Investigations on Asymptotic Safety of Metric, Tetrad, and Einstein-Cartan Gravity

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Kurzfassung

Unter den verschiedenen Zugängen zur Konstruktion einer fundamentalen Quantentheorie der Gravitation beruht das Prinzip der Asymptotischen Sicherheit auf der Annahme, dass die Quantengravitation im Rahmen einer üblichen Quantenfeldtheorie formuliert werden kann, die allerdings nicht-perturbativ konstruiert werden muss. In diesem Fall wird das Hochenergieverhalten der Theorie durch einen nicht-Gaußchen Fixpunkt des Renormierungsgruppenflusses kontrolliert, so dass der Limes eines unendlichen Cutoffs wohldefiniert ist. Eine solche Theorie wird als nicht-perturbativ renormierbar bezeichnet. Im letzten Jahrzehnt wurden zahlreiche Hinweise gefunden, dass in vierdimensionaler, metrischer Gravitation ein solcher, für die Konstruktion einer asymptotisch sicheren Theorie geeigneter Fixpunkt in der Tat existiert.
Die vorliegende Arbeit erweitert das Programm der Asymptotischen Sicherheit um drei voneinander unabhängige Studien, die sich im Feldgehalt der untersuchten Quantentheorien unterscheiden, aber in der durch ein semi-direktes Produkt gegebenen Struktur der zugrundeliegenden Eichgruppe gleichen. Die Arbeit erlaubt damit zum ersten Mal den direkten Vergleich von drei asymptotisch sicheren Theorien der Gravitation, die auf unterschiedlichen fundamentalen Feldern beruhen.
Die erste Studie untersucht dabei das gekoppelte System von metrischer Gravitation und $\mathsf{SU}(N)$ Yang-Mills Theorie. Insbesondere wird dabei der Einfluss der Gravitation auf das Laufen der Yang-Mills Kopplungskonstante analysiert und dessen Folgen für QED und das Standardmodell diskutiert. In der zweiten Untersuchung wird zum ersten Mal eine asymptotisch sichere Theorie der Gravitation betrachtet, die allein durch das Vielbein beschrieben wird. Ihr Renormierungsgruppenfluß wird mit der entsprechenden Approximation der metrischen Theorie verglichen und der Einfluß der vergrößerten Eichgruppe analysiert. Die dritte Studie untersucht die Asymptotische Sicherheit der Gravitation im Einstein-Cartan Zugang. Dabei dient der Spinzusammenhang neben dem Vielbein als zweite fundamentale Feldvariable. Aufgrund der höheren Anzahl unabhängiger Feldkomponenten und der größeren Eichgruppe ist jede Renormierungsgruppenanalyse dieses Systems ungleich schwieriger als die analoge Rechnung im metrischen Zugang. Um die technische Komplexität der Aufgabe zu verringern wird in dieser Arbeit eine neuartige funktionale Renormierungsgruppengleichung eingeführt, die die Auswertung des Flusses auf ein rein algebraisches Problem reduziert. Um ein erstes Beispiel ihrer Eignung zu geben, wird die neue Gleichung auf eine dreidimensionale Trunkierung von Form der Holst Wirkung angewendet, die die Newton-Konstante, die kosmologische Konstante und den Immirzi-Parameter als laufende Kopplungen enthält. In einem detaillierten Vergleich mit einer früheren Studie desselben Systems wird dabei die Zuverlässigkeit der neuen Gleichung demonstriert, die sie für künftige Untersuchungen von allgemeineren Trunkierungen qualifiziert.

Abstract

Among the different approaches for a construction of a fundamental quantum theory of gravity the Asymptotic Safety scenario conjectures that quantum gravity can be defined within the framework of conventional quantum field theory, but only non-perturbatively. In this case its high energy behavior is controlled by a non-Gaussian fixed point of the renormalization group flow, such that its infinite cutoff limit can be taken in a well defined way. A theory of this kind is referred to as non-perturbatively renormalizable. In the last decade a considerable amount of evidence has been collected that in four dimensional metric gravity such a fixed point, suitable for the Asymptotic Safety construction, indeed exists.
This thesis extends the Asymptotic Safety program of quantum gravity by three independent studies, that differ in the fundamental field variables the investigated quantum theory is based on, but all exhibit a gauge group of equivalent semi-direct product structure. It allows for the first time for a direct comparison of three asymptotically safe theories of gravity constructed from different field variables.
The first study investigates metric gravity coupled to $\mathsf{SU}(N)$ Yang-Mills theory. In particular the gravitational effects to the running of the gauge coupling are analyzed and its implications for QED and the Standard Model are discussed. The second analysis amounts to the first investigation on an asymptotically safe theory of gravity in a pure tetrad formulation. Its renormalization group flow is compared to the corresponding approximation of the metric theory and the influence of its enlarged gauge group on the UV behavior of the theory is analyzed. The third study explores Asymptotic Safety of gravity in the Einstein-Cartan setting. Here, besides the tetrad, the spin connection is considered a second fundamental field. The larger number of independent field components and the enlarged gauge group render any RG analysis of this system much more difficult than the analog metric analysis. In order to reduce the complexity of this task a novel functional renormalization group equation is proposed, that allows for an evaluation of the flow in a purely algebraic manner. As a first example of its suitability it is applied to a three dimensional truncation of the form of the Holst action, with the Newton constant, the cosmological constant and the Immirzi parameter as its running couplings. A detailed comparison of the resulting renormalization group flow to a previous study of the same system demonstrates the reliability of the new equation and suggests its use for future studies of extended truncations in this framework.

Contents

1 Introduction

In the search for an adequate description of the fundamental principles of nature the conception of quantum field theory (QFT) has been developed over the last century. After an elementary understanding of this new type of quantum theory had been achieved, QFT has proven to be an immensely successful concept, and it is by now a widely accepted fact that the physics of all particles discovered so far and their interactions is best described in this framework by a model known as the Standard Model (SM) of particle physics.

Together with the development of the SM a generic procedure for the quantization of a given classical field theory has been worked out. Using the so-called path integral quantization the transition from classical to quantum theory is achieved by a functional integration over the exponentiated classical action. This functional integration arises as we do not only have to consider the path in configuration space that is a solution to the classical field equations, but rather sum over all paths weighted by their probability amplitude, in order to compute the time evolution of a quantum theory.

For realistic field theories, however, this path integral is an extremely complicated object, such that its exact evaluation seems an unattainable goal. For that reason one typically evaluates the path integral of the corresponding free theory and takes into account all interactions of the fields only as a perturbative series in the coupling constants.

Already in the early days of quantum electrodynamics (QED) it was noted that the new theory leads to infinite energy shifts [Opp30, Wal30b, Wal30a] of observable quantities. Later on, it was understood that the computation of observable quantities in the perturbative setting generically leads to divergent results, as soon as (virtual) photons of arbitrarily high momentum are taken into account in the path integral. Without a deeper understanding of their meaning the divergences arising in QED could be classified and the technique of perturbative renormalization was invented. These achievements were mainly due to Bethe, Dyson, Feynman, Schwinger and Tomonaga (see their articles in [Sch58]).

This technique amounts to a prescription how the divergences can be hidden in unphysical (counter-)terms of the bare, classical action, such that the resulting quantum effective action, from which all observable quantities are to be computed, stays finite. At first only the tremendous success of QED predictions using this prescription silenced the doubts concerning this procedure [Wei].

Later on, a classification of perturbatively renormalizable theories was carried out, and it was understood, that the absorption of divergences into a finite number of bare coupling constants is generically possible, if the coupling constants have positive mass dimension. On this ground, the SM of particle physics was built. It incorporates the three forces of electromagnetic, weak and strong interaction, that at the classical level can all be described as Yang-Mills (YM) theories, together with the known matter content in one renormalizable QFT.

Simultaneously, it was tried to apply this successful procedure of quantization to the Einstein-Hilbert action describing classical general relativity. The main motivation for a quantization of gravity thereby lied in the prospect to find a universal (quantum) framework that describes all fundamental forces of nature, rather than to make more accurate predictions, as classical general relativity describes gravity very well at all energy scales that are accessible even to present-day experiments. If gravity was not quantized, Einstein's equation would necessarily combine classical quantities describing the geometry of spacetime with quantum objects, that describe its matter content causing the curvature of spacetime. For most theoreticians this would amount to a very unsatisfactory model of nature most probably hinting at a lack of understanding of the genuinely fundamental principles of nature.

However, it soon became clear that Einstein-Hilbert gravity belongs to the class of perturbatively non-renormalizable theories. While at 1-loop pure gravity is found on-shell finite [HV74] the inclusion of matter spoils renormalizability already at this level [HV74]. Later on, it was shown that at 2-loop level even pure gravity is non-renormalizable in the perturbative sense [GS86, Ven92]. The uncontrollable appearance of ever new divergences at every higher loop order in perturbative quantum gravity, that renders it impossible to hide them in *finitely* many bare coupling constants, can thereby be traced back to the negative mass dimension of its coupling, the Newton constant.

These discouraging results led to the search for alternate theories that can be quantized properly and give rise to the well tested properties of classical general relativity only in their low momentum regime as an effective field theory. One idea to eliminate the problematic divergences was to introduce an additional symmetry between bosonic and fermionic degrees of freedom, the supersymmetry, such that each divergent contribution from a certain graviton diagram is exactly canceled by its fermionic partner particle. Besides the fact that all supersymmetric theories predict particles that have not been observed so far, it turned out an extremely involved task—still to be completed—to find out whether in a maximally supersymmetric field theory of gravity indeed all divergences can be removed this way. String theory, a different approach to quantum gravity most often combined with the concept of supersymmetry, leaves the framework of QFT in order to remove the divergences. Here, the fundamental degrees of freedom are changed from pointlike excitations of the metric field to excitations of a string of finite length, which results in the absence of UV divergent integrals. However, string theory in its present first quantized form is based on perturbation theory and it is far from clear how a non-perturbative definition of string theory or a string field field theory would look like.

In addition we want to mention two further approaches to quantum gravity, namely loop quantum gravity (LQG) and causal dynamical triangulations (CDT). Contrary to string theory, these theories avoid the use of perturbative methods from the outset, as they assume that the immanent divergences in perturbative quantum gravity are only due to a non-applicability of perturbation theory.

In LQG the so-called Holst action, which describes gravity in a first order formulation and will also play a major role in this work, serves as the starting point. In contrast to the approach we employ later on, LQG relies on a Hamiltonian formalism that uses a special choice of variables to parametrize phase space [Ash91], that are then quantized by canonical quantization. Here the main difficulty lies in correctly imposing a Hamiltonian constraint, that describes the time evolution of the theory.

In CDT, on the other hand, the Euclidean path integral of the Einstein-Hilbert action is evaluated by Monte-Carlo simulations. Thereby, the sum over all spacetimes is discretized by summing over all possible triangulations with a given edge length and a fixed maximal number of simplices. As a crucial result it was found that only if the sum over all geometries is restricted to a subset that satisfies certain causality conditions the resulting quantum theory shows indeed a phase with the hoped-for but nontrivial property that spacetime is smooth and four dimensional on macroscopic scales [AJL04].

For more details on these alternate approaches to quantum gravity we refer to the standard textbooks [Kie07, Rov04] and the review article [Woo09].

In the meantime, while all these ideas were developed, a deeper understanding of the necessity of the renormalization procedure and conditions for the existence of fundamental field theories was obtained. Pioneered by Wilson [WK74, Wil75] renormalization group (RG) techniques were developed that are based on the idea of evaluating the path integral not all at once, but in a piecewise manner. Starting from this idea an RG flow of action functionals can be defined, that describes the same system at different momentum scales. In the following it was understood that the general condition for a theory to be considered fundamental is satisfied whenever its RG flow approaches a fixed point (FP) in the high momentum (UV) limit. This condition ensures that no divergences occur in observable quantities. Moreover, the fundamental theory turns out predictive if the FP shows only finitely many UV attractive directions, that correspond to the number of external parameters of the theory. Once these parameters are determined by (finitely many) measurements, the theory is uniquely defined by the requirement that its UV limit is taken at the FP under consideration. Therefore all other observables can then be predicted.

This non-perturbative definition of renormalizability also applies to all perturbatively renormalizable, fundamental theories—like Yang-Mills theory or quantum chromodynamics (QCD)—as these approach the Gaussian fixed point (GFP) in the UV limit, which corresponds to an asymptotically free, non-interacting theory. In case of the GFP it is particularly easy to determine its UV attractive directions: They correspond to all field monomials, whose coupling constants have a positive mass dimension.

This insight provided a deeper understanding why theories with couplings of positive mass dimesions generically turn out perturbatively renormalizable, but at the same time raised the question whether perturbatively non-renormalizable theories could turn out non-perturbatively renormalizable. In this case the RG flow approaches a non-Gaussian FP (NGFP), that corresponds to an interacting theory, which may occur outside the realm of perturbation theory. Theories whose continuum limit is defined at such a NGFP are referred to as asymptotically safe. Indeed toy models, like the

Gross-Neveu model in three dimensions, could be constructed that were shown to belong to this class [GK85a, GK85b].

These considerations also renewed the interest in metric gravity, that might as well be defined as a QFT only non-perturbatively. From results available from an ε-expansion for gravity in $d = 2 + \varepsilon$-dimensions, which indicated the existence of a NGFP near two dimensions, Weinberg conjectured in 1979 [Wei79] that this fixed point should not be destroyed by dimensional continuation and therefore might also exist in four dimensions. Unfortunately, at that time no computational tool was known that would have allowed for a thorough investigation of this conjecture.

It took until the '90s that such an appropriate tool was found: The functional renormalization group equation (FRGE) for the effective average action (EAA) which was first derived for scalar [Wet93] and Yang-Mills theory [RW93a, RW93b, RW94a, RW94b], and later on generalized such that it could be applied to gravity [Reu98]. It is an exact functional equation for a running effective action functional, that typically cannot be solved in full generality but allows for non-perturbative approximations by choosing an ansatz for the form of the action functional, a so-called truncation. This FRGE rendered the investigation of the RG flow of metric gravity possible in arbitrary dimensions of spacetime. Since then numerous studies of different approximations have been carried out[1], all of which indicate the existence of a NGFP for metric gravity in $d = 4$ spacetime dimensions. Also the inclusion of matter fields, that causes first divergences in the perturbative approach, has been explored in some detail [DP98, PP03b, PP03a, VZ10], and it was found that Asymptotic Safety of gravity is compatible with the matter content of the SM, although general bounds on the number of matter fields were found to exist. Taken together an impressive amount of evidence for Asymptotic Safety of metric gravity at a NGFP in the space of diffeomorphism invariant action functionals has been collected, such that it is very likely to be a true feature of this theory space and not merely an artifact of the approximations applied. Theories of metric gravity whose continuum limit is taken at this NGFP are called Quantum Einstein Gravity (QEG).

Despite its success, it is important to be aware of the fact that the Asymptotic Safety scenario for quantum gravity is by no means restricted to the metric formulation of classical general relativity. In the search for an asymptotically safe theory of gravity the only restriction one has to obey is that the space of action functionals on which the

[1]For an overview of this field of research we refer to the review articles [NR06, Per09, RS12].

RG flow is considered, the "theory space", contains an action functional that gives rise to Einstein's equation. Otherwise the resulting quantum theory will lack the classical regime of general relativity. As there is a remarkable number of Lagrangian variational principles that all give rise to Einstein's equation, we find a variety of different theory spaces, that are equally plausible to be investigated. Besides the metric theory space, containing the Einstein-Hilbert action, we will explore in this thesis the tetrad theory space, comprising the Einstein-Hilbert action reexpressed in terms of the tetrad, and the Einstein-Cartan theory space of action functionals constructed from the tetrad and an independent spin connection. It contains the Hilbert-Palatini and the Holst action that are suitable to describe the classical regime of the resulting quantum theory.

It should be noted that these spaces of action functionals do not have very much in common, except for the fact that they contain points which give rise to equivalent field equations: In particular, they are of different "size" (the action functionals contained are not in one-to-one correspondence to each other), the constituent fields differ in the number of independent field components, and the symmetry group is not the same, although all action functionals considered should, of course, retain diffeomorphism invariance. For this reason the RG flows on these spaces and their fixed point structure are a priori completely unrelated from each other such that *the investigation of the Asymptotic Safety scenario of quantum gravity has to be carried out separately for each of these spaces.*

Even if two or more theory spaces were found to contain a fixed point suitable for the Asymptotic Safety construction, we do not expect the quantum theories defined at these fixed points to be similar or even equivalent. Nonetheless, two such theories might give rise to certain similar predictions, but these can only be assessed on the level of observable quantities.

But why should we not stick to the metric formulation as long as no results are known that disfavor the Asymptotic Safety construction in metric theory space?

Although the effect of fermionic matter in QEG has been investigated in [DP98, PP03b, PP03a, VZ10], these results can only claim the status of an approximation, not only because the studies are restricted to truncations of the metric theory space, but mainly because fermionic actions require the introduction of a vielbein field in order to generalize spinors—being representations of the Lorentz transformations in Minkowski spacetime—to curved spacetimes. For that reason fermionic matter terms naturally extend theory spaces that contain the tetrad rather than the metric as their fundamental field variable. In this respect both, tetrad and Einstein-Cartan theory space, appear as the more adequate framework to describe a fundamental theory of gravity coupled to fermionic matter, and are certainly most natural to be explored.

The first step for the investigation of the tetrad theory space is undertaken in this thesis. We thereby consider the truncation obtained by reexpressing the metric in the Einstein-Hilbert action in terms of the tetrad field. Using an otherwise equivalent setting to the original work on metric gravity in this truncation [Reu98] we are able to trace the differences due to the change of theory space quite explicitly. There are two main reasons to expect both RG flows to differ.

First, it is well-known that the exact FRGE is not covariant under diffeomorphisms in field space. Therefore, the resulting RG flow will change under field reparametrizations, but at the level of observables this parametrization dependence should be canceled out. This expectation is due to the fact that we do *not* change the theory space by a simple reparametrization; it rather corresponds to a different choice of basis in the same theory space.

Second, conceptually more interesting than the additional contributions due to the field reparametrization is the effect of the change in symmetry group, that inevitably accompanies the change of field variables from metric to tetrad gravity. As we will see later on the group of gauge transformations in tetrad gravity is enlarged from the group of diffeomorphisms underlying metric gravity to a group with the semi-direct product structure

$$\mathbf{G} = \mathsf{Diff}(\mathcal{M}) \ltimes \mathsf{O}(d)_{\text{loc}} \,. \tag{1.1}$$

It reflects the fact that in (Euclidean) tetrad gravity we can choose among an $\mathsf{O}(d)$-manifold of equivalent frames in each point of spacetime. It is mainly this change of symmetry group why the transition from metric to tetrad gravity amounts to a change of theory space, and we will find that this change indeed shows significant effects on the RG flow: First, we shall find that the dependence of the resulting RG flow on the precise cutoff procedure becomes more pronounced—an effect that is probably due to the larger ratio of gauge to physical field components in the tetrad formulation—and, second, the contributions of the additional Faddeev-Popov ghost fields related to the $\mathsf{O}(d)_{\text{loc}}$ part of the gauge transformations will turn out crucial to the UV behavior of the flow.

In a second study presented in this thesis, the Einstein-Cartan theory space is explored using a three dimensional Holst-type truncation, that also has been investigated in the first non-perturbative study of this theory space [Dau, DR]. In addition to the vielbein, with its 16 field components, we encounter here a spin connection field of 24 independent components. Taken together this amounts to a quadruplication of independent field components compared to the metric case. This is the main technical reason why RG studies on this theory space are immensely more involved than those in metric theory space when truncations containing the same number of invariants are considered.

Although the first investigation on this space [Dau] could be carried out using a conventional, well-tested FRGE, an inclusion of additional fermionic matter terms, that would be the most interesting extension of this analysis, seems currently out of reach in this very same setting due to its enormous technical complexity.

As a solution to this problem we propose, construct, and test a novel approximative FRGE in this thesis, that simplifies the evaluation of the RG flow as it reduces its computation to a mere algebraic task. In order to assess the reliability of this new FRGE we will carefully compare our results with the previous study [Dau, DR]. We shall indeed find a sufficient amount of structural similarities of the RG flows to recommend this new functional flow equation for the investigation of more general truncations including e. g. fermionic matter terms.

Before we could start to investigate the RG flow on tetrad and Einstein-Cartan theory space, we had to overcome another conceptual problem that is related to the semi-direct product structure of its symmetry group (1.1):

If one employs the background field formalism [Abb81, Abb82] and uses the standard Faddeev-Popov procedure to construct the ghost action from the gauge fixing condition, one finds that the resulting ghost action is *not background gauge invariant* and thus spoils the central idea of the background field method. For this reason a generalization of the background field method had to be developed, which meets the requirement of a background gauge invariant ghost action.

As an example of manageable complexity this new, generalized method was first applied to the system of metric gravity coupled to $\mathsf{SU}(N)$ Yang-Mills theory. The corresponding theory space shows a structurally equivalent group of gauge transformations, namely $\mathbf{G} = \mathsf{Diff}(\mathcal{M}) \ltimes \mathsf{SU}(N)_{\text{loc}}$. As a result of this study we were able to compute gravitational corrections to the running of the Yang-Mills gauge coupling, first published in [DHR10], that are presented in this thesis as well.

Thus, in total this thesis is dedicated to the investigation of theories of gravity with a gauge group of the above semi-direct product structure. We will start in metric theory space extended by a Yang-Mills gauge field, shift to the pure tetrad description of gravity, and then also include the spin connection as an independent field variable. For each of these three investigations we will employ an appropriate FRGE adapted to the respective problem.

In detail, the rest of this thesis is organized as follows:

In Chapter 2 we review the field theoretical concepts for the RG analysis of gravity theories, that were qualitatively described above, in more mathematical terms. We thereby concentrate on the correct extension of the RG framework to gauge groups of the semi-direct product type (1.1) and the introduction of the different FRGEs that are applied in the course of the thesis.

In Chapter 3 we study the effect of gravitational contributions to the Yang-Mills coupling constant and discuss its implications for QED and the SM coupled to asymptotically safe gravity. Chapter 4 contains the first study of the tetrad theory space using a truncation obtained from the Einstein-Hilbert action of metric gravity translated to the vielbein fields. In particular, we investigate the effect of the enlarged gauge group and the additional $\mathsf{O}(d)$-ghost fields it involves.

Chapter 5 is devoted to an RG analysis of the Holst action, that describes gravity in a first-order formulation and considers the spin connection, besides the tetrad, as a second independent field variable. Here, we present the first application of the new flow equation, that we characterize as a Wegner-Houghton(WH)-like flow equation, due to its structural similarity to this well known equation. We discuss in some detail the properties of the resulting RG flow and compare them to the existing study [Dau, DR] of the same theory space in order to judge the applicability of the new WH-like FRGE.

Chapter 6 contains a summary of the most important results obtained and points out some open questions that are left for future investigation.

The main body of the thesis is amended by several appendices. In Appendix A we present the notational conventions used throughout this work and summarize the often used abbreviations. Appendix B contains a derivation of the original Wegner-Houghton RG equation, and its structural similarity to the new WH-like FRGE is discussed. Appendices C–E display results obtained in Chapters 3–5 in their general form, that would otherwise impair the reading fluency in these chapters. Finally, Appendix F discusses the relevant classical aspects of spacetimes exhibiting torsion, concentrating on quadratic torsion invariants, action functionals for gravity in spacetimes with torsion, and the field equations they give rise to.

Remark: The main body of this thesis is written in such a way that the content of the three studies we report on in Chapters 3, 4 and 5 is essentially self-contained. Therefore each of these chapters includes its own introductory and concluding section, and refers only to the background material presented in Chapter 2.

2 Field Theoretical Background

In this chapter we introduce the field theoretical foundations that form the basis of the functional renormalization group techniques in general and their application to gravity, in particular. Of course, this rather short chapter cannot treat the subject in a comprehensive manner; therefore we will emphasize the aspects most important for our investigations in the subsequent chapters and will otherwise refer to the extensive literature on the subject for further details. Most of the background material on the topic is covered in the classic review articles [BB01] on exact functional renormalization group equations (FRGEs) in general, [BTW02] on the FRGE for the effective average action and [NR06] on the Asymptotic Safety scenario for gravity. The present state of research on this field is reviewed in the recent article [RS12].

In the first section we present the conception of the Asymptotic Safety scenario and its relation to the very fundamental notion of theory space. In a second section we concentrate on some peculiarities of gauge theories that come into play when we want to apply renormalization group techniques to gravity or other gauge theories. In the last section the four different flow equations that play a major role in this work are introduced and we will discuss in some detail how they are related to each other.

2.1 Theory space and Asymptotic Safety

2.1.1 Renormalization group

The central idea at the very heart of the renormalization group is that the quantum fluctuations to be integrated over in the path integral can be taken into account in a piecewise manner. This approach was pioneered by Wilson [WK74, Wil75], who applied this idea directly at the level of the path integral, dividing it into several integrations each corresponding to a certain "momentum shell" of quantum fluctuations.

To be more concrete, let $S_\Lambda[\phi]$ denote the bare action of a dynamical system describing a scalar field ϕ with UV momentum cutoff Λ. We may decompose ϕ into the eigenmodes of the operator $-\Box = -\partial_\mu \partial^\mu$, the momentum modes, carrying momentum p. In consequence, the quantum field ϕ may contain modes of all momenta in the interval $|p| \in [0, \Lambda]$. Now we split ϕ according to $\phi = \Phi + \varphi$, such that Φ contains only the modes of ϕ with $|p| \in [0, \Lambda']$ and φ the ones with $|p| \in [\Lambda', \Lambda]$. In addition, we split the path integral into two functional integrations over these two domains. On this basis we are able to define a second, "coarse grained" bare action $S_{\Lambda'}[\Phi]$ that only depends on the lower momentum modes but describes the same quantum system as we demand that the partition function both actions give rise to stays the same. Thus we have

$$Z = \int [\mathcal{D}\phi]_{[0,\Lambda]} \, e^{-S_\Lambda[\phi]} = \int [\mathcal{D}\Phi]_{[0,\Lambda']} \left[\int [\mathcal{D}\varphi]_{[\Lambda',\Lambda]} \, e^{-S_\Lambda[\Phi+\varphi]} \right] \stackrel{!}{=} \int [\mathcal{D}\Phi]_{[0,\Lambda']} \, e^{-S_{\Lambda'}[\Phi]}, \quad (2.1)$$

so that the coarse grained action is defined by

$$e^{-S_{\Lambda'}[\Phi]} = \int [\mathcal{D}\varphi]_{[\Lambda',\Lambda]} \, e^{-S_\Lambda[\Phi+\varphi]}. \quad (2.2)$$

Here, $\int [\mathcal{D}\phi]_{[a,b]}$ denotes the functional integral over momentum modes $|p| \in [a, b]$.

By considering an infinitesimal momentum shell integration of the above type it is possible to derive the Wegner-Houghton renormalization group equation for the bare action S_Λ, that expresses the scale derivative $\partial_\Lambda S_\Lambda$ in terms of functional derivatives $\delta S_\Lambda / \delta \phi$ of the action at that scale. This derivation is carried out in detail in Appendix B.

Instead of using a sharp momentum cutoff splitting the functional integration into two domains one might also use smooth regulator functions, that essentially have the same effect. Different choices of these functions lead to different renormalization group equations [BB01, Mor94, SSA$^+$], as e. g. the Polchinski equation, all of which have in common that their solutions relate bare actions describing the same quantum system with different UV cutoff scales. Thus, in all these cases a further functional integration of the low momentum quantum fluctuations has to be carried out in order to obtain the corresponding Green's functions, from which all physical observables are to be calculated.

In contrast to this "bare action" approach to the renormalization group it is possible to implement the underlying idea only at the level of the effective action Γ, leading to the exact FRGE for the EAA Γ_k [Wet93][1].

In this case we achieve the coarse graining effect by the addition of a (scale-dependent) mode suppression term $\Delta_k S[\varphi] = \frac{1}{2}\int \varphi \mathcal{R}_k(p^2)\,\varphi$ to the bare action $S[\varphi]$. The mode suppression kernel $\mathcal{R}_k(p^2)$ thereby is subject to certain conditions: It must be a continuous function in p^2 and k, which is monotonically decreasing in p^2 for fixed k and increasing in k for a fixed argument p^2. Moreover it satisfies

$$\lim_{p^2\to 0} \mathcal{R}_k(p^2) > 0, \quad \mathcal{R}_k(p^2) \approx 0 \quad \text{for} \quad p^2 \gg k^2 \quad \text{and} \quad \lim_{k\to 0} \mathcal{R}_k(p^2) = 0\,. \tag{2.3}$$

Usually these properties are achieved by choosing the mode suppression kernel to be of the form

$$\mathcal{R}_k(p^2) = \mathcal{Z}_k k^2 R^{(0)}(p^2/k^2), \tag{2.4}$$

where the shape function $R^{(0)}$ is dimensionless and interpolates smoothly between $R^{(0)}(0) = 1$ and $\lim_{z\to\infty} R^{(0)}(z) = 0$, and $\mathcal{Z}_k$ is, in the general case, a matrix in field space, which is to be adapted to the system under consideration as we will describe below. Physically speaking the addition of the mode suppression term can be thought of as assigning a mass to all modes of momentum smaller than k, thus suppressing their propagation in the path integral.

The one parameter family of bare actions $S + \Delta_k S$ together with a source term $\int J\cdot\varphi$ gives rise to a family of generating functionals $W_k[J] = \ln Z_k[J]$. The corresponding scale-dependent generalization of the effective action, the EAA $\Gamma_k[\phi]$, depending on the classical field $\phi = \langle\varphi\rangle$, is given by

$$\Gamma_k[\phi] = \tilde{\Gamma}_k[\phi] - \Delta_k S[\phi]\,, \tag{2.5}$$

where $\tilde{\Gamma}_k[\phi]$ denotes the Legendre transform of $W_k[J]$. It was first shown in [Wet93] that it satisfies the following exact FRGE

$$\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\,\frac{\partial_t \mathcal{R}_k}{\Gamma_k^{(2)} + \mathcal{R}_k}\,, \tag{2.6}$$

[1] For reviews see: [BTW02, Wet01].

where $\Gamma^{(2)}$ denotes the Hessian of the effective action functional, Tr is a functional trace over all fluctuation field modes and $t \equiv \ln k$.

A convenient property of this flow equation is that it does not rely on the UV regularization of the original path integral any more: Due to the scale derivative of the mode suppression kernel $\mathcal{R}_k$ the contributions to the trace on the RHS quickly tend to zero in the UV, while its occurrence in the denominator of (2.6) acts as an IR regulator. Thus, effectively, the trace only takes into account modes of momenta $|p| \approx k$, and we can safely take the infinite cutoff limit of the functional trace on the RHS of the flow equation. This has the direct advantage that in the search for a fundamental, UV complete quantum field theory all difficulties related with a properly defined UV regularization of the path integral are circumvented.

The solutions to (2.6) are one-parameter families of (effective) action functionals, which we refer to as RG trajectories. The RG flow along these trajectories—reflecting the underlying coarse graining procedure—is thereby directed from the UV (large k) to the IR (small k). The limit $k \to 0$ corresponds to removing the mode suppression term as $\Delta_{k=0}S = 0$, such that we obtain an ordinary effective action $\Gamma = \Gamma_{k=0}$ at the IR endpoint of each trajectory. From this effective action all Green's functions of the quantum theory can be computed directly and no additional functional integration is needed. If we can find a complete trajectory, that connects the IR endpoint continuously with a well defined UV limit, i. e. it runs into a UV attractive fixed point of the RG flow, this amounts to a proper definition of a fundamental quantum field theory.

However, from the fixed point action $\Gamma_{k=\infty}$ we cannot directly infer the bare action S_Λ for $\Lambda \to \infty$, that when being quantized gives rise to the effective action $\Gamma_{k=0}$, since its exact form will depend on the precise regularization of the path integral. As noted above this information is not encoded in the running Γ_k, since the FRGE is independent of the specific regularization of the path integral. Thus, only in a second step it is possible to reconstruct the bare action S_Λ from the complete trajectory Γ_k, which has been shown explicitly for the case of QEG in [MR09]. Indeed it turns out that the difference $\Gamma_\infty - S_\Lambda$ is a simple, explicitly calculable functional.

Taken together the flow of the EAA, compared to the running bare action, is much more closely related to the physics at scale k as the functional Γ_k directly corresponds to an effective action at the scale k such that no further functional integral has to be carried out in order to connect it with physical observables. Note however, that the trajectory Γ_k for $k \neq 0$ depends on the precise form of the mode suppression kernel $\mathcal{R}_k(p^2)$, while only its IR endpoint, corresponding to a vanishing kernel, is universal. This variability is referred to as RG scheme dependence. Exact predictions for observable quantities at any momentum scale should therefore, in principle, be calculated from the universal effective action $\Gamma = \Gamma_{k=0}$. Direct predictions from the course of the trajectory Γ_k at $k \neq 0$ can only approximate the exact ones as good as the action $S + \Delta_k S$ effectively describes the bare action of the system at scale k.

The more direct relation of the EAA, Γ_k, to actual physics at a given momentum scale k, compared to the Wilsonian flow of the bare action, S_Λ, is paid for by a less direct relation between Γ_k and the classical action $S_{\Lambda\to\infty}$ that, by a quantization procedure, gives rise to the quantum effective action $\Gamma_{k=0}$.

2.1.2 Asymptotic Safety

As we have seen that the existence of fundamental theories is closely related to the existence of complete RG trajectories of the EAA it becomes clear that in this setting also the notion of renormalizability gets a generalized meaning compared to the perturbative approach.

Instead of asking whether a certain given action, whose form is taken from some classical dynamical system, is renormalizable in the sense that we are able to prevent divergences from appearing in physical observables by adding the correct counterterms, the corresponding question in the functional RG approach must be whether the space of all action functionals, the theory space $\mathcal{T}$, contains UV attractive fixed points and complete trajectories emanating from them, each of which is a candidate for a well-defined fundamental quantum field theory.

The set of all trajectories pulled into a certain fixed point under the inverse RG flow is called the UV attractive hypersurface $\mathscr{S}_{\text{UV}}$ of the fixed point. If $\mathscr{S}_{\text{UV}}$ is finite dimensional, with $s = \dim(\mathscr{S}_{\text{UV}})$, we can pin down the one trajectory realized in nature by measuring s independent observables at a fixed scale k_0. These measured observables correspond to one point on $\mathscr{S}_{\text{UV}}$ that is taken as the initial condition for $\Gamma_{k=k_0}$ and thus specifies the trajectory completely. Hence only finitely many measurements are needed as input parameters of our theory. They give rise to a complete trajectory inside $\mathscr{S}_{\text{UV}}$ such that all other observables are predictions of the theory. Moreover, also the corresponding fundamental bare action is a prediction of the RG flow; it can be reconstructed from the complete trajectory, and does *not* serve as an input in this setting.

This generalized notion of renormalizability, that only depends on the existence of UV attractive fixed points in theory space, comprises the possibility of asymptotically free theories, i. e. UV well-behaved, renormalizable theories in the perturbative sense of the term. An asymptotically free theory, like QCD, corresponds to an RG trajectory whose UV limit is taken at the Gaussian fixed point. This FP is defined as the point in theory space that corresponds to an action functional of a non-interacting, free theory. For the case of QCD the approach of the GFP amounts to a vanishing of the coupling constant, and it has been verified that both, perturbative methods and non-perturbative RG techniques, yield equivalent results [RW94a, RW97] for the running of coupling constant in the deep UV.

Besides this class of theories we should also expect to find fundamental theories whose UV limit is taken at a non-Gaussian, interacting fixed point in theory space. Although their behavior does not become trivial in the deep UV, it is controlled by the scaling properties of the fixed point, such that no divergences arise. Thus, from a global perspective this kind of UV limit seems equally plausible as the asymptotically free case. Theories of this second class are called asymptotically safe or non-perturbatively renormalizable as they typically elude a purely perturbative analysis if the fixed point is located outside the part of theory space that is accessible by perturbation theory.

Thus, non-perturbative methods as the exact FRGE (2.6) for the EAA or related (approximate) functional RG equations are needed in order to explore the Asymptotic Safety scenario.

2.1.3 Exploring Asymptotic Safety in truncated theory space

Theory space. The theory space is the most important ingredient to any RG study as it indirectly defines the system under consideration. Its defining properties are the field content Φ of which all its points, i. e. all action functionals $A : \Phi \mapsto A[\Phi]$ are built from, and the symmetry group $\mathbf{G}$ under whose action these functionals are left invariant.

Generally we will assume that the theory space

$$\mathcal{T} = \{A : \Phi \mapsto A[\Phi] \,|\, A \text{ invariant under } \mathbf{G}\} \tag{2.7}$$

allows for a (typically infinite) set $\{P_\alpha[\Phi]\}$ of basis functionals, such that any point in theory space can be expanded as a unique linear combination of these functionals according to

$$A[\Phi] = \sum_{\alpha=1}^{\infty} \bar{u}_\alpha P_\alpha[\Phi]. \tag{2.8}$$

Thus, the generalized couplings $\bar{u}_\alpha$ serve as coordinates in theory space and any trajectory in theory space is therefore parametrized by an infinite set of coordinate functions, or "running couplings", $\bar{u}_\alpha(k)$,

$$\Gamma_k[\Phi] = \sum_{\alpha=1}^{\infty} \bar{u}_\alpha(k) P_\alpha[\Phi]. \tag{2.9}$$

Any FRGE we could use to explore a given theory space defines a vector field on that theory space whose integral curves are the trajectories solving the FRGE. By expanding both sides of the FRGE in terms of a set of basis functionals, replacing the couplings by their dimensionless counterparts $u_\alpha(k) = \bar{u}_\alpha(k) k^{-d_\alpha}$ and equating the coefficients we are able to transform the FRGE into an equivalent system of autonomous differential equations

$$\partial_t u_\alpha(k) = \beta_\alpha(u_1, u_2, \cdots)\,, \tag{2.10}$$

where the RHS defines a β-function that does not depend on k explicitly, β_α, for each coupling u_α.

Truncations. Usually it is not possible to solve or even derive this complete system of differential equations for a given theory space. Thus, one has to approximate the exact flow (the exact FRGE gives rise to) in a non-perturbative way, which can be done by reducing the basis $\{P_\alpha[\Phi]\}$ to a finite (or infinite) subset $\{P_i[\Phi]\} \subset \{P_\alpha[\Phi]\}$, in order to render the calculation technically feasible. This kind of approximation is known as a truncation of theory space.

Geometrically a truncation of theory space corresponds to neglecting all components of the vector field generated by the FRGE that do not lie inside the subspace of the truncation. As the vector field is first projected onto the subspace of the truncation and then its integral curves are determined, the resulting trajectories do not coincide with the exact trajectories projected onto the subspace under consideration: The difference is exactly the error in the predictions of the running couplings that we pick up by truncating the theory space.

Obviously, the magnitude of this error will depend very much on the truncation chosen. We should expect that extending a given truncation by adding additional invariants will decrease the error of the truncation. However, truncations of the same number of invariants will differ largely in reliability depending on the extend to which they cover the physics content of the full theory. Thus, in any functional RG study it is a major challenge to find a truncation large enough to reliably describe the physics of the model while at the same time being small enough to allow for an explicit computation.

Note that in gravity, due to the large number of independent field components, RG studies are particularly complicated: In metric gravity already the two dimensional truncation $\{\int\sqrt{g}, \int\sqrt{g}R\}$ and its three dimensional extension $\{\int\sqrt{g}, \int\sqrt{g}R, \int\sqrt{g}R^2\}$ truncation amount to demanding calculations [Reu98, LR02]. With the help of computer algebra systems it has been possible to extent these truncations up to $\int\sqrt{g}R^8$ [CPR08, CPR09, MS08] and only recently first studies of the infinite dimensional $f(R)$-truncation have been published [BC12, DSZ]. However, an analysis of the complete (curvature)^2-subspace is still awaited to be carried out. Being aware of this fact helps to understand why in the even more involved setting of Einstein-Cartan gravity, as yet only up to three dimensional truncations could be analyzed [Dau, DR].

Symmetries of the theory space and the choice of FRGE. Generally, the RG equation employed to analyze a theory space

$$\mathcal{T} = \{A : \Phi \mapsto A[\Phi] \,|\, A \text{ invariant under } \mathbf{G}\} \tag{2.11}$$

should be invariant under the symmetry group $\mathbf{G}$, i. e. it should retain the symmetries, as otherwise the flow generated by the FRGE will leave $\mathcal{T}$. This consideration is particularly important when gauge theories like gravity are considered. Here, the theory space can be reduced to gauge invariant functionals of the fields, if it is possible to construct a FRGE that preserves gauge invariance. We will show below how the FRGE (2.6) can be generalized to a gauge invariant setting by the use of the background field method. The resulting exact FRGE is the best known tool to investigate the RG flow of gauge theories in a completely gauge covariant setting.

On the other hand, any theory space whose functionals exhibit a certain symmetry can be seen as embedded in, and hence a truncation of, the larger theory space constructed from the same field content but without the additional symmetry requirement. Using an exact FRGE that retains the symmetry on this larger space will lead to an RG flow in which the symmetric subspace forms a self-consistent truncation, i. e. the resulting trajectories lie entirely either inside the subspace or outside of it. Thus, if we find a UV attractive FP in the symmetric subspace, it gives rise to a complete RG trajectory whose IR endpoint is, as well, symmetric. Using a different exact FRGE in the larger space that, however, does not retain the symmetry, we should expect to find the same FP as its existence should be a property of the underlying theory space rather than relying to the exact form of the RG equation. If we take an asymptotically safe trajectory emanating from this FP and follow it to the IR, it will leave the symmetric subspace at some point, but its IR endpoint, eventually, will be symmetric again, as we then have integrated out all fluctuation modes of a symmetric theory, albeit in a non-covariant manner. Thus, it should be possible to describe and analyze the RG flow of a symmetric theory, using both types of FRGE, one that retains the symmetry throughout and one that restores it only at the level of the effective action $\Gamma_{k=0}$.

On the other hand, when considering practical calculations that always involve a truncation of theory space, the covariant approach, besides being the most natural to choose for a symmetric problem, becomes even more advantageous. Using a truncation of a fixed number of invariants we can always approximate the symmetric subspace better than the larger total theory space. Thus, the quality of the truncation can be increased the more, the larger the symmetry group is, such that with a comparable calculational effort the covariant approach is expected to lead to more accurate results.

Nonetheless it should be kept in mind that using a non-covariant approach for a symmetric theory is comparable to choosing a worse truncation, but should not be despised as being inappropriate from the outset.

Relating different studies within the Asymptotic Safety program. From the above we have learned that particularly well suited tools to investigate the Asymptotic Safety scenario are the exact FRGE for the EAA (2.6), its generalization to gauge theories, to be introduced later, or related (approximate) functional RG equations on the level of the effective action. Using such a FRGE the program to find and investigate asymptotically safe theories consists of the following major steps:

(i) Define a theory space of action functionals.

(ii) For practical reasons: Choose a truncation of theory space to be analyzed.

(iii) Choose an appropriate FRGE and compute the resulting RG flow in this truncation.

(iv) Analyze the structure of the resulting flow, search for UV attractive fixed points and find complete RG trajectories.

Within this program it is obvious that different theory spaces should lead to different quantum theories, as their fixed point structures are generally not related to each other. Even if two theory spaces are motivated by the same classical theory, it could well be that one of them exhibits a FP suitable for the Asymptotic Safety construction while the other does not, indicating that the corresponding fundamental QFT can only be formulated in terms of the field content of one of the theory spaces. Nonetheless, similarities between fundamental theories formulated in different theory spaces might exist. However, these can only be evaluated on the basis of the quantum properties both theories imply; a question that is beyond the present scope of research in asymptotically safe quantum gravity, that primarily focusses on the mere existence of a fundamental QFT of gravity in different theory spaces. Hence, we keep in mind that *the existence of NGFPs has to be investigated independently for every theory space under consideration.*

Considering the same theory space the application of different FRGEs to the same truncation should lead, however, to comparable results. The most trustable results should be obtained by the FRGE (2.6) and its generalization to gauge theories, as it is exact on the untruncated level. Often approximate FRGEs are applied, whose approximations usually can only be justified in comparison to this exact equation or to other well-approved approximations. In particular, any new approximate RG equation should be tested this way.

Similarly, one should expect that different truncations of the action functional Γ_k lead to consistent results, if the essential physics is captured by both truncations. Extending a given truncation should obviously lead to a better approximation of the exact, untruncated flow, such that the stability of fundamental properties, as e. g. the critical exponents of a given fixed point, under extension of the truncation can be considered a measure for its quality. Usually this kind of quality check requires a second complete RG analysis of a more involved truncation. In Chapter 5 we introduce a method that allows for a similar test of the truncation within only one computation.

Note that, in most cases, we are bound to consider a finite or infinite dimensional truncation of theory space, as an untruncated calculation is not feasible. Thus a strict proof of existence of a NGFP in theory space within a usual RG analysis is out of reach, as it could always be an artifact of the truncation considered, in principle, no matter how highly dimensional it is chosen. Nonetheless, every RG study on the same theory space, using different truncations or a different RG equation, helps to understand the structure of the underlying theory space and may add further evidence for the existence of fixed points found in previous calculations.

In the next subsection we introduce the four theory spaces that are investigated in course of this work, all of which can be considered to construct a well-defined quantum theory of gravity. After having generalized (2.6) to the case of gauge theories in Section 2.2, we move on and discuss the second main ingredient to the RG studies of this work, namely the different approximations to the exact FRGE that will be employed.

2.1.4 Theory spaces of metric, tetrad and Einstein-Cartan gravity

In classical General Relativity there exists a remarkably rich variety of different variational principles which give rise to Einstein's equation, or equations equivalent to it but expressed in terms of different field variables. The best known examples are the Einstein-Hilbert action expressed in terms of the metric, $S_{\text{EH}}[g_{\mu\nu}]$, or the tetrad, respectively, $S_{\text{EH}}[e^a{}_\mu]$. The latter action functional is obtained by inserting the representation of the metric in terms of vielbeins into the former: $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$.

Another classically equivalent formulation, at least in absence of spinning matter, is provided by the first order Hilbert-Palatini action $S_{\text{HP}}[e^a{}_\mu, \omega^{ab}{}_\mu]$ which, besides the tetrad, depends on the spin connection $\omega^{ab}{}_\mu$ assuming values in the Lie algebra of $\mathsf{O}(1,3)$. Variation of S_{HP} with respect to $\omega^{ab}{}_\mu$ leads, in vacuo, to an equation of motion which expresses that this connection has vanishing torsion. It can be solved algebraically as $\omega = \omega(e)$ which, when inserted into S_{HP}, brings us back to $S_{\text{EH}}[e] \equiv S_{\text{HP}}[e, \omega(e)]$.

Another equivalent formulation is based upon the self-dual Hilbert-Palatini action $S^{\text{sd}}_{\text{HP}}[e^a{}_\mu, \omega^{(+)}{}^{ab}{}_\mu]$, which only depends on the (complex, in the Lorentzian case) self-dual projection of the spin connection, $\omega^{(+)}{}^{ab}{}_\mu$ [Ash91, AL04, Rov04, Thi07, Kie07]. This action in turn is closely related to the Plebanski action [Ple77], containing additional 2-form fields, and to the Capovilla-Dell-Jacobson action [CJD89, CJD91] which involves essentially only a self-dual connection. Similarly, Krasnov's diffeomorphism invariant

Yang-Mills theories [Kra11a, Kra11b] allow for a "pure connection" reformulation of General Relativity as well as deformations thereof.

The above variational principles are Lagrangian in nature; the fields employed provide a parametrization of configuration space. The corresponding Legendre transformation yields a Hamiltonian description in which the "carrier fields" of the gravitational interaction parametrize a phase-space now. In this way the ADM-Hamiltonian [ADM62] and Ashtekar's Hamiltonian [Ash87], for instance, make their appearance.

Regarding the ongoing search for a quantum (field) theory of gravity this multitude of classical formalisms offers many equally plausible possibilities to explore. A priori it is not clear which one of the above hamiltonian systems, if any, is linked to the as yet unknown fundamental quantum theory in the simplest or most easy to guess way.

The most prominent advantage of the functional RG approach to quantum gravity compared to other approaches is now, that we do not have to specify the classical dynamical system we are going to quantize from the outset. In this sense, it depends on the classical input data to a lesser extent. The only inspiration we draw from the classical system is its field content Φ and its group of symmetry transformations $\mathbf{G}$ that together form the theory space, the respective system gives rise to. The RG analysis then allows us to search for fixed points in this space, which, if found, predicts the fundamental action of the system being quantized. The correct classical limit at low momentum scales k is achieved by choosing among the asymptotically safe trajectories the one that runs through the point in theory space that corresponds to the action of the classical system.

The above examples of classical actions for gravity are motivate the investigation of the following theory spaces:

(i) Einstein gravity. In case of the common metric description of gravity we have $\Phi = g_{\mu\nu}$, and the gauge group is given by the diffeomorphisms of the manifold $\mathcal{M}$, $\mathbf{G} = \mathsf{Diff}(\mathcal{M})$. We denote the corresponding theory space by

$$\mathcal{T}_{\text{E}} = \left\{ A[g_{\mu\nu}, \cdots] \,|\, \text{inv. under } \mathbf{G} = \mathsf{Diff}(\mathcal{M}) \right\} . \tag{2.12}$$

Most of the investigations on asymptotically safe gravity have been carried out within this theory space or extensions thereof, when additional matter fields are coupled to gravity. All of these studies have found a NGFP suitable for the Asymptotic Safety construction and, taken together, they form an impressive amount of evidence for the existence of a NGFP in this theory space.

We will encounter an extension of this space, $\mathcal{T}_{\mathrm{E,YM}}$, in Chapter 3, when we discuss the gravitational effects on the running of the Yang-Mills gauge coupling. This theory space is obtained by the inclusion of an additional Yang-Mills field A^a_μ coupled to gravity, which at the same time enlarges the total gauge group according to

$$\mathcal{T}_{\mathrm{E,YM}} = \left\{ A[g_{\mu\nu}, A^a_\mu, \cdots] \,|\, \text{inv. under } \mathbf{G} = \mathsf{Diff}(\mathcal{M}) \ltimes \mathsf{SU}(N)_{\mathrm{loc}} \right\} . \tag{2.13}$$

(ii) "Tetrad only" gravity. Expressing the metric in terms of the tetrad does not merely amount to a change of field variables on the level of theory space. As (in Euclidean spacetimes, that we consider throughout) there exists an $\mathsf{O}(d)$ manifold of tetrads that correspond to the same metric, this ambiguity is usually treated as an additional $\mathsf{O}(d)_{\mathrm{loc}}$ gauge freedom. Thus, the theory space of "tetrad only" gravity is given by

$$\mathcal{T}_{\mathrm{tet}} = \left\{ A[e^a{}_\mu, \cdots] \,|\, \text{inv. under } \mathbf{G} = \mathsf{Diff}(\mathcal{M}) \ltimes \mathsf{O}(d)_{\mathrm{loc}} \right\} . \tag{2.14}$$

Note thereby the structural similarity between the gauge groups of $\mathcal{T}_{\mathrm{tet}}$ and $\mathcal{T}_{\mathrm{E,YM}}$.

This theory space $\mathcal{T}_{\mathrm{tet}}$ was first explored in the article [HR12], that forms the basis of Chapter 4 of this work. It is intermediate between $\mathcal{T}_{\mathrm{E}}$ and the Einstein-Cartan theory space to be introduced next, in the sense that the gauge group is already enlarged compared to metric gravity, while the connection is still fixed to the Levi-Civita choice. Exploring $\mathcal{T}_{\mathrm{tet}}$ thus can help to understand the cause of differences found between RG studies of metric and Einstein-Cartan gravity.

(iii) Einstein-Cartan gravity. In comparison to the "tetrad only" case in Einstein-Cartan gravity we introduce the spin connection $\omega^{ab}{}_\mu$ as an additional independent field variable. The corresponding theory space is, hence, defined by

$$\mathcal{T}_{\mathrm{EC}} = \left\{ A[e^a{}_\mu, \omega^{ab}_{\ \ \mu}, \cdots] \,|\, \text{inv. under } \mathbf{G} = \mathsf{Diff}(\mathcal{M}) \ltimes \mathsf{O}(d)_{\mathrm{loc}} \right\} . \tag{2.15}$$

First fully non-perturbative investigations of this space with $d = 4$ have been published recently [Dau, DR12, DR]. We consider this theory space in Chapter 5 using the same truncation and $d = 4$ as in this previous study, but we employ a new, approximative RG equation whose reliability can be tested in direct comparison to the findings of [DR].

(iv) Chiral gravity. As noted before, also the restriction of the Hilbert-Palatini action to spin connections of a defined chirality gives rise to Einstein's equation. A first RG analysis of the corresponding theory space

$$\mathcal{T}_{\mathrm{EC}}^{\pm} = \left\{ A[e^{a}{}_{\mu}, (\omega^{(\pm)})^{ab}{}_{\mu}, \cdots] \,|\, \text{inv. under } \mathbf{G} = \mathsf{Diff}(\mathcal{M}) \ltimes \mathsf{O}(4)_{\mathrm{loc}} \right\} . \tag{2.16}$$

is carried out in this work in Chapter 5.5.

2.2 Generalization of the FRGE to gauge theories

In this section we generalize the exact FRGE for scalar fields, (2.6), to the case of gauge theories. The main challenge here is to find a formulation of the FRGE that retains the gauge symmetry, such that the resulting flow does not leave the theory space of gauge invariant action functionals, while at the same time allows for a well defined propagator $(\Gamma^{(2)})^{-1}$ being the central object from which the RG flow is calculated.

These two seemingly irreconcilable goals can be reached by the use of the background field method that is outlined in a first subsection. With the help of this method we are able to write down the sought for generalization of (2.6) in a second subsection. The last subsection is devoted to the construction of an appropriate ghost action, that meets the requirements set by the background field formalism even if the total gauge group $\mathbf{G}$ has the structure of a semi-direct product as is the case in the theory spaces $\mathcal{T}_{\mathrm{E,YM}}$, $\mathcal{T}_{\mathrm{tet}}$ and $\mathcal{T}_{\mathrm{EC}}$ from above.

2.2.1 Background field method

Let us shortly review the main ideas underlying the background field method, that is crucial to the functional RG of gauge theories. For more details on the method in general and its application to gravity in particular we refer to the extensive literature on the subject as e. g. [Abb81, Abb82, DeW67, GNW75, CDRM85, Adl82].

The partition function of gauge theories is given by a path integral of the following schematical form:

$$Z[J] = \int \mathcal{D}\hat{\Phi}\, \mathcal{D}\hat{\Xi}\, \mathcal{D}\hat{\bar{\Xi}}\, e^{-S[\hat{\Phi}] - S_{\mathrm{gf}}[\hat{\Phi}] - S_{\mathrm{gh}}[\hat{\Phi}, \hat{\Xi}, \hat{\bar{\Xi}}] + \int J \cdot (\hat{\Phi}, \hat{\Xi}, \hat{\bar{\Xi}})} . \tag{2.17}$$

Here, $\hat{\Phi}$ and $\{\hat{\Xi}, \hat{\bar{\Xi}}\}$ collectively denote all quantum fields and (quantum) ghost fields, respectively, that are present in the theory under consideration, and J denotes their sources. Throughout this work we encounter the quantum fields

$$\hat{\Phi} \subset \left\{\gamma_{\mu\nu}, \mathcal{A}^a_\mu, \hat{e}^a{}_\mu, \hat{\omega}^{ab}{}_\mu\right\}, \tag{2.18}$$

i. e. the metric, an $\mathsf{SU}(N)$ gauge field, the tetrad and the spin connection. The (anti-) ghost fields corresponding to the gauge symmetries of spacetime diffeomorphisms, $\mathsf{SU}(N)_{\text{loc}}$ and $\mathsf{O}(d)_{\text{loc}}$-transformations are denoted by

$$\hat{\Xi} \subset \left\{\mathcal{C}^\mu, \Sigma^a, \Sigma^{ab}\right\} \qquad \hat{\bar{\Xi}} \subset \left\{\bar{\mathcal{C}}_\mu, \bar{\Sigma}_a, \bar{\Sigma}_{ab}\right\}, \tag{2.19}$$

respectively. The fact that we denote the $\mathsf{SU}(N)$ ghosts and the $\mathsf{O}(d)$ ghosts by the same symbol will not lead to any confusion, as we do not consider theories that obey both symmetries at a time.

In the above path integral the bare action $S[\hat{\Phi}]$ is invariant under gauge transformations of $\hat{\Phi}$. For the diffeomorphisms these are given by the Lie derivatives

$$\delta_{\text{D}}(v)\phi = \mathcal{L}_v\phi \qquad \text{for any} \qquad \phi \in \hat{\Phi}\,. \tag{2.20}$$

Under $\mathsf{SU}(N)_{\text{loc}}$ and $\mathsf{O}(d)_{\text{loc}}$ transformations we have

$$\delta_{\text{YM}}(\lambda)\mathcal{A}^a_\mu = -\partial_\mu\lambda^a - f^{abc}\mathcal{A}^b_\mu\lambda^c \equiv -\hat{\nabla}_\mu\lambda^a \tag{2.21}$$

and

$$\delta_{\text{L}}(\lambda)\hat{e}^a{}_\mu = \lambda^a{}_b\hat{e}^b{}_\mu, \quad \delta_{\text{L}}(\lambda)\hat{\omega}^{ab}{}_\mu = -\partial_\mu\lambda^{ab} - \hat{\omega}^a{}_{c\mu}\lambda^{cb} - \hat{\omega}^b{}_{c\mu}\lambda^{ac} \equiv -\hat{\nabla}_\mu\lambda^{ab}\,, \tag{2.22}$$

respectively. Here, $\hat{\nabla}$ denotes the covariant derivative formed from the respective quantum connection $\mathcal{A}^a_\mu$ or $\hat{\omega}^{ab}{}_\mu$ and f^{abc} are the antisymmetric structure constants of $\mathsf{SU}(N)$. The quantum metric $\gamma_{\mu\nu}$ is a scalar w. r. t. these transformations. The ghost fields transform under all gauge transformations as tensors of the rank their index structure implies.

The gauge invariant action S, however, has to be supplemented by a gauge fixing action S_{gf} and a ghost action S_{gh} that is obtained by the usual Faddeev-Popov method, in order to avoid an overcounting of gauge equivalent configurations in the path integral. To serve its purpose $S_{\rm gf}$ necessarily has to single out one configuration in each gauge orbit, such that it cannot be chosen invariant under the above gauge transformations.

The starting point of the background field formalism is a split of the quantum fields into a sum of a (classical) background field and a quantum fluctuation $\hat{\Phi} = \bar{\Phi} + \varphi$. We denote the individual fields by

$$\bar{\Phi} = \left\{ \bar{g}_{\mu\nu}, \bar{A}^a_\mu, \bar{e}^a{}_\mu, \bar{\omega}^{ab}{}_\mu \right\} \quad \text{and} \quad \varphi = \left\{ h_{\mu\nu}, a^a_\mu, \varepsilon^a{}_\mu, \tau^{ab}{}_\mu \right\} . \tag{2.23}$$

It is important to stress that this split is completely arbitrary, so that in particular the fluctuations do not have to be "small" in any perturbative sense.

Together with the field split we have to specify how the variation of the full quantum fields $\hat{\Phi}$ under gauge transformations is distributed over their components $(\bar{\Phi}, \varphi)$, such that $\delta\bar{\Phi} + \delta\varphi = \delta\hat{\Phi}$. Let us consider the following two possibilities:

(i) Background gauge transformations $\delta^{\rm B}$. In this case the background fields transform in the same manner as the respective full quantum field, while all fluctuation fields transform as tensors. In particular, we thus have

$$\delta^{\rm B}_{\rm D}(v)\phi = \mathcal{L}_v \phi \qquad \forall \phi \in \bar{\Phi} \cup \varphi \tag{2.24}$$

for the diffeomorphisms and

$$\begin{aligned} \delta^{\rm B}_{\rm YM}(\lambda)\bar{A}^a_\mu &= -\partial_\mu \lambda^a - f^{abc}\bar{A}^b_\mu \lambda^c, & \delta^{\rm B}_{\rm YM}(\lambda) a^a_\mu &= f^{abc}\lambda^b a^c_\mu \\ \delta^{\rm B}_{\rm L}(\lambda)\bar{\omega}^{ab}{}_\mu &= -\partial_\mu \lambda^{ab} - \bar{\omega}^a{}_{c\mu}\lambda^{cb} - \bar{\omega}^b{}_{c\mu}\lambda^{ac}, & \delta^{\rm B}_{\rm L}(\lambda)\tau^{ab}{}_\mu &= \lambda^a{}_c \tau^{cb}{}_\mu + \lambda^b{}_c \tau^{ac}{}_\mu \\ \delta^{\rm B}_{\rm L}(\lambda)\bar{e}^a{}_\mu &= \lambda^a{}_b \bar{e}^b{}_\mu, & \delta^{\rm B}_{\rm L}(\lambda)\varepsilon^a{}_\mu &= \lambda^a{}_b \varepsilon^b{}_\mu \end{aligned} \tag{2.25}$$

for $\mathsf{SU}(N)_{\rm loc}$ and $\mathsf{O}(d)_{\rm loc}$ transformations.

(ii) True gauge transformations $\delta^{\rm G}$. Here, the complete transformation is shifted to the fluctuation fields, such that only these are affected by the transformation, while the background fields stay constant. Concretely they read

$$\delta^{\rm G}_{\rm D}(v)\phi = \delta^{\rm G}_{\rm YM}(\lambda)\phi = \delta^{\rm G}_{\rm L}(\lambda)\phi = 0, \qquad \phi \in \bar{\Phi} \tag{2.26}$$

for the background fields, and for the fluctuation fields we have

$$\begin{aligned} \delta^{\mathrm{G}}_{\mathrm{D}}(v)h_{\mu\nu} &= \delta_{\mathrm{D}}(v)\gamma_{\mu\nu}, & & \\ \delta^{\mathrm{G}}_{\mathrm{D}}(v)a^a_\mu &= \delta_{\mathrm{D}}(v)\mathcal{A}^a_\mu, & \delta^{\mathrm{G}}_{\mathrm{YM}}(\lambda)a^a_\mu &= \delta_{\mathrm{YM}}(\lambda)\mathcal{A}^a_\mu \\ \delta^{\mathrm{G}}_{\mathrm{D}}(v)\varepsilon^a{}_\mu &= \delta_{\mathrm{D}}(v)\hat{e}^a{}_\mu, & \delta^{\mathrm{G}}_{\mathrm{L}}(\lambda)\varepsilon^a{}_\mu &= \delta_{\mathrm{L}}(\lambda)\hat{e}^a{}_\mu, \\ \delta^{\mathrm{G}}_{\mathrm{D}}(v)\tau^{ab}{}_\mu &= \delta_{\mathrm{D}}(v)\hat{\omega}^{ab}{}_\mu, & \delta^{\mathrm{G}}_{\mathrm{L}}(\lambda)\tau^{ab}{}_\mu &= \delta_{\mathrm{L}}(\lambda)\hat{\omega}^{ab}{}_\mu\,. \end{aligned} \tag{2.27}$$

After these technical considerations the crucial observation at the heart of the background field method is now that we may use a different partition function $Z[J;\bar\Phi]$ to calculate the effective action that the partition function $Z[J]$ gives rise to. This second partition function parametrically depends on the background fields and is given by

$$Z[J;\bar\Phi] = \int \mathcal{D}\varphi\,\mathcal{D}\hat\Xi\,\mathcal{D}\hat{\bar\Xi}\; e^{-S[\bar\Phi+\varphi]-S_{\mathrm{gf}}[\varphi;\bar\Phi]-S_{\mathrm{gh}}[\varphi,\hat\Xi,\hat{\bar\Xi};\bar\Phi]+\int J\cdot(\varphi,\hat\Xi,\hat{\bar\Xi})}\,. \tag{2.28}$$

Here, the ghost action is constructed by the Faddeev-Popov method using the variation of the gauge fixing condition w. r. t. the above *true gauge transformations* δ^{G}.

By a shift of the integration variable, assuming a translationally invariant path integral measure, it can be shown [Abb82] that it is related to the ordinary partition function (2.17) according to

$$Z[J;\bar\Phi] = Z[J]\cdot e^{-\int J\cdot\bar\Phi} \tag{2.29}$$

under the condition that on the RHS the gauge fixing action $S^{\bar\Phi}_{\mathrm{gf}}[\hat\Phi] = S_{\mathrm{gf}}[\hat\Phi-\bar\Phi;\bar\Phi]$ is used in the definition of the partition function $Z[J]$. This choice of gauge fixing may seem a bit unorthodox in view of the conventional approach as it involves an external field $\bar\Phi$, but turns out perfectly viable from the technical point of view [Abb82].

The most important consequence of (2.29) is obtained at the level of the effective actions that are implied by both partition functions and have the expectation values of the fields $\bar\varphi = \langle\varphi\rangle$ and $\Phi = \bar\Phi + \bar\varphi = \langle\hat\Phi\rangle$ as well as $\Xi = \langle\hat\Xi\rangle$, $\bar\Xi = \langle\hat{\bar\Xi}\rangle$ as their arguments. The individual fields used in this work are denoted by

$$\begin{aligned} \bar\varphi &= \{\bar h_{\mu\nu}, \bar a^a_\mu, \bar\varepsilon^a{}_\mu, \bar\tau^{ab}{}_\mu\}, & \Phi &= \{g_{\mu\nu}, A^a_\mu, e^a{}_\mu, \omega^{ab}{}_\mu\}, \\ \Xi &= \{\xi^\mu, \Upsilon^a, \Upsilon^{ab}\}, & \bar\Xi &= \{\bar\xi_\mu, \bar\Upsilon_a, \bar\Upsilon_{ab}\}\,. \end{aligned} \tag{2.30}$$

For the effective actions we then find the relation [Abb82]

$$\Gamma[\bar{\varphi}, \Xi, \bar{\Xi}; \bar{\Phi}] = \Gamma[\bar{\Phi} + \bar{\varphi}, \Xi, \bar{\Xi}] \;, \tag{2.31}$$

and thus, by setting the fluctuations to zero at this point, we obtain the usual effective action from the path integral (2.28) according to

$$\Gamma[\bar{\Phi}, \Xi, \bar{\Xi}] = \Gamma[0, \Xi, \bar{\Xi}; \bar{\Phi}] \;. \tag{2.32}$$

Moreover, and most importantly for our purposes, we can learn from (2.32) that if we construct $Z[J; \bar{\Phi}]$ in such a way that it is invariant under the *background field transformations* δ^{B} then also the resulting effective action $\Gamma[\bar{\Phi}, \Xi, \bar{\Xi}]$ is a background gauge invariant functional. Inspecting (2.28) we find that we can achieve this goal by choosing a background gauge covariant gauge fixing condition, that renders S_{gf} invariant, and in turn results in a background gauge invariant ghost action S_{gh} as well.

Thus we conclude that the partition function (2.28) in conjunction with a gauge-fixing condition that breaks the invariance under true gauge transformations but retains background gauge invariance is a suitable starting point for the generalization of the exact FRGE (2.6) to the case of gauge theories.

2.2.2 Exact FRGE for gauge theories

In order to construct an exact renormalization group equation for gauge theories we start with (2.28) and add, as in the scalar case, a mode suppression term to the exponent. It is of the form

$$\begin{aligned} \Delta_k S &= \frac{1}{2} \int \mathrm{d}^d x \, \sqrt{g} \, \big(\varphi, \hat{\Xi}, \hat{\bar{\Xi}}\big) \mathcal{R}_k[\bar{\Phi}] \begin{pmatrix} \varphi \\ \hat{\Xi} \\ \hat{\bar{\Xi}} \end{pmatrix} \\ &= \frac{1}{2} \int \mathrm{d}^d x \, \sqrt{g} \, \varphi \, \breve{\mathcal{R}}_k[\bar{\Phi}] \, \varphi + \int \mathrm{d}^d x \, \sqrt{g} \, \hat{\bar{\Xi}} \, \mathcal{R}_k^{\mathrm{gh}}[\bar{\Phi}] \, \hat{\Xi} \, . \end{aligned} \tag{2.33}$$

The mode suppression kernel $\mathcal{R}_k[\bar{\Phi}]$ is block diagonal in field space and decomposes into a diagonal part $\breve{\mathcal{R}}_k[\bar{\Phi}]$ that cuts off the Grassmann-even fields φ and a skewsymmetric block that takes care of ghost-fields $\{\hat{\Xi}, \hat{\bar{\Xi}}\}$ and can be expressed in terms of $\mathcal{R}_k^{\text{gh}}[\bar{\Phi}]$.

Note that according to the conception of the RG the momentum modes of all fields have to be cut off in an equivalent manner, so that ghost fields and all physical fields are treated on the same footing here. Later on we will explain in detail how the exact form of the mode suppression kernel should be adapted to a given truncation.

With this mode suppression kernel the derivation of the flow equation for the EAA is completely analogous to the scalar case [Wet93] and can be found e. g. in [RW94a, Reu98, RS10]. The resulting FRGE, which governs the scale dependence of the EAA $\Gamma_k[\Phi, \bar{\Phi}, \Xi, \bar{\Xi}] \equiv \Gamma_k[\Phi - \bar{\Phi}, \Xi, \bar{\Xi}; \bar{\Phi}]$, reads

$$\partial_t \Gamma_k = \frac{1}{2} \operatorname{STr}\Big[\Big(\Gamma_k^{(2)} + \mathcal{R}_k(\Delta)\Big)^{-1} \partial_t \mathcal{R}_k(\Delta)\Big] \,. \tag{2.34}$$

Here, $\Gamma_k^{(2)}$ denotes the Hessian of Γ_k with respect to all fluctuation and ghost expectation values, and as before $t \equiv \ln k$ is the "RG time". The operator Δ denotes some suitably chosen generalized Laplacian w. r. t. whose spectrum the modes of the fluctuation fields in the trace are cut off. The most straightforward generalization of the coarse graining idea in momentum space therefore is to choose $\Delta = \bar{\mathcal{D}}^2$, i. e. the Laplacian constructed from the **G**-covariant derivative $\bar{\mathcal{D}}_\mu$ based on the background connection. This choice corresponds to a type I cutoff operator in the classification of [CPR09]. Furthermore, in (2.34) the supertrace "STr" refers to an operator trace, that sums over all independent field components and takes into account the Grassmann-odd fields with an additional minus sign.

At this point we encounter a second argument for the need of a background field: At the heart of the RG lies the idea to classify the modes of the quantum fields into high momenta, that are integrated out, and low momentum modes, that remain and effectively describe the theory. In gauge theories this notion should be defined covariantly, such that the cutoff operator Δ has to be constructed from covariant derivatives that necessarily contain a fixed background connection. Nonetheless this approach retains background field independence, which is particularly important in the case of gravity, by the fact that the background field never has to be specified throughout the calculation, such that all results are independent of its exact form; see [RW09, MR10] for details.

As we will not consider any renormalization effects in the ghost sector throughout this work, the part of Γ_k that corresponds to the ghost action with the quantum fields replaced by their expectation values, $S_{\text{gh}}[\bar{\varphi}, \Xi, \bar{\Xi}; \bar{\Phi}]$, is left unrenormalized in this class of approximations. This has the effect, that we can set the fluctuations $\bar{\varphi} = 0$ in the ghost action even before computing the Hessian $\Gamma_k^{(2)}$, which in turn results in a decomposition of the FRGE (2.34) into the two blocks

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr}\Big[\Big(\breve{\Gamma}_k^{(2)} + \breve{\mathcal{R}}_k(\Delta)\Big)^{-1} \partial_t \breve{\mathcal{R}}_k(\Delta)\Big] - \text{Tr}\Big[\Big(S_{\text{gh}}^{(2)} + \mathcal{R}_k^{\text{gh}}(\Delta)\Big)^{-1} \partial_t \mathcal{R}_k^{\text{gh}}(\Delta)\Big] , \quad (2.35)$$

where $\Gamma \equiv \breve{\Gamma} + S_{\text{gh}}$.

This is the form of the exact FRGE that is used in Chapter 4 to investigate the RG flow of "tetrad only" gravity, while the more compact form of (2.34) serves as the starting point for several approximations to the exact FRGE that are compared in Section 2.3.

2.2.3 Construction of the ghost action

According to the standard Faddeev-Popov method the ghost action S_{gh} that represents the Faddeev-Popov determinant $\det \delta\mathcal{F}^J/\delta v^I$ in the path integral, is obtained by applying an infinitesimal gauge transformation $\delta(v)$ to the gauge condition $\mathcal{F}^J$, replacing the parameter of the transformation by the corresponding ghost field and contracting the expression with its anti-ghost field. Here, the indices I, J stand for an arbitrary general index structure of the objects under consideration. Schematically the ghost action is thus of the form

$$S_{\text{gh}}[\hat{\Phi}, \hat{\Xi}, \hat{\bar{\Xi}}] = -\int \text{d}^d x \, \hat{\bar{\Xi}}_J \frac{\partial \mathcal{F}^J}{\partial \hat{\Phi}^I} \delta(\hat{\Xi}) \hat{\Phi}^I \, . \quad (2.36)$$

Employing the background field method as described above we should use the true gauge transformations in the construction of the ghost action. As then only the fluctuation fields transform and their infinitesimal variation equals the one of the full quantum field, we find in this case

$$S_{\text{gh}}[\varphi, \hat{\Xi}, \hat{\bar{\Xi}}; \bar{\Phi}] = -\int \text{d}^d x \, \hat{\bar{\Xi}}_J \frac{\partial \mathcal{F}^J}{\partial \varphi^I} \delta^{\text{G}}(\hat{\Xi}) \varphi^I = -\int \text{d}^d x \, \hat{\bar{\Xi}}_J \frac{\partial \mathcal{F}^J}{\partial \varphi^I} \delta(\hat{\Xi}) \hat{\Phi}^I \, . \quad (2.37)$$

If the total gauge group consists only of a single component as e. g. only diffeomorphisms in metric gravity or the $\mathsf{SU}(N)$ in case of pure YM theory, this ghost action is background gauge invariant if and only if the gauge condition $\mathcal{F}^I$ transforms as a tensor under the group action. To see this, let us consider the different factors in the ghost action: Clearly $\hat{\bar{\Xi}}_J$ transforms as a tensor, and also the derivative $\partial\mathcal{F}^J/\partial\varphi^I$ is a tensor, as under background gauge transformations all fluctuation fields transform tensorially. Finally, also $\delta(\hat{\Xi})\hat{\Phi}^I$ is a tensor since every gauge transformation of a full quantum field preserves the character of the field (tensor, connection), such that its infinitesimal variation must transform as a tensor w. r. t. the gauge group under consideration.

If we consider YM theory coupled to gravity or gravity in the vielbein formalism, the total gauge group has a product structure and consists of the spacetime diffeomorphisms and some inner group $\mathsf{SU}(N)_{\text{loc}}$ or $\mathsf{O}(4)_{\text{loc}}$, respectively. The naive generalization of the above approach would be to introduce a second gauge fixing condition $\mathcal{G}^I$ for the inner group and apply both types of gauge transformations to each gauge fixing condition. The corresponding ghost action would then be of the form

$$\begin{aligned} S_{\text{gh}}[h,a,\mathcal{C},\bar{\mathcal{C}},\Sigma,\bar{\Sigma};\bar{g},\bar{A}] = -\int \mathrm{d}^d x\,\sqrt{\bar{g}}\,\Big(&\kappa^{-1}\,\bar{\mathcal{C}}_\mu \bar{g}^{\mu\nu}\frac{\partial\mathcal{F}_\nu}{\partial h_{\rho\sigma}}\delta_{\text{D}}(\mathcal{C})\gamma_{\rho\sigma}+ \\ &+\kappa^{-1}\,\bar{\mathcal{C}}_\mu \bar{g}^{\mu\nu}\frac{\partial\mathcal{F}_\nu}{\partial h_{\rho\sigma}}\delta_{\text{YM}}(\Sigma)\gamma_{\rho\sigma} + \hat{g}\,\bar{\Sigma}^a\frac{\partial\mathcal{G}^a}{\partial a^b_\mu}\delta_{\text{D}}(\mathcal{C})\mathcal{A}^b_\mu + \hat{g}\,\bar{\Sigma}^a\frac{\partial\mathcal{G}^a}{\partial a^b_\mu}\delta_{\text{YM}}(\Sigma)\mathcal{A}^b_\mu\Big), \end{aligned} \tag{2.38}$$

where we specialized for the case of YM theory coupled to gravity (formulated in a spacetime of vanishing torsion) and considered gauge fixing conditions of the form $\mathcal{F}_\mu(h_{\mu\nu};\bar{g}_{\mu\nu},\bar{A}^a_\mu)$ and $\mathcal{G}^a(a^a_\mu;\bar{g}_{\mu\nu},\bar{A}^a_\mu)$ for the diffeomorphisms and the $\mathsf{SU}(N)$ transformations, respectively. We will stick to this example for the rest of this section. The treatment of tetrad gravity is completely analogous due to the structural equivalence of its gauge group; see [DR, DR10] for the explicit formulae.

The problem with the ghost action (2.38) lies in the last factor of the third term, that reads explicitly:

$$\delta_{\text{D}}(\mathcal{C})\mathcal{A}^b_\mu = \mathcal{L}_{\mathcal{C}}\mathcal{A}^b_\mu = \mathcal{C}^\rho\partial_\rho\mathcal{A}^b_\mu + (\partial_\mu\mathcal{C}^\rho)\mathcal{A}^b_\rho = \mathcal{C}^\rho D_\rho\mathcal{A}^b_\mu + (D_\mu\mathcal{C}^\rho)\mathcal{A}^b_\rho\,. \tag{2.39}$$

From the last expression we observe the manifestly covariant character of the term w. r. t. diffeomorphisms.[2] From the same expression it becomes obvious that the tensorial character of $\mathcal{A}_\mu^b$ w. r. t. the $\mathsf{SU}(N)$ transformations is not retained by the diffeomorphisms: The Lie derivative is not covariant under $\mathsf{SU}(N)$ transformations, and hence we see that the diffeomorphisms do not map $\mathsf{SU}(N)$ tensors onto $\mathsf{SU}(N)$ tensors. Thus the ghost action (2.38) spoils the background gauge invariance crucial to our approach and is therefore not suitable for the construction of an invariant effective average action Γ_k.

In order to find the proper generalization of the ghost action (2.37) that retains background gauge invariance it is helpful to analyze the structure of the total gauge group in a bit more detail. To this end we introduce the so-called Ward operators, the generators of the gauge transformations in the space of functionals which depend on the full quantum fields,

$$\begin{aligned}\mathcal{W}_{\mathrm{D}}(v) &= -\int \mathrm{d}^d x \left[\delta_{\mathrm{D}}(v)\hat{\Phi}^I(x)\frac{\delta}{\delta\hat{\Phi}^I(x)} + \delta_{\mathrm{D}}(v)\hat{\Xi}^I(x)\frac{\delta}{\delta\hat{\Xi}^I(x)} + \delta_{\mathrm{D}}(v)\hat{\bar{\Xi}}^I(x)\frac{\delta}{\delta\hat{\bar{\Xi}}^I(x)}\right]\\ \mathcal{W}_{\mathrm{YM}}(\lambda) &= -\int \mathrm{d}^d x \left[\delta_{\mathrm{YM}}(\lambda)\hat{\Phi}^I(x)\frac{\delta}{\delta\hat{\Phi}^I(x)} + \delta_{\mathrm{YM}}(\lambda)\hat{\Xi}^I(x)\frac{\delta}{\delta\hat{\Xi}^I(x)} + \delta_{\mathrm{YM}}(\lambda)\hat{\bar{\Xi}}^I(x)\frac{\delta}{\delta\hat{\bar{\Xi}}^I(x)}\right].\end{aligned} \tag{2.40}$$

They satisfy the following algebra

$$\begin{aligned}[\mathcal{W}_{\mathrm{D}}(v_1), \mathcal{W}_{\mathrm{D}}(v_2)] &= \mathcal{W}_{\mathrm{D}}([v_1, v_2])\,,\\ [\mathcal{W}_{\mathrm{YM}}(\lambda_1), \mathcal{W}_{\mathrm{YM}}(\lambda_2)] &= \mathcal{W}_{\mathrm{YM}}(f\lambda_1\lambda_2)\,,\\ [\mathcal{W}_{\mathrm{D}}(v), \mathcal{W}_{\mathrm{YM}}(\lambda)] &= \mathcal{W}_{\mathrm{YM}}(\mathcal{L}_v\lambda)\,.\end{aligned} \tag{2.41}$$

The first two relations state the usual composition rules for diffeomorphisms and $\mathsf{SU}(N)$ transformations. The fact that the mixed commutator of the Ward operators does not vanish but results in an $\mathsf{SU}(N)$ transformation shows that only the

[2]Notational remark: Since we deal in this work with theories exhibiting two gauge invariances we have to distinguish three different kinds of covariant derivatives. We write $D \equiv \partial+\Gamma$ for the covariant derivatives that are constructed by means of the spacetime connection $\Gamma^\rho_{\mu\nu}$ and $\nabla \equiv \partial+A$ or $\nabla \equiv \partial+\omega$ for those constructed from the $\mathsf{SU}(N)$ connection A^a_μ or the spin connection $\omega^{ab}{}_\mu$, respectively; the covariant derivative containing both of these connections is denoted by $\mathcal{D} \equiv \partial + A/\omega + \Gamma$. By adding a bar, we denote their analogs evaluated on the background configurations.

$\mathsf{SU}(N)_{\rm loc}$ is a normal subgroup of the total gauge $\mathbf{G}$. Hence, it has the structure of a semi-direct product of the two subgroups

$$\mathbf{G} = \mathsf{Diff} \ltimes \mathsf{SU(N)}_{\rm loc}. \tag{2.42}$$

Also the spoiled gauge covariance in the ghost action can be traced back to this non-vanishing commutator. If we explicitly write down the infinitesimal transformation of the problematic term $\delta_{\rm D}(\mathcal{C})\mathcal{A}^a_\mu$ under $\mathsf{SU}(N)_{\rm loc}$ transformations we find

$$\begin{aligned}\delta_{\rm YM}(\lambda)\delta_{\rm D}(\mathcal{C})\mathcal{A}^a_\mu &= \mathcal{W}_{\rm YM}(\lambda)\mathcal{W}_{\rm D}(\mathcal{C})\mathcal{A}^a_\mu \\ &= \mathcal{C}^\rho\partial_\rho(-\partial_\mu\lambda^a - f^{abc}\mathcal{A}^b_\mu\lambda^c) + (\partial_\mu\mathcal{C}^\rho)(-\partial_\rho\lambda^a - f^{abc}\mathcal{A}^b_\rho\lambda^c) \\ &\neq f^{abc}\lambda^b\left(\mathcal{C}^\rho\partial_\rho\mathcal{A}^c_\mu + (\partial_\mu\mathcal{C}^\rho)\mathcal{A}^c_\rho\right) \\ &= \mathcal{W}_{\rm D}(\mathcal{C})\mathcal{W}_{\rm YM}(\lambda)\mathcal{A}^a_\mu\,,\end{aligned} \tag{2.43}$$

and conclude that $\delta_{\rm D}(\mathcal{C})\mathcal{A}^a_\mu$ would transform as an $\mathsf{SU}(N)_{\rm loc}$ tensor if the two types of Ward operators commuted.

This observation evokes the question whether it is possible to reparametrize the total group of gauge transformations by a linear combination of its generators, such that these modified Ward operators have a vanishing mixed commutator.

Indeed this turns out possible and there even arise two different choices for a modification of the diffeomorphisms, that make the mixed commutator vanish. Let us present these two possibilities in the following. In both cases the new, modified diffeomorphisms include an additional $\mathsf{SU}(N)_{\rm loc}$ transformation whose parameter depends on the vector field v^μ generating the diffeomorphism. Loosely speaking, the $\mathsf{SU}(N)_{\rm loc}$ covariantization of the Lie derivative in (2.39) is achieved by shifting a certain v-dependent part of $\mathsf{SU(N)}_{\rm loc}$ into the diffeomorphism sector [Jac78].

(i) In the first case the parameter of the $\mathsf{SU}(N)_{\rm loc}$ transformation is obtained as the contraction of the vector field v^μ with the *full quantum field* $\mathcal{A}^a_\mu$: $\lambda^a = \mathcal{A}^a_\mu v^\mu$. The modified diffeomorphisms are then defined as

$$\widetilde{\mathcal{W}_{\rm D}}(v) \equiv \mathcal{W}_{\rm D}(v) + \mathcal{W}_{\rm YM}(\mathcal{A}\cdot v)\,. \tag{2.44}$$

The notion of an "invariant functional" and therefore the total theory space obviously remains unchanged by this reparametrization

$$\begin{aligned}\mathcal{F}_{\text{inv}} &= \{F \mid \widetilde{\mathcal{W}_{\text{D}}}(v)F = 0 \;\wedge\; \mathcal{W}_{\text{YM}}(\lambda)F = 0 \;\forall\; v^\mu,\; \lambda^a\} \\ &= \{F \mid \mathcal{W}_{\text{D}}(v)F = 0 \;\wedge\; \mathcal{W}_{\text{YM}}(\lambda)F = 0 \;\forall\; v^\mu,\; \lambda^a\}\;,\end{aligned} \tag{2.45}$$

while the algebra is modified according to $((v_1 v_2 \cdot \hat{F})^a \equiv v_1^\mu v_2^\nu \hat{F}^a_{\mu\nu})$

$$\begin{aligned}[\widetilde{\mathcal{W}_{\text{D}}}(v_1), \widetilde{\mathcal{W}_{\text{D}}}(v_2)] &= \widetilde{\mathcal{W}_{\text{D}}}([v_1, v_2]) - \mathcal{W}_{\text{YM}}(v_1 v_2 \cdot \hat{F})\;, \\ [\mathcal{W}_{\text{YM}}(\lambda_1), \mathcal{W}_{\text{YM}}(\lambda_2)] &= \mathcal{W}_{\text{YM}}(f \lambda_1 \lambda_2)\;, \\ [\widetilde{\mathcal{W}_{\text{D}}}(v), \mathcal{W}_{\text{YM}}(\lambda)] &= 0\;,\end{aligned} \tag{2.46}$$

where $\hat{F}^a_{\mu\nu}$ is the field strength tensor of the quantum gauge field $\mathcal{A}^a_\mu$.

By replacing the original true diffeomorphisms δ_{D} in (2.38) by these new, modified diffeomorphisms $\widetilde{\delta_{\text{D}}}$ we finally arrive at a background gauge invariant ghost action of the form

$$\begin{aligned}S_{\text{gh}}[h, a, \mathcal{C}, \bar{\mathcal{C}}, \Sigma, \bar{\Sigma}; \bar{g}, \bar{A}] = -\int \mathrm{d}^d x\, \sqrt{\bar{g}}\, \Big(\kappa^{-1}\, \bar{\mathcal{C}}_\mu \bar{g}^{\mu\nu} \frac{\partial \mathcal{F}_\nu}{\partial h_{\rho\sigma}} \widetilde{\delta_{\text{D}}}(\mathcal{C}) \gamma_{\rho\sigma} + \\ + \kappa^{-1}\, \bar{\mathcal{C}}_\mu \bar{g}^{\mu\nu} \frac{\partial \mathcal{F}_\nu}{\partial h_{\rho\sigma}} \delta_{\text{YM}}(\Sigma) \gamma_{\rho\sigma} + \hat{g}\, \bar{\Sigma}^a \frac{\partial \mathcal{G}^a}{\partial a^b_\mu} \widetilde{\delta_{\text{D}}}(\mathcal{C}) \mathcal{A}^b_\mu + \hat{g}\, \bar{\Sigma}^a \frac{\partial \mathcal{G}^a}{\partial a^b_\mu} \delta_{\text{YM}}(\Sigma) \mathcal{A}^b_\mu \Big)\;.\end{aligned} \tag{2.47}$$

Compared to (2.38) the first term did not change since

$$\widetilde{\delta_{\text{D}}}(\mathcal{C}) \gamma_{\rho\sigma} = \delta_{\text{D}}(\mathcal{C}) \gamma_{\rho\sigma} = \mathcal{L}_{\mathcal{C}} \gamma_{\rho\sigma}, \tag{2.48}$$

but the term that caused all trouble, (2.39), now reads

$$\widetilde{\delta_{\text{D}}}(\mathcal{C}) \mathcal{A}^b_\mu = \hat{F}^b_{\rho\mu} \mathcal{C}^\rho\;, \tag{2.49}$$

and has hence a manifestly tensorial character w. r. t. both, diffeomorphisms and $\mathsf{SU}(N)_{\text{loc}}$ transformations.

(ii) The second choice of reparametrization differs from the first one only in the $\mathsf{SU}(N)_{\text{loc}}$ parameter used for the definition of the modified diffeomorphisms. Here it is obtained by contracting the vector field v^μ with the *background field* $\bar{A}^a_\mu$: $\lambda^a = \bar{A}^a_\mu v^\mu$. Then we define the modified diffeomorphisms as

$$\widetilde{\widetilde{\mathcal{W}_{\rm D}}}(v) \equiv \mathcal{W}_{\rm D}(v) + \mathcal{W}_{\rm YM}(\bar{A}\cdot v)\ . \tag{2.50}$$

As before the notion of an "invariant functional" and the total theory space remains unchanged

$$\begin{aligned}\mathcal{F}_{\rm inv} &= \{F \mid \widetilde{\widetilde{\mathcal{W}_{\rm D}}}(v)F = 0\ \wedge\ \mathcal{W}_{\rm YM}(\lambda)F = 0\ \forall\ v^\mu,\ \lambda^a\} \\ &= \{F \mid \mathcal{W}_{\rm D}(v)F = 0\ \wedge\ \mathcal{W}_{\rm YM}(\lambda)F = 0\ \forall\ v^\mu,\ \lambda^a\}\ ,\end{aligned} \tag{2.51}$$

while the algebra now is modified according to $((v_1 v_2\cdot\bar{F})^a \equiv v_1^\mu v_2^\nu \bar{F}^a_{\mu\nu})$

$$\begin{aligned}[\widetilde{\widetilde{\mathcal{W}_{\rm D}}}(v_1), \widetilde{\widetilde{\mathcal{W}_{\rm D}}}(v_2)] &= \widetilde{\widetilde{\mathcal{W}_{\rm D}}}([v_1, v_2]) + \mathcal{W}_{\rm YM}(v_1 v_2\cdot\bar{F})\ , \\ [\mathcal{W}_{\rm YM}(\lambda_1), \mathcal{W}_{\rm YM}(\lambda_2)] &= \mathcal{W}_{\rm YM}(f\lambda_1\lambda_2)\ , \\ [\widetilde{\widetilde{\mathcal{W}_{\rm D}}}(v), \mathcal{W}_{\rm YM}(\lambda)] &= \mathcal{W}_{\rm YM}(v\cdot\bar{\nabla}\lambda)\ ,\end{aligned} \tag{2.52}$$

where $\bar{F}^a_{\mu\nu}$ is the field strength tensor of the background gauge field $\bar{A}^a_\mu$. The difference compared to the algebra (2.46) is explained by the fact that the parametric background field dependence of the argument is not subject to the action of the Ward operators.

However, if we consider the corresponding background gauge transformations we re-obtain an algebra of the former type, i. e.

$$\begin{aligned}[\widetilde{\widetilde{\mathcal{W}^{\rm B}_{\rm D}}}(v_1), \widetilde{\widetilde{\mathcal{W}^{\rm B}_{\rm D}}}(v_2)] &= \widetilde{\widetilde{\mathcal{W}^{\rm B}_{\rm D}}}([v_1, v_2]) - \mathcal{W}^{\rm B}_{\rm YM}(v_1 v_2\cdot\bar{F})\ , \\ [\mathcal{W}^{\rm B}_{\rm YM}(\lambda_1), \mathcal{W}^{\rm B}_{\rm YM}(\lambda_2)] &= \mathcal{W}^{\rm B}_{\rm YM}(f\lambda_1\lambda_2)\ , \\ [\widetilde{\widetilde{\mathcal{W}^{\rm B}_{\rm D}}}(v), \mathcal{W}^{\rm B}_{\rm YM}(\lambda)] &= 0\ ,\end{aligned} \tag{2.53}$$

and we will see below, that the vanishing mixed commutator of the background Ward operators is sufficient to construct a background gauge invariant ghost action.

The formal expression for the ghost action differs only by an additional tilde on top of the infinitesimal diffeomorphisms

$$\begin{aligned}S_{\rm gh}[h, a, \mathcal{C}, \bar{\mathcal{C}}, \Sigma, \bar{\Sigma}; \bar{g}, \bar{A}] = -\int {\rm d}^d x\, \sqrt{\bar{g}}\, \Big(&\kappa^{-1}\, \bar{\mathcal{C}}_\mu \bar{g}^{\mu\nu} \frac{\partial\mathcal{F}_\nu}{\partial h_{\rho\sigma}} \widetilde{\widetilde{\delta_{\rm D}}}(\mathcal{C})\gamma_{\rho\sigma} + \\ + \kappa^{-1}\, \bar{\mathcal{C}}_\mu \bar{g}^{\mu\nu} \frac{\partial\mathcal{F}_\nu}{\partial h_{\rho\sigma}} \delta_{\rm YM}(\Sigma)\gamma_{\rho\sigma} + \hat{g}\, \bar{\Sigma}^a \frac{\partial\mathcal{G}^a}{\partial a^b_\mu} &\widetilde{\widetilde{\delta_{\rm D}}}(\mathcal{C})\mathcal{A}^b_\mu + \hat{g}\, \bar{\Sigma}^a \frac{\partial\mathcal{G}^a}{\partial a^b_\mu} \delta_{\rm YM}(\Sigma)\mathcal{A}^b_\mu\Big)\ ,\end{aligned} \tag{2.54}$$

but the explicit expression of the formerly problematic term (2.39) now reads

$$\widetilde{\widetilde{\delta_{\mathrm{D}}}}(\mathcal{C})\mathcal{A}^b_\mu = \bar{F}^b_{\rho\mu}\mathcal{C}^\rho + \big(\bar{\mathcal{D}}_\rho a^b_\mu\big)\mathcal{C}^\rho + a^b_\rho\big(\bar{D}_\mu\mathcal{C}^\rho\big)\,. \tag{2.55}$$

Thus we obtain again a tensor under background gauge transformations and, hence, we succeeded in constructing a suitable ghost action, but its explicit form is more complicated than before.

It is difficult to assess which of the above two possibilities should generally be used in an RG analysis. The formalism seems to suggest the first choice: As in the expression for the ghost action only transformations of the undecomposed quantum fields occur, it seems only natural also to choose the full quantum field $\mathcal{A}^a_\mu$ as the parameter in the modified diffeomorphisms. Moreover, it generally leads to a ghost-action of a simpler form as all fluctuation fields will combine to full quantum fields therein.

From the conceptual point of view, however, several questions arise when opting for the first possibility, that require a deeper analysis. Primarily one would have to find out how to deal with infinitesimal gauge transformations that contain the quantum field itself as their parameter. A first important observation is that these are no longer linear in the quantum fields, such that it is not clear, whether or how they are related to finite transformations via an exponential map. Moreover, one would have to carefully check whether the Faddeev-Popov method itself remains unchanged, i. e. that the ghost-action we constructed gives rise to the correct functional determinant in the path integral.

Thus, conceptually, the second alternative seems more attractive. Although introducing the background field into the parameter of the gauge transformation may seem a bit artificial, it has the virtue that it serves merely as an external parameter at this point. Hence, the conceptually problematic mixing of the gauge transformations with the actual configuration of the quantum fields is avoided.

Luckily, within our approximation we do not have to make a definite choice between the two possibilities. As we will neglect the renormalization effects in the ghost sector in all our studies, we can set the fluctuation fields to zero in the ghost action S_{gh} even before the Hessian is computed. In this case both possibilities from above become equivalent, as eqns. (2.49) and (2.55) differ only in terms containing fluctuation fields.

Whenever an RG study including the ghost sector is carried out, however, one should be aware of the fact that the two possibilities of constructing the ghost action will lead to different results. Before opting for the first alternative, in particular, the issues raised above should be solved.

With this construction of an appropriate ghost action we have defined all ingredients of the exact FRGE for gauge theories (2.35). Next we are going to introduce a number of different approximations to this exact equation that will be applied to the different theory spaces, later on.

2.3 Comparison of RG flow equations

In this section we want to introduce certain approximations to the exact FRGE for the effective average action, which will be used in the subsequent chapters on to analyze the renormalization group flow of quantum gravity in the different theory spaces introduced above.

The derivation of the flow equations presented here, that are related to each other by three successive approximative steps, clearly has the logical status of a motivation only, since the validity of each of these steps can eventually be proven only by a comparison to results obtained from the exact equation. The main reason for using an approximative instead of the exact RG equation is that the computational effort for a given truncation can be reduced in consequence. This is especially useful when considering the Einstein-Cartan theory space $\mathcal{T}_{\text{EC}}$: RG studies on this space show a dramatically increased complexity mainly due to the quadruplication of independent field components compared to the metric formulation. If we further want to include fermionic matter to the model in future calculations, the exact treatment is rendered a hopelessly involved task. Here, a well-tested simplified approximative FRGE would be more than welcome.

We start from the FRGE for the effective average action, (2.34),

$$\partial_t \Gamma_k = \frac{1}{2}\,\text{STr}\left[\frac{\partial_t \mathcal{R}_k(\Delta)}{\Gamma_k^{(2)} + \mathcal{R}_k(\Delta)}\right] \quad \text{with} \quad \mathcal{R}_k(\Delta) = \mathcal{Z}_k k^2 R^{(0)}\left(\frac{\Delta}{k^2}\right) . \tag{2.56}$$

As introduced above, $R^{(0)}(x)$ is a dimensionless shape function, which interpolates smoothly between 1 and 0 according to

$$R^{(0)}(0) = 1 \quad \text{and} \quad \lim_{x\to\infty} R^{(0)}(x) = 0, \tag{2.57}$$

but is arbitrary apart from that. The argument Δ is called cutoff operator since the discrimination between "low" and "high" momentum fluctuations is obtained with respect to its eigenmodes with eigenvalues $\lambda < k^2$ and $\lambda > k^2$, respectively. Therefore, the cutoff operator Δ is usually chosen to a generalized momentum squared operator e. g. the flat Laplacian $-\Box$ for a scalar theory or the (background) covariant Laplacian $-\bar{D}^2$ in the case of metric gravity. For the equation to be exact any positive operator is permissable as a cutoff operator, whose eigenmodes form a complete set of functions in Hilbert space. Following the RG flow from the scale k to $k - \delta k$ corresponds to integrating out the modes of the cutoff operator of approximately this range of eigenvalues. Therefore, the conditions the cutoff operator is subject to guarantee that we end up at $k = 0$ having integrated out the fluctuations at all scales.

At this point we want to explain how the factor $\mathcal{Z}_k$, being a matrix in field space, is conventionally adapted to a given truncation by the so-called $\mathcal{Z}_k = \zeta_k$-rule: $\mathcal{Z}_k$ should be chosen such that for any eigenmode of $\Gamma_k^{(2)}$ with eigenvalue $\zeta_k p^2$ the sum $\Gamma_k^{(2)} + \mathcal{R}_k$ has eigenvalue $\zeta_k(p^2 + k^2 R^{(0)}(p^2/k^2))$. It has proven most useful to apply this kind of adaptation even for $\zeta_k < 0$: In case of metric gravity in the Einstein-Hilbert truncation [Reu98], divergences that may arise due to the conformal factor problem, can be circumvented by this choice. In higher order $f(R)$-truncations and for polynomials of even order in R, the conformal factor problem ceases to exist as in the UV all ζ_k become positive. It has been shown [LR02, CPR09] that these truncations show a very similar UV behavior to the one found in [Reu98], which can been seen as a justification of the $\mathcal{Z}_k = \zeta_k$-rule even for $\zeta_k < 0$. For a more detailed discussion of this point see [LR02].

In a first step of approximation we employ a "spectrally adjusted" or "type III" [CPR09] cutoff operator, namely we choose $\Delta = \mathcal{Z}_k^{-1}\Gamma_k^{(2)}[\bar{\Phi}, \bar{\Phi}]$, depending on the background fields $\bar{\Phi}$ only (i. e. with the fluctuations $\bar{\varphi}$ set to zero). This choice of operator can be seen as an approximation to the covariant Laplacian $\Delta = -\bar{\mathcal{D}}^2$ by the following argument: Since the matrix $\mathcal{Z}_k$ is adapted to the inverse propagator $\Gamma_k^{(2)}$ such that the addition $\Gamma_k^{(2)} + \mathcal{R}_k$ shifts its modes according to $\zeta_k p^2 \to \zeta_k(p^2 + k^2 R^{(0)}(p^2/k^2))$, we obtain for the background field independent part $\Gamma_{0\,k}^{(2)}$ of the inverse propagator $\Gamma_k^{(2)}$ the identity $\mathcal{Z}_k^{-1}\Gamma_{0\,k}^{(2)} = -\partial^2 \mathbb{1}$. Differences to using the covariant Laplacian therefore only occur in the background field dependent terms of the cutoff operator.

In general $\mathcal{Z}_k^{-1}\Gamma_k^{(2)}$ is not necessarily a positive operator, and moreover a new dependence on the scale parameter k is introduced into the flow equation by this choice of cutoff operator. Thus the picture of integrating out the eigenmodes of the cutoff operator shell by shell becomes somewhat unclear by this approximation. On the other hand we gain that the RHS of the flow equation now allows for a simple spectral representation as it only depends on a single differential operator. Inserting the spectrally adjusted cutoff operator into the flow equation we obtain

$$\begin{aligned}\partial_t\Gamma_k &= \frac{1}{2}\,\mathrm{STr}\left[\frac{2\left(R^{(0)}\left(\frac{x}{k^2}\right)-\frac{x}{k^2}R^{(0)\prime}\left(\frac{x}{k^2}\right)\right)-\eta R^{(0)}\left(\frac{x}{k^2}\right)+\frac{\partial_t x}{k^2}R^{(0)\prime}\left(\frac{x}{k^2}\right)}{\frac{x}{k^2}+R^{(0)}\left(\frac{x}{k^2}\right)}\right]\Bigg|_{x=\mathcal{Z}_k^{-1}\Gamma_k^{(2)}}\\ &= \frac{1}{2}\,\mathrm{STr}\left[W_1(x)-\eta W_2(x)+\frac{\partial_t\left(\mathcal{Z}_k^{-1}\Gamma_k^{(2)}\right)}{k^2}W_3(x)\right]\Bigg|_{x=\mathcal{Z}_k^{-1}\Gamma_k^{(2)}},\end{aligned} \tag{2.58}$$

where $\eta = -\mathcal{Z}_k^{-1}(\partial_t\mathcal{Z}_k)$. The last term in the square brackets reflects the newly introduced scale dependence of the cutoff operator. Deriving this equation one has to keep in mind that the operators $\partial_t(\mathcal{Z}_k^{-1}\Gamma_k^{(2)})$ and $\mathcal{Z}_k^{-1}\Gamma_k^{(2)}$ as well as the matrix η in general do not commute. But since $\partial_t(\mathcal{Z}_k^{-1}\Gamma_k^{(2)})$ and η only occur (at most) once in each summand under the trace, all possible orderings of the operators in these terms become equivalent due to the cyclicity of the trace. Only because of that we arrive at an equation of such a simple form.

As a next step we perform an inverse Laplace transform with respect to the variable x resulting in

$$\begin{aligned}\partial_t\Gamma_k &= \frac{1}{2}\,\mathrm{STr}\left[\int_0^\infty \mathrm{d}s\left(\widetilde{W}_1(s)-\eta\widetilde{W}_2(s)+\frac{\partial_t\left(\mathcal{Z}_k^{-1}\Gamma_k^{(2)}\right)}{k^2}\widetilde{W}_3(s)\right)e^{-s\mathcal{Z}_k^{-1}\Gamma_k^{(2)}}\right]\\ &= \frac{1}{2}\int_0^\infty \mathrm{d}s\left[\widetilde{W}_1(s)\,\mathrm{STr}\left[e^{-s\mathcal{Z}_k^{-1}\Gamma_k^{(2)}}\right]-\widetilde{W}_2(s)\,\mathrm{STr}\left[\eta\, e^{-s\mathcal{Z}_k^{-1}\Gamma_k^{(2)}}\right]\right.\\ &\qquad\left.+\frac{\widetilde{W}_3(s)}{k^2}\,\mathrm{STr}\left[\partial_t\left(\mathcal{Z}_k^{-1}\Gamma_k^{(2)}\right)e^{-s\mathcal{Z}_k^{-1}\Gamma_k^{(2)}}\right]\right].\end{aligned} \tag{2.59}$$

This FRGE is used in Chapter 3 to analyze the gravitational contributions to running gauge couplings. The three different terms in (2.59) correspond to different approximations: If we only take into account the first term, the computation amounts to a usual 1-loop calculation with a non-standard regulator. Including the second trace is a first step of RG improvement: The η matrix corresponds to the running couplings being fed back into the RHS of the flow equation. Finally, the third term takes into account the additional k-dependence of the RHS, due to the running of the couplings in the type III cutoff operator; it can be seen as a second step of RG improvement.

In view of the further approximations to be applied it is useful to rewrite (2.59) in yet another form:

$$\begin{aligned}\partial_t \Gamma_k &= \frac{1}{2}\,\mathrm{STr}\left[\int_0^\infty \mathrm{d}s \left(\widetilde{W}_1(s) - \eta \widetilde{W}_2(s) + \frac{\partial_t\big(\mathcal{Z}_k^{-1}\Gamma_k^{(2)}\big)}{k^2}\widetilde{W}_3(s)\right) e^{-s\mathcal{Z}_k^{-1}\Gamma_k^{(2)}}\right] \\ &= \frac{1}{2}\,\mathrm{STr}\left[\int_0^\infty \mathrm{d}s \left(-\frac{\partial_t f_k(s)}{s}\right) e^{-s\mathcal{Z}_k^{-1}\Gamma_k^{(2)}}\right] \\ &= \frac{1}{2}\,\mathrm{STr}\left[\int_0^\infty \mathrm{d}s \left(-\frac{D_t f_k(\mathcal{Z}_k s)}{s}\right) e^{-s\Gamma_k^{(2)}}\right],\end{aligned} \tag{2.60}$$

where the second equality sign is meant as a *definition* for the matrix valued function $f_k(s)$, and the scale derivative D_t only acts on the explicit k-dependence of the function f_k (i. e. not on its argument $\mathcal{Z}_k s$).

In a second step of approximation we substitute $f_k(s)$ in the integrand by a simpler function of s, which is chosen such that the above definition nonetheless qualitatively reproduces the desired properties of $\mathcal{R}_k$. In particular, the chosen function $f_k(s)$ should regularize the integral in (2.60) both, in the UV and the IR. Thus, it should fall off to zero quickly for arguments $s > k^{-2}$ and $s < \Lambda^{-2}$, where Λ denotes some UV cutoff scale. A common choice is the one parameter set of functions $f_k^m(s)$ given by [BR05]

$$f_k^m(s) = \frac{\Gamma(m+1, sk^2) - \Gamma(m+1, s\Lambda^2)}{\Gamma(m+1)}, \tag{2.61}$$

where m is an arbitrary real, positive parameter, and $\Gamma(\alpha, x) = \int_x^\infty \mathrm{d}t\; t^{\alpha-1}e^{-t}$ denotes the incomplete Gamma function. Note that Λ, thanks to the scale derivative in the integrand, can safely be taken to infinity at the level of the flow equation. As a result this yields a FRGE of the form

$$\partial_t \Gamma_k = -\frac{1}{2}\,\mathrm{STr}\left[\int_0^\infty \frac{\mathrm{d}s}{s}\; D_t f_k^m(\mathcal{Z}_k s)\; e^{-s\Gamma_k^{(2)}}\right], \tag{2.62}$$

which is known as the “proper-time” (PT) flow equation [Reub].

A flow equation of this type has widely been used in the literature [Lia96, Lia97, SBW02, SP99, PSPW00]; it is well known to give accurate results for several scalar theories [BZ01] and has also been applied to metric gravity [BR05].

The first fully non-perturbative study of the theory space $\mathcal{T}_{\text{EC}}$ has also been carried out using a flow equation of the proper-time type [Dau, DR12, DR]. Therein first evidence for the existence of asymptotically safe theories, called Quantum Einstein Cartan Gravity (QECG), in this space was found, and we will compare our findings on QECG in Chapter 5 carefully to it.

As a small aside we note that the proper-time flow equation can also be derived by "RG improvement" of the one-loop effective action. At first we have to regularize the UV divergences and introduce an IR cutoff scale k in the one-loop expression. This can be done, using Schwinger's proper-time representation for the logarithm, by cutting off the proper-time variable s which results in a restriction of the integration to the interval $s \in [\Lambda^{-2}, k^{-2}]$.

$$\Gamma^{\text{1-loop}}\Big|_{\text{REG}} = S + \frac{1}{2}\,\text{STr}\ln S^{(2)}\Big|_{\text{REG}} = S - \frac{1}{2}\,\text{STr}\int_{\Lambda^{-2}}^{k^{-2}} \frac{\text{d}s}{s}\; e^{-sS^{(2)}} \tag{2.63}$$

In a more general setting the same effect can be obtained by inserting the "smearing functions" $f_k^m(\mathcal{Z}s)$ into the integrand, whose domain is essentially restricted to this interval:

$$\Gamma^{\text{1-loop}}\Big|_{\text{REG}} = S - \frac{1}{2}\,\text{STr}\int_0^\infty \frac{\text{d}s}{s} f_k^m(\mathcal{Z}s)\; e^{-sS^{(2)}}. \tag{2.64}$$

Taking the scale derivative ∂_t of this equation and subsequently employing RG improvement ($S^{(2)} \to \Gamma_k^{(2)}$, $\mathcal{Z} \to \mathcal{Z}_k$) we get back to equation (2.62).

Let us come back to the proper-time flow equation (2.62) and perform the last step of approximation. The main idea here is to use different smearing functions f_k^m depending on the momentum dependence of the exponentiated operator. At this point we specialize for the case of constant background fields $\bar{\Phi}$, such that $\Gamma_k^{(2)}$ is diagonal in momentum space. Then we can represent the functional trace by a momentum integral according to

$$\begin{aligned}
\partial_t\Gamma_k &= -\frac{1}{2}\,\text{STr}\left[\int_0^\infty \frac{\text{d}s}{s}\partial_t f_k^m(s) e^{-s\mathcal{Z}_k^{-1}\Gamma_k^{(2)}}\right] \\
&= -\frac{1}{2}\,\text{str}\left[\int_0^\infty \frac{\text{d}s}{s}\partial_t f_k^m(s)\int \text{d}^d x\;\text{d}^d p\;\langle x|p\rangle\langle p|e^{-s\mathcal{Z}_k^{-1}\Gamma_k^{(2)}}|x\rangle\right] \\
&\overset{\bar{\Phi}=\text{const}}{=} -\frac{1}{2}\int_0^\infty \frac{\text{d}s}{s}\partial_t f_k^m(s)\int \text{d}^d x \int \frac{\text{d}^d p}{(2\pi)^d}\;\text{str}\left[e^{-s\mathcal{Z}_k^{-1}\Gamma_k^{(2)}(p)}\right].
\end{aligned} \tag{2.65}$$

Here str denotes the remaining algebraic supertrace which is due to the matrix structure of the operator. Now we make use of the fact, that $\mathcal{Z}_k$ is adapted to $\Gamma^{(2)}_{0\,k}$ as described above, such that we can pull out the dominant quadratic momentum dependence in the exponent and expand the remainder as a power series in the momentum variable p:

$$\begin{aligned} \mathcal{Z}_k^{-1}\Gamma_k^{(2)} &= \mathcal{Z}_k^{-1}\Gamma^{(2)}_{0\,k} + \mathcal{Z}_k^{-1}\Gamma^{(2)}_{\bar{\Phi}\,k} = p^2\mathbb{1} + \mathcal{Z}_k^{-1}\Gamma^{(2)}_{\bar{\Phi}\,k} \\ \Rightarrow e^{-s\mathcal{Z}_k^{-1}\Gamma_k^{(2)}} &= e^{-sp^2} e^{\mathcal{Z}_k^{-1}\Gamma^{(2)}_{\bar{\Phi}\,k}} = e^{-sp^2}\left(\sum_{n=0}^{\infty} p^n A_n\right). \end{aligned} \tag{2.66}$$

In this series expansion the quantities A_n carry the matrix character of the expression. In principle we have to sum over different possible momentum dependences in each order (e.g. p^2 and $p_\mu p_\nu$ in second order), but due to the symmetric momentum integration those will combine to a single contribution for every even power p^{2n}, while the odd powers vanish. Therefore we have

$$\begin{aligned} \partial_t\Gamma_k &= -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}s}{s}\partial_t f_k^m(s)\int \mathrm{d}^d x\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{str}\left[e^{-sp^2}\sum_{n=0}^{\infty}p^{2n}A_{2n}\right] \\ &= -\frac{1}{2}\,\mathrm{str}\left[\int_0^\infty\frac{\mathrm{d}s}{s}\int\mathrm{d}^d x\sum_{n=0}^{\infty}\partial_t\left(f_k^m(s)\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,e^{-sp^2}p^{2n}\right)A_{2n}\right]. \end{aligned} \tag{2.67}$$

At this point we employ our last approximation, namely to choose the parameter m of the smearing function for each term in the expansion to the value $m = n + d/2 - 1$. We will see in a moment how this specific choice of m changes the character of the cutoff procedure [Lia96]. Inserting the explicit form of the family $f_k^m(s)$ of smearing functions (2.61) we obtain

$$\begin{aligned} f_k^{n+d/2-1}(s)\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,e^{-sp^2}p^{2n} &= \frac{\Gamma(n+d/2, sk^2) - \Gamma(n+d/2, s\Lambda^2)}{\Gamma(n+d/2)}\cdot \\ &\qquad\qquad \cdot\, 2v_d\int_0^\infty \mathrm{d}y\; e^{-sy}\; y^{n+d/2-1} \\ &= \frac{2\,v_d}{s^{d/2+n}}\big(\Gamma(n+d/2, sk^2) - \Gamma(n+d/2, s\Lambda^2)\big) \\ &= \int_{|p|=k}^{\Lambda}\frac{\mathrm{d}^d p}{(2\pi)^d}\,e^{-sp^2}p^{2n}, \end{aligned} \tag{2.68}$$

where $v_d = (2^{d+1}\pi^{d/2}\Gamma(d/2))^{-1}$.

From (2.68) we observe that this last step of approximation actually corresponds to a transition from proper-time regularization to momentum cutoff regularization [Lia96]: Using the proper-time cutoff with one smearing function f^m only, a specific power of the momentum p^{2m-d+2} is cut off sharply, while the others are cut off smoothly in such a way that background gauge invariance is retained. Adapting the smearing functions for each power of momentum p^{2n} we obtain a sharp momentum cutoff in all orders of p.

A principal drawback of this step of approximation is that we spoil the background gauge invariance of our approach, since we now cut off the gauge dependent quantity p instead of the gauge invariant proper-time parameter s.

Recall, however, from Section 2.1.3 that breaking the covariance of an FRGE applied to a gauge theory should have a similar effect as choosing a less good truncation of the theory space. Hence, we will have to assess the reliability of this approximative step at the level of the resulting RG flow and its universal properties.

Inserting relation (2.68) into the flow equation (2.67) and resumming the series to an exponential we obtain

$$\begin{aligned}\partial_t \Gamma_k &= -\frac{1}{2}\mathrm{str}\left[\int_0^\infty \frac{\mathrm{d}s}{s}\int \mathrm{d}^d x\, \sum_{n=0}^{\infty} \partial_t \left(\int_{|p|=k}^{\Lambda} \frac{\mathrm{d}^d p}{(2\pi)^d}\, e^{-sp^2} p^{2n}\right) A_{2n}\right] \\ &= -\frac{1}{2}\mathrm{str}\left[\int_0^\infty \frac{\mathrm{d}s}{s}\int \mathrm{d}^d x\, D_t \int_{|p|=k}^{\Lambda} \frac{\mathrm{d}^d p}{(2\pi)^d}\, e^{-s\mathcal{Z}_k^{-1}\Gamma_k^{(2)}}\right] \\ &= \frac{1}{2}\, D_t\, \mathrm{STr}\Big|_k \ln\left(\Gamma_k^{(2)}\right),\end{aligned} \tag{2.69}$$

where $\mathrm{STr}|_k$ denotes the infrared (IR) regularization of the trace by a sharp cutoff of the momentum integral and D_t is a scale derivative acting only on the explicit scale dependence of this IR cutoff. Note that at this level we can safely take the limit $\Lambda \to \infty$ as the expression is UV finite due to the scale derivative D_t. It is this form of the FRGE that we use to study the RG flow of quantum gravity in the first order formulation (Chapter 5).

Discussion of the new approximative flow equation. As we have seen above, the derivation of the new flow equation, (2.69), involves the following three approximative steps

- Use the complete Hessian $\Gamma_k^{(2)}$ as the cutoff operator Δ in the exact FRGE (2.34) in order to obtain a spectral representation of its RHS by a Laplace transformation.
- Replace the exact Laplace transform of the RHS by one member (fixed m) of the set of functions $f_k^m(s)/s$ that is chosen to reproduce certain convergence properties of the exact Laplace transform.
- Choose a different value for m for the terms of different momentum dependence in the integrand. Explicitly we choose $m = n+d/2-1$ for the terms proportional to p^{2n}.

These seemingly rather crude approximations of the exact RG equation become necessary since the Hessian in the first order (e,ω)-formulation of pure gravity is a 40 x 40-matrix operator, compared to the corresponding 10 x 10 operator in metric gravity. Using the approximative FRGE (2.69) we can reduce the evaluation of its RHS to a completely algebraic task, as we show in Section 5.2.1. Compared to the proper-time equation the major simplification of the resulting β-functions lies in the fact, that here the proper-time integral can be carried out explicitly, such that only one dimensionful scaling parameter k is left over.

Since the derivation of this approximation, unavoidably, lacks a strict mathematical justification we shall demonstrate its validity by a comparison to the result obtained from the proper-time equation for pure gravity [DR]. If it turns out reliable, the new FRGE (2.69) suggests itself for the analysis of even more involved systems, like first order gravity coupled to fermionic matter, in future investigations.

As a final comment it is worth noting that our approximation to the exact FRGE, (2.69), for the effective average action Γ_k is now formally equivalent to the Wegner-Houghton equation (for the Wilsonian running bare action S_k) in the special case of constant background fields. In this case only the term corresponding to 1 PI contributions on the RHS of the WH equation is non-vanishing. Therefore, it is plausible that the same equation may serve as an approximation for both S_k and Γ_k. In Appendix B the WH equation is introduced and this correspondence is demonstrated in more detail. Due to this formal equivalence we will refer to the new FRGE (2.69) also as WH-like flow equation.

3 Effects of Quantum Gravity in Gauge Theories

3.1 Motivation

This chapter is devoted to the effects of asymptotically safe gravity, when it is coupled to gauge theories. We thereby focus onto the two cases of $\mathsf{SU}(N)$ Yang-Mills theory and QED, which divide the chapter two main parts.

The first part deals with a question that has caused considerable debate over the last few years, namely the existence and explicit form of gravitational corrections to the β-function of the Yang-Mills gauge coupling.

In a first study on this topic Robinson and Wilczek [RW06, Rob] found a non-zero correction at the one-loop level within an effective field theory setting. It shows the same negative sign as the well-known gluon contribution and thus supports the approach of asymptotic freedom in the deep UV. Later on, Pietrykowski [Pie07] realized that this result is gauge fixing dependent, such that also a vanishing one-loop contribution can be obtained when considering a certain gauge condition. This observation motivated a manifestly gauge invariant as well as gauge fixing independent study of the effect by Toms [Tom07] using the Vilkovisky-deWitt method. Therein, a vanishing gravitational contribution was found that only gets modified when in addition a cosmological constant is present [Tom08]. However, this result was put into question by Ebert et al. [EPR09], who noted that the use of the dimensional regularization technique applied in [Tom07, Tom08] is not suitable to the problem, as it does not keep track of the quadratic divergences, that are expected to cause the effect. Nonetheless, in their study [EPR09] Ebert et al. obtain a vanishing correction, as well, by applying a cutoff regularization. In [TW10] Tang and Wu argued that this regularization technique is not permissable either, as it does not respect gauge invariance. Performing a calculation in a scheme which both retains quadratic divergences and preserves gauge invariance ("loop regularization") they obtained a non-zero gravitational correction to the one-loop β-function.

At this point of the discussion our RG study of the running gauge coupling [DHR10] was published, that forms the basis of the first part of this chapter. In contrast to all studies mentioned before, it is carried out within the framework of the Asymptotic Safety approach to quantum gravity and thus takes into account metric gravity not only as an effective but rather a fundamental quantum field theory. Besides that functional RG methods for effective actions are a perfectly well suited tool to circumvent the shortcomings of other regularization schemes that were criticized in previous calculations, being in this sense most comparable the study by Tang and Wu [TW10]: When the construction of the ghost action is correctly generalized to the semi-direct product structure of the pertinent gauge group, as described in Section 2.2.3, the FRGE approach is manifestly (background) gauge invariant and at the same time retains all quadratic divergences that are crucial to the problem. At the perturbative level the non-zero result we obtain turns out to be in line with the original computation [RW06] and [TW10], accelerating the approach of asymptotic freedom.

In view of these new findings Toms reanalyzed the problem [Tom10] for the case of gravity coupled to QED, using the Vilkovisky-deWitt method, but this time employing a proper-time cutoff. As this scheme is gauge fixing independent, keeps track of powerlike divergences and preserves gauge invariance, the study [Tom10] amounts to the most complete perturbative analysis of the problem to date. It leads to the following one-loop RG equation for the running of the electric charge with the energy scale E:

$$E\,\frac{\mathrm{d}e(E)}{\mathrm{d}E} = \frac{e^3}{12\pi^2} - \frac{e}{\pi}\left(GE^2 + \frac{3}{2}G\Lambda\right) \,. \tag{3.1}$$

The first term on the RHS of (3.1) is the familiar one due to the fermion loops and tends to increase e at large energies. The second term amounts to the gravity correction and involves Newton's constant G as well as the cosmological constant Λ. It has a negative sign and tries to drive e to smaller values as E increases. In fact, it has been claimed [Tom10] that the electric charge vanishes at high energies and may be regarded an *asymptotically free* coupling therefore. This result follows directly from (3.1) as in the perturbative context G is considered fixed, such that the gravity contribution increases with the energy squared and will at some finite scale outweigh the fermionic contributions. Of course, it remains questionable whether at this scale the perturbative setting is still applicable.

In the second part of this chapter we therefore reconsider this picture in the light of asymptotically safe gravity. In this context, the running of the dimensionful Newton constant is taken into account. Using the result of the first part of the chapter we shall demonstrate that QED coupled to QEG is asymptotically safe, and that, besides asymptotic freedom of the electric charge, there exists a second option for its behavior at high energies: it may assume a non-zero fixed point value $e^* \neq 0$. If this option is realized the asymptotic behavior of QED + QEG is governed by a non-Gaussian fixed point whose hypersurface $\mathscr{S}_{\rm UV}$ is likely to have a lower dimension than in the corresponding asymptotically *free* case $e^* = 0$. We shall exploit this larger degree of predictivity to show within a simple truncation of theory space that in the asymptotically safe theory with $e^* \neq 0$ the infrared value of the electron charge, or the fine-structure constant $\alpha \equiv e^2/(4\pi)$, is a *computable number* which is completely fixed by the electron mass in Planck units.

No matter which option for the UV limit of the coupled system is realized in nature our results suggest that the triviality problem of pure QED will eventually be solved by the onset of quantum gravitational effects.

3.2 The running Yang-Mills coupling constant

In order to determine the gravitational corrections to the running Yang-Mills gauge coupling we explicitly evaluate the FRGE (2.59) on the theory space $\mathcal{T}_{\rm E,YM}$ in this section. We thereby truncate $\mathcal{T}_{\rm E,YM}$ to a set of invariants which is general enough to allow for an approximate determination of the β-function for the scale dependent Yang-Mills coupling $g_{\rm YM}(k)$. This truncation is given by the following ansatz:

$$\begin{aligned}\Gamma_k[g,\bar{g},A,\bar{A},\xi,\bar{\xi},\Upsilon,\bar{\Upsilon}] = \Gamma_k^{\rm EH}[g] + \Gamma_k^{\rm YM}[g,A] + \Gamma_k^{\rm gf}[g-\bar{g},A-\bar{A};\bar{g},\bar{A}] \\ +S_{\rm gh}[g-\bar{g},A-\bar{A},\xi,\bar{\xi},\Upsilon,\bar{\Upsilon};\bar{g},\bar{A}]\end{aligned} \tag{3.2}$$

Here,

$$\Gamma_k^{\rm EH}[g] = 2\kappa^2 Z_N(k)\int {\rm d}^d x\,\sqrt{g}\,\left(-R(g)+2\Lambda(k)\right) \tag{3.3}$$

is a k-dependent form of the Einstein-Hilbert action. The corresponding dimensionful running parameters are the cosmological constant $\Lambda(k)$ and Newton's constant $G(k) \equiv \hat{G}/Z_N(k)$, where $\hat{G}$ is a fixed reference value.

Furthermore,

$$\Gamma_k^{\rm YM}[g,A] = \frac{Z_F(k)}{4\,\hat{g}_{\rm YM}^2}\int {\rm d}^d x\,\sqrt{g}\,g^{\mu\rho}g^{\nu\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \tag{3.4}$$

is the standard second-order Yang-Mills action, with a k-dependent prefactor $Z_F(k)$ though. Hence the (dimensionful, except in d=4) running gauge coupling is $\bar{g}_{\rm YM}(k) = \hat{g}_{\rm YM} Z_F(k)^{-1/2}$ with some constant $\hat{g}_{\rm YM}$. Finally,

$$\Gamma_k^{\rm gf}[g-\bar{g}, A-\bar{A};\bar{g},\bar{A}] = \int {\rm d}^d x\,\sqrt{\bar{g}}\,\left(\frac{Z_N(k)}{2\alpha_{\rm D}}\bar{g}^{\mu\nu}\mathcal{F}_\mu\mathcal{F}_\nu + \frac{Z_F(k)}{2\alpha_{\rm YM}}\mathcal{G}^a\mathcal{G}^a\right) \tag{3.5}$$

implements the gauge fixing conditions for the diffeomorphisms, $\mathcal{F}_\mu$, and the $\mathsf{SU}(N)$ gauge transformations, $\mathcal{G}^a$. As in [Reu98] and [RW94a] we factored out the wave function renormalizations Z_N and Z_F from the gauge fixing parameters $\alpha_{\rm D}$ and $\alpha_{\rm YM}$, respectively. In principle the latter are still k-dependent but we shall neglect their running here. In fact, later on we set $\alpha_{\rm D} = \alpha_{\rm YM} = 1$. Our choice for the gauge conditions complies with the requirements set by the background field method that were discussed in Section 2.2.1:

$$\mathcal{F}_\mu(\bar{h};\bar{g}) = \sqrt{2}\kappa\left(\delta^\beta_\mu\bar{g}^{\alpha\gamma}\bar{D}_\gamma - \frac{1}{2}\bar{g}^{\alpha\beta}\bar{D}_\mu\right)\bar{h}_{\alpha\beta}\,, \tag{3.6}$$

$$\mathcal{G}^a(\bar{a};\bar{g},\bar{A}) = \hat{g}^{-1}\bar{g}^{\mu\nu}\bar{\mathcal{D}}_\mu\bar{a}^a_\nu\,. \tag{3.7}$$

The resulting ghost action constructed according to the method introduced in Section 2.2.3, with $a^a_\mu \neq 0$ and $h_{\mu\nu} \neq 0$ still and opting for the second choice of modified diffeomorphisms, $\widetilde{\widetilde{\delta_{\rm D}}}$, reads

$$\begin{aligned} S_{\rm gh}&[h,a,\xi,\bar{\xi},\Upsilon,\bar{\Upsilon};\bar{g},\bar{A}] = \\ &-\int {\rm d}^d x\,\sqrt{\bar{g}}\,\Big(\sqrt{2}\bar{\xi}_\mu\left(\bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda\left(g_{\rho\nu}D_\sigma + g_{\sigma\nu}D_\rho\right) - \bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda g_{\sigma\nu}D_\rho\right)\xi^\nu \\ &\quad + \bar{\Upsilon}^a\bar{g}^{\mu\nu}\bar{\mathcal{D}}_\mu\big(\bar{F}^a_{\rho\nu}\xi^\rho + (\bar{\mathcal{D}}_\rho a^a_\nu)\xi^\rho + a^a_\rho(\bar{D}_\nu\xi^\rho)\big) + \bar{\Upsilon}^a\left(\bar{g}^{\mu\rho}\delta^{ab}\bar{\mathcal{D}}_\mu\nabla_\rho\right)\Upsilon^b\Big)\,. \end{aligned} \tag{3.8}$$

It can be checked that $S_{\rm gh}$ of eq. (3.8) is invariant under background gauge transformations: $\mathcal{W}^{\rm B}_{\rm YM}S_{\rm gh} = 0 = \mathcal{W}^{\rm B}_{\rm D}S_{\rm gh}$. While this is true even for non-vanishing fluctuations h and a, in the present calculation we shall need $S_{\rm gh}$ only for $h = 0 = a$, as we are going to neglect all renormalization effects of ghost couplings.

At this point a remark concerning the expected reliability of this truncation ansatz might be in order. As for its gravitational part, all generalizations of the Einstein-

Hilbert truncation explored during the past decade did not change the qualitative picture it gives rise to, at least close to the non-Gaussian fixed point. In the Yang-Mills sector we retained only the first monomial of a systematic derivative expansion. From the analysis in [RW94a] without gravity we know that this truncation is not only sufficient to reproduce one-loop perturbation theory exactly, but even approximates the two-loop result for the β-function with a small error of a few percent. Therefore we may expect that this truncation, too, is perfectly sufficient as long as k is sufficiently large (well above dynamically generated "confinement" scales, say). In particular, it should thus reliably describe the FP regime of the coupled system.

Let us briefly review at this point how we arrive at a FRGE of the type (2.59) and how it subsequently decomposes into a Grassmann-even and a ghost part for the specific truncation that we study here. First, we insert the truncation ansatz (3.2) into the exact FRGE (2.35) resulting in a decomposition of the supertrace into a "bosonic" and a ghost contribution:

$$\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\left[\frac{\partial_t \breve{\mathcal{R}}_k\big(\mathcal{Z}_k^{-1}\breve{\Gamma}_k^{(2)}\big)}{\breve{\Gamma}_k^{(2)} + \breve{\mathcal{R}}_k\big(\mathcal{Z}_k^{-1}\breve{\Gamma}_k^{(2)}\big)}\right] - \mathrm{Tr}\left[\frac{\partial_t \mathcal{R}_k^{\mathrm{gh}}\big(\mathcal{Z}_{\mathrm{gh}}^{-1}S_{\mathrm{gh}}^{(2)}\big)}{S_{\mathrm{gh}}^{(2)} + \mathcal{R}_k^{\mathrm{gh}}\big(\mathcal{Z}_{\mathrm{gh}}^{-1}S_{\mathrm{gh}}^{(2)}\big)}\right] \tag{3.9}$$

Here, the bosonic part of the action is given by $\breve{\Gamma}_k \equiv \Gamma_k^{\mathrm{EH}} + \Gamma_k^{\mathrm{YM}} + \Gamma_k^{\mathrm{gf}}$, and $\breve{\Gamma}_k^{(2)}$ is its Hessian. The coarse graining operators in the two sectors of (3.9) have the structure

$$\breve{\mathcal{R}}_k(x) = \breve{\mathcal{Z}}_k k^2 R^{(0)}(x/k^2) \qquad \mathcal{R}_k^{\mathrm{gh}}(x) = \mathcal{Z}_k^{\mathrm{gh}} k^2 R^{(0)}(x/k^2)\,. \tag{3.10}$$

As introduced in Section 2.3 $R^{(0)}(y)$ is a "shape function" continuously interpolating between $R^{(0)}(0) = 1$ and $\lim_{y\to\infty} R^{(0)}(y) = 0$. The constants $\breve{\mathcal{Z}}_k$ and $\mathcal{Z}_k^{\mathrm{gh}}$ are matrices in field space that are adapted to the truncation according to the $\mathcal{Z} = \zeta$-rule (cf. Section 2.3). We shall see that this requirement is met if $\breve{\mathcal{Z}}_k$ and $\mathcal{Z}_k^{\mathrm{gh}}$ have the following block structure in $(\bar{h}, \bar{a}, \xi, \bar{\xi}, \Upsilon, \bar{\Upsilon})$-space:

$$\begin{aligned}
&\Big[\big(\breve{\mathcal{Z}}_k\big)_{\bar{h}\bar{h}}\Big]^{\mu\nu}{}_{\rho\sigma} = \frac{Z_N(k)\kappa^2}{2}\big(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\nu_\rho\delta^\mu_\sigma - \bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}\big) \qquad &&\Big[\big(\breve{\mathcal{Z}}_k\big)_{\bar{a}\bar{a}}\Big]^{a\mu b}{}_{\nu} = \frac{Z_F(k)}{\hat{g}^2}\,\delta^{ab}\delta^\mu_\nu \\
&\Big[\big(\mathcal{Z}_k^{\mathrm{gh}}\big)_{\bar{\xi}\xi}\Big]^{\mu}{}_{\nu} = \sqrt{2}\,\delta^\mu_\nu &&\Big[\big(\mathcal{Z}_k^{\mathrm{gh}}\big)_{\bar{\Upsilon}\Upsilon}\Big]^{ab} = \delta^{ab}
\end{aligned} \tag{3.11}$$

Note that $\mathcal{Z}_k^{\mathrm{gh}}$ is actually k-independent.

In setting up eq. (3.9) we opted for the complete Hessian operator $\Gamma_k^{(2)}$ to play the role of Δ. More precisely, we set $\Delta = \breve{\mathcal{Z}}_k^{-1}\breve{\Gamma}_k^{(2)}$ and $\Delta = \mathcal{Z}_{\text{gh}}^{-1}S_{\text{gh}}^{(2)}$ in the $(\bar{h},\bar{a})$- and the ghost-sectors, respectively. The multiplication by the inverse $\mathcal{Z}$ matrices brings Δ closer to an ordinary (covariant) Laplacian; symbolically, if $\Gamma_k^{(2)} = -\zeta_k\partial^2 + \cdots$, $\mathcal{Z}_k = \zeta_k$, we employ $\Delta = -\partial^2 + \cdots$ rather than $\Delta = -\zeta_k\partial^2 + \cdots$.

As described in Section 2.3 this approximation of the exact FRGE is useful as it allows for a simple spectral representation of the RHS of the flow equation (3.9) as given in (2.59). Its decomposition into a Grassmann-even and a ghost sector reads

$$\begin{aligned}\partial_t\Gamma_k = \int_0^\infty \mathrm{d}s \Bigg[\frac{1}{2}\widetilde{\breve{W}}_1(s)\,\mathrm{Tr}\left[e^{-s\breve{\mathcal{Z}}_k^{-1}\breve{\Gamma}_k^{(2)}}\right] - \widetilde{W}_1^{\text{gh}}(s)\,\mathrm{Tr}\left[e^{-s\mathcal{Z}_k^{\text{gh}\,-1}S_{\text{gh}}^{(2)}}\right] \\ -\frac{1}{2}\widetilde{\breve{W}}_2(s)\,\mathrm{Tr}\left[\breve{\eta}\,e^{-s\breve{\mathcal{Z}}_k^{-1}\breve{\Gamma}_k^{(2)}}\right] + \frac{\widetilde{\breve{W}}_3(s)}{2k^2}\,\mathrm{Tr}\left[\partial_t(\breve{\mathcal{Z}}_k^{-1}\breve{\Gamma}_k^{(2)})e^{-s\breve{\mathcal{Z}}_k^{-1}\breve{\Gamma}_k^{(2)}}\right]\Bigg]. \quad (3.12)\end{aligned}$$

We observe that due to the k-independence of $\mathcal{Z}_k^{\text{gh}}$ and $S_{\text{gh}}^{(2)}$ in our approximation, the ghost sector enters only the first part of the expression that corresponds to one-loop contributions, while the "RG improvement" terms remain free from it.

The strategy to further evaluate this FRGE is as follows. After having calculated $\Gamma_k^{(2)}$ we may set $\bar{g}_{\mu\nu} = g_{\mu\nu}$ and $\bar{A}_\mu^a = A_\mu^a$ as we are not going to extract any "extra" background field dependence [MR10]. Then we expand the traces up to a certain order in derivatives and in the fields, which allows us to determine the β-functions as the prefactors of the invariants present in our truncation. At this point it turns out possible to carry out the integration over the proper-time variable s explicitly, leading to "threshold functions" $\{\Phi_n^p, \tilde{\Phi}_n^p, \tilde{\tilde{\Phi}}_n^p\}$ that only depend on the exact form of the shape functions $R^{(0)}$. The resulting β-functions are then obtained as functions of the couplings depending on the threshold functions.

Following this general strategy the first step is to calculate the complete Hessian $\Gamma^{(2)}$. Its most complicated part is the Hessian of the bosonic action $\breve{\Gamma}_k^{(2)}$, i. e. the matrix of its second functional derivatives with respect to the dynamical fields $(\bar{h},\bar{a})$, or equivalently (g, A), at fixed backgrounds $(\bar{g},\bar{A})$. This Hessian is most transparently displayed by means of the associated quadratic form Γ_k^{quad}, which appears in the expansion

$$\breve{\Gamma}_k[\bar{g}+\bar{h},\bar{A}+\bar{a},\bar{g},\bar{A}] = \breve{\Gamma}_k[\bar{g},\bar{A},\bar{g},\bar{A}] + O(\bar{h},\bar{a}) + \Gamma_k^{\text{quad}}[\bar{h},\bar{a};\bar{g},\bar{A}] + \mathcal{O}(\{\bar{h},\bar{a}\}^3)\,. \quad (3.13)$$

Explicitly, Γ_k^{quad} is the sum of the following terms, that reflect the block structure of $\breve{\Gamma}_k^{(2)}$ in $(\bar{h}, \bar{a})$-space:

$$\left(\Gamma_k^{\mathrm{quad}}\right)_{\bar{h}\bar{h}} = Z_N \kappa^2 \int \mathrm{d}^d x \, \sqrt{\bar{g}}\, \bar{h}_{\chi\xi} \Big(\Big(U^{\chi\xi}{}_{\eta\zeta} - \bar{g}^{\rho\sigma} \bar{D}_\rho \bar{D}_\sigma K^{\chi\xi}{}_{\eta\zeta} \Big) + \\ + \left(1 - \frac{1}{\alpha_{\mathrm{D}}}\right) L^{\rho\sigma\chi\xi}{}_{\eta\zeta} \bar{D}_\rho \bar{D}_\sigma + \frac{Z_F}{2 Z_N \kappa^2} N^{\mu\nu\rho\sigma\chi\xi}{}_{\eta\zeta} \frac{1}{4} \bar{F}^a_{\mu\nu} \bar{F}^a_{\rho\sigma} \Big) \bar{h}^{\eta\zeta}, \quad (3.14)$$

$$\left(\Gamma_k^{\mathrm{quad}}\right)_{\bar{a}\bar{a}} = \frac{Z_F}{2\, \hat{g}^2_{\mathrm{YM}}} \int \mathrm{d}^d x \, \sqrt{\bar{g}}\, \bar{a}^a_\xi \Big(- \delta^{ab} \delta^\xi_\eta \bar{g}^{\rho\mu} \bar{\mathcal{D}}_\rho \bar{\mathcal{D}}_\mu + 2 \bar{g}^{\xi\rho} f^{abc} \bar{F}^c_{\rho\eta} + \delta^{ab} \bar{g}^{\xi\rho} \bar{R}_{\rho\eta} + \\ + \left(1 - \frac{1}{\alpha_{\mathrm{YM}}}\right) \delta^{ab} \bar{g}^{\xi\rho} \bar{\mathcal{D}}_\rho \bar{\mathcal{D}}_\eta \Big) \bar{a}^{b\eta}, \quad (3.15)$$

$$\left(\Gamma_k^{\mathrm{quad}}\right)_{\bar{h}\bar{a}} = \frac{Z_F}{2 \hat{g}^2_{\mathrm{YM}}} \int \mathrm{d}^d x \, \sqrt{\bar{g}}\, \bar{h}_{\eta\zeta} \Big(\Big(\frac{1}{2} \delta^\sigma_\xi \bar{g}^{\eta\zeta} \bar{g}^{\mu\rho} + \delta^\eta_\xi \bar{g}^{\zeta\rho} \bar{g}^{\sigma\mu} + \delta^\rho_\xi \bar{g}^{\sigma\zeta} \bar{g}^{\mu\eta} \Big) \bar{F}^a_{\rho\sigma} \bar{\mathcal{D}}_\mu \Big) \bar{a}^{a\xi}, \quad (3.16)$$

$$\left(\Gamma_k^{\mathrm{quad}}\right)_{\bar{a}\bar{h}} = \left(\Gamma^{\mathrm{quad}}\right)_{\bar{h}\bar{a}} \,. \quad (3.17)$$

The above quadratic functionals contain the kernels

$$K^{\chi\xi}{}_{\eta\zeta} = \frac{1}{4} \left(\delta^\chi_\eta \delta^\xi_\zeta + \delta^\xi_\eta \delta^\chi_\zeta - \bar{g}^{\chi\xi} \bar{g}_{\eta\zeta} \right), \quad (3.18)$$

$$U^{\chi\xi}{}_{\eta\zeta} = \frac{1}{4} \left(\delta^\chi_\eta \delta^\xi_\zeta + \delta^\xi_\eta \delta^\chi_\zeta - \bar{g}^{\chi\xi} \bar{g}_{\eta\zeta} \right) \left(\bar{R} - 2\Lambda \right) + \bar{g}^{\chi\xi} \bar{R}_{\eta\zeta} - \delta^\chi_\eta \bar{R}^\xi_\zeta - \bar{R}_\zeta{}^\chi{}_\eta{}^\xi, \quad (3.19)$$

$$L^{\rho\sigma\chi\xi}{}_{\eta\zeta} = \left(\frac{1}{4} \bar{g}^{\chi\xi} \bar{g}^{\rho\sigma} \bar{g}_{\eta\zeta} - \frac{1}{2} \delta^\rho_\eta \delta^\sigma_\zeta \bar{g}^{\chi\xi} - \frac{1}{2} \bar{g}^{\chi\rho} \bar{g}^{\xi\sigma} \bar{g}_{\eta\zeta} + \delta^\chi_\eta \delta^\sigma_\zeta \bar{g}^{\xi\rho} \right), \quad (3.20)$$

$$N^{\mu\nu\rho\sigma\chi\xi}{}_{\eta\zeta} = \frac{1}{2} \left(\frac{1}{2} \bar{g}^{\chi\xi} \bar{g}_{\eta\zeta} - \delta^\chi_\eta \delta^\xi_\zeta \right) \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} + \\ + 2 \left(\delta^\mu_\eta \delta^\rho_\zeta \bar{g}^{\nu\chi} \bar{g}^{\sigma\xi} - \delta^\mu_\eta \delta^\rho_\zeta \bar{g}^{\chi\xi} \bar{g}^{\nu\sigma} + 2 \delta^\xi_\eta \delta^\rho_\zeta \bar{g}^{\mu\chi} \bar{g}^{\sigma\nu} \right) . \quad (3.21)$$

The corresponding part of the Hessian in the ghost sector $S^{(2)}_{\mathrm{gh}}$ can be read off directly from the ghost action (3.8) by setting $a = 0 = h$, as it is purely quadratic in the ghost fields.

Using these formulae it can be checked that the $\mathcal{Z}$-factors (3.11) are correctly chosen.

The truncation contains three running couplings, $\bar{g}_{\mathrm{YM}}(k)$, $G(k)$ and $\Lambda(k)$. As described above, their β-functions can be found from the FRGE (3.12) by "projecting out" the corresponding invariants in the derivative expansion of the traces and equating them to the corresponding field monomials on the LHS of the flow equation. The resulting system of differential equations becomes autonomous if we employ the dimensionless counterparts of $\bar{g}_{\text{YM}}$, G and Λ respectively:

$$g^2_{\text{YM}}(k) \equiv \frac{k^{d-4}}{Z_F(k)}\hat{g}^2_{\text{YM}}, \quad g(k) \equiv \frac{k^{d-2}}{32\pi Z_N(k)\kappa^2} = k^{d-2}G(k), \quad \lambda(k) \equiv k^{-2}\Lambda(k)\,. \tag{3.22}$$

In terms of these variables the three coupled RG equations have the structure

$$\begin{aligned} \partial_t g^2_{\text{YM}} &= \beta_{\text{YM}} \equiv (d-4+\eta_F)\, g^2_{\text{YM}}\,, \\ \partial_t g &= \beta_g \equiv (d-2+\eta_N)\, g\,, \\ \partial_t \lambda &= \beta_\lambda\,. \end{aligned} \tag{3.23}$$

Here, we introduced the anomalous dimensions related to the Yang-Mills and the gravitational field, respectively:

$$\eta_F = -\partial_t \ln Z_F, \qquad \eta_N = -\partial_t \ln Z_N\,. \tag{3.24}$$

The scope of this chapter is restricted to the gravitationally corrected Yang-Mills β-function β_{YM}. Therefore it is sufficient to extract the $F^2_{\mu\nu}$-term from the derivative expansion of the traces. For identifying this monomial and reading off its prefactor we may insert any metric. We shall employ the most convenient choice, $g_{\mu\nu} = \bar{g}_{\mu\nu} = \delta_{\mu\nu}$. Furthermore, we set $\alpha_{\text{D}} = \alpha_{\text{YM}} = 1$ from now on. The remaining calculation is in principle straightforward, but rather lengthy. One has to expand the traces up to terms with two fields $A^a_\mu(x)$ and two derivatives acting on them. Because of the built-in background gauge invariance those terms should combine to $F^a_{\mu\nu}F^{a\mu\nu}$. As a check we verified that this indeed happens.

Let us now discuss the result. Here we specialize for $d = 4$ spacetime dimensions; for general d the reader is referred to Appendix C. We present three different formulae for η_F; they differ with respect to the degree of "RG improvement" they take into account.

To start with, we "switch off" all RG improvements. This means that we discard all terms in $\partial_t \mathcal{R}_k$ on the RHS of the flow equation where ∂_t hits either a $\mathcal{Z}_k$-factor or the $\Gamma^{(2)}_k$ in the argument of $\mathcal{R}_k$ which corresponds to only considering the first traces in (3.12) containing contributions from the "bosonic" and the ghost sector, or equivalently setting $\breve{\widetilde{W}}_2 = 0 = \breve{\widetilde{W}}_3$. In this way the evaluation of the FRGE amounts to a one-loop calculation, with a non-standard regulator though. Expanding the result in λ we find

$$\eta_F = -\frac{6}{\pi}\, g\, \Phi_1^1(0) - \frac{11}{24\pi^2} N g_{\mathrm{YM}}^2 + \mathcal{O}(\lambda^2)\,, \tag{3.25}$$

so that in this approximation

$$\partial_t g_{\mathrm{YM}}^2 = -\frac{6}{\pi}\, g\, g_{\mathrm{YM}}^2\, \Phi_1^1(0) - \frac{11}{24\pi^2} N g_{\mathrm{YM}}^4\,. \tag{3.26}$$

Here, $\Phi_1^1(0)$ is one of the threshold functions which were encountered in the pure gravity calculation [Reu98] already:

$$\begin{aligned} \Phi_n^p(w) &= \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1} \frac{R^{(0)}(z) - z R^{(0)\prime}(z)}{[z + R^{(0)}(z) + w]^p} && (n > 0)\,, \\ \Phi_0^p(w) &= (1+w)^{-p} && (n = 0)\,. \end{aligned} \tag{3.27}$$

The second contribution on the RHS of (3.26) is the familiar “asymptotic freedom” term due to the self-interaction of the gauge bosons, while the first one, due to the virtual gravitons, is new.

Several comments are in order here.

(A) The gravitational correction is manifestly cutoff scheme dependent, i. e. it depends, via $\Phi_1^1(0)$, on the shape function $R^{(0)}$. However, for any admissible choice of $R^{(0)}$ the constant $\Phi_1^1(0)$ is positive. As a result, the gravity term has a qualitatively similar impact on $g_{\mathrm{YM}}(k)$ as the gauge boson loops, namely to drive $g_{\mathrm{YM}}(k)$ smaller at larger k. It tends to speed up the approach of asymptotic freedom.

For the exponential cutoff $R^{(0)}(y) = y/(e^y - 1)$, for instance, one finds $\Phi_1^1(0) = \pi^2/6$, while the “optimized” one [Lit00, Lit01], $R^{(0)}(y) = (1-y)\Theta(1-y)$, yields $\Phi_1^1(0) = 1$.

(B) The gravitational correction, in perturbative language, originates from a quadratic divergence or, in FRGE language, a quadratic running with k. For this reason its scheme dependence is by no means surprising or alarming. Rather, it is the usual situation which is always encountered when the effective average action is applied to matter theories with a quadratic running of parameters, masses, say. However, one should note that the couplings in Γ_k as such are not observable or “physical” quantities. Only the latter must be $R^{(0)}$-independent, and this independence comes about by a compensation of the scheme dependence among different running couplings. (In truncations this compensation might not be perfect.) In general there will be compensations between effective propagators and vertices, for instance. Analogous remarks apply to the gauge fixing dependence.

(C) The β-function for $g^2_{\rm YM}$ depends on all three couplings, $g^2_{\rm YM}$, g and λ. In the approximation of (3.26) it happens to be independent of λ, but it does depend on $g(k) \equiv k^2 G(k)$, the dimensionless Newton constant. Hence the differential equation for $g_{\rm YM}$ cannot be solved in isolation. In principle the full system (3.23) should be considered, and this would include the backreaction of the matter fields on the running of the gravitational parameters g and λ. We shall not study this backreaction within this work. Instead, let us assume that the complete RG trajectory $k \mapsto (g_{\rm YM}(k), g(k), \lambda(k))$ admits a classical regime [RW04] in which Newton's constant does not run appreciably so that we may approximate

$$G(k) \approx G_0 = \text{const}, \quad g(k) = G_0 k^2 \,. \tag{3.28}$$

This approximation, implicitly, has been made in all perturbative studies [RW06, Rob, Pie07, Tom07, EPR09, TW10, Tom08]. With (3.28), for an abelian field ($N = 0$), say,

$$\partial_t g^2_{\rm YM} = -\frac{6}{\pi}\, \Phi^1_1(0)\, G_0\, k^2\, g^2_{\rm YM} \,. \tag{3.29}$$

Incidentally this β-function has the same general structure as the result by Robinson and Wilczek [RW06]; it is proportional to $G_0 g^2_{\rm YM}$ and depends explicitly on the energy scale k. Its k^2-dependence indicates that the underlying quantum effect is related to a quadratic divergence.

Eq. (3.29) is easily solved: $g^2_{\rm YM}(k) = g^2_{\rm YM}(0) \cdot \exp\left(-\omega_{\rm YM}(k/m_{\rm Pl})^2\right)$. Here $\omega_{\rm YM} \equiv 3\Phi^1_1(0)/\pi$ and $m_{\rm Pl} \equiv G_0^{-1/2}$ is the (ordinary, constant) Planck mass. To first order in the $k/m_{\rm Pl}$-expansion we obtain

$$g^2_{\rm YM}(k) = g^2_{\rm YM}(0) \left[1 - \omega_{\rm YM} \cdot \left(\frac{k}{m_{\rm Pl}}\right)^2 + \mathcal{O}\left(\frac{k^4}{m^4_{\rm Pl}}\right)\right] . \tag{3.30}$$

We note that to leading order Newton's constant itself [Reu98] has an analogous scale dependence, including the sign of the correction: $G(k) = G_0 \left[1 - \omega \cdot (k/m_{\rm Pl})^2 + \cdots\right]$.

(D) In order to illustrate how the above result fits into the Asymptotic Safety picture of Quantum Einstein Gravity [Wei79, RS10, LR07a, LR07b, NR06, Per09, RS12] we consider a free Maxwell field again. It is known that, in the Einstein-Hilbert truncation, the RG flow of the average action possesses a non-Gaussian fixed point for the two gravitational couplings, (g^*, λ^*), both in pure gravity [Reu98, Sou99, LR01] and in presence of a free Maxwell field [PP03b, PP03a]. At this fixed point the *dimensionless* Newton constant equals a positive constant, $g(k) = g^*$, while the dimensionful one runs to zero quadratically: $G(k) = g^*/k^2 \to 0$ for $k \to \infty$. In this regime we have

$$\partial_t g_{\mathrm{YM}}^2 = -\frac{6}{\pi}\,\Phi_1^1(0)\,g^*\,g_{\mathrm{YM}}^2\,. \tag{3.31}$$

The solution to this equation reads

$$g_{\mathrm{YM}}^2(k) \propto k^{-\Theta_{\mathrm{YM}}}, \quad \Theta_{\mathrm{YM}} = \frac{6}{\pi}\,g^*\,\Phi_1^1(0)\,. \tag{3.32}$$

At the fixed point the gauge coupling approaches zero according to a power law with a critical exponent Θ_{YM}, a positive number of order unity.[1] Thus the total system has a non-trivial fixed point of the form $(g_{\mathrm{YM}}^* = 0, g^* > 0, \lambda^* > 0)$. Obviously the approach of $g_{\mathrm{YM}} = 0$ is much faster than without gravity where $g_{\mathrm{YM}}(k) \propto 1/\ln(k)$. Note that g_{YM} is a *relevant* parameter, it grows when k is lowered, and hence it contributes one unit to the dimensionality of the fixed point's UV critical manifold.

Next we present the results for η_F with the RG improvements included. In a first step we retain only the terms which arise when ∂_t hits the $\mathcal{Z}_k$-factors in $\mathcal{R}_k$, thus including the term containing $\breve{\widetilde{W}}(s)$ in (3.12). These additional terms are proportional to η_F and η_N, respectively. As now η_F appears also on the RHS of the RG equation we obtain an implicit equation for it. Its solution reads

$$\eta_F = \frac{-\frac{6}{\pi}\,g\,\Phi_1^1(0) - \frac{11}{24\pi^2}N g_{\mathrm{YM}}^2 - \frac{2}{\pi}\,\eta_N\,\lambda\,g}{1 - \frac{3}{\pi}\,g\,\tilde{\Phi}_1^1(0) - \frac{5}{24\pi^2}N g_{\mathrm{YM}}^2 - \frac{2}{\pi}\,\lambda\,g} + \mathcal{O}(\lambda^2)\,. \tag{3.33}$$

In this approximation η_F depends not only on Newton's but also on the cosmological constant already at linear order of the λ-expansion. Eq. (3.33) resums terms of arbitrary order both in g_{YM} and g; it generalizes a known result [RW94a] for pure

[1]One cannot easily extract the precise numerical value of Θ_{YM} from existing calculations since the determination of g^* in the Einstein-Maxwell system in ref. [PP03b, PP03a] employs a cutoff different from the present one.

Yang-Mills theory. If we go on and include also the terms coming from the scale derivative of $\Gamma_k^{(2)}$ in the argument of $\mathcal{R}_k$ up to terms of order $\mathcal{O}(\lambda^3)$ we obtain

$$\eta_F = \frac{-\frac{6}{\pi}g\Phi^1_1 - \frac{11}{24\pi^2}Ng^2_{\text{YM}} + \frac{1}{\pi}\eta_N g\Big(3\tilde{\tilde{\Phi}}{}^1_2 - 2\lambda - (2\tilde{\tilde{\Phi}}{}^1_0 + 3\tilde{\Phi}^1_1)\lambda^2\Big) - \frac{4}{\pi}g\lambda(2\lambda+\beta_\lambda)\tilde{\tilde{\Phi}}{}^1_0 + \frac{4}{\pi}\lambda^2 g\Phi^1_{-1}}{1 - \frac{3}{\pi}\,g\left(\tilde{\Phi}^1_1 - \tilde{\tilde{\Phi}}{}^1_2\right) - \frac{5}{24\pi^2}Ng^2_{\text{YM}} - \frac{2}{\pi}\,\lambda\, g - \frac{2}{\pi}\,\lambda^2\, g\,\tilde{\tilde{\Phi}}{}^1_0 + \frac{1}{\pi}\lambda^2 g\tilde{\Phi}^1_{-1}} \tag{3.34}$$

The eqs. (3.33) and (3.34) contain the following integrals involving $R^{(0)}$:

$$\begin{aligned}
\tilde{\Phi}^p_n(w) &= \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}z\; z^{n-1}\frac{R^{(0)}(z)}{[z+R^{(0)}(z)+w]^p} && (n>0)\\
\tilde{\Phi}^p_0(w) &= (1+w)^{-p} && (n=0)\\
\tilde{\tilde{\Phi}}{}^p_n(w) &= \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}z\; z^{n-1}\frac{R^{(0)\prime}(z)}{[z+R^{(0)}(z)+w]^p} && (n>0)\\
\tilde{\tilde{\Phi}}{}^p_0(w) &= \frac{R^{(0)\prime}(0)}{(1+w)^p} && (n=0)
\end{aligned} \tag{3.35}$$

For $n<0$ we define in addition

$$\begin{aligned}
\Phi^1_n(0) &= (-1)^{-n}\,\frac{\mathrm{d}^{-n}}{\mathrm{d}z^{-n}}\frac{R^{(0)}(z) - zR^{(0)\prime}(z)}{z+R^{(0)}(z)}\bigg|_{z=0}\\
\tilde{\Phi}^1_n(0) &= (-1)^{-n}\,\frac{\mathrm{d}^{-n}}{\mathrm{d}z^{-n}}\frac{R^{(0)}(z)}{z+R^{(0)}(z)}\bigg|_{z=0}\\
\tilde{\tilde{\Phi}}{}^1_n(0) &= (-1)^{-n}\,\frac{\mathrm{d}^{-n}}{\mathrm{d}z^{-n}}\frac{R^{(0)\prime}(z)}{z+R^{(0)}(z)}\bigg|_{z=0}
\end{aligned} \tag{3.36}$$

The threshold functions $\tilde{\Phi}^p_n(w)$ appeared already in the Einstein-Hilbert truncation of pure gravity [Reu98]. In eq. (3.34) we abbreviated $\Phi^p_n \equiv \Phi^p_n(0)$, $\tilde{\Phi}^p_n \equiv \tilde{\Phi}^p_n(0)$ and $\tilde{\tilde{\Phi}}{}^p_n \equiv \tilde{\tilde{\Phi}}{}^p_n(0)$, respectively. What is new in (3.34) is the occurrence of the complete β-function of the cosmological constant, β_λ. It originates from the differentiation of the Λ-term contained in $\Gamma_k^{(2)}$.

To conclude this section let us mention a first application of our results: By a standard argument, knowledge about the k-dependence of wave function normalization constants such as $Z_F(k)$ can be used in order to deduce information about the related fully dressed propagator implied by $\Gamma \equiv \Gamma_{k=0}$. In the case at hand the running inverse propagator of the gauge field, on a flat background, has the form $Z_F(k)p^2 \propto g_{\text{YM}}^{-2}(k)p^2$. At high momenta, if there is no other relevant physical cutoff scale but the momentum itself, the dressed propagator $D(p)$ obtains by setting $k = |p|$, whence $D(p)^{-1} \propto g_{\mathrm{YM}}^{-2}(|p|)p^2$. For the example of eq. (3.30), for instance, this leads us to expect that the photon propagator gets modified by a p^4-term when p approaches the Planck scale:

$$D(p)^{-1} = p^2 + \omega_{\mathrm{YM}}\, p^4/m_{\mathrm{Pl}}^2 + O(p^6/m_{\mathrm{Pl}}^4)\,. \tag{3.37}$$

Likewise the fixed point running of (3.32) implies the following behavior for $p^2 \to \infty$:

$$D(p) \propto 1/p^{2(1+\Theta_{\mathrm{YM}}/2)}\,. \tag{3.38}$$

As Θ_{YM} is positive the gauge field propagator falls off faster than $1/p^2$, thanks to the quantum gravity corrections. In fact, the same argument when applied to the graviton propagator leads to a $1/p^4$-behavior for $p^2 \to \infty$ [LR01]. The asymptotic propagator (3.38) suggests that the quantum gravity corrections improve the finiteness properties of the matter field theory, and this precisely fits into the picture of Asymptotic Safety. It is also interesting to note that (3.38) leads to a modified static electromagnetic potential $A_0(r)$ of a classical point charge.[2] The 3-dimensional Fourier transform of (3.38) yields the potential $A_0(r) \propto r^{\Theta_{\mathrm{YM}}-1}$ which, if Θ_{YM} is large enough, could even be regular at $r = 0$. This makes it obvious that the gravity induced running of the gauge coupling is closely related to the old problem of divergent self energies.

3.3 Effects of asymptotically safe gravity on QED

3.3.1 The RG equations of the coupled system

In this section we shall use a projected form of the gravitational average action [Reu98] to describe the non-perturbative RG behavior of QED coupled to QEG in terms of a simple 3-dimensional theory space, treating the charge $e(k)$, or equivalently the fine-structure constant $\alpha(k) \equiv e(k)^2/(4\pi)$, along with Newton's constant and the cosmological constant as running quantities. Combining the results of [Reu98] for pure gravity in the Einstein-Hilbert truncation with the findings of the previous section for

[2] Treating the source dynamically, also form factor effects need to be included [Reua].

the gravity corrections to the running of α we are led to the following "caricature" of the flow equations:

$$\partial_t g = \beta_g \equiv \big[2 + \eta_N(g,\lambda)\big] g \tag{3.39a}$$

$$\partial_t \lambda = \beta_\lambda(g,\lambda) \tag{3.39b}$$

$$\partial_t \alpha = \beta_\alpha \equiv \left(A h_2(\alpha) - \frac{6}{\pi}\Phi_1^1(0) g \right) \alpha \tag{3.39c}$$

with the coefficient

$$A \equiv \frac{2}{3\pi} n_F. \tag{3.40}$$

Here, we consider for illustrative purposes a variant of quantum electrodynamics with n_F "flavors" of electrons.

Several comments are in order now.

(A) The equations are written in terms of the dimensionless running couplings $g(k) \equiv k^2\, G(k)$, $\lambda(k) \equiv \Lambda(k)/k^2$ and $\alpha(k)$, where k is the IR cutoff built into the average action. As usual, the dimensionless "RG time" is denoted by $t \equiv \ln(k/k_0)$.

(B) The first two equations, (3.39a) and (3.39b), are taken to be those of pure gravity in the Einstein-Hilbert truncation. The anomalous dimension of Newton's constant, $\eta_N(g,\lambda)$, and the β-function for the cosmological constant, $\beta_\lambda(g,\lambda)$, were found in ref. [Reu98].[3] Neglecting the backreaction of the matter fields on the renormalization in the gravity sector is (at least partially) justified by the investigations in [PP03b, PP03a], where is was found that a Maxwell field and one or a few Dirac fields do not qualitatively alter the RG flow of g and λ; this calculation had assumed free matter fields though.

(C) For small g the anomalous dimension η_N can be expanded in a power series in the Newton constant according to

$$\eta_N = B_1(\lambda) h_1(\lambda, g) = B_1(\lambda)\left(g + B_2(\lambda) g^2 + \ldots\right) \tag{3.41}$$

with functions B_1 and B_2 given in [Reu98][4]. From several non-perturbative calculations [RS12, NR06] we know the function $h_1(g,\lambda)$ rather precisely and we find that

[3] For the explicit formulae see eqs. (4.41) and (4.43) in ref. [Reu98].
[4] See eqs. (4.40) in ref. [Reu98].

$B_1(\lambda) < 0$ for all λ. Those calculations show in particular that the running of $g(k)$ does not change very much if one approximates $\lambda(k) \approx 0$, $B_1(\lambda) \approx B_1(0), B_2(\lambda) \approx 0$, whence

$$\eta_N(g, \lambda) \approx B_1(0)\, g < 0. \tag{3.42}$$

In terms of the standard threshold functions $\Phi^n_p(w)$ introduced in the last section we have explicitly $B_1(0) = -1/(3\pi)\big[24\Phi^2_2(0) - \Phi^1_1(0)\big]$.

(D) The β-function for α in the third equation, eq. (3.39c), involves a pure matter contribution, written as $A\, h_2(\alpha)\, \alpha$, and the gravity correction $\propto g$ taken from the Yang-Mills case (cf. Section 3.2). The former has been computed in perturbation theory, with the first two terms being (for $n_F = 1$)

$$\beta_\alpha(\alpha)\big|_{g=0} \equiv A\, h_2(\alpha)\, \alpha = \alpha \left[\frac{2}{3}\left(\frac{\alpha}{\pi}\right) + \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 + \mathcal{O}(\alpha^3)\right]. \tag{3.43}$$

To obtain a qualitative understanding it will be sufficient to employ the 1-loop approximation

$$h_2(\alpha) = \alpha. \tag{3.44}$$

Indeed, several lattice and flow equation studies [GHL+98, KKL01, KKL02, GJ04] indicate that there exists no non-trivial continuum limit for QED (without gravity), and this means that $\beta_\alpha(\alpha)\big|_{g=0}$ has no zero at any $\alpha > 0$. Therefore, $h_2(\alpha) = \beta_\alpha/(A\alpha)$ starts out as $h_2(\alpha) = \alpha$ in the perturbative regime $\alpha \lesssim 1$, and for larger α it is still known to be an increasing function: $h_2'(\alpha) > 0$. To be able to solve the RG equations analytically we shall set $h_2(\alpha) = \alpha$ for all values of α. This is a qualitatively reliable approximation since, as we shall see, at most a zero of $h_2(\alpha)$ could change the general picture.

(E) As it stands, β_α applies only above the threshold due to the mass of the electron at $k = m_e$. At $k \lesssim m_e$ the fermion loops no longer renormalize α. In the full fledged average action formalism this decoupling is described by a certain threshold function. Here a simplified description will be sufficient where we set $A = 0$ if $k < m_e$.

(F) The gravity contribution on the right hand side of (3.39c) was derived in Section 3.2 within a truncation of the theory space $\mathcal{T}_{\text{E,YM}}$ that included the gauge field action $\frac{1}{4e^2(k)} \int \mathrm{d}^d x \sqrt{g} F_{\mu\nu} F^{\mu\nu}$ besides the Einstein-Hilbert terms. Within the approximation considered there the gravity correction to $\partial_t g_{\text{YM}}$ is seen to be independent of the interactions within the matter sector, if any. Therefore it is the same for QED and the non-abelian Yang-Mills field considered before so that we may obtain (3.39c) by simply replacing in (3.26) the non-abelian gauge boson contribution with the corresponding fermion term (of the opposite sign!). Besides the leading 1-loop correction also subleading corrections to $\partial_t g_{\mathrm{YM}}$ involving the cosmological constant were found in the Yang-Mills case. They, too, within their domain of reliability do not change the qualitative picture we are interested in here and are omitted therefore.

(G) Identifying the scales E and k, the two terms inside the brackets on the RHS of the perturbative result (3.1), in our notation, translate to $g + \frac{3}{2} g\lambda$. Thus, for $\lambda = 0$, the perturbative gravitational correction has the same structure as (3.39c) from the average action. Since λ is small in the applications below, subleading corrections such as the term $\frac{3}{2} g\lambda$ are inessential for the qualitative properties of the flow.

We close the discussion of the β-functions (3.39) with a word of warning. In view of the non-universality of the gravitational corrections found in the last section it is important to stress that a priori all our results hold true only for the very definition of $e(k)$ used here, namely via the prefactor of the $\int F^2$-term in Γ_k. As such $e(k)$ is not a directly observable quantity and since the gravitational corrections to it are due to quadratic divergences, we expect them to be of a lesser degree of universality compared to those contributions stemming from logarithmic divergences. This enhanced scheme dependence, however, is expected to drop out e. g. by a compensation of effective propagators and vertices, when observables are calculated from Γ_k. Thus, the considerable debate in the literature about the gravitational corrections to the β-function of gauge couplings, on their precise form [RW06, Rob, Pie07, Tom07, EPR09, TW10, Tom08, Tom10], as well as their usefulness [ADEH11] and observability [EM12] in scattering experiments may not be directly applicable to our approach as every comparison with different definitions of $e(k)$ in other settings or schemes requires a separate analysis. We emphasize that within the Asymptotic Safety program these issues are of secondary interest, anyway, and not (yet) relevant. Our goal is first of all to *construct* a quantum field theory by devising a way to take the infinite cutoff limit of the corresponding functional integral; we do this by replacing the functional integral computation by the task of solving an exact RG equation for the effective average action Γ_k and trying to take the continuum limit at a fixed point of its flow. Only once this is achieved one can start to *analyze and interpret* the resulting theory, and only then questions such as those above on scattering experiments can (and should) be asked.

For the time being we are still in the first phase, and so the RG equations used here should be seen as a tool towards understanding the flow of Γ_k and the possible continuum limits it might hint at.

3.3.2 The fixed points

Let us start the analysis of the system (3.39) by finding its fixed points, i. e. common zeros $(g^*, \lambda^*, \alpha^*)$ of all three β functions. Obviously there is a trivial or Gaussian fixed point **GFP** at $g^* = \lambda^* = \alpha^* = 0$.

Furthermore we know that the subsystem of flow equations for pure gravity in the Einstein-Hilbert truncation, eqs. (3.39a, 3.39b), admits for a NGFP at $(g_0^*, \lambda_0^*) \neq 0$. This fixed point lifts to a NGFP of the full system located at $(g_0^*, \lambda_0^*, \alpha^* = 0)$. We will denote it by $\mathbf{NGFP_1}$. This fixed point is trivial from the QED perspective, the electromagnetic interaction is "switched off" there, while the gravitational self-interaction is the same as in pure gravity.

There exists a second NGFP, non-trivial also in the QED sense, if the equation

$$h_2(\alpha) = \frac{6}{\pi}\frac{g^*}{A}\Phi_1^1(0) \tag{3.45}$$

has a solution for some $\alpha = \alpha^* \neq 0$. We shall see in a moment that this is indeed the case. This fixed point we will call $\mathbf{NGFP_2}$.

In the following we will be particularly interested in an Asymptotic Safety scenario with respect to this NGFP. Near the fixed point the (linearized) flow is governed by the stability matrix $\mathcal{B}_{ij} = \partial_{u_j}\beta_{u_i}(u^*)$ according to

$$\partial_t u_i(k) = \sum_j \mathcal{B}_{ij}\left(u_j(k) - u_j^*\right), \tag{3.46}$$

where $u = (g, \lambda, \alpha)$. For the system under consideration the stability matrix is of the form

$$\mathcal{B} = \begin{pmatrix} \partial_g\beta_g & \partial_\lambda\beta_g & 0 \\ \partial_g\beta_\lambda & \partial_\lambda\beta_\lambda & 0 \\ \partial_g\beta_\alpha & \partial_\lambda\beta_\alpha & \partial_\alpha\beta_\alpha \end{pmatrix}\Bigg|_{u=u^*} \tag{3.47}$$

Two of its eigenvalues are therefore identical to the case of pure gravity in the Einstein-Hilbert truncation, giving rise to the familiar two *UV attractive* directions [RS02, LR01]. The third eigenvalue is given by

$$\partial_\alpha \beta_\alpha(u^*) = \left(A h_2(\alpha^*) - \frac{6}{\pi} \Phi_1^1(0) g^* \right) + A h_2'(\alpha^*) \alpha^* = A h_2'(\alpha^*) \alpha^* . \tag{3.48}$$

If α^* is positive, which will actually turn out to be the case, the sign of $\partial_\alpha \beta_\alpha(u^*)$ agrees with the sign of $h_2'(\alpha^*)$.

At this point we take advantage of the information from the lattice and flow equations studies [GHL$^+$98, KKL01, KKL02, GJ04] trying to find a non-trivial continuum limit of QED without gravity. In **(D)** of Section 3.3.1 we saw that their negative results suggest that $h_2'(\alpha^*) > 0$ holds true even beyond perturbation theory. As a consequence, the third eigenvalue $\partial_\alpha \beta_\alpha(u^*)$, corresponding to the α direction in the 3-dimensional g-λ-α–theory space, is *UV repulsive*. With two UV attractive and one repulsive direction the UV critical hypersurface $\mathscr{S}_{\text{UV}}$ pertaining to $\mathbf{NGFP_2}$ is a two-dimensional surface in a 3-dimensional space, i. e. $s_2 = \dim \mathscr{S}_{\text{UV}}(\mathbf{NGFP_2}) = 2$.

In comparison, let us also analyze the eigenvalues of the stability matrix of the other fixed point $\mathbf{NGFP_1}$. As β_g and β_λ do not depend on α in our approximation the first two eigenvalues remain the same as for $\mathbf{NGFP_2}$. However, for the third eigenvalue we obtain

$$\partial_\alpha \beta_\alpha(u^*) = \left(A h_2(\alpha^*) - \frac{6}{\pi} \Phi_1^1(0) g^* \right) + A h_2'(\alpha^*) \alpha^* \overset{\alpha^*=0}{=} -\frac{6}{\pi} \Phi_1^1(0) g^* < 0, \tag{3.49}$$

such that the third direction turns out to be *UV attractive* as well. Hence, $\mathbf{NGFP_1}$ has a 3-dimensional UV critical hypersurface, i. e. $s_1 = \dim \mathscr{S}_{\text{UV}}(\mathbf{NGFP_1}) = 3$. The fact that $s_2 < s_1$ reflects the enhanced predictivity of an Asymptotic Safety scenario with respect to $\mathbf{NGFP_2}$ compared to $\mathbf{NGFP_1}$.

3.3.3 Explicit RG trajectories

Let us now analyze the flow in a simple analytically tractable approximation. For that we expand the functions h_1 and h_2 to first order in g and α, respectively,

$$h_1(g) = g + \mathcal{O}(g^2) \qquad \text{and} \qquad h_2(\alpha) = \alpha + \mathcal{O}(\alpha^2). \tag{3.50}$$

Furthermore, we neglect the running of the cosmological constant and fix $\lambda = \lambda_0$ to a constant value. The remaining system of flow equations reads

$$\partial_t g = \big[2 + B_1(\lambda_0)\, g\big] g\,, \tag{3.51a}$$

$$\partial_t \alpha = \left(A\alpha - \frac{6}{\pi}\Phi_1^1(0)\, g\right)\alpha. \tag{3.51b}$$

In this approximation there clearly exists a $\mathbf{NGFP_2}$ with fixed point values

$$g^* = -\frac{2}{B_1(\lambda_0)} \qquad \text{and} \qquad \alpha^* = \frac{6}{\pi}\frac{g^*}{A}\Phi_1^1(0). \tag{3.52}$$

In the following we will express the constant $B_1(\lambda_0)$ in terms of the fixed point value g^* according to $B_1(\lambda_0) = -2/g^*$.

The approximation allows us to solve (3.51a) in separation; its solution is given by

$$g(k) = \frac{G_0 k^2}{1 + \frac{G_0 k^2}{g^*}}. \tag{3.53}$$

The constant of integration $G_0 \equiv \lim_{k\to 0} g(k)/k^2$ can be interpreted as the IR value of the running Newton constant. The simple RG trajectory (3.53) for g shares a crucial feature with any asymptotically safe trajectory of the exact system for pure gravity, namely that it connects the classical regime $g(k) \approx G_0 k^2$ for $k \ll m_{\mathrm{Pl}} \equiv G_0^{-1/2}$ and the fixed point regime $g(k) \approx g^*$ for $k \gg m_{\mathrm{Pl}}$. Note that the Planck mass is defined in terms of the constant G_0.

Due to the simplified form of (3.51b), the RG equation for α is now an ordinary differential equation of Riccati type, which can therefore be solved in closed form without the need for a specification of the function $g(k)$.

Its general solution reads, with $\Phi_1^1 \equiv \Phi_1^1(0)$,

$$\frac{1}{\alpha(k)} = \frac{1}{\alpha_0} \exp\left(\frac{6}{\pi}\Phi_1^1 \int_{k_0}^{k} \frac{g(k')}{k'} \mathrm{d}k'\right) - A \int_{k_0}^{k} \exp\left(\frac{6}{\pi}\Phi_1^1 \int_{k'}^{k} \frac{g(k'')}{k''} \mathrm{d}k''\right) \frac{\mathrm{d}k'}{k'}, \quad (3.54)$$

where $\alpha_0 = \alpha(k_0)$ is the value of the fine-structure constant at a fixed reference scale k_0. If we now specialize for the function $g(k)$ of eq. (3.53) we can perform the integrations in (3.54) and we find

$$\begin{aligned}\frac{1}{\alpha(k)} = \left[\frac{g^* + G_0 k^2}{g^* + G_0 k_0^2}\right]^{\frac{3}{\pi}\Phi_1^1 g^*} & \left[\frac{1}{\alpha_0} - \frac{1}{\alpha^*}\left(1 + \frac{g^*}{G_0 k_0^2}\right) {}_2F_1\left(1, 1, 1 + \frac{3}{\pi}\Phi_1^1 g^*; -\frac{g^*}{G_0 k_0^2}\right)\right] \\ & + \frac{1}{\alpha^*}\left(1 + \frac{g^*}{G_0 k^2}\right) {}_2F_1\left(1, 1, 1 + \frac{3}{\pi}\Phi_1^1 g^*; -\frac{g^*}{G_0 k^2}\right),\end{aligned} \quad (3.55)$$

where ${}_2F_1(a, b, c; z)$ denotes the (ordinary) hypergeometric function.

From eq. (3.55) we infer that there exist three kinds of possible UV behavior for $\alpha(k)$. They differ by the value of the terms inside the square brackets $[\cdots]$ on the RHS of (3.55). This value is independent of the scale k. As the prefactor of $[\cdots]$ diverges proportional to $k^{6\Phi_1^1 g^*/\pi}$ when $k \to \infty$, we find the limit $\lim_{k\to\infty} \alpha(k) = 0$ for every strictly positive value $[\cdots] > 0$. This corresponds to asymptotic freedom of the fine-structure constant, and is similar to the behavior found by Robinson and Wilczek [RW06] and Toms [Tom10], but here with a concomitant running of Newton's constant. The corresponding RG trajectories of the full system are asymptotically safe with respect to $\mathbf{NGFP_1}$.

For a negative value $[\cdots] < 0$ there will be a scale k_{LP} at which the two terms on the RHS of (3.55) cancel, so that α diverges at finite energies, corresponding to a Landau type singularity.

A third type of limiting behavior is obtained for the case that the bracket vanishes exactly: $[\cdots] = 0$. As we are then left only with the second term of the RHS of (3.55), and since ${}_2F_1(a, b, c; 0) = 1$, we find $\lim_{k\to\infty} \alpha(k) = \alpha^*$, corresponding to an asymptotically safe trajectory with a non-zero coupling at the FP. This is precisely the behavior to be expected due to the UV repulsive direction of the fixed point. Since for $k \to \infty$ the trajectory will only flow into $\mathbf{NGFP_2}$ for one specific value of α_0, this value of α_0, and hence the whole trajectory $\alpha(k)$, can be predicted under the assumption of Asymptotic Safety.

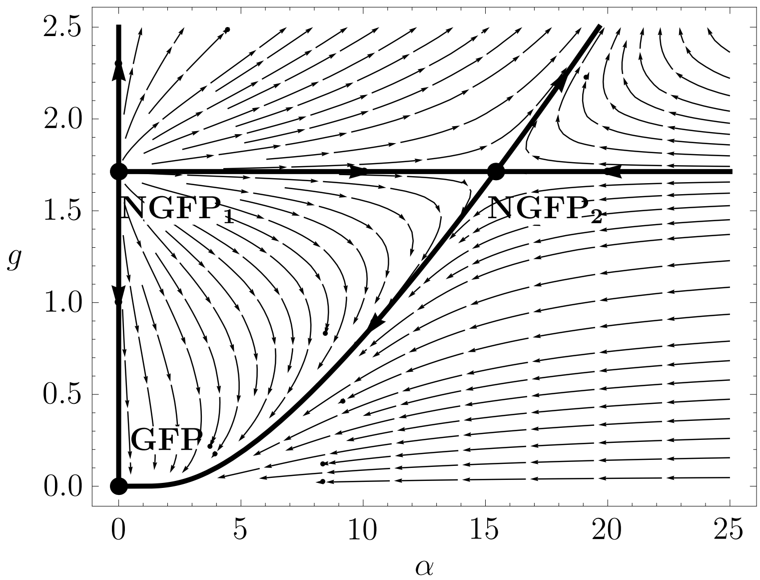


Figure 3.1. The RG flow on the g-α–plane implied by the simplified equations (3.51). It is dominated by two non-Gaussian fixed points. Their respective value of $\dim \mathscr{S}_{\text{UV}}$ differs by one unit. (The arrows point in the direction of decreasing k.)

The situation is illustrated by the (α,g)–phase portrait in Fig. 3.1. Bearing in mind that the arrows always point towards the IR, we see that $\mathbf{NGFP_1}$ is IR repulsive in both directions shown, while $\mathbf{NGFP_2}$ is IR attractive in one direction. This is consistent with our earlier discussion which showed that in the 3-dimensional (λ,g,α)–space $\mathbf{NGFP_1}$ has 3 and $\mathbf{NGFP_2}$ has only 2 IR repulsive (or equivalently, UV attractive) eigendirections.

In Fig. 3.1, the trajectories inside the triangle $\mathbf{GFP}$–$\mathbf{NGFP_1}$–$\mathbf{NGFP_2}$ are those corresponding to the case $[\cdots] > 0$ above; they are asymptotically safe with respect to $\mathbf{NGFP_1}$. The $\mathbf{NGFP_2} \to \mathbf{GFP}$ boundary of this triangle is the unique trajectory (heading towards smaller g and α values) which is asymptotically safe with respect to the second non-trivial fixed point, $\mathbf{NGFP_2}$.

The diagram in Fig. 3.1 corresponds to a massless electron for which A keeps its non-zero value at arbitrarily small scales. In reality the α-running due to the fermions stops near m_e, of course.

3.3.4 Asymptotic Safety construction at $\mathbf{NGFP_2}$

Let us investigate the unique asymptotically safe trajectory emanating from $\mathbf{NGFP_2}$ in more detail. First, we note that the condition of a vanishing bracket $[\cdots]$ in (3.55) is self-consistent in the sense, that the resulting function $\alpha_0(k_0)$ is of identical form as the remaining function $\alpha(k)$:

$$\frac{1}{\alpha(k)} = \frac{1}{\alpha^*}\left(1+\frac{g^*}{G_0 k^2}\right)\,{}_2F_1\left(1,1,1+\frac{3}{\pi}\Phi_1^1 g^*;-\frac{g^*}{G_0 k^2}\right). \tag{3.56}$$

Second, let us approximate this function for scales $k \ll m_{\rm Pl}$ much below the Planck scale. Later on we shall need it at $k = m_e$, for instance, where m_e is the mass of the electron. Then the argument $\frac{g^*}{G_0 m_e^2} = g^*\left(\frac{m_{\rm Pl}}{m_e}\right)^2 \approx 10^{44}$ is extremely large and this will be an excellent approximation. Hence we may safely truncate the general series expansion of the hypergeometric function,

$$\begin{aligned}{}_2F_1(a,a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}(-z)^{-a}\sum_{n=0}^{\infty}\frac{(a)_n(1-c+a)_n}{(n!)^2}z^{-n}\cdot \\ \cdot\Big(\ln(-z)+2\,\psi(n+1)-\psi(a+n)-\psi(c-a-n)\Big),\end{aligned} \tag{3.57}$$

after its first term, and approximate the resulting factor $1+G_0k^2/g^* \approx 1$, such that our final result for scales $k \ll m_{\rm Pl}$ reads

$$\frac{1}{\alpha(k)} = \frac{g^*}{\alpha^*}\cdot\frac{3}{\pi}\Phi_1^1\cdot\left[\ln\left(\frac{g^*}{G_0k^2}\right)-\gamma-\psi\left(\frac{3}{\pi}\Phi_1^1 g^*\right)\right]. \tag{3.58}$$

Here ψ denotes the Digamma function and γ is Euler's constant. Using (3.52) in order to reexpress the ratio g^*/α^* we can write (3.58) also in the following form:

$$\frac{1}{\alpha(k)} = \frac{A}{2}\left[\ln\left(\frac{g^*}{G_0k^2}\right)-\gamma-\psi\left(\frac{3}{\pi}\Phi_1^1 g^*\right)\right]. \tag{3.59}$$

Recall that $A \equiv \frac{2}{3\pi}n_F$ is a completely universal constant, sensitive only to the number of (hypothetical) electron species. Hence, for $k \ll m_{\rm Pl}$, we recover the logarithmic running $\alpha(k)^{-1} = -A\ln k + \text{const}$ familiar from pure QED.

In the opposite extreme of k comparable to, or larger than the Planck mass the gravity corrections set in, stop this logarithmic behavior, and cause the coupling to freeze at a finite value $\alpha(k \to \infty) = \alpha^*$. Obviously, along this RG trajectory no Landau pole singularity is encountered!

Note also that according to eq. (3.59) we have $\alpha(k) \propto 1/n_F$ for every value of k. As a consequence, if we consider a toy model with a large number of electron flavors, all α-values that appear along the RG trajectory can be made as small as we like, and this renders perturbation theory in α increasingly precise. At the fixed point we have for instance

$$\alpha^* = 9\,\Phi_1^1(0)\frac{g^*}{n_F}. \tag{3.60}$$

In an Asymptotic Safety scenario based upon the fixed point $\mathbf{NGFP_2}$, within the truncation considered, the infrared value of the fine-structure constant $\alpha_{\text{IR}} \equiv \lim_{k\to 0}\alpha(k)$ is a computable number. Using eq. (3.59) to calculate α_{IR} we must remember however that as it stands it holds true only for $k \gtrsim m_e$. When k drops below the electron mass the standard QED contribution to the running of $\alpha(k)$ goes to zero, and the gravity corrections are zero there anyhow. Hence approximately, $\partial_t\alpha(k) = 0$ for $0 \leq k \lesssim m_e$. Thus eq. (3.59) leads to the following prediction for $\alpha_{\text{IR}} \approx \alpha(m_e)$:

$$\frac{1}{\alpha_{\text{IR}}} = \frac{A}{2}\left[2\,\ln\left(\frac{m_{\text{Pl}}}{m_e}\right) + \ln(g^*) - \gamma - \psi\left(\frac{3}{\pi}\Phi_1^1 g^*\right)\right]. \tag{3.61}$$

As the fixed point coordinates are an output of the RG equations, the only input parameter needed to predict α_{IR} in this approximation is the electron mass in Planck units, m_e/m_{Pl}.

It is tempting to insert numbers into eq. (3.61). With $m_e = 5.11 \cdot 10^{-4}\,\text{GeV}$ and $m_{\text{Pl}} = 1.22 \cdot 10^{19}\,\text{GeV}$ one finds $m_e/m_{\text{Pl}} = 4.19 \cdot 10^{-23}$, and for the optimized cutoff [Lit00, Lit01] we have $\Phi_1^1 = 1$. The value of $g^* = -2/B_1(\lambda_0)$ depends on the value chosen for λ_0. For $\lambda_0 = 0$ or $\lambda_0 = \lambda^* \approx 0.193$, the fixed point value of λ in the Einstein-Hilbert truncation, we get $g^* \approx 1.71$ or $g^* \approx 0.83$, respectively. From that we obtain

$$\frac{1}{\alpha_{\text{IR}}} \overset{\lambda_0=0}{\approx} 10.91\,n_F \qquad \text{or} \qquad \frac{1}{\alpha_{\text{IR}}} \overset{\lambda_0=\lambda^*}{\approx} 10.96\,n_F. \tag{3.62}$$

We observe that the result is relatively insensitive to the value of g^* and/or Φ_1^1, but it scales linearly with the number of electron species, n_F.

Obviously, for $n_F = 1$, this estimate differs from the fine-structure constant measured in real nature, $\alpha \approx 1/137$, by a factor of roughly 13. However, even within the limits of our crude approximation (3.50), a serious comparison with experiment must include the renormalization effects due to the other particles besides the electron, all those of the Standard Model, and possibly beyond. Within the "n_F flavor QED" considered here we could mimic their effect by appropriately choosing n_F. It would then follow that the observed $\alpha_{\rm IR}$ is consistent with Asymptotic Safety at $\mathbf{NGFP_2}$ if $n_F = 13$.

It is reassuring that for this large number the value of the natural expansion parameter of QED perturbation theory, (α/π), is rather small already. At $\mathbf{NGFP_2}$, for example, one has $(\alpha^*/\pi) \approx 0.38$ and $(\alpha^*/\pi) \approx 0.18$, respectively.

Next let us try the full Standard Model and its minimal supersymmetric extension (MSSM).[5] Applying the above discussion to the weak hypercharge rather than the electromagnetic $U(1)$ one again has a one loop flow equation of the type $\partial_t \alpha_1 = A\alpha_1^2$, this time with $A = 41/(20\pi)$ for the SM and $A = 33/(10\pi)$ for the MSSM, respectively [ABF+92]. Here $\alpha_1 \equiv 5\alpha/(3\cos^2\theta_W)$, where θ_W is the Weinberg angle. It is most convenient to compare the prediction of Asymptotic Safety to the experimental value at the Z mass. From eq. (3.61) with the new value of A and m_e replaced by M_Z we obtain (with $\lambda_0 = 0$):

$$\alpha_1^{\rm SM}(M_Z) \approx 1/25.7 \tag{3.63a}$$

$$\alpha_1^{\rm MSSM}(M_Z) \approx 1/41.3 \tag{3.63b}$$

As compared to the experimental value $\alpha_1^{\rm exp}(M_Z) \approx 1/59.5$ both of these predictions are too high, the supersymmetric one less so. Clearly we may not take these numbers too seriously. After all, while for the reasons discussed above we believe that the one loop form of the matter β-functions is a reliable guide with respect to the general structure of the RG flow, its quantitative status is questionable.

Nevertheless the following observation might be of interest. The predictions (3.63) turn out larger than the experimental value since in the SM and MSSM the coefficient A is *too small*. As a consequence, the matter driven renormalizations which reduce $\alpha_1(k)$ when k is lowered are too weak. If we could take the RG equations seriously at the quantitative level the conclusion would be that Asymptotic Safety at $\mathbf{NGFP_2}$ is possible if there exist more particles with a $U(1)$ charge than those of the SM or MSSM. We find it remarkable that not very many more seem to be needed; it is sufficient to increase A by a small factor of order unity.

[5] For a related discussion see [SW10].

On the other hand, if ultimately it turns out that the Standard Model coupled to QEG is not asymptotically safe with respect to $\mathbf{NGFP_2}$ then its RG trajectory would be one of those inside the $\mathbf{GFP}$–$\mathbf{NGFP_1}$–$\mathbf{NGFP_2}$ triangle in Fig. 3.1. In this case it is asymptotically safe with respect to the other non-trivial fixed point, $\mathbf{NGFP_1}$. As for being free from divergences and predictive at all energies this is still not too much of a drawback, though. It only means that the $U(1)$ coupling is not a prediction but necessarily an experimental input.

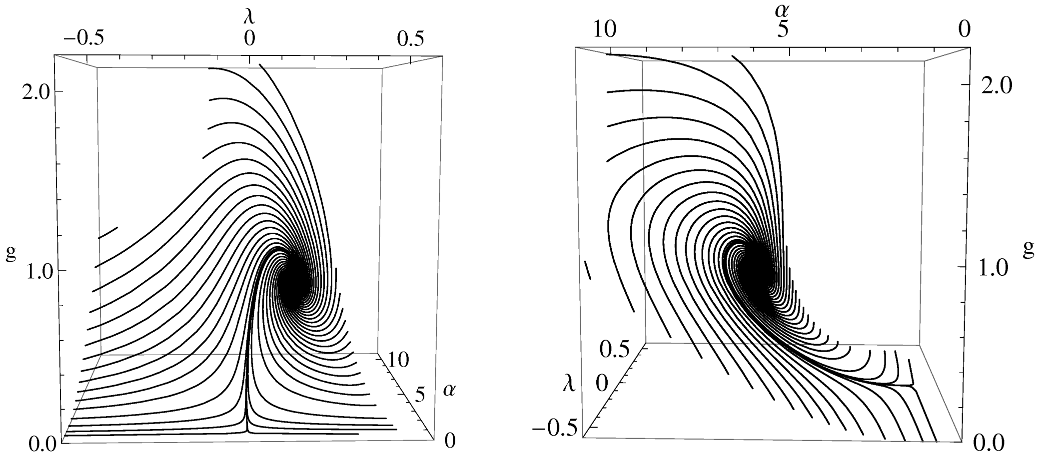
3.3.5 Numerical results

Returning to QED coupled to QEG we shall now go beyond the analytically tractable approximation of the previous subsection and employ *exact* numerical solutions $g(k)$, $\lambda(k)$ for the pure gravity subsystem of eqs. (3.39a) and (3.39b). Thereby the exact form of the functions $\eta_N = B_1(\lambda)h_1(g,\lambda)$ and $\beta_\lambda(g,\lambda)$ as implied by the Einstein-Hilbert truncation [Reu98] are used, and then the two coupled equations for g and λ are solved numerically as in [RS02]. Then, for every given RG trajectory $k \mapsto (g(k), \lambda(k))$, we calculate the corresponding $\alpha(k)$ by inserting $g(k)$ into (3.39c) and solving this decoupled differential equation numerically, too.

Staying within the one-loop approximation of β_α we thus confirm the existence of both non-Gaussian fixed points, $\mathbf{NGFP_1}$ and $\mathbf{NGFP_2}$. In accord with the general discussion above, the latter is seen to have two UV attractive and one repulsive direction. All RG trajectories heading for $k \to \infty$ towards $\mathbf{NGFP_2}$ lie in its two dimensional UV critical surface $\mathscr{S}_{\text{UV}}$. It is visualized in Fig. 3.2 by a family of trajectories starting on $\mathscr{S}_{\text{UV}}$ close to the FP, which were traced down to lower scales k.

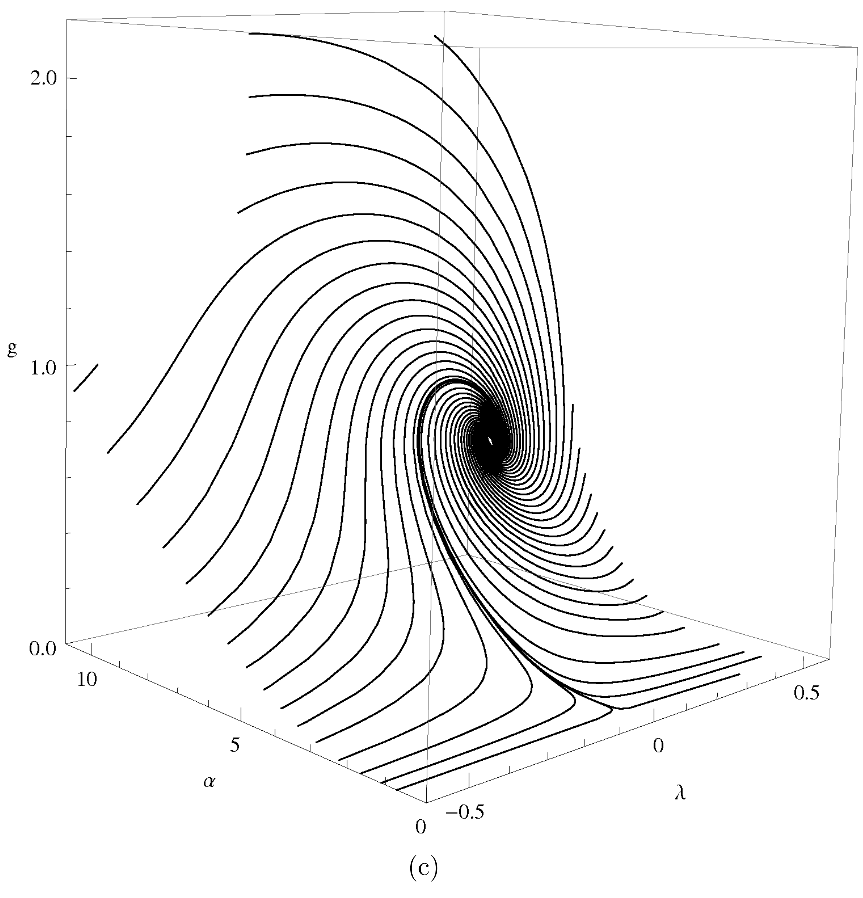
As the backreaction of the matter on the gravity sector is neglected, the flow in a projection onto the (λ,g)–plane is identical to the one of pure gravity (Fig. 3.2(a)). We can therefore classify the trajectories as in [RS02], being of type Ia, IIa, and IIIa, when the IR value of the cosmological constant is negative, zero, or positive, respectively.

As we rotate the coordinate frame (Fig. 3.2(c),3.2(b)), we see how the critical surface is bent in coupling space. Especially we note that the fine-structure constant only gets renormalized to small values $\alpha \ll 1$, if the (λ,g)–projection of the trajectory is sufficiently close to the type IIa trajectory of pure gravity (the "separatrix" [RS02]). This is because only these trajectories give rise to a long classical regime with $G, \Lambda \approx$ const [BR07, BR08, RW04]. They spend a tremendous amount of renormalization



(a)

(b)



(c)

Figure 3.2. Trajectories running inside the UV critical surface $\mathscr{S}_{\rm UV}$ of $\mathbf{NGFP_2}$ in (λ,g,α)–theory space.

group time close to the Gaussian fixed point of the gravity sector. The classical regime of gravity is needed for the logarithmic running of α to be of effect.

As a concrete example of a trajectory with a long classical regime we consider the "realistic" RG trajectory discussed in [RW04] and [BR07, BR08]. In these references a specific (λ,g)–trajectory has been identified which matches the observed values of G and Λ. It is of type IIIa and can be characterized by its turning point (the point of smallest λ) whose coordinates are

$$(g_T, \lambda_T) = \left(g_T, \frac{\Phi_2^1(0)}{2\pi} g_T \right) \qquad \text{with} \qquad g_T \approx 10^{-60}. \tag{3.64}$$

The turning point is passed at the scale $k_T \approx 10^{-30} m_{\text{Pl}}$. To make the numerical solution of the RG equations feasible we transform the equations to double logarithmic variables using $\tau(k) \equiv \ln(k/k_T) = \ln(k/m_{\text{Pl}}) + 30 \ln(10)$ as the RG time variable. The transition at the Planck scale between the classical and fixed point scaling regime therefore takes place at about $\tau(k = m_{\text{Pl}}) = 30 \ln(10) \approx 69$.

Having fixed the (λ,g)–trajectory to be the "realistic" one, there is a unique asymptotically safe trajectory relative to $\mathbf{NGFP_2}$ in the three dimensional coupling space. The corresponding $\alpha(k_T)$ can now be found by a shooting method: If we start slightly above $\mathscr{S}_{\text{UV}} \equiv \mathscr{S}_{\text{UV}}(\mathbf{NGFP_2})$ and evolve towards the UV the coupling $\alpha(k)$ will head to infinity at a finite scale, while starting below $\mathscr{S}_{\text{UV}}$ will result in an asymptotically free trajectory: $\alpha(k \to \infty) = 0$. This trajectory is asymptotically safe with respect to $\mathbf{NGFP_1}$, and in Fig. 3.1 it corresponds to one of those *inside* the triangle **GFP**–$\mathbf{NGFP_1}$–$\mathbf{NGFP_2}$. The closer we get to $\mathscr{S}_{\text{UV}}$ the more the trajectory gets "squeezed" into the corner of the triangle at $\mathbf{NGFP_2}$, ultimately leading to two separate pieces, the $\mathbf{NGFP_1} \to \mathbf{NGFP}_2$ and the $\mathbf{NGFP_2} \to \mathbf{GFP}$ branch, respectively. Due to unavoidable numerical errors any starting value $\alpha(k_T)$ will eventually opt for one of the two cases, but if we fine tune it to happen at sufficiently large τ, we will end up with a good estimate for the trajectory which is asymptotically safe with respect to $\mathbf{NGFP_2}$, the boundary line $\mathbf{NGFP_2} \to \mathbf{GFP}$ of the triangle in Fig. 3.1.

The result of this procedure is depicted in Fig. 3.3 where we set $n_F = 1$. As can be seen, there is a rapid transition to the fixed point scaling regime at the Planck scale ($\tau \approx 69$), above which all three dimensionless couplings remain constant at their fixed point values. By fine tuning we were able to choose the initial parameter to

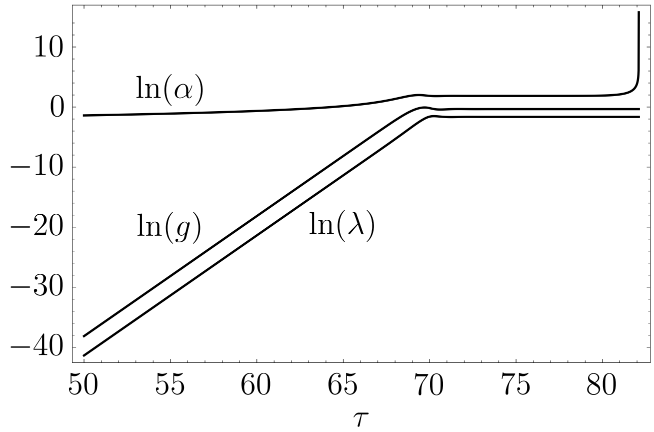


Figure 3.3. Double logarithmic plot of the running couplings of the "realistic" trajectory.

$1/\alpha(k_T) = 14.65$ which ensures that the trajectory stays at the fixed point value for about five orders of magnitude in k before it shoots away to infinity.

The scale of the electron mass corresponds to a τ-value $\tau(k = m_e) = \ln(4.19 \cdot 10^{-23}) + 30\ln(10) \approx 17.55$, which is far above the turning point scale. At this scale the asymptotically safe trajectory predicts a value $\alpha(m_e)^{-1} \approx 10.93$ which is in perfect agreement with (3.62). Hence, we can conclude that the running of λ, as well as the exact functional form of $h_1(g, \lambda)$, are of little effect to the IR value of α.

3.4 Discussion and Conclusion

In the first part of this chapter we presented a non-perturbative RG study computing the gravitational corrections to the running of the YM gauge coupling constant. This way we obtained a β-function for $g_{\text{YM}}(k)$, which in our setting is defined as a coefficient in the derivative expansion of the effective average action. Our approach has two features which are essential here: First, it retains all quadratic divergences (as opposed to dimensional regularization, say), and second, by the background field technique, the regularization (the cutoff $\mathcal{R}_k$) preserves gauge invariance.[6] In this setting, we do get a non-zero gravitational correction. This correction is scheme and gauge fixing dependent but, as we explained, this is by no means unexpected but rather the usual

[6]As in all traditional applications of the average action, Γ_k is not gauge fixing independent, though, i. e. we do not use the Vilkovisky-deWitt method.

situation. When observable quantities are computed from Γ_k the scheme and gauge fixing dependences will cancel among the different running couplings involved.

In the recent literature on the gravitational corrections to the Yang-Mills β-function [RW06, Rob, Pie07, Tom07, EPR09, TW10, Tom08] there has been a certain amount of confusion as some of the computations do get a non-zero result while others do not. However, we believe that different calculations have no reason to yield the same result unless they agree on virtually all details of the regularization and renormalization procedure. The quantum effects of interest are related to quadratic divergences (or a k^2-running), and so we should not expect the same high degree of universality as in the case of the familiar gauge boson contribution which is related to a logarithmic divergence.

Among the above mentioned perturbative calculations the two studies, [TW10] by Tang and Wu and the later published [Tom10] by Toms, are directly comparable to ours. Both of them employ a regulator which retains quadratic divergences and treats them in a gauge invariant manner. Moreover, ref. [Tom10] obtains a gauge fixing independent result by employing the Vilkovisky-deWitt method. It is gratifying to see that these, too, get a non-zero gravitational correction which has the same structure as ours when we omit the RG improvements.

In a subsequent RG study Folkerts et al. [FLP12] conjectured that the gravitational correction to the running Yang-Mills coupling is substantially influenced by the extra background field dependence that is introduced by the mode suppression kernel. In our view this question can only be assessed thoroughly in a future full fledged “bi-metric” computation [MR10], that keeps track of the extra background field dependence of both fields $\bar{g}_{\mu\nu}$ and $\bar{A}^a_\mu$.

In the second part of this chapter we coupled Quantum Electrodynamics to quantized gravity and explored the possibility of an asymptotically safe UV limit of the combined system. Using a simple truncation of the corresponding effective average action we found evidence indicating that this is indeed possible. There exist two non-trivial fixed points which lend themselves for an Asymptotic Safety construction. Using the first one, $\mathbf{NGFP_1}$, the fixed point value of the fine-structure constant is zero, and its infrared value α_{IR} is a free parameter which is not fixed by the theory itself but has to be taken from experiment. Basing the theory on the second non-Gaussian fixed point $\mathbf{NGFP_2}$ instead, the fixed point value α^* is non-zero, and the (“renormalized”) low energy value of the fine-structure constant α_{IR} can be predicted in terms of the electron mass in Planck units. In either case the coupled theory QED + QEG is well behaved in the ultraviolet, there is no Landau singularity in particular, and it is not trivial, i. e. the continuum limit is an (electromagnetically and gravitationally) interacting theory.

The key ingredient in the RG equations considered is the quantum gravity contribution to β_α that was adopted from the YM result obtained in the first part of the chapter. It is proportional to $g\,\alpha \equiv G(k)k^2\alpha$, and its sign is such that for increasing k it counteracts the growth of $\alpha(k)$ caused by the fermions. This can lead to two qualitatively different scenarios for the high energy behavior of QED + QEG. In the first one, which is also seen in perturbation theory [RW06, Tom10], the gravitational effects win over the fermionic ones and $\alpha(k)$ is driven to zero in the UV: it has become an asymptotically free coupling. For RG trajectories of this type the k-dependence of Newton's constant plays no essential role; the decrease of $G(k)$ becomes substantial only after $\alpha(k)$ is almost zero already. Instead, in the second scenario, the initial (low energy) value of $\alpha(k)$ is such that it has not yet become very small when the weakening of gravity due to the decrease of $G(k)$ sets in. In particular in the asymptotic scaling regime it decreases rapidly, $G(k) = g^*/k^2$, so that the fermions, still trying to increase α for $k \to \infty$, have a better chance to win over the gravitons now. Along certain trajectories they indeed do, but what is more interesting is the possibility of an exact compensation of the two trends. This is exactly what happens at the second non-trivial fixed point, $\mathbf{NGFP_2}$, which is characterized by a non-zero α^*.

Note that this second possibility is closely related to the Asymptotic Safety of (pure) gravity. It could not be found in perturbation theory which, while using a similar gravity correction to β_α, treats the factor of G it contains as a constant and therefore misses the weakening of gravity at high scales.

The most remarkable feature of the second fixed point is the reduced dimensionality of its UV critical manifold and the resulting higher degree of predictivity than in perturbation theory. We take this as a first hint indicating that after coupling the Standard Model to asymptotically safe gravity it might perhaps be possible to compute some of its as to yet free parameters from first principles.

4 Tetrad Gravity

4.1 Motivation

In this chapter we perform a first RG study of the “tetrad only” theory space $\mathcal{T}_{\text{tet}}$, whose main results have previously been published in [HR12]. We thereby investigate a truncation of $\mathcal{T}_{\text{tet}}$ that is obtained by “translating” the Einstein-Hilbert truncation in $\mathcal{T}_{\text{E}}$, i. e. by expressing the metric in terms of the vielbein according to $g_{\mu\nu} = e^{a}{}_{mu}\eta_{ab}e^{b}{}_{\nu}$. Thus, a direct comparison to the results of the metric computations [Reu98, LR01, RS02] using the same exact FRGE is possible, that allows for a detailed analysis of differences due to the change of theory space.

There are three main reasons that motivated our investigation:

(A) The first functional RG based results obtained on the Einstein-Cartan theory space $\mathcal{T}_{\text{EC}}$, in a truncation with a scale dependent Hilbert-Palatini action (including a running Immirzi term), show certain characteristic differences in comparison with the familiar case of $\mathcal{T}_{\text{E}}$ truncated with a running Einstein-Hilbert action; in particular, the $\mathcal{T}_{\text{EC}}$ results show a stronger RG scheme and gauge fixing dependence than the older ones on the “Einstein” case [DR12, DR]. It would be interesting to know whether these differences are mainly due to the use of the different truncations, different field variables, or both. Especially when we are going to evaluate the new WH-like FRGE on $\mathcal{T}_{\text{EC}}$ in the next chapter, this knowledge will become important in order to judge the applicability of this new approximation. In the following investigation we shall therefore change only the field variable (and the group $\mathbf{G}$ correspondingly), but not the truncation, and so it should be possible to disentangle the two sources of deviations to some extent.

We note here that, like most settings of quantum field theory, the flow equation of the average action is *not* invariant under diffeomorphisms in field space, $\Phi \mapsto \Phi'(\Phi)$. Thus, at intermediate steps, as long as one does not compute observables, there is no reason to expect any field parametrization independence. Moreover, and perhaps this is even more important, the gauge fixing and ghost sectors are quite different for $\mathsf{Diff}(\mathcal{M})$ and $\mathsf{Diff}(\mathcal{M}) \ltimes \mathsf{O}(d)_{\text{loc}}$, respectively. Therefore the β-functions for the running Newton constant G_k or cosmological constant Λ_k, for instance, may well depend on whether the FRGE is formulated in terms of the metric or tetrad. Similar remarks apply also to a recent study of the perturbative RG running of G_k and the Immirzi parameter [BS11].

(B) On theory spaces involving fermions coupled to gravity introducing vielbeins is compulsory (as far as we do not consider the special case of Dirac-Kähler fermions). Besides the pure gravity couplings, such as G_k, Λ_k, etc. the average action will then depend on additional couplings related to the matter field monomials. If we collectively denote these couplings by u_{grav} and u_{mat}, respectively, their β-functions are of the form

$$\beta_{\text{grav}} = \beta^{\text{grav}}_{\text{grav}}(u_{\text{grav}}) + \beta^{\text{mat}}_{\text{grav}}(u_{\text{grav}}, u_{\text{mat}}) \tag{4.1}$$

$$\beta_{\text{mat}} = \beta^{\text{mat}}_{\text{mat}}(u_{\text{mat}}) + \beta^{\text{grav}}_{\text{mat}}(u_{\text{grav}}, u_{\text{mat}}) \tag{4.2}$$

Diagrammatically speaking, the two parts $\beta^{\text{grav}}_{\text{grav}}$ and $\beta^{\text{mat}}_{\text{grav}}$ of the pure gravity β-functions stem from the graviton and matter loops, respectively. Conversely, the running of the matter couplings has a part due to pure matter loops, $\beta^{\text{mat}}_{\text{mat}}$, plus mixed matter-gravity contributions, $\beta^{\text{grav}}_{\text{mat}}$.

In order to get a first impression of the impact the fermions have on the gravitational RG flow one might neglect the running of the matter couplings, and try to compute β_{grav} only. While the evaluation of $\beta^{\text{mat}}_{\text{grav}}$ from the fermion loops clearly requires a vielbein and a spin connection, the pure gravity part $\beta^{\text{grav}}_{\text{grav}}$ does not obviously do so. From a pragmatic point of view it is therefore tempting to take the $\beta^{\text{grav}}_{\text{grav}}$ part from a (much simpler, and already available) computation in the metric formalism. The invariants $I[g_{\mu\nu}]$ occurring in the latter one would interpret as $I[e] \equiv I[g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu]$. For some (but not all) field monomials in $\mathcal{T}_{\text{tet}}$ this establishes a correspondence to monomials in $\mathcal{T}_{\text{E}}$, and one can try to identify their running prefactors; for instance, $\Lambda_k \int \mathrm{d}^d x \sqrt{g} \in \mathcal{T}_{\text{E}} \;\leftrightarrow\; \Lambda_k \int \mathrm{d}^d x\, e \in \mathcal{T}_{\text{tet}}$, when g and e denote the determinants of $g_{\mu\nu}$ and $e^a{}_\mu$, respectively.

Thus, it seems that only the fermion loops, $\beta^{\text{mat}}_{\text{grav}}$, need to be calculated. This requires fixing a Lorentz gauge in order to associate a unique $e^a{}_\mu$ to a given $g_{\mu\nu}$, and for $\omega^{ab}{}_\mu$ one might take the unique Levi-Civita connection associated to this vielbein, ${\omega_{\text{LC}}}^{ab}{}_\mu(e)$.

We shall refer to this procedure as a *hybrid calculation.* Clearly it can be meaningful at most within a truncation of $\mathcal{T}_\text{E}$ and $\mathcal{T}_\text{tet}$ that allows an identification of monomials; an example is the Einstein-Hilbert action regarded as a functional of $g_{\mu\nu}$ and $e^a{}_\mu$, respectively, with the same two couplings G_k and Λ_k occurring in both cases. At the exact level there exists certainly no such one-to-one correspondence between action monomials in $\mathcal{T}_\text{E}$ and $\mathcal{T}_\text{tet}$. Nevertheless, if it was possible to establish the "hybrid" scheme as a reliable approximation, this would be of considerable importance for the feasibility of practical calculations.

As to yet, all investigations for the gravity + fermions theory space, in particular in the Asymptotic Safety context, are, in fact, hybrid computations of this form [DP98, PP03b, PP03a, VZ10, EG11]. They combine the metric-formalism β-functions for G_k and Λ_k in the Einstein-Hilbert truncation with certain matter contributions $\beta^\text{mat}_\text{grav}$, solve for $u_\text{grav}(k)$, and insert the result into (4.2) to obtain the running of the matter couplings. In ref. [EG11] the gravity corrections to certain 4-fermion couplings u_mat were studied in this way.

A necessary condition for the consistency of the hybrid approach is that the pure gravity part $\beta^\text{grav}_\text{grav}$ does not change much when we switch from $g_{\mu\nu}$ to $e^a{}_\mu$ as the fundamental field variable in the Einstein-Hilbert truncation. Only in direct comparison to the corresponding RG study in $\mathcal{T}_\text{tet}$, as presented in this chapter, we are able to explicitly test whether or not this is actually the case. Eventually, the results of our investigation indicate, that the hybrid scheme is very hard, if not impossible to justify, at least at the quantitative level. We shall demonstrate in detail that if one aims at some degree of numerical precision, one should consistently work with the vielbein and its corresponding ghost system already at the pure gravity level.

(C) Picking the vielbein as the fundamental field variable requires fixing an $\mathsf{O}(d)_\text{loc}$ gauge. In perturbation theory, a popular choice is the Deser-van Nieuwenhuizen algebraic gauge fixing condition where the antisymmetric part of the $d \times d$ matrix $e^a{}_\mu$ is required to vanish [DN74]. As $\mathsf{O}(d)$ has $\frac{1}{2}\,d\,(d-1)$ parameters, this reduces the d^2 independent components of $e^a{}_\mu$ to $\frac{1}{2}d(d+1)$, which is precisely the number of independent fields $g_{\mu\nu}$ has in d dimensions.

It was shown that, for this gauge, and in perturbation theory, no Faddeev-Popov ghosts need to be introduced for the $\mathsf{O}(d)_\text{loc}$ factor of $\mathbf{G}$, and that it allows to explicitly express vielbein fluctuations purely in terms of metric fluctuations [Woo84]. Therefore the point of view was advocated that even in presence of fermions the vielbein can be eliminated in favor of the metric.

While this method was proven to be correct in a well defined perturbative context, recently it has been proposed to use this same procedure, in particular the omission of the $\mathsf{O}(d)_{\text{loc}}$ ghosts, also in the context of a nonperturbative flow equation for the gravity–fermion system [VZ10, EG11]. If applicable, it would provide a very economic framework for hybrid computations of the type sketched above.

However, as we are going to discuss in detail, there are reasons to doubt that the perturbative arguments justifying the omission of the $\mathsf{O}(d)_{\text{loc}}$ ghosts carry over to the nonperturbative setting of the FRGE. In fact, in perturbation theory the ghosts are omitted since their inverse propagator contains no derivatives, they are non-propagating, leading to a trivial Faddeev-Popov determinant. In the FRGE, instead, a straightforward evaluation of the functional traces cuts off all field modes in a uniform fashion, no matter if their kinetic term contains 2, or more, or no derivatives at all.

In the following RG study we shall therefore explicitly evaluate the contributions to β_{grav} from the non-propagating $\mathsf{O}(d)_{\text{loc}}$ ghosts pertaining to the symmetric vielbein gauge, and we shall analyze whether they really can be discarded in setting up the flow equation for the average action.

In the next two sections we will first present the derivation the β-functions of G_k and Λ_k in tetrad theory space followed by a detailed numerical analysis of the properties of the resulting RG flow. The last section of this chapter contains a discussion of the results obtained focussing in particular on the issues raised above.

4.2 RG flow on $\mathcal{T}_{\text{tet}}$ in Einstein-Hilbert truncation

4.2.1 The RG framework

Throughout this chapter we will consider the theory space $\mathcal{T}_{\text{tet}}$ (cf. Section 2.1.4). It contains all functionals $A[e^a{}_\mu, \bar{e}^a{}_\mu, \xi^\mu, \bar{\xi}_\mu, \Upsilon^{ab}, \bar{\Upsilon}_{ab}]$ that are invariant under the background gauge transformations of the total gauge group $\mathbf{G} = \mathsf{Diff} \ltimes \mathsf{O}(d)_{\text{loc}}$. Besides the expectation value of the vielbein field and its background configuration they depend on the diffeomorphism ghosts $(\xi^\mu, \bar{\xi}_\mu)$ and the $\mathsf{O}(d)_{\text{loc}}$ ghosts $(\Upsilon^{ab}, \bar{\Upsilon}_{ab})$. Instead of $e^a{}_\mu$ we shall often consider the vielbein fluctuation $\bar{\varepsilon}^a{}_\mu \equiv e^a{}_\mu - \bar{e}^a{}_\mu$ the independent argument of the action.

In this theory space we will consider a truncation of the form

$$\Gamma_k = \Gamma_k^{\text{EH}} + \Gamma_k^{\text{gf}} + S_{\text{gh}} \equiv \breve{\Gamma} + S_{\text{gh}} \; . \tag{4.3}$$

In particular, the only invariant containing ghost fields is the classical ghost action S_{gh}, whose renormalization, moreover, will be neglected. In addition the truncation will be of the single-metric type [MR10], such no extra background field dependence is determined and we can set the fluctuations to zero once the Hessian of the truncation ansatz has been computed.

In order to compute the RG flow on $\mathcal{T}_{\text{tet}}$ we decide for the exact FRGE (2.34) that decomposes for a truncation of the above type into the two blocks

$$\partial_t \Gamma_k = \frac{1}{2}\text{Tr}\Big[\Big(\breve{\Gamma}_k^{(2)} + \breve{\mathcal{R}}_k(\Delta)\Big)^{-1} \partial_t \breve{\mathcal{R}}_k(\Delta)\Big] - \text{Tr}\Big[\Big(S_{\text{gh}}^{(2)} + \mathcal{R}_k^{\text{gh}}(\Delta)\Big)^{-1} \partial_t \mathcal{R}_k^{\text{gh}}(\Delta)\Big] \; . \tag{4.4}$$

Further we will employ a cutoff of type Ia [CPR09], i. e. we choose $\Delta = -\bar{D}^2$ and do not decompose the vielbein fluctuations into their spin components, before the cutoff kernel is adapted to the truncation.

Hence, our setting is completely analogous to the metric calculations [Reu98] so that all differences we shall find can be attributed to the change of theory space.

4.2.2 The truncation

Let us now introduce the details of our truncation. Its Einstein-Hilbert part is found by expressing the metric in terms of the tetrad in the well known Einstein-Hilbert action for metric gravity:

$$\Gamma_k^{\text{EH}}[e] = -\frac{1}{16\pi G_k} \int \mathrm{d}^d x \, \sqrt{g(e)} \Big(R(g(e)) - 2\Lambda_k \Big) \, . \tag{4.5}$$

This action involves two running couplings, the cosmological constant Λ_k and Newton's constant G_k; the latter is frequently expressed in terms of the dimensionless function Z_{Nk} according to $G_k \equiv Z_{Nk}^{-1} \hat{G}$ with a constant $\hat{G}$.

To be as general as possible we re-express the metric in terms of the new field variable $e^a{}_\mu$ in the following way:

$$g_{\mu\nu} = \xi^{-1} e^a{}_\mu e^b{}_\nu \eta_{ab}. \tag{4.6}$$

This representation resembles the usual vielbein decomposition of the metric, except for the additional free parameter $\xi > 0$. For this reason we will refer to the field $e^a{}_\mu$ as a generalized vielbein for a given $g_{\mu\nu}$. Treating $e^a{}_\mu$ as the independent variable we assume that the basis 1-forms $e^a = e^a{}_\mu \mathrm{d}x^\mu$ indeed form a non-degenerate co-frame. The parameter ξ is merely a mathematical tool that enables us to study a continuous class of field redefinitions at a time.

As for the usual vielbein this generalized decomposition of the metric is not unique, but there exists an $\mathsf{O}(d)$ manifold of vielbein fields corresponding to the same metric. This arbitrariness is treated as an additional gauge freedom, such that the total group of gauge transformations is enlarged compared to the metric case: $\mathbf{G} = \mathsf{Diff}(\mathcal{M}) \ltimes \mathsf{O}(d)_{\mathrm{loc}}$. Thus we are also in need of a second gauge fixing term and the corresponding background gauge invariant ghost-action is constructed according to the formalism presented in Section 2.2.3.

If we decompose both the metric $g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \bar{h}_{\mu\nu}$ and the vielbein $e^a{}_\mu \equiv \bar{e}^a{}_\mu + \bar{\varepsilon}^a{}_\mu$ into background fields and fluctuations, we find[1]

$$\bar{g}_{\mu\nu} + \bar{h}_{\mu\nu} = g_{\mu\nu} = \xi^{-1}(\bar{e}^a{}_\mu + \bar{\varepsilon}^a{}_\mu)(\bar{e}^b{}_\nu + \bar{\varepsilon}^b{}_\nu)\eta_{ab} = \xi^{-1}\bar{e}^a{}_\mu \bar{e}^b{}_\nu \eta_{ab} + \xi^{-1}\bar{\varepsilon}_{(\mu\nu)} + \mathcal{O}(\bar{\varepsilon}^2). \quad (4.7)$$

Here and in the following we use the background vielbein $\bar{e}^a{}_\mu$ to change the type of the first (i. e., frame) index of the vielbein fluctuation: $\bar{\varepsilon}_{\mu\nu} = \eta_{ab}\bar{e}^a{}_\mu \bar{\varepsilon}^b{}_\nu$. We see that the symmetric part of the vielbein fluctuations, $\frac{1}{2}\bar{\varepsilon}_{(\mu\nu)}$, is proportional to the metric fluctuations $\bar{h}_{\mu\nu}$ in lowest order, while we can relate the additional $d(d-1)/2$ gauge degrees of freedom carried by $e^a{}_\mu$ to the antisymmetric part of the fluctuations, $\frac{1}{2}\bar{\varepsilon}_{[\mu\nu]}$.

This observation motivates the following choice of gauge conditions. For the diffeomorphisms we choose the usual harmonic gauge fixing function for metric fluctuations, replacing $\bar{h}_{\mu\nu} \mapsto \xi^{-1}\bar{\varepsilon}_{(\mu\nu)}$, with $\kappa \equiv (32\pi\bar{G})^{-1/2}$:

$$\mathcal{F}_\mu = \sqrt{2}\kappa\, \xi^{-1}\Big(\bar{D}^\nu \bar{\varepsilon}_{(\mu\nu)} - \bar{D}_\mu \bar{\varepsilon}^\nu{}_\nu\Big). \quad (4.8)$$

The $\mathsf{O}(d)$ transformations are gauge fixed using

$$\mathcal{G}^{ab} = \xi^{-\frac{1}{2}}\, \bar{g}^{\mu\nu}\bar{\varepsilon}^{[a}{}_\mu \bar{e}^{b]}{}_\nu = \xi^{-\frac{1}{2}}\bar{\varepsilon}^{[ab]}, \quad (4.9)$$

corresponding to a suppression of the antisymmetric vielbein fluctuations.

[1] Note that in our notation the (anti-)symmetrization brackets do not contain any weighting factor, i. e. . symbolically $(ab) = ab + ab$. More on our conventions can be found in Appendix A.

With these gauge conditions the gauge fixing term in the effective average action assumes the usual form, involving parameters α_{D} and α_{L}:

$$\Gamma_k^{\text{gf}}[e,\bar{e}] = \frac{1}{2\alpha_{\text{D}}} \int \text{d}^d x\, \sqrt{\bar{g}}\bar{g}^{\mu\nu}\mathcal{F}_\mu\mathcal{F}_\nu + \frac{1}{2\alpha_{\text{L}}} \int \text{d}^d x\, \sqrt{\bar{g}}\, \mathcal{G}^{ab}\mathcal{G}_{ab}. \tag{4.10}$$

In the following we fix the diffeomorphism gauge parameter α_{D} to $\alpha_{\text{D}} = 1/Z_{Nk}$ which leads to the same cancelation in the kinetic operator as in metric gravity [Reu98].

In order to obtain a *background* **G**-*invariant* ghost action with respect to both $\mathsf{O}(d)_{\text{loc}}$ transformations and diffeomorphisms, we can make use of the Faddeev-Popov construction only if we first reparametrize the gauge transformations in such a way, that the new generators of diffeomorphisms and $\mathsf{O}(d)$ transformations commute. This corresponds to an $\mathsf{O}(d)$ covariantization of the Lie derivative (cf. Section 2.2.3). Following this procedure, while treating the ghost sector classically (i. e. we can set $e = \bar{e}$ already at the level of the ghost action) we arrive at

$$S_{\text{gh}}[\xi,\bar{\xi},\Upsilon',\bar{\Upsilon}';\bar{e}] = -\!\int\!\text{d}^d x\,\bar{e} \begin{pmatrix} \bar{\xi}_\mu \\ \bar{\Upsilon}'_{ab} \end{pmatrix}^{\!T} \begin{pmatrix} \sqrt{2}\xi^{-1}\big(\delta^\mu{}_\rho\bar{D}^2 + \bar{R}^\mu{}_\rho\big) & 0 \\ 2\,\xi^{-\frac{1}{2}}\bar{\mu}\,\bar{e}^{b\mu}\bar{e}^a{}_\rho\bar{D}_\mu & \xi^{-\frac{1}{2}}\bar{\mu}^2\delta_c^{[a}\delta_d^{b]} \end{pmatrix} \begin{pmatrix} \xi^\rho \\ \Upsilon'^{cd} \end{pmatrix}. \tag{4.11}$$

Here $\bar{\xi}_\mu$, ξ^μ represent the diffeomorphism ghosts and $\bar{\Upsilon}_{ab}$, Υ^{ab} the $\mathsf{O}(d)$ ghost fields.

As the infinitesimal transformation under diffeomorphisms contains a derivative, while the corresponding $\mathsf{O}(d)$ transformation does not, the diffeomorphism ghosts have a canonical mass dimension of one unit less compared to the $\mathsf{O}(d)$ ghosts. In order to obtain a Hessian operator of a well-defined mass dimension we have rescaled the fields $\bar{\Upsilon}_{ab} = \bar{\mu}\bar{\Upsilon}'_{ab}$, $\Upsilon^{ab} = \bar{\mu}\Upsilon'^{ab}$ with an arbitrary mass parameter $\bar{\mu}$; consequently the Hessian operator obtains a mass dimension of 2.

4.2.3 Structure of the vielbein sector

After having presented the details of our truncation we can now pass on to the evaluation of the FRGE (4.4) in this truncation. On the LHS of the equation, after setting $\bar{e} = e$, we obtain the same result as in the metric version of the Einstein-Hilbert truncation [Reu98]:

$$\partial_t\Gamma_k[e,e] = 2\kappa^2 \int \text{d}^d x\, \sqrt{g(e)}\big[-R(g(e))\partial_t Z_{Nk} + 2\partial_t\big(Z_{Nk}\Lambda_k\big)\big] \tag{4.12}$$

On the RHS of the FRGE, however, we find two types of additional contributions to the supertrace as compared to those already present in the metric description. While the second type of contributions is due to the extended gauge group of the theory, the first type is closely linked to the off-shell character of the FRGE. This can be seen as follows.

In order to obtain $\breve{\Gamma}^{(2)}$ we expand $\breve{\Gamma}_k$ to second order in the vielbein fluctuations and read off the operator from the quadratic term $\breve{\Gamma}_k^{\text{quad}}$. As Γ_{gf} is already quadratic in the fluctuations we only have to expand $\Gamma_{\text{EH},k}$. For

$$\Gamma_{\text{EH}}^{\text{quad}} = \frac{1}{2}\delta_e^2 \Gamma_{\text{EH}}\Big|_{e=\bar{e}} \tag{4.13}$$

we find

$$\begin{aligned}\Gamma_{\text{EH}}^{\text{quad}} = & \frac{1}{2}\cdot\frac{1}{\xi^2}\int \mathrm{d}^d x_1 \mathrm{d}^d x_2 \, \frac{\delta^2 \Gamma_{\text{EH}}}{\delta g_{\rho\sigma}(x_2)\delta g_{\mu\nu}(x_1)}\Bigg|_{g=\bar{g}} \bar{\varepsilon}_{(\mu\nu)}(x_1)\bar{\varepsilon}_{(\rho\sigma)}(x_2) \\ & + \frac{1}{2}\cdot\frac{1}{\xi}\int \mathrm{d}^d x \, \frac{\delta \Gamma_{\text{EH}}}{\delta g_{\mu\nu}(x)}\Bigg|_{g=\bar{g}} \bar{\varepsilon}_{a(\nu}(x)\bar{\varepsilon}^a{}_{\mu)}(x)\,.\end{aligned} \tag{4.14}$$

Here we have used the chain rule for functional derivatives. Obviously, the first term on the RHS of (4.14) corresponds exactly to the one known from the metric calculation, while the second term is due to the field redefinition. We note that those two terms come with different powers of ξ, which enables us to keep track of their respective origin during the entire calculation and in the final result. This was in fact our main motivation for introducing this book-keeping device.

Note also that in (4.14) the term due to the field redefinition is proportional to the first variation $\delta\Gamma_{\text{EH}}/\delta g_{\mu\nu}$. So it would vanish if we were to go "on shell", i. e. to insert a special metric or vielbein which happens to be a stationary point of Γ_{EH}. We emphasize that in the process of computing β-functions this would be a severe mistake. To see this, consider an (exact) average action expanded as

$$\Gamma_k[\Phi,\bar{\Phi}] = \sum_\alpha \bar{u}_\alpha(k) P_\alpha[\Phi,\bar{\Phi}], \tag{4.15}$$

where $\bar{u}_\alpha(k)$ denote the running couplings and the P_α's are $\mathbf{G}$-invariant basis functionals (integrated field monomials, say) independent of k. When represented in this fashion one may think of Γ_k as a "generating function" for the set of running couplings, $\{\bar{u}_\alpha(k)\}$, which are "projected out" by expanding Γ_k in the basis $\{P_\alpha[\,\cdot\,,\,\cdot\,]\}$. In this picture the fields Φ, $\bar{\Phi}$ have a subordinate status only. They serve as arguments of the P_α's, and their only role is that of a dummy variable needed in order to define the basis functionals P_α. Therefore, in order for the set $\{P_\alpha\}$ to remain *complete* it is in general not possible to narrow down the function space Φ, $\bar{\Phi}$ are drawn from in any way, for instance by stationary point conditions or the like. In this sense, the average action and its associated FRGE are intrinsically "off shell" in nature.

At most at the level of truncations where the set $\{P_\alpha\}$ is incomplete anyhow we may opt for special choices of the fields (e. g. satisfying convenient symmetry conditions) as long as the invariants in the truncation ansatz when calculated for these fields can still be distinguished from all other invariants and from each other. This is an often used computational trick that simplifies practical calculations without affecting the result in any way.

For the total quadratic part of the action $\breve{\Gamma}_k$ we obtain, with $\sqrt{\bar{g}} \equiv \bar{e}$,

$$\begin{aligned}\breve{\Gamma}_k^{\text{quad}}[\bar{\varepsilon};\bar{e}] = &\frac{Z_{Nk}\kappa^2}{\xi^2}\int \mathrm{d}^dx\,\sqrt{\bar{g}}\;\bar{\varepsilon}_{(\mu\nu)}\big[-K^{\mu\nu}{}_{\rho\sigma}\bar{D}^2 + U^{\mu\nu}{}_{\rho\sigma}\big]\bar{\varepsilon}^{(\rho\sigma)}\\ &+\frac{Z_{Nk}\kappa^2}{\xi}\int \mathrm{d}^dx\,\sqrt{\bar{g}}\left(\bar{R}^{\mu\nu} + \Lambda_k\bar{g}^{\mu\nu} - \frac{\bar{R}}{2}\bar{g}^{\mu\nu}\right)\bar{\varepsilon}_{a(\nu}\bar{\varepsilon}^a{}_{\mu)}\\ &+\frac{1}{2\alpha_{\mathrm{L}}}\frac{1}{\xi}\int \mathrm{d}^dx\,\sqrt{\bar{g}}\;\bar{\varepsilon}^{[ab]}\bar{\varepsilon}_{[ab]}\end{aligned} \tag{4.16}$$

where

$$K^{\mu\nu}{}_{\rho\sigma} \equiv \frac{1}{4}\Big(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho - \bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}\Big) \tag{4.17}$$

and

$$\begin{aligned}U^{\mu\nu}{}_{\rho\sigma} \equiv \frac{1}{4}\Big(&\big[\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho - \bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}\big]\big(\bar{R} - 2\Lambda_k\big)\\ &+ 2\big[\bar{g}^{\mu\nu}\bar{R}_{\rho\sigma} + \bar{g}_{\rho\sigma}\bar{R}^{\mu\nu}\big] - \delta^{(\mu}_{(\rho}\bar{R}^{\nu)}{}_{\sigma)} - \bar{R}^{(\nu}{}_{(\rho}{}^{\mu)}{}_{\sigma)}\Big).\end{aligned} \tag{4.18}$$

We observe that the first term on the RHS of (4.16) is exactly the contribution known from the metric computation [Reu98]; in particular thanks to $\alpha_{\mathrm{D}} = 1/Z_{Nk}$ all non-minimal terms in the differential operator canceled. The second and third terms in (4.16) correspond to the already mentioned first and second type of new contributions, respectively.

In a next step we decompose the vielbein fluctuations $\bar{\varepsilon}_{\mu\nu}$ into their symmetric traceless part $\widehat{\varepsilon}_{\mu\nu}$, antisymmetric part $\tilde{\varepsilon}_{\mu\nu}$, and trace part ϕ_ε, according to

$$\bar{\varepsilon}_{\mu\nu} = \widehat{\varepsilon}_{\mu\nu} + \tilde{\varepsilon}_{\mu\nu} + \frac{1}{d}\,\bar{g}_{\mu\nu}\phi_\varepsilon \tag{4.19}$$

with $\widehat{\varepsilon}_{\mu\nu} = \widehat{\varepsilon}_{\nu\mu}$, $\widehat{\varepsilon}^{\mu}{}_{\mu} = 0$ and $\tilde{\varepsilon}_{\mu\nu} = \frac{1}{2}\bar{\varepsilon}_{[\mu\nu]}$.

In addition we specify the background spacetime to be a maximally symmetric Einstein space with

$$\bar{R}_{\mu\nu\rho\sigma} = \frac{1}{d(d-1)}\big[\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}\big]\bar{R} \quad \text{and} \quad \bar{R}_{\mu\nu} = \frac{1}{d}\,\bar{g}_{\mu\nu}\bar{R}. \tag{4.20}$$

This spacetime is still sufficiently general to identify the contributions to the relevant invariants $\int\sqrt{g}$ and $\int\sqrt{g}\bar{R}$ unambiguously. Within the present truncation it is thus a permissable restriction of the function space of the metric; it does not affect the generality of the calculation and so is an example of the computational trick mentioned above.

Using the relations (4.19) and (4.20) the quadratic part of the action reads

$$\begin{aligned}\breve{\Gamma}_k^{\text{quad}}[\bar{\varepsilon};\bar{e}] = \frac{Z_{Nk}\kappa^2}{2}\frac{4}{\xi^2}\int \mathrm{d}^d x\,\sqrt{\bar{g}}\Bigg\{&\widehat{\varepsilon}_{\mu\nu}\Big[-\bar{D}^2 + (\xi-2)\Lambda_k + C_T(\xi)\bar{R}\Big]\widehat{\varepsilon}^{\mu\nu}\\ &+\tilde{\varepsilon}_{\mu\nu}\,\xi\left[\frac{1}{Z_{Nk}\kappa^2\alpha_{\mathrm{L}}} + \Lambda_k - \frac{d-2}{2d}\bar{R}\right]\tilde{\varepsilon}^{\mu\nu}\\ &-\frac{d-2}{2d}\phi_\varepsilon\left[-\bar{D}^2 - \left(2+\frac{2\xi}{d-2}\right)\Lambda_k + C_S(\xi)\bar{R}\right]\phi_\varepsilon\Bigg\}\end{aligned} \tag{4.21}$$

with the constants

$$C_T(\xi) \equiv \frac{d(d-3)+4}{d(d-1)} - \frac{d-2}{2d}\xi, \qquad C_S(\xi) \equiv \frac{d-4+\xi}{d}. \tag{4.22}$$

Note that whereas the symmetric tensor $\widehat{\varepsilon}_{\mu\nu}$ has a standard positive definite kinetic term, its antisymmetric counterpart is non-propagating; the $\tilde{\varepsilon}_{\mu\nu}$-bilinear contains no derivatives at all, but only a (gauge dependent) mass term. Note also that in $d > 2$ the trace part ϕ_ε has a "wrong sign" kinetic term, reflecting the well known conformal factor instability [Reu98].

Let us now fix the precise form of the cutoff operator $\mathcal{R}_k$ in the various sectors of field space. We choose it to be of the generic form

$$\mathcal{R}_k = \mathcal{Z}_k k^2 R^{(0)}(-\bar{D}^2/k^2), \tag{4.23}$$

where $\mathcal{Z}_k$ is a matrix in field space, and $R^{(0)}(u)$ is a dimensionless "shape function" that interpolates smoothly between $R^{(0)}(0) = 1$ and $\lim_{u\to\infty} R^{(0)}(u) = 0$. The matrix $\mathcal{Z}_k$ is chosen by the $\mathcal{Z} = \zeta$-rule introduced in Section 2.3, that also has been applied in the metric calculation in [Reu98]: If a certain field mode has a kinetic operator of the form $[-\bar{D}^2 + \cdots]$, the $\mathcal{Z}_k$ is fixed in such a way that in the sum $\Gamma_k + \Delta_k S$ this operator gets replaced by $[-\bar{D}^2 + k^2 R^{(0)}(-\bar{D}^2/k^2) + \cdots]$.

In the case at hand it is straightforward to implement this rule for $\widehat{\varepsilon}_{\mu\nu}$ and ϕ_ε. In the different sectors we choose

$$(\mathcal{Z}_k)_{\widehat{\varepsilon}\widehat{\varepsilon}} = 2Z_{Nk}\xi^{-2}\kappa^2, \quad (\mathcal{Z}_k)_{\tilde{\varepsilon}\tilde{\varepsilon}} = 2\xi^{-1}Z_{Nk}\kappa^2, \quad (\mathcal{Z}_k)_{\phi_\varepsilon\phi_\varepsilon} = -2\xi^{-2}Z_{Nk}\kappa^2\frac{d-2}{2d}. \tag{4.24}$$

As for the antisymmetric tensor $\tilde{\varepsilon}_{\mu\nu}$, we fixed the corresponding $\mathcal{Z}_k$ in such a way that, taking the overall prefactor into account, the addition of $\mathcal{R}_k$ to the inverse propagator replaces the square brackets in the $\tilde{\varepsilon}_{\mu\nu}$-bilinear of (4.21) by

$$\left[k^2 R^{(0)}(-\bar{D}^2/k^2) + \frac{1}{Z_{Nk}\kappa^2\alpha_{\text{L}}} + \Lambda_k - \frac{d-2}{2d}\bar{R}\right]. \tag{4.25}$$

Now we have specified all ingredients entering the supertrace on the RHS of (4.4) in the different sectors.

First of all we note that the contributions of the antisymmetric sector vanish in the limit of $\alpha_{\text{L}} \to 0$, as this part of the trace is given by

$$\begin{aligned}
&\frac{1}{2}\text{Tr}_{\tilde{\varepsilon}\tilde{\varepsilon}}\left[\frac{\partial_t\big(Z_{Nk}k^2R^{(0)}(-\bar{D}^2/k^2)\big)}{Z_{Nk}\big(k^2R^{(0)}(-\bar{D}^2/k^2) + \Lambda_k + \frac{1}{Z_{Nk}\kappa^2\alpha_{\text{L}}} - \bar{R}\frac{d-2}{2d}\big)}\right] \\
&\qquad = \frac{\alpha_{\text{L}}}{2}\text{Tr}_{\tilde{\varepsilon}\tilde{\varepsilon}}\left[\frac{\partial_t\big(Z_{Nk}k^2R^{(0)}(-\bar{D}^2/k^2)\big)}{Z_{Nk}\big(\alpha_{\text{L}}k^2R^{(0)}(-\bar{D}^2/k^2) + \alpha_{\text{L}}\Lambda_k + \frac{1}{Z_{Nk}\kappa^2} - \alpha_{\text{L}}\bar{R}\frac{d-2}{2d}\big)}\right] \\
&\qquad\qquad \xrightarrow{\alpha_{\text{L}}\to 0} 0.
\end{aligned} \tag{4.26}$$

This behavior is easy to understand as the limit $\alpha_{\mathrm{L}} \to 0$ corresponds to a sharp implementation of the $\mathsf{O}(d)$ gauge condition that introduces a delta functional $\delta[\tilde{\varepsilon}_{\mu\nu}]$ into the path integral. Since the domain of tensors with $\tilde{\varepsilon}_{\mu\nu} = 0$ is invariant under the coarse graining operation it is obvious that the antisymmetric fluctuations should not contribute to any RG running in this limit. From now on we will choose the gauge $\alpha_{\mathrm{L}} = 0$ in order to simplify the discussion.

In this particularly simple gauge the quadratic form (4.21) is structurally similar to the corresponding equation in the metric formalism, see eq. (4.12) in [Reu98]. However, the prefactors of Λ_k in the various terms of $\breve{\Gamma}_k^{\mathrm{quad}}$ and the now ξ-dependent coefficients $C_S(\xi)$, $C_T(\xi)$ of the curvature scalar $\bar{R}$ are different and this will have a rather significant impact on the resulting RG flow. Replacing these constants appropriately in the original metric calculation we can obtain the "bosonic" contributions to the β-functions without a new calculation from those of [Reu98].

4.2.4 Propagating and non-propagating ghosts

Let us move on and discuss the ghost sector. Here we choose the cutoff operator to be

$$\mathcal{R}_k^{\mathrm{gh}} = \begin{pmatrix} \sqrt{2}\xi^{-1}\delta^{\mu}{}_{\rho}k^2 R^{(0)}(-\bar{D}^2/k^2) & 0 \\ 0 & \frac{1}{2} Z_{Lk}^{\mathrm{gh}} \delta_c^{[a}\delta_d^{b]} k^2 R^{(0)}(-\bar{D}^2/k^2) \end{pmatrix}. \tag{4.27}$$

In the diffeomorphism-ghost sector we have adjusted $\mathcal{Z}_k^{\mathrm{gh}}$ to the kinetic term according to the above rule.

In the $\mathsf{O}(d)$ ghost sector, however, there is no kinetic term; the ghosts do not propagate. Nevertheless, a consistent application of the FRGE requires us not to ignore, but to systematically integrate out these non-propagating modes in the same way as all the others, i. e. ordered, and eventually cut off according to their $\bar{D}^2$-eigenvalue. Therefore we introduce a cutoff-operator (with a prefactor unrelated to the couplings in Γ_k, denoted by $Z_{L\,k}^{\mathrm{gh}}$) in this sector as well.[2]

In the gauge chosen, the inverse ghost propagator $S_{\mathrm{gh}}^{(2)} + \mathcal{R}_k^{\mathrm{gh}}$ is a triangular matrix, such that the contributions of the different sectors to the trace decouple.

[2]Recall that ideally, at the exact level, the cutoff action $\Delta_k S$ would be independent of the running couplings present in Γ_k [Wet93].

For any constant choice of $Z^{\text{gh}}_{L\,k} = Z^{\text{gh}}_{L}$ we obtain contributions of the $\mathsf{O}(d)$ ghost sector of the form

$$\text{Tr}\left[\frac{\partial_t(Z^{\text{gh}}_L k^2 R^{(0)})}{-M^2 + Z^{\text{gh}}_L k^2 R^{(0)}}\,\frac{1}{2}\delta^{[a}_c\delta^{b]}_d\right] = \text{Tr}\left[\frac{k^{-2}\partial_t(k^2 R^{(0)})}{-\frac{M^2}{Z^{\text{gh}}_L k^2} + R^{(0)}}\,\frac{1}{2}\delta^{[a}_c\delta^{b]}_d\right] \tag{4.28}$$

with the abbreviation $M^2 \equiv 2\bar{\mu}^2\xi^{-1/2}$. Introducing the dimensionless mass parameter $\mu \equiv \bar{\mu}/k$, and then neglecting any further running of μ, we observe that the trace (4.28) depends only on the k-independent dimensionless quantity

$$-\frac{M^2}{Z^{\text{gh}}_L k^2} \equiv -\frac{2\mu^2}{Z^{\text{gh}}_L \xi^{1/2}}. \tag{4.29}$$

In order to avoid divergences due to a vanishing denominator in (4.28) we have to choose a negative value for Z^{gh}_L, as known from the conformal sector. Since both parameters, μ and Z^{gh}_L, occur only in the combination (4.29) we can mimic any choice of $Z^{\text{gh}}_L < 0$ by choosing a suitable μ. (In particular $Z^{\text{gh}}_L \mapsto -1$, upon replacing $\mu^2 \mapsto -\mu^2/Z^{\text{gh}}_L$.)

In the following we will discuss three distinguished choices of Z^{gh}_L:

(i) $Z^{\text{gh}}_L = -1$: the cutoff term is unrelated to Γ_k, the $\mathsf{O}(d)$ ghost contribution will therefore depend on μ and ξ.

(ii) $Z^{\text{gh}}_L = -M^2/k^2 = -2\mu^2\xi^{-1/2}$: the cutoff is optimally adapted to the form of Γ_k leading to a cancelation of the parameters μ and ξ. This procedure is closest to the above rule for usual kinetic term adaptation and we therefore expect the most reliable results for this choice.

(iii) $Z^{\text{gh}}_L \to 0$: no cutoff term introduced. This choice corresponds to neglecting the $\mathsf{O}(d)$ ghost modes completely; the trace (4.28) vanishes.

As explained above, these three choices are equivalent to using $Z^{\text{gh}}_L = -1$ and setting μ^2 equal to μ^2, $\xi^{1/2}/2$, and $\mu \to \infty$, respectively. We shall refer to them as the *ghost adaptation schemes (i)–(iii)* from now on.

4.2.5 The interpolating beta functions

The remaining part of the calculation consists of projecting out the invariants $\int\sqrt{\bar{g}}$ and $\int\sqrt{\bar{g}}\bar{R}$ from the supertrace in order to find the β-functions for G_k and Λ_k; it follows exactly the metric calculation in [Reu98].

If we turn over to dimensionless couplings

$$g_k = \frac{k^{d-2}}{32\pi Z_{Nk}\kappa^2} = k^{d-2} G_k, \qquad \lambda_k = k^{-2}\Lambda_k \tag{4.30}$$

the resulting system of coupled RG equations is autonomous and has the structure

$$\partial_t g_k = \beta_g(g_k, \lambda_k) \equiv \big[d - 2 + \eta_N(g_k, \lambda_k)\big] g_k, \tag{4.31}$$
$$\partial_t \lambda_k = \beta_\lambda(g_k, \lambda_k) \tag{4.32}$$

with the anomalous dimension $\eta_N = -\partial_t \ln Z_{Nk}$. We shall employ the standard threshold functions Φ, $\tilde{\Phi}$ of [Reu98] along with a new type of threshold function, $\check{\Phi}$, defined according to

$$\Phi^p_n(w) = \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1} \frac{R^{(0)}(z) - z R^{(0)\prime}(z)}{[z + R^{(0)}(z) + w]^p} \tag{4.33}$$
$$\tilde{\Phi}^p_n(w) = \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1} \frac{R^{(0)}(z)}{[z + R^{(0)}(z) + w]^p} \qquad \text{for} \quad n > 0 \tag{4.34}$$
$$\check{\Phi}^p_n(w) = \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1} \frac{R^{(0)}(z) - z R^{(0)\prime}(z)}{[R^{(0)}(z) + w]^p} \tag{4.35}$$

and $\check{\Phi}^p_0(w) = \tilde{\Phi}^p_0(w) = \Phi^p_0(w) = (1+w)^{-p}$. We can write down an explicit expression for η_N in terms of the couplings g, λ then:

$$\eta_N(g, \lambda) = \frac{g \bar{B}_1(\lambda)}{1 - g \bar{B}_2(\lambda)}\,. \tag{4.36}$$

The functions $\bar{B}_1$ and $\bar{B}_2$ are ξ-dependent generalizations of similar ones occurring in [Reu98]:

$$\begin{aligned}
\bar{B}_1(\lambda) = \frac{1}{3}(4\pi)^{1-d/2} \Bigg[&(d-1)(d+2)\, \Phi^1_{d/2-1}\big((\xi-2)\lambda\big) \\
&+ 2\Phi^1_{d/2-1}\Big(-2\lambda \frac{d-2+\xi}{d-2}\Big) - 4d\, \Phi^1_{d/2-1}(0) \\
&- 2d(d-1)\, \check{\Phi}^1_{d/2-1}\Big(\frac{2\mu^2}{\sqrt{\xi}}\Big) - 6(d-1)(d+2)\, C_T(\xi)\, \Phi^2_{d/2}\big((\xi-2)\lambda\big) \\
&- 12\, C_S(\xi)\, \Phi^2_{d/2}\Big(-2\lambda \frac{d-2+\xi}{d-2}\Big) - 24\Phi^2_{d/2}(0) \Bigg]
\end{aligned} \tag{4.37}$$

and

$$
\begin{aligned}
\bar{B}_2(\lambda) = -\frac{1}{6}(4\pi)^{1-d/2}\Big[&(d-1)(d+2)\,\tilde{\Phi}^1_{d/2-1}\big((\xi-2)\lambda\big) \\
&+ 2\tilde{\Phi}^1_{d/2-1}\Big(-2\lambda\frac{d-2+\xi}{d-2}\Big) - 6(d-1)(d+2)\,C_T(\xi)\,\tilde{\Phi}^2_{d/2}\big((\xi-2)\lambda\big) \\
&- 12\,C_S(\xi)\,\tilde{\Phi}^2_{d/2}\Big(-2\lambda\frac{d-2+\xi}{d-2}\Big)\Big]. \qquad (4.38)
\end{aligned}
$$

For the β-function of the cosmological constant we obtain

$$
\begin{aligned}
\beta_\lambda = &-(2-\eta_N)\lambda \\
&+ \frac{1}{2}g_k(4\pi)^{1-d/2}\Big[2(d-1)(d+2)\,\Phi^1_{d/2}\big((\xi-2)\lambda\big) + 4\Phi^1_{d/2}\Big(-2\lambda\frac{d-2+\xi}{d-2}\Big) \\
&- 8d\,\Phi^1_{d/2}(0) - 4d(d-1)\,\check{\Phi}^1_{d/2}\Big(\frac{2\mu^2}{\sqrt{\xi}}\Big) - (d-1)(d+2)\,\eta_N\,\tilde{\Phi}^1_{d/2}\big((\xi-2)\lambda\big) \\
&- 2\,\eta_N\,\tilde{\Phi}^1_{d/2}\Big(-2\lambda\frac{d-2+\xi}{d-2}\Big)\Big].
\end{aligned} \qquad (4.39)
$$

These general ξ-dependent expressions are in exact correspondence to the eqns. (4.40) and (4.43) of ref. [Reu98] for metric gravity. Analyzing the ξ-dependence of the RG flow they give rise to is a convenient way of exploring the field parametrization dependence of the flow.

An important observation is that for a constant, ξ-independent choice of μ (i. e. in the ghost adaptation schemes (i) and (iii)) *the above β-functions reduce precisely to those of the metric result in the limit $\xi \to 0$.* All prefactors and arguments of the threshold functions coincide and the function $\check{\Phi}$ vanishes in this limit. Although this result is far from obvious when considering the definition of ξ in eq. (4.6), we can now regard this one parameter family of field redefinitions as an interpolation between the metric description (for $\xi \to 0$) and the usual vielbein decomposition (for $\xi = 1$).

In scheme (ii) however, the argument of $\check{\Phi}$ is constant, so that in the limit $\xi \to 0$ the β-functions match the metric result except for the additional $\check{\Phi}$ contributions, which are precisely the terms due to the $\mathsf{O}(d)$ ghosts.

4.3 Numerical analysis of the RG flow

In this section we will analyze the RG flow in $d = 4$ dimensions. We will compare results of different cutoff schemes, namely with the optimized shape function, $R^{(0)}(z) = (1-z)\theta(1-z)$, and the exponential one, $R^{(0)}(z) = sz/(e^{sz}-1)$, for shape parameters s ranging from 2 to 20.

4.3.1 The standard vielbein case $\xi = 1$

To start with, let us consider the usual vielbein representation of the metric in (4.6) and set $\xi = 1$ for the time being. With ξ fixed the flow continues to depend on the mass parameter $\mu \equiv \bar{\mu}/k$. We shall analyze this dependence in the following, highlighting especially the implications of those choices of μ that correspond to the three ghost adaptation schemes (i)–(iii).

A first encouraging result is that there exists a non-Gaussian fixed point, for any value of the dimensionless constant $\mu \neq 0$, and in all cutoff schemes we studied.

(A) Fixed point properties. Figure 4.1 shows, for the case of the optimized cutoff, the μ-dependence of three quantities one might expect to be universal, namely the critical exponents θ_i at the fixed point and the product $g^*\lambda^*$. We notice that, while the very existence of the fixed point is indeed universal, its properties heavily depend on the value of μ: For $\mu \lesssim 0.8$ we find a UV attractive FP with two real critical exponents, which then turn into a complex conjugated pair. At $\mu \approx 1.35$ the FP changes its character and becomes UV repulsive in both directions. For large μ-values the dependence on μ weakens for all three "universal" quantities.

Employing the exponential cutoff (not shown here) essentially leads to the same picture: real critical exponents turn into a complex pair before the otherwise UV attractive FP gets UV repulsive for large μ. In all cases the product $g^*\lambda^*$ changes its sign from negative to positive within the interval of μ, in which the FP is attractive and has complex critical exponents.

It is important to stress that even in a much better truncation with many more invariants we would not expect these quantities to become independent of μ: The parameter μ should not be considered a free parameter corresponding e. g. to different cutoff schemes, but it rather corresponds to an additional *coupling*. In principle its running is prescribed by an additional β-function which however is not determined

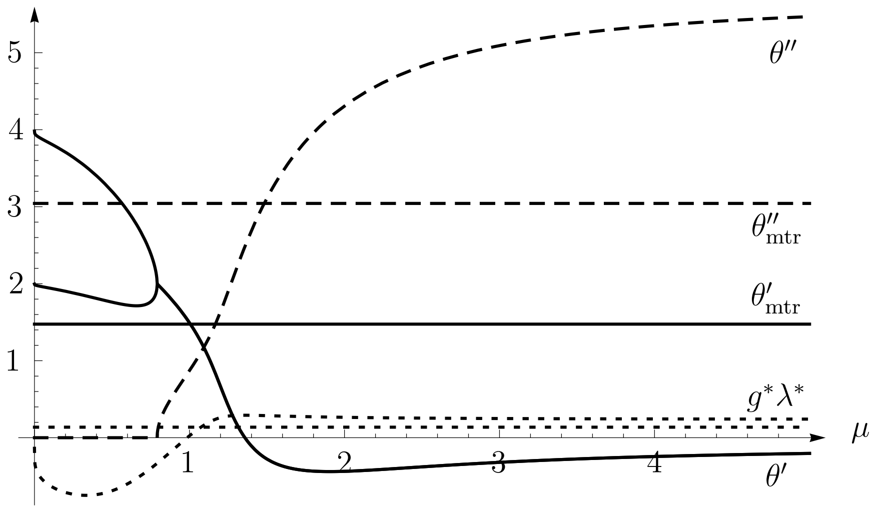


Figure 4.1. The critical exponents $\theta_i = \theta_i' + i\theta_i''$ split into real and imaginary part (solid and dashed line, respectively) of the NGFP and the product $g^*\lambda^*$ (dotted line) as a function of the mass parameter μ for the optimized cutoff. The straight horizontal lines represent the values of the corresponding quantities in the metric calculation.

by the present calculation. Therefore one should not worry too much about the μ-dependence of the "universal" quantities.

In the **ghost adaptation scheme (i)** the best we can do, as we did not calculate the running of μ in our truncation, is to sensibly choose a fixed value for the constant μ. Most naturally we would choose a value of the order of 1 as any other choice would correspond to the introduction of an additional unmotivated physical scale other than k.

Strikingly, in all cutoff schemes studied there exists indeed a μ-interval including, or at least close to $\boldsymbol{\mu} = \mathbf{1}$ in which the situation is similar to the metric theory: We find the NGFP, it is UV attractive, has $g^*\lambda^* > 0$, and a pair of complex conjugate critical exponents.

As an alternative to choosing $\mu = 1$ it is therefore tempting to find the "best fit" to the metric calculation by selecting a μ-value such that there is also a quantitative agreement of the universal quantities.

In Fig. 4.1 the values corresponding to the metric calculation are given by the horizontal lines. We observe that the crossings of the lines of the same type are quite close to each other and are all located at a μ of the order of 1. For the optimized cutoff we find the crossing for the real part of the critical exponent θ_i' very much at $\mu \approx 1$

and for the product $g^*\lambda^*$ at about $\mu \approx 1.1$; the imaginary part θ_i'' takes on its metric value at $\mu \approx 1.45$ in a region where the FP turned UV repulsive already. Taking the average of these values we arrive at $\boldsymbol{\mu \approx 1.2}$, for which we expect the best agreement between the vielbein and the metric theory.

The fact that we find this agreement of metric and vielbein values in a relatively small μ-interval close to the most natural value of $\mu = 1$ can be interpreted as an indication that also the full quantum theories are similar to each other or perhaps equivalent.

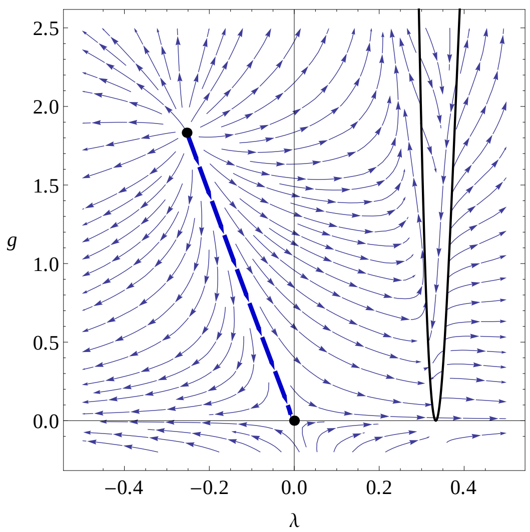
The **adaptation scheme (ii)** is expected to be the most reliable one. It yields the value $\mu = 1/\sqrt{2} \approx 0.7$ for $\xi = 1$. However, in this scheme the results of scheme (i) are not confirmed: For the smaller value of the parameter μ we find a UV attractive NGFP at $g^*\lambda^* < 0$, with two real critical exponents.

The **adaptation scheme (iii)** corresponds to large $\mu \to \infty$, so that we find a UV repulsive FP in this scheme.

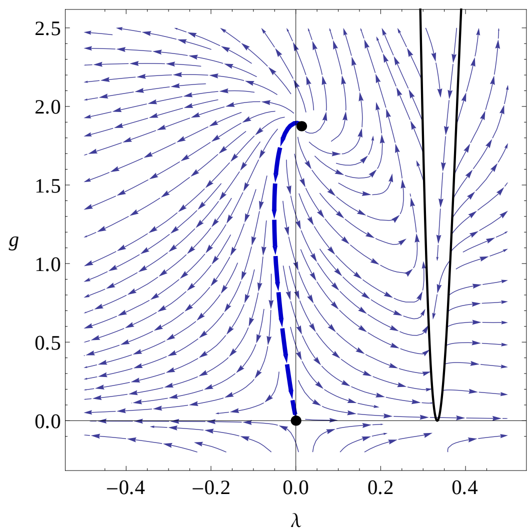
(B) The phase portrait. Let us now discuss the entire RG flow. In Fig. 4.2 we have plotted its phase portrait for different values of μ. Figs. 4.2(b) and 4.2(c) correspond to the first adaptation scheme **(i)**. We observe that the best fit case $\mu = 1.2$ is indeed most similar to the metric flow known from Quantum Einstein Gravity in the Einstein-Hilbert truncation [Reu98, RS02, LR01]: We find a NGFP in the positive (λ, g)-quadrant with two attractive directions; the trajectories spiral into it due to the nonzero imaginary part of the critical exponents. It is in an interplay with the Gaussian fixed point and there exists a "separatrix" that separates trajectories with positive and negative IR values for the cosmological constant λ. Also a major difference to the metric case is to be noted: The UV repulsive direction of the GFP has changed and points now into the negative λ-halfplane. Therefore the separatrix starts off with negative λ before heading to the NGFP at $\lambda^* > 0$. This effect can be traced back to be due to the $\mathsf{O}(d)$ ghost contributions.

For smaller μ (as e. g. $\mu = 1/\sqrt{2}$ in adaptation scheme **(ii)**) these contributions are enhanced in such a way, that the NGFP itself lies at $\lambda^* < 0$ (cf. Fig. 4.2(a)). Now the fixed point has real critical exponents, but is still UV attractive. Qualitatively this picture resembles much the RG flow of Quantum Einstein Cartan Gravity (QECG) in the planes of vanishing and infinite Immirzi parameter as found in [DR, DR12].

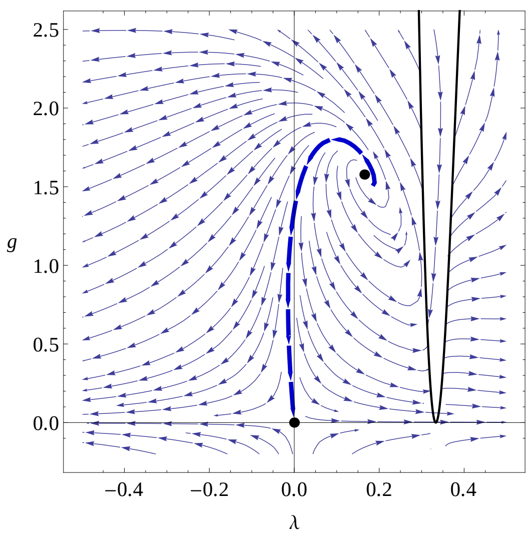
For large μ (scheme **(iii)**), as exemplarily shown for the case of $\mu = 2$ in Fig. 4.2(d), we find a rather different behavior. Although the flow looks similar to the metric case



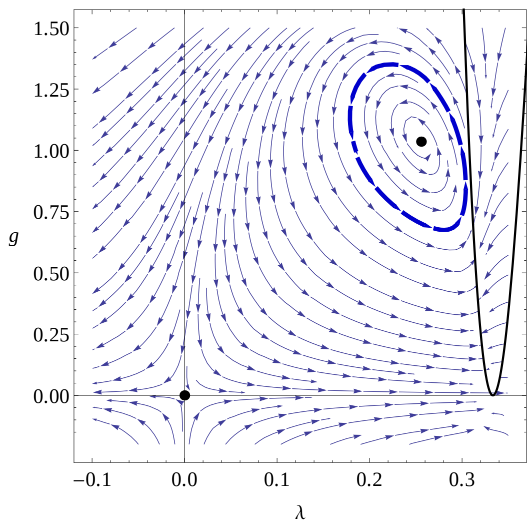
(a) $\mu = 1/\sqrt{2}$, ghost adaptation scheme (ii): The phase portrait resembles the one in QECG.



(b) μ=1: A value close to $\mu = 1$ seems the most natural choice when using ghost adaptation scheme (i).



(c) μ=1.2: For this value we obtain a situation most similar to the metric theory in scheme (i).



(d) μ=2: The limit cycle arises as a qualitatively new feature of the phase portrait when scheme (iii) is applied.

Figure 4.2. RG phase portraits for different values of the mass parameter μ at $\xi = 1$. The figures show the impact of the $\mathsf{O}(d)$ ghost contribution: While for large μ, when it is suppressed, we obtain a limit cycle, for smaller μ we find flow diagrams similar to the ones known from QEG and QECG, respectively.

in large parts of the (λ, g)-plane, the NGFP is repulsive now; the critical exponents form a complex conjugate pair with a negative real part. These two circumstances lead to the formation of a *limit cycle* around the NGFP. This limit cycle is UV attractive for trajectories approaching it both from outside and from the interior.

Clearly such a limit cycle is an interesting and intriguing new possibility for the nonperturbative UV completion of a quantum field theory. It is "asymptotically safe" in a novel sense. However, in this concrete case the picture of a limit cycle is hardly credible against the background of all RG flow studies of gravity to date. Nevertheless it is inspiring to see its formation for the first time in quantum gravity.

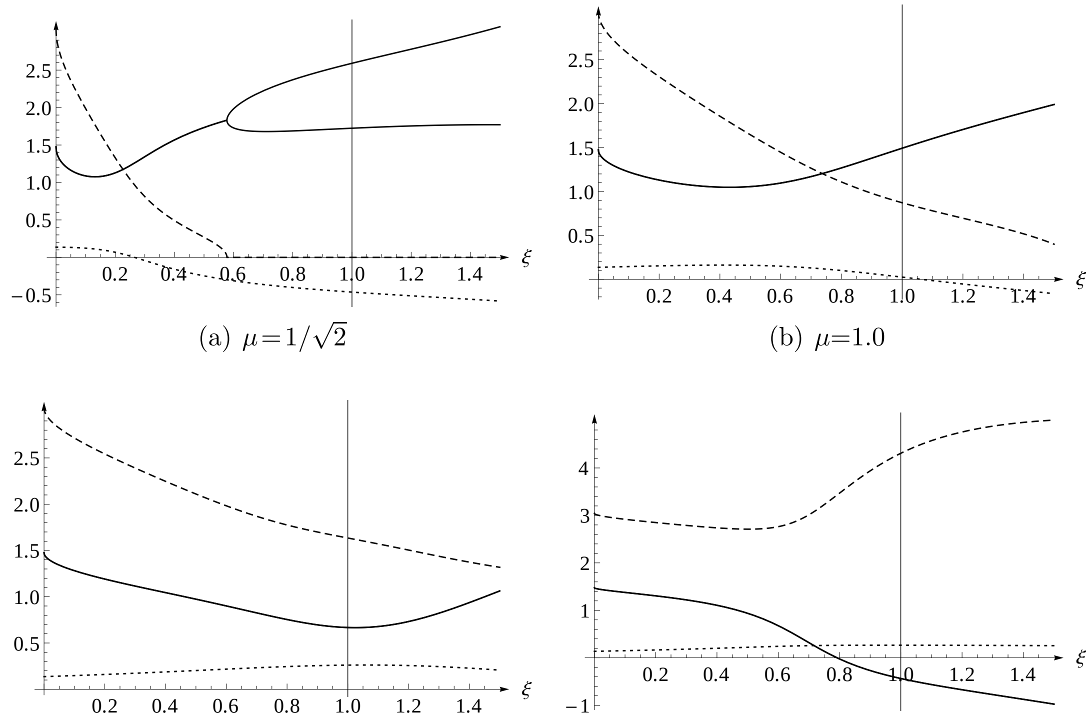
(C) Non-propagating ghosts. The fact that we should not choose the parameter μ too large teaches us another important lesson: Consider the β-functions as given in the previous section. They involve the new threshold function $\check{\Phi}^1_{d/2}(w)$ that vanishes for $w \to \infty$ and diverges for $w \to 0$. In both β-functions, the terms with $\check{\Phi}^1_{d/2}(w)$ are exactly the ghost contributions of the $\mathsf{O}(d)$ gauge group. Since the $\check{\Phi}$ argument is always $w = 2\mu^2/\sqrt{\xi}$ we can control the magnitude of these contributions by changing μ: We obtain a suppression for large μ and an infinite enhancement in the limit $\mu \to 0$. If we had not added a cutoff for the $\mathsf{O}(d)$ ghosts, the situation would correspond to the limit $\mu \to \infty$, i. e. adaptation scheme (iii). In this case we find a *UV repulsive* fixed point, quite different from all results known from metric calculations. We therefore conclude that contrary to the situation of perturbation theory [Woo84] *it is crucial to include all modes of the non-background fields into the renormalization procedure, whether they are propagating or not, by introducing a cutoff-operator for all of them and retaining their contribution to the supertrace in the FRGE.* This implies that we should choose adaptation scheme (i) or (ii) but *not* (iii).

Similar remarks might also apply to perturbative calculations with regularization schemes which retain power divergences.[3]

4.3.2 Field parametrizations with $\xi \neq 1$

When altering the value of ξ, we do not change theory space as both field content and symmetry group remain the same. Therefore we expect to find the same fixed point properties in the RG flow for all values of ξ, resulting in universal quantities that,

[3] It might be interesting to reconsider the calculation [BS11] in this light since there a proper-time regulator has been used.



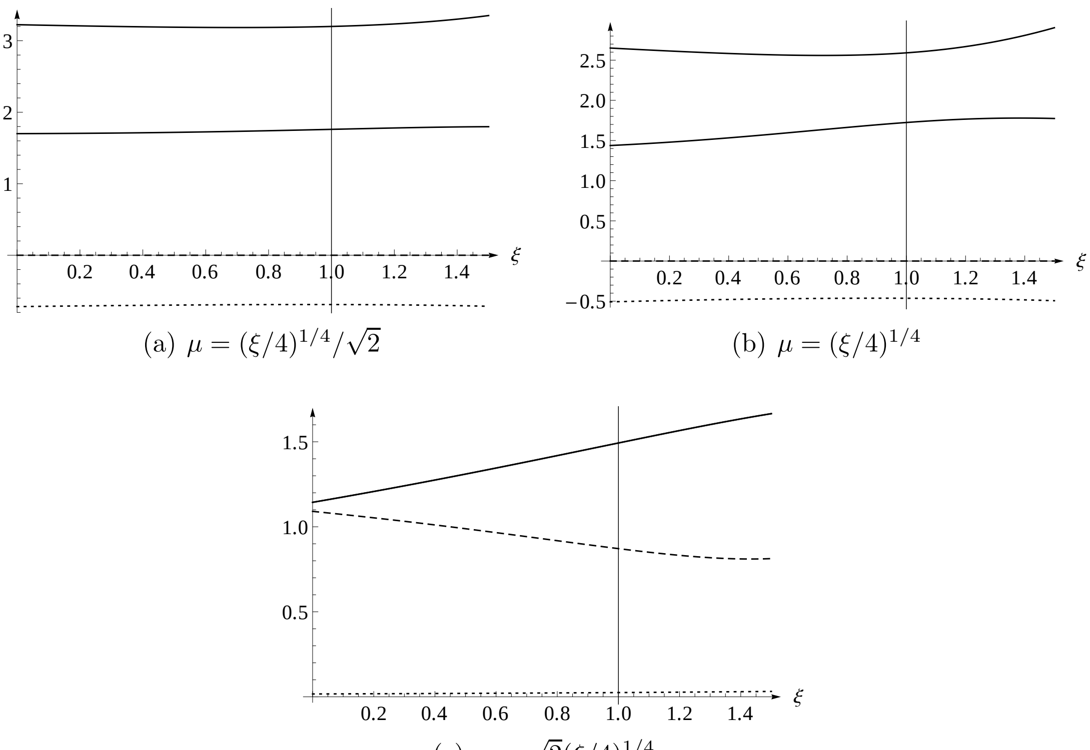
(a) $\mu = 1/\sqrt{2}$ (b) $\mu = 1.0$ (c) $\mu = 1.2$ (d) $\mu = 2$

Figure 4.3. Critical exponents and $g^*\lambda^*$ calculated using the optimized cutoff, for different values of the mass parameter μ, as functions of ξ: The real part of the critical exponents θ_i' (solid), its imaginary part θ_i'' (dashed) and the product of the fixed point coordinates $g^*\lambda^*$ (dotted).

in case of a good approximation to the exact flow, are largely independent of ξ. We will use this criterion in order to test the reliability of the different ghost adaptation schemes in this section.

(A) Adaptation scheme (i). In Fig. 4.3 we have plotted the universal quantities (critical exponents and the product of the fixed point coordinates) for various values of the mass parameter μ as functions of ξ. As μ does not depend on ξ in these examples, all of them correspond the adaptation scheme (i), although Fig. 4.3(d) already shows typical characteristics of the large μ limit and can therefore also be seen as an example of scheme (iii).

In all four cases we start with the values of the metric theory at $\xi = 0$ and find for each of them a quite pronounced dependence on ξ. While for small μ as shown in Fig. 4.3(a) the critical exponents turn from complex to real and $g^*\lambda^*$ turns negative as we move towards $\xi = 1$, for large μ (Fig. 4.3(d)) $g^*\lambda^*$ stays positive but the fixed



(a) $\mu = (\xi/4)^{1/4}/\sqrt{2}$

(b) $\mu = (\xi/4)^{1/4}$

(c) $\mu = \sqrt{2}(\xi/4)^{1/4}$

Figure 4.4. Critical exponents and $g^*\lambda^*$ for different values of an adapted mass parameter μ as a function of ξ (θ' solid, θ'' dashed, $g^*\lambda^*$ dotted), calculated with the optimized cutoff.

point gets repulsive. Only in the region of $\mu \approx 1$ the situation improves a little, as no quantity changes its sign in the interval of $\xi \in [0, 1]$. However the quantities plotted are far from being constant with respect to ξ; furthermore if we compare the analogous results obtained with the family of s-dependent exponential cutoffs (as is done in Appendix D) we find that these results still show a substantial cutoff scheme dependence. Can we do better than this?

(B) Adaptation scheme (ii). If we employ the optimally adapted cutoff (ii) instead (Fig. 4.4(b)), the $\mathsf{O}(d)$ ghost contribution is now independent of ξ. Therefore $\xi = 0$ does not correspond to the metric theory any more. In this case we find the universal quantities almost independent of ξ.

Variants of this cutoff adaptation differing by a factor of $\sqrt{2}$ (Figs. 4.4(a), 4.4(c)) show that the universality can even be improved when choosing a smaller μ. This effect, however, does not really improve the reliability of the flow.: In the limit $\mu \to 0$

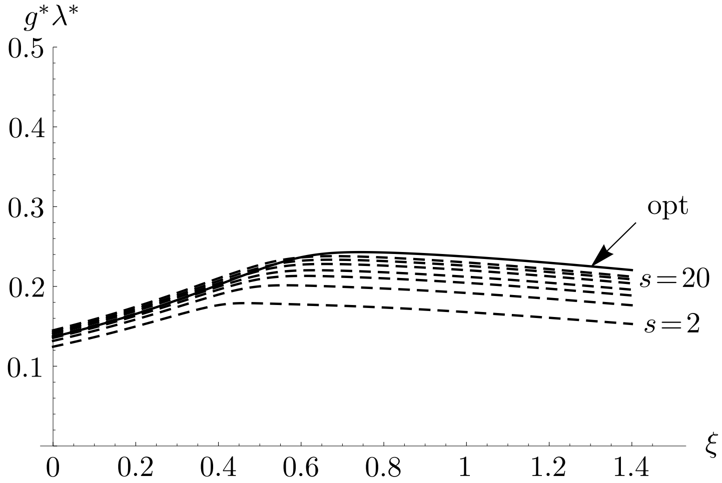


Figure 4.5. The product $g^*\lambda^*$ in ghost cutoff scheme (iii) applying the optimized and the exponential cutoff for various values of the shape parameter s. We observe that, as expected for an on-shell quantity, its RG scheme dependence and its ξ dependence is fairly small.

the constant $\mathsf{O}(d)$ ghost contribution diverges and governs the RG flow, so that the effect of the physical field modes becomes negligible. Therefore it is evident that the ξ-dependence weakens when going to smaller values of μ, but only at the cost of losing the physics content of the flow.

(C) Adaptation scheme (iii). Neglecting the ghost contributions, the typical picture for the optimized as well as for the exponential cutoff looks very similar to Fig. 4.3(d). In particular we always find that in the vielbein case (at $\xi = 1$) the FP turned repulsive, such that a limit cycle has formed. This drastic change in the UV behavior of the theory disfavors the neglection of the ghosts compared the other adaptation schemes.

From Fig. 4.3 we also observe that the product $g^*\lambda^*$ becomes the more independent of ξ the more the ghost contributions are suppressed. This behavior is analyzed in more detail in Fig. 4.5, where we plotted the product $g^*\lambda^*$ in the limit $\mu \to \infty$ for the optimized and various shape parameters of the exponential cutoff. We find that the product is almost scheme independent in the metric limit $\xi = 0$, but its scheme dependence for non-zero ξ stays fairly small as well. Moreover, within each scheme it depends far less on ξ than for any finite fixed value of μ.

Recall that in cutoff scheme (iii) the only difference to the metric calculation lies in the terms proportional to the equations of motion that additionally appear in the Hessian w. r. t. the vielbeins (cf. eq. (4.14)). The product $g^*\lambda^*$, however, is considered an on-shell quantity [LR01], such that it should be virtually independent of these off-shell contributions. Thus, the stability observed in Fig. 4.5 is actually to be expected and impressively verifies the on-shell character of $g^*\lambda^*$.

(D) Discussion. The properties of the universal quantities calculated in this section show, that the influence of the $\mathsf{O}(d)$ ghosts on the fixed point properties is quite significant. While neglecting these contributions (adaptation scheme (iii)) leads to the implausible result that the fixed point changes its character and gets UV repulsive for some $\xi \in [0,1]$, the simple unadapted cutoff (scheme (i)) leads to universal quantities strongly dependent on ξ.

Only the optimally adapted ghost cutoff (scheme (ii)) predicts relatively stable values for the universal quantities. These values indicate a fixed point at $\lambda^* < 0$ with real critical exponents, that therefore may not be the one known from the metric theory.

If this picture is correct, part of the ξ-dependence found in scheme (i) is clearly due to the fact, that in this scheme the quantities are forced to take on their metric values at $\xi = 0$. This way we would have constructed an interpolation between theories of different universality class which obviously leads to a ξ-dependence of the "universal quantities".

Nevertheless, all results show a cutoff scheme, i. e. $R^{(0)}(\,\cdot\,)$-dependence that is more severe than in the metric case. It is analyzed further in Appendix D to which the reader might turn at this point.

Apparently the truncation chosen is less reliable than the Einstein-Hilbert truncation of metric gravity, although it can be considered as its exact "translation" to the tetrad theory space. Together with the different FP properties this indicates that the quantum theories of metric and tetrad gravity (if both should turn out nonperturbatively renormalizable) are perhaps not similar to each other. For this reason it is crucial to use tetrads as fundamental field variables whenever an RG study of fermions coupled to gravity is performed even if only the pure gravity β-functions are investigated. Our results can be considered a warning that in a nonperturbative RG analysis the $\mathsf{O}(d)$ ghost sector cannot be ignored (as opposed to perturbation theory [Woo84]). Seen in this light, the status of hybrid calculations which add fermionic contributions to metric QEG seems questionable.

4.4 Summary and Conclusions

In this chapter we performed a first analysis of the renormalization group flow on the "tetrad only" theory space $\mathcal{T}_{\text{tet}} = \{A[e^a{}_\mu, \cdots]\}$. Its points are action functionals which, besides the indispensable background and ghost fields, depend on the vielbein $e^a{}_\mu$ only, and which are invariant under the semidirect product of spacetime diffeomorphisms and local Lorentz transformations. Contrary to Einstein-Cartan theory, that is studied in the next chapter, the spin connection is not an independent field, but rather is identified with the Levi-Civita connection implied by $e^a{}_\mu$. This excludes the possibility of field configurations with torsion. We truncated $\mathcal{T}_{\text{tet}}$ so as to consist of a running Einstein-Hilbert term, along with the classical gauge fixing and ghost terms. As a result, the only difference in comparison to QEG in the Einstein-Hilbert truncation [Reu98, LR01, RS02] is the use of $e^a{}_\mu$ rather than the metric $g_{\mu\nu}$ as the fundamental field variable and the larger group of gauge transformations $\mathsf{Diff}(\mathcal{M}) \ltimes \mathsf{O}(d)_{\text{loc}}$ replacing $\mathsf{Diff}(\mathcal{M})$. In the present treatment the latter has the status of a composite field: $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b_\nu$. Our main tool was the gravitational effective average action on $\mathcal{T}_{\text{tet}}$, and in particular the exact FRGE which governs its scale dependence. Since this framework is not covariant under field reparametrizations, and since the respective groups $\mathbf{G}$ are different, the RG flow on $\mathcal{T}_{\text{tet}}$ is likely to differ from the one of QEG, even at the exact level.

This expectation was confirmed by our explicit calculation. The details of the Einstein-Hilbert flows with $e^a{}_\mu$ and $g_{\mu\nu}$, respectively, as fundamental fields are indeed different in a significant way. However, their gross topological features are still similar, nevertheless. In particular we found on $\mathcal{T}_{\text{tet}}$ one, and only one, non-Gaussian fixed point, exactly as in QEG. Provided it is not a truncation artifact it seems suitable for taking a nonperturbative continuum limit there, thus defining an asymptotically safe field theory.

To assess the reliability of the approximations made we investigated the dependence of "universal" quantities such as critical exponents and the product $g^*\lambda^*$ on the cutoff shape function $R^{(0)}(\cdot)$ and the mass parameter μ. The latter had to be introduced in order to give equal canonical dimensions to diffeomorphism and $\mathsf{O}(d)_{\text{loc}}$ ghosts. While in a more complete truncation it would be treated as a running coupling with its own β-function, we neglected its running in the present investigation. The upshot of the analysis is that the very existence of the NGFP indeed seems to be a universal feature, in the sense that it exists for all admissible cutoffs and values of μ.

However, further details, even the critical exponents show a variability with $R^{(0)}(\cdot)$ and μ which is significantly larger than in QEG with the same truncation. (In particular, in QEG there is no analog of the parameter μ.) Thus we must conclude that, using the same type of flow equation and the same (Einstein-Hilbert) truncation, the use of the vielbein instead of the metric leads to a less robust RG flow.

Can we understand on general grounds why the flow of the metric theory might have better robustness properties than the one based upon the tetrad? A possible explanation is as follows.

The running couplings parametrizing a general functional Γ_k are, per se, not measurable quantities, that is, typical observables are complicated combinations of these couplings, and in forming these combinations the scheme dependence which the individual couplings have (even in the exact theory!) cancels among them. Consider now a theory space whose actions are constrained to be invariant under a group $\mathbf{G}$ of gauge transformations which we make larger and larger. As a result, more and more excitations carried by the (fixed) set of fields considered are declared "unphysical" gauge modes. Nevertheless all those modes continue to contribute to the supertrace in the FRGE, but are counteracted by an increasing number of ghosts needed to gauge-fix $\mathbf{G}$. Loosely speaking, increasing the size of $\mathbf{G}$ reduces the amount of "physical" (in the sense of "non-gauge") or "observable" contents encoded in the running couplings. In diagrammatic terms, the ratio of physical excitations relative to gauge excitations gets smaller when $\mathbf{G}$ grows. However, since those features of the RG flow which are due to the gauge modes have no reason to be scheme independent, one can expect that the larger is $\mathbf{G}$ the more scheme dependent is even the exact RG flow.

While, in $d = 4$, metric gravity has 4 gauge parameters per spacetime point related to the diffeomorphisms, this number increases to 4+6 in tetrad gravity since local Lorentz invariance is demanded in addition. If we assume that both theories have the same number of physical degrees of freedom, it is clear that tetrad gravity has a smaller ratio of physical to unphysical field modes, and this might explain to some extent why its RG flow has the more delicate scheme dependence we observed.

To close with, let us come back to the issues raised at the beginning of this chapter which motivated the present analysis.

(A) In ref. [DR12, DR] the RG flow was computed for a 3D truncation of $\mathcal{T}_{\text{EC}} = \{A[e, \omega, \cdots]\}$ which has the same gauge transformations $\mathsf{Diff}(\mathcal{M}) \ltimes \mathsf{O}(d)_{\text{loc}}$ as $\mathcal{T}_{\text{tet}} = \{A[e, \cdots]\}$, but treats the spin connection as an independent field. There, too, the very existence of a NGFP is a robust feature which is obtained for all cutoff and gauge choices, but the quantitative details are more scheme dependent then we are used to from QEG. In this respect the results of [DR12] are very reminiscent of what we found in the present investigation. In [DR12] both the truncated action and the fundamental variables are different from QEG ("Holst" instead of "Einstein-Hilbert", and (e, ω) instead of the metric). In the light of our present results we can say that the hitherto unexplained relatively strong scheme dependence seen in [DR12] could be entirely due to the different variables used and the related larger group of gauge transformations; even though the running actions used were quite different in the two cases (first vs. second order in derivatives, etc.) this is not necessarily the cause for the observed differences.

With respect to our RG study on $\mathcal{T}_{\text{EC}}$ carried out using the WH-like RG equation in the next chapter, we keep in mind that even for a good approximation of the exact equation the details of the flow are expected to differ from those found in [DR12] due to this enhanced scheme dependence. Thus, for judging the applicability of the new approximate FRGE we should only consult the gross features of the flow for a direct comparison.

(B) In the literature [EG11, VZ10, ZZVP10, PP03b, PP03a, DP98] "hybrid" calculations were proposed in order to avoid re-calculating parts of the β-functions for the gravitational couplings in presence of fermionic matter. The idea is to use the tetrad formalism only when it comes to evaluating fermion loops, but to keep the metric as the fundamental variable for the gravity loops. While this can be legitimate in perturbation theory [Woo84], the present investigation revealed that the quantitative details of the flow of Newton's constant and the cosmological constant are significantly different in the metric and the vielbein formalism. Hence, adding the fermionic loops to the "old" metric β-functions does not seem a consistent procedure, even within the limited scope of a truncation. Thus we must conclude that one should refrain from such hybrid calculations when one aims at quantitative results.

After the publication of [HR12] on which this section of the thesis is based, a second RG study of $\mathcal{T}_{\text{tet}}$ appeared [DP], which independently verifies our results in [HR12]. In addition, for the same truncation of $\mathcal{T}_{\text{tet}}$ a type I and a type II cutoff is applied after the fluctuation fields have been decomposed into their spin components. This procedure allows to investigate values of the gauge fixing parameter α_{D} other than $\alpha_{\text{D}} = 1$. It is observed that the pronounced μ-dependence of the critical exponents considerably weakens in the (most physical) limit of $\alpha_{\rm D} \to 0$ and their value is thus close to the metric result for all μ. Moreover, for the type II cutoff the critical exponents are insensitive of $\alpha_{\rm D}$ to a good approximation such that the weak μ-dependence is obtained for all values $0 < \alpha_{\rm D} < 1$. Hence, it is explicitly demonstrated, that among the various RG schemes there are some which lead to results most similar to the metric theory. In these schemes also the correct treatment of the Lorentz ghosts does not seem to play an as crucial role as it does in our computation, since the complete neglection of these ghost contributions, corresponding to the limit $\mu \to \infty$, does not generally lead to a change of the fixed point's character from UV attractive to repulsive any more.

The authors draw the conclusion, that metric and tetrad gravity do lead to comparable results, but due to the enhanced gauge and RG scheme dependence one should refrain from explicit computations on $\mathcal{T}_{\rm tet}$ "as a matter of practical convenience". Instead one should try to analyze the effects of fermions within the metric theory space $\mathcal{T}_{\rm E}$ as far as possible, by hiding the explicit tetrad dependence of their kinetic term using the computational trick of squaring the Dirac operator.

We do not entirely share this conclusion. While the proposed procedure is certainly useful concerning its practicability, the results obtained will lack ultimate legitimation from the conceptional point of view: Any action containing dynamical spinorial fermions will always be an element of a suitable extension of $\mathcal{T}_{\rm tet}$, but it is alien to $\mathcal{T}_{\rm E}$. On the exact level it is therefore clear that any RG study of gravity + fermions naturally has to be carried out in the extended $\mathcal{T}_{\rm tet}$. Here, the question arises whether we can take over results from metric gravity to the tetrad theory, since every element of $\mathcal{T}_{\rm E}$ is in direct correspondence to an invariant of $\mathcal{T}_{\rm tet}$. This is done in the "hybrid computations", and the new results [DP] indicate, that there might exist certain RG schemes and gauges, that justify this as an approximation. The inverse procedure is far more indirect, as we then relate the square of a field monomial in $\mathcal{T}_{\rm tet}$ to an element of $\mathcal{T}_{\rm E}$ in order to take into account fermions effectively. Technically we may achieve a reduced gauge and RG scheme dependence by this step, but we cannot expect the accuracy of our predictions to improve. It might be that we trade an unstable result for a stable one that, however, is subject to an approximation error of at least that size hidden in the result.

(C) In the symmetric vielbein gauge the $\mathsf{O}(d)_{\rm loc}$ ghosts are non-propagating. It was therefore argued, in perturbation theory, that they simply may be ignored in practical calculations [Woo84]. As we saw quite explicitly, the same is not true in the FRGE framework. The Lorentz ghosts do have a considerable impact on the RG flow we found, and moreover the arguments put forward in [Woo84] are easily seen not to carry over to Γ_k at $k > 0$.

Several semi-quantitative calculations, like [SW10] or the one presented in the Chapter 3 ([HR11]), have shown that the Standard Model coupled to asymptotically safe gravity may lead to a theory with enhanced predictivity, that is some of the perturbatively undetermined parameters of the Standard Model (like the mass of the Higgs boson [SW10] or the fine-structure constant [HR11]) can be calculated in the coupled gravity + matter theory. The RG study of tetrad gravity presented in this chapter has identified possible pitfalls in RG calculations of such coupled systems of gravity and fermions and indicated how to avoid them. This paves the way for a fully quantitative treatment of the considerations in refs. [SW10] and [HR11]. Even though this might require more work than thought before, the chance to *compute* the Higgs mass or the fine-structure constant clearly will be worth the effort.

5 QECG in Holst Truncation

In this chapter we analyze the RG flow on the Einstein-Cartan theory space $\mathcal{T}_{\text{EC}}$ using the new Wegner-Houghton type RG equation for the effective action Γ_k that was introduced in Section 2.3.

The main difference to "tetrad only" gravity, that we dealt with in the last chapter, lies in the fact, that in $\mathcal{T}_{\text{EC}}$ the spin connection ${\omega^{ab}}_\mu$ is considered an independent field variable. Thus, the spacetime connection $\Gamma^\lambda_{\mu\nu}$ is no longer restricted to the Levi-Civita choice, but is allowed to carry torsion. A fundamental QFT of gravity whose UV limit is taken at a NGFP in this theory space is called Quantum Einstein Cartan Gravity (QECG). First evidence for the existence of a suitable NGFP has been collected in [Dau].

The truncation of $\mathcal{T}_{\text{EC}}$ which we are going to employ is motivated by the Holst action

$$S_{\text{Ho}}[e,\omega] = -\frac{1}{16\pi\hat{G}}\int \mathrm{d}^d x\, e\Big[{e_a}^\mu {e_b}^\nu\Big({F^{ab}}_{\mu\nu} - \frac{1}{2\gamma}{\varepsilon^{ab}}_{cd}{F^{cd}}_{\mu\nu}\Big) - 2\Lambda\Big]. \tag{5.1}$$

It generalizes the Hilbert-Palatini action of classical Einstein-Cartan gravity by the addition of the Immirzi term (with the Immirzi parameter γ as its coupling), that vanishes on spacetimes without torsion and hence does not have a counterpart in $\mathcal{T}_{\text{tet}}$. Thus, by the presence of this term, our truncation explicitly reflects the fact that $\mathcal{T}_{\text{EC}}$ is a generalization of $\mathcal{T}_{\text{tet}}$. Note, however, that also the other two terms in S_{Ho} differ from the torsionless case $\omega = \omega(e)$ (cf. Appendix F.3).

In addition, the inclusion of the Immirzi term is motivated by two further reasons.

The first more conceptual reason being that the Holst action is the starting point for several other approaches to the quantization of gravity as there are canonical quantum gravity in Ashtekar's variables [Ash91, AL04, Rov04], LQG [Thi07] or spin foam models [Per03]. Thus, the RG analysis of the Holst action may help to find and understand relations between the Asymptotic Safety scenario and these alternative approaches to quantum gravity. In particular, in LQG the coupling associated to the Immirzi term, the dimensionless Immirzi parameter γ, is of interest since it is present in the spectrum of area and volume operators and thus enters also the expression for the entropy of black holes [Rov04]. Usually in LQG γ is considered a parameter of fixed value parametrizing different quantum theories of gravity. In the RG approach, however, γ has the status of an additional coupling and we will find that as such it will generically be subject to a non-trivial RG running. In order to directly compare the two approaches to Quantum Gravity it would be important to relate these two quite different conceptions of the Immirzi parameter in some way.

The second reason is of more phenomenological nature. When we include fermionic matter in the truncation, the fermions will act as a source of torsion in the classical field equations. Here the presence of the Immirzi term gives rise to a CP violating four-fermion interaction, whose coupling constant depends on the value of the Immirzi parameter. Thus in this case the Immirzi parameter leads to an observable effect [PR06], at least in principle, that may have had consequences for the evolution of the early universe [FMT05]. The study of pure gravity, that we present here, can be seen as a first step in this direction as we try to simplify the calculation as far as possible by the application of the new WH-like flow equation. If this procedure turns out reliable it may be generalized to include fermions, rendering this even more complex task technically feasible. By identifying the domain of validity of the new WH-like equation we therefore provide substantial preliminary work for future projects on this field of research.

The study presented in the following is the second fully non-perturbative RG analysis of Einstein-Cartan gravity, while the first RG study on $\mathcal{T}_{\text{EC}}$ has been carried out in [Dau, DR, DR10, DR12]. For the present calculation we deliberately chose a similar setting, using the same truncation with comparable gauge-fixing and ghost terms, so that, except for minor details, the only difference between the two calculations lies in the RG equation used to study the flow. While in [Dau] a well-tested RG equation which makes use of the PT approximation (similar to (2.62)) has been employed, we present here the first application of the new WH-like flow equation. Due to the similar framework we are able to directly compare both RG equations; as neither of them is an exact equation we cannot judge the absolute validity of the respective approximations they make use of, but searching for common features of the two resulting RG flows both calculations may support each other, such that the credibility of these results is strengthened.

This chapter is organized as follows: In a first section (5.1) we introduce the truncation studied and give details about the gauge-fixing conditions used as well as the resulting ghost action. Section 5.2 contains the main part of the calculation, namely the computation of $\Gamma^{(2)}$ and the corresponding traces on the RHS of the flow equation up to second order in the spin connection ω, evaluated for constant background fields. Before we can derive the exact form of the resulting β-functions at the end of Section 5.3 we discuss different possibilities for projecting the RHS onto the invariants under consideration in this section. The last section (5.4) contains a detailed analysis of the resulting RG flow as well as a comparison to the similar study carried out in [Dau, DR].

5.1 Truncating the theory space

In Chapter 2.1 we introduced the notion of theory space and discussed its role as a fundamental ingredient to the Asymptotic Safety program. In this chapter we deal with gravity in a formulation where the tetrad e^a and the spin connection ω^{ab} form the set of independent fundamental fields and functionals thereof span the theory space. This choice clearly is inspired by the classical theory of Einstein-Cartan gravity; however, as we want to stress once more, this does not mean that we try to "quantize" classical Einstein-Cartan gravity: By taking into account the full theory space of *all* gauge invariant functionals of e^a and ω^{ab} the classical action S_Λ (which can be reconstructed from the fixed point effective action $\Gamma_{k\to\infty}$ [MR09, MR11]) corresponding to the quantum theory $\Gamma_{k=0}$ we wish to construct almost certainly does not coincide with the classical Einstein-Cartan action.

Besides the field content, theory space is defined by the group of gauge transformations one wants to impose. Here our choice is lead as well by the classical theory of Einstein-Cartan gravity: In addition to the group of diffeomorphisms, that forms the gauge group of metric gravity, in the tetrad formulation we demand invariance under a second group of gauge transformations, namely the $\mathsf{O}(4)_{\text{loc}}$. As we already explained in the last chapter this is due to the fact, that at each point of spacetime an $\mathsf{O}(4)$-manifold of tetrads $e^a{}_\mu$ gives rise to the same metric $g_{\mu\nu} = e^a{}_\mu \eta_{ab} e^b{}_\nu = e^c{}_\mu \Lambda^a{}_c \eta_{ab} \Lambda^b{}_d e^d{}_\nu$, $\forall \Lambda \in \mathsf{O}(4)$, and this arbitrariness shall be treated as an additional gauge freedom. However, as the diffeomorphisms relate properties of the spacetime at different points, while the $\mathsf{O}(4)$-transformations act ultralocally, only the $\mathsf{O}(4)_{\text{loc}}$ forms a normal subgroup of the total group of gauge transformations $\mathbf{G}$. The structure of $\mathbf{G}$ is therefore given as the semi-direct product $\mathbf{G} = \mathsf{Diff} \ltimes \mathsf{O}(4)_{\text{loc}}$. The theory space inferred from classical Einstein-Cartan gravity is thus defined as

$$\mathcal{T}_{\text{EC}} = \left\{ A[e^a{}_\mu, \omega^{ab}{}_\mu, \cdots]; \mathbf{G} = \mathsf{Diff}(\mathcal{M}) \ltimes \mathsf{O}(4)_{\text{loc}} \right\} , \tag{5.2}$$

where A denotes an arbitrary (action) functional that is invariant under the action of the gauge transformations $\mathbf{G}$, while the dots stand for ghost and background fields that eventually have to be introduced in our approach.

In comparison to the theory space $\mathcal{T}_{\text{tet}}$ that was analyzed in Chapter 4, the only difference lies in the spin connection $\omega^{ab}{}_\mu$ that is an independent field variable here, whereas it could be expressed in terms of the vielbein before.

In the following subsections we discuss the nature of the space $\mathcal{T}_{\text{EC}}$ and introduce the details of the truncation we use to study the RG flow on it.

5.1.1 A Holst type truncation

In order to construct a “basis” of the theory space $\mathcal{T}_{\text{EC}}$ from its field content we have to find a complete set of linearly independent integrated field monomials that can be composed from the basic fields $\{e^a, \omega^{ab}\}$ and are invariant under the action of the total symmetry group $\mathbf{G}$.

This problem can be reformulated in a more geometrical fashion: In our setting we deal with a four dimensional Euclidean manifold $\mathcal{M}$ without boundary ($\partial\mathcal{M} = 0$). Thus we may ask equivalently: How many different 4-forms can be constructed from a generic (not necessarily invertible) tetrad one-form e^a and the spin connection one-form ω^{ab} that form a (pseudo-)scalar w. r. t. the $\mathsf{O}(4)_{\text{loc}}$-transformations?

As basis building blocks we can use the tetrad e^a and its covariant exterior derivative, the torsion two-form $T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b$. Since the spin connection itself transforms inhomogeneously under $\mathsf{O}(4)_{\text{loc}}$-transformations and we want to construct an $\mathsf{O}(4)_{\text{loc}}$-tensor, we can only use its exterior covariant derivative, the curvature two-form or field strength tensor F^{ab}, as a third building block

$$F^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} . \tag{5.3}$$

From these three building blocks we can construct 4-forms using the wedge product and form $\mathsf{O}(4)$-(pseudo)-scalars by contracting the $\mathsf{O}(4)$-indices with the flat Euclidean metric η_{ab} or the fully anti-symmetric ε-tensor ε_{abcd}. Following this procedure we find 6 different field monomials, that, however, are partly related to each other according to

$$\begin{aligned}
&(\text{a}) \quad \varepsilon_{abcd}\, e^a \wedge e^b \wedge e^c \wedge e^d \\
&(\text{b}) \quad \varepsilon_{abcd}\, e^a \wedge e^b \wedge F^{cd} \\
&(\text{c}) \quad e^a \wedge e^b \wedge F_{ab} \\
&(\text{d}) \quad \varepsilon_{abcd}\, F^{ab} \wedge F^{cd} = \varepsilon_{abcd}\, \mathrm{d}\left(\omega^{ab} \wedge \left(\mathrm{d}\omega^{cd} + \tfrac{2}{3}\, \omega^{ce} \wedge \omega_e{}^d\right)\right) \\
&(\text{e}) \quad F^{ab} \wedge F_{ab} = \mathrm{d}\left(\omega^{ab} \wedge \left(\mathrm{d}\omega_{ab} + \tfrac{2}{3}\, \omega_a{}^c \wedge \omega_{cb}\right)\right) \\
&(\text{f}) \quad T^a \wedge T_a = e^a \wedge e^b \wedge F_{ab} + \mathrm{d}(e^a \wedge T_a)\, .
\end{aligned} \tag{5.4}$$

All monomials that are contracted by the ε-tensor correspond to $\mathsf{O}(4)$-scalars, while the others form $\mathsf{O}(4)$-pseudo-scalars. This is due to the fact that any 4-form, being a maximal form on the manifold, contains another ε-density ($\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = \varepsilon^{\mu\nu\rho\sigma}\mathrm{d}^4x$).

The first monomial (a) is proportional to the volume form of the manifold such that its corresponding coupling is the cosmological constant. The second term (b) is a scalar curvature term, with the Newton constant as its related coupling; together the first two terms form the Hilbert-Palatini action of classical gravity in the first order formalism. The third monomial (c) is known as the Immirzi term that is connected to the Immirzi parameter γ as its coupling. It is parity-odd and from relation (f) we find that it vanishes on torsionless manifolds. When added to the Hilbert-Palatini action the terms (a)-(c), including their related couplings, form yet another action that leads to Einstein's equation for any value of the parameter γ; we will refer to this action as the Holst action [Hol96].

We observe that only the first three monomials contain local information about the manifold, while the field strength squared terms (d,e) turn out as being of topological nature, giving rise to the Euler and the Pontryagin invariant, respectively. The last equation (f) reveals a relation between the torsion squared invariant and the Immirzi term. On any manifold of a given topology this relation is fixed as the exterior derivative term in (f) corresponds to a third topological invariant, the Nieh-Yan invariant. If we restrict ourselves to considering local invariants only, the Holst action therefore amounts to the most general ansatz possible, as long as the invertibility of the tetrad is not postulated.

Unfortunately, the formalism we use to compute the RG flow requires us to add a gauge-fixing term S_{gf} and a cutoff action ΔS_k to the truncation ansatz, which, in the background field formalism, both can only be constructed if we assume the invertibility of the background vielbein $\bar{e}^a$. Once introduced, these terms couple the RG flow of the above 3 invariants to the infinitely many additional invariants, that can be constructed using the inverse background vielbein. Thus, the three-dimensional theory space described by an ansatz of the form of the Holst action

$$\begin{aligned}\Gamma_{\text{Ho}\,k}[e,\omega] &= -\frac{1}{16\pi G_k}\left(\frac{1}{2}\int \varepsilon_{abcd}\, e^a \wedge e^b \wedge F^{cd} - \frac{1}{\gamma_k}\int e^a \wedge e^b \wedge F_{ab}\right.\\ &\qquad\left. -\frac{\Lambda_k}{12}\int \varepsilon_{abcd}\, e^a \wedge e^b \wedge e^c \wedge e^d\right) \\ &= -\frac{1}{16\pi G_k}\int \mathrm{d}^4x\, e\left[e_a{}^{\mu} e_b{}^{\nu}\left(F^{ab}{}_{\mu\nu} - \frac{1}{2\gamma_k}\varepsilon^{ab}{}_{cd}F^{cd}{}_{\mu\nu}\right) - 2\Lambda_k\right]\end{aligned} \tag{5.5}$$

necessarily becomes a three-dimensional truncation of an infinite dimensional theory space, when its RG flow is calculated using the exact FRGE (2.34) or related functional RG techniques. The truncation (5.5) forms the starting point of our investigations of the RG flow of QECG.

5.1.2 Logical relation of the RG flows on $\mathcal{T}_{\text{EC}}$ and $\mathcal{T}_{\text{E}}$

Relation between the classical theories. It is often stated that, loosely speaking, the classical theories of metric gravity resulting from the Einstein-Hilbert action S_{EH} and gravity in the first-order formalism based on the Holst action S_{Ho} are equivalent. Let us carefully review this statement to see which implications we may expect for the resulting quantum theories.

For $\gamma \neq \pm 1$ the Holst action S_{Ho} contains 24 independent field components due to the new field $\omega^{ab}{}_{\mu}$ and 16 due to the field $e^a{}_{\mu}$. As the, compared to the metric formulation, additional $\mathsf{O}(4)$ gauge group is 6 dimensional we can employ a gauge fixing condition that requires the antisymmetric part of $e^a{}_{\mu}$ to vanish. Then the remaining 10 field components of its symmetric part correspond to the usual 10 metric field components. This gauge is known as the Deser–van Nieuwenhuizen algebraic vielbein gauge [DN74].

From a variation of the action w. r. t. the spin connection $\omega^{ab}{}_{\mu}$ we obtain 24 equations of motion, that are linear in $\omega^{ab}{}_{\mu}$ as the action is at most quadratic in $\omega^{ab}{}_{\mu}$. If we assume the invertibility of the tetrad $e^{a}{}_{\mu}$, we can solve these linear equations for the 24 components of $\omega^{ab}{}_{\mu}(e)$. One can show [Giu94, Dau] that these solutions of the equations of motion are equivalent to requiring the 24 components of the torsion tensor to vanish ($T^{a}{}_{\mu\nu} = 0$). Hence, $\omega^{ab}{}_{\mu}(e)$ corresponds to the unique metric and torsionless connection, the Levi-Civita connection $\omega^{ab}{}_{\mu}(e) = \omega_{\mathrm{LC}}{}^{ab}{}_{\mu}$.

Substituting this solution into the equations of motion for $e^{a}{}_{\mu}$ and thus assuming the invertibility of the tetrad, we are led to 10 independent equations that are equivalent to Einstein's equation [Giu94].

The case of $\gamma = \pm 1$ is special, as the action S_{Ho} then only depends on one chirality of the spin connection $\omega^{ab}{}_{\mu}$, i. e. on its self-dual or anti-selfdual part. This can be seen as follows:

First, let us define the notion of duality w. r. t. the $\mathsf{O}(4)$-indices. The duality operator is defined as $(\star)^{ab}{}_{cd} = \frac{1}{2}\varepsilon^{ab}{}_{cd}$; antisymmetric objects with eigenvalues $+1$ w. r. t. this operator are called selfdual, those with eigenvalues -1 anti-selfdual. In addition one can define a projector on the (anti-)selfdual part of any antisymmetric second rank $\mathsf{O}(4)$-tensor by $(P^{\pm})^{ab}{}_{cd} = \frac{1}{4}(\delta^{a}_{[c}\delta^{b}_{d]} \pm \varepsilon^{ab}{}_{cd})$ and decompose it into a selfdual and anti-selfdual part, e.g. $F^{ab} = F^{(+)\,ab} + F^{(-)\,ab}$.

Second, we observe that the combination of the scalar curvature term and the Immirzi term in the case of $\gamma = \pm 1$ leaves us with

$$\frac{1}{2}\int \mathrm{d}^4x\, e\Big[e_a{}^{\mu} e_b{}^{\nu}\Big(F^{ab}{}_{\mu\nu} \mp \frac{1}{2}\varepsilon^{ab}{}_{cd}F^{cd}{}_{\mu\nu}\Big)\Big] = \int \mathrm{d}^4x\, e\big[e_a{}^{\mu} e_b{}^{\nu}\left(P^{\mp\,ab}{}_{cd}F^{cd}{}_{\mu\nu}\right)\big]\,. \tag{5.6}$$

In a last step, one can show by direct computation that (anti-)selfdual part of the field strength tensor of a generic spin connection equals the field strength tensor of the (anti-)selfdual part of that spin connection:

$$(P^{\pm}F)^{ab}(\omega) = F^{(\pm)\,ab}(\omega) = F^{ab}(\omega^{(\pm)}) = F^{ab}(P^{\pm}\omega)\,. \tag{5.7}$$

Due to this fact we only have 12 independent components of the spin connection ($\omega^{(+)}$ or $\omega^{(-)}$) the action depends on in the case $\gamma = \mp 1$, leading to 12 equations of motion, that can be solved for $\omega^{(\pm)}(e)$, when the invertibility of the tetrad is assumed. One can show that $\omega^{(\pm)}(e)$ is the (anti-)selfdual projection of the spin connection corresponding to the Levi-Civita connection, i. e. $\omega^{(\pm)}(e)^{ab}{}_{\mu} = (P^{\pm}\omega_{\mathrm{LC}})^{ab}{}_{\mu}$. Note that this spin connection necessarily gives rise to a spacetime that exhibits a non-vanishing torsion. Nonetheless, substituted to the tetrad equations of motions we, again, arrive at Einstein's equation.[1] Thus we find the field equations of general relativity among the classical solutions of chiral gravity, albeit formulated in a spacetime with a connection differing from the usual Levi-Civita connection. Within classical gravity this difference in the connection cannot be observed; it will only become observable if we couple (fermionic) matter to the system.

As we have seen above the Holst action comprises a theory of gravity in (anti-) selfdual variables for the case $\gamma = \pm 1$, that depends on less independent field components. When we try to formulate a quantum theory based on a path integral over this action for a general value of γ, we have to expect divergences in the limit $\gamma \to \pm 1$, as the integration over the other duality component will not be suppressed at all. In order to study the RG-flow of the (anti-)selfdual case we thus have to eliminate the redundant field components before the operator traces on the RHS of the flow equation are evaluated. It will turn out that this elimination is rather simple if we employ an appropriate decomposition of the fluctuation fields before. This way we were able to study the RG flow of gravity in selfdual variables in this thesis for the first time (cf. Section 5.5).

Let us come back to the equations of motion of the Holst action. As each term in the action (5.1) is at least quadratic in the vielbein $e^{a}{}_{\mu}$ we can infer that the action as well as its variation w. r. t. $\omega^{ab}{}_{\mu}$ and $e^{a}{}_{\mu}$ is zero for a vanishing tetrad. Hence, ($e^{a}{}_{\mu} = 0,\ \omega^{ab}{}_{\mu} = \text{arbitrary}$) is an additional solution to the equations of motion of the Holst action, such that the stationary points of S_{EH} and S_{Ho} are not in a one-to-one correspondence. It may be speculated that a path integral of the Holst action receives essential contributions from field configurations close to this additional stationary point, that corresponds to "no spacetime at all", while they do not contribute in the case of metric gravity. Obviously, this would lead to a significant differences between the quantum theories constructed from the two actions, S_{EH} and S_{Ho}.

In a final remark let us stress that even the equivalence of a subset of solutions to the equations of motions of the Holst action with those of the Einstein-Hilbert action breaks down when fermions are present in the truncation. If we consider just

[1]The field equations of the tetrad and the spin connection for the chiral case are discussed in more detail in Appendix F.4.

a standard fermionic kinetic term of the form $\bar{\psi}\gamma^a e_a{}^\mu \nabla_\mu \psi - \overline{\nabla_\mu \psi}\gamma^a e_a{}^\mu \psi$ with the covariant derivative acting on spinors according to $\nabla_\mu \psi = (\partial_\mu + \frac{1}{4}\omega^{ab}{}_\mu \gamma_a \gamma_b)\psi$, we find that it is linear in $\omega^{ab}{}_\mu$ and thus enters its equations of motion as a constant (i. e. $\omega^{ab}{}_\mu$-independent) term. Since these equations demand for a vanishing torsion of the spacetime in absence of fermions, we can infer that fermions act as a source of torsion. Again, it is possible to solve these equations for $\omega(e, \bar{\psi}, \psi)^{ab}{}_\mu = \omega(e)^{ab}{}_\mu + e^{b\nu} K(\bar{\psi}, \psi)^a{}_{\mu\nu}$, that now decomposes into the torsion-free part from before and a contorsion tensor K depending on the fermion fields. Substituting this solution into the action, the torsionless contributions add up to the Einstein-Hilbert equivalent action depending on the tetrad $e^a{}_\mu$ only, while the contorsion terms constitute an effective four fermion interaction term. In contrast to pure gravity, where the Immirzi parameter drops out of the equations of motion, this 4-fermion interaction comes with a coupling constant proportional to $\frac{\gamma^2}{\gamma^2+1}$. Hence, it offers, at least in principle, the possibility to measure the value of the Immirzi parameter γ. However, as the interaction term is suppressed by one power of the Newton constant, it is compatible with all experiments to date, that it has not been observed so far. For further details see e. g. [PR06, FMT05, FS, BSV07].

Relation between QEG and QECG. We have seen in the last paragraph that the similarity of the classical theories resulting from the Holst and the Einstein-Hilbert action, respectively, relies on the equations of motion and therefore on the notion of "on-shell-ness". In stark contrast to this, the calculation of the RG flow on a theory space is a fundamentally off-shell procedure; indeed, the notion of "on-shell-ness" is not even defined until a UV fixed point in the infinite dimensional theory space is found. The fixed point action, being a prediction of the RG analysis, serves (up to a simple explicitly constructible functional[MR09]) as the fundamental action of the theory. Before the form of it is known, we cannot derive the fundamental equations of motion, and even the number of fundamental degrees of freedom of the theory is unknown. Thus, similarities between theories stemming from different classical actions based on their field equations cannot be expected to carry over to quantum theories constructed by the Asymptotic Safety program.

In fact, following the Asymptotic Safety program, the only inspiration we draw from the classical theories is their field content and their symmetries in order to construct a theory space whose fixed point structure is to be computed. Only when their theory spaces coincide and two theories are constructed from the same UV fixed point we can consider them similar: In this case they belong to the same universality class. However, inspired by classical metric gravity and Einstein-Cartan gravity we were led to different theory spaces $\mathcal{T}_{\mathrm{E}}$ and $\mathcal{T}_{\mathrm{EC}}$, that differ in both, field content and symmetry requirements. For $\mathcal{T}_{\mathrm{E}}$ numerous RG studies have been carried out, all of them indicating the existence of a UV fixed point in this space, most probably allowing for the construction of a predictive quantum theory called Quantum Einstein Gravity (QEG). The theory space $\mathcal{T}_{\mathrm{EC}}$ is far less well explored: Up to now, there exists only one fully non-perturbative study of its RG flow [Dau, DR], which is encouraging though, as it found first evidence for the existence of UV fixed points in this space, as well. Thus, it might be possible to construct a quantum theory, called Quantum Einstein-Cartan Gravity (QECG), at these points. It has to be stressed that the results of [Dau, DR] and the ones obtained in this chapter of the thesis are conceptually independent from all QEG results as the fixed point structures of different theory spaces are completely independent from each other.

Of course, this reasoning does not exclude that similarities between the quantum theories QEG and QECG might nevertheless exist. However, it is important to be aware of the fact, that the existence of both theories has to be verified independently of each other and that we cannot expect any similarities from the outset. Only when both theories have been constructed, we can analyze their similarities and differences on the basis of the quantum properties they imply.

5.1.3 Gauge fixing conditions

Let us return to more practical considerations and properly define the ingredients still missing in our truncation. As we are dealing with a gauge theory we need to fix a gauge and add the corresponding ghost action to the truncation ansatz 5.5. From this point on we have to assume the invertibility of the background vielbein $\bar{e}^a{}_\mu$.

In complete analogy to the study [Dau] we will choose a gauge fixing action of the form

$$\Gamma_k^{\mathrm{gf}}[e, \bar{\omega}; \bar{\varepsilon}] = \frac{1}{2}\frac{1}{16\pi G_k}\int \mathrm{d}^4x\, \bar{e}\, \bar{g}^{\mu\nu}\mathcal{F}_\mu\mathcal{F}_\nu + \frac{1}{2}\int \mathrm{d}^4x\, \bar{e}\, \mathcal{G}^{ab}\mathcal{G}_{ab}\,, \tag{5.8}$$

where $\mathcal{F}_\mu$ is the diffeomorphism and $\mathcal{G}_{ab}$ the Lorentz/$\mathsf{O}(4)$-gauge fixing condition.

Explicitly, we have chosen the gauge fixing conditions

$$\mathcal{F}_\mu = \frac{1}{\sqrt{\alpha_\mathrm{D}}} \bar{e}_a{}^\nu \left(\bar{\mathcal{D}}_\nu \bar{\varepsilon}^a{}_\mu + \beta_\mathrm{D} \bar{\mathcal{D}}_\mu \bar{\varepsilon}^a{}_\nu \right) \tag{5.9}$$

and

$$\mathcal{G}^{ab} = \frac{1}{\sqrt{\alpha_\mathrm{L}}} \bar{g}^{\mu\nu} \left(\bar{e}^b{}_\nu \bar{\varepsilon}^a{}_\mu - \bar{e}^a{}_\nu \bar{\varepsilon}^b{}_\mu \right) \; . \tag{5.10}$$

Let us now discuss the form of the gauge fixing action. The prefactors occurring in Γ_k^{gf} containing the running Newton constant G_k but no gauge fixing parameters may seem unusual at first sight: First, note that the usual gauge fixing parameters for the diffeomorphisms, α_D, and the $\mathsf{O}(4)$-transformations, α_L, are shifted to the definition of the gauge fixing condition. In consequence the ghost action derived from these gauge conditions will depend on the gauge parameters and we will see later that only this choice allows us to discuss the limit of Landau gauge, $\alpha_\mathrm{L}, \alpha_\mathrm{D} \to 0$. As it can be argued that this limit is a fixed point of the generically running gauge fixing parameters [EHW96], this gauge is considered a specifically reliable choice in truncations that do not treat α_D and α_L as running couplings.

The prefactor in the diffeomorphism part containing the Newton constant, on the other hand, is useful as it simplifies the combination of contributions to the Hessian from the action and the gauge fixing term, while it renders α_D dimensionless at the same time; for this reason it is widely used in the literature.

For the $\mathsf{O}(4)$-part we start with a gauge fixing action of the standard form. Note that α_L has mass dimension -4 leading to a dimensionless action Γ_k^{gf}. In order to establish an overall prefactor of $(16\pi G_k)^{-1}$ of the gauge fixing action we redefine

$$\alpha_\mathrm{L} = 16\pi G_k \alpha'_\mathrm{L} \tag{5.11}$$

with α'_L having a remaining mass dimension of -2.

In the diffeomorphism gauge condition (5.9) we encounter an additional gauge parameter β_D; it has been shown in [Dau] that this choice of gauge conditions indeed fixes the gauge of the 10 dimensional total gauge group $\mathbf{G} = \mathsf{Diff} \ltimes \mathsf{O}(4)_\mathrm{loc}$ completely for all $\beta_\mathrm{D} \neq -1$. It is hence confirmed that the gauge fixing term breaks the invariance of the action under true gauge transformations, $\delta^\mathrm{G}(\lambda, w)$, such that the resulting propagator is well-defined.

On the other hand, it can be found by inspection that the above gauge fixing action retains background gauge invariance: As the tetrad fluctuation $\bar{\varepsilon}^a{}_\mu$ transforms as a tensor w. r. t. background transformations and all other objects appearing contain only background quantities, the gauge fixing conditions are tensors under background transformations $\delta^{\mathrm{B}}(\lambda, w)$ of the rank their index structure suggests. These are combined to scalars in the gauge fixing action Γ_k^{gf}, such that it turns out invariant under background gauge transformations.

Thus, the above set of gauge conditions satisfies the requirements set by the general background field method as introduced in Chapter 2.2.1.

5.1.4 Construction of the ghost action

As explained in Chapter 2.2.3 the usual Faddeev-Popov construction for the ghost action, that contains the true gauge transformations of the gauge fixing condition, does not lead to a background gauge invariant ghost action. This was shown to be rooted in the non-trivial structure of the gauge group $\mathbf{G} = \mathsf{Diff} \ltimes \mathsf{O}(4)_{\mathrm{loc}}$ of the system: Pure diffeomorphisms do not map $\mathsf{O}(4)$ tensors onto $\mathsf{O}(4)$ tensors. For that reason we have to reparametrize the gauge group $\mathbf{G}$ in such a way that the pure diffeomorphisms are augmented by an $\mathsf{O}(4)_{\mathrm{loc}}$ transformation that $\mathsf{O}(4)$-covariantizes the Lie derivative generating the diffeomorphisms; in this case the covariantization reads

$$\begin{aligned}\widetilde{\widetilde{\delta_{\mathrm{D}}^{\mathrm{G}}}}(w)\varepsilon^a{}_\mu &= \left(\delta_{\mathrm{D}}^{\mathrm{G}}(w) + \delta_{\mathrm{L}}^{\mathrm{G}}(w\cdot\bar{\omega})\right)\varepsilon^a{}_\mu \\ &= \mathscr{L}_w e^a{}_\mu + w^\rho \bar{\omega}^a{}_{b\rho} e^b{}_\mu \\ &= e^a{}_\rho \bar{D}_\mu w^\rho + w^\rho \bar{\mathcal{D}}_\rho e^a{}_\mu + w^\rho \bar{T}^\nu_{\rho\mu} e^a{}_\nu\,.\end{aligned} \tag{5.12}$$

Only in the last form the background covariant tensor character of the expression becomes obvious.

With this covariant reparametrization of the group of gauge transformations $\mathbf{G}$ at hand the ghost action is constructed, as usual, by replacing the transformation parameters of diffeomorphisms and $\mathsf{O}(4)_{\mathrm{loc}}$-transformations with the corresponding diffeomorphism and $\mathsf{O}(4)$-ghost fields, $(w^\mu, \lambda^{ab}) \mapsto (\xi^\mu, \Upsilon^{ab})$. As the ghost sector is treated classically, i. e. all renormalization effects of the ghost couplings are neglected, we can set all tetrad and spin connection fluctuations to zero, even before the Hessian is computed. The resulting ghost action (where we already have replaced the quantum fields by their expectation values) reads,

$$\begin{aligned}
S_{\text{gh}}[\bar{\varepsilon}, \bar{\tau}, \bar{\xi}, \xi, \bar{\Upsilon}, \Upsilon; \bar{e}, \bar{\omega}]\Big|_{\substack{\bar{\varepsilon}=0\\ \bar{\tau}=0}} &= \\
&= -\int \mathrm{d}^4x\, \bar{e}\, \bigg(\bar{\xi}_\mu \bar{g}^{\mu\rho} \frac{\partial \mathcal{F}_\rho}{\partial \varepsilon^a{}_\nu} \widetilde{\widetilde{\delta^{\mathrm{G}}_{\mathrm{D}}}}(\xi) \varepsilon^a{}_\nu + \bar{\xi}_\mu \bar{g}^{\mu\rho} \frac{\partial \mathcal{F}_\rho}{\partial \varepsilon^a{}_\nu} \delta^{\mathrm{G}}_{\mathrm{L}}(\Upsilon) \varepsilon^a{}_\nu \\
&\qquad + \bar{\Upsilon}_{ab} \frac{\partial \mathcal{G}^{ab}}{\partial \varepsilon^c{}_\nu} \widetilde{\widetilde{\delta^{\mathrm{G}}_{\mathrm{D}}}}(\xi) \varepsilon^c{}_\nu + \bar{\Upsilon}_{ab} \frac{\partial \mathcal{G}^{ab}}{\partial \varepsilon^c{}_\nu} \delta^{\mathrm{G}}_{\mathrm{L}}(\Upsilon) \varepsilon^a{}_\nu \bigg) \bigg|_{\substack{\varepsilon=0\\ \tau=0}} \\
&= -\int \mathrm{d}^4x\, \bar{e}\, \big[\bar{\xi}_\mu \bar{g}^{\mu\nu} \big(\bar{D}_\sigma \bar{D}_\nu + \beta_{\mathrm{D}} \bar{D}_\nu \bar{D}_\sigma - \bar{D}_\alpha \bar{T}^\alpha_{\nu\sigma} - \beta_{\mathrm{D}} \bar{D}_\nu \bar{T}^\alpha_{\alpha\sigma} \big) \xi^\sigma \\
&\qquad + \bar{\xi}_\mu (\bar{e}_b{}^\mu \bar{e}_a{}^\rho \bar{\nabla}_\rho) \Upsilon^{ab} \\
&\qquad + \bar{\Upsilon}_{ab}\, 2\, \bar{g}^{\mu\nu} \bar{e}^b{}_\nu \bar{e}^a{}_\alpha (\delta^\alpha{}_\sigma \bar{D}_\mu - \bar{T}^\alpha_{\mu\sigma}) \xi^\sigma \\
&\qquad + \bar{\Upsilon}_{ab} (2\, \delta^a{}_c \delta^b{}_d) \Upsilon^{cd} \big] \,,
\end{aligned} \tag{5.13}$$

where the different covariant derivatives act on all objects to their right.

5.2 Application of the WH-like flow equation to the Holst truncation

This section describes how the WH-like flow equation (2.69) is applied to the Holst truncation introduced in the previous section. We will thereby focus on the general method applied to evaluate the traces on the RHS as well as conceptual difficulties that arise in comparison to metric gravity. The actual calculation is, due to the quadruplication of the number of independent field components compared to the metric case, an extremely tedious task that was, to a large part, carried out using the tensor manipulation package MathTensor for Mathematica [PC94].

The calculation described in this section consists of the following major steps. Starting point is the WH-like equation (cf. Chapter 2.3) of the form

$$\partial_t \Gamma_k[\bar{e}, \bar{\omega}] = \frac{1}{2} D_t\, \mathrm{STr} \bigg|_k \ln \Gamma_k^{(2)} \,. \tag{5.14}$$

In a first subsection we discuss how this rather formal equation has to be interpreted concretely and how its RHS will be expanded in the constant background fields $\bar{e}$ and

$\bar{\omega}$. In a second step we compute the main basic ingredient to the equation, $\Gamma^{(2)}$, from our general truncation ansatz

$$\Gamma_k = \Gamma_k^{\mathrm{Ho}} + \Gamma_k^{\mathrm{gf}} + S_{\mathrm{gh}} \equiv \breve{\Gamma}_k + S_{\mathrm{gh}} \,, \tag{5.15}$$

whose constituents were introduced in the last section. In order to calculate the trace we will introduce a decomposition of the fluctuation fields $(\bar{\varepsilon}^a{}_\mu, \bar{\tau}^{ab}{}_\mu)$ that retains the value of the trace, thus giving $\Gamma_k^{(2)}$ in a different basis, in the third subsection. In a last subsection we discuss the structure of the result obtained, when the RHS is evaluated up to second order in the background field $\bar{\omega}$, but before it is projected onto the invariants of the Holst truncation.

5.2.1 General procedure to evaluate the flow equation

In this section we give a concrete meaning to equation (5.14) by explaining how its different constituents will be evaluated.

Definition of the Hessian. The Hessian $\Gamma_k^{(2)}$ is defined as the second variation of the effective action Γ_k w. r. t. the (generalized) fluctuation fields $\bar{\varphi}_i$ evaluated on the background field configuration $\bar{\Phi}_i = \Phi_i\big|_{\bar{\varphi}_j=0} = \bar{\Phi}_i + \bar{\varphi}_i|_{\bar{\varphi}_j=0}$:

$$\left[\Gamma_{ij}^{(2)}(x,y)\right]^{a_i b_j} = \frac{1}{\bar{e}}\frac{\delta}{\delta\bar{\varphi}_j^{b_j}(y)}\frac{1}{\bar{e}}\frac{\delta}{\delta\bar{\varphi}_i^{a_i}(x)}\Gamma\bigg|_{\varphi_k=0} = \langle x,i,a_i|\Gamma^{(2)}|y,j,b_j\rangle \tag{5.16}$$

Here, the indices $i, j, \cdots$ label different field variables and the indices $a_i, b_j, \cdots$ denote their respective index structure. Thus $\bar{\varphi}$ is the set of all fluctuation fields considered; as long as the fluctuations are not further decomposed we simply have

$$\bar{\varphi} = \{\bar{\varepsilon}, \bar{\tau}\}, \quad \bar{\Phi} = \{\bar{e}, \bar{\omega}\} \quad \text{and} \quad \Phi = \bar{\Phi} + \bar{\varphi} = \{e, \omega\} \tag{5.17}$$

in the Grassmann even sector. If the value of i is fixed, $\bar{\varphi}_i$ denotes a specific fluctuation field. Similarly $\bar{\varphi}_i^{a_i}$ denotes a specific component of this field.

Note that we have suppressed the dependence of Γ on the "RG-time" t in the above formula in order to avoid unnecessary notational complexity.

The second equality introduces a notation, that resembles the bra-ket notation of quantum mechanics and should remind us of the fact, that the Hessian is a Hermitian operator, that can be defined by matrix elements in a certain basis. Here it is given in a basis labeled by the field types i, j, their components a_i, b_j and the position variables x, y. Later on we will see how a basis transformation to momentum space and decomposed field variables leads to the most simple form of this Hessian operator.

For the further evaluation of (5.14) it is crucial that the Hessian is computed for an x-independent background field configuration $\{\bar{e}^a{}_\mu, \bar{\omega}^{ab}{}_\mu\} = \text{const}$. Only in this case the position dependence of the operator can be separated off, leading to the following form of the matrix elements in field space (with the component indices a_i, b_j suppressed):

$$\Gamma^{(2)}_{ij}(x,y) = \Gamma^{(2)}_{ij}(\partial_{x^\mu}) \frac{\delta^{(4)}(x-y)}{\bar{e}} \,. \tag{5.18}$$

As we will see below, this structure corresponds to the fact, that the operator is diagonal in a generalized momentum space. Since this special form is preserved if the operator is applied multiple times, functions of this operator decompose according to

$$\Big[f\big(\Gamma^{(2)}\big)\Big]_{ij}(x,y) = \Big[f\big(\Gamma^{(2)}(\partial_{x^\mu})\big)\Big]_{ij} \frac{\delta^{(4)}(x-y)}{\bar{e}} \,, \tag{5.19}$$

where the function f is understood as a power series in its (matrix-valued) argument.

A generalized basis in momentum space. For the evaluation of the trace it is convenient to choose a basis that is adapted to the form of the Hessian operator and that at the same time allows for a simple IR cut-off procedure for modes of momentum smaller than the cutoff scale k.

In a covariant setting the notion of momentum is typically defined by a generalized covariant Laplacian operator $-\bar{D}^2$: Its eigenfunctions with eigenvalue p^2 correspond to fluctuations carrying momentum p^2. The (IR regulated) trace is then obtained by summing over its eigenfunctions with $p^2 \geq k^2$.

Using the WH-like flow equation we give up background gauge covariance in order to obtain a structurally simpler form of the RHS, where all integrations can be carried out explicitly, leading to β-functions that are (except for logarithmic contributions to the running of Λ_k) rational functions in all couplings. To this end we take the trace by integrating over the complete set of eigenfunctions of the flat Laplacian $-\Box = -\bar{g}^{\mu\nu}\partial_\mu\partial_\nu$ and using an IR cutoff w. r. t. its eigenvalues $p^2 \geq k^2$. The precise form of its eigenfunctions depends, of course, on the background spacetime chosen: As already mentioned, it is crucial to our approach that we choose constant background fields $\{\bar{e},\bar{\omega}\}$, resulting in a flat manifold with non-vanishing torsion. As we want to consider the case of a boundaryless manifold, we choose the flat torus T^4 as background manifold, or equivalently a 4-cube in $\mathbb{R}^4$ of volume L^4 with periodic boundary conditions.

One could naively think that on a flat torus the covariant derivative reduces to the partial derivative, but here this is not the case as torsion is induced by the presence of the constant spin connection $\bar{\omega}^{ab}{}_{\mu}$. Explicitly, we can split the spacetime connection into its Levi-Civita part and the contorsion $K^{\lambda}{}_{\mu\nu}$ according to

$$\bar{\Gamma}^{\lambda}_{\mu\nu} = \underbrace{\big(\Gamma_{\mathrm{LC}}(\bar{e})\big)^{\lambda}_{\mu\nu}}_{=0} + \bar{K}^{\lambda}{}_{\mu\nu} \tag{5.20}$$

such that

$$\bar{K}^{\lambda}{}_{\mu\nu} = \bar{\Gamma}^{\lambda}_{\mu\nu} = \bar{e}_a{}^{\lambda}\bar{\omega}^a{}_{b\mu}\bar{e}^b{}_{\nu}\,. \tag{5.21}$$

The torsion on the torus is hence given by the antisymmetric part of the contorsion, $\bar{T}^{\lambda}{}_{\mu\nu} = \bar{K}^{\lambda}{}_{[\mu\nu]}$, but also the corresponding symmetric part of the contorsion is generically non-vanishing.

Now we are in the position to express the action of the covariant Laplacian $-\bar{D}^2$ in terms of the d'Alembertian $-\Box$ and contorsion terms. Acting on a scalar we obtain

$$-\bar{D}^2\phi = -\partial^2\phi + \bar{g}^{\mu\nu}\bar{K}^{\tau}{}_{\mu\nu}\partial_\tau\phi\,, \tag{5.22}$$

while for a vector field we find

$$-\bar{D}^2 v^\rho = -\Box v^\rho - 2\bar{g}^{\mu\nu}\bar{K}^{\rho}{}_{\mu\tau}\partial_\nu v^\tau + \bar{g}^{\mu\nu}\bar{K}^{\tau}{}_{\mu\nu}\partial_\tau v^\rho + \bar{g}^{\mu\nu}\bar{K}^{\alpha}{}_{\mu\nu}\bar{K}^{\rho}{}_{\alpha\tau}v^\tau - \bar{g}^{\mu\nu}\bar{K}^{\rho}{}_{\mu\alpha}\bar{K}^{\alpha}{}_{\nu\tau}v^\tau. \tag{5.23}$$

Thus, we see explicitly how the two cutoff operators $-\bar{D}^2$ and $-\Box$ differ by contorsion terms. We observe that the additional terms are at most first order in derivatives, while in the highest (second) order of derivatives both choices coincide. Hence, we might expect that both cutoff operators will lead to comparable results in the high momentum regime. In this respect the change of cutoff operator, $\bar{D}^2 \to \Box$, is similar to switching from a type I to a type II or III cutoff operator [CPR09]. While the latter still amount to manifestly covariant choices, here we have to face the additional approximation that lies in the abandoning of background gauge invariance.

On the flat torus T^4 the eigenfunctions of $-\Box$, that form a complete set of basis functions on that manifold, are well-known to be the periodic ("plane wave") functions

$$\psi_n(x) = \langle x|n\rangle = \frac{1}{L^2}\exp\left[i\frac{2\pi}{L}n_a \bar{e}^a{}_\mu x^\mu\right], \tag{5.24}$$

where n_a is a four-component vector of (dimensionless) integers. All eigenvalues of the $-\Box$-operator are thus of the form $p^2 = (2\pi/L)^2 n_a n^a$.

If we now trace an operator $\hat{O}$ of the form

$$\langle x|\hat{O}|y\rangle = O(\partial_{x^\mu})\frac{\delta^{(4)}(x-y)}{\bar{e}} \tag{5.25}$$

applying a UV cutoff at $N(L) = \frac{L}{2\pi}P$ to the components n_a we find

$$\begin{aligned}
\mathrm{Tr}\,\hat{O}\Big|^P &= \sum_{n_1,\cdots,n_4=-N}^{N}\langle n|\hat{O}|n\rangle = \sum_{n_1,\cdots,n_4=-N}^{N}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{e}^2\langle n|x\rangle\langle x|\hat{O}|y\rangle\langle y|n\rangle \\
&= \frac{1}{(2\pi)^4}\int \mathrm{d}^4x\,\bar{e}\left[\left(\frac{2\pi}{L}\right)^4\sum_{n_1,\cdots,n_4=-N}^{N} O\left(i\frac{2\pi}{L}n_a\bar{e}^a{}_\mu\right)\right] \\
&= \int \mathrm{d}^4x\,\bar{e}\frac{1}{(2\pi)^4}\left[\int_{-P}^{P}\mathrm{d}^4p\,O(ip_\mu) + \mathcal{O}\Big(\frac{1}{L}\Big)\right].
\end{aligned} \tag{5.26}$$

In the last step we have approximated the sum over equidistant sampling points by the corresponding integral over p_μ. This approximation gets exact in the limit of infinitely many sampling points, i. e. $L \to \infty$. Alternatively, this approximation is obtained by employing the Euler-MacLaurin formula, using its first term only.

In our calculation we can use the prefactor of the momentum integral in order to identify the running of the couplings $(\Lambda_k, \gamma_k, G_k)$ under consideration: Since the corresponding invariants evaluated at constant background fields are proportional to the spacetime volume $\int \mathrm{d}^4x$ in front and therefore scale with L^4, we know that, even in the case of a finite spacetime volume L^4, the correction terms $\mathcal{O}(1/L)$, that scale at most like L^3, do not contribute to the running of the couplings. We thus conclude that the RG flow of the three couplings $(\Lambda_k, \gamma_k, G_k)$ on a finite torus is equivalent to the limit of an infinite torus, although the value of the invariants diverges in this (formal) limit. In the following we therefore choose to work in this infinite volume limit. In this limit the basis functions of $-\Box$ are given by

$$\psi_p(x) = \langle x|p\rangle = \frac{1}{\sqrt{2\pi}^4} \exp\left[ip_\mu x^\mu\right], \tag{5.27}$$

with $p_\mu \in \mathbb{R}^4$, and the functional trace can then be evaluated using formula

$$\mathrm{Tr}\,\hat{O} = \int \mathrm{d}^4x\,\bar{e} \int \frac{\mathrm{d}^4p}{(2\pi)^4} \mathrm{tr}\,O(ip_\mu) \tag{5.28}$$

for any operator of the form (5.25). Here, tr denotes the remaining trace over the algebraic part of the operator $\hat{O}$.

Irreducible basis in field space. For the computation of the algebraic part of the trace we have to be careful not to overcount the number of independent field components of $\varphi_i^{a_i}$, that may be less than the range of the index a_i due to imposed symmetry or transversality conditions. For this reason we introduce a new basis for the algebraic part of the operator, given by I_i basis fields Φ^{I_i} for each type of field i. The transformation matrices $v_i{}^{I_i}{}_{a_i}(p)$, that depend on the direction of the momentum variable p^μ for transversal fields $\varphi_i^{a_i}$, connect the two bases. The new index I_i runs from 1 up to the number of independent field components of the field φ_i. We will refer to this new basis as "irreducible".

As an example consider the symmetric metric fluctuation field $h_{\mu\nu}$. The above reparametrization corresponds to the introduction of a new field H_I with $I = 1, \cdots, 10$, that corresponds to the 10 independent components of the metric fluctuation. The transformation between the field types is then given by the relation

$$h_{\mu\nu} = \sum_{I=1}^{10} (v_h)^I{}_{\mu\nu} H_I. \tag{5.29}$$

The irreducible basis fields are chosen orthonormal such that the transformation matrices satisfy

$$\sum_{a_i} v_i{}^{I_i}{}_{a_i} v_{iJ_i}{}^{a_i} = \delta^{I_i}{}_{J_i} \quad \text{or with suppressed indices} \quad v_i\, v_i^T = \mathbb{1}. \tag{5.30}$$

Moreover, we can construct the operators

$$(P_i)^{a_i}{}_{b_i} = \sum_{I_i} v_{iI_i}{}^{a_i} v_i{}^{I_i}{}_{b_i} \quad \text{or} \quad \widetilde{P}_i = v_i^T\, v_i\,, \tag{5.31}$$

that are defined on the space of tensors with index structure a_i and are projectors onto the subspace spanned by the basis fields Φ^{I_i}, i. e. project a tensor onto its part exhibiting the symmetry and transversality properties of the field φ_i.

Of course, we can transform any matrix that shows the symmetry and transversality properties of the fields φ_i from one basis to the other. When the indices are suppressed we will denote the matrix expressed in the "reducible" basis by a tilde on top. This also explains the notation in (5.31). For a general matrix M we thus have:

$$\begin{aligned}
(M_{ij})^{I_i}{}_{J_j} &= \sum_{a_i,b_j} (v_i)^{I_i}{}_{a_i} (M_{ij})^{a_i}_{b_j} (v_j)^{b_j}_{J_j} \quad \Leftrightarrow \quad M_{ij} = v_i \widetilde{M}_{ij} v_j^T\,,\\
(M_{ij})^{a_i}_{b_j} &= \sum_{I_i,J_j} (v_i)_{I_i}{}^{a_i} (M_{ij})^{I_i}{}_{J_j} (v_j)^{J_j}_{b_j} \quad \Leftrightarrow \quad \widetilde{M}_{ij} = v_i^T M_{ij} v_j\,.
\end{aligned} \tag{5.32}$$

The trace of the algebraic part of an operator, tr , is independent of the basis as

$$\begin{aligned}
\operatorname{tr} M = \sum_i (M_{ii})^{I_i}{}_{I_i} &= \sum_i \left(v_i \widetilde{M}_{ii} v_i^T\right)^{I_i}{}_{I_i} = \sum_i \left(v_i^T v_i \widetilde{M}_{ii}\right)^{a_i}{}_{a_i}\\
&= \sum_i \left(\widetilde{P}_i \widetilde{M}_{ii}\right)^{a_i}{}_{a_i} = \sum_i \left(\widetilde{M}_{ii}\right)^{a_i}{}_{a_i} = \operatorname{tr} \widetilde{M}\,,
\end{aligned} \tag{5.33}$$

where the symmetry properties of $\widetilde{M}$ amount to the fact that $\widetilde{P}_i \widetilde{M}_{ii} = \widetilde{M}_{ii} = \widetilde{M}_{ii} \widetilde{P}_i$.

The new basis of the total field space is thus given by the set of basis elements that are characterized by the 4-momentum p and the index I_i and will be abstractly denoted by $|p, I_i\rangle$. Their representation in the previous "position space" basis is given by

$$\langle x, i, a_i | p, j, I_j\rangle = \delta_{ij}\, v_i{}^{I_j}{}_{a_i}(p) \frac{e^{ip_\mu x^\mu}}{\sqrt{2\pi}^{\,4}}\,. \tag{5.34}$$

Evaluation of the trace. Now we are able to evaluate the trace in the generalized momentum space basis $|p, i, I_i\rangle$ introduced above, by expressing the matrix elements of the Hessian in terms of the position space basis $|x, i, a_i\rangle$. For notational simplicity we discuss the Grassmann-even part of the trace in detail and include the ghost sector, that should be treated in complete analogy, only at the very end:

$$\begin{aligned}
\mathrm{Tr}\left[\ln\breve{\Gamma}^{(2)}\right] &= \sum_{i,I_i}\int \mathrm{d}^4p\,\langle p,i,I_i|\ln\breve{\Gamma}^{(2)}|p,i,I_i\rangle \\
&= \sum_{\substack{i,j,k,\\ I_i,a_j,b_k}}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{e}^2\mathrm{d}^4p\,\langle p,i,I_i|x,j,a_j\rangle\langle x,j,a_j|\ln\breve{\Gamma}^{(2)}|y,k,b_k\rangle \\
&\qquad\cdot\langle y,k,b_k|p,i,I_i\rangle \\
&= \sum_{\substack{i,j,k,\\ I_i,a_j,b_k}}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4p\,\bar{e}^2\;\delta_{ij}v_i{}^{I_i}{}_{a_j}\frac{e^{-ip_\mu x^\mu}}{\sqrt{2\pi}^4}\left[\ln\breve{\Gamma}^{(2)}(\partial_{x^\mu})\right]^{a_j}_{jk\,b_k}\frac{\delta^{(4)}(x-y)}{\bar{e}} \\
&\qquad\cdot\delta_{ki}v_{iI_i}{}^{b_k}\frac{e^{ip_\mu y^\mu}}{\sqrt{2\pi}^4} \\
&= \int\mathrm{d}^4x\,\bar{e}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\sum_{i,I_i,a_i,b_i}v_{iI_i}{}^{b_i}v_i{}^{I_i}{}_{a_i}\left[\ln\breve{\Gamma}^{(2)}(ip_\mu)\right]^{a_i}_{ii\,b_i} \\
&= \int\mathrm{d}^4x\,\bar{e}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\sum_{i,a_i,b_i}(P_i)^{b_i}{}_{a_i}\left[\ln\breve{\Gamma}^{(2)}(ip_\mu)\right]^{a_i}_{ii\,b_i}\,.
\end{aligned} \tag{5.35}$$

In the next step we rewrite the momentum integration in spherical coordinates splitting it into a radial and an angular part, where the former is cut off at the IR momentum scale k:

$$\mathrm{STr}\Big|_k\left[\ln\breve{\Gamma}^{(2)}\right] = \frac{1}{(2\pi)^4}\int\mathrm{d}^4x\,\bar{e}\int_k^\infty\mathrm{d}p\,(p^2)^{3/2}\int\mathrm{d}\Omega_p\sum_{i,a_i,b_i}(P_i)^{b_i}{}_{a_i}\left[\ln\breve{\Gamma}^{(2)}(ip_\mu)\right]^{a_i}_{ii\,b_i}. \tag{5.36}$$

The angular part of the momentum integration can be carried out using the rules of symmetric integration, such that odd powers of p_μ in the integrand vanish and even ones are replaced by certain combinations of the metric (see e. g. Appendix B in [Ynd06]), e. g.:

$$\int\mathrm{d}\Omega_p\,p_\mu p_\nu f(p^2) = 2\pi^2\,\frac{\bar{g}_{\mu\nu}}{4}p^2 f(p^2) \tag{5.37}$$

or

$$\int\mathrm{d}\Omega_p\,p_\mu p_\nu p_\rho p_\sigma f(p^2) = 2\pi^2\,\frac{\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma}+\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma}+\bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}}{4\cdot(4+2)}p^4 f(p^2)\,. \tag{5.38}$$

Performing this integration is obviously a simple algebraic task leaving us with an integrand of the radial integration depending only on p^2.

In a last step we include the scale derivative D_t in our discussion, denoting a scale derivative that only acts onto the IR cutoff of the trace and does not act onto the k-dependent couplings inside the Hessian $\breve{\Gamma}_k^{(2)}$. (At this point we restore the as yet suppressed k dependence of $\breve{\Gamma}_k^{(2)}$.) It thus removes the radial momentum integration leaving us with minus the integrand evaluated at the lower boundary $p = k$.

If we finally include the ghost sector with the usual minus sign the full WH-like flow equation (5.14) can hence be written in the form

$$\begin{aligned}\partial_t \Gamma_k[\bar{e}, \bar{\omega}] = -\frac{k^4}{(2\pi)^4} \int \mathrm{d}^4 x\, \bar{e} \Bigg[\int \mathrm{d}\Omega_p \sum_{i, a_i, b_i} \frac{1}{2} (P_i)^{b_i}{}_{a_i} \big[\ln \breve{\Gamma}_k^{(2)}(ip_\mu) \big]^{a_i}_{ii\, b_i} \\ - \int \mathrm{d}\Omega_p \sum_{\substack{i \in \\ \text{ghosts}}, a_i, b_i} (P_i)^{b_i}{}_{a_i} \big[\ln S^{(2)}_{\mathrm{gh}\, k}(ip_\mu) \big]^{a_i}_{ii\, b_i} \Bigg]_{p=k} .\end{aligned} \tag{5.39}$$

At this point only the logarithm of the Hessian is left awaiting evaluation. First of all, we should mention that the logarithm $\ln M$ is only defined for a matrix M given in the "irreducible" basis, i. e. with capital indices I_i, J_j. This is due to the fact that the determinant of any matrix fulfilling e. g. transversality conditions is zero, $\det \widetilde{M} = 0$, as its rows are not independent. Hence it is not invertible in the larger matrix space with index structure a_i, b_j and thus its logarithm $\ln \widetilde{M}$ is not defined.

As we want to project the RHS onto invariants that are at most quadratic in the (constant) background field $\bar{\omega}^{ab}{}_\mu$, we will expand the logarithm up to second order in this field. To this end we split the Hessian into a part independent of $\bar{\omega}$ and a part containing all $\bar{\omega}$-dependence,

$$\breve{\Gamma}^{(2)} = H_0 + V(\bar{\omega}) \,, \tag{5.40}$$

with V being a matrix whose elements are at least linear in $\bar{\omega}$. Thus, the logarithm can be expanded in V according to

$$\begin{aligned}\ln \breve{\Gamma}^{(2)} &= \ln\big[H_0 + V(\bar{\omega})\big] = \ln\big[H_0\big(\mathbb{1} + H_0^{-1} V(\bar{\omega})\big)\big] = \ln\big[H_0\big] + \ln\big[\mathbb{1} + H_0^{-1} V(\bar{\omega})\big] \\ &= \ln\big[H_0\big] + H_0^{-1} V(\bar{\omega}) - \frac{1}{2}\big(H_0^{-1} V(\bar{\omega})\big)^2 + \mathcal{O}(\bar{\omega}^3) \,.\end{aligned} \tag{5.41}$$

While the parts depending on $\bar{\omega}$ can be computed by simple matrix multiplication, the logarithm of H_0 remains. Here we will employ the well-known matrix identity tr ln M = ln det M in order to simplify the evaluation of this term. We thus find in the two bases

$$\begin{aligned}\mathrm{tr}\,\ln\breve{\Gamma}^{(2)} &= \sum_{i,a_i,b_i} (P_i)^{b_i}{}_{a_i}\big[\ln\breve{\Gamma}^{(2)}(ip_\mu)\big]_{ii}{}^{a_i}{}_{b_i} \\ &= \ln\det H_0 + \mathrm{tr}\,H_0^{-1}V - \frac{1}{2}\,\mathrm{tr}\,(H_0^{-1}V)^2 + \mathcal{O}(\bar{\omega}^3) \\ &= \ln\det v\widetilde{H_0}v^T + \mathrm{tr}\,\widetilde{P}\widetilde{H_0^{-1}}\widetilde{V} - \frac{1}{2}\,\mathrm{tr}\,\widetilde{P}(\widetilde{H_0^{-1}}\widetilde{V})^2 + \mathcal{O}(\bar{\omega}^3)\,.\end{aligned} \tag{5.42}$$

To be able to compute the inverse of H_0 we will have to perform a complete transverse-traceless decomposition of the fluctuation fields $\bar{\varepsilon}, \bar{\tau}$, that will be introduced in Section 5.2.3. In this field basis H_0 will have a very simple structure: It is block diagonal in the field space (indices i, j) and diagonal in the space of field components a_i, b_j, i. e. we can write

$$\begin{aligned}(H_0)_{ij}{}^{a_i}{}_{b_j} &= (H_0)_{ij} \otimes P_i{}^{a_i}{}_{b_j} \quad &&\text{for} \quad a_i \sim b_j\,, \\ (H_0)_{ij}{}^{a_i}{}_{b_j} &= 0 &&\text{for} \quad a_i \nsim b_j\,.\end{aligned} \tag{5.43}$$

The block-diagonality is here reflected in the fact, that the elements of H_0 are only non-zero if the corresponding fields φ_i and φ_j have an index structure of the same type ($a_i \sim b_j$), e. g. both are scalars, divergence-free vectors etc. Hence, in this basis H_0 can be inverted easily by inverting the first factor of the tensor product

$$(H_0^{-1})_{ij}{}^{a_i}{}_{b_j} = (H_0^{-1})_{ij} \otimes P_i{}^{a_i}{}_{b_j} \tag{5.44}$$

and the above determinant simplifies to

$$\det v\widetilde{H_0}v^T = \det\big[(H_0)_{ij} \otimes v^{I_i}{}_{a_i} P_i{}^{a_i}{}_{b_j} v_{J_j}{}^{b_j}\big] = \det\big[(H_0)_{ij} \otimes \mathbb{1}^{I_i}{}_{J_j}\big]\,. \tag{5.45}$$

In a last step we can apply the formula for the determinant of a tensor product of matrices

$$\det\big[M \otimes N\big] = (\det M)^{\mathrm{rank}\,N} \cdot (\det N)^{\mathrm{rank}\,M} \tag{5.46}$$

and arrive at

$$\det v\widetilde{H_0}v^T = \prod \big(\det(H_0)_{ij}\big)^{\delta^{I_i}_{J_j}}\,, \tag{5.47}$$

where the symbolic product sign denotes the product over the different blocks of the matrix $(H_0)_{ij}$ in field space. In each block the fields have the same index structure, and $\delta^{I_i}{}_{J_j}$ therefore results in the number of independent field components, the fields in each block have.

Concluding remarks. Taken together eqns. (5.39), (5.42) and (5.47) give a concrete meaning to the RHS of the WH-like flow equation (5.14) and show up the way we are going to evaluate it. Note that, using the WH-like flow equation, we have reduced this task to purely algebraic manipulations of matrices in field space, which amounts to a great simplification compared to the use of the FRGE or a proper-time flow equation. Moreover, we note that using the last line of (5.42) together with (5.47), we never have to construct the basis of independent field components, and in particular the transformation matrices v_i, explicitly as we were able to reformulate all algebraic matrix contributions in terms of the "reducible" basis using only the projectors to the irreducible subspaces.

However, this procedure has some limitations. First, as our technique requires the use of constant background fields, we can only project out invariants that do not vanish in this case and thus have at least some part that stays non-zero for algebraic reasons (in our case the $\bar{\omega}^2$-term of the field strength $\bar{F}$). Nonetheless, even if this is the case it might not be possible to uniquely map the remaining constant parts onto the invariants one started with. Projecting out the flow of couplings that correspond to certain invariants from the RHS then becomes ambiguous; a problem which we will address in more detail in Section 5.3.

Second, our approach does not retain background gauge invariance. Thus, we do not quantize the system w. r. t. all possible background spacetimes, but the flat background plays a distinct role. Note that the loss of background gauge invariance is not due to the very use of a constant background field to evaluate the RHS; when employing the FRGE, for example, it is also possible to choose a specific background spacetime to project out the invariants one is interested in. The important difference here is, that we use a cutoff operator $-\Box$ to evaluate and cut off the trace, that does not correspond to the covariant momentum operator evaluated on that background, i. e. $-\bar{D}^2\big|_{\bar{e},\bar{\omega}=\text{const}} \neq -\Box$.

However, as pointed out before, the loss of background gauge invariance is not too much a drawback. At the level of the effective action $\Gamma = \Gamma_{k=0}$, where all fluctuations have been integrated out and from which all observables are to be calculated, it does not matter which background was used to classify the fluctuation fields according to their momentum. Hence, the exact untruncated flow in the backround invariant approach leads to the same effective action as the one using a distinct background spacetime. The difference is the theory space the trajectory lies in: In the background invariant approach the fields $\{\bar{e}, \bar{\omega}\}$ arrange themselves to gauge invariant field monomials, that amount to a subspace of the theory space spanned by all possible field monomials of $\{\bar{e}, \bar{\omega}\}$, where the trajectory in our case lives in. As we should expect that the existence of a UV fixed point does not depend on the details of the cutoff procedure chosen, both approaches should in principle be suitable to investigate the UV behavior of the theory. Seen in this light, the loss of background invariance is equivalent to an additional approximation, as we truncate a larger theory space to the same number of invariants in the truncation ansatz.

5.2.2 The Hessian $\breve{\Gamma}_k^{(2)}$ in $(\bar{\varepsilon}, \bar{\tau})$-basis

In this section we give the result of the second variation $\breve{\Gamma}_k^{\text{quad}}$ of our truncation ansatz Γ_k with respect to the fields $e^a{}_\mu$ and $\omega^{ab}{}_\mu$, where $\delta e^a{}_\mu = \bar{\varepsilon}^a{}_\mu$ and $\delta\omega^{ab}{}_\mu = \bar{\tau}^{ab}_\mu$. Suppressing all indices, it is connected to the Hessian $\breve{\Gamma}_k^{(2)}$ according to

$$\breve{\Gamma}_k^{\text{quad}} = \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, \bar{e}^2 \, \Big(\bar{\varepsilon}(x), \bar{\tau}(x)\Big)\, \breve{\Gamma}_k^{(2)}(x,y) \begin{pmatrix} \bar{\varepsilon}(y) \\ \bar{\tau}(y) \end{pmatrix} . \tag{5.48}$$

Thus, the Hessian can be obtained from $\breve{\Gamma}_k^{\text{quad}}$ by first symmetrizing the quadratic form w. r. t. both fluctuation fields and their index symmetries, and then cancelling the $\frac{1}{2}$ factor, the integrations and the leftmost fluctuation field, while replacing the rightmost fluctuation field by $\delta^{(4)}(x-y)/\bar{e}$.

We will present $\breve{\Gamma}_k^{\text{quad}} = \Gamma_{\text{Ho}\,k}^{\text{quad}} + \Gamma_{\text{gf}\,k}^{\text{quad}}$ as a sum of the contributions stemming from the Holst action and those from the gauge fixing terms. Note that the latter only contributes to the $(\bar{\varepsilon}, \bar{\varepsilon})$-block of the quadratic form and that $\Gamma_{\text{gf}\,k}$ is already quadratic in the fluctuations such that $\Gamma_{\text{gf}\,k}^{\text{quad}} = \Gamma_{\text{gf}\,k}$. The evaluation of $\Gamma_{\text{Ho}\,k}^{\text{quad}}$ is straightforward. From the basic building blocks

$$\begin{aligned} \delta F^{ab}{}_{\mu\nu} &= \nabla_{[\mu}\bar{\tau}^{ab}{}_{\nu]}, & \delta^2 F^{ab}{}_{\mu\nu} &= 2\bar{\tau}^a{}_{c[\mu}\bar{\tau}^{cb}{}_{\nu]}\,, \\ \delta(e\, e_a{}^{[\mu} e_b{}^{\nu]}) &= \varepsilon_{abcd}\varepsilon^{\mu\nu\rho\sigma} e^c{}_\rho \bar{\varepsilon}^d{}_\sigma, & \delta^2(e\, e_a{}^{[\mu} e_b{}^{\nu]}) &= \varepsilon_{abcd}\varepsilon^{\mu\nu\rho\sigma} \bar{\varepsilon}^c{}_\rho \bar{\varepsilon}^d{}_\sigma\,, \\ & & \delta^2(e) &= \frac{1}{2}\varepsilon_{abcd}\varepsilon^{\mu\nu\rho\sigma} e^c{}_\rho e^d{}_\sigma \bar{\varepsilon}^a{}_\mu \bar{\varepsilon}^b{}_\nu\,, \end{aligned} \tag{5.49}$$

we find

$$\Gamma^{\text{quad}}_{\text{Ho}\,k} = -\frac{1}{2}\frac{Z_{\text{N}}}{16\pi\hat{G}}\int \mathrm{d}^4x \Big[\bar{\varepsilon}^c{}_\rho\Big[\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abcd}\Big(\frac{1}{4}\Big(\delta^c_{[e}\delta^d_{f]} - \frac{1}{\gamma_k}\varepsilon^{cd}{}_{ef}\Big)\bar{F}^{ef}{}_{\mu\nu} - \bar{e}^a{}_\mu\bar{e}^b{}_\nu\Lambda_k\Big)\Big]\bar{\varepsilon}^d{}_\sigma$$
$$+\frac{1}{2}\,\bar{\varepsilon}^e{}_\rho\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abef}\bar{e}^f{}_\sigma\Big(\delta^a{}_{[c}\delta^b{}_{d]} - \frac{1}{\gamma_k}\varepsilon^{ab}{}_{cd}\Big)\bar{\nabla}_\mu\bar{\tau}^{cd}{}_\nu \quad (5.50)$$
$$+2\,\bar{\tau}^{ce}{}_\mu\Big[\bar{e}\,\bar{e}_a{}^\mu\bar{e}_b{}^\nu\Big(\delta^a{}_{[c}\delta^b{}_{d]} - \frac{1}{\gamma_k}\varepsilon^{ab}{}_{cd}\Big)\eta_{ef}\Big]\,\bar{\tau}^{fd}{}_\nu\,.$$

In principle, we also need to construct the Hessian in the ghost sector. But as the ghost action S_{gh} (cf. eq. (5.13)) is by construction quadratic in the ghost fields (ξ_μ, Υ^{ab}) we find here, as for the gauge fixing part, $S^{\text{quad}}_{\text{gh}} = S_{\text{gh}}$ such that the Hessian can be directly read off from the ghost action.

5.2.3 Decomposition of the fluctuations and ghost fields

Decomposition of $\bar{\varepsilon}$ and $\bar{\tau}$. In the above undecomposed form it turns out technically not feasible to directly invert the total $\bar{\omega}$-independent part of the bosonic Hessian operator $H_0 = \breve{\Gamma}^{(2)}_k|_{\bar{\omega}=0}$ even for constant background fields $\bar{e}$. This is owed to the fact, that H_0 is a 40 x 40-matrix operator (16+24 independent components of $\bar{\varepsilon}^a{}_\mu$ and $\bar{\tau}^{ab}{}_\mu$) with a complicated matrix structure.

For that reason it is helpful to decompose the fluctuation fields into transverse and longitudinal parts w. r. t. the generalized "plane wave" basis introduced before which we are going to employ in order to evaluate the trace later on. For the vielbein fluctuation we write

$$\bar{\mu}^{\frac{1}{2}}\bar{\varepsilon}^a{}_\mu(x) = \frac{\partial^a\partial_\mu}{-\Box}a(x) + \frac{\partial_\mu}{\sqrt{-\Box}}b^a(x) + \frac{\partial^a}{\sqrt{-\Box}}c_\mu(x) + d^a{}_\mu(x)\,, \quad (5.51)$$

while the spin connection fluctuation is decomposed according to

$$\bar{\tau}^{ab}_\mu(x) = \frac{\bar{\mu}^{\frac{1}{2}}}{\sqrt{2}}\Big[\frac{\partial_\mu\partial^{[a}}{-\Box}A^{b]}(x) + \frac{\partial^{[a}}{\sqrt{-\Box}}B^{b]}{}_\mu(x) + \varepsilon^{ab}{}_{cd}\frac{\partial_\mu\partial^c}{-\Box}C^d(x) + \varepsilon^{ab}{}_{cd}\frac{\partial^c}{\sqrt{-\Box}}D^d{}_\mu(x)\Big] \quad (5.52)$$

Several comments on this decomposition are in order:

(A) All fields occurring in the decomposition have a vanishing divergence in all their indices, i. e.

$$\partial_a b^a = 0 = \partial^\mu c_\mu \, , \quad \partial_a d^a{}_\mu = 0 \, , \quad \partial^\mu d^a{}_\mu = 0 \tag{5.53}$$

and

$$\partial_a A^a = 0 = \partial_a C^a \, , \quad \partial_a B^a{}_\mu = 0 = \partial_a D^a{}_\mu \, , \quad \partial^\mu B^a{}_\mu = 0 = \partial^\mu D^a{}_\mu \, . \tag{5.54}$$

Here, partial derivatives with an $\mathsf{O}(4)$-index contain the background vielbein implicitly, $\partial_a = \bar{e}_a{}^\mu \partial_\mu$, $\partial^a = \bar{e}^a{}_\mu \partial^\mu = \bar{e}^a{}_\mu \bar{g}^{\mu\nu} \partial_\nu$.

(B) We have introduced a rescaling of the fluctuations using the positive definite operator $-\Box = -\bar{g}^{\mu\nu}\partial_\mu\partial_\nu$; the powers of $\sqrt{-\Box}$ present in the different terms hereby correspond to the number of partial derivatives acting on the respective component field. Thus, we achieve that, first, all component fields have the same mass dimension and, second, all $\Box$ operators appearing in the Hessian that are only due to the decomposition will be canceled by the denominators, such that they cannot be confused with "true" kinetic terms. Hence, the Hessian operator $\Gamma^{(2)}_{\text{Ho}}$ will be first order in the derivatives, before and after the decomposition. Note that when passing to the generalized plane wave basis, we find the simple replacement rules $\partial_\mu \to i p_\mu$, $\sqrt{-\Box} \to \sqrt{p^2}$, for derivatives acting to the right.

(C) As it may not be evident that (5.51) and (5.52) indeed form a complete decomposition of the full original field space, let us construct the system of projectors onto the contributions of each component field. To this end we define the projectors onto the longitudinal and the transversal part of a vector (under diffeomorphisms) according to

$$\big(P_L\big)^\mu{}_\nu = -\frac{\partial^\mu \partial_\nu}{(-\Box)}, \qquad \big(P_T\big)^\mu{}_\nu = \delta^\mu_\nu + \frac{\partial^\mu \partial_\nu}{(-\Box)} \, . \tag{5.55}$$

The analogous objects with Greek indices replaced by Latin ones are used to decompose $\mathsf{O}(4)$-vectors.

For the component fields of the vielbein we then find

$$\begin{aligned} \big(P_{\mathbf{a}}\big)^a{}_{\mu b}{}^\nu &= \big(P_L\big)^\nu{}_\mu \big(P_L\big)^a{}_b, \quad \big(P_{\mathbf{b}}\big)^a{}_{\mu b}{}^\nu = \big(P_L\big)^\nu{}_\mu \big(P_T\big)^a{}_b, \\ \big(P_{\mathbf{c}}\big)^a{}_{\mu b}{}^\nu &= \big(P_T\big)^\nu{}_\mu \big(P_L\big)^a{}_b, \quad \big(P_{\mathbf{d}}\big)^a{}_{\mu b}{}^\nu = \big(P_T\big)^\nu{}_\mu \big(P_T\big)^a{}_b. \end{aligned} \tag{5.56}$$

When acting on a vielbein fluctuation $\bar{\varepsilon}^b{}_\nu$ these projectors map onto the summand of the respective component field. From (5.56) we directly see that (5.51) is indeed a decomposition into longitudinal and transverse parts w. r. t. both indices. The projectors (5.56) form a complete set of projectors, i. e. $P_\mathbf{a} + P_\mathbf{b} + P_\mathbf{c} + P_\mathbf{d} = \mathbb{1}\big(\,\widehat{=}\,\delta^a{}_b\delta_\mu{}^\nu\big)$ and are orthogonal to each other, i. e. $P_\mathbf{a}P_\mathbf{b} = 0$ etc.

For the spin connection fluctuation we work in a field space, that only contains tensors antisymmetric w. r. t. the $\mathsf{O}(4)$-indices. Projecting onto the longitudinal part of one index, directly affects the other index via anti-symmetrization. In this space the above projectors therefore generalize to

$$\big(P_L\big)^{ab}{}_{cd} = -\frac{1}{2}\frac{\delta^{[a}_{\ [c}\partial^{b]}\partial_{d]}}{(-\Box)}, \qquad \big(P_T\big)^{ab}{}_{cd} = \frac{1}{2}\left(\delta^{[a}_{\ c}\delta^{b]}_{\ d} + \frac{\delta^{[a}_{\ [c}\partial^{b]}\partial_{d]}}{(-\Box)}\right). \tag{5.57}$$

We denote them by the same symbols as we can distinguish them by the number of indices they carry. With these definitions the projectors onto the components of the spin connection read

$$\begin{aligned} \big(P_\mathbf{A}\big)^{ab}{}_{\mu cd}{}^{\nu} &= \big(P_L\big)^{\nu}{}_{\mu}\big(P_L\big)^{ab}{}_{cd}, & \big(P_\mathbf{B}\big)^{ab}{}_{\mu cd}{}^{\nu} &= \big(P_T\big)^{\nu}{}_{\mu}\big(P_L\big)^{ab}{}_{cd}, \\ \big(P_\mathbf{C}\big)^{ab}{}_{\mu cd}{}^{\nu} &= \big(P_L\big)^{\nu}{}_{\mu}\big(P_T\big)^{ab}{}_{cd}, & \big(P_\mathbf{D}\big)^{ab}{}_{\mu cd}{}^{\nu} &= \big(P_T\big)^{\nu}{}_{\mu}\big(P_T\big)^{ab}{}_{cd}. \end{aligned} \tag{5.58}$$

Again, one can check that they form an orthogonal set, fulfilling the completeness relation $P_\mathbf{A} + P_\mathbf{B} + P_\mathbf{C} + P_\mathbf{D} = \mathbb{1}\ \big(\,\widehat{=}\,\frac{1}{2}\delta_\mu{}^\nu\delta^{[a}_{\ c}\delta^{b]}_{\ d}\big)$.

By taking the trace of the above projectors we can count the number of independent field components in each sector. With $\big(P_L\big)^{\mu}{}_{\mu} = 1$, $\big(P_T\big)^{\mu}{}_{\mu} = 3$, $\big(P_L\big)^{ab}{}_{ab} = 3$ and $\big(P_T\big)^{ab}{}_{ab} = 3$ we find that the vielbein degrees of freedom decompose according to $16 = 1 + 3 + 3 + 9$ and the spin connection according to $24 = 3 + 9 + 3 + 9$. These counts make it plausible that the independent field components of each sector can be represented by a scalar field a, divergence-free vector fields $\{b, c, A, D\}$ and divergence-free tensor fields $\{d, B, D\}$.

(D) In the decompositions (5.51), (5.52) we have introduced a mass scale $\bar{\mu}$, that may appear artificial at first sight, but is crucial in order to define a Hessian operator of definite mass dimension from the quadratic form Γ^{quad}: As the tetrad is dimensionless but the spin connection (being a connection) has mass dimension 1, the different blocks of the matrix resulting from splitting off the fluctuations differ in dimension.

Such a matrix, however, does not give rise to an operator whose spectrum and trace are well defined, as we then have to sum up quantities of different mass dimension. By introducing the additional mass scale $\bar{\mu}$ we equalize the mass dimensions of all component fields. In principle, this can be done in an arbitrary way, rescaling for example only the vielbein fluctuation by $\bar{\mu}^1$, but as in [Dau] we opted for the symmetric scheme.

In contrast to [Dau] we used a different numerical prefactor in the decomposition (5.52). Only with this choice the trace of an operator is invariant under the decomposition. Otherwise, the component fields contribute with different weights to the trace. As a simple check of the decomposition the following equation should hold without any additional numerical prefactors in front of the component fields:

$$\int \mathrm{d}^4x\, \bar{\tau}_{ab}{}^{\mu} \bar{\tau}^{ab}{}_{\mu} = \int \mathrm{d}^4x \left(A^a A_a + B_a{}^{\mu} B^a{}_{\mu} + C^a C_a + D_a{}^{\mu} D^a{}_{\mu} \right) \tag{5.59}$$

For the decomposition of the tetrad an analog relation holds.

In a second step we want to further decompose all tensor fields in the above decompositions, $\{d, B, D\}$, into trace, antisymmetric and symmetric-traceless part. Using the $d^a{}_{\mu}$-field as an example the decomposition reads

$$d^a{}_{\mu} = -\frac{1}{\sqrt{3}} \left(\bar{e}^a{}_{\mu} + \frac{\partial^a \partial_{\mu}}{(-\Box)} \right) d + \frac{1}{\sqrt{2}}\, \varepsilon^a{}_{bcd} \bar{e}^b{}_{\mu} \frac{\partial^c}{\sqrt{-\Box}}\, d^d + \widehat{d}^a{}_{\mu}\,. \tag{5.60}$$

Here, d describes the scalar trace mode, d^a is a divergence-less vector field representing the antisymmetric part of $d^a{}_{\mu}$ and $\widehat{d}^a{}_{\mu}$ is a traceless symmetric tensor ($\bar{e}_a{}^{\mu} \widehat{d}^a{}_{\mu} = 0$, $\bar{e}^{b\mu} \widehat{d}^a{}_{\mu} = \bar{e}^{a\mu} \widehat{d}^b{}_{\mu}$), with vanishing divergence in both indices separately.

As for the first decomposition analogous remarks to point **(B)** and **(D)** apply here, concerning the rescaling with the operator $-\Box$ and the numerical prefactors chosen.

Again, it is instructive to analyze the corresponding projectors for each of the parts. If we introduce the trace projector P_{tr} and the antisymmetry projector P_{as} according to

$$\left(P_{\mathrm{tr}}\right)^a{}_{\mu b}{}^{\nu} = \frac{1}{4} \bar{e}^a{}_{\mu} \bar{e}_b{}^{\nu}, \qquad \left(P_{\mathrm{as}}\right)^a{}_{\mu b}{}^{\nu} = \frac{1}{2} \left(\delta^a_b \delta^{\nu}_{\mu} - \bar{e}^{a\nu} \bar{e}_{b\mu} \right) \tag{5.61}$$

we can decompose the full d-subspace by the three orthogonal projectors $P_{d,\mathrm{tr}}$, $P_{d,\mathrm{as}}$, and $P_{d,\mathrm{st}}$ defined by

$$P_{d,\mathrm{tr}} := \frac{4}{3} P_d P_{\mathrm{tr}} P_d , \quad P_{d,\mathrm{as}} := P_d P_{\mathrm{as}} P_d , \quad \text{and} \quad P_{d,\mathrm{st}} := P_d \big(\mathbb{1} - \tfrac{4}{3} P_{\mathrm{tr}} - P_{\mathrm{as}} \big) P_d \, . \tag{5.62}$$

Indeed,

$$P_d = P_{d,\mathrm{tr}} + P_{d,\mathrm{as}} + P_{d,\mathrm{st}} \, . \tag{5.63}$$

Note that we have to rescale the trace projector by a factor of $\frac{4}{3}$ in order to define an idempotent operator $P_{d,\mathrm{tr}}$. This is owed to the fact that due to the transversality conditions the d-subspace has a lower dimension than the full rank-two tensor space, in particular we have $\bar{e}_a{}^\mu \big(P_d\big)^a{}_{\mu b}{}^\nu \bar{e}^b{}_\nu = 3$ as opposed to $\bar{e}_a{}^\mu \mathbb{1}^a{}_{\mu b}{}^\nu \bar{e}^b{}_\nu = 4$. One can check that the three operators defined in (5.62) indeed project onto the three parts of the decomposition (5.60) splitting the independent field components of $d^a{}_\mu$ according to $9 = 1 + 3 + 5$.

For the other tensor fields, $\{B, D\}$, we proceed in complete analogy. As we will work only in the completely decomposed setting, the tensor fields $\{d, B, D\}$ will not be of importance any more. For that reason, we will drop the hat on the last component field $\{\widehat{d}, \widehat{B}, \widehat{D}\} \to \{d, B, D\}$ such that the latter denote divergence-less, symmetric and traceless tensors, from now on.

Decomposition of the ghost fields. The decomposition of the ghost fields follows the same logic as for the commuting fluctuations, but due to their simpler tensor structure the ghost fields decompose into far less component fields. In principle, the remarks concerning the previous decompositions also apply in the ghost sector so that we will merely state the corresponding formulas here.

The diffeomorphism ghost ξ_μ can be split into a transverse part g_μ and a longitudinal component f according to

$$\xi_\mu = \frac{\partial_\mu}{\sqrt{-\Box}} f + g_\mu \, , \tag{5.64}$$

with $\partial^\mu g_\mu = 0$.

The O(4)-ghost Υ^{ab} is split into two parts $\{F, G\}$ that are dual to each other and are rescaled by a factor of $\bar{\mu}$:

$$\bar{\mu}^{-1}\Upsilon^{ab} = \frac{1}{\sqrt{2}}\left(\frac{\partial^{[a}}{\sqrt{-\Box}}F^{b]} + \varepsilon^{ab}{}_{cd}\frac{\partial^{c}}{\sqrt{-\Box}}G^{d}\right) , \tag{5.65}$$

where $\partial_a F^a = 0 = \partial_a G^a$.

The decomposition of the anti-ghosts $\bar{\xi}$ and $\bar{\Upsilon}$ is completely analogous and can thus be obtained by putting a bar on every field in eqns. (5.64) and (5.65).

5.2.4 The Hessian in the decomposed field basis

By substituting the decompositions of the fluctuations introduced in the last subsection into the expression of the quadratic forms (5.50), (5.8) and for the ghost sector (5.13) we can transform the quadratic forms to the component field basis. After this substitution we use integration by parts until all derivatives act to the right. As we work with a closed background manifold (flat torus) in doing so no boundary terms have to be considered. A large fraction of the occurring terms vanishes due to the transversality conditions that all component fields satisfy. Nonetheless, the decomposition to the component fields generates a huge number of terms: The bosonic sector consists in total of 14 component fields, so that the corresponding Hessian is a 14 x 14 matrix in field space whose 196 entries are, due to their remaining index structure, still complicated expressions in terms of $\bar{e}^a{}_\mu$, $\bar{\omega}^{ab}{}_\mu$ and the partial derivative operator ∂_μ. For that reason we calculated the Hessian in the component field basis using the tensor manipulation package MathTensor for Mathematica [PC94].

For the evaluation of the flow equation we have to split the Hessian according to $\breve{\Gamma}^{(2)} = H_0 + V(\bar{\omega})$ and $S^{(2)}_{\text{gh}} = H^{\text{gh}}_0 + V^{\text{gh}}(\bar{\omega})$ into a "free" part H_0 and an interaction part $V(\bar{\omega})$ (cf. Section 5.2.1). The next two paragraphs will be devoted to these building blocks.

The H_0 part of the Hessian. While the total Hessian is a considerably complicated object in the decomposed field basis, its free part H_0 is comparatively simple although it can not be written in an as compact form as in (5.50). The free part of the total quadratic form stemming from the Holst action (5.50) and the gauge fixing term (5.8) in the component field basis reads

$$
\begin{aligned}
\breve{\Gamma}_k^{\text{quad}}\Big|_{\bar{e}=\text{const},\bar{\omega}=0} &= \frac{1}{2}\frac{Z_{\text{N}\,k}}{16\pi\hat{G}}\int \mathrm{d}^4x\,\bar{e}\,\times \\
\times\Bigg[&\begin{pmatrix} a \\ d \\ B \\ D \end{pmatrix}^{\mathrm{T}}
\begin{pmatrix}
\frac{(1+\beta_{\text{D}})^2}{\alpha_{\text{D}}\bar{\mu}}\hat{\text{P}}^2 & \frac{\sqrt{3}}{\bar{\mu}}\left(2\Lambda_k+\frac{\beta_{\text{D}}(1+\beta_{\text{D}})}{\alpha_{\text{D}}}\hat{\text{P}}^2\right) & 0 & 0 \\
\frac{\sqrt{3}}{\bar{\mu}}\left(2\Lambda_k+\frac{\beta_{\text{D}}(1+\beta_{\text{D}})}{\alpha_{\text{D}}}\hat{\text{P}}^2\right) & \frac{1}{\bar{\mu}}\left(4\Lambda_k+3\frac{\beta_{\text{D}}^2}{\alpha_{\text{D}}}\hat{\text{P}}^2\right) & 2\sqrt{2}\hat{\text{P}} & -\frac{2\sqrt{2}}{\gamma_k}\hat{\text{P}} \\
0 & 2\sqrt{2}\hat{\text{P}} & 2\bar{\mu} & -\frac{2}{\gamma_k}\bar{\mu} \\
0 & -\frac{2\sqrt{2}}{\gamma_k}\hat{\text{P}} & -\frac{2}{\gamma_k}\bar{\mu} & 2\bar{\mu}
\end{pmatrix}
\begin{pmatrix} a \\ d \\ B \\ D \end{pmatrix} \\
+ &\begin{pmatrix} b_a \\ c_a \\ d_a \\ A_a \\ B_a \\ C_a \\ D_a \end{pmatrix}^{\mathrm{T}}
\begin{pmatrix}
\frac{2}{\alpha'_{\text{L}}\bar{\mu}} & -\frac{2}{\bar{\mu}}\left(\Lambda_k+\frac{1}{\alpha'_{\text{L}}}\right) & 0 & 0 & 0 & 0 & 0 \\
-\frac{2}{\bar{\mu}}\left(\Lambda_k+\frac{1}{\alpha'_{\text{L}}}\right) & \frac{1}{\bar{\mu}}\left(\frac{\hat{\text{P}}^2}{\alpha_{\text{D}}}+\frac{2}{\alpha'_{\text{L}}}\right) & 0 & 0 & -\frac{2}{\gamma_k}\hat{\text{P}} & 0 & 2\hat{\text{P}} \\
0 & 0 & \frac{1}{\bar{\mu}}\left(2\Lambda_k+\frac{4}{\alpha'_{\text{L}}}\right) & 0 & \sqrt{2}\hat{\text{P}} & 0 & -\frac{\sqrt{2}}{\gamma_k}\hat{\text{P}} \\
0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{\gamma_k}\bar{\mu} & 0 & \sqrt{2}\bar{\mu} \\
0 & -\frac{2}{\gamma_k}\hat{\text{P}} & \sqrt{2}\hat{\text{P}} & -\frac{\sqrt{2}}{\gamma_k}\bar{\mu} & \bar{\mu} & \sqrt{2}\bar{\mu} & -\frac{1}{\gamma_k}\bar{\mu} \\
0 & 0 & 0 & 0 & \sqrt{2}\bar{\mu} & 0 & -\frac{\sqrt{2}}{\gamma_k}\bar{\mu} \\
0 & 2\hat{\text{P}} & -\frac{\sqrt{2}}{\gamma_k}\hat{\text{P}} & \sqrt{2}\bar{\mu} & -\frac{1}{\gamma_k}\bar{\mu} & -\frac{\sqrt{2}}{\gamma_k}\bar{\mu} & \bar{\mu}
\end{pmatrix}
\begin{pmatrix} b^a \\ c^a \\ d^a \\ A^a \\ B^a \\ C^a \\ D^a \end{pmatrix} \\
+ &\begin{pmatrix} d_a{}^\mu \\ B_a{}^\mu \\ D_a{}^\mu \end{pmatrix}^{\mathrm{T}}
\begin{pmatrix}
-\frac{2}{\bar{\mu}}\Lambda_k & -\sqrt{2}\hat{\text{P}} & \frac{\sqrt{2}}{\gamma_k}\hat{\text{P}} \\
-\sqrt{2}\hat{\text{P}} & -\bar{\mu} & \frac{\bar{\mu}}{\gamma_k} \\
\frac{\sqrt{2}}{\gamma_k}\hat{\text{P}} & \frac{\bar{\mu}}{\gamma_k} & -\bar{\mu}
\end{pmatrix}
\begin{pmatrix} d^a{}_\mu \\ B^a{}_\mu \\ D^a{}_\mu \end{pmatrix}\Bigg]\,.
\end{aligned}
\tag{5.66}
$$

Here, we have introduced the abbreviatory notations $c^a = \bar{e}^a{}_\mu c^\mu$ and for the absolute momentum operator $\hat{\text{P}} := \sqrt{-\Box}$.

H_0 can be read off from this result, being the 14 x 14 block diagonal matrix operator in field space, whose blocks are given by the above matrices, including the overall prefactor $Z_{\text{N}\,k}/(16\pi\hat{G})$.

Let us comment on the form of H_0:

(A) While the above form of H_0 is notationally more complicated than in the undecomposed basis, structurally it is the most simple form: We observe that it is diagonal in the remaining index structure and block diagonal in field space, such that scalar, vector and tensor components only couple to themselves.

This simple form can be expected from the outset, as all component fields are transverse and the partial derivative ∂_μ in the free part H_0 is the only object carrying an index (the constant tetrads $\bar{e}^a{}_\mu$ are being absorbed by partial derivatives changing the type of their index). Thus, they can only combine to $-\Box$-operators in H_0 as all free indices give a vanishing contribution due to the transversality of the fields. For the same reason, no coupling between component fields of different tensor type can occur in H_0.

Due to this simple structure H_0 can be inverted by inverting its matrix structure in field space and taking the tensor product with the identity operators on the respective field space i. e. scalars, transverse vectors and transverse symmetric traceless tensors (cf. eqns. (5.43), (5.44)). Thus, by the decomposition of the fields we have reached our main goal, namely the explicit invertibility of H_0.

(B) One can check the plausibility of the result (5.66) by several consistency checks [Dau]. As the component fields with small/capital letters correspond to the fluctuations $\bar{\varepsilon}/\bar{\tau}$ we can easily compare with the $\bar{\varepsilon}/\bar{\tau}$-block structure in the undecomposed basis.

First of all we see that all elements stemming from the $\bar{\varepsilon}\bar{\varepsilon}$-block are proportional to $\bar{\mu}^{-1}$, while those from the $\bar{\tau}\bar{\tau}$-block come with $\bar{\mu}$ and the $\bar{\varepsilon}\bar{\tau}$-blocks are not affected. Thus the $\bar{\mu}$-rescaling has the desired effect of giving H_0 a well defined mass dimension.

Second, the first order derivatives of the Holst action appearing in the $\bar{\varepsilon}\bar{\tau}$-block are here reflected in the $\hat{\mathrm{P}}$-operators in the corresponding blocks. All $\hat{\mathrm{P}}^2$-operators in H_0 appear in the $\bar{\varepsilon}\bar{\varepsilon}$-block and correspond to the second order derivatives that are part of the diffeomorphism gauge fixing action.

At last, one can divide the component fields into tensors and pseudo tensors. Since the complete fluctuation $\bar{\tau}$ transforms as a tensor under parity, the tensor components come with an even number and the pseudo-tensor components with an odd number of ε_{abcd} pseudo-tensors as prefactors in the field decomposition (5.52), (5.60). As noted before, the Immirzi term in the Holst action corresponds to its pseudo scalar part; thus, all pseudo scalar contributions to $\breve{\Gamma}^{\text{quad}}$ are proportional to $\frac{1}{\gamma_k}$, while the scalar contributions do not contain the Immirzi parameter. As one can check, the Immirzi parameter occurs in all matrix elements coupling tensors with pseudo tensors (and only there) forming pseudo scalar contributions.

(C) As the decomposition of the fields was carried out with the help of a computer algebra system, we have cross-checked our result for H_0 with the corresponding result from [Dau] in order to ensure the functionality of our algorithm. We were able to verify that all differences occurring are due to the different numerical prefactors in the decomposition of the fields and a factor of 2 traced back to a different definition of the $\mathsf{O}(4)$-gauge condition.

In the ghost sector the decomposition is much simpler. For the free part H_0^{gh} of the quadratic form S_{gh} we find:

$$S_{\text{gh}}\Big|_{\substack{\bar{e}=\text{const},\\ \bar{\omega}=0}} = \int \mathrm{d}^4x\,\bar{e} \begin{pmatrix} \bar{f} \\ \bar{g}_a \\ \bar{F}_a \\ \bar{G}_a \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -\frac{(1+\beta_{\text{D}})}{\sqrt{\alpha_{\text{D}}}}\hat{\mathrm{P}}^2 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\bar{\mu}}{\sqrt{2\alpha_{\text{D}}}}\hat{\mathrm{P}} & 0 \\ 0 & -\frac{\sqrt{2}\bar{\mu}}{\sqrt{\alpha_{\text{L}}}}\hat{\mathrm{P}} & \frac{2\bar{\mu}^2}{\sqrt{\alpha_{\text{L}}}} & 0 \\ 0 & 0 & 0 & \frac{2\bar{\mu}^2}{\sqrt{\alpha_{\text{L}}}} \end{pmatrix} \begin{pmatrix} f \\ g^a \\ F^a \\ G^a \end{pmatrix} \tag{5.67}$$

Again the correspondence to the analogous expression in [Dau] has been verified. From (5.67) we see explicitly that for $\beta_{\text{D}} = -1$ the diffeomorphism gauge fixing condition breaks down and the ghost operator develops a zero mode in the scalar sector, while it is invertible for all other values of β_{D}.

The interaction part $V(\bar{\omega})$. For the interaction part $V(\bar{\omega})$ of the Hessian the index structure of its elements does not simplify in a comparable manner: Here we have the objects $\bar{\omega}^{ab}{}_{\mu}$, ε_{abcd} and ∂_μ carrying indices that can be contracted in various ways using the background vielbein $\bar{e}^a{}_\mu$ and its inverse. As long as the free indices of the resulting expressions do not belong to partial derivatives, they may contribute to $V(\bar{\omega})$. For this reason, most of the matrix elements of $V(\bar{\omega})$ are so complicated expressions that it is not instructive to write down $V(\bar{\omega})$ in the decomposed basis. (The total expression would fill many pages.)

Let us mention, however, that if we split $V(\bar{\omega}) = V^1(\bar{\omega}) + V^2(\bar{\omega})$, where V^1 is linear and V^2 quadratic in $\bar{\omega}$ (note that there are no higher order terms in the action), these matrices have a certain block structure in field space, that can already be read off from the $(\bar{\varepsilon}, \bar{\tau})$ representation (5.50) and (5.8), and this can be exploited to simplify the calculation of the traces later on. For constant background fields $\{\bar{e}, \bar{\omega}\}$ we find schematically:

$$V^1(\bar{\omega}) = \begin{pmatrix} \left(\Gamma^{(2)}_{\text{gf}}\right)_{\bar{\varepsilon}\bar{\varepsilon}} & \left(\Gamma^{(2)}_{\text{Ho}}\right)_{\bar{\varepsilon}\bar{\tau}} \\ \left(\Gamma^{(2)}_{\text{Ho}}\right)_{\bar{\tau}\bar{\varepsilon}} & 0 \end{pmatrix}\Bigg|_{\substack{\text{linear}\\ \text{in }\bar{\omega}}}, \quad V^2(\bar{\omega}) = \begin{pmatrix} \left(\Gamma^{(2)}_{\text{gf}} + \Gamma^{(2)}_{\text{Ho}}\right)_{\bar{\varepsilon}\bar{\varepsilon}} & 0 \\ 0 & 0 \end{pmatrix}\Bigg|_{\substack{\text{quadratic}\\ \text{in }\bar{\omega}}} \tag{5.68}$$

5.2.5 Evaluation of the flow equation

After having defined all ingredients to the flow equation

$$\partial_t \Gamma_k[\bar{e},\bar{\omega}] = -\frac{k^4}{(2\pi)^4}\int \mathrm{d}^4x\,\bar{e}\Big[\int \mathrm{d}\Omega_p \sum_{i,a_i,b_i} \frac{1}{2}(P_i)^{b_i}{}_{a_i}\big[\ln \breve{\Gamma}^{(2)}(ip_\mu)\big]_{ii\,b_i}^{a_i} - \int \mathrm{d}\Omega_p \sum_{\substack{i\in\\ \text{ghosts}},a_i,b_i} (P_i)^{b_i}{}_{a_i}\big[\ln S^{(2)}_{\text{gh}}(ip_\mu)\big]_{ii\,b_i}^{a_i}\Big]_{p=k} \tag{5.69}$$

we are now able to evaluate both of its sides.

Let us first consider the left hand side of the equation. If we substitute the Holst truncation (5.5) into it and switch to dimensionless couplings,

$$g_k = k^2 G_k, \qquad \lambda_k = k^{-2}\Lambda_k, \tag{5.70}$$

in order to obtain an autonomous system of β-functions, we find

$$\begin{aligned}\partial_t \Gamma_k[\bar{e},\bar{\omega}] = &-\frac{k^2}{16\pi g_k}\left(2-\frac{\partial_t g_k}{g_k}\right) &&\cdot\int \mathrm{d}^4x\,\bar{e}\,\bar{e}_a{}^\mu \bar{e}_b{}^\nu \bar{F}^{ab}{}_{\mu\nu}\\ &+\frac{k^2}{16\pi g_k}\left(2-\frac{\partial_t g_k}{g_k}-\frac{\partial_t \gamma_k}{\gamma_k}\right)\frac{1}{\gamma_k} &&\cdot\int \mathrm{d}^4x\,\bar{e}\,\frac{1}{2}\bar{e}_a{}^\mu \bar{e}_b{}^\nu \varepsilon^{ab}{}_{cd}\bar{F}^{cd}{}_{\mu\nu}\\ &+\frac{k^2}{16\pi g_k}\left(2-\frac{\partial_t g_k}{g_k}+2+\frac{\partial_t \lambda_k}{\lambda_k}\right)2\lambda_k k^2 &&\cdot\int \mathrm{d}^4x\,\bar{e}\,.\end{aligned} \tag{5.71}$$

We can further evaluate the field monomials for the constant background fields $\bar{e}$ and $\bar{\omega}$ leading to

$$\bar{e}\,\bar{e}_a{}^\mu \bar{e}_b{}^\nu \bar{F}^{ab}{}_{\mu\nu} = \bar{e}\,(\bar{\omega}_{abc}\bar{\omega}^{acb} - \bar{\omega}^a{}_{ca}\bar{\omega}^{bc}{}_b), \quad \frac{1}{2}\,\bar{e}\,\bar{e}_a{}^\mu \bar{e}_b{}^\nu \varepsilon^{ab}{}_{cd}\bar{F}^{cd}{}_{\mu\nu} = \bar{e}\varepsilon_{abcd}\bar{\omega}_e{}^{ab}\bar{\omega}^{ecd}\,. \tag{5.72}$$

Here, we have used the background vielbein to formally change the spacetime index of $\bar{\omega}$ to an $\mathsf{O}(4)$-index. This is only done for notational simplicity; if needed, the tetrads can be restored at any point of the calculation.

Now, let us turn to the right hand side of (5.69). We have to extract all terms from the logarithms that are independent of or second order in $\bar{\omega}$, as the invariants we want to project on are either of zeroth or of second order in $\bar{\omega}$. For this case the expansion of the logarithms (5.42) simplifies to

$$\mathrm{tr}\,\ln\breve{\Gamma}^{(2)} = \frac{1}{2}\ln\left[\prod\big(\det(H_0)_{ij}\big)^{\delta^{I_i}_{J_j}}\right]^2 + \mathrm{tr}\,\widetilde{P}\widetilde{H_0^{-1}}\widetilde{V^2} - \frac{1}{2}\mathrm{tr}\,\widetilde{P}(\widetilde{H_0^{-1}}\widetilde{V^1})^2 + \cdots \quad (5.73)$$

(and analogous for the ghost sector). The dots stand for terms linear in $\bar{\omega}$ and $\mathcal{O}(\bar{\omega}^3)$-terms. Note, that all terms linear in $\bar{\omega}$ vanish anyway when the momentum integration is carried out, as no field monomials exist that have a linear part for a constant field $\bar{\omega}$. Thus, (5.73) still represents a full expansion of the right hand side up to second order in $\bar{\omega}$. In the $\bar{\omega}$-independent part of (5.73) we have applied another computational trick: As the spectrum of H_0 is not necessarily positive, the logarithm of its determinant could be ill-defined. For that reason we replace $\ln\det H_0 \to \frac{1}{2}\ln\det H_0^2$, defining a method how to treat the negative part of its spectrum.

As a next step we now substitute H_0, V^1 and V^2 in the decomposed field basis into (5.73) and the result into (5.69). Then we are left with an expression containing all possible monomials that are quadratic in $\bar{\omega}$; in order to prepare for a comparison of their coefficients on both sides of the flow equation (5.69) we would like to cast it into the form

$$\begin{aligned}\partial_t\Gamma_k = \mathrm{rhsF}\cdot k^2\int\mathrm{d}^4x\,\bar{e}\,(\bar{\omega}_{abc}\bar{\omega}^{acb} - \bar{\omega}^a{}_{ca}\bar{\omega}^{bc}{}_b) + \mathrm{rhsF}^*\cdot k^2\int\mathrm{d}^4x\,\bar{e}\,\varepsilon_{abcd}\bar{\omega}_e{}^{ab}\bar{\omega}^{ecd}\\ + \mathrm{rhs}\Lambda\cdot k^4\int\mathrm{d}^4x\,\bar{e} + k^2\int\mathrm{d}^4x\,\bar{e}\,(\text{further, independent }\bar{\omega}^2\text{-monomials})\,.\end{aligned} \quad (5.74)$$

Here, rhsF, rhsF*, and rhsΛ are (dimensionless) functions of the dimensionless couplings (λ, γ, g) and the gauge fixing parameters $(\alpha_\mathrm{D}, \alpha'_\mathrm{L}, \beta_\mathrm{D})$.

If we analyze the expression (5.74) in more detail, we find that the exact form of the coefficients rhsF and rhsF* depends on the basis of independent field monomials quadratic in $\bar{\omega}$ we choose; i. e. only if we fix the form of the remaining independent terms in (5.74), corresponding to a particular choice of basis in theory space, we can define the coefficients of the invariants we are interested in.[2] The details of this projec-

[2] The explicit form of the complete RHS containing all independent $\bar{\omega}^2$-monomials does not depend on the basis of theory space chosen, but only on the gauge parameters. Since the result for general gauge parameters fills about 20 pages without being illuminative, we only display this unambiguous result for the $(\alpha_\mathrm{D}, \alpha'_\mathrm{L}, \beta_\mathrm{D}) = (0,0,0)$ gauge in Appendix E.

tion ambiguity are discussed in the next section. Let us only note here that this effect is neither astonishing nor alarming as the couplings are not directly measurable quantities. Observable quantities (up to truncation artifacts) only contain combinations of couplings that do not depend on the choice of basis in theory space any more.

While the exact expressions for the functions rhsF and rhsF* is basis dependent, we can, however, discuss the general form of the result, i. e. how these functions structurally depend on the couplings. This is possible as the generic form of all prefactors of the $\bar{\omega}^2$-expressions corresponding to the same parity is equal. In any basis chosen the functions rhsF and rhsF* are linear combinations of these prefactors and, thus, of the same form.

We find that neither rhsF nor rhsF* depends on the Newton constant g, and their functional form in terms of the Immirzi parameter is very simple. This can be understood as $\breve{\Gamma}^{\text{quad}}$ contains the Newton constant only as a global prefactor that drops out in the expansion of the logarithm and is only present in the $\bar{\omega}$-independent determinant i. e. in rhsΛ. On the other hand $\breve{\Gamma}^{\text{quad}}$ contains a factor γ^{-1} in every parity-odd element, leading to a simple γ-dependence of its inverse. We therefore expect, that the parity-even prefactor rhsF only contains even powers and, in contrast, rhsF* only odd powers of γ. The dependence on the cosmological constant is, however, very involved and comprises a complicated dependence on the gauge fixing parameters as well, whose details depend on the basis chosen. Explicitly the coefficient functions are of the form

$$\begin{aligned} \text{rhsF}(\lambda, \gamma) &= \frac{1}{\gamma^2}\frac{P_8(\lambda)}{N(\lambda)} + \frac{P_{10}(\lambda)}{N(\lambda)}\,, \\ \text{rhsF}^*(\lambda, \gamma) &= \frac{1}{\gamma}\frac{P_9(\lambda)}{N(\lambda)}\,. \end{aligned} \tag{5.75}$$

Here, $N(\lambda)$ is a common denominator, which is a polynomial in λ of degree 10 given by

$$\begin{aligned} N(\lambda) = (\lambda-1)^2(\alpha'_{\text{L}}\lambda+2)^2\big(3\alpha_{\text{D}}\lambda^2 + (2\beta_{\text{D}}^2+\beta_{\text{D}}-1)\lambda + \beta_{\text{D}}^2 + 2\beta_{\text{D}} + 1\big)\cdot \\ \cdot\big(2\alpha_{\text{D}}\alpha'_{\text{L}}\lambda^2 + 4\alpha_{\text{D}}\lambda - 1\big)^2\big(1+\beta_{\text{D}}\big)^2 \end{aligned} \tag{5.76}$$

and turns out basis independent. The functions $P_n(\lambda)$ are polynomials in λ of degree n. Their explicit form is basis dependent and each of them fills several pages as long as the gauge parameters are not chosen to specific values.

The coefficient function rhsΛ is basis independent. It is given by

$$\begin{aligned}\text{rhs}\Lambda(\lambda,\gamma,g) = -\frac{1}{32\pi^2}\bigg[&\ln\Big(\frac{\gamma^2-1}{\gamma^2}\Big)^{24} + \ln\big[(\lambda-1)^{10}(2+\alpha_{\rm L}'\lambda)^6\big] \\ &+ \ln\big[(3\alpha_{\rm D}\lambda^2+(2\beta_{\rm D}^2+\beta_{\rm D}-1)\lambda+\beta_{\rm D}^2+2\beta_{\rm D}+1)^2(2\alpha_{\rm D}\alpha_{\rm L}'\lambda^2+4\alpha_{\rm D}\lambda-1)^6\big] \\ &- \ln\big[g^{68}\pi^{68}M^{160}\mu^{32}(1+\beta_{\rm D})^4\big] - \ln\mathcal{N}\bigg] \end{aligned} \tag{5.77}$$

Here, μ and M are dimensionless mass parameters: $\mu = \bar{\mu}/k$ stems from the rescaling of the fluctuation fields performed earlier and $M = \bar{M}/k$ is a mass parameter, that has to be introduced in order to render the argument of the logarithm dimensionless. As both parameters only occur in the combination μM^5 we can substitute them by a single parameter $m = M = \mu$ in the following. $\mathcal{N}$ is a purely numerical factor with $\ln\mathcal{N} \approx 167.74$.

Note that in the above expression for rhsΛ the limit of $(\alpha_{\rm L}', \alpha_{\rm D}) \to 0$ is well-defined as no argument of the logarithm vanishes completely. If we, however, decompose $\text{rhs}\Lambda = \text{rhs}\Lambda_{\rm grav} + \text{rhs}\Lambda_{\rm gh}$ into graviton and ghost contributions we find that both parts contain logarithmic terms that diverge in this limit, but cancel each other in the sum. Here we see explicitly that this cancelation of divergences is due to the incorporation of the gauge parameters into the gauge conditions; otherwise the ghost contributions would have been $(\alpha_{\rm L}', \alpha_{\rm D})$-independent and the limit of the $(\alpha_{\rm L}', \alpha_{\rm D}) = (0,0)$-gauge would not be well defined.

Independent of the exact expressions for the coefficients in (5.75) we can derive the form of the resulting β-functions for the couplings (λ, γ, g) by equating the coefficients in (5.71) and (5.74), leading to

$$\begin{aligned} \beta_g &= \partial_t g = g\big[2 + 16\pi g\ \text{rhsF}(\lambda,\gamma)\big] \\ \beta_\gamma &= \partial_t \gamma = -16\pi g\gamma\big[\gamma\ \text{rhsF}^*(\lambda,\gamma) +\ \text{rhsF}(\lambda,\gamma)\big] \\ \beta_\lambda &= \partial_t\lambda = 16\pi g\lambda\ \text{rhsF}(\lambda,\gamma) + 8\pi g\ \text{rhs}\Lambda(\lambda,\gamma,g) - 2\lambda \end{aligned} \tag{5.78}$$

This system of β-functions states the main result of this section. We have studied the resulting RG flow for various gauge choices and projection schemes that will be introduced in the next section. The properties of the flow are discussed in detail in Section 5.4.

5.3 Projection schemes in theory space

5.3.1 Beta-functions and the choice of basis

It is clear from the outset that β-functions generically depend on the basis chosen in theory space, since couplings serve as coordinates in theory space and the β-functions obtain as their scale derivatives. Thus, β-functions themselves and the running couplings they give rise to can not be considered physical quantities. All observable quantities therefore must depend on the couplings in such a way that their basis dependence drops out.

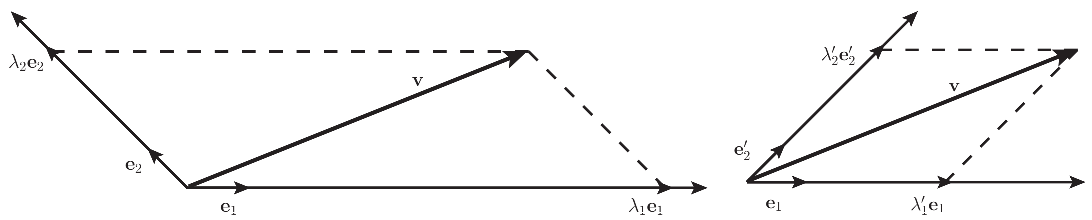


Figure 5.1. Illustrative example: The vector $\mathbf{v}$ is decomposed w. r. t. the two bases $(\mathbf{e}_1, \mathbf{e}_2)$ and $(\mathbf{e}_1, \mathbf{e}_2')$. Although the basis vector $\mathbf{e}_1$ does not change under the basis transformation, the corresponding coordinate $\lambda_1 \to \lambda_1'$ transforms non-trivially. This is because the spaces spanned by $\mathbf{e_2}$ and $\mathbf{e_2'}$ that form the kernel of the projection do not coincide.

A more subtle consequence of the basis dependence is that β-functions of a given invariant may transform non-trivially even if we perform a change of basis involving only the other basis invariants.

In a general vector space the coordinate of a fixed basis element only stays constant if the space spanned by the other basis elements is invariant under the basis transformation (cf. Fig. 5.1). Thus, in order to fix a coordinate of some vector w. r. t. a given basis element we also have to specify the space spanned by all other basis elements. This corresponds to the definition of a projection operator onto the basis element: Its kernel defines the space spanned by the other basis elements while its range is spanned by the element we want to project on. Note that only if the vector space is equipped with a scalar product, orthogonal projections can be defined, that allow the construction of the kernel from the range of the projector. Thus, for the definition of a projection scheme in a vector space lacking the notion of orthogonality, like theory space, we not only need to specify the invariants we want to project on, but also those which should be projected out.

If we use a systematic expansion of the invariants in theory space, as e.g. the derivative expansion, the β-functions of couplings corresponding to invariants up to a given order will not change any more once all basis invariants up to that order are fixed. From the above general considerations this is easily understood since in this case the kernel of the projection scheme used is implicitly defined as the space of higher derivative invariants. As this kernel does not change no matter which invariants are used as its basis, the β-functions of the lower derivative couplings are defined unambiguously once a basis in the range of the projection scheme has been chosen.

Within a subspace of a fixed order of derivatives, however, we have to define the kernel of the projection scheme explicitly before the β-function of a single coupling can be determined.

This effect can be illustrated by a well-known example from metric gravity: In $d \neq 4$ we find 3 independent field monomials ($R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, $R_{\mu\nu}R^{\mu\nu}$, R^2) in the curvature-squared subspace, that may serve as its basis. Even if we only want to compute the β-function of the R^2 coupling, we have to define the space spanned by the other two basis elements and thus give some information about the choice of basis in the 3-dimensional subspace before it can be determined.

Note that a projection scheme can also be defined implicitly by using a specific background spacetime: In [LR02] a truncation of metric gravity has been considered that only includes the R^2 term and none of the other curvature-squared invariants. In order to determine the β-function of its coupling a maximally symmetric background spacetime has been used with

$$R_{\mu\nu} = \frac{1}{d} g_{\mu\nu} R \quad \text{and} \quad R_{\mu\nu\rho\sigma} = \frac{1}{d(d-1)} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) R \, . \tag{5.79}$$

Using these relations one can see immediately that the combinations

$$R_{\mu\nu}R^{\mu\nu} - \frac{1}{d}R^2 \quad \text{and} \quad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{2}{d(d-1)}R^2 \tag{5.80}$$

of the curvature-squared invariants (and all linear combinations of them) are mapped to zero once we employ a maximally symmetric spacetime. Thus they define the kernel of the projection scheme that is implicitly defined by the choice of the maximally symmetric background spacetime.

In $d = 4$ the three curvature squared monomials from above can be combined to a topological term, the Euler invariant χ_{E} of the manifold using the Gauss-Bonnet theorem (see e. g. [Ort07]). Thus, on spacetimes with $\chi_{\mathrm{E}} = 0$ the three invariants form an overcomplete basis of the curvature squared subspace. Clearly, the β-functions will depend on our choice which of the three monomials is considered as linearly dependent on the other two and thus is excluded from the basis. Reducing an overcomplete basis to a complete one hence also amounts to the definition of a projection scheme and can been seen as an example of the projection ambiguity of the β-functions, albeit a more trivial one.

In the case of tetrad gravity, we find an analogous ambiguity: We expanded the right hand side of the flow equation up to second order in the spin connection and from this expression we want to extract the coefficient of the terms that correspond to the Immirzi term. On any manifold of the same topology as the flat torus we work on, the value of the Nieh-Yan invariant is zero and therefore $\int e_a \wedge e_b \wedge F^{ab} = \int T^a \wedge T_a$. Hence, the Immirzi term lies within the subspace of parity-odd torsion squared monomials. In Appendix F we show that of the four different contractions of two torsion tensors with the ε-symbol $(T_1^{2(-)}, \cdots, T_4^{2(-)})$ only two are linearly independent. Thus, the subspace is two dimensional and in order to determine the prefactor of the Immirzi term, $\mathrm{rhsF}^*(\lambda, \gamma)$, we have to first choose the second basis monomial in this space.

Up this point, all observations apply even for an ideal untruncated calculation and are thus independent of the details of our calculation. Working with the WH-like flow equation (5.14), however, requires us to use constant background fields $\{\bar{e}, \bar{\omega}\}$. For this reason a second ambiguity of the above kind arises for the curvature term: As from the field strength tensor evaluated on a constant background only the $\bar{\omega}^2$-part remains $(\bar{F}^{ab}{}_{\mu\nu} = \bar{\omega}^a{}_{c[\mu}\bar{\omega}^{cb}{}_{\nu]})$, the curvature term in the action becomes indistinguishable from a certain combination of parity-even torsion squared monomials $(T_1^{2(+)}, T_2^{2(+)}, T_3^{2(+)})$ evaluated on the same background. This space is three dimensional (cf. Appendix F) such that we have to first specify the space spanned by the other two basis vectors before we can extract the corresponding prefactor $\mathrm{rhsF}(\lambda, \gamma)$.

In the remaining two subsections we describe which choices of basis have been considered in out study and how the inherent ambiguity can be exploited to find an optimized gauge condition. In the discussion we will make use of the decomposition of the torsion tensor into irreducible components (cf. [BH11, Sha02]) according to

$$T^\lambda{}_{\mu\nu} = \frac{1}{3}\big(\delta^\lambda{}_\nu T_\mu - \delta^\lambda{}_\mu T_\nu\big) + \frac{1}{6e}\varepsilon^\lambda{}_{\mu\nu\sigma}S^\sigma + q^\lambda{}_{\mu\nu} \tag{5.81}$$

with ${q^\lambda}_{\mu\lambda} = 0$, ${q^\lambda}_{\mu\nu} = -{q^\lambda}_{\nu\mu}$ and $\varepsilon^{\mu\nu\rho\sigma} q_{\nu\rho\sigma} = 0$. The details of this decomposition are further explained in Appendix F.1.

5.3.2 Subspace of parity-odd T^2-invariants

As already mentioned above, the subspace of parity-odd torsion squared monomials turns out two dimensional and with the help of the decomposition (5.81) a quite natural choice of basis of this space arises, which is given by the two monomials

$$I_4 = S_\mu T^\mu \quad \text{and} \quad I_5 = e^{-1} \varepsilon_{\alpha\beta\gamma\delta} q^{\alpha\beta\mu} {q^{\gamma\delta}}_\mu \,. \tag{5.82}$$

If one wanted to carry out a more general calculation including all the subspace in the truncation considered, one would probably choose to work with this basis, as the choice I_4, I_5 seems less arbitrary than any choice of 2 invariants among the $T_i^{2(-)}$, $i = 1, \cdots, 4$.

However, as our truncation only contains the Immirzi term, which is given as the linear combination

$$\int e_a \wedge e_b \wedge F^{ab} = \frac{1}{4} \int \mathrm{d}^4 x \, e \, T_1^{2(-)} = \int \mathrm{d}^4 x \, e \left(-\frac{1}{3} I_4 + I_5 \right) , \tag{5.83}$$

we are bound to choose this combination as one of our basis elements. For the second basis element we have a free choice among all other linear independent combinations of I_4 and I_5.

In the following we will motivate how this freedom can be used in order to optimize the truncation and the choice of gauge parameters. The very idea of choosing a good truncation is that all RG trajectories of the exact, untruncated flow that start in the subspace defined by the truncation invariants lie almost completely within this subspace of theory space anyway, such that the truncation captures all essential features of the flow. This requires that the components of the exact RG flow causing the departure from the subspace are considered in some sense “small”. A truncation becomes exact only when the corresponding subspace in theory space is mapped onto itself under RG transformations.

Usually it is not possible to judge the reliability of a given truncation by this criterion without performing a completely independent second RG analysis considering a more general truncation. This is because the right hand side of the flow equation is usually directly projected onto the invariants contained in the truncation such that all information about other invariants generated by the flow is lost at that point.

In our calculation we have projected the right hand side onto the space of torsion squared invariants and by doing that, we have computed how the RG flow starting within the Holst truncation “leaks” from the truncation into this larger part of theory space. If the Holst truncation was an exact truncation, the right hand side would contain the invariants I_4 and I_5 only in the linear combination corresponding to the Immirzi term. In this case the choice of a second basis invariant would be obsolete, as the coordinate of the Immirzi invariant would not depend on it, while this second coordinate stays zero.

This consideration suggests that in an inexact truncation the ratio of the two coordinates may serve as an indicator for the reliability of the truncation. Pictorially speaking, this ratio of the coordinates, which generically depends on all couplings, is related to the angle at which the RG flow departs from the subspace of the truncation at this point in theory space.

Unfortunately one would need to define a scalar product in theory space in order to give this illustrative interpretation of the ratio a quantitative meaning: As long as we cannot normalize the basis elements, we can rescale one of them resulting in an inverse scaling of the respective coordinate which alters the ratio of the two coordinates and as long as we cannot choose the basis orthogonally, the ratio will also depend on the angle between the basis elements.

Thus, with no scalar product at hand, we can only draw qualitative conclusions from the ratio of coordinates. These conclusions will implicitly assume that if we were to introduce a scalar product in theory space, the norm of the most natural basis elements $\{I_4, I_5\}$ should be of the same order of magnitude and they should not turn out practically collinear. Taken together, we assume that the elements of theory space parametrized by φ according to

$$\mathbf{v}(\varphi) = \sin(\varphi) I_4 + \cos(\varphi) I_5 \tag{5.84}$$

should all have a norm within the same order of magnitude. If we choose the monomial $\mathbf{v}(\varphi)$, besides the Immirzi term, as a second basis element, we can discuss a continuous class of bases which we will denote by $\mathcal{B}^{(-)}_{\mathbf{v}(\varphi)}$.

Besides this family of bases we introduce a discrete set of bases according to

$$\mathcal{B}_i^{(-)} := \left\{ \frac{1}{4} \int \mathrm{d}^4x\, e\, T_1^{2(-)}, \frac{1}{2} \int \mathrm{d}^4x\, e\, T_i^{2(-)} \right\} \quad \text{with} \quad i = 2, 3, 4 \tag{5.85}$$

and

$$\mathcal{B}_{I_5}^{(-)} := \left\{ \frac{1}{4} \int \mathrm{d}^4x\, e\, T_1^{2(-)}, \frac{3}{2} \int \mathrm{d}^4x\, e\, I_5 \right\}. \tag{5.86}$$

Here, $T_i^{2(-)}$ denote the four different parity-odd torsion squared monomials, that result from contracting two torsion tensors with the ε-symbol. Their explicit form is given in Appendix F.

As $\mathbf{v}(\varphi)$ describes a rotation in the (I_4, I_5)-plane, the basis $\mathcal{B}_{\mathbf{v}(\varphi)}^{(-)}$ is, up to a rescaling of the second basis element, equivalent to each of the discrete bases for a certain value of φ. Explicitly these values are given by $\tan(\varphi) = \{\infty, \frac{1}{6}, \frac{2}{3}, 0\}$ for the bases $\{\mathcal{B}_i^{(-)}, \mathcal{B}_b^{(-)}\}$.

With these bases defined we can discuss the coordinate ratios as a measure for the reliability of our truncation explicitly. To this end we decompose the result for the RHS of the flow equation (5.74) first into its parity-even and -odd parts

$$\mathrm{rhs} = \mathrm{rhs}^{(+)} + \mathrm{rhs}^{(-)} \tag{5.87}$$

and the parity-odd part $\mathrm{rhs}^{(-)}$ is further decomposed into a linear combination of the basis invariants. In the basis $\mathcal{B}_i^{(-)}$ we introduce the notation

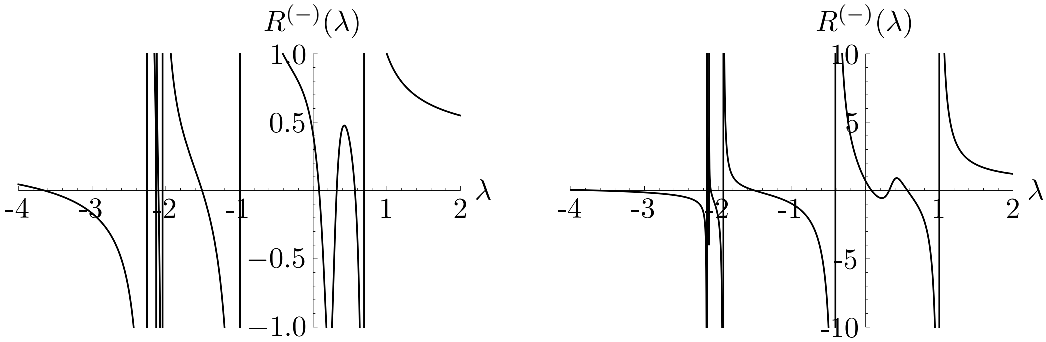
$$\mathrm{rhs}^{(-)} = \big(\mathrm{rhs^*F}\big)_{\mathcal{B}_i^{(-)}} \cdot \frac{1}{4} \int \mathrm{d}^4x\, e\, T_1^{2(-)} + \mathrm{rhsT}_i^{2(-)} \cdot \frac{1}{2} \int \mathrm{d}^4x\, e\, T_i^{2(-)}\,, \tag{5.88}$$

for $i = 2, 3, 4$.

Thus, $\big(\mathrm{rhs^*F}\big)_{\mathcal{B}_i^{(-)}}$ and $\mathrm{rhsT}_i^{2(-)}$ play the role of coordinate functions in the basis $\mathcal{B}_i^{(-)}$. From the discussion of the general form of the RHS (cf. eq. (5.75)) we know that these are functions of the couplings λ and γ, but that the γ-dependence drops out when we compute the ratio of both functions. Thus we are led to a function of the form

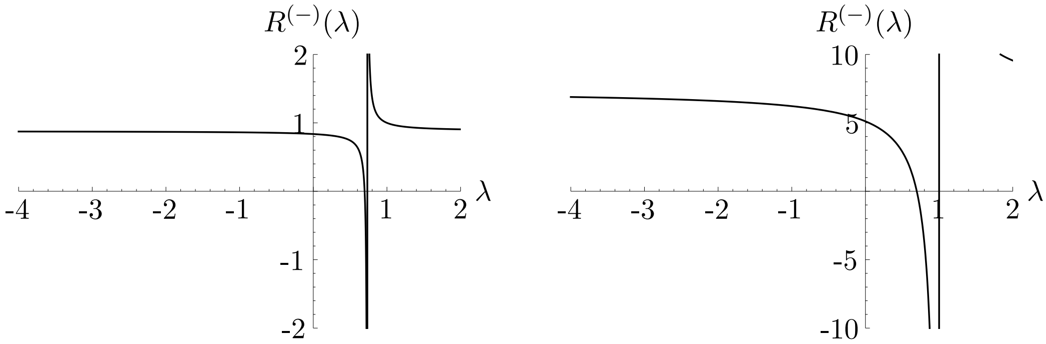
$$R^{(-)}(\lambda) = \frac{\big(\mathrm{rhs^*F}\big)_{\mathcal{B}_i^{(-)}}}{\mathrm{rhsT}_i^{2(-)}} = \frac{P_9^{\mathcal{B}_i^{(-)}}(\lambda)}{P_9^{T_i^{2(-)}}(\lambda)}\,. \tag{5.89}$$

As it is the ratio of two polynomials of degree 9 and a polynomial of degree n generally has n zeroes, not all of which have to be real, we expect these functions to exhibit up to 9 zeros and 9 poles on the real axis.



(a) Basis $\mathcal{B}_2^{(-)}$ (b) Basis $\mathcal{B}_3^{(-)}$

Figure 5.2. Typical plots of the coordinate ratio $R^{(-)}(\lambda)$ for two different bases as a function of the cosmological constant λ, shown here for the $(\alpha_\text{D}, \alpha'_\text{L}, \beta_\text{D}) = (1, 1, 0)$ gauge.



(a) Basis $\mathcal{B}_2^{(-)}$ (b) Basis $\mathcal{B}_3^{(-)}$

Figure 5.3. The coordinate ratio $R^{(-)}(\lambda)$ for the same bases as in Fig. 5.2 as a function of λ in the $(\alpha_\text{D}, \alpha'_\text{L}, \beta_\text{D}) = (0, 0, 0)$ gauge.

In Fig. 5.2 we have plotted the coordinate ratio as a function of λ for the bases $\mathcal{B}_2^{(-)}$ and $\mathcal{B}_3^{(-)}$. From the figure we observe that many of these poles and zeros generically occur in the most interesting part of the phase diagram, namely at small (positive or negative) cosmological constant λ. For that reason the ratio function is wildly fluctuating at small λ indicating that the reliability of the truncation strongly depends on the value of λ. Taking this into account it does not make sense to simply choose one basis and further discuss specific properties of the resulting RG flow in different gauges, as the reliability of these results would be highly questionable.

Instead we should try to find a specific gauge that improves the situation. As most of the zeros of the polynomials $P_9(\lambda)$ depend on the gauge parameters, we can try to move them such that they are situated outside the region of small λ. It turns out that almost all movable zeros tend to larger values of λ for small values of $\alpha'_\text{L}, \alpha_\text{D}$. In the limit $\alpha'_{\mathrm{L}}, \alpha_{\mathrm{D}} \to 0$ the polynomials simplify: Because the prefactors of all higher orders in λ vanish, only polynomials of degree two remain. Stated differently, all but two zeros are removed by moving them to infinity. If we, in addition, send $\beta_{\mathrm{D}} \to 0$ the ratio of the polynomials further simplifies as in this limit a common factor $(1-\lambda)$ of numerator and denominator can be canceled, such that we are left with a single pole.

In Fig. 5.3 the coordinate ratios are shown for the same bases as in Fig. 5.2 but for the $(\alpha_{\mathrm{D}}, \alpha'_{\mathrm{L}}, \beta_{\mathrm{D}}) = (0,0,0)$-gauge. We observe a tremendous simplification in the graphs and find that the ratio now is a constant function except for the vicinity of the one remaining pole, which, unfortunately, cannot be removed within our gauge freedom. Moreover, we find that the asymptotic value and the width of the pole remain basis depend quantities (note the different scales of the two plots). Nonetheless, the comparison between Figs. 5.2 and 5.3 shows strikingly that the $(0,0,0)$-gauge should be preferred compared to other gauges in any basis. Besides the fact that the $(0,0,0)$-point in gauge parameter space can be argued to be a fixed point of the RG flow in this space [EHW96], we find here a new argument in favor of this gauge, namely that *it optimizes the consistency of a given truncation.*

In a last step of basis optimization we consider the asymptotic value of the ratio functions $R^{(-)}_{\infty} = \lim_{\lambda\to\pm\infty} R^{(-)}(\lambda)$. As we can freely scale each of the corresponding basis functionals, which leads to an inverse scaling of the ratio, it is difficult to compare the different discrete bases. For that reason, we now turn over to the continuous set of bases $\mathcal{B}^{(-)}_{\mathbf{v}(\varphi)}$ and discuss the asymptotic value as a function of its parameter φ. The corresponding graph is shown in Fig. 5.4.

The resulting function is of the expected form: If we decompose a fixed vector $\mathbf{v}$ w. r. t. different bases whose first element is held fixed and the second rotates, we should find a 2π-periodic function for the coordinate ratio. It has poles at those points where the second basis element points into the direction of the vector $\mathbf{v}$, such that the first coordinate (the denominator) vanishes. The minima of the absolute value of the function occur when the second basis element is chosen orthogonal to the vector $\mathbf{v}$ we wish to decompose. At the angle at which both basis elements are collinear, both coordinates diverge in opposite direction to $\pm\infty$ resulting in a ratio of -1; this angle corresponds to the point labeled by $T_1^{2(-)}$ in Fig. 5.4.

Under the assumption that the basis elements $\mathbf{v}(\varphi)$ are of approximately the same norm all these observations also apply to the graph in Fig. 5.4. Here we have also marked the points that correspond to the directions of the discrete torsion squared

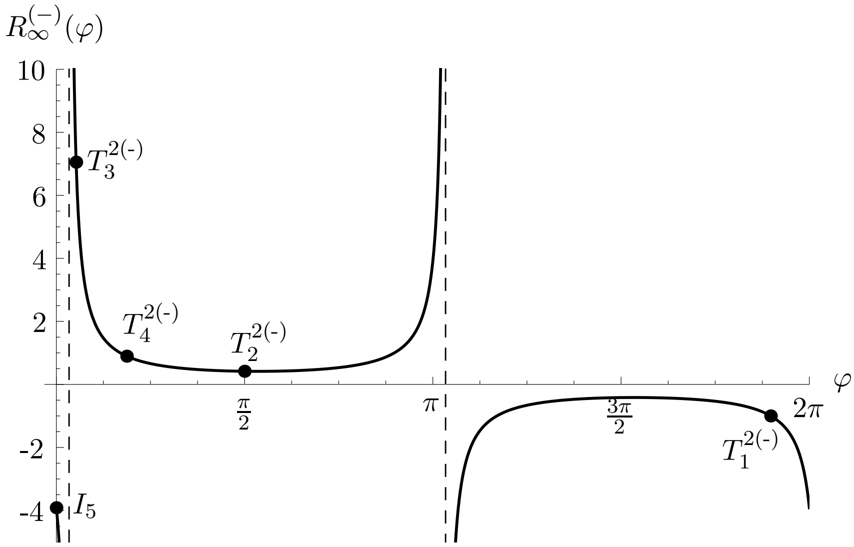


Figure 5.4. Asymptotic value $R_\infty^{(-)} = \lim_{\lambda\to\pm\infty} R^{(-)}(\lambda)$ of the coordinate ratio for the continuous set of bases $\mathcal{B}^{(-)}_{\mathbf{v}(\varphi)}$ in $(\alpha_\mathrm{D}, \alpha'_\mathrm{L}, \beta_\mathrm{D}) = (0,0,0)$ gauge as a function of the parameter φ.

monomials $T_i^{2(-)}$ and I_5 as second basis elements. We find that the monomial $T_3^{2(-)}$ is almost collinear with our result for right hand side of the flow equation, as it lies very close to the pole of the function. On the other hand, $T_2^{2(-)}$, lies in the vicinity of the minimum of the function, indicating that it is "almost orthogonal" to our right hand side expression. For this reason, we have chosen the basis $\mathcal{B}_2^{(-)}$ in the parity-odd subspace for the further detailed analysis of the resulting RG flow.

As a last remark let us mention that the width of the pole, given by the angle difference between the point $T_1^{2(-)}$ and the pole, may serve as a measure for the reliability of the truncation. In our case we find that it is of order $\pi/8$ and thus not particularly small, indicating that an extension of the Holst truncation to the full torsion squared subspace could be sensible in order to improve the stability of the results.

5.3.3 Subspace of parity-even T^2-invariants

In the subspace of parity-even T^2-invariants the situation is more complicated as the space is 3-dimensional. Let us stress again, that the basis ambiguity here is only due to the choice of constant background fields, which, however is inevitable if one wants to make use of the WH-like flow equation. In an RG analysis that uses a general background the curvature term could be identified unambiguously from the terms that are first order in derivatives and the spin connection. Evaluated on a constant background, however, we cannot distinguish the curvature term from a certain combination of torsion squared invariants. This kind of approximation is well known and often used to make the most complicated computations feasible; choosing e. g. maximally symmetric spacetimes as a background in metric gravity results in an indistinguishability of the curvature squared monomials [LR02]. In contrast to this example from metric gravity, here the choice of constant background does not completely fix the projection scheme that maps onto our truncation space, but it only maps the curvature term into the torsion squared subspace. Thus we pick up an additional basis ambiguity here, that has to be fixed by completing the projection scheme.

As for the parity-odd case the basis monomials constructed from the irreducible torsion components,

$$I_1 = T_\mu T^\mu, \quad I_2 = S_\mu S^\mu \quad \text{and} \quad I_3 = q^{\mu\nu\rho} q_{\mu\nu\rho}, \tag{5.90}$$

appear as the most natural choice for a basis in the parity-even subspace. Alternatively, one could choose the three torsion squares $T_i^{2(+)}$, $i = 1, 2, 3$, defined in Appendix F.2. However, if we evaluate the curvature term on the constant background $(\bar{e}, \bar{\omega})$ we find:

$$\begin{aligned} \bar{e}_a \wedge \bar{e}_b \wedge * \bar{F}^{ab} &= \left(\bar{\omega}_{abc}\bar{\omega}^{acb} - \bar{\omega}_{ab}{}^{a}\bar{\omega}_{c}{}^{bc}\right)\bar{e}\,\mathrm{d}^4x \\ &= \frac{1}{4}\left(\bar{T}_1^{2(+)} + 2\bar{T}_2^{2(+)} - 4\bar{T}_3^{2(+)}\right)\bar{e}\,\mathrm{d}^4x \\ &= \left(-\frac{2}{3}\bar{I}_1 - \frac{1}{24}\bar{I}_2 + \frac{1}{2}\bar{I}_3\right)\bar{e}\,\mathrm{d}^4x\,. \end{aligned} \tag{5.91}$$

Thus, it does not coincide with any of the basis vectors but is a combination of all three basis elements in both natural bases. As this combination has to be held fixed as the first basis element that we want to project on, arbitrary choices of the other two basis invariants among the $T_i^{2(+)}$ and the I_i, but also among the spin connection squares $\bar{\omega}_{abc}\bar{\omega}^{abc}, \bar{\omega}_{abc}\bar{\omega}^{acb}, \bar{\omega}_{ab}{}^{a}\bar{\omega}_{c}{}^{bc}$ are equally plausible.

Unfortunately also the discussion of the coordinate ratios for the different bases does not lead to a unique preference of one specific basis: Although also in this subspace an enormous simplification of the ratio functions can be reached by using the (0,0,0)-

gauge, the structure of the right hand side expression (5.75) causes the ratios to be functions of the two couplings (λ, γ), which complicates a systematic analysis.

Since we were not able to single out one specific basis as being optimal w. r. t. the consistency of the truncation, we analyzed the RG flow for four different, more or less arbitrarily chosen bases, which are defined by their elements according to

$$\begin{aligned}
\mathcal{B}_1^{(+)} &= \Big\{ \int \bar{e}(\bar{\omega}_{abc}\bar{\omega}^{acb} - \bar{\omega}_{ab}{}^{a}\bar{\omega}^{cb}{}_{c}), \int \bar{e}(\bar{\omega}_{abc}\bar{\omega}^{acb} + \bar{\omega}_{ab}{}^{a}\bar{\omega}^{cb}{}_{c}), 2\int \bar{e}\,\bar{\omega}_{abc}\bar{\omega}^{abc} \Big\},\\
\mathcal{B}_2^{(+)} &= \Big\{ \int \bar{e}(\bar{\omega}_{abc}\bar{\omega}^{acb} - \bar{\omega}_{ab}{}^{a}\bar{\omega}^{cb}{}_{c}), 2\int \bar{e}\,\bar{\omega}_{ab}{}^{a}\bar{\omega}^{cb}{}_{c}, 2\int \bar{e}\,\bar{\omega}_{abc}\bar{\omega}^{abc} \Big\},\\
\mathcal{B}_3^{(+)} &= \Big\{ \int \bar{e}(\bar{\omega}_{abc}\bar{\omega}^{acb} - \bar{\omega}_{ab}{}^{a}\bar{\omega}^{cb}{}_{c}), 2\int \bar{e}\,\bar{\omega}_{abc}\bar{\omega}^{acb}, 2\int \bar{e}\,\bar{\omega}_{abc}\bar{\omega}^{abc} \Big\},\\
\mathcal{B}_4^{(+)} &= \Big\{ \int \bar{e}(\bar{\omega}_{abc}\bar{\omega}^{acb} - \bar{\omega}_{ab}{}^{a}\bar{\omega}^{cb}{}_{c}), -\frac{1}{4}\int \bar{e}\, I_2,\ \ \frac{3}{2}\int \bar{e}\, I_3 \Big\}\,,
\end{aligned} \tag{5.92}$$

where $\int = \int \mathrm{d}^4x$. The prefactors of the different invariants are chosen such, that the bases are related to each other by transformation matrices of unit determinant.

With these bases specified we are now able to discuss the properties of the RG flow of the Holst truncation in the next section.

5.4 Analysis of the RG flow

After having introduced the different projection schemes we are going to analyze, we are now able to write down the explicit form of the β-functions for the three couplings of the Holst action. From (5.78), (5.75), and (5.77) we find that for all the bases the β-functions are of the general form

$$\beta_g(\lambda,\gamma,g) = g\,[2 + \eta_N(\lambda,\gamma,g)]\,, \tag{5.93a}$$

$$\beta_\gamma(\lambda,\gamma,g) = -\frac{16\pi g\gamma}{N(\lambda)}\left[P_9(\lambda) + \frac{1}{\gamma^2}P_8(\lambda) + P_{10}(\lambda)\right], \tag{5.93b}$$

$$\begin{aligned}\beta_\lambda(\lambda,\gamma,g) = -2\lambda &+ \frac{16\pi g\lambda}{N(\lambda)}\left[\frac{1}{\gamma^2}P_8(\lambda) + P_{10}(\lambda)\right]\\
&- \frac{g}{4\pi}\left[12\ln\left(\frac{\gamma^2-1}{\gamma^2}\right)^2 + 5\ln(1-\lambda)^2 - 96\ln m^2 - 34\ln g^2 - \ln\mathcal{N}'\right],\end{aligned} \tag{5.93c}$$

with the anomalous dimension of Newton's constant

$$\eta_N(\lambda, \gamma, g) = \frac{16\pi g}{N(\lambda)} \left(\frac{1}{\gamma^2} P_8(\lambda) + P_{10}(\lambda) \right) . \tag{5.94}$$

Here, we have already specialized the logarithmic terms in β_λ to the case of the preferred $(0, 0, 0)$-gauge, as we will restrict the discussion to this case. $\mathcal{N}'$ is a purely numerical quantity given by

$$\ln \mathcal{N}' = \ln \mathcal{N} + 68 \ln \pi - 6 \ln 2 \approx 241.42 \, . \tag{5.95}$$

In the (0,0,0)-gauge also the $P_n(\lambda)$ polynomials simplify and their degree is then smaller than n, but for notational consistency we will stick to this notation. In addition it reminds us of true complexity of the β-functions in terms of λ for a general choice of gauge.

As we have discussed before, the virtue of choosing $\beta_\mathrm{D} = 0$ in addition to $(\alpha_\mathrm{D}, \alpha'_\mathrm{L}) = (0, 0)$ lies in the fact, that the polynomials $P_n(\lambda)$ then contain a factor of $(1-\lambda)$. In the β-functions every polynomial $P_n(\lambda)$ is divided by the denominator $N(\lambda)$ such that this factor is canceled. In the $(0, 0, 0)$-gauge the new common denominator is thus given by $N(\lambda)/(1 - \lambda)$ and the numerators are composed of the polynomials $P_n(\lambda)/(1 - \lambda)$. Since the following discussion is restricted to this preferred gauge, we will present only the explicit expressions of these “rescaled” quantities.

The denominator $N(\lambda)$ simplifies considerably in the $(0, 0, 0)$-gauge and takes on the explicit form

$$\frac{N(\lambda)}{1-\lambda} \overset{(0,0,0)}{=} 4(\lambda - 1)^2 \, . \tag{5.96}$$

For the rest of this section we will consider four distinct bases in the subspace of torsion-squared invariants. As we have shown in the last section the optimal basis in the parity-odd sector is $\mathcal{B}_2^{(-)}$, while for the parity-even part we have introduced the four bases $\mathcal{B}^{(+)}_{\{1,2,3,4\}}$. As a shorthand notation for the bases of the complete torsion squared subspace we will denote to the combinations of these bases by $\mathcal{B}_i = (\mathcal{B}_2^{(-)}, \mathcal{B}_i^{(+)})$, $i = 1, 2, 3, 4$.

For each of these bases $\mathcal{B}_i$ the polynomials $P_n(\lambda)$ take on different explicit forms. As the polynomial $P_9(\lambda)$ only depends on the basis in the parity-odd sector, it is equal for all four bases considered. In the $(0,0,0)$-gauge it assumes the simple linear form

$$\frac{P_9(\lambda)}{1-\lambda} = -\frac{5}{64\pi^2}(15\lambda - 11)\,. \tag{5.97}$$

The polynomials $P_8(\lambda)$ and $P_{10}(\lambda)$ take on different forms in the different bases. Their explicit form for the (0,0,0)-gauge is given in Table 5.1. We observe that basis $\mathcal{B}_1$ plays a special role as the polynomial P_8 vanishes in this case. We will investigate the deep implications of this fact during the rest of this section.

	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$
$\frac{P_8(\lambda)}{1-\lambda}$	0	$\frac{5}{96\pi^2}$	$-\frac{5}{96\pi^2}$	$-\frac{5}{32\pi^2}$
$\frac{P_{10}(\lambda)}{1-\lambda}$	$\frac{55\lambda^2-42\lambda-18}{128\pi^2}$	$-\frac{43\lambda^2-186\lambda+178}{384\pi^2}$	$\frac{373\lambda^2-438\lambda+70}{384\pi^2}$	$\frac{(\lambda-1)(52\lambda-25)}{32\pi^2}$

Table 5.1. Explicit forms of the polynomials $P_8(\lambda)$ and $P_{10}(\lambda)$ in $(\alpha_{\text{D}}, \alpha'_{\text{L}}, \beta_{\text{D}}) = (0,0,0)$-gauge for the four bases $\mathcal{B}_i$, which we will explore in detail.

The analysis of the system of flow equations (5.93) in this section is organized as follows: In the first two subsections we will explore the RG flow in two two-dimensional truncations beginning with the (λ, g)-truncation for a fixed value of the Immirzi parameter γ followed by the analysis of the (γ, g)-system for a fixed cosmological constant λ. In the third subsection we discuss the full three dimensional RG flow of the Holst truncation, including its fixed point structure and its phase portrait. In all these subsections we will first compare the results for the four distinct bases $\mathcal{B}_i$, $i = 1, \cdots, 4$, before we evaluate the qualitative similarities with the analysis carried out in [DR]. We thereby keep in mind that the ultimate justification for the applicability of our new, structurally simplified WH-like flow equation can only lie in the accordance of its predictions with other—either exact or at least well approved approximate— RG flow equations. In a final subsection we will analyze the RG flow of chiral gravity for the first time, which in our formalism requires only a slight modification of the derivation for the general case.

5.4.1 The (λ, g)-subsystem

The system of flow equations we are going to discuss in this subsection is given by the two equations (5.93a) and (5.93c),

$$\begin{aligned} \beta_g(\lambda,\gamma,g) &= g\left[2+\eta_N(\lambda,\gamma,g)\right], \\ \beta_\lambda(\lambda,\gamma,g) &= -2\lambda + \frac{16\pi g\lambda}{N(\lambda)}\left[\frac{1}{\gamma^2}P_8(\lambda)+P_{10}(\lambda)\right] \\ &\quad - \frac{g}{4\pi}\left[12\ln\left(\frac{\gamma^2-1}{\gamma^2}\right)^2 + 5\ln(1-\lambda)^2 - 96\ln m^2 - 34\ln g^2 - \ln\mathcal{N}'\right]. \end{aligned} \tag{5.98}$$

Throughout this subsection we treat γ as an external parameter, that can be chosen to an arbitrary fixed value; in particular we will discuss how the RG flow depends on the value of γ. Before we analyze its fixed point structure let us first discuss the limitations of this truncation due to divergences inherent in the β-functions.

We find that the β-functions are not well defined for the values $\gamma = 0$ and $\gamma = \pm 1$. While the latter divergence was expected from the outset, as the action depends in this limit only on one chiral component of the spin connection, the reason for the former remains somewhat unclear. It should be noted that the quadratic divergence in both β-functions at $\gamma \to 0$ is basis dependent since it ceases to exist in basis $\mathcal{B}_1$ where $P_8 = 0$. Nevertheless a logarithmic divergence of β_λ at $\gamma \to 0$ still remains even in this basis. The limit $\gamma \to \pm\infty$ is, however, perfectly well defined in all bases. When we discuss properties of the RG flow that apply for any value of γ in the following, we implicitly exclude these pathological cases of $\gamma = 0, \pm 1$.

Within the phase portrait in all (λ, g)-planes we find a barrier of the flow at $\lambda = 1$ for any value of γ. As not only the logarithmic term in β_λ diverges on this line, but also the denominator $N(\lambda)$, the ratio β_λ/β_g stays finite in the limit $\lambda \to 1$. Hence, the RG flow simply stops at this line, similar to the situation at $\lambda = 0.5$ in metric gravity [Reu98, RS02]. In contrast to the metric case the singularity here is a pole of even order, such that the flow does not change direction at this line.

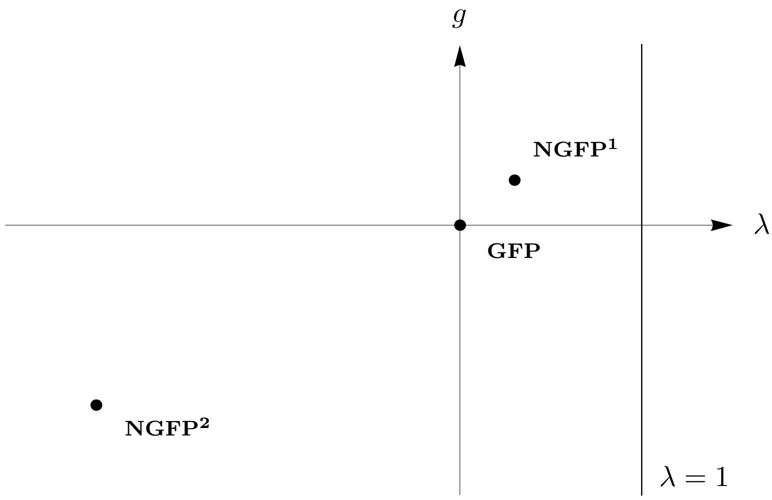


Figure 5.5. Sketch of the fixed point structure of the (λ, g)-system. Besides the position of the three fixed points, the barrier of the flow at $\lambda = 1$ is depicted.

Fixed point structure. As we will see shortly the (λ, g)-system exhibits in total three FPs, the Gaussian FP (**GFP**) and two non-Gaussian fixed points ($\mathbf{NGFP^1}$, $\mathbf{NGFP^2}$). Their approximate location in theory space is depicted in Fig. 5.5 in order to illustrate the situation, before the details of their properties are discussed.

The Gaussian fixed point. First we observe that the system (5.98) allows for a Gaussian fixed point at $(\lambda, g) = (0, 0)$. However, the critical exponents, defined as the negative eigenvalues of the stability matrix at the fixed point

$$\mathscr{B}\big|_{g=g^*,\lambda=\lambda^*} = \begin{pmatrix} \partial_g \beta_g & \partial_\lambda \beta_g \\ \partial_g \beta_\lambda & \partial_\lambda \beta_\lambda \end{pmatrix}\Bigg|_{g=g^*,\lambda=\lambda^*} \tag{5.99}$$

cannot be determined as the flow cannot be expanded at the GFP: Although β_λ is well defined at the GFP its partial derivative w. r. t. g is not. Thus, it is not possible to linearize the RG flow in the vicinity of the GFP in the usual way. Nonetheless it possible to approximate the system in this region by taking into account only the terms linear in the couplings and the logarithmic term that causes the divergence of the derivative at the GFP. The resulting approximate system of flow equations reads

$$\begin{aligned} \beta_g &= 2g \\ \beta_\lambda &= -2\lambda + \frac{17}{2\pi} g \ln g^2 - Cg \, , \end{aligned} \tag{5.100}$$

where $C = (4\pi)^{-1}[12\ln(1-1/\gamma^2)^2 - 96\ln m^2 - \ln\mathcal{N}']$ is a constant. It turns out feasible to integrate this system of β-functions, and its solution subject to the initial condition (λ_0, g_0) at the scale $k = k_0$ is given by

$$g(k) = g_0\left(\tfrac{k}{k_0}\right)^2$$
$$\lambda(k) = \lambda_0\left(\tfrac{k}{k_0}\right)^{-2} + \frac{g_0}{4}\Big(C + \tfrac{17}{2\pi}\big(1 - \ln g_0^2\big)\Big)\Big[\left(\tfrac{k}{k_0}\right)^{-2} - \left(\tfrac{k}{k_0}\right)^2\Big] + \tfrac{17}{2\pi}\, g_0\left(\tfrac{k}{k_0}\right)^2 \ln\left(\tfrac{k}{k_0}\right)\,. \tag{5.101}$$

Although the flow cannot be linearized in the usual sense, we observe that it is perfectly well-defined in the vicinity of the GFP, exhibiting non-polynomic terms, though. In metric gravity contributions of this form are known from the running bare action S_Λ that can be reconstructed from any asymptotically safe trajectory of the effective average action Γ_k [MR09]. We suppose that the logarithmic contributions here are rather typical for the new WH-like equation, as it is structurally similar to the Wegner-Houghton equation, being an RG equation for the running bare action (cf. Appendix B).

Non-Gaussian fixed points. Besides the Gaussian fixed point the system (5.98) generically allows for non-Gaussian fixed points as well. This is seen as follows: We can solve the condition $\beta_g(\lambda, g) = 0$ for

$$g^*(\lambda) = -\frac{1}{8\pi}\frac{N(\lambda)}{P_8/\gamma^2 + P_{10}(\lambda)}\,. \tag{5.102}$$

This solution is substituted into the second fixed point condition $\beta_\lambda(\lambda, g^*(\lambda)) = 0$. The zeros of this function of λ correspond to all non-Gaussian fixed points in the (λ, g)-plane. Explicitly this condition amounts to the solution of the following equation for λ:

$$4\lambda = -\frac{1}{4\pi}\, g^*(\lambda)\left[12\ln\Big[\frac{\gamma^2-1}{\gamma^2}\Big]^2 + 5\ln(\lambda-1)^2 - 96\ln m^2 - 34\ln g^*(\lambda)^2 - \ln\mathcal{N}'\right]. \tag{5.103}$$

From the asymptotic behavior of this equation, which is linear on the LHS while it is logarithmic on the RHS ($g^*(\lambda)$ tends to a constant value for large $|\lambda|$), we expect that there is at least one solution to equation (5.103). In addition, further solutions can be generated by the variation of $g^*(\lambda)$. Especially there will appear solutions in the vicinity of the poles of $g^*(\lambda)$. These latter solutions, however, were observed to give rise to fixed points with a tiny basin of attraction, such that they do not at all

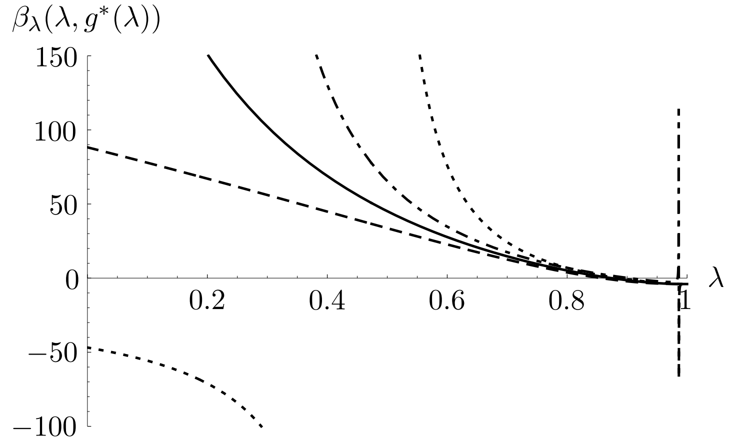


Figure 5.6. Fixed point condition $\beta_\lambda(\lambda, g^*(\lambda))$ of the (λ, g)-system: At $\gamma = 5$ we find a fixed point at $\lambda \approx 0.9$ for all four bases ($\mathcal{B}_1$ solid, $\mathcal{B}_2$ dashed, $\mathcal{B}_3$ dot-dashed, $\mathcal{B}_4$ dotted). In basis $\mathcal{B}_3$ the function $\beta_\lambda(\lambda, g^*(\lambda))$ exhibits a pole close to $\lambda = 1$ giving rise to another, however unphysical, fixed point.

influence the overall structure of the flow. Therefore these fixed point solutions should not be considered physical.

Typically we find two solutions of eq. (5.103) that can be considered physical. We denote the corresponding fixed points by $\mathbf{NGFP^1}$ and $\mathbf{NGFP^2}$ and discuss their properties separately in the subsequent paragraphs.

(A) The fixed point $\mathbf{NGFP^1}$. The most stable solution that gives rise to the fixed point $\mathbf{NGFP^1}$ occurs at $\lambda \approx 0.9$: As the RHS falls to zero at $\lambda = 1$ and the LHS grows linearly to 4 at that point, we generically find a solution to the fixed point condition in the interval $\lambda \in [0, 1]$. This behavior is depicted for the four bases $\mathcal{B}_i$ and $\gamma = 5$ in Fig. 5.6.

For other values of γ the situation is very similar. For $\gamma > 1$ we generally find that $0.8 < \lambda^* < 0.9$ while for $\gamma < 1$ the FP moves towards smaller λ^*. Only in basis $\mathcal{B}_2$ it ceases to exist for $\gamma \lesssim 0.75$.

In Fig. 5.7 the fixed point position is plotted as a function of γ for all four basis considered. We find that except for basis $\mathcal{B}_2$ the fixed point exists in all bases for all values of γ. For $\gamma > 1$ the fixed point lies at almost the same position for all bases. For small γ, however, the g^* coordinate diverges in basis $\mathcal{B}_1$, while in the bases $\mathcal{B}_3$ and $\mathcal{B}_4$ the fixed point merges with the Gaussian one.

Let us move on and analyze the stability properties of the fixed point $\mathbf{NGFP^1}$. In Fig. 5.8 the real parts of the critical exponents for the four bases are plotted as a

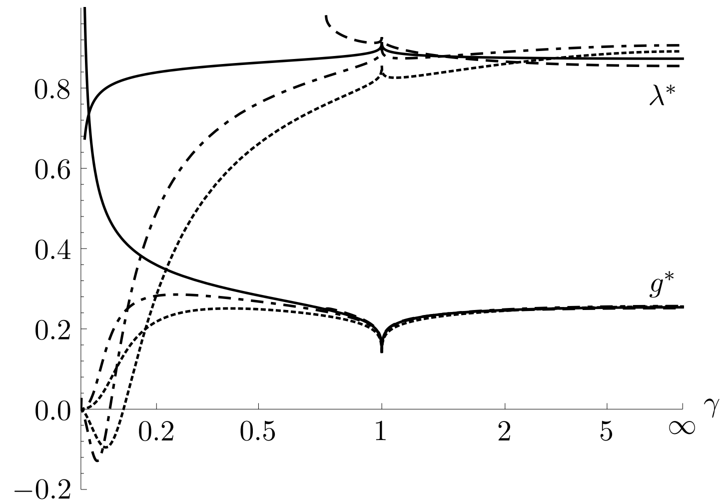


Figure 5.7. Fixed point position for all four bases ($\mathcal{B}_1$ solid, $\mathcal{B}_2$ dashed, $\mathcal{B}_3$ dot-dashed, $\mathcal{B}_4$ dotted) and $m = 1$ as a function of γ. While for $\gamma > 1$ the fixed point position is almost independent of the basis, for small λ we find their behavior differing: In basis $\mathcal{B}_1$ the g^* coordinate diverges, in basis $\mathcal{B}_2$ the FP ceases to exist at $\gamma \approx 0.75$ and in $\mathcal{B}_{3,4}$ the FP merges with the GFP.

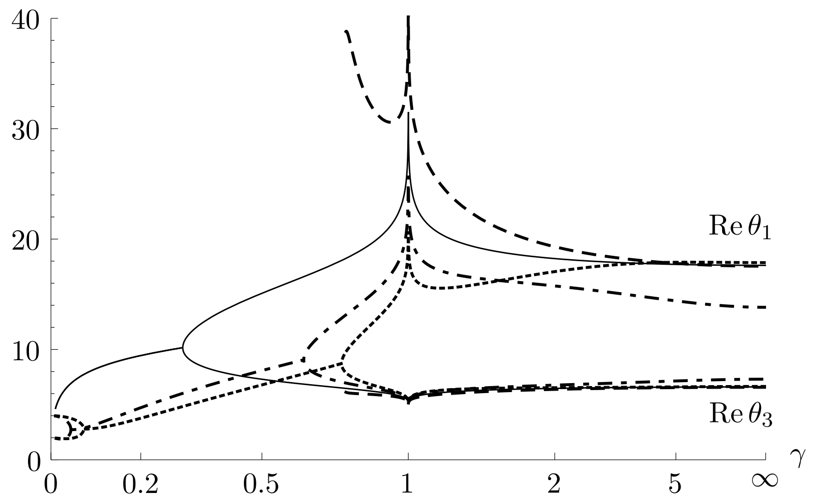


Figure 5.8. Real part of the critical exponents for all four bases ($\mathcal{B}_1$ solid, $\mathcal{B}_2$ dashed, $\mathcal{B}_3$ dot-dashed, $\mathcal{B}_4$ dotted) and $m = 1$ as a function of γ.

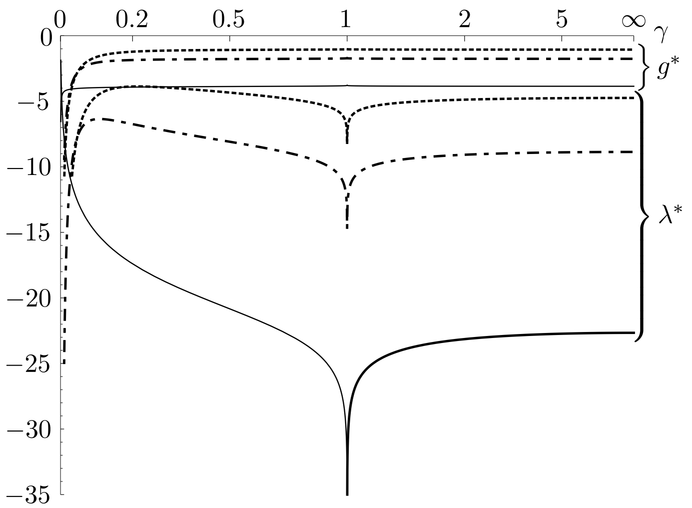


Figure 5.9. Fixed point position for the three bases $\mathcal{B}_1$ (solid), $\mathcal{B}_3$ (dot-dashed) and $\mathcal{B}_4$ (dotted) and $m = 1$ as a function of γ. Compared to the first fixed point, the FP position here depends severely on the basis chosen. However, for each basis the g^* coordinate is almost independent of γ, although λ^* varies significantly.

function of γ. At the bifurcation points the critical exponents become real, while in the range of γ with a single line only the critical exponents form a complex conjugated pair, whose imaginary part is not depicted.

As a first observation we find that for all bases and all values of γ both critical exponents are positive, i. e. that the fixed point is always UV attractive in both directions. Moreover, the qualitative dependence of the critical exponents on γ turns out similar for the different bases: At infinity they start real and we find a peak of the curves close to $\gamma = 1$, that probably is an artefact of the singularity of the β-functions at that point. Shortly after that the critical exponents turn complex, at least for some interval in γ. The dashed line, corresponding to basis $\mathcal{B}_2$, stops at $\gamma \approx 0.75$ as the FP vanishes for smaller γ. Note that the absolute value of the critical exponents is fairly large over the whole range of γ, which might cast some doubt on the physical significance of this FP. For large γ, on the other hand, the functions corresponding to 3 out of 4 bases coincide to a remarkably good degree, showing a special robustness of the result in the limit $\gamma \to \infty$.

(B) The fixed point NGFP$^\mathbf{2}$. Let us turn over to the discussion of the second FP solution. It is only present in the bases $\mathcal{B}_1$, $\mathcal{B}_3$ and $\mathcal{B}_4$, while for basis $\mathcal{B}_2$ we generally find only the first physical fixed point **NGFP$^\mathbf{1}$** from above. The fixed point **NGFP$^\mathbf{2}$**

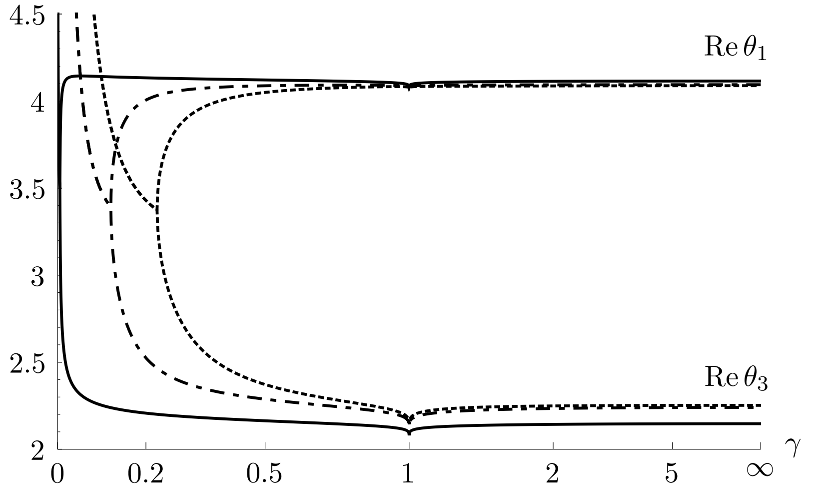


Figure 5.10. Real part of the critical exponents for the three bases $\mathcal{B}_1$ (solid), $\mathcal{B}_3$ (dot-dashed) and $\mathcal{B}_4$ (dotted), and $m = 1$ as a function of γ. For all γ the fixed point is attractive in both directions. The critical exponents are almost constant and independent of the basis in a large part of total range in γ.

is located at large negative λ^* and the corresponding g^* coordinate typically lies within the range $g^* \in [-4, -1]$.

The graphs showing the fixed point coordinates as functions of γ are shown in Fig. 5.9. We find that, except for a small region close to $\gamma \approx 0$, that is dominated by the logarithmic divergence, λ^* starts off decreasing in value up to $\gamma = 1$ where it shows a significant peak, while it stays approximately constant for $\gamma > 1$. What catches the eye, however, is that $g^*(\gamma, \lambda^*)$ is constant to a good approximation, although λ^* shows such a pronounced variation. Hence, a remarkable compensation in the function $g^*(\lambda^*)$ seems to take place.

In comparison to the first fixed point we discussed, here, the FP position is quite variable, depending on both γ and the choice of basis. Also the coordinates take on large absolute values. However, the fixed point position is not a physical observable so that we should not be concerned about these results but first analyze the absolute value and stability of the critical exponents, which are considered physically more meaningful.

In Fig. 5.10 the real parts of the critical exponents corresponding to the second fixed point are depicted. We observe that the fixed point is UV attractive in both directions as well and that the absolute value of the critical exponents lies in a perfectly reasonable range. Moreover, the critical exponents are real, constant and almost independent of the basis chosen for all $\gamma > 0.5$. Taken together these findings strongly support the physical significance of the fixed point.

(C) Discussion. We conclude that both fixed points in principle allow for the asymptotic safety construction and show the same predictivity as both critical hypersurfaces are two dimensional. In comparison it is difficult to judge which of the two fixed points should be considered most reliable. While the first one lies in the positive (λ, g)-quadrant at a position similar to the one known from metric gravity, but its large critical exponents question its reliability, the second one occurs in the negative (λ, g)-quadrant at large coordinate values—which is hard to reconcile with the phenomenological reasoning in favor of $g^* > 0$—, but shows a remarkable stability of its critical exponents. In the following we will therefore assign the same level of credibility to both fixed points, $\mathbf{NGFP^1}$ and $\mathbf{NGFP^2}$, and treat them on the same footing.

The above discussion and Fig. 5.7 refers to the case with the mass parameter m chosen to $m = 1$. As there is no physical mechanism which could generate a second momentum/mass scale besides k, a choice of $m \approx 1$ seems most natural. We tested that qualitatively the situation does not change for other choices of $m \in [0.5, 5]$. As in basis $\mathcal{B}_1$ all γ-dependence besides the logarithmic term drops out, we can infer that changing m in this case is equivalent to choosing a different γ: Small m then correspond to $\gamma \ll 1$ and larger m to $\gamma \approx 1$. Qualitatively this correspondence is also found for the other bases. The explicit m dependence of the FP properties will be analyzed in more detail when we discuss the corresponding "lifts" of the fixed points in the 3-dimensional truncation.

The phase portrait. In this paragraph we discuss the resulting phase portrait of the (λ, g)-truncation. As the two non-Gaussian fixed points occur at very different coordinate scales, it is not possible to depict the resulting flow in the vicinity of both FPs equally good in only one diagram. For that reason the two Figs. 5.11 and 5.12 each focus on one of the fixed points and its interplay with the Gaussian FP. The figures are restricted to the $\mathcal{B}_1$ basis, but the qualitative features of the flow were found to be similar for all bases that show the respective fixed point.

Fig. 5.11 pictures the $\mathbf{NGFP^1}$ in the positive (λ, g)-quadrant. Qualitatively all panels of different γ resemble each other, while only the fixed point position moves slightly.

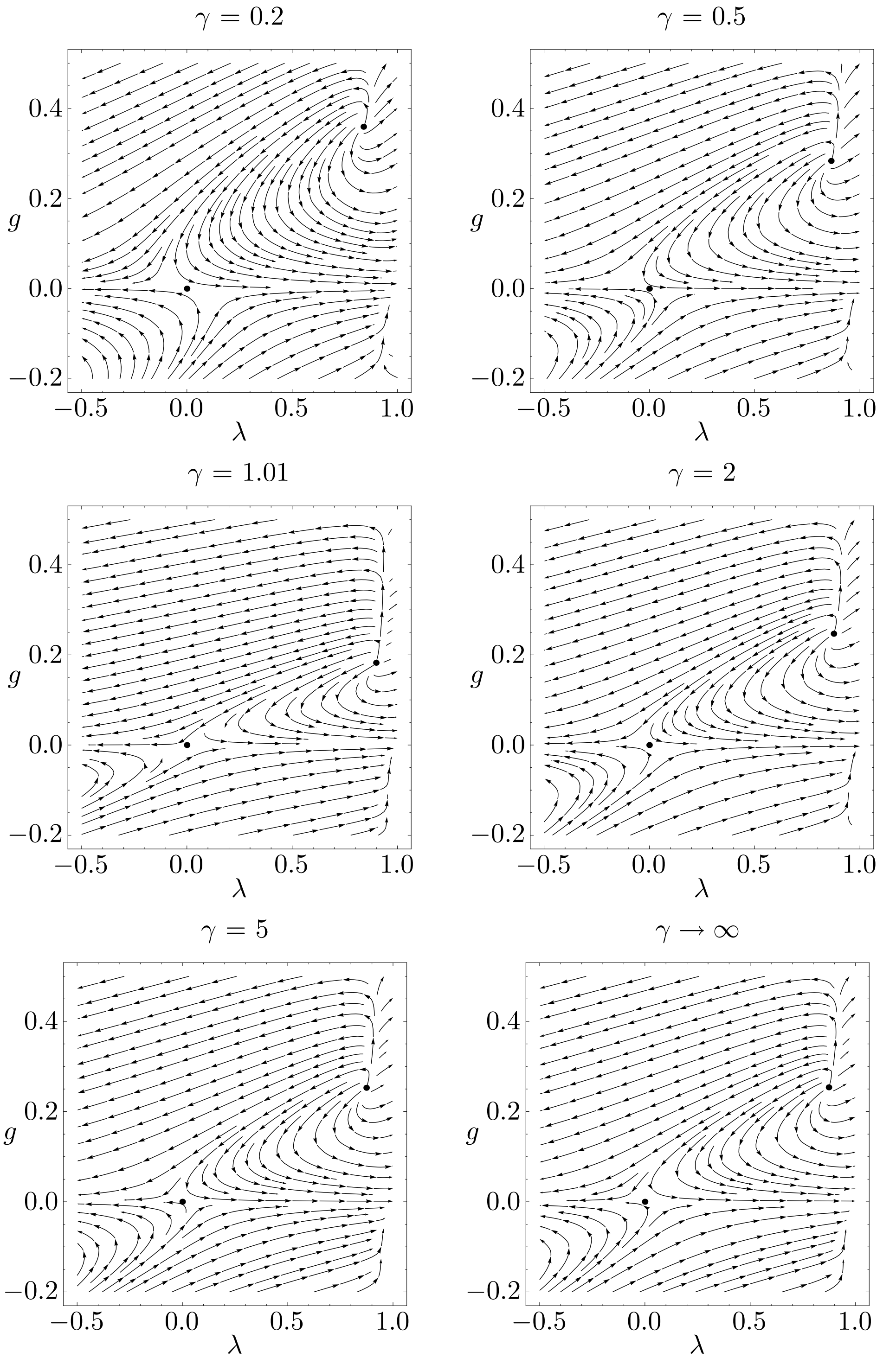


Figure 5.11. Phase portraits of the (λ, g)-truncation for different fixed values of γ in basis $\mathcal{B}_1$ focussing on $\mathbf{NGFP^1}$ in the positive (λ, g)-quadrant. While the fixed point position changes slightly, the qualitative features of the flow are remarkably similar for all choices of γ.

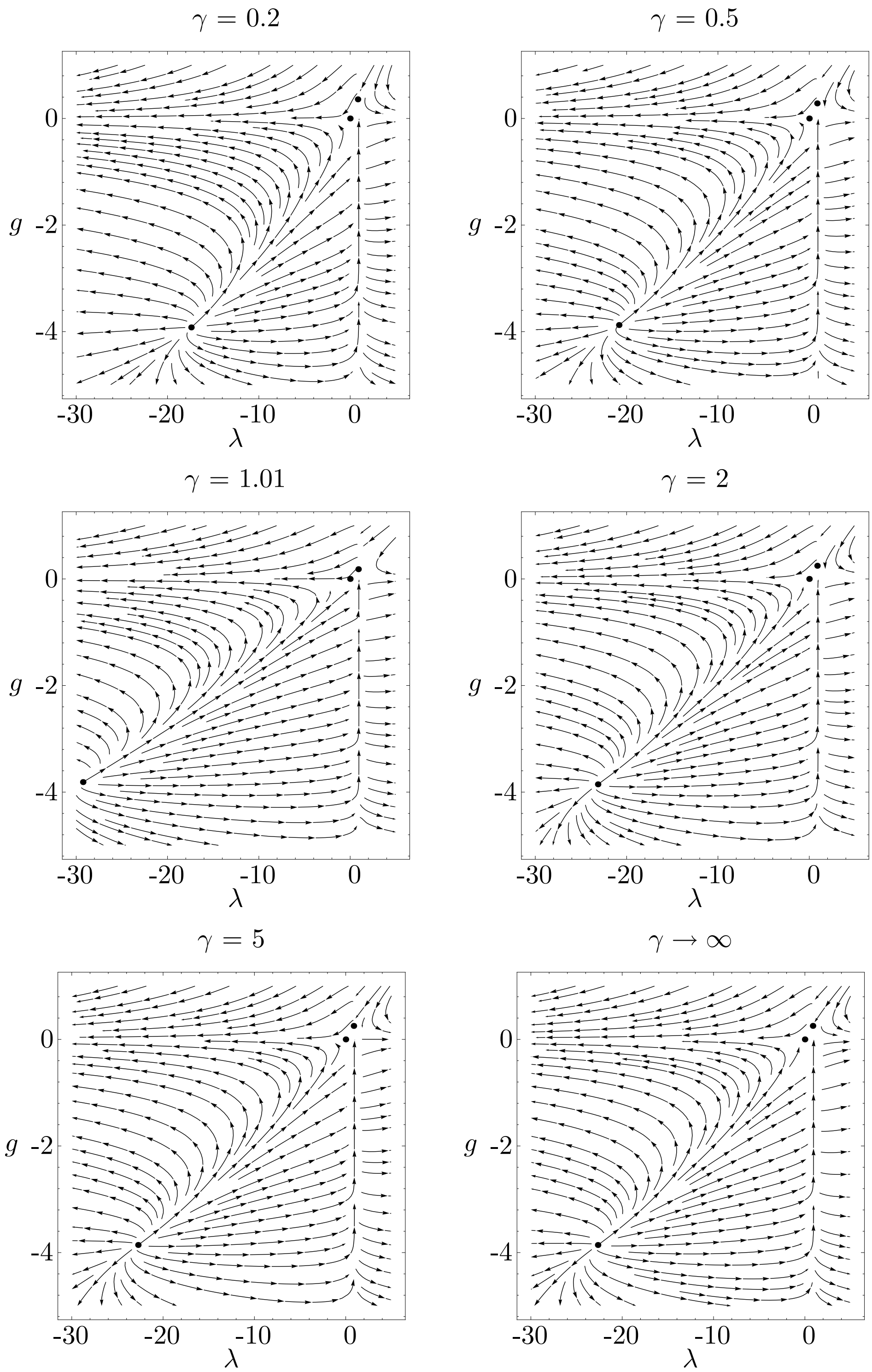


Figure 5.12. Phase portraits of the (λ, g)-truncation for different fixed values of γ in basis $\mathcal{B}_1$ focussing on $\mathbf{NGFP^2}$ in the negative (λ, g)-quadrant. Again only the fixed point moves, leaving the RG flow qualitatively unchanged for all choices of γ.

The λ-axis, being the critical surface $\mathscr{S}_{\text{UV}}$ of the GFP is an IR attractive line that cannot be crossed by any trajectory. All points above this axis with $\lambda < 1$ lie on trajectories that are asymptotically safe w. r. t. $\mathbf{NGFP^1}$. The divergence of the β-functions at $\lambda = 1$ is approached in a controlled way, such that the flow stops on this line.

The similarity to the phase portrait of metric gravity in Einstein-Hilbert truncation is striking. In complete analogy to the metric case a classification of the asymptotically safe RG trajectories is possible: We find the NGFP connected to the GFP by a separatrix (type IIa trajectory) that separates the trajectories with a negative IR cosmological constant (type Ia) from those with a positive one (type IIIa). However, one significant difference to the metric description should be noted: Following the separatrix from $\mathbf{NGFP^1}$ at positive λ to the IR we find that the cosmological constant turns negative before ending in the **GFP**. In general such a situation occurs if the second eigendirection of stability matrix at the **GFP** points in a direction of negative λ. Here, due to the logarithmic divergence of the stability matrix at the **GFP** the separatrix even ends up parallel to the λ-axis in the **GFP**.

In Fig. 5.12 the second NGFP in the negative (λ, g)-quadrant is depicted. Again, qualitatively all panels of different γ resemble each other, but the fixed point position changes. We see that the points in the negative g-halfplane with $\lambda < 1$ are all attracted to $\mathbf{NGFP^2}$ in the UV. Also here, we find a separatrix connecting the FP and the **GFP** that separates positive from negative IR cosmological constants. Due to the limiting nature of λ-axis no trajectory that is asymptotically safe w. r. t. $\mathbf{NGFP^2}$ will run to positive g in the IR. Hence, the physical significance of these trajectories is questionable, as we should expect a positive Newton constant in the IR for phenomenological reasons.

Comparison to the proper-time flow. If we compare the results obtained here with the RG study using a proper-time flow equation in [Dau, DR] we find that the fixed point structures do not coincide. In [DR] the (λ, g)-truncation shows three non-Gaussian fixed points, all of which occur at $g^* > 0$. Two of them have one UV attractive and one repulsive direction, while the third one is attractive in both directions. Thus, only this last one is comparable to our two NGFPs. If we were to identify it with one of our FPs, we would choose our second fixed point at negative g: Both have large absolute values of their (λ^*, g^*)-coordinates, that are of the same order of magnitude and the critical exponents are similar. In [DR] they are given as $\{3.4, 1.8\}$ while we find in the limit $\gamma \to \infty$ on average $\{4.1, 2.2\}$. However, while in Chapter 4 a simple mechanism capable of switching the sign of the λ^* coordinate was found, we do not know of such a mechanism concerning the g^* coordinate. For that reason more evidence should be collected before one can rely on this identification of the fixed points.

5.4.2 The (γ, g)-subsystem

The (γ, g)-truncation, that we are about to discuss next, is based on the two β-functions (5.93a) and (5.93b),

$$\begin{aligned} \beta_g(\lambda, \gamma, g) &= g\,[2 + \eta_N(\lambda, \gamma, g)]\,, \\ \beta_\gamma(\lambda, \gamma, g) &= -\frac{16\pi g \gamma}{N(\lambda)} \left[P_9(\lambda) + \frac{1}{\gamma^2} P_8(\lambda) + P_{10}(\lambda) \right]. \end{aligned} \tag{5.104}$$

Now λ is thought of as a fixed external parameter, and we can study the properties of the flow in dependence of its value. We observe that the system (5.104) does not suffer from the limitations due to logarithmic divergences as these only occur in β_λ; nevertheless we again find the $\lambda = 1$-barrier of the flow due to the (double) zero of the denominator $N(\lambda)$. In the bases $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$, where the polynomial $P_8(\lambda)$ is non-vanishing an additional divergence at $\gamma = 0$ is found. Surprisingly, the limit of chiral gravity, $\gamma = \pm 1$, is perfectly well defined in this two-dimensional truncation.

Besides these limitations we find that β_g is an even function of γ, while β_γ is odd. Together this leads to a symmetry of the flow under $\gamma \mapsto -\gamma$, under which the flow remains unchanged.

Fixed point structure. Let us, first of all, illuminate the general situation in theory space by a schematical plot of the fixed points we are going to discuss and the names given to them (cf. Fig. 5.13). We find, that in the (γ, g)-subsystem the fixed point structure depends on the basis chosen. For basis $\mathcal{B}_1$ there is a NGFP at $\gamma = 0$ ($\mathbf{NGFP'_0}$), while in the bases $\mathcal{B}_{2,3,4}$ we find a pair of NGFPs at finite non-zero γ ($\mathbf{NGFP'_{fin}}$). Besides that a second NGFP ($\mathbf{NGFP'_\infty}$) and a fixed line at $g = 0$ is present in all bases, that includes the **GFP**.

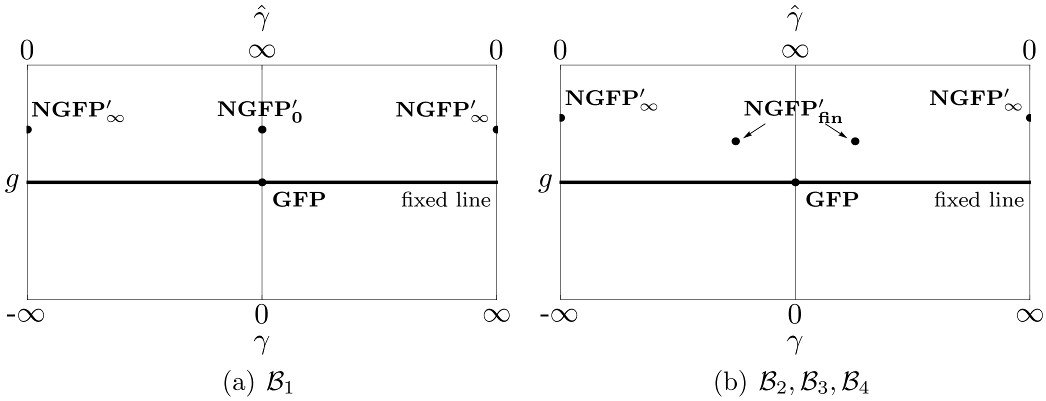


Figure 5.13. Sketch of the fixed point structure of the (γ, g)-system.

(A) The Gaussian fixed point. The first and most obvious solution of the fixed point condition $(\beta_\gamma, \beta_g) = (0,0)$ is the limit of vanishing Newton constant g. It gives rise to a **fixed line** $g = 0$ with γ arbitrary that comprises the **GFP** at $(\gamma, g) = (0,0)$. This fixed line turns out IR attractive (in g-direction) for all values of λ as the stability matrix along this line is of the form

$$\mathscr{B}\Big|_{g=0} = \begin{pmatrix} 2 & 0 \\ \partial_g \beta_\gamma(\lambda, \gamma, g\!=\!0) & 0 \end{pmatrix} . \tag{5.105}$$

For the search of non-Gaussian fixed points we have to distinguish basis $\mathcal{B}_1$ from the other bases $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$ as the absence of $P_8(\lambda)$ changes the mechanism how the non-Gaussian fixed point at finite γ is established. For this reason the fixed point $\mathbf{NGFP'_0}$ which is found in $\mathcal{B}_1$ exhibits properties that differ from those obtained by taking the limit $P_8 \to 0$ of the fixed point $\mathbf{NGFP'_{fin}}$ found for the other three bases.

(B) The fixed point $\mathbf{NGFP'_{fin}}$. Let us first discuss $\mathbf{NGFP'_{fin}}$, that exists in the bases $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$. Since P_8 is non-zero in this case, the only possibility to satisfy the condition $\beta_\gamma = 0$ for $g \neq 0$ is that the terms in the square brackets add up to zero (cf. (5.104)). Hence the fixed point coordinate γ^* is determined by the condition

$$\big(\gamma^*(\lambda)\big)^2 = -\frac{P_8}{P_9(\lambda) + P_{10}(\lambda)} . \tag{5.106}$$

We observe that the condition can only be satisfied by real γ if the right hand side is positive. As P_8 takes on a constant value in $(0,0,0)$-gauge for all three bases, sign changes of the RHS only occur at the zeros of $P_9 + P_{10}$, which therefore limit the domain of existence of the FP in λ. At these limiting values of λ the fixed point coordinate γ^* diverges. Moreover, as $P_9 + P_{10}$ does not have any poles, $\left(\gamma^*\right)^2 = 0$ is only obtained in the limit $\lambda \to \infty$. For any fixed value of λ, except for the zeros of $P_9 + P_{10}$, we thus find a fixed point solution $\mathbf{NGFP'_{fin}}$ arising at finite non-zero γ^*. Note that the solutions to (5.106) come in pairs of $\pm\gamma^*$, which is not surprising as the total RG flow is invariant under the transition $\gamma \mapsto -\gamma$.

The corresponding g^*-coordinate is found from $\beta_g(\gamma^*(\lambda), g) = 0$ resulting in

$$g^*(\lambda) = \frac{1}{8\pi} \frac{N(\lambda)}{P_9(\lambda)} \,. \tag{5.107}$$

Since $N(\lambda) \geq 0$ the fixed point lies at positive g^* if $P_9(\lambda)$ is positive.

With analytical expressions for the fixed point coordinates at hand, we can also find an analytical expression for the critical exponents of the fixed point $\mathbf{NGFP'_{fin}}$. As $\partial_g \beta_\gamma|_{\mathbf{NGFP'_{fin}}} = 0$ the stability matrix at the fixed point is triangular, such that one of its eigenvectors points in the direction of the g-axis. The corresponding critical exponents are found to

$$\theta_g = 2, \qquad \theta_2 = 4\frac{P_9(\lambda) + P_{10}(\lambda)}{P_9(\lambda)} \,. \tag{5.108}$$

Note that the divergence of the critical exponent at $P_9 = 0$ occurs at the same position as the pole of $g^*(\lambda)$, such that both quantities change sign there. The zeros of θ_2, on the other hand, occur at the poles of $\gamma^*(\lambda)$.

In Fig. 5.14 we have plotted the position of $\mathbf{NGFP'_{fin}}$ as functions of λ for the three bases $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$. However, $g^*(\lambda)$ is found basis independent because it only depends on $P_9(\lambda)$, whose explicit form is fixed by our choice of basis in the parity-odd torsion squared subspace.

We find that $g^* > 0$ for all $\lambda \leq 0.733$ and therefore in particular also for small $|\lambda| \ll 1$. The domain of existence of the FP differs for the three bases: While for $\mathcal{B}_3$ and $\mathcal{B}_4$ the FP exists up to $\lambda \leq 0.603$ and $\lambda \leq 0.651$, respectively, for $\mathcal{B}_2$ is ceases to exist for $\lambda \leq 0.53$. Thus, there is a small interval in λ in which the FP exists for all three bases. Nonetheless, we find confirmed the observation from the (λ, g)-truncation, that predictions concerning fixed point existence and stability in basis $\mathcal{B}_2$ differ considerably from the other bases. For that reason we will consider the β-

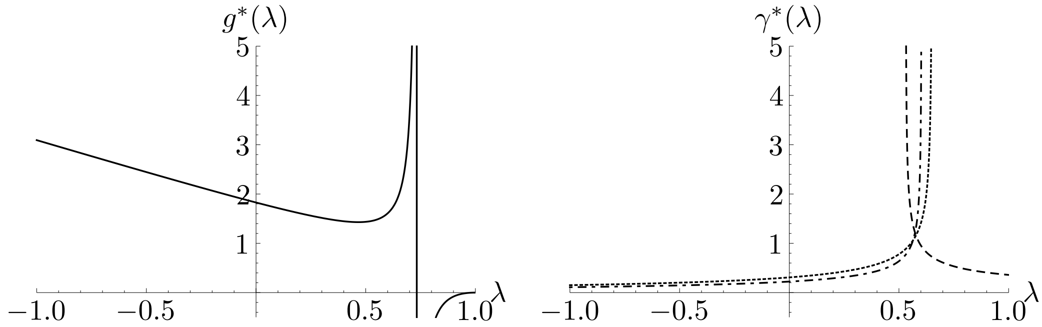


Figure 5.14. Position of the fixed point $\mathbf{NGFP'_{fin}}$ as a function of λ. While g^* only depends on the (in our case fixed) choice of basis in the parity-odd T^2 sector, γ^* differs for the three bases $\mathcal{B}_2$ (dashed), $\mathcal{B}_3$ (dot-dashed) and $\mathcal{B}_4$ (dotted).

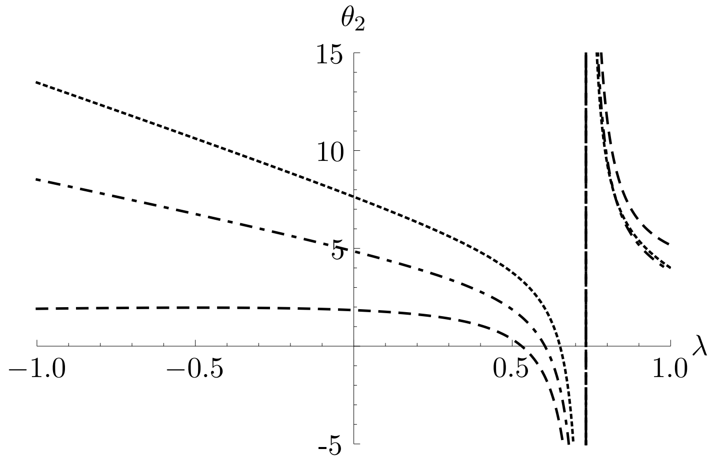


Figure 5.15. The critical exponent θ_2 of the fixed point $\mathbf{NGFP'_{fin}}$ as a function of λ. In the region $-1 \leq \lambda \leq 0.5$ the function is positive for all three bases $\mathcal{B}_2$ (dashed), $\mathcal{B}_3$ (dot-dashed) and $\mathcal{B}_4$ (dotted), corresponding to a second UV attractive direction.

functions in basis $\mathcal{B}_2$ as the exception from the rule, which can be traced back to the fact, that all coefficients in $P_{10}(\lambda)$ for this basis have the opposite sign compared to the other bases.

Considering basis $\mathcal{B}_2$ less credible, we find that the results of bases $\mathcal{B}_3$ and $\mathcal{B}_4$ lie perfectly in line with each other. Both predict a pair of fixed points which lies for $\lambda < 0.5$ at very small $|\gamma^*|$. In the range $0.5 < \gamma < 0.733$ both coordinates diverge, which is considered a hint that the results in this range become less trustworthy for these large values of the cosmological constant. In Fig. 5.15 we have plotted the critical exponent θ_2 of the fixed point. For the bases $\mathcal{B}_3$ and $\mathcal{B}_4$ we find that it corresponds to a UV attractive direction ($\theta_2 > 0$) exactly over the whole range in λ in which the fixed point exists.

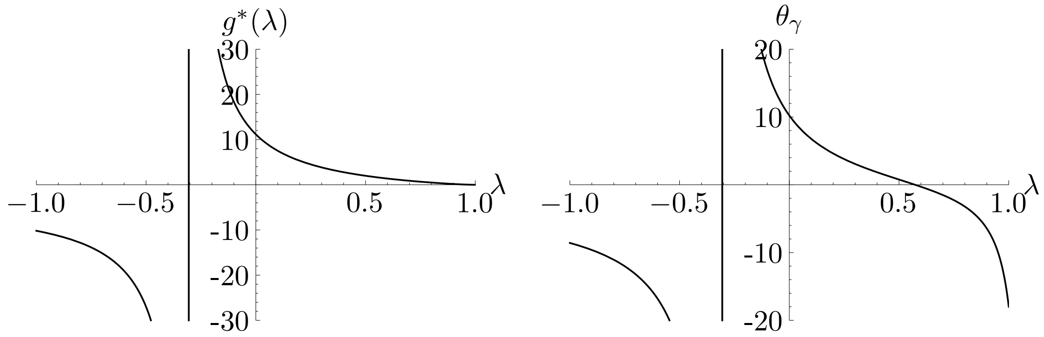


Figure 5.16. g^*-coordinate and critical exponent θ_γ of the fixed point $\mathbf{NGFP'_0}$ that is only present in basis $\mathcal{B}_1$ as functions of λ.

Taken together with the result $\theta_g = 2$ we conclude that for small λ both bases predict a pair of fixed points $\mathbf{NGFP'_{fin}}$ at finite γ, that is attractive in both directions. The absolute value of θ_2, as a little blemish, turns out fairly large.

(C) The fixed point $\mathbf{NGFP'_0}$. As already pointed out above, for basis $\mathcal{B}_1$ we find a different fixed point solution $\mathbf{NGFP'_0}$. With all P_8-terms absent in the β-functions (5.104) there is only one non-Gaussian fixed point solution given by

$$g^*(\lambda) = -\frac{1}{8\pi}\frac{N(\lambda)}{P_{10}(\lambda)}, \qquad \gamma^* = 0\,. \tag{5.109}$$

In this case the stability matrix at the fixed point $\mathbf{NGFP'_0}$ is diagonal as also $\partial_\gamma \beta_g = 0$. Thus, both critical exponents can be associated with the g and γ axes in theory space and we find

$$\theta_g = 2, \qquad \theta_\gamma = -2\,\frac{P_9(\lambda) + P_{10}(\lambda)}{P_{10}(\lambda)}\,. \tag{5.110}$$

We observe that both g^* and θ_γ of this new fixed point $\mathbf{NGFP'_0}$ cannot be obtained by taking the limit $P_8 \to 0$ of the corresponding quantities of $\mathbf{NGFP'_{fin}}$. Thus, in basis $\mathcal{B}_1$ a truly different fixed point is present.

In Fig. 5.16 (left panel) the position of the fixed point is depicted. While $\gamma^* = 0$ for all λ we find that the fixed point value of the Newton constant is positive for all $-0.306 \leq \lambda \leq 1$, where g^* diverges at the lower boundary. In the right panel we see, that θ_γ diverges at the same point and is positive up to a value of $\lambda \approx 0.573$.

We conclude that in basis $\mathcal{B}_1$ the pair of fixed points $\mathbf{NGFP'_{fin}}$ at finite non-zero γ^* gets replaced by $\mathbf{NGFP'_0}$, located at $\gamma^* = 0$ for all λ. Despite this difference both fixed points have in common, that for small $|\lambda|$ they are UV attractive in both directions of the (γ, g)-plane.

(D) Absence of the Immirzi term and duality map. In order to explore the full part of theory space that is spanned by the Immirzi and the curvature term, we certainly also have to consider its 1-dimensional subspace where the Immirzi term is absent. This space corresponds to the limit $\gamma \to \infty$ with g arbitrary and is thus not contained in the coordinate chart we have chosen so far. For this reason it turns out useful to introduce a different coordinate chart $\hat{\gamma}$, which in the overlapping region is related to γ by $\hat{\gamma} = \frac{1}{\gamma}$. Hence, similar to the stereographic projection of a sphere S^1, the charts overlap at all values of γ except for the two points $\gamma = 0$ and $\hat{\gamma} = 0$, that are covered by only one of the two charts.

If we analyze the behavior of the β-functions under this change of coordinates, we are not only enabled to examine the $\gamma \to \infty$-limit properly, but we also find a most remarkable property of the RG flow in the (γ, g)-truncation. Under the coordinate change $\gamma \mapsto 1/\hat{\gamma}$ the β-functions are transformed according to $\beta_g(\lambda, \gamma, g) \mapsto \beta_g(\lambda, 1/\hat{\gamma}, g)$ and $\beta_\gamma(\lambda, \gamma, g) \mapsto \beta_{\hat{\gamma}}(\lambda, \hat{\gamma}, g) = -\hat{\gamma}^2 \beta_\gamma(\lambda, 1/\hat{\gamma}, g)$ giving rise to the explicit form

$$\begin{aligned} \beta_g(\lambda, \hat{\gamma}, g) &= g \left[2 + \frac{16\pi g}{N(\lambda)} \big(\hat{\gamma}^2 P_8(\lambda) + P_{10}(\lambda) \big) \right], \\ \beta_{\hat{\gamma}}(\lambda, \hat{\gamma}, g) &= \frac{16\pi g \hat{\gamma}}{N(\lambda)} \left[P_9(\lambda) + \hat{\gamma}^2 P_8(\lambda) + P_{10}(\lambda) \right]. \end{aligned} \tag{5.111}$$

If we now compare the system (5.111) with the original β-functions (5.104), that read

$$\begin{aligned} \beta_g(\lambda, \gamma, g) &= g \left[2 + \frac{16\pi g}{N(\lambda)} \left(\frac{1}{\gamma^2} P_8(\lambda) + P_{10}(\lambda) \right) \right], \\ \beta_\gamma(\lambda, \gamma, g) &= -\frac{16\pi g \gamma}{N(\lambda)} \left[P_9(\lambda) + \frac{1}{\gamma^2} P_8(\lambda) + P_{10}(\lambda) \right], \end{aligned} \tag{5.112}$$

we observe that in the case $P_8(\lambda) = 0$ (i. e. in basis $\mathcal{B}_1$) β_g is left invariant under the coordinate change and the flow of the Immirzi parameter satisfies

$$\beta_{\hat{\gamma}}(\lambda, \gamma, g) = -\beta_\gamma(\lambda, \gamma, g). \tag{5.113}$$

Thus, for the *same arguments* the β-functions coincide up to a minus sign. We conclude that in the case $P_8(\lambda) = 0$ the RG flow at large values of the Immirzi parameter γ contains the same information as at small values $(1/\gamma)$. Therefore we refer to the transformation $\gamma \mapsto 1/\gamma$ as a duality map.

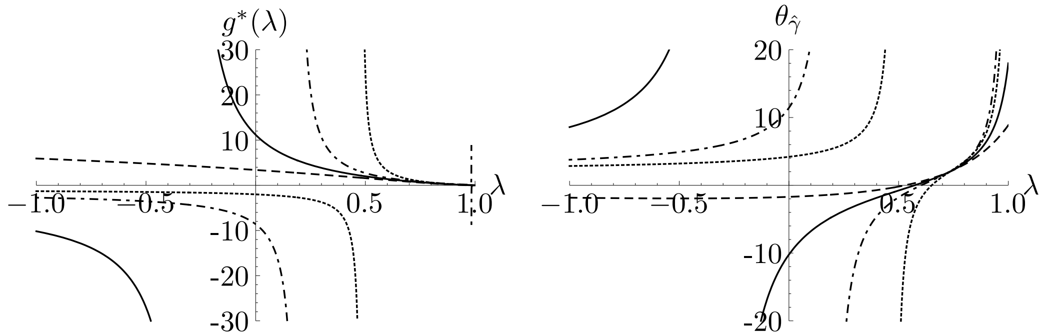


Figure 5.17. g^*-coordinate and critical exponent θ_γ of the fixed point $\mathbf{NGFP}'_\infty$ present in all bases ($\mathcal{B}_1$ solid, $\mathcal{B}_2$ dashed, $\mathcal{B}_3$ dot-dashed, $\mathcal{B}_4$ dotted) as a function of λ. While the functions for small λ differ considerably due to the basis-dependent position of their pole, there is a region of remarkably good agreement of all bases at $\lambda \in [0.7, 0.85]$.

(E) The fixed point $\mathbf{NGFP}'_\infty$. Let us now discuss the additional fixed point that arises at $\hat{\gamma} = 0$ and which we refer to as $\mathbf{NGFP}'_\infty$. It is found in all four bases at the coordinate values

$$\hat{\gamma}^* = 0, \qquad g^* = -\frac{1}{8\pi}\frac{N(\lambda)}{P_{10}(\lambda)}\,. \tag{5.114}$$

Also at this fixed point we find that the stability matrix is diagonal such that the critical exponents can be associated with the coordinate directions. Explicitly we find for them

$$\theta_g = 2, \qquad \theta_{\hat{\gamma}} = 2\frac{P_9(\lambda) + P_{10}(\lambda)}{P_{10}(\lambda)}\,. \tag{5.115}$$

Although this fixed point exists in all bases, we find (cf. Fig. 5.17) that the predictions concerning its position and attractivity properties are severely basis-dependent for small $|\lambda|$. However there is an interval $\lambda \in [0.65, 0.85]$ where the functions $g^*(\lambda)$ and $\theta_{\hat{\gamma}}(\lambda)$ coincide surprisingly well for all four bases. Also the limits at large negative λ (< -10) are qualitatively similar at least for the bases $\mathcal{B}_1$, $\mathcal{B}_3$ and $\mathcal{B}_4$. Our findings in the previous subsection on the (λ, g)-truncation suggest already that it is precisely these regions of λ where 3-dimensional "lifts" of the FPs in the (λ, g)-truncation will be found. Hence, we conclude that the (γ, g)-truncation in the limit $\gamma \to \infty$ is particularly good at those fixed λ that correspond to the 3-dimensional fixed point values, not only because β_λ vanishes there, but also due to the enhanced basis-independence.

(F) The interplay of the fixed points. A final important observation concerning the properties of the fixed points is that the zeros of $\theta_{\hat{\gamma}}$ of $\mathbf{NGFP}'_\infty$ coincide exactly with those of θ_γ or θ_2 for $\mathbf{NGFP}'_0$ and $\mathbf{NGFP}'_{\mathbf{fin}}$, respectively, and the functions cross zero in such a way, that the fixed points are antagonistic for all bases and all λ as long as both exist, i. e. one is UV attractive while the other is repulsive along the second direction. Recall, however, that the zeros of θ_2 are also the points at which $\mathbf{NGFP'_{fin}}$ ceases to exist, such that in each basis $\mathbf{NGFP'_{fin}}$ does not change its attractivity properties.

In basis $\mathcal{B}_1$, however, this antagonism of the fixed points $\mathbf{NGFP'_\infty}$ and $\mathbf{NGFP'_0}$ is taken to another level: Besides having the same fixed point value $g^*(\lambda)$ we find that the critical exponents satisfy $\theta_\gamma = -\theta_{\hat{\gamma}}$ for all λ. This is because for $\mathcal{B}_1$ the β-functions in the two coordinate charts are of the same form, except for a sign change in β_γ compared to $\beta_{\hat{\gamma}}$. Thus under the duality map $\gamma \mapsto 1/\gamma$ only the γ component of flow switches its sign and the fixed points are mapped onto each other.

Note that for $\lambda = 0.573$ we have $\theta_\gamma = 0 = \theta_{\hat{\gamma}}$. As this value of λ corresponds to the zero of the sum $P_9(\lambda) + P_{10}(\lambda)$, $\beta_\gamma(\gamma, g)$ vanishes for all γ and g. Hence, in this special case, the Immirzi parameter stops running and the straight line connecting $\mathbf{NGFP'_\infty}$ and $\mathbf{NGFP'_0}$ at fixed g^* becomes a UV attractive fixed line. The corresponding phase portrait will be discussed below.

Phase portraits. In Fig. 5.18 we have plotted the phase portraits of the (γ, g)-truncation in basis $\mathcal{B}_3$ for three different values of the cosmological constant. We used an *arctangent* rescaling of the γ-axis in order to compactify it such that the fixed point at $\hat{\gamma} = 0$ can be depicted in the same diagram. Clearly, this results in a highly non-linear scale of the γ-axis.

We observe that $\mathbf{NGFP'_\infty}$ is present in all three panels of Fig. 5.18, and it always gives rise to a complete RG trajectory on the $\hat{\gamma} = 0$ line with the fixed point as the UV limit and the γ-axis as its IR endpoint.

The pair of fixed points $\mathbf{NGFP'_{fin}}$ only exists in the first panel with $\lambda = 0.45$ and it is UV attractive in both directions as discussed above, while $\mathbf{NGFP'_\infty}$ only has one attractive direction in this case. We find a trajectory that connects both non-Gaussian fixed points and one linking $\mathbf{NGFP'_{fin}}$ with the origin. These trajectories act as separatrices: All trajectories below them are complete: In the UV they approach $\mathbf{NGFP'_{fin}}$ while in the IR they end on the fixed line $g = 0$. Therefore we find an asymptotically safe trajectory for any IR value of the Immirzi parameter. This statement remains true for all phase portraits of the (γ, g)-truncation we analyzed. Note also that the flow in γ direction changes sign on the line $\gamma = \gamma^*$ of $\mathbf{NGFP'_{fin}}$. For that reason each trajectory lies completely either in the region of larger or of smaller γ compared to the fixed point value. In the region $|\gamma| < |\gamma^*|$ the Immirzi parameter runs to smaller (absolute) values in the IR, while for $|\gamma| > |\gamma^*|$ it runs to larger values. In the first panel we may assert the latter to be the predominant direction of the γ-flow as $|\gamma^*| < 1$. (In the $g < 0$-halfplane the running is just reversed.)

For larger values of λ (as e. g. $\lambda = 0.8$ in the second panel of 5.18) this is no longer true: The pair of fixed points $\mathbf{NGFP'_{fin}}$ moves to larger values $|\gamma^*|$ until it merges with $\mathbf{NGFP'_{\infty}}$ and ceases to exist for even larger values of λ. By moving the fixed points outwards the predominant direction of the γ flow changes until in the whole upper halfplane the γ flow points to smaller values in the IR. Then, as depicted in the second panel, all trajectories in the $g > 0$-halfplane are asymptotically safe w. r. t. $\mathbf{NGFP'_{\infty}}$.

To conclude the discussion of the phase portraits in basis $\mathcal{B}_3$ we want to point out an interesting mechanism that is present in all bases. As $P_8(\lambda)$ is, for any basis (and any gauge), a polynomial in λ whose degree is smaller by 2 compared to the denominator $N(\lambda)$, the corresponding terms in the β-functions are suppressed quadratically for large λ, while the P_9 terms are suppressed only linearly and the P_{10} terms become constant. Thus, at large $|\lambda|$ the duality of the β-functions known from basis $\mathcal{B}_1$ is established approximately in the whole (γ, g)-plane, except for a narrow strip around the origin of the order $|\gamma| \lesssim 1/|\lambda|$. For this reason the third panel of Fig. 5.18 is very similar to the phase portraits in basis $\mathcal{B}_1$, which we will discuss next. The only qualitative difference is that no fixed point on the g-axis arises, as the γ-divergence in the β-functions persists.

Let us turn over to the RG flow resulting from basis $\mathcal{B}_1$, which is depicted in Fig. 5.19 for three different values of λ. We find the two fixed points $\mathbf{NGFP'_{\infty}}$ and $\mathbf{NGFP'_{0}}$ as deduced above, which govern the flow in the whole $g > 0$-halfplane. Depending on the value of λ, one of the fixed points has two attractive directions while the other shows only one. This antagonistic behavior prescribes the predominant direction of the γ-flow. As in basis $\mathcal{B}_3$ we find for small λ ($\lambda = 0$, upper panel) that it is directed to larger values of the Immirzi parameter in the IR while for larger values ($\lambda = 0.8$, lower panel) it reverses its direction. Different to the case of $\mathcal{B}_3$ this change does not happen due to a movement of one of the fixed points but only due to a change of their attractivity properties. For that reason we find a specific value of the cosmological constant ($\lambda \approx 0.57$, centered panel) for which the flow of the Immirzi parameter stops

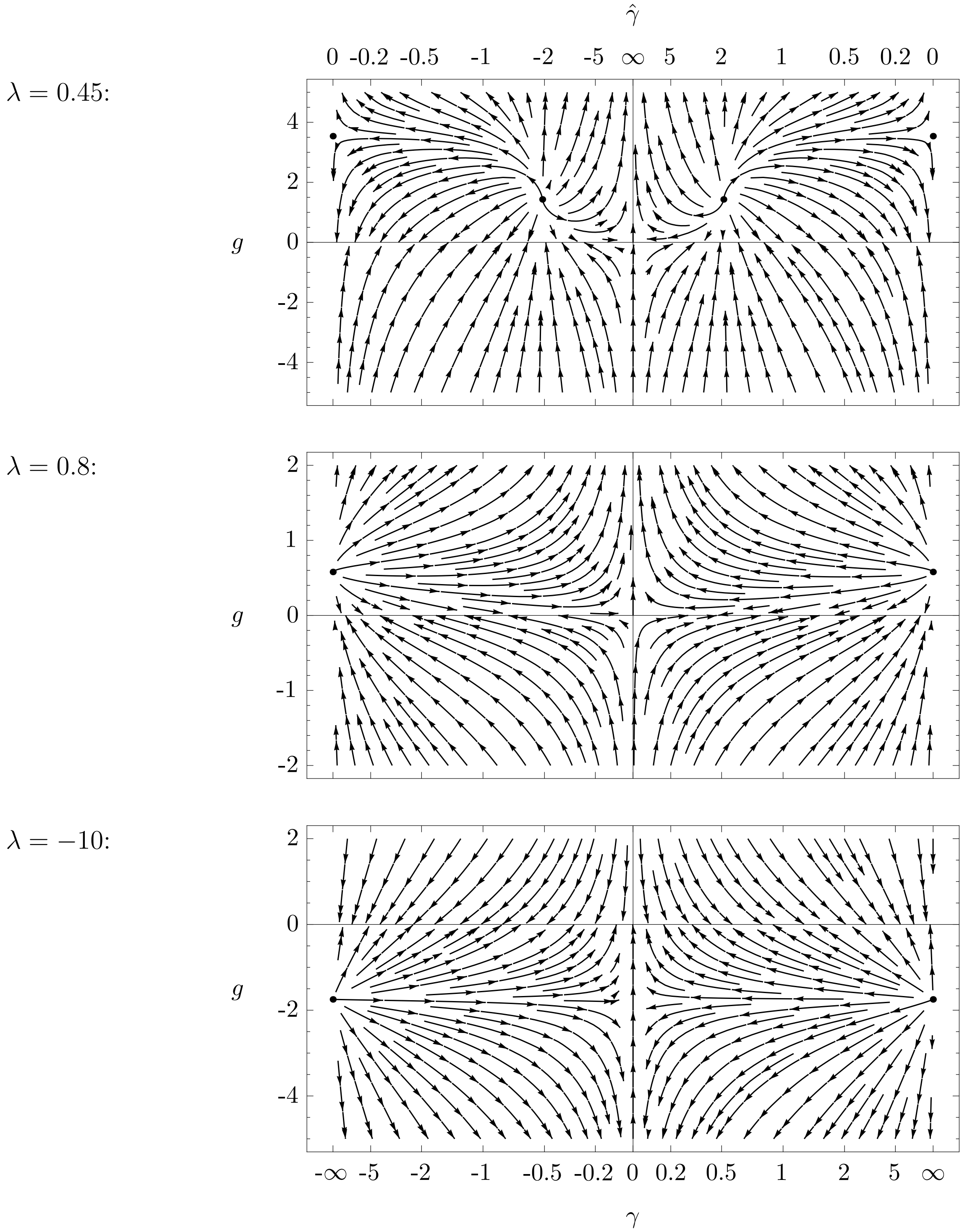


Figure 5.18. Phase portraits of the (γ, g)-flow in basis $\mathcal{B}_3$ for various fixed values of λ. At small λ both $\mathbf{NGFP'_\infty}$ and $\mathbf{NGFP'_{fin}}$ exist. For increasing λ $\mathbf{NGFP'_{fin}}$ moves to larger $|\gamma|$ until it merges with $\mathbf{NGFP'_\infty}$ and ceases to exist. For large (negative) λ the duality found in basis $\mathcal{B}_1$ is approximately established.

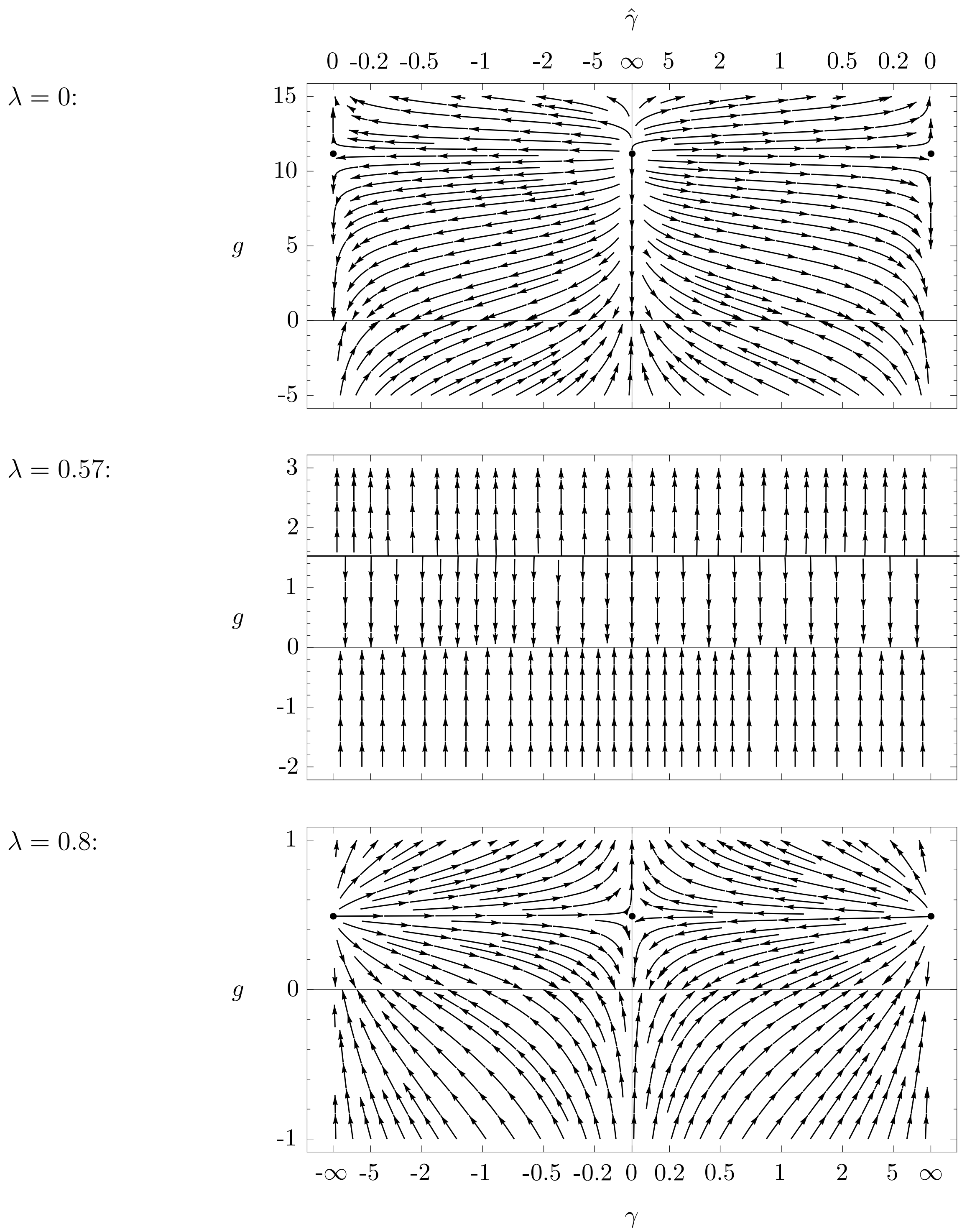


Figure 5.19. Phase portraits of the (γ, g)-flow in basis $\mathcal{B}_1$ for various fixed values of λ. We always find the fixed points $\mathbf{NGFP}'_\infty$ and $\mathbf{NGFP}'_0$, while λ determines the direction of the γ-flow. For $\lambda \approx 0.57$ it vanishes such that all trajectories run vertically.

completely. In this special case all complete RG trajectories lie between the fixed lines $g = 0$ (IR limit) and $g = g^*$ (UV limit). If the exact RG flow showed this behavior we could construct a quantum theory of gravity with a prescribed fixed value of the Immirzi parameter, that would not change under RG transformations.

For all other values of the cosmological constant we find an asymptotically safe trajectory for any prescribed IR value of γ_{IR} (with $g_{\text{IR}} = 0$) which in the UV either runs to $\gamma = 0$ or $\hat{\gamma} = 0$, i. e. to the value of the fixed point with two UV attractive directions. In this case we only find γ unrenormalized if we start with the fixed point values $\gamma_{\text{IR}} = 0$ or $\hat{\gamma}_{\text{IR}} = 0$ in the IR.

Comparison to the PT flow. When we compare the results of this subsection with the corresponding results of the proper-time flow analysis of [DR] we find indeed a lot of similarities. Besides the fixed line at $g = 0$, which is present in both studies, our most stable result is the existence of a NGFP at $\hat{\gamma} = 0$, which shows one attractive and one repulsive direction at small λ, but is attractive in both directions if we prescribe the physical fixed point values λ^* of the (λ, g)-truncation. This qualitative behavior was also found in [DR].

Besides that, the proper-time RG study reports a second NGFP at $\gamma = 0$, which shows an antagonistic behavior to the first NGFP that results in a predominant direction of the RG flow of the Immirzi parameter. While this direction coincides with our findings in both bases $\mathcal{B}_3$ and $\mathcal{B}_1$, the second fixed point is only found at $\gamma = 0$ if we decide for basis $\mathcal{B}_1$. In this case, however, the analogies between the results of both studies can be extended even further, as we will point out in the following.

In [DR] the β-functions are too complicated to be written down explicitly. Nevertheless to a good approximation they showed a very simple dependence on the Immirzi parameter γ which gave rise to a conjecture concerning their form in an exact treatment, with the conclusion being that β_g was expected to be independent of γ and β_γ being proportional to γ. In basis $\mathcal{B}_1$ we find that our β-functions are exactly of this predicted form (cf. (5.104) for $P_8 = 0$)!

Moreover, the complete halt of the RG running of γ is only possible if β_γ shows such a simple γ dependence. Although the Immirzi parameter staying unrenormalized under RG transformations appears as a very special case in the above discussion (only in basis $\mathcal{B}_1$ and only at $\lambda \approx 0.57$), one can argue that this behavior should actually be expected for an RG study of the Holst truncation with *vanishing* cosmological constant, that properly treats parity-even and -odd contributions on the same footing: Since the Immirzi parameter is a relative coupling between the parity-even curvature term and the parity-odd Immirzi term, a vanishing of its flow means that both terms are renormalized exactly the same way and therefore share the Newton constant as their mutual coupling, while the Immirzi parameter only sets a fixed (k-independent) ratio between parity-even and -odd terms in Γ_k.

We can only expect such a symmetry between both sorts of terms, if we study a truncation ansatz which exhibits the same symmetry between its scalar and pseudo-scalar constituents and make sure that the method applied to calculate the RG flow respects this symmetry throughout. With this in mind it is clear that our method cannot maintain such a symmetry even if it is present in the original truncation. This is due to the gauge-fixing action (and resulting ghost action) we used, that only contains scalar and no pseudo-scalar terms.

It has been shown in [Dau] that the terms in β_γ corresponding to the square bracket $[P_9(\lambda) + P_{10}(\lambda)]$ in (5.104) (for basis $\mathcal{B}_1$) are proportional to the difference of scalar and pseudo-scalar contributions to the running of γ. If the symmetry of the truncation was maintained, this bracket would therefore vanish for $\lambda = 0$. By choosing $\lambda \neq 0$ we deliberately break the symmetry of the underlying truncation, as the cosmological constant term corresponds to an additional scalar constituent, that has no pseudo-scalar counterpart.

On the other hand it seems quite natural that we can use the value of the cosmological constant to control the amount of scalar contributions to the renormalization of the Immirzi parameter. From this point of view we find that it is possible to effectively restore the symmetry between parity-even and -odd terms in Γ_k, that was broken by the gauge-fixing procedure, by choosing $\lambda \approx 0.57$. Thus, our result corroborates impressively the conjecture of an inherent symmetry between scalar and pseudo-scalar terms in the (γ, g)-truncation that was put forward in [Dau] referring to the simple form of β_γ, as we are able to restore this symmetry for a specific value of λ in basis $\mathcal{B}_1$.

Conclusion. In summary, the results of this subsection have shown that the RG analysis of the (γ, g)-truncation using the new WH-like flow equation and the proper-time RG study of the same system carried out in [Dau, DR] reinforce each other. Moreover, among the a priori equally suitable bases $\mathcal{B}_i$ in theory space, $\mathcal{B}_1$ is singled out as the one that, first, maximizes the physically significant similarities between both studies and, second, incorporates a symmetry property of the truncation, that leads to a β-function of the Immirzi parameter of the simple form

$$\beta_\gamma = g\,\gamma\, f(\lambda), \tag{5.116}$$

which is lost otherwise. For these reasons we shall consider basis $\mathcal{B}_1$ as being the most physical from now on; hence, we will concentrate on this basis in the discussion of the RG flow in the full 3-dimensional truncation covered in the next subsection.

5.4.3 The complete (λ, γ, g)-system

In this subsection we analyze the RG flow in the complete 3-dimensional coupling space of the Holst truncation. As before we will first discuss its fixed point structure for all four bases $\mathcal{B}_i$ and study the dependence of their properties on the mass parameter m that was held fixed to unity up to now. From the above we keep in mind that $\mathcal{B}_1$ amounts to the most physical basis in theory space. Therefore the subsequent analysis of the phase diagram will be restricted to this basis.

The system of flow equations under consideration in this subsection is given by (5.93). Concerning its global properties we find that the $\gamma \mapsto -\gamma$ symmetry of the flow is preserved as β_λ is an even function of γ as well. Moreover, we observe that the 3-dimensional coupling space is divided by three planes, which no trajectory of the flow can cross: The $g = 0$-plane, where $\beta_g = 0$, the $\gamma = 0$-plane (with $\beta_\gamma = 0$) and the $\lambda = 1$-plane, where all three β-functions diverge. The line $g = 0 = \lambda$ can, by inspection, be identified as a fixed line and it corresponds to the GFP known from the (λ, g)-truncation for all values of γ. As all trajectories in the half-space $\lambda > 1$ are separated from the classical regime close to this Gaussian fixed line, we will not consider this part of the coupling space any further.

Fixed point structure. Since $\beta_{\hat\gamma}$ vanishes in the $\hat\gamma = 0$-plane, it amounts to a self-consistent 2-dimensional (λ, g)-truncation, i. e. trajectories starting in this plane will never leave it. Thus, the trajectories of the three dimensional flow in this plane, coincide with the flow of the (λ, g)-system in the $\gamma \to \infty$ limit (cf. Section 5.4.1). From this observation we can already conclude that in the $\hat\gamma = 0$-plane we find two NGFPs, $\mathbf{NGFP^1_\infty}$ and $\mathbf{NGFP^2_\infty}$, that are UV attractive in both directions of the

(λ, g) \ (γ, g)	$\mathbf{NGFP'_0}$ ($\mathcal{B}_1$)	$\mathbf{NGFP'_{fin}}$ ($\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$)	$\mathbf{NGFP'_\infty}$
$\mathbf{NGFP^1}$	$\mathbf{NGFP^1_0}$?	-	$\mathbf{NGFP^1_\infty}$
$\mathbf{NGFP^2}$	$\mathbf{NGFP^2_0}$?	-	$\mathbf{NGFP^2_\infty}$

Table 5.2. Overview of the fixed points present in the different truncations: The first row contains the fixed points of the (γ, g)-truncation, while the first column displays the ones of the (λ, g)-truncation. The main body of the table contains the names given to the corresponding fixed points in the (λ, γ, g)-truncation, in case they exist.

plane, exhibiting the critical exponents found in Section 5.4.1. Therefore, we only have to analyze their third critical exponent, that describes the attractivity property in $\hat{\gamma}$-direction. We conclude that the fixed points $\mathbf{NGFP'_\infty}$ and $\{\mathbf{NGFP^1}, \mathbf{NGFP^2}\}$ that we found in the 2-dimensional truncations are the projections of the 3-dimensional fixed points $\{\mathbf{NGFP^1_\infty}, \mathbf{NGFP^2_\infty}\}$ for fixed λ and γ, respectively.

Similarly the $\gamma = 0$-plane amounts to a self-consistent truncation as β_γ vanishes in this plane. However, since β_λ diverges here, this limit eludes further investigation. Nonetheless we want to stress that the mechanism giving rise to $\{\mathbf{NGFP^1}, \mathbf{NGFP^2}\}$ works for arbitrarily small γ, such that in basis $\mathcal{B}_1$ we should expect the existence of the three dimensional lifts $\{\mathbf{NGFP^1_0}, \mathbf{NGFP^2_0}\}$ in the $\gamma = 0$-plane if only the—probably unphysical—logarithmic divergence of β_λ at $\gamma \to 0$ could be removed.

In contrast to this situation, the fixed point solutions $\mathbf{NGFP'_{fin}}$, that were found in bases $\mathcal{B}_{\{2,3,4\}}$, are lost when the third coupling λ is subject to renormalization as well. Thus, in the 3-dimensional truncation for these bases no fixed point at finite γ^* is found.

We have summarized the above discussion on the different fixed points and their presence in the different truncations in Table 5.2.

In the next two paragraphs we will analyze the fixed points $\mathbf{NGFP^1_\infty}$ and $\mathbf{NGFP^2_\infty}$ and will thereby focus on their third critical exponent, that describes the attractivity property in $\hat{\gamma}$-direction. In addition we will investigate the dependence of all fixed point properties on the mass parameter m, that has been discussed only qualitatively in Section 5.4.1 so far.

In a third paragraph, we will explore an additional mechanism present only in the full three dimensional truncation that in basis $\mathcal{B}_1$ gives rise to a pair of fixed points

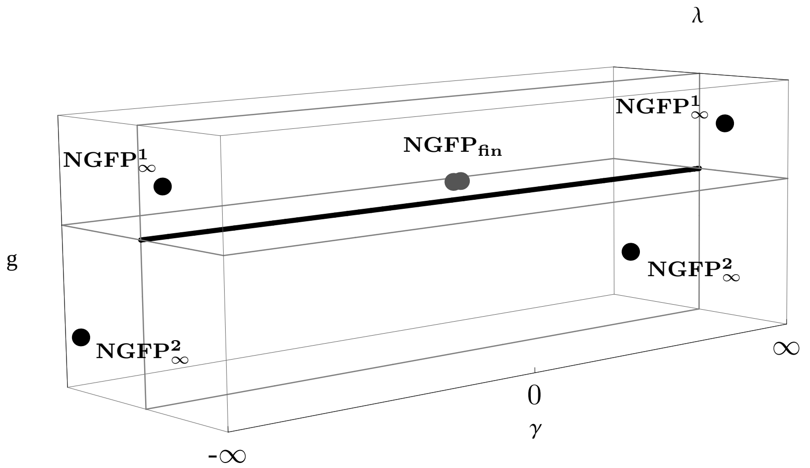


Figure 5.20. Sketch of the fixed point structure of the (λ, γ, g)-system. The pair of fixed points $\mathbf{NGFP_{fin}}$ is shaded in gray as it is only present in basis $\mathcal{B}_1$, and absent in the other bases.

at finite $|\gamma^*|$. We will denote this fixed point by $\mathbf{NGFP_{fin}}$. In the other bases no equivalent to this fixed point is observed.

Before we start to discuss the details of the individual fixed points let us again schematically depict the situation in the 3-dimensional theory space in Fig. 5.20.

(A) The fixed point $\mathbf{NGFP^1_\infty}$. The first NGFP in the $\hat{\gamma}=0$-plane lies in the positive (λ, g)-quadrant. Its position as a function of the mass parameter m is depicted in Fig. 5.21. We find the results for the 4 different bases perfectly aligned. Moreover, the position does not depend very much on the value of m, at least for $m \gtrsim 1$. At small m we find a rapid variation of the FP position with m, which seems, in the light of the almost flat curves for larger m, unphysical. Thus this is a first indication that our truncation should be trusted in only for $m \gtrsim 0.5$.

In Fig. 5.22 we have plotted the corresponding critical exponents. In the left panel we find depicted the real parts of θ_1 and θ_3 that correspond to the eigendirection lying inside the $\hat{\gamma} = 0$-plane. They start off as a complex pair and turn real at $m \approx 0.8$. While one of the critical exponents becomes unreasonably large the other one stays at moderate values of about 5-8.

In the right panel the critical exponent corresponding to the $\hat{\gamma}$-direction is plotted and we observe that it is increasing with m.

For all bases we find that, in the trusted region of $m \gtrsim 0.5$, the fixed point is attractive in all three directions. Besides this qualitative similarity found in all bases,

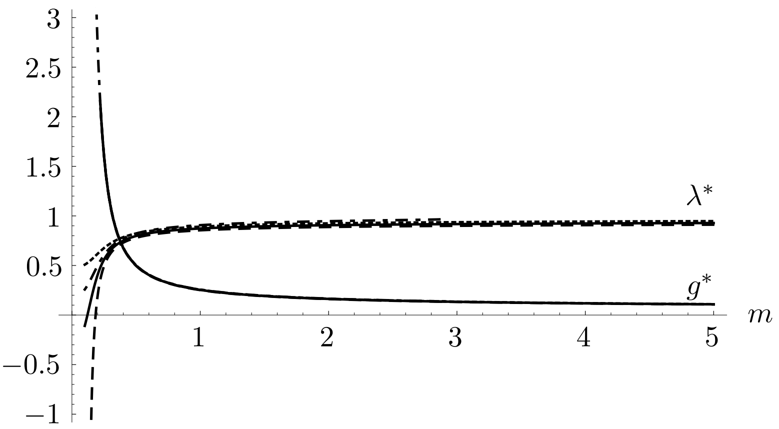


Figure 5.21. The fixed point coordinates of $\mathbf{NGFP}^{\mathbf{1}}_{\infty}$ as functions of m. For not to small values of m the curves are almost constant functions and match each other perfectly for the four bases ($\mathcal{B}_1$ solid, $\mathcal{B}_2$ dashed, $\mathcal{B}_3$ dot-dashed, $\mathcal{B}_4$ dotted).

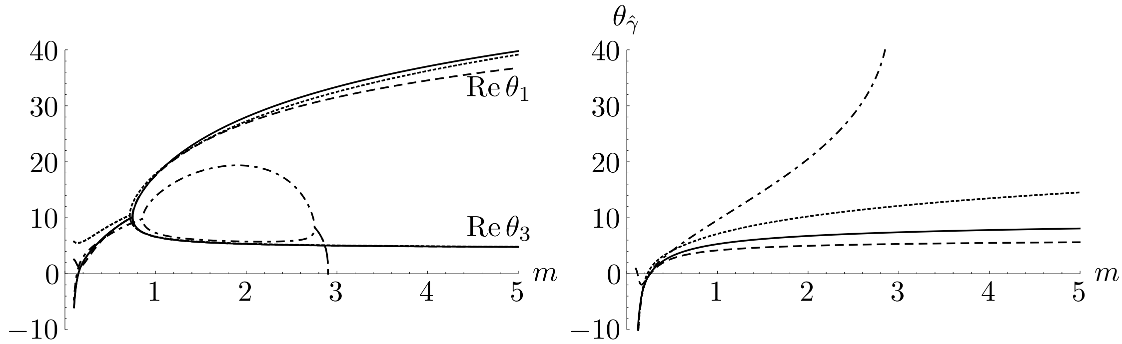


Figure 5.22. The critical exponents of $\mathbf{NGFP}^{\mathbf{1}}_{\infty}$ as functions of m. Here the results for the bases $\mathcal{B}_1$ (solid), $\mathcal{B}_2$ (dashed) and $\mathcal{B}_4$ (dotted) are well aligned. While for these bases the FP exists up to large m, in basis $\mathcal{B}_3$ (dot-dashed) it vanishes at $m \approx 3$.

the attractivity properties in the bases $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_4$ are very similar as functions of m, even quantitatively. In basis $\mathcal{B}_3$, on the other hand, the fixed point vanishes at $m \approx 3$ and the critical exponents $\theta_{\hat{\gamma}}$ gets considerably larger than in the other bases.

(B) The fixed point $\mathbf{NGFP}^{\mathbf{2}}_{\infty}$. In Fig. 5.23 the position of the second fixed point, situated in the negative (λ, g)-quadrant, is shown. Note that this FP does not exist in basis $\mathcal{B}_2$ such that we are left with three different types of lines in Figs. 5.23 and 5.24. We find that λ^* is decreasing to very large negative values, for increasing m, while g^* approaches small absolute values. At small m the g^* coordinate begins to

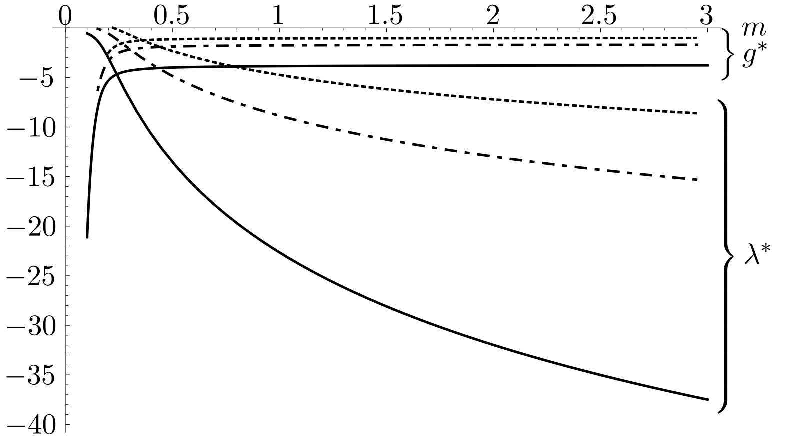


Figure 5.23. The fixed point coordinates of $\mathbf{NGFP}^{\mathbf{2}}_{\boldsymbol{\infty}}$ that exists in the three bases $\mathcal{B}_1$ (solid), $\mathcal{B}_3$ (dot-dashed) and $\mathcal{B}_4$ (dotted) as functions of m. While the λ^*-coordinate varies with m and the choice of basis, for the g^*-coordinate this dependence is far less pronounced.

diverge, which we should interpret as the boundary of the trusted region in m which here occurs at $m \approx 0.3$.

The corresponding critical exponents are plotted in Fig. 5.24. We find a very stable prediction of three real and positive critical exponents whose values do not depend on the choice of basis and that show only a slight variation with m, for $m \gtrsim 0.7$. Moreover, in contrast to $\mathbf{NGFP}^{\mathbf{1}}_{\boldsymbol{\infty}}$ all critical exponents take on reasonably small absolute values.

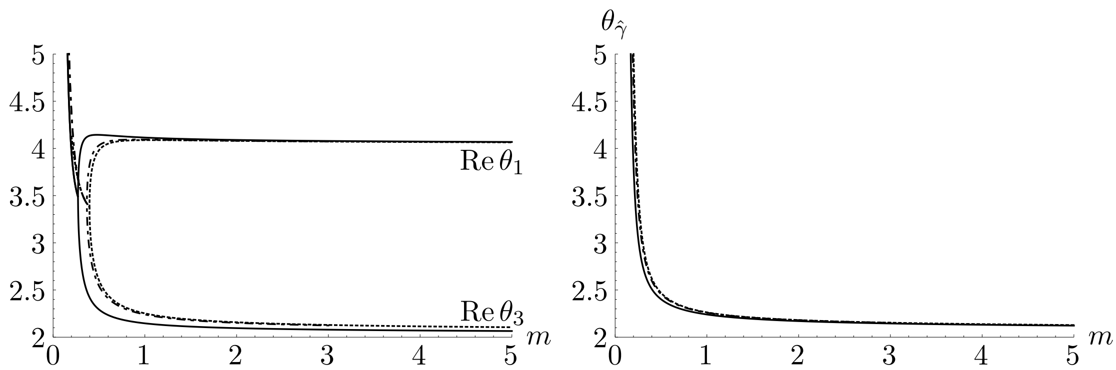


Figure 5.24. The critical exponents of $\mathbf{NGFP}^{\mathbf{2}}_{\boldsymbol{\infty}}$ as functions of m. Here the results for the bases $\mathcal{B}_1$ (solid), $\mathcal{B}_2$ (dashed) and $\mathcal{B}_4$ (dotted) are well aligned and are almost constant for $m \geq 0.8$ and stay in a reasonable range.

(C) The fixed point $\mathbf{NGFP_{fin}}$. In basis $\mathcal{B}_1$, besides the above fixed points in the $\hat{\gamma}=0$-plane, an additional NGFP at finite γ exists. Its λ^* coordinate is given by the condition $\beta_\gamma = 0$ and thus corresponds to the zeros of $P_9(\lambda) + P_{10}(\lambda)$ at $\lambda \approx 0.573$ and $\lambda \approx 2.918$. As the latter lies behind the $\lambda = 1$ divergence of the β-functions, we will only consider the former. Its corresponding g^*-coordinate is found from $\beta_g = 0$ to $g^* = 1.525$.

These two coordinates are substituted into $\beta_\lambda(\lambda^*, \gamma, g^*) = 0$, and each solution of this equation for γ amounts to a fixed point of the 3-dimensional flow. This last fixed point condition has the following general form:

$$\beta_\lambda = 0 = C_1(g^*, \lambda^*, m) + C_2(g^*) \ln \left(\frac{\gamma^2 - 1}{\gamma^2} \right)^2 , \tag{5.117}$$

where C_1 and $C_2 < 0$ are constants that only depend on the other (already fixed) FP coordinates and the mass parameter m. From this form we can infer that there is always at least one solution to the condition (5.117) in the interval $\gamma^2 \in (0, 1)$, as the logarithm is a smooth function interpolating between $\pm\infty$ for the limits $\gamma \to 0$ and $\gamma \to 1$, respectively. We refer to the fixed point corresponding to this solution as $\mathbf{NGFP_{fin}}$. Whether or not a second solution with $\gamma^2 > 1$ exists, depends on the value of m: For $\gamma > 1$ the RHS is a monotonically decreasing function and in the limit $\gamma \to \infty$ the logarithmic term vanishes. Thus, a second fixed point solution with $|\tilde{\gamma}^*| > 1$ exists for those values of m for which $C_1(g^*, \lambda^*, m) < 0$.

Note that for the other bases this mechanism is not at work as the dependence of β_g and β_γ on γ is different. In this case we can only infer functions $\gamma^*(\lambda)$ and $g^*(\lambda)$ from β_g and β_γ. However, the last condition $\beta_\lambda(\lambda, \gamma^*(\lambda), g^*(\lambda)) = 0$ does not have a solution, such that we do not find a fixed point at finite γ in the other bases.

The exact values of the coordinate γ^* and the critical exponents can only be obtained numerically. The resulting functions of m are depicted in Fig. 5.25. In the left panel we find the fixed values g^* and λ^* represented by constant functions, as well as the function $\gamma^*(m)$, corresponding to the solution that exists for all m. It starts off at $\gamma^*(0) = \pm 1$ and approaches 0 rapidly for $m \gtrsim 0.8$. The second solution $\tilde{\gamma}^*$ exists only for $m \lesssim 0.35$; as this region of m has been proven to obtain questionable results before, we will not consider this second pair of fixed point solutions any further.

In the right panel the real parts of the critical exponents corresponding to the fixed point $\mathbf{NGFP_{fin}}$ are shown. We observe that there is one UV repulsive eigendirection

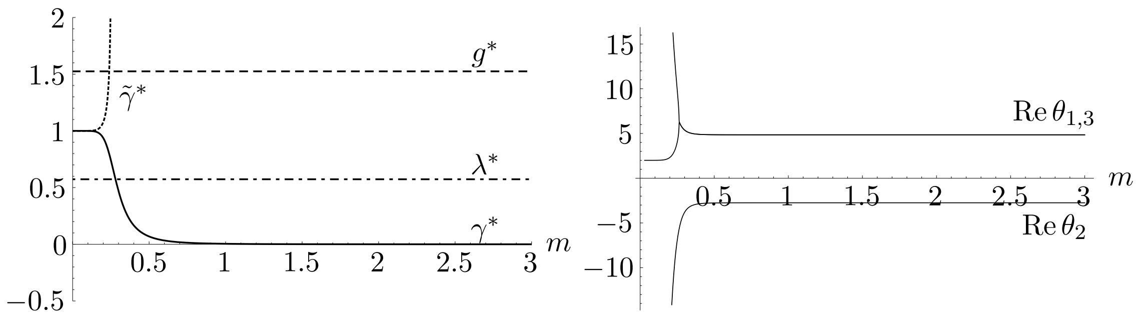


Figure 5.25. Position and critical exponents of $\mathbf{NGFP_{fin}}$ as a function of m. For $m > 0.8$ the fixed points have moved to $\gamma^* \approx 0$ and the functions are constant from there on. At small $m < 0.35$ a second pair of fixed points (with coordinate $\tilde{\gamma}^*$) exists, which is not further analyzed.

for all m. The other two critical exponents, that are real and positive for small m, become a complex conjugated pair at $m \approx 0.4$ with positive real part. Once the fixed point coordinate γ^* approaches 0 with increasing m, the values of the critical exponents do not change any more.

Phase portrait. The resulting phase portrait of the full 3-dimensional RG flow is depicted in Figs. 5.26 and 5.27. As it is difficult to visualize a 3-dimensional vector field in every point of space, we have decided to display sets of trajectories, whose starting points lie in planes of fixed $|\gamma|$. The red trajectories start close to the $\gamma=0$-plane at $\gamma = \pm 0.06$, the blue ones at $\gamma = \pm 1.38$ and the black ones lie entirely in the $\hat{\gamma}=0$-plane. In addition to the three-dimensional view in (a) we have given a frontal view onto the (γ, g)-plane in (b) and the (λ, γ)-plane in (c) for a better understanding of the spatial course of the trajectories in theory space.

In Fig. 5.26 we have plotted a region of theory space with $g > 0$. The fixed point $\mathbf{NGFP^1_\infty}$ is clearly visible in (a) and (b), lying on the lateral faces of the box that correspond to the $|\gamma| \to \infty$ limit. All trajectories shown have this fixed point as their UV limit. The second fixed point $\mathbf{NGFP_{fin}}$ is not visible, as its UV critical hypersurface is only two dimensional.

Note that the flow in the lateral faces, due to the self-consistency of the $\hat{\gamma} = 0$-truncation, corresponds exactly to the flow depicted in the lower right panel of Fig. 5.11.

From Fig. 5.26 (c) we observe that the flow first stays very much in the γ-plane of its starting points. Only when λ exceeds its fixed point value, the running of γ

becomes predominant. For that reason (b) looks very similar to the last panel of Fig. 5.19 as most of the visible part of the trajectories in this projection lies in the small region $\lambda^* < \lambda < 1$. This behavior of the flow can be related to the critical exponents of the fixed point $\mathbf{NGFP^1_\infty}$: The huge value of $\theta_1 \approx 20$ results in a predominantly fast running of λ. Only when it has become close to its fixed point value, the other couplings, which are related to critical exponents of much smaller magnitude, are seen to run as well.

Fig. 5.27 shows a similar set of plots, only for the $g < 0$ part of coupling space. Again we find that the fixed point at $\hat{\gamma} = 0$, $\mathbf{NGFP^2_\infty}$, is dominating the flow. This time the plane $\hat{\gamma} = 0$ (black trajectories) corresponds to the flow depicted in the lower right panel of Fig. 5.12, but we observe that the other trajectories (blue, red), in contrast to the above case, do not stay in their starting plane of fixed γ, as the fixed point $\mathbf{NGFP^2_\infty}$ has three positive critical exponents of the same order of magnitude.

We conclude that in our truncation both FPs, $\mathbf{NGFP^1_\infty}$ and $\mathbf{NGFP^2_\infty}$, allow for the construction of an asymptotically safe quantum theory and show a basin of attraction that spans the whole part of coupling space with $g > 0$ and $g < 0$, respectively, that we have analyzed.

Let us finally comment on the influence of the logarithmic divergences in β_λ on the gross properties of the flow:

We have seen from the above examples that the divergence of β_λ at $\gamma = 1$ is crossed by the trajectories smoothly. Indeed this divergence only has a very local effect on the trajectories, that pass the $\gamma = 1$ plane being tangential to it, similarly to the function $x \ln x^2$ crossing $x = 0$. Fig. 5.27 (c) gives an idea of this behavior by the sharp bend in the red trajectories at $\gamma = 1$, while in Fig. 5.26 (c) the effect is to localized for being visible at the given scale.

Similarly, the fixed point $\mathbf{NGFP'_0}$ from the (γ, g)-truncation vanishes due to a logarithmic divergence of β_λ. Again this effect is very local, such that e. g. Fig. 5.26 (b) would not look much different, if an additional fixed point at $\gamma = 0$ was present.

If we take into account that the logarithmic contributions to β_λ correspond to prefactors of a quartic momentum divergence of the path integral, that are most sensitive to all details of the renormalization procedure, we should not take their exact form too seriously. Therefore we should not exclude the existence of one (or more) additional fixed point(s) at $\gamma = 0$ with two UV attractive directions in the three dimensional coupling space on the basis of the above results.

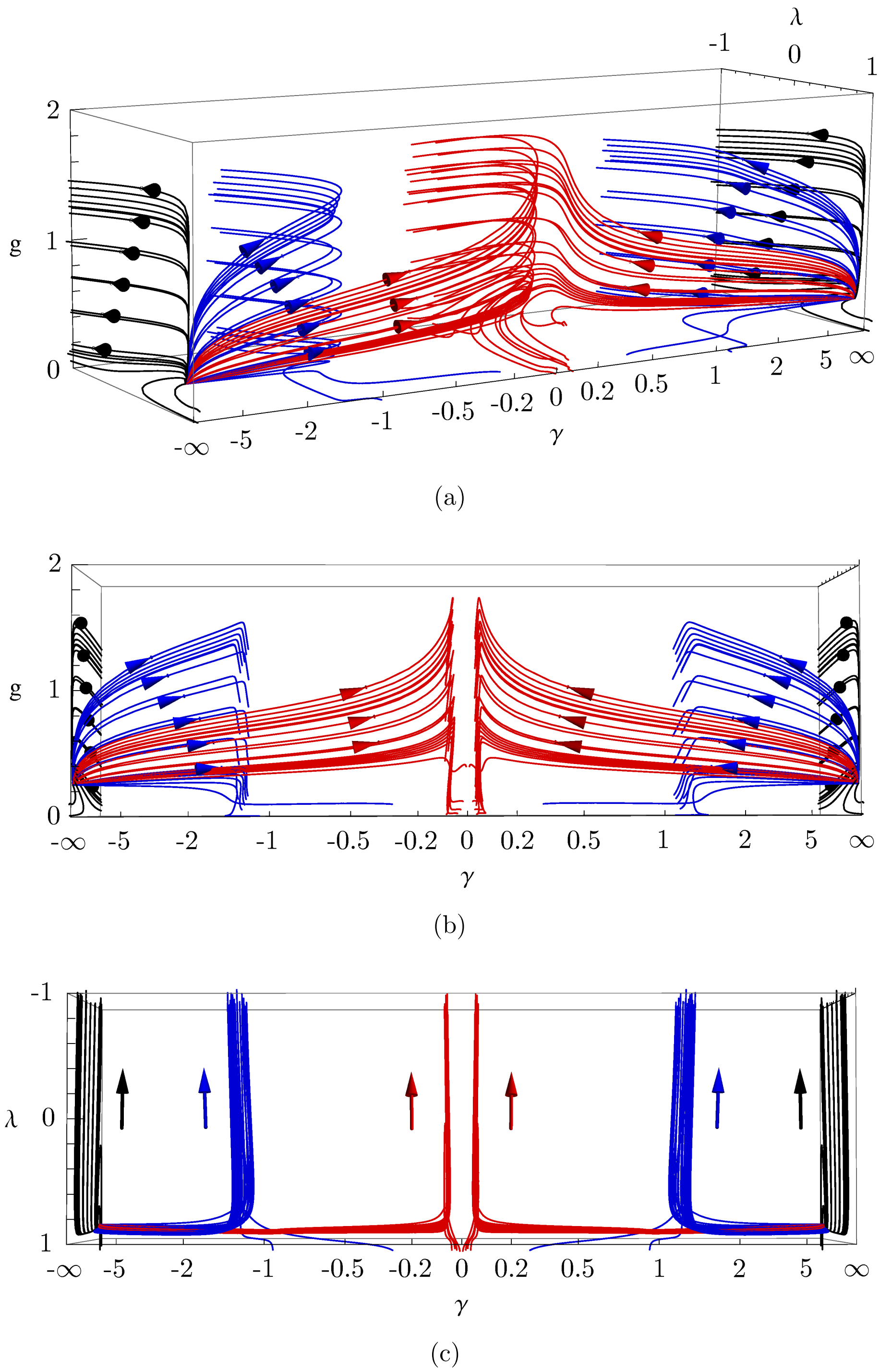


Figure 5.26. RG flow in the part of the 3-dimensional coupling space with $g > 0$. All trajectories of the same color pass through points with the same $|\gamma|$: $|\gamma| = 0.06$ (red), $|\gamma| = 1.38$ (blue) and $|\hat{\gamma}| = 0$ (black). The flow is directed such that all trajectories share the fixed point $\mathbf{NGFP}^{\mathbf{1}}_{\infty}$ as their UV limit.

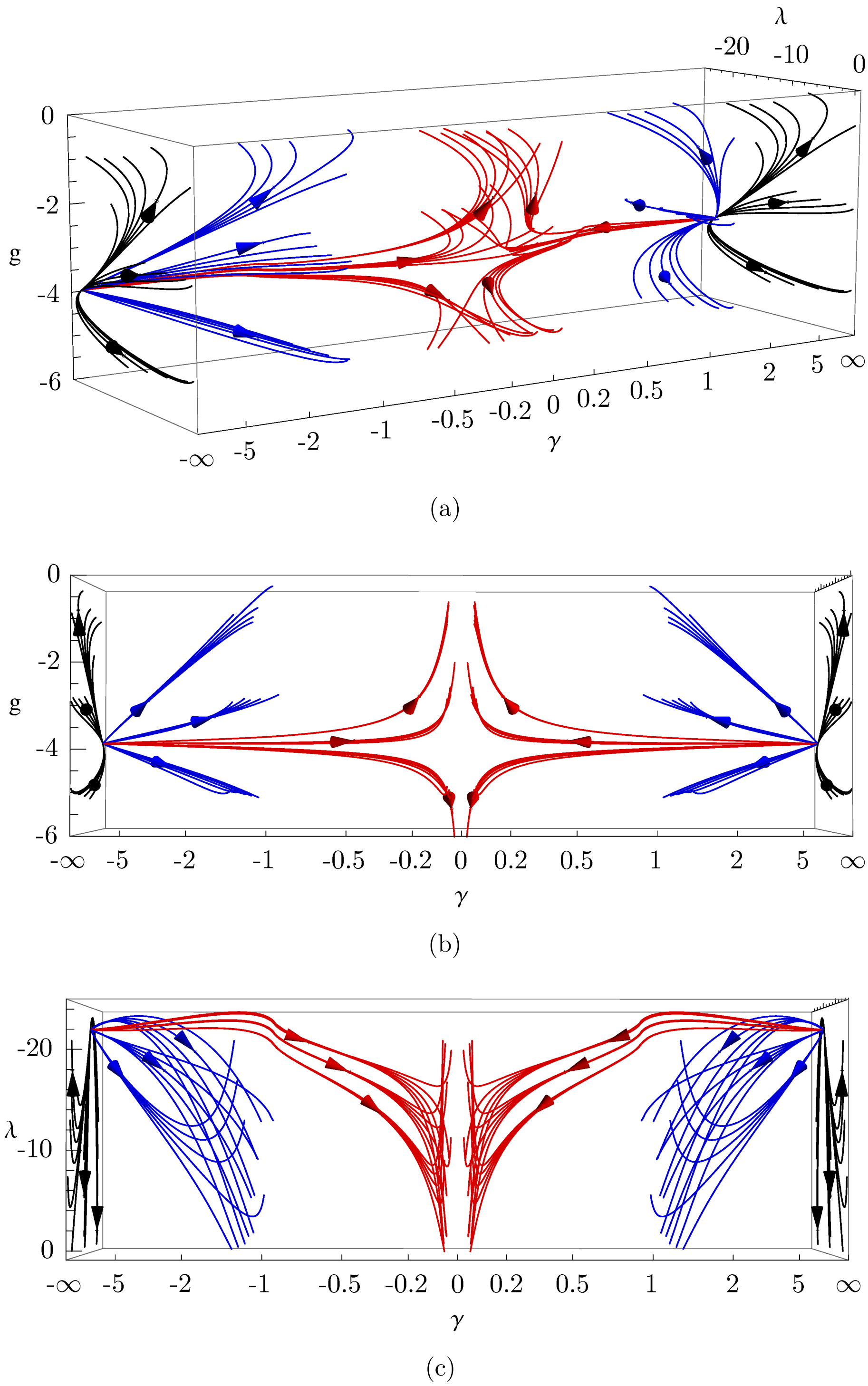


Figure 5.27. RG flow in the part of the 3-dimensional coupling space with $g < 0$. All trajectories of the same color pass through points with the same fixed $|\gamma|$: $|\gamma| = 0.06$ (red), $|\gamma| = 1.38$ (blue) and $|\hat{\gamma}| = 0$ (black). The flow is directed such that all trajectories share the fixed point $\mathbf{NGFP}^{\mathbf{2}}_{\infty}$ as their UV limit.

Comparison to the PT flow. In summary we find that also the 3-dimensional flow shows certain characteristic similarities to the flow obtained in the PT approximation [DR]. Besides the existence of a NGFP in the $\hat{\gamma}=0$-plane that is UV attractive in all three directions we also find the predominant direction of the γ-flow towards larger absolute values in the UV that was reported in [DR].

The most prominent differences to the PT study are the exact position of the fixed points in the $\hat{\gamma} = 0$-plane and the existence of a fixed point at $\gamma = 0$. Both features heavily depend on the exact form of the contributions from quartic momentum divergences, that are notoriously unstable under different renormalization procedures.

Thus, both studies support each other concerning the form of the β-functions β_g and β_γ as well as most properties of the fixed points in the $\hat{\gamma}=0$-plane. Another common feature of both calculations is the absence of a reliable fixed point at finite γ. Thus all asymptotically safe trajectories either take on the value $\hat{\gamma} = 0$, corresponding to freely fluctuating torsion, or possibly also $\gamma = 0$, where certain torsion components are suppressed completely, in the deep UV. Note that none of these limits can be considered as being equivalent to metric gravity, since some torsion components fluctuate freely in both cases.

5.5 RG flow of chiral gravity

This section contains the first RG analysis of chiral gravity. It is carried out using the WH-like flow equation, whose evaluation was performed in a completely analogous manner to the study of the full QECG presented in the previous sections of this chapter. Thus, we will only highlight the differences compared to QECG in a first subsection before we directly proceed with the presentation of the resulting RG flow in the second subsection.

5.5.1 Modifications compared to the RG study of QECG

Field content. The most obvious modification in comparison to QECG is that we restrict the field space of spin connections to one chirality. Thus we only allow spin connections that are either selfdual or anti-selfdual, which, in the Euclidean setting, corresponds to eigenvectors of the duality operator $\star = \frac{1}{2}\varepsilon^{ab}{}_{cd}$ with eigenvalues $+1$ or -1, respectively.

As any generic spin connection ω can be decomposed into a selfdual and an anti-selfdual part $\omega^{(\pm)}$ by the projectors $P^{\pm} = \frac{1}{2}(\mathbb{1} \pm \star)$ according to $\omega = (P^{+} + P^{-})\omega = \omega^{(+)} + \omega^{(-)}$, this restriction corresponds to halving the number of independent components of the spin connection field variable. Thus we are left with $28 = 16 + 12$ independent field components of vielbein and spin connection, respectively. This different counting will be reflected in the dimension of the Hessian, that in the chiral case corresponds to a 28 x 28-matrix operator. We will see later on how an adapted decomposition of the fluctuation fields gives rise to a simple reduction of the Hessian that was obtained for the general Holst truncation to the chiral case.

Gauge symmetry. As the gauge field $\omega^{(\pm)}$ contains only half the number of independent field components compared to the full spin connection ω, that is related to the $\mathsf{O}(4)$-gauge symmetry, one can naively expect that this part of the gauge group is reduced to half of its size in the chiral case. To make this statement more precise let us shortly discuss how the chirality projectors $P^{\pm}$ decompose the total $\mathsf{O}(4)_{\text{loc}}$ into two chiral components.

If we denote the six generators of the full $\mathsf{O}(4)$-gauge group by M_{ab}, with $M_{ab} = -M_{ba}$, by definition they satisfy the algebra

$$[M_{ab}, M_{cd}] = i(\delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \delta_{bc}M_{ad} - \delta_{ad}M_{bc}) \,. \tag{5.118}$$

One can show by direct computation that the 3 generators $M^{\pm}_{ab} = (P^{\pm}M)_{ab}$ of each sign satisfy an algebra of the same form, individually. Moreover, the generators of different duality commute with each other, i. e. $[M^{\pm}_{ab}, M^{\mp}_{cd}] = 0$. Using the t'Hooft η-symbols [Hoo76] that map (anti-)selfdual $\mathsf{O}(4)$-tensors onto $\mathsf{SO}(3)$-vectors it is an easy task to show that the generators $L^{\pm}_{i} = \frac{1}{4}\eta_{i}{}^{ab}M^{\pm}_{ab}$ satisfy the usual angular momentum algebra $[L_i, L_j] = i\varepsilon_{ij}{}^{k}L_k$. Thus we have shown that the $\mathsf{O}(4)$-algebra decomposes into two chiral factors of $\mathsf{SO}(3)$ such that locally also the groups satisfy

$$\mathsf{O}(4) \cong \mathsf{SO}^{+}(3) \times \mathsf{SO}^{-}(3). \tag{5.119}$$

This construction in Euclidean spacetime is analogous to the decomposition of the group of Lorentz transformations in Minkowski spacetime, $\mathsf{SO}(3,1)$, into two chiral $\mathsf{SU}(2)$ components which is used in order to classify all its representations. In contrast to our case and due to the distinct role of the time direction in Minkowski space the generators of boosts and rotations in the Lorentz group $\mathsf{SO}(3{,}1)$ obtain as complex combinations of the $\mathsf{SU}(2)$ components. Moreover, there, the Lorentz duality star squares to -1, $\star^2 = -1$, such that its eigenvalues are given by $\mp i$, corresponding to (anti-)selfdual components.

When we restrict ourselves to spin connections of one chirality we, thus, also reduce the gauge group to one chiral component of the above decomposition. In summary, we therefore conclude that the theory space of chiral gravity is reduced in both, the field content and the total symmetry group $\mathbf{G}$, and is hence is given by

$$\mathcal{T}^{\pm}_{\text{EC}} = \left\{ A[e^a{}_\mu, \omega^{\pm ab}{}_\mu, \cdots] \,|\, \text{inv. under } \mathbf{G} = \mathsf{Diff}(\mathcal{M}) \ltimes \mathsf{SO}^{\pm}(3)_{\text{loc}} \right\}, \tag{5.120}$$

where the dots stand for additional background- and ghost-field dependence.

Gauge conditions and ghost fields. With the reduced gauge group at hand also the 6 gauge fixing conditions $\mathcal{G}_{ab}$ of the former $\mathsf{O}(4)_{\text{loc}}$-group have to be reduced to only 3 that are needed to gauge-fix the remaining $\mathsf{SO}^{\pm}(3)_{\text{loc}}$ component. Most easily this is done by a projection of $\mathcal{G}_{ab}$ to its (anti-)selfdual part, such that we will use the conditions

$$\mathcal{G}^{\pm}_{ab} = (P^{\pm}\mathcal{G})_{ab}\,, \tag{5.121}$$

with the explicit form of $\mathcal{G}_{ab}$ given in (5.10). In consequence of this projection also the $\mathsf{O}(4)$-ghost fields $\bar{\Upsilon}_{ab}, \Upsilon_{ab}$ get replaced by their (anti-)selfdual components $\bar{\Upsilon}^{\pm}_{ab}, \Upsilon^{\pm}_{ab}$.

As a final comment we want to mention that due to the restriction to one chirality of the spin connection, of course, also the diffeomorphism gauge-condition $\mathcal{F}_\mu$ gets modified slightly, since the covariant derivative inside it now is constructed from the chiral spin connection.

Decomposition of the fluctuation fields. Analyzing the decomposition of the spin connection fluctuations that was used for the Holst truncation, eq. (5.52), we observe that the sets of fields (A, B) and (C, D), respectively, switch their roles under $\mathsf{O}(4)$-dualization and thus describe fluctuations dual to each other:

$$(A, B) \;\stackrel{\star}{\longleftrightarrow}\; (C, D). \tag{5.122}$$

For the chiral case it is most useful to employ a different decomposition splitting the fluctuations into selfdual and anti-selfdual components. It is obtained from the decomposition (5.52) according to

$$\bar{\tau}^{ab}_{\ \mu}(x) = \frac{\bar{\mu}^{\frac{1}{2}}}{\sqrt{2}}\left[\frac{\partial_\mu \partial^{[a}}{-\Box} A^{b]}(x) + \frac{\partial^{[a}}{\sqrt{-\Box}} B^{b]}_{\ \ \mu}(x) + \varepsilon^{ab}_{\ \ cd}\frac{\partial_\mu\partial^c}{-\Box} C^d(x) + \varepsilon^{ab}_{\ \ cd}\frac{\partial^c}{\sqrt{-\Box}} D^d_{\ \mu}(x)\right] \quad (5.123)$$

$$= \sqrt{2}\sum_{\pm}\Big(\left(P^{\pm}\right)^{ab}_{\ \ cd}\frac{\partial_\mu\partial^c}{-\Box}\left(A^d \pm C^d\right) + \left(P^{\pm}\right)^{ab}_{\ \ cd}\frac{\partial^c}{\sqrt{-\Box}}\left(B^d_{\ \mu} \pm D^d_{\ \mu}\right)\Big)$$

$$=: 2\sum_{\pm}\left(\left(P^{\pm}\right)^{ab}_{\ \ cd}\frac{\partial_\mu\partial^c}{-\Box}A^d_{\pm} + \left(P^{\pm}\right)^{ab}_{\ \ cd}\frac{\partial^c}{\sqrt{-\Box}}B_{\pm\ \mu}^{\ d}\right). \quad (5.124)$$

The last equality defines the four new component fields by

$$A_{\pm} := \frac{1}{\sqrt{2}}(A \pm C) \quad \text{and} \quad B_{\pm} := \frac{1}{\sqrt{2}}(B \pm D). \quad (5.125)$$

We observe that now the fields with the plus/minus index describe the selfdual/anti-selfdual components of the fluctuation field $\bar{\tau}$, respectively.

It is important to note that up to this point both decompositions, (5.123) and (5.124), are completely equivalent and that the same RG flow of the full Holst truncation with a running Immirzi parameter can be obtained using either decomposition. In the chiral case, however, we fix the Immirzi parameter to $\gamma = \mp 1$ such that the action only depends on the selfdual/anti-selfdual part of the spin connection. This fact results in a quadratic form $\Gamma^{\text{quad}}_{\pm}$ that only depends on the fields $A_{\pm}, B_{\pm}$ of the respective sign index, leading to vanishing rows and columns in the Hessian, that correspond to the fields of the other sign index.

The reason for the vanishing rows and columns, and the obvious zero modes of the Hessian they give rise to, is that using the decomposition (5.124) we did not restrict the field space of fluctuations to one chirality. This restriction can be carried out at this stage by simply discarding the (vanishing) rows and columns of the Hessian that correspond to the other chirality. Thus we have a simple method at hand that reduces the 40 x 40 matrix operator of QECG to the 28 x 28 Hessian of chiral gravity, reflecting the reduced number of independent field components in the spin connection.

For the reduction of the O(4)-ghosts we proceed in complete analogy. First we define a general chiral decomposition

$$\begin{aligned}\bar{\mu}^{-1}\Upsilon^{ab} &= \frac{1}{\sqrt{2}}\left(\frac{\partial^{[a}}{\sqrt{-\Box}}F^{b]} + \varepsilon^{ab}{}_{cd}\frac{\partial^{c}}{\sqrt{-\Box}}G^{d}\right) \\ &= 2\sum_{\pm}\left(\left(P^{+}\right)^{ab}{}_{cd}\partial^{c}F_{+}^{d} + \left(P^{-}\right)^{ab}{}_{cd}\partial^{c}F_{-}^{d}\right) ,\end{aligned} \tag{5.126}$$

where $F_{\pm} := \frac{1}{\sqrt{2}}(F \pm G)$. After having calculated the Hessian w. r. t. this decomposition in the chiral case we skip the rows and columns corresponding to the opposite chirality.

With this reduced Hessian operator at hand, we proceed with the evaluation of the WH-like flow equation the same way as described for the full Holst truncation in Section 5.2.

5.5.2 Derivation of the β-functions

Keeping the differences to the full Holst truncation in mind we are now in the position to derive the β-functions of Newton's constant and the cosmological constant in chiral gravity. We thus start with a truncation of the form

$$\Gamma_{k}^{\pm} = -\frac{1}{8\pi G_{k}}\int \mathrm{d}^{4}x e\Big[e_{a}{}^{\mu}e_{b}{}^{\nu}F(\omega^{(\pm)})^{ab}{}_{\mu,\nu} - \Lambda_{k}\Big] + \Gamma_{\mathrm{gf}}^{\pm} + S_{\mathrm{gh}}^{\pm}, \tag{5.127}$$

which corresponds to the Holst truncation (5.5) with $\gamma = \mp 1$ and the gauge fixing and ghost terms modified as discussed in the last subsection.

The left hand side of the flow equation, with the fluctuations set to zero, reads

$$\begin{aligned}\partial_{t}\Gamma_{k}^{\pm}[\bar{e},\bar{\omega}^{\pm}] = &-\frac{k^{2}}{8\pi g_{k}}\left(2-\frac{\partial_{t}g_{k}}{g_{k}}\right) \qquad\qquad \cdot\int \mathrm{d}^{d}x\,\bar{e}\,\bar{e}_{a}{}^{\mu}\bar{e}_{b}{}^{\nu}\bar{F}(\omega^{(\pm)})^{ab}{}_{\mu\nu} \\ &+\frac{k^{2}}{8\pi g_{k}}\left(2-\frac{\partial_{t}g_{k}}{g_{k}}+2+\frac{\partial_{t}\lambda_{k}}{\lambda_{k}}\right)\lambda_{k}k^{2}\cdot\int \mathrm{d}^{d}x\,\bar{e}\,.\end{aligned} \tag{5.128}$$

Inserting the constant background fields $\bar{e}$ and $\bar{\omega}^{(\pm)}$ we will identify the prefactor of the field strength term on the right hand side, denoted rhsF$^{\pm}$, by the combination of $\left(\bar{\omega}^{(\pm)}\right)^{2}$-contractions:

$$\bar{e}\,\bar{e}_{a}{}^{\mu}\bar{e}_{b}{}^{\nu}\bar{F}^{ab}{}_{\mu\nu} = \bar{e}\left[\left(\bar{\omega}^{(\pm)}\right)_{abc}\left(\bar{\omega}^{(\pm)}\right)^{acb} - \left(\bar{\omega}^{(\pm)}\right)^{a}{}_{ca}\left(\bar{\omega}^{(\pm)}\right)^{bc}{}_{b}\right] . \tag{5.129}$$

As for the full Holst truncation, this can not be done unambiguously. In Appendix F.2 we show that for an (anti-)selfdual background spin connection all contractions quadratic in $\bar{\omega}^{(\pm)}$ can be expressed in terms of the above two and thus of all 5 independent torsion squared invariants of the general case, on this background only two remain linearly independent. Hence, we need to specify exactly one additional basis invariant besides the curvature term in order to identify its prefactor unambiguously.

Following this reasoning we evaluate the RHS of the flow equation and finally cast it into the form

$$\begin{aligned}\partial_t \Gamma_k^{\pm} =& \mathrm{rhsF}^{\pm} \cdot k^2 \int \mathrm{d}^d x\, \bar{e} \left(\big(\bar{\omega}^{(\pm)}\big)_{abc} \big(\bar{\omega}^{(\pm)}\big)^{acb} - \big(\bar{\omega}^{(\pm)}\big)^{a}{}_{ca} \big(\bar{\omega}^{(\pm)}\big)^{bc}{}_{b} \right) \\ &+ \mathrm{rhs}\Lambda^{\pm} \cdot k^4 \int \mathrm{d}^d x\, \bar{e} + \mathrm{rhsI}_i^{\pm} \cdot k^2 \int \mathrm{d}^d x\, \bar{e}\, \bar{I}_i^{(\pm)} \,. \end{aligned} \tag{5.130}$$

Here, $\bar{I}_i^{(\pm)}$ is the additional field monomial that completes the basis in the projected part of theory space. In particular we will consider the choices $\bar{I}_1^{(\pm)}$ and $\bar{I}_3^{(\pm)}$ defined in Appendix F.2 along with the one parameter family

$$\bar{I}_\varphi^{(\pm)} = \sin(\varphi) \big(\bar{\omega}^{(\pm)}\big)^{abc} \big(\bar{\omega}^{(\pm)}\big)_{acb} + \cos(\varphi) \big(\bar{\omega}^{(\pm)}\big)^{ab}{}_{a} \big(\bar{\omega}^{(\pm)}\big)^{c}{}_{bc} \tag{5.131}$$

of $(\bar{\omega}^{(\pm)})^2$-contractions. Note that for $\varphi = 0$, $\bar{I}_\varphi^{(\pm)}$ and $\bar{I}_1^{(\pm)}$ coincide, while for $\varphi = \arctan(3)$, $\bar{I}_\varphi^{(\pm)}$ is proportional to $\bar{I}_1^{(\pm)}$. Thus the continuous family of monomials contains the two discrete choices in a certain sense. Together with the curvature invariant as a first basis element we denote the corresponding (chiral) bases by $\mathcal{B}_1^{\mathrm{ch}}$, $\mathcal{B}_3^{\mathrm{ch}}$ and $\mathcal{B}_\varphi^{\mathrm{ch}}$.

Let us move on and discuss the general form of the functions $\mathrm{rhsF}^{\pm}$, $\mathrm{rhsI}_i^{\pm}$ and $\mathrm{rhs}\Lambda^{\pm}$. For a general choice of gauge parameters we find that $\mathrm{rhsF}^{\pm}(\lambda)$, $\mathrm{rhsI}_i^{\pm}(\lambda)$ only depend on the cosmological constant λ, and that these functions in λ are given as the ratio of two polynomials of degree 10, with a common denominator. In the $(\alpha_{\mathrm{D}}, \alpha'_{\mathrm{L}}, \beta_{\mathrm{D}}) = (0, 0, 0)$-gauge these polynomials simplify, such that the remainder is a ratio of polynomials of degree 4. Unfortunately, the simplification is not as impressive as in the case of the full Holst truncation, where a reduction to degree 1 was obtained. Nonetheless this gauge leads to the most extensive simplification possible and we will thus stick to this preferred gauge for the rest of the discussion.

In order to judge the reliability of the different choices of bases we could discuss the ratio functions $\mathrm{rhsF}^{\pm}(\lambda)/\mathrm{rhsI}_i^{\pm}(\lambda)$, as in the QECG case. Being a ratio of degree

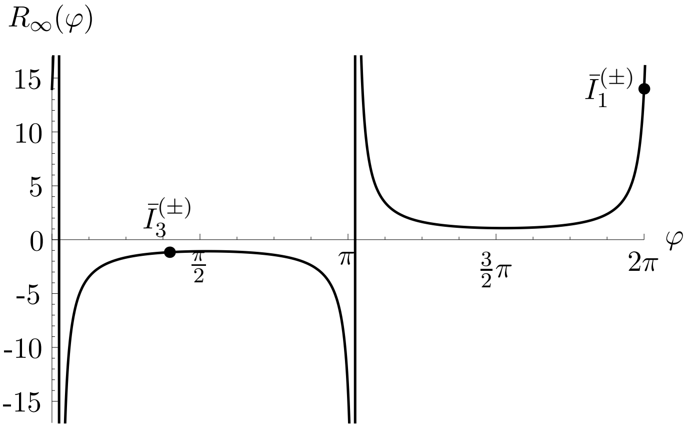


Figure 5.28. Asymptotic ratio of the coordinate functions $R_\infty = \lim_{\lambda\to\infty} \mathrm{rhsF}^\pm/\mathrm{rhs}\bar{I}_i^\pm$ as a function of the basis parameter φ.

4 polynomials even in $(0,0,0)$-gauge, this is, however, not very instructive. For the same reason one could argue that also the dependence of the limiting value $R_\infty(\varphi) = \lim_{\lambda\to\infty} \mathrm{rhsF}^\pm(\lambda)/\mathrm{rhsI}^\pm_\varphi(\lambda)$ on φ is only of limited value for measuring the quality of the truncation in a given basis. As we will find an astonishingly good correspondence between the form of its graph and the existence of NGFPs in our truncation of chiral gravity, we nevertheless have plotted the function in Fig. 5.28. We find that the direction of monomial $I_1^{(\pm)}$ ($\varphi = 0$) results in a much larger limiting value than that of $I_3^{(\pm)}$ ($\varphi = \arctan(3) \approx 0.4\,\pi$). For the time being we will not classify the quality of the basis choice according to this observation, but analyze the RG flow for all bases on the same footing.

The last function $\mathrm{rhs}\Lambda^\pm(\lambda, g)$ depends on both couplings (λ, g) and, independent of the basis chosen, takes on the form

$$\mathrm{rhs}\Lambda^\pm(\lambda, g) = -\frac{1}{32\pi^2}\left(\ln\left[\frac{(\lambda-1)^{12}\lambda^6}{g^{50}m^{50}}\right] - \ln\mathcal{N}^\pm\right), \tag{5.132}$$

in (0,0,0)-gauge, with $\ln\mathcal{N}^\pm \approx 151.5$.

As the main result of this subsection let us now write down the β-functions obtained for chiral gravity

$$\begin{aligned}\beta_g(\lambda, g) &= +2g + 8\pi g^2\,\mathrm{rhsF}^\pm(\lambda)\\ \beta_\lambda(\lambda, g) &= -2\lambda + 8\pi g\lambda\,\mathrm{rhsF}^\pm(\lambda) + 8\pi g\,\mathrm{rhs}\Lambda^\pm(\lambda, g)\,.\end{aligned} \tag{5.133}$$

Therein, the explicit form of the function $\mathrm{rhsF}^{\pm}(\lambda)$ is given by

$$\mathcal{B}_1^{\mathrm{ch}}: \quad \mathrm{rhsF}^{\pm}(\lambda) = -\frac{12\lambda^4 - 277\lambda^3 + 244\lambda^2 - 40\lambda - 4}{512\pi^2(\lambda-1)^2\lambda^2} \tag{5.134a}$$

$$\mathcal{B}_3^{\mathrm{ch}}: \quad \mathrm{rhsF}^{\pm}(\lambda) = \frac{57\lambda^4 - 49\lambda^3 - 80\lambda^2 + 56\lambda - 4}{256\pi^2(\lambda-1)^2\lambda^2} \tag{5.134b}$$

$$\mathcal{B}_\varphi^{\mathrm{ch}}: \quad \mathrm{rhsF}^{\pm}(\lambda) = -\frac{(-156\lambda^4 + 223\lambda^3 + 132\lambda^2 - 136\lambda + 12)\sin(\varphi)}{512\pi^2(\lambda-1)^2\lambda^2(\sin(\varphi)+\cos(\varphi))} - \frac{(12\lambda^4 - 277\lambda^3 + 244\lambda^2 - 40\lambda - 4)\cos(\varphi)}{512\pi^2(\lambda-1)^2\lambda^2(\sin(\varphi)+\cos(\varphi))} \tag{5.134c}$$

depending on the basis chosen. As expected from the symmetry of the RG flow of the Holst truncation under $\gamma \mapsto -\gamma$ in each basis the result obtained is the same for both chiralities.

5.5.3 Analysis of the RG flow

In this subsection we are going to analyze the RG flow of chiral gravity resulting from the system of β-functions (5.133), whose explicit form depends on the basis chosen (cf. (5.134)).

A first look onto the β-functions reveals a divergence of both function on the line $\lambda = 0$, which comes in addition to the divergence at $\lambda = 1$, that we know already from the QECG case. Keep in mind that for a generic choice of gauge parameters there are usually more divergences for fixed λ, all of which are moved to infinity when approaching the $(\alpha_{\mathrm{D}}, \alpha'_{\mathrm{L}}, \beta_{\mathrm{D}}) = (0, 0, 0)$ limit. Thus the "new" divergence at $\lambda = 0$ just corresponds to a zero of the denominator that approaches zero in this limit.

However, this divergence has an interesting effect: As the pole in β_g is of one degree higher than the one in β_λ, the trajectories of the flow do not reach this line. Thus all trajectories in the positive quadrant are confined to this quadrant and we will see that in the IR they either run to small values of g and large values of λ (which we know as type IIIa trajectories from metric gravity and QECG) or to small λ and large values of g, which amounts to a completely new IR behavior seen for the first time in gravity.

For the discussion of the RG flow in the rest of this subsection we set the dimensionless mass parameter to its most natural value, $m = 1$, in all concrete examples.

Fixed point structure. An immediate consequence of the second divergence line is that the origin in coupling space now lies on this line, such that a Gaussian fixed point cannot be properly defined there.

Besides that, we can find all non-Gaussian fixed points by solving the condition $\beta_g(\lambda, g) = 0$ for

$$g^*(\lambda) = -\frac{1}{4\pi\,\mathrm{rhsF}^{\pm}(\lambda)}, \tag{5.135}$$

substituting this solution into the second condition $\beta_\lambda(\lambda, g^*(\lambda)) = 0$ and searching for its zeros. This final step can only be carried out numerically, due to the logarithmic terms in $\mathrm{rhs}\Lambda^{\pm}(\lambda)$.

We have performed this task for the continuous set of bases $\mathcal{B}^{\mathrm{ch}}_{\varphi}$ and generically found two fixed points at $\lambda < 1$, one at small positive λ, which we will denote by $\mathbf{NGFP^1_\pm}$, and the second at large negative λ ($\mathbf{NGFP^2_\pm}$), resembling very much to the situation in any (λ, g)-plane of fixed $\gamma \neq \pm 1$ of the full Holst truncation. However, we find that the existence of the fixed points depends on the value of φ, i. e. on the basis chosen. We will discuss this issue in more detail below, after having analyzed the properties of the fixed points in the next two paragraphs.

Besides these two most stable fixed point solutions we found additional solutions, that were considered unphysical, as they occur very close to singularities of the function $\beta_\lambda(\lambda, g^*(\lambda))$ and and the influence on the RG flow of the fixed points they give rise to is very localized.

The position of the two fixed points we are going to discuss in detail is plotted in Fig. 5.29, together with the two barriers of the flow at $\lambda = 0$ and $\lambda = 1$.

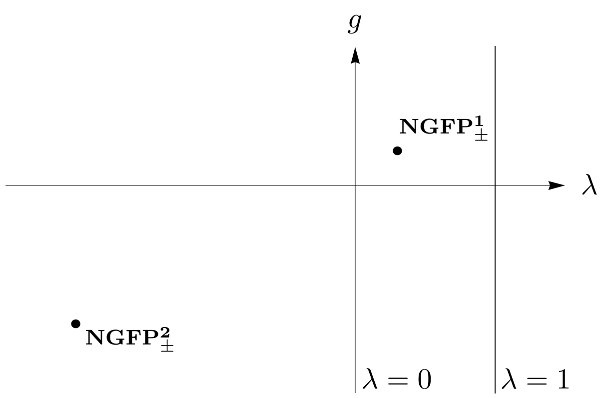


Figure 5.29. Sketch of the fixed point structure in the chiral case. The two barriers of the RG flow at $\lambda = 0$ and $\lambda = 1$ are depicted schematically as well.

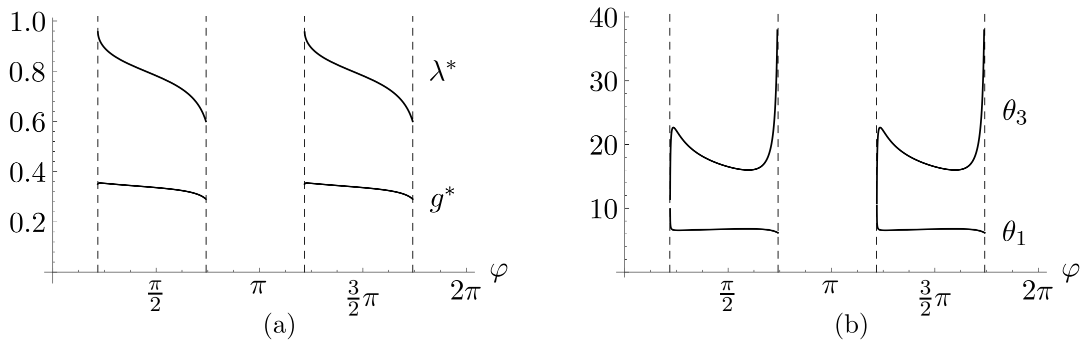


Figure 5.30. Position and critical exponents of $\mathbf{NGFP}^{\mathbf{1}}_{\pm}$ as functions of the continuous parameter φ of the basis $\mathcal{B}^{\rm ch}_{\varphi}$.

(A) The fixed point $\mathbf{NGFP}^{\mathbf{1}}_{\pm}$. In Fig. 5.30(a) we have plotted the fixed point position of the first NGFP as a function of the basis parameter φ. As the coordinate function $\text{rhsI}_i^{\pm}$ switches its sign, while $\text{rhsF}^{\pm}$ stays constant under $\varphi \mapsto \varphi + \pi$, this and the following figures are π-periodic in φ. We observe that the fixed point is only present in the interval $\pi/4 \lesssim \varphi \lesssim 3/4\pi$ (and its π-periodic counterpart). Moreover, both FP coordinates decrease with increasing φ although both coordinate values stay within one order of magnitude and, in particular, g^* turns out remarkably stable.

From the figure we can also conclude, that the fixed point does not exist in the discrete basis $\mathcal{B}^{\rm ch}_1$ ($\varphi = 0$). For the basis $\mathcal{B}^{\rm ch}_3$ the explicit fixed point coordinates read

$$\lambda^* = 0.818, \qquad g^* = 0.343\,. \tag{5.136}$$

Fig. 5.30(b) shows the corresponding plot of the critical exponents θ_1 and θ_3. At the lower boundary of the interval in which the FP exists, there seems to be a bifurcation point, where the critical exponents become real. Quickly thereafter θ_1 approaches a value of about $6 - 7$ while θ_3 fluctuates around 20, before it diverges at the upper boundary of the interval. Most importantly both critical exponents are positive, such that the fixed point is UV attractive and allows for the Asymptotic Safety construction.

For the discrete basis $\mathcal{B}^{\rm ch}_3$ the critical exponents are given by

$$\theta_1 = 6.66, \qquad \theta_3 = 17.78\,. \tag{5.137}$$

Qualitatively, but also quantitatively this fixed point resembles much the fixed point $\mathbf{NGFP}^{\mathbf{1}}_{\infty}$ of the full Holst truncation. Both fixed point coordinates are smaller than

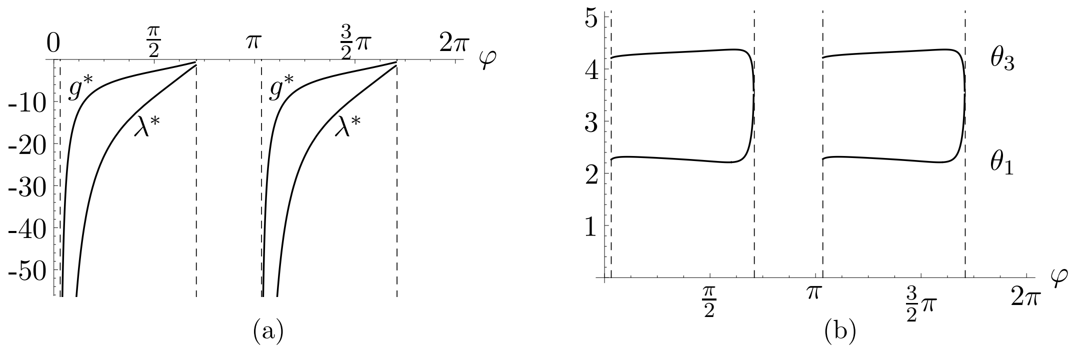


Figure 5.31. Position and critical exponents of $\mathbf{NGFP}^{\mathbf{2}}_{\pm}$ as functions of the continuous parameter φ of the basis $\mathcal{B}^{\text{ch}}_{\varphi}$.

one and relatively stable, but especially one critical exponent takes on a fairly large value.

(B) The fixed point $\mathbf{NGFP}^{\mathbf{2}}_{\pm}$. Let us turn over to the second fixed point. Its position as a function of φ is depicted in Fig. 5.31(a). We observe that it exists for a slightly larger interval in φ: The lower boundary is shifted to $\varphi \approx 0.1$, while the upper boundary occurs at virtually the same value $\varphi \approx 3/4\pi$ as in the case of $\mathbf{NGFP}^{\mathbf{1}}_{\pm}$. We find that the fixed point position heavily depends on the value of φ: It starts at infinite negative values at the lower boundary and moves close to the origin at the upper boundary. In between it always stays within the negative (λ, g) quadrant.

As for the first fixed point, $\mathbf{NGFP}^{\mathbf{2}}_{\pm}$ does not exist in basis $\mathcal{B}^{\text{ch}}_{1}$; in basis $\mathcal{B}^{\text{ch}}_{3}$ it occurs at

$$\lambda^* = -13.2, \qquad g^* = -3.86\,. \tag{5.138}$$

The corresponding critical exponents are depicted in Fig. 5.31(b). Their almost perfect independence on φ, taking into account the huge variation of the fixed point position is most striking: Both critical exponents are approximately constant with $\theta_1 \approx 2.3$ and $\theta_3 \approx 4.3$. In particular, both exponents are real and positive, giving rise to a UV attractivity of the FP in both directions. In basis $\mathcal{B}^{\text{ch}}_{3}$ the critical exponents read explicitly

$$\theta_1 = 2.26, \qquad \theta_2 = 4.33\,. \tag{5.139}$$

Thus we can conclude that also the properties of $\mathbf{NGFP}^{\mathbf{2}}_{\pm}$ are comparable to those of its "counterpart" $\mathbf{NGFP}^{\mathbf{2}}_{\infty}$ in the full Holst truncation.

(C) Discussion. Let us finally comment in more detail on the interval of existence of both fixed points. Naively one could think that universal properties of the flow like the existence of fixed points, should also be independent of the basis chosen. This is not the case, as the projection method clearly can be chosen in a particularly disadvantageous way, such that the physical content of the theory is projected out. While it is impossible to identify the best projection of the flow without knowing its exact untruncated form, in the case at hand we know two of these disadvantageous choices for φ:

(i) Those φ at which the poles in Fig. 5.28 occur, correspond to a basis, where the second invariant points exactly into the direction of the expanded RHS of the flow equation. Hence, $\text{rhsF}^{\pm}$ gets zero in this case and the information we are interested in is projected out.

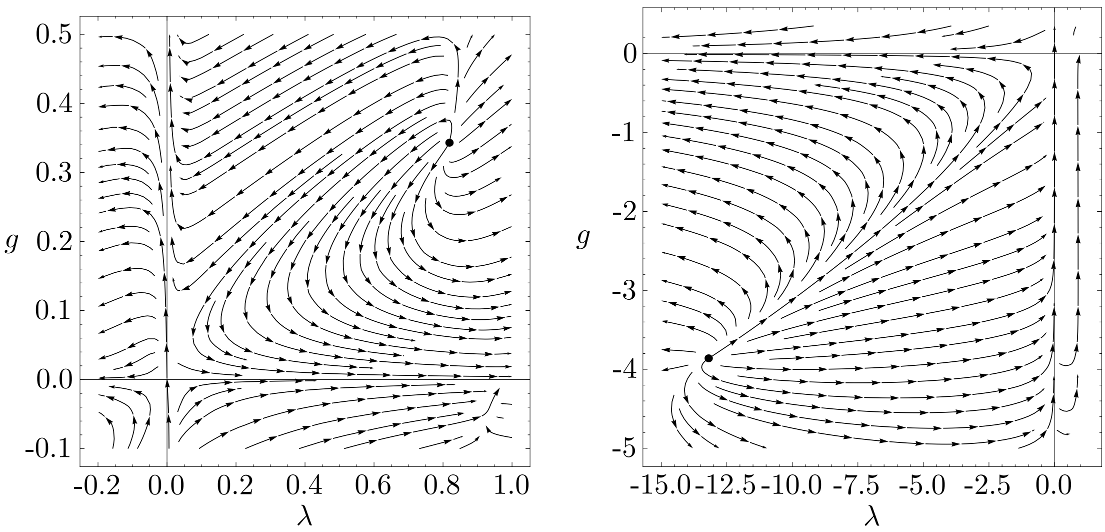
(ii) At $\varphi = 3/4\pi$ both invariants of the basis point in the same direction, i. e. are linearly dependent. Thus, in this limit, both coordinate functions $\text{rhsF}^{\pm}$ and $\text{rhs}\bar{I}^{\pm}_{\varphi}$ diverge and, although their ratio stays finite, the extracted RG flow becomes questionable.

It is certainly no mere coincidence that the boundaries of the interval of existence of $\mathbf{NGFP}^{\mathbf{2}}_{\infty}$ (and also the upper boundary for $\mathbf{NGFP}^{\mathbf{1}}_{\infty}$) lie very close to these extreme cases. From this point of view one should consider a basis in the middle of this interval as most reliable. Hence, we favor basis $\mathcal{B}^{\text{ch}}_3$ over $\mathcal{B}^{\text{ch}}_1$ and will discuss the phase portrait using the specific example of this basis in the next paragraph.

The phase portrait. In Fig. 5.32 we plot the phase portrait of the RG flow of chiral gravity obtained for the basis choice $\mathcal{B}^{\text{ch}}_3$. In subfigure (a) of Fig. 5.32 the vicinity of $\mathbf{NGFP}^{\mathbf{1}}_{\pm}$ and the flow towards the origin is depicted. We observe that the trajectories shortly before arriving at the origin are bent to one side or the other, such that they either run towards large values of λ and small g (as known from metric gravity) or to large g and small λ in the IR. This new behavior is clearly due to the existence of the additional divergence at $\lambda = 0$ compared to both metric gravity and the QECG case.

Subfigure (b) focusses on $\mathbf{NGFP}^{\mathbf{2}}_{\pm}$ and the negative (λ, g) quadrant. It shows no particular differences compared to the (λ, g)-truncations for fixed $\gamma \neq \pm 1$ of the Holst action, except for the divergence at $\lambda = 0$ (cf. Fig. 5.12).

This additional divergence, however, should probably not be taken to seriously. We were able to trace back its origin to the modified gauge condition $\mathcal{G}^{\pm}_{ab}$ in eq. (5.121).



(a) (b)

Figure 5.32. Phase portrait of chiral gravity obtained in basis $\mathcal{B}_3^{\text{ch}}$.

Picking the "+" chirality for the sake of the argument, it leads to the gauge fixing term

$$\mathcal{G}^+_{ab}\mathcal{G}^{+\,ab} = (P^+\mathcal{G})_{ab}(P^+\mathcal{G})^{ab}\,. \tag{5.140}$$

Instead it should also be admissible to use the *complete* gauge condition $\mathcal{G}_{ab}$ in the gauge fixing action $S^{\pm}_{\text{gf}}$. It decomposes according to

$$\mathcal{G}_{ab}\mathcal{G}^{ab} = (P^+\mathcal{G})_{ab}(P^+\mathcal{G})^{ab} + (P^-\mathcal{G})_{ab}(P^-\mathcal{G})^{ab}\,, \tag{5.141}$$

where the second term on the RHS is simply invariant under selfdual, i. e. $\mathsf{SO}^+(3)_{\text{loc}}$ transformations, while the first still gauge fixes them, exactly as in (5.140). Using this second gauge condition, the divergence at $\lambda = 0$ is not present in the β-functions. However, it has the disadvantage that it is not possible to take the limit of the preferred $(0, 0, 0)$-gauge in this case. For that reason we opted for the chiral gauge condition, which irrespective of the practical considerations seems the most natural choice. Nonetheless this observation puts the physical meaning of the divergence arising at $\lambda = 0$ into question.

Taking together all our findings on the RG flow of chiral gravity, we conclude that the cases $\gamma = \mp 1$ result in an additional self-consistent "sub-truncation" within the general Holst action ansatz. Most strikingly the resulting phase portrait and the properties of the two NGFPs we found correspond, qualitatively and quantitatively, very well to the other self-consistent sub-truncation, namely the $\hat{\gamma} = 0$-plane, in which the Immirzi parameter is not renormalized, too.

5.6 Discussion and Conclusion

Recall that the primary motivation for the RG analysis on $\mathcal{T}_{\text{EC}}$ was to assess the viability of the approximations that the WH-like flow equation relies on in direct comparison to the proper-time flow study carried out in [Dau, DR]. After having analyzed the resulting flow in some detail we are now in a position to answer this question to a large part affirmatively. Already in the general setting we find β-functions for g and γ that are of comparable form to those found in [Dau], in the sense that they share a similar and simple dependence on the couplings g and γ, while as a function of λ the β-functions are most complicated.

In the following we have seen that the consistency of our truncation is optimized by choosing the preferred $(\alpha_{\text{D}}, \alpha'_{\text{L}}, \beta_{\text{D}}) = (0, 0, 0)$ gauge, which in turn leads to a simplified dependence of the β-functions on λ. Besides the freedom of choosing the gauge fixing parameters, we had to deal with an additional projection ambiguity compared to the PT study, that somehow is the price to pay for the purely algebraic character of the new WH-like flow equation: Since we are bound to use constant background fields, the flow of the curvature and the Immirzi term can not be disentangled from the flow of certain torsion squared invariants. We therefore analyzed a set of different projection schemes, which were defined by introducing different bases in theory space, and we were able to show that the similarities to the PT study even grow further if we opt for basis $\mathcal{B}_1$.

Disregarding the flow of the cosmological constant and its notorious instabilities (stemming from quartic divergences), we find a system of β-functions $\{\beta_g, \beta_\gamma\}$ that is of the same structure as the corresponding one in [Dau]. In particular, it satisfies exactly the in [Dau] only conjectured duality property under the mapping $\gamma \mapsto 1/\gamma$ of the Immirzi parameter. Under this duality transformation the non-Gaussian fixed points $\mathbf{NGFP'_0}$ at $\gamma = 0$ and $\mathbf{NGFP'_\infty}$ at $\hat{\gamma} = 0$ are mapped onto each other. These fixed points generically show an antagonistic behavior, one being UV attractive in two and the other in one direction, leading to a predominant direction of the γ-flow, that depends on the fixed value of λ.

For the special choice of $\lambda \approx 0.57$ it was observed that the γ-flow stops completely. We argued that by this value the symmetry of scalar and pseudo-scalar terms in the truncation ansatz, that is spoiled by the introduction of gauge fixing and ghost terms, can be effectively restored, leading to an equal renormalization of the curvature and the Immirzi term. In a setting that respects this symmetry throughout, we therefore expect the flow of the Immirzi parameter to vanish, as long as no volume term is present in the (Holst-)truncation (i. e. $\lambda = 0$).

In this setting the different conceptions of the Immirzi parameter used in LQG as a fixed external quantity and in the RG approach as a running coupling are easy to reconcile: In both cases the value of γ is not scale dependent and therefore also in the RG setting the different quantum theories can be parametrized by fixed values of the Immirzi parameter. However, the cosmological constant is known to be non-zero, which inevitably leads to a non-trivial running of the Immirzi parameter and in consequence renders a comparison of the role of the Immirzi parameter in LQG and in asymptotically safe gravity more complicated.

At this point we also want to make contact to a related perturbative study on the running of the Immirzi parameter. In [BS11, BS12] a one loop renormalization of the Holst action is carried out, keeping track of the powerlike divergences by employing a proper-time cutoff regularization. In this setting the following β-functions are obtained

$$\beta_g = g\Big(2 - \frac{17}{3\pi} g\Big), \qquad \beta_\gamma = \frac{4}{3\pi}\, g\,\gamma\,, \tag{5.142}$$

that are to be compared with our findings at $\lambda = 0$

$$\beta_g = g\Big(2 - \frac{9}{16\pi} g\Big), \qquad \beta_\gamma = -\frac{23}{8\pi}\, g\,\gamma\,. \tag{5.143}$$

We observe that both calculations lead to β-functions of the same form. Moreover, the sign of the anomalous dimension η_N of the Newton constant coincides for both studies and allows for a fixed point value $g^* > 0$.

The predominant direction of the γ-flow is found opposite to our result; however, as a vanishing result should be expected in a symmetry preserving setting, the sign of the outcome of a calculation that spoils this symmetry may depend on the very details of the calculation. Unfortunately the analysis [BS11] is restricted to the case of a vanishing cosmological constant, such that we can not investigate the expected sign change of the γ-flow depending on λ. The main differences, apart from being a one-loop calculation, in the setting of [BS11] compared to our calculation are a different gauge-fixing condition for the diffeomorphisms and the complete neglection of $\mathsf{O}(4)$-ghost contributions, that may have a crucial influence as we have pointed out in Chapter 3. On the technical side also the regularization procedure is different (proper-time cutoff vs. sharp momentum cutoff). Taken together these differences might very well account for the sign difference in the γ-flow at $\lambda = 0$.

When including λ as a running coupling to our truncation, the fixed point at $\hat{\gamma} = 0$ gets lifted to the higher dimensional theory space and we find two NGFPs in the $\hat{\gamma} = 0$-plane. The fixed point at $\gamma = 0$, however, ceases to exist due to a, probably unphysical, divergence of β_λ that arises at $\gamma = 0$. Thus, the similarities to [DR] are weaker in the three dimensional truncation, but nonetheless our computation independently supports the existence of NGFPs suitable for the Asymptotic Safety scenario in the $\hat{\gamma} = 0$-plane.

In the last section we used the WH-like setting to carry out a first RG study of chiral gravity. Here we found little to no qualitative difference to the RG flow of g and λ in other planes of fixed $\gamma \neq \pm 1$. Especially the positive quadrant shows a striking similarity to the metric Einstein-Hilbert truncation, although the critical exponents of the fixed point are real here and their absolute value is quite large.

Thus, in total we conclude that the approximations the WH-like flow equation relies on lead only to minor changes of the flow as far as only its gross topological features are considered. Within this setting we were able to single out a preferred choice of gauge parameters, $(\alpha_\mathrm{D}, \alpha'_\mathrm{L}, \beta_\mathrm{D}) = (0, 0, 0)$, and of the projection scheme, namely to use basis $\mathcal{B}_1$ in the torsion squared subspace. Due to the immense reduction in computational effort, which we gain when opting for the WH-like equation instead of the PT flow equation, we can only recommend to employ this preferred setting in future RG studies of enlarged truncations that in addition may also include fermionic matter.

6 Summary and Conclusions

In this thesis three independent RG studies of asymptotically safe gravity have been carried out. While the structure of the group of gauge transformations is of semi-direct product type in all three cases, the field content of the three theories differs and also three different types of FRGE have been employed for the analysis of their respective RG flow. A major achievement of this thesis is hence that for the first time the RG flows of metric, tetrad and Einstein-Cartan gravity could be directly compared and the relation between the different theory spaces could be explored. Moreover, a new approximative functional RG equation, the "WH-like flow equation", was developed, tested, and applied; it allowed for a purely algebraic evaluation of the RG flow in the Einstein-Cartan case.

Let us summarize here the principal achievements of each of the three studies separately. For the more detailed description of the results obtained the reader is referred to the concluding section of the respective chapter of this thesis.

Asymptotically safe gravity and gauge theories. The study presented in Chapter 3 amounts to a first fully non-perturbative analysis of the corrections of metric gravity to the running of the Yang-Mills coupling constant. Previously numerous perturbative studies on the subject had appeared that employed different computational techniques which had partly led to contradicting results. The analysis in this thesis is a valuable contribution to this lively debate, as it helps to clarify the situation by circumventing two major shortcomings of previous perturbative studies: It retains all contributions due to quadratic divergences as well as (background) gauge invariance. In order to ensure the latter it turned out crucial to employ a generalization of the usual Faddeev-Popov method for the construction of the ghost-action [DHR10, Dau] that takes into account the semi-direct product structure of the symmetry group of the underlying theory space $\mathcal{T}_{\text{E,YM}}$.

As the main result of this study we showed that gravitons and gluons have a similar effect on the running of the gauge coupling, i. e. that both support the approach of asymptotic freedom in the UV, and that this effect of the gravitons is already present at one-loop level. It is encouraging that coupling Yang-Mills theory to asymptotically safe gravity does not destroy asymptotic freedom of the gauge field sector, but rather improves its UV properties.

This result awakened the interest in how other parts of the SM are affected by the presence of asymptotically safe gravity. In particular, we studied the effects on QED and on the $\mathsf{U}(1)$-sector of the electroweak interaction. Here, we found that gravitons and fermions have a competing effect on the running of the fine-structure constant/weak hypercharge, such that the quantum gravitational effects may prevent the couplings from running into a Landau type singularity. In this case the coupled theory can be considered fundamental. Concerning the UV limit of this fundamental theory we found two different scenarios: In the first case the graviton contributions outweigh the fermionic ones leading to asymptotic freedom of the respective gauge coupling of the coupled system. The more interesting possibility, however, lies in the exact balancing of the competing contributions in the UV limit, that results in a finite non-zero fixed point value of the gauge coupling. We were able to show that this scenario generically corresponds to a fundamental theory with a lesser number of free parameters. In our concrete examples we demonstrated this fact by computing a first estimate of the fine-structure constant and the weak-hypercharge in terms of the electron/Z-boson mass in Planck units, which turned out in the correct order of magnitude.

As these results indicate, it may be possible to “heal” the ill-defined UV behavior of QED by coupling it to QEG. This raises the hope that also the 100-year-old problem of the divergent electron self-energy can finally be solved by the effects of (asymptotically safe) quantum gravity. Our result seems to confirm the longstanding speculation that it is the fluctuations of spacetime taken into account in quantum gravity that effectively smear out the problematic singular points.

Taken together the new investigations on gauge theories coupled to asymptotically safe gravity carried out in this thesis suggest that also the Standard Model coupled to gravity may turn out asymptotically safe once Asymptotic Safety of pure gravity is fully established. Moreover, we found an interesting mechanism that demonstrates how Asymptotic Safety may even enhance the predictive power of the coupled system.

"Tetrad only" gravity. In Chapter 4 we reported on the first RG study of gravity in the theory space constructed from the tetrad field only [HR12]. Here, the application of the exact FRGE for the EAA on a truncation derived from the Einstein-Hilbert truncation of metric gravity allowed us to directly compare the results obtained to the corresponding metric analysis in [Reu98]. We pointed out that the resulting RG flow differs from its metric counterpart for two reasons: First, due to the off-shell character of the FRGE every field reparametrization may lead to additional contributions to its RHS, but second, and most importantly, the transition to the tetrad as the fundamental field variable amounts to a change of theory space, $\mathcal{T}_{\text{E}} \to \mathcal{T}_{\text{tet}}$, that enlarges the group of gauge transformations from $\mathbf{G} = \mathsf{Diff}(\mathcal{M})$ in the metric case to

$$\mathbf{G} = \mathsf{Diff}(\mathcal{M}) \ltimes \mathsf{O}(d)_{\text{loc}} \tag{6.1}$$

in tetrad gravity. Moreover, it can be shown that $\mathcal{T}_{\text{tet}}$ is larger than $\mathcal{T}_{\text{E}}$ such that the couplings in both spaces are not in a one-to-one correspondence. At the level of our truncation, the effect of the change of theory space was encoded in the contributions of the additional $\mathsf{O}(d)$ ghost fields, that crucially influenced the UV properties of the theory. While a complete neglection of the additional ghost fields rendered the non-Gaussian fixed point UV repulsive, in turn giving rise to a limit cycle, a consistent treatment of the ghost fields restored the existence of a UV attractive fixed point. Thus, we found first evidence for the existence of a NGFP suitable for the Asymptotic Safety construction in $\mathcal{T}_{\text{tet}}$, while demonstrating at the same time that in a non-perturbative RG approach the $\mathsf{O}(d)$ ghost contributions cannot be neglected, unlike in perturbation theory [Woo84].

In direct comparison to the metric RG flow it became clear that the tetrad formulation of the Einstein-Hilbert flow is much more sensitive to the value of external parameters like the mass parameter μ, that had to be introduced to define a Faddeev-Popov operator of definite mass dimension, or the explicit RG scheme used. This behavior was attributed to the larger ratio of gauge to physical field components compared to the metric case. For that reason a future analysis aiming at quantitative evidence whether and how the theories of metric and tetrad gravity defined at the respective fixed points of their theory spaces are related to each other can ultimately only be based on predictions of observable quantities both theories give rise to, as only at this level gauge and RG dependences are expected to be compensated.

QECG in the Holst truncation. Chapter 5 was devoted to the RG analysis of the Holst truncation of gravity formulated in Einstein-Cartan theory space $\mathcal{T}_{\text{EC}}$. In this analysis three main methodological advancements have been achieved:

- A novel approximative WH-like flow equation for a running effective action has been developed. It is structurally equivalent to the Wegner-Houghton flow equation for the running bare action for the special case of constant background fields. We showed how this new flow equation is related to the exact FRGE by a series of approximations and that it renders the evaluation of the RG flow a purely algebraic task. In consequence also the resulting β-functions simplify considerably as the momentum integration contained in the trace on the RHS can be carried out explicitly.

- A projection technique allowing for the optimization of a truncation containing a fixed set of invariants was introduced. In order to do that, it was pointed out that a truncation is only properly defined, if we, in addition to the set of invariants, concretely specify a projection scheme. This can be done by defining the space of field monomials that form the kernel of the projection. It was shown that it may be advantageous to first only specify part of the projection kernel and thereby project the RHS of the flow equation onto a subspace of theory space that is larger than the space spanned by the set of field monomials of the truncation. (In our case this larger space contained *all* invariants that are at most quadratic in the spin connection.) In a second step it is then possible to choose the remaining kernel elements among the invariants of this larger subspace in such a way that the resulting RG flow stays to a good approximation inside the subspace spanned by the truncation invariants. Using this technique the Holst truncation could be optimized by a choice of basis. Moreover, it was understood that at least part of the strong gauge parameter dependence that is generically found in RG studies of $\mathcal{T}_{\text{EC}}$ is due to fact that the quality of the truncation in the above sense depends very much on the gauge chosen. In this respect the $(\alpha_{\text{D}}, \alpha'_{\text{L}}, \beta_{\text{D}}) = (0, 0, 0)$-gauge was singled out as the preferred choice.

- The decomposition of the fluctuation fields that had been introduced in [Dau] was modified in such a way that the component fields describe (anti-)selfdual fluctuation modes. By this reparametrization of the fluctuations it was possible to analyze the RG flow of chiral gravity for the first time, as it arises as a simple modification of the general Einstein-Cartan case.

Applying the above developments to the Holst truncation of $\mathcal{T}_{\text{EC}}$ we arrived at a system of β-functions for its three couplings λ, γ and g as a major result of the chapter. The RG flow they give rise to was analyzed in detail, and to a large part this analysis could be carried out analytically due to the enormous simplification of the β-functions compared to the study of [Dau, DR]. In direct comparison to [Dau, DR] many structural similarities of the RG flow have been observed, which at the same time can be seen as an independent verification of these previous results and an approval of the applicability of the new WH-like flow equation to the Einstein-Cartan theory space.

Most strikingly the similarity of the resulting RG flows was seen in the (γ, g)-subsystem, that does not suffer from the notorious instabilities of the λ-flow. The above mentioned optimization procedure of the truncated flow led to a choice of four bases $\mathcal{B}_i$ $(i = 1, \cdots, 4)$ in theory space, from which $\mathcal{B}_1$ was singled out as the one maximizing the structural similarities in comparison to [DR]. In this most physical basis $\mathcal{B}_1$ not only the position and stability properties of the fixed points coincide but also the conjectured duality map $\gamma \mapsto 1/\gamma$ was verified analytically at the level of the β-functions.

The first RG study of chiral gravity ($\gamma = \pm 1$), that in our approach could be carried out by only minor modifications of the general Einstein-Cartan case, using the reparametrized decomposition of the fluctuation fields, revealed a picture very similar to the (λ, g)-truncations of other fixed values of $\gamma \neq \pm 1$ with two UV attractive NGFPs, in the positive and the negative (λ, g)-quadrant, the latter exhibiting a remarkable stability of its critical exponents. The only evident difference between the chiral case and other fixed values of γ, namely the existence of an additional barrier of the RG flow at $\lambda = 0$, was traced back to the different gauge condition used, and is hence probably unphysical. Among the two-dimensional truncations of fixed γ, the planes $\gamma \to \infty$ and $\gamma = 0$ as well as the chiral case $\gamma \to \pm 1$ play a special role as they amount to self-consistent truncations, in which the Immirzi parameter is not renormalized.

In view of future calculations in the Einstein-Cartan theory space it seems very desirable to search for a gauge-fixing condition that in the general case retains the symmetry of the Holst truncation w. r. t. scalar and pseudo-scalar contributions, and/or in the chiral case allows for the $(\alpha_{\mathrm{D}}, \alpha'_{\mathrm{L}}, \beta_{\mathrm{D}}) = (0,0,0)$-limit without exhibiting the $\lambda = 0$ divergence.

Apart from an optimization of the gauge fixing condition, the inclusion of fermionic matter seems most interesting for a future project. With the simplified WH-like flow equation, that nonetheless preserves many structural properties of the flow, we paved the way for such an even more involved computation, that would allow for the investigation of the CP-violating effect of the Immirzi term in theories of fermions coupled to asymptotically safe gravity and its possible implications for the cosmology of the early universe.

A Abbreviations and Conventions

In this appendix we list the abbreviations used in the thesis and introduce our notation conventions. We thereby define the notation of various physical quantities and list the most important relations among them.

Abbreviation	Meaning
EAA	effective average action
FP	fixed point
FRGE	functional renormalization group equation
GFP	Gaussian fixed point
IR	infrared
LHS	left hand side
LQG	loop quantum gravity
MSSM	minimal supersymmetric standard model
NGFP	non-Gaussian fixed point
PT	proper-time
QCD	quantum chromodynamics
QED	quantum electrodynamics
QEG	quantum Einstein gravity
QFT	quantum field theory
RG	renormalization group
RHS	right hand side
SM	Standard Model
UV	ultraviolet
WH	Wegner-Houghton
YM	Yang-Mills

Object	Notation/Convention/Relations
(Anti-)Symmetrization	$\delta^{\mu}_{[\rho}\delta^{\nu}_{\sigma]} = \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}$, $\delta^{\mu}_{(\rho}\delta^{\nu}_{\sigma)} = \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} + \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}$, etc.
Euclidean (curved) metric	$g_{\mu\nu}$
Euclidean (flat) metric	η_{ab}, $\eta_{ab} \equiv \delta_{ab}$
Tetrad/vielbein field	$e^{a}{}_{\mu}$
Determinant of the metric	$g \equiv \det(g_{\mu\nu})$
Determinant of the tetrad	$e \equiv \det(e^{a}{}_{\mu})$
Levi-Civita connection	$\Gamma_{\rm LC}(g)^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)$
	$\Gamma_{\rm LC}(e)^{\lambda}_{\mu\nu} = e_{a}{}^{\lambda}(\partial_\mu e^{a}{}_{\nu} + \omega(e)^{a}{}_{b\mu} e^{b}{}_{\nu}) = \Gamma_{\rm LC}(g(e))^{\lambda}_{\mu\nu}$
Levi-Civita spin connection	$\omega(e)^{ab}{}_{\mu} = \frac{1}{2} e^{a\lambda}(e_{c\nu}\partial_{[\mu} e^{c}{}_{\lambda]} + e_{c\lambda}\partial_{[\nu} e^{c}{}_{\mu]} - e_{c\mu}\partial_{[\lambda} e^{c}{}_{\nu]}) e^{b\nu}$
General spacetime connection	$\Gamma^{\lambda}_{\mu\nu} = e_{a}{}^{\lambda}(\partial_\mu e^{a}{}_{\nu} + \omega^{a}{}_{b\mu} e^{b}{}_{\nu}) = e_{a}{}^{\lambda}\nabla_\mu e^{a}{}_{\nu}$
	$\Gamma^{\lambda}_{\mu\nu} = \Gamma_{\rm LC}(e)^{\lambda}_{\mu\nu} + K^{\lambda}{}_{\mu\nu}$
Riemann curvature tensor	$R_{\mu\nu\rho}{}^{\lambda} = \partial_{[\nu}\Gamma^{\lambda}_{\mu]\rho} + \Gamma^{\sigma}_{[\mu\|\rho}\Gamma^{\lambda}_{\nu]\sigma}$
Ricci tensor	$R_{\mu\nu} = R_{\mu\lambda\nu}{}^{\lambda}$
Ricci scalar	$R = g^{\mu\nu} R_{\mu\nu}$
Field strength tensors	$F^{a}_{\mu\nu} = \partial_{[\mu} A^{a}_{\nu]} + f^{abc} A^{b}_{\mu} A^{c}_{\nu}$
	$F^{ab}{}_{\mu\nu} = \partial_{[\mu}\omega^{ab}{}_{\nu]} + \omega^{a}{}_{c[\mu}\omega^{cb}{}_{\nu]}$
Torsion tensor	$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{[\mu\nu]}$
Contorsion tensor	$K^{\lambda}{}_{\mu\nu} = \frac{1}{2}(T^{\lambda}{}_{\mu\nu} - T_{\mu\nu}{}^{\lambda} + T_{\nu}{}^{\lambda}{}_{\mu})$
Covariant derivatives	$D = \partial + \Gamma$
(symbolically)	$\nabla = \partial + \omega/A$ $\mathsf{O}(d)/\mathsf{SU}(N)$ connection
	$\mathcal{D} = \partial + \Gamma + \omega/A$
Curvature operator:	
a) for vectors A_λ	$[D_\mu, D_\nu] A_\lambda = R_{\mu\nu\lambda}{}^{\rho} A_\rho - T^{\rho}{}_{\mu\nu} D_\rho A_\lambda$
b) for tensors $H_{\lambda\sigma}$	$[D_\mu, D_\nu] H_{\lambda\sigma} = R_{\mu\nu\lambda}{}^{\rho} H_{\rho\sigma} + R_{\mu\nu\sigma}{}^{\rho} H_{\lambda\rho} - T^{\rho}{}_{\mu\nu} D_\rho H_{\lambda\sigma}$

Remark: In the Chapters 3 and 4 we only consider torsionless spacetimes with $T^{\lambda}{}_{\mu\nu} = 0 = K^{\lambda}{}_{\mu\nu}$. Therefore we have $\Gamma^{\lambda}_{\mu\nu} = \Gamma_{\rm LC}{}^{\lambda}_{\mu\nu}$ in these cases, and we drop the subscript LC throughout these Chapters.

B The Wegner-Houghton Equation

In this appendix we derive the Wegner-Houghton RG equation for the running bare action and discuss its structural similarity to the new WH-like flow equation derived in Section 2.3 and first applied in Chapter 5. The below presentation is based on the original work [WH73] as well as [HH96, Mor94, SSA$^+$], but is aimed to be self-contained.

The Wegner-Houghton equation is an exact renormalization group equation for the Wilsonian effective action. Starting point for its derivation is the general idea behind Kadanoff-Wilson renormalization, namely to compute the path integral over fluctuations on all scales piecewise, taking into account only fluctuations in a shell of momenta $k' \leq |p| \leq k$. This way one obtains a "running bare action" S_k, defined by the property that the partition function stays the same, when fast fluctuating modes are intergrated out:

$$Z = \int [\mathcal{D}\Phi]_{[0,k']} \, e^{-S_{k'}[\Phi]} = \int [\mathcal{D}\phi]_{[0,k]} \, e^{-S_k[\phi]} = \int [\mathcal{D}\Phi]_{[0,k']} \left[\int [\mathcal{D}\varphi]_{[k',k]} \, e^{-S_k[\Phi+\varphi]} \right] \quad \text{(B.1)}$$

so that

$$e^{-S_{k'}[\Phi]} = \int \, [\mathcal{D}\varphi]_{[k',k]} \, e^{-S_k[\Phi+\varphi]}, \quad \text{(B.2)}$$

where the notation $[\mathcal{D}\phi]_{[a,b]}$ denotes the path integral for Fourier modes of the field ϕ in the momentum interval $[a, b]$.

From this result we can derive a differential equation relating the bare actions at different scales by taking into account only an infinitesimal momentum shell $k' = k - \delta k$. To this end we have to analyze the right hand side of equation (B.2) and truncate the action S_k to those terms, that contribute to the differential of the action $S_k - S_{k-\delta k}$ to at most linear order in δk. With this truncated action, $S_k^{\delta k}$, the path integral in (B.2) turns out to be computable, leading to a closed form differential equation for the Wilson effective action, the Wegner-Houghton equation.

Let us work out this general procedure in a bit more detail. Consider a real scalar field $\phi(x)$ and its Fourier transform

$$\phi(p) = \frac{1}{(2\pi)^{d/2}} \int \mathrm{d}^d x \, \phi(x) e^{-ipx} \quad \text{with} \quad \phi(-p) = \phi^*(p). \tag{B.3}$$

A generic action functional $S[\phi]$ can be expanded in momentum space according to

$$\begin{aligned} S[\phi] =& \frac{1}{2} \int \mathrm{d}^d p \, v_2(p) \, \phi(p)\phi(-p) \\ &+ \sum_{n=3}^{\infty} \frac{1}{n!} \int \mathrm{d}^d p_1 \ldots \mathrm{d}^d p_n \, v_n(p_1, \ldots, p_n) \, \phi(p_1) \ldots \phi(p_n) \, \delta(p_1 + \ldots + p_n), \end{aligned} \tag{B.4}$$

where the functions v_n are symmetric in their arguments and $v_2(p) = v_2(-p)$, without loss of generality. In the case of a running bare action S_k the scale dependence of the action is described as a scale dependence of the functions v_n. Fluctuations with momenta above this scale are already integrated out, so that $\phi(p) = 0$ for $|p| > k$.

Now we split the field ϕ into two fields Φ and φ corresponding to the low and high momentum modes, respectively:

$$\phi(p) = \Phi(p) + \varphi(p) \quad \text{with} \quad \begin{cases} \Phi(p) = 0 & \text{for } |p| \geq k - \delta k \\ \varphi(p) = 0 & \text{unless } k - \delta k \leq |p| \leq k. \end{cases} \tag{B.5}$$

The first term in (B.4), corresponding to a generalized kinetic term, can be separated off according to

$$S' = S - S^0 \qquad \text{with} \qquad S^0 = \frac{1}{2} \int \mathrm{d}^d p \, v_2(p) \, \phi(p)\phi(-p). \tag{B.6}$$

Since, by definition, the product $\Phi(p)\varphi(-p)$ vanishes for all momenta p, S^0 has the property

$$\begin{aligned} S^0[\phi] = S^0[\Phi + \varphi] &= \frac{1}{2} \int \mathrm{d}^d p \, v_2(p) \, (\Phi(p) + \varphi(p))(\Phi(-p) + \varphi(-p)) \\ &= \frac{1}{2} \int \mathrm{d}^d p \, v_2(p) \, \Phi(p)\Phi(-p) + \frac{1}{2} \int \mathrm{d}^d p \, v_2(p) \, \varphi(p)\varphi(-p) \\ &= S^0[\Phi] + S^0[\varphi]. \end{aligned} \tag{B.7}$$

S' does not have this additivity property and therefore we define a new quantity $\hat{S}$ as the difference

$$
\begin{aligned}
\hat{S}[\Phi + \varphi] &= S'[\Phi + \varphi] - S'[\Phi] \\
&= \int \mathrm{d}^d p \, \frac{\delta S'[\Phi]}{\delta \varphi(p)} \, \varphi(p) + \frac{1}{2} \int \mathrm{d}^d p_1 \, \mathrm{d}^d p_2 \, \varphi(p_1) \, \frac{\delta^2 S'[\Phi]}{\delta \varphi(p_1) \delta \varphi(p_2)} \, \varphi(p_2) + \dots \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}^d p_1 \dots \mathrm{d}^d p_n \, \hat{S}^{(n)}[\Phi] \, \varphi(p_1) \dots \varphi(p_n),
\end{aligned}
\tag{B.8}
$$

so that the expansion of $\hat{S}[\Phi + \varphi]$ contains no term independent of φ. Note that the integrations are effectively integrations over the momentum shell $k - \delta k \leq |p| \leq k$, since $\varphi(p) = 0$ otherwise.

Using these definitions, we now have

$$
\begin{aligned}
S[\Phi + \varphi] &= S^0[\Phi + \varphi] + S'[\Phi + \varphi] \\
&= S^0[\Phi] + S'[\Phi] + S^0[\varphi] + \hat{S}[\Phi + \varphi] \\
&= S[\Phi] + S^0[\varphi] + \hat{S}[\Phi + \varphi].
\end{aligned}
\tag{B.9}
$$

Since the path integral will only be carried out over the fluctuation field φ, this identity corresponds to a separation of the action $S[\Phi + \varphi]$ into a fixed part $S[\Phi]$ depending only on the "external field" Φ, a kinetic term of the fluctuations $S^0[\varphi]$, and an interaction terms $\hat{S}[\Phi + \varphi]$. Inserting the identity into (B.2), with $k' = k - \delta k$, gives

$$
\begin{aligned}
e^{S_k[\Phi] - S_{k-\delta k}[\Phi]} &= \int [\mathcal{D}\varphi]_{[k-\delta k, k]} \, e^{-S_k^0[\varphi] - \hat{S}_k[\Phi + \varphi]} \\
&= \left[e^{-\hat{S}_k[\Phi + \frac{\delta}{\delta J}]} \int [\mathcal{D}\varphi]_{[k-\delta k, k]} \, e^{-S_k^0[\varphi] + \int \mathrm{d}^d p \, J(-p) \varphi(p)} \right]_{J=0},
\end{aligned}
\tag{B.10}
$$

where the last step is only valid in the sense of an asymptotic expansion. J shall only couple to modes in the shell, so that $J(p) = 0$ unless $k - \delta k \leq |p| \leq k$.

Now the functional integration can be carried out. As a first step, by a shifting the integration function according to $\psi(p) = \varphi(p) - \frac{1}{v_2(p)} J(p)$ with $[\mathcal{D}\psi] = [\mathcal{D}\varphi]$, the exponent reduces to a term quadratic in the field ψ:

$$\int [\mathcal{D}\varphi]_{[k-\delta k,k]}\, e^{-S_k^0[\varphi] + \int \mathrm{d}^d p\, J(-p)\varphi(p)}$$
$$= \int [\mathcal{D}\psi]_{[k-\delta k,k]}\, e^{-\frac{1}{2}\int \mathrm{d}^d p\, \psi(p) v_2(p) \psi(-p) + \frac{1}{2}\int \mathrm{d}^d p\, J(p) \frac{1}{v_2(p)} J(-p)} \,. \tag{B.11}$$

Second, we split the integration over the modes in the momentum shell

$$B_{k-\delta k,k} = \{p \,\big|\, |p| \in [k-\delta k, k]\}$$

into two half shells

$$B^{\pm}_{k-\delta k,k} = \{p \,\big|\, |p| \in [k-\delta k, k], p_0 \gtrless 0\}$$

such that $\int [\mathcal{D}\varphi]_{[k-\delta k,k]} = \int [\mathcal{D}\psi]_{p\in B^+_{k-\delta k,k}} [\mathcal{D}\psi]_{p\in B^-_{k-\delta k,k}} = \int [\mathcal{D}\psi]_{p\in B^+_{k-\delta k,k}} [\mathcal{D}\psi]_{-p\in B^+_{k-\delta k,k}} = \int [\mathcal{D}\psi]_{p\in B^+_{k-\delta k,k}} [\mathcal{D}\psi^*]_{p\in B^+_{k-\delta k,k}}$.
In addition we note that, due to the symmetry $v_2(-p) = v_2(p)$ and $\psi(-p) = \psi^*(p)$, we can rewrite the exponent according to

$$\int \mathrm{d}^d p\, \psi(p) v_2(p) \psi(-p) = 2 \int_{B^+_{k-\delta k,k}} \mathrm{d}^d p\, \psi(p) v_2(p) \psi^*(p)$$

such that the complete path integral reads:

$$= \int [\mathcal{D}\psi]_{p\in B^+_{k-\delta k,k}} [\mathcal{D}\psi^*]_{p\in B^+_{k-\delta k,k}}\, e^{-\int_{B^+} \mathrm{d}^d p\, \psi(p) v_2(p) \psi^*(p) + \frac{1}{2}\int \mathrm{d}^d p\, J(p) \frac{1}{v_2(p)} J(-p)} \,. \tag{B.12}$$

Now we make a further change of integration variables and integrate separately over the real and imaginary part of ψ

$$= \int [\mathcal{D}\mathrm{Re}\psi]_{p\in B^+_{k-\delta k,k}} [\mathcal{D}\mathrm{Im}\psi]_{p\in B^+_{k-\delta k,k}}\, e^{-\int_{B^+} \mathrm{d}^d p\, v_2(p) \left((\mathrm{Re}\psi(p))^2 + (\mathrm{Im}\psi(p))^2 \right)}$$
$$\cdot\, e^{\frac{1}{2}\int \mathrm{d}^d p\, J(p) \frac{1}{v_2(p)} J(-p)} \,. \tag{B.13}$$

At last, we can restore the integration over the full momentum shell $B_{k-\delta k,k}$ by noting that the analogous integration over the modes in $B^-_{k-\delta k,k}$ leads the same result:

$$= \left[\int [\mathcal{D}\mathrm{Re}\psi]_{[k-\delta k,k]}\,[\mathcal{D}\mathrm{Im}\psi]_{[k-\delta k,k]}\,e^{-\int \mathrm{d}^d p\, v_2(p)\left((\mathrm{Re}\psi(p))^2+(\mathrm{Im}\psi(p))^2\right)}\right]^{1/2} \cdot e^{\frac{1}{2}\int \mathrm{d}^d p\, J(p)\frac{1}{v_2(p)}J(-p)}$$

$$= \left[\prod_{k-\delta k\leq p\leq k}\frac{1}{\sqrt{v_2(p)}}\right]\cdot e^{\frac{1}{2}\int \mathrm{d}^d p\, J(p)\frac{1}{v_2(p)}J(-p)}\,, \tag{B.14}$$

where a (infinite) constant coming from the Gaussian integrals was absorbed in the functional measure. After the functional integration eq. (B.10) therefore becomes

$$e^{S_k[\Phi]-S_{k-\delta k}[\Phi]} = \left[\prod_{k-\delta k\leq p\leq k}\frac{1}{\sqrt{v_2(p)}}\right]\left[e^{-\hat{S}_k[\Phi+\frac{\delta}{\delta J}]}\cdot e^{\frac{1}{2}\int \mathrm{d}^d p\, J(p)\frac{1}{v_2(p)}J(-p)}\right]_{J=0}. \tag{B.15}$$

If we now expand $\hat{S}_k$, as in (B.8), and the exponential $\exp(-\hat{S}_k)$, we can think of (B.15) as a series of Feynman diagrams:

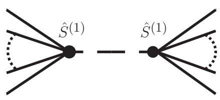
Each coefficient $\hat{S}_k^{(n)}$ gives rise to a vertex with an arbitrary number of external lines whose momentum is smaller than $k-\delta k$ described by its Φ-dependence and n internal lines carrying momenta in the shell. There is momentum conservation at each vertex due to the delta distributions in (B.4). All internal lines are connected by propagator terms $1/v_2(p)$ and each internal line picks up an integration over the momentum shell $k-\delta k\leq |p|\leq k$.

To find an expression for $S_k[\Phi]-S_{k-\delta k}[\Phi]$ we just have to take the logarithm of equation (B.15)

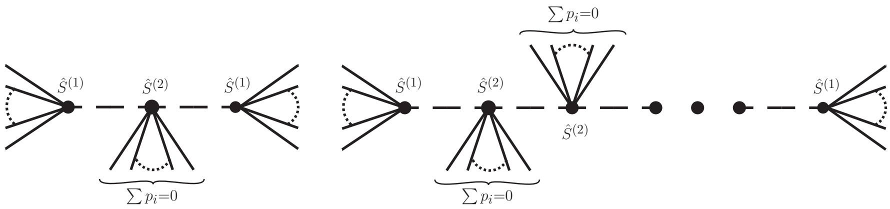
$$S_k[\Phi]-S_{k-\delta k}[\Phi] = -\frac{1}{2}\int' \mathrm{d}^d p\,\ln v_2(p)+\ln\left[e^{-\hat{S}_k[\Phi+\frac{\delta}{\delta J}]}\cdot e^{\frac{1}{2}\int \mathrm{d}^d p\, J(p)\frac{1}{v_2(p)}J(-p)}\right]_{J=0}, \tag{B.16}$$

where $\int' \mathrm{d}^d p$ denotes the integration over the momentum shell $k-\delta k\leq |p|\leq k$. As usual, the logarithm amounts to the sum of the *connected* diagrams of the above type.

Up to now, we have not made use of the fact that δk is an infinitesimal quantity, so that we only need to take into account the connected diagrams contributing to linear order in δk. As each internal line corresponds to a shell integration of order δk, diagrams with only one internal line will contribute. These diagrams are built from two vertices connected by one internal line, which have arbitrarily many external lines



(a) Diagram with one internal line



(b) Insertion of additional vertices to the diagram from Fig. B.1(a)

Figure B.1. Tree-level diagrams contributing to linear order in δk.

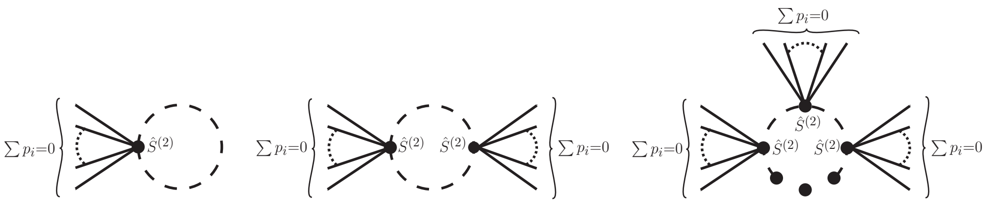


Figure B.2. One-loop diagrams contributing to linear order in δk.

at each vertex (cf. Fig. B.1(a)). Diagrams with more than one internal line will only contribute, when the internal momenta on all lines are constrained to the same value, so that only one independent shell integration remains. Diagrams of this type can be obtained in two ways. First, we can insert one or more vertices with two internal lines into the diagrams from above. If the sum of external momenta vanishes at each of these vertices, the internal lines are constrained to the same momentum due to momentum conservation at the vertex (cf. Fig. B.1(b)). Second, we can built up a one-loop diagram of these vertices with two internal lines containing an arbitrary number of vertices (cf. Fig. B.2).

In summary all connected diagrams which contain only two types of vertices contribute: the vertex with one internal line, which is due to $\hat{S}_k^{(1)}$, and the vertex with two internal lines carrying the same momentum, due to $\hat{S}_k^{(2)}$. Restricting the right hand side of (B.16) to terms linear in δk is therefore equivalent to truncating $\hat{S}$ to

$$\hat{S}_k^{\delta k}[\Phi+\varphi] = \int \mathrm{d}^d p \, \frac{\delta S_k'}{\delta\varphi(p)}[\Phi]\,\varphi(p) + \frac{1}{2}\int \mathrm{d}^d p\,\varphi(p)\,\frac{\delta^2 S_k'}{\delta\varphi(p)\delta\varphi(-p)}[\Phi]\,\varphi(-p). \quad \text{(B.17)}$$

These diagrammatic considerations only aimed at identifying all those terms in $\hat{S}_k$ that contribute to (B.10) to at most linear order in δk. We may thus replace $\hat{S}_k$ in (B.10) by $\hat{S}_k^{\delta k}$, and with $S = S' + S^0$ the first line of (B.10) reads:

$$\begin{aligned} e^{S_k[\Phi]-S_{k-\delta k}[\Phi]} = \int [\mathcal{D}\varphi]_{[k-\delta k,k]} \exp\Big\{ -\!\int \mathrm{d}^d p \, \frac{\delta S_k[\Phi]}{\delta\varphi(p)}\,\varphi(p) \\ - \frac{1}{2}\!\int \mathrm{d}^d p\,\varphi(p)\,\frac{\delta^2 S_k[\Phi]}{\delta\varphi(p)\delta\varphi(-p)}\,\varphi(-p)\Big\} + \mathcal{O}(\delta k^2)\,. \end{aligned} \quad \text{(B.18)}$$

Substituting

$$\psi(p) = \varphi(p) + \left(\frac{\delta^2 S_k[\Phi]}{\delta\varphi(p)\delta\varphi(-p)}\right)^{-1} \frac{\delta S_k}{\delta\varphi(-p)}[\Phi], \quad \text{(B.19)}$$

we are left with a Gaussian functional integral, which can be evaluated easily. Taking the logarithm, we end up with

$$\begin{aligned} S_k[\Phi] - S_{k-\delta k}[\Phi] = &-\frac{1}{2}\int' \mathrm{d}^d p\,\ln\left[\frac{\delta^2 S_k[\Phi]}{\delta\varphi(p)\delta\varphi(-p)}\right] \\ &+ \frac{1}{2}\int' \mathrm{d}^d p\,\frac{\delta S_k}{\delta\varphi(p)}[\Phi]\left(\frac{\delta^2 S_k[\Phi]}{\delta\varphi(p)\delta\varphi(-p)}\right)^{-1}\frac{\delta S_k}{\delta\varphi(-p)}[\Phi]. \end{aligned} \quad \text{(B.20)}$$

In a last step we take the limit of infinitesimal δk, in which we can separate off the δk-dependence of the shell integration by a transformation to spherical coordinates

$$\int' \mathrm{d}^d p = \int \mathrm{d}\Omega \int' p^{d-1}\mathrm{d}p = k^{d-1}\delta k \int \mathrm{d}\Omega \quad \text{(B.21)}$$

and we thus obtain the Wegner-Houghton equation

$$\partial_t S_k[\Phi] = \frac{k^d}{2}\!\int \mathrm{d}\Omega \left(\frac{\delta S_k[\Phi]}{\delta\varphi(k)}\left[\frac{\delta^2 S_k[\Phi]}{\delta\varphi(k)\delta\varphi(-k)}\right]^{-1}\frac{\delta S_k[\Phi]}{\delta\varphi(-k)} - \ln\left[\frac{\delta^2 S_k[\Phi]}{\delta\varphi(k)\delta\varphi(-k)}\right]\right), \quad \text{(B.22)}$$

where $t = \ln k$. Here, the first term describes the contribution from the tree-level graphs, while the second term is due to the loop diagrams.

Despite its similarity to the usual one-loop contribution to the effective action

$$\Gamma^{\text{1-loop}}[\Phi] - S[\Phi] = \frac{1}{2}\text{Tr}\left[\ln\left(\delta^2 S[\Phi]\right)\right] = \frac{1}{2}\int \mathrm{d}^d p \ \ln\left(\delta^2 S[\Phi]\right), \tag{B.23}$$

it is important to keep in mind, that in the case of the WH equation, we have to sum up the logarithms of the diagonal elements of $\delta^2 S$, while in the case of the one-loop effective action, we have to sum up the diagonal elements of the logarithm of the operator $\delta^2 S$. This can be traced back to the fact, that in the Wegner-Houghton one-loop diagrams all lines are confined to the same momentum, while in a general one-loop diagram the momenta on the lines are arbitrary.

Let us examine the special case of a constant field Φ. Then the operator $\delta^2 S[\Phi]$ is diagonal in momentum space, since it is a function of derivatives only. Therefore the operations of taking the logarithm and taking the trace of $\delta^2 S[\Phi]$ commute. In addition, the first term in the WH equation (B.22) vanishes for a constant field Φ, due to momentum conservation at the vertex with one internal line. Thus, in this special case, the WH equation for the Wilson effective action S_k,

$$\partial_t S_k[\Phi] = -\frac{k^d}{2}\left[\int \mathrm{d}\Omega_p \ \ln(\delta^2 S_k[\Phi])\right]_{|p|=k}, \tag{B.24}$$

becomes formally equivalent to the new WH-like flow equation, that was derived as an approximation of the exact FRGE for the EAA in Section 2.3,

$$\partial_t \Gamma_k = \frac{1}{2} D_t \,\mathrm{STr}\Big|_k \ln\left(\Gamma_k^{(2)}\right) = -\frac{k^d}{2}\left[\int \mathrm{d}\Omega_p \ \ln(\delta^2 \Gamma_k[\Phi])\right]_{|p|=k}. \tag{B.25}$$

As the effective action Γ is the generating functional of the 1 PI Green's functions we expect for any (approximate) flow equation for a running effective action Γ_k that its RHS contains only terms corresponding to 1 PI graphs, while any Wilsonian flow equation additionally contains tree graph terms. We conclude that it is only due to the vanishing contributions of the tree graphs in the WH equation for constant background fields, that the same flow equation approximates both the Wilsonian and the running effective action in the subspace spanned by those invariants that are non-vanishing on this class of backgrounds.

C Running Gauge Coupling in Arbitrary Dimensions

This appendix contains supplementary material to Section 3.2 of the main body of this thesis. We display here the general result for the anomalous dimension η_F of the gauge coupling g_{YM} in d spacetime dimensions. With all RG improvements included, it assumes the form:

$$
\begin{aligned}
(4\pi)^{d/2}\eta_F = &-32\pi g\Big[A\,\chi_{\frac{d}{2}-1}(-2\lambda) + C\,(2\lambda)^{-1}\Big(\chi_{\frac{d}{2}}(-2\lambda) - \chi_{\frac{d}{2}}(0)\Big)\Big] \\
&- g_{\mathrm{YM}}^2 N\left(\frac{26-d}{3}\right)\chi_{\frac{d}{2}-2}(0) \\
&+ 16\pi\eta_N g\Big[A\tilde{\chi}_{\frac{d}{2}-1}(-2\lambda) \\
&\qquad + C\Big((2\lambda)^{-1}\tilde{\chi}_{\frac{d}{2}}(-2\lambda) - (2\lambda)^{-2}\Big(\tilde{\chi}_{\frac{d}{2}+1}(-2\lambda) - \tilde{\chi}_{\frac{d}{2}+1}(0)\Big)\Big)\Big] \\
&+ 16\pi\eta_F g\Big[C\Big(-(2\lambda)^{-1}\tilde{\chi}_{\frac{d}{2}}(0) + (2\lambda)^{-2}\Big(\tilde{\chi}_{\frac{d}{2}+1}(-2\lambda) - \tilde{\chi}_{\frac{d}{2}+1}(0)\Big)\Big)\Big] \\
&+ \eta_F g_{\mathrm{YM}}^2 N\left(\frac{24-d}{6}\right)\tilde{\chi}_{\frac{d}{2}-2}(0) \\
&- 16\pi(\eta_F - \eta_N)g\Big[A\,\tilde{\tilde{\chi}}_{\frac{d}{2}}(-2\lambda) + C\,(2\lambda)^{-1}\Big(\tilde{\tilde{\chi}}_{\frac{d}{2}+1}(-2\lambda) - \tilde{\tilde{\chi}}_{\frac{d}{2}+1}(0)\Big)\Big] \\
&+ 32\pi g(2\lambda + \beta_\lambda)\Big[A\,\tilde{\tilde{\chi}}_{\frac{d}{2}-1}(-2\lambda) \\
&\qquad + C\Big((2\lambda)^{-1}\tilde{\tilde{\chi}}_{\frac{d}{2}}(-2\lambda) - (2\lambda)^{-2}\Big(\tilde{\tilde{\chi}}_{\frac{d}{2}+1}(-2\lambda) - \tilde{\tilde{\chi}}_{\frac{d}{2}+1}(0)\Big)\Big)\Big].
\end{aligned}
\tag{C.1}
$$

Eq. (C.1) contains the following functions of d:

$$
A = \frac{d^3 - 9d^2 + 10d + 16}{4(d-2)}, \qquad C = \frac{2d^2 - 5d}{d-2}\,. \tag{C.2}
$$

Furthermore we introduced the following new threshold functions:

$$\begin{aligned}
\chi_n(w) &= \frac{1}{\Gamma(n)} \int_w^\infty \mathrm{d}z\, (z-w)^{n-1} \frac{R^{(0)}(z) - zR^{(0)\prime}(z)}{z + R^{(0)}(z)} \\
\tilde{\chi}_n(w) &= \frac{1}{\Gamma(n)} \int_w^\infty \mathrm{d}z\, (z-w)^{n-1} \frac{R^{(0)}(z)}{z + R^{(0)}(z)} \qquad\qquad (n > 0) \\
\tilde{\tilde{\chi}}_n(w) &= \frac{1}{\Gamma(n)} \int_w^\infty \mathrm{d}z\, (z-w)^{n-1} \frac{R^{(0)\prime}(z)}{z + R^{(0)}(z)}
\end{aligned} \tag{C.3}$$

and

$$\begin{aligned}
\chi_n(w) &= (-1)^{-n} \left.\frac{\mathrm{d}^{-n}}{\mathrm{d}z^{-n}} \frac{R^{(0)}(z) - zR^{(0)\prime}(z)}{z + R^{(0)}(z)}\right|_{z=w} \\
\tilde{\chi}_n(w) &= (-1)^{-n} \left.\frac{\mathrm{d}^{-n}}{\mathrm{d}z^{-n}} \frac{R^{(0)}(z)}{z + R^{(0)}(z)}\right|_{z=w} \qquad\qquad (n \leq 0) \\
\tilde{\tilde{\chi}}_n(w) &= (-1)^{-n} \left.\frac{\mathrm{d}^{-n}}{\mathrm{d}z^{-n}} \frac{R^{(0)\prime}(z)}{z + R^{(0)}(z)}\right|_{z=w}
\end{aligned} \tag{C.4}$$

In particular we find the relations

$$\chi_n(0) = \Phi^1_n(0), \qquad \tilde{\chi}_n(0) = \tilde{\Phi}^1_n(0) \quad \text{and} \quad \tilde{\tilde{\chi}}_n(0) = \tilde{\tilde{\Phi}}^1_n(0) \tag{C.5}$$

between the χ- and the Φ-functions, that hold for all n. Moreover, one can check that the χ-functions satisfy the following differential equation for all $n \in \mathbb{Z}$:

$$\chi_{n-1}(w) = -\frac{\mathrm{d}}{\mathrm{d}w} \chi_n(w)\,. \tag{C.6}$$

We can exploit these two facts in order to expand the χ-integrals in their argument w into an infinite sum of the corresponding Φ-integrals according to

$$\chi_n(w) = \sum_{k=0}^{\infty} \frac{(-w)^k}{k!} \Phi^1_{n-k}(0), \tag{C.7}$$

and analogously for $\tilde{\chi}$ and $\tilde{\tilde{\chi}}$. In practice this expansion corresponds to an expansion in the cosmological constant λ. It has been performed in the main part of this thesis in order to derive the various approximations (3.26), (3.33) and (3.34) in $d = 4$ from the general result (C.1).

The 1-loop result for a general spacetime dimension d is obtained from Eq. (C.1) by omitting from its RHS all terms containing η_F, η_N, or any $\tilde{\tilde{\chi}}$ integral. The latter terms arise from the differentiation of $\breve{\Gamma}_k^{(2)}$ in the arguments of $\mathcal{R}_k$.

Note, that the result (C.1) differs from the corresponding equation (A.1) in the previously published study [DHR10]. This is due to an inconsistency in [DHR10] as the result (A.1) therein actually corresponds to a different choice of $\breve{\mathcal{Z}}_k$ than given in (3.11) (or equivalently eq. (4.10) in [DHR10]). If the calculation is carried out with the choice (3.11), that is consistent with the $\mathcal{Z}_k = \zeta_k$-rule, we obtain (C.1) as the result for an arbitrary spacetime dimension d. Evidently (C.1) amounts to a major simplification compared to (A.1) in [DHR10]. Note, however, that both results coincide in the most important case of $d = 4$, such that the results in the main body of the article are not affected by this modification.

A second correction compared to [DHR10] should be remarked: In (A.4) of [DHR10] it was claimed that the expansion of the χ-functions ends after finitely many terms. This is, however, not true: The correct form of the expansion contains an infinite number terms as it is given in (C.7). In consequence this leads to the fact that the eqns. (3.33) and (3.34) correspond to a truncated series expansion in λ and therefore are only valid up to the neglected higher order terms.

D Exponential Cutoff in Tetrad Gravity

In this appendix we want to further explore the cutoff scheme dependence of the "universal" quantities that were studied in Section 4.3 for the optimized cutoff. We do this by employing a one parameter family of exponential cutoff functions $R^{(0)}(z) = sz/(e^{sz}-1)$ for shape parameters s ranging from 2 to 20. The resulting dependence of $g^*\lambda^*$ and the critical exponents on the shape parameter s and the mass parameter μ is obtained numerically and is depicted in two sets of figures.

The first set of figures, Figs. D.1 to D.3, corresponds to the application of ghost adaptation scheme (i), with each figure representing a different choice of μ ($\mu = 0.5,\ 1,\ 5$ in Figs. D.1, D.2, D.3, respectively). Fig. D.3 where μ is already rather large can also be seen as representing ghost adaptation scheme (iii). Each of the figures contains a series of plots ordered from small to large shape parameters employed in the exponential cutoff function.

In a good approximation of the exact flow we would expect the plots to show only little ξ-dependence of the universal quantities, resulting in horizontal lines, as well as only small variations of the same picture for different shape parameters, i. e. almost equal plots within each of the figures. The dependence on μ, on the other hand, could be more pronounced, as it should be seen as an additional coupling set to a fixed value.

We see, however, that there is a severe dependence on the parameter ξ in all three figures; it leads to a change of sign of $g^*\lambda^*$, changing critical exponents from complex to real, and even a change of character of the fixed point from UV attractive to repulsive. Although we already made similar observations for the optimized cutoff function (cf. Fig. 4.3), only the exponential cutoff functions reveal the full degree of scheme dependence in these results: While for the optimized cutoff we were able to choose a value of $\mu \approx 1$ such that none of the above problematic changes occurred in the interval $\xi \in [0, 1]$ (cf. Fig. 4.3 b, c), we now find that *changing s has an* *effect similar to choosing a different μ.* For that reason we find qualitatively the same plots (Fig. 4.3 a, b, c, d) obtained for the optimized cutoff function and distinguished choices of μ all within the family of exponential cutoff functions for the same value of $\mu = 1$ (Fig. D.2 a, c, d, f).

Only for large μ the variation of the plots within Fig. D.3 is relatively weak. But here we find a large ξ-dependence of the critical exponents leading to a change of character of the fixed point in all plots, as we already found for the optimized cutoff in the same limit.

Taken together these observations show, that both adaptation schemes (i) and (iii) lead to severely scheme dependent results, that make it almost impossible draw any universally valid conclusion besides the existence of a NGFP.

Let us therefore go on and discuss the second set of figures (Figs. D.4 to D.6). Again each figure represents a certain choice of the parameter μ and contains a series of plots showing values of the same quantities obtained for different shape parameters s of the exponential cutoff function. In this case however, we employed the three variants of the ghost adaptation scheme (ii) differing by a factor of $\sqrt{2}$, that we already introduced when we discussed this scheme for the optimized cutoff function in the main part of this paper (cf. Fig. 4.4).

The first and most prominent observation is that *the ξ-dependence in the plots is considerably weaker for all three variants of scheme (ii) compared to schemes (i) and (iii).* While we still find some dependence on the shape parameter s (in all the three figures there are real critical exponents for small s turning complex for larger s), except for the plots D.6(e) and D.6(f), all plots show almost horizontal lines, i. e. virtually no ξ-dependence of the universal quantities.

Secondly, as for the optimized cutoff, we find the weakest scheme dependence for the variant of adaptation scheme (ii) with the smallest value of μ (cf. Fig. D.4). This, however, is probably due to the effective suppression of the physical degrees of freedom in the limit of small μ, as explained in Section 4.3.2.

For these reasons we conclude that, *within the limits of the present truncation, the most reliable results are from the plots employing adaptation scheme (ii)* as shown in Fig. D.5. They suggest, in accordance with the optimized cutoff result in Fig. 4.4(b), a UV attractive FP in the $\lambda^* < 0$ region, presumably with real critical exponents.

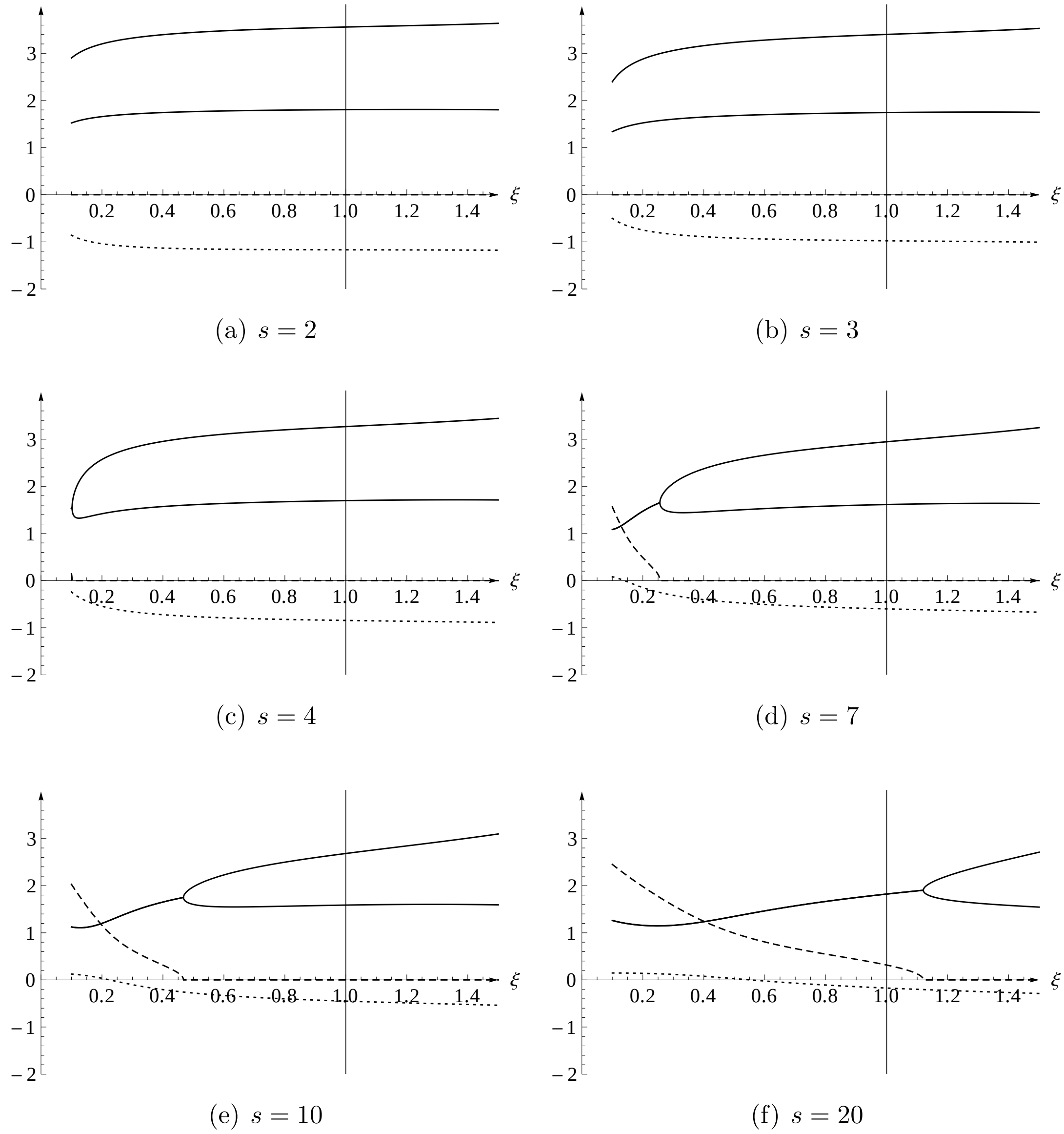
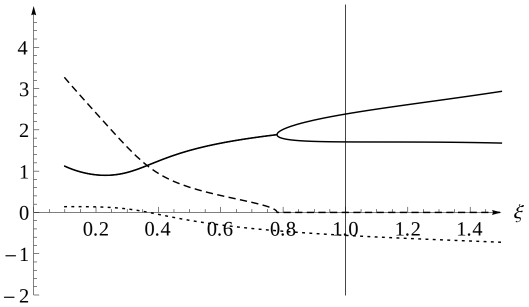
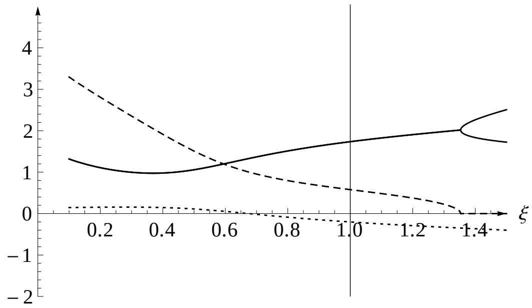


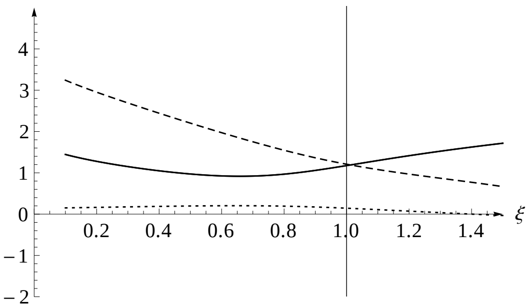
Figure D.1. Critical exponents and $g^*\lambda^*$ for different shape parameters s depending on ξ with mass parameter $\mu = 0.5$ (θ' solid, θ'' dashed, $g^*\lambda^*$ dotted).



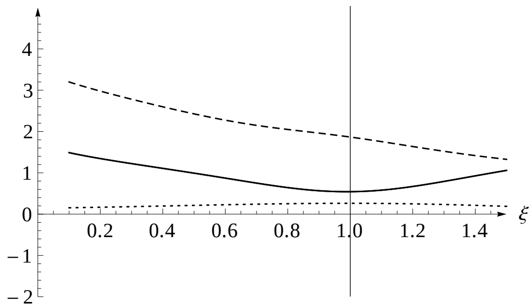
(a) $s = 2$



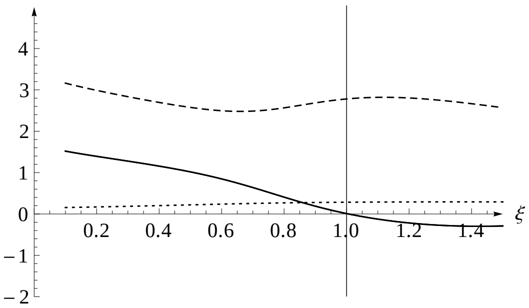
(b) $s = 3$



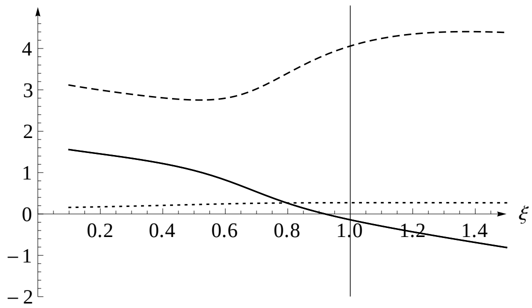
(c) $s = 5$



(d) $s = 7$

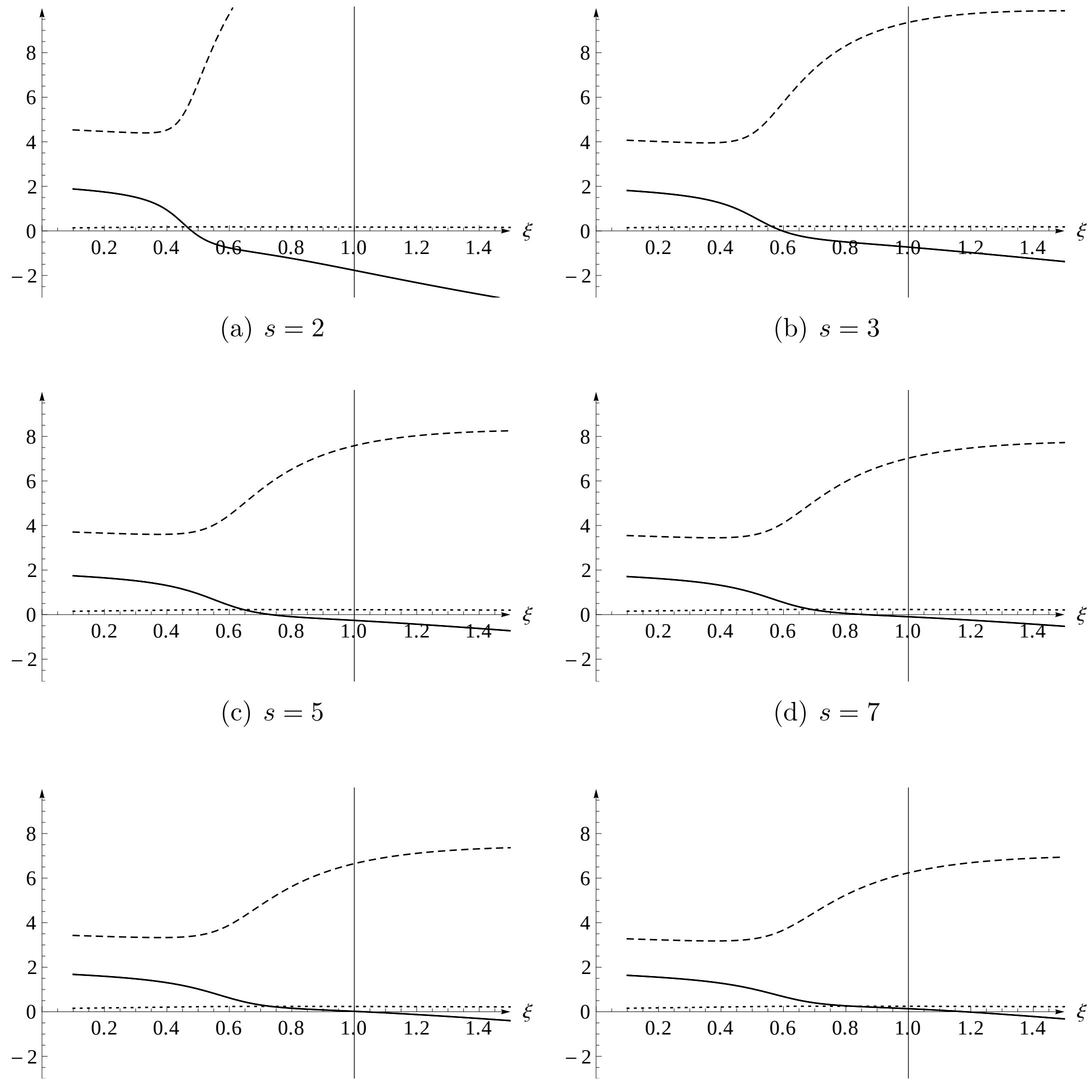


(e) $s = 10$



(f) $s = 20$

Figure D.2. Critical exponents and $g^*\lambda^*$ for different shape parameters s depending on ξ with mass parameter $\mu = 1$ (θ' solid, θ'' dashed, $g^*\lambda^*$ dotted).

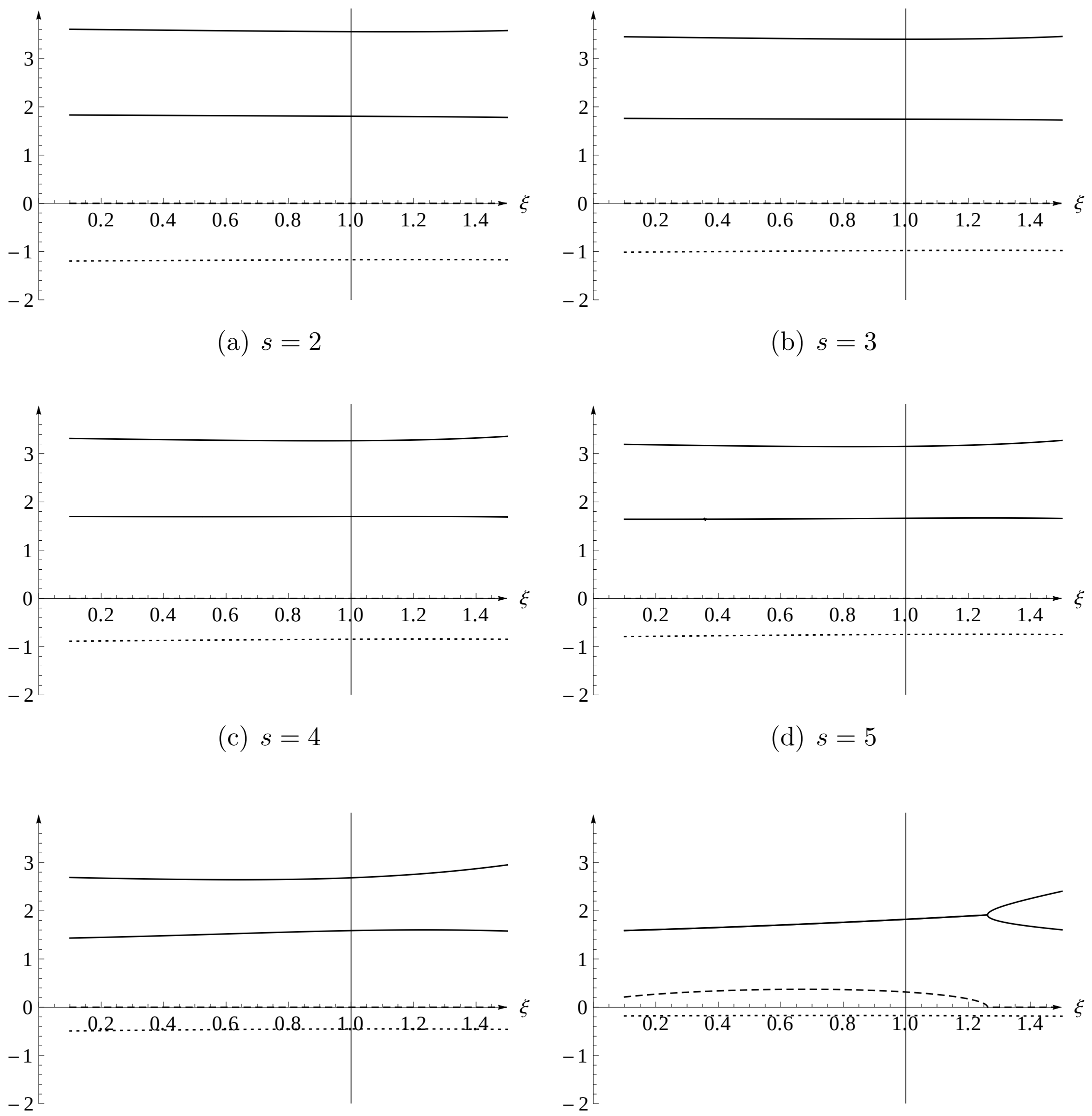


(a) $s = 2$ (b) $s = 3$

(c) $s = 5$ (d) $s = 7$

(e) $s = 10$ (f) $s = 20$

Figure D.3. Critical exponents and $g^*\lambda^*$ for different shape parameters s depending on ξ with mass parameter $\mu = 5$ (θ' solid, θ'' dashed, $g^*\lambda^*$ dotted).

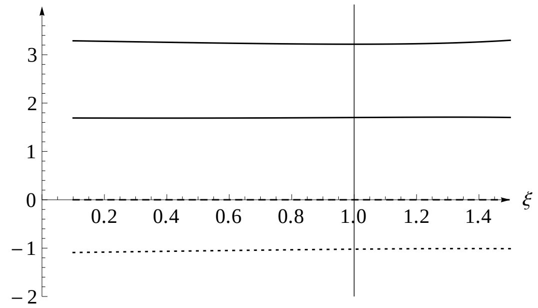


(a) $s = 2$ (b) $s = 3$

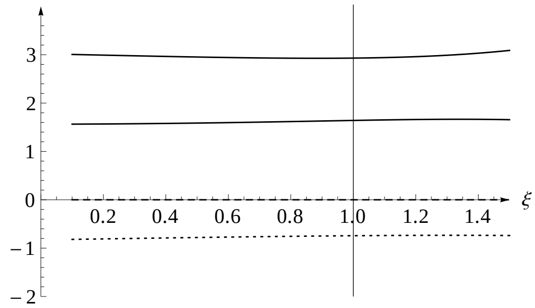
(c) $s = 4$ (d) $s = 5$

(e) $s = 10$ (f) $s = 20$

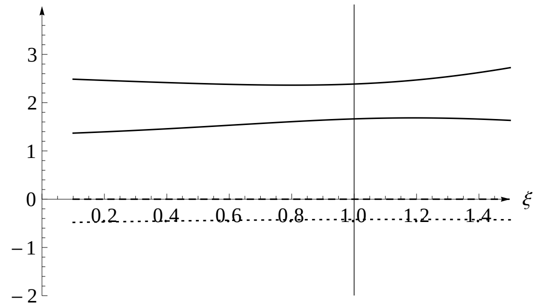
Figure D.4. Critical exponents and $g^*\lambda^*$ for different shape parameters s depending on ξ with an adapted mass parameter $\mu = (\xi/4)^{1/4}/\sqrt{2}$ (θ' solid, θ'' dashed, $g^*\lambda^*$ dotted).



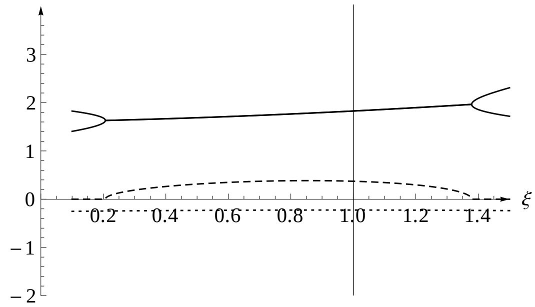
(a) $s = 2$



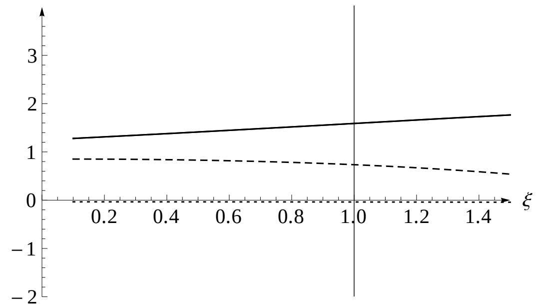
(b) $s = 3$



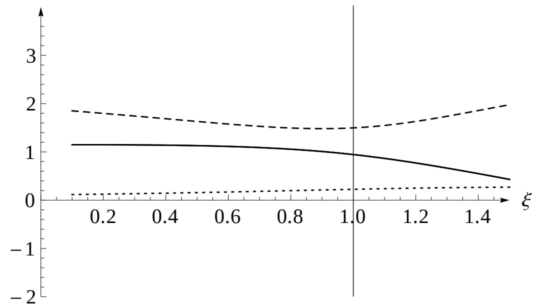
(c) $s = 5$



(d) $s = 7$



(e) $s = 10$



(f) $s = 20$

Figure D.5. Critical exponents and $g^*\lambda^*$ for different shape parameters s depending on ξ with an adapted mass parameter $\mu = (\xi/4)^{1/4}$ (θ' solid, θ'' dashed, $g^*\lambda^*$ dotted).

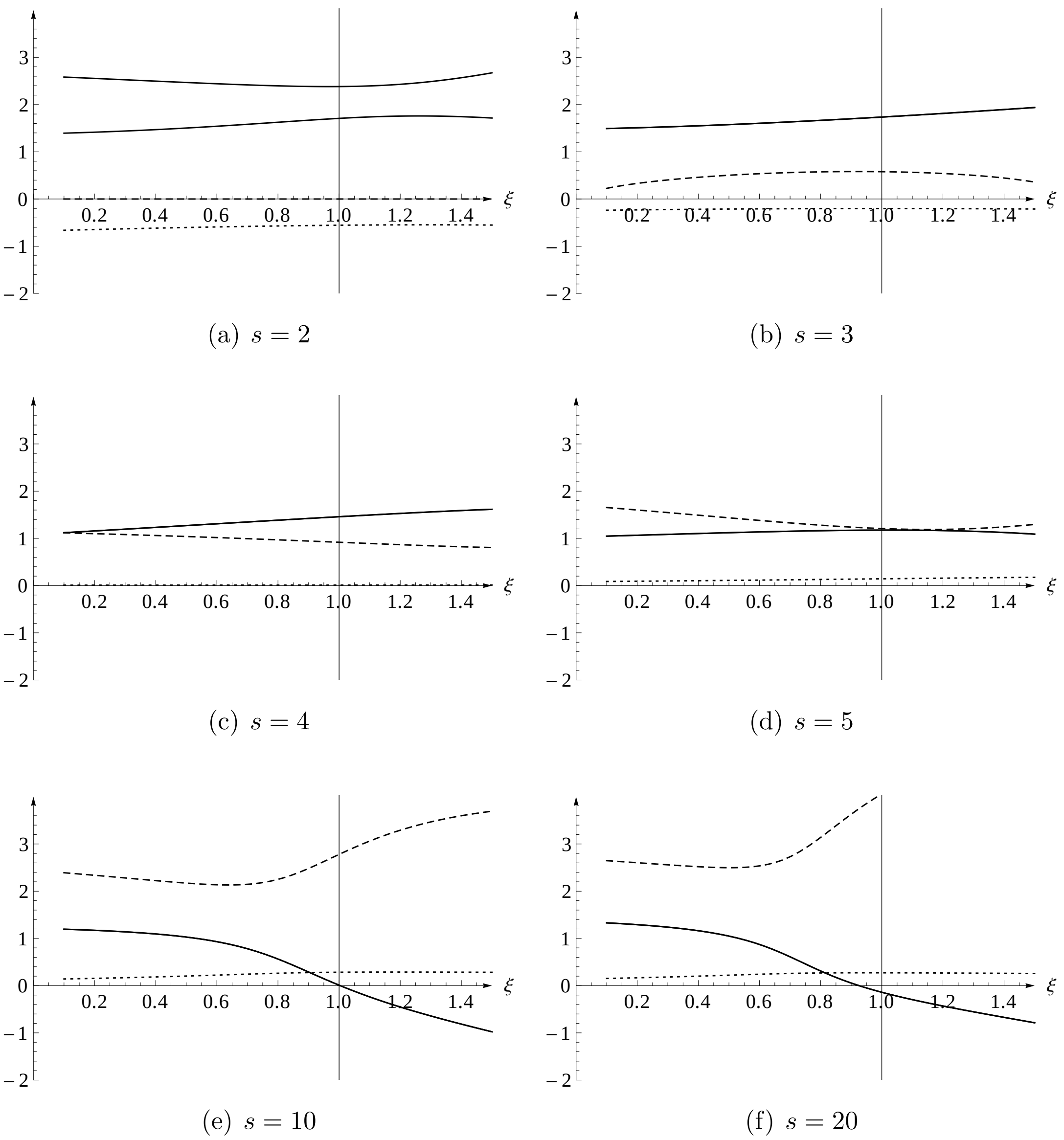


Figure D.6. Critical exponents and $g^*\lambda^*$ for different shape parameters s depending on ξ with an adapted mass parameter $\mu = \sqrt{2} \cdot (\xi/4)^{1/4}$ (θ' solid, θ'' dashed, $g^*\lambda^*$ dotted).

E RHS of the QECG Flow Equation prior to Projection

In this appendix we display the RHS of the WH-like flow equation (5.69) applied to the Holst truncation (5.15), that is studied in Chapter 5.

Expanding the RHS of the flow equation, evaluated for a constant background field configuration, $\{\bar{e}, \bar{\omega}\}$, in terms of $\bar{\omega}$ up to second order we find

$$
\begin{aligned}
\partial_t \Gamma_k \Big|_{\substack{\bar{\omega},\bar{e}= \\ \text{const}}} = & -\frac{p^4}{32\pi^2}\left(\ln \frac{(1-\gamma^2)^{24}(1-\lambda)^{12}}{\gamma^{48} g^{68}} - \ln M^{160}\mu^{32}\mathcal{N}'\right)\int \mathrm{d}^4x\, \bar{e} \\
& -\frac{p^2}{1536\pi^2}\frac{1}{(1-\lambda)^2}\Bigg[\left(\left(70-438\lambda+373\lambda^2\right)-\frac{20}{\gamma^2}\right)\int \mathrm{d}^4x\, \bar{e}\, \bar{\omega}_{pq}{}^{p}\bar{\omega}_{r}{}^{qr} \\
& \qquad +\left(\left(106-174\lambda+43\lambda^2\right)-\frac{20}{\gamma^2}\right)\int \mathrm{d}^4x\, \bar{e}\, \bar{\omega}_{pqr}\bar{\omega}^{pqr} \\
& \qquad +\left(\left(178-186\lambda+43\lambda^2\right)-\frac{20}{\gamma^2}\right)\int \mathrm{d}^4x\, \bar{e}\, \bar{\omega}_{pqr}\bar{\omega}^{prq}\Bigg] \\
& -\frac{p^2}{128\pi^2}\frac{23-33\lambda}{(1-\lambda)^2}\,\frac{1}{\gamma}\int \mathrm{d}^4x\, \bar{e}\, \varepsilon_{pqrs}\bar{\omega}_t{}^{pt}\bar{\omega}^{qrs} \\
& -\frac{5\,p^2}{256\pi^2}\frac{11-15\lambda}{(1-\lambda)^2}\,\frac{1}{\gamma}\int \mathrm{d}^4x\, \bar{e}\, \varepsilon_{pqrs}\bar{\omega}_t{}^{pq}\bar{\omega}^{rts} \\
& + \mathcal{O}(\bar{\omega}^3),
\end{aligned}
\tag{E.1}
$$

where $\mathcal{N}'$ is a pure number with $\ln \mathcal{N}' \approx 241.42$.

Here we have specialized the result for the choice of gauge parameters $(\alpha_{\mathrm{D}}, \alpha'_{\mathrm{L}}, \beta_{\mathrm{D}}) = (0,0,0)$, which considerably simplifies the result. The analogous expression for general gauge parameters fills many pages, due to its complicated polynomial structure in λ and the gauge parameters.

At this level, the result does not show any ambiguity. However, in order to identify the coefficient functions rhsF and rhsF* as defined in Chapter 5.2.5, we have to define a projection scheme onto the invariants of the Holst truncation. This can be done by specifying a basis in the space of $\bar{\omega}^2$-monomials. Depending on the choice of basis we are led to different explicit expressions of the functions rhsF and rhsF*; an effect which we refer to as a projection ambiguity. The explicit expressions for rhsF and rhsF* projected using the four bases $\mathcal{B}_i$ $(i = 1, \cdots, 4)$ which are given in Chapter 5.4 in fact all have been derived from (E.1).

F Classical Aspects of Torsion

In this appendix we discuss several classical aspects of spacetimes exhibiting torsion. In a first section we introduce the torsion tensor and its decomposition into irreducible components. Then we investigate the subspace of theory space spanned by torsion squared monomials and introduce different bases monomials in this space. The third section contains the Holst action rewritten in several equivalent forms, depending on the metric and various choices of torsion field variables. In the last section we derive the classical field equations of the spin connection and the tetrad in chiral gravity.

F.1 Preliminaries

Identities of the Levi-Civita symbol. The Levi-Civita symbol ε^{abcd} denotes the unique totally antisymmetric $\mathsf{O}(4)$-tensor with $\varepsilon^{0123} = 1 \ \Rightarrow \varepsilon_{0123} = 1$ that is defined in every local frame $e^a{}_\mu(x)$. The above implication, however, does only hold for a spacetime of Euclidean signature; in the Lorentzian case we pick up a minus sign here. For that reason the following identities only hold for a Euclidean spacetime, but differ from their Lorentzian counterparts only by corresponding minus signs.

Due to its antisymmetry one immediately infers the property

$$\varepsilon^{abcd}\varepsilon_{stuv} = \delta_s^{[a}\delta_t^b\delta_u^c\delta_v^{d]} \tag{F.1}$$

of the product of two ε-tensors. From that one can deduce by contraction of index pairs the following identities

$$\begin{aligned}
\varepsilon^{abcd}\varepsilon_{stud} &= \delta_s^{[a}\delta_t^b\delta_u^{c]}\,, \\
\varepsilon^{abcd}\varepsilon_{stcd} &= 2\delta_s^{[a}\delta_t^{b]}\,, \\
\varepsilon^{abcd}\varepsilon_{sbcd} &= 6\delta_s^a\,, \\
\varepsilon^{abcd}\varepsilon_{abcd} &= 24\,.
\end{aligned} \tag{F.2}$$

Note that throughout this work the (anti-)symmetrization brackets do not contain a weighting factor in our conventions, i. e. symbolically we have $[ab] = ab - ba$ and $(ab) = ab + ba$ (cf. Appendix A).

Another algebraic identity for the Levi-Civita symbol can be derived from the product of three ε-symbols, using the above identities:

$$-\delta_s^{[e}\delta_t^f\delta_u^{g]}\varepsilon_{ghi}{}^a = \varepsilon^a{}_{bcd}\varepsilon^{def}{}_g\varepsilon^g{}_{hia} = -\varepsilon^a{}_{bcd}\delta_h^{[d}\delta_i^e\delta_a^{f]} \tag{F.3}$$

$$\Leftrightarrow \qquad \varepsilon^{[f}{}_{bc[h}\delta^{e]}{}_{i]} = -\varepsilon^{[f}{}_{hi[b}\delta^{e]}{}_{c]}\,. \tag{F.4}$$

Using the Levi-Civita symbol we can express the determinant of the inverse vielbein by

$$e^{-1} = e_a{}^0 e_b{}^1 e_c{}^2 e_d{}^3 \varepsilon^{abcd}\,. \tag{F.5}$$

Multiplying this equation by the vielbein determinant motivates the definition of the ε-tensor density on the spacetime by

$$\varepsilon^{\mu\nu\rho\sigma} = e\, e_a{}^\mu e_b{}^\nu e_c{}^\rho e_d{}^\sigma \varepsilon^{abcd}\,. \tag{F.6}$$

It obviously inherits the total antisymmetry from the Levi-Civita tensor and satisfies $\varepsilon^{0123} = 1$ as well, but it transforms as a tensor *density* under diffeomorphisms of spacetime.

The ε-tensor density satisfies the identities (F.2) with a factor of e^2 on the right hand side as well as (F.4), both with the (flat) $\mathsf{O}(4)$-indices substituted by (curved) spacetime ones.

We often apply the identities (F.2) and (F.4) in this appendix but also throughout the main body of this thesis.

Torsion and contorsion. The torsion tensor on an affinely connected spacetime is defined as the vector valued two-form

$$T(X,Y) = D_X Y - D_Y X - [X,Y], \tag{F.7}$$

where the brackets denote the Lie-bracket of vector fields. In components we thus find

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{[\mu\nu]} \quad \Rightarrow \quad \Gamma^\lambda_{\mu\nu} = \frac{1}{2}\left(\Gamma^\lambda_{(\mu\nu)} + T^\lambda{}_{\mu\nu}\right)\,. \tag{F.8}$$

Using the Ricci condition (metricity of the connection) and (F.7) one can derive a generalized Koszul formula, that results in an equation relating the general connection and the Levi-Civita connection according to

$$\Gamma^{\lambda}_{\mu\nu} = \left(\Gamma_{\mathrm{LC}}\right)^{\lambda}_{\mu\nu} + \frac{1}{2}\left(T^{\lambda}{}_{\mu\nu} - T_{\mu\nu}{}^{\lambda} + T_{\nu}{}^{\lambda}{}_{\mu}\right) \equiv \left(\Gamma_{\mathrm{LC}}\right)^{\lambda}_{\mu\nu} + K^{\lambda}{}_{\mu\nu}\,. \tag{F.9}$$

Here, we have defined the contorsion tensor $K^{\lambda}{}_{\mu\nu}$ as the difference between the Levi-Civita connection Γ_{LC} and the metric-compatible connection Γ exhibiting torsion T. In comparison to (F.8) we note that the symmetric part of $\Gamma^{\lambda}_{(\mu\nu)}$ does not coincide with $\left(\Gamma_{\mathrm{LC}}\right)^{\lambda}_{\mu\nu}$.

If we consider the vielbein $e^{a}{}_{\mu}$ and the spin connection $\omega^{ab}{}_{\mu}$ as fundamental variables, it is obvious that the Levi-Civita connection, being a function of the metric, can be expressed purely in terms of the vielbein, such that only the contorsion part of the connection depends on $\omega^{ab}{}_{\mu}$:

$$\Gamma(e,\omega)^{\lambda}_{\mu\nu} = e_{a}{}^{\lambda}\left(\partial_{\mu} e^{a}{}_{\nu} + \omega^{a}{}_{c\mu} e^{c}{}_{\nu}\right) = \left(\Gamma_{\mathrm{LC}}(e)\right)^{\lambda}_{\mu\nu} + K(e,\omega)^{\lambda}{}_{\mu\nu}\,. \tag{F.10}$$

If one calculates the Riemann curvature tensor from the above connection $\Gamma(e,\omega)$, one finds that it is related to the field strength tensor F in the following way:

$$F^{ab}{}_{\mu\nu} = e^{a}{}_{\rho} e^{b\sigma} R(e,\omega)_{\mu\nu}{}^{\rho}{}_{\sigma} \tag{F.11}$$

Moreover, the Riemann tensor can be decomposed into the curvature tensor of the Levi-Civita connection and contorsion terms

$$R(e,\omega)_{\mu\nu}{}^{\rho}{}_{\sigma} = \left(R_{\mathrm{LC}}\right)_{\mu\nu}{}^{\rho}{}_{\sigma} + D^{\mathrm{LC}}_{[\mu} K^{\rho}{}_{\nu]\sigma} + K^{\rho}{}_{[\mu|\tau} K^{\tau}{}_{\nu]\sigma}\,. \tag{F.12}$$

Irreducible decomposition of the torsion tensor. In order to identify the independent invariants quadratic in the torsion tensor, we will decompose the torsion tensor into its irreducible components. The resulting orthogonal decomposition reads [BH11, Sha02]

$$T^{\lambda}{}_{\mu\nu} = \frac{1}{3}\left(\delta^{\lambda}{}_{\nu} T_{\mu} - \delta^{\lambda}{}_{\mu} T_{\nu}\right) + \frac{1}{6e}\varepsilon^{\lambda}{}_{\mu\nu\sigma} S^{\sigma} + q^{\lambda}{}_{\mu\nu} \tag{F.13}$$

with $q^{\lambda}{}_{\mu\lambda} = 0$, $\varepsilon^{\mu\nu\rho\sigma} q_{\nu\rho\sigma} = 0$, and $q^{\lambda}{}_{\mu\nu} = -q^{\lambda}{}_{\nu\mu}$. Here, $T_{\mu} = T^{\lambda}{}_{\mu\lambda}$ is the trace of the torsion tensor and $S^{\mu} = \frac{1}{e}\varepsilon_{\nu}{}^{\rho\sigma\mu} T^{\nu}{}_{\rho\sigma}$ is its totally antisymmetric part. The vielbein determinant e in the decomposition ensures that all torsion components defined here transform as tensors under spacetime diffeomorphisms.

Let us further investigate the symmetry properties of q. First, we note that all contractions of two of its indices vanish due to $q^{\lambda}{}_{\mu\lambda} = 0$ and its antisymmetry in the last two indices. Second, as its totally antisymmetric part vanishes, $\varepsilon^{\mu\nu\rho\sigma} q_{\nu\rho\sigma} = 0 \Leftrightarrow q^{[\nu\rho\sigma]} = 0$, we obtain with the antisymmetry in the last two indices

$$0 = q^{[\mu\nu\rho]} = q^{[\mu\nu]\rho} - q^{[\mu|\rho|\nu]} - q^{\rho[\nu\mu]} = 2\big(q^{[\mu\nu]\rho} + q^{\rho\mu\nu}\big) \quad \Leftrightarrow \quad q^{[\mu\nu]\rho} = -q^{\rho\mu\nu} \,. \tag{F.14}$$

Now it is easy to classify all independent invariants quadratic in these irreducible torsion components.

For the parity-even invariants we are left with three possible independent combinations: As S^μ is a pseudo-vector and T^μ as well as $q^{\mu\nu\rho}$ are (true) tensors that can only couple to itself to form a scalar. If we contract T^μ with $q^{\mu\nu\rho}$ the other two indices of q have to be contracted and thus the combination vanishes. In principle, we could also combine S and q using an additional ε-density, but these combinations vanish as q has no totally antisymmetric part. Hence, we are left with the three parity-even invariants

$$I_1 = T^\mu T_\mu, \qquad I_2 = S^\mu S_\mu, \qquad I_3 = q^{\mu\nu\rho} q_{\mu\nu\rho} \,. \tag{F.15}$$

At first sight one could wonder whether there are additional independent q^2-contractions. This is, however, not the case: In total we start with 6 contractions that correspond to the 6 permutations of the indices of the second q-factor. The terms with odd permutations are related to the remaining cyclic permutations by the antisymmetry of q in the last two indices. For the cyclic permutations we find with (F.14)

$$q_{\mu\nu\rho} q^{\nu\rho\mu} = \frac{1}{2} q_{\mu\nu\rho} q^{[\nu\rho]\mu} = -\frac{1}{2} q_{\mu\nu\rho} q^{\mu\nu\rho}, \quad q_{\mu\nu\rho} q^{\rho\mu\nu} = \frac{1}{2} q_{[\mu\nu]\rho} q^{\rho\mu\nu} = -\frac{1}{2} q_{\rho\mu\nu} q^{\rho\mu\nu} \,, \tag{F.16}$$

such that only I_3 remains independent.

For the parity-odd combinations there are only two independent invariants, namely

$$I_4 = S_\mu T^\mu, \qquad I_5 = \frac{1}{e}\, \varepsilon_{\alpha\beta\gamma\delta}\, q^{\alpha\beta\mu}\, q^{\gamma\delta}{}_{\mu} \,. \tag{F.17}$$

The two other εq^2 combinations one might think of as independent, namely those where either both first indices of the q tensors are contracted or the first index of the first q factor is contracted with the last index of the second factor, are related to I_5 according to

$$\frac{1}{e}\,\varepsilon_{\alpha\beta\gamma\delta}\,q^{\mu\alpha\beta}\,q_{\mu}{}^{\gamma\delta} = 4I_5, \qquad \frac{1}{e}\,\varepsilon_{\alpha\beta\gamma\delta}\,q^{\mu\alpha\beta}\,q^{\gamma\delta}{}_{\mu} = -2I_5\,. \tag{F.18}$$

These relations can be shown using the identity (F.4) of the ε-symbol.

F.2 Invariants quadratic in the torsion tensor

In this section we discuss all possible field monomials quadratic in the torsion tensor and give their decomposition in the irreducible components introduced above as well as the expressions in terms of the spin connection that remain when the monomials are evaluated for constant background fields $\{\bar{e},\bar{\omega}\}$.

Parity-even monomials. There are three different contractions of the torsion tensor with itself that are not related to each other by its symmetries. As we know already that the space of parity-even monomials is three dimensional and spanned by $I_{\{1,2,3\}}$ we can conclude that they correspond to a different basis of this space. In detail we find that the two bases are related by

$$\begin{aligned} T_1^{2(+)} &= T^{\mu\nu\rho}T_{\mu\nu\rho} = \frac{2}{3}I_1 + \frac{1}{6}I_2 + I_3 &&\widehat{=}\; 2\,\bar{\omega}^{abc}\bar{\omega}_{abc} - 2\,\bar{\omega}^{abc}\bar{\omega}_{acb} \\ T_2^{2(+)} &= T^{\mu\nu\rho}T_{\nu\mu\rho} = \frac{1}{3}I_1 - \frac{1}{6}I_2 + \frac{1}{2}I_3 &&\widehat{=} -\bar{\omega}^{abc}\bar{\omega}_{abc} + 3\,\bar{\omega}^{abc}\bar{\omega}_{acb} \\ T_3^{2(+)} &= T^{\mu\nu}{}_{\mu}T^{\rho}{}_{\nu\rho} = I_1 &&\widehat{=}\; \bar{\omega}^{ab}{}_{a}\bar{\omega}^{c}{}_{bc}\,. \end{aligned} \tag{F.19}$$

On the right hand side we have evaluated the monomials on constant background fields $\{\bar{e},\bar{\omega}\}$. In addition we have incorporated the vielbein into the spin connection, changing its index structure. The explicit vielbein expressions can be reconstructed uniquely.

Parity-odd monomials. In the parity-odd sector we find four torsion squared monomials contracted with the ε-symbol. Those four expressions are not linearly independent from each other, as the corresponding subspace of theory space is two dimensional (spanned by I_4 and I_5); while in the decomposed setting the linear relation between the four monomials becomes obvious, it can also be shown for the undecomposed torsion tensor using the identity (F.4). In the irreducible component basis the parity-odd torsion squared monomials read

$$\begin{aligned}
T_1^{2(-)} &= e^{-1}\varepsilon^{\mu\nu\rho\sigma}T^\tau{}_{\mu\nu}T_{\tau\rho\sigma} = -\frac{4}{3}I_4 + 4I_5 \mathrel{\widehat{=}} && 4\,\varepsilon_{pqrs}\,\bar{\omega}_t{}^{pq}\bar{\omega}^{trs}\\
T_2^{2(-)} &= e^{-1}\varepsilon^\rho{}_\nu{}^{\sigma\tau}T^\mu{}_{\mu\rho}T^\nu{}_{\sigma\tau} = \quad I_4 \mathrel{\widehat{=}} -2\,\varepsilon_{pqrs}\,\bar{\omega}_t{}^{pt}\bar{\omega}^{qrs}\\
T_3^{2(-)} &= e^{-1}\varepsilon^{\mu\nu}{}_\rho{}^\sigma T^\tau{}_{\mu\nu}T^\rho{}_{\tau\sigma} = \frac{1}{3}I_4 + 2I_5 \mathrel{\widehat{=}} -2\,\varepsilon_{pqrs}\,\bar{\omega}_t{}^{pt}\bar{\omega}^{qrs} + 2\,\varepsilon_{pqrs}\,\bar{\omega}_t{}^{pq}\bar{\omega}^{trs}\\
T_4^{2(-)} &= e^{-1}\varepsilon_{\mu\nu}{}^{\rho\sigma}T^\mu{}_{\tau\rho}T^{\nu\tau}{}_\sigma = -\frac{2}{3}I_4 - I_5 \mathrel{\widehat{=}} 2\,\varepsilon_{pqrs}\,\bar{\omega}_t{}^{pt}\bar{\omega}^{qrs} - \varepsilon_{pqrs}\,\bar{\omega}_t{}^{pq}\bar{\omega}^{trs}.
\end{aligned} \tag{F.20}$$

As for the torsion tensor there exist four contractions of the spin connection, that are related via (F.4); in the above we have used in particular

$$\begin{aligned}
\varepsilon_{pqrs}\bar{\omega}^{pqt}\bar{\omega}^{rs}{}_t &= -4\,\varepsilon_{pqrs}\bar{\omega}_t{}^{pt}\bar{\omega}^{qrs} + 4\,\varepsilon_{pqrs}\bar{\omega}_t{}^{pq}\bar{\omega}^{trs}\\
\varepsilon_{pqrs}\bar{\omega}^{pqt}\bar{\omega}_t{}^{rs} &= \quad \varepsilon_{pqrs}\bar{\omega}_t{}^{pt}\bar{\omega}^{qrs} - 2\,\varepsilon_{pqrs}\bar{\omega}_t{}^{pq}\bar{\omega}^{trs}
\end{aligned} \tag{F.21}$$

in order to reexpress the right hand side of (F.20) in terms of the other two independent contractions only.

Modifications in the (anti-)selfdual case. If we consider a torsion tensor that is constructed from an (anti-)selfdual spin connection $\omega^{(\pm)}$ we could expect that there are less independent quadratic torsion invariants, as the spin connection has only half the number of independent components. However, this does not seem to be the case as the following consideration shows.

A selfdual spin connection, $\omega^{(+)}$, can be represented as the sum of the selfdual part of the spin connection that corresponds to the Levi-Civita connection, $\omega_{\text{LC}}^{(+)}$, and a selfdual remainder $\Lambda^{(+)}$. In a second step we substitute $\omega_{\text{LC}}^{(+)} = \omega_{\text{LC}} - \omega_{\text{LC}}^{(-)}$ such that we arrive at

$$\omega^{(\pm)} = \omega_{\text{LC}}^{(\pm)} + \Lambda^{(\pm)} = \omega_{\text{LC}} - \omega_{\text{LC}}^{(\mp)} + \Lambda^{(\pm)}\,. \tag{F.22}$$

The connection constructed from this selfdual spin connection can, hence, be written as

$$\begin{aligned} \Gamma^{\lambda}_{\mu\nu} = e_a{}^{\lambda}\, {}^{(\pm)}\nabla_{\mu} e^a{}_{\nu} &= e_a{}^{\lambda}\, {}^{\text{LC}}\nabla_{\mu} e^a{}_{\nu} + e_a{}^{\lambda}\Big(- \big(\omega^{(\mp)}_{\text{LC}}\big)^a{}_{b\mu} + \big(\Lambda^{(\pm)}\big)^a{}_{b\mu}\Big) e^b{}_{\nu} \\ &= \quad \big(\Gamma_{\text{LC}}\big)^{\lambda}_{\mu\nu} \quad + \qquad\qquad K^{\lambda}{}_{\mu\nu}\,, \end{aligned} \tag{F.23}$$

such that the resulting torsion tensor reads

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{[\mu\nu]} = K^{\lambda}{}_{[\mu\nu]} = e_a{}^{\lambda}\Big(- \big(\omega^{(\mp)}_{\text{LC}}\big)^a{}_{b[\mu} + \big(\Lambda^{(\pm)}\big)^a{}_{b[\mu}\Big) e^b{}_{\nu]}\,. \tag{F.24}$$

We observe that this torsion tensor is constructed from a contorsion K that always contains both, a self-dual and an anti-selfdual component. In this respect the torsion tensor of a selfdual spin connection is completely generic; in particular we therefore cannot simplify any contraction of the torsion tensor with the ε-symbol, which would give rise to relations between the parity-even and -odd invariants from above in the chiral case.

The only fact that is special in the case of a selfdual spin connection, is that the anti-selfdual part of the contorsion is given by the anti-selfdual projection of the Levi-Civita connection, and is thus fixed by the vielbein e. Only if it is possible to make use of this fact in an explicit calculation of the above invariants $I_1, \cdots, I_5$ a reduction of the number of independent torsion squared monomials in the chiral case would arise.

We do not want to discuss this possibility in more detail as the more important question for our concrete RG study is which invariants we can distinguish when they are evaluated on the background spacetime chosen. As we used the WH-like flow equation for our RG study of chiral gravity that requires us to employ constant background fields $\{\bar{e}, \bar{\omega}^{(\pm)}\}$, we can directly infer that the Levi-Civita connection vanishes for any constant choice of background fields: $\Gamma_{\text{LC}} = 0 = \omega_{\text{LC}}$. Hence, in this case we have $\Lambda^{(\pm)} = \bar{\omega}^{(\pm)}$ and the contorsion tensor only contains one spin connection of definite chirality. Therefore we can now use the selfduality of $\bar{\omega}^{(\pm)}$ to simplify the contraction of the torsion tensor with the ε-symbol and find that the components $\bar{T}^{(\pm)}_{\mu}$ and $\bar{S}^{(\pm)}_{\mu}$ of the torsion tensor (evaluated on constant background fields) become proportional to each other

$$\bar{S}^{(\pm)}_{\mu} = \frac{1}{e}\varepsilon_{\lambda}{}^{\rho\sigma}{}_{\mu}(\bar{T}^{(\pm)})^{\lambda}{}_{\rho\sigma} = \mp 4 \bar{e}^{\nu}_{a}\bar{e}_{b\mu}(\bar{\omega}^{(\pm)})^{ab}{}_{\nu} = \pm 4 (\bar{T}^{(\pm)})^{\nu}{}_{\mu\nu} = \pm 4 \bar{T}^{(\pm)}_{\mu}\,. \tag{F.25}$$

From that we conclude that the invariants I_1, I_2 and I_4 from above are related to each other when evaluated on the constant background field configuration $\{\bar{e}, \bar{\omega}^{(\pm)}\}$ according to

$$\bar{I}_1^{(\pm)} = \frac{1}{16}\,\bar{I}_2^{(\pm)} = \pm\frac{1}{4}\,\bar{I}_4^{(\pm)} = (\bar{\omega}^{(\pm)})^{ab}{}_a(\bar{\omega}^{(\pm)})^c{}_{bc}\,. \tag{F.26}$$

Similarly also the invariants I_3 and I_5 are related by

$$\bar{I}_3^{(\pm)} = \pm 2\bar{I}_5^{(\pm)} = \frac{2}{3}\Big((\bar{\omega}^{(\pm)})^{ab}{}_a(\bar{\omega}^{(\pm)})^c{}_{bc} + 3(\bar{\omega}^{(\pm)})^{abc}(\bar{\omega}^{(\pm)})_{acb}\Big)\,. \tag{F.27}$$

This can be seen by direct computation of the invariants on the background field configuration using the following identities in order to express all possible contractions of the (anti-)selfdual spin connection in terms of the two used above:

$$\begin{aligned}
\varepsilon_{pqrs}(\bar{\omega}^{(\pm)})^{tp}{}_t(\bar{\omega}^{(\pm)})^{qrs} &= \mp 2(\bar{\omega}^{(\pm)})^{tp}{}_t(\bar{\omega}^{(\pm)})^s{}_{ps}\\
\varepsilon_{pqrs}(\bar{\omega}^{(\pm)})^{pqt}(\bar{\omega}^{(\pm)})^{rs}{}_t &= \pm 4\Big((\bar{\omega}^{(\pm)})^{tp}{}_t(\bar{\omega}^{(\pm)})^s{}_{ps} + (\bar{\omega}^{(\pm)})^{pqr}(\bar{\omega}^{(\pm)})_{prq}\Big)\\
\varepsilon_{pqrs}(\bar{\omega}^{(\pm)})^{tpq}(\bar{\omega}^{(\pm)})^{rs}{}_t &= \mp 2(\bar{\omega}^{(\pm)})^{pqr}(\bar{\omega}^{(\pm)})_{prq}\\
\varepsilon_{pqrs}(\bar{\omega}^{(\pm)})_t{}^{pq}(\bar{\omega}^{(\pm)})^{trs} &= \mp\Big((\bar{\omega}^{(\pm)})^{tp}{}_t(\bar{\omega}^{(\pm)})^s{}_{ps} - (\bar{\omega}^{(\pm)})^{pqr}(\bar{\omega}^{(\pm)})_{prq}\Big)\\
(\bar{\omega}^{(\pm)})^{pqr}(\bar{\omega}^{(\pm)})_{pqr} &= 2(\bar{\omega}^{(\pm)})^{pq}{}_p(\bar{\omega}^{(\pm)})^r{}_{qr} + 2(\bar{\omega}^{(\pm)})^{pqr}(\bar{\omega}^{(\pm)})_{prq}
\end{aligned} \tag{F.28}$$

We conclude that in the case of chiral gravity by employing constant background fields all torsion squared monomials are projected onto two-dimensional space spanned by the monomials $\bar{I}_1^{(\pm)}$ and $\bar{I}_3^{(\pm)}$.

F.3 Holst action in metric and torsion variables

In this section starting from the Holst action with the tetrad and the spin connection as field variables we deduce several equivalent actions that depend on the metric and one independent field, that is either a general connection Γ (allowing for torsion but satisfying the metricity condition), the contorsion K, the torsion T or its irreducible components T, S, q.

The Holst action in terms of tetrads and the spin connection is given by

$$S_{\mathrm{Ho}} = -\frac{1}{16\pi G}\int \mathrm{d}^4x\, e\left[e_a{}^{\mu}e_b{}^{\nu}\left(F(e,\omega)^{ab}{}_{\mu\nu} - \frac{1}{2\gamma}\varepsilon^{ab}{}_{cd}F(e,\omega)^{cd}{}_{\mu\nu}\right) - 2\Lambda\right]. \tag{F.29}$$

Using (F.11) we can reexpress the field strength $F(e,\omega)$ by a Riemann tensor $R(\Gamma)$, with a connection Γ exhibiting torsion, according to

$$= -\frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{g}\left[R(\Gamma) - \frac{1}{2\gamma\sqrt{g}}\varepsilon^{\mu\nu}{}_{\rho}{}^{\sigma}R(\Gamma)_{\mu\nu}{}^{\rho}{}_{\sigma} - 2\Lambda\right]. \tag{F.30}$$

In a next step the Riemann tensor $R(\Gamma)$ is written in terms of its Levi-Civita counterpart R_{LC} and contorsion terms as given in (F.12) resulting in

$$\begin{aligned} = -\frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{g}\Big[R_{\mathrm{LC}} + D^{\mathrm{LC}}_{[\rho}K^{\rho}{}_{\nu]}{}^{\nu} + K^{\rho}{}_{[\rho|\tau}K^{\tau}{}_{\nu]}{}^{\nu} \\ - \frac{1}{\gamma}\frac{\varepsilon^{\mu\nu}{}_{\rho}{}^{\sigma}}{\sqrt{g}}(D^{\mathrm{LC}}_{\mu}K^{\rho}{}_{\nu\sigma} + K^{\rho}{}_{\mu\tau}K^{\tau}{}_{\nu\sigma}) - 2\Lambda\Big]. \end{aligned} \tag{F.31}$$

Now we can switch from the contorsion variable K to the torsion tensor T using (F.9)

$$\begin{aligned} = -\frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{g}\Big[R_{\mathrm{LC}} + 2D^{\mathrm{LC}}_{\rho}T^{\nu\rho}{}_{\nu} + \frac{1}{4}T_{\mu\nu\rho}T^{\mu\nu\rho} + \frac{1}{2}T_{\mu\nu\rho}T^{\nu\mu\rho} - T^{\mu\nu}{}_{\mu}T^{\rho}{}_{\nu\rho} \\ - \frac{1}{\gamma}\frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}\left(-\frac{1}{2}D^{\mathrm{LC}}_{\mu}T_{\nu\rho\sigma} + T^{\tau}{}_{\mu\nu}T_{\tau\rho\sigma}\right) - 2\Lambda\Big]. \end{aligned} \tag{F.32}$$

In a last step we decompose the torsion tensor into its irreducible components (F.13) and obtain

$$\begin{aligned} = -\frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{g}\Big[R_{\mathrm{LC}} + 2D^{\mathrm{LC}}_{\mu}T^{\mu} - \frac{2}{3}T_{\mu}T^{\mu} - \frac{1}{24}S_{\mu}S^{\mu} + \frac{1}{2}q_{\mu\nu\rho}q^{\mu\nu\rho} \\ - \frac{1}{\gamma}\left(\frac{1}{2}D^{\mathrm{LC}}_{\mu}S^{\mu} - \frac{1}{3}T_{\mu}S^{\mu} + \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}q_{\mu\nu}{}^{\tau}q_{\rho\sigma\tau}\right) - 2\Lambda\Big]. \end{aligned} \tag{F.33}$$

F.4 Field equations of chiral gravity

In this section we derive the equations of motion for the tetrad and the chiral spin connection the Holst action gives rise to in case of $\gamma = \mp 1$ in order to show that they comprise Einstein's equation. A similar consideration applies to the case $\gamma \neq \mp 1$, which was carried out in detail in [Dau]. Our presentation here follows the reasoning of [Giu94], where the same has been shown, the only difference being, that we adapt the notation to our conventions and work throughout with explicit component expressions.

Starting from the chiral Holst action ($\gamma = \mp 1$)

$$\begin{aligned} S^{\pm}[e^a{}_\mu, \omega^{(\pm)\,ab}{}_\mu] &= -\frac{1}{8\pi G}\int \mathrm{d}^4x\, e\big(e_a{}^\mu e_b{}^\nu F^{(\pm)\,ab}{}_{\mu\nu} - \Lambda\big) \\ &= -\frac{1}{8\pi G}\int \mathrm{d}^4x\big(\pm 2\, e^c{}_\rho e^d{}_\sigma \varepsilon^{\mu\nu\rho\sigma} P^{\pm}{}_{cdab} F^{(\pm)\,ab}{}_{\mu\nu} - e\,\Lambda\big) \end{aligned} \tag{F.34}$$

we obtain the equations of motion for the chiral spin connection, using $\delta_{\omega^{(\pm)}} F^{(\pm)\,ab}{}_{\mu\nu} = \nabla^{\pm}_{[\mu}\tau^{ab}{}_{\mu]}$ and partial integration, which read

$$\delta_{\omega^{(\pm)}} S^{(\pm)}[e^a{}_\mu, \omega^{(\pm)\,ab}{}_\mu] = 0 \quad \Leftrightarrow \quad \varepsilon^{\mu\nu\rho\sigma} P^{\pm\,ab}{}_{cd} \nabla^{\pm}_\nu \big(e^c{}_\rho e^d{}_\sigma\big) = 0\,, \tag{F.35}$$

where $\nabla^{\pm}_\mu$ denotes the $\mathsf{SO}(3)^{\pm}$-covariant derivative constructed from the chiral spin connection $\omega^{(\pm)}$. In the following we will show that (F.35) has a solution, $\omega^{(\pm)}(e)$, that turns out to be the chiral projection of the spin connection $\omega(e)$ which corresponds to the Levi-Civita spacetime connection Γ_{LC}, i. e. $\omega^{(\pm)}(e) = P^{\pm}\omega(e)$, if the vielbein is invertible.

In order to prove this assertion we decompose $\omega^{(\pm)}$ in (F.35) without loss of generality according to

$$\omega^{(\pm)} = \omega^{(\pm)}(e) + \Lambda^{(\pm)} = \omega(e) - \omega^{(\mp)}(e) + \Lambda^{(\pm)}\,, \tag{F.36}$$

where $\Lambda^{(\pm)}$ is a tensor with the same index structure as ω, which is antisymmetric and of the respective chirality in its first two indices, but arbitrary apart from that. Eventually we will show that (F.35) implies $\Lambda^{(\pm)} = 0$.

Substituting (F.36) into (F.35) yields

$$\varepsilon^{\mu\nu\rho\sigma} P^{\pm\,ab}{}_{cd}\big({}^{\rm LC}\nabla_\nu(e^c{}_\rho e^d{}_\sigma) + (\Lambda^{(\pm)\,c}{}_{e\nu} - \omega^{(\mp)}(e)^c{}_{e\nu})e^e{}_\rho e^d{}_\sigma + \\ + (\Lambda^{(\pm)\,d}{}_{e\nu} - \omega^{(\mp)}(e)^d{}_{e\nu})e^c{}_\rho e^e{}_\sigma\big) = 0\,. \quad (F.37)$$

The first term on the LHS vanishes due to the metricity condition and the symmetry of the Levi-Civita connection, as we can write

$$\varepsilon^{\mu\nu\rho\sigma\,{\rm LC}}\nabla_\nu(e^c{}_\rho e^d{}_\sigma) \\ = \varepsilon^{\mu\nu\rho\sigma}\big(\underbrace{{}^{\rm LC}\mathcal{D}_\nu(e^c{}_\rho e^d{}_\sigma)}_{=0} + \big(\Gamma_{\rm LC}\big)^\tau_{\nu\rho} e^c{}_\tau e^d{}_\sigma + \big(\Gamma_{\rm LC}\big)^\tau_{\nu\sigma} e^c{}_\rho e^d{}_\tau\big) = 0\,. \quad (F.38)$$

The rest of equation (F.37) can be cast in the form

$$\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} P^{\pm\,ab[c}{}_{[d}\delta^{f]}{}_{e]}\big(\Lambda^{(\pm)}{}_{cf\nu} - \omega^{(\mp)}(e)_{cf\nu}\big)e^e{}_\rho e^d{}_\sigma = 0\,. \quad (F.39)$$

Now we can apply a property of the projector $P^\pm$, that is directly inferred from the analog identity (F.4) of the ε-tensor,

$$P^{\pm\,ab[c}{}_{[d}\delta^{f]}{}_{e]} = -P^{\pm\,cf[a}{}_{[d}\delta^{b]}{}_{e]}\,, \quad (F.40)$$

such that (F.39) now reads

$$-\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\delta^{[b}{}_{[e}\Big[\big(\underbrace{P^\pm\Lambda^{(\pm)}}_{=\Lambda^{(\pm)}}\big)^{a]}{}_{d]\nu} - \big(\underbrace{P^\pm\omega^{(\mp)}(e)}_{=0}\big)^{a]}{}_{d]\nu}\Big]e^e{}_\rho e^d{}_\sigma = 0\,. \quad (F.41)$$

Hence, we are left with a condition for the tensor $\Lambda^{(\pm)}$, that, by contraction with another ε-tensor density, can be brought to the form

$$\Lambda^{(\pm)\,[a}{}_{d[\nu}e^{b]}{}_\rho e^d{}_{\sigma]} = 0\,. \quad (F.42)$$

From (F.42) we derive two further conditions for $\Lambda^{(\pm)}$ by contraction with the inverse vielbein. If we first contract with $e_b{}^\rho$ we find

$$\Lambda^{(\pm)\,a}{}_{d[\nu}e^d{}_{\sigma]} - e^a{}_\sigma e_b{}^\rho \Lambda^{(\pm)\,b}{}_{d[\rho}e^d{}_{\nu]} - e^a{}_\nu e_b{}^\rho \Lambda^{(\pm)\,b}{}_{d[\sigma}e^d{}_{\rho]} = 0\,. \quad (F.43)$$

A further contraction with $e_a{}^\sigma$ yields

$$e_b{}^\rho \Lambda^{(\pm)\,b}{}_{d[\rho} e^d{}_{\nu]} = 0\,. \tag{F.44}$$

Finally, we insert (F.44) back into (F.43), and obtain

$$\Lambda^{(\pm)}{}_{ad[\nu} e^d{}_{\sigma]} = 0 \quad \Leftrightarrow \quad \Lambda^{(\pm)}{}_{ad\nu} = e^e{}_\nu \Lambda^{(\pm)}{}_{ae\mu} e_d{}^\mu \tag{F.45}$$

Hence we have shown that $\Lambda^{(\pm)}$ satisfies a second symmetry condition (apart from being anti-symmetric in its first two indices), that concerns its last two indices. Now, by successively switching the last and the first two indices for three times, we find

$$\begin{aligned}
\Lambda^{(\pm)}{}_{ad\nu} &= e_d{}^\sigma \Lambda^{(\pm)}{}_{ae\sigma} e^e{}_\nu = -e_d{}^\sigma \Lambda^{(\pm)}{}_{ea\sigma} e^e{}_\nu \\
&= -e_d{}^\sigma e_a{}^\tau \Lambda^{(\pm)}{}_{ef\tau} e^f{}_\sigma e^e{}_\nu = e_a{}^\tau \Lambda^{(\pm)}{}_{de\tau} e^e{}_\nu \\
&= e_a{}^\tau e_e{}^\alpha \Lambda^{(\pm)}{}_{dg\alpha} e^g{}_\tau e^e{}_\nu \qquad = -\Lambda^{(\pm)}{}_{ad\nu}\,.
\end{aligned} \tag{F.46}$$

and therefore

$$\Lambda^{(\pm)}{}_{ad\nu} = 0\,. \tag{F.47}$$

Thus, it is proven that $\omega^{(\pm)}(e) = P^\pm \omega(e)$ solves the equations of motion of the spin connection $\delta_{\omega^{(\pm)}} S^\pm = 0$.

As a second step we want to substitute this solution into the action and hence compute $S^\pm[\omega^{(\pm)}(e), e]$. To this end we first compute the contorsion $K^{(\pm)}$ the solution $\omega^{(\pm)}(e)$ gives rise to. We have

$$\begin{aligned}
\Gamma^{(\pm)}(e)^\lambda_{\mu\nu} &= e_a{}^\lambda \big(\partial_\mu e^a{}_\nu + \omega^{(\pm)}(e)^a{}_{b\mu} e^b{}_\nu\big) \\
&= e_a{}^\lambda \big(\partial_\mu e^a{}_\nu + \omega(e)^a{}_{b\mu} e^b{}_\nu - \omega^{(\mp)}(e)^a{}_{b\mu} e^b{}_\nu\big) \\
&= \Gamma_{\mathrm{LC}}{}^\lambda_{\mu\nu} - e_a{}^\lambda \omega^{(\mp)}(e)^a{}_{b\mu} e^b{}_\nu\,.
\end{aligned} \tag{F.48}$$

From the last line we can simply read off the contorsion as

$$K^{(\pm)}(e)^\lambda{}_{\mu\nu} = -e_a{}^\lambda \omega^{(\mp)}(e)^a{}_{b\mu} e^b{}_\nu\,. \tag{F.49}$$

Substituting this contorsion into the chiral Holst action expressed in vielbein and contorsion variable (cf. (F.31)) we find

$$\begin{aligned} S^{\pm}[e,\omega^{(\pm)}(e)] &= -\frac{1}{16\pi G}\int \mathrm{d}^4x\, e\Big[R_{\mathrm{LC}} + D^{\mathrm{LC}}_{[\rho}K^{(\pm)\rho}{}_{\nu]}{}^{\nu} + K^{(\pm)\rho}{}_{[\rho|\tau}K^{(\pm)\tau}{}_{\nu]}{}^{\nu} \\ &\qquad \pm \frac{\varepsilon^{\mu\nu}{}_{\rho}{}^{\sigma}}{e}\big(D^{\mathrm{LC}}_{\mu}K^{(\pm)\rho}{}_{\nu\sigma} + K^{(\pm)\rho}{}_{\mu\tau}K^{(\pm)\tau}{}_{\nu\sigma}\big) - 2\Lambda\Big] \\ &= -\frac{1}{16\pi G}\int \mathrm{d}^4x\, e\big[R_{\mathrm{LC}} - 2\Lambda\big]\,, \end{aligned} \tag{F.50}$$

as it is easy to show that the contorsion terms cancel due to their chirality property. Since we are left with the well-known Einstein-Hilbert action expressed in terms of the vielbein it is now obvious that the field equation for the vielbein resulting from this action is Einstein's equation.

Finally, we want to mention that, in principle, we should first vary w. r. t. $e^a{}_\mu$ and then substitute the solution into the field equation of the spin connection in order to obtain the correct tetrad field equation. Obviously substitution and variation generally do not commute at this point. However, at a stationary point $\delta_{\omega^{(\pm)}}S^{\pm} = 0$ of the action they do:

$$\begin{aligned} &\delta_e S^{\pm}[e,\omega^{\pm}(e)] = 0 \\ \Leftrightarrow\quad &\underbrace{\frac{\delta S^{\pm}}{\delta\omega^{(\pm)}}\bigg|_{\omega^{(\pm)}=\omega^{(\pm)}(e)}}_{=0}\frac{\delta\omega^{(\pm)}}{\delta e} + \frac{\delta S^{\pm}}{\delta e}[\omega^{(\pm)},e]\bigg|_{\omega^{(\pm)}=\omega^{(\pm)}(e)} = 0 \\ \Leftrightarrow\quad &\delta_e S^{\pm}[\omega^{(\pm)},e]\bigg|_{\omega^{(\pm)}=\omega^{(\pm)}(e)} = 0 \end{aligned} \tag{F.51}$$

Thus we have shown that the action $S^{\pm}[e,\omega^{\pm}]$ gives rise to the Einstein equation, when varied w. r. t. $e^a{}_\mu$, if we substitute for $\omega^{(\pm)}$ the solution to its field equation, $\omega^{(\pm)}(e)$, before or after the variation.

It is interesting to note that the way classical metric gravity is contained in chiral gravity is different to the cases of all other values of the Immirzi parameter. For $\gamma \neq \pm 1$ the analog solution to the field equations of the spin connection $\omega(e)$ corresponds to the torsionless Levi-Civita connection. Hence, the manifold defined by $e^a{}_\mu$ and equipped with the spin connection $\omega(e)^{ab}{}_\mu$ is itself a Riemann spacetime ($T^\lambda{}_{\mu\nu} = 0$) that could be described by the metric $g(e)$ alone. In chiral gravity, however, the spacetime specified by $(e, \omega^{(\pm)}(e))$ exhibits torsion, and thus cannot be described equivalently as a solution of Einsteins equation in metric gravity. Nevertheless, as we have seen above, the action principle leads to Einsteins equation for the vielbein $e^a{}_\mu$ on this spacetime, and we can reconstruct the corresponding spacetime of classical metric gravity using its solution $e^a{}_\mu$ together with the unique Levi-Civita connection $\Gamma_{\mathrm{LC}}(e)$ this vielbein gives rise to.

Parts of the results of Chapter 3, and especially the generalization of the ghost action to gauge groups of semi-direct product type as presented in Chapter 2, have already been published in [DHR10], while most of the results presented in Chapter 3 and 4 appeared already in [DHR10, DHR11, HR11] and [HR12], respectively. Chapter 5 contains as yet unpublished material.

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