Forum of Mathematics, Sigma (2020), **Vol. 8**:e58, 1–55
doi:10.1017/fms.2020.54

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RESEARCH ARTICLE

Tropically constructed Lagrangians in mirror quintic threefolds

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Received: 31 May 2019; **Revised:** 17 September 2020; **Accepted:** 9 October 2020

2020 Mathematics Subject Classification: 53D12 (primary); 14J32; 14T20; 14T90; 53D37; 57R17 (secondary)

Abstract

We use tropical curves and toric degeneration techniques to construct closed embedded Lagrangian rational homology spheres in a lot of Calabi-Yau threefolds. The homology spheres are mirror dual to the holomorphic curves contributing to the Gromov-Witten (GW) invariants. In view of Joyce's conjecture, these Lagrangians are expected to have special Lagrangian representatives and hence solve a special Lagrangian enumerative problem in Calabi-Yau threefolds.

We apply this construction to the tropical curves obtained from the 2,875 lines on the quintic Calabi-Yau threefold. Each admissible tropical curve gives a Lagrangian rational homology sphere in the corresponding mirror quintic threefold and the Joyce's weight of each of these Lagrangians equals the multiplicity of the corresponding tropical curve.

As applications, we show that disjoint curves give pairwise homologous but non-Hamiltonian isotopic Lagrangians and we check in an example that > 300 mutually disjoint curves (and hence Lagrangians) arise. Dehn twists along these Lagrangians generate an abelian subgroup of the symplectic mapping class group with that rank.

Contents

1. Introduction

Special Lagrangian submanifolds of Calabi-Yau threefolds have received much attention due to their role in mirror symmetry. Based on Thomas [57] and Thomas and Yau [58], Dominic Joyce [31] conjectured that a Lagrangian submanifold L admits a special Lagrangian representative (after surgery at a discrete set of times under Lagrangian mean curvature flow) if L is a stable object in the derived Fukaya category with respect to an appropriate Bridgeland stability condition. Therefore, roughly speaking, special Lagrangians correspond to stable objects. In [29], Joyce proposed a counting invariant for rigid special Lagrangians (i.e., special Lagrangian rational homology spheres) so that each of these Lagrangians L is weighted by $w(L) := |H_1(L,\mathbb{Z})|$ when it is counted and, under the conjectural correspondence between special Lagrangians and stable objects, Joyce's counting invariant is conjectured to be mirror to the Donaldson-Thomas invariant. One possible explanation of the weight $w(L) = |H_1(L,\mathbb{Z})|$ is that objects in the Fukaya category are Lagrangians with local systems and $|H_1(L,\mathbb{Z})|$ is exactly the number of rank one local systems on L, giving $|H_1(L,\mathbb{Z})|$ many different objects in the Fukaya category. (The original explanation in [29] is different.)

Even before counting, finding special Lagrangians is a challenging problem ([28], [30] etc). The main source of examples is given by the set of real points. Making a given Lagrangian special is hard. Even without the specialty assumption, there are not many explicit methods to construct closed embedded Lagrangian submanifolds in Calabi-Yau threefolds in the literature, especially when the Calabi-Yau is assumed to be compact and the Lagrangians spherical. In this article, we provide a new method to address the latter difficulty using toric degeneration techniques and tropical curves.

The Lagrangians are constructed by dualizing tropical curves that contribute to the Gromov-Witten (GW) invariant of the mirror. Therefore, even though we do not show that the Lagrangians we construct are (Hamiltonian isotopic to) special Lagrangians, their quasi-isomorphism classes in the Fukaya category are conjecturally mirror dual to the stable sheaves contributing to the Donaldson-Thomas invariants (via DT/GW correspondence). It is expected that these Lagrangians would give the full set of stable objects in a fixed K-class in the Fukaya category with respect to a stability condition and hence play an important role in the enumerative study of stable objects in the Fukaya category. In particular, we find that the weight $w(L)$ indeed coincides with the multiplicity of the corresponding tropical curve, which is also how it enters the mirror dual GW count. We communicated this result to Mikhalkin, who then also confirmed it in his approach [41].

The idea of construction is motivated by Strominger, Yau and Zaslow's (SYZ) conjecture [56] and the construction of cycles in [47]. Parallel results without connection to enumerative geometry have very recently been achieved independent from us in the situation where the symplectic manifold is noncompact [38], [37], [25, 26], a toric variety [41], [27] or a torus bundle over torus [54].

Roughly speaking, if there is a Lagrangian torus fibration for a Calabi-Yau manifold and a tropical curve γ in the base integral affine manifold such that all edges of γ have weight one, then it is easy to construct for each edge e of γ a Lagrangian torus times interval L_e lying above e, and for each trivalent vertex v of γ a Lagrangian pairs of pants times torus L_v lying above a small neighbourhood of v. Moreover, these local pieces can be constructed in a way that can be patched together smoothly, resulting in a Lagrangian submanifold L°_γ. Furthermore, if γ hits the discriminant at the end points appropriately, then L°_γ can be closed up to a closed embedded Lagrangian L_γ, whose diffeomorphism type is determined by the combinatorial type of γ *and* the local monodromy at points where the discriminant is hit. We will explain this in more details in Subsection 2.6. We call L_γ a tropical Lagrangian over γ. The key point is that this construction is straightforward only when we have been given a Lagrangian torus fibration. However, the only compact Kähler Calabi-Yau threefolds that knowingly admit a Lagrangian torus fibration are torus bundles over a torus.

Our actual construction starts with a family of smooth threefold hypersurfaces $M_t \subset \mathbb{P}_\Delta$ in a toric 4-orbifold $\mathbb{P}_\Delta$ degenerating to $M_0 = \partial\mathbb{P}_\Delta$, the toric boundary divisor of $\mathbb{P}_\Delta$ with the reduced scheme structure. Let $(\partial\mathbb{P}_\Delta)_{\mathrm{Sing}}$ be the locus of singular points of $\partial\mathbb{P}_\Delta$, $(\partial\mathbb{P}_\Delta)_{\mathrm{Smooth}} := \partial\mathbb{P}_\Delta \setminus (\partial\mathbb{P}_\Delta)_{\mathrm{Sing}}$, $\mathrm{Disc} := M_t \cap (\partial\mathbb{P}_\Delta)_{\mathrm{Sing}}$ be the discriminant, $\pi_\Delta : \mathbb{P}_\Delta \to \Delta$ be the moment map and $\mathcal{A} := \pi_\Delta(\mathrm{Disc})$. Suppose that $\mathbb{P}_\Delta$ has at worst isolated Gorenstein orbifold singularities. The singularities are necessarily at the preimage of vertices of Δ under π_Δ, and thus M_t is a smooth threefold for $|t| > 0$ small.

Starting with a reflexive polytope Δ_X, [21] exhibited a Minkowski summand Δ' so that $\Delta = \Delta_X + \Delta'$ has the property that $(\partial\Delta, \mathcal{A})$ is *simple* ([23], Definition 1.60). We equip $\partial\Delta \setminus \mathcal{A}$ with an integral affine structure using the integral affine structure on $\pi_\Delta((\partial\mathbb{P}_\Delta)_{\mathrm{Smooth}})$ and the fan structure at the vertices; see Definition 3.13 in [21], Example 1.18 in [23]. Therefore, we can define tropical curves γ in $(\partial\Delta, \mathcal{A})$; see Definition 1 below. We require $\gamma \cap \mathcal{A}$ to be the set of univalent vertices of γ. Let Λ be the local system of integral tangent vectors on $\partial\Delta \setminus \mathcal{A}$. When γ is rigid and Λ is trivialisable over γ, we can associate to γ its multiplicity $\mathrm{mult}(\gamma)$ defined in [36] (cf. [35], [42]), which we recall in Subsection 2.7. The multiplicity depends on the directions of edges of γ as well as the monodromy action around $\mathcal{A}$ near the univalent vertices of γ. We call a tropical curve γ *admissible* if for each univalent vertex v, there is a neighbourhood O_v of v such that $O_v \cap \mathcal{A}$ is an embedded curve (rather than two-dimensional). The tropical lines that end on the 'internal edges of the quintic curves' in a mirror quintic threefold are admissible; see Lemma 2.5 – in the example of Subsection 2.5, more than half of the lines are admissible. Moreover, an admissible tropical curve determines a diffeomorphism type of a 3-manifold in the way that the diffeomorphism type of a tropical Lagrangian over γ is determined by γ. By slight abuse of terminology, we call the diffeomorphism type determined by γ a Lagrangian lift of γ (see Subsection 2.6). We denote the ϵ-neighbourhood of γ with respect to the Euclidean distance on Δ by $W_\epsilon(\gamma)$. Our main theorem is the following.

Theorem 1.1. *Let γ be an admissible tropical curve in $\partial\Delta$. For any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $0 < |t| < \delta$, there is a closed embedded Lagrangian $L \subset M_t$ such that $\pi_\Delta(L) \subset W_\epsilon(\gamma)$ and L is diffeomorphic to a Lagrangian lift of γ. Moreover, whenever* $\mathrm{mult}(\gamma)$ *is well defined, we have* $w(L) = \mathrm{mult}(\gamma)$.

Remark 1.2. For a discussion of nonadmissible tropical curves, see Remark 6.3, 6.20.

Remark 1.3. Though our main examples are mirror quintics, Theorem 1.1 applies to all admissible tropical curves that arise from the setup in [21] explained above. For example, the tropical curves are not necessarily simply connected.

Remark 1.4. Though $\mathrm{mult}(\gamma)$ is only defined when Λ is trivialisable over γ [36], in view of Theorem 1.1, it is tempting to define $\mathrm{mult}(\gamma)$ by $w(L)$ when Λ is not trivialisable. We believe that this definition will have application to enumerative problems in algebraic/tropical geometry.

Remark 1.5. One can easily generalise $w(L) = \mathrm{mult}(\gamma)$ to all dimensions (see Remark 2.8). However, it was pointed out to us by Joyce that we do not expect a special Lagrangian counting invariant in dimensions higher than 4.

Proof (Sketch of proof of Theorem 1.1). The construction is divided into two parts: for the geometry away from the discriminant and near the discriminant. Both constructions rely heavily on the fact that we can isotope M_t symplectically to a nice symplectic hypersurface in local coordinates, as long as the isotopy is away from the discriminant and does not produce new discriminant (see Lemma 4.1).

For the construction away from the discriminant, we isotope M_t to a standard form (Lemma 5.3) in a chart such that M_t admits a local Lagrangian torus fibration and we can construct a local Lagrangian from the torus fibration (Proposition 5.4). We have to deal with compatibility of standard forms (Lemma 5.8), transition of symplectic charts (Corollary 5.11) and the trivalent vertices of γ (Lemma 5.15). The outcome will be an embedded Lagrangian with toroidal boundaries such that the π_Δ-image lies in a small neighbourhood of γ.

Then we need to close up the Lagrangian with toroidal boundaries by Lagrangian solid tori near the discriminant, which is the essential part of the construction. The basic idea is that we can deform M_t to a particular M such that we have complete control away from the discriminant. We find an appropriate open subset V of M that is an exact symplectic manifold with contact boundary (Proposition 6.28) and we have complete control near the contact boundary of V. We show that V is a symplectic bundle over an annulus and we use our control near ∂V to show that the boundaries of the fibres are standard contact S^3. By a famous result of Gromov, each fibre is symplectomorphic to an open 4-ball (Theorem 6.8). There is a Legendrian T^2 inside ∂V, which is an S^1-bundle over S^1 with respect to the symplectic 4-ball fibre bundle structure on V. This Legendrian T^2 can be filled by a Lagrangian solid torus in V by a soft symplectic method (Proposition 6.18), which gives the Lagrangian solid torus we need.

Once the Lagrangian is constructed, the statement that $w(L) = \mathrm{mult}(\gamma)$ follows from a simple calculation using Čech cohomology (see Subsection 2.7) that was independently obtained in [41] by a different argument after a presentation of our result given by the second author in 2017. □

Application to symplectic topology

Let $\mathcal{S}$ be the set of admissible tropical lines in $(\partial\Delta, \mathcal{A})$ associated with a pencil of mirror quintics. We can show that the Lagrangians constructed by Theorem 1.1 are homologous and non-Hamiltonian isotopic in the following sense.

Theorem 1.6. *Let $\gamma \in \mathcal{S}$ and L_γ be a Lagrangian obtained by Theorem 1.1. For any $\gamma' \in \mathcal{S}$, we can get a Lagrangian $L_{\gamma'}$ by Theorem 1.1 such that $[L_\gamma] = [L_{\gamma'}] \in H_3(M, \mathbb{Z})$. Moreover, if $\gamma \cap \gamma' = \emptyset$, then L_γ is not Hamiltonian isotopic to $L_{\gamma'}$.*

Theorem 1.6 gives a large number of pairwise homologous but non-Hamiltonian isotopic Lagrangian rational homology spheres, which is a rare application to symplectic topology in the literature.

When L is diffeomorphic to a free quotient of a sphere by a finite subgroup of $SO(4)$, we can define a Dehn twist along L, which is an element in the symplectomorphism group $\mathrm{Symp}(M)$ of M. It is easy to deduce the following from Theorem 1.1.

Corollary 1.7. *Let $k_{\max}$ be the maximum number of disjoint tropical curves satisfying Theorem 1.1 such that for each $i = 1, \ldots, k_{\max}$, the corresponding L_i is a spherical manifold. Then $\pi_0(\mathrm{Symp}(M))$ contains an abelian subgroup isomorphic to $\mathbb{Z}^{k_{\max}}$.*

Note that, in generic situations, most tropical curves are disjoint from the others, so Corollary 1.7 gives a large rank of abelian subgroup in $\pi_0(\mathrm{Symp}(M))$.

By a computer-aided search for a particular symplectic mirror quintic, we found 354 pairwise disjoint admissible tropical lines giving 312 Lagrangian S^3 and 42 Lagrangian $\mathbb{RP}^3$ in the mirror quintic, all of which are pairwise disjoint. The total number of admissible tropical lines in our example is 1, 451, out of which 1, 406 have multiplicity one (giving Lagrangian S^3s) and 45 have multiplicity two (giving Lagrangian $\mathbb{RP}^3$s). The total number of tropical lines in our example is, however, 2, 785, out of which 2, 695 have multiplicity one and 90 have multiplicity two, so the weighted sum is indeed 2,875. Their adjacency matrix has full rank, which implies that every tropical line intersects some other tropical line. Inspection of the centre of Figure 3 gives an impression of the meeting of tropical lines, yet tropical lines also meet others across components of the degenerate Calabi-Yau unlike what is possibly expected. We do not know whether this is a general phenomenon or due to possibly not having picked the most general deformation. We chose a random small perturbation of the subdivision given in Subsection 2.4.

Structure of the article

In Section 2 we give some background of SYZ mirror symmetry and the tropical curves in the affine base. We also explain the topology of the Lagrangians and derive some consequences, including Theorem 1.6, by assuming Theorem 1.1, which is proved in the subsequent sections. In Section 3, we review toric geometry from a symplectic perspective. In Section 4, we explain how to perform symplectic isotopy away from the discriminant for our pencil of hypersurfaces. Then we explain the construction of the Lagrangians away from the discriminant and near the discriminant in Sections 5 and 6, respectively. We conclude the proof of Theorem 1.1 in Subsection 6.8.

2. From 2,875 lines on the quintic to Lagrangians in the quintic mirror

The toy model of the SYZ mirror symmetry conjecture is the following. Set $V = \mathbb{R}^n$ and let TV and T^*V denote the tangent and cotangent bundle. Let $T_{\mathbb{Z}}V$ denote the local system on V of integral tangent vectors (using the lattice $2\pi\mathbb{Z}^n$ in $\mathbb{R}^n$). The quotient $TV/T_{\mathbb{Z}}V$ is an $(S^1)^n$-bundle over V. Similarly, we can define $T^*_{\mathbb{Z}}V$ and another $(S^1)^n$-bundle $T^*V/T^*_{\mathbb{Z}}V$. We arrive at dual torus fibrations over V,

$$X := TV/T_{\mathbb{Z}}V \to V \leftarrow T^*V/T^*_{\mathbb{Z}}V =: \check{X},$$

where the left one carries a natural complex structure with complex coordinates $(z_j)_{j=1,\ldots,n}$ given by $(x, \alpha = \sum_j y_j \partial/\partial x_j) \mapsto (z_j = x_j + \sqrt{-1}y_j)_{j=1,\ldots,n}$ and x_j the jth coordinate on V. The right one carries a natural symplectic structure inherited from the canonical one of T^*V. Part of the conjecture of SYZ is that mirror symmetry is locally of this form. Unless we are talking about complex tori, in practice there are also singular torus fibres in these bundles for Euler characteristic reasons, and we will get back to this.

Note that this toy model gives insight on how a complex submanifold ought to become a Lagrangian submanifold of the mirror dual (see Subsection 6.3 of [3]). If W is an integrally generated linear subspace of V, then $TW/T_{\mathbb{Z}}W$ is naturally a complex submanifold of X. On the other hand, $W^\perp/(W^\perp \cap T^*_{\mathbb{Z}}V)$ as a subbundle of $T^*V/T^*_{\mathbb{Z}}V$ supported over W is a Lagrangian submanifold of $\check{X}$. To reach sufficient generality, one needs to run this construction for the situation where W is a tropical variety; that is, a polyhedral complex. At a general point it still looks just like the above but then pieces are glued nontrivially when polyhedral parts meet another. However, this does not produce a differentiable submanifold, let alone complex or Lagrangian. Improvements on the symplectic side can be made by thickening the tropical W to an amoeba (see [37]). In this article, we are only interested in the situation where W is one-dimensional, so a tropical curve, and the focus will be put on constructing closed Lagrangian submanifolds in Calabi-Yau threefolds using tropical curves. Whenever $\check{X}$ compactifies to a projective toric variety and the tropical curve attaches to the codimension two strata in the moment polytope in particular ways, Mikhalkin recently gave a construction of closed Lagrangian submanifolds in the projective toric variety [41]. On the other hand, no Lagrangian torus fibration is known for any simply connected compact Calabi-Yau threefold. This is the situation that we are interested in, which is also the subject of the SYZ conjecture. Luckily, most Calabi-Yau threefolds permit degenerations to a reducible union of toric varieties, introducing the toric techniques we lay out for the quintic and its mirror dual in the next sections.

2.1. The quintic threefold and its symplectic mirror duals

The most famous Calabi-Yau threefold is the quintic X in $\mathbb{CP}^4$. Its mirror dual $\check{X}$ is a crepant resolution of an anticanonical hypersurface in the weighted projective space $\mathbb{P}_{\Delta_{\check{X}}}$ associated to the lattice simplex

$$\Delta_{\check{X}} = \operatorname{conv}\left\{e_1, \dots, e_4, -\sum_j e_j\right\}.$$

One finds $\mathbb{P}_{\Delta_{\check{X}}} \cong \mathbb{CP}^4/(\mathbb{Z}_5)^3$. As progress towards nailing the SYZ conjecture for the quintic, Mark Gross [20, Theorem 4.4] gave a topological torus fibration on a space that is diffeomorphic to X and Matessi and Castaño-Bernard [6] showed that this one can be upgraded to a piecewise smooth Lagrangian fibration for some symplectic structure and a similar approach works for $\check{X}$. Recently, Evans-Mauri [15] gave a Lagrangian fibration local model for parts of the fibration that are most difficult to deal with in dimension three. Whether this can be used for global compactifications of fibrations or tropical Lagrangian attachment problems presumably requires a similarly careful analysis as for the situations that we are going to consider. For recent progress on the SYZ topology, we refer to [49, 48, 50].

We are not working on the diffeomorphic model but on the actual symplectic quintic mirror $\check{X}$; in fact, our construction applies to Lagrangians in each of the many possibilities resulting from different choices of a crepant resolution for the quintic mirror ($h^{1,1}(\check{X}) = 101$). For our construction, it suffices to have a Lagrangian torus fibration *locally* around the Lagrangian L that we wish to construct from a tropical curve γ. To say where γ lives, we make use of the construction of the real affine base space of the torus fibration from [20].

The Newton polytope of the quintic is the polar dual to $\Delta_{\check{X}}$; that is, the convex hull $\operatorname{conv}\{0, 5e_1, \dots, 5e_4\}$ translated by $(-1,-1,-1,-1)$ so that its unique interior lattice point $(1,1,1,1)$ becomes the origin. We call the resulting polytope Δ_X. Choosing $a_m \in \mathbb{R}_{\geq 0}$ for each $m \in \partial\Delta_X \cap \mathbb{Z}^4$ yields a cone in $\mathbb{R}^4 \oplus \mathbb{R}$ generated by the set of (m, a_m) and its boundary gives the graph of a piecewise linear convex function $\varphi : \mathbb{R}^4 \to \mathbb{R}$. We require that every face in the boundary is a simplicial cone and we assume that each (m, a_m) generates a ray of this cone; in particular, $\varphi(m) = a_m$ for all m. There are lots of φ satisfying these properties, and each one gives a toric projective crepant partial resolution

$$\operatorname{res} : \mathbb{P}_\Delta \to \mathbb{P}_{\Delta_{\check{X}}},$$

where $\mathbb{P}_\Delta$ is given by the fan in $\mathbb{R}^4$ whose maximal cones are the maximal regions of linearity of φ. Equivalently, $\mathbb{P}_\Delta$ is given by the polytope

$$\Delta = \bigcap_{m \in \partial\Delta_X \cap \mathbb{Z}^4} \{n \in \mathbb{R}^4 | \langle n, m\rangle \geq -a_m\} = \{n \in \mathbb{R}^4 | \langle n, m\rangle \leq \varphi(m) \text{ for all } m \in \mathbb{R}^4\}. \tag{1}$$

Lemma 2.1. $\mathbb{P}_\Delta$ *has at worst isolated Gorenstein orbifold singularities.*

Proof. If σ is a maximal cone in the fan – that is, a maximal region of linearity of φ – then it is simplicial by assumption and hence generated by $m_1, \dots, m_4 \in \partial\Delta_X$, say. Moreover, these generators are all contained in a single facet F of Δ_X because the fan refines the normal fan of $\Delta_{\check{X}}$. By the assumption that each $(m, \varphi(m))$ for $m \in \partial\Delta_X \cap \mathbb{Z}^4$ is a ray generator, we find $F \cap \sigma \cap \mathbb{Z}^4 = \{m_1, \dots, m_4\}$. So σ is a cone over the elementary lattice simplex given by the convex hull of $m_1, \dots, m_4$, and thus gives a terminal toric Gorenstein orbifold singularity, and these have codimension four. □

Let w_j be the monomial associated to e_j. Consider the (singular) hypersurface in $\mathbb{P}_{\check{X}}$ given by an anticanonical section written as a Laurent polynomial on $(\mathbb{C}^*)^4$,

$$h = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \alpha_4 w_4 + \alpha_0 + \alpha_5 (w_1 w_2 w_3 w_4)^{-1},$$

with $\alpha_j \in \mathbb{C}^*$ and $-5\alpha_0^5 \neq \prod_{i=1}^5 \alpha_i$ (so that $h = 0$ gives a submanifold of $(\mathbb{C}^*)^4$; cf. [4, §2]). The monomial exponents of h are precisely the lattice points of $\Delta_{\check{X}}$. The closure of $h = 0$ in $\mathbb{P}_\Delta$ misses the isolated orbifold points at zero-dimensional strata (Lemma 2.1) and gives a symplectic 6-manifold $\check{X}$ with symplectic structure induced from $\mathbb{P}_\Delta$. Furthermore, $\check{X}$ is a Calabi-Yau manifold because it agrees with the crepant resolution under res of the anticanonical hypersurface, the closure of $h = 0$ inside $\mathbb{P}_{\Delta_{\check{X}}}$. By deforming the $a_m \in \mathbb{R}_{\geq 0}$, one can study continuous deformations of the symplectic structure. The space of crepant symplectic resolutions acquires an interesting chamber structure with a point on a wall given by a set of a_m that violates the simplicialness of φ. Just as a remark: the wall geometry is governed by the *secondary polytope* of Δ_X.

2.2. *The real affine manifold and tropical curves*

Following [21], we next explain how to give a real integral affine structure on a large open subset of $\partial\Delta$ where Δ is a lattice polytope obtained from a φ as in Subsection 2.1. We split Δ as a Minkowski sum

$$\Delta = \Delta_{\check{X}} + \Delta',$$

where Δ' is the polytope associated to the piecewise linear function φ' that takes value $\varphi(m) - 1$ at $m \in \partial\Delta_X \cap \mathbb{Z}^4$. Indeed, this gives a decomposition as claimed because $\Delta_{\check{X}}$ is the polytope of the function $\varphi_{\check{X}}$ taking value 1 on all of $\partial\Delta_X$, so $\varphi = \varphi_{\check{X}} + \varphi'$. By the decomposition, every vertex v of Δ is uniquely expressible as $v = v_{\check{X}} + v'$ for $v_{\check{X}}$ a vertex of $\Delta_{\check{X}}$ and v' a vertex of Δ'. We project a small neighbourhood $W_v \subset \partial\Delta$ of v onto the quotient of the affine four-space $[v + \mathbb{R}^4]$ that contains W_v by the affine line $[v + \mathbb{R}v_{\check{X}}]$ resulting an affine three-space. The projection is thus injective and thereby gives a real affine chart for W_v. There is also an integral structure obtained by complementing $v_{\check{X}}$ to a lattice basis of $\mathbb{R}^4$ to find a lattice for the quotient. We do this for each vertex v of Δ. Furthermore, for each facet F of Δ, its interior $\mathrm{Int}(F)$ carries a natural integral affine structure from the tangent space to the facet. Combining the resulting charts W_v with the interiors of facet $\mathrm{Int}(F)$ yields an atlas on the union of these charts for an integral affine structure; that is, transitions in $\mathrm{GL}_3(\mathbb{Z}) \ltimes \mathbb{R}^3$. By choosing W_v suitably, the complement of the union of charts can be made to be

$$\mathcal{A} := \pi_\Delta(\check{X} \cap \mathbb{P}_\Delta^{[2]}) = \pi_\Delta(\check{X}) \cap \partial\Delta^{[2]},$$

where $\mathbb{P}_\Delta^{[2]}$ is the union of complex two-dimensional strata, $\partial\Delta^{[2]}$ is the union of two-cells of Δ and $\pi_\Delta : \mathbb{P}_\Delta \to \Delta$ is the moment map for the Hamiltonian $(S^1)^4$-action on $\mathbb{P}_\Delta$. We do not need Δ to be a lattice polytope for this construction. The affine structure is integral affine because $\Delta_{\check{X}}$ is a lattice polytope. Let Λ denote the local system of integral tangent vectors on $\partial\Delta \setminus \mathcal{A}$ (we also used $T_\mathbb{Z}$ before).

Definition 1. A *tropical curve* in $(\partial\Delta, \mathcal{A})$ is a graph γ (realised as a topological space) together with a continuous injection $h : \gamma \to \partial\Delta$ such that

1. a vertex of γ is either univalent or trivalent,
2. $h(v) \in \mathcal{A} \iff v$ is a univalent vertex of γ,
3. the image of the interior of an edge e is a straight line segment in the affine structure of $\partial\Delta \setminus \mathcal{A}$ of rational tangent direction,
4. for v with $h(v) \in \mathcal{A}$, the primitive tangent vector of the adjacent edge e generates the image of $T_\nu - \mathrm{id}$ for T_ν the monodromy of Λ along any nontrivial simple loop ν around $\mathcal{A}$ in a small neighbourhood of v,

5. for every trivalent vertex v, and $e_1, e_2, e_3 \in \Lambda_v$ the primitive tangent vectors into the outgoing edges, we have $e_1 + e_2 + e_3 = 0$ and e_1, e_2 span a saturated sublattice of Λ_v.

We consider two tropical curves Γ_1, Γ_2 the same if there exists a homeomorphism $\Gamma_1 \to \Gamma_2$ that commutes with h_1, h_2. By slight abuse of notation, we also use γ to refer to the image of h. The following lemma guarantees that we can always satisfy (4) above as long as the tropical curve approaches $\mathcal{A}$ from the right direction. The lemma directly follows from the aforementioned simplicity of $(\partial\Delta, \mathcal{A})$; cf. [20].

Lemma 2.2. *Let $x \in \partial\Delta \setminus \mathcal{A}$ be a point contained in a small neighbourhood V so that $(V, V \cap \mathcal{A})$ is homotopic to (D, p) as a pair, where D is an open disc and p is a point in D. Let $\nu \in \pi_1(V \setminus \mathcal{A}) = \pi_1(D \setminus p) \cong \mathbb{Z}$ be a generator and $T_\nu : \Lambda_x \to \Lambda_x$ the monodromy of Λ along ν. In a suitable basis of $\Lambda_x \cong \mathbb{Z}^3$, T_ν is given by $\left(\begin{smallmatrix} 1&0&0\\ 0&1&1\\ 0&0&1 \end{smallmatrix}\right)$. In particular, the image of $T_\nu - \mathrm{id}$ is saturated of rank one; that is generated by a primitive vector.*

2.3. Katz's methods for finding lines on a quintic

The quintic X permits a flat degeneration to the union of coordinate hyperplanes simply by interpolation. If X is given by the homogeneous quintic equation f_5 in the variables $u_0, \ldots, u_4$ then we define the family of hypersurfaces in $\mathbb{CP}^4$ varying with t by

$$u_0 \cdot \ldots \cdot u_4 + t f_5 = 0$$

and denote by X_0 the fibre with $t = 0$. Because X_0 is the union of five projective spaces, it contains infinitely many lines. However, only a finite number of them deform to the nearby fibres, worked out by Katz in [53]. Assuming f_5 is general, the intersection of X with each coordinate two-plane is a smooth complex quintic curve. There are 10 of these.

Theorem 2.3 (Katz). *A line in X_0 deforms into the nearby fibre if and only if it does not meet any coordinate line of $\mathbb{P}^4$ but meets four of the* 10 *quintic curves.*

Note that it follows that a line that deforms needs to be contained in a unique irreducible component $H \cong \mathbb{P}^3$ of X_0 and needs to meet the four quintic curves that are contained in this component, namely, the intersections of the four coordinate planes of H with X. A general quintic hosts $2,875 = 5 \cdot 575$ many lines and in the degeneration, each H contains 575 deformable lines [53]. On the dense algebraic torus $(\mathbb{C}^*)^3$ of $\mathbb{CP}^3$, we may apply the map $(\mathbb{C}^*)^3 \to \mathbb{R}^3$ given by $\log|\cdot|$ for each coordinate. Each line maps to an amoeba with four legs going off to infinity in the directions of the rays in the fan of the toric variety $\mathbb{P}^3$. Furthermore, these legs 'meet the amoeba of the quintic plane curves at infinity'. We are not going to make this more precise because we only use this idea as inspiration. There is a closely related theorem that was our main motivation combined with Katz's findings.

Theorem 2.4 ([36]). *The number of tropical lines in $\mathbb{R}^3$ meeting* 5 *general quintic tropical curves at tropical infinity each in one of the four directions of the rays of the fan of $\mathbb{P}^3$ when counted with their tropical multiplicities agrees with the number of complex lines in $\mathbb{P}^3$ meeting five general quintic plane curves.*

So we may almost deduce from Katz's count of complex lines a count of tropical lines via this theorem. The only issue here is the attribute 'general'. Indeed, the quintic curves in Katz's situation are *not* in general position. If they were, the count would be $2 \cdot 5^4$ by standard Schubert calculus, but this number is way bigger than 575. Indeed, any pair of quintic curves meets each other in five points, which would not happen if they were in general position. They meet each other because they arise from the same equation $f_5 = 0$ restricted to each coordinate plane.

We expect that in the more special position where the tropical quintics meet each other, after removing degenerate tropical lines (meaning those that move in positive-dimensional families, meet vertices of the discriminant curve or do not have the expected combinatorial type $\rtimes\!\!\!\ltimes$), then one actually finds 575

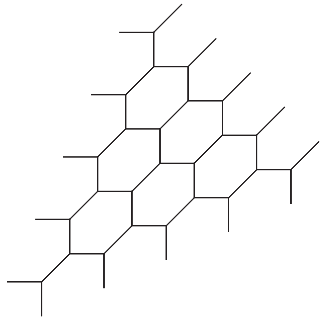


Figure 1. *The tropical plane quintic curve that describes part of the discriminant $\mathcal{A}$ of $\partial\Delta$.*

when counting these with multiplicity. We verify this below in a global example. Before going into its details, let us clarify why tropical lines in $\mathbb{R}^3$ that meet tropical quintics at infinity relate to $(\partial\Delta, \mathcal{A})$ in the sense of Definition 1. For $s \in [0,1]$, setting $\Delta_s = \Delta_{\check{X}} + s\Delta'$, we observe $\Delta_0 = \Delta_{\check{X}}$ and $\Delta_1 = \Delta$. In this sense, Δ is a deformation of $\Delta_{\check{X}}$ and note that Δ_s has the same combinatorial type for all $s > 0$. Recall the notion of the discrete Legendre transform from [22, 23, 46]. Because Δ_X and $\Delta_{\check{X}}$ are polar duals, their boundaries are discrete Legendre dual [23, Example 1.18]. The subdivided boundary of Δ_X by means of φ is the discrete Legendre dual to $\partial\Delta$. For a 2-cell τ in $\partial\Delta$, there are three possibilities for what its deformation $\bar{\tau}$ in $\Delta_0 = \Delta_{\check{X}}$ can be, namely, zero-, one- or two-dimensional. These cases match with whether its dual (one-dimensional) face $\check{\tau}$ in the subdivision of $\partial\Delta_X$ lies in a 3-, 2- or 1-cell of Δ_X. Most important, because the subdivision of Δ_X by φ governs the composition $\mathbb{P}_\Delta \xrightarrow{\text{res}} \mathbb{P}_{\Delta_{\check{X}}} \to \Delta_{\check{X}}$, the following holds.

Lemma 2.5. *Let $\tau \subset \partial\Delta$ be a 2-cell. Recall the monomials $w_1, \dots, w_4$. We set $w_5 := (w_1 w_2 w_3 w_4)^{-1}$ and $i \in \bar{\tau}$ means that the vertex of $\Delta_{\check{X}}$ corresponding to w_i is contained in $\bar{\tau}$. The amoeba part $\mathcal{A} \cap \tau$ is given by*

$$g_{\tau,0} := \sum_{i \in \bar{\tau}} \alpha_i w_i$$

as an equation on the torus orbit dense in the stratum of $\mathbb{P}_\Delta$ given by τ.

In particular, for τ deforming to an edge of $\Delta_{\check{X}}$, $g_{\tau,0}$ is a binomial. Also note that $\mathcal{A} \cap \tau = \emptyset$ if $\bar{\tau}$ is a 0*-cell.*

Note that $g_{\tau,0}$ is a binomial if and only if the corresponding amoeba is one-dimensional and hence tropical curves ending on it will be admissible (see also Remark 6.2).

If φ is a unimodular subdivision – for example, as in Figure 2 – then most 2-cells of $\partial\Delta$ have $\mathcal{A} \cap \tau = \emptyset$, there are $5 \cdot 10$ many 2-cells of $\partial\Delta$ that deform to triangles in $\Delta_{\check{X}}$ but most interesting for us, $10 \cdot 30$ 2-cells deform to edges; hence their amoeba is given by a binomial. These amoeba pieces arrange as 10 plane quintic curves; for example, as in Figure 1. (One verifies that indeed the number of interior edges is 30 here.) Each quintic curve is dual to the triangulation of a 2-face of Δ_X; for example, consider the front face in the right-hand part of Figure 2. Each facet of Δ_X contains four triangle faces; hence, dually, four quintic curves arrange together as the boundary of a space tropical quintic surface in $\mathbb{R}^3$. In particular, we can view them as lying at infinity and because they make up the discriminant in $\partial\Delta$, a tropical line in $\mathbb{R}^3$ with ends on the four quintics thus gives a tropical curve in $\partial\Delta$. There are a lot of these (see Figure 3), and most of them are admissible; that is, they meet one of the 30 inner edges of each quintic, rather than the 15 outer ones. Also note that this configuration appears five times in the boundary of $\partial\Delta$.

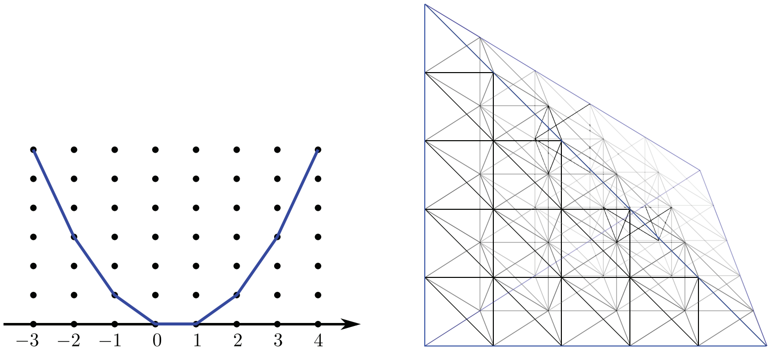


Figure 2. *Graph of φ_0 and Weyl-A_3 subdivision of a facet of the moment polytope of the quintic threefold (obtained from φ_1).*

2.4. A very symmetric subdivision and resolution of the quintic mirror

We next give an example for $\mathbb{P}_\Delta$ that is even a manifold (see also [20, p. 122: Figure 4.6]). The subdivision of each facet of $\partial\Delta_X$ is obtained from the affine Weyl chambers of type A_3; cf. [32, III,§2]. Concretely, let $\varphi_0 : \mathbb{R} \to \mathbb{R}$ be the unique continuous convex function that is linear on each connected component of $\mathbb{R} \setminus \mathbb{Z}$, changes slope by 1 at each point in $\mathbb{Z}$ and is constantly zero on $[0, 1]$ (see Figure 2). One finds $\varphi_0(n) = n(n-1)/2$ for $n \in \mathbb{Z}$ ('discrete parabola'). Now consider the piecewise affine function $\varphi_1 : \mathbb{R}^4 \to \mathbb{R}$ given by

$$\begin{aligned}
\varphi_1(x_1, x_2, x_3, x_4) = {} & \varphi_0(x_1) + \varphi_0(x_2) + \varphi_0(x_3) + \varphi_0(x_4) \\
& + \varphi_0(x_1 + x_2) + \varphi_0(x_2 + x_3) + \varphi_0(x_3 + x_4) \\
& + \varphi_0(x_1 + x_2 + x_3) + \varphi_0(x_2 + x_3 + x_4) \\
& + \varphi_0(x_1 + x_2 + x_3 + x_4),
\end{aligned}$$

and finally define φ as the unique piecewise linear function on $\mathbb{R}^4$ that coincides with φ_1 on $\partial\Delta_X$. For $m \in \partial\Delta_X \cap \mathbb{Z}^4$, set $a_m = \varphi(m)$ and recall from Subsection 2.1 that φ is entirely determined from the set of a_m.

The induced subdivisions of any two facets are isomorphic and look like the right-hand side of Figure 2. One checks that each four-dimensional cone in the fan given by φ is lattice-isomorphic to the standard cone $\mathbb{R}^4_{\geq 0} \subset \mathbb{R}^4$, so the resulting $\mathbb{P}_\Delta$ is smooth. In our explicit example below, we will use a slight perturbation replacing a_m by $a_m + \varepsilon_m$ for random ε_m to increase our chance of being in a generic situation. Plugging the perturbed a_m into (1) yields a slightly deformed Δ, and though the complex manifold $\mathbb{P}_\Delta$ does not change, this slightly perturbs the symplectic form in a well-understood manner. The best way to understand what $(\partial\Delta, \mathcal{A})$ looks like is by considering its discrete Legendre dual. The subdivision of Δ_X by φ is five copies of the right-hand side of Figure 2 glued along facets. Therefore, after identification, there are ten 2-faces, each carrying a subdivision that is dual to that of a quintic curve in its most symmetric form shown in Figure 1.

2.5. The findings of a computer search for the tropical lines

As described in the previous section, we obtained a particular $\partial\Delta$ as a small perturbation of φ that gave the very symmetric subdivision of Δ_X. From Katz's work as described in Subsection 2.3, we are looking for tropical lines in $\partial\Delta$ that meet the quadruple of tropical quintic curves where each tropical quintic is dual to the subdivision of one of the 10 triangle faces of Δ_X. We used a computer for this search following the pseudo code[1].

[1]For more details, the complete code with instructions and results, see https://arxiv.org/src/1904.11780v3/anc

 https://doi.org/10.1017/fms.2020.54

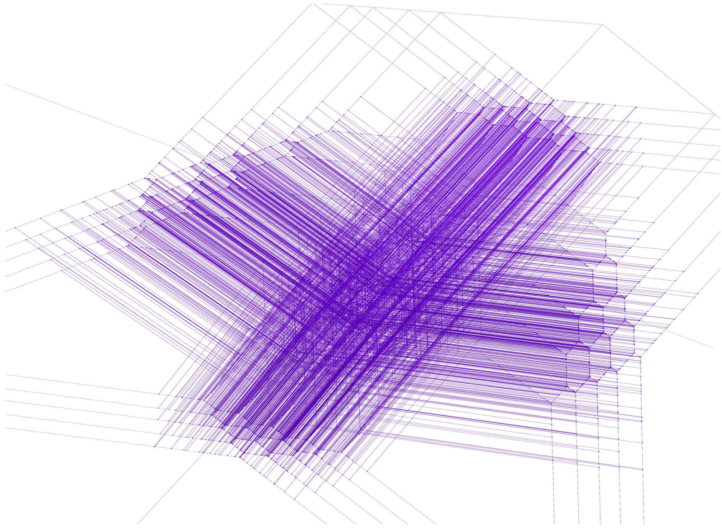


Figure 3. *Tropical lines in* $\mathbb{R}^3$ *meeting four tropical quintics at infinity, of total multiplicity* 575.

Algorithm 1: Tropical line search

Data: Unimodular regular subdivision of the convex hull Δ_X of $0, 5e_1, \ldots, 5e_4$ in $\mathbb{R}^4$ by piecewise affine height function that is integral at integral points.
Result: Findings of all tropical lines.
compute the tropical quintic threefold inside $\mathbb{R}^4$ associated to the height function;
compute the five tropical quintic surfaces S_i ($i = 1, \ldots, 5$) at the five asymptotic directions of infinity (each sitting inside an $\mathbb{R}^3$);
compute for each of these quintic surfaces S_i the quintic curves C_{ij} ($j = 1, \ldots, 4$) at the respective four directions of infinity. Each C_{ij} has 45 edges;
for $i \leftarrow 1, \ldots, 5$ **do**
    for *each of the* 45^4 *tuples* $(a_1, \ldots, a_4)$ *with* a_j *an edge of* C_{ij} **do**
        for *each of the three generic combinatorial types of a tropical line in* $\mathbb{R}^3$ **do**
            Check whether there exists a tropical line of the given type meeting $a_1, \ldots, a_4$ and, if so, record it.
        end
    end
end
From the recorded tropical lines, remove all those that are nonrigid (i.e., they are part of a positive-dimensional family) or are special (they meet vertices of the tropical quintics C_{ij}). The remaining ones are the result of the search.

After removing all lines that meet vertices of the quintics, that have only one internal vertex or are nonrigid (that is move in families), we did actually get the expected count – when counting with multiplicity (which is remarkable in view of [60, 44, 8]). That is, maybe surprisingly, the lines were not all of multiplicity one. We give the definition of the multiplicity in Subsection 2.7. For each of the five facets of Δ_X, the count with multiplicity of the tropical curves gave indeed 575, so in total 2,875 as expected. We found 2,695 curves of multiplicity one and 90 of multiplicity two. These 90 did not evenly distribute over the five facets: $15+16+18+20+21$. Though 90 is a number that has not appeared yet in the context of the quintic to our knowledge, one may speculate that relates to the count of real lines, which was found to be 15 in [55]: for rational curves on an elliptic surface, the presence of higher

Figure 4. *A pair of pants.*

multiplicity tropical curves is implied from the Welschinger invariant to differ from the GW invariant (see, e.g., [59, Section 4.2.2]).

The goal is to construct Lagrangian threefolds from these tropical curves. The remainder of this article carries this out for admissible curves. Recall that the requirement is that the tropical curve meets the discriminant amoeba $\mathcal{A}$ in points where this amoeba is one-dimensional. By Lemma 2.5, this holds true if the tropical line meets the internal edges of the quintic curves; that is, no outer edges. A bit more than half the curves feature this: we get $1,451$ admissible lines, out of which 45 have multiplicity two (multiplicity weighted account is $1,496$). Interestingly, the admissible curves do not meet curves of other facets (unlike nonadmissible ones), though possibly still other curves in their own facet. We found a set of 354 admissible lines that are pairwise disjoint, out of which 42 have multiplicity two.

2.6. Lagrangian lift of a tropical curve

In this section, we give the definition of the diffeomorphism type of a Lagrangian lift of a tropical curve γ in $(\partial\Delta, \mathcal{A})$ to the Calabi-Yau given by $\partial\Delta$; for example, the mirror quintic as before. Using the integral affine structure on $\partial\Delta \setminus \mathcal{A}$, we can define a Lagrangian torus bundle by

$$\check{X}^\circ := T^*(\partial\Delta \setminus \mathcal{A})/T^*_{\mathbb{Z}}(\partial\Delta \setminus \mathcal{A}). \tag{2}$$

Recall the notation $\Lambda = T_{\mathbb{Z}}(\partial\Delta \setminus \mathcal{A})$ and note that $\partial\Delta$ is an orientable topological manifold and so $\bigwedge^3 \Lambda \cong \mathbb{Z}$. Fixing an orientation once and for all, we can talk about oriented bases of stalks of Λ.

For each edge e of γ and a point x in the interior e° of e, we get the two-dimensional subspace $e^\perp$ of $T^*_x(\partial\Delta \setminus \mathcal{A})$ consisting of co-vectors that are perpendicular to the direction of e. By Definition 1 (3), every translation $e^\perp + a$ descends to an embedded 2-torus in $\check{X}^\circ$. A smooth family of these 2-tori over $x \in e^\circ$ defines a (trivial) torus bundle L_{e° over e° and the total space L_{e° is a Lagrangian submanifold in $\check{X}^\circ$. It extends over the vertices of e that do not lie in $\mathcal{A}$, and we let L_e denote the extension.

Remark 2.6. Let $f : e^\circ \to \mathbb{R}$ be a smooth function such that it descends to a compactly supported function $f' : e^\circ \to \mathbb{R}/2\pi\mathbb{Z}$. Given a smooth family of 2-tori over $x \in e^\circ$ as above, we can define a new family by fibrewise translating the 2-tori by $f(x)$. The resulting Lagrangian is a different embedding of a 2-torus times interval to $\check{X}^\circ$. The function f' being compactly supported corresponds to the fact that the two embeddings coincide near the ends of e°.

For each trivalent vertex v of γ, by Definition 1 (5), we can identify the primitive tangent vector of the outgoing edges as $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (-1,-1,0)$ with respect to a $\mathbb{Z}$-basis of $\Lambda_v \cong \mathbb{Z}^3$. Let $\tilde{L}_v$ be the subset of $T_v\partial\Delta/(T_v\partial\Delta)_{\mathbb{Z}} = (\mathbb{R}/2\pi\mathbb{Z})^3$ consisting of all points (q_1, q_2, q_3) such that

$$\begin{aligned} &\{q_1, q_2 \geq 0 \text{ and } q_1 + q_2 \leq \pi\} \text{ or} \\ &\{q_1, q_2 \leq 0 \text{ and } q_1 + q_2 \geq -\pi\}. \end{aligned} \tag{3}$$

Equipping $\tilde{L}_v$ with the subspace topology yields a finite CW complex of the same homotopy type as a pairs of pants times a circle. More explicitly, $\tilde{L}_v$ has a trivial circle factor given by the q_3-coordinate, and (3) defines two triangles in the q_1, q_2-coordinates and the vertices of the triangles are glued at $(q_1, q_2) = (0,0), (0,\pi), (\pi, 0)$, respectively (see Figure 4).

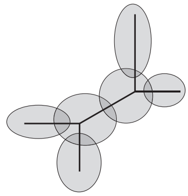


Figure 5. *An admissible Čech covering of a tropical curve.*

If we equip the two triangles in (3) with opposite orientations, then the boundary of them is exactly given by the circles $C_1 := \{q_1 = 0\}$, $C_2 := \{q_2 = 0\}$ and $C_3 := \{q_1 + q_2 = \pi\}$. For $i = 1, 2, 3$, the product of C_i with the circle in q_3-coordinate is exactly $e_i^\perp + a_i \subset T_v^*(\partial\Delta \setminus \mathcal{A})$, where $a_1 = a_2 = 0$ and $a_3 = \pi$. For an appropriate choice of orientations, one can see that the boundary of $\tilde{L}_v$ cancels the boundary of $\cup_{i=1}^3 L_{e_i}$ lying above v, yet $\tilde{L}_v \cup \bigcup_{i=1}^3 L_{e_i}$ is only a Lagrangian cell complex instead of a manifold. In Subsection 5.2, we explain how to replace the union of the triangles by a pair of pants and obtain a Lagrangian pair of pants times circle L_v that can be glued with $\cup_{i=1}^3 L_{e_i}$ smoothly.

Every univalent vertex v of γ lies in $\mathcal{A}$ by Definition 1(2). Let v and T_v be as in Definition 1(4), so, by Lemma 2.2, $T_v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ for a suitable basis. The primitive direction of the edge e adjacent to v is by assumption given by $\pm(0, 1, 0)$ so the 2-tori in L_e lying above e are generated by $\partial_{q_1}, \partial_{q_3}$. We can glue a solid torus L_v to the toroidal boundary component of L_e lying above v to cap off this boundary component. Moreover, we require that the circle generated by ∂_{q_3} is a meridian of L_v. It is useful to observe that ∂_{q_3} is characterised by being perpendicular to the invariant plane $\ker(T_v - \mathrm{id})$.

Definition 2. The *diffeomorphism type of a Lagrangian lift* of a tropical curve γ is the diffeomorphism type of the closed 3-manifold obtained by gluing L_v and L_e as above over all vertices v and edges e of γ.

2.7. Lagrangian weight versus tropical multiplicity

Following Joyce, we define the weight of a Lagrangian rational homology sphere L to be $w(L) := |H_1(L, \mathbb{Z})|$ and, more generally, $w(L) := |H_1(L, \mathbb{Z})_{\mathrm{tor}}|$. Let γ be a tropical curve in $(\partial\Delta, \mathcal{A})$. In this subsection, we explain how $w(L_\gamma)$ of a tropical Lagrangian L_γ can be computed by a Čech covering of the corresponding tropical curve γ. Because our Lagrangian L_γ is homotopic to the Lagrangian cell complex $\tilde{L}_\gamma$ that is built by $\tilde{L}_v$ instead of L_v at the trivalent vertices v (see Subsection 2.6), it suffices to compute the first homology of $\tilde{L}_\gamma$. For simplicity, we denote $\tilde{L}_\gamma$ by L_γ in this subsection. The universal coefficient theorem gives $(H_1(L_\gamma, \mathbb{Z}))_{\mathrm{tor}} = (H^2(L_\gamma, \mathbb{Z}))_{\mathrm{tor}}$, so we may compute $w(L_\gamma)$ via Čech cohomology.

A collection $\{U_j\}_{j=1}^m$ of open sets in $\partial\Delta$ that covers γ is called *admissible* (see Figure 5) if

1. $U_{j_1} \cap U_{j_2} \cap U_{j_3} = \emptyset$ whenever j_1, j_2, j_3 are pairwise distinct,
2. for all j, $\gamma \cap U_j$ is connected and it contains exactly one vertex of γ that is, by definition, either trivalent or univalent, and
3. for $j_1 \neq j_2$, $\gamma \cap U_{j_1} \cap U_{j_2}$ (which may be empty) contains no vertex.

For $\{U_j\}_{j=1}^m$ admissible, $H_i(U_j \cap \gamma, \mathbb{Z})$ and $H_i(U_{j_1} \cap U_{j_2} \cap \gamma, \mathbb{Z})$ are torsion free for all i, j, j_1, j_2 and therefore

$$w(L_\gamma) = \left| \mathrm{coker}\left(\bigoplus_j H^1(\pi_\Delta^{-1}(U_j) \cap L_\gamma, \mathbb{Z}) \xrightarrow{\Phi_\gamma} \bigoplus_{i<j} H^1(\pi_\Delta^{-1}(U_i \cap U_j) \cap L_\gamma, \mathbb{Z}) \right)_{\mathrm{tor}} \right|,$$

where the map Φ_γ is the Čech map on H^1 (restriction with sign). Recall from [36, 35, 42] the definition of multiplicity $\mathrm{mult}(\gamma)$ of a tropical curve γ. Applicable for us is [36, Equation (13)] because we need to consider tropical curves with constraints on unbounded edges (i.e., univalent vertices for us). Let γ° be the interior of γ and assume that we can trivialise Λ on γ°; that is, set $N := \Gamma(\gamma^\circ, \Lambda)$ and $N \cong \mathbb{Z}^3$. Furthermore, each univalent vertex v of γ gives a saturated rank two subspace A_v in N as the kernel of $T_v - \mathrm{id}$ near v. We view this as a constraint for the tropical curve γ in $N_\mathbb{R}$ in the sense of [36]. Given these constraints, [36, Equation (13)] provides a map of lattices Φ whose cokernel torsion gives the *tropical multiplicity* $\mathrm{mult}(\gamma)$ of γ.

Proposition 2.7. *The Čech map Φ_γ is quasi-isomorphic to the tropical multiplicity computing map Φ from [36, Equation (13)] and thus $w(L_\gamma) = \mathrm{mult}(\gamma)$.*

Proof. The assertion follows if one shows that there is a natural isomorphism

$$H^1(\pi_\Delta^{-1}(U_i \cap U_j) \cap L_\gamma, \mathbb{Z}) \cong N/\mathbb{Z}v$$

whenever $U_i \cap U_j \neq \emptyset$ and v is the primitive generator of the edge of γ that meets $U_i \cap U_j$ and an isomorphism

$$H^1(\pi_\Delta^{-1}(U_i) \cap L_\gamma, \mathbb{Z}) \cong N$$

whenever U_i contains a trivalent vertex and an isomorphism

$$H^1(\pi_\Delta^{-1}(U_i) \cap L_\gamma, \mathbb{Z}) \cong w^\perp/\mathbb{Z}v$$

whenever U_i contains a univalent vertex of γ, $w^\perp = \ker(T_v - \mathrm{id})$ and v the primitive generator of the image of $T_v - \mathrm{id}$. Furthermore, the restriction maps $H^1(\pi_\Delta^{-1}(U_j) \cap L_\gamma, \mathbb{Z}) \to H^1(\pi_\Delta^{-1}(U_i \cap U_j) \cap L_\gamma, \mathbb{Z})$ are supposed to be the natural maps under these isomorphisms. It is straightforward to check the isomorphisms and naturality of restriction maps from the local descriptions of L_v and L_e given in Subsection 2.6. □

Remark 2.8. Proposition 2.7 can be generalised to all dimensions for all tropical curves γ satisfying exactly the same set of conditions in Definition 1. The main reason is that, in higher dimensions, $\pi_\Delta^{-1}(U_i \cap U_j) \cap L_\gamma$ and $\pi_\Delta^{-1}(U_i) \cap L_\gamma$ split as a product and there is a trivial factor accounting for the extra dimensions. Moreover, the universal coefficient theorem gives $(H_1(L_\gamma, \mathbb{Z}))_{\mathrm{tor}} = (H^2(L_\gamma, \mathbb{Z}))_{\mathrm{tor}}$ no matter what the dimension is, so the same Čech cohomology calculation applies to conclude that $w(L_\gamma) := |(H_1(L_\gamma, \mathbb{Z}))_{\mathrm{tor}}| = \mathrm{mult}(\gamma)$.

2.8. Homology class of the Lagrangians

Recall from Section 4 in [45] that a tropical 2-cycle in an affine manifold B with singularities $\mathcal{A}$ is simply a sheaf homology cycle representing a class in $H_2(B, \iota_* \bigwedge^2 \Lambda)$ for $\iota : B \setminus \mathcal{A} \to B$ the inclusion of the regular part. Moreover, by (0.6) in [45], there is a homomorphism $r_2 : H_2(B, \iota_* \bigwedge^2 \Lambda) \to H_3(\check{X}, \mathbb{Z})/W_2$ with $W_2 = \mathrm{Im}(r_1) + \mathrm{Im}(r_0)$ for similar maps r_0, r_1, c.f. [49, 48]. For $\Gamma \in H_2(B, \iota_* \bigwedge^2 \Lambda)$, we simply refer to any lift of $r_2(\Gamma)$ from $H_3(\check{X}, \mathbb{Z})/W_2$ to $H_3(\check{X}, \mathbb{Z})$ by L_Γ.

Lemma 2.9. *There is a tropical 2-cycle $\Gamma \subset \partial\Delta$ whose associated 3-cycle L_Γ inside M_t has intersection number ± 1 with each Lagrangian L_γ constructed from a tropical line γ. Changing the orientation of L_γ if needed, we can thus assume this intersection number is $+1$.*

Proof. Recall from Subsection 2.3 that the subdivided boundary of Δ_X, call it $\check{B}$, is discrete Legendre dual to $B := \partial\Delta$. In particular, $\check{B}$ and B are homeomorphic with dual linear parts of their affine structures. This means the homeomorphism $\mathfrak{D} : B \to \check{B}$ identifies the local system $\check{\Lambda}_{\check{B}}$ of integral tangent vectors on $\check{B} \setminus \mathfrak{D}(\mathcal{A})$ with the similar local system Λ on $B \setminus \mathcal{A}$. We remark that $\mathfrak{D}(\mathcal{A})$ is contained in a neighbourhood of the union of 2-faces of $\partial\Delta_X$. Making use of $\mathfrak{D}$, in order to produce the desired tropical 2-cycle Γ, it therefore suffices to give a cycle for $H_2(\check{B}, \check{\iota}_* \bigwedge^2 \check{\Lambda}_{\check{B}})$ where $\check{\iota}$ is the inclusion $\check{B} \setminus \mathfrak{D}(\mathcal{A}) \hookrightarrow \check{B}$. Because

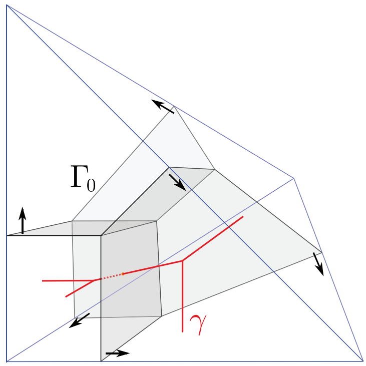


Figure 6. *Intersection of a tropical line γ and a tropical 2-cycle Γ in $\partial\Delta_X$ where Γ consists of five copies of the depicted cycle Γ_0, one copy for each facet of $\partial\Delta_X$. The line γ, however, is contained in a unique facet of $\partial\Delta_X$.*

$\check{B}$ is orientable, we have an isomorphism $\check{\iota}_* \bigwedge^2 \check{\Lambda}_{\check{B}} \cong \check{\iota}_*\Lambda_{\check{B}}$ for $\Lambda_{\check{B}}$ the dual of $\check{\Lambda}_{\check{B}}$. We may thus give a suitable cycle Γ representing a class in $H_2(\check{B}, \check{\iota}_*\Lambda_{\check{B}})$ in order to prove the lemma.

Recall that $\partial\Delta_X$ consists of five tetrahedra. Figure 6 shows one such tetrahedron containing a union Γ_0 of six polyhedral disks. The configuration Γ_0 can be described as a homeomorphic version of the compactification of the union of two-dimensional cones in the fan of $\mathbb{P}^3$. Each of the five facets of Δ_X contains such a configuration Γ_0, and we may move the five copies of Γ_0 so that they fit together to a cycle Γ. That is, Γ is actually a union of only 10 disks, each of which is glued from 3 disks that stem from different copies of Γ_0. The 10 disks of Γ are naturally in bijection with the edges of $\partial\Delta_X$; indeed, we simply match a disk with the edge that it meets (transversely).

As the next step, we need to attach a section in $\Gamma(D, \check{\iota}_*\Lambda_{\check{B}})$ to each of the 10 disks D so that the 10 sections satisfy the cycle condition at the 1-cells where disks meet (three at a time). We make use of the fact that the tangent space to a cell of the polyhedral decomposition of Δ_X is always monodromy-invariant for all monodromy transformations along loops in $U \setminus (\mathfrak{D}(\mathcal{A}) \cap U)$ for U a neighbourhood of the interior of the cell. In the case of a pair (D, e) of a disk D of Γ and the corresponding transverse edge e of Δ_X, we may choose a primitive generator v_D of the tangent direction to e as the section of $\Gamma(D, \check{\iota}_*\Lambda_{\check{B}})$ that we associate with D. Making use of the existence of an orientation of $\check{B}$, the sign of v_D and orientations of D can be chosen so that the cocycle condition on $\{v_D\}_D$ is satisfied and we have thus produced a valid cycle Γ as desired.

It remains to show that L_Γ satisfies the claimed intersection-theoretic property. For this purpose, we take the image of γ along $\mathfrak{D}$ and view γ as a cycle in $H_1(\check{B}, \check{\iota}_*\check{\Lambda})$. Theorem 7 in [45] says that the intersection number $L_\gamma.L_\Gamma$ agrees with the tropical intersection of γ and Γ. The tropical intersection number in turn is defined in item (3) of Theorem 6 in [45]. Note that γ and Γ have a unique point of physical intersection. We are left with verifying that the sections carried by γ and Γ at this point respectively pair to ± 1. The sections of γ carried by the outer legs are precisely generators for the perp space of the 2-cells of Δ_X that they meet. The balancing condition then implies what the section at the central edge of γ is. With this information and the knowledge that a disk D of Γ carries the section v_D that is a generator for the tangent space to the edge of Δ_X that is met by D, it is easy to see from Figure 6 that the tropical intersection of γ and Γ is indeed ± 1 (and invariance of the intersection number under deforming the cycles being given by Theorem 6 in [45]). □

For a fixed tropical line γ, there are more than one L_γ that can be constructed from Theorem 1.1 due to the freedom of choices in the construction. In particular, for each L_γ and any integer a, we can construct another Lagrangian $(L_\gamma)'$ by Theorem 1.1 such that the difference of their homology classes $[(L_\gamma)'] - [L_\gamma]$ is a times the torus fibre class. Using this freedom, we can prove the following.

Proposition 2.10. *If γ, γ' are two disjoint tropical lines, Lagrangians $L_\gamma, L_{\gamma'}$ can be constructed via Theorem 1.1 so that they are homologous.*

Proof. We use the well-known fact that the vanishing cycle $\alpha \cong T^3$ of the quintic mirror degeneration is a primitive nontrivial homology class (it generates $W_0 \cong \mathbb{Z}$ of the monodromy weight filtration) with $\alpha.\alpha = 0$. Using the cycle L_Γ from Lemma 2.9, we find the following intersection numbers:

$$L_\Gamma.L_\gamma = 1, \qquad L_\Gamma.\alpha = 0, \qquad L_\gamma.\alpha = 0, \qquad L_\gamma.L_\gamma = 0, \tag{4}$$

where the middle ones follow from the fact that α can be supported in the complement of L_Γ and L_γ and the last one follows because the intersection pairing is antisymmetric on $H_3(M_t)$. We have equations (4) similarly for $L_{\gamma'}$ in place of L_γ. Because the middle cohomology of the mirror quintic has rank four, we can complement $[\alpha], [L_\gamma], [L_\Gamma]$ to a basis of $H_3(M_t, \mathbb{Z})$ by adding a fourth cycle S. Moreover, because the restriction of the intersection pairing to the span of L_γ, L_Γ is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, we can require S to be in its orthogonal complement. We write $[L_{\gamma'}] = a[\alpha] + b[L_\gamma] + c[L_\Gamma] + dS$ and want to determine the coefficients a, b, c, d. From the analogue of (4) for $L_{\gamma'}$, we find $d = 0$ by pairing $L_{\gamma'}$ with α because necessarily $\alpha.S \neq 0$ for $[\alpha]$ being nonzero. Because γ, γ' do not meet, $L_\gamma.L_{\gamma'} = 0$, which yields $c = 0$. Consequently, $1 = L_\Gamma.L_{\gamma'} = b$ and hence $[L_{\gamma'}] = a[\alpha] + [L_\gamma]$ for some a.

As explained in Remark 2.6, for the construction of the Lagrangian torus bundle over an edge e of γ', there is a freedom given by translating the 2-tori fibres by a function on e. Note that $[\alpha]$ is exactly the fundamental class of the trace of the 2π translation by a 2-torus in a 3-torus fibre. By applying the freedom in the construction and wrapping around $-a$ times, we can construct $L_{\gamma'}$ such that $[L_{\gamma'}] = [L_\gamma]$. □

Proof (Proof of Theorem 1.6). The Lagrangians L_γ and $L_{\gamma'}$ being homologous is the content of Proposition 2.10. Because they are rational homology spheres, they have unobstructed Floer cohomology over characteristic 0 [16] and we have $HF(L_\gamma, L_\gamma) = HF(L_{\gamma'}, L_{\gamma'}) = H^*(S^3)$ by the degeneration of the spectral sequence in the second page. Moreover, when $\gamma \cap \gamma' = \emptyset$, we have $HF(L_\gamma, L_{\gamma'}) = 0$. By Hamiltonian invariance of Floer cohomology, we conclude that L_γ is not Hamiltonian isotopic to $L_{\gamma'}$. □

Remark 2.11. Theorem 1.6 also works when $\gamma \cap \gamma'$ is a single point. In this case, if L_γ is Hamiltonian isotopic to $L_{\gamma'}$, then $HF(L_\gamma, L_{\gamma'})$ would be well defined but, one can see from the local model that L_γ intersects cleanly with $L_{\gamma'}$ along a circle, so $HF(L_\gamma, L_{\gamma'})$ is either 0 or concentrated on consecutive degree. This gives a contradiction.

It is less clear what $HF(L_\gamma, L_{\gamma'})$ is when γ overlaps with γ' along a codimension 0 subset. These cases arise in our computer-aided search.

2.9. Symplectomorphism group

Proof (Proof of Corollary 1.7). Each spherical Lagrangian submanifold L_i gives rise to a symplectomorphism $\tau_{L_i} : M \to M$, called the Dehn twist along L_i, supported inside an arbitrarily small neighbourhood of L_i (see [51], [34]). Therefore, it is clear that $\{\tau_{L_i}\}_{i=1}^{k_{\max}}$ generates an abelian subgroup in $\mathrm{Symp}(M)$ that descends to an abelian subgroup G of $\pi_0(\mathrm{Symp}(M)) = \mathrm{Symp}(M)/\mathrm{Ham}(M)$ (the equality uses the fact that $\pi_1(M)$ is trivial).

We recall from [51] that each τ_{L_i} can be lifted canonically to a $\mathbb{Z}$-graded symplectomorphism because $c_1(M) = 0$. Moreover, we know that $\tau_{L_i}(L_i) = L_i[-2]$ and $\tau_{L_i}(L_j) = L_j$ are $\mathbb{Z}$-graded Lagrangians, for all $i \neq j$. Therefore, $g \in G$ is completely determined by $(HF(L_i, g(L_j)))_{i,j=1}^{k_{\max}}$ and G is isomorphic to $\mathbb{Z}^{k_{\max}}$. □

Remark 2.12. If L_i is a spherical Lagrangian with $|\pi_1(L_i)| = m$, then $(\tau_{L_1})_* A = A + m([L_i] \cdot A)[L_i]$ for $A \in H_3(M, \mathbb{Z})$. Because $[L_i] = [L_j]$ for all i, j (Theorem 1.1(2)), the natural map $G \subset$

$\pi_0(\mathrm{Symp}(M)) \to \mathrm{Aut}(H_3(M,\mathbb{Z}))$ has a large kernel. It is less clear what the kernel of the natural map $G \subset \pi_0(\mathrm{Symp}(M)) \to \pi_0(\mathrm{Diff}(M))$ is.

3. Toric geometry in symplectic coordinates

We review some material about complex toric orbifolds. The presentation below is extracted from [1] and [2] (see also [24], [33] and [5]). Any projective complex toric orbifold X is Kähler and can be equipped with a Kähler form ω_X such that, for $i = \sqrt{-1}$, the action of the real torus

$$T^n := i\mathbb{R}^n/2\pi i\mathbb{Z}^n \subset \mathbb{C}^n/2\pi i\mathbb{Z}^n =: T^n_{\mathbb{C}}$$

is effective and Hamiltonian with respect to ω_X. The effective Hamiltonian action induces a moment map $\pi_\Delta : X \to \mathbb{R}^n$ with image $\Delta := \pi_\Delta(X)$ being a **simple** and **rational** convex polytope. This means that Δ is a convex polytope such that

- there are precisely n edges meeting at each vertex p;
- each edge meeting a vertex p is of the form $\{p + rv_j | r \in [0, r_j]\}$ for some $v_j \in \mathbb{Z}^n$, $r_j \geq 0$ for $1 \leq j \leq n$;
- $\{v_j\}_{j=1}^n$ form a $\mathbb{Q}$-basis of the lattice $\mathbb{Z}^n$.

If the last bullet is replaced by that $\{v_j\}_{j=1}^n$ can be chosen to be a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^n$, then Δ is called a Delzant polytope and X is a smooth manifold.

We call a face of codimension one of Δ a *facet*.

Definition 3. A labeled polytope is a simple rational convex polytope Δ plus a positive integer m (label) attached to each facet of Δ.

The label m of a facet F is the order of the orbifold structure group of the generic points in $(\pi_\Delta)^{-1}(F)$. If not mentioned, we assume all labels to be 1.

Lerman and Tolman [33] proved that a labeled simple rational convex polytope Δ determines a unique (up to equivariant symplectomorphism) compact symplectic orbifold (X, ω_X) with effective Hamiltonian torus action and moment map image Δ, which is a generalisation of Delzant's result on Delzant polytope and compact symplectic manifold (X, ω_X) with effective Hamiltonian torus action [9]. They also proved that if J_1 and J_2 are torus-invariant complex structures on X that are compatible with ω_X, then (X, J_1) and (X, J_2) are equivariantly biholomorphic ([33, Theorem 9.4]; see also [1, Section 2]). However, because there can be different torus-invariant Kähler structures on X, we need to go into details about the transition between complex and symplectic coordinates.

3.1. Complex coordinates

Let $X^\circ := \{x \in X \,|\, T^n \text{ acts freely on } x\}$. There is a biholomorphic identification

$$X^\circ = \mathbb{C}^n/2\pi i\mathbb{Z}^n = \{u + iv \,|\, u \in \mathbb{R}^n, v \in \mathbb{R}^n/2\pi\mathbb{Z}^n\}$$

such that $t \in T^n$ acts by

$$t \cdot (u + iv) = u + i(v + t).$$

The Kähler form ω_X is given by $\omega_X := 2i\partial\bar{\partial} f_\omega$ for a potential $f_\omega(u, v) = f_\omega(u) \in C^\infty(X^\circ)$, depending only on u (see [24] or [2, Exercise 3.5] for the definition of $f_\omega(u)$).

3.2. Symplectic coordinates

Dually, we have the symplectic identification $X^\circ = \Delta^\circ \times T^n$, where Δ° is the interior of Δ. The torus acts on $(p, q) \in \Delta^\circ \times T^n$ by

$$t \cdot (p, q) = (p, q + t)$$

and the symplectic form is $\omega_X := dp \wedge dq$. The complex structure J is determined by a function $f_J(p,q) = f_J(p) \in C^\infty(X^\circ)$ according to the following procedure. Let $F_J := \mathrm{Hess}_p(f_J)$ be the Hessian of f_J in the p coordinates (f_J and F_J are denoted by g and G, respectively, in [1]). The complex structure in (p,q) coordinates is given by

$$J = \begin{bmatrix} 0 & -F_J^{-1} \\ F_J & 0 \end{bmatrix}.$$

The transition maps between the complex and symplectic coordinates are given by

$$\begin{cases} p = \frac{\partial f_\omega}{\partial u}, & q = v, \\ u = \frac{\partial f_J}{\partial p}, & v = q. \end{cases} \tag{5}$$

There are restrictions for f_ω and f_J to satisfy near infinity so that we have a well-defined Kähler structure on X.

A canonical choice of complex structure is given by Guillemin as follows. The simple rational convex polytope Δ can be described by a set of inequalities of the form

$$\langle p, \mu_r \rangle - \rho_r \geq 0 \text{ for } r = 1, \ldots, d,$$

where d is the number of facets, each μ_r is a primitive element of $\mathbb{Z}^n$ and $\rho_r \in \mathbb{R}$. We define affine linear functions $l_r : \mathbb{R}^n \to \mathbb{R}$, $r = 1, \ldots, d$,

$$l_r(p) := \langle p, m_r\mu_r \rangle - \lambda_r,$$

where m_r is the label of the rth facet and $\lambda_r = m_r\rho_r$, so $p \in \Delta$ if and only if $l_r(p) \geq 0$ for all $r = 1, \ldots, d$.

Theorem 3.1 ([1], [2], [24]). *The 'canonical' compatible complex structure J_Δ on $\Delta^\circ \times T^n$ is given (in (p,q)-coordinates) by*

$$J_\Delta = \begin{bmatrix} 0 & -F_{J,can}^{-1} \\ F_{J,can} & 0 \end{bmatrix}, \tag{6}$$

where $F_{J,can} = \mathrm{Hess}_p(f_{J,can})$ and

$$f_{J,can}(p) := \frac{1}{2}\sum_{r=1}^{d} l_r(p)\log(l_r(p)). \tag{7}$$

Remark 3.2. Fixing ω_X, all torus-invariant complex structures J on X compatible with ω_X are classified in [1, Theorem 2].

Example 3.3 (Extending charts). We consider the following important noncompact example. Let $X = \mathbb{C}^n$ with moment polytope $\Delta = \mathbb{R}^n_{\geq 0}$ and $l_r(p_1, \ldots, p_n) = p_r$. We have symplectic coordinates $(p_j, q_j) \in X^\circ = (\mathbb{C}^*)^n \subset X$. Define $z_j = x_j + iy_j = \sqrt{2p_j}\exp(iq_j) \in \mathbb{C}^*$, so that we have $\sum_{j=1}^n dx_j \wedge dy_j = \sum_{j=1}^n dp_j \wedge dq_j$. We can extend the domain of z_j from $\mathbb{C}^*$ to $\mathbb{C}$ and thus provide a symplectic chart to X and moment map $X \to \mathbb{R}^n_{\geq 0}$ is given by $(z_1, \ldots, z_n) \mapsto \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2)$.

For the complex coordinates, (7) yields $f_{J,can} = \frac{1}{2}\sum_{j=1}^n p_j \log(p_j)$ and $\frac{\partial}{\partial p_j} f_{J,can} = \frac{1}{2}(1+\log(p_j))$, so the Hessian of $f_{J,can}$ is given by

$$F_{J,can} = \begin{bmatrix} \frac{1}{2p_1} & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & \frac{1}{2p_n} \end{bmatrix}.$$

We define J_Δ by Equation (6). Then a direct calculation gives

$$J_\Delta(\partial_x) = \partial_y.$$

Let $u_j = \frac{\partial f_{J,can}(p)}{\partial p_j}$, $v_j = q_j$ and $w_j = e^{u_j + iv_j}$ be the holomorphic coordinates on $(\mathbb{C}^*)^n$ (see (5)). Then $u_j = \frac{1}{2}(1 + \log(p_j))$ and $w_j = e^{\frac{1}{2}}\sqrt{p_j}e^{iq_j} = (\frac{e}{2})^{\frac{1}{2}} z_j$. The holomorphic coordinates $(w_1, \dots, w_n)$ on $(\mathbb{C}^*)^n$ naturally extend to holomorphic coordinates on $\mathbb{C}^n$.

Lemma 3.4 (Integral linear transformation). *Let Δ_1 be a labeled polytope and $\Delta_2 = A\Delta_1 + v$ where $A \in GL_n(\mathbb{Z})$ and $v \in \mathbb{Z}^n$. Let X_1 and X_2 be the canonical Kähler toric orbifold with moment polytope being Δ_1 and Δ_2, respectively. Then X_1 and X_2 are Kähler isomorphic.*

Proof. This follows from realising that neither the definition of the symplectic nor complex structures needs coordinates, because the μ_r are intrinsic to the integral affine structure and hence are the l_r. □

Example 3.5 (Transforming hypersurfaces). Let X be a toric manifold with moment image a Delzant polyhedron Δ. By picking a vertex v and replacing Δ by $A(\Delta - v)$ for some $A \in GL_n(\mathbb{Z})$ (see Lemma 3.4), we can assume $l_r(p_1, \dots, p_n) = p_r$ for $r = 1, \dots, n$ and the remaining facets of Δ are contained respectively in $l_r = 0$ for $r = n+1, \dots, d$. Let $w_j = \exp(u_j + iv_j) = \exp(\frac{\partial f_{J,can}(p)}{\partial p_j} + iq_j) \in \mathbb{C}^*$, which gives a $T^n_{\mathbb{C}}$ equivariant identification between $(\mathbb{C}^*)^n \subset \mathbb{C}^n$ and X°. We know that (see (7))

$$f_{J,can} := \frac{1}{2}\sum_{j=1}^{n} p_j \log(p_j) + R, \tag{8}$$

where R is the contribution from other facets. Assume now we are given a family of hypersurfaces via

$$w_1 \dots w_n = tg(w) \tag{9}$$

for some polynomial g in holomorphic coordinates and $t \in \mathbb{C}$ a family parameter. The logarithm of this hypersurface equation is transformed to

$$\frac{n}{2} + \log\left(\sqrt{\prod_{j=1}^{n} p_j}\right) + i\left(\sum_{j=1}^{n} q_j\right) + \sum_{j=1}^{n}\frac{\partial R(p)}{\partial p_j} = \log(t) + \log(g(w(p,q))) \tag{10}$$

in symplectic coordinates. Notice that R can be smoothly extended to the origin, so by exponentiating and setting $z_j := \sqrt{2p_j}\exp(iq_j)$, we may write this equation as

$$\prod_{j=1}^{n} z_j = tf(p,q), \tag{11}$$

where

$$f(p,q) = g(w(p,q))h(p) \tag{12}$$

for $h = \sqrt{2}^n \exp(-\frac{n}{2} - \sum_{j=1}^{n}\frac{\partial R(p)}{\partial p_j})$. Most important, later on, h is a nonvanishing C^∞-function depending only on p.

With the above example, we know how to transform a complex hypersurface defined by the equation $w_1 \dots w_n = tg(w)$ into a symplectic hypersurface in symplectic coordinates (p,q) for a toric manifold X. To cover a large range of applications, we need an analogue for toric orbifolds.

3.3. Isolated Gorenstein toric orbifold singularities

Now consider a cone $\Delta \subset \mathbb{R}^n$ generated by $v_1, \ldots, v_n \in \mathbb{Z}^n$. The ring $\mathbb{C}[\Delta \cap \mathbb{Z}^n]$ is the coordinate ring of an Abelian quotient singularity X_Δ as follows. The ring is regular if and only if the v_i form a lattice basis. Let σ be the dual cone of Δ. It is also integrally generated, so let N be the sublattice generated by the primitive ray generators of σ as a sublattice of $(\mathbb{Z}^n)^*$, the dual lattice $M = \mathrm{Hom}(N, \mathbb{Z})$ contains the original lattice $\mathbb{Z}^n$ and the cone Δ is a standard cone when viewed with respect to M; that is, $\mathbb{C}[\Delta \cap M] = \mathbb{C}[w_1, \ldots, w_n]$, where w_j is the monomial given by the primitive generator of $(\mathbb{R}_{\geq 0} v_j) \cap M$. The subring $\mathbb{C}[\Delta \cap \mathbb{Z}^n] \subseteq \mathbb{C}[\Delta \cap M]$ is the ring of invariants of the group action $K = (\mathbb{Z}^n)^*/N$ that acts on a monomial z^m via $g.z^m = \exp(2\pi i \langle g, m\rangle) z^m$ (see [17, Section 2.2, p. 34]). We need this a bit more explicit and also want to make further assumptions. We require the singularity to be isolated. Because then K necessarily acts faithfully on the subring $\mathbb{C}[w_1] = \mathbb{C}[(\mathbb{R}_{\geq 0} v_1) \cap M]$, we conclude that K is cyclic, say, K is the group of kth roots of unity. Let ζ be a primitive generator. The action is

$$\zeta.(w_1, \ldots, w_n) = (\zeta^{a_1} w_1, \ldots, \zeta^{a_n} w_n)$$

for some integers a_j with $\gcd(a_j, k) = 1$ for all j, which is equivalent to the isolatedness of the singularity. One can check the following result.

Lemma 3.6. *Under the given assumptions, the cover $\mathbb{C}^n \to \mathbb{C}^n/K$ is unbranched away from the origin.*

We want to further assume that the singularity is Gorenstein, which is equivalent to the statement that the Gorenstein monomial $w_1 w_2 \ldots w_n$ is invariant under K; that is,

$$\sum_j a_j \in k\mathbb{Z}.$$

We now address the symplectic coordinates. Let $T_0 = (S^1)^n$ and consider the standard T_0-action on $\mathbb{C}^n$ by $\theta \cdot (w_1, \ldots, w_n) = (e^{\theta_1 i} w_1, \ldots, e^{\theta_n i} w_n)$. Recall from Example 3.3 that the standard symplectic coordinates of the toric variety $\mathbb{C}^n$ are $z_j = e^{-\frac{1}{2}} w_j$ and $z_j = \sqrt{2p_j}\exp(iq_j)$ giving the moment map $\mathbb{C}^n \to \mathbb{R}^n_{\geq 0}$. Note that K is a subgroup of T_0, so $T_1 = T_0/K$ acts faithfully on the orbifold singularity $\mathbb{C}^n/K$. We claim that the moment map of $\mathbb{C}^n$ factors through that of $\mathbb{C}^n/K$; that is,

$$\begin{tikzcd} \mathbb{C}^n \arrow[r] \arrow[d] & \mathbb{C}^n/K \arrow[d, "\pi"] \\ \mathbb{R}^n_{\geq 0} & \Delta \arrow[l] \end{tikzcd}$$

where the bottom horizontal map is the real affine isomorphism given by the fact that Δ becomes a standard cone with respect to M. The right vertical map is the moment map of the orbifold singularity. The diagram clearly commutes and because the symplectic structures can be defined using the moment maps, the diagram is compatible with symplectic structures. The only thing to check is that the complex structures used in the diagram coincide with the canonical ones obtained from the complex potential $f_{J,can}$ in Theorem 3.1. By Example 3.3 this is true for the left vertical map. Because the μ_j are actually the primitive generators of the rays of σ, and are therefore contained in N, we find that the potential $f_{J,can}$ for $(\Delta, \mathbb{Z}^n)$ is identical to the one for (Δ, M), which gives the desired compatibility.

We finally want to consider the situation where the Gorenstein singularity appears locally at the vertex of a compact polytope Δ. Let $\mathbb{P}_\Delta$ be the compact Kähler orbifold obtained from Δ and $\pi_\Delta : \mathbb{P}_\Delta \to \Delta$ the moment map. Let $v \in \Delta$ be a vertex. Replacing Δ by $\Delta - v$ and invoking Lemma 3.4, we may assume $v = 0$. Compared to the local study above, there is no difference for the complex structure; however, the compact polytope Δ gives a different symplectic structure on the local model $\mathbb{C}^n$.

Consider a neighbourhood O_v of v in Δ that is then also a neighbourhood of v in the cone $\mathbb{R}_{\geq 0}\Delta$. The two inverse images under the moments maps $\pi_\Delta^{-1}(O_v)$ and $\pi^{-1}(O_v)$ resulting from this are naturally

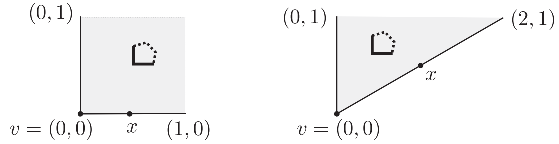


Figure 7. *Left: two-dimensional analogue when $\pi_\Delta^{-1}(v)$ is a smooth point. Right: two-dimensional analogue when $\pi_\Delta^{-1}(v)$ is an orbifold point.*

symplectomorphic. Assume now that we have a family of hypersurfaces in $\mathbb{C}^n/K$ as given by (9); that is,

$$w_1 \cdot \ldots \cdot w_n = tg(w),$$

where we use the coordinates of $\mathbb{C}^n$ and so $g(w)$ is now a K-invariant polynomial. By the Gorenstein assumption, the monomial $w_1 w_2 \ldots w_n$ is K-invariant. The same analysis as in Example 3.5 gives (10) as the equation for the family of hypersurfaces in symplectic coordinates with the only difference that now f and h are K-invariant.

3.4. Corner charts in 4-orbifolds

Let $\mathbb{P}_\Delta$ be a four-dimensional Gorenstein-projective toric orbifold with isolated singularities and moment polytope Δ. For each point z of $\mathbb{P}_\Delta$, we can choose a vertex v of Δ lying in the face containing $\pi_\Delta(z)$. Let

$$\llcorner = \Delta \setminus \cup_{v \notin F} F \tag{13}$$

where F are facets of Δ. If $\pi_\Delta^{-1}(v)$ is a smooth point of $\mathbb{P}_\Delta$, then we can, by an integral affine linear transform, assume v is the origin and the primitive edge directions emerging from v coincide with the positive real axes in $\mathbb{R}^4$. If Δ has integral points in its interior, after the transform, $(1, 1, 1, 1)$ must be one of them (in fact the only one if Δ is reflexive). We can give a symplectic chart $U \subset \mathbb{R}^8$ to $\pi_\Delta^{-1}(\llcorner)$ as in Example 3.3, which is $T^4_{\mathbb{C}}$-equivariantly biholomorphic to $\mathbb{C}^4$ (see Figure 7). More generally, if $\pi_\Delta^{-1}(v)$ is an orbifold point of $\mathbb{P}_\Delta$, then we have just shown in Subsection 3.3 that $\pi_\Delta^{-1}(\llcorner)$ is equivariantly symplectomorphic to the model $X_v = \mathbb{C}^4/K$ with the symplectic structure induced from π_Δ. The smooth case can be viewed like the situation $K = \{\mathrm{id}\}$. In both cases, we call X_v a **symplectic corner chart** for $\mathbb{P}_\Delta$ associated to the vertex v. All mirror quintic threefolds are hosted inside a toric variety $\mathbb{P}_\Delta$ of the type considered here.

4. Geometric setup

Let $\mathbb{P}_\Delta$ be a complex projective toric orbifold of complex dimension four with moment polytope Δ. Recall that $-K_{\mathbb{P}_\Delta} = \partial\mathbb{P}_\Delta$; we assume this is nef or, equivalently (for a toric variety), that $\mathcal{O}(-K_{\mathbb{P}_\Delta})$ is generated by global sections ([43], Theorem 2.7). Let Δ_K denote the corresponding lattice polytope. We have a birational morphism $\mathbb{P}_\Delta \to \mathbb{P}_{\Delta_K}$ that we will use to pull back an anticanonical hypersurface. We equip $\mathbb{P}_\Delta$ with the canonical Kähler structure. Set $\mathcal{L} := \mathcal{O}(-K_{\mathbb{P}_\Delta})$ and let $s_0 \in H^0(\mathbb{P}_\Delta, \mathcal{L})$ such that $s_0^{-1}(0) = \partial\mathbb{P}_\Delta$.

Let $C^\infty(\mathbb{P}_\Delta, \mathcal{L})$ denote the vector space of C^∞-sections of $\mathcal{L}$. For every $s \in C^\infty(\mathbb{P}_\Delta, \mathcal{L})$ and $t \in \mathbb{C}$, we define

$$M_t^s := \{s_0 = ts\} \subset \mathbb{P}_\Delta. \tag{14}$$

The total family of M_t^s is denoted by M^s. Let $(\partial\mathbb{P}_\Delta)_{\mathrm{Sing}}$ denote the locus of singular points of $\partial\mathbb{P}_\Delta$ (we also used $\mathbb{P}_\Delta^{[2]}$ before). We define the discriminant of s via

$$\mathrm{Disc}(s) := s^{-1}(0) \cap (\partial\mathbb{P}_\Delta)_{\mathrm{Sing}}. \tag{15}$$

As explained in Subsection 3.4, a symplectic corner chart U/K comes together with the quotient map $\Pi_{U/K} : U \to U/K$ and the diffeomorphism $\Phi_U : U \to \mathbb{C}^4$. In a symplectic corner chart, we define

$$\widetilde{M}_t^s := \Pi_{U/K}^{-1}(M_t^s \cap U/K) \tag{16}$$

$$= \Phi_U^{-1}(\{w \in \mathbb{C}^4 | w_1 w_2 w_3 w_4 = t f(w)\}) \tag{17}$$

for some K-invariant function $f \in C^\infty(\mathbb{C}^4, \mathbb{C})$. The second equality comes from the fact that, with respect to a choice of trivialisation, $s_0 = w_1 w_2 w_3 w_4 h(w)$ for some nonvanishing K-invariant function h on $\mathbb{C}^4$. It is clear that if $s \neq 0$ at the orbifold points of $\mathbb{P}_\Delta$, then M_t^s does not contain any orbifold point whenever $t \neq 0$.

When $s_1 \in H^0(\mathbb{P}_\Delta, \mathcal{L})$, we get a family of complex subvarieties $M_t^{s_1}$ parametrized by $t \in \mathbb{C}$. Let

$$H^0(\mathbb{P}_\Delta, \mathcal{L})_{\mathrm{Reg}} := \{s_1 \in H^0(\mathbb{P}_\Delta, \mathcal{L}) | M_t^{s_1} \text{ is smooth for all } |t| > 0 \text{ small}\}. \tag{18}$$

When $M_t^{s_1}$ is a smooth manifold, it is a symplectic hypersurface in $\mathbb{P}_\Delta$ and the symplectomorphism type is independent of t by Moser's argument. For smooth but not necessarily holomorphic sections, we have the following sufficient condition to guarantee that M_t^s is symplectic (when t is sufficiently close to 0).

Lemma 4.1 (Good deformation). *Let $s_1 \in H^0(\mathbb{P}_\Delta, \mathcal{L})_{\mathrm{Reg}}$. Suppose we have a smooth family $(s^u)_{u \in [0,1]} \in C^\infty(\mathbb{P}_\Delta, \mathcal{L})$ such that*

- $s^u = s_1$ *near* $\mathrm{Disc}(s_1)$ *for all u,*
- $\mathrm{Disc}(s^u) = \mathrm{Disc}(s_1)$ *for all u,*

then there exist $\delta > 0$ such that $M_t^u := M_t^{s^u}$ is a smooth symplectic hypersurface in $\mathbb{P}_\Delta$ for all $0 < |t| < \delta$ and all u.

Proof. For any regular neighbourhood N of $\partial \mathbb{P}_\Delta$, there exists $\delta' > 0$ such that $M_t^u \subset N$ for all $|t| < \delta'$ for all u. This is because M_t^u C^0-converges to $\partial \mathbb{P}_\Delta$ uniformly as $|t|$ goes to 0. Therefore, if for each point $x \in \partial \mathbb{P}_\Delta$, we can find a neighbourhood O_x of x such that $M_t^u \cap O_x$ is symplectic for all $|t| > 0$ small and all $u \in [0,1]$, then M_t^u is symplectic for all $|t| > 0$ small and all $u \in [0,1]$.

Because M_t^u is independent of u in a neighbourhood O_{Disc} of $\mathrm{Disc}(s_1)$ (by the first bullet), we can take $O_x = O_{\mathrm{Disc}}$ if $x \in O_{\mathrm{Disc}}$. Now we assume that $x \in \partial \mathbb{P}_\Delta \setminus O_{\mathrm{Disc}}$.

First suppose $\pi_\Delta(x)$ lies in the interior of a 3-cell. There exists a symplectic corner chart U/K and an open subset $V \subset U$ such that $x \in \Pi_{U/K}(V)$, $\Pi_{U/K}(V) \cap \mathrm{Disc}(s_1) = \emptyset$ and

$$\widetilde{M}_t^u \cap V = \Phi_U^{-1}(\{w_1 = t f^u(w)\})$$

for some smooth family of functions $f^u : \Phi_U(V) \to \mathbb{C}$. This is because we can assume w_2, w_3, w_4 are invertible in $\Phi_U(V)$ and absorbed by f^u. Let $F_t^u = w_1 - t f^u(w)$. The differential is given by

$$DF_t^u = [1, 0, 0, 0] - t D f^u.$$

Because $\ker(DF_t^u) = T\widetilde{M}_t^u$ and the first term of DF_t^u dominates (say, with respect to the Euclidean norm in the chart) when $|t|$ small, $\widetilde{M}_t^u \cap V$ is symplectic for all $|t| > 0$ small and all $u \in [0,1]$. Therefore, we can take $O_x = \Pi_{U/K}(V)$.

Now suppose $\pi_\Delta(x)$ lies in the interior of a 2-cell. There exists a symplectic corner chart U/K and an open subset $V \subset U$ such that $x \in \Pi_{U/K}(V)$ and

$$\widetilde{M}_t^u \cap V = \Phi_U^{-1}(\{w_1 w_2 = t f^u(w)\})$$

for some smooth family of functions $f^u : \Phi_U(V) \to \mathbb{C}$ such that $f^u(0, 0, w_3, w_4) \neq 0$ (by the second bullet and the assumption that $x \in \partial \mathbb{P}_\Delta \setminus O_{\mathrm{Disc}}$). This is because we can assume w_3, w_4 are invertible in

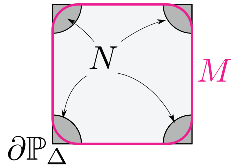


Figure 8. *The symplectic model M close to the boundary of $\mathbb{P}_\Delta$.*

$\Phi_U(V)$ and $\Phi_U(\Pi^{-1}_{U/K}((\partial\mathbb{P}_\Delta)_{\mathrm{Sing}}) \cap V) = \Phi_U(V) \cap \{w_1 = w_2 = 0\}$. Therefore, there exists $c > 0$ such that $\max\{|w_1|, |w_2|, |f^u(w)|\} > c$ for all points in $\Phi_U(V)$. Let $F^u_t = w_1w_2 - tf^u(w)$. The differential is given by

$$DF^u_t = [w_2, w_1, 0, 0] - tDf^u.$$

Again, we want to show that the first term of DF^u_t dominates for $w \in \{F^u_t = 0\}$ for all u when $|t| > 0$ small.

Because $\|Df^u\|$ is bounded, the norm of the second vector is of order $|t|$. At points where $|w_1| > c$ or $|w_2| > c$, the first term clearly dominates when $|t| > 0$ small. By the assumption, all other points satisfy $|f^u| > c$. As a result, for $w \in \{F^u_t = 0\}$, we have $|w_1|^2 + |w_2|^2 \geq 2|w_1w_2| = 2|tf^u| > 2|t|c$ so the norm of $[w_2, w_1, 0]$ is of order at least $\sqrt{|t|}$ and hence dominates when $|t| > 0$ small. This implies that there exist $\delta > 0$ such that $\widetilde{M}^u_t \cap V$ is a symplectic manifold for all $0 < |t| < \delta$ and all u.

Similarly, when $\pi_\Delta(x)$ lies in the interior of a 1-cell, we have

$$\widetilde{M}^u_t \cap V = \Phi^{-1}_U(\{w_1w_2w_3 = tf^u(w)\})$$

for some V and f^u. There exists $c > 0$ such that

$$\max\{|w_1w_2|, |w_2w_3|, |w_1w_3|, |f^u(w)|\} > c.$$

At points where $|w_1w_2| > c$ or $|w_2w_3| > c$ or $|w_1w_3| > c$, the first term of DF^u_t, which is given by $[w_2w_3, w_1w_3, w_1w_2, 0]$, dominates when $|t| > 0$ small. At points where $|f^u| > c$, we have $|w_1w_2|^2 + |w_2w_3|^2 + |w_1w_3|^2 \geq 3|w_1w_2w_3|^{\frac{4}{3}} > 3|t|^{\frac{4}{3}}c$ so the norm of the first term of DF^u_t is of order $|t|^{\frac{2}{3}}$ and the second term of DF^u_t is of order $|t|$ so the first term dominates when $|t| > 0$ small.

One can do the same analysis when $\pi_\Delta(x)$ is a vertex of Δ. In this case, the norm of the first term and second term of DF^u_t is of order $|t|^{\frac{3}{4}}$ and $|t|$, respectively.

□

We remark that $\mathrm{Disc}(s^u) = \mathrm{Disc}(s_1)$ implies that s^u does not vanish at the orbifold points. In view of Lemma 4.1, it is convenient to have the following definition.

Definition 4. Let $s_1 \in H^0(\mathbb{P}_\Delta, \mathcal{L})_{\mathrm{Reg}}$. We say that $s \in C^\infty(\mathbb{P}_\Delta, \mathcal{L})$ is s_1-*admissible* if $s = s_1$ in a neighbourhood of $\mathrm{Disc}(s_1)$ and $\mathrm{Disc}(s) = \mathrm{Disc}(s_1)$.

We say that $s \in C^\infty(\mathbb{P}_\Delta, \mathcal{L})$ is *admissible* if it is s_1-admissible for some $s_1 \in H^0(\mathbb{P}_\Delta, \mathcal{L})_{\mathrm{Reg}}$.

Corollary 4.2. *For $s_1 \in H^0(\mathbb{P}_\Delta, \mathcal{L})_{\mathrm{Reg}}$ and any regular neighbourhood N of $(\partial\mathbb{P}_\Delta)_{\mathrm{Sing}}$, there is a symplectic hypersurface $M \subset \mathbb{P}_\Delta$ such that M is symplectic isotopic to $M^{s_1}_t$ for some $|t| > 0$ small, and $(\partial\mathbb{P}_\Delta \setminus N) \subset M \subset (\partial\mathbb{P}_\Delta \cup N)$; see Figure 8.*

Proof. Let $N' \subset N$ be a smaller neighbourhood of $(\partial\mathbb{P}_\Delta)_{\mathrm{Sing}}$. Let $\chi : \mathbb{P}_\Delta \to \mathbb{R}$ be a smooth function that has values 1 in N'
and 0 outside N. Then $s := \chi s_1$ is an s_1-admissible section. Moreover, $s^u := (1-u)s_1 + us$ is a smooth family of s_1-admissible sections, so we can apply Lemma 4.1. Let the resulting family be M^u_t. By Moser's argument, $M^0_t = M^{s_1}_t$ is symplectic isotopic to $M := M^1_t$ when $0 < |t| < \delta$.

It follows from the definition of s that for $x \in (\mathbb{P}_\Delta \setminus N)$, we have

$$x \in M \quad \Leftrightarrow \quad s_0(x) - ts(x) = 0 \quad \Leftrightarrow \quad s_0(x) = 0 \quad \Leftrightarrow \quad x \in \partial\mathbb{P}_\Delta \setminus N.$$

This gives the assertion. □

An important consequence of Corollary 4.2 is that we can transfer the Lagrangian torus fibre bundle structure of $\partial\mathbb{P}_\Delta \setminus (\partial\mathbb{P}_\Delta)_{\mathrm{Sing}}$ to a Lagrangian torus fibre bundle structure in a large open subset of M, and hence a large open subset of $M_t^{s_1}$.

Lemma 4.3. *If $(s^u)_{u\in[0,1]}$ is a family of s_1-admissible sections such that, for some open subset $V \subset \mathbb{P}_\Delta$, $s^u|_{\mathbb{P}_\Delta \setminus V}$ is independent of u, then there exists $\delta > 0$ such that for all $0 < |t| < \delta$ there is a symplectomorphism $\phi_{V,t} : M_t^{s^0} \to M_t^{s^1}$ such that $\phi_{V,t}|_{M_t^{s^0} \setminus V}$ is the identity.*

Proof. By Lemma 4.1, $M_t^{s^u}$ is a family of symplectic hypersurfaces for $0 < |t| < \delta$. By assumption, $M_t^{s^u} \cap (\mathbb{P}_\Delta \setminus V)$ is independent of u. The existence of $\phi_{V,t}$ follows from a standard application of Moser's argument. □

Outlook: recall γ, $W_\epsilon(\gamma)$ and M_t from Theorem 1.1. In its proof, for all $\epsilon > 0$, we will construct a family of admissible sections $(s^u)_{u\in[0,1]}$ such that $M_t^{s^0} = M_t$ and $M_t^{s^1} \cap \pi_\Delta^{-1}(W_\epsilon(\gamma))$ contains a Lagrangian L that is diffeomorphic to a Lagrangian lift of γ for all $|t| > 0$ small. Moreover, for $V := \pi_\Delta^{-1}(W_\epsilon(\gamma))$, $s^u|_{\mathbb{P}_\Delta \setminus V}$ will be independent of u. We can apply Lemma 4.3 to get a symplectomorphism $\phi_{V,t} : M_t \to M_t^{s^1}$ and $\phi_{V,t}^{-1}(L)$ will be our desired Lagrangian in $M_t \cap \pi_\Delta^{-1}(W_\epsilon(\gamma))$.

5. Away from discriminant

This section gives the construction of Lagrangians away from the discriminant. In Subsection 5.1, we give a local Lagrangian model and explain how to glue these Lagrangian models away from the discriminant. We will complete our Lagrangian construction away from the discriminant after the discussion in Subsection 5.2, which concerns trivalent vertices of a tropical curve. We conclude the proof of Theorem 1.1 in Subsection 6.8. For simplicity of notation, in the rest of the article, we only consider M_t^s for $t \in \mathbb{R}$, $t > 0$ instead of $t \in \mathbb{C}^*$.

5.1. Standard Lagrangian model

There are four tasks to be completed in this subsection, which will be accomplished in the four subsequent sub-subsections, respectively. Firstly, points on a tropical curve γ can lie in different strata of $\partial\Delta$, so we want to enumerate all possibilities and describe the neighbourhood of points in different strata. Then, for each point $x \in \gamma$ and a neighbourhood $O_x \subset \Delta$ of x, we want to isotope $M_t^s \cap \pi_\Delta^{-1}(O_x)$ to a standard form for constructing a local Lagrangian in $M_t^s \cap \pi_\Delta^{-1}(O_x)$. After that, we explain how to glue the local Lagrangians in $M_t^s \cap \pi_\Delta^{-1}(O_{x_1})$ and $M_t^s \cap \pi_\Delta^{-1}(O_{x_2})$ when $O_{x_1} \cap O_{x_2} \neq \emptyset$. Finally, because the local Lagrangians are constructed with respect to a symplectic corner chart, we will deal with the transition of symplectic corner charts so that all of the local Lagrangian models in different symplectic corner charts can be glued together.

In sub-subsections 5.1.1, 5.1.2 and 5.1.3, we work inside a single symplectic corner chart U/K with moment map image $\Box := \pi_\Delta(U/K)$. There is an induced moment map $\pi_{\widetilde{\Delta}} : U \to \mathbb{R}^4$ and we denote the image by $\widetilde{\Box}$. Recall from Subsection 3.3 that the images $\Box$ and $\widetilde{\Box}$ are related by a rational linear affine transformation (in particular, a bijective map) so there are corresponding subsets $\widetilde{x}$, $\widetilde{\gamma}$, $\widetilde{\mathcal{A}}$, $\widetilde{O_x}$, etc., in $\widetilde{\Box}$. On the other hand, subsets in U/K (e.g., $M_t^s \cap U/K$) can be lifted to K-invariant sets in U (e.g., $\widetilde{M_t^s}$) that are compatible with the moment maps. For simplicity of notation, we omit all of the $\widetilde{(-)}$ in Subsections 5.1.1, 5.1.2 and 5.1.3 and work K-equivariantly in U (U/K will also be denoted by U). By possibly adding a translation, we identify $\Box$ with an open subset of $\mathbb{R}^4_{\geq 0}$ that contains the origin.

5.1.1. Neighbourhood of a point in a tropical curve

We define a function Type : $\partial\Delta \to \{0, 1, 2, 3\}$ such that $\mathrm{Type}(x) = n$ if x is in the interior of an n-cell. In other words, Type specifies the stratum that x lies in.

For a neighbourhood $O_0 \subset \square \setminus \mathcal{A}$ of the origin such that $O_0 \cap \partial\square$ is contractible, the integral affine structure on $O_0 \cap \partial\square$ is inherited from the C^0-embedding $O_0 \cap \partial\square \hookrightarrow \square/\mathbb{R}(1, 1, 1, 1)$ (see the definition of the chart W_v in Subsection 2.2).

Let $x \in \partial\square$ and $c(r) : [-1, 1] \to \partial\square\setminus\mathcal{A}$ be a straight line (regarded as a closed segment in a tropical curve γ that contains x) in a small neighbourhood of x such that $c(0) = x$. We have the following situations using that $\partial\Delta$ is simple.

(A) If $\mathrm{Type}(x) = 0$, then $(\mathrm{Type}(c(-1)), \mathrm{Type}(c(0)), \mathrm{Type}(c(1)))$ can only take values (modulo the symmetry $r \mapsto -r$) $(2, 0, 2)$, $(1, 0, 3)$, $(2, 0, 3)$ and $(3, 0, 3)$.
(B) If $\mathrm{Type}(x) = 1$, then $(\mathrm{Type}(c(-1)), \mathrm{Type}(c(0)), \mathrm{Type}(c(1)))$ can only take values (modulo the symmetry $r \mapsto -r$) $(1, 1, 1)$, $(2, 1, 3)$ and $(3, 1, 3)$.
(C) If $\mathrm{Type}(x) = 2$, then $(\mathrm{Type}(c(-1)), \mathrm{Type}(c(0)), \mathrm{Type}(c(1)))$ can only take values $(2, 2, 2)$ and $(3, 2, 3)$.
(D) If $\mathrm{Type}(x) = 3$, then $\mathrm{Type}(c(r)) = 3$ for all r.

Remark 5.1. From the enumeration above, we can see that for any straight line $c\colon [-1, 1] \to \partial\Delta\setminus\mathcal{A}$, if $r = r_0$ is a discontinuity of $\mathrm{Type}(c(r))$, then $\mathrm{Type}(c(r_0)) < \mathrm{Type}(c(r))$ for all r close to but not equal to r_0.

5.1.2. Local Lagrangian models at points in different strata

For each point $x \in \gamma$ and each open subset $c \subset \gamma$ containing x, we want to isotope $M_t^s \cap \pi_\Delta^{-1}(O_x)$ to another K-invariant hypersurface so that we can build a K-invariant Lagrangian in $M_t^s \cap \pi_\Delta^{-1}(O_x)$ whose π_Δ-image is close to c. First we describe a class of symplectic manifolds in $\pi_\Delta^{-1}(O_x)$ to which we would like to isotope $M_t^s \cap \pi_\Delta^{-1}(O_x)$.

Definition 5. For a point $x = (x_1, \dots, x_4) \in (\partial\square) \setminus \mathcal{A}$ and an s_1-admissible section $s \in C^\infty(\mathbb{P}_\Delta, \mathcal{L})$, we say that M^s is x**-standard with respect to** U if there is a neighbourhood $O_x \subset \square$ of x that does not meet any facet that does not contain x, and furthermore such that $M_t^s \cap \pi_\Delta^{-1}(O_x)$ is given by

$$\left(\prod_{j, x_j=0} \sqrt{p_j}\right) e^{i(q_1+q_2+q_3+q_4)} = tc \tag{19}$$

for some constant $c \in \mathbb{C}$. If $\mathrm{Type}(x) = 0, 1, 2$, we require $c \neq 0$.

For a point $x \in \partial\Delta \setminus \mathcal{A}$, we say that M^s is x**-standard** if there is a symplectic corner chart U such that M^s is x-standard with respect to U.

Because the K action on U acts only on the q_i coordinates and the Gorenstein coordinate $q_1+q_2+q_3+q_4$ is invariant under the K action, M^s is K-invariant if M^s is x-standard with respect to U. To see that this is a sensible notion, we at least need to observe the following.

Lemma 5.2. *If M^s is x-standard with respect to U, then $(M_{t\neq 0}^s \cap \pi_\Delta^{-1}(O_x)) \cap (\partial\mathbb{P}_\Delta)_{\mathrm{Sing}} = \emptyset$. In other words, $M_t^s \cap \pi_\Delta^{-1}(O_x)$ is disjoint from the discriminant for all $t > 0$.*

Proof. Notice that we can rewrite equation (19) as

$$z_1z_2z_3z_4 = \sqrt{16p_1p_2p_3p_4}e^{i(q_1+q_2+q_3+q_4)} = 4tc \prod_{j, x_j\neq 0} \sqrt{p_j}.$$

To prove the lemma, it suffices to show that the zero locus of $4tc \prod_{j,x_j\neq 0} \sqrt{p_j}$ does not intersect with $(\partial\mathbb{P}_\Delta)_{\mathrm{Sing}} \cap \pi_\Delta^{-1}(O_x)$. When $\mathrm{Type}(x) = 0, 1, 2$, by Definition 5, we have $c \neq 0$. Moreover, inside $\pi_\Delta^{-1}(O_x)$, we have $p_j > 0$ when $x_j \neq 0$. Altogether this implies that $4tc \prod_{j,x_j\neq 0} \sqrt{p_j}$ never vanishes in $\pi_\Delta^{-1}(O_x)$.

Because $(\partial\mathbb{P}_\Delta)_{Sing} = \pi_\Delta^{-1}(\text{codim-2-strata})$, when $\text{Type}(x) = 3$, $(\partial\mathbb{P}_\Delta)_{\text{Sing}} \cap \pi_\Delta^{-1}(O_x) = \emptyset$. □

The next lemma addresses that we can always isotope $M_t^s \cap \pi_\Delta^{-1}(O_x)$ to an x-standard one through admissible sections that are K-invariant in U.

Lemma 5.3. *Let s be an s_1-admissible section. Let $x \in \partial\Delta \backslash \mathcal{A}$ be a point and N_x be a neighbourhood of x in Δ such that $N_x \cap \mathcal{A} = \emptyset$. Then there is a symplectic corner chart U containing x and a family of s_1-admissible section $(s^u)_{u\in[0,1]}$ such that $s^0 = s$, for all u, $s^u = s$ outside $\pi_\Delta^{-1}(N_x)$, s^u is K-invariant in U and $M_t^{s^1}$ is x-standard with respect to U.*

Proof. If $\text{Type}(x) = 0$, then x is a vertex and we take the symplectic corner chart U associated to x. Because $x \notin \mathcal{A}$, there exists a neighbourhood $O_x \subset N_x$ of x such that $\pi_\Delta^{-1}(O_x)$ is contractible and, by (11), $M_t^s \cap \pi_\Delta^{-1}(O_x)$ is given by

$$z_1z_2z_3z_4 = \sqrt{16p_1p_2p_3p_4}e^{i(q_1+q_2+q_3+q_4)} = tf \tag{20}$$

for some $f \in C^\infty(\pi_\Delta^{-1}(O_x), \mathbb{C}^*)$. Because $\pi_\Delta^{-1}(O_x)$ is contractible, f is null-homotopic. For any subset $O \subset O_x \setminus \{x\}$, because f is null-homotopic, we know that $f|_{\pi_\Delta^{-1}(O)}$ is null-homotopic and it descends to a null-homotopic function in the quotient by K. Therefore, for any neighbourhood $O'_x \subset O_x$ of x, we can deform f, through K-invariant nonvanishing functions inside $\pi_\Delta^{-1}(O_x)$, to a function that is constant in $\pi_\Delta^{-1}(O'_x)$. Moreover, the deformation can be chosen to be compactly supported. There is no new discriminant created during the deformation because it is through nonvanishing functions (cf. Lemma 5.2). The deformation is constant near the discriminant $\mathcal{A}$ because $N_x \cap \mathcal{A} = \emptyset$. Because the deformation is compactly supported, it can patch with f outside a compact set in $\pi_\Delta^{-1}(O_x)$ to give a family of K-invariant s_1-admissible sections with required properties.

If $\text{Type}(x) = 1$, let δ be the 1-cell in Δ containing x. By simplicity of $(\partial\Delta, \mathcal{A})$ (see introduction and [23], Definition 1.60), there is a vertex v in δ that can be connected to x by a path δ' in δ that does not intersect with $\mathcal{A}$. Let U be the symplectic corner chart associated to v. Without loss of generality, we assume $x_1 \neq 0$ and $x_j = 0$ for $j = 2, 3, 4$. Notice that $\delta' \cap \mathcal{A} = \emptyset$ implies that there exists a neighbourhood $N_{\delta'} \subset \partial\Delta$ of δ' such that $\pi_\Delta^{-1}(N_{\delta'}) \cap \text{Disc}(s_1) = \emptyset$, $\pi_\Delta^{-1}(N_{\delta'})$ is contractible and $\pi_\Delta^{-1}(N_{\delta'}) \cap M_t^s$ is given by equation (20) for some $f \in C^\infty(\pi_\Delta^{-1}(N_{\delta'}), \mathbb{C}^*)$. This implies that for a neighbourhood $O_x \subset N_x \cap N_{\delta'}$ of x, $f|_{\pi_\Delta^{-1}(O_x)}$ is null-homotopic even though $\pi_\Delta^{-1}(O_x)$ is not contractible. In other words,

$$(f|_{\pi_\Delta^{-1}(O_x)})_* : \pi_1(\pi_\Delta^{-1}(O_x)) \to \pi_1(\mathbb{C}^*) = \mathbb{Z} \text{ is zero} \tag{21}$$

and the same is true when $f|_{\pi_\Delta^{-1}(O_x)}$ is descended to the quotient by K. On the other hand, for O_x not containing v, $p_1 > 0$ in $\pi_\Delta^{-1}(O_x)$, so it gives a map $\sqrt{p_1}|_{\pi_\Delta^{-1}(O_x)} \to \mathbb{R}_{>0} \subset \mathbb{C}^*$. Moreover, $(\sqrt{p_1})_* : \pi_1(\pi_\Delta^{-1}(O_x)) \to \pi_1(\mathbb{C}^*)$ is also zero. Therefore, there is no topological obstruction to deform $f|_{\pi_\Delta^{-1}(O_x)}$ to $\sqrt{p_1}|_{\pi_\Delta^{-1}(O_x)}$ inside $\pi_\Delta^{-1}(O_x)$ via K-invariant $\mathbb{C}^*$-valued functions. Most notable, $\mathbb{C}^*$-valued functions are nonvanishing functions. Similar to the previous case, we can assume that the deformation is compactly supported and it gives a family of s_1-admissible sections with required properties by patching with f outside $\pi_\Delta^{-1}(O_x)$.

If $\text{Type}(x) = 2$, we use simplicity of $(\partial\Delta, \mathcal{A})$ again to find a vertex v and a path such that it lies inside a 2-cell of $\partial\Delta$, connects v and x, and does not intersect with $\mathcal{A}$. Let U be the symplectic corner chart associated to v. The equation of M_t^s is again locally given by Equation (20) for some f. Moreover, f is again null-homotopic. If $x_1, x_2 \neq 0$ and $x_3 = x_4 = 0$, we can deform f to $\sqrt{p_1p_2}$ inside $\pi_\Delta^{-1}(O_x)$ for some small neighbourhood O_x of x. This gives our desired family of s_1-admissible sections as in the previous case.

If $\text{Type}(x) = 3$, then we can take O_x such that it does not intersect $0, 1, 2$-cells of Δ. Therefore, we can do any compactly supported deformation of the corresponding f without creating/destroying discriminant loci (i.e., we allow deformation of f via functions that vanish somewhere). It is instructive to compare it with the proof of Lemma 5.2. The outcome is that the lemma is trivially true when $\mathrm{Type}(x) = 3$. □

We are now ready to give the local Lagrangians in $M_t^s \cap \pi_\Delta^{-1}(O_x)$ when M_t^s is x-standard.

Proposition 5.4 (Standard Lagrangian model). *Let $M^s \subset U$ be x-standard for some $x \in \partial \Delta$ and O_x be a neighbourhood of x such that (19) holds. Let W be a rationally generated two-dimensional affine plane in $\mathbb{R}^4$ containing x and $(1,1,1,1) \in TW$. Let $c := W \cap O_x \cap \partial\Delta$ (regarded as a straight line segment in $\partial\Delta$). Then there is a family of proper K-invariant (possibly disconnected) Lagrangian submanifolds L_t in $\pi_\Delta^{-1}(O_x) \cap M_t^s$, for $t > 0$, such that*

(I) *$W^\perp \subset T_{(p,q)} L_t$ for all $(p,q) \in L_t$, and*
(II) *$\pi_\Delta(L_t) \subset W$.*

Moreover, every family of proper K-invariant Lagrangian submanifolds L_t' in $\pi_\Delta^{-1}(O_x) \cap M_t^s$ satisfying $(I), (II)$ can be given one of the following parametrizations (either Case A or Case B).

Let $W_T^\perp$ be the quotient of $W^\perp$ by the lattice $W^\perp \cap (2\pi\mathbb{Z})^4$. Under the natural identification between $W_T^\perp$ and the T^2 subgroup of $(\mathbb{R}/2\pi\mathbb{Z})^4$ in the q-variables, the cyclic group K is either contained in $W_T^\perp$ or $K \cap W_T^\perp = \{0\}$.

Case A. *If K is contained in $W_T^\perp$, then L_t is connected and there exists an $\mathbb{R}^4$-valued function $P(r,t)$ and an $(\mathbb{R}/2\pi\mathbb{Z})^4$-valued function $Q(r,\theta_1,\theta_2,t)$, for $(r,\theta_1,\theta_2) \in (0,1) \times (\mathbb{R}/2\pi\mathbb{Z})^2$, parametrizing L_t. In this case, L_t' is given by*

$$\{(p,q) \in \pi_\Delta^{-1}(O_x) \cap M_t^s | p = P(r,t), q = Q(r,\theta_1,\theta_2,t) + H(r)\} \tag{22}$$

for some $H(r) \in C^\infty((0,1),(\mathbb{R}/2\pi\mathbb{Z})^4)$ satisfying $\sum_j H_j(r) = 0$ for all $r \in (0,1)$, where H_j is the j^{th} component of H.

Case B. *If $K \cap W_T^\perp = \{0\}$, then L_t has $|K|$ connected components and there exists $P(r,t)$ and $Q(r,\theta_1,\theta_2,t)$ as above parametrizing one of the connected components so that the other connected components are parametrized by $p = P(r,t)$ and $q = Q(r,\theta_1,\theta_2,t) + \kappa$ for $\kappa \in K$. In this case, one of the components of L_t' can be parametrized by $p = P(r,t)$ and $q = Q(r,\theta_1,\theta_2,t) + H(r)$ for some $H(r)$ as above and the other components are obtained by adding $\kappa \in K$ in the q coordinates.*

Furthermore, in either Case A or Case B, the family of $\{P(r,t)|r \in (0,1)\}$ Hausdorff converges to c when t approaches 0.

Definition 6. A proper Lagrangian submanifold L_t in $\pi_\Delta^{-1}(O_x) \cap M_t^s$ satisfying Proposition 5.4 (I), (II) is called c**-standard**.

Before giving the proof, it would be helpful to have an intuitive understanding of what L_t looks like. For fixed (r,t), $\{Q(r,\theta,\theta_2,t)|\theta,\theta_2 \in \mathbb{R}/2\pi\mathbb{Z}\}$ is a 2-torus lying inside L_t with p-coordinates being $p = P(r,t)$, so when K is contained in $W_T^\perp$, L_t is a 2-torus bundle over the curve $\{p = P(r,t)\}$ and when $K \cap W_T^\perp = \{0\}$, L_t has $|K|$ connected components and each of them is a 2-torus bundle over the curve. Moreover, condition (I) describes the tangent directions of the 2-torus. Condition (II) implies that the curve $\{p = P(r,t)\}$ is a subset of $W \cap O_x$, which Hausdorff converges to c when t approaches to 0.

Also note that the function $H(r)$ in (22) plays exactly the same role as f in Remark 2.6.

Proof. We have enumerated the possibilities of c in Subsection 5.1.1. Existence of L_t is a simple case by case calculation.

For cases of type A, we have $x = (0,0,0,0)$ and $\pi_\Delta^{-1}(O_x) \cap M_t^s$ is given by

$$(\sqrt{p_1p_2p_3p_4})e^{i(q_1+q_2+q_3+q_4)} = tc_pe^{ic_q}$$

for some constants $c_p > 0$ and $c_q \in \mathbb{R}/2\pi\mathbb{Z}$. In particular, a point $(p,q) \in \pi_\Delta^{-1}(O_x) \cap M_t^s$ has to satisfy $p_1p_2p_3p_4 = t^2c_p^2$. Notice that, for each fixed $t > 0$, $H_t := \{p_1p_2p_3p_4 = t^2c_p^2\}$ is a hyperbola so $H_t \cap W \cap \square$ is a smoothly embedded curve. More rigorously, let $g : \mathbb{R}^4_{\geq 0} \to \mathbb{R}_{\geq 0}$ be $g(p) = p_1p_2p_3p_4$. For each $p \in W \cap (\partial\square)$, the ray $\{p + \lambda(1,1,1,1) | \lambda \geq 0\}$ lies in W and the function $g_p(\lambda) := g(p+\lambda(1,1,1,1))$ is a **strictly** monotonic increasing function on the ray because, for $v := \sum_{j=1}^4 \partial_{p_j}$, we have $v(g) > 0$ over $\mathbb{R}^4_{>0}$. Because $g_p(0) = 0$, for each fixed $t > 0$ there is exactly one $\lambda > 0$ such that $g_p(\lambda) = t^2c_p^2$. This means that for each $p \in W \cap \partial\square$, there is at most one $p' \in W \cap \square$ such that $p' \in H_t$ and $p' = p + \lambda(1,1,1,1)$ for some $\lambda > 0$. Because $W \cap \partial\square$ is a continuous curve, $H_t \cap W \cap \square$ is a smoothing of it and hence a smoothly embedded curve. We define $P_t := H_t \cap W \cap O_x$, which is smooth because it is an open subset of a smooth curve. It is clear that P_t Hausdorff converges to c when t goes to 0.

For each $t > 0$ and $p \in P_t$, we can pick 2-tori $Q_{p,t} \subset \pi_\Delta^{-1}(p)$ such that $Q_{p,t}$ varies smoothly with respect to p, $Q_{p,t}$ is parallel to $W^\perp$ and $e^{i(q_1+q_2+q_3+q_4)} = e^{ic_q}$ for all $q \in Q_{p,t}$. This family of 2-tori gives a submanifold $L_t \subset \pi_\Delta^{-1}(O_x) \cap M_t^s$. The fact that $P_t \subset W$ and $TQ_{p,t} = W^\perp$ for all $p \in P_t$ implies that L_t is a Lagrangian submanifold.

When $K \subset W_T^\perp$, it is easy to see that (22) gives all proper K-invariant Lagrangians satisfying (I), (II).

On the other hand, when $K \cap W_T^\perp = \{0\}$, we replace L_t by its K-orbit. It is also easy to see that any other proper K-invariant Lagrangian satisfying (I), (II) is given by adding a function $H(r)$ to the q-coordinates of all components simultaneously.

For cases of type B, we have $x = (0,0,0,a)$ for some $a > 0$ and we need to consider the set of $p \in W \cap O_x$ that solves $\sqrt{p_1p_2p_3} = tc_p$. This time, we can take $g(p) = p_1p_2p_3$ for $p \in \mathbb{R}^4_{\geq 0}$ and $g_p(\lambda) = g(p + \lambda(1,1,1,1))$ for $p \in W \cap \{p_1p_2p_3 = 0\}$, and $\lambda \geq 0$. Let $v = \partial_{p_1} + \partial_{p_2} + \partial_{p_3} + \partial_{p_4}$ and we have $v(g) > 0$ over $\mathbb{R}^4_{>0}$. Similar to the previous case, it means that for each $p \in W \cap \{p_1p_2p_3 = 0\}$, there is at most one $p' \in W \cap \square$ such that $p' \in H_t$ and $p' = p + \lambda(1,1,1,1)$ for some $\lambda > 0$. The rest of the argument is the same.

For cases of type C or $\mathrm{Type}(x) = 3$, we need to consider $p \in W \cap O_x$ that solves $\sqrt{p_1p_2} = tc_p$ and $\sqrt{p_1} = tc_p$, respectively. The rest of the argument is the same. □

5.1.3. Gluing local Lagrangians

In the previous sub-subsection, we explained how to construct a local Lagrangian when M^s is x-standard. Now, suppose $c : [-1,1] \to \partial\square$ (again, $\mathrm{Im}(c)$ is regarded as a closed segment of a tropical curve γ) has the property that $\mathrm{Type}(c(r))$ is discontinuous at $r = 0$ and M^s is $c(0)$-standard with respect to U. Then M^s is not $c(r)$ standard with respect to U for any r close to but not equal to 0. Therefore, we need to generalise Proposition 5.4 and explain how to glue the local Lagrangian models together.

Definition 7. Let U be a symplectic corner chart and $\square = \pi_\Delta(U)$. Let $c^\circ \subset \partial\square \setminus \mathcal{A}$ be an open straight line segment. Let $c : [0,1] \to \partial\square \setminus \mathcal{A}$ be a straight line such that $c((0,1)) = c^\circ$ and $\mathrm{Type}(c(0)) \leq \mathrm{Type}(c(1)) = \mathrm{Type}(c(r))$ for $r \in (0,1]$. Given an admissible section $s \in C^\infty(\mathbb{P}_\Delta, \mathcal{L})$, we say that M^s is c°**-transition-standard with respect to** U if M^s is $c(0)$-standard and $c(1)$-standard with respect to U and there is a neighbourhood $O_{c^\circ} \subset \square \setminus \mathcal{A}$ of c° such that c° is proper inside O_{c° and $\pi_\Delta^{-1}(O_{c^\circ}) \cap M_t^s$ is given by

$$\left(\prod_{j, c(0)_j = 0} \sqrt{p_j}\right) e^{i(q_1+q_2+q_3+q_4)} = tf_p \left(\prod_{j, c(0)_j = 0, c(1)_j \neq 0} p_j\right) e^{if_q(p)} \tag{23}$$

for some function $f_q \in C^\infty(U, \mathbb{R}/2\pi\mathbb{Z})$ depending only on p (in particular, K-invariant), and some $f_p \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}_{>0})$ is such that $\frac{u}{f_p^2(u)}$ is a monotonic increasing function and f_p is an interpolation from c_p to $c'_p\sqrt{u}$ for some constants $c_p, c'_p > 0$. In (23), $c(k)_j$ is the p_j-coordinate of $c(k)$ for $k = 0, 1$, and, whenever $\mathrm{Type}(c(0)) = \mathrm{Type}(c(1))$, $\prod_{j, c(0)_j = 0, c(1)_j \neq 0} p_j$ (which is a product over the empty set) is interpreted as 1.

We say that M^s is c°**-transition-standard** if M^s is c°-transition-standard with respect to some symplectic corner chart.

Remark 5.5. Note that for M^s to be $c(0)$-standard and $c(1)$-standard simultaneously, it is necessary for f_p to be an interpolation from c_p to $c'_p\sqrt{u}$. The monotonicity of $\frac{u}{f_p^2(u)}$ is imposed to achieve Lemma 5.8 below.

Lemma 5.6. *Let s be an s_1-admissible section. Let $c : [0,1] \to \partial\square\backslash\mathcal{A}$ be a straight line such that* $\mathrm{Type}(c(0)) \le \mathrm{Type}(c(1)) = \mathrm{Type}(c(r))$ *for all $r \in (0,1]$. Let $c^\circ := c((0,1))$ and N_c be a neighbourhood of* $\mathrm{Im}(c)$ *in $\square\backslash\mathcal{A}$. Then there is a symplectic corner chart U and a family of s_1-admissible section $(s^u)_{u\in[0,1]}$ such that $s^0 = s$, for all u, $s^u = s$ outside $\pi_\Delta^{-1}(N_c)$, s^u is K-invariant in U, and M^{s^1} is c°-transition-standard with respect to U.*

Proof. The proof is in parallel to Lemma 5.3. We give the details when $\mathrm{Type}(c(0)) = 1 < \mathrm{Type}(c(1))$ and leave the remaining to the readers.

Let $x = c(0)$. We pick a vertex v and the corresponding symplectic corner chart U as in the proof of Lemma 5.3. We can find a neighbourhood O_c of $\mathrm{Im}(c)$ such that $M_t^s \cap \pi_\Delta^{-1}(O_c)$ is given by (20) for some $f \in C^\infty(\pi_\Delta^{-1}(O_c), \mathbb{C}^*)$ and $(f)_* : \pi_1(\pi_\Delta^{-1}(O_c)) \to \pi_1(\mathbb{C}^*)$ is the zero map (see (21)). Say $x_j \neq 0$ exactly when $j = 1$ and $c(1)_j \neq 0$ exactly when $j = 1, \dots, n_c$ (here $n_c \in \{2,3\}$). Let $g(r) = \prod_{j, c(0)_j=0, c(1)_j\neq 0} c(r)_j = c(r)_2 \dots c(r)_{n_c}$, where $c(r)_j$ is the jth-coordinate of $c(r)$ for $r \in [0,1]$. Note that $g(0) = 0$ and $g(r)$ is strictly increasing.

Inside $\pi_\Delta^{-1}(O_c)$, there is no topological obstruction to deform f through K-invariant nonvanishing functions to $\sqrt{p_1} f_p(p_2 \dots p_{n_c}) e^{i f_q(p)}$ for some $f_p \in C^\infty(\mathbb{R}_{\ge 0}, \mathbb{R}_{>0})$ such that $\frac{u}{f_p^2(u)}$ is a monotonic increasing function, and there are constants $c_p, c'_p > 0$ such that $f_p(u) = c_p$ near $u = 0$ and $f_p(u) = c'_p\sqrt{u}$ near $u = g(1)$. The conditions on f_p near $u = 0$ and $u = g(1)$ imply that M^s is $c(0)$-standard and $c(1)$-standard simultaneously. Moreover, there exists $O_{c^\circ} \subset O_c$ such that c° is proper inside O_{c° and $\pi_\Delta^{-1}(O_{c^\circ}) \cap M_t^s$ satisfies (23). Therefore, the result follows. □

A simple but crucial observation is that we can extend the ‘standard region’ by a further isotopy without destroying the previously established standard region in the following sense.

Lemma 5.7. *Let $c_1, c_2 : [0,1] \to \partial\square\backslash\mathcal{A}$ be two straight lines as in Lemma 5.6 such that $c_1(0) = c_2(0)$ or $c_1(0) = c_2(1)$ or $c_1(1) = c_2(1)$. Suppose we have applied Lemma 5.6 to c_1 and denote the resulting s^1 as s. Let N_2 be a neighbourhood of* $\mathrm{Im}(c_2)$ *in $\square\backslash\mathcal{A}$. Then there is a family of s_1-admissible sections $(s^u)_{u\in[0,1]}$ such that $s^0 = s$, for all u, $s^u = s$ outside $\pi_\Delta^{-1}(N_2)$, s^u is K-invariant in U, and M^{s^1} is simultaneously c_1°-transition-standard and c_2°-transition-standard with respect to U.*

Proof. We want to apply (the proof) of Lemma 5.6 to c_2. The key point is that, inside $\pi_\Delta^{-1}(O_{c_2})$, there is no topological obstruction to deform f through K-invariant nonvanishing functions to a function as in Lemma 5.6 and, in addition, we are free to choose the deformation to be trivial inside $\pi_\Delta^{-1}(O_{c_1})$ for some small neighbourhood O_{c_1} of $\mathrm{Im}(c_1)$. In this case, the corresponding section s^1 will make M^{s^1} simultaneously c_1°-transition-standard and c_2°-transition-standard. □

Lemma 5.8. *Let M^s be c°-transition-standard with respect to U. Let $O_{c^\circ} \subset \square$ be a neighbourhood of c° such that (23) holds. Let W be the rationally generated two-dimensional plane in $\mathbb{R}^4$ that contains c° and $(1,1,1,1) \in TW$. Then there exists a family of proper K-invariant (possibly disconnected) Lagrangian submanifold L_t in $\pi_\Delta^{-1}(O_{c^\circ}) \cap M_t^s$, for $t > 0$, such that*

(I) *$W^\perp \subset T_{(p,q)}L_t$ for all $(p,q) \in L_t$, and*
(II) *$\pi_\Delta(L_t) \subset W$.*

Moreover, $\pi_\Delta(L_t)$ Hausdorff converges to c° when t approaches 0.

Proof. Similar to Proposition 5.4, because f_q only depends on p, it suffices to show that for each $t > 0$, the set of $p \in W \cap O_{c^\circ}$ that solves

$$\prod_{j,c(0)_j=0} \sqrt{p_j} = t f_p \left(\prod_{j,c(0)_j=0,c(1)_j\neq 0} p_j \right) \tag{24}$$

is an open subset of a smoothly embedded curve.

We only consider the case that $\mathrm{Type}(c(0)) = 0$ and $\mathrm{Type}(c(1)) = 3$. The other cases can be dealt with similarly. In this case $\pi_\Delta^{-1}(O_c) \cap M_t^s$ is given by

$$(\sqrt{p_1 p_2 p_3 p_4}) e^{i(q_1+q_2+q_3+q_4)} = t f_p (p_2 p_3 p_4) e^{i f_q(p)}.$$

Notice that $\nu := \sum_{j=1}^4 \partial_{p_j}$ satisfies $\nu(\frac{p_1p_2p_3p_4}{f_p^2(p_2p_3p_4)}) > 0$ for all $p \in \square \backslash \partial \square$ because we assumed that $\frac{u}{f_p^2(u)}$ is monotonic increasing and $\partial_{p_1}(\frac{p_1p_2p_3p_4}{f_p^2(p_2p_3p_4)}) = \frac{p_2p_3p_4}{f_p^2(p_2p_3p_4)} > 0$ for all $p \in \square \backslash \partial \square$. The rest of the argument is the same. $\square$

Definition 8. A proper K-invariant Lagrangian submanifold L_t in $\pi_\Delta^{-1}(O_{c^\circ}) \cap M_t^s$ satisfying Lemma 5.8 (I), (II) is called c°**-transition-standard**.

We summarise the steps taken so far.

Proposition 5.9. *Let $c : [0,1] \to \partial \square \backslash \mathcal{A}$ be a straight line and N_c be a neighbourhood of* $\mathrm{Im}(c)$ *in $\square \setminus \mathcal{A}$. Let $c^\circ = c((0,1))$. Then for any s_1-admissible section s, there is a family of s_1-admissible section $(s^u)_{u\in[0,1]}$ such that $s^0 = s$, for all u, $s^u = s$ outside $\pi_\Delta^{-1}(N_c)$, s^u is K-invariant, and for all $x \in$* $\mathrm{Im}(c)$*, M^{s^1} is either x-standard with respect to U or there exists an open line segment $c_x^\circ \subset c^\circ$ containing x such that M^{s^1} is c_x°-transition-standard with respect to U.*

Moreover, there is a neighbourhood $O_{c^\circ} \subset \square \backslash \mathcal{A}$ of c° and a family of proper K-invariant Lagrangian L_t in $M_t^{s^1} \cap \pi_\Delta^{-1}(O_{c^\circ})$, for $t > 0$, such that c° is proper inside O_{c°, L_t is a 2-torus bundle (or union of $|K|$ disjoint 2-torus bundles) with respect to π_Δ and $\pi_\Delta(L_t)$ Hausdorff converges to c° as t goes to 0.

Proof. The function $\mathrm{Type}(c(r))$ is discontinuous at finitely many points, say, at $0 \le r_1 < \cdots < r_k \le 1$. By extending c slightly, we assume $r_1 > 0$ and $r_k < 1$. Pick a $d_j \in (r_j, r_{j+1})$ for $j = 1, \ldots, k-1$. Let $d_0 = 0$ and $d_k = 1$. For $j = 1, \ldots, k$, let $c_j^+(r) = c|_{[r_j,d_j]}(r)$ and $c_j^-(-r) = c|_{[d_{j-1},r_j]}(r)$. By reparametrizing the domain of $c_j^\pm$, we can assume that they satisfy the assumption of Lemma 5.6. We can apply Lemma 5.6 and 5.7 to the neighbourhoods of $\{\mathrm{Im}(c_j^\pm)\}_{j=1}^k$ to find a family of s_1-admissible section $(s^u)_{u\in[0,1]}$ such that $s^0 = s$, for all u, $s^u = s$ outside $\pi_\Delta^{-1}(N_c)$, s^u is K-invariant, and M^{s^1} is $(c_j^\pm)^\circ$-transition-standard with respect to U for all $(c_j^\pm)^\circ$, where $(c_j^\pm)^\circ$ is the set of interior points of $\mathrm{Im}(c_j^\pm)$.

For each $c_j^\pm$, we obtain a K-invariant Lagrangian $(L_t)_{c_j^\pm}$ by Lemma 5.8 such that (I) and (II) are satisfied. By definition of transition-standard, M^{s^1} is x-standard with respect to U for $x = d_0, \ldots, d_k, r_1, \ldots, r_k$. Therefore, by Proposition 5.4, there exist neighbourhoods O_{d_j} of d_j such that $(L_t)_{c_j^+} \cap \pi_\Delta^{-1}(O_x)$ and $(L_t)_{c_{j+1}^-} \cap \pi_\Delta^{-1}(O_x)$ are given by (22) for some appropriate P, Q, H. By interpolating the H, we can concatenate the K-invariant Lagrangians $(L_t)_{c_j^+} \cap \pi_\Delta^{-1}(O_x)$ and $(L_t)_{c_{j+1}^-} \cap \pi_\Delta^{-1}(O_x)$ (for all $j = 1, \ldots, k-1$), so the result follows. $\square$

5.1.4. Transition between symplectic corner charts

Because tropical curves considered in Theorem 1.1 are not necessarily contained in a single $\square$, we now want to explain the transition between different symplectic corner charts and then how the Lagrangians from different symplectic corner charts can be glued together. The key conclusion we want to draw is that being x-standard is independent of choice of symplectic corner charts when x is suitably far away from $\mathcal{A}$ (see Corollary 5.11).

Let U_0, U_1 be symplectic corner charts at vertices v_0 and v_1 of Δ, respectively. We assume that v_0 and v_1 are connected by a 1-cell δ in Δ. Recall that $\Delta = \Delta_X + \Delta'$ and thus both v_0 and v_1 decompose accordingly, say, as $v_0 = v_0^X + v_0'$ and $v_1 = v_1^X + v_1'$. From the description of the monodromy in [21, Proposition 3.15] that involves the vector $v_0^X - v_1^X$, we see that $v_0^X = v_1^X$ holds if and only if δ does not meet the discriminant $\mathcal{A}$. Let us assume that the reflexive polytope Δ_X has been translated so that its unique interior lattice point coincides with the origin. By the Gorenstein assumption, the monoid $\big(\mathbb{R}_{\geq 0}(\Delta - v_0)\big) \cap \mathbb{Z}^4$ is Gorenstein; that is, the ideal of integral points in its interior is generated by a single element and this element is $-v_0^X$ (the Gorenstein character). A similar statement holds if we replace v_0 by v_1. We conclude the following consequence from this observation.

Lemma 5.10. *Let U_0/K_0 and U_1/K_1 be corner charts at v_0 and v_1 respectively connected by an edge δ so that $\delta \cap \mathcal{A} = \emptyset$, then the Gorenstein characters of U_0/K_0 and U_1/K_1 agree on the overlap $U_1/K_1 \cap U_0/K_0$.*

Lemma 5.11. *Let $x \in \partial\Delta$ and Π be the cell of $\partial\Delta$ whose interior contains x. Let v_0, v_1 be vertices of Π such that there exists a union of 1-cells $\{\delta_i\}_i$ in Π connecting v_0 and v_1 and $\delta_i \cap \mathcal{A} = \emptyset$ for all i. Let $U_0/K_0, U_1/K_1$ be the symplectic corner charts at v_0 and v_1, respectively. If M^s is x-standard with respect to U_0 then M^s is x-standard with respect to U_1.*

Proof. It suffices to assume that v_0 and v_1 are the endpoints of a single edge δ with $\delta \cap \mathcal{A} = \emptyset$. By the standardness assumption on U_0, there is a neighbourhood O_x of x such that the hypersurface $M_t^s \cap \pi_\Delta^{-1}(O_x)$ in the coordinates of U_0 is given by

$$\left(\prod_{j, x_j = 0} \sqrt{p_j}\right) e^{i(q_1+q_2+q_3+q_4)} = tc. \tag{25}$$

The Gorenstein character $e^{i(q_1+q_2+q_3+q_4)}$ descends to U_0/K_0 and by Lemma 5.11, it agrees with the one in U_1. The coordinate p_j measures the distance from the corresponding facet of Δ that contains x. In other words, the map $p \mapsto p_j$ is given by pairing with a dual vector that is an inward normal to the facet. This is true for both U_1 and U_0. Note that the interior of the facet corresponding to $x_j = 0$ is necessarily contained in both $\pi_\Delta(U_0/K_0)$ and $\pi_\Delta(U_1/K_1)$ because both charts contain x and x lies in that facet. Because the coordinate transformation of the p-coordinates from U_0 to U_1 is affine $\mathbb{Q}$-linear and identifies the respective placements of the polytope, it follows that, if $x_j = 0$, the respective coordinates p_j for U_0 and U_1 are constant multiples of one another. If we transform (25) from the coordinates of U_0 to the coordinates of U_1, the left-hand side takes the same shape up to multiplication by a constant that we can absorb into the constant c on the right, so we see that M^s is also x-standard with respect to U_1. □

When $\mathrm{Type}(x) = 3$, we can remove the assumption that v_0 and v_1 are connected by a union of 1-cells, in the following sense.

Lemma 5.12. *Let Π and x be as in Lemma 5.11 but we assume that* $\mathrm{Type}(x) = 3$ *(so* $\dim(\Pi) = 3$*). Let v_0, v_1 be vertices of Π and U_0, U_1 be the corresponding corner charts. If M^s is x-standard such that equation (19) holds for $c = 0$ with respect to U_0, then the same is true with respect to U_1.*

Proof. That equation (19) holds for $c = 0$ implies that $M_t^s \cap \pi_\Delta^{-1}(O_x)$ coincides with $\pi_\Delta^{-1}(O_x \cap \Pi)$, which is independent of coordinates. Therefore, it is true with respect to U_0 if and only if it is true with respect to U_1. □

5.2. *Trivalent vertex*

In this subsection, we construct a local Lagrangian modeled on a trivalent vertex of a tropical curve γ. Near the trivalent vertex, γ is contained in a two-dimensional plane, so we start our construction in T^*T^2.

Lemma 5.13. *In T^*T^2, there is a Lagrangian pair of pants L such that outside a compact set, L coincides with the union of the negative co-normal bundles of a* $(1, 0)$ *and* $(0, 1)$ *curve and the positive co-normal bundle of a* $(1, 1)$ *curve.*

Proof. Let r_i, θ_i be the polar coordinates of $\mathbb{R}^2\backslash\{0\}$ for $i = 1, 2$. For a symplectic form on $(\mathbb{R}^2\backslash\{0\})^2$ we use $\omega := \sum_i d(\log(r_i)) \wedge d\theta_i$. Now, T^*T^2 is symplectomorphic to $((\mathbb{R}^2\backslash\{0\})^2, \omega)$ by the identification $(p_i, q_i) = (\log(r_i), \theta_i)$, where the q_i are the base coordinates of T^*T^2. In the complex coordinate $z_j = r_j e^{i\theta_j}$, the holomorphic pair of pants $H = \{(z_1, z_2) \mid z_1 + z_2 = 1\}$ is given by

$$r_1 \cos(\theta_1) + r_2 \cos(\theta_2) = 1, \quad r_1 \sin(\theta_1) + r_2 \sin(\theta_2) = 0.$$

To obtain a Lagrangian pair of pants, we use hyperkähler rotation. Concretely, by transforming $\theta_1 \mapsto \theta_2, \theta_2 \mapsto -\theta_1$ keeping r_1, r_2 fixed, we know that

$$L := \left\{ (r_1, \theta_1, r_2, \theta_2) \in (\mathbb{R}^2\backslash\{0\})^2 \;\middle|\; \begin{matrix} r_1 \cos(\theta_2) + r_2 \cos(\theta_1) = 1, \\ -r_2 \sin(\theta_1) + r_1 \sin(\theta_2) = 0 \end{matrix} \right\}$$

is diffeomorphic to a pair of pants. The three punctures correspond to $r_1 = 0$, $r_2 = 0$ and $r_1 = r_2 = \infty$, respectively. We next check that L is Lagrangian.

The tangent space of L is spanned by

$$\cos(\theta_1)\partial_{r_1} - \frac{\sin(\theta_1)}{r_1}\partial_{\theta_2} - \cos(\theta_2)\partial_{r_2} + \frac{\sin(\theta_2)}{r_2}\partial_{\theta_1}, \tag{26}$$

$$\sin(\theta_1)\partial_{r_1} + \frac{\cos(\theta_1)}{r_1}\partial_{\theta_2} + \sin(\theta_2)\partial_{r_2} + \frac{\cos(\theta_2)}{r_2}\partial_{\theta_1} \tag{27}$$

as can be checked by applying these to the defining equations of L. Computing $\omega((26), (27))$ gives zero; hence L is Lagrangian.

Let $\pi : T^*T^2 \to \mathbb{R}^2$ be the projection $\pi(p_i, q_i) = (p_1, p_2)$, which is a Lagrangian torus fibre bundle. Note that $\pi(L) = \pi(H)$, which is an amoeba with three legs asymptotic to the negative p_1 axis, the negative p_2 axis and the line $\{p_1 = p_2 | p_1 > 0\}$. More precisely, when $r_1 > 0$ is sufficiently small, θ_1 is close to 0 and r_2 is close to 1. The situation is similar when $r_2 > 0$ is sufficiently small. When r_1, r_2 are sufficiently large, we consider the equation $r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 + \theta_2) = 1$ obtained by sum of squares of two defining equations of L. This implies that $1 \geq (r_1 - r_2)^2$ and $\cos(\theta_1 + \theta_2)$ is close to -1 when r_1, r_2 large, which in turn implies that $\frac{r_1}{r_2}$ is close to 1 and $\theta_1 + \theta_2$ is close to $-\pi$. To complete the proof, it suffices to deform L to another Lagrangian L' such that the three legs of $\pi(L')$ completely coincide with the asymptotic lines outside a compact set.

We now explain the deformation procedure. One can check that $\alpha := p_i dq_i$ is exact when restricted to L by showing that $\int_{c_i} \alpha = 0$, where c_i are simple closed loops wrapping around the asymptotes $r_i = 0$ for $i = 1, 2$. Define

$$E_1 := \{(p_1, q_1, p_2, q_2) \,|q_1 = 0, p_2 = 0\},$$

which is the co-normal bundle of $\{q_1 = p_1 = p_2 = 0\} \subset \{p_1 = p_2 = 0\}$ when we identify $\{p_1 = p_2 = 0\}$ with the zero section of T^*T^2. In particular, E_1 is a Lagrangian. The projection $\pi_1 : L \to E_1$ defined by $\pi_1(p_1, q_1, p_2, q_2) = (p_1, 0, 0, q_2)$ is injective and submersive near the end corresponding to $p_1 = -\infty$. By locally identifying a neighbourhood of the zero section of T^*E_1 with an open subset of T^*T^2, L can be identified as a section of $T^*E_1 \to E_1$ near $p_1 = -\infty$. Because we checked that L is exact for α, one can find a Hamiltonian isotopy to move this end of L to E_1. For the end of L corresponding to $p_2 = -\infty$ and $p_1 = p_2 = \infty$, we can take $E_2 := \{(p_1, q_1, p_2, q_2)|q_2 = 0, p_1 = 0\}$ and $E_3 := \{(p_1, q_1, p_2, q_2)|p_1 = p_2, q_1 = -\pi - q_2\}$ to substitute E_1, and $\pi_2(p_1, q_1, p_2, q_2) = (0, q_1, p_2, 0)$ and $\pi_3(p_1, q_1, p_2, q_2) = (p_1, -\pi - q_2, p_1, q_2)$ to substitute π_1, respectively. This completes the proof. □

By multiplying Lemma 5.13 with a trivial T^*S^1 factor, we have the following.

Corollary 5.14. *In T^*T^3, there is a Lagrangian pair of pants times circle L such that outside a compact set, L coincides with the union of the negative co-normal bundles of a $(1,0,0)$-curve times a $(0,0,1)$-curve, of a $(0,1,0)$-curve times a $(0,0,1)$-curve, and the positive co-normal bundle of a $(1,1,0)$ curve times a $(0,0,1)$-curve.*

By applying backward Liouville flow for the standard Liouville structure on T^*T^3, we can assume L to lie inside a small open neighbourhood of the union of the zero section T^3 and the negative/positive co-normal bundles, and the neighbourhood is as small as we want.

Lemma 5.15. *Let M^s be x-standard for some $x \in \partial\square\setminus\mathcal{A}$ so that $\pi_\Delta^{-1}(O'_x)\cap M_t^s$ is given by equation (19) for a small neighbourhood O'_x of x. Let $c_j : [0,1) \to O'_x\cap\partial\square$ for $j = 1,2,3$ be proper straight lines such that $c_j(0) = x$ for all j. Assume the directions of c_j is integral linearly equivalent to $\{e_1, e_2, -e_1-e_2\}$ with respect to the integral affine structure on $\partial\square\setminus\mathcal{A}$. Then there exists a small neighbourhood $O_x \subset O'_x$ of x, small neighbourhoods $O_{c_j} \subset O'_x$ of $\mathrm{Im}(c_j)$ and a family of proper Lagrangian pair of pants times circle $L_t \subset \pi_\Delta^{-1}(O'_x)\cap M_t^s$, for $t > 0$, such that $L_t \cap \pi_\Delta^{-1}(O_{c_j})$ is $\mathrm{Im}(c_j)$-standard outside $\pi_\Delta^{-1}(O_x)$ for $j = 1,2,3$.*

Proof. By Proposition 5.4, we can construct $\mathrm{Im}(c_j)$-standard Lagrangian L_j in $\pi_\Delta^{-1}(O'_x)\cap M_t^s$. The set of p-coordinates of L_j is determined by condition (II) in Proposition 5.4. Let the set of p-coordinates of L_j be P_j. Notice that $\cap_{j=1,2,3}P_j$ is a singleton given by the unique element in $\pi_\Delta(M_t^s)$ such that $p_1 - x_1 = \cdots = p_4 - x_4$. Let p^* be the unique element in $\cap_{j=1,2,3}P_j$ and $T_{p^*} := \pi_\Delta^{-1}(p^*)\cap M_t^s$ be the Lagrangian T^3 in M_t^s.

The assumption of the directions of c_j implies that, for some choice of coordinates in T_{p^*}, the intersection pattern of L_j with T_{p^*} is exactly given by $(1,0,0)$-curve times $(0,0,1)$-curve, $(0,1,0)$-curve times $(0,0,1)$-curve and $(1,1,0)$ curve times $(0,0,1)$-curve. We can do a Hamiltonian perturbation of L_j such that, with respect to a choice of Weinstein neighbourhood of T_{p^*}, L_j coincides with the negative co-normal bundles of a $(1,0,0)$-curve times $(0,0,1)$-curve, $(0,1,0)$-curve times $(0,0,1)$-curve, and the positive co-normal bundle of a $(1,1,0)$ curve times $(0,0,1)$-curve.

We can also adjust $T_{p^*}\cap L_j$ by parallel translation of the 2-tori using $H(u)$ in Proposition 5.4 if necessary. Therefore, we can apply Corollary 5.14 to glue the L_j together and obtain a proper Lagrangian pair of pants times circle L_t. It is clear that $L_t\cap\pi_\Delta^{-1}(O_{c_j})$ is $\mathrm{Im}(c_j)$-standard outside $\pi_\Delta^{-1}(O_x)$ for some small neighbourhood O_x of x. □

5.3. Assembling local Lagrangian pieces away from the discriminant

We apply the results in the previous two subsections and conclude the construction of the Lagrangian away from the discriminant.

Terminology 5.16. *A solid torus is a manifold diffeomorphic to $S^1 \times \{z \in \mathbb{C} \mid |z| \le 1\}$. An open solid torus is a manifold diffeomorphic to the interior of a solid torus.*

Let γ be an admissible tropical curve (see the assumption of Theorem 1.1). Let N be a neighbourhood of γ and $B' \subset B \subset N$ be small open tubular neighbourhoods of the ends of γ such that the closure $\overline{B}'$ of B' lies inside B. In particular, we can write $B = \cup_e B_e$ and $B' = \cup_e B'_e$ where the union is taken over all the ends e of γ and B_e, B'_e are small topological balls containing e.

Proposition 5.17. *Suppose there exists an s_1-admissible section s and, for all $t > 0$ small and for each end e, a Lagrangian open solid torus L_t^e in $\pi_\Delta^{-1}(B)\cap M_t^s$ such that L_t^e is $(B_e\setminus\overline{B}'_e)\cap\gamma$-standard, and the directions of the meridian and longitude of L_t^e with respect to the integral affine structure are as in L_v in Subsection 2.6. Then, for all $t > 0$ sufficiently small, there is a closed Lagrangian $L_t \subset M_t^s$ such that L_t is diffeomorphic to a Lagrangian lift of γ and $\pi_\Delta(L_t) \subset N$. Moreover, we have $w(L_t) = \mathrm{mult}(\gamma)$.*

Proof. We first explain the construction of L_t and the proof concept is the same as for Proposition 5.9. Let $D := \{d_i\}_{i=1}^K \subset \gamma\setminus B$ be a finite collection of points such that it contains all of the trivalent points of γ and all of the points in $\partial\overline{B} \cap \gamma$. By adding more points to D if necessary, we can assume that every point x on γ is contained in the image of a curve $c : [0,1] \to \square \setminus \mathcal{A}$, for some $\square$, such that $\mathrm{Type}(c(0)) \leq \mathrm{Type}(c(r)) = \mathrm{Type}(c(1))$ for all $r \in (0,1]$. In particular, it implies that the interval between two adjacent points d, d' of D (adjacent with respect to the topology on γ) is the image of such a curve c. We denote the open (respectively closed) interval between two adjacent points d, d' by (d, d') (respectively $[d, d']$).

By repeatedly applying Lemma 5.7, we get a new s_1-admissible section s' such that $M^{s'}$ is (d, d')-transition-standard for all adjacent points $d, d' \in D$ and $s' = s$ outside $\pi_\Delta^{-1}(N)$. Moreover, because M^s is standard for points in $(B \setminus \overline{B}') \cap \gamma$ a priori, when we apply Lemma 5.7, we can assume that the outcome s' equals s inside $\pi_\Delta^{-1}(B)$.

As a consequence of $M^{s'}$ being (d, d')-transition-standard, $M^{s'}$ is x-standard for all $x \in D$ (here, we use Corollary 5.11 and Lemma 5.12 to guarantee that being x-standard is independent of corner charts: if $y \in (d, d')$ has $\mathrm{Type}(y) = 3$, we apply Lemma 5.12; if $y \in (d, d')$ has $\mathrm{Type}(y) < 3$, the assumption of Corollary 5.11 will be satisfied, so we can apply Corollary 5.11). In particular, $M^{s'}$ is x-standard for all trivalent points x of γ. Let x be a trivalent point of γ and let $d_{i_1}, d_{i_2}, d_{i_3} \in D$ be the three adjacent points of x on the three incident edges of x, respectively. We can apply Lemma 5.15 at x. The result is a point $b_{i_k} \in (x, d_{i_k})$ for each $k = 1, 2, 3$ such that, for all $t > 0$ small, there exists a Lagrangian pair of pants times circle $L_t^x \subset M^{s'}$ such that L_t^x is (x, b_{i_k})-standard outside the preimage of a small neighbourhood O_x of x under π_Δ. Because $M^{s'}$ is (x, d_{i_k})-transition-standard for all $k = 1, 2, 3$, we can apply Lemma 5.8 to extend L_t^x so that it becomes (x, d_{i_k})-transition-standard outside $\pi_\Delta^{-1}(O_x)$.

Now, as in the proof of Proposition 5.9, for all adjacent $d, d' \in D$ such that d, d' are not trivalent points of γ, we can also construct Lagrangian local pieces in $M^{s'}$ that are (d, d')-transition-standard. Moreover, we can glue these local pieces together smoothly to get, for all $t > 0$ small, a closed Lagrangian L_t.

Because s and s' are interpolated by a family of s_1-admissible sections that is unchanged outside $\pi_\Delta^{-1}(N)$, we can apply Lemma 4.3 to conclude that $L_t \subset M_t^{s'}$ can be brought back, via a symplectic isotopy, to a closed embedded Lagrangian inside $M_t^s \cap \pi_\Delta^{-1}(N)$.

Finally, for the diffeomorphism type and topology of L_t, it is clear from the construction that the diffeomorphism type of L_t is governed by γ and coincides with Definition 2. In particular, for a rigid γ of genus zero, L_t is a rational homology sphere and $w(L) = \mathrm{mult}(\gamma)$. $\square$

6. Near the discriminant

In this section, we explain the construction of a local Lagrangian solid torus that serves as capping off the Lagrangian 3-folds near the discriminant. We first explain the case where $\mathbb{P}_\Delta$ is a toric manifold; Subsection 6.8.1 reduces the more general orbifold situation to the manifold case.

Let U be a symplectic corner chart. As explained in Section 3 (see (5)), we have an explicit diffeomorphism $\Phi_U : U \to \mathbb{C}^4$ given by

$$w_j = \exp(u_j + iv_j) = \Phi_{U,j}(z) = \exp\left(\frac{\partial f_J(p)}{\partial p_j}\right)\exp(iq_j), \tag{28}$$

where $(w_1, \ldots, w_4) \in \mathbb{C}^4$, $z = (z_1, \ldots, z_4)$, $z_j = \sqrt{2p_j}\exp(iq_j)$ and $\Phi_U = (\Phi_{U,1}, \ldots, \Phi_{U,4})$.

Let $s_1 \in H^0(\mathbb{P}_\Delta, \mathcal{L})_{\mathrm{Reg}}$ and $M_t := M_t^{s_1}$ (see (18)). For the purpose of capping off the Lagrangian, we will make an assumption on the shape of the discriminant near the ending. Say the piece of the discriminant that we want to cap off the Lagrangian at is contained in the complex two-dimensional stratum $T = \{w_1 = w_2 = 0, w_3 w_4 \neq 0\}$.

Assumption 6.1. *In this section, we assume*

$$M_t \cap U = \Phi_U^{-1}(\{w_1 w_2 w_3 w_4 = tg(w)\}), \tag{29}$$

where $g(w) := c(b - w_3) + w_1 h_1(w) + w_2 h_2(w)$ *for some polynomial functions* $h_1, h_2 : \mathbb{C}^4 \to \mathbb{C}$ *and constants* $b, c \in \mathbb{C}^*$*. In other words, the restriction of* g *to* T *is constant in* w_4 *and degree one in* w_3*.*

Remark 6.2. As a consequence of Lemma 2.5, the situation of Assumption 6.1 is equivalent to the corner chart U being based at a vertex with an adjacent 2-cell that deforms to a 1-cell in $\Delta_{\check{X}}$. Furthermore, g being locally of this form is equivalent to its amoeba $\mathcal{A}$ locally being one-dimensional. By the admissibility assumption on the tropical curves that we build Lagrangians for, its univalent vertices permit a nearby vertex of Δ that is contained in a 2-cell so that the associated chart Φ_U gives the hypersurface the form of (29) for T the toric 2-stratum associated to the 2-cell in Δ that contains the univalent vertex of γ. $\mathcal{A}$ is locally of dimension one if and only if (29) holds.

Remark 6.3 (Trivalent vertex). It is natural to ask whether Theorem 1.1 can be generalised to tropical curves whose univalent vertices end at a codimension one part of $\mathcal{A}$. In this case, the local model is

$$M_t \cap U = \Phi_U^{-1}(\{w_1 w_2 w_3 w_4 = t g(w)\}) \tag{30}$$

and $g(w) = c(b - b_3 w_3 - b_4 w_4) + w_1 h_1(w) + w_2 h_2(w)$ for some polynomial functions $h_1, h_2 : \mathbb{C}^4 \to \mathbb{C}$ and constants $b, c, b_3, b_4 \in \mathbb{C}^*$.

The key difficulty for this generalisation is whether one can straighten the discriminant as in Proposition 6.19. More details will be explained in Remark 6.20.

Let $g_0 := g|_T$. Because $g_0(w_3, w_4) = g(0, 0, w_3, w_4) = c(b - w_3)$, the discriminant $\mathrm{Disc}(s_1)$ intersected with the stratum T is

$$\mathrm{Disc}^0(s_1) := \mathrm{Disc}(s_1) \cap T = \{g_0 = 0\} = \{w_3 = b\} \cap T. \tag{31}$$

Let $\pi : U \to \square$ be the moment map restricted to U and $\mathcal{A} := \pi(\mathrm{Disc}^0(s_1))$.

Lemma 6.4. $\pi|_{\mathrm{Disc}^0(s_1)}$ *is an* S^1*-fibre bundle over* $\mathcal{A}$ *and the tangent space of each* S^1*-fibre is generated by* $\partial_{q_4} = \partial_{v_4}$*. Moreover,* $\mathcal{A}$ *is an open embedded curve inside the 2-cell* $\{p_1 = p_2 = 0\} \subset \square$ *such that* $\mathcal{A}$ *is transverse to the slices* $\{p_4 = const\}$*.*

Proof. Because $\mathrm{Disc}(s_1) \cap T$ is connected, so is its projection $\mathcal{A}$. Inserting $w_j = e^{u_j + i v_j}$ into $g_0(w_3, w_4) = g(0, 0, w_3, w_4)$ and taking logarithm yields that $\mathrm{Disc}(s_1) \cap T$ is given by $u_3 = const$ and $v_3 = const$, so it is invariant under the subtorus action $\{(0, 0, 0, \vartheta) \in T^4 | \vartheta \in S^1\}$. This proves the first statement of the lemma.

The curve $\mathcal{A}$ in p-coordinates is found by inserting $u_3 = \partial f_J(p)/\partial p_3$ into $u_3 = const$ and because f_J is a smooth function, $\mathcal{A}$ is a smooth connected curve. Let $p = (0, 0, f_1(r), f_2(r))$ be a parametrization of $\mathcal{A}$. Note that $\mathrm{Disc}(s_1) \cap T$ is symplectic with tangent space generated by $\{f_1'(r)\partial_{p_3} + f_2'(r)\partial_{p_4}, \partial_{q_4}\}$. This means that $\omega(f_1'(r)\partial_{p_3} + f_2'(r)\partial_{p_4}, \partial_{q_4}) = f_2'(r) \neq 0$ for all r, so $\mathcal{A}$ is transverse to the slices $\{p_4 = const\}$. $\square$

Remark 6.5. An alternative proof of Lemma 6.4 suggested by an anonymous referee is as follows: the Hessian of f_J is positive definite so $\partial^2 f_J(p)/\partial p_3^2 > 0$, and therefore $\mathcal{A} = \{u_3 = const\}$ is a curve as claimed.

We consider a straight line segment $\gamma(r) = (0, 0, r, R) \in \square$ for some fixed $R \in \mathbb{R}_{>0}$ parametrized by $r \in (r_0, r_1]$, inside the 2-cell $\{p_1 = p_2 = 0\} = \pi(T)$, such that $0 < r_0 < r_1$ and $\gamma(r) \in \mathcal{A} \iff r = r_1$ (see Figure 9).

The main result we want to prove in this section is the following.

Theorem 6.6 (Lagrangian solid tori). *Let* s *be an* s_1*-admissible section. For any neighbourhood* $N \subset \square$ *of* $\gamma(r_1)$*, there exist* $r' < r'' < r_1$ *with* $\gamma([r', r_1]) \subset N$*, and a family of* s_1*-admissible section* $(s^u)_{u \in [0,1]}$ *such that* $s^0 = s$*, for all* u*,* $s^u = s$ *outside* $\pi_\Delta^{-1}(N)$ *and* M^{s^1} *is* x*-standard with respect to* U *for all* $x \in \gamma([r', r''])$*.*

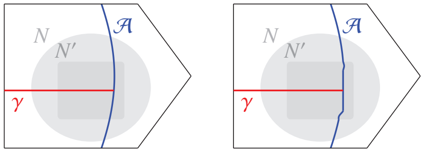


Figure 9. *The straight line segment γ inside the two cell $\{p_1 = p_2 = 0\}$. The transition from the left to the right image is addressed in Subsection 6.3.*

Moreover there exists a neighbourhood $N' \subset N$ of $\gamma((r', r_1])$ such that $\gamma((r', r_1])$ is proper in N' and there exists a family of proper Lagrangian open solid tori $L_t \subset (M_t^{s^1} \cap \pi^{-1}(N'))$, for all $t > 0$ sufficiently small, such that L_t is $(\gamma((r', r'')))$-standard (see Definition 6 and Terminology 5.16).

Note that, in Theorem 6.6, L_t being $(\gamma((r', r'')))$-standard and proper in $(M_t^{s^1} \cap \pi^{-1}(N'))$ implies that the infinite end of L_t is contained in $\pi^{-1}(\gamma((r', r'')))$. We will use this property to glue L_t with the standard Lagrangian models constructed in Section 5 to conclude the proof of Theorem 1.1, eventually.

6.1. Lagrangian construction near the discriminant under assumptions

In this section, we give the construction of a Lagrangian solid torus under two additional assumptions on M_t near $\mathrm{Disc}(s_1) \cap T$, and we later show how to reduce the general case to this case. We start with some preliminaries about contact geometry and Legendrian submanifolds.

6.1.1. Digression into contact geometry

Let (P, ω) be a compact symplectic manifold with boundary. A Liouville structure on (P, ω) is a choice of $\alpha \in \Omega^1(P)$ such that $d\alpha = \omega$ and that the vector field Z, ω-dual to α (i.e., $\iota_Z \omega = \alpha$), points outward along ∂P. The triple (P, ω, α) is called a Liouville domain.

Example 6.7. Let $(B^{2n}, \sum r_j dr_j \wedge d\theta_j)$ be the standard symplectic closed ball. We can pick $\alpha = \sum \frac{r_j^2}{2} d\theta_j$. In this case, $Z = \sum \frac{r_j}{2} \partial_{r_j}$ points outward along ∂B^{2n}.

Given a Liouville domain (P, ω, α), $(\partial P, \ker(\alpha|_{\partial P}))$ is a contact manifold (see, e.g., [18], [40]) and we call it the contact boundary of (P, ω, α). The contact boundary of the Liouville domain in Example 6.7 is called the standard contact sphere (S^{2n-1}, ξ_{std}). In general, there are many contact structures one can put on an odd-dimensional manifold even if one restricts to those that arise as the contact boundary of a Liouville domain. In contrast, there is a unique contact structure on the three-dimensional sphere (up to contactomorphisms) that can be the contact boundary of a Liouville domain, namely, the standard one (see [12]).

Theorem 6.8 (see [11] and also Theorem 1.7 of [39]). *If (P, ω, α) is a Liouville domain with its contact boundary being the standard contact 3-sphere, then (P, ω, α) is symplectic deformation equivalent to the standard symplectic closed 4-ball.*

A knot K in (S^3, ξ_{std}) is called Legendrian if $T_p K \subset \xi_{std}$ for every point $p \in K$. A Legendrian unknot is a Legendrian knot such that its underlying smooth knot type is an unknot.

Example 6.9. Let $K \subset (S^3, \xi_{std}) \subset \mathbb{R}^4$ be the intersection of (S^3, ξ_{std}) with a Lagrangian vector subspace of $(\mathbb{R}^4, \omega_{std})$. Then K is a Legendrian unknot and we call it a standard Legendrian unknot.

The Legendrian isotopy type of a Legendrian unknot is classified by its Thurston-Bennequin number and rotation number (see [13] and also [14, Section 5] for more about these background materials). There is exactly one Legendrian unknot with Thurston-Bennequin number -1 up to Legendrian isotopy and it is realised by the standard Legendrian unknot. By the Thurston-Bennequin inequality, a Legendrian unknot can bound an embedded Lagrangian disk in (B^4, ω_{std}) only if its Thurston-Bennequin number is -1. The converse is also well known to be true.

Lemma 6.10 (Bounding a Lagrangian disk). *Let (P, ω, α) be a Liouville domain with contact boundary (S^3, ξ_{std}). If $\Lambda \subset (\partial P, \ker(\alpha))$ is Legendrian isotopic to the standard Legendrian unknot, then there is an embedded Lagrangian disk $D \subset (P, \omega)$ such that $\partial D = D \cap \partial P = \Lambda$.*

Proof. By Theorem 6.8, it suffices to assume that (P, ω, α) is a star-shaped domain in $(\mathbb{R}^4, \omega_{std})$. By [7, Theorem 1.2], there is a small Darboux ball $B^4 \subset P$ and an embedded Lagrangian $L \subset P \setminus \mathrm{Int}(B^4)$ such that $L \cap \partial P = \Lambda$ and $L \cap \partial B^4$ is a standard Legendrian unknot. Moreover, we can assume that L is invariant with respect to radial direction near ∂B^4. Therefore, we can close up L by a Lagrangian plane in B^4 by Example 6.9. □

Let $M_t' := \{(z_1, z_2, z_3) \in \mathbb{C}^3 | z_1 z_2 = t z_3\}$, which is a complex and hence symplectic hypersurface for all $t \neq 0$. With positive ϵ, let $Y_\epsilon := \{z \in \mathbb{C}^3 | |z_1|^2 + |z_2|^2 + |z_3|^2 = \epsilon\}$ be the 5-sphere equipped with the standard contact structure and contact form $\alpha|_{Y_\epsilon} = \sum \frac{r_i^2}{2} d\theta_i$ (see Example 6.7).

Lemma 6.11. *For $t \in \mathbb{R}_{>0}$, the contact form $\alpha|_{Y_\epsilon}$ restricts to a contact form on $Y_{\epsilon,t} := M_t' \cap Y_\epsilon$ such that $(Y_{\epsilon,t}, \ker(\alpha|_{Y_{\epsilon,t}}))$ is contactomorphic to the standard contact 3-sphere.*

Proof. This result is well known (see Remark 6.12), but we still want to give some details. Without loss of generality, we assume t is real positive. Note that $Y_{\epsilon,t}$ is the union of $Y_{\epsilon,t} \setminus \{z_1 = 0\}$ and $Y_{\epsilon,t} \setminus \{z_2 = 0\}$. We parametrize $Y_{\epsilon,t} \setminus \{z_1 = 0\}$ and $Y_{\epsilon,t} \setminus \{z_2 = 0\}$ by

$$\left\{(r_1, \theta_1, r_2, \theta_2, r_3, \theta_3) = \left(r, \theta_1, \frac{\rho(\epsilon, t, r)t}{r}, \theta_2, \rho(\epsilon, t, r), \theta_1 + \theta_2\right) \middle| r \in (0, \sqrt{\epsilon}], \theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z}\right\},$$
$$\left\{(r_1, \theta_1, r_2, \theta_2, r_3, \theta_3) = \left(\frac{\rho(\epsilon, t, r)t}{r}, \theta_1, r, \theta_2, \rho(\epsilon, t, r), \theta_1 + \theta_2\right) \middle| r \in (0, \sqrt{\epsilon}], \theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z}\right\},$$

where $\rho(\epsilon, t, r) := \sqrt{\frac{r^2(\epsilon - r^2)}{r^2 + t^2}}$, so when $r = \sqrt{\epsilon}$, we have $\rho(\epsilon, t, \sqrt{\epsilon}) = 0$ and the corresponding angular variable (i.e., θ_2 for the first equation and θ_1 for the second equation) collapses. In particular, the parametrizations of $Y_{\epsilon,t} \setminus \{z_1 = 0\}$ and $Y_{\epsilon,t} \setminus \{z_2 = 0\}$ exactly give a Heegaard decomposition of $Y_{\epsilon,t}$. The collapsing circles at the ends have intersection pairing one in the Heegaard surface (a 2-torus) so $Y_{\epsilon,t} = S^3$.

Let $\Phi(s_1, \vartheta_1, s_2, \vartheta_2) := (s_1\sqrt{t}\exp(i\vartheta_1), s_2\sqrt{t}\exp(i\vartheta_2), s_1 s_2 \exp(i(\vartheta_1 + \vartheta_2)))$ be a chart for M_t' and let $\alpha := \sum \frac{r_j^2}{2} d\theta_j|_{M_t}$ and recall that $\omega = \sum r_j dr_j \wedge \theta_j$. Then we have

$$\begin{aligned}
\Phi^*\alpha &= \frac{s_1^2}{2}(t + s_2^2)d\vartheta_1 + \frac{s_2^2}{2}(t + s_1^2)d\vartheta_2,\\
\Phi^*\omega &= s_1(t + s_2^2)ds_1 \wedge d\vartheta_1 + s_1^2 s_2 ds_2 \wedge d\vartheta_1 + s_1 s_2^2 ds_1 \wedge d\vartheta_2 + s_2(t + s_1^2)ds_2 \wedge d\vartheta_2, \text{ so}\\
Z_t &= \frac{1}{2(t + s_1^2 + s_2^2)}(s_1(t + s_1^2)\partial_{s_1} + s_2(t + s_2^2)\partial_{s_2})
\end{aligned}$$

is checked to be the dual of $\Phi^*\alpha$ with respect to $\Phi^*\omega|_{M_t}$. In particular, the Liouville vector field Z_t points outward along $\partial(M_t' \cap \{|z| \leq \epsilon\})$. Therefore, $M_t' \cap \{|z| \leq \epsilon\}$ is a Liouville domain with contact boundary $(Y_{\epsilon,t}, \ker(\alpha|_{Y_{\epsilon,t}}))$. Because the standard contact 3-sphere is the only contact 3-sphere that arises as the boundary of a Liouville domain, the result follows. □

Remark 6.12. $Y_{\epsilon,t}$ is called the link of the 'singularity' of M_t' at the origin. Because M_t' is smooth at the origin for $t \neq 0$, the link of the origin is contactomorphic to the standard contact 3-sphere.

By translating the z_3 coordinate, we know that $Y_{a,\epsilon,t} := \{z \in \mathbb{C}^3 | z_1 z_2 = t(z_3 - a)\} \cap \{z \in \mathbb{C} | |z_1|^2 + |z_2|^2 + |z_3 - a|^2 = \epsilon\}$ is naturally equipped with a contact structure making it a standard contact 3-sphere for $a \in \mathbb{C}$ and $\epsilon > 0$.

Lemma 6.13. *With $t \in \mathbb{R}_{>0}$, the following is a Lagrangian disk in $M'_{a,t} := \{z_1 z_2 = t(z_3 - a)\}$,*

$$L := \{(re^{i\theta}, re^{-i\theta}, \frac{r^2}{t} + a) | r \in [0, \infty), \theta \in \mathbb{R}/2\pi\mathbb{Z}\}. \tag{32}$$

Moreover, $\partial L := L \cap Y_{a,\epsilon,t}$ is a Legendrian and has the Legendrian isotopy type of a standard Legendrian unknot in $Y_{a,\epsilon,t}$.

Proof. Being a Lagrangian disk is an easy check. Using the chart Φ for M'_t from above, shifted by $(0, 0, a)$, we have

$$\Phi^{-1}(L) = \left\{ \left(\frac{r}{\sqrt{t}}, \theta, \frac{r}{\sqrt{t}}, -\theta \right) \right\}.$$

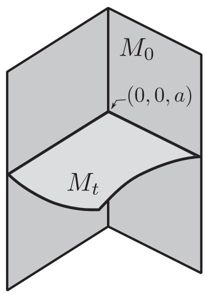
By the proof of Lemma 6.11, $Z_t = \frac{1}{2(t+s_1^2+s_2^2)}(s_1(t+s_1^2)\partial_{s_1} + s_2(t+s_2^2)\partial_{s_2})$, so we have $Z_t|_x \in T_x(\Phi^{-1}(L))$ for all $x \in \Phi^{-1}(L)$. Therefore, ∂L is a Legendrian. The only Legendrian isotopy type that can bound a Lagrangian disk is the standard one, so ∂L is Legendrian isotopic to the standard Legendrian unknot. □

Remark 6.14. For every $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, there is a symplectomorphism $M'_{a,t} \to M'_{a,t}$ given by $z_1 \mapsto e^{i\phi} z_1$, $z_2 \mapsto e^{i\phi} z_2$, $z_3 \mapsto e^{i(2\phi)}(z_3 - a) + a$. Therefore, if the domain of r in (32) is replaced by $e^{i\phi}[0, \infty)$ for some $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, Lemma 6.13 still holds.

Having reviewed some basic contact geometry, now we explain the construction of Lagrangian solid tori under Assumptions 6.15 and 6.17 below.

6.1.2. Overview of the construction

In one dimension lower like the situation we just considered, let z_1, z_2, z_3 be symplectic coordinates of $\mathbb{C}^3$ and suppose we have a family M_t of hypersurfaces in $\mathbb{C}^3$ with $M_0 = \{z_1 z_2 = 0\}$, M_t a symplectic submanifold for $t \neq 0$, for all $t \neq 0$ the discriminant $M_t \cap \mathrm{Sing}(M_0)$ equals $\{(0, 0, a)\}$ for fixed $a \in \mathbb{C}^*$. Say we have two balls V, U centred at $\{(0, 0, a)\}$ with $\bar{V} \subset U$ so that



$$M_t \cap (U \setminus V) = \{z_1 z_2 = t(z_3 - a)\}, \tag{33}$$

$$M_t \cap U \text{ is a Liouville domain.} \tag{34}$$

By (33) and Lemma 6.11, we know that $\partial(M_t \cap U)$ is the standard contact 3-sphere. By Lemma 6.13, we have a Legendrian unknot

$$\Lambda_r := \left\{ \left(re^{i\theta}, re^{-i\theta}, \frac{r^2}{t} + a \right) \middle| \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\} \tag{35}$$

inside $\partial(M_t \cap U)$ for some appropriate r. Furthermore, by (34) and Theorem 6.8, we know that $M_t \cap U$ is symplectic deformation equivalent to the standard symplectic ball when $t \neq 0$. Moreover, by Lemma 6.10, we know that we can fill Λ_r by a Lagrangian disk in $M_t \cap U$. This Lagrangian disk will generally allow us to construct closed Lagrangian surfaces for a tropical curve ending at the discriminant with such a disk closing up the ending.

In the situation that interests us one dimension higher, the ending needs to be given by a solid 3-torus. This situation is considerably harder for the following reason. Ideally, we would like to have a product situation locally. This means that there is a symplectic annulus (A, ω_A) such that the family is simply given by $M'_t \times A$ where M'_t is as above, the discriminant is then $\{(0, 0, a)\} \times A$, we obtain a Lagrangian disk D in the first factor M'_t as before and then for any circle C in A that generates the fundamental group of A, we find $D \times C$ as the desired solid torus in $M'_t \times A$. However, it is very hard to understand the symplectic form near the discriminant, not to mention to try to deform it to a product situation, so this easy setup will not be achievable for us. The next weaker concept from a product is a fibration, which is what we will be using instead, as follows.

Let z_1, z_2, z_3, z_4 be symplectic coordinates of $\mathbb{C}^4$, M_t a family of hypersurfaces in $\mathbb{C}^4$ with $M_0 = \{z_1z_2 = 0\}$, M_t a symplectic submanifold for $t \neq 0$, and for all $t \neq 0$ the discriminant $M_t \cap \mathrm{Sing}(M_0)$ equals $\{(0, 0, a)\} \times A$ for fixed $a \in \mathbb{C}^*$ and some 0-centred annulus $A \subset \mathbb{C}$. Again, say we have two balls $V', U' \subset \mathbb{C}^3$ centred at $\{(0, 0, a)\}$ with $\bar{V}' \subset U'$ so that setting $U = U' \times A, V = V' \times A$,

$$M_t \cap (U \setminus V) = \{z_1z_2 = t(z_3 - a)\}, \tag{36}$$

$$M_t \cap U \text{ is a Liouville domain.} \tag{37}$$

We will show below that the restriction of the projection $U \longrightarrow A$ to M_t gives a 'nice' exact symplectic fibration $\pi : M_t \cap U \longrightarrow A$. Every fibre of π is the lower-dimensional situation as above. After symplectic completion, we get an exact symplectic fibration $\mathrm{Comp}(\pi) : \mathrm{Comp}(M_t \cap U) \longrightarrow T^*S^1$ such that fibres are standard symplectic $\mathbb{R}^4$. Because the compactly supported symplectomorphism group of standard $\mathbb{R}^4$ is trivial, we can find a compactly supported exact symplectic deformation from $\mathrm{Comp}(M_t \cap U)$ to $\mathrm{Comp}(M_t \cap U)'$ such that $\mathrm{Comp}(\pi)$ is still an exact symplectic fibration and the symplectic monodromy around a simple loop $C \subset A$ is the identity. Therefore, we can construct a Lagrangian disk as above in a fibre of a point of C and apply symplectic parallel transport along C to get a Lagrangian solid torus in $\mathrm{Comp}(M_t \cap U)'$. Because $\mathrm{Comp}(M_t \cap U)'$ and $\mathrm{Comp}(M_t \cap U)$ are related by a compactly supported exact symplectic deformation, we get a corresponding Lagrangian torus in $\mathrm{Comp}(M_t \cap U)$ and we can apply the backward Liouville flow to obtain a Lagrangian solid torus in $M_t \cap U$.

In the sections below, we will explain this construction in more details.

6.1.3. Main construction

Let U be a symplectic corner chart such that Assumption 6.1 holds. Let s be an s_1-admissible section and let $T, \mathcal{A}, \gamma$ and $\pi : U \longrightarrow ⛶$ be the ones from the beginning of Section 6. By Lemma 6.4, the q_3-coordinate of $\mathrm{Disc}(s) \cap T$ is a constant and we denote it simply by q. We are interested in the circle in the discriminant that lies above the point where γ hits its amoeba image $\mathcal{A}$. Also by Lemma 6.4, we find $C := \pi^{-1}(\gamma(r_1)) \cap \mathrm{Disc}(s_1) \cap T$ to be a circle with constant radial coordinate, say, given by $|p_4| = R \iff |z_4| = \sqrt{2R}$. So in z-coordinates, by setting $a = \sqrt{2r_1}e^{iq}$, the circle C is given by

$$C = \left\{ z = (0, 0, a, z_4) \,\middle|\, |z_4| = \sqrt{2R} \right\}. \tag{38}$$

We will construct a Lagrangian solid torus inside $M_t \cap U_C$ for an appropriate closed neighbourhood U_C of C. From now on, every tubular neighbourhood of C that we choose will be **closed** and of the form

$$U_C := B \times B \times D \times A \subset (\mathbb{R}^2)^4 \tag{39}$$

for B a 0-centred disk, D an a-centred disk and A a 0-centred annulus (shrinking and then taking closure of the one we had before) so that the circle of radius $\sqrt{2R}$ is contained in A. We will make two further assumptions for which we will show in later sections how these can be achieved. The first assumption is that $\mathrm{Disc}(s) \cap T$ depends only on the z_4-coordinate near C as illustrated on the right in Figure 9.

Assumption 6.15. *There exists a tubular neighbourhood U_C of C such that*

$$\mathrm{Disc}(s) \cap T \cap U_C = \{(0, 0, a, z_4) \in U_C | z_4 \in A\} \tag{40}$$

for A a shrinking of the previous annulus A still containing the circle of radius $\sqrt{2R}$. In particular, $\mathcal{A} \cap \pi(U_C) = \{(0, 0, r_1)\} \times I$ where I is a straight line segment in the affine p-coordinates of ⬠ and I is given by projecting the radial part of A.

To construct the Lagrangian solid torus, we also need to make an assumption on the restriction of π to $U_C \cap M_t$, and for that we introduce the following notion.

Definition 9. Let (E, ω_E) be a symplectic manifold with corners and (Σ, ω_Σ) be a symplectic surface with boundary. Let $\pi : E \to \Sigma$ be a symplectic fibration. The **vertical boundary** of π is $\partial^v E := \pi^{-1}(\partial\Sigma)$. The **horizontal boundary** $\partial^h E$ of π is the closure of $\partial E \setminus \partial^v E$. The fibration π is called a **smoothly trivial exact symplectic fibration** if

1. π is a smoothly trivial fibre bundle,
2. there is a one form α_E such that $d\alpha_E = \omega_E$ and the induced Liouville vector field points outward along $\partial^v E$ and $\partial^h E$,
3. there exists a neighbourhood N of $\partial^h E$ and a symplectic manifold (F, ω_F) with smooth boundary such that there is a symplectomorphism $\Psi : (N, \omega_E|_N) \simeq (F \times \Sigma, \omega_F \oplus \omega_\Sigma)$ and $\pi_\Sigma \circ \Psi = \pi|_N$, where $\pi_\Sigma : F \times \Sigma \to \Sigma$ is the projection to the second factor.

The last condition is also referred to as π being *symplectically trivial near the horizontal boundary*.

Remark 6.16. A smoothly trivial exact symplectic fibration is a strictly more restrictive notion than that of an exact symplectic fibration as given in [52, Section 15].

Assumption 6.17. *There exist tubular neighbourhoods*

$$U_C := B \times B \times D \times A \quad \textit{and} \quad V_C := B' \times B' \times D' \times A \tag{41}$$

with U_C as in (39) and $B' \subset B$ a 0-centred closed disk of smaller radius and $D' \subset D$ an a-centred closed disk of smaller radius but U_C and V_C have notably the same A-factors such that

$$\begin{cases} M_t^s \cap (U_C \backslash V_C) = \{z_1 z_2 = t(z_3 - a)\} \textit{ and} \\ \pi : M_t^s \cap U_C \to A \textit{ is a smoothly trivial exact symplectic fibration}, \end{cases} \tag{42}$$

where, as usual, $z_j = \sqrt{2p_j} e^{iq_j} = x_j + iy_j$ for $j = 1, 2, 3, 4$.

We next carry out the Lagrangian solid torus construction under Assumptions 6.15 and 6.17 (in fact, we only use Assumption 6.17 for the Lagrangian construction and we will see in Subsections 6.4-6.6 that Assumption 6.15 is used to obtain Assumption 6.17). By Lemma 6.11, the contact boundaries of fibres of the projection $\pi : M_t \cap U_C \to A$ are contactomorphic to the standard contact 3-sphere. This implies that π is actually a symplectic 4-ball bundle over A by Theorem 6.8. Moreover, by Lemma 6.13, for every $z_4 \in A$, there is a unique $r > 0$ such that $r^2 + a \in \partial D$ for D the third factor of U_C. When $t \in \mathbb{R}_{>0}$ is small,

$$\Lambda_{z_4, r} := \left\{ (\sqrt{t} r e^{i\theta}, \sqrt{t} r e^{-i\theta}, r^2 + a, z_4) \middle| \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\} \tag{43}$$

is a Legendrian unknot in $\partial(\pi^{-1}(z_4))$. (By Remark 6.14, we can also take $r \in \mathbb{C}^*$ with nonzero argument.) Recall that R is the p_4-coordinate of $\gamma(r_1)$. Because π is assumed to be symplectically trivial near the horizontal boundary, it is clear that

$$\Lambda_{C,r} := \bigcup_{z_4 : |z_4| = \sqrt{2}R} \Lambda_{z_4,r} \tag{44}$$

is a Legendrian torus in the contact boundary of $M_t \cap U_C$ (after rounding corners to be able to call $M_t \cap U_C$ a Liouville domain, even though $\Lambda_{C,r}$ does not meet any corners because it projects to the interior of A).

Proposition 6.18. *The Legendrian torus $\Lambda_{C,r}$ bounds an embedded Lagrangian solid torus $L_{C,r}$ in $M_t \cap U_C$ such that every $\Lambda_{z_4,r}$ is a meridian.*

Proof. We use the notation $M'_t := M_t \cap U_C$ in this proof. Let $\mathrm{Comp}(M'_t)$ be the symplectic completion of M'_t. In other words,

$$\mathrm{Comp}(M'_t) := M'_t \cup_{\partial M'_t} ([1, \infty) \times \partial M'_t) \tag{45}$$

and the symplectic form on $([1, \infty) \times \partial M'_t)$ is given by $d(\rho\alpha|_{\partial M'_t})$ for ρ the linear coordinate on $[1, \infty)$ and α the one-form on M'_t defining its Liouville structure. Because π is trivial near the horizontal boundary (third item of Definition 9), $\mathrm{Comp}(M'_t)$ can be obtained by first performing symplectic completion along the fibres of π and then completing along the base direction. Therefore, we have a symplectic $\mathbb{R}^4$-bundle $\mathrm{Comp}(\pi) : \mathrm{Comp}(M'_t) \to \mathbb{C}^*$ extended from π. We also have a Lagrangian submanifold $[1, \infty) \times \Lambda_{C,r} \subset [1, \infty) \times \partial M'_t \subset \mathrm{Comp}(M'_t)$ fibring over the circle $\{|z_4| = \sqrt{2}R\}$ with respect to $\mathrm{Comp}(\pi)$.

Gromov showed that the compactly supported symplectomorphism group of $(\mathbb{R}^4, \omega_{std})$ is contractible [19]. Therefore, there exists an exact symplectic deformation $\mathrm{Comp}(M'_t)'$ of $\mathrm{Comp}(M'_t)$ supported inside a compact set $K \subset \mathrm{Comp}(M'_t)$ such that after the deformation, $\mathrm{Comp}(\pi) : \mathrm{Comp}(M'_t)' \to \mathbb{C}^*$ is still a symplectic $\mathbb{R}^4$-bundle and the symplectic monodromy along $\{|z_4| = \sqrt{2}R\}$ defined by symplectic parallel transport becomes the identity (see [52, Lemma 15.3]).

Pick a point $z_4 \in A$ such that $|z_4| = \sqrt{2}R$. There exists $\rho_0 > 1$ sufficiently large such that $[\rho_0, \infty) \times \Lambda_{z_4,r} \subset [1, \infty) \times \partial M'_t$ is disjoint from K. Because $\{\rho_0\} \times \Lambda_{z_4,r}$ is a Legendrian isotopic to the standard Legendrian unknot in the relevant contact hypersurface S^3 of $\mathrm{Comp}(\pi)^{-1}(z_4) = (\mathbb{R}^4, \omega_{std})$, the proper annulus $[\rho_0, \infty) \times \Lambda_{z_4,r}$ can be extended to a smooth proper Lagrangian disk $L_{z_4,r}$ in $\mathrm{Comp}(\pi)^{-1}(z_4)$, by Lemma 6.10. (Note that when a Legendrian is Lagrangian fillable, one can always perturb the Lagrangian filling near the Legendrian boundary to get another Lagrangian filling that is cylindrical near its Legendrian boundary; therefore, the Lagrangian disk $L_{z_4,r}$ can be made to be smooth.) We engage $L_{z_4,r}$ in symplectic parallel transport along $\{|z_4| = \sqrt{2}R\}$. The fact that the monodromy is the identity implies that the trace of $L_{z_4,r}$ is an embedded proper Lagrangian open solid torus, denoted by $L'_{C,r}$, with a cylindrical end $[\rho_0, \infty) \times \Lambda_{C,r}$. Because $\{\rho_0\} \times \Lambda_{z_4,r}$ bounds a disk in $L_{z_4,r}$, it is a meridian of $L'_{C,r}$.

Finally, because $\mathrm{Comp}(M'_t)'$ is a compactly supported exact symplectic deformation of $\mathrm{Comp}(M'_t)$, there is also an embedded proper Lagrangian solid torus $L''_{C,r} \subset \mathrm{Comp}(M'_t)$ with the cylindrical end $[\rho_1, \infty) \times \Lambda_{C,r}$ for some sufficiently large ρ_1. Therefore, one can argue using backward Liouville flow as in the proof of Lemma 6.10 to rescale $L''_{C,r}$ and make its cylindrical part as long as we like. We consequently obtain a Lagrangian filling $L_{C,r}$ of $\Lambda_{C,r}$ inside M'_t with the properties required in the proposition. □

6.1.4. Plan for the remaining part

In the following subsections, we will generalise Proposition 6.18. In Subsections 6.2 and 6.3, we explain how to isotope the discriminant of s so that Assumption 6.15 holds. In Subsection 6.4, 6.5 and 6.6, we construct a smoothly trivial exact symplectic fibration such that Assumption 6.17 holds. We conclude the proof of Theorem 6.6 in Subsection 6.7. The proof of Theorem 1.1 is given in Subsection 6.8.

6.2. Integral linear transform

We go back to the general setup in Theorem 6.6. In particular, we have a parametrized straight line segment γ with $\gamma(r_1) \in \mathcal{A}$. In this section, we want to apply an integral linear transformation to transform the (p, q)-coordinates to obtain new $(\hat{p}, \hat{q})$-coordinates so that $\sum_{i=1}^4 q_i = \hat{q}_1 + \hat{q}_2$. This will help us to get rid of the fourth-coordinate in the defining equation of $M_t^s \cap U_C$ later on. Define

$$\hat{A} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \text{ so the inverse transpose } \quad \hat{A}^{-T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider a change of symplectic coordinates $\Psi^\circ : U^\circ = U \setminus \{p_1 \dots p_4 = 0\} \to \hat{U}^\circ := \operatorname{Im}(\Psi^\circ) \subset \mathbb{R}^4 \times \mathbb{R}^4/2\pi\mathbb{Z}^4$ given by the integral linear transform $\Psi^\circ(p, q) = (\hat{p}, \hat{q}) = (\hat{A}p, \hat{A}^{-T}q)$. Note that, for $j = 1, 2$, we have $\hat{p}_j = p_j$ so we can define $\hat{z}_j := \sqrt{2\hat{p}_j}e^{i\hat{q}_j}$ and partially compactify $\hat{U}^\circ$ to $\hat{U}$ by allowing $\hat{z}_j = 0$ for $j = 1, 2$. We can smoothly extend Ψ° to $\Psi : U \setminus \{p_3 p_4 = 0\} \to \hat{U} = \operatorname{Im}(\Psi) \subset \mathbb{C}^2 \times \mathbb{R}^2 \times \mathbb{R}^2/2\pi\mathbb{Z}^2$. More explicitly,

$$\Psi(z) = (\sqrt{2p_1}e^{iq_1}, \sqrt{2p_2}e^{i(q_2+q_3+q_4)}, -p_2 + p_3, -p_2 + p_4, q_3, q_4). \tag{46}$$

Note that $\Psi|_{\{z_1=z_2=0\}}$ is the identity map. Therefore, just like before, by Lemma 6.4, $\operatorname{Disc}(s_1) \cap T$ has the constant $\hat{q}_3$-coordinate $\arg(a)$ and $\mathcal{A}$ is transverse to the slices $\{\hat{p}_4 = const\}$. The straight line $\hat{\gamma}(r) := \Psi(\gamma(r))$ is still given by $(0, 0, r, R)$ for $r_0 < r \le r_1$, and $C = \{\hat{z} = (0, 0, a, \hat{z}_4) | \, |\hat{z}_4| = \sqrt{2}R\}$.

For use in the next section, we now apply the transformation to the pencil. Observe that we achieved $\sum_{i=1}^4 q_i = \hat{q}_1 + \hat{q}_2$ and have

$$p_3 = \hat{p}_2 + \hat{p}_3, \qquad p_4 = \hat{p}_2 + \hat{p}_4. \tag{47}$$

Inserting this and more broadly $z_j = \Psi^{-1}(\hat{z}_j)$ into equation (12) in Example 3.5 yields

$$2\hat{z}_1\hat{z}_2\sqrt{(\hat{p}_2 + \hat{p}_3)(\hat{p}_2 + \hat{p}_4)} = th(\hat{p})g(w(\hat{p}, \hat{q})). \tag{48}$$

6.3. Straightening the discriminant

We assume from now until Subsection 6.7 that we have performed the transformation Ψ given in the previous subsection (Subsection 6.2). For better readability, we will use the notation p instead of $\hat{p}$, z for $\hat{z}$ and so forth.

Our next step is to apply a compactly supported Hamiltonian diffeomorphism to deform $\operatorname{Disc}(s_1) \cap T$ such that the p_3-coordinate of $\operatorname{Disc}(s_1) \cap T$ becomes independent of the p_4-coordinate near C. In other words, we want that Assumption 6.15 holds after deforming $\operatorname{Disc}(s_1) \cap T$.

Similar to (39), we use a tubular neighbourhood of C of the form

$$U_C := \{((p_1, q_1), \dots, (p_4, q_4)) \in B \times B \times D \times A \subset U \tag{49}$$

and taken small enough so that p_3 and p_4 take positive values in U_C, which works by (46). It is then sensible to define $z_j = \sqrt{2p_j}e^{iq_j} = x_j + iy_j$ in U_C for $j = 1, 2, 3, 4$.

Proposition 6.19. *For any tubular neighbourhood N of C, there is a Hamiltonian diffeomorphism $\phi_H : \mathbb{P}_\Delta \to \mathbb{P}_\Delta$ supported inside N and a tubular neighbourhood $U_C \subset N$ of C given by (49) such that*

- *ϕ_H preserves all the toric strata of $\mathbb{P}_\Delta$ setwise,*
- *$\phi_H(\operatorname{Disc}(s_1) \cap T) \cap U_C = \{(0, 0, a, z_4) | z_4 \in A\}$, and*
- *$z_j = z_j \circ \phi_H^{-1}$ inside U_C for $j = 1, 2$.*

After establishing Proposition 6.19, we push forward all of the data and define

$$\hat{J}_\Delta := (\phi_H)_* J_\Delta \tag{50}$$

$$\hat{s}_i := s_i \circ \phi_H^{-1} \tag{51}$$

$$\hat{\mathcal{L}} := (\phi_H^{-1})^* \mathcal{L} \tag{52}$$

$$\mathrm{Disc}(\hat{s}_1) := (\hat{s}_1)^{-1}(0) \cap (\partial \mathbb{P}_\Delta)_{\mathrm{Sing}} = \phi_H(\mathrm{Disc}(s_1)) \tag{53}$$

$$\hat{M}_t^s := \{\hat{s}_0 = ts\} \text{ for } s \in C^\infty(\mathbb{P}_\Delta, \hat{\mathcal{L}}) \tag{54}$$

$$\hat{M}_t := \hat{M}_t^{\hat{s}_1} = \phi_H(M_t). \tag{55}$$

In particular, $\hat{J}_\Delta$ is a complex structure on $\mathbb{P}_\Delta$, and $\hat{s}_i$ are holomorphic sections of the holomorphic bundle $\hat{\mathcal{L}}$. Therefore, it makes sense to talk about $\hat{s}_1$-admissible sections (which are the same as s_1-admissible sections precomposed by ϕ_H^{-1}). Most notable, by the second bullet of Proposition 6.19, $\mathrm{Disc}(\hat{s}_1) \cap T \cap U_C$ satisfies Assumption 6.15 in (p, q)-coordinates.

Remark 6.20 (Trivalent vertex). As mentioned in Remark 6.3, the key difficulty to generalise Theorem 1.1 to tropical curves with ends on a codimension one part of $\mathcal{A}$ is whether one can establish the corresponding result of Proposition 6.19.

More precisely, suppose we are given the local model (30) and a straight line segment $\gamma(r) = (0, 0, r, R)$ parametrized by $r \in (r_0, r_1]$ such that $\gamma(r) \in \mathcal{A}$ if and only if $r = r_1$. Let $(0, 0, a, b) \in \mathrm{Disc}(s_1)$ such that $\pi_\Delta(0, 0, a, b) = \gamma(r_1)$ and let $C = \{(0, 0, a, z_4) : |z_4| = |b|\}$. We define neighbourhood U_C of C as above. If Proposition 6.19 is true in this setup, which means that it is true for all the ends of a tropical curve, then the Lagrangian construction in Theorem 1.1 applies to the tropical curve.

With that said, it is tempting to try to mimic the proof of Proposition 6.19 below to make $\mathcal{A}$ to be very close to a trivalent graph and if $\gamma(r_1)$ is not the trivalent point of the graph, we would be able to get a Hamiltonian ϕ_H satisfying all three bullets of Proposition 6.19. However, such a ϕ_H is not supported inside N. For ϕ_H to be supported inside N, we can only perturb $\mathrm{Disc}(s_1)$ in N and hence cannot shrink $\mathcal{A}$ to a trivalent graph.

If one uses a ϕ_H that is not supported inside N to run the rest of the argument, one can still get a closed Lagrangian that is diffeomorphic to a Lagrangian lift of the tropical curve but one cannot control the π_Δ-image of the Lagrangian to be in a small neighbourhood of the tropical curve.

It is very possible that Proposition 6.19 for appropriate $\gamma(r)$ is true in this setup. Even though it is a very explicit local question, we are not able to write down a clean condition on γ for it to work, especially when $\gamma(r_1)$ is very close to 'the trivalent point of $\mathcal{A}$'.

Before giving the proof of Proposition 6.19, we first conclude the resulting local model of $\hat{M}_t \cap U_C$. We remind the reader that $\hat{z}$ and $\hat{p}$ in the previous section are denoted by z and p in this section.

Lemma 6.21. *Let ϕ_H and U_C be chosen as in Proposition 6.19. Then we have*

$$\hat{M}_t \cap U_C = \{z_1 z_2 = t g_U(z)\}$$

for some smooth function $g_U : U_C \to \mathbb{C}$ such that

- *$g_U = g$ up to a change of coordinates and a multiplication by a nonvanishing function: more precisely, $g_U = \rho(z) g(\Phi_U \circ \Psi^{-1}(\phi_H^{-1}(z)))$ for some $\rho(z) : U_C \to \mathbb{C}^*$,*
- *the zero locus of $(g_U)_0 := g_U|_{\{z_1 = z_2 = 0\}}$ is given by $\mathrm{Disc}(\hat{s}_1) \cap T \cap U_C$,*
- *$(g_U)_0$ is submersive (i.e., $D(g_U)_0$ surjective) near $\mathrm{Disc}(\hat{s}_1) \cap T \cap U_C$,*
- *$(g_U)_0|_{(D \setminus \{a\}) \times A}$ is homotopic to $(z_3, z_4) \mapsto z_3 - a$ as $\mathbb{C}^*$-valued functions.*

Proof. By Assumption 6.1 and equation (48), we have

$$M_t \cap U_C = \{2z_1z_2\sqrt{(p_2+p_3)(p_2+p_4)} = th(p)g(w(p,q))\}.$$

Because $(p_2+p_3)(p_2+p_4) > 0$, we can rearrange the terms to get

$$M_t \cap U_C = \{z_1z_2 = t\tilde{h}(p)g(w(p,q))\},$$

where $\tilde{h}(p) = \frac{h(p)}{2\sqrt{(p_2+p_3)(p_2+p_4)}}$. Note that $\tilde{h}(p)$ is a nonvanishing positive function because the numerator and denominator are both positive. On the other hand, by tracing back the definitions, we have $w(p,q) = \Phi_U \circ \Psi^{-1}(p,q)$. Applying ϕ_H to M_t corresponds to precomposing the coordinates in the defining equation by ϕ_H^{-1}, so we have

$$\hat{M}_t \cap U_C = \{\tilde{z}_1\tilde{z}_2 = t\tilde{h}(\phi_H^{-1}(z))g(\Phi_U \circ \Psi^{-1}(\phi_H^{-1}(z)))\}, \tag{56}$$

where $\tilde{z}_i := z_i \circ \phi_H^{-1}$ for $i = 1, 2$. If we define $\rho(z) := \tilde{h}(\phi_H^{-1}(z))$, then by the third bullet of Proposition 6.19, we get the first bullet of this lemma.

In U_C, from the first bullet of Proposition 6.19 and the discussion above, it is clear that $(g_U)_0 = g|_T$ up to a change of coordinates and a multiplication by a nonvanishing function. Therefore, $(g_U)_0^{-1}(0) = \phi_H \circ \Psi \circ \Phi_U^{-1}((g|_T)^{-1}(0)) \cap U_C = \mathrm{Disc}(\hat{s}_1) \cap T \cap U_C$, which is exactly the second bullet.

We now consider the third bullet. Because Φ_U, Ψ and ϕ_H are diffeomorphisms, it suffices to check that $D(g|_T)$ is submersive near $\mathrm{Disc}(s_1) \cap T$. We can check it in the complex chart where $g|_T = a(b - w_3)$ and $\mathrm{Disc}(\hat{s}_1) \cap T = \{b = w_3, w_1 = w_2 = 0, w_4 \neq 0\}$. Therefore, the third bullet follows.

Finally, because ϕ_H is isotopic to the identity, in order to understand the homotopy class of $(g_U)_0|_{(D\setminus\{a\})\times A}$, in view of (56), it suffices to understand the homotopy class of

$$\tilde{h}(p)g(\Phi_U \circ \Psi^{-1}(z))|_{\{0\}\times\{0\}\times(D\setminus\{a\})\times A}. \tag{57}$$

It is clear that $\tilde{h}(p)$ is null-homotopic because, on one hand, it is well defined and nonvanishing on the whole D factor and, on the other, it is independent of the q_4-coordinate. The homotopy class of the remaining term, $g(\Phi_U \circ \Psi^{-1}(z))$, can be understood by combining the fact that, away from the zero locus, $g|_T$ is homotopic to $(w_3, w_4) \mapsto w_3 - b \in \mathbb{C}^*$ and $q_3 = v_3$ is preserved under Ψ (see (46)). □

6.3.1. Proof of Proposition 6.19

Let $N \subset U_C$ be a tubular neighbourhood of C of a similar form as U_C. Under abuse of repeating notation, N is thus given by

$$N = \{((p_1,q_1),\ldots,(p_4,q_4)) \in B \times B \times D \times A\}.$$

Let a_0, a_1 be the radii of A in the p_4-coordinate with $a_0 < R < a_1$ for R the radius of C. By Lemma 6.4, there exists a smooth $p_{\mathrm{Disc}} : [a_0, a_1] \to \mathbb{R}_{>0}$ and a constant $q_{\mathrm{Disc}} \in [0, 2\pi)$ such that

$$\mathrm{Disc}(s_1) \cap T \cap N = \{z = (0, 0, \sqrt{2p_{\mathrm{Disc}}(p_4)}e^{iq_{\mathrm{Disc}}}, z_4) | z_4 \in A\}.$$

In particular, by $C \subset \mathrm{Disc}(s_1) \cap T \cap N$, we have $\sqrt{2p_{\mathrm{Disc}}(R)} = |a|$ and $q_{\mathrm{Disc}} = \arg(a)$.

By ignoring the first two factors, we can view $\mathrm{Disc}(s_1) \cap T \cap N$ as a symplectic section of the projection $\pi : D \times A \to A$. In the following lemma, we explain how to deform this symplectic section (denoted by Z in the lemma) to another symplectic section that is locally constant near $\pi(C)$. After that, we will explain in Lemma 6.23 how to thicken this Hamiltonian isotopy inside T to be a Hamiltonian isotopy in U_C to achieve Proposition 6.19.

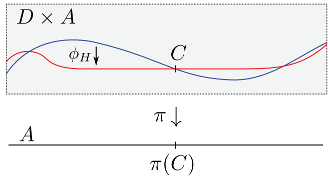


Figure 10. *The symplectic section Z (blue) is deformed to a section locally constant near $\pi(C)$ (red) after a compactly supported Hamiltonian diffeomorphism.*

Lemma 6.22. *Let $Z \subset D \times A$ be the image of the symplectic section*

$$A \longrightarrow D \times A, \quad z_4 \mapsto (\sqrt{2p_{\mathrm{Disc}}(p_4)}e^{iq_{\mathrm{Disc}}}, z_4)$$

of π. There exists a Hamiltonian $H : D \times A \longrightarrow \mathbb{R}$, supported inside the interior of the domain $D \times A$, and a neighbourhood W of $\pi(C) = \{p_4 = R\}$ in A such that $\pi^{-1}(W) \cap \phi_H(Z) = \{(a, z_4)|z_4 \in W\}$ (see Figure 10).

Proof. Consider the Hamiltonian

$$H = (p_{\mathrm{Disc}}(p_4) - p_{\mathrm{Disc}}(R))(q_3 - q_{\mathrm{Disc}}) \in C^\infty(D \times A).$$

This is well defined because the q_3-coordinate on D is bounded in an interval for D is a-centred with $a \neq 0$ and D does not meet $z_3 = 0$. The corresponding Hamiltonian vector field is (with the sign convention $dH = -\iota_{X_H}\omega$) given by

$$X_H = -(p_{\mathrm{Disc}}(p_4) - p_{\mathrm{Disc}}(R))\partial_{p_3} + p'_{\mathrm{Disc}}(p_4)(q_3 - q_{\mathrm{Disc}})\partial_{q_4}.$$

In particular, $X_H|_C = 0$ and when the time 1 flow ϕ_H is well defined, we have

$$\phi_H(p_3, q_3, p_4, q_4) = (p_3 + p_{\mathrm{Disc}}(R) - p_{\mathrm{Disc}}(p_4), q_3, p_4, q_4 + p'_{\mathrm{Disc}}(p_4)(q_3 - q_{\mathrm{Disc}})),$$

so $\phi_H(C) = C$ and $\phi_H(Z)$ is a section over A with (p_3, q_3)-coordinates equal to $(p_{\mathrm{Disc}}(R), q_{\mathrm{Disc}})$.

Note that H is not compactly supported (and ϕ_H is not everywhere well defined). In order to get a compactly supported Hamiltonian $\tilde{H}$, we need to multiply a cutoff function to H of the form $\rho_1(p_4)\rho_2(p_3, q_3)$ such that $\rho_1(p_4) : A \longrightarrow \mathbb{R}$ equals 1 near R and $\rho_2(p_3, q_3) : D \longrightarrow \mathbb{R}$ equals 1 near $(p_{\mathrm{Disc}}(R), q_{\mathrm{Disc}})$. Now, for $\tilde{H} = \rho_1(p_4)\rho_2(p_3, q_3)H$, it follows that for a sufficiently small neighbourhood $W \subset A$ of $\{p_4 = R\}$, we will get $\pi^{-1}(W) \cap \phi_{\tilde{H}}(Z) = \{(a, z_4)|z_4 \in W\}$. □

We can thicken the constructed Hamiltonian as follows.

Lemma 6.23. *As before, except with two extra ball factors B, let $Z := \mathrm{Disc}(s_1) \cap N$ be the symplectic section of the fibre bundle $\pi : N \longrightarrow A$ given by projection and $C = \{(0, 0, a, z_4))||z_4| = \sqrt{2R}\}$. There exists a Hamiltonian $H : N \longrightarrow \mathbb{R}$, supported inside the interior of N, and a neighbourhood W of $\pi(C)$ in A such that $\pi^{-1}(W) \cap \phi_H(Z) = \{(0, 0, a, z_4)|z_4 \in W\}$. Moreover, ϕ_H preserves $\{0\} \times B \times D \times A$, $B \times \{0\} \times D \times A$ and $\{0\} \times \{0\} \times D \times A$ setwise.*

Proof. Let $h : B \longrightarrow \mathbb{R}$ be a function supported inside the interior of B such that $h \equiv 1$ near the origin. Let $H^0 : D \times A \longrightarrow \mathbb{R}$ be the Hamiltonian obtained via Lemma 6.22. We define $H = h(z_1)h(z_2)H^0(z_3, z_4)$, so H is supported inside the interior of N. Moreover, the Hamiltonian vector field satisfies

$$X_H|_{\{0\}\times\{0\}\times D\times A} = X_{H^0} \tag{58}$$

$$X_H|_{\{0\}\times B\times D\times A} = H^0(z_3, z_4)X_{h_2} + h(z_2)X_{H^0} \tag{59}$$

$$X_H|_{B\times\{0\}\times D\times A} = H^0(z_3, z_4)X_{h_1} + h(z_1)X_{H^0}, \tag{60}$$

where X_{h_i} denotes the unique vector field that pushes down to X_h in the ith B-factor and trivial to the other factors. We conclude the assertion. □

Proof (Proof of Proposition 6.19). Given a tubular neighbourhood N of C, we can apply Lemma 6.23 to get a Hamiltonian diffeomorphism $\phi_H : \mathbb{P}_\Delta \to \mathbb{P}_\Delta$ supported inside N such that ϕ_H preserves all tori strata setwise (so the first bullet of Proposition 6.19 holds).

If U_C is a small tubular neighbourhood of C such that $\pi(U_C) \subset W$, where W is obtained in Lemma 6.23, then the second bullet of Proposition 6.19 holds.

Finally, a simple but crucial observation is that Equation (58) is true *near* $\{0\} \times \{0\} \times D \times A$. Therefore, $\tilde{z}_i = z_i$ near $\{0\} \times \{0\} \times D \times A$ for $i = 1, 2$. By shrinking U_C, we obtain the third bullet of Proposition 6.19. □

6.4. *A symplectic fibration*

We assume from now until Subsection 6.7 that we have applied the diffeomorphism ϕ_H given in Proposition 6.19. For better readability, we will drop the 'hat' notations.

In this subsection and the next two, we want to equip $\pi : M_t^s \cap U_C \to A$ with a smoothly trivial exact symplectic fibration structure for some appropriate s_1-admissible section s. After that, Assumption 6.17 will be justified and we can apply Proposition 6.18 to get some Lagrangian solid torus. As a first step towards this, we consider $s = s_1$ and equip $M_t \cap U_C$ with a symplectic fibre bundle structure over A (see Proposition 6.26 below). The main tool is the following linear algebra observation first made by Simon Donaldson (and known by the slogan 'almost holomorphic implies symplectic').

Proposition 6.24 ([10], Proposition 3)**.** *Let $\alpha : \mathbb{C}^n \to \mathbb{C}$ be an $\mathbb{R}$-linear map. Let $\alpha^{1,0}$ and $\alpha^{0,1}$ be the complex linear and the anti-complex linear parts of α, respectively. If $|\alpha^{0,1}| < |\alpha^{1,0}|$, then* $\ker(\alpha)$ *is symplectic of rank* $2n-2$ *in* $\mathbb{C}^n$.

Let $G^t := z_1z_2 - tg_U(z) : U_C \to \mathbb{C}$ where g_U and U_C are obtained in Lemma 6.21 and Proposition 6.19. Because the tangent space of $M_t \cap U_C$ is given by $\ker(DG^t)$ for $t \neq 0$, analysing DG^t and how it is related to the projection π will be the heart of this subsection.

In (x, y)-coordinates (see the paragraph after (49)), we have

$$Dg_U = [\partial_{x_1} g_U(x, y), \partial_{y_1} g_U(x, y), \ldots, \partial_{x_4} g_U(x, y), \partial_{y_4} g_U(x, y)].$$

Let

$$D_3 g_U := [0, \ldots, 0, \partial_{x_3} g_U(z), \partial_{y_3} g_U(z), 0, 0],$$

$$D_4 g_U := [0, \ldots, 0, \partial_{x_4} g_U(z), \partial_{y_4} g_U(z)],$$

which taken together form a 2×8 real matrix-valued function on U_C. For $t \neq 0$, we know that $M_t \cap U_C$ is symplectic or, equivalently, $\ker(DG^t)$ is symplectic at all points where $G^t = 0$, because M_t is a holomorphic submanifold. If $D_4 g_U \equiv 0$, then G^t is independent of the z_4-coordinate, so factors as $U_C \to B \times B \times D \xrightarrow{(G^t)'} \mathbb{C}$. If N_t denotes the fibre of $(G^t)'$ over 0, then $M_t \cap U_C = N_t \times A$, a symplectic product. Though there is no reason to have $D_4 g_U \equiv 0$, we are in fact going to show that if we 'remove' the term $tD_4 g_U$ from DG^t, then $\ker(DG^t + tD_4 g_U)$ is still symplectic near C, and we show that this implies that π is a symplectic fibre bundle for U_C sufficiently small.

Lemma 6.25. *There exists a tubular neighbourhood $U'_C \subset U_C$ of C such that* $\ker((DG^t + tD_4 g_U)|_{T_z U'_C})$ *is symplectic of rank* 6 *for all $z \in M_t \cap U'_C$ and all $t > 0$.*

Proof. With the transformation after Proposition 6.19 implicit, we denote $\hat{J}_\Delta$ just by J_Δ etc. in the following. We have $M_t = \{s_0 = ts_1\}$ and both s_0 and s_1 are holomorphic section and thus

$$DG^t \circ J_\Delta|_{T_z U_C} = J_\mathbb{C} \circ DG^t|_{T_z U_C}$$

for all $z \in M_t$ and all $t > 0$, where $J_\mathbb{C}$ is the standard complex structure of $\mathbb{C}$.

As a result, for $z \in M_t$ and any $\mathbb{R}$-linear matrix $A : T_z U_C \to \mathbb{C}$, we have

$$\begin{aligned}(DG^t + A)^{1,0} &= DG^t + A^{1,0}\\ (DG^t + A)^{0,1} &= A^{0,1},\end{aligned}$$

where superscripts $(1, 0)$ and $(0, 1)$ are the $(J_\Delta, J_\mathbb{C})$ complex linear part and anti-complex linear part, respectively, so

$$2A^{1,0} = A - J_\mathbb{C} A J_\Delta \quad \text{and} \quad 2A^{0,1} = A + J_\mathbb{C} A J_\Delta. \tag{61}$$

Using the fact that $\|J_\Delta\|$ is uniformly bounded, applying triangle inequality to (61) gives a $c_0 > 0$ such that $\|A^{1,0}\|, \|A^{0,1}\| < c_0\|A\|$ for every $\mathbb{R}$-linear matrix A. Now assume additionally that $\|A\| < c_1\|DG^t(z)\|$ for some $c_1 > 0$, then

$$\begin{aligned}\|(DG^t + A)^{1,0}\| &\geq \|DG^t\| - \|A^{1,0}\|\\ &> \|DG^t\| - c_0\|A\|\\ &> \|DG^t\| - c_0 c_1\|DG^t\|\\ &> \frac{(1 - c_0 c_1)}{c_0 c_1}\|A^{0,1}\|\\ &= \frac{(1 - c_0 c_1)}{c_0 c_1}\|(DG^t + A)^{0,1}\|.\end{aligned}$$

Hence, given $c_1 > 0$ (independent of t) such that $c_0 c_1 < \frac{1}{2}$, we have for all A satisfying $\|A\| < c_1\|DG^t(z)\|$ for all t that $\|(DG^t + A)^{1,0}\| > \|(DG^t + A)^{0,1}\|$. In this case, $\ker(DG^t + A)$ is symplectic of rank 6 for all t by Proposition 6.24.

By the second and third bullets of Lemma 6.21, we know that $\partial_{x_4} g_U(z) = \partial_{y_4} g_U(z) = 0$, and $\partial_{x_3} g_U(z), \partial_{y_3} g_U(z) \neq 0$ for $z \in \mathrm{Disc}(s_1)$. Therefore, for any $c_1 > 0$ such that $c_0 c_1 < \frac{1}{2}$, there exist small neighbourhood U'_C of $C \subset \mathrm{Disc}(s_1)$ such that

$$\|D_4 g_U\| < c_1\|D_3 g_U\|$$

for all $z \in U'_C$. As a result, we have $c_1\|DG^t(z)\| \geq c_1\|tD_3 g_U\| > \|tD_4 g_U\|$ so $\ker(DG^t + tD_4 g_U)$ is symplectic for all $z \in M_t \cap U'_C$ and for all $t > 0$. □

By shrinking the U_C we chose in Proposition 6.19 if necessary, we can assume U_C is small enough such that Lemma 6.25 is satisfied, and we will do so in the following.

Proposition 6.26. *Let C and U_C be as before and let $\pi : M_t \cap U_C \to A$ be the restriction of the projection $\pi_4 : U_C \to A$. We find that π is a symplectic fibration without singularities.*

Proof. Let $H^t(z) = (G^t(z), \pi_4(z)) : U_C \to \mathbb{C} \times A$. For all $z_4 \in A$ we get $F_{z_4} := (H^t)^{-1}(0, z_4) = \pi^{-1}(z_4)$. Along F_{z_4}, we have

$$\ker(DH^t|_{F_{z_4}}) = \{v \in \ker(DG^t|_{F_{z_4}}) \mid v_7 = v_8 = 0\} \subset \ker((DG^t + tD_4 g_U)|_{F_{z_4}}),$$

where v_7, v_8 are the seventh and eighth entries of the vector v, respectively. Notice that $\ker((DG^t + tD_4 g_U)_{F_{z_4}}) = \ker(DH^t|_{F_{z_4}}) \oplus \mathbb{R}\langle v_7, v_8\rangle$ and the left-hand side has rank 6 by Lemma 6.25; hence $\dim_\mathbb{R}(\ker(DH^t|_{F_{z_4}})) = 4$ and therefore π has smooth fibres.

Moreover, $\ker((DG^t + tD_4 g_U)_{F_{z_4}})$ is symplectic by Lemma 6.25. It is clear that $\mathbb{R}\langle v_7, v_8\rangle$ is symplectic and its symplectic orthogonal complement is $\ker(DH^t|_{F_{z_4}})$, so $\ker(DH^t|_{F_{z_4}})$ is also symplectic. □

6.5. Liouville vector field

We recall that $U_C = B \times B \times D \times A$. For $j = 1, 2$, let $(\varrho_j, \vartheta_j) = (\sqrt{2p_j}, q_j)$ be the polar coordinates of the first two factors, respectively. Using $z_3 = \sqrt{2p_3}e^{iq_3}$, we can symplectically identify D with a closed disk in $\mathbb{C}$ centred at a. Translating by a, the polar coordinates on $\mathbb{C}$ induce a polar coordinate (ϱ_3, ϑ_3) on D with $\{\varrho_3 = 0\} = \{a\}$. We identify A with a S^1-equivariant neighbourhood of the zero section in T^*S^1 such that $\{|z_4| = R\}$ is mapped to the zero section. Let ϱ_4 and ϑ_4 be the fibre and base coordinates of T^*S^1 and hence coordinates on A. With these new notations, the symplectic form on U_C can be rewritten as $d\alpha$, where

$$\alpha := \sum_{i=1}^{3} \frac{\varrho_i^2}{2} d\vartheta_i + \varrho_4 d\vartheta_4.$$

We also have a Liouville vector field (see Subsection 6.1 for some background)

$$Z_{U_C} := \sum_{i=1}^{3} \frac{\varrho_i}{2} \partial_{\varrho_i} + \varrho_4 \partial_{\varrho_4}$$

pointing outward along ∂U_C making U_C a convex exact symplectic manifold (or, equivalently, a Liouville domain). The restriction of α to $M_t \cap U_C$ induces a Liouville vector field Z_{M_t} on it. We want to show that Z_{M_t} points outward along the vertical boundary $\pi^{-1}(\partial A)$ of $M_t \cap U_C$.

Proposition 6.27. *Given U'_C as in Proposition 6.26, there exists a shrinking of the $B \times B \times D$-factor of U'_C to obtain an open set U_C such that Z_{M_t} points outward along the vertical boundary of the fibration $\pi : M_t \cap U_C \to A$.*

Proof. The Liouville vector field $Z_{U'_C}$ decomposes with respect to $TM_t \oplus (TM_t)^\omega$ in, say, $Z_1 + Z_2$. For $v \in TM_t$, we have

$$\alpha|_{M_t}(v) = \alpha(v, 0) = \omega_{U'_C}(Z_1 + Z_2, (v, 0)) = \omega_{U'_C}(Z_1, (v, 0)) = \omega_{M_t}(Z_1, v)$$

because $\omega_{U'_C}(Z_2, (v, 0)) = 0$ by M_t being symplectic in U'_C. This being true for all v, we conclude that $Z_{M_t} = Z_1$.

Let $A_0 := \{\varrho_1 = \varrho_2 = \varrho_3 = 0\} \times A$, which lies inside M_t for all t. Note that $Z_{U'_C} = \varrho_4 \partial_{\varrho_4}$ on A_0 so it points outward along ∂A_0. Note also that $Z_{U'_C}|_{A_0} \in TM_t$, so the $(TM_t)^\omega$-component of $Z_{U'_C}|_{A_0}$ is 0, which in turn implies that $Z_{M_t}|_{A_0}$ points outward along ∂A_0. Because pointing outward along $M_t \cap (B \times B \times D \times \partial A)$ is an open condition, by shrinking the $B \times B \times D$ factor, we can ensure that Z_{M_t} points outward along $M_t \cap (B \times B \times D \times \partial A)$. □

6.6. A good deformation

We are going to construct a smoothly trivial exact symplectic fibration and justify Assumption 6.17 in this subsection. Ideally, we would like the symplectic fibration $\pi : M_t \cap U_C \to A$ to be a smoothly trivial exact symplectic fibration, but it is not true in general that π is trivial near the horizontal boundary even if we assume U_C to be very small. However, we can show that it is true after appropriately deforming s_1 to another s_1-admissible section, which has been the whole purpose of introducing the notion of admissible sections.

Proposition 6.28 (Homotoping into Assumption 6.17). *For any open neighbourhood N of C, there are tubular neighbourhoods $U_C, V_C \subset N$ as in (41) so that $V_C \subsetneq U_C$ is a closed neighbourhood of* $\mathrm{Disc}(s_1) \cap U_C$. *The neighbourhood U_C satisfies Propositions 6.19, 6.26 and 6.27. There is also a smooth* *family* $(s^u)_{u\in[0,1]}$ *of* s_1*-admissible sections with* $s^0 = s_1$ *and for all* u, $s^u = s_1$ *outside* N *and*

$$M_t^{s^1} \cap (U_C \backslash V_C) = \{z_1 z_2 = t(z_3 - a)\}. \tag{62}$$

Moreover, the projection to A, $\pi : M_t^{s^1} \cap U_C \to A$ *is a smoothly trivial exact symplectic fibration for all* $t > 0$ *small.*

Recall that every s_1-admissible section equals s_1 near $\mathrm{Disc}(s_1)$ (hence is more messy than (62)) and recall that $\mathrm{Disc}(s_1) \cap U_C = \{0\} \times \{0\} \times \{a\} \times A$, so we cannot hope for (62) to be true if $V_C \subset U_C$ is not a neighbourhood of $\mathrm{Disc}(s_1) \cap U_C$, which is why the A-factors of V_C and U_C agree in (41).

The proof of Proposition 6.28 is divided into two steps. The first step is the construction of $(s^u)_{u\in[0,1]}$ and V_C, and the second step is to justify that π is a smoothly trivial exact symplectic fibration for all $t > 0$ small.

Proof (Proof of Proposition 6.28: Step 1). Pick $U'_C \subset N$ sufficiently small such that Propositions 6.19, 6.26 and 6.27 are satisfied. We shrink the A-factor of U'_C to obtain an open set U''_C that still satisfies Propositions 6.19 and 6.26. Finally, apply Proposition 6.27 to U''_C to shrink its $B\times B\times D$-factor and arrive at an open set U_C that also satisfies all three propositions like U'_C and, furthermore, U_C is contained in the interior of U'_C, which we will need later.

We work on U_C now. By the last bullet of Lemma 6.21, we know that $(g_U)_0|_{(D\backslash\{a\})\times A} \to \mathbb{C}^*$ is homotopic to $z_3 - a : (D \setminus \{a\}) \times A \to \mathbb{C}^*$. Therefore, there is no obstruction to constructing a smooth family of functions $(h^u)_{u\in[0,1]} : D \times A \to \mathbb{C}$ such that

- $h^0 = (g_U)_0$,
- h^u is independent of u near the discriminant $\{a\} \times A$,
- $(h^u)^{-1}(0) = \{a\} \times A$ for all u, and
- there is a neighbourhood of V_0 of $\{a\} \times A$ inside $D \times A$ such that $h^1|_{(D\times A)\backslash V_0} = z_3 - a$.

The second and third bullets above correspond to admissibility of sections, and the last bullet corresponds to (62).

After h^u is constructed, there is no obstruction to extend it to $g^u : U_C \to \mathbb{C}$ such that $g^0 = g_U$, for all u, $g^u|_{\{z_1=z_2=0\}} = h^u$, g^u is independent of u near the discriminant $\mathrm{Disc}(s_1) \cap U_C$ and there exists a closed neighbourhood $V_C \subset U_C$ of $\mathrm{Disc}(s_1) \cap U_C$ such that $g^1|_{U_C\backslash V_C} = z_3 - a$. Indeed, note that we permit g^u to take value 0 outside $\{z_1 = z_2 = 0\} \cap U_C$. This is because $\{z_1 = z_2 = 0\} \cap U_C$ is exactly the intersection between the two-dimensional toric strata and U_C, so even if g^u is 0 somewhere in $U_C \setminus \{z_1 = z_2 = 0\}$, it will not create new discriminant (cf. the proof of Corollary 4.2).

With this understood, we can extend the isotopy $(g^u)_{u\in[0,1]}$ from U_C to U'_C so that it equals g_U for all u near the boundary of U'_C as well as near the discriminant. Recall that $M_t \cap U'_C = \{s_0 = ts_1\} \cap U'_C = \{z_1 z_2 = tg_U\} \cap U'_C$. We can patch g^u with g_U outside U'_C to obtain a family of s_1-admissible sections $(s^u)_{u\in[0,1]}$ such that $s^0 = s_1$, for all u, $s^u = s_1$ outside N and (62) is satisfied on U_C. □

Proof (Proof of Proposition 6.28: Step two). Now, we want to address why π is a smoothly trivial exact symplectic fibration for all $t > 0$ small. Let $\pi_4 : U_C \to A$ be the obvious projection (note the difference with the π above, namely, π is the restriction of π_4 to $M_t^s \cap U_C$ for some s but π_4 is defined on the entire U_C). We use the notation $M_t^u := M_t^{s^u} \cap U_C$ in this proof. We will choose a subset $V'_C \subset V_C$ (for V_C defined in step one), so the vertical boundary of the fibration $\pi_4|_{M_t^u}$ is divided into two parts, namely, (a) $\pi_4|_{M_t^u \cap V'_C}$, b) $\pi_4|_{M_t^u \cap (U_C\backslash V'_C)}$ where we use different arguments. We first choose V'_C.

Let $G^{t,u} := z_1 z_2 - tg^u(z)$ for $g^u : U_C \to \mathbb{C}$ constructed in step 1. In particular, we have $M_t^u = (G^{t,u})^{-1}(0)$. Near $\mathrm{Disc}(s_1) \cap U_C$, $G^{t,u}$ is independent of u, so by Proposition 6.26, there exists a neighbourhood $V'_C \subset V_C$ of $\mathrm{Disc}(s_1) \cap U_C$ such that $\pi_4|_{M_t^u \cap V'_C}$ is a symplectic fibration without singularity for all $0 < |t| < \delta$ and all u. Moreover, by Proposition 6.27, $Z_{M_t^u}$ points outward along the vertical boundary of $\pi_4|_{M_t^u \cap V'_C}$, so we are done with (a)).

 https://doi.org/10.1017/fms.2020.54

For (b), as argued in the proof of Lemma 4.1, $D(z_1z_2)$ dominates $tDg^u(z)$ outside V'_C when t is small. Therefore, $\ker(G^{t,u})$ is an arbitrarily small perturbation of $\ker(D(z_1z_2))$ outside V'_C for all u when t is small. More precisely, $\ker(G^{t,u})|_{M^u_t\setminus V'_C}$ converges uniformly to $\ker(D(z_1z_2))$ outside V'_C for all u when t goes to 0. Because $\ker(D(z_1z_2)) \cap \ker(D(z_4))$ is symplectic, the fact that $\ker(G^{t,u})|_{M^u_t\setminus V'_C}$ converges uniformly to $\ker(D(z_1z_2))$ implies that for t small, $\pi_4|_{M^u_t\setminus V'_C}$ is a symplectic fibration without singularity. On the other hand, by the local model in Lemma 6.11, we also know that $\ker(G^{t,u})|_{M^u_t\setminus V'_C}$ converges uniformly to $\ker(D(z_1z_2))$ which implies that for t small, $Z_{M^u_t}$ points outward along the vertical boundary of $\pi_4|_{M^u_t\setminus V'_C}$. We conclude that there exists $\delta > 0$ such that $\pi_4|_{M^u_t}$ is a symplectic fibration without singularity, and $Z_{M^u_t}$ points outward along the vertical boundary of the fibration $\pi_4|_{M^u_t}$, for all $0 < |t| < \delta$ and all u.

Finally, we need to deal with the outward-pointing vector field along the horizontal boundary. We have $g^1|_{U_C\setminus V_C} = z_3 - a$, so we have $M^1_t \setminus V_C = \{z_1z_2 = t(z_3 - a)\}$. Because the horizontal boundary of $\pi : M^1_t \to A$ lies inside $M^1_t \setminus V_C$ and $M^1_t \setminus V_C$ is independent of the z_4-coordinate, π is symplectically trivial near the horizontal boundary. By Lemma 6.11, we know that $Z_{M^u_t}$ also points outward along the horizontal boundary of π. □

As a consequence of Proposition 6.18, we get the following corollary.

Corollary 6.29. *Under Proposition 6.28, there exist a family of proper Lagrangian solid tori $L_t \subset M^{s^1}_t \cap U_C$, for $t > 0$ small, such that $L_t \subset M^{s^1}_t \cap (U_C\setminus V_C)$ is a cylindrical Lagrangian over the Legendrian* (44).

6.7. Proof of Theorem 6.6

Recall our convention to write $\hat{p}_j$ as p_j, etc. We undo this convention now to distinguish between the two sets of coordinates. The last section used the $\hat{p}_j$-coordinates. Recall the transformation Ψ between the two sets of coordinates from (46). In this section, we apply Ψ^{-1} to transform the Lagrangian solid tori obtained in Corollary 6.29 back to (p, q)-coordinates. After that, we will conclude the proof of Theorem 6.6.

We start with the situation as in Proposition 6.28, so we have U_C, V_C of the form (49) and a family $(s^u)_{u\in[0,1]}$ of s_1-admissible sections. Recall that Ψ is merely a change of coordinates and that it preserves the coordinates on $\{z_1 = z_2 = 0\} = \{\hat{z}_1 = \hat{z}_2 = 0\}$. By applying Ψ^{-1} to $M^{s^1}_t$ – that is, inserting (46) – we get the following:

$$\begin{aligned} &M^{s^1}_t \cap (U_C \setminus V_C) = \{\hat{z}_1\hat{z}_2 = t(\hat{z}_3 - a)\} \\ =&\{\sqrt{4p_1p_2}e^{i(q_1+q_2+q_3+q_4)} = t(\sqrt{2(p_3 - p_2)}e^{iq_3} - a)\}. \end{aligned} \tag{63}$$

Recall that the third factor of U_C and V_C is disks centred at a, say, D_U and D_V, respectively ($D_V \subsetneq D_U$). Recall also that we have a straight line $\gamma(r) = (0, 0, r, R)$ in p-coordinates for $r_0 < r \le r_1$ (see the paragraph before Theorem 6.6) and $a = \sqrt{2r_1}e^{iq}$, so γ ends at a. We choose $r_0 < r' < r'' < r''' < r_1$ such that if A_1 is the 0-centred annulus with radii r', r'' and A_2 is the 0-centred annulus with radii r'', r''', then

$$A_1 \cap D_U = \emptyset, \quad A_2 \cap D_V = \emptyset, \quad \text{but} \quad A_2 \cap D_U \neq \emptyset;$$

see Figure 11. We want to perform an additional symplectic isotopy for $M^{s^1}_t$ so that the new symplectic hypersurface is x-standard for all $x \in \gamma([r', r''])$ (see Definition 5), as explained in the following lemma. For $\epsilon > 0$, let B_ϵ denote the closed 0-centred ϵ-ball in $\mathbb{R}^2$. We set

$$W_{1,\epsilon} = B_\epsilon \times B_\epsilon \times A_1 \times A,$$

$$W_{2,\epsilon} = B_\epsilon \times B_\epsilon \times A_2 \times A.$$

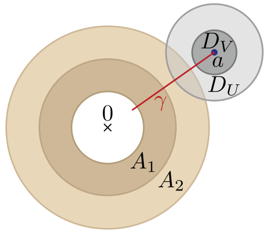


Figure 11. *Disks and annuli in the z_3-plane.*

Lemma 6.30. *Let $N \subset \mathbb{P}_\Delta$ be an open set such that there exists $\epsilon > 0$ with $U_C, W_{1,\epsilon}, W_{2,\epsilon} \subset N$. Then there exists a smooth family $(s^u_{\text{tran}})_{u\in[0,1]}$ of s_1-admissible sections such that $s^0_{\text{tran}} = s^1$ and, for all u, $s^u_{\text{tran}} = s^1$ inside U_C and outside N. Furthermore,*

$$M_t^{s^1_{\text{tran}}} \cap (W_{1,\epsilon} \cup W_{2,\epsilon}) = \{\sqrt{p_1p_2}e^{i(q_1+\cdots+q_4)} = tg_{\text{tran}}(p_3 - p_2, q_3)\} \tag{64}$$

for some $g_{\text{tran}}(p_3 - p_2, q_3) \in C^\infty(W_{1,\epsilon} \cup W_{2,\epsilon}, \mathbb{C})$ such that $g_{\text{tran}}|_{W_{1,\epsilon}}$ is a nonzero constant.

Note that $g_{\text{tran}}|_{W_{1,\epsilon}}$ being a nonzero constant implies that $M_t^{s^1_{\text{tran}}}$ is x-standard for $x \in \gamma([r', r''])$.

Proof. This proof is very similar to the proof of Lemma 5.3 and step 1 of the proof of Proposition 6.28. We know that $M_t^{s^1} \cap (W_{1,\epsilon} \cup W_{2,\epsilon})$ is given by

$$\sqrt{p_1p_2p_3p_4}e^{i(q_1+\cdots+q_4)} = tf \tag{65}$$

for some $f \in C^\infty(W_{1,\epsilon} \cup W_{2,\epsilon}, \mathbb{C})$ (cf. (20)). Let B_A denote the smallest 0-centred ball containing A (i.e., of radius the larger radius of A). By the fact that s^1 is s_1-admissible, we have $\operatorname{Im}(f|_{\{p_1=p_2=0\}}) \subset \mathbb{C}^*$. Moreover, because there is an open subset G of $\left(\{0\} \times \{0\} \times B_{r'''} \times B_A\right)$ that is homeomorphic to a ball, contains both the origin and $(A_1 \cup A_2) \times A$ and such that $G \cap \operatorname{Disc}(s_1) = \emptyset$, we have that

$$(f|_{\{p_1=p_2=0\}})_* : \pi_1((A_1 \cup A_2) \times A) \longrightarrow \pi_1(\mathbb{C}^*) \tag{66}$$

is the zero map (cf. (21)). Thus, there is no obstruction to constructing a smooth family $f^u_{\text{tran},0} : (A_1 \cup A_2) \times A \longrightarrow \mathbb{C}^*$, for $u \in [0,1]$, such that $f^0_{\text{tran},0} = f|_{\{p_1=p_2=0\}}$, $f^u_{\text{tran},0}$ is independent of u inside $((A_1 \cup A_2) \cap D_U) \times A$, $f^1_{\text{tran},0} = \sqrt{p_3p_4}g_{\text{tran},0}(p_3, q_3)$ for some $g_{\text{tran},0} : (A_1 \cup A_2) \times A \longrightarrow \mathbb{C}^*$ and $g_{\text{tran},0}|_{A_1\times A}$ is a nonzero constant.

Finally, as in the step 1 of the proof of Proposition 6.28, we can extend $f^u_{\text{tran},0}$ and $g_{\text{tran},0}$ to f^u_{tran} and g_{tran}, which are defined over the whole $W_{\epsilon,1} \cup W_{\epsilon,2}$ such that, by patching, f^u_{tran} induces a family $(s^u_{\text{tran}})_{u\in[0,1]}$ of s_1-admissible sections with all of the properties listed in the proposition satisfied. In particular, g_{tran} satisfies (64). □

Next, we want to describe the family (for $t > 0$ small) of proper Lagrangian solid tori $L_t \subset M_t^{s^1} \cap U_C$ in Corollary 6.29 in (p, q)-coordinates. Recall that $a = \sqrt{2r_1}e^{iq}$ and that Remark 6.14 permits us to choose any argument for the Legendrian. For our purpose, we pick r to be the map $r \mapsto re^{i\phi}$ with $\phi = \frac{\pi+q}{2}$. This way, $r^2 + a$ parametrizes a curve that starts at a and moves straight towards the origin. We find $L_t \cap (U_C \setminus V_C)$ in $\hat{z}$-coordinates (as in (32)) given by

$$\left\{\hat{z} = \left(re^{i\theta_1}, re^{-i\theta_1}, \frac{r^2}{t} + a, \sqrt{2}Re^{i\theta_2}\right) \in M_t^{s^1} \cap (U_C \setminus V_C) \,\middle|\, \begin{array}{l} r \in e^{i\frac{\pi+q}{2}}(0,\infty), \\ \theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z} \end{array}\right\}$$

and in the (p, q)-coordinates (applying (47) alias inserting (46)) this is described by the following equations:

$$p_1 = p_2 = \frac{|r|^2}{2}, \qquad p_3 = \frac{\left(\sqrt{2r_1} - \frac{|r|^2}{t}\right)^2}{2} + \frac{|r|^2}{2}, \qquad p_4 = R + \frac{|r|^2}{2}, \tag{67}$$

$$q_1 = \frac{\pi + q}{2} + \theta_1, \qquad q_2 = \frac{\pi + q}{2} - \theta_1 - \theta_2 - q, \qquad q_3 = q, \qquad q_4 = \theta_2. \tag{68}$$

The tropical curve γ is contained in the line through $(0, 0, 1, R)$ and $(0, 0, 0, R)$, so in view of Proposition 5.4, we define W to be the affine 2-plane in $\mathbb{R}^4$ containing the points $(1, 1, 1, 1 + R)$, $(0, 0, 1, R)$ and $(0, 0, 0, R)$, so $\gamma \subseteq W \cap \partial\square$. By inspecting (67), we see that $p \in W$ for all $(p, q) \in L_t \cap (U_C \setminus V_C)$ and, by deriving (68), we find $\partial_{\theta_1}, \partial_{\theta_2} \in W^\perp$.

The following lemma gives a family (for $t > 0$ small) of Lagrangian solid tori (with boundary) in $M_t^{s^1_{\mathrm{tran}}}$ that are $\gamma((r', r''))$-standard (see Definition 6).

Lemma 6.31. *For s^1_{tran} in Lemma 6.30, there is a family of Lagrangian solid tori with boundary, for $t > 0$ sufficiently small, $L_t^{\mathrm{tran}} \subset M_t^{s^1_{\mathrm{tran}}} \cap (W_{1,\epsilon} \cup W_{2,\epsilon} \cup U_C)$ such that*

1. *$p \in W \cup \pi_\Delta(V_C)$ for all $(p, q) \in L_t^{\mathrm{tran}}$, and*
2. *$W^\perp \subset T_{(p,q)} L_t$ for all $(p, q) \in L_t^{\mathrm{tran}}$ satisfying $p \in W \backslash \pi_\Delta(V_C)$, and*
3. *the p_3-coordinate of all points in the torus boundary $\partial L_t^{\mathrm{tran}}$ is r'.*

Proof. By the construction in Lemma 6.30, $s^1_{\mathrm{tran}}|_{U_C} = s^1|_{U_C}$ so the L_t constructed in Corollary 6.29 are Lagrangian inside $M_t^{s^1_{\mathrm{tran}}} \cap U_C$. Inspecting (44) and (43), for a fixed $t > 0$ sufficiently small, by Proposition 6.18 and the paragraph before, the (p_3, q_3)-coordinates of all of the points in ∂L_t are the same and they lie in ∂D_U. Therefore, we need to explain how to 'extend' L_t to $L_t^{\mathrm{tran}} \subset M_t^{s^1_{\mathrm{tran}}} \cap (W_{1,\epsilon} \cup W_{2,\epsilon} \cup U_C)$ so that, in particular, the p_3-coordinate of all of the points in the torus boundary $\partial L_t^{\mathrm{tran}}$ equal r'.

The proof strategy is the same as Proposition 5.4 and Lemma 5.8. By Lemma 6.30, $M_t^{s^1_{\mathrm{tran}}} \cap (W_{1,\epsilon} \cup W_{2,\epsilon})$ is given by

$$\sqrt{p_1 p_2} e^{i(q_1 + \cdots + q_4)} = t g_{\mathrm{tran}}(p_3 - p_2, q_3). \tag{69}$$

We want to construct a Lagrangian in $M_t^{s^1_{\mathrm{tran}}} \cap (W_{1,\epsilon} \cup W_{2,\epsilon})$ such that $q_3 = q$ is a constant. We move γ inside W from the toric boundary to the nearby fibres as follows: Consider the function $\rho := \frac{p_1 p_2}{|g_{\mathrm{tran}}(p_3 - p_2, q)|^2}$ on $\pi_\Delta(W_{1,\epsilon} \cup W_{2,\epsilon})$, so $\rho = t^2$ is the moment map image of the hypersurface (69). Let $\nu := \sum_{j=1}^4 \partial_{p_j}$, so $\nu(g_{\mathrm{tran}}(p_3 - p_2, q_3)) = 0$. This implies $\nu(\rho) > 0$ for all $p \in \pi_\Delta(W_{1,\epsilon} \cup W_{2,\epsilon}) \backslash \{p_1 = p_2 = 0\}$, so ρ is strictly increasing in the direction $(1, 1, 1, 1)$ and zero on the boundary $\{p_1 = 0\} \cup \{p_2 = 0\}$. For small $t > 0$, for all $r \in (r', r''')$, there exists a unique λ such that $p = \gamma(r) + \lambda(1, 1, 1, 1)$ satisfies $\sqrt{p_1 p_2} = t|g_{\mathrm{tran}}(p_3 - p_2, q)|$. Therefore, by the reasoning in Proposition 5.4 and Lemma 5.8, we get a family of Lagrangians in $M_t^{s^1_{\mathrm{tran}}} \cap (W_{1,\epsilon} \cup W_{2,\epsilon})$ that is $\gamma((r', r''))$-standard. By choosing q_1, q_2, q_4-coordinates appropriately (cf. (22)), this family can be smoothly attached to L_t to give L_t^{tran} as desired. □

Now recall that we applied a Hamiltonian isotopy ϕ in Subsection 6.3 to modify s_1 so that the discriminant became straight in the sense of Proposition 6.19 at the endpoint of the tropical curve. We will account for this step in the following and conclude the proof of Theorem 6.6 where we carefully distinguish between s_1 and $\hat{s}_1$, etc.; see (50)-(55).

Proof (Proof of Theorem 6.6). Let s be an s_1-admissible section and N be a neighbourhood of $\gamma(r_1)$. Let $r' < r'' < r_1$ be such that $\gamma([r', r_1]) \subset N$. Consequently, $W_{1,\epsilon} \subset \pi_\Delta^{-1}(N)$ for $\epsilon > 0$ small (indeed, recall that $W_{1,\epsilon}$ is defined with respect to (p, q)-coordinates). Now, after applying the integral linear transformation Ψ, we let $N_1 \subset \pi_\Delta^{-1}(N)$ be a tubular neighbourhood of C such that $W_{1,\epsilon} \cap N_1 = \emptyset$, where N_1 is defined in $(\hat{p}, \hat{q})$-coordinates. We apply Proposition 6.19 to N_1 so that we get a Hamiltonian isotopy ϕ_H supported inside N_1 to straighten the discriminant. By Corollary 6.29, there exist neighbourhoods $V_C \subset U_C \subset N_1$ of C, $\hat{s_1}$-admissible sections $(s^u)_{u\in[0,1]}$ and for all $t > 0$ small, Lagrangian solid tori $L_t \subset \hat{M}_t^{s^1} \cap U_C$. Let $r''' < r_1$ such that the corresponding $W_{2,\epsilon}$ satisfies $W_{2,\epsilon} \cap U_C \neq \emptyset$ and $W_{2,\epsilon} \cap V_C = \emptyset$ as before. We can now apply Lemma 6.31 to obtain $L_t^{\mathrm{tran}} \subset \hat{M}_t^{s^1_{\mathrm{tran}}} \cap (W_{1,\epsilon} \cup W_{2,\epsilon} \cup U_C)$.

Finally, we apply the inverse of the Hamiltonian isotopy ϕ_H to get $\phi_H^{-1}(L_t^{\mathrm{tran}}) \subset \phi_H^{-1}(\hat{M}_t^{s^1_{\mathrm{tran}}}) \cap \phi_H^{-1}(W_{1,\epsilon} \cup W_{2,\epsilon} \cup U_C)$. First note that $\phi_H^{-1}(\hat{M}_t^{s^1_{\mathrm{tran}}}) = M_t^{s^1_{\mathrm{tran}} \circ \phi_H}$ and $s^1_{\mathrm{tran}} \circ \phi_H$ is s_1-admissible. By definition, ϕ_H^{-1} is the identity outside N_1. As a result, $\phi_H^{-1}(\hat{M}_t^{s^1_{\mathrm{tran}}}) = M_t^{s^1_{\mathrm{tran}} \circ \phi_H}$ remains x standard in $W_{1,\epsilon}$ (because $W_{1,\epsilon} \cap N_1 = \emptyset$). Moreover, $\phi_H^{-1}(L_t^{\mathrm{tran}})$ remains $\gamma((r', r''))$-standard for the same reason. This finishes the proof. □

6.8. *Concluding the proof of Theorem 1.1*

Proof (Proof of Theorem 1.1). Let γ be a tropical curve satisfying the assumptions of Theorem 1.1. Let N be a neighbourhood of γ. We can apply Theorem 6.6 to construct open Lagrangian solid tori for the endings of the tropical curve near the discriminant such that the noncompact ends of the tori are standard with respect to an open subset of γ. Therefore, we can apply Proposition 5.17 to obtain, for all $t > 0$ small, a closed Lagrangian $L_t \subset M_t^s$ such that $\pi_\Delta(L_t) \subset N$.

Again, as explained in the proof of Proposition 5.17, we can assume the families of s_1-admissible sections we have constructed are constant outside $\pi_\Delta^{-1}(N)$. Therefore, we can apply Lemma 4.3 to conclude that $L_t \subset M_t^s$ can be brought back, via a symplectic isotopy, to a closed embedded Lagrangian inside $M_t \cap \pi_\Delta^{-1}(N)$.

The statement regarding multiplicity is proved in Proposition 2.7. □

6.8.1. Orbifold case

When $\mathbb{P}_\Delta$ is a toric orbifold, the proof of Theorem 1.1 goes very similar. First, by Lemma 3.6, the cover $\mathbb{C}^n \to \mathbb{C}^n/K$ is unbranched away from the origin. This means that if U/K is a symplectic corner chart for $\mathbb{P}_\Delta$, then $U\backslash\{0\} \to (U\backslash\{0\})/K$ is an unbranched cyclic covering. Note that near the discriminant, the tropical curve is in the direction $(0, 0, 1, 0)$ with respect to the symplectic corner chart U. Because the cyclic group K is generated by an element in $(\mathbb{R}/2\pi\mathbb{Z})^4$ with nonzero components (otherwise, the orbifold points will not be isolated), it implies that we are necessarily in the case $K \cap W_T^\perp = \{0\}$ in Proposition 5.4. Therefore, if we denote the lift of Disc $\subset U/G$ to U by $\widetilde{\mathrm{Disc}}$, then we can apply Theorem 6.6 in U to get a family of solid Lagrangian tori L_t near $\widetilde{\mathrm{Disc}}$ and its K-orbit is a disjoint union of $|K|$ solid Lagrangian tori. Because L_t are away from the origin in U, it descends to a family of solid Lagrangian tori in U/K near Disc. Therefore, the result follows.

Acknowledgements. We thank Paul Seidel for suggesting to study mirror duals of lines and for bringing the authors together at the Institute for Advanced Study in Princeton. Further thanks for helpful discussions go to Mohammed Abouzaid, Ivan Smith, Travis Mandel, Johannes Walcher, Hans Jockers and Penka Georgieva. We also thank the two anonymous referees.

The first author was funded by the National Science Foundation under agreement No. DMS-1128155 and by EPSRC (Establish Career Fellowship EP/N01815X/1). The second author was funded by DFG grant RU 1629/4-1. The authors also gratefully acknowledge the support of the Hausdorff Research Institute for Mathematics (Bonn), through the Junior Trimester Program on Symplectic Geometry and Representation Theory.

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