A novel functional renormalization group framework for gauge theories and gravity

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Abstract

In this thesis we develop further the functional renormalization group (RG) approach to quantum field theory (QFT) based on the effective average action (EAA) and on the exact flow equation that it satisfies. The EAA is a generalization of the standard effective action that interpolates smoothly between the bare action for $k \to \infty$ and the standard effective action for $k \to 0$. In this way, the problem of performing the functional integral is converted into the problem of integrating the exact flow of the EAA from the UV to the IR. The EAA formalism deals naturally with several different aspects of a QFT. One aspect is related to the discovery of non-Gaussian fixed points of the RG flow that can be used to construct continuum limits. In particular, the EAA framework is a useful setting to search for Asymptotically Safe theories, i.e. theories valid up to arbitrarily high energies. A second aspect in which the EAA reveals its usefulness are non-perturbative calculations. In fact, the exact flow that it satisfies is a valuable starting point for devising new approximation schemes.

In the first part of this thesis we review and extend the formalism, in particular we derive the exact RG flow equation for the EAA and the related hierarchy of coupled flow equations for the proper-vertices. We show how standard perturbation theory emerges as a particular way to iteratively solve the flow equation, if the starting point is the bare action. Next, we explore both technical and conceptual issues by means of three different applications of the formalism, to QED, to general non-linear sigma models (NLσM) and to matter fields on curved spacetimes.

In the main part of this thesis we construct the EAA for non-abelian gauge theories and for quantum Einstein gravity (QEG), using the background field method to implement the coarse-graining procedure in a gauge invariant way. We propose a new truncation scheme where the EAA is expanded in powers of the curvature or field strength. Crucial to the practical use of this expansion is the development of new techniques to manage functional traces such as the algorithm proposed in this thesis. This allows to project the flow of all terms in the EAA which are analytic in the fields. As an application we show how the low energy effective action for quantum gravity emerges as the result of integrating the RG flow.

In any treatment of theories with local symmetries that introduces a reference scale, the question of preserving gauge invariance along the flow emerges as predominant. In the EAA framework this problem is dealt with the use of the background field formalism. This comes at the cost of enlarging the theory space where the EAA lives to the space of functionals of both fluctuation and background fields. In this thesis, we study how the identities dictated by the symmetries are modified by the introduction of the cutoff and we study so called

bimetric truncations of the EAA that contain both fluctuation and background couplings. In particular, we confirm the existence of a non-Gaussian fixed point for QEG, that is at the heart of the Asymptotic Safety scenario in quantum gravity; in the enlarged bimetric theory space where the running of the cosmological constant and of Newton's constant is influenced by fluctuation couplings.

Zusammenfassung

In dieser Doktorarbeit wird der Funktionale Renormierungsgruppen (RG) - Zugang zur Quantenfeldtheorie (QFT), basierend auf der 'Effective Average Action' (EAA) und der exakten Flussgleichung, die diese erfüllt, weiterentwickelt. Die EAA ist eine Verallgemeinerung der gewöhnlichen effektiven Wirkung, die stetig zwischen der nackten Wirkung für $k \to \infty$ und der gewöhnlichen effektiven Wirkung für $k \to 0$ interpoliert. Hierdurch wird das eigentliche Problem des Auswertens des Funktionalintegrals überführt in die Bestimmungen des Flusses der EAA vom UV- bis hin zum IR-Bereich. Der EAA-Formalismus eignet sich auf natürliche Art und Weise zur Lösung verschieder Probleme der QFT. Ein Aspekt ist verknüpft mit der Suche nach einem Nicht-Gaußschen Fixpunkt des RG Flusses, der dazu verwendet werden kann, den Kontinuums-Limes zu konstruieren. Insbesondere bietet der EAA-Formalismus einen hilfreichen Rahmen, um asymptotisch sichere Theorien zu finden, d.h. Theorien, die bis zu beliebig hohen Energien ihre Gültigkeit bewahren. Ein zweiter Aspekt, bei dem der EAA-Zugang sich als besonders hilfreich erweist, liegt im Bereich nicht-störungstheoretischer Berechnungen. Die exakte Flussgleichung, die von der EAA erfüllt wird, ist in der Tat ein wichtiger Ausgangspunkt um neue Näherungsmethoden zu entwickeln.

Im ersten Teil der Arbeit geben wir eine Einführung in den Formalismus. Insbesondere wird die exakte RG Flussgleichung für die EAA und die damit verbundene Hierarchie der gekoppelten Flussgleichungen für eigentliche Vertizes hergeleitet. Wir zeigen, wie sich die gewöhnliche Störungstheorie als spezielle Form einer iterativen Lösung der RG Flussgleichung ergibt, sofern als Ausgangspunkt die nackte Wirkung herangezogen wird. Anschließend werden sowohl konzeptionelle als auch technische Fragen am Beispiel von drei verschiedenen Anwendungen des Formalismus erörtert: des QED, der allgemeinen nichtlinearen Sigma-Modells (NLσM) und von Materiefeldern im gekrümmten Raum.

Im Hauptteil dieser Arbeit widmen wir uns der Konstruktion der EAA für nicht-abelsche Eichtheorien und für die Quanten-Einsteingravitation (QEG), wobei wir hier den Hintergrund feld-Formalismus verwenden, um auf eichinvariante Weise das Coarse-Graining Verfahren zu implementieren. Wir schlagen ein neues Trunkierungsverfahren vor, wobei die EAA nach Ordnungen der Feldstärke bzw. Krümmung entwickelt wird. Der wesentliche Aspekt bei der praktischen Anwendung dieses Algorithmus liegt in der Konstruktion eines neuen Hilfsmittels, um die Berechnung von funktionalen Spuren handhabbar zu machen. Dies erlaubt den Fluss aller in den Feldern analytischen Terme der EAA herauszuprojizieren. Als Anwendung zeigen wir, wie die niederenergetische effektive Wirkung der QEG als Ergebnis des integrierten RG-Flusses zu Tage tritt.

Bei der Betrachtung von Theorien mit lokaler Symmetrie, welche eine Referenzskala mit sich tragen, ist die Frage nach Erhaltung der Eichsymmetrie entlang des Flusses von großer Bedeutung. Im Rahmen der EAA wird dieses Problem durch Benutzung des Hintergrundfeld-Formalismus berücksichtigt. Dies erfordert eine Erweiterung des Theorienraums in dem die EAA definiert ist, zum Raum sg. "bimetrischer" Funktionale, der sowohl von den Fluktuations- als auch den Hintergrund-Feldern abhängt. In dieser Arbeit untersuchen wir, wie die Identitäten, die aus den Symmetrien entstehen, durch die Einführung des Cutoffs modifiziert werden. Weiterhin studieren wir bimetrische Trunkierungen der EAA, die neben Fluktuations- auch Hintergrund-Kopplungen enthalten. Insbesondere bestätigen wir die Existenz des Nicht-Gaußschen Fixpunkts für die QEG, der für das Programm der Asymptotischen Sicherheit von zentraler Bedeutung ist. Wir lassen dabei erstmals zu, dass das Laufen der Kosmologischen- und der Newton-Konstante durch die Fluktuationskopplungen beeinflusst wird.

Contents

Chapter 1

Introduction

In the last century, theoretical physics has witnessed extraordinary achievements, ranging from the discovery of the quantum world to the formulation of general relativity. In particular, in the last sixty years, quantum field theory (QFT) has been developed and now we posses a wide theoretical framework that we use to understand the fundamental interactions and the cosmos. Equally important developments have been made in the understanding of critical behavior of systems composed of many interacting parts, as those studied in statistical mechanics and in condensed matter. We also made deep progress in the study of non-equilibrium systems and we discovered that deterministic systems may become chaotic.

In all these discoveries we often followed guideline principles that turned out not to be completely correct and a different justification of the assumptions leading to correct answers was formulated only in a second time. The main example of this kind of situation is the perturbative renormalization principle that was used to formulate the standard model of particle physics. It was trough the synthesis that K. Wilson made at the beginning of the seventies that we started to understand the reasons why the standard model of particle physics was working so well and the real nature of the UV divergences that affect QFT. This conceptual framework is now known as the Renormalization Group (RG) theory. A deep connection between QFT and statistical mechanics emerged, and ultimately between all theories where fluctuations play a consistent role. We discovered that there is a close relation between the ability to construct fundamental theories and second-order phase transitions, the concept of universality emerged as a fundamental theoretical tool. Another important conceptual development, not unrelated with RG ideas, was the development of effective field theory (EFT). Theoretical physics is now some how more mature, in the sense that we are now able to understand the reasons why our theories are predictive and the extent to which they are. The unreasonable success of mathematics in physics is not anymore such deep mystery.

1.0.1 Renormalization Group theory

Several different approaches to quantum field theory (QFT) have been developed so far. In all cases one starts from a regularized version of the theory, which is mathematically well defined, and successively tries to remove the regularization so to obtain unambiguous physical predictions.

One common starting point to develop a QFT is the functional integral construction of the effective action $\Gamma[\varphi]$. This can be defined as in equation (B.17) of Appendix B:

$$e^{-\Gamma[\varphi]} = \int D\chi \, e^{-S[\varphi+\chi]+\int \frac{\delta\Gamma[\varphi]}{\delta\varphi}\chi} \qquad \langle\chi\rangle = 0 \,. \tag{1.1}$$

In equation (1.1) we introduced the average field, denoted by φ, and the classical or bare action $S[\phi]$ that we are quantizing[1]. In this section we will consider the (on shell) effective action as a prototype for all physical observables. For example, the partition function of a QFT can be calculated from the effective action using the following relation,

$$\Gamma[\varphi_*] = -\log Z \,, \tag{1.2}$$

where the field configuration φ_* is the quantum vacuum state and is the solution of the following quantum equations of motion:

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = 0 \,. \tag{1.3}$$

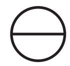
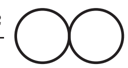
Equation (1.3) represents the quantum action principle.

There are two main types of regularization procedures that are commonly employed to give a mathematically grounded definition of the functional integral in (1.1), making it a well defined finite dimensional integral. In the first case space (or spacetime) is discretized by defining the theory on a lattice, while in the second case continuity is preserved but a cutoff is imposed on the field modes that are integrated in the functional integral. Examples of the first kind are lattice approaches, as lattice gauge theories [38] and discrete approaches to quantum gravity [73, 75]. Examples of the second type are the functional RG approaches

[1] The relation between the quantum field ϕ, the fluctuation field χ and the average field φ is $\phi = \varphi + \chi$.

to QFT, among them is the effective average action (EAA) approach studied in this thesis. The fact that both these regularization procedures act at the level of the functional integral make them non-perturbative in nature.

This is obviously in contrast with perturbation theory [10, 9, 12], where a formal series expansion is regularized term by term. Perturbation theory is based on our ability of exactly evaluating Gaussian functional integrals. The functional integral is expanded around a quadratic part of the action and all the other terms are treated as interactions. In this way we generate the formal loop expansion for the effective action:

$$\Gamma[\varphi] \;=\; S[\varphi] + \tfrac{\hbar}{2}\mathrm{Tr}\log S^{(2)}[\varphi] - \tfrac{\hbar^2}{12}$$  $$+ \tfrac{\hbar^2}{8}$$  $$+ O(\hbar^3)\,.$$

Here we introduced $\hbar$ as a loop counting parameter. The loop expansion needs first to be regularized and then to be (perturbatively) renormalized[2], as we said, term by term. But the series so constructed is not usually convergent and is thus not enough to reconstruct the original functional integral. Still, the perturbative expansion is one of the most valuable tools at our disposal, in particular in the framework of effective field theory (EFT). As we will see later, EFT is the most general way of making physical prediction using Gaussian integration techniques.

Every regularization scheme introduces an arbitrary cutoff scale. In lattice regularizations this is given by the lattice spacing a, while in cutoff formulations this is given by the ultraviolet (UV) cutoff Λ. We can consider these two scales as related by $\Lambda = \frac{1}{a}$ and we implicitly assume this relation in the following considerations. The theory of renormalization studies when and how it is possible to remove the regularization, or equivalently how to take the limit $\Lambda \to \infty$ if it exists. This limit is commonly called "continuum limit". We say that a theory is renormalizable when we can remove the regularization, by taking the continuum limit, in such a way that all physical quantities remain finite and just a finite number of parameters need to be fixed by experiments to make the theory predictive. The study of how it is possible to construct the continuum limit is one of the two faces of the renormalization group (RG) theory, the other being the understanding of critical phenomena, i.e. of universality[3]. This two apparently different problems are strictly related in the RG theory and we due to K. Wilson the formulation of this general framework [1, 2]. If a theory is renormalizable in the above mentioned way, then it is in principle predictive at arbitrary high energy scales, or

[2]We assume that the reader is familiar with standard perturbative renormalization [9].

[3]Since we are interested here in the UV aspects of renormalization we refer the reader to the literature for more details on critical phenomena and on the concept of universality [12].

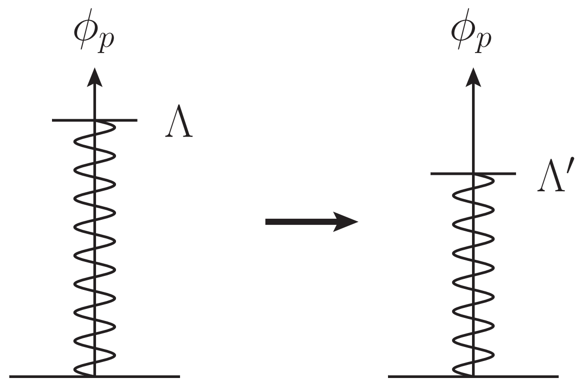


Figure 1.1: In the Wilsonian framework the functional integral is defined as a sum over all field modes ϕ_p of momentum p smaller then the UV cutoff Λ. The RG flow of the WEA is generated by varying the cutoff scaleΛ.

equivalently at arbitrary small spacetime scales, and is thus considered to be a fundamental physical theory. This picture is invalidated if, for example, new physics emerges above a certain energy scale.

In the standard Wilsonian RG framework one studies what happens when the UV cutoff is varied. If we lower the UV cutoff to $\Lambda' < \Lambda$ we can define a new action $S_{\Lambda'}[\phi]$ as the resultant of the integration over the momentum shell defined by these two scales. It is important to realize that even if the initial bare action $S_\Lambda[\phi]$ is local the resulting $S_{\Lambda'}[\phi]$ is an extremely complicated non-local action containing all possible invariants compatible with the symmetries of the theory. The action so generated is called the Wilsonian effective action (WEA) and the flow which relates the WEAs at different values of the UV cutoff is the RG flow. In this way the RG theory naturally introduces the space of all possible actions compatible with the symmetries of the theory. This is called "theory space" and is the place where the RG flow takes place.

The variation of the UV cutoff can be implemented in a smooth way by introducing a damping factor in the action, in place of sharply cutting-off the modes in the functional measure, and by codifying the flow in a differential equation for the WEA [3].

If the field modes we integrated out pertain to a massive excitation of mass M, then we can expand the WEA at the lower scale in a local series of invariants suppressed by inverse powers of M. This is the decoupling mechanism. But if the field modes we integrated out correspond to mass-less excitation then there is no low energy scale in which we can expand the WEA and thus non-local term can become significant. We will come back to this point later when discussing EFTs.

Modulo the remarks just made, the general WEA can be expanded in a basis of local invariants[4] $I_i[\phi]$, parametrized by Λ-dependent dimensionless coupling constants $g_i(\Lambda)$, of the following form:

$$S_\Lambda[\phi] = \sum_{i=0}^{\infty} \Lambda^{\Delta_i} g_i(\Lambda)\, I_i[\phi]\,. \tag{1.4}$$

The Δ_i in (1.4) are the dimensions of the (dimensionful) coupling constants. Note that we will consider the masses as just part of the set of coupling constants[5]. Using the expansion in terms of local invariants (1.4), one can describe the RG flow by the following system of differential equations for the Λ-dependent dimensionless coupling constants:

$$\Lambda \frac{d}{d\Lambda} g_i(\Lambda) = -\Delta_i g_i + \text{quantum corrections} = \beta_i\left(g\right)\,. \tag{1.5}$$

The functions $\beta_i(g)$ are the "beta functions" for the dimensionless couplings. The problem of constructing a continuum limit can now be translates into the related problem of searching for solutions g_i^* of the system:

$$\beta_i(g) = 0\,, \tag{1.6}$$

with particular properties. The g_i^* represent the fixed points of the RG flow.

The dimensions Δ_i in (1.4) are defined as minus the dimensions of the invariants $I_i[\phi]$, in such a way to make the action dimensionless. Since there is no a priory way to choose the dimensions Δ_i, they are commonly chosen to be the canonical ones, i.e. they are fixed by requiring the Gaussian action[6]

$$S_G[\phi] = \frac{1}{2}\int d^dx\, \partial_\mu\phi\partial^\mu\phi \tag{1.7}$$

to be dimensionless. Assigning mass dimension in units of the cutoff Λ to the coordinates fixes $[d^dx] = \Lambda^{-d}$ and $[\partial_\mu] = \Lambda$. Thus we must have $[\phi] = \Lambda^{d/2-1}$. With these definitions all the dimensions of the invariants $I_i[\phi]$ are fixed. But this is just a conventional choice. It is important to realize that the dimension of the field is a fundamental fractal property of the theory and, as we will see, every universality class has a distinctive spectrum of "anomalous dimensions" for the field and for all composite operators $I_i[\phi]$. To account for this, we must allow for scale dependent wave-function renormalization constants $Z_\phi(g)$ that are introduced

[4] If our aim is to study the continuum limit of a QFT, we can focus on local invariants since the dangerous divergent terms in Λ are expected to be so.

[5] Not all coupling constants are "essential", the "inessential" one can be removed by field redefinitions [86, 80].

[6] We consider here a scalar field as an example.

by the field redefinitions $\phi \to Z_\phi^{1/2}\phi$. If at a given fixed point the anomalous dimension, defined by

$$\eta_\phi(g) = -\Lambda \frac{d}{d\Lambda} \log\, Z_\phi(g)\,, \tag{1.8}$$

is non-zero $\eta_\phi(g^*) \neq 0$, then the factor $Z_\phi(g^*)$ in the Gaussian action (1.7) acquires a non-zero dimension $\Lambda^{-\eta_\phi^*}$. For dimensional consistency the dimension of field must change in order to compensate for this:

$$[\phi] = \Lambda^{\frac{d}{2}-1+\frac{\eta_{\phi^*}}{2}}\,. \tag{1.9}$$

In this way the RG theory "corrects" the scaling properties of the theory. The field and all the composite operators dimensionalities have thus a proper meaning only near a fixed point and strongly depend on it. The ability to understand this phenomenon together with the capacity of actually calculating the non-trivial scaling spectrum of a theory around a fixed point are the most important success of the RG theory [1, 2].

To qualitatively understand the varieties of asymptotic behavior the running coupling constant may have as $\Lambda \to \infty$, we consider the simple situation where theory space is reduced to a one dimensional subspace of only one coupling constant $g(\Lambda)$. In this case the full flow is encoded in one beta function $\beta(g)$. Integrating the flow equation (1.5) for this particular case gives:

$$\Lambda = M\, \exp \int_{g_M}^{g_\Lambda} \frac{dg}{\beta(g)}\,, \tag{1.10}$$

where M is a mass scale arising as an integration constant. Integrating instead equation (1.8) gives:

$$Z_\Lambda = Z_M\, \exp\left\{-\int_M^\Lambda \eta(g_{\Lambda'})\frac{d\Lambda'}{\Lambda'}\right\}\,. \tag{1.11}$$

There are basically three possible forms for the function $\beta(g)$ and these are represented in Figure 2. In the first case the beta function is always positive for positive coupling and grows rapidly enough to make the integral in (1.10) convergent in the upper limit for $g_\Lambda \to \infty$. Therefore the bare coupling g_Λ is driven away from $g = 0$ and diverges at a finite scale. This is the so-called "Landau pole" case and is generally considered unphysical. In the second case the beta function is always negative for non-zero coupling so that $g = 0$ is an UV attractive fixed point and the theory is said to be "asymptotically free". Since the coupling is small, perturbation theory can be used to calculate the beta function, and is usually of the form $\beta(g) = -bg^2$ with $b > 0$. The integral in (1.10) can then be easily calculated to yield:

$$g_\Lambda = \frac{g_M}{1 + b\, g_M \log \frac{\Lambda}{M}}\,. \tag{1.12}$$

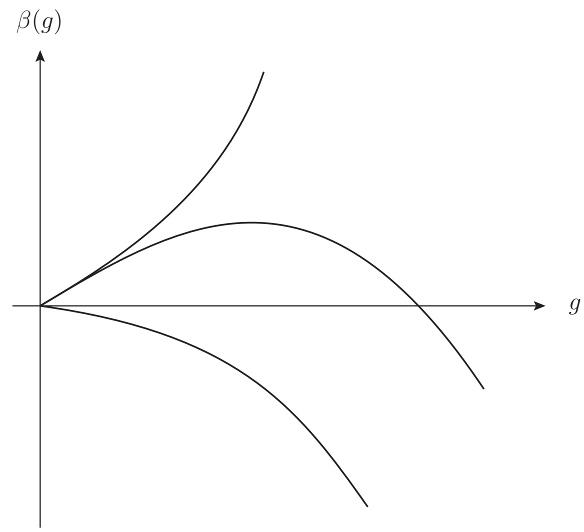


Figure 1.2: Possible forms of the beta function $\beta(g)$ for the coupling g.

For $\Lambda \to \infty$ this approaches the g_M independent limit $\left(b \log \frac{\Lambda}{M}\right)^{-1}$ and then becomes zero. Also, the anomalous dimension of a given invariant $I_i[\phi]$ has the perturbative form $\eta(g) = Cg^n$ for some constant C and for some positive integer n. Inserting in (1.11) the g_M independent limit for g_Λ gives, for the ratio of the wave-function renormalization, the following result:

$$\frac{Z_\Lambda}{Z_M} = \begin{cases} \left(\log \frac{\Lambda}{M}\right)^{-C/b} & n = 1 \\ \exp\left\{-\frac{C}{b^n(1-n)}\left(\log \frac{\Lambda}{M}\right)^{1-n}\right\} & n > 1 \end{cases} . \tag{1.13}$$

In this case the Gaussian scaling is modified by logarithmic corrections. In the last case the beta function is positive in the vicinity of the origin but has a (simple) zero at the UV fixed point g_*. The coupling constant will grow if smaller then fixed point value, otherwise it will decrease. Taylor expanding the beta function for $g < g_*$ and inserting in (1.10) gives the following important relation:

$$M \sim \Lambda |g_* - g_\Lambda|^\nu \qquad\qquad \nu = -\frac{1}{\beta'(g_*)} . \tag{1.14}$$

The critical exponent ν is the mass critical exponent [12, 75]. To obtain a non-trivial continuum limit we have to send $\Lambda \to \infty$ so that M, which plays the role of the inverse correlation length, remains finite or vanishes. From (1.11) we find instead

$$Z_\Lambda \sim \Lambda^{-\eta(g_*)} , \tag{1.15}$$

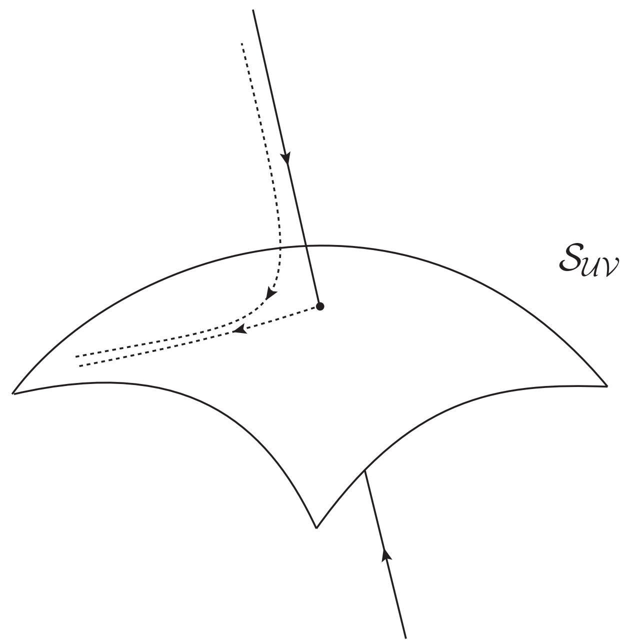


Figure 1.3: The RG flow in theory space. Two nearby trajectories on the UV critical surface of a fixed point are shown. To obtain a finite theory as we take the limit $\Lambda \to \infty$ we need to tune (non-perturbatively renormalize) the theory to lie on the UV critical surface $\mathcal{S}_{UV}$.

which shows that the Gaussian operator, or equivalently the field, acquires a new scaling dimension $\frac{d-2+\eta(g_*)}{2}$. Thus at non-Gaussian fixed point the scaling spectrum of the theory is non-trivial. Non-Gaussian fixed points of the RG represents scale invariant, or even conformal invariant, interacting theories that are representative of non-trivial universality classes [13].

In presence of more couplings, which are infinite in the general situations, the asymptotic behavior for $\Lambda \to \infty$ can be of different types: trajectories can go to infinity for finite or infinite Λ, there can be fixed points or even limit cycles. Even the very speculative possibility that the RG flow may be chaotic has to be considered, opening the road to the question of what such a case could mean. Of all these situations the most interesting one is when fixed points, in particular non-Gaussian ones, are present. In this case it is useful to linearize the system (1.5) around a solution g_i^* of (1.6):

$$\Lambda \frac{d}{d\Lambda} \delta g_i = M_{ij}\, \delta g_i \,, \tag{1.16}$$

where the stability matrix M_{ij} is defined by

$$M_{ij} = \left.\frac{\partial \beta_i}{\partial g_j}\right|_{g=g_*} . \tag{1.17}$$

If we consider the eigenvectors of the stability matrix $\sum_j M_{ij} v_j^I = \lambda^I v_i^I$ we can write the perturbations as a linear combination of the eigenvectors $\delta g_i = \sum_I C_I v_i^I \Lambda^{\lambda^I}$ so that we can write

$$g_i = g_i^* + \sum_I C_I v_i^I \Lambda^{\lambda^I} .$$

The couplings will reach the fixed point g_i^* for $\Lambda \to \infty$ only if the coefficients are zero $C_I = 0$ for all I with positive eigenvalues $\lambda^I > 0$. The linear combinations of coupling that are attracted to the fixed point are called irrelevant, those which are repelled are termed relevant while if $\lambda^I = 0$ they are marginal. The set of all points in theory space attracted to the fixed point for $\Lambda \to \infty$, i.e. the basin of attraction of the fixed point, is the UV critical surface $\mathcal{S}_{UV}$. The dimension of this surface $\dim \mathcal{S}_{UV}$, if finite, sets the number of independent parameters to by fixed by experiment to make the theory predictive. Note that if $\dim \mathcal{S}_{UV} < \infty$ then $\dim \mathcal{S}_{IR} = \infty$ and viceversa[7]. If a fixed point with finite dimensional UV critical surface can be found in a give theory space, then it is possible to construct a non-perturbative renormalizable QFT that depends on $\dim \mathcal{S}_{UV}$ independent parameters. This scenario is termed asymptotic safety after S. Weinberg [86]. This generalizes the more know case of asymptotic freedom in which the fixed point in question is Gaussian.

When instead the fixed point is attractive in the IR and has there a finite dimensional critical surface $\dim \mathcal{S}_{IR} < \infty$, then all those microscopic models, who's classical action can be taken to lie on $\mathcal{S}_{IR}$ by tuning a finite number of parameters, have the same macroscopic physics. This is what we need to describe critical phenomena and is the IR manifestation of universality [12].

If a UV attractive fixed point with a finite dimensional UV attractive critical surface exists then by fixing the bare theory to lie on it we can take the limit $\Lambda \to \infty$ resulting in a theory where only $\dim \mathcal{S}_{UV}$ couplings and masses have to be taken from experiments to have a predictive theory. Even if we are able to find such a fixed point, that can be in principle found by studying the beta function system (1.6), we still need to calculate the path integral with bare action arbitrary close to the fixed point action $S_*[\phi]$. This can be done if we are able to solve the fixed point theory, as is possible to do in the context of conformal field theory (CFT) in $d = 2$, then we can define any theory around this fixed point by perturbing $S_*[\phi]$ [13]. But this kind of exact results are basically never available for realistic theories

[7] We are assuming that fluctuations lifts all marginal operators to be relevant or irrelevant.

away from two dimensions.

In the framework of perturbative or effective QFT the RG theory is usually introduced in a different way. As we have seen, the effective action is constructed using the loop expansion where every term is made finite by a perturbative renormalization; i.e. infinities are removed by subtractions after they have been regularized, usually by dimensional regularization. After the theory has been made finite, we still need to impose renormalization conditions to fix the renormalized parameter with the observed ones. It is at this point that the RG comes into play in the form of renormalization conditions imposed at the arbitrary reference scale μ. This defines renormalized coupling constants $g_i(\mu)$ depending on this "sliding scale" μ [11]. By the request that renormalized physical quantities, like the (on-shell) effective action, must be independent of this arbitrary chosen scale, we obtain the RG equations for the couplings:

$$\beta_i^{PT}(g) = \mu \frac{d}{d\mu} g_i(\mu) \qquad\qquad \eta_\phi^{PT}(g) = -\mu \frac{d}{d\mu} \log\, Z_\phi(g)\,. \tag{1.18}$$

where we used a superscript to remark that these are perturbative beta functions and anomalous dimensions. Remember also that we are considering the masses as part of the coupling constants set. In the same way we can also write down a flow equation for the renormalized effective action:

$$\left(\mu \frac{\partial}{\partial \mu} + \sum_i \beta_i^{PT}(g) \frac{\partial}{\partial g_i} - \eta_\phi^{PT}(g) \int d^d x \varphi(x) \frac{\delta}{\delta \varphi(x)}\right) \Gamma_\mu[Z_{\phi,\mu}^{1/2} \varphi, m_{i,\mu}, g_{i,\mu}] = 0\,. \tag{1.19}$$

This is the well known Callan-Symanzik homogeneous RG equation [9, 12]. This implementation of the general RG theory is useful when we dispose of a computation and renormalization framework able to give use, within some expansion scheme, the full effective action on which we can impose the renormalization conditions at the arbitrary scale μ. When we are able to do this, then this is a good way to observe the physical system we are analyzing at different resolutions, which are in a way or the other related to μ, in this way obtaining the physical RG flow. This approach is less useful when we don't have a proper way to renormalize the theory and we do not dispose of an efficient way to calculate the contributions of fluctuation to the effective action. Perturbation theory is the optimal setting were to study asymptotically free theories, since the question of if a theory is actually so can be settled by perturbative calculation of the beta functions. But this is not the proper setting where to investigate for universality classes that cannot be uncovered by perturbative expansions and in particular is not the proper setting where to search for asymptotically safe theories.

We have to mention that the perturbative RG approach was actually the first one developed. Even if the full RG conceptual framework was not yet formulated, the work of Stueckelberg and Petermann [4] and the better known work of Gell-Mann and Low [5] introduced the beta functions for the running electric charge in QED. In fact equations (1.18) are know after this last two authors. The aim that motivated this first studies was to improve the range of validity of perturbation theory, in particular when large logarithms were invalidating it [11]. The perturbative approach is the one commonly used in high energy physics since perturbative and effective theories are efficient tools to deal with the standard model of particle physics. All the general properties of the RG flow discussed in the previous section carry over to the perturbative RG just discussed by replacing Λ with μ.

When are the Wilsonian RG and the perturbative RG comparable? In the UV were the beta are mass-less and depend only on dimensionless couplings, in this case the first two terms of the Taylor expansion of both $\beta_{g,\Lambda}$ and $\beta_{g,\mu}$ can be shown to be equal [11]. This implies that if a theory is perturbatively asymptotically free then it is possible to take the continuum limit on the lattice. In more general conditions instead, as in presence of dimensionful couplings or non-zero masses, there are no general results that relate the RG flow in these two formulations.

Today we don't consider any more the property of QFT to be perturbatively renormalizable as a fundamental physical requirement. Instead we are more interested to understand why QFTs are sometimes very good models of natural phenomena. The answer to this question is encoded into the effective field theory (EFT) formalisms. As we already mentioned, EFT is the most general way of making physical predictions using our ability to perform Gaussian integrals. In this framework the loop expansion for the effective action is used in a very clever way: the saddle point expansion is constructed around the quadratic part of the action which correctly describes the low energy fluctuations of the quantum fields and all the other terms allowed by symmetries, both (perturbatively) renormalizable and non-renormalizable ones, are considered as interactions. It is assumed that the description in terms of the low energy degrees of freedom will break down at the characteristic high energy scale M and the action for the theory is expanded in inverse powers of this scale:

$$S[\phi] = \sum_{i=0}^{\infty} M^{\Delta_i} g_i \, I_i[\phi] \,. \tag{1.20}$$

The EFT reasoning is particularly simple if no other characteristic scales other than M are present, i.e. if the theory has been regularized using a mass-independent scheme as

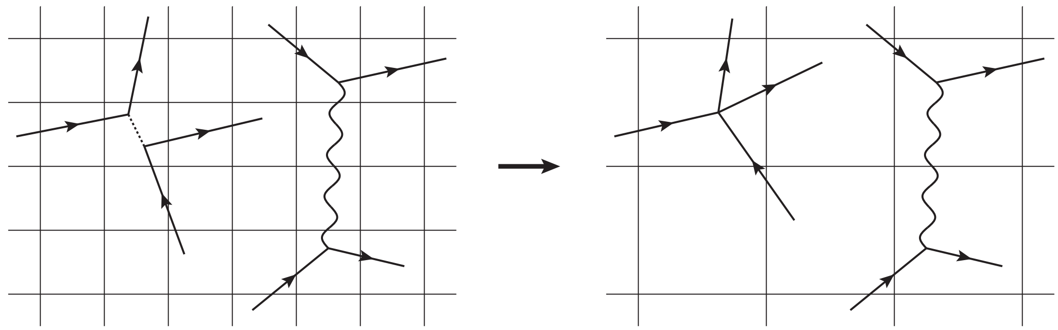


Figure 1.4: After a RG transformation from the scale $\Lambda = \frac{1}{a}$ to the scale $\Lambda' = \frac{1}{2}\Lambda$ heavy fields (dot) of mass $\Lambda > M > \Lambda'$ do not propagate any more and are effectively described as a local interaction among light field (continuous). Mass-less fields (wavy) can generate instead non-local interactions among the light fields. The WEA $S_{\Lambda'}[\phi]$ can be expanded in an inverse power series in M where only a finite number of local invariants, describing the light-heavy fields interactions, need to be retained. To describe the light-mass-less interactions instead non-local terms may be needed.

dimensional regularization. The basic idea of EFT is to define the error up to which we want to calculate the effective action, i.e. we fix the ratio p/M, where p is the characteristic energy scale of the process under investigation. For example p is the mass or the momenta of the light fields. If we now insert the action (1.20) in the loop expansion for the effective action, every diagram will be proportional the following to the factor:

$$\left(\frac{p}{M}\right)^w \qquad w = \sum_i n_i \Delta_i \,, \tag{1.21}$$

where n_i is the number of vertices steaming from the operator $I_i[\phi]$ and $\Delta_i = n_i^\phi d_\phi - n_i^\partial + d$ with n_i^ϕ and n_i^∂ the numbers of fields and derivatives in this invariant. d_ϕ is the canonical dimension of the field, for a scalar field we have as before $d_\phi = \frac{d}{2} - 1$. Using simple arguments about the topology of Feynman diagrams we can prove the following relation:

$$w = -d + dL + \sum_i \left[\left(d_\phi - \frac{d}{2} \right) n_i^\phi + n_i^\partial \right] n_i \,. \tag{1.22}$$

where L is the number of loops in the diagram. We see that once we have fixed the error to which we want to know the effective action, i.e the value of w, the maximum value of L we need to consider a given interaction $I_i[\phi]$ is fixed by (1.22). In particular, to any fixed value for w we need to consider only a finite number of interactions and only a finite number

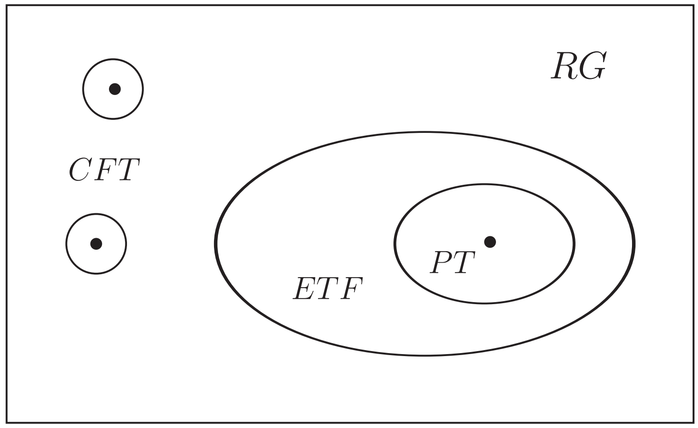


Figure 1.5: Within the general renormalization group theory (RG) we can understand all the approaches to QFT. Perturbative (PT) renormalizable theories are a subclass of effective field theories (EFT) and both are based on the Gaussian fixed point of the RG flow. Conformal field theories are used to describe interacting non-Gaussian fixed point theories but only the full RG theory is capable of describing the full theory space.

of loops. Thus to make unambiguous predictions we need to fix from experiments only the values of the dimensionless coupling constants for the interactions we are considering, if we are not able to calculate them from the high energy theory. This is a fundamental result and ultimately the explanation of the success of theories like QED or the standard model of particle physics is related to the fact that the respective fundamental mass scales are much higher then the energy probed by the relative experiments [10, 7]. From this perspective, perturbative renormalizable theories are just those theories where to improve the precision in the predictions, i.e. we increase the value of w, we don't need to consider new interactions. Note that this does not mean that a perturbatively renormalizable theory is fundamental but just that it is a convenient theory to work with.

We remark that mass-independent regularization schemes are very useful in EFT because they allow the simple power counting we just made but they fail to describe thresholds, i.e. they fail to describe decoupling. When using this kind of regularizations threshold phenomena have to be introduced by hand as matching conditions. The results of EFT can obviously be obtained also using a mass-dependent regularization scheme, since physics cannot depend on this choice, but the arguments become more involved due to the presence of the cutoff scale.

The origin of the fundamental mass scale M can be understood in terms of the WEA: it is just the mass of the heavy fields which have already been integrated out. The same remark we made before about the decoupling of the high energy degrees of freedom applies again: only massive fields decouple completely while mass-less fields can contribute to the

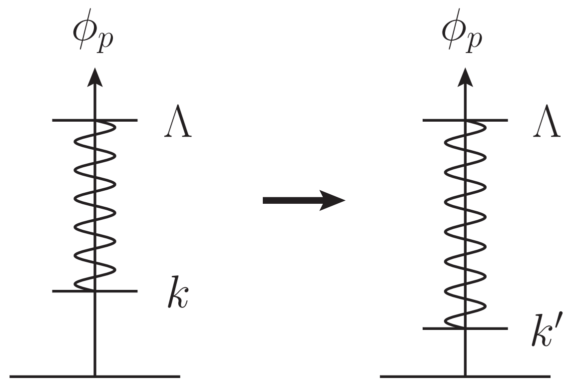


Figure 1.6: In the effective average action (EAA) framework an IR cutoff k is introduced and the functional integral is defined over the field modes ϕ_p of momenta $k < p < \Lambda$.

low energy physics with non-local terms in the effective action of the EFT. In particular these terms are related, and can be obtained, by the analysis of anomalies [8]. In theory there is the possibility that high energy physics can have a signature on the low energy domain. A cartoon of this fact is shown in Figure 1.4.

We can understand now the physics behind standard renormalization: the UV cutoff Λ is just an arbitrary scale smaller then a fundamental mass scale M and the bare action is just the result of the integration of the heavy fields between the scales M and Λ. Physical observables cannot depend on Λ since they cannot depend on the fact that we decided to integrated out the heavy fields first and successively the light ones. The procedure of perturbative renormalization reflects this fact: the UV scale that enters both the loop integrals and the coupling constants, i.e. the vertices, must drop out from observables since this dependence arises by the arbitrary introduction of the cutoff [7].

Thus we can understand both perturbative renormalization and EFT as part of the general Wilsonian RG theory. How can we unveil the other regions of this general theory? More precisely, how can we make physical predictions that go beyond the scale at which the EFT description brakes down? How can we discover those fixed points of the RG flow that are out of the domain where perturbation theory is applicable? How do we construct asymptotically safe theories? The answer is in principle simple: by doing the momentum shell integration that relates the WEA at different scales step by step, and in doing so we extend the RG flow to cover theory space completely.

Only in recent years it has finally come to use a practical way to do this. Instead of considering the functional integral with a floating UV cutoff, we introduce an additional IR cutoff scale k to restrict the integration of the field modes to the range $k < p < \Lambda$ and we

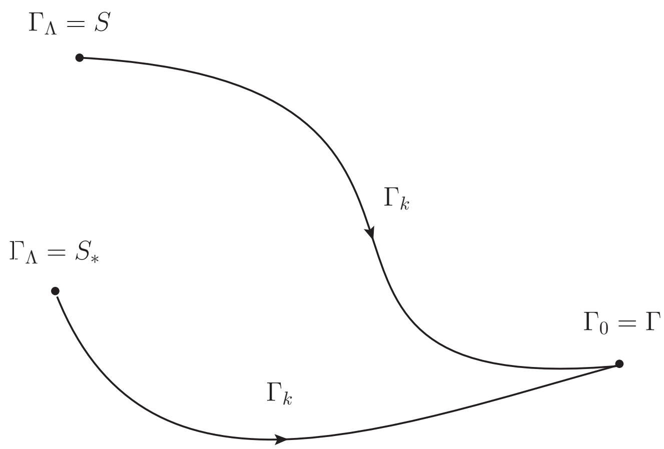


Figure 1.7: The flow of the EAA in theory space. At the UV scale Λ the EAA can be either a given microscopic action S or a fixed point action S_*. In the limit $k \to 0$ we recover the full effective action Γ as the result of the integration of the flow.

shift our the attention to the effective action in place of the bare action. The functional defined in this way is called effective average action (EAA) [15, 18, 17]. But why considering an effective action that now depends on two cutoff scales? The answer is that the scale dependence of the effective average action on the IR scale k can be encoded in an exact RG flow equation [14]. This equation is the fundamental tool used in this approach and reads as follows:

$$k\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2 \Gamma_k[\varphi]}{\delta\varphi\delta\varphi} + R_k\right)^{-1} \partial_t R_k \,. \tag{1.23}$$

This flow is well defined both in the IR as well as in the UV and we can thus forget about the original UV regularization of the functional integral. Also, it is easy to show that the EAA interpolates smoothly between the bare action for $k \to \Lambda$ and the full effective action for $k \to 0$. This properties allow us to translate the problem of computing the functional integral into the problem of integrating the RG flow described by the exact flow equation (1.23). In particular we can use (1.23) to devise new computational tools to use to approximate the effective action seen as the $k = 0$ limit of the EAA. All the general RG analysis described before carries over to this formalisms by the identification[8] $\Lambda \to k$. We have thus an unified framework where to search for continuum limits and at the same time where to calculate the effective action that can follow from these.

The EAA joins the conceptual virtues of WEA with new computational possibilities

[8]As we remarked before, there is no direct relation between the explicit flow in the three different RG implementations we are considering.

offered by the exact flow it satisfies. It is left to demonstrate that the EAA is really capable to maintain these promises. It is possible, in practice, to use the exact flow of the EAA to answer real non-perturbative questions? Is quantum gravity asymptotically safe? Can the EAA formalisms deal with theories with local symmetries? These are the main question we try to answer in this thesis.

1.0.2 Outline of the thesis

In this thesis we develop further the functional renormalization group (RG) approach to quantum field theory (QFT) based of the effective average action (EAA) and on the exact flow equation that it satisfies. As we said in the previous section, the EAA is a generalization of the standard effective action that interpolates smoothly between the bare action for $k \to \infty$ and the standard effective action for $k \to 0$. In this way, the problem of performing the functional integral is converted into the problem of integrating the flow of the EAA from the UV to the IR. The technical reason why this reformulation is useful is the fact that the EAA average action satisfies an exact flow equation that can be used as the starting point to define the QFT in theory space.

The EAA formalism deals naturally with several different aspects of a QFT. One aspect is related to the discovery of non-Gaussian fixed points of the RG flow that can be used to construct continuum limits. In particular, the EAA framework is a useful setting where to search for Asymptotically Safe theories, i.e. theories valid up to arbitrary high energies. A second aspect in which the EAA reveals its usefulness is to non-perturbative calculations. In fact, the exact flow that it satisfies, is a valuable starting point for devising new approximation schemes.

In the second chapter of this thesis we introduce the formalism. We state the basic definitions in section 2.2, in particular we derive the exact RG flow equation for the EAA in section 2.2.1 and the related hierarchy of coupled flow equations for the proper-vertices in section 2.2.2. In section 2.2.3, we show how standard perturbation theory emerges as just a particular way to iteratively solve the flow equation if the starting point is the bare action. We propose that more general initial points in the iterative process, in particularly scale dependent ones, may be useful new approximation scheme to the solution of the flow equation. Next, in section 2.3 we explore both technical and conceptual issues by means of three different applications of the formalism. In section 2.3.1 we treat some basic QED, in section 2.3.2 we apply the formalism to general non-linear sigma models (NLσM) and in section 2.3.3 we study matter fields on curved spaces.

In the main part of this thesis, Chapters 3 and Chapter 4, we construct the EAA for non-abelian gauge theories and for quantum gravity, using the background field method to implement the coarse-graining procedure in a gauge invariant way. The construction of the background effective average action (bEAA) is done in section 3.3 for non-abelian gauge theories and in section 4.3 for quantum gravity. In sections 3.2 and 4.2, after briefly reviewing the classical theory, we study the quantum theory through the bEAA approach respectively for non-abelian gauge theories and for quantum gravity. We state the basic definition and properties and then we study both local, sections 3.2.2.1 and 4.2.2.1, and non-local truncations, sections 3.2.2.2 and 4.2.2.2, of the EAA. When treating non-local truncations we propose a new truncation scheme where the bEAA is expanded in powers of the curvature or field strength that we call "curvature expansion". Crucial to the practical use of this expansion is the development of new techniques to manage functional traces, that are developed in section 3.3.4, and that furnish the basis for a general algorithm, firstly proposed in this thesis, that allows to project the flow of all terms in the bEAA which are analytic in the fields. As an application, in section 4.2.2.2 we show how the low energy effective action for quantum gravity emerges as the result of integrating the RG flow.

In any treatment of theories with local symmetries that introduces a reference scale, the question of preserving gauge invariance along the flow emerges as predominant. As we said, in the bEAA framework this problem is dealt with the use of the background field formalism. This comes at the cost of enlarging the theory space where the EAA lives to the space of functionals of both fluctuation and background fields. We study how the identities dictated by the symmetries are modified by the introduction of the cutoff by deriving the modified Ward-Takahashi identities in section 3.3.2 for non-abelian gauge theories and in section 4.3.2. for quantum gravity. We study truncations of the bEAA that contain both fluctuation and background couplings when we deal with local truncations in non-abelian gauge theories in section 3.2.2.1 and in quantum gravity in section 4.2.2.1. In particular, in this last section, we confirm the existence of a non-Gaussian fixed point for quantum gravity, which is at the basis of the Asymptotic Safety scenario in quantum gravity, in the enlarged theory space where the running of the cosmological constant and of Newton's constant is influenced by fluctuation couplings. All the derivations and technical details pertaining to these two chapters are collected in the respective Appendix to the Chapter.

In the first appendix we treat the heat kernel expansion and we propose a new way to derive the non-local expansion for its trace. We also introduce the Q-functional technology used to compute functional traces. In the second appendix we review the basic of QFT that we use all over the thesis. In the third appendix we fix the formalism for non-abelian gauge theories, in particular the functional quantization via Faddeev-Popov and the background field method. In the last one, after reviewing the some basic differential geometry we introduce the functional integral for quantum gravity.

Chapter 2

Introduction to the functional RG

2.1 Introduction

There are many different ways to implement the general RG coarse-graining procedure. The first one developed were Migdal-Kadanoff's block-spin real space RG and Wilson's original momentum shell mode elimination. The functional RG approach focuses on the mode elimination procedure of Wilson, but in place of integrating finite momentum shells, one encodes the integration over an infinitesimal momentum shell in a differential equation describing how the effective action changes as the cutoff is varied. The striking and fundamental point is that it is possible to write exact functional equations describing this process. These are the exact RG flow equations that characterize the functional RG framework.

In particular, it turns out to be convenient to study a scale dependent generalization of the effective action, called effective average action (EAA). In this way, we can work directly with the mean or average fields, which have a clear and direct physical interpretation. Also, the exact flow equation for the EAA turns out to be extremely compact and powerful. Still, the flow equation is a very complicated functional integro-differential equation, which can be treated only at the cost of making truncations of the full EAA.

As we will see in this chapter, the EAA is also suited to be applied to physical systems in presence of background gauge fields and can be extended to treat matter fields on arbitrary curved manifolds. In this last case, the matter fields are interacting with the background geometry. All this is possible because the mode elimination is performed by separating the slow modes, to be integrated out, from the fast modes in a covariant way. To do this we introduce a cutoff action constructed employing the covariant Laplacian that respects the symmetries of the underling theory. In the following chapters, and as the main topic of this thesis, the EAA formalism will be extended to treat dynamical gauge fields and even dynamical geometries.

In this first chapter we review the EAA formalism for matter fields on general backgrounds. We derive the exact flow equation the EAA satisfies and other basic properties. Through three applications in three quite different contexts we will learn several important conceptual and technical facts about the formalism. For a general reference about the EAA we refer to [15, 16], while for an introduction to the EAA formalism see [18, 17].

2.2 Effective average action (EAA)

The effective average action (EAA) is a functional that interpolates smoothly between the bare, or classical, action and the effective action. We will define the EAA $\Gamma_k[\varphi]$ for theories on a non-trivial gauge or gravitational background. In this chapter we will quantize only matter fields, gauge theories will be treated starting from the next chapter. For expository simplicity we will consider only a scalar field ϕ, on the manifold $\mathcal{M}$ with Riemannian metric $g_{\mu\nu}$ and in presence of a gauge connection (abelian or non-abelian) A_μ, to state the basic definitions. It is then straightforward to extend these definitions to a more general matter content.

Starting from the functional integral (B.4) from Appendix B, which defines the generating functional of correlation functions

$$Z[J;A,g] = \int D_g\phi \, \exp\left(-S[\phi;A,g] + \int d^dx\sqrt{g}J\phi\right) , \tag{2.1}$$

we add to the bare action $S[\phi;A,g]$ an infrared (IR) “cutoff“ or “regulator” term $\Delta S_k[\phi;A,g]$ of the form:

$$\Delta S_k[\phi;A,g] = \frac{1}{2}\int d^dx\sqrt{g}\phi\, R_k(\Delta)\,\phi\,. \tag{2.2}$$

In (2.2) the operator kernel $R_k(\Delta)$ is chosen so to suppress the field modes ϕ_n, eigenfunctions $\Delta\phi_n = \lambda_n\phi_n$ of the covariant Laplacian Δ, with eigenvalues smaller than the cutoff scale $\lambda_n < k^2$. We will call $\Delta S_k[\phi;A,g]$ the cutoff action and Δ the cutoff operator.

For example, in condensed matter applications the cutoff operator is usually chosen to be the flat space Laplacian $\Delta = -\partial^2$. When we will consider QED in section 2.3.1 we will cutoff the modes with the gauge Laplacian $\Delta = -D^2 = -\left(\partial_\mu + eiA_\mu\right)\left(\partial^\mu + eiA^\mu\right)$, while in section 2.3.3 we will choose the Laplace-Beltrami operator acting on scalars, $\Delta\phi = -\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right)$, as cutoff operator. In the next chapters, when dealing with non-abelian gauge theories and quantum gravity, we will consider more general choices.

The functional form of $R_k(z)$ is arbitrary except for the requirements that it should be a monotonically decreasing function in both z and k, i.e. that $R_k(z) \to 0$ for $z \gg k^2$ and that $R_k(z) \to k^2$ for $z \ll k^2$.

In this way we obtain the scale dependent generalization of (2.1):

$$Z_k[J; A, g] = \int D_g \phi \, \exp\left(-S[\phi; A, g] - \Delta S_k[\phi; A, g] + \int d^d x \sqrt{g} J \phi \right) . \tag{2.3}$$

Next we define the scale dependent generalization of the generating functional of connected correlation functions as:

$$W_k[J; A, g] = \log Z_k[J; A, g] . \tag{2.4}$$

The EAA is the scale dependent generalization of the effective action, defined in equation (B.13) of Appendix B. It is defined by the Lagrange transform of (2.4):

$$\Gamma_k[\varphi; A, g] + \Delta S_k[\varphi; A, g] = \int d^d x \sqrt{g} J_\varphi \varphi - W_k[J_\varphi; A, g] , \tag{2.5}$$

where $\varphi = \langle \phi \rangle$ is the mean or average field. In (2.5) we have solved the equation $\frac{\delta W_k[J;A,g]}{\delta J} = \varphi_J$ to obtain the current as function of the mean field $J \equiv J_\varphi$.

Note that the Legendre transform of the generating functional of the connected correlation functions $W_k[J; A, g]$ is the combination $\Gamma_k[\varphi; A, g] + \Delta S_k[\varphi; A, g]$ and not just the EAA; thus $\Gamma_k[\varphi; A, g]$ needs not to be a convex functional of φ for non zero k.

We can derive the integro-differential equation satisfied by the EAA by the same steps we did to derive the integro-differential representation of the effective action, equation (B.17) in Appendix B. We find:[1]

$$\begin{aligned} e^{-\Gamma_k[\varphi]} \;\; = \;\; & \int D\chi \exp \Big[-S[\varphi + \chi] - \Delta S_k[\varphi + \chi] \\ & + \int d^d x \sqrt{g} \left(\frac{\delta \Gamma_k[\varphi]}{\delta \varphi} + \frac{\delta \Delta S_k[\varphi]}{\delta \varphi} \right) \chi + \Delta S_k[\varphi] \Big] . \end{aligned} \tag{2.6}$$

Equation (2.6) has to be considered together with the condition of vanishing vacuum expectation value of the fluctuation field $\langle \chi \rangle = 0$. We can rearrange the terms in (2.6), containing

[1] We omit the arguments A_μ and $g_{\mu\nu}$ of Γ_k and ΔS_k as soon as they are understood.

the cutoff action, in the following way:

$$\begin{aligned} -\Delta S_k[\varphi+\chi]+\Delta S_k[\varphi]+\int d^dx\sqrt{g}\frac{\delta\Delta S_k[\varphi]}{\delta\varphi}\chi &= -\frac{1}{2}\int d^dx\sqrt{g}\,\chi\frac{\delta^2\Delta S_k[\varphi]}{\delta\varphi\delta\varphi}\chi+O(\chi^3) \\ &= -\Delta S_k[\chi]\,. \end{aligned} \tag{2.7}$$

In the second step in equation (2.7), we used the fact that the cutoff action, as defined in (2.2), is quadratic in the argument field. Inserting (2.7) into (2.6) gives:

$$e^{-\Gamma_k[\varphi]}=\int D\chi\exp\left[-S[\varphi+\chi]-\Delta S_k[\chi]+\int d^dx\sqrt{g}\frac{\delta\Gamma_k[\varphi]}{\delta\varphi}\chi\right]\,. \tag{2.8}$$

In (2.8) the condition $\langle\chi\rangle=0$ is understood. Relation (2.8) is the integro-differential equation that is satisfied by the EAA. Note that we can also use (2.8) as the starting point to define the EAA in place of (2.5).

As we said before, the EAA interpolates smoothly between the bare (or classical) action at UV scale and the effective action at the IR scale. To study the limit $k\to\infty$, we notice that the cutoff action behaves as $Ck^2\int d^dx\sqrt{g}\varphi^2$ for $k\to\infty$, with C a cutoff shape dependent constant. If we redefine the fluctuation field as $\chi\to\chi/k$ and we use the relation $\Delta S_k[\chi/k]=\Delta S_k[\chi]/k^2$, which follows from the definition of the cutoff action (2.2), then equation (2.8), in the $k\to\infty$ limit, becomes:

$$\begin{aligned} e^{-\Gamma_k[\varphi]} &= \int D\chi\exp\left[-S[\varphi+\chi/k]-\frac{1}{k^2}\Delta S_k[\chi]+\frac{1}{k}\int d^dx\sqrt{g}\frac{\delta\Gamma_k[\varphi]}{\delta\varphi}\chi\right] \\ &\underset{k\to\infty}{\to} e^{-S[\varphi]}\int D\chi\exp\left[-\frac{1}{2}C\int d^dx\sqrt{g}\chi^2\right]\,. \end{aligned} \tag{2.9}$$

In (2.9) we assumed that $\frac{\delta\Gamma_k[\varphi]}{\delta\varphi}$ is finite in the limit $k\to\infty$. The functional integral so obtained is Gaussian and is thus just a constant[2]. We thus arrive to the relation

$$\Gamma_\infty[\varphi]=S[\varphi]+\text{const}\,, \tag{2.10}$$

which can be seen as a boundary condition for the EAA as $k\to\infty$. It is easy to see that in the opposite limit $k\to0$ the EAA becomes the effective action since the cutoff kernel $R_k(z)$ vanishes. We have thus shown that the EAA interpolates between the bare action in the UV,

[2]If we rescale the fluctuation field as $\chi\to\chi/\sqrt{C}$ we obtain the Gaussian integral that actually defines the path integral and is normalized to be equal to one.

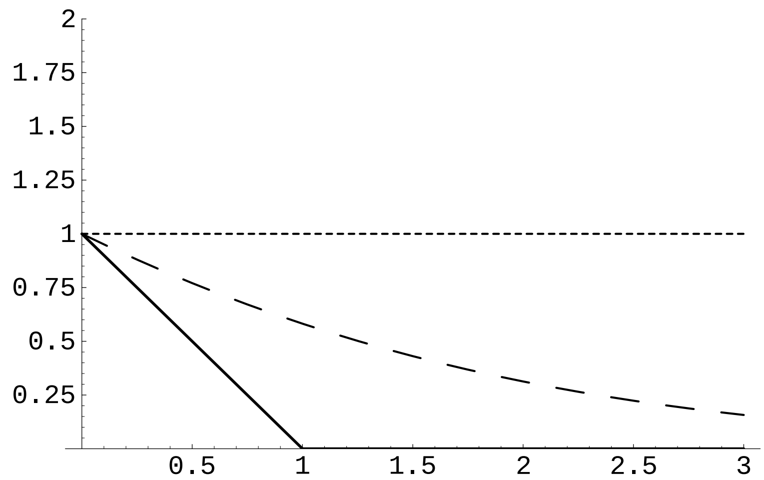


Figure 2.1: The regulator shape functions form (2.12). The mass-type cutoff shape function (short dashed), the exponential cutoff shape function (long dashed) and the optimized cutoff shape function (thick) are plotted in units of k^2 as a function of z/k^2.

for $k \to \infty$, and the effective action in the IR, for $k \to 0$:

$$\lim_{k\to 0} \Gamma_k[\varphi; A, g] = \Gamma[\varphi; A, g] \qquad\qquad \lim_{k\to\infty} \Gamma_k[\varphi; A, g] = S[\varphi; A, g]\,. \tag{2.11}$$

Once we have a way to compute the EAA for all k, we can use the properties (2.11) to give a new construction of the effective action. In this way, the EAA framework emerges as a new promising setting to define and calculate functional integrals. As we will see in the next section, the flow of $\Gamma_k[\varphi; A, g]$ is well defined for every finite non-zero k. Renormalization aspects are related to the limit $k \to \infty$ while the full effective action is recovered in the limit $k \to 0$.

There is a considerable freedom in the choice of the functional form of the regulator kernel $R_k(z)$. In this thesis we will use one of the following three examples of cutoff shape functions. The "optimized" [19], the "exponential", and the "mass-type" cutoffs are, respectively:

$$\begin{aligned} R_k^{mass}(z) &= k^2 \\ R_k^{opt}(z) &= (k^2 - z)\theta(k^2 - z) \\ R_k^{exp}(z) &= \frac{z}{e^{z/k^2} - 1}\,. \end{aligned} \tag{2.12}$$

In Figure 2.1 we show a plot of the cutoff shape functions in (2.12). The mass cutoff is not properly a cutoff because for $z \gg k^2$ it does not go to zero. Hence it does not guarantee the UV finiteness of the flow. Still it is useful, since it often allows for analytical computations

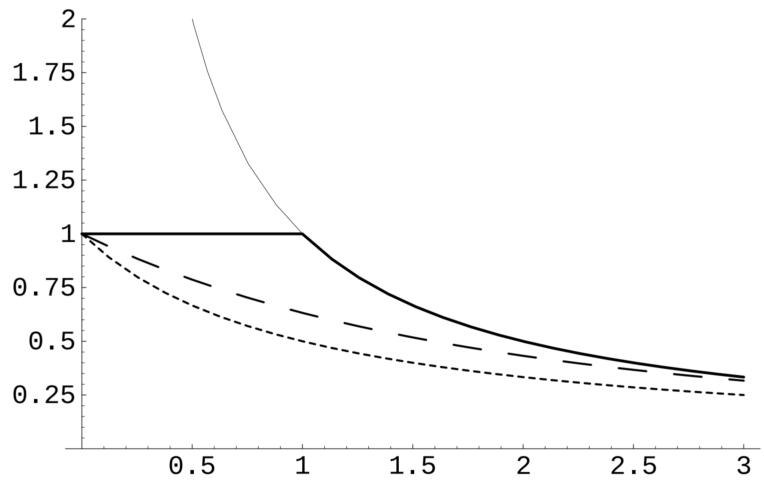


Figure 2.2: The bare propagator (continuous) is plotted in units of k^2 as a functions of z/k^2 together with the regularized propagators obtained using in (2.13), respectively, the optimized (thick), the exponential (long dashed) and the mass-type (short dashed) cutoff shapes.

of UV finite quantities.

It is easy to understand how the regulator acts when the bare propagator goes like $\frac{1}{p^2} \equiv \frac{1}{z}$. The regularized propagator

$$G_k(z) = \frac{1}{z + R_k(z)} \tag{2.13}$$

is shown in Figure 2.2, for the three different cutoff shapes in (2.12), plotted together with the bare one. For modes of momentum eigenvalues greater then the RG scale $z \gg k^2$ the propagation is unaffected, while starting at the cutoff scale, their propagation is successively suppressed as if they were massive particles of constantly growing mass k.

2.2.1 Exact flow equation for the EAA

The major virtue of the EAA is that it is possible to write down an exact equation describing how it varies when the cutoff scale k is changed. This relation is the exact functional RG equation satisfied by $\Gamma_k[\varphi]$. To be general here we consider φ as a multiplet of matter fields, the components of which we indicate with capital letters.

If we differentiate the integro-differential equation (2.8) with respect to the "RG time" or

"RG parameter" $t = \log k/k_0$[3] we find:

$$e^{-\Gamma_k[\varphi]}\partial_t\Gamma_k[\varphi] = \int D\chi \left[\partial_t \Delta S_k[\chi] - \int d^dx\sqrt{g}\partial_t \frac{\delta\Gamma_k[\varphi]}{\delta\varphi_A}\chi_A\right] e^{-S[\varphi+\chi]-\Delta S_k[\chi]+\int\sqrt{g}\frac{\delta\Gamma_k[\varphi]}{\delta\varphi}\chi} . \tag{2.14}$$

Expressing the terms on the rhs of (2.14) as expectation values, using equation (B.2) and equation (2.2), we can rewrite (2.14) as:

$$\begin{aligned}\partial_t\Gamma_k[\varphi] &= \langle\partial_t\Delta S_k[\chi]\rangle - \int d^dx\sqrt{g}\,\partial_t\frac{\delta\Gamma_k[\varphi]}{\delta\varphi_A}\langle\chi_A\rangle \\ &= \frac{1}{2}\int d^dx\sqrt{g}\,\langle\chi_A\chi_B\rangle\,\partial_t R_{k\,BA} . \end{aligned} \tag{2.15}$$

In (2.15) we used the fact that the field χ_A has vanishing vacuum expectation value $\langle\chi_A\rangle = 0$ and the symmetry of the cutoff kernel in the indices A and B. Since the fluctuation field has zero average, the two-point function in (2.15) can be considered to be the connected one. Using the standard relation, equation (B.20) from Appendix B, we can express it in terms of the inverse Hessian of the EAA plus the cutoff action[4] :

$$\langle\chi_A\chi_B\rangle = \left(\frac{\delta^2\left(\Gamma_k[\varphi]+\Delta S_k[\varphi]\right)}{\delta\varphi_A\delta\varphi_B}\right)^{-1} = \left(\frac{\delta^2\Gamma_k[\varphi]}{\delta\varphi_A\delta\varphi_B} + R_{k\,AB}\right)^{-1} . \tag{2.16}$$

Inserting (2.16) into (2.15) and writing a functional trace in place of the integral, gives:

$$\partial_t\Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k[\varphi]}{\delta\varphi\delta\varphi} + R_k\right)^{-1}\partial_t R_k . \tag{2.17}$$

Equation (2.17) is a closed equation that describes the RG flow of the EAA. This equation is exact since no approximations where made in its derivation and is the main technical tool employed within the functional RG approach to quantum field theory. It was first derived in this form in [14].

Note that to obtain a one-loop like flow equation, as is equation (2.17), it is crucial that the cutoff action (2.2) is quadratic in the argument field. Otherwise we would had found higher order vertices of the EAA on the rhs of (2.17) that spoil the one-loop structure of the flow equation.

The flow generated by equation (2.17) is both UV finite, due to the presence of the term

[3] Here k_0 is an arbitrary scale.

[4] Remember that it is $\Gamma_k[\varphi] + \Delta S_k[\varphi]$ to be the Legendre transform of $W_k[J]$, see the definition (2.5) of the EAA.

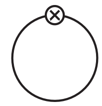
$$\partial_t \Gamma_k[\varphi] \;=\; \tfrac{1}{2}$$ 

Figure 2.3: Graphical representation of the exact RG flow equation (2.17). The continuous line represents the regularized propagator defined in equation (2.18) while the cross represents the insertion of $\partial_t R_k$.

$\partial_t R_k$, which constrains the momentum in the integral to be between zero and k, and IR finite, due to the presence of the regulator R_k in the propagator that behaves like a mass term for $k \to 0$. Thus the EAA obtained by integrating the flow equation (2.17) is a finite functional for non-zero k.

We will often use a notation involving the "regularized propagator" defined by:

$$G_k[\varphi] = \left(\frac{\delta^2 \Gamma_k[\varphi]}{\delta\varphi\delta\varphi} + R_k \right)^{-1} . \tag{2.18}$$

With this notation the flow equation (2.17) becomes simply:

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} \mathrm{Tr}\, G_k[\varphi] \partial_t R_k \, . \tag{2.19}$$

We can represent graphically the flow equation as in Figure 2.3.

The exact flow equation (2.17) can be re-derived as an RG improvement of the one-loop EAA. If we apply standard perturbation theory, equations (B.58) and (B.61) from Appendix B, to the EAA, as defined in equation (2.8), we find to one-loop order:

$$\Gamma_k[\varphi] = S[\varphi] + \frac{1}{2} \mathrm{Tr} \log \left(S^{(2)}[\varphi] + R_k \right) + ... \tag{2.20}$$

If we differentiate (2.20) with respect to the RG parameter t, we are lead to:

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} \mathrm{Tr} \left(\frac{\delta^2 S[\varphi]}{\delta\varphi\delta\varphi} + R_k \right)^{-1} \partial_t R_k \, . \tag{2.21}$$

This is the one-loop flow equation that the EAA satisfies. If we now RG improve it by replacing the Hessian of the bare action with the Hessian of the EAA, we recover the exact flow equation (2.17). As we mentioned before, this happens because we have chosen the cutoff action (2.2) to be quadratic in the argument field.

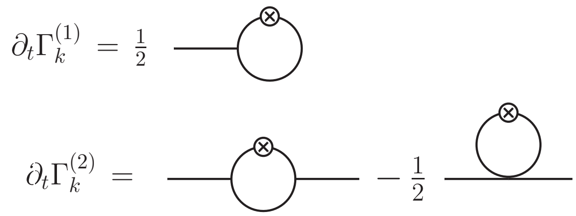


Figure 2.4: Graphical representation of the first two equations of the hierarchy of flow equations for the proper-vertices of the EAA from equation (2.22).

2.2.2 Flow equations for the proper-vertices of the EAA

The exact flow equation (2.17) is equivalent to a hierarchy of flow equations for the proper-vertices of the EAA. To derive this hierarchy, we simply differentiate (2.17) with respect to the field φ.

The first two equations in the hierarchy are:

$$\begin{aligned}
\partial_t \Gamma^{(1)}_{k,x}[\varphi; A, g] &= -\frac{1}{2} \mathrm{Tr}\, G_k[\varphi; A, g]\, \Gamma^{(3)}_{k,x}[\varphi; A, g] G_k[\varphi; A, g] \partial_t R_k[A, g] \\
\partial_t \Gamma^{(2)}_{k,xy}[\varphi; A, g] &= \mathrm{Tr}\, G_k[\varphi; A, g]\, \Gamma^{(3)}_{k,x}[\varphi; A, g] G_k[\varphi; A, g]\, \Gamma^{(3)}_{k,y}[\varphi; A, g] G_k[\varphi; A, g] \partial_t R_k[A, g] \\
&\quad -\frac{1}{2} \mathrm{Tr}\, G_k[\varphi; A, g]\, \Gamma^{(4)}_{k,xy}[\varphi; A, g] G_k[\varphi; A, g] \partial_t R_k[A, g]
\end{aligned} \tag{2.22}$$

These equations can be represented graphically as in Figure 2.4. The flow equations for higher vertices can be easily calculated by taking further functional derivatives. For example, the flow for the three-point function is given in Figure 2.5. In equation (2.22) we used the abbreviation $\Gamma^{(n)}_{k,x_1 \ldots x_n}[\varphi] \equiv \frac{\delta^n \Gamma_k[\varphi]}{\delta\phi(x_1)\ldots\delta\phi(x_n)}$, that we will often adopt, to indicate functional derivatives of the EAA or of other functionals.

Note that in the flow equations (2.17) or (2.22), the cutoff kernel depends on the background gauge connection and on the background metric. Here we tare taking only functional derivatives with respect to the average field φ, only in this case the equations in (2.22) are correct. In the case we were differentiating with respect to the background fields we must have cared about how to differentiate the cutoff kernels. This can be necessary for computational purposes, but will not be dealt with in this chapter. We will see that this will become necessary when treating non-abelian gauge theories in Chapter 3 and quantum gravity in Chapter 4.

Actually, every equation of the hierarchy, if considered for a general field configuration $\varphi(x)$, is equivalent to the basic flow equation (2.17) and gives thus no new information. The hierarchy becomes useful when the EAA can be expanded in functional Taylor series around

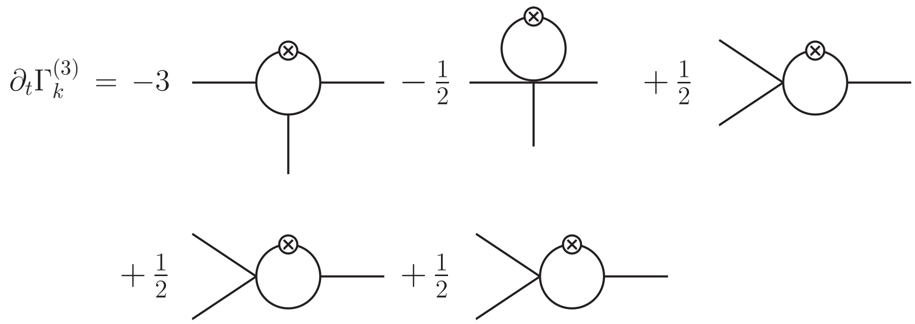


Figure 2.5: Flow equation for the three point function of the EAA. Repeated diagrams stands for different permutations of the coordinates of the external legs.

a given field configuration $\bar{\varphi}(x)$, using the abbreviation $\int_{x_1 \ldots x_n} \equiv \int dx_1^d \ldots d^d x_n \sqrt{g_{x_1}} \ldots \sqrt{g_{x_n}}$ we can write:

$$\Gamma_k[\varphi; A, g] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{x_1 \ldots x_n} \Gamma^{(n)}_{k, x_1 \ldots x_n}[\bar{\varphi}; A, g] \left[\varphi(x_1) - \bar{\varphi}(x_1)\right] \ldots \left[\varphi(x_n) - \bar{\varphi}(x_n)\right] . \tag{2.23}$$

The expansion (2.23) makes the hierarchy of flow equations for the proper-vertices an infinite system of equations for the vertices $\Gamma^{(n)}_{k, x_1 \ldots x_n}[\bar{\varphi}; A, g]$. We can now truncate the hierarchy to a finite order n, obtaining in this way a finite system for the first n proper-vertices of the EAA, that can be used as a starting point for useful applications [20]. This kind of truncation of the EAA is usually called vertex expansion. Up to now, it has been employed only to treat the EAA of matter fields on flat space with zero gauge backgrounds. In this thesis we will develop the tools to construct the analogous of the vertex expansion for non-abelian gauge theories in Chapter 3 and for quantum gravity in Chapter 4.

The fact that we Taylor expand the EAA does not mean that we cannot study truncations of the EAA which are non-analytical in the fields. For example, $\bar{\varphi}$ can be a constant field configuration. This is what is done to calculate the flow of the functions $V_k(\varphi)$ and $Z_k(\varphi)$ (to be defined in a moment) at order ∂^2 of the derivative expansion [21, 25, 22]. The derivative expansion is a truncation scheme where the EAA is expanded in powers of the derivatives. This scheme is usually employed for matter field theories on flat space, where the expansion becomes an expansion in powers of the momentum. It is manly used to study critical phenomena and to calculate the related critical exponents. If we consider a one component scalar field, in d-dimensional flat space, with a $\mathbb{Z}_2$ symmetry, the derivative expansion for the

EAA to order ∂^4 reads [26]:

$$\begin{aligned}\Gamma_k[\varphi] &= \int d^dx \left\{ \frac{1}{2} Z_k(\varphi)(\partial\varphi)^2 + V_k(\varphi) + \frac{1}{2} W_{1,k}(\varphi)\left(\partial^2\varphi\right)^2 + \right. \\ & \left. -\frac{1}{2} W_{2,k}(\varphi)(\partial\varphi)^2 \varphi \partial^2\varphi + \frac{1}{4} W_{3,k}(\varphi)(\partial\varphi)^4 \right\} + O\left(\partial^6\right) . \qquad (2.24)\end{aligned}$$

The effective potential $V_k(\varphi)$, the wave-function renormalization function $Z_k(\varphi)$ and the fourth order functions $W_{i,k}(\varphi)$, $i = 1, 2, 3$, are arbitrary functions of φ. To derive the flow for these functions one needs to consider the running of the n-th proper-vertex of the EAA evaluated at the constant field configuration $\bar{\varphi}$. In particular, to extract the running of the wave-function renormalization function and of $W_{1,k}(\varphi)$ one needs to consider the running of the two-point function of the EAA. To extract the running of $W_{2,k}(\varphi)$ one considers the running of the three-point function, while to extract the running of $W_{3,k}(\varphi)$ the running of the four-point function is needed. These equations can be evaluated in momentum space after $\bar{\varphi}$ has been taken to be constant.

We derive now the flow equation for the effective potential $V_k(\varphi)$ and for the wave-function renormalization function $Z_k(\varphi)$, dropping the fourth order functions $W_{i,k}(\varphi)$. It is not difficult to show that the flow equation for the effective potential is given by:

$$\partial_t V_k(\bar{\varphi}) = \frac{1}{(4\pi)^{d/2}} Q_{\frac{d}{2}} \left[\left(\partial_t R_k - \eta R_k \right) G_k(\bar{\varphi}) \right] , \qquad (2.25)$$

where in (2.25) the regularized propagator at the constant field configuration $\bar{\varphi}$ is defined as:

$$G_k(z, \bar{\varphi}) = \frac{1}{Z_k(\bar{\varphi})\, z + V_k''(\bar{\varphi}) + R_k(z)} . \qquad (2.26)$$

In (2.25) we wrote the rhs side as a "Q-functional" as defined in equation (A.39) of Appendix A. In this thesis we will always write the beta functions, before a particular cutoff shape function as been chosen, in terms of Q-functionals. In this sense a Q-functional is a functional that maps a cutoff shape function to the explicit form of the beta function. The anomalous dimension of the scalar field in (2.25) is defined by $\eta = -\partial_t \log Z_k(\bar{\varphi}_0)$, where $\bar{\varphi}_0$ is the minimum of the effective potential $V'(\bar{\varphi}_0) = 0$. This definition of the anomalous dimension is valid both in the symmetric and in the broken phase [15]. It is a little more involved to derive the flow equation for the wave-function renormalization function. From the graphical representation of Figure 2.4 we can readily write down the flow equation for the two-point function of the EAA in momentum space. We have:

$$\begin{aligned}\partial_t \Gamma^{(2)}_{p,-p}(\bar{\varphi}) &= \int_q G_q(\bar{\varphi})\Gamma^{(3)}_{q,p,-q-p}(\bar{\varphi})G_{q+p}(\bar{\varphi})\Gamma^{(3)}_{q+p,-p,-q}(\bar{\varphi})G_q(\bar{\varphi})\partial_t R_q + \\ &-\frac{1}{2}\int_q G_q(\bar{\varphi})\Gamma^{(4)}_{q,p,-p,-q}(\bar{\varphi})G_q(\bar{\varphi})\partial_t R_q \,. \end{aligned} \tag{2.27}$$

Introducing on the rhs of equation (2.27) the vertices of the EAA (2.24) at order ∂^2 and extracting the p^2 terms gives, after some algebra, the following flow equation for the wave-function renormalization function:

$$\begin{aligned}(4\pi)^{d/2}\,\partial_t Z_k &= (V_k''')^2 \left\{ Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta R_k)\, G_k G_k'\right] + Q_{\frac{d}{2}+1}\left[(\partial_t R_k - \eta R_k)\, G_k^2 G_k''\right]\right\} \\ &+Z_k' V_k''' \left\{ Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta R_k)\, G_k^3\right] + (d+2) Q_{\frac{d}{2}+1}\left[(\partial_t R_k - \eta R_k)\, G_k^2 G_k'\right]\right. \\ &\left.(d+2) Q_{\frac{d}{2}+2}\left[(\partial_t R_k - \eta R_k)\, G_k^2 G_k''\right]\right\} \\ &+ (Z_k')^2 \left\{ \frac{2d+1}{2} Q_{\frac{d}{2}+1}\left[(\partial_t R_k - \eta R_k)\, G_k^3\right]\right. \\ &+ \frac{(d+4)(d+2)}{4} Q_{\frac{d}{2}+2}\left[(\partial_t R_k - \eta R_k)\, G_k^2 G_k'\right] \\ &\left.\frac{(d+4)(d+2)}{4} Q_{\frac{d}{2}+3}\left[(\partial_t R_k - \eta R_k)\, G_k^2 G_k''\right]\right\} \\ &-\frac{1}{2} Z_k'' Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta R_k)\, G_k\right] \,. \end{aligned} \tag{2.28}$$

Equations (2.25) and (2.28) represent the flow equations for $V_k(\bar{\varphi})$ and $Z_k(\bar{\varphi})$ for general cutoff shape function at order ∂^2 of the derivative expansion.

Once an appropriate cutoff shape function as been chosen, the integrals in (2.25) and (2.28) can be done analytically. In this way we obtain a system of partial differential equations for $V_k(\bar{\varphi})$ and $Z_k(\bar{\varphi})$ in the variables k and $\bar{\varphi}$. After re-writing the flow equations (2.25) and (2.28) in terms of the dimensionless field $\tilde{\varphi} = k^{\frac{d}{2}-1} Z_k^{-1}(\bar{\varphi}_0)\bar{\varphi}$ and of the dimensionless functions $\tilde{V}_k(\tilde{\varphi})$ and $\tilde{Z}_k(\tilde{\varphi})$, we can study the system

$$\partial_t \tilde{V}_k(\tilde{\varphi}) = 0 \qquad\qquad \partial_t \tilde{Z}_k(\tilde{\varphi}) = 0\,, \tag{2.29}$$

to find the fixed point effective potential and fixed point wave-function renormalization function. It is possible, for example, to show that in $d = 3$ there is only one scaling solution to the system (2.29). This scaling solution corresponds to what is know as the Wilson-Fisher fixed point and correspond to the scalar $\mathbb{Z}_2$ symmetric universality class in $d = 3$. By linearizing around this solution, it is possible to calculate the critical exponents that characterize this particular universality class to a good numerical precision.

As these critical exponents can be accurately calculated with various perturbative expansions, as for example the ϵ-expansion [12], the fact that they are very well reproduced by the derivative expansion of the EAA [26] is not a critical test for the non-perturbative strength of the EAA approach. It is in $d = 2$, where every perturbative approach fails to describe correctly the various universality classes, first constructed exactly using conformal field theory (CFT) methods [13], that the system (2.29) reveals its non-perturbative potentialities. It was shown in [24] that the system (2.29), in $d = 2$, has several scaling solutions, each of which corresponds to one of the universality classes know from CFT. It was also found in [24] that the derivative expansion (2.24) breaks down as the anomalous dimension of the scalar field grows, making the contributions of the neglected higher derivative terms, increasingly more important.

2.2.3 Perturbation theory form the EAA

The flow equation for the EAA (2.17) is an integro-differential equation and, as it stands, it is very difficult to solve. We have seen that one way to proceed is to truncate the EAA using the vertex expansion or the derivative expansion. But there are other possible strategies to follow, for example we can try an iterative solution of the flow equation. This means that we have to chose an initial ansatz $\Gamma_{k,0}[\varphi]$ for the EAA, with possibly some given scale dependence, to plug it into the rhs of the flow equation (2.17), while we consider the lhs as the flow of the first approximation $\Gamma_{k,1}[\varphi]$ to the EAA. Then we integrate the flow to obtain $\Gamma_{0,1}[\varphi]$, after we imposed the initial condition $\Gamma_{\Lambda,1}[\varphi] = S_\Lambda[\varphi]$ as found in (2.11), and we re-insert it in the rhs of the flow equation; in this way generating the flow of the second approximation $\Gamma_{k,2}[\varphi]$. We generate a sequence $\Gamma_{0,n}[\varphi]$ of approximate actions that may converge in the limit $n \to \infty$ to the full effective action. Obviously the issue of convergence of this procedure is also a very hard mathematical problem, but depending on the initial ansatz $\Gamma_{k,0}[\varphi]$, we could generate sequences (or series) that can be useful approximations. In fact, if we use as our initial ansatz the bare action $\Gamma_{\Lambda,0}[\varphi] = S_\Lambda[\varphi]$ we generate the standard loop expansion of perturbation theory as we will shortly show. As it is well known [12], perturbative series, when not Borel summable, are just asymptotic expansions, but still they turn out to be very useful. From this point of view this iterative procedure can be of valuable use also for initial ansatz different from the bare action, in particular those ansatz where $\Gamma_{k,0}[\varphi]$ has already some non-trivial scale dependence. As an example of this, we can take the initial ansatz

$$\Gamma_{k,0}[\varphi] = \int d^d x \left[\frac{1}{2} Z_k(\varphi_0)\partial_\mu\varphi\partial^\mu\varphi + V_k(\varphi)\right] ,$$

where $Z_k(\varphi_0)$ and $V_k(\varphi)$ are the explicit known functions of k and φ already calculated from the flow equations (2.25) and (2.28). Already this simple initial ansatz has never been considered and deserves further study.

To recover the loop expansion, as an iterative solution of the flow equation, we introduce $\hbar$ in (2.17) as a loop counting parameter:

$$\partial_t \Gamma_k[\varphi] = \frac{\hbar}{2}\mathrm{Tr}\frac{\partial_t R_k}{\Gamma_k^{(2)}[\varphi] + R_k} , \tag{2.30}$$

and we write the following expansion for the EAA:

$$\Gamma_k[\varphi] = S_\Lambda[\varphi] + \sum_{L=1}^{n} \hbar^L \Gamma_{L,k}[\varphi] . \tag{2.31}$$

With the definition made in (2.31) the actions $\Gamma_{n,k}[\varphi]$ of the previous paragraph are actually the finite sums $\sum_{L=1}^{n} \hbar^L \Gamma_{L,k}[\varphi]$. Inserting (2.31) in the the flow equation (2.30) gives:

$$\hbar\,\partial_t\Gamma_{1,k}[\varphi] + \hbar^2\partial_t\Gamma_{2,k}[\varphi] + ... = \frac{\hbar}{2}\mathrm{Tr}\frac{\partial_t R_k}{S_\Lambda^{(2)}[\varphi] + R_k + \hbar\,\Gamma_{1,k}^{(2)}[\varphi] + \hbar^2\Gamma_{2,k}^{(2)}[\varphi] + ...} . \tag{2.32}$$

In the regularized propagator in (2.32) the Hessians of the terms $\Gamma_{n,k}[\varphi]$ need to be renormalized to make the denominator finite. This is done by writing the bare action $S_\Lambda[\varphi]$ as the sum of the renormalized action[5] $S_0[\varphi]$ and the Λ dependent counterterm action $\delta S_\Lambda[\varphi]$ as:

$$S_\Lambda[\varphi] = S_0[\varphi] + \delta S_\Lambda[\varphi] = S_0[\varphi] + \sum_{L=1}^{\infty} \hbar^L \delta S_{L,\Lambda}[\varphi] . \tag{2.33}$$

Inserting (2.33) into (2.32) transforms the denominator to the form:

$$S_0^{(2)}[\varphi] + R_k + \hbar\left(\Gamma_{1,k}^{(2)}[\varphi] + \delta S_{2,\Lambda}^{(2)}[\varphi]\right) + \hbar^2\left(\Gamma_{2,k}^{(2)}[\varphi] + \delta S_{2,\Lambda}^{(2)}[\varphi]\right) + O\left(\hbar^3\right) . \tag{2.34}$$

In this way the rhs of (2.32) will be finite, after the counterterms have been chosen properly,

[5]Here we use the notation S_0 for the renormalized action to be consistent with the notations used in this thesis where renormalized quantities are obtained as the $k \to 0$ limit of scale dependent quantities.

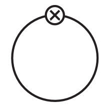
$\partial_t \Gamma_{1,k}[\varphi] \; = \; \frac{1}{2}$ 

Figure 2.6: Graphical representation for the flow equation of the one-loop contribution $\Gamma_{1,k}[\varphi]$ from equation (2.38). The continuous line represents the regularized renormalized propagator. Due to the one-loop structure of the exact flow, this representation is similar to the one in Figure 2.3.

to order $L-1$ when we are calculating the flow of the L-th order contribution. Each loop contribution can be written as the sum of a divergent part and a renormalized part as:

$$\Gamma_{L,k}[\varphi] = [\Gamma_{L,k}[\varphi]]_{\text{div}} + [\Gamma_{L,k}[\varphi]]_{\text{ren}} \;. \tag{2.35}$$

We chose the counterterms so to cancel the divergent parts by defining:

$$\delta S_{L,\Lambda}[\varphi] = -\,[\Gamma_{L,0}[\varphi]]_{\text{div}} \;. \tag{2.36}$$

Note that we defined the counterterm in (2.36) as minus the divergent part of the loop contribution at $k=0$. Due to the separation of scales this makes finite also the contribution at $k \neq 0$ since $[\Gamma_{L,k}[\varphi]]_{\text{div}} = [\Gamma_{L,0}[\varphi]]_{\text{div}}$. Note also that it can be shown that the divergent part of every loop contribution is a local action in the fields [12] and so every counterterm $\delta S_{L,\Lambda}[\varphi]$ is also local.

The flow of the L-th loop contribution to the EAA action can be extracted from (2.32) as follows:

$$\partial_t \Gamma_{L,k}[\varphi] = \frac{1}{(L-1)!} \frac{\partial^{L-1}}{\partial \hbar^{L-1}} \frac{\partial_t \Gamma_k[\varphi]}{\hbar}\bigg|_{\hbar=0} \;. \tag{2.37}$$

We will now show that the first two contributions, $L=1,2$, to the effective action generated using (2.37) reproduce the loop expansion, given in Appendix B, to two-loop order. This fact is a manifestation of the "exactness" of the flow [27] for EAA described by (2.30).

The flow of the one-loop contribution obtained from (2.37) is simply:

$$\partial_t \Gamma_{1,k}[\varphi] = \frac{1}{2} \text{Tr}\, G_k[\varphi] \partial_t R_k = \frac{1}{2} \text{Tr}\, \partial_t \log\, G_k^{-1}[\varphi] \,, \tag{2.38}$$

where we defined

$$G_k[\varphi] = \frac{1}{S_0^{(2)}[\varphi] + R_k} \,, \tag{2.39}$$

as the (IR) regularized renormalized propagator. In the second step of (2.38) we used the relation $\partial_t (\log G_k) = G_k \partial_t R_k$, which is valid in this case because the only dependence of the regularized renormalized propagator on k is through the cutoff kernel R_k. As already noticed, the one-loop flow equation has the same structure as the flow equation (2.17) for the full EAA.

It is tempting in (2.38) to exchange the t-derivative with the trace integral, but if we do this the flow will not be anymore UV finite. We should first calculate explicitly the trace in (2.38) and then integrate it from k and Λ. Here we are only interested to show that we correctly recover the loop expansion, so we introduce an additional UV cutoff in the trace integral using Λ to interchange it with the t-derivative. We will write this as $\mathrm{Tr} \to \mathrm{Tr}_\Lambda$.

We integrate now the one-loop flow (2.38) from the UV scale Λ, where we impose the boundary condition $\Gamma_{1,\Lambda}[\varphi] = 0$, to the IR scale k. We find:

$$\Gamma_{1,k}[\varphi] = -\frac{1}{2}\int_k^\Lambda \frac{dk'}{k'}\partial_{t'}\Gamma_{1,k'}[\varphi] = -\frac{1}{2}\int_k^\Lambda \frac{dk'}{k'}\partial_{t'}\mathrm{Tr}_\Lambda \log\, G_{k'}^{-1} = \frac{1}{2}\,\mathrm{Tr}_\Lambda \log G_k^{-1}\Big|_\Lambda^k\,. \tag{2.40}$$

If we send $k \to 0$ in (2.40) we obtain the following one-loop contribution:

$$\Gamma_{1,0}[\varphi] = \frac{1}{2}\mathrm{Tr}_\Lambda \log\, S_0^{(2)}[\varphi] - \frac{1}{2}\mathrm{Tr}_\Lambda \log\left(S_0^{(2)}[\varphi] + R_\Lambda\right)\,. \tag{2.41}$$

If now the bare theory is perturbatively renormalizable, then the divergent part $[\Gamma_{1,0}[\varphi]]_{\mathrm{div}}$ of (2.41) can be reabsorbed by the counterterms in $S_\Lambda[\varphi]$. When we insert (2.41) in (2.31) and we combine it with the appropriate counterterm in (2.33), fixed as in (2.36), we find the perturbatively renormalized one-loop contribution to the effective action:

$$[\Gamma_{1,0}[\varphi]]_{\mathrm{ren}} = \Gamma_{1,0}[\varphi] + \delta S_{1,\Lambda}[\varphi] \qquad\qquad \Lambda \to \infty\,. \tag{2.42}$$

In the following we will consider all renormalized quantities to be in the limit $\Lambda \to \infty$.

As an example we consider a massless scalar field in $d = 4$ with interaction $V(\phi) = \frac{\lambda}{4!}\phi^4$. The $k \neq 0$ one-loop contribution (2.40) becomes:

$$\Gamma_{1,k}[\varphi] = \frac{1}{2}\mathrm{Tr}_\Lambda \log \frac{-\partial^2 + V''(\varphi) + R_k(-\partial^2)}{-\partial^2 + V''(\varphi) + R_\Lambda(-\partial^2)}\,. \tag{2.43}$$

For a constant field configuration $\varphi = \bar\varphi$ the trace in (2.43) can be written as a momentum space integral with cutoff:

$$\Gamma_{1,k}[\bar\varphi] = \frac{S_4}{2}\int_0^\Lambda dq\, q^3 \log \frac{q^2 + V''(\bar\varphi) + R_k(q^2)}{q^2 + V''(\bar\varphi) + R_\Lambda(q^2)}\,, \tag{2.44}$$

where $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$. If we insert the optimized cutoff shape function $R_k^{opt}(z) = (k^2 - z)\theta(k^2 - z)$ in (2.44) we find:

$$\Gamma_{1,k}[\bar{\varphi}] = \frac{S_4}{16} \left\{ k^4 - \Lambda^4 - 2\left(k^2 - \Lambda^2\right) V''(\bar{\varphi}) + 2\left[V''(\bar{\varphi})\right]^2 \log \frac{k^2 + V''(\bar{\varphi})}{\Lambda^2 + V''(\bar{\varphi})} \right\} . \tag{2.45}$$

Note that the two scales k and Λ do not mix and so $[\Gamma_{1,k}[\varphi]]_{\text{div}} = [\Gamma_{1,0}[\varphi]]_{\text{div}}$ as we assumed earlier. In particular we have:

$$[\Gamma_{1,0}[\bar{\varphi}]]_{\text{div}} = \frac{1}{16\pi^2} \left\{ -\frac{\Lambda^4}{8} + \frac{\Lambda^2}{4} V''(\bar{\varphi}) - \frac{1}{2}\left[V''(\bar{\varphi})\right]^2 \log \Lambda \right\} . \tag{2.46}$$

Equations (2.36) and (2.46) tell us that we have to choose the one-loop counterterm action as:

$$\delta S_{1,\Lambda}[\phi] = \frac{1}{16\pi^2} \int d^4x \left[\frac{\Lambda^4}{8} - \frac{\Lambda^2}{8} \lambda_0 \phi^2 + \frac{1}{8} \lambda_0^2 \phi^4 \log \Lambda \right] . \tag{2.47}$$

To one-loop order the bare action is thus

$$S_\Lambda[\phi] = \int d^4x \left[E_\Lambda + \frac{1}{2} Z_\Lambda \left(\partial\phi\right)^2 + \frac{m_\Lambda^2}{2}\phi^2 + \frac{\lambda_\Lambda}{4!}\phi^4 \right] , \tag{2.48}$$

where the bare parameters are related to the renormalized ones by the following relations:

$$E_\Lambda = \frac{\Lambda^4}{128\pi^2} \qquad Z_\Lambda = 1 \qquad m_\Lambda^2 = \frac{\Lambda^2}{64\pi^2}\lambda_0$$

$$\lambda_\Lambda = \lambda_0 + \frac{3}{16\pi^2} \lambda_0^2 \log \Lambda \, . \tag{2.49}$$

We can solve the last relation of (2.49) for the renormalized coupling to find:

$$\lambda_0 = \lambda_\Lambda - \frac{3}{16\pi^2} \lambda_\Lambda^2 \log \Lambda + O\left(\lambda_\Lambda^3\right) . \tag{2.50}$$

From the condition $\Lambda \partial_\Lambda \lambda_0 = 0$ that the renormalized coupling is independent of the cutoff Λ and from relation (2.50) we find Wilson's beta function for the bare coupling:

$$\Lambda \partial_\Lambda \lambda_\Lambda = \frac{3}{16\pi^2} \lambda_\Lambda^2 + O\left(\lambda_\Lambda^3\right) . \tag{2.51}$$

Equation (2.51) represent the standard one-loop beta function for the scalar ϕ^4 coupling [9].

We go back now to the general case. The running of the two-loop contribution to the

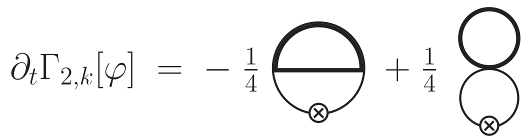


Figure 2.7: Flow of the two-loop contribution $\Gamma_{2,k}[\varphi]$ flow from equation (2.55). The thick loops are the renormalized ones, while the other ones are regularized by an insertion of $\partial_t R_k$.

EAA from (2.37) reads:

$$\partial_t \Gamma_{2,k}[\varphi] = -\frac{1}{2} \mathrm{Tr}\, G_k[\varphi] \left[\Gamma^{(2)}_{1,k}[\varphi]\right]_{\mathrm{ren}} G_k[\varphi] \partial_t R_k = \frac{1}{2} \mathrm{Tr} \left[\Gamma^{(2)}_{1,k}[\varphi]\right]_{\mathrm{ren}} \partial_t G_k[\varphi]\,, \tag{2.52}$$

where we used the cyclicity of the trace and the relation

$$\partial_t G_k = -G_k \partial_t R_k G_k\,, \tag{2.53}$$

again valid only because G_k depends on k only through the cutoff kernel R_k. Note that the flow of the two-loop contribution is given in terms of the Hessian of the renormalized one-loop contribution at $k \neq 0$. This is valid in general: the flow of the L-loop contribution is given in terms of $[\Gamma_{n,k}[\varphi]]_{\mathrm{ren}}$ for $n = 1, ..., L-1$. The loop expansion is therefore constructed loop by loop. We can calculate the Hessian $\left[\Gamma^{(2)}_{1,k}[\varphi]\right]_{\mathrm{ren}}$ needed in the flow (2.52) by taking functional derivatives of (2.42) but for $k \neq 0$. We find:

$$\left[\Gamma^{(2)}_{1,k}[\varphi]_{xy}\right]_{\mathrm{ren}} = \frac{1}{2} \left[S^{(3)}_{0,axb} G_{k,bc} S^{(3)}_{0,cyd} G_{k,da} - S^{(4)}_{0,axyb} G_{k,ba}\right]_{\mathrm{ren}}\,. \tag{2.54}$$

In (2.54) we used a condensed matrix notation in place of writing the integrals explicitly. The two-loop flow (2.52) then becomes:

$$\partial_t \Gamma_{2,k}[\varphi] = \frac{1}{4} \left[S^{(3)}_{0,axb} G_{k,bc} S^{(3)}_{0,cyd} G_{k,da} - S^{(4)}_{0,axyb} G_{k,ba}\right]_{\mathrm{ren}} \partial_t G_{k,yx}\,. \tag{2.55}$$

This is represented graphically as in Figure 2.7. The argument of the trace can be re-written using relation (2.53) as:

$$\frac{1}{4} \left[S^{(3)}_{0,axb} G_{k,bc} S^{(3)}_{0,cyd} G_{k,da} - S^{(4)}_{0,axyb} G_{k,ba}\right]_{\mathrm{ren}} \partial_t G_{k,yx} =$$

$$\partial_t \left[-\frac{1}{12} G_{k,ab} S^{(3)}_{0,bxc} G_{k,cd} S^{(3)}_{0,dya} G_{k,yx} + \frac{1}{8} G_{k,ab} S^{(4)}_{0,axyb} G_{k,yx}\right]\,. \tag{2.56}$$

Inserting (2.56) in (2.55) and integrating from 0 to Λ gives the two-loop contribution:

$$\Gamma_{2,0}[\varphi] = -\frac{1}{12}\left[G_{0,ab}S^{(3)}_{0,bxc}G_{0,cd}S^{(3)}_{0,dye}G_{0,ea}\right]_{\text{ren}} + \frac{1}{8}\left[G_{0,ab}S^{(4)}_{0,bxyc}G_{0,ca}\right]_{\text{ren}} . \tag{2.57}$$

Note that the two-loop contribution (2.57) is the expected perturbative one. The divergent part $[\Gamma_{2,0}[\varphi]]_{\text{div}}$ of (2.57) originates from the xy loop integral while the $abcde$ loop integral in the first term, or the abc loop integral in the second term, are finite due to the one-loop subtractions. Combining (2.57) with the two-loop counterterm $\delta S_{2,\Lambda}[\varphi]$ gives, in the limit $\Lambda \to \infty$, the renormalized two-loop contribution:

$$[\Gamma_{2,0}[\varphi]]_{\text{ren}} = \Gamma_{2,0}[\varphi] + \delta S_{2,\Lambda}[\varphi] . \tag{2.58}$$

Combining (2.31), (2.42) and (2.58) we obtain the renormalized effective action to two-loop order:

$$\begin{aligned} [\Gamma_0[\varphi]]_{\text{ren}} \;=\;& S_0[\varphi] + \frac{\hbar}{2}\left[\text{Tr}\log G_0\right]_{\text{ren}} + \\ & -\frac{\hbar^2}{12}\left[\left[G_{0,ab}S^{(3)}_{0,bxc}G_{0,cd}S^{(3)}_{0,dye}G_{0,ea}\right]_{\text{ren}}\right]_{\text{ren}} + \\ & +\frac{\hbar^2}{8}\left[\left[G_{0,ab}S^{(4)}_{0,bxyc}G_{0,ca}\right]_{\text{ren}}\right]_{\text{ren}} + O(\hbar^3) . \end{aligned} \tag{2.59}$$

Note that in (2.59) all quantities are the renormalized ones as it should in a perturbative expansion of the effective action.

In summary, the renormalized perturbative expansion emerges, loop by loop, as a particular way of solving the exact flow equation (2.30) satisfied by the EAA. This is a manifestation of the exactness of the flow equation for the EAA.

2.3 Applications of the EAA

In this section we discuss three different applications of the EAA formalism in order to understand how it works is specific examples. In this way we touch several important technical and conceptual points. In the first application, we show how some basic results of quantum electrodynamics (QED) are recovered, as the running of the electric charge and the low energy Euler-Heisenberg effective action. We also make the explicit calculation of the one-loop polarization function and see how renormalization is introduced. In the second application we study general non-linear sigma models (NLσM) and understand how the geometric flow that characterizes these models is obtained within the EAA approach. We study the UV behavior of these models, which are know to be asymptotically free in $d = 2$ and discuss the possibility that this extends to asymptotic safety in higher dimensions, in particular $d = 4$. In the last section we discuss matter fields on curved space-times. We construct the EAA for a minimally coupled scalar field and we use it to calculate the effective action finding the non-local form first discovered by Polyakov.

2.3.1 Quantum electrodynamics

Quantum electrodynamics (QED) is the most successful of all physical theories of natural phenomena. The agreement between theory and experiment has reached astonishing levels. Still, today it is regarded only as an effective theory. This is mostly because, when not embedded in a larger theory such as the Standard Model, the theory is "trivial". If we fix a non-zero value for the bare electric charge at the UV scale and send $\Lambda \to \infty$, we find a zero renormalized charge. It was also in the QED context that the first effective, low energy, action was calculate by Euler and Heisenberg [6]. In this section we will see how these properties of QED emerge within the EAA approach. We will see how the "flow quantization" actually works and we will provide a first example of how calculations on non-trivial backgrounds are done within the EAA framework, by calculating the photon polarization function.

QED is an abelian gauge theory with $SO(2) \simeq U(1)$ as gauge group. We consider the complex $U(1)$ representation where the group elements are simply given by $R = e^{i\theta} = 1 + i\theta + ...$ The abelian gauge connection A_μ transforms as:

$$A_\mu \to A_\mu + \partial_\mu \theta \,. \tag{2.60}$$

The covariant derivative is constructed as $D_\mu = \partial_\mu + ieA_\mu$, with e the bare electric charge. The commutator of covariant derivatives defines the field strength $F_{\mu\nu}$ as follows:

$$[D_\mu, D_\nu]\, f = ie(\partial_\mu A_\nu - \partial_\nu A_\mu) f = ieF_{\mu\nu}\, f \,. \tag{2.61}$$

Electrons are described by Dirac spinors $\bar{\psi}$ and ψ; in the $U(1)$ picture they transforms as:

$$\psi \to e^{i\theta}\psi \qquad\qquad \bar{\psi} \to e^{-i\theta}\bar{\psi} \,. \tag{2.62}$$

The classical (Euclidean) theory is defined as the minimal gauge invariant action that can be written down:

$$S[A,\bar{\psi},\psi] = \int d^dx \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\not{D} + m_e)\psi\right] . \tag{2.63}$$

In (2.63) m_e is the bare electron mass and the covariant derivative acting on spinors is defined by $\not{D} = \gamma^\mu D_\mu$.

In the quantum path integral formulation, we have to introduce the gauge-fixing action:

$$S_{gf}[A] = \frac{1}{2\alpha}\int d^dx \, (\partial_\mu A^\mu)^2 . \tag{2.64}$$

On flat d-dimensional space, the Faddeev-Popov ghosts decouple and can thus be discarded. The effective action can be defined using its integro-differential definition:

$$\begin{aligned} e^{-\Gamma[A,\bar{\psi},\psi]} &= \int DAD\bar{\psi}D\psi \exp\Big\{-S[A+a,\bar{\psi}+\bar{\xi},\psi+\xi] + S_{gf}[A+a] \\ &\quad + \int d^dx \left[\frac{\delta\Gamma[A,\bar{\psi},\psi]}{\delta A^\mu}J^\mu + \bar{\xi}\frac{\delta\Gamma[A,\bar{\psi},\psi]}{\delta\psi} - \frac{\delta\Gamma[A,\bar{\psi},\psi]}{\delta\bar{\psi}}\xi\right]\Big\} . \end{aligned} \tag{2.65}$$

The effective action defined (2.65) can be calculated to one-loop order using relation (B.60) from Appendix B for Gaussian functional integrals over mixed bosonic and fermionic fields. In this way, the one-loop effective action can be reduced to the following functional traces:

$$\begin{aligned} \Gamma_1[A,\bar{\psi},\psi] &= \frac{1}{2}\mathrm{Tr}\log\left[-\partial^2 g^{\mu\nu} + \left(1-\frac{1}{\alpha}\right)\partial^\mu\partial^\nu - \bar{\psi}\gamma^\mu\frac{1}{\not{D}+m_e}\gamma^\nu\psi\right] \\ &\quad -\mathrm{Tr}\log\left(\not{D}+m_e\right) . \end{aligned} \tag{2.66}$$

Using the following relation

$$\det\left(\not{D}+m_e\right) = \det\gamma_5\left(\not{D}+m_e\right)\gamma_5 = \det\left(-\not{D}+m_e\right) , \tag{2.67}$$

we can rewrite the second trace in (2.66) in the following way:

$$\mathrm{Tr}\log\left(\not{D}+m_e\right) = \frac{1}{2}\log\left[\det\left(\not{D}+m_e\right)\det\left(-\not{D}+m_e\right)\right] = \frac{1}{2}\mathrm{Tr}\log\left(-\not{D}^2+m_e^2\right) . \tag{2.68}$$

Relation (2.67) follows from the properties of the gamma matrix γ_5, which anti-commutes with all the γ^μ and has square equal to one $(\gamma^5)^2 = 1$.

The differential operator $-\not{D}^2$ appearing in (2.68) can be rewritten as:

$$\begin{aligned} -\not{D}^2 &= -\gamma^\mu\gamma^\nu D_\mu D_\nu \\ &= -\frac{1}{2}\left(\gamma^\mu\gamma^\nu D_\mu D_\nu + \gamma^\mu\gamma^\nu D_\nu D_\mu + ieF_{\mu\nu}\right) \\ &= -D^2 - \frac{ie}{4}\left[\gamma^\mu,\gamma^\nu\right] F_{\mu\nu} \\ &= -D^2 - e\,\sigma\cdot F\,. \end{aligned} \tag{2.69}$$

In (2.69) we used (2.61), the commutation relations for Dirac matrices $\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}$ and we defined the tensor $\sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu,\gamma^\nu\right]$.

Here we will construct the EAA for QED introducing the cutoff kernels directly in the one-loop equation for the effective action (2.66) to obtain:

$$\begin{aligned} \Gamma_{1,k}[A,\bar\psi,\psi] &= \frac{1}{2}\mathrm{Tr}\log\left[-\partial^2 g^{\mu\nu} + \left(1-\frac{1}{\alpha}\right)\partial^\mu\partial^\nu - \bar\psi\gamma^\mu\frac{1}{\not{D}+m_e}\gamma^\nu\psi + R_k(-\partial^2)^{\mu\nu}\right] \\ &\quad -\frac{1}{2}\mathrm{Tr}\log\left[-\not{D}^2 + m_e^2 + R_k(-\not{D}^2)\right]\,. \end{aligned} \tag{2.70}$$

Note that in (2.70) we choose as the cutoff operator for the photon field the flat-space Laplacian $-\partial^2$, while for the fermion fields we used the covariant operator $-\not{D}^2$.

In the quantum theory the photon and fermion fields are renormalized by their respective wave-functions renormalizations:

$$A_\mu \to Z_A^{1/2} A_\mu \qquad\qquad \psi \to Z_\psi^{1/2}\psi \qquad\qquad \bar\psi \to Z_\psi^{1/2}\bar\psi\,. \tag{2.71}$$

The running electric charge e_k is introduced using the non-renormalization of the covariant derivative: the partial derivative in $D_\mu = \partial_\mu + ie_k\, Z_{A,k}^{1/2} A_\mu$ does not renormalize, so we must have:

$$e_k = Z_{A,k}^{-1/2}\,. \tag{2.72}$$

This implies that to calculate the beta function of the running electric charge e_k it is enough to calculate $\partial_t Z_{A,k}$. From now on we impose (2.72), in this way the covariant derivative does not contain anymore the running electric charge.

We will start studying the photon part of the EAA, i. e. $\Gamma_{1,k}[A,0,0]$. Differentiating

(2.70) at $\bar{\psi} = \psi = 0$ with respect to the RG parameter t gives the following flow equation:

$$\begin{aligned} \partial_t \Gamma_{1,k}[A,0,0] &= \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(-\partial^2)^{\mu\nu}}{-\partial^2 g^{\mu\nu} + \left(1-\frac{1}{\alpha}\right)\partial^\mu\partial^\nu + R_k(-\partial^2)^{\mu\nu}} \\ &\quad -\frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(-\not{D}^2)}{-\not{D}^2 + m_e^2 + R_k(-\not{D}^2)} \, . \end{aligned} \tag{2.73}$$

The first trace in (2.73) does not depend on the photon field and thus will not generate any A_μ contribution to $\partial_t \Gamma_k[A,0,0]$. This reflects the fact that QED is an abelian gauge theory with no photon self-interactions. Thus to one-loop order, all the contributions to the running of the gauge part of the EAA stems from the fermionic trace in (2.73). This one can be evaluated using the non-local heat kernel expansion for the operator $-\not{D}^2$ described in Appendix A. But first we need to choose a truncation ansatz for the EAA to insert in the lhs of equation (2.73). We will expand $\Gamma_{1,k}[A,0,0]$ in powers of the field strength; up to second power we have:

$$\Gamma_{1,k}[A,0,0] = \int d^d x \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}F_{\mu\nu}\Pi_k\left(-D^2\right)F^{\mu\nu} + O\left(F^3\right)\right] . \tag{2.74}$$

Here $\Pi_k(x)$ is the running polarization function, it is defined as a function of the gauge covariant Laplacian. Inserting (2.74) in the lhs of the flow equation (2.73) gives:

$$\begin{aligned} \partial_t \Gamma_{1,k}[Z_{A,k}^{1/2}A,0,0] &= \partial_t Z_{A,k}\frac{1}{4}\int d^d x \, F_{\mu\nu}F^{\mu\nu} \\ &\quad + \frac{1}{4}\int d^d x \, F_{\mu\nu}\partial_t\left[Z_{A,k}\Pi_k\left(-D^2\right)\right]F^{\mu\nu} + O\left(F^4\right) . \end{aligned} \tag{2.75}$$

Note that due to the non-renormalization of the covariant derivative, we have in the second term of (2.75) just an overall factor of the wave-function renormalization of the photon field.

From now on we will work in the physical dimension $d = 4$. We now calculate the rhs of the flow equation (2.73) using the non-local heat kernel expansion, equation (A.8) from Appendix A with the identification $\Delta = -\not{D}^2$. The operator (2.69) is of the general Laplacian type (A.6) with $U = \sigma \cdot F$ and $\Omega_{\mu\nu} = ieF_{\mu\nu}$. Using (A.38) we find:

$$\mathrm{Tr}\, h_k(-\not{D}^2) = \frac{1}{(4\pi)^2}\int d^4 x \left\{\mathrm{tr}\,\mathbf{1} + F_{\mu\nu}\left[\int_0^\infty ds\, \tilde{h}_k(s)\, f_{F^2}\left(-sD^2\right)\right]F^{\mu\nu} + O\left(F^3\right)\right\} . \tag{2.76}$$

In deriving (2.76) we used the following traces:

$$\mathrm{tr}\,\sigma\cdot F=0 \qquad\qquad \mathrm{tr}\,(\sigma\cdot F)^2=2F_{\mu\nu}F^{\mu\nu} \qquad\qquad \mathrm{tr}\,\Omega_{\mu\nu}\Omega^{\mu\nu}=-F_{\mu\nu}F^{\mu\nu}$$

and we defined the function

$$h_k(z)=\frac{\partial_t R_k(z)}{z+m_e^2+R_k(z)}\,, \tag{2.77}$$

of which $\tilde{h}_k(s)$ is the inverse Laplace transform. Also, $f_{F^2}(x)$ is the following non-local heat kernel structure function:

$$f_{F^2}(x)=-4f_\Omega(x)+2f_U(x)=f(x)+\frac{2}{x}\left[f(x)-1\right]=4\int_0^1 d\xi\,\xi(1-\xi)\,e^{-\xi(1-\xi)x}\,. \tag{2.78}$$

In (2.78) $f(x)$ is the basic heat kernel structure function (A.10). If we Taylor expand (2.78) to lowest order we find:

$$f_{F^2}(x)=\frac{2}{3}-\frac{2}{15}x+O\left(x^2\right)\,. \tag{2.79}$$

Inserting the expansion (2.79) in equation (2.76) and equating the first term with the first term on the rhs of (2.75) gives the following beta function for the wave-function renormalization of the photon field:

$$\partial_t Z_{A,k}=-\frac{1}{12\pi^2}Q_0[h_k] \tag{2.80}$$

If we set the electron mass to zero $m_e=0$, the Q-functional in (2.80) becomes independent of the cutoff shape function. From relation (A.40) we know that $Q_0[h_k]=h_k(0)$. From the definition (2.77) and from the general properties of the cutoff shape functions we have $h_k(0)=2$ independently of the choice we make for $R_k(z)$. Setting $Q_0[h_k]=2$ in (2.80) we find the following flow equation for the running electric charge:

$$\partial_t e_k=\frac{e_k^3}{12\pi^2}\,. \tag{2.81}$$

This is the standard beta function found in perturbation theory when a mass independent regularization scheme, as dimensional regularization MS (or $\overline{\text{MS}}$) scheme [9, 6], is used. For $m_e\neq 0$ and using the optimized cutoff shape function (2.12), equation (2.80) becomes instead:

$$\partial_t Z_{A,k}=-\frac{1}{6\pi^2}\frac{1}{1+\frac{m_e^2}{k^2}}\,; \tag{2.82}$$

while (2.81) is modified to the form:

$$\partial_t e_k = \frac{1}{12\pi^2}\frac{e_k^3}{1+\frac{m_e^2}{k^2}}\,. \tag{2.83}$$

From (2.82) we can read-off the scale dependent anomalous dimension of the photon field $\eta_{A,k} = -\partial_t Z_{A,k}/Z_{A,k}$:

$$\eta_{A,k} = \frac{1}{6\pi^2}\frac{e_k^2}{1+\frac{m_e^2}{k^2}}\,. \tag{2.84}$$

Note that in terms of (2.84) the beta function for the electric charge can also be written as $\partial_t e_k^2 = \eta_{A,k} e_k^2$. Within the EAA approach, the anomalous dimension for the photon field has been calculated beyond one-loop order in [28].

The presence of the denominator term m_e^2/k^2 in equation (2.83) is of key importance to describe the decoupling of the electron at scales much smaller the electron mass $k \ll m_e$. At these scales the rhs of (2.83) becomes smaller and smaller, until the flow of the electric charge effectively stops.

It is interesting to compare the beta function (2.83) with one calculated using standard perturbation theory but employing a mass dependent regularization scheme, as the dimensional regularization μ-scheme. After the identification $k \equiv \mu$ we find, from equation (3.25) of [6], the beta function:

$$\partial_t e_k = \frac{e_k^3}{2\pi^2}\int_0^1 dx\, \frac{x^2(1-x)^2}{x(1-x)+\frac{m_e^2}{k^2}}\,. \tag{2.85}$$

We here see in action the same decoupling mechanism. A comparison of the three beta functions (2.81), (2.83) and (2.85), shows that only mass dependent regularization schemes are capable of describing threshold phenomena while mass independent schemes generate the same running at all scales: heavy particles do not decouple. The mass dependence of the beta functions (2.83) and (2.85) is shown in Figure 2.8.

We now integrate beta function (2.83) for the electric charge from the UV scale Λ to the IR scale k. We find the following formula relating the bare electric charge e_Λ to the running electric charge e_k:

$$\frac{1}{e_k^2} - \frac{1}{e_\Lambda^2} = \frac{1}{12\pi^2}\log\frac{1+\frac{\Lambda^2}{m_e^2}}{1+\frac{k^2}{m_e^2}}\,. \tag{2.86}$$

Solving (2.86) for e_k gives:

$$e_k^2 = \frac{e_\Lambda^2}{1+\frac{e_\Lambda^2}{12\pi^2}\log\frac{1+\Lambda^2/m_e^2}{1+k^2/m_e^2}}\,. \tag{2.87}$$

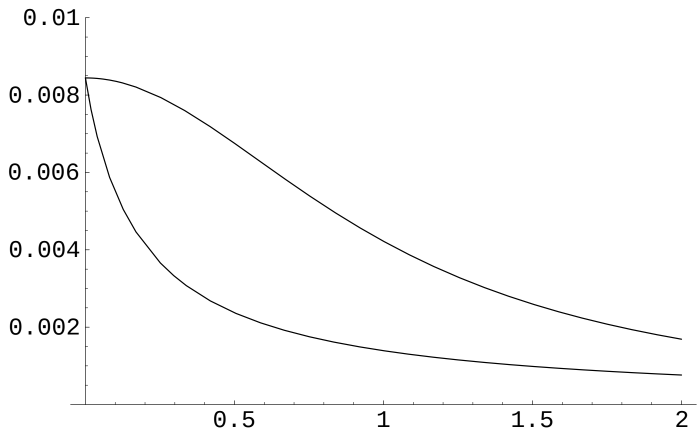


Figure 2.8: Mass dependent beta functions for the electric charge in QED. Comparison between dimensional regularization μ-scheme with $k \equiv \mu$ (lower curve) from equation (2.85) and EAA scheme (upper curve) from equation (2.83). $\partial_t e_k / e_k^3$ is plotted as a function of m_e/k.

Relation (2.87) is interesting for several reasons. First, it shows the screening effect of vacuum: electron-positron pairs polarize the vacuum around an electric charge so that the effective electric charge e_k, at the scale k, is smaller than the electric charge e_Λ at the higher scale Λ. This is shown in Figure 2.9. Second, for $k \to 0$ it relates the bare electric charge e_Λ to the renormalized electric charge e_0:

$$e_0^2 = \frac{e_\Lambda^2}{1 + \frac{e_\Lambda^2}{12\pi^2} \log\left(1 + \frac{\Lambda^2}{m_e^2}\right)} . \tag{2.88}$$

Third, it shows that QED, as defined by the bare action (2.63), is a trivial quantum field theory: if we take the limit $\Lambda \to \infty$ in equation (2.88), at fixed finite e_Λ, we get a zero renormalized electric charge e_0! The other way around: if we solve (2.88) for the bare charge e_Λ^2 and we set the renormalized charge e_0^2 to some fixed value, then the bare coupling will diverge at the finite “Landau pole” scale

$$\Lambda_L = m_e^2 \left(e^{12\pi^2/e_0^2} - 1\right) .$$

These are the two faces of QED’s triviality. So, even if the theory is perturbatively renormalizable, it cannot be a fundamental theory valid at arbitrary high energy scales. This example, among others, shows that the perturbative renormalization principle, which was used to successfully construct the Standard Model of particle physics, is just a practically

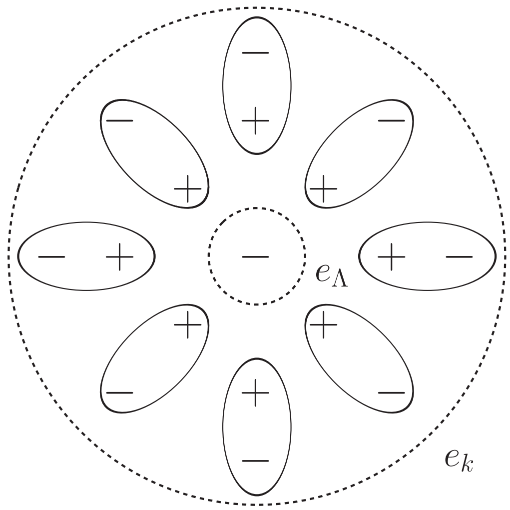


Figure 2.9: Polarization of the vacuum by electron-positron pairs. The effective electric charge e_k at the energy scale k appears smaller than the electric charge e_Λ at the bigger energy scale Λ.

convenient assumption void of any true physical meaning. To find an explanation for the success of QED, we have to take the effective field theory point of view, see [10] for a clear analysis.

We turn back to relation (2.76). By inserting the full heat kernel structure function (2.78), we can read of the flow equation for the polarization function by comparing with equation (2.75):

$$\begin{aligned} \partial_t Z_{A,k} + \partial_t\left[Z_{A,k}\Pi_k(x)\right] &= -\frac{1}{8\pi^2}\int_0^\infty ds\,\tilde{h}_k(s)\,f_{F^2}(sx) \\ &= -\frac{1}{2\pi^2}\int_0^1 d\xi\,\xi(1-\xi)\,Q_0[h_k(z+x\xi(1-\xi))] \\ &= -\frac{1}{2\pi^2}\int_0^1 d\xi\,\xi(1-\xi)\,h_k(x\xi(1-\xi))\,. \end{aligned} \tag{2.89}$$

In (2.89) the variable x stands for the gauge Laplacian and we used the properties of the Q-functional, as explained in Appendix B, in particular the relation $Q_0[f(z+a)] = f(a)$. Using $\partial_t\left[Z_{A,k}\Pi_k(x)\right] = Z_{A,k}\left[-\eta_{A,k}\Pi_k(x) + \partial_t\Pi_k(x)\right]$, relation (2.72) and (2.84) we can rewrite the flow equation (2.89) in the following way:

$$\partial_t\Pi_k(x) = \frac{1}{6\pi^2}\frac{e_k^2}{1+\frac{m_e^2}{k^2}}\left[1+\Pi_k(x)\right] - \frac{e_k^2}{2\pi^2}\int_0^1 d\xi\,\xi(1-\xi)\,h_k(x\xi(1-\xi))\,. \tag{2.90}$$

Relation (2.90) expresses the flow equation for the one-loop photon polarization function.

We can find the renormalized polarization function $\Pi_0(x)$ integrating the flow (2.90) from the UV scale Λ, where we impose the boundary condition $\Pi_\Lambda(x) = 0$, to $k = 0$. Note first that the second term in (2.90) is at least of order e_k^4. We will discard it here since we are interested to show how the standard one-loop result is derived in this framework. We also set the running charge in (2.90) to its renormalized value e_0. Integrating from k to Λ, using the property $h_k(z) = \partial_t \log\left[z + m_e^2 + R_k(z)\right]$ and equation (2.86), we find:

$$\begin{aligned} \Pi_\Lambda(x) - \Pi_k(x) &= \frac{e_0^2}{12\pi^2} \log \frac{1 + \frac{\Lambda^2}{m_e^2}}{1 + \frac{k^2}{m_e^2}} \\ &- \frac{e_0^2}{2\pi^2} \int_0^1 d\xi \, \xi(1-\xi) \, \log \frac{x\xi(1-\xi) + m_e^2 + R_\Lambda\left(x\xi(1-\xi)\right)}{x\xi(1-\xi) + m_e^2 + R_k\left(x\xi(1-\xi)\right)} \,. \end{aligned} \tag{2.91}$$

Consider now the optimized cutoff shape function $R_k^{opt}(z) = (k^2 - z)\theta(k^2 - z)$ from (2.12). For $\Lambda \to \infty$ we have $R_\Lambda(z) \sim \Lambda^2 - z$, while for $k \to 0$ we have $R_k(z) \sim 0$. For $k = 0$ equation (2.91) takes the following form:

$$\begin{aligned} \Pi_\Lambda(x) - \Pi_0(x) &= \frac{e_0^2}{12\pi^2} \log\left(1 + \frac{\Lambda^2}{m_e^2}\right) - \frac{e_0^2}{12\pi^2} \log\left(1 + \frac{\Lambda^2}{m_e^2}\right) + \\ &+ \frac{e_0^2}{2\pi^2} \int_0^1 d\xi \, \xi(1-\xi) \, \log\left(1 + \xi(1-\xi)\frac{x}{m_e^2}\right) . \end{aligned} \tag{2.92}$$

Note now that the renormalization of the electric charge in (2.92) is enough to allow us to take the limit $\Lambda \to \infty$ in (2.92). At $k = 0$ we find the following renormalized vacuum polarization function:

$$\Pi_0(x) = -\frac{e_0^2}{8\pi^2} \int_0^1 d\xi \, \xi(1-\xi) \log\left[1 + \xi(1-\xi)\frac{x}{m_e^2}\right] . \tag{2.93}$$

Inserting (2.93) in equation (2.74) finally gives the one-loop photon part of the effective action:

$$\begin{aligned} \Gamma_{1,0}[A,0,0] &= \frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu} + \\ &- \frac{e_0^2}{8\pi^2} \int d^4x \, F_{\mu\nu} \left(\int_0^1 d\xi \, \xi(1-\xi) \log\left[1 + \xi(1-\xi)\frac{-D^2}{m_e^2}\right] \right) F^{\mu\nu} \\ &+ O\left(F^4\right) . \end{aligned} \tag{2.94}$$

Equation (2.94) is the expression for the photon part of the EAA in QED to second order in the field strength and in e_0^2. Although the polarization function in (2.94) is a function of the gauge Laplacian, in an abelian theory like QED it boils to a function of just the flat Laplacian $-\partial^2$ and thus does not give non-zero contribution to higher vertices of the effective action. If instead we were considering a non-abelian gauge theory, the contribution to the effective action would had been of the same form as in (2.94) and thus to all orders in gauge field.

We look now at the next terms in (2.90). They are the following F^4 invariants (parity invariance forbids F^3 terms in the EAA for QED):

$$a_k \int d^dx\ (F_{\mu\nu}F^{\mu\nu})^2 + b_k \int d^dx\, F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}\,. \tag{2.95}$$

From the fermion trace in the flow equation (2.73), following the standard heat kernel procedure, we find the renormalized value:

$$a_0 = -\frac{1}{18}\frac{e^4}{8\pi^2}\int_0^\infty \frac{dk}{k}\frac{k^2}{(k^2+m_e^2)^3} = -\frac{1}{18}\frac{1}{32\pi^2}\frac{e^4}{m_e^4}\,,$$

where we imposed the initial condition $a_\Lambda = 0$. Doing the same for the other term, we recover the Euler-Heisenberg contribution to the photon part of the effective action [6]:

$$\Gamma_{1,0}[A,0,0]|_{F^4} = \frac{1}{(4\pi)^2}\frac{e^4}{m_e^4}\int d^4x \left[-\frac{1}{36}\,(F_{\mu\nu}F^{\mu\nu})^2 + \frac{7}{90}F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}\right]. \tag{2.96}$$

The terms in (2.96) describes the low energy effective scattering of photons mediated by virtual electrons. Note that from the effective theory point of view we have been calculating the effective action to order p^4/m_e^4 with p the photon momentum. The power of effective field theory is that, even without deriving equation (2.96) from the bare action (2.63) by integrating the flow, we could had constructed it only on dimensional grounds. To be able to make predictions about light by light scattering to order p^4/m_e^4 we just need to fit from experiments the two numbers $-\frac{1}{36}$ and $\frac{7}{90}$.

Along the same lines the one-loop fermion-photon part of the effective action can be constructed. Obviously also the full non-perturbative effective action for QED can be studied within the EAA framework. We will not do this here since our aim was only to show how to use the EAA framework in a well known setting as is QED.

2.3.2 Non-linear sigma models (NLσM)

Non-linear sigma models (NLσM) are a rich class of theories [29], describing the dynamics of a map φ from a d-dimensional manifold $\mathcal{M}$ to a D-dimensional manifold $\mathcal{N}$. They have been applied to phenomenological models of high energy physics, such as chiral perturbation theory in low energy QCD [6]; to phenomenological models of condensed matter physics, as magnetic systems [30]; to the statistical mechanics of surfaces, both in realistic systems like membranes [31] and more abstractly to string theories [32]. NLσM have interesting supersymmetric generalizations and also applications in mathematics. Recently even the RG flow of NLσMs has found a mathematical application in the proof of the long standing Poincaré conjecture [33].

Given a coordinate system $\{x^\mu\}$ on $\mathcal{M}$ and $\{y^A\}$ on $\mathcal{N}$, one can describe the map φ by D scalar fields $\varphi^A(x)$. Physics must be independent of the choice of coordinates on $\mathcal{N}$, forcing the action to be a functional constructed with tensorial structures on $\mathcal{N}$. Only derivative interactions are allowed. Linear scalar theories correspond to the case when $\mathcal{N}$ is linear. In this case, and only in this case, one can choose the action to describe free fields. General NLσM are profoundly different from linear scalar theories.

We will study general NLσM via the EAA approach following [34]. The bare action for the general NLσM is usually taken to be:

$$S[\phi] = \frac{1}{2}\int d^d x\, h_{AB}(\phi)\partial_\mu\phi^A\partial^\mu\phi^B\,. \tag{2.97}$$

Here $h_{\alpha\beta}$ is a dimensionless metric on $\mathcal{N}$ and the wave-function renormalization ζ has dimensions k^{d-2} and is related to the coupling constant by $\zeta = 1/g^2$. We will consider here only the case where the manifold $\mathcal{M}$ is flat.

Applying the formalism of quantum field theory to these models requires some adaptation. The simplest treatment is based on the assumption that the ground state of the theory is a constant map $\bar{\varphi}$. There exists a local diffeomorphism $\exp_{\bar{\varphi}}$ of the tangent space $T_{\bar{\varphi}}N$ to a neighborhood U of $\bar{\varphi}$, given by mapping a vector ξ to the point lying a distance $||\xi||$ along the geodesic emanating from $\bar{\varphi}$ in the direction of ξ. The components ξ^A can be used as coordinates on U, called normal coordinates. Fluctuations around the vacuum are described by the fields $\xi^A(x)$, which can be quantized by path integral methods. When the action is thus expanded around $\bar{\varphi}$ and the fields are canonically normalized, one recognizes that g plays the role of coupling constant, and since it has dimension of $k^{\frac{2-d}{2}}$, this perturbative expansion is non-renormalizable for $d > 2$. As a consequence, phenomenological applications

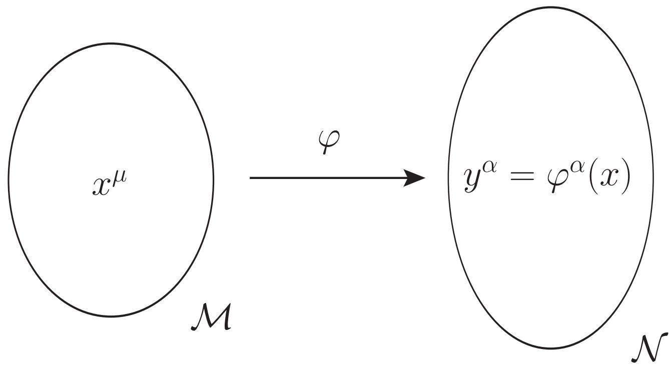


Figure 2.10: φ is a map from a d-dimensional manifold $\mathcal{M}$ to a D-dimensional manifold $\mathcal{N}$.

of the NLσM in $d = 4$ are usually regarded as effective field theories with a cutoff.

To construct the EAA we use the background field method[6] where we expand the full quantum field around a background configuration $\bar{\varphi}$ as $\varphi^A(x) = \bar{\varphi}^A(x) + \eta^A(x)$. For reasons that will become clear soon, it will not be sufficient to consider constant backgrounds, so the simple procedure described above will have to be generalized. Furthermore, the field η is a difference of coordinates and does not have good transformation properties. At each point $x \in \mathcal{M}$ one evaluates the EAA using the normal coordinates centered at $\bar{\varphi}(x)$. They are the components of a section ξ of $\bar{\varphi}^* T\mathcal{N}$, such that $\exp_{\bar{\varphi}(x)}(\xi(x)) = \varphi(x)$. Using normal coordinate at $\bar{\varphi}$ we can easily derive the following expansion for the field derivative[7]:

$$\partial_\mu \varphi^A = \partial_\mu \bar{\varphi}^A + \bar{\nabla}_\mu \xi^A - \frac{1}{3} \partial_\mu \bar{\varphi}^B \bar{R}^A{}_{CBD} \xi^C \xi^D + O(\xi^3)\,, \tag{2.98}$$

and for the metric tensor:

$$h_{AB}(\varphi) = h_{AB}(\bar{\varphi}) - \frac{1}{3} \bar{R}_{ACBD} \xi^C \xi^D + O(\xi^3)\,. \tag{2.99}$$

In (2.98) and (2.99) we constructed the Riemann tensor, at the background point $\bar{\varphi}$, using the Christoffel symbols Γ^A_{BC} of the metric h_{AB}. The covariant derivative in (2.98) is as following:

$$\nabla_\mu \xi^A = \partial_\mu \xi^A + \omega^A_{\mu C} \xi^C \qquad\qquad \omega^A_{\mu B} = \partial_\mu \varphi^C \Gamma^A_{CB}\,. \tag{2.100}$$

[6]The EAA for theories with local symmetries is constructed using the background field method starting from the next chapter. In this section we skip a detailed development of the background EAA since we are interested in the NLσM as an application of the general EAA formalism. The reader is referred to Chapter 3 for more details on the background field formalisms within the EAA approach.

[7]A bar over the covariant derivatives or over the curvature tensor means that we are evaluating the quantity at the point $\bar{\varphi}$.

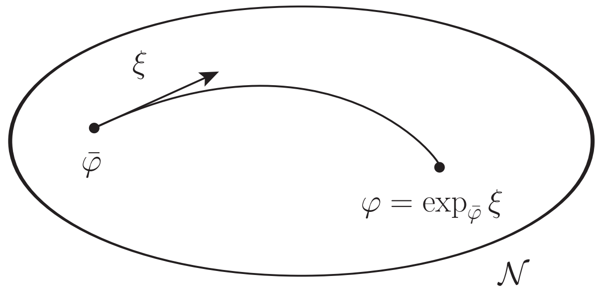


Figure 2.11: The exponential mapping used to define the background field expansion.

The field η can be written as a function of ξ, which is taken as the fluctuation field in the background field method.

The EAA for the general NLσM can be expanded in powers of the derivatives, the lowest term being:

$$\Gamma_k[\varphi] = \frac{1}{2}\zeta_k \int d^dx\, h_{AB}(\varphi)\partial_\mu\varphi^A\partial^\mu\varphi^B + O\left(\partial^4\right) , \tag{2.101}$$

where ζ_k is the running wave-function renormalization. One can now expand the EAA (2.101) in powers of ξ as follows:

$$\begin{aligned} \Gamma_k[\varphi] &= \Gamma_k[\bar{\varphi}] + \int d^dx\, \Gamma^{(1,0)}_{k,x}[0;\bar{\varphi}]_A \xi^A_x(x) \\ &+\frac{1}{2}\int d^dx d^dy\, \Gamma^{(2;0)}_{k,xy}[0;\bar{\varphi}]_{AB}\xi^A_x\xi^B_y + O(\xi^3) , \end{aligned} \tag{2.102}$$

and write the result in a tensorial form, in such a way that invariance under background coordinate transformations is manifest.

The cutoff action is constructed using the background field $\bar{\varphi}$ to define the cutoff kernel:

$$\Delta_k S[\bar{\varphi}, \xi] = \frac{1}{2}\zeta_k \int d^dx\, \xi^A R_{k\,AB}[\bar{\varphi}]\xi^B . \tag{2.103}$$

Note that in (2.103) we inserted a factor of the running wave-function renormalization ζ_k. The flow equation for the EAA takes the following form[8]:

$$\partial_t\Gamma_k[\bar{\varphi}] = \frac{1}{2}\mathrm{Tr}\left(\Gamma^{(2,0)}_k[0;\bar{\varphi}] + R_k[\bar{\varphi}]\right)^{-1} \partial_t R_k[\bar{\varphi}] . \tag{2.104}$$

Using relations (2.98) and (2.99) one can extract from the expansion (2.102) the following

[8]See Chapter 3 for more details on this point

Hessian with respect to the fluctuation field ξ:

$$\Gamma_k^{(2,0)}[0;\bar{\varphi}]_{AB} = \zeta_k \left(-\bar{\nabla}^2\delta_{AB} + \bar{R}_{ACBD}\partial_\mu\bar{\varphi}^C\partial^\mu\bar{\varphi}^D\right) , \tag{2.105}$$

where $-\bar{\nabla}^2 = -\bar{\nabla}_\mu\bar{\nabla}^\mu$ is the Laplacian of the covariant derivative (2.100) at the point $\bar{\varphi}$. We have to choose the cutoff operator we use to separate the fast modes from the slow ones, here we take following operator[9]:

$$\bar{\Delta} = -\bar{\nabla}^2\delta_{AB} + \bar{R}_{ACBD}\partial_\mu\bar{\varphi}^C\partial^\mu\bar{\varphi}^D , \tag{2.106}$$

which is of the general Laplacian type we discuss the heat kernel expansion in Appendix A. With this choice the flow equation (2.104) becomes simply:

$$\partial_t\Gamma_k[\xi;\bar{\varphi}] = \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\bar{\Delta}) - \eta_k R_k(\bar{\Delta})}{\bar{\Delta} + R_k(\bar{\Delta})} , \tag{2.107}$$

where we introduced the field anomalous dimension $\eta_k = -\partial_t \log\zeta_k$.

We can now use the local heat kernel expansion to evaluate the functional trace in (2.107). Using equation (A.38) from Appendix A, we find the following term of order ∂^2:

$$\partial_t\Gamma_k[\bar{\varphi}]|_{\partial^2} = -\frac{1}{2(4\pi)^{d/2}}Q_{\frac{d}{2}-1}[(\partial_t R_k - \eta_k R_k)G_k]\int d^dx\, \bar{R}_{AB}\partial_\mu\bar{\varphi}^A\partial^\mu\bar{\varphi}^B . \tag{2.108}$$

In (2.108) we defined the regularized propagator as:

$$G_k(z) = \frac{1}{z + R_k(z)} . \tag{2.109}$$

Comparing the t-derivative of (2.101) with (2.108) gives the following flow equation[10]:

$$\partial_t\left[\zeta_k h_{AB}\right] = 2c_d k^{d-2}\left(1 + \frac{\eta_k}{d+2}\right) R_{AB} . \tag{2.110}$$

In (2.108) we employed the optimized cutoff shape function in the Q-functional and we defined $c_d = \frac{1}{(4\pi)^{d/2}\Gamma(d/2+1)}$. If we set $\eta_k = 0$ in equation (2.110), we find the famous "Geometric flow" that characterizes the RG flow of general NLσM: the beta functional of the full metric $G_{AB} = \zeta_k h_{AB}$ is given by the Ricci tensor R_{AB} of G_{AB} on $\mathcal{N}$. It is possible to check that for $d = 2$ the numerical coefficient c_2 is cutoff shape independent. For $d > 2$ the coefficient

[9] This corresponds to a type II cutoff in the nomenclature of the next chapters.

[10] We omit to the bar over background quantities in some equations for clarity.

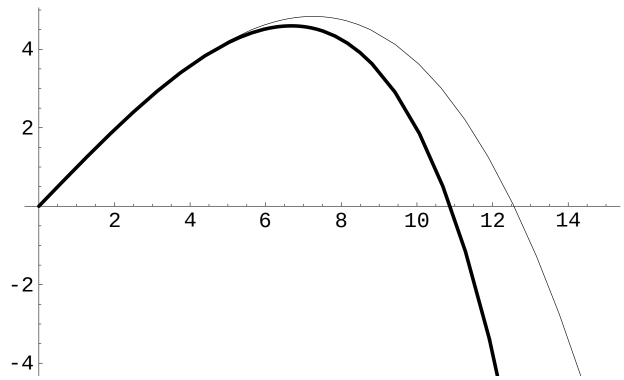


Figure 2.12: Beta function (2.112) for the dimensionless coupling constant $\tilde{g}_k$ (dashed) for the $\mathcal{N} = S^3$ NLσM in $d = 4$ together with its one-loop approximation (continuous).

c_d depends on the choice of cutoff shape function, but one can show that the qualitative properties of the beta function are the same for any cutoff.

Let us now suppose that the metric h_{AB} has some Killing vectors, generating a Lie group G. Since the cutoff is defined by means of the G-invariant Laplacian $-\nabla^2$, it preserves the G invariance. Therefore if the initial point of the flow is an invariant metric, the flow takes place within the restricted class of invariant metrics. From now on we shall restrict ourselves to homogeneous spaces $\mathcal{N} = G/H$ admitting a single invariant Einstein metric h_{AB}, up to scalings. In this case in equation (2.110) it is convenient to think of h_{AB} as being fixed and we interpret the RG flow as affecting only ζ_k. The Ricci tensor of h_{AB} is $R_{AB} = \frac{R}{D} h_{AB}$, where R is the Ricci scalar, therefore:

$$\partial_t \zeta_k = 2 c_d k^{d-2} \left(1 + \frac{\eta_k}{d+2}\right) \frac{R}{D} \,. \tag{2.111}$$

When (2.111) is solved for $\partial_t \zeta_k$ one obtains a rational beta function which, in terms of the dimensionless coupling $\tilde{g}_k = k^{\frac{d-2}{2}} g_k$, reads:

$$\partial_t \tilde{g}_k = \frac{d-2}{2} \tilde{g}_k - \frac{c_d \frac{R}{D} \tilde{g}_k^3}{1 - 2 c_d \frac{R}{D(d+2)} \tilde{g}_k^2} \,. \tag{2.112}$$

This beta function is the main result of this section and is shown in Figure 2.12.

We first look at the one-loop flow, where we consider the lhs of (2.112) to order $\tilde{g}_k^2$. If $d > 2$ and $R > 0$ there is a non-Gaussian fixed point at $\tilde{g}_*^2 = \frac{d-2}{2} \frac{D}{c_d R}$. For large R it occurs at small coupling, where perturbation theory is reliable. The derivative of the beta function at the fixed point is

$$\frac{\partial}{\partial \tilde{g}_k} \partial_t \tilde{g}_k \bigg|_{\tilde{g}_*} = 2 - d < 0 \qquad \Rightarrow \qquad \nu = \frac{1}{d-2}\,, \tag{2.113}$$

so this fixed point is UV attractive and the mass critical exponent [75] is mean-field like as should be in a one-loop approximation. This shows that a NLσM with positive Ricci curvature is an asymptotically safe theory. In particular for $N = S^D$ and $R = D(D-1)$ we reproduce the results of the $2+\epsilon$ expansion for the $SO(D+1)$ model [32].

When we consider the full beta function (2.112), the fixed point is shifted at the value $\tilde{g}_*^2 = \frac{1}{2}\frac{D(d^2-4)}{c_d d R}$ and is still UV attractive in $d > 2$. The mass critical exponent is now smaller then the mean-field value:

$$\nu = \frac{d+2}{2d(d-2)} < \frac{1}{d-2}\,, \tag{2.114}$$

the anomalous dimension as the value $\eta = d - 2$. In particular, in $d = 4$ we have a non-Gaussian fixed point with critical exponents $\nu = \frac{3}{8}$ and $\eta = 2$. Numerically, the results do not differ very much from the one-loop ones, but since their derivation is not based on perturbation theory, their validity does not depend on the coupling being small. This indicates that general NLσM may be asymptotically safe even in $d = 4$. Since the truncation of the EAA (2.101) we are considering is very crude, this result has to be considered more as a hint for future research then a definitive answer. But if this scenario turns out to be correct, then general NLσM are inherently different from the linear theory. We should mention here that according to lattice calculations the triviality of ϕ^4 theory in $d = 4$ is expected to extend also to the corresponding NLσSM [37, 38]. It will be interesting to understand how these results fit with this expectation. In this connection we observe that a non-Gaussian fixed point in the NLσM is not ruled out by a recent investigation of the triviality issue using functional RG methods [39]. It may also be useful to repeat and improve the numerical simulations of [40]. For further studies of general NLσM within the EAA approach see [41].

Every manifold can be isometrically embedded in a linear space of sufficiently high dimension and it is sometimes convenient to regard the NLσM as a constrained linear theory. For example, in the $SO(D+1)$ model, one can start from a linear theory with action

$$\int d^d x \, \left[\frac{1}{2} Z_k \sum_{a=1}^{D+1} \partial_\mu \phi^a \partial^\mu \phi^a + \frac{1}{2} \lambda_k (\rho - \bar{\rho}_k)^2 \right] \,, \tag{2.115}$$

where $\rho = \frac{1}{2}\sum_{a=1}^{D+1} \phi^a \phi^a$ and Z_k, λ_k, $\bar{\rho}_k$ are running couplings. The action (2.97) can be

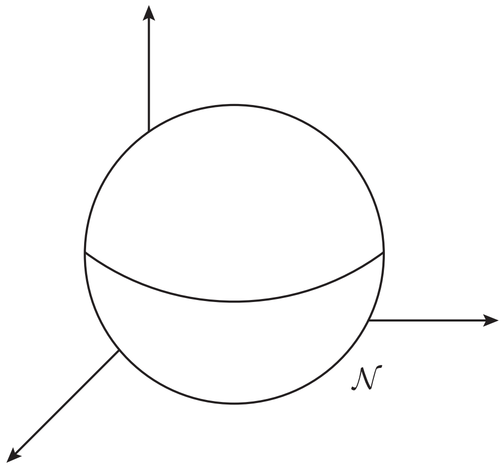


Figure 2.13: The manifold $\mathcal{N}$ can be be isometrically embedded in a linear space of sufficiently high dimension.

obtained in the limit $\lambda_k \to \infty$, with the identification $\zeta_k = 2Z_k\bar{\rho}_k$. It is therefore of some interest to derive the beta function of the NLσM from the one of the linear theory. The beta functions of Z_k, λ_k and $\bar{\rho}_k$ are given e.g. in [15], where the notation $\kappa_k \equiv Z_k\bar{\rho}_k k^{d-2} = \frac{1}{2}\zeta_k k^{d-2}$ is used. Evaluating these beta functions with the optimized cutoff and taking the limit $\lambda_k \to \infty$, the anomalous dimension $\eta_{Z,k} \equiv \partial_t Z_k/Z_k \to c_d/\kappa_k$, whereas $\partial_t\kappa_k \to (2-d-\eta_{Z,k})\kappa_k + Dc_d = (2-d)\kappa_k + (D-1)c_d$, in complete accordance with (2.111). Since the beta function (2.111) implies a (power law) divergence for $k \to \infty$, this means that the divergence is the same in the NLσM and in the $\lambda_k \to \infty$ limit of the linear theory, in agreement with [15].

As a further check of the formalism, we can compute also the effect of $\tilde{g}_k$ on the running of the four derivative terms. There are two such contributions in the heat kernel expansion of the lhs of the flow equation (2.107) coming from the heat kernel coefficient $b_4(\Delta)$, given in equation A.7 of Appendix A, for the cutoff operator (2.106). One is the term $\frac{1}{2}U_{AB}U^{BA}$, where $U_{AB} = \bar{R}_{ACBD}\partial_\mu\bar{\varphi}^C\partial^\mu\bar{\varphi}^D$, while the other is $\frac{1}{12}\Omega^{AB}_{\mu\nu}\Omega^{\mu\nu}_{BA}$, where $\Omega^{AB}_{\mu\nu}$ is the commutator of the covariant derivative (2.100) and is explicitly given by

$$\Omega_{\mu\nu}{}^A{}_B = \partial_\mu\omega_\nu{}^A{}_B - \partial_\nu\omega_\mu{}^A{}_B + \omega_\mu{}^A{}_C\omega_\nu{}^C{}_B - \omega_\nu{}^A{}_C\omega_\mu{}^C{}_B\,. \tag{2.116}$$

The first contribution is easily evaluated to yield:

$$U_{AB}U^{BA} = \bar{R}_{EAFB}\bar{R}^{F\ E}_{\ C\ D}\partial_\mu\bar{\varphi}^A\partial^\mu\bar{\varphi}^B\partial_\nu\bar{\varphi}^C\partial^\nu\bar{\varphi}^D\,, \tag{2.117}$$

while using

$$\partial_\mu \omega_{\nu B}^A = \partial_\mu \left(\partial_\nu \bar{\varphi}^B \Gamma_{BC}^A \right) = \partial_\mu \partial_\nu \bar{\varphi}^B \Gamma_{BC}^A + \partial_\nu \bar{\varphi}^B \partial_\mu \Gamma_{BC}^A \,,$$

(the first term is symmetric in $\mu\nu$ and will cancel out) and $\partial_\nu \bar{\varphi}^B \partial_\mu \Gamma_{BC}^A = \partial_\nu \bar{\varphi}^B \partial_\mu \bar{\varphi}^D \partial_D \Gamma_{BC}^A$, we find after some manipulations:

$$\begin{aligned}
\Omega_{\mu\nu}{}^A{}_B &= \partial_\nu \bar{\varphi}^C \partial_\mu \bar{\varphi}^D \partial_D \Gamma_{CB}^A - \partial_\mu \bar{\varphi}^C \partial_\nu \bar{\varphi}^D \partial_D \Gamma_{CB}^A + \\
&\quad + \partial_\mu \bar{\varphi}^D \Gamma_{DC}^A \partial_\nu \bar{\varphi}^E \Gamma_{EB}^C - \partial_\nu \bar{\varphi}^D \Gamma_{DC}^A \partial_\mu \bar{\varphi}^E \Gamma_{EB}^C \\
&= \partial_\nu \bar{\varphi}^E \partial_\mu \bar{\varphi}^D \left[\partial_D \Gamma_{EB}^A - \partial_E \Gamma_{DB}^A + \Gamma_{DC}^A \Gamma_{EB}^C - \Gamma_{EC}^A \Gamma_{DB}^C \right] \\
&= \partial_\mu \bar{\varphi}^D \partial_\nu \bar{\varphi}^E \bar{R}^A{}_{BDE} \,.
\end{aligned}$$

Thus we finally have:

$$\frac{1}{12} \Omega_{\mu\nu}^{AB} \Omega_{BA}^{\mu\nu} = \frac{1}{12} \bar{R}_{AB}{}^{EF} \bar{R}_{CDEF} \partial_\mu \bar{\varphi}^A \partial_\nu \bar{\varphi}^B \partial^\mu \bar{\varphi}^C \partial^\nu \bar{\varphi}^D \,. \tag{2.118}$$

The terms of order ∂^4 on rhs of the flow equation (2.107) are, using (A.38) from Appendix A and (2.117)-(2.118), the following:

$$\begin{aligned}
\partial_t \Gamma_k[\bar{\varphi}]|_{\partial^4} &= \frac{1}{2(4\pi)^{d/2}} \int d^d x \left\{ Q_{\frac{d}{2}-2} \left[\partial_t R_k G_k \right] b_4(-D^2) + Q_{\frac{d}{2}} \left[\partial_t R_k G_k^3 \right] U_{AB} U^{BA} \right\} \\
&= c_d \int d^d x \left\{ \frac{d(d-2)}{4} \frac{1}{6} \bar{R}_{AB}{}^{EF} \bar{R}_{CDEF} + \bar{R}_{EAFB} \bar{R}^F{}_C{}^E{}_D \right\} \\
&\quad \times \partial_\mu \bar{\varphi}^A \partial^\mu \bar{\varphi}^B \partial_\nu \bar{\varphi}^C \partial^\nu \bar{\varphi}^D \,.
\end{aligned} \tag{2.119}$$

Note that the Q-functionals in (2.119) are cutoff shape independent. In the case of the $SO(4)$ model in four dimensions ($d = 4$, $D = 3$, $\mathcal{N} = S^3$) the allowed four derivative terms in the EAA are

$$(l_{1,k} h_{AB} h_{CD} + l_{2,k} h_{AC} h_{BD}) \partial_\mu \varphi^A \partial^\mu \varphi^B \partial_\nu \varphi^C \partial^\nu \varphi^D \,, \tag{2.120}$$

with $l_{1,k}$ and $l_{2,k}$ running coupling constants. The Riemann tensor is of the form $R_{ABCD} = h_{AC} h_{BD} - h_{AD} h_{BC}$ and one obtains the following beta functions:

$$\partial_t l_{1,k} = \frac{2}{3} c_4 \qquad \partial_t l_{2,k} = \frac{4}{3} c_4 \,. \tag{2.121}$$

When one solves (2.121) for $l_{1,k}$ and $l_{2,k}$, the results diverge logarithmically for $k \to \infty$; using the identification $\log k^2 = \frac{1}{d-4}$, the coefficients of the divergence agree with the dimensionally regulated one-loop calculation in [35].

We conclude with some comments. The NLσM has many features in common with gravity, discussed in Chapter 4, already at the kinematical level [36], and comparison between the two theories may be useful. Also the structure of the dynamics is very similar: except for the factor $\sqrt{\det g}$ and for the different contractions of the indices, the action (2.97) for a group-valued NLσM and the Einstein-Hilbert action for gravity both have the structure $\zeta \int (g^{-1}\partial g)^2$ where g is either a G-valued scalar field or the metric, and ζ has dimension k^{d-2}. The results of the present section confirms that these analogies extend also to the properties of the RG flow.

An "asymptotically safe" NLσM could be more useful in weak interaction physics. In fact, the $SO(4)$ NLσM can be regarded as the strong coupling limit of the scalar sector of the standard model. Replacing the complex Higgs doublet by a S^3 NLσM results in a "Higgsless" theory. Normally this is regarded only as an approximate description valid below some cutoff of the order of the mass of the Higgs particle, but if there is a fixed point, and assuming that there are no resonances, then the Higgsless theory could hold up to much higher energies [42].

2.3.3 Matter field theories on curved backgrounds

In this section we discuss matter theories on curved manifolds equipped with a metric $g_{\mu\nu}$. In particular, we will consider a minimally coupled scalar field on a two dimensional manifold and we will follow [43]. We will show how to derive the Polyakov effective action [44, 81],

$$\Gamma[g] = -\frac{1}{96\pi}\int d^2x\sqrt{g}R\frac{1}{\Delta}R\,, \tag{2.122}$$

by integrating the flow of the EAA. It is quite instructive to consider matter contributions to the gravitational EAA, especially because ambiguities related to the construction of a gauge invariant flow are not involved. The closely related Liouville field theory in two dimensions has been studied within the EAA approach in [45].

The classical action for a minimally coupled scalar field ϕ is given by

$$S[\phi, g] = \frac{1}{2}\int d^2x\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{1}{2}\int d^2x\sqrt{g}\phi\Delta\phi\,, \tag{2.123}$$

where we introduced the covariant Laplacian operator Δ acting on scalar fields defined by:

$$\Delta\phi = -\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right)\,.$$

Being that the action (2.123) is quadratic in the fields, the one-loop EAA (2.20) is already exact and it is thus enough to consider the flow equation (2.124):

$$\partial_t \Gamma_k[\varphi, g] = \frac{1}{2} \mathrm{Tr} \left(\frac{\delta^2 S[\varphi, g]}{\delta\varphi\delta\varphi} + R_k[g] \right)^{-1} \partial_t R_k[g] \,. \tag{2.124}$$

Note that when we insert the bare action (2.123) in equation (2.124), the rhs becomes independent of φ, so the flow will not generate any non-trivial dependence of the EAA on φ. It is thus enough to concentrate on the flow of the gravitational part of the EAA, i.e. $\Gamma_k[0, g]$, which is the only non-trivial part of the EAA.

The Hessian of the bare action (2.123) is just the covariant Laplacian $S^{(2,0)}[0, g] = \Delta$ and so equation (2.124) reduces to

$$\partial_t \Gamma_k[0, g] = \frac{1}{2} \mathrm{Tr} \frac{\partial_t R_k(\Delta)}{\Delta + R_k(\Delta)} \,. \tag{2.125}$$

We can evaluate the functional trace in equation (2.125) using the technology developed in Appendix A. Defining, as before, the function $h_k(z) = \frac{\partial_t R_k(z)}{z+R_k(z)}$ and using the non-local heat kernel expansion (A.12) in equation (A.3) we find:

$$\begin{aligned}
\partial_t \Gamma_k[0, g] \;\; &= \;\; \frac{1}{8\pi} Q_1[h_k] \int d^2x \sqrt{g} + \frac{1}{48\pi} Q_0[h_k] \int d^2x \sqrt{g} R + \\
&\quad + \frac{1}{8\pi} \int d^2x \sqrt{g} R \left[\int_0^\infty ds \, \tilde{h}_k(s) \, s \, f_{R2d}(s\Delta) \right] R + O(R^3) \,.
\end{aligned} \tag{2.126}$$

Here $\tilde{h}_k(s)$ is the inverse Laplace transform of the function $h_k(x)$ and the Q-functionals are defined in equation (A.38) of Appendix B. In (2.126) $f_{R2d}(x)$ is the non-local structure function defined in (A.13).

To make progress we need to devise a truncation ansatz for the EAA to insert into the lhs of equation (2.125). We will consider an ansatz where the EAA is local in the curvature, but non-local in the covariant momentum square, i.e. in Δ. We are lead to the following truncation ansatz, which comprises the first terms of the curvature expansion:

$$\Gamma_k[0, g] = \int d^2x \sqrt{g} \left(a_k + b_k R + R \, c_k(\Delta) R \right) + O(R^3) \,. \tag{2.127}$$

Here $c_k(x)$ is any function of the covariant Laplacian. We are working in two dimensions so this is the only structure function at second order in the curvature to be considered.

By comparing equation (2.126) to (2.127), the beta functions for the first two couplings in (2.127) are immediately found:

$$\partial_t a_k = \frac{1}{8\pi} Q_1[h_k] \qquad\qquad \partial_t b_k = \frac{1}{48\pi} Q_0[h_k]\,. \tag{2.128}$$

From the curvature square term in (2.126) we find the following flow equation for the non-local structure function:

$$\partial_t c_k(x) = \frac{1}{8\pi} \int_0^\infty ds\, \tilde{h}_k(s)\, s\, f_{R2d}(sx)\,. \tag{2.129}$$

If we now insert the explicit form of the heat kernel structure function $f_{R2d}(x)$ from equation (A.13) of Appendix B, written in terms of the basic parameter integral (A.10), and expressing everything in terms of Q-functionals, we find:

$$\begin{aligned} 8\pi\, \partial_t c_k(x) \;\; = \;\; & \frac{1}{32} \int_0^1 d\xi\, Q_{-1}\left[h_k(z + x\xi(1-\xi))\right] + \\ & + \frac{1}{8x} \int_0^1 d\xi\, Q_0\left[h_k(z + x\xi(1-\xi))\right] - \frac{1}{16x} Q_0[h_k] + \\ & + \frac{3}{8x^2} \int_0^1 d\xi\, Q_1\left[h_k(z + x\xi(1-\xi))\right] - \frac{3}{8x^2} Q_1[h_k]\,. \end{aligned} \tag{2.130}$$

In the last equation the dummy index z is shown to indicate that the Q-functionals are to be evaluated at the shifted point $z + x\xi(1-\xi)$. The next step is to use the properties of the Q-functionals to find:

$$\begin{aligned} 8\pi\, \partial_t c_k(x) \;\; = \;\; & -\frac{1}{32} \int_0^1 d\xi\, h_k'\left(x\xi(1-\xi)\right) + \frac{1}{8x} \int_0^1 d\xi\, h_k\left(x\xi(1-\xi)\right) + \\ & - \frac{1}{16x} h_k(0) - \frac{3}{8x^2} \int_0^1 d\xi \int_0^{x\xi(1-\xi)} dz\, h_k\left(z\right)\,. \end{aligned} \tag{2.131}$$

Note that we combined the last two terms of equation (2.130) into a single z integral.

Equation (2.131) is the explicit flow equation for the structure function $c_k(x)$. It should be possible to integrate equation (2.131) from the UV to the IR scale to recover the Polyakov effective action (2.122) without specifying the cutoff shape function $R_k(z)$. Here we will show how this can be done by explicitly using the cutoff shapes in (2.12).

First we use the "optimized" cutoff to evaluate the beta functions (2.128):

$$\partial_t a_k = \frac{k^2}{4\pi} \qquad\qquad \partial_t b_k = \frac{1}{24\pi}\,. \tag{2.132}$$

After collecting the overall power k^{-2} and writing the parameter integrals in terms of the

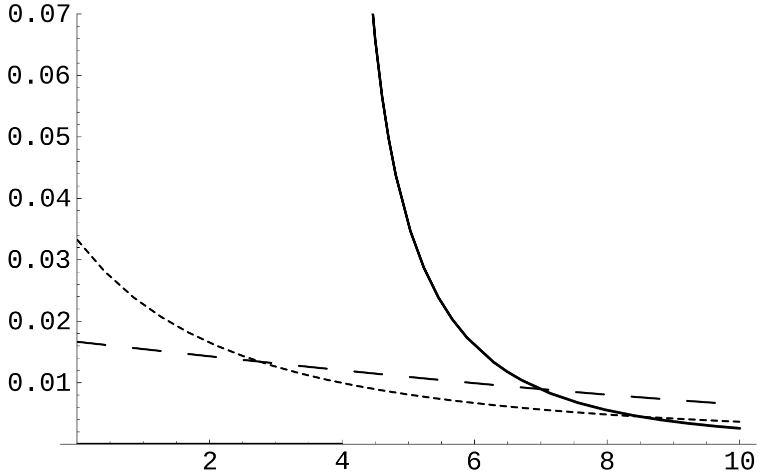


Figure 2.14: The function $f(u)$ evaluated using the exponential cutoff (long dashed), the mass cutoff (short dashed) and the optimized cutoff (thick). Note that all three functions are analytic around the origin and that $f_{opt}(u)$ develop a pole at $u = 4$.

dimensionless variable $u = x/k^2$, the flow equation (2.131) can be rewritten in the form

$$\partial_t c_k(x) = \frac{1}{8\pi\, k^2} f\left(\frac{x}{k^2}\right) . \tag{2.133}$$

The function $f(u)$ depends explicitly on the cutoff shape function used. In the case of the optimized and mass cutoffs we find, after some elementary integrations, respectively:

$$\begin{aligned} f_{opt}(u) &= \frac{1}{8u}\left[\sqrt{\frac{u}{u-4}} - \frac{u+4}{u}\sqrt{\frac{u-4}{u}}\right]\theta(u-4) \\ f_{mass}(u) &= \frac{\sqrt{u(u+4)}(u+6) + 8(u+3)\,\mathrm{artanh}\sqrt{\frac{u}{u+4}}}{(u+4)^{3/2}u^{5/2}} . \end{aligned} \tag{2.134}$$

The parameter integrals in equation (2.131) cannot be evaluated analytically for the exponential cutoff, but this can still be done numerically. The functions $f(u)$ evaluated for the three different cutoff shape functions are plotted in Figure 2.14. Note that they are all analytic in a neighborhood of the origin, $f_{opt}(u)$ is even zero in the entire interval $[0, 4)$.

If we were to interpret $f(u)$ as a power series in u about $u = 0$, it follows that we have a non zero running of local terms of the form $c_k^{(n)} \int \sqrt{g} R \Delta^n R$ only for the exponential and the mass cutoff. For example, we can expand for small u

$$f_{mass}(u) = \frac{1}{30} - \frac{u}{70} + \frac{u^2}{210} + O(u^3) ,$$

and read off the resulting beta functions for the couplings $c_k^{(n)}$ in the mass cutoff case. For the optimized cutoff none of the couplings $c_k^{(n)}$ has a non-zero beta function. Any finite truncation of the EAA containing some of the couplings $c_k^{(n)}$ will never reproduce the correct IR behavior and will lead to IR divergences. More importantly, the running of the couplings $c_k^{(-n)}$, $n > 0$, which multiply non-local terms involving inverse powers of Δ, is zero for all three cutoff choices. In particular, the beta function of the coupling $c_k^{(-1)}$ pertaining to the operator $\int \sqrt{g} R \frac{1}{\Delta} R$ is zero, even if this is the form the EAA is expected to reach at $k = 0$! We conclude that, at least for the cutoff shapes we considered, to capture the non-local features of the EAA we need to consider the running of the whole structure function $c_k(x)$.

We now integrate the flow equations from the UV scale Λ, where we impose the initial conditions $\Gamma_\Lambda[\varphi, g] = S_\Lambda[\varphi, g]$, to the IR scale k. We will see that imposing the initial conditions not only selects which theory we are quantizing, but also implements the renormalization conditions. In the limit $k \to 0$ we will find the full effective action.

We start solving the differential equations (2.132). Integrating from k to Λ gives

$$\begin{aligned} a_k &= a_\Lambda - \frac{1}{4\pi}(\Lambda^2 - k^2) \\ b_k &= b_\Lambda - \frac{1}{24\pi} \log \frac{\Lambda}{k} \,. \end{aligned} \tag{2.135}$$

The coupling a_k and b_k have to be renormalized. This can be done by setting $a_\Lambda = \frac{\Lambda^2}{4\pi}$, so that the renormalized a_0 vanishes and conformal invariance of the effective action is preserved, and by setting $b_\Lambda = \frac{1}{24\pi} \log \frac{\Lambda}{k_0}$ with k_0 an arbitrary scale.

Integrating the RG equation (2.133) of the structure function gives

$$c_k(x) = c_\Lambda(x) - \frac{1}{8\pi} \int_k^\Lambda \frac{dk'}{k'^3} \, f\left(\frac{x}{k'^2}\right) \,.$$

If we use the variable $u = x/k^2$ we have $dk/k^3 = -du/2x$ and we come to:

$$c_k(x) = c_\Lambda(x) - \frac{1}{16\pi x} \int_{x/\Lambda^2}^{x/k^2} du \, f\,(u) \,. \tag{2.136}$$

If the integral in (2.136) is convergent at both the lower and upper limits it becomes a pure number in the limit $\Lambda \to \infty$ and $k \to 0$. The functional form of $c_0(x)$ will be in agreement with the Polyakov effective action (2.122) if

$$\int_0^\infty du\, f(u) = \frac{1}{6}\,.$$

This condition should be met for any cutoff choice. A simple integration of (2.134) shows that this is so for the optimized and mass cutoffs. A numerical evaluation shows that also the exponential cutoff gives the desired result. The functions plotted in Figure 2.14 have thus all the same area. This is a non-trivial check of the cutoff shape independence of physical quantities in the EAA formalism.

Imposing the boundary condition $c_\infty(x) = 0$, we thus recover Polyakov's non-local effective action, equation (2.122), as the result of the integration of the RG flow:

$$c_0(x) = -\frac{1}{96\pi x}\,.$$

We are now in a position to write down the full EAA within truncation (2.127) and using the optimized cutoff shape function. At any given scale k we have:

$$\begin{aligned}\Gamma_k[0,g] &= \frac{k^2}{4\pi}\int d^2x\sqrt{g} + \frac{\chi}{6}\log\frac{k}{k_0} + \\ &\quad -\frac{1}{96\pi}\int d^2x\sqrt{g}R\left[\frac{\sqrt{\Delta/k^2-4}(\Delta/k^2+2)}{\Delta\,(\Delta/k^2)^{3/2}}\theta(\Delta/k^2-4)\right]R + O(R^3)\,.\end{aligned} \tag{2.137}$$

This relation is the main result of this section. It shows how the EAA interpolates smoothly between the classical action at the scale $k = \Lambda$ and the EA at the scale $k = 0$. The evolution of the structure function $c_k(\Delta)$ in its final form (2.137) is plotted in Figure 2.14.

Note that the convergence is non-uniform, $c_k(x) \sim c_0(x)$ for $x < 4k^2$, and that the singularity in the structure function is obtained only at $k = 0$. Note also that we have written the linear term in the curvature in terms of the Euler topological characteristic of the manifold $\chi = \frac{1}{4\pi}\int d^2x\sqrt{g}R$. Only when the topology of the manifold $\mathcal{M}$ is the one of a torus and the Euler characteristic is zero $\chi = 0$, there is no problem in taking the limit $k \to 0$. In the spherical case, $\chi = 2$, or in all higher genus topologies, the limit $k \to 0$ can be taken only if we also send $k_0 \to 0$ in such a way that $\frac{k}{k_0}$ remains constant. A similar phenomenon is encountered in what is called the "double scaling limit" in two dimensional quantum gravity [73].

In principle we still have to show that all higher terms, that would extend the truncation

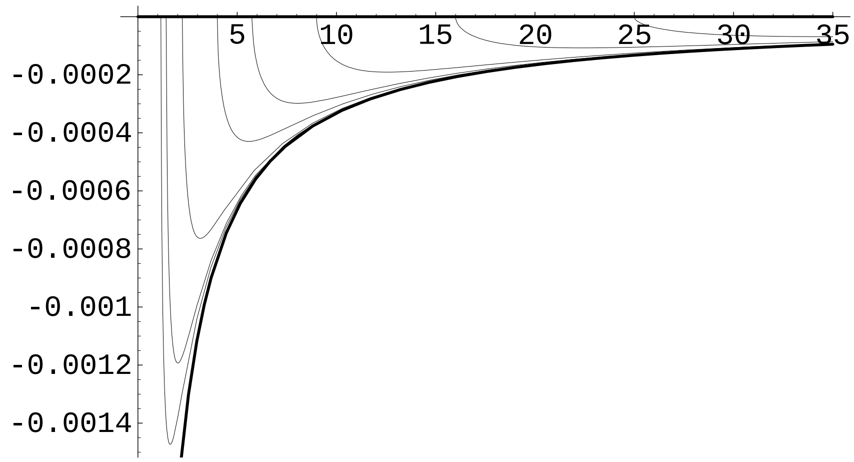


Figure 2.15: Flow of the structure function $c_k(x)$ from $c_\infty(x) = 0$ (upper thick curve) to $c_0(x) = -\frac{1}{96\pi x}$ (lower thick curve). The structure function $c_k(x)$ is plotted as a function of x for different values of the IR cutoff in the range $\infty \geq k \geq 0$.

(2.127) to higher curvatures, and are in principle present in the EAA, vanish at $k = 0$. Only then we would had completely recovered Polyakov's result. We shall not embark on such a proof since this issue is special to two dimensions and, contrary to the discussion above, it does not generalize to the higher dimensional case to which we will turn in Chapter 4.

In summary, in this section we explained how the Polyakov effective action for a minimally coupled scalar field on a curved two dimensional manifold emerges within the functional RG approach. To do this we calculated the RG flow of the structure function $c_k(\Delta)$ using the non-local heat kernel expansion. We learned that in order to be able to recover, at the IR scale, special non-local terms in the EAA, $\int \sqrt{g} R \frac{1}{\Delta} R$ in our example, it is necessary to include the running of the complete structure function which allows for an arbitrary dependence on Δ. We also saw that, quite remarkably, individual non-local terms in a Laurent series expansion, $\int \sqrt{g} R \Delta^{-n} R$, $n > 0$, have no RG running, even though the $k \to 0$ limit of the EAA is precisely of this type. This is an important observation in view of the applications of this framework to quantized gravity, that we start to develop in Chapter 4.

2.4 Summary

In this chapter we introduced the EAA together with its basic properties. In section 2.2.1, we derived the exact RG flow equation, given in (2.17), that the EAA satisfies. From this equation we constructed in section 2.2.2 a hierarchy of coupled equations for the proper-vertices of the EAA and we explained how these equations can be used to devise useful approximation schemes for calculating the full EAA.

Next we showed in section 2.2.3 how the standard renormalized perturbative expansion is recovered as a particular way to iteratively solve the exact flow equation for the EAA if the starting point of the iteration is the bare action..

In the last part of the chapter we exposed three different first applications of the EAA. This in order to understand and learn the basic properties of the formalisms through concrete examples.

In the first application, we studied one-loop QED focusing on the photon part of the EAA. We derived the running of the electric charge ad used this to explain the reasons why we consider QED as just an effective theory: triviality, or equivalently, the existence of a Landau pole in its flow. As a first example of the use of the EAA to make explicit calculations in QFT we calculated the running vacuum polarization in the one-loop approximation and we showed that for $k \to 0$ we recover the standard result from perturbation theory. We also prove that, as it should, this last result is independent of the cutoff we employ. Finally we calculate the next terms in the photon part of the effective action and find the old result of Euler and Heisenberg.

In the second application we construct the EAA for general Non-Linear-σ-Models (NLσM) by constructing the cutoff action in such a way that the underling symmetries of the model are preserved. To be able to do so we use the background field method (this is the topic of the next two chapters and will be fully developed there) together with the geometric geodesic expansion. We study a simple truncation, the second order derivative expansion, and within the one-loop approximation we recover the geometric flow that characterizes the quantum behavior of these models. Next we consider homogeneous spaces, so that the effect of RG flow is just to rescale the geometry of the model, beyond the one-loop approximation. Our results point in the direction that general NLσM can be asymptotically safe in $d \geq 2$, in particular in four dimensions. We speculate the role that such a scenario can have for particle physics.

The last application is to matter fields in curved space. In particular we study a minimally coupled scalar on a general two dimensional manifold. We learn that, in order to calculate the full effective action for $k \to 0$, it is mandatory to consider truncations of the EAA that contain an infinite number of coupling constants. We encode these in the form of running "structure functions" and we devise a curvature expansion of the EAA. We derive the flow of the structure function relative to the model we studied and we show that for $k = 0$ we recover the effective action first found by Polyakov.

Chapter 3

Functional RG for gauge theories

3.1 Introduction

Quantum field theories based on a gauge or local symmetry group G are the fundamental building blocks out of which the standard model of particle interactions is constructed. QED is a gauge theory with abelian gauge group $G = U(1)$, while the electroweak and strong forces are described using a non-abelian gauge group. The interesting property of gauge theories with non-abelian groups is that they are self-interacting: the vector bosons which mediate the forces interact with each other. This phenomenon has the important consequence that screening or anti screening effects may suppress or enhance the interaction strength at certain energy scales. This is indeed what happens for gauge theories with gauge group $G = SU(N)$, as those used to construct the standard model of particle interactions. This phenomenon is particularly relevant in the case of strong interactions, where at high energies, the coupling constant, which quantifies the force strength, becomes smaller and smaller, meaning that the force strength becomes weaker and weaker. This property is called asymptotic freedom because the vector bosons, become free particles as the energy scale grows towards infinity. This important discovery was made in 1974 by Wilzeck, Polizer and Gross [46, 47] and firmly established non-abelian gauge theories as the fundamental theory able to describe strong interaction. This theory is what we call QCD. Despite the extraordinary success obtained by QCD in the high energy, ultraviolet (UV), regime with the use of perturbation theory, put on firm ground by asymptotic freedom, it is still a difficult problem to describe the low energy, infrared (IR) physics of gluons and quarks, in particular at energies where the coupling constant grows to be of order one and perturbation theory is no more a useful tool. This happens around the energy scale Λ_{QDC} where the confinement transition takes place and quarks become bounded together by gluons to form mesons and hadrons. This is physically one of the most interesting regimes and up to now it has mainly been treated using effective field theory ideas as chiral perturbation theory [6]. What we would really like to have is a complete theoretical framework where the degrees of freedom used at a particular energy scale can be continued to cover all the interesting physical scales involved. To be able to do so, non-perturbative methods have to be employed. The most developed up to now is lattice gauge theory, where spacetime is discretized in order to make the functional integral and the partition functional well defined, finite dimensional integrals. These are then evaluated numerically by Monte Carlo techniques at the cost of hard computations which require intense use of dedicated workstations.

The functional renormalization group (fRG) approach to non-abelian gauge theories tries to introduces non-perturbative methods that can still be treated as much as it is possible analytically. Here the basic approximation scheme are the truncations of theory space, the hope is to device wise truncations able to capture in few running couplings or running functions enough information to understand the basic physics involved and to make quantitative predictions. Considering that, at the practical level, the method deals with coupled differential equations for the running coupling constants or integro-differential equations for the running functions, numerical techniques are still necessary. The hope is that they come in a less prominent role then in lattice approaches.

The crucial point that we have not yet mentioned is that implementing the RG coarse-graining procedure to gauge theories is a difficult task which generally comes at a price. In the implementation followed in this thesis we use the background field method and the cost is that we have to enlarge theory space to include functionals not only of the dynamical fluctuating fields but also of the background gauge fields. This is because we are "softly" breaking physical gauge invariance by making the theory in presence of the RG cutoff invariant under background plus physical gauge transformations. The important point is that we are able to control this breaking of gauge invariance with the aid of modified Ward-Takahashi identities that ensure that when the cutoff is removed we recover full gauge invariance. The functional RG approach is some how dual to the lattice one: the first preserves the spacetime symmetries while breaking the gauge ones while the second breaks spacetime symmetry going on the lattice but preserves gauge symmetry using gauge invariant variables as Wilson loops.

In section 3.2 of this chapter, after rapidly introducing the classical theory, we turn to the EAA quantization of non-abelian gauge theories based on the formalism developed in section 3.3. We explore the approach by means, first of local truncations where we study the running of the gauge coupling within various approximations, and subsequently by introducing non-local ones in terms of a curvature expansion where we look at the running of the covariant polarization function. In section 3.3 the formalism is developed, the exact flow equation is derived for both the EAA and its proper-vertices. The modified Ward-Takahashi identities and the modified Zinn-Justin equation are treated. Finally we introduce new diagrammatic and momentum space techniques that allow to project the flow of any truncation of the EAA which is analytic in the fields. In the Appendix to the Chapter we report all the technical details about the derivations.

3.2 EAA approach to non-abelian gauge theories

3.2.1 Classical theory

The geometric idea behind gauge theories is roughly that the gauge field is a connection in a fiber bundle over spacetime where the fiber is the gauge group G. We will consider here only the case where $G = SU(N)$. We use the gauge connection or gauge field[1] A_μ, defined in each local trivialization of the fiber bundle, to construct a covariant derivative:

$$D_\mu = \partial_\mu + gA_\mu\,, \tag{3.1}$$

where g is the coupling constant, and the field strength or curvature:[2]

$$F_{\mu\nu} = [D_\mu, D_\nu]\,. \tag{3.2}$$

Two gauge connections written in two different trivializations of the fiber bundle A'_μ and A_μ are related by a "gauge transformation"

$$A'_\mu = g\,A_\mu g^{-1} - (\partial_\mu g)g^{-1}\,, \tag{3.3}$$

where g is the fiber bundle structure function which relates the two trivializations. The field strength (3.2) transforms homogeneously under gauge transformations (3.3):

$$F'_{\mu\nu} = gF_{\mu\nu}g^{-1} \tag{3.4}$$

[1] We will often use the term "gluons" to indicate the gauge fields of a non-abelian gauge theory and the term "photon" to indicate the gauge field of an abelian gauge theory.

[2] For more details refer to Appendix C at this point.

We can now write down an action for the gauge field integrating over spacetime the simplest invariant under the gauge transformations (3.3) we can construct:

$$S[A] = \frac{1}{2} \int d^d x \, \mathrm{tr} F_{\mu\nu} F^{\mu\nu} = \frac{1}{4} \int d^d x \, F^a_{\mu\nu} F^{a\mu\nu} \, . \tag{3.5}$$

This action is taken as the bare action for non-abelian gauge theories. That (3.5) is invariant under gauge transformations is easily checked using (3.4). The equations of motion following from the action principle

$$\frac{\delta S[A]}{\delta A_\mu} = 0 \, ,$$

are

$$\partial_\mu F^{\mu\nu} + g[A_\mu, F^{\mu\nu}] = 0 \, . \tag{3.6}$$

The commutator in (3.6) vanishes in the case of an abelian gauge group thus giving one of Maxwell's equations. The term proportional to g is the self-interaction of the gauge field. When $g \to 0$, as happens in the quantum theory in the high energy regime, the gluons become free particles.

3.2.2 Quantum theory

We carry over the quantization of the theory using the EAA approach. This means that we need to construct a complete RG trajectory in theory space that connects the classical or bare action for $k \to \infty$ and the full effective action for $k \to 0$. The EAA is constructed using the background field method introduced in section 3.3, to which we remand the reader at this moment.

The background effective average action (bEAA), as is defined in (3.59), has the general form (3.65):

$$\Gamma_k[a, \bar{c}, c; \bar{A}] = \bar{\Gamma}_k[\bar{A} + a] + \hat{\Gamma}_k[a, \bar{c}, c; \bar{A}] \, , \tag{3.7}$$

where a_μ is the gauge fluctuation field, $\bar{c}$ and c are the ghost fields[3] and $\bar{A}_\mu$ is the background gauge field. The full quantum field $A_\mu = \bar{A}_\mu + a_\mu$ is, in this approach, linearly split into the background and the fluctuation field. The bEAA is invariant under combined physical and background gauge transformations (3.63):

$$(\delta + \bar{\delta})\Gamma_k[a, \bar{c}, c; \bar{A}] = 0 \, . \tag{3.8}$$

[3]We often refer to the fields in the multiplet $\varphi = (a_\mu, \bar{c}, c)$ as the "fluctuation fields".

The gauge covariant effective average action (gEAA), denoted by $\bar{\Gamma}_k[A]$, is that part of the bEAA which is invariant under physical gauge transformations (3.66),

$$\delta\bar{\Gamma}_k[A] = 0\,, \tag{3.9}$$

and is a functional of the full quantum field, at least for $k \to 0$. In fact, at intermediate scales we must introduce the scale dependent wave-function renormalization for all the fields involved:

$$\begin{aligned} a_\mu &\to Z_{a,k}^{1/2} a_\mu & \bar{A}_\mu &\to Z_{\bar{A},k}^{1/2}\bar{A}_\mu \\ \bar{c} &\to Z_{c,k}^{1/2}\bar{c} & c &\to Z_{c,k}^{1/2} c\,. \end{aligned} \tag{3.10}$$

In particular, this implies that the linear split of the full quantum field

$$A_\mu = Z_{\bar{A},k}^{1/2}\bar{A}_\mu + Z_{a,k}^{1/2} a_\mu\,, \tag{3.11}$$

can be broken by renormalization if the wave-function renormalization of the gauge fluctuation field $Z_{a,k}$ runs differently then the wave-function renormalization of the background gauge field $Z_{\bar{A},k}$. We will see that this is indeed what generally happens and this implies that for $k \neq 0$ we need to consider the running of the full bEAA.

The beta function of the running gauge coupling g_k is related to the running wave-function renormalization of the background field $Z_{\bar{A},k}$. Using the non-renormalization of the background covariant derivative we find:

$$g_k = Z_{\bar{A},k}^{-1/2}\,. \tag{3.12}$$

Equation (3.12) tells us that we can calculate the running of the gauge coupling by looking at the running of the wave function renormalization of the background field. This was the original reason why the background field method was introduced [48]: only diagrams with no external fluctuation legs need to be considered in perturbation theory to calculate the beta function of of the gauge coupling. We see that, in view of (3.12), the covariant derivative constructed with the full quantum field (3.11) becomes:

$$D_\mu = \partial_\mu + \bar{A}_\mu + g_k Z_{a,k}^{1/2} a_\mu\,, \tag{3.13}$$

and is thus scale dependent. As a consequence of (3.12) the background covariant derivative does not renormalize.

The functional $\hat{\Gamma}_k[a,\bar{c},c;\bar{A}]$ is the remainder effective average action (rEAA) and plays the role of a generalized gauge-fixing and ghost action as, in the limit $k\to\infty$, it flows to the classical gauge-fixing (3.61) and ghost (3.62) actions:

$$\lim_{k\to\infty}\hat{\Gamma}_k[a,\bar{c},c;\bar{A}] = S_{gf}[a;\bar{A}] + S_{gh}[a,\bar{c},c;\bar{A}]\,. \tag{3.14}$$

The exact RG flow equation for the bEAA can be easily derived and reads (3.70):

$$\partial_t\Gamma_k[\varphi;\bar{A}] = \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2;0)}[\varphi;\bar{A}] + R_k[\bar{A}]\right)^{-1}\partial_t R_k[\bar{A}]\,, \tag{3.15}$$

where $\varphi=(a_\mu,\bar{c},c)$ is the field fluctuation multiplet. As explained in section 3.3 the cutoff kernel in (3.15) is constructed using a differential operator, for example the covariant Laplacian, constructed using the background field.

It is tempting to set $\varphi=0$ in (3.15) and hoping in this way to get a closed flow equation for the gEAA since $\partial_t\bar{\Gamma}_k[\bar{A}]=\partial_t\Gamma_k[0;\bar{A}]$. The subtlety is that the flow (3.15) is "driven" by the Hessian of the bEAA taken with respect to the fluctuation multiplet and $\Gamma_k^{(2;0)}[0;\bar{A}]$ is not equal to $\bar{\Gamma}_k^{(2)}[\bar{A}]$. Thus, in general, we have to consider the full flow of the bEAA that takes place in the enlarged theory space of functionals of the fields a_μ, $\bar{c}$, c and $\bar{A}_\mu$. In the next section we will study the flow of the gauge coupling, which is part of the gEAA, and we will see how this is influenced by couplings which are part of a truncation of the rEAA. In particular we will present different ways to "close" the flow of g_k, which naturally depends on the fluctuation couplings, i.e. $Z_{a,k}$, $Z_{c,k}$ and the gluon and ghost masses as well as the guage-fixing parameter.

When we consider a truncation ansatz for the bEAA which is bilinear in the ghost fields, the flow equation for the gEAA becomes (3.75):

$$\begin{aligned}\partial_t\bar{\Gamma}_k[\bar{A}] \;&=\; \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2,0,0;0)}[0;\bar{A}] + R_{k,aa}[\bar{A}]\right)^{-1}\partial_t R_{k,aa}[\bar{A}]\\ &\quad -\mathrm{Tr}\left(\Gamma_k^{(0,1,1;0)}[0;\bar{A}] + R_{k,\bar{c}c}[\bar{A}]\right)^{-1}\partial_t R_{k,\bar{c}c}[\bar{A}]\,.\end{aligned} \tag{3.16}$$

The flow equation (3.16) can be seen as the RG improvement of the one-loop effective action (C.50) of Appendix B.

3.2.2.1 Local truncations

We will consider first a truncation ansatz for the gEAA in (3.7) which is the RG improvement of the classical action (3.5):

$$\bar{\Gamma}_k[A] = \frac{1}{4}\int d^dx\, F^a_{\mu\nu}F^{a\mu\nu}\,. \tag{3.17}$$

We have to consider the flow in the enlarged functional space which means that we have to insert (3.11) in (3.17). Using the basic variations (3.142) we find the following expansion:

$$\begin{aligned}
\bar{\Gamma}_k[Z^{1/2}_{\bar{A},k}\bar{A} + Z^{1/2}_{a,k}a] \;=\;& Z_{\bar{A},k}\bar{\Gamma}_k[\bar{A}] + Z^{1/2}_{\bar{A},k}Z^{1/2}_{a,k}\int d^dx\,\bar{F}^{\mu\nu}\bar{D}_\mu a_\nu + \\
&+\frac{1}{2}Z_{a,k}\int d^dx\, a_\mu\left[-\bar{D}^2 g^{\mu\nu} + 2i\bar{F}^{\mu\nu} + \bar{D}^\mu\bar{D}^\nu\right]a_\nu + \\
&+g_k Z^{3/2}_{a,k} f^{abc}\int d^dx\,\bar{D}_\mu a^a_\nu\, a^b_\mu a^c_\nu + \\
&+\frac{1}{4}g_k^2 Z^2_{a,k} f^{abc}f^{ade}\int d^dx\, a^{b\mu}a^{c\nu}a^d_\mu a^e_\nu\,.
\end{aligned} \tag{3.18}$$

Note here the factors of g_k steaming from the fact that the action (3.17) is constructed with the covariant derivative of the full quantum field (3.13). We see from (3.18) that the flow equation for $\bar{\Gamma}_k[\bar{A}]$ is enough to extract the beta function of the wave-function renormalization of the background field. This is done in section 3.5.4. To find the running of the wave-function renormalization of the fluctuation field we have to use the full flow equation for the bEAA (3.15), in particular it is clear from (3.18) that this can be done by looking at the running of the zero-field proper-vertex $\gamma_k^{(2,0,0;0)} = \Gamma_k^{(2,0,0;0)}[0,0,0;0]$. This is done in section 3.5.6.

Second, we have to consider a truncation ansatz for the rEAA. We expand the rEAA in powers of the fluctuation field a_μ and we consider those terms that are not already present in (3.18). These are the running masses $m_{a,k}$ and $m_{c,k}$ of respectively the gauge fluctuation field and the ghost fields, the running gauge-fixing parameter α_k and the scale dependent ghost wave-function renormalization $Z_{c,k}$. We take the following truncation ansatz:

$$\begin{aligned}
\hat{\Gamma}_k[Z^{1/2}_{a,k}a, Z^{1/2}_{c,k}\bar{c}, Z^{1/2}_{c,k}c; Z^{1/2}_{\bar{A},k}\bar{A}] =& \frac{1}{2}Z_{c,k}\int d^dx\left[\frac{1}{\alpha_k}\bar{D}_\mu a^\mu \bar{D}_\nu a^\nu + m^2_{a,k}\, a_\mu a^\mu\right] \\
&+Z_{c,k}\int d^dx\,\left[\bar{D}_\mu\bar{c}\left(\bar{D}^\mu + g_k Z^{1/2}_{a,k}a^\mu\right)c + m_{c,k}\,\bar{c}c\right]\,.
\end{aligned} \tag{3.19}$$

The ansatz (3.19) amounts to an RG improvement of the classical gauge-fixing and ghost actions (3.61) and (3.62). Note also that in this part of the truncation the gauge coupling is present due the structure of the background ghost action (3.62) we are improving.

From the flow equation (3.16) for the gEAA we can calculate the RG running of the wave-function renormalization of the background field. As explained in section 3.5.3 there are two possible ways to choose the cutoff operator that we employ to separate the field modes to integrate out from those we don't integrate out. In one case, that we call type I, we use the covariant Laplacian $-\bar{D}^2$ as cutoff operator in both the gauge and ghost sectors. In the other case, that we call type II, we use the differential operator $\bar{D}_T^{\mu\nu} = -\bar{D}^2 g^{\mu\nu} + 2i\bar{F}^{\mu\nu}$ to cutoff the gauge fluctuation modes in place of the covariant Laplacian. The general form of $\partial_t Z_{\bar{A},k}$, in these two cases, is derived is section 3.5.3 and the results are given in equation (3.180) for type I and in equation (3.183) for type II.

For general values of the masses of the fluctuation field we can only evaluate explicitly the beta function of the wave-function renormalization of the background field if we employ the optimized cutoff. Here we are working with the gauge-fixing parameter fixed to $\alpha_k = 1$ for convenience. For general d and for type I cutoff we find (3.184):

$$\partial_t Z_{\bar{A},k} = \frac{N\, k^{d-4}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)} \left[-\frac{d}{6}\frac{d-2-\eta_{a,k}}{1+\tilde{m}^2_{a,k}} + \frac{32}{d(d+2)}\frac{d+2-\eta_{a,k}}{(1+\tilde{m}^2_{a,k})^3} + \frac{1}{3}\frac{d-2-\eta_{c,k}}{1+\tilde{m}^2_{c,k}} \right] , \tag{3.20}$$

where we defined $\tilde{m}_{a,k} = k^{-2} m_{a,k}$ and $\tilde{m}_{c,k} = k^{-2} m_{c,k}$. For type II cutoff (3.185) we find instead:

$$\partial_t Z_{\bar{A},k} = \frac{N\, k^{d-4}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)} \left[\frac{24-d}{6}\frac{d-2-\eta_{a,k}}{1+\tilde{m}^2_{a,k}} + \frac{1}{3}\frac{d-2-\eta_{c,k}}{1+\tilde{m}^2_{c,k}} \right] . \tag{3.21}$$

These equations show that the running of $Z_{\bar{A},k}$ is determined by the running of the dynamical set of couplings $\{Z_{a,k}, Z_{c,k}, m^2_{a,k}, m^2_{c,k}\}$. The anomalous dimensions in (3.20) and (3.21) are defined by

$$\eta_{a,k} = -\partial_t \log\, Z_{a,k} \qquad\qquad \eta_{c,k} = -\partial_t \log\, Z_{c,k} \, . \tag{3.22}$$

If we set $d = 4$ in equation (3.20) and if we neglect the masses, the beta function becomes cutoff shape independent:

$$\partial_t Z_{\bar{A},k} = \frac{N}{(4\pi)^2} \left(\frac{22}{3} - \frac{2}{3}\eta_{a,k} - \frac{1}{3}\eta_{c,k} \right) . \tag{3.23}$$

The beta function for the gauge coupling is readily found differentiating (3.23) with respect to the RG parameter, i.e. $\partial_t Z_{\bar{A},k} = -\partial_t g_k / 2g_k^3$, thus:

$$\partial_t g_k = \frac{N}{(4\pi)^2}\left(-\frac{11}{3} + \frac{1}{3}\eta_{a,k} + \frac{1}{6}\eta_{c,k}\right) g_k^3 \,. \tag{3.24}$$

Later we will see that the anomalous dimensions of the fluctuation fields are proportional to g_k^2, so in (3.24) only the first term goes like g_k^3: this is just the standard one-loop running of the gauge coupling [9, 11, 12]. Note that the beta function is negative for small coupling: this show that non-abelian gauge theories are asymptotically free. If instead we set $d = 4$ in equation (3.21) and if we neglect the masses, we find the following beta functions for type II cutoff:

$$\partial_t Z_{\bar{A},k} = \frac{N}{(4\pi)^2}\left(\frac{22}{3} - \frac{10}{3}\eta_{a,k} - \frac{1}{3}\eta_{c,k}\right) \tag{3.25}$$

and

$$\partial_t g_k = \frac{N}{(4\pi)^2}\left(-\frac{11}{3} + \frac{5}{3}\eta_{a,k} + \frac{1}{6}\eta_{c,k}\right) g_k^3 \,. \tag{3.26}$$

The beta functions are still cutoff shape independent, but a comparison between (3.23-3.24) and (3.25-3.26) reveals that the coefficients that multiply the anomalous dimensions depend on the cutoff operator we choose. However, these coefficients are also gauge dependent and for $\alpha_k = 0$ they may have closer values.

The flow equations for the gauge coupling (3.24) and (3.26) need the specification of the anomalous dimensions of the fluctuation field and of the ghost fields to be completely determined. This is the actual manifestation of the fact that the flow of the gEAA, here represented by g_k, is not closed but is given in terms of the flow of the fluctuation couplings, here represented only by $\eta_{a,k}$ and $\eta_{c,k}$ since we fixed $m_{a,k} = m_{c,k} = 0$ and $\alpha_k = 1$. We propose now three different approximations that allows us to obtain closed beta functions for g_k. The first approximation is just the one-loop approximation that consists in setting $\eta_{a,k} = \eta_{c,k} = 0$. The second approximation consists in choosing:

$$\eta_{\bar{A},k} = \eta_{a,k} \qquad\qquad \eta_{c,k} = 0 \,, \tag{3.27}$$

where the anomalous dimension of the background field is obtained from (3.23) or (3.25) using:

$$\eta_{\bar{A},k} = -\partial_t \log Z_{\bar{A},k} \,. \tag{3.28}$$

In all previous works on the bEAA for non-abelian gauge theories [51, 56, 57] this approximation was used. Inserting the approximation (3.27) in (3.24) or (3.26) transforms these equations in linear systems for $\partial_t g_k$. These can be easily solved to yield:

$$\partial_t g_k = -\frac{1}{(4\pi)^2}\frac{11N}{3}\frac{g_k^3}{1-\frac{2N}{3}\frac{g_k^2}{(4\pi)^2}}\,, \tag{3.29}$$

in the case of type I cutoff, and

$$\partial_t g_k = -\frac{1}{(4\pi)^2}\frac{11N}{3}\frac{g_k^3}{1-\frac{10N}{3}\frac{g_k^2}{(4\pi)^2}}\,, \tag{3.30}$$

in the case of type II cutoff. Equation (3.29) or (3.30) are rational functions of the gauge coupling. This shows how the approximation (3.27) implements the resummation of an infinite number of perturbative diagrams. At this point is not clear which contributions are actually resummed by the identification (3.27) between the fluctuation and background anomalous dimensions and thus the validity of this approximation is questionable.

For small coupling $g_k \ll 1$ we can expand these equations. For type I we find:

$$\partial_t g_k = -\frac{g_k^3}{(4\pi)^2}\frac{11}{3}N - \frac{g_k^5}{(4\pi)^4}\frac{22}{9}N^2 + O\left(g_k^7\right)\,, \tag{3.31}$$

while for type II we get:

$$\partial_t g_k = -\frac{g_k^3}{(4\pi)^2}\frac{11}{3}N - \frac{g_k^5}{(4\pi)^4}\frac{110}{9}N^2 + O\left(g_k^7\right)\,. \tag{3.32}$$

As we already noticed, the one-loop contributions are equal and agree with the perturbative result, the two-loop contributions are instead quite different. If we compare them with the perturbative result $-\frac{g_k^5}{(4\pi)^4}\frac{102}{9}N^2$ [48] we see that type I is 79% smaller while the type II is just 8% bigger. What we learn is that the approximation made in (3.27) implements an RG improvement which strongly depends, at least, on the cutoff operator employed and is in general not under control.

The third way to obtain a closed beta function for the gauge coupling, that we propose here for the first time, is to first calculate the anomalous dimensions $\eta_{a,k}$ and $\eta_{c,k}$ and then reinsert them back in (3.24) or (3.26). This means that we are considering the flow of the full bEAA which takes place in the enlarged theory space of all functionals of the fluctuation gauge field, of the ghost fields and of the background gauge field. In the truncation we are considering, given by equations (3.18) and (3.19), this is the sixth-dimensional space parametrized by the coupling constants $\{g_k, Z_{a,k}, Z_{c,k}, m_{a,k}, m_{c,k}, \alpha_k\}$. We will see that the anomalous dimensions of the fluctuation gauge field and of the ghost fields are determined by a linear system that can be solved to give them as a function of the gauge coupling, the gauge-fixing parameter and of the masses. Since it turns out that the ghost running mass $m_{c,k}$ has vanishing beta function and can thus be fixed to be zero at all scales, we set it to this value from now on. Contrary to the beta function of the gauge coupling, all the other are the same for both cutoff operator choices. The explicit derivation of these beta functions is done in section 3.5.6 and 3.5.7 using the approach introduced in section 3.3.4 based on the flow equations for the zero-field proper-vertices of the bEAA.

We start by considering the linear system satisfied by the fluctuation and ghost anomalous dimensions. We will solve for these anomalous dimensions and we will insert the result back into the beta functions for the gauge coupling (3.24) or (3.26). In this way we obtain a closed form for $\partial_t g_k$.

The general form of $\eta_{a,k}$ and $\eta_{c,k}$ are derived in section 3.5.6 and 3.5.7 for general cutoff shape function, value of the gauge-fixing parameter and dimension. The explicit forms are given in equation (3.221) and (3.242) respectively. Since previously we calculated the beta function for the gauge coupling in the gauge $\alpha_k = 1$, we will consider these anomalous dimensions in this gauge also. For this choice and employing the optimized cutoff shape function they are given in equation (3.222) from section 3.5.6 and in equation (3.243) from section 3.5.7,

$$\begin{aligned}\eta_{a,k} &= \frac{g_k^2 N\, k^{d-4}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left[\frac{4\,(3d-1)}{d(d+2)}\frac{1}{(1+\tilde{m}_{a,k}^2)^4}\right. \\ &\quad \left. -\frac{20}{d(d+2)}\frac{d+2-\eta_{a,k}}{(1+\tilde{m}_{a,k}^2)^3} - \frac{4}{d(d+2)}\frac{1}{(1+\tilde{m}_{c,k}^2)^4}\right] \end{aligned} \tag{3.33}$$

and

$$\eta_{c,k} = -\frac{g_k^2 N\, k^{d-4}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\frac{4}{d(d+2)}\frac{2+d-\eta_{a,k}}{(1+\tilde{m}_{a,k}^2)^2(1+\tilde{m}_{c,k}^2)}\,. \tag{3.34}$$

In particular, setting $d=4$ in (3.33) and (3.34) gives the following linear system:

$$\begin{aligned}\eta_{a,k} &= -\frac{g_k^2 N}{(4\pi)^{d/2}}\left[\frac{1}{6} - \frac{11}{6}\frac{1}{(1+\tilde{m}_{a,k}^2)^4} + \frac{5}{6}\frac{6-\eta_{a,k}}{(1+\tilde{m}_{a,k}^2)^3}\right] \\ \eta_{c,k} &= -\frac{g_k^2 N}{(4\pi)^{d/2}}\frac{6-\eta_{a,k}}{6(1+\tilde{m}_{a,k}^2)^2}\,. \end{aligned} \tag{3.35}$$

Note first that the anomalous dimension of the ghost field is entirely determined by the gauge fluctuation anomalous dimension and that this can be obtained from the first equation in (3.35). This fact depends on the gauge, if we consider these anomalous dimensions for $m_{a,k}=0$ but for general value of the gauge-fixing parameter α_k we find, equation (3.224) of section 3.5.6 and equation (3.245) of section 3.5.7, the following forms:

$$\begin{aligned}\eta_{a,k} &= \frac{g_k^2 N}{(4\pi)^2}\left[-\frac{13-3\alpha_k}{3}+\frac{31-102\alpha_k+144\alpha_k^2-58\alpha_k^3-15\alpha_k^4+48\alpha_k^3\log\alpha_k}{36(1-\alpha_k)^3}\eta_{a,k}\right]\\ \eta_{c,k} &= \frac{g_k^2 N}{(4\pi)^2}\left[\frac{\alpha_k-3}{2}+\frac{(1+\alpha_k)(1-4\alpha_k+3\alpha_k^2-2\alpha_k^2\log\alpha_k)}{8(1-\alpha_k)^3}\eta_{a,k}\right.\\ &\left.\quad+\frac{1-4\alpha_k+11\alpha_k^2-8\alpha_k^3+2\alpha_k^2(1+2\alpha_k)\log\alpha_k}{8(1-\alpha_k)^3}\eta_{c,k}\right]. \end{aligned} \tag{3.36}$$

We see from (3.36) that for a general value of the gauge-fixing parameter, the anomalous dimensions are determined by solving a linear system. An analogous system, within a different implementation of the cutoff has been obtained in the non-background EAA approach to non-abelian gauge theories in [59]. Considering that the anomalous dimensions in the rhs of (3.36) are at least of order g_k^2 we find to lowest order the following forms:

$$\begin{aligned}\eta_{a,k} &= -\frac{g_k^2 N}{(4\pi)^2}\frac{13-3\alpha_k}{3}+O\left(g_k^4,\tilde{m}_{a,k}^2\right)\\ \eta_{c,k} &= -\frac{g_k^2 N}{(4\pi)^2}\frac{\alpha_k-3}{2}+O\left(g_k^4,\tilde{m}_{a,k}^2\right). \end{aligned} \tag{3.37}$$

The terms on the rhs of (3.37) are scheme independent and agree with the perturbative ones [59, 9].

By solving the first equation in (3.35) we obtain the anomalous dimension of the gauge fluctuation field as a function of $m_{a,k}$ and g_k alone:

$$\eta_{a,k} = -\frac{g_k^2 N}{(4\pi)^2}\frac{\frac{1}{6}+\frac{5}{(1+\tilde{m}_{a,k}^2)^3}-\frac{11}{6(1+\tilde{m}_{a,k}^2)^4}}{1-\frac{g_k^2 N}{(4\pi)^2}\frac{5}{6(1+\tilde{m}_{a,k}^2)^3}}. \tag{3.38}$$

The ghost anomalous dimensions is obtained as a function of $m_{a,k}$ and g_k by inserting (3.38) in (3.35):

$$\eta_{c,k} = -\frac{g_k^2 N}{(4\pi)^{d/2}}\frac{1}{(1+\tilde{m}_{a,k}^2)^2}-\frac{g_k^4 N^2}{(4\pi)^4}\frac{\frac{1}{6(1+\tilde{m}_{a,k}^2)^2}+\frac{5}{(1+\tilde{m}_{a,k}^2)^5}-\frac{11}{6(1+\tilde{m}_{a,k}^2)^6}}{1-\frac{g_k^2 N}{(4\pi)^2}\frac{5}{6(1+\tilde{m}_{a,k}^2)^3}}. \tag{3.39}$$

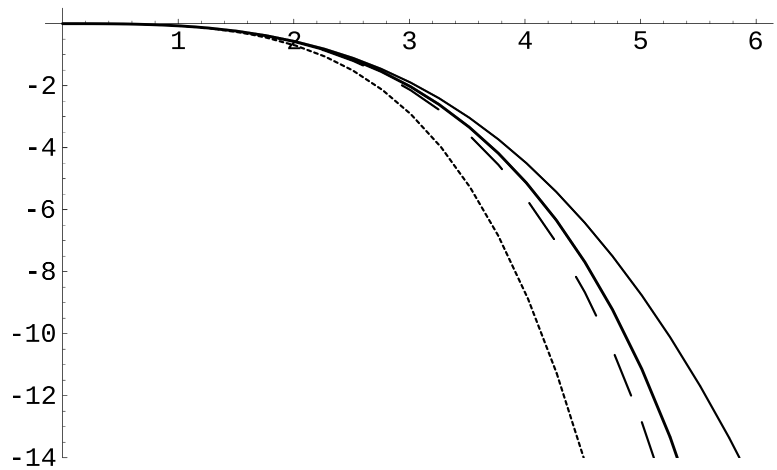


Figure 3.1: The three forms for the type I beta function for the gauge coupling together with the two-loop one. One-loop (continuous), two-loop (short dashed), improved as in (3.29) (long dashed) and improved as in (3.40) (thick).

For general value of the gauge-fixing parameter the form for these anomalous dimensions can also be obtained analytically but the resulting forms are quite cumbersome and so we do not report them here.

We can now turn back to the beta functions for the gauge coupling and close them by inserting the functions (3.38) and (3.39) in (3.24) or in (3.26). In particular, if we set $m_{a,k} = 0$, we find for the type I beta function the following RG improved result:

$$\partial_t g_k = -\frac{g_k^3 N}{(4\pi)^2} \frac{\frac{11}{3} - \frac{16}{9}\frac{g_k^2 N}{(4\pi)^2} - \frac{5}{108}\frac{g_k^4 N^2}{(4\pi)^4}}{1 - \frac{5}{6}\frac{g_k^2 N}{(4\pi)^2}} \,. \tag{3.40}$$

For the type II beta function we find instead the following RG improved form:

$$\partial_t g_k = -\frac{g_k^3 N}{(4\pi)^2} \frac{\frac{11}{3} - \frac{16}{6}\frac{g_k^2 N}{(4\pi)^2} - \frac{5}{108}\frac{g_k^4 N^2}{(4\pi)^4}}{1 - \frac{5}{6}\frac{g_k^2 N}{(4\pi)^2}} \,. \tag{3.41}$$

Note that the beta function (3.40) and (3.41) differ now only in the coefficient of the second term in the denominator. These beta functions clearly show how the influence of the fluctuation couplings, here $Z_{a,k}$ and $Z_{c,k}$, is encoded in the flow of the gauge coupling as a particular kind of improvement. The type I beta functions (3.29) and (3.40) are given in Figure 3.1 together with the one-loop and two-loops results. The analogous figure for the type II beta functions (3.30) and (3.41) is Figure 3.2. We note that for both cutoff types, the

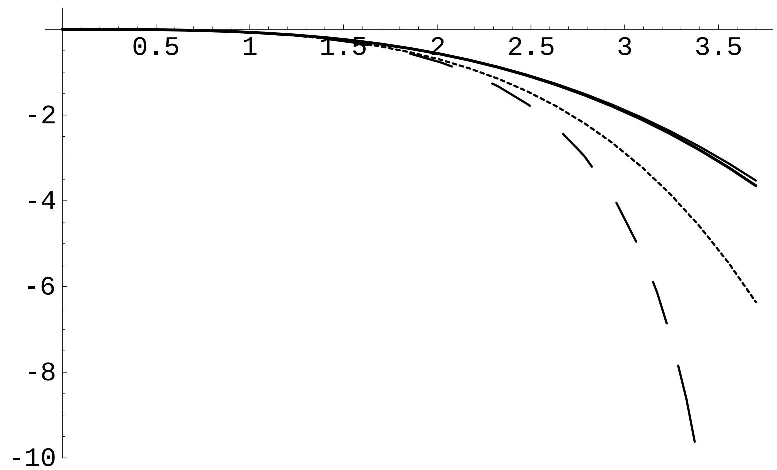


Figure 3.2: The three forms for the type II beta function for the gauge coupling together with the two-loop one. One-loop (continuous), two-loop (short dashed), improved as in (3.30) (long dashed) and improved as in (3.41) (thick).

beta functions that account for the non-trivial form of the fluctuation and ghost anomalous dimensions are in between the one-loop and two-loop beta functions. We interpret this fact as the indication that flow described by these beta functions is closer to the exact one, even if in the type II case the beta function differs slightly from the one-loop one. Considering that the anomalous dimensions we used to obtain a closed form for these beta functions have been calcualted in the gauge $\alpha_k = 1$ we speculate that the situation may improve if we consider instead the Landau gauge [68]. Note also that the beta functions obtined employing the identification (3.27) are those which approximate in a better way the two-loop beta function even if we don't have a proper justification for this approxiamtion. All the improved beta functions we considered, (3.29), (3.30), (3.40) and (3.41) are reliable only in the high energy asymptotic free regime since they all diverge for finite non-zero g_k. To be able to extend the flow of the guage coupling toward the IR, a more general class of truncations has to be considered [57, 58].

We integrate now the one-loop beta function for the gauge coupling from the UV scale Λ to the IR scale k. After an elementary integration we find:

$$\frac{1}{g_\Lambda^2} - \frac{1}{g_k^2} = \frac{1}{(4\pi)^2}\frac{22N}{3}\log\frac{\Lambda}{k}\,. \tag{3.42}$$

We solve (3.42) for g_Λ^2 as we are interested to study the theory in the $\Lambda \to \infty$ regime where

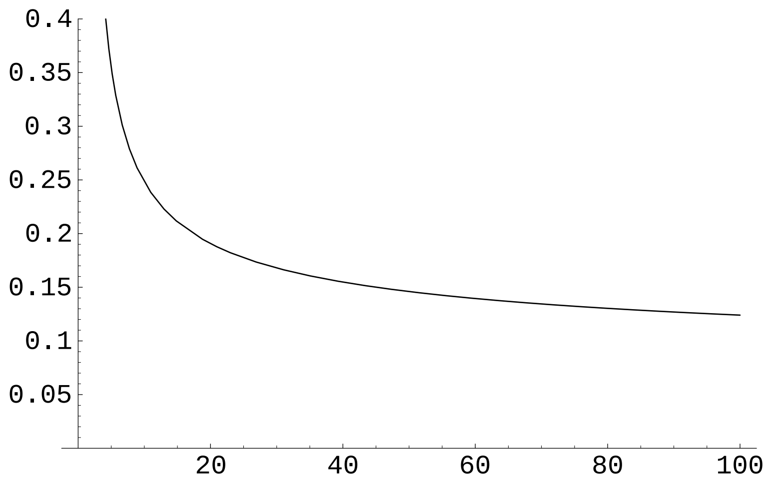


Figure 3.3: Running non-abelian coupling constant $\alpha_\Lambda = \frac{g_\Lambda^2}{4\pi}$ as a function of the UV scale Λ from (3.45).

the truncation ansatz for the gEAA (3.17) is valid due to asymptotic freedom:

$$g_\Lambda^2 = \frac{g_k^2}{1 + \frac{g_k^2}{(4\pi)^2}\frac{22N}{3}\log\frac{\Lambda}{k}}\,. \tag{3.43}$$

A mass scale M can be defined by the relation:

$$1 = \frac{g_k^2}{(4\pi)^2}\frac{22N}{3}\log\frac{k}{M}, \tag{3.44}$$

if we insert (3.44) in (3.43) we can write:

$$\alpha_\Lambda = \frac{2\pi}{\frac{11}{3}N\,\log\frac{\Lambda}{M}}\,, \tag{3.45}$$

where we defined $\alpha_\Lambda = \frac{g_\Lambda^2}{4\pi}$. This is the standard result found in perturbation theory and this derivation shows how this fundamental physical phenomena arises within the bEAA formalism. The scale dependent QCD parameter α_Λ is plotted in Figure 3.3 as a function of Λ. The picture clearly shows that the coupling constant is decreasing in magnitude as we go to higher and higher scales.

Next, we consider the flow of the gauge fluctuation mass. This beta function has been calculated in section 3.5.6 and is given in equations (3.216) and (3.217) for general cutoff shape function, value of the gauge-fixing parameter and dimension. In the case of interest,

$d = 4$ and $\alpha_k = 1$, if we employ the optimized cutoff shape function we finds equation (3.218) from section 3.5.6. In terms of the dimensionless variable $m^2_{a,k} = k^2 \tilde{m}^2_{a,k}$, we find the following form:

$$\begin{aligned}\partial_t \tilde{m}^2_{a,k} &= -(2 - \eta_{a,k})\,\tilde{m}^2_{a,k} + \frac{g_k^2 N}{(4\pi)^2}\left[-\frac{6 - \eta_{a,k}}{(1 + \tilde{m}^2_{a,k})^2}\right. \\ &\qquad \left. + \frac{3}{8}\frac{8 - \eta_{a,k}}{(1 + \tilde{m}^2_{a,k})^3} + \frac{8 - \eta_{c,k}}{24}\right] . \end{aligned} \tag{3.46}$$

We are now in the position to study the full closed system for $\partial_t g_k$ and $\partial_t \tilde{m}_{a,k}$ obtained by closing (3.24) or (3.26) and (3.46) using (3.38) and (3.39) .The beta functions so obtained are quite complicated polynomial functions of $\tilde{m}^2_{a,k}$ and we give them here only to lowest order in g_k. They read:

$$\begin{aligned}\partial_t \tilde{m}^2_{a,k} &= -2\tilde{m}^2_{a,k} - \frac{g_k^2 N}{(4\pi)^2}\frac{16 + 66\,\tilde{m}^2_{a,k} + 58\tilde{m}^4_{a,k} - 2\tilde{m}^6_{a,k} + 2\tilde{m}^8_{a,k} + \tilde{m}^{10}_{a,k}}{6(1 + \tilde{m}^2_{a,k})^4} + O\left(g_k^4\right) \\ \partial_t g_k &= -\frac{g_k^3 N}{(4\pi)^2}\frac{11 - \tilde{m}^2_{a,k} + \tilde{m}^4_{a,k} + \tilde{m}^6_{a,k}}{3(1 + \tilde{m}^2_{a,k})^3} + O\left(g_k^4\right) . \end{aligned} \tag{3.47}$$

We can solve both the approximate system (3.47) and the full one numerically. The results of these integrations are given in Figure 3.4 and Figure 3.5. By the numerical study we made, we learned that in order to obtain a vanishing gluon mass in the IR for $k \to 0$, so that the full effective action is gauge invariant, we need to tune the initial value for the gluon mass to the value $m^2_{a,\Lambda} \sim -\Lambda^2$. This indicates that the flucutation couplings, i.e non-trivial truncations of the rEAA, not only exert an influence on the running of the physical couplings, but need also a non-trivial renormalization in the UV. This shows again how important it is to study the flow of the full bEAA.

Finally we consider the running of the gauge-fixing parameter. The beta function for α_k is derived in section 3.5.6 and is given for general cutoff shape function and dimension in equations (3.229) and (3.230). In $d = 4$, for $\tilde{m}_{a,k} = 0$ and employing the optimized cutoff shape function, we find the following form:

$$\partial_t \alpha_k = -\eta_{a,k}\alpha_k + \frac{g_k^2 N}{(4\pi)^2}\frac{\alpha_k^2\left(5 - 15\alpha_k + 18\alpha_k^2 - 5\alpha_k^3 - 3\alpha_k^4 + 6\alpha_k^3 \log \alpha_k\right)}{12\left(1 - \alpha_k\right)^3}\eta_{a,k} . \tag{3.48}$$

In we insert (3.38) in (3.48) we obtain the complete beta function for α_k in the case $\tilde{m}_{a,k} = 0$. This beta function has a fixed point at $\alpha_k = 0$ where it has slope $\frac{1}{(4\pi)^2}\frac{10N}{3}g_k^2\left(1 - \frac{5Ng_k^2}{6(4\pi)^2}\right)^{-1}$

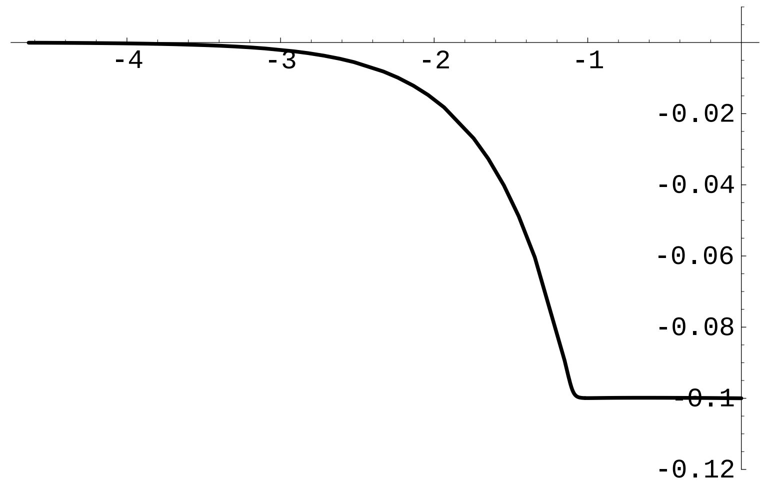


Figure 3.4: Flow of the gauge fluctuation mass obtained by solving the complete version of system (3.47). $m^2_{a,k}/\Lambda^2$ is plotted as a function of $\log\frac{k}{\Lambda}$. Note that to obtain a vanishing mass for $k \to 0$ we need to tune $m^2_{a,\Lambda} \sim -\Lambda^2$.

and so the fixed point is IR attractive if $g_k^2 < \frac{6(4\pi)^2}{5N}$ near the fixed point. For $N = 3$ this means $g_k < 63$ and thus the gauge-fixing parameter is attracted to zero, i.e. to the Landau gauge, in the IR. This confirms the expectation that Landau gauge represents a fixed point of the flow of the guage-fixing parameter [16].

3.2.2.2 Non-local truncations

To be able to explore the IR physics of non-abelian gauge theories we need to consider a more general class of truncations that retain at least an infinite number of invariants and coupling constants. A way to do so is to generalize the truncation (3.17) for the gEAA to what we call a "curvature expansion". For non-abelian gauge theories this expansion has the following form:

$$\bar{\Gamma}_k[A] = \frac{1}{4}\int d^dx\, F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{4}\int d^dx\, F^a_{\mu\nu}\Pi_k\left(-D^2\right)^{ab}F^{b\mu\nu} + O\left(F^3\right)\,, \tag{3.49}$$

where $\Pi_k(x)$ is the running vacuum polarization function. If we expand (3.49) in powers of the fluctuation field we find to lowest order

$$\bar{\Gamma}_k[Z^{1/2}_{\bar{A},k}\bar{A} + Z^{1/2}_{a,k}a] = Z_{\bar{A},k}\bar{\Gamma}_k[\bar{A}] + O(a)\,, \tag{3.50}$$

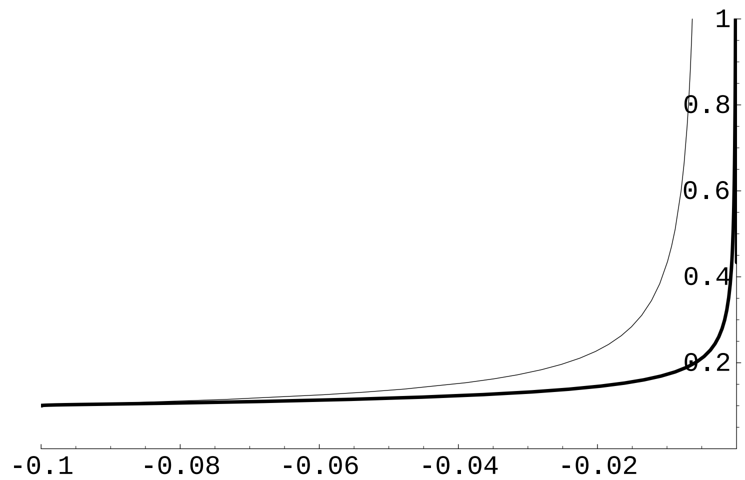


Figure 3.5: Flow of the gauge fluctuation mass in units of the UV cutoff $m_{a,k}^2/\Lambda^2$ (x-axis) and of the gauge coupling g_k obtained by solving the system (3.47) (continuous line) and the full version of it (thick line). Tuning $m_{a,\Lambda}^2 \sim -\Lambda^2$ the fluctuation mass rapidly goes to zero and the gauge coupling starts to grow.

since the background covariant Laplacian does not renormalize. Thus the background wave-function renormalization and so the guage coupling enters the flow of $\Pi_k(x)$ only as an overall factor.

To calculate the complete non-perturbative running of the vacuum polarization function obtained by inserting (3.49) in the flow equation (3.16) for the gEAA is a difficult task. In principle, using the techniques of sect 3.3.4, it is possible to project out the flow equation for $\Pi_k(x)$, but this is still quite complicated. Also, a truncation ansatz involving a structure function has to be considered in the ghost sector. This is particularly relevant in view of the ghost IR enhancement found in Dyson-Schwinger (DS) equation studies in Landau gauge $\alpha_k = 0$ non-abelian gauge theories [61, 62]. Within the EAA approach, several works in this direction have already been done. These consider truncations similar to the one in (3.49), but all these works are within the non-background approach to non-abelian gauge theories where gauge invariance is explicitly broken along the flow and the field modes are regulated using the flat space Laplacian [60, 63, 67, 64, 65, 66]. On the contrary, the ansatz (3.49) is covariant and thus gives, in the proper-vertex hierarchy described in section 3.3.3, non-trivial contributions to all vertices. In the works just mentioned, all vertices are considered as bare. Still, the results found in this more naive approach are stimulating and in general accordance with DS studies. The question is whether the covariant background approach can be more efficient in penetrating into the IR physics than the non-background ones. This kind of applications of the bEAA are now practicable using the formalism developed in section

3.3.4 and will be subject of future investigations. For the moment we will consider only the one-loop running of the vacuum polarization function $\Pi_k(x)$ induced by the local action (3.17).

To extract the one-loop running of the vacuum polarization function, we employ the non-local heat kernel expansion to calculate the functional trace in the rhs of the flow equation (3.16) for the gEAA. The details are given in section 3.5.5. In $d = 4$ and by employing the type II cutoff operator introduced in the previous section together with the optimized shape function, we find (3.202):

$$\partial_t \left[Z_{\bar{A},k} \Pi_k(x) \right] = -\partial_t Z_{\bar{A},k} + \frac{N}{(4\pi)^2} \left[\frac{22}{3} - \left(\frac{22}{3} + \frac{8k^2}{3x} \right) \sqrt{1 - \frac{4k^2}{x}} \theta(x - 4k^2) \right] , \tag{3.51}$$

where x stands for the covariant Laplacian Δ. Inserting into equation (3.51) the beta function for the wave-function renormalization of the background field, from equation (3.25) with $\eta_{a,k} = \eta_{c,k} = 0$, the constant term in the rhs cancels and we are left with the following simple formula:

$$\partial_t \Pi_k(x) = \eta_{\bar{A},k} \Pi_k(x) + \frac{g_k^2 N}{(4\pi)^2} f\left(\frac{x}{k^2} \right) . \tag{3.52}$$

In (3.52) we used the relation $g_k^2 = Z_{\bar{A},k}^{-1}$ and the definition of the anomalous dimension of the background field (3.28). The function $f(u)$ is found to be:

$$f(u) = -\left(\frac{22}{3} + \frac{8}{3u} \right) \sqrt{1 - \frac{4}{u}} \theta(u - 4) . \tag{3.53}$$

Note that the k dependence in (3.52) enters only via the combination $u = x/k^2$. In (3.53), $\eta_{\bar{A},k} \Pi_k(x)$ is at least of order g_k^4 and we discard it here. Also, we set the running coupling to its renormalized value g_0. We can integrate now the flow equation (3.52) from the IR scale k to the UV scale Λ thus finding:

$$\Pi_\Lambda(x) - \Pi_k(x) = \frac{g_0^2 N}{(4\pi)^2} \int_{x/\Lambda^2}^{x/k^2} \frac{du}{u} f(u) , \tag{3.54}$$

where we changed variable to $u = x/k^2$. Thanks to the cancellation between the constant term in (3.51) and the one-loop term of the beta function of the wave-function renormalization of the background field, the integral in (3.54) is finite in the limit $\Lambda \to \infty$. In this limit, the vacuum polarization function goes to its boundary value, i.e. $\Pi_\Lambda(x) = 0$. The vacuum polarization function at the scale k is finally found to be:

$$\begin{aligned}\Pi_k(x) &= -\frac{g_0^2 N}{(4\pi)^2}\left\{\left(\frac{128}{9}+\frac{16k^2}{9x}\right)\sqrt{1-\frac{4k^2}{x}}\right.\\ &\left.-\frac{22}{3}\left[\log\frac{x}{k^2}+\log\frac{1}{2}\left(1+\sqrt{1-\frac{4k^2}{x}}\right)\right]\right\}\theta(x-4k^2)\,. \qquad (3.55)\end{aligned}$$

From (3.55) we see that we cannot extend (3.55) down to arbitrary low energies, i.e. we cannot send $k \to 0$ since the first logarithm term diverges in this limit. For $k^2 \ll x$ we find the following contribution to the gEAA:

$$g_0^2 \frac{N}{64\pi^2}\int d^dx\, F^a_{\mu\nu}\left[\frac{22}{3}\log\frac{(-D^2)^{ab}}{k^2}-\frac{128}{9}\delta^{ab}\right]F^{b\mu\nu}\,. \qquad (3.56)$$

We interpret the obstruction to the limit $k \to 0$ in (3.55) as a signal of the breakdown of the approximation used in its derivation, where we considered the flow of $\Pi_k(x)$ as driven only by the operator $\frac{1}{4}\int d^dx F^2$. In order to be able to continue the flow of the gEAA in the deep IR, we need the full non-perturbative power of the flow equation (3.16) that becomes available if we insert the complete ansatz (3.49) in the rhs side of it [68].

3.3 Background effective average action (bEAA)

In this section we generalize the construction of the EAA done in Chapter 2 to the case of gauge theories, in particular to non-abelian gauge theories. The important point in the construction is obviously that gauge invariance has to be preserved after the introduction of the cutoff. We have seen in the case of matter fields how it is possible, within the EAA framework, to covariantly cutoff field modes by defining a cutoff action using covariant operators in the external fields, like the gauge Laplacian in QED or the Beltrami-Laplace operator in the case of matter fields on curved backgrounds. We learned also the importance, for the cutoff action, to be quadratic in the fields in order to obtain a one-loop like flow equation. This means that if we try to introduce in gauge theories a cutoff by simply taking as cutoff kernel a function of the covariant Laplacian, this will spoil the simple one loop structure of the flow. Still the EAA will not be gauge invariant because of the non-covariant coupling of the gauge field to the source. The way out is to employ the background field method as was first done in [51]. The quantum field A_μ is linearly split between the background field $\bar{A}_\mu$

and the fluctuation field a_μ:

$$A_\mu = \bar{A}_\mu + a_\mu \tag{3.57}$$

and the cutoff action is taken to be quadratic in the fluctuation field, while the cutoff operator is constructed with the background field. We consider thus a cutoff action of the following form:

$$\Delta S_k[\varphi; \bar{A}] = \frac{1}{2}\int d^dx\, \varphi\, R_k[\bar{A}]\varphi\,, \tag{3.58}$$

here $\varphi = (a, \bar{c}, c)$ is the field multiplet, combining the fluctuating field a_μ and the ghost fields $\bar{c}$ and c. The background effective average action (bEAA) is constructed introducing the cutoff action (3.58) into the integro-differential definition of the background effective action, equation (C.39) from Appendix C, to obtain:[4]

$$e^{-\Gamma_k[\varphi;\bar{A}]} = \int D\chi \exp\left(-S[\chi+\varphi; \bar{A}] - \Delta S_k[\chi; \bar{A}] + \int d^dx\, \Gamma_k^{(1;0)}[\varphi; \bar{A}]\chi\right)\,. \tag{3.59}$$

The multiplet field χ in (3.59) has zero vacuum expectation value $\langle\chi\rangle = 0$. The bare action, as constructed in Appendix C, is

$$S[\varphi; \bar{A}] = S[\bar{A} + a] + S_{gf}[a; \bar{A}] + S_{gh}[a, \bar{c}, c; \bar{A}]\,, \tag{3.60}$$

with the following background gauge-fixing action

$$S_{gf}[a; \bar{A}] = \frac{1}{2\alpha}\int d^dx\, \bar{D}_\mu a^\mu \bar{D}_\nu a^\nu \tag{3.61}$$

and the following background ghost action

$$S_{gh}[a, \bar{c}, c; \bar{A}] = \int d^dx\, \bar{D}_\mu \bar{c}\, D^\mu c = \int d^dx\, \bar{D}_\mu \bar{c}\left(\bar{D}^\mu + g a^\mu\right) c\,. \tag{3.62}$$

The bEAA is invariant under combined physical plus background gauge transformations:

$$(\delta + \bar{\delta})\Gamma_k[\varphi; \bar{A}] = 0\,. \tag{3.63}$$

We can now define a gauge covariant functional that we will call gauge covariant effective average action (gEAA). This is defined by setting in the bEAA $\varphi = 0$, or equivalently

[4]Functional derivatives of a functionals depending on many arguments are indicated by the notation $\Gamma^{(n_1,n_2,\ldots)}[....]$.

$A_\mu = \bar{A}_\mu$ and $\bar{c} = c = 0$:

$$\bar{\Gamma}_k[\bar{A}] = \Gamma_k[0;\bar{A}] \,. \tag{3.64}$$

This is equivalent to the parametrization of the bEAA as the sum of a functional of the full quantum field $A = \bar{A} + a$, the gEAA, and a "remainder functional" $\hat{\Gamma}_k[\varphi;\bar{A}]$ (rEAA) which is still a functional of both the fluctuation multiplet and the background gauge field:

$$\Gamma_k[\varphi;\bar{A}] = \bar{\Gamma}_k[\bar{A}+a] + \hat{\Gamma}_k[\varphi;\bar{A}] \,. \tag{3.65}$$

The gEAA constructed in this way is invariant under physical gauge transformations:

$$\delta\bar{\Gamma}_k[\bar{A}] = 0 \,, \tag{3.66}$$

while the rEAA is invariant under simultaneous physical and background gauge transformations as the full bEAA.

3.3.1 Exact flow equations for the bEAA

In this section we derive the exact flow equation that the bEAA satisfies. We will follow the derivation of the exact flow equation for the standard EAA given in section 2.2.1 of Chapter 2. Differentiating the integro-differential equation for the bEAA (3.59) with respect to the "RG time" $t = \log k/k_0$ we find:

$$\begin{aligned} e^{-\Gamma_k[\varphi;\bar{A}]}\partial_t\Gamma_k[\varphi;\bar{A}] &= \int D\chi \left(\partial_t \Delta S_k[\chi;\bar{A}] - \int d^dx \partial_t \Gamma_k^{(1;0)}[\varphi;\bar{A}]\chi\right) \times \\ &\quad \times e^{-S[\varphi+\chi;\bar{A}] - \Delta S_k[\chi;\bar{A}] + \int \Gamma_k^{(1;0)}[\varphi;\bar{A}]\chi} \,. \end{aligned} \tag{3.67}$$

Expressing the terms on the rhs of (3.67) as expectation values using (B.2) and (3.58) we can rewrite (3.67) as[5]

$$\begin{aligned} \partial_t\Gamma_k[\varphi;\bar{A}] &= \left\langle \partial_t \Delta S_k[\chi;\bar{A}] \right\rangle - \int d^dx\, \partial_t \Gamma_k^{(1;0)}[\varphi;\bar{A}]\, \langle \chi_A \rangle \\ &= \frac{1}{2} \int d^dx\, \langle \chi_A \chi_B \rangle\, \partial_t R_{k,BA} \,, \end{aligned} \tag{3.68}$$

where we used the vanishing vacuum expectation value $\langle \chi_A \rangle = 0$ and the symmetry of the cutoff kernel in $A \leftrightarrow B$. The two-point function of the fluctuation field can be written in

[5] We introduce here the multiplet indices $A, B, \ldots$

terms of the inverse Hessian of the bEAA, where the functional derivative are taken with respect to fluctuation fields, plus the cutoff action:

$$\langle \chi_A \chi_B \rangle = \left(\Gamma_k^{(2;0)}[\varphi;\bar{A}] + \Delta S_k^{(2;0)}[\varphi;\bar{A}] \right)^{-1} = \left(\Gamma_k^{(2;0)}[\varphi;\bar{A}] + R_k[\bar{A}] \right)^{-1} . \tag{3.69}$$

Inserting (3.69) into (3.68) and writing a functional trace in place of the integral, gives:

$$\partial_t \Gamma_k[\varphi;\bar{A}] = \frac{1}{2} \mathrm{Tr} \left(\Gamma_k^{(2;0)}[\varphi;\bar{A}] + R_k[\bar{A}] \right)^{-1} \partial_t R_k[\bar{A}] . \tag{3.70}$$

This is the exact flow equation for the bEAA for non-abelian gauge theories [51] and is the main result of this section. The flow generated by (3.70) has the same general properties as the flow for matter fields described in Chapter 2. If we define the "regularized propagator" as

$$G_k[\varphi;\bar{A}] = \left(\Gamma_k^{(2;0)}[\varphi;\bar{A}] + R_k[\bar{A}] \right)^{-1} , \tag{3.71}$$

then the flow equation for the bEAA (3.70) can be rewritten in the compact form:

$$\partial_t \Gamma_k[\varphi;\bar{A}] = \frac{1}{2} \mathrm{Tr}\, G_k[\varphi;\bar{A}] \partial_t R_k[\bar{A}] . \tag{3.72}$$

As before, the flow equation has a one-loop structure and can be derived as an RG improvement of the one-loop bEAA calculated from the integro-differential equation (3.59)

From (3.65) and (3.72) we can readily write down the flow equation for the gEAA:

$$\partial_t \bar{\Gamma}_k[\bar{A}] = \partial_t \Gamma_k[0;\bar{A}] = \frac{1}{2} \mathrm{Tr}\, G_k[0;\bar{A}] \partial_t R_k[\bar{A}] . \tag{3.73}$$

Note that $\Gamma_k^{(2;0)}[0,\bar{A}]$ is "super-diagonal" if the ghost action is bilinear. In this case we can immediately perform the multiplet trace in the flow equation (3.70). Using the notations $\Gamma_{k,aa} = \Gamma_k^{(2,0,0;0)}[0,0,0;\bar{A}]$, $\Gamma_{k,\bar{c}c} = \Gamma_k^{(0,1,1;0)}[0,0,0;\bar{A}]$, $R_{k,aa} = \Delta S_k^{(2,0,0;0)}[0,0,0;\bar{A}]$ and $R_{k,\bar{c}c} =$

$\Delta S_k^{(0,1,1;0)}[0,0,0;\bar{A}]$ we can write the flow equation in the following matrix form:

$$
\begin{aligned}
\partial_t \bar{\Gamma}_k[\bar{A}] &= \frac{1}{2}\mathrm{Tr}\begin{pmatrix} \Gamma_{k,aa}+R_{k,aa} & 0 & 0 \\ 0 & 0 & \Gamma_{k,\bar{c}c}+R_{k,\bar{c}c} \\ 0 & -\left(\Gamma_{k,\bar{c}c}+R_{k,\bar{c}c}\right) & 0 \end{pmatrix}^{-1} \\
&\quad\times \begin{pmatrix} \partial_t R_{k,aa} & 0 & 0 \\ 0 & 0 & \partial_t R_{k,\bar{c}c} \\ 0 & -\partial_t \dot{R}_{k,\bar{c}c} & 0 \end{pmatrix} \\
&= \frac{1}{2}\mathrm{Tr}\begin{pmatrix} G_{k,aa} & 0 & 0 \\ 0 & 0 & G_{k,\bar{c}c} \\ 0 & -G_{k,\bar{c}c} & 0 \end{pmatrix}\begin{pmatrix} \partial_t R_{k,aa} & 0 & 0 \\ 0 & 0 & \partial_t R_{k,\bar{c}c} \\ 0 & -\partial_t R_{k,\bar{c}c} & 0 \end{pmatrix} \\
&= \frac{1}{2}\mathrm{Tr}\begin{pmatrix} G_{k,aa}\partial_t R_{k,aa} & 0 & 0 \\ 0 & G_{k,\bar{c}c}\partial_t R_{k,\bar{c}c} & 0 \\ 0 & 0 & G_{k,\bar{c}c}\partial_t R_{k,\bar{c}c} \end{pmatrix} \\
&= \frac{1}{2}\mathrm{Tr}\, G_{k,aa}\partial_t R_{k,aa} - \mathrm{Tr}\, G_{k,\bar{c}c}\partial_t R_{k,\bar{c}c}\,. \qquad (3.74)
\end{aligned}
$$

In (3.74) we used the property that the trace over anti-commuting fields carries an additional minus sign. We have found the flow equation for the gEAA in its commonly used form [51]:

$$
\begin{aligned}
\partial_t \bar{\Gamma}_k[\bar{A}] &= \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2,0,0;0)}[0;\bar{A}] + R_{k,aa}[\bar{A}]\right)^{-1}\partial_t R_{k,aa}[\bar{A}] + \\
&\quad -\mathrm{Tr}\left(\Gamma_k^{(0,1,1;0)}[0;\bar{A}] + R_{k,\bar{c}c}[\bar{A}]\right)^{-1}\partial_t R_{k,\bar{c}c}[\bar{A}]\,. \qquad (3.75)
\end{aligned}
$$

It is important to realize that equation (3.75) is not a closed equation for the gEAA since it involves the Hessian taken with respect to the fluctuation field of the bEAA. This implies that for $k \neq 0$ it is necessary to consider the flow in the extended theory space of all functionals of the fields φ and $\bar{A}_\mu$ invariant under simultaneous physical and background gauge transformations. Equation (3.75) can be obtained as an RG improvement of the one-loop effective action (C.50) for non-abelian gauge theories given in Appendix C.

The flow equation for rEAA can be deduced differentiating equation (3.65) and using (3.70) and (3.75):

$$
\partial_t \hat{\Gamma}_k[\varphi;\bar{A}] = \partial_t \bar{\Gamma}_k[\bar{A}+a] - \partial_t \Gamma_k[\varphi;\bar{A}]\,. \qquad (3.76)
$$

In the Appendix to the Chapter we will use the flow equation (3.75) to calculate the beta function for the wave-function renormalization of the background field and for the running vacuum polarization function $\Pi_k(x)$.

3.3.2 Modified Ward-Takahashi identities

The bEAA is invariant under combined physical δ and background $\bar{\delta}$ gauge transformations (3.63). The gauge-fixing action, the ghost action and, more importantly, the cutoff action spoil the physical gauge invariance of the bEAA. This means that the bEAA should obey modified Ward-Takahashi identities under physical gauge transformation [69, 54, 56, 71, 70, 18, 16]. In presence of a non-invariant term in the bare action, the effective action obeys the following Ward-Takahashi (WT) identity, equation (B.69) from Appendix B. In the case we are considering this impies the following relation:

$$\int d^dx\,\langle\delta\phi\rangle\left(\Gamma_k^{(1;0)}[\varphi;\bar{A}]+\Delta S_k[\varphi;\bar{A}]\right)=\left\langle\delta S_{gf}[a;\bar{A}]+\delta S_{gh}[\phi;\bar{A}]+\delta\Delta S_k[\phi;\bar{A}]\right\rangle\,. \tag{3.77}$$

Note that on the lhs of (3.77) we have the sum of bEAA and of the cutoff action, as this combination is the Legendre transform of the generating functional of connected correlations of φ, which enters the general WT identity (B.69).

Physical gauge transformations act linearly on the fields and thus we have $\langle\delta\phi\rangle=\delta\varphi$. We can write them as $\delta\phi=\theta^A\mathcal{G}_A\phi$ where $\mathcal{G}_A$ are the symmetry generators and the θ_A are the parameters of the gauge transformations. See Appendix B at this point for more details. The first term on the lhs of equation (3.77) can be rewritten as

$$\int d^dx\,\Gamma_k^{(1;0)}[\varphi;\bar{A}]\delta\varphi=\delta\Gamma_k[\varphi;\bar{A}]\,, \tag{3.78}$$

in this way we obtain:

$$\mathcal{G}\Gamma_k[\varphi;\bar{A}]=\left\langle\mathcal{G}\left(S_{gf}[a;\bar{A}]+S_{gh}[\phi;\bar{A}]\right)\right\rangle+\left\langle\mathcal{G}\Delta S_k[\phi;\bar{A}]\right\rangle-\mathcal{G}\Delta S_k[\varphi;\bar{A}]\,. \tag{3.79}$$

The first one-loop term on the rhs of (3.79) is the standard one of non-abelian gauge theories [12], while the second one-loop term is the RG modification induced by the cutoff action. Using (3.58), we can rewrite the second term of (3.79) as

$$\left\langle\mathcal{G}\Delta S_k[\phi;\bar{A}]\right\rangle=\int d^dx\,R_{k,AB}[\bar{A}]\,\langle\mathcal{G}\phi_A\phi_B\rangle=\frac{1}{2}\int d^dx\,R_{k,AB}[\bar{A}]\mathcal{G}\,\langle\phi_A\phi_B\rangle\;, \tag{3.80}$$

and the third as:

$$\mathcal{G}\Delta S_k[\varphi;\bar{A}] = \frac{1}{2}\int d^dx\, R_{AB}[\bar{A}]\mathcal{G}(\varphi_A\varphi_B)\,. \tag{3.81}$$

We can combine equation (3.80) and equation (3.81) in the following way:

$$\begin{aligned}\left\langle\mathcal{G}\Delta S_k[\phi;\bar{A}]\right\rangle - \mathcal{G}\Delta S_k[\varphi;\bar{A}] &= \frac{1}{2}\int d^dx\, R_{k,AB}[\bar{A}]\mathcal{G}\left(\langle\phi_A\phi_B\rangle - \varphi_A\varphi_B\right)\\ &= \frac{1}{2}\int d^dx\, R_{k,AB}[\bar{A}]\mathcal{G}G_{k,BA}[\varphi;\bar{A}]\,,\end{aligned} \tag{3.82}$$

where in the second step we used (3.69) and the definition of the regularized propagator (3.71). Finally, using (3.82), the identity (3.79) boils down to:

$$\mathcal{G}\Gamma_k[\varphi;\bar{A}] = \left\langle\mathcal{G}\left(S_{gf}[a;\bar{A}] + S_{gh}[\phi;\bar{A}]\right)\right\rangle + \frac{1}{2}\int d^dx\, R_{k,AB}[\bar{A}]\mathcal{G}G_{k,BA}[\varphi;\bar{A}]\,. \tag{3.83}$$

Equation (3.83) represents the modified WT identity (mWT) the bEAA for non-abelian gauge theories satisfies. The modifying term in (3.83) stems from the introduction of the cutoff, the important point is that it vanishes in the $k \to 0$ limit as the cutoff kernel $R_k[\bar{A}]$ goes to zero. Thus the standard WT identity is recovered in the IR and is satisfied by $k \to 0$ limit of the bEAA $\Gamma_0[\varphi;\bar{A}]$. This property is of fundamental importance for the approach, since it shows that a fully gauge invariant theory is recovered as result of the integration of the flow.

The gEAA does not depend on the fluctuation fields and so we simply have:

$$\mathcal{G}\bar{\Gamma}_k[\bar{A}] = 0\,. \tag{3.84}$$

Also, the gEAA is invariant background gauge transformations $\bar{\delta}\phi = \bar{\theta}^A\bar{\mathcal{G}}_A\phi$:

$$\bar{\mathcal{G}}\bar{\Gamma}_k[\bar{A}] = 0\,. \tag{3.85}$$

This implies that the mWT (3.83) is a constrain only on the form of the rEAA $\hat{\Gamma}_k[\varphi;\bar{A}]$. Also, the linear split symmetry of the quantum is recovered at $k = 0$.

Gauge fixed non-abelian gauge theories are no more invariant under physical gauge symmetries but posses a new global symmetry: BRST invariance. The BRST transformation $\delta_{BRST}\phi$, defined in equation (C.29) of Appendix C, is not linear in the fields but it is nilpotent $\delta_{BRST}^2 = 0$ as proved in Appendix C. Thus the BRST variations $\delta_{BRST}\phi$ are composite operators. To deal with this complication it is useful to introduce additional currents K, the BRST currents, that couple to the BRST variations. The bEAA becomes a functional also of K, which is just a spectator in the Legendre transform that leads to the bEAA, and the following relation holds:

$$\frac{\delta\Gamma_k[\varphi,K;\bar{A}]}{\delta K} = -\frac{\delta W_k[\varphi,K;\bar{A}]}{\delta K} = \langle\delta\phi\rangle \ . \tag{3.86}$$

The general form of the Zinn-Justin equation, equation (B.76) from Appendix B, when applied to the bEAA becomes:

$$\int d^dx \frac{\delta\Gamma_k[\varphi,K;\bar{A}]}{\delta K}\frac{\delta\Gamma_k[\varphi,K;\bar{A}]}{\delta\varphi} = \left\langle\delta\Delta S_k[\phi;\bar{A}]\right\rangle - \int d^dx \frac{\delta\Gamma_k[\varphi,K;\bar{A}]}{\delta K}\frac{\delta\Delta S_k[\varphi;\bar{A}]}{\delta\varphi} \ . \tag{3.87}$$

Using (3.86) we can further manipulate the cutoff terms in (3.87) to get

$$\left\langle\delta\Delta S_k[\phi;\bar{A}]\right\rangle = \int d^dx\, R_{k,AB}[\bar{A}]\left\langle\delta\phi_A\phi_B\right\rangle \qquad \frac{\delta\Delta S_k[\varphi;\bar{A}]}{\delta\varphi_A} = \int d^dx\, R_{k,AB}[\bar{A}]\varphi_B \ ,$$

inserting these relations in (3.87) gives

$$\frac{\delta\Gamma_k[\varphi,K;\bar{A}]}{\delta K_A}\frac{\delta\Gamma_k[\varphi,K;\bar{A}]}{\delta\varphi_A} = \int d^dx\, R_{k,AB}[\bar{A}]\left(\langle\delta\phi_A\phi_B\rangle - \langle\delta\phi_A\rangle\langle\phi_B\rangle\right) \ . \tag{3.88}$$

In equation (3.88) the connected correlation between the fields and the BRST variations appears on the rhs, this can be expressed as a mixed second functional derivative of the generator functional of connected correlations in presence of the BRST currents as follows:

$$\frac{\delta^2 W_k[J,K;\bar{A}]}{\delta J_A\delta K_B} = \langle\delta\phi_A\phi_B\rangle - \langle\delta\phi_A\rangle\langle\phi_B\rangle \ . \tag{3.89}$$

We can write (3.89) in terms of mixed functional derivatives of the bEAA in presence of the BRST currents as:

$$\frac{\delta^2 W_k[J,K;\bar{A}]}{\delta J_A\delta K_B} = -\int d^dx\, \frac{\delta\varphi_C}{\delta J_A}\frac{\delta^2\Gamma_k[\varphi,K;\bar{A}]}{\delta\varphi_C\delta K_B} = -\int d^dx\, G_{k,AC}[\varphi,K;\bar{A}]\frac{\delta^2\Gamma_k[\varphi,K;\bar{A}]}{\delta\varphi_C\delta K_B} .re \tag{3.90}$$

Inserting (3.90) in equation (3.88) finally gives:

$$\frac{\delta\Gamma_k[\varphi,K;\bar{A}]}{\delta K_A}\frac{\delta\Gamma_k[\varphi,K;\bar{A}]}{\delta\varphi_A} = -\int d^dx\, R_{k,AB}[\bar{A}]\frac{\delta^2\Gamma_k[\varphi,K;\bar{A}]}{\delta K_B\delta\varphi_C}G_{k,CA}[\varphi,K;\bar{A}] \ . \tag{3.91}$$

This is the modified Zinn-Justin (mZJ) equation which generalize the ZJ equation (B.75) in presence of the cutoff. Note that the lhs, usually denoted as “star product” or “Batalin-

Vilkovisky" bracket [12], is no more vanishing and thus the advantage of using the BRST approach is not clear anymore when we are in presence of a cutoff of the type that is employed here. It is worth noting that the mZJ equation has more the structure of the "master equation", arising in the Batalin-Vilkovisky quantization of gauge theories, where the "Laplacian" on the rhs is replaced by a sort of "regularized Laplacian" as is in the rhs of (3.91). Further study in this direction may be useful to better understand the role of the cutoff in the approach to quantization offered by the exact flow equation for the bEAA.

3.3.3 Flow equations for proper vertices of the bEAA

In this section we derive the hierarchy of equations governing the flow of the proper-vertices of both the full bEAA and the gEAA. Starting from the flow equation for the bEAA (3.70), we can derive a hierarchy of flow equations for the proper-vertices of the bEAA simply by taking functional derivatives with respect to the fields.

In the background formalism we are employing, we can take functional derivatives with respect to both the fluctuation multiplet φ and the background field $\bar{A}_\mu$. In this second case we have to remember that the cutoff terms in the flow equation depend explicitly on the background field. This adds new additional terms to the flow equations for the proper-vertices that are not present in the flow equations for the proper vertices of the EAA in the non-background formalism. We will see that these terms are crucial in preserving the gauge covariance of the gEAA along the flow.

Taking a functional derivative of the flow equation (3.70) with respect to the fluctuation multiplet or with respect to the background field gives the following flow equations for the one-vertices of the bEAA:

$$\begin{aligned}
\partial_t \Gamma_k^{(1;0)}[\varphi;\bar{A}] &= -\frac{1}{2}\mathrm{Tr}\, G_k[\varphi;\bar{A}]\, \Gamma_k^{(3;0)}[\varphi;\bar{A}] G_k[\varphi;\bar{A}]\partial_t R_k[\bar{A}] \\
\partial_t \Gamma_k^{(0;1)}[\varphi;\bar{A}] &= -\frac{1}{2}\mathrm{Tr}\, G_k[\varphi;\bar{A}] \left(\Gamma_k^{(2;1)}[\varphi;\bar{A}] + R_k^{(1)}[\bar{A}]\right) G_k[\varphi;\bar{A}]\partial_t R_k[\bar{A}] \\
&\quad +\frac{1}{2}\mathrm{Tr}\, G_k[\varphi;\bar{A}]\partial_t R_k^{(1)}[\bar{A}]\,.
\end{aligned} \tag{3.92}$$

Note that in the second equation of (3.92) there are terms containing functional derivatives of the cutoff kernel $R_k[\bar{A}]$.

Taking a further derivative of equation (3.92) with respect to both the fluctuation and background field gives the following flow equations for the two-vertices of the bEAA[6]:

$$\begin{aligned}
\partial_t\Gamma_k^{(2;0)} &= \mathrm{Tr}\, G_k\, \Gamma_k^{(3;0)} G_k\, \Gamma_k^{(3;0)} G_k \partial_t R_k - \frac{1}{2}\mathrm{Tr}\, G_k\, \Gamma_k^{(4;0)} G_k \partial_t R_k \\
\partial_t\Gamma_k^{(1;1)} &= \mathrm{Tr}\, G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k\, \Gamma_k^{(3;0)} G_k \partial_t R_k \\
&\quad -\frac{1}{2}\mathrm{Tr}\, G_k \left(\Gamma_k^{(3;1)} + R_k^{(1)}\right) G_k \partial_t R_k - \frac{1}{2}\mathrm{Tr}\, G_k\, \Gamma_k^{(3;0)} \partial_t R_k^{(1)} \\
\partial_t\Gamma_k^{(0;2)} &= \mathrm{Tr}\, G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \partial_t R_k \\
&\quad -\frac{1}{2}\mathrm{Tr}\, G_k \left(\Gamma_k^{(2;2)} + R_k^{(2)}\right) G_k \partial_t R_k \\
&\quad -\mathrm{Tr}\, G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \partial_t R_k^{(1)} + \frac{1}{2}\mathrm{Tr}\, G_k \partial_t R_k^{(2)} \,.
\end{aligned} \tag{3.93}$$

Proceeding in this way, we generate the full hierarchy of flow equations for the proper-vertices $\Gamma_k^{(n;m)}$of the bEAA. In general the flow of the proper-vertex $\Gamma_k^{(n;m)}$ involves the proper-vertex up to $\Gamma_k^{(n+2;m)}$ and functional derivatives of the cutoff kernel up to $R_k^{(m)}$.

Note that, as they stand in equation (3.92) and (3.93), every equation of the hierarchy has the same information content as the original flow equation (3.70). To make profit of the above derived equations, we perform now a Taylor expansion of the two argument functional $\Gamma_k[\varphi;\bar{A}]$, that we express in the following way:

$$\Gamma_k[\varphi;\bar{A}] = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \int_{x_1...x_n y_1...y_m} \gamma_{k,x_1...x_n y_1...y_m}^{(n;m)} \varphi_{x_1}...\varphi_{x_n} \bar{A}_{y_1}...\bar{A}_{y_m} \,, \tag{3.94}$$

In (3.94) we defined the zero-field proper-vertices as:

$$\gamma_{k,x_1...x_n y_1...y_m}^{(n;m)} \equiv \Gamma_{k,x_1...x_n y_1...y_m}^{(n;m)}[0;0] \,. \tag{3.95}$$

If we evaluate now the hierarchy of flow equations, the first of which are equation (3.92) and (3.93), for $\varphi = 0$ and $\bar{A}_\mu = 0$, they become and infinite system of coupled equations for the zero-field proper-vertices $\gamma_k^{(n;m)}$. From the expansion (3.94), we see that this system we can be used to extract the RG running of all terms in the bEAA that are analytic in the fields. In particular these terms can be of non-local character.

To handle possible general non-analytical operators, a scheme different from this has to be developed. Up to now some calculations have been done for truncations like $\bar{\Gamma}_k[A] =$

[6]Here, as in several other equations of section, we omit for clarity to explicitly write the arguments of the functionals.

$\int W\left(\frac{1}{4}F^2\right)$ in non-abelian gauge theories [56, 57, 58, 63], or like $\bar{\Gamma}_k[g] = \int \sqrt{g} f(R)$ in the gravitational contest [104, 105, 96]. This studies employed a specific choice for the background fields and the calculation tools used are not general enough to handle all possible general non-analytic terms that may be present in the bEAA. Still, as several example show, it seems that the non-trivial part of the bEAA has a non-local structure instead of a non-analytical one. For this reason, among others, the computational strategy based on the hierarchy of flow equations for the zero-field proper-vertices of the bEAA is a promising research route to study the bEAA in its full generality.

As for the bEAA, we can derive a hierarchy of flow equations for the proper-vertices of the gEAA. In this case the functional depends only on the background field. Taking a functional derivative of (3.75) with respect to this field gives the following flow equation for the one-vertex of the gEAA:

$$\begin{aligned}\partial_t \bar{\Gamma}_k^{(1)}[\bar{A}] &= -\frac{1}{2}\mathrm{Tr}\, G_k[0,\bar{A}]\left(\Gamma_k^{(2;1)}[0,\bar{A}] + R_k^{(1)}[\bar{A}]\right) G_k[0,\bar{A}]\partial_t R_k[\bar{A}] \\ &\quad +\frac{1}{2}\mathrm{Tr}\, G_k[0,\bar{A}]\partial_t R_k^{(1)}[\bar{A}]\,. \end{aligned} \tag{3.96}$$

The second term in (3.96) contains a functional derivative of the cutoff kernel as in the case of the second equation in (3.92). Actually, equation (3.96) is the same as equation (3.92) if we set $\varphi = 0$ and if we consider the relation $\partial_t \Gamma_k^{(0;1)}[0;\bar{A}] = \partial_t \bar{\Gamma}_k[\bar{A}]$.

A further derivative of (3.96) with respect to $\bar{A}_\mu$ gives the following flow equation for the two-vertex of the gEAA:

$$\begin{aligned}\partial_t \bar{\Gamma}_k^{(2)} &= \mathrm{Tr}\, G_k\left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \partial_t R_k \\ &\quad -\frac{1}{2}\mathrm{Tr}\, G_k\left(\Gamma_k^{(2;2)} + R_k^{(2)}\right) G_k \partial_t R_k \\ &\quad -\mathrm{Tr}\, G_k\left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \partial_t R_k^{(1)} + \frac{1}{2}\mathrm{Tr}\, G_k \partial_t R_k^{(2)}\,. \end{aligned} \tag{3.97}$$

As for (3.96), this equation is equal to the last equation in (3.93) if we set $\varphi = 0$ and if we use that $\partial_t \Gamma_k^{(0;2)}[0;\bar{A}] = \partial_t \bar{\Gamma}_k^{(2)}[\bar{A}]$. As already said, the terms coming from functional derivatives of the cutoff kernel, that are present in the background formalism, but not in the ordinary one, are crucial in preserving the covariant character of the flow of the gEAA and its vertices. As we did for the full bEAA, we can perform a Taylor expansion of the gEAA analogous to (3.94) and define the zero-field proper-vertices

$$\bar{\gamma}_{k,x_1\ldots x_n}^{(n)} \equiv \bar{\Gamma}_{k,x_1\ldots x_n}[0]\,, \tag{3.98}$$

to turn the hierarchy of flow equations for the proper vertices of the gEAA in an infinite dimensional coupled system for this just defined vertices.

We notice now that there is a more compact form in which we can rewrite the flow equations for the proper-vertices we just derived. If we introduce the formal operator defined as

$$\tilde{\partial}_t = (\dot{R}_k - \eta_k R_k)\frac{\partial}{\partial R_k}\,, \tag{3.99}$$

where here η_k is a multiplet matrix of anomalous dimensions, we can rewrite the flow equation for the bEAA (3.70) as:

$$\partial_t \Gamma_k[\varphi;\bar{A}] = \frac{1}{2}\mathrm{Tr}\, G_k[\varphi;\bar{A}]\partial_t R_k[\bar{A}] = -\frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t \log G_k[\varphi;\bar{A}]\,. \tag{3.100}$$

In (3.100) we used the following simple relations:

$$\tilde{\partial}_t G_k = -G_k \partial_t R_k G_k \qquad\qquad \tilde{\partial}_t \log G_k = G_k^{-1}\tilde{\partial}_t G_k = G_k \partial_t R_k\,.$$

In this way, we can rewrite the flow equation for the one-vertices of the bEAA (3.92) in the compact form:

$$\begin{aligned}
\partial_t \Gamma_k^{(1;0)}[\varphi;\bar{A}] &= -\frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t \left\{\Gamma_k^{(3;0)}[\varphi;\bar{A}]G_k[\varphi;\bar{A}]\right\} \\
\partial_t \Gamma_k^{(0;1)}[\varphi;\bar{A}] &= -\frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t \left\{\left(\Gamma_k^{(2;1)}[\varphi;\bar{A}] + R_k^{(1)}[\bar{A}]\right) G_k[\varphi;\bar{A}]\right\}\,,
\end{aligned} \tag{3.101}$$

while the flow equations for the two-vertices of the bEAA (3.93) read now:

$$\begin{aligned}
\partial_t \Gamma_k^{(2;0)} &= \frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t \left\{\Gamma_k^{(3;0)} G_k\, \Gamma_k^{(3;0)} G_k\right\} - \frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t \left\{\Gamma_k^{(4;0)} G_k\right\} \\
\partial_t \Gamma_k^{(1;1)} &= \frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t \left\{\left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k\, \Gamma_k^{(3;0)} G_k\right\} - \frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t \left\{\Gamma_k^{(3;1)} G_k\right\} \\
\partial_t \Gamma_k^{(0;2)} &= \frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t \left\{\left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k\right\} \\
&\quad -\frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t \left\{\left(\Gamma_k^{(2;2)} + R_k^{(2)}\right) G_k\right\}\,.
\end{aligned} \tag{3.102}$$

This notation turns out to be useful since the flow equations (3.101) and (3.102) contain less terms then their counter parts (3.92) and (3.93) and are thus much more manageable when employed in actual computations. The same reasoning applies to all subsequent equations of the hierarchy and extend to the flow equations for the zero-field proper-vertices $\gamma_k^{(n;m)}$. Also in this case the flow equations for the proper-vertices of the gEAA are just those for the

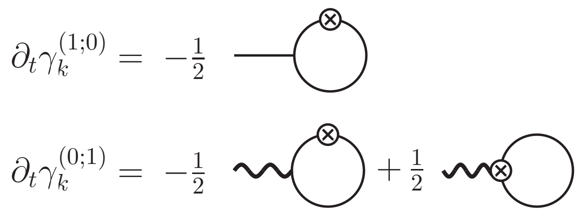


Figure 3.6: Diagrammatic representation of the flow equations for $\partial_t \gamma_k^{(1;0)}$ and $\partial_t \gamma_k^{(0;1)}$ as given in (3.92).

bEAA evaluated at $\varphi = 0$ and we don't need to repeat them here. The same applies to the flow equation for the proper-vertices $\bar{\gamma}_k^{(n)}$.

We will see in the next section how the flow equation for the zero-field proper-vertices just defined can be turned into a powerful computational device to perform actual computations in the bEAA framework.

3.3.4 Diagrammatic and momentum space techniques

In this section we expose the important technical tools developed in this thesis in order to make calculations possible in a large new class of truncations of the bEAA. These include those related to the truncation (3.17) and (3.19) studied in the main part of this chapter. All the results of this section can be extended to treat the bEAA for quantum gravity and more generally the bEAA of any theory with local gauge invariance. For this reason we try to maintain our notation as general as possible.

We first develop some useful diagrammatic rules as a device to organize the various contributions to the flow equation for the zero-field proper-vertices $\gamma_k^{(n;m)}$ of the bEAA derived in the previous section. Then we introduce the momentum space representation that enables us to explicitly calculate these contributions.

As we will see, there are some non-trivial technical steps that have to be done in order to be able to write explicitly, in momentum space, the flow equations for zero-field proper-vertices with some background legs, i.e. the vertices $\gamma^{(n;m)}$ for $m > 0$. This issue is related to the dependence of the cutoff kernel $R_k[\bar{A}]$ on the cutoff operator constructed with the use of the background field. The functional derivatives of the cutoff kernel, $R_k^{(m)}[\bar{A}]$, can be calculated as the terms of a Taylor series expansion of the cutoff kernel with respect to the background field. This expansion will be performed using the perturbative expansion for the un-traced heat kernel developed is Appendix A.

We start to introduce the diagrammatic rules used to represent the hierarchy of flow

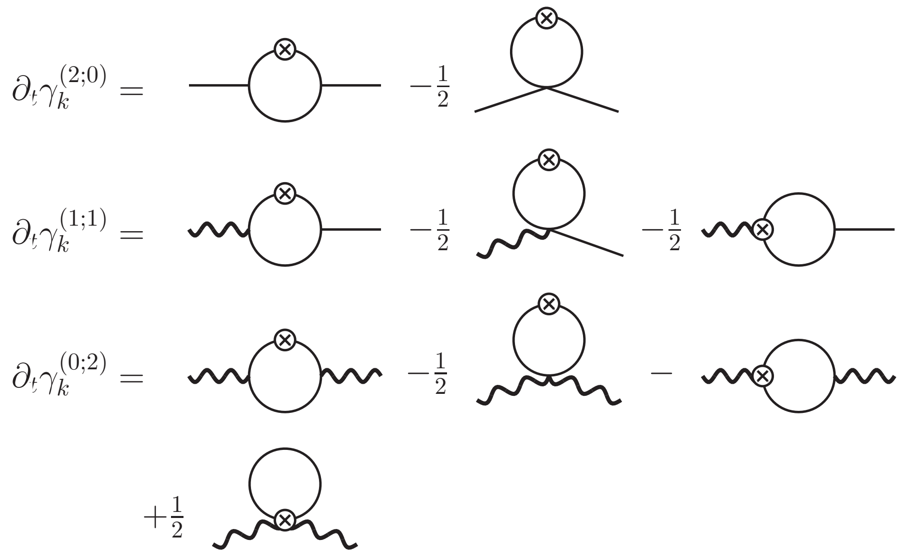
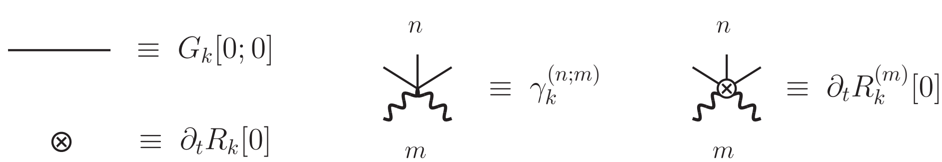


Figure 3.7: Diagrammatic representation of the flow equations for the vertices $\partial_t \gamma_k^{(2;0)}$, $\partial_t \gamma_k^{(1;1)}$ and $\partial_t \gamma_k^{(0;2)}$ as in equation (3.93).

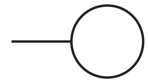
equations for the zero-field proper-vertices. In particular, we show how to represent equations (3.92) and (3.93). The virtue of diagrammatic techniques is that they allow to switch from coordinate space to momentum space straightforwardly.

We represent the zero-field regularized propagator $G_k[0;0]$ with an internal continuous line, the cutoff insertions $\partial_t R_k$ are indicated with a crossed circle and the zero-field proper-vertices $\gamma_k^{(n;m)}$ are represented as vertex with n external continuous lines and m external thick wavy lines. The diagrammatic rules are summarized as follows:



Finally, to every closed loop, and in all flow equations there is just one as the flow equation has a one-loop structure, we associate a coordinate or momentum integral together with the factor $\partial_t R_k - \eta R_k$. Here the anomalous dimension η pertains to the field the cutoff kernel of which is considered.

Following these diagrammatic rules, the flow equations (3.92) for the zero-field one-vertices $\partial_t \gamma_k^{(1;0)}$ and $\partial_t \gamma_k^{(0;1)}$ can be represented as in Figure 3.6, while the flow equations (3.93) for the zero-field two-vertices $\partial_t \gamma_k^{(2;0)}$, $\partial_t \gamma_k^{(1;1)}$ and $\partial_t \gamma_k^{(0;2)}$ can be represented as in Figure 3.7.

$$\partial_t \gamma_k^{(1;0)} = -\tfrac{1}{2}$$ 

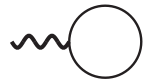
$$\partial_t \gamma_k^{(0;1)} = -\tfrac{1}{2}$$ 

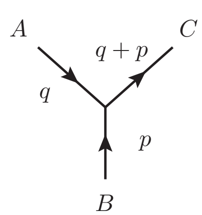
Figure 3.8: Diagrammatic representation of the flow equation for the one-vertices $\partial_t \gamma_k^{(1;0)}$ and $\partial_t \gamma_k^{(0;1)}$ as given in (3.101).

As explained in the previous section, it is useful to work also with the set of flow equations for the bEAA (3.101) and (3.102) written employing the formal operator $\tilde{\partial}_t$ defined in (3.99). In this case there are no explicit insertion of the cutoff term $\partial_t R_k$ in the loops, but to every loop is now associated an integration together with the action of this formal operator, i.e. $\int_x \tilde{\partial}_t$ in coordinate space or $\int_q \tilde{\partial}_t$ in momentum space. In this way, the flow equations (3.101) for the zero-field one-vertices $\partial_t \gamma_k^{(1;0)}$ and $\partial_t \gamma_k^{(0;1)}$ can be represented as in Figure 3.8, while the equations (3.102) for the zero-field two-vertices $\partial_t \gamma_k^{(2;0)}$, $\partial_t \gamma_k^{(1;1)}$ and $\partial_t \gamma_k^{(0;2)}$ can be represented as in Figure 3.9.

Note that in this last representation, the flow equations for the different one-vertices or two-vertices assume a more symmetric form with respect to each other. This reflects in a computational advantage, especially when considering the flow equation for the two-vertex $\gamma_k^{(0;2)}$, where the additional terms involving cutoff kernel vertices are accounted for by the action of the operator $\tilde{\partial}_t$. As we will see in section 3.5.4, this fact is very useful.

We are now in the position to write down the flow equations for the zero-field proper-vertices $\gamma_k^{(n;m)}$ of the bEAA in momentum space. The only non-standard step is to write the momentum space representation of the terms involving functional derivatives of the cutoff kernel $R_k[\bar{A}]$ with respect to the background field, i.e. the momentum representation of the vertices $\gamma_k^{(2;1)}$ and $\gamma_k^{(2;2)}$. We start by stating the final result and postpone the details of the derivation to the next section.

The zero-field proper-vertex $\gamma_k^{(3;0)}$ is represented graphically by the following diagram:



$$= [\gamma^{(2;1)}_{q,p,-q-p}]^{ABC}$$

Here we use capital letters to indicate general composite indices $A = a\,\alpha, B = b\,\beta, C = c\,\gamma$, so

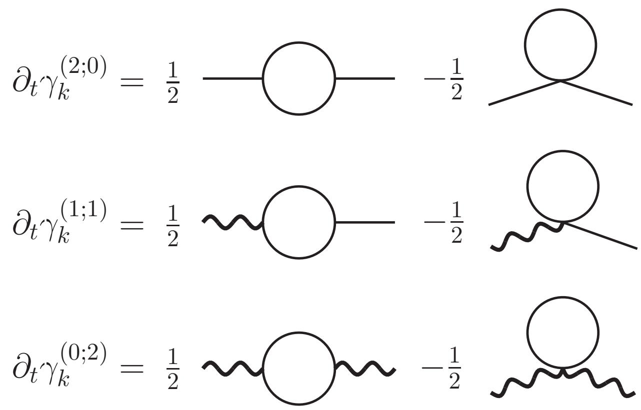


Figure 3.9: Diagrammatic representation of the flow equations for the two-vertices $\partial_t \gamma_k^{(2;0)}$, $\partial_t \gamma_k^{(1;1)}$ and $\partial_t \gamma_k^{(0;2)}$ as in equation (3.102).

that the definitions we make can be easily extended to more general situations. For example, in the gravitational context we will have $A = \alpha\beta, B = \gamma\delta, C = \epsilon\kappa$. Note that each index is associated with a momentum variable, so that A, B, C are the indices of the related momenta $q, p, -q-p$ respectively. Note also that we always define ingoing momenta as being positive.

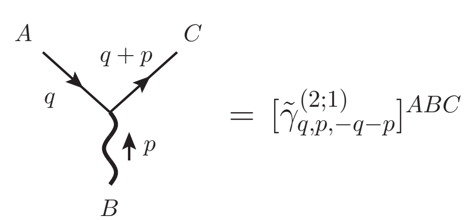
When considering the vertices with background lines, it is useful to consider the "tilde" bEAA[7] defined by the relation

$$\tilde{\Gamma}_k[\varphi; \bar{A}] = \Gamma_k[\varphi; \bar{A}] + \Delta S_k[\varphi; \bar{A}] , \tag{3.103}$$

and to define the related "tilde" zero-field proper-vertices

$$\tilde{\gamma}_k^{(n;m)} = \gamma_k^{(n;m)} + \Delta S_k^{(n;m)}[0; 0] . \tag{3.104}$$

In terms of these, the two-fluctuations one-background zero-field proper-vertex $\tilde{\gamma}_k^{(2;1)} = \gamma_k^{(2;1)} + \Delta S_k^{(2;1)}[0; 0]$ is represented graphically by the diagram:



[7] Note that $\tilde{\Gamma}_k[\varphi; \bar{A}]$ is the actual Legendre transform of the functional generator of connected correlation functions $W_k[J; \bar{A}]$.

The non-trivial technical step is to write explicitly the momentum space representation of the tilde zero-field proper-vertex $\tilde{\gamma}_k^{(2;1)}$. In the next section we show that this representation turns out to be:

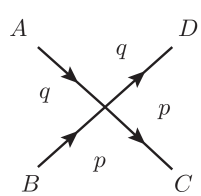
$$[\tilde{\gamma}^{(2;1)}_{q,p,-q-p}]^{ABC} = [\gamma^{(2;1)}_{q,p,-q-p}]^{ABC} + [L^{(2;1)}_{q,p,-q-p}]^{ABC} R^{(1)}_{q+p,p} \,. \tag{3.105}$$

In (3.105) the functional $L[\varphi;\bar{A}]$ is the "cutoff operator action", defined in the next section, which is the action whose Hessian with respect to φ is the cutoff operator. Furthermore,

$$R^{(1)}_{q+p,q} \equiv \frac{R_{q+p} - R_q}{(q+p)^2 - q^2} \,, \tag{3.106}$$

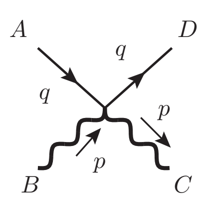
represents the first finite-difference derivative of the cutoff shape function. The momentum space representation (3.105) is of crucial importance since it makes, together with the generalization to higher vertices given in the following, access to the computational use of the flow equations for the zero-field proper-vertices $\gamma_k^{(n:m)}$. Obviously, for every n the relation $\tilde{\gamma}_k^{(n;0)} = \gamma_k^{(n;0)}$ holds. Hence we can re-phrase the flow equations for the zero-field proper-vertices solely in terms of the tilde vertices $\tilde{\gamma}^{(n;m)}$ alone.

The four-fluctuation vertex $\gamma_k^{(4;0)}$ is represented graphically as:



$$= [\gamma^{(4:0)}_{q,p,-p,-q}]^{ABCD}$$

Note that here we are giving the four-vertex only for a particular combination of momenta, which is not the most general one, since this will be the case we will treat in this thesis. The general case can be easily derived. The two-fluctuations two-backgrounds vertex $\tilde{\gamma}_k^{(2;2)} = \gamma_k^{(2;2)} + \Delta S_k^{(2;2)}[0;0]$ is represented instead by the diagram:



$$= [\tilde{\gamma}^{(2;2)}_{q,p,-p,-q}]^{ABCD}$$

The momentum space representation of $\tilde{\gamma}_k^{(2;2)}$, as shown in the next section, turns out to be:

$$\begin{aligned}[\tilde{\gamma}^{(2;2)}_{q,p,-p,-q}]^{ABCD} &= [\gamma^{(2;1)}_{q,p,-p,-q}]^{ABC} + [L^{(2;2)}_{q,p,-p,-q}]^{ABCD} R^{(2)}_{q+p,p} \\ &\quad + [L^{(2;1)}_{q,p,-q-p}]^{AB1} [L^{(2;1)}_{q+p,-p,-q}]^{1CD} R'_q \,. \end{aligned} \tag{3.107}$$

Note that now in (3.107) the cutoff action enters both as a four-vertex and as a product of two three-vertices. This time we need to consider the second finite-difference derivative of the cutoff shape function defined by:

$$R^{(2)}_{q+p,q} \equiv \frac{2}{(q+p)^2 - q^2} \left[\frac{R_{q+p} - R_q}{(q+p)^2 - q^2} - R'_q \right] . \tag{3.108}$$

The relation (3.107) is the second example of the general rule needed to represent explicitly in momentum space the flow equations for the zero-field proper-vertices $\gamma_k^{(n;m)}$. Note that the second finite-difference derivative (3.108) equals the first finite difference derivative of $R^{(1)}_{q+p,q}$, as it should be. The finite-difference derivatives can be expanded for small external momenta as follows:

$$\begin{aligned} R^{(1)}_{q+p,q} &= R'_q + p\cdot q\, R''_q + \frac{1}{2} p^2\, R''_q + \frac{2}{3} (p\cdot q)^2\, R^{(3)}_q + O\left(p^3\right) \\ R^{(2)}_{q+p,q} &= R''_q + \frac{2}{3} p\cdot q\, R^{(3)}_q + \frac{1}{3} p^2\, R^{(4)}_q + \frac{1}{3} (p\cdot q)^2\, R^{(4)}_q + O\left(p^3\right) \,. \end{aligned} \tag{3.109}$$

The correction terms in (3.109) proportional to p or p^2 are those needed to make the flow of the vertices $\bar{\gamma}_k^{(m)} = \gamma_k^{(0;m)}$ covariant, as they should be by construction. It is worth noticing that these rules actually implement a kind of mass regularization of the effective action which respects gauge symmetry.

We are ready now to write the flow equations for the zero-field proper vertices of the bEAA in their momentum space representation. We will be interested solely in the flow equations for the two-vertices, equations (3.93) and (3.102), since it is from these equations that in the next sections we will extract the beta functions of the coupling we are considering in truncation (3.17) and (3.19). Using the rules introduced in the previous paragraph, we find for the first equation in (3.93), describing the flow of the zero-filed fluctuation-fluctuation two-vertex, the following momentum space representation:

$$\begin{aligned}[\partial_t \gamma^{(2;0)}_{p,-p}]^{AB} &= \int_q (\partial_t R_q - \eta R_q)\, [\gamma^{(3;0)}_{q,p,-q-p}]^{4A1}[G_{q+p}]^{12}[\gamma^{(3;0)}_{q+p,-p,-q}]^{2B3}[G_q]^{34} \\ &\quad -\frac{1}{2}\int_q (\partial_t R_q - \eta R_q)\, [\gamma^{(4;0)}_{q,p,-p,-q}]^{1AB2}[G_q]^{21}\,. \end{aligned} \tag{3.110}$$

In (3.110) and in the following relations η is the multiplet matrix of anomalous dimension of the multiplet fluctuation field φ. Note also that we are using the generalized notation for the indices introduced before. With respect to the first equation in Figure 3.7, the first line in (3.110) is the contribution from the first diagram, while the second line is the contribution from the second one. The second equation in (3.93), describing the flow of the zero-field fluctuation-background two-vertex, takes the form:

$$\begin{aligned}[\partial_t \gamma^{(1;1)}_{p,-p}]^{AB} &= \int_q (\partial_t R_q - \eta R_q)\, [\tilde{\gamma}^{(2;1)}_{q,p,-q-p}]^{4A1}[G_{q+p}]^{12}[\gamma^{(3;0)}_{q+p,-p,-q}]^{2B3}[G_q]^{34} \\ &\quad -\frac{1}{2}\int_q (\partial_t R_q - \eta R_q)\, [\tilde{\gamma}^{(3;1)}_{q,p,-p,-q}]^{1AB2}[G_q]^{21} \\ &\quad -\int_q [L^{(2;1)}_{q,p,-q-p}(\partial_t R^{(1)}_{q+p,q} - \eta R^{(1)}_{q+p,q})]^{4A1} \\ &\quad \times [G_{q+p}]^{12}[\gamma^{(3;0)}_{q+p,-p,-q}]^{2B3}[G_q]^{34}\,. \end{aligned} \tag{3.111}$$

In (3.111) there are now two vertices of the tilde zero-field proper-vertex since, referring to the second equation in Figure 3.7, there is a background line attached to respectively a three-vertex, a four-vertex and to the factor $\partial_t R_k[\bar{A}]$. The contribution from these three diagrams are respectively the first, second and third lines of (3.111). The last of equations (3.93) takes the following form:

$$\begin{aligned}[\partial_t \gamma^{(0;2)}_{p,-p}]^{AB} &= \int_q (\partial_t R_q - \eta R_q)\, [\tilde{\gamma}^{(2;1)}_{q,p,-q-p}]^{4A1}[G_{q+p}]^{12}[\tilde{\gamma}^{(2;1)}_{q+p,-p,-q}]^{2B3}[G_q]^{34} \\ &\quad -\frac{1}{2}\int_q (\partial_t R_q - \eta R_q)\, [\tilde{\gamma}^{(2;2)}_{q,p,-p,-q}]^{1AB2}[G_q]_{21} \\ &\quad -\int_q [L^{(2;1)}_{q,p,-q-p}\left(\partial_t R^{(1)}_{q+p,q} - \eta R^{(1)}_{q+p,q}\right)]^{4A1}[G_{q+p}]^{12}[\tilde{\gamma}^{(2;1)}_{q+p,-p,-q}]^{2B3}[G_q]^{34} \\ &\quad +\frac{1}{2}\int_q \Big\{[L^{(2;2)}_{q,p,-p,-q}\left(\partial_t R'_q - \eta R'_q\right)]^{1AB2}[G_q]_{21} \\ &\quad +[L^{(2;1)}_{q,p,-q-p}]^{1A3}[L^{(2;1)}_{q+p,-p,-q}]^{3B2}(\partial_t R^{(2)}_{q+p,q} - \eta R^{(2)}_{q+p,q})[G_q]_{21}\Big\}\,. \end{aligned} \tag{3.112}$$

Equation (3.112) represents the flow of the zero-field background-background two-vertex of the bEAA and thus every term is written in terms of the tilde zero-field vertex and of the cutoff action $L[\varphi;\bar{A}]$. As we explained earlier, these equations are very general and can be adapted to every theory with local gauge symmetry treated in the bEAA framework.

In terms of the compact representation introduced earlier using the formal operator $\tilde{\partial}_t$ defined in (3.99), the flow equations for the zero-field two-vertices of the bEAA are given in equation (3.102). These are represented graphically in Figure 3.9 and all three have the same overall structure. The flow of the zero-filed fluctuation-fluctuation two-vertex has the following momentum space representation:

$$\begin{aligned}[][\partial_t \gamma^{(2;0)}_{p,-p}]^{AB} &= \frac{1}{2}\int_q \tilde{\partial}_t \left\{ [\gamma^{(3;0)}_{q,p,-q-p}]^{4A1} [G_{q+p}]^{12} [\gamma^{(3;0)}_{q+p,-p,-q}]^{2B3} [G_q]^{34} \right\} \\ &\quad -\frac{1}{2}\int_q \tilde{\partial}_t \left\{ [\gamma^{(4;0)}_{q,p,-p,-q}]^{1AB2} [G_q]^{21} \right\} . \end{aligned} \tag{3.113}$$

The second equation in (3.102) of in Figure 3.9 expresses the flow of the zero-field fluctuation-background two-vertex and differs from (3.113) in two tilde vertices:

$$\begin{aligned}[][\partial_t \gamma^{(1;1)}_{p,-p}]^{AB} &= \frac{1}{2}\int_q \tilde{\partial}_t \left\{ [\tilde{\gamma}^{(2;1)}_{q,p,-q-p}]^{4A1} [G_{q+p}]^{12} [\gamma^{(3;0)}_{q+p,-p,-q}]^{2B3} [G_q]^{34} \right\} \\ &\quad -\frac{1}{2}\int_q \tilde{\partial}_t \left\{ [\tilde{\gamma}^{(3;1)}_{q,p,-p,-q}]^{1AB2} [G_q]^{21} \right\} . \end{aligned} \tag{3.114}$$

Finally, the compact form for the flow of the zero-field background-background two-vertex is:

$$\begin{aligned}[][\partial_t \gamma^{(0;2)}_{p,-p}]^{AB} &= \frac{1}{2}\int_q \tilde{\partial}_t \left\{ [\tilde{\gamma}^{(2;1)}_{q,p,-q-p}]^{4A1} [G_{q+p}]^{12} [\tilde{\gamma}^{(2;1)}_{q+p,-p,-q}]^{2B3} [G_q]^{34} \right\} \\ &\quad -\frac{1}{2}\int_q \tilde{\partial}_t \left\{ [\tilde{\gamma}^{(2;2)}_{q,p,-p,-q}]^{1AB2} [G_q]_{21} \right\} . \end{aligned} \tag{3.115}$$

Note that in equation (3.115) all zero-field proper-vertices are tilde vertices. Thus the flow equation for the zero-field proper-vertices of the bEAA are formally as those of the standard EAA when written in terms of the formal operator $\tilde{\partial}_t$ but with tilde vertices in place of the standard vertices. All the non-trivial dependence of the cutoff kernel is in this way hided in the dependence of the tilde vertices on it. This turns out to be very useful property in actual computations.

As already said, equation (3.115), as equation (3.112), represents also the flow of the zero-field proper-vertex $\bar{\gamma}_k^{(2)}$ of the gEAA since we have that $\partial_t \bar{\gamma}_k^{(2)} = \partial_t \gamma_k^{(0;2)}$.

Clearly, these are only the first equations of the respective hierarchies and the results exposed in this section are valid for all the subsequent equations of the hierarchy for both the zero-field proper-vertices of the bEAA and of the gEAA.

The equations (3.110-3.115) are the main result of this section. A lot of information about the flow of both the bEAA and of the gEAA can be extracted already from the flow of the zero-field two-vertices described by these equations. In particular, in this thesis we will use equation (3.110), in sections 3.5.6 and 3.5.7 to calculate the beta functions of the running gauge fluctuation and ghost masses, of the wave-functions renormalizations of the gauge fluctuation and ghost fields and of the gauge-fixing parameter. We will use (3.115), in section 3.5.4, to calculate the beta function for the running wave-function renormalization of the background field from which we will find the beta function of the non-abelian coupling constant.

The results of this section, when combined with the flow equations of the previous one, constitute the basis for a concrete framework in which all truncations of the bEAA, which are analytic in the fields, can be handled.

3.3.4.1 Momentum space representation of background vertices

In this subsection we show how to derive the momentum space representation of background vertices, in particular we will derive the two relations (3.105) and (3.107) used in the previous section to explicitly write the momentum space representation of the flow equations for the zero-field proper-vertices of the bEAA.

What we need to do is to calculate the momentum space representation of the cutoff vertices $\Delta S_k^{(2;1)}[0;0]$ and $\Delta S_k^{(2;2)}[0;0]$. From the definition of the cutoff action (3.58) we see that:

$$\Delta S_k^{(2;0)}[0;\bar{A}]_{xy}^{AB} = \int_{zw} \delta_{xz} R_k[\bar{A}]_{zw}^{AB} \delta_{wy} = R_k[\bar{A}]_{xy}^{AB} \,. \tag{3.116}$$

The cutoff kernel $R_k[\bar{A}]$ is a function of the cutoff operator $L^{(2;0)}[0;\bar{A}]$, constructed as the Hessian of the cutoff operator action $L[a;\bar{A}]$. For example, if the cutoff operator we consider is just the gauge Laplacian, then we have the following relation:

$$L^{(2;0)}[0;\bar{A}]_{xy} = \int_z \bar{D}_{z\mu} \delta_{zx} \bar{D}_z^{\mu} \delta_{zy} = -\int_z \delta_{zx} \bar{D}_{z\mu} \bar{D}_z^{\mu} \delta_{zy} = -\bar{D}_{\mu x} \bar{D}^{\mu y} \delta_{xy} = -\bar{D}_x^2 \delta_{xy} \,.$$

Thus, after making a Laplace transform, we can write the cutoff kernel in terms of the un-traced heat kernel of the cutoff operator:

$$R_k[\bar{A}]^{AB}_{xy} = \int_0^\infty ds\, \tilde{R}_k(s)\, K^s[\bar{A}]^{AB}_{xy}\,, \tag{3.117}$$

where the un-traced heat kernel can be written in terms of the Hessian of the Laplacian action:

$$K^s[\bar{A}] = \exp\left\{-sL^{(2;0)}[0;\bar{A}]\right\}\,. \tag{3.118}$$

Inserting (3.117) in equation (3.116) and setting the background field to zero after having differentiated with respect to it one time, gives the following representation in terms of the un-traced heat kernel for the cutoff vertex with one background leg :

$$\Delta S_k^{(2;1)}[0;0]^{ABC}_{xyz} = \left.\frac{\delta R_k[\bar{A}]^{AB}_{xy}}{\delta \bar{A}^C_z}\right|_{\bar{A}=0} = \int_0^\infty ds\, \tilde{R}_k(s)\, \left.\frac{\delta K^s[\bar{A}]^{AB}_{xy}}{\delta \bar{A}^C_z}\right|_{\bar{A}=0}\,. \tag{3.119}$$

We can now use the perturbative expansion in for the un-traced heat kernel developed in Appendix A, equation (A.21), to write the last term of (3.119) as follows:

$$\int_0^\infty ds\, \tilde{R}_k(s)\, \left.\frac{\delta K^s[\bar{A}]^{AB}_{xy}}{\delta \bar{A}^C_z}\right|_{\bar{A}=0} = \int_0^\infty ds\, \tilde{R}_k(s)(-s)\int_0^1 dt\, K^{s(1-t)}_{0,xw} L^{(2;1)}[0;0]^{ABC}_{wuz} K^{st}_{0,uy}\,. \tag{3.120}$$

In (3.120) we omitted to write explicitly the coordinate integrals and we wrote the flat space un-traced heat kernels as $K^s[0]^{AB}_{xy} = K^s_{0,xy}\delta^{AB}$, where $K^s_{0,xy}$ is given in equation (A.18) of Appendix A. Going to momentum space and inserting (3.120) in (3.119) gives the following representation for the cutoff vertex:

$$\begin{aligned}
\Delta S_k^{(2;1)}[0;0]^{ABC}_{p_1,p_2,p_3} &= (2\pi)^d\delta_{p_1+p_2+p_3}\int_0^\infty ds\, \tilde{R}_k(s)(-s)\int_0^1 dt\, K^{s(1-t)}_{0,p_1}[L^{(2;1)}_{p_1,p_2,p_3}]^{ABC}K^{st}_{0,p_2} \\
&= (2\pi)^d\delta_{p_1+p_2+p_3}\int_0^\infty ds\, \tilde{R}_k(s)(-s)\int_0^1 dt\, e^{-s(1-t)p_1^2}[L^{(2;1)}_{p_1,p_2,p_3}]^{ABC}e^{-stp_2^2} \\
&= (2\pi)^d\delta_{p_1+p_2+p_3}[L^{(2;1)}_{p_1,p_2,p_3}]^{ABC} \\
&\quad\times\int_0^1 dt\int_0^\infty ds\, \tilde{R}_k(s)(-s)e^{-s(1-t)p_1^2-stp_2^2}\,.
\end{aligned} \tag{3.121}$$

Here we used the following simple momentum space representation for the flat space un-traced heat kernel $K^s_{0,p} = e^{-sp^2}$. It is left to evaluate the double integral in (3.121). This can be done with the aid of the Q-functionals relations of Appendix A, in particular using (A.39) we find:

$$\begin{aligned}\int_0^\infty ds\,\tilde{R}_k(s)(-s)e^{-s(1-t)p_1^2-stp_2^2} &= -Q_{-1}[R_k(z+s(1-t)p_1^2+stp_2^2)] \\ &= R_k'(s(1-t)p_1^2+stp_2^2)\,. \end{aligned} \tag{3.122}$$

Now the parameter integral is easily evaluated:

$$\int_0^1 dt\,R_k'(s(1-t)p_1^2+stp_2^2) = \frac{R_k(p_2^2)-R_k(p_1^2)}{p_2^2-p_1^2}\,. \tag{3.123}$$

If we introduce the first finite-difference derivative, defined as

$$f^{(1)}_{p_1,p_2} = \frac{f(p_2^2)-f(p_1^2)}{p_2^2-p_1^2}\,,$$

we can finally write, for the cutoff vertex with one external background leg (3.121), the following momentum space representation:

$$\Delta S_k^{(2;1)}[0;0]^{ABC}_{p_1,p_2,p_3} = (2\pi)^d\delta_{p_1+p_2+p_3}[L^{(2;1)}_{p_1,p_2,p_3}]^{ABC}R^{(1)}_{p_1,p_2}\,. \tag{3.124}$$

We just need now to consider (3.124) for the momentum values $p_1=q$, $p_2=-q-p$ and $p_3=p$ to prove the relation given in equation (3.105):

$$\Delta S_k^{(2;1)}[0;0]^{ABC}_{q,-q-p,p} = \Omega\,[L^{(2;1)}_{q,-q-p,p}]^{ABC}R^{(1)}_{q+p,p}\,. \tag{3.125}$$

Along the same lines we can derive the momentum space representation for the cutoff vertex with two external legs. In place of (3.119) we have now

$$\Delta S_k^{(2;2)}[0;0]^{ABCD}_{xyzw} = \left.\frac{\delta^2 R_k[\bar{A}]^{AB}_{xy}}{\delta\bar{A}^D_w\delta\bar{A}^C_z}\right|_{\bar{A}=0} = \int_0^\infty ds\,\tilde{R}_k(s)\left.\frac{\delta^2 K^s[\bar{A}]^{AB}_{xy}}{\delta\bar{A}^D_w\delta\bar{A}^C_z}\right|_{\bar{A}=0}\,. \tag{3.126}$$

Using the perturbative expansion (A.21) gives now the following expansion for the second functional derivative of the cutoff kernel:

$$\begin{aligned}\left.\frac{\delta^2 R_k[\bar{A}]^{AB}_{xy}}{\delta \bar{A}^D_w \delta \bar{A}^C_z}\right|_{\bar{A}=0} &= \int_0^\infty ds\, \tilde{R}_k(s)\ \left.\frac{\delta^2 K^s[\bar{A}]^{AB}_{xy}}{\delta \bar{A}^D_w \delta \bar{A}^C_z}\right|_{\bar{A}=0} \\ &= \int_0^\infty ds\, \tilde{R}_k(s)(-s)\int_0^1 dt\, K^{s(1-t)}_{0,xu} L^{(2;2)}[0;0]^{ABCD}_{uvzw} K^{st}_{0,vy} \\ &\quad +2\int_0^\infty ds\, \tilde{R}_k(s)\, s^2 \int_0^1 dt_1 \int_0^{t_1} dt_2\, K^{s(1-t_1)}_{0,xu} L^{(2;1)}[0;0]^{AMB}_{uvz} \\ &\quad \times K^{s(t_1-t_2)}_{0,vr} L^{(2;1)}[0;0]^{CMD}_{rtw} K^{st_2}_{0,ty}\,. \end{aligned} \tag{3.127}$$

When we insert (3.127) into (3.126) and shift to momentum space, the first contribution in (3.127) becomes, like in the previous case, of the form:

$$(2\pi)^d \delta_{p_1+p_2+p_3+p_4} [L^{(2;2)}_{p_1,p_2,p_3,p_4}]^{ABCD} R^{(1)}_{p_4,p_1}\,. \tag{3.128}$$

The second contribution takes instead the following form:

$$\begin{aligned} &2(2\pi)^d \delta_{p_1+p_2+p_3+p_4} [L^{(2;1)}_{p_1,-p_1-p_2,p_2}]^{AMB} [L^{(2;1)}_{p_3,p_1+p_2,p_4}]^{CMD} \\ &\times \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^\infty ds\, \tilde{R}_k(s)\, s^2\, e^{-s(1-t_1)p_1^2 - s(t_1-t_2)(p_1+p_2)^2 - s_2 t_2 p_4^2}\,. \end{aligned} \tag{3.129}$$

We can calculate the double integral in (3.129) using the properties of the Q-functionals as before:

$$\begin{aligned} &\int_0^\infty ds\, \tilde{R}_k(s)\, s^2\, e^{-s(1-t_1)p_1^2 - s(t_1-t_2)(p_1+p_2)^2 - s_2 t_2 p_4^2} = \\ &= Q_{-2}[R_k(z + s(1-t_1)p_1^2 + s(t_1-t_2)(p_1+p_2)^2 + s_2 t_2 p_4^2)] \\ &= R_k''(s(1-t_1)p_1^2 + s(t_1-t_2)(p_1+p_2)^2 + s_2 t_2 p_4^2)\,, \end{aligned} \tag{3.130}$$

and

$$\begin{aligned} &2\int_0^1 dt_1 \int_0^{t_1} dt_2\, R_k''(s(1-t_1)p_1^2 + s(t_1-t_2)(p_1+p_2)^2 + s_2 t_2 p_4^2) = \\ &= \frac{2}{(p_1+p_2)^2 - p_4^2}\left[\frac{R_k((p_1+p_2)^2) - R_k(p_1^2)}{(p_1+p_2)^2 - p_1^2} - \frac{R_k(p_4^2) - R_k(p_1^2)}{p_4^2 - p_1^2}\right]\,. \end{aligned} \tag{3.131}$$

Finally, inserting (3.131) in (3.129) and combining with (3.128), gives the following momentum space representation for (3.126):

$$\Delta S^{(2;2)}_k[0;0]^{ABCD}_{p_1,p_2,p_3,p_4} = (2\pi)^d \delta_{p_1-p_2+q_1+q_2} \left\{ [L^{(2;2)}_{p_1,p_2,p_3,p_4}]^{ABCD} R^{(1)}_{p_4,p_1} + \right.$$

$$+[L^{(2;1)}_{p_1,-p_1-p_2,p_2}]^{AMB}[L^{(2;1)}_{p_3,p_1+p_2,p_4}]^{CMD}\frac{2}{(p_1+p_2)^2-p_4^2}\left[R^{(1)}_{p_1+p_2,p_1}-R^{(1)}_{p_4,p_1}\right]\Bigg\} \,. \tag{3.132}$$

To recover relation (3.107) we set $p_1 = -p_4 = q$ and $p_2 = -p_3 = p$ in (3.132) so that:

$$\Delta S^{(2;2)}_k[0;0]^{ABCD}_{p_1,p_2,p_3,p_4} = \Omega\left\{[L^{(2;2)}_{p_1,p_2,p_3,p_4}]^{ABCD}R'_p + [L^{(2;1)}_{p_1,-p_1-p_2,p_2}]^{AMB}[L^{(2;1)}_{p_3,p_1+p_2,p_4}]^{CMD}R^{(2)}_{p+q,p}\right\} \,. \tag{3.133}$$

In (3.133) we defined the second finite-difference derivative of a function as:

$$f^{(2)}_{p+q,q} = \frac{2}{(p+q)^2-p^2}\left[f^{(1)}_{p+q,p} - f'(p^2)\right] \,. \tag{3.134}$$

This concludes the derivation of the realtions (3.105) and (3.107) needed in the previous section to explicitly write down the momentum space representaion for the flow equaitons of the zero-field proper-vertices of the bEAA.

3.4 Summary

In this chapter we introduced the background effective average action (bEAA) and we applied it to non-abelian gauge theories with gauge group $G = SU(N)$. The details of this construction, in particular the issue of preserving gauge invariance along the flow, are covered in section 3.3. The background field is used to construct the cutoff operator that we used to separate the slow field modes from the fast filed modes. These last ones are integrated out. The cost of this construction is that the flow takes place in the enlarged theory space of all functionals of the fluctuation fields and of the background field. The functional defined by setting to zero the former, which we call gauge covariant EAA (gEAA), is invariant under physical gauge transformations, while the original bEAA is invariant only under combined physical plus background gauge transformations. The flow of both the bEAA and the gEAA has the simple one-loop structure of the flow of the standard non-background EAA introduced in Chapter 2. The main subtlety is that the flow of the gauge invariant gEAA is not closed because is determined by the Hessian of the full bEAA. This fact forces us, even if the modified Ward-Takahashi identities derived in section 3.3.2 show that the full physical gauge symmetry is restored at $k = 0$, to consider the flow in the enlarged theory space, i.e. the flow of the full bEAA. This is done in section 3.2.2.1, where we start studying the quantum theory by a local truncation of the bEAA that involves together with the running of the gauge coupling constant, the running gluon and ghost masses, the running wave-function renormalization of the fluctuation gauge field and of the ghost fields and the running gauge-fixing parameter. We derive the running for all these couplings and learn from the system of the beta functions several important facts. First we show how to account for the contribution of the non-physical couplings, i.e. $\eta_{a,k}$ and $\eta_{c,k}$, in the flow of the gauge coupling by eliminating these anomalous dimensions after we determined them by solving a linear system. We study the running of the fluctuation gluon masses and we show that it vanishes for $k = 0$ as it should. This is a non-trivial check of the formalism. Finally we show that the running gauge-fixing parameter flows to zero in the IR, thus confirming the expectation that Landau gauge is the most physical gauge.

In section 3.2.2.2 we propose a new approximation scheme, that we call curvature expansion, in which the gEAA is expanded in powers of the field strength and the gEAA is parametrized in terms of running structure functions. We make the first step in the direction of studying the flow of these non-local truncations by analyzing the one-loop flow of the field strength square running structure function, i.e. the running vacuum polarization function. We propose this truncation scheme as a promising approach to unveil the IR physics of non-abelian gauge theories.

Finally we introduced new diagrammatic and momentum space techniques, exposed in section 3.3.4, based on the hierarchy of flow equations of the bEAA. In particular we show how all the results obtained with the aid of the local and non-local expansion of the heat kernel are obtained by the proposed method. The most promising applications of this formalism will be material for future work.

3.5 Appendix to Chapter 3

In this Appendix to the Chapter we perform all the calculations needed in the main part of the chapter. We first calculate the variations and the functional derivatives of the action functionals that compose the bEAA in the truncation we are considering. Then we construct the regularized propagator by choosing the explicit form of the cutoff kernel and operator. In the last three sections we use the heat kernel expansion and the techniques developed in section 3.3.4 to calculate the beta functions studied in the main part of the chapter.

3.5.1 Variations and functional derivatives

In this section we calculate the variations and the functional derivatives that are used in the next sections. The basic variation of the field strength is:

$$\begin{aligned}\delta F^a_{\mu\nu} &= \delta(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu) \\ &= \partial_\mu a^a_\nu - \partial_\nu a^a_\mu + f^{abc} a^b_\mu A^c_\nu + f^{abc} A^b_\mu a^c_\nu \\ &= D^{ab}_\mu a^b_\nu - D^{ab}_\nu a^b_\mu \,. \end{aligned} \tag{3.135}$$

Here we are using the notation $\delta A^a_\mu = a^a_\mu$. The second variation is simply:

$$\delta^2 F^a_{\mu\nu} = 2 f^{abc} a^b_\mu a^c_\nu \,, \tag{3.136}$$

while all higher variations of the field strength vanish since the field strength is quadratic in A_μ. Next, we consider the basic invariant:

$$I[A] = \frac{1}{4}\int d^d x\, F^a_{\mu\nu} F^{a\mu\nu} \,. \tag{3.137}$$

The first and second variations of (3.137) are:

$$\delta I[A] = \frac{1}{2}\int d^d x\, F^{a\mu\nu} \delta F^a_{\mu\nu} = \frac{1}{2}\int d^d x\, F^{a\mu\nu}\left(D_\mu a^a_\nu - D_\nu a^a_\mu\right) = \int d^d x\, F^{a\mu\nu} D_\mu a^a_\nu \tag{3.138}$$

and

$$\begin{aligned}\delta^2 I[A] &= \frac{1}{2}\int d^d x \left[\delta F^{a\mu\nu}\delta F^a_{\mu\nu} + F^{a\mu\nu}\delta^2 F^a_{\mu\nu}\right] \\ &= \int d^d x \left[\left(D_\mu a^a_\nu - D_\nu a^a_\mu\right) D_\mu a^a_\nu + F^{a\mu\nu} f^{abc} a^b_\mu a^c_\nu\right] \\ &= \int d^d x \left[D_\mu a^a_\nu\, D_\mu a^a_\nu - D_\nu a^a_\mu\, D_\mu a^a_\nu + F^{a\mu\nu} f^{abc} a^b_\mu a^c_\nu\right] \,. \end{aligned} \tag{3.139}$$

Being the action (3.137) quartic in the gauge field, we can go on to calculate the other two non-vanishing variations. We have:

$$\delta^3 I[A] = \frac{3}{2}\int d^d x\, \delta F^{a\mu\nu}\delta^2 F^a_{\mu\nu} = 6\int d^d x\, D_\mu a^a_\nu\, f^{abc} a^b_\mu a^c_\nu \tag{3.140}$$

and

$$\delta^4 I[A] = \frac{3}{2}\int d^dx\, \delta^2 F^{a\mu\nu}\delta^2 F^a_{\mu\nu} = 6 f^{abc} f^{ade}\int d^dx\, a^{b\mu} a^{c\nu} a^d_\mu a^e_\nu\,. \tag{3.141}$$

We can insert the relations (3.137-3.141) in the following expansion

$$I[\bar{A}+a] = I[\bar{A}] + \delta I[\bar{A}] + \frac{1}{2}\delta^2 I[\bar{A}] + \frac{1}{3!}\delta^3 I[\bar{A}] + \frac{1}{4!}\delta^4 I[\bar{A}]\,,$$

to find the exact relation:

$$\begin{aligned} I[\bar{A}+a] &= I[\bar{A}] + \int d^dx\, \bar{F}^{\mu\nu}\bar{D}_\mu a_\nu + \frac{1}{2}\int d^dx\, a_\mu\left[-\bar{D}^2 g^{\mu\nu} + 2i\bar{F}^{\mu\nu} + \bar{D}^\mu\bar{D}^\nu\right] a_\nu \\ &\quad + g\int d^dx\, \bar{D}_\mu a^a_\nu\, a^b_\mu a^c_\nu + \frac{1}{4} g^2 f^{abc} f^{ade}\int d^dx\, a^{b\mu} a^{c\nu} a^d_\mu a^e_\nu\,. \end{aligned} \tag{3.142}$$

Note that the background gauge-fixing action (3.61) and the background ghost action (3.62) are already in their "varied form" since they are by construction quadratic in the fields.

We now calculate the functional derivatives of the functional (3.137), of the gauge-fixing action (3.61) and of the ghost action (3.62) that are needed in the flow equations for both the bEAA and the gEAA. We start with the functional $I[A]$ defined in (3.137). It is useful to expand the integrand of (3.137) as follows:

$$\begin{aligned} \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} &= \frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu\right)\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + f^{ade}A^{d\mu}A^{e\nu}\right) \\ &= \frac{1}{2}\left(\partial_\mu A^a_\nu \partial^\mu A^{a\nu} - \partial_\mu A^a_\nu \partial^\nu A^{a\mu}\right) + f^{abc}\partial_\mu A^a_\nu A^{b\mu} A^{c\nu} + \\ &\quad + \frac{1}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu}\,. \end{aligned} \tag{3.143}$$

Using (3.143) as reference, it is easy to write the second functional derivative of (3.137):

$$\begin{aligned} \left.\frac{\delta^2 I[A]}{\delta A^{a_2\alpha_2}_{x_2}\delta A^{a_1\alpha_1}_{x_1}}\right|_{A=0} &= \frac{1}{2}\int_x \left(2\partial_\mu g_{\alpha_1\nu}\delta^{aa_1}\delta_{xx_1}\partial^\mu\delta^\nu_{\alpha_2}\delta^{aa_2}\delta_{xx_2} - \partial_\mu g_{\alpha_1\nu}\delta^{aa_1}\delta_{xx_1}\partial^\nu\delta^\mu_{\alpha_2}\delta^{aa_2}\delta_{xx_2} + \right. \\ &\quad \left. -\partial_\mu g_{\alpha_2\nu}\delta^{aa_2}\delta_{xx_2}\partial^\nu\delta^\mu_{\alpha_1}\delta^{aa_1}\delta_{xx_1}\right) \\ &= \delta^{a_1a_2}\int_x \left[g_{\alpha_1\alpha_2}\partial_\mu\delta_{xx_1}\partial^\mu\delta_{xx_2} - \partial_{\alpha_2}\delta_{xx_1}\partial_{\alpha_1}\delta_{xx_2}\right] \\ &\to \delta^{a_1a_2}\left[-g_{\alpha_1\alpha_2}(p_1\cdot p_2) + \frac{1}{2}\left(p_{1\alpha_2}p_{2\alpha_1} + p_{1\alpha_1}p_{2\alpha_2}\right)\right]\,, \end{aligned} \tag{3.144}$$

where we integrated by parts the term $\int_x \partial_{\alpha_2}\delta_{xx_1}\partial_{\alpha_1}\delta_{xx_2} = \int_x \partial_{\alpha_1}\delta_{xx_1}\partial_{\alpha_2}\delta_{xx_2}$ to symmetrize the final expression. The arrow in the last step means that we are "translating" the expression to momentum space following the basic rule $\partial_\mu \delta_{xx_i} \to i p_{i\mu}$. We can write (3.144) as:

$$[I^{(2)}_{p_1,p_2}]^{\alpha\beta\, ab} = (2\pi)^d \delta_{p_1+p_2} \delta^{ab} \left[-g^{\alpha\beta}(p_1 \cdot p_2) + \frac{1}{2}\left(p_1^\alpha p_2^\beta + p_1^\beta p_2^\alpha \right) \right] . \qquad (3.145)$$

The third functional derivative of (3.137), which gives the gauge three-vertex, is:

$$\left.\frac{\delta^3 I[A]}{\delta A^{a_3\alpha_3}_{x_3} \delta A^{a_2\alpha_2}_{x_2} \delta A^{a_1\alpha_1}_{x_1}}\right|_{A=0} = f^{abc} \int_x \left[(\partial_\mu g_{\alpha_1 \nu} \delta^{aa_1} \delta_{xx_1}) \left(\delta^\mu_{\alpha_2} \delta^{ba_2} \delta_{xx_2} \right) \left(\delta^\nu_{\alpha_3} \delta^{ca_3} \delta_{xx_3} \right) + \ldots \right]$$
$$\to f^{a_1 a_2 a_3} i p_{1\alpha_2} g_{\alpha_1 \alpha_3} + \ldots$$

where the dots stand for permutations of the indices and momenta. This can be rewrite as:

$$[I^{(3)}_{p_1,p_2,p_3}]^{\alpha\beta\gamma\, abc} = (2\pi)^d \delta_{p_1+p_2+p_3} i f^{abc} \left[g^{\alpha\beta}(p_2 - p_1)^\gamma + g^{\beta\gamma}(p_3 - p_2)^\alpha + g^{\gamma\alpha}(p_1 - p_3)^\beta \right] . \qquad (3.146)$$

The fourth functional derivative of (3.137), which gives the gauge four-vertex, is:

$$\left.\frac{\delta^4 I[A]}{\delta A^{a_4\alpha_4}_{x_4} \delta A^{a_3\alpha_3}_{x_3} \delta A^{a_2\alpha_2}_{x_2} \delta A^{a_1\alpha_1}_{x_1}}\right|_{A=0} = \frac{1}{4} f^{abc} f^{ade} \int_x \left(g_{\alpha_1 \mu} \delta^{ba_1} \delta_{xx_1} \right) \left(g_{\alpha_2 \nu} \delta^{ca_2} \delta_{xx_2} \right) \times$$
$$\times \left(\delta^\mu_{\alpha_3} \delta^{da_3} \delta_{xx_3} \right) \left(\delta^\nu_{\alpha_4} \delta^{ea_4} \delta_{xx_4} \right) + \ldots$$
$$\to \frac{1}{4} f^{aa_1a_2} f^{aa_3a_4} g_{\alpha_1\alpha_3} g_{\alpha_2\alpha_4} + \ldots$$

or as:

$$\begin{aligned} [I^{(4)}_{p_1,p_2,p_3,p_4}]^{\alpha_1\alpha_2\alpha_3\alpha_4\, a_1a_2a_3a_4} = \; & (2\pi)^d \delta_{p_1+p_2+p_3+p_4} \left[f^{aa_1a_2} f^{aa_3a_4} \left(g^{\alpha_1\alpha_3} g^{\alpha_2\alpha_4} - g^{\alpha_1\alpha_4} g^{\alpha_2\alpha_3} \right) + \right. \\ & f^{aa_1a_3} f^{aa_2a_4} \left(g^{\alpha_1\alpha_2} g^{\alpha_3\alpha_4} - g^{\alpha_1\alpha_4} g^{\alpha_2\alpha_3} \right) + \\ & \left. f^{aa_1a_4} f^{aa_2a_3} \left(g^{\alpha_1\alpha_2} g^{\alpha_3\alpha_4} - g^{\alpha_1\alpha_3} g^{\alpha_2\alpha_4} \right) \right] . \end{aligned} \qquad (3.147)$$

Note that in the next sections we will use the vertices (3.146) and (3.147) to construct the zero-field proper vertices of the bEAA with both gauge fluctuation or gauge background legs. This is possible, since for a gauge invariant action as is (3.137), the following property holds:

$$\frac{\delta I[A]}{\delta A^a_\mu} = \frac{\delta I[\bar{A} + a]}{\delta a^a_\mu} = \frac{\delta I[\bar{A} + a]}{\delta \bar{A}^a_\mu} . \qquad (3.148)$$

Within truncation (3.17), the vertices of the action (3.137) just considered are all we need.

Next, we need the vertices coming from the gauge-fixing action (3.61). The two gauge fluctuation functional derivative is:

$$\left.\frac{\delta^2 S_{gf}}{\delta a_{x_2}^{a_2\alpha_2}\delta a_{x_1}^{a_1\alpha_1}}\right|_{a=\bar{A}=0} = \frac{1}{\alpha}\int_x \partial_\mu\delta^{ab}\delta^\mu_{\alpha_1}\delta^{ba_1}\delta_{xx_1}\partial_\nu\delta^{ac}\delta^\nu_{\alpha_2}\delta^{ca_2}\delta_{xx_2}$$
$$\to -\frac{1}{\alpha}\delta^{a_1a_2}p_{1\alpha_1}p_{2\alpha_2}\,,$$

thus we have

$$[S^{(2,0,0;0)}_{gf\;p,-p}]^{\alpha\beta\,ab} = -\frac{1}{\alpha}\delta^{ab}p^\alpha p^\beta\,. \tag{3.149}$$

The mixed functional derivatives give the gauge fluctuation-fluctuation-background vertex:

$$\left.\frac{\delta^3 S_{gf}}{\delta\bar{A}_y^{b\beta}\delta a_{x_2}^{a_2\alpha_2}\delta a_{x_1}^{a_1\alpha_1}}\right|_{a=\bar{A}=0} = \frac{1}{\alpha}\int_x\left(f^{aa_1b}g_{\alpha_1\beta}\delta_{xy}\delta_{xx_1}\partial_{\alpha_2}\delta^{aa_2}\delta_{xx_2}+\partial_{\alpha_1}\delta^{aa_1}\delta_{xx_1}f^{aa_2b}g_{\alpha_2\beta}\delta_{xy}\delta_{xx_2}\right)$$
$$\to \frac{1}{\alpha}\left(f^{a_2a_1b}g_{\alpha_1\beta}ip_{2\alpha_2}+f^{a_1a_2b}g_{\alpha_2\beta}ip_{1\alpha_1}\right)\,,$$

and so

$$[S^{(2,0,0;1)}_{gf\;p_1,p_2,p_3}]^{\alpha\beta\gamma\,abc} = (2\pi)^d\delta_{p_1+p_2+p_3}\frac{1}{\alpha}if^{abc}\left(g^{\beta\gamma}p_1^\alpha - g^{\alpha\gamma}p_2^\beta\right)\,. \tag{3.150}$$

Four mixed functional derivatives give the fluctuation-fluctuation-background-background vertex:

$$\left.\frac{\delta^4 S_{gf}}{\delta\bar{A}_{y_2}^{b_2\beta_2}\delta\bar{A}_{y_1}^{b_1\beta_1}\delta a_{x_2}^{a_2\alpha_2}\delta a_{x_1}^{a_1\alpha_1}}\right|_{a=\bar{A}=0} = \frac{1}{\alpha}\int_x\left(f^{aa_1b_1}g_{\alpha_1\beta_1}\delta_{xy_1}\delta_{xx_1}f^{aa_2b_2}g_{\alpha_2\beta_2}\delta_{xy_2}\delta_{xx_2}+\right.$$
$$\left.+f^{aa_1b_2}g_{\alpha_1\beta_2}\delta_{xy_2}\delta_{xx_1}f^{aa_2b_1}g_{\alpha_2\beta_1}\delta_{xy_1}\delta_{xx_2}\right)$$
$$\to \frac{1}{\alpha}\left(f^{aa_1b_1}f^{aa_2b_2}g_{\alpha_1\beta_1}g_{\alpha_2\beta_2}+f^{aa_1b_2}f^{aa_2b_1}g_{\alpha_1\beta_2}g_{\alpha_2\beta_1}\right)\,,$$

thus

$$[S^{(2,0,0;2)}_{gf\;p_1,p_2,p_3,p_4}]^{\alpha\beta\mu\nu\,abcd} = (2\pi)^d\delta_{p_1+p_2+p_3+p_4}\frac{1}{\alpha}\left(f^{abe}f^{cde}g_{\alpha\beta}g_{\mu\nu}+f^{ade}f^{bce}g_{\alpha\nu}g_{\beta\mu}\right)\,. \tag{3.151}$$

The vertices (3.150) and (3.151) will be used in section 3.5.4. The ghost action (3.62), when we write out explicitly the covariant derivatives, reads:

$$S_{gh}[a,\bar{c},c;\bar{A}] = \int d^dx\left(\partial_\mu\bar{c}^a+f^{abc}\bar{A}^b_\mu\bar{c}^c\right)\left(\partial^\mu c^a+f^{ade}\bar{A}^{d\mu}c^e+f^{ade}a^{d\mu}c^e\right)\,. \tag{3.152}$$

Note first that (3.152) will generate the three-vertices ghost-ghost-fluctuation and ghost-ghost-background but only the four-vertex ghost-ghost-background-background. The two three-vertices differ by a factor of two since the background field enters both covariant derivatives while the fluctuation field does not. We have:

$$\left.\frac{\delta^3 S_{gh}}{\delta a_{x_3}^{a_3\alpha}\delta c_{x_2}^{a_2}\delta \bar{c}_{x_1}^{a_1}}\right|_{\varphi=\bar{A}=0} = \int_x \partial_\mu \delta_{xx_1}\delta^{aa_1} f^{ade}\delta_{xx_3}\delta^{da_3}\delta^\mu_\alpha \delta_{xx_3}\delta^{ea_2} \to if^{a_1a_3a_2}p_{1\alpha}\,,$$

or

$$[S^{(1,1,1;0)}_{gh\;p_1,p_2,p_3}]^{\alpha\,abc} = -(2\pi)^d \delta_{p_1+p_2+p_3} i f^{abc} p_{2\alpha}\,. \tag{3.153}$$

We find also:

$$\begin{aligned}\left.\frac{\delta^3 S_{gh}}{\delta \bar{A}_{x_3}^{a_3\alpha}\delta c_{x_2}^{a_2}\delta \bar{c}_{x_1}^{a_1}}\right|_{\varphi=\bar{A}=0} &= \int \big[\partial_\mu \delta_{xx_1}\delta^{aa_1} f^{ade}\delta_{xx_3}\delta^{da_3}\delta^\mu_\alpha \delta_{xx_3}\delta^{ea_2} \\ &\quad + f^{abc}\delta_{xx_3}\delta^{ba_3} g_{\mu\alpha}\delta_{xx_1}\delta^{ca_1}\partial^\mu \delta_{xx_2}\delta^{aa_2}\big] \\ &\to -if^{a_1a_2a_3}p_{1\alpha} + if^{a_2a_3a_1}p_{2\alpha}\,,\end{aligned}$$

and thus

$$[S^{(0,1,1;1)}_{gh\;p_1,p_2,p_3}]^{\alpha\,abc} = (2\pi)^d \delta_{p_1+p_2+p_3} i f^{abc} (p_2 - p_1)_\alpha\,. \tag{3.154}$$

Note that in both (3.153) and (3.154) the indices and the momentum variables are related in the precise order they appear. Finally the four-vertex is:

$$\begin{aligned}\left.\frac{\delta^4 S_{gh}}{\delta \bar{A}_{x_4}^{a_4\beta}\delta \bar{A}_{x_3}^{a_3\alpha}\delta c_{x_2}^{a_2}\delta \bar{c}_{x_1}^{a_1}}\right|_{\varphi=\bar{A}=0} &= \int d^d x\, \big[f^{abc}\delta_{xx_3}\delta^{ba_3} g_{\mu\alpha}\delta_{xx_1}\delta^{ca_1} f^{ade}\delta_{xx_4}\delta^{da_4}\delta^\mu_\beta \delta_{xx_2}\delta^{ea_2} \\ &\quad + f^{abc}\delta_{xx_4}\delta^{ba_4} g_{\mu\beta}\delta_{xx_1}\delta^{ca_1} f^{ade}\delta_{xx_3}\delta^{da_3}\delta^\mu_\alpha \delta_{xx_2}\delta^{ea_2}\big] \\ &\to g_{\alpha\beta}\left(f^{aa_3a_1}f^{aa_4a_2} + f^{aa_4a_1}f^{aa_3a_2}\right)\,,\end{aligned}$$

and so:

$$[S^{(0,1,1;2)}_{gh\;p_1,p_2,p_3,p_4}]^{\alpha\beta\,abcd} = (2\pi)^d \delta_{p_1+p_2+p_3+p_4} g^{\alpha\beta}\left(f^{eac}f^{ebd} + f^{ead}f^{ebc}\right)\,. \tag{3.155}$$

With this we derived all the variations and all the vertices that we will use in the next section to explicitly calculate the running of the coupling present in truncation (3.17) and (3.19) that we considered in the main part of the chapter.

3.5.2 Regularized Propagator

In this section we construct the regularized propagators and we choose the cutoff kernels and the cutoff operators.

The Hessian that enters both the flow equation for the bEAA and the flow equation for gEAA, within truncation (3.18) and (3.19) is:[8]

$$\begin{aligned}[\gamma^{(2,0,0;0)}_{p,-p}]^{\alpha\beta\, ab} &= [\bar{\gamma}^{(2)}_{p,-p}+\hat{\gamma}^{(2,0,0;0)}_{p,-p}]^{\alpha\beta\, ab} \\ &= \Omega\, Z_{a,k}\,\delta^{ab}\left\{g^{\alpha\beta}m^2_{a,k}+p^2\left[(1-P)^{\alpha\beta}+\frac{1}{\alpha_k}P^{\alpha\beta}\right]\right\}, \end{aligned} \tag{3.156}$$

where we introduced the orthogonal longitudinal and transverse projectors $P^{\alpha\beta}=p^\alpha p^\beta/p^2$ and $(1-P)^{\alpha\beta}$. Here $\Omega=(2\pi)^d\delta(0)$ is a spacetime volume factor. The fluctuation field regularized inverse propagator, defined by

$$G_p=\Omega\frac{Z_{a,k}}{\gamma^{(2,0,0;0)}_{p,-p}+R_p}\,, \tag{3.157}$$

that follows from (3.156) is:

$$[G^{-1}_p]^{\alpha\beta\, ab}=\delta^{ab}\left\{g^{\alpha\beta}m^2_{a,k}+p^2\left[(1-P)^{\alpha\beta}+\frac{1}{\alpha_k}P^{\alpha\beta}\right]\right\}+[R_p]^{\alpha\beta\, ab}\,. \tag{3.158}$$

Note that we have factored out $\Omega Z_{a,k}$ in the regularized propagator and in the cutoff kernel. We have now to choose the tensor structure of the cutoff kernel, there are two basic ways to do this:

$$[R_p]^{\alpha\beta\, ab} = \delta^{ab}\left[(1-P)^{\alpha\beta}+\frac{1}{\alpha_k}P^{\alpha\beta}\right]R_p \tag{3.159}$$

$$[R_p]^{\alpha\beta\, ab} = \delta^{ab}g^{\alpha\beta}R_q\,. \tag{3.160}$$

Note that in the first case, (3.159), we are introducing in the cutoff kernel the running gauge coupling α_k: this will give rise to additional terms on the rhs of the flow equation, generated by the $\partial_t R_k$ factor, proportional to $\partial_t\alpha_k$. It is not clear how this terms should be interpreted. Using the relation

$$[\mathbf{1}a+(\mathbf{1}-\mathbf{P})b+\mathbf{P}c]^{-1}=\frac{1}{a+b}(\mathbf{1}-\mathbf{P})+\frac{1}{a+c}\mathbf{P}\,,$$

we can invert equation (3.158) to obtain the regularized propagator. In the case that we are

[8] In this technical section, as in the next we omit to write the scale index k on the functionals to simplify the notation.

using the cutoff kernel as defined in (3.159), we find the form:

$$[G_p]^{\alpha\beta\,ab} = \delta^{ab}\frac{1}{p^2+m_{a,k}^2+R_p}(1-P)^{\alpha\beta} + \delta^{ab}\frac{\alpha_k}{p^2+\alpha_k m_{a,k}^2+R_p}P^{\alpha\beta}\,, \tag{3.161}$$

while in the case we are using the cutoff kernel as defined in (3.160), we get instead:

$$[G_p]^{\alpha\beta\,ab} = \delta^{ab}\frac{1}{p^2+m_{a,k}^2+R_p}(1-P)^{\alpha\beta} + \delta^{ab}\frac{\alpha_k}{p^2+\alpha_k\left(m_{a,k}^2+R_p\right)}P^{\alpha\beta}\,. \tag{3.162}$$

In this thesis we will consider the second case corresponding to (3.160). This is the minimal cutoff choice we can make. We define the transverse and longitudinal regularized propagators as

$$G_{T,k}(z) = \frac{1}{z+m_{a,k}^2+R_k(z)} \tag{3.163}$$

$$G_{L,k}(z) = \frac{\alpha_k}{z+\alpha_k\left(m_{a,k}^2+R_k(z)\right)}\,. \tag{3.164}$$

We can now write (3.162) as follows:

$$[G_p]^{\alpha\beta\,ab} = \delta^{ab}(1-P)^{\alpha\beta}G_p^T + \delta^{ab}P^{\alpha\beta}G_p^L\,.$$

Note that $G_{L,k}(z) = 0$ if $\alpha_k = 0$ and that $G_{L,k}(z) = G_{T,k}(z)$ if $\alpha_k = 1$.

The inverse regularized ghost propagator is easily obtained:

$$[\gamma^{(0,1,1;0)}_{p,-p}]^{ab} = Z_{c,k}[S^{(0,1,1;0)}_{gf\;p,-p}]^{ab} = Z_{c,k}\delta^{ab}(p^2+m_{c,k}^2)\,, \tag{3.165}$$

with the minimal cutoff kernel choice:

$$[R^c_p]^{ab} = \delta^{ab}R_p\,, \tag{3.166}$$

we find the form:

$$[G^c_p]^{ab} = \delta^{ab}\frac{1}{p^2+m_{c,k}^2+R_p}\,. \tag{3.167}$$

We will often write:

$$G_{c,k}(z) = \frac{1}{z+m_{c,k}^2+R_k(z)}\,. \tag{3.168}$$

This completes the construction of the regularized propagators needed in the flow equations for the bEAA.

3.5.3 Derivation of $\partial_t Z_{\bar{A},k}$

In this section we derive the beta function for the background wave-function renormalization. This will also give the RG flow of the gauge coupling since these are related by $g = Z_{\bar{A}}^{-1}$. In this section we omit to write the k-dependence of the coupling constants explicitly.

We will set the gauge-fixing parameter to $\alpha = 1$ and first show how to calculate the running of the wave-function renormalization of the background field $Z_{\bar{A}}$ using the heat kernel expansion. From (3.18) we see that to extract $\partial_t Z_{\bar{A}}$ we need to consider the flow equation (3.75) for the gEAA:

$$\begin{aligned}\partial_t \bar{\Gamma}_k[\bar{A}] &= \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2,0,0,0)}[0,0,0;\bar{A}] + R_{k,aa}[\bar{A}]\right)^{-1} \partial_t R_{k,aa}[\bar{A}] + \\ &\quad -\mathrm{Tr}\left(\Gamma_k^{(0,1,1,0)}[0,0,0;\bar{A}] + R_{k,\bar{c}c,}[\bar{A}]\right)^{-1} \partial_t R_{k,\bar{c}c}[\bar{A}]\,, \end{aligned} \tag{3.169}$$

where, consistently with the previous section, we make the following definitions:

$$R_{k,aa}[\bar{A}]^{ab\,\mu\nu} = Z_a R_k \delta^{ab} g^{\mu\nu} \qquad\qquad R_{k,\bar{c}c}[\bar{A}]^{ab} = Z_c R_k \delta^{ab}\,. \tag{3.170}$$

We need to calculate the Hessian's involved in the flow equation (3.169), using the general decomposition of the bEAA (3.65) they can be written as follows:

$$\Gamma_k^{(2,0,0;0)}[a,\bar{c},c;\bar{A}] = \bar{\Gamma}_k^{(2)}[\bar{A}+a] + \hat{\Gamma}_k^{(2,0,0;0)}[a,\bar{c},c;\bar{A}] \tag{3.171}$$

and

$$\Gamma_k^{(0,1,1;0)}[a,\bar{c},c;\bar{A}] = \hat{\Gamma}_k^{(0,1,1;0)}[a,\bar{c},c;\bar{A}]\,. \tag{3.172}$$

Within the truncation we are considering, (3.18) and (3.19), we find:

$$\begin{aligned}\bar{\Gamma}_k^{(2)}[\bar{A}+a]^{ab\,\mu\nu} &= Z_a\left(-D^2\delta^{ab}g^{\mu\nu} + 2f^{abc}F^{c\mu\nu} + D^{ac\mu}D^{cb\nu}\right) \\ \hat{\Gamma}_k^{(2,0,0;0)}[a,\bar{c},c;\bar{A}]^{ab\,\mu\nu} &= Z_a\left(-\frac{1}{\alpha}\bar{D}^{ac\mu}\bar{D}^{cb\nu} + m_a^2\delta^{ab}g^{\mu\nu}\right) \\ \hat{\Gamma}_k^{(0,1,1;0)}[a,\bar{c},c;\bar{A}]^{ab} &= Z_c\left(-\bar{D}_\mu^{ac}D^{cb\mu} + m_c^2\delta^{ab}\right)\,. \end{aligned} \tag{3.173}$$

Note that is we set $Z_a = 1$ and $m_a^2 = Z_c = m_c^2 = 0$ we have $\hat{\Gamma}_k^{(2,0,0;0)}[a,\bar{c},c;\bar{A}] = S_{gf}^{(2,0,0;0)}[a,\bar{c},c;\bar{A}]$ and $\hat{\Gamma}_k^{(0,1,1;0)}[a,\bar{c},c;\bar{A}] = S_{gh}^{(0,1,1;0)}[a,\bar{c},c;\bar{A}]$. At zero fluctuation fields $a = \bar{c} = c = 0$ the Hessian's (3.171) and (3.172) become:

$$\begin{aligned}\Gamma_k^{(2,0,0;0)}[0,0,0;\bar{A}]^{ab\,\mu\nu} &= Z_a\left[-\bar{D}^2\delta^{ab}g^{\mu\nu}+2f^{abc}\bar{F}^{c\mu\nu}+m_a^2\delta^{ab}g^{\mu\nu}+\left(1-\frac{1}{\alpha}\right)\bar{D}^{ac\mu}\bar{D}^{cb\nu}\right]\\ \Gamma_k^{(0,1,1;0)}[0,0,0;\bar{A}]^{ab} &= Z_c\delta^{ab}\left(-\bar{D}^2+m_c^2\right)\,. \end{aligned} \tag{3.174}$$

From now on we set $\alpha = 1$ so that the fluctuation Hessian in (3.174) becomes the Laplace-type operator $D_T^{\mu\nu} = -D^2 g^{\mu\nu} + 2iF^{\mu\nu}$. We need now to choose the cutoff operator, the eigenvalues of which, we compare to the RG scale k to separate the fast from the slow field modes. Without introducing running couplings in the cutoff action, apart for the wave-function renormalization of the fluctuation fields, there are two possible choices in the gauge sector. Looking at equation (3.174), we see that we can take as cutoff operator simply the covariant Laplacian $-\bar{D}^2 g^{\mu\nu}$, or instead the full the Laplace-type differential operator $D_T^{\mu\nu}$. Cutoff actions constructed in this way are called respectively type I and type II. In the ghost sector we consider the covariant Laplacian as cutoff operator in both cases.

We start by deriving the beta functions employing the type II cutoff. Considering (3.174), the flow equation (3.169) can be written as:

$$\partial_t\bar{\Gamma}_k[\bar{A}] = \frac{1}{2}\mathrm{Tr}_{1c}\frac{\partial_t R_k\left(\bar{D}_T\right)-\eta_a R_k\left(\bar{D}_T\right)}{\bar{D}_T^{\mu\nu}+R_k\left(\bar{D}_T\right)^{\mu\nu}+m_a^2 g^{\mu\nu}} - \mathrm{Tr}_{0c}\frac{\partial_t R_k\left(-\bar{D}^2\right)-\eta_c R_k\left(-\bar{D}^2\right)}{-\bar{D}^2+R_k\left(-\bar{D}^2\right)+m_c^2}\,. \tag{3.175}$$

In (3.175) we emphasize that the traces are also over spacetime as well as color indices. If on the lhs of the flow equation (3.175) we insert the truncation ansatz (3.17) we find:

$$\partial_t\bar{\Gamma}_k[\bar{A}] = \partial_t Z_{\bar{A}}\frac{1}{4}\int d^dx\,\bar{F}^a_{\mu\nu}\bar{F}^{a\mu\nu}\,. \tag{3.176}$$

This will be compared with the expansion of the trace terms. We now use the local heat kernel expansion, equations (A.4) and (A.38) from Appendix A, for the operators $\bar{D}_T$ and $-\bar{D}^2$, to expand the traces on the rhs side of equation (3.175) in terms of gauge invariant operators. In particular, we are interested to the terms proportional to $\frac{1}{4}\int\bar{F}^2$, to compare with (3.176) to extract the beta function $\partial_t Z_{\bar{A},k}$. They are:

$$\begin{aligned}\partial_t\bar{\Gamma}_k[\bar{A}]\big|_{\frac{1}{4}\int\bar{F}^2} &= \frac{1}{(4\pi)^{d/2}}\left\{\frac{1}{2}B_2\left(\bar{D}_T\right)Q_{\frac{d}{2}-2}\left[\left(\partial_t R_k-\eta_a R_k\right)G_{T,k}\right]\right.\\ &\qquad\left.-B_2\left(-\bar{D}^2\right)Q_{\frac{d}{2}-2}\left[\left(\partial_t R_k-\eta_c R_k\right)G_{c,k}\right]\right\}\,. \end{aligned}\tag{3.177}$$

The heat kernel coefficients in (3.177) are given in equation (A.5) of Appendix A. For the differential operator $\bar{D}_T$ we have the following heat kernel coefficient:

$$
\begin{aligned}
B_2\left(\bar{D}_T\right) &= \int d^dx \left[\frac{1}{2}\mathrm{tr}\, U^2 + \frac{1}{12}\mathrm{tr}\,\Omega^2\right] \\
&= \int d^dx \left[\frac{1}{2}\left(2f^{abc}\bar{F}^{c\mu\nu}\right)\left(2f^{abd}\bar{F}^d_{\mu\nu}\right) + \frac{1}{12}\mathrm{tr}\,\left(-i\bar{F}^{\mu\nu}\right)\left(-i\bar{F}_{\mu\nu}\right)\right] \\
&= \frac{24-d}{12} N \int d^dx \bar{F}^2 \,. \qquad (3.178)
\end{aligned}
$$

In (3.178) we used the relations $U^{ab\mu\nu} = 2f^{abc}\bar{F}^{c\mu\nu}$, $f^{abc}f^{abd} = N\delta^{ab}$ and $\Omega_{\mu\nu} = -iF_{\mu\nu}$. For the ghost operator $-\bar{D}^2$ we find the following heat kernel coefficient:

$$
B_2\left(-\bar{D}^2\right) = \int d^dx \frac{1}{12}\mathrm{tr}\,\Omega^2 = -\frac{N}{12}\int d^dx \bar{F}^2 \,. \qquad (3.179)
$$

Inserting (3.178) and (3.179) in (3.177) and comparing with (3.176) finally gives:

$$
\partial_t Z_{\bar{A}} = \frac{N}{(4\pi)^{d/2}} \left\{ \frac{24-d}{6} Q_{\frac{d}{2}-2}\left[\left(\partial_t R_k - \eta_a R_k\right) G_{T,k}\right] + \frac{1}{3} Q_{\frac{d}{2}-2}\left[\left(\partial_t R_k - \eta_c R_k\right) G_{c,k}\right] \right\} . \qquad (3.180)
$$

Equation (3.180) represents the beta function for the wave-function renormalization of the background field within truncation (3.18-3.19) in the gauge for $\alpha = 1$ and within a type II cutoff.

If instead we employ a type I cutoff we need to recalculate only the gauge contribution to the flow equation (3.175). Now it reads:

$$
\frac{1}{2}\mathrm{Tr}_{1c} \frac{\partial_t R_k\left(-\bar{D}^2\right) - \eta_a R_k\left(-\bar{D}^2\right)}{-\bar{D}^2 g^{\mu\nu} + 2i\bar{F}^{\mu\nu} + R_k(-\bar{D}^2) g^{\mu\nu} + m_a^2 g^{\mu\nu}} \,. \qquad (3.181)
$$

To collect all terms proportional to $\frac{1}{4}\int \bar{F}^2$ in (3.181), we expand the denominator in powers of the curvature:

$$
\begin{aligned}
\frac{1}{-\bar{D}^2 + 2i\bar{F} + R_k(-\bar{D}^2) + m_a^2} &= G_{T,k}(-\bar{D}^2) - 2i\, G_{T,k}(-\bar{D}^2)\bar{F}\, G_{T,k}(-\bar{D}^2) \\
&\quad + 4G_{T,k}(-\bar{D}^2)\bar{F}\, G_{T,k}(-\bar{D}^2)\bar{F}\, G_{T,k}(-\bar{D}^2) \\
&\quad + O\left(\bar{F}^3\right) \,. \qquad (3.182)
\end{aligned}
$$

The first term in (3.182) generates a contribution proportional to $\frac{1}{4}\int \bar{F}^2$ when we expand the trace using the heat kernel expansion, while the third term is already proportional to $\frac{1}{4}\int \bar{F}^2$.

Collecting the contributions of these terms we are lead to the following beta function for the wave-function renormalization within a type I cutoff choice:

$$\begin{aligned}\partial_t Z_{\bar{A}} &= \frac{N}{(4\pi)^{d/2}}\left\{-\frac{d}{6}Q_{\frac{d}{2}-2}\left[(\partial_t R_k-\eta_a R_k)G_{T,k}\right]+8Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_a R_k)G_{T,k}^3\right]\right.\\&\left.+\frac{1}{3}Q_{\frac{d}{2}-2}\left[(\partial_t R_k-\eta_c R_k)G_{c,k}\right]\right\}\,. \qquad (3.183)\end{aligned}$$

Given a cutoff choice we can evaluate the beta functions (3.180) and (3.183) explicitly. We first consider the optimized cutoff (2.12), we find for type I:

$$\partial_t Z_{\bar{A}} = \frac{N\,k^{d-4}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left[-\frac{d}{6}\frac{d-2-\eta_a}{1+m_a^2/k^2}+\frac{32}{d(d+2)}\frac{d+2-\eta_a}{(1+m_a^2/k^2)^3}+\frac{1}{3}\frac{d-2-\eta_c}{1+m_c^2/k^2}\right]\,, \qquad (3.184)$$

while for type II:

$$\partial_t Z_{\bar{A}} = \frac{N\,k^{d-4}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left[\frac{24-d}{6}\frac{d-2-\eta_a}{1+m_a^2/k^2}+\frac{1}{3}\frac{d-2-\eta_c}{1+m_c^2/k^2}\right]\,. \qquad (3.185)$$

Equation (3.184) and (3.185) are the main results of this section.

Note that to calculate $\partial_t Z_{\bar{A}}$ for general α we need to know the heat kernel expansion for the operator

$$D_T^{ab,\mu\nu} = -D^2\delta^{ab}g^{\mu\nu}+\left(1-\frac{1}{\alpha}\right)D^{a\mu}D^{b\nu}+2f^{abc}F^{c\mu\nu}\,,$$

which we don't know. Also if we use a type II cutoff with this operator we will insert in the cutoff kernel an additional running of α. One way of performing this calculation is to perform a decomposition of the gauge field into its spin components [56, 57].

3.5.4 Derivation of $\partial_t Z_{\bar{A},k}$ from $\partial_t \bar{\gamma}_k^{(2)}$

We now show how we can derive the beta function for the wave-function renormalization of the background field from the flow equation for the zero-field two-point function of the gEAA employing the techniques exposed in section 3.3.4. The flow equation (3.115) reads

$$\begin{aligned}[\partial_t \gamma^{(0;2)}_{p,-p}]^{AB} &= \frac{1}{2}\int_q \tilde{\partial}_t\left\{[\tilde{\gamma}^{(2;1)}_{q,p,-q-p}]^{4A1}[G_{q+p}]^{12}[\tilde{\gamma}^{(2;1)}_{q+p,-p,-q}]^{2B3}[G_q]^{34}\right\}\\&-\frac{1}{2}\int_q \tilde{\partial}_t\left\{[\tilde{\gamma}^{(2;2)}_{q,p,-p,-q}]^{1AB2}[G_q]_{21}\right\}\,, \qquad (3.186)\end{aligned}$$

where the tilde vertices are given in terms of the zero-field proper-vertices of the bEAA and of the cutoff operator action (3.104). The flow equation (3.186) is represented graphically in Figure 3.10. Remember that the the cutoff action is just that action whose Hessian is the cutoff operator. First, we need to specify the cutoff operator we employ to cutoff the field modes. As already defined in the previous section, there are two basic possibilities, which we called type I and type II, where the cutoff operator is taken, respectively, as the gauge Laplacian $-D^2$ or as the operator D_T defined previously.

We start now by considering type I, where the cutoff action is just:

$$L[a;\bar{A}] = \frac{1}{2}\int d^dx D_\mu a_\nu D^\mu a^\nu \,, \tag{3.187}$$

so that

$$L^{(2;0)}[0;\bar{A}]^{ab\,\alpha\beta}_{xy} = g^{\alpha\beta}\int d^dz D^{ac}_{z\mu}\delta_{zx} D^{cb\mu}_z\delta_{zy} = -g^{\alpha\beta}D^{ac}_{x\mu}D^{cb\mu}_y\delta_{xy} = g^{\alpha\beta}(-D^2)^{ab}\delta_{xy}\,. \tag{3.188}$$

In the flow equation (3.186) we need the vertices for the stemming from the cutoff action (3.188). They are just:

$$\begin{aligned}
[L^{(2;1)}_{p_1,p_2,p_3}]^{abc\,\alpha\beta\gamma} &= (2\pi)^d\delta_{p_1+p_2+p_3}i(p_2-p_1)^\alpha g^{\beta\gamma}f^{abc}\\
[L^{(2;2)}_{p_1,p_2,p_3,p_4}]^{abcd\,\alpha\beta\gamma\delta} &= (2\pi)^d\delta_{p_1+p_2+p_3+p_4}2g^{\alpha\beta}g^{\gamma\delta}f^{ade}f^{bec}\,.
\end{aligned}\tag{3.189}$$

For the ghosts the cutoff operator is just the gauge Laplacian and the cutoff operator action reads simply:

$$L[\bar{c},c;\bar{A}] = \int d^dx D_\mu\bar{c}D^\mu c\,. \tag{3.190}$$

The zero-field vertices implied by (3.190) are the the following:

$$\begin{aligned}
[L^{(1,1;1)}_{p_1,p_2,p_3}]^{abc} &= (2\pi)^d\delta_{p_1+p_2+p_3}i(p_2-p_1)^\alpha f^{abc}\\
[L^{(1,1;2)}_{p_1,p_2,p_3,p_4}]^{abcd} &= (2\pi)^d\delta_{p_1+p_2+p_3+p_4}2g^{\alpha\beta}g^{\gamma\delta}f^{ade}f^{bec}\,.
\end{aligned}\tag{3.191}$$

To extract the beta function for the wave-function renormalization of the background field we project the lhs of the flow equation (3.186) to obtain:

$$(1-P)_{\alpha\beta}[\partial_t\bar{\gamma}^{(2)}_{p,-p}]^{ab\,\alpha\beta} = \delta^{ab}(d-1)\partial_t Z_{\bar{A}}\,p^2\,. \tag{3.192}$$

From (3.192) we see that from the transverse component of the flow equation (3.186) we need

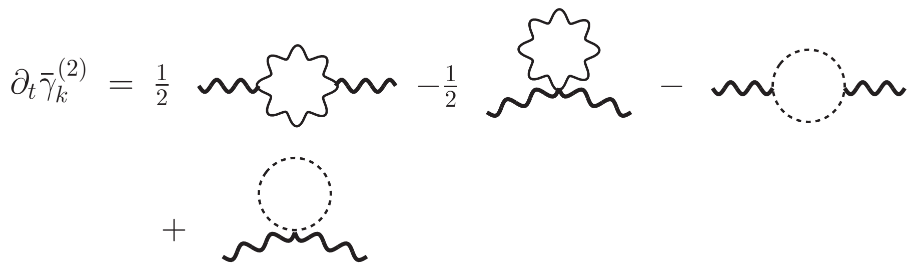


Figure 3.10: Diagrammatic representation of the flow equation for the two point function of the gEAA after the field multiplet decomposition. Thick lines are the represent the background field while light line represent gauge fluctuation and ghost fields.

to extract those terms proportional to p^2. We find:

$$
\begin{aligned}
(1-P)_{\alpha\beta}[\partial_t \bar{\gamma}^{(2)}_{p,-p}]^{ab\,\alpha\beta} \;=\; & -4(d-1)p^2 N\delta^{ab}\int_q \tilde{\partial}_t \left\{G_q G_{q+p}\right\} + \\
& -2dN\delta^{ab}\int_q q^2(1-x^2)\tilde{\partial}_t \left\{G_q G_{q+p}\left(1+R^{(1)}_{q+p,q}\right)^2\right\} + \\
& +2dN\delta^{ab}\int_q q^2(1-x^2)\tilde{\partial}_t \left\{G_q R^{(2)}_{q+p,q}\right\} + \\
& +4N\delta^{ab}\int_q q^2(1-x^2)\tilde{\partial}_t \left\{G^c_q G^c_{q+p}\left(1+R^{(1)}_{q+p,q}\right)^2\right\} + \\
& -4N\delta^{ab}\int_q q^2(1-x^2)\tilde{\partial}_t \left\{G^c_q R^{(2)}_{q+p,q}\right\} , \qquad (3.193)
\end{aligned}
$$

where the contributions in the first two lines come from the first diagram of Figure 3.10 while the contributions in the other three lines come, in order, from the other three diagrams of Figure 3.10. The first integral in equation (3.193) is needed only for $p \to 0$, since the term is already proportional to p^2, and it easy to see that it can be rewritten as a Q-functional in the following way:

$$
\int_q \tilde{\partial}_t G_q^2 = -\frac{2}{(4\pi)^{d/2}} Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_a R_k)G^3\right] . \qquad (3.194)
$$

Note here the important fact that the contributions from the second and and third line combine to give a term of the form:

$$
\int_q q^2(1-x^2)\tilde{\partial}_t \left\{G_q\left[G_{q+p}\left(1+R^{(1)}_{q+p,q}\right)^2 - R^{(2)}_{q+p,q}\right]\right\} . \qquad (3.195)
$$

It can be shown that the following relation is valid in arbitrary dimension:

$$\int_q q^2(1-x^2)\tilde{\partial}_t \left\{ G_q \left[G_{q+p} \left(1+R^{(1)}_{q+p,q}\right)^2 - R^{(2)}_{q+p,q} \right] \right\} =$$

$$= \frac{1}{(4\pi)^{d/2}} Q_{\frac{d}{2}-2}\left[(\partial_t R_k - \eta_a R_k) G_{T,k}\right] \left\{ -1 + \frac{d-1}{12} p^2 \right\} + O\left(p^4\right) . \qquad (3.196)$$

This relation can be easily verified by inserting a sufficiently smooth cutoff shape function and calculating both sides in a Taylor expansion in p. Note also that the ghost contributions in (3.193) combine in the same way. Using the relation (3.196) to extract from the rhs of equation (3.193) the terms of order p^2 and by comparing with (3.192) we finally find the following beta function for the wave function-renormalization of the background field:

$$\begin{aligned} \partial_t Z_{\bar{A}} \;&=\; \frac{N}{(4\pi)^{d/2}} \left\{ 8 Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_a R_k) G^3_{T,k}\right] - \frac{d}{6} Q_{\frac{d}{2}-2}\left[(\partial_t R_k - \eta_a R_k) G_{T,k}\right] + \right. \\ &\quad \left. + \frac{1}{3} Q_{\frac{d}{2}-2}\left[(\partial_t R_k - \eta_a R_k) G_{c,k}\right] \right\} , \end{aligned}$$

which is precisely (3.183).

Within this framework, to consider a type II cutoff means to consider in the above derivation the cutoff action as the following action:

$$L[a;\bar{A}] = \Gamma_k[\bar{A}+a] + S_{gf}[a;\bar{A}] \, , \qquad (3.197)$$

where the actions in (3.197) are the gEAA of our truncation ansatz (3.17) and the gauge fixing action (3.61) for $\alpha = 1$. In the ghost sector we don't make any changes. The gauge contributions to equation (3.186) combine now completely to the following form:

$$-N\delta^{ab} \int_q \left[4(d-1)p^2 + 2dq^2(1-x^2)\right] \tilde{\partial}_t \left\{ G_q \left[G_{q+p} \left(1+R^{(1)}_{q+p,q}\right)^2 - R^{(2)}_{q+p,q} \right] \right\} . \qquad (3.198)$$

Using again the expansion (3.196), this time considering both the p^0 and p^2 terms, to expand the rhs of the flow equation and comparing with (3.192) gives now the following expression for the beta function for the wave-function renormalization of the background gauge field:

$$\partial_t Z_{\bar{A}} = \frac{N}{(4\pi)^{d/2}} \left\{ \frac{24-d}{6} Q_{\frac{d}{2}-2}\left[(\partial_t R_k - \eta_a R_k) G_{T,k}\right] + \frac{1}{3} Q_{\frac{d}{2}-2}\left[(\partial_t R_k - \eta_a R_k) G_{c,k}\right] \right\} .$$

Again, we have re-derived beta function (3.180) found before when employing a type II cutoff.

We have thus shown how the techniques developed in section 3.3.4 can be used to calculate the beta function in the truncation we are considering. This way of evaluating the flow equation for the bEAA is very general and can be applied to any general truncation ansatz, in particular to those that cannot be treated with the aid of the local heat kernel expansion. For example, we are now in the position to calculate the running of the wave-function renormalization of the background field for a general value of the gauge-fixing parameter α_k without any further obstacle and for the cutoff choices we made in (3.170). The aim of this section was just to show how the method introduced in section 3.3.4 can be used to reproduce the results obtained with the aid of the local heat kernel expansion in the previous section and how the various cutoff choices are reflected here. In section 3.5.6 we use this approach to calculate the running of the gauge fluctuation and ghost masses, of the wave-function renormalization of the fluctuation and of the ghost fields and of the gauge-fixing parameter. All other further applications to local truncations of the gEAA are left to future studies.

3.5.5 Derivation of $\partial_t \Pi_k(x)$

In this subsection we derive the running of the vacuum polarization function $\Pi_k(x)$. We set $\eta_{a,k} = \eta_{c,k} = 0$ and $m_{a,k} = m_{c,k} = 0$ in equation (3.175) and we use the non-local heat kernel expansion form Appendix A. The curvature square term in the expansion is:

$$\partial_t \bar{\Gamma}_k[A]\big|_{F^2} = \frac{N}{(4\pi)^{d/2}} \int d^d x F^a_{\mu\nu} \left[\int_0^\infty ds\, \tilde{h}_k(s)\, s^{2-d/2}\, f_{F^2}\left(-sD^2\right) \right] F^{a\mu\nu} \,, \tag{3.199}$$

where the structure function $f_{F^2}(x)$ is found to be:

$$f_{F^2}(x) = f(x) + \frac{d-2}{4x} \left[f(x) - 1 \right] \,, \tag{3.200}$$

and $h_k(z) = G_{T,k}(z) \partial_t R_k(z)$. Here $f(x)$ is the basic form factor (A.10). Comparing the truncation ansatz (3.49) with (3.199) gives the flow equation for the running vacuum polarization function:

$$\partial_t \left[Z_{\bar{A}} \Pi_k(x) \right] + \partial_t Z_{\bar{A},k} = \frac{4N}{(4\pi)^{d/2}} \int_0^\infty ds\, \tilde{h}_k(s)\, s^{2-d/2}\, f_{F^2}(sx) \,. \tag{3.201}$$

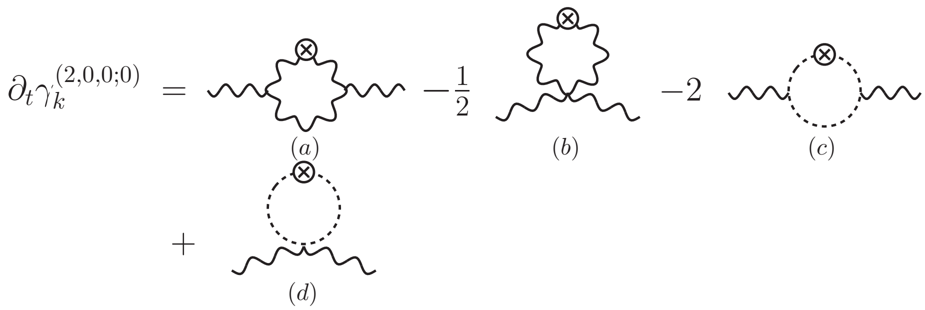


Figure 3.11: Diagrammatic representation of the flow equation for the proper vertex of the bEAA used to calculate the beta functions $\partial_t Z_{a,k}$, $\partial_t m^2_{a,k}$ and $\partial_t \alpha_k$. Note that within truncation (3.18-3.19) the vertex $\gamma_k^{(2,1,1;0)}$ is zero and thus diagram (d) gives no contribution.

Writing every thing in terms of the Q-functionals gives:

$$\begin{aligned}\partial_t\left[Z_{\bar{A}}\Pi_k(x)\right] &= -\partial_t Z_{\bar{A}} + \frac{N}{(4\pi)^{d/2}}\left\{4\int_0^1 d\xi\, Q_{\frac{d}{2}-2}\left[h_k\left(z+x\xi(1-\xi)\right)\right]+\right.\\ &\left. +\frac{d-2}{x}\left(\int_0^1 d\xi\, Q_{\frac{d}{2}-1}\left[h_k\left(z+x\xi(1-\xi)\right)\right]-Q_{\frac{d}{2}-1}[h_k]\right)\right\}\,. \qquad (3.202)\end{aligned}$$

Equation (3.202) when combined with equation (3.180) is the one-loop RG running for the vacuum polarization function within type II cutoff. In the case of type I cutoff the calculation is similar and we don't report it here.

3.5.6 Derivation of $\partial_t m^2_{a,k}$, $\partial_t Z_{a,k}$ and $\partial_t \alpha_k$.

In this section we calculate the beta functions of the gauge fluctuation mass $m_{a,k}$, of the gauge fluctuation wave-function renormalization $Z_{a,k}$ and of the gauge-fixing parameter α_k. We will extract these beta functions form the flow equation for the two-point function $\gamma_k^{(2,0,0;0)}$ of the bEAA. In this section, as in the following, we will omit, for clarity, to explicitly write the scale dependence of the running couplings.

After the multiplet decomposition, and within truncation (3.18-3.19), the flow equation for $\gamma_k^{(2,0,0;0)}$ becomes as in Fig. 3.11. In formulas this can be written as:

$$\begin{aligned}\left[\partial_t\gamma^{(2,0,0;0)}_{p,-p}\right]^{\mu\nu\, mn} &= g^2 Z_a\int_q(\partial_t R_q-\eta_a R_q)\,[a_{p,q}]^{\mu\nu\, mn}-\frac{1}{2}g^2 Z_a\int_q(\partial_t R_q-\eta_a R_q)\,[b_{p,q}]^{\mu\nu\, mn}\\ &\quad -2g^2 Z_a\int_q(\partial_t R_q-\eta_c R_q)\,[c_{p,q}]^{\mu\nu\, mn}\,. \qquad (3.203)\end{aligned}$$

Note first that the last diagram in Figure 3.11, diagram d, is identically zero, since in our truncation there is no term bilinear in the ghost and in the gauge fluctuation fields. Every other diagram is proportional to $g^2 Z_a$ since the gauge fluctuation three-vertex comes with a factor $gZ_a^{3/2}$, the four-vertex with a factor $g^2 Z_a^2$, while the regularized gauge propagators come with a power of Z_a^{-1} and a gauge cutoff kernel insertion with a power of Z_a. In the ghost diagrams the three-vertex gives a factor $gZ_a^{1/2} Z_c$, there is no four-vertex, the regularized ghost propagator has a factor of Z_c^{-1} and the ghost cutoff kernel insertion has a power of Z_c. Also, all the volume factors Ω on both sides of equation (3.203) delete each other. Finally, the tensor products entering the flow equation (3.203) are:

$$
\begin{aligned}
[a_{p,q}]^{\mu\nu\, mn} &= [G_q]^{\alpha\beta\, ab}[G_q]^{\beta\gamma\, bc}[\gamma^{(3,0,0;0)}_{q,p,-q-p}]^{\gamma\mu\delta\, cmd}[G_{q+p}]^{\delta\kappa\, dk}[\gamma^{(3,0,0;0)}_{q+p,-p,-q}]^{\kappa\nu\alpha\, kna} && (3.204)\\
[b_{p,q}]^{\mu\nu\, mn} &= [G_q]^{\alpha\beta\, ab}[\gamma^{(4,0,0;0)}_{q,p,-q-p}]^{\beta\mu\nu\gamma\, bmnc}[G_q]^{\gamma\alpha\, ca} && (3.205)\\
[c_{p,q}]^{\mu\nu\, mn} &= [G^c_q]^{ab}[\gamma^{(1,1,1;0)}_{q,p,-q-p}]^{\mu\, bmc}[G^c_q]^{cd}[\gamma^{(1,1,1;0)}_{q+p,-p,-q}]^{\nu\, dne}[G^c_q]^{ea}\,, && (3.206)
\end{aligned}
$$

together with the following vertices:

$$
\gamma^{(3,0,0;0)}_{p_1,p_2,p_3} = I^{(3)}_{p_1,p_2,p_3} \qquad \gamma^{(4,0,0;0)}_{p_1,p_2,p_3,p_4} = I^{(4)}_{p_1,p_2,p_3,p_4} \qquad \gamma^{(1,1,1;0)}_{p_1,p_2,p_3} = S^{(1,1,1;0)}_{gh,p_1,p_2,p_3}\,. \tag{3.207}
$$

The first vertex in (3.207) is given in equation (3.146), the second in equation (3.147) while the third in equation (3.153). The group factors are calculated like in the standard diagrammatic of non-abelian gauge theories.

To deal with a scalar equation we can project equation (3.203) using the orthogonal projectors $(1-P)^{\mu\nu}$ and $P^{\mu\nu}$. In this way we obtain to independent equations, describing the flow of respectively the transverse and longitudinal components of $\gamma_k^{(2,0,0;0)}$. The transverse equation can be used to extract the running of m_a and Z_a, while the longitudinal equation to extract the running of α. This can be seen by applying the projectors to the lhs of the flow equation (3.203). We find respectively

$$
(1-P)_{\alpha\beta}[\partial_t \bar{\gamma}^{(2)}_{p,-p} + \partial_t \hat{\gamma}^{(2,0,0;0)}_{p,-p}]^{\alpha\beta\, ab} = \Omega\delta^{ab}(d-1)\left\{\partial_t\left(Z_a m_a^2\right) + p^2 \partial_t Z_a\right\} \tag{3.208}
$$

and

$$
P_{\alpha\beta}[\partial_t \bar{\gamma}^{(2)}_{p,-p} + \partial_t \hat{\gamma}^{(2,0,0;0)}_{p,-p}]^{\alpha\beta\, ab} = \Omega\delta^{ab}\left\{\partial_t\left(Z_a m_a^2\right) + p^2 \partial_t\left(\frac{Z_a}{\alpha}\right)\right\}\,. \tag{3.209}
$$

In (3.208) we used the trace $(1-P)^{\alpha}_{\alpha} = d-1$, while in (3.209) we used $P^{\alpha}_{\alpha} = 1$. Equation (3.208) shows that we can find, respectively, $\partial_t\left(Z_a m_a^2\right)$ as the term of order p^0 and $\partial_t Z_a$ as the term of order p^2 of the transverse equation. $\partial_t\left(Z_a m_a^2\right)$ can be extracted also as the term of order p^0 of the longitudinal equation, we will show that these two ways to obtain the running of the gauge fluctuation mass lead to an equal result. Finally, $\partial_t\left(\frac{Z_a}{\alpha}\right)$ is found as the term of order p^2 of the longitudinal equation.

We start studying the transverse component of the flow equation (3.203). To do this we need the projections of the tensors defined previously. Acting on (3.204) gives:

$$
\begin{aligned}
(1-P)^{\mu\nu}[a_{p,q}]^{mn}_{\mu\nu} \;=\; & -N\delta^{mn}\Big\{\left(G^T_q\right)^2 G^T_{q+p}\left[2\left(d-2+x^2\right)p\cdot q\right.\\
& \left.-\left(5d-6-(4d-5)x^2\right)\left(p^2+q^2\right)\right]\\
& -\left(G^L_q\right)^2 G^T_{q+p}\left[\left(d-2+x^2\right)q^2+2\left(2d-3+x^2\right)p\cdot q\right.\\
& \left.+\left(1+(4d-5)\,x^2\right)p^2\right]\\
& +\left(G^T_q\right)^2\left[G^T_{q+p}-G^L_{q+p}\right]\left(d-2+x^2\right)\frac{\left(p^2-q^2\right)^2}{(p+q)^2}\\
& +\left(G^L_q\right)^2\left[G^T_{q+p}-G^L_{q+p}\right]\left(1-x^2\right)\frac{p^4}{(p+q)^2}\Big\}\,. \qquad (3.210)
\end{aligned}
$$

The first factor in (3.210) is the group factor $f^{amb}f^{bnc}=-N\delta^{mn}$. In equation (3.210) and in the following, the variable x is the cosine of the angle between p and q. Note that equation (3.210) simplifies considerably in the two gauges $\alpha=0$, where $G_L=0$, and $\alpha=1$, where $G_L=G_T$. The transverse contribution from (3.205) is:

$$
(1-P)^{\mu\nu}[b_{p,q}]^{mn}_{\mu\nu}=-2N\delta^{mn}\left\{-(d-2+x^2)\left(G^L_q\right)^2-(d^2-3d+3-x^2)\left(G^T_q\right)^2\right\}\,, \qquad (3.211)
$$

here the group factor is $f^{cam}f^{cbn}+f^{cab}f^{cmn}=-2N\delta^{mn}$. From (3.206) we find the transverse contribution from ghost diagram (c):

$$
(1-P)^{\mu\nu}[c_{p,q}]^{mn}_{\mu\nu}=-N\delta^{mn}\left\{-q^2(1-x^2)\left(G^c_q\right)^2 G^c_{q+p}\right\}\,. \qquad (3.212)
$$

In (3.212) the group factor is $f^{abm}f^{ban}=-N\delta^{mn}$. Once we insert equations (3.210), (3.211) and (3.212) back in (3.203) we obtain, within truncation (3.18-3.19), the explicit flow of the vertex $\gamma^{(2,0,0;0)}_{p,-p}$, to all orders in the external momenta p. The momentum integrals in (3.203) can be performed if we use spherical coordinates with the z-axis along p:

$$
\int_q \to \frac{S_{d-1}}{(2\pi)^d}\int_0^\infty dq\, q^{d-1}\int_{-1}^1 dx\left(1-x^2\right)^{\frac{d-3}{2}}\,. \qquad (3.213)
$$

Here $S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ is the volume of the d-dimensional sphere and $x = \cos\theta$ with θ the angle between p and q. We also shift the radial integral to the variable $z = q^2$, so that:

$$\int_0^\infty dq\, q^{d-1} \to \frac{1}{2}\int_0^\infty dz\, z^{\frac{d}{2}-1} . \tag{3.214}$$

After expanding equation (3.203) in powers of p, the x-integrals can be easily performed. After matching the appropriate powers of p the resulting beta functions are expressed as z-integrals.

We start to calculate the running of the fluctuation field mass. If we define

$$\partial_t \left(Z_a m_a^2\right) = \beta_{Z_a m_a^2}\left(g, \eta_a, \eta_c, m_a, m_c, \alpha\right) , \tag{3.215}$$

we have that the beta function for the gauge fluctuation mass squared can be written as:

$$\partial_t m_a^2 = \eta_a\, m_a^2 + Z_a^{-1}\beta_{Z_a m_a^2} . \tag{3.216}$$

The anomalous dimension of the gauge fluctuation field, $\eta_a = -\partial_t \log Z_a$, is given later in equation (3.221). After expressing everything in terms of Q-functionals, defined in equation (A.38) of Appendix A, we finally find:

$$\begin{aligned} Z_a^{-1}\beta_{Z_a m_a^2} \;=\; & \frac{g^2 N}{(4\pi)^{d/2}}\left\{\frac{d-1}{2}Q_{\frac{d}{2}+1}\left[\left(\partial_t R - \eta_a R\right)\left(4\,G_T^3 + G_T^2 G_L + G_T G_L^2\right)\right]\right. \\ & -\frac{2(d-1)}{d}Q_{\frac{d}{2}}\left[\left(\partial_t R - \eta_a R\right)\left((d-1)G_T^2 + G_L^2\right)\right] \\ & \left. +Q_{\frac{d}{2}+1}\left[\left(\partial_t R - \eta_c R\right)G_c^3\right]\right\} . \end{aligned} \tag{3.217}$$

Inserting (3.217) in (3.216) gives the beta function for the gauge fluctuation mass for arbitrary cutoff shape function, gauge-fixing parameter and dimension. In equation (3.217) the contribution in the first line can be traced back to diagram (a), in the second line to diagram (b) and in the last line to diagram (c). The Q-functionals in equation (3.217) can all be evaluated analytically if we employ the optimized cutoff shape function. This is even true for arbitrary value of the gauge-fixing parameter α but the resulting equations are not very illuminating. We will consider here only the cases $\alpha = 1$ and $\alpha = 0$. Using the Q-functionals integrals from Appendix A, we find the following forms:

$$Z_a^{-1}\beta_{Z_a m_a^2}\left(g, \eta_a, \eta_c, m_a, m_c, 1\right) = \frac{g^2 N\, k^{d-2}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left[-\frac{8(d-1)}{d(d+2)}\frac{d+2-\eta_a}{\left(1+m_a^2/k^2\right)^2}\right.$$

$$+\frac{24(d-1)}{d(d+2)(d+4)}\frac{d+4-\eta_a}{(1+m_a^2/k^2)^3}+\frac{8}{d(d+2)(d+4)}\frac{d+4-\eta_c}{(1+m_c^2/k^2)^3}\Bigg] \tag{3.218}$$

and

$$Z_a^{-1}\beta_{Z_a m_a^2}(g,\eta_a,\eta_c,m_a,m_c,0)=\frac{g^2N\,k^{d-2}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\Bigg[-\frac{8(d-1)^2}{d^2(d+2)}\frac{d+2-\eta_a}{(1+m_a^2/k^2)^2}$$

$$+\frac{16(d-1)}{d(d+2)(d+4)}\frac{d+4-\eta_a}{(1+m_a^2/k^2)^3}+\frac{8}{d(d+2)(d+4)}\frac{d+4-\eta_c}{(1+m_c^2/k^2)^3}\,. \tag{3.219}$$

From the terms proportional to p^2 of the transverse component of (3.203) we can extract the beta function of the gauge fluctuation wave-function renormalization which is better accounted for in the form of anomalous dimension:

$$\eta_a(g,\eta_a,\eta_c,m_a,m_c,\alpha)=-\partial_t\log Z_a=-Z_a^{-1}\partial_t Z_a\,. \tag{3.220}$$

As before, we can write everything in terms of Q-functionals, to obtain:

$$\begin{aligned}\eta_a \;=\;& \frac{g^2N}{(4\pi)^{d/2}}\Bigg\{-\frac{8d^2+4d-20}{d(d+2)}Q_{\frac{d}{2}}\left[(\partial_t R-\eta_a R)\,G_T^3\right]\\ &-2(d-1)Q_{\frac{d}{2}+1}\left[(\partial_t R-\eta_a R)\,G_T^2G_T'\right]\\ &-2(d-1)Q_{\frac{d}{2}+2}\left[(\partial_t R-\eta_a R)\,G_T^2G_T''\right]+\frac{(3d+5)(d-2)}{d(d+2)}Q_{\frac{d}{2}}\left[(\partial_t R-\eta_a R)\,G_T^2G_L\right]\\ \\ &-\frac{d^2-3d-6}{2(d+2)}Q_{\frac{d}{2}+1}\left[(\partial_t R-\eta_a R)\,G_T^2G_L'\right]-\frac{d+1}{2}Q_{\frac{d}{2}+2}\left[(\partial_t R-\eta_a R)\,G_T^2G_L''\right]\\ &-\frac{5}{d}Q_{\frac{d}{2}}\left[(\partial_t R-\eta_a R)\,G_L^2G_T\right]-\frac{d^2+9d+10}{2(d+2)}Q_{\frac{d}{2}+1}\left[(\partial_t R-\eta_a R)\,G_L^2G_T'\right]\\ &-\frac{d+1}{2}Q_{\frac{d}{2}+2}\left[(\partial_t R-\eta_a R)\,G_L^2G_T''\right]+Q_{\frac{d}{2}+1}\left[(\partial_t R-\eta_c R)\,G_c^2G_c'\right]\\ &+Q_{\frac{d}{2}+2}\left[(\partial_t R-\eta_c R)\,G_c^2G_c''\right]\Bigg\}\,.\end{aligned} \tag{3.221}$$

Equation (3.221) gives implicitly the anomalous dimension of the gauge fluctuation field as part of a linear system for the variables η_a and η_c. It is valid for arbitrary cutoff shape function, gauge and dimension. In the next section we will calculate and analogous equation for the anomalous dimension η_c of the ghost field that together with (3.221) can be used to solve for both $\eta_a(g,m_a,m_c,\alpha)$ and $\eta_c(g,m_a,m_c,\alpha)$. It is possible to calculate analytically the Q-functionals in (3.221) if we employ the optimized cutoff shape function. In particular, for the gauge-fixing parameters values $\alpha = 1$ and $\alpha = 0$ we have, respectively,

$$\eta_a\left(g,\eta_a,\eta_c,m_a,m_c,1\right) = \frac{g^2 N\, k^{d-4}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left[\frac{4\,(3d-1)}{d(d+2)}\frac{1}{\left(1+m_a^2/k^2\right)^4}\right.$$
$$\left. -\frac{20}{d(d+2)}\frac{d+2-\eta_a}{\left(1+m_a^2/k^2\right)^3} - \frac{4}{d(d+2)}\frac{1}{\left(1+m_c^2/k^2\right)^4}\right] \tag{3.222}$$

and

$$\eta_a\left(g,\eta_a,\eta_c,m_a,m_c,0\right) = \frac{g^2 N\, k^{d-4}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left[\frac{8\,(d-1)}{d(d+2)}\frac{1}{\left(1+m_a^2/k^2\right)^4}+\right.$$
$$\left. -\frac{16\,(2d^2+d-5)}{d^2(d+2)^2}\frac{d+2-\eta_a}{\left(1+m_a^2/k^2\right)^3} - \frac{4}{d(d+2)}\frac{1}{\left(1+m_c^2/k^2\right)^4}\right]. \tag{3.223}$$

The dependence of the anomalous dimension of the fluctuation field on the gauge-fixing parameter can be seen from the simple case where we set $m_a = m_c = 0$ in the rhs of equation (3.221) and we evaluate the Q-functionals employing the optimized cutoff in $d = 4$:

$$\eta_a\left(g,\eta_a,\eta_c,0,0,\alpha\right) = \frac{g^2 N}{(4\pi)^2}\left[-\frac{13-3\alpha}{3}\right.$$
$$\left. +\frac{31-102\alpha+144\alpha^2-58\alpha^3-15\alpha^4+48\alpha^3\log\alpha}{36\,(1-\alpha)^3}\eta_a\right]. \tag{3.224}$$

This shows the strong gauge dependence of the anomalous dimension of the fluctuation field even in $d = 4$, where instead the anomalous dimension of the background field is gauge independent.

We now look at the longitudinal component of the flow equation (3.203). Projecting with $P^{\mu\nu}$ the tensor structure of diagram (a), equation (3.204), gives:

$$\begin{aligned} P^{\mu\nu}[a_{p,q}]^{mn}_{\mu\nu} \;=\; & -N\delta^{mn}\Big\{-\left(G^T_q\right)^2 G^T_{q+p}\left[p^2\left(d-2+x^2\right)+2p\cdot q\left(2d-3+x^2\right)\right.\\ & \left.+q^2\left(1+(4d-5)\,x^2\right)\right] - \left(G^L_q\right)^2 G^T_{q+p}(p+q)^2\left(1-x^2\right)\\ & +\left(G^T_q\right)^2\left[G^T_{q+p}-G^L_{q+p}\right]\left(1-x^2\right)\frac{q^4}{(p+q)^2}\Big\}\,. \end{aligned} \tag{3.225}$$

Here the group factor is as before $-N\delta^{mn}$. Form diagram (b), projecting equation (3.205) gives:

$$P^{\mu\nu}[b_{p,q}]^{mn}_{\mu\nu} = -2N\delta^{mn}\left\{-(1-x^2)\left(G^L_q\right)^2 - (d-2+x^2)\left(G^T_q\right)^2\right\}, \tag{3.226}$$

where the group factor is as before $-2N\delta^{mn}$. For ghost diagram (c) we find, by projecting equation (3.206), the following contribution:

$$P^{\mu\nu}[c_{p,q}]^{mn}_{\mu\nu} = -N\delta^{mn}\left\{2qx\,(p+qx)\left(G^c_q\right)^2 G^c_{q+p}\right\}. \tag{3.227}$$

We can extract now the running of the gauge-fixing parameter α from the terms proportional to p^2 of the longitudinal components of the flow equation (3.203). In particular, if we define

$$\partial_t\left(Z_a/\alpha\right) = \beta_{Z_a/\alpha}\left(g,\eta_a,\eta_c,m_a,m_c,\alpha\right), \tag{3.228}$$

we can write the beta function of the gauge-fixing parameter as:

$$\partial_t\alpha = -\alpha\eta_a - \alpha^2 Z_a^{-1}\beta_{Z_a/\alpha}\,. \tag{3.229}$$

In terms of Q-functionals we find for the beta function (3.228) the following form:

$$\begin{aligned}
Z_a^{-1}\beta_{Z_a/\alpha} \;=\; & \frac{g^2N}{(4\pi)^{d/2}}\left\{\frac{(d-1)\,(d^2+2d-4)}{d\,(d+2)}Q_{\frac{d}{2}}\left[(\partial_t R-\eta_a R)\,G_T^3\right]\right. \\
& +6\,(d-1)\,Q_{\frac{d}{2}+1}\left[(\partial_t R-\eta_a R)\,G_T^2 G_T'\right] + 6(d-1)Q_{\frac{d}{2}+2}\left[(\partial_t R-\eta_a R)\,G_T^2 G_T''\right] \\
& -\frac{(d-1)(d-2)}{d\,(d+2)}Q_{\frac{d}{2}}\left[(\partial_t R-\eta_a R)\,G_T^2 G_L\right] \\
& +\frac{(d-1)(d-2)}{2(d+2)}Q_{\frac{d}{2}+1}\left[(\partial_t R-\eta_a R)\,G_T^2 G_L'\right] \\
& +\frac{d-1}{2}Q_{\frac{d}{2}+2}\left[(\partial_t R-\eta_a R)\,G_T^2 G_L''\right] + \frac{d-1}{d}Q_{\frac{d}{2}}\left[(\partial_t R-\eta_a R)\,G_L^2 G_T\right] \\
& +\frac{(d-1)(d+6)}{2(d+2)}Q_{\frac{d}{2}+1}\left[(\partial_t R-\eta_a R)\,G_L^2 G_T'\right] \\
& +\frac{d-1}{2}Q_{\frac{d}{2}+2}\left[(\partial_t R-\eta_a R)\,G_L^2 G_T''\right] \\
& \left. -3Q_{\frac{d}{2}+1}\left[(\partial_t R-\eta_c R)\,G_c^2 G_c'\right] - 3Q_{\frac{d}{2}+2}\left[(\partial_t R-\eta_c R)\,G_c^2 G_c''\right]\right\}.
\end{aligned} \tag{3.230}$$

Inserting equation (3.230) in (3.229) after we have calculated $\eta_a(g,m_a,m_c,\alpha)$, gives the beta function for the gauge-fixing parameter for arbitrary cutoff shape function and dimension. The general form can be calculated analytically employing the optimized cutoff shape function but it is quite cumbersome. Here we consider only the case $m_a = m_c = 0$:

$$\partial_t \alpha = -\eta_a \alpha + \frac{g^2 N}{(4\pi)^2} \frac{\alpha^2 \left(5 - 15\alpha + 18\alpha^2 - 5\alpha^3 - 3\alpha^4 + 6\alpha^3 \log \alpha\right)}{12\,(1-\alpha)^3} \eta_a\, v \tag{3.231}$$

or the following Taylor expansion:

$$\partial_t \alpha = -\eta_a \alpha - \frac{g^2 N}{(4\pi)^2} \left[\frac{1}{2} - \frac{3}{\left(1 + m_a^2/k^2\right)^4} + \frac{5}{12} \frac{6 - \eta_a}{\left(1 + m_a^2/k^2\right)^3}\right] \alpha^2 + O\left(\alpha^4\right) . \tag{3.232}$$

This concludes the derivations of the beta functions of our truncation which can be extracted from the flow equation (3.203) for the vertex $\gamma^{(2,0,0;0)}_{p,-p}$ of the bEAA. In the next section we calculate the remaining flow of the mass and anomalous dimension of the ghost fields.

3.5.7 Derivation of $\partial_t m^2_{c,k}$ and $\partial_t Z_{c,k}$

In this section we calculate the beta functions for the wave-function renormalization of the ghost fields and for the ghost mass. The only term in the truncation (3.19) that we are considering that contains the ghost fields and the related couplings is the following:

$$Z_c \int d^d x \left[\bar{D}_\mu \bar{c} \left(\bar{D}^\mu + g Z_a^{1/2} a^\mu\right) c + m_c^2 \bar{c} c\right] . \tag{3.233}$$

We can extract the beta functions for the coupling in (3.233) from the flow equation for the ghost-ghost zero-field proper-vertex $\gamma^{(0,1,1;0)}_k$. This equation is obtained from the flow equation for the zero-field proper-vertex $\gamma^{(2;0)}_k$ after we make the multiplet decomposition and reads:

$$[\partial_t \gamma^{(0,1,1;0)}_{p,-p}]^{ab} = g^2 Z_c \int_q (\partial_t R_q - \eta_a R_q)\, [e_{p,q}]^{ab} + g^2 Z_c \int_q (\partial_t R_q - \eta_c R_q)\, [f_{p,q}]^{ab} . \tag{3.234}$$

In equation (3.234) there is only the ghost-ghost-gluon vertex since the action (3.233) is at most trilinear in the ghost and gauge fluctuation fields. This flow equation is represented graphically in Figure 3.12. Could had been writing down the flow equation (3.234) by just considering all the possible diagrams entering in Figure 3.12. Here we show how this equation is derived by starting directly from the flow equation for the bEAA. As we just said, the only non-zero vertices are:

$$\gamma_k^{(2,1,0;0)} = \begin{pmatrix} 0 & 0 & \gamma_{ac\bar{c}} \\ 0 & 0 & 0 \\ \gamma_{ca\bar{c}} & 0 & 0 \end{pmatrix} \qquad \gamma_k^{(2,0,1;0)} = \begin{pmatrix} 0 & \gamma_{a\bar{c}c} & 0 \\ \gamma_{\bar{c}ac} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3.235}$$

The regularized propagators G_{aa} and $G_{\bar{c}c}$ have been defined in section 3.5.2. The multiplet trace involved in the flow equation is reduced as follows:

$$\begin{aligned}
&\mathrm{Tr} \begin{pmatrix} G_{aa} & 0 & 0 \\ 0 & 0 & -G_{\bar{c}c} \\ 0 & G_{\bar{c}c} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \gamma_{ac\bar{c}} \\ 0 & 0 & 0 \\ \gamma_{ca\bar{c}} & 0 & 0 \end{pmatrix} \begin{pmatrix} G_{aa} & 0 & 0 \\ 0 & 0 & -G_{\bar{c}c} \\ 0 & G_{\bar{c}c} & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 & \gamma_{a\bar{c}c} & 0 \\ \gamma_{\bar{c}ac} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_t R_{k,aa} & 0 & 0 \\ 0 & 0 & \partial_t R_{k,\bar{c}c} \\ 0 & -\partial_t R_{k,\bar{c}c} & 0 \end{pmatrix} \\
&= \mathrm{Tr} \begin{pmatrix} 0 & 0 & G_{aa}\gamma_{ac\bar{c}} \\ -G_{\bar{c}c}\gamma_{ca\bar{c}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & G_{aa}\gamma_{a\bar{c}c} & 0 \\ 0 & 0 & 0 \\ G_{\bar{c}c}\gamma_{\bar{c}ac} & 0 & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} \partial_t R_{k,aa} & 0 & 0 \\ 0 & 0 & \partial_t R_{k,\bar{c}c} \\ 0 & -\partial_t R_{k,\bar{c}c} & 0 \end{pmatrix} \\
&= \mathrm{Tr} \begin{pmatrix} G_{aa}\gamma_{ac\bar{c}}G_{\bar{c}c}\gamma_{\bar{c}ac}\partial_t R_{k,aa} & 0 & 0 \\ 0 & -G_{\bar{c}c}\gamma_{ca\bar{c}}G_{aa}\gamma_{a\bar{c}c}\partial_t R_{k,\bar{c}c} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (3.236) \\
&= -G_{aa}\gamma_{a\bar{c}c}G_{\bar{c}c}\gamma_{\bar{c}ca}\partial_t R_{k,aa} - G_{\bar{c}c}\gamma_{c\bar{c}a}G_{aa}\gamma_{ac\bar{c}}\partial_t R_{k,\bar{c}c} \, , \qquad (3.237)
\end{aligned}$$

which corresponds to the rhs of (3.234) when written in momentum space. Note that in the last step of (3.237) we have interchanged the order in which we have taken the functional derivatives with respect to the ghost fields thus generating a minus sign. Inserting (3.233) in the lhs of (3.234) gives:

$$[\partial_t \gamma_{p,-p}^{(0,1,1;0)}]^{ab} = \delta^{ab} \left[\partial_t \left(m_c^2 Z_c \right) + p^2 \, \partial_t Z_c \right] . \tag{3.238}$$

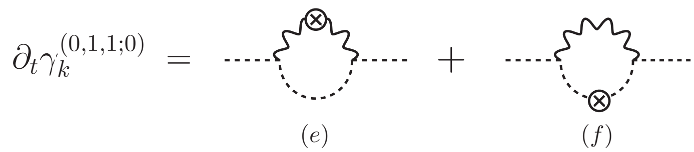


Figure 3.12: Graphical representation of the flow equation for the ghost-ghost zero-field proper vertex $\gamma_k^{(0,1,1;0)}$ from (3.234).

We see that we can extract $\partial_t\left(m_c^2 Z_c\right)$ as the term of order p^0 of (3.234) and $\partial_t Z_c$ as the term of order p^2. Skipping the following steps in the derivation we find that

$$\partial_t\left(m_c^2 Z_c\right) \equiv \beta_{Z_c m_c^2}\left(g, \eta_a, \eta_c, m_a, m_c, \alpha\right) , \tag{3.239}$$

turns out to be zero in every gauge:

$$\beta_{Z_c m_c^2}\left(g, \eta_a, \eta_c, m_a, m_c, \alpha\right) = 0 \, . \tag{3.240}$$

The beta function for the wave function-renormalization of the ghost fields is accounted for by defining the ghost anomalous dimension:

$$\eta_c\left(g, \eta_a, \eta_c, m_a, m_c, \alpha\right) = -\partial_t \log Z_c = -Z_c^{-1} \partial_t Z_c \, . \tag{3.241}$$

In terms of Q-functionals we find the following result:

$$\begin{aligned} \eta_c \;\; = \;\; & \frac{g^2 N}{(4\pi)^{d/2}} \left\{ -\frac{1}{d} Q_{\frac{d}{2}} \left[(\partial_t R - \eta_a R)((d-1) G_T^2 - G_L^2) G_c \right] \right. \\ & -\frac{d-1}{d} Q_{\frac{d}{2}} \left[(\partial_t R - \eta_c R) G_T G_c^2 \right] + Q_{\frac{d}{2}+1} \left[(\partial_t R - \eta_a R) G_c^2 G_L' \right] \\ & \left. + Q_{\frac{d}{2}+1} \left[(\partial_t R - \eta_a R) G_L^2 G_c' \right] \right\} \, . \end{aligned} \tag{3.242}$$

Equation (3.242) represents the ghost anomalous dimensions within the truncation (3.17) and (3.19) for general cutoff shape function, value of the gauge-fixing parameter and dimension. Employing the optimized cutoff in (3.242) we find for the gauge-fixing value $\alpha = 1$ the following form:

$$\eta_c\left(g, \eta_a, \eta_c, m_a, m_c, 1\right) = -\frac{g^2 N \, k^{d-4}}{(4\pi)^{d/2} \Gamma\left(\frac{d}{2}\right)} \frac{4}{d(d+2)} \frac{2 + d - \eta_a}{\left(1 + m_a^2/k^2\right)^2 \left(1 + m_c^2/k^2\right)} \, . \tag{3.243}$$

In the gauge $\alpha = 0$ we find instead:

$$\begin{aligned}\eta_c\left(g,\eta_a,\eta_c,m_a,m_c,0\right) &= \frac{g^2 N\, k^{d-4}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left\{-\frac{4(d-1)}{d^2(d+2)}\frac{2+d-\eta_a}{\left(1+m_a^2/k^2\right)^2\left(1+m_c^2/k^2\right)}\right. \\ &\left. -\frac{4(d-1)}{d^2(d+2)}\frac{d+2-\eta_c}{\left(1+m_a^2/k^2\right)^2\left(1+m_c^2/k^2\right)^2}\right\}. \end{aligned} \tag{3.244}$$

Considering (3.242) at $m_a = m_c = 0$ and in the physical dimension $d = 4$ but for general value of the gauge-fixing parameter shows that:

$$\begin{aligned}\eta_c\left(g,\eta_a,\eta_c,0,0,\alpha\right) &= \frac{g^2 N}{(4\pi)^2}\left[\frac{\alpha-3}{2}+\frac{(1+\alpha)\left(1-4\alpha+3\alpha^2-2\alpha^2\log\alpha\right)}{8\left(1-\alpha\right)^3}\eta_a\right. \\ &\left.+\frac{1-4\alpha+11\alpha^2-8\alpha^3+2\alpha^2\left(1+2\alpha\right)\log\alpha}{8\left(1-\alpha\right)^3}\eta_c\right]. \end{aligned} \tag{3.245}$$

The first term in equation (3.245) is in agreement with [59]. Equations (3.240) and (3.242) are the main result of this section, while the physical implications of equations (3.243) and (3.245) are discussed in the main part of this chapter.

Chapter 4

Functional RG for quantum gravity

4.1 Introduction

General relativity and quantum mechanics are not yet unified in a coherent theory as we do not have yet a fully successful theory of quantum gravity. But this does not mean that we lack any kind of quantum gravitational predictions: at least at low energy, quantum gravity can be described by an effective field theory based on metric degrees of freedom, as was first shown by Donoghue and others [82, 83]. At scales much smaller than the characteristic scale, which is here the Planck mass, effective field theory predictions are possible and calculable. Examples of this kind are the calculation of the first quantum corrections to the gravitational interaction potential between two masses [82, 84] and the low-energy graviton scattering cross-section [85]. The important point about these predictions is that no matter which theory actually describes high-energy quantum gravity - a string theory, a spin foam model, or other approaches - in the infrared (IR) limit any physically valid theory must reproduce the results found in the effective field theory framework.

In the last years, the hypothesis that the high energy completion of quantum gravity can still be described using the metric as fundamental degrees of freedom has gained some new support. In particular, the possibility that gravity may be asymptotically safe, a proposal first made by S. Weinberg [86], has been investigated within the functional RG approach. Progress in this direction has been possible thanks to the development of the EAA for quantum gravity made by M. Reuter in [87].

In this chapter we show how to construct the EAA for quantum gravity, first considering the truncation where the gauge invariant part of the EAA is taken to be the RG improved classical Einstein-Hilbert action. We show that in $d = 4$ the theory appears to asymptotically safe, we calculate the critical exponents with two different cutoff choices in a given gauge. We make contact both with the $2+\epsilon$ expansion results [89, 90, 91] and with the one-loop perturbative divergences first calculated by t'Hooft and Veltmann [88]. Then we study a truncation of the full background EAA where also the gauge-fixing and ghost sector have a non-trivial running, we calculate the anomalous dimensions of the fluctuation metric and of the ghost fields and we consider also the running of the Pauli-Fierz mass [93].

Next, we show how the low energy effective field theory predictions naturally arise in the effective average action approach to quantum gravity. To be able to recover the known results we devise a new approximation scheme to the EAA, the curvature expansion. This was introduced in the previous chapter in the context of non-abelian gauge theories and is now extended to the gravitational case. In this way we start to delineate a picture able to describe gravitational phenomena at all scales: from the UV physics of the non-trivial fixed point down to the IR physics of the low energy effective action. We then look at the $d=2$ case where we try to make contact with the solution of two dimensional quantum gravity.

4.2 EAA approach to quantum gravity

4.2.1 Classical theory

General Relativity, the classical theory of gravitational phenomena, is described by the Einstein-Hilbert action:

$$S_{EH}[g] = \frac{1}{16\pi G}\int d^dx\sqrt{g}\,(2\Lambda - R)\ . \tag{4.1}$$

Here $G = 6.67428\times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ is Newton's gravitational constant and Λ is the cosmological constant. In units of the IR cutoff they have dimensions $[G] = k^{2-d}$ and $[\Lambda] = k^2$. Newton's constant is related to the Planck mass by the following relation:

$$M_{Planck} = G^{-1/2}\,. \tag{4.2}$$

The Planck mass is the fundamental mass scale of gravitational interactions. The classical equations of motion are derived minimizing the Einstein-Hilbert action (4.1) with respect to the metric:

$$\frac{\delta S_{EH}[g]}{\delta g_{\mu\nu}} + \frac{\delta S_m[\phi,\psi,A_\mu;g]}{\delta g_{\mu\nu}} = 0\,. \tag{4.3}$$

In (4.3) we included general matter fields in the form of the action $S_m[\phi, \psi, A_\mu; g]$ for scalar fields ϕ, fermion fields ψ and gauge fields A_μ. The variation of the gravitational action (4.1) with respect to the metric can be written, considering (4.144) and (4.145), as follows:

$$\delta S_{EH} = \frac{1}{16\pi G}\int d^dx\sqrt{g}\left[\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right)h^{\mu\nu} + \nabla^2 h - \nabla^\mu\nabla^\nu h_{\mu\nu}\right], \tag{4.4}$$

where $h_{\mu\nu} = \delta g_{\mu\nu}$. We can drop the last two terms in (4.4) since they are total derivatives and contribute only to boundary terms. The matter action variation is related to the classical energy momentum tensor $T_{\mu\nu}$ by the relation:

$$\delta S_m = \frac{1}{2}\int d^dx\sqrt{g}T_{\mu\nu}h^{\mu\nu}\,. \tag{4.5}$$

Inserting the variations (4.4) and (4.5) in equation (4.3) gives Einstein's equations for general relativity:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G\, T^{\mu\nu}\,. \tag{4.6}$$

Note that the gravitational coupling constant G does enter in equation (4.6) only if matter is present.

From an effective field theory point of view many more terms can be added at the classical action (4.1), the first candidates being the curvature squared terms:

$$\int d^dx\sqrt{g}R^2 \qquad \int d^dx\sqrt{g}R_{\mu\nu}R^{\mu\nu} \qquad \int d^dx\sqrt{g}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\,.$$

There are severe bounds on the magnitude of the value of the couplings of these invariants [72]. So in a classical setting, this terms can be considered absent. We will see instead that in the quantum theory they will be generated by quantum fluctuations together with more complicated structures.

4.2.2 Quantum Theory

We now start to study the quantum theory using the EAA approach. Quantum gravity, if based on the Einstein-Hilbert action (4.1), is not perturbatively renormalizable, as first shown in [88], but as we will see later, there is now evidence that the theory may be asymptotically safe [100, 76, 77, 80]. This means, that to quantize the theory, we need to construct a complete RG trajectory in theory space that connects the fixed point action for $k \to \infty$ to the full effective action for $k \to 0$. As for non-abelian gauge theories, the EAA for quantum gravity is constructed using the background field method in section 4.3, to which we remand the reader at this moment.

The general decomposition of the bEAA is as in equation (4.84):

$$\Gamma_k[h,\bar{C},C;\bar{g}] = \bar{\Gamma}_k[\bar{g}+h] + \hat{\Gamma}_k[h,\bar{C},C;\bar{g}]\,, \tag{4.7}$$

where $h_{\mu\nu}$ is the fluctuation metric, $\bar{C}_\mu$ and C^ν are the ghost fields and $\bar{g}_{\mu\nu}$ is the background metric. The full quantum metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ is linearly split into the background and the fluctuation metric. The bEAA is invariant under combined physical and background diffeomorphisms (4.82):

$$(\delta+\bar{\delta})\Gamma_k[h,\bar{C},C;\bar{g}] = 0\,. \tag{4.8}$$

The diffeomorphism covariant effective average action (gEAA), denoted by $\bar{\Gamma}_k[g]$ in (4.7), is that part of the bEAA which is invariant under physical diffeomorphisms (4.85),

$$\delta\bar{\Gamma}_k[g] = 0\,, \tag{4.9}$$

and is a functional of the full quantum metric, at least for $k \to 0$. In fact, at intermediate scales $k \neq 0$ we must study the flow of the full bEAA. This is done here by introducing scale dependent wave-function renormalization for all the fields present in the cutoff action (4.76), i.e. the fields of which we cutoff the modes in defining the bEAA,

$$h_{\mu\nu} \to Z_{h,k}^{1/2} h_{\mu\nu} \qquad \bar{C}_\mu \to Z_{C,k}^{1/2}\bar{C}_\mu \qquad C^\nu \to Z_{C,k}^{1/2} C^\nu\,, \tag{4.10}$$

and by considering non-trivial truncations of the functional $\hat{\Gamma}_k[h,\bar{C},C;\bar{g}]$ in (4.7). This last functional is the remainder effective average action (rEAA) and plays the role of a generalized gauge-fixing and ghost action as, in the limit $k \to \infty$, it flows to the classical gauge-fixing (4.80) and ghost (4.81) actions. It is defined by the property $\hat{\Gamma}_k[0,0,0;\bar{g}] = 0$.

Another interesting thing to notice is that both the background and the full covariant derivative do not renormalize. This is due to the fact that in the definition of the Christoffel symbols, equation (D.31) of Appendix D, both the metric and its inverse enter and thus they are invariant under global rescaling.

The exact RG flow equation for the bEAA is derived in section 4.3.1 and reads (4.89):

$$\partial_t \Gamma_k[\varphi;\bar{g}] = \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2;0)}[\varphi;\bar{g}] + R_k[\bar{g}]\right)^{-1} \partial_t R_k[\bar{g}]\,, \tag{4.11}$$

where $\varphi = (h_{\mu\nu}, \bar{C}_\mu, C^\nu)$ is the field fluctuation multiplet. As explained in section 4.3 the cutoff kernel in (4.11) is constructed using a cutoff operator constructed using the background metric. In particular, since in the definition of the bEAA we are cutting off the field modes of the fluctuation metric and of the ghost fields, the cutoff term $\partial_t R_k[\bar{g}]$ in the flow equation (4.11) will contain terms proportional to their anomalous dimensions,

$$\eta_{h,k} = -\partial_t \log Z_{h,k} \qquad\qquad \eta_{C,k} = -\partial_t \log Z_{C,k}\,, \tag{4.12}$$

steaming from the redefinitions (4.10).

It is tempting to set $\varphi = 0$ in (4.11) and hope in this way to obtain a closed flow equation for the gEAA, since $\partial_t \bar{\Gamma}_k[\bar{g}] = \partial_t \Gamma_k[0;\bar{g}]$. The subtlety is that the flow (4.11) is "driven" by the Hessian of the bEAA taken with respect to the fluctuation multiplet and $\Gamma_k^{(2;0)}[0;\bar{g}]$ is not equal to $\bar{\Gamma}_k^{(2)}[\bar{g}]$. This remarks the fact that in general we have to consider the flow of the full bEAA that takes place in the enlarged theory space of functionals of the fields in φ and of $\bar{g}_{\mu\nu}$. When we consider a truncation ansatz for the bEAA which is bilinear in the ghost fields, the flow equation for the gEAA becomes (4.93):

$$\begin{aligned}\partial_t \bar{\Gamma}_k[\bar{g}] &= \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2,0,0;0)}[0;\bar{g}] + R_{k,hh}[\bar{g}]\right)^{-1} \partial_t R_{k,hh}[\bar{g}] \\ &\quad -\mathrm{Tr}\left(\Gamma_k^{(0,1,1;0)}[0;\bar{g}] + R_{k,\bar{C}C}[\bar{g}]\right)^{-1} \partial_t R_{k,\bar{C}C}[\bar{g}]\,.\end{aligned} \tag{4.13}$$

The flow equation (4.13) can be seen as the RG improvement of the one-loop flow, obtained from the one-loop effective action (D.106) derived in Appendix D.

In section 4.2.2.1 we will study the flow of the cosmological constant Λ_k and of Newton's constant G_k, which are part of a truncation of the gEAA, and we will see how the flow of these couplings is influenced by couplings which are part of a truncation of the rEAA. In particular we will compare three different ways to "close" the flow of Λ_k and G_k, i.e. how to obtain a non-trivial RG improved form for these beta functions, which naturally depend on the anomalous dimensions (4.12) and on couplings pertaining to a truncation of the rEAA. In section 4.2.2.1 we will consider a truncation of the rEAA comprising the running Pauli-Fierz mass and the running gauge-fixing parameters. We will find that even in this enlarged theory space quantum gravity still appears to be asymptotically safe, thus giving new support to this interesting scenario.

In section 4.2.2.2 we study non-local truncations of the bEAA, this class of truncations is necessary if we want to calculate the full effective action for quantum gravity as the $k \to 0$ limit of the bEAA. In particular we show how the low energy effective action for quantum gravity is recovered within this formalism [113].

4.2.2.1 Local truncations

We start considering as a truncation ansatz for the gEAA the RG improved version of the Einstein-Hilbert action (4.1) where Newton's constant and the cosmological constant become scale dependent quantities:

$$\bar{\Gamma}_k[g] = \frac{1}{16\pi G_k}\int d^dx\sqrt{g}\,(2\Lambda_k - R)\ , \tag{4.14}$$

The full quantum metric $g_{\mu\nu}$ entering in (4.14) is then linearly split into a background metric $\bar{g}_{\mu\nu}$ and a fluctuation metric $h_{\mu\nu}$ in the following way:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa_k h_{\mu\nu}\,. \tag{4.15}$$

In (4.15) we redefined the fluctuation metric by introducing the gravitational coupling constant κ_k, which is related to Newton's constant by the relation $\kappa_k = \sqrt{16\pi G_k}$. Inserting in (4.14) the full quantum metric as in (4.15) and expanding in powers of the fluctuation metric using the variations (4.144), (4.145) , (4.151) from section 4.5.1, gives to order $\kappa_k^{3/2}h^3$ the following relation:

$$\begin{aligned}
\bar{\Gamma}_k[\bar{g}+\kappa_k Z_{h,k}^{1/2}h] \;=\; & \frac{1}{\kappa_k^2}\int d^dx\sqrt{\bar{g}}\left(2\Lambda_k - \bar{R}\right) \\
& +\frac{Z_{h,k}^{1/2}}{\kappa_k}\int d^dx\sqrt{\bar{g}}\left[-\bar{\Delta}h - \bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu} + h_{\mu\nu}\bar{R}^{\mu\nu} + \frac{1}{2}h\left(2\Lambda_k - \bar{R}\right)\right] \\
& +\frac{1}{2}Z_{h,k}\int d^dx\left[\frac{1}{2}h^{\mu\nu}\bar{\Delta}h_{\mu\nu} - \frac{1}{2}h\bar{\Delta}h + h^{\mu\nu}\bar{\nabla}_\nu\bar{\nabla}_\alpha h^\alpha_\mu - h\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu}\right. \\
& -h^{\mu\nu}h^\alpha_\mu\bar{R}_{\nu\alpha} - h^{\mu\nu}h^{\alpha\beta}\bar{R}_{\alpha\mu\beta\nu} - h\bar{R}^{\mu\nu}h_{\mu\nu} \\
& \left.+\left(\frac{1}{4}h^2 - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)\left(2\Lambda_k - \bar{R}\right)\right] \\
& +O\left(\kappa_k^{3/2}h^3\right)\,.
\end{aligned} \tag{4.16}$$

In (4.16) we introduced the wave-function renormalization of the fluctuation metric $Z_{h,k}$ as in (4.10). Note that in (4.16) the kinetic term of the metric fluctuation trace h comes with the wrong sign; this is the signal that the Einstein-Hilbert action (4.1) is unstable in the conformal sector. See [81] for a more detailed discussion of this point.

An important difference between quantum gravity and non-abelian gauge theories is that any ansatz for the gEAA of the first is necessarily non-polynomial in the full quantum metric, because it involves both the inverse metric and the square root of the determinant of the metric. Therefore, the expansion around any background metric does involve an infinite number of terms. Already in the full version of (4.16) all powers of the metric fluctuation are present, giving rise to non-zero contributions to arbitrary high vertices. This is indeed a peculiar property of gravity.

As we said already in the analysis of non-abelian gauge theories in Chapter 3, the flow of the full bEAA takes place in the enlarged theory space, in this case, of functionals of the fluctuation metric, of the ghost fields and of the background metric, which are invariant under combined physical and background diffeomorphisms. To consistently study the flow of the bEAA in quantum gravity, we must consider also the running of the couplings present in the rEAA $\hat{\Gamma}_k[h,\bar{C},C;\bar{g}]$. We consider here an expansion of the rEAA in powers of the fluctuation metric and of the ghost fields, as we have already done in Chapter 3 for non-abelian gauge theories. To second power in $h_{\mu\nu}$ and first in $\bar{C}_\mu, C^\mu$, we consider a truncation ansatz comprising the running wave-function renormalization of the fluctuation metric $Z_{h,k}$ and of the ghost fields $Z_{C,k}$, the running Pauli-Fierz mass $m_{h,k}$ [93, 72] and the running gauge-fixing parameters α_k and β_k. We study the following ansatz:

$$\begin{aligned}
\hat{\Gamma}_k[Z_{h,k}^{1/2}h, Z_{C,k}^{1/2}\bar{C}, Z_{C,k}^{1/2}C;\bar{g}] \;=\;& \frac{1}{2}Z_{h,k}\int d^dx\sqrt{\bar{g}}\left(h_{\mu\nu}h^{\mu\nu}-h^2\right)m_{h,k}^2 \\
&+\frac{1}{2\alpha_k}Z_{h,k}\int d^dx\sqrt{\bar{g}}\bar{g}^{\mu\nu}\left(\bar{\nabla}^\alpha h_{\alpha\mu}-\frac{\beta_k^2}{2}\bar{\nabla}_\mu h\right)^2 \\
&-Z_{C,k}\int d^dx\sqrt{\bar{g}}\,\bar{C}^\mu\left[\bar{\nabla}^\alpha g_{\nu\alpha}\nabla_\mu+\bar{\nabla}^\alpha g_{\mu\nu}\nabla_\alpha\right. \\
&\left.-\beta_k\bar{\nabla}_\mu g_{\nu\alpha}\nabla^\alpha\right]C^\nu\,.
\end{aligned} \tag{4.17}$$

Note that in (4.17) the ghost action involves both covariant derivatives in the full quantum metric ∇_μ and in the background metric $\bar{\nabla}_\mu$. The action (4.17) amounts to an RG improvement of the classical gauge-fixing (4.80) and ghost (4.81) actions. We are thus considering the sub-space of theory space parametrized by the following set of couplings $\{\Lambda_k, G_k, m_{h,k}, Z_{h,k}, Z_{C,k}, \alpha_k, \beta_k\}$.

Even if the methods developed in this thesis are capable of treating the full truncation composed of the functionals (4.14) and (4.17), we will limit ourselves to consider the case where the gauge-fixing parameters are fixed to the values $\alpha_k=\beta_k=1$. This choice is dictated by technical reasons since in this gauge the standard heat kernel techniques can be used. The running of Newton's constant and of the cosmological constant have already been studied for general gauge-fixing parameters in [94, 95]. It is worth noting that the most natural choice for the gauge-fixing parameters should be $\alpha_k = 0$ and $\beta_k = 2/d$, where only traceless transverse gravitons and the conformal factor propagate. It is believed that this values correspond to a fixed point of the RG flow, as is in the case of non-abelian gauge theories. The full truncation (4.14) and (4.17) will be analyzed in [122]. We do not consider here a running ghost mass since we have seen in Chapter 3 that the analogous term for non-abelian gauge theories plays no important role.

To fourth order in the derivatives, we can add to the ansatz for the gEAA (4.14) the following curvature terms:

$$\bar{\Gamma}_k[g]\big|_{\mathcal{R}^2} = \int d^dx\sqrt{g}\left(\frac{1}{2\lambda_k}C^2 + \frac{1}{\xi_k}R^2 + \frac{1}{\rho_k}E + \frac{1}{\tau_k}\Delta R\right) . \tag{4.18}$$

In (4.18) we expressed the four derivative invariants in the basis $\{C^2, R^2, E, \Delta R\}$. C^2 is the square of the conformal invariant Weyl tensor given in $d = 4$ by

$$C^2 = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2 ,$$

while E is the integrand of the Euler topological invariant in four dimension:

$$E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2 .$$

In Appendix D we study the geometric interpretation of these curvature invariants. The RG running of the couplings $\{\Lambda_k, G_k, \lambda_k, \xi_k, \rho_k, \tau_k\}$ has been studied, within the bEAA approach, under several different approximations in [97, 98, 99]. In this thesis we will only look at the running of these couplings as induced by the Einstein-Hilbert truncation (4.14).

We start now to study the running of the gEAA (4.14) under the condition $\alpha_k = \beta_k = 1$. As we did in Chapter 3 for non-abelian gauge theories, we consider two different cutoff operator choices. The first case, that we call type I, considers both $R_{k,hh}[\bar{g}]$ and $R_{k,\bar{C}C}[\bar{g}]$ as functions of the covariant Laplacian, while the second case, type II, considers the graviton and ghost cutoff kernels as a function, respectively, of the operators $\mathbf{\Delta}_2$ and Δ_1 defined in section 4.5.4. Other possible cutoff procedures are extensively studied in [96].

We discuss first the beta functions of Newton's constant and of the cosmological constant for $m_{h,k} = 0$. These are derived, for general cutoff shape function and dimension, in section 4.5.4. In particular, for type I cutoff these are determined by the system (4.209), while for type

II cutoff they are determined by (4.226). After introducing the dimensionless cosmological constant $\tilde{\Lambda}_k = k^{-2}\Lambda_k$ and the dimensionless Newton's constant $\tilde{G}_k = k^{d-2}G_k$ and employing the optimized cutoff shape function, we find the following type I beta functions:

$$\begin{aligned}\partial_t\tilde{\Lambda}_k \;=\;& -2\tilde{\Lambda}_k + \frac{8\pi}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}\left\{\frac{d(d+1)}{4}\frac{d+2-\eta_{h,k}}{1-2\tilde{\Lambda}_k} - d(d+2-\eta_{C,k})\right.\\ & -2\tilde{\Lambda}_k\left[\frac{d(d+1)(d+2)}{48}\frac{d-\eta_{h,k}}{1-2\tilde{\Lambda}_k} - \frac{d(d+2)}{12}(d-\eta_{C,k})\right.\\ & \left.\left. -\frac{d(d-1)}{4}\frac{2+d-\eta_{h,k}}{(1-2\tilde{\Lambda}_k)^2} - (d+2-\eta_{C,k})\right]\right\}\tilde{G}_k\,, \end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\partial_t\tilde{G}_k \;=\;& (d-2)\tilde{G}_k + \frac{16\pi}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}\left\{\frac{d(d+1)(d+2)}{48}\frac{d-\eta_{h,k}}{1-2\tilde{\Lambda}_k}\right.\\ & \left.-\frac{d(d+2)}{12}(d-\eta_{C,k}) - \frac{d(d-1)}{4}\frac{2+d-\eta_{h,k}}{(1-2\tilde{\Lambda}_k)^2} - (d+2-\eta_{C,k})\right\}\tilde{G}_k^2\,. \end{aligned} \tag{4.20}$$

When we employ the type II cutoff, we find instead the following beta functions for the dimensionless coupling constants:

$$\begin{aligned}\partial_t\tilde{\Lambda}_k \;=\;& -2\tilde{\Lambda}_k + \frac{8\pi}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}\left\{\frac{d(d+1)}{4}\frac{d+2-\eta_{h,k}}{1-2\tilde{\Lambda}_k} - d(d+2-\eta_{C,k})\right.\\ & \left.-2\tilde{\Lambda}_k\left[\frac{d(5d-7)}{24}\frac{d-\eta_{h,k}}{1-2\tilde{\Lambda}_k} + \frac{d+6}{6}\left(d-\eta_{C,k}\right)\right]\right\}\tilde{G}_k \end{aligned} \tag{4.21}$$

and

$$\partial_t\tilde{G}_k = (d-2)\tilde{G}_k - \frac{16\pi\tilde{G}_k^2}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)}\left\{\frac{d(5d-7)}{24}\frac{d-\eta_{h,k}}{1-2\tilde{\Lambda}_k} + \frac{d+6}{6}\left(d-\eta_{C,k}\right)\right\}\,. \tag{4.22}$$

These beta functions are the fundamental result of this section and constitute the basis of the bEAA approach to quantum gravity. They are valid for $d \geq 2$ but note that for general dimension they depend both on the cutoff shape function and on the cutoff operator employed.

We start now to study the flow in physical dimension $d = 4$, where from equations (4.19) and (4.20) we find the following set of type I beta functions:

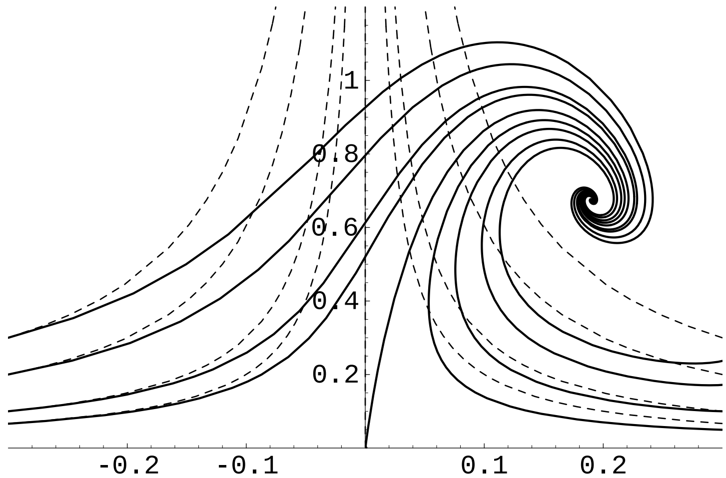


Figure 4.1: Renormalization group flow in the $\tilde{G}_k$, $\tilde{\Lambda}_k$ obtained by integrating the type I beta functions (4.26).

$$\begin{aligned}\partial_t\tilde{\Lambda}_k &= -2\tilde{\Lambda}_k+\frac{\tilde{G}_k}{12\pi}\left\{\frac{6-8\tilde{\Lambda}_k-24\tilde{\Lambda}_k^2-112\tilde{\Lambda}_k^3}{(1-2\tilde{\Lambda}_k)^2}\right.\\ &\qquad\left.-\frac{5-11\tilde{\Lambda}_k-10\tilde{\Lambda}_k^2}{(1-2\tilde{\Lambda}_k)^2}\eta_{h,k}+(4+6\tilde{\Lambda}_k)\eta_{C,k}\right\}\\ \partial_t\tilde{G}_k &= 2\tilde{G}_k+\frac{\tilde{G}_k^2}{12\pi}\left\{-\frac{44-72\tilde{\Lambda}_k+112\tilde{\Lambda}_k^2}{(1-2\tilde{\Lambda}_k)^2}+\frac{1+10\tilde{\Lambda}_k}{(1-2\tilde{\Lambda}_k)^2}\eta_{h,k}+6\eta_{C,k}\right\}. \qquad (4.23)\end{aligned}$$

From equation (4.21) and equation (4.22) we find the following system for the type II beta functions:

$$\begin{aligned}\partial_t\tilde{\Lambda}_k &= -2\tilde{\Lambda}_k+\frac{\tilde{G}_k}{12\pi}\left\{\frac{6-44\tilde{\Lambda}_k+80\tilde{\Lambda}_k^2}{1-2\tilde{\Lambda}_k}-\frac{5-13\tilde{\Lambda}_k}{1-2\tilde{\Lambda}_k}\eta_{h,k}+\frac{4+2\tilde{\Lambda}_k-20\tilde{\Lambda}_k^2}{1-2\tilde{\Lambda}_k}\eta_{C,k}\right\}\\ \partial_t\tilde{G}_k &= 2\tilde{G}_k+\frac{\tilde{G}_k^2}{12\pi}\left\{-\frac{92-80\tilde{\Lambda}_k}{1-2\tilde{\Lambda}_k}+\frac{13}{1-2\tilde{\Lambda}_k}\eta_{h,k}+10\eta_{C,k}\right\}. \qquad (4.24)\end{aligned}$$

As we noticed already in Chapter 3 for non-abelian gauge theories, the beta functions for the physical couplings, here Λ_k and G_k, are not determined by a closed system. In (4.23) and (4.24) this happens because of the presence, on the rhs, of the anomalous dimensions of the fluctuation metric $\eta_{h,k}$ and of the ghost fields $\eta_{C,k}$. This reflects the fact noticed in the previous section that for $k\neq 0$ the flow of the gEAA is not closed and the flow of the full bEAA has to be considered. As we did in Chapter 3, we present now three different ways to close the beta function system (4.23) or (4.24).

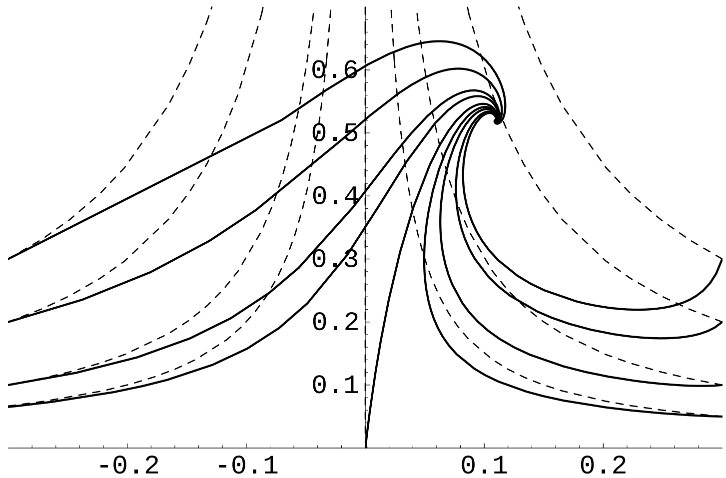


Figure 4.2: Renormalization group flow in the $\tilde{G}_k$, $\tilde{\Lambda}_k$ obtained by integrating the type II beta functions (4.27).

The first way is the trivial one where we set $\eta_{h,k} = \eta_{C,k} = 0$. This amounts to a one-loop approximation where all the non-perturbative information contained in the flow is discarded. Within this approximation the beta functions are just (4.23) and (4.24) where only the first terms inside the parenthesis are retained. These one-loop beta functions have been analyzed in [96].

In the second way to close the beta function system, we employ an approximation adopted in all previous studies of these equations [87, 94, 100, 101, 95, 102, 103, 96]. The system (4.23) or (4.24) is here closed by imposing the following relations:

$$Z_{h,k} = \kappa_k^{-1} \qquad\qquad Z_{C,k} = 0\,. \tag{4.25}$$

The identification in (4.25) implies a non-trivial, but difficult to interpret, RG improvement of the beta functions. We will call this procedure the "standard improvement" of the beta functions (4.23) and (4.24). For type I cutoff, we find the following standard improvement of (4.23):

$$\begin{aligned}
\partial_t \tilde{\Lambda}_k &= -2\tilde{\Lambda}_k + \frac{1}{6\pi}\frac{(3-4\tilde{\Lambda}_k - 12\tilde{\Lambda}_k^2 - 56\tilde{\Lambda}_k^3)\tilde{G}_k + \frac{1}{12\pi}(107-20\tilde{\Lambda}_k)\tilde{G}_k^2}{(1-2\tilde{\Lambda}_k)^2 - \frac{1}{12\pi}(1+10\tilde{\Lambda}_k)\tilde{G}_k} \\
\partial_t \tilde{G}_k &= 2\tilde{G}_k - \frac{1}{3\pi}\frac{(11-18\tilde{\Lambda}_k + 28\tilde{\Lambda}_k^2)\tilde{G}_k^2}{(1-2\tilde{\Lambda}_k)^2 - \frac{1}{12\pi}(1+10\tilde{\Lambda}_k)\tilde{G}_k}\,.
\end{aligned} \tag{4.26}$$

The beta functions in (4.26) are exactly those first obtained in [87]. For type II cutoff we

	$\tilde{\Lambda}_*$	$\tilde{G}_*$	$\theta' \pm i\theta''$	$\tilde{\Lambda}_*\tilde{G}_*$
Type I one-loop	0.121	1.172	$-1.868 \pm 1.398i$	0.142
Type II one-loop	0.047	0.775	$-2.310 \pm 0.382i$	0.036
Type I Std	0.193	0.707	$-1.475 \pm 3.043i$	0.137
Type II Std	0.092	0.555	$-2.425 \pm 1.270i$	0.051
Type I New	0.082	1.162	$-3.117 \pm 0.564i$	0.095
Type II New	0.039	0.696	$-2.617 \pm 0.255i$	0.027

Table 4.1: Fixed points and critical exponents for the various closures of the flow of Λ_k and G_k.

find instead the following standard improvement of (4.24):

$$\begin{aligned} \partial_t \tilde{\Lambda}_k &= -2\tilde{\Lambda}_k + \frac{1}{6\pi}\frac{(3-28\tilde{\Lambda}_k+84\tilde{\Lambda}_k^2-80\tilde{\Lambda}_k^3)\tilde{G}_k+\frac{1}{12\pi}(191-512\tilde{\Lambda}_k)\tilde{G}_k^2}{(1-2\tilde{\Lambda}_k)^2-\frac{13}{12\pi}(1-2\tilde{\Lambda}_k)\tilde{G}_k} . \\ \partial_t \tilde{G}_k &= 2\tilde{G}_k - \frac{1}{3\pi}\frac{(23-20\tilde{\Lambda}_k)\tilde{G}_k^2}{1-2\tilde{\Lambda}_k-\frac{13}{12\pi}\tilde{G}_k} . \end{aligned} \tag{4.27}$$

The system (4.27) as been proposed in [96] together with some variants of it. Note that the beta function (4.26) and (4.27) are rational functions of both $\tilde{G}_k$ and $\tilde{\Lambda}_k$, this can be interpreted as a resummation of an infinite number of perturbative diagrams implemented by the RG improvement implied by (4.25). The outcome of the numerical integration of (4.26) and (4.27) is shown in Figure 4.1 and Figure 4.2 respectively. The presence of a non-Gaussian fixed point is clearly visible in these pictures and the values for $\tilde{\Lambda}_*$ and $\tilde{G}_*$ for both cutoff types are reported in Table 4.1, together with the respective one-loop values. The important point is that the non-Gaussian fixed point is UV attractive in both directions. The flow near to the fixed point is spiraling towards it, i.e. there is a pair of complex conjugated critical exponents with negative real part [1]. These are also reported in Table 4.1. Thus within the truncation we are considering and the standard improvement of the relative beta functions, quantum gravity is asymptotically safe. Actually, we still need to show that the critical surface is finite dimensional. Evidence for this to be true has been given in [104, 105]. For more details see [96].

The third way to close the beta functions system, that we propose here for the first time, is to separately calculate the anomalous dimensions $\eta_{h,k}$, $\eta_{C,k}$ as functions of $\tilde{\Lambda}_k$, $\tilde{G}_k$ and successively reinsert these back in the beta functions (4.23) or (4.24). In this way we

[1] Here we follow the convention of [96] that a negative value for the critical exponent implies that the relative eigendirection is UV attractive.

obtain closed beta functions (within the truncation considered) that account for the flow of the wave-function renormalizations $Z_{h,k}$ and $Z_{C,k}$. In doing so we make a step further in considering the flow in the enlarged theory space where the bEAA lives. We are adopting the point of view that Λ_k and G_k are physical couplings while $Z_{h,k}$ and $Z_{C,k}$ are not, but the influence of these last couplings is non-trivial and it is important to account for it. The calculations of the anomalous dimensions $\eta_{h,k}$ and $\eta_{C,k}$ are done explicitly in sections 4.5.6 and 4.5.7 using the flow equations for the zero-field proper-vertices of the bEAA, $\gamma_k^{(2,0,0;0)}$ and $\gamma_k^{(0,1,1;0)}$. These results are given, for general cutoff shape function and dimension, in equations (4.255) and (4.256). Also, $\eta_{h,k}$ and $\eta_{C,k}$ turn out not to depend on the cutoff operator type. If we insert now the optimized cutoff shape function in (4.255) we find the following form for the anomalous dimension of the fluctuation metric:

$$
\begin{aligned}
\eta_{h,k} = & -\frac{16\pi\tilde{G}_k}{2(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)}\Bigg\{-\frac{5d^2+44d+116}{(d+4)\,(d^2-4)\,(d+1)}\frac{20}{(1-2\tilde{\Lambda}_k)^4} \\
& +\frac{80(d+6)}{(d^2-4)(d+1)}\frac{\tilde{\Lambda}_k}{(1-2\tilde{\Lambda}_k)^4}+\frac{d^2-5d+4}{d+2}\frac{d+2-\eta_{h,k}}{(1-2\tilde{\Lambda}_k)^2} \\
& -\frac{320\tilde{\Lambda}_k^2}{(d^2-4)(d+1)}\frac{d+2-\eta_{h,k}}{(1-2\tilde{\Lambda}_k)^4}-\frac{480\tilde{\Lambda}_k}{(d^2-4)(d+1)}\frac{d+2-\eta_{h,k}}{(1-2\tilde{\Lambda}_k)^3} \\
& +\frac{20(31d+120)}{3(d+4)(d^2-4)(d+1)}\frac{d+4-\eta_{h,k}}{(1-2\tilde{\Lambda}_k)^3}-\frac{32}{(d+4)(d+2)}(d+4-\eta_{C,k})\Bigg\}\,. \qquad (4.28)
\end{aligned}
$$

By inserting the optimized cutoff shape function in (4.256) we find the following form for the anomalous dimensions of the ghost fields:

$$
\begin{aligned}
\eta_{C,k} = & -\frac{16\pi\tilde{G}_k}{2(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)}\Bigg\{4\frac{3d^4-3d^3-12d^2+20d-12}{d^2(d+4)(d^2-4)}\frac{4+d-\eta_{h,k}}{(1-2\tilde{\Lambda}_k)^2} \\
& -\frac{2(d^2-d-8)}{(d+4)(d^2-4)}\frac{d+4-\eta_{C,k}}{1-2\tilde{\Lambda}_k}\Bigg\}\,. \qquad (4.29)
\end{aligned}
$$

Note that equations (4.28) and (4.29) constitute a linear system for the indeterminates $\eta_{h,k}$ and $\eta_{C,k}$ that can be solved to yield these anomalous dimensions as functions solely of $\tilde{\Lambda}_k$ and $\tilde{G}_k$. In general, the anomalous dimensions (4.28) and (4.29), like in non-abelian gauge theories, are strongly dependent on the gauge-fixing parameters α_k and β_k. This general case, along with the conjecture that these anomalous dimensions may obey a scaling relation, analogous to the one obeyed by non-abelian gauge theories at the IR fixed point [64, 66], will be investigate further in [112]. By solving the linear system composed of equations (4.28) and

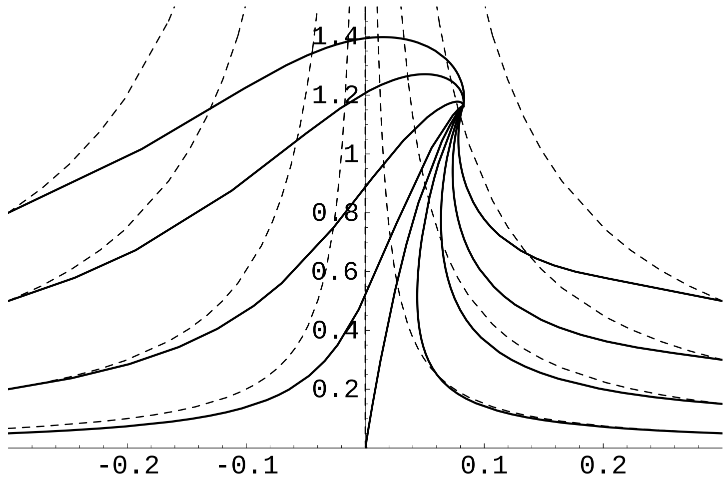


Figure 4.3: Renormalization group flow in the $\tilde{G}_k$, $\tilde{\Lambda}_k$ plane for the type I beta functions obtained by employing the explicit expressions for η_h and η_C.

(4.29) in $d = 4$, we find the following form for the anomalous dimension of the fluctuation field as a function of $\tilde{\Lambda}_k$ and $\tilde{G}_k$ alone:

$$\begin{aligned}\eta_{h,k} &= -\frac{1}{24\pi}\Big[(39 - 358\tilde{\Lambda}_k + 176\tilde{\Lambda}_k^2 + 1792\tilde{\Lambda}_k^3 - 2560\tilde{\Lambda}_k^4 + 1024\tilde{\Lambda}_k^5)\tilde{G}_k \\ &\quad - \frac{1}{48\pi}(381 - 2176\tilde{\Lambda}_k + 5040\tilde{\Lambda}_k^2 - 616\tilde{\Lambda}_k^3)\tilde{G}_k^2\Big] \\ &\quad \times \Big[(1-2\tilde{\Lambda}_k)^5 - \frac{1}{12\pi}(10 - 63\tilde{\Lambda}_k + 115\tilde{\Lambda}_k^2 - 56\tilde{\Lambda}_k^3 - 4\tilde{\Lambda}_k^4)\tilde{G}_k \\ &\quad + \frac{1}{576\pi^2}(18 - 125\tilde{\Lambda}_k + 307\tilde{\Lambda}_k^2 - 226\tilde{\Lambda}_k^3)\tilde{G}_k^2\Big]^{-1} \end{aligned} \tag{4.30}$$

and the following form for the anomalous dimension of the ghost field as a function of $\tilde{\Lambda}_k$ and $\tilde{G}_k$ alone:

$$\begin{aligned}\eta_{C,k} &= -\frac{1}{48\pi(1-2\tilde{\Lambda}_k)}\Big[(1-2\tilde{\Lambda}_k)^4(105 + 16\tilde{\Lambda}_k)\tilde{G}_k \\ &\quad - \frac{1}{192\pi}(12813 - 40496\tilde{\Lambda}_k + 85760\tilde{\Lambda}_k^2 - 107520\tilde{\Lambda}_k^3 + 57856\tilde{\Lambda}_k^4)\tilde{G}_k^2\Big] \\ &\quad \times \Big[(1-2\tilde{\Lambda}_k)^5 - \frac{1}{12\pi}(10 - 63\tilde{\Lambda}_k + 115\tilde{\Lambda}_k^2 - 56\tilde{\Lambda}_k^3 - 4\tilde{\Lambda}_k^4)\tilde{G}_k \\ &\quad + \frac{1}{576\pi^2}(18 - 125\tilde{\Lambda}_k + 307\tilde{\Lambda}_k^2 - 226\tilde{\Lambda}_k^3)\tilde{G}_k^2\Big]^{-1} . \end{aligned} \tag{4.31}$$

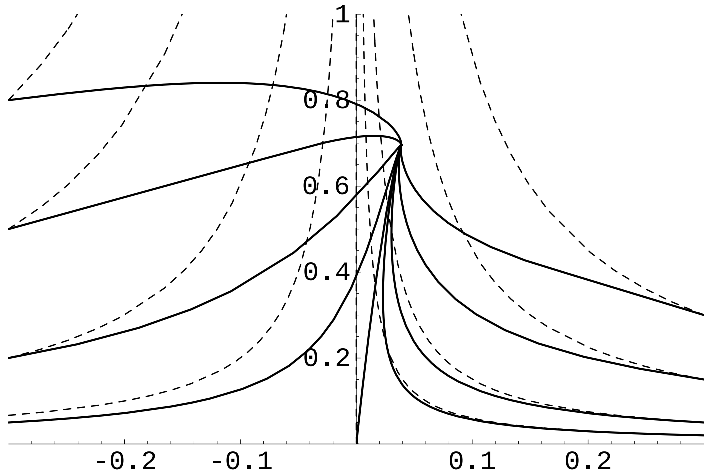


Figure 4.4: Renormalization group flow in the $\tilde{G}_k$, $\tilde{\Lambda}_k$ plane for the type II beta functions obtained by employing the explicit expressions for η_h and η_C.

Inserting back (4.30) and (4.31) in the beta functions (4.23) or (4.24) finally gives the "new improved" form of $\partial_t\tilde{\Lambda}_k$ and $\partial_t\tilde{G}_k$ that accounts for the non-trivial influence that $Z_{h,k}$ and $Z_{C,k}$ have on the flow of the dimensionless cosmological and Newton's constants. The result of the numerical integration of these beta functions is plotted in Figure 4.3 and Figure 4.4 for type I cutoff and type II cutoff respectively. Note that, despite these new beta functions differ non-trivially from the one-loop and from the standard improved ones, the overall picture of the flow is unchanged. In particular there is still one non-Gaussian fixed point which is attractive toward the UV. The fixed point values for the dimensionless couplings and for the critical exponents for this new improved case are also given in Table 4.1. The critical exponents still form a complex conjugate pair, but now the imaginary part is smaller in comparison to the real one. This is reflected in the fact that the flow next to the non-Gaussian fixed point is now less spiraling and one can speculate that in a more complete truncation the critical exponents may become real. If we insert the fixed point values for the cosmological constant and for Newton's constant in (4.30) and (4.31) we find the values $\eta^I_{h*} = -0.637$ and $\eta^I_{C*} = -1.262$ for type I cutoff while for we find $\eta^{II}_{h*} = -0.442$ and $\eta^{II}_{C*} = -0.602$ for type II cutoff. This values depend also on the values of the gauge fixing-parameters and it is expected that for $\alpha_k = 0$ and $\beta_k = \frac{2}{d}$ they become cutoff independent [112]. We have to mention here that the anomalous dimension of the ghost fields has already been calculated in [106, 108] and has been used to improve the standard closure based on the relations in (4.25). From the point of view of this section, this procedure is an hybrid between the standard and the new improvements that is not completely justified since the arbitrary relation $Z_{h,k} = \kappa_k^{-1}$

is still employed. Another truncation of the ghost sector has been considered in [107]. We have to mention that a complementary strategy to explore the full flow of the bEAA as been developed in [109, 110, 111].

We have thus shown that accounting for the non-trivial influence that the anomalous dimensions $\eta_{h,k}$ and $\eta_{C,k}$ have on the flow has changed the properties of the non-Gaussian fixed point only quantitatively but not qualitatively. In light of these results the asymptotic safety scenario in quantum gravity is strongly reinforced, since they show that the non-Gaussian fixed point is still unique and UV attractive even when we consider truncations of the full bEAA. As a proposal for future work, it will be interesting to improve the beta functions [98, 99] for the higher derivative couplings in (4.18) by the method proposed here.

To better understand the kind of RG improvements, or resummations, implemented by the three different procedures presented here to close the beta functions of Newton's constant and of the cosmological constant, we focus on the running of $\tilde{G}_k$ and we set $\tilde{\Lambda}_k = 0$. In this case the one-loop beta functions for the dimensionless Newton's constant, for both cutoff types, become simply as follows:

$$\partial_t \tilde{G}_k = 2\tilde{G}_k - \frac{11}{3\pi}\tilde{G}_k^2 \qquad \text{type I} \tag{4.32}$$

$$\partial_t \tilde{G}_k = 2\tilde{G}_k - \frac{23}{3\pi}\tilde{G}_k^2 \qquad \text{type II}\,. \tag{4.33}$$

Note that the one-loop coefficient in (4.32) and (4.33) are not universal as expected in $d = 4$. The beta functions (4.32) and (4.33) have a non-Gaussian fixed point for $\tilde{G}_*^I = 1.714$ and $\tilde{G}_*^{II} = 0.820$ respectively. For a theory with only one coupling constant, the mass critical exponent is given by minus the first derivative of the beta function evaluated at the fixed point [75]:

$$\nu^{-1} = -\left.\frac{\partial}{\partial \tilde{G}_k}\partial_t \tilde{G}_k\right|_{\tilde{G}_*}\,. \tag{4.34}$$

From (4.32) and (4.33) we find the mean field like values $\nu^I = \nu^{II} = \frac{1}{2}$, as expected in a one-loop calculation. When instead we consider the standard improved version of the beta function for the dimensionless Newton's constant, equations (4.26) and (4.27) for $\tilde{\Lambda}_k = 0$, we find the following forms:

$$\partial_t \tilde{G}_k = 2\tilde{G}_k - \frac{11}{3\pi}\frac{\tilde{G}_k^2}{1 - \frac{1}{12\pi}\tilde{G}_k} \qquad \text{type I} \tag{4.35}$$

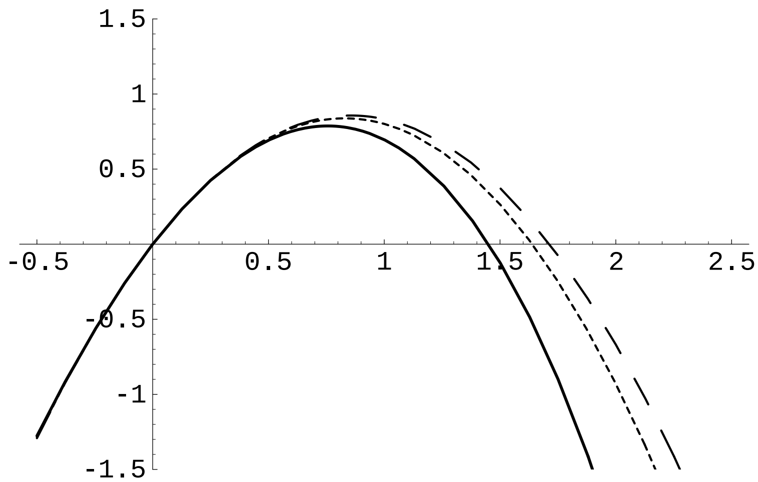


Figure 4.5: Beta functions $\partial_t \tilde{G}$ for the dimensionless Newton's constant within type I cutoff for the three different closures schemes as a function of $\tilde{G}$ in $d = 4$. One-loop form (long dashed) from equation (4.32), standard improved form (short dashed) from equation (4.35) and new improved form (thick) from equation (4.37).

$$\partial_t \tilde{G}_k = 2\tilde{G}_k - \frac{23}{3\pi}\frac{\tilde{G}_k^2}{1 - \frac{13}{12\pi}\tilde{G}_k} \qquad \text{type II}\,. \tag{4.36}$$

Note that the effect of the standard RG improvement has been to make the beta functions (4.35) and (4.36) rational functions of $\tilde{G}_k$. The non-Gaussian fixed points are now at the values $\tilde{G}_*^I = 1.639$ and $\tilde{G}_*^{II} = 0.639$, while the mass critical exponent (4.34) has the values $\nu^I = 0.478$ and $\nu^{II} = 0.390$. Note that these values differ by each other for about 20%. Finally, when we close the beta function of the dimensionless Newton's constant, equations (4.26) and (4.27) for $\tilde{\Lambda}_k = 0$, by inserting the calculated anomalous dimensions of the fluctuation metric and of the ghost fields, equations (4.30) and (4.31) now function of $\tilde{G}_k$ alone, we find the forms:

$$\partial_t \tilde{G}_k = 2\tilde{G}_k - \frac{11}{3\pi}\frac{1 - \frac{263}{528\pi}\tilde{G}_k - \frac{1019}{6144\pi^2}\tilde{G}_k^2}{1 - \frac{5}{6\pi}\tilde{G}_k + \frac{1}{32\pi^2}\tilde{G}_k^2}\tilde{G}_k^2 \qquad \text{type I} \tag{4.37}$$

$$\partial_t \tilde{G}_k = 2\tilde{G}_k - \frac{23}{3\pi}\frac{1 - \frac{101}{276\pi}\tilde{G}_k - \frac{23543}{141312\pi^2}\tilde{G}_k^2}{1 - \frac{5}{6\pi}\tilde{G}_k + \frac{1}{32\pi^2}\tilde{G}_k^2}\tilde{G}_k^2 \qquad \text{type II}\,. \tag{4.38}$$

Note that the RG improvement implied in this last procedure made the beta functions (4.37) and (4.38) rational functions of higher order than the previous ones (4.35) and (4.36) obtained by the standard RG improvement. Also, the denominators in (4.37) and (4.38) are now equal. In this last case the values for the zero of the beta functions are $G_*^I = 1.451$

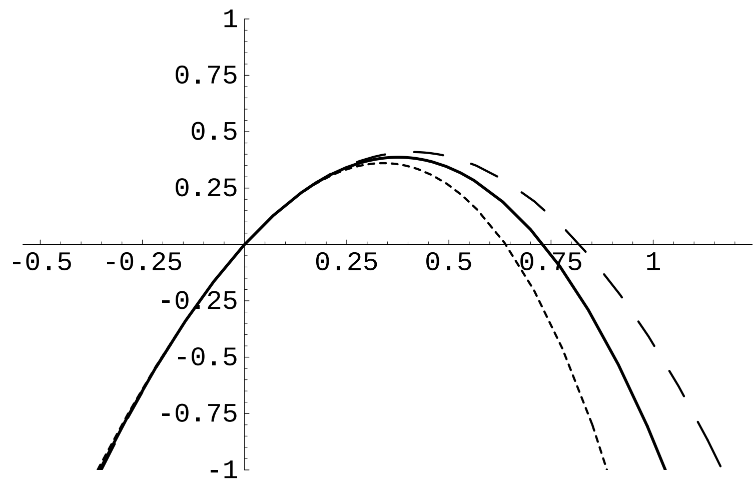


Figure 4.6: Beta functions $\partial_t\tilde{G}$ for the dimensionless Newton's constant within type II cutoff for the three different closures schemes as a function of $\tilde{G}$ in $d = 4$. One-loop form (long dashed) from equation (4.33), standard improved form (short dashed) from equation (4.36) and new improved form (thick) from equation (4.38).

and $G_*^{II} = 0.731$, while the values of the mass critical exponent are now $\nu_I = 0.421$ and $\nu_{II} = 0.446$. It is interesting that these values for the mass critical exponent are now closer as they differ only for about 6%. This can be considered has an indication that the accounted flow of $Z_{h,k}$ and $Z_{C,k}$ has made the flow of the dimensionless Newton's constant closer to the exact one, and thus less sensible to truncation artifacts. The three beta functions for both cutoff types are shown in Figure 4.5 and Figure 4.6. It is interesting to note that in the case of type II cutoff, the new improved beta function lies between the standard improved and one-loop ones and is not always smaller [96].

We turn now to consider the flow of the dimensionless Newton's constant near two dimensions where we can make direct contact with the standard $d = 2+\epsilon$ perturbative studies. In all cases, i.e. for both cutoff types and for any cutoff shape function, we find the following form:

$$\partial_t\tilde{G}_k = \epsilon\tilde{G}_k - \frac{38}{3}\tilde{G}_k^2 + O(\tilde{G}_k^3)\,. \tag{4.39}$$

The scheme independent one-loop coefficient in (4.39) matches the one calculated using the ϵ-expansion [89, 90, 91]. The beta function (4.39) as a UV attractive non-Gaussian fixed point at $\tilde{G}_* = \frac{3\epsilon}{38}$ and the theory is thus asymptotically safe. It is important to notice that this fixed point is continuously related to the one in four dimensions. For more details on this point see [96]. We note that subsequent analysis of $d = 2 + \epsilon$ quantum gravity [92] have found a different coefficient, precisely $\frac{50}{3}$ instead $\frac{38}{3}$. This discrepancy as to be imputed to the different way the gravitational conformal factor has been quantized in these last studies.

It is not difficult to calculate the running of the fourth order invariants in (4.18) induced by the truncation we are considering. In particular, for $\tilde{\Lambda}_k = 0$ we find, equation (4.223) from section 4.5.4, the following result:

$$\partial_t \bar{\Gamma}_k[g]\big|_{\mathcal{R}^2} = \frac{1}{(4\pi)^2} \int d^d x \sqrt{g} \left(\frac{7}{10} R_{\mu\nu} R^{\mu\nu} + \frac{1}{60} R^2 + \frac{53}{45} E + \frac{19}{15} \Delta R \right) , \tag{4.40}$$

which is independent of the cutoff operator and of the cutoff shape function employed. This contributions to the flow of tho couplings in (4.18) are related to the UV divergences found in perturbation theory [88]. For a discussion of this point see [96]. We will recover the first two terms of (4.40), which are not total-derivatives, in the next section when treating a non-local truncation of the gEAA.

We now switch on the Pauli-Fierz mass term and we will consider only the type I cutoff case for simplicity. The resulting beta functions are quite cumbersome for general values of Λ_k and $m_{h,k}$, so we consider only the $\Lambda_k = 0$ case here. The general form of the beta function for the Pauli-Fierz mass is derived in section 4.5.6 for general cutoff shape function and dimension in the gauge $\alpha_k = \beta_k = 1$ and is given in equations (4.251) and (4.252). In physical dimension and employing the optimized cutoff shape function, we find, for the dimensionless squared mass $\tilde{m}_{h,k}^2 = k^{-2} m_{h,k}$, the following form:

$$\begin{aligned} \partial_t \tilde{m}_{h,k}^2 \;=\; & (-2 + \eta_{h,k}) \tilde{m}_{h,k}^2 + \frac{\tilde{G}_k}{2\pi} \left\{ \frac{79}{96} \frac{8 - \eta_{h,k}}{1 + \tilde{m}_{h,k}^2} - \frac{9}{32} \frac{8 - \eta_{h,k}}{1 + 3\tilde{m}_{h,k}^2} \right. \\ & \left. + \frac{207}{320} \frac{10 - \eta_{h,k}}{(1 + \tilde{m}_{h,k}^2)^3} + \frac{1}{960} \frac{10 - \eta_{h,k}}{(1 + 3\tilde{m}_{h,k}^2)^3} - \frac{7}{960} \frac{10 - \eta_{h,k}}{(1 + \tilde{m}_{h,k}^2)(1 + 3\tilde{m}_{h,k}^2)^2} \right\} \end{aligned} \tag{4.41}$$

For non-zero Pauli-Fierz mass, the beta function for Newton's constant is given in equation (4.218) of section 4.5.4 for arbitrary cutoff shape function and dimension. Employing the optimized cutoff and setting $d = 4$ we get equation (4.220) which reads:

$$\partial_t \tilde{G}_k = 2\tilde{G}_k + \frac{\tilde{G}_k^2}{6\pi} \left\{ \frac{9}{4} \frac{4 - \eta_{h,k}}{1 + \tilde{m}_{h,k}^2} + \frac{1}{4} \frac{4 - \eta_{h,k}}{1 + 3\tilde{m}_{h,k}^2} - 3 \frac{6 - \eta_{h,k}}{(1 + \tilde{m}_{h,k}^2)^2} - 14 + 3\eta_{C,k} \right\} . \tag{4.42}$$

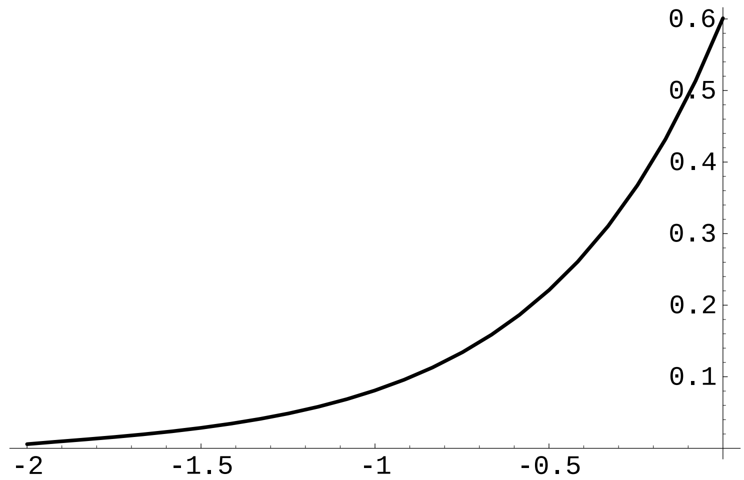


Figure 4.7: Flow of the dimensionful Pauli-Fierz mass in units of the UV scale $m^2_{h,k}/\Lambda$ with initial condition $m^2_{h,\Lambda} \sim \Lambda\, \tilde{m}^2_{h*} = 0.601$.

We can now insert the explicit forms for the anomalous dimensions of the fluctuation metric and of the ghost fields for general values of the Pauli-Fierz mass into (4.41) and (4.42) to obtain a closed system for $\partial_t \tilde{m}^2_{h,k}$ and $\partial_t \tilde{G}_k$. We can solve this system numerically and we find only a non-Gaussian fixed point at the values $\tilde{m}^2_{h*} = 0.601$ and $\tilde{G}_* = 2.275$. This fixed point is UV attractive with eigenvalues -4.051 and -1.602, while at the Gaussian fixed point we find the canonical eigenvalues -2 and 2. More importantly, if we follow the flow of the dimensionful mass from the non-Gaussian fixed point to the IR we find that it goes to zero as expected. A plot of the solution of the system (4.41) and (4.42) is given in Figure 4.7.

4.2.2.2 Non-local truncations

We now start to use the flow equation for the bEAA as a tool to actually compute the full effective action. We learned in section 2.3.3 of the Chapter 2 that in order to be able to extend the flow of the EAA down to $k = 0$ we need to consider truncation ansatz with at least an infinite number of terms. We proposed a truncation scheme that we called "curvature expansion" where we consider an ansatz containing running structure functions. In the present contest we consider the following ansatz for the gEAA:

$$\bar{\Gamma}_k[g] = \int d^dx \sqrt{g} \left[\frac{1}{16\pi G_k} (2\Lambda_k - R) + R F_{1,k}(\Delta) R + R_{\mu\nu} F_{2,k}(\Delta) R^{\mu\nu} \right] . \qquad (4.43)$$

In (4.43) we added to the truncation (4.14) all possible curvature square terms, in the $\{R_{\mu\nu}R^{\mu\nu}, R^2, E\}$ basis, in the form of functions of the full covariant Laplacian $F_{i,k}(x)$, $i = 1, 2$, sandwiched between Ricci scalar and tensor curvatures.

As for the case of non-abelian gauge theories in Chapter 3, the task of inserting the full truncation (4.43) in the flow equation for the gEAA in a difficult one, even if in principle possible using the techniques developed in section 3.3.4 of Chapter 3. Here we will consider only the flow of the form factors[2] in (4.43) induced by the Einstein-Hilbert operator $\int \sqrt{g}R$ and we set $\Lambda_k = m_{h,k} = \eta_{h,k} = \eta_{C,k} = 0$. We will continue to use the gauge $\alpha_k = \beta_k = 1$ and we will work with type II cutoff operators. In section 4.5.5 we employ the non-local heat kernel expansion, derived in Appendix A, to the extract the flow equations for the structure functions under these conditions. They are given in arbitrary dimension and for general cutoff shape function in equations (4.234) and (4.235). This section follows [113].

We start to consider the $d = 4$ case, the flow equations (4.236) and (4.237) can be casted in the following form:

$$\partial_t F_{i,k}(x) = \frac{1}{(4\pi)^2} g_i\left(\frac{x}{k^2}\right) , \tag{4.44}$$

where the functions $g_i(u)$ can be calculated once a cutoff shape function has been chosen. Note that, in $d = 4$ as here, all the k-dependence is through the $u = x/k^2$ dependence of the functions $g_i(u)$. If we employ the optimized cutoff shape function and by using the relevant parametric integrals of Q-functionals from Appendix A, we find:

$$g_1(u) = \frac{1}{60} + \left(-\frac{1}{60} + \frac{19}{5u} + \frac{1}{15u^2}\right)\sqrt{1-\frac{4}{u}}\theta(u-4) \tag{4.45}$$

$$g_2(u) = \frac{7}{10} - \left(\frac{7}{10} + \frac{76}{15u} + \frac{8}{15u^2}\right)\sqrt{1-\frac{4}{u}}\theta(u-4)\,. \tag{4.46}$$

These are the beta functions for the non-local form factors and they are plotted in Figure 4.8.

The functions (4.45) and (4.46) are constant in a neighborhood of the origin: if instead of considering the full functional dependence of the form factors $F_{i,k}(x)$ on the covariant Laplacian, we had considered only a local expansion to a polynomial, we would had found zero beta functions for the running couplings of all derivative terms of the form $\int \sqrt{g}R_{\mu\nu}\Delta^n R^{\mu\nu}$ or $\int \sqrt{g}R\Delta^n R$. As already noticed in section 2.3.3 of Chapter 2, this shows that truncations to a finite number of local terms are generally not powerful enough to correctly describe IR physics. Truncations of the gEAA need at least to project the flow onto an infinite dimensional subspace of theory space, as here the one of the functions $F_{i,k}(x)$.

We now integrate the flow equations (4.44) from a UV scale Λ down to a generic IR scale

[2] We consider the words "structure function" and "form factor" as synonymous.

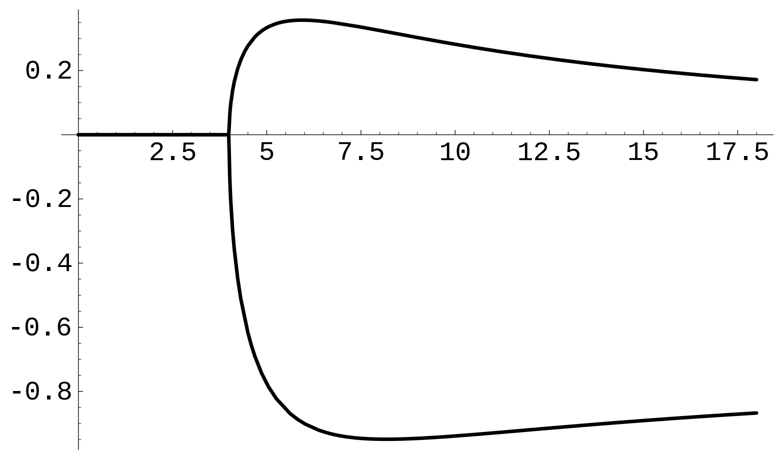


Figure 4.8: The functions $g_1(u) - \frac{1}{60}$ (upper curve) and $g_2(u) - \frac{7}{10}$ (lower curve) representing the flow of the form factors in (2.127) after we imposed the UV boundary conditions (4.50). Note that the flow stops for $x < 4k^2$: only IR or slow modes contribute effectively to the RG running of the form factors.

k. It is expected that for $\Lambda \to \infty$ we will encounter the usual ultraviolet divergences that are found in one-loop effective field theory. Using $\partial_t = k\partial_k$ in (4.44), we find

$$F_{i,\Lambda}(x) - F_{i,k}(x) = \frac{1}{(4\pi)^2} \int_k^\Lambda \frac{dk'}{k'} g_i\left(\frac{x}{k'^2}\right) ,$$

and after going to the variable $u = x/k^2$ (with $dk/k = -du/2u$) we get:

$$F_{i,\Lambda}(x) - F_{i,k}(x) = \frac{1}{2(4\pi)^2} \int_{x/\Lambda^2}^{x/k^2} \frac{du}{u} g_i(u) . \tag{4.47}$$

The constant terms in the flow functions (4.45) and (4.46) make the integrals in (4.47) logarithmically divergent at the lower limit when $\Lambda \to \infty$. We can isolate these divergences in the following way:

$$\begin{aligned} F_{1,\Lambda}(x) - F_{1,k}(x) &= \frac{1}{(4\pi)^2}\frac{1}{60}\left(\log\frac{\Lambda}{k_0} + \log\frac{k_0}{k}\right) \\ &\quad + \frac{1}{2(4\pi)^2}\int_{x/\Lambda^2}^{x/k^2}\frac{du}{u}\left[g_1(u) - \frac{1}{60}\right] \end{aligned} \tag{4.48}$$

$$
\begin{aligned}
F_{2,\Lambda}(x) - F_{2,k}(x) &= \frac{1}{(4\pi)^2}\frac{7}{10}\left(\log\frac{\Lambda}{k_0} + \log\frac{k_0}{k}\right) \\
&\quad + \frac{1}{2(4\pi)^2}\int_{x/\Lambda^2}^{x/k^2}\frac{du}{u}\left[g_2(u) - \frac{7}{10}\right], \qquad (4.49)
\end{aligned}
$$

where k_0 is an arbitrary reference scale which plays a role akin to μ in effective field theory. The logarithmic Λ terms in (4.48) and (4.49) correspond to the UV divergences (4.40) found earlier in the previous section. We can renormalize the theory imposing the following UV boundary conditions:

$$
\begin{aligned}
F_{1,\Lambda}(x) &= \frac{1}{(4\pi)^2}\frac{1}{60}\log\frac{\Lambda}{k_0} + c_1 \\
F_{2,\Lambda}(x) &= \frac{1}{(4\pi)^2}\frac{7}{10}\log\frac{\Lambda}{k_0} + c_2, \qquad (4.50)
\end{aligned}
$$

where the c_i are possible finite renormalizations. The important point in equations (4.48) and (4.49) is that the integrals are now convergent in the lower limit when we take $\Lambda \to \infty$. The scale-dependent form factors at the scale k turn out to be:

$$
\begin{aligned}
F_{1,k}(x) &= \frac{1}{32\pi^2}\left\{\left[\frac{1}{30}\log + \left(-\frac{601}{900} + \frac{1139k^2}{450x} + \frac{2k^4}{75x^2}\right)\sqrt{1-\frac{4k^2}{x}}\right.\right. \\
&\quad \left.\left. + \frac{1}{60}\log\left(\frac{x}{k_0^2}\right)\right]\theta(x-4k^2) + \frac{1}{60}\log\left(\frac{k^2}{k_0^2}\right)\theta(4k^2-x)\right\} \\
F_{1,k}(x) &= \frac{1}{32\pi^2}\left\{\left[\frac{7}{5}\log\frac{1}{2}\left(1+\sqrt{1-\frac{4k^2}{x}}\right) - \left(\frac{41}{75} + \frac{84k^2}{25x} + \frac{16k^4}{75x^2}\right)\sqrt{1-\frac{4k^2}{x}}\right.\right. \\
&\quad \left.\left. + \frac{7}{10}\log\left(\frac{x}{k_0^2}\right)\right]\theta(x-4k^2) + \frac{7}{10}\log\left(\frac{k^2}{k_0^2}\right)\theta(4k^2-x)\right\}. \qquad (4.51)
\end{aligned}
$$

These results are now to be reinserted in the $O(\mathcal{R}^2)$ part of the truncation ansatz (4.43):

$$
\bar{\Gamma}_k[g]\big|_{\mathcal{R}^2} = \int d^4x\,\sqrt{g}\left[RF_{1,k}(\Delta)R + R_{\mu\nu}F_{2,k}(\Delta)R^{\mu\nu}\right]. \qquad (4.52)
$$

This is the result we were looking for. Note that the form factors are continuous at $x = 4k^2$, and that for $k \to 0$ we obtain a well-defined limit, namely the action:

$$
\bar{\Gamma}_0[g]\big|_{\mathcal{R}^2} = \frac{1}{32\pi^2}\int d^4x\sqrt{g}\left[\frac{1}{60}R\,\log\left(\frac{\Delta}{k_0^2}\right)R + \frac{7}{10}R_{\mu\nu}\,\log\left(\frac{\Delta}{k_0^2}\right)R^{\mu\nu}\right], \qquad (4.53)
$$

where we fixed the finite renormalizations in (4.50) to $c_1 = -\frac{1}{(4\pi)^2}\frac{601}{1800}$ and $c_2 = -\frac{1}{(4\pi)^2}\frac{41}{150}$. The resulting non-local terms equal the part of standard one-loop quantum gravity effective action that arises solely from graviton and ghosts vacuum polarization [114].

We have obtained a non-local gEAA, equations (4.51) and (4.52), that flows from an ultraviolet scale $k = \Lambda$ to the infrared limit $k = 0$, and in the latter limit equals the expected effective field theory result. We discuss now the possible physical effects arising from the gEAA just calculated for $k \neq 0$ by computing the quantum corrections to the Newtonian potential stemming from it. We couple (4.43) to a classical matter source:

$$\begin{aligned}\bar{\Gamma}_k[g] &= \frac{1}{16\pi G_k}\int d^4x\sqrt{g}\,(2\Lambda_k - R) + \int d^4x\sqrt{g}\,[RF_{1,k}(\Delta)R + R_{\mu\nu}F_{2,k}(\Delta)R^{\mu\nu}] \\ &+ S_m[\phi,\psi,A_\mu;g]\,. \end{aligned} \tag{4.54}$$

Note that the matter action is taken to be scale independent. From (4.54) we will derive the equations of motion under the assumption that the gravitational field is weak throughout space in addition to static. Before presenting the explicit calculations, we would like to clarify two important points. Firstly, we will switch to work in a 3+1 static spacetime, analytically continuing our Euclidean expressions for the form factors into the Lorentzian sector. With the usual definition of the "in-out" effective action, this would imply a replacement of the Euclidean propagators by the corresponding Feynman propagators in the form factors, and the resulting field equations would be neither real nor causal. In order to get real and causal equations, one can introduce a "Closed Time Path" (CTP) or "in-in" effective action. As shown in [126], when the quantum fluctuations are in the vacuum state, the CTP procedure is equivalent to the replacement of the Euclidean propagator by the retarded propagators in the form factors appearing in the field equations. Due to the staticity assumption, this is equivalent to the replacement $\Delta \to -\nabla^2$ (the 3-Laplacian) in the form factors. Secondly, as pointed out in [114], the solutions of the field equations derived from the effective action will depend on the gauge fixing parameters, and therefore they are not physical. Indeed, for our calculations we considered the particular values $\alpha_k = \beta_k = 1$ in the gauge fixing condition. In the general case, the effective action and the quantum corrections to the metric will depend explicitly on α_k and β_k. In order to obtain physical results, it is necessary to define an observable from the quantum corrected metric, as proposed in [114]. Though important, this issue will not be relevant in the discussion that follows.

We write $g_{\mu\nu}(\mathbf{x}) = \eta_{\mu\nu} + h_{\mu\nu}(\mathbf{x})$ with $h_{\mu\nu}$ small everywhere, and we will proceed assuming k to be a fixed parameter, ignoring for the moment the possibility (discussed e.g. in [115, 117, 118, 119], and also later in this section) that it should depend on position. The equations of motion derived from (4.54) are:

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} - \Lambda_k g_{\mu\nu} = 8\pi G_k \left[T^{\rm mat}_{\mu\nu} + F_{1,k}(-\nabla^2) H^{(1)}_{\mu\nu} + F_{2,k}(-\nabla^2) H^{(2)}_{\mu\nu} \right] \tag{4.55}$$

where $H^{(1)}_{\mu\nu} = 4\nabla_\mu\nabla_\nu R - 4\eta_{\mu\nu}\nabla^2 R$ and $H^{(2)}_{\mu\nu} = 2\nabla_\mu\nabla_\nu R - \eta_{\mu\nu}\nabla^2 R - 2\nabla^2 R_{\mu\nu}$ are the variations of the squared curvature scalar and squared Ricci tensor. All curvature tensors are evaluated at first order in $h_{\mu\nu}$. We will assume in what follows that we are in a non-cosmological regime where Λ_k can be neglected.

We choose the classical matter to be static and non-relativistic so that there exists a quasi-Cartesian coordinate system in which

$$T^{\rm mat}_{\mu\nu} = {\rm diag}(\rho(\mathbf{x}), 0, 0, 0)\,. \tag{4.56}$$

We write for the metric perturbation $h_{\mu\nu} = h^c_{\mu\nu} + h^q_{\mu\nu}$ where $h^c_{\mu\nu}$ solves the classical equations of motion and $h^q_{\mu\nu}$ is $O(\hbar)$; also, we write $G_k = G_0(1+\delta G_k)$ with G_0 being the experimental value of G, assumed to be measured at $k=0$, and δG_k being $O(\hbar)$. Expanding to the first order in $\hbar$, the equations for the classical and quantum parts of the metric read:

$$\nabla^2 h^c_{\mu\nu} = 16\pi G_0 \left[T^{\rm mat}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\kappa} T^{\rm mat}_{\lambda\kappa} \right] = 8\pi G_0\, \rho\, {\rm diag}(1,1,1,1) \tag{4.57}$$

$$\begin{aligned} \nabla^2 h^q_{\mu\nu} = 16\pi G_0 \Big\{ &\left[4F_{1,k}(-\nabla^2) + 2F_{2,k}(-\nabla^2) \right] \partial_\mu\partial_\nu R + \eta_{\mu\nu} \left[2F_{1,k}(-\nabla^2) + F_{2,k}(-\nabla^2) \right] \nabla^2 R \\ &-2F_{2,k}(-\nabla^2)\nabla^2 R_{\mu\nu} + \frac{1}{2}\delta G_k\, \rho\, {\rm diag}(1,1,1,1) \Big\} \end{aligned} \tag{4.58}$$

In (4.58) the Ricci tensor and its trace are understood to be computed from $h^c_{\mu\nu}$ exclusively. The classical Newtonian potential $\phi(\mathbf{x})$ is equal to $-\frac{1}{2}h^c_{00}$ and, per (4.57), is found solving Poisson's equation as usual. Its quantum correction, bearing the same relation to h^q_{00}, will be found from

$$h^q_{00}(\mathbf{x}) = \delta G_k h^c_{00}(\mathbf{x}) + 256\pi^2 G_0^2 \left[F_{1,k}(-\nabla^2) + F_{2,k}(-\nabla^2) \right] \rho(\mathbf{x})\,, \tag{4.59}$$

which is obtained replacing in (4.58) $R_{\mu\nu}$ by $-\frac{1}{2}\nabla^2 h^c_{\mu\nu}$, using (4.57), and canceling Laplacians. Therefore, the quantum correction to the Newtonian potential consists of two terms: a trivial shift due to renormalization of G, plus a non-trivial part that is found by direct application of the non-local form factor $F_1(x) + F_2(x)$ to the classical matter distribution. From now on we take h^q_{00} to refer only to the nontrivial part, absorbing the first term in a redefinition of h^c_{00}.

Since the Laplacian in all the preceding expressions is flat (to keep only the first order in the metric perturbation), the action of the form factors in (4.59) can be computed with an ordinary Fourier transform:

$$
\begin{aligned}
h^q_{00}(\mathbf{x}) &= 256\pi^2 G_0^2 \frac{1}{(2\pi)^3} \int \mathrm{d}^3x' \rho(\mathbf{x}') \int \mathrm{d}^3p \, \left[F_{1,k}(p^2) + F_{2,k}(p^2)\right] \mathrm{e}^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \\
&= -\frac{4G^2}{\pi^2} \int \mathrm{d}^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \int_0^\infty \mathrm{d}p \; pF_k(p) \sin(p\,|\mathbf{x}-\mathbf{x}'|)\,, \qquad (4.60)
\end{aligned}
$$

where we have defined:

$$
\begin{aligned}
F_k(p) &= -32\pi^2 \left[F_{1,k}(p^2) + F_{2,k}(p^2)\right] \\
&= \frac{43}{60} \log\left(\frac{p^2 + (k^2 - p^2)\theta(4k^2 - p^2)}{k_0^2}\right) + \Bigg[\left(-\frac{1093}{900} - \frac{373}{450}\frac{k^2}{p^2} - \frac{14}{75}\frac{k^4}{p^4}\right)\sqrt{1 - \frac{4k^2}{p^2}} \\
&\quad + \frac{43}{30} \log \frac{1}{2}\left(1 + \sqrt{1 - \frac{4k^2}{p^2}}\right)\Bigg]\theta(p^2 - 4k^2)\,. \qquad (4.61)
\end{aligned}
$$

Equations (4.60) and (4.61) comprise the result we wanted for the quantum correction. For further analysis we call I_1 the term of the p-integral in (4.60) that comes from the first term of (4.61), and I_2 the p-integral of the remaining terms. I_1 can be evaluated exactly as a combination of elementary integrals and distributional Fourier transforms, whereas in I_2 no such closed form can be found. We have for I_1:

$$
\begin{aligned}
I_1 = \frac{43}{60} \Bigg\{ &-\frac{\pi}{X^2} + \delta'(X) \log k_0 + \left[\frac{\sin(2kX)}{X^2} - \frac{2k\cos(2kX)}{X}\right] \log 4 \\
&+ \frac{2\mathrm{Si}(2kX)}{X^2} - \frac{2\sin(2kX)}{X^2} \Bigg\}\,, \qquad (4.62)
\end{aligned}
$$

where we defined $X = |\mathbf{x} - \mathbf{x}'|$.

The remaining terms, comprising I_2, can also be rewritten as a combination of distributional Fourier transforms and convergent integrals, but for them the convergent part cannot usually be computed in closed form (though it can be investigated numerically if so desired). For this reason, we restrict ourselves to evaluating the large X asymptotic expansion of the result. If the matter distribution is a point source, i.e. $\rho(\mathbf{x}) = M\delta^3(\mathbf{x})$, this will give us the long-distance quantum corrections to the Newtonian potential of a point source in an asymptotic series. However, note that we will later call into question the physical validity of such an asymptotic expansion, so the following calculation needs to be taken with a grain of salt. We find that for large X we have:

$$\begin{aligned} I_2 = & -\frac{43}{30}\log 4\,\frac{k\cos(2kX)}{X} - \frac{7}{200}\sqrt{\pi k}\frac{\sin\left(2kX+\frac{3\pi}{4}\right)}{X^{3/2}} \\ & + \frac{43}{30}\left[1+\log 2\right]\frac{\sin(2kX)}{X^2} + o\left(\frac{1}{(kX)^2}\right) . \end{aligned} \tag{4.63}$$

Joining this with the asymptotic expansion for the result we obtained for I_1, we conclude that the quantum correction for the Newtonian potential of a point source of mass M is given at long distances $r = |\mathbf{x}|$ by

$$\begin{aligned} h^q_{00}(r) \quad = \quad & -\frac{8MG_0^2 k}{\pi r^2}\left[\frac{43}{60} - \frac{7}{400\sqrt{\pi}}\frac{\sin\left(2kr+3\pi/4\right)}{\sqrt{kr}}\right. \\ & \left. -\frac{43}{120\pi}\frac{\left(2\cos(2kr)+\pi\right)}{kr} + o\left(\frac{1}{kr}\right)\right] . \end{aligned} \tag{4.64}$$

The full Newtonian potential of the point source would therefore be given asymptotically by:

$$\begin{aligned} \phi(r) = & -\frac{MG_0}{r}\left[1+\delta G_k - \frac{43G_0k}{15\pi r} + \frac{7G_0\sqrt{k}}{100\pi^{3/2}}\frac{\sin\left(2kr+3\pi/4\right)}{r^{3/2}}\right. \\ & \left. +\frac{43G_0}{30\pi^2}\frac{2\cos(2kr)+\pi}{r^2} + o\left(\frac{1}{r^2}\right)\right] . \end{aligned} \tag{4.65}$$

One might find the asymptotic result (4.65) peculiar in several respects: the leading order correction is dominant over the effective field theory ones (k/r^2 versus $1/r^3$), and the sub-leading terms are oscillatory, which is difficult to interpret physically. We suggest that this is due to the large X expansion being unphysical for the problem under consideration. Recall that the equations of motion derived from the effective action predict the expectation values of quantities such as the metric. From the effective average action, which is defined by a functional integral with an infrared cutoff at scale k, we would expect to obtain, if k is treated as fixed, equations of motion for "approximate expectation values", that do not include the information about low frequency field modes. These might be a good approximation in the $kr \ll 1$ regime (where r is the radial coordinate for us, or more in general the greatest physical length scale involved in the problem) but we would expect the results to be incorrect for $kr \gg 1$. If this reasoning is accepted, the result in (4.65) is not physical, since it is obtained as a large r expansion for fixed k; the physical quantum correction to the potential is to be obtained instead from the $k \to 0$ limit at fixed r.

Taking the limit $k \to 0$ in (4.60) in I_2 gives zero up to a purely local term involving $\delta'(r)$. In I_1, this results in a similar local term plus $43\pi/(60r^2)$. This latter term implies a nontrivial long-distance correction for the Newtonian potential

$$\phi(r) = -\frac{MG_0}{r}\left[1 + \frac{43G_0}{30\pi r^2}\right] \tag{4.66}$$

(because δG_k, obtained by integrating the flow the flow of G_k, is $\sim G_0 k^2$ and thus vanishes at $k = 0$). This is the same result obtained in effective quantum gravity for the contribution of the graviton and ghost vacuum polarization diagrams [82, 114]. Since we have treated the matter source in a purely classical way, we are not obtaining the terms due to vertex corrections[3]. This confirms that the effective field theory quantum corrections to Newton's law are indeed recovered in the right limit from the effective average action.

These remarks above concord with the general philosophy of interpreting the EAA as a useful device to follow the flow of the renormalization group (and hopefully discover a UV fixed point) but not as an action from which physics can be directly extracted by solving equations of motion: these, predicting expectation values of observables, should be computed from the $k = 0$ usual effective action. On the other hand, there have been several attempts [115, 117, 118, 119] to extract physics from the effective average action itself by identifying k with an inverse distance scale of the spacetime under consideration, instead of as a fixed parameter. For a static and spherically symmetric situation like the one we are considering (with just a point source) this proposal means $k = \zeta/r$ with ζ a numerical constant. (Note that above we argued that the effective average action with cutoff k is to be trusted only for $k \ll r^{-1}$, whereas this approach conjectures that it can be trusted only for $k = \zeta r^{-1}$.). If we do this replacement before computing I_1 and I_2, we see that in the case of a point source the variable x is just 2ζ, and so all the functions of it are numerical constants. The Newtonian potential would therefore be

$$\phi(r) = -\frac{MG_0}{r}\left[1 + \frac{\xi\, G_0}{r^2}\right], \tag{4.67}$$

with ξ a numerical constant (the δG_k term gives a similar contribution, as has often been noted). This agrees with the general form of the effective field theory correction discussed above. However, this agreement in form is rather trivial, following just from dimensional

[3] A local term with $\delta'(r)$ is also found in effective quantum gravity, with an arbitrary constant involving the renormalization scale μ.

analysis and the lack of other length scales in the problem once we make the conjectured identification for k. A more interesting test of the conjectured identification in the case of an extended source with spherical symmetry, such as a spherical star. Here a naive application of this conjecture that equates k with the inverse distance to the center of the star, at the level of (4.60), leads to

$$h^q_{00}(\mathbf{x}) = -\frac{4\zeta^2 G^2}{\pi^2}\int \mathrm{d}^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|\, r^2} f\left(\frac{\zeta\,|\mathbf{x}-\mathbf{x}'|}{r}\right), \tag{4.68}$$

which is obtained from (4.60) by making the replacement $k = \zeta/r$, changing variables from p to $v = pr$, and defining f as the result of the v-integrals $I_1 + I_2$ with $1/r^2$ taken out. On the other hand, the result obtained from effective theory (and from our formulas as $k \to 0$) is

$$h^q_{00}(\mathbf{x}) = \frac{43G^2}{15\pi}\int \mathrm{d}^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{4.69}$$

These two formulas agree asymptotically for large r (up to a factor related to ζ), but will in general be different at closer distances where results depend on new length scales implicit in the function $\rho(r)$ for the density within the star (see [120] for a discussion of how (4.69) depends on this).

The qualitative agreement with effective field theory for a general source could be obtained if (instead of r^{-1}) we identified k with $|\mathbf{x}-\mathbf{x}'|^{-1}$ before integrating over $\mathbf{x}'$, in effect defining a $k(\mathbf{x},\mathbf{x}')$ as the inverse distance to each point source $\mathbf{x}'$ instead of the inverse distance to the global center of symmetry of the spacetime. We leave open the question of whether this kind of identification can be motivated or justified. The alternative position that we are suggesting is that the equations of motion derived from the effective average action are to be trusted only in a small k regime, and ultimately are only correct when $k \to 0$ and we recover the usual effective action.

It would obviously be desirable to go beyond our approximations. One-loop computations of the flow of higher order non-local terms seems possible in principle by extension of the methods of this thesis, though computationally very intensive. On the other hand, going beyond the one-loop approximation and obtaining information about the non-perturbative flow is a difficult and open research question. A first step towards it would be to improve the running of the form factors by computing the beta function not from the bare action but from an action with running G_k and Λ_k, which can be found from the beta functions of the previous section.

We now switch our attention to two dimensions where, due to the fact that the Ricci tensor is proportional to the Ricci scalar $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$, there is only one structure function at the order curvature square and this is given by the following linear combination:

$$F_k(x) = F_{1,k}(x) + \frac{1}{2} F_{2,k}(x) \,. \tag{4.70}$$

The flow equation for $F_k(x)$, given in (4.239), can be written as in (4.44):

$$\partial_t F_k(x) = \frac{1}{8\pi k^2} g\left(\frac{x}{k^2}\right) \,. \tag{4.71}$$

Note the overall power of k^{-2} in (4.71). If we employ the optimized cutoff shape function and if we use the parametric integrals of Q-functionals from Appendix A, we find for the function $g(u)$ the following form:

$$\begin{aligned} g(u) \;\; &= \;\; \frac{1}{16u^2}\left[-(12+27u)\sqrt{1-\frac{4}{u}} + 11u\sqrt{\frac{u}{u-4}}\right]\theta(u-4) \\ &\quad + \frac{1}{4u^2}\left[(4+9u)\sqrt{1-\frac{4}{u}} - 9u\sqrt{\frac{u}{u-4}}\right]\theta(u-4) \,. \end{aligned} \tag{4.72}$$

In equation (4.72) we separated the contributions coming from the graviton trace, in the first line, from the ghost contribution, in the second line. Note that now there are no UV divergences.

Integrating the flow from the UV scale Λ to the IR scale k and shifting to the variable $u = x/k^2$ gives:

$$F_k(x) = F_\Lambda(x) - \frac{1}{16\pi x}\int_{x/\Lambda^2}^{x/k^2} du\, g(u) \,. \tag{4.73}$$

The integral in (4.73) is finite for $\Lambda \to \infty$, so we can take the limit and impose the boundary condition $F_\Lambda(x) = 0$, to find:

$$\begin{aligned} F_k(x) \;\; &= \;\; -\frac{1}{16\pi x}\left\{\frac{\sqrt{x/k^2-4}\,(13x/k^2+2)}{4(x/k^2)^{3/2}} - \log\frac{x}{k^2} - 2\log\frac{1}{2}\left(1+\sqrt{1-\frac{4k^2}{x}}\right)\right. \\ &\quad \left. - \frac{\sqrt{x/k^2-4}\,(13x/k^2+2)}{3(x/k^2)^{3/2}}\right\}\theta(x/k^2-4) \,. \end{aligned} \tag{4.74}$$

The first line in (4.74) is the graviton contribution while the second line is the ghost contribution. We see that the first logarithmic term in the graviton contribution to $F_k(x)$ in

(4.74) do not have a finite limit for $k \to 0$. The ghost term flows to $\frac{26}{96\pi x}$, which is know to be the correct contribution to the effective action of two dimensional quantum gravity from the ghost sector. Usually this result is obtained employing the conformal gauge-fixing and considering the relative conformal field theory [44, 81]. The ghost effective action has the same form as the Polyakov effective action for matter fields that we calculated in section 2.3.3 of Chapter 2, but contributes with a different sign. Also, the first term in the graviton contribution has a similar form and flows to $-\frac{39}{96\pi x}$, which is far from being the known correct answer $-\frac{1}{96\pi x}$.

We have learned that the ghost sector is correctly described in our formalism, at least in $d = 2$, and we can speculate that it will be possible to treat quantum gravity implementing the EAA formalism only on the gravitational part of the effective action. This two dimensional example clearly shows that to be able to calculate the effective action in quantum gravity we must consider the full non-perturbative flow equation where we insert the structure functions $F_k(x)$ on both sides of the flow equation (4.13) for the gEAA. Knowing that the complete effective action for two dimensional quantum gravity has precisely the form (4.43), this will represent an exact truncation and is thus a crucial test for the formalisms to reproduce the results of two dimensional quantum gravity. This is a very promising avenue of future research [122].

4.3 EAA for quantum gravity

In this section we extend the construction of the background effective average action (bEAA), done in Chapter 3 for non-abelian gauge theories, to quantum gravity. In Chapter 3 we employed the background field method to introduce a cutoff action in such a way to preserve gauge invariance and the simple one-loop structure of the flow. Here we have to preserve diffeomorphism invariance, or invariance under general coordinate transformations. If we consider only small fluctuations around a fixed background, it is clear that we can use the fixed background metric to construct the differential operator to use to separate the slow from the fast field modes. The problem arises as soon as we start to consider strong fluctuations and quantum regimes where a background does not exist any more. In the quantum gravitational contest, one usually speaks about "background independence" as the concept that any theory of quantum gravity needs a formulation where no privileged metric is employed. In the functional RG contest, background independence is realized through the background field method, where an auxiliary arbitrary background metric is introduced together with the fluctuation metric and the cutoff operator in constructed with the former. In this way, the one-loop structure of the flow is preserved and a functional of the two metrics is defined. Only when we equate these two metrics we arrive at a diffeomorphism invariant functional that plays the role of the effective action for quantum gravity.

The metric that we integrate over in the path integral is linearly split as the sum of the background metric $\bar{g}_{\mu\nu}$ and the fluctuation metric $h_{\mu\nu}$:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,. \tag{4.75}$$

The cutoff action is taken to be quadratic in the fluctuation metric, while the cutoff operator is constructed with the background metric and can be, for example, the covariant Laplacian. The general form of the cutoff action is:

$$\Delta S_k[\varphi;\bar{g}] = \frac{1}{2}\int d^dx\sqrt{\bar{g}}\varphi\, R_k[\bar{g}]\varphi \,. \tag{4.76}$$

Here $\varphi = (h, \bar{C}, C)$ is the field multiplet combining the fluctuating metric with the ghost vector fields $\bar{C}_\mu, C^\mu$. The background effective average action (bEAA) is defined introducing in the integro-differential equation for the background effective action, defined in (D.98) in Appendix D, the cutoff action (4.76):

$$e^{-\Gamma_k[\varphi;\bar{g}]} = \int D\chi \exp\left(-S[\chi+\varphi;\bar{g}] - \Delta S_k[\chi;\bar{g}] + \int d^dx\sqrt{\bar{g}}\Gamma_k^{(1;0)}[\varphi;\bar{g}]\chi\right) , \tag{4.77}$$

where the field multiplet χ has zero vacuum expectation value $\langle\chi\rangle = 0$. The bare action in (4.77) is defined in equation (D.95) of Appendix D and reads:

$$S[\varphi;\bar{g}] = S[\bar{g}+h] + S_{gf}[h;\bar{g}] + S_{gh}[h,\bar{C},C;\bar{g}] \,. \tag{4.78}$$

The background gauge-fixing condition is $f_\mu[h,\bar{g}] = 0$, where

$$f_\mu[h,\bar{g}] = \left(\delta_\mu^\alpha \bar{\nabla}^\beta - \frac{\beta}{2}\bar{g}^{\alpha\beta}\bar{\nabla}_\mu\right) h_{\alpha\beta} \,, \tag{4.79}$$

while the background gauge-fixing action reads:

$$S_{gf}[h;\bar{g}] = \frac{1}{2\alpha}\int d^dx\sqrt{\bar{g}}\bar{g}^{\mu\nu} f_\mu[h,\bar{g}] f_\nu[h,\bar{g}] \,, \tag{4.80}$$

Here α and β are gauge-fixing parameters. The ghost action in (4.78), related to the gauge-fixing condition (4.79), is readily found to be:

$$S_{gh}[h,\bar{C},C;\bar{g}] = -\int d^dx\sqrt{\bar{g}}\,\bar{C}^\mu \left(\bar{\nabla}^\alpha g_{\nu\alpha}\nabla_\mu + \bar{\nabla}^\alpha g_{\mu\nu}\nabla_\alpha - \beta\bar{\nabla}_\mu g_{\nu\alpha}\nabla^\alpha\right) C^\nu \,. \tag{4.81}$$

The bEAA defined in this way is invariant under combined physical plus background diffeomorphisms:

$$(\delta+\bar{\delta})\Gamma_k[\varphi;\bar{g}] = 0\,. \tag{4.82}$$

We can now define a diffeomorphism covariant functional, that we will call gauge covariant EAA (gEAA), by setting in the bEAA the fluctuation multiplet to zero $\varphi = 0$, or equivalently $g_{\mu\nu} = \bar{g}_{\mu\nu}$ and $\bar{C}_\mu = C^\mu = 0$:

$$\bar{\Gamma}_k[g] = \Gamma_k[0,0,0;g]\,. \tag{4.83}$$

This is equivalent to the parametrization of the bEAA as the sum of a functional of the full quantum metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, the gEAA, and a "reminder functional" $\hat{\Gamma}_k[\varphi;\bar{g}]$ (rEAA) which remains functional of both the fluctuation multiplet and the background metric separately:

$$\Gamma_k[\varphi;\bar{g}] = \bar{\Gamma}_k[\bar{g}+h] + \hat{\Gamma}_k[\varphi;\bar{g}]\,. \tag{4.84}$$

The gEAA (4.83) in this way is invariant under physical diffeomorphisms:

$$\delta\bar{\Gamma}_k[\bar{g}] = 0\,, \tag{4.85}$$

while the rEAA is invariant under simultaneous physical and background diffeomorphisms as the full bEAA.

4.3.1 Exact flow equations for the bEAA in quantum gravity

We will follow the derivation of the flow equation for the bEAA done in Chapter 3 for non-abelian gauge theories. Differentiating the integro-differential definition of the bEAA (4.77), with respect to the "RG time" $t = \log k/k_0$, gives:

$$\begin{aligned} e^{-\Gamma_k[\varphi;\bar{g}]}\partial_t\Gamma_k[\varphi;\bar{g}] \;\; = \;\; & \int D\chi \left(\partial_t\Delta S_k[\chi;\bar{g}] - \int d^dx\sqrt{\bar{g}}\,\partial_t\Gamma_k^{(1;0)}[\varphi;\bar{g}]\chi\right) \times \\ & \times e^{-S[\varphi+\chi;\bar{g}]-\Delta S_k[\chi;\bar{g}]+\int\sqrt{\bar{g}}\,\Gamma_k^{(1;0)}[\varphi;\bar{g}]\chi}\,. \end{aligned} \tag{4.86}$$

Expressing the terms on the right hand side of (4.86) as expectation values, using relation (B.2) form Appendix B and (4.76), we can rewrite (4.86) as following:[4]

$$\begin{aligned}\partial_t \Gamma_k[\varphi;\bar{g}] &= \langle \partial_t \Delta S_k[\chi;\bar{g}]\rangle - \int d^d x \sqrt{\bar{g}}\, \partial_t \Gamma_k^{(1;0)}[\varphi;\bar{g}]\, \langle \chi_A \rangle \\ &= \frac{1}{2}\int d^d x \sqrt{\bar{g}}\, \langle \chi_A \chi_B \rangle\, \partial_t R_{k,BA}[\bar{g}]\,. \end{aligned} \tag{4.87}$$

In (4.87) we used the vanishing vacuum expectation value of the field multiplet $\langle \chi_A \rangle = 0$ and the symmetry of the cutoff kernel in the indices A and B. The two-point function of the fluctuation field can be written in terms of the inverse Hessian of the bEAA, where the functional derivatives are taken with respect to fluctuation fields, plus the cutoff action:

$$\langle \chi_A \chi_B \rangle = \left(\Gamma_k^{(2;0)}[\varphi;\bar{g}] + \Delta S_k^{(2;0)}[\varphi;\bar{g}]\right)^{-1} = \left(\Gamma_k^{(2;0)}[\varphi;\bar{g}] + R_k[\bar{g}]\right)^{-1}. \tag{4.88}$$

Inserting (4.88) into (4.87) and writing a functional trace in place of the integral, gives:

$$\partial_t \Gamma_k[\varphi;\bar{g}] = \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2;0)}[\varphi;\bar{g}] + R_k[\bar{g}]\right)^{-1} \partial_t R_k[\bar{g}]\,. \tag{4.89}$$

The flow equation (4.89) is the exact flow equation for the bEAA for quantum gravity [87] and is the main result of this section. The flow generated by (4.89) has the same general properties as the flow for non-abelian gauge theories described in Chapter 3. As we did before, if we define the "regularized propagator" as

$$G_k[\varphi;\bar{g}] = \left(\Gamma_k^{(2;0)}[\varphi;\bar{g}] + R_k[\bar{g}]\right)^{-1}, \tag{4.90}$$

then the flow equation for the bEAA (4.89) can be rewritten in the compact form:

$$\partial_t \Gamma_k[\varphi;\bar{g}] = \frac{1}{2}\mathrm{Tr}\, G_k[\varphi;\bar{g}]\partial_t R_k[\bar{g}]\,. \tag{4.91}$$

As for the other cases, the flow equation has a one-loop structure and can be derived as an RG improvement of the one-loop bEAA calculated from the integro-differential equation (4.77).

Using (4.83) and (4.91) we can readily write down the flow equation for the gEAA:

$$\partial_t \bar{\Gamma}_k[\bar{g}] = \partial_t \Gamma_k[0;\bar{g}] = \frac{1}{2}\mathrm{Tr}\, G_k[0;\bar{g}]\partial_t R_k[\bar{g}]\,. \tag{4.92}$$

[4] We introduce here the the multiplet indices $A, B, \ldots$

Note that $\Gamma_k^{(2;0)}[0,\bar{g}]$ is "super-diagonal" if the ghost action is bilinear in the ghosts and in this case we can immediately do the multiplet trace in (4.92). Following the same steps of done in section 3.3.1 of Chapter 3, we find the flow equation for the gEAA in its commonly used form:

$$\begin{aligned}\partial_t\bar{\Gamma}_k[\bar{g}] &= \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2,0,0;0)}[0,0,0;\bar{g}]+R_{k,hh}[\bar{g}]\right)^{-1}\partial_t R_{k,hh}[\bar{g}]+\\ &\quad -\mathrm{Tr}\left(\Gamma_k^{(0,1,1;0)}[0,0,0;\bar{g}]+R_{k,\bar{C}C}[\bar{g}]\right)^{-1}\partial_t R_{k,\bar{C}C}[\bar{g}]\,. \end{aligned} \tag{4.93}$$

In (4.93) we defined the cutoff kernels by the relations $R_{k,hh}[\bar{g}]\equiv\Delta S_k^{(2,0,0;0)}[\bar{g}]$ and $R_{k,\bar{C}C}[\bar{g}]\equiv\Delta S_k^{(0,1,1;0)}[\bar{g}]$. As in the case of non-abelian gauge theories, it is important to realize that equation (4.93) is not a closed equation for the gEAA, since it involves the Hessian of the bEAA taken with respect to the fluctuation metric and the ghost fields. This implies that for $k\neq 0$ it is necessary to consider the flow in the extended theory space of all functionals of the fields $h_{\mu\nu},\bar{C}_\mu,C^\nu$ and $\bar{g}_{\mu\nu}$ invariant under simultaneous physical and background diffeomorphisms, i.e. the flow of $\Gamma_k[h,\bar{C},C;\bar{g}]$.

The flow equation for rEAA can be deduced from directly from (4.84):

$$\partial_t\hat{\Gamma}_k[\varphi;\bar{g}]=\partial_t\bar{\Gamma}_k[\bar{g}+h]-\partial_t\Gamma_k[\varphi;\bar{g}]\,. \tag{4.94}$$

In section 4.3.3 we study the flow equations for the running proper-vertices of both the gEAA and the bEAA starting from equations (4.89) and (4.93) respectively.

4.3.2 Modified Ward-Takahashi identities

As for non-abelian gauge theories, the bEAA in quantum gravity obeys modified Ward-Takahashi (WT) identities.

The bEAA is invariant under combined physical δ and background $\bar{\delta}$ gauge transformations (4.82). The gauge-fixing action (4.80), the ghost action (4.81) and, more importantly, the cutoff action (4.76) spoil physical diffeomorphism invariance of the bEAA. This means that the bEAA should obey modified Ward-Takahashi identities under physical diffeomorphisms [87].

Physical diffeomorphisms act linearly on the fields in φ and so we have $\langle\delta\phi\rangle=\delta\varphi$. We can write them as $\delta\phi=\epsilon^A\mathcal{G}_A\phi$ where $\mathcal{G}_A$ are the symmetry generators and the ϵ_A are the parameters of an infinitesimal diffeomorphism (see Appendix D for more details).

Following the same steps done in section 3.3.2 of Chapter 3 we find the following relation:

$$\mathcal{G}\Gamma_k[\varphi;\bar{g}] = \langle \mathcal{G}\left(S_{gf}[h;\bar{g}] + S_{gh}[\phi;\bar{g}]\right)\rangle + \frac{1}{2}R_{k,AB}[\bar{g}]\mathcal{G}G_{k,BA}[\varphi;\bar{g}]\,. \tag{4.95}$$

Equation (4.95) represents the modified WT identity (mWT) the bEAA for quantum gravity satisfies. The modifying term in (4.95) stems from the introduction of the cutoff, the important point is that it vanishes in the $k \to 0$ limit as the cutoff kernel $R_k[\bar{g}]$ goes to zero. Thus the standard WT identity is recovered in the IR and is satisfied by $k \to 0$ limit of the bEAA $\Gamma_0[\varphi;\bar{g}]$. This property is of fundamental importance for the approach, since it shows that a fully diffeomorphism invariant theory is recovered as result of the integration of the flow.

The gEAA does not depend on the fluctuation fields and we simply have:

$$\mathcal{G}\bar{\Gamma}_k[\bar{A}] = 0\,. \tag{4.96}$$

Also, the gEAA is invariant background diffeomorphisms $\bar{\delta}\phi = \bar{\epsilon}^A\bar{\mathcal{G}}_A\phi$ and so:

$$\bar{\mathcal{G}}\bar{\Gamma}_k[\bar{A}] = 0\,. \tag{4.97}$$

This implies that the mWT (4.95) is a constrain only for the form of the rEAA.

4.3.3 Flow equations for the proper vertices

In complete analogy to what we did for non-abelian gauge theories in section 3.3.3 of Chapter 3, we derive the hierarchy of equations governing the flow of the proper-vertices of both the full bEAA and the gEAA.

Starting from the flow equation for the bEAA (4.91) we can derive a hierarchy of flow equations for the proper-vertices of the bEAA simply by taking functional derivatives with respect to the fields. Within the background formalism we are employing, we can take functional derivatives with respect to both the fluctuation multiplet $\varphi = (h_{\mu\nu}, \bar{C}_\mu C^\nu)$ and the background metric $\bar{g}_{\mu\nu}$. In this second case, we have to remember that the cutoff terms in the flow equation depend explicitly on the background metric and so there are new additional terms in the flow equations for the proper-vertices that are not present in the flow equations for the proper-vertices of the EAA in the non-background formalism. We will see that these terms are crucial in preserving the diffeomorphism covariance of the gEAA along the flow.

Taking a functional derivative of the flow equation (4.91) with respect to the fluctuation multiplet or with respect to the background metric gives the following flow equations for the one-vertices of the bEAA:

$$
\begin{aligned}
\partial_t \Gamma_k^{(1;0)}[\varphi;\bar{g}] &= -\frac{1}{2}\mathrm{Tr}\, G_k[\varphi;\bar{g}]\,\Gamma_k^{(3;0)}[\varphi;\bar{g}] G_k[\varphi;\bar{g}]\partial_t R_k[\bar{g}] \\
\partial_t \Gamma_k^{(0;1)}[\varphi;\bar{g}] &= -\frac{1}{2}\mathrm{Tr}\, G_k[\varphi;\bar{g}] \left(\Gamma_k^{(2;1)}[\varphi;\bar{g}] + R_k^{(1)}[\bar{g}]\right) G_k[\varphi;\bar{g}]\partial_t R_k[\bar{g}] \\
&\quad +\frac{1}{2}\mathrm{Tr}\, G_k[\varphi;\bar{g}]\partial_t R_k^{(1)}[\bar{g}]\,.
\end{aligned} \tag{4.98}
$$

Note that in the second equation of (4.98) there are terms containing functional derivatives of the cutoff kernel $R_k[\bar{g}]$ with respect to the background metric. Taking a further derivative of equation (4.98) with respect to both the fluctuation multiplet and the background metric gives the following flow equations for the two-vertices of the bEAA[5]:

$$
\begin{aligned}
\partial_t \Gamma_k^{(2;0)} &= \mathrm{Tr}\, G_k\,\Gamma_k^{(3;0)} G_k\,\Gamma_k^{(3;0)} G_k \partial_t R_k - \frac{1}{2}\mathrm{Tr}\, G_k\,\Gamma_k^{(4;0)} G_k \partial_t R_k \\
\partial_t \Gamma_k^{(1;1)} &= \mathrm{Tr}\, G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k\,\Gamma_k^{(3;0)} G_k \partial_t R_k \\
&\quad -\frac{1}{2}\mathrm{Tr}\, G_k \left(\Gamma_k^{(3;1)} + R_k^{(1)}\right) G_k \partial_t R_k - \frac{1}{2}\mathrm{Tr}\, G_k\,\Gamma_k^{(3;0)} \partial_t R_k^{(1)} \\
\partial_t \Gamma_k^{(0;2)} &= \mathrm{Tr}\, G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \partial_t R_k \\
&\quad -\frac{1}{2}\mathrm{Tr}\, G_k \left(\Gamma_k^{(2;2)} + R_k^{(2)}\right) G_k \partial_t R_k \\
&\quad -\mathrm{Tr}\, G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \partial_t R_k^{(1)} + \frac{1}{2}\mathrm{Tr}\, G_k \partial_t R_k^{(2)}\,.
\end{aligned} \tag{4.99}
$$

Proceeding in this way, we generate the full hierarchy of flow equations for the proper-vertices $\Gamma_k^{(n;m)}$ of the bEAA. In general the flow of the proper-vertex $\Gamma_k^{(n;m)}$ involves proper-vertices up to $\Gamma_k^{(n+2;m)}$ and functional derivatives of the cutoff kernel up to $R_k^{(m)}$.

Note that, as they stand in equation (4.98) and (4.99), every equation of the hierarchy has the same information content as the original flow equation (4.91). To make profit of the above derived equations, we perform now a Taylor expansion of the two argument functional $\Gamma_k[\varphi;\bar{g}]$, that we express in the following way:[6]

$$
\Gamma_k[\varphi;\bar{g}] = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \int_{x_1\ldots x_n y_1\ldots y_m} \gamma^{(n;m)}_{k,x_1\ldots x_n y_1 \ldots y_m} \varphi_{x_1}\cdots\varphi_{x_n}\bar{g}_{y_1}\cdots\bar{g}_{y_m}\,, \tag{4.100}
$$

[5]Here, as in several other equations of this section, we omit for clarity to explicitly write the arguments of the functionals.

[6]We omit to explicitly write Lorentz indices for clarity.

In (4.100) we defined the zero-field proper-vertices:

$$\gamma^{(n;m)}_{k,x_1\ldots x_n y_1\ldots y_m} \equiv \Gamma^{(n;m)}_{k,x_1\ldots x_n y_1\ldots y_m}[0;0]\,. \tag{4.101}$$

If we evaluate now the hierarchy of flow equations, the first of which are equation (4.98) and (4.99), for $\varphi = 0$ and $\bar{g}_{\mu\nu} = 0$, they become an infinite system of coupled equations for the zero-field proper-vertices $\gamma^{(n;m)}_k$. From the expansion (4.100), we see that this system can be used to extract the RG running of all terms in the bEAA that are analytic in the fields. In particular these terms can be of non-local character.

As for the bEAA, we can derive a hierarchy of flow equations for the proper vertices of the gEAA. In this case the functional depends only on the background metric. Taking a functional derivative of (4.92) with respect to $\bar{g}_{\mu\nu}$ gives the following flow equation for the one-vertex of the gEAA:

$$\begin{aligned} \partial_t \bar{\Gamma}^{(1)}_k[\bar{g}] \;&=\; -\frac{1}{2}\mathrm{Tr}\, G_k[0,\bar{g}]\left(\Gamma^{(2;1)}_k[0,\bar{g}] + R^{(1)}_k[\bar{g}]\right) G_k[0,\bar{g}]\partial_t R_k[\bar{g}] \\ &\quad +\frac{1}{2}\mathrm{Tr}\, G_k[0,\bar{g}]\partial_t R^{(1)}_k[\bar{g}]\,. \end{aligned} \tag{4.102}$$

The second term in (4.102) contains a functional derivative of the cutoff kernel as in the case of the second equation in (4.98). Actually, equation (4.102) is the same as equation (4.98) if we set $\varphi = 0$ and if we consider the relation $\partial_t \Gamma^{(0;1)}_k[0;\bar{g}] = \partial_t \bar{\Gamma}_k[\bar{g}]$.

A further derivative of (4.102) with respect to $\bar{g}_{\mu\nu}$ gives the following flow equation for the two-vertex of the gEAA:

$$\begin{aligned} \partial_t \bar{\Gamma}^{(2)}_k \;&=\; \mathrm{Tr}\, G_k\left(\Gamma^{(2;1)}_k + R^{(1)}_k\right) G_k \left(\Gamma^{(2;1)}_k + R^{(1)}_k\right) G_k \partial_t R_k \\ &\quad -\frac{1}{2}\mathrm{Tr}\, G_k\left(\Gamma^{(2;2)}_k + R^{(2)}_k\right) G_k \partial_t R_k \\ &\quad -\mathrm{Tr}\, G_k\left(\Gamma^{(2;1)}_k + R^{(1)}_k\right) G_k \partial_t R^{(1)}_k + \frac{1}{2}\mathrm{Tr}\, G_k \partial_t R^{(2)}_k\,. \end{aligned} \tag{4.103}$$

As for (4.102), this equation is equal to the last equation in (4.99) if we set $\varphi = 0$ and if we use the relation $\partial_t \Gamma^{(0;2)}_k[0;\bar{g}] = \partial_t \bar{\Gamma}^{(2)}_k[\bar{g}]$. As already said, the terms coming from functional derivatives of the cutoff kernel, that are present in the background formalism but not in the ordinary one, are crucial in preserving the covariant character of the flow of the gEAA and of its vertices. As we did for the full bEAA, we can perform a Taylor expansion of the gEAA analogous to (4.100) and define the zero-field proper-vertices

$$\bar{\gamma}^{(n)}_{k,x_1\ldots x_n} \equiv \bar{\Gamma}_{k,x_1\ldots x_n}[0]\,, \tag{4.104}$$

to turn the hierarchy of flow equations for the proper-vertices of the gEAA in an infinite dimensional coupled system for the $\bar{\gamma}^{(n)}_k$.

As was noticed in Chapter 3, there is a more compact form in which we can rewrite the flow equations for the proper-vertices we just derived. If we introduce the formal operator defined as

$$\tilde{\partial}_t = (\dot{R}_k - \eta_k R_k)\frac{\partial}{\partial R_k}\,, \tag{4.105}$$

where here η_k is a multiplet matrix of anomalous dimensions, we can rewrite the flow equation for the bEAA (4.91) as:

$$\partial_t \Gamma_k[\varphi;\bar{g}] = \frac{1}{2}\mathrm{Tr}\, G_k[\varphi;\bar{g}]\partial_t R_k[\bar{g}] = -\frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t \log G_k[\varphi;\bar{g}]\,. \tag{4.106}$$

In this way, we can rewrite the flow equation for the one-vertices of the bEAA (4.98) in the compact form:

$$\begin{aligned}
\partial_t \Gamma^{(1;0)}_k[\varphi;\bar{A}] &= -\frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t\left\{\Gamma^{(3;0)}_k[\varphi;\bar{A}]G_k[\varphi;\bar{A}]\right\}\\
\partial_t \Gamma^{(0;1)}_k[\varphi;\bar{A}] &= -\frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t\left\{\left(\Gamma^{(2;1)}_k[\varphi;\bar{A}] + R^{(1)}_k[\bar{A}]\right)G_k[\varphi;\bar{A}]\right\}\,,
\end{aligned} \tag{4.107}$$

while the flow equations for the two-vertices of the bEAA (4.99) read now:

$$\begin{aligned}
\partial_t \Gamma^{(2;0)}_k &= \frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t\left\{\Gamma^{(3;0)}_k G_k \Gamma^{(3;0)}_k G_k\right\} - \frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t\left\{\Gamma^{(4;0)}_k G_k\right\}\\
\partial_t \Gamma^{(1;1)}_k &= \frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t\left\{\left(\Gamma^{(2;1)}_k + R^{(1)}_k\right)G_k \Gamma^{(3;0)}_k G_k\right\} - \frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t\left\{\Gamma^{(3;1)}_k G_k\right\}\\
\partial_t \Gamma^{(0;2)}_k &= \frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t\left\{\left(\Gamma^{(2;1)}_k + R^{(1)}_k\right)G_k\left(\Gamma^{(2;1)}_k + R^{(1)}_k\right)G_k\right\}\\
&\quad -\frac{1}{2}\mathrm{Tr}\,\tilde{\partial}_t\left\{\left(\Gamma^{(2;2)}_k + R^{(2)}_k\right)G_k\right\}\,.
\end{aligned} \tag{4.108}$$

This notation turns out to be useful since the flow equations (4.107) and (4.108) contain less terms then their counter parts (4.98) and (4.99) and are thus much more manageable when employed in actual computations. The same reasoning applies to all subsequent equations of the hierarchy and extend to the flow equations for the zero-field proper-vertices $\gamma^{(n;m)}_k$. Also, in this case the flow equations for the proper-vertices of the gEAA are just those for the bEAA evaluated at $\varphi = 0$ and we don't need to repeat them here. The same applies to the flow equation for the zero-field proper-vertices $\bar{\gamma}_k^{(n)}$.

As for non-abelian gauge theories, the flow equation for the zero-field proper-vertices in quantum gravity can be turned into a powerful computational device to perform actual computations in the bEAA framework.

The diagrammatic representation of the flow equations for the zero-field proper-vertices of the bEAA and the momentum space techniques developed in section 3.3.4 of Chapter 3 carry over straightforwardly to the quantum gravity contest here studied and for this reason we do not repeat their derivation here. Section 3.3.4 can be read at this moment with the only modification that $\bar{A}_\mu \to \bar{g}_{\mu\nu}$ and by interpreting wavy line as graviton ones. In sections 4.5.6 and 4.5.7 we will employ the flow equations for the zero-field proper-vertices $\gamma_k^{(2,0,0;0)}$ and $\gamma_k^{(0,1,1;0)}$ to extract the RG running of the Pauli-Fierz mass m_h and of the fluctuation metric and ghost wave-function renormalizations.

4.4 Summary

In this chapter we developed the EAA approach to quantum gravity. The main problem in the construction of the RG flow for gravitational theories is to preserve background independence, i.e. the fact that the quantum theory of geometry should be constructed with no reference to any pre-fixed background metric. The way in which we deal with this problem is by implementing the coarse-graining procedure using an arbitrary background metric to distinguish the fast field modes from the slow ones that we never specify. The bEAA so constructed is a functional of both the fluctuation metric, along with the ghost fields, and of the background metric. The functional defined by setting the former to zero, that we call gauge invariant EAA (gEAA), is a diffeomorphism invariant functional of only one metric. As for non-abelian gauge theories, the diffeomorphism invariance of the bEAA is controlled by modified symmetry identities derived in section 4.3.2. From the exact flow equation that the bEAA satisfies we derive a hierarchy of flow equations, that together with the general techniques introduced in section 3.3.4 of Chapter 3, enables us to project the flow of both the bEAA or gEAA in any given truncation in which the action is analytic in the fields. These comprise both local and non-local truncations. The first ones are treated in section 4.2.2.1, where we consider the flow of the cosmological constant and of Newton's constant in three different approximations. The novelty is that we close the system that determines $\partial_t\tilde{\Lambda}$ and $\partial_t\tilde{G}$ using the anomalous dimensions of the fluctuation metric and of ghost fields that naturally enter this system. These anomalous dimensions are obtained by solving a linear system. The flow in these three different approximations is discussed and compared. A fundamental result is that the non-Gaussian fixed point, on which the asymptotic safety scenario in quantum gravity is based, is preserved in this non-trivial extension. We treat also the running Pauli-Fierz mass. The method used to derive these beta functions is based on the flow equations for the zero-field proper-vertices of the bEAA, introduced in section 4.3.3, and enables future extension to cover also the running of the gauge-fixing parameters [122]. Next, we make the first step toward the use of the EAA approach to actually compute the full effective action by considering a non-local truncation of the gEAA. This is in the spirit of the curvature expansion already introduced for matter fields in section 2.3.3 of Chapter 2 and in the contest of non-abelian gauge theories in section 3.2.2.2 of Chapter 3. In section 4.2.2.2 we showed how the running structure functions that encode the flow of the gEAA to order curvature square of the curvature expansion can be used to yield, in the limit $k \to 0$, the effective action for quantum gravity first obtained by Donoghue [113]. We investigated the possible physical interpretation of the gEAA for small but non-zero k. Finally, we looked at two dimensional quantum gravity and we concluded that it is necessary to consider the full flow of the structure functions in order to extract the known non-perturbative part of the theory. The future extensions that are now under reach are a promising avenue of research [122].

4.5 Appendix to Chapter 4

In this Appendix to the Chapter we expose the derivations of all the beta functions investigated in the main part of the chapter. In the first section we study the basic curvature invariants and we calculate their variations and functional derivatives. In the second and third sections we define a basis of projection operators and we use them to construct the regularized graviton propagator. In the fourth section we calculate the beta functions of the cosmological constant and of Newton's constant using heat kernel techniques. In the fifth section we compute the RG running of the curvature square structure functions induced by the Einstein-Hilbert operator $\int \sqrt{g}R$. In the following two sections we calculate the beta functions for the fluctuation metric Pauli-Fierz mass and for the wave-function renormalization of the fluctuation metric $Z_{h,k}$ and of the ghosts $Z_{C,k}$ appearing in the beta functions for Λ_k and G_k.

4.5.1 Variations and functional derivatives

We start introducing the basic curvature invariants. The basic invariants in the Einstein-Hilbert action are the volume and the integral of the Ricci scalar:

$$I_0[g] = \int d^dx\sqrt{g} \qquad\qquad I_1[g] = \int d^dx\sqrt{g}R\,. \tag{4.109}$$

Note that in $d = 2$ the integrand of the Ricci scalar is proportional to the Euler characteristic for a two dimensional manifold (D.71):

$$\chi(\mathcal{M}) = \frac{1}{4\pi}\int_{\mathcal{M}} d^2x\sqrt{g}\,R\,. \tag{4.110}$$

Up to two curvatures, or four derivatives, the invariants we can construct are:

$$I_{2,1}[g] = \int d^dx\sqrt{g}R^2 \qquad\qquad I_{2,2}[g] = \int d^dx\sqrt{g}R_{\mu\nu}R^{\mu\nu}$$

$$I_{2,3}[g] = \int d^dx\sqrt{g}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \qquad\qquad I_{2,4}[g] = \int d^dx\sqrt{g}\Box R\,. \tag{4.111}$$

The last invariant in (4.111) is a total derivative and is usually dropped. In $d = 4$ the three curvature square invariants are not independent since the linear combination

$$E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2\,, \tag{4.112}$$

is the integrand of the the Euler characteristic for a four dimensional manifold (D.77):

$$\chi(\mathcal{M}) = \frac{1}{32\pi^2}\int_{\mathcal{M}} d^4x\sqrt{g}\,E\,. \tag{4.113}$$

Relation (4.110) and (4.113) are proven in Appendix D using heat kernel methods. We define the invariant:

$$I_E[g] = \int d^dx\sqrt{g}\,E\,. \tag{4.114}$$

There is another interesting combination of the four derivatives invariants, this defines the Weyl conformal tensor (D.60), the square of which is (D.61):

$$C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - \frac{4}{d-2}R_{\mu\nu}R^{\mu\nu} + \frac{2}{(d-1)(d-2)}R^2\,. \tag{4.115}$$

The Weyl tensor is completely traceless and the action

$$I_C[g] = \int d^dx\sqrt{g}\, C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}\,, \tag{4.116}$$

is invariant under local conformal transformations, i.e. $I_C[e^\sigma g] = I_C[g]$ for any $\sigma(x)$. See Appendix D for a proof.

We now calculate the variations of the basic invariants just defined. We define $h_{\mu\nu} = \delta g_{\mu\nu}$ to be the first variation of the metric tensor. The first variations of the inverse metric can be deduced from the following relations, valid for any invertible matrix M,

$$M^{-1}M = 1 \qquad \Rightarrow \qquad \delta M^{-1}M + M^{-1}\delta M = 0 \qquad \Rightarrow \qquad \delta M^{-1} = -M^{-1}\delta M\, M^{-1}\,. \tag{4.117}$$

Setting $M_{\mu\nu} = g_{\mu\nu}$ and $\delta M_{\mu\nu} = h_{\mu\nu}$ in (4.117) gives:

$$\delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu} = -h^{\alpha\beta}\,. \tag{4.118}$$

The second variation can be calculated iterating (4.118):

$$\begin{aligned}
\delta^2 g^{\alpha\beta} &= -\delta g^{\alpha\mu}g^{\beta\nu}h_{\mu\nu} - g^{\alpha\mu}\delta g^{\beta\nu}h_{\mu\nu} \\
&= g^{\alpha\lambda}g^{\mu\rho}g^{\beta\nu}h_{\lambda\rho}h_{\mu\nu} + g^{\alpha\mu}g^{\beta\lambda}g^{\nu\rho}h_{\lambda\rho}h_{\mu\nu} \\
&= 2h^{\alpha\lambda}h_\lambda^\beta\,.
\end{aligned} \tag{4.119}$$

The third variation is similarly found to be:

$$\delta^3 g^{\alpha\beta} = -3!h_\rho^\alpha h_\sigma^\rho h^{\sigma\beta}\,. \tag{4.120}$$

Combining (4.118), (4.119) and (4.120) gives the following expansion for the inverse metric around the background metric $\bar{g}_{\mu\nu}$:

$$\begin{aligned}
g^{\alpha\beta} &= \bar{g}^{\alpha\beta} + \delta g^{\alpha\beta} + \frac{1}{2}\delta^2 g^{\alpha\beta} + \frac{1}{3!}\delta^3 g^{\alpha\beta} + O\left(h^4\right) \\
&= \bar{g}^{\alpha\beta} - h^{\alpha\beta} + h^{\alpha\lambda}h_\lambda^\beta - h_\rho^\alpha h_\sigma^\rho h^{\sigma\beta} + O\left(h^4\right)\,.
\end{aligned} \tag{4.121}$$

It is not difficult to write the general n-th variation of the inverse metric tensor, it can be proven by induction that:

$$\delta^n g^{\alpha\beta} = (-1)^n n!h_{\lambda_1}^\alpha h_{\lambda_2}^{\lambda_1}\cdots h_{\lambda_{n-1}}^{\lambda_{n-2}}h^{\lambda_{n-1}\beta}\,. \tag{4.122}$$

The variations of the determinant of the metric tensor can be easily found using the following relation, valid again for any invertible matrix M,

$$\log\det M = \mathrm{tr}\log M\,. \tag{4.123}$$

A variation of equation (4.123) gives:

$$\delta\det M = \delta e^{\log\det M} = \det M\,\delta\mathrm{tr}\log M = \det M\,\mathrm{tr}\left(M^{-1}\delta M\right)\,. \tag{4.124}$$

Inserting in (4.124) $M_{\mu\nu} = g_{\mu\nu}$ and $\delta M_{\mu\nu} = h_{\mu\nu}$ brings to

$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{\alpha\beta}\delta g_{\alpha\beta} = \frac{1}{2}\sqrt{g}h\,. \tag{4.125}$$

The second variation follows easily:

$$\delta^2\sqrt{g} = \frac{1}{4}\sqrt{g}\delta g^\alpha_\alpha\delta g^\beta_\beta - \frac{1}{2}\sqrt{g}\delta g^{\alpha\beta}\delta g_{\alpha\beta} = \sqrt{g}\left(\frac{1}{4}h^2 - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)\,. \tag{4.126}$$

For completeness the third variation of the metric determinant is found to be:

$$\delta^3\sqrt{g} = \sqrt{g}\left(\frac{1}{8}h^3 - \frac{3}{4}hh_{\mu\nu}h^{\mu\nu} + h_{\mu\nu}h^{\nu\alpha}h_\alpha^{\ \mu}\right)\,. \tag{4.127}$$

We don't have a closed formula for the n-th variation of the square root of the determinant of the metric, but for any given n these can be easily determined. We find now the variations of the Christoffel symbols, defined in equation (D.31) of Appendix D:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right)\,. \tag{4.128}$$

Using geodesic coordinates, it can be proven that the first variation of the Christoffel symbols is:

$$\delta\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\nabla_\mu h_{\nu\beta} + \nabla_\nu h_{\mu\beta} - \nabla_\beta h_{\mu\nu}\right)\,. \tag{4.129}$$

More generally we have the fundamental relation, that can again be proven by induction on n, for the n-th variation of the Christoffel symbols:

$$\delta^n\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\left(\delta^{n-1}g^{\alpha\beta}\right)\left(\nabla_\mu h_{\nu\beta} + \nabla_\nu h_{\mu\beta} - \nabla_\beta h_{\mu\nu}\right)\,. \tag{4.130}$$

All the non-linearities of the Christoffel symbols are due the inverse metric of which we know exactly the n-variation (4.122). Introducing the tensor:

$$G_{\mu\nu\alpha} = \frac{1}{2}\left(\nabla_\mu h_{\nu\alpha} + \nabla_\nu h_{\mu\alpha} - \nabla_\alpha h_{\mu\nu}\right) \,, \tag{4.131}$$

we can rewrite the n-th variation of the Christoffel symbols simply as:

$$\delta^n \Gamma^\alpha_{\mu\nu} = \delta^{n-1} g^{\alpha\beta}\, G_{\mu\nu\beta} \,. \tag{4.132}$$

Note that the tensor (4.131) is symmetric in the first two indices $G_{\mu\nu\alpha} = G_{\nu\mu\alpha}$. In particular we have the useful contractions:

$$G^{\alpha}{}_{\alpha}{}^{\mu} = \nabla^\alpha h^\mu_\alpha - \frac{1}{2}\nabla^\mu h \qquad\qquad G^{\alpha}_{\mu\alpha} = \frac{1}{2}\nabla_\mu h \,. \tag{4.133}$$

We turn now the variations of the fundamental building block of all gravitational invariants: the Riemann tensor. It is defined in equation (D.42) of Appendix D:

$$R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\lambda\nu}\Gamma^\lambda_{\beta\mu} - \Gamma^\alpha_{\lambda\mu}\Gamma^\lambda_{\beta\nu} \,. \tag{4.134}$$

The Ricci tensor and the Ricci scalar are defined by the following contractions:

$$R_{\beta\nu} = R^\alpha_{\beta\alpha\nu} \qquad\qquad R = g^{\beta\nu} R_{\beta\nu} \,. \tag{4.135}$$

All the properties of the curvature tensors are derived and reviewed in Appendix D.

The n-th variation of the Riemann tensor is found directly from the definition (4.134) and using the binomial theorem for the variation of a product:

$$\delta^n R^\alpha_{\beta\mu\nu} = \nabla_\mu \delta^n \Gamma^\alpha_{\beta\nu} - \nabla_\nu \delta^n \Gamma^\alpha_{\beta\mu} + \sum_{i=1}^{n-1} \begin{pmatrix} n \\ i \end{pmatrix} \left(\delta^{n-i}\Gamma^\alpha_{\mu\lambda}\delta^i \Gamma^\lambda_{\beta\nu} - \delta^{n-i}\Gamma^\alpha_{\nu\lambda}\delta^i\Gamma^\lambda_{\beta\mu}\right) \,. \tag{4.136}$$

This relation together with equation (4.130) or (4.132) and (4.122) gives us, in a closed form, all possible variations of the Riemann tensor. This is a fundamental result. The n-th variations of the Ricci tensor (4.135) are obtained straightforwardly from (4.136) by contraction:

$$\delta^n R_{\beta\nu} = \delta^n R^\alpha_{\beta\alpha\nu} \,. \tag{4.137}$$

The n-variation of the Ricci scalar follows from (4.135) and is:

$$\delta^n R = \sum_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \delta^{n-i} g^{\beta\nu} \delta^i R_{\beta\nu} \,. \tag{4.138}$$

We can now study some particular examples. From the fundamental relation (4.136), for $i = 1$, we find

$$\delta R^{\alpha}_{\beta\mu\nu} = \nabla_\mu G^{\alpha}_{\beta\nu} - \nabla_\nu G^{\alpha}_{\beta\mu} \,.$$

Using (4.137) and the second relation in (4.133) gives the first variation of the Ricci tensor:

$$\begin{aligned} \delta R_{\mu\nu} &= \nabla_\alpha G^{\alpha}_{\mu\nu} - \nabla_\nu G^{\alpha}_{\mu\alpha} \\ &= \frac{1}{2}\left[\nabla_\alpha \left(\nabla_\mu h^{\alpha}_{\nu} + \nabla_\nu h^{\alpha}_{\mu} - \nabla^\alpha h_{\mu\nu}\right) - \nabla_\nu \nabla_\mu h\right] \\ &= \frac{1}{2}\left(-\nabla^2 h_{\mu\nu} - \nabla_\nu \nabla_\mu h + \nabla_\alpha \nabla_\mu h^{\alpha}_{\nu} + \nabla_\alpha \nabla_\nu h^{\alpha}_{\mu}\right) \,. \end{aligned} \tag{4.139}$$

Combining (4.139) with (4.138) gives the first variation of the Ricci scalar:

$$\begin{aligned} \delta R &= g^{\mu\nu} \delta R_{\mu\nu} + \delta g^{\mu\nu} R_{\mu\nu} \\ &= -\nabla^2 h + \nabla^\mu \nabla^\nu h_{\mu\nu} - h_{\mu\nu} R^{\mu\nu} \,. \end{aligned} \tag{4.140}$$

From (4.136) with $n = 2$ we get the second variation of the Riemann tensor

$$\delta^2 R^{\alpha}_{\beta\mu\nu} = -\nabla_\mu \left(h^{\alpha\gamma} G_{\gamma\beta\nu}\right) + \nabla_\nu \left(h^{\alpha\gamma} G_{\gamma\beta\mu}\right) + 2\left(G^{\alpha}_{\mu\gamma} G^{\gamma}_{\beta\nu} - G^{\alpha}_{\nu\gamma} G^{\gamma}_{\beta\mu}\right) \,, \tag{4.141}$$

while the second variation of the Ricci tensor is again just the contraction of (4.141):

$$\delta^2 R_{\mu\nu} = -\nabla_\alpha \left(h^{\alpha\beta} G_{\beta\mu\nu}\right) + \nabla_\nu \left(h^{\alpha\beta} G_{\beta\mu\alpha}\right) + 2\left(G^{\alpha}_{\alpha\beta} G^{\beta}_{\mu\nu} - G^{\alpha}_{\nu\beta} G^{\beta}_{\mu\alpha}\right) \,. \tag{4.142}$$

The second variation of the Ricci scalar is given in terms of (4.118), (4.119), (4.139) and (4.142):

$$\delta^2 R = \delta^2 g^{\mu\nu} R_{\mu\nu} + 2\delta g^{\mu\nu} \delta R_{\mu\nu} + g^{\mu\nu} \delta^2 R_{\mu\nu} \,. \tag{4.143}$$

We can now find the variations of the curvature invariants $I_i[g]$. Using (4.125) and (4.126) we find:

$$\delta I_0[g] = \frac{1}{2}\int d^d x \sqrt{g} h \qquad \delta^2 I_0[g] = \int d^d x \sqrt{g}\left(\frac{1}{4}h^2 - \frac{1}{2}h^{\alpha\beta} h_{\alpha\beta}\right) \,. \tag{4.144}$$

Using (4.125) and (4.140) we find:

$$\delta I_1[g] = \int d^dx\, (\delta\sqrt{g}R + \sqrt{g}\delta R) = \int d^dx\sqrt{g}\left(-\nabla^2 h + \nabla^\mu\nabla^\nu h_{\mu\nu} - h_{\mu\nu}R^{\mu\nu} + \frac{1}{2}hR\right) . \tag{4.145}$$

For the second variation we have:

$$\delta^2 I_1[g] = \int d^dx \left(\delta^2\sqrt{g}R + 2\delta\sqrt{g}\delta R + \sqrt{g}\delta^2 R\right) , \tag{4.146}$$

the first two terms in (4.146) are rapidly evaluated using (4.125), (4.126) and (4.140). The last term in (4.146) can be expanded as:

$$\int d^dx\sqrt{g}\delta^2 R = \int d^dx\sqrt{g}\left(\delta^2 g^{\mu\nu}R_{\mu\nu} + 2\delta g^{\mu\nu}\delta R_{\mu\nu} + g^{\mu\nu}\delta^2 R_{\mu\nu}\right) . \tag{4.147}$$

Again, the first two terms in (4.147) need just the relations (4.118), (4.119) and (4.139), the last can be written employing (4.142). Modulo a total derivative, we have:

$$\int d^dx\sqrt{g}g^{\mu\nu}\delta^2 R_{\mu\nu} = 2\int d^dx\sqrt{g}\left(G^\alpha_{\alpha\beta}G^{\beta\gamma}_{\ \ \gamma} - G^{\alpha\gamma}_{\ \ \beta}G^\beta_{\ \gamma\alpha}\right) , \tag{4.148}$$

using in (4.148) the relations (4.133) and the product

$$G^{\alpha\gamma}_{\ \ \beta}G^\beta_{\ \gamma\alpha} = \frac{1}{4}\left(-\nabla^\gamma h_{\alpha\beta}\nabla_\gamma h^{\alpha\beta} + 2\nabla^\gamma h^{\alpha\beta}\nabla_\alpha h_{\beta\gamma}\right) ,$$

we find

$$\begin{aligned}\int d^dx\sqrt{g}g^{\mu\nu}\delta^2 R_{\mu\nu} &= 2\int d^dx\sqrt{g}\left(G^\alpha_{\alpha\beta}G^{\beta\gamma}_{\ \ \gamma} - G^{\alpha\gamma}_{\ \ \beta}G^\beta_{\ \gamma\alpha}\right) ,\\ &= \int d^dx\sqrt{g}\left(\nabla_\mu h^{\mu\nu}\nabla_\nu h - \frac{1}{2}\nabla_\mu h\nabla^\mu h\right.\\ &\qquad \left. + \frac{1}{2}\nabla^\alpha h_{\mu\nu}\nabla_\alpha h^{\mu\nu} - \nabla^\alpha h^{\mu\nu}\nabla_\mu h_{\nu\alpha}\right) .\end{aligned} \tag{4.149}$$

Inserting in (4.146) the variation (4.147) and (4.149) finally gives:

$$\begin{aligned}\delta^2 I_1[g] &= \int d^dx\sqrt{g}\left[-\frac{1}{2}h\nabla^2 h + \frac{1}{2}h^{\mu\nu}\nabla^2 h_{\mu\nu} - h^{\mu\nu}\nabla_\alpha\nabla_\mu h^\alpha_\nu + h\nabla^\mu\nabla^\nu h_{\mu\nu} +\right.\\ &\qquad \left. + 2h^{\mu\nu}h^\alpha_\mu R_{\nu\alpha} - hR^{\mu\nu}h_{\mu\nu} + \left(\frac{1}{4}h^2 - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)R\right] .\end{aligned} \tag{4.150}$$

Commuting covariant derivatives in the third term of (4.150) as

$$\nabla_\alpha \nabla_\mu h^\alpha_\nu = \nabla_\mu \nabla_\alpha h^\alpha_\nu + R_{\mu\alpha} h^\alpha_\nu - R_{\alpha\mu\beta\nu} h^{\alpha\beta} \,,$$

we can recast (4.150) to the form:

$$\begin{aligned} \delta^2 I_1[g] &= \int d^dx \sqrt{g} \left[-\frac{1}{2} h^{\mu\nu} \Delta h_{\mu\nu} + \frac{1}{2} h \Delta h - h^{\mu\nu} \nabla_\nu \nabla_\alpha h^\alpha_\mu + h \nabla^\mu \nabla^\nu h_{\mu\nu} \right. \\ & \left. + h^{\mu\nu} h^\alpha_\mu R_{\nu\alpha} + h^{\mu\nu} h^{\alpha\beta} R_{\alpha\mu\beta\nu} - h R^{\mu\nu} h_{\mu\nu} + \left(\frac{1}{4} h^2 - \frac{1}{2} h^{\alpha\beta} h_{\alpha\beta} \right) R \right] , \end{aligned} \quad (4.151)$$

which can be later combined with the gauge-fixing action. It is straightforward now to calculate higher order variations of both the actions $I_0[g]$ and $I_1[g]$, since their variations can always be reduced to combinations of variations of the inverse metric, of the metric determinant and of the Christoffel symbols, which are all known exactly. In the same way, we can easily calculate the variations of the higher curvature invariants (4.111). We will not do this here since, in this thesis, we will concentrate to truncations where only variations of $I_0[g]$ and $I_1[g]$ are needed.

The background gauge fixing action (4.80) is already quadratic in the metric fluctuation, when expanded reads:

$$S_{gf}[h; g] = \frac{1}{2\alpha} \int d^dx \sqrt{g} \left(-h^{\mu\nu} \nabla_\nu \nabla_\alpha h^\alpha_\mu + \beta h \nabla^\mu \nabla^\nu h_{\mu\nu} + \frac{\beta^2}{4} h \Delta h \right) . \quad (4.152)$$

Combining (4.152) with (4.151) gives:

$$\begin{aligned} -\frac{1}{2} \delta I_1[g] + S_{gf}[h; g] &= \frac{1}{2} \int d^dx \sqrt{g} \left[\frac{1}{2} h^{\mu\nu} \Delta h_{\mu\nu} - \frac{1}{2} \left(1 - \frac{\beta^2}{2\alpha} \right) h \Delta h \right. \\ & + \left(1 - \frac{1}{\alpha} \right) h^{\mu\nu} \nabla_\nu \nabla_\alpha h^\alpha_\mu - \left(1 - \frac{\beta}{\alpha} \right) h \nabla^\mu \nabla^\nu h_{\mu\nu} \\ & - h^{\mu\nu} h^\alpha_\mu R_{\nu\alpha} - h^{\mu\nu} h^{\rho\alpha} R_{\rho\nu\alpha\mu} + h R^{\mu\nu} h_{\mu\nu} \\ & \left. - \left(\frac{1}{4} h^2 - \frac{1}{2} h^{\alpha\beta} h_{\alpha\beta} \right) R \right] . \end{aligned} \quad (4.153)$$

We will use (4.153) in section 4.5.4 to construct the Hessian's needed in the flow equation for the bEAA. Note that the gauge choice $\alpha = \beta = 1$ diagonalizes the Hessian (4.153).

From the variations just obtained we can calculate all the functional derivatives of the previous defined invariants by employing the following relation between variations and functional derivatives[7]:

$$\delta^{(n)}(...)(x) = \frac{1}{n!}\int_{x_1...x_n} \left[(...)^{(n)}(x)\right]^{\mu_1\nu_1...\mu_n\nu_n}(x_1,...,x_n)h_{\mu_1\nu_1}(x_1)...h_{\mu_n\nu_n}(x_n). \tag{4.154}$$

Using (4.154) we can derive all the gravitational vertices needed in the flow equations for the zero-field proper-vertices used in sections 4.5.6 and 4.5.7.

4.5.2 Decomposition and projectors

In this section we study the different degrees of freedom that are contained in the fluctuation metric, we understand which degrees of freedom are physical and which are pure gauge. We use this knowledge to construct the projector basis that we will use in the next section to construct the regularized graviton propagator.

We start decomposing the metric fluctuation in transverse $h^T_{\mu\nu}$ and longitudinal $h^L_{\mu\nu}$ components:

$$h_{\mu\nu} = h^T_{\mu\nu} + h^L_{\mu\nu}\,, \tag{4.155}$$

with the following transversality condition $\nabla^\mu h^T_{\mu\nu} = 0$. The longitudinal part can be written in terms of the vector ξ_μ as:

$$h^L_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = \nabla_\mu\xi^T_\nu + \nabla_\nu\xi^T_\mu + 2\nabla_\mu\nabla_\nu\sigma\,. \tag{4.156}$$

In (4.156) we decomposed the vector into a transverse ξ^T_μ vector and the gradient of the scalar σ as $\xi_\mu = \xi^T_\mu + \nabla_\mu\sigma$, with the transversality condition $\nabla^\mu\xi^T_\mu = 0$. We can extract the trace of the fluctuation metric

$$h = g^{\mu\nu}h_{\mu\nu} = g^{\mu\nu}h^T_{\mu\nu} - 2\Delta\sigma\,, \tag{4.157}$$

writing the transverse component of $h_{\mu\nu}$ in the following way:

$$h^T_{\mu\nu} = h^{TT}_{\mu\nu} + \frac{1}{d}g_{\mu\nu}(h + 2\Delta\sigma)\,, \tag{4.158}$$

with $h^{TT}_{\mu\nu}$ the transverse-traceless metric satisfying $g^{\mu\nu}h^{TT}_{\mu\nu} = 0$. Inserting (4.156) and (4.158) in (4.155) gives:

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + \nabla_\mu\xi^T_\nu + \nabla_\nu\xi^T_\mu + 2\nabla_\mu\nabla_\nu\sigma + \frac{1}{d}g_{\mu\nu}(h + 2\Delta\sigma)\,. \tag{4.159}$$

[7]We use the convention $\int_x \equiv \int d^dx\sqrt{g_x}$.

In (4.159) the metric fluctuation is decomposed into a transverse-traceless symmetric tensor, a transverse vector and two scalar degrees of freedom, the trace and the longitudinal component of the vector. To see which of these degrees of freedom are physical and which are pure gauge we can insert in (4.159) the gauge transformation of the metric fluctuation parametrized by the vector χ_μ:

$$\delta h_{\mu\nu} = \nabla_\mu \chi_\nu + \nabla_\nu \chi_\mu = \nabla_\mu \chi_\nu^T + \nabla_\nu \chi_\mu^T + 2\nabla_\mu \nabla_\nu \chi \,. \tag{4.160}$$

In (4.160) with decomposed the gauge transformation vector as $\chi_\mu = \chi_\mu^T + \nabla_\mu \chi$ with as usual $\nabla^\mu \chi_\mu^T = 0$. Matching (4.160) to

$$\delta h_{\mu\nu} = \delta h_{\mu\nu}^{TT} + \nabla_\mu \delta\xi_\nu^T + \nabla_\nu \delta\xi_\mu^T + 2\nabla_\mu \nabla_\nu \delta\sigma + \frac{1}{d} g_{\mu\nu} (\delta h + 2\Delta\delta\sigma) \,,$$

we find:

$$\delta h_{\mu\nu}^{TT} = 0 \qquad \delta\xi_\mu^T = \chi_\mu^T \qquad \delta\sigma = \chi \qquad \delta h = -2\Delta\chi \,. \tag{4.161}$$

These are the gauge transformation properties of the metric fluctuation components. We see that the transverse-traceless symmetric tensor is a physical degree of freedom, which can be associated with the graviton. Also the following combination of the two scalar degrees of freedom

$$S = h + 2\Delta\sigma \,, \tag{4.162}$$

is gauge invariant $\delta S = 0$ and is as well physical. It correspond to the conformal mode that in the path integral formulation of gravity is dynamical as the graviton. Instead, the transverse vector ξ_μ^T and the scalar field σ are pure gauge fields [81].

When we will work with the flow equations for the zero-field proper-vertices of the bEAA in sections 4.5.6 and 4.5.7, we will work in flat space where the decomposition (4.159) naturally gives rise to a set of projectors operators that we will use as a base to express the regularized inverse gravitational propagator $\gamma_k^{(2,0,0;0)} + \mathbf{R}_k[\delta]$ where $\gamma_k^{(2,0,0;0)} = \mathbf{\Gamma}_k^{(2,0,0;0)}[0,0,0;\delta]$. Using the properties of these projectors we can then easily obtain the regularized gravitational propagator $\mathbf{G}_k[0;\delta] = \left(\gamma_k^{(2,0,0;0)} + \mathbf{R}_k[\delta]\right)^{-1}$. The basic longitudinal projector is defined by $P^{\mu\nu ij} = \partial^\mu \partial^\nu / \partial^2$ and projects out the longitudinal component of a vector field, $\delta^{\mu\nu} - P^{\mu\nu}$ instead projects out the transverse component of a vector field. The graviton is the transverse part of the traceless component of the metric, in flat space we can define it as follows:

$$\begin{aligned} h^{TT}_{\mu\nu} &= \left[\frac{1}{2}\left(\delta^\alpha_\mu - P^\alpha_\mu\right)\left(\delta^\beta_\nu - P^\beta_\nu\right) + \frac{1}{2}\left(\delta^\alpha_\mu - P^\alpha_\mu\right)\left(\delta^\beta_\nu - P^\beta_\nu\right) + \right. \\ & \left. -\frac{1}{d-1}\left(g_{\mu\nu} - P_{\mu\nu}\right)\left(g^{\alpha\beta} - P^{\alpha\beta}\right)\right] h_{\alpha\beta} \\ &= \left[\tilde{\delta}^{\alpha\beta}_{\mu\nu} - \frac{1}{d-1}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\right] h_{\alpha\beta}\,, \end{aligned} \tag{4.163}$$

where we defined $\tilde{g}^{\mu\nu} = g^{\mu\nu} - P^{\mu\nu}$. We also have the following relations for the scalar degrees of freedom:

$$S = \frac{1}{d-1}\tilde{g}^{kl} h_{kl} \qquad\qquad \Box\sigma = \frac{d}{d-1}\left(P^{kl} - \frac{1}{d}g^{kl}\right) h_{kl}\,. \tag{4.164}$$

Inspired by (4.163) and (4.164) we define the following projectors:

$$\begin{aligned} P_2^{\mu\nu,\alpha\beta} &= \tilde{\delta}^{\mu\nu,\alpha\beta} - \frac{1}{d-1}\tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta} \\ P_1^{\mu\nu,\alpha\beta} &= \frac{1}{2}\left(\tilde{g}^{\mu\alpha}P^{\nu\beta} + \tilde{g}^{\mu\beta}P^{\nu\alpha} + \tilde{g}^{\nu\alpha}P^{\mu\beta} + \tilde{g}^{\nu\beta}P^{\mu\alpha}\right) \\ P_S^{\mu\nu,\alpha\beta} &= \frac{1}{d-1}\tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta} \\ P_\sigma^{\mu\nu,\alpha\beta} &= P^{\mu\nu}P^{\alpha\beta} \\ P_{S\sigma}^{\mu\nu,\alpha\beta} &= \frac{1}{\sqrt{d-1}}\left(\tilde{g}^{\mu\nu}P^{\alpha\beta} + P^{\mu\nu}\tilde{g}^{\alpha\beta}\right)\,. \end{aligned} \tag{4.165}$$

The projectors in (4.165) have the following traces (where we use the notation $A = \mu\nu$ and $B = \alpha\beta$ and hats mean contractions):

$$\begin{aligned} P_2^{\hat{A}B} &= \frac{d^2-d-2}{2} & \qquad P_2^{\hat{A}\hat{B}} &= 0 \\ P_1^{\hat{A}B} &= d-1 & P_1^{\hat{A}\hat{B}} &= 0 \\ P_S^{\hat{A}B} &= 1 & P_S^{\hat{A}\hat{B}} &= d-1 \\ P_\sigma^{\hat{A}B} &= 1 & P_\sigma^{\hat{A}\hat{B}} &= 1 \\ P_{S\sigma}^{\hat{A}B} &= 0 & P_{S\sigma}^{\hat{A}\hat{B}} &= 2\sqrt{d-1} \end{aligned} \tag{4.166}$$

and satisfy the following relations:

$$
\begin{aligned}
[P_2 + P_1 + P_S + P_\sigma]^{\mu\nu,\alpha\beta} &= \delta^{\mu\nu,\alpha\beta} \\
\left[(d-1)P_S + P_\sigma + \sqrt{d-1}P_{S\sigma}\right]^{\mu\nu,\alpha\beta} &= g^{\mu\nu}g^{\alpha\beta} \\
\left[2P_\sigma + \sqrt{d-1}P_{S\sigma}\right]^{\mu\nu,\alpha\beta} &= g^{\mu\nu}P^{\alpha\beta} + P^{\mu\nu}g^{\alpha\beta} \\
[P_1 + 2P_\sigma]^{\mu\nu,\alpha\beta} &= \frac{1}{2}\left(g^{\mu\alpha}P^{\nu\beta} + g^{\mu\beta}P^{\nu\alpha} + g^{\nu\alpha}P^{\mu\beta} + g^{\nu\beta}P^{\mu\alpha}\right) \\
P_\sigma^{\mu\nu,\alpha\beta} &= P^{\mu\nu}P^{\alpha\beta}\,.
\end{aligned}
\tag{4.167}
$$

We can also introduce the trace projection operator as follows:

$$
P^{\mu\nu,\alpha\beta} = \frac{1}{d}g^{\mu\nu}g^{\alpha\beta}\,, \tag{4.168}
$$

and from (4.167) this can be expressed in terms of the other projection operators as[8]:

$$
\mathbf{P} = \frac{d-1}{d}\mathbf{P}_S + \frac{1}{d}\mathbf{P}_\sigma + \frac{\sqrt{d-1}}{d}\mathbf{P}_{S\sigma}\,, \tag{4.169}
$$

so that

$$
\mathbf{1} - \mathbf{P} = \mathbf{P}_2 + \mathbf{P}_1 + \frac{1}{d}\mathbf{P}_S + \frac{d-1}{d}\mathbf{P}_\sigma - \frac{\sqrt{d-1}}{d}\mathbf{P}_{S\sigma}\,. \tag{4.170}
$$

The non-zero products between these projection operators are:

$$
\begin{aligned}
&\mathbf{P}_S\mathbf{P}_{S\sigma} + \mathbf{P}_\sigma\mathbf{P}_{S\sigma} = \mathbf{P}_{S\sigma} \qquad && \mathbf{P}_S\mathbf{P}_{S\sigma} = \mathbf{P}_{S\sigma}\mathbf{P}_\sigma \\
&\mathbf{P}_{S\sigma}\mathbf{P}_S = \mathbf{P}_\sigma\mathbf{P}_{S\sigma} \qquad && \mathbf{P}_{S\sigma}\mathbf{P}_{S\sigma} = \mathbf{P}_S + \mathbf{P}_\sigma\,.
\end{aligned}
\tag{4.171}
$$

There is a useful isomorphisms that encodes (4.171) and that can be used to simplify the operations with these projectors. This reads:

$$
\mathbf{P}_S \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{P}_\sigma \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{P}_{S\sigma} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{4.172}
$$

The general structure of the inverse propagator that we will encounter in the next section is as follows:

$$
\mathbf{M} = \lambda_2\mathbf{P}_2 + \lambda_1\mathbf{P}_1 + \lambda_S\mathbf{P}_S + \lambda_\sigma\mathbf{P}_\sigma + \lambda_{S\sigma}\mathbf{P}_{S\sigma}\,, \tag{4.173}
$$

[8] We will sometimes suppress indices for notation clarity and we will use boldface symbols to indicate linear operators in the space of symmetric tensors.

We can invert (4.173) to obtain:

$$\mathbf{M}^{-1} = \frac{1}{\lambda_2}\mathbf{P}_2 + \frac{1}{\lambda_1}\mathbf{P}_1 + \frac{\lambda_\sigma}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2}\mathbf{P}_S + \frac{\lambda_S}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2}\mathbf{P}_\sigma - \frac{\lambda_{S\sigma}}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2}\mathbf{P}_{S\sigma}\,. \qquad (4.174)$$

The scalar part of (4.174) has been derived by using the isomorphisms (4.172) in the following way:

$$\begin{aligned}
(\lambda_S\mathbf{P}_S + \lambda_\sigma\mathbf{P}_\sigma + \lambda_{S\sigma}\mathbf{P}_{S\sigma})^{-1} \quad &\to \quad \begin{pmatrix} \lambda_S & \lambda_{S\sigma} \\ \lambda_{S\sigma} & \lambda_\sigma \end{pmatrix}^{-1} \\
&= \quad \frac{1}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2}\begin{pmatrix} \lambda_\sigma & -\lambda_{S\sigma} \\ -\lambda_{S\sigma} & \lambda_S \end{pmatrix} \\
&\to \quad \frac{1}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2}\left(\lambda_\sigma\mathbf{P}_S + \lambda_S\mathbf{P}_\sigma - \lambda_{S\sigma}\mathbf{P}_{S\sigma}\right)\,.
\end{aligned}$$

Equation (4.174) is the fundamental relation used in the next section to construct the regularized graviton propagator.

4.5.3 Regularized propagator

In this section we construct the regularized graviton propagator that enters the flow equation for the bEAA for quantum gravity. We will use the flow equations for the zero-field proper-vertices of the bEAA, $\gamma_k^{(2,0,0;0)}$ and $\gamma_k^{(0,1,1;0)}$, to calculate the running of the wave-function renormalization of the fluctuation metric, of the fluctuation metric Pauli-Fierz mass and of the ghost wave-function renormalization in section 4.5.6 and 4.5.7. In both cases we need the regularized graviton propagator evaluated for flat background metric. For the truncations we are considering in this chapter, we need the functional derivatives of the basic invariants (4.109) evaluated at $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$. In momentum space, these are given by the following relations:

$$\begin{aligned}
I_0^{(2)}(p,-p)^{\mu\nu,\alpha\beta} \quad &= \quad -\frac{1}{2}\delta^{\mu\nu,\alpha\beta} + \frac{1}{4}g^{\mu\nu}g^{\alpha\beta} \\
-\frac{1}{2}I_1^{(2)}(p,-p)^{\mu\nu,\alpha\beta} + S_{gf}(p,-p)^{\mu\nu,\alpha\beta} \quad &= \quad \frac{1}{2}p^2\delta^{\mu\nu,\alpha\beta} - \frac{1}{2}\left(1 - \frac{\beta^2}{2\alpha}\right)p^2 g^{\mu\nu}g^{\alpha\beta} \\
&\quad -\frac{1}{4}\left(1 - \frac{1}{\alpha}\right)\left(g^{\mu\alpha}p^\nu p^\beta + g^{\mu\beta}p^\nu p^\alpha\right. \\
&\quad \left.+g^{\nu\alpha}p^\mu p^\beta + g^{\nu\beta}p^\mu p^\alpha\right) \\
&\quad +\frac{1}{2}\left(1 - \frac{\beta}{\alpha}\right)\left(g^{\mu\nu}p^\alpha p^\beta + g^{\alpha\beta}p^\mu p^\nu\right)\,, \qquad (4.175)
\end{aligned}$$

where in the second line we added the contribution from the gauge-fixing term (4.152). Note that the second relation in (4.175) agrees with the variation (4.153) if we evaluate it on flat momentum space. We can now write down, in momentum space, the Hessian of the bEAA we are considering using the projection operators introduced in the previous section. We find the following form:[9]

$$\begin{aligned}\gamma_k^{(2,0,0;0)}(p,-p) &= Z_h \Bigg\{ \frac{1}{2}\left(p^2+2m_h^2-2\Lambda\right)\mathbf{P}_2 + \left(\frac{1}{2\alpha}p^2+m_h^2-\Lambda\right)\mathbf{P}_1 \\ &+ \left[-\left(\frac{d-2}{2}-\frac{(d-1)\beta^2}{4\alpha}\right)p^2+\frac{\Lambda}{2}\right]\mathbf{P}_S + \left(\frac{(2-\beta)^2}{4\alpha}p^2-\frac{\Lambda}{2}\right)\mathbf{P}_\sigma \\ &+ \frac{\sqrt{d-1}}{2}\left[\frac{\beta(\beta-2)}{2\alpha}p^2+2m_h^2+\Lambda\right]\mathbf{P}_{S\sigma}\Bigg\}\,. \end{aligned} \tag{4.176}$$

It is now the moment to chose the tensor structure of the cutoff kernel. Here we will consider the following form:

$$\mathbf{R}_k[\delta] = \frac{1}{2}Z_h\left[\mathbf{P}_2+\mathbf{P}_1-\frac{d-3}{2}\mathbf{P}_S+\frac{1}{2}\mathbf{P}_\sigma-\frac{\sqrt{d-1}}{2}\mathbf{P}_{S\sigma}\right]R_k(p^2)\,, \tag{4.177}$$

that corresponds to the choice that we will make in the next section when we derive the beta function of Λ and G. The inverse regularized graviton propagator can thus be written by summing (4.176) and (4.177):

$$\gamma_k^{(2,0,0;0)} + \mathbf{R}_k[\delta] = Z_h\left[\gamma_2\mathbf{P}_2+\gamma_1\mathbf{P}_1+\gamma_S\mathbf{P}_S+\gamma_{S\sigma}\mathbf{P}_{s\sigma}+\gamma_\sigma\mathbf{P}_\sigma\right]\,, \tag{4.178}$$

where the various spin components in (4.178) are:

$$\begin{aligned}\gamma_2(p^2) &= \frac{1}{2}(p^2-2\Lambda)+m_h^2+\frac{1}{2}R_k(p^2) \\ \gamma_1(p^2) &= \frac{1}{2\alpha}p^2-\Lambda+m_h^2+R_k(p^2) \\ \gamma_S(p^2) &= -\left[\frac{d-2}{2}-\frac{(d-1)\beta^2}{4\alpha}\right]p^2-dm_h^2+\frac{d-3}{2}\Lambda+\frac{d-1}{4}R_k(p^2) \\ \gamma_\sigma(p^2) &= \frac{(2-\beta)^2}{4\alpha}p^2-\frac{\Lambda}{2}+\frac{d-1}{4}R_k(p^2) \\ \gamma_{S\sigma}(p^2) &= \sqrt{d-1}\left[\frac{\beta(\beta-2)}{4\alpha}p^2+m_h^2+\frac{\Lambda}{2}\right]\,. \end{aligned} \tag{4.179}$$

[9]In this section as in the following we will omit to write the k dependence of the running coupling constants for clarity.

The regularized graviton propagator is defined as the inverse regularized Hessian of the bEAA, i.e the inverse of (4.178):

$$\mathbf{G}_k[0;\delta] = \left(\gamma_k^{(2,0,0;0)} + \mathbf{R}_k[\delta]\right)^{-1} . \tag{4.180}$$

Using the isomorphism described in the previous section to invert (4.178) we find the following general form:

$$\mathbf{G}_k[0;\delta] = G_2\mathbf{P}_2 + G_1\mathbf{P}_1 + G_S\mathbf{P}_S + G_{S\sigma}\mathbf{P}_{S\sigma} + G_\sigma\mathbf{P}_\sigma \,, \tag{4.181}$$

together with the following spin components:

$$\begin{aligned}
G_2(p^2) &= \frac{2}{p^2 + 2m_h^2 - 2\Lambda + R_k(p^2)} \\
G_1(p^2) &= \frac{2\alpha}{p^2 + 2\alpha\left[m_h^2 - \Lambda + R_k(p^2)\right]} \\
G_S(p^2) &= \frac{\gamma_\sigma}{\gamma_S\gamma_\sigma - \gamma_{S\sigma}^2} \\
G_\sigma(p^2) &= \frac{\gamma_S}{\gamma_S\gamma_\sigma - \gamma_{S\sigma}^2} \\
G_{S\sigma}(p^2) &= -\frac{\gamma_{S\sigma}}{\gamma_S\gamma_\sigma - \gamma_{S\sigma}^2} \,.
\end{aligned} \tag{4.182}$$

Equations (4.181) and (4.182) represent the general for of the regularized graviton propagator on flat space for general values of Λ, m_h, α and β.

There are particular gauge-fixing parameter choices that simplify the graviton regularized propagator. One case, that correspond to the gauge used in sections 4.5.4, 4.5.6 and 4.5.7, is the choice $\alpha = \beta = 1$. In this case we have:

$$\mathbf{G}_k[0;\delta] = (\mathbf{1} - \mathbf{P})\, G_{TF,k}(p^2) - \frac{2}{d-2}\mathbf{P}\, G_{T,k}(p^2) \,, \tag{4.183}$$

where $\mathbf{P}$ is the trace projector (4.169). In (4.183) $G_{TF,k}$ and $G_{T,k}$ are, respectively, the trace-free and trace parts of the regularized graviton propagator and are defined in equation (4.215) of section 4.5.4. The cutoff kernel (4.177), when written in terms of $\mathbf{P}$ using (4.169) and (4.170) reads as follows:

$$\mathbf{R}_k[\delta] = \frac{1}{2} Z_h \left[\mathbf{1} - \mathbf{P} - \frac{d-2}{2}\mathbf{P}\right] R_k(p^2) \,. \tag{4.184}$$

This shows that the cutoff kernel (4.177) is as the one employed in [87, 96]. Another interesting case, that correspond to the physical decomposition of the degrees of freedom described in section 4.5.2, is the gauge $\alpha = 0$ and $\beta = 2/d$ where we find the following form for the various spin components:

$$\begin{aligned} G_2(p^2) &= \frac{2}{p^2 + 2m_h^2 - 2\Lambda + R_k(p^2)} \\ G_1(p^2) &= 0 \\ G_S(p^2) &= \frac{1}{\frac{d-2}{2}\left(-p^2 + 2m_h^2\right) + \frac{d(d-2)}{2(d-1)}\Lambda + R_k(p^2)} \\ G_\sigma(p^2) &= \frac{1}{d-1} G_S(p^2) \\ G_{S\sigma}(p^2) &= \frac{1}{\sqrt{d-1}} G_S(p^2) \,. \end{aligned} \tag{4.185}$$

Note that in this gauge the scalars projectors come in the following combination:

$$\frac{1}{d-1}\left[(d-1)\mathbf{P}_S + \mathbf{P}_\sigma + \sqrt{d-1}\mathbf{P}_{S\sigma}\right] = \frac{d}{d-1}\mathbf{P} \,. \tag{4.186}$$

This shows that in this gauge the propagating degrees of freedom are $h^{TT}_{\mu\nu}$ and S. Note that in $d = 4$ the regularized propagator for the scalar modes becomes simply:

$$G_S(p^2) = \frac{1}{-p^2 - 2m_h^2 + \frac{4}{3}\Lambda + R_k(p^2)} \,. \tag{4.187}$$

It is important to notice that only for the value $\beta = \frac{d}{2}$, the limit $\alpha \to 0$ gives the same form of the regularized propagators for all three scalar components apart an overall factor. We will use the regularized graviton propagator (4.183) both in section 4.5.6 and section 4.5.7.

4.5.4 Derivation of $\partial_t G_k$ and $\partial_t \Lambda_k$

In this section we calculate the beta function of Newton's constant $\partial_t G$ and the beta function of the cosmological constant $\partial_t \Lambda$ for non-zero Pauli-Fierz mass m_h and in the gauge $\alpha = \beta = 1$, where heat kernel techniques can be used. We will employ both a type I and a type II cutoff operator. The truncation ansatz for the gEAA that we are considering, equation (4.14) is:

$$\bar\Gamma_k[g] = \frac{1}{16\pi G}\int d^dx \sqrt{g}\,(2\Lambda - R) \,. \tag{4.188}$$

Differentiating (4.188) with respect to the RG time gives:

$$\partial_t \bar{\Gamma}_k[\bar{g}] = \partial_t \left(\frac{\Lambda}{8\pi G}\right) \int d^dx \sqrt{g} - \partial_t \left(\frac{1}{16\pi G}\right) \int d^dx \sqrt{g} R\,. \tag{4.189}$$

From (4.189) we see that we can extract the beta functions of the cosmological constant and of Newton's constant from those terms proportional to the invariants $I_0[g]$ and $I_1[g]$ steaming from the expansion of functional traces on the rhs of the flow equation

$$\begin{aligned}\partial_t \bar{\Gamma}_k[\bar{g}] \quad &= \quad \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2,0,0,0)}[0,0,0;\bar{g}]^{\mu\nu}_{\alpha\beta} + R_k[\bar{g}]^{\mu\nu}_{\alpha\beta}\right)^{-1} \partial_t R_k[\bar{g}]^{\alpha\beta}_{\mu\nu} + \\ &\quad -\mathrm{Tr}\left(\Gamma_k^{(0,1,1,0)}[0,0,0;\bar{g}]^{\mu}_{\nu} + R_k[\bar{g}]^{\mu}_{\nu}\right)^{-1} \partial_t R_k[\bar{g}]^{\mu}_{\nu}\end{aligned} \tag{4.190}$$

for the gEAA. Note that in (4.190) the cutoff kernels in the graviton and ghost sectors are distinguished by the indices. From here on we will consider only the gauge $\alpha = \beta = 1$ that allows us to employ heat kernel methods. We use the general decomposition of the bEAA given in (4.84) to write:

$$\Gamma_k^{(2,0,0;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu}_{\alpha\beta} = \bar{\Gamma}_k^{(2)}[\bar{g}+h]^{\mu\nu}_{\alpha\beta} + \hat{\Gamma}_k^{(2,0,0;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu}_{\alpha\beta} \tag{4.191}$$

and

$$\Gamma_k^{(0,1,1;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu} = Z_C S_{gh}^{(0,1,1;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu}\,. \tag{4.192}$$

In (4.192) we used our ansatz for the rEAA given in equation (4.17). To calculate the gravitational Hessian needed in equation (4.190), we can extract the quadratic part in the fluctuation metric of the action (4.188) using equation (4.153):

$$\begin{aligned}\frac{1}{2}\int d^dx\sqrt{\bar{g}}h_{\mu\nu}\Gamma_k^{(2,0,0;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu}_{\alpha\beta}h^{\alpha\beta} &= \frac{1}{2}Z_h\int d^dx\sqrt{\bar{g}}\left[\frac{1}{2}h^{\mu\nu}\bar{\Delta}h_{\mu\nu} - \frac{1}{4}h\bar{\Delta}h\right.\\ &\quad + m_h^2\left(h^{\alpha\beta}h_{\alpha\beta} - h^2\right) - h^{\mu\nu}h^{\alpha}_{\mu}\bar{R}_{\nu\alpha} - h^{\mu\nu}h^{\alpha\beta}\bar{R}_{\alpha\mu\beta\nu}\\ &\quad \left. + h\bar{R}^{\mu\nu}h_{\mu\nu} + \left(\frac{1}{4}h^2 - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)(2\Lambda - \bar{R})\right]\,.\end{aligned} \tag{4.193}$$

The gravitational Hessian can now be easily extracted from (4.193) and reads:

$$\Gamma_k^{(2,0,0;0)}[0,0,0;g]^{\mu\nu}_{\rho\sigma} = \frac{Z_h}{2}\left[\delta^{\mu\nu}_{\alpha\beta} - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\right]\left[\delta^{\alpha\beta}_{\rho\sigma}(\Delta + m_h^2 - 2\Lambda) + \frac{m_h^2}{d-2}g^{\alpha\beta}g_{\rho\sigma} + U^{\alpha\beta}_{\rho\sigma}\right]\,, \tag{4.194}$$

where the symmetric spin two tensor identity $\delta^{\mu\nu}_{\rho\sigma} = \frac{1}{2}\left(\delta^\mu_\rho \delta^\nu_\sigma + \delta^\mu_\sigma \delta^\nu_\rho\right)$ and the trace projector $P^{\mu\nu}_{\rho\sigma} = \frac{1}{d} g^{\mu\nu} g_{\rho\sigma}$ have been defined in section 4.5.3 and we defined the following tensor:

$$
\begin{aligned}
U^{\alpha\beta}_{\rho\sigma} &= \left(\delta^{\alpha\beta}_{\rho\sigma} - \frac{1}{2} g^{\alpha\beta} g_{\rho\sigma}\right) R + g^{\alpha\beta} R_{\rho\sigma} + R^{\alpha\beta} g_{\rho\sigma} + \\
&\quad -\frac{1}{2}\left(\delta^\alpha_\rho R^\beta_\sigma + \delta^\alpha_\sigma R^\beta_\rho + R^\alpha_\rho \delta^\beta_\sigma + R^\alpha_\sigma \delta^\beta_\rho\right) - \left(R^{\beta\ \alpha}_{\ \rho\ \sigma} + R^{\beta\ \alpha}_{\ \sigma\ \rho}\right) + \\
&\quad -\frac{d-4}{2(d-2)}\left(R\, g^{\alpha\beta} g_{\rho\sigma} + g^{\alpha\beta} R_{\rho\sigma} + R^{\alpha\beta} g_{\rho\sigma}\right) .
\end{aligned} \tag{4.195}
$$

Note, for later use, that the tensor (4.195) is linear in the curvatures.

We will sometimes suppress indices for notation clarity and we will use boldface symbols to indicate linear operators in the space of symmetric tensors. For example, the operators just defined will be indicated as $\mathbf{1}$, $\mathbf{P}$ and $\mathbf{U}$. Note that $\mathbf{1}-\mathbf{P}$ and $\mathbf{P}$ are orthogonal projectors into the trace and trace free subspaces in the space of symmetric tensors. With this notation we can rewrite (4.194) in the following way:

$$
\mathbf{\Gamma}^{(2,0,0;0)}_k[0,0,0;g] = \frac{1}{2} Z_h \left[(\mathbf{1}-\mathbf{P}) - \frac{d-2}{2}\mathbf{P}\right]\left[\mathbf{1}(\Delta + m_h^2 - 2\Lambda) + m_h^2 \frac{d}{d-2}\mathbf{P} + \mathbf{U}\right] . \tag{4.196}
$$

The ghost action (4.81) when evaluated at zero fluctuation metric becomes:

$$
S_{gh}[0,\bar{C},C;\bar{g}] = \int d^dx \sqrt{\bar{g}}\, \bar{C}^\mu \left[\bar{\Delta}\bar{g}_{\mu\nu} - (1-\beta)\bar{\nabla}_\mu \bar{\nabla}_\nu - \bar{R}_{\mu\nu}\right] C^\nu . \tag{4.197}
$$

If we then set $\beta = 1$ in (4.197) we find the following ghost Hessian:

$$
\Gamma^{(0,1,1;0)}_k[0,0,0;g]^\mu_\nu = Z_C\left(\Delta \delta^\mu_\nu - R^\mu_\nu\right) . \tag{4.198}
$$

For later use we report here the following traces of the tensors defined in (4.195) and before:

$$
\mathrm{tr}\,\mathbf{1} = \frac{d(d+1)}{2} \qquad \mathrm{tr}\,\mathbf{P} = 1 \qquad \mathrm{tr}\,\mathbf{U} = \frac{d(d-1)}{2} R
$$

$$
\mathrm{tr}\,\mathbf{U}^2 = \frac{d^3 - 5d^2 + 8d + 4}{2(d-2)} R^2 + \frac{d^2 - 8d + 4}{d-2} R_{\mu\nu} R^{\mu\nu} + 3 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} . \tag{4.199}
$$

We have now to choose the cutoff operator that is used to separate the slow modes from the fast modes in the functional integral. As we did for non-abelian gauge theories in Chapter 3, we will employ two different cutoff operator choices. In this way we can study the dependence of the resulting beta functions on this choice. In the first case, that we call type I, we use in both the graviton and ghost sectors the covariant Laplacian as cutoff operator. In the second case, we employ the differential operator $\mathbf{\Delta}_2 = \Delta\mathbf{1} + \mathbf{U}$ for the graviton modes and $\Delta_1 = \Delta\delta^\mu_\nu - R^\mu_\nu$ for the ghost modes.

We star to consider type I cutoff. We define the graviton cutoff kernel as:

$$\mathbf{R}_k(\Delta) = \frac{Z_h}{2}\left[(\mathbf{1}-\mathbf{P}) - \frac{d-2}{2}\mathbf{P}\right] R_k(\Delta)\,, \tag{4.200}$$

which corresponds to the flat space expression (4.184) of the previous section. For the ghost cutoff kernel we take:

$$R_k(\Delta)^\mu_\nu = Z_C \delta^\mu_\nu R_k(\Delta)\,. \tag{4.201}$$

Remembering that the anomalous dimension of the fluctuation metric is defined by $\eta_h = -\partial_t \log Z_h$, we see that the flow equation (4.190) for the gEAA becomes:

$$\partial_t \bar{\Gamma}_k[g] = \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\Delta) - \eta_h R_k(\Delta)}{\mathbf{1}\left(\Delta + m_h^2 - 2\Lambda\right) + m_h^2 \frac{d}{d-2}\mathbf{P} + \mathbf{U}} - \mathrm{Tr}\frac{\partial_t R_k(\Delta) - \eta_C R_k(\Delta)}{\Delta\delta^{\mu\nu} - R^{\mu\nu}}\,. \tag{4.202}$$

Note that the wave-function renormalization factors in (4.202) have deleted each other leaving terms proportional to the anomalous dimension of the fluctuation metric and of the ghost fields. Note also that the possible troublesome conformal instability does not affect (4.202) due to our cutoff choice (4.200).

We now set $m_h = 0$. On a general background it is impossible to invert the operator $\mathbf{1}\left(\Delta - 2\Lambda\right) + \mathbf{U}$, so we expand it in powers of $\mathbf{U}$, remembering that this last is proportional to R, to find:

$$\left[\mathbf{1}\left(\Delta - 2\Lambda + R_k(\Delta)\right) + \mathbf{U}\right]^{-1} = G_k(\Delta)\left[\mathbf{1} - G_k(\Delta)\mathbf{U} + G_k^2(\Delta)\mathbf{U}^2 + O\left(R^3\right)\right]\,. \tag{4.203}$$

In (4.203) we defined the regularized graviton propagator as follows:

$$G_k(z) = \frac{1}{z + R_k(z) - 2\Lambda_k}\,. \tag{4.204}$$

We can do the same in the ghost sector, we find:

$$\left(\Delta\delta^{\mu\nu} - R^{\mu\nu}\right)^{-1} = G_{C,k}(\Delta)\left[g^{\mu\nu} - G_{C,k}(\Delta)R^{\mu\nu} + G_{C,k}^2(\Delta)\,R^\mu_\rho R^{\rho\nu} + O\left(R^3\right)\right]\,, \tag{4.205}$$

where the regularized ghost propagator is now:

$$G_{C,k}(z) = \frac{1}{z + R_k(z)} \,. \tag{4.206}$$

Inserting back in the flow equation (4.202) for the gEAA the expansions (4.203) and (4.205) gives, to second order in the curvatures, the following result:

$$\begin{aligned}
\partial_t \bar{\Gamma}_k[g] \;=\; & \frac{d(d+1)}{4}\mathrm{Tr}\left[(\partial_t R_k - \eta_h R_k) G_k\right] - d\,\mathrm{Tr}\left[(\partial_t R_k - \eta_C R_k) G_{C,k}\right] \\
& - \left\{ \frac{d(d-1)}{4}\mathrm{Tr}\left[(\partial_t R_k - \eta_h R_k) G_k^2\right] + \mathrm{Tr}\left[(\partial_t R_k - \eta_C R_k) G_{C,k}^2\right] \right\} R \\
& + \frac{d^3 - 5d^2 + 8d + 4}{4(d-2)}\mathrm{Tr}\left[(\partial_t R_k - \eta_h R_k) G_k^3\right] R^2 \\
& + \left\{ \frac{d^2 - 8d + 4}{2(d-2)}\mathrm{Tr}\left[(\partial_t R_k - \eta_h R_k) G_k^3\right] - \mathrm{Tr}\left[(\partial_t R_k - \eta_C R_k) G_{C,k}^3\right] \right\} R_{\mu\nu} R^{\mu\nu} \\
& + \frac{3}{2}\mathrm{Tr}\left[(\partial_t R_k - \eta_h R_k) G_k^3\right] R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + O\left(\mathcal{R}^3\right) \,.
\end{aligned} \tag{4.207}$$

In (4.207) we calculated the traces over spacetime indices using the traces in (4.199). The remaining functional traces in (4.207) can be calculated, as usual, using the local heat kernel expansion and the trace technology from Appendix A. Collecting all terms of order zero and one in the scalar curvature and writing everything in terms of Q-functionals finally gives the following expansion:

$$\begin{aligned}
\partial_t \bar{\Gamma}_k[g] \;=\; & \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{g} \left\{ \frac{d(d+1)}{4} Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_h R_k) G_k\right] - d Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_C R_k) G_{C,k}\right] \right. \\
& + \left[\frac{d(d+1)}{24} Q_{\frac{d}{2}-1}\left[(\partial_t R_k - \eta_h R_k) G_k\right] - \frac{d}{6} Q_{\frac{d}{2}-1}\left[(\partial_t R_k - \eta_C R_k) G_{C,k}\right] \right. \\
& \left.\left. - \frac{d(d-1)}{4} Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_h R_k) G_k^2\right] - Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_C R_k) G_{C,k}^2\right] \right] R \right\} \\
& + O\left(\mathcal{R}^2\right) \,.
\end{aligned} \tag{4.208}$$

By comparing (4.208) with (4.189) we find the following relations,

$$\begin{aligned}
\partial_t \left(\frac{\Lambda_k}{8\pi G_k} \right) \;=\; & \frac{1}{(4\pi)^{d/2}} \left\{ \frac{d(d+1)}{4} Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_h R_k) G_k\right] \right. \\
& \left. - d Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_C R_k) G_{C,k}\right] \right\}
\end{aligned}$$

$$\begin{aligned}\partial_t\left(\frac{1}{16\pi G_k}\right) &= \frac{1}{(4\pi)^{d/2}}\left\{\frac{d(d+1)}{24}Q_{\frac{d}{2}-1}\left[(\partial_t R_k-\eta_h R_k)G_k\right]-\frac{d}{6}Q_{\frac{d}{2}-1}\left[(\partial_t R_k-\eta_C R_k)G_{C,k}\right]\right.\\ &\left.-\frac{d(d-1)}{4}Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_h R_k)G_k^2\right]-Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_C R_k)G_{C,k}^2\right]\right\}, \quad (4.209)\end{aligned}$$

from which we can solve for the beta functions for the cosmological and Newton's constants. Inserting the optimized cutoff shape function into (4.209) permits the analytical evaluation of the Q-functionals and we find the following explicit system:

$$\begin{aligned}\partial_t\left(\frac{\Lambda}{8\pi G}\right) &= \frac{k^d}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}\left\{\frac{d(d+1)}{4}\frac{d+2-\eta_h}{1-2\tilde{\Lambda}}-d(d+2-\eta_C)\right\}\\ \partial_t\left(\frac{1}{16\pi G}\right) &= -\frac{k^{d-2}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}\left\{\frac{d(d+1)(d+2)}{48}\frac{d-\eta_h}{1-2\tilde{\Lambda}}-\frac{d(d+2)}{12}(d-\eta_C)\right.\\ &\left.-\frac{d(d-1)}{4}\frac{2+d-\eta_h}{(1-2\tilde{\Lambda})^2}-(d+2-\eta_C)\right\}. \quad (4.210)\end{aligned}$$

In the original calculation of the beta functions (4.209) done in [87] the background metric was chosen to be a metric on the d-dimensional sphere in order to be able to invert the operator $\mathbf{1}(\Delta-2\Lambda)+\mathbf{U}$. The above calculation, made instead by considering an arbitrary background metric, shows that the beta functions in (4.209) are independent of this last choice, as it should be for a background in depended RG flow [96].

We consider now a non-zero value for the Pauli-Fierz mass m_h. To calculate the contributions to the beta functions (4.209) given by m_h, we could follow the steps just made with minor modifications. It is interesting to do this by considering instead the background metric as being the one of a d-dimensional sphere as in the original derivation. On the sphere the Riemann and Ricci tensors are proportional to the Ricci scalar:

$$R_{\mu\nu}=\frac{R}{d}g_{\mu\nu} \qquad\qquad R_{\mu\nu\rho\sigma}=\frac{R}{d(d-1)}\left(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}\right). \quad (4.211)$$

Considering (4.211), the $\mathbf{U}$ tensor in (4.195) becomes simply:

$$\mathbf{U}=(\mathbf{1}-\mathbf{P})\frac{d^2-3d+4}{d(d-1)}R-\mathbf{P}\frac{d-4}{d}R. \quad (4.212)$$

Using the fact that now (4.212) is decomposed in the orthogonal basis of the trace and trace-free projectors, we can easily re-express the Hessian (4.196) in the following way:

$$\begin{aligned}\mathbf{\Gamma}_k^{(2,0,0;0)}[0,0,0;g] &= \frac{1}{2}Z_h\left[(\mathbf{1}-\mathbf{P})\left(\Delta+m_h^2-2\Lambda+\frac{d^2-3d+4}{d(d-1)}R\right)\right.\\&\quad\left.-\frac{d-2}{2}\mathbf{P}\left(\Delta+2\frac{d-1}{d-2}m_h^2-2\Lambda+\frac{d-4}{d}R\right)\right]. \qquad (4.213)\end{aligned}$$

It is easy now to write down explicitly the full regularized graviton propagator:

$$\begin{aligned}&\left[\mathbf{1}\left(\Delta+m_h^2-2\Lambda+R_k(\Delta)\right)+m_h^2\frac{d}{d-2}\mathbf{P}+\mathbf{U}\right]^{-1}=\\&=(\mathbf{1}-\mathbf{P})\frac{1}{\Delta+R_k(\Delta)+m_h^2-2\Lambda+\frac{d^2-3d+4}{d(d-1)}R}\\&\quad-\frac{2}{d-2}\mathbf{P}\frac{1}{\Delta+R_k(\Delta)+2\frac{d-1}{d-2}m_h^2-2\Lambda+\frac{d-4}{d}R}. \qquad (4.214)\end{aligned}$$

Equation (4.214) expresses the full regularized graviton propagator steaming from our truncation (4.14) and (4.17) in the gauge $\alpha=\beta=1$ when the background metric is a metric on the d-dimensional sphere. Note also, that there is a kinematical singularity in the regularized propagator (4.214) for $d=2$. We can now define the trace and trace-free parts of the regularized graviton propagator on the d-dimensional sphere as follows:

$$\begin{aligned}G_{TF,k}(z) &= \frac{1}{z+R_k(z)+m_h^2-2\Lambda_k+\frac{d^2-3d+4}{d(d-1)}R}\\G_{T,k}(z) &= \frac{1}{z+R_k(z)+2\frac{d-1}{d-2}m_h^2-2\Lambda_k+\frac{d-4}{d}R}. \qquad (4.215)\end{aligned}$$

Note that due to the presence of the Pauli-Fierz mass term, the trace and trace-free regularized propagators in (4.215) are different even at $R=0$. The ghost regularized propagator on the d-dimensional sphere becomes simply:

$$G_{C,k}=\frac{1}{z+R_k(z)-\frac{R}{d}}. \qquad (4.216)$$

To proceed, we insert in the graviton part of the flow equation (4.202) the identity in the space of symmetric rank two tensor in the form $\mathbf{1}=(\mathbf{1}-\mathbf{P})+\mathbf{P}$. This gives the following relation:

$$\begin{aligned}
\partial_t \bar{\Gamma}_k[g] &= \frac{1}{2}\mathrm{Tr}(\mathbf{1}-\mathbf{P})(\partial_t R_k - \eta_h R_k)G_{TF,k} + \frac{1}{2}\mathrm{Tr}\mathbf{P}(\partial_t R_k - \eta_h R_k)G_{T,k} \\
&\quad -\mathrm{Tr}\delta^\mu_\nu(\partial_t R_k - \eta_C R_k)G_{C,k} \\
&= \frac{d^2+d-2}{4}\mathrm{Tr}_x(\partial_t R_k - \eta_h R_k)G_{TF,k} + \frac{1}{2}\mathrm{Tr}_x(\partial_t R_k - \eta_h R_k)G_{T,k} \\
&\quad -d\,\mathrm{Tr}_x(\partial_t R_k - \eta_C R_k)G_{C,k}\,.
\end{aligned} \tag{4.217}$$

We evaluated the Lorentz traces in (4.217) with the help of (4.199). Note that both the kinematical singularity and the conformal instability are gone due to our choice for the cutoff kernel. Equation (4.217) is the flow equation for the gEAA induced by the Einstein-Hilbert truncation (4.14) with spherical background metric to all orders in the curvature scalar R.

Being the spectrum of the covariant Laplacian explicitly know on the d-dimensional sphere, we could evaluate the functional traces in (4.217) exactly by summing the relative series over the eigenvalues spectrum. Otherwise, we could use the Euler-Mclauren formula to obtain an asymptotic expansion for these traces. It's easy to verify that this last procedure gives the same expansion as if we were using the standard local heat kernel expansion to evaluate the traces in (4.217). In both cases, we recover, to order R, the expansion found earlier on a general background (4.208). This happens because to linear order in the curvature there is only one curvature invariant, i.e. R, and thus a spherical background is enough to unambiguously extract the beta functions for Λ and G. This is no more true for higher order invariants and the advantage of working on a symmetric backgrounds, as here is the sphere, is lost due to inability to disentangle the different tensor structures.

Collecting all terms of zeroth and first order in the scalar curvature that are present on the rhs of (4.217), steaming from the expansion of $G_{TF,k}$, $G_{T,k}$ and from the heat kernel expansion, we find the generalization of (4.209) to the case of non-zero m_h:

$$\begin{aligned}
\partial_t\left(\frac{\Lambda_k}{8\pi G_k}\right) &= \frac{1}{(4\pi)^{d/2}}\Bigg\{\frac{d^2+d-2}{4}Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_h R_k)G_{TF,k}\right] + \frac{1}{2}Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_h R_k)G_{T,k}\right] \\
&\quad -dQ_{\frac{d}{2}}\left[(\partial_t R_k - \eta_C R_k)G_{C,k}\right]\Bigg\}
\end{aligned}$$

$$\begin{aligned}\partial_t\left(\frac{1}{16\pi G_k}\right) &= \frac{1}{(4\pi)^{d/2}}\left\{\frac{d^2+d-2}{24}Q_{\frac{d}{2}-1}\left[(\partial_t R_k-\eta_h R_k)G_{TF,k}\right]\right.\\ &+\frac{1}{12}Q_{\frac{d}{2}-1}\left[(\partial_t R_k-\eta_h R_k)G_{T,k}\right]-\frac{d}{6}Q_{\frac{d}{2}-1}\left[(\partial_t R_k-\eta_C R_k)G_{C,k}\right]\\ &-\frac{d^2-3d+4}{d(d-1)}\frac{d^2+d-2}{4}Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_h R_k)G^2_{TF,k}\right]\\ &\left.-\frac{d-4}{2d}Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_h R_k)G^2_{T,k}\right]-Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_C R_k)G^2_{C,k}\right]\right\}\end{aligned} \tag{4.218}$$

We can evaluate the beta functions (4.218) using the optimized cutoff shape function. In terms of the dimensionless couplings $\tilde{\Lambda}=k^{-2}\Lambda$, $\tilde{G}=k^{d-2}G$ and $\tilde{m}_h^2=k^{-2}m_h$ we find the following forms:

$$\begin{aligned}\partial_t\tilde{\Lambda} &= -2\tilde{\Lambda}+\frac{8\pi\tilde{G}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left\{-4+\frac{d-1}{d}\frac{d+2-\eta_h}{1-2\tilde{\Lambda}+\tilde{m}_h^2}-\right.\\ &\frac{2\left(d^2-3d+4\right)\tilde{\Lambda}}{d^2}\frac{2+d-\eta_h}{\left(1-2\tilde{\Lambda}+\tilde{m}_h^2\right)^2}+\frac{2}{d(d+2)}\frac{2+d-\eta_h}{1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2}\\ &\left.-\frac{4\left(d-4\right)\tilde{\Lambda}}{d^2(d+2)}\frac{2+d-\eta_h}{\left(1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2\right)^2}-\frac{8\tilde{\Lambda}}{d(d+2)}\left(d+2-\eta_C\right)+\frac{4}{2+d}\eta_C\right\}\\ &+\frac{4\pi\tilde{G}\tilde{\Lambda}}{3(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left\{-\frac{d^2+d-2}{d}\frac{d-\eta_h}{1-2\tilde{\Lambda}-\tilde{m}_h^2}\right.\\ &\left.+\frac{2}{d}\frac{d-\eta_h}{1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2}-4\left(d-\eta_C\right)\right\}\end{aligned} \tag{4.219}$$

and

$$\begin{aligned}\partial_t\tilde{G} &= (d-2)\tilde{G}+\frac{16\pi}{(4\pi)^{d/2}d^2\Gamma\left(\frac{d}{2}\right)}\left\{-\frac{4d}{d+2}\left(d+2-\eta_C\right)\right.\\ &\left.-\left(d^2-3d+4\right)\frac{d+2-\eta_h}{(1-2\Lambda+m_h^2)^2}-\frac{2\left(d-4\right)}{d+2}\frac{d+2-\eta_h}{\left(1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2\right)^2}\right\}\tilde{G}^2\\ &+\frac{4\pi}{3(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left\{\frac{d^2+d-2}{d}\frac{d-\eta_h}{1-2\tilde{\Lambda}+m_h^2}\right.\\ &\left.+\frac{2}{d}\frac{d-\eta_h}{1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2}-4\left(d-\eta_C\right)\right\}\tilde{G}^2\,.\end{aligned} \tag{4.220}$$

These beta functions represent the generalization of the beta functions for the dimensionless cosmological and Newton's constant in presence of a non-zero Pauli-Fierz mass.

We analyze now the contributions from our truncation, i.e. from the Einstein-Hilbert operator $\int \sqrt{g}R$, to the terms (4.18) in the gEAA which are quadratic in the curvatures. These terms can be calculated either using the local heat kernel expansion, as we do in this section, or as the first terms in a non-local heat kernel expansion, as we do in the next section. We will see that these two procedures give equal results, modulo total derivatives, for the reasons explained in Appendix A. The terms of second order in the curvatures in equation (4.207) are, when evaluated explicitly using the local heat kernel expansion, the following:

$$\begin{aligned}\partial_t\bar{\Gamma}_k[g]\big|_{\mathcal{R}^2} &= \frac{1}{(4\pi)^{d/2}}\int d^dx\sqrt{g}\Big\{Q_{\frac{d}{2}-2}[\partial_t R_k G_k]\Big[-\frac{d(33-d)}{720}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \\ &\quad -\frac{d(d-3)}{720}R_{\mu\nu}R^{\mu\nu}+\frac{d(d-3)}{288}R^2+\frac{2d^2-d+10d}{60}\Delta R\Big] \\ &\quad +Q_{\frac{d}{2}-1}[\partial_t R_k G_k^2]\frac{d^2+d-4}{24}R^2+Q_{\frac{d}{2}}[\partial_t R_k G_k^3]\Big[\frac{3}{2}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \\ &\quad +\frac{d^2-10d+8}{2(d-2)}R_{\mu\nu}R^{\mu\nu}+\frac{d^3-5d^2+8d+4}{4(d-2)}R^2\Big]\Big\}\,. \end{aligned} \tag{4.221}$$

In (4.221) we set $\Lambda = m_h = \eta_h = \eta_C = 0$. Also, we used $\mathrm{Tr}_L\Omega_{\mu\nu}\Omega^{\mu\nu} = -R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ for vectors and $\mathrm{Tr}_L\Omega_{\mu\nu}\Omega^{\mu\nu} = -(d+2)R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ for symmetric tensors. All the Q-functionals in (4.221) are cutoff shape independent in $d=4$ and when evaluated give rise to the following form:

$$\partial_t\bar{\Gamma}_k[g]\big|_{\mathcal{R}^2} = \frac{1}{(4\pi)^2}\int d^dx\sqrt{g}\left(\frac{53}{45}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-\frac{361}{90}R_{\mu\nu}R^{\mu\nu}+\frac{43}{36}R^2+\frac{19}{15}\Delta R\right)\,. \tag{4.222}$$

To compare later with the result obtained with the non-local heat kernel expansion, we rewrite (4.223) in the $\{R^2, R_{\mu\nu}R^{\mu\nu}, E, \Delta R\}$ basis of curvature square invariants:

$$\partial_t\bar{\Gamma}_k[g]\big|_{\mathcal{R}^2} = \frac{1}{(4\pi)^2}\int d^dx\sqrt{g}\left(\frac{7}{10}R_{\mu\nu}R^{\mu\nu}+\frac{1}{60}R^2+\frac{53}{45}E+\frac{19}{15}\Delta R\right)\,. \tag{4.223}$$

We explain in the main part of the chapter how these actions are related to the perturbative UV divergences first calculated in [88, 96].

We now turn to consider type II cutoff where we take as cutoff operators $\mathbf{\Delta}_2 = \Delta\mathbf{1}+\mathbf{U}$ for the gravitons and $(\Delta_1)^\mu_\nu = \Delta\delta^\mu_\nu - R^\mu_\nu$ for the ghosts. The flow equation for the gEAA

(4.223), at $m_h = 0$, becomes now simply the following:

$$\partial_t \bar{\Gamma}_k[g] = \frac{1}{2} \mathrm{Tr}_{xL} \mathbf{G}_k(\boldsymbol{\Delta}_2) \partial_t \mathbf{R}_k(\boldsymbol{\Delta}_2) - \mathrm{Tr}_{xL} G_k(\Delta_1)^\mu_\nu \partial_t R_k(\Delta_1)^\nu_\mu . \tag{4.224}$$

It is now easy to evaluate the traces in (4.224) using the local heat kernel expansion. Using the following heat kernel coefficients for the cutoff operators we are considering

$$\mathrm{tr}\, b_2(\boldsymbol{\Delta}_2) = \mathrm{tr}\left[\mathbf{1}\frac{R}{6} - \mathbf{U}\right] = -\frac{d(5d-7)}{12} R \qquad \mathrm{tr}\, b_2(\Delta_1) = \mathrm{tr}\left[\delta^\mu_\nu \frac{R}{6} + R^\mu_\nu\right] = \frac{d+6}{d} R ,$$

we find, to linear order in the curvature, the following expansion:

$$\begin{aligned} \partial_t \bar{\Gamma}_k[g] \;&=\; \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{g} \left\{ \frac{d(d+1)}{4} Q_{\frac{d}{2}} \left[(\partial_t R_k - \eta_h R_k) G_k\right] \right. \\ & -d\, Q_{\frac{d}{2}} \left[(\partial_t R_k - \eta_C R_k) G_{C,k}\right] - \left[\frac{d(5d-7)}{24} Q_{\frac{d}{2}-1} \left[(\partial_t R_k - \eta_h R_k) G_k\right] \right. \\ & \left. \frac{d+6}{6} Q_{\frac{d}{2}-1} \left[(\partial_t R_k - \eta_C R_k) G_{C,k}\right] R \right\} + O\left(\mathcal{R}^2\right) . \end{aligned} \tag{4.225}$$

From (4.225) we can extract the following relations that determine the beta functions of Λ and G:

$$\begin{aligned} \partial_t \left(\frac{\Lambda_k}{8\pi G_k} \right) &= \frac{1}{(4\pi)^{d/2}} \left\{ \frac{d(d+1)}{4} Q_{\frac{d}{2}} \left[(\partial_t R_k - \eta_h R_k) G_k\right] \right. \\ & \qquad \left. -d\, Q_{\frac{d}{2}} \left[(\partial_t R_k - \eta_C R_k) G_{C,k}\right] \right\} \\ \partial_t \left(\frac{1}{16\pi G_k} \right) &= \frac{1}{(4\pi)^{d/2}} \left\{ \frac{d(5d-7)}{24} Q_{\frac{d}{2}-1} \left[(\partial_t R_k - \eta_h R_k) G_k\right] \right. \\ & \qquad \left. + \frac{d+6}{6} Q_{\frac{d}{2}-1} \left[(\partial_t R_k - \eta_C R_k) G_{C,k}\right] \right\} . \end{aligned} \tag{4.226}$$

Inserting in (4.226) the optimized cutoff function we find:

$$\begin{aligned} \partial_t \left(\frac{\Lambda}{8\pi G} \right) &= \frac{k^d}{(4\pi)^{d/2} \Gamma\left(\frac{d}{2}+2\right)} \left\{ \frac{d(d+1)}{4} \frac{d+2-\eta_h}{1-2\tilde{\Lambda}} - d(d+2-\eta_C) \right\} \\ \partial_t \left(\frac{1}{16\pi G} \right) &= \frac{k^{d-2}}{(4\pi)^{d/2} \Gamma\left(\frac{d}{2}+1\right)} \left\{ \frac{d(5d-7)}{24} \frac{d-\eta_h}{1-2\tilde{\Lambda}} + \frac{d+6}{6} (d-\eta_C) \right\} . \end{aligned} \tag{4.227}$$

As we did before, we can look at the running, induced by the Einstein-Hilbert operator $\int \sqrt{g} R$, of the curvature square terms in (4.18) for $\Lambda = m_h = \eta_h = \eta_C = 0$. This time it is much easier, we find:

$$\partial_t \bar{\Gamma}_k[g]\big|_{\mathcal{R}^2} = \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{g}\, Q_{\frac{d}{2}-2} \left[(\partial_t R_k - \eta_h R_k) G_k \right] \left\{ \frac{1}{2} \mathrm{tr}\, b_4(\mathbf{\Delta}_2) - \mathrm{tr}\, b_4(\Delta_1) \right\} . \quad (4.228)$$

Using the explicit expressions for the heat kernel coefficients, equation (A.7) from Appendix A, we find:

$$\begin{aligned} \mathrm{tr}\, b_4(\mathbf{\Delta}_2) \;\; = \;\; & \frac{d^2 - 29d + 480}{360} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - \frac{d^3 - 181d^2 + 1438d - 720}{360(d-2)} R_{\mu\nu} R^{\mu\nu} \\ & + \frac{25d^3 - 145d^2 + 262d + 144}{144(d-2)} R^2 + \frac{d(2d-3)}{30} \Delta R \end{aligned} \quad (4.229)$$

and

$$\mathrm{tr}\, b_4(\Delta_1) = \frac{d-15}{180} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \frac{90-d}{180} R_{\mu\nu} R^{\mu\nu} + \frac{d+12}{72} R^2 - \frac{d+5}{30} \Delta R \, . \quad (4.230)$$

In $d = 4$ the Q-functional in equation (4.228) is cutoff shape independent. Inserting (4.229) and (4.230) in (4.228) gives back equation (4.223). This shows that both cutoff types reproduce correctly the contributions to the flow of the curvature square terms. This is analogous to what we found for non-abelian gauge theories in Chapter 3.

It is possible to use the general momentum space technique introduced in section 3.3.4 of Chapter 3 to relax the choice $\alpha = \beta = 1$ for the gauge-fixing parameters but we will not do this here. This as been done for $m_h = 0$ by employing a slight modification of type I cutoff in [94, 95]. This amounts to use the general decomposition of the metric tensor introduced in section 4.5.2 to diagonalize the Hessian's entering the flow equation (4.190) for the gEAA. For $m_h \neq 0$ this will be considered in [112].

4.5.5 Derivation of $\partial_t F_{i,k}(x)$

We now derive the flow equations for the structure functions $F_{i,k}(x)$ in (4.43) induced by the operator $\int \sqrt{g} R$. We will use a the type II cutoff operator and work with arbitrary cutoff shape function. Also, we will consider the flow of the structure functions in arbitrary dimension.

In place of using the local heat kernel expansion to calculate the $O(\mathcal{R}^2)$ contributions to the flow (4.190), as we did in the previous section in equation (4.228), we employ instead the non-local heat kernel expansion described in Appendix A. We find:

$$\begin{aligned}\partial_t \bar{\Gamma}_k[g]\big|_{\mathcal{R}^2} &= \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{g} \left\{ R \left[\int_0^\infty ds\, \tilde{h}(s)\, s^{2-\frac{d}{2}}\, f_1(s\Delta) \right] R \right. \\ &\left. + R_{\mu\nu} \left[\int_0^\infty ds\, \tilde{h}(s)\, s^{2-\frac{d}{2}}\, f_2(s\Delta) \right] R^{\mu\nu} \right\} . \qquad (4.231)\end{aligned}$$

Here we defined the function $h_k(z) = \partial_t R_k(z) G_k(z)$, of which $\tilde{h}_k(s)$ is the inverse-Laplace transform. The functions $f_1(x)$ and $f_2(x)$ are derived combining the non-local heat kernel functions for the operators $\mathbf{\Delta}_2$, Δ_1 and expressing them in the basis $\{R^2, R_{\mu\nu}R^{\mu\nu}, E, \Delta R\}$, they read:

$$\begin{aligned} f_1(x) &= \frac{9d^3 - 61d^2 - 10d + 320}{128(d-2)} f(x) - \frac{3d^2 + 7d + 16}{32x} f(x) \\ &+ \frac{17d^2 + 45d + 96}{192x} - \frac{d(d-3)}{32x^2} \left[f(x) - 1 \right] \\ f_2(x) &= \frac{d^2 + 2d - 16}{4(d-2)} f(x) + \frac{d}{x} f(x) - \frac{(27-d)d}{24x} + \frac{d(d-3)}{4x^2} \left[f(x) - 1 \right] . \qquad (4.232)\end{aligned}$$

From (4.231) we can extract the running of the structure functions

$$\partial_t F_{i,k}(x) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty ds\, \tilde{h}_k(s)\, s^{2-\frac{d}{2}}\, f_i(s\Delta) \,, \qquad (4.233)$$

for $i = 1, 2$. For each i this can be rewritten in terms of a combination of Q-functionals inside parameter integrals. Inserting (4.232) in (4.233) and using the definitions of the Q-functionals from Appendix A, we find the following forms:

$$\begin{aligned} (4\pi)^{d/2}\, \partial_t F_{1,k}(x) &= \frac{9d^3 - 61d^2 - 10d + 320}{128(d-2)} \int_0^1 d\xi\, Q_{\frac{d}{2}-2} \left[h_k \left(z + x\xi(1-\xi) \right) \right] \\ &- \frac{3d^2 + 7d + 16}{32x} \int_0^1 d\xi\, Q_{\frac{d}{2}-1} \left[h_k \left(z + x\xi(1-\xi) \right) \right] \\ &+ \frac{17d^2 + 45d + 96}{192x} Q_{\frac{d}{2}-1}[h_k] \\ &- \frac{d(d-3)}{32x^2} \left\{ \int_0^1 d\xi\, Q_{\frac{d}{2}} \left[h_k \left(z + x\xi(1-\xi) \right) \right] - Q_{\frac{d}{2}}[h_k] \right\} \qquad (4.234)\end{aligned}$$

and

$$
\begin{aligned}
(4\pi)^{d/2}\,\partial_t F_{2,k}(x) &= \frac{d^2+2d-16}{4(d-2)}\int_0^1 d\xi\, Q_{\frac{d}{2}-2}\left[h_k\left(z+x\xi(1-\xi)\right)\right] \\
&\quad +\frac{d}{x}\int_0^1 d\xi\, Q_{\frac{d}{2}-1}\left[h_k\left(z+x\xi(1-\xi)\right)\right]-\frac{(27-d)d}{24x}Q_{\frac{d}{2}-1}[h_k] \\
&\quad +\frac{d(d-3)}{4x^2}\left\{\int_0^1 d\xi\, Q_{\frac{d}{2}}\left[h_k\left(z+x\xi(1-\xi)\right)\right]-Q_{\frac{d}{2}}[h_k]\right\}\,, \qquad (4.235)
\end{aligned}
$$

where x stands for Δ. Equations (4.234) and (4.235) are the contributions induced by the operator $\int\sqrt{g}R$, within a type II cutoff, to the flow of the curvature square structure functions in truncation (4.43). They are valid in arbitrary dimension and for general cutoff shape function. These are the main results of this section.

We first study the flow equations (4.234) and (4.235) in the physical dimension $d=4$. They read:

$$
\begin{aligned}
(4\pi)^2\,\partial_t F_{1,k}(x) &= -\frac{15}{32}\int_0^1 d\xi\, Q_0\left[h_k\left(z+x\xi(1-\xi)\right)\right]-\frac{23}{8x}\int_0^1 d\xi\, Q_1\left[h_k\left(z+x\xi(1-\xi)\right)\right]+ \\
&\quad +\frac{137}{48x}Q_1[h_k]-\frac{1}{8x^2}\left\{\int_0^1 d\xi\, Q_2\left[h_k\left(z+x\xi(1-\xi)\right)\right]-Q_2[h_k]\right\} \qquad (4.236)
\end{aligned}
$$

and

$$
\begin{aligned}
(4\pi)^2\,\partial_t F_{2,k}(x) &= \int_0^1 d\xi\, Q_0\left[h_k\left(z+x\xi(1-\xi)\right)\right]+\frac{4}{x}\int_0^1 d\xi\, Q_1\left[h_k\left(z+x\xi(1-\xi)\right)\right]+ \\
&\quad -\frac{23}{6x}Q_1[h_k]+\frac{1}{x^2}\left\{\int_0^1 d\xi\, Q_2\left[h_k\left(z+x\xi(1-\xi)\right)\right]-Q_2[h_k]\right\}\,. \qquad (4.237)
\end{aligned}
$$

Inserting the optimized cutoff shape function in the equations (4.236) and (4.237), using the Q-functionals (A.59), (A.63) and (A.64) from Appendix A, we find equations (4.44), (4.45) and (4.46).

In the two dimensional case the Ricci tensor is proportional to the Ricci scalar $R_{\mu\nu}=\frac{1}{2}g_{\mu\nu}R$. To order curvature square we have thus to consider only the structure function given by the following linear combination:

$$
F_k(x)=F_{1,k}(x)+\frac{1}{2}F_{2,k}(x)\,. \qquad (4.238)
$$

The running of the two dimensional structure function (4.238) can be deduced from and the flow equations (4.234) and (4.235) evaluated at $d=2$. Note that the poles in the first terms of (4.234) and (4.235) cancel each other. We find the following form:

$$\begin{aligned} 4\pi\,\partial_t F_k(x) &= \frac{11}{64}\int_0^1 d\xi\, Q_{-1}\left[h_k\left(z+x\xi(1-\xi)\right)\right] + \frac{15}{16x}\int_0^1 d\xi\, Q_0\left[h_k\left(z+x\xi(1-\xi)\right)\right] \\ &\quad -\frac{27}{32x}Q_0[h_k] + \frac{9}{16x^2}\left\{\int_0^1 d\xi\, Q_1\left[h_k\left(z+x\xi(1-\xi)\right)\right] - Q_1[h_k]\right\} \\ &\quad -\frac{9}{16}\int_0^1 d\xi\, Q_{-1}\left[h_k\left(z+x\xi(1-\xi)\right)\right] - \frac{5}{4x}\int_0^1 d\xi\, Q_0\left[h_k\left(z+x\xi(1-\xi)\right)\right] \\ &\quad +\frac{9}{8x}Q_0[h_k] - \frac{3}{4x^2}\left\{\int_0^1 d\xi\, Q_1\left[h_k\left(z+x\xi(1-\xi)\right)\right] - Q_1[h_k]\right\}. \end{aligned} \tag{4.239}$$

In (4.239) we have separated the graviton contributions, in the first two lines, from the ghost contributions, in the second two lines.

4.5.6 Derivation of $\partial_t Z_{h,k}$ and $\partial_t m_{h,k}$

In this section we calculate the beta functions of the fluctuation metric Pauli-Fierz mass m_h and of the fluctuation metric wave-function renormalization Z_h. We will extract the beta functions $\partial_t m_h^2$ and $\partial_t Z_h$ form the flow equation for the zero-field proper-vertex $\gamma_k^{(2,0,0;0)}$ of the bEAA. The derivations of this section are similar to those made in section 3.5.6.

After the multiplet decomposition, and within the truncation we are considering in this chapter, the flow equation becomes as in Fig. 4.9. In formulas we have:

$$\begin{aligned} [\partial_t\gamma^{(2)}_{p,-p}]^{\mu\nu\,\alpha\beta} &= \kappa^2 Z_h\int_q(\partial_t R_q-\eta_h R_q)\,[a_{p,q}]^{\mu\nu\,\alpha\beta} - \frac{1}{2}\kappa^2 Z_h\int_q(\partial_t R_q-\eta_h R_q)\,[b_{p,q}]^{\mu\nu\,\alpha\beta} \\ &\quad -2\kappa^2 Z_h\int_q(\partial_t R_q-\eta_C R_q)\,[c_{p,q}]^{\mu\nu\,\alpha\beta} \\ &\quad -2\kappa^2 Z_h\int_q(\partial_t R_q-\eta_C R_q)\,[d_{p,q}]^{\mu\nu\,\alpha\beta}. \end{aligned} \tag{4.240}$$

Every diagram in Figure 4.9 is proportional to $\kappa^2 Z_h$ since the metric fluctuation three-vertex come with a power $\kappa Z_h^{3/2}$, the four-vertex with a power $\kappa^2 Z_h^2$ while the regularized graviton propagators with a factor Z_h^{-1} and graviton cutoff insertion with a factor Z_h. In the ghost diagrams the three-vertex has a power $\kappa Z_h^{1/2} Z_C$, the four-vertex a power $\kappa Z_h Z_C$, the regularized ghost propagators has a power Z_C^{-1} and the ghost cutoff insertion has power Z_C. Also all the volume factors Ω delete each other. The tensor products entering (4.240) are:

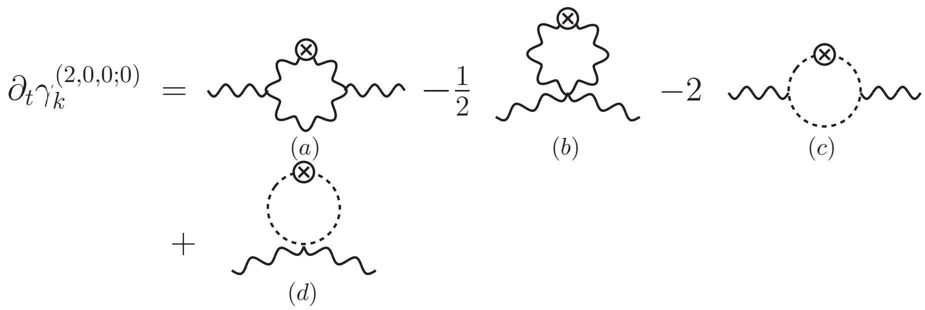


Figure 4.9: Diagrammatic representation of the flow equation for the zero-field proper-vertex $\gamma_k^{(2,0,0;0)}$ of the bEAA used to calculate the beta functions $\partial_t Z_{h,k}$ and $\partial_t m^2_{h,k}$.

$$
\begin{aligned}
[a_{p,q}]^{MN} &= [G_q]^{AB}[\gamma^{(3,0,0;0)}_{q,p,-q-p}]^{BMC}[G_{q+p}]^{CD}[\gamma^{(3,0,0;0)}_{q+p,-p,-q}]^{DNE}[G_q]^{EF}[R]^{FA} && (4.241)\\
[b_{p,q}]^{MN} &= [G_q]^{AB}[\gamma^{(4,0,0;0)}_{q,p,-q-p}]^{BMNC}[G_q]^{CD}[R]^{DA} && (4.242)\\
[c_{p,q}]^{MN} &= [G^{gh}_q]^{\alpha\beta}[\gamma^{(1,1,1;0)}_{q,p,-q-p}]^{\beta M\gamma}[G^{gh}_q]^{\gamma}[\gamma^{(1,1,1;0)}_{q+p,-p,-q}]^{\gamma N\delta}[G^{gh}_q]^{\delta\alpha}\,, && (4.243)\\
[d_{p,q}]^{MN} &= [G^{gh}_q]^{\alpha\beta}[\gamma^{(2,1,1;0)}_{q,p,-q-p}]^{\beta M\gamma}[G^{gh}_q]^{\gamma\alpha} && (4.244)
\end{aligned}
$$

where the vertices entering (4.241) are:

$$
\begin{aligned}
\gamma^{(3,0,0;0)}_{p_1,p_2,p_3} &= 2\Lambda I_0^{(3)}[\delta]_{p_1,p_2,p_3} - I_1^{(3)}[\delta]_{p_1,p_2,p_3}\\
\gamma^{(4,0,0;0)}_{p_1,p_2,p_3,p_4} &= 2\Lambda I_0^{(3)}[\delta]_{p_1,p_2,p_3} - I_1^{(3)}[\delta]_{p_1,p_2,p_3}\\
\gamma^{(1,1,1;0)}_{p_1,p_2,p_3} &= S^{(1,1,1;0)}_{gh}[0,0,0;\delta]\\
\gamma^{(2,1,1;0)}_{p_1,p_2,p_3} &= S^{(2,1,1;0)}_{gh}[0,0,0;\delta]\,. && (4.245)
\end{aligned}
$$

The vertices entering (4.245) can be deduced from the relations of section 4.5.1. The momentum integrals in (4.240) can be written in spherical coordinates:

$$
\int_q \to \frac{S_{d-1}}{(2\pi)^d}\int_0^\infty dq\, q^{d-1}\int_{-1}^{1} dx\,\left(1-x^2\right)^{\frac{d-3}{2}}\,, \tag{4.246}
$$

where $S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ is the volume of the d-dimensional sphere and $x = \cos\theta$ with θ the angle between p and q. We can also shift to the variable $z = q^2$ so that

$$
\int_0^\infty dq\, q^{d-1} \to \frac{1}{2}\int_0^\infty dz\, z^{\frac{d}{2}-1}\,.
$$

We will give the results only in the gauge $\alpha = \beta = 1$ and in $d = 4$. The first term in equation

(4.240), when projected with $\mathbf{P}_2$, is:

$$
\begin{aligned}
P_2^{\mu\nu\,\alpha\beta}[a_{p,q}]_{\mu\nu\,\alpha\beta} \;=\; & \frac{1}{24}(1-x^2)^2 q^4 \left(G_q^T\right)^2 G_{q+p}^T \\
& -\frac{1}{48}\left[15p^4 + 210p^3qx + 10pq^3x\left(17+4x^2\right)\right. \\
& \left.+5p^2q^2\left(52+101x^2\right) + q^4\left(7+6x^2+2x^4\right)\right]\left(G_q^{TF}\right)^2 G_{q+p}^T \\
& +\frac{1}{48}\left[375p^4 + 750p^3qx - 10pq^3x\left(13+2x^2\right)\right. \\
& \left.+5p^2q^2\left(22+23x^2\right) + q^4\left(7+6x^2+2x^4\right)\right]\left(G_q^T\right)^2 G_{q+p}^{TF} \\
& +\frac{1}{8}\left[225p^4 + 450p^3qx + 150pq^3x\left(2+x^2\right)\right. \\
& +15p^2q^2\left(11+34x^2\right) + 3q^4\left(34+32x^2+9x^4\right) \\
& -80\left(3p^2+3pqx+q^2\left(1+2x^2\right)\right)\Lambda \\
& \left.+160\Lambda^2\right]\left(G_q^{TF}\right)^2 G_{q+p}^{TF}\,.
\end{aligned}
\tag{4.247}
$$

From diagram (b) we have:

$$
\begin{aligned}
P_2^{\mu\nu\,\alpha\beta}[b_{p,q}]_{\mu\nu\,\alpha\beta} \;=\; & \frac{5}{12}\left[21p^2 + 19q^2 + 5q^2x^2\right]\left(G_q^T\right)^2 + \\
& +\frac{5}{12}\left[-21p^2 - 55q^2 - 17q^2x^2 - 48\Lambda\right]\left(G_q^{TF}\right)^2 ,
\end{aligned}
\tag{4.248}
$$

while for the ghost diagram (c) we find:

$$
P_2^{\mu\nu\,\alpha\beta}[c_{p,q}]_{\mu\nu\,\alpha\beta} = \frac{5}{3}p^2\left[3p^2 + 6pqx + q^2\left(1+2x^2\right)\right]\left(G_q^C\right)^2 G_{q+p}^C\,. \tag{4.249}
$$

Note that diagram (c) is already proportional to p^2. Thus it does not give any contribution to the running of the Pauli-Fierz mass. Finally for diagram (d) we have:

$$
P_2^{\mu\nu\,\alpha\beta}[d_{p,q}]_{\mu\nu\,\alpha\beta} = \frac{5}{2}pqx\left(G_q^C\right)^2\,.
$$

Being linear in x it gives zero when integrate over the angular integral. Diagram (d) can thus be omitted.

Once we insert equations (4.247), (4.248) and (4.249) back in (4.240) we obtain within our truncation, the explicit flow of the zero-field proper-vertex $\gamma_{p,-p}^{(2,0,0;0)}$, to all orders in the external momenta p. Here we will be interested only in the lower orders: from the p^0 term we extract the beta function $\partial_t\left(Z_h m_h^2\right)$ while from the p^2 term we will extract the beta function

$\partial_t Z_h$.

We start to calculate the running of the fluctuation metric Pauli-Fierz mass. All diagrams in Figure 4.9 give a non-zero contribution to order p^0. Define

$$\partial_t \left(Z_h m_h^2\right) = \beta_{Z_h m_h^2} \left(\kappa, \Lambda, Z_h, Z_C, m_h\right) , \tag{4.250}$$

so that we have:

$$\partial_t m_h^2 = \eta_h \, m_h^2 + \beta_{Z_h m_h^2}/Z_h \, , \tag{4.251}$$

where the anomalous dimension of the fluctuation field is given by $\eta_h = -\partial_t \log Z_h$. After expressing everything in terms of Q-functionals we finally find:

$$\begin{aligned}
\frac{\beta_{Z_h m_h^2}}{Z_h} \;=\; \frac{\kappa^2}{2(4\pi)^{d/2}} \Bigg\{ & \frac{320\Lambda^2}{(d-2)\,(d+1)} Q_{\frac{d}{2}}[(\partial_t R_k - \eta_h R_k) G_{TF}^3] \\
& - \frac{160\,(d+2)\,\Lambda}{d\,(d-2)\,(d+1)} Q_{\frac{d}{2}+1}[(\partial_t R_k - \eta_h R_k) G_{TF}^3] \\
& + \frac{(d-1)\,(d+2)}{6\,(d^2-4)} Q_{\frac{d}{2}+2}[(\partial_t R_k - \eta_h R_k) G_T^3] \\
& + \frac{3}{2}\frac{34d^2+100d+91}{(d-2)\,(d+1)} Q_{\frac{d}{2}+2}[(\partial_t R_k - \eta_h R_k) G_{TF}^3] \\
& - \frac{7d^2+20d+18}{12\,(d-2)\,(d+1)} Q_{\frac{d}{2}+2}[(\partial_t R_k - \eta_h R_k)\left(G_T^2 G_{TF} + G_T G_{TF}^2\right)] \\
& + \frac{d^4-7d^3+30d^2+34d-108}{d^2} Q_{\frac{d}{2}+1}[(\partial_t R_k - \eta_h R_k) G_{TF}^2] \\
& - \frac{(d^3-7d^2-10d+24)\Lambda}{d} Q_{\frac{d}{2}}[(\partial_t R_k - \eta_h R_k) G_{TF}^2] \\
& - 4\frac{(13-d)\,(d-4)\,d+54}{d^2\,(d-2)} Q_{\frac{d}{2}+1}[(\partial_t R_k - \eta_h R_k) G_T^2] \\
& + 2\frac{(6-d)\,(d-4)\,\Lambda}{d\,(d-2)} Q_{\frac{d}{2}}[(\partial_t R_k - \eta_h R_k) G_T^2] \Bigg\} \, .
\end{aligned} \tag{4.252}$$

Equation (4.252) together with equation (4.251) gives the beta function for the Pauli-Fierz mass in general dimension and for arbitrary cutoff shape function in the gauge $\alpha = \beta = 1$. The Q-functional in equation (4.252) can be evaluated analytically if we employ the optimized cutoff shape function.

From the terms proportional to p^2 in (4.240) we can extract the beta function of the wave-function renormalization of the fluctuation metric. If we define

$$\partial_t Z_h \equiv \beta_{Z_h}\left(\kappa, \Lambda, Z_h, Z_C, m_h\right) \,, \tag{4.253}$$

we can write the anomalous dimension of the fluctuation metric as:

$$\eta_h\left(\kappa, Z_h, Z_C, m_h\right) = -\partial_t \log Z_h = -\beta_{Z_h}/Z_h \,. \tag{4.254}$$

When we write everything in terms of Q-functionals we finally find:

$$\begin{aligned}
\eta_h \;=\; & \frac{\kappa^2}{2(4\pi)^{d/2}}\Bigg\{-\frac{480\Lambda}{(d-2)(d+1)}Q_{\frac{d}{2}}\left[\left(\partial_t R-\eta_h R\right)G_{TF}^3\right] \\
&+\frac{320\Lambda^2}{(d-2)(d+1)}Q_{\frac{d}{2}}\left[\left(\partial_t R-\eta_h R\right)G_{TF}^2 G'_{TF}\right] \\
&-\frac{5\left(22d+23\right)}{6\left(d-2\right)\left(d+1\right)}Q_{\frac{d}{2}+1}\left[\left(\partial_t R-\eta_h R\right)G_T^2 G_{TF}\right] \\
&-\frac{5\left(52d+101\right)}{6\left(d-2\right)\left(d+1\right)}Q_{\frac{d}{2}+1}\left[\left(\partial_t R-\eta_h R\right)G_{TF}^2 G_T\right] \\
&+\frac{15\left(11d+34\right)}{(d-2)(d+1)}Q_{\frac{d}{2}+1}\left[\left(\partial_t R-\eta_h R\right)G_{TF}^3\right] \\
&-\frac{80\left(d+8\right)\Lambda}{(d-2)(d+1)}Q_{\frac{d}{2}+1}\left[\left(\partial_t R-\eta_h R\right)G_{TF}^2 G'_{TF}\right] \\
&+\frac{320\Lambda^2}{(d-2)(d+1)}Q_{\frac{d}{2}+1}\left[\left(\partial_t R-\eta_h R\right)G_{TF}^2 G''_{TF}\right] \\
&+\frac{d-1}{6\left(d-2\right)}Q_{\frac{d}{2}+2}\left[\left(\partial_t R-\eta_h R\right)G_T^2 G'_T\right] \\
&-\frac{7d^2+360d+938}{12\left(d-2\right)\left(d+1\right)}Q_{\frac{d}{2}+2}\left[\left(\partial_t R-\eta_h R\right)G_{TF}^2 G'_T\right] \\
&-\frac{7d^2-240d-622}{12\left(d-2\right)\left(d+1\right)}Q_{\frac{d}{2}+2}\left[\left(\partial_t R-\eta_h R\right)G_T^2 G'_{TF}\right] \\
&-\frac{80\left(d+8\right)\Lambda}{(d-2)(d+1)}Q_{\frac{d}{2}+2}\left[\left(\partial_t R-\eta_h R\right)G_{TF}^2 G''_{TF}\right] \\
&+\frac{3}{2}\frac{34d^2+300d+791}{(d-2)(d+1)}Q_{\frac{d}{2}+2}\left[\left(\partial_t R-\eta_h R\right)G_{TF}^2 G'_{TF}\right] \\
&+\frac{d-1}{6\left(d-2\right)}Q_{\frac{d}{2}+3}\left[\left(\partial_t R-\eta_h R\right)G_T^2 G''_T\right] \\
&-\frac{7d^2+60d+158}{12\left(d-2\right)\left(d+1\right)}Q_{\frac{d}{2}+3}\left[\left(\partial_t R-\eta_h R\right)\left(G_{TF}^2 G''_T+G_T^2 G''_{TF}\right)\right]
\end{aligned}$$

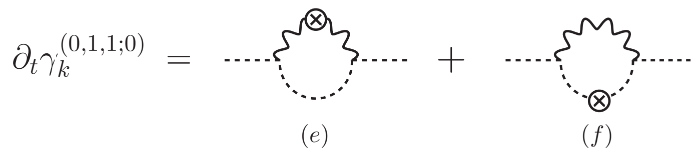


Figure 4.10: Graphical representation of the flow equation for the ghost-ghost zero-field proper vertex $\gamma_k^{(0,1,1;0)}$ of the bEAA used to calculate $\partial_t Z_h$.

$$
\begin{aligned}
&-\frac{3}{2}\frac{34d^2+300d+791}{(d-2)(d+1)}Q_{\frac{d}{2}+3}\left[(\partial_t R-\eta_h R)\,G_{TF}^2 G_{TF}''\right]\\
&-\frac{36-2d}{d}Q_{\frac{d}{2}}\left[(\partial_t R-\eta_h R)\,G_T^2\right]\\
&+\frac{d^3-5d^2+2d+36}{d}Q_{\frac{d}{2}}\left[(\partial_t R-\eta_h R)\,G_{TF}^2\right]\\
&-\frac{16}{d}Q_{\frac{d}{2}+1}\left[(\partial_t R-\eta_C R)\,G_C^2\right]\Bigg\}\,. \qquad (4.255)
\end{aligned}
$$

Equation (4.255) gives the anomalous dimension of the fluctuation metric in d-dimensions, for an arbitrary cutoff shape function in the gauge $\alpha=\beta=1$. It is possible to calculate the Q-functionals in (4.255) analytically if we employ the optimized cutoff shape function. For $m_h=0$ the result is given in equation (4.28).

This completes the derivation of the beta functions of the coupling we are considering that can be extracted from the flow of the zero-field proper-vertex $\gamma_k^{(2,0,0;0)}$. Equations (4.252) and (4.255) are the main results of this section.

4.5.7 Derivation of $\partial_t Z_{C,k}$

This derivation is analogous to the one made in section 3.5.7 for non-abelian gauge theories. The truncation ansatz is quadratic in the ghost fields and so the flow equation for the zero-field proper-vertex $\gamma_k^{(0,1,1;0)}$ is decomposed as in Figure 4.10. The diagrams in Figure 4.10 involve only one type of vertex: the ghost-ghost-graviton vertex. Note that the cutoff kernel has a different structure in the two sectors and involves different wave-function renormalization. Thus the two diagrams in Figure 4.10 give different contributions.

As usual we evaluate the tensor contractions, we Taylor expand in powers of the external momentum, we retain the term of order p^2 and we do the angular integrals. In terms of Q-functionals we find the following general expression for the anomalous dimension of the ghost fields:

$$\begin{aligned}
\eta_C &= \frac{\kappa^2}{(4\pi)^{d/2}}\Bigg\{-\frac{2d-3}{d^2\,(d-2)}Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)G_C G_T^2\right] \\
&+\frac{3(d^2+d-2)}{4d^2}Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)G_C G_{TF}^2\right]-\frac{d+2}{d^2\,(d-2)}Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_h R_k)G_T^2 G_C'\right] \\
&+\frac{(d-1)\,(d+2)^2}{4d^2}Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_h R_k)G_{TF}^2 G_C'\right] \\
&-\frac{d+1}{d^2\,(d-2)}Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_C R_k)G_C^2 G_T\right]+\frac{d^2+d-2}{2d}Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_C R_k)G_C^2 G_{TF}\right] \\
&-\frac{d+2}{d^2\,(d-2)}Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_C R_k)G_C^2 G_T'\right] \\
&+\frac{(d-1)\,(d+2)^2}{4d^2}Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_C R_k)G_C^2 G_{TF}'\right]\Bigg\}\,. \qquad (4.256)
\end{aligned}$$

When we set $m_h = 0$ in 4.256 we find instead:

$$\begin{aligned}
\eta_C &= \frac{\kappa^2}{(4\pi)^{d/2}}\Bigg\{-\frac{3d^2-3d-4}{4d\,(d-2)}Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)G^2 G_C\right] \\
&-\frac{(d+2)\,(d^2-d-4)}{4d\,(d-2)}Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_h R_k)G^2 G_C'\right] \\
&+\frac{d^2-d-8}{4d\,(d-2)}Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_C R_k)G_C^2 G\right] \\
&+\frac{(d+2)\,(d^2-d-4)}{4d\,(d-2)}Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_C R_k)G_C^2 G'\right]\Bigg\}\,. \qquad (4.257)
\end{aligned}$$

A similar form for the anomalous dimension of the ghost fields has been found, using slightly different implementations of the cutoff, in [106, 108]. Equations (4.256) and (4.257) are the main results of this section.

Chapter 5

Conclusions

In this thesis developed the functional RG approach to quantum field theory based of the effective average action (EAA) and on the exact flow equations that it satisfies. Precisely the fact that the flow of the EAA is exact offers the opportunity to develop a new approach quantum field theory, and more generally to any theory where fluctuations are relevant. In particular this functional reformulation does not have any reference to a bare action and is focused on the properties of theory space.

This point of view has two main virtues. First, we can use the formalism to search for new continuum limits, i.e. attractive UV fixed points of the RG flow, on which we can construct a proper mathematical definition of the functional integral. Second, even if a Gaussian saddle point expansion is not available, we can use the exact flow equation, combined with some kind of truncation of the EAA, to find new approximations to the effective action. These can contain valuable non-perturbative information.

About the first point, in this thesis we have analyzed both non-abelian gauge theories and quantum gravity. For the first we have recovered asymptotic freedom and the universal properties of the UV flow, as expected. In the second case we have extended the treatments of local truncations and in this way we have given new support to the asymptotic safety scenario in quantum gravity.

To the second point we started to study truncation schemes able to put the exactness of the flow to profit. In particular we proposed, for both gauge and gravitational theories, a new expansion scheme that we called "curvature expansion". This consists in expansion the EAA in powers of the curvatures or field strength where we retain all the momentum structure in the form of running structure functions. Crucial to the practical use of this expansion is the development of new calculation tools to manage functional traces. In this thesis we developed such scheme in the form of flow equations describing the running of the proper-vertices of the EAA. When the background field method is involved, as in the EAA formulation of gauge and gravitational theories, these equations have non-trivial contributions steaming from the fact that the cutoff action is constructed using the background fields. This terms are responsible to maintain gauge covariance in the flow of the gauge invariant part of the EAA. In this way the the standard heat kernel techniques are emended and truncations schemes, as the above mentioned curvature expansion, can be put to profit. In the thesis we construct several examples, that have to be intended as first applications, and we show how results previously obtained by heat kernel techniques are recovered and easily generalized. In particular we have at our disposal a computation algorithm able to explicitly project the flow of any truncation of the EAA which is analytic in the fields. It is admissible to think that with some developments these calculations can become straightforward. We have to mention, that in theories tensorially complicated as is gravity, the aid of symbolic manipulation software becomes, if not indispensable, strictly recommended.

This techniques can be employed also, as we did, to project the flow of truncation of the full bEAA, i.e. bi-gauge field truncations in non-abelian gauge theories and bi-metric truncations in quantum gravity. As we said, in any RG treatment of theories with local symmetries the need to introduce a reference scale may spoil gauge invariance along the flow. In the EAA framework we are employing, this problem is dealt with the use of the background field formalism. This comes at the cost of enlarging theory space to functionals of both the fluctuation and background fields. To this issues we dedicated lot of efforts and we studied how the identities dictated by the local symmetries are modified by the introduction of the cutoff. We systematically studied the flow in this larger space for both non-abelian gauge theories, in Chapter 3, and for quantum gravity in Chapter 4.

In summary, the EAA framework emerges as capable of treating both conceptual and computational issues. To the first, the possibility to map the calculation of the functional integral to a flow problem gives an adequate setting where to study foundational problems as the mathematical definition of the path integral based on non-Gaussian fixed points of the RG flow. To the second, the exact flow equation that the EAA satisfies comprise a novel setting where diverse expansions can be constructed. In particular the non-perturbative nature of the exact flow equation offers to both issues a new interesting avenue of research that year after year is taken more and more seriously.

Appendix A

Heat kernel techniques

This appendix is dedicated to the development of the heat kernel techniques used in this thesis. In the first section we review the definition of the heat kernel and the asymptotic expansion of its trace. We will consider both the local and the non-local expansions for second order covariant Laplacians on a general d-dimensional manifold with arbitrary gauge connection. In the second section we expose a perturbative scheme, firstly developed in this thesis, to calculate the un-traced heat kernel. This perturbative expansion will be used in Chapter 3 to construct the momentum space representation of the flow equations for the zero-field proper-vertices of the bEAA for theories with general local gauge invariance. We will also show how this method can be used to give a new and independent derivation of the non-local heat kernel expansion first derived in [126, 127, 128]. Finally, we show how to use the heat kernel trace expansion to calculate functional traces of functions of covariant Laplacians by introducing the "Q-functional" technology that is used throughout this thesis. For a review of the more mathematical and geometrical aspects of the heat kernel see [130], while for a physicist perspective see [123, 124].

A.1 Basic definitions

The heat kernel $K(s;x,y)$ satisfies the following partial differential equation with boundary condition[1]:

$$(\partial_s + \Delta_x)\, K_{xy}(s) = 0 \qquad\qquad K_{xy}(0) = \delta_{xy}\,. \tag{A.1}$$

In (A.1), as in the following, the covariant Laplacian in denoted by $\Delta \equiv -\nabla_\mu \nabla^\mu$, where ∇_μ is the general covariant derivative comprising both the Levi-Civita and gauge connections.

[1] Here and in the following we use the compact notation $K_{xy}(s) \equiv K(s;x,y)$ and $\delta_{xy} \equiv \delta^{(d)}(x-y)$.

When we deal with flat space situations where only the gauge connection is present, we will indicate the covariant derivative as D_μ. The heat kernel as defined in (A.1) can be interpreted as describing a continuous diffusion process on a manifold, where the diffusing particles are all concentrated at the origin when the heat kernel "proper-time" parameter, related to the diffusion constant D and to time t as $s = Dt$, is zero. Equation (A.1) is immediately solved as $K^s_{xy} = e^{-s\Delta_x}\delta_{xy}$, thus the trace of the heat kernel is equal to:

$$\mathrm{Tr}\, K(s) = \mathrm{Tr}\, e^{-s\Delta} \,. \tag{A.2}$$

The usefulness of the heat kernel stems from the fact that every functional trace of a function $h(\Delta)$ of the covariant Laplacian can be related to it by a Laplace transform:

$$\mathrm{Tr}\, h(\Delta) = \int_0^\infty ds\, \tilde{h}(s)\, \mathrm{Tr}\, e^{-s\Delta} \,. \tag{A.3}$$

Here $\tilde{h}(s)$ is the inverse-Laplace transform of $h(x)$. To compute such a functional trace, one just needs to know the expansion of the trace of the heat kernel to the desired accuracy. There are two basic expansions available, these are respectively, the local and the non-local one. We review these two in the next two sections.

A.1.1 Local heat kernel expansion

For the trace of the heat kernel there exists a standard asymptotic series expansion in local curvature polynomials [123, 125]. This reads:

$$\mathrm{Tr}\, K(s) = \frac{1}{(4\pi s)^{d/2}} \sum_{n=0}^{\infty} B_{2n}(\Delta) s^n \,. \tag{A.4}$$

In (A.4) the $B_{2n}(\Delta)$ are the integrated heat kernel coefficients for the covariant Laplacian Δ, they are related to the un-integrated coefficients $b_{2n}(\Delta)$ by the following relation:

$$B_{2n}(\Delta) = \int d^d x \sqrt{g}\, \mathrm{tr}\, b_{2n}(\Delta) \,. \tag{A.5}$$

Note that in (A.4) the only explicit dependence on the dimension is trough the overall factor $(4\pi)^{d/2}$, while other dependence on the dimension is generated by the trace operation in (A.5). The un-integrated heat kernel coefficients do not depend on the dimension. In this thesis we will be interested only on covariant Laplacians of the following type:

$$\Delta = -\nabla^2 \mathbf{1} + \mathbf{U} \,. \tag{A.6}$$

In (A.6), as in the following, we use bold face notation to remember the matrix structure that the covariant Laplacians may have. The first few coefficients, which have been calculated using various techniques by different authors, are as follows [123, 125]:

$$\begin{aligned} b_0(\Delta) &= \mathbf{1} \\ b_2(\Delta) &= \mathbf{1}\frac{R}{6} - \mathbf{U} \\ b_4(\Delta) &= \frac{1}{2}\mathbf{U}^2 + \frac{1}{6}\nabla^2\mathbf{U} + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} - \frac{R}{6}\mathbf{U} \\ &\quad + \frac{\mathbf{1}}{180}R^2_{\mu\nu\alpha\beta} - \frac{\mathbf{1}}{180}R^2_{\mu\nu} + \frac{\mathbf{1}}{72}R^2 - \frac{\mathbf{1}}{30}\nabla^2 R \,. \end{aligned} \tag{A.7}$$

In (A.7) the curvature tensors are constructed using the Levi-Civita connection in ∇_μ while the tensor $\Omega_{\mu\nu}$ represents the field strength of the gauge connection in ∇_μ.

A.1.2 Non-local heat kernel expansion

In general we need a more sophisticated version of the heat kernel expansion which includes an infinite number of heat kernel coefficients. This expansion has been developed in [126, 127, 128] and retains the infinite number of heat kernel coefficients in the form of non-local "structure functions" or "form factors". The generalized expansion that replaces (A.4) reads as follows:

$$\begin{aligned} \mathrm{Tr}\, K(s) &= \frac{1}{(4\pi s)^{d/2}} \int d^d x \sqrt{g}\, \mathrm{tr} \Big\{ \mathbf{1} + s\mathbf{1}\frac{R}{6} + s^2 \left[\mathbf{1} R_{\mu\nu} f_{Ric}(s\Delta) R^{\mu\nu} + \mathbf{1} R\, f_R(s\Delta) R + \right. \\ &\quad \left. + R\, f_{RU}(s\Delta)\mathbf{U} + \mathbf{U} f_U(s\Delta)\mathbf{U} + \Omega_{\mu\nu} f_\Omega(s\Delta)\Omega^{\mu\nu} \right] + O(\mathcal{R}^3) \Big\} \,. \end{aligned} \tag{A.8}$$

The heat kernel structure functions in (A.8) are found to be:

$$\begin{aligned} f_{Ric}(x) &= \frac{1}{6x} + \frac{1}{x^2}\left[f(x) - 1\right] \\ f_R(x) &= \frac{1}{288}f(x) + \frac{1}{24x}f(x) - \frac{1}{16x} - \frac{1}{8x^2}\left[f(x) - 1\right] \\ f_{RU}(x) &= -\frac{1}{4}f(x) - \frac{1}{2x}\left[f(x) - 1\right] \end{aligned}$$

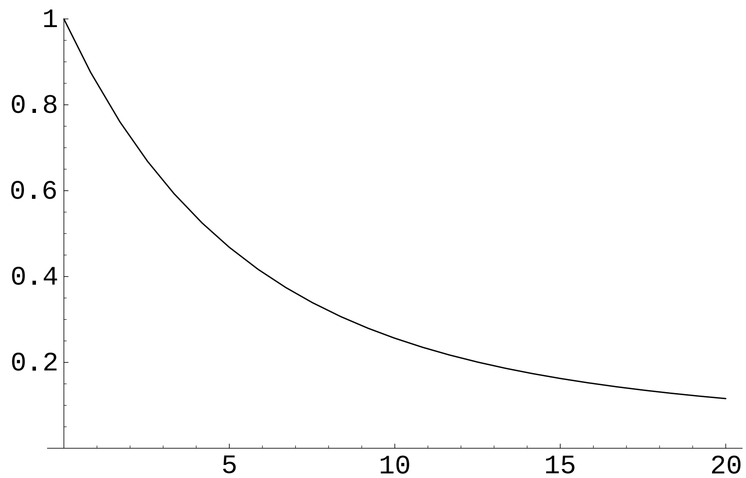


Figure A.1: The basic heat kernel structure function $f(x)$ as defined in equation (A.10).

$$\begin{aligned} f_U(x) &= \frac{1}{2} f(x) \\ f_\Omega(x) &= -\frac{1}{2x}\left[f(x) - 1\right], \end{aligned} \tag{A.9}$$

where the basic heat kernel structure function $f(x)$ is defined in terms of the parameter integral:

$$f(x) = \int_0^1 d\xi \, e^{-x\xi(1-\xi)} . \tag{A.10}$$

A plot of (A.10) is given in Figure A.1. Inserting in (A.9) the Taylor expansion of the basic structure function, $f(x) = 1 - \frac{x}{6} + \frac{x^2}{60} + O(x^4)$, gives the following “short time” expansion for the structure functions defined in (A.9):

$$\begin{aligned} f_{Ric}(x) &= \frac{1}{60} - \frac{x}{840} + \frac{x^2}{15120} + O(x^4) \\ f_R(x) &= \frac{1}{120} - \frac{17x}{15120} + \frac{x^2}{10080} + O(x^4) \\ f_{RU}(x) &= -\frac{1}{6} + \frac{x}{30} - \frac{x^2}{280} + O(x^4) \\ f_U(x) &= \frac{1}{2} - \frac{x}{12} + \frac{x^2}{120} + O(x^4) \\ f_\Omega(x) &= \frac{1}{12} - \frac{x}{120} + \frac{x^2}{1680} + O(x^4) . \end{aligned} \tag{A.11}$$

If we insert (A.11) in (A.8) we recover the first coefficients of local heat kernel expansion of the previous section. In particular, if we compare with (A.7) we see that not all coefficients match exactly. This is because the local heat kernel expansion is derived by calculating the un-integrated coefficients while the non-local heat kernel expansion is derived by calculating the integrated ones. So the coefficients derived by expanding the structure functions (A.9) may differ from the local ones (A.7) by a total derivative or a boundary term. For example, only two of the three possible curvature square invariants present in (A.7) appear in (A.8), the third one has been eliminated using Bianchi's identities and discarding a boundary term. For this reason also the total derivative terms in the coefficient $B_4(\Delta)$ are not present in the non-local expansion. Thus, in general, a straightforward series expansion of the non-local heat kernel structure functions will not reproduce exactly the same heat kernel coefficients of the local expansion. See [127] for more details on this point. But the series in (A.11), when inserted back in (A.8) generate an infinite number of local heat kernel coefficients, all these of the form $\int \sqrt{g}\mathbf{U}\Delta^n\mathbf{U}$, $\int \sqrt{g}R\Delta^n\mathbf{U}$, $\int \sqrt{g}\mathbf{1}R_{\mu\nu}\Delta^n R^{\mu\nu}$, $\int \sqrt{g}\mathbf{1}R\Delta^n R$ and $\int \sqrt{g}\Omega_{\mu\nu}\Delta^n\Omega^{\mu\nu}$ contributing to the integrated coefficients $B_{2n}(\Delta)$. Actually, the heat kernel structure functions (A.9) can be obtained by resumming these coefficients [128].

In $d = 2$ there is only one curvature square invariant in (A.8) because of the relation $R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$, so the gravitational part of the non-local heat kernel expansion becomes just:

$$\mathrm{Tr}\, K(s) = \frac{1}{4\pi s}\int d^2x\sqrt{g}\,\mathrm{tr}\mathbf{1}\left[1 + s\frac{R}{6} + s^2 R\, f_{R2d}(s\Delta)R + O(R^3)\right] , \tag{A.12}$$

where:

$$f_{R2d}(x) = \frac{1}{32}f(x) + \frac{1}{16x}\left[2f(x) - 1\right] + \frac{3}{8x^2}\left[f(x) - 1\right] . \tag{A.13}$$

We will show in the next section how the non-local heat kernel expansion (A.8) can be derived.

A.2 Perturbative expansion of the heat kernel

In this section we develop a perturbative expansion for the un-traced heat kernel where the covariant Laplacian is decomposed as the sum of the flat space Laplacian $-\partial^2$ and an interaction part V in the following way[2]:

$$\Delta = -\partial^2 + V . \tag{A.14}$$

[2]In this section we suppress to use boldface characters to indicate matrix structures for clarity.

The potential V contains U, all terms proportional to the gauge connection and all terms obtained by expanding $g_{\mu\nu}$ around $\delta_{\mu\nu}$. For example, consider the abelian gauge Laplacian, then the potential term contains all the terms that vanish for $A_\mu = 0$:

$$\Delta = -D_\mu D^\mu = -\left(\partial_\mu + iA_\mu\right)\left(\partial^\mu + iA^\mu\right) = -\partial^2 \underbrace{-2A_\mu\partial^\mu - \partial_\mu A^\mu + A_\mu A^\mu}_{V} .$$

We start to calculate the flat space heat kernel $K^t_{0,xy}$ around which we will perform the perturbative expansion. From equation (A.1) we see that it satisfies the following equation with boundary condition:

$$\left(\partial_t - \partial_x^2\right) K^t_{0,xy} = 0 \qquad\qquad K^0_{0,xy} = \delta_{xy} . \tag{A.15}$$

Equation (A.15) can be easily solved in momentum space, we Fourier transform to

$$K^t_{0,xy} = \int_{qq'} K^t_{0,qq'}\, e^{-i(xq+yq')} , \tag{A.16}$$

so that equation (A.15) becomes simply:

$$\left(\partial_t + q^2\right) K^t_{0,qq'} = 0 \qquad\qquad K^t_{0,qq'} = \delta_{qq'} . \tag{A.17}$$

The solution of (A.17) is trivially seen to be $K^t_{0,qq'} = \delta_{qq'} e^{-tq^2}$. Transforming this solution back to coordinate space, completing the square and using the basic Gaussian integral $\int_q e^{-q^2} = (4\pi)^{d/2}$, gives the following result:

$$K^t_{0,xy} = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{(x-y)^2}{4t}} . \tag{A.18}$$

Equation (A.18) is the fundamental solution around which we will construct the perturbative expansion.

To derive the perturbative expansion[3] for K^t around K^t_0 we define the product $U^t =$

[3]In the following we omit to write explicitly the dependence on coordinates for clarity.

$K_0^{-t}K^t$. Using (A.1) and (A.17) we find it satisfies the following equation:

$$\begin{aligned} \partial_t U^t &= \partial_t K_0^{-t} K^t + K_0^{-t}\partial_t K^t \\ &= K_0^{-t}(-\partial^2)K^t - K_0^{-t}\Delta K^t \\ &= -K_0^{-t} V K^t \\ &= -K_0^{-t} V K_0^t U^t \,. \end{aligned} \tag{A.19}$$

We know that equation (A.19) can be solved using Dyson's series, thus we find for U^t and K^t the solutions:

$$U^t = T\exp\left\{-\int_0^t d\tau K_0^{-\tau} V K_0^{\tau}\right\} \quad \Rightarrow \quad K^t = K_0^t\, T\exp\left\{-\int_0^t d\tau K_0^{-\tau} V K_0^{\tau}\right\}. \tag{A.20}$$

In (A.20) the exponentials are time-ordered with respect to t. Rescaling the integration variable in (A.20) as $\tau \to \tau/s$ gives the final formula for the perturbative expansion of the un-traced heat kernel:

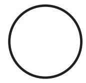
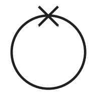
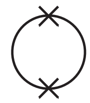
$$\begin{aligned} K^s &= K_0^s\, T\exp\left\{-s\int_0^1 dt K_0^{-st} V K_0^{st}\right\} \\ &= K_0^s - s\int_0^1 dt\, K_0^{s(1-t)} V\, K_0^{st} + \\ &\quad + s^2 \int_0^1 dt_1 \int_0^{t_1} dt_2\, K_0^{s(1-t_1)} V\, K_0^{s(t_1-t_2)} V\, K_0^{st_2} + O(V^3)\,. \end{aligned} \tag{A.21}$$

The expansion (A.21) is conveniently represented graphically as:

where a continuous line represents a K_0 factor and a cross an insertion of the interaction potential V. The parameter integral of the general term is the straightforward generalization of the parameter integrals in (A.21).

The perturbative expansion for the un-traced heat kernel just derived is the fundamental technical tool used in section 3.3.4 of Chapter 3 to derive the momentum space representation for the flow equation of the zero-field proper vertices of the bEAA.

We use now the expansion (A.21) to derive the perturbative expansion for the trace of the heat kernel. To do this we simply trace equation (A.21). This gives, in graphical form, the following expansion for the heat kernel trace:

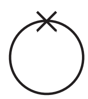
$\mathrm{Tr}\, K^s =$  $- s$  $+ s^2$  $+ \cdots$

The general term in this expansion is now of the form:

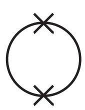
$$(-s)^n \int_0^1 dt_1 \cdots \int_0^{t_{n-1}} dt_n \,\mathrm{Tr}\, K_0^{s(1-t_1+t_n)} V K_0^{s(t_1-t_2)} \cdots V\, K_0^{s(t_{n-2}-t_{n-1})} V\,. \tag{A.22}$$

Note that in (A.22) we used the cyclicity of the trace to combine the last flat space heat kernel with the first.

As a first example of how this expansion can be used, we consider the Laplacian $\Delta = -\partial^2 + U$ acting on scalar fields. The first two contributions to the heat kernel trace are:

 $= \int_0^1 dt\, \mathrm{Tr}\, K_0^s U \; = \; \frac{1}{(4\pi s)^{d/2}} \int_x U_x$

and

 $= \int_0^1 dt_1 \int_0^{t_1} dt_2 \,\mathrm{Tr}\, K_0^{s(1-t_1+t_2)} U K_0^{s(t_1-t_2)} U$

The first contribution is already in its final form. The second one can be simplified by changing variables to $\xi = t_2 - t_1$ so that it can be rewritten as:

$$\frac{1}{2}\int_0^1 d\xi \int d^d x d^d y\, K_{xy,0}^{s(1-\xi)}\, K_{xy,0}^{-s\xi} U_x U_y\,, \tag{A.23}$$

The combination of flat space heat kernels in (A.23) can be simplified using the following identity:

$$K_{0,xy}^{s(1-\xi)} K_{0,xy}^{s\xi} = \frac{1}{(4\pi s)^{d/2}} K_{0,xy}^{s\xi(1-\xi)}\,. \tag{A.24}$$

This relation can be easily verified inserting the explicit form of the flat space heat kernel (A.18). Using (A.24) in (A.23) gives:

$$\frac{1}{(4\pi s)^{d/2}} \int d^d x d^d y\, U_x \left(\frac{1}{2} \int_0^1 d\xi\, e^{-s\xi(1-\xi)(-\partial_x^2)} \delta_{xy} \right) U_y\,. \tag{A.25}$$

We see that in (A.25) the basic structure function (A.10) arises naturally in its parametric form. We can finally write (A.23) as:

$$\frac{1}{(4\pi s)^{d/2}}\frac{1}{2}\int d^dx\, U_x f\left(s\Delta_x\right) U_x\,. \tag{A.26}$$

Thus, to second order in powers of U, the trace of the heat kernel is given by the following non-local expansion:

$$\mathrm{Tr}\, K^s = \frac{1}{(4\pi s)^{d/2}}\int d^dx \left(1 - sU + s^2 U\, f_U\left(s\Delta\right) U + ...\right)\,. \tag{A.27}$$

This is just the expansion (A.8) adapted to the Laplacian operator we are considering. Note, as we said before, that the term

$$\frac{s^2}{(4\pi s)^{d/2}}\int d^dx\left(-\frac{1}{6}\partial^2 U\right)\,,$$

is absent in the expansion (A.27) but present in the local heat kernel coefficient (A.7) since it is a total derivative.

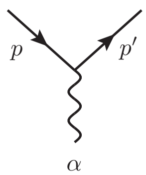
As a second example of how to use the perturbative expansion for the trace of the heat kernel, we use it to derive the pure gauge part of the non-local heat kernel expansion (A.8), i.e. we calculate $f_\Omega(x)$. Due to symmetry arguments, it is enough to consider the simplest case where $f_\Omega(x)$ is not vanishing, i. e. we consider the abelian gauge Laplacian. The non-local structure function so calculated will be then valid for the general case where the $\Omega_{\mu\nu}$ can be arbitrary complicated. This is one of the advantages of the heat kernel expansion: just calculate the coefficients, or the structure functions, in the simplest case and then use them to treat the general case.

First we note that the Laplacian can be written as the Hessian of the following "Laplacian action":

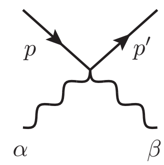
$$L[\phi^*,\phi;A] = \int d^dx\left(D_\mu\phi\right)^*\left(D^\mu\phi\right)\,. \tag{A.28}$$

The Hessian of (A.28) is in fact:

$$L^{(1,1;0)}_{xy}[0,0;A] = \int d^dz D^*_{z\mu}\delta_{xz}D^\mu_z\delta_{yz} = -D^*_{\mu x}D^\mu_y\delta_{xy}\,. \tag{A.29}$$



$= (p+p')^\alpha$



$= 2g^{\alpha\beta}$

Figure A.2: Feynman rules from the Laplacian action (A.28).

If we expand the Hessian of the Laplacian action (A.29) in powers of the gauge field as

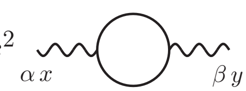
$$L_{xy}^{(1,1;0)}[0,0;A] = L_{xy}^{(1,1;0)}[0,0;0] + \int_z L_{xyz}^{(1,1;1)}[0,0;0]A_z + \frac{1}{2}\int_{zw} L_{xyzw}^{(1,1;2)}[0,0;0]A_zA_w + O(A^3)\,, \tag{A.30}$$

and we note that $L_{xy}^{(1,1;0)}[0,0;0] = -\partial_x^2\delta_{xy}$ is the flat space Laplacian, we can write the interaction potential as follows:

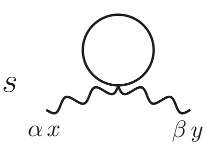
$$V = \int_z L_{xyz}^{(1,1;1)}[0,0;0]A_z + \frac{1}{2}\int_{zw} L_{xyzw}^{(1,1;2)}[0,0;0]A_zA_w + O(A^3)\,. \tag{A.31}$$

From symmetry considerations alone we know that the heat kernel expansion for the gauge Laplacian (A.29) has the general form (A.8) and that the first non-trivial contribution is of order curvature square, i. e. is the function $f_\Omega(x)$. To find an equation satisfied by this term we simply differentiate (A.8) two times with respect to the gauge field to obtain:

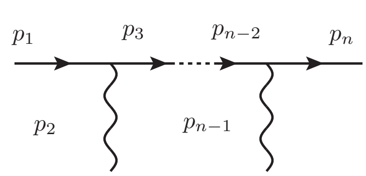
$$\left.\frac{\delta^2 \mathrm{Tr}\, K^s}{\delta A_x^\alpha \delta A_y^\beta}\right|_{A=0} = 2s^2$$



$-s$



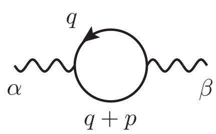
We just need to evaluate this equation in momentum space. The vertices of the Laplacian action are the standard ones for a scalar field interacting with the electromagnetic interaction and they are shown in Figure A.2. Internal lines are treated according to:



$=$

$$= \int_0^1 dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-2}} dt_{n-1}\, e^{-s(1-t_1)p_1^2} e^{-s(t_1-t_2)p_2^2} \dots$$
$$\dots\, e^{-s(t_{n-2}-t_{n-1})p_{n-1}^2} e^{-st_{n-1}p_n^2}\,.$$

As usual we integrate over closed loop momentum. The first diagram gives the following contribution:



$$= \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_q (2q+p)^\alpha (2q+p)^\beta e^{-s(1-t_1+t_2)q^2 - s(t_1-t_2)(q+p)^2} \,.$$

Terms linear in q integrate to zero so we can set $(2q+p)^\alpha(2q+p)^\beta = 4q^\alpha q^\beta + p^\alpha p^\beta$ under the integral. Re-writing the exponent as

$$\begin{aligned}(1-t_1+t_2)q^2 - s(t_1-t_2)(q+p)^2 &= q^2 - \xi q^2 + \xi(q^2+p^2+2q\cdot p) \\ &= (q+\xi p)^2 + \xi(1-\xi)p^2 \,,\end{aligned}$$

shifting the momentum variable $q^\alpha \to q^\alpha + \xi^\alpha$, dropping terms linear in q^α and using the relation $q^\alpha q^\beta \to \frac{1}{d} g^{\alpha\beta} q^2$ under the integral gives:

$$\frac{1}{2}\int_0^1 d\xi\, e^{-s\xi(1-\xi)p^2} \left\{ \frac{4}{d} g^{\alpha\beta} \int_q q^2 e^{-sq^2} + p^\alpha p^\beta \left(1-4\xi^2\right) \int_q e^{-sq^2} \right\} . \tag{A.32}$$

Using the basic Gaussian integrals

$$\int_k e^{-sk^2} = \frac{1}{(4\pi s)^{d/2}} \qquad\qquad \int_k k^2 e^{-sk^2} = \frac{1}{(4\pi s)^{d/2}} \frac{d}{2s} \,,$$

in (A.32) gives:

$$\frac{1}{(4\pi s)^{d/2}} \frac{1}{2} \int_0^1 d\xi\, e^{-s\xi(1-\xi)p^2} \left\{ \frac{2}{s} g^{\alpha\beta} + \left(1-4\xi^2\right) p^\alpha p^\beta \right\} . \tag{A.33}$$

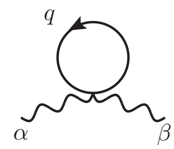
The parameter integral in (A.33) can be written in terms of the basic form factor (A.10) as:

$$\frac{1}{2}\int_0^1 d\xi\, e^{-\xi(1-\xi)\alpha} \left(1-4\xi^2\right) = \frac{1-f(\alpha)}{\alpha} \,,$$

thus we get the following result for the first diagram:

$$\frac{1}{(4\pi s)^{d/2}} \left\{ \frac{1}{s} f\left(sp^2\right) g^{\alpha\beta} + \frac{1-f\left(sp^2\right)}{sp^2} p^\alpha p^\beta \right\} . \tag{A.34}$$

Note that (A.34) is not transverse. The second diagram gives instead:



$$= 2g^{\alpha\beta}\int_q e^{-sq^2} = \frac{1}{(4\pi s)^{d/2}}\frac{2}{s}g^{\alpha\beta}$$

The two diagrams combine now to give the following transverse structure:

$$-2\frac{s^2}{(4\pi s)^{d/2}}\left(p^2 g^{\alpha\beta} - p^\alpha p^\beta\right)\frac{1-f(sp^2)}{sp^2}\,,$$

which correspond to the momentum space representation of the term:

$$\frac{s^2}{(4\pi s)^{d/2}}\int dx\, F_{\mu\nu} f_\Omega\left(s\Delta\right) F^{\mu\nu}\,, \tag{A.35}$$

with the following form for the structure function:

$$f_\Omega\left(x\right) = -\frac{f(x)-1}{2x}\,. \tag{A.36}$$

We see that (A.36) agrees with (A.9).

Along the same lines it is possible to derive the gravitational form factors in the non-local heat kernel expansion (A.8) by considering the Laplacian action as equal to the action of a minimally coupled scalar field $L[\phi; g] = \frac{1}{2}\int\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. We do not do this here and refer to [129] for further details.

The derivations of this section show that the perturbative expansion here developed is able to re-derive the non-local expansion for the trace of the heat kernel exposed in section A.1.2. More importantly, this approach can be easily applied to general differential operators other then second order Laplacians [129].

A.3 Trace technology

One of the most useful applications of the heat kernel expansion is to the calculation of functional traces. Consider a function of the Laplacian operator $f(\Delta)$, we want to calculate its trace. With the aid of a Laplace transform we can reduce the trace of $f(\Delta)$ to the trace of the heat kernel as following:

$$\mathrm{Tr}\, f(\Delta) = \int_0^\infty ds\, \tilde{f}(s)\mathrm{Tr}\, e^{-t\Delta} = \int_0^\infty dt\, \tilde{f}(t)\mathrm{Tr}K^s\,. \tag{A.37}$$

Here $\tilde{f}(x)$ is the inverse-Laplace transform of $f(x)$. Inserting in the last equation the local heat kernel expansion (A.4) we find:

$$\mathrm{Tr}\, f(\Delta) = \sum_{n=0}^{\infty} Q_{\frac{d}{2}-n}[f] B_{2n}(\Delta) \,, \tag{A.38}$$

where B_{2n} are the integrated local coefficients (A.5) and

$$Q_n[f] = \int_0^{\infty} dt\, \tilde{f}(t) t^{-n} \tag{A.39}$$

are the "Q-functionals". These can be stated in terms of $f(x)$ as following:

$$Q_n[f] = \begin{cases} \frac{1}{\Gamma(n)} \int_0^{\infty} dz\, z^{n-1} f(z) & n > 0 \\ (-1) f^{(n)}(0) & n \leq 0 \end{cases} . \tag{A.40}$$

It is possible to extend (A.40) also to half integers values of n. For more details see the appendix of [96]. With the aid of (A.38) and (A.40) we can now calculate the local expansion for the trace of any function of the Laplacian operator.

A.3.1 Threshold integrals

In this section we explicitly calculate the Q-functionals or threshold integrals needed in this thesis. We will employ the optimized cutoff shape function

$$R_k(z) = (k^2 - z)\theta(k^2 - z) \,, \tag{A.41}$$

that enables the analytical calculation of all the threshold integrals we will encounter.

As already defined in the main text, the general form of the regularized propagator is as follows:

$$G_k(z) = \frac{1}{z + R_k(z) + \omega} \,, \tag{A.42}$$

were ω is generally a squared mass but can even be something else. In all the beta functions we will study there are three basic Q-functionals to be evaluated. The first one is of the form:

$$Q_n \left[(\partial_t R_k - \eta R_k) G_k^m \right] = \frac{k^{2(n+1-m)}}{\Gamma(n+2)} \frac{2(n+1) - \eta}{(1 + \omega/k^2)^m} \,, \tag{A.43}$$

where η is a given anomalous dimension. The second form involves a derivative of the regularized propagator with respect to the argument z and turns out to be zero if we employ the optimized cutoff:

$$Q_n\left[(\partial_t R_k - \eta R_k)G_k^m G_k'\right] = 0\,. \tag{A.44}$$

To see why (A.44) vanishes, we just need to notice that

$$G_k'(z) = -G_k^2(z)(1 + R_k'(z))\,, \tag{A.45}$$

vanishes since $R_k'(z) = -\theta(k^2 - z)$ becomes -1 inside the integral in the Q-functional. The last combination we are interested is when a factor of the second derivative of the regularized propagator is present:

$$Q_n\left[(\partial_t R_k - \eta R_k)G_k^m G_k''\right] = -\frac{k^{2(n+2-m)}}{\Gamma(n)}\frac{1}{(1+\omega/k^2)^m}\,. \tag{A.46}$$

In (A.46) we used the following relation:

$$G_k''(z) = 2G_k^3(1 + R_k'(z))^2 - G_k^2 R_k''(z)\,, \tag{A.47}$$

where only the second term is non-zero when using the optimized cutoff shape function. We used also the relation $R_k''(z) = \delta(k^2 - z)$. It is obvious how to generalize (A.43), (A.44) and (A.46) when different regularized propagator of the general form (A.42) are present. In studying non-abelian gauge theories in Chapter 3 we will encounter a regularized propagator which is not of the form (A.42) but reads:

$$G_{L,k}(z) = \frac{\alpha}{z + \alpha(\omega + R_k(z))}\,, \tag{A.48}$$

where α is the gauge-fixing parameter. In this case the Q-functionals turn out to be more complicated since, due to the presence of the gauge-fixing parameter in the denominator of (A.48), the usual simplification that occurs when employing the optimized cutoff does not occur any more. We find a similar form for the regularized propagator also in Chapter 2 when dealing with the derivative expansion at order ∂^2. In that case we have $\alpha = Z_k(\varphi)$ in the denominator of (A.48) while $\alpha = 1$ in the numerator. We find the following result:

$$\begin{aligned} Q_n\left[(\partial_t R_k - \eta R_k) G_k^m\right] &= \frac{k^{2(n+1-m)}}{\Gamma(n+2)} \frac{1}{\alpha^m \left(1+\omega/k^2\right)^m} \left[-(1+n)(-2+\eta)\right. \\ &\quad \times {}_2F_1\left(n, m, n+1, \frac{\alpha-1}{\alpha\left(1+\omega/k^2\right)}\right) \\ &\quad \left. + n\eta\, {}_2F_1\left(n+1, m, n+2, \frac{\alpha-1}{\alpha\left(1+\omega/k^2\right)}\right)\right]. \end{aligned} \tag{A.49}$$

Also

$$Q_n\left[(\partial_t R_k - \eta R_k) G_k^m G_k'\right] = (\alpha-1) Q_n\left[(\partial_t R_k - \eta R_k) G_k^m\right], \tag{A.50}$$

while

$$\begin{aligned} Q_n\left[(\partial_t R_k - \eta R_k) G_k^m G_k''\right] &= (\alpha-1)^2 Q_n\left[(\partial_t R_k - \eta R_k) G_k^m\right] \\ &\quad - \frac{k^{2(n+2-m)}}{\Gamma(n)} \frac{\alpha}{\left(1+\alpha\omega/k^2\right)^m}. \end{aligned} \tag{A.51}$$

If instead we consider the non-local heat kernel expansion (A.8) in (A.37) we encounter Q-functionals inside parameter integrals of the following form:

$$\int_0^1 d\xi\, Q_n\left[h_k\left(z + x\xi(1-\xi)\right)\right] \tag{A.52}$$

where we defined the function $h_k(z) = \partial_t R_k(z) G_k(z)$ for $\omega = 0$. The integrals (A.52) can be calculated analytically if we employ the optimized cutoff shape function (A.41). We will need these integrals in the cases $n = -1, 0, 1, 2$. We start to consider the case $n = -1$, where, using (A.40), we have to evaluate

$$\int_0^1 d\xi\, Q_{-1}\left[h_k\left(z + x\xi(1-\xi)\right)\right] = -\int_0^1 d\xi\, h_k'\left(x\xi(1-\xi)\right). \tag{A.53}$$

We have $h_k'\left(x\xi(1-\xi)\right) = \frac{2}{x}\delta\left(\frac{1}{\tilde{x}} - \xi(1-\xi)\right)$ where $\tilde{x} = x/k^2$. We need to study the following quadratic equation:

$$f(\xi) = \xi^2 - \xi + \frac{1}{\tilde{x}} \qquad\qquad f(\xi) = 0 \qquad \Rightarrow \qquad \xi_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4}{\tilde{x}}}. \tag{A.54}$$

Using the properties of the delta function and the fact that $f'(\xi_\pm) = \pm\sqrt{1 - \frac{4}{\tilde{x}}}$ we find that

(A.53) becomes:

$$\begin{aligned}\int_0^1 d\xi\, h_k'\left(x\xi(1-\xi)\right) &= \frac{2}{x}\int_0^1 d\xi \left\{\frac{\delta(\xi-\xi_+)}{\sqrt{1-\frac{4}{\tilde{x}}}}+\frac{\delta(\xi-\xi_-)}{\sqrt{1-\frac{4}{\tilde{x}}}}\right\} \\ &= \frac{4}{x}\sqrt{1+\frac{4}{\tilde{x}}}\theta(\tilde{x}-4)\,, \end{aligned} \tag{A.55}$$

since only for $\tilde{x} \leq 4$ the roots $\xi_\pm$ are real. Inserting back (A.55) in (A.53) gives the following result:

$$\int_0^1 d\xi\, Q_{-1}\left[h_k\left(z+x\xi(1-\xi)\right)\right] = \frac{4}{x}\sqrt{1+\frac{4k^2}{x}}\theta(x-4k^2)\,. \tag{A.56}$$

For the case where $n = 0$, using (A.40), we have that:

$$\int_0^1 d\xi\, Q_0\left[h_k\left(z+x\xi(1-\xi)\right)\right] = \int_0^1 d\xi\, h_k\left(x\xi(1-\xi)\right)\,. \tag{A.57}$$

We have now $h_k\left(x\xi(1-\xi)\right) = 2\theta\left(\frac{1}{\tilde{x}}-\xi(1-\xi)\right)$ and we need to find out when $f(\xi) \geq 0$. From (A.54) we see that this happens for $0 \leq \xi \leq \xi_-$ and $\xi_+ \leq \xi \leq 1$ when $\tilde{x} \geq 4$ and always when $\tilde{x} < 4$. The parameter integral integral in (A.57) becomes thus:

$$\int_0^1 d\xi\, h_k\left(x\xi(1-\xi)\right) = \begin{cases} 2 & \tilde{x} < 4 \\ 2-2(\xi_+-\xi_-) & \tilde{x} \geq 4 \end{cases}\,. \tag{A.58}$$

Inserting (A.58) in (A.57) and using (A.54) finally gives:

$$\int_0^1 d\xi\, Q_0\left[h_k\left(z+x\xi(1-\xi)\right)\right] = 2\left[1-\sqrt{1-\frac{4k^2}{x}}\theta(x-4k^2)\right]\,. \tag{A.59}$$

In the remaining case $n \geq 0$, using (A.40), we have to evaluate:

$$\int_0^1 d\xi\, Q_n\left[h_k\left(z+x\xi(1-\xi)\right)\right] = \frac{1}{\Gamma(n)}\int_0^1 d\xi \int_0^\infty dz\, z^{n-1} h_k\left(z+x\xi(1-\xi)\right)\,. \tag{A.60}$$

Now we have $h_k\left(x\xi(1-\xi)\right) = 2\theta\left(\frac{1}{\tilde{x}}-\xi(1-\xi)-z\right)$ so that the z-integral in (A.60) is reduced to:

$$\int_0^\infty dz\, z^{n-1} h_k\left(z + x\xi(1-\xi)\right) = 2\int_0^{k^2 - x\xi(1-\xi)} dz\, z^{n-1} = \frac{2}{n}\left[k^2 - x\xi(1-\xi)\right]^n \,, \tag{A.61}$$

with the condition $k^2 - x\xi(1-\xi) \geq 0$, otherwise it vanishes. As for the $n = 0$ case, this condition is satisfied for $0 \leq \xi \leq \xi_-$ and $\xi_+ \leq \xi \leq 1$ when $\tilde{x} \geq 4$ and always when $\tilde{x} < 4$. Inserting back (A.61) in (A.60) thus gives:

$$\int_0^\infty dz\, z^{n-1} h_k\left(z + x\xi(1-\xi)\right) = \begin{cases} \frac{2}{\Gamma(n+1)} \int_0^1 d\xi\, [k^2 - x\xi(1-\xi)]^n & \tilde{x} < 4 \\ \frac{2}{\Gamma(n+1)} \left\{\int_0^{\xi_-} d\xi + \int_{\xi_+}^1 d\xi\right\} [k^2 - x\xi(1-\xi)]^n & \tilde{x} \geq 4 \end{cases} . \tag{A.62}$$

In the cases of interest, $n = 1, 2$, performing the integrals in (A.62) gives the following results:

$$\int_0^1 d\xi\, Q_1\left[h_k\left(z + x\xi(1-\xi)\right)\right] = 2k^2\left[1 - \frac{x}{6k^2} + \frac{x}{6k^2}\left(1 - \frac{4k^2}{x}\right)^{\frac{3}{2}} \theta(x - 4k^2)\right] \,, \tag{A.63}$$

and

$$\begin{aligned} \int_0^1 d\xi\, Q_2\left[h_k\left(z + x\xi(1-\xi)\right)\right] \;&=\; k^4\left[1 - \frac{x}{3k^2} + \frac{x^2}{30k^4}\right. \\ &\left. - \frac{x^2}{30k^4}\left(1 - \frac{4k^2}{x}\right)^{\frac{5}{2}} \theta(x - 4k^2)\right] . \end{aligned} \tag{A.64}$$

This concludes the explicit evaluation of the Q-functionals needed in the applications exposed in the main part of the thesis.

Appendix B

Basic quantum field theory

In this Appendix we give a short review of the basic concepts of the functional formulation of Quantum Field Theory (QFT). We will work with a Euclidean signature, so we are actually doing Statistical Field Theory (STF). For a general introduction see the textbooks [10, 11, 12, 9].

B.1 Functional formulation of QFT

The fundamental object of a QFT, as defined in the functional formalism, is the partition function. In a theory with classical or bare action $S[\phi]$ the partition function is defined by:

$$Z = \int D\phi \, e^{-S[\phi]} \,, \tag{B.1}$$

where $\phi(x)$ is the field variable. The field can be either bosonic or fermionic. In the first case we integrate over classical fields while in the second case we use anti-commuting Grassmann variables. Given an observable $\mathcal{O}\left[\phi(x)\right]$, the expectation value is defined to be

$$\langle \mathcal{O}\left[\phi\right] \rangle = \frac{1}{Z} \int D\phi \, \mathcal{O}\left[\phi\right] \, e^{-S[\phi]} \,. \tag{B.2}$$

An important role is played by the expectation value of the product of n fields at different points. This defines correlations or Green's functions:

$$G(x_1, \ldots, x_n) = \langle \phi(x_1), \ldots, \phi(x_n) \rangle \ . \tag{B.3}$$

B.1.1 Generating functionals

It is convenient to introduce auxiliary currents $J(x)$ and to define the partition functional as follows[1]:

$$Z[J] = \int D\phi \, e^{-S[\phi]+\int J\phi} . \tag{B.4}$$

In this way we can write the correlation functions as:

$$\langle\phi(x_1),\ldots,\phi(x_n)\rangle = \frac{1}{Z[J]}\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z[J]\bigg|_{J=0} . \tag{B.5}$$

Equation (B.5) says that the partition functional (B.4) is the generating functional of correlation functions. We are more interested in connected correlation functions, these are generated by the following functional:

$$W[J] = \log\, Z[J] . \tag{B.6}$$

We can see this by calculating the first functional derivatives of (B.6). A first derivative gives:

$$\frac{\delta W[J]}{\delta J(x)} = \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)} = \langle\phi(x)\rangle_J , \tag{B.7}$$

where $\langle\phi(x)\rangle_J$ is the vacuum expectation value of the field in presence of the current $J(x)$. For the two-point function of $W[J]$ we find the following:

$$\begin{aligned}\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)} &= \frac{1}{Z[J]}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)} - \frac{1}{Z[J]^2}\frac{\delta Z[J]}{\delta J(x_1)}\frac{\delta Z[J]}{\delta J(x_2)} \\ &= \langle\phi(x_1)\phi(x_2)\rangle_J - \langle\phi(x_1)\rangle_J\langle\phi(x_2)\rangle_J ,\end{aligned} \tag{B.8}$$

and we see that this corresponds to the connected two-point correlation function, i. e. the propagator. For the three-point function of $W[J]$ we find instead:

$$\begin{aligned}\frac{\delta^3 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} &= \frac{1}{Z[J]}\frac{\delta^3 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} - \frac{1}{Z[J]^2}\frac{\delta Z[J]}{\delta J(x_3)}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)} \\ &\quad - \frac{1}{Z[J]^2}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_3)}\frac{\delta Z[J]}{\delta J(x_2)} - \frac{1}{Z[J]^2}\frac{\delta Z[J]}{\delta J(x_1)}\frac{\delta^2 Z[J]}{\delta J(x_2)\delta J(x_3)} \\ &\quad + 2\frac{1}{Z[J]^3}\frac{\delta Z[J]}{\delta J(x_1)}\frac{\delta Z[J]}{\delta J(x_2)}\frac{\delta Z[J]}{\delta J(x_3)}\end{aligned}$$

[1] Obviously the partition function is given by $Z = Z[0]$.

$$
\begin{aligned}
&= \langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle_J - \langle\phi(x_1)\rangle_J \langle\phi(x_2)\phi(x_3)\rangle_J \\
&\quad - \langle\phi(x_2)\rangle_J \langle\phi(x_1)\phi(x_3)\rangle_J - \langle\phi(x_3)\rangle_J \langle\phi(x_1)\phi(x_2)\rangle_J \\
&\quad +2 \langle\phi(x_1)\rangle_J \langle\phi(x_2)\rangle_J \langle\phi(x_3)\rangle_J \,. \qquad \text{(B.9)}
\end{aligned}
$$

Equations (B.7-B.9) show that $W[J]$ is the generating functional of connected correlation functions. In general we have:

$$\langle\phi(x_1), \ldots, \phi(x_n)\rangle_C = \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} W[J]\bigg|_{J=0} \,. \qquad \text{(B.10)}$$

These are called also connected Green's functions $G_C(\phi(x_1), \ldots, \phi(x_n)) = \langle\phi(x_1), \ldots, \phi(x_n)\rangle_C$. These relations between connected and non-connected correlation functions can be inverted to give the following:

$$
\begin{aligned}
\langle\phi(x_1)\rangle &= \langle\phi(x_1)\rangle_C \\
\langle\phi(x_1)\phi(x_2)\rangle &= \langle\phi(x_1)\phi(x_2)\rangle_C + \langle\phi(x_1)\rangle_C \langle\phi(x_2)\rangle_C \\
\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle &= \langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle_C + \langle\phi(x_1)\phi(x_2)\rangle_C \langle\phi(x_3)\rangle_C + \\
&\quad \langle\phi(x_1)\phi(x_3)\rangle_C \langle\phi(x_2)\rangle_C + \langle\phi(x_2)\phi(x_3)\rangle_C \langle\phi(x_1)\rangle_C + \\
&\quad + \langle\phi(x_1)\rangle_C \langle\phi(x_2)\rangle_C \langle\phi(x_3)\rangle_C \,. \qquad \text{(B.11)}
\end{aligned}
$$

In a QFT, the natural variable to use is the vacuum expectation value of the field $\varphi_J(x) = \langle\phi(x)\rangle_J$. From equation (B.7) we have that

$$\frac{\delta W[J]}{\delta J(x)} = \varphi_J(x)\,. \qquad \text{(B.12)}$$

To construct a functional of $\varphi_J(x)$ we can solve (B.12) to obtain $J_\varphi(x)$ and take a Legendre transform of the functional $W[J]$:

$$\Gamma[\varphi] = \int d^d x J_\varphi(x)\varphi(x) - W[J_\varphi]\,. \qquad \text{(B.13)}$$

This is the definition of the effective action. Differentiating equation (B.13) with respect to φ gives:

$$\begin{aligned}\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} &= \int d^dy \left(\frac{\delta J_\varphi(y)}{\delta\varphi(x)}\varphi(y) + J_\varphi(y)\frac{\delta\varphi(y)}{\delta\varphi(x)}\right) - \frac{\delta W[J_\varphi]}{\delta\varphi(x)} \\ &= \int d^dy \left(\frac{\delta J_\varphi(y)}{\delta\varphi(x)}\varphi(y) + J_\varphi(y)\delta_{xy}\right) - \int d^dy \underbrace{\frac{\delta W[J_\varphi]}{\delta J(y)}}_{=\varphi(y)} \frac{\delta J(y)}{\delta\varphi(x)} \\ &= J_\varphi(x)\,. \end{aligned} \tag{B.14}$$

If we set $J = 0$ in (B.14) we obtain the following equation:

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = 0\,, \tag{B.15}$$

which is the quantum generalization of the principle of least action. Using (B.6), (B.13) and (B.14) in (B.4) gives the following integral representation for the effective action:

$$e^{-\Gamma[\varphi]} = \int D\phi\, e^{-S[\phi]+\int \frac{\delta\Gamma[\varphi]}{\delta\varphi}(\phi-\varphi)}\,. \tag{B.16}$$

Shifting to the fluctuation field $\chi = \phi - \varphi$ in the functional integral (B.16) finally gives:

$$e^{-\Gamma[\varphi]} = \int D\chi\, e^{-S[\varphi+\chi]+\int \frac{\delta\Gamma[\varphi]}{\delta\varphi}\chi} \qquad\qquad \langle\chi\rangle = 0\,. \tag{B.17}$$

The solution of equation (B.15) φ_* is the quantum vacuum expectation value of the field, if we insert it in the integral representation of the effective action (B.17) we find the following relation for the on-shell effective action:

$$e^{-\Gamma[\varphi_*]} = \int D\chi\, e^{-S[\varphi_*+\chi]}\,. \tag{B.18}$$

We can write the relation between the on-shell effective action (B.18) and the zero-source partition function (B.1) as follows:

$$\Gamma[\varphi_*] = -\log Z\,. \tag{B.19}$$

Equation (B.19) shows how the effective action formalism can be used to calculate the zero-source partition function: first calculate the effective action, second solve equation (B.15) to find the vacuum expectation value of the field φ_* and third use relation (B.19).

The effective action is the generating functional of proper-vertices. The following relations, found by differentiating (B.12) and (B.14),

$$\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)} = \frac{\delta\varphi(x_1)}{\delta J(x_2)} \qquad \frac{\delta^2\Gamma[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)} = \frac{\delta J(x_2)}{\delta\varphi(x_1)}\,,$$

show that the Hessian's of $W[J]$ and $\Gamma[\varphi]$ are inverse to each other:

$$\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)} = \left(\frac{\delta^2\Gamma[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right)^{-1} = G_C(x_1,x_2)\,. \tag{B.20}$$

For the connected three point function we have instead[2]:

$$\frac{\delta^3 W}{\delta J_1\delta J_2\delta J_3} = \frac{\delta}{\delta J_1}G_{23} = \frac{\delta\varphi_A}{\delta J_1}\frac{\delta G_{23}}{\delta\varphi_A}\,. \tag{B.21}$$

Using the following relation

$$\frac{\delta G_{23}}{\delta\varphi_A} = -G_{2B}G_{3C}\frac{\delta^3\Gamma}{\delta\varphi_A\delta\varphi_B\delta\varphi_C}\,,$$

in (B.21) gives:

$$\frac{\delta^3 W}{\delta J_1\delta J_2\delta J_3} = -G_{1A}G_{2B}G_{3C}\frac{\delta^3\Gamma}{\delta\varphi_A\delta\varphi_B\delta\varphi_C}\,. \tag{B.22}$$

For the connected four point function we find:

$$\begin{aligned}
\frac{\delta^4 W}{\delta J_1\delta J_2\delta J_3\delta J_4} &= -\frac{\delta}{\delta J_4}\left[G_{1A}G_{2B}G_{3C}\frac{\delta^3\Gamma}{\delta\varphi_A\delta\varphi_B\delta\varphi_C}\right] \\
&= -G_{4D}\left[\frac{\delta G_{1A}}{\delta\varphi_D}G_{2B}G_{3C} + G_{1A}\frac{\delta G_{2B}}{\delta\varphi_D}G_{3C} + \right. \\
&\quad \left. + G_{1A}G_{2B}\frac{\delta G_{3C}}{\delta\varphi_D}\right]\frac{\delta^3\Gamma}{\delta\varphi_A\delta\varphi_B\delta\varphi_C} + \\
&\quad -G_{1A}G_{2B}G_{3C}G_{4D}\frac{\delta^4\Gamma}{\delta\varphi_A\delta\varphi_B\delta\varphi_C\delta\varphi_D}\,,
\end{aligned}$$

[2] We use now a condensed notation for the arguments of the fields $\varphi(x) = \varphi_x$ and so on.

reordering some indices gives:

$$\begin{aligned}\frac{\delta^4 W}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} &= [G_{1A}G_{2B}G_{3C}G_{4D} + G_{1A}G_{2C}G_{3B}G_{4D} + \\ &+ G_{1A}G_{2D}G_{3C}G_{4B}] \frac{\delta^3 \Gamma}{\delta\varphi_A \delta\varphi_B \delta\varphi_X} G_{XY} \frac{\delta^3 \Gamma}{\delta\varphi_Y \delta\varphi_C \delta\varphi_D} + \\ &- G_{1A}G_{2B}G_{3C}G_{4D} \frac{\delta^4 \Gamma}{\delta\varphi_A \delta\varphi_B \delta\varphi_C \delta\varphi_D} \,. \end{aligned} \tag{B.23}$$

Equations (B.20), (B.22) and (B.23) are the first of a hierarchy of relations expressing connected correlation functions in terms of the proper-vertices and propagators.

B.1.2 Gaussian integrals

The ability to perform Gaussian integrals is at the base of the perturbative approach to QFT. It is also the basic tool used in effective field theory. For this reasons we review here the basics relations.

We start with the finite dimensional Gaussian integral for bosonic variables ϕ_i, $i = 1, .., N$. The Gaussian action takes the following form:

$$S[\phi] = \frac{1}{2} \sum_{ij} \phi_i M_{ij} \phi_j \,, \tag{B.24}$$

where M_{ij} is a positive definite matrix, i.e. M_{ij} is diagonalize and all eigenvalues satisfy $\lambda_i > 0$. The partition function is defined as the finite dimensional version of (B.1):

$$Z = \int d\phi \, e^{-\frac{1}{2} \sum_{ij} \phi_i M_{ij} \phi_j} \,, \tag{B.25}$$

with measure $d\phi = \Pi_i d\phi_i$. Eventually by a redefinition of the ϕ_i, they can be taken to be an eigenbasis, i.e. to satisfy $\sum_j M_{ij}\phi_j = \lambda_i \delta_{ij} \phi_j$. Equation (B.25) becomes:

$$Z = \int d\phi \, e^{-\frac{1}{2} \sum_i \lambda_i \phi_i^2} = \prod_i \int d\phi_i \, e^{-\frac{1}{2}\lambda_i \phi_i^2} \,. \tag{B.26}$$

Using the basic Gaussian integral

$$\int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2} a x^2} = \sqrt{\frac{2\pi}{a}} \,,$$

in (B.26) finally gives:

$$Z = \prod_i \left(\frac{2\pi}{\lambda_i}\right)^{1/2} = \frac{(2\pi)^{N/2}}{\det^{1/2} M} \,. \tag{B.27}$$

We can extend this result to calculate the full generating functional of correlation functions $Z(J)$. Completing the square and shifting the integration variables to $\phi \to \phi - J\, M^{-1}$ gives:

$$\begin{aligned} Z(J) &= \int d\phi\, e^{-\frac{1}{2}\phi M \phi + J\phi} \\ &= e^{\frac{1}{2}J\, M^{-1} J} \int d\phi\, e^{-\frac{1}{2}\left(\phi - J\, M^{-1}\right) M \left(\phi - M^{-1} J\right)} \\ &= Z(0)\, e^{\frac{1}{2}J\, M^{-1} J} \,, \end{aligned} \tag{B.28}$$

with obviously $Z = Z(0)$. Using (B.28) we can calculate all correlation functions:

$$\begin{aligned} \langle \phi_1 \phi_2 \rangle &= \frac{1}{Z(J)} \frac{\delta^2 Z(J)}{\delta J_1 \delta J_2}\bigg|_{J=0} \\ &= M^{-1}_{12} \\ \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle &= \frac{1}{Z(J)} \frac{\delta^4 Z(J)}{\delta J_1 \delta J_2 \delta J_3 \delta J_4}\bigg|_{J=0} \\ &= M^{-1}_{12} M^{-1}_{34} + M^{-1}_{13} M^{-1}_{24} + M^{-1}_{14} M^{-1}_{23} \,. \end{aligned} \tag{B.29}$$

By induction we can easily prove the general relation:

$$\langle \phi_1 ... \phi_N \rangle = \sum_{i_1 < i_2 \neq ... \neq i_{N-1} < i_N} M^{-1}_{i_1 i_2} ... M^{-1}_{i_{N-1} i_N} \,. \tag{B.30}$$

Relation (B.30) is usually called Wick theorem and here it follows as combinatorial property of Gaussian integrals. The generating functional of connected correlations functions is defined by

$$W(J) = \log\, Z(J) = \frac{1}{2} J\, M^{-1} J \,. \tag{B.31}$$

In (B.31) we used (B.28) and we fixed the normalization $Z(0) = 1$. The average field is determined by:

$$\varphi_i(J) = \frac{\delta W(J)}{\delta J_i} = M^{-1}_{ij} J_j \,, \tag{B.32}$$

thus $J_i(\varphi) = M_{ij}\varphi_j$ and the effective action becomes:

$$\Gamma(\varphi) = J_i(\varphi)\varphi_i - W(J(\varphi)) = \frac{1}{2}\varphi_i M_{ij} \varphi_j \,. \tag{B.33}$$

In particular, this shows that for a Gaussian theory the bare action and the effective action are equal $\Gamma(\varphi) = S(\varphi)$.

We can also consider the case of complex fields. In this case we integrate over ϕ_i and ϕ_i^* separately. Using the basic complex bosonic Gaussian integral

$$\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz^* \, e^{-a|z|^2} = \frac{\pi}{a}\,, \tag{B.34}$$

we find the following form for the generating functional of connected correlation functions:

$$Z(J, J^*) = \int d\phi d\phi^* \, e^{-\phi^* M\phi + J\phi^* + J^*\phi} = Z(0,0)\, e^{J^* M^{-1} J}\,, \tag{B.35}$$

where now

$$Z(0,0) = \frac{\pi^N}{\det\, M}\,. \tag{B.36}$$

As before, we can show that the bare action and the effective action are equal.

Fermionic variables are elements of a Grassmann algebra, with elements θ_i, defined by the following anti-commutation relations:

$$\theta_i\theta_j + \theta_j\theta_i = 0 \qquad \Rightarrow \qquad \theta_i^2 = 0\,. \tag{B.37}$$

A function of Grassmann variables has a truncated Taylor expansion:

$$f(\theta_1, ..., \theta_N) = f_0 + f_i\theta_i + f_{ij}\theta_i\theta_j + f_{ijk}\theta_i\theta_j\theta_k + ... + f_{i_1...i_N}\theta_{i_1}\cdots\theta_{i_N}\,, \tag{B.38}$$

where the coefficients $f_{i_1...i_N}$ are totally antisymmetric and the indices run from one to N. Integration over Grassmann variables is defined by the rule:

$$\int_\theta f(\theta) = \int d\theta_1...d\theta_N\, f(\theta_1, ..., \theta_N) = f_{N...1}\,. \tag{B.39}$$

The property (B.39) can be seen as a manifestation of the following anti-commutation rules for the differentials:

$$d\theta_i d\theta_j + d\theta_j d\theta_i = 0 \qquad\qquad d\theta_i\theta_j + \theta_j d\theta_i = 0\,, \tag{B.40}$$

together with the basic integration rule:

$$\int d\theta = 0 \qquad\qquad \int d\theta\,\theta = 1\,. \tag{B.41}$$

The integral over Grassmann variables so defined is called "Berezin integral". The integral is translation invariant:

$$\int d(\theta+\xi)\,f(\theta) = \int d\theta\,f(\theta)\,, \tag{B.42}$$

and has the following property under rescaling:

$$1 = \int d\theta'\,\theta' = \int d(c\theta)\,c\theta = c\int d(c\theta)\,\theta \qquad \Rightarrow \qquad d(c\theta) = \frac{1}{c}d\theta\,. \tag{B.43}$$

Employing the rules (B.38) and (B.41) we can calculate the basic Grassmann Gaussian integral:

$$\int d\bar{\theta}d\theta\,e^{-a\bar{\theta}\theta} = \int d\bar{\theta}d\theta(1-a\bar{\theta}\theta) = -a\int d\bar{\theta}d\theta\bar{\theta}\theta = a\,. \tag{B.44}$$

Note that the fermionic Gaussian integral (B.44) is linear in a while the bosonic Gaussian integral (B.34) goes as $\frac{\pi}{a}$. The general fermionic Gaussian action reads:

$$S(\bar{\theta},\theta) = \bar{\theta}_i M_{ij}\theta_j\,, \tag{B.45}$$

where summation over repeated indices is understood. The partition function is defined by

$$Z = \int d\bar{\theta}d\theta\,e^{-\bar{\theta}_i M_{ij}\theta_j}\,, \tag{B.46}$$

where the measure is defined as $d\bar{\theta}d\theta = d\bar{\theta}_1...d\bar{\theta}_N d\theta_1...d\theta_N$. We can evaluate (B.46) in the following way:

$$\begin{aligned} Z &= \frac{(-1)^N}{N!}\int d\bar{\theta}d\theta\,\bar{\theta}_{i_1}M_{i_1j_1}\theta_{j_1}...\bar{\theta}_{i_N}M_{i_Nj_N}\theta_{j_N} \\ &= \frac{(-1)^N}{N!}\underbrace{\int d\bar{\theta}d\theta\,\bar{\theta}_{i_1}\theta_{j_1}...\bar{\theta}_{i_N}\theta_{j_N}}_{=(-1)^N\epsilon_{i_1...i_N}\epsilon_{j_1...j_N}}M_{i_1j_1}...M_{i_Nj_N} \\ &= \frac{1}{N!}\epsilon_{i_1...i_N}\epsilon_{j_1...j_N}M_{i_1j_1}...M_{i_Nj_N} \\ &= \det M\,. \end{aligned} \tag{B.47}$$

We can now calculate the generating functional of fermionic Gaussian correlations with the help of the shifts $\theta \to \theta + M^{-1}\eta$ and $\bar{\theta} \to \bar{\theta} + \bar{\eta} M^{-1}$, we find:

$$Z(\bar{\eta}, \eta) = \int d\bar{\theta} d\theta \, e^{-\bar{\theta}_i M_{ij} \theta_j + \bar{\eta}_i \theta_i + \bar{\theta}_i \eta_i} = Z(0,0) \, e^{\bar{\eta}_i M_{ij}^{-1} \eta_j} \,, \tag{B.48}$$

where again $Z(0,0) = Z$. From (B.48) we can derive Wick's theorem for fermionic Gaussian variables as we did before for the bosonic case.

In the general case, where both bosonic and fermionic fields are present, we can combine them in a super-field multiplet $\Phi = (\phi, \theta)$. We can define the super-determinant as follows:

$$(\mathrm{sdet}\, M)^{-1} = \frac{1}{(2\pi)^N} \int d\bar{\Phi} d\Phi \, e^{-\bar{\Phi}^T M \Phi} \,, \tag{B.49}$$

where the factor in front is a convenient normalization and where M is a super-matrix of the form:

$$M = \begin{pmatrix} M_{BB} & M_{BF} \\ M_{FB} & M_{FF} \end{pmatrix} . \tag{B.50}$$

To calculate the integral, we seek for a change of variables that diagonalize the quadratic form:

$$\begin{aligned} \bar{\Phi}^T M \Phi &= \left(\bar{\phi}^T, \bar{\theta}^T \right) \begin{pmatrix} M_{BB} & M_{BF} \\ M_{FB} & M_{FF} \end{pmatrix} \begin{pmatrix} \phi \\ \theta \end{pmatrix} \\ &= \bar{\phi}^T M_{BB} \phi + \bar{\phi}^T M_{BF} \theta + \bar{\theta}^T M_{FB} \phi + \bar{\theta}^T M_{FF} \theta \,. \end{aligned} \tag{B.51}$$

If we shift $\phi \to \phi + X\theta$ and $\bar{\phi}^T \to \bar{\phi}^T + \bar{\theta}^T \bar{X}$ in (B.51) we get:

$$\begin{aligned} \bar{\Phi}^T M \Phi &= \bar{\phi}^T M_{BB} \phi + \bar{\phi}^T \left(M_{BB} X + M_{BF} \right) \theta + \bar{\theta}^T \left(\bar{X} M_{BB} + M_{FB} \right) \phi + \\ &\quad + \bar{\theta}^T \left(M_{FF} + \bar{X} M_{BB} X + \bar{X} M_{BF} + M_{FC} X \right) \theta \,. \end{aligned} \tag{B.52}$$

To diagonalize the quadratic form (B.51) we impose $M_{BB} X + M_{BF} = 0$ and $\bar{X} M_{BB} + M_{FB} = 0$. This gives $X = -M_{BB}^{-1} M_{BF}$ and $\bar{X} = -M_{FB} M_{BB}^{-1}$. Inserting these relation back in (B.52) finally gives:

$$\bar{\Phi}^T M \Phi = \bar{\phi}^T M_{BB} \phi + \bar{\theta}^T \left(M_{FF} - M_{FB} M_{BB}^{-1} M_{BF} \right) \theta \,. \tag{B.53}$$

We can now compute the mixed Gaussian integral (B.49). We find the following value for the super-determinant:

$$(\mathrm{sdet}\, M)^{-1} = \frac{\det^{1/2} M_{BB}}{\det\left(M_{FF} - M_{FB} M_{BB}^{-1} M_{BF}\right)} \tag{B.54}$$

We had instead made the variable change on the fermionic variables, we would had found the following equivalent expression for the super-determinant:

$$(\mathrm{sdet}\, M)^{-1} = \frac{\det^{1/2}\left(M_{BB} - M_{BF} M_{FF}^{-1} M_{FB}\right)}{\det M_{FF}} \,. \tag{B.55}$$

We will use (B.54) and (B.55) to derive the explicit expression for the one-loop effective action for non-abelian gauge theories in Appendix C and for quantum gravity in Appendix D.

B.2 Perturbative expansion of the effective action

It is possible to calculate the effective action perturbatively in a loop expansion by the saddle point expansion. We start from the integro-differential equation (B.17) that the effective action satisfies by reintroducing $\hbar$ and shifting $\chi \to \sqrt{\hbar}\,\chi$. We find:

$$e^{-\frac{1}{\hbar}\Gamma[\varphi]} = \int D\chi\, e^{-\frac{1}{\hbar}S[\varphi+\sqrt{\hbar}\,\chi] + \frac{1}{\sqrt{\hbar}}\Gamma^{(1)}[\varphi]\chi} \qquad\qquad \langle\chi\rangle = 0\,. \tag{B.56}$$

Next, we expand the action in powers of the fluctuation (integrations are understood):

$$\begin{aligned} \frac{1}{\hbar}S[\varphi+\sqrt{\hbar}\,\chi] \;\; &= \;\; \frac{1}{\hbar}S[\varphi] + \frac{1}{\sqrt{\hbar}}S^{(1)}[\varphi]\chi + \frac{1}{2}S^{(2)}[\varphi]\chi\chi + \\ &\quad + \frac{\sqrt{\hbar}}{3!}S^{(3)}[\varphi]\chi\chi\chi + \frac{\hbar}{4!}S^{(4)}[\varphi]\chi\chi\chi\chi + O\left(\hbar^{3/2}\chi^5\right) \,, \end{aligned} \tag{B.57}$$

and we expand the effective action in powers of $\hbar$ as follows:

$$\Gamma[\varphi] = S[\varphi] + \hbar\Gamma_1[\varphi] + \hbar^2\Gamma_2[\varphi] + O\left(\hbar^3\right) \,. \tag{B.58}$$

Inserting (B.57) and (B.58) back into (B.56) and expanding the rhs in powers of the fluctuation χ gives:

$$\begin{aligned} e^{-\Gamma_1[\varphi]-\hbar\Gamma_2[\varphi]+\dots} &= \int D\chi\, e^{-\frac{1}{2}S^{(2)}[\varphi]\chi\chi-\sqrt{\hbar}\left(\frac{1}{3!}S^{(3)}[\varphi]\chi\chi\chi-\Gamma_1^{(1)}[\varphi]\chi\right)-\frac{\hbar}{4!}S^{(4)}[\varphi]\chi\chi\chi\chi+\dots} \\ &= \int D\chi\, e^{-\frac{1}{2}S^{(2)}\chi\chi}\left[1-\sqrt{\hbar}\left(\frac{1}{3!}S^{(3)}[\varphi]\chi\chi\chi-\Gamma_1^{(1)}[\varphi]\chi\right)+\right. \\ &\left.+\frac{\hbar}{2}\left(\frac{1}{3!}S^{(3)}[\varphi]\chi\chi\chi-\Gamma_1^{(1)}[\varphi]\chi\right)^2-\frac{\hbar}{4!}S^{(4)}[\varphi]\chi\chi\chi\chi\right] \\ &+O\left(\hbar^{3/2}\chi^5\right)\,. \end{aligned} \tag{B.59}$$

The integrals in (B.59) are now all Gaussian and can be thus performed. Remembering that $\langle\chi\rangle = 0$, only even correlations in (B.59) are non-zero. We can extract various loop contributions to the effective action by equating, on both sides of (B.59), those terms of the same order in $\hbar$. The one-loop contribution is simply:

$$e^{-\Gamma_1[\varphi]} = \int D\chi\, e^{-\frac{1}{2}S^{(2)}\chi\chi} = \left(\det S^{(2)}[\varphi]\right)^{-\frac{1}{2}}\,, \tag{B.60}$$

or, using the relation $\log\det S^{(2)}[\varphi] = \mathrm{Tr}\,\log S^{(2)}[\varphi]$,

$$\Gamma_1[\varphi] = \frac{1}{2}\mathrm{Tr}\log S^{(2)}[\varphi]\,. \tag{B.61}$$

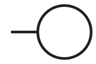
In terms of the following Gaussian correlation functions for the field χ,

$$\langle\chi_1 \dots \chi_n\rangle = \frac{\int D\chi\, e^{-\frac{1}{2}S^{(2)}\chi\chi}\,\chi_1\dots\chi_n}{\int D\chi\, e^{-\frac{1}{2}S^{(2)}\chi\chi}}\,,$$

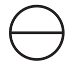
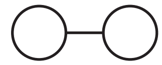
the two-loop contribution can be written as:

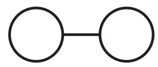
$$\begin{aligned} \Gamma_2[\varphi] &= -\frac{1}{2}\left(\frac{1}{3!}\right)^2 S^{(3)}_{123}[\varphi]S^{(3)}_{456}[\varphi]\,\langle\chi_1\chi_2\chi_3\chi_4\chi_5\chi_6\rangle + \\ &+\frac{1}{3!}S^{(3)}_{123}[\varphi]\Gamma^{(1)}_{1,4}[\varphi]\,\langle\chi_1\chi_2\chi_3\chi_4\rangle - \frac{1}{2}\Gamma^{(1)}_{1,1}[\varphi]\Gamma^{(1)}_{1,2}[\varphi]\,\langle\chi_1\chi_2\rangle + \\ &+\frac{1}{4!}S^{(4)}_{1234}[\varphi]\,\langle\chi_1\chi_2\chi_3\chi_4\rangle\,. \end{aligned} \tag{B.62}$$

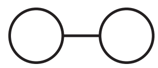
Note that the two-loop contribution in (B.60) involves the first functional derivative of the one-loop contribution. Differentiating equation (B.59) once with respect to φ gives:

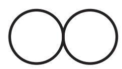
$$\Gamma^{(1)}_{1,1}[\varphi] = \tfrac{1}{2} S^{(3)}_{123}[\varphi] G_{23}[\varphi] =$$ 

where we defined the field dependent propagator $G_{12}[\varphi] = S^{(2)}_{12}[\varphi]^{-1} = \langle \chi_1 \chi_2 \rangle$. Using Wick's theorem (B.52) to reduce the Gaussian correlations in (B.62), we find, in obvious diagrammatic notation, the following contributions:

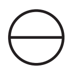
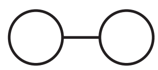
$$S^{(3)}_{123}[\varphi] S^{(3)}_{456}[\varphi] \langle \chi_1 \chi_2 \chi_3 \chi_4 \chi_5 \chi_6 \rangle = 6$$  $+9$ 

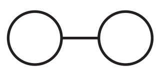
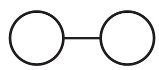
$$S^{(3)}_{123}[\varphi] \Gamma^{(1)}_{1,4}[\varphi] \langle \chi_1 \chi_2 \chi_3 \chi_4 \rangle = \tfrac{3}{2}$$ 

$$\Gamma^{(1)}_{1,1} \Gamma^{(1)}_{1,2}[\varphi] G_{12}[\varphi] = \tfrac{1}{4}$$ 

$$S^{(4)}_{1234}[\varphi] \langle \chi_1 \chi_2 \chi_3 \chi_4 \rangle = 3$$ 

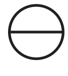
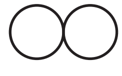
Inserting these terms in (B.62) gives the two-loop contribution in the form:

$$\Gamma_2[\varphi] \ = \ -\tfrac{1}{2} \left(\tfrac{1}{3!}\right)^2 \Big[\, 6$$  $+9$  $\Big] +$

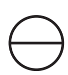
$$+ \tfrac{1}{3!} \tfrac{3}{2}$$  $- \tfrac{1}{2} \tfrac{1}{4}$  $+$

$$+ \tfrac{1}{4!} 3$$ 

By simplifying this expression, we see that only one-particle irreducible diagrams are left at the end:

$$\Gamma_2[\varphi] \ = \ -\tfrac{1}{12}$$  $+ \tfrac{1}{8}$ 

Finally, the effective action to order $\hbar^2$ is thus given by the following formula:

$$\Gamma[\varphi] \ = \ S[\varphi] + \tfrac{\hbar}{2} \mathrm{Tr} \log S^{(2)}[\varphi] - \tfrac{\hbar^2}{12}$$  $+ \tfrac{\hbar^2}{8}$  $+ O(\hbar^3)\,.$

Three loop terms, or higher, can be deduced by drawing diagrams and by figuring out the appropriate coefficients.

In general, we can calculate the loop contributions to the effective action by employing the heat kernel expansion. We just need to use the relation $\frac{1}{x} = \int_0^\infty ds\, e^{-sx}$ so that we can relate the propagator to the heat kernel of the operator $S^{(2)}[\varphi]$ as follows:

$$G_{xy}[\varphi] = \int_0^\infty ds\, e^{-sS^{(2)}_{xy}[\varphi]} = \int_0^\infty ds\, K^s_{xy}[\varphi]\,. \tag{B.63}$$

In this way we can evaluate the one-loop contribution using the non-local expansion for the heat kernel trace, while we can evaluate all higher contributions employing the expansion for the un-traced heat kernel both described in Appendix A.

B.3 Ward-Takahashi identities

Consider a symmetry of the action and of the measure such that for $\phi \to \phi + \delta\phi$ we have:

$$S[\phi + \delta\phi] = S[\phi] \qquad\qquad D(\phi+\delta\phi) = D\phi\,. \tag{B.64}$$

For a general operator $\mathcal{O}[\phi]$, which is not invariant under the symmetry, we must have:

$$\begin{aligned} \langle\mathcal{O}[\phi]\rangle &= \int D\phi\,\mathcal{O}[\phi]e^{-S[\phi]} \\ &= \int D(\phi+\delta\phi)\,\mathcal{O}[\phi+\delta\phi]\,e^{-S[\phi+\delta\phi]} \\ &= \int D\phi\,\mathcal{O}[\phi+\delta\phi]\,e^{-S[\phi]} \\ &= \int D\phi\,\left(\mathcal{O}[\phi]+\delta\mathcal{O}[\phi]\right)\,e^{-S[\phi]} \\ &= \langle\mathcal{O}[\phi]\rangle + \langle\delta\mathcal{O}[\phi]\rangle\,, \end{aligned}$$

so we find:

$$\langle\delta\mathcal{O}[\phi]\rangle = 0\,. \tag{B.65}$$

Equation (B.65) is the basic Ward-Takahashi (WT) identity. To rephrase the WT identity in terms of the generating functionals we remember that $Z[J] = Z[0]\left\langle e^{\int J\phi}\right\rangle$. We have to insert in (B.65) the operator $\mathcal{O}[\phi] = e^{\int J\phi}$, who's variation is $\delta e^{\int J\phi} = e^{\int J\phi} \int J\delta\phi$, to find:

$$\int d^d x J(x) \langle \delta\phi(x) \rangle_J = 0 \,. \tag{B.66}$$

In terms of the effective action we can rewrite (B.66) as follows:

$$\int d^d x \, \langle \delta\phi(x) \rangle \frac{\delta \Gamma[\varphi]}{\delta\varphi(x)} = 0 \,. \tag{B.67}$$

Equation (B.67) is the WT identity for the effective action. If $\delta\phi$ is linear in the fields, then $\langle \delta\phi \rangle = \delta\varphi$, and the lhs of equation (B.67) becomes just:

$$\int d^d x \, \langle \delta\phi(x) \rangle \frac{\delta \Gamma[\varphi]}{\delta\varphi(x)} = \int d^d x \delta\varphi(x) \frac{\delta \Gamma[\varphi]}{\delta\varphi(x)} = \delta\Gamma[\varphi].$$

In this case the effective action satisfies the same symmetries of the bare action:

$$\delta\Gamma[\varphi] = 0 \,. \tag{B.68}$$

In the case in which the bare action is not invariant under the symmetry $\phi \to \phi + \delta\phi$ the WT identity (B.67) becomes:

$$\int d^d x \, \langle \delta\phi(x) \rangle \frac{\delta \Gamma[\varphi]}{\delta\varphi(x)} = \langle \delta S[\phi] \rangle \,. \tag{B.69}$$

Equation (B.69) represents the general WT identity. It will be used in Chapter 3 and 4 to derive the modified WT identities that the EAA for non-abelian gauge theories and quantum gravity satisfy.

When $\delta\phi$ is not linear in the fields but the symmetry is nilpotent $\delta^2 = 0$, as in the case of BRST symmetry, it is useful to introduce additional currents K, the BRST currents, that couple to $\delta\phi$. Note that $\delta\phi$ is now a composite operator. The generator functional of connected correlation functions becomes now:

$$e^{W[J,K]} = \int D\phi \, e^{-S[\phi] + \int J\phi + \int K\delta\phi} \,, \tag{B.70}$$

with

$$\frac{\delta W[J,K]}{\delta K(x)} = \langle \delta\phi(x) \rangle \,. \tag{B.71}$$

The modified bare action $S[\phi] + \int K\delta\phi$ is still invariant under the symmetry transformation because of the nilpotency of δ. Thus the modified effective action:

$$\Gamma[\varphi, K] = \int d^d x \, J_\varphi(x)\varphi(x) - W[J, K]\,, \tag{B.72}$$

where K is only a spectator in the Legendre transform, satisfies the WT identity:

$$\int \langle \delta\phi(x) \rangle \, \frac{\delta \Gamma[\varphi, K]}{\delta \varphi(x)} = 0\,. \tag{B.73}$$

Considering that

$$\frac{\delta \Gamma[\varphi, K]}{\delta K(x)} = -\frac{\delta W[J, K]}{\delta K(x)}\,, \tag{B.74}$$

the WT identity (B.73) can be recast in the compact form:

$$\int d^d x \frac{\delta \Gamma[\varphi, K]}{\delta K(x)} \frac{\delta \Gamma[\varphi, K]}{\delta \varphi(x)} = 0\,. \tag{B.75}$$

Equation (B.75) is the Zinn-Justin (ZJ) equation. If the bare action is not invariant under the symmetry transformation we get instead:

$$\int d^d x \frac{\delta \Gamma[\varphi, K]}{\delta K(x)} \frac{\delta \Gamma[\varphi, K]}{\delta \varphi(x)} = \langle \delta S[\phi] \rangle \,. \tag{B.76}$$

We will use the general form of the ZJ equation (B.76) in Chapter 3 to derive the modified ZJ equation valid for the EAA of non-abelian gauge theories.

Appendix C

Basic non-abelian gauge theory

In this appendix we review the basic material needed to study non-abelian gauge theories in Chapter 3. For a general reference see the textbooks [11, 12, 9].

C.1 Definitions

Given a Lie group G we pick up a representation such that the group elements are represented by matrices of the form:

$$R = e^{-\theta}\,. \tag{C.1}$$

Here $\theta = -it^a\theta^a$ are the group parameters, the indices $a, b, ...$ run from one to $\dim G$, and the matrices t^a are the generators of the Lie algebra of G in the given representation. The generators satisfy the following commutation relations:

$$\left[t^a, t^b\right] = if^{abc}t^c\,. \tag{C.2}$$

The structure constants f^{abc} in a general representation are antisymmetric in the first two indices $f^{abc} = -f^{bac}$. The generators satisfy also the Jacoby identity:

$$\left[\left[t^a, t^b\right], t^c\right] + \left[\left[t^b, t^c\right], t^a\right] + \left[\left[t^c, t^a\right], t^b\right] = 0\,. \tag{C.3}$$

We can rewrite the Jacoby identity (C.3) in terms of the structure constants as follows:

$$f^{cal}f^{cbk} + f^{cak}f^{clb} + f^{cab}f^{ckl} = 0\,. \tag{C.4}$$

Matter fields ϕ are sections of the fiber bundle $\mathcal{B}$ with base the spacetime manifold $\mathcal{M}$ and fiber the gauge group G. Under a gauge transformation (or under a change of local trivialization), they transform as $\phi' = R\,\phi$. Covariant differentiation is introduced asking the transformation property $(D_\mu\phi)' = R\,D_\mu\phi$ (or more generally $D'_\mu = RD_\mu R^{-1}$) to hold, together with the definition:

$$D_\mu = \partial_\mu + A_\mu\,, \tag{C.5}$$

where $A_\mu = -iA^a_\mu t^a$ is the Lie algebra valued gauge connection. Thus we have:

$$\begin{aligned}(D_\mu\phi)' &= (\partial_\mu + A'_\mu)R\,\phi \\ &= (\partial_\mu R)\phi + R\,\partial_\mu\phi + A'_\mu R\,\phi \\ &= R\,\partial_\mu\phi + R\,A_\mu\phi\,,\end{aligned}$$

and so the gauge field must transforms as:

$$A'_\mu = R\,A_\mu\,R^{-1} - (\partial_\mu R)R^{-1}\,. \tag{C.6}$$

Note that we can rewrite the inhomogeneous term in (C.6) using the relation:

$$RR^{-1} = 1 \;\Rightarrow\; (\partial_\mu R)R^{-1} = -R\partial_\mu R^{-1}\,.$$

The gauge field strength $F_{\mu\nu}$ is the curvature of the gauge connection. It can be defined as the commutator of covariant derivatives acting on matter fields:

$$[D_\mu, D_\nu]\,\phi = F_{\mu\nu}\phi\,. \tag{C.7}$$

To find the explicit form of the field strength we just need few steps:

$$\begin{aligned}[D_\mu, D_\nu]\,\phi &= (\partial_\mu + A_\mu)(\partial_\nu + A_\nu)\phi - (\mu \leftrightarrow \nu) \\ &= \partial_\mu\partial_\nu\phi + A_\nu\partial_\mu\phi + \partial_\mu A_\nu\phi + A_\mu\partial_\nu\phi + A_\mu A_\nu - (\mu \leftrightarrow \nu) \\ &= (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])\,\phi\,,\end{aligned}$$

so we find

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = D_\mu A_\nu - D_\nu A_\mu\,. \tag{C.8}$$

The commutator term in (C.8) will vanish in the case we are considering an abelian gauge group, in this case we recover the the field strength of electrodynamics.

In components, the covariant derivative (C.5), reads

$$D_\mu^{ab} = \delta^{ab}\partial_\mu + f^{abc}A_\mu^c\,. \tag{C.9}$$

In components we have

$$[A_\mu, A_\nu] = (-i)^2 A_\mu^b A_\nu^c \left[t^b, t^c\right] = -A_\mu^b A_\nu^c\, i f^{bca} t^a\,, \tag{C.10}$$

if we set $F_{\mu\nu} = -iF_{\mu\nu}^a t^a$ we get

$$F_{\mu\nu}^a \;=\; \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{bca}A_\mu^b A_\nu^c\,. \tag{C.11}$$

From the transformation properties of the connection it is possible to find out how the gauge field transforms:

$$\begin{aligned} F'_{\mu\nu} &= D_\mu A'_\nu - D'_\nu A'_\mu \\ &= R D_\mu R^{-1}\left(R\,A_\mu\,R^{-1} - (\partial_\mu R)R^{-1}\right) - (\mu \leftrightarrow \nu) \\ &= R\,(D_\mu A_\nu - D_\nu A_\mu)\,R^{-1}\,, \end{aligned}$$

that gives

$$F'_{\mu\nu} = R F_{\mu\nu} R^{-1}\,, \tag{C.12}$$

showing that the field strength transforms homogeneously. Using the transformation property (C.12) we see that we can construct the following gauge invariant combination:

$$\begin{aligned} \operatorname{tr} F'_{\mu\nu}F'^{\mu\nu} &= \operatorname{tr}\left(R\,F_{\mu\nu}\,R^{-1}R\,F^{\mu\nu}\,R^{-1}\right) \\ &= \operatorname{tr}\left(F_{\mu\nu}F^{\mu\nu}\,R^{-1}R\right) \\ &= \operatorname{tr} F_{\mu\nu}F^{\mu\nu}\,. \end{aligned} \tag{C.13}$$

The integral of the invariant (C.13) over spacetime is the classical action for non-abelian gauge theories:

$$S[A] = \frac{1}{2}\int d^dx \operatorname{tr} F_{\mu\nu}F^{\mu\nu} = \frac{1}{4}\int d^dx F_{\mu\nu}^a F^{a\mu\nu}\,. \tag{C.14}$$

In (C.14) we used the normalization of the generators $\mathrm{tr} t^a t^b = \frac{1}{2}\delta^{ab}$. The field equations for non-abelian gauge theories are obtained by a variation of the action (C.14) with respect to the gauge field A_μ. We find:

$$
\begin{aligned}
\delta S &= \int d^d x \, \mathrm{tr}\, \delta F_{\mu\nu} F^{\mu\nu} \\
&= 2 \int d^d x \, \mathrm{tr}\left(\partial_\mu \delta A_\nu + A_\mu \delta A_\nu - \delta A_\nu A_\mu\right) F^{\mu\nu} \\
&= -2 \int d^d x \, \mathrm{tr}\left(\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}]\right) \delta A_\nu \,,
\end{aligned}
$$

the condition $\delta S = 0$ implies the following non-abelian or "Yang-Mills" equations of motion:

$$\partial_\mu F^{\mu\nu} + g\,[A_\mu, F^{\mu\nu}] = 0\,. \tag{C.15}$$

In (C.15) g is the non-abelian charge that is present since we have redefined the covariant derivative as $D_\mu = \partial_\mu + gA_\mu$. Equation (C.15) can be written as $D_\mu F^{\mu\nu} = 0$ if we introduce the covariant derivative acting on fields with one group index a:

$$D_\mu V = \partial_\mu V + [A_\mu, V]\ . \tag{C.16}$$

We can prove that (C.16) transforms homogeneously $D'_\mu V' = R D_\mu V R^{-1}$. Finally, from the Jacoby identity (C.3) it follows the following relation:

$$D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0\,, \tag{C.17}$$

which is called Bianchi identity.

For infinitesimal values of the gauge parameter, the group elements can be expanded as $R = e^{-\theta} \approx 1 - \theta$. Matter fields transform infinitesimally as:

$$\phi' = R\,\phi \approx \phi - \theta\phi \Rightarrow \delta\phi \ = \ -\theta\phi\,. \tag{C.18}$$

The gauge connection transforms infinitesimally as:

$$
\begin{aligned}
A'_\mu &= R\,A_\mu\,R^{-1} - (\partial_\mu R)\,R^{-1} \\
&\approx (1-\theta)A_\mu(1+\theta) + (\partial_\mu\theta)(1+\theta) \\
&\approx A_\mu - \theta A_\mu + A_\mu\theta + \partial_\mu\theta \\
&= A_\mu + \partial_\mu\theta + [A_\mu, \theta] \\
&\Rightarrow \delta A_\mu \ = \ D_\mu\theta\,,
\end{aligned} \tag{C.19}
$$

while the field strength transforms as:

$$\begin{aligned} F'_{\mu\nu} &= RF_{\mu\nu}R^{-1} \\ &\approx (1-\theta)F_{\mu\nu}(1+\theta) \\ &= F_{\mu\nu} - \theta F_{\mu\nu} + \theta F_{\mu\nu} \\ &\Rightarrow \delta F_{\mu\nu} \;=\; [F_{\mu\nu}, \theta] \; . \end{aligned} \tag{C.20}$$

In component form, we have the following relations

$$\delta\phi^i = i\theta^a (t^a_{\mathrm{r}})^{ij}\phi^j \tag{C.21}$$

$$\delta A^a_\mu \;=\; \partial_\mu \theta^a + f^{abc} A^b_\mu \theta^c \tag{C.22}$$

$$\delta F^a_{\mu\nu} \;=\; f^{abc} F^b_{\mu\nu}\theta^c \, , \tag{C.23}$$

where we used the relation $f^{abc} = i(t^c_{\mathrm{ad}})^{ab}$ valid in the adjoint representation.

C.2 Functional integral quantization of non-abelian gauge theories

In this section we quantize non-abelian gauge theories by functional methods. We first review the Faddeev-Popov quantization and the related BRST symmetry. Then we introduce the background field method and we construct the background effective action.

C.2.1 Faddeev-Popov and BRST symmetry

The partition function is defined by the functional integral:

$$Z = \int \frac{DA}{V_{\mathrm{gauge}}}\, e^{-S[A]} \, . \tag{C.24}$$

In (C.24) the action $S[A]$ and the measure DA are gauge invariant: $DA_\theta = DA$ and $S[A_\theta] = S[A]$ for any gauge transformation parametrized by θ. To omit over-counting due to gauge equivalent connections in (C.24) we divided out from the measure a factor of the volume of the gauge group V_{gauge}. The functional integral (C.24) is over the space of connections modulo gauge transformations. We follow the Faddeev-Popov method [Peskin, Weinberg] to factor out the volume of the gauge group. The first step is to insert in the partition function (C.24) the following identity:

$$1 = \int Df\,\delta[f] = \int D\theta\,\delta[f_\theta]\,\det\frac{\delta f_\theta}{\delta\theta}\,, \tag{C.25}$$

where $f^a(A) = 0$ is a given gauge-fixing condition. Inserting (C.25) in (C.24) and manipulating further gives:

$$\begin{aligned} Z &= \int \frac{DA}{V_{\text{gauge}}} D\theta\,\delta[f_\theta]\,\det\frac{\delta f_\theta}{\delta\theta} e^{-S[A]} \\ &= \frac{1}{V_{\text{gauge}}}\int DA_\theta D\theta\,\delta[f_\theta]\,\det\frac{\delta f_\theta}{\delta\theta} e^{-S[A_\theta]} \\ &= \frac{1}{V_{\text{gauge}}}\int D\theta \int DA\,\delta[f]\,\det\left.\frac{\delta f_\theta}{\delta\theta}\right|_{\theta=0} e^{-S[A]} \\ &= \int DA\,\delta[f]\,\det\left.\frac{\delta f_\theta}{\delta\theta}\right|_{\theta=0} e^{-S[A]}\,. \end{aligned} \tag{C.26}$$

We explain now step by step the manipulations done in (C.26): in the first step we used the gauge invariance of the action and of the me sure; in the second we shifted the integration variable from A_θ to A and we set $\theta = 0$ in the remaining Faddeev-Popov determinant; in the last step we isolated the integration over the gauge parameters θ and we identified it as the gauge volume $V_{\text{gauge}} = \int D\theta$. It is possible to repeat the same steps with a more general gauge fixing functional $B[f]$ in place of $\delta[f]$ and functionally Fourier transform with respect to a field b. At last, we write the Faddeev-Popov determinant using ghost fields $\bar{c}$ and c. Finally we get:

$$Z = \int DADbD\bar{c}Dc\, e^{-S_{BRST}[A,b,\bar{c},c]}\,, \tag{C.27}$$

where the final action is of the form:

$$S_{BRST}[A,b,\bar{c},c] = S[A] + \int d^dx\left(-\frac{\alpha}{2}b^ab^a + b^af^a + \bar{c}^a\frac{\delta f^a}{\delta\theta^b}c^b\right)\,. \tag{C.28}$$

We can eliminate the auxiliary field b^a by the relation

$$\frac{\delta S_{BRST}}{\delta b^a} = -\alpha b^a + f^a = 0 \qquad \Rightarrow \qquad b^a = \frac{1}{\alpha}f^a\,,$$

so that the first two terms in (C.28) give back the standard gauge-fixing term:

$$\int d^dx \left(-\frac{\alpha}{2} b^a b^a + b^a f^a\right) = \frac{1}{2\alpha} \int d^dx f^a f^a \,.$$

We have a new symmetry replacing the original gauge symmetry which is called BRST symmetry:

$$\begin{aligned} \delta_{BRST} A^a_\mu &= \epsilon D_\mu c^a \\ &= \epsilon \left(\partial_\mu c^a + f^{abc} A^b_\mu c^c\right) \\ \delta_{BRST} c^a &= -\frac{1}{2} \epsilon f^{abc} c^b c^c \\ \delta_{BRST} \bar{c}^a &= -\epsilon b^a \\ \delta_{BRST} b^a &= 0 \,, \end{aligned} \tag{C.29}$$

where ϵ is a Grassmann parameter. The BRST operator s, defined by $\delta_{BRST} = \epsilon s$, is nilpotent $s^2 = 0$. To prove this, we start by showing the effect it has on the gauge connection

$$\begin{aligned} s^2 A^a_\mu &= s\left(\partial_\mu c^a + f^{abc} A^b_\mu c^c\right) \\ &= \partial_\mu (s c^a) + f^{abc} s A^b_\mu c^c + f^{abc} A^b_\mu s c^c \\ &= -\frac{1}{2} f^{abc} \partial_\mu (c^b c^c) + f^{abc} \partial_\mu c^b c^c + \\ &\quad + f^{abc} f^{bkl} A^k_\mu c^l c^c - \frac{1}{2} f^{abc} f^{ckl} A^b_\mu c^k c^l \,. \end{aligned} \tag{C.30}$$

Due to the Grassmann nature of the ghost and the antisymmetry of the structure constants the first two terms in (C.31) cancel each other, the other terms can be manipulated until we factor one term that vanishes due to the Jacoby identity (C.4) in terms of the structure constants:

$$\begin{aligned} s^2 A^a_\mu &= \left(-f^{ack} f^{cbl} - \frac{1}{2} f^{abc} f^{ckl}\right) A^b_\mu c^k c^l \\ &= -\frac{1}{2} \left(f^{cal} f^{cbk} + f^{cak} f^{clb} + f^{cab} f^{ckl}\right) A^b_\mu c^k c^l \\ &= 0 \,. \end{aligned}$$

For the ghost we have:

$$\begin{aligned}
s^2 c^a &= -\frac{1}{2} f^{abc} s c^b c^c - \frac{1}{2} f^{abc} c^b s c^c \\
&= -\frac{1}{2} f^{abc} \left(-\frac{1}{2} f^{bkl} c^k c^l c^c - \frac{1}{2} f^{ckl} c^b c^k c^l \right) \\
&= \frac{1}{4} \left(f^{acb} f^{ckl} + f^{abc} f^{ckl} \right) c^b c^k c^l \\
&= \frac{1}{4} \left(-f^{abc} + f^{abc} \right) f^{ckl} c^b c^k c^l \\
&= 0 \,.
\end{aligned}$$

The action of the BRST operator is trivial on the other fields:

$$s^2 \bar{c}^a = -s b^a = 0 \qquad\qquad s^2 b^a = 0 \,.$$

To prove now that the action $S_{BRST}[A, \bar{c}, c, b]$ is BRST invariant we just have to note that:

$$\int d^d x \left(-\frac{\alpha}{2} b^a b^a + b^a f^a + \bar{c}^a \frac{\delta f^a}{\delta \theta^b} c^b \right) = s \int d^d x\, \bar{c}^b \left(-\frac{\alpha}{2} b^a + f^a \right) ,$$

and use the nilpotency of s. The BRST symmetry has thus replaced the original gauge symmetry.

C.2.2 Background field method

To construct a "gauge invariant" effective action we introduce another gauge field $\bar{A}_\mu$, the background field, which transforms under a background gauge transformation $\bar{\delta}$ as

$$\bar{\delta} \bar{A}_\mu = \bar{D}_\mu \theta \,, \tag{C.31}$$

where the covariant derivative in (C.31) is constructed using the background field $\bar{D}_\mu = \partial_\mu + g\bar{A}_\mu$. Under a gauge transformation the background field transforms as $\delta \bar{A}_\mu = 0$. If we consider the combination $a_\mu = A_\mu - \bar{A}_\mu$, which is called the fluctuation field, we see that under a combined physical and background gauge transformation it transforms homogeneously:

$$(\delta + \bar{\delta}) a_\mu = D_\mu \theta - \bar{D}_\mu \theta = [A_\mu, \theta] - \left[\bar{A}_\mu, \theta \right] = [a_\mu, \theta] \ . \tag{C.32}$$

We can now use the fluctuation field to construct a covariant contraction with the current J^μ under this combined transformation. The background gauge-fixing condition is defined as follows:

$$f = \bar{D}_\mu a^\mu \,, \tag{C.33}$$

and so the gauge-fixing action becomes:

$$S_{gf}[a;\bar{A}] = \frac{1}{2\alpha}\int d^dx\, f_\mu f^\mu = \frac{1}{2\alpha}\int d^dx \bar{D}_\mu a^\mu \bar{D}_\nu a^\nu \,. \tag{C.34}$$

The Faddeev-Popov operator is obtained by varying (C.33) with respect to a physical gauge transformation:

$$f_\theta = \bar{D}_\mu (a^\mu + D^\mu\theta) \qquad \Rightarrow \qquad \left.\frac{\delta f_\theta}{\delta\theta}\right|_{\theta=0} = \bar{D}_\mu D^\mu \,,$$

giving the following ghost action:

$$S_{gh}[a,\bar{c},c;\bar{A}] = -\int d^dx\, \bar{c}\bar{D}_\mu D^\mu c = \int d^dx \bar{D}_\mu \bar{c} D^\mu c = \int d^dx \bar{D}_\mu \bar{c}\left(\bar{D}^\mu + g a^\mu\right) c \,. \tag{C.35}$$

Note that both the background gauge-fixing and background ghost actions are invariant under combined physical and background gauge transformations:

$$(\delta + \bar{\delta}) S_{gf}[a;\bar{A}] = 0 \qquad\qquad (\delta+\bar{\delta}) S_{gh}[a,\bar{c},c;\bar{A}] = 0 \,. \tag{C.36}$$

Thus the "background classical action"

$$S[\varphi;\bar{A}] = S[A] + S_{gf}[a;\bar{A}] + S_{gh}[a,\bar{c},c;\bar{A}] \,. \tag{C.37}$$

is also invariant:

$$(\delta+\bar{\delta}) S[\varphi;\bar{A}] = 0 \,. \tag{C.38}$$

The background effective action (bEA) is defined by the following integro-differential equation:

$$e^{-\Gamma[\varphi;\bar{A}]} = \int D\chi \,\exp\left\{-S[\varphi+\chi;\bar{A}] + \int d^dx\, \Gamma^{(1)}[\varphi;\bar{A}]\chi\right\} \,, \tag{C.39}$$

From (C.36) and (C.38) it follows the bEA, as defined in (C.39), is also invariant under combined physical and background gauge transformations:

$$(\delta+\bar{\delta})\Gamma[\varphi;\bar{A}] = 0 \,. \tag{C.40}$$

We can define the full quantum gauge field as:

$$A_\mu = \bar{A}_\mu + a_\mu \,. \tag{C.41}$$

The main reason to employ the background field method is that we can define a gauge invariant functional $\bar{\Gamma}[\bar{A}]$, that we call gauge invariant effective action (gEA), by setting $\varphi = 0$, or equivalently $A_\mu = \bar{A}_\mu$ and $\bar{c} = c = 0$, in the bEA (C.39):

$$\bar{\Gamma}[\bar{A}] = \Gamma[0; \bar{A}] \,. \tag{C.42}$$

The integro-differential equation satisfied by the gEA is just (C.39) for $\varphi = 0$:

$$e^{-\bar{\Gamma}[\bar{A}]} = \int D\chi \, \exp\left\{ -S[\chi; \bar{A}] + \int d^d x \, \Gamma^{(1)}[0; \bar{A}]\chi \right\} . \tag{C.43}$$

It is important to notice that this definition for the gEA is not closed since on the rhs of (C.43) there is the first functional derivative of the bEA and not of the gEA. Most importantly, the invariance (C.40) now becomes:

$$\delta \bar{\Gamma}[\bar{A}] = 0 \,. \tag{C.44}$$

Equation (C.44) is a fundamental result since it shows that a gauge invariant construction of the effective action is possible.

We consider now the perturbative expansion of the bEA and of the gEA. From Appendix B we have that $\Gamma_0[\varphi; \bar{A}] = S[\varphi; \bar{A}]$ and using (C.42) we have:

$$\bar{\Gamma}_0[\bar{A}] = S[\bar{A}] + \underbrace{S_{gf}[0; \bar{A}]}_{=0} + \underbrace{S_{gh}[0, 0, 0; \bar{A}]}_{=0} = S[\bar{A}] \,. \tag{C.45}$$

The one-loop contribution is given by (B.61):

$$\Gamma_1[\varphi; \bar{A}] = \frac{1}{2} \mathrm{Tr} \log S^{(2;0)}[\varphi; \bar{A}] \,, \tag{C.46}$$

The Hessian of (C.37) is

$$S^{(2;0)}[\varphi; \bar{A}] = \begin{pmatrix} M_{BB} & M_{BF} \\ M_{FB} & M_{FF} \end{pmatrix} , \tag{C.47}$$

where

$$
\begin{aligned}
M_{BB} &= S^{(2)}[\bar{A}+a] + S_{gf}^{(2,0)}[a;\bar{A}] + S_{gh}^{(2,0,0,0)}[a,\bar{c},c;\bar{A}] \\
M_{BF} &= S_{gh}^{(1,0,1,0)}[a,\bar{c},c;\bar{A}] \\
M_{FB} &= S_{gh}^{(1,1,0,0)}[a,\bar{c},c;\bar{A}] \\
M_{FF} &= S_{gh}^{(0,1,1,0)}[a,\bar{c},c;\bar{A}]\,.
\end{aligned}
\tag{C.48}
$$

Using the properties of the super-determinant and super-trace (B.54) we find:

$$
\mathrm{Tr}\log S^{(2;0)}[\varphi;\bar{A}] = \log\det\left(M_{FF} - M_{FB}M_{BB}^{-1}M_{BF}\right) - \frac{1}{2}\log\det\, M_{BB}\,,
$$

and thus:

$$
\begin{aligned}
\Gamma_1[a,\bar{c},c;\bar{A}] = \frac{1}{2}\mathrm{Tr}\log\Big(S^{(2)}[\bar{A}+a] + S_{gf}^{(2,0)}[a;\bar{A}] + S_{gh}^{(2,0,0,0)}[a,\bar{c},c;\bar{A}]\Big) \\
-\mathrm{Tr}\log\left(S_{gh}^{(0,1,1,0)}[a,\bar{c},c;\bar{A}] - \frac{S_{gh}^{(1,1,0,0)}[a,\bar{c},c;\bar{A}]S_{gh}^{(1,0,1,0)}[a,\bar{c},c;\bar{A}]}{S^{(2)}[a+\bar{A}] + S_{gf}^{(2,0)}[a;\bar{A}] + S_{gh}^{(2,0)}[a,\bar{c},c;\bar{A}]}\right)\,.
\end{aligned}
\tag{C.49}
$$

The gEA at one-loop is found from (C.49) from $\varphi = 0$ and reads:

$$
\begin{aligned}
\bar{\Gamma}_1[\bar{A}] &= \Gamma_1[0,0,0;\bar{A}] \\
&= \frac{1}{2}\mathrm{Tr}\log\Big(S^{(2)}[\bar{A}] + S_{gf}^{(2,0)}[0;\bar{A}] + S_{gh}^{(2,0,0,0)}[0,0,0;\bar{A}]\Big) \\
&\quad -\mathrm{Tr}\log\, S_{gh}^{(0,1,1,0)}[0,0,0;\bar{A}]\,.
\end{aligned}
\tag{C.50}
$$

The elaborate construction of the bEA ensures that all contributions steaming from the functional traces in the lhs of (C.50) are gauge invariant. We will use (C.50) in Chapter 3 to obtain the flow equation for the gEAA as the RG improvement of the one-loop flow.

Appendix D

Basic gravity

In the first part of this appendix we review some basic notions about differential geometry which are needed to study both classical gravity, i.e. general relativity, and quantum gravity. In the second part we introduce the functional integral quantization of gravity, in particular we employ the background field method to construct a diffeomorphism invariant effective action.

D.1 Differential geometry

In this section we review basics facts about differential geometry of manifolds. We assume that the reader is familiar with the fundamental notions about manifolds and suggest [130] for details.

D.1.0.1 Differential structures

In a coordinate basis ∂_μ at the point p of $\mathcal{M}$, a vector is the differential operator $V = V^\mu \partial_\mu$ belonging to the tangent space $T\mathcal{M}_p$. A one-form is an element of the cotangent space $T\mathcal{M}_p^*$, the vector space dual to $T\mathcal{M}_p$. In terms of the dual basis dx^μ, defined by the relation $dx^\mu(\partial_\nu) = \delta^\mu_\nu$, we have $\omega = \omega_\mu dx^\mu$. A one-form field is an element of $\Omega^1(\mathcal{M})$ and a function is an element of $\Omega^0(\mathcal{M})$. The action of a one-form on a vector is given by the following:

$$\omega(V) = \omega_\mu dx^\mu(V^\nu \partial_\nu) = \omega_\mu V^\nu dx^\mu(\partial_\nu) = \omega_\mu V^\mu \,, \tag{D.1}$$

while the action of a vector field on a function is defined as $V\{f\} = V^\mu \partial_\mu f$. The "external differential" of a function is defined to be the one-form $df = \partial_\mu f\, dx^\mu$. We have thus:

$$V\{f\} = df(V)\,. \tag{D.2}$$

The Lie bracket of two vector fields V and W is defined as the commutator vector field:

$$\begin{aligned} [V,W]f &= V\{W\{f\}\} - W\{V\{f\}\} \\ &= V^\mu \partial_\mu (W^\nu \partial_\nu f) - W^\mu \partial_\mu (V^\nu \partial_\nu f) \\ &= (V^\mu \partial_\mu W^\nu - W^\mu \partial_\mu V^\nu)\partial_\nu f\,, \end{aligned} \tag{D.3}$$

thus we have $[V,W] = (V^\mu \partial_\mu W^\nu - W^\mu \partial_\mu V^\nu)\partial_\nu$. In particular we have $[\partial_\mu, \partial_\nu] = 0$ for any coordinate basis. We can prove the Jacoby identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0\,. \tag{D.4}$$

We can define the Lie derivative of the vector field W along W as $\mathcal{L}_V W = [V,W]$. We can easily prove the following properties:

$$\begin{aligned} \mathcal{L}_V W &= -\mathcal{L}_W V \\ \mathcal{L}_V [W,U] &= [\mathcal{L}_V W, U] + [W, \mathcal{L}_V U] \\ \mathcal{L}_{[V,W]} U &= \mathcal{L}_V \mathcal{L}_W U - \mathcal{L}_W \mathcal{L}_V U \\ \mathcal{L}_V (fW) &= V\{f\}W + f\mathcal{L}_V W\,. \end{aligned} \tag{D.5}$$

A differential form of degree k is an element of $\Omega^k(\mathcal{M})$ and can be constructed from one-forms using the wedge product. This is defined as the anti-symmetrization of the tensor product of two forms:

$$\omega_k \wedge \eta_l = \frac{(k+l)!}{k!l!}\mathrm{Alt}(\omega_k \otimes \eta_l)\,. \tag{D.6}$$

The wedge product satisfies the following properties:

$$\begin{aligned} \omega \wedge (\eta \wedge \xi) &= \omega \wedge (\eta \wedge \xi) \\ \omega_k \wedge \eta_l &= (-1)^{kl} \eta_l \wedge \omega_k\,. \end{aligned} \tag{D.7}$$

The "exterior differential" of a form is the map $d : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ defined by the relations:

$$\begin{aligned} df(V) &= V\{f\} \\ d(\omega_k \wedge \eta_l) &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \\ d^2 &= 0\,. \end{aligned} \tag{D.8}$$

Note that the exterior differential is nilpotent. In a coordinate chart we can write the following representation:

$$d\omega = d(\omega_{\mu_1 \dots \mu_k} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}) = \partial_\nu \omega_{\mu_1 \dots \mu_k} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}\,. \tag{D.9}$$

There exists an invariant formula to express the exterior differential. For the differential of a one-form this reads:

$$d\omega(X,Y) = X\left\{\omega(Y)\right\} - Y\left\{\omega(X)\right\} - \omega\left([X,Y]\right)\,, \tag{D.10}$$

while for the differential of a two-form we have:

$$\begin{aligned} d\omega(X,Y,Z) &= X\left\{\omega(Y,Z)\right\} - Y\left\{\omega(X,Z)\right\} + Z\{\omega(X,Y)\} + \\ &\quad -\omega\left([X,Y]\right) + \omega\left([X,Z]\right) - \omega\left([Y,Z]\right)\,, \end{aligned} \tag{D.11}$$

and so on. The Lie derivative can be defined to act on forms, we have to fundamental relations:

$$\mathcal{L}_V \omega = i_V(d\omega) + d(i_V \omega) \qquad\qquad \mathcal{L}_V d\omega = d(\mathcal{L}_V \omega)\,. \tag{D.12}$$

The first relation in (D.12) is called Cartan's formula. Forms are the natural objects to be integrated. A k-form has to be integrated over a k-dimensional (sub)manifold. The fundamental generalization of Stoke's theorem is the following:

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega\,, \tag{D.13}$$

where $\partial\mathcal{M}$ is the boundary of the manifold $\mathcal{M}$.

A metric tensor is a positive non-degenerate bilinear form g that acts on two vector fields X and Y as follows:

$$g(X,Y) = g_{\mu\nu} X^\mu Y^\nu\,. \tag{D.14}$$

The vector spaces $\Omega^p(\mathcal{M})$ and $\Omega^{d-p}(\mathcal{M})$ have the same dimension, with the aid of the metric tensor it is possible to construct an isomorphism between then by defining:

$$*dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} = \frac{1}{(d-p)!} g^{\mu_1\nu_1} ... g^{\mu_p\nu_p} \epsilon_{\nu_1...\nu_d} dx^{\mu_{p+1}} \wedge ... \wedge dx^{\mu_d}\,, \tag{D.15}$$

$*$ is the Hodge star operator and $\epsilon_{\mu\nu...}$ is the totally antisymmetric tensor. The dual a p-form is defined by:

$$*\omega_p = \frac{1}{p!} \omega_{\mu_1...\mu_p} * dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}\,. \tag{D.16}$$

In particular, we can define the volume d-form associated to the metric g by the relation:

$$dV = *1 = \frac{1}{d!} \epsilon_{\mu_1...\mu_d} dx^{\mu_1} \wedge ... \wedge dx^{\mu_d} = dx^1 \wedge ... \wedge dx^d\,. \tag{D.17}$$

It is easy to prove the following properties of the Hodge star operator:

$$*^2 = \begin{cases} (-1)^{p(d-p)} & \text{Euclidean} \\ (-1)^{p(d-p)+1} & \text{Lorentzian} \end{cases} \tag{D.18}$$

and

$$*^{-1} = \begin{cases} (-1)^{p(d-p)} * & \text{Euclidean} \\ (-1)^{p(d-p)+1} * & \text{Lorentzian} \end{cases}\,. \tag{D.19}$$

We can define an inner product between two p-forms by:

$$(\alpha_p, \beta_p) = \int_{\mathcal{M}} \alpha \wedge *\beta = \frac{1}{d!} \int_{\mathcal{M}} \alpha_{\mu_1...\mu_p} \beta^{\mu_1...\mu_p} = (\beta_p, \alpha_p)\,. \tag{D.20}$$

On an Euclidean manifold we always have $(\alpha, \alpha) \geq 0$, the equality is obtained only if $\alpha = 0$ and so the inner product is non-degenerate. On a closed manifold ($\partial\mathcal{M} = 0$) we can define the co-exterior differential as the adjoint operator to the external differential by the following relations:

$$\begin{aligned} (d\alpha_{p-1}, \beta_p) &= \int d\alpha \wedge *\beta \\ &= \underbrace{\int d(\alpha \wedge \beta)}_{=0} + (-1)^p \int \alpha \wedge d * \beta \\ &= \int \alpha \wedge * \left[(-1)^p *^{-1} d * \beta\right] \\ &= (\alpha_{p-1}, \delta\beta_p)\,. \end{aligned} \tag{D.21}$$

From (D.21) we see that the co-exterior differential is maps p-forms to $(p-1)$-forms $\delta_p : \Omega^p(\mathcal{M}) \to \Omega^{p-1}(\mathcal{M})$ and that can be expressed in terms of the exterior differential and the Hodge star operator as:

$$\begin{aligned} \delta_p &= (-1)^p *^{-1} d * \\ &= (-1)^p (-1)^{(d-p+1)(d-(d-p+1))} * d * \\ &= (-1)^{d(p+1)+1} * d_{d-p} * \, . \end{aligned} \tag{D.22}$$

The co-exterior differential is nilpotent as is the exterior differential: $\delta^2 \sim *d**d* \sim *d^2* = 0$. We can now define the Laplacian acting on a p-form $\Delta_p : \Omega^p(\mathcal{M}) \to \Omega^p(\mathcal{M})$ by the following composition:

$$\Delta_p = \delta_{p+1} d_p + d_{p-1}\delta_p = (d+\delta)^2 \, . \tag{D.23}$$

The Laplacian on forms satisfies:

$$*\Delta = \Delta * \qquad\qquad d\Delta = \Delta d \qquad\qquad \delta\Delta = \delta\Delta \, . \tag{D.24}$$

On Euclidean manifolds the Laplacian is positive:

$$(\omega, \Delta\omega) = (\omega, \delta d\omega) + (\omega, d\delta\omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega) \geq 0 \tag{D.25}$$

and a form is harmonic only and only if it is both closed and co-closed: $\Delta\omega = 0 \quad \Leftrightarrow \quad d\omega = 0 \quad \text{and} \quad \delta\omega = 0$. The de Rham cohomology groups are defined as the quotients:

$$H^p_{dR}(\mathcal{M}) = \frac{\ker\, d_p}{\operatorname{Im} d_{p-1}} \, . \tag{D.26}$$

The fundamental theorem of de Rham states that the Euler characteristic of a manifold $\chi(\mathcal{M})$ can be written in terms of the dimensions of the de Rham cohomology groups as follows:

$$\chi(\mathcal{M}) = \sum_{p=0}^{d} (-1)^p \dim H^p_{dR}(\mathcal{M}) \, . \tag{D.27}$$

The fundamental theorem of Hodge states instead that in every cohomology class there is one and only one harmonic form:

$$\ker\, \Delta_p = H^p_{dR}(\mathcal{M}) \, . \tag{D.28}$$

We will use de Rham's and Hodge's theorem to prove the Chern-Gauss-Bonnet theorem in section D.1.0.4.

D.1.0.2 Connection and covariant derivative

Geodesic are those curves that have minimal length. The geodesic equation can be obtained has the solution of the following variational problem:

$$\delta \int ds = 0\,, \tag{D.29}$$

where the infinitesimal line element is given by $ds = g_{\mu\nu} dx^{\mu} dx^{v}$. We obtain the following equation geodesic equation:

$$\frac{d^2 x^{\gamma}}{ds^2} + \Gamma^{\gamma}_{\mu\beta} \frac{dx^{\mu}}{ds} \frac{dx^{\beta}}{ds} = 0\,, \tag{D.30}$$

where the Christoffel symbols are related to the metric tensor by the relation:

$$\Gamma^{\gamma}_{\mu\beta} = \frac{1}{2} g^{\alpha\gamma} \left(\partial_{\beta} g + \partial_{\mu} g_{\beta\alpha} - \partial_{\alpha} g_{\mu\beta} \right)\,. \tag{D.31}$$

Note that the Christoffel symbols do not transform as a tensor. We say that a vector V tangent to the curve parametrized by $x^{\mu}(s)$, with tangent vector filed $T^{\mu} = \frac{dx^{\mu}}{ds}$, is parallel transported if:

$$\frac{d}{ds} \left(g_{\mu\nu} T^{\mu} V^{\nu} \right) = 0 \qquad \Rightarrow \qquad \frac{dV^{\mu}}{ds} + \Gamma^{\mu}_{\alpha\beta} T^{\alpha} V^{\beta} = 0\,. \tag{D.32}$$

A geodesic is thus a curve that parallel transports its own tangent vectors. We can solve equation (D.32) as follows:

$$V^{\mu}(s) = P^{\mu}_{\ \nu}(s, s_0) V^{\nu}(s_0)\,, \tag{D.33}$$

where the parallel transport kernel is given by the following path-ordered expression:

$$P(\lambda, \lambda_0) = \mathcal{P} \exp \left\{ - \int_{\lambda_0}^{\lambda} d\lambda \, \Gamma(X) \right\}\,, \tag{D.34}$$

where $X^{\mu} = \frac{dx^{\mu}}{d\lambda}$ and $\Gamma(X)^{\alpha}_{\beta} \equiv \Gamma^{\alpha}_{\mu\beta} X^{\mu}$. We can define the covariant derivative along X^{μ}, of a vector field V^{μ}, as the following limit:

$$\nabla_X V = \lim_{\Delta\lambda \to 0} \frac{P^{-1}(\lambda + \Delta\lambda, \lambda) V(\lambda + \Delta\lambda) - V(\lambda)}{\Delta\lambda}\,. \tag{D.35}$$

From (D.34) we see that

$$P(\lambda+\Delta\lambda,\lambda) = 1 - \int_{\lambda}^{\lambda+\Delta\lambda} d\eta\,\Gamma(\eta) + O(\Delta\lambda^2) = 1 - \Delta\lambda\Gamma(\lambda) + O(\Delta\lambda^2)\,,$$

and thus we find the following relation:

$$\begin{aligned}
\nabla_X V &= \lim_{\Delta\lambda\to 0}\frac{1}{\Delta\lambda}\left[\left(1+\Delta\lambda\Gamma(\lambda)+O(\Delta\lambda^2)\right)V(\lambda+\Delta\lambda) - V(\lambda)\right] \\
&= \lim_{\Delta\lambda\to 0}\frac{V(\lambda+\Delta\lambda)-V(\lambda)}{\Delta\lambda} + \Gamma(\lambda)V(\lambda) \\
&= \frac{d}{d\lambda}V(\lambda) + \Gamma(\lambda)V(\lambda)\,.
\end{aligned} \tag{D.36}$$

In a coordinate chart we can write (D.36) in the following way:

$$\begin{aligned}
\nabla_X V &= \left(\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\alpha\beta}X^\alpha V^\beta\right)\partial_\mu \\
&= X^\alpha\left(\partial_\alpha V^\mu + \Gamma^\mu_{\alpha\beta}V^\beta\right)\partial_\mu \\
&= X^\alpha \nabla_\alpha V^\mu \partial_\mu\,,
\end{aligned}$$

so the components of the covariant derivative acting on vectors are given by the following expression:

$$\nabla_\alpha V^\mu = \partial_\alpha V^\mu + \Gamma^\mu_{\alpha\beta}V^\beta\,. \tag{D.37}$$

We say that a connection is compatible with the metric when $\nabla_X g = 0$ for every X or if $\nabla_Z g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$, i.e. if parallel transported vector fields have constant scalar product. On a Riemannian manifold $(\mathcal{M}, g)$ there exist a unique connection ∇ which is symmetric and compatible with the metric g. This is called Levi-Civita connection and the explicit coordinate representation is given by the Christoffel symbols (D.31).

D.1.0.3 Curvature

The Riemann curvature $(1,3)$ tensor measures the degree of non-commutativity of covariant derivatives. It can be defined in a coordinate invariant way as follows:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\,. \tag{D.38}$$

The torsion $(1,2)$ tensor is defined as:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]\,. \tag{D.39}$$

We still have to show that the Riemann tensor is actually so, using the properties of the covariant derivative and of vector fields we find:

$$\begin{aligned}
R(aX,bY)cZ &= \nabla_{aX}\nabla_{bY}(cZ) - \nabla_{bY}\nabla_{aX}(cZ) - \nabla_{[aX,bY]}(cZ) \\
&= a\nabla_X\left(bc\nabla_Y Z + bY\{c\}Z\right) - b\nabla_Y\left(ac\nabla_X Z + aX\{c\}Z\right) + \\
&\quad + \left(-aX\{b\}\nabla_Y + bY\{a\}\nabla_X - ab\nabla_{[X,T]}\right)(cZ) \\
&= aX\{bc\}\nabla_Y Z + abc\nabla_X\nabla_Y Z + aX\{bY\{c\}\}Z + abY\{c\}\nabla_X Z + \\
&\quad -bY\{ac\}\nabla_X Z - abc\nabla_Y\nabla_X Z - bY\{aX\{c\}\}Z - abX\{c\}\nabla_Y Z + \\
&\quad -aX\{b\}Y\{c\}Z - acX\{b\}\nabla_Y Z + bY\{a\}X\{c\}Z + bcY\{a\}\nabla_X Z + \\
&\quad -ab[X,Y]\{c\}Z - abc\nabla_{[X,T]}Z \\
&= abcR(X,Y)Z + \\
&\quad + \left(aX\{bc\} - abX\{c\} - acX\{b\}\right)\nabla_Y Z + \\
&\quad + \left(abY\{c\} - bY\{ac\} + bcY\{a\}\right)\nabla_X Z + \\
&\quad + \left(aX\{bY\{c\}\} - bY\{aX\{c\}\} + bY\{a\}X\{c\} + \right. \\
&\quad \left. -aX\{b\}Y\{c\} - ab[X,Y]\{c\}\right) Z \\
&= abcR(X,Y)Z\,,
\end{aligned} \tag{D.40}$$

where we used

$$\begin{aligned}
[aX,bY]\{f\} &= aX\{bY\{f\}\} - bY\{aX\{f\}\} \\
&= aX\{b\}Y\{f\} + abX\{Y\{f\}\} - bY\{a\}X\{f\} - abY\{X\{f\}\} \\
&= aX\{b\}Y\{f\} - bY\{a\}X\{f\} + ab[X,Y]\{f\}\,.
\end{aligned}$$

Note that in the last line of (D.40) all the inhomogeneous terms cancels out. In the same way we can show that also the torsion is a tensor:

$$\begin{aligned} T(aX,bY) &= \nabla_{aX}bY - \nabla_{bY}aX - [aX,bY] \\ &= a\left(X\{b\}Y + b\nabla_X Y\right) - b\left(Y\{a\}X + a\nabla_Y X\right) + \\ &\quad -aX\{b\}Y + bY\{a\}X - ab[X,Y] \\ &= abT(X,Y)\,. \end{aligned} \tag{D.41}$$

Using (D.40) we can now find the coordinate form of the Riemann tensor. From

$$R(X,Y)Z = X^\mu Y^\nu Z^\beta R(\partial_\mu,\partial_\nu)\partial_\beta$$

and[1]

$$\begin{aligned} dx^\alpha\left(R(\partial_\mu,\partial_\nu)\partial_\beta\right) &= dx^\alpha\left(R(\partial_\mu,\partial_\nu)\partial_\beta\right) \\ &= dx^\alpha\left(\nabla_\mu\nabla_\nu\partial_\beta - \nabla_\nu\nabla_\mu\partial_\beta\right) \\ &= dx^\alpha\left(\nabla_\mu\Gamma^\gamma_{\nu\beta}\partial_\gamma - \nabla_\nu\Gamma^\gamma_{\mu\beta}\partial_\gamma\right) \\ &= dx^\alpha\left(\partial_\mu\Gamma^\gamma_{\nu\beta}\partial_\gamma + \Gamma^\gamma_{\nu\beta}\Gamma^\delta_{\mu\gamma}\partial_\delta - (\mu\leftrightarrow\nu)\right) \\ &= \partial_\mu\Gamma^\alpha_{\nu\beta} + \Gamma^\gamma_{\nu\beta}\Gamma^\alpha_{\mu\gamma} - (\mu\leftrightarrow\nu)\,, \end{aligned}$$

we find the expression:

$$R^\alpha_{\mu\nu\beta} = \partial_\mu\Gamma^\alpha_{\nu\beta} - \partial_\nu\Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\nu\beta} - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\beta}\,. \tag{D.42}$$

The torsion tensor instead has the following coordinate expression:

$$\begin{aligned} dx^\alpha\left(T(\partial_\mu,\partial_\nu)\right) &= dx^\alpha\left(T(\partial_\mu,\partial_\nu)\right) \\ &= dx^\alpha\left(\nabla_\mu\partial_\nu - \nabla_\nu\partial_\mu\right) \\ &= dx^\alpha\left(\Gamma^\beta_{\mu\nu}\partial_\beta - \Gamma^\beta_{\nu\mu}\partial_\beta\right) \\ &= \left(\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}\right)\,, \end{aligned}$$

so

$$T^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}\,. \tag{D.43}$$

[1] We used the relation $\nabla_\mu\partial_\nu = \Gamma^\alpha_{\mu\nu}\partial_\alpha$.

The torsion tensor vanishes in a coordinate basis since the Christoffel symbols are symmetric in the lower indices. We can define the one-form (matrix valued) connection $\Gamma^\alpha{}_\beta = \Gamma^\alpha_{\mu\beta} dx^\mu$ and two-form (matrix valued) curvature

$$R^\alpha{}_\beta = \frac{1}{2} R^\alpha{}_{\mu\nu\beta} dx^\mu \wedge dx^\nu = d\Gamma^\alpha{}_\beta + \Gamma^\alpha{}_\gamma \wedge \Gamma^\gamma{}_\beta \,. \tag{D.44}$$

The symmetry of the Christoffel symbols in the lower indices implies

$$\Gamma^\alpha{}_\beta \wedge dx^\beta = \Gamma^\alpha_{\mu\beta} dx^\mu \wedge dx^\beta = 0\,, \tag{D.45}$$

taking an external derivative of (D.45) gives the first of Bianchi's identities:

$$\begin{aligned} 0 &= d\Gamma^\alpha{}_\beta \wedge dx^\beta \\ &= d\Gamma^\alpha{}_\beta \wedge dx^\beta + \Gamma^\alpha{}_\gamma \wedge \Gamma^\gamma{}_\beta \wedge dx^\beta \\ &= R^\alpha{}_\beta \wedge dx^\beta \,. \end{aligned} \tag{D.46}$$

The component form of (D.46) is:

$$R^\alpha{}_{\beta\mu\nu} + R^\alpha{}_{\mu\nu\beta} + R^\alpha{}_{\nu\beta\mu} = 0\,. \tag{D.47}$$

The (0,4) Riemann tensor has the following symmetries properties:

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} \qquad\qquad R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}\,. \tag{D.48}$$

If we permute cyclically the indices in the first Bianchi identity (D.47) we find the relations:

$$\begin{aligned} R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} &= 0 \\ R_{\beta\mu\nu\alpha} + R_{\beta\nu\alpha\mu} + R_{\beta\alpha\mu\nu} &= 0 \\ R_{\mu\nu\alpha\beta} + R_{\mu\alpha\beta\nu} + R_{\mu\beta\nu\alpha} &= 0 \\ R_{\nu\alpha\beta\mu} + R_{\nu\beta\mu\alpha} + R_{\nu\mu\alpha\beta} &= 0\,, \end{aligned}$$

summing these four equations shows that the $(0,4)$ Riemann tensor is symmetric in the exchange of the first two indices with the second two:

$$2R_{\alpha\mu\nu\beta} + 2R_{\beta\nu\alpha\mu} = 0 \qquad \Rightarrow \qquad R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}\,. \tag{D.49}$$

Considering these symmetries, we can count the number of independent components of the Riemann tensor. We find:

$$\underbrace{\frac{d(d-1)}{2}}_{\text{antisymmetry in } \alpha\beta} \cdot \underbrace{\frac{d(d-1)}{2}}_{\text{antisymmetry in } \mu\nu} \underbrace{-\frac{1}{3!}d \cdot d \cdot (d-1) \cdot (d-1)}_{\text{first Bianchi identity}} = \frac{d^2(d^2-1)}{12}\,,$$

in $d = 2$ we find 1 component (the Gaussian curvature), in $d = 3$ we find 6 components and in $d = 4$ there are 20 components. The symmetric $(0, 2)$ Ricci tensor is defined as the following contraction:

$$R_{\beta\nu} = R^{\alpha}_{\ \beta\alpha\nu} = g^{\mu\alpha} R_{\alpha\beta\mu\nu} = g^{\mu\alpha} R_{\mu\nu\alpha\beta} = R^{\alpha}_{\ \nu\alpha\beta} = R_{\nu\beta}\,, \tag{D.50}$$

while the Ricci scalar, or scalar curvature, is the trace of the Ricci tensor:

$$R = g^{\mu\nu} R_{\mu\nu}\,. \tag{D.51}$$

By taking an external derivative of the first Bianchi identity (D.46) we find:

$$\begin{aligned} dR &= d\Gamma \wedge \Gamma - \Gamma \wedge d\Gamma \\ &= R \wedge \Gamma - \Gamma \wedge \Gamma \wedge \Gamma - \Gamma \wedge R + \Gamma \wedge \Gamma \wedge \Gamma \\ &= R \wedge \Gamma - \Gamma \wedge R\,, \end{aligned}$$

which gives the second Bianchi identity:

$$dR + \Gamma \wedge R - R \wedge \Gamma = 0\,. \tag{D.52}$$

The component form of the second Bianchi identity is obtained by expanding (D.52) as

$$\partial_\delta R^{\alpha}_{\ \beta\mu\nu} dx^\delta \wedge dx^\mu \wedge dx^\nu + \Gamma^{\alpha}_{\delta\gamma} R^{\gamma}_{\ \beta\mu\nu} dx^\delta \wedge dx^\mu \wedge dx^\nu - \Gamma^{\gamma}_{\delta\beta} R^{\alpha}_{\ \gamma\mu\nu} dx^\mu \wedge dx^\nu \wedge dx^\delta = 0$$

and by summing the zero contribution $\left(\Gamma^{\gamma}_{\delta\mu} R^{\alpha}_{\ \beta\gamma\nu} + \Gamma^{\gamma}_{\delta\nu} R^{\alpha}_{\ \beta\mu\gamma}\right) dx^\delta \wedge dx^\mu \wedge dx^\nu = 0$, so to find:

$$\nabla_\delta R^{\alpha}_{\ \beta\mu\nu} dx^\delta \wedge dx^\mu \wedge dx^\nu = 0\,,$$

that implies:

$$\nabla_\delta R^{\alpha}_{\ \beta\mu\nu} + \nabla_\mu R^{\alpha}_{\ \beta\nu\delta} + \nabla_\nu R^{\alpha}_{\ \beta\delta\mu} = 0\,. \tag{D.53}$$

Equation (D.53) is the component form of the second Bianchi identity.

Two metrics are conformally equivalent if they can be related by $g = e^{2\sigma}\hat{g}$, with σ an arbitrary function. We investigate now the transformation properties of the Riemann tensor under conformal transformations. If we define the $(1,1)$ tensor

$$BX = \hat{\nabla}_X S + S\{\sigma\}X - \frac{1}{2}X\{\sigma\}S\,, \tag{D.54}$$

we can write the conformal transformation of the Riemann tensor as following:

$$R(X,Y)Z - \hat{R}(X,Y)Z = g(BX,Z)Y - g(BY,Z)X + g(X,Z)BY - g(Y,Z)BX\,. \tag{D.55}$$

The component form for the tensor (D.54) is:

$$B^\mu_\nu = \nabla_\nu \partial^\mu \sigma + \partial^\mu\sigma\partial_\nu\sigma - \frac{1}{2}\partial^\lambda\sigma\partial_\lambda\sigma\,\delta^\mu_\nu\,, \tag{D.56}$$

while the component form of (D.55) is:

$$\bar{R}^\alpha_{\ \mu\nu\beta} - R^\alpha_{\ \mu\nu\beta} = g_{\mu\beta}B^\alpha_{\ \nu} - g_{\nu\beta}B^\alpha_{\ \mu} + \delta^\alpha_\nu B^\gamma_{\ \mu}g_{\gamma\beta} - \delta^\alpha_\mu B^\gamma_{\ \nu}g_{\gamma\beta}\,. \tag{D.57}$$

Contracting gives:

$$\begin{aligned} \bar{R}_{\nu\beta} &= R_{\nu\beta} + B_{\beta\nu} - g_{\nu\beta}B + B_{\beta\nu} - dB_{\beta\nu} \\ &= R_{\nu\beta} - (d-2)B_{\nu\beta} - g_{\nu\beta}B \end{aligned} \tag{D.58}$$

and

$$e^{2\sigma}\bar{R} = R - 2(d-1)B\,,$$

or

$$\bar{g}_{\mu\nu}\bar{R} = g_{\mu\nu}\left[R - 2(d-1)B\right]\,. \tag{D.59}$$

By eliminating the tensor B from equation (D.57) by using (D.58) and (D.59) we find the conformal invariant combination which defines the Weyl conformal tensor:

$$\begin{aligned} C_{\alpha\beta\mu\nu} &= R_{\alpha\beta\mu\nu} - \frac{1}{d-2}\left(g_{\alpha\mu}R_{\beta\nu} - g_{\alpha\nu}R_{\beta\mu} - g_{\beta\mu}R_{\alpha\nu} + g_{\beta\nu}R_{\alpha\mu}\right) + \\ &\quad + \frac{R}{(d-2)(d-1)}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right)\,, \end{aligned} \tag{D.60}$$

with

$$\bar{C}_{\alpha\beta\mu\nu} = e^{2\sigma} C_{\alpha\beta\mu\nu} \,. \tag{D.61}$$

The square of the Weyl tensor is related to the other curvature square invariants as following:

$$C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - \frac{4}{d-2}R_{\mu\nu}R^{\mu\nu} + \frac{2}{(d-1)(d-2)}R^2 \,. \tag{D.62}$$

Note that the Weyl tensor is non-vanishing only for $d \geq 4$.

D.1.0.4 Chern-Gauss-Bonnet theorem

There exists a generalization to higher dimensions of the Gauss-Bonnet theorem. We will first relate the Euler characteristic to the trace of the heat kernel and then we will use the asymptotic expansion of the heat kernel to find the analytic expression for $\chi(\mathcal{M})$. Consider the eigenspaces E^p_λ of p-forms relative to the eigenvalue $\lambda > 0$ of Δ_p (this vector spaces are zero dimensional for a general value of λ). For all $\omega \in E^p_\lambda$ we have

$$\Delta\omega = \lambda\omega \qquad \Rightarrow \qquad \Delta d\omega = d\Delta\omega = \lambda d\omega \qquad \Rightarrow \qquad d\omega \in E^{p+1}_\lambda \,. \tag{D.63}$$

We can consider the following chain complex which is well defined thanks to (D.63):

$$0 \;\overset{i}{\to}\; E^0_\lambda \;\overset{d}{\to}\; E^1_\lambda \;\overset{d}{\to}\; \cdots \;\overset{d}{\to}\; E^d_\lambda \;\overset{d}{\to}\; 0 \,. \tag{D.64}$$

Take a closed form $\eta \in E^{p+1}_\lambda$, it is exact:

$$\lambda\eta = \Delta\eta = (d\delta + \delta d)\eta = d\delta\eta \qquad \Rightarrow \qquad \eta = d\left(\frac{1}{\lambda}\delta\eta\right) \,.$$

Thus $\mathrm{Im}\, d_{p-1} = \ker d_p$ and so the complex (D.64) is exact. This implies the following relation:

$$\sum_{p=0}^{d} (-1)^p \dim E^p_\lambda = 0 \qquad \text{for all} \quad \lambda > 0 \,. \tag{D.65}$$

Using de Rham's theorem (D.27), Hodge's theorem (D.28) and relation (D.65) we can write the Euler characteristic in terms of the heat kernel trace by the following steps:

$$\begin{aligned}
\chi(\mathcal{M}) &= \sum_{p=0}^{d}(-1)^p \dim H^p_{dR}(\mathcal{M}) \\
&= \sum_{p=0}^{d}(-1)^p \dim \ker \Delta_p \\
&= \sum_{p=0}^{d}(-1)^p \dim E^p_0 + \sum_{\lambda_p>0}\sum_{p=0}^{d}(-1)^p \dim E^p_{\lambda_p} \\
&= \sum_{\lambda_p}\sum_{p=0}^{d}(-1)^p \dim E^p_{\lambda_p} \\
&= \lim_{t\to 0}\sum_{p=0}^{d}(-1)^p \sum_{\lambda_p} \dim E^p_{\lambda_p}\, e^{-t\lambda_p} \\
&= \lim_{t\to 0}\sum_{p=0}^{d}(-1)^p \,\mathrm{Tr}\, e^{-t\Delta_p}\,.
\end{aligned} \tag{D.66}$$

Inserting the local heat kernel expansion, equation (A.4) form Appendix A, in (D.66) we find:

$$\chi(\mathcal{M}) = \frac{1}{(4\pi)^{d/2}} \lim_{t\to 0}\sum_{n=0}^{\infty}\left[\sum_{q=0}^{d}(-1)^q B_{2n}(\Delta_q)\right] t^{n-d/2}\,. \tag{D.67}$$

In (D.67) the $B_n(\Delta_q)$ are the integrated local heat kernel coefficients, given in (A.5), of the Laplacian on q-forms. If we can show, as we do later for $d = 2, 4$, that all the terms proportional to negative powers of t in (D.67) vanish, then only the $n = d/2$ term survives the $t \to 0$ limit and we finally find the fundamental result:

$$\chi(\mathcal{M}) = \frac{1}{(4\pi)^{d/2}} \sum_{q=0}^{d}(-1)^q B_d(\Delta_q)\,. \tag{D.68}$$

Equation (D.68) represents the fundamental theorem of Chern-Gauss-Bonnet that gives the Euler characteristic as the integral of curvature invariants of the manifold.

We now explicitly calculate the rhs of (D.68) for $d = 2$ and $d = 4$. In components, the

Laplacian on zero-forms, one-forms and two-forms can be written, respectively, as follows:

$$
\begin{aligned}
\Delta_0 &= -\nabla^2 \\
(\Delta_1)^\mu_\nu &= -\nabla^2 \delta^\mu_\nu + R^\mu_\nu \\
(\Delta_2)^{\alpha\beta}_{\mu\nu} &= -\nabla^2 I^{\alpha\beta}_{\mu\nu} + R^\gamma{}_\mu I^{\alpha\beta}_{\gamma\nu} + R^\gamma{}_\mu I^{\alpha\beta}_{\mu\gamma} + 2R^{\alpha\beta}{}_{\mu\nu} \,,
\end{aligned} \tag{D.69}
$$

where the identity on the space of antisymmetric $(0,2)$ tensors is $I^{\alpha\beta}_{\mu\nu} = \frac{1}{2}\left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu\right)$. The Laplacians in (D.69) have the general form (A.6) $\Delta = -\nabla^2 \mathbf{1} + \mathbf{U}$ and the first heat kernel coefficients of this operator are given in (A.7).

In $d = 2$ we have from (D.68):

$$
\chi(\mathcal{M}) = \frac{1}{4\pi}\left[B_2(\Delta_0) - B_2(\Delta_1) + B_2(\Delta_2)\right] = \frac{1}{4\pi}\left[2B_2(\Delta_0) - B_2(\Delta_1)\right] . \tag{D.70}
$$

We find the following coefficients:

$$
B_2(\Delta_0) = \frac{1}{6}\int_{\mathcal{M}} \sqrt{g} R \qquad\qquad B_2(\Delta_1) = \int_{\mathcal{M}} \sqrt{g}\left(d\frac{R}{6} - R\right) \underset{d=2}{=} -\frac{2}{3}\int_{\mathcal{M}} \sqrt{g} R \,,
$$

so (D.70) becomes:

$$
\chi(\mathcal{M}) = \frac{1}{4\pi}\left(\frac{1}{3} + \frac{2}{3}\right)\int_{\mathcal{M}} \sqrt{g} R \underset{d=2}{=} \frac{1}{4\pi}\int_{\mathcal{M}} \sqrt{g} R \,, \tag{D.71}
$$

which reproduces the well know Gauss-Bonnet theorem. We must check also that the following terms sum to zero:

$$
2B_0(\Delta_0) - B_0(\Delta_1) = \int_{\mathcal{M}} \sqrt{g}\,(2-d)\Big|_{d=2} = 0 \,.
$$

This is the first example of non-trivial cancellation that is at the base of (D.68).

In $d = 4$ we find the relation:

$$
\begin{aligned}
\chi(M) &= \frac{1}{(4\pi)^2}\left[B_4(\Delta_0) - B_4(\Delta_1) + B_4(\Delta_2) - B_4(\Delta_3) + B_4(\Delta_4)\right] \\
&= \frac{1}{(4\pi)^2}\left[2B_4(\Delta_0) - 2B_4(\Delta_1) + B_4(\Delta_2)\right] ,
\end{aligned} \tag{D.72}
$$

for the Laplacian on functions we find:

$$\operatorname{tr} b_4(\Delta_0) = \frac{1}{180}R^2_{\mu\nu\alpha\beta} - \frac{1}{180}R^2_{\mu\nu} + \frac{1}{72}R^2 - \frac{1}{30}\nabla^2 R\,. \tag{D.73}$$

For a one-form we have

$$\operatorname{tr}\mathbf{1} = d \qquad \operatorname{tr}\mathbf{U} = R \qquad \operatorname{tr}\mathbf{U}^2 = R^2_{\mu\nu} \qquad \operatorname{tr}\Omega_{\mu\nu}\Omega^{\mu\nu} = -R^2_{\mu\nu\alpha\beta}\,,$$

so we find:

$$\begin{aligned}\operatorname{tr} b_4(\Delta_1) &= \frac{1}{2}R^2_{\mu\nu} - \frac{1}{6}R^2 + \frac{1}{6}\nabla^2 R + d\left[\frac{1}{180}R^2_{\mu\nu\alpha\beta} - \frac{1}{180}R^2_{\mu\nu}\right.\\ &\quad \left.+\frac{1}{72}R^2 - \frac{1}{30}\nabla^2 R\right] - \frac{1}{12}R^2_{\mu\nu\alpha\beta}\\ &\underset{d=4}{=} -\frac{11}{180}R^2_{\mu\nu\alpha\beta} + \frac{43}{90}R^2_{\mu\nu} - \frac{1}{9}R^2 + \frac{1}{30}\nabla^2 R\,.\end{aligned} \tag{D.74}$$

For a two-form we have:

$$\operatorname{tr}\mathbf{1} = \frac{d(d-1)}{2} \qquad \operatorname{tr}\mathbf{U} = 2R \qquad \operatorname{tr}\mathbf{U}^2 = R^2 - 2R^2_{\mu\nu} + R^2_{\mu\nu\alpha\beta}$$

$$\operatorname{tr}\Omega_{\mu\nu}\Omega^{\mu\nu} = -(d-2)R^2_{\mu\nu\alpha\beta}\,,$$

so:

$$\begin{aligned}\operatorname{tr} b_4(\Delta_2) &= \frac{1}{2}R^2 - R^2_{\mu\nu} + R^2_{\mu\nu\alpha\beta} - \frac{1}{3}R^2 + \frac{1}{3}\nabla^2 R + \frac{d(d-1)}{2}\left[\frac{1}{180}R^2_{\mu\nu\alpha\beta}\right.\\ &\quad \left.-\frac{1}{180}R^2_{\mu\nu} + \frac{1}{72}R^2 - \frac{1}{30}\nabla^2 R\right] - \frac{d-2}{12}R^2_{\mu\nu\alpha\beta}\\ &\underset{d=4}{=} \frac{11}{30}R^2_{\mu\nu\alpha\beta} - \frac{31}{30}R^2_{\mu\nu} + \frac{1}{4}R^2 + \frac{2}{15}\nabla^2 R\,.\end{aligned} \tag{D.75}$$

Collecting (D.73), (D.74) and (D.75) gives:

$$\begin{aligned}2B_4(\Delta_0) - 2B_4(\Delta_1) + B_4(\Delta_2) &\underset{d=4}{=} \left(\frac{1}{90} + \frac{11}{90} + \frac{11}{30}\right)R^2_{\mu\nu\alpha\beta} + \left(-\frac{1}{90} - \frac{43}{45} - \frac{31}{30}\right)R^2_{\mu\nu}\\ &\quad + \left(\frac{1}{36} + \frac{2}{9} + \frac{1}{4}\right)R^2 + \left(-\frac{1}{15} - \frac{1}{15} + \frac{2}{15}\right)\nabla^2 R\\ &= \frac{1}{2}R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + \frac{1}{2}R^2\,.\end{aligned} \tag{D.76}$$

Note that the derivative terms in (D.76) cancel! The Chern-Gauss-Bonnet theorem in $d = 4$ is obtained by inserting (D.76) in (D.72) and reads:

$$\chi(\mathcal{M}) = \frac{1}{32\pi^2}\int_{\mathcal{M}} \sqrt{g}E \qquad E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2\,. \tag{D.77}$$

As a last thing, we must check that the following combinations add to zero in $d = 4$:

$$\begin{aligned} 2B_0(\Delta_0) - 2B_0(\Delta_1) + B_0(\Delta_2) &= \int_{\mathcal{M}} \sqrt{g}\left[2 - 2d + \frac{d(d-1)}{2}\right]_{d=4} \\ &\underset{d=4}{=} \int_M \sqrt{g}\,(2 - 8 + 6) \\ &= 0 \end{aligned} \tag{D.78}$$

and

$$\begin{aligned} 2B_2(\Delta_0) - 2B_2(\Delta_1) + B_2(\Delta_2) &= \int_M \sqrt{g}\left\{2\frac{R}{6} - 2\left(d\frac{R}{6} - R\right) + \left(\frac{d(d-1)}{2}\frac{R}{6} - 2R\right)\right\} \\ &\underset{d=4}{=} \int_M \sqrt{g}\left(\frac{1}{3} - \frac{4}{3} + 2 + 1 - 2\right) R \\ &= 0\,. \end{aligned} \tag{D.79}$$

Equations (D.78) and (D.79) are a second example of the non-trivial cancellation at the base of (D.68).

D.2 Functional integral quantization of gravity

In this section we quantize the gravity using the background field method, for a general introduction see [81]. The basic symmetry of the theory is diffeomorphism invariance, this is expressed infinitesimally as:

$$\delta g_{\mu\nu} = g^{\epsilon}_{\mu\nu} - g_{\mu\nu} = \mathcal{L}_{\epsilon} g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu\,, \tag{D.80}$$

where ϵ^μ is an infinitesimal vector. The partition function is defined by the following functional integral:

$$Z = \frac{1}{V_{Diff}}\int Dg_{\mu\nu}\, e^{-S[g]}\,, \tag{D.81}$$

where to omit over-counting do to gauge equivalent metrics we divided out the volume of the diffeomorphism group V_{Diff}. We decompose the integration variable according to

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \tag{D.82}$$

where $\bar{g}_{\mu\nu}$ is a general background metric and $h_{\mu\nu}$ is the metric fluctuation. The integration over the metric $g_{\mu\nu}$ can be replaced by one over $h_{\mu\nu}$, i.e. $Dg_{\mu\nu} = Dh_{\mu\nu}$. With respect to following physical transformations,

$$\delta h_{\mu\nu} = h^{\epsilon}_{\mu\nu} - h_{\mu\nu} = \nabla_{\mu}\epsilon_{\nu} + \nabla_{\nu}\epsilon_{\mu} \tag{D.83}$$

and

$$\delta\bar{g} = \bar{g}^{\epsilon}_{\mu\nu} - \bar{g}_{\mu\nu} = 0\,, \tag{D.84}$$

the action and the measure are invariant: $S[g^{\epsilon}] = S[g]$ and $Dh^{\epsilon}_{\mu\nu} = Dh_{\mu\nu}$. Background diffeomorphism are defined as:

$$\bar{\delta}h_{\mu\nu} = 0 \qquad\qquad \bar{\delta}\bar{g}_{\mu\nu} = \bar{\mathcal{L}}_{\epsilon}\bar{g}_{\mu\nu} = \bar{\nabla}_{\mu}\epsilon_{\nu} + \bar{\nabla}_{\nu}\epsilon_{\mu}\,. \tag{D.85}$$

The functional integral is over the space of metrics modulo diffeomorphism, we follow the Faddeev-Popov method to factor out the diffeomorphism. The first step is to insert in the partition function the following identity:

$$1 = \int Df\,\delta[f] = \int D\epsilon^{\mu}\,\delta[f^{\epsilon}]\,\det\frac{\delta f^{\epsilon}}{\delta\epsilon}\,, \tag{D.86}$$

where $f_{\mu}[h,\bar{g}] = 0$ is the background gauge-fixing condition. We follow the same steps done for non-abelian gauge theories in Appendix C:

$$\begin{aligned} Z &= \frac{1}{V_{Diff}}\int Dh_{\mu\nu}D\epsilon^{\mu}\,\delta[f^{\epsilon}]\,\det\frac{\delta f^{\epsilon}}{\delta\epsilon}\,e^{-S[\bar{g}+h]} \\ &= \frac{1}{V_{Diff}}\int Dh^{\epsilon}_{\mu\nu}D\epsilon^{\mu}\,\delta[f^{\epsilon}]\,\det\frac{\delta f^{\epsilon}}{\delta\epsilon}\,e^{-S[\bar{g}+h^{\epsilon}]} \\ &= \frac{1}{V_{Diff}}\int D\epsilon^{\mu}\int Dh_{\mu\nu}\,\delta[f]\,\det\mathcal{M}\,e^{-S[\bar{g}+h]} \\ &= \int Dh_{\mu\nu}\,\delta[f]\,\det\mathcal{M}\,e^{-S[g]}\,. \end{aligned} \tag{D.87}$$

Step by step: in the first step we used the gauge invariance of the action and of the measure; in the second we shifted the integration variable from $h^{\epsilon}_{\mu\nu}$ to $h_{\mu\nu}$ and we set (we use a linear gauge-fixing condition):

$$\mathcal{M}[h,\bar{g}]^{\mu}{}_{\nu} = \left.\frac{\delta f_{\mu}[h^{\epsilon},\bar{g}]}{\delta\epsilon^{\nu}}\right|_{\epsilon=0}\;; \tag{D.88}$$

in the last step we isolated the integration over the vector ϵ^{μ} and we identified it as the volume of the diffeomorphism group. It is possible to repeat the same steps with a more general gauge fixing functional $B[f]$ in place of $\delta[f]$ and functionally Fourier transform with respect to a field $b^{\mu\nu}$. Last we write the Faddeev-Popov determinant using the ghost fields as:

$$\det\mathcal{M} = \int DC^{\mu}D\bar{C}_{\mu}\exp\left(\int d^dx\sqrt{\bar{g}}\,\bar{C}_{\mu}\mathcal{M}^{\mu}{}_{\nu}C^{\nu}\right)\,. \tag{D.89}$$

We will consider the following linear gauge-fixing condition:

$$f_{\mu}[h,\bar{g}] = \left(\delta^{\alpha}_{\mu}\bar{\nabla}^{\beta} - \frac{\beta}{2}\bar{g}^{\alpha\beta}\bar{\nabla}_{\mu}\right)h_{\alpha\beta}\,, \tag{D.90}$$

and the following background gauge-fixing action:

$$S_{gf}[h;\bar{g}] = \frac{1}{2\alpha}\int d^dx\sqrt{\bar{g}}\bar{g}^{\mu\nu}f_{\mu}[h,\bar{g}]f_{\nu}[h,\bar{g}]\,, \tag{D.91}$$

where α and β are gauge-fixing parameters. The ghost action related to the gauge condition (D.89) can be found using (D.83) as follows:

$$\begin{aligned}\frac{\delta f_{\mu}[h^{\epsilon},\bar{g}]}{\delta\epsilon^{\nu}} &= \frac{\delta f_{\mu}[h^{\epsilon},\bar{g}]}{\delta h^{\epsilon}_{\alpha\beta}}\frac{\delta h^{\epsilon}_{\alpha\beta}}{\delta\epsilon^{\nu}}\\ &= \left(\delta^{\alpha}_{\mu}\bar{\nabla}^{\beta} - \frac{\beta}{2}\bar{g}^{\alpha\beta}\bar{\nabla}_{\mu}\right)(g_{\nu\beta}\nabla_{\alpha} + g_{\nu\alpha}\nabla_{\beta})\\ &= \bar{\nabla}^{\alpha}g_{\alpha\nu}\nabla_{\mu} + \bar{\nabla}^{\alpha}g_{\mu\nu}\nabla_{\alpha} - \beta\bar{\nabla}_{\mu}g_{\nu\alpha}\nabla^{\alpha}\,.\end{aligned} \tag{D.92}$$

The ghost action is thus

$$S_{gh}[h,\bar{C},C;\bar{g}] = -\int d^dx\sqrt{\bar{g}}\,\bar{C}^{\mu}\left(\bar{\nabla}^{\alpha}g_{\nu\alpha}\nabla_{\mu} + \bar{\nabla}^{\alpha}g_{\mu\nu}\nabla_{\alpha} - \beta\bar{\nabla}_{\mu}g_{\nu\alpha}\nabla^{\alpha}\right)C^{\nu}\,. \tag{D.93}$$

Finally we find:

$$Z = \int Dh_{\mu\nu}D\bar{C}_{\mu}DC^{\mu}\,e^{-S[h,\bar{C},C;\bar{g}]}\,, \tag{D.94}$$

with the action

$$S[h,\bar{C},C;\bar{g}] = S[\bar{g}+h] + S_{gf}[h;\bar{g}] + S_{gh}[h,\bar{C},C;\bar{g}]\,. \tag{D.95}$$

Note that both the background gauge-fixing and background ghost actions are invariant under combined physical and background diffeomorphism:

$$(\delta+\bar{\delta})S_{gf}[h;\bar{g}] = 0 \qquad\qquad (\delta+\bar{\delta})S_{gh}[h,\bar{C},C;\bar{g}] = 0\,. \tag{D.96}$$

Thus the "background classical action" is also invariant:

$$(\delta+\bar{\delta})S[h,\bar{C},C;\bar{g}] = 0\,. \tag{D.97}$$

The background effective action (bEA) is defined by the following integro-differential equation:

$$e^{-\Gamma[\varphi;\bar{g}]} = \int D\chi\, \exp\left\{-S[\varphi+\chi;\bar{g}] + \int d^dx\sqrt{\bar{g}}\,\Gamma^{(1)}[\varphi;\bar{g}]\chi\right\}\,, \tag{D.98}$$

where $\varphi = (h_{\mu\nu}, \bar{C}_\mu, C^\nu)$. From (D.97) it follows the bEA as defined in (D.98) is invariant under combined physical and background diffeomorphism:

$$(\delta+\bar{\delta})\Gamma[\varphi;\bar{g}] = 0\,. \tag{D.99}$$

We can define the full quantum metric as follows[2]:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,. \tag{D.100}$$

The main reason to employ the background field method is that we can define a diffeomorphism invariant functional $\bar{\Gamma}[\bar{g}]$, that we call gauge invariant effective action (gEA), by setting $\varphi = 0$, or equivalently $g_{\mu\nu} = \bar{g}_{\mu\nu}$ and $\bar{C}_\mu = C_\mu = 0$, in the bEA (D.98):

$$\bar{\Gamma}[\bar{g}] = \Gamma[0;\bar{g}]\,. \tag{D.101}$$

The integro-differential equation satisfied by the gEA is just (D.98) for $\varphi = 0$:

$$e^{-\bar{\Gamma}[\bar{g}]} = \int D\chi\, \exp\left\{-S[\chi;\bar{g}] + \int d^dx\sqrt{\bar{g}}\,\Gamma^{(1)}[0;\bar{g}]\chi\right\}\,. \tag{D.102}$$

[2] No confusion should arise from the fact that we are using $h_{\mu\nu}$ to indicate both the dummy integration variable and the mean fluctuation metric.

It is important to note that this definition for the gEA is not close since on the lhs of (D.102) there is the first functional derivative of the bEA and not of the gEA. Most importantly, the invariance (D.99) now becomes:

$$\bar{\delta}\bar{\Gamma}[\bar{g}] = 0\,. \tag{D.103}$$

Thus we succeeded in constructing a diffeomorphism invariant effective action.

We consider now the perturbative expansion of the bEA and of the gEA. From Appendix B we have that $\Gamma_0[\varphi;\bar{g}] = S[\varphi;\bar{g}]$ and using (D.101) we have:

$$\bar{\Gamma}_0[\bar{g}] = S[\bar{g}] + \underbrace{S_{gf}[0;\bar{g}]}_{=0} + \underbrace{S_{gh}[0,0,0;\bar{g}]}_{=0} = S[\bar{g}]\,. \tag{D.104}$$

The one loop contribution is given by (B.61):

$$\Gamma_1[\varphi;\bar{g}] = \frac{1}{2}\mathrm{Tr}\log S^{(2;0)}[\varphi;\bar{g}]\,,$$

Using the properties of the super-determinant and super-trace we find, as for gauge theories in Appendix C, the following relation:

$$\Gamma_1[h,\bar{C},C;\bar{g}] = \frac{1}{2}\mathrm{Tr}\log\left(S^{(2)}[\bar{g}+h] + S^{(2,0)}_{gf}[h;\bar{g}] + S^{(2,0,0,0)}_{gh}[h,\bar{C},C;\bar{g}]\right)$$
$$-\mathrm{Tr}\log\left(S^{(0,1,1,0)}_{gh}[h,\bar{C},C;\bar{g}] - \frac{S^{(1,1,0,0)}_{gh}[h,\bar{C},C;\bar{g}]S^{(1,0,1,0)}_{gh}[h,\bar{C},C;\bar{gA}]}{S^{(2)}[\bar{g}+h] + S^{(2,0)}_{gf}[h;\bar{g}] + S^{(2,0)}_{gh}[h,\bar{C},C;\bar{g}]}\right)\,. \tag{D.105}$$

The gauge invariant EA at one-loop is found from (D.105) for $\varphi = 0$ and reads:

$$\begin{aligned}\bar{\Gamma}_1[\bar{g}] &= \Gamma_1[0,0,0;\bar{g}] \\ &= \frac{1}{2}\mathrm{Tr}\log\left(S^{(2)}[\bar{g}] + S^{(2,0)}_{gf}[0;\bar{g}] + S^{(2,0,0,0)}_{gh}[0,0,0;\bar{g}]\right) \\ &\quad -\mathrm{Tr}\log\, S^{(0,1,1,0)}_{gh}[0,0,0;\bar{g}]\,.\end{aligned} \tag{D.106}$$

The elaborate construction of the bEA ensures that all contributions steaming from the functional traces in the lhs of (D.106) are diffeomorphism invariant. We will use (D.106) in Chapter 4 to obtain the flow equation for the gEAA as the RG improvement of the one-loop flow.

Bibliography

[1] K.G. Wilson, Phys. Rev. B 4, 3174 (1971) 3184.

[2] K.G. Wilson, Rev. Mod. Phys. 47 (1975) 773.

[3] J. Polchinski, Nucl. Phys. B 231 (1984) 269.

[4] E.C.G. Stueckelberg and A. Petermann, Helv. Phys. Acta 26 (1953) 499.

[5] M. Gell-Mann and F.E. Low, Phys. Rev. 95 (1954) 1300.

[6] A. Pich, "Effective field theory", arXiv:hep-ph/9806303.

[7] C.P. Burgess, Ann. Rev. Nucl. Part. Sci. 57 (2007) 329, arXiv:hep-th/0701053.

[8] E. Mottola, arXiv:1008.5006 [gr-qc].

[9] M.E. Peskin and D.V. Schroeder, *An Introduction To Quantum Field Theory*, Addison-Wesley (1995).

[10] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations*, Cambridge University Press (1995).

[11] S. Weinberg, *The quantum theory of fields. Vol. 2: Modern applications*, Cambridge University Press (1996).

[12] J. Zinn-Justin, *Quantum field theory and critical phenomena*, Int. Ser. Monogr. Phys. 113, 1 (2002).

[13] G. Mussardo, *Statistical field theory*, Oxford University Press (2010).

[14] C. Wetterich, Phys. Lett. B 301 (1993) 90.

[15] J. Berges, N. Tetradis and C. Wetterich, Phys. Rep. 363 (2002) 223.

[16] J.M. Pawlowski, Annals Phys. 322 (2007) 2831, arXiv:hep-th/0512261.

[17] B. Delamotte, "An introduction to the nonperturbative renormalization group", arXiv:cond-mat/0702365.

[18] H. Gies, "Introduction to the functional RG and applications to gauge theories", arXiv:hep-ph/0611146.

[19] D. Litim, Phys. Rev. D 64 (2001) 105007, arXiv:hep-th/0103195.

[20] J.P. Blaizot, R. Mendez Galain and N. Wschebor, Phys. Lett. B 632 (2006) 571, arXiv:hep-th/0503103.

[21] C. Wetterich, Z. Phys. C 57 (1993) 451.

[22] T.R. Morris, Phys. Lett. B 329 (1994) 241, arXiv:hep-ph/9403340.

[23] T.R. Morris, Phys. Lett. B 334 (1994) 355, arXiv:hep-th/9405190.

[24] T.R.Morris, Phys. Lett. B 345 (1995) 139, arXiv:hep-th/9410141.

[25] L. Canet, B. Delamotte, D. Mouhanna and J. Vidal, Phys. Rev. D 67 (2003) 065004, arXiv:hep-th/0211055.

[26] L. Canet, B. Delamotte, D. Mouhanna and J. Vidal, Phys. Rev. B 68 (2003) 064421, arXiv:hep-th/0302227.

[27] D.F. Litim and J.M. Pawlowski, Phys. Rev. D 65 (2002) 081701, arXiv:hep-th/0111191.

[28] H. Gies and J. Jaeckel, Phys. Rev. Lett. 93 (2004) 110405, arXiv:hep-ph/0405183.

[29] S.V. Ketov, Quantum Non-linear Sigma-Models, Springer (2000).

[30] E.H. Fradkin, "Field theories of condensed matter systems", Addison-Wesley (1991).

[31] D. Nelson, T. Piran and S. Weinberg, “"Statistical Mechanics of Membranes and Surfaces: 2nd Edition", World Scientific Publishing Company (2004).

[32] A.M. Polyakov, Phys. Lett. B 59 (1975) 79-81.

[33] G. Perelman, arXiv:math.DG/0211159, arXiv:math.DG/0303109.

[34] A. Codello and R. Percacci, Phys. Lett. B 672 (2009) 280, arXiv:0810.0715 [hep-th].

[35] J. Gasser, H. Leutwyler, Annals Phys. (1984) 158, 142.

[36] R. Percacci "Geometry of nonlinear field theories", World Scientific (1986).

[37] M. Luscher, P. Weisz, Phys. Lett. B 212 (1988) 472.

[38] J. Smit, "Introduction to quantum fields on a lattice", Cambridge (2002).

[39] O.J. Rosten, "Fundamentals of the Exact Renormalization Group", arXiv:0808.0082 [hep-th].

[40] B.E. Baaquie, G. Bhanot, Nucl. Phys. B 382 (1992) 409-418.

[41] R. Percacci and O. Zanusso, Phys. Rev. D 81 (2010) 065012, arXiv:0910.0851 [hep-th].

[42] M. Fabbrichesi, R. Percacci, A. Tonero and O. Zanusso, arXiv:1010.0912 [hep-ph].

[43] A. Codello, Annals Phys. 325 (2010) 1727, arXiv:1004.2171 [hep-th].

[44] A. M. Polyakov, Phys. Lett. B 103 (1981) 207.

[45] M. Reuter and C. Wetterich, Nucl. Phys. B 506 (1997) 483, arXiv:hep-th/9605039.

[46] D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343.

[47] H. D. Polizer, Phys. Rev. 30 (1973) 1346.

[48] L.F. Abbott, Nucl. Phys. B 185 (1981) 189.

[49] M. Reuter and C. Wetterich, Nucl. Phys. B 391 (1993) 147.

[50] M. Reuter and C. Wetterich, Nucl. Phys. B 408 (1993) 91.

[51] M. Reuter and C. Wetterich, Nucl. Phys. B 417 (1994) 181.

[52] M. Reuter and C. Wetterich, Nucl. Phys. B 427 (1994) 291.

[53] M. Reuter, Phys. Rev. D 53 (1996) 4430, arXiv:hep-th/9511128.

[54] M. Reuter, "Effective Average Actions and Nonperturbative Evolution Equations", arXiv:hep-th/9602012.

[55] M. Reuter, Mod. Phys. Lett. A 12 (1997) 2777, arXiv:hep-th/9604124.

[56] M. Reuter and C. Wetterich, Phys. Rev. D 56 (1997) 7893, arXiv:hep-th/9708051.

[57] H. Gies, Phys. Rev. D 66 (2002) 025006, arXiv:hep-th/0202207.

[58] H. Gies, Phys. Rev. D 68 (2003) 085015, arXiv:hep-th/0305208.

[59] U. Ellwanger, M. Hirsch and A. Weber, Z. Phys. C 69 (1996) 687, arXiv:hep-th/9506019.

[60] U. Ellwanger, M. Hirsch and A. Weber, Eur. Phys. J. C 1 (1998) 563, arXiv:hep-ph/9606468.

[61] L. von Smekal, R. Alkofer and A. Hauck, Phys. Rev. Lett. 79 (1997) 3591, arXiv:hep-ph/9705242.

[62] R. Alkofer and L. von Smekal, Phys. Rept. 353 (2001) 281, arXiv:hep-ph/0007355.

[63] C.S. Fischer and H. Gies, JHEP 0410 (2004) 048, arXiv:hep-ph/0408089.

[64] C.S. Fischer and J.M. Pawlowski, Phys. Rev. D 75 (2007) 025012, arXiv:hep-th/0609009.

[65] C.S. Fischer, A. Maas and J.M. Pawlowski, Annals Phys. 324 (2009) 2408, arXiv:0810.1987 [hep-ph].

[66] C.S. Fischer and J.M. Pawlowski, Phys. Rev. D 80 (2009) 025023, arXiv:0903.2193 [hep-th].

[67] J.M. Pawlowski, D.F. Litim, S. Nedelko and L. von Smekal, Phys. Rev. Lett. 93 (2004) 152002, arXiv:hep-th/0312324.

[68] A. Codello, in preparation

[69] U. Ellwanger, Phys. Lett. B 335 (1994) 364, arXiv:hep-th/9402077.

[70] D.F. Litim and J.M. Pawlowski, Phys. Lett. B 546 (2002) 279, arXiv:hep-th/0208216.

[71] F. Freire, D.F. Litim and J.M. Pawlowski, Phys. Lett. B 495 (2000) 256, arXiv:hep-th/0009110.

[72] M. Maggiore, *Gravitational Waves. Vol. 1: Theory and Experiments*, Oxford University Press (2007).

[73] J. Ambjorn, B. Durhuus and T. Jonsson, *Quantum geometry. A statistical field theory approach*, Cambridge University Press (1997).

[74] C. Kiefer, *Quantum Gravity*, Second Edition, Oxford Science Publications, Oxford (2007).

[75] H. Hamber, *Quantum Gravitation*, Springer, Berlin (2008).

[76] M. Niedermaier and M. Reuter, Living Rev. Rel. 9 (2006) 5.

[77] M. Reuter and F. Saueressig, "Functional Renormalization Group Equations, Asymptotic Safety, and Quantum Einstein Gravity", arXiv:0708.1317 [hep-th].

[78] O. Lauscher and M. Reuter in *Quantum Gravity*, B. Fauser, J. Tolksdorf and E. Zeidler (Eds.), Birkhäuser, Basel (2007), hep-th/0511260.

[79] O. Lauscher and M. Reuter in *Approaches to Fundamental Physics*, I. O. Stamatescu and E. Seiler (Eds.), Springer, Berlin (2007).

[80] R. Percacci in *Approaches to Quantum Gravity*, D. Oriti (Ed.), Cambridge University Press (2009), arXiv:0709.3851 [hep-th].

[81] E. Mottola, J. Math. Phys. 36 (1995) 2470, arXiv:hep-th/9502109.

[82] J.F. Donoghue, Phys. Rev. D 50 (1994) 3874, arXiv:gr-qc/9405057.

[83] C.P. Burgess, Living Rev. Rel. 7 (2004) 5, arXiv:gr-qc/0311082.

[84] N.E.J. Bjerrum-Bohr, J.F. Donoghue and B.R. Holstein, Phys. Rev. D 67 (2003) 084033, arXiv:hep-th/0211072.

[85] J.F. Donoghue and T. Torma, Phys. Rev. D 60 (1999) 024003, arXiv:hep-th/9901156.

[86] S. Weinberg in *General Relativity, an Einstein Centenary Survey*, S. W. Hawking and W. Israel (Eds.), Cambridge University Press (1979).

[87] M. Reuter, Phys. Rev. D 57 (1998) 971, hep-th/9605030.

[88] G. 't Hooft and M.J.G. Veltman, Annales Poincare Phys. Theor. A 20 (1974) 69.

[89] R. Gastmans, R. Kallosh and C. Truffin, Nucl. Phys. B 133 (1978) 417.

[90] S.M. Christensen and M.J.Duff, Phys. Lett. B 79 (1978) 213.

[91] H. Kawai and M. Ninomiya, Nucl. Phys. B 336 (1990) 115.

[92] H. Kawai, Y. Kitazawa and M. Ninomiya, Nucl. Phys. B 404 (1993) 684, arXiv:hep-th/9303123.

[93] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A 173 (1939) 211.

[94] D. Dou and R. Percacci, Class. Quant. Grav. 15 (1998) 3449.

[95] O. Lauscher and M. Reuter, Phys. Rev. D 65 (2002) 025013, hep-th/0108040.

[96] A. Codello, R. Percacci and C. Rahmede, Annals Phys. 324 (2009) 414, arXiv:0805.2909 [hep-th].

[97] A. Codello and R. Percacci, Phys. Rev. Lett. 97 (2006) 221301.

[98] D. Benedetti, P. Machado and F. Saueressig, Mod. Phys. Lett. A 24 (2009) 2233, arXiv:0901.2984.

[99] D. Benedetti, P. Machado and F. Saueressig, Nucl. Phys. B 824 (2010) 168, arXiv:0902.4630.

[100] W. Souma, Prog. Theor. Phys. 102 (1999) 181.

[101] M. Reuter and F. Saueressig, Phys. Rev. D 65 (2002) 065016, arXiv:hep-th/0110054.

[102] D. Litim, Phys. Rev. Lett. 92 (2004) 201301, hep-th/0312114.

[103] P. Fischer and D. Litim, Phys. Lett. B 638 (2006) 497, hep-th/0602203.

[104] A. Codello, R. Percacci and C. Rahmede, Int. J. Mod. Phys. A 23 (2008) 143-150, arXiv:0705.1769.

[105] P. Machado and F. Saueressig, Phys. Rev. D 77 (2008) 124045, arXiv:0712.0445.

[106] K. Groh and F. Saueressig, J. Phys. A 43 (2010) 365403, arXiv:1001.5032 [hep-th].

[107] A. Eichhorn, H. Gies and M.M. Scherer, Phys. Rev. D 80 (2009) 104003, arXiv:0907.1828 [hep-th].

[108] A. Eichhorn and H. Gies, Phys. Rev. D 81 (2010) 104010, arXiv:1001.5033 [hep-th].

[109] E. Manrique and M. Reuter, Annals Phys. 325 (2010) 785, arXiv:0907.2617 [gr-qc].

[110] E. Manrique, M. Reuter and F. Saueressig, arXiv:1003.5129 [hep-th].

[111] E. Manrique, M. Reuter and F. Saueressig, arXiv:1006.0099 [hep-th].

[112] A. Codello and A. Eichhorn, in preparation.

[113] A. Satz, A. Codello and F.D. Mazzitelli, Phys. Rev. D 82 (2010) 084011, arXiv:1006.3808 [hep-th].

[114] D.A.R. Dalvit and F.D. Mazzitelli, Phys. Rev. D 56 (1997) 7779, arXiv:hep-th/9708102.

[115] A. Bonanno and M. Reuter, Phys. Rev. D 62 (2000) 043008, hep-th/0002196.

[116] A. Bonanno and M. Reuter, JCAP 08 (2007) 024, arXiv:0706.0174 [hep-th].

[117] M. Reuter and H. Weyer, Phys. Rev. D 69 (2004) 104022, hep-th/0311196.

[118] M. Reuter and H. Weyer, Phys. Rev. D 70 (2004) 124028, hep-th/0410117.

[119] K. Falls, D.F. Litim and A. Raghuraman, arXiv:1002.0260 [hep-th].

[120] A. Satz, F.D. Mazzitelli and E. Alvarez, Phys. Rev. D 71 (2005) 064001, arXiv:gr-qc/0411046.

[121] A. Bonanno, A. Contillo and R. Percacci, arXiv:1006.0192 [gr-qc].

[122] A. Codello, in preparation.

[123] D. V. Vassilevich, Phys. Rept. 388 (2003) 279, arXiv:hep-th/0306138.

[124] G. A. Vilkovisky, "Heat kernel: Rencontre entre physiciens et mathematiciens", (1992).

[125] A. O. Barvinsky and G. A. Vilkovisky, Phys. Rept. 119 (1985) 1..

[126] A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. B 282 (1987) 163.

[127] A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. B 333 (1990) 471.

[128] I. G. Avramidi, Lect. Notes Phys. M 64 (2000) 1.

[129] A. Codello, in preparation.

[130] S. Rosenberg, *The Laplacian On A Riemannian Manifold*, Cambridge University Press (1997).

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Pubblications

[97], [104], [95], [34], [43], [120] from the Bibliography and:

A. Codello J. Phys. A: Math. Theor. 43 (2010) 385002, arXiv:1008.4720.