\bullet_{U/S}$ since $\Omega^m_{U/S}$ is finite locally free. Thus $W^\bullet_{X/S}$ depends only on the equivalence class of $f$. It is clear that it is quasi-coherent. For every sheaf $\mathcal{G}$ on $U$, $j_*\mathcal{G}$ is $Z$-closed, so in particular $W^m_{X/S}$ is $Z$-closed. Set $\mathcal{F}^\vee := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. By Lemma 2.3, $\mathcal{F}^\vee$ is $Z$-closed for all $\mathcal{F}$, so in particular $(W^m_{X/S})^{\vee\vee}$ is a $Z$-closed sheaf and it coincides with $W^m_{X/S}$ on $U$, hence $(W^m_{X/S})^{\vee\vee} = W^m_{X/S}$ and $W^m_{X/S}$ is reflexive. By the extension theorem [17, 9.4.8], there is a coherent $\mathcal{G}$ that restricts to $W^m_{X/S}$ on $U$. Now $\mathcal{G}^{\vee\vee} = W^m_{X/S}$ since both are $Z$-closed and agree on $U$, hence $W^m_{X/S}$ is also coherent. The argument for $\Theta^m_{X/S}$ is similar. □

**Lemma 2.5.** $W^m_{X/S} = \mathcal{H}om(\Theta^m_{X/S}, \mathcal{O}_X)$ *and* $\Theta^m_{X/S} = \mathcal{H}om(W^m_{X/S}, \mathcal{O}_X)$*.*

*Proof.* The statement is clear on $U$ where all sheaves are locally free and then it follows since all sheaves are $Z$-closed. □

*Remark* 2.6. The pushforward $j_*\mathcal{M}_U \to j_*\mathcal{O}_U = \mathcal{O}_X$ to $X$ yields a log structure which is compatible with $S$, so every generically log smooth family is canonically a log morphism $X \to S$. We don't know whether this pushforward is compatible with base change (and we don't care).

*Remark* 2.7. In view of Remark 2.6, neither the so defined log structure $\mathcal{M}_X$ nor the associated sheaf of log differentials $\Omega_{X/S}$ will be coherent in general, see Example 2.11. On the the other hand, $W^m_{X/S}$ and $\Theta^m_{X/S}$ are coherent and have further good properties in the case of log toroidal families as we will see.

Let $X \to S$ be a generically log smooth family. One defines for the log smooth morphism $U \to S$ the *horizontal divisor* $D_U \subset U$ (see e.g. [41, Definition 2.4], also Remark 3.2 below). This is only a Weil divisor in general. We denote by $D$ its closure in $X$ and by $I_D$ the corresponding ideal sheaf. We define $W^m_{X/S}(-D) := j_*((I_D W^m_{X/S})|_U)$. (This doesn't need to agree with $I_D W^m_{X/S}$.)

**Proposition 2.8.** *Let $S = \mathrm{Spec}(\mathbb{N} \to \Bbbk)$ for $\Bbbk$ a field where $1 \mapsto 0$. Let $f : X \to S$ be a generically log smooth family of relative dimension $d$ and let $\omega_f = f^!\mathcal{O}_S$ denote the (globally normalized) relative dualizing sheaf, then*

$$W^d_{X/S}(-D) = \omega_f.$$

*Proof.* On $U$, this is [41, Theorem 2.21, (ii)] and since both sheaves are $Z$-closed, the statement follows. □

**Example 2.9.** Let $f : X \to S$ be a log smooth and saturated morphism of Noetherian fine saturated log schemes. Then $f$ is flat by [28, 4.5] and has Cohen-Macaulay fibers by [42, II.4.1]. We see that $f : X \to S$ gives a generically log smooth family for $U = X$ and $W^\bullet_{X/S}$ is the ordinary log de Rham complex.

Not every log smooth morphism is saturated, e.g. see [26, Rem. 9.1] for a log smooth morphism that is not even integral.

**Example 2.10.** Let $X/\operatorname{Spec} R$ be a toric variety over a Noetherian base ring $R$. The fibers over points in $\operatorname{Spec} R$ are normal (and Cohen-Macaulay), so there is a regular open $U \subset X$ whose complement has relative codimension $\geq 2$ over $\operatorname{Spec} R$. For every divisorial log structure on $X$ coming from a torus invariant divisor $D$ on $X$, the map $U \to \operatorname{Spec} R$ is log smooth and saturated when using the trivial log structure on $\operatorname{Spec} R$. Hence $X \to \operatorname{Spec} R$ is a generically log smooth family. The differentials $W^\bullet_{X/S}$ coincide with what is called *reflexive* or *Danilov* or *Zariski-Steenbrink differentials* with log poles in $D$. This example extends to toroidal pairs $(X, D)$ over $\operatorname{Spec} R$.

**Example 2.11.** The $\mathbb{Z}[t]$-algebra $A = \mathbb{Z}[x, y, t, w]/(xy - tw)$ defines a map $f : \operatorname{Spec} A \to \mathbb{A}^1$ that is log smooth and saturated away from the origin when using the divisorial log structure given by $t = 0$ on source and target, hence a generically log smooth family. The log structure on $\operatorname{Spec} A$ is not coherent at the origin, so $f$ is not log smooth. Even worse, $\Omega_f$ is not a coherent sheaf at the origin, see [15, Example 1.11].

2.1. **Analytification.** Given a generically log smooth family $f : X \to S$ of finite type over $\mathbb{C}$, we denote the associated family of complex analytic spaces by $f^{an} : X^{an} \to S^{an}$. Induced by $f$, the open $U^{an} \subset X^{an}$ carries an fs log structure so that $U^{an} \to S^{an}$ is a log smooth and saturated morphism of fs log analytic spaces. The analogue of Lemma 2.3 holds if $X^{an}, S^{an}$ are Cohen-Macaulay by [5, Thm. 3.6]. For $S = \operatorname{Spec}(Q \to A)$ with $A$ an Artin ring and

$$W^{\bullet,an}_{X/S} := j^{an}_* \Omega^\bullet_{U^{an}/S^{an}},$$

we have $W^{m,an}_{X/S} \cong (W^m_{X/S})^{an}$ since both are reflexive coherent $\mathcal{O}_{X^{an}}$-modules that coincide on $U^{an}$. If $f$ is proper then GAGA gives $H^q(X^{an}, W^{p,an}_{X/S}) \cong H^q(X, W^p_{X/S})$ and also

$$\mathbb{H}^{p+q}(X^{an}, W^{\bullet,an}_{X/S}) \cong \mathbb{H}^{p+q}(X, W^\bullet_{X/S}),$$

e.g. via the comparison of the Hodge-to-de Rham spectral sequences.

## 3. Elementary Log Toroidal Families

For basic notions and constructions of monoids, see [34].

**Definition 3.1.** An *elementary (log) toroidal datum* $(Q \subset P, \mathcal{F})$ (ETD for short) consists of an injection $Q \hookrightarrow P$ of sharp toric monoids that turns $P$ into a free $Q$-set

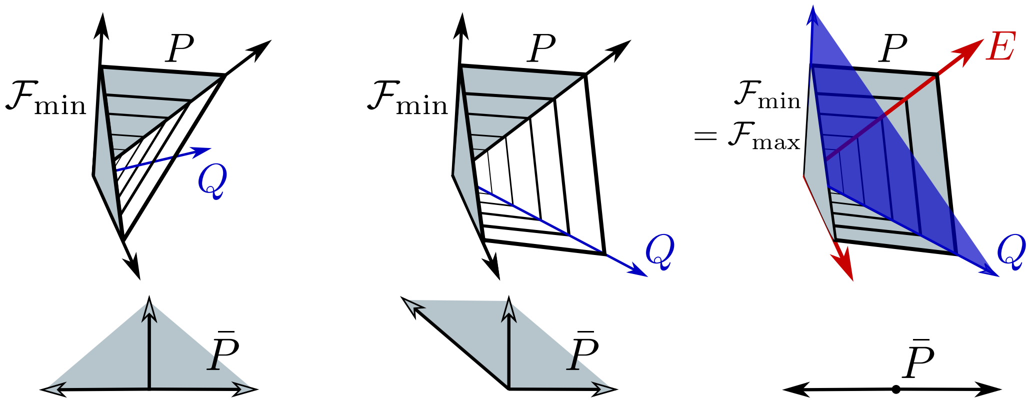


FIGURE 3.1. Three examples of a saturated injection $Q \subset P$ and the projection $\bar{P}$, the outer two are log smooth, the middle one gives Example 2.11.

whose canonical basis is a union of faces of $P$. We furthermore record a set $\mathcal{F}$ of facets of $P$ containing all facets that don't contain $Q$. Set

$$\mathcal{F}_{\min} := \underbrace{\{F \subset P \text{ a facet} \,|\, Q \not\subset F\}}_{\text{vertical facets}},$$

so $\mathcal{F}_{\min} \subset \mathcal{F} \subset \mathcal{F}_{\max}$ where $\mathcal{F}_{\max}$ is the set of all facets.

*Remark* 3.2. The facets in $\mathcal{F} \setminus \mathcal{F}_{\min}$ will give the *horizontal divisor* that we referred to as $D$ before.

**Lemma 3.3.** *([34, Corollary I.4.6.11, Theorem I.4.8.14, Corollary I.1.4.3]) The requirement on the injection $Q \hookrightarrow P$ in Definition 3.1 is equivalent to saying this map is saturated.*

See Figure 3.1 for examples. Even the case $Q = 0$ can be interesting since then $\mathcal{F}_{\min} = \emptyset$. We denote the union of faces of $P$ that gives the generating set of the free $Q$-action by $E$. A face $F$ of $P$ contained in $E$ we call an *essential face*. Every $p \in P$ has a unique decomposition $p = e + q$ with $e \in E, q \in Q$, hence

$$E \times Q \to P, \quad (e, q) \mapsto e + q, \tag{3.1}$$

is bijective ([34, Theorem I.4.8.14], cf. [27, Lemma 1.1]). Furthermore, we see that $E = P \setminus (Q^+ + P)$ where $Q^+ = Q \setminus 0$ is the maximal ideal. Moreover, projecting $E$ to $P^{\mathrm{gp}}/Q^{\mathrm{gp}}$ is injective and the set of essential faces gives a fan in $P^{\mathrm{gp}}/Q^{\mathrm{gp}}$ whose support $\bar{P}$ is convex in $(P^{\mathrm{gp}}/Q^{\mathrm{gp}}) \otimes_{\mathbb{Z}} \mathbb{R}$ since it is the convex hull of the projection of $P$. Note that $\bar{P}^{\mathrm{gp}} = P^{\mathrm{gp}}/Q^{\mathrm{gp}}$. A choice of splitting $P^{\mathrm{gp}} \cong \bar{P}^{\mathrm{gp}} \oplus Q^{\mathrm{gp}}$ yields a unique map of sets $\varphi : \bar{P} \to Q^{\mathrm{gp}}$ so that $\mathrm{id} \times \varphi : \bar{P} \to \bar{P} \oplus Q^{\mathrm{gp}}$ is a section of the projection $P \to \bar{P}$ with the property that its image is $E$, so

$$P = \{(\bar{p}, q) \in \bar{P} \oplus Q^{\mathrm{gp}} \,|\, \exists \tilde{q} \in Q : q = \varphi(\bar{p}) + \tilde{q}\}. \tag{3.2}$$

**Lemma 3.4.** *The injection $Q \subset P$ induces a Cohen-Macaulay morphism $\underline{f}$ : $\operatorname{Spec}\mathbb{Z}[P] \to \operatorname{Spec}\mathbb{Z}[Q]$ with fiber dimension $d = \operatorname{rk}(P^{\mathrm{gp}}/Q^{\mathrm{gp}})$.*

*Proof.* Since $P$ is free as a $Q$-set (generated by $E$), $\operatorname{Spec}\mathbb{Z}[P]$ is a flat $\operatorname{Spec}\mathbb{Z}[Q]$-module. By [18, Cor.6.3.5] the total space of a faithfully flat morphism of Noetherian schemes is Cohen-Macaulay if and only if the base and all fibers are. By Hoechster's theorem, the fibers of $\operatorname{Spec}\mathbb{Z}[P] \to \operatorname{Spec}\mathbb{Z}$ are Cohen-Macaulay, hence $\operatorname{Spec}\mathbb{Z}[P]$ and $\operatorname{Spec}\mathbb{Z}[Q]$ are Cohen-Macaulay. Now flatness of $\underline{f}$ implies it is Cohen-Macaulay. □

We next want to define an open set $U$ in the domain of $\underline{f}$ that satisfies (CC). We will actually define its complement and for this we need a good understanding of the faces of $P$.

**Lemma 3.5.** *Let $F \subseteq P$ be a face. Set $\bar{F} := F \cap E$, $Q' := Q \cap F$, then*

$$F = \bar{F} + Q' := \{\bar{f} + q' | \bar{f} \in \bar{F}, q' \in Q'\}.$$

*Since $E$ is a union of faces of $P$, so is $\bar{F}$. Note also that $Q'$ is a face of $Q$.*

*Proof.* By the decomposition (3.1), any element in $F$ has the form $\bar{f} + q$ with $\bar{f} \in E, q \in Q$. Since $F$ is a face, $\bar{f}, q$ are both in $F$, hence $F \subseteq \bar{F} + Q'$. The reverse inclusion is clear. □

Consider the set of *bad faces* of $P$ defined as

$$\mathcal{B} = \left\{ \bar{F} + Q' \;\middle|\; \begin{array}{c} \bar{F} \text{ is a union of essential faces of rank at most } d-2 \\ Q' \text{ is a face of } Q,\ \bar{F} + Q' \text{ is a face of } P \end{array} \right\}.$$

Recall that there is a 1-1 correspondence between faces $F$ of $P$ and torus orbits closures $V_F := \operatorname{Spec}\mathbb{Z}[F]$ in $\operatorname{Spec}\mathbb{Z}[P]$. Similarly, for $Q'$ a face of $Q$, we have a torus orbit closure $V_{Q'} := \operatorname{Spec}\mathbb{Z}[Q'] \subseteq \operatorname{Spec}\mathbb{Z}[Q]$.

**Lemma 3.6.** *Given $\bar{F} + Q' \in \mathcal{B}$, we find that $V_{\bar{F}+Q'}$ is flat over $V_{Q'} \subset \operatorname{Spec}\mathbb{Z}[Q]$. Furthermore, if $X$ is a fiber of $\underline{f}$, then $\operatorname{codim}(X \cap V_{\bar{F}+Q'}, X) \geq 2$.*

*Proof.* Since $\bar{F} + Q'$ is free as a $Q'$-set, $\mathbb{Z}[\bar{F} + Q']$ is a free $\mathbb{Z}[Q']$-module, so the flatness statement follows. The origin 0 given by the prime ideal $(z^q | q \in Q^+)$ is contained in $V_{Q'}$, let $X_0$ be the fiber over it. It suffices to check the codimension condition for this particular fiber. But note that $X_0 \cap V_{\bar{F}+Q'} = \bigcup_{F \subset \bar{F}} V_F$ where the union runs over faces $F$ of $P$ contained in $\bar{F}$ and we have $\dim V_F \leq d-2$ by the assumption on $\bar{F}$. □

Set

$$U_P := \operatorname{Spec}\mathbb{Z}[P] \setminus \left( \bigcup_{B \in \mathcal{B}} V_B \right). \tag{3.3}$$

For every face $F$ of $P$, we have an open subset $\operatorname{Spec}\mathbb{Z}[P_F]$ of $\operatorname{Spec}\mathbb{Z}[P]$ where $P_F$ is the localization of $P$ in $F$, i.e. $P_F$ is the submonoid of $P^{\mathrm{gp}}$ generated by $P$ and $-F$.

**Lemma 3.7.** *We find $U_P = \bigcup_F U_F$ where the union is over the essential faces $F$ of rank $d-1$.*

*Proof.* Since $U_P$ is a union of torus orbits, it suffices to check that any torus orbit contained in $U_P$ is contained in some $U_F$ for $F$ essential of rank $d-1$. Every torus orbit is given by $O_G := \operatorname{Spec}\mathbb{Z}[G^{\mathrm{gp}}]$ for $G$ a face of $P$. Assume $O_G \subseteq U$. We use Lemma 3.5 to write $G = \bar{G} + Q'$. If $\operatorname{rk}\bar{G} \leq d-2$, then $G \in \mathcal{B}$, so $O_G \not\subseteq U$. Hence, $\dim \bar{G} \geq d-1$ and $\bar{G}$ contains some essential face $F$ of rank $d-1$. Then $F$ is also contained in $G$ and thus $O_G$ is contained in $U_F$. Conversely, since $O_F$ is not in any $V_B$, the assertion follows. □

Let $A_Q := \operatorname{Spec}(Q \to \mathbb{Z}[Q])$ denote the log scheme with standard toric log structure and let $A_{P,\mathcal{F}}$ be the log scheme with underlying scheme $\operatorname{Spec}\mathbb{Z}[Q]$ and divisorial log structure given by the divisor $\bigcup_{F\in\mathcal{F}} \operatorname{Spec}\mathbb{Z}[F]$. The map $f : A_{P,\mathcal{F}} \to A_Q$ induced by $\theta$ is naturally a log morphism by the condition on $\mathcal{F}$ to contain the vertical faces. We work here with Zariski log structures which however coincide with the pushforward of the corresponding étale log structures by [34, Prop. III.1.6.5].

**Lemma 3.8** (Theorem 3.5 in [28] or Theorem 4.1 in [26])**.** *If $\mathcal{F} = \mathcal{F}_{\max}$, then $f$ is log smooth.*

**Proposition 3.9.** *The map $f : A_{P,\mathcal{F}} \to A_Q$ is a generically log smooth family with $U_P$ serving as the specified dense open of log smoothness.*

*Proof.* If $\mathcal{F} = \mathcal{F}_{\max}$ then $f$ is saturated since $\theta$ is saturated. More generally, since $A_{P,\mathcal{F}_{\max}} \to A_{P,\mathcal{F}}$ is locally given by embedding a face, it is exact. Now by [34, I.4.8.5(2)], $f$ is saturated.

The assertion is clear if $d = 0 \iff P = Q$, so assume $d > 0$. Given Lemma 3.4, we still need to verify that $U$ satisfies (CC) and that $f$ is log smooth on $U_P$. Note that Lemma 3.6 implies that $U_P$ satisfies (CC) since the complement of $U_P$ is the union of closed sets each of which has codimension at least two in each fiber.

To see that $f$ is log smooth on $U_P$, by Lemma 3.7, it suffices to check that $f$ is log smooth on $U_F$ for $F$ essential of rank $d-1$. Let $F$ be such a face. Set $\bar{P}_F := P_F/F^{\mathrm{gp}}$ and note that the projection of $Q$ to $\bar{P}_F$ is injective because $F^{\mathrm{gp}} \cap Q = \{0\}$. There is an isomorphism $P_F \cong F^{\mathrm{gp}} \times \bar{P}_F$ commuting with the injection of $Q$ that is $\{0\} \times Q$ on the right.

The log structure on $U_F$ is a divisorial log structure given by a set of divisors each of which pulls back from $\operatorname{Spec}\mathbb{Z}[\bar{P}_F]$, so we may consider the corresponding divisorial log structure on $\operatorname{Spec}\mathbb{Z}[\bar{P}_F]$ to upgrade this to a log scheme $\bar{U}_F$. We have a factorization $U_F \to \bar{U}_F \to A_Q$ with the first map a smooth projection from a product that is therefore strict, hence log smooth. It thus suffices to show that $\bar{U}_F \to A_Q$ is log smooth. Note that $\bar{U}_F \to A_Q$ is the log morphism of an ETD with $d = 1$. The following lemma finishes the proof. □

**Lemma 3.10.** *Assume that $f : A_{P,\mathcal{F}} \to A_Q$ has one-dimensional fibers (i.e. $d = 1$), then $f$ is log smooth. (The third situation of Figure 3.1 is an example.)*

*Proof.* We are done by Lemma 3.8 if $\mathcal{F} = \mathcal{F}_{\max}$ and this always holds if $Q$ meets the interior of $P$. So assume $Q$ is contained in a proper face of $P$, then by Lemma 3.5 it is in fact a facet of $P$ and then $\bar{P} = \mathbb{N}$ and consequently $P = \mathbb{N} \times Q$. A facet of $P$ that is not $Q$ is in $\mathcal{F}_{\min} = \{\mathbb{N} \times F \mid F \text{ is a facet of Q}\}$. Hence $\mathcal{F} \subsetneq \mathcal{F}_{\max}$ implies $\mathcal{F} = \mathcal{F}_{\min}$ and thus $f$ is strict. Since $\underline{f}$ is smooth, we find $f$ is log smooth. $\square$

**Corollary 3.11.** *It is possible to find open subsets $U_1$ and $U_2$ so that $U_P = U_1 \cup U_2$ and $A_P|_{U_1} = A_{P,\mathcal{F}}|_{U_1}$ and $f : U_2 \subset A_{P,\mathcal{F}} \to A_Q$ is strict and smooth.*

*Proof.* Let $\mathcal{E}_1$ be the set of essential faces of rank $d-1$ such that when applying the proof of Lemma 3.10 to $\bar{U}_F \to A_Q$ from the proof of the proposition, we are in the case $\mathcal{F} = \mathcal{F}_{max}$, and let $\mathcal{E}_2$ be the set of faces where we are in case $\mathcal{F} = \mathcal{F}_{min}$. Then for $F \in \mathcal{E}_1$ we have $A_P|_{U_F} = A_{P,\mathcal{F}}|_{U_F}$, and for $F \in \mathcal{E}_2$, the morphism $U_F \to A_Q$ is strict and smooth. Now we define $U_1 = \bigcup_{F \in \mathcal{E}_1} U_F$ and $U_2 = \bigcup_{F \in \mathcal{E}_2} U_F$. $\square$

**Example 3.12.** If $(Q \subset P, \mathcal{F})$ is an ETD and $r \geq 0$, then we obtain another ETD $(Q \times \{0\} \subset P \times \mathbb{N}^r, \mathcal{F}')$ where $\mathcal{F}' = \{F \times \mathbb{N}^r \mid F \in \mathcal{F}\}$.

## 4. Log Toroidal Families

We define log toroidal families and investigate their basic properties.

**Definition 4.1.** We say that a generically log smooth family $f : X \to S$ is *log toroidal* if for every geometric point $\bar{x} \to X$, we have a commutative diagram

(LM)
$$\begin{tikzcd}
 & & (V, g^{-1}(U)) \arrow[dll, "g"'] \arrow[dd] \arrow[drr, dotted, "h"] & & & & \\
(X,U) \arrow[dd, "f"'] & & & & (L,U_L) \arrow[dll] \arrow[drr, "c"] & & \\
 & & \tilde{S} \arrow[dll] \arrow[drr, "a"] & & & & (A_{P,\mathcal{F}}, U_P) \arrow[dll] \\
S & & & & A_Q & &
\end{tikzcd}$$

where $g : V \to X$ is an étale neighborhood of $\bar{x}$, $\tilde{S} \to S$ is a strict étale neighborhood of $f(\bar{x})$ and $a$ is given by a chart $Q \to \mathcal{M}_{\tilde{S}}$ of $\tilde{S}$. The bottom right diagonal map is required to be given by an ETD $(Q \subset P, \mathcal{F})$ and $U_P \subset A_{P,\mathcal{F}}$ denotes the open set from (3.3). The solid arrows are morphisms of schemes and log morphisms on the specified opens, whereas $h : V \to L$ is an étale morphism only of underlying schemes. The bottom right diamond is Cartesian, in particular $U_L = c^{-1}(U_P)$. Moreover, we have an open $\tilde{U} \subset V$ satisfying (CC), such that $\tilde{U} \subset g^{-1}(U) \cap h^{-1}(U_L)$ and there is an isomorphism $g^*\mathcal{M}_X \cong h^*\mathcal{M}_L$ of the two log structures on $\tilde{U}$ compatible with the maps to $S$.

The diagram (LM) is called a *local model* for $f : X \to S$ at $\bar{x}$. If $S \cong \mathrm{Spec}(Q \to B)$, every point has a local model with $\tilde{S} = S$ and $a$ is given by the chart $Q \to B$ then we say $f : X \to S$ is *log toroidal with respect to* $a : S \to A_Q$.

Log toroidal families are stable under strict base change.

*Remark* 4.2. Note that Definition 4.1 only requires a covering of $X$ by (LM) but does not say that an arbitrary geometric point $\bar{x} \in X$ permits a diagram (LM) that identifies $\bar{x}$ with the origin in $A_P$. However, if $\Bbbk$ is algebraically closed, one can show that by localizing the ETD in (LM) and using Example 3.12 one can assume that $\bar{x} \in X$ becomes the origin in $A_P$. We will make use of this fact in the proof of Theorem 1.10.

**Example 4.3.** Every elementary log toroidal family $f : A_{P,\mathcal{F}} \to A_Q$ is log toroidal.

**Example 4.4.** The generically log smooth families given in Example 2.10 are log toroidal families with $Q = 0$ in every ETD.

**Example 4.5.** A saturated log smooth morphism $f : X \to S$ is log toroidal with $U = X$. Indeed, locally starting from a neat chart of $f$, set $\mathcal{F} = \mathcal{F}_{\max}$ and then apply Example 3.12 to have local models. That this works is *not* a trivial consequence of Theorem 3.5 in [28]. Instead, use [34, Theorem VI.3.3.3].

**Example 4.6.** In the setting and notation of the Gross-Siebert program, [15, Theorem 2.6] shows that if $(B, \mathcal{P})$ is positive and simple, and $s$ is lifted open gluing data, then $X_0^\dagger(B, \mathcal{P}, s) \to \mathrm{Spec}(\mathbb{N} \to k)$ is log toroidal. More generally, it was shown in [35, Proposition 2.8] that c.i.t. log Calabi-Yau spaces are log toroidal over $\mathrm{Spec}(\mathbb{N} \to k)$. The divisorial deformations defined in [15] are also log toroidal families.

## 5. Log Structures and Infinitesimal Deformations

Let $X$ be a toroidal crossing space over a field $\Bbbk$. As mentioned in the introduction, $X$ can be equipped with a sheaf of sets $\mathcal{LS}_X$ which we recall now. Let $S = \mathrm{Spec}(\mathbb{N} \overset{1 \mapsto 0}{\to} \Bbbk)$ be the standard log point. The pair $(\mathcal{P}, \mathbb{1})$ gives a *ghost structure* in the sense of [14, Definition 3.16]. Indeed, the *type* of the ghost structure is fixed by requiring it to be the one given by the local chart that comes with the definition of a toroidal crossing space. We will refer to this type as the *given type* below. By [14, Definition 3.19 and Proposition 3.20], there is a sheaf $\mathcal{LS}_X$ (denoted $\mathcal{LS}_{X^g}$ in loc.cit.) with the property that for every étale open $U \subset X$, there is a natural bijection

$$\Gamma(U, \mathcal{LS}_X) = \left\{ \begin{array}{c|c} \mathcal{M}_U \to \mathcal{O}_U \text{ a log structure of} & (U, \mathcal{M}_U) \to S \text{ via } 1 \mapsto \tilde{\mathbb{1}} \text{ is a} \\ \text{the given type, } \tilde{\mathbb{1}} \in \Gamma(U, \mathcal{M}_U), & \text{log smooth morphism and} \\ \overline{\mathcal{M}}_U \overset{\sim}{\to} \mathcal{P} \text{ an isomorphism} & \mathcal{M}_U \to \overline{\mathcal{M}}_U \to \mathcal{P} \text{ sends } \tilde{\mathbb{1}} \text{ to } \mathbb{1} \end{array} \right\}$$

where the set on the right is to be taken modulo isomorphisms. The support of $\mathcal{P}/\mathbb{1}$ agrees with $X_{\mathrm{sing}}$, so the sheaf $\mathcal{LS}_X$ is supported on $X_{\mathrm{sing}}$.

Set $S_\varepsilon := \mathrm{Spec}(\mathbb{N} \xrightarrow{1 \mapsto \varepsilon} \Bbbk[\varepsilon]/(\varepsilon^2))$. If $V \to S$ is a log smooth morphism with $V$ affine, then there is a unique log smooth lifting $V_\varepsilon \to S_\varepsilon$ up to isomorphism. For $(\mathcal{M}, \tilde{\mathbb{1}}) \in \mathcal{LS}_X(U)$ and an affine $V \subset U$, the deformation $i : V \to V_\varepsilon$ yields an extension

$$0 \to \mathcal{O}_V \to i^*\Omega^1_{\underline{V_\varepsilon}} \to \Omega^1_{\underline{V}} \to 0 \tag{5.1}$$

where on the left $1 \mapsto i^* d\varepsilon$. The classes of such local extensions glue to a well-defined class in $\mathcal{E}xt^1(\Omega^1_U, \mathcal{O}_U)$ (though neither the extensions nor the deformations need to glue). We have thus defined a map of sheaves of sets

$$\eta : \mathcal{LS}_X \to \mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X) = \mathcal{T}^1_X. \tag{5.2}$$

A relationship between log structures and infinitesimal deformations had been observed before [31, Prop. 1.1], [39, Remark (3.11)], [26, Thm 11.7], [14, Example 3.30] though the existence of the map $\eta$ seems not to have been noticed so far. Both sheaves in (5.2) have a natural action of $\mathcal{O}_X^\times$: indeed, $\mathcal{T}^1_X$ because it is coherent and $\mathcal{LS}_X$ for we let a section $\lambda$ of $\mathcal{O}_X^\times$ act by $\tilde{\mathbb{1}} \mapsto \lambda^{-1}\tilde{\mathbb{1}}$.

**Proposition 5.1.** *The map $\eta$ is $\mathcal{O}_X^\times$-equivariant.*

*Proof.* At a geometric point $\bar{x} \in X$ with $M = (\mathcal{M}, \tilde{\mathbb{1}}) \in \mathcal{LS}_{X,\bar{x}}$ for $\mathcal{M}$ defined on some étale $U \to X$ that contains $\bar{x}$, let $\mu_M : \mathcal{O}_{X,\bar{x}} \to \mathcal{T}^1_{X,\bar{x}}$ denote the connecting homomorphism at $\bar{x}$ in the long exact sequence obtained from applying $\mathcal{H}om(-, \mathcal{O}_X)$ to (5.1). By a general fact for extensions, we have $\mu_M(1) = \eta(M)$. For $\lambda \in \mathcal{O}^\times_{X,\bar{x}}$, let $M_\lambda \in \mathcal{LS}_{X,\bar{x}}$ denote the element $(\mathcal{M}, \lambda^{-1}\tilde{\mathbb{1}})$. The statement of the lemma comes down to the following claim.

$$\begin{tikzcd}
& U_1 \arrow[r, "i_1"] \arrow[ddl, dotted] \arrow[ddr] & (U_1)_\varepsilon \arrow[ddr] & \\
& U_\lambda \arrow[u, dotted, "\chi"'] \arrow[r, "i_\lambda"] \arrow[dl, dotted] \arrow[dr] & (U_\lambda)_\varepsilon \arrow[d] \arrow[dr] & \\
\underline{S} & & S \arrow[ll] \arrow[r] & S_\varepsilon \arrow[lll, dotted, bend left]
\end{tikzcd}$$

**Claim 1.** $\mu_M(\lambda) = \eta(M_\lambda)$.

To prove the claim, let $U_1, U_\lambda$ denote the log smooth schemes over $S$ respectively obtained from the log scheme $U$ and the map to $S$ given by $1 \mapsto \mathbb{1}$ and $1 \mapsto \lambda^{-1}\mathbb{1}$ respectively. Let $(U_1)_\varepsilon$ and $(U_\lambda)_\varepsilon$ be the unique deformations of $U_1, U_\lambda$ over $S_\varepsilon$ respectively. Let $\chi : U_\lambda \to U_1$ be the canonical isomorphism over $\underline{S} = \mathrm{Spec}\big(0 \to \Bbbk\big)$.

We are now going to use facts about idealized log schemes, see [34, III.1.3 & Variant 3.1.21] for an introduction. We give $S_\varepsilon$ the ideal $\langle 2 \rangle$ generated by $2 \in \mathbb{N}$ and $(U_1)_\varepsilon$ and $(U_\lambda)_\varepsilon$ the pullback ideals $K_1, K_\lambda$ respectively so that $((U_1)_\varepsilon, K_1)$ and $((U_\lambda)_\varepsilon, K_\lambda)$ are ideally log smooth over $(S_\varepsilon, \langle 2 \rangle)$. The map $(S_\varepsilon, \langle 2 \rangle) \to (A_\mathbb{N}, \emptyset)$ is an étale map of idealized log schemes and $A_\mathbb{N} \to \underline{S}$ is log smooth, hence the composition $\pi : ((U_1)_\varepsilon, K_1) \to (S_\varepsilon, \langle 2 \rangle) \to \underline{S}$ is log smooth. We apply the infinitesimal lifting property to the diagram

$$\begin{tikzcd}
(U,K) \arrow[r, "i_1"] \arrow[d, "i_\lambda"] & ((U_1)_\varepsilon, K_1) \arrow[d, "\pi"] \\
((U_\lambda)_\varepsilon, K_\lambda) \arrow[r] & \underline{S}
\end{tikzcd}$$

where $(U,K)$ is the idealized log scheme $U = U_1 \overset{\chi}{=} U_\lambda$ with ideal given by $(\tilde{\mathbb{1}})^2$ or equivalently $(\lambda^{-1}\tilde{\mathbb{1}})^2$. The left vertical map $i_\lambda$ is strict for the log structure and ideal and given by a square-zero-ideal. We obtain a morphism $\tilde{\chi} : (U_\lambda)_\varepsilon \to (U_1)_\varepsilon$ of log schemes that preserves the ideals and is an isomorphism on ghost sheaves. Consequently, with $\rho_\lambda \in \mathcal{M}_{(U_\lambda)_\varepsilon,\bar{x}}$ and $\rho_1 \in \mathcal{M}_{(U_1)_\varepsilon,\bar{x}}$ the images of the generator $1 \in \mathcal{M}_{S_\varepsilon}$ respectively, we have $\tilde{\chi}^*\rho_1 = \tilde{\lambda} \cdot \rho_\lambda$ for some $\tilde{\lambda} \in \mathcal{O}_{(U_\lambda)_\varepsilon^\times,\bar{x}}$ that restricts to $\lambda \in \mathcal{O}_{U,\bar{x}}^\times$. This implies that $\tilde{\chi}$ becomes an isomorphism after shrinking $(U_\lambda)_\varepsilon$, $(U_1)_\varepsilon$ if needed. Using $i_1 \circ \chi = \tilde{\chi} \circ i_\lambda$, we obtain the commutative diagram

$$\begin{tikzcd}
\eta(M_\lambda): & 0 \arrow[r] & \mathcal{O}_U \arrow[r, "1\mapsto i_\lambda^* d\alpha(\rho_\lambda)"] \arrow[d, "\lambda^{-1}\cdot"] & i_\lambda^*\Omega^1_{\underline{(U_\lambda)_\varepsilon}} \arrow[r] \arrow[d, equal] & \Omega^1_{\underline{U}} \arrow[r] \arrow[d, equal] & 0 \\
\eta(M_1): & 0 \arrow[r] & \mathcal{O}_U \arrow[r, "1\mapsto i_\lambda^* d(\tilde{\lambda}\alpha(\rho_\lambda))"] & i_\lambda^*\Omega^1_{\underline{(U_\lambda)_\varepsilon}} \arrow[r] & \Omega^1_{\underline{U}} \arrow[r] & 0
\end{tikzcd}$$

and we conclude $\eta(M_\lambda) = \mu_M(\lambda)$ via standard homological algebra. □

**Lemma 5.2.** *Let $\bar{x} \in X$ be a geometric point with $\Bbbk[\mathcal{P}_{\bar{x}}]$ smooth, then*

*(1) for $M \in \mathcal{LS}_{X,\bar{x}}$, the map $\mu_{M,\bar{x}} : \mathcal{O}_{X,\bar{x}} \to \mathcal{T}_{X,\bar{x}}$ is surjective,*
*(2) $\mathcal{O}_{X,\bar{x}}^\times$ acts transitively on $\mathcal{LS}_{X,\bar{x}}$,*
*(3) $\eta_{\bar{x}} : \mathcal{LS}_{X,\bar{x}} \to \mathcal{T}_{X,\bar{x}}$ is injective.*

*Proof.* Set $P := \mathcal{P}_{\bar{x}}$. For (1), let $U \to X$ be an étale affine neighborhood of $\bar{x}$ where $M = (\mathcal{M}_U, \mathbb{1}_U)$ is defined and $h : (U, \mathcal{M}_U) \to \operatorname{Spec}\big(P \to \Bbbk[P]/(z^{\mathbb{1}_{\bar{x}}})\big)$ the strict $S$-morphism whose underlying map is smooth. Possibly after shrinking $U$, via $\varepsilon \mapsto \mathbb{1}_{\bar{x}}$, we obtain a strict map of extensions over $S_\varepsilon$,

$$h_\varepsilon : (U_\varepsilon, \mathcal{M}_{U_\varepsilon}) \to \operatorname{Spec}\big(P \to \Bbbk[P]/(z^{(\mathbb{1}_{\bar{x}}+\mathbb{1}_{\bar{x}})})\big)$$

whose underlying morphism is also smooth and hence $\Omega_{\underline{U_\varepsilon}}$ is locally free. This implies that the corresponding term $\mathcal{E}xt^1(\Omega_{\underline{U_\varepsilon}}, \mathcal{O}_X)_{\bar{x}}$ in the long exact sequence for (5.1) vanishes and thus $\mu_{M,\bar{x}}$ is surjective.

To show (2), note that it suffices to show that any two elements in $\mathcal{LS}_{X,\bar{x}}$ are isomorphic over $\underline{S}$. Equivalently by [14, Definition 3.19 & Corollary 3.12], the composition

$$\mathcal{LS}_{X,\bar{x}} \subset \mathcal{E}xt^1(\mathcal{P}_{\bar{x}}^{\mathrm{gp}}/\mathbb{Z}\mathbb{1}_{\bar{x}}, \mathcal{O}_{X,\bar{x}}^\times) \to \mathcal{E}xt^1(\mathcal{P}_{\bar{x}}^{\mathrm{gp}}, \mathcal{O}_{X,\bar{x}}^\times)$$

needs to be the constant map. By assumption, $P$ is free and then (2) follows from the description of $\mathcal{E}xt^1(\mathcal{P}_{\bar{x}}^{\mathrm{gp}}, \mathcal{O}_{X,\bar{x}}^\times)$ in [14, Proposition 3.14].

For (3), if $\bar{x} \notin X_{\text{sing}}$ both stalks are trivial and there is noting to show, so assume $\bar{x} \in X_{\text{sing}}$. By [12, Proposition 1.10], we have $\mathcal{T}^1_{X,\bar{x}} \cong \mathcal{O}_{X_{\text{sing}},\bar{x}}$, so the kernel of the action of $\mathcal{O}^\times_{X,\bar{x}}$ on $\mathcal{T}^1_{X,\bar{x}}$ is $K := \ker\big(\mathcal{O}^\times_{X,\bar{x}} \to \mathcal{O}^\times_{X_{\text{sing}},\bar{x}}\big)$. If we show that $K$ is contained in the kernel of the action of $\mathcal{O}^\times_{X,\bar{x}}$ on $\mathcal{LS}_{X,\bar{x}}$, then (3) follows from (2) and Prop. 5.1. By assumption, $X$ is normal crossings at $\bar{x}$. Let $X_1, ..., X_r$ be the local components of $X$ at $\bar{x}$, $r \geq 2$. Let $\lambda \in K$ be given and write $\lambda = 1 + \sum_{i=1}^r f_i$ where $f_i|_{X_j} = 0$ for $i \neq j$. We observe that $\lambda = \prod_i (1 + f_i)$ because $f_i f_j = 0$ for $i \neq j$. If $\mathbb{N}^r \to \mathcal{O}_{X,\bar{x}}$, $e_i \mapsto h_i$ is a chart of $X$ at $\bar{x}$ representing an element of $\mathcal{LS}_{X,\bar{x}}$ with $\mathbb{1} = \sum_i e_i$ and $V(h_i) = X_i$, then $e_i \mapsto (1 + f_i)e_i$ defines an automorphism of $\mathcal{M}_{X,\bar{x}}$ compatible with the map to $\mathcal{O}_{X,\bar{x}}$ because $(1 + f_i)h_i = h_i$. It takes $\mathbb{1}$ to $\lambda\mathbb{1}$, so $\lambda$ acts trivially on $\mathcal{LS}_{X,\bar{x}}$. □

*Remark* 5.3. For $\kappa \geq 2$, consider the monoid $P_\kappa = \langle e_1, e_2, \mathbb{1} | e_1 + e_2 = \kappa\mathbb{1}\rangle$ and the toroidal crossing space $X = \operatorname{Spec}\big(P_\kappa \to \Bbbk[P_\kappa]/(z^{\mathbb{1}})\big)$. The map $\eta : \mathcal{LS}_X \to \mathcal{T}^1_X$ is the zero map $\Bbbk^\times \to \Bbbk$, so the smoothness assumption in Lemma 5.2 is necessary.

**Theorem 5.4.** *Let $X$ be a toroidal crossing space with $P_{\bar{x}} \cong \mathbb{N}^2$ whenever $\bar{x}$ is the generic point of a component of $X_{\text{sing}}$ then $\eta : \mathcal{LS}_X \to \mathcal{T}^1_X$ is injective. On the open set $V \subset X$ of points $\bar{x}$ with $\mathcal{P}_{\bar{x}} \cong \mathbb{N}^r$ for some $r$, we have $\eta(\mathcal{LS}_V) = (\mathcal{T}^1_V)^\times$ where $(\mathcal{T}^1_X)^\times \subset \mathcal{T}^1_X$ denotes the subsheaf of those elements that generate $\mathcal{T}^1_X$ as an $\mathcal{O}_X$-module.*

*Proof.* The second statement is Lemma 5.2. For the first statement also follows from the Lemma combined with the fact that for every open $U \subset X$, the restriction map $\mathcal{LS}_X(U) \to \mathcal{LS}_X(U \cap V)$ is injective which is a consequence of Corollary 6.2 below. Indeed, in view of the diagram on the right, that the composition of the left vertical and bottom horizontal arrow is injective implies the injectivity of the top horizontal arrow.

$$\begin{tikzcd} \mathcal{LS}_X(U) \arrow[r, "\eta"] \arrow[d] & \mathcal{T}^1_X(U) \arrow[d] \\ \mathcal{LS}_X(U \cap V) \arrow[r, "\eta"] & \mathcal{T}^1_X(U \cap V). \end{tikzcd}$$

□

## 6. Toroidal Crossing Spaces as Log Toroidal Families

Let $X$ be a toroidal crossing space. Let $\bar{x}$ be geometric point and $V_{\bar{x}}$ the étale neighborhood with a smooth map $V_{\bar{x}} \to \operatorname{Spec} \Bbbk[\mathcal{P}_{\bar{x}}]/z^{\mathbb{1}}$ that exists by the definition of $X$. Set $N = \mathcal{P}^{\text{gp}}_{\bar{x}}$ and $M_\mathbb{R} = \operatorname{Hom}(N, \mathbb{R})$. We obtain a lattice polytope $\sigma_{\bar{x}} = \{m \in M_\mathbb{R} \,|\, m|_{\mathcal{P}_{\bar{x}}} \geq 0, \mathbb{1}(m) = 1\}$ (we use that $X$ is reduced here). For a face $\tau \subset \sigma_{\bar{x}}$, we denote by $V_\tau$ the inverse image of the closed subset $\operatorname{Spec} \Bbbk[\tau^\perp \cap \mathcal{P}_{\bar{x}}]$ of $\operatorname{Spec} \Bbbk[\mathcal{P}_{\bar{x}}]/z^{\mathbb{1}}$ in $V_{\bar{x}}$. Theorem 3.22 in [14] says the following.

**Theorem 6.1** (Gross-Siebert)**.** *$\mathcal{LS}_X|_{V_{\bar{x}}}$ is isomorphic to a subsheaf of $\oplus_\omega \mathcal{O}^\times_{V_\omega}$ where the sum is over the edges of $\sigma_{\bar{x}}$. The sections of the subsheaf on an open $V \subset V_{\bar{x}}$*

*are given as the tuples $(f_\omega)_\omega$ so that for every two-face $\tau$ of $\sigma_{\bar{x}}$ holds*

$$\prod_{\omega\subset\tau} d_\omega \otimes f_\omega^{\epsilon_\tau(\omega)}|_{V_\tau} = 1. \tag{6.1}$$

*as an equality in $M\otimes_{\mathbb{Z}}\Gamma(V,\mathcal{O}_{V_\tau}^\times)$ where $d_\omega$ is a primitive generator of the tangent space to $\omega$ and $\epsilon_\tau(\omega)\in\{-1,+1\}$ is such that $(\epsilon_\tau(\omega)d_\omega)_{\omega\subset\tau}$ gives an oriented boundary of $\tau$.*

**Corollary 6.2.** *Given an étale open $U\to X$, the natural map $\mathcal{LS}_X(U)\to\prod_{\bar{x}}\mathcal{LS}_{X,\bar{x}}$, for the product running over the generic points $\bar{x}$ of the components of $U_{\mathrm{sing}}$, is injective.*

The isomorphism in the theorem naturally depends on the morphism $V_{\bar{x}}\to \operatorname{Spec}\Bbbk[\mathcal{P}_{\bar{x}}]/z^{\mathbb{1}}$ in a way that enables the following result.

**Corollary 6.3.** *For each irreducible component $X_\omega$ of $X_{\mathrm{sing}}$ there is an $\mathcal{O}_{\tilde{X}_\omega}^\times$-torsor $\mathcal{N}_\omega^\times$ on its normalization $\tilde{X}_\omega$ so that*

$$\mathcal{LS}_X\subseteq\bigoplus_{X_\omega} q_{\omega,*}\mathcal{N}_\omega^\times$$

*for $q_\omega:\tilde{X}_\omega\to X_\omega$ the normalization and the subsheaf is locally characterized by Theorem 6.1 when using suitable local trivializations of the torsors.*

Let $\mathcal{N}_\omega$ denote the associated line bundle so that $\mathcal{N}_\omega^\times$ is its $\mathcal{O}_{\tilde{X}_\omega}^\times$-torsor of generating sections. We therefore obtain an injection of $\mathcal{LS}_X$ in the coherent sheaf $\bigoplus_{X_\omega} q_{\omega,*}\mathcal{N}_\omega$.

**Lemma 6.4.** *Under the hypothesis of Theorem 5.4, the injection $\mathcal{LS}_X\hookrightarrow\bigoplus_{X_\omega} q_{\omega,*}\mathcal{N}_\omega$ is $\mathcal{O}_X^\times$-equivariant.*

*Proof.* We borrow the notation $P_\kappa$ from Remark 5.3. From a careful analysis of the proof of [14, Theorem 3.22] one finds that the action $\mathbb{1}\mapsto\lambda^{-1}\mathbb{1}$ becomes $f_\omega\mapsto\lambda^{\kappa_\omega}f_\omega$ where $\kappa_\omega$ is such that $\mathcal{P}_{\bar{x}}\cong P_{\kappa_\omega}$ at the generic point $\bar{x}$ of $X_\omega$. Indeed, if a local model at $\bar{x}$ is given by $xy=f_\omega(z^{\mathbb{1}})^{\kappa_\omega}$, this is equivalent to $xy=\lambda^{\kappa_\omega}f_\omega(\lambda^{-1}z^{\mathbb{1}})^{\kappa_\omega}$ which explains the action. The hypothesis of Theorem 5.4 says that $\kappa_\omega=1$ for all $\omega$, so indeed the action of $\mathcal{O}_X^\times$ on $\mathcal{LS}_X$ is compatible with the ordinary action on the coherent sheaf $\bigoplus_{X_\omega} q_{\omega,*}\mathcal{N}_\omega$. □

**Theorem 6.5.** *If $X$ is a normal crossing space, then the injection in Lemma 6.4 factors as the composition of $\eta:\mathcal{LS}_X\to\mathcal{T}_X^1$ and a uniquely determined injection of coherent sheaves $\mathcal{T}_X^1\hookrightarrow\bigoplus_{X_\omega} q_{\omega,*}\mathcal{N}_\omega$.*

*Proof.* Given Lemma 6.4 and Theorem 5.4 and noting that $V=X$ for a normal crossing space and that the annihilator of $\mathcal{T}_X^1$ is contained in the annihilator of $\bigoplus_{X_\omega} q_{\omega,*}\mathcal{N}_\omega$, the statement becomes an elementary lemma about a cyclic module whose proof we omit. □

**Definition 6.6.** For a point $\bar{x} \in X$, let $X^\circ_{\bar{x}} \subset X$ genote the Zariski locally closed subset where $\mathcal{P}$ is locally constant with stalk $\mathcal{P}_{\bar{x}}$, so that $X$ is the disjoint union of $X^\circ_{\bar{y}}$ for suitable points $\bar{y}$. We call the closure $X_{\bar{x}}$ of $X^\circ_{\bar{x}}$ the *stratum* of $\bar{x}$ which again decomposes into $X^\circ_{\bar{y}}$. We infer the notion of strata to the normalization of $X$.

A section of $s \in \Gamma(U, \mathcal{LS}_X)$ for a Zariski open $U \subset X$ is called *schön* if it extends to a section $(s_\omega)_\omega \in \Gamma(X, \bigoplus_{X_\omega} q_{\omega,*}\mathcal{N}_\omega)$ so that, for each $\omega$, the vanishing locus $\tilde{Z}_\omega$ of $s_\omega$ in $\tilde{X}_\omega$ is reduced, doesn't contain any strata and has regular intersection with each stratum inside $\tilde{X}_\omega$ (in particular $\tilde{Z}_\omega \cap X^\circ_\omega$ is smooth). We also assume that $Z = \bigcup_\omega q_\omega(\tilde{Z}_\omega)$ is the complement of $U$ (otherwise $U$ can be enlarged).

**Definition 6.7.** A schön section is called *simple* if for every closed point $\bar{x} \in X$ with $V_{\bar{x}} \to \operatorname{Spec} \Bbbk[\mathcal{P}_{\bar{x}}]/z^{\mathbb{1}}$ the smooth map from a neighborhood, we have the following situation. Let $Z \cap V_{\bar{x}} = \bigcup_{\omega \in \Omega} Z_\omega$ be the local decomposition of $Z$ into irreducible components where we may assume each $Z_\omega$ contains $\bar{x}$.

(1) There is a disjoint union $\Omega = \Omega_1 \sqcup ... \sqcup \Omega_q$ with $q < \operatorname{rk} \mathcal{P}_{\bar{x}}$ such that $Z_i := Z_\omega \cap X_{\bar{x}} = Z_{\omega'} \cap X_{\bar{x}}$ whenever $\omega, \omega'$ are in the same $\Omega_i$.
(2) $Z_1, ..., Z_q$ form a collection of normal crossing divisors in $X_{\bar{x}}$ at $\bar{x}$.
(3) for each $i$, the primitive vectors $d_\omega$ for $\omega \in \Omega_i$ are the set of edge vectors of an elementary simplex $\Delta_i \subset N_{\mathbb{R}}$. (A lattice simplex is *elementary* if its vertices are the only lattice points contained in it.)

We remark that if $q_\omega : \tilde{X}_\omega \to X_\omega$ is not an embedding, the zero set $\tilde{Z}_\omega$ of $s_\omega$ may locally contribute two or more components of $Z$ at a point $\bar{x}$ which may or may not lie in different $\Omega_i$.

**Theorem 6.8** (Gross-Siebert)**.** *A toroidal crossing space $X$ over an algebraically closed field $\Bbbk$ together with simple section $s \in \Gamma(U, \mathcal{LS}_X)$ gives $X$ the structure of a log toroidal family over $S = \operatorname{Spec}\big(\mathbb{N} \to \Bbbk\big)$ with $U$ the locus of log smoothness.*

*Proof.* Using assumptions in Definition 6.7, the proof is the same as the one of [15, Theorem 2.6]. See also Example 4.6. □

We remark that the $\Delta_i$ give the local structure of the singularities in the nearby fiber, cf. [15, Proposition 2.2]. We also remark that all ETDs have $\mathcal{F} = \mathcal{F}_{\min}$, i.e. there is no horizontal divisor. Proposition 2.8 implies $W^{\dim X}_{X/S} = \omega_{X/S}$.

**Proposition 6.9.** *A normal crossing space $X$ with $X_{\text{sing}}$ projective and $\mathcal{T}^1_X$ generated by global sections permits a dense open $U$ and a simple section $s \in \Gamma(U, \mathcal{LS}_X)$. In view of 6.7, we have $q = 1$ at every point in $Z$ and $\Delta_1$ in each ETD is a standard simplex which means all ETDs have smooth nearby fibers.*

*Proof.* Applying Bertini's theorem to the line bundle $\mathcal{T}^1_X$ on $X_{\text{sing}}$, we obtain a section $\hat{s} \in \Gamma(X_{\text{sing}}, \mathcal{T}^1_X)$ that gives a simple section $s \in \Gamma(X \setminus V(\hat{s}), \mathcal{LS}_X)$ by Theorem 6.5. □

**Lemma 6.10.** *Let $f : X \to S$ be a log toroidal family with empty horizontal divisor. Let $E \subset \underline{X}$ be a Cartier divisor that meets all strata and $Z$ transversely, i.e. locally along $E$ the triple $(X, Z, E)$ is étale equivalent to $(E \times \mathbb{A}^1, (E \cap Z) \times \mathbb{A}^1, E \times \{0\})$. There is a new log toroidal family $X(\log E) \to S$ that has $E$ as its horizontal divisor and factors through $f$ (by forgetting $E$), so in particular $W^{\dim X}_{X(\log E)/S}(-E) = \omega_{X/S}$.*

*Proof.* On $U$ the result is straightforward and along $Z$ we use the product description to make $E$ the horizontal divisor in the ETDs by adding a summand $\mathbb{N}$ to $P$ and the unique new facet gets included in $\mathcal{F}$. That these give local models follows the same proof as [15, Theorem 2.6] noting that we may treat the local equation for $E$ as one of the $f_i$ in the notation of loc.cit.. □

**Lemma 6.11.** *Let $f : X \to S$ be a projective log toroidal family with empty horizontal divisor and assume that $\omega^{-1}_{X/S}$ is generated by global sections, then $\omega^{-1}_{X/S} \cong \mathcal{O}_X(E)$ for a divisor $E$ that satisfies the assumption of Lemma 6.10. In particular, $W^{\dim X}_{X(\log E)/S} \cong \mathcal{O}_X$.*

*Proof.* This follows via an application of Bertini's theorem. □

**Proposition 6.12.** *Theorem 1.1 follows from Theorem 1.7.*

*Proof.* We are given $E$ that is transverse to the strata of $X$. We apply a slight variant of Proposition 6.9 by making sure the zero locus $Z$ of the section $\hat{s}$ generated by Bertini is transverse also to $E$. Theorem 1.7 gives an orbifold smoothing but we know it is an actual smoothing from the fact that each $\Delta_1$ is standard. □

**Theorem 6.13** (Gross-Siebert)**.** *Let $Y := X(\log E) \to S$ be a log toroidal family obtained from a toroidal crossing space $\underline{X}$ via a simple section $s \in \Gamma(U, \mathcal{LS}_{\underline{X}})$ and a divisor $E$ as in Lemma 6.10. The analogue of Theorem 2.11 in [15] holds for $Y \to S$, in particular if $V \subset Y$ is affine open, then any two infinitesimal deformations of $V/S$ are isomorphic.*

*Proof.* The proof works precisely as in loc.cit. We remark that in Lemma 2.14, the exact sequence in (2) becomes $0 \to \Theta_{Y/S} \to \Theta_{\underline{X}/\Bbbk}(\log E) \to \mathcal{B} \to 0$ where $\Theta_{\underline{X}/\Bbbk}(\log E)$ denotes ordinary derivations that preserve the ideal of $E$. In other words, for the ordinary deformations, we consider the ones of the pair $(\underline{X}, E)$ rather than just $\underline{X}$. □

## 7. Differentials for Elementary Log Toroidal Families

We fix a principal ideal domain $R$ as base ring. The constructions in §3 carry through when replacing $\mathbb{Z}$ by $R$. We will use the following elementary lemma.

**Lemma 7.1.** *Let $n, m \geq 0$ and $G_1, ..., G_r \subset R^n$ be submodules each of which is a direct summand, then the natural map $\bigwedge^m_R(\bigcap_i G_i) \to \bigcap_i \bigwedge^m_R G_i$ is an isomorphism.*

First consider the absolute case, i.e. an ETD $(Q \subset P, \mathcal{F})$ with $Q = 0$ and let $f : A_{P,\mathcal{F}} \to \operatorname{Spec} R$ be the associated log morphism. One checks that $U$ from (3.3) is simply the complement of codimension two strata. Recall from Example 2.10 that $W^m := W^m_{A_{P,\mathcal{F}}/\operatorname{Spec} R}$ are just the Danilov differentials with log poles in the divisor given by the facets in $\mathcal{F}$. Danilov already computed these in [9, Proposition 15.5] over a field but because of Lemma 7.1 the same calculation works over $R$ and we obtain the following.

**Proposition 7.2** (absolute case)**.** *We have a grading* $\Gamma(A_P, W^m) = \bigoplus_{p\in P}(W^m)_p$ *with*

$$(W^m)_p = \bigwedge_R^m \left( \bigcap_{\substack{F\in\mathcal{F}_{max}\setminus\mathcal{F}\\ p\in F}} F^{gp} \otimes_{\mathbb{Z}} R \right)$$

*where the intersection is* $P^{gp} \otimes_{\mathbb{Z}} R$ *if the index set is empty.*

Let us next assume we have a general ETD $(Q \subset P, \mathcal{F})$ and let again $f$ denote the associated log toroidal family and $W^m_f := W^m_{A_{P,\mathcal{F}}/\operatorname{Spec} A_Q}$ the differentials. Note that since $\mathcal{F}$ contains all vertical facets, every facet in $\mathcal{F}_{\max} \setminus \mathcal{F}$ contains $Q$. We obtain the following generalization.

**Proposition 7.3** (general case)**.** *We have a grading* $\Gamma(A_P, W^m_f) = \bigoplus_{p\in P}(W^m_f)_p$ *with*

$$(W^m_f)_p = \bigwedge_R^m \left( \left( \bigcap_{\substack{F\in\mathcal{F}_{\max}\setminus\mathcal{F}\\ p\in F}} F^{gp} \otimes_{\mathbb{Z}} R \right) \Big/ (Q^{gp} \otimes_{\mathbb{Z}} R) \right)$$

*where the intersection is* $P^{gp} \otimes_{\mathbb{Z}} R$ *if the index set is empty. Since* $Q^{\mathrm{gp}} \subset P^{\mathrm{gp}}$ *splits, we can equivalently take the quotient before the intersection.*

*Proof.* We can compose $f$ with the projection to $\operatorname{Spec} R$ to relate the current situation to that of Proposition 7.2. The open set $U^{\mathrm{abs}}$ in the absolute case is the complement of $Z^{\mathrm{abs}}$, the union of all codimension two strata. Hence, $U^{\mathrm{abs}}$ is covered by $U_F$ where $F$ runs over the facets of $P$. On the other hand the open set $U$ for $f$ as given in (3.3) has a cover $U_F$ where $F$ runs over the essential faces of rank $d-1$ by Lemma 3.7. Obviously, $U^{\mathrm{abs}} \subseteq U$. Note that since $W^m_f$ is locally free on $U$ and $\mathcal{O}_U$ is $Z^{\mathrm{abs}}$-closed, we find that $W^m_f$ is not only $Z$-closed but also $Z^{\mathrm{abs}}$-closed. Consider the commutative diagram of solid arrows

$$\begin{tikzcd}
0 \arrow[r] & f^*\Omega_{A_Q/\operatorname{Spec} R} \arrow[r, "\iota"] \arrow[d, equal] & W^1_{A_{P,\mathcal{F}}/\operatorname{Spec} R} \arrow[r] \arrow[d] & W^1_f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f^*\Omega_{A_Q/\operatorname{Spec} R} \arrow[r] & W^1_{A_P/\operatorname{Spec} R} \arrow[r] \arrow[ul, dotted] & W^1_{A_P/A_Q} \arrow[r] & 0
\end{tikzcd} \tag{7.1}$$

where the top row is obtained by pushing it forward from $U^{\mathrm{abs}}$. The bottom sequence is obtained from tensoring the sequence $0 \to Q^{\mathrm{gp}} \to P^{\mathrm{gp}} \to P^{\mathrm{gp}}/Q^{\mathrm{gp}} \to 0$ with

$\mathcal{O}_{A_P}$, in particular, it is exact and splits. Hence the dotted diagonal arrow exists and commutes with the other maps. Therefore, $\mathrm{coker}(\iota)$ is a direct summand of $W^1_{A_{P,\mathcal{F}}/\operatorname{Spec} R}$, in particular $Z^{\mathrm{abs}}$-closed. Moreover, $\mathrm{coker}(\iota) \to W^1_f$ is an isomorphism on $U^{\mathrm{abs}}$ and since both sheaves are $Z^{\mathrm{abs}}$-closed, we have $\mathrm{coker}(\iota) = W^1_f$ and thus the top row is exact and splits.

Let $\langle f^*\Omega_{A_Q/\operatorname{Spec} R}\rangle$ denote the homogeneous ideal in the sheaf of exterior algebras $W^\bullet_{A_{P,\mathcal{F}}/\operatorname{Spec} R}$ generated by $f^*\Omega_{A_Q/\operatorname{Spec} R}$. The split exactness above gives the split exactness of the following sequence

$$0 \to \langle f^*\Omega_{A_Q/\operatorname{Spec} R}\rangle_m \to W^m_{A_{P,\mathcal{F}}/\operatorname{Spec} R} \to W^m_f \to 0.$$

Since $A_P$ is affine and $\langle f^*\Omega_{A_Q/\operatorname{Spec} R}\rangle$ coherent, applying $\Gamma(A_P, \cdot)$ to this sequence yields another exact sequence which already gives that $\Gamma(A_P, W^m_f)$ is $P$-graded. We have $\Gamma(A_P, f^*\Omega_{A_Q/\operatorname{Spec} R}) = Q^{\mathrm{gp}} \otimes_{\mathbb{Z}} R[P]$. Set $\mathbf{F}_p := \left(\bigcap_{\substack{F\in\mathcal{F}_{max}\setminus\mathcal{F}\\ p\in F}} F^{gp} \otimes_{\mathbb{Z}} R\right)$ and let $\langle Q^{\mathrm{gp}} \otimes R\rangle \subseteq \bigwedge^\bullet_R \mathbf{F}_p$ be the homogeneous ideal generated by $Q^{\mathrm{gp}} \otimes R$. One computes $\Gamma(A_P, \langle f^*\Omega_{A_Q/\operatorname{Spec} R}\rangle_m)_p = \langle Q^{\mathrm{gp}} \otimes R\rangle_m$. Using Proposition 7.2, in degree $p \in P$, we obtain the exact sequence

$$0 \to \langle Q^{\mathrm{gp}} \otimes R\rangle_m \to \bigwedge^m_R \mathbf{F}_p \to (W^m_f)_p \to 0.$$

Using a splitting of the injection $(Q^{\mathrm{gp}} \otimes R) \subseteq \mathbf{F}_p$ and comparing leads to the assertion. $\square$

**Corollary 7.4.** *For all $m$, $W^m_f$ is flat over $A_Q$.*

*Proof.* Inspecting the result in Proposition 7.3, we find $\Gamma(A_P, W^m_f)$ is a free $R[Q]$-module. $\square$

7.1. **Change of Base.** Let $(Q \subset P, \mathcal{F})$ be an ETD, $\mathcal{T}$ be a Noetherian ring and $T = \operatorname{Spec} \mathcal{T} \to \operatorname{Spec} R[Q]$ be any morphism. Denote by $\sigma$ the composition $Q \to R[Q] \to \mathcal{T}$ which turns $T$ into a coherent log scheme. Define $Y$ by the fiber diagram of log toroidal families

$$\begin{tikzcd} Y \arrow[r, "c"] \arrow[d] & A_{P,\mathcal{F}} \arrow[d, "f"] \\ T \arrow[r] & A_Q \end{tikzcd} \tag{7.2}$$

We want to study when the natural map $c^*W^m_f \to W^m_{Y/T}$ is an isomorphism. This holds if $f$ is log smooth since then $W^m_f = \Omega^m_f$ are the ordinary log differentials which satisfy this isomorphism property by their universal property. In particular, $c^*W^m_f \to W^m_{Y/T}$ is always an isomorphism on the open set $V := c^{-1}(U)$. The following example shows that it is not an isomorphism in general. For a subset $I \subset P$, let $\langle I\rangle$ be the smallest face of $P$ containing $I$.

**Example 7.5.** Let $P$ be the submonoid of $\mathbb{Z}^2$ generated by $(1,0),(1,1),(1,2)$ and let $Q = 0$. The monoid $P$ has two facets $H_1 = \langle(1,0)\rangle$ and $H_2 = \langle(1,2)\rangle$ and setting $\mathcal{F} = \emptyset$ yields an ETD. Let $f : A_{P,\mathcal{F}} \to A_Q = \operatorname{Spec}\mathbb{Z}$ be the corresponding map. Now set $\mathcal{T} = \mathbb{Z}/2\mathbb{Z}$ inducing the natural map $T = \operatorname{Spec}\mathcal{T} \to \operatorname{Spec}\mathbb{Z}$ and a fiber diagram as above. One checks that $c^*W^1_f \to W^1_{Y/T}$ is not an isomorphism by computing both terms via Proposition 7.2. It suffices to check the degree $p = 0$, indeed, $(W^1_f)_0 = H_1^{\mathrm{gp}} \cap H_2^{\mathrm{gp}} = 0$ but

$$(W^1_{Y/T})_0 = (H_1^{\mathrm{gp}} \otimes \mathbb{Z}/2\mathbb{Z}) \cap (H_2^{\mathrm{gp}} \otimes \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \cdot (1,0) \subset (\mathbb{Z}/2\mathbb{Z})^2.$$

Hence, $((W^1_f) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z})_0 = 0$ but $(W^1_{Y/T})_0 \neq 0$.

The example teaches that base change is related to the (non-)commuting of intersection and tensor product. The following lemma (that is an elementary exercise in Tor groups) will help us. We say a ring $\mathcal{T}$ is of *characteristic* $\geq p_0$ if for the residue field $\kappa_{\mathfrak{p}}$ of every point $\mathfrak{p}$ holds $\operatorname{char}\kappa_{\mathfrak{p}} \geq p_0$ or $\operatorname{char}\kappa_{\mathfrak{p}} = 0$.

**Lemma 7.6.** *Let $G$ be a finitely generated $\mathbb{Z}$-module and $H, H' \subset G$ be two submodules. Then there is $p_0$ such that for every ring $\mathcal{T}$ of characteristic $\geq p_0$ we have*

$$(H \cap H') \otimes \mathcal{T} = (H \otimes \mathcal{T}) \cap (H' \otimes \mathcal{T})$$

*and each term here is a submodule of $G \otimes \mathcal{T}$.*

In the general situation, observe we have $\Gamma(Y, \mathcal{O}_Y) = \bigoplus_{e\in E} z^e \cdot \mathcal{T}$ with multiplication

$$z^{e_1} \cdot z^{e_2} = z^e \cdot \sigma(q) \quad \text{whenever} \quad e_1 + e_2 = e + q$$

with $e \in E, q \in Q$ under the canonical decomposition from (3.1). Similarly, Proposition 7.3 gives

$$\Gamma(Y, c^*W^m_f) = \bigoplus_{e\in E} z^e \cdot ((W^m_f)_e \otimes_R \mathcal{T}). \tag{7.3}$$

**Lemma 7.7.** *Recall $V = c^{-1}(U)$. Equivalent are*

*(1) the map $c^*W^m_f \to W^m_{Y/T}$ is an isomorphism,*
*(2) $c^*W^m_f$ is reflexive,*
*(3) the restriction map $\rho : \Gamma(Y, c^*W^m_f) \to \Gamma(V, c^*W^m_f)$ is surjective.*

*Proof. (1)⇒(2)*: $W^m_{Y/T}$ is reflexive; *(2)⇒(3)*: $c^*W^m_f$ is $(Y \setminus V)$-closed; *(3)⇒(1)*: Consider the commutative square

$$\begin{tikzcd} \Gamma(Y, c^*W^m_f) \arrow[r] \arrow[d, "\rho"] & \Gamma(Y, W^m_{Y/T}) \arrow[d] \\ \Gamma(V, c^*W^m_f) \arrow[r] & \Gamma(V, W^m_{Y/T}) \end{tikzcd}$$

where the right vertical map is an isomorphism because $W^m_{Y/T}$ is reflexive by Lemma 2.4. The bottom horizontal map is an isomorphism by what we said just before Example 7.5. Now *(1)* holds if the top horizontal map is an isomorphism which follows from *(3)* if $\rho$ is additionally injective. This injectivity is a general fact that we prove next. Recall that $A_{P,\mathcal{F}_{\max}} = A_P$ and we have a map $A_P \to A_{P,\mathcal{F}}$ that gives us another commutative square

$$\begin{tikzcd} \Gamma(Y, c^*W^m_f) \arrow[r] \arrow[d, "\rho"] & \Gamma(Y, c^*W^m_{A_P/A_Q}) \arrow[d] \\ \Gamma(V, c^*W^m_f) \arrow[r] & \Gamma(V, c^*W^m_{A_P/A_Q}). \end{tikzcd} \tag{7.4}$$

Since $A_P \to A_Q$ is log smooth and $W^m_{A_P/A_Q} = \Omega^m_{A_P/A_Q}$ a free sheaf, the right vertical map is an isomorphism. We get that $\rho$ is injective if the top horizontal map is injective. The latter can be computed from Proposition 7.3. Indeed, this follows from (7.3) since for every $e \in E$, the cokernel of $(W^m_f)_e \to (W^m_{A_P/A_Q})_e$ is a free $R$-module. $\square$

We next provide a useful criterion for the surjectivity of $\rho$. Let $\mathcal{E}$ be the set of essential faces of $P$ of rank $d-1$. By Lemma 3.7, $U$ is covered by $\{U_F | F \in \mathcal{E}\}$. Set $V_F = c^{-1}(U_F)$ so these cover $V$. For each $F \in \mathcal{E}$, choose $e_F \in F$ in the relative interior, i.e. $\langle e_F \rangle = F$.

**Theorem 7.8.** *Write $M_p := (W^m_f)_p$ for short, and assume that for every subset $\mathcal{E}' \subset \mathcal{E}$ and every $e \in E$ the natural map*

$$\left( \bigcap_{F\in\mathcal{E}'} M_{e+e_F} \right) \otimes_R \mathcal{T} \to \bigcap_{F\in\mathcal{E}'} (M_{e+e_F} \otimes_R \mathcal{T})$$

*is an isomorphism. Then $\rho$ is surjective.*

*Proof.* We write $M = \Gamma(A_P, W^m_f)$, $N = \Gamma(A_P, W^m_{A_P/A_Q})$ and $N_p$ for the degree $p$ part of $N$. By proposition 7.3, $M_p$ and $N_p$ only depend on $\langle p \rangle$. We are going to use that for $p_1, p_2 \in P$ holds

$$\langle p_1 + p_2 \rangle = \langle \langle p_1 \rangle \cup \langle p_2 \rangle \rangle. \tag{7.5}$$

We have a natural injection $M \subseteq N$ by Proposition 7.3. Given $\mu \in \Gamma(V, c^*W^m_f)$, we want to show it has a preimage under $\rho$. We do have a unique preimage $\nu$ under the right vertical map of (7.4), so in $N \otimes_{R[Q]} \mathcal{T}$ and we are going to show that this preimage lies in $M \otimes_{R[Q]} \mathcal{T}$. Say $\nu = \sum_e z^e \cdot n_e$ with $n_e \in N_e \otimes \mathcal{T}$ is such that $\nu|_V = \mu$. In particular $\nu|_{V_F} = \mu|_{V_F}$ for all $F \in \mathcal{E}$. There is some large $a \geq 1$ so that for each $F \in \mathcal{E}$ there are $m_{F,e} \in M_e \otimes \mathcal{T}$ such that

$$\mu|_{V_F} = z^{-ae_F} \sum_e z^e \cdot m_{F,e}$$

and therefore $\nu|_{V_F} = \mu|_{V_F}$ implies

$$z^{ae_F} \sum_e z^e \cdot n_e \ \in \ \bigoplus_{e \in E} z^e \cdot (M_e \otimes_R \mathcal{T}) \ \subset \ \bigoplus_{e \in E} z^e \cdot (N_e \otimes_R \mathcal{T}).$$

If $e + ae_F = \tilde{e} + q$ is the decomposition $P = E \times Q$, then $n_e \cdot \sigma(q) \in M_{\tilde{e}} \otimes_R \mathcal{T}$. By (7.5),

$$e + ae_F \in E \iff \langle e + e_F \rangle \subset E \iff e + e_F \in E$$

and if this holds, then $\sigma(q) = 1$, so setting

$$\mathcal{E}_e := \{F \in \mathcal{E} \mid e + e_F \in E\}$$

we obtain $n_e \in \bigcap_{F \in \mathcal{E}_e} (M_{e+ae_F} \otimes_R \mathcal{T})$ and $M_{e+ae_F} = M_{e+e_F}$. Note that $\mathcal{E}_e$ does not depend on the chosen $e_F$. Using the assumption, we get

$$n_e \in \bigcap_{F \in \mathcal{E}_e} (M_{e+e_F} \otimes_R \mathcal{T}) = \left( \bigcap_{F \in \mathcal{E}_e} M_{e+e_F} \right) \otimes_R \mathcal{T}.$$

For the next step, define $\mathcal{F}_e = \{H \in \mathcal{F}_{max} \setminus \mathcal{F} \mid \exists F \in \mathcal{E}_e : e + e_F \in H\}$. We use Lemma 7.1 to compute

$$\bigcap_{F \in \mathcal{E}_e} M_{e+e_F} = \bigwedge_R^m \left( \bigcap_{H \in \mathcal{F}_e} \frac{H^{gp} \otimes_{\mathbb{Z}} R}{Q^{gp} \otimes_{\mathbb{Z}} R} \right).$$

We finally claim that $\mathcal{F}_e = \{H \in \mathcal{F}_{max} \setminus \mathcal{F} \mid e \in H\}$, indeed given an $H$ in the latter, we just need to exhibit an $F \in \mathcal{E}$ that is also contained in $H$ with $\langle e, F \rangle \subset E$ which can be done since $H \cap E$ is a union of faces in $\mathcal{E}$. Thus, $n_e \in M_e \otimes_R \mathcal{T}$, so indeed $\nu \in M \otimes_{R[Q]} \mathcal{T}$ and we are done. □

**Corollary 7.9.** *Let $(Q \subset P, \mathcal{F})$ be an ETD, $\mathcal{T}$ a Noetherian ring and $T = \operatorname{Spec} \mathcal{T} \to A_Q$ a strict morphism of log schemes. Then $c^* W_f^m$ is reflexive and $c^* W_f^m \to W_{Y/T}^m$ is an isomorphism provided that the composition*

$$R \to R[Q] \to \mathcal{T}$$

*is flat, e.g. when $R$ is a field.*

As Example 7.5 shows, the conditions of Lemma 7.8 are not always satisfied in case $R = \mathbb{Z}$. However, we do get close:

**Corollary 7.10.** *Let $(Q \subset P, \mathcal{F})$ be an ETD, and assume $f : A_{P,\mathcal{F}} \to A_Q$ to be defined over $R = \mathbb{Z}$. Then there is a $p_0 = p_0(Q \subset P, \mathcal{F})$ such that for $T = \operatorname{Spec} \mathcal{T} \to A_Q$ with a Noetherian ring $\mathcal{T}$ of characteristic $\geq p_0$, the sheaf $c^* W_f^m$ is reflexive, and $c^* W_f^m \to W_{Y/T}^m$ is an isomorphism.*

*Proof.* Apply Lemma 7.6 recursively and use that the modules $M_p \subset \bigwedge_{\mathbb{Z}}^m (P^{gp}/Q^{gp})$ are free direct summands and that the set of situations to consider for the assumption of Theorem 7.8 is finite. □

For a field $\Bbbk$, consider a monoid ideal $K \subset Q$, let $(K) \subset \Bbbk[Q]$ denote the corresponding monomial ideal of $\Bbbk[Q]$ and set $\mathcal{T} = \Bbbk[Q]/(K)$. The map $T = \operatorname{Spec} \mathcal{T} \to \Bbbk[Q]$ is the natural one and $Y \to T$ is defined by (7.2) as before. We set $E_K := P \setminus (P + K)$ and note this generalizes the union of essential faces $E$ from §3, indeed $E = E_{Q\setminus\{0\}}$. Combining Proposition 7.3 with Corollary 7.9 (for $R = \Bbbk$) gives the following.

**Corollary 7.11.** $\Gamma(Y, W^m_{Y/T}) \cong \bigoplus_{e\in E_K} z^e {\cdot} \bigwedge^m \left(\bigcap_{H\in\mathcal{F}_{\max}\setminus\mathcal{F}:e\in H}(H^{gp}\otimes\Bbbk)/(Q^{gp}\otimes\Bbbk)\right)$ *with differential* $d(z^e \cdot n) = z^e \cdot [e] \wedge n$.

With $c : Y \to A_{P,\mathcal{F}}$ the notation from before, we apply $c^*$ to the split exact sequence given by the top row of (7.1) and obtain another split exact sequence. The left term is free and $c^*W^m_f$ is reflexive by Corollary 7.9. Hence, $c^*W^m_{A_{P,\mathcal{F}}/\Bbbk}$ is also reflexive. With $V = c^{-1}(U)$, we find the natural surjection $c^*\Omega^\bullet_{U/\Bbbk} \twoheadrightarrow \Omega^\bullet_{V/\Bbbk}$ to be an isomorphism (e.g. by local freeness of both). For $j : V \hookrightarrow Y$ the inclusion and $W^\bullet_Y := j_*\Omega^\bullet_{V/\Bbbk}$ we thus have $c^*W^m_{A_{P,\mathcal{F}}/\Bbbk} \cong W^m_Y$. Plugging this into Proposition 7.2 yields the following.

**Corollary 7.12.** $\Gamma(Y, W^m_Y) \cong \bigoplus_{e\in E_K} z^e \cdot \bigwedge^m \left(\bigcap_{H\in\mathcal{F}_{\max}\setminus\mathcal{F}:e\in H} H^{gp}\otimes\Bbbk\right)$ *with differential* $d(z^e \cdot n) = z^e \cdot e \wedge n$.

7.2. **Local Analytic Theory.** We keep the setup and notation from before (with $\Bbbk = \mathbb{C}$), so $(Q \subset P, \mathcal{F})$ is an ETD and $K \subset Q$ a monoid ideal. We additionally assume that $Q \setminus K$ is finite, so $\mathcal{T} = \mathbb{C}[Q]/(K)$ is an Artinian local ring. For $P^+ = P \setminus \{0\}$, let $\mathbb{C}[\![P]\!]$ be the completion of $\mathbb{C}[P]$ in $(P^+)$.

**Lemma 7.13.** *([34, Prop. V.1.1.3.]) For every local homomorphism $h : P \to \mathbb{N}$, i.e. $h^{-1}(0) = \{0\}$ and we may view $h$ as a grading, it holds*

$$\mathcal{O}_{A^{an}_P,0} = \left\{ \sum_{p\in P} \alpha_p z^p \;\middle|\; \alpha_p \in \mathbb{C},\ \sup_{p\in P^+}\left\{\frac{\log\,|\alpha_p|}{h(p)}\right\} < \infty \right\} \subset \mathbb{C}[\![P]\!].$$

We have $\Gamma(Y, \mathcal{O}_Y) \cong \mathbb{C}[E_K] := \bigoplus_{e\in E_K} \mathbb{C}\cdot z^e$ with $z^e \cdot z^{e'} = z^{e+e'}$ if $e + e' \in E_K$ and $z^e \cdot z^{e'} = 0$ otherwise. By [20, Cor. 3.2.] and Lemma 7.13, the complete local ring at the origin in $Y^{an}$ is

$$\hat{\mathcal{O}}_{Y,0} \cong (\mathbb{C}[Q]/(K)) \otimes_{\mathbb{C}[\![Q]\!]} \mathbb{C}[\![P]\!] \cong \left\{ \sum_{e\in E_K} \alpha_e z^e \right\} =: \mathbb{C}[\![E_K]\!]$$

Lemma 7.13 together with Krull's intersection theorem and the surjectivity of $\mathcal{O}_{A^{an}_P,0} \to \mathcal{O}_{Y^{an},0}$ yields

$$\mathcal{O}_{Y^{an},0} = \left\{ \sum_{e\in E_K} \alpha_e z^e \in \mathbb{C}[\![E_K]\!] \;\middle|\; \sup_{e\in E_K\setminus 0}\left\{\frac{\log|\alpha_e|}{h(e)}\right\} < \infty \right\}. \tag{7.6}$$

**Lemma 7.14.** *Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional $\mathbb{C}$-vector space with a Hermitian inner product. For every $e \in E_K$, let $V_e \subset V$ be subvector spaces so that*

$$\tilde{V} := \bigoplus_{e\in E_K} z^e \cdot V_e \subset V[E_K]$$

*is a $\mathbb{C}[E_K]$-module. Assume moreover that $V_e \subset V$ depends only on the set $F(e) := \{H \subset P \text{ a facet} \mid Q \subset H, e \in H\}$. Set $V[\![E_K]\!] := \prod_{e\in E_K} z^e \cdot V_e$ and $\mathcal{V}^{an} := \tilde{V} \otimes_{\mathbb{C}[E_K]} \mathcal{O}_{Y^{an}}$. We find its stalk at the origin to be*

$$\mathcal{V}_0^{an} \cong \left\{ \sum_{e\in E_K} z^e \cdot v_e \in V[\![E_K]\!] \;\middle|\; \sup_{e\in E_K\setminus 0} \left\{ \frac{\log \|v_e\|}{h(e)} \right\} < \infty \right\}$$

*Proof.* The set of possible $F(e)$ is finite, so there is only a finite set of different $V_e$. Choosing orthonormal bases for all $V_e$ allows to reduce the assertion to (7.6). We leave the technical details to the reader. □

*Remark* 7.15. We can use Lemma 7.14 to compute the stalk at 0 of the analytification of $W^m_{Y/T}$ and $W^m_Y$ by using Corollary 7.11 and Corollary 7.12 respectively.

## 8. Base Change of Differentials for Log Toroidal Families

**Definition 8.1** (BC)**.** We say that a generically log smooth morphism $f : X \to S$ satisfies the *basechange property* if for every strict morphism $T \to S$ of Noetherian fs log schemes, $m \in \mathbb{Z}$ and $c$ the map given by the Cartesian diagram

$$\text{(BC)} \qquad \begin{array}{ccc} Y & \xrightarrow{c} & X \\ {\scriptstyle g}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ T & \xrightarrow{b} & S, \end{array}$$

the sheaf $c^*W^m_{X/S}$ is reflexive or equivalently, the natural map $c^*W^m_{X/S} \to W^m_{Y/T}$ is an isomorphism.

**Theorem 8.2** (Base Change over Fields)**.** *Let $f : X \to S$ be a log toroidal family over a field $\Bbbk$, then $f$ satisfies* (BC)*.*

*Proof.* This follows directly from the local statement Corollary 7.9. □

**Theorem 8.3** (Generic Base Change)**.** *Let $f : X \to S$ be a log toroidal family. Then there is a finite set of prime numbers $p_1, ..., p_N \in \mathbb{Z}$ so that if $f^\circ : X^\circ \to S^\circ$ is obtained from $f$ by inverting $p_1, ..., p_N$ (i.e. basechange to* $\operatorname{Spec} \mathbb{Z}_{p_1\cdots p_N}$*), then $f^\circ$ satisfies* (BC)*.*

*Proof.* Again, this follows directly from the local statement Corollary 7.10 combined with the fact that we can use a finite cover by local models. □

An application of the above theorems is the following lemma which is crucial for the degeneration of the Hodge-to-de-Rham spectral sequence.

**Lemma 8.4.** *(cf. [24, Prop. 6.6]) Let $f : X \to S$ be a proper log toroidal family with $S$ affine, and let $b : T \to S$ with $T$ affine. Assume $c^*W^m_{X/S} = W^m_{Y/T}$ holds for all $m$. Then we have isomorphisms*

$$Lb^*Rf_*W^p_{X/S} \to Rg_*W^p_{Y/T} \tag{8.1}$$

$$Lb^*Rf_*W^\bullet_{X/S} \to Rg_*W^\bullet_{Y/T} \tag{8.2}$$

*in $D^b(T)$. If, for fixed $p$, all $R^q f_*W^p_{X/S}$ are locally free of constant rank, then* (8.1) *induces an isomorphism*

$$b^*R^qf_*W^p_{X/S} \xrightarrow{\cong} R^qg_*W^p_{Y/T}$$

*If, for all $n$, the sheaf $R^nf_*W^\bullet_{X/S}$ is locally free of constant rank, then* (8.2) *induces an isomorphism*

$$b^*R^nf_*W^\bullet_{X/S} \xrightarrow{\cong} R^ng_*W^\bullet_{X/S}$$

*Proof.* Knowing the flatness of $W^m_{X/S}$ over $S$ which is Corollary 7.4, the proof becomes identical to that in [24, Prop. 6.6]. □

## 9. Spreading Out Log Toroidal Families

We fix a sharp toric monoid $Q$, a field $\Bbbk \supset \mathbb{Q}$ and set $S = \mathrm{Spec}(Q \to \Bbbk)$ where the map $Q \to \Bbbk$ is $q \mapsto 0$ except $0 \mapsto 1$. We choose distinct subrings $B_\lambda \subseteq \Bbbk$ for all $\lambda$ in some index set $\Lambda$ so that any two $B_{\lambda_1}, B_{\lambda_2}$ are both contained in a third $B_\lambda$. We say $\lambda_1 \leq \lambda_2$ if $B_{\lambda_1} \subseteq B_{\lambda_2}$. Furthermore, we require $\varinjlim_\lambda B_\lambda = \Bbbk$ and that each $B_\lambda$ is of finite type over $\mathbb{Z}$. We get log schemes $S_\lambda = \mathrm{Spec}(Q \to B_\lambda)$ each with a strict map $S \to S_\lambda$ and in fact $S = \varprojlim_\lambda S_\lambda$.

**Proposition 9.1.** *Let $f : X \to S$ be a log toroidal family of relative dimension $d = \mathrm{rk}\,\Omega^1_{U/S}$. Then there is $\lambda \in \Lambda$ and a log toroidal family $f_\lambda : X_\lambda \to S_\lambda$, so that $f$ is obtained by base change from $f_\lambda$, i.e. there is a Cartesian square*

$$\begin{array}{ccc} X & \longrightarrow & X_\lambda \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_\lambda} \\ S & \longrightarrow & S_\lambda. \end{array}$$

*If $f$ is separated and/or proper, we can assume $f_\lambda$ to be so, too.*

*Proof.* By [19, Thm. 8.8.2 (ii)], [19, Thm. 8.10.5] and [19, Thm. 11.2.6 (ii)] we can find a $\lambda \in \Lambda$ and a morphism $f_\lambda : X_\lambda \to S_\lambda$ that is finitely presented and flat, and an isomorphism $S \times_{S_\lambda} X_\lambda \cong X$ over $S$. If $f : X \to S$ is separated respective proper, we can choose $f_\lambda$ moreover separated respective proper. Using [19, Corollaire 12.1.7(iii)] and [19, Thm. 8.10.5], we can choose $\lambda$ such that $f_\lambda$ is a Cohen-Macaulay morphism. Since these decompose disjointly over the relative codimension, again by increasing $\lambda$ if needed, we may assume that $f_\lambda$ has relative dimension $d$.

We next spread out $U$ such that $U_\lambda \subset X_\lambda$ satisfies (CC). We do this by spreading out its complement $Z$. Indeed, by [2, 05M5, Lemma 31.16.1], we can increase $\lambda$ so that every fiber of $Z_\lambda \to S_\lambda$ has dimension $\leq d-2$ and then define $U_\lambda := X_\lambda \setminus Z_\lambda$.

Now a straightforward generalization of the method employed in [40, 4.11.1] yields that for appropriate $\lambda$, we can find a log structure on $U_\lambda$ and upgrade $f_\lambda$ to a log morphism such that $U_\lambda$ is fs and $f_\lambda$ is log smooth and saturated. While Tsuji uses absolute charts to construct the log structure, we choose relative charts $A_{P_i} \to A_Q$ with saturated injections $Q \subset P_i$.

Finally - again by possibly increasing $\lambda$ - we show the family $f_\lambda : X_\lambda \to S_\lambda$ log toroidal. We fix a finite covering $\{V_i \to X\}$ with local models $(Q \subset P_i, \mathcal{F}_i)$ as in Definition 4.1, and for each of them, we construct a diagram

$$\begin{tikzcd}
 & V_i \arrow[rr] \arrow[dl, "g"'] \arrow[dd, dotted, "h"] & & V_{i,\lambda} \arrow[dl, "g_\lambda"'] \arrow[dd, dotted, "h_\lambda"] & & \\
X \arrow[rr, "p_\lambda"'] \arrow[dd, "f"] & & X_\lambda \arrow[dd, "f_\lambda"] & & & \\
 & L_i \arrow[rr] \arrow[dl, "r"'] & & L_{i,\lambda} \arrow[rr] \arrow[dl, "r_\lambda"'] & & A_{P_i,\mathcal{F}_i} \arrow[dl] \\
S \arrow[rr, "q_\lambda"'] & & S_\lambda \arrow[rr, "a_\lambda"'] & & A_Q &
\end{tikzcd}$$

Namely we first spread out $V_i \to S$ to $V_{i,\lambda} \to S_\lambda$. Then $L_{i,\lambda}$ is defined by base change, and we construct the étale morphisms of schemes $g_\lambda : V_{i,\lambda} \to X_\lambda$ and $h_\lambda : V_{i,\lambda} \to L_{i,\lambda}$ also by spreading out. We can assume that $X_\lambda$ is covered by $\{V_{i,\lambda} \to X_\lambda\}$ and that $\tilde{U}_i \subset V_i$ spreads out to an open $\tilde{U}_{i,\lambda} \subset V_{i,\lambda}$ satisfying (CC). We get two log structures $(g_\lambda)^*_{log}\mathcal{M}_{X_\lambda}$ and $(h_\lambda)^*_{log}\mathcal{M}_{L_{i,\lambda}}$ on $\tilde{U}_{i,\lambda}$ which we identify by [40, 4.11.3]. By the same Lemma, the two morphisms $(g \circ f)^*_{log}\mathcal{M}_{S_\lambda} \to \mathcal{M}_{\tilde{U}_{i,\lambda}}$ coming from $f_\lambda \circ g_\lambda$ respective $r_\lambda \circ h_\lambda$ coincide. Since $\{V_i \to X\}$ is a finite covering, we can find $\lambda$ that admits the above construction for all $V_i$ simultaneously. $\square$

## 10. The Cartier Isomorphism

In this section, we define the Cartier homomorphism for a generically log smooth family $f : X \to S$ in characteristic $p > 0$. We then prove that it is an isomorphism if $f$ is log toroidal. Similar to [4], we first study the situation on $U$ and then examine its extension to all of $X$. Let $F_S : S \to S$ be the absolute log Frobenius on the base, i.e. given by taking $p$th power in $\mathcal{M}_S$ and $\mathcal{O}_S$ respectively, we similarly define $F_X : X \to X$. We define $f' : X' \to S$ and the relative Frobenius $F$ by the Cartesian square

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[rr, bend left, "F_X"] \arrow[dr, "f"'] & X' \arrow[r, "s"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
 & S \arrow[r, "F_S"] & S.
\end{tikzcd}$$

Set $U' := s^{-1}(U)$ and $Z' = X' \setminus U'$.

**Theorem 10.1** ([28]). *We have a canonical (Cartier) isomorphism of $\mathcal{O}_{U'}$-modules*

$$C_U^{-1} : \Omega^m_{U'/S} \to \mathcal{H}^m(F_* \Omega^\bullet_{U/S})$$

*which is compatible with $\wedge$ and satisfies $C^{-1}(a) = F^*(a)$ for $a \in \mathcal{O}_{X'}$ and $C^{-1}(\mathrm{dlog}(s^* q)) = \mathrm{dlog}(q)$ for $q \in \mathcal{M}_U$.*

*Proof.* This is [28, Theorem 4.12(1)] once we identify $U'' = U'$: Kato considers the factorization $U \xrightarrow{g} U'' \xrightarrow{h} (U')^{\mathrm{int}} \xrightarrow{i} U'$ of $F|_U$ where $i$ is the integralization of $U'$ and $g \circ h$ is the unique factorization of this weakly purely inseparable morphism where $h$ is étale and $g$ purely inseparable, using [28, Proposition 4.10(2)]. Now $i$ is an isomorphism because $f$ is integral. By [34, Cor. III.2.5.4], since $f : U \to S$ is saturated, $F : U \to U'$ is exact. The uniqueness of the factorization $g \circ h$ now implies that $h$ is an isomorphism. $\square$

Since $W^m_{X'/S}$ is $Z'$-closed, pushing forward the inverse of $C_U^{-1}$ to $X'$, we obtain a homomorphism

$$C : \mathcal{H}^m(F_* W^\bullet_{X/S}) \to W^m_{X'/S}$$

which is an isomorphism on $U'$. We obtain the following lemma.

**Lemma 10.2.** *The map $C$ is an isomorphism if and only if $\mathcal{H}^m(F_* W^\bullet_{X/S})$ is $Z'$-closed.*

**Definition 10.3.** We say that a generically log smooth family $f : X \to S$ in positive characteristic has the *Cartier isomorphism property* if $C$ is an isomorphism for all $m \geq 0$.

By Theorem 10.1, $\mathcal{H}^m(F_* W^\bullet_{X/S})$ is locally free on $U'$, hence it is $Z'$-closed if and only if it is reflexive. Reflexivity can be checked étale locally.

**Lemma 10.4.** *Let $(Q \subset P, \mathcal{F})$ be an ETD, let $b : T \to A_Q$ be strict with $\underline{T} = \operatorname{Spec} \mathcal{T}$ and consider the Cartesian diagram*

$$\begin{tikzcd} Y \arrow[r, "c"] \arrow[d, "g"'] & A_{P,\mathcal{F}} \arrow[d, "f"] \\ T \arrow[r, "b"] & A_Q \end{tikzcd}$$

*Then $\mathcal{H}^m(F_* W^\bullet_{Y/T})$ is reflexive.*

**Corollary 10.5.** *Every log toroidal family $f : X \to S$ over $\mathbb{F}_p$ has the Cartier isomorphism property.*

*Proof of Lemma 10.4.* Set $V := c^{-1}(U_P)$ and let $Y', V'$ be the base changes by the absolute Frobenius $F_T$. Let $F : Y \to Y'$ be the relative Frobenius. Inspired by the Frobenius decomposition [11, Theorem 2.1.], we construct a homomorphism

$\phi^\bullet : \bigoplus_m W^m_{Y'/T}[-m] \to F_* W^\bullet_{Y/T}$ of *complexes of $\mathcal{O}_{Y'}$-modules* which induces an isomorphism in cohomology. Since the left hand side has zero differentials, the assertion then follows from the reflexivity of $W^m_{Y'/T}$ given by Lemma 2.4.

Similar to §7.1, we find explicitly that $R' := \Gamma(Y', \mathcal{O}_{Y'}) = \bigoplus_{e\in E} z^e \cdot \mathcal{T}$ with

$$z^{e_1} \cdot z^{e_2} = z^e \cdot \sigma(q)^p \quad \text{whenever} \quad e_1 + e_2 = e + q$$

with $e \in E, q \in Q$. We have $s^*(z^e \cdot t) = z^e \cdot t^p$ and $F^*(z^e \cdot t) = z^{p\cdot e} \cdot t$. Writing $W^m_e := (W^m_f)_e \otimes_{\mathbb{F}_p} \mathcal{T}$, the module $\Gamma(Y', W^m_{Y'/T})$ is given by the $\mathcal{T}$-module $\bigoplus_{e\in E} z^e \cdot W^m_e$ on which $R'$ acts as

$$(z^{e_1} \cdot t_1) \cdot [z^{e_2} \cdot (w \otimes t_2)] = z^e \cdot (w \otimes \sigma(q)^p t_1 t_2) \quad \text{whenever} \quad e_1 + e_2 = e + q$$

with $e \in E, q \in Q$. Similarly, $\Gamma(Y', F_* W^m_{Y/T})$ is given by the same $\mathcal{T}$-module, however now $R'$ acting via $F^*$ as

$$(z^{e_1} \cdot t_1) \cdot [z^{e_2} \cdot (w \otimes t_2)] = z^e \cdot (w \otimes \sigma(q) t_1 t_2) \quad \text{whenever} \quad p \cdot e_1 + e_2 = e + q.$$

Note the subtle difference. The differential on $F_* W^\bullet_{Y/T}$ is given by $d(z^e \cdot (w \otimes t)) = z^e \cdot ([e] \wedge w \otimes t)$. We define

$$\phi^\bullet : \bigoplus_m W^m_{Y'/T}[-m] \to F_* W^\bullet_{Y/T}, \quad z^e \cdot (w \otimes t) \mapsto z^{p\cdot e} \cdot (w \otimes t)$$

and claim $\mathcal{H}^m(\phi^\bullet)$ is an isomorphism. Indeed, first note that $\phi^\bullet$ itself is injective. Then set $E_p = \{p \cdot e | e \in E\}$. We have $\mathrm{im}(\phi^m) = \bigoplus_{e\in E_p} z^e \cdot W^m_e$ because $W^m_e = W^m_{e/p}$ for $e \in E_p$ by Prop. 7.3. Denoting the coboundaries of $F_* W^m_{Y/T}$ by $B^m$, we have $\mathrm{im}(\phi^m) \cap B^m = 0$ since $0 = [e] \in W^1_e$ for $e \in E_p$ because $e = pe'$ and $p$ is zero in $\mathcal{T}$. This readily gives that $\mathcal{H}^m(\phi^\bullet)$ is injective. For surjectivity, if $e \notin E_p$, observe that $[e] \neq 0$, so if $w \in W^m_e$, then $[e] \wedge w = 0$ if and only if there is some $w' \in W^{m-1}_e$ with $[e] \wedge w' = w$. □

*Remark* 10.6. We believe that $\mathcal{H}^m(\phi^\bullet)$ is the log Cartier isomorphism on $V'$.

## 11. The Decomposition of $F_* W^\bullet_{X_0/S_0}$

We prove a log version of the decomposition theorem [11, Thm. 2.1] in the setting of generically log smooth families. (We noticed that [11, Cor. 3.7] alias [24] doesn't generalize well to the generically log smooth setting.) The assumption for $f : X \to S$ to be saturated on the log smooth locus allows a simpler approach than [28, Thm. 4.12]. Our setting is as follows: let $k$ be a perfect field with char $k = p$ (thus $\mathbb{Z}/p^2\mathbb{Z} \to W_2(k)$ is flat), and let $Q$ be a sharp toric monoid. Set $S_0 = \mathrm{Spec}(Q \to k)$ and $S = \mathrm{Spec}(Q \to W_2(k))$ where in both cases $Q \ni q \mapsto 0$ except $0 \mapsto 1$. The Frobenius endomorphism on $k$ becomes an endomorphism $F_0$ of $S_0$ via $Q \ni q \mapsto pq$. Similarly, its lift to $W_2(k)$ defined via $(a_1, a_2) \mapsto (a_1^p, a_2^p)$ becomes[1] an endomorphism $F_S$ of $S$ that restricts to $F_0$ on $S_0$. Let $f : X \to S$ be a generically log smooth family

[1]Warning: This is *not* the $p$th power map on $W_2(k)$ and thus depends on the chosen chart.

and let $f_0 : X_0 \to S_0$ be its restriction to $S_0$. We consider the commutative diagram of generically log smooth families

$$\begin{tikzcd}
 & X_0 \arrow[rr, "F"] \arrow[dl, "i"'] \arrow[ddrr, bend left, "f_0"'] & & X_0' \arrow[rr, "s"] \arrow[dl, "i'"'] \arrow[dd, "f_0'"] & & X_0 \arrow[dl] \arrow[dd, "f_0"] & & \\
X \arrow[rr, dotted, "G"] \arrow[ddrr, bend right, "f"'] & & X' \arrow[rr] \arrow[dd, "f'"] & & X \arrow[dd, "f"] & & & \\
 & & & S_0 \arrow[rr, "F_0"'] \arrow[dl] & & S_0 \arrow[rr] \arrow[dl] & & \operatorname{Spec} \mathbb{F}_p \arrow[dl] \\
 & & S \arrow[rr, "F_S"] & & S \arrow[rr] & & \operatorname{Spec} \mathbb{Z}/p^2\mathbb{Z} &
\end{tikzcd}$$

where $X_0', X'$ are defined by requiring the front and back square to be Cartesian and $F$ is the relative Frobenius, i.e. $F$ is induced by the back square's Cartesianness using the Frobenius endomorphisms on $X_0$ and $S_0$. Since $X$ doesn't have a Frobenius, we *don't* easily obtain the dotted arrow $G$ in a similar way and in general it does not exist globally. We call a locally defined morphism $G$ that fits into the diagram a *local Frobenius lifting*. Because the (Zariski or étale) topologies are identified along $F$ and $i$, we can define Frobenius liftings simply at the level of sheaves:

**Definition 11.1.** Let $Y' \to X'$ be an étale open. Then a Frobenius lifting $G : Y \to Y'$ on $Y'$ consists of a ring homomorphism $G^* : \mathcal{O}_{Y'} \to G_*\mathcal{O}_Y$ yielding a morphism of schemes and a monoid homomorphism $G^* : \mathcal{M}_{Y'}|_{V'} \to G_*\mathcal{M}_Y|_{V'}$ defined on some $V' \subset Y'$ satisfying $(CC)$, yielding a log morphism. Two Frobenius liftings are considered equal if they are equal on some smaller (Zariski) open satisfying $(CC)$. The Frobenius liftings form an étale sheaf of sets $\mathcal{F}rob(X, X')$.

*Remark* 11.2. We need the flexibility of $V'$ in the definition of $\mathcal{F}rob(X, X')$ to construct Frobenius liftings from local models as they occur for log toroidal families. We will see below that we could have as well required the log part to be defined on $Y' \cap U'$, see the proof of Proposition 11.4.

Let $j : U' \hookrightarrow X'$ denote the pullback of $U \subset X$ and $Z' = X' \setminus U'$. By Lemma 2.3, $\mathcal{F}rob(X, X') = j_*(\mathcal{F}rob(X, X')|_{U'})$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf defining $X_0 \subset X$, flatness gives $\mathcal{I} = p \cdot \mathcal{O}_X \cong \mathcal{O}_{X_0}$. Using $\mathcal{I}^2 = 0$, one checks that $F_*\mathcal{I}$ is an $\mathcal{O}_{X'}$-module. Considering derivations on $U'$ with values in $F_*\mathcal{I}$, we obtain a sheaf of groups $\mathcal{G} := j_*\mathcal{D}er_{U'/S}(F_*\mathcal{I}) = j_*\mathcal{H}om(\Omega^1_{U'/S}, F_*\mathcal{I})$ which agrees with $\mathcal{H}om(W^1_{X'/S}, F_*\mathcal{I})$ because $F_*\mathcal{I}$ is $Z'$-closed by Lemma 2.3.

**Lemma 11.3.** *The restriction $\mathcal{F}rob(X, X')|_{U'}$ is a $\mathcal{G}|_{U'}$-torsor and hence $\mathcal{F}rob(X, X')$ is a $\mathcal{G}$-pseudo-torsor.*

*Proof.* Let $\mathcal{D}$ be the sheaf of sets on $U'$ given by étale local deformations of the diagram

$$\begin{array}{ccc} U_0 & \xrightarrow{i'\circ F} & U' \\ {\scriptstyle i}\big\downarrow & & {\scriptstyle f'}\big\downarrow \\ U & \xrightarrow{f} & S \end{array}$$

in the sense of [34, Def. IV.2.2.1], i.e. $\mathcal{D}$ is the sheaf of morphisms $U \to U'$ making the diagram commute. The sheaf $\mathcal{D}$ is a $\mathcal{G}|_{U'}$-pseudo-torsor by [34, Thm. IV.2.2.2] and because $f' : U' \to S$ is smooth, it is a torsor. Because $\Omega^1_{U'/S}$ is locally free, $\mathcal{D}$ is locally isomorphic to $(F_*\mathcal{I})^{\oplus d}$. By Lemma 2.3, $\mathcal{D}$ is $\tilde{Z}$-closed for every $\tilde{Z} \subset X'$ satisfying $\mathrm{codim}(\tilde{Z}, X') \geq 2$. By this property, the obvious homomorphism $\mathcal{D} \to \mathcal{F}rob(X, X')|_{U'}$ is an isomorphism of sheaves of sets making $\mathcal{F}rob(X, X')|_{U'}$ a $\mathcal{G}|_{U'}$-torsor. □

**Proposition 11.4.** *Let $Y' \to X'$ be an étale open and $G : Y \to Y'$ a local Frobenius lifting. Then there is a canonical homomorphism of complexes*

$$\phi_G : W^1_{Y_0'/S_0}[-1] \to F_* W^\bullet_{Y_0/S_0}$$

*inducing the Cartier isomorphism in first cohomology on $U_0' \cap Y_0'$. If $h \in \mathcal{G}(Y')$, then $\phi_G$ and $\phi_{h\cdot G}$ are related by*

$$\phi_{h\cdot G} = \phi_G + (F_*d) \circ \tilde{h}$$

*where $\tilde{h} : W^1_{Y_0'/S_0} \to F_*\mathcal{I} \cong F_* W^0_{Y_0/S_0}$ is the induced homomorphism.*

*Proof.* We choose $V' = U' \cap Y'$ for the representative of $G$. The straightforward log version of the construction of [24, Prop. 3.8] yields a homomorphism $\Omega^1_{V_0'/S_0} \to F_*\Omega^1_{V_0/S_0}$ and this has also been used implicitly by Kato in [28, Thm. 4.12]. Applying $j_*$ yields $(\phi_G)^1$, and we define the other $(\phi_G)^m$ to be 0. The resulting $\phi_G$ does not depend on $V'$ since the involved sheaves are $\tilde{Z}$-closed for every $\tilde{Z} \subset Y_0'$ satisfying $\mathrm{codim}(\tilde{Z}, Y_0') \geq 2$, so $\phi_G$ is well-defined. The construction yields that $\mathcal{H}^1(\phi_G)$ is the Cartier isomorphism of Theorem 10.1 on $V_0' = U_0' \cap Y_0'$. The second statement is similar to [24, Lemma 5.4,(5.4.1)] except that we use the more elegant language of torsors (as already remarked in [11, Rem. 2.2 (iii)]) which renders the analogue of [24, Lemma 5.4,(5.4.2)] trivial. □

**Theorem 11.5.** *Let $f : X \to S$ be a generically log smooth family, assume that $f_0 : X_0 \to S_0$ has the Cartier isomorphism property (Def. 10.3), and assume that $\mathcal{F}rob(X, X')$ is a $\mathcal{G}$-torsor. Then we have a quasi-isomorphism*

$$\bigoplus_{m<p} W^m_{X_0'/S_0}[-m] \to \tau_{<p} F_* W^\bullet_{X_0/S_0}$$

*in $D^b(X_0')$ where $\tau_{<p}$ means the truncation of a complex.*

*Proof.* Because $\mathcal{F}rob(X, X')$ is a torsor, we can find an étale cover $\mathfrak{Y} = \{Y'_\alpha\}$ of $X'$ such that we have a local Frobenius lifting $G_\alpha : Y_\alpha \to Y'_\alpha$. We obtain an induced cover $\mathfrak{Y}_0$ of $X'_0$. On the log smooth locus $U'_0 \subset X'_0$, we can apply an argument as implicitly used in [28, Thm. 4.12]: using Proposition 11.4, the gluing method of Step **B** in the proof of [24, Thm. 5.1] yields a homomorphism

$$\varphi : \Omega^1_{U'_0/S_0}[-1] \to \check{\mathcal{C}}^\bullet(\mathfrak{Y}_0 \cap U'_0, F_*\Omega^\bullet_{U_0/S_0}) =: \check{\mathcal{C}}^\bullet_U$$

of complexes of sheaves where $\check{\mathcal{C}}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet)$ refers to the total sheaf Čech complex for a cover $\mathfrak{U}$ and a complex of sheaves $\mathcal{F}^\bullet$. We also have the natural quasi-isomorphism

$$\psi : F_*W^\bullet_{X_0/S_0} \to \check{\mathcal{C}}^\bullet(\mathfrak{Y}_0, F_*W^\bullet_{X_0/S_0}).$$

Using $\psi$ and that the question is local, Prop. 11.4 gives that $\varphi$ induces the Cartier isomorphism on $U'_0$ for $\mathcal{H}^1$. Now let $0 \leq m < p$. With the antisymmetrization map $a_m : \Omega^m_{U'_0/S_0}[-m] \to (\Omega^1_{U'_0/S_0}[-1])^{\otimes m}$ defined by $a_m(\omega_1 \wedge ... \wedge \omega_m) = \frac{1}{m!}\sum_{\sigma\in S_m} \mathrm{sgn}(\sigma)\omega_{\sigma(1)} \otimes ... \otimes \omega_{\sigma(m)}$, we obtain a morphism

$$\varphi^m : \Omega^m_{U'_0/S_0}[-m] \xrightarrow{a_m} (\Omega^1_{U'_0/S_0}[-1])^{\otimes m} \xrightarrow{\varphi^{\otimes m}} (\check{\mathcal{C}}^\bullet_U)^{\otimes m} \to \check{\mathcal{C}}^\bullet_U$$

where the last map is induced by the wedge product on $F_*\Omega^\bullet_{U_0/S_0}$. Note that the various $\varphi^m$ are compatible with the wedge product of $\Omega^\bullet_{U'_0/S_0}$ and of the cohomology of $F_*\Omega^\bullet_{U_0/S_0}$ hence $\varphi^m$ induces the Cartier isomorphism in cohomology. Taking the sum, we obtain a quasi-isomorphism

$$\varphi^\bullet : \bigoplus_{m<p} \Omega^m_{U'_0/S_0}[-m] \to \tau_{<p}\check{\mathcal{C}}^\bullet_U.$$

Since $j_*\check{\mathcal{C}}^\bullet_U = \check{\mathcal{C}}^\bullet(\mathfrak{Y}_0, F_*W^\bullet_{X_0/S_0})$, we obtain the desired homomorphism in $D^b(X'_0)$ as $\psi^{-1} \circ j_*\varphi^\bullet$. It is a quasi-isomorphism because $f_0 : X_0 \to S_0$ has the Cartier isomorphism property by assumption. ☐

We like to apply this theorem to the case of a log toroidal family. It remains only to show that $\mathcal{F}rob(X, X')$ is a torsor:

**Proposition 11.6.** *In the above situation assume $f : X \to S$ log toroidal with respect to $S \to A_Q$. Then $\mathcal{F}rob(X, X')$ is a $\mathcal{G}$-torsor, i.e. Frobenius liftings exist locally.*

*Proof.* Let $(Q \subset P, \mathcal{F})$ be an ETD from a local model of $f : X \to S$, as given in (LM) with $S = \tilde{S}$. Consider the diagram

$$\begin{tikzcd} L \arrow[r, dotted, "F"] \arrow[d] & L \arrow[r, "c"] \arrow[d] & A_{P,\mathcal{F}} \arrow[d] \\ S \arrow[r, "F_S"] & S \arrow[r, "a"] & A_Q. \end{tikzcd}$$

We claim that for the local existence of a Frobenius lifting, it suffices to show that there is a scheme morphism $F : \underline{L} \to \underline{L}$ that is the underlying morphism of a log morphism on $c^{-1}(U_P)$ such that the diagram commutes and the induced map $F \times_S S_0$ on $L_0 = L \times_S S_0$ is the absolute Frobenius. Indeed, then $F$ plays the role of an absolute Frobenius on $L$, and its induced relative Frobenius gives rise to a local Frobenius lifting on $X'$ via the local model.

The scheme $\underline{L}$ is affine with $\mathcal{O}(L) = \bigoplus_{e \in E} z^e \cdot W_2(k)$ allowing us to define $F : \underline{L} \to \underline{L}$ via $F^*(z^e \cdot w) := z^{pe} \cdot F_S^*(w)$. It remains to extend $F$ to the log structure on $c^{-1}(U_P)$. Consider the maps of log schemes

$$M := \mathrm{Spec}(P \to \mathcal{O}(L)) \to L \to \mathrm{Spec}(Q \to \mathcal{O}(L)) =: N.$$

With the notation of Corollary 3.11, we define $W_i := c^{-1}(U_i)$. Observe that $M|_{W_1} = L|_{W_1}$ and $L|_{W_2} = N|_{W_2}$. On $N$ and $M$, we get morphisms $F_N : N \to N$ and $F_M : M \to M$ by mapping $q \mapsto p \cdot q$ on the monoids and using $F^*$ on the rings. They are compatible with each other and with the maps to $S$, and moreover $F_N \times_S S_0$ and $F_M \times_S S_0$ are the absolute Frobenii on $N_0, M_0$. We define partially $F|_{W_1} := F_M|_{W_1}$ and $F|_{W_2} := F_N|_{W_2}$. Because $N|_{W_1 \cap W_2} = L|_{W_1 \cap W_2} = M|_{W_1 \cap W_2}$ these definitions agree on $W_1 \cap W_2$ and we obtain a log morphism defined on $c^{-1}(U_P) = W_1 \cup W_2$ which gives the desired map. □

## 12. The Hodge-to-de-Rham Spectral Sequence

We put the pieces together to prove Theorem 1.9 from the introduction. Let $S = \mathrm{Spec}(Q \to \Bbbk)$ for a field $\Bbbk \supset \mathbb{Q}$ with $Q \ni q \mapsto \delta_{q0}$, and let $f : X \to S$ be a proper log toroidal family of relative dimension $d$ with respect to $S \to A_Q$. Setting $h^{pq} = \dim_{\Bbbk} R^q f_* W^p_{X/S}$ and $h^n = \dim_{\Bbbk} R^n f_* W^\bullet_{X/S}$, it suffices to prove $\sum_{p+q=n} h^{pq} = h^n$.

By Proposition 9.1, we can find an $S_\lambda = \mathrm{Spec}(Q \to B_\lambda)$ and a proper log toroidal family with respect to $S_\lambda \to A_Q$. Since $B_\lambda$ is integral, by shrinking $S_\lambda$, we can find a spreading out $\phi : \mathfrak{X} \to \mathcal{S}$ such that $R^q \phi_* W^p_{\mathfrak{X}/\mathcal{S}}$ and $R^n \phi_* W^\bullet_{\mathfrak{X}/\mathcal{S}}$ are locally free of constant rank $r^{pq}$ respective $r^n$ and such that $\mathcal{S}/\mathbb{Z}$ is smooth as schemes. By Theorem 8.3 we can furthermore assume that $W^m_{\mathfrak{X}/\mathcal{S}}$ is compatible with any base change, and we can assume that $\mathrm{char}\, \kappa(s) > d$ for the residue field $\kappa(s)$ of every closed point $s \in \mathcal{S}$. Now let $\mathrm{Spec}\, k \to \mathcal{S}$ be a closed point. Since $\mathcal{S}/\mathbb{Z}$ is smooth, we can find a factorization

$$\mathrm{Spec}\, k \to \mathrm{Spec}\, W_2(k) \to \mathcal{S}$$

which induces diagram (SO) from the introduction by strict base change. Setting $g^{pq} := \dim_k R^q (\phi_k)_* W^p_{\mathfrak{X}_k/k}$ and $g^{pq} := \dim_k R^n (\phi_k)_* W^\bullet_{\mathfrak{X}_k/k}$, Lemma 8.4 yields $h^{pq} = r^{pq} = g^{pq}$ and $h^n = r^n = g^n$ hence it suffices to show $\sum_{p+q=n} g^{pq} = g^n$. Note that in diagram (SO) on the right, we are in the situation of Proposition 11.6, so by Theorem 11.5 we have a quasi-isomorphism

$$\bigoplus_m W^m_{\mathfrak{X}'_k/k}[-m] \simeq (F_0)_* W^\bullet_{\mathfrak{X}_k/k}.$$

Now a computation as in [11, Cor. 2.4] yields $\sum_{p+q=n} g^{pq} = g^n$ concluding the proof of Theorem 1.9.

12.1. **The Relative Spectral Sequence.** *Proof of Theorem 1.10.* By Corollary 7.9, the formation of $W^p_{X/S}$ commutes with base change which is an ingredient for the classical base change theorem, e.g. [10, §3], [29, Theorem (8.0)]. It thereby suffices to show the surjectivity of

$$\mathbb{H}^k(X, W^\bullet_{X/S}) \to \mathbb{H}^k(X_0, W^\bullet_{X_0/S_0}).$$

We prove this with the idea of [38, section (2.6)], cf. [31, Lemma 4.1] and [15, Thm. 4.1]. We define a complex

$$\mathcal{L}^\bullet := W^{\bullet,an}_X[u] = \bigoplus_{s=0}^{\infty} W^{\bullet,an}_X \cdot u^s, \qquad d(\alpha_s u^s) = d\alpha_s \cdot u^s + s\delta(\rho) \wedge \alpha_s \cdot u^{s-1}$$

of analytic sheaves where $\rho = f^*(1) \in \mathcal{M}_{X^{an}}$ and $\delta : \mathcal{M}_{X^{an}} \to W^{1,an}_X$ is the log part of the universal derivation. Here $W^{\bullet,an}_X$ denotes absolute differentials as in Corollary 7.12. Projection to the $u^0$-summand composed with $W^{\bullet,an}_X \to W^{\bullet,an}_{X/S}$ yields a map $\mathcal{L}^\bullet \to W^{\bullet,an}_{X/S}$ whose composition with $W^{\bullet,an}_{X/S} \to W^{\bullet,an}_{X_0/S_0}$ fits into an exact sequence

$$0 \to \mathcal{K}^\bullet \to \mathcal{L}^\bullet \xrightarrow{\phi^\bullet} W^{\bullet,an}_{X_0/S_0} \to 0$$

of complexes that defines $\mathcal{K}^\bullet$. Since $f : X \to S$ has ETD local models, we may use Corollaries 7.12,7.12 and Remark 7.15 to have a local description of this sequence. Lemma 12.1 below verifies that $\mathcal{K}^\bullet$ is acyclic for all ETDs with one-dimensional base, so $\phi^\bullet$ is a quasi-isomorphism and Theorem 1.10 follows by the discussion in §2.1. □

**Lemma 12.1.** *Let* $(\mathbb{N} \subset P, \mathcal{F})$ *be an ETD, and let* $f : X \to S = S_m$ *be the base change of* $A_{P,\mathcal{F}} \to A_{\mathbb{N}}$ *along* $S_m \to A_{\mathbb{N}}$. *With* $0 \in A_{P,\mathcal{F}}$ *denoting the origin, we have* $\mathcal{H}^k(\mathcal{K}^\bullet)_0 = 0$ *for all* $k$.

*Proof.* We choose Hermitian inner products on the vector spaces $L := P^{gp} \otimes \mathbb{C}$ and $W := (P^{gp} \otimes \mathbb{C})/(\mathbb{N}^{gp} \otimes \mathbb{C})$. With $K = (m+1) + \mathbb{N} \subset \mathbb{N}$, we recall $E_K$ from §7.2. For $e \in E_K$, we define

$$L_e := \bigcap_{H \in \mathcal{F}_{\max} \setminus \mathcal{F} : e \in H} H^{gp} \otimes \mathbb{C} \qquad \text{and} \qquad W_e := \bigcap_{H \in \mathcal{F}_{\max} \setminus \mathcal{F} : e \in H} (H^{gp} \otimes \mathbb{C})/(\mathbb{N}^{gp} \otimes \mathbb{C}).$$

By Remark 7.15 and Lemma 7.14, elements of $\mathcal{L}^k_0$ are formal sums

$$(\ell_{e,s}) := \sum_{s=0}^{N} \sum_{e \in E_K} u^s z^e \ell_{e,s} \ , \quad \ell_{e,s} \in \bigwedge^k L_e \ , \quad \sup_{\substack{e \in E_K \setminus 0 \\ 1 \le s \le N}} \{\log \|\ell_{e,s}\| / h(e)\} < \infty$$

and elements of $W^{k,an}_{X_0/S_0,0}$ are formal sums

$$(w_e) := \sum_{e\in E} z^e \cdot w_e, \quad w_e \in \bigwedge^k W_e \ , \quad \sup_{e\in E\setminus 0}\{\log\|w_e\|/h(e)\} < \infty$$

Note that $(\ell_{e,s})$ is summed over $E_K$ whereas $(w_e)$ is summed over $E$. We denote the kernel of $\pi : \bigwedge^k L_e \to \bigwedge^k W_e$ by $K^k_e$ and observe $\phi((\ell_{e,s})) = (\pi(\ell_{e,0}))$, so $(\ell_{e,s}) \in \mathcal{K}^k_0$ if and only if $\ell_{e,0} \in K^k_e$ for all $e \in E$. With $\bar\rho := 1 \otimes 1 \in \mathbb{N}^{gp} \otimes \mathbb{C}$ we have $\delta(\rho) = z^0 \cdot \bar\rho \in W^1_X$ and thus

$$d((\ell_{e,s})) = (e \wedge \ell_{e,s} + (s+1)\bar\rho \wedge \ell_{e,s+1}) \tag{12.1}$$

Let $(\ell_{e,s}) \in \mathcal{K}^0_0$ and assume $d((\ell_{e,s})) = 0$. Since $\ell_{e,s} \in \mathbb{C}$, for $e \neq 0$ by descending induction in $s$ starting from $\ell_{e,N}$ we find $\ell_{e,s} = 0$. We have $\ell_{0,0} = 0$ and ascending induction yields $\ell_{0,s} = 0$. Thus $\mathcal{H}^0(\mathcal{K}^\bullet)_0 = 0$.

Next, let $(\ell_{e,s}) \in \mathcal{K}^{k+1}_0$ for $k \geq 0$ with $d((\ell_{e,s})) = 0$. Starting with $e = 0$, we construct $(\tau_{e,s}) \in \mathcal{K}^k_0$ with $d((\tau_{e,s})) = (\ell_{e,s})$ using the following claim.

**Claim 2.** *Let $(L, \langle\cdot,\cdot\rangle)$ be a $\mathbb{C}$-vector space of finite dimension with a Hermitian inner product. Let $0 \neq p \in L$ and $k \geq 0$, and assume $\ell \in \bigwedge^{k+1} L$ with $p \wedge \ell = 0$. Then there is a $\tilde\ell \in \bigwedge^k L$ with $p \wedge \tilde\ell = \ell$ and $\|p\| \cdot \|\tilde\ell\| = \|\ell\|$.*

*Proof.* Let $\ell_1 := \frac{p}{\|p\|}, \ell_2, ..., \ell_n$ be an orthonormal basis of $L$, and $\{\ell_{i_1...i_k}\}$ the induced basis of $\bigwedge^k L$. If $\ell = \sum \alpha_{i_1...i_{k+1}} \ell_{i_1...i_{k+1}}$ satisfies the assumption, then $\tilde\ell = \frac{1}{\|p\|}\sum \alpha_{1 i_2...i_{k+1}} \ell_{i_2...i_{k+1}}$ is a solution. □

We set $\tau_{0,0} = 0$. Writing out (12.1) for $e = 0$ yields

$$d\big(\ell_{0,0} + \ell_{0,1}u + \ell_{0,2}u^2 + ...\big) = \bar\rho \wedge \ell_{0,1} \ + \ 2\bar\rho \wedge \ell_{0,2}u \ + \ 3\bar\rho \wedge \ell_{0,3}u^2 \ + \ ...$$

and therefore $\bar\rho \wedge \ell_{0,i} = 0$ for $i > 0$. Since $\ell_{0,0} \in K^0_0$, we also have $\bar\rho \wedge \ell_{0,0} = 0$. By Claim 2, there is $\tau_{0,s+1} \in \bigwedge^k L_0$ with $\bar\rho \wedge \tau_{0,s+1} = \ell_{0,s}$ and we are done with the case $e = 0$. For $e \neq 0$ we need to care about convergence. Without loss of generality, $N \geq 1$. Since $e \wedge \ell_{e,N} = 0$, we can find by Claim 2 $\tau_{e,N} \in \bigwedge^k L_e$ with $e \wedge \tau_{e,N} = \ell_{e,N}$ and $\|\tau_{e,N}\| \cdot \|e\| = \|\ell_{e,N}\|$. For $s \geq 1$, we construct $\tau_{e,s} \in \bigwedge^k L_e$ by descending induction. Because of $e \wedge (\ell_{e,s} - (s+1)\bar\rho \wedge \tau_{e,s+1}) = 0$, there is $\tau_{e,s}$ with $e \wedge \tau_{e,s} = \ell_{e,s} - (s+1)\bar\rho \wedge \tau_{e,s+1}$ and

$$\|\tau_{e,s}\| \cdot \|e\| = \|\ell_{e,s} - (s+1)\bar\rho \wedge \tau_{e,s+1}\|. \tag{12.2}$$

For $e \notin E$, we go one step further and construct $\tau_{e,0} \in \bigwedge^k L_e$ with the same method, but for $e \in E$, the construction of $\tau_{e,0} \in K^k_e$ is more intricate. We need another claim:

**Claim 3.** *Let $(L, \langle\cdot,\cdot\rangle)$ be a $\mathbb{C}$-vector space of finite dimension with a Hermitian inner product. Let $0 \neq V, Y \subset L$ be subspaces with $V \cap Y = 0$. Then there is a constant $\gamma > 0$ with the following property: for every subspace $H$ with $V \subset H \subset L$*

*and $k \geq 0$, let $K_H^k$ be the kernel of $\bigwedge^k H \to \bigwedge^k(H/V)$. Then for every $0 \neq p \in Y \cap H$ and every $\ell \in K_H^{k+1}$ with $p \wedge \ell = 0$, there is a $\tilde{\ell} \in K_H^k$ with $p \wedge \tilde{\ell} = \ell$ and $\gamma \cdot \|p\| \cdot \|\tilde{\ell}\| \leq \|\ell\|$.*

*Proof.* Let $p = (p_1, p_2)$ be the decomposition of $p$ under $L = V \oplus V^\perp$, so $\|p\|^2 = \|p_1\|^2 + \|p_2\|^2$. Since $V \cap Y = 0$, we have for $\gamma^2 := \inf_{0 \neq p \in Y} \|p_2\|^2/\|p\|^2$ that $0 < \gamma \leq 1$. Let $\ell_0 := \frac{p_2}{\|p_2\|}, \ell_1, \ell_2 ...$ be an orthonormal basis of $H$ and then $\bar{\ell}_0 = \frac{p}{\|p\|}$, $\bar{\ell}_i := \ell_i$ for $i > 0$ is an ordinary basis of $H$. For $\ell = \sum \alpha_{i_0 \ldots i_k} \bar{\ell}_{i_0 \ldots i_k} \in K_H^{k+1}$ with $p \wedge \ell = 0$, we define $\tilde{\ell} := \frac{1}{\|p\|} \sum \alpha_{0 i_1 \ldots i_k} \bar{\ell}_{i_1 \ldots i_k} \in K_H^k$ to have $p \wedge \tilde{\ell} = \ell$. We also find

$$\|\ell\|^2 = \left\| \sum \alpha_{0 i_1 \ldots i_k} \frac{p}{\|p\|} \wedge \ell_{i_1 \ldots i_k} \right\|^2 \geq \left\| \sum \alpha_{0 i_1 \ldots i_k} \frac{p_2}{\|p\|} \wedge \ell_{i_1 \ldots i_k} \right\|^2 \geq \gamma^2 \cdot \|p\|^2 \cdot \|\tilde{\ell}\|^2$$

□

We apply Claim 3 to $L = P^{gp} \otimes \mathbb{C}$. Let $F_e \subset P$ be the face generated by $e$ and $Y = F_e^{gp} \otimes \mathbb{C}$. Let $V = \mathbb{N}^{gp} \otimes \mathbb{C}$ and $H = L_e$, so $K_H^k = K_e^k$. Then $e \wedge (\ell_{e,0} - \bar{\rho} \wedge \tau_{e,1}) = 0$, so we find $\tau_{e,0} \in K_e^k$ with $e \wedge \tau_{e,0} = \ell_{e,0} - \bar{\rho} \wedge \tau_{e,1}$ and

$$\gamma \cdot \|\tau_{e,0}\| \cdot \|e\| \leq \|\ell_{e,0} - \bar{\rho} \wedge \tau_{e,1}\| \tag{12.3}$$

The factor $\gamma$ depends on $Y$, but there are only finitely many faces generated by elements $e \in E$, so we take for $\gamma$ the minimum over them and furthermore $\gamma < 1$. Applying the triangle inequality to the right hand side of (12.3) and using induction and (12.2) yields

$$\|\tau_{e,s}\| \leq \frac{1}{\gamma} \cdot \frac{1}{\|e\|} \sum_{k=s}^{N} \left( \frac{\|\bar{\rho}\|}{\|e\|} \right)^{k-s} \cdot \frac{k!}{s!} \cdot \|\ell_{e,k}\|$$

for all $e \neq 0$. Because $\inf_{e \neq 0}\{\|e\|\} > 0$, there is a bound $M > 1$ independent of $e$ such that $\|\tau_{e,s}\| \leq M \cdot \max_k\{\|\ell_{e,k}\|\}$ which proves

$$\sup_{e \in E_K \setminus 0} \{\log \|\tau_{e,s}\| / h(e)\} < \infty$$

and thus $(\tau_{e,s}) \in \mathcal{K}_0^k$. By construction, $d((\tau_{e,s})) = (\ell_{e,s})$, so $\mathcal{H}^k(\mathcal{K}^\bullet)_0 = 0$. □

## 13. Smoothings via Maurer-Cartan Solutions

We adapt the methods of [25] to our setup in §13.1 and §13.2 and then argue how to obtain an analytic smoothing from a formal one in §13.3. The combination of all these sections gives a proof of Theorem 1.7. The main ingredients are Theorem 6.13, Theorem 1.9 and Theorem 1.10. A key ingredient is also Lemma 6.11 to know that $W^d_{X/S}$ is trivial for $d = \dim X$.

13.1. **Constructing a Formal Deformation from a Solution to the Maurer-Cartan Equation.** We define ${}^kS = \operatorname{Spec}(\mathbb{N} \xrightarrow{1 \mapsto t} \mathbb{C}[t]/t^{k+1})$ and assume to be given a proper log toroidal family ${}^0X \to {}^0S$. Let $\{{}^0V_\alpha\}_\alpha$ be an affine cover of ${}^0X$. For fixed $\alpha$, let $\{{}^kV_\alpha \to {}^kS\}_k$ be a system of deformations, compatible with restriction from $k$ to $k-1$ as obtained from Theorem 6.13. Note that $V_{\alpha\beta} := {}^0V_\alpha \cap {}^0V_\beta$ is affine because ${}^0X$ is separated. We give names to the restrictions of thickenings via ${}^kV_{\alpha;\alpha\beta} := {}^kV_\alpha|_{V_{\alpha\beta}}$. Again by Theorem 6.13, we find isomorphisms

$$ {}^k\phi_{\alpha\beta} : {}^kV_{\alpha;\alpha\beta} \to {}^kV_{\beta;\alpha\beta} $$

of generically log smooth families over ${}^kS$ which are compatible with the restrictions to the base changes via ${}^{k-1}S \to {}^kS$ but do not necessarily satisfy a cocycle condition.

We now analytify ${}^kX \to {}^kS$ as well as ${}^kV_\alpha, {}^kV_{\alpha;\alpha\beta}$. We keep using the same symbols though now refer to the analytifications respectively.

Let $\{U_i\}_{i\in I}$ be a cover of ${}^0X$ by Stein open sets that is also a basis for the analytic topology of ${}^0X$ with $I$ countable and totally ordered. Set $U_{i_0\dots i_l} := \bigcap_{k=0}^l U_{i_k}$. We obtain the sheaves of Gerstenhaber algebras

$$ {}^k\mathcal{G}^p_\alpha := \Theta^{-p}_{({}^kV_\alpha)/{}^kS} $$

concentrated in non-positive degrees via the negative Schouten-Nijenhuis bracket $-[\cdot,\cdot]$ and $\wedge$. Set $\blacktriangle_l = \mathrm{Spec}(\mathbb{C}[x_0,\dots,x_n]/(x_0+\dots+x_n-1))$ and $\mathcal{A}^q(\blacktriangle_l) = \Omega^q_{\blacktriangle_l}$ and let $d_{j,l} : \blacktriangle_{l-1} \to \blacktriangle_l$ be given by $x_j \mapsto 0$. One constructs the Thom-Whitney bicomplex

$$ \text{(TW)} \qquad {}^kTW^{p,q}_{\alpha;\alpha_0\dots\alpha_l} = \left\{ (\varphi_{i_0\dots i_l})_{i_0<\dots<i_l} \;\middle|\; \begin{array}{c} U_{i_j} \subset V_{\alpha_0} \cap \dots \cap V_{\alpha_l} \text{ for } 0 \le j \le l, \\ \varphi_{i_0\dots i_l} \in \mathcal{A}^q(\blacktriangle_l) \otimes_{\mathbb{C}} {}^k\mathcal{G}^p_\alpha(U_{i_0\dots i_l}), \\ d^*_{j,l}(\varphi_{i_0\dots i_l}) = \varphi_{i_0\dots\hat{i}_j\dots i_l}|_{U_{i_0\dots i_l}} \end{array} \right\}. $$

The differential for the index $p$ is trivial and the differential $\bar\partial_\alpha$ for the index $q$ is induced by the de Rham differential on $\mathcal{A}^q(\blacktriangle_l)$. Furthermore, $-[\cdot,\cdot]$ and $\wedge$ turn $TW$ into a Gerstenhaber algebra. For $W \subset V_\alpha$, let ${}^kTW^{p,q}_{\alpha;\alpha}|_W$ be given by (TW) but with the additional requirement to have $U_{i_j} \subset W$. The presheaf $W \mapsto {}^kTW^{p,\bullet}_{\alpha;\alpha}|_W$ gives a resolution of the sheaf ${}^k\mathcal{G}^p_\alpha$ on $V_\alpha$, so ${}^k\mathcal{G}^p_\alpha(W) = H^0_{\bar\partial_\alpha}({}^kTW^{p,\bullet}_{\alpha;\alpha}|_W)$.

The isomorphisms ${}^k\phi_{\alpha\beta}$ induce isomorphisms ${}^k\psi_{\alpha\beta} : {}^k\mathcal{G}^\bullet_\alpha|_{V_{\alpha\beta}} \to {}^k\mathcal{G}^\bullet_\beta|_{V_{\alpha\beta}}$ of sheaves of Gerstenhaber algebras which can be used ([25, Key Lemma 3.21]) to construct isomorphisms

$$ {}^kg_{\alpha\beta} : {}^kTW^{p,q}_{\alpha;\alpha\beta} \to {}^kTW^{p,q}_{\beta;\alpha\beta} $$

that satisfy the cocycle condition ${}^kg_{\gamma\alpha}{}^kg_{\beta\gamma}{}^kg_{\alpha\beta} = \mathrm{id}$ and are compatible with restriction from $k$ to $k-1$ and with $-[\cdot,\cdot]$ and $\wedge$. The cocycle condition allows one to glue $\{{}^kTW^{p,q}_\alpha\}_\alpha$ to a presheaf ${}^k\mathrm{PV}^{p,q}$ on ${}^0X$ compatible with restricting from $k$ to $k-1$. We set ${}^k\mathrm{PV}^n := \bigoplus_{p+q=n} {}^k\mathrm{PV}^{p,q}$.

While ${}^kg_{\alpha\beta}$ are not necessarily compatible with the differentials $\bar\partial_\alpha, \bar\partial_\beta$, there exist ${}^k\mathfrak{d}_\alpha \in {}^kTW^{-1,1}_\alpha$ such that $(\bar\partial_\alpha + [{}^k\mathfrak{d}_\alpha,\cdot])_\alpha$ gives a system of maps compatible with ${}^kg_{\alpha\beta}$ ([25, Theorem 3.34]). This system glues to an operator $\bar\partial$ on ${}^k\mathrm{PV}^{p,q}$ compatible with restriction from $k$ to $k-1$. However, $\bar\partial$ is not a differential because

$$ \bar\partial^2 = [{}^k\mathfrak{l}_\alpha,\,\cdot\,] \qquad \text{for} \qquad {}^k\mathfrak{l}_\alpha := \bar\partial_\alpha({}^k\mathfrak{d}_\alpha) + \frac{1}{2}[{}^k\mathfrak{d}_\alpha, {}^k\mathfrak{d}_\alpha] \in {}^kTW^{-1,2}_\alpha. $$

The $\{{}^k\mathfrak{l}_\alpha\}_\alpha$ glue to a global element ${}^k\mathfrak{l} \in {}^k\mathrm{PV}^{-1,2}$ that is compatible with restricting from $k$ to $k-1$. If ${}^k\phi \in {}^k\mathrm{PV}^{-1,1}$ solves the Maurer-Cartan equation

$$\bar\partial({}^k\phi) + \frac{1}{2}[{}^k\phi, {}^k\phi] + {}^k\mathfrak{l} = 0 \tag{MC1}$$

then $(\bar\partial + [{}^k\phi, \cdot])^2 = 0$. In this case the cohomology $H^\bullet_{(\bar\partial+[{}^k\phi,\cdot])}({}^k\mathrm{PV}^\bullet)$ is a presheaf of Gerstenhaber algebras on ${}^0X$ that is locally isomorphic to ${}^k\mathcal{G}^\bullet_\alpha$. The sheafification of its degree zero part gives a sheaf $\mathcal{O}_{X_k}$ of $\mathbb{C}[t]/t^{k+1}$-algebras on ${}^0X$ which we take as the $k$th order deformation of ${}^0X$. Taking the limit $\mathcal{O}_{\mathfrak{X}} := \varprojlim_k \mathcal{O}_{X_k}$ yields a flat and proper morphism $\mathfrak{X} \to \mathfrak{S}$ with $\mathfrak{S} := \mathrm{Spf}(\mathbb{C}[\![t]\!])$.

13.2. **Constructing a Solution to the Maurer-Cartan Equation using the Batalin-Vilkovisky Operator.** Let ${}^k\omega_\alpha \in \Gamma({}^0V_\alpha, W^d_{{}^kV_\alpha/{}^kS})$ be a choice of generator, each being a lift to $k$ of a global generator ${}^0\omega \in \Gamma({}^0X, W^d_{{}^0X/{}^kS})$. The Batalin-Vilkovisky operator ${}^k\Delta_\alpha$ is the transfer of the de Rham differential $\mathtt{d}$ to the polyvector fields, i.e. ${}^k\Delta_\alpha$ is the composition

$$\Theta^p_{({}^kV_\alpha)/{}^kS} \xrightarrow{\llcorner({}^k\omega_\alpha)} W^{d-p}_{({}^kV_\alpha)/{}^kS} \xrightarrow{\mathtt{d}} W^{d-p+1}_{({}^kV_\alpha)/{}^kS} \xrightarrow{\llcorner({}^k\omega_\alpha)^{-1}} \Theta^{p-1}_{({}^kV_\alpha)/{}^kS}$$

and thus a differential ${}^k\mathcal{G}^p_\alpha \to {}^k\mathcal{G}^{p+1}_\alpha$. Choosing ${}^k\omega_\alpha$ compatible with restricting from $k$ to $k-1$, also the ${}^k\Delta_\alpha$ share this property. For $W \subset {}^0V_\alpha \cap {}^0V_\beta$ there is $\lambda_{\alpha\beta} \in \Gamma(W, {}^k\mathcal{G}^0_\alpha)$ with ${}^k\omega_\alpha|_W = \lambda_{\alpha\beta} \cdot {}^k\omega_\beta|_W$. Setting ${}^k\mathfrak{w}_{\alpha\beta} := \log(\lambda_{\alpha\beta})$ yields

$${}^k\psi_{\beta\alpha} \circ {}^k\Delta_\beta \circ {}^k\psi_{\alpha\beta} - {}^k\Delta_\alpha = [{}^k\mathfrak{w}_{\alpha\beta}, \cdot\,]$$

and then $\{{}^k\mathfrak{w}_{\alpha\beta}\}_{\alpha\beta}$ can be upgraded ([25, Theorem 3.34]) to a Čech cocycle for ${}^kTW^{0,0}_{\alpha;\alpha\beta}$ which by exactness lifts to a collection ${}^k\mathfrak{f}_\alpha \in TW^{0,0}_\alpha$. The collection is compatible with restricting from $k$ to $k-1$ and satisfies

$${}^kg_{\beta\alpha} \circ ({}^k\Delta_\beta + [{}^k\mathfrak{f}_\beta, \cdot]) \circ {}^kg_{\alpha\beta} = ({}^k\Delta_\alpha + [{}^k\mathfrak{f}_\alpha, \cdot]).$$

Since ${}^k\mathfrak{f}_\alpha$ lives in degree $(0,0)$, one has $({}^k\Delta_\alpha + [{}^k\mathfrak{f}_\alpha, \cdot])^2 = 0$, so we can glue the collection $\{{}^k\Delta_\alpha + [{}^k\mathfrak{f}_\alpha, \cdot]\}_\alpha$ to an operator $\Delta : {}^k\,\mathrm{PV}^{p,q} \to {}^k\,\mathrm{PV}^{p+1,q}$ with $\Delta^2 = 0$. Now,

$$\Delta\bar\partial + \bar\partial\Delta = [{}^k\mathfrak{y}, \cdot] \quad \text{for} \quad {}^k\mathfrak{y}_\alpha := {}^k\Delta_\alpha({}^k\mathfrak{d}_\alpha) + {}^k\bar\partial_\alpha({}^k\mathfrak{f}_\alpha) + [{}^k\mathfrak{d}_\alpha, {}^k\mathfrak{f}_\alpha]$$

and ${}^k\mathfrak{y} \in {}^k\mathrm{PV}^{0,1}$ is glued from the collection ${}^k\mathfrak{y}_\alpha$. By construction,

$$\breve{d} := \bar\partial + \Delta + (\mathfrak{l} + \mathfrak{y})\wedge$$

satisfies $\breve{d}^2 = 0$ and furthermore $(\mathfrak{l} + \mathfrak{y}) \equiv 0 \mod (t)$.

**Theorem 13.1.** *The natural maps $H^i_{\breve{d}}({}^k\mathrm{PV}^\bullet) \to H^i_{\breve{d}}({}^{k-1}\mathrm{PV}^\bullet)$ are surjective for all $i$ and $k$.*

*Proof.* As in [25, Proposition 4.8], the elements $\exp({}^k\mathfrak{f}_\alpha \llcorner){}^k\omega_\alpha$ glue to a global element ${}^k\omega$ in the Thom-Whitney de Rham complex $({}^k_{\Vert}\mathcal{A}^\bullet, d)$ (constructed from $W^\bullet_{{}^kV_\alpha/{}^kS}$ in our case) compatible with restricting from $k$ to $k-1$. Contracting ${}^k\omega$ gives an isomorphism of complexes ${}^k\mathrm{PV}^\bullet \to {}^k_{\|}\mathcal{A}^\bullet$, so it suffices to prove surjectivity of $H^i_d({}^k_{\|}\mathcal{A}^\bullet) \to H^i_d({}^{k-1}_{\|}\mathcal{A}^\bullet)$. This follows from Theorem 1.10 as argued in [25, Lemma 4.17]. □

*Remark* 13.2. For a formal variable $u^{\frac{1}{2}}$, consider on $\mathrm{PV}^\bullet[\![u^{\frac{1}{2}}]\!]$ the differential $\breve{d}_u := \bar\partial + u\Delta + u^{-1}(\mathfrak{l} + u\mathfrak{y})\wedge$. A direct computation gives $\breve{d}_u = u^{\frac{1}{2}} I_u^{-1} \circ \breve{d} \circ I_u$ where $I_u$ is defined by $I_u(\varphi) = u^{\frac{p-q-2}{2}}\varphi$ for $\varphi \in \mathrm{PV}^{p,q}[\![u^{\frac{1}{2}}]\!][u^{-\frac{1}{2}}]$ (cf. [25, Notation 5.1]). Theorem 13.1 thus implies that

$$H^i_{\breve{d}_u}({}^k\mathrm{PV}^\bullet[\![u^{\frac{1}{2}}]\!][u^{-\frac{1}{2}}]) \to H^i_{\breve{d}_u}({}^{k-1}\mathrm{PV}^\bullet[\![u^{\frac{1}{2}}]\!][u^{-\frac{1}{2}}]) \tag{13.1}$$

is surjective for all $i, k$.

**Theorem 13.3.** *For all $i$, $H^i_{\bar\partial+u\Delta}({}^0\mathrm{PV}^\bullet[\![u]\!])$ is a free $\mathbb{C}[\![u]\!]$-module of finite rank.*

*Proof.* Note that $k = 0$. With $\bar\partial$ the Čech differential for the cover $\{V_\alpha\}_\alpha$, the degeneration of the Hodge to de Rham spectral sequence for $(W^\bullet_{{}^0X/{}^0S}, d)$ at $E_1$ by Theorem 1.9 is equivalent to $H^i_{\bar\partial+ud}(\{V_\alpha\}_\alpha, W^\bullet_{{}^0X/{}^0S}[\![u]\!])$ being a free $\mathbb{C}[\![u]\!]$-module of finite rank. The quasi-isomorphisms $W^\bullet_{{}^0X/{}^0S}[\![u]\!] \to {}^0_{\|}\mathcal{A}^\bullet[\![u]\!]$ and ${}^0\mathrm{PV}^\bullet[\![u]\!] \to {}^0_{\|}\mathcal{A}^\bullet[\![u]\!]$ yield the assertion. □

**Theorem 13.4.** *For every $\psi \in {}^0\mathrm{PV}^\bullet[\![u]\!]$ with $(\bar\partial + u\Delta)(\psi) = 0$, there exist ${}^k\varphi \in {}^k\mathrm{PV}^0[\![u]\!]$ for all $k \geq 0$ with ${}^k\varphi \equiv {}^{k+1}\varphi \mod t^{k+1}$ and ${}^0\varphi = \psi$ solving*

$$(\bar\partial + u\Delta)({}^k\varphi) + \frac{1}{2}[{}^k\varphi, {}^k\varphi] + ({}^k\mathfrak{l} + u\,{}^k\mathfrak{y}) = 0. \tag{MC2}$$

*Furthermore, the solution can be given so that, setting ${}^k\phi := ({}^k\varphi \mod u)$ with ${}^k\phi = \sum_j {}^k\phi_j$ and ${}^k\phi_j \in {}^k\mathrm{PV}^{-j,j}$, it holds ${}^k\phi_0 = 0$ and thus ${}^k\phi_1 \in {}^k\mathrm{PV}^{-1,1}$ solves* (MC1).

*Proof.* The first assertion becomes [25, Theorem 5.5] after checking that we have the ingredients for its proof available. The proof goes by induction over $k$ and uses (i) the surjectivity in Theorem 13.1 for $k = 0$, (ii) the surjectivity in Equation (13.1) for all $k$ and (iii) Theorem 13.3 in each step to get rid of negative powers of $u$ in ${}^k\varphi$. The second statement is [25, Lemma 5.11]. □

13.3. **From a Formal Deformation to an Analytic Deformation.** Let $\mathfrak{S}$ be the completion of an analytic variety $S$ in a non-zerodivisor $t \in \Gamma(S, \mathcal{O}_S)$. Let $S_k$ be the closed analytic subvariety defined by $t^k$. If $X \to S$ is flat, we denote by $X_k \to S_k$ the base change to $S_k$, similarly for a flat map $\mathfrak{X} \to \mathfrak{S}$.

**Theorem 13.5** ([21], Theorem B.1)**.** *Given a proper and flat formal analytic morphism $\hat\varphi : \mathfrak{X} \to \mathfrak{S}$, for every $k > 0$ there is a proper flat analytic morphism $\varphi : X \to S$ together with an $S_k$-isomorphism $\mathfrak{X}_k \to X_k$ of the base changes of $\hat\varphi$ and $\varphi$ to $S_k$.*

**Theorem 13.6** ([36], Theorem 5.5 (1)). *In the situation of Theorem 13.5, given $s \in S_0$ and $X_s = \varphi^{-1}(s)$, there exists an integer $K > 0$ such that whenever $\varphi : X \to S$ is obtained for $k > K$ then every point $x \in X_s$ has a neighborhood in $X$ whose $t$-completion is formally isomorphic to a neighborhood of $x$ in $\mathfrak{X}$, in particular if $\mathfrak{X}$ is a smoothing of a fiber $X_s$ for $t \neq 0$ then so is $X$.*

**Theorem 13.7** ([36], Theorem 5.5 (3)). *In the situation of Theorem 13.6, for $X_0$ the base change to $S_0$, the maps of pairs $(X, X_0) \to (S, S_0)$ and $(\mathfrak{X}, \mathfrak{X}_0) \to (\mathfrak{S}, S_0)$ turn $\hat{\varphi}$ and $\varphi$ into log morphisms via the divisorial log structures. There is an isomorphism of the log fibers over $s \in S$ whose underlying morphism is restriction to the fiber $X_s$ of the $S_k$-isomorphism $\mathfrak{X}_k \to X_k$.*

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