On the difference of spectral projections

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Abstract

The aim of the present Ph.D. thesis is to investigate the relationship between differences of spectral projections and Hankel integral operators. This leads us to the following question: Is the difference of two spectral projections $E_{(-\infty,\lambda)}(A+B)$ and $E_{(-\infty,\lambda)}(A)$ associated with an open interval $(-\infty,\lambda)$ unitarily equivalent to a Hankel integral operator, provided that A and B are self-adjoint operators on a complex separable Hilbert space of infinite dimension, where A is semibounded and B is of rank 1?

We show that, roughly speaking, the answer to this question is positive for all but at most countably many $\lambda \in \mathbb{R}$. Further, we prove a similar result in the more general case when B is compact.

The above question is motivated by the following classical example given by M. Krein: The difference of the resolvents of the Neumann and Dirichlet Laplacians on the semi-axis at the spectral point -1 is a rank one operator, but the difference of the spectral projections of these resolvents associated with $(-\infty,\lambda)$ is not even Hilbert Schmidt, for all $0<\lambda<1$. The latter difference is a Hankel integral operator that can be computed explicitly (and is not even compact, as was shown more than fifty years later by Kostrykin and Makarov who diagonalized this Hankel integral operator).

With this example, M. Krein showed that the "naive" definition (proposed by Lifshits)

$$\xi(\lambda) \text{ ``}=\text{''} \ \operatorname{trace}\big(E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A)\big)$$

of the spectral shift function $\xi \in \mathsf{L}^1(\mathbb{R})$ for a pair A, $A+B$ need not work in general. The spectral shift function was introduced at a formal level by Lifshits; M. Krein presented a rigorous definition.

In the final chapter of the present thesis, we generalize M. Krein's example to operators of the type

$$\Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big)^{\mathrm{N/D}} \otimes I + I \otimes L \quad \text{in} \quad \mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G},$$

where L is a self-adjoint nonnegative operator on a complex separable Hilbert space $\mathfrak{G}$. In particular, we observe that the difference of the spectral projections is again unitarily equivalent to a Hankel integral operator.

Zusammenfassung

Das Ziel der vorliegenden Dissertation ist es, die Verbindung zwischen Differenzen von Spektralprojektionen und Hankel-Integraloperatoren zu erforschen. Dies führt uns zu der folgenden Frage: Ist die Differenz von zwei Spektralprojektionen $E_{(-\infty,\lambda)}(A+B)$ und $E_{(-\infty,\lambda)}(A)$ bzgl. des offenen Intervalls $(-\infty,\lambda)$ unitär äquivalent zu einem Hankel-Integraloperator, wenn A und B selbstadjungierte Operatoren auf einem unendlichdimensionalen komplexen separablen Hilbertraum sind, wobei A halbbeschränkt ist und B vom Rang 1?

Wir zeigen (grob gesagt), dass diese Frage für alle bis auf höchstens abzählbar viele $\lambda \in \mathbb{R}$ positiv beantwortet werden kann. Ferner beweisen wir ein ähnliches Resultat im allgemeineren Fall, wenn B kompakt ist.

Die obige Frage ist motiviert durch das folgende klassische Beispiel von M. Krein: Die Differenz der Resolventen der Neumann- und Dirichlet-Laplace-Operatoren auf der Halbachse im Punkt -1 ist ein Rang-1-Operator, aber die Differenz der Spektralprojektionen dieser Resolventen bzgl. $(-\infty,\lambda)$ ist nicht einmal Hilbert Schmidt, für alle $0<\lambda<1$. Die letztere Differenz ist ein Hankel-Integraloperator, der explizit berechnet werden kann (und der nicht einmal kompakt ist, wie Kostrykin und Makarov, die diesen Hankel-Integraloperator mehr als fünfzig Jahre später diagonalisiert haben, zeigen konnten).

Mit diesem Beispiel hat M. Krein gezeigt, dass die „naive“ Definition (vorgeschlagen von Lifshits)

$$\xi(\lambda) \text{ „=“ } \operatorname{Spur}\big(E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A)\big)$$

der spektralen Verschiebungsfunktion $\xi \in \mathsf{L}^1(\mathbb{R})$ für ein Paar A, $A+B$ im Allgemeinen nicht funktioniert. Die spektrale Verschiebungsfunktion wurde auf formaler Ebene von Lifshits eingeführt; M. Krein hat eine rigorose Definition präsentiert.

Im abschließenden Kapitel der vorliegenden Dissertation verallgemeinern wir M. Kreins Beispiel für Operatoren vom Typ

$$\Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big)^{\mathrm{N/D}} \otimes I + I \otimes L \quad \text{in} \quad \mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G},$$

wobei L ein selbstadjungierter nichtnegativer Operator auf einem komplexen separablen Hilbertraum $\mathfrak{G}$ ist. Insbesondere beobachten wir, dass die Differenz der Spektralprojektionen wieder unitär äquivalent zu einem Hankel-Integraloperator ist.

Contents

Introduction

Can one find the maximal subclass of scalar-valued Borel functions such that for all self-adjoint operators whose difference is of trace class, the *Lifshits–M. Krein trace formula* (1) holds? (cf. [**45**, p. 141]) This question of M. Krein is very natural to pose. He showed in [**44**] that if A and B are self-adjoint operators acting on a separable Hilbert space $\mathfrak{H} \neq \{0\}$, then there exists an L^1 function ξ on $\mathbb{R}$ (the *spectral shift function*) such that for sufficiently nice functions f, one has

$$\operatorname{trace}\big(f(A+B) - f(A)\big) = \int_{\mathbb{R}} \xi(t)\dot{f}(t)\,\mathrm{d}t. \tag{1}$$

At a formal level, the spectral shift function was introduced by Lifshits [**51**]; M. Krein presented in [**44**] a rigorous definition.

Recently, Peller [**63**] proved that the maximal class of functions on $\mathbb{R}$ so that the Lifshits–M. Krein trace formula (1) holds (for all self-adjoint operators A and B, where B is of trace class) coincides with the class of *operator Lipschitz functions*. These are, by definition, continuous scalar-valued functions f on $\mathbb{R}$ such that there exists a constant $c > 0$ with

$$\|f(A+B) - f(A)\|_{\mathrm{op}} \leq c\,\|B\|_{\mathrm{op}}$$

for all self-adjoint operators A and B, where B is of trace class and $\|\bullet\|_{\mathrm{op}}$ denotes the operator norm.

Let us now consider the case when $f = \mathbb{1}_{(-\infty,\lambda)}$ is the characteristic function associated with an open interval $(-\infty, \lambda)$, $\lambda \in \mathbb{R}$. Formally, we obtain from (1) that

$$\operatorname{trace}\big(E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A)\big) = \xi(\lambda), \tag{2}$$

where $E_{(-\infty,\lambda)}(A)$ denotes the spectral projection of A with respect to $(-\infty, \lambda)$. However, M. Krein [**44**] gave an example in which (2) does not hold even though B is of rank 1. He considered the Neumann Laplacian $H = (-\,\mathrm{d}^2/\,\mathrm{d}t^2)^{\mathrm{N}}$ and the Dirichlet Laplacian $H^{\mathrm{D}} = (-\,\mathrm{d}^2/\,\mathrm{d}t^2)^{\mathrm{D}}$ in $\mathsf{L}^2(\mathbb{R}_+)$, where $\mathbb{R}_+ = (0, \infty)$. They both have a simple purely absolutely continuous spectrum filling in $[0, \infty)$. Let us denote the resolvent of H^{D} at the spectral point -1 by A_0 and the resolvent of H at the spectral point -1 by A_1. M. Krein showed in [**44**, pp. 622–624] that, on one hand,

$$A_1 - A_0 = \langle \bullet, \varphi \rangle_{\mathsf{L}^2(\mathbb{R}_+)} \varphi \quad \text{with} \quad \varphi(x) = \mathrm{e}^{-x}$$

and, on the other hand,

$$\Big(\big[E_{(-\infty,\lambda)}(A_0) - E_{(-\infty,\lambda)}(A_1)\big]\psi\Big)(x) = \frac{2}{\pi} \int_{\mathbb{R}_+} \frac{\sin\big((\frac{1}{\lambda} - 1)^{1/2}(x+y)\big)}{x+y} \psi(y)\,\mathrm{d}y \tag{3}$$

for all $0 < \lambda < 1$ and every scalar-valued continuous function ψ with compact support. He concluded that, in this situation, $E_{(-\infty,\lambda)}(A_0) - E_{(-\infty,\lambda)}(A_1)$ is not Hilbert Schmidt.

More than fifty years later, Kostrykin and Makarov [**43**] explicitly diagonalized the operator $E_{(-\infty,\lambda)}(A_0) - E_{(-\infty,\lambda)}(A_1)$ (using results of Rosenblum [**74**]):

Theorem 1 (see [43, Theorem 1]).
If $0 < \lambda < 1$, *then* $E_{(-\infty,\lambda)}(A_0) - E_{(-\infty,\lambda)}(A_1)$ *has a simple purely absolutely continuous spectrum filling in the interval* $[-1,1]$. *In particular,* $E_{(-\infty,\lambda)}(A_0) - E_{(-\infty,\lambda)}(A_1)$ *is not compact.*

We observe that equation (3) defines a bounded integral operator on $\mathsf{L}^2(\mathbb{R}_+)$ whose kernel function depends only on the sum of the variables. Formally speaking, such integral operators on $\mathsf{L}^2(\mathbb{R}_+)$ are called *Hankel (integral) operators*; we present a brief introduction to the theory of Hankel operators in Section I.1.

In the paper [**82**] by the author of the present Ph.D. thesis (CU), a "natural" generalization of (3) is considered, and the corresponding Hankel operators on $\mathsf{L}^2(\mathbb{R}_+)$ are explicitly diagonalized; each of these Hankel operators has a simple purely absolutely continuous spectrum filling in the interval $[-1,1]$ and is therefore unitarily equivalent to the operator defined by (3). It is to emphasize that the results of [**82**] are *not* part of the present Ph.D. thesis.

It is known that there is a relationship between operators of the type $f(A+B) - f(A)$ and Hankel operators, see M. Krein's example and a result of Peller [**60**] as well as some recent results of Pushnitski [**67**–**69**] and together with Yafaev [**71**, **72**] that we will discuss in Chapter II. The present thesis is intended to add some more knowledge on this subject.

Let us consider the difference

$$D(\lambda) = E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A) \qquad (\lambda \in \mathbb{R})$$

of the spectral projections in the case when A and B are self-adjoint operators acting on a complex separable Hilbert space $\mathfrak{H} \neq \{0\}$, where A is semibounded and B is at least compact.

Inspired by M. Krein's example, we pose the following question.

Question 2 (see [83, Question 1]).
Let $\lambda \in \mathbb{R}$. *Is it true that the difference of the spectral projections acting on* $\mathfrak{H}$,

$$D(\lambda) = E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A),$$

is unitarily equivalent to a bounded self-adjoint Hankel operator, provided that A *is semibounded and* B *is of rank* 1*?*

It is to emphasize that paper [**83**] was written by CU in the framework of the present Ph.D. thesis. In particular, the results of [**83**] are part of the present Ph.D. thesis.

Roughly speaking, we can answer Question 2 in the affirmative for all but at most countably many λ. More precisely:

Theorem 3 (see [83, Theorem 2]).
Let A and B be two self-adjoint operators acting on $\mathfrak{H}$, where A is semibounded and B is of rank 1. Then there exists a number k in $\mathbb{N}_0$ such that for all λ in $\mathbb{R}$ except for at most countably many λ in $\sigma_{\mathrm{ess}}(A)$, the operator $D(\lambda)$ on $\mathfrak{H}$ is unitarily equivalent to a block diagonal operator $T(\lambda) \oplus 0$ on $\mathsf{L}^2(\mathbb{R}_+) \oplus \mathbb{C}^k$, where $T(\lambda)$ is a bounded self-adjoint Hankel operator on $\mathsf{L}^2(\mathbb{R}_+)$.

We emphasize that the number k in Theorem 3 does not depend on λ.

In the case when B is compact (not necessarily of rank 1), we can show the following version of Theorem 3.

Theorem 4 (see [83, Theorem 3]).
Let A and B be two self-adjoint operators acting on $\mathfrak{H}$, where A is semibounded and B is compact. Let $1/4 > a_1 > a_2 > \cdots > 0$ be an arbitrary decreasing null sequence of real numbers. Then for all λ in $\mathbb{R}$ except for at most countably many λ in $\sigma_{\mathrm{ess}}(A)$, there exist a compact self-adjoint operator $K(\lambda)$ on $\mathfrak{H}$ and a bounded self-adjoint Hankel operator $T(\lambda)$ on $\mathsf{L}^2(\mathbb{R}_+)$ with the following properties:

(1) $D(\lambda) + K(\lambda)$ on $\mathfrak{H}$ is unitarily equivalent to $T(\lambda)$ on $\mathsf{L}^2(\mathbb{R}_+)$;
(2) either $K(\lambda)$ is a finite rank operator or $\nu_m(\lambda)/a_m \to 0$ as $m \to \infty$, where $\nu_1(\lambda), \nu_2(\lambda), \ldots$ denote the nonzero eigenvalues of $K(\lambda)$ ordered by decreasing modulus (with multiplicity taken into account).

Moreover, we can always choose $K(\lambda)$ of finite rank if B is of finite rank.

Our main tool for the proofs of Theorems 3 and 4 is the characterization theorem for bounded self-adjoint Hankel operators (see Theorem III.4); it was shown in 1995 by Megretskii, Peller, and Treil [**56**].

The results presented Theorems 3 and 4 are of abstract nature. It is generally hard to compute differences of spectral projections explicitly. However, in the joint work [**66**] of Olaf Post and CU, a generalization of M. Krein's example is considered in which the computation can be performed. The results of [**66**] are part of the present Ph.D. thesis; please note that the contribution of CU to [**66**] is declared on p. 113.

We generalize M. Krein's example by considering operators of the type

$$H = \Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big)^{\mathrm{N}} \otimes I + I \otimes L \quad \text{and} \quad H^{\mathrm{D}} = \Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big)^{\mathrm{D}} \otimes I + I \otimes L \quad \text{in} \quad \mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G}, \tag{4}$$

where $\mathfrak{G} \neq \{0\}$ is a complex separable Hilbert space and L is a self-adjoint nonnegative operator on $\mathfrak{G}$. We call H resp. H^{D} the *(abstract) Neumann* resp. *Dirichlet operator*. In particular, this framework includes:

(1) M. Krein's example of the half-line $\mathbb{R}_+$ with $L = 0$ and $\mathfrak{G} = \mathbb{C}$;
(2) the example of the classical half-space $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ with $L = -\Delta_{\mathbb{R}^{n-1}}$ and $n \geq 2$;
(3) the case when L is (minus) the Laplacian on a generally noncompact manifold $\mathcal{Y}$, e. g., on the cylinder $\mathbb{R}_+ \times \mathcal{Y}$ with Neumann resp. Dirichlet boundary conditions on $\{0\} \times \mathcal{Y}$.

Remark 5. The domains of H and H^{D} were computed explicitly by Malamud and Neidhardt in [**52**, Proposition 5.2]:

$$\operatorname{Dom}(H) = \{u \in \mathsf{W}^{2,2}(\mathbb{R}_+;\mathfrak{G}) \cap \mathsf{L}^2(\mathbb{R}_+;\operatorname{Dom}(L)) : \dot{u}(0) = 0\},$$
$$\operatorname{Dom}(H^{\mathrm{D}}) = \{u \in \mathsf{W}^{2,2}(\mathbb{R}_+;\mathfrak{G}) \cap \mathsf{L}^2(\mathbb{R}_+;\operatorname{Dom}(L)) : u(0) = 0\}.$$

Here $\mathsf{W}^{2,2}(\mathbb{R}_+;\mathfrak{G})$ denotes the Sobolev space of the second order of all $\mathfrak{G}$-valued square integrable functions on $\mathbb{R}_+$ (see Section I.3 for more information on Banach-valued functions), the domain $\operatorname{Dom}(L)$ is equipped with the graph norm of L, and $\dot{u}$ stands for the (weak) derivative of u.

We consider the resolvents

$$A_0 = (H^{\mathrm{D}} + I)^{-1} \quad \text{and} \quad A_1 = (H + I)^{-1} \tag{5}$$

of the operators H^{D} and H defined in (4) at the spectral point -1. The difference $A_1 - A_0$ of the resolvents can be computed with the help of an M. Krein-type resolvent formula from the theory of boundary pairs [**65**], see Theorem 7 (1) below.

Remark 6. Using the theory of boundary triplets, related results were obtained by Malamud and Neidhardt [**52**] and together with Boitsev, Brasche, and Popov [**12**]; in particular, for the case when L is bounded. Let us note that in [**12, 52**] one has to "regularize" the boundary triplet (i. e., one has to modify the boundary map and spectrally decompose L into bounded operators) in order to treat also unbounded operators L. We can directly apply the theory of boundary pairs [**65**] to unbounded operators L.

Let us denote by $\mathsf{C}_{\mathrm{c}}(\mathbb{R}_+)$ the class of all continuous scalar-valued functions on $\mathbb{R}_+$. We can show:

Theorem 7 (see [66, Theorem 1.1]).

(1) The resolvent difference $A_1 - A_0$ acts on elementary tensors $\psi \otimes \chi$ as follows:

$$\big([A_1 - A_0](\psi \otimes \chi)\big)(t) = \int_{\mathbb{R}_+} \psi(\tau) \exp\big(-(L+I)^{1/2}(t+\tau)\big)(L+I)^{-1/2}\chi \, \mathrm{d}\tau$$

for all $\psi \in \mathsf{C}_{\mathrm{c}}(\mathbb{R}_+)$ and all $\chi \in \mathfrak{G}$.

(2) Let $0 < \vartheta < 1$ and let $\alpha(\vartheta) = \frac{1}{\vartheta} - 1 > 0$. Then the difference of the spectral projections of A_0 and A_1 associated with the open interval $(-\infty, \vartheta)$ acts on elementary tensors $\psi \otimes \chi$ as follows:

$$\Big(\big[E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)\big](\psi \otimes \chi)\Big)(t)$$
$$= \frac{2}{\pi} \int_{\mathbb{R}_+} \psi(\tau) E_{[0,\alpha(\vartheta))}(L) \frac{\sin\big((\alpha(\vartheta)I - L)^{1/2}(t+\tau)\big)}{t+\tau} \chi \, \mathrm{d}\tau$$

for all $\psi \in \mathsf{C}_{\mathrm{c}}(\mathbb{R}_+)$ and all $\chi \in \mathfrak{G}$.

We recall that L can be represented as multiplication operator by the independent variable on a von Neumann direct integral $\int_\sigma^\oplus \mathfrak{G}(\lambda) \, \mathrm{d}\mathfrak{p}(\lambda)$, where $\sigma = \sigma(L)$ denotes the spectrum of L (see Section I.4 for more information on direct integrals). In this situation, a scaling transformation yields the following beautiful representation with separated variables for the resolvent difference $A_1 - A_0$:

Theorem 8 (see [66, Theorem 1.2]).
The resolvent difference $A_1 - A_0$ is unitarily equivalent to

$$\left(\left[\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)^{\mathrm{N}} + I\right]^{-1} - \left[\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)^{\mathrm{D}} + I\right]^{-1}\right) \otimes (L+I)^{-1} \quad \text{on } \mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G}.$$

Remark 9. In particular, Theorem 8 implies that $A_1 - A_0$ is trace class if and only if $(L+I)^{-1}$ is trace class. This result was obtained earlier by Gorbachuk and Kutovoi [**35**]. Their proof relies on the resolvent identities and the ideal properties of trace class operators; the resolvent difference is not computed explicitly in [**35**].

The spectral decomposition of the difference of the spectral projections looks as follows:

Theorem 10 (see [66, Theorem 1.4]).
Let $0 < \vartheta < 1$ and let $\alpha(\vartheta) = \frac{1}{\vartheta} - 1 > 0$. Then one has:

(1) $\sigma\big(E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)\big) = \begin{cases} [-1,1] & \text{if } \mu\big(\sigma \cap [0,\alpha(\vartheta))\big) > 0 \\ \{0\} & \text{if } \mu\big(\sigma \cap [0,\alpha(\vartheta))\big) = 0. \end{cases}$

(2) $\sigma_{\mathrm{p}}\big(E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)\big) = \begin{cases} \emptyset & \text{if } \mu(\sigma \cap [\alpha(\vartheta),\infty)) = 0 \\ \{0\} & \text{if } \mu(\sigma \cap [\alpha(\vartheta),\infty)) > 0. \end{cases}$

If $\mu(\sigma \cap [\alpha(\vartheta),\infty)) > 0$, then the multiplicity of the eigenvalue 0 is infinite.

(3) $\sigma_{\mathrm{ac}}\big(E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)\big) = \begin{cases} [-1,1] & \text{if } \mu\big(\sigma \cap [0,\alpha(\vartheta))\big) > 0 \\ \emptyset & \text{if } \mu\big(\sigma \cap [0,\alpha(\vartheta))\big) = 0. \end{cases}$

If $\mu\big(\sigma \cap [0,\alpha(\vartheta))\big) > 0$, then the (uniform) multiplicity of the absolutely continuous spectrum equals the dimension of $\int_{\sigma\cap[0,\alpha(\vartheta))}^{\oplus} \mathfrak{G}(\lambda)\,\mathrm{d}\mu(\lambda)$.

(4) The singular continuous spectrum is empty.

In particular, we re-obtain the results of M. Krein's example in the case when $L = 0$ and $\mathfrak{G} = \mathbb{C}$. Moreover, let us note:

Remark 11 (Link to Hankel operators; see [66, Remark 1.5]).
We observe that $E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)$ is unitarily equivalent to its negative, that its kernel is either trivial or infinite dimensional, and that 0 belongs to its spectrum, for all $0 < \vartheta < 1$. Consequently, the characterization theorem for bounded self-adjoint Hankel operators (see Theorem III.4) implies that $E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)$ is always unitarily equivalent to a Hankel integral operator on $\mathsf{L}^2(\mathbb{R}_+)$.

Example 12 (Classical half-space; see [66, Example 1.6]).
If L is the free Laplacian on $\mathbb{R}^{n-1}$ for some $n \geq 2$, then the difference of the spectral projections associated with $(-\infty,\vartheta)$ has infinite dimensional kernel, and its (absolutely continuous) spectrum equals $[-1,1]$ and is of infinite multiplicity, for all $0 < \vartheta < 1$.

Let us describe the structure of the present thesis.

We start with a preliminary chapter. First, we introduce bounded Hankel operators. Next, we recall some basic facts on invariant and reducing subspaces as well as the spectral theorem for self-adjoint operators. Subsequently, we study the Bochner integral and Sobolev spaces (of Banach-valued functions on an interval). Finally, we introduce the von Neumann direct integral of separable Hilbert spaces.

In Chapter II, we review literature on differences of the type $f(A+B)-f(A)$. First, we compute the spectral shift function in the (elementary) situation when $\mathfrak{H}$ is finite dimensional, following M. Krein's lecture notes from [**45**]. Afterwards, we briefly discuss Peller's recent result (presented in Theorem II.5) that the maximal class of functions on $\mathbb{R}$ so that the Lifshits–M. Krein trace formula (1) holds (for all self-adjoint operators A and B, where B is of trace class) coincides with the class of operator Lipschitz functions. We introduce "double operator integrals" and present a link between $f(A+B)-f(A)$ and Hankel operators that was found by Peller, see Remark II.27. Subsequently, we collect some facts on M. Krein's example from [**44**] and [**43**]. Finally, we introduce some basic notions from scattering theory and then discuss three results of Pushnitski and Yafaev, namely Theorems II.51, II.53, and II.59.

In Chapter III, we present the main tools for the proofs of Theorems 3 and 4. These tools are based on two characterization theorems. The first one, presented in Theorem III.4, is the above-mentioned characterization theorem for bounded self-adjoint Hankel operators due to Megretskii, Peller, and Treil [**56**]. The second one, by Davis [**21**], provides necessary and sufficient conditions for an operator to be the difference of two orthogonal projections, see Proposition III.13. Combining Proposition III.13 with Theorem III.4, we obtain necessary and sufficient conditions for the difference of two spectral projections to be unitarily equivalent to a bounded self-adjoint Hankel operator, see Theorem III.14. Moreover, we sketch one direction of the proof of Theorem III.14 under natural additional conditions formulated in Hypothesis III.21.

Chapter IV contains the proofs of Theorems 3 and 4. Furthermore, we discuss in Section IV.2 why and in which situation we need to set $k \neq 0$ in Theorem 3. In particular, we formulate and prove a more detailed version of Theorem 3, namely Theorem IV.1$'$. Moreover, we show that Question 2 can be answered in the affirmative whenever the kernel of $D(\lambda)$ is infinite dimensional. For this, we present sufficient conditions in Propositions IV.20 and IV.22. In Section IV.7, we discuss some examples, including the almost Mathieu operator. Finally, we discuss two open problems. It is to emphasize that:

- Chapter IV is based on the paper [**83**] by CU;
- the results of [**83**] constitute the first pillar of the research of the present thesis.

Chapter V contains the proofs of Theorems 7, 8, and 10. Moreover, we discuss two ideas for further research. It is to emphasize that:

- Chapter V is based on the paper [**66**] which is a joint work of Olaf Post and CU;
- the results of [**66**] constitute the second pillar of the research of the present thesis;
- the contribution of CU to [**66**] is declared on p. 113.

CHAPTER I

Preliminaries

In this preliminary chapter, we recall some results from (functional) analysis.

I.1. Bounded Hankel (integral) operators

Let us start with a brief introduction to the theory of bounded Hankel operators. In this section, we follow Peller's monograph [**62**].

Formally, a Hankel operator T_κ on $\mathsf{L}^2(\mathbb{R}_+)$ is an integral operator such that the kernel function κ depends only on the sum of the variables:

$$(T_\kappa\psi)(x) = \int_{\mathbb{R}_+} \kappa(x+y)\psi(y)\mathrm{d}y, \quad \psi \in \mathsf{L}^2(\mathbb{R}_+).$$

In the case when κ is in $\mathsf{L}^1(\mathbb{R}_+)$, it is easy to show that T_κ is bounded on $\mathsf{L}^2(\mathbb{R}_+)$ with operator norm $\leq \|\kappa\|_{\mathsf{L}^1(\mathbb{R}_+)}$.

It turns out that the operator T_κ can be bounded for certain distributions κ. We follow here Peller's approach and denote by $\mathscr{D}'(\mathcal{I})$ the space of all continuous *antilinear* functionals on $\mathscr{D}(\mathcal{I}) = \mathsf{C}_\mathrm{c}^\infty(\mathcal{I})$, where $\mathcal{I} \subset \mathbb{R}$ is an open (nonempty) interval (see [**62**, pp. 47–48]).

Let us introduce some notation:

Notation I.1. We define the "reflection" operator

$$R : \mathsf{L}^2(\mathbb{R}_-) \to \mathsf{L}^2(\mathbb{R}_+), \quad (R\eta)(x) = \eta(-x) \quad (x \in \mathbb{R}_+).$$

Let us extend every function $\eta \in \mathsf{L}^2(\mathbb{R}_-)$ by 0 to the whole of $\mathbb{R}$ and denote this extension by $\boldsymbol{\eta}$. This way, we naturally embed $\mathsf{L}^2(\mathbb{R}_-)$ into $\mathsf{L}^2(\mathbb{R})$.

Analogously, we extend every function $\psi \in \mathsf{L}^2(\mathbb{R}_+)$ by 0 to the whole of $\mathbb{R}$ and denote this extension by $\boldsymbol{\psi}$.

Finally, $\boldsymbol{\psi} * \boldsymbol{\eta}$ stands for the convolution of $\boldsymbol{\psi}$ and $\boldsymbol{\eta}$,

$$(\boldsymbol{\psi} * \boldsymbol{\eta})(t) = \int_{\mathbb{R}} \boldsymbol{\psi}(t-s)\boldsymbol{\eta}(s)\,\mathrm{d}s \quad (t \in \mathbb{R}).$$

Let $q \in \mathscr{D}'(\mathbb{R}_-)$. We define $\boldsymbol{q}(\boldsymbol{\eta}) = q(\eta)$ for every $\eta \in \mathscr{D}(\mathbb{R}_-)$. Let us note that for all $\eta, \chi \in \mathscr{D}(\mathbb{R}_-)$, the convolution of $\boldsymbol{\eta}$ and $\boldsymbol{\chi}$ is supported in $\mathbb{R}_-$ and thus

$$\boldsymbol{q}(\boldsymbol{\eta} * \boldsymbol{\chi}) = q\big((\boldsymbol{\eta} * \boldsymbol{\chi})\restriction_{\mathbb{R}_-}\big).$$

We define the sesquilinear form

$$\mathfrak{g}_q : \mathscr{D}(\mathbb{R}_+) \times \mathscr{D}(\mathbb{R}_-) \to \mathbb{C}, \quad \mathfrak{g}_q[\psi, \eta] = \boldsymbol{q}\big((\boldsymbol{R}\overline{\boldsymbol{\psi}}) * \boldsymbol{\eta}\big),$$

where $(\boldsymbol{R}\overline{\boldsymbol{\psi}})(t) = \overline{\boldsymbol{\psi}}(-t)$.

Preliminary consideration I.2. Let us assume that $\mathfrak{g}_q$ is bounded, i.e., there exists $c > 0$ with

$$\big|\mathfrak{g}_q[\psi,\eta]\big| \le c\|\psi\|_{\mathsf{L}^2(\mathbb{R}_+)}\|\eta\|_{\mathsf{L}^2(\mathbb{R}_-)} \tag{I.1}$$

for all $\psi \in \mathscr{D}(\mathbb{R}_+)$ and all $\eta \in \mathscr{D}(\mathbb{R}_-)$. Then we can extend $\mathfrak{g}_q$ uniquely to the whole of $\mathsf{L}^2(\mathbb{R}_+) \times \mathsf{L}^2(\mathbb{R}_-)$ such that inequality (I.1) still holds for every $\psi \in \mathsf{L}^2(\mathbb{R}_+)$ and every $\eta \in \mathsf{L}^2(\mathbb{R}_-)$, with the same constant c. Therefore, as is well known (see, e.g., Kato [**40**, p. 256]), there exists a unique bounded operator $G_q : \mathsf{L}^2(\mathbb{R}_+) \to \mathsf{L}^2(\mathbb{R}_-)$ satisfying

$$\mathfrak{g}_q[\psi,\eta] = \langle G_q\psi,\eta\rangle_{\mathsf{L}^2(\mathbb{R}_-)} \quad \text{for all } \psi \in \mathsf{L}^2(\mathbb{R}_+) \text{ and all } \eta \in \mathsf{L}^2(\mathbb{R}_-).$$

Conversely, let $G_q : \mathsf{L}^2(\mathbb{R}_+) \to \mathsf{L}^2(\mathbb{R}_-)$ be a bounded operator such that

$$\langle G_q\psi,\eta\rangle_{\mathsf{L}^2(\mathbb{R}_-)} = \boldsymbol{q}\big((\boldsymbol{R}\overline{\boldsymbol{\psi}}) * \boldsymbol{\eta}\big) \quad \text{for all } \psi \in \mathscr{D}(\mathbb{R}_+) \text{ and all } \eta \in \mathscr{D}(\mathbb{R}_-).$$

Then the sesquilinear form

$$\mathfrak{g}_q : \mathsf{L}^2(\mathbb{R}_+) \times \mathsf{L}^2(\mathbb{R}_-) \to \mathbb{C}, \quad \mathfrak{g}_q[\psi,\eta] = \langle G_q\psi,\eta\rangle_{\mathsf{L}^2(\mathbb{R}_-)},$$

is obviously bounded and satisfies

$$\mathfrak{g}_q[\psi,\eta] = \boldsymbol{q}\big((\boldsymbol{R}\overline{\boldsymbol{\psi}}) * \boldsymbol{\eta}\big) \quad \text{for all } \psi \in \mathscr{D}(\mathbb{R}_+) \text{ and all } \eta \in \mathscr{D}(\mathbb{R}_-).$$

Let us note that in the case when $q \in \mathsf{L}^1(\mathbb{R}_-)$ and $\kappa(x) = q(-x)$, we have

$$\begin{aligned}\boldsymbol{q}\big((\boldsymbol{R}\overline{\boldsymbol{\psi}}) * \boldsymbol{\eta}\big) &= \int_{\mathbb{R}_+} \Big(\int_{\mathbb{R}_+} q(-x-y)\psi(y)\,\mathrm{d}y\Big)\overline{\eta(-x)}\,\mathrm{d}x \\ &= \int_{\mathbb{R}_+} \Big(\int_{\mathbb{R}_+} \kappa(x+y)\psi(y)\,\mathrm{d}y\Big)\overline{\eta(-x)}\,\mathrm{d}x\end{aligned}$$

for all $\psi \in \mathsf{L}^2(\mathbb{R}_+)$ and all $\eta \in \mathsf{L}^2(\mathbb{R}_-)$. This motivates the following definition:

Definition I.3. Let $q \in \mathscr{D}'(\mathbb{R}_-)$. We set

$$\kappa(\psi) = q(R^{-1}\psi) \quad \text{for all } \psi \in \mathscr{D}(\mathbb{R}_+).$$

If the operator G_q of Preliminary consideration I.2 is bounded, then we define the bounded *Hankel (integral) operator* T_κ on $\mathsf{L}^2(\mathbb{R}_+)$ by RG_q.

We can characterize when G_q is bounded in terms of the distribution q. For this, we need the Fourier transform $\mathscr{F}$; we will use the same definition as Peller [**62**, p. 48]:

(1) if $g \in \mathsf{L}^1(\mathbb{R})$, then

$$\big(\mathscr{F}g\big)(s) = \int_{\mathbb{R}} \exp(-2\pi\mathrm{i}ts)g(t)\,\mathrm{d}t;$$

(2) if ϕ is a Schwartz function on $\mathbb{R}$ and g is a continuous *antilinear* functional on the Schwartz space on $\mathbb{R}$, then

$$\big(\mathscr{F}g\big)(\phi) = g\big(\mathscr{F}^*\phi\big).$$

Proposition I.4 (see [62, Theorem 8.1, p. 49]).
Let $q \in \mathscr{D}'(\mathbb{R}_-)$. Then the operator G_q of Preliminary consideration I.2 is bounded if and only if there exists a function $g \in \mathsf{L}^\infty(\mathbb{R})$ such that

$$\big(\mathscr{F} g\big)(\boldsymbol{\eta}) = q(\eta) \quad \text{for all } \eta \in \mathscr{D}(\mathbb{R}_-). \tag{I.2}$$

In this case, the operator norm of G_q is given by

$$\|G_q\|_{\mathrm{op}} = \inf \|g\|_{\mathsf{L}^\infty(\mathbb{R})},$$

where the infimum is taken over all $g \in \mathsf{L}^\infty(\mathbb{R})$ satisfying (I.2).

Let us reformulate Proposition I.4.

Corollary I.5 (cf. [62, Theorem 8.8, p. 52]).
Let $\kappa \in \mathscr{D}'(\mathbb{R}_+)$. Then the operator T_κ of Definition I.3 is bounded if and only if there exists a function $\breve{g} \in \mathsf{L}^\infty(\mathbb{R})$ such that

$$\big(\mathscr{F} \breve{g}\big)(\boldsymbol{\psi}) = \kappa(\psi) \quad \text{for all } \psi \in \mathscr{D}(\mathbb{R}_+). \tag{I.3}$$

In this case, the operator norm of T_κ is given by

$$\|T_\kappa\|_{\mathrm{op}} = \inf \|\breve{g}\|_{\mathsf{L}^\infty(\mathbb{R})},$$

where the infimum is taken over all $\breve{g} \in \mathsf{L}^\infty(\mathbb{R})$ satisfying (I.3).

Remark I.6. A straightforward computation shows that if $g \in \mathsf{L}^\infty(\mathbb{R})$ satisfies (I.2), then $\breve{g}(x) = g(-x)$ is such that (I.3) holds.

Remark I.7 (Hankel integral operators on $\mathsf{L}^2(\mathbb{R}_+;\mathbb{C}^N)$; cf. [62, Remark on p. 71]).
Let $N \in \mathbb{N}$ and let $q_{jk} \in \mathscr{D}'(\mathbb{R}_-)$ for all $j,k = 1,\dots,N$. We set $q = (q_{jk})_{j,k=1,\dots,N}$. Let us proceed componentwise. According to Proposition I.4, we can define the bounded operator $G_{q_{jk}} : \mathsf{L}^2(\mathbb{R}_+) \to \mathsf{L}^2(\mathbb{R}_-)$ as in Preliminary consideration I.2 if and only if q_{jk} satisfies (I.2) for some $g_{jk} \in \mathsf{L}^\infty(\mathbb{R})$. Let us assume that this is the case for each $j,k = 1,\dots,N$. We set

$$G_q\psi = \left(\sum_{k=1}^N G_{q_{jk}}\psi_k\right)_{j=1,\dots,N} \qquad \big(\psi = (\psi_k)_{k=1,\dots,N} \in \mathscr{D}(\mathbb{R}_+;\mathbb{C}^N)\big).$$

Then we have

$$\langle G_q\psi, \eta\rangle_{\mathsf{L}^2(\mathbb{R}_-;\mathbb{C}^N)} = \sum_{j,k=1}^N \boldsymbol{q}_{jk}\big((\boldsymbol{R}\,\overline{\boldsymbol{\psi}_k}) * \boldsymbol{\eta}_j\big)$$

for all $\psi = (\psi_k)_{k=1,\dots,N} \in \mathscr{D}(\mathbb{R}_+;\mathbb{C}^N)$ and all $\eta = (\eta_j)_{j=1,\dots,N} \in \mathscr{D}(\mathbb{R}_-;\mathbb{C}^N)$.

Just like in Definition I.3, we now obtain a bounded Hankel integral operator $T_{\kappa_{jk}}$ on $\mathsf{L}^2(\mathbb{R}_+)$, for each $j,k = 1,\dots,N$. For all $\psi = (\psi_k)_{k=1,\dots,N} \in \mathscr{D}(\mathbb{R}_+;\mathbb{C}^N)$, we set

$$T_\kappa\psi = \left(\sum_{k=1}^N T_{\kappa_{jk}}\psi_k\right)_{j=1,\dots,N};$$

T_κ is called a *Hankel (integral) operator on* $\mathsf{L}^2(\mathbb{R}_+;\mathbb{C}^N)$.

Examples of Hankel integral operators will be discussed in Chapters II, IV, and V.

Relation to Hankel operators on the Hardy class on the torus. In the present thesis, it is sufficient to consider bounded Hankel integral operators – except in one case. Namely, we will discuss a link between "Hankel operators on the Hardy class on the torus" and differences of functions of operators observed by Peller, see Remark II.27 below.

Let us introduce the Hardy class on the torus (see, e. g., Mashreghi [**54**, p. 104]). Let $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. For $h \in \mathsf{L}^2(\mathbb{T})$ and $n \in \mathbb{Z}$, we denote by

$$\hat{h}(n) = \int_{-\pi}^{\pi} h\big(\exp(\mathrm{i}t)\big) \exp(-\mathrm{i}nt) \, \frac{\mathrm{d}t}{2\pi}$$

the nth Fourier coefficient of h.

Definition I.8 (Hardy class; see [54, equation (5.2)]).
We define by

$$\mathsf{H}^2(\mathbb{T}) = \big\{h \in \mathsf{L}^2(\mathbb{T}) : \hat{h}(-1) = \hat{h}(-2) = \cdots = 0\big\}$$

the *Hardy class on* $\mathbb{T}$.

Remark I.9. Let us note that every function of $\mathsf{H}^2(\mathbb{T})$ can be interpreted as a boundary value of an analytic function on $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$. For a self-contained and clearly written introduction to Hardy spaces on the torus and on the upper half-plane, the author of the present thesis recommends the monograph [**54**]. In particular, "A panoramic view of the representation theorems" is presented in [**54**, Appendix B].

Clearly, we can decompose $\mathsf{L}^2(\mathbb{T})$ as the orthogonal sum of $\mathsf{H}^2(\mathbb{T})$ and

$$\mathsf{H}^2_-(\mathbb{T}) = \big\{h \in \mathsf{L}^2(\mathbb{T}) : \hat{h}(0) = \hat{h}(1) = \cdots = 0\big\}.$$

We denote the orthogonal projection of $\mathsf{L}^2(\mathbb{T})$ onto $\mathsf{H}^2_-(\mathbb{T})$ by P_-.

Definition I.10 (Hankel operators on $\mathsf{H}^2(\mathbb{T})$; see [62, p. 6]).
Let $\phi \in \mathsf{L}^2(\mathbb{T})$. We define the *Hankel operator* H_ϕ from $\mathsf{H}^2(\mathbb{T})$ to $\mathsf{H}^2_-(\mathbb{T})$ on the dense subspace of polynomials in $\mathsf{H}^2(\mathbb{T})$ by

$$H_\phi h = P_-(\phi h),$$

where $(\phi h)(\zeta) = \phi(\zeta)h(\zeta)$. The function ϕ is called a *symbol* of H_ϕ.

If H_ϕ is bounded on the dense subspace of polynomials in $\mathsf{H}^2(\mathbb{T})$, then (by the **B**ounded **L**inear **T**ransformation theorem) we can uniquely extend it to a bounded operator on the whole of $\mathsf{H}^2(\mathbb{T})$ that we also denote by H_ϕ.

Proposition I.11 (see [62, Theorem 1.3, p. 7]).
Let $\phi \in \mathsf{L}^2(\mathbb{T})$. Then H_ϕ is bounded on $\mathsf{H}^2(\mathbb{T})$ if and only if there exists $\psi \in \mathsf{L}^\infty(\mathbb{T})$ with

$$\hat{\psi}(n) = \hat{\phi}(n) \quad \textit{for all } n = -1, -2, \ldots. \tag{I.4}$$

In this case, the operator norm of H_ϕ is given by

$$\big\|H_\phi\big\|_{\mathrm{op}} = \inf \|\psi\|_{\mathsf{L}^\infty(\mathbb{T})},$$

where the infimum is taken over all $\psi \in \mathsf{L}^\infty(\mathbb{T})$ satisfying (I.4).

Remark I.12.

(I) A Hankel operator on the Hardy class on $\mathbb{T}$ is bounded if and only if it has a bounded symbol.

(II) "A Hankel operator has many different symbols" (see [**62**, p. 6]). However, if $H = H_\phi$ is a bounded Hankel operator, then there exists $\eta \in \mathsf{L}^\infty(\mathbb{T}) \cap \mathsf{H}^2(\mathbb{T})$ with

$$\|H\|_{\mathrm{op}} = \|\phi - \eta\|_{\mathsf{L}^\infty(\mathbb{T})},$$

see [**62**, p. 7]. We have $H = H_{\phi-\eta}$ and call $\phi - \eta$ a *symbol of minimal norm* of H. In the case when H attains its norm for some $h \in \mathsf{H}^2(\mathbb{T}) \setminus \{0\}$, i. e.,

$$\|Hh\|_{\mathsf{L}^2(\mathbb{T})} = \|H\|_{\mathrm{op}} \|h\|_{\mathsf{L}^2(\mathbb{T})},$$

[**62**, Theorem 1.4, p. 8] implies that $\phi - \eta$ is the unique symbol of minimal norm of H.

By direct computation, we obtain:

Lemma I.13 (cf. [62, p. 723]).
The operator $U : \mathsf{L}^2(\mathbb{T}) \to \mathsf{L}^2(\mathbb{R})$ defined by

$$(Uh)(x) = \frac{1}{\pi^{1/2}(x + \mathrm{i})}\, h\Big(\frac{x - \mathrm{i}}{x + \mathrm{i}}\Big) \qquad (x \in \mathbb{R})$$

is unitary.

Proof. We substitute

$$x = \tan(t/2) \qquad (-\pi < t < \pi).$$

Then we have

$$\frac{x - \mathrm{i}}{x + \mathrm{i}} = -\exp(\mathrm{i}t).$$

Thus, for every $h \in \mathsf{L}^2(\mathbb{T})$,

$$\begin{aligned} \|Uh\|^2_{\mathsf{L}^2(\mathbb{R})} &= \frac{1}{\pi} \int_{\mathbb{R}} \Big| h\Big(\frac{x - \mathrm{i}}{x + \mathrm{i}}\Big)\Big|^2 \frac{\mathrm{d}x}{1 + x^2} \\ &= \int_{-\pi}^{\pi} \big| h\big(-\exp(\mathrm{i}t)\big)\big|^2 \frac{\mathrm{d}t}{2\pi} \\ &= \|h\|^2_{\mathsf{L}^2(\mathbb{T})}. \end{aligned}$$

Consequently, U is an isometric operator from $\mathsf{L}^2(\mathbb{T})$ to $\mathsf{L}^2(\mathbb{R})$. Since $V : \mathsf{L}^2(\mathbb{R}) \to \mathsf{L}^2(\mathbb{T})$ given by

$$(Vg)(\zeta) = \pi^{1/2} \frac{2\mathrm{i}}{1 - \zeta}\, g\Big(\mathrm{i}\frac{1 + \zeta}{1 - \zeta}\Big) \qquad (\zeta \in \mathbb{T} \setminus \{1\})$$

is the inverse operator of U, it follows that U is unitary and the lemma is proved. □

We can relate Hankel integral operators on $\mathsf{L}^2(\mathbb{R}_+)$ to Hankel operators on $\mathsf{H}^2(\mathbb{T})$.

Proposition I.14 (cf. [62, Lemmas 8.2–8.3, pp. 50–51]).
Let $\kappa \in \mathscr{D}'(\mathbb{R}_+)$ be such that the Hankel integral operator T_κ on $\mathsf{L}^2(\mathbb{R}_+)$ is bounded. Then

$$H_\phi = U^{-1}\mathscr{F}^{-1}R^{-1}T_\kappa \mathscr{F} U$$

is a bounded Hankel operator on $\mathsf{H}^2(\mathbb{T})$ *with symbol* $\phi \in \mathsf{L}^\infty(\mathbb{T})$ *given by*

$$\phi(\zeta) = \breve{g}\Big(\mathrm{i}\frac{\zeta+1}{\zeta-1}\Big) \qquad (\zeta \in \mathbb{T} \setminus \{1\}),$$

where $\breve{g} \in \mathsf{L}^\infty(\mathbb{R})$ *satisfies (I.3).*

I.2. Some basic facts

In this section, we briefly recall some basic facts on invariant and reducing subspaces as well as the spectral theorem for self-adjoint operators. We follow the monographs of Schmüdgen [**75**], Weidmann [**85**], and Birman and Solomyak [**11**].

Definition I.15 (Invariant subspaces; cf. [75, Definition 1.7, p. 20]).
Let T be an operator on a Hilbert space $\mathfrak{H}$ with domain $\mathrm{Dom}(T)$. If a closed subspace $\mathfrak{M}$ of $\mathfrak{H}$ satisfies

$$\{T\psi : \psi \in \mathrm{Dom}(T) \cap \mathfrak{M}\} \subset \mathfrak{M},$$

then $\mathfrak{M}$ is called an *invariant subspace* of T. We will also say that $\mathfrak{M}$ is *invariant under* T.

Definition I.16 (Reducing subspaces; cf. [85, Exercise 5.39]).
Let T be an operator on a Hilbert space $\mathfrak{H}$ with domain $\mathrm{Dom}(T)$, and let $\mathfrak{M}$ be a closed subspace of $\mathfrak{H}$. If $\mathfrak{M}$ and the orthogonal complement of $\mathfrak{M}$ in $\mathfrak{H}$, $\mathfrak{M}^\perp$, are invariant under T and if

$$\mathrm{Dom}(T) = \{\psi + \eta : \psi \in \mathrm{Dom}(T) \cap \mathfrak{M}, \eta \in \mathrm{Dom}(T) \cap \mathfrak{M}^\perp\},$$

then we call $\mathfrak{M}$ a *reducing subspace* of T.

Remark I.17. Clearly, for a bounded self-adjoint operator defined on the whole Hilbert space, a closed subspace is reducing if and only if it is invariant.

Let us recall the following

Definition I.18 (Spectral measure; see [75, Definition 4.2, p. 66]).
Let $\mathcal{Y}$ be a nonempty set, and let $\mathscr{A}$ be a sigma-algebra of subsets of $\mathcal{Y}$. We call a mapping E from $\mathscr{A}$ into the orthogonal projections on a Hilbert space $\mathfrak{H}$ a *spectral measure* if:

(1) $E_{\mathcal{Y}} = I$;
(2) $E_\delta\psi = \lim_{N\to\infty}\sum_{n=1}^N E_{\delta_n}\psi$ for every sequence $(\delta_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from $\mathscr{A}$ whose union $\delta = \cup_{n=1}^\infty \delta_n$ is also in $\mathscr{A}$ and for all $\psi \in \mathfrak{H}$.

Remark I.19.

(1) In view of Definition I.18 (2), one says that E is *countably additive.*
(2) Every $\delta \in \mathscr{A}$ with $E_\delta = 0$ is called an *E-null set.*

Notation I.20. We denote by $\mathsf{S}(\mathcal{Y}, E)$ the set of all equivalence classes of E-measurable scalar-valued functions defined E-almost everywhere on $\mathcal{Y}$, where we identify functions that coincide up to an E-null set.

As usual, we can integrate every function $f \in \mathsf{S}(\mathcal{Y}, E)$ with respect to E and thus obtain a densely defined operator, denoted by $\int_{\mathcal{Y}} f \, \mathrm{d}E$, with domain

$$\operatorname{Dom}\Big(\int_{\mathcal{Y}} f \, \mathrm{d}E\Big) = \Big\{\psi \in \mathfrak{H} : \int_{\mathcal{Y}} |f(y)|^2 \, \mathrm{d}\mu_\psi(y) < \infty\Big\},$$

where $\mu_\psi(\delta) = \langle E_\delta \psi, \psi\rangle_{\mathfrak{H}}$ for all $\psi \in \mathfrak{H}$ and all $\delta \in \mathscr{A}$. We recall the following important

Theorem I.21 (Spectral theorem; see [75, Theorem 5.7, p. 89]).
Let T be a self-adjoint operator on a Hilbert space $\mathfrak{H}$. Then there exists a unique spectral measure $E = E(T)$ on the Borel sigma-algebra $\mathcal{B}(\mathbb{R})$ such that

$$T = \int_{\mathbb{R}} y \, \mathrm{d}E_y(T).$$

The following well-known criterion will be useful.

Lemma I.22 (see [85, Theorem 7.28]).
Let T be a self-adjoint operator with spectral measure $E(T)$ on a Hilbert space $\mathfrak{H}$. Then a closed subspace $\mathfrak{M}$ of $\mathfrak{H}$ reduces T if and only if the orthogonal projection onto $\mathfrak{M}$ commutes with $E_{(-\infty,t]}(T)$ for every $t \in \mathbb{R}$.

We close this section with a brief discussion on orthogonal decompositions and orthogonal sums of separable Hilbert spaces.

Let us adopt the following notation from [**11**, p. 159]:

Notation I.23. For a given $m \in \mathbb{N} \cup \{\infty\}$, we set $[1, m\rangle = \{1, \dots, m\}$ if $m \in \mathbb{N}$ and $[1, \infty\rangle = \mathbb{N}$.

Now, we follow [**11**, p. 29]. Let $(\mathfrak{G}_n)_{n\in[1,m\rangle}$ be a sequence of separable Hilbert spaces. We define $\widetilde{\mathfrak{H}}$ to be the set of all sequences $(g_n)_{n\in[1,m\rangle}$ such that $\sum_{n\in[1,m\rangle} \|g_n\|^2_{\mathfrak{G}_n} < \infty$. Then $\widetilde{\mathfrak{H}}$ is a normed space with linear operations defined componentwise. Moreover, $\widetilde{\mathfrak{H}}$ is a pre-Hilbert space with inner product

$$\langle \tilde{h}^{(1)}, \tilde{h}^{(2)}\rangle_{\widetilde{\mathfrak{H}}} = \sum_{n\in[1,m\rangle} \langle g_n^{(1)}, g_n^{(2)}\rangle_{\mathfrak{G}_n}.$$

We write

$$\widetilde{\mathfrak{H}} = \bigoplus_{n\in[1,m\rangle} \mathfrak{G}_n. \tag{I.5}$$

One has:

Lemma I.24 (see [11, Theorem 5, p. 29]).
The pre-Hilbert space $\widetilde{\mathfrak{H}}$ defined in (I.5) is complete and separable.

Remark I.25. Let $(\mathfrak{M}_n)_{n\in[1,m\rangle}$ be a sequence of pairwise orthogonal closed subspaces of a separable Hilbert space $\mathfrak{H}$ such that

$$\overline{\operatorname{span}}\,\{\psi_n : \psi_n \in \mathfrak{M}_n\} = \mathfrak{H},$$

where $m \in \mathbb{N} \cup \{\infty\}$. Then every $\mathfrak{M}_n$ is a separable Hilbert space and the mapping

$$\mathfrak{H} \ni \psi = \sum_{n\in[1,m\rangle} \psi_n \mapsto (\psi_n)_{n\in[1,m\rangle} \in \widetilde{\mathfrak{H}}$$

is unitary (see [**11**, pp. 29–30]). We therefore identify $\mathfrak{H}$ and $\widetilde{\mathfrak{H}}$ and write

$$\mathfrak{H} = \widetilde{\mathfrak{H}} = \bigoplus_{n \in [1,m\rangle} \mathfrak{M}_n.$$

I.3. Banach-valued functions on an interval

In this section, we study functions on an open interval $\mathfrak{I} \subset \mathbb{R}$ that take values in a complex Banach space $\mathfrak{X}$. We follow Cazenave and Haraux [**17**].

I.3.1. Measurability.

Definition I.26 (see [17, Definition 1.4.1]).
A function $u : \mathfrak{I} \to \mathfrak{X}$ is called *Bochner measurable* if there exist a set $\delta \subset \mathfrak{I}$ of Lebesgue measure 0 and a sequence $(u_n)_{n \in \mathbb{N}_0} \subset \mathsf{C}_{\mathrm{c}}(\mathfrak{I}; \mathfrak{X})$ such that

$$\lim_{n \to \infty} u_n(t) = u(t) \quad \text{for all } t \in \mathfrak{I} \setminus \delta.$$

One has:

Proposition I.27 (see [17, Proposition 1.4.3]).
If outside a subset of $\mathfrak{I}$ of Lebesgue measure 0, $u : \mathfrak{I} \to \mathfrak{X}$ is the pointwise limit of a sequence of Bochner measurable functions, then u is Bochner measurable.

Let us note two consequences of Proposition I.27:

Remark I.28 (cf. [17, Remarks 1.4.4–1.4.5]).

(I) If $u : \mathfrak{I} \to \mathfrak{X}$ is Bochner measurable and $\psi : \mathfrak{I} \to \mathbb{R}$ is Lebesgue measurable, then $t \mapsto \psi(t)u(t)$ is Bochner measurable.

(II) Let x be a vector in $\mathfrak{X}$, and let $\Delta \subset \mathfrak{I}$ be Lebesgue measurable. Then the function $t \mapsto \mathbb{1}_\Delta(t)x$ is Bochner measurable.

Next, we present Pettis' measurability theorem.

Theorem I.29 (see [17, Proposition 1.4.6]).
A function $u : \mathfrak{I} \to \mathfrak{X}$ is Bochner measurable if and only if the following two conditions are satisfied:

(1) for every bounded linear functional $\Phi \in \mathfrak{X}'$, the scalar-valued function $t \mapsto \Phi(u(t))$ is Lebesgue measurable;

(2) there exists a set $\delta \subset \mathfrak{I}$ of Lebesgue measure 0 such that $u(\mathfrak{I} \setminus \delta)$ is separable.

Corollary I.30 (cf. [17, Corollary 1.4.8]).
If $u : \mathfrak{I} \to \mathfrak{X}$ is continuous, then it is Bochner measurable.

I.3.2. The Bochner integral. Let us describe the concept of Bochner integration of Banach-valued functions on an interval.

First, we note that if $u : \mathfrak{I} \to \mathfrak{X}$ is Bochner measurable and $v \in \mathsf{C}_{\mathrm{c}}(\mathfrak{I}; \mathfrak{X})$, then $t \mapsto \|u(t) - v(t)\|_{\mathfrak{X}}$ is Lebesgue measurable and nonnegative; therefore, $\int_{\mathfrak{I}} \|u(t) - v(t)\|_{\mathfrak{X}} \, \mathrm{d}t$ exists.

Definition I.31 (see [17, Definition 1.4.10]).
We call a Bochner measurable function $u : \mathfrak{I} \to \mathfrak{X}$ *Bochner integrable* if there exists a sequence $(u_n)_{n\in\mathbb{N}_0} \subset \mathsf{C}_\mathrm{c}(\mathfrak{I};\mathfrak{X})$ such that

$$\lim_{n\to\infty} \int_{\mathfrak{I}} \|u(t) - u_n(t)\|_{\mathfrak{X}} \,\mathrm{d}t = 0.$$

Remark I.32. For every $v \in \mathsf{C}_\mathrm{c}(\mathfrak{I};\mathfrak{X})$, we can choose a compact interval $\mathfrak{J} \subset \mathfrak{I}$ such that v vanishes everywhere on $\mathfrak{I} \setminus \mathfrak{J}$. Then the Riemann integral (defined as the limit of Riemann sums) of v on $\mathfrak{J}$ exists and is denoted by

$$\int_{\mathfrak{J}} v(t) \,\mathrm{d}t.$$

One has

$$\Big\| \int_{\mathfrak{J}} v(t) \,\mathrm{d}t \Big\|_{\mathfrak{X}} \le \int_{\mathfrak{J}} \|v(t)\|_{\mathfrak{X}} \,\mathrm{d}t \le \sup_{t\in\mathfrak{J}} \|v(t)\|_{\mathfrak{X}} \text{ (length of } \mathfrak{J}).$$

We write

$$\int_{\mathfrak{I}} v(t) \,\mathrm{d}t = \int_{\mathfrak{J}} v(t) \,\mathrm{d}t;$$

obviously, this is independent of the choice of $\mathfrak{J}$.

Proposition I.33 (see [17, Proposition 1.4.12]).
Let $u : \mathfrak{I} \to \mathfrak{X}$ be Bochner integrable. Then there exists a unique vector x in $\mathfrak{X}$ such that for every sequence $(u_n)_{n\in\mathbb{N}_0} \subset \mathsf{C}_\mathrm{c}(\mathfrak{I};\mathfrak{X})$ satisfying

$$\lim_{n\to\infty} \int_{\mathfrak{I}} \|u(t) - u_n(t)\|_{\mathfrak{X}} \,\mathrm{d}t = 0,$$

we have

$$\lim_{n\to\infty} \int_{\mathfrak{I}} u_n(t) \,\mathrm{d}t = x. \tag{I.6}$$

Definition I.34 (see [17, Definition 1.4.13]).
If $u : \mathfrak{I} \to \mathfrak{X}$ is Bochner integrable, then the vector x in $\mathfrak{X}$ from Proposition I.33 is called the *Bochner integral of* u *on* $\mathfrak{I}$ and is denoted by $\int_{\mathfrak{I}} u(t) \,\mathrm{d}t$.

Next, we present Bochner's theorem.

Theorem I.35 (see [17, Proposition 1.4.14]).
A Bochner measurable function $u : \mathfrak{I} \to \mathfrak{X}$ is Bochner integrable if and only if the scalar-valued function $t \mapsto \|u(t)\|_{\mathfrak{X}}$ is Lebesgue integrable. In this case,

$$\Big\| \int_{\mathfrak{I}} u(t) \,\mathrm{d}t \Big\|_{\mathfrak{X}} \le \int_{\mathfrak{I}} \|u(t)\|_{\mathfrak{X}} \,\mathrm{d}t.$$

As a corollary, we obtain the dominated convergence theorem for Bochner integrable functions.

Corollary I.36 (see [17, Corollary 1.4.15]).
Let $(u_n)_{n\in\mathbb{N}_0}$ be a sequence of Bochner integrable functions from $\mathfrak{I}$ to $\mathfrak{X}$, let $\psi : \mathfrak{I} \to \mathbb{R}$ be Lebesgue integrable with $\|u_n(t)\|_{\mathfrak{X}} \le \psi(t)$ for all $n \in \mathbb{N}_0$ and for every t outside a subset of $\mathfrak{I}$ of Lebesgue measure 0, and let $u : \mathfrak{I} \to \mathfrak{X}$ be such that

$$\lim_{n\to\infty} u_n(t) = u(t) \quad \textit{for almost every } t \in \mathfrak{I}.$$

Then u is Bochner integrable and we have

$$\int_{\mathcal{J}} u(t)\,\mathrm{d}t = \lim_{n\to\infty} \int_{\mathcal{J}} u_n(t)\,\mathrm{d}t.$$

Definition I.37 (see [17, Definition 1.4.16]).
Let $1 \le p < \infty$. We denote by $\mathsf{L}^p(\mathcal{J};\mathfrak{X})$ the set of all (equivalence classes of) Bochner measurable functions $u : \mathcal{J} \to \mathfrak{X}$ such that

$$\int_{\mathcal{J}} \|u(t)\|_{\mathfrak{X}}^p \,\mathrm{d}t < \infty.$$

One has:

Proposition I.38 (see [25, Theorem 5, p. 121] and [25, Theorem 6, p. 146]).
If $1 \le p < \infty$, then $\mathsf{L}^p(\mathcal{J};\mathfrak{X})$ equipped with

$$\|u\|_{\mathsf{L}^p(\mathcal{J};\mathfrak{X})} = \Big(\int_{\mathcal{J}} \|u(t)\|_{\mathfrak{X}}^p \,\mathrm{d}t\Big)^{1/p}$$

is a Banach space.

Definition I.39 (Simple functions; cf. [25, Definition 9, p. 105]).
If $x_1,\dots,x_n$ are pairwise distinct vectors in $\mathfrak{X}$ and $u : \mathcal{J} \to \{x_1,\dots,x_n\} \subset \mathfrak{X}$ is such that the sets $u^{-1}(\{x_1\}),\dots,u^{-1}(\{x_n\})$ are Lebesgue measurable, then we call u a *simple function*.

Remark I.40. Let $u : \mathcal{J} \to \{x_1,\dots,x_n\} \subset \mathfrak{X}$ be a simple function. By Proposition I.27 and Remark I.28, we know that u is Bochner measurable. According to Bochner's theorem, u is Bochner integrable if and only if each of the sets $u^{-1}(\{x_1\}),\dots,u^{-1}(\{x_n\})$ has finite Lebesgue measure.

One has:

Lemma I.41 (Dense subsets of $\mathsf{L}^p(\mathcal{J};\mathfrak{X})$).
If $1 \le p < \infty$, then the set of all Bochner integrable simple functions from $\mathcal{J}$ to $\mathfrak{X}$ is dense in $\mathsf{L}^p(\mathcal{J};\mathfrak{X})$. Consequently, $\mathsf{C}_c^\infty(\mathcal{J};\mathfrak{X})$ lies dense in $\mathsf{L}^p(\mathcal{J};\mathfrak{X})$.

Proof. According to [**25**, Corollary 8, p. 125], the set of all (equivalence classes of) Bochner integrable simple functions from $\mathcal{J}$ to $\mathfrak{X}$ is dense in $\mathsf{L}^p(\mathcal{J};\mathfrak{X})$. Since we know that $\mathsf{C}_c^\infty(\mathcal{J})$ lies dense in $\mathsf{L}^p(\mathcal{J})$, it follows that for each set $\Delta \subset \mathcal{J}$ of finite Lebesgue measure, there exists a sequence $(\psi_n)_{n\in\mathbb{N}_0} \subset \mathsf{C}_c^\infty(\mathcal{J})$ such that

$$\lim_{n\to\infty} \|\mathbb{1}_\Delta - \psi_n\|_{\mathsf{L}^p(\mathcal{J})} = 0.$$

Consequently, if $x \in \mathfrak{X}$, then $(x\,\psi_n)_{n\in\mathbb{N}_0} \subset \mathsf{C}_c^\infty(\mathcal{J};\mathfrak{X})$ with

$$\lim_{n\to\infty} \|x\,\mathbb{1}_\Delta - x\,\psi_n\|_{\mathsf{L}^p(\mathcal{J},\mathfrak{X})} = 0.$$

This completes the proof of the lemma. □

Let us note the following useful result.

Lemma I.42 (see [17, Proposition 1.4.22]).
Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two complex Banach spaces, and let T be a bounded operator from $\mathfrak{X}$ to $\mathfrak{Y}$. If $u \in \mathsf{L}^p(\mathcal{I};\mathfrak{X})$, where $1 \le p < \infty$, then $t \mapsto T(u(t))$ is in $\mathsf{L}^p(\mathcal{I};\mathfrak{Y})$ and

$$\|Tu\|_{\mathsf{L}^p(\mathcal{I};\mathfrak{Y})} \le \|T\|_{\mathrm{op}} \|u\|_{\mathsf{L}^p(\mathcal{I};\mathfrak{X})}.$$

Moreover, if $u \in \mathsf{L}^1(\mathcal{I};\mathfrak{X})$, then

$$T\Big(\int_{\mathcal{I}} u(t)\,\mathrm{d}t\Big) = \int_{\mathcal{I}} T(u(t))\,\mathrm{d}t.$$

Definition I.43 ($\mathsf{L}^p_{\mathrm{loc}}(\mathcal{I};\mathfrak{X})$; see [17, Definition 1.4.21]).
For every $1 \le p < \infty$, we denote by $\mathsf{L}^p_{\mathrm{loc}}(\mathcal{I};\mathfrak{X})$ the set of all (equivalence classes of) Bochner measurable functions $u : \mathcal{I} \to \mathfrak{X}$ such that $u{\restriction}_{\mathcal{J}} \in \mathsf{L}^p(\mathcal{J};\mathfrak{X})$ for each compact interval $\mathcal{J} \subset \mathcal{I}$.

I.3.3. Sobolev spaces (of Banach-valued functions on an interval). Let us describe the Sobolev spaces of first and second order of Banach-valued functions on an interval.

Definition I.44 (Weak derivative).
Let $u \in \mathsf{L}^1_{\mathrm{loc}}(\mathcal{I};\mathfrak{X})$. If there exists $v \in \mathsf{L}^1_{\mathrm{loc}}(\mathcal{I};\mathfrak{X})$ such that

$$\int_{\mathcal{I}} v(t)\psi(t)\,\mathrm{d}t = -\int_{\mathcal{I}} u(t)\dot{\psi}(t)\,\mathrm{d}t \quad \text{for all } \psi \in \mathsf{C}^\infty_{\mathrm{c}}(\mathcal{I}),$$

then we call v the *weak derivative* of u and denote it by $\dot{u}$.

Let us note:

Remark I.45.

(I) If it exists, then the weak derivative is unique. Indeed, let v and $\tilde{v}$ be two weak derivatives of u. Then $v - \tilde{v} \in \mathsf{L}^1_{\mathrm{loc}}(\mathcal{I};\mathfrak{X})$ and, according to Lemma I.42,

$$\int_{\mathcal{I}} T\big(v(t) - \tilde{v}(t)\big)\psi(t)\,\mathrm{d}t = 0 \quad \text{for all } \psi \in \mathsf{C}^\infty_{\mathrm{c}}(\mathcal{I}) \tag{I.7}$$

and every bounded linear functional T from $\mathfrak{X}$ to $\mathbb{C}$. Using the fundamental lemma of the calculus of variations and the Hahn–Banach theorem, one can show that $v(t) = \tilde{v}(t)$ for almost all $t \in \mathcal{I}$.

(II) Integration by parts shows that each continuously differentiable function from $\mathcal{I}$ to $\mathfrak{X}$ is weakly differentiable, and the weak derivative coincides (almost everywhere) with the classical derivative.

Lemma I.46 (see [17, Corollary 1.4.31]).
Let $v \in \mathsf{L}^1_{\mathrm{loc}}(\mathcal{I};\mathfrak{X})$ and $t_0 \in \mathcal{I}$. Then

$$u : t \mapsto \int_{t_0}^{t} v(\tau)\,\mathrm{d}\tau \tag{I.8}$$

is continuous on $\mathcal{I}$. Moreover:

(1) v is the weak derivative of u;

(2) u is differentiable almost everywhere and

$$\lim_{s\to t} \frac{u(s) - u(t)}{s - t} = v(t) \quad \textit{for almost every } t \in \mathcal{I}.$$

Remark I.47. The function u defined in (I.8) is thus absolutely continuous.

Definition I.48 (Sobolev space of first order).
For every $1 \leq p < \infty$, we denote the $\mathfrak{X}$-valued *Sobolev space of first order* on $\mathcal{I}$, consisting of all $u \in \mathsf{L}^p(\mathcal{I};\mathfrak{X})$ such that u is weakly differentiable with $\dot{u} \in \mathsf{L}^p(\mathcal{I};\mathfrak{X})$, by $\mathsf{W}^{1,p}(\mathcal{I};\mathfrak{X})$.

Proposition I.49. *For every* $1 \leq p < \infty$, $\mathsf{W}^{1,p}(\mathcal{I};\mathfrak{X})$ *equipped with*

$$\|u\|_{\mathsf{W}^{1,p}(\mathcal{I};\mathfrak{X})} = \Big(\|u\|^p_{\mathsf{L}^p(\mathcal{I};\mathfrak{X})} + \|\dot{u}\|^p_{\mathsf{L}^p(\mathcal{I};\mathfrak{X})}\Big)^{1/p}$$

is a Banach space.

Proof. It is easy to see that $\mathsf{W}^{1,p}(\mathcal{I};\mathfrak{X})$ is a vector space with norm $\|\bullet\|_{\mathsf{W}^{1,p}(\mathcal{I};\mathfrak{X})}$.

Analogously to the scalar-valued case (cf. [**15**, pp. 203–204]), one can show that $\mathsf{W}^{1,p}(\mathcal{I};\mathfrak{X})$ is complete and thus a Banach space. Indeed, let $(u_n)_{n\in\mathbb{N}_0} \subset \mathsf{W}^{1,p}(\mathcal{I};\mathfrak{X})$ be a Cauchy sequence. Since $\mathsf{L}^p(\mathcal{I};\mathfrak{X})$ is a Banach space, there exist $u, v \in \mathsf{L}^p(\mathcal{I};\mathfrak{X})$ such that

$$\|u - u_n\|_{\mathsf{L}^p(\mathcal{I};\mathfrak{X})} \xrightarrow{n\to\infty} 0 \quad \text{and} \quad \|v - \dot{u}_n\|_{\mathsf{L}^p(\mathcal{I};\mathfrak{X})} \xrightarrow{n\to\infty} 0.$$

Let $\psi \in \mathsf{C}^\infty_\mathrm{c}(\mathcal{I})$. We have

$$\int_\mathcal{I} u_n(t)\dot{\psi}(t)\,\mathrm{d}t = -\int_\mathcal{I} \dot{u}_n(t)\psi(t)\,\mathrm{d}t \quad \text{for all } n \in \mathbb{N}_0$$

and therefore, by Hölder's inequality with $\frac{1}{p} + \frac{1}{p^*} = 1$,

$$\begin{aligned}
&\Big|\int_\mathcal{I} u(t)\dot{\psi}(t)\,\mathrm{d}t + \int_\mathcal{I} v(t)\psi(t)\,\mathrm{d}t\Big| \\
&\leq \int_\mathcal{I} \|u(t) - u_n(t)\|_\mathfrak{X}\,|\dot{\psi}(t)|\,\mathrm{d}t + \int_\mathcal{I} \|v(t) - \dot{u}_n(t)\|_\mathfrak{X}\,|\psi(t)|\,\mathrm{d}t \\
&\leq \|u - u_n\|_{\mathsf{L}^p(\mathcal{I};\mathfrak{X})}\|\dot{\psi}\|_{\mathsf{L}^{p^*}(\mathcal{I})} + \|v - \dot{u}_n\|_{\mathsf{L}^p(\mathcal{I};\mathfrak{X})}\|\psi\|_{\mathsf{L}^{p^*}(\mathcal{I})} \xrightarrow{n\to\infty} 0.
\end{aligned}$$

It follows that

$$\int_\mathcal{I} u(t)\dot{\psi}(t)\,\mathrm{d}t = -\int_\mathcal{I} v(t)\psi(t)\,\mathrm{d}t$$

and thus, since $\psi \in \mathsf{C}^\infty_\mathrm{c}(\mathcal{I})$ was arbitrary, v is the weak derivative of u. This completes the proof. □

Let us characterize the functions in $\mathsf{W}^{1,p}(\mathcal{I};\mathfrak{X})$.

Proposition I.50 (see [17, Theorem 1.4.35]).
For every $1 \leq p < \infty$ *and each* $u \in \mathsf{L}^p(\mathcal{I};\mathfrak{X})$, *the following properties are equivalent:*

(1) $u \in \mathsf{W}^{1,p}(\mathcal{I};\mathfrak{X})$.

(2) There exists $v \in \mathsf{L}^p(\mathcal{I};\mathfrak{X})$ *such that*

$$u(t) = u(t_0) + \int_{t_0}^t v(\tau)\,\mathrm{d}\tau$$

for almost all $t_0, t \in \mathcal{I}$.

(3) There exist $v \in \mathsf{L}^p(\mathcal{I};\mathfrak{X})$, $x_0 \in \mathfrak{X}$, *and* $t_0 \in \mathcal{I}$, *such that*

$$u(t) = x_0 + \int_{t_0}^t v(\tau)\,\mathrm{d}\tau$$

for almost all $t \in \mathcal{J}$*.*

(4) u *is absolutely continuous and therefore differentiable almost everywhere on* $\mathcal{J}$*. Moreover,* $\dot{u} \in \mathsf{L}^p(\mathcal{J};\mathfrak{X})$*.*

Remark I.51 (to Proposition I.50 (4)).
More precisely, every $u \in \mathsf{W}^{1,p}(\mathcal{J};\mathfrak{X})$ admits an absolutely continuous representative $\tilde{u}$. In view of Lemma I.46, we very often identify u and $\tilde{u}$.

Corollary I.52 (see [17, Corollary 1.4.36]).
Let $1 \leq p < \infty$*. Then every* $u \in \mathsf{W}^{1,p}(\mathcal{J};\mathfrak{X})$ *admits a bounded uniformly continuous representative.*

Definition I.53 (Weak derivative of second order).
Let $u \in \mathsf{L}^1_{\mathrm{loc}}(\mathcal{J};\mathfrak{X})$. If there exists $w \in \mathsf{L}^1_{\mathrm{loc}}(\mathcal{J};\mathfrak{X})$ such that

$$\int_{\mathcal{J}} w(t)\psi(t)\,\mathrm{d}t = \int_{\mathcal{J}} u(t)\ddot{\psi}(t)\,\mathrm{d}t \quad \text{for all } \psi \in \mathsf{C}^\infty_{\mathrm{c}}(\mathcal{J}),$$

then we call w the *second weak derivative* of u and denote it by $\ddot{u}$.

Definition I.54 (Sobolev space of second order).
For every $1 \leq p < \infty$, we denote the $\mathfrak{X}$-valued *Sobolev space of second order* on $\mathcal{J}$, consisting of all $u \in \mathsf{L}^p(\mathcal{J};\mathfrak{X})$ such that $\dot{u}, \ddot{u} \in \mathsf{L}^p(\mathcal{J};\mathfrak{X})$, by $\mathsf{W}^{2,p}(\mathcal{J};\mathfrak{X})$.

We have:

Proposition I.55. *For every* $1 \leq p < \infty$*,* $\mathsf{W}^{2,p}(\mathcal{J};\mathfrak{X})$ *equipped with*

$$\|u\|_{\mathsf{W}^{1,p}(\mathcal{J};\mathfrak{X})} = \left(\|u\|^p_{\mathsf{L}^p(\mathcal{J};\mathfrak{X})} + \|\dot{u}\|^p_{\mathsf{L}^p(\mathcal{J};\mathfrak{X})} + \|\ddot{u}\|^p_{\mathsf{L}^p(\mathcal{J};\mathfrak{X})}\right)^{1/p}$$

is a Banach space.

Proof. Analogously to Proposition I.49. □

Let us note:

Remark I.56. Let $1 \leq p < \infty$. Every $u \in \mathsf{W}^{2,p}(\mathcal{J};\mathfrak{X})$ admits a continuously differentiable representative. To see this, we choose an absolutely continuous representative of u (that we also call u) and a bounded continuous representative v of $\dot{u} \in \mathsf{W}^{1,p}(\mathcal{J};\mathfrak{X})$. Then for all $t_0, t \in \mathcal{J}$,

$$u(t) - u(t_0) = \int_{t_0}^{t} v(\tau)\,\mathrm{d}\tau.$$

Since v is continuous and $[\min\{t_0,t\}, \max\{t_0,t\}]$ is compact, it follows from the fundamental theorem of calculus that u is continuously differentiable with $\dot{u} = v$.

I.4. The von Neumann direct integral of separable Hilbert spaces

In Section I.2 above, we briefly recalled the construction of the orthogonal sum of at most countably many separable Hilbert spaces. This concept can be generalized by the so-called “von Neumann direct integral of separable Hilbert spaces” which is an important tool in the present thesis. For an introduction to this technique, we follow the monograph **[11]** of Birman and Solomyak.

Definition I.57 (Separable measure space; cf. [11, p. 5]).
Let $(\mathcal{Y},\ \mathscr{A},\ \mu)$ be a measure space with a sigma-finite measure μ. We call $(\mathcal{Y},\ \mathscr{A},\ \mu)$ *separable* if there exists $\mathscr{M} \subset \mathscr{A}$ such that:

(1) $\mathscr{M}$ is countable;
(2) for every $\delta \in \mathscr{A}$ with $\mu(\delta) < \infty$ and for every $\varepsilon > 0$ there exists $\delta_\varepsilon \in \mathscr{M}$ satisfying

$$\mu\big((\delta \setminus \delta_\varepsilon) \bigcup (\delta_\varepsilon \setminus \delta)\big) < \varepsilon.$$

It is not hard to show that:

Lemma I.58 (see [11, Theorem 4, p. 21]).
For a sigma-finite measure μ, the measure space $(\mathcal{Y},\ \mathscr{A},\ \mu)$ is separable if and only if $\mathsf{L}^2(\mathcal{Y},\mu)$ is separable.

Remark I.59.

(I) If μ is a finite Borel measure on a complete separable metric space $\mathcal{Y}$, then $(\mathcal{Y},\ \mathcal{B}(\mathcal{Y}),\ \mu)$ is a separable measure space (see [**11**, pp. 6–7]).
(II) Under the assumptions of (I), let $\mathcal{Y}'$ be a μ-measurable subset of $\mathcal{Y}$. By considering $\mathsf{L}^2(\mathcal{Y}',\mu)$ as a subspace of $\mathsf{L}^2(\mathcal{Y},\mu)$, we obtain that $\mathsf{L}^2(\mathcal{Y}',\mu)$ is separable (cf. [**85**, p. 27]) and thus, by Lemma I.58, the measure space $(\mathcal{Y}',\ \mathcal{B}(\mathcal{Y}'),\ \mu)$ is separable.

Throughout this section, $(\mathcal{Y},\ \mathscr{A},\ \mu)$ denotes a sigma-finite separable measure space. Furthermore, we assume that $\mu(\mathcal{Y}) > 0$ (cf. Remark I.78 below).

I.4.1. The Hilbert space $\mathsf{L}^2(\mathcal{Y},\mu;\mathfrak{G})$.

Definition I.60 (see [11, p. 30]).
We say that a vector-valued function h defined μ-almost everywhere on $\mathcal{Y}$ that takes values in a separable Hilbert space $\mathfrak{G}$ is *measurable* if for every $g \in \mathfrak{G}$, the scalar-valued function $y \mapsto \langle h(y), g\rangle_{\mathfrak{G}}$ is μ-measurable.

Remark I.61. If μ is Lebesgue measure and $\mathcal{Y}$ is an open interval $\mathcal{I} \subset \mathbb{R}$, then a vector-valued function defined almost everywhere on $\mathcal{I}$ is Bochner measurable if and only if it is measurable in the sense of Definition I.60. This follows from Pettis' measurability theorem (see Theorem I.29) because $\mathfrak{G}$ is separable.

It is easy to see that the set of all measurable vector-valued functions h satisfying

$$\int_{\mathcal{Y}} \|h(y)\|^2_{\mathfrak{G}}\, \mathrm{d}\mu(y) < \infty$$

is a vector space; we denote it by $\mathfrak{H}$. Let us identify vector-valued measurable functions that coincide μ-almost everywhere. Then $\mathfrak{H}$ is a pre-Hilbert space with the inner product

$$\langle h_1, h_2\rangle_{\mathfrak{H}} = \int_{\mathcal{Y}} \langle h_1(y), h_2(y)\rangle_{\mathfrak{G}}\, \mathrm{d}\mu(y).$$

Furthermore, one has:

Lemma I.62 (see [11, Theorem 6, p. 30]).
The pre-Hilbert space $\mathfrak{H}$ is complete and separable.

Let us denote the Hilbert space $\mathfrak{H}$ of Lemma I.62 by $\mathsf{L}^2(\mathcal{Y}, \mu; \mathfrak{G})$.

Remark I.63. Clearly, $\mathsf{L}^2(\mathcal{Y}, \mu; \mathbb{C})$ coincides with the usual L^2-space $\mathsf{L}^2(\mathcal{Y}, \mu)$.

I.4.2. Measurable families of Hilbert spaces. Let the mapping $y \mapsto \mathfrak{G}(y)$ be defined μ-almost everywhere on $\mathcal{Y}$, where the complex Hilbert spaces $\mathfrak{G}(y) \neq \{0\}$ are assumed to be separable. We denote the inner product in $\mathfrak{G}(y)$ by $\langle \bullet, \bullet \rangle_{\mathfrak{G}(y)}$ and the induced norm by $\| \bullet \|_{\mathfrak{G}(y)}$. We assume that the *multiplicity function*

$$\nu : y \mapsto \dim \mathfrak{G}(y) \in \mathbb{N} \cup \{\infty\}$$

is μ-measurable. That is, for every $m \in \mathbb{N} \cup \{\infty\}$, the subset

$$\mathcal{Y}_m = \{y \in \mathcal{Y} : \nu(y) = m\}$$

of $\mathcal{Y}$ is supposed to be measurable. We follow the axiomatic definition in [**11**, pp. 159–161] of measurable structure for the class of vector-valued functions g that are defined μ-almost everywhere on $\mathcal{Y}$ and take values $g(y) \in \mathfrak{G}(y)$.

Definition I.64 (Base of measurability; see [11, p. 160]).
A finite or countable set Ω_0 of vector-valued functions on $\mathcal{Y}$ is called a *base of measurablity* if:

(1) $\overline{\operatorname{span}}\,\{g(y) : g \in \Omega_0\} = \mathfrak{G}(y)$ for μ-almost every $y \in \mathcal{Y}$;
(2) the scalar-valued function $y \mapsto \langle g_1(y), g_2(y) \rangle_{\mathfrak{G}(y)}$ is μ-measurable for all $g_1, g_2 \in \Omega_0$.

For the rest of this section, we suppose that such a base of measurablity exists.

Definition I.65 (Measurability with respect to Ω_0; see [11, p. 160]).
Let u be a vector-valued function defined μ-almost everywhere on $\mathcal{Y}$ and taking values $u(y) \in \mathfrak{G}(y)$. Then u is called *measurable* with respect to Ω_0 if for every $g \in \Omega_0$, the scalar-valued function

$$y \mapsto \langle u(y), g(y) \rangle_{\mathfrak{G}(y)}$$

is μ-measurable. We denote by $\widehat{\Omega}_0$ the set of all vector-valued functions that are measurable with respect to Ω_0.

Clearly (as noted in [**11**, p. 160]), we have:

Lemma I.66. *$\widehat{\Omega}_0$ is a vector space with $\Omega_0 \subset \widehat{\Omega}_0$. Moreover, if $u \in \widehat{\Omega}_0$ and f is a scalar-valued μ-measurable function on $\mathcal{Y}$, then the product $y \mapsto f(y)u(y)$ belongs to $\widehat{\Omega}_0$.*

Let us recall that for a given $m \in \mathbb{N} \cup \{\infty\}$, we set $[1, m\rangle = \{1, \ldots, m\}$ if $m \in \mathbb{N}$ and $[1, \infty\rangle = \mathbb{N}$.

Definition I.67 (Orthogonal base of measurability; see [11, p. 160]).
Let $\mu\text{-}\sup\ \nu$ be the essential supremum of ν with respect to μ,

$$\mu\text{-}\sup\ \nu = \inf_{\delta \in \mathscr{A},\, \mu(\delta)=0}\ \sup_{y \in \mathcal{Y} \setminus \delta} \nu(y) = n \in \mathbb{N} \cup \{\infty\}.$$

If $\Omega_0 = \{e_m : m \in [1, n\rangle\}$, where, outside a μ-null set, $\{e_m(y) : m \in [1, \nu(y)\rangle\}$ is an orthogonal basis of $\mathfrak{G}(y)$ and $e_m(y) = 0$ for $m > \nu(y)$, then Ω_0 is called an *orthogonal base of measurability.*

Remark I.68.

(I) Obviously, the set Ω_0 in Definition I.67 is a base of measurability in the sense of Definition I.64.

(II) In Definition I.67, we can replace "orthogonal basis" by "orthonormal basis." Indeed, if a scalar-valued nonnegative function is μ-measurable, then so is its square root. Consequently, outside a set of μ-measure 0, we can replace the vector-valued functions e_m by

$$\tilde{e}_m : y \mapsto \begin{cases} \frac{e_m(y)}{\|e_m(y)\|_{\mathfrak{G}(y)}} & \text{if } m \in [1, \nu(y)\rangle \\ 0 & \text{if } m > \nu(y) \end{cases}.$$

In particular, we have $\|\tilde{e}_1(y)\|_{\mathfrak{G}(y)} = 1$ for μ-almost every $y \in \mathfrak{Y}$.

The set of measurable vector-valued functions with respect to a given base of measurability can always be generated by an orthogonal base of measurability:

Lemma I.69 (see [11, Lemma 1, p. 160]).
If Ω_0 is a base of measurability and $\Omega = \widehat{\Omega}_0$, then there exists an orthogonal base of measurability $\Omega_1 \subset \Omega$ such that $\widehat{\Omega}_1 = \Omega$.

Definition I.70 (Measurable structure/Hilbert family; see [11, p. 161]).
Let $\Omega = \widehat{\Omega}_0$, where Ω_0 is a base of measurability.

(1) We call Ω a *measurable structure* on the family of Hilbert spaces $\mathfrak{G}(y)$.

(2) The family of Hilbert spaces $\mathfrak{G}(y)$ endowed with the measurable structure Ω is called a *measurable Hilbert family* on the sigma-finite separable measure space $(\mathfrak{Y}, \mathscr{A}, \mu)$ and is denoted by $(\mathfrak{G}(\bullet), \Omega)$.

Definition I.71 (Measurable operator-valued functions; see [11, p. 161]).
Let $(\mathfrak{G}(\bullet), \Omega)$ and $(\mathfrak{G}'(\bullet), \Omega')$ be two measurable Hilbert families on the sigma-finite separable measure space $(\mathfrak{Y}, \mathscr{A}, \mu)$, and let T be an operator-valued function defined μ-almost everywhere on $\mathfrak{Y}$ that takes values in the set of bounded operators from $\mathfrak{G}(y)$ to $\mathfrak{G}'(y)$. Then T is called *measurable* if for all $g \in \Omega$ and all $g' \in \Omega'$, the scalar-valued functions

$$y \mapsto \langle T(y)g(y), g'(y)\rangle_{\mathfrak{G}'(y)}$$

are μ-measurable.

Remark I.72 (see [11, p. 161]).
In Definition I.71, it is sufficient to consider elements of the bases of measurability Ω_0 and Ω_0'.

I.4.3. The von Neumann direct integral. In the present subsection, we define the von Neumann direct integral of separable Hilbert spaces and show that it is again a separable Hilbert space.

Definition I.73 (The von Neumann direct integral; see [11, pp. 161–162]).
Let $(\mathfrak{G}(\bullet), \Omega)$ be a measurable Hilbert family on the sigma-finite separable measure space $(\mathcal{Y}, \mathscr{A}, \mathbb{\mu})$. We identify measurable vector-valued functions that coincide $\mathbb{\mu}$-almost everywhere. Then the set of all $h \in \Omega$ satisfying

$$\int_{\mathcal{Y}} \|h(y)\|^2_{\mathfrak{G}(y)} \, \mathrm{d}\mathbb{\mu}(y) < \infty$$

is denoted by

$$\mathfrak{H} = \int_{\mathcal{Y}}^{\oplus \Omega} \mathfrak{G}(y) \, \mathrm{d}\mathbb{\mu}(y) \tag{I.9}$$

and is called the *von Neumann direct integral* of the Hilbert spaces $\mathfrak{G}(y)$ with respect to the measurable structure Ω. For $h_1, h_2 \in \mathfrak{H}$, let us define

$$\langle h_1, h_2 \rangle_{\mathfrak{H}} = \int_{\mathcal{Y}} \langle h_1(y), h_2(y) \rangle_{\mathfrak{G}(y)} \, \mathrm{d}\mathbb{\mu}(y). \tag{I.10}$$

Clearly, $\mathfrak{H} = \int_{\mathcal{Y}}^{\oplus \Omega} \mathfrak{G}(y) \, \mathrm{d}\mathbb{\mu}(y)$ is a vector space. The function $\langle \bullet, \bullet \rangle_{\mathfrak{H}}$ defined in (I.10) is easily seen to be an inner product on $\mathfrak{H}$; we denote the induced norm by $\| \bullet \|_{\mathfrak{H}}$. Consequently, $\mathfrak{H}$ is a pre-Hilbert space. Moreover, one has:

Proposition I.74 (see [11, p. 162]).
$\mathfrak{H} = \int_{\mathcal{Y}}^{\oplus \Omega} \mathfrak{G}(y) \, \mathrm{d}\mathbb{\mu}(y)$ *is a separable Hilbert space.*

For the proof of Proposition I.74 (see below), it is sufficient to show that the pre-Hilbert space $\mathfrak{H} = \int_{\mathcal{Y}}^{\oplus \Omega} \mathfrak{G}(y) \, \mathrm{d}\mathbb{\mu}(y)$ is isometrically isomorphic to a separable Hilbert space. First, let us construct a "model space" $\mathfrak{H}' = \int_{\mathcal{Y}}^{\oplus \Omega'} \mathfrak{G}'(y) \, \mathrm{d}\mathbb{\mu}(y)$:

Example I.75 (see [11, p. 162]).
Let $(\mathfrak{G}(\bullet), \Omega)$ be a measurable Hilbert family on the sigma-finite separable measure space $(\mathcal{Y}, \mathscr{A}, \mathbb{\mu})$. As above, we set $n = \mathbb{\mu}\text{-}\sup\ \nu$. We choose a $\mathbb{\mu}$-null set $\delta \in \mathscr{A}$ such that the multiplicity function ν is defined everywhere on $\mathcal{Y} \setminus \delta$:

$$\nu(y) = \dim \mathfrak{G}(y) \in \mathbb{N} \cup \{\infty\} \qquad (y \in \mathcal{Y} \setminus \delta).$$

Let $m \in [1, n\rangle$. We choose a separable Hilbert space $\mathfrak{G}'_m$ of dimension m with an orthonormal basis $\{e_j^m : j \in [1, m\rangle\}$. Then for every $y \in \mathcal{Y} \setminus \delta$, we set

$$e_j(y) = \begin{cases} e_j^m & \text{if } j \in [1, \nu(y)\rangle \\ 0 & \text{if } j > \nu(y) \end{cases}.$$

Clearly, $\Omega'_0 = \{e_j(\bullet) : j \in [1, n\rangle\}$ is an orthogonal base of measurability of the Hilbert family $\mathfrak{G}'(\bullet)$, where we put

$$\mathfrak{G}'(y) = \mathfrak{G}'_{\nu(y)} \quad \text{on } \mathcal{Y} \setminus \delta, \qquad \text{i.e.,} \qquad \mathfrak{G}'(y) = \mathfrak{G}'_m \quad \text{for all } y \in \mathcal{Y}_m \setminus \delta.$$

Thus, we can construct the von Neumann direct integral

$$\mathfrak{H}' = \int_{\mathcal{Y}}^{\oplus \Omega'} \mathfrak{G}'(y) \, \mathrm{d}\mathbb{\mu}(y)$$

of the Hilbert spaces $\mathfrak{G}'(y)$ with respect to the measurable structure Ω' (which is generated by Ω'_0).

For every $m \in [1,n\rangle$, let us identify $\mathsf{L}^2(\mathcal{Y}_m, \mu; \mathfrak{G}'_m)$ with the subspace

$$\{h' \in \mathfrak{H}' : h'(y) = 0 \text{ for } \mu\text{-almost every } y \in \mathcal{Y} \setminus \mathcal{Y}_m\} \subset \mathfrak{H}'.$$

It is easily seen that

$$\mathfrak{H}' \to \bigoplus_{m\in[1,n\rangle} \mathsf{L}^2(\mathcal{Y}_m, \mu; \mathfrak{G}'_m), \qquad h' \mapsto \big(h'\restriction_{\mathcal{Y}_m}\big)_{m\in[1,n\rangle},$$

is a unitary operator. Since we know that $\bigoplus_{m\in[1,n\rangle} \mathsf{L}^2(\mathcal{Y}_m, \mu; \mathfrak{G}'_m)$ is a separable Hilbert space (see Section I.2), it follows that the pre-Hilbert space $\mathfrak{H}'$ is separable and complete.

Remark I.76. In Example I.75, we can even choose a single fixed separable Hilbert space of infinite dimension, $\widehat{\mathfrak{G}}$, with orthonormal basis $\{e_j : j \in \mathbb{N}\}$ and then set

$$\mathfrak{G}'(y) = \overline{\operatorname{span}}\{e_j : j \in [1, \nu(y)\rangle\} \subset \widehat{\mathfrak{G}} \qquad (y \in \mathcal{Y} \setminus \delta).$$

We will use this fact in Chapter III below.

Next, let us formulate the following result:

Proposition I.77 (see [11, Lemma 3, p. 161] and [11, Theorem 4, p. 162]). *Let $(\mathfrak{G}(\bullet), \Omega)$ and $(\mathfrak{G}'(\bullet), \Omega')$ be two measurable Hilbert families on the sigma-finite separable measure space $(\mathcal{Y}, \mathscr{A}, \mu)$ satisfying*

$$\dim \mathfrak{G}(y) = \dim \mathfrak{G}'(y) \quad \textit{for } \mu\textit{-almost all } y \in \mathcal{Y}.$$

Then there exists a measurable operator-valued function W defined μ-almost everywhere on $\mathcal{Y}$ such that:

(1) $W(y)$ is a unitary operator from $\mathfrak{G}(y)$ onto $\mathfrak{G}'(y)$ (μ-almost everywhere);
(2) W transfers the measurable structure Ω onto Ω' in the sense that

$$\Omega' = \{h' : h'(y) = W(y)h(y) \ \mu\text{-almost everywhere on } \mathcal{Y},\ h \in \Omega\}.$$

Furthermore, $\widehat{W} : h \mapsto h'$ is a unitary operator from $\mathfrak{H} = \int_{\mathcal{Y}}^{\oplus\Omega} \mathfrak{G}(y)\, d\mu(y)$ onto $\mathfrak{H}' = \int_{\mathcal{Y}}^{\oplus\Omega'} \mathfrak{G}'(y)\, d\mu(y)$, where $h'(y) = W(y)h(y)$ μ-almost everywhere on $\mathcal{Y}$.

Finally, we conclude:

Proof of Proposition I.74. Combining Example I.75 and Proposition I.77, it follows that if $(\mathfrak{G}(\bullet), \Omega)$ is a measurable Hilbert family on the sigma-finite separable measure space $(\mathcal{Y}, \mathscr{A}, \mu)$, then the pre-Hilbert space $\mathfrak{H} = \int_{\mathcal{Y}}^{\oplus\Omega} \mathfrak{G}(y)\, d\mu(y)$ is separable and complete. □

Remark I.78. If $\mu(\mathcal{Y}) = 0$, then we set $\int_{\mathcal{Y}} \mathfrak{G}(y)\, d\mu(y) = \{0\}$.

I.4.4. Multiplication operators and decomposable operators. As before, let $(\mathcal{Y}, \mathscr{A}, \mu)$ be a sigma-finite separable measure space. Further, let $\mathfrak{H} = \int_{\mathcal{Y}}^{\oplus\Omega} \mathfrak{G}(y)\, d\mu(y)$ be a von Neumann direct integral of Hilbert spaces $\mathfrak{G}(y)$ with respect to a measurable structure Ω.

If $\delta \in \mathscr{A}$, then we denote by X_δ the operator on $\mathfrak{H}$ that acts by multiplication with the characteristic function $\mathbb{1}_\delta$. That is, for all $\delta \in \mathscr{A}$ and all $h \in \mathfrak{H}$, one has

$$\big(X_\delta h\big)(y) = \mathbb{1}_\delta(y)h(y) \tag{I.11}$$

$\mathfrak{p}$-almost everywhere on $\mathfrak{Y}$. We have:

Lemma I.79 (see [11, p. 164]).
X is a spectral measure on $\mathfrak{H}$. Furthermore, for all $\delta \in \mathscr{A}$, one has $X_\delta = 0$ if and only if $\mathfrak{p}(\delta) = 0$.

Notation I.80. We denote by $\mathsf{S}(\mathfrak{Y}, \mathfrak{p})$ the set of all equivalence classes of $\mathfrak{p}$-measurable scalar-valued functions defined $\mathfrak{p}$-almost everywhere on $\mathfrak{Y}$, where we identify functions that coincide up to a $\mathfrak{p}$-null set.

For the analogous definition of $\mathsf{S}(\mathfrak{Y}, X)$, see Notation I.20 above.

It is a direct consequence of Lemma I.79 that $\mathsf{S}(\mathfrak{Y}, X) = \mathsf{S}(\mathfrak{Y}, \mathfrak{p})$. As usual, we can integrate every function $f \in \mathsf{S}(\mathfrak{Y}, X)$ with respect to X and thus obtain a densely defined operator, denoted by $\int_{\mathfrak{Y}} f \,\mathrm{d}X$, with domain

$$\operatorname{Dom}\Big(\int_{\mathfrak{Y}} f \,\mathrm{d}X\Big) = \Big\{h \in \mathfrak{H} : \int_{\mathfrak{Y}} |f(y)|^2 \,\mathrm{d}\mathfrak{p}_h(y) < \infty\Big\},$$

where $\mathfrak{p}_h(\delta) = \langle X_\delta h, h\rangle_{\mathfrak{H}}$ for all $h \in \mathfrak{H}$ and all $\delta \in \mathscr{A}$.

Let us compare the operator $\int_{\mathfrak{Y}} f \,\mathrm{d}X$ with the multiplication operator M_f defined on

$$\operatorname{Dom}(M_f) = \Big\{h \in \mathfrak{H} : \int_{\mathfrak{Y}} |f(y)|^2 \|h(y)\|^2_{\mathfrak{G}(y)} \,\mathrm{d}\mathfrak{p}(y) < \infty\Big\}$$

by

$$\big(M_f h\big)(y) = f(y)h(y) \quad \mathfrak{p}\text{-almost everywhere on } \mathfrak{Y}.$$

We have:

Proposition I.81 (see [11, Theorem 1, p. 164]).
The sets $\operatorname{Dom}\big(\int_{\mathfrak{Y}} f \,\mathrm{d}X\big)$ and $\operatorname{Dom}(M_f)$ coincide and the operators $\int_{\mathfrak{Y}} f \,\mathrm{d}X$ and M_f are equal.

Let $\mathfrak{p}'$ be a sigma-finite measure such that $(\mathfrak{Y}, \mathscr{A}, \mathfrak{p}')$ is a separable measure space; let further $(\mathfrak{G}'(\bullet), \Omega')$ be a measurable Hilbert family. We consider the two von Neumann direct integrals of separable Hilbert spaces given by

$$\mathfrak{H} = \int_{\mathfrak{Y}}^{\oplus_\Omega} \mathfrak{G}(y) \,\mathrm{d}\mathfrak{p}(y) \quad \text{and} \quad \mathfrak{H}' = \int_{\mathfrak{Y}}^{\oplus_{\Omega'}} \mathfrak{G}'(y) \,\mathrm{d}\mathfrak{p}'(y). \tag{I.12}$$

Analogously to X on $\mathfrak{H}$, we obtain a spectral measure on $\mathfrak{H}'$ that we denote by X'. Moreover, M'_f stands for the multiplication operator by f on $\mathfrak{H}'$, whenever $f \in \mathsf{S}(\mathfrak{Y}, \mathfrak{p}')$. If the measures $\mathfrak{p}$ and $\mathfrak{p}'$ are equivalent, then $\mathsf{S}(\mathfrak{Y}, \mathfrak{p}) = \mathsf{S}(\mathfrak{Y}, \mathfrak{p}')$ and we can compare M_f and M'_f. First, we introduce the following

Notation I.82. We write $\frac{\mathrm{d}\mathfrak{p}}{\mathrm{d}\mathfrak{p}'}$ for the Radon–Nikodym derivative of $\mathfrak{p}$ with respect to $\mathfrak{p}'$.

One has:

Proposition I.83 (see [11, Theorem 2, p. 165]).
Let the von Neumann direct integrals $\mathfrak{H}$ and $\mathfrak{H}'$ be defined as in (I.12).

(1) We suppose that

$$\mathfrak{p} \text{ is equivalent to } \mathfrak{p}' \quad \text{and} \quad \dim \mathfrak{G}'(y) = \dim \mathfrak{G}(y) \ \mathfrak{p}\text{-almost everywhere} \tag{I.13}$$

as well as that V is a measurable operator-valued function defined $\mathfrak{p}$-almost everywhere on $\mathfrak{Y}$ such that $V(y) : \mathfrak{G}(y) \to \mathfrak{G}'(y)$ is unitary. Then

$$\widehat{V} : \big(\widehat{V}h\big)(y) = p(y)V(y)h(y), \quad p = \Big(\frac{\mathrm{d}\mathfrak{p}}{\mathrm{d}\mathfrak{p}'}\Big)^{1/2}, \tag{I.14}$$

is a unitary operator from $\mathfrak{H}$ onto $\mathfrak{H}'$ satisfying $\widehat{V}M_f = M'_f\widehat{V}$ for each $f \in \mathsf{S}(\mathfrak{Y}, \mathfrak{p})$. In particular, we have

$$\widehat{V}X_\delta = X'_\delta\widehat{V} \quad \text{for all } \delta \in \mathscr{A}. \tag{I.15}$$

(2) If $\widehat{V}$ is a unitary operator from $\mathfrak{H}$ onto $\mathfrak{H}'$ satisfying (I.15), then (I.13) holds and $\widehat{V}$ admits the representation (I.14).

Remark I.84 (see [11, p. 174]).
Proposition I.83 shows that we can always replace $\mathfrak{p}$ by an equivalent *finite* measure. Indeed, we choose any function $q \in \mathsf{L}^1(\mathfrak{Y}, \mathfrak{p})$ with $q(y) > 0$ for $\mathfrak{p}$-almost every $y \in \mathfrak{Y}$; then

$$\mathfrak{p}'(\delta) = \int_\delta q(y)\,\mathrm{d}\mathfrak{p}(y) \qquad (\delta \in \mathscr{A})$$

does the job.

Notation I.85 (see [11, pp. 163–164]).
Following common convention, we write $\int_{\mathfrak{Y}}^{\oplus} \mathfrak{G}(y)\,\mathrm{d}\mathfrak{p}(y)$ instead of $\int_{\mathfrak{Y}}^{\oplus\Omega} \mathfrak{G}(y)\,\mathrm{d}\mathfrak{p}(y)$.

Next, we consider operators (acting on a von Neumann direct integral of separable Hilbert spaces) that commute with the spectral measure X.

Proposition I.86 (see [11, Theorem 3, p. 166]).
Let $\mathfrak{H} = \int_{\mathfrak{Y}}^{\oplus} \mathfrak{G}(y)\,\mathrm{d}\mathfrak{p}(y)$ be a von Neumann direct integral of separable Hilbert spaces.

(1) Let T be a measurable operator-valued function defined $\mathfrak{p}$-almost everywhere on $\mathfrak{Y}$ that takes values in the set of bounded operators on $\mathfrak{G}(y)$. We assume that the essential supremum with respect to $\mathfrak{p}$ over the operator norms $\|T(y)\|_{\mathrm{op}}$,

$$\mathfrak{p}\text{-}\sup_{y\in\mathfrak{Y}}\|T(y)\|_{\mathrm{op}} = \inf_{\delta\in\mathscr{A},\,\mathfrak{p}(\delta)=0}\ \sup_{y\in\mathfrak{Y}\setminus\delta}\|T(y)\|_{\mathrm{op}},$$

is finite. Then

$$\widehat{T} : h \mapsto \big(y \mapsto T(y)h(y)\big) \tag{I.16}$$

is a bounded operator on $\mathfrak{H}$ commuting with M_f for all $f \in \mathsf{S}(\mathfrak{Y}, \mathfrak{p})$. In particular, $\widehat{T}$ commutes with X_δ for every $\delta \in \mathscr{A}$. Besides,

$$\big\|\widehat{T}\big\|_{\mathrm{op}} = \mathfrak{p}\text{-}\sup_{y\in\mathfrak{Y}}\|T(y)\|_{\mathrm{op}}. \tag{I.17}$$

(2) If $\widehat{T}$ is a bounded operator on $\mathfrak{H}$ that commutes with X_δ for every $\delta \in \mathscr{A}$, then $\widehat{T}$ admits the representation (I.16).

Definition I.87 (Decomposable operators; see [11, p. 168]).
Bounded operators on a von Neumann direct integral of separable Hilbert spaces acting by (I.16) are called *decomposable*.

Notation I.88. We write $h = \int_{\mathcal{Y}}^{\oplus} h(y)\,\mathrm{d}\mu(y) \in \mathfrak{H}$. Accordingly, since any decomposable operator $\widehat{T}$ on $\mathfrak{H} = \int_{\mathcal{Y}}^{\oplus} \mathfrak{G}(y)\,\mathrm{d}\mu(y)$ acts fiberwise, we use the notation

$$\widehat{T} = \int_{\mathcal{Y}}^{\oplus} T(y)\,\mathrm{d}\mu(y).$$

Now (I.16) reads as follows:

$$\widehat{T}h = \int_{\mathcal{Y}}^{\oplus} T(y)h(y)\,\mathrm{d}\mu(y).$$

As noted in [**11**, p. 168], it is clear that if $\widehat{T_1} = \int_{\mathcal{Y}}^{\oplus} T_1(y)\,\mathrm{d}\mu(y)$ and $\widehat{T_2} = \int_{\mathcal{Y}}^{\oplus} T_2(y)\,\mathrm{d}\mu(y)$ are decomposable operators on $\mathfrak{H} = \int_{\mathcal{Y}}^{\oplus} \mathfrak{G}(y)\,\mathrm{d}\mu(y)$, then so are $\widehat{T_1}+\widehat{T_2}$, $\widehat{T_1}\widehat{T_2}$, and $(\widehat{T_i})^*$ ($i = 1, 2$); we have

$$\widehat{T_1} + \widehat{T_2} = \int_{\mathcal{Y}}^{\oplus} \big(T_1(y) + T_2(y)\big)\,\mathrm{d}\mu(y), \quad \widehat{T_1}\widehat{T_2} = \int_{\mathcal{Y}}^{\oplus} T_1(y)T_2(y)\,\mathrm{d}\mu(y), \quad \text{and}$$

$$\Big(\widehat{T_i}\Big)^* = \int_{\mathcal{Y}}^{\oplus} \big(T_i(y)\big)^*\,\mathrm{d}\mu(y) \quad (i = 1, 2).$$

Moreover, one has:

Proposition I.89 (see [11, Theorem 5, p. 168]).
Let $\widehat{T} = \int_{\mathcal{Y}}^{\oplus} T(y)\,\mathrm{d}\mu(y)$ be a decomposable operator on $\mathfrak{H} = \int_{\mathcal{Y}}^{\oplus} \mathfrak{G}(y)\,\mathrm{d}\mu(y)$. Then:

(1) $\widehat{T}$ is self-adjoint if and only if $T(y)$ is self-adjoint (for μ-almost every $y \in \mathcal{Y}$);
(2) $\widehat{T}$ is unitary if and only if $T(y)$ is unitary (for μ-almost every $y \in \mathcal{Y}$);
(3) $\widehat{T}$ is normal if and only if $T(y)$ is normal (for μ-almost every $y \in \mathcal{Y}$);
(4) $\widehat{T}$ is an orthogonal projection on $\mathfrak{H}$ if and only if $T(y)$ is an orthogonal projection on $\mathfrak{G}(y)$ (for μ-almost every $y \in \mathcal{Y}$).

I.4.5. Unitary invariants of spectral measure. As before, let $(\mathcal{Y}, \mathscr{A}, \mu)$ be a separable measure space. Let further E be a spectral measure (with respect to $(\mathcal{Y}, \mathscr{A})$) on the complex separable Hilbert space $\widetilde{\mathfrak{H}}$. We say that E and μ are *type-equivalent* if for every $\delta \in \mathscr{A}$, $E_\delta = 0$ if and only if $\mu(\delta) = 0$. Let us formulate the main results of this section.

Theorem I.90 (see [11, Theorem 1, p. 173]).
Let E be a spectral measure on $\widetilde{\mathfrak{H}}$. If E and μ are type-equivalent, then there exist a von Neumann direct integral $\mathfrak{H} = \int_{\mathcal{Y}} \mathfrak{G}(y)\,\mathrm{d}\mu(y)$ and a unitary operator V from $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}$ such that

$$V\Big(\int_{\mathcal{Y}} f\,\mathrm{d}E\Big) = M_f V \quad \textit{for all } f \in \mathsf{S}(\mathcal{Y}, E).$$

In particular,

$$VE_\delta = X_\delta V \quad \textit{for every } \delta \in \mathscr{A}.$$

Moreover, one has:

Theorem I.91 (cf. [11, Theorem 2, p. 173]).
Let E be a spectral measure on $\widetilde{\mathfrak{H}}$. Let further $\mathfrak{H}$ and $\mathfrak{H}'$ be as in (I.12) with corresponding spectral measures X and X'. If there exist unitary operators V from $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}$ as well as V' from $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}'$ such that

$$VE_\delta = X_\delta V \quad \text{and} \quad V'E_\delta = X'_\delta V' \quad \text{for every } \delta \in \mathscr{A},$$

then the measures μ and μ' are equivalent and the multiplicity functions ν and ν' coincide μ-almost everywhere. In particular, μ and E and hence also μ' and E are type-equivalent.

An important special case: Borel spectral measures on $\mathbb{R}$. In the following, let $(\mathcal{Y}, \mathscr{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ be a sigma-finite separable measure space. By the spectral theorem (see Theorem I.21), we know that to every self-adjoint operator A on $\widetilde{\mathfrak{H}}$, there corresponds a unique Borel spectral measure $E(A)$ on $\widetilde{\mathfrak{H}}$. Very often, we simply call $E(A)$ the spectral measure of A.

For self-adjoint operators, we have the following functional model:

Theorem I.92 (see [11, Theorem 1, p. 177]).
Let A be a self-adjoint operator on $\widetilde{\mathfrak{H}}$ with spectral measure $E(A)$. If $E(A)$ and μ are type-equivalent, then A is unitarily equivalent to the multiplication operator by the independent variable on a von Neumann direct integral

$$\mathfrak{H} = \int_{\mathbb{R}} \mathfrak{G}(y)\, \mathrm{d}\mu(y).$$

Definition I.93 (Scalar spectral measures).
A measure μ as in Theorem I.92 is called a *scalar spectral measure* of A.

Remark I.94.

(I) For every self-adjoint operator, we can choose a *finite* scalar spectral measure. Indeed, if we are given a self-adjoint operator A, then there exists a finite Borel measure μ' on $\mathbb{R}$ which is type-equivalent to $E(A)$ (see [**11**, Theorem 4, p. 171]); according to Remark I.59, $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu')$ is a separable measure space.

(II) By Theorem I.91, any two scalar spectral measures of a self-adjoint operator are equivalent.

We recall now well-known decompositions of self-adjoint operators into absolutely continuous and singular parts (see, e. g, [**75**, pp. 189–192]).

Notation I.95. Let A be a self-adjoint operator on $\widetilde{\mathfrak{H}}$. For every $\tilde{h} \in \widetilde{\mathfrak{H}}$, let us denote the finite Borel measure $\mathcal{B}(\mathbb{R}) \ni \Delta \mapsto \langle E_\Delta(A)\tilde{h}, \tilde{h}\rangle_{\widetilde{\mathfrak{H}}}$ by $\mu_{\tilde{h}}$. Further, λ stands for the Borel–Lebesgue measure on $\mathbb{R}$.

We set:

$$\widetilde{\mathfrak{H}}^{(\mathrm{ac})} = \big\{\tilde{h} : \mu_{\tilde{h}}(\Delta) = 0 \text{ for every } \lambda\text{-null set } \Delta \in \mathcal{B}(\mathbb{R})\big\},$$

$$\widetilde{\mathfrak{H}}^{(\mathrm{p})} = \begin{cases} \{0\} & \text{if } A \text{ has no eigenvalues} \\ \overline{\operatorname{span}}\big\{\tilde{h} \in \widetilde{\mathfrak{H}} : \tilde{h} \text{ is an eigenvector of } A\big\} & \text{otherwise} \end{cases},$$

$$\widetilde{\mathfrak{H}}^{(\mathrm{s})} = \big(\widetilde{\mathfrak{H}}^{(\mathrm{ac})}\big)^{\perp},$$

$$\widetilde{\mathfrak{H}}^{(\mathrm{sc})} = \big(\widetilde{\mathfrak{H}}^{(\mathrm{ac})} \oplus \widetilde{\mathfrak{H}}^{(\mathrm{p})}\big)^{\perp}.$$

It is well known (see, e. g., [**75**, pp. 190–191]) that for each $\bullet \in \{\mathrm{ac},\ \mathrm{p},\ \mathrm{s},\ \mathrm{sc}\}$, $\widetilde{\mathfrak{H}}^{(\bullet)}$ is a closed subspace of $\widetilde{\mathfrak{H}}$ that reduces the operator A. We can therefore represent A on $\widetilde{\mathfrak{H}}$ as a block diagonal operator

$$\big(A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{ac})}}\big) \oplus \big(A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{s})}}\big) \quad \text{on} \quad \widetilde{\mathfrak{H}}^{(\mathrm{ac})} \oplus \widetilde{\mathfrak{H}}^{(\mathrm{s})}$$

or

$$\big(A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{ac})}}\big) \oplus \big(A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{p})}}\big) \oplus \big(A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{sc})}}\big) \quad \text{on} \quad \widetilde{\mathfrak{H}}^{(\mathrm{ac})} \oplus \widetilde{\mathfrak{H}}^{(\mathrm{p})} \oplus \widetilde{\mathfrak{H}}^{(\mathrm{sc})}.$$

Furthermore, we have (see, e. g., [**75**, p. 191])

$$\sigma(A) = \sigma\big(A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{ac})}}\big) \bigcup \sigma\big(A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{s})}}\big) \tag{I.18}$$

and

$$\sigma(A) = \sigma\big(A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{ac})}}\big) \bigcup \sigma\big(A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{p})}}\big) \bigcup \sigma\big(A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{sc})}}\big). \tag{I.19}$$

Let us note that the spectra on the right hand side of (I.18) (of (I.19)) need not be (pairwise) disjoint in general.

Definition I.96 (see [75, p. 191]).
We call:

(1) $A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{ac})}}$ the *absolutely continuous part* of A and $\sigma(A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{ac})}})$ the *absolutely continuous spectrum* of A;
(2) $A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{p})}}$ the *discontinuous part* of A;
(3) $A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{s})}}$ the *singular part* of A and $\sigma(A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{s})}})$ the *singular spectrum* of A;
(4) $A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{sc})}}$ the *singular continuous part* of A and $\sigma(A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{sc})}})$ the *singular continuous spectrum* of A.

Let us note that the spectrum of $A{\restriction}_{\widetilde{\mathfrak{H}}^{(\mathrm{p})}}$ is the closure of the set of all eigenvalues of A.

In view of Remark I.94, we can assume without loss of generality that μ is a finite scalar spectral measure of A. Moreover, [**11**, Theorem 4, p. 171] shows that we can choose $\mu = \mu_{\tilde{h}}$ for some $\tilde{h} \in \widetilde{\mathfrak{H}}$.

Remark I.97. By applying the Lebesgue decomposition theorem (see, e. g., [**18**, Theorem 4.3.2, p. 130]) to μ with respect to λ, we obtain finite Borel measures μ_{ac} and μ_{s} on $\mathbb{R}$ such that:

(1) μ_{ac} is absolutely continuous with respect to λ;
(2) μ_{s} is singular with respect to λ;
(3) $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{s}}$.

The decomposition in (3) is unique.

The measures μ_{ac} and μ_{s} are called the *absolutely continuous part* and the *singular part* of μ, respectively.

Remark I.98. We can choose Borel sets $\mathcal{Y}_{\mathrm{ac}}$ and $\mathcal{Y}_{\mathrm{s}}$ such that:

(1) $\mathbb{R} = \mathcal{Y}_{\mathrm{ac}} \bigcup \mathcal{Y}_{\mathrm{s}}$ and $\mathcal{Y}_{\mathrm{ac}} \bigcap \mathcal{Y}_{\mathrm{s}} = \emptyset$;

(2) $\mu_{\mathrm{ac}}(\Delta) = \mu(\Delta \cap \mathcal{Y}_{\mathrm{ac}})$ and $\mu_{\mathrm{s}}(\Delta) = \mu(\Delta \cap \mathcal{Y}_{\mathrm{s}})$ for all $\Delta \in \mathcal{B}(\mathbb{R})$.

Let us note that $\mathcal{Y}_{\mathrm{ac}}$ and $\mathcal{Y}_{\mathrm{s}}$ are only unique up to Borel sets of μ-measure 0.

Definition I.99 (Pure point measures; cf. [75, p. 400]).
A Borel measure ρ on $\mathbb{R}$ which is finite on compact sets is called *pure point* if there exists an at most countable set $\mathcal{N} \subset \mathbb{R}$ such that $\rho(\mathbb{R} \setminus \mathcal{N}) = 0$.

Remark I.100. One can show that every Borel measure on $\mathbb{R}$ which is finite on compact sets is sigma-finite (see, e. g., [**18**, Propositions 7.2.3 and 7.2.5]).

Let us now refine the decomposition of μ in Remark I.97.

Lemma I.101 (cf. [18, p. 132]).
Let μ be a finite Borel measure on $\mathbb{R}$ and let μ_{ac} as in Remark I.97. Then there exist a finite pure point Borel measure μ_{p} and a finite Borel measure μ_{sc} on $\mathbb{R}$ such that:

(1) μ_{ac} is absolutely continuous with respect to λ;

(2) μ_{sc} is singular with respect to λ as well as $\mu_{\mathrm{sc}}(\{y\}) = 0$ for every $y \in \mathbb{R}$;

(3) $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{p}} + \mu_{\mathrm{sc}}$.

The decomposition in (3) is unique.

Proof. We put

$$\mathcal{Y}_{\mathrm{p}} = \{y \in \mathbb{R} : \mu(\{y\}) > 0\}. \tag{I.20}$$

Then the set $\mathcal{Y}_{\mathrm{p}}$ is at most countable. We define the finite pure point Borel measure μ_{p} on $\mathbb{R}$ by

$$\mu_{\mathrm{p}}(\Delta) = \mu(\Delta \cap \mathcal{Y}_{\mathrm{p}}) \quad \text{for all } \Delta \in \mathcal{B}(\mathbb{R}). \tag{I.21}$$

Let μ_{s} be as in Remark I.97. We now define μ_{sc} by

$$\mu_{\mathrm{sc}}(\Delta) = \mu_{\mathrm{s}}(\Delta \setminus \mathcal{Y}_{\mathrm{p}}) \quad \text{for all } \Delta \in \mathcal{B}(\mathbb{R}). \tag{I.22}$$

Then (1)–(3) of Lemma I.101 hold.

It remains to show that the decomposition in (3) is unique. Let $\mu = \mu'_{\mathrm{ac}} + \mu'_{\mathrm{p}} + \mu'_{\mathrm{sc}}$ be such that μ'_{ac} is absolutely continuous with respect to λ, μ'_{sc} is singular with respect to λ as well as $\mu'_{\mathrm{sc}}(\{y\}) = 0$ for every $y \in \mathbb{R}$, and μ'_{p} is pure point. Let $\mathcal{N} \subset \mathbb{R}$ be at most countable with $\mu'_{\mathrm{p}}(\mathbb{R} \setminus \mathcal{N}) = 0$. Since for every $y \in \mathbb{R}$,

$$\mu(\{y\}) = \mu_{\mathrm{p}}(\{y\}) = \mu'_{\mathrm{p}}(\{y\}),$$

we obtain $\mathcal{Y}_{\mathrm{p}} \subset \mathcal{N}$ and $\mu'_{\mathrm{p}}(\mathcal{N} \setminus \mathcal{Y}_{\mathrm{p}}) = \mu_{\mathrm{p}}(\mathcal{N} \setminus \mathcal{Y}_{\mathrm{p}}) = 0$ as well as $\mu_{\mathrm{p}}(\Delta \cap \mathcal{Y}_{\mathrm{p}}) = \mu'_{\mathrm{p}}(\Delta \cap \mathcal{Y}_{\mathrm{p}})$ for all $\Delta \in \mathcal{B}(\mathbb{R})$. We conclude that $\mu_{\mathrm{p}} = \mu'_{\mathrm{p}}$ and thus $\mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} = \mu'_{\mathrm{ac}} + \mu'_{\mathrm{sc}}$. The uniqueness in the Lebesgue decomposition theorem now yields $\mu_{\mathrm{ac}} = \mu'_{\mathrm{ac}}$ and $\mu_{\mathrm{sc}} = \mu'_{\mathrm{sc}}$. This finishes the proof. □

The measures μ_{p} and μ_{sc} are called the *pure point part* and the *singular continuous part* of μ, respectively.

Remark I.102. Let $\mathcal{Y}_{\mathrm{p}}$ be as in (I.20). We choose $\mathcal{Y}_{\mathrm{ac}}$ and $\mathcal{Y}_{\mathrm{s}}$ according to Remark I.98. Then one has $\mathcal{Y}_{\mathrm{p}} \subset \mathcal{Y}_{\mathrm{s}}$. We set

$$\mathcal{Y}_{\mathrm{sc}} = \mathbb{R} \setminus \Big(\mathcal{Y}_{\mathrm{ac}} \bigcup \mathcal{Y}_{\mathrm{p}}\Big).$$

Then we have:

(1) $\mathbb{R} = \mathcal{Y}_{\mathrm{ac}} \bigcup \mathcal{Y}_{\mathrm{p}} \bigcup \mathcal{Y}_{\mathrm{sc}}$;

(2) the sets $\mathcal{Y}_{\mathrm{ac}}$, $\mathcal{Y}_{\mathrm{p}}$, and $\mathcal{Y}_{\mathrm{sc}}$ are pairwise disjoint;

(3) $\mu_\bullet(\Delta) = \mu(\Delta \cap \mathcal{Y}_\bullet)$ for all $\Delta \in \mathcal{B}(\mathbb{R})$ and each $\bullet \in \{\mathrm{ac, p, sc}\}$.

Let us note that $\mathcal{Y}_{\mathrm{ac}}$, $\mathcal{Y}_{\mathrm{p}}$, and $\mathcal{Y}_{\mathrm{sc}}$ are only unique up to Borel sets of μ-measure 0.

We have:

Lemma I.103. *Let A be a self-adjoint operator on $\widetilde{\mathfrak{H}}$ with spectral measure $E(A)$ and finite scalar spectral measure μ. Then for each $\bullet \in \{\mathrm{ac, p, s, sc}\}$, we have*

$$\widetilde{\mathfrak{H}}^{(\bullet)} = \operatorname{Ran} E_{\mathcal{Y}_\bullet}(A).$$

Proof. It is clear that $\widetilde{\mathfrak{H}}^{(\mathrm{p})} = \operatorname{Ran} E_{\mathcal{Y}_{\mathrm{p}}}(A)$.

By [**11**, pp. 179–180], one has

$$\widetilde{\mathfrak{H}}^{(\mathrm{ac})} = \operatorname{Ran} E_{\mathcal{Y}_{\mathrm{ac}}}(A) \quad \text{and} \quad \widetilde{\mathfrak{H}}^{(\mathrm{s})} = \operatorname{Ran} E_{\mathcal{Y}_{\mathrm{s}}}(A).$$

Since $\widetilde{\mathfrak{H}}^{(\mathrm{sc})} = \big(\widetilde{\mathfrak{H}}^{(\mathrm{ac})} \oplus \widetilde{\mathfrak{H}}^{(\mathrm{p})}\big)^\perp$ and $\mathcal{Y}_{\mathrm{sc}} = \mathbb{R} \setminus \big(\mathcal{Y}_{\mathrm{ac}} \bigcup \mathcal{Y}_{\mathrm{p}}\big)$, we obtain $\widetilde{\mathfrak{H}}^{(\mathrm{sc})} = \operatorname{Ran} E_{\mathcal{Y}_{\mathrm{sc}}}(A)$. This finishes the proof. □

Remark I.104. Given a self-adjoint operator A on $\widetilde{\mathfrak{H}}$ with spectral measure $E = E(A)$ and finite scalar spectral measure μ, there exists, by Theorem I.92, a von Neumann direct integral $\mathfrak{H} = \int_{\mathbb{R}}^{\oplus} \mathfrak{G}(y)\, \mathrm{d}\mu(y)$ such that A is unitarily equivalent to the multiplication operator M_y on $\mathfrak{H}$. The spectral measure X on $\mathfrak{H}$ defined as in (I.11) corresponds to M_y. Now we can decompose $\mathfrak{H}$ with respect to the absolutely continuous part and the singular part(s) of M_y:

$$\mathfrak{H} = \mathfrak{H}^{(\mathrm{ac})} \oplus \mathfrak{H}^{(\mathrm{s})} = \mathfrak{H}^{(\mathrm{ac})} \oplus \mathfrak{H}^{(\mathrm{p})} \oplus \mathfrak{H}^{(\mathrm{sc})},$$

cf. Notation I.95.

The above decompositions of $\widetilde{\mathfrak{H}}$ are invariant under unitary transformations:

Lemma I.105. *Let A be a self-adjoint operator on $\widetilde{\mathfrak{H}}$ with spectral measure $E = E(A)$ and finite scalar spectral measure μ. Let further $V : \widetilde{\mathfrak{H}} \to \mathfrak{H} = \int_{\mathbb{R}}^{\oplus} \mathfrak{G}(y)\, \mathrm{d}\mu(y)$ be unitary such that A is unitarily equivalent to the multiplication operator M_y on $\mathfrak{H}$:*

$$VA = M_y V.$$

Finally, let $X = X(M_y)$ be the spectral measure associated with M_y. Then for each $\bullet \in \{\mathrm{ac, p, s, sc}\}$, we have

$$\operatorname{Ran}\big(V\restriction_{\widetilde{\mathfrak{H}}^{(\bullet)}}\big) = \mathfrak{H}^{(\bullet)} \quad \textit{and} \quad \big(V\restriction_{\widetilde{\mathfrak{H}}^{(\bullet)}}\big)\big(E\restriction_{\widetilde{\mathfrak{H}}^{(\bullet)}}\big) = \big(X\restriction_{\mathfrak{H}^{(\bullet)}}\big)\big(V\restriction_{\widetilde{\mathfrak{H}}^{(\bullet)}}\big).$$

Proof. It is clear that

$$\operatorname{Ran}\big(V\restriction_{\widetilde{\mathfrak{H}}^{(\mathrm{p})}}\big) = \mathfrak{H}^{(\mathrm{p})} \quad \text{and} \quad \big(V\restriction_{\widetilde{\mathfrak{H}}^{(\mathrm{p})}}\big)\big(E\restriction_{\widetilde{\mathfrak{H}}^{(\mathrm{p})}}\big) = \big(X\restriction_{\mathfrak{H}^{(\mathrm{p})}}\big)\big(V\restriction_{\widetilde{\mathfrak{H}}^{(\mathrm{p})}}\big).$$

Let $\bullet \in \{\mathrm{ac, s}\}$. Then, by [**11**, Theorem 3, p. 181], one has

$$\operatorname{Ran}\big(V\restriction_{\widetilde{\mathfrak{H}}^{(\bullet)}}\big) = \mathfrak{H}^{(\bullet)} \quad \text{and} \quad \big(V\restriction_{\widetilde{\mathfrak{H}}^{(\bullet)}}\big)\big(E\restriction_{\widetilde{\mathfrak{H}}^{(\bullet)}}\big) = \big(X\restriction_{\mathfrak{H}^{(\bullet)}}\big)\big(V\restriction_{\widetilde{\mathfrak{H}}^{(\bullet)}}\big).$$

Since $\widetilde{\mathfrak{H}}^{(\mathrm{sc})} = \big(\widetilde{\mathfrak{H}}^{(\mathrm{ac})} \oplus \widetilde{\mathfrak{H}}^{(\mathrm{p})}\big)^{\perp}$ and $\mathfrak{H}^{(\mathrm{sc})} = \big(\mathfrak{H}^{(\mathrm{ac})} \oplus \mathfrak{H}^{(\mathrm{p})}\big)^{\perp}$, we obtain

$$\operatorname{Ran}\big(V{\restriction_{\widetilde{\mathfrak{H}}^{(\mathrm{sc})}}}\big) = \mathfrak{H}^{(\mathrm{sc})} \quad \text{and} \quad \big(V{\restriction_{\widetilde{\mathfrak{H}}^{(\mathrm{sc})}}}\big)\big(E{\restriction_{\widetilde{\mathfrak{H}}^{(\mathrm{sc})}}}\big) = \big(X{\restriction_{\mathfrak{H}^{(\mathrm{sc})}}}\big)\big(V{\restriction_{\widetilde{\mathfrak{H}}^{(\mathrm{sc})}}}\big).$$

This finishes the proof. □

Let us formulate the following useful result.

Proposition I.106. *Let A be a self-adjoint operator on $\widetilde{\mathfrak{H}}$ with spectral measure $E(A)$ and finite scalar spectral measure $\mathfrak{u}$. Let further $\mathfrak{H} = \int_{\mathbb{R}}^{\oplus} \mathfrak{G}(y)\,\mathrm{d}\mathfrak{u}(y)$ be such that A is unitarily equivalent to the multiplication operator by the independent variable on $\mathfrak{H}$, M_y. Then for each $\bullet \in \{\mathrm{ac, p, s, sc}\}$, we have*

$$\mathfrak{H}^{(\bullet)} = \int_{\mathbb{R}}^{\oplus} \mathfrak{G}(y)\,\mathrm{d}\mathfrak{u}_{\bullet}(y).$$

In particular, $\int_{\mathbb{R}}^{\oplus} \mathfrak{G}(y)\,\mathrm{d}\mathfrak{u}_{\bullet}(y)$ is a closed subspace of $\mathfrak{H}$ that reduces M_y.

Proof. We choose $\mathcal{Y}_{\mathrm{ac}}$, $\mathcal{Y}_{\mathrm{p}}$, $\mathcal{Y}_{\mathrm{s}}$, and $\mathcal{Y}_{\mathrm{sc}}$ according to Remarks I.98 and I.102. As usual, let $X = X(M_y)$ on $\mathfrak{H}$ defined as in (I.11) be the spectral measure associated with M_y. Let $\bullet \in \{\mathrm{ac, p, s, sc}\}$. It follows from Lemmas I.103 and I.105 that

$$\mathfrak{H}^{(\bullet)} = \operatorname{Ran} X_{\mathcal{Y}_{\bullet}}.$$

Since

$$\operatorname{Ran} X_{\mathcal{Y}_{\bullet}} = \big\{h \in \mathfrak{H} : h = 0 \ \mathfrak{u}\text{-almost everywhere on } \mathbb{R} \setminus \mathcal{Y}_{\bullet}\big\}$$

and $\mathfrak{u}(\Delta \cap \mathcal{Y}_{\bullet}) = \mathfrak{u}_{\bullet}(\Delta)$ for all $\Delta \in \mathcal{B}(\mathbb{R})$, we obtain

$$\mathfrak{H}^{(\bullet)} = \int_{\mathbb{R}}^{\oplus} \mathfrak{G}(y)\,\mathrm{d}\mathfrak{u}_{\bullet}(y),$$

as claimed. □

Let us relate the spectral multiplicity of a self-adjoint operator to the spectral multiplicities of its parts.

Proposition I.107. *Let A be a self-adjoint operator on $\widetilde{\mathfrak{H}}$ with spectral measure $E(A)$, finite scalar spectral measure $\mathfrak{u}$, and multiplicity function ν. Let further $\mathfrak{H} = \int_{\mathbb{R}}^{\oplus} \mathfrak{G}(y)\,\mathrm{d}\mathfrak{u}(y)$ be such that A is unitarily equivalent to the multiplication operator by the independent variable on $\mathfrak{H}$. We choose $\mathcal{Y}_{\mathrm{ac}}$, $\mathcal{Y}_{\mathrm{p}}$, $\mathcal{Y}_{\mathrm{s}}$, and $\mathcal{Y}_{\mathrm{sc}}$ according to Remarks I.98 and I.102. One has:*

(1) For each $\bullet \in \{\mathrm{ac, p, s, sc}\}$, $y \mapsto \nu_{\bullet}(y) = \mathbb{1}_{\mathcal{Y}_{\bullet}}(y)\nu(y)$ is a multiplicity function of $\mathfrak{H}^{(\bullet)} = \int_{\mathbb{R}}^{\oplus} \mathfrak{G}(y)\,\mathrm{d}\mathfrak{u}_{\bullet}(y)$. In particular, for $\mathfrak{u}$-almost every $y \in \mathbb{R}$, we have

$$\nu(y) = \sum_{\bullet \in \{\mathrm{ac,s}\}} \nu_{\bullet}(y) \quad \text{and} \quad \nu(y) = \sum_{\bullet \in \{\mathrm{ac,\,p,\,sc}\}} \nu_{\bullet}(y).$$

(2) For all $\bullet \in \{\mathrm{ac, p, s, sc}\}$, let $\tilde{\nu}_{\bullet}$ be a multiplicity function of $\mathfrak{H}^{(\bullet)} = \int_{\mathbb{R}}^{\oplus} \mathfrak{G}(y)\,\mathrm{d}\mathfrak{u}_{\bullet}(y)$. Then for $\mathfrak{u}$-almost every $y \in \mathbb{R}$, we have

$$\nu(y) = \sum_{\bullet \in \{\mathrm{ac,s}\}} \mathbb{1}_{\mathcal{Y}_{\bullet}}(y)\tilde{\nu}_{\bullet}(y) \quad \text{and} \quad \nu(y) = \sum_{\bullet \in \{\mathrm{ac,\,p,\,sc}\}} \mathbb{1}_{\mathcal{Y}_{\bullet}}(y)\tilde{\nu}_{\bullet}(y). \tag{I.23}$$

Proof. (1) For each $\bullet \in \{\mathrm{ac, p, s, sc}\}$, we have

$$\dim \mathfrak{G}(y) = \nu(y) = \mathbb{1}_{\mathcal{Y}_\bullet}(y)\nu(y) \quad \text{for } \upmu_\bullet\text{-almost every } y \in \mathbb{R}$$

because

$$\upmu_\bullet\big(\mathbb{R} \setminus \mathcal{Y}_\bullet\big) = 0.$$

(2) Let $\bullet \in \{\mathrm{ac, p, s, sc}\}$. Since $\upmu_\bullet\big(\mathbb{R} \setminus \mathcal{Y}_\bullet\big) = 0$ and

$$\upmu_\bullet(\Delta) = \upmu(\Delta \cap \mathcal{Y}_\bullet) \qquad \text{for all } \Delta \in \mathcal{B}(\mathbb{R}),$$

we have

$$\nu(y) = \dim \mathfrak{G}(y) = \tilde{\nu}_\bullet(y) \qquad \text{for } \upmu\text{-almost every } y \in \mathcal{Y}_\bullet.$$

Consequently, (I.23) holds for $\upmu$-almost all $y \in \mathbb{R}$. □

Remark I.108 (to Proposition I.107).

(I) Since $\mathcal{Y}_{\mathrm{ac}}$ and $\mathcal{Y}_{\mathrm{s}}$ (resp. $\mathcal{Y}_{\mathrm{ac}}$, $\mathcal{Y}_{\mathrm{p}}$, and $\mathcal{Y}_{\mathrm{sc}}$) are unique up to Borel sets of $\upmu$-measure 0, the decompositions in (I.23) are $\upmu$-almost everywhere uniquely determined.

(II) If a self-adjoint operator A with finite scalar spectral measure $\upmu$ and multiplicity function ν possesses an embedded eigenvalue at y_0, then we have $\upmu(\{y_0\}) > 0$ and

$$\nu(y_0) = \dim\ \mathrm{Ker}(A - y_0 I).$$

CHAPTER II

On differences of the type $f(A+B)-f(A)$

In this chapter, we review literature on differences of the type $f(A+B)-f(A)$, where A and B are self-adjoint operators acting on a complex separable Hilbert space $\mathfrak{H}$ (not necessarily of infinite dimension) and f is a Borel function on $\mathbb{R}$. Let us assume that B is (at least) compact.

In the literature, there is a particular interest in the case when B and $f(A+B)-f(A)$ are both of trace class, see, e. g., M. Krein [**44**, **45**], Birman and Solomyak [**7**, **10**], and Peller [**60**, **61**, **63**] together with Aleksandrov [**1**].

M. Krein showed in [**44**] that if B is of trace class, then there exists an L^1 function ξ on $\mathbb{R}$ (the *spectral shift function*) such that for sufficiently nice functions f, the *Lifshits–M. Krein trace formula* holds:

$$\operatorname{trace}\big(f(A+B)-f(A)\big)=\int_{\mathbb{R}}\xi(t)\dot{f}(t)\,\mathrm{d}t. \tag{II.1}$$

At a formal level, the spectral shift function was introduced by Lifshits [**51**]; M. Krein presented in [**44**] a rigorous definition.

In Section II.1, we follow M. Krein's lecture notes from [**45**] to compute the spectral shift function in the (elementary) situation when $\mathfrak{H}$ is finite dimensional. Afterwards, we briefly discuss a question of M. Krein and a recent result of Peller [**63**]. We introduce "double operator integrals" and formulate in Remark II.27 a link between $f(A+B)-f(A)$ and Hankel operators that was found by Peller [**60**]. In Section II.3, we collect some facts on a classical example given by M. Krein in [**44**]. Finally, we introduce some basic notions from scattering theory and then discuss three results of Pushnitski [**67**] and together with Yafaev [**71**].

II.1. The case when $\mathfrak{H}$ is finite dimensional

In this section, we follow M. Krein [**45**, pp. 108–109]. We consider here the case when $\mathfrak{H}$ is finite dimensional, say, $\dim\mathfrak{H}=m\in\mathbb{N}$. Let A be a self-adjoint operator on $\mathfrak{H}$ with eigenvalues $\lambda_1,\dots,\lambda_m$. We denote by $n_A:\mathbb{R}\to\mathbb{Z}$ the eigenvalue counting function, where $n_A(\lambda)$ equals the number of eigenvalues of A that are less than *or equal to* λ.

Lemma II.1. *If f is an absolutely continuous scalar-valued function on $\mathbb{R}$, then the Lifshits–M. Krein trace formula holds for all self-adjoint operators A and B on $\mathfrak{H}$, and the spectral shift function is given by $\xi=n_A-n_{A+B}$.*

Proof. Without loss of generality, we can assume that f takes values in $[0,\infty)$ (otherwise, we write $f=(\operatorname{Re}f)^+-(\operatorname{Re}f)^-+\mathrm{i}((\operatorname{Im}f)^+-(\operatorname{Im}f)^-)$ and consider each term separately). Let us denote the eigenvalues of A by $\lambda_1,\dots,\lambda_m$ and the eigenvalues of $A+B$ by $\mu_1,\dots,\mu_m$.

Then the trace of $f(A)$ is given by

$$\operatorname{trace} f(A) = \sum_{j=1}^{m} f(\lambda_j) = \int_{\mathbb{R}} f(t)\, \mathrm{d}n_A(t);$$

analogously, we have

$$\operatorname{trace} f(A+B) = \sum_{j=1}^{m} f(\mu_j) = \int_{\mathbb{R}} f(t)\, \mathrm{d}n_{A+B}(t).$$

Hence, we can represent the trace of $f(A+B)-f(A)$ as a Riemann–Stieltjes integral:

$$\operatorname{trace}\big(f(A+B)-f(A)\big) = \int_{\mathbb{R}} f(t)\, \mathrm{d}\big(n_{A+B}(t)-n_A(t)\big).$$

Integration by parts yields

$$\operatorname{trace}\big(f(A+B)-f(A)\big) = -\int_{\mathbb{R}} \big(n_{A+B}(t)-n_A(t)\big)\, \mathrm{d}f(t);$$

we note that $n_{A+B}(t)-n_A(t) = 0$ for all $t < \min\big(\sigma(A)\cup\sigma(A+B)\big)$ and also for all $t > \max\big(\sigma(A)\cup\sigma(A+B)\big)$. We observe that the Lebesgue–Stieltjes integral

$$\text{L-S}\int_{\mathbb{R}} \big(n_{A+B}(t)-n_A(t)\big)\, \mathrm{d}f(t)$$

coincides, on one hand, with the Riemann–Stieltjes integral

$$\int_{\mathbb{R}} \big(n_{A+B}(t)-n_A(t)\big)\, \mathrm{d}f(t)$$

and, on the other hand, with the Lebesgue integral $\int_{\mathbb{R}} \big(n_{A+B}(t)-n_A(t)\big)\dot{f}(t)\,\mathrm{d}t$. This finishes the proof. □

II.2. A question of M. Krein and a recent result of Peller

In this section, $\mathfrak{H}$ is a complex separable Hilbert space.

The following question naturally arises.

Question II.2 (cf. [45, p. 141]).
Can one find necessary and sufficient conditions on f such that (II.1) holds for all self-adjoint operators A and B, where B is of trace class?

We need the following definition.

Definition II.3 (Operator Lipschitz functions; see [1, p. 606]).
Let f be a continuous scalar-valued function on $\mathbb{R}$. If there exists a constant $c > 0$ such that the following inequality holds for all bounded self-adjoint operators A and B acting on $\mathfrak{H}$, then f is called *operator Lipschitz*:

$$\|f(A+B)-f(A)\|_{\mathrm{op}} \le c\,\|B\|_{\mathrm{op}}. \tag{II.2}$$

Recently, Peller [**63**] showed that the maximal class of functions on $\mathbb{R}$ so that the Lifshits–M. Krein trace formula holds (for all self-adjoint operators A and B, where B is of trace class) coincides with the class of operator Lipschitz functions.

Definition II.4 (Trace class Lipschitz functions; see [1, p. 650]).
A continuous scalar-valued function f on $\mathbb{R}$ is called *trace class Lipschitz* if inequality (II.2) holds with the operator norm replaced by the trace norm.

Theorem II.5 (see [1, Theorem 3.6.5] and [63, Theorem 6.1]).
Let f be a continuous scalar-valued function on $\mathbb{R}$. Then the following assertions are equivalent:

(1) f is operator Lipschitz;
(2) f is trace class Lipschitz;
(3) $f(A+B)-f(A)$ is of trace class whenever A and B are self-adjoint operators acting on $\mathfrak{H}$, where B is of trace class.

Moreover, if f is operator Lipschitz, then (II.1) holds for all self-adjoint operators A and B acting on $\mathfrak{H}$, where B is of trace class.

Remark II.6. Due to results of Farforovskaya [**27**, **28**], being Lipschitz continuous is not sufficient for f to ensure that (II.1) holds. Moreover, one can prove that the modulus function $t \mapsto |t|$ is not operator Lipschitz, see Kato [**41**] and McIntosh [**55**].

In view of Theorem II.5, let us note that Peller [**60**,**61**] previously gave a necessary and also a sufficient condition on f (namely, being in certain Besov spaces, see Proposition II.26 below) such that (II.1) holds.

In his proofs, Peller uses the theory of "double operator integrals."

Double operator integrals in a Hilbert space. Formally, a double operator integral (DOI) is an expression

$$\int_{\mathbb{R}} \int_{\mathbb{R}} g(\lambda,\mu)\,\mathrm{d}E_\lambda\, T\, \mathrm{d}F_\mu, \tag{II.3}$$

where T is a bounded operator on a complex separable Hilbert space $\mathfrak{H}$, E and F are two Borel spectral measures on $\mathbb{R}$, and g is a bounded scalar-valued Borel function on $\mathbb{R}^2$.

DOI in a Hilbert space are a useful tool in operator theory. They were first considered by Daletskii and S. Krein (see [**19**]). In the case when $E=F$ is supported on a compact interval and g is bounded, continuous, and possesses a continuous partial derivative with respect to μ, the expression (II.3) can be defined as the double Riemann–Stieltjes integral

$$\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} g(\lambda,\mu)\,\mathrm{d}E_\lambda \Big)\, T\, \mathrm{d}E_\mu, \tag{II.4}$$

see [**19**, Theorem 1.4]. Daletskii and S. Krein computed the derivative of an operator-valued function $t \mapsto f(H(t))$:

Proposition II.7 (see [19, Theorem 2.1]).
Let $H(t)$ be a bounded self-adjoint operator on $\mathfrak{H}$ with spectral measure $E(t)=E(H(t))$ for every t contained in the compact interval $[a,b]$. We suppose that $t \mapsto H(t)$ is continuously differentiable (in the operator norm). Let f be a scalar-valued C^2 function on the compact interval $[\alpha,\beta]$. We suppose that the spectrum of every operator $H(t)$ is included in $[\alpha,\beta]$. Then the operator-valued function $[a,b] \ni t \mapsto f(H(t))$ is continuously differentiable (in

the operator norm) with

$$\frac{\mathrm{d}f\big(H(t)\big)}{\mathrm{d}t}=\int_\alpha^\beta\int_\alpha^\beta\frac{f(\lambda)-f(\mu)}{\lambda-\mu}\,\mathrm{d}E_\lambda(t)\frac{\mathrm{d}H(t)}{\mathrm{d}t}\,\mathrm{d}E_\mu(t), \tag{II.5}$$

where we set $\big(f(\lambda)-f(\mu)\big)/(\lambda-\mu)=\dot f(\lambda)$ *if* $\lambda=\mu$.

Equation (II.5) is often called the *Daletskii–S. Krein formula.* Provided that the functions H and f are sufficiently regular, Daletskii and S. Krein also computed the higher derivatives of $t\mapsto f(H(t))$ and estimated the remainder to obtain a Taylor series expansion.

Later, Birman and Solomyak developed a systematic theory of DOI in a series of papers, see [**6**, **8**–**10**]. In particular, the smoothness assumption on f in the Daletskii–S. Krein formula can be relaxed; namely, (II.5) still holds if we replace the condition $f\in\mathsf{C}^2$ by

$$f\in\mathsf{C}_\mathrm{b}(\mathbb{R})\quad\text{and}\quad\int_{\mathbb{R}_+}\Big(\sup_{x\in\mathbb{R}}|f(x+t)-2f(x)+f(x-t)|\Big)\frac{\mathrm{d}t}{t^2}<\infty,$$

see [**7**, Theorem 8.7]. This leads us to the following definition and subsequent lemma.

Definition II.8 (cf. [80, Remark 2, p. 181]).
We define the *Besov space*

$$\mathsf{B}^1_{\infty,1}(\mathbb{R})=\Big\{f\in\mathsf{L}^\infty(\mathbb{R}):\int_{\mathbb{R}_+}\|\tilde f_t\|_{\mathsf{L}^\infty(\mathbb{R})}\frac{\mathrm{d}t}{t^2}<\infty\Big\},$$

where $\tilde f_t(x)=f(x+t)-2f(x)+f(x-t)$.

Lemma II.9 (see [80, Remark 2, p. 103]).
Every $f\in\mathsf{B}^1_{\infty,1}(\mathbb{R})$ *admits a bounded uniformly continuous representative which, moreover, possesses a bounded uniformly continuous derivative.*

Let us also introduce the following classes of functions.

Definition II.10 (cf. [80, Remark 2, p. 181]).
We define the *Besov space*

$$\mathsf{B}^1_{1,1}(\mathbb{R})=\Big\{f\in\mathsf{L}^1(\mathbb{R}):\int_{\mathbb{R}_+}\|\tilde f_t\|_{\mathsf{L}^1(\mathbb{R})}\frac{\mathrm{d}t}{t^2}<\infty\Big\},$$

where $\tilde f_t(x)=f(x+t)-2f(x)+f(x-t)$.

Lemma II.11 (see [81, formula (10), p. 90]).
$\mathsf{B}^1_{1,1}(\mathbb{R})$ *can be continuously imbedded into the Sobolev space* $\mathsf{W}^{1,1}(\mathbb{R})$.

Definition II.12 (cf. [60, p. 122]).
Let us denote by $\widetilde{\mathsf{B}^1_{1,1}}(\mathbb{R})$ the class of all $f\in\mathsf{L}^1_\mathrm{loc}(\mathbb{R})$ such that for every bounded interval $\mathcal{I}\subset\mathbb{R}$, the restriction of f to $\mathcal{I}$ can be extended to an element of $\mathsf{B}^1_{1,1}(\mathbb{R})$.

Furthermore, let us define the Besov space $\mathsf{B}^1_{1,1}$ on the torus.

Definition II.13 (see [2, p. 475]).
The *Besov space* $\mathsf{B}^1_{1,1}(\mathbb{T})$ consists of all (equivalence classes of) functions $h:\mathbb{T}\to\mathbb{C}$ with

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\big|h\big(\mathrm{e}^{\mathrm{i}(s+t)}\big)-2h\big(\mathrm{e}^{\mathrm{i}s}\big)+h\big(\mathrm{e}^{\mathrm{i}(s-t)}\big)\big|\,\mathrm{d}s\,\frac{\mathrm{d}t}{t^2}<\infty.$$

For more information on Besov spaces, we refer to the books [**80**] and [**81**] of Triebel.

Let us now turn to the construction of DOI due to Birman and Solomyak. Essentially, we follow their survey paper [**7**].

Notation II.14. Let $\mathfrak{S}_2(\mathfrak{H})$ be the space of all Hilbert Schmidt operators on $\mathfrak{H}$.

Birman and Solomyak started constructing DOI on $\mathfrak{S}_2(\mathfrak{H})$, i. e., in the framework of separable Hilbert spaces. Let us describe their approach in a simple situation. Let E and F be two Borel spectral measures on $\mathbb{R}$. If δ and ∂ are Borel subsets of $\mathbb{R}$, then we define

$$\mathcal{E}_\delta : T \mapsto E_\delta T \quad \text{and} \quad \mathcal{F}_\partial : T \mapsto T F_\partial \qquad \big(T \in \mathfrak{S}_2(\mathfrak{H})\big).$$

It is easy to see (cf. [**7**, p. 140]) that $\mathcal{E}$ and $\mathcal{F}$ are commuting Borel spectral measures on $\mathbb{R}$. We have:

Lemma II.15 (see [11, Theorem 6, p. 129]).
If $\mathcal{E}$ and $\mathcal{F}$ are two commuting Borel spectral measures on $\mathbb{R}$, then there exists a unique Borel spectral measure $\mathcal{G}$ on $\mathbb{R}^2$ such that

$$\mathcal{G}_{\delta\times\mathbb{R}} = \mathcal{E}_\delta \quad \textit{and} \quad \mathcal{G}_{\mathbb{R}\times\partial} = \mathcal{F}_\partial$$

for all $\delta, \partial \in \mathcal{B}(\mathbb{R})$.

We call $\mathcal{G}$ the *product* of $\mathcal{E}$ and $\mathcal{F}$. By construction, one has

$$\mathcal{G}_{\delta\times\partial} = \mathcal{E}_\delta \mathcal{F}_\partial \tag{II.6}$$

for measurable rectangles $\delta \times \partial$ (see [**11**, p. 129]).

By the standard Carathéodory extension procedure (see [**11**, pp. 127–128]), we obtain a sigma-algebra $\mathcal{B}^*(\mathbb{R}^2)$ containing all subsets Ω of Borel sets $\Delta \subset \mathbb{R}^2$ with $\mathcal{G}_\Delta = 0$; setting $\mathcal{G}_\Omega = 0$ for every such Ω, we extend $\mathcal{G}$ to a spectral measure on the whole of $\mathcal{B}^*(\mathbb{R}^2)$. We call this extension *null set complete* and denote it again by $\mathcal{G}$.

For the rest of this section, we assume that $\mathcal{G}$ is null set complete.

Notation II.16. Let us write $\mathsf{L}^\infty(\mathbb{R}^2, \mathcal{G})$ for the set of all (equivalence classes of) $\mathcal{G}$-measurable functions $g : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\|g\|_{\mathsf{L}^\infty(\mathbb{R}^2,\mathcal{G})} = \mathcal{G}\text{-}\sup_{(\lambda,\mu)\in\mathbb{R}^2} |g(\lambda,\mu)| = \inf_{\Delta\in\mathcal{B}^*(\mathbb{R}^2),\, \mathcal{G}_\Delta=0} \ \sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\Delta} |g(\lambda,\mu)|$$

is finite.

We obtain a bounded functional calculus.

Proposition II.17 (see [11, Theorem 1, p. 132]).
The mapping

$$\mathcal{T} : \mathsf{L}^\infty(\mathbb{R}^2, \mathcal{G}) \ni g \mapsto \int_{\mathbb{R}^2} g(\lambda,\mu)\, \mathrm{d}\mathcal{G}_{\lambda,\mu}$$

is linear, multiplicative, involutive, and isometric with values in the set of bounded operators on $\mathfrak{S}_2(\mathfrak{H})$. That is, for all $g, g_1, g_2 \in \mathsf{L}^\infty(\mathbb{R}^2, \mathcal{G})$ and all $\alpha \in \mathbb{C}$, we have:

- $\mathcal{T}_{g_1+\alpha g_2} = \mathcal{T}_{g_1} + \alpha\mathcal{T}_{g_2}$,
- $\mathcal{T}_{g_1 g_2} = \mathcal{T}_{g_1}\ \mathcal{T}_{g_2}$,
- $\mathcal{T}_{\bar{g}} = \left(\mathcal{T}_g\right)^*$,
- $\|\mathcal{T}_g\|_{\mathrm{op}} = \|g\|_{\mathsf{L}^\infty(\mathbb{R}^2,\mathcal{G})}$.

Definition II.18 (cf. [7, p. 141] and [1, p. 608]).
For $g \in \mathsf{L}^\infty(\mathbb{R}^2, \mathcal{G})$, the mapping $\mathcal{T}_g$ from Proposition II.17 is called a *transformer.* In order to emphasize that this transformer depends on E and F, we sometimes write $\mathcal{T}_g^{E,F}$ instead of $\mathcal{T}_g$. For every $T \in \mathfrak{S}_2(\mathfrak{H})$, we write

$$\mathcal{T}_g^{E,F} T = \int_{\mathbb{R}} \int_{\mathbb{R}} g(\lambda,\mu)\,\mathrm{d}E_\lambda\, T\ \mathrm{d}F_\mu.$$

We call $\mathcal{T}_g^{E,F} T$ the *double operator integral* associated with g, E and F as well as T.

Remark II.19. Under the additional assumptions that $E = F$ is supported on a compact interval and g is bounded, continuous, and possesses a continuous partial derivative with respect to μ, the DOI $\mathcal{T}_g^{E,E} T$ coincides with the expression in (II.4).

Notation II.20. Let $\mathfrak{S}_1(\mathfrak{H})$ be the space of all trace class operators on $\mathfrak{H}$.

Definition II.21 (Schur multipliers; cf. [60, p. 112 and p. 115]).
Let us denote by $\mathsf{M}(E,F)$ the class of all $g \in \mathsf{L}^\infty(\mathbb{R}^2, \mathcal{G})$ such that the transformer $\mathcal{T}_g^{E,F}$ is bounded on $\mathfrak{S}_1(\mathfrak{H})$ (i. e., from $\mathfrak{S}_1(\mathfrak{H})$ to itself). The elements of $\mathsf{M}(E,F)$ are called *Schur multipliers* (for the pair E, F).

Moreover, if $\mathcal{I} \subset \mathbb{R}$ is an interval, then we write $\mathsf{M}_{\mathcal{I}}$ for the class of all $g \in \mathsf{L}^\infty(\mathbb{R}^2, \mathcal{G})$ that are Schur multipliers for every pair E, F of spectral measures supported in $\mathcal{I}$.

Below, Schur multipliers that admit a representation as a "divided difference" will be of special interest. More precisely, if $f : \mathbb{R} \to \mathbb{C}$ is a Lipschitz continuous function, then we define

$$g_f(\lambda,\mu) = \frac{f(\lambda) - f(\mu)}{\lambda - \mu} \quad \text{if } \lambda \neq \mu.$$

The following question arises.

Question II.22 (cf. [7, p. 151]).
Is every bounded extension of g_f to the whole of $\mathbb{R}^2$ an element of $\mathsf{L}^\infty(\mathbb{R}^2, \mathcal{G})$?

It turns out that this is indeed the case. For the proof, we need the following result.

Lemma II.23 (cf. [7, equation (7.3)]).
The diagonal $\mathbf{diag} = \{(\lambda,\lambda) : \lambda \in \mathbb{R}\} \subset \mathbb{R}^2$ *is $\mathcal{G}$-measurable and*

$$\int_{\mathbb{R}^2} \mathbb{1}_{\mathbf{diag}}(\lambda,\mu)\,\mathrm{d}\mathcal{G}_{\lambda,\mu} = \sum_{\lambda\in\mathcal{N}} \mathcal{E}_{\{\lambda\}}\mathcal{F}_{\{\lambda\}},$$

where $\mathcal{N} = \{\lambda \in \mathbb{R} : \mathcal{E}_{\{\lambda\}} \neq 0,\ \mathcal{F}_{\{\lambda\}} \neq 0\}$ *is at most countable.*

Proof. Being a closed subset of $\mathbb{R}^2$, **diag** is Borelian and thus $\mathcal{G}$-measurable.

It is well known that spectral measures that take values in a separable Hilbert space possess at most countably many point masses. Hence, $\mathcal{N}$ is at most countable.

Let us choose finite scalar spectral measures $\mu_{\mathcal{E}}$ of $\int_{\mathbb{R}} \lambda \, d\mathcal{E}_\lambda$ and $\mu_{\mathcal{F}}$ of $\int_{\mathbb{R}} \mu \, d\mathcal{F}_\mu$ (cf. Remark I.94). It is not hard to show that if $\Delta \in \mathcal{B}(\mathbb{R}^2)$ with $(\mu_{\mathcal{E}} \otimes \mu_{\mathcal{F}})(\Delta) = 0$, then $\mathcal{G}_\Delta = 0$. By Fubini's theorem, we obtain

$$(\mu_{\mathcal{E}} \otimes \mu_{\mathcal{F}})(\mathbf{diag} \setminus \{(\lambda, \lambda) : \lambda \in \mathcal{N}\}) = 0.$$

Hence, $\mathcal{G}_{\mathbf{diag}} = \mathcal{G}_{\{(\lambda,\lambda):\lambda\in\mathcal{N}\}}$ and, since $\mathcal{N}$ is at most countable, the sigma-additivity of $\mathcal{G}$ yields the desired formula. □

We can now answer Question II.22 in the affirmative.

Lemma II.24 (cf. [7, p. 152]).
If $f : \mathbb{R} \to \mathbb{C}$ is a Lipschitz continuous function, then every bounded extension of g_f to the whole of $\mathbb{R}^2$ is an element of $\mathsf{L}^\infty(\mathbb{R}^2, \mathcal{G})$.

Proof. Since the spectral measure $\mathcal{G}$ is null set complete, the assertion follows from Lemma II.23. □

Let us note that if the function f is continuously differentiable, then it is natural to set $g_f(\lambda, \lambda) = \dot{f}(\lambda)$.

In view of perturbation theory, it is important to study differences that are of the type $f(A+B) - f(A)$, where A and B are self-adjoint operators and f is a scalar-valued Borel function on $\mathbb{R}$. Under certain assumptions, we can represent such differences as DOI with respect to the spectral measures $E(A)$ and $E(A+B)$. Moreover, we are interested in necessary as well as sufficient conditions on f such that $f(A+B) - f(A)$ is of trace class whenever B is of trace class, cf. Question II.2. Birman and Solomyak showed:

Theorem II.25 (see [10, Theorem 4.3]).
Let us suppose that A and B are self-adjoint operators acting on $\mathfrak{H}$, where B is of trace class. Then we have

$$f(A+B) - f(A) = \mathcal{T}_{g_f}^{E(A),E(A+B)} B \in \mathfrak{S}_1(\mathfrak{H})$$

for every f such that $g_f \in \mathsf{M}\big(E(A), E(A+B)\big)$ is a Schur multiplier.

Peller proved the following:

Proposition II.26.

(1) If $f \in \mathsf{B}^1_{\infty,1}(\mathbb{R})$, then $g_f \in \mathsf{M}_{\mathbb{R}}$.

(2) If $g_f \in \mathsf{M}_{\mathcal{J}}$ for every bounded interval $\mathcal{J} \subset \mathbb{R}$, then $f \in \widetilde{\mathsf{B}^1_{1,1}}(\mathbb{R})$.

Proof. See [**61**, Theorem 2] for part (1) and [**60**, Theorem 8] for part (2). □

Remark II.27 (Link to Hankel operators).
We take a closer look at the proof of Proposition II.26 (2).

Let $\mathcal{J} \subset \mathbb{R}$ be a bounded interval. We have $g_f \in \mathsf{M}_{\mathcal{J}}$. By defining

$$h\big(e^{2\pi i t/\mathrm{length}(\mathcal{J})}\big) = f(t) \qquad (t \in \mathcal{J}),$$

we obtain a function $h : \mathbb{T} \to \mathbb{C}$. Peller showed (see [**60**, p. 122] and [**60**, Theorem 4]) that h belongs to $\mathsf{B}^1_{1,1}(\mathbb{T})$; for this, he used the following characterization theorem for trace class

Hankel operators (see [**62**, Corollary 1.2, p. 233]): *A Hankel operator H_ϕ on $\mathsf{H}^2(\mathbb{T})$ with symbol $\phi \in \mathsf{L}^\infty(\mathbb{T})$ is of trace class if and only if the orthogonal projection of ϕ on $\mathsf{H}^2_-(\mathbb{T})$ belongs to $\mathsf{B}^1_{1,1}(\mathbb{T})$.*

Since $h \in \mathsf{B}^1_{1,1}(\mathbb{T})$ and $\mathcal{I} \subset \mathbb{R}$ was an arbitrary bounded interval, it follows that $f \in \widetilde{\mathsf{B}^1_{1,1}}(\mathbb{R})$, cf. [**60**, p. 122].

Consequently, there is a relationship between differences of the type $f(A+B)-f(A)$ and Hankel operators. The present thesis is intended to add some more knowledge on this subject.

We finish this section with a further remark.

Remark II.28. Let us mention two more facts on DOI (see [**7**]):

(I) For the construction of DOI, the spectral measures need not be Borelian. In this general situation, one uses [**6**, Theorem 1] to obtain that the additive projection-valued function in (II.6) is sigma-additive.

(II) Transformers can be interpreted as "multipliers" in the framework of integral operators, multiplying the integral kernel with a bounded function.

II.3. On a classical example given by M. Krein

We follow here M. Krein [**44**, pp. 622–624]. Let us consider the Neumann Laplacian $H = (-\,\mathrm{d}^2/\,\mathrm{d}t^2)^{\mathrm{N}}$ and the Dirichlet Laplacian $H^{\mathrm{D}} = (-\,\mathrm{d}^2/\,\mathrm{d}t^2)^{\mathrm{D}}$ in $\mathsf{L}^2(\mathbb{R}_+)$. They both have a simple purely absolutely continuous spectrum filling in $[0,\infty)$. We denote the resolvent of H^{D} at the spectral point -1 by A_0 and the resolvent of H at the spectral point -1 by A_1. Then A_0 and A_1 are bounded self-adjoint integral operators on $\mathsf{L}^2(\mathbb{R}_+)$ with kernel functions

$$a_0(x,y) = \begin{cases} \sinh(x)\mathrm{e}^{-y} & \text{if } x \le y \\ \sinh(y)\mathrm{e}^{-x} & \text{if } x \ge y \end{cases} \quad \text{and} \quad a_1(x,y) = \begin{cases} \cosh(x)\mathrm{e}^{-y} & \text{if } x \le y \\ \cosh(y)\mathrm{e}^{-x} & \text{if } x \ge y \end{cases}. \tag{II.7}$$

We compute

$$A_1 - A_0 = \langle \bullet, \varphi \rangle_{\mathsf{L}^2(\mathbb{R}_+)} \varphi, \qquad \text{where } \varphi(x) = \mathrm{e}^{-x}; \tag{II.8}$$

in particular, the resolvent difference $A_1 - A_0$ is of rank 1.

Let $\psi \in \mathsf{C}_{\mathrm{c}}(\mathbb{R}_+)$ and let $\mu > 0$. M. Krein calculated

$$\Big(\Big[E_{(-\infty,\mu)}(H) - E_{(-\infty,\mu)}\big(H^{\mathrm{D}}\big)\Big]\psi\Big)(x) = \frac{2}{\pi}\int_{\mathbb{R}_+} \frac{\sin\big(\mu^{1/2}(x+y)\big)}{x+y}\psi(y)\,\mathrm{d}y \tag{II.9}$$

and, on the other hand,

$$E_{(-\infty,\mu(\lambda))}(H) - E_{(-\infty,\mu(\lambda))}\big(H^{\mathrm{D}}\big) = E_{(-\infty,\lambda)}(A_0) - E_{(-\infty,\lambda)}(A_1),$$

where $\mu(\lambda) = \frac{1}{\lambda} - 1 > 0$. Consequently, for every $0 < \lambda < 1$,

$$\Big(\Big[E_{(-\infty,\lambda)}(A_0) - E_{(-\infty,\lambda)}(A_1)\Big]\psi\Big)(x) = \frac{2}{\pi}\int_{\mathbb{R}_+} \frac{\sin\big((\frac{1}{\lambda}-1)^{1/2}(x+y)\big)}{x+y}\psi(y)\,\mathrm{d}y. \tag{II.10}$$

M. Krein concluded that $E_{(-\infty,\lambda)}(A_0) - E_{(-\infty,\lambda)}(A_1)$ is not Hilbert Schmidt if $0 < \lambda < 1$.

More than fifty years later, Kostrykin and Makarov [**43**] explicitly diagonalized the operator $E_{(-\infty,\lambda)}(A_0) - E_{(-\infty,\lambda)}(A_1)$:

Theorem II.29 (see [43, Theorem 1]).
If $0 < \lambda < 1$, then $E_{(-\infty,\lambda)}(A_0) - E_{(-\infty,\lambda)}(A_1)$ has a simple purely absolutely continuous spectrum filling in the interval $[-1,1]$. In particular, $E_{(-\infty,\lambda)}(A_0) - E_{(-\infty,\lambda)}(A_1)$ is not compact.

Remark II.30 (to M. Krein's example).

(I) Since the difference of the spectral projections is not compact, the left hand side of the Lifshits–M. Krein trace formula is ill-defined.
(II) The difference of the spectral projections is a Hankel operator.

II.4. On three results of Pushnitski and Yafaev

The operator $D(\lambda) = E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A)$, $\lambda \in \mathbb{R}$, has been studied by Pushnitski [**67–70**] and together with Yafaev [**71, 72**]. Before reviewing some of their results, we need to define the "wave operators" and the "scattering matrix." Then we present the Birman–M. Krein formula which connects the "scattering matrix" and the spectral shift function. For this, we follow Yafaev's monograph [**86**]. Finally, we discuss three results of Pushnitski and Yafaev.

Throughout this section, let A and B be two self-adjoint operators acting on a complex separable Hilbert space $\mathfrak{H}$.

II.4.1. The wave operators. The scattering operator and matrix. The Birman–M. Krein formula. Let us introduce some notation (cf. [**86**, p. 67]). We denote by $P_A^{(\mathrm{ac})}$ the orthogonal projection onto the absolutely continuous subspace $\mathfrak{H}_A^{(\mathrm{ac})}$ of $\mathfrak{H}$ with respect to the initial operator A. Furthermore, we write $\mathfrak{H}_A^{(\mathrm{s})}$ for the orthogonal complement of $\mathfrak{H}_A^{(\mathrm{ac})}$ in $\mathfrak{H}$. For the perturbed operator $A+B$, we omit the subscript "$A+B$," i. e., we denote by $P^{(\mathrm{ac})}$ the orthogonal projection onto the absolutely continuous subspace $\mathfrak{H}^{(\mathrm{ac})}$ of $\mathfrak{H}$ with respect to $A+B$, etc.

Definition II.31 (Wave operators; see [86, Definition 1, p. 67]).
The *wave operators* for the pair A, $A+B$ and the bounded operator $\mathscr{I} : \mathfrak{H} \to \mathfrak{H}$ are defined by

$$W_\pm = W_\pm(A+B,\ A;\ \mathscr{I}) = \operatorname*{s-lim}_{t\to\pm\infty} \exp\big(\mathrm{i}(A+B)t\big)\mathscr{I}\exp(-\mathrm{i}At)P_A^{(\mathrm{ac})}, \tag{II.11}$$

provided that these strong limits exist.

For the rest of this subsection, let us assume that the wave operators for the pair A, $A+B$ and the bounded operator $\mathscr{I} : \mathfrak{H} \to \mathfrak{H}$ exist.

Definition II.32 (Completeness of wave operators; see [86, Definition 1, p. 78]).
The wave operators $W_\pm = W_\pm(A+B,\ A;\ \mathscr{I})$ are called *complete* if:

(1) $\mathrm{Ker}(W_\pm) = \mathfrak{H}_A^{(\mathrm{s})}$,
(2) $\mathrm{Ran}(W_\pm) = \mathfrak{H}^{(\mathrm{ac})}$.

Definition II.33 (Scattering operator; see [86, p. 82]).
The *scattering operator* is defined by

$$\mathscr{S}(A+B,\ A;\ \mathscr{I}) = \mathscr{S} = W_+^* W_-.$$

Remark II.34. It follows from (II.11) that

$$\operatorname{Ker}\big(\mathscr{S}(A+B,\ A;\ \mathscr{I})\big) \supset \mathfrak{H}_A^{(\mathrm{s})} \quad \text{and} \quad \operatorname{Ran}\big(\mathscr{S}(A+B,\ A;\ \mathscr{I})\big) \subset \mathfrak{H}_A^{(\mathrm{ac})}.$$

We therefore consider the scattering operator $\mathscr{S}(A+B,\ A;\ \mathscr{I})$ only on the subspace $\mathfrak{H}_A^{(\mathrm{ac})}$ of $\mathfrak{H}$, as Yafaev does (see [**86**, p. 82]).

Here is a list of properties of the scattering operator:

Lemma II.35 (cf. [86, p. 82]).
The scattering operator $\mathscr{S}(A+B,\ A;\ \mathscr{I})$:

(1) is bounded with operator norm $\leq \|\mathscr{I}\|_{\mathrm{op}}^2$;
(2) commutes with the absolutely continuous part of A;
(3) is unitary if the wave operators are isometric on $\mathfrak{H}_A^{(\mathrm{ac})}$ and are complete.

Proof. Part (1) follows from [**86**, Lemma 2, p. 68]; part (2) is a consequence of [**86**, Theorem 4, p. 69]; part (3) can be found in [**86**, Corollary 2, p. 82]. □

Definition II.36 (Core of the spectrum; see [86, Definition 8, p. 25]).
A set $\hat{\sigma} \in \mathcal{B}(\mathbb{R})$ of full $E(A)$-measure (i. e., $E_{\mathbb{R}\setminus\hat{\sigma}}(A) = 0$) is called a *core of the spectrum of A* if:

(1) for every other $\tilde{\sigma} \in \mathcal{B}(\mathbb{R})$ of full $E(A)$-measure, $\hat{\sigma} \setminus \tilde{\sigma}$ has Lebesgue measure 0;
(2) $\hat{\sigma} \subset \sigma(A)$.

Lemma II.37 (cf. [86, p. 82]).
The absolutely continuous part of A is unitarily equivalent to the multiplication operator by the independent variable on a von Neumann direct integral

$$\int_{\hat{\sigma}_A}^{\oplus} \mathfrak{H}(\lambda)\, \mathrm{d}\lambda, \tag{II.12}$$

where $\hat{\sigma}_A = \hat{\sigma}(A)$ is a core of the spectrum of A.

Proof. Let μ be a finite scalar spectral measure of A. Then by Lemma I.105 and Proposition I.106, we know that the absolutely continuous part of A is unitarily equivalent to the multiplication operator M_λ by the independent variable on a von Neumann direct integral

$$\int_{\mathbb{R}}^{\oplus} \mathfrak{H}(\lambda)\, \mathrm{d}\mu_{\mathrm{ac}}(\lambda).$$

Being of full $E(A)$-measure, $\hat{\sigma}_A$ is of full μ_{ac}-measure and thus we can identify

$$\int_{\hat{\sigma}_A}^{\oplus} \mathfrak{H}(\lambda)\, \mathrm{d}\mu_{\mathrm{ac}}(\lambda) = \int_{\mathbb{R}}^{\oplus} \mathfrak{H}(\lambda)\, \mathrm{d}\mu_{\mathrm{ac}}(\lambda).$$

It follows from [**78**, Theorem 2.5.2] (cf. also [**75**, Theorem B.9, p. 401] or [**86**, pp. 14–16]) that the restrictions of Borel–Lebsgue measure λ and μ_{ac} to $\hat{\sigma}_A$ are equivalent. In view of Proposition I.83, M_λ on $\int_{\hat{\sigma}_A}^{\oplus} \mathfrak{H}(\lambda)\, \mathrm{d}\mu_{\mathrm{ac}}(\lambda)$ is therefore unitarily equivalent to M_λ

on $\int_{\hat{\sigma}_A}^{\oplus} \mathfrak{H}(\lambda)\, \mathrm{d}\mathbb{\lambda}(\lambda)$. Since the latter operator is unitarily equivalent to M_λ on $\int_{\hat{\sigma}_A}^{\oplus} \mathfrak{H}(\lambda)\, \mathrm{d}\lambda$, the proof is complete. □

Remark II.38 (cf. [86, p. 16]).
In general, we cannot replace $\hat{\sigma}_A$ by $\sigma_{\mathrm{ac}}(A)$ in (II.12). Indeed, let $A = M_\lambda$ be the multiplication operator by the independent variable on $\mathsf{L}^2(\mathbb{R}, \mu)$ with $\mu(\Delta) = \mathbb{\lambda}(\Delta \cap ([0,1] \setminus \mathcal{C}))$ for all $\Delta \in \mathcal{B}(\mathbb{R})$, where $\mathcal{C}$ is a compact subset of $[0,1]$. Then $\mu_{\mathrm{ac}}(\mathcal{C}) = \mathbb{\lambda}(\mathcal{C} \cap ([0,1] \setminus \mathcal{C})) = 0$. So if $\mathcal{C}$ is a modified Cantor set (see, e. g, [**16**, pp. 67–68]) of Lebesgue measure $1/2$, then the restrictions of Lebesgue measure and μ_{ac} to $\sigma_{\mathrm{ac}}(A) = [0,1]$ (more precisely, to $\mathcal{B}([0,1])$) are not equivalent.

Lemma II.39 (see [86, p. 82]).
The scattering operator $\mathscr{S}(A+B,\ A;\ \mathscr{I})$ is unitarily equivalent to multiplication by an operator-valued function $\lambda \mapsto \Sigma(\lambda;\ A+B,\ A;\ \mathscr{I})$ on the space (II.12).

Proof. We combine Lemma II.35 (1)–(2) and Lemma II.37. □

Definition II.40 (Scattering matrix; cf. [86, p. 82]).
We call the operator $\Sigma(\lambda) = \Sigma(\lambda;\ A+B,\ A;\ \mathscr{I})$ of Lemma II.39 the *scattering matrix.*

The famous Birman–M. Krein formula connects, under certain assumptions, the scattering matrix with the spectral shift function:

$$\det \Sigma(\lambda;\ A+B,\ A;\ I) = \exp\big(-2\pi \mathrm{i} \xi(\lambda)\big) \tag{II.13}$$

for almost every $\lambda \in \hat{\sigma}_A$. For instance, one has:

Theorem II.41 (see [86, Theorem 1, p. 282]).
If B is of trace class, then the Birman–M. Krein formula (II.13) holds for almost every $\lambda \in \hat{\sigma}_A$.

II.4.2. The local wave operators. The local scattering operator and matrix. The considerations of the preceding subsection can be "localized" in the following sense:

Definition II.42 (Local wave operators; see [86, p. 74]).
We put $P_A^{(\mathrm{ac})}(\delta) = E_\delta(A) P_A^{(\mathrm{ac})}$ for every $\delta \in \mathcal{B}(\mathbb{R})$. Then the *local wave operators* for the pair A, $A+B$, a bounded operator $\mathscr{I} : \mathfrak{H} \to \mathfrak{H}$, and a Borel set $\delta \in \mathcal{B}(\mathbb{R})$ are defined by

$$W_\pm(A+B,\ A;\ \mathscr{I},\ \delta) = \operatorname*{s-lim}_{t \to \pm\infty} \exp\big(\mathrm{i}(A+B)t\big) \mathscr{I} \exp(-\mathrm{i}At) P_A^{(\mathrm{ac})}(\delta), \tag{II.14}$$

provided that these strong limits exist.

If the wave operators for a pair A, $A+B$ and a bounded operator $\mathscr{I} : \mathfrak{H} \to \mathfrak{H}$ do not exist, then there is still a chance that the local wave operators exist.

Let $\delta \in \mathcal{B}(\mathbb{R})$. By functional calculus (with respect to A), we obtain (cf. [**86**, p. 75]):

$$\begin{aligned} W_\pm(A+B,\ A;\ I,\ \delta) &= \operatorname*{s-lim}_{t \to \pm\infty} \exp\big(\mathrm{i}(A+B)t\big) \exp(-\mathrm{i}At) E_\delta(A) P_A^{(\mathrm{ac})} \\ &= \operatorname*{s-lim}_{t \to \pm\infty} \exp\big(\mathrm{i}(A+B)t\big) E_\delta(A) \exp(-\mathrm{i}At) P_A^{(\mathrm{ac})} \\ &= W_\pm(A+B,\ A;\ E_\delta(A)), \end{aligned}$$

assuming that the strong limits exist. Therefore, we can identify the local wave operators $W_\pm(A+B,\ A;\ I,\ \delta)$ with the "global" wave operators $W_\pm(A+B,\ A;\ E_\delta(A))$.

For the rest of this subsection, we assume that the local wave operators for the pair A, $A+B$, the bounded operator $\mathscr{I}:\mathfrak{H}\to\mathfrak{H}$, and the Borel set $\delta\in\mathcal{B}(\mathbb{R})$ exist.

Definition II.43 (Completeness of local wave operators; see [86, p. 81]).
The local wave operators $W_\pm = W_\pm(A+B,\ A;\ \mathscr{I},\ \delta)$ are called *complete* if:

(1) $\operatorname{Ker}(W_\pm)$ is the orthogonal complement of $\operatorname{Ran}\big(P_A^{(\mathrm{ac})}(\delta)\big)$ in $\mathfrak{H}$;
(2) $\operatorname{Ran}(W_\pm)=\operatorname{Ran}\big(P^{(\mathrm{ac})}(\delta)\big)$.

Definition II.44 (Local scattering operator; cf. [86, p. 83]).
The *local scattering operator* is defined by

$$\mathscr{S}(A+B,\ A;\ \mathscr{I},\ \delta)=\mathscr{S}=W_+^*W_-.$$

Remark II.45. It follows from (II.14) that $\operatorname{Ker}\big(\mathscr{S}(A+B,\ A;\ \mathscr{I},\ \delta)\big)$ includes the orthogonal complement of $\operatorname{Ran}\big(P_A^{(\mathrm{ac})}(\delta)\big)$ in $\mathfrak{H}$ and that $\operatorname{Ran}\big(\mathscr{S}(A+B,\ A;\ \mathscr{I},\ \delta)\big)\big)\subset\operatorname{Ran}\big(P_A^{(\mathrm{ac})}(\delta)\big)$. We therefore consider the local scattering operator $\mathscr{S}(A+B,\ A;\ \mathscr{I},\ \delta)$ only on the subspace $\operatorname{Ran}\big(P_A^{(\mathrm{ac})}(\delta)\big)$ of $\mathfrak{H}$.

Here is a list of properties of the local scattering operator:

Lemma II.46 (cf. [86, pp. 75 and 83]).
The local scattering operator $\mathscr{S}(A+B,\ A;\ \mathscr{I},\ \delta)\big)$:

(1) is bounded with operator norm $\leq\|\mathscr{I}\|_{\mathrm{op}}^2$;
(2) commutes with the part of A with respect to $\operatorname{Ran}\big(P_A^{(\mathrm{ac})}(\delta)\big)$;
(3) is unitary if the local wave operators are isometric on $\operatorname{Ran}\big(P_A^{(\mathrm{ac})}(\delta)\big)$ and are complete.

Proof. Analogously to Lemma II.35. □

From this, we deduce:

Lemma II.47. *The local scattering operator $\mathscr{S}(A+B,\ A;\ \mathscr{I},\ \delta)\big)$ is unitarily equivalent to multiplication by an operator-valued function $\lambda\mapsto\Sigma(\lambda;\ A+B,\ A;\ \mathscr{I},\ \delta)$ on a von Neumann direct integral*

$$\int_{\delta\cap\hat{\sigma}_A}^{\oplus}\mathfrak{H}(\lambda)\,\mathrm{d}\lambda, \tag{II.15}$$

where $\hat{\sigma}_A=\hat{\sigma}(A)$ is a core of the spectrum of A.

Proof. Analogously to Lemma II.39. □

Definition II.48 (Local scattering matrix; see [71, p. 1955]).
We call the operator $\Sigma(\lambda)=\Sigma(\lambda;\ A+B,\ A;\ \mathscr{I},\ \delta)$ of Lemma II.47 the *local scattering matrix*.

II.4.3. Three results of Pushnitski and Yafaev. Let us proceed chronologically. We first discuss two results of Pushnitski from [**67**] and then one result of Pushnitski together with Yafaev from [**71**].

Pushnitski and Yafaev assumed that the operator B is factorized as $B = G^*B_0G$. In order to keep the technical details simple, we consider here the following special case (cf. [**67**, p. 228]) of such factorization, which is sufficient for the present work: Let us define the compact self-adjoint operator $G = |B|^{\frac{1}{2}} : \mathfrak{H} \to \mathfrak{H}$ and the bounded self-adjoint operator $B_0 = \operatorname{sign}(B) : \mathfrak{H} \to \mathfrak{H}$. Then one has $B = G^*B_0G$.

Let us now define the operator-valued functions h_0 and h on $\mathbb{R}$ by (cf. [**67**, p. 228])

$$h_0(\lambda) = GE_{(-\infty,\lambda)}(A)G^*, \quad h(\lambda) = GE_{(-\infty,\lambda)}(A+B)G^* \qquad (\lambda \in \mathbb{R}).$$

We will need the following assumptions:

Hypothesis II.49 (cf. [67, Hypothesis 1.1]).
We suppose that there exists an open interval δ included in the absolutely continuous spectrum of A. Next, we assume that the derivatives

$$\dot{h}_0(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}h_0(\lambda) \quad \text{and} \quad \dot{h}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}h(\lambda)$$

exist in operator norm for all $\lambda \in \delta$, and that the maps $\delta \ni \lambda \mapsto \dot{h}_0(\lambda)$ and $\delta \ni \lambda \mapsto \dot{h}(\lambda)$ are Hölder continuous (with some positive exponent) in the operator norm.

Remark II.50. Let us assume Hypothesis II.49. Then $\Sigma(\lambda;\ A+B,\ A;\ I,\ \delta) - I(\lambda)$ is compact for every $\lambda \in \delta$ (see [**67**, pp. 228–229]), where $I(\lambda)$ stands for the identity operator in the fiber $\mathfrak{H}(\lambda)$ of the von Neumann direct integral (II.15).

The following result of Pushnitski describes the essential spectrum of $D(\lambda)$.

Theorem II.51 (see [67, Theorem 1.1]).
Let us assume Hypothesis II.49. Then for all $\lambda \in \delta$,

$$\sigma_{\mathrm{ess}}\big(D(\lambda)\big) = [-\rho, \rho], \quad \rho = \frac{1}{2}\left\|\Sigma(\lambda;\ A+B,\ A;\ I,\ \delta) - I(\lambda)\right\|_{\mathrm{op}},$$

where $I(\lambda)$ stands for the identity operator in the fiber $\mathfrak{H}(\lambda)$ of the von Neumann direct integral (II.15).

For the next result from [**67**], we need the following hypothesis (stronger than Hypothesis II.49).

Hypothesis II.52 (cf. [67, Hypothesis 1.2]).
We suppose that there exists an open interval δ included in the absolutely continuous spectrum of A. Next, we assume that the operator G is Hilbert Schmidt. We further suppose that the derivatives

$$\dot{h}_0(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}h_0(\lambda) \quad \text{and} \quad \dot{h}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}h(\lambda)$$

exist in the trace norm for all $\lambda \in \delta$, and that the maps $\delta \ni \lambda \mapsto \dot{h}_0(\lambda)$ and $\delta \ni \lambda \mapsto \dot{h}(\lambda)$ are Hölder continuous (with some positive exponent) in the trace norm.

The following result of Pushnitski holds:

Theorem II.53 (see [67, Theorem 1.2]).
Let us assume Hypothesis II.52. Then for all $\lambda \in \delta$, the absolutely continuous part of $D(\lambda)$ is unitarily equivalent to a direct sum of multiplication operators by the independent variable in

$$\mathsf{L}^2([-\rho_n, \rho_n]) \quad \textit{with} \quad \rho_n = \frac{1}{2}\left|\mathrm{e}^{\mathrm{i}\theta_n(\lambda)} - 1\right| = \sin\big(\theta_n(\lambda)/2\big),$$

where the numbers $\mathrm{e}^{\mathrm{i}\theta_n(\lambda)}$ denote the eigenvalues of the scattering matrix $\Sigma(\lambda)$ distinct from 1 (with multiplicity taken into account). There may be finitely or infinitely many of these eigenvalues.

We will need the following assumptions:

Hypothesis II.54 (cf. [71, Assumption 2.2]).
Let $\Delta \subset \mathbb{R}$ be a compact interval. We assume that the spectrum of A in Δ is purely absolutely continuous with a constant multiplicity $N_0 \in \mathbb{N} \cup \{\infty\}$. Moreover, we suppose that G is "strongly A-smooth" (see Definition II.55) on Δ with some exponent $\alpha \in (0,1]$.

Definition II.55 (Strong A-smoothness; cf. [71, p. 1954]).
Let us assume that Δ and N_0 are as in Hypothesis II.54. Let $U : \mathrm{Ran}E_\Delta(A) \to \mathsf{L}^2(\Delta; \mathfrak{G})$, $\dim \mathfrak{G} = N_0$, be a unitary operator such that for all $\eta \in \mathrm{Ran}E_\Delta(A)$,

$$(UA\eta)(\lambda) = \lambda\,(U\eta)(\lambda) \qquad (\lambda \in \Delta).$$

Then G is called *strongly A-smooth* on Δ with some exponent $\alpha \in (0,1]$ if the operator

$$G_\Delta = GE_\Delta(A) : \mathrm{Ran}E_\Delta(A) \to \mathfrak{H}$$

satisfies the equation

$$\big(UG_\Delta^*\psi\big)(\lambda) = Z(\lambda)\psi \quad \text{for all } \psi \in \mathfrak{H},\ \lambda \in \Delta,$$

where $Z(\lambda) : \mathfrak{H} \to \mathfrak{G}$, $\lambda \in \Delta$, is a bounded family (with respect to the operator norm) of compact operators such that the map $\Delta \ni \lambda \mapsto Z(\lambda)$ is Hölder continuous with exponent α in the operator norm.

Before we state [**71**, Theorem 2.6], we cite two auxiliary results:

Proposition II.56 (see [71, Proposition 2.3]; cf. also [72, Proposition 2.4]).
Let us assume Hypothesis II.54. Then the operator-valued function $T(z) = G(A - zI)^{-1}G^*$ is Hölder continuous in the operator norm for all z with $\mathrm{Re}\, z$ in the interior of Δ and $\mathrm{Im}\, z \geq 0$. The set $\mathcal{M} \subset \Delta$ where the equation

$$\psi + \lim_{\varepsilon \to 0+} T(\lambda + \mathrm{i}\varepsilon)\mathrm{sign}(B)\psi = 0$$

has a nontrivial solution is closed and has Lebesgue measure 0. Let Ω be the complement of $\mathcal{M}$ in the interior of Δ. Then the operator

$$I + \lim_{\varepsilon \to 0+} T(\lambda + \mathrm{i}\varepsilon)\mathrm{sign}(B),$$

where $\lim_{\varepsilon \to 0+} T(\lambda + \mathrm{i}\varepsilon)$ converges with respect to the operator norm, is invertible for all $\lambda \in \Omega$.

Proof. We combine [**86**, Theorem 7, p. 137], [**86**, Theorem 2, p. 146–147], and [**86**, Theorem 3, p. 147] □

Proposition II.57 (see [71, Proposition 2.4]).
Let us assume Hypothesis II.54. Then the local wave operators $W_\pm(A+B,\ A;\ I,\ \Delta)$ exist and are complete. Moreover, the spectrum of $A+B$ in Ω is purely absolutely continuous.

Proof. We combine [**86**, Theorem 7, p. 137], [**86**, Theorem 2, p. 144], and [**71**, Lemma A.1]. □

Remark II.58. Let us assume Hypothesis II.54. Then $\Sigma(\lambda;\ A+B,\ A;\ I,\ \Delta) - I(\lambda)$ is compact for every $\lambda \in \Omega$ (Pushnitski and Yafaev deduce that from a stationary representation for the scattering matrix, see [**71**, Proposition 2.5]). Here $I(\lambda)$ stands for the identity operator in the fiber $\mathfrak{H}(\lambda)$ of the von Neumann direct integral (II.15) with Δ in place of δ.

The following result of Pushnitski and Yafaev holds:

Theorem II.59 (see [71, Theorem 2.6]).
Let us assume Hypothesis II.54. Let $\lambda \in \Omega$, where Ω is as in Proposition II.56. We denote by $\mathrm{e}^{\mathrm{i}\theta_n(\lambda)}$ the eigenvalues of the scattering matrix $\Sigma(\lambda)$ distinct from 1 (with multiplicity taken into account). There may be finitely or infinitely many of these eigenvalues. Then we have:

(1) The absolutely continuous part of $D(\lambda)$ is unitarily equivalent to a direct sum of multiplication operators by the independent variable in
$$\mathsf{L}^2([-\rho_n, \rho_n]) \quad \textit{with} \quad \rho_n = \frac{1}{2}\left|\mathrm{e}^{\mathrm{i}\theta_n(\lambda)} - 1\right| = \sin\bigl(\theta_n(\lambda)/2\bigr).$$
(2) The eigenvalues of $D(\lambda)$ can accumulate only to 0 and to the points $\pm\frac{1}{2}\left|\mathrm{e}^{\mathrm{i}\theta_n(\lambda)}-1\right|$. All eigenvalues of $D(\lambda)$ distinct from 0 and $\pm\frac{1}{2}\left|\mathrm{e}^{\mathrm{i}\theta_n(\lambda)}-1\right|$ have finite multiplicties.
(3) The singular continuous spectrum of $D(\lambda)$ is empty.

Remark II.60. Pushnitski and Yafaev also investigated (in the scattering theory framework) the spectral properties of differences $f(A+B) - f(A)$ in the case when f has finitely many jump discontinuities, see [**72**, Theorem 7.2].

The proofs of Theorems II.51, II.53, and II.59 use scattering theory and "model operators" related to Hankel operators.

Idea of proof of Theorem II.59. We start with:

First conclusions from Hypothesis II.54. The operator $\Sigma(\lambda) - I(\lambda)$ is compact for every $\lambda \in \Omega$ (see Remark II.58). As above, we denote by $\mathrm{e}^{\mathrm{i}\theta_n(\lambda)}$ the eigenvalues of the scattering matrix $\Sigma(\lambda)$ distinct from 1 (with multiplicity taken into account); there may be finitely (say, $N \in \mathbb{N}$) or infinitely many ($N = \infty$) of these eigenvalues.

We assume without loss of generality that $\Delta = [-1, 1]$ and $\lambda = 0 \in \Omega$ (cf. [**71**, p. 1957]). Next, we choose $a > 0$ such that $[-a, a] \subset \Omega$. By Hypothesis II.54 and Proposition II.57, we have
$$E_{\{0\}}(A) = E_{\{0\}}(A+B) = 0. \tag{II.16}$$

The spectrum of $D(0)^2$ (almost) determines the spectrum of $D(0)$. Let us decompose $\mathfrak{H}=\mathfrak{H}_0\oplus\mathfrak{H}_0^\perp$, where

$$\mathfrak{H}_0=\big[\mathrm{Ker}\big(D(0)-I\big)\oplus\mathrm{Ker}\big(D(0)+I\big)\big]^\perp.$$

It is a classical result (see, e. g., Proposition III.13 for a formulation of Davis' version [**21**, Theorem 6.1]) – which will also be used in Chapter IV of the present thesis – that:

(1) $\mathfrak{H}_0$ is reducing for $D(0)$;

(2) the restricted operators $D(0)\restriction_{\mathfrak{H}_0}$ and $(-D(0))\restriction_{\mathfrak{H}_0}$ are unitarily equivalent.

In order to analyze the spectral properties of $D(0)$, it is therefore sufficient to investigate the spectral properties of $D(0)^2$ and to compute the dimensions of $\mathrm{Ker}\big(D(0)-I\big)$ and $\mathrm{Ker}\big(D(0)+I\big)$, cf. [**71**, p. 1961].

A block diagonal decomposition of $D(0)^2$. In view of (II.16), a direct computation shows the following block diagonal representation of $D(0)^2$ (cf. [**71**, p. 1961]):

$$D(0)^2=\big((M_-)\restriction_{\mathrm{Ran}E_{\mathbb{R}_-}(A)}\big)\oplus\big((M_+)\restriction_{\mathrm{Ran}E_{\mathbb{R}_+}(A)}\big)\quad\text{on}\quad \mathrm{Ran}E_{\mathbb{R}_-}(A)\oplus\mathrm{Ran}E_{\mathbb{R}_+}(A),$$

where $\mathbb{R}_-=(-\infty,0)$, $\mathbb{R}_+=(0,\infty)$, and

$$M_+=E_{\mathbb{R}_+}(A)E_{\mathbb{R}_-}(A+B)E_{\mathbb{R}_+}(A)\quad\text{as well as}\quad M_-=E_{\mathbb{R}_-}(A)E_{\mathbb{R}_+}(A+B)E_{\mathbb{R}_-}(A).$$

Therefore, we can describe the spectral properties of $D(0)^2$ by investigating the spectral properties of M_+ and M_- separately.

Application of scattering theory to M_+. It turns out (see [**71**, pp. 1961–1962]) that the spectral properties of M_- and M_+ can be analyzed in a similar way; let us describe the idea of proof for M_+ (cf. also [**71**, p. 1957]). For this, a "model operator" M is contructed (see next paragraph). It is shown in [**71**, Sections 4–5] that M_+-M can be factorized with a strongly M-smooth operator (with some exponent $>1/2$) and a compact operator. By methods of scattering theory, Pushnitski and Yafaev conclude (see [**71**, p. 1962]) that the wave operators $W_\pm(M_+,M)$ exist and are complete. Subsequently, the spectral properties of M_+ can be described (see again [**71**, p. 1962]). Finally, let us note that the dimensions of $\mathrm{Ker}\big(D(0)-I\big)$ and of $\mathrm{Ker}\big(D(0)+I\big)$ cannot be infinite unless there exists an $n\in[1,N\rangle$ with $\frac{1}{2}\left|\mathrm{e}^{\mathrm{i}\theta_n}-1\right|=1$, see [**71**, p. 1961].

The "model operator" M and the role of Hankel operator theory. First, let us briefly recall the following:

Remark II.61 (Tensor product of bounded operators).
Let X_i be a bounded operator on a complex separable Hilbert space $\mathfrak{G}_i$, where $i=1,2$. We write $\mathfrak{G}_1\otimes\mathfrak{G}_2$ for the usual Hilbert space tensor product and $\mathfrak{G}_1\odot\mathfrak{G}_2$ for the algebraic tensor product of $\mathfrak{G}_1$ and $\mathfrak{G}_2$. In this situation,

$$(X_1\odot X_2)\bigg(\sum_{j=1}^{r}\chi_1^{(j)}\otimes\chi_2^{(j)}\bigg)=\sum_{j=1}^{r}\Big(X_1\chi_1^{(j)}\Big)\otimes\Big(X_2\chi_2^{(j)}\Big)\qquad\big(\chi_i^{(j)}\in\mathfrak{G}_i,\ r\in\mathbb{N}\big)$$

is a well-defined operator on $\mathfrak{G}_1\odot\mathfrak{G}_2\subset\mathfrak{G}_1\otimes\mathfrak{G}_2$ which is bounded with operator norm equal to $\|X_1\|_{\mathrm{op}}\|X_2\|_{\mathrm{op}}$ (see [**75**, Proposition 7.20, p. 156]). According to the **B**ounded **L**inear **T**ransformation theorem, we can thus uniquely extend $X_1\odot X_2$ to a bounded operator, denoted by $X_1\otimes X_2$ and called the *tensor product of* X_1 and X_2, on the whole of $\mathfrak{G}_1\otimes\mathfrak{G}_2$.

The Carleman operator, i. e., the Hankel integral operator on $\mathsf{L}^2(\mathbb{R}_+)$ with kernel function $1/(x+y)$, can be explicitly diagonalized. Let C_a be the integral operator on $\mathsf{L}^2(0,a)$ with kernel function $1/(\pi(x+y))$. Pushnitski and Yafaev call C_a a *half-Carleman operator* (see [**71**, p. 1957]). They explicitly diagonalize C_a; it has a simple purely absolutely continuous spectrum filling in the interval $[0,1]$, see [**71**, Lemma 3.1].

Let us note that the operator C_a^2 also has a simple purely absolutely continuous spectrum filling in the interval $[0,1]$, see [**71**, Lemma 3.4].

Essentially, the "model operator" is given by (see [**71**, p. 1960]) the tensor product

$$M_1 = C_a^2 \otimes \left[\frac{1}{4}\big(\Sigma(0) - I\big)\big(\Sigma(0)^* - I\big)\right] \quad \text{on} \quad \mathsf{L}^2((0,a);\mathfrak{G}) = \mathsf{L}^2(0,a) \otimes \mathfrak{G}.$$

We set $\mathfrak{H}(a) = \mathrm{Ran} E_{(0,a)}(A)$ and $U_a = U\restriction_{\mathfrak{H}(a)} : \mathfrak{H}(a) \to \mathsf{L}^2((0,a);\mathfrak{G})$. Now we define the "model operator" M by (see again [**71**, p. 1960]) the block diagonal operator

$$M = (U_a^* M_1 U_a) \oplus 0 \quad \text{on} \quad \mathfrak{H}(a) \oplus \mathfrak{H}(a)^{\perp}.$$

Since U_a is unitary, the spectral properties of M are as follows (see [**71**, Theorem 3.5]): *Apart from the eigenvalue* 0*, the spectrum of* M *is absolutely continuous. It is given by*

$$\sigma_{\mathrm{ac}}(M) = \bigcup_{n=1}^{N} \left[0, \frac{1}{4}\left|\mathrm{e}^{\mathrm{i}\theta_n} - 1\right|^2\right]; \tag{II.17}$$

each of the intervals in the union on right hand side of (II.17) contributes multiplicity 1 *to the spectrum.*

This completes the description of the idea of proof of Theorem II.59.

Remark II.62 (The role of Hankel operator theory).
In the next chapter, we will present (in Theorem III.4) the characterization theorem [**56**, Theorem 1] for bounded self-adjoint Hankel operators, which is an important tool for the research of the author of the present thesis. According to that characterization theorem, C_a and C_a^2 are unitarily equivalent to a bounded self-adjoint Hankel operator.

Finally, we note:

Remark II.63. The results of Theorems II.51, II.53, and II.59 are related to some results of Chapter IV of the present thesis. However, no scattering theory is used in Chapter IV.

CHAPTER III

The main question. Presentation of the main tools

As we have seen in Chapter II, it is very important to investigate differences of the type $f(A+B)-f(A)$, where A and B are self-adjoint operators on Hilbert space and f is a scalar-valued Borel function on $\mathbb{R}$. In the present thesis, we focus on the case when $f=\mathbb{1}_{(-\infty,\lambda)}$ is the characteristic function of the open interval $(-\infty,\lambda)$ (in particular, f is not continuous).

In Section III.1, we pose the main question. This question is taken from the paper [**83**] by the present author (the results of [**83**] will be discussed in Chapter IV). In Sections III.2 and III.3, we present the main tools to investigate the properties of $E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A)$.

III.1. The main question

Throughout this chapter, $\mathfrak{H}$ stands for a complex separable Hilbert space of infinite dimension.

Inspired by M. Krein's example (see Section II.3 above), we pose the following question.

Question III.1 (see [83, Question 1]).
Let $\lambda\in\mathbb{R}$. Is it true that the difference of the spectral projections acting on $\mathfrak{H}$,

$$D(\lambda)=E_{(-\infty,\lambda)}(A+B)-E_{(-\infty,\lambda)}(A),$$

is unitarily equivalent to a bounded self-adjoint Hankel operator, provided that A is semibounded and B is of rank 1?

III.2. On a characterization theorem of Megretskii, Peller, and Treil

In this section, we present the characterization theorem for bounded self-adjoint Hankel operators; it was proved in 1995 by Megretskii, Peller, and Treil. We first need a preliminary consideration.

Preliminary consideration III.2 (cf. [56, p. 245] or [62, p. 490]).
Let T be a bounded self-adjoint operator acting on $\mathfrak{H}$ with finite scalar spectral measure μ and multiplicity function ν. We extend and modify ν as follows:

(1) The function ν is defined μ-almost everywhere on $\mathbb{R}$; we extend it by 0 to the whole of $\mathbb{R}$ and denote this extension again by ν.

(2) Let us define the Borel measure $\tilde{\mu}$ on $\mathbb{R}$ by $\tilde{\mu}(\Delta)=\mu(\Delta)+\mu(-\Delta)$, where $-\Delta=\{-t : t\in\Delta\}$. Clearly, μ is absolutely continuous with respect to $\tilde{\mu}$ so by the Radon–Nikodym theorem, there exists a nonnegative function $\rho\in\mathsf{L}^1(\mathbb{R},\tilde{\mu})$ such

that $\mu(\Delta) = \int_\Delta \rho(t)\, \mathrm{d}\tilde{\mu}(t)$ for all $\Delta \in \mathcal{B}(\mathbb{R})$. Evidently, $\rho^{-1}(\{0\}) \in \mathcal{B}(\mathbb{R})$ with

$$\mu\big(\rho^{-1}(\{0\})\big) = \int_{\rho^{-1}(\{0\})} \rho(t)\, \mathrm{d}\tilde{\mu}(t) = 0.$$

We define $\tilde{\nu}$ by modifying ν on the μ-null set $\rho^{-1}(\{0\})$ by 0:

$$\tilde{\nu}(t) = \begin{cases} \nu(t) & \text{if} \quad t \notin \rho^{-1}(\{0\}) \\ 0 & \text{if} \quad t \in \rho^{-1}(\{0\}) \end{cases}. \tag{III.1}$$

The function $\tilde{\nu}$ has the property that if $\delta \in \mathcal{B}(\mathbb{R})$ is of μ-measure 0, then $\tilde{\nu}$ vanishes $\tilde{\mu}$-almost everywhere on δ.

Remark III.3. Let $\tilde{\mu}$ be as in Preliminary consideration III.2. Then

$$\mathcal{B}(\mathbb{R}) \ni \Delta \mapsto \tilde{\mu}\big(\Delta \cap [0,\infty)\big)$$

is a finite scalar spectral measure of $|T|$. Indeed, we have $\tilde{\mu}(\mathbb{R} \cap [0,\infty)) \le 2\,\mu(\mathbb{R}) < \infty$ by assumption, and since it is well known (see, e.g., [**11**, Theorem 4, p. 158]) that the spectral measure of $|T|$ is given by

$$E_\Delta(|T|) = E_{\big(\Delta\cap[0,\infty)\big) \cup \big(-(\Delta\cap[0,\infty))\big)}(T) \qquad (\Delta \in \mathcal{B}(\mathbb{R})),$$

we easily obtain that $E(|T|)$ and $\mathcal{B}(\mathbb{R}) \ni \Delta \mapsto \tilde{\mu}(\Delta\cap[0,\infty))$ are type-equivalent (see p. 21).

We can now present the announced characterization theorem.

Theorem III.4 (see [56, Theorem 1]).
Let T be a bounded self-adjoint operator acting on $\mathfrak{H}$ with finite scalar spectral measure μ and multiplicity function ν. Let $\tilde{\nu}$ be the function obtained by extension and modification of ν according to Preliminary consideration III.2. Then T is unitarily equivalent to a Hankel operator if and only if the following three conditions are satisfied:

- *(C1) either* Ker $T = \{0\}$ *or* dim Ker $T = \infty$*;*
- *(C2) T is not boundedly invertible;*
- *($\overline{\text{C3}}$)* $|\tilde{\nu}(t) - \tilde{\nu}(-t)| \le 2$ μ_{ac}*-almost everywhere and* $|\tilde{\nu}(t) - \tilde{\nu}(-t)| \le 1$ μ_{s}*-almost everywhere.*

We have to explain the precise meaning of the inequalities in condition ($\overline{\text{C3}}$).

Remark III.5.

(I) If $\tilde{\nu}(t) = \infty$ or $\tilde{\nu}(-t) = \infty$, then ($\overline{\text{C3}}$) has to be understood as $\tilde{\nu}(t) = \tilde{\nu}(-t) = \infty$ (cf. [**56**, Remark on p. 249]).

(II) Let us show that *if ν_1 and ν_2 are two spectral multiplicity functions with respect to μ, then Theorem III.4 ($\overline{\text{C3}}$) is satisfied for $\tilde{\nu}_1$ if and only if it is satisfied for $\tilde{\nu}_2$.*

In order to prove this, we first choose μ-null sets δ_i $(i = 1,2,3)$ such that ν_1 is defined everywhere on $\mathbb{R} \setminus \delta_1$, ν_2 is defined everywhere on $\mathbb{R} \setminus \delta_2$, and

$$\nu_1(t) = \nu_2(t) \quad \text{for all } t \in \big(\mathbb{R} \setminus (\delta_1 \cup \delta_2)\big) \setminus \delta_3 = \mathbb{R} \setminus (\delta_1 \cup \delta_2 \cup \delta_3).$$

We set $\delta = \delta_1 \cup \delta_2 \cup \delta_3$. Then $\tilde{\nu}_1$ and $\tilde{\nu}_2$ coincide everywhere on $(\mathbb{R} \setminus \delta) \bigcup \rho^{-1}(\{0\})$; evidently, $(\mathbb{R} \setminus \delta) \bigcup \rho^{-1}(\{0\}) = \mathbb{R} \setminus \big(\delta \setminus \rho^{-1}(\{0\})\big)$. Since

$$0 = \mu\big(\delta \setminus \rho^{-1}(\{0\})\big) = \int_{\delta \setminus \rho^{-1}(\{0\})} \rho(t) \, \mathrm{d}\tilde{\mu}(t),$$

we know that $\delta \setminus \rho^{-1}(\{0\})$ has $\tilde{\mu}$-measure 0. We set

$$\tilde{\delta} = \Big(-\big(\delta \setminus \rho^{-1}(\{0\})\big)\Big) \bigcup \big(\delta \setminus \rho^{-1}(\{0\})\big);$$

then $\tilde{\delta} = -\tilde{\delta}$, $\mu\big(\tilde{\delta}\big) = 0$, and

$$\tilde{\nu}_1(t) = \tilde{\nu}_2(t) \quad \text{for every } t \in \mathbb{R} \setminus \tilde{\delta}.$$

Consequently, Theorem III.4 ($\overline{\text{C3}}$) is satisfied for $\tilde{\nu}_1$ if and only if it is satisfied for $\tilde{\nu}_2$, as claimed.

The following example shows that the modification made in Preliminary consideration III.2 (2) is indeed necessary.

Example III.6. Let ν be a multiplicity function with respect to the scalar spectral measure μ. Just as in Preliminary consideration III.2 (1), we extend ν by 0 to the whole of $\mathbb{R}$ and denote this extension again by ν.

We now define $\tilde{\nu}$ as in (III.1) and $\hat{\nu}$ by

$$\hat{\nu}(t) = \begin{cases} \nu(t) & \text{if} \quad t \notin \rho^{-1}(\{0\}) \\ 4711 & \text{if} \quad t \in \rho^{-1}(\{0\}) \end{cases}.$$

Let us assume that $\tilde{\mu}\big(\rho^{-1}(\{0\})\big) > 0$. Since $\mu\big(\rho^{-1}(\{0\})\big) = 0$, we have $\mu\big(-\rho^{-1}(\{0\})\big) > 0$ and thus

$$\Delta = \big(-\rho^{-1}(\{0\})\big) \setminus \rho^{-1}(\{0\})$$

has positive μ-measure. We note that

$$\tilde{\nu}(t) = \hat{\nu}(t) = \nu(t) \quad \text{for every } t \in \Delta \tag{III.2}$$

and

$$\tilde{\nu}(-t) = 0 \quad \text{as well as} \quad \hat{\nu}(-t) = 4711 \quad \text{for all } t \in \Delta. \tag{III.3}$$

Let us assume that condition ($\overline{\text{C3}}$) of Theorem III.4 is satisfied for $\tilde{\nu}$. Since $\mu(\Delta) > 0$, we can choose $\bullet \in \{\text{ac, s}\}$ such that $\mu_\bullet(\Delta) > 0$. By assumption, there exists a $\mu_\bullet$-null set $\tilde{\partial}$ such that

$$|\tilde{\nu}(t) - \tilde{\nu}(-t)| \leq 2 \quad \text{for every } t \in \Delta \setminus \tilde{\partial}.$$

Using (III.2) and (III.3), we obtain

$$0 \leq \tilde{\nu}(t) = \hat{\nu}(t) = \nu(t) \leq 2 \quad \text{for all } t \in \Delta \setminus \tilde{\partial}. \tag{III.4}$$

Let now $\hat{\partial}$ be an arbitrary $\mu_\bullet$-null set. Then (III.4) and (III.3) yield

$$|\hat{\nu}(t) - \hat{\nu}(-t)| = |\nu(t) - 4711| > 4000 \quad \text{for every } t \in \Delta \setminus \big(\tilde{\partial} \cup \hat{\partial}\big).$$

Consequently, condition ($\overline{\text{C3}}$) of Theorem III.4 is violated for $\hat{\nu}$.

In order to avoid such ambiguities, we need the modification in Preliminary consideration III.2 (2).

The next lemma shows that it is natural to modify ν as in (III.1).

Lemma III.7 (cf. [62, p. 491]).
Let T be a bounded self-adjoint operator acting on $\mathfrak{H}$ with finite scalar spectral measure $\mathfrak{p}$ and multiplicity function ν. Let $\tilde{\mathfrak{p}}$ and $\tilde{\nu}$ be as in Preliminary consideration III.2. Then if $\nu_{|T|}$ is a multiplicity function of $|T|$, we have

$$\nu_{|T|}(t) = \tilde{\nu}(t) + \tilde{\nu}(-t) \qquad \text{for } \tilde{\mathfrak{p}}\text{-almost all } t > 0. \tag{III.5}$$

Proof. We show the lemma in three steps.

Step 1. By Theorem I.92, the operator T on $\mathfrak{H}$ is unitarily equivalent to the multiplication operator M_t by the independent variable on a von Neumann direct integral $\int_{\mathbb{R}}^{\oplus} \mathfrak{H}(t)\,\mathrm{d}\mathfrak{p}(t)$. Let ρ be as in Preliminary consideration III.2. For brevity, we set

$$\Omega_+ = \big(\mathbb{R} \setminus \rho^{-1}(\{0\})\big) \cap [0,\infty) \qquad \text{and} \qquad \Omega_- = \big(\mathbb{R} \setminus \rho^{-1}(\{0\})\big) \cap (-\infty, 0).$$

Then we can identify

$$\int_{\mathbb{R}}^{\oplus} \mathfrak{H}(t)\,\mathrm{d}\mathfrak{p}(t) = \Big[\int_{\Omega_-}^{\oplus} \mathfrak{H}(t)\,\mathrm{d}\mathfrak{p}(t)\Big] \bigoplus \Big[\int_{\Omega_+}^{\oplus} \mathfrak{H}(t)\,\mathrm{d}\mathfrak{p}(t)\Big].$$

Let us note that on $\mathbb{R} \setminus \rho^{-1}(\{0\})$, the measures $\tilde{\mathfrak{p}}$ and $\mathfrak{p}$ are equivalent. Thus, according to Proposition I.83, M_t on $\int_{\Omega_\pm}^{\oplus} \mathfrak{H}(t)\,\mathrm{d}\mathfrak{p}(t)$ is unitarily equivalent to M_t on $\int_{\Omega_\pm}^{\oplus} \mathfrak{H}(t)\,\mathrm{d}\tilde{\mathfrak{p}}(t)$.

Step 2. Since $|T|$ on $\mathfrak{H}$ is unitarily equivalent to the block diagonal operator

$$|M_\tau| \oplus |M_t| \quad \text{on} \quad \Big[\int_{\Omega_-}^{\oplus} \mathfrak{H}(\tau)\,\mathrm{d}\tilde{\mathfrak{p}}(\tau)\Big] \bigoplus \Big[\int_{\Omega_+}^{\oplus} \mathfrak{H}(t)\,\mathrm{d}\tilde{\mathfrak{p}}(t)\Big],$$

we consider the components (1) $|M_\tau|$ and (2) $|M_t|$ separately.

(1) We choose a $\tilde{\mathfrak{p}}$-null set $\partial \subset \Omega_-$ such that $\mathfrak{H}(\tau)$ is defined for all $\tau \in \Omega_- \setminus \partial$ and set $\breve{\mathfrak{H}}(t) = \mathfrak{H}(-t)$ for each $t \in -(\Omega_- \setminus \partial)$. We obtain that $|M_\tau| = -M_\tau$ on $\int_{\Omega_-}^{\oplus} \mathfrak{H}(\tau)\,\mathrm{d}\tilde{\mathfrak{p}}(\tau)$ is unitarily equivalent to M_t on $\int_{-\Omega_-}^{\oplus} \breve{\mathfrak{H}}(t)\,\mathrm{d}\tilde{\mathfrak{p}}(t)$. One has $\dim \breve{\mathfrak{H}}(t) = \tilde{\nu}(-t)$ for $\tilde{\mathfrak{p}}$-almost every $t \in -\Omega_-$.

(2) Clearly, $|M_t| = M_t$ on $\int_{\Omega_+}^{\oplus} \mathfrak{H}(t)\,\mathrm{d}\tilde{\mathfrak{p}}(t)$. We have $\dim \mathfrak{H}(t) = \tilde{\nu}(t)$ for $\tilde{\mathfrak{p}}$-almost every $t \in \Omega_+$.

Step 3. We set

$$\mathfrak{K}(t) = \begin{cases} \mathfrak{H}(t) & \text{if } t \in \Omega_+ \setminus (-\Omega_-) \\ \breve{\mathfrak{H}}(t) & \text{if } t \in (-\Omega_-) \setminus \Omega_+ \\ \mathfrak{H}(t) \oplus \breve{\mathfrak{H}}(t) & \text{if } t \in \Omega_+ \cap (-\Omega_-) \end{cases}.$$

Then $|T|$ on $\mathfrak{H}$ is unitarily equivalent to M_t on $\int_{\Omega_+ \cup (-\Omega_-)}^{\oplus} \mathfrak{K}(t)\,\mathrm{d}\tilde{\mathfrak{p}}(t)$. Let us note that $\Omega_+ \cup (-\Omega_-)$ is a subset of $[0,\infty)$ with $\tilde{\mathfrak{p}}([0,\infty) \setminus (\Omega_+ \cup (-\Omega_-))) = 0$. In view of Remark III.3 and Step 2, we therefore conclude that (III.5) holds. □

Let us formulate the following "vector-valued" version of Theorem III.4.

Theorem III.8 (see [56, Theorem 2]).
Let T be a bounded self-adjoint operator acting on $\mathfrak{H}$ with finite scalar spectral measure μ and multiplicity function ν. Let $\tilde{\nu}$ be the function obtained by extension and modification of ν according to Preliminary consideration III.2. Then T is unitarily equivalent to a Hankel operator on $\mathsf{L}^2(\mathbb{R}_+;\mathbb{C}^N)$ if and only if the following three conditions are satisfied:

- *(C1)* *either* $\operatorname{Ker} T = \{0\}$ *or* $\dim \operatorname{Ker} T = \infty$*;*
- *(C2)* *T is not boundedly invertible;*
- *($\overline{\text{C3}_{\boldsymbol{N}}}$)* $|\tilde{\nu}(t) - \tilde{\nu}(-t)| \leq 2N$ *μ_{ac}-almost everywhere and* $|\tilde{\nu}(t) - \tilde{\nu}(-t)| \leq N$ *μ_{s}-almost everywhere.*

Remark III.9. If $\tilde{\nu}(t) = \infty$ or $\tilde{\nu}(-t) = \infty$, then ($\overline{\text{C3}_N}$) has to be understood as $\tilde{\nu}(t) = \tilde{\nu}(-t) = \infty$.

III.3. The main tools

Let us first briefly discuss a result which is often called *Halmos' decomposition theorem.* Our presentation follows Böttcher and Spitkovsky [**14**] (rather than Halmos [**37**]). Let P and Q be orthogonal projections on $\mathfrak{H}$. We write (cf. [**14**, p. 1413])

$$\operatorname{Ran} P = \big((\operatorname{Ran} P) \cap (\operatorname{Ran} Q)\big) \oplus \big((\operatorname{Ran} P) \cap (\operatorname{Ker} Q)\big) \oplus \mathfrak{M}_0$$

with some closed subspace $\mathfrak{M}_0$ of $\operatorname{Ran} P$ and, analogously,

$$\operatorname{Ker} P = \big((\operatorname{Ker} P) \cap (\operatorname{Ran} Q)\big) \oplus \big((\operatorname{Ker} P) \cap (\operatorname{Ker} Q)\big) \oplus \mathfrak{M}_1$$

with some closed subspace $\mathfrak{M}_1$ of $\operatorname{Ker} P$. We therefore obtain the following orthogonal decomposition (cf. [**14**, formula (1)]) of $\mathfrak{H}$:

$$\left.\begin{aligned} \mathfrak{H} = \big((\operatorname{Ran} P) \cap (\operatorname{Ran} Q)\big) &\oplus \big((\operatorname{Ran} P) \cap (\operatorname{Ker} Q)\big) \\ \oplus \big((\operatorname{Ker} P) \cap (\operatorname{Ran} Q)\big) &\oplus \big((\operatorname{Ker} P) \cap (\operatorname{Ker} Q)\big) \oplus \mathfrak{M}_0 \oplus \mathfrak{M}_1. \end{aligned}\right\} \tag{III.6}$$

In the formulation of Halmos' decomposition theorem (see Proposition III.11 below), we use the following common notation:

Notation III.10 (cf. [14, p. 1413]).
Let S be a self-adjoint operator on $\mathfrak{H}$, and let α and β be real numbers. We write

$$S \geq \alpha I \quad \text{if} \quad \langle S\psi, \psi\rangle_{\mathfrak{H}} \geq \alpha \langle \psi, \psi\rangle_{\mathfrak{H}} \quad \text{for all} \quad \psi \in \operatorname{Dom}(S)$$

and

$$S \leq \beta I \quad \text{if} \quad \langle S\psi, \psi\rangle_{\mathfrak{H}} \leq \beta \langle \psi, \psi\rangle_{\mathfrak{H}} \quad \text{for all} \quad \psi \in \operatorname{Dom}(S).$$

Moreover, $\alpha I \leq S \leq \beta I$ means that $\alpha \leq \beta$, $S \geq \alpha I$, and $S \leq \beta I$.

Proposition III.11 (see [14, Theorem 1.1]).
If one of the spaces $\mathfrak{M}_0$ and $\mathfrak{M}_1$ is nontrivial, then $\mathfrak{M}_0$ and $\mathfrak{M}_1$ have the same dimension and there exist a unitary operator R from $\mathfrak{M}_1$ onto $\mathfrak{M}_0$ and self-adjoint operators S, C on $\mathfrak{M}_0$ such that $0 \leq S \leq I$, $0 \leq C \leq I$, $S^2 + C^2 = I$, $\operatorname{Ker} S = \operatorname{Ker} C = \{0\}$, as well as, with respect to the decomposition (III.6),

$$P = I \oplus I \oplus 0 \oplus 0 \oplus \begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix},$$

$$Q = I \oplus 0 \oplus I \oplus 0 \oplus \begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix}.$$

Remark III.12 (to Proposition III.11).
"S and C are called the operator sine and cosine of the pair $(\mathfrak{M}_0, \mathfrak{M}_1)$" (see [**14**, p. 1415]). For more (also historical) information on this subject, we refer to [**14**, Section 1].

Most of the following material is taken, almost verbatim, from the author's paper [**83**, pp. 4–6].

We will need the following result of Davis on the difference of two orthogonal projections:

Proposition III.13 (see [21, Theorem 6.1]).
Let D be a bounded self-adjoint operator acting on $\mathfrak{H}$. Then D is the difference of two orthogonal projections if and only if the following two properties hold:

(1) $\sigma(D) \subset [-1, 1]$;

(2) the restricted operators $D\restriction_{\mathfrak{H}_0}$ and $(-D)\restriction_{\mathfrak{H}_0}$ are unitarily equivalent, where the closed subspace

$$\mathfrak{H}_0 = \big[\big(\text{Ker } (D - I)\big) \oplus \big(\text{Ker } (D + I)\big)\big]^{\perp} \tag{III.7}$$

of $\mathfrak{H}$ is reducing for D.

Combining Proposition III.13 with Theorem III.4, we obtain:

Theorem III.14 (cf. the formulation of [83, Theorem 2.2]).
The difference D of two orthogonal projections is unitarily equivalent to a bounded self-adjoint Hankel operator if and only if the following three conditions hold:

(C1) either $\text{Ker } D = \{0\}$ *or* $\dim \text{Ker } D = \infty$;

(C2) D is not boundedly invertible;

(C3) $|\dim \text{Ker}(D - I) - \dim \text{Ker}(D + I)| \leq 1$.

If $\dim \text{Ker}(D - I) = \infty$ or $\dim \text{Ker}(D + I) = \infty$, then (C3) has to be understood as $\dim \text{Ker}(D - I) = \dim \text{Ker}(D + I) = \infty$.

Remark III.15 (see [83, Remark 2.3]).
In order to answer Question III.1, we will investigate when the operator $D(\lambda)$ fulfills conditions (C1)–(C3) of Theorem III.14.

We have:

Proposition III.16 (see [83, Proposition 2.4]).
Let D be the difference of two orthogonal projections acting on $\mathfrak{H}$. Let

$$K = \widetilde{K} \oplus 0 \quad \text{on} \quad \widetilde{\mathfrak{H}} \oplus \mathfrak{H}_{\mathrm{g}}$$

be a compact self-adjoint block diagonal operator such that $\sigma(D+K) \subset [-1,1]$, where $\widetilde{\mathfrak{H}} = (\operatorname{Ker} D) \oplus \big(\operatorname{Ker}(D-I)\big) \oplus \big(\operatorname{Ker}(D+I)\big)$ and $\mathfrak{H}_{\mathrm{g}}$ is the orthogonal complement of $\widetilde{\mathfrak{H}}$ in $\mathfrak{H}$. Then $D+K$ is unitarily equivalent to a bounded self-adjoint Hankel operator if the following three conditions hold:

- *(C1) either* $\operatorname{Ker}(D+K) = \{0\}$ *or* $\dim \operatorname{Ker}(D+K) = \infty$*;*
- *(C2) $D+K$ is not boundedly invertible;*
- *(C3′)* $|\dim \operatorname{Ker}(D{\restriction}_{\widetilde{\mathfrak{H}}} + \widetilde{K} - tI) - \dim \operatorname{Ker}(D{\restriction}_{\widetilde{\mathfrak{H}}} + \widetilde{K} + tI)| \leq 1$ *for every* $0 < t \leq 1$.

If $\dim \operatorname{Ker}(D{\restriction}_{\widetilde{\mathfrak{H}}} + \widetilde{K} - tI) = \infty$ or $\dim \operatorname{Ker}(D{\restriction}_{\widetilde{\mathfrak{H}}} + \widetilde{K} + tI) = \infty$, then (C3′) has to be understood as $\dim \operatorname{Ker}(D{\restriction}_{\widetilde{\mathfrak{H}}} + \widetilde{K} - tI) = \dim \operatorname{Ker}(D{\restriction}_{\widetilde{\mathfrak{H}}} + \widetilde{K} + tI) = \infty$.

Proof of Proposition III.16. We decompose $D+K$ as follows:

$$D + K = \big(D{\restriction}_{\widetilde{\mathfrak{H}}} + \widetilde{K}\big) \oplus \big(D{\restriction}_{\mathfrak{H}_{\mathrm{g}}}\big) \quad \text{on} \quad \widetilde{\mathfrak{H}} \oplus \mathfrak{H}_{\mathrm{g}}.$$

First, let us consider the components (1) $D{\restriction}_{\widetilde{\mathfrak{H}}} + \widetilde{K}$ and (2) $D{\restriction}_{\mathfrak{H}_{\mathrm{g}}}$ separately.

(1) Since the essential spectrum of $D{\restriction}_{\widetilde{\mathfrak{H}}}$ consists of at most three points and $\widetilde{K}$ is compact, we know that both the absolutely continuous and the singular continuous spectrum of $D{\restriction}_{\widetilde{\mathfrak{H}}} + \widetilde{K}$ are empty.

(2) We observe that

$$D{\restriction}_{\mathfrak{H}_0} = \big(D{\restriction}_{\mathfrak{H}_{\mathrm{g}}}\big) \oplus 0 \quad \text{on} \quad \mathfrak{H}_{\mathrm{g}} \oplus (\operatorname{Ker} D),$$

where $\mathfrak{H}_0$ is defined as in (III.7) above. Consequently, Proposition III.13 yields that $D{\restriction}_{\mathfrak{H}_{\mathrm{g}}}$ and $(-D){\restriction}_{\mathfrak{H}_{\mathrm{g}}}$ are unitarily equivalent.

Now, an application of Theorem III.4 completes the proof. ☐

Remark III.17 (see [83, Remark 2.5]).
In order to show Theorem IV.2 (see below), we will construct an operator $K(\lambda)$ such that $D(\lambda) + K(\lambda)$ fulfills conditions (C1)–(C3′) of Proposition III.16.

More generally, we will need necessary and sufficient conditions for differences of two orthogonal projections to be unitarily equivalent to Hankel operators on $\mathsf{L}^2\big(\mathbb{R}_+; \mathbb{C}^N\big)$, where $N \in \mathbb{N}$. Combining Proposition III.13 with Theorem III.8, we obtain:

Theorem III.18 (cf. the formulation of [83, Theorem 2.6]).
The difference D of two orthogonal projections is unitarily equivalent to a bounded self-adjoint Hankel operator on $\mathsf{L}^2\big(\mathbb{R}_+; \mathbb{C}^N\big)$ if and only if the following three conditions hold:

- *(C1) either* $\operatorname{Ker} D = \{0\}$ *or* $\dim \operatorname{Ker} D = \infty$*;*
- *(C2) D is not boundedly invertible;*
- *(C3$_{\boldsymbol{N}}$)* $|\dim \operatorname{Ker}(D-I) - \dim \operatorname{Ker}(D+I)| \leq N$.

If $\dim \operatorname{Ker}(D-I) = \infty$ or $\dim \operatorname{Ker}(D+I) = \infty$, then (C3$_N$) has to be understood as $\dim \operatorname{Ker}(D-I) = \dim \operatorname{Ker}(D+I) = \infty$.

III.4. On the sufficiency of conditions (C1)–(C3) of Theorem III.14

If D is the difference of two orthogonal projections on $\mathfrak{H}$ such that conditions (C1)–(C3) of Theorem III.14 hold, then Megretskii, Peller, and Treil construct in [**56**, Section II.4] a Hankel operator T on $\mathsf{L}^2(\mathbb{R}_+)$ such that D and T are unitarily equivalent. Let us sketch the proof in the following special situation.

Hypothesis III.19. We suppose that D is the difference of two spectral projections $E_{(-\infty,\lambda)}(A+B)$ and $E_{(-\infty,\lambda)}(A)$ associated with the open interval $(-\infty,\lambda)$, where A is a semibounded self-adjoint operator on $\mathfrak{H}$, B is a self-adjoint rank one operator on $\mathfrak{H}$, and $\lambda \in \mathbb{R}$. We choose a finite scalar spectral measure $\mathbb{\mu}$ and a multiplicity function ν of D.

Let us note:

Remark III.20. In view of Proposition III.13, we have $\tilde{\nu}(t) = \tilde{\nu}(-t)$ for $\tilde{\mathbb{\mu}}$-almost all $t \in \sigma(|D|) \setminus \{1\}$, where $\tilde{\nu}$ is as in Preliminary consideration III.2. We can therefore assume without loss of generality that $\tilde{\nu}(t) \geq \tilde{\nu}(-t)$ for $\tilde{\mathbb{\mu}}$-almost every $t \in \sigma(|D|)$ (otherwise, we consider $-D$ instead of D).

Moreover, it turns out that the case when $\operatorname{Ker} D = \{0\}$ is of special importance, see Theorem IV.13 below. We therefore assume throughout the present section:

Hypothesis III.21. The operator D from Hypothesis III.19 has the following three additional properties:

(1) $\operatorname{Ker} D = \{0\}$;
(2) D fulfills conditions (C2)–(C3) of Theorem III.14;
(3) $\tilde{\nu}(t) \geq \tilde{\nu}(-t)$ for $\tilde{\mathbb{\mu}}$-almost every $t \in \sigma(|D|)$.

The first auxiliary operator. In this paragraph, we follow [**56**, pp. 262–263].

According to Remark III.3, $\mathcal{B}(\mathbb{R}) \ni \Delta \mapsto \tilde{\mathbb{\mu}}(\Delta \cap [0,\infty))$ is a finite scalar spectral measure of $|D|$. Since $\operatorname{Ker}|D| = \{0\}$ and $\sigma(|D|) \subset [0,1]$,

$$\mathcal{B}(\mathbb{R}) \ni \Delta \mapsto \mathbb{\rho}(\Delta) = \int_{\Delta\cap[0,\infty)} t^2 \, \mathrm{d}\tilde{\mathbb{\mu}}(t)$$

is also a finite scalar spectral measure of $|D|$. We have

$$\int_{\sigma(|D|)} \int_{\sigma(|D|)} \frac{1}{(t+\tau)^2} \, \mathrm{d}\mathbb{\rho}(t) \, \mathrm{d}\mathbb{\rho}(\tau) = \int_{\sigma(|D|)} \int_{\sigma(|D|)} \frac{t^2\tau^2}{(t+\tau)^2} \, \mathrm{d}\tilde{\mathbb{\mu}}(t) \, \mathrm{d}\tilde{\mathbb{\mu}}(\tau) \leq \frac{1}{4}\, \tilde{\mathbb{\mu}}(\mathbb{R})^2 < \infty.$$

One can show:

Lemma III.22 (see [56, Lemma 5.2, p. 268]).
We can multiply $\mathbb{\rho}$ by a positive weight function from $\mathsf{L}^1(\mathbb{R},\mathbb{\rho})$ to obtain an equivalent finite Borel measure $\mathbb{\rho}'$ on $\mathbb{R}$ such that

$$\int_{\sigma(|D|)} \int_{\sigma(|D|)} \frac{1}{(t+\tau)^2} \, \mathrm{d}\mathbb{\rho}'(t) \, \mathrm{d}\mathbb{\rho}'(\tau) < \infty \quad \textit{and} \quad \int_{\sigma(|D|)} \frac{1}{t} \, \mathrm{d}\mathbb{\rho}'(t) = \infty.$$

Notation III.23. We will write $\wp$ in place of $\wp'$. That is,

$$\int_{\sigma(|D|)} \int_{\sigma(|D|)} \frac{1}{(t+\tau)^2} \,\mathrm{d}\wp(t)\,\mathrm{d}\wp(\tau) < \infty \tag{III.8}$$

and

$$\int_{\sigma(|D|)} \frac{1}{t} \,\mathrm{d}\wp(t) = \infty. \tag{III.9}$$

By Theorem I.92, $|D|$ on $\mathfrak{H}$ is unitarily equivalent to the multiplication operator M_t by the independent variable on a von Neumann direct integral $\mathfrak{K} = \int_{\sigma(|D|)}^{\oplus} \mathfrak{G}(t)\,\mathrm{d}\wp(t)$. Let $\nu_{|D|}$ be a multiplicity function of $|D|$. We choose a $\wp$-null set δ such that $\dim \mathfrak{G}(t) = \nu_{|D|}(t)$ for every $t \in \sigma(|D|) \setminus \delta$. According to Remark I.76 and Proposition I.83, we can assume that the fibers $\mathfrak{G}(t)$ are imbedded into a single fixed separable Hilbert space of infinite dimension, $\widehat{\mathfrak{G}}$, with orthonormal basis $\{e_j : j \in \mathbb{N}\}$:

$$\mathfrak{G}(t) = \mathfrak{G}_{\nu_{|D|}(t)} = \overline{\operatorname{span}}\{e_j : j \in [1, \nu_{|D|}(t)\rangle\} \subset \widehat{\mathfrak{G}} \qquad (t \in \sigma(|D|) \setminus \delta).$$

Since $\operatorname{Ker} D = \{0\}$, Lemma III.7 yields (cf. also [**56**, p. 262])

$$\nu_{|D|}(t) = \tilde{\nu}(t) + \tilde{\nu}(-t) \qquad \text{for } \wp\text{-almost all } t \in \sigma(|D|).$$

According to (III.8), the integral operator (cf. [**56**, p. 263])

$$X_1 : \mathsf{L}^2(\sigma(|D|), \wp) \to \mathsf{L}^2(\sigma(|D|), \wp), \qquad \big(X_1\psi\big)(t) = \int_{\sigma(|D|)} \frac{-1}{t+\tau} \psi(t)\,\mathrm{d}\wp(\tau),$$

is Hilbert Schmidt. Let

$$\mathfrak{K} \supset \mathfrak{K}_1 = \left\{ \int_{\sigma(|D|)}^{\oplus} \psi(t) e_1 \,\mathrm{d}\wp(t) : \psi \in \mathsf{L}^2(\sigma(|D|), \wp) \right\}.$$

We define the operator X_0 on $\mathfrak{K} = \mathfrak{K}_1 \oplus (\mathfrak{K}_1)^{\perp}$ by (cf. again [**56**, p. 263])

$$\big(X_0\!\restriction_{\mathfrak{K}_1}\big)\left(\int_{\sigma(|D|)}^{\oplus} \psi(t) e_1 \,\mathrm{d}\wp(t)\right) = \int_{\sigma(|D|)}^{\oplus} \big(X_1\psi\big)(t) e_1 \,\mathrm{d}\wp(t) \quad \text{and} \quad X_0\!\restriction_{(\mathfrak{K}_1)^{\perp}} = 0. \tag{III.10}$$

Clearly, X_0 is Hilbert Schmidt.

Remark III.24. X_0 defined as in (III.10) is the first auxiliary operator.

The second auxiliary operator. In this paragraph, we follow [**56**, pp. 264 and 267].

For each $n \in \mathbb{N} \cup \{\infty\}$, let us choose strictly positive real numbers $a_j^{(n)}$, $j \in [1, n-1\rangle$, such that (see [**56**, (4.4), p. 264])

$$\sum_{j=1}^{n-1} \big(a_j^{(n)}\big)^2 < \frac{1}{2n^2} \quad \text{if } n \in \mathbb{N} \quad \text{and} \quad \sum_{j=1}^{\infty} \big(a_j^{(\infty)}\big)^2 < \infty. \tag{III.11}$$

Next, we define the matrices Y_n, $n \in \mathbb{N} \cup \{\infty\}$, with respect to the bases $(e_j)_{j \in [1,n\rangle}$ by (cf. [**56**, p. 267])

$$Y_1 = (0), \quad Y_2 = \begin{pmatrix} 0 & a_1^{(2)} \\ -a_1^{(2)} & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & a_1^{(3)} & 0 \\ -a_1^{(3)} & 0 & a_2^{(3)} \\ 0 & -a_2^{(3)} & 0 \end{pmatrix}, \quad \ldots$$

In view of (III.11), Y_n is a Hilbert Schmidt operator on $\overline{\operatorname{span}}\{e_j : j \in [1,n\rangle\}$ for every $n \in \mathbb{N}\cup\{\infty\}$ and, if $n \in \mathbb{N}$, the Hilbert Schmidt norm of Y_n is at most $1/n$, see [**56**, p. 267]. Hence, an application of Proposition I.86 yields that

$$Y = \int_{\sigma(|D|)}^{\oplus} Y_{\nu_{|D|}(t)} \,\mathrm{d}\wp(t) \tag{III.12}$$

is a bounded operator on $\mathfrak{K} = \int_{\sigma(|D|)}^{\oplus} \mathfrak{G}_{\nu_{|D|}(t)} \,\mathrm{d}\wp(t)$.

Remark III.25.

(I) Y defined as in (III.12) is the second auxiliary operator. Megretskii, Peller, and Treil showed that $\sigma(Y)$ consists of at most countably many points that can accumulate only at 0 (see [**56**, p. 267]).

(II) Let us note that Y need *not* be compact. In order to see this, we identify

$$\int_{\sigma(|D|)}^{\oplus} Y_{\nu_{|D|}(t)} \,\mathrm{d}\wp(t) = \bigoplus_{n\in\operatorname{Ran}(\nu_{|D|})} M_{Y_n} \quad \text{on} \quad \bigoplus_{n\in\operatorname{Ran}(\nu_{|D|})} \mathsf{L}^2(\mathcal{Y}_n, \wp; \mathfrak{G}_n),$$

where M_{Y_n} denotes the multiplication operator by Y_n on $\mathsf{L}^2(\mathcal{Y}_n, \wp; \mathfrak{G}_n)$ with $\mathcal{Y}_n = \{t \in \sigma(|D|) : \nu_{|D|}(t) = n\}$. Let us assume that $\mathcal{Y}_2 = [0,1]$, $\mathsf{L}^2([0,1],\wp)$ is infinite dimensional, and

$$Y_2 = \begin{pmatrix} 0 & \frac{1}{3} \\ -\frac{1}{3} & 0 \end{pmatrix} \quad \text{on} \quad \mathfrak{G}_2 = \mathbb{C}^2.$$

Then for every function $\psi \in \mathsf{L}^2([0,1],\wp) \setminus \{0\}$, $\begin{pmatrix} -\mathrm{i}\psi \\ \psi \end{pmatrix}$ is an eigenvector of M_{Y_2} associated with the eigenvalue $\frac{\mathrm{i}}{3}$. Therefore, Y is not compact.

The integral kernel of the Hankel operator. Let $\mathfrak{K} = \int_{\sigma(|D|)}^{\oplus} \mathfrak{G}_{\nu_{|D|}(t)} \,\mathrm{d}\wp(t)$ be as before. If X is a bounded operator on $\mathfrak{K}$ such that

$$\lim_{y\to\infty} \|\exp(yX)g\|_{\mathfrak{K}} = 0$$

for every $g \in \mathfrak{K}$, then we call X *asymptotically stable.* Megretskii, Peller, and Treil showed:

Proposition III.26 (see [56, Theorem 4.2, p. 264]).
The operator $X = X_0 + Y$ on $\mathfrak{K}$ is asymptotically stable, where X_0 is the first and Y is the second auxiliary operator.

Megretskii, Peller, and Treil proved:

Proposition III.27 (cf. [56, Corollary on p. 264]).
Let D satisfy Hypothesis III.21. Then D is unitarily equivalent to $(T_\kappa)\restriction_{(\operatorname{Ker} T_\kappa)^\perp}$, where T_κ is the bounded self-adjoint Hankel operator on $\mathsf{L}^2(\mathbb{R}_+)$ with

$$\kappa : \mathbb{R}_+ \to \mathbb{R}, \qquad \kappa(y) = \big\langle \exp(yX)e_1, e_1\big\rangle_{\mathfrak{K}}, \tag{III.13}$$

and X is defined as in Proposition III.26.

Remark III.28. The function κ defined in (III.13) (cf. [**56**, p. 256]) is continuous and one has

$$\lim_{y\to 0+}\kappa(y)=\mathbb{p}(\sigma(|D|))<\infty \quad \text{as well as} \quad \lim_{y\to\infty}\kappa(y)=0.$$

We need one further result of Megretskii, Peller, and Treil.

Proposition III.29 (see [56, Theorem 5.1, p. 268]).
Let T_κ be as in Proposition III.27. Then $\operatorname{Ker} T_\kappa=\{0\}$ *if and only if* $\int_{\sigma(|D|)}\frac{1}{t}\,\mathrm{d}\mathbb{p}(t)=\infty$.

We can now conclude:

Corollary III.30. *If D satisfies Hypothesis III.21, then it is unitarily equivalent to a bounded self-adjoint Hankel operator.*

Proof. We combine Proposition III.27 with Proposition III.29 and (III.9). ☐

Remark III.31. One can associate to the operators

$$X=X_0+Y \text{ on } \mathfrak{K}, \quad M:\mathbb{C}\ni u\mapsto ue_1\in\mathfrak{K}, \quad \text{and} \quad C:\mathfrak{K}\ni g\mapsto\langle g,e_1\rangle_{\mathfrak{K}}\in\mathbb{C}$$

a so-called “linear dynamical system with continuous time” (cf. [**62**, pp. 466–467], see also [**56**, p. 256]):

$$\begin{cases}\dot g(y)=Xg(y)+Mu(y)\\[2ex] z(y)=Cg(y)\end{cases} \qquad (y\in\mathbb{R}).$$

Here u is interpreted as “input signal” and z as “output signal.” We refer to [**62**, Chapter 11] for more information on this subject.

CHAPTER IV

On the difference of spectral projections

This chapter is based on the paper [**83**] by the author of the present thesis. It is to emphasize that the results of [**83**] constitute the first pillar of the research of the present thesis.

Most of the following material is taken, almost verbatim, from the author's paper [**83**, pp. 1–4].

The setting is as follows. For a semibounded self-adjoint operator A and a compact self-adjoint operator B acting on a complex separable Hilbert space of infinite dimension, we study the difference $D(\lambda) = E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A)$, $\lambda \in \mathbb{R}$, of the spectral projections associated with the open interval $(-\infty, \lambda)$. As we discussed in Chapter II above, it is known that there is a relationship between the operator $D(\lambda)$ and Hankel operators. The present thesis is intended to add some more knowledge on this subject.

In the case when B is of rank 1, we show that $D(\lambda)$ is unitarily equivalent to a block diagonal operator $T(\lambda) \oplus 0$, where $T(\lambda)$ is a bounded self-adjoint Hankel operator, for all but at most countably many $\lambda \in \mathbb{R}$.

If, more generally, B is compact, then we obtain that $D(\lambda)$ is unitarily equivalent to $T(\lambda)+C(\lambda)$ for all but at most countably many $\lambda \in \mathbb{R}$, where $T(\lambda)$ is a bounded self-adjoint Hankel operator and $C(\lambda)$ is a compact self-adjoint operator.

Let us describe the structure of the present chapter. We start by presenting the main results, Theorems IV.1–IV.2. Next, in Section IV.2, we show a useful block diagonal decomposition of the operator $D(\lambda)$ and discuss why we cannot in general set $k = 0$ in the statement of Theorem IV.1. In particular, we formulate a more detailed version of Theorem IV.1, namely Theorem IV.1$'$. Then, in Sections IV.3–IV.6, we investigate whether the operator $D(\lambda)$ fulfills the conditions of Theorem III.14 or Proposition III.16. For this:

- we show that for every $\lambda \in \mathbb{R}$, the dimensions of $\operatorname{Ker}\big(D(\lambda) \pm I\big)$ do not exceed the rank of B, where I denotes the identity operator, see Theorem IV.8;
- we prove, under suitable assumptions on A and B, that $\operatorname{Ker} D(\lambda)$ is either trivial or infinite dimensional for all but at most two $\lambda \in \mathbb{R}$, see Theorem IV.13;
- we show that zero belongs to the essential spectrum of $D(\lambda)$ for all but at most countably many $\lambda \in \mathbb{R}$, see Theorem IV.24.

The proofs of Theorems IV.1$'$ and IV.2 are then performed in Subsection IV.6.3.

Moreover, it turns out that in view of Theorems III.14 and IV.8, we can answer Question III.1 in the affirmative whenever the kernel of $D(\lambda)$ is infinite dimensional. For this, we present sufficient conditions in Propositions IV.20 and IV.22.

In Section IV.7, we discuss some examples, including the almost Mathieu operator. Finally, we discuss two open problems.

IV.1. Main results

Most of the following material is taken, almost verbatim, from the author's paper [**83**, pp. 2–3].

Throughout this chapter, A stands for a semibounded self-adjoint operator acting on a complex separable Hilbert space $\mathfrak{H}$ of infinite dimension. As before, we denote the spectrum and the essential spectrum of A by $\sigma(A)$ and $\sigma_{\mathrm{ess}}(A)$, respectively. Furthermore, let us recall that $\operatorname{span}\{\varphi_j \in \mathfrak{H} : j \in \mathcal{J}\}$ is the (not necessarily closed) linear span generated by the vectors φ_j, $j \in \mathcal{J}$, where $\mathcal{J}$ is some index set. If there exists a vector $\varphi \in \mathfrak{H}$ such that

$$\overline{\operatorname{span}}\,\{E_\Omega(A)\varphi : \Omega \in \mathcal{B}(\mathbb{R})\} = \overline{\operatorname{span}\,\{E_\Omega(A)\varphi : \Omega \in \mathcal{B}(\mathbb{R})\}} = \mathfrak{H},$$

then φ is called *cyclic* for A. As before, $\mathcal{B}(\mathbb{R})$ stands for the sigma-algebra of Borel sets of $\mathbb{R}$. We note that in the case when A is bounded, φ is cyclic for A if and only if $\overline{\operatorname{span}}\,\{A^n\varphi : n \in \mathbb{N}_0\} = \mathfrak{H}$.

The following theorem is the main result of this chapter.

Theorem IV.1 (see [83, Theorem 2]).
Let A and B be two self-adjoint operators acting on $\mathfrak{H}$, where A is semibounded and B is of rank 1. Then there exists a number k in $\mathbb{N}_0$ such that for all λ in $\mathbb{R}$ except for at most countably many λ in $\sigma_{\mathrm{ess}}(A)$, the operator $D(\lambda)$ on $\mathfrak{H}$ is unitarily equivalent to a block diagonal operator $T(\lambda) \oplus 0$ on $\mathsf{L}^2(\mathbb{R}_+) \oplus \mathbb{C}^k$, where $T(\lambda)$ is a bounded self-adjoint Hankel operator on $\mathsf{L}^2(\mathbb{R}_+)$.

We emphasize that the number k in Theorem IV.1 does not depend on λ. We will discuss in Section IV.2 below why and in which situation we need to set $k \neq 0$ in Theorem IV.1.

If, more generally, B is compact, then we show the following version of Theorem IV.1.

Theorem IV.2 (see [83, Theorem 3]).
Let A and B be two self-adjoint operators acting on $\mathfrak{H}$, where A is semibounded and B is compact. Let $1/4 > a_1 > a_2 > \cdots > 0$ be an arbitrary decreasing null sequence of real numbers. Then for all λ in $\mathbb{R}$ except for at most countably many λ in $\sigma_{\mathrm{ess}}(A)$, there exist a compact self-adjoint operator $K(\lambda)$ on $\mathfrak{H}$ and a bounded self-adjoint Hankel operator $T(\lambda)$ on $\mathsf{L}^2(\mathbb{R}_+)$ with the following properties:

(1) $D(\lambda) + K(\lambda)$ on $\mathfrak{H}$ is unitarily equivalent to $T(\lambda)$ on $\mathsf{L}^2(\mathbb{R}_+)$;

(2) either $K(\lambda)$ is a finite rank operator or $\nu_m(\lambda)/a_m \to 0$ as $m \to \infty$, where $\nu_1(\lambda), \nu_2(\lambda), \ldots$ denote the nonzero eigenvalues of $K(\lambda)$ ordered by decreasing modulus (with multiplicity taken into account).

Moreover, we can always choose $K(\lambda)$ of finite rank if B is of finite rank.

Remark IV.3 (see [83, Remark 4]).
By putting $C(\lambda) = -W(\lambda)K(\lambda)\big(W(\lambda)\big)^*$, where $W(\lambda) : \mathfrak{H} \to \mathsf{L}^2(\mathbb{R}_+)$ is unitary such that $W(\lambda)\big(D(\lambda) + K(\lambda)\big)\big(W(\lambda)\big)^* = T(\lambda)$, we can easily reformulate Theorem IV.2 to obtain the assertion stated on p. 56.

In the main body of the chapter, however, we prefer to use the present formulation of Theorem IV.2 because this is more suitable for proving.

IV.2. On a useful block diagonal decomposition of $D(\lambda)$

The following material is taken, almost verbatim, from the author's paper [**83**, pp. 6–7].

In this section, we show a useful block diagonal decomposition of $D(\lambda)$ and discuss why we cannot in general set $k = 0$ in the statement of Theorem IV.1 above. In particular, we formulate a more detailed version of Theorem IV.1, namely Theorem IV.1′.

First, let us prove the announced block diagonal decomposition of $D(\lambda)$:

Lemma IV.4 (see [83, Lemma 3.1]).
Let A and B be two self-adjoint operators acting on $\mathfrak{H}$, where A is semibounded and B is of rank $N \in \mathbb{N}$. We write $B = \sum_{j=1}^{N} \alpha_j \langle \bullet, \varphi_j \rangle_{\mathfrak{H}} \varphi_j$, where $\varphi_1, \ldots, \varphi_N$ are pairwise orthogonal nonzero vectors and $\alpha_1, \ldots, \alpha_N$ are nonzero real numbers. Further, we set $\mathfrak{N} = \overline{\operatorname{span}}\,\{E_\Omega(A)\varphi_j : \Omega \in \mathcal{B}(\mathbb{R}),\, j = 1, \ldots, N\}$. Then for all $\lambda \in \mathbb{R}$, we can represent $D(\lambda)$ as follows:

$$D(\lambda) = \Big(E_{(-\infty,\lambda)}\big(A\restriction_{\mathfrak{N}} + B\restriction_{\mathfrak{N}}\big) - E_{(-\infty,\lambda)}\big(A\restriction_{\mathfrak{N}}\big)\Big) \oplus 0 \quad \text{on} \quad \mathfrak{N} \oplus \mathfrak{N}^{\perp}. \tag{IV.1}$$

In particular, $\mathfrak{N}^{\perp}$ is included in the kernel of $D(\lambda)$ for all $\lambda \in \mathbb{R}$.

Proof. It is well known that $\mathfrak{N}$ reduces the operator A if and only if the orthogonal projection onto $\mathfrak{N}$ commutes with the spectral projection $E_{(-\infty,t]}(A)$ for every $t \in \mathbb{R}$, see Lemma I.22. By definition of $\mathfrak{N}$ and the functional calculus, we thus obtain that $\mathfrak{N}$ reduces A.

Obviously, B is bounded and $B\restriction_{\mathfrak{N}^{\perp}} = 0$. Consequently, $\mathfrak{N}$ reduces B and thus also $A + B$. By the functional calculus, we see that (IV.1) holds. □

Remark IV.5. Let us note that the type of decomposition from Lemma IV.4 is well known, cf. [**39**, § 2].

Now, let us formulate the announced more detailed version of Theorem IV.1 above.

Theorem IV.1′ (see [83, Theorem 2′]).
Let A, B, and $\mathfrak{N}$ be as in Lemma IV.4 with $N = 1$. Then we have:

(1) if $\dim(\mathfrak{N}^{\perp}) = \infty$, then $D(\lambda)$ is unitarily equivalent to a bounded self-adjoint Hankel operator with infinite dimensional kernel for all λ in $\mathbb{R}$;

(2) if $\mathfrak{N}^{\perp} = \{0\}$, then $D(\lambda)$ is unitarily equivalent to a bounded self-adjoint Hankel operator for all λ in $\mathbb{R}$ except for at most countably many λ in $\sigma_{\mathrm{ess}}(A)$;

(3) if $\dim(\mathfrak{N}^{\perp}) = k \in \mathbb{N}$, then the operator

$$E_{(-\infty,\lambda)}\big(A\restriction_{\mathfrak{N}} + B\restriction_{\mathfrak{N}}\big) - E_{(-\infty,\lambda)}\big(A\restriction_{\mathfrak{N}}\big) \quad \text{on} \quad \mathfrak{N}$$

from the block diagonal decomposition (IV.1) of $D(\lambda)$ is unitarily equivalent to a bounded self-adjoint Hankel operator for all λ in $\mathbb{R}$ except for at most countably many λ in $\sigma_{\mathrm{ess}}(A)$.

In particular, we can answer Question III.1 in the affirmative for all but at most countably many λ whenever $\dim(\mathfrak{N}^{\perp}) \in \{0, \infty\}$.

Remark IV.6 (see [83, Remark 3.2]).

(I) Obviously, Theorem IV.1′ implies Theorem IV.1.

(II) In view of Theorem IV.1′, we observe that the case when we need to set $k \neq 0$ in Theorem IV.1 can only occur if $\dim(\mathfrak{N}^{\perp}) \in \mathbb{N}$.

(III) If A has no eigenvalues, then $\dim(\mathfrak{N}^{\perp}) \in \{0, \infty\}$ and thus we can put $k = 0$ in Theorem IV.1.

The following example illustrates that the case when we need to set $k \neq 0$ in Theorem IV.1 indeed occurs for every $k \in \mathbb{N}$.

Example IV.7 (see [83, Example 3.3]).
Essentially, this is an application of M. Krein's example (see Section II.3).

Let $0 < \lambda < 1$. We consider the bounded self-adjoint integral operators A_i, $i = 0, 1$, on $\mathsf{L}^2(\mathbb{R}_+)$ with kernel functions defined as in (II.7). We know that $A_0 - A_1$ is of rank 1 and that the difference $E_{(-\infty,\lambda)}(A_0) - E_{(-\infty,\lambda)}(A_1)$ is a Hankel operator with a simple purely absolutely continuous spectrum filling in the interval $[-1, 1]$. In particular, one has

$$\operatorname{Ker}\big(E_{(-\infty,\lambda)}(A_0) - E_{(-\infty,\lambda)}(A_1)\big) = \{0\}.$$

Let $k \in \mathbb{N}$. Now we consider block diagonal operators

$$X_i = A_i \oplus M : \mathsf{L}^2(\mathbb{R}_+) \oplus \mathbb{C}^k \to \mathsf{L}^2(\mathbb{R}_+) \oplus \mathbb{C}^k \qquad (i = 0, 1), \tag{IV.2}$$

where $M \in \mathbb{C}^{k \times k}$ is an arbitrary fixed self-adjoint matrix. Then one has

$$\dim \operatorname{Ker}\big(E_{(-\infty,\lambda)}(X_0) - E_{(-\infty,\lambda)}(X_1)\big) = k.$$

IV.3. On the dimensions of Ker$\big(\boldsymbol{D(\lambda)} \pm \boldsymbol{I}\big)$

The following material is taken, almost verbatim, from the author's paper [**83**, pp. 7–9].

We will show that the dimensions of $\operatorname{Ker}\big(D(\lambda) \pm I\big)$ do not exceed the rank of the perturbation B:

Theorem IV.8 (see [83, Theorem 4.1]).
Let A and B be two self-adjoint operators acting on $\mathfrak{H}$, where A is semibounded and B is of rank $N \in \mathbb{N}$. Then for all λ in $\mathbb{R}$, one has

$$\dim \operatorname{Ker}\big(D(\lambda) \pm I\big) \leq N. \tag{IV.3}$$

In particular, the operator $D(\lambda)$ satisfies condition ($C3_N$) of Theorem III.18 above for all $\lambda \in \mathbb{R}$.

Remark IV.9 (see [83, Remark 4.2]).
In view of Theorems III.18 and IV.8, $D(\lambda)$ is unitarily equivalent to a bounded self-adjoint Hankel operator on $\mathsf{L}^2\big(\mathbb{R}_+; \mathbb{C}^N\big)$ whenever the kernel of $D(\lambda)$ is infinite dimensional and B is of rank $N \in \mathbb{N}$. For this, we present sufficient conditions in Propositions IV.20 and IV.22.

First, we prove Theorem IV.8 in the case when A is bounded, see Lemma IV.10. Further below, in Subsection IV.6.2, we will trace the case when A is semibounded and unbounded back to the situation when A is bounded by means of resolvents.

At the end of the present section, we give an example which shows that the following case can occur:

$$\dim \operatorname{Ker}\big(D(\lambda) - I\big) = \operatorname{rank} B \quad \text{and} \quad \operatorname{Ker}\big(D(\lambda) + I\big) = \{0\}$$

for some $\lambda \in \mathbb{R}$. Then, in particular, equality holds in the inequality in condition (C3$_N$) of Theorem III.18 if B is of rank $N \in \mathbb{N}$.

Lemma IV.10 (see [83, Lemma 4.3]).
The statement of Theorem IV.8 holds in the case when A is bounded.

Proof. Let us write $P(\lambda) = E_{(-\infty,\lambda)}(A + B)$ and $Q(\lambda) = E_{(-\infty,\lambda)}(A)$.

We only show that $\dim \operatorname{Ker}(P(\lambda) - Q(\lambda) - I) \leq N$; the other inequality is proved analogously.

Let us assume for contradiction that there exists an orthonormal system $\psi_1, \ldots, \psi_{N+1}$ in $\operatorname{Ker}(P(\lambda) - Q(\lambda) - I)$. Then we can choose a normalized vector $\tilde{\psi}$ in

$$\operatorname{span}\{\psi_1, \ldots, \psi_{N+1}\} \cap (\operatorname{Ran} B)^{\perp} \neq \{0\}.$$

Hence, $P(\lambda)\tilde{\psi} = \tilde{\psi}$ and $Q(\lambda)\tilde{\psi} = 0$. This implies

$$\big\langle (A + B)\tilde{\psi}, \tilde{\psi} \big\rangle_{\mathfrak{H}} < \lambda \quad \text{and} \quad \big\langle A\tilde{\psi}, \tilde{\psi} \big\rangle_{\mathfrak{H}} \geq \lambda$$

so that

$$\lambda > \big\langle (A + B)\tilde{\psi}, \tilde{\psi} \big\rangle_{\mathfrak{H}} = \big\langle A\tilde{\psi}, \tilde{\psi} \big\rangle_{\mathfrak{H}} \geq \lambda,$$

which is a contradiction. □

Remark IV.11 (see [83, Remark 4.4]).
If we consider an unbounded self-adjoint operator A, then the proof of Lemma IV.10 does not work, because $\tilde{\psi}$ might not belong to the domain of A.

Here is the announced example:

Example IV.12 (see [83, Example 4.5]).

(1) We consider the bounded self-adjoint diagonal operator

$$A = \operatorname{diag}(-1, -1/2, -1/3, -1/4, \ldots) : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$$

and, for $N \in \mathbb{N}$, the self-adjoint diagonal operator

$$B = \operatorname{diag}(\underbrace{-1, \ldots, -1}_{N \text{ times}}, 0, \ldots) : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0).$$

Then B is of rank N, and we see that

$$\begin{aligned} &\dim \operatorname{Ker}\big(E_{(-\infty,\lambda)}(A + B) - E_{(-\infty,\lambda)}(A) - I\big) = N \\ &\text{and} \quad \operatorname{Ker}\big(E_{(-\infty,\lambda)}(A + B) - E_{(-\infty,\lambda)}(A) + I\big) = \{0\} \end{aligned}$$

for all $\lambda \in (-1 - 1/N, -1)$.

(2) Let $a_0 = -1$ and $a_1 = -1/2$. We consider the bounded self-adjoint diagonal operator

$$A = \operatorname{diag}\left(a_0, a_0 + \frac{1/2}{4}, a_1, a_1 + \frac{1/6}{4}, a_0 + \frac{1/2}{5}, a_1 + \frac{1/6}{5}, a_0 + \frac{1/2}{6}, \dots\right)$$

on $\ell^2(\mathbb{N}_0)$. Since $|a_0 - a_1| = 1/2$, it follows that the compact self-adjoint diagonal operator

$$B = -2\ \operatorname{diag}\left(0, \frac{1/2}{4}, 0, \frac{1/6}{4}, \frac{1/2}{5}, \frac{1/6}{5}, \frac{1/2}{6}, \dots\right) : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$$

is such that

$$(+) \quad \begin{cases} \dim \operatorname{Ker}\big(E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A) - I\big) = \infty \\ \text{and} \quad \operatorname{Ker}\big(E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A) + I\big) = \{0\} \end{cases}$$

for $\lambda \in \{-1, -1/2\}$.

Clearly, this example can be extended such that (+) holds for all λ contained in $\{-1, -1/2, -1/3, \dots\}$.

IV.4. On the dimension of Ker $D(\lambda)$

Most of the following material is taken, almost verbatim, from the author's paper [**83**, pp. 9–14].

In this section, we deal with the question whether the kernel of $D(\lambda)$ is either trivial or infinite dimensional (cf. condition (C1) of Theorems III.14 and III.18). We will show:

Theorem IV.13 (see [83, Theorem 5.1]).
Let A and B be two self-adjoint operators acting on $\mathfrak{H}$, where A is semibounded and B is of rank 1. We write $B = \alpha\langle\bullet, \varphi\rangle_{\mathfrak{H}}\, \varphi$, where $\alpha \in \mathbb{R}\setminus\{0\}$, and assume that the vector $\varphi \in \mathfrak{H}$ is cyclic for A. Then we have:

(1) If A is bounded, then the kernel of $D(\lambda)$ is:
  - *(a) infinite dimensional for all $\lambda \in \mathbb{R}\setminus[\min\sigma_{\mathrm{ess}}(A), \max\sigma_{\mathrm{ess}}(A)]$;*
  - *(b) trivial for every $\lambda \in (\min\sigma_{\mathrm{ess}}(A), \max\sigma_{\mathrm{ess}}(A))$.*

(2) If A has a purely discrete spectrum, then the kernel of $D(\lambda)$ is infinite dimensional for all $\lambda \in \mathbb{R}$.

(3) If A is bounded from below and unbounded from above and if the spectrum of A is not purely discrete, then the kernel of $D(\lambda)$ is:
  - *(a) infinite dimensional for all $\lambda < \min\sigma_{\mathrm{ess}}(A)$;*
  - *(b) trivial for every $\lambda > \min\sigma_{\mathrm{ess}}(A)$.*

(4) If A is bounded from above and unbounded from below and if the spectrum of A is not purely discrete, then the kernel of $D(\lambda)$ is:
  - *(a) infinite dimensional for all $\lambda > \max\sigma_{\mathrm{ess}}(A)$;*
  - *(b) trivial for every $\lambda < \max\sigma_{\mathrm{ess}}(A)$.*

In particular, we have

$$\textit{either} \quad \text{Ker } D(\lambda) = \{0\} \quad \textit{or} \quad \dim \text{Ker } D(\lambda) = \infty$$

for all $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(A)$ *and for all but at most two* $\lambda \in \sigma_{\text{ess}}(A)$.

Remark IV.14 (see [83, Remark 5.2]).

(I) Condition (C1) of Theorems III.14 and III.18 always holds if A has a purely discrete spectrum.

(II) According to Theorem IV.13 (1), the dimension of $\operatorname{Ker} D(\lambda)$ depends only on the position of λ with respect to the points $\min \sigma_{\text{ess}}(A)$ and $\max \sigma_{\text{ess}}(A)$. In particular, the dimension of $\operatorname{Ker} D(\lambda)$ does not depend on whether λ is a resolvent point of A, an eigenvalue of A, etc.

(III) The points $\min \sigma_{\text{ess}}(A)$ and $\max \sigma_{\text{ess}}(A)$ may both exist in $\mathbb{R}$ even if the operator A is semibounded and unbounded. We note that the dimension of $\operatorname{Ker} D(\lambda)$ depends only on the position of λ with respect to a *single* point if A is semibounded and unbounded and if the spectrum of A is not purely discrete, see Theorem IV.13 (3) and (4).

We prove Theorem IV.13 (1) in Subsection IV.4.1. Then, we present sufficient conditions such that the kernel of $D(\lambda)$ is infinite dimensional for all λ in $\mathbb{R}$, provided that the self-adjoint operator B is of finite rank, see Propositions IV.20 and IV.22. In particular, Proposition IV.20 implies Theorem IV.13 (2). Further below, in Subsection IV.6.2, we will show Theorem IV.13 (3)–(4).

IV.4.1. Proof of Theorem IV.13 (1). We start with some preparations.

We write (as in Section IV.3 above)

$$P(\lambda) = E_{(-\infty,\lambda)}(A+B) \quad \text{and} \quad Q(\lambda) = E_{(-\infty,\lambda)}(A),$$

where $\lambda \in \mathbb{R}$. We observe (cf. Proposition III.11) that the kernel of $D(\lambda) = P(\lambda) - Q(\lambda)$ is equal to the orthogonal sum of $(\text{Ran } P(\lambda)) \cap (\text{Ran } Q(\lambda))$ and $(\text{Ker } P(\lambda)) \cap (\text{Ker } Q(\lambda))$. Therefore, we investigate the dimensions of $(\text{Ran } P(\lambda)) \cap (\text{Ran } Q(\lambda))$ and $(\text{Ker } P(\lambda)) \cap (\text{Ker } Q(\lambda))$ separately.

In order to prove Theorem IV.13 (1), it suffices to show the following two lemmas.

Lemma IV.15 (see [83, Lemma 5.3]).
Let A and B be as in Theorem IV.13 (1), and let $\lambda \in \mathbb{R} \setminus \{\max \sigma_{\text{ess}}(A)\}$. *Then the dimension of* $(\text{Ran } P(\lambda)) \cap (\text{Ran } Q(\lambda))$ *is:*

(1) infinite if and only if $\lambda > \max \sigma_{\text{ess}}(A)$;

(2) zero if and only if $\lambda < \max \sigma_{\text{ess}}(A)$.

Lemma IV.16 (see [83, Lemma 5.4]).
Let A and B be as in Theorem IV.13 (1), and let $\lambda \in \mathbb{R} \setminus \{\min \sigma_{\text{ess}}(A)\}$. *Then the dimension of* $(\text{Ker } P(\lambda)) \cap (\text{Ker } Q(\lambda))$ *is:*

(1) infinite if and only if $\lambda < \min \sigma_{\text{ess}}(A)$;

(2) zero if and only if $\lambda > \min \sigma_{\text{ess}}(A)$.

Lemma IV.16 can be proved analogously to Lemma IV.15, so we will only show Lemma IV.15. For this, the following two results turn out to be very useful.

Proposition IV.17 (see [49, Theorem 2.1]).
Let $A = M_t$ be the multiplication operator by the independent variable on $\mathsf{L}^2(\mathbb{R}, \mathbb{\mu})$, where $\mathbb{\mu}$ is a Borel probability measure on $\mathbb{R}$. For $\varphi(t) = 1$ on $\mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$, we consider the perturbed operator $A + B_\alpha = M_t + \alpha\langle \bullet, \varphi\rangle_{\mathsf{L}^2(\mathbb{R},\mathbb{\mu})}\, \varphi$ on $\mathsf{L}^2(\mathbb{R}, \mathbb{\mu})$ and the Borel probability measure

$$\mathbb{\mu}_\alpha(\Omega) = \langle E_\Omega(A + B_\alpha)\varphi, \varphi\rangle_{\mathsf{L}^2(\mathbb{R},\mathbb{\mu})} \qquad \big(\Omega \in \mathcal{B}(\mathbb{R})\big).$$

Then $A + B_\alpha$ is unitarily equivalent to the multiplication operator by the independent variable on $\mathsf{L}^2(\mathbb{R}, \mathbb{\mu}_\alpha)$, $M_x = V_\alpha(A + B_\alpha)V_\alpha^*$, where $V_\alpha : \mathsf{L}^2(\mathbb{R}, \mathbb{\mu}) \to \mathsf{L}^2(\mathbb{R}, \mathbb{\mu}_\alpha)$ is given by

$$(V_\alpha f)(x) = f(x) - \alpha \int_{\mathbb{R}} \frac{f(x) - f(t)}{x - t} \mathrm{d}\mathbb{\mu}(t) \tag{IV.4}$$

on the dense subspace of continuously differentiable functions $f : \mathbb{R} \to \mathbb{C}$ with compact support.

Proposition IV.18 (see [38, Proposition 2.2]).
Let $\mathbb{\rho}_1$ be a finite Borel measure on $\mathbb{R}$ and let f be a real-valued function that belongs to $\mathsf{L}^1(\mathbb{R}, \mathbb{\rho}_1)$. We set $\mathbb{\rho}_2(\Omega) = \int_\Omega f(t)\, \mathrm{d}\mathbb{\rho}_1(t)$ for every $\Omega \in \mathcal{B}(\mathbb{R})$. Then we have

$$\lim_{\varepsilon \searrow 0} \frac{p_{\mathbb{\rho}_2}(\tau + \mathrm{i}\varepsilon)}{p_{\mathbb{\rho}_1}(\tau + \mathrm{i}\varepsilon)} = f(\tau) \quad \text{for } \mathbb{\rho}_1\text{-almost all } \tau \in \mathbb{R},$$

where the Poisson transform $p_{\mathbb{\rho}_\ell}$ of $\mathbb{\rho}_\ell$ is given by

$$p_{\mathbb{\rho}_\ell}(\tau + \mathrm{i}\varepsilon) = \varepsilon \int_{\mathbb{R}} \frac{\mathrm{d}\mathbb{\rho}_\ell(t)}{(\tau - t)^2 + \varepsilon^2} \qquad (\tau \in \mathbb{R},\ \varepsilon > 0,\ \ell = 1, 2).$$

Let us now show Lemma IV.15.

Proof of Lemma IV.15. The idea of this proof is essentially due to the author's supervisor, Vadim Kostrykin.

It will be useful to represent the rank one operator $B = \alpha\langle \bullet, \varphi\rangle_{\mathfrak{H}}\, \varphi$ such that the vector φ is normalized. This determines α uniquely since B is fixed. However, for consistency with the notation below ($\mathbb{\mu}_\alpha$, U_α, V_α), let us write B_α instead of B for the rest of this proof. Further, we will write $\mathsf{L}^2(\mathbb{\mu}_\alpha)$ in place of $\mathsf{L}^2(\mathbb{R}, \mathbb{\mu}_\alpha)$.

We show Lemma IV.15 in two steps. First, we follow Liaw and Treil [**49**, pp. 1948–1949] to represent the operators A and $A + B_\alpha$ such that Proposition IV.17 is applicable. Then, using the representations from Step 1 and some results from harmonic analysis, we can perform the main part of the proof of Lemma IV.15.

Step 1. We define the Borel probability measures $\mathbb{\mu}$ and $\mathbb{\mu}_\alpha$ on $\mathbb{R}$ by

$$\mathbb{\mu}(\Omega) = \langle E_\Omega(A)\varphi, \varphi\rangle_{\mathfrak{H}} \quad \text{and} \quad \mathbb{\mu}_\alpha(\Omega) = \langle E_\Omega(A + B_\alpha)\varphi, \varphi\rangle_{\mathfrak{H}} \qquad \big(\Omega \in \mathcal{B}(\mathbb{R})\big),$$

respectively. It is well known (see, e. g., [**75**, Proposition 5.18]) that there exist unitary operators $U : \mathfrak{H} \to \mathsf{L}^2(\mathbb{\mu})$ and $U_\alpha : \mathfrak{H} \to \mathsf{L}^2(\mathbb{\mu}_\alpha)$ such that $UAU^* = M_t$ is the multiplication operator by the independent variable on $\mathsf{L}^2(\mathbb{\mu})$, $U_\alpha(A + B_\alpha)U_\alpha^* = M_x$ is the multiplication operator by the independent variable on $\mathsf{L}^2(\mathbb{\mu}_\alpha)$, and one has both $(U\varphi)(t) = 1$ on $\mathbb{R}$ and

$(U_\alpha\varphi)(x) = 1$ on $\mathbb{R}$. Clearly, U and U_α are uniquely determined by these properties. By Proposition IV.17, the unitary operator $V_\alpha = U_\alpha U^* : \mathsf{L}^2(\mu) \to \mathsf{L}^2(\mu_\alpha)$ is given by

$$(V_\alpha f)(x) = f(x) - \alpha \int \frac{f(x)-f(t)}{x-t}\mathrm{d}\mu(t)$$

for all continuously differentiable functions $f : \mathbb{R} \to \mathbb{C}$ with compact support (see (IV.4)). Without loss of generality, we may assume that A is already the multiplication operator by the independent variable on $\mathsf{L}^2(\mu)$, i. e., we identify $\mathfrak{H}$ with $\mathsf{L}^2(\mu)$, A with UAU^*, as well as $A + B_\alpha$ with $U(A + B_\alpha)U^*$.

Step 2. The well-known fact (see, e. g., [**75**, Example 5.4]) that

$$\operatorname{supp} \mu_\alpha = \sigma(A + B_\alpha)$$

implies that the cardinality of $(\lambda, \infty) \cap \operatorname{supp} \mu_\alpha$ is infinite [resp. finite] if and only if $\lambda < \max \sigma_{\mathrm{ess}}(A)$ [resp. $\lambda > \max \sigma_{\mathrm{ess}}(A)$].

Case 1. The cardinality of $(\lambda, \infty) \cap \operatorname{supp} \mu_\alpha$ is finite.
Since $\lambda > \max \sigma_{\mathrm{ess}}(A)$, it follows that

$$\dim \operatorname{Ran} E_{[\lambda,\infty)}(A + B_\alpha) < \infty \quad \text{and} \quad \dim \operatorname{Ran} E_{[\lambda,\infty)}(A) < \infty.$$

Therefore, $\operatorname{Ran} E_{(-\infty,\lambda)}(A + B_\alpha) \cap \operatorname{Ran} E_{(-\infty,\lambda)}(A)$ is infinite dimensional.

Case 2. The cardinality of $(\lambda, \infty) \cap \operatorname{supp} \mu_\alpha$ is infinite.
If $\lambda \le \min \sigma(A)$ or $\lambda \le \min \sigma(A + B_\alpha)$, then $(\operatorname{Ran} P(\lambda)) \cap (\operatorname{Ran} Q(\lambda)) = \{0\}$, as claimed. Now we suppose that $\lambda > \min \sigma(A)$ and $\lambda > \min \sigma(A + B_\alpha)$.

Let $f \in (\operatorname{Ran} P(\lambda)) \cap (\operatorname{Ran} Q(\lambda))$. Then one has

$$f(t) = 0 \text{ for } \mu\text{-almost all } t \ge \lambda \quad \text{and} \quad (V_\alpha f)(x) = 0 \text{ for } \mu_\alpha\text{-almost all } x \ge \lambda.$$

We would like to show that $f = 0$. This is done in three steps.

Step 2.1. We choose a representative $\tilde{f}$ in the equivalence class of f such that $\tilde{f}(t) = 0$ for *all* $t \ge \lambda$. Let $r \in \left(0, \frac{\max \sigma_{\mathrm{ess}}(A) - \lambda}{3}\right)$. Since μ is a finite Borel measure on $\mathbb{R}$, we know that the set of continuously differentiable scalar-valued functions on $\mathbb{R}$ with compact support is dense in $\mathsf{L}^2(\mu)$ with respect to $\|\bullet\|_{\mathsf{L}^2(\mu)}$. Thus, a standard mollification argument shows that we can choose continuously differentiable functions $\tilde{f}_n : \mathbb{R} \to \mathbb{C}$ with compact support such that

$$\|\tilde{f}_n - \tilde{f}\|_{\mathsf{L}^2(\mu)} < 1/n \quad \text{and} \quad \tilde{f}_n(t) = 0 \text{ for all } t \ge \lambda + r, \quad n \in \mathbb{N}.$$

In particular, we may insert $\tilde{f}_n$ into formula (IV.4) and obtain

$$\left(V_\alpha \tilde{f}_n\right)(x) = \alpha \int_{(-\infty,\lambda+r)} \frac{\tilde{f}_n(t)}{x-t}\mathrm{d}\mu(t) \quad \text{for all } x \ge \lambda + 2r.$$

It is readily seen that

$$(Sg)(x) = \int_{(-\infty,\lambda+r)} \frac{g(t)}{x-t}\mathrm{d}\mu(t) \qquad (x \ge \lambda + 2r)$$

defines a bounded operator $S : \mathsf{L}^2\left(\mathbb{1}_{(-\infty,\lambda+r)}\mathrm{d}\mu\right) \to \mathsf{L}^2\left(\mathbb{1}_{[\lambda+2r,\infty)}\mathrm{d}\mu_\alpha\right)$ with operator norm at most $1/r$.

It is now easy to show that

$$\int_{(-\infty,\lambda]} \frac{\tilde{f}(t)}{x-t} \mathrm{d}\mu(t) = 0 \quad \text{for } \mu_\alpha\text{-almost all } x \geq \lambda + 2r. \tag{IV.5}$$

As $r \in \left(0, \frac{\max \sigma_{\mathrm{ess}}(A)-\lambda}{3}\right)$ in (IV.5) was arbitrary, we obtain that

$$\int_{(-\infty,\lambda]} \frac{\tilde{f}(t)}{x-t} \mathrm{d}\mu(t) = 0 \quad \text{for } \mu_\alpha\text{-almost all } x > \lambda.$$

From now on, we may assume without loss of generality that $\tilde{f}$ is real-valued.

Step 2.2. Let us consider the holomorphic function from $\mathbb{C} \setminus (-\infty, \lambda]$ to $\mathbb{C}$ defined by

$$z \mapsto \int_{(-\infty,\lambda]} \frac{\tilde{f}(t)}{z-t} \mathrm{d}\mu(t).$$

Since $\lambda < \max \sigma_{\mathrm{ess}}(A)$, the identity theorem for holomorphic functions implies that

$$\int_{(-\infty,\lambda]} \frac{\tilde{f}(t)}{z-t} \mathrm{d}\mu(t) = 0 \quad \text{for all } z \in \mathbb{C} \setminus (-\infty, \lambda]. \tag{IV.6}$$

In particular, the imaginary part of the left hand side of (IV.6) vanishes for every $z = \tau - \mathrm{i}\varepsilon$, where $\tau \in \mathbb{R}$ and $\varepsilon > 0$. This yields

$$\varepsilon \int_{(-\infty,\lambda]} \frac{\tilde{f}(t)}{(\tau-t)^2 + \varepsilon^2} \mathrm{d}\mu(t) = 0 \quad \text{for all } \tau \in \mathbb{R},\ \varepsilon > 0. \tag{IV.7}$$

Step 2.3. We define the finite positive Borel measure $\rho_1 : \mathcal{B}(\mathbb{R}) \to [0, \infty)$ and the finite signed Borel measure $\rho_2 : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ by

$$\rho_1(\Omega) = \int_{\Omega \cap (-\infty,\lambda]} \mathrm{d}\mu(t), \quad \rho_2(\Omega) = \int_{\Omega \cap (-\infty,\lambda]} \tilde{f}(t) \mathrm{d}\mu(t);$$

note that $\tilde{f}$ belongs to $\mathsf{L}^1(\mu)$. Let us denote the Poisson transform of ρ_ℓ by p_{ρ_ℓ},

$$p_{\rho_\ell}(\tau + \mathrm{i}\varepsilon) = \varepsilon \int_{\mathbb{R}} \frac{\mathrm{d}\rho_\ell(t)}{(\tau-t)^2 + \varepsilon^2} \qquad (\tau \in \mathbb{R},\ \varepsilon > 0,\ \ell = 1, 2).$$

By (IV.7), we know that

$$p_{\rho_2}(\tau + \mathrm{i}\varepsilon) = 0 \quad \text{for all } \tau \in \mathbb{R},\ \varepsilon > 0.$$

Furthermore, since ρ_1 is not the trivial measure, one has

$$p_{\rho_1}(\tau + \mathrm{i}\varepsilon) > 0 \quad \text{for every } \tau \in \mathbb{R},\ \varepsilon > 0.$$

Now Proposition IV.18 implies that

$$0 = \lim_{\varepsilon \searrow 0} \frac{p_{\rho_2}(\tau + \mathrm{i}\varepsilon)}{p_{\rho_1}(\tau + \mathrm{i}\varepsilon)} = \tilde{f}(\tau) \quad \text{for } \mu\text{-almost all } \tau \leq \lambda.$$

Hence, $\tilde{f}(\tau) = 0$ for μ-almost all $\tau \in \mathbb{R}$. We conclude that $(\operatorname{Ran} P(\lambda)) \cap (\operatorname{Ran} Q(\lambda))$ is trivial. This shows Lemma IV.15. □

Remark IV.19 (see [83, Remark 5.5]).
The proof of Lemma IV.15 only works if the self-adjoint operator A is bounded, because we use that the spectra of A and $A + B$ are sequentially compact to ensure that every

subset of infinite cardinality has an accumulation point. Subsequently, we can apply the identity theorem for holomorphic functions.

Proof of Theorem IV.13 (1). We combine Lemmas IV.15 and IV.16. □

IV.4.2. Sufficient conditions such that dim Ker $D(\lambda) = \infty$. Let us first show:

Proposition IV.20 (see [83, Proposition 5.6]).
Let A and B be two self-adjoint operators acting on $\mathfrak{H}$, where A is semibounded with a purely discrete spectrum and B is of rank $N \in \mathbb{N}$. Then the range of $D(\lambda)$ is finite dimensional for all $\lambda \in \mathbb{R}$. In particular, the statement of Theorem IV.13 (2) holds.

Remark IV.21 (see [83, Remark 5.7]).
In view of Theorem IV.8 (whose proof will be completed in Subsection IV.6.2 below), Proposition IV.20 and Theorem III.18 imply that $D(\lambda)$ is then unitarily equivalent to a finite rank self-adjoint Hankel operator on $\mathsf{L}^2(\mathbb{R}_+;\mathbb{C}^N)$ for *all* $\lambda \in \mathbb{R}$.

Proof of Proposition IV.20. We need to show that the range of

$$D(\lambda) = E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A) \tag{IV.8}$$

$$= E_{[\lambda,\infty)}(A) - E_{[\lambda,\infty)}(A+B) \tag{IV.9}$$

is finite dimensional for all $\lambda \in \mathbb{R}$.

First, we assume that A is bounded from below. By the invariance of the essential spectrum under compact perturbations, it is clear that the operator $A+B$ also has a purely discrete spectrum. Moreover, $A+B$ is bounded from below as well. Consequently, by (IV.8), the range of $D(\lambda)$ is finite dimensional for all $\lambda \in \mathbb{R}$.

In the case when A is bounded from above, the proof runs analogously, except that we now use (IV.9) instead of (IV.8). This shows Proposition IV.20. □

The following result provides more sufficient conditions such that the kernel of $D(\lambda)$ is infinite dimensional.

Proposition IV.22 (see [83, Proposition 5.8]).
Let A and B be two self-adjoint operators acting on $\mathfrak{H}$, where A is semibounded and B is of rank $N \in \mathbb{N}$. Then the kernel of $D(\lambda)$ is infinite dimensional for all $\lambda \in \mathbb{R}$ whenever at least one of the following three cases occurs for $X = A$ or for $X = A+B$:

(1) The spectrum of X contains an eigenvalue of infinite multiplicity. In particular, this pertains to the case when the range of X is finite dimensional.
(2) The spectrum of X contains infinitely many eigenvalues with multiplicity at least $N+1$.
(3) The spectrum of the restricted operator $X\restriction_{\mathfrak{E}^\perp}$ has multiplicity at least $N+1$ (not necessarily uniform), where $\mathfrak{E}$ is the set of all eigenvectors of X.

Remark IV.23 (see [83, Remark 5.9]).
In view of Theorem IV.8 (whose proof will be completed in Subsection IV.6.2 below), Proposition IV.22 and Theorem III.18 imply that $D(\lambda)$ is then unitarily equivalent to a

bounded self-adjoint Hankel operator on $\mathsf{L}^2(\mathbb{R}_+;\mathbb{C}^N)$ with infinite dimensional kernel for *all* $\lambda \in \mathbb{R}$.

Proof of Proposition IV.22. First, we suppose that there exists an eigenvalue λ_0 of $X = A$ with multiplicity $m \geq N+1$, i. e., $m \in \{N+1, N+2, \dots\} \cup \{\infty\}$. We set

$$\mathfrak{M} = \big(\operatorname{Ker}(A - \lambda_0 I)\big) \cap (\operatorname{Ran} B)^{\perp} \neq \{0\}.$$

It is easy to show that $\mathfrak{M}$ is a closed subspace of $\mathfrak{H}$ such that dim $\mathfrak{M} \geq m - N$.

It is well known that $\mathfrak{M}$ reduces the operator A if and only if the orthogonal projection onto $\mathfrak{M}$ commutes with the spectral projection $E_{(-\infty,t]}(A)$ for all $t \in \mathbb{R}$, see Lemma I.22. By definition of $\mathfrak{M}$ and the functional calculus, we thus obtain that $\mathfrak{M}$ reduces A.

Obviously, B is bounded and $B\restriction_{\mathfrak{M}} = 0$. Consequently, $\mathfrak{M}$ reduces B and thus also $A + B$. By the functional calculus, we see that $\mathfrak{M}$ is included in the kernel of $D(\lambda)$ for all $\lambda \in \mathbb{R}$.

It follows that the kernel of $D(\lambda)$ is infinite dimensional for all $\lambda \in \mathbb{R}$ whenever cases (1) or (2) occur for the operator $X = A$; if $X = A + B$, then the proof runs analogously.

Now, we suppose that case (3) occurs for $X = A$. Let us write

$$B = \sum_{j=1}^{N} \alpha_j \langle \bullet, \varphi_j \rangle_{\mathfrak{H}}\, \varphi_j : \mathfrak{H} \to \mathfrak{H},$$

where $\varphi_1, \dots, \varphi_N$ form an orthonormal system in $\mathfrak{H}$ and $\alpha_1, \dots, \alpha_N$ are nonzero real numbers. We set

$$\mathfrak{N} = \overline{\operatorname{span}}\,\{E_\Omega(A)\varphi_j : \Omega \in \mathcal{B}(\mathbb{R}),\ j = 1, \dots, N\} \subset \mathfrak{H}.$$

By Lemma IV.4, we know that the closed subspace $\mathfrak{N}^{\perp}$ is included in the kernel of $D(\lambda)$ for all $\lambda \in \mathbb{R}$. A standard proof using the theory of von Neumann direct integrals (see Section I.4, see in particular Theorem I.92) shows that $\mathfrak{N}^{\perp}$ is infinite dimensional.

If case (3) occurs for $X = A + B$, then one can proceed analogously. This completes the proof of Proposition IV.22. ☐

IV.5. On invertibility of $D(\lambda)$

The following material is taken, almost verbatim, from the author's paper [**83**, pp. 14–20].

We will prove that for all but at most countably many $\lambda \in \mathbb{R}$, the operator $D(\lambda)$ is not boundedly invertible:

Theorem IV.24 (see [83, Theorem 6.1]).
Let A and B be two self-adjoint operators acting on $\mathfrak{H}$, where A is semibounded and B is compact. Then the following assertions hold:

(1) If $\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(A)$, then $D(\lambda)$ is a compact operator. In particular, zero belongs to the essential spectrum of $D(\lambda)$.

(2) Zero belongs to the essential spectrum of $D(\lambda)$ for all but at most countably many λ in $\sigma_{\mathrm{ess}}(A)$.

Note that we cannot exclude the case that the exceptional set is dense in $\sigma_{\mathrm{ess}}(A)$.

Remark IV.25 (see [83, Remark 6.2]).

(I) Martínez-Avendaño and Treil showed "that given any compact subset of the complex plane containing zero, there exists a Hankel operator having this set as its spectrum" (see [**53**, p. 83]). Thus, Theorem IV.24 and [**53**, Theorem 1.1] lead to the following result: *For all λ in $\mathbb{R}$ except for at most countably many λ in $\sigma_{\mathrm{ess}}(A)$, there exists a Hankel operator $T(\lambda)$ such that $\sigma\big(T(\lambda)\big) = \sigma\big(D(\lambda)\big)$.*

(II) Radjavi showed in [**73**, Theorem 6] that every bounded self-adjoint operator Y (acting on a complex separable Hilbert space) with zero in the essential spectrum is a *self-commutator*, i. e., one has $Y = X^*X - XX^*$ for some bounded operator X. In view of Theorem IV.24, the difference $D(\lambda)$ is therefore a self-commutator for all but at most countably many λ in $\sigma_{\mathrm{ess}}(A)$.

In this section, we prove Theorem IV.24 in the case when the operator A is bounded. Further below, in Subsection IV.6.2, we will trace the case when A is semibounded and unbounded back to the situation when A is bounded by means of resolvents.

Clearly, in order to prove that Theorem IV.24 holds in the case when A is bounded, it suffices to show the following result:

Proposition IV.26 (see [83, Proposition 6.3]).
Let A and B be two self-adjoint operators acting on $\mathfrak{H}$, where A is bounded and B is compact. Then the following assertions hold:

(1) if $\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(A)$, then $D(\lambda)$ is a compact operator;

(2′) zero belongs to the essential spectrum of $D(\lambda)$ for all but at most countably many λ in $\mathbb{R}$.

In Subsection IV.5.1, we show Proposition IV.26 in the case when the range of B is finite dimensional. Then, if B is compact with infinite dimensional range, we modify the proof.

Under additional assumptions (see Subsection II.4.3), we can deduce from results due to Pushnitski and Yafaev that $D(\lambda)$ is not boundedly invertible.

IV.5.1. The case when the range of B is finite dimensional. Throughout this subsection, we consider a self-adjoint finite rank operator

$$B = \sum_{j=1}^{N} \alpha_j \langle \bullet, \varphi_j \rangle_{\mathfrak{H}} \varphi_j : \mathfrak{H} \to \mathfrak{H} \qquad (N \in \mathbb{N}),$$

where $\varphi_1, \ldots, \varphi_N$ form an orthonormal system in $\mathfrak{H}$ and $\alpha_1, \ldots, \alpha_N$ are nonzero real numbers.

For $X = A$ or $X = A + B$, we define the sets $\mathcal{M}(X)$ and $\mathcal{M}_-(X)$ by:

$$\mathcal{M}(X) = \{\lambda \in \sigma_{\mathrm{ess}}(X) : \text{there exist } \lambda_k^{\pm} \neq \lambda \text{ in } \sigma(X) \text{ such that } \lambda_k^- \nearrow \lambda,\ \lambda_k^+ \searrow \lambda\},$$
$$\mathcal{M}_-(X) = \{\lambda \in \sigma_{\mathrm{ess}}(X) : \text{there exist } \lambda_k^- \neq \lambda \text{ in } \sigma(X) \text{ such that } \lambda_k^- \nearrow \lambda\} \setminus \mathcal{M}(X).$$

The following well-known result (cf. the formulation of [**83**, Lemma 6.4]) shows that these sets do not depend on whether $X = A$ or $X = A + B$.

Lemma IV.27 (see [3, Proposition 2.1]; see also [5, p. 83]).
Let $\widetilde{A}$ and $\widetilde{B}$ be bounded self-adjoint operators acting on $\mathfrak{H}$. If $N = \dim \operatorname{Ran} \widetilde{B}$ is in $\mathbb{N}$ and $\mathcal{I} \subset \mathbb{R}$ is a nonempty interval included in the resolvent set of $\widetilde{A}$, then $\mathcal{I}$ contains no more than N eigenvalues of the operator $\widetilde{A} + \widetilde{B}$ (taking into account their multiplicities).

In view of this lemma and the fact that the essential spectrum is invariant under compact perturbations, we will write $\mathcal{M}$ instead of $\mathcal{M}(X)$ and $\mathcal{M}_-$ instead of $\mathcal{M}_-(X)$, where $X = A$ or $X = A + B$.

We have:

Lemma IV.28 (see [83, Lemma 6.5]).
Let $\lambda \in \mathbb{R} \setminus (\mathcal{M} \cup \mathcal{M}_-)$. Then $D(\lambda)$ is a trace class operator. In particular, Proposition IV.26 (1) holds in the case when the range of B is finite dimensional.

Proof. Since $\lambda \in \mathbb{R} \setminus (\mathcal{M} \cup \mathcal{M}_-)$, there exists an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ with compact support such that

$$E_{(-\infty,\lambda)}(A + B) - E_{(-\infty,\lambda)}(A) = f(A + B) - f(A).$$

In particular, we have $f \in \mathsf{B}^1_{\infty,1}(\mathbb{R})$ (cf. Definition II.8). Combining Proposition II.26 (1) with Theorem II.25, it follows that $D(\lambda)$ is a trace class operator. □

An analogous proof shows that $D(\lambda)$ is a trace class operator for λ in $\mathcal{M}_-$, provided that $E_{\{\lambda\}}(A + B) - E_{\{\lambda\}}(A)$ is of trace class.

Let us now prove:

Proposition IV.29 (see [83, Proposition 6.6]).
Assertion (2′) of Proposition IV.26 holds in the case when the range of B is finite dimensional, i. e., zero belongs to the essential spectrum of $D(\lambda)$ for all but at most countably many λ in $\mathbb{R}$.

In the proof of Proposition IV.29, we will use the notion of weak convergence for sequences of probability measures.

Definition IV.30 (see [83, Definition 6.7]; cf. also [42, Definition 13.12]).
Let $\mathfrak{X}$ be a metric space. A sequence $\nu_1, \nu_2, \dots$ of Borel probability measures on $\mathfrak{X}$ is said to converge *weakly* to a Borel probability measure ν on $\mathfrak{X}$ if

$$\lim_{n\to\infty} \int f \mathrm{d}\nu_n = \int f \mathrm{d}\nu \quad \text{for every bounded continuous function } f : \mathfrak{X} \to \mathbb{R}.$$

If $\nu_1, \nu_2, \dots$ converges weakly to ν, then we write $\nu_n \xrightarrow{\mathrm{w}} \nu$, $n \to \infty$.

In the proof of Proposition IV.29, we will also use the following equivalent formulation of weak convergence for sequences of probability measures.

Proposition IV.31 (see [42, Theorem 13.16 (Portmanteau)]).
Let $\mathfrak{X}$ be a metric space, and let $\nu, \nu_1, \nu_2, \dots$ be Borel probability measures on $\mathfrak{X}$. Then $\nu_n \xrightarrow{\mathrm{w}} \nu$ as $n \to \infty$ if and only if $\lim_{n\to\infty} \int f \mathrm{d}\nu_n = \int f \mathrm{d}\nu$ for every bounded Borel function $f : \mathfrak{X} \to \mathbb{R}$ with $\nu(\mathcal{U}_f) = 0$, where $\mathcal{U}_f$ denotes the set of points of discontinuity of f.

Proof of Proposition IV.29. First, we note that if $\lambda < \min\big(\sigma(A) \cup \sigma(A+B)\big)$ or $\lambda > \max\big(\sigma(A) \cup \sigma(A+B)\big)$, then $D(\lambda)$ is the zero operator, and there is nothing to show. So let us henceforth assume that $\lambda \geq \min\big(\sigma(A) \cup \sigma(A+B)\big)$ and $\lambda \leq \max\big(\sigma(A) \cup \sigma(A+B)\big)$.

We show Proposition IV.29 in four steps. The idea is to apply Weyl's criterion (see, e. g., [**75**, Proposition 8.11]) to a suitable sequence of normalized vectors. In this proof, we denote by $\|g\|_{\infty,\mathcal{K}}$ the supremum norm of a function $g : \mathcal{K} \to \mathbb{R}$, where $\mathcal{K}$ is a compact subset of $\mathbb{R}$, and by $\|\bullet\|_{\mathrm{op}}$ the usual operator norm on $\mathfrak{H}$.

Step 1. We choose a sequence $(\psi_n)_{n\in\mathbb{N}}$ of normalized vectors in $\mathfrak{H}$ such that

$$\begin{aligned}&\psi_1 \perp \{\varphi_j : j = 1, \dots, N\}, \ \psi_2 \perp \{\psi_1, \varphi_j, A\varphi_j : j = 1, \dots, N\}, \ \dots \ , \\ &\psi_n \perp \big\{\psi_1, \dots, \psi_{n-1}, \varphi_j, A\varphi_j, \dots, A^{n-1}\varphi_j : j = 1, \dots, N\big\}, \ \dots\end{aligned}$$

For every $n \in \mathbb{N}$, let us now define the Borel probability measures ν_n and $\tilde{\nu}_n$ on $\mathbb{R}$ by

$$\nu_n(\Omega) = \langle E_\Omega(A)\psi_n, \psi_n\rangle_{\mathfrak{H}}, \quad \tilde{\nu}_n(\Omega) = \langle E_\Omega(A+B)\psi_n, \psi_n\rangle_{\mathfrak{H}} \qquad \big(\Omega \in \mathcal{B}(\mathbb{R})\big).$$

It is easy to see that by Prohorov's theorem (see, e. g., [**59**, Proposition 7.2.3]), there exist a subsubsequence $\big(\psi_{n_{k_\ell}}\big)_{\ell\in\mathbb{N}}$ of $(\psi_n)_{n\in\mathbb{N}}$ and Borel probability measures ν and $\tilde{\nu}$ with support included in $\sigma(A)$ and $\sigma(A+B)$, respectively, such that

$$\nu_{n_{k_\ell}} \xrightarrow{\mathrm{w}} \nu \quad \text{and} \quad \tilde{\nu}_{n_{k_\ell}} \xrightarrow{\mathrm{w}} \tilde{\nu} \ \text{ as } \ \ell \to \infty.$$

In order to simplify our notation, let us denote the subsubsequences $\big(\psi_{n_{k_\ell}}\big)_{\ell\in\mathbb{N}}$, $\big(\nu_{n_{k_\ell}}\big)_{\ell\in\mathbb{N}}$, and $\big(\tilde{\nu}_{n_{k_\ell}}\big)_{\ell\in\mathbb{N}}$ again by $(\psi_n)_{n\in\mathbb{N}}$, $(\nu_n)_{n\in\mathbb{N}}$, and $(\tilde{\nu}_n)_{n\in\mathbb{N}}$.

Step 2. We put

$$\mathcal{N}_A = \{\mu \in \mathbb{R} : \nu(\{\mu\}) > 0\} \quad \text{and} \quad \mathcal{N}_{A+B} = \{\mu \in \mathbb{R} : \tilde{\nu}(\{\mu\}) > 0\}.$$

Then the set $\mathcal{N}_A \cup \mathcal{N}_{A+B}$ is at most countable. We henceforth assume that λ does not belong to $\mathcal{N}_A \cup \mathcal{N}_{A+B}$.

Step 3. Let us put $s = \min\big(\sigma(A) \cup \sigma(A+B)\big) - 1$. We consider the bounded continuous functions $f_m : \mathbb{R} \to \mathbb{R}$, $m \in \mathbb{N}$, defined by

$$f_m(t) = \big(1 + m(t-s)\big)\, \mathbb{1}_{[s-1/m,\, s]}(t) + \mathbb{1}_{(s,\, \lambda)}(t) + \big(1 - m(t-\lambda)\big)\, \mathbb{1}_{[\lambda,\, \lambda+1/m]}(t).$$

The figure below shows (qualitatively) the graph of f_m.

FIGURE 1. The graph of f_m (see [**83**, figure 1]).

For all $m \in \mathbb{N}$, we can choose polynomials $p_{m,k}$, $k \in \mathbb{N}$, such that

$$\|f_m - p_{m,k}\|_{\infty,\mathcal{K}} \to 0 \quad \text{as } k \to \infty, \tag{IV.10}$$

where $\mathcal{K} = \big[\min\big(\sigma(A) \cup \sigma(A+B)\big) - 10, \max\big(\sigma(A) \cup \sigma(A+B)\big) + 10\big]$. By construction of $(\psi_n)_{n\in\mathbb{N}}$ in Step 1, one has

$$p_{m,k}(A+B)\psi_n = p_{m,k}(A)\psi_n \quad \text{for all } n > \text{degree of } p_{m,k}. \tag{IV.11}$$

Let us note that $|\mathbb{1}_{(-\infty,\lambda)} - f_m|^2$ is a bounded Borel function which is continuous except for the set $\{\lambda\}$ with $\nu(\{\lambda\}) = \tilde{\nu}(\{\lambda\}) = 0$, for every $m \in \mathbb{N}$.

Step 4. Taken together, equation (IV.11) and the Portmanteau theorem (see, e. g., Proposition IV.31) imply

$$\begin{aligned}
&\limsup_{n\to\infty} \left\| \left(E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A)\right) \psi_n \right\|_{\mathfrak{H}} \\
&\leq \left(\int_{\mathbb{R}} |\mathbb{1}_{(-\infty,\lambda)}(t) - f_m(t)|^2 \mathrm{d}\nu(t) \right)^{1/2} \\
&\quad + \left(\int_{\mathbb{R}} |\mathbb{1}_{(-\infty,\lambda)}(t) - f_m(t)|^2 \mathrm{d}\tilde{\nu}(t) \right)^{1/2} \\
&\quad + \|f_m(A) - p_{m,k}(A)\|_{\mathrm{op}} \\
&\quad + \|f_m(A+B) - p_{m,k}(A+B)\|_{\mathrm{op}}
\end{aligned}$$

for all $m \in \mathbb{N}$ and all $k \in \mathbb{N}$. First, we send $k \to \infty$ and then we take the limit $m \to \infty$. As $m \to \infty$, the sequence $(|\mathbb{1}_{(-\infty,\lambda)} - f_m|^2)_{m\in\mathbb{N}}$ converges to 0 pointwise almost everywhere with respect to both ν and $\tilde{\nu}$. Therefore, (IV.10) and the dominated convergence theorem imply that

$$\limsup_{n\to\infty} \left\| \left(E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A)\right) \psi_n \right\|_{\mathfrak{H}} \leq 0$$

and thus

$$\lim_{n\to\infty} \left\| \left(E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A)\right) \psi_n \right\|_{\mathfrak{H}} = 0.$$

Recall that $(\psi_n)_{n\in\mathbb{N}}$ is an orthonormal sequence. Thus, an application of Weyl's criterion (see, e. g., [**75**, Proposition 8.11]) yields that $0 \in \sigma_{\mathrm{ess}}\big(D(\lambda)\big)$. This shows Proposition IV.29. ☐

Remark IV.32 (see [83, Remark 6.8]).
The proof of Proposition IV.29 only works if the self-adjoint operator A is bounded. For instance, we use the compactness of $\sigma(A)$ to uniformly approximate f_m by polynomials.

Moreover, if we consider an unbounded self-adjoint operator A, then it is unclear whether an orthonormal sequence $(\psi_n)_{n\in\mathbb{N}}$ as in Step 1 can be found in the domain of A.

IV.5.2. The case when the range of *B* is infinite dimensional. Throughout this subsection, we assume that B is a compact self-adjoint operator with infinite dimensional range.

Lemma IV.33 (see [83, Lemma 6.9]).
Assertion (1) of Proposition IV.26 holds in the case when B is compact with infinite dimensional range, i. e., if $\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(A)$, then $D(\lambda)$ is compact.

Proof. Since $D(\lambda) = 0$ for all $\lambda \le \zeta_0 = \min\big(\sigma(A) \cup \sigma(A+B)\big)$, we will henceforth assume that $\lambda > \zeta_0$. As $\lambda \notin \sigma_{\mathrm{ess}}(A)$, we can choose $\varepsilon > 0$ such that:

- $r_\lambda = \lambda - \zeta_0 - \varepsilon > 0$;
- $(\lambda - 2\varepsilon, \lambda + 2\varepsilon) \setminus \{\lambda\}$ is included in the resolvent sets of A and $A + B$.

Then we represent $D(\lambda)$ as follows:

$$D(\lambda) = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - \zeta_0| = r_\lambda} \big[(\zeta I - A - B)^{-1} - (\zeta I - A)^{-1}\big] \mathrm{d}\zeta, \tag{IV.12}$$

where we integrate counterclockwise around the circle centered at ζ_0 with radius $r_\lambda > 0$. Recall that we can approximate the integral in (IV.12) by Riemann–Stieltjes sums (in the operator norm).

By the second resolvent equation, we have

$$(\zeta I - A - B)^{-1} - (\zeta I - A)^{-1} = (\zeta I - A - B)^{-1} B (\zeta I - A)^{-1}$$

for every $\zeta \in \mathbb{C}$ with $|\zeta - \zeta_0| = r_\lambda$. Since the compact operators form a closed ideal (with respect to the operator norm) in the algebra of bounded operators, it thus follows that $D(\lambda)$ is compact, as claimed. □

Remark IV.34. We note that the result of Lemma IV.33 is well known, see [**70**, Proof of Proposition 2.1] for (almost) the same proof.

Let us now prove:

Proposition IV.35 (see [83, Proposition 6.10]).
Assertion (2′) of Proposition IV.26 holds in the case when B is compact with infinite dimensional range, i. e., zero belongs to the essential spectrum of $D(\lambda)$ for all but at most countably many λ in $\mathbb{R}$.

Proof. Let us write $B = \sum_{j=1}^\infty \alpha_j \langle \bullet, \varphi_j \rangle_{\mathfrak{H}}\, \varphi_j$, where $\varphi_1, \varphi_2, \dots$ is an orthonormal system in $\mathfrak{H}$ and $\alpha_1, \alpha_2, \dots$ are nonzero real numbers with $\alpha_j \to 0$ as $j \to \infty$. Since it might happen that $\{\varphi_1, \varphi_2, \dots\}^\perp$ is trivial, we need to modify the proof of Proposition IV.29 above. However, we can show the present proposition similarly to Proposition IV.29, so we will focus on the differences.

Difference 1. We choose an orthonormal sequence $(\psi_n)_{n \in \mathbb{N}}$ in $\mathfrak{H}$ as follows:

$$\begin{aligned} &\psi_1 \perp \varphi_1,\ \psi_2 \perp \{\psi_1, \varphi_1, \varphi_2, A\varphi_1, A\varphi_2\},\ \dots\ , \\ &\psi_n \perp \{\psi_1, \dots, \psi_{n-1}, \varphi_j, A\varphi_j, \dots, A^{n-1}\varphi_j : j = 1, \dots, n\},\ \dots \end{aligned}$$

By construction, we have

$$p(A + F_\ell)\psi_n = p(A)\psi_n \quad \text{for all } n > \max(\ell,\ \text{degree of } p), \tag{IV.11′}$$

where p is a polynomial, $\ell \in \mathbb{N}$, and $F_\ell = \sum_{j=1}^\ell \alpha_j \langle \bullet, \varphi_j \rangle_{\mathfrak{H}}\, \varphi_j$.

Difference 2. Now we continue as in the proof of Proposition IV.29 and estimate as follows, using (IV.11′) instead of (IV.11):

$$\begin{aligned}&\limsup_{n\to\infty}\left\|\big(E_{(-\infty,\lambda)}(A+B)-E_{(-\infty,\lambda)}(A)\big)\,\psi_n\right\|_{\mathfrak{H}}\\&\leq\left(\int_{\mathbb{R}}|\mathbb{1}_{(-\infty,\lambda)}(t)-f_m(t)|^2\mathrm{d}\nu(t)\right)^{1/2}\\&\quad+\left(\int_{\mathbb{R}}|\mathbb{1}_{(-\infty,\lambda)}(t)-f_m(t)|^2\mathrm{d}\tilde{\nu}(t)\right)^{1/2}\\&\quad+\|f_m(A)-p_{m,k}(A)\|_{\mathrm{op}}\\&\quad+\|f_m(A+B)-p_{m,k}(A+B)\|_{\mathrm{op}}\\&\quad+\|p_{m,k}(A+B)-p_{m,k}(A+F_\ell)\|_{\mathrm{op}}\end{aligned}$$

for all $k,\ell,m\in\mathbb{N}$, where $\|\bullet\|_{\mathrm{op}}$ denotes the operator norm. Clearly, $\|B-F_\ell\|_{\mathrm{op}}\to 0$ as $\ell\to\infty$. Therefore, $\|p_{m,k}(A+B)-p_{m,k}(A+F_\ell)\|_{\mathrm{op}}\to 0$ as $\ell\to\infty$.

Thus, analogously to the proof of Proposition IV.29, it follows that

$$\lim_{n\to\infty}\left\|\big(E_{(-\infty,\lambda)}(A+B)-E_{(-\infty,\lambda)}(A)\big)\,\psi_n\right\|_{\mathfrak{H}}=0.$$

Hence, an application of Weyl's criterion (see, e.g., [**75**, Proposition 8.11]) yields that $0\in\sigma_{\mathrm{ess}}\big(D(\lambda)\big)$. This completes the proof. □

IV.5.3. The smooth situation. Theorem IV.24 (whose proof will be completed in Subsection IV.6.2 below) ensures that, in the general situation, $D(\lambda)$ is not boundedly invertible, for all but at most countably many λ in $\sigma_{\mathrm{ess}}(A)$. Under certain additional assumptions (see Hypothesis II.49), we may deduce from Theorem II.51 that $D(\lambda)$ is not boundedly invertible.

Example IV.36 (see [83, Example 6.12]).
Again, let us consider M. Krein's example (see Section II.3). That is, $\mathfrak{H}=\mathsf{L}^2(\mathbb{R}_+)$, the initial operator $A=A_1$ is the integral operator with kernel function a_1 defined as in (II.7), and $B=-\langle\bullet,\varphi\rangle_{\mathsf{L}^2(\mathbb{R}_+)}\,\varphi$ with $\varphi(x)=\mathrm{e}^{-x}$. Put $\delta=(0,1)$. Then Pushnitski showed (see [**67**, pp. 229–230]) that, by Theorem II.51, one has $\sigma_{\mathrm{ess}}\big(D(\lambda)\big)=[-1,1]$ for all $0<\lambda<1$.

In particular, the operator $D(\lambda)$ fulfills condition (C2) of Theorem III.14 for every $0<\lambda<1$.

IV.6. Proofs of the main results

The following material is taken, almost verbatim, from the author's paper [**83**, pp. 20–25].

This section is devoted to the proofs of Theorems IV.1′ and IV.2. As we noted in Remark IV.6 above, Theorem IV.1′ immediately implies Theorem IV.1.

First, we observe that our key results (that we have shown so far) still hold if we consider $E_{(-\infty,\lambda]}(A)-E_{(-\infty,\lambda]}(A+B)$, the difference of the spectral projections associated with the closed interval $(-\infty,\lambda]$ instead of the open interval $(-\infty,\lambda)$, see Subsection IV.6.1. Then, we complete the proofs of Theorems IV.8, IV.13, and IV.24. Finally, in Subsection IV.6.3, we show Theorems IV.1′ and IV.2.

IV.6.1. An important remark.

Remark IV.37 (see [83, Remark 7.1]).
If we consider $E_{(-\infty,\lambda]}(A) - E_{(-\infty,\lambda]}(A+B)$, the difference of the spectral projections associated with the closed interval $(-\infty,\lambda]$ instead of the open interval $(-\infty,\lambda)$, then all assertions in Lemma IV.10, Theorem IV.13 (1), and Propositions IV.20, IV.22, and IV.26 remain true. All proofs can easily be modified.

IV.6.2. Completion of the proofs of Theorems IV.8, IV.13, and IV.24. In this subsection, we assume that the self-adjoint operator A is semibounded *and unbounded.* As before, we write

$$D(\lambda) = E_{(-\infty,\lambda)}(A+B) - E_{(-\infty,\lambda)}(A)$$

if B is a compact self-adjoint operator and $\lambda \in \mathbb{R}$. In the following, we use M. Krein's approach from [**44**, pp. 622–623] to trace the case when A is semibounded and unbounded back to the situation when A is bounded by means of resolvents.

First, let us consider the case when A is bounded from below. Then we can choose $c \in \mathbb{R}$ such that

$$A + cI \geq 0 \quad \text{and} \quad A + B + cI \geq 0. \tag{IV.13}$$

We have:

Lemma IV.38 (see [83, Lemma 7.2]).
Let A be a self-adjoint operator which is bounded from below and unbounded from above, let B be a compact self-adjoint operator, and let $c \in \mathbb{R}$ be such that (IV.13) holds. Then $D(\lambda) = 0$ for all $\lambda < -c$ and

$$D(\lambda) = E_{(-\infty,\mu(\lambda)]}\big(A'\big) - E_{(-\infty,\mu(\lambda)]}\big(A'+B'\big) \quad \text{for all } \lambda \geq -c. \tag{IV.14}$$

Here $\mu(\lambda) = \frac{1}{\lambda+1+c}$, $A' = \big(A + (1+c)I\big)^{-1}$, and

$$B' = -\big(A+B+(1+c)I\big)^{-1} B \big(A+(1+c)I\big)^{-1}.$$

Moreover, we have $\operatorname{rank} B' = \operatorname{rank} B$.

Proof. Clearly, $D(\lambda) = 0$ for all $\lambda < -c$. Now let $\lambda \geq -c$. We compute

$$\begin{aligned} D(\lambda) &= E_{[\lambda,\infty)}(A) - E_{[\lambda,\infty)}(A+B) \\ &= E_{(-\infty,\mu(\lambda)]}\big((A+(1+c)I)^{-1}\big) - E_{(-\infty,\mu(\lambda)]}\big((A+B+(1+c)I)^{-1}\big). \end{aligned}$$

By the second resolvent equation, one has

$$\big(A+B+(1+c)I\big)^{-1} = \big(A+(1+c)I\big)^{-1} - \big(A+B+(1+c)I\big)^{-1} B \big(A+(1+c)I\big)^{-1}.$$

Therefore, (IV.14) holds.

Obviously, the operator B' is compact and self-adjoint with $\operatorname{rank} B' \leq \operatorname{rank} B$. Let us show that also $\operatorname{rank} B' \geq \operatorname{rank} B$. Since the operators $\big(A+B+(1+c)I\big)^{-1}$ and $B\restriction_{\overline{\operatorname{Ran} B}}$

are one-to-one, it suffices to prove that

$$\operatorname{rank}\Big(P_{\overline{\operatorname{Ran} B}}\big(A+(1+c)I\big)^{-1}\Big) \geq \operatorname{rank} B, \tag{IV.15}$$

where $P_{\overline{\operatorname{Ran} B}}$ is the orthogonal projection of $\mathfrak{H}$ onto $\overline{\operatorname{Ran} B}$. Assume for contradiction that $\operatorname{rank}\Big(P_{\overline{\operatorname{Ran} B}}\big(A+(1+c)I\big)^{-1}\Big) < \operatorname{rank} B$. Then we can choose a nonzero vector in $\overline{\operatorname{Ran} B}$ that is orthogonal to $\operatorname{Ran}\Big(P_{\overline{\operatorname{Ran} B}}\big(A+(1+c)I\big)^{-1}\Big)$ and thus also orthogonal to $\operatorname{Ran}\big(A+(1+c)I\big)^{-1}$, contradicting the fact that $\operatorname{Ran}\big(A+(1+c)I\big)^{-1}$ is dense in $\mathfrak{H}$. Therefore, (IV.15) holds. This completes the proof of the lemma. □

Combining Lemma IV.38 with Remark IV.37 above, we obtain:

Lemma IV.39 (see [83, Lemma 7.3]).
The statement of Theorem IV.8 holds in the case when A is bounded from below and unbounded from above.

We will need the following auxiliary result:

Lemma IV.40 (see [83, Lemma 7.4]).
Let A be a self-adjoint operator which is bounded from below and unbounded from above, and let $c \in \mathbb{R}$ be such that $A + cI \geq 0$. We write $A' = \big(A+(1+c)I\big)^{-1}$. Then we have:

(1) the function $t \mapsto \frac{1}{t+1+c}$ is one-to-one from $\sigma_{\text{ess}}(A)$ onto $\sigma_{\text{ess}}(A') \setminus \{0\}$;
(2) if a vector φ is cyclic for A, then the vector $A'\varphi$ is cyclic for A'.

Proof. Standard arguments from spectral theory (see Section I.4) show that:

- $0 \in \sigma_{\text{ess}}(A')$ is not an eigenvalue of A';
- the function $t \mapsto \frac{1}{t+1+c}$ is one-to-one from $\sigma(A)$ onto $\sigma(A') \setminus \{0\}$;
- the function $t \mapsto \frac{1}{t+1+c}$ is one-to-one from $\sigma_{\text{ess}}(A)$ onto $\sigma_{\text{ess}}(A') \setminus \{0\}$, i. e., (1) holds.

Consequently, we have

$$\begin{aligned}\operatorname{span}\big\{E_{\Omega'}(A')A'\varphi : \Omega' \in \mathcal{B}(\mathbb{R})\big\} &= \operatorname{span}\big\{A'E_{\Omega'\setminus\{0\}}(A')\varphi : \Omega' \in \mathcal{B}(\mathbb{R})\big\} \\ &= \operatorname{span}\big\{A'E_{\Omega}(A)\varphi : \Omega \in \mathcal{B}(\mathbb{R})\big\} \\ &= A'\big(\operatorname{span}\{E_{\Omega}(A)\varphi : \Omega \in \mathcal{B}(\mathbb{R})\}\big).\end{aligned}$$

Since $A' = \big(A+(1+c)I\big)^{-1}$ is a bounded operator with dense range, it follows that $\operatorname{span}\{E_{\Omega'}(A')A'\varphi : \Omega' \in \mathcal{B}(\mathbb{R})\}$ is dense in $\mathfrak{H}$ if $\operatorname{span}\{E_{\Omega}(A)\varphi : \Omega \in \mathcal{B}(\mathbb{R})\}$ is dense in $\mathfrak{H}$, i. e., (2) holds. This completes the proof of the lemma. □

We can now show:

Lemma IV.41 (see [83, Lemma 7.5]).
The statements of Theorem IV.13 (3) hold.

Proof. Let c be such that (IV.13) holds. By Lemma IV.40 (1), we know that the function $t \mapsto \frac{1}{t+1+c}$ is one-to-one from $\sigma_{\text{ess}}(A)$ onto $\sigma_{\text{ess}}\big((A+(1+c)I)^{-1}\big) \setminus \{0\}$.

We have $\min \sigma_{\text{ess}}\big((A+(1+c)I)^{-1}\big) = 0$ and, since the spectrum of A is not purely discrete, $\max \sigma_{\text{ess}}\big((A+(1+c)I)^{-1}\big) = \frac{1}{\lambda_0+1+c} > 0$, where $\lambda_0 = \min \sigma_{\text{ess}}(A)$. Therefore, $\mu(\lambda) = \frac{1}{\lambda+1+c}$ belongs to the open interval $\Big(0, \frac{1}{\lambda_0+1+c}\Big)$ if and only if $\lambda > \lambda_0$.

By assumption of Theorem IV.13 (3), the vector φ is cyclic for A. It follows from Lemma IV.40 (2) that $(A+(1+c)I)^{-1}\varphi$ is cyclic for $(A+(1+c)I)^{-1}$.

In view of Lemma IV.38 and Remark IV.37, the claim follows. □

Combining Lemma IV.38, Remark IV.37, and Lemma IV.40 (1), we obtain:

Lemma IV.42 (see [83, Lemma 7.6]).
The statements of Theorem IV.24 hold in the case when A is bounded from below and unbounded from above.

Now, we consider the case when A is bounded from above. Then we can choose $c \in \mathbb{R}$ such that

$$A - cI \le 0 \quad \text{and} \quad A + B - cI \le 0.$$

It suffices to consider $D(\lambda)$ for $\lambda \le c$. We compute

$$\begin{aligned} D(\lambda) &= E_{(\mu(\lambda),\infty)}\big((A+B-(1+c)I)^{-1}\big) - E_{(\mu(\lambda),\infty)}\big((A-(1+c)I)^{-1}\big) \\ &= E_{(-\infty,\mu(\lambda)]}\big((A-(1+c)I)^{-1}\big) - E_{(-\infty,\mu(\lambda)]}\big((A+B-(1+c)I)^{-1}\big), \end{aligned}$$

where $\mu(\lambda) = \frac{1}{\lambda-(1+c)}$. By the second resolvent equation, one has

$$\big(A+B-(1+c)I\big)^{-1} = \big(A-(1+c)I\big)^{-1} - \big(A+B-(1+c)I\big)^{-1}B\big(A-(1+c)I\big)^{-1}.$$

The operator $B'' = -\big(A+B-(1+c)I\big)^{-1}B\big(A-(1+c)I\big)^{-1}$ is compact and self-adjoint with rank $B'' =$ rank B. We can now proceed analogously to the case when A is bounded from below to obtain the following three lemmas:

Lemma IV.43 (see [83, Lemma 7.7]).
The statement of Theorem IV.8 holds in the case when A is bounded from above and unbounded from below.

Lemma IV.44 (see [83, Lemma 7.8]).
The statements of Theorem IV.13 (4) hold.

Lemma IV.45 (see [83, Lemma 7.9]).
The statements of Theorem IV.24 hold in the case when A is bounded from above and unbounded from below.

IV.6.3. Proofs of Theorems IV.1′ and IV.2. Let us first show Theorem IV.1′.

Proof of Theorem IV.1′. Let us recall that $B = \alpha\langle \bullet, \varphi\rangle_{\mathfrak{H}}\, \varphi$, where $\alpha \in \mathbb{R}\setminus\{0\}$ and $\varphi \in \mathfrak{H}\setminus\{0\}$, and $\mathfrak{N} = \overline{\operatorname{span}}\,\{E_\Omega(A)\varphi : \Omega \in \mathcal{B}(\mathbb{R})\}$.

(1) Let $\dim(\mathfrak{N}^\perp) = \infty$. Then, by Lemma IV.4, $\dim \operatorname{Ker} D(\lambda) = \infty$ for all λ in $\mathbb{R}$. Consequently, Theorems IV.8 and III.14 imply that $D(\lambda)$ is unitarily equivalent to a bounded self-adjoint Hankel operator with infinite dimensional kernel for all λ in $\mathbb{R}$.

(2) In the case when $\mathfrak{N}^\perp = \{0\}$, we know that φ is cyclic for A. Consequently, Theorems IV.8, IV.13, IV.24, and III.14 imply that $D(\lambda)$ is unitarily equivalent to a bounded self-adjoint Hankel operator for all λ in $\mathbb{R}$ except for at most countably many λ in $\sigma_{\mathrm{ess}}(A)$.

(3) Let $\dim(\mathfrak{N}^\perp) = k \in \mathbb{N}$. Clearly, $\mathfrak{N}$ is a separable Hilbert space of infinite dimension, and φ is cyclic for $A\restriction_{\mathfrak{N}}$. By (IV.1), we know that

$$D(\lambda) = \Big(E_{(-\infty,\lambda)}\big(A\restriction_{\mathfrak{N}} + B\restriction_{\mathfrak{N}}\big) - E_{(-\infty,\lambda)}\big(A\restriction_{\mathfrak{N}}\big)\Big) \oplus 0 \quad \text{on} \quad \mathfrak{N} \oplus \mathfrak{N}^\perp$$

for all $\lambda \in \mathbb{R}$. Analogously to the preceding case, Theorems IV.8, IV.13, IV.24, and III.14 imply that $E_{(-\infty,\lambda)}\big(A\restriction_{\mathfrak{N}} + B\restriction_{\mathfrak{N}}\big) - E_{(-\infty,\lambda)}\big(A\restriction_{\mathfrak{N}}\big)$ is unitarily equivalent to a bounded self-adjoint Hankel operator for all λ in $\mathbb{R}$ except for at most countably many λ in $\sigma_{\mathrm{ess}}(A\restriction_{\mathfrak{N}}) \subset \sigma_{\mathrm{ess}}(A)$. This completes the proof. $\square$

Let us now show Theorem IV.2.

Proof of Theorem IV.2. Let $\lambda \in \mathbb{R}$. We rewrite the orthogonal decomposition of $\mathfrak{H}$ from the Halmos' decomposition theorem (see, e. g., Proposition III.11) with respect to the difference $D(\lambda)$ of $E_{(-\infty,\lambda)}(A+B)$ and $E_{(-\infty,\lambda)}(A)$:

$$\mathfrak{H} = \Big(\operatorname{Ker} D(\lambda)\Big) \oplus \Big(\operatorname{Ker}\big(D(\lambda) - I\big)\Big) \oplus \Big(\operatorname{Ker}\big(D(\lambda) + I\big)\Big) \oplus \mathfrak{H}_{\mathrm{g}}(\lambda). \qquad \text{(IV.16)}$$

Here $\mathfrak{H}_{\mathrm{g}}(\lambda)$ is the orthogonal complement of

$$\widetilde{\mathfrak{H}}(\lambda) = \Big(\operatorname{Ker} D(\lambda)\Big) \oplus \Big(\operatorname{Ker}\big(D(\lambda) - I\big)\Big) \oplus \Big(\operatorname{Ker}\big(D(\lambda) + I\big)\Big)$$

in $\mathfrak{H}$. Clearly, $\mathfrak{H}_{\mathrm{g}}(\lambda)$ is reducing for the operator $D(\lambda)$.

Subclaim. For every $\lambda \in \mathbb{R}$, there exists a compact self-adjoint block diagonal operator $K(\lambda) = \widetilde{K}(\lambda) \oplus 0$ on $\widetilde{\mathfrak{H}}(\lambda) \oplus \mathfrak{H}_{\mathrm{g}}(\lambda)$ with the following properties:

- (a) $K(\lambda)$ satisfies assertion (2) of Theorem IV.2;
- (b) $\sigma\big(D(\lambda) + K(\lambda)\big) \subset [-1, 1]$;
- (c) we have either $\operatorname{Ker}\big(D(\lambda) + K(\lambda)\big) = \{0\}$ or $\dim \operatorname{Ker}\big(D(\lambda) + K(\lambda)\big) = \infty$, i. e., $D(\lambda) + K(\lambda)$ fulfills condition (C1) of Proposition III.16;
- (d) the dimensions of $\operatorname{Ker}\big(D(\lambda)\restriction_{\widetilde{\mathfrak{H}}(\lambda)} + \widetilde{K}(\lambda) - tI\big)$ and $\operatorname{Ker}\big(D(\lambda)\restriction_{\widetilde{\mathfrak{H}}(\lambda)} + \widetilde{K}(\lambda) + tI\big)$ differ by at most one for every $0 < t \leq 1$, i. e., condition (C3′) of Proposition III.16 is satisfied;
- (e) if B of finite rank, then $K(\lambda)$ is of finite rank.

We continue with the proof of Theorem IV.2 and show the subclaim further below.

The block diagonal operator $\widetilde{K}(\lambda) \oplus 0$ serves as a correction term for $D(\lambda)$. In particular, no correction term is needed if $\widetilde{\mathfrak{H}}(\lambda) = \{0\}$.

Theorem IV.24 and the invariance of the essential spectrum under compact perturbations imply that zero belongs to the essential spectrum of $D(\lambda) + K(\lambda)$ for all λ in $\mathbb{R}$ except for at most countably many λ in $\sigma_{\mathrm{ess}}(A)$. Consequently, in view of the subclaim, an application of Proposition III.16 yields that $D(\lambda) + K(\lambda)$ is unitarily equivalent to a bounded self-adjoint Hankel operator $T(\lambda)$ on $\mathsf{L}^2(\mathbb{R}_+)$ for all λ in $\mathbb{R}$ except for at most countably many λ in $\sigma_{\mathrm{ess}}(A)$. Thus, Theorem IV.2 is proved when we show the subclaim above.

Proof of the subclaim. $\widetilde{K}(\lambda)$ will be the sum of two operators $\widetilde{K}_0(\lambda)$ and $\widetilde{K}_1(\lambda)$. Before we construct these operators, let us introduce some notation: We write

$$n_\beta(\lambda) = \dim\ \operatorname{Ker}\big(D(\lambda) - \beta I\big), \quad \text{where } \beta \in \{-1, 0, 1\}, \quad \text{and}$$

$$r_0(\lambda) = \begin{cases} n_0(\lambda) & \text{if } 0 < n_0(\lambda) < \infty \\ 0 & \text{otherwise} \end{cases}.$$

Let $1/4 > a_1 > a_2 > \cdots > 0$ be a decreasing null sequence of real numbers.

Construction of $\widetilde{K_0}(\lambda)$. If $0 < n_0(\lambda) < \infty$, then we choose an orthonormal basis $\psi_1(\lambda), \ldots, \psi_{n_0(\lambda)}(\lambda)$ of Ker $D(\lambda)$ and set

$$\widetilde{K}_0(\lambda) = -\sum_{m=1}^{n_0(\lambda)} a_m \langle \bullet, \psi_m(\lambda)\rangle \restriction_{\widetilde{\mathfrak{H}}(\lambda)} \psi_m(\lambda);$$

otherwise we put $\widetilde{K}_0(\lambda) = 0$.

Construction of $\widetilde{K_1}(\lambda)$. In the case when $n_{-1}(\lambda)$ and $n_1(\lambda)$ differ by at most one (i. e., $D(\lambda)$ fulfills condition (C3) of Theorem III.14), we set $\widetilde{K}_1(\lambda) = 0$.

Let us now suppose that $n_{-1}(\lambda)$ and $n_1(\lambda)$ differ by at least two. Without loss of generality, let $n_1(\lambda) > n_{-1}(\lambda)$ (in particular, $n_{-1}(\lambda)$ is then finite). We set

$$\widetilde{K}_1(\lambda) = -\sum_{m=1}^{n_1(\lambda)-n_{-1}(\lambda)} \frac{a_{m+r_0(\lambda)}}{2^m} \langle \bullet, \eta_{m+n_{-1}(\lambda)}(\lambda)\rangle \restriction_{\widetilde{\mathfrak{H}}(\lambda)} \eta_{m+n_{-1}(\lambda)}(\lambda),$$

where $\{\eta_m(\lambda) : m \in [1, n_1(\lambda)\rangle\}$ is an orthonormal basis of Ker $\big(D(\lambda) - I\big)$.

Finally, we put $\widetilde{K}(\lambda) = \widetilde{K}_0(\lambda) + \widetilde{K}_1(\lambda)$. Since $\sigma\big(D(\lambda)\big) \subset [-1, 1]$, it follows from (IV.16) and the construction of $K(\lambda) = \widetilde{K}(\lambda) \oplus 0$ that parts (a)–(d) of the subclaim hold. In the case when B is of finite rank, Theorem IV.8 implies that $n_{\pm 1}(\lambda) < \infty$, so by construction, $K(\lambda)$ is of finite rank. This concludes the proof of the subclaim and thus, as noted above, of Theorem IV.2. □

IV.7. Some examples

The following material is taken, almost verbatim, from the author's paper [**83**, pp. 25–28].

In this section, we apply the above theory in the context of operators that are of particular interest in various fields of (applied) mathematics, such as Schrödinger operators.

In any of the following examples, there exists an $N \in \mathbb{N}$ such that $D(\lambda)$ is unitarily equivalent to a bounded self-adjoint Hankel operator on $\mathsf{L}^2\big(\mathbb{R}_+; \mathbb{C}^N\big)$ for all λ in $\mathbb{R}$.

First, we consider the case when A has a purely discrete spectrum.

Example IV.46 (see [83, Example 8.1]).
Let $\mathfrak{H} = \mathsf{L}^2(\mathbb{R}^n)$ and suppose that $V \geq 0$ is in $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that Lebesgue measure of $\{x \in \mathbb{R}^n : 0 \leq V(x) < M\}$ is finite for all $M > 0$. Then the self-adjoint Schrödinger operator $A \geq 0$ defined by the form sum of $-\Delta$ and V has a purely discrete spectrum, see [**84**, Example 4.1]; see also [**77**, Theorem 1]. Therefore, if B is any self-adjoint operator of rank $N \in \mathbb{N}$, Proposition IV.20 implies that $D(\lambda)$ is of finite rank and thus,

by Remark IV.9, unitarily equivalent to a (finite rank) self-adjoint Hankel operator on $\mathsf{L}^2(\mathbb{R}_+;\mathbb{C}^N)$ for all $\lambda \in \mathbb{R}$.

Next, we consider the case when $B = \alpha\langle \bullet, \varphi\rangle_{\mathfrak{H}}\,\varphi$ is of rank 1, where $\alpha \in \mathbb{R}\setminus\{0\}$ and φ is cyclic for A.

Example IV.47 (see [83, Example 8.2]).
Once again, let us consider M. Krein's example (see Section II.3).

The operators $A = A_1$ and $A + B = A_0$, where $B = -\langle \bullet, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}_+)}\,\varphi$ with $\varphi(x) = \mathrm{e}^{-x}$, from Section II.3 both have a simple purely absolutely continuous spectrum filling in the interval $[0,1]$. Therefore, $D(\lambda) = 0$ for all $\lambda \in \mathbb{R}\setminus(0,1)$.

Claim $(*)$ The vector $\varphi \in \mathsf{L}^2(\mathbb{R}_+)$ is cyclic for A_i, $i = 0, 1$.

Hence, Theorem IV.13 (1) implies that the kernel of $D(\lambda)$ is trivial for all $0 < \lambda < 1$.

Furthermore, an application of Theorem IV.1′ (2) yields that $D(\lambda)$ is unitarily equivalent to a bounded self-adjoint Hankel operator for all λ in $\mathbb{R}$ except for at most countably many λ in $[0,1]$.

Let us note that, in this example, explicit computations show that there are no exceptional points (see Section II.3).

Proof of Claim $(*)$. Let $n \in \mathbb{N}_0$. We denote the nth Laguerre polynomial by L_n, i. e., $L_n(x) = \frac{\mathrm{e}^x}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n\mathrm{e}^{-x})$. Let us define ψ_n on $\mathbb{R}_+$ by $\psi_n(x) = x^n\mathrm{e}^{-x}$. A straightforward computation shows that

$$\big(A_0\psi_n\big)(x) = \frac{1}{2}\,\mathrm{e}^{-x}\left\{\frac{x^{n+1}}{n+1} + \frac{1}{2^{n+1}}\sum_{\ell=0}^{n-1}(2x)^{n-\ell}\,\frac{n!}{(n-\ell)!}\right\}.$$

By induction on $n \in \mathbb{N}_0$, it easily follows that $p\cdot\varphi$ belongs to the linear span of $A_0^\ell\varphi$, $\ell \in \mathbb{N}_0$, $\ell \le n$, for all polynomials p of degree $\le n$.

In particular, the functions η_m defined on $\mathbb{R}_+$ by $\eta_m(x) = 2^{1/2}\,L_m(2x)\mathrm{e}^{-x}$ are elements of $\operatorname{span}\big\{A_0^\ell\varphi : \ell \in \mathbb{N}_0, \ell \le n\big\}$ for all $m \in \mathbb{N}_0$ with $m \le n$. Since $(\eta_m)_{m\in\mathbb{N}_0}$ is an orthonormal basis of $\mathsf{L}^2(\mathbb{R}_+)$, it follows that φ is cyclic for A_0 (and hence for A_1). ☐

Example IV.47 suggests the conjecture that Theorem IV.1′ (2) can be strengthened to hold up to a finite exceptional set.

Next, let us consider different examples where the multiplicity in the spectrum of A is such that, for suitable B, we can apply Proposition IV.22.

Example IV.48 (see [83, Example 8.3]).

(1) Let A be an arbitrary orthogonal projection on $\mathfrak{H}$, and let B be a self-adjoint operator of rank $N \in \mathbb{N}$. Then zero or one is an eigenvalue of A with infinite multiplicity. Consequently, Proposition IV.22 (1) implies that $D(\lambda)$ has infinite dimensional kernel and is thus, by Remark IV.9, unitarily equivalent to a self-adjoint Hankel operator on $\mathsf{L}^2(\mathbb{R}_+;\mathbb{C}^N)$ for all $\lambda \in \mathbb{R}$.

(2) Set $\mathfrak{H} = \mathsf{L}^2(\mathbb{R}_+)$. Let A be the Carleman operator, i. e., the bounded self-adjoint Hankel operator on $\mathsf{L}^2(\mathbb{R}_+)$ defined by

$$(A\psi)(x) = \int_{\mathbb{R}_+} \frac{\psi(y)}{x+y} \mathrm{d}y$$

for all continuous functions $\psi : \mathbb{R}_+ \to \mathbb{C}$ with compact support. Since it is well known (see, e. g., [**62**, Chapter 10, Theorem 2.3]) that the Carleman operator has a purely absolutely continuous spectrum of uniform multiplicity two filling in the interval $[0, \pi]$, we obtain: If B is any self-adjoint operator of rank one, then Proposition IV.22 (3) implies that $D(\lambda)$ has infinite dimensional kernel and is thus, by Remark IV.9, unitarily equivalent to a self-adjoint Hankel operator for all $\lambda \in \mathbb{R}$.

Jacobi operators. Set $\mathfrak{H} = \ell^2(\mathbb{Z})$. So far, we have often denoted vectors in $\mathfrak{H}$ by ψ or ψ_1, ψ_2, etc. In order to avoid ambiguities, we will write

$$\ell^2(\mathbb{Z}) \ni \psi = (\omega_n)_{n\in\mathbb{Z}}.$$

Let us consider a bounded self-adjoint Jacobi operator H acting on $\ell^2(\mathbb{Z})$. More precisely, we suppose that there exist bounded real-valued sequences $a = (a_n)_{n\in\mathbb{Z}}$ and $b = (b_n)_{n\in\mathbb{Z}}$ with $a_n > 0$ for all $n \in \mathbb{Z}$ such that

$$H(\omega_n)_{n\in\mathbb{Z}} = \big(a_n\omega_{n+1} + a_{n-1}\omega_{n-1} + b_n\omega_n\big)_{n\in\mathbb{Z}},$$

cf. [**79**, Theorem 1.5 and Lemma 1.6]. The following result is well known (cf. the formulation of [**83**, Lemma 8.4]).

Lemma IV.49 (see [79, Lemma 3.6]).
Let H be a bounded self-adjoint Jacobi operator on $\ell^2(\mathbb{Z})$. Then the singular spectrum of H has spectral multiplicity one, and the absolutely continuous spectrum of H has multiplicity at most two.

In the case where H has a simple spectrum, we can choose a cyclic vector φ for H. We set $A = H$ and consider the rank one perturbation $B = \alpha\langle\bullet, \varphi\rangle_{\ell^2(\mathbb{Z})}\,\varphi$ for any $\alpha \in \mathbb{R}\setminus\{0\}$. Then, by Theorem IV.1′ (2), $D(\lambda)$ is unitarily equivalent to a bounded self-adjoint Hankel operator for all λ in $\mathbb{R}$ except for at most countably many λ in $\sigma_{\mathrm{ess}}(A)$.

Now, let us consider an example where the absolutely continuous spectrum of the Jacobi operator has multiplicity two.

Example IV.50 (see [83, Example 8.5]).
Let H_V be the discrete Schrödinger operator on $\ell^2(\mathbb{Z})$ with bounded potential $V : \mathbb{Z} \to \mathbb{R}$,

$$H_V(\omega_n)_{n\in\mathbb{Z}} = \big(\omega_{n+1} + \omega_{n-1} + V_n\omega_n\big)_{n\in\mathbb{Z}}.$$

If the spectrum of H_V contains only finitely many points outside of the interval $[-2, 2]$, then [**20**, Theorem 2] implies that H_V has a purely absolutely continuous spectrum of multiplicity two on $[-2, 2]$. (In particular, it is well known that the free Jacobi operator H_0 with $V = 0$ has a purely absolutely continuous spectrum of multiplicity two filling in the interval $[-2, 2]$.) In this case, we set $A = H_V$. Then if B is any self-adjoint operator of rank one, Proposition IV.22 (3) implies that $D(\lambda)$ has infinite dimensional kernel and is thus, by Remark IV.9, unitarily equivalent to a self-adjoint Hankel operator for all $\lambda \in \mathbb{R}$.

Last, let us consider the almost Mathieu operator $H_{\kappa,\beta,\theta} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$H_{\kappa,\beta,\theta}(\omega_n)_{n\in\mathbb{Z}} = \Big(\omega_{n+1} + \omega_{n-1} + 2\kappa\cos\big(2\pi(\theta + n\beta)\big)\omega_n\Big)_{n\in\mathbb{Z}},$$

where $\kappa \in \mathbb{R} \setminus \{0\}$ and $\beta, \theta \in \mathbb{R}$. In fact, it suffices to consider $\beta, \theta \in \mathbb{R}/\mathbb{Z}$.

The almost Mathieu operator plays an important role in solid state physics, see, e. g., the review [**48**] and the references therein.

Here, we are interested in cases where Proposition IV.22 can be applied to the almost Mathieu operator with an arbitrary self-adjoint rank one perturbation, see Example IV.53 below. Sufficient conditions for this purpose are provided by the following lemma.

Lemma IV.51 (cf. the formulation of [83, Lemma 8.6]).

(1) If β is rational, then for all κ and θ the almost Mathieu operator $H_{\kappa,\beta,\theta}$ is periodic and has a purely absolutely continuous spectrum of uniform multiplicity two.

(2) If β is irrational and $|\kappa| < 1$, then for all θ the almost Mathieu operator $H_{\kappa,\beta,\theta}$ has a purely absolutely continuous spectrum of uniform multiplicity two.

Proof. (1) If β is rational, then $H_{\kappa,\beta,\theta}$ is a periodic Jacobi operator. Hence, it is well known (see, e. g., [**79**, p. 122]) that the spectrum of $H_{\kappa,\beta,\theta}$ is purely absolutely continuous. According to [**22**, Theorem 9.1], we know that the absolutely continuous spectrum of $H_{\kappa,\beta,\theta}$ is uniformly of multiplicity two. This proves (1).

(2) Suppose that β is irrational. Avila showed (see [**4**, Main Theorem]) that the almost Mathieu operator $H_{\kappa,\beta,\theta}$ has a purely absolutely continuous spectrum if and only if $|\kappa| < 1$. Again, [**22**, Theorem 9.1] implies that the absolutely continuous spectrum of $H_{\kappa,\beta,\theta}$ is uniformly of multiplicity two. This completes the proof. □

Remark IV.52 (see [83, Remark 8.7]).
Problems 4–6 of Simon's list [**76**] are concerned with the almost Mathieu operator. Avila's result [**4**, Main Theorem], which we used in the above proof, is a solution for Problem 6 in [**76**].

Here is the announced example.

Example IV.53 (see [83, Example 8.8]).
Assume that the parameters κ, β, and θ are such that Lemma IV.51 is applicable to the almost Mathieu operator $A = H_{\kappa,\beta,\theta}$. Then if B is any self-adjoint operator of rank one, Proposition IV.22 (3) implies that $D(\lambda)$ has infinite dimensional kernel and is thus, by Remark IV.9, unitarily equivalent to a self-adjoint Hankel operator for all $\lambda \in \mathbb{R}$.

IV.8. Two open problems

We start with

Open problem IV.54. Describe the asymptotical behaviour of the function $\mathbb{R}_+ \ni y \mapsto \kappa(y) \in \mathbb{R}$ defined in (III.13) as $y \to \infty$.

Remark IV.55 (to Open problem IV.54).
As noted in Remark III.28, κ defined in (III.13) is continuous,

$$\lim_{y\to 0+} \kappa(y) \text{ exists in } \mathbb{R}, \qquad \text{and} \qquad \lim_{y\to\infty} \kappa(y) = 0.$$

On the other hand, it is well known (see, e. g., [**62**, Corollary 8.11, p. 54]) that the Hankel operator T_κ on $\mathsf{L}^2(\mathbb{R}_+)$ is compact if κ belongs to $\mathsf{L}^1(\mathbb{R}_+)$. So if we had

$$\limsup_{y\to\infty} \left|\kappa(y)\, y^{1+\varepsilon}\right| < \infty \quad \text{for some } \varepsilon > 0,$$

then we would know that T_κ is compact.

Next, we consider

Open problem IV.56. Prove a result similar to Theorem IV.13 in the case when the perturbation B is of rank $N \geq 2$.

IV.8.1. Discussion of Open problem IV.56. The idea is to use a result like Proposition IV.17 (see [**49**, Theorem 2.1]) in the case when $\operatorname{rank} B = N \geq 2$. Recently, in [**50**], Liaw and Treil generalized their result [**49**, Theorem 2.1]; for simplicity, we describe [**50**, Theorem 6.1] (see Proposition IV.69 below) in the following special situation.

Hypothesis IV.57. Let A be a bounded self-adjoint operator on a complex separable Hilbert space $\mathfrak{H}$, and let $B_Y = GYG^*$ on $\mathfrak{H}$ be of finite rank $N \geq 2$, where $G : \mathbb{C}^N \to \mathfrak{H}$ satisfies $\operatorname{Ran}(B_Y) = \operatorname{Ran}(G)$ and Y is a self-adjoint $N \times N$ matrix. We set (cf. [**50**, p. 4])

$$g_1 = G(1, 0, \dots, 0), \ \dots, \ g_N = G(0, \dots, 0, 1).$$

Remark IV.58. Since G is of rank N, the vectors $g_1, \dots, g_N$ are linearly independent.

We need the following notion (see [**50**, p. 5]; cf. also [**26**, Definition 12, p. 1349]):

Definition IV.59 (Positive matrix measure).
Let $\mathcal{I} \subset \mathbb{R}$ be a nonempty bounded open interval. Given complex Borel measures $\mu^{(mn)}$, $m, n = 1, \dots, N$, on $\mathcal{I}$ such that each $\mu^{(mm)}$, $m = 1, \dots, N$, is positive, we call

$$\mathbf{M} : \mathcal{B}(\mathcal{I}) \to \mathbb{C}^{N\times N}, \quad \mathbf{M}(\Delta) = \big(\mu^{(mn)}(\Delta)\big)_{m,n=1,\dots,N},$$

a *matrix measure* on $\mathcal{I}$. If, moreover, $\mathbf{M}(\Delta) \in \mathbb{C}^{N\times N}$ is self-adjoint and positive semidefinite for every $\Delta \in \mathcal{B}(\mathcal{I})$, then we say that $\mathbf{M}$ is a *positive matrix measure* on $\mathcal{I}$.

Example IV.60. Let A and B_Y be as in Hypothesis IV.57. We choose a bounded open interval $\mathcal{I} \subset \mathbb{R}$ that includes $\sigma(A) \cup \sigma(A + B_Y)$ and define (cf. [**50**, formula (2.3)])

$$\mathbf{M} : \mathcal{B}(\mathcal{I}) \to \mathbb{C}^{N\times N}, \quad \mathbf{M}(\Delta) = G^* E_\Delta(A) G, \tag{IV.17}$$

as well as (cf. [**50**, p. 7])

$$\mathbf{M}_Y : \mathcal{B}(\mathcal{I}) \to \mathbb{C}^{N\times N}, \quad \mathbf{M}_Y(\Delta) = G^* E_\Delta(A + B_Y) G. \tag{IV.18}$$

It is easy to verify that $\mathbf{M}$ and $\mathbf{M}_Y$ are positive matrix measures on $\mathcal{I}$.

Next, let us recall some facts taken from [**29**, Section II.6] (cf. also [**26**, pp. 1349–1350] and [**50**, p. 5]). Let $\mathbf{M}$ be a positive matrix measure on $\mathcal{I}$. Using the representation

$\mathbf{M}(\Delta) = \mathbf{M}(\Delta)^{1/2}\mathbf{M}(\Delta)^{1/2}$ and the Cauchy–Schwarz inequality, we obtain

$$\left|\mu^{(mn)}(\Delta)\right| \leq \frac{1}{2}\big(\mu^{(mm)}(\Delta) + \mu^{(nn)}(\Delta)\big) \qquad (\Delta \in \mathcal{B}(\mathcal{I});\ m, n = 1, \ldots, N). \tag{IV.19}$$

Let us fix $m, n = 1, \ldots, N$. It follows from (IV.19) that $\mu^{(mn)}$ is absolutely continuous with respect to

$$\mu(\Delta) = \operatorname{trace}\mathbf{M}(\Delta) \qquad (\Delta \in \mathcal{B}(\mathcal{I}));$$

we note that μ is a positive finite scalar measure. By the Radon–Nikodym theorem, there exists a μ-measurable complex-valued function $w^{(mn)}$ such that

$$\mu^{(mn)}(\Delta) = \int_{\Delta} w^{(mn)}(t)\,\mathrm{d}\mu(t) \qquad (\Delta \in \mathcal{B}(\mathcal{I})). \tag{IV.20}$$

One can show:

Lemma IV.61 (see [29, Lemma 6-1, p. 107]).
Let $W = \big(w^{(mn)}\big)_{m,n=1,\ldots,N}$, where $w^{(mn)}$, $m, n = 1, \ldots, N$, are as in (IV.20). Then $W(t) \in \mathbb{C}^{N\times N}$ is positive semidefinite for μ-almost every $t \in \mathcal{I}$.

Following [**29**, p. 107] and [**50**, p. 5], we define:

(1) by $\mathsf{L}^2_0(\mathcal{I}, \mathbf{M}; \mathbb{C}^N)$ the set of all N-tuples $f = (f_1, \ldots, f_N)$ of Borel functions such that

$$\|f\|^2 = \int_{\mathcal{I}} \langle W(t)f(t), f(t)\rangle_{\mathbb{C}^N}\,\mathrm{d}\mu(t) < \infty;$$

(2) by $\mathsf{L}^2(\mathcal{I}, \mathbf{M}; \mathbb{C}^N)$ the set of all equivalence classes in $\mathsf{L}^2_0(\mathcal{I}, \mathbf{M}; \mathbb{C}^N)$ modulo functions $f \in \mathsf{L}^2_0(\mathcal{I}, \mathbf{M}; \mathbb{C}^N)$ for which $\|f\| = 0$.

As usual, one can show that $\mathsf{L}^2(\mathcal{I}, \mathbf{M}; \mathbb{C}^N)$ endowed with the inner product

$$\big\langle f^{(1)}, f^{(2)}\big\rangle_{\mathsf{L}^2(\mathcal{I},\mathbf{M};\mathbb{C}^N)} = \int_{\mathcal{I}} \big\langle W(t)f^{(1)}(t), f^{(2)}(t)\big\rangle_{\mathbb{C}^N}\,\mathrm{d}\mu(t) \tag{IV.21}$$

is a pre-Hilbert space. Moreover, we have:

Proposition IV.62 (see [29, Theorem 6-4, p. 111]).
Let $\mathbf{M}$ be a positive matrix measure on $\mathcal{I}$. Then the pre-Hilbert space $\mathsf{L}^2(\mathbf{M}) = \mathsf{L}^2(\mathcal{I}, \mathbf{M}; \mathbb{C}^N)$ endowed with the inner product given by (IV.21) is complete and thus a Hilbert space.

Remark IV.63 (to Proposition IV.62).
Let $f \in \mathsf{L}^2(\mathbf{M})$. It is easy to see that $\langle f, f\rangle_{\mathsf{L}^2(\mathbf{M})} = 0$ if and only if $W(t)f(t) = 0$ for μ-almost every t. Therefore, we can naturally define the vector-valued integral

$$\int_{\mathcal{I}} [\,\mathrm{d}\mathbf{M}] f = \int_{\mathcal{I}} W(t)f(t)\,\mathrm{d}\mu(t),$$

cf. [**50**, p. 5].

One has the following (abstract) spectral representation result.

Proposition IV.64 (see [29, Theorem 6-5, p. 111]).
Let A be a bounded self-adjoint operator on $\mathfrak{H}$ such that there exist linearly independent vectors $b_1, \dots, b_N$ with

$$\overline{\operatorname{span}}\,\{A^n b_i : n \in \mathbb{N}_0,\, i = 1, \dots, N\} = \mathfrak{H}.$$

Then A is unitarily equivalent to the multiplication operator by the independent variable on an $\mathsf{L}^2(\mathbf{M}) = \mathsf{L}^2(\mathfrak{I}, \mathbf{M}; \mathbb{C}^N)$ space for some positive matrix measure $\mathbf{M}$ on a bounded open interval $\mathfrak{I}$ that includes the spectrum of A.

Remark IV.65. Let A and B_Y be as in Hypothesis IV.57. In [**50**, Theorem 6.1] (see Proposition IV.69 below), Liaw and Treil found a formula for the spectral representation of $A+B_Y$. In view of Lemma IV.4, we can assume without loss of generality that (cf. [**50**, p. 4])

$$\overline{\operatorname{span}}\,\{A^n g_i : n \in \mathbb{N}_0,\, i = 1, \dots, N\} = \mathfrak{H}, \tag{IV.22}$$

where we recall that the vectors $g_1, \dots, g_N$ defined in Hypothesis IV.57 are linearly independent and generate $\operatorname{Ran}(B_Y) = \operatorname{Ran}(G)$. In this situation, one also has (see, e. g., [**50**, Lemma 2.5])

$$\overline{\operatorname{span}}\,\big\{\big(A + B_Y\big)^n g_i : n \in \mathbb{N}_0,\, i = 1, \dots, N\big\} = \mathfrak{H}.$$

We conclude from Proposition IV.64:

Corollary IV.66. *Let A and B_Y from Hypothesis IV.57 be such that (IV.22) holds. Then there exist positive matrix measures $\mathbf{M}$ and $\mathbf{M}_Y$ as well as unitary operators*

$$V : \mathfrak{H} \to \mathsf{L}^2(\mathbf{M}) \quad \textit{and} \quad V_Y : \mathfrak{H} \to \mathsf{L}^2\big(\mathbf{M}_Y\big)$$

such that

$$VAV^* = M_t \quad \textit{on} \quad \mathsf{L}^2(\mathbf{M}) \qquad \textit{and} \qquad V_Y(A + B_Y)V_Y^* = M_x \quad \textit{on} \quad \mathsf{L}^2\big(\mathbf{M}_Y\big).$$

Remark IV.67. Let A and B_Y from Hypothesis IV.57 be such that (IV.22) holds. Without loss of generality, we assume that A acts by multiplication by the independent variable on a von Neumann direct integral of separable Hilbert spaces (cf. Theorem I.92). We define $\mathbf{M}$ as in (IV.17) and $\mathbf{M}_Y$ as in (IV.18). Then the unitary operator V from Corollary IV.66 can be chosen such that $VG : \mathbb{C}^N \to \mathsf{L}^2(\mathbf{M})$ is given by $(VG\mathbf{c})(t) = \mathbf{c}$ for all t, see [**50**, pp. 6 and 17]. As noted in [**50**, p. 17], the adjoint operator G^*V^* then acts as follows:

$$\big(G^*V^*\big)f = \int [\,\mathrm{d}\mathbf{M}(t)]f(t) \qquad (f \in \mathsf{L}^2(\mathbf{M})).$$

Hypothesis IV.68. We henceforth assume that the operator A is given in its spectral representation described in Remark IV.67 (cf. [**50**, p. 17]). That is, we identify $\mathfrak{H}$ with $\mathsf{L}^2(\mathbf{M})$, A with M_t, G with VG, and G^* with G^*V^*.

The announced recent result of Liaw and Treil reads as follows.

Proposition IV.69 (see [50, Theorem 6.1]).
Let us assume Hypotheses IV.57 and IV.68. Then $A + B_Y$ *is unitarily equivalent to the multiplication operator by the independent variable on* $\mathsf{L}^2\big(\mathbf{M}_Y\big)$*,* $M_x = V_Y(A + B_Y)V_Y^*$*, where* $V_Y : \mathsf{L}^2(\mathbf{M}) \to \mathsf{L}^2\big(\mathbf{M}_Y\big)$ *is given by*

$$\big(V_Y(h\mathbf{c})\big)(x) = h(x)\mathbf{c} - Y\int [\,\mathrm{d}\mathbf{M}(t)]\frac{h(t) - h(x)}{t - x}\mathbf{c}$$

for all $\mathbf{c} \in \mathbb{C}^N$ *and all continuously differentiable functions* $h : \mathbb{R} \to \mathbb{C}$ *with compact support.*

Maybe one can use Proposition IV.69 to solve Open problem IV.56.

CHAPTER V

On a generalization of M. Krein's example

This chapter is based on the paper [**66**] which is a joint work of Olaf Post and the author of the present thesis (CU); please note that the contribution of CU to [**66**] is declared on p. 113. It is to emphasize that the results of [**66**] constitute the second pillar of the research of the present thesis.

Notation V.1. In this chapter, we index all sequences by the set $\mathbb{N}$ of natural numbers and therefore write (u_m) in place of $(u_m)_{m\in\mathbb{N}}$.

The following material is taken, almost verbatim, from the paper [**66**, pp. 292–294] by Olaf Post and the present author.

In this chapter, we generalize M. Krein's example (see Section II.3) by considering operators of the type

$$H = \Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big)^{\mathrm{N}} \otimes I + I \otimes L \text{ and } H^{\mathrm{D}} = \Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big)^{\mathrm{D}} \otimes I + I \otimes L \quad \text{in} \quad \mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G}, \quad \text{(V.1)}$$

where $\mathfrak{G} \neq \{0\}$ is a complex separable Hilbert space and L is a self-adjoint nonnegative operator on $\mathfrak{G}$ (precise definitions are given in Section V.3). We call H resp. H^{D} the *(abstract) Neumann* resp. *Dirichlet operator*. In particular, this framework includes:

(1) M. Krein's example of the half-line $\mathbb{R}_+$ with $L = 0$ and $\mathfrak{G} = \mathbb{C}$;
(2) the example of the classical half-space $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ with $L = -\Delta_{\mathbb{R}^{n-1}}$ and $n \geq 2$;
(3) the case when L is (minus) the Laplacian on a generally noncompact manifold $\mathfrak{Y}$, e. g., on the cylinder $\mathbb{R}_+ \times \mathfrak{Y}$ with Neumann resp. Dirichlet boundary conditions on $\{0\} \times \mathfrak{Y}$.

We consider the resolvents

$$A_0 = (H^{\mathrm{D}} + I)^{-1} \quad \text{and} \quad A_1 = (H + I)^{-1} \qquad \text{(V.2)}$$

of the operators H^{D} and H defined in (V.1) at the spectral point -1. The difference $A_1 - A_0$ of the resolvents will be computed with the help of an M. Krein-type resolvent formula from the theory of boundary pairs [**65**].

Next, we would like to calculate the difference $E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)$ of the spectral projections for all $0 < \vartheta < 1$. It is generally hard to compute differences of spectral projections explicitly. In our example, however, the computation can be performed, using the transformation formula for spectral measures (this idea is borrowed from M. Krein's example) and a convolution-type formula from [**85**].

We give a full description of the unitary invariants of the resolvent difference and of the difference of the spectral projections. Moreover, the spectral properties establish a link between the difference of the spectral projections and Hankel operators.

Operators of the type (V.1) have been studied before; criteria for self-adjointness (see, e. g., Schmüdgen's monograph [**75**]), the spectrum (see, e. g., [**75**] or Weidmann's monograph [**85**]), and a convolution-type formula for the spectral projection (see [**85**]) are known and will be very useful in this chapter. We present these results in Subsection V.2.2.

There are classical works on spectral theory of self-adjoint boundary value problems with operator-valued potential as in (V.1), see, e. g., Gorbachuk and Kutovoi [**32**, **33**, **35**, **36**, **46**] and the monograph [**34**].

Remark V.2. Gorbachuk and Kutovoi showed in [**35**] that $A_1 - A_0$ is trace class if and only if (in the present notation) $(L+I)^{-1}$ is trace class. Sufficient criteria for $A_1 - A_0$ to belong to Schatten classes can be found in [**36**]. The proofs rely on the resolvent identities and the ideal properties of Schatten classes; the resolvent difference is not computed explicitly in [**35**, **36**].

Abstract boundary value problems have often been treated using operator theory. A boundary triplet for the Schrödinger operator on the semi-axis with operator-valued potential was constructed in [**24**, Section 9.6] for the first time; note that the boundary operators of this boundary triplet depend on rational powers of the potential. We refer to the review article [**23**] for an overview on boundary triplets and also to [**65**] for the concept of boundary pairs, see also the references therein. Such concepts allow for example to calculate differences of resolvents of operators with different boundary conditions. There are related works by Boitsev, Neidhardt, and Popov [**13**] on tensor products of boundary triplets (with bounded operator L), Malamud and Neidhardt [**52**] for unitary equivalence and regularity properties of different self-adjoint realizations, Gesztesy, Weikard, and Zinchenko [**30**, **31**] for a general spectral theory of Schrödinger operators with bounded operator potentials, and Mogilevskii [**58**], see also the references therein. Moreover, when finishing the paper [**66**], Olaf Post and the author of the present thesis have learned about the recent paper [**12**] in which Boitsev, Brasche, Malamud, Neidhardt, and Popov construct a boundary triplet for the adjoint of the symmetric operator $T \otimes I + I \otimes L$, where T is symmetric and L is self-adjoint. This generalizes the situation of (V.1), where $T = -\mathrm{d}^2/\mathrm{d}t^2$ on $\mathsf{L}^2(\mathbb{R}_+)$. The focus in [**12**] is on self-adjoint extensions which do not respect the tensor structure (V.1) as models for quantum systems coupled to a reservoir. Note that in [**12**, **52**] one has to "regularize" the boundary triplet (i.e., one has to modify the boundary map and spectrally decompose L into bounded operators) in order to treat also unbounded operators L. In our approach, we can directly treat unbounded operators L without changing the boundary map or decomposing L. The special case of operators L with purely discrete spectrum has been treated, e. g., in [**65**, Section 6.4] or, in a slightly different setting, in [**64**, Section 3.5.1].

V.1. Main results

The following material is taken, almost verbatim, from the paper [**66**, pp. 294–296] by Olaf Post and the present author.

Let A_0 and A_1 be the resolvents defined in (V.2) of the (abstract) Dirichlet and Neumann operators given in (V.1) above.

Theorem V.3 (see [66, Theorem 1.1]).

(1) The resolvent difference $A_1 - A_0$ *acts on elementary tensors* $\psi \otimes \chi$ *as follows:*

$$\big([A_1 - A_0](\psi \otimes \chi)\big)(t) = \int_{\mathbb{R}_+} \psi(\tau) \exp\big(-(L+I)^{1/2}(t+\tau)\big)(L+I)^{-1/2}\chi \,\mathrm{d}\tau$$

for all $\psi \in \mathsf{C}_\mathrm{c}(\mathbb{R}_+)$ *and all* $\chi \in \mathfrak{G}$.

(2) Let $0 < \vartheta < 1$ *and let* $\alpha(\vartheta) = \frac{1}{\vartheta} - 1 > 0$. *Then the difference of the spectral projections of* A_0 *and* A_1 *associated with the open interval* $(-\infty, \vartheta)$ *acts on elementary tensors* $\psi \otimes \chi$ *as follows:*

$$\Big(\big[E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)\big](\psi \otimes \chi)\Big)(t) = \frac{2}{\pi}\int_{\mathbb{R}_+} \psi(\tau) E_{[0,\alpha(\vartheta))}(L) \frac{\sin\big((\alpha(\vartheta)I - L)^{1/2}(t+\tau)\big)}{t+\tau}\chi \,\mathrm{d}\tau$$

for all $\psi \in \mathsf{C}_\mathrm{c}(\mathbb{R}_+)$ *and all* $\chi \in \mathfrak{G}$.

Consequently, the operators $A_1 - A_0$ and $E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)$ can be evaluated explicitly on the dense subspace of $\mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G}$ that is given by the linear hull of all elementary tensors of the type $\psi \otimes \chi$ with $\psi \in \mathsf{C}_\mathrm{c}(\mathbb{R}_+)$ and $\chi \in \mathfrak{G}$.

If we represent L as multiplication operator by the independent variable on a von Neumann direct integral (see Section I.4), then a scaling transformation yields the following beautiful representation with separated variables for the resolvent difference $A_1 - A_0$:

Theorem V.4 (see [66, Theorem 1.2]).
The resolvent difference $A_1 - A_0$ *is unitarily equivalent to*

$$\left(\left[\Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big)^{\mathrm{N}} + I\right]^{-1} - \left[\Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big)^{\mathrm{D}} + I\right]^{-1}\right) \otimes (L+I)^{-1} \quad \text{on } \mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G}.$$

For brevity, let us write $\sigma = \sigma(L)$ for the spectrum of L. By Theorem I.92, we know that L is unitarily equivalent to the multiplication operator by the independent variable on a von Neumann direct integral $\int_\sigma^\oplus \mathfrak{G}(\lambda)\,\mathrm{d}\mu(\lambda)$. Moreover, the first factor (the difference of the Neumann and Dirichlet resolvents) in the previous theorem is a rank 1 operator with eigenvalue 0 of infinite multiplicity and simple eigenvalue 1/2, see (II.8). Hence, we conclude:

Corollary V.5 (see [66, Corollary 1.3]).
One has

$$\sigma(A_1 - A_0) = \{0\} \cup \Big\{\frac{1}{2(\lambda+1)} : \lambda \in \sigma\Big\},$$

and the spectral decomposition of $A_1 - A_0$ *is as follows:*

(1) 0 *is an eigenvalue of infinite multiplicity;*

(2) for $\bullet \in \{\mathrm{p}, \mathrm{ac}, \mathrm{sc}\}$ *one has* $\sigma_\bullet(A_1 - A_0) \setminus \{0\} = \big\{\frac{1}{2(\lambda+1)} : \lambda \in \sigma_\bullet\big\}$, *and the multiplicity of* $\frac{1}{2(\lambda+1)}$ *(with respect to* $A_1 - A_0$*) coincides with the multiplicity of* λ *(with respect to* L*) for* $\mu_\bullet$*-almost all* λ.

In particular, $A_1 - A_0$ *is compact if and only if* L *has a purely discrete spectrum.*[1]

[1]This is equivalent to $(L+I)^{-1}$ being compact, cf. Remark V.2.

The spectral decomposition of the difference of the spectral projections looks as follows:

Theorem V.6 (see [66, Theorem 1.4]).
Let $0 < \vartheta < 1$ *and let* $\alpha(\vartheta) = \frac{1}{\vartheta} - 1 > 0$*. Then one has:*

(1) $\sigma\big(E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)\big) = \begin{cases} [-1,1] & \textit{if } \mu\big(\sigma \cap [0,\alpha(\vartheta))\big) > 0 \\ \{0\} & \textit{if } \mu\big(\sigma \cap [0,\alpha(\vartheta))\big) = 0. \end{cases}$

(2) $\sigma_{\mathrm{p}}\big(E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)\big) = \begin{cases} \emptyset & \textit{if } \mu(\sigma \cap [\alpha(\vartheta),\infty)) = 0 \\ \{0\} & \textit{if } \mu(\sigma \cap [\alpha(\vartheta),\infty)) > 0. \end{cases}$

If $\mu(\sigma \cap [\alpha(\vartheta),\infty)) > 0$*, then the multiplicity of the eigenvalue* 0 *is infinite.*

(3) $\sigma_{\mathrm{ac}}\big(E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)\big) = \begin{cases} [-1,1] & \textit{if } \mu\big(\sigma \cap [0,\alpha(\vartheta))\big) > 0 \\ \emptyset & \textit{if } \mu\big(\sigma \cap [0,\alpha(\vartheta))\big) = 0. \end{cases}$

If $\mu\big(\sigma \cap [0,\alpha(\vartheta))\big) > 0$*, then the (uniform) multiplicity of the absolutely continuous spectrum equals the dimension of* $\int^{\oplus}_{\sigma\cap[0,\alpha(\vartheta))} \mathfrak{G}(\lambda)\,\mathrm{d}\mu(\lambda)$*.*

(4) The singular continuous spectrum is empty.

Let us close this section with a remark and an example.

Remark V.7 (Link to Hankel operators; see [66, Remark 1.5]).
We observe that $E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)$ is unitarily equivalent to its negative, that its kernel is either trivial or infinite dimensional, and that 0 belongs to its spectrum, for all $0 < \vartheta < 1$. Consequently, the characterization theorem for bounded self-adjoint Hankel operators (see Theorem III.4) implies that $E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1)$ is always unitarily equivalent to a Hankel integral operator on $\mathsf{L}^2(\mathbb{R}_+)$.

Example V.8 (Classical half-space; see [66, Example 1.6]).
If L is the free Laplacian on $\mathbb{R}^{n-1}$ for some $n \geq 2$, then the difference of the spectral projections associated with $(-\infty,\vartheta)$ has infinite dimensional kernel, and its (absolutely continuous) spectrum equals $[-1,1]$ and is of infinite multiplicity, for all $0 < \vartheta < 1$.

The further structure of this chapter is as follows. First, we briefly present the main tool of our analysis, namely the concept of boundary pairs and some facts on the tensor product of operators as well as the von Neumann direct integral decomposition of a self-adjoint operator (for a theoretical background, we refer to Section I.4). In Section V.3, we apply the theory of boundary pairs to our example and calculate the related objects explicitly. In particular, we establish Theorem V.3 (1). Section V.4 contains the proof of Theorem V.4. In Section V.5, we establish Theorem V.3 (2) and Theorem V.6. Finally, we discuss two ideas for further research.

V.2. Tools

Most of the following material is taken, almost verbatim, from the paper [**66**, pp. 296–298] by Olaf Post and the present author.

V.2.1. Boundary pairs. Let us briefly explain the concept of boundary pairs which is used to solve certain abstract boundary value problems for operators via their associated forms. Details can be found in [**65**].

Let $\mathfrak{H} \neq \{0\}$ be a complex Hilbert space and $\mathfrak{h}$ a closed and densely defined nonnegative form on $\mathfrak{H}$ with domain $\mathrm{Dom}(\mathfrak{h}) = \mathfrak{H}^1$. In particular, $\mathfrak{H}^1$ with its intrinsic norm defined by $\|u\|_{\mathfrak{h}}^2 = \mathfrak{h}(u) + \|u\|_{\mathfrak{H}}^2$ is complete. Further, let $\mathfrak{G} \neq \{0\}$ be another complex Hilbert space and $\Gamma : \mathfrak{H}^1 \to \mathfrak{G}$ a bounded operator.

Definition V.9 (Boundary pair/operator; see [65, Definition 2.1]).
We say that $(\Gamma, \mathfrak{G})$ is a *boundary pair* (associated with $\mathfrak{h}$) if the kernel of Γ is dense in $\mathfrak{H}$ with respect to the norm $\| \bullet \|_{\mathfrak{H}}$ and if the range $\mathrm{Ran}(\Gamma) = \mathfrak{G}^{1/2}$ is dense in $\mathfrak{G}$.

In this case, we call Γ a *boundary operator* (or *boundary map*).

Given a boundary pair $(\Gamma, \mathfrak{G})$, we can define the following objects:

- the *(abstract) Neumann operator* H as the self-adjoint nonnegative operator associated with $\mathfrak{h}$;
- the *(abstract) Dirichlet operator* H^{D} as the self-adjoint nonnegative operator associated with $\mathfrak{h}\restriction_{\mathrm{Ker}(\Gamma)}$ (note that $\mathrm{Ker}(\Gamma)$ is a closed subspace of $\mathfrak{H}^1$);
- the *space of weak solutions* in $z \in \mathbb{C}$,

$$\mathfrak{N}^1(z) = \big\{ h \in \mathfrak{H}^1 \; : \; \mathfrak{h}(h, u) = z\langle h, u\rangle_{\mathfrak{H}} \text{ for all } u \in \mathrm{Ker}(\Gamma) = \mathfrak{H}^{1,\mathrm{D}} \big\}.$$

One has:

Lemma V.10 (see [65, Proposition 2.9]).
Let $z \in \mathbb{C} \setminus \sigma(H^{\mathrm{D}})$.

(1) If $h_1, h_2 \in \mathfrak{N}^1(z)$ satisfy $\Gamma h_1 = \Gamma h_2$, then $h_1 = h_2$.
(2) We have $\mathfrak{H}^1 = \mathfrak{H}^{1,\mathrm{D}} \dotplus \mathfrak{N}^1(z)$ (direct sum with closed subspaces); the sum is orthogonal if $z = -1$.

In view of Lemma V.10, the *(Dirichlet) solution operator*

$$S(z) = (\Gamma\restriction_{\mathfrak{N}^1(z)})^{-1} : \mathrm{Ran}(\Gamma) = \mathfrak{G}^{1/2} \to \mathfrak{N}^1(z) \subset \mathfrak{H}^1$$

is well-defined.

Remark V.11. Let us note that $\mathfrak{G}^{1/2}$ endowed with the inner product $\langle \chi, \eta\rangle_{\mathfrak{G}^{1/2}} = \mathfrak{h}\big(S(-1)\chi, S(-1)\eta\big) + \langle S(-1)\chi, S(-1)\eta\rangle_{\mathfrak{H}}$ and the induced norm $\|\chi\|_{\mathfrak{G}^{1/2}}$ is a Hilbert space, see [**65**, p. 1062]

Let us recall some notions; we follow Kato [**40**].

Definition V.12 (Accretive operators; see [40, p. 279]).
An operator Λ on $\mathfrak{G}$ with domain $\mathrm{Dom}(\Lambda)$ is called *accretive* if $\mathrm{Re}\,\big(\langle \Lambda\chi, \chi\rangle_{\mathfrak{G}}\big) \geq 0$ for all $\chi \in \mathrm{Dom}(\Lambda)$.

Definition V.13 (m-accretive operators; see [40, p. 279]; cf. also [57, Thm. 2.3.2]).
An operator Λ on $\mathfrak{G}$ with domain $\mathrm{Dom}(\Lambda)$ is said to be *m-accretive* if for every $\zeta \in \mathbb{C}$ with $\mathrm{Re}\,\zeta < 0$,

$$\Lambda - \zeta I \text{ is boundedly invertible} \qquad \text{and} \qquad \|(\Lambda - \zeta I)^{-1}\|_{\mathrm{op}} \leq \frac{1}{|\mathrm{Re}\,\zeta|}.$$

Remark V.14 (to Definition V.13).
Every m-accretive operator is closed, accretive, and densely defined, see [**40**, p. 279]; cf. also Miklavčič [**57**, p. 58].

Definition V.15 (Quasi-m-accretive operators; see [40, p. 279]).
An operator Λ on $\mathfrak{G}$ with domain $\mathrm{Dom}(\Lambda)$ is called *quasi-m-accretive* if there exists an $\alpha \in \mathbb{C}$ such that $\Lambda + \alpha I$ is m-accretive.

Definition V.16 (Sectorial operators; see [40, p. 280]; cf. also [75, Definition 3.7]).
An operator Λ on $\mathfrak{G}$ with domain $\mathrm{Dom}(\Lambda)$ is said to be *sectorial* if there exist $\vartheta \in [0, \pi/2)$ and $\gamma \in \mathbb{R}$ such that the numerical range of Λ,

$$\Theta(\Lambda) = \big\{\langle \Lambda\chi, \chi\rangle_{\mathfrak{G}} : \chi \in \mathrm{Dom}(\Lambda),\ \|\chi\|_{\mathfrak{G}} = 1\big\},$$

is included in a sector

$$\mathcal{S}_{\gamma,\vartheta} = \big\{\zeta \in \mathbb{C} : \mathrm{Re}\,(\zeta - \gamma) > 0,\ |\arg(\zeta - \gamma)| \leq \vartheta\big\}. \tag{V.3}$$

Remark V.17. In (V.3), $-\pi < \arg(\zeta - \gamma) < \pi$ denotes the angle of $\zeta - \gamma$.

Definition V.18 (m-sectorial operators; see [40, p. 280]).
An operator on $\mathfrak{G}$ is called *m-sectorial* if it is sectorial and quasi-m-accretive.

Before we can define "sectorial forms," we need to recall some more terminology.

Definition V.19 (see [40, pp. 309–310]).
Let $\mathfrak{l}$ be a form on $\mathfrak{G}$ with domain $\mathrm{Dom}(\mathfrak{l})$.

(1) The *adjoint* $\mathfrak{l}^*$ of $\mathfrak{l}$ is given by

$$\mathfrak{l}^*(\chi, \eta) = \overline{\mathfrak{l}(\eta, \chi)} \quad \text{for all } \chi, \eta \in \mathrm{Dom}(\mathfrak{l}^*) = \mathrm{Dom}(\mathfrak{l}).$$

(2) The *real part* of $\mathfrak{l}$ is defined by

$$\mathrm{Re}\,(\mathfrak{l}) = \frac{1}{2}\big(\mathfrak{l} + \mathfrak{l}^*\big).$$

Definition V.20 (Sectorial forms; see [40, p. 310]).
A form $\mathfrak{l}$ on $\mathfrak{G}$ with domain $\mathrm{Dom}(\mathfrak{l})$ is called *sectorial* if there exist $\vartheta \in [0, \pi/2)$ and $\gamma \in \mathbb{R}$ such that the numerical range of $\mathfrak{l}$,

$$\big\{\mathfrak{l}(\chi) : \chi \in \mathrm{Dom}(\mathfrak{l}),\ \|\chi\|_{\mathfrak{G}} = 1\big\},$$

is included in a sector

$$\big\{\zeta \in \mathbb{C} : \mathrm{Re}\,(\zeta - \gamma) > 0,\ |\arg(\zeta - \gamma)| \leq \vartheta\big\}. \tag{V.4}$$

Remark V.21.

(I) In (V.4), $-\pi < \arg(\zeta - \gamma) < \pi$ denotes the angle of $\zeta - \gamma$.
(II) A sectorial form $\mathfrak{l}$ is said to be *closed* if $\mathrm{Re}\,(\mathfrak{l})$ is closed (cf. [**40**, p. 313]).

The following representation theorem holds.

Proposition V.22 (see [40, Theorem 2.1, p. 322]; cf. also [75, Theorem 11.8]). *Let $\mathfrak{l}$ be a densely defined closed sectorial form on $\mathfrak{G}$ with domain* $\mathrm{Dom}(\mathfrak{l})$. *Then there exists a unique associated m-sectorial operator $\Lambda_{\mathfrak{l}}$ with domain*

$$\mathrm{Dom}\left(\Lambda_{\mathfrak{l}}\right) = \left\{\chi \in \mathrm{Dom}(\mathfrak{l}) : \exists \varphi \in \mathfrak{G}\, \forall \eta \in \mathrm{Dom}(\mathfrak{l}),\, \mathfrak{l}(\chi, \eta) = \langle \varphi, \eta \rangle_{\mathfrak{G}}\right\}$$
$$\text{and} \qquad \Lambda_{\mathfrak{l}}\, \chi = \varphi.$$

Furthermore, $\mathrm{Dom}(\Lambda_{\mathfrak{l}})$ *is a dense subspace of the Hilbert space* $(\mathrm{Dom}(\mathfrak{l}), \|\bullet\|_{\mathrm{Re}(\mathfrak{l})})$.

We even have:

Corollary V.23 (see [40, Theorem 2.7, p. 323]; see also [75, Corollary 11.9]). *The map $\mathfrak{l} \mapsto \Lambda_{\mathfrak{l}}$ gives a one-to-one correspondence between densely defined closed sectorial forms and m-sectorial operators on $\mathfrak{G}$.*

Let us now continue with the theory of boundary pairs.

Proposition V.24 (see [65, Theorem 2.12]). *Let $(\Gamma, \mathfrak{G})$ be a boundary pair associated with $\mathfrak{h}$, and let $z \in \mathbb{C} \setminus \sigma(H^{\mathrm{D}})$. Then*

$$\mathfrak{l}_z : \mathfrak{G}^{1/2} \times \mathfrak{G}^{1/2} \to \mathbb{C}, \quad \mathfrak{l}_z(\chi, \eta) = \mathfrak{h}\big(S(z)\chi, S(-1)\eta\big) - z\langle S(z)\chi, S(-1)\eta\rangle_{\mathfrak{H}},$$

is a well-defined bounded form (with respect to the norm $\|\bullet\|_{\mathfrak{G}^{1/2}}$).

We call $\mathfrak{l}_z$ from Proposition V.24 the *Dirichlet-to-Neumann form*.

Notation V.25. For brevity, we will often write S in place of $S(-1)$.

Definition V.26 (Elliptic regularity; see [65, Definition 3.1]). We say that a boundary pair $(\Gamma, \mathfrak{G})$ is *elliptically regular* if there exists a constant $c > 0$ such that

$$\|S\chi\|_{\mathfrak{H}} \leq c\|\chi\|_{\mathfrak{G}} \quad \text{for all } \chi \in \mathfrak{G}^{1/2}.$$

Let us note:

Remark V.27 (to Definition V.26; cf. [66, p. 297]). A boundary pair $(\Gamma, \mathfrak{G})$ is elliptically regular if and only if the associated solution operator $S : \mathfrak{G}^{1/2} \to \mathfrak{H}^1$ extends to a bounded operator $\bar{S} : \mathfrak{G} \to \mathfrak{H}$.

In this case, we call $\bar{S}$ the *extended solution operator*.

One has:

Proposition V.28 (see [65, Theorem 3.8]). *Let $(\Gamma, \mathfrak{G})$ be an elliptically regular boundary pair, and let $z \in \mathbb{C} \setminus \sigma(H^{\mathrm{D}})$. Then the Dirichlet-to-Neumann form $\mathfrak{l}_z$ is closed, densely defined, and sectorial (as form on $\mathfrak{G}$ with* $\mathrm{Dom}(\mathfrak{l}_z) = \mathfrak{G}^{1/2}$*). Moreover, the domain of the associated m-sectorial operator $\Lambda_{\mathfrak{l}_z}$ is independent of z.*

Notation V.29. We will write $\Lambda(z)$ in place of $\Lambda_{\mathfrak{l}_z}$ and call it the *Dirichlet-to-Neumann operator*.

The main example is the following:

Example V.30 (see [66, p. 297]).
Let $\mathcal{X}$ be an open subset of $\mathbb{R}^n$ with smooth boundary $\mathcal{Y} = \partial\mathcal{X}$. Let $\mathfrak{H} = \mathsf{L}^2(\mathcal{X})$, $\mathfrak{h}(u) = \int_{\mathcal{X}} |\nabla u(x)|^2 \,\mathrm{d}x$, $\mathrm{Dom}(\mathfrak{h}) = \mathsf{W}^{1,2}(\mathcal{X})$. Moreover, let $\Gamma u = u{\restriction}_{\mathcal{Y}}$, i. e., Γ is the (Sobolev) trace map. Under suitable conditions (e. g., $\mathcal{Y}$ is compact or some curvature assumptions of $\mathcal{Y}$), $\Gamma : \mathsf{W}^{1,2}(\mathcal{X}) \to \mathsf{L}^2(\mathcal{Y})$ is bounded, where we consider $\mathcal{Y}$ as Riemannian manifold with its natural $(n-1)$-dimensional measure. In our example above, we have $\mathcal{X} = \mathbb{R}^n_+$ and $\mathcal{Y} = \{0\} \times \mathbb{R}^{n-1}$. Then H resp. H^{D} is the Neumann resp. Dirichlet Laplacian; $\mathfrak{N}^1(z)$ the space of weak solutions of $(-\Delta - zI)h = 0$ with $h \in \mathsf{W}^{1,2}(\mathcal{X})$; $S(z)$ is the solution operator, associating to $\chi \in \mathrm{Ran}(\Gamma)$ the weak solution h with $\Gamma h = \chi$. Moreover, $\Lambda(z)$ is the classical Dirichlet-to-Neumann operator (see, e. g., Lassas, Cheney, and Uhlmann [**47**]), associating to a boundary function $\chi : \mathcal{Y} \to \mathbb{C}$ the normal derivative of the function $h \in \mathfrak{N}^1(z)$ with $\Gamma h = \chi$.

One of the main results of [**65**] is the following.

Theorem V.31 (An M. Krein-type formula; see [65, Theorem 1.2]).
Let $(\Gamma, \mathfrak{G})$ be an elliptically regular boundary pair, and let $z \in \mathbb{C} \setminus \big(\sigma(H^{\mathrm{D}}) \cup \sigma(H)\big)$. Then the Dirichlet-to-Neumann operator $\Lambda(z)$ has a bounded inverse $\Lambda(z)^{-1} : \mathfrak{G} \to \mathfrak{G}$, and the following M. Krein-type formula holds:

$$(H - zI)^{-1} - (H^{\mathrm{D}} - zI)^{-1} = \bar{S}(z)\Lambda(z)^{-1}\bar{S}(\overline{z})^*.$$

V.2.2. Tensor product of operators. In this subsection, we fix some notation and briefly discuss how a result from [**75**] about cores for certain self-adjoint product type operators carries over to the forms associated with these operators; furthermore, we present three facts on operators of this product type.

We start with some preparations.

Remark V.32 (Tensor product of operators).
Let T_i be a densely defined and closable operator on a complex separable Hilbert space $\mathfrak{G}_i$ with domain $\mathrm{Dom}(T_i)$, where $i = 1, 2$. We write $\mathfrak{G}_1 \otimes \mathfrak{G}_2$ for the usual Hilbert space tensor product and $\mathfrak{G}_1 \odot \mathfrak{G}_2$ for the algebraic tensor product of $\mathfrak{G}_1$ and $\mathfrak{G}_2$. In this situation, one has (see [**75**, Proposition 7.20 (i)]):

$$(T_1 \odot T_2)\Bigg(\sum_{j=1}^{r} \chi_1^{(j)} \otimes \chi_2^{(j)}\Bigg) = \sum_{j=1}^{r} \Big(T_1\chi_1^{(j)}\Big) \otimes \Big(T_2\chi_2^{(j)}\Big) \qquad \Big(\chi_i^{(j)} \in \mathrm{Dom}(T_i),\ r \in \mathbb{N}\Big)$$

is a well-defined operator on $\mathfrak{G}_1 \otimes \mathfrak{G}_2$ with domain

$$\mathrm{Dom}(T_1 \odot T_2) = \Bigg\{ \sum_{j=1}^{r} \chi_1^{(j)} \otimes \chi_2^{(j)} : \chi_i^{(j)} \in \mathrm{Dom}(T_i),\ r \in \mathbb{N}\Bigg\}.$$

Moreover, $T_1 \odot T_2$ is also densely defined and closable (see [**75**, Lemma 7.21]). Its closure is denoted by $T_1 \otimes T_2$ and called the *tensor product of* T_1 and T_2.

If, in addition, T_1 and T_2 are symmetric and nonnegative (i. e., $\langle T_i\chi, \chi\rangle_{\mathfrak{G}_i} \geq 0$ for all $\chi \in \mathrm{Dom}(T_i)$), then $T_1 \odot T_2$ is also symmetric and nonnegative, see [**75**, Proposition 7.20 (iii)].

Let us note that in the case when T_1 and T_2 from Remark V.32 are bounded with $\mathrm{Dom}(T_1) = \mathfrak{G}_1$ and $\mathrm{Dom}(T_2) = \mathfrak{G}_2$, then $T_1 \otimes T_2$ clearly coincides with the *tensor product of bounded operators* discussed in Remark II.61 above.

In the following, we will assume that the operators T_1 and T_2 are self-adjoint and nonnegative. Let $T \in \{T_1, T_2\}$. We recall (see [**75**, p. 145]) that a vector $\chi \in \bigcap_{m=1}^{\infty} \mathrm{Dom}(T^m)$ is called *bounded for* T if there exists a constant $B_\chi > 0$ such that $\|T^m \chi\| \leq B_\chi^m$ for every $m \in \mathbb{N}$. In this case, we write $\chi \in \mathfrak{D}^{\mathrm{b}}(T)$. The following result will be very useful.

Proposition V.33. *Let $T_i \geq 0$ be a self-adjoint operator on $\mathfrak{G}_i$, where $i = 1, 2$. Then the operators $T_1 \otimes T_2$, $T_1 \otimes I$, $I \otimes T_2$, and $T_1 \otimes I + I \otimes T_2$ are nonnegative and self-adjoint. The dense subspace*

$$\mathfrak{D}_{\mathrm{b}} = \mathrm{span}\{\chi_1 \otimes \chi_2 : \chi_1 \in \mathfrak{D}^{\mathrm{b}}(T_1), \chi_2 \in \mathfrak{D}^{\mathrm{b}}(T_2)\} \tag{V.5}$$

of $\mathfrak{G}_1 \otimes \mathfrak{G}_2$ is an invariant core for each of these four operators.

Proposition V.33 is a consequence of the following two lemmas.

Lemma V.34 (see [75, Theorem 7.23]).
Let $T_i \geq 0$ be a self-adjoint operator on $\mathfrak{G}_i$, where $i = 1, 2$. Then the operators $T_1 \otimes T_2$, $T_1 \otimes I$, and $I \otimes T_2$ are self-adjoint and nonnegative; $T_1 \otimes I + I \otimes T_2$ is essentially *self-adjoint and nonnegative. The dense subspace*

$$\mathfrak{D}_{\mathrm{b}} = \mathrm{span}\{\chi_1 \otimes \chi_2 : \chi_1 \in \mathfrak{D}^{\mathrm{b}}(T_1), \chi_2 \in \mathfrak{D}^{\mathrm{b}}(T_2)\}$$

of $\mathfrak{G}_1 \otimes \mathfrak{G}_2$ is an invariant core for each of these four operators.

Lemma V.35 (see [75, Exercise 17.a, p. 163]).
The operator $T_1 \otimes I + I \otimes T_2$ from Lemma V.34 is closed and hence self-adjoint.

Proof. Qualitatively, the assertion is a consequence of the first binomial formula.

For brevity, let us write $H = T_1 \otimes I + I \otimes T_2$ and $\mathfrak{H} = \mathfrak{G}_1 \otimes \mathfrak{G}_2$. Let (u_m) be a sequence in $\mathfrak{D}_{\mathrm{b}}$ such that

$$u_m \xrightarrow{m\to\infty} u \in \mathfrak{H} \quad \text{and} \quad Hu_m \xrightarrow{m\to\infty} v \in \mathfrak{H}.$$

Then, on one hand,

$$\|Hu_n - Hu_m\|_{\mathfrak{H}}^2 \xrightarrow{n,m\to\infty} 0.$$

On the other hand, as $T_1 \otimes T_2 \geq 0$ (see Lemma V.34), we have

$$\begin{aligned}\|Hu_n - Hu_m\|_{\mathfrak{H}}^2 &= \|(T_1 \otimes I)u_n - (T_1 \otimes I)u_m\|_{\mathfrak{H}}^2 + \|(I \otimes T_2)u_n - (I \otimes T_2)u_m\|_{\mathfrak{H}}^2 \\ &\quad + 2\langle (T_1 \otimes T_2)(u_n - u_m), u_n - u_m\rangle_{\mathfrak{H}} \\ &\geq \|(T_1 \otimes I)u_n - (T_1 \otimes I)u_m\|_{\mathfrak{H}}^2 + \|(I \otimes T_2)u_n - (I \otimes T_2)u_m\|_{\mathfrak{H}}^2.\end{aligned}$$

Since $T_1 \otimes I$ and $I \otimes T_2$ are self-adjoint (hence closed), we thus obtain

$$u \in \mathrm{Dom}(T_1 \otimes I) \quad \text{and} \quad u \in \mathrm{Dom}(I \otimes T_2), \quad \text{i.e.,} \quad u \in \mathrm{Dom}(H).$$

We conclude that

$$\begin{aligned}\|Hu - v\|_{\mathfrak{H}} &\leq \|(T_1 \otimes I)u - (T_1 \otimes I)u_m\|_{\mathfrak{H}} \\ &\quad + \|(I \otimes T_2)u - (I \otimes T_2)u_m\|_{\mathfrak{H}} + \|Hu_m - v\|_{\mathfrak{H}} \\ &\xrightarrow{m\to\infty} 0.\end{aligned}$$

It follows that H is closed. □

Proof of Proposition V.33. We combine Lemmas V.34 and V.35. □

The following result is an easy consequence of Proposition V.33 and will be very useful.

Proposition V.36 (see [66, Proposition 2.1]).
The subspace $\mathfrak{D}_\mathrm{b}$ of $\mathfrak{G}_1 \otimes \mathfrak{G}_2$ defined in (V.5) *is a core for the form associated with the operator $T_1 \otimes I + I \otimes T_2$.*

Proof. For brevity, let us write $H = T_1 \otimes I + I \otimes T_2$ and $\mathfrak{H} = \mathfrak{G}_1 \otimes \mathfrak{G}_2$. It suffices to show that $\mathfrak{D}_\mathrm{b}$ is a core for the self-adjoint operator $H^{1/2}$, see [**75**, Proposition 10.5].

It is well known (see, e. g., [**75**, Corollary 4.14]) that the domain of H is a core for $H^{1/2}$. Let $u \in \mathrm{Dom}(H)$. Since $\mathfrak{D}_\mathrm{b}$ is a core for H (see Proposition V.33), we can choose a sequence $(u_m) \subset \mathfrak{D}_\mathrm{b}$ such that $u_m \to u$ in $\mathfrak{H}$ and $Hu_m \to Hu$ in $\mathfrak{H}$ as $m \to \infty$. It follows directly from the functional calculus for self-adjoint operators and the obvious inequality $\lambda \leq 1 + \lambda^2$ for all $\lambda \in \mathbb{R}$ that $H^{1/2}u_m \to H^{1/2}u$ in $\mathfrak{H}$ as $m \to \infty$. Consequently, $\mathfrak{D}_\mathrm{b}$ is a core for $H^{1/2}$, as claimed. □

Here are three more facts on operators of the type $T_1 \otimes I + I \otimes T_2$.

Proposition V.37 (cf. the formulation of [66, Proposition 2.2]).
Let, as above, T_1 and T_2 be nonnegative self-adjoint operators.

(1) $\sigma(T_1 \otimes I + I \otimes T_2) = \{t_1 + t_2 : t_i \in \sigma(T_i),\ i = 1, 2\}$.

(2) For every $\alpha \in \mathbb{R}$, all $\psi, \rho \in \mathfrak{G}_1$, and all $\chi, \eta \in \mathfrak{G}_2$, one has

$$\begin{aligned}&\langle E_{(-\infty,\alpha)}(T_1 \otimes I + I \otimes T_2)(\psi \otimes \chi), \rho \otimes \eta\rangle_{\mathfrak{G}_1\otimes\mathfrak{G}_2} \\ &\qquad = \int_{\mathbb{R}} \langle E_{(-\infty,\alpha-\lambda)}(T_1)\psi, \rho\rangle_{\mathfrak{G}_1}\, \mathrm{d}\langle E_\lambda(T_2)\chi, \eta\rangle.\end{aligned}$$

(3) The operator $T_1 \otimes I + I \otimes T_2$ has a purely absolutely continuous spectrum if T_1 has a purely absolutely continuous spectrum.

Proof. (1) In view of [**75**, Corollary 7.25] and the fact that $T_1 \otimes I + I \otimes T_2$ is self-adjoint, we have

$$\sigma(T_1 \otimes I + I \otimes T_2) = \overline{\{t_1 + t_2 : t_i \in \sigma(T_i),\ i = 1, 2\}}.$$

It only remains to show that $\{t_1 + t_2 : t_i \in \sigma(T_i),\ i = 1, 2\}$ is a closed subset of $\mathbb{R}$ (cf. [**75**, Exercise 18.a, p. 163]). This follows easily from the Bolzano–Weierstraß theorem.

For part (2), see [**85**, Theorem 8.34] and for part (3), see [**52**, Proposition A.2 (iv)]. □

V.2.3. The von Neumann direct integral. The theory of von Neumann direct integrals is one of the main tools in this chapter; for a theoretical background, we refer to Section I.4. In the present subsection, we recall some notation and discuss how the theory of von Neumann direct integrals can be applied in our example.

Let $\mathfrak{p}$ be a finite Borel measure on $\mathbb{R}$. We denote the von Neumann direct integral of complex separable Hilbert spaces $\mathfrak{G}(\lambda)$ by $\mathfrak{G} = \int_{\mathbb{R}}^{\oplus} \mathfrak{G}(\lambda)\, \mathrm{d}\mathfrak{p}(\lambda)$. Any element $\chi \in \mathfrak{G}$ corresponds to the values $\chi(\lambda) \in \mathfrak{G}(\lambda)$, defined for $\mathfrak{p}$-almost all $\lambda \in \mathbb{R}$. We will use the notation $\chi = \int_{\mathbb{R}}^{\oplus} \chi(\lambda)\, \mathrm{d}\mathfrak{p}(\lambda)$. The von Neumann direct integral $\mathfrak{G}$ together with the inner product

$$\langle \chi_1, \chi_2 \rangle_{\mathfrak{G}} = \int_{\mathbb{R}} \langle \chi_1(\lambda), \chi_2(\lambda) \rangle_{\mathfrak{G}(\lambda)}\, \mathrm{d}\mathfrak{p}(\lambda) \qquad (\chi_1, \chi_2 \in \mathfrak{G})$$

is a Hilbert space. The induced norm is denoted by $\|\bullet\|_{\mathfrak{G}}$. We assume without loss of generality that $\mathfrak{G}(\lambda) \neq \{0\}$ for $\mathfrak{p}$-almost every λ. Further, we identify the Hilbert spaces $\int_{\mathbb{R}}^{\oplus} \mathfrak{G}(\lambda)\, \mathrm{d}\mathfrak{p}(\lambda)$ and $\int_{\mathrm{supp}(\mathfrak{p})}^{\oplus} \mathfrak{G}(\lambda)\, \mathrm{d}\mathfrak{p}(\lambda)$, where $\mathrm{supp}(\mathfrak{p})$ denotes the support of the measure $\mathfrak{p}$. We will make use of the well-known fact that *every self-adjoint operator on a complex separable Hilbert space is unitarily equivalent to the multiplication operator by the independent variable on a von Neumann direct integral* (see Theorem I.92).

Except for Subsection V.3.7, we will suppose in Sections V.3–V.5:

Assumption V.38 (see [66, Assumption 2.4]).
The operator L in (V.1) acts by multiplication by the independent variable on a von Neumann direct integral $\mathfrak{G} = \int_{\mathbb{R}}^{\oplus} \mathfrak{G}(\lambda)\, \mathrm{d}\mathfrak{p}(\lambda) \neq \{0\}$.

Remark V.39 (see [66, Remark 2.5]).
With this assumption, we do not forfeit generality. This is clear in view of Theorem V.4, Corollary V.5, and Theorem V.6. In view of Theorem V.3, we will show in Subsections V.3.7 and V.5.1 below that the corresponding results from Proposition V.59 and Lemma V.65 naturally carry over to the situation when L is not necessarily a multiplication operator.

V.3. The boundary pair of the generalized half-space problem

The following material is taken, almost verbatim, from the paper [**66**, pp. 299–308] by Olaf Post and the present author.

Let $\mathfrak{G} \neq \{0\}$ be a complex separable Hilbert space and $\mathfrak{H} = \mathsf{L}^2(\mathbb{R}_+; \mathfrak{G})$. As $\mathfrak{H}$ and $\mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G}$ are naturally isometrically isomorphic, we will very often identify $\psi(\bullet)\chi$ with $\psi \otimes \chi$ for all $\psi \in \mathsf{L}^2(\mathbb{R}_+)$ and $\chi \in \mathfrak{G}$.

V.3.1. The form and its associated operator. Let us consider the nonnegative form $\mathfrak{h}$ on $\mathfrak{H}$, with domain $\mathfrak{H}^1 = \mathsf{W}^{1,2}(\mathbb{R}_+; \mathfrak{G}) \cap \mathsf{L}^2(\mathbb{R}_+; \mathrm{Dom}(L^{1/2}))$, defined by

$$\mathfrak{h}(u, v) = \int_{\mathbb{R}_+} \Big(\langle \dot{u}(t), \dot{v}(t) \rangle_{\mathfrak{G}} + \big\langle L^{1/2}\big(u(t)\big), L^{1/2}\big(v(t)\big) \big\rangle_{\mathfrak{G}} \Big)\, \mathrm{d}t,$$

where $\mathrm{Dom}(L^{1/2})$ is equipped with the graph norm of $L^{1/2}$; see Section I.3 for a brief introduction to Sobolev spaces on an interval. It is easy to see that $\mathfrak{h}$ is closed and densely defined.

Let H be the self-adjoint nonnegative operator

$$H = \Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big)^{\mathrm{N}} \otimes I + I \otimes L \quad \text{on } \mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G}.$$

Using the above-mentioned identification of $\mathfrak{H} = \mathsf{L}^2(\mathbb{R}_+;\mathfrak{G})$ with $\mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G}$, one can show:

Proposition V.40 (see [52, Proposition 5.2 (iii)]).
The domain of H is given by

$$\mathrm{Dom}(H) = \{u \in \mathsf{W}^{2,2}(\mathbb{R}_+;\mathfrak{G}) \cap \mathsf{L}^2(\mathbb{R}_+;\mathrm{Dom}(L)) : \dot{u}(0) = 0\},$$

where $\mathrm{Dom}(L)$ *is equipped with the graph norm of L.*

We conclude:

Lemma V.41 (see [66, Lemma 3.1]).
The operator H is associated with the form $\mathfrak{h}$.

Proof. For all $u \in \mathrm{Dom}(H)$ and all $v \in \mathfrak{H}^1$, we have

$$\begin{aligned}\mathfrak{h}(u,v) &= \int_{\mathbb{R}_+} \Big(\langle \dot{u}(t), \dot{v}(t)\rangle_{\mathfrak{G}} + \langle L^{1/2}(u(t)), L^{1/2}(v(t))\rangle_{\mathfrak{G}}\Big)\,\mathrm{d}t \\ &= \int_{\mathbb{R}_+} \Big(\langle -\ddot{u}(t), v(t)\rangle_{\mathfrak{G}} + \langle L(u(t)), v(t)\rangle_{\mathfrak{G}}\Big)\,\mathrm{d}t = \langle Hu, v\rangle_{\mathfrak{H}},\end{aligned}$$

where we used integration by parts and the self-adjointness of $L^{1/2}$. Since H is self-adjoint, the claim follows. □

Let us recall that

$$\mathfrak{D}_{\mathrm{b}} = \mathrm{span}\{\psi \otimes \chi : \psi \in \mathfrak{D}^{\mathrm{b}}\big((-\mathrm{d}^2/\mathrm{d}t^2)^{\mathrm{N}}\big), \chi \in \mathfrak{D}^{\mathrm{b}}(L)\} \subset \mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G}$$

is a core for H as well as for $\mathfrak{h}$ by Subsection V.2.2, where the functions $\chi = \int_\sigma^{\oplus} \chi(\lambda)\,\mathrm{d}\mathfrak{p}(\lambda)$ are $\mathfrak{p}$-measurable and satisfy $\int_\sigma \langle \chi(\lambda), \chi(\lambda)\rangle_{\mathfrak{G}(\lambda)}\,\mathrm{d}\mathfrak{p}(\lambda) < \infty$.

Notation V.42. We denote by $\sqrt{z}$ the branch of the square root defined on the plane cut along the positive half-axis.

Functions of the type

$$h : \mathbb{R}_+ \to \mathfrak{G}, \qquad t \mapsto h(t) = \int_\sigma^{\oplus} \exp(\mathrm{i}\sqrt{z-\lambda}\,t)\chi(\lambda)\,\mathrm{d}\mathfrak{p}(\lambda), \tag{V.6}$$

will play an important role in this chapter. First of all, we have to check that h is in $\mathfrak{H}$ for all $z \in \mathbb{C} \setminus [\min\sigma, \infty)$ and all $\chi \in \mathfrak{G}$.

Lemma V.43 (see [66, Lemma 3.2]).
Let $z \in \mathbb{C} \setminus [\min\sigma, \infty)$ and let $\chi \in \mathfrak{G}$. Then the function $h : \mathbb{R}_+ \to \mathfrak{G}$ defined in (V.6) *is continuous and $h \in \mathfrak{H}$.*

Proof. For every $t \in \mathbb{R}_+$, one has $\|h(t)\|_{\mathfrak{G}} \le \|\chi\|_{\mathfrak{G}} < \infty$ so h is $\mathfrak{G}$-valued. By the dominated convergence theorem, we see that $\mathbb{R}_+ \ni t \mapsto h(t) \in \mathfrak{G}$ is continuous. Consequently, h is measurable and we compute

$$\begin{aligned} \|h\|_{\mathfrak{H}}^2 &\le \int_{\mathbb{R}_+} \mathrm{d}t \int_\sigma \mathrm{d}\mu(\lambda) \exp\left(-2^{1/2}\left(|z| - \mathrm{Re}\,(z)\right)^{1/2} t\right) \|\chi(\lambda)\|_{\mathfrak{G}(\lambda)}^2 \\ &= \frac{1}{2^{1/2}\left(|z| - \mathrm{Re}\,(z)\right)^{1/2}} \|\chi\|_{\mathfrak{G}}^2 < \infty \qquad \text{if } z \in \mathbb{C} \setminus \mathbb{R} \end{aligned} \tag{V.7}$$

as well as

$$\begin{aligned} \|h\|_{\mathfrak{H}}^2 &\le \int_{\mathbb{R}_+} \mathrm{d}t \int_\sigma \mathrm{d}\mu(\lambda) \exp\left(-2\left(\min\sigma - z\right)^{1/2} t\right) \|\chi(\lambda)\|_{\mathfrak{G}(\lambda)}^2 \\ &= \frac{1}{2\left(\min\sigma - z\right)^{1/2}} \|\chi\|_{\mathfrak{G}}^2 < \infty \qquad \text{if } z \in (-\infty, \min\sigma). \end{aligned} \tag{V.8}$$

□

Next, we show:

Lemma V.44 (see [66, Lemma 3.3]).
Let $z \in \mathbb{C} \setminus [\min\sigma, \infty)$ and let $\chi \in \mathrm{Dom}(L^{1/4})$. Then the function $h : \mathbb{R}_+ \to \mathfrak{G}$ defined in (V.6) *is also in $\mathfrak{H}^1$.*

Proof. First, we consider the case when $\chi \in \mathrm{Dom}(L)$. By Lemma V.43, we know that $h \in \mathfrak{H}$, and it is straightforward to show that $h \in \mathfrak{H}^1$; note that $\dot{h}$ exists in the strong sense.

Now we consider the case when $\chi \in \mathrm{Dom}(L^{1/4})$. Again, Lemma V.43 shows that h is in $\mathfrak{H}$. Since $\mathrm{Dom}(L)$ is a core for $L^{1/4}$, we can approximate χ by a sequence $(\chi_m) \subset \mathrm{Dom}(L)$ with respect to the graph norm of $L^{1/4}$. Straightforward computations show that $\|h - h_m\|_{\mathfrak{H}} \xrightarrow{m\to\infty} 0$ and $\mathfrak{h}(h_k - h_m) \xrightarrow{k,m\to\infty} 0$, where $h_m = \int_\sigma^\oplus \exp(\mathrm{i}\sqrt{z-\lambda}\,\bullet)\chi_m(\lambda)\,\mathrm{d}\mu(\lambda)$ for all $m \in \mathbb{N}$. Consequently, the closedness of $\mathfrak{h}$ yields:

$$h \in \mathfrak{H}^1 \quad \text{and} \quad \|h - h_m\|_{\mathfrak{h}} \xrightarrow{m\to\infty} 0. \tag{V.9}$$

This completes the proof of the lemma. □

V.3.2. The boundary operator. As boundary operator, we will choose the restriction to $\mathfrak{H}^1$ of the usual boundary operator on the Sobolev space $\mathsf{W}^{1,2}(\mathbb{R}_+; \mathfrak{G})$ that evaluates a given function at zero, i. e., we define $\Gamma : \mathfrak{H}^1 \to \mathfrak{G}$ by $\Gamma u = u(0)$. The next two lemmas show that Γ is indeed a boundary operator in the sense of Definition V.9.

Lemma V.45 (see [66, Lemma 3.4]).
One has $\|\Gamma\|_{\mathrm{op}} \le 2$.

Proof. Let $u \in \mathfrak{H}^1$. We define the Lipschitz continuous function $f : [0,\infty) \to [0,1]$ by

$$f(t) = 1 - t \text{ if } 0 \le t < 1 \quad \text{and} \quad f(t) = 0 \text{ if } t \ge 1.$$

Then one has

$$u(0) = -\big((f\cdot u)(1) - (f\cdot u)(0)\big) = -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(f\cdot u)(t)\,\mathrm{d}t = -\int_0^1 \big(\dot{f}(t)u(t) + f(t)\dot{u}(t)\big)\,\mathrm{d}t.$$

The result now follows from

$$\begin{aligned}\|\Gamma u\|_{\mathfrak{G}}^2 &\le 2\int_0^1 \big\|\dot f(t)u(t)+f(t)\dot u(t)\big\|_{\mathfrak{G}}^2\,\mathrm{d}t\\ &\le 4\int_0^1 \big(\|u(t)\|_{\mathfrak{G}}^2+\|\dot u(t)\|_{\mathfrak{G}}^2\big)\,\mathrm{d}t\\ &\le 4\|u\|_{\mathfrak{h}}^2.\end{aligned}$$

□

The proof of the following lemma is straightforward:

Lemma V.46 (see [66, Lemma 3.5]).
The kernel of Γ *is dense in* $\mathfrak{H}$ *with respect to the norm* $\|\bullet\|_{\mathfrak{H}}$*, and the range of* Γ *is dense in* $\mathfrak{G}$*.*

Next, we define the form $\mathfrak{h}^{\mathrm{D}} = \mathfrak{h}\restriction_{\mathfrak{H}^{1,\mathrm{D}}}$ on the closed subspace $\mathfrak{H}^{1,\mathrm{D}} = \mathrm{Ker}(\Gamma)$ of $\mathfrak{H}^1$. Then $\mathfrak{h}^{\mathrm{D}}$ is densely defined, closed, and nonnegative (as form on $\mathfrak{H}$ with domain $\mathfrak{H}^{1,\mathrm{D}}$). We call H^{D}, the self-adjoint nonnegative operator associated with $\mathfrak{h}^{\mathrm{D}}$, the *Dirichlet operator.* We will show (see Lemma V.47) that the Dirichlet operator coincides with

$$\Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big)^{\mathrm{D}}\otimes I + I\otimes L \quad \text{on } \mathsf{L}^2(\mathbb{R}_+)\otimes\mathfrak{G}.$$

We know (see Subsection V.2.2 above) that

$$\mathfrak{D}_{\mathrm{b}}^{\mathrm{D}} = \mathrm{span}\{\psi\otimes\chi : \psi\in\mathfrak{D}^{\mathrm{b}}\big((-\,\mathrm{d}^2/\mathrm{d}t^2)^{\mathrm{D}}\big), \chi\in\mathfrak{D}^{\mathrm{b}}(L)\}\subset\mathsf{L}^2(\mathbb{R}_+)\otimes\mathfrak{G}$$

is an invariant core for $\big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big)^{\mathrm{D}}\otimes I + I\otimes L$. Let us note that $\mathfrak{D}_{\mathrm{b}}^{\mathrm{D}}\subset\mathrm{Ker}(\Gamma)$. We have the following expected result:

Lemma V.47 (see [66, Lemma 3.6]; cf. also [12, Proposition 5.6]).
The Dirichlet operator is given by

$$H^{\mathrm{D}} = \Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big)^{\mathrm{D}}\otimes I + I\otimes L.$$

Proof. For brevity, we write $\widetilde{H}^{\mathrm{D}} = \big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big)^{\mathrm{D}}\otimes I + I\otimes L$. We will show that $\widetilde{H}^{\mathrm{D}}$ is associated with $\mathfrak{h}^{\mathrm{D}}$. This is proved in three steps:

Step 1. Integration by parts yields $\mathfrak{h}^{\mathrm{D}}(u,v) = \big\langle \widetilde{H}^{\mathrm{D}}u, v\big\rangle_{\mathfrak{H}}$ for all $u,v\in\mathfrak{D}_{\mathrm{b}}^{\mathrm{D}}$.

Step 2. Let $u\in\mathfrak{D}_{\mathrm{b}}^{\mathrm{D}}$ and let $\tilde v\in\mathrm{Ker}(\Gamma)$. We choose $(v_k)\subset\mathfrak{D}_{\mathrm{b}}$ with $\|\tilde v - v_k\|_{\mathfrak{h}} \xrightarrow[k\to\infty]{} 0$. Integration by parts yields $\mathfrak{h}(u,v_k) = \big\langle \widetilde{H}^{\mathrm{D}}u, v_k\big\rangle_{\mathfrak{H}} - \langle\Gamma(\dot u),\Gamma v_k\rangle_{\mathfrak{G}}$, where $\Gamma(\dot u)\in\mathrm{Dom}(L)\subset\mathfrak{G}$. As $k\to\infty$ we obtain that, on one hand, $\mathfrak{h}(u,v_k)\to\mathfrak{h}(u,\tilde v) = \mathfrak{h}^{\mathrm{D}}(u,\tilde v)$ and, on the other hand, $\big\langle \widetilde{H}^{\mathrm{D}}u, v_k\big\rangle_{\mathfrak{H}} - \langle\Gamma(\dot u),\Gamma v_k\rangle_{\mathfrak{G}} \to \big\langle \widetilde{H}^{\mathrm{D}}u, \tilde v\big\rangle_{\mathfrak{H}} - \langle\Gamma(\dot u),\Gamma \tilde v\rangle_{\mathfrak{G}} = \big\langle \widetilde{H}^{\mathrm{D}}u, \tilde v\big\rangle_{\mathfrak{H}}$.

Step 3. Let $\tilde u\in\mathrm{Dom}\big(\widetilde{H}^{\mathrm{D}}\big)$ and let $\tilde v\in\mathrm{Ker}(\Gamma)$. We choose $(u_m)\subset\mathfrak{D}_{\mathrm{b}}^{\mathrm{D}}$ with $\|\tilde u - u_m\|_{\widetilde{H}^{\mathrm{D}}} \xrightarrow[m\to\infty]{} 0$. Then, by Step 1 and the positivity of $\widetilde{H}^{\mathrm{D}}$, one has

$$\mathfrak{h}^{\mathrm{D}}(u_k-u_m) = \big|\big\langle \widetilde{H}^{\mathrm{D}}(u_k-u_m), u_k-u_m\big\rangle_{\mathfrak{H}}\big| \le \|u_k-u_m\|_{\widetilde{H}^{\mathrm{D}}}^2 \quad \text{for all } k,m\in\mathbb{N}$$

so (u_m) is Cauchy with respect to $\|\bullet\|_{\mathfrak{h}^{\mathrm{D}}}$. Since $\mathfrak{h}^{\mathrm{D}}$ is closed, it follows that $\tilde u\in\mathrm{Ker}(\Gamma)$ and $\|\tilde u - u_m\|_{\mathfrak{h}^{\mathrm{D}}} \xrightarrow[m\to\infty]{} 0$. As $m\to\infty$ we obtain that, on one hand, $\mathfrak{h}^{\mathrm{D}}(u_m,\tilde v)\to\mathfrak{h}^{\mathrm{D}}(\tilde u,\tilde v)$

and, on the other hand, $\langle \widetilde{H}^{\mathrm{D}} u_m, \tilde{v} \rangle_{\mathfrak{H}} \to \langle \widetilde{H}^{\mathrm{D}} \tilde{u}, \tilde{v} \rangle_{\mathfrak{H}}$. Consequently,

$$\mathfrak{h}^{\mathrm{D}}(\tilde{u}, \tilde{v}) = \langle \widetilde{H}^{\mathrm{D}} \tilde{u}, \tilde{v} \rangle_{\mathfrak{H}}$$

and thus $\widetilde{H}^{\mathrm{D}} \subset H^{\mathrm{D}}$. Since $\widetilde{H}^{\mathrm{D}}$ and H^{D} are both self-adjoint, we conclude that $\widetilde{H}^{\mathrm{D}} = H^{\mathrm{D}}$. □

Lemma V.48 (see [66, Lemma 3.7]).

(1) The operators H and H^{D} are unitarily equivalent.
(2) The spectrum of H is purely absolutely continuous filling in the interval $[\min \sigma, \infty)$; the same is true for H^{D}.

Proof. (1) Since the Neumann and Dirichlet Laplacians on $\mathsf{L}^2(\mathbb{R}_+)$ are unitarily equivalent, it follows that H and H^{D} are also unitarily equivalent. Part (2) is a consequence of Proposition V.37. □

Remark V.49 (A regularity result; cf. the formulation of [66, Remark 3.8]). One can actually show that the domain of H^{D} is given by (see [**52**, Proposition 5.2 (ii)])

$$\mathrm{Dom}(H^{\mathrm{D}}) = \{u \in \mathsf{W}^{2,2}(\mathbb{R}_+; \mathfrak{G}) \cap \mathsf{L}^2(\mathbb{R}_+; \mathrm{Dom}(L)) : u(0) = 0\},$$

where $\mathrm{Dom}(L)$ is equipped with the graph norm of L.

V.3.3. The solution operator and the range of the boundary operator. Let $z \in \mathbb{C} \setminus [\min \sigma, \infty)$. We define

$$\mathfrak{N}^1(z) = \{h \in \mathfrak{H}^1 : \mathfrak{h}(h, u) = z\langle h, u\rangle_{\mathfrak{H}} \text{ for all } u \in \mathrm{Ker}(\Gamma)\}.$$

The solution operator $S(z) = \left(\Gamma\restriction_{\mathfrak{N}^1(z)}\right)^{-1}$ associates to a boundary value $\chi \in \mathrm{Ran}(\Gamma)$ the unique element $h \in \mathfrak{N}^1(z)$ such that $\Gamma h = \chi$ (see Lemma V.10).

Lemma V.50 (see [66, Lemma 3.9]).
One has $\mathrm{Dom}(L^{1/4}) \subset \mathrm{Ran}(\Gamma)$ and, for every $z \in \mathbb{C} \setminus [\min \sigma, \infty)$,

$$S(z)\restriction_{\mathrm{Dom}(L^{1/4})}\chi = \int_{\sigma}^{\oplus} \exp\bigl(\mathrm{i}\sqrt{z-\lambda}\,\bullet\bigr)\chi(\lambda)\,\mathrm{d}\mu(\lambda). \tag{V.10}$$

Proof. The lemma is proved in two steps. First, we show that $\mathrm{Dom}(L) \subset \mathrm{Ran}(\Gamma)$ and (V.10) holds on $\mathrm{Dom}(L)$. Then, by approximation, we obtain that $\mathrm{Dom}(L^{1/4}) \subset \mathrm{Ran}(\Gamma)$ and (V.10) holds on $\mathrm{Dom}(L^{1/4})$.

Step 1. Let $\chi \in \mathrm{Dom}(L)$ and let $h = \int_{\sigma}^{\oplus} \exp\left(\mathrm{i}\sqrt{z-\lambda}\,\bullet\right)\chi(\lambda)\,\mathrm{d}\mu(\lambda)$. By Lemma V.44, we know that $h \in \mathfrak{H}^1$ and hence $\Gamma h = \chi$. It remains to show that $h \in \mathfrak{N}^1(z)$. This is proved as follows:

Let $v \in \mathfrak{D}_{\mathrm{b}}$. A straightforward computation shows that

$$\begin{aligned}\mathfrak{h}(h, v) &= \langle \dot{h}, \dot{v}\rangle_{\mathfrak{H}} + \int_{\mathbb{R}_+} \langle L\bigl(h(t)\bigr), v(t)\rangle_{\mathfrak{G}}\,\mathrm{d}t \\ &= z\langle h, v\rangle_{\mathfrak{H}} - \mathrm{i}\int_{\sigma} \langle \sqrt{z-\lambda}\,\chi(\lambda), (\Gamma v)(\lambda)\rangle_{\mathfrak{G}(\lambda)}\,\mathrm{d}\mu(\lambda).\end{aligned}$$

Now let $u \in \operatorname{Ker}(\Gamma)$. We choose a sequence $(v_m) \subset \mathfrak{D}_{\mathrm{b}}$ with $\|u - v_m\|_{\mathfrak{h}} \xrightarrow{m\to\infty} 0$. Clearly, $\mathfrak{h}(h, v_m) \xrightarrow{m\to\infty} \mathfrak{h}(h, u)$ and $z\,\langle h, v_m\rangle_{\mathfrak{H}} \xrightarrow{m\to\infty} z\,\langle h, u\rangle_{\mathfrak{H}}$, and an easy computation shows that $|-\mathrm{i}\int_\sigma \langle \sqrt{z-\lambda}\,\chi(\lambda), (\Gamma v_m)(\lambda)\rangle_{\mathfrak{G}(\lambda)}\,\mathrm{d}\mu(\lambda)| \xrightarrow{m\to\infty} 0$. It follows that h is in $\mathfrak{N}^1(z)$.

Step 2. Let $\chi \in \operatorname{Dom}(L^{1/4})$ and let $h = \int_\sigma^\oplus \exp\big(\mathrm{i}\sqrt{z-\lambda}\,\bullet\big)\chi(\lambda)\,\mathrm{d}\mu(\lambda)$. Again, we know by Lemma V.44 that $h \in \mathfrak{H}^1$ and hence $\Gamma h = \chi$.

Now we choose a sequence $(\chi_m) \subset \operatorname{Dom}(L)$ with $\|\chi - \chi_m\|_{L^{1/4}} \xrightarrow{m\to\infty} 0$. By Step 1, we know that $h_m = \int_\sigma^\oplus \exp\big(\mathrm{i}\sqrt{z-\lambda}\,\bullet\big)\chi_m(\lambda)\,\mathrm{d}\mu(\lambda) \in \mathfrak{N}^1(z)$ for all $m \in \mathbb{N}$, and (V.9) implies that $\|h - h_m\|_{\mathfrak{h}} \xrightarrow{m\to\infty} 0$. Consequently, $h \in \mathfrak{N}^1(z)$. This completes the proof of the lemma. $\square$

The following proposition shows that $\operatorname{Ran}(\Gamma) \subset \operatorname{Dom}(L^{1/4})$ so, in fact, $S(z)\restriction_{\operatorname{Dom}(L^{1/4})} = S(z)$.

Proposition V.51 (see [66, Proposition 3.10]).
One has $\operatorname{Ran}(\Gamma) \subset \operatorname{Dom}(L^{1/4})$.

Proof. We decompose $\mathfrak{H}^1$ into the orthogonal sum of $\mathfrak{N}^1 = \mathfrak{N}^1(-1)$ and $\operatorname{Ker}(\Gamma)$. Since Γ is linear, it suffices to show that $\Gamma h \in \operatorname{Dom}(L^{1/4})$ for all $h \in \mathfrak{N}^1$. This is proved in four steps:

Step 1. Let $h \in \mathfrak{N}^1$. We choose a sequence $(\tilde h_m) \subset \mathfrak{D}_{\mathrm{b}}$ with $\|h - \tilde h_m\|_{\mathfrak{h}} \xrightarrow{m\to\infty} 0$. Put

$$h_m = P_{\mathfrak{N}^1}\tilde h_m \qquad (m \in \mathbb{N}),$$

where $P_{\mathfrak{N}^1}$ denotes the orthogonal projection of $\mathfrak{H}^1$ onto $\mathfrak{N}^1$.

Step 2. Let $m \in \mathbb{N}$ and set $\chi_m = \Gamma h_m$. Then one has:

$$\chi_m = \Gamma P_{\mathfrak{N}^1}\tilde h_m = \Gamma P_{\mathfrak{N}^1}\tilde h_m + \Gamma P_{\operatorname{Ker}(\Gamma)}\tilde h_m = \Gamma \tilde h_m \in \operatorname{Dom}(L),$$

where $P_{\operatorname{Ker}(\Gamma)}$ denotes the orthogonal projection of $\mathfrak{H}^1$ onto $\operatorname{Ker}(\Gamma)$. By Lemma V.50, we know that

$$\int_\sigma^\oplus \exp\big(-(1+\lambda)^{1/2}\,\bullet\big)\chi_m(\lambda)\,\mathrm{d}\mu(\lambda) \in \mathfrak{N}^1 \qquad \text{and}$$
$$\Gamma\Big(\int_\sigma^\oplus \exp\big(-(1+\lambda)^{1/2}\,\bullet\big)\chi_m(\lambda)\,\mathrm{d}\mu(\lambda)\Big) = \chi_m.$$

Since $\Gamma\restriction_{\mathfrak{N}^1}$ is injective, we thus obtain:

$$h_m = \int_\sigma^\oplus \exp\big(-(1+\lambda)^{1/2}\,\bullet\big)\chi_m(\lambda)\,\mathrm{d}\mu(\lambda).$$

Step 3. Clearly, $\|h - h_m\|_{\mathfrak{h}} = \|P_{\mathfrak{N}^1}(h - \tilde h_m)\|_{\mathfrak{h}} \le \|h - \tilde h_m\|_{\mathfrak{h}} \xrightarrow{m\to\infty} 0$. It follows that

$$\|\Gamma h - \chi_m\|_{\mathfrak{G}} = \|\Gamma h - \Gamma h_m\|_{\mathfrak{G}} \xrightarrow[m\to\infty]{} 0.$$

Step 4. We already know that (h_m) is a Cauchy sequence with respect to $\|\bullet\|_{\mathfrak{h}}$. A straightforward computation shows that

$$\begin{aligned}\|h_k - h_m\|_{\mathfrak{h}}^2 &\ge \int_\sigma \lambda\|\chi_k(\lambda) - \chi_m(\lambda)\|_{\mathfrak{G}(\lambda)}^2 \int_{\mathbb{R}_+} \exp\big(-2(1+\lambda)^{1/2}\,t\big)\,\mathrm{d}t\,\mathrm{d}\mu(\lambda)\\ &= \frac{1}{2}\int_\sigma \Big(\frac{\lambda}{1+\lambda}\Big)^{1/2}\cdot \lambda^{1/2}\|\chi_k(\lambda) - \chi_m(\lambda)\|_{\mathfrak{G}(\lambda)}^2\,\mathrm{d}\mu(\lambda)\end{aligned}$$

for all $k, m \in \mathbb{N}$. Choosing $\lambda_0 > 0$ so large that $(\lambda/(1+\lambda))^{1/2} \geq 1/2$ for all $\lambda \geq \lambda_0$, we thus obtain:

$$\|\chi_k - \chi_m\|^2_{L^{1/4}} \leq \Big(1 + \lambda_0^{1/2}\Big)\|\chi_k - \chi_m\|^2_{\mathfrak{G}} + 4\|h_k - h_m\|^2_{\mathfrak{h}} \xrightarrow{k,m\to\infty} 0.$$

Since $\mathrm{Dom}(L^{1/4})$ is complete with respect to $\|\bullet\|_{L^{1/4}}$, there exists $\chi \in \mathrm{Dom}(L^{1/4})$ such that $\|\chi - \chi_m\|_{L^{1/4}} \xrightarrow{m\to\infty} 0$. Consequently, one has $\Gamma h = \chi \in \mathrm{Dom}(L^{1/4})$, as claimed. ☐

Remark V.52 (see [66, Remark 3.11]).
Γ is surjective if and only if L is bounded.

We have thus computed the solution operator $S(z)$ at every point $z \in \mathbb{C} \setminus [\min\sigma, \infty)$. In particular, if $z = -1$ and $\chi \in \mathrm{Dom}(L^{1/4})$, then (V.8) tells us that

$$\|S(-1)\chi\|^2_{\mathfrak{H}} \leq \frac{1}{2}\|\chi\|^2_{\mathfrak{G}}.$$

This inequality proves (recall Definition V.26):

Lemma V.53 (see [66, Lemma 3.12]).
The boundary pair $(\Gamma, \mathfrak{G})$ is elliptically regular.

V.3.4. The extended solution operator and its adjoint. Let $z \in \mathbb{C} \setminus [\min\sigma, \infty)$. According to (V.7) and (V.8), we know that

$$\mathfrak{G} \ni \chi \mapsto \int_\sigma^\oplus \exp(\mathrm{i}\sqrt{z-\lambda}\,\bullet)\chi(\lambda)\,\mathrm{d}\mu(\lambda) \in \mathfrak{H}$$

defines a bounded operator. In the preceding subsection, we have shown that the solution operator $S(z) : \mathrm{Ran}(\Gamma) \to \mathfrak{H}^1 \subset \mathfrak{H}$ is given by $S(z)\chi = \int_\sigma^\oplus \exp(\mathrm{i}\sqrt{z-\lambda}\,\bullet)\chi(\lambda)\,\mathrm{d}\mu(\lambda)$. As, by Lemma V.46, $\mathrm{Ran}(\Gamma)$ is dense in $\mathfrak{G}$ we can extend this formula to all of $\mathfrak{G}$:

Lemma V.54 (see [66, Lemma 3.13]).
If $z \in \mathbb{C} \setminus [\min\sigma, \infty)$, then the unique bounded extension of $S(z)$ to $\mathfrak{G}$ is given by

$$\bar{S}(z) : \mathfrak{G} \to \mathfrak{H}, \quad \bar{S}(z)\chi = \int_\sigma^\oplus \exp(\mathrm{i}\sqrt{z-\lambda}\,\bullet)\chi(\lambda)\,\mathrm{d}\mu(\lambda).$$

Next, we compute the adjoint of the extended solution operator.

Lemma V.55 (see [66, Lemma 3.14]).
If $z \in \mathbb{C}\setminus[\min\sigma, \infty)$, then the bounded operator $(\bar{S}(\bar{z}))^* : \mathfrak{H} \to \mathfrak{G}$ acts on elementary tensors as follows:

$$\big(\bar{S}(\bar{z})\big)^*(\psi \otimes \eta) = \int_\sigma^\oplus \Big(\int_{\mathbb{R}_+} \psi(t)\exp\big(\mathrm{i}\sqrt{z-\lambda}\,t\big)\,\mathrm{d}t\Big)\eta(\lambda)\,\mathrm{d}\mu(\lambda)$$

for all $\psi \in \mathsf{C}_\mathrm{c}(\mathbb{R}_+)$ and all $\eta \in \mathfrak{G}$. Consequently, $(\bar{S}(\bar{z}))^*$ can be evaluated explicitly on the dense subspace $\mathsf{C}_\mathrm{c}(\mathbb{R}_+) \odot \mathfrak{G}$ of $\mathfrak{H}$.

Proof. Standard arguments show that

$$\int_\sigma^\oplus \Big(\int_{\mathbb{R}_+} \psi(t)\exp\big(\mathrm{i}\sqrt{z-\lambda}\,t\big)\,\mathrm{d}t\Big)\eta(\lambda)\,\mathrm{d}\mu(\lambda) \in \mathfrak{G}. \tag{V.11}$$

Let $\chi \in \mathfrak{G}$. By Fubini's theorem,

$$\begin{aligned}\big\langle \big(\bar{S}(\bar{z})\big)^*(\psi \otimes \eta), \chi \big\rangle_{\mathfrak{G}} &= \big\langle \psi \otimes \eta, \bar{S}(\bar{z})\chi \big\rangle_{\mathfrak{H}} \\ &= \int_\sigma \int_{\mathbb{R}_+} \Big\langle \psi(t)\, \overline{\exp(\mathrm{i}\sqrt{\bar{z}-\lambda}\, t)}\, \eta(\lambda), \chi(\lambda) \Big\rangle_{\mathfrak{G}(\lambda)} \mathrm{d}t \, \mathrm{d}\mu(\lambda).\end{aligned}$$

It is easily seen that $\overline{\exp(\mathrm{i}\sqrt{\bar{z}-\lambda}\, t)} = \exp(\mathrm{i}\sqrt{z-\lambda}\, t)$. Therefore, (V.11) implies

$$\big\langle \big(\bar{S}(\bar{z})\big)^*(\psi \otimes \eta), \chi \big\rangle_{\mathfrak{G}} = \Big\langle \int_\sigma^\oplus \Big(\int_{\mathbb{R}_+} \psi(t) \exp\big(\mathrm{i}\sqrt{z-\lambda}\, t\big)\, \mathrm{d}t \Big) \eta(\lambda)\, \mathrm{d}\mu(\lambda), \chi \Big\rangle_{\mathfrak{G}}.$$

Since $\chi \in \mathfrak{G}$ was arbitrary, this proves the lemma. □

V.3.5. The Dirichlet-to-Neumann operator. We can think of the Dirichlet-to-Neumann operator $\Lambda(z)$ as follows (see [**65**, top of p. 1053]): it maps certain boundary values $\chi \in \mathrm{Dom}(\Lambda(z)) \subset \mathrm{Dom}\left(L^{1/4}\right)$ to the "normal" derivative $\partial_{\mathrm{n}} h$ of the corresponding Dirichlet solution $h = S(z)\chi$. In our situation, this means:

$$\begin{aligned}\Lambda(z)\chi &= -\frac{\partial}{\partial t}\Big(S(z)\chi\Big)\Big|_{t=0} \\ &= -\int_\sigma^\oplus \mathrm{i}\sqrt{z-\lambda}\; \exp\big(\mathrm{i}\sqrt{z-\lambda}\, t\big)\chi(\lambda)\, \mathrm{d}\mu(\lambda)\Big|_{t=0} \\ &= -\int_\sigma^\oplus \mathrm{i}\sqrt{z-\lambda}\; \chi(\lambda)\, \mathrm{d}\mu(\lambda).\end{aligned}$$

As we will show in Lemmas V.57–V.58 below, this formal computation indeed gives us the correct result.

Let $z \in \mathbb{C} \setminus [\min \sigma, \infty)$. Then the Dirichlet-to-Neumann form

$$\mathfrak{l}_z : \mathrm{Dom}\left(L^{1/4}\right) \times \mathrm{Dom}\left(L^{1/4}\right) \to \mathbb{C}, \quad \mathfrak{l}_z(\chi, \eta) = \mathfrak{h}\big(S(z)\chi, S(-1)\eta\big) - z\langle S(z)\chi, S(-1)\eta\rangle_{\mathfrak{H}},$$

is well-defined (and bounded with respect to the norm $\|\bullet\|_{\mathfrak{G}^{1/2}}$), see Proposition V.24 (and Remark V.11). One has:

Lemma V.56 (see [66, Lemma 3.15]).
If $z \in \mathbb{C} \setminus [\min \sigma, \infty)$, then $\mathfrak{l}_z$ is given by

$$\mathfrak{l}_z(\chi, \eta) = \int_\sigma \big(-\mathrm{i}\sqrt{z-\lambda}\big) \langle \chi(\lambda), \eta(\lambda)\rangle_{\mathfrak{G}(\lambda)}\, \mathrm{d}\mu(\lambda) \tag{V.12}$$

for all $\chi, \eta \in \mathrm{Dom}(L^{1/4})$.

Proof. The lemma is proved in two steps. First, we show (V.12) for $\chi, \eta \in \mathrm{Dom}(L)$, and then we complete the proof by approximation.

Step 1. Let $\chi, \eta \in \mathrm{Dom}(L)$. Using Lemmas V.44 and V.50 and Fubini's theorem, we compute:

$$\begin{aligned}\mathfrak{l}_z(\chi, \eta) = \int_\sigma &\langle \chi(\lambda), \eta(\lambda)\rangle_{\mathfrak{G}(\lambda)} \cdot \\ &\cdot \int_{\mathbb{R}_+} \exp\Big(\big(\mathrm{i}\sqrt{z-\lambda} - (1+\lambda)^{1/2}\big)t\Big)\, \mathrm{d}t \,\Big(\mathrm{i}\sqrt{z-\lambda}\,(-(1+\lambda)^{1/2}) + \lambda - z\Big)\, \mathrm{d}\mu(\lambda) \\ = \int_\sigma &\big(-\mathrm{i}\sqrt{z-\lambda}\big)\langle \chi(\lambda), \eta(\lambda)\rangle_{\mathfrak{G}(\lambda)}\, \mathrm{d}\mu(\lambda).\end{aligned}$$

Step 2. Let $\chi, \eta \in \mathrm{Dom}(L^{1/4})$. We choose two sequences $(\chi_m) \subset \mathrm{Dom}(L)$ and $(\eta_m) \subset \mathrm{Dom}(L)$ such that $\|\chi - \chi_m\|_{L^{1/4}} \xrightarrow{m\to\infty} 0$ and $\|\eta - \eta_m\|_{L^{1/4}} \xrightarrow{m\to\infty} 0$. By (V.9), we know that

$$\|S(z)\chi - S(z)\chi_m\|_{\mathfrak{h}} \xrightarrow{m\to\infty} 0 \quad \text{and} \quad \|S(z)\eta - S(z)\eta_m\|_{\mathfrak{h}} \xrightarrow{m\to\infty} 0.$$

Consequently,

$$\mathfrak{h}\big(S(z)\chi_m, S(-1)\eta_m\big) - z\langle S(z)\chi_m, S(-1)\eta_m\rangle_{\mathfrak{H}} \xrightarrow{m\to\infty} \mathfrak{h}\big(S(z)\chi, S(-1)\eta\big) - z\langle S(z)\chi, S(-1)\eta\rangle_{\mathfrak{H}}.$$

Furthermore, a straightforward computation shows that

$$\int_\sigma \big(-\mathrm{i}\sqrt{z-\lambda}\big)\langle \chi_m(\lambda), \eta_m(\lambda)\rangle_{\mathfrak{G}(\lambda)}\,\mathrm{d}\mu(\lambda) \xrightarrow{m\to\infty} \int_\sigma \big(-\mathrm{i}\sqrt{z-\lambda}\big)\langle \chi(\lambda), \eta(\lambda)\rangle_{\mathfrak{G}(\lambda)}\,\mathrm{d}\mu(\lambda).$$

Thus, (V.12) holds and the lemma is proved. □

As the boundary pair $(\Gamma, \mathfrak{G})$ is elliptically regular, it follows from Proposition V.28 that the Dirichlet-to-Neumann form is (as form on $\mathfrak{G}$ with domain $\mathfrak{G}^{1/2}$) closed, densely defined, and sectorial for all $z \in \mathbb{C}\setminus[\min\sigma, \infty)$. Consequently, the associated Dirichlet-to-Neumann operator $\Lambda(z)$ is m-sectorial (see Proposition V.22). We know that

$$\mathrm{Dom}\big(\Lambda(z)\big) = \big\{\chi \in \mathrm{Dom}(L^{1/4}) : \exists \varphi \in \mathfrak{G}\ \forall \eta \in \mathrm{Dom}(L^{1/4}),\ \mathfrak{l}_z(\chi,\eta) = \langle \varphi, \eta\rangle_{\mathfrak{G}}\big\} \tag{V.13}$$

and $\Lambda(z)\chi = \varphi$. One has:

Lemma V.57 (see [66, Lemma 3.16]).
If $z \in \mathbb{C}\setminus[\min\sigma,\infty)$, *then* $\mathrm{Dom}\big(\Lambda(z)\big) \supset \mathrm{Dom}(L^{1/2})$ *and*

$$\Lambda(z)\restriction_{\mathrm{Dom}(L^{1/2})}\chi = \int_\sigma^\oplus (-\mathrm{i}\sqrt{z-\lambda})\chi(\lambda)\,\mathrm{d}\mu(\lambda).$$

Proof. Let $\chi \in \mathrm{Dom}(L^{1/2})$. Then

$$\varphi = \int_\sigma^\oplus (-\mathrm{i}\sqrt{z-\lambda})\chi(\lambda)\,\mathrm{d}\mu(\lambda) \quad \text{is in } \mathfrak{G}.$$

Therefore, Lemma V.56 implies that $\mathfrak{l}_z(\chi,\eta) = \langle\varphi,\eta\rangle_{\mathfrak{G}}$ for all $\eta \in \mathrm{Dom}(L^{1/4})$. This proves the lemma. □

Furthermore, it follows from Proposition V.28 that $\mathrm{Dom}\big(\Lambda(z)\big) = \mathrm{Dom}\big(\Lambda(-1)\big)$ is independent of $z \in \mathbb{C}\setminus[\min\sigma,\infty)$. The next lemma shows that $\mathrm{Dom}\big(\Lambda(-1)\big) \subset \mathrm{Dom}(L^{1/2})$ so, in fact, $\Lambda(z)\restriction_{\mathrm{Dom}(L^{1/2})} = \Lambda(z)$.

Lemma V.58 (see [66, Lemma 3.17]).
One has $\mathrm{Dom}\big(\Lambda(-1)\big) \subset \mathrm{Dom}(L^{1/2})$.

Proof. First, we observe that for all $\eta \in \mathrm{Dom}(L^{1/4})$, we have

$$\int_\sigma^\oplus (1+\lambda)^{-1/4}\eta(\lambda)\,\mathrm{d}\mu(\lambda) \in \mathrm{Dom}(L^{1/4}).$$

Now let $\chi \in \mathrm{Dom}\big(\Lambda(-1)\big)$. We choose $\varphi \in \mathfrak{G}$ according to (V.13). Then clearly

$$\int_\sigma^\oplus (1+\lambda)^{-1/4}\varphi(\lambda)\,\mathrm{d}\mu(\lambda) \in \mathfrak{G}$$

and, since $\operatorname{Dom}\big(\Lambda(-1)\big) \subset \operatorname{Dom}(L^{1/4})$,

$$\int_\sigma^\oplus (1+\lambda)^{1/4}\chi(\lambda)\,\mathrm{d}\mu(\lambda) \in \mathfrak{G}.$$

Consequently, for all $\eta \in \operatorname{Dom}(L^{1/4})$, Lemma V.56 implies:

$$\begin{aligned} 0 &= \mathfrak{l}_{-1}\Big(\chi, \int_\sigma^\oplus (1+\lambda)^{-1/4}\eta(\lambda)\,\mathrm{d}\mu(\lambda)\Big) - \Big\langle \varphi, \int_\sigma^\oplus (1+\lambda)^{-1/4}\eta(\lambda)\,\mathrm{d}\mu(\lambda)\Big\rangle_{\mathfrak{G}} \\ &= \Big\langle \int_\sigma^\oplus (1+\lambda)^{1/4}\chi(\lambda)\,\mathrm{d}\mu(\lambda) - \int_\sigma^\oplus (1+\lambda)^{-1/4}\varphi(\lambda)\,\mathrm{d}\mu(\lambda), \eta\Big\rangle_{\mathfrak{G}}. \end{aligned}$$

As $\operatorname{Dom}(L^{1/4})$ is dense in $\mathfrak{G}$, we obtain that, for μ-almost all λ in σ,

$$(1+\lambda)^{1/4}\chi(\lambda) = (1+\lambda)^{-1/4}\varphi(\lambda).$$

Therefore, $\int_\sigma^\oplus (1+\lambda)^{1/2}\,\chi(\lambda)\,\mathrm{d}\mu(\lambda) = \varphi \in \mathfrak{G}$ and thus $\chi \in \operatorname{Dom}(L^{1/2})$, as claimed. □

In particular, for all $z \in \mathbb{C} \setminus [\min\sigma, \infty)$, the *Neumann-to-Dirichlet operator*

$$\Lambda(z)^{-1} \colon \mathfrak{G} \to \mathfrak{G}, \quad \Lambda(z)^{-1}\chi = \int_\sigma^\oplus \frac{\mathrm{i}}{\sqrt{z-\lambda}}\chi(\lambda)\,\mathrm{d}\mu(\lambda), \tag{V.14}$$

is bounded.

V.3.6. An M. Krein-type resolvent formula. We have now computed the extended solution operator as well as its adjoint and the Neumann-to-Dirichlet operator. Putting these results together, we obtain, since the boundary pair $(\Gamma, \mathfrak{G})$ is elliptically regular, the following M. Krein-type resolvent formula for $(H - zI)^{-1} - (H^{\mathrm{D}} - zI)^{-1}$.

Proposition V.59 (see [66, Proposition 3.18]).
Let $z \in \mathbb{C} \setminus [\min\sigma, \infty)$. Then $(H - zI)^{-1} - (H^{\mathrm{D}} - zI)^{-1} : \mathfrak{H} \to \mathfrak{H}$ satisfies

$$(H - zI)^{-1} - (H^{\mathrm{D}} - zI)^{-1} = \bar{S}(z)\Lambda(z)^{-1}\big(\bar{S}(\bar{z})\big)^*. \tag{V.15}$$

This operator acts on elementary tensors as follows:

$$\begin{aligned} &\Big(\bar{S}(z)\Lambda(z)^{-1}\big(\bar{S}(\bar{z})\big)^*(\psi\otimes\chi)\Big)(t) \\ &\qquad = \int_\sigma^\oplus \frac{\mathrm{i}}{\sqrt{z-\lambda}}\chi(\lambda)\int_{\mathbb{R}_+}\psi(\tau)\exp\big(\mathrm{i}\sqrt{z-\lambda}\,(t+\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}\mu(\lambda) \end{aligned}$$

for all $\psi \in \mathsf{C}_{\mathrm{c}}(\mathbb{R}_+)$ and all $\chi \in \mathfrak{G}$. Consequently, the difference of the resolvents from (V.15) *can be evaluated explicitly on the dense subspace $\mathsf{C}_{\mathrm{c}}(\mathbb{R}_+) \odot \mathfrak{G}$ of $\mathfrak{H}$.*

Proof. By Lemma V.53, we know that the boundary pair $(\Gamma, \mathfrak{G})$ is elliptically regular. Hence, Theorem V.31 implies (V.15). The explicit representation of (V.15) on $\mathsf{C}_{\mathrm{c}}(\mathbb{R}_+) \odot \mathfrak{G}$ follows directly from Lemma V.54, (V.14), and Lemma V.55. □

V.3.7. Explicit formulas for the boundary pair of the generalized half-space problem. Let us summarize the explicit formulas we have found for the boundary pair of the generalized half-space problem, written in a more handy version without referring to the direct integral representation of L:

Proposition V.60 (see [66, Proposition 3.19]).
Let $z \in \mathbb{C} \setminus [\min \sigma, \infty)$. One has:

(1) *The solution operator $S(z) : \mathrm{Dom}(L^{1/4}) \to \mathfrak{H}^1$ is given by*

$$\big(S(z)\chi\big)(t) = \exp\big(\mathrm{i}\sqrt{zI - L}\,t\big)\chi. \tag{V.16}$$

In particular, $\|S(-1)\chi\|_{\mathfrak{H}}^2 \leq \frac{1}{2}\|\chi\|_{\mathfrak{G}}^2$ for every $\chi \in \mathrm{Dom}(L^{1/4})$ so $(\Gamma, \mathfrak{G})$ is an elliptically regular boundary pair.

(2) *The Dirichlet-to-Neumann operator $\Lambda(z) : \mathrm{Dom}(L^{1/2}) \to \mathfrak{G}$ is given by*

$$\Lambda(z)\chi = \mathrm{i}\sqrt{zI - L}\,\chi. \tag{V.17}$$

(3) *The difference of the resolvents of H and H^{D} acts on elementary tensors as follows:*

$$\Big(\big[(H - zI)^{-1} - (H^{\mathrm{D}} - zI)^{-1}\big](\psi \otimes \chi)\Big)(t)$$
$$= \mathrm{i}\int_{\mathbb{R}_+} \psi(\tau) \exp\big(\mathrm{i}\sqrt{zI - L}(t + \tau)\big)\big(\sqrt{zI - L}\big)^{-1}\chi \, \mathrm{d}\tau$$

for all $\psi \in \mathsf{C}_{\mathrm{c}}(\mathbb{R}_+)$ and all $\chi \in \mathfrak{G}$.

Remark V.61 (to Proposition V.60 (1)–(2)).
In the case when L is bounded, cf. (V.16) with [**52**, equation (4.3)] as well as (V.17) with [**52**, equation (4.5)].

Proof of Proposition V.60. The results from Lemma V.50, Proposition V.51, Lemma V.53, Lemma V.57, Lemma V.58, and Proposition V.59 carry over to the situation when L is not necessarily a multiplication operator, using Theorem I.92 and the functional calculus. □

Proof of Theorem V.3 (1)**.** Set $z = -1$ in Proposition V.60 (3). □

V.4. A formula with separated variables for the difference of the resolvents

The following material is taken, almost verbatim, from the paper [**66**, pp. 309–311] by Olaf Post and the present author.

In this section, we establish Theorem V.4 and Corollary V.5.

The outline of the proof of Theorem V.4 is as follows:

Step 1. We change the order of evaluation with respect to the variables $t \in \mathbb{R}_+$ and $\lambda \in \sigma$ in the representation formula from Proposition V.59. Then, for μ-almost all λ in σ, we will obtain a vector-valued Hankel-type integral operator.

Step 2. The application of a scaling transformation will lead to a unitarily equivalent representation of (V.15) with separated variables, as claimed.

We perform Step 1 in Subsection V.4.1 and Step 2 in Subsection V.4.2. Finally, we deduce Corollary V.5 from Theorem V.4.

V.4.1. Proof of Theorem V.4. Step 1.

First, we observe that

$$W : \mathsf{C}_{\mathrm{c}}(\mathbb{R}_+) \odot \mathfrak{G} \subset \mathfrak{H} \to \int_\sigma^\oplus \mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G}(\lambda)\,\mathrm{d}\mu(\lambda), \quad W(\psi \otimes \chi) = \int_\sigma^\oplus \psi \otimes \chi(\lambda)\,\mathrm{d}\mu(\lambda),$$

defines an isometric operator with dense range. We denote the unique bounded extension of W to $\mathfrak{H}$ by the same symbol W. Obviously, W is a unitary operator from $\mathfrak{H}$ onto $\int_\sigma^\oplus \mathsf{L}^2(\mathbb{R}_+)\otimes\mathfrak{G}(\lambda)\,\mathrm{d}\mathfrak{p}(\lambda)$. The similarity transformation with respect to the natural unitary operator W leads to the expected result:

Lemma V.62 (see [66, Lemma 4.1]).
If $z \in \mathbb{C}\setminus[\min\sigma,\infty)$, then, for all $\psi \in \mathsf{C}_\mathrm{c}(\mathbb{R}_+)$ and all $\chi \in \mathfrak{G}$, one has

$$\Big(W\bar{S}(z)\Lambda(z)^{-1}\big(\bar{S}(\bar{z})\big)^*(\psi\otimes\chi)\Big)(\lambda) = \frac{\mathrm{i}}{\sqrt{z-\lambda}}\int_{\mathbb{R}_+}\psi(\tau)\exp\big(\mathrm{i}\sqrt{z-\lambda}\,(\bullet+\tau)\big)\,\mathrm{d}\tau\otimes\chi(\lambda)$$

for $\mathfrak{p}$-almost every λ in σ.

Proof. This is a consequence of Proposition V.59 and Fubini's theorem. □

In particular, Lemma V.62 shows that

$$W\big[(H-zI)^{-1}-(H^\mathrm{D}-zI)^{-1}\big]W^{-1} = \int_\sigma^\oplus T(\lambda)\,\mathrm{d}\mathfrak{p}(\lambda),$$

where for every fixed $\lambda\in\sigma$ outside a set of $\mathfrak{p}$-measure 0,

$$T(\lambda)\big(\psi\otimes\chi(\lambda)\big) = \frac{\mathrm{i}}{\sqrt{z-\lambda}}\int_{\mathbb{R}_+}\psi(\tau)\exp\big(\mathrm{i}\sqrt{z-\lambda}\,(\bullet+\tau)\big)\,\mathrm{d}\tau\otimes\chi(\lambda) \tag{V.18}$$

for all $\psi\in\mathsf{C}_\mathrm{c}(\mathbb{R}_+)$ and all $\chi(\lambda)\in\mathfrak{G}(\lambda)$. We will write $T=\int_\sigma^\oplus T(\lambda)\,\mathrm{d}\mathfrak{p}(\lambda)$.

Remark V.63 (see [66, Remark 4.2]).
$T(\lambda)$ defined in (V.18) is a vector-valued Hankel-type integral operator, as the first factor is an integral operator on $\mathsf{L}^2(\mathbb{R}_+)$ with kernel function depending only on the sum $t+\tau$ of the variables $t,\tau\in\mathbb{R}_+$.

V.4.2. Proof of Theorem V.4. Step 2. For the rest of this subsection, we assume that

$$z\in(-\infty,\min\sigma).$$

It is then clear that $\lambda-z>0$ and hence $\mathrm{i}\sqrt{z-\lambda}=-(\lambda-z)^{1/2}$ for all $\lambda\in\sigma$. Therefore,

$$U(\lambda):\mathsf{L}^2\big(\mathbb{R}_+;\mathfrak{G}(\lambda)\big)\to\mathsf{L}^2\big(\mathbb{R}_+;\mathfrak{G}(\lambda)\big),\quad \big(U(\lambda)f\big)(t)=(\lambda-z)^{1/4}f\big((\lambda-z)^{1/2}\,t\big),$$

is unitary for every fixed λ outside a set of $\mathfrak{p}$-measure 0, and $U=\int_\sigma^\oplus U(\lambda)\,\mathrm{d}\mathfrak{p}(\lambda)$ defines a unitary operator on $\int_\sigma^\oplus \mathsf{L}^2\big(\mathbb{R}_+;\mathfrak{G}(\lambda)\big)\,\mathrm{d}\mathfrak{p}(\lambda)$. Note that U depends on z, but we will suppress this dependency in our notation (as we already did for T in the previous subsection).

Let us now perform the scaling transformation of T with respect to U. As both operators are fibered with respect to the direct integral over λ, we have

$$U^{-1}TU=\int_\sigma^\oplus U(\lambda)^{-1}T(\lambda)U(\lambda)\,\mathrm{d}\mathfrak{p}(\lambda).$$

Moreover, for every fixed $\lambda\in\sigma$ outside a set of $\mathfrak{p}$-measure 0, we calculate

$$\big(U(\lambda)^{-1}T(\lambda)U(\lambda)\big)\big(\psi\otimes\chi(\lambda)\big)=\int_{\mathbb{R}_+}\exp\big(-(\bullet+\tau)\big)\psi(\tau)\,\mathrm{d}\tau\otimes\frac{\chi(\lambda)}{\lambda-z}$$

for all $\psi\in\mathsf{C}_\mathrm{c}(\mathbb{R}_+)$ and all $\chi(\lambda)\in\mathfrak{G}(\lambda)$.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by $\varphi(t) = \exp(-t)$. By (II.8), we know that the difference of the resolvents (at -1) of the Neumann and Dirichlet Laplacians on the semi-axis is given by

$$\left[\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)^{\mathrm{N}} + I\right]^{-1} - \left[\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)^{\mathrm{D}} + I\right]^{-1} = \langle \bullet, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}_+)}\varphi. \tag{V.19}$$

Since L is the multiplication operator by the independent variable on $\mathfrak{G}$, one has

$$(L - zI)^{-1}\chi = \int_\sigma^\oplus \frac{\chi(\lambda)}{\lambda - z}\,\mathrm{d}\mu(\lambda).$$

We have thus shown Theorem V.4 (set $z = -1$). □

V.4.3. The spectral properties of the difference of the resolvents. Theorem V.4 allows us to determine the spectral properties of the difference of the resolvents as stated in Corollary V.5.

Proof of Corollary V.5. We denote by B_φ the self-adjoint rank one operator on $\mathsf{L}^2(\mathbb{R}_+)$ from equation (V.19), where $\varphi(t) = \exp(-t)$. By Theorem V.4, we know that the resolvent difference $(H + I)^{-1} - (H^{\mathrm{D}} + I)^{-1}$ on $\mathfrak{H}$ is unitarily equivalent to

$$B_\varphi \otimes (L + I)^{-1} \quad \text{on } \mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G}. \tag{V.20}$$

Let us denote by $\{\varphi\}^\perp$ the orthogonal complement of $\mathbb{C}\varphi$ in $\mathsf{L}^2(\mathbb{R}_+)$. Then the operator from (V.20) is unitarily equivalent to the block diagonal operator

$$0 \oplus \left[\frac{1}{2}(L + I)^{-1}\right] \quad \text{on } \left[\{\varphi\}^\perp \otimes \mathfrak{G}\right] \oplus \mathfrak{G},$$

because the range of B_φ is spanned by φ and $\langle \varphi, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}_+)} = \frac{1}{2}$. Now, standard arguments from spectral theory (see Section I.4) complete the proof. □

V.5. The difference of the spectral projections

The following material is taken, almost verbatim, from the paper [**66**, pp. 311–313] by Olaf Post and the present author.

In this section, we establish Theorem V.3 (2) and Theorem V.6. First, we use Proposition V.37 to compute the difference $E_{(-\infty,\alpha)}(H) - E_{(-\infty,\alpha)}(H^{\mathrm{D}})$ of the spectral projections for every $\alpha > 0$. Then, we can prove Theorem V.3 (2). In Subsections V.5.2 and V.5.3, we show Theorem V.6 in two steps. The outline of the proof of Theorem V.6 is as follows:

Step 1. We change the order of evaluation with respect to the variables $t \in \mathbb{R}_+$ and $\lambda \in \sigma$ in the formula for $E_{(-\infty,\alpha)}(H) - E_{(-\infty,\alpha)}(H^{\mathrm{D}})$. We will obtain, for μ-almost all λ in σ, a vector-valued Hankel-type integral operator.

Step 2. We will see that these vector-valued Hankel-type integral operators are closely related to the Hankel integral operator from (II.9). After this observation, we will be able to complete the proof of Theorem V.6, using the result of Kostrykin and Makarov [**43**] formulated in Theorem II.29.

V.5.1. Proof of Theorem V.3 (2)**.** Since $H \geq 0$ and $H^{\mathrm{D}} \geq 0$ both have a purely absolutely continuous spectrum, we may, without loss of generality, assume that $\alpha > 0$. By Proposition V.37 (2), formula (II.9), and Fubini's theorem, we obtain that

$$\begin{aligned}&\big\langle \big(E_{(-\infty,\alpha)}(H) - E_{(-\infty,\alpha)}(H^{\mathrm{D}})\big)(\psi \otimes \chi), \rho \otimes \eta\big\rangle_{\mathfrak{H}} \\ &\quad = \frac{2}{\pi} \int_{\mathbb{R}_+} \mathrm{d}t \int_{\sigma} \mathrm{d}\mu(\lambda) \int_{\mathbb{R}_+} \mathrm{d}\tau \, \langle \psi(\tau)\mathbb{1}_{[0,\alpha)}(\lambda)\chi(\lambda), \rho(t)\eta(\lambda)\rangle_{\mathfrak{G}(\lambda)} \frac{\sin\big((\alpha-\lambda)^{1/2}\,(t+\tau)\big)}{t+\tau}\end{aligned}$$

for all $\psi, \rho \in \mathsf{C}_{\mathrm{c}}(\mathbb{R}_+)$ and all $\chi \in \mathfrak{G}$, $\eta \in \mathrm{Dom}(L)$.

Remark V.64 (see [66, Remark 5.1]).
Alternatively, this can also be computed using Proposition V.59 and Stone's formula for spectral projections.

Further, one proves for all t in $\mathbb{R}_+$ that

$$h(t) = \frac{2}{\pi} \int_{\sigma}^{\oplus} \int_{\mathbb{R}_+} \psi(\tau)\mathbb{1}_{[0,\alpha)}(\lambda) \frac{\sin\big((\alpha-\lambda)^{1/2}(t+\tau)\big)}{t+\tau} \,\mathrm{d}\tau\, \chi(\lambda)\,\mathrm{d}\mu(\lambda) \in \mathfrak{G}. \tag{V.21}$$

By the dominated convergence theorem, we obtain that $\mathbb{R}_+ \ni t \mapsto h(t) \in \mathfrak{G}$ is continuous. Consequently, h is measurable and we compute

$$\|h\|_{\mathfrak{H}} \leq \|\chi\|_{\mathfrak{G}} \frac{1}{\tau_0^{1/2}} \max_{\tau \in \mathbb{R}_+} |\psi(\tau)| \int_{\mathrm{supp}(\psi)} \mathrm{d}\tau < \infty,$$

where $\tau_0 = \min\big(\mathrm{supp}(\psi)\big) > 0$. We have shown that

$$\big\langle \big(E_{(-\infty,\alpha)}(H) - E_{(-\infty,\alpha)}(H^{\mathrm{D}})\big)(\psi \otimes \chi), \rho \otimes \eta\big\rangle_{\mathfrak{H}} = \langle h, \rho \otimes \eta\rangle_{\mathfrak{H}}$$

for all $\rho \in \mathsf{C}_{\mathrm{c}}(\mathbb{R}_+)$ and all $\eta \in \mathrm{Dom}(L)$. Since $\mathsf{C}_{\mathrm{c}}(\mathbb{R}_+) \odot \mathrm{Dom}(L)$ is dense in $\mathfrak{H}$, we have established the following result:

Lemma V.65 (see [66, Lemma 5.2]).
If $\alpha > 0$*, then* $\big(E_{(-\infty,\alpha)}(H) - E_{(-\infty,\alpha)}(H^{\mathrm{D}})\big)(\psi \otimes \chi) = h$ *for all* $\psi \in \mathsf{C}_{\mathrm{c}}(\mathbb{R}_+)$ *and all* $\chi \in \mathfrak{G}$*, where* $h \in \mathfrak{H}$ *is defined as in* (V.21) *above.*

We can now prove Theorem V.3 (2).

Proof of Theorem V.3 (2)**.** The result from Lemma V.65 carries over to the situation when L is not necessarily a multiplication operator, using Theorem I.92 and the functional calculus:

$$\begin{aligned}&\Big(\big[E_{(-\infty,\alpha(\vartheta))}(H) - E_{(-\infty,\alpha(\vartheta))}(H^{\mathrm{D}})\big](\psi \otimes \chi)\Big)(t) \\ &\qquad = \frac{2}{\pi} \int_{\mathbb{R}_+} \psi(\tau) E_{[0,\alpha(\vartheta))}(L) \frac{\sin\big((\alpha(\vartheta)I - L)^{1/2}(t+\tau)\big)}{t+\tau} \chi \,\mathrm{d}\tau\end{aligned}$$

for all $\psi \in \mathsf{C}_{\mathrm{c}}(\mathbb{R}_+)$ and all $\chi \in \mathfrak{G}$, where $0 < \vartheta < 1$ and $\alpha(\vartheta) = \frac{1}{\vartheta} - 1$. Last, we observe that

$$E_{(-\infty,\alpha(\vartheta))}(H) - E_{(-\infty,\alpha(\vartheta))}(H^{\mathrm{D}}) = E_{(-\infty,\vartheta)}(A_0) - E_{(-\infty,\vartheta)}(A_1). \tag{V.22}$$

Now the proof of Theorem V.3 (2) is complete. □

V.5.2. Proof of Theorem V.6. Step 1. Analogously to Lemma V.62, one shows:

Lemma V.66 (see [66, Lemma 5.3]).
Let $\alpha > 0$ and let $\psi \in \mathsf{C}_c(\mathbb{R}_+)$, $\chi \in \mathfrak{G}$. Then one has

$$\Big(W\big(E_{(-\infty,\alpha)}(H) - E_{(-\infty,\alpha)}(H^{\mathrm{D}})\big)(\psi \otimes \chi)\Big)(\lambda) = \frac{2}{\pi}\int_{\mathbb{R}_+} \mathbb{1}_{[0,\alpha)}(\lambda)\psi(\tau)\frac{\sin\big((\alpha-\lambda)^{1/2}(\bullet+\tau)\big)}{\bullet+\tau}\,\mathrm{d}\tau \otimes \chi(\lambda)$$

for μ-almost all λ in σ, where $W : \mathfrak{H} \to \int_\sigma^\oplus \mathsf{L}^2(\mathbb{R}_+) \otimes \mathfrak{G}(\lambda)\,\mathrm{d}\mu(\lambda)$ is the unitary operator defined in Subsection V.4.1 above.

V.5.3. Proof of Theorem V.6. Step 2. Lemma V.65 shows that if $\mu\big(\sigma\cap[0,\alpha)\big) = 0$, then $E_{(-\infty,\alpha)}(H) - E_{(-\infty,\alpha)}(H^{\mathrm{D}}) = 0$. Let us now consider the more interesting case when $\mu\big(\sigma\cap[0,\alpha)\big) > 0$. Lemma V.66 implies in this case that $E_{(-\infty,\alpha)}(H) - E_{(-\infty,\alpha)}(H^{\mathrm{D}})$ is unitarily equivalent to the block diagonal operator

$$\Big[\int_{\sigma\cap[0,\alpha)}^\oplus \widetilde{T}(\lambda)\,\mathrm{d}\mu(\lambda)\Big] \oplus 0$$

$$\text{on} \quad \Big[\int_{\sigma\cap[0,\alpha)}^\oplus \mathsf{L}^2(\mathbb{R}_+;\mathfrak{G}(\lambda))\,\mathrm{d}\mu(\lambda)\Big] \oplus \Big[\int_{\sigma\cap[\alpha,\infty)}^\oplus \mathsf{L}^2(\mathbb{R}_+;\mathfrak{G}(\lambda))\,\mathrm{d}\mu(\lambda)\Big],$$

where for every fixed $\lambda \in \sigma\cap[0,\alpha)$ outside a set of μ-measure 0,

$$\widetilde{T}(\lambda)\big(\psi\otimes\chi(\lambda)\big) = \frac{2}{\pi}\int_{\mathbb{R}_+}\psi(\tau)\frac{\sin\big((\alpha-\lambda)^{1/2}\,(\bullet+\tau)\big)}{\bullet+\tau}\,\mathrm{d}\tau\otimes\chi(\lambda)$$

for all $\psi \in \mathsf{C}_c(\mathbb{R}_+)$ and all vectors $\chi(\lambda) \in \mathfrak{G}(\lambda)$. We will write $\widetilde{T} = \int_{\sigma\cap[0,\alpha)}^\oplus \widetilde{T}(\lambda)\,\mathrm{d}\mu(\lambda)$.

Next, we define the unitary operator

$$\widetilde{U} = \int_{\sigma\cap[0,\alpha)}^\oplus \widetilde{U}(\lambda)\,\mathrm{d}\mu(\lambda) \quad \text{on} \quad \int_{\sigma\cap[0,\alpha)}^\oplus \mathsf{L}^2(\mathbb{R}_+;\mathfrak{G}(\lambda))\,\mathrm{d}\mu(\lambda),$$

where $\widetilde{U}(\lambda)$ is the unitary scaling operator on $\mathsf{L}^2\big(\mathbb{R}_+;\mathfrak{G}(\lambda)\big)$ given by

$$\big(\widetilde{U}(\lambda)f\big)(t) = (\alpha-\lambda)^{1/4} f\big((\alpha-\lambda)^{1/2}\,t\big)$$

for μ-almost all $\lambda \in \sigma\cap[0,\alpha)$. Note that $\widetilde{U}$ depends also on α, but as before for U, we suppress this dependency. Again, both operators $\widetilde{U}$ and $\widetilde{T}$ are fibered with respect to the direct integral over λ, hence $\widetilde{U}^{-1}\widetilde{T}\widetilde{U} = \int_{\sigma\cap[0,\alpha)}^\oplus \widetilde{U}(\lambda)^{-1}\widetilde{T}(\lambda)\widetilde{U}(\lambda)\,\mathrm{d}\mu(\lambda)$. Moreover, for every fixed $\lambda \in \sigma\cap[0,\alpha)$ outside a set of μ-measure 0, we compute

$$\begin{aligned}\big(\widetilde{U}(\lambda)^{-1}\widetilde{T}(\lambda)\widetilde{U}(\lambda)\big)\big(\psi\otimes\chi(\lambda)\big) &= \frac{2}{\pi}\int_{\mathbb{R}_+}\frac{\sin(\bullet+\tau)}{\bullet+\tau}\psi(\tau)\,\mathrm{d}\tau\otimes\chi(\lambda)\\ &= K\psi\otimes\chi(\lambda)\end{aligned}$$

for all $\psi \in \mathsf{C}_c(\mathbb{R}_+)$ and all $\chi(\lambda) \in \mathfrak{G}(\lambda)$, where

$$(K\psi)(t) = \frac{2}{\pi}\int_{\mathbb{R}_+}\frac{\sin(t+\tau)}{t+\tau}\psi(\tau)\,\mathrm{d}\tau.$$

By Theorem II.29, we know that K has a simple and purely absolutely continuous spectrum filling in the interval $[-1, 1]$. Consequently, the operator

$$\widetilde{U}^{-1}\widetilde{T}\widetilde{U} \quad \text{on} \quad \int_{\sigma\cap[0,\alpha)}^{\oplus} \mathsf{L}^2(\mathbb{R}_+; \mathfrak{G}(\lambda))\, \mathrm{d}\mu(\lambda)$$

is unitarily equivalent to the multiplication operator by the independent variable on

$$\mathsf{L}^2\Big([-1, 1]; \int_{\sigma\cap[0,\alpha)}^{\oplus} \mathfrak{G}(\lambda)\, \mathrm{d}\mu(\lambda)\Big).$$

Now, an application of the transformation rule for spectral measures (see (V.22) above) completes the proof of Theorem V.6. ☐

Remark V.67 (see [66, Remark 5.4]).
Let us note that K defined above is the Hankel integral operator on $\mathsf{L}^2(\mathbb{R}_+)$ from M. Krein's example with parameter 1/2, see (II.10).

V.6. Two ideas for further research

We would like to find other interesting examples such that the resolvent difference as well as the difference of the spectral projections can be computed explicitly. Let us discuss two ideas.

(1) We "deform the half-space." Let $\mathfrak{U} \neq \{0\}$ be a complex vector space, and let $\mathfrak{G}(0) = (\mathfrak{U}, \langle \bullet, \bullet \rangle_0)$ be a separable Hilbert space. Following [**64**, Appendix A.2], we assume that

$$\mathfrak{H} = \int_{\mathbb{R}_+}^{\oplus} \mathfrak{G}(t)\, \mathrm{d}t$$

is a von Neumann direct integral of separable Hilbert spaces $\mathfrak{G}(t) = (\mathfrak{U}, \langle \bullet, \bullet \rangle_t)$ that are defined for almost every $t \in \mathbb{R}_+$ (with respect to Lebesgue measure). Further, we suppose that there exists a measurable function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ (in particular, ρ is strictly positive) such that for almost all $t \in \mathbb{R}_+$ and for every $u \in \mathfrak{U}$,

$$\langle u, u \rangle_t = \rho(t) \langle u, u \rangle_0.$$

In this situation, $\mathfrak{H}$ is called a *warped product* with *distortion function* ρ, see [**64**, Definition A.2.2].

Now we would like to construct closed and densely defined nonnegative forms $\mathfrak{h}$ and $\mathfrak{h}^{\mathrm{D}}$ with associated self-adjoint nonnegative operators H and H^{D} that generalize the setting from (V.1) and then compute the resolvent difference (at the spectral point -1) as well as the difference of the spectral projections explicitly.

(2) In (V.1), we can replace the differential expression $-\,\mathrm{d}^2/\,\mathrm{d}t^2$ by $(-\,\mathrm{d}^2/\,\mathrm{d}t^2)^m$ for some $m \geq 2$ or by a Sturm–Liouville differential expression (with suitable boundary conditions).

Who is who in the bibliography?

Some authors published under different spellings of their name. In the bibliography of the present thesis, we have used the spelling from the English version of the quote (if available). Here is a list of the authors in question, cf. the category "published as" at MathSciNet. Please note that the spellings used in the present thesis are typeset bold.

- M. **Birman** ≡ M. Sh. Birman ≡ M. Š. Birman
- Yu. L. **Daletskii** ≡ Ju. L. Daleckiĭ
- Yu. B. **Farforovskaya** ≡ Ju. B. Farforovskaja
- M. L. **Gorbachuk** ≡ M. L. Gorbačuk
- **M. Krein** ≡ M. G. Krein ≡ M. G. Kreĭn
- **S. Krein** ≡ S. G. Kreĭn
- M. **Malamud** ≡ M. M. Malamud
- M. **Solomyak** ≡ M. Z. Solomjak

Declaration

Hereby I affirm that I have written the submitted Ph.D. thesis "On the difference of spectral projections" independently and without unauthorized assistance. I have marked all passages taken literally or by analogy from other authors. Further, I have not used any other tools or texts than those specified in the Ph.D. thesis.

The submitted Ph.D. thesis "On the difference of spectral projections" is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or other qualification at the Johannes Gutenberg University Mainz or any other University or similar institution.

During my Ph.D. thesis, I published my research findings in papers [**83**] and [**66**]. The paper [**66**] is a joint work of Professor Olaf Post from the University of Trier and myself; my contribution includes (but is not limited to):

- [**66**, Theorem 1.2];
- [**66**, Theorem 1.4];
- the proof of [**66**, Proposition 3.10];
- the proof of [**66**, Lemma 3.17].

In the present Ph.D. thesis, this list corresponds to:

- Theorem V.4;
- Theorem V.6;
- the proof of Proposition V.51;
- the proof of Lemma V.58.

Mainz, 02/21/2019 Christoph Uebersohn

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