The geometry of Lagrangian fibres

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Abstract

If the generic fibre $f^{-1}(c)$ of a Lagrangian fibration $f : X \to B$ on a complex Poisson–variety X is smooth, compact, and connected, it is isomorphic to the compactification of a complex abelian Lie–group. For affine Lagrangian fibres it is not clear what the structure of the fibre is. Adler and van Moerbeke developed a strategy to prove that the generic fibre of a Lagrangian fibration is isomorphic to the affine part of an abelian variety.
We extend their strategy to verify that the generic fibre of a given Lagrangian fibration is the affine part of a $(\mathbb{C}^*)^r$–extension of an abelian variety. This strategy turned out to be successful for all examples we studied. Additionally we studied examples of Lagrangian fibrations that have the affine part of a ramified cyclic cover of an abelian variety as generic fibre. We obtained an embedding in a Lagrangian fibration that has the affine part of a $\mathbb{C}^*$–extension of an abelian variety as generic fibre. This embedding is not an embedding in the category of Lagrangian fibrations. The $\mathbb{C}^*$–quotient of the new Lagrangian fibration defines in a natural way a deformation of the cyclic quotient of the original Lagrangian fibration.

Zusammenfassung

Die generische Faser einer Lagrangeschen Faserung $f : X \to B$ ist isomorph zu der Kompaktifizierung einer komplexen abelschen Lie–Gruppe, wenn sie kompakt und zusammenhängend ist. Sind die Fasern affien ist im Allgemeinen nicht klar, was die Struktur der Faser ist. Adler und van Moerbeke haben eine Strategie entwickelt nach zu weisen, dass die Faser isomorph zum affinen Teil einer Abelschen Varietät ist.
Wir konnten ihre Strategie für die Fälle in denen die Faser der affine Teil einer $(\mathbb{C}^*)^r$–Erweiterung einer Abelschen Varietät ist erweitern und diese erfolgreich in mehreren Beispielen anwenden. In den Beispielen die wir untersucht haben stellten wir fest, dass Lagrangesche Faserungen, deren generische Faser der affine Teil einer verzweigten, zyklischen Überlagerung einer Abelschen Varietät ist in eine Lagrangesche Faserung eingebettet werden können, deren generische Faser der affine Teil einer $\mathbb{C}^*$–Erweiterung einer Abelschen Varietät ist. Der $\mathbb{C}^*$ Quotient der neuen Lagrangeschen Faserung definiert auf natürliche Weise eine Deformation des zyklischen Quotienten der ursprünglichen Lagrangeschen Faserung.

Contents

Introduction

An important source of examples of Lagrangian fibrations are integrable systems. A family of simple examples of integrable systems is obtained from the Hamilton function

$$H := \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(q_1^2 + q_2^2) \ .$$

This mechanical system has two degrees of freedom and is always integrable. The second integral is given by the angular momentum:

$$G := p_1 q_2 - p_2 q_1 \ .$$

The equations of motion can be solved in terms of quasiperiodic functions as one would expect as G defines a *rotational symmetry*. Surprisingly many other integrable systems have equations of motion that can be integrated in terms of quasiperiodic functions that have no explicit symmetry. On example is the integrable spinning top Kovalevskaya discovered [Kov89]. She showed that the equations of motion can be integrated in terms of hyperelliptic functions. To find her integrable system Kovalevskaya computed Laurent series solutions to the equations of motion.
Later in the early 1980s Adler and van Moerbeke developed a strategy, based on Kovalevskayas Laurent series method, to prove that the generic fibre of a given example of a Lagrangian fibration is isomorphic to the affine part of an abelian variety $\mathcal{A}$ and that the Hamiltonian vector fields extend to holomorphic vector fields on $\mathcal{A}$. It turned out that this strategy is very fruitful, as it could be applied to various examples.
In the late 1980s an interesting example of a Lagrangian fibration that has no complete Hamiltonian vector fields, but is not isomorphic to the affine part of an abelian variety was found [BvM87]. This example yields an action of a cyclic group and the Lagrangian fibration of the quotient has the affine part of an abelian variety as generic fibre. Piovan [Pio92] showed that in this case the generic fibre is isomorphic to the affine part of a ramified cyclic covering of an abelian variety, but the vector fields do not extend to holomorphic vector fields on the completion.
1999 Fedorov [Fed99] showed that the flow of a Hamiltonian vector field of a concrete integrable system can be integrated in terms of generalised theta functions, which indicates that the generic fibre is isomorphic to the affine part of a complex abelian Lie–group, that is a $\mathbb{C}^*$–extension of an abelian variety. At the same time Gavrilov constructed an integrable system on a family of affine parts of generalised jacobian varieties of a given family of singular curves [Gav99].
In this thesis we give an extension of the strategy of Adler and van Moerbeke to prove that the generic fibre of a given Lagrangian fibration is isomorphic to an affine part of a $(\mathbb{C}^*)^r$–extension of an abelian variety. We apply this strategy successfully to several

examples. For all examples we have studied we observe an interesting phenomenon: Lagrangian fibrations that have the affine part of a cyclic ramified cover of an abelian variety as generic fibre can be *embedded* in a Lagrangian fibration that have a $\mathbb{C}^*$–extension of this abelian variety as generic fibre. This construction even works in the case of an integrable system discovered by Duistermaat, where the generic fibre is isomorphic to a cyclic ramified cover of a $\mathbb{C}^*$–extension of an abelian surface. For this example we obtain an embedding in a Lagrangian fibration that has the affine part of a $(\mathbb{C}^*)^2$–extension of an abelian surface as generic fibre.
All these strategies can be used to study the geometry of the general fibre. To obtain a complete description of the fibration, we have to study the special fibres as well. We hope that our analysis of the examples can be extended to include the singular fibres.

The first chapter is devoted to Kovalevskayas method and the strategy of Adler and van Moerbeke. We give a short introduction to the basic definitions and results of Lagrangian fibrations and integrable systems. In the sections 1.6 and 1.7 we extend the strategy of Adler and van Moerbeke to the case where the generic fibre of a Lagrangian fibration is isomorphic to the affine part of a $(\mathbb{C}^*)^r$–extension of an abelian variety.
In the chapters 2 and 3 we apply this strategy to the cases of integrable potentials of degree 3 in two variables, that were not investigated before. Together with the already known Hénon–Heiles example, we now have results for all these cases. We study the Duistermaat–system in chapter 4 this yields the first example of an integrable system whose generic fibre completes in to a cyclic ramified cover of a $\mathbb{C}^*$–extension of an abelian surface, as far as we know. An integrable potential of degree 4 in two variables is studied in chapter 5. It was already known that the generic fibre of this integrable system is isomorphic to the affine part of a cyclic ramified cover of the product $\mathcal{F} \times \mathcal{E}$ of two elliptic curves. From the embedding in an integrable system that has the $\mathbb{C}^*$–extension of an abelian surface we obtain a deformation of the cyclic quotient of the original system.

1 Integrable systems

1.1 Poisson varieties and integrable systems

We review some basic definitions and results on Poisson–geometry. In the following we work in the category of complex analytic spaces. In most examples we are dealing with affine or projective varieties over $\mathbb{C}$.

Definition 1 (Poisson–variety) A *Poisson–variety* is a complex analytic space X together with a $\mathcal{O}_X$–biderivation, the so called *Poisson–bracket*

$$\{.,.\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X \ , \quad (f,g) \mapsto \{f,g\}$$

that puts the structure of a sheaf of Lie algebras on $\mathcal{O}_X$. So $\{.,.\}$ is skew–symmetric and we have for all $f,h,g \in \mathcal{O}_X$ the Jacobi identity:

$$\{f,\{h,g\}\} + \{h,\{g,f\}\} + \{g,\{f,h\}\} = 0 \ .$$

The Poisson–bracket induces a map:

$$\chi : \mathcal{O}_X \to \Theta_X \ , \quad f \mapsto \chi_f := \{f,.\} \ .$$

We call the vector field χ_f the *Hamiltonian vector field of* f. From the Poisson–bracket on X we obtain an element $\theta \in \bigwedge^2 \Theta_{X^{\mathrm{reg}}}$ on the regular part X^{reg} of X by:

$$\theta(df,dg) := \{f,g\} \ .$$

On a normal variety it is equivalent to define a Poisson–bracket via a biderivation $\{.,.\}$ or by an element $\theta \in \bigwedge^2 \Theta_{X^{\mathrm{reg}}}$ that induces a Poisson–bracket on X^{reg}. On the regular part of X we obtain a morphism of $\mathcal{O}_{X^{\mathrm{reg}}}$–modules:

$$\Omega_{X^{\mathrm{reg}}} \to \Theta_{X^{\mathrm{reg}}} \ , \quad \alpha \mapsto \theta(\alpha,.) \ .$$

The rank of this $\mathcal{O}_{X^{\mathrm{reg}}}$–morphism has to be even as θ is skew symmetric and we call this the *rank of the Poisson–bracket.*

Definition 2 (Poisson–morphism) A morphism $\varphi : X \to Y$ between Poisson–varieties is called *Poisson–morphism* if for all $f,g \in \mathcal{O}_Y$:

$$\{\varphi^* f, \varphi^* g\}_X = \varphi^*\{f,g\}_Y \ .$$

Here we denoted with $\{.,.\}_X$ the Poisson–bracket on X and with $\{.,.\}_Y$ the Poisson–bracket on Y.

The product $X \times Y$ of two Poisson–varieties X and Y carries a natural structure of a Poisson–variety in such a way that the projections to X and Y are Poisson–morphisms. It is the product in the category of Poisson–varieties.

Definition 3 (Symplectic variety) A $2n$–dimensional complex normal variety with a non–degenerate, closed two form $\omega \in H^0(\Omega^2_{X^{\mathrm{reg}}})$ is called *symplectic variety* (X, ω).

The symplectic form ω induces an isomorphism between $\Omega_{X^{\mathrm{reg}}}$ and $\Theta_{X^{\mathrm{reg}}}$ which is usually denoted by:

$$\Omega_{X^{\mathrm{reg}}} \to \Theta_{X^{\mathrm{reg}}} \ , \quad \alpha \mapsto \alpha^{\#} \ .$$

We can define a Poisson–bracket on X^{reg} by:

$$\{f, g\} := dg(df^{\#}) \ ,$$

and as X is normal, it extends to a Poisson–bracket on X. A simple but important example of a symplectic variety and hence a Poisson–variety is $\mathbb{C}^{2n}$, together with the *standard symplectic form*:

$$\omega := \sum_{i=1}^{n} dp_i \wedge dq_i \ ,$$

where we denote by $q_1, \ldots, q_n, p_1, \ldots, p_n$ the coordinates of $\mathbb{C}^{2n}$. The symplectic form ω induces the following Poisson–bracket:

$$\{f, g\} := \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \ .$$

Via the next theorem we obtain Poisson–varieties from quotients by reductive groups.

Theorem 1 *The quotient of an affine Poisson–variety X by a reductive group $\mathcal{G}$ is again a Poisson–variety if $\mathcal{G}$ carries a Poisson–structure that turns the action of $\mathcal{G}$ on X:*

$$\mathcal{G} \times X \to X$$

in to a morphism of Poisson–varieties.

A proof is given in proposition 2.23 in [Van96]. We obtain two interesting examples of reductive quotients.

Corollary 1 *The quotient of an affine Poisson–variety X by a finite group Σ is again a Poisson–variety if for all $f, g \in \mathcal{O}_X$ and $\sigma \in \Sigma$:*

$$\sigma^*\{f, g\} = \{\sigma^* f, \sigma^* g\} \ .$$

Proof. A finite group carries a trivial Poisson–structure and the condition of the corollary just states that the multiplication with all elements of Σ defines a Poisson–morphism from X to X and so the multiplication map turns in to a Poisson–morphism. □

If $\mathcal{G}$ denotes a complex Lie–group that acts on X

$$\mathcal{G} \times X \to X \ .$$

For each element $v \in T_e\mathcal{G}$ we obtain the *adjoint vector field* $\mathcal{V}_v \in H^0(X, \Theta_X)$. If a collection of vector fields $\mathcal{V}_1, \ldots, \mathcal{V}_r \in H^0(X, \Theta_X)$ obtained as the adjoint vector fields of a basis of the Lie–algebra $\mathfrak{g}$ of a Lie–group $\mathcal{G}$ acting on X, we say that the group action of $\mathcal{G}$ on $\mathcal{X}$ is *induced by the flow* of $\mathcal{V}_1, \ldots, \mathcal{V}_r$.

Corollary 2 *Let X be an affine Poisson–variety and $\mathcal{G}$ a connected reductive Lie–group acting on X. The quotient $X//\mathcal{G}$ is a Poisson–variety, if for all adjoint vector fields $\mathcal{V} \in H^0(X, \Theta_X)$ and $f, g \in \mathcal{O}_X$:*

$$\mathcal{V}\{f, g\} = \{\mathcal{V}f, g\} + \{f, \mathcal{V}g\} \ .$$

Proof. We have to show that for all $g \in \mathcal{G}$ the multiplication by g defines a Poisson–morphism from X to X. We can write the multiplication by g as the flow of a adjoint vector field $\mathcal{V}$ by some value $t \in \mathbb{C}$ and the Poisson–bracket is invariant under the action of g if the Lie–derivation by the adjoint vector field $\mathcal{V}$ is zero. This yields

$$L_{\mathcal{V}}\theta(df, dg) = \mathcal{V}\{f, g\} - \{\mathcal{V}f, g\} - \{f, \mathcal{V}g\} \ ,$$

evaluated at some df, dg for $f, g \in \mathcal{O}_X$. □

Corollary 3 *Let X be an affine Poisson–variety and χ_G a Hamiltonian vector field defined by the global section $G \in \mathcal{O}_X(X)$. If χ_G induces via its flow a $\mathbb{C}^*$–action on X the quotient $X//\mathbb{C}^* = spec(\mathcal{O}_X(X)^{\mathbb{C}^*})$ is again a Poisson–variety.*

Proof. To apply theorem 1 we have to show, that the $\mathbb{C}^*$–action is a Poisson–action which is equivalent to show, that the Lie–derivation of the Poisson–bracket by χ_G vanishes, which is a direct computation:

$$\begin{aligned} L_{\chi_G}\theta(df, dg) &= \chi_G\{f, g\} - \{\chi_G f, g\} - \{f, \chi_G g\} \\ &= \{G, \{f, g\}\} - \{\{G, f\}, g\} - \{f, \{G, g\}\} \\ &= \{G, \{f, g\}\} + \{g, \{G, f\}\} + \{f, \{g, G\}\} \\ &= 0 \ . \end{aligned}$$

□

A Poisson–variety caries a special classes of subvarieties.

Definition 4 (Involutive subvariety) An ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ is called *involutive*, if $\{\mathcal{I}, \mathcal{I}\} \subset \mathcal{I}$. We call a subvariety $Y \subset X$ *involutive* if its ideal sheaf $\mathcal{I}_Y$ is involutive.

The dimension of an involutive subvariety $Y \subset X$ is at least half the rank of the Poisson–structure at the generic point of Y.

Definition 5 (Lagrangian subvariety) An involutive subvariety $L \subset X$ of a Poisson–variety is called *Lagrangian* if the dimension of L equals half the rank of the Poisson–structure at the generic point of L.

If $Y \subset X$ is involutive and $\mathcal{I}$ its ideal sheaf, we obtain for each $f \in \mathcal{I}$ an element $\chi_f \in \Theta_X$. As the ideal sheaf is involutive we have:

$$\chi_f \mathcal{I} = \{f, \mathcal{I}\} \subset \mathcal{I}$$

so χ_f is tangent to L and yields an element in Θ_Y. We denote the image of $\mathcal{I}$ under the map $\chi : \mathcal{O}_X \to \Theta_X$ by $Ham(Y) \subset \Theta_Y$. In the case of a Lagrangian subvariety $L \subset X$ the rank of $Ham(L)$ as an $\mathcal{O}_L$–module equals the dimension of L.

Definition 6 (Lagrangian fibration) A flat morphism $f : X \to B$ from a Poisson–variety X to a variety B is called *Lagrangian fibration* if the generic fibre is Lagrangian.

Definition 7 (Morphism of Lagrangian fibrations) A morphism of Lagrangian fibrations from $f : X \to B_X$ to $g : Y \to B_Y$ is a pair of a Poisson–morphism $\varphi : X \to Y$ and a morphism $\psi : B_X \to B_Y$, such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ B_X & \xrightarrow{\psi} & B_Y \end{array}$$

commutes.

An important class of examples of Lagrangian fibrations is given by integrable systems:

Definition 8 (Affine integrable system) A subalgebra $A \subset \mathcal{O}_X(X)$, where X is an affine Poisson–variety, is called *integrable system* if $\{A, A\} = 0$ and $dim(X) - dim(A) = r$ half the rank of the Poisson–bracket.

The inclusion $A \subset \mathcal{O}_X(X)$ induces a morphism $f : X \to spec(A)$ whose fibres are involutive subvarieties and by the condition $dim(X) - dim(A) = r$ the generic fibre is Lagrangian. So we obtain a Lagrangian fibration.

Definition 9 (Integrable system) A Lagrangian fibration $f : X \to B$ is called integrable system if for each point $b \in B$ we have an affine neighbourhood $b \in A \subset B$, such that $\{f^*(\mathcal{O}_B(A)), f^*(\mathcal{O}_B(A))\} = 0$.

1.2 The geometry of Lagrangian fibres

In this section we collect a few results on Lagrangian fibres that can be found for example in [AMV04]. We will in the following always assume that the generic fibre of a Lagrangian fibration is smooth, connected and that the Hamiltonian vector fields span the tangent space at each point. This is a natural assumption as the following statement shows.

Theorem 2 *The Hamiltonian vector fields of a smooth fibre $f^{-1}(b)$ of a Lagrangian fibration $f : X \to B$ span the tangent space at each point of the fibre if $f^{-1}(b) \cap X^{sing} = \emptyset$ and $f^{-1}(b) \cap R_\theta = \emptyset$. Here we denote by R_θ the algebraic subset of X where the rank of the Poisson–structure θ is lower than at the generic point.*

If we replace X with X^{reg} in the theorem above we are in the situation of [Van96]. A natural question is: 'What is the geometry of a Lagrangian fibre?'. This question has of course only partial answers. An important tool to study this question is the *flow* to the Hamiltonian vector fields. To avoid fancy notation we will formulate all statements for a space X, which in our application will be the generic fibre of a Lagrangian fibration (rather than the total space of the fibration).
For a global vector field $\mathcal{V} \in H^0(X, \Theta_X)$ on some analytic variety X we obtain the flow of $\mathcal{V}$:

$$\Phi_{\mathcal{V}} : \mathcal{X} \to X$$

where $\mathcal{X} \subset \mathbb{C} \times X$ denotes an analytic open subset that contains $\{0\} \times X$. The existence is guarantied by Picards theorem for ODE's (see theorem 2.1 in [AMV04]). We call a vector field $\mathcal{V} \in H^0(X, \Theta_X)$ *complete* if the flow is defined on $\mathbb{C} \times X$. If all Hamiltonian vector fields on a Lagrangian fibre are complete, we get an answer to our question from the next theorem, which is a complex analogue of the Arnold–Liouville theorem [Arn89]:

Theorem 3 *Let X be a d–dimensional connected complex manifold and $\mathcal{V}_1, \ldots, \mathcal{V}_d$ pairwise commuting complete holomorphic vector fields such that $\mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_d$ vanishes nowhere. Then X is isomorphic to an abelian complex Lie–group.*

Proof. The vector fields define a $\mathbb{C}^d$–action on X by:

$$\Psi : \mathbb{C}^d \times X \to X \ , \quad ((t_1, \ldots, t_d), x) \mapsto \Phi^{t_1}_{\mathcal{V}_1} \circ \Phi^{t_2}_{\mathcal{V}_2} \circ \cdots \circ \Phi^{t_d}_{\mathcal{V}_d}(x)$$

where $\Phi^{t_i}_{\mathcal{V}_i} : X \to X$ denote the flow of $\mathcal{V}_i$ by the *time* t_i. This is well defined since the flows commute as the vector fields commute. The action of $\mathbb{C}^d$ is locally free as the vector fields span the tangent space at each point of X. Furthermore, the action of $\mathbb{C}^d$ is transitive: the $\mathbb{C}^d$–orbit of a point $x \in X$ is both open and closed and as X is connected it must coincide with X. From the locally freeness of the action we obtain that the stabiliser $\Gamma \subset \mathbb{C}^d$ of a point $x \in X$ of the action is a finitely generated free subgroup. The choice of a point in $x \in X$ gives an isomorphism $X \simeq \mathbb{C}^d/\Gamma$. □

Note that all connected complex abelian Lie–groups are isomorphic to $\mathbb{C}^g/\Gamma$ for some finitely generated free $\mathbb{Z}$–module Γ.
If $\mathbb{R} \otimes \Gamma = \mathbb{C}^g$ we find that X is a complex torus. If $\mathbb{R} \otimes \Gamma \neq \mathbb{C}^g$ the Lie–group is not compact.
On a d–dimensional analytic space X the sections $\mathcal{V}_1, \ldots, \mathcal{V}_d \in H^0(X, \Theta_X)$ define a section $\mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_d \in H^0(X, \wedge^d \Theta_X)$ in the rank 1 sheaf $\wedge^d \Theta_X$.

Proposition 1 *Let $\mathcal{V}_1, \ldots, \mathcal{V}_d \in H^0(X, \Theta_X)$ be commuting vector fields on the d–dimensional analytic space X. The vanishing locus Σ of $\mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_d \in H^0(X, \wedge^d \Theta_X)$ is invariant under the flow of all $\mathcal{V}_i$.*

Proof. On the smooth part X^{reg} of X the vanishing locus $\Sigma \cap X^{reg}$ is invariant under the flow of $\mathcal{V}_i$ if the Lie–derivation of $\mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_d$ vanishes. This can be done by a direct computation:

$$\mathcal{L}_{\mathcal{V}_i}(\mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_d) = \sum_{j=1}^{d} \mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_{j-1} \wedge [\mathcal{V}_i, \mathcal{V}_j] \wedge \mathcal{V}_{j+1} \wedge \cdots \wedge \mathcal{V}_d = 0 \ .$$

On the singular locus $\mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_d$ vanishes and the singular locus is invariant under the flow of any vector field. □

If X is compact all global vector fields are automatically complete and we obtain a variation of theorem 3.

Theorem 4 *Let $\mathcal{V}_1, \dots, \mathcal{V}_d \in H^0(X, \Theta_X)$ be commuting vector fields on a d–dimensional compact analytic space X commute. Let Σ denote the vanishing locus of $\mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_d$. Then $X \setminus \Sigma$ is isomorphic to a complex abelian Lie–group.*

Proof. Since X is compact all $\mathcal{V}_i$ are complete. From proposition 1 we obtain that the restriction of $\mathcal{V}_i$ to $X \setminus \Sigma$ is complete and as $\mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_d$ vanishes nowhere on $X \setminus \Sigma$ we obtain the smoothness of $X \setminus \Sigma$. Then theorem 3 yields that $X \setminus \Sigma$ is isomorphic to a complex abelian Lie–group. □

If $X \subset \mathbb{P}^n$ is quasi–projective X is an abelian algebraic group.

Definition 10 (Algebraic completely integrable) We call a Lagrangian fibration $f : X \to B$ *algebraic completely integrable* (a.c.i.) if the generic fibre of f is the affine part of a complex abelian variety and the Hamiltonian vector fields extend to global vector fields on the abelian variety.

The definition of algebraic completely integrability used by Adler and van Moerbeke is formulated for integrable systems on a Poisson–manifold X induced by a collection of Poisson–commuting Hamilton functions $H_1, \dots, H_g$ that induce an integrable system in our sense via the polynomial momentum map:

$$\mathbb{F} : X \to \mathbb{C}^g \ , \quad x \mapsto (H_1(x), \dots, H_g(x)) \ .$$

For the examples studied in this work we need this slightly more general definition. Beside the examples studied by Adler and van Moerbeke, Hitchin [Hit94] gave a large class of examples of integrable systems that are almost *by construction* algebraic completely integrable. He showed that the cotangent bundle of moduli spaces of stable vector bundles on Riemann surfaces carry the structure of integrable systems and are indeed algebraic completely integrable. Explicit equations for these examples have been given later by Vanhaecke [Van96] and by Donagi and Markman [DM96]. The fibres of these examples are affine parts of Jacobian varieties or Prym varieties [Don95].
Adler, van Moerbeke and Vanhaecke [AMV04] gave a criterion when the generic fibre of a Lagrangian fibration is isomorphic to the affine part of an abelian variety, which we will review.

Definition 11 A vector field $\mathcal{V}$ on an analytic space X has the *flow–through property* along a closed subset $Y \subset X$ if for every point $x \in Y$ there exists an open subset $U \in \mathbb{C}$ containing the zero such that

$$\Phi(U,x) \cap Y = x \quad \text{and} \quad \Phi(U,x) \cap X \setminus Y \neq \emptyset \ .$$

As before $\Phi : \mathcal{X} \to X$ is the flow of $\mathcal{V}$ for some $\mathcal{X} \subset \mathbb{C} \times X$ containing $\{0\} \times X$.

We can use this property to prove the smoothness of a projective completion:

Proposition 2 *The compactification $\overline{X}$ of a smooth analytic space X is smooth if it admits a vector field $\overline{\mathcal{V}} \in H^0(\overline{X}, \Theta_{\overline{X}})$ that has the flow–through property along $D := \overline{X} \setminus X$.*

Proof. The flow of a global vector field on a compact variety is always complete. As $\overline{\mathcal{V}}$ has the flow–through property along D we can find for each $x \in D$ an analytic open neighbourhood U_x, such that the flow of $\mathcal{V}$ defines an isomorphism between an analytic subsets $V_x \subset X$ and U_x. Now all points in X are smooth points and as the flow of $\overline{\mathcal{V}}$ is holomorphic we obtain that the points of D have to be smooth points of $\overline{X}$ too. □

Theorem 5 *Let $\mathcal{V}_1, \ldots, \mathcal{V}_d \in H^0(X, \Theta_X)$ be commuting vector fields that span the tangent space of X at each point. If $\mathcal{V}_1$ extends to $\overline{\mathcal{V}}_1 \in H^0(\overline{X}, \Theta_{\overline{X}})$ with the flow–through property along $D := \overline{X} \setminus X$, then*

- *all $\mathcal{V}_i$ extend to $\overline{\mathcal{V}}_i \in H^0(\overline{X}, \Theta_{\overline{X}})$*
- *$\overline{\mathcal{V}}_1 \wedge \cdots \wedge \overline{\mathcal{V}}_d$ vanishes nowhere on $\overline{X}$*
- *$\overline{X}$ is isomorphic to a complex torus.*

Proof. The flow of $\overline{\mathcal{V}}_1$ defines for each point $x \in D$ an isomorphism:

$$\Phi : U_x \to V_x$$

from an analytic open neighbourhood U_x of x to an analytic open subset $V_x \subset X$. For all $2 \leq i \leq g$ we define:

$$\overline{\mathcal{V}}_i : \mathcal{O}_{\overline{X}}(U_x) \to \mathcal{O}_{\overline{X}}(U_x) \ , \quad s \mapsto \big(\mathcal{V}_i(s \circ \Phi^{-1})\big) \circ \Phi \ .$$

As $L_{\overline{\mathcal{V}}_1} \mathcal{V}_i = 0$, this is well defined and since Φ is an isomorphism, $\overline{\mathcal{V}}_1 \wedge \cdots \wedge \overline{\mathcal{V}}_g$ vanishes nowhere on U_x. From theorem 3 we obtain the last point. □

In our applications $X \subset \mathbb{C}^n$ is affine and $\overline{X} \subset \mathbb{P}^n$ its projective closure, then $\overline{X}$ is an abelian variety.

1.3 Polynomial integrable systems

On $\mathbb{C}^{2n}$ an integrable system can be defined by n algebraically independent elements $H_1, \ldots, H_n \in \mathbb{C}[q_1, \ldots, q_n, p_1, \ldots, p_n]$ that are in involution:

$$\{H_i, H_j\} = 0$$

for all i and j. The morphism:

$$\mathbb{F} : \mathbb{C}^{2n} \to \mathbb{C}^n \ , \quad x \mapsto (H_1(x), \ldots, H_n(x))$$

defines a Lagrangian fibration. The following one–parameter family of Hamiltonians was introduced by Hénon and Heiles [HH63] and used in a model of a galactic system:

$$H_\varepsilon := \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{\varepsilon}{3} q_1^3 + q_1 q_2^2 \ . \tag{1.1}$$

It is famous for exhibiting chaotic motion for generic ε. Ziglin developed a criterion for non–integrability of Hamiltonian systems [Zig82], [Zig83] and proved that the Hénon–Heiles system is not integrable for $\varepsilon \in \mathbb{C} \setminus \{1, 2, 6, 16\}$. Morales–Ruiz and Ramis [MR99] developed further criteria of non–integrability extending Ziglins method. Using the results of Morales–Ruiz and Ramis, Maciejewski and Przybylska [MP04] gave a proof that the case $\varepsilon = 2$ is not integrable.
For $\varepsilon = 1$ it has a second integral, that is a polynomial $G_1 \in \mathbb{C}[q_1, q_2, p_1, p_2]$ independent of H_1 that Poisson commutes $\{H_1, G_1\} = 0$:

$$H_1 := \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{1}{3} q_1^3 + q_1 q_2^2 \ , \quad G_1 := 2p_1 p_2 + q_1^2 q_2 + \frac{1}{3} q_2^3 \ .$$

However, if we perform a linear change of coordinates by:

$$q_+ := q_1 + q_2 \ , \quad p_+ := p_1 + p_2 \ , \quad q_- := q_1 - q_2 \ , \quad p_- := p_1 - p_2 \ ,$$

and set

$$H_+ := \frac{1}{2} p_+^2 + \frac{1}{3} q_+^3 \ , \quad H_- := \frac{1}{2} p_-^2 + \frac{1}{3} q_-^3$$

then a simple computation shows:

$$H_1 = \frac{1}{2}\left(H_+ + H_-\right) \ , \quad G_1 = \frac{1}{2}\left(H_+ - H_-\right) \ .$$

So we get a separation of variables and the fibre $H_1 = h$ and $G_1 = g$ is the product of the two elliptic curves $H_+ = \frac{1}{2}(h+g)$ and $H_- = \frac{1}{2}(h-g)$.
For $\varepsilon = 6$ we obtain an integrable system that is not a product. Adler and van Moerbeke [AvM88] proved that the generic fibre is a $(1,2)$–polarised abelian surface minus a smooth, hyperelliptic curve of genus 3. It is isomorphic to the Prym of a $2:1$–cover of an elliptic curve by this genus 3 curve and the Hamiltonian vector fields extend to linear vector fields on the Prym.

Besides this example and the cases where the integrable system is isomorphic to a product, Dorizzi, Grammaticos, and Ramani [DGR82] discovered two additional examples of integrable systems on $\mathbb{C}^4$ with a homogenous potential of degree 3. The first example is given by (1.1) for $\varepsilon = 16$, the second one is the polynomial:

$$H := \frac{1}{2}(p_1^2 + p_2^2) + q_1^3 + \frac{1}{2}q_1 q_2^2 + \frac{\sqrt{-3}}{18}q_2^3 \ . \tag{1.2}$$

Maciejewski and Przybylska [MP04] gave a proof that for a Hamilton function of the type:

$$H := \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2)$$

with a homogenous potential $V(q_1, q_2) \in \mathbb{C}[q_1, q_2]$ of degree 3, there are beside the cases that are isomorphic to products only three integrable examples, namely the Hénon–Heiles system for $\varepsilon = 6$, $\varepsilon = 16$ and the case of (1.2). We study the geometry of the generic fibre of the second example in chapter 2 and the third example in chapter 3.

1.4 Proving algebraic complete integrability

We now review some results of Adler, van Moerbeke, and Vanhaecke that can be found in [AMV04]. These can be used to show that the generic fibre of a given Lagrangian fibration fulfils the criteria of theorem 5. Later it will be our standard situation that we are given an embedding of a smooth affine variety X in $\mathbb{C}^n$ and we want to prove that its completion in $\mathbb{P}^n$ is an abelian variety

$$\begin{array}{ccc} X & \xhookrightarrow{\varphi} & \mathbb{C}^n \\ \big\downarrow & & \big\downarrow \\ \overline{X} & \xhookrightarrow{\overline{\varphi}} & \mathbb{P}^n \end{array} \ .$$

We will denote the compactifying divisor by $D := \overline{X} \setminus X$.

Proposition 3 *A vector field $\mathcal{V} \in H^0(X, \Theta_X)$ extends to a vector field $\overline{\mathcal{V}} \in H^0(\overline{X}, \Theta_{\overline{X}})$, if for all coordinate functions $y_0 = 1, y_1, \dots, y_n$ on $\mathbb{C}^n$ there are constants $c_{\alpha\beta}^{(ij)} \in \mathbb{C}$, such that:*

$$(\mathcal{V}y_i)y_j - y_i(\mathcal{V}y_j) = \sum_{\alpha,\beta=0}^{n} c_{\alpha\beta}^{(ij)} y_\alpha y_\beta$$

modulo the ideal $\mathcal{I}(X) \subset \mathbb{C}[y_1, \dots, y_n]$ of X.

Proof. We choose homogenous coordinates $(z_0 : \cdots : z_n)$ on $\mathbb{P}^n$ in such a way that $y_i = \frac{z_i}{z_0}$. We set $U_j = \overline{X} \cap \{z_j \neq 0\}$ so $U_0 = X$. Then $\mathcal{O}_{\overline{X}}(U_j)$ is generated as algebra by $\frac{z_0}{z_j}, \dots, \frac{z_n}{z_j}$. On the open subset U_j we define a vector field $\overline{\mathcal{V}}_j$ as derivation of $\mathcal{O}_{\overline{X}}(U_j)$:

$$\overline{\mathcal{V}}_j\left(\frac{z_i}{z_j}\right) := \sum_{\alpha,\beta=0}^{n} c_{\alpha\beta}^{(ij)} \frac{z_\alpha}{z_j}\frac{z_\beta}{z_j} \ .$$

On the intersection $U_0 \cap U_j$ the vector fields $\mathcal{V}$ and $\overline{\mathcal{V}}_j$ coincide since:

$$\begin{aligned}\mathcal{V}|_{U_0\cap U_j}\left(\frac{z_i}{z_j}\right) &= \mathcal{V}|_{U_0\cap U_j}\left(\frac{y_i}{y_j}\right) = \frac{\mathcal{V}(y_i)y_j - \mathcal{V}(y_j)y_i}{y_j^2} \\ &= \sum_{\alpha,\beta=0}^{n} c_{\alpha\beta}^{(ij)} \frac{y_\alpha}{y_j}\frac{y_\beta}{y_j} = \sum_{\alpha,\beta=0}^{n} c_{\alpha\beta}^{(ij)} \frac{z_\alpha}{z_j}\frac{z_\beta}{z_j} \\ &= \overline{\mathcal{V}}_j|_{U_0\cap U_j}\left(\frac{z_i}{z_j}\right) .\end{aligned}$$

As a consequence we find that:

$$\overline{\mathcal{V}}_i|_{U_i\cap U_j\cap U_0} = \overline{\mathcal{V}}_j|_{U_i\cap U_j\cap U_0}$$

so all $\overline{\mathcal{V}}_i \in \Theta_X(U_i)$ coincide on a Zariski open subset of X and thus defines an element $\overline{\mathcal{V}} \in H^0(\overline{X}, \Theta_{\overline{X}})$. □

Corollary 4 *Let $\varphi : X \to \mathbb{P}^n$ denote an embedding of an affine variety X defined by the functions $\varphi_0 = 1, \varphi_1, \ldots, \varphi_n \in \mathcal{O}_X(X)$ via:*

$$x \mapsto (\varphi_0(x) = 1 : \varphi_1(x) : \cdots : \varphi_n(x))$$

and $\mathcal{V} \in H^0(X, \Theta_X)$ a vector field. The vector field $\varphi_*\mathcal{V}$ on $\varphi(X)$ extends to a vector field $\overline{\mathcal{V}} \in H^0(\overline{X}, \Theta_{\overline{X}})$, if there are constants $c_{\alpha\beta}^{(ij)} \in \mathbb{C}$, such that:

$$(\mathcal{V}\varphi_i)\varphi_j - \varphi_i(\mathcal{V}\varphi_j) = \sum_{\alpha,\beta=0}^{n} c_{\alpha\beta}^{(ij)} \varphi_\alpha\varphi_\beta .$$

Proof. We denote $\varphi^\# : \mathcal{O}_{\varphi(X)} \to \mathcal{O}_X$ the map of sheafs of rings induced by φ. We obtain:

$$\begin{aligned}&\varphi^\# \left((\varphi_*\mathcal{V}y_i)y_j - y_i(\varphi_*\mathcal{V}y_j)\right) = (\mathcal{V}\varphi_i)\varphi_j - \varphi_i(\mathcal{V}\varphi_j) \\ &= \sum_{\alpha,\beta=0}^{n} c_{\alpha\beta}^{(ij)} \varphi_\alpha\varphi_\beta = \varphi^\# \left(\sum_{\alpha,\beta=0}^{n} c_{\alpha\beta}^{(ij)} y_\alpha y_\beta\right) .\end{aligned}$$

As φ defines an embedding, $\varphi^\#$ is injective and we obtain:

$$(\varphi_*\mathcal{V}y_i)y_j - y_i(\varphi_*\mathcal{V}y_j) = \sum_{\alpha,\beta=0}^{n} c_{\alpha\beta}^{(ij)} y_\alpha y_\beta$$

and by proposition 3 we are done. □

For a vector field on a projective variety it is possible to verify the flow–through property in a direct way:

Proposition 4 *A vector field $\mathcal{V} \in H^0(X, \Theta_X)$ on a compact surface X has the flow–through property along a divisor D if $\mathcal{V}$ is transversal at the generic point of D and vanishes nowhere on D.*

Proof. As the vector field is transversal at the generic point of D, it has flow–through property at the generic point. The set of points in D where $\mathcal{V}$ is tangent is finite as the dimension of D is 1. If $\mathcal{V}$ has not the flow–through property at one of these points and $\mathcal{V}$ is not zero at this point, the flow of $\mathcal{V}$ defines an isomorphism between an analytic open subset $U_0 \subset \mathbb{C}$ and an analytic open subset $U_d \subset D$. Now $\mathcal{V}$ is tangent at all points of U_d and if U_d is not empty, $\mathcal{V}$ is tangent at infinitely many points of D which yields a contradiction. □

The condition of proposition 4 is easy to check. In our situation the embedding of X in $\mathbb{P}^n$ is obtained from the line bundle corresponding to the divisor $D = X \cap H$, where $H \subset \mathbb{P}^n$ is the hyperplane defined by $\{z_0 = 0\}$. For higher dimensional X we obtain a recursive version of proposition 4.

Proposition 5 *A vector field $\mathcal{V} \in H^0(X, \Theta_X)$ on a compact variety X has the flow–through property along a subvariety $Y \subset X$ if $\mathcal{V}$ is transversal at the generic point of D and has the flow–through property along the subvariety of Y where $\mathcal{V}$ is not transversal.*

The proof is some howe obvious, but we can use this statement to give a recursive algorithm to check the flow–through property for an given vector field $\mathcal{V} \in H^0(X, \Theta_X)$ along a subvariety $Y \subset X$. First we compute the subvariety Y_1 of Y where $\mathcal{V}$ is tangent to Y. Now we compute in each step a subvariety Y_{i+1} of codimension at lest 1 in Y_i where the vector field $\mathcal{V}$ is tangent to Y_i and continue with Y_{i+1}. The condition of being transversal at points is just the non–vanishing.

1.5 Kovalevskayas method of computing Laurent solutions

Sofia Kovalevskaya used families of Laurent series to find her famous integrable system and proved that the equations of motion can be integrated in terms of theta functions [Kov89]. For historical aspects see [Coo84].
Adler and van Moerbeke used this method to find candidates for the components of the divisor that compactifies the generic fibre of an integrable system. If the general fibre is indeed the affine part of an abelian variety, these Laurent series can be used to compute a basis of the global sections of a line bundle defined by a divisor B supported on the divisor D that compactifies the generic fibre in to an abelian variety.
We will illustrate this procedure, which is known as *Kovalevskayas method* or *Painlevé analysis*, in a special situation. For more details we refer to [AMV04].
We take $X = \mathbb{C}^n$ equipped with a polynomial Poisson–structure $\{.,.\}$ and $H_1, \ldots, H_r \in \mathbb{C}[x_1, \ldots, x_n]$ algebraically independent constants of motion defining an integrable system. The momentum map is given by:

$$\mathbb{F} : \mathbb{C}^n \to \mathbb{C}^r, \quad x \mapsto (H_1(x), \ldots, H_r(x)) \ .$$

The Hamiltonian H_1 defines the equations of motion:

$$\dot{x}_i = \{H_1, x_i\} \ .$$

We assume that the constants of motion H_i are weighted homogenous for some weights $\varpi(x_i) = \nu_i$ and that the Hamiltonian vector field χ_{H_1} has the property:

$$\varpi\left(\chi_{H_1} x_j\right) = \varpi(x_j) + 1$$

for all $1 \leq j \leq n$. Under these assumptions the ansatz:

$$x_i(t) = x_i^{(0)} \frac{1}{t^{\nu_i}}$$

yields:

$$\frac{d}{dt} x_i(t) = \frac{-\nu_i}{t^{\nu_i+1}} x_i^{(0)} = f_i(x_1(t), \ldots, x_n(t)) = \frac{1}{t^{\nu_i+1}} f_i(x_1^{(0)}, \ldots, x_n^{(0)})$$

where we denote by $f_i := \{H_1, x_i\}$ the i–th component of the vector field χ_{H_i}. From the last equation we obtain n equations on the $x_i^{(0)}$ namely

$$f_i(x_1^{(0)}, \ldots, x_n^{(0)}) + \nu_i x_i^{(0)} = 0$$

called the *initial equations*. The zero locus in $\mathbb{C}^n$ of this set of equations is called the *initial locus*. The origin $0 \in \mathbb{C}^n$ is always contained in the initial locus, as the f_i are homogenous of nonzero weight. For each point p in the initial locus we can try to extended this solution to a Laurent series:

$$x_i(t) = \frac{1}{t^{\nu_i}} \left(x_i^{(0)}(p) + x_i^{(1)}(p)t + \ldots \right)$$

and we obtain equations for the higher coefficients $x_i^{(k)}(p)$ of the form:

$$\left(k\mathrm{Id} - K(x_1^{(0)}(p), \ldots, x_n^{(0)}(p)) \right) x^{(k)}(p) = R^{(k)}$$

where $x^{(k)}(p)$ is the n–tuple of the k–th coefficients, $R^{(k)}$ depends only on the coefficients $x^{(1)}(p), \ldots, x^{(k-1)}(p)$, and K denotes the so called *Kovalevskaya matrix* defined by:

$$K_{ij} := \frac{\partial f_i}{\partial x_j} + \nu_i \delta_{ij} \ .$$

If K has a positive integral eigenvalue λ with a corresponding eigenvector v_λ it is possible to add a parameter to the Laurent series in the λ–th step. We call the eigenvalues of the Kovalevskaya matrix *Kovalevskaya–exponents*. As K depends on the point p of the initial locus, the Kovalevskaya exponents in general also will depend on p.
If the Kovalevskaya exponents are *constant* on a component Y_i of the initial locus and we can add for all the positive integer Kovalevskaya exponents a parameter, we obtain Laurent series parametrised by $\Gamma^{(i)} := Y_i \times \mathbb{C}^r$. The affine variety $\Gamma^{(i)}$ is called the

Painlevé wall and denote $x^{\Gamma^{(i)}}(t)$ the Laurent series parametrised by $\Gamma^{(i)}$. Laurent series parametrised by $n-1$ dimensional Painlevé walls are called *principal balances* and Laurent series that are parametrised by lower dimensional Painlevé walls are called lower balances. If we substitute the principal balances $x^{\Gamma^{(i)}}(t)$ in the constants of motion $H_1, \dots, H_r$ we expect that all terms in t will vanish except the constant one and we obtain r equations by setting the values of $H_1, \dots, H_r$ to some generic values $h_1, \dots, h_r$. These r equations define a subvariety in $\Gamma^{(i)}$ which is called the i–th *Painlevé–divisor* D_i. The Painlevé divisors are good candidates for the divisors that compactify the generic fibre since the Laurent series solutions corresponding to principal balances define a morphism:

$$x^{\Gamma^{(i)}} : \mathcal{Y}_i \to \mathbb{C}^n$$

that is locally an isomorphism, where $\mathcal{Y}_i \subset \mathbb{C}^* \times \Gamma^{(i)}$. For a formal linear combination of the components D_i of the Painlevé–divisor $B = \sum_{i=1}^m m_i D_i$ we define:

Definition 12 Let $B = \sum_{i=1}^m m_i D_i$ for $m_i \in \mathbb{N}_0$ be a formal linear combination of the components of the Painlevé–divisor we call the vector space:

$$\mathcal{P}(B) = \left\{ f \in \mathbb{C}[x_1, \dots, x_n] \mid \nu\left(f(x^{\Gamma^{(i)}}(t)) \right) \leq m_i \ \forall\ 1 \leq i \leq m \right\} ,$$

of polynomials of pole order less than B.

Remark 1 This construction can be applied to a more general situation for an affine Lagrangian fibration $f : X \to B$ where the ideal $\mathcal{I}_X$ of $X \subset \mathbb{C}^n$ is generated by weighted homogenous polynomials. As X is affine the vector field χ_{H_1} extends to a vector field on $\mathbb{C}^n$ and from the generators of $\mathcal{I}_X$ we obtain extra relations for the Painlevé–divisors.

It turns out that several integrable systems do not have enough positive integral Kovalevskaya exponents, but they yield enough *rational* positive Kovalevskaya exponents. The first example of an integrable system with this property is the Goryachev–Chaplygin top [Gor00], [Cha48], which was studied by Bechlivandis and van Moerbeke [BvM87]. They discovered that the integrable system admits a $\mathbb{Z}/2\mathbb{Z}$ action, such that the quotient by this action is algebraic completely integrable and gave the following definition of this new class of examples.

Definition 13 **Almost algebraic integrable** A Lagrangian fibration $f : X \to B$ is called *almost algebraic integrable* (a.a.i.) if there is a Poisson–action of a cyclic group Σ on X and f induces a Lagrangian fibration $\tilde{f} : X/\Sigma \to B$ that is algebraic completely integrable.

Piovan [Pio92] showed that in these examples a formal Laurent series solution in a m–th root of t can be written down and he gave a criterion when the general fibre completes in to a cyclic ramified cover of an abelian variety.
During the late 1990s an other class of integrable systems was studied. Gavrilov [Gav99] constructed an integrable system on a family of affine parts of *generalised* jacobian varieties of singular curves. This system has the affine part of a non compact abelian algebraic group as generic fibre. Almost at the same time Fedorov [Fed99] integrated

the differential equations of an specific integrable system in terms of generalised theta functions of a non–compact abelian algebraic group, which indicates that the generic fibre of this system is the affine part of a non–compact abelian algebraic group.
A simple example for an integrable system that has a generic fibre that completes not in an abelian variety is the harmonic oscillator. In two degrees of freedom the harmonic oscillator is defined by:

$$H := H_1 + H_2 \ ,$$

for:

$$H_1 := G = \frac{1}{2}p_1^2 + q_1^2 \ , \quad H_2 := H - G = \frac{1}{2}p_2^2 + q_2^2 \ .$$

As H_1 and H_2 Poisson–commute one of them defines a second integral and we find that the generic fibre is the product of two affine smooth rational curves $H_1 = h_1$ and $H_2 = h_2$. The vector fields χ_{H_i} can be integrated explicitly in terms of exponential functions, which are the generalised theta functions of $\mathbb{C}^*$.
Motivated by these examples it is natural to give the following definition.

Definition 14 Generalised algebraic completely integrable A Lagrangian fibration $f : X \to B$ is called *generalised algebraic completely integrable* (g.a.c.i.) if the generic fibre is isomorphic to the affine part of a non compact complex abelian Lie–group $\mathcal{G}$ and the Hamiltonian vector fields extend to complete vector fields on $\mathcal{G}$.

If we assume that X is smooth and $f : X \to B$ is an integrable system defined by Poisson commuting global sections $H_1, \ldots, H_g \in \mathcal{O}_X(X)$ we obtain an integrable system that is generalised algebraic integrable in the sense of [AMV04]. Their definition is however more general. They require that the general fibre is stratified by components that are *birational* to abelian algebraic groups. Generalised algebraic completely integrable Lagrangian fibrations often yield algebraic completely integrable Lagrangian fibrations in a natural way.

Theorem 6 *Let $f : X \to \mathbb{C}^n$ be an affine, generalised algebraic completely integrable system, such that the generic fibre is isomorphic to the affine part of a $(\mathbb{C}^*)^r$ extension $\mathcal{G}$ of an abelian variety. Then we obtain a $(\mathbb{C}^*)^r$–action on X in such a way that the quotient system $\tilde{f} : X//(\mathbb{C}^*)^r \to \mathbb{C}^n$ is algebraic completely integrable.*

Proof. The morphism f is defined by n elements $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ via:

$$f : X \to \mathbb{C}^n, \quad x \mapsto (f_1(x), \ldots, f_n(x)) \ .$$

The generic fibre is isomorphic to the affine part of a $(\mathbb{C}^*)^r$ group–extension $\mathcal{G}$ of an abelian variety $\mathcal{A}$:

$$e \to (\mathbb{C}^*)^r \to \mathcal{G} \to \mathcal{A} \to e \ .$$

The hamiltonian vector fields $\chi_{f_1}, \ldots, \chi_{f_n}$ extend to complete vector fields $\overline{\chi}_{f_1}, \ldots, \overline{\chi}_{f_n}$ on $\mathcal{G}$ and induce via their flow the group action on $\mathcal{G}$. After some linear coordinate change in $\mathbb{C}^n$ we can assume, that $\overline{\chi}_{f_1}, \ldots, \overline{\chi}_{f_r}$ induce the $(\mathbb{C}^*)^r$–action on the generic fibre. As the vector fields χ_{f_i} are global vector fields on X they induce a $(\mathbb{C}^*)^r$–action on X. The quotient $X//(\mathbb{C}^*)^r = spec(\mathcal{O}_X(X)^{(\mathbb{C}^*)^r})$ is obtained from:

$$\mathcal{O}_X(X)^{(\mathbb{C}^*)^r} = \{\varphi \in \mathcal{O}_X(X) \mid \chi_{f_1}\varphi = \chi_{f_2}\varphi = \cdots = \chi_{f_r}\varphi = 0\}$$

and by Corollary 3 $X//(\mathbb{C}^*)^r$ is a Poisson–variety. The f_i are $(\mathbb{C}^*)^r$–invariant since all $\chi_{f_i} f_j = \{f_i, f_j\} = 0$ and we obtain $f_i \in \mathcal{O}_X(X)^{(\mathbb{C}^*)^r}$ and thus f induces a morphism:

$$\tilde{f} : X//(\mathbb{C}^*)^r \to \mathbb{C}^n \ ,$$

that has involutive fibres. To prove that $\tilde{f}$ is an integrable system, we have to prove that the dimension of the generic fibre equals half the rank of the induced Poisson–structure on $X//(\mathbb{C}^*)^r$. In each step of taking a quotient by the $\mathbb{C}^*$–action of a Hamilton vector field χ_{f_i}, the dimension of the generic fibre lowers by one and the rank of the induced Poisson–structure lowers by exactly two since the Hamiltonian vector field χ_{f_i} gets mapped to zero, the other vector fields stay independent, and the rank of the induced Poisson–structure has to stay even. But now we are done, since the generic fibre of the quotient system $\tilde{f} : X//(\mathbb{C}^*)^r \to \mathbb{C}^n$ is the $(\mathbb{C}^*)^r$–quotient of the generic fibre of the original system $f : X \to \mathbb{C}^n$ and the quotient of $\mathcal{G}$ by the $(\mathbb{C}^*)^r$–action induced by:

$$e \to (\mathbb{C}^*)^r \to \mathcal{G} \to \mathcal{A} \to e$$

is the abelian variety $\mathcal{A}$. □

There remains the question: 'If such an $(\mathbb{C}^*)^r$ quotient is algebraic completely integrable, was the original system generalised algebraic completely integrable?'. We will answer this question in the next section.

1.6 Proving generalised algebraic complete integrability

The previous sections provide results that allow us to complete an affine variety and prove that its completion is an abelian variety. Theorem 6 shows that an integrable system that has the affine part of a $(\mathbb{C}^*)^r$–extension of an abelian variety as generic fibre yields in a natural way an integrable system that has the affine part of an abelian variety as generic fibre.
Now we want to give an answer to the question 'when is a Lagrangian fibration $f : X \to B$ generalised algebraic completely integrable if its quotient system $\tilde{f} : X//(\mathbb{C}^*)^r \to B$ is algebraic completely integrable?'. In general the original Lagrangian fibration need not be g.a.c.i. if its $(\mathbb{C}^*)^r$–quotient is a.c.i.. The following simple example illustrates what can go wrong.

Example 1 We consider an elliptic curve minus a point $\mathcal{E} \setminus \{p\}$. On $X := \mathcal{E} \setminus \{p\} \times \mathbb{C}^*$ we have two vector fields $\mathcal{V}_1 = \frac{d}{dt}$ and $\mathcal{V}_2 = \frac{d}{dz} + s\frac{d}{dt}$, where $\frac{d}{dt}$ denotes the vector field induced by the $\mathbb{C}^*$–action, $\frac{d}{dz}$ a vector field induced by the holomorphic vector field on $\mathcal{E}$, and $s \in \mathcal{O}_{\mathcal{E}}(*p)$ non–constant. A direct computation shows that $\mathcal{V}_1 \wedge \mathcal{V}_2$ vanishes nowhere on X which is the affine part of a trivial $\mathbb{C}^*$–extension of $\mathcal{E}$, but $\mathcal{V}_2$ does not extend to a holomorphic vector field on $\mathbb{C}^* \times \mathcal{E}$.

We will assume in the following that X is a smooth affine variety of dimension $g+r$. On X there will be r complete vector fields $\lambda_1, \ldots, \lambda_r \in H^0(X, \Theta_X)$ that induce via their flow a $(\mathbb{C}^*)^r$–action on X and g vector fields $\mathcal{V}_1, \ldots, \mathcal{V}_g \in H^0(X, \Theta_X)$ such that:

$$\mathcal{V}_1 \wedge \cdots \wedge \mathcal{V}_g \wedge \lambda_1 \wedge \cdots \wedge \lambda_r$$

vanishes nowhere on X and all λ_i and $\mathcal{V}_j$ commute. Furthermore we assume that the quotient by the $(\mathbb{C}^*)^r$–action:

$$\pi : X \to X//(\mathbb{C}^*)^r$$

is isomorphic to the affine part of an abelian variety $X//(\mathbb{C}^*)^r \simeq \mathcal{A} \setminus D$ and the vector fields $\mathcal{W}_i := \pi_* \mathcal{V}_i$ are obtained as restrictions of global vector fields $\overline{\mathcal{W}}_i \in H^0(\mathcal{A}, \Theta_{\mathcal{A}})$. The morphism π defines a $(\mathbb{C}^*)^r$–fibre bundle over $\mathcal{A} \setminus D$. The main question is when it is possible to extend the $(\mathbb{C}^*)^r$–bundle over $\mathcal{A} \setminus D$ to $\mathcal{A}$. The embedding of X in $\mathbb{P}^n \times \mathbb{P}^m$ defines a fibration over $\mathcal{A}$ by the induced projection on $\mathbb{P}^n$:

$$\tilde{\pi} : \overline{X} \to \mathcal{A} \ ,$$

where $\overline{X}$ denotes the completion in $\mathbb{P}^n \times \mathbb{P}^m$.

Proposition 6 *A vector field $\mathcal{V} \in H^0(X, \Theta_X)$ extends to a vector field $\overline{\mathcal{V}} \in H^0(\overline{X}, \Theta_{\overline{X}})$, if for all $0 \leq i, j \leq n$ and $0 \leq k, l \leq m$ there are constants $c^{(ijkl)}_{\alpha\beta\gamma\delta} \in \mathbb{C}$, such that:*

$$x_k y_i \left(x_l(\mathcal{V} y_j) + y_j(\mathcal{V} x_l) \right) - x_l y_j \left(x_k(\mathcal{V} y_i) + y_i(\mathcal{V} x_k) \right) = \sum_{\gamma,\delta=0}^{n} \sum_{\alpha,\beta=0}^{m} c^{(ijkl)}_{\alpha\beta\gamma\delta} x_\alpha x_\beta y_\gamma y_\delta$$

modulo the ideal $\mathcal{I}(X) \subset \mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ of X. Here we denote $y_1, \ldots, y_n$ the coordinates of $\mathbb{C}^n$, $x_1, \ldots, x_m$ the coordinates of $\mathbb{C}^m$ and set $x_0 = y_0 = 1$.

Proof. We use the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$ defined by:

$$((y_0 : \cdots : y_n), (x_0 : \cdots : x_m)) \mapsto (y_0 x_0 : y_0 x_1 : \cdots : y_n x_{m-1} : y_n x_m) \ ,$$

and denote by z_{ij} the homogenous coordinates of $\mathbb{P}^{(n+1)(m+1)-1}$. As in the proof of proposition 3 we have to show that the vector fields $\overline{\mathcal{V}}_{ki}$ defined on $\mathcal{O}_{\overline{X}}(\overline{X} \cap \{z_{ki} \neq 0\})$ by:

$$\overline{\mathcal{V}}_{ki} \left(\frac{z_{lj}}{z_{ki}} \right) = \sum_{\gamma,\delta=0}^{n} \sum_{\alpha,\beta=0}^{m} c^{(ijkl)}_{\alpha\beta\gamma\delta} \frac{z_{\alpha\gamma}}{z_{ki}} \frac{z_{\beta\delta}}{z_{ki}}$$

coincide on an zariski open subset of X. A direct computation shows:

$$\begin{aligned} \mathcal{V} \left(\frac{z_{lj}}{z_{ki}} \right) = \mathcal{V} \left(\frac{x_l y_j}{x_k y_i} \right) &= \frac{x_k y_i \left(x_l(\mathcal{V} y_j) + y_j(\mathcal{V} x_l) \right) - x_l y_j \left(x_k(\mathcal{V} y_i) + y_i(\mathcal{V} x_k) \right)}{(x_k y_i)^2} \\ &= \frac{\sum_{\gamma,\delta=0}^{n} \sum_{\alpha,\beta=0}^{m} c^{(ijkl)}_{\alpha\beta\gamma\delta} x_\alpha x_\beta y_\gamma y_\delta}{(x_k y_i)^2} = \sum_{\gamma,\delta=0}^{n} \sum_{\alpha,\beta=0}^{m} c^{(ijkl)}_{\alpha\beta\gamma\delta} \frac{z_{\alpha\gamma}}{z_{ki}} \frac{z_{\beta\delta}}{z_{ki}} \ . \end{aligned}$$

that $\mathcal{V}$ and $\overline{V}_j$ coincide on $U_0 \cap U_j$. □

It is quite complicated to check the conditions of the last proposition for a given example, thus we will give a criterion that is easier to verify. First of all we need not check that $\mathcal{V}$ is holomorphic in all affine parts $\{z_{ij} \neq 0\}$ of $\mathbb{P}^{(n+1)(m+1)-1}$. The subset where $\mathcal{V}$ is not holomorphic defines a divisor on $\overline{X}$ and thus has codimension 1 in $\overline{X}$.

Theorem 7 *If $\overline{X} \cap \{y_0 = 0\} \cap \{y_r = 0\}$ has codimension 2 in $\overline{X}$, the vector field $\mathcal{V} \in H^0(X, \Theta_X)$ extends to a vector field $\overline{\mathcal{V}} \in H^0(\overline{X}, \Theta_{\overline{X}})$, if for all $0 \leq i \leq n$ and $0 \leq j \leq m$ there are constants $a^{(i)}_{\alpha\beta}, c^{(j)}_{\gamma\alpha\beta} \in \mathbb{C}$, such that:*

$$\mathcal{V}y_i = \sum_{\alpha,\beta=0}^{n} a^{(i)}_{\alpha\beta} y_\alpha y_\beta$$

$$(\mathcal{V}x_j)y_r - x_j(\mathcal{V}y_r) = \sum_{\gamma=0}^{m} \sum_{\alpha,\beta=0}^{n} c^{(j)}_{\gamma\alpha\beta} x_\gamma y_\alpha y_\beta \ .$$

Here x_j denote by the coordinate functions on $\mathbb{C}^m$ and y_i the coordinate functions on $\mathbb{C}^n$, where we set $x_0 = y_0 = 1$.

Proof. We choose an embedding of X in $\mathbb{P}^{(n+1)^2(m+1)-1}$ via an embedding of $\mathbb{P}^n \times \mathbb{P}^m$. We denote by z_{ijk} the homogenous coordinates of $\mathbb{P}^{(n+1)^2(m+1)-1}$ where $0 \leq i \leq m$ and $0 \leq j, k \leq n$. The embedding of $\mathbb{P}^n \times \mathbb{P}^m$ is defined by:

$$z_{ijk} := x_i y_j y_k \ .$$

Since $\overline{X} \cap \{z_0 = 0\} \cap \{z_r = 0\}$ has codimension 2 in $\overline{X}$ it is sufficient to show that $\mathcal{V}$ extends to a vector field $\overline{\mathcal{V}} \in \Theta_{\overline{X}}(\overline{X} \setminus (\{z_0 = 0\} \cap \{z_r = 0\}))$. On $\overline{X} \cap \{z_{00r} \neq 0\}$ we define the vector field $\overline{\mathcal{V}}_r \in \Theta_{\overline{X}}(\overline{X} \cap \{z_r = 0\})$ by:

$$\overline{\mathcal{V}}_r\left(\frac{z_{ijk}}{z_{00r}}\right) = \sum_{\gamma=0}^{m} \sum_{\alpha,\beta=0}^{n} c^{(j)}_{\gamma\alpha\beta} \frac{z_{0jk}}{z_{00r}} \frac{z_{\gamma\alpha\beta}}{z_{00r}} + \sum_{\alpha,\beta=0}^{n} a^{(j)}_{\alpha\beta} \frac{z_{0rk}}{z_{00r}} \frac{z_{i\alpha\beta}}{z_{00r}} + \sum_{\alpha,\beta=0}^{n} a^{(k)}_{\alpha\beta} \frac{z_{0rj}}{z_{00r}} \frac{z_{i\alpha\beta}}{z_{00r}} \ .$$

A direct computation shows that $\mathcal{V}|_{\overline{X}\cap\{z_0=0\}\cap\{z_r=0\}} = \overline{\mathcal{V}}_r|_{\overline{X}\cap\{z_0=0\}\cap\{z_r=0\}}$:

$$\begin{aligned}
\mathcal{V}\left(\frac{z_{ijk}}{z_{00r}}\right) = \mathcal{V}\left(\frac{x_iy_jy_k}{y_r}\right) &= \frac{y_r\mathcal{V}(x_iy_jy_k) - x_iy_jy_k\mathcal{V}(y_r)}{y_r^2} \\
&= \frac{y_r(\mathcal{V}x_i)y_jy_k + y_rx_i(\mathcal{V}y_j)y_k + y_rx_iy_j(\mathcal{V}y_k) - x_iy_jy_k\mathcal{V}(y_r)}{y_r^2} \\
&= \frac{y_jy_k\left(y_r(\mathcal{V}x_i) - x_i\mathcal{V}(y_r)\right) + y_rx_i(\mathcal{V}y_j)y_k + y_rx_iy_j(\mathcal{V}y_k)}{y_r^2} \\
&= \frac{y_jy_k\left(\sum_{\gamma=0}^m\sum_{\alpha,\beta=0}^n c_{\gamma\alpha\beta}^{(j)}x_\gamma y_\alpha y_\beta\right)}{y_r^2} \\
&+ \frac{x_iy_ry_k\left(\sum_{\alpha,\beta=0}^n a_{\alpha\beta}^{(j)}y_\alpha y_\beta\right) + x_iy_ry_j\left(\sum_{\alpha,\beta=0}^n a_{\alpha\beta}^{(k)}y_\alpha y_\beta\right)}{y_r^2} \\
&= \sum_{\gamma=0}^m\sum_{\alpha,\beta=0}^n c_{\gamma\alpha\beta}^{(j)}\frac{z_{0jk}}{z_{00r}}\frac{z_{\gamma\alpha\beta}}{z_{00r}} \\
&+ \sum_{\alpha,\beta=0}^n a_{\alpha\beta}^{(j)}\frac{z_{0rk}}{z_{00r}}\frac{z_{i\alpha\beta}}{z_{00r}} + \sum_{\alpha,\beta=0}^n a_{\alpha\beta}^{(k)}\frac{z_{0rj}}{z_{00r}}\frac{z_{i\alpha\beta}}{z_{00r}}\ .
\end{aligned}$$

So $\mathcal{V}$ is holomorphic in the Zariski open subset $\overline{X}\cap\{z_0=0\}\cap\{z_r=0\}$. □

Theorem 7 induces a statement on the holomorphic functions on X defining the embedding in $\mathbb{C}^n\times\mathbb{C}^m\subset\mathbb{P}^n\times\mathbb{P}^m$:

Corollary 5 *Let $\varphi_0 = 1, \varphi_1, \dots, \varphi_n, \psi_0 = 1, \psi_1, \dots, \psi_m \in \mathcal{O}_X(X)$ defining an embedding $\Psi : X \to \mathbb{P}^n\times\mathbb{P}^m$ by:*

$$x \mapsto (\varphi_0(x) : \varphi_1(x) : \dots : \varphi_n(x)), (\psi_0(x), \psi_1(x), \dots, \psi_m(x))\ .$$

A vector field $\mathcal{V}\in H^0(X,\Theta_X)$ extends to a vector field $\overline{\mathcal{V}}\in H^0(\overline{X},\Theta_{\overline{X}})$ via $\Psi_*\mathcal{V} = \overline{\mathcal{V}}|_{\Psi(X)}$, where $\overline{X}$ denotes the closure of $\Psi(X)$ in $\mathbb{P}^n\times\mathbb{P}^m$, if for all $0\le i\le n$ and $0\le j\le m$:

$$\begin{aligned}
\mathcal{V}\varphi_i &= \sum_{\alpha,\beta=0}^n a_{\alpha\beta}^{(i)}\varphi_\alpha\varphi_\beta \\
(\mathcal{V}\psi_j)\varphi_r - \psi_j(\mathcal{V}\varphi_r) &= \sum_{\gamma=0}^m\sum_{\alpha,\beta=0}^n c_{\gamma\alpha\beta}^{(j)}\psi_\gamma\varphi_\alpha\varphi_\beta\ .
\end{aligned}$$

Proof. The proof of this corollary is completely analogues to the proof of corollary 4. □

If the embedding $\varphi : X//(\mathbb{C}^*)^r \to \mathbb{P}^n$ is projectively normal, the first equation of corollary 5 is automatically fulfilled if $\pi_*\mathcal{V}$ is obtained from a global vector field on $\mathcal{A}$. This can be seen very easily. Let $\varphi_i \in H^0(\mathcal{A},\mathcal{O}_{\mathcal{A}}(D))$ denote a global section in a line bundle that defines a projectively normal embedding, then $\mathcal{V}\varphi_i \in H^0(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(2D))$ but from the projective normality we obtain that the multiplication map:

$$\mu : H^0(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(D)) \times H^0(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(D)) \to H^0(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(2D))$$

is surjective and so we can write $\mathcal{V}\varphi_i$ as a linear combination of $\varphi_k\varphi_l$. The second equation can be checked using a Singular script.
In all examples we have studied it was possible to choose as the functions that define the embedding $\psi_0 = 1, \psi_1, \ldots, \psi_m \in H^0(X, \mathcal{O}_X)$ the original coordinate functions of X.

1.7 The strategy of studying Lagrangian fibres

The results from Adler, van Moerbeke, and Vanhaecke [AMV04] together with the results from Piovan [Pio92] and the last section can be used to study the geometry of Lagrangian fibres and give a proof of their algebraic integrability. The following steps do not give an algorithm, because there is no guaranty that they will give a result, but in all examples we have studied it worked well.
We start with an affine Lagrangian fibration:

$$f : X \to B \ .$$

We assume that the generic fibre is smooth and the Hamiltonian vector fields $\chi_{H_1}, \ldots, \chi_{H_n}$ span the tangent space at each point of the generic fibre.

Step 1 the $(\mathbb{C}^*)^r$–quotient

In the first step we have to find the maximal linear $(\mathbb{C}^*)^r$–action. Therefore we start with a basis of the Hamiltonian vector fields $\chi_{H_1}, \ldots, \chi_{H_s}$. We try to find a linear combination χ_{G_1} of the χ_{H_i} that defines a linear action via its flow. We obtain a new basis $\chi_{G_1}, \chi_{H_1}, \ldots, \chi_{H_{s-1}}$ where we assumed, that the first $s-1$ original Hamiltonian vector fields extend χ_{G_1} to a basis. We continue recursively until we have a basis $\chi_{G_1}, \ldots, \chi_{G_r}, \chi_{H_1}, \ldots, \chi_{H_{s-r}}$ where the χ_{G_i} induce the action of a linear abelian group $(\mathbb{C}^*)^r$ and there is no linear combination of the remaining $\chi_{H_1}, \ldots, \chi_{H_{s-r}}$ that induces a linear action.
The quotient $X//(\mathbb{C}^*)^r$ is obtained as the spectrum of the $(\mathbb{C}^*)^r$ invariant ring:

$$\mathcal{O}_X(X)^{(\mathbb{C}^*)^r} = \{s \in \mathcal{O}_X(X) \mid \chi_{G_i} s = 0 \ \forall \ 1 \leq i \leq r\} \ .$$

As the pullback $f_i = f^* x_i$ of the generators x_i of $\mathcal{O}_B(B)$ are $(\mathbb{C}^*)^r$–invariant the quotient yields a Lagrangian fibration:

$$\tilde{f} : X//(\mathbb{C}^*)^r \to B$$

and proceed with the second step.

Step 2 the Laurent series solutions

We compute the initial locus to one of the vector fields χ_{H_1} of the Lagrangian fibration $\tilde{f} : X//(\mathbb{C}^*)^r \to B$. To each component of the initial locus we compute the Kovalevskaya matrix and its eigenvalues. Then we try to find for each component a principal balance. Therefore we set $m \in \mathbb{N}$ the minimal integer, such that all positive rational eigenvalues to all components of the initial locus are fractions of the type $\frac{a}{m}$ with $a \in \mathbb{N}$. From each component of the initial locus we obtain a formal Laurent series solution in $t^{\frac{1}{m}}$. Again we are interested in series that are parametrised by a $\dim(X//(\mathbb{C}^*)^r) - 1$ dimensional Painlevé–wall.
The galois action of exchanging $t^{\frac{1}{m}} \mapsto \xi_m t^{\frac{1}{m}}$ where ξ_m denotes the m–th unit root defines a $\mathbb{Z}/m\mathbb{Z}$ action on the Laurent series solutions. If this $\mathbb{Z}/m\mathbb{Z}$ action is induced by a $\mathbb{Z}/m\mathbb{Z}$ Poisson–action on X that leaves $f : X \to B$ invariant, we obtain a Lagrangian fibration $\check{f} : (X//(\mathbb{C}^*)^r)/(\mathbb{Z}/m\mathbb{Z}) \to B$.

Step 3 the cyclic quotient

If we substitute the formal Laurent series solutions of step 2 in the generators of the cyclic quotient $(X//(\mathbb{C}^*)^r)/(\mathbb{Z}/m\mathbb{Z})$ we obtain Laurent series solutions in t. One has to be careful with the Painlevé–walls parameterising the Laurent series in the new variables since the finite morphism $X//(\mathbb{C}^*)^r \to (X//(\mathbb{C}^*)^r)/(\mathbb{Z}/m\mathbb{Z})$ induces a finite morphism on the Painlevé–walls. As a consequence the new Painlevé–wall is usually a $\mathbb{Z}/m\mathbb{Z}$–quotient of the old one, which can be checked by computing the Laurent series in the new variables and compare the coefficients. As a consequence the formal Painlevé–divisor computed with the Painlevé–walls from the formal Laurent series is usually a cyclic $\mathbb{Z}/m\mathbb{Z}$ cover of the Painlevé–divisor.

Step 4 morphisms to $\mathbb{P}^N$

The Lagrangian fibration $\check{f} : (X//(\mathbb{C}^*)^r)/(\mathbb{Z}/m\mathbb{Z}) \to B$ is a good candidate for an algebraic complete integrable Lagrangian fibration. We can use the Laurent series $x^{\Gamma^{(i)}}(t)$ to compute the global sections of the divisor $B = \sum_{i=1}^m m_i D_i$, where $\Gamma^{(i)}$ denote the Painlevé–walls of the principal balances and the D_i denote the corresponding components of the Painlevé–divisor:

$$\mathcal{P}\left(\sum_{i=1}^m m_i D_i\right) = \left\{ f \in \mathbb{C}[x_1, \ldots, x_n] \mid \nu\left(f(x^{\Gamma^{(i)}}(t))\right) \leq m_i \ \forall\ 1 \leq i \leq m \right\} .$$

We start with one component of the Painlevé–divisor $B = D_i$ and compute a basis of $\mathcal{P}(B)$. This basis defines a morphism to $\mathbb{P}^N$ where $N + 1$ is the dimension of the vector space $\mathcal{P}(B)$

$$x \mapsto (\varphi_0(x) = 1 : \varphi_1(X) : \cdots : \varphi_N(x)) .$$

If the induced morphism defines an embedding of the general fibre we are done. Otherwise we add a component of the Painlevé–divisor to B. We should check that the intersection of the closure of the image intersected with the hyperplane $z_0 = 0$ defines a divisor that is given by the compactifications of the components of the Painlevé–divisor D_i. If the divisor has more components, we have missed a principal balance.

Step 5 extending the vector fields

The vector fields $\chi_{H_1}, \ldots, \chi_{H_{s-r}}$ extend to holomorphic vector fields on the completion $\overline{\varphi_{\mathcal{P}(B)}}(\mathcal{F}_c)$ if all *Wronskians*:

$$W_{ijk} := \{H_i, \varphi_j\}\varphi_k - \{H_i, \varphi_k\}\varphi_j$$

can be written as polynomials of maximum degree 2 in the φ_i with coefficients in the field $\mathbb{C}(c_1, \ldots, c_d)$ where $c_1, \ldots, c_d$ are the coordinates of the generic point in B. If this step fails, the reason can be that $\mathcal{P}(B)$ is not projectively normal and we can add another component D_i to B until $\mathcal{P}(B)$ is projectively normal. To prove that $\overline{\varphi_{\mathcal{P}(B)}}(\mathcal{F}_c)$ is an abelian variety we can use the statements from section 1.2.

The geometry of the generic fibre

If the Lagrangian fibration $f : X \to B$ does neither admit a non trivial linear action as in step 1, nor a finite action as in step 2 and the steps 3 to 5 are successful, the $f : X \to B$ is algebraic completely integrable.
We can use theorem 4 to prove that the Lagrangian fibration is generalised algebraic integrable if we find a linear action in step 1 but no cyclic action in step 2.
An almost algebraic integrable Lagrangian fibration is obtained if f admits a non trivial cyclic action in step 2, but no linear action in step 1 and step 3 to 5 are successful.
If $f : X \to B$ admits a linear action $(\mathbb{C}^*)^r$ and a cyclic action $\mathbb{Z}/m\mathbb{Z}$ we have to check if the $\mathbb{Z}/m\mathbb{Z}$ action on on the $(\mathbb{C}^*)^r$ quotient $\tilde{f} : X//(\mathbb{C}^*)^r \to B$ lifts to an $\mathbb{Z}/m\mathbb{Z}$ action on $f : X \to B$. In this case the action of $(\mathbb{C}^*)^r$ and $\mathbb{Z}/m\mathbb{Z}$ on X commute as $(\mathbb{C}^*)^r$ is induced by the flow of $\chi_{G_1}, \ldots, \chi_{G_r}$ which are $\mathbb{Z}/m\mathbb{Z}$ invariant. Now we can use theorem 4 to show that the quotient $g : X/(\mathbb{Z}/m\mathbb{Z}) \to B$ is generalised algebraic integrable. The general fibre in this case completes in a ramified $\mathbb{Z}/m\mathbb{Z}$ cover of the compactification of an abelian algebraic group.
It is possible that we are successful with the steps 1 to 3, but fail in step 4 or 5. All examples with this behaviour that are known to us have either a general fibre that completes in a *subvariety* of an abelian variety or the vector fields do not extend to a completion. In section 6.1 we give examples for these cases.

2 The Hénon–Heiles System

In this chapter we will study the geometry of the $\varepsilon = 16$ Hénon–Heiles integrable system. The Hamiltonian function is given by:

$$H := \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{16}{3}q_1^3 + q_1 q_2^2 \ ,$$

and has a second constant of motion, discovered by Dorizzi, Grammaticos, and Ramani [DGR82]:

$$G := p_2^4 + 4q_1 q_2^2 p_2^2 - \frac{4}{3}q_2^3 p_1 p_2 - \frac{4}{3}q_1^2 q_2^4 - \frac{2}{9}q_2^6 \ .$$

Ravoson, Gavrillov and Caboz showed that the flow of χ_H is given in terms of theta functions of elliptic curves [RGC93]. This suggests that the system is almost algebraically integrable.

Proposition 7 *The fibres of the momentum map $\mathbb{F}_2 = (H, G)$ are smooth away from the discriminant:*

$$\Delta = \{(4h^2 - g)g = 0\} \ .$$

Proof. This can be done by a direct computation using Singular. □

Proposition 8 *The general fibre $\mathcal{F}_{hg} := \mathbb{F}_2^{-1}(h, g)$ contains four elliptic curves.*

Proof. The intersection of the generic fibre $\mathcal{F}_{hg}$ with $q_2 = 0$ yields the equations:

$$0 = \frac{16}{3}q_1^3 + \frac{1}{2}p_1^2 - h + \frac{1}{2}p_2^2 \ , \quad p_2^4 = g \ , \quad q_2 = 0 \ .$$

This defines four elliptic curves. □

So if $\mathcal{F}_{hg}$ is the affine part of an abelian variety $\mathcal{A}_{hg}$ it has to be isogenous to the product of two elliptic curves by the splitting property of abelian varieties theorem 23.

Proposition 9 *Kovalevskayas method yields for the Hamilton vector field χ_H one principal balance with the positive Kovalevskaya exponents $\{\frac{3}{2}, \frac{7}{2}, 6\}$. The formal Laurent* *series in $\sqrt{t}$ are given by:*

$$
\begin{aligned}
q_1(t) &= -\frac{3}{8}\frac{1}{t^2} + \frac{1}{3}\alpha_1^2 t - \frac{4}{9}\alpha_1\alpha_2 t^3 + \frac{1}{4}\alpha_3 t^4 + \dots \\
q_2(t) &= -2\alpha_1 \frac{1}{\sqrt{t}} + \frac{2}{3}\alpha_2 \sqrt{t}^3 + \frac{4}{9}\alpha_1^3 \sqrt{t}^5 + \dots \\
p_1(t) &= \frac{3}{4}\frac{1}{t^3} + \frac{1}{3}\alpha_1^2 - \frac{4}{3}\alpha_1\alpha_2 t^2 + \alpha_3 t^3 + \dots \\
p_2(t) &= \alpha_1 \frac{1}{\sqrt{t}^3} + \alpha_2\sqrt{t} + \frac{10}{9}\alpha_1^3\sqrt{t}^3 + \dots \ .
\end{aligned}
$$

Proof. The initial locus consists of the origin and three additional points. Two points are given by:

$$p_1^{(0)} = -3 \ , \quad q_1^{(0)} = \frac{3}{2} \ , \quad (q_2^{(0)})^2 = -126 \ , \quad p_2^{(0)} = -2q_2^{(0)} \ .$$

These two points have Kovalevskaya–exponents $\{6, 12, -1, -7\}$, so the Laurent–series to these points are parametrised by two parameters and thus yield a lower balance. The remaining point is given by:

$$p_1^{(0)} = \frac{3}{4} \ , \quad q_1^{(0)} = -\frac{3}{8} \ , \quad q_2^{(0)} = 0 \ , \quad p_2^{(0)} = 0 \ .$$

It yields the Kovalevskaya–exponents $\{\frac{3}{2}, \frac{7}{2}, 6, -1\}$ and so it could define a principal balance as formal Laurent–series in $\sqrt{t}$. A direct computation yields the Laurent series in $\sqrt{t}$. □

2.1 The quotient system

The $\mathbb{Z}/2\mathbb{Z}$ action of exchanging $\sqrt{t}$ with $-\sqrt{t}$ is induced by a $\mathbb{Z}/2\mathbb{Z}$ action on $\mathbb{C}^4$ via

$$\sigma : (q_1, q_2, p_1, p_2) \mapsto (q_1, -q_2, p_1, -p_2) \ .$$

We obtain an integrable system on $\mathbb{C}^4/\langle\sigma\rangle$ from the $\varepsilon = 16$ Hénon–Heiles system via the next proposition:

Proposition 10 *The symplectic structure on $\mathbb{C}^4$ induces a symplectic structure on the quotient $X := \mathbb{C}^4/\langle\sigma\rangle$. We obtain X as the spectrum of:*

$$\mathbb{C}[q_1, q_2, p_1, p_2]^\sigma \simeq \mathbb{C}[q, p, u, w, v]/(uv - w^2) \ .$$

The polynomials H and G are σ invariant and induce an integrable system on X.

Proof. It is sufficient to show that the Poisson–bracket is invariant under the action of σ for the generators of q_1, q_2, p_1, p_2. The only non vanishing Poisson–brackets are:

$$
\begin{aligned}
\{\sigma p_1, \sigma q_1\} &= \{p_1, q_1\} = 1 = \sigma 1 \\
\{\sigma p_2, \sigma q_2\} &= \{-p_2, -q_2\} = 1 = \sigma 1 \ .
\end{aligned}
$$

The quotient X is normal and away from the A_1 singular plane $\{u = v = w = 0\}$ the symplectic structure is not degenerate and hence X defines a symplectic variety. The isomorphism of the rings is induced by

$$(q, p, u, w, v) \mapsto (q_1, p_1, q_2^2, q_2 p_2, p_2^2) \ .$$

A simple computation shows that H and G are σ invariant and so they define an integrable system on X. The Poisson–brackets in the new variables are:

$$\{p, q\} = 1 \quad \{v, w\} = 2v \quad \{v, u\} = 4w \quad \{w, u\} = 2u$$

and all other vanish. We can write the constants of motion in these variables:

$$H := \frac{1}{2}\left(p^2 + v\right) + \frac{16}{3}q^3 + qu$$
$$G := v^2 + 4quv - \frac{4}{3}puw - \frac{4}{3}q^2u^2 - \frac{2}{9}u^3 \ .$$

□

The action of σ on the general fibre defines a cyclic covering $\mathcal{F}_{hg} \to \mathcal{F}_{hg}/\langle\sigma\rangle$ that is unramified. If we intersect the general fibre $\mathcal{F}_{hg}/\langle\sigma\rangle$ with the hyperplane $\{u = 0\}$ we obtain two elliptic curves:

$$\frac{1}{2}p^2 + \frac{16}{3}q^3 - h - \frac{1}{2}v = 0 \ , \quad u = 0 \ , \quad v^2 = g \ , \quad w = 0 \ . \tag{2.1}$$

Hence if the general fibre of $(X, \mathbb{F}_X)$ is the affine part of an abelian surface, it has to be also isogenous to the product of two elliptic curves.

Proposition 11 *Kovalevskayas method for χ_H on X yields only one principal balance with the positive Kovalevskaya exponents $\{3, 5, 6, 7\}$. The Laurent series are given by:*

$$q(t) := -\frac{3}{8}\frac{1}{t^2} + \frac{1}{3}\gamma_1 t - \frac{2}{9}\gamma_2 t^3 - \frac{1}{4}\gamma_3 t^4 - \frac{1}{18}\gamma_4 t^5 + \dots$$
$$p(t) := -\frac{3}{4}\frac{1}{t^3} - \frac{1}{3}\gamma_1 + \frac{2}{3}\gamma_2 t^2 + \gamma_3 t^3 + \frac{5}{18}\gamma_4 t^4 + \dots$$
$$u(t) := 4\gamma_1\frac{1}{t} - \frac{4}{3}\gamma_2 t + \frac{4}{9}\gamma_4 t^3 + \dots$$
$$w(t) := 2\gamma_1\frac{1}{t^2} + \frac{2}{3}\gamma_2 - \frac{2}{3}\gamma_4 t^2 + \dots$$
$$v(t) := \frac{1}{t^3}\gamma_1 + \frac{1}{t}\gamma_2 + \gamma_4 t + \dots \ .$$

An equation for a plane model of the Painlevé–divisor $\mathcal{C}_4$ is:

$$x^5 + c_1x^3 + c_2x - y^3 = 0 \ ,$$

which defines a smooth affine curve of genus 4. This curve admits a $\mathbb{Z}/2\mathbb{Z}$ action via

$$\tau : (x, y) \mapsto (-x, -y)$$

and the quotient is smooth of genus 2. The action of τ lifts to an action of X:

$$\tau(q, p, u, w, v) \mapsto (q, -p, u, -w, v) \ .$$

Proof. The Laurent series can be computed using SINGULAR. If we substitute the Laurent series in H, G and set H and G to the generic values h and g we obtain, together with the equation $uv - w^2 = 0$, the equations of the Painlevé–divisor:

$$\begin{aligned} h &= \frac{5}{2}\gamma_1^2 - \frac{21}{16}\gamma_3 \\ g &= \frac{1856}{9}\gamma_1^4 - 168\gamma_1^2\gamma_3 + \frac{128}{3}\gamma_2\gamma_4 \\ 0 &= \gamma_2^2 - 4\gamma_1\gamma_4 \ . \end{aligned}$$

We obtain the equation in x and y if we set $x := 4\gamma_1$, $y := 2\sqrt[3]{12}\gamma_2$, $c_1 := -18h$, and $c_2 := \frac{81}{4}g$. The projection on the coordinate x defines a $3:1$ cover of $\mathbb{P}^1$ which is double ramified at 0, ∞ and the zeros of:

$$x^4 + c_1 x^2 + c_2 = 0 \ .$$

From the Riemann–Hurwitz formula we obtain the genus via:

$$\chi(\mathcal{C}) = 2 - 2g = 3\chi(\mathbb{P}^1) - \deg(R) = 6 - 12 = -6 \ .$$

The quotient of the Painlevé–divisor $\mathcal{C}_4/\langle\tau\rangle$ admits a $3:1$ cover over $\mathbb{P}^1$ double ramified at 4 points, which shows that the genus is 2. We used SINGULAR to check that the curves are smooth away from its discriminant:

$$\Delta = \{(4h^2 - g)g^2 = 0\} \ .$$

The action of τ is induced by the action on the Laurent series:

$$\tau : (t, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \mapsto (-t, -\gamma_1, -\gamma_2, \gamma_3, -\gamma_4)$$

and lifts to an action on X. □

There is an interesting aspect of the action τ. It acts on the Laurent series by the *time inversion* $\tau(t) = -t$, what we expect since $\tau_*\chi_H = -\chi_H$ and $\tau_*\chi_G = -\chi_G$. The next step is to find a candidate for a basis of the line bundle defined by the Painlevé–divisor.

Proposition 12 *The four polynomials:*

$$\begin{aligned} \varphi_0 &:= 1 \\ \varphi_1 &:= u \\ \varphi_2 &:= 32q^3 + 3p^2 + 4qu \\ \varphi_3 &:= 8q^2u + u^2 + 6pw - 12qv \end{aligned}$$

define a basis of the elements of $\mathcal{P}(\mathcal{C}_4)$ *of weighted degree less or equal than* 36.

Proof. The proof is a direct computation using SINGULAR. □

The dimension of $\mathcal{P}(\mathcal{C}_4)$ is 4. The image of the generic fibre under the morphism

$$x \mapsto (\varphi_0(x) = 1 : \varphi_1(x) : \varphi_2(x) : \varphi_3(x)) \tag{2.2}$$

is given by the quadratic equation:

$$0 = -9z_2^2 + 2z_1z_3 + (36h)z_2 + (-36h^2 + g) \ .$$

It could be expected for an abelian surface that is isogenous to the product of two elliptic curves.

2.2 Embedding in a 6–dimensional integrable system

The integrable $\varepsilon = 16$ Hénon-Heiles system $(\mathbb{C}^4, \mathbb{F}_2)$ can be *embedded* in an integrable system on $\mathbb{C}^6$ with the momentum map $\mathbb{F}_3 = (H, G, F)$ where:

$$\begin{aligned}
H &:= \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{16}{3}q_1^3 + q_1(q_2^2 + q_3^2) \\
G &:= (p_2^2 + p_3^2)^2 + 4q_1(q_2^2 + q_3^2)(p_2^2 + p_3^2) - \frac{4}{3}p_1(q_2^2 + q_3^2)(q_2p_2 + q_3p_3) \\
&\quad - \frac{4}{3}q_1^2(q_2^2 + q_3^2)^2 - \frac{2}{9}(q_2^2 + q_3^2)^3 + \frac{4}{3}q_1(q_2p_3 - q_3p_2)^2 \\
F &:= q_2p_3 - q_3p_2 \ .
\end{aligned}$$

We obtain this embedding of $\mathbb{F}_2 : \mathbb{C}^4 \to \mathbb{C}^2$ in $\mathbb{F}_3 : \mathbb{C}^6 \to \mathbb{C}^3$ from the embedding $i : \mathbb{C}^4 \hookrightarrow \mathbb{C}^6$ defined by:

$$(q_1, q_2, p_1, p_2) \mapsto (q_1, q_2, 0, p_1, p_2, 0) \ .$$

The polynomial F vanishes on $q_3 = p_3 = 0$ and we obtain the morphism $\mathbb{F}_2 : \mathbb{C}^4 \to \mathbb{C}^2$ from the restriction of $\mathbb{F}_3$ to $q_3 = p_3 = 0$ in $\mathbb{C}^6$ and compose it with the projection on the first two coordinates on $\mathbb{C}^3$. This embedding is not a Poisson–morphism since:

$$\{i^*p_3, i^*q_3\} = \{0, 0\} = 0 \neq i^*\{p_3, q_3\} = i^*1 = 1 \ .$$

Proposition 13 *The fibres of $\mathbb{F}_3 : \mathbb{C}^6 \to \mathbb{C}^3$ are smooth away from the discriminant:*

$$12500f^{12}+3359232f^6h^5-1749600f^6h^3g+243000f^6hg^2+944784h^4g^3-472392h^2g^4+59049g^5 = 0 \ .$$

Proof. This can be done by a direct computation using Singular □

As the equation of the discriminant depends only on powers of f^6 we obtain a $\mathbb{Z}/6\mathbb{Z}$ cyclic cover of the surface defined by the equation:

$$12500f^2+3359232fh^5-1749600fh^3g+243000fhg^2+944784h^4g^3-472392h^2g^4+59049g^5 = 0 \ .$$

This cover is ramified along the two rational curves:

$$\{f = 0, g - 4h^2 = 0\} \ , \quad \{f = 0, g = 0\} \ ,$$

which coincides with the discriminant of the undeformed Hénon–Heiles system. The singular locus of the quotient of the discriminant is:

$$\{45g - 144h^2 = 0, 250f + 243hg^2 = 0\} \ , \quad \{45g - 324h^2 = 0, 125f + 324hg^2 = 0\} \ .$$

2.3 The $\mathbb{C}^*$–quotient

The $\mathbb{C}^*$–quotient yields an integrable system on a singular variety.

Proposition 14 *The quotient by the $\mathbb{C}^*$–action induced by the flow of χ_F $\mathcal{X} := \mathbb{C}^6//\mathbb{C}^*$ is given by the spectrum of:*

$$\begin{aligned}\mathcal{O}_{\mathcal{X}}(\mathcal{X}) = \mathbb{C}[q_1,q_2,q_3,p_1,p_2,p_3]^{\Phi_{\chi_F}} &= \left\{f \in \mathbb{C}[q_1,q_2,q_3,p_1,p_2,p_3] \mid \chi_F f = \{F,f\} = 0\right\} \\ &= \mathbb{C}[Q,P,U,V,W,L]/(UV - W^2 - L^2)\ .\end{aligned}$$

The $\mathbb{C}^*$–quotient $\mathbb{C}^6 \to \mathcal{X}$ is induced by the inclusion $\mathcal{O}_{\mathcal{X}}(\mathcal{X}) \hookrightarrow \mathbb{C}[q_1,q_2,q_3,p_1,p_2,p_3]$ defined by:

$$\begin{aligned} Q &\mapsto q_1 \\ P &\mapsto p_1 \\ U &\mapsto q_2^2 + q_3^2 \\ V &\mapsto p_2^2 + p_3^2 \\ W &\mapsto q_2p_2 + q_3p_3 \\ L &\mapsto q_2p_3 - q_3p_2\ . \end{aligned}$$

For the capital Q,P,U,V,W we obtain the same Poisson bracket as for the small q,p,u,v,w of the finite quotient:

$$\{P,Q\} = 1 \quad \{V,W\} = 2V \quad \{V,U\} = 4W \quad \{W,U\} = 2U$$

and L commutes with all the other variables.

Proof. From a SINGULAR computation we obtain the elements of $\mathbb{C}[q_1,q_2,q_3,p_1,p_2,p_3]^{\Phi_{\chi_F}}$. □

We can write H, G and F in the new variables:

$$\begin{aligned} H &:= \frac{1}{2}\left(P^2 + V\right) + \frac{16}{3}Q^3 + QU \\ G &:= V^2 + 4QUV - \frac{4}{3}PUW - \frac{4}{3}Q^2U^2 - \frac{2}{9}U^3 + \frac{4}{3}QL^2 \\ F &:= L. \end{aligned}$$

Since L commutes with all the other variables $L : \mathcal{X} \to \mathbb{C}$ defines a family of integrable systems $(\mathcal{X}_l = \mathcal{X} \cap \{L = l\}, \mathbb{F}_{\mathcal{X}_l} = (H|_{L=l}, G|_{L=l}))$, where the quotient of the Hénon–Heiles system $(X, \mathbb{F}_X) = (\mathcal{X}_0, \mathbb{F}_0)$. By this construction we obtain a 3 parameter family of surfaces which contains the 2 parameter family from $(X, \mathbb{F}_X)$ as a subfamily:

$$\begin{array}{ccccc} X & \hookrightarrow & \mathcal{X} & \hookleftarrow & \mathcal{X}_l \\ \downarrow{\scriptstyle \mathbb{F}_X} & & \downarrow{\scriptstyle \mathbb{F}_{\mathcal{X}}} & & \downarrow{\scriptstyle \mathbb{F}_{\mathcal{X}_l}} \\ \mathbb{C}^2 & \hookrightarrow & \mathbb{C}^3 & \hookleftarrow & \mathbb{C}^2 \end{array}\ .$$

This defines a deformation of the integrable system on the singular symplectic variety X. The general member $\mathcal{X}_l$ is smooth.
Note that we also have the following picture:

$$\begin{array}{ccc} (\mathbb{C}^4, \mathbb{F}_2) & \hookrightarrow & (\mathbb{C}^6, \mathbb{F}_3) \\ \big\downarrow{\scriptstyle 2:1} & & \big\downarrow{\scriptstyle \mathbb{C}^*} \\ (X, \mathbb{F}_X) & \hookrightarrow & (\mathcal{X}, \mathbb{F}_{\mathcal{X}}) \end{array} \quad .$$

On the left hand side we get a $\mathbb{Z}/2\mathbb{Z}$ quotient and on the right side a $\mathbb{C}^*$–quotient.

Proposition 15 *The intersection of the general fibre $\mathcal{F}_{hgf}$ of $(\mathcal{X}, \mathbb{F}_{\mathcal{X}})$ with $\{U = 0\}$ yields a curve of genus 4 that has the plane model:*

$$1024Q^6 + P^4 + 192Q^3P^2 - 384hQ^3 - 36hP^2 + 24f^2Q + (36h^2 - g) = 0 \ . \tag{2.3}$$

For $f = 0$ it degenerates in to two elliptic curves:

$$\begin{aligned} 0 &= 1024Q^6 + P^4 + 192Q^3P^2 - 384hQ^3 - 36hP^2 + (36h^2 - g) \\ &= \big(32Q^3 + 3P^2 - 6h - V\big)\big(32Q^3 + 3P^2 - 6h + V\big) \\ V^2 &= g \ . \end{aligned}$$

Proof. We obtain this equations if we set $U = 0$ in H, G, and F. □

Proposition 16 *On the quotient system we find the positive Kovalevskaya exponents $\{3, 5, 5, 6, 7\}$, which yields the Laurent series:*

$$\begin{aligned} Q(t) &:= -\frac{3}{8}\frac{1}{t^2} + \frac{1}{3}\gamma_1 t - \frac{2}{9}\gamma_2 t^3 - \frac{1}{4}\gamma_4 t^4 - \frac{1}{18}\gamma_5 t^5 + \dots \\ P(t) &:= -\frac{3}{4}\frac{1}{t^3} - \frac{1}{3}\gamma_1 + \frac{2}{3}\gamma_2 t^2 + \gamma_4 t^3 + \frac{5}{18}\gamma_5 t^6 + \dots \\ U(t) &:= 4\gamma_1\frac{1}{t} - \frac{4}{3}\gamma_2 t + \frac{4}{9}\gamma_5 t^3 + \dots \\ W(t) &:= 2\gamma_1\frac{1}{t^2} + \frac{2}{3}\gamma_2 - \frac{2}{3}\gamma_5 t^2 + \dots \\ L(t) &:= \gamma_3 \\ V(t) &:= \frac{1}{t^3}\gamma_1 + \frac{1}{t}\gamma_2 + \gamma_5 t + \dots \ . \end{aligned}$$

Proof. We obtain only one component of the initial locus that yields enough positive integer Kovalevskaya exponents and obtain the Laurent series parametrised by $\mathbb{C}^5$. □

Next we can use the Laurent series to compute the equations of the Painlevé–divisor.

Proposition 17 *A plane model of the Painlevé–divisor is given by the affine equation:*

$$x^5 + c_1x^3 + c_2x + y^3 + c_3^2y = 0 \ . \tag{2.4}$$

Proof. Substituting the Laurent series in the constants of motion yields:

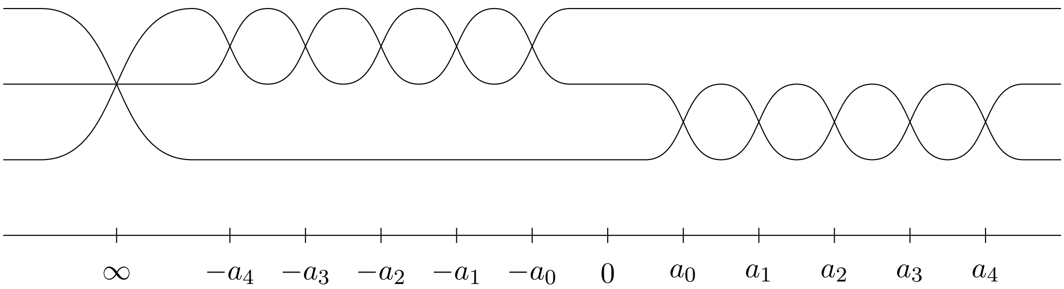
$$
\begin{aligned}
0 &= \frac{13}{18}\gamma_1^2 - \frac{21}{16}\gamma_4 - h \\
0 &= -\frac{64}{27}\gamma_1^4 - \frac{168}{9}\gamma_1^2\gamma_4 + \frac{128}{27}\gamma_2\gamma_5 - g \\
0 &= \gamma_3 - f \\
0 &= 16\gamma_2^2 + 9\gamma_3^2 - 64\gamma_1\gamma_5 .
\end{aligned}
$$

The last equation is obtained from the relation on the variables $UV - W^2 - L^2 = 0$. After eliminating γ_5, γ_4, γ_3 we obtain the plane equations for the Painlevé divisor. Where we set $x := 4\gamma_1$, $y := 2\sqrt[3]{12}\gamma_2$ and

$$c_1 := -18h \ , \quad c_2 := \frac{81}{4}g \ , \quad c_3 := \frac{\sqrt{27}}{\sqrt[6]{12}}f \ .$$

□

Proposition 18 *The Painlevé curve has genus* 4 *and yields a* $3:1$ *cover of* $\mathbb{P}^1$ *with a double ramification at* ∞ *and simple ramification in* 8 *points as indicated below:*



By a rescaling we can set $a_0 = 1$ *and obtain* 4 *points controlled by* f, h, g.

Proof. A direct computation shows that the projection to the x coordinate is double ramified at ∞ and simple ramified along the zeros of:

$$0 = 27x^{10} + 54hx^8 + \left(27h^2 + 54g\right)x^6 + 54hgx^4 + 27g^2x^2 + 4f^6 \ .$$

As the equation depends only on x^2, the points of ramification are the square roots of the zeros of

$$0 = 27z^5 + 54hz^4 + \left(27h^2 + 54g\right)z^3 + 54hgz^2 + 27g^2z + 4f^6 \ . \tag{2.5}$$

We obtain the genus from the Riemann-Hurwitz formula:

$$\chi(\mathcal{C}) = 2 - 2g = 3.\chi(\mathbb{P}^1) - \#R = 6 - 12 = -6.$$

□

We obtain a $\mathbb{Z}/2\mathbb{Z}$ symmetry of the Painlevé curve $\mathcal{C}$ defined by the action:

$$\tau : (x, y) \mapsto (-x, -y) \ .$$

Quoting out this action yields the curve $\mathcal{C}' := \Gamma/\langle\tau\rangle$ defined by the equations:

$$\begin{aligned} 0 &= y^2 - xz \\ 0 &= x^3 + c_1x^2 + c_2x + yz + c_3y \\ 0 &= x^2y + c_1xy + c_2y + z^2 + c_3z \ . \end{aligned}$$

It is a curve of geometric genus 2. Hence the Prym variety $\mathrm{Prym}(\mathcal{C}/\mathcal{C}')$ is a principally polarised abelian surface by proposition 38. The curves $\mathcal{C}$ and $\mathcal{C}'$ turn singular if the parameters lie on the discriminant of the momentum map.

2.4 Algebraic integrability

With the use of the Laurent series solutions to χ_H of $(\mathcal{X}, \mathbb{F}_{\mathcal{X}})$ we are able to compute a basis of $\mathcal{P}(\mathcal{C})$:

$$\begin{aligned} \varphi_0 &:= 1 \\ \varphi_1 &:= U \\ \varphi_2 &:= 32Q^3 + 3P^2 + 4QU \\ \varphi_3 &:= 8Q^2U + U^2 + 6PW - 12QV \ . \end{aligned}$$

The basis of $\mathcal{P}(2\mathcal{C})$ and $\mathcal{P}(3\mathcal{C})$ can be found in the appendix 6.3. We observe that $\dim(\mathcal{P}(2\mathcal{C})) = 4\dim(\mathcal{P}(\mathcal{C})) = 16$ and $\dim(\mathcal{P}(3\mathcal{C})) = 9\dim(\mathcal{P}(\mathcal{C})) = 36$ like we expect for an abelian surface if $\mathcal{P}(n\mathcal{C}) = H^0(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(n\mathcal{C}))$.

Proposition 19 *The image of the general fibre $\mathcal{F}_{hgf}$ under the map define by the basis of $\mathcal{P}(\mathcal{C})$ is the following Kummer surface in $\mathbb{P}^3$:*

$$\begin{aligned} 0 =&(-324f^2)z_2^3z_3 + (72f^2)z_1z_2z_3^2 + (81g)z_2^4 + (-36g)z_1z_2^2z_3 \\ &+ (4g)z_1^2z_3^2 + (648hf^2)z_1^2z_2^2 + (-144hf^2)z_1^3z_3 + (1944hf^2)z_0z_2^2z_3 \\ &+ (-144hf^2)z_0z_1z_3^2 + (-648hg)z_0z_2^3 + (144hg)z_0z_1z_2z_3 + (36f^4)z_1^4 \\ &+ (648f^4)z_0z_1z_2^2 + (72f^4)z_0z_1^2z_3 + (36f^4)z_0^2z_3^2 + (-2592h^2f^2)z_0z_1^2z_2 \\ &+ (-3888h^2f^2 + 36gf^2)z_0^2z_2z_3 + (1944h^2g - 18g^2)z_0^2z_2^2 \\ &+ (-144h^2g + 4g^2)z_0^2z_1z_3 + (2592h^3f^2 - 72hgf^2)z_0^2z_1^2 \\ &+ (2592h^3f^2 - 72hgf^2)z_0^3z_3 + (-2592h^3g + 72hg^2 + 1296f^6)z_0^3z_2 \\ &+ (-2592h^2f^4 + 72gf^4)z_0^3z_1 + (1296h^4g - 72h^2g^2 + g^3)z_0^4. \end{aligned}$$

For $f = 0$ it degenerates into the square of the conic we already computed in equation (2.2)!

Proof. This is a direct computation using SINGULAR. The quartic surface has 16 double points and hence defines a Kummer surface. □

The intersection of the Kummer with $z_0 = 0$ is given by the curve $\mathcal{D}$:

$$\begin{aligned} 0 =& (36f^4)z_1^4 + (648hf^2)z_1^2z_2^2 + (81g)z_2^4 + (-144hf^2)z_1^3z_3 \\ &+ (-36g)z_1z_2^2z_3 + (-324f^2)z_2^3z_3 + (4g)z_1^2z_3^2 + (72f^2)z_1z_2z_3^2 \ . \end{aligned}$$

It has a double point at $(0:0:1)$ and the projection on z_1, z_2 induces a $2:1$ cover of $\mathbb{P}^1$ ramified along the zeros of:

$$(144h^2f^4 - 4gf^4)z_1^6 - 72z_1^5z_2f^6 - 648z_1^3z_2^3hf^4 + 729z_2^6f^4 \ .$$

If we set the parameters $(f, h, g) = (1, 1, 1)$, we are able to compute the image of $\mathcal{F}_c$ under $\varphi_{\mathcal{L}(2\mathcal{C})}$ in $\mathbb{P}^{15}$. The image has degree 32 and its ideal is generated by 72 quadratic equations. The intersection $\overline{\varphi_{\mathcal{L}(2\mathcal{C})}(\mathcal{F}_c)} \cap \{z_0 = 0\}$ is twice a genus 4 curve of degree 16 having a double point, which coincides with the computations from section 6.2.3.
The $\mathbb{Z}/2\mathbb{Z}$ action on the Painlevé curve (2.4) is induced by a $\mathbb{Z}/2\mathbb{Z}$ action on the variable t of the Laurent series and the parameters γ_i of the Painlevé wall:

$$\tau : (t, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) \mapsto (-t, -\gamma_1, -\gamma_2, \gamma_3, \gamma_4, -\gamma_5) \ .$$

The induced action on the variables reads as:

$$\tau : (Q, P, U, W, L, V) \mapsto (Q, -P, U, -W, L, V) \ .$$

If the completion of the generic fibre is an abelian surface and τ extends to the completion it has to be the action of -1 since τ maps the Hamiltonian vector fields χ_H and χ_G to $-\chi_H$ and $-\chi_G$. The sections of $H^0(\mathcal{L}(\mathcal{C}))$ are τ–invariant. On $H^0(\mathcal{L}(2\mathcal{C}))$ the action of τ is given by:

$$\tau : (\varphi_0 = 1 : \varphi_1 : \cdots : \varphi_{15}) \mapsto \tag{2.6}$$
$$(\varphi_0 : \varphi_2 : -\varphi_3 : \varphi_4 : -\varphi_5 : \varphi_6 : \varphi_7 : -\varphi_8 : \varphi_9 : -\varphi_{10} : \varphi_{11} : -\varphi_{12} : \varphi_{13} : -\varphi_{14} : \varphi_{15}) \ . \tag{2.7}$$

Hence we find $h^0(\mathcal{L}(2\mathcal{C})^-) = 6$ and $h^0(\mathcal{L}(2\mathcal{C})^+) = 10$. Furthermore, this shows that the $\mathbb{Z}/2\mathbb{Z}$ action lifts to the embedding of $\mathcal{F}_{fhg}$ in $\mathbb{P}^{15}$.

Theorem 8 *The integrable system $(\mathcal{X}, \mathbb{F}_{\mathcal{X}})$ and its subsystem $(X, \mathbb{F}_X)$ are algebraic complete integrable. The divisor $\mathcal{C}_c := \overline{\mathcal{F}_c} \setminus \mathcal{F}_c$ is a curve of geometric genus 4 with one double point. It induces a $(2, 2)$ polarisation on $\overline{\mathcal{F}}_c$. The general fibre is isogenous to the Prym$(\tilde{\mathcal{C}}/\tilde{\mathcal{C}}')$, where $\tilde{\mathcal{C}}$ denotes the normalisation of $\mathcal{C}_c$ and $\tilde{\mathcal{C}}'$ its $\mathbb{Z}/2\mathbb{Z}$–quotient induced by the $-1_{\overline{\mathcal{F}}_c}$ action on $\mathcal{C}_c$.*

Proof. We use $\mathcal{P}(2\mathcal{C}_c)$ to compute an embedding of $\mathcal{F}_c$ in $\mathbb{P}^{15}$. In the first step we computed the Wronskians:

$$W_{ijk} := \{H_k, \varphi_i\}\varphi_j - \{H_k, \varphi_j\}\varphi_i \ .$$

Using the computer algebra program Singular we could check that all W_{ijk} can be written as polynomials of degree less or equal 2 in the φ_i with coefficients in $\mathbb{C}(f,h,g)$. Hence the vector fields χ_H and χ_G extend to holomorphic vector fields on the completion of $\mathcal{F}_c$. In the next step we have to show that the vector field χ_H vanishes nowhere on $\overline{\mathcal{F}}_c = \mathcal{A}_c$. This can be done by a computation for the fibre $\mathbb{F}^{-1}(0,1,1)$ and is tangent at the generic point of the divisor $\mathcal{C}_c$.
As the property of the non vanishing of a vector field on a fibre is open, this holds as well for the general fibre of $(\mathcal{X},\mathbb{F}_{\mathcal{X}})$ as for the general fibre of $(X,\mathbb{F}_X)$. Theorem 5 implies that the general fibres of both integrable systems are the affine parts of an abelian surface. Since $h^0(\mathcal{P}(\mathcal{C}_c)) = 4$ the induced polarisation has to be of type $(1,4)$ or $(2,2)$. The image of $\varphi_{\mathcal{P}(\mathcal{C}_c)}$ is a Kummer surface and hence the induced polarisation is of type $(2,2)$. The $\mathbb{Z}/2\mathbb{Z}$ action on $\mathcal{C}$ lifts to an $\mathbb{Z}/2\mathbb{Z}$ action τ on $\mathbb{P}^{15}$ defined in (2.6) inducing the $-1_{\mathcal{A}_c}$ action as it maps the vector fields χ_H and χ_G to $-\chi_H$ and $-\chi_G$. Thus by remark 27 the abelian surface $\mathcal{A}_c$ is isogenous to $Prym(\tilde{\mathcal{C}}/\tilde{\mathcal{C}}')$. □

If we set $l = 0$ the generic fibre of $(\mathbb{F}_{\mathcal{X}_0})$ contains an elliptic curve and so is isomorphic to the product of two elliptic curves. As a consequence of the last theorem we find:

Corollary 6 *The integrable system $(\mathbb{C}^4,\mathbb{F}_2)$ is almost algebraic integrable. The general fibre completes in to a surface $\mathcal{B}_c$ and the $\mathbb{Z}/2\mathbb{Z}$ action defines a cyclic $2:1$ cover of $\mathcal{A}_c$ ramified along the nodal genus 4 curve $\mathcal{C}$. The surface $\mathcal{A}_c$ is isogenous to the product of two elliptic curves.*

Proof. We already proved that the $\mathbb{Z}/2\mathbb{Z}$ quotient $(X,\mathbb{F}_X)$ is algebraic completely integrable and hence $(\mathbb{C}^4,\mathbb{F}_2)$ is almost algebraic integrable. The second statement is a direct consequence of the attachment Lemma of [Pio92] and the last one is a consequence of the splitting property since $\mathcal{A}_c$ contains elliptic curves. □

Theorem 9 *The integrable system $(\mathbb{C}^6,\mathbb{F}_3)$ is generalised algebraic integrable. The general fibre is isomorphic to the affine part of a $\mathbb{C}^*$–extension of the abelian surface $\mathcal{A}_c$, the completion of general fibre of $(\mathcal{X},\mathbb{F}_{\mathcal{X}})$.*

Proof. The flow of χ_F induces a $\mathbb{C}^*$ action on the fibre of $(\mathbb{C}^6,\mathbb{F}_3)$ the $\mathbb{C}^*$ quotient is the algebraic completely integrable system $(\mathcal{X},\mathbb{F}_{\mathcal{X}})$, hence the $\mathbb{C}^*$ quotient of the general fibre is the affine part of an abelian surface. The projectively normal embedding of the general fibre of $\mathbb{F}_{\mathcal{X}}$ obtains an extension to an embedding of the general fibre of $\mathbb{F}_3$ in $\mathbb{P}^{15}\times\mathbb{P}^6$:

$$\begin{array}{ccc} \mathbb{F}_3^{-1}(c) & \stackrel{\Psi}{\hookrightarrow} & \mathbb{P}^{15}\times\mathbb{P}^6 \\ \pi\downarrow & & \downarrow \\ \mathbb{F}_{\mathcal{X}}^{-1}(c) & \stackrel{\varphi}{\hookrightarrow} & \mathbb{P}^{15} \end{array}$$

where we just added the holomorphic sections

$$\psi_1 := q_1\ ,\quad \psi_2 := q_2\ ,\quad \psi_3 := q_3\ ,\quad \psi_4 := p_1\ ,\quad \psi_5 := p_2\ ,\quad \psi_6 := p_3 \in H^0(\mathcal{F}_c,\mathcal{O}_{\mathcal{F}_c})\ .$$

Again using SINGULAR we could verify that:

$$\{H_k,\psi_i\}\varphi_j-\{H_k,\varphi_j\}\psi_i=\sum_{\alpha,\beta=0}^{15}\sum_{\gamma=0}^{6}a_{\gamma\alpha\beta}^{(kij)}\psi_\gamma\varphi_\alpha\varphi_\beta$$

for some $a_{\gamma\alpha\beta}^{(kij)}\in\mathbb{C}(f,h,g)$. Here $H_1=H$ and $H_2=G$. The statement follows then by Theorem 4. □

The example of this Hénon–Heiles system shows something interesting. For an algebraic completely integrable or generalised algebraic completely integrable system the general fibre is the affine part of an abelian algebraic group and the hamiltonian vector fields induce the group action. In the case of almost algebraic integrable or almost generalised algebraic integrable system this is not true. But in the previous example we have seen that we can imbed the almost algebraic integrable system $(\mathbb{C}^4,\mathbb{F}_2)$ in an generalised algebraic completely integrable system $(\mathbb{C}^6,\mathbb{F}_3)$. Thus the general fibre of $\mathbb{F}_2$ is a subvariety of an abelian algebraic group and the hamiltonian vector fields are induced by the group action on this algebraic group.

3 Grammaticos integrable potential of degree 3

Dorizzi, Grammaticos, and Ramani [DGR82] discovered, that besides the Hénon–Heiles integrable systems, there is a further example of a homogenous potential in two variables of degree 3, defined by the Hamilton function:

$$H := \frac{1}{2}(p_1^2 + p_2^2) + q_1^3 + \frac{1}{2}q_1q_2^2 + \frac{\sqrt{-3}}{18}q_2^3 \ . \tag{3.1}$$

Its second integral reads as:

$$G := p_1p_2^3 - \frac{\sqrt{-3}}{2}p_2^4 + \frac{1}{2}q_2^3p_1^2 - \left(\frac{3}{2}q_1q_2^2 - \frac{\sqrt{-3}}{2}q_2^3\right)p_1p_2 + \left(3q_1^2q_2 - \sqrt{-3}q_1q_2^2 + \frac{1}{2}q_2^3\right)p_2^2$$
$$+ \frac{1}{2}q_1^3q_2^3 + \frac{\sqrt{-3}}{8}q_1^2q_2^4 + \frac{1}{4}q_1q_2^5 + 5\frac{\sqrt{-3}}{72}q_2^6 \ .$$

The square root of -3 can be eliminated by the symplectic transformation:

$$(q_1, q_2, p_1, p_2) \mapsto \left(q_1, \sqrt{-3}q_2, p_1, \frac{1}{\sqrt{-3}}p_2\right) \ .$$

After this transformation we obtain:

$$\begin{aligned} H &= \frac{1}{2}\left(p_1^2 - \frac{1}{3}p_2^2\right) + q_1^3 - \frac{3}{2}q_1q_2^2 + \frac{1}{2}q_2^3 \\ G &= -\frac{3}{2}q_1^3q_2^3 + \frac{9}{8}q_1^2q_2^4 + \frac{9}{4}q_1q_2^5 - \frac{15}{8}q_2^6 - \frac{3}{2}q_2^3p_1^2 - \frac{3}{2}q_1q_2^2p_1p_2 - \frac{3}{2}q_2^3p_1p_2 - q_1^2q_2p_2^2 \\ &\quad - q_1q_2^2p_2^2 + \frac{1}{2}q_2^3p_2^2 + \frac{1}{9}p_1p_2^3 - \frac{1}{18}p_2^4 \ . \end{aligned}$$

We will now analyse this system and it will turn out that the generic fibre is the affine part of a $(1, 6)$ polarised abelian surface. The compactification divisor is a curve of genus 4 with a singularity of type D_4.

Proposition 20 *The fibres of the moment map $\mathbb{F} = (H, G)$ are smooth away from the discriminant:*

$$\Delta = \{h^2g - 2g^2 = 0\} \ .$$

Proof. The proof is a direct computation. □

3.1 Laurent series solutions to χ_H

The first step to analyse the complex geometry of the generic momentum map fibre is to compute all principal balances to the flow of χ_H.

Proposition 21 *Kovalevskaya's method yields only one principal balance with the positive Kovalevskaya–exponents* $\{1,4,6\}$.

$$\begin{aligned}
q_1(t) &= -\frac{2}{t^2} - \frac{1}{8}\gamma_1^2 + \frac{1}{16}\gamma_1^3 t - \frac{3}{128}\gamma_1^4 t^2 + \frac{1}{12}\gamma_1\gamma_2 t^3 + \frac{1}{4}\gamma_3 t^4 + \dots\\
q_2(t) &= \gamma_1\frac{1}{t} - \frac{1}{4}\gamma_1^2 + \frac{1}{16}\gamma_1^3 t - \frac{1}{6}\gamma_2 t^2 + \left(\frac{1}{128}\gamma_1^5 - \frac{1}{24}\gamma_1\gamma_2\right) t^3 - \frac{1}{96}\gamma_1^2\gamma_2 t^4 + \dots\\
p_1(t) &= +\frac{4}{t^3} + \frac{1}{16}\gamma_1^3 - \frac{3}{64}\gamma_1^4 t + \frac{1}{4}\gamma_1\gamma_2 t^2 + \gamma_3 t^3 + \dots\\
p_2(t) &= 3\gamma_1\frac{1}{t^2} - \frac{3}{16}\gamma_1^3 + \gamma_2 t + \left(\frac{3}{8}\gamma_1\gamma_2 - \frac{9}{128}\gamma_1^5\right) t^2 + \frac{1}{8}\gamma_1^2\gamma_2 t^3 + \dots\ .
\end{aligned}$$

Proof. Only one component of the initial locus yields enough positive Kovalevskaya exponents. This component is zero dimensional and we obtain the Kovalevskaya exponents $\{1,4,6,-1\}$. □

Using this Laurent series we are able to compute the affine equations of the Painlevé–divisor:

Proposition 22 *A plane model for the Painlevé–divisor is given by the affine equation:*

$$0 = \frac{1}{3}y^3 + 3x^4y^2 + (x^2h - 27x^8)y + 45x^{12} - 3y^6h + g\ .$$

The Painlevé–curve is smooth of genus 4.

Proof. If we substitute the Laurent series in H and G we find:

$$\begin{aligned}
h &= \gamma_1^6 - \frac{20}{13}\gamma_1^2\gamma_2 + \gamma_3\\
g &= \gamma_1^{12} - \frac{68}{15}\gamma_1^8\gamma_2 + \frac{13}{10}\gamma_1^6\gamma_3 + \frac{14}{3}\gamma_1^4\gamma_2^2 - \frac{13}{3}\gamma_1^2\gamma_2\gamma_3 - \frac{20}{9}\gamma_2^3\ .
\end{aligned}$$

After eliminating γ_3 and rescaling h, g, γ_1 and γ_2 we get the plain equation if we replace γ_1 by x and γ_2 by y. □

The projection on the x coordinate defines a $3:1$ cover of $\mathbb{P}^1$. It is simply ramified along the 12 points:

$$0 = (27h^2 - 18g)x^{12} + (6hg - 8h^3)x^6 - g^2\ .$$

As the equation depends only on x^6, the 12 points are the sixth roots of the two zeros of the equation:

$$0 = (27h^2 - 18g)z^2 + (6hg - 8h^3)z - g^2\ .$$

For generic choices of h and g this equation has two different zeros that are parametrised by h and g. From the degree of the ramification divisor R we obtain the geometric genus of $\mathcal{C}_P$ via:

$$2 - 2g = \chi(\mathcal{C}_P) = 3\chi(\mathbb{P}^1) - \#R = 6 - 12 = -6$$

and hence $g = 4$.

3.2 Algebraic integrability

With the use of the Laurent series solutions of the flow of χ_H, we are able to compute a basis for $\mathcal{P}(\mathcal{C})$:

$$\begin{aligned}
\varphi_0 &:= 1\\
\varphi_1 &:= q_2\\
\varphi_2 &:= q_2p_1 + \tfrac{2}{3}q_1p_2 + \tfrac{1}{3}q_2p_2\\
\varphi_3 &:= -\tfrac{3}{2}q_1q_2^2 + \tfrac{3}{2}q_2^3 - \tfrac{1}{3}p_2^2\\
\varphi_4 &:= q_1^3q_2 - \tfrac{3}{2}q_1^2q_2^2 - \tfrac{3}{4}q_1q_2^3 + \tfrac{5}{4}q_2^4 + \tfrac{1}{2}q_2p_1^2 + q_2p_1p_2 + \tfrac{1}{3}q_1p_2^2\\
\varphi_5 &:= \tfrac{3}{2}q_1q_2^2p_1 + 2q_1^2q_2p_2 + \tfrac{1}{2}q_1q_2^2p_2 - q_2^3p_2 - \tfrac{1}{3}p_1p_2^2 + \tfrac{1}{9}p_2^3\ .
\end{aligned}$$

From a basis of $\mathcal{P}(2\mathcal{C}_P)$ we obtain an embedding of the generic fibre in $\mathbb{P}^{23}$. If the general fibre completes in to an abelian surface the Painlevé divisor $\mathcal{C}_P$ should induce a $(1,6)$ polarisation as $h^0(\mathcal{L}(\mathcal{C}_P)) = 6 = \delta_1\delta_2$ and $\delta_1|\delta_2$. We also computed a basis for $H^0(2\mathcal{C}_P)$ and found $h^0(\mathcal{L}(2\mathcal{C}_P)) = 24$, as expected, since $2\mathcal{C}_P$ should induce a $(2,12)$ polarisation. The explicit basis is written down in the appendix 6.4.
Ramanan [Ram84] proved that a $(1,6)$ polarisation on a general abelian surface is very ample.

Proposition 23 *The ideal of the general fibre under the embedding defined by the basis of $\mathcal{P}(\mathcal{C})$ in $\mathbb{P}^5$ is given by the following 5 cubics and 6 quadrics:*

$$\begin{aligned}
0 &= 9z_1^2z_3g - 15z_2^2z_3h - 28z_1z_3z_4h - 2z_3z_4^2 - 36z_1z_2z_5h + 6z_2z_4z_5 - 20z_1^2z_0h^3\\
&\quad - 36z_1^2z_0hg + 9z_2^2z_0g + 40z_1z_4z_0h^2 - 12z_1z_4z_0g - 20z_4^2z_0h\\
0 &= 2z_1^2z_3h + 3z_2^2z_3 + 4z_1z_3z_4 + 6z_1z_2z_5 + 4z_1^2z_0h^2 + 6z_1^2z_0g - 8z_1z_4z_0h + 4z_4^2z_0\\
0 &= 27z_1^3g^2 + 36z_1z_2^2h^3 - 351z_1z_2^2hg - 56z_3^3h^2 + 6z_3^3g - 120z_1^2z_4h^3 - 234z_1^2z_4hg\\
&\quad - 36z_2^2z_4h^2 + 27z_2^2z_4g + 168z_1z_4^2h^2 - 18z_1z_4^2g - 48z_4^3h - 72z_3z_5^2h\\
&\quad + 48z_3^2z_0h^3 - 180z_3^2z_0hg - 24z_5^2z_0h^2 + 18z_5^2z_0g + 48z_3z_0^2h^2g - 36z_3z_0^2g^2 - 288z_0^3hg^2\\
0 &= 9z_1^3hg - 9z_1z_2^2h^2 + 81z_1z_2^2g + 14z_3^3h + 30z_1^2z_4h^2 + 54z_1^2z_4g + 9z_2^2z_4h - 42z_1z_4^2h\\
&\quad + 12z_4^3 + 18z_3z_5^2 - 12z_3^2z_0h^2 + 36z_3^2z_0g + 6z_5^2z_0h - 12z_3z_0^2hg + 72z_0^3g^2\\
0 &= 12z_1^3h^2 + 9z_1^3g - 9z_1z_2^2h + 2z_3^3 - 6z_1^2z_4h + 9z_2^2z_4 - 6z_1z_4^2 - 12z_3^2z_0h + 6z_5^2z_0 - 12z_3z_0^2g
\end{aligned}$$

$$
\begin{aligned}
0 &= 18z_1z_2^2z_4h - 3z_3^3z_4 - 18z_2^2z_4^2 - 10z_1z_3^2z_0h^2 - 9z_1z_3^2z_0g + 22z_3^2z_4z_0h + 6z_2z_3z_5z_0h \\
&\quad + 12z_1z_3z_0^2hg + 30z_3z_4z_0^2g + 18z_2z_5z_0^2g + 12z_1z_0^3h^2g + 18z_1z_0^3g^2 - 12z_4z_0^3hg \\
0 &= 162z_2^4g + 30z_1z_3^3h^2 + 45z_1z_3^3g - 240z_1^3z_4h^3 - 144z_1^3z_4hg + 702z_1z_2^2z_4g + 26z_3^3z_4h \\
&\quad + 336z_1^2z_4^2h^2 + 288z_1^2z_4^2g - 96z_1z_4^3h - 78z_2z_3^2z_5h - 24z_1z_3z_5^2h + 24z_3z_4z_5^2 - 72z_2z_5^3 \\
&\quad + 40z_1z_3^2z_0h^3 - 108z_1z_3^2z_0hg + 8z_3^2z_4z_0h^2 + 30z_3^2z_4z_0g - 24z_2z_3z_5z_0h^2 - 234z_2z_3z_5z_0g \\
&\quad - 48z_1z_5^2z_0h^2 - 72z_1z_5^2z_0g - 372z_1z_3z_0^2h^2g - 378z_1z_3z_0^2g^2 + 444z_3z_4z_0^2hg - 48z_1z_0^3h^3g \\
&\quad + 48z_4z_0^3h^2g + 756z_4z_0^3g^2 \\
0 &= 27z_1z_2^3g + 5z_2z_3^3h + 6z_1^2z_2z_4h^2 + 9z_1^2z_2z_4g - 12z_1z_2z_4^2h + 6z_2z_4^3 + 2z_1z_3^2z_5h + z_3^2z_4z_5 \\
&\quad + 6z_2z_3z_5^2 + 15z_2z_3^2z_0g + 4z_1z_3z_5z_0h^2 + 6z_1z_3z_5z_0g - 4z_3z_4z_5z_0h - 6z_2z_3z_0^2hg \\
&\quad - 6z_4z_5z_0^2g + 18z_2z_0^3g^2 \\
0 &= 27z_1^2z_2^2g + 5z_1z_3^3h + z_3^3z_4 - 3z_2z_3^2z_5 + 15z_1z_3^2g - 6z_1z_3z_0^2hg + 6z_3z_4z_0^2g + 18z_1z_0^3g^2 \\
0 &= 18z_1^2z_2^2h - 3z_1z_3^3 - 18z_1z_2^2z_4 + 14z_1z_3^2z_0h - 2z_3^2z_4z_0 + 6z_2z_3z_5z_0 + 6z_1z_3z_0^2g \\
&\quad + 12z_1z_0^3hg - 12z_4z_0^3g \\
0 &= 9z_1^3z_2g + 9z_1z_2^3h - 4z_2z_3^3 - 18z_1^2z_2z_4h - 9z_2^3z_4 + 18z_1z_2z_4^2 - 6z_1z_3^2z_5 + 12z_2z_3^2z_0h \\
&\quad + 8z_1z_3z_5z_0h - 8z_3z_4z_5z_0 - 6z_2z_5^2z_0 - 12z_1z_5z_0^2g\ .
\end{aligned}
$$

Proof. This can be done by a direct computation using SINGULAR. □

Theorem 10 *The integrable system defined by the Hamilton function* (3.1) *is algebraic completely integrable. The general fibre $\mathcal{F}_c$ is isomorphic to $\mathcal{A}_c \setminus \mathcal{C}_c$ where $\mathcal{A}_c$ is an abelian surface and $\mathcal{C}_c$ is a curve of geometric genus* 4 *with a D_4 singularity that induces a polarisation of type* $(1,6)$ *on $\mathcal{A}_c$.*

Proof. The first step of the prove is to show, that the vector fields χ_H and χ_G extends to all of $\overline{\varphi(\mathcal{F}_c)}$. This can not be done using the embedding in $\mathbb{P}^5$. From $\mathcal{P}(2\mathcal{C}_c)$ we obtain an embedding in $\mathbb{P}^{23}$. The intersection of $\overline{\varphi(\mathcal{F}_c)} \subset \mathbb{P}^{23}$ with the hyperplanes $\{z_1 = 0\}$, $\{z_2 = 0\}$, and $\{z_3 = 0\}$ is empty. In order to show that χ_H and χ_G extend to a holomorphic vector fields we used a SINGULAR script to compute the wronskians:

$$W_{ij}^H := \{H, \varphi_i\}\varphi_j - \{H, \varphi_j\}\varphi_i$$

and

$$W_{ij}^G := \{G, \varphi_i\}\varphi_j - \{G, \varphi_j\}\varphi_i$$

as polynomials in the φ_i with coefficients in $\mathbb{C}(h,g)$ of degree 2.
If we set h and g to 1 we can use the embedding in $\mathbb{P}^5$ to show, that $\chi_H \wedge \chi_G$ does not vanish on the generic point of the divisor Γ. As $\chi_H \wedge \chi_G$ defines the canonical bundle and Γ is irreducible $\chi_H \wedge \chi_G$ vanishes nowhere on the completion of the generic fibre. A smooth surface with two commuting global vector fields, that span the tangent space at each point is an abelian surface.
The dimension $h^0(\mathcal{L}(\mathcal{C})) = 6 = \delta_1\delta_2$ and as $\delta_1|\delta_2$ we obtain $(\delta_1, \delta_2) = (1,6)$. The

intersection of the image of $\mathcal{F}_c$ in $\mathbb{P}^5$ with $\{z_0 = 0\}$ is a singular curve of geometric genus 4. The only singular point is the point $(0 : 0 : 0 : 0 : 0 : 1)$ a generic projection shows that it is a D_4–singularity and thus the arithmetic genus is 7. $\square$

4 The Duistermaat System

Duistermaat studied perturbations of the $1:1:2$ resonance:

$$H = q_1^2 + p_1^2 + q_2^2 + p_2^2 + 2q_3^2 + 2p_3^2 \ .$$

In Birkhoff normal form a perturbation of H of degree 3 is of the form $F_\beta := \beta_1 f_1 + \beta_2 f_2$. That is a general linear combination of the two polynomials of degree 3 that are in involution with H:

$$f_1 := q_3(q_1^2 - p_1^2) + 2p_3(q_1 p_1) \ , \quad f_2 := q_3(q_2^2 - p_2^2) + 2p_3(q_2 p_2) \ .$$

We may think of F_β as a family of polynomials parametrised by a point $\beta = (\beta_1 : \beta_2) \in \mathbb{P}^1$. Duistermaat proved that there is no third integral if

$$\beta \notin \{(1:1), (0:1), (1:0), (2:1), (1:2)\} \ .$$

The cases where $\beta = (1:0)$ and $\beta = (0:1)$ are reducible and we obtain that the integrable system is isomorphic to a product system on $\mathbb{C}^4 \times \mathbb{C}^2$. By the symmetry of the situation we obtain the $(1:2)$ case from the $(2:1)$ case if we exchange q_1, p_1 with q_2, p_2.
The geometrically most interesting case is $\beta = (2:1)$, because the other cases yield a third integral of weight 2 and the corresponding Hamiltonian vector field induces a $\mathbb{C}^*$–action. For $\beta = (2:1)$ we obtain the integrable system:

$$\begin{aligned} H &= q_1^2 + p_1^2 + q_2^2 + p_2^2 + 2q_3^2 + 2p_3^2 \\ F &= q_3(2(q_1^2 - p_1^2) + (q_2^2 - p_2^2)) + 2p_3(2q_1 p_1 + q_2 p_2) \\ G &= (p_1 q_2 - q_1 p_2)^2 (p_2^2 + q_2^2) + 2(\frac{1}{2} q_3 (q_2^2 - p_2^2) + p_3 q_2 p_2)^2 \ . \end{aligned}$$

In the following we will call this choice of β the *Duistermaat–system.* Obviously the generic fibre of this system can not be the affine part of an abelian variety, since the intersection of $\mathcal{F}_c$ with $\{q_3 = 1, p_3 = 0\}$ is a rational curve. Another reason is that the vectorfield χ_H induces an $\mathbb{C}^*$–action on the general fibre $\mathcal{F}_c$.

Proposition 24 *Kovalevskayas method yields for the Hamiltonian vector field χ_F only* *one principal balance with the positive Kovalevskaya–exponents* $\{\frac{1}{2}, \frac{3}{2}, 2, 3\}$*:*

$$\begin{aligned}
q_1 &= \frac{\alpha}{t} - \frac{1}{4}c_1^2\alpha + (\frac{1}{32}c_1^4\alpha + \frac{1}{2}c_1c_2\beta + c_3\alpha)t \\
&\quad - (\frac{1}{256}c_1^6\alpha - \frac{1}{8}c_1^3c_2\beta + \frac{3}{8}c_1^2c_3\alpha + \frac{1}{8}c_2^2\alpha - c_4\beta)t^2 + \dots \\
q_2 &= \frac{c_1\alpha}{\sqrt{t}} - (\frac{1}{8}c_1^3\alpha + c_2\beta)\sqrt{t} + (\frac{1}{128}c_1^5\alpha - \frac{1}{8}c_1^2c_2\beta + \frac{1}{2}c_1c_3\alpha)\sqrt{t}^3 + \dots \\
q_3 &= \frac{4\alpha\beta}{t} + c_1^2\alpha\beta - 8c_3\alpha\beta t + (\frac{1}{64}c_1^6\alpha\beta + \frac{3}{2}c_1^2c_3\alpha\beta + \frac{3}{2}c_2^2\alpha\beta - 4c_4\beta^2 - \frac{1}{4}c_4)t^2 + \dots \\
p_1 &= \frac{\beta}{t} - \frac{1}{4}c_1^2\beta + (\frac{1}{32}c_1^4\beta - \frac{1}{2}c_1c_2\alpha + c_3\beta)t \\
&\quad - (\frac{1}{256}c_1^6\beta + \frac{1}{8}c_1^3c_2\alpha + \frac{3}{8}c_1^2c_3\beta + \frac{1}{8}c_2^2\beta + c_4\alpha)t^2 + \dots \\
p_2 &= \frac{c_1\beta}{\sqrt{t}} + (-\frac{1}{8}c_1^3\beta + c_2\alpha)\sqrt{t} + (\frac{1}{128}c_1^5\beta + \frac{1}{8}c_1^2c_2\alpha + \frac{1}{2}c_1c_3\beta)\sqrt{t}^3 + \dots \\
p_3 &= \frac{4\beta^2 + \frac{1}{4}}{t} + (c_1^2\beta^2 + \frac{1}{1}6c_1^2) - (8c_3\beta^2 + \frac{1}{2}c_3)t \\
&\quad + (\frac{1}{64}c_1^6\beta^2 + \frac{1}{1024}c_1^6 + \frac{3}{2}c_1^2c_3\beta^2 + \frac{3}{32}c_1^2c_3 + \frac{3}{2}c_2^2\beta^2 + \frac{3}{32}c_2^2 + 4c_4\alpha\beta)t^2 + \dots \ .
\end{aligned}$$

Proof. Only one component of the initial locus yields enough positive *rational* eigenvalues. These Kovalevskaya exponents are $\{\frac{1}{2}, \frac{3}{2}, 2, 3, 0, -1\}$. We renamed the variables of the initial locus $q_1^{(0)} := \alpha$ and $p_1^{(0)} := \beta$ which fulfil the relation:

$$8\alpha^2 + 8\beta^2 + 1 = 0 \ .$$

□

The $\mathbb{Z}/2\mathbb{Z}$ action of replacing $\sqrt{t}$ with $-\sqrt{t}$ in the Laurent series is induced by:

$$\sigma : (q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (q_1, -q_2, q_3, p_1, -p_2, p_3) \ .$$

The constants of motion H, F, and G are invariant under the action of σ as well as the Poisson–bracket so we get an integrable system on the quotient $Y_1 := \mathbb{C}^6/\langle\sigma\rangle$. The quotient by the $\mathbb{C}^*$–action induced by the flow of χ_H yields an other integrable system on $Y_2 = \mathbb{C}^6//\mathbb{C}^*$. Since χ_H is σ invariant, we get the following commutative diagram of integrable systems:

$$\begin{array}{ccc}
(\mathbb{C}^6, \mathbb{F}_3) & \xrightarrow{\mathbb{Z}/2\mathbb{Z}} & (Y_1 := \mathbb{C}^6/(\mathbb{Z}/2\mathbb{Z}), \mathbb{F}_{Y_1}) \\
\downarrow{\scriptstyle \mathbb{C}^*} & & \downarrow{\scriptstyle \mathbb{C}^*} \\
(Y_2 := \mathbb{C}^6//\mathbb{C}^*, \mathbb{F}_{Y_2}) & \xrightarrow[\mathbb{Z}/2\mathbb{Z}]{} & (X := \mathbb{C}^6/(\mathbb{C}^* \times \mathbb{Z}/2\mathbb{Z}), \mathbb{F}_X)
\end{array} .$$

Where the maps from left to the right are taking the $\mathbb{Z}/2\mathbb{Z}$ quotient, the maps from up to down are defined by quoting out the $\mathbb{C}^*$ action of the flow of χ_H.

Proposition 25 *The morphism*

$$\mathbb{C}^6 \to \mathbb{C}^9 \ , \qquad x \mapsto (z_1(x), \ldots, z_9(x))$$

induces an embedding of $(X, \mathbb{F}_X)$ *in* $\mathbb{C}^9$*, where:*

$$\begin{aligned}
&z_1 = q_3^2 + p_3^2 \ , \quad z_2 = q_2^2 + p_2^2 \ , \quad z_3 = q_1^2 + p_1^2 \\
&z_4 = q_2 q_3 p_2 - \tfrac{1}{2} q_2^2 p_3 + \tfrac{1}{2} p_2^2 p_3 \ , \quad z_5 = q_1 q_3 p_1 - \tfrac{1}{2} q_1^2 p_3 + \tfrac{1}{2} p_1^2 p_3 \\
&z_6 = q_2^2 q_3 - q_3 p_2^2 + 2 q_2 p_2 p_3 \ , \quad z_7 = q_1^2 q_3 - q_3 p_1^2 + 2 q_1 p_1 p_3 \\
&z_8 = q_2^2 p_1^2 - 2 q_1 q_2 p_1 p_2 + q_1^2 p_2^2 \ , \quad z_9 = q_1 q_2^2 p_1 - q_1^2 q_2 p_2 + q_2 p_1^2 p_2 - q_1 p_1 p_2^2 \ .
\end{aligned}$$

The constants of motion read in these variables:

$$\begin{aligned}
H &= 2z_1 + z_2 + z_3 \\
F &= z_6 + 2z_7 \\
G &= \frac{1}{2} z_6^2 + z_2 z_8 \ .
\end{aligned}$$

Proof. A direct computation using SINGULAR shows that $\chi_H z_i = 0$ and so $z_i \in \mathbb{C}[q_1, q_2, q_3, p_1, p_2, p_3]^{\mathbb{C}^*}$. The Poisson–brackets $\{z_i, z_j\}$ are polynomials in $\mathbb{C}[z_1, \ldots, z_9]$. □

4.1 Integrable systems related to the Duistermaat system

Like in the Hénon–Heiles $\varepsilon = 16$ example, we obtain a deformation of $(X, \mathbb{F}_X)$ from the embedding $i : \mathbb{C}^6 \to \mathbb{C}^8$ defined by $(q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (q_1, q_2, q_3, 0, p_1, p_2, p_3, 0)$. On $\mathbb{C}^8$ we define an integrable system by the constants of motion:

$$\begin{aligned}
L &:= q_4 p_2 - q_2 p_4 \\
H &:= q_1^2 + p_1^2 + q_2^2 + p_2^2 + 2q_3^2 + 2p_3^2 + q_4^2 + p_4^2 \\
F &:= q_3 \big(2(q_1^2 - p_1^2) + (q_2^2 + q_4^2 - p_2^2 - p_4^2) \big) + 2p_3 (2q_1 p_1 + q_2 p_2 + q_4 p_4) \\
G &:= \big(p_1^2 (q_2^2 + q_4^2) - 2p_1 q_1 (p_2 q_2 + q_4 p_4) + q_1^2 (p_2^2 + p_4^2) \big) \left(p_2^2 + p_4^2 + q_2^2 + q_4^2 \right) \\
&\quad + 2 \big(1/2 q_3 (q_2^2 + q_4^2 - p_2^2 - p_4^2) + p_3 (q_2 p_2 + q_4 p_4) \big)^2 - (q_1^2 + p_1^2) L^2.
\end{aligned}$$

If we restrict these polynomials to $q_4 = p_4 = 0$ we obtain the polynomials H, F, and G from the Duistermaat–system and L turns to zero. As before the embedding is no Poisson–morphism. Besides the $\mathbb{C}^*$ action induced by the flow of χ_H, the flow of χ_L induces a $\mathbb{C}^*$–action, so we get a $(\mathbb{C}^*)^2$–action on the fibres of $\mathbb{F}_4$. The $\mathbb{Z}/2\mathbb{Z}$ action on the fibres of $\mathbb{F}_3$ is induced via the embedding $(\mathbb{C}^6, \mathbb{F}_3) \hookrightarrow (\mathbb{C}^8, \mathbb{F}_4)$ by the $\mathbb{C}^*$ action of χ_L.

Proposition 26 *The quotient* $\mathcal{X} := \mathbb{C}^8 // (\mathbb{C}^*)^2$ *can be embedded in* $\mathbb{C}^{10}$ *via the morphism:*

$$\mathbb{C}^8 \to \mathbb{C}^{10} \ , \qquad x \mapsto (z_1(x), z_2(x), \ldots, z_{10}(x)) \ ,$$

where the z_i are defined by:

$$\begin{aligned}
z_1 &= q_4p_2 - q_2p_4 \\
z_2 &= q_3^2 + p_3^2 \\
z_3 &= q_2^2 + q_4^2 + p_2^2 + p_4^2 \\
z_4 &= q_1^2 + p_1^2 \\
z_5 &= q_2q_3p_2 - \frac{1}{2}q_2^2p_3 - \frac{1}{2}q_4^2p_3 + \frac{1}{2}p_2^2p_3 + q_3q_4p_4 + \frac{1}{2}p_3p_4^2 \\
z_6 &= q_1q_3p_1 - \frac{1}{2}q_1^2p_3 + \frac{1}{2}p_1^2p_3 \\
z_7 &= q_2^2q_3 + q_3q_4^2 - q_3p_2^2 + 2q_2p_2p_3 + 2q_4p_3p_4 - q_3p_4^2 \\
z_8 &= q_1^2q_3 - q_3p_1^2 + 2q_1p_1p_3 \\
z_9 &= q_2^2p_1^2 + q_4^2p_1^2 - 2q_1q_2p_1p_2 + q_1^2p_2^2 - 2q_1q_4p_1p_4 + q_1^2p_4^2 \\
z_{10} &= q_1q_2^2p_1 + q_1q_4^2p_1 - q_1^2q_2p_2 + q_2p_1^2p_2 - q_1p_1p_2^2 - q_1^2q_4p_4 + q_4p_1^2p_4 - q_1p_1p_4^2 \ .
\end{aligned}$$

The constants of motion read in the new coordinates:

$$\begin{aligned}
L &= z_1 \\
H &= 2z_2 + z_3 + z_4 \\
F &= z_7 + 2z_8 \\
G &= -z_1^2z_4 + \frac{1}{2}z_7^2 + z_3z_9 \ ,
\end{aligned}$$

Proof. This can be done by a direct computation. A SINGULAR computation shows that the z_i lie in:

$$\mathcal{O}(\mathbb{C}^8)^{(\mathbb{C}^*)^2} = \left\{ f \in \mathcal{O}(\mathbb{C}^8) \mid \chi_L f = \chi_H f = 0 \right\} \ .$$

The Poisson–brackets of the $\{z_i, z_j\}$ are polynomials in $\mathbb{C}[z_1, \ldots, z_{10}]$. □

On $\mathbb{C}^8$ we obtain two $\mathbb{C}^*$–actions induced by the flow of χ_H and χ_L. If we take the quotient by the χ_H action π_H and the χ_L action π_L and vice versa, we get the commutative diagram of integrable systems:

$$\begin{array}{ccc}
(\mathbb{C}^8, \mathbb{F}_4) & \xrightarrow{\pi_L} & (\mathcal{Y}_1 := \mathbb{C}^8//\mathbb{C}^*, \mathbb{F}_{\mathcal{Y}_1}) \\
\downarrow{\scriptstyle \pi_H} & & \downarrow{\scriptstyle p_H} \\
(\mathcal{Y}_2 := \mathbb{C}^8//\mathbb{C}^*, \mathbb{F}_{\mathcal{Y}_2}) & \xrightarrow[p_L]{} & (\mathcal{X} := \mathbb{C}^8/(\mathbb{C}^*)^2, \mathbb{F}_{\mathcal{X}})
\end{array}$$

Proposition 27 *The polynomial $L \in \mathcal{O}(\mathbb{C}^8)^{(\mathbb{C}^*)^2} = \mathcal{O}_{\mathcal{X}}(\mathcal{X})$ defines a family of integrable systems:*

$$L : \mathcal{X} \to \mathbb{C} \ ,$$

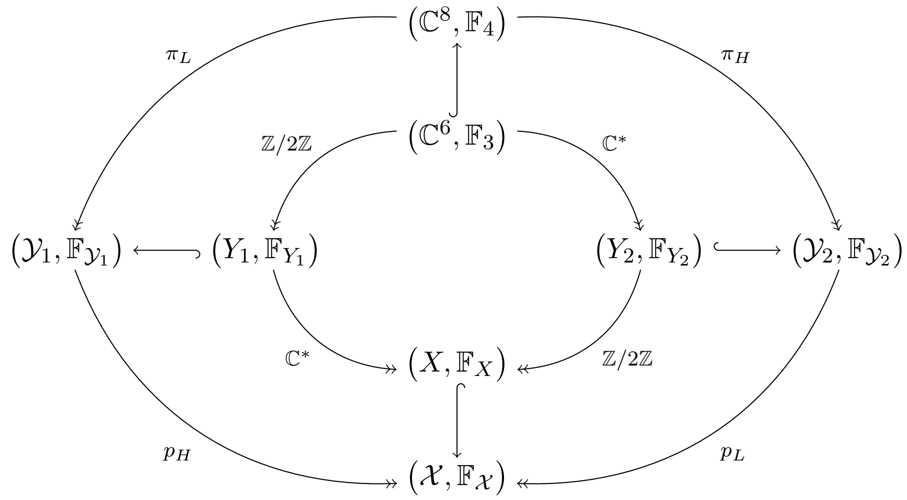
where the integrable system on $\mathcal{X}_l := L^{-1}(l)$ is defined by:

$$\mathbb{F}_{\mathcal{X}_l} := (H|_{\mathcal{X}_l}, F|_{\mathcal{X}_l}, G|_{\mathcal{X}_l}) \ .$$

The fibre $(\mathcal{X}_0, \mathbb{F}_{\mathcal{X}_0}) = (X, \mathbb{F}_X)$ *and so we obtain a deformation of the integrable system* $(X, \mathbb{F}_X)$ *from* $L : \mathcal{X} \to \mathbb{C}$*.*

Proof. The element $L \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$ Poisson–commutes with all all elements of $\mathcal{O}_{\mathcal{X}}(\mathcal{X})$ and so the restriction to $L = l$ defines an integrable system. □

Together with the embedding $(\mathbb{C}^6, \mathbb{F}_3) \hookrightarrow (\mathbb{C}^8, \mathbb{F}_4)$ we get the following commutative diagram:



.

Except the embedding $(\mathbb{C}^6, \mathbb{F}_3) \hookrightarrow (\mathbb{C}^8, \mathbb{F}_4)$ all arrows are morphisms of integrable systems.

4.2 Algebraic integrability

Applying Kovalevkayas method we computed the Laurent series solutions to χ_F for $(\mathbb{C}^8, \mathbb{F}_4)$. Again we find one principal balance having half integral Kovalevskaya exponents. Hence we get a Laurent series in $\sqrt{t}$. Substituting this series into the new variables z_i we end up with a Laurent series solution to χ_F for $(\mathcal{X}, \mathbb{F}_{\mathcal{X}})$ in $\sqrt{t}$ which has only even exponents an hence gives a Laurent series in t. These series can be used to compute the global sections of $\mathcal{P}(D)$:

$$\begin{aligned}
\varphi_0 &= 1\\
\varphi_1 &= z_3\\
\varphi_2 &= z_7\\
\varphi_3 &= -z_9.
\end{aligned}$$

Proposition 28 *The image of* $\mathcal{F}_c$ *under the map* $\varphi_{\mathcal{P}(D)}$ *is the following Kummer sur-*

face:

$$
\begin{aligned}
0 = &(4l^4)z_1^2z_2^2 + (-16l^2f)z_1z_2^3 + (32hl^2+32g)z_2^4 + (8l^2f)z_1^2z_2z_3 \\
&+ (-16hl^2-32g)z_1z_2^2z_3 + (8g)z_1^2z_3^2 + (-16l^2)z_2^2z_3^2 + (8l^2)z_1z_3^3 \\
&+ (-4l^4f)z_0z_1^2z_2 + (16l^2g)z_0z_1z_2^2 + (-8l^2g)z_0z_1^2z_3 + (32l^4)z_0z_2^2z_3 \\
&+ (-8l^4)z_0z_1z_3^2 + (l^4f^2)z_0^2z_1^2 + (8l^2fg)z_0^2z_1z_2 + (-32hl^2g-16l^6-32g^2)z_0^2z_2^2 \\
&+ (8hl^2g+16g^2)z_0^2z_1z_3 + (-16l^4f)z_0^2z_2z_3 + (8hl^4+8l^2g)z_0^2z_3^2 + (-8l^2g^2)z_0^3z_1 \\
&+ (16l^6f)z_0^3z_2 + (8hl^2g^2-4l^6f^2+8g^3)z_0^4.
\end{aligned}
$$

The parameters l, h, f, g denote the values of L, H, F, G. For $l = 0$ the quartic degenerates into the square of a quadric which has the equation:

$$0 = -z_2^2 + 2z_1z_3 + 2g \ .$$

Proof. This can be done by a direct computation. Since the surface is of degree 4 in $\mathbb{P}^3$ and has 16 double points, it has to be a Kummer surface. □

The intersection of the Kummer surface with the hyperplane $\{z_0 = 0$ is given by the equation:

$$
\begin{aligned}
0 = &(l^4)z_1^2z_2^2 + (-2l^2f)z_1z_2^3 + (2l^2h+2g)z_2^4 + (4l^2f)z_1^2z_2z_3 + (-4l^2h-8g)z_1z_2^2z_3 \\
&+ (8g)z_1^2z_3^2 + (-4l^2)z_2^2z_3^2 + (8l^2)z_1z_3^3 \ .
\end{aligned}
$$

This curve has geometric genus 2 and an A_1-singularity at the point $(1 : 0 : 0)$. The projection on the last two coordinates defines a $2 : 1$ cover of $\mathbb{P}^1$ ramified along the 6 zeros of the projective equation:

$$
\begin{aligned}
0 = &(2l^6h - l^4f^2 + 2l^4g)z_2^6 + (4l^4hf)z_2^5z_3 + (-4l^6 - 4l^4h^2)z_2^4z_3^2 \\
&+ (-8l^4f)z_2^3z_3^3 + (16l^4h)z_2^2z_3^4 + (-16l^4)z_3^6 \ .
\end{aligned}
$$

From the Riemann–Hurwiz formula we obtained the genus:

$$\chi(\mathcal{C}_2) = 2 - 2g = 2\chi(\mathbb{P}^1) - \deg(R) = 4 - 6 = -2 \ .$$

In the next step we want to apply Kovalevkayas method to get an embedding of the generic fibre of $(\mathcal{X}, \mathbb{F}_{\mathcal{X}})$ from a sufficient high multiple of the Painlevé–divisor D.

Proposition 29 *Up to a weighted degree of 36 a basis of $\mathcal{P}(2D)$ is given by:*

$$
\begin{aligned}
&\varphi_0 = 1 \ , \quad \varphi_1 = z_2 \ , \quad \varphi_2 = z_3 \ , \quad \varphi_3 = z_5 \ , \quad \varphi_4 = z_7 \\
&\varphi_5 = z_2z_3 + \frac{1}{2}z_3z_4 \ , \quad \varphi_6 = z_9 \ , \quad \varphi_7 = z_{10} \ , \quad \varphi_8 = z_3z_8 \\
&\varphi_9 = 2z_2z_5 + z_3z_5 + z_3z_6 \ , \quad \varphi_{10} = -z_3z_9 \ , \quad \varphi_{11} = z_5z_8 - \frac{1}{4}z_3z_{10} \\
&\varphi_{12} = z_2z_4z_5 + \frac{1}{2}z_4^2z_5 - z_2z_3z_6 - \frac{1}{2}z_3z_4z_6 - z_5z_9 \ , \quad \varphi_{13} = -z_8z_9 \\
&\varphi_{14} = -z_9^2 \ , \quad \varphi_{15} = -2z_4z_5z_8 + 2z_3z_6z_8 - z_9z_{10} \ .
\end{aligned}
$$

Proof. This can be done by a direct computation. □

Using this basis we obtain a morphism:

$$\varphi_{2D} : \mathcal{F}_c \to \mathbb{P}^{15} \ , \quad x \mapsto (\varphi_0(x) : \varphi_1(x) : \cdots : \varphi_{15}(x)),$$

which imbeds $\mathcal{F}_c$ into $\mathbb{P}^{15}$. If we set the parameters to $l = h = f = g = 1$ we are able to compute $\overline{\mathcal{F}_1} \subset \mathbb{P}^{15}$. The embedding is projectively normal. The ideal of the surface is generated by 72 quadratic equations. The degree of $\overline{\mathcal{F}_1}$ is 32. The intersection $\overline{\mathcal{F}_1} \cap \{z_0 = 0\}$ is twice a degree 16 curve of genus 4 with a double point at $z_5 = 1$ and all the other $z_i = 0$. The curve is contained in the hyperplane section $\{z_0 = z_2 = z_4 = z_6 = 0\}$. Hence φ_{2D} imbeds D in $\mathbb{P}^{11}$, which coincides with the result of Theorem 27.
If we intersect $\{z_0 = 0\}$ with $\overline{\mathcal{F}_c}$ we get twice a singular curve $\mathcal{C}$ of geometric genus 4 and arithmetic genus $p_a = 5$, it has one ordinary double point. If we project it on the coordinates z_1, z_3, z_5, z_7 we get after renaming the variables z_0, z_1, z_2, z_3 the equations for $\mathcal{C}$ in $\mathbb{P}^3$:

$$\begin{aligned}
&0 = 2z_1^2 + z_0 z_2\\
&0 = (16l^4)z_0^4 + (16hl^2 + 32g)z_0^2 z_1^2 + (8hl^2)z_0^3 z_2 + (8l^2)z_0 z_1^2 z_2 + (4l^2)z_0^2 z_2^2\\
&\quad + (4h)z_1^2 z_2^2 + (2h)z_0 z_2^3 + (8f)z_0 z_1 z_2 z_3 + (8l^2)z_0^2 z_3^2 + (-4h)z_1^2 z_3^2\\
&\quad + (2h)z_0 z_2 z_3^2 + z_2^2 z_3^2 + z_3^4
\end{aligned}$$

The coordinates of the double point are $(0 : 0 : 1 : 0)$. This curve has two extra singularities, which are a product of the projection. This curve caries a $\mathbb{Z}/2\mathbb{Z}$ symmetry induced by $\sigma : (z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : -z_1 : z_2 : -z_3)$. The quotient $\mathcal{C}' := \mathcal{C}/\langle\sigma\rangle$ is a genus 2 curve with an double point and the $2:1$ map $\mathcal{C} \to \mathcal{C}'$ is ramified along this double point. The $\mathbb{Z}/2\mathbb{Z}$ action lifts to an action on $\mathbb{P}^{15}$ by:

$$\begin{aligned}
&\tau : (\varphi_0 = 1 : \varphi_1 : \cdots : \varphi_{15}) \mapsto\\
&(\varphi_0 : \varphi_2 : -\varphi_3 : \varphi_4 : \varphi_5 : \varphi_6 : -\varphi_7 : \varphi_8 : -\varphi_9 : \varphi_{10} : -\varphi_{11} : -\varphi_{12} : \varphi_{13} : \varphi_{14} : -\varphi_{15}) \ .
\end{aligned}$$

We find that $h^0(\mathcal{L}(2\mathcal{C})^+) = 10$ and $h^0(\mathcal{L}(2\mathcal{C})^-) = 6$.

Theorem 11 *The integrable system $(\mathcal{X}, \mathbb{F}_{\mathcal{X}})$ and its subsystem $(X, \mathbb{F}_X)$ are algebraic completely integrable. The general fibre of $\mathbb{F}_{\mathcal{X}}$ is the affine part of an abelian surface $\mathcal{A}_c$ minus a curve D of geometric genus 4 having a A_1–singular point. The line bundle $\mathcal{L}(D)$ puts a polarisation of type $(2,2)$ on $\mathcal{A}$. The abelian surface $\mathcal{A}_c$ is isogenous to $Prym(\tilde{\mathcal{C}}/\tilde{\mathcal{C}}')$, where $\tilde{\mathcal{C}}$ is the normalisation of $\mathcal{C}$ and $\tilde{\mathcal{C}}'$ its quotient by the $\mathbb{Z}/2\mathbb{Z}$ action induced by $-1_{\mathcal{A}_c}$.*

Proof. In the first step we check that the vector fields χ_F and χ_G extend to holomorphic vector fields on $\overline{\varphi_{2D}(\mathcal{F}_c)}$. When we try to write the down the vector fields one realises, that $\chi_F\varphi_1 = \{F, \varphi_1\} \notin \mathbb{C}[l,h,f,g]\mathbb{C}[\varphi_0, \ldots, \varphi_{15}]$. But $G\chi_F\varphi_1 \in \mathbb{C}[l,h,f,g]\mathbb{C}[\varphi_0, \ldots, \varphi_{15}]$ which tells us, that $\chi_F\varphi_1$ lies in $\mathbb{C}(l,h,f,g)[\varphi_0, \ldots, \varphi_{15}]$. Knowing this we could check that χ_F and χ_G are holomorphic in the affine pieces $y_2 \neq 0$, $y_3 \neq 0$ and $y_4 \neq 0$. Since

$\{y_2 = 0, y_3 = 0, y_4 = 0\} \cap \overline{\mathcal{F}_c} = \emptyset$ the vector fields extend to holomorphic vector fields on $\overline{\mathcal{F}_c}$. It remains to show, that $\chi_F \wedge \chi_G$ vanishes nowhere on $\overline{\mathcal{F}_c}$. This can be done by proving the flow–through property using theorem 5. Using this embedding we are able to compute the equations of $D = \overline{\mathcal{F}_c} \setminus \mathcal{F}_c$ and find that D has a A_1 singularity and the geometric genus 4, hence it has arithmetic genus $p_a = 5$. Since $h^0(\mathcal{L}(D)) = p_a - 1 = 4 = \delta_1\delta_2$ the polarisation is of typ $(1, 4)$ or $(2, 2)$. As the image in $\mathbb{P}^3$ is a Kummer surface the polarisation has to be of type $(2, 2)$. The action of τ is induced by the -1–action on $\mathcal{A}_c$ since $\tau_*\chi_H = -\chi_H$ and $\tau_*\chi_G = -\chi_G$. From remark 27 we obtain the last statement. □

Since $(\mathbb{C}^8, \mathbb{F}_4)$ admits a $(\mathbb{C}^*)^2$–action, it is an interesting candidate for an generalised algebraic integrable system. We have already proved in the last theorem, that the $(\mathbb{C}^*)^2$ quotient of $(\mathbb{C}^8, \mathbb{F}_4)$ is algebraic completely integrable, hence we can use theorem 4 to verify that $(\mathbb{C}^8, \mathbb{F}_4)$ is g.a.c.i..

Theorem 12 *The integrable system* $(\mathbb{C}^8, \mathbb{F}_4)$ *is generalised algebraic completely integrable. The general fibre of* $\mathbb{F}_4$ *is the affine part of an abelian algebraic group* $\mathcal{G}$ *which is a group extension of the principal polarised abelian surface, whose affine piece is the general fibre of* $\mathbb{F}_{\mathcal{X}}$ *by* $(\mathbb{C}^*)^2$*:*

$$0 \to (\mathbb{C}^*)^2 \to \mathcal{G} \to \mathcal{A} \to 0.$$

The affine piece is the preimage of $\mathcal{A} \setminus D$ *in* $\mathcal{G}$*.*

Proof. Let us denote by π the map $\pi : \mathbb{F}_4^{-1}(c) \to \mathbb{F}_{\mathcal{X}}^{-1}(c)$. We get from the embedding $\varphi_{2D} : \mathbb{F}_{\mathcal{X}}^{-1}(c) \hookrightarrow \mathbb{P}^{15}$ a map $\varphi_{\pi^*(\mathcal{L}(2D))} : \mathbb{F}_4^{-1}(c) \to \mathbb{P}^{15}$ which we can extend to an embedding of $\mathbb{F}_4^{-1}(c)$ in $\mathbb{P}^{15} \times \mathbb{P}^8$ by adding the holomorphic sections

$$\begin{aligned}
&\psi_1 := q_1 \ , \quad \psi_2 := q_2 \ , \quad \psi_3 := q_3 \ , \quad \psi_4 := q_4 \ , \\
&\psi_5 := p_1 \ , \quad \psi_6 := p_2 \ , \quad \psi_7 := p_3 \ , \quad \psi_8 := p_4 \in H^0(\mathcal{F}_c, \mathcal{O}_{\mathcal{F}_c}) \ ,
\end{aligned}$$

and we obtain the commutative diagram:

$$\begin{array}{ccc}
\mathbb{F}_4 & \longrightarrow & \mathbb{P}^{15} \times \mathbb{P}^8 \\
\downarrow & & \downarrow \\
\mathbb{F}_{\mathcal{X}} & \hookrightarrow & \mathbb{P}^{15}
\end{array} .$$

With the use of theorem 4 we can complete the proof. □

The Duistermaat–system yields as generic fibre a $\mathbb{Z}/2\mathbb{Z}$ cyclic cover of a $\mathbb{C}^*$–extension of an abelian surface. This is an example of a type of algebraic integrable systems that has not been discussed or observed before. In analogy to the almost algebraic integrable system we define:

Definition 15 Almost generalised algebraic integrable A Lagrangian fibration $f : X \to B$ is called *almost generalised algebraic integrable* (a.g.a.i.) if it yields the action of a finite cyclic group Σ such that the induced quotient $\tilde{f} : X/\Sigma \to B$ is generalised algebraic completely integrable.

As a consequence we get:

Theorem 13 *(i) The integrable systems $(\mathcal{X}, \mathbb{F}_{\mathcal{X}})$ and $(X, \mathbb{F}_X)$ are a.c.i..*

(ii) The integrable system $(Y_2, \mathbb{F}_{Y_2})$ is a.a.i..

(iii) The integrable systems $(\mathbb{C}^8, \mathbb{F}_4)$, $(\mathcal{Y}_1, \mathbb{F}_{\mathcal{Y}_1})$, $(\mathcal{Y}_2, \mathbb{F}_{\mathcal{Y}_2})$ and $(Y_1, \mathbb{F}_{Y_1})$ are g.a.c.i..

(iv) The Duistermaat–ystem $(\mathbb{C}^6, \mathbb{F}_3)$ is a.g.a.i..

Proof. We proved (i) in Theorem 11. (ii) follows direct from the definition since the general fibre of $\mathbb{F}_{Y_2}$ is a cyclic cover of the general fibre of the general fibre $\mathbb{F}_X$. The first system of the *(iii)* is the result of Theorem 12. The systems $(\mathcal{Y}_1, \mathbb{F}_{\mathcal{Y}_1})$ and $(\mathcal{Y}_2, \mathbb{F}_{\mathcal{Y}_2})$ are $\mathbb{C}^*$–quotients of $(\mathbb{C}^8, \mathbb{F}_4)$ and hence generalised algebraic integrable. The system $(Y_1, \mathbb{F}_{Y_1})$ is a subfamily of $(\mathcal{Y}_1, \mathbb{F}_{\mathcal{Y}_1})$ and is thus also generalised algebraic integrable. The last point is almost generalised integrable since its $\mathbb{Z}/2\mathbb{Z}$ quotient is generalised algebraic integrable. □

One interesting point about the last theorem is, that we got an example of an almost generalised integrable system and that in analogy to the almost algebraic integrable system we find an embedding in an generalised algebraic integrable system. The singular genus 2 curve defining the family of Prym varieties, which are isogenous to the family of abelian surfaces from theorem 11, gets extra singularities if the parameters (l, h, f, g) lie on the threefold:

$$\begin{aligned}
0 = {} & 128l^{18}h - 64l^{16}h^3 + 8l^{14}h^5 - 64l^{16}f^2 - 1200l^{14}h^2f^2 + 528l^{12}h^4f^2 \\
& - 64l^{10}h^6f^2 + 2106l^{12}hf^4 + 432l^{10}h^3f^4 + 128l^{16}g + 1088l^{14}h^2g \\
& - 536l^{12}h^4g + 64l^{10}h^6g - 729l^{10}f^6 - 5760l^{12}hf^2g + 504l^{10}h^3f^2g \\
& + 3402l^{10}f^4g + 2880l^{12}hg^2 + 1616l^{10}h^3g^2 - 1088l^8h^5g^2 + 128l^6h^7g^2 \\
& - 4752l^{10}f^2g^2 - 4104l^8h^2f^2g^2 - 864l^6h^4f^2g^2 + 1458l^6hf^4g^2 \\
& + 1728l^{10}g^3 + 9936l^8h^2g^3 - 2880l^6h^4g^3 + 128l^4h^6g^3 - 9720l^6hf^2g^3 \\
& - 864l^4h^3f^2g^3 + 1458l^4f^4g^3 + 13608l^6hg^4 - 1728l^4h^3g^4 - 5832l^4f^2g^4 \\
& + 5832l^4g^5 \ .
\end{aligned}$$

We were not able to compute the discriminant of the momentum map $\mathbb{F}_4$ but we think that they should coincide.

5 Grammaticos integrable potential of degree 4

Dorizzi, Grammaticos, and Ramani [DGR82] discovered, that besides their four families of integrable systems, there is a further example of a homogenous potential in two variables of degree 4, defined by the Hamilton function:

$$H := \frac{1}{2}(p_1^2 + p_2^2) + q_1^4 + \frac{3}{4}q_1^2 q_2^2 + \frac{1}{8}q_2^4 \ . \tag{5.1}$$

Its second integral reads as:

$$G := p_2^4 + \frac{1}{2}q_2^4 p_1^2 - 2q_1 q_2^3 p_1 p_2 + \left(3q_1^2 q_2^2 + \frac{1}{2}q_2^4\right) p_2^2 + \frac{1}{4}q_1^4 q_2^4 + \frac{1}{4}q_1^2 q_2^6 + \frac{1}{16}q_2^8 \ .$$

Proposition 30 *The fibre of the momentum map is smooth away from the discriminant:*

$$\Delta = \{(4h^2 - g)g = 0\} \ .$$

Proof. This can be done by a direct computation. □

This system has been studied by Jacob Baltuch in his PhD thesis [Bal92]. He proved that the generic fibre of the momentum map completes in to a ramified cover of two elliptic curves $\mathcal{F} \times \mathcal{E}$ and that the Hamilton vector field χ_H can be integrated in terms of elliptic functions. In section 5.3 we show, that this integrable system is closely related to an integrable system on $\mathbb{C}^6$ defining a deformation of this one.

5.1 Laurent series solutions to χ_H

The first step to analyse the complex geometry of the generic momentum map fibre is to compute all principal balances to the flow of χ_H.

Proposition 31 *Kovalevkayas method yields two principal balances with a zero dimensional initial locus and the positive Kovalevskaya exponents* $\{\frac{1}{2}, \frac{5}{2}, 4\}$. *We obtain the two* *following Laurent series solution for $\varepsilon = \pm\frac{1}{\sqrt{-2}}$:*

$$
\begin{aligned}
q_1(t) &:= \varepsilon\frac{1}{t} + \varepsilon\gamma_1^2 + \varepsilon\gamma_1^4 t - \left(\frac{1}{8}\varepsilon\gamma_1^6 + \varepsilon\gamma_1\gamma_2\right)t^2 + \frac{1}{3}\gamma_3 t^3 + \dots \\
q_2(t) &:= -2\gamma_1\frac{1}{\sqrt{t}} - \gamma_1^3\sqrt{t} + \frac{2}{3}\gamma_2\sqrt{t}^3 - \frac{5}{8}\gamma_1^7\sqrt{t}^5 + \dots \\
p_1(t) &:= -\varepsilon\frac{1}{t^2} + \varepsilon\gamma_1^4 - (\frac{1}{4}\varepsilon\gamma_1^6 + 2\varepsilon\gamma_1\gamma_2)t + \gamma_3 t^2 + \dots \\
p_2(t) &:= +\gamma_1\frac{1}{\sqrt{t}^3} - \frac{1}{2}\gamma_1^3\frac{1}{\sqrt{t}} + \varepsilon\sqrt{t} - \frac{25}{16}\gamma_1^7\sqrt{t}^3 + \dots \ .
\end{aligned}
$$

Proof. Only two points of the initial locus yield enough positive *rational* Kovalevskaya–exponents namely $\{\frac{1}{2}, \frac{5}{2}, 4-1\}$. From this points we obtain the Laurent–series above for the two square roots of -2. □

5.2 The quotient system

The $\mathbb{Z}/2\mathbb{Z}$–action on the Laurent series of exchanging $\sqrt{t}$ with $-\sqrt{t}$ is induced by the $\mathbb{Z}/2\mathbb{Z}$–action $\sigma : (q_1, q_2, p_1, p_2) \mapsto (q_1, -q_2, p_1, -p_2)$.

Proposition 32 *The quotient $\mathbb{C}^4/\langle\sigma\rangle$ is obtained from:*

$$\mathbb{C}[q_1, q_2, p_1, p_2]^\sigma \simeq \mathbb{C}[q, p, u, w, v]/(uv - w^2) \ .$$

Proof. We obtain the isomorphism of the rings from:

$$q := q_1, \qquad p := p_1 \ , \qquad u := q_2^2 \ , \qquad w := q_2 p_2 \ , \qquad v := p_2^2 \ .$$

□

Substituting the Laurent series in the new variables yields:

$$
\begin{aligned}
q(t) &:= \varepsilon\frac{1}{t} + \varepsilon\gamma_1^2 + \varepsilon\gamma_1^4 t - \left(\frac{1}{8}\varepsilon\gamma_1^6 + \varepsilon\gamma_1\gamma_2\right)t^2 + \frac{1}{3}\gamma_3 t^3 + \dots \\
p(t) &:= -\varepsilon\frac{1}{t^2} + \varepsilon\gamma_1^4 - \left(\frac{1}{4}\varepsilon\gamma_1^6 + 2\varepsilon\gamma_1\gamma_2\right)t + \gamma_3 t^2 + \dots \\
u(t) &:= 4\gamma_1^2\frac{1}{t} + 4\gamma_1^4 + \left(\gamma_1^6 - \frac{8}{3}\gamma_1\gamma_2\right)t + \left(\frac{5}{2}\gamma_1^8 - \frac{4}{3}\gamma_1^3\gamma_2\right)t^2 + \dots \\
w(t) &:= -2\gamma_1^2\frac{1}{t^2} + \left(\frac{1}{2}\gamma_1^6 - \frac{4}{3}\gamma_1\gamma_2\right) + \left(\frac{5}{2}\gamma_1^8 - \frac{4}{3}\gamma_1^3\gamma_2\right)t + \dots \\
v(t) &:= \gamma_1^2\frac{1}{t^3} - \gamma_1^4\frac{1}{t^2} + \left(\frac{1}{4}\gamma_1^6 + 2\gamma_1\gamma_2\right)\frac{1}{t} - \left(\frac{25}{8}\gamma_1^8 + \gamma_1^3\gamma_2\right) + \dots \ .
\end{aligned}
$$

We observe that the Laurent series for the new variables depend only on γ_1^2, $\gamma_1\gamma_2$, γ_2^2, and γ_3.

5.3 Embedding in an integrable system on $\mathbb{C}^6$

The embedding of $\mathbb{C}^4$ in $\mathbb{C}^6$ defined by $(q_1, q_2, p_1, p_2) \mapsto (q_1, q_2, 0, p_1, p_2, 0)$ induces an embedding of the integrable system on $\mathbb{C}^4$ in an integrable system on $\mathbb{C}^6$. This embedding is again no embedding of integrable systems, it is even no Poisson–morphism. The integrable system on $\mathbb{C}^6$ is defined by:

$$\begin{aligned}
H &:= \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + q_1^4 + \frac{3}{4}q_1^2(q_2^2 + q_3^2) + \frac{1}{8}(q_2^2 + q_3^2)^2 \\
G &:= \frac{1}{4}q_1^4(q_2^2 + q_3^2)^2 + \frac{1}{4}q_1^2(q_2^2 + q_3^2)^3 - 2q_1^2(q_3p_2 - q_2p_3)^2 \\
&\quad + 3q_1^2(q_2^2 + q_3^2)(p_2^2 + p_3^2) - 2q_1p_1(q_2^2 + q_3^2)(q_2p_2 + q_3p_3) \\
&\quad + \frac{1}{2}p_1^2(q_2^2 + q_3^2)^2 + \frac{1}{16}(q_2^2 + q_3^2)^4 \\
&\quad + \frac{1}{2}(q_2^2 + q_3^2)^2(p_2^2 + p_3^2) + (p_2^2 + p_3^2)^2 \\
L &:= q_3p_2 - q_2p_3 \ .
\end{aligned}$$

The vector field χ_L induces a $\mathbb{C}^*$–action on the fibre of the momentum map. Hence this system is a candidate for an generalised algebraic completely integrable system.

Proposition 33 *The integrable system $(\mathbb{C}^6, \mathbb{F}_3)$ induces an integrable system on the quotient $X := \mathbb{C}^6//\mathbb{C}^*$ defined by:*

$$\begin{aligned}
\mathcal{O}_X(X) &:= \mathbb{C}[q_q, q_2, q_3, p_1, p_2, p_3]^{\mathbb{C}^*} = \left\{ f \in \mathbb{C}[q_q, q_2, q_3, p_1, p_2, p_3] \mid \chi_L.f = 0 \right\} \\
&= \mathbb{C}[Q, P, U, W, L, V]/(W^2 + L^2 - UV) \ .
\end{aligned}$$

The constants of motion read in the capital letter coordinates:

$$\begin{aligned}
H &= Q^4 + \frac{3}{4}Q^2U + \frac{1}{2}P^2 + \frac{1}{8}U^2 + \frac{1}{2}V \\
G &= \frac{1}{4}Q^4U^2 + \frac{1}{4}Q^2U^3 + \frac{1}{2}P^2U^2 + \frac{1}{16}U^4 - 2QPUW - 2Q^2L^2 + 3Q^2UV + \frac{1}{2}U^2V + V^2 \\
F &= L \ .
\end{aligned}$$

Proof. We obtain the isomorphism of rings from:

$$Q := q_1 \ , \ P := p_1 \ , \ U := q_2^2 + q_3^2 \ , \ W := q_2p_2 + q_3p_3 \ , \ L := q_3p_2 - q_2p_3 \ .$$

□

From the $\mathbb{C}^*$ and the $\mathbb{Z}/2\mathbb{Z}$–quotients we obtain the commutative diagram:

$$\begin{array}{ccc}
(\mathbb{C}^4, \mathbb{F}_2) & \hookrightarrow & (\mathbb{C}^6, \mathbb{F}_3) \\
\downarrow{\scriptstyle 2:1} & & \downarrow{\scriptstyle \mathbb{C}^*} \\
(X, \mathbb{F}_X) & \hookrightarrow & (\mathcal{X}, \mathbb{F}_{\mathcal{X}})
\end{array} \ .$$

The embedding of $(\mathbb{C}^4, \mathbb{F}_2)$ in $(\mathbb{C}^6, \mathbb{F}_3)$ is obtained from the embedding of $\mathbb{C}^4$ in $\mathbb{C}^6$ defined by:

$$i : (q_1, q_2, p_1, p_2) \mapsto (q_1, q_2, 0, p_1, p_2, 0)$$

and we obtain $\mathbb{F}_2$ from $\mathbb{F}_3$ by a projection on the first two coordinates in $\mathbb{C}^3$. The embedding is as before no Poisson–morphism. Again we obtain a deformation of the singular X from $L : \mathbb{C}$ that defines a deformation of the integrable system $(X, \mathbb{F}_X)$:

$$\begin{array}{ccccc} X & \hookrightarrow & \mathcal{X} & \hookleftarrow & \mathcal{X}_l \\ \downarrow{\scriptstyle \mathbb{F}_X} & & \downarrow{\scriptstyle \mathbb{F}_{\mathcal{X}}} & & \downarrow{\scriptstyle \mathbb{F}_{\mathcal{X}_l}} \\ \mathbb{C}^2 & \hookrightarrow & \mathbb{C}^3 & \hookleftarrow & \mathbb{C}^2 \end{array} .$$

Proposition 34 *Kovalevkayas method yields two Laurent series solutions to the flow of χ_H for $\varepsilon = \pm\frac{1}{\sqrt{-2}}$:*

$$\begin{aligned}
Q(t) &:= \frac{\varepsilon}{t} + \varepsilon\gamma_1 + \varepsilon\gamma_1^2 t - \frac{1}{2}\varepsilon\gamma_2 t^2 + \frac{1}{3}\gamma_4 t^3 + \frac{1}{9}\left(\gamma_1\gamma_4 - \varepsilon\gamma_5 - \frac{5}{2}\varepsilon\gamma_1^2\gamma_2\right) t^4 + \ldots \\
P(t) &:= -\varepsilon\frac{1}{t^2} + \varepsilon\gamma_1^2 - \varepsilon\gamma_2 t + \gamma_4 t^2 + \frac{1}{9}\left(4\gamma_1\gamma_4 - 4\varepsilon\gamma_5 - 10\varepsilon\gamma_1^2\gamma_2\right) t^3 + \ldots \\
U(t) &:= 4\gamma_1\frac{1}{t} + 4\gamma_1^2 + \frac{4}{3}\left(\gamma_1^3 - \gamma_2\right) t + \frac{2}{3}\left(+4\gamma_1^4 - \gamma_1\gamma_2\right) t^2 + \frac{4}{9}\left(\gamma_5 - 4\varepsilon\gamma_1\gamma_4 - 2\gamma_1^2\gamma_2\right) t^3 + \ldots \\
W(t) &:= -2\gamma_1\frac{1}{t^2} + \frac{2}{3}\left(\gamma_1^3 - \gamma_2\right) + \frac{2}{3}\left(4\gamma_1^4 - \gamma_1\gamma_2\right) t + \frac{2}{3}\left(\gamma_5 - 4\varepsilon\gamma_1\gamma_4 - 2\gamma_1^2\gamma_2\right) t^2 + \ldots \\
L(t) &:= \gamma_3 \\
V(t) &:= \gamma_1\frac{1}{t^3} - \gamma_1^2\frac{1}{t^2} + \gamma_2\frac{1}{t} - \left(3\gamma_1^4 + \frac{1}{2}\gamma_1\gamma_2\right) + \gamma_5 t + \ldots \ .
\end{aligned}$$

Proof. We find two zero dimensional components of the initial locus that yield 5 positive integer Kovalevskaya exponents $\{2, 4, 5, 5, 6\}$. □

Proposition 35 *The Laurent series yields two Painlevé–divisors $\mathcal{C}_+$ and $\mathcal{C}_-$ for $\varepsilon = \pm\frac{1}{\sqrt{-2}}$ having both the plane affine equation:*

$$0 = 2x^9 + c_1x^5 - c_3^2x^3 + c_2x + \left(3x^6 + c_1x^2 - c_3^2\right) y - y^3 \ . \tag{5.2}$$

The genus of the Painlevé–curves is 3.

Proof. Substituting this series in to the constants of motion yields after rescaling the variables:

$$\begin{aligned}
H &= -\frac{19}{6}\gamma_1^4 - \frac{1}{4}\gamma_1\gamma_2 - \frac{5}{3}\varepsilon\gamma_4 \\
G &= -\frac{64}{153}\gamma_1^5\gamma_2 - \frac{2048}{459}\varepsilon\gamma_1^4\gamma_4 - \frac{164}{51}\gamma_1^2\gamma_2^2 - \frac{32}{17}\gamma_1^2\gamma_3^2 + \frac{328}{51}\gamma_1^3\gamma_5 + \frac{64}{9}\varepsilon\gamma_1\gamma_2\gamma_4 + \gamma_2\gamma_5 \\
F &= \gamma_3 \ .
\end{aligned}$$

The equation $0 = W^2 + L^2 - UV$ reads as:

$$0 = \frac{136}{9}\gamma_1^6 + 4\gamma_1^3\gamma_2 + \frac{112}{9}\varepsilon\gamma_1^2\gamma_4 + \gamma_2^2 + \gamma_3^2 - 2\gamma_1\gamma_5 \ .$$

If we set H, G, and F to the generic values h, g, and f we obtain the equation of the two isomorphic Pailevé curves $\mathcal{C}_+$ and $\mathcal{C}_-$ for $\varepsilon = \pm\frac{1}{\sqrt{-2}}$. After eliminating γ_3, γ_4, and γ_5 we set $x := 2\gamma_1$ and $y := 3\Gamma_2$ and obtain the new parameters:

$$c_1 := 36h \ , \quad c_2 = 27g \ , \quad c_3 = 3f \ .$$

The projection on x induces a $3:1$ cover of $\mathbb{P}^1$ ramified along the zeros of the equation:

$$\begin{aligned} 0 &= (9c_1^2 - 108c_2)x^{10} - 18c_1c_3^2x^8 + (9c_3^4 + 4c_1^3 - 54c_1c_2)x^6 \\ &+ (54c_2c_3^2 - 12c_1^2c_3^2)x^4 + (12c_1c_3^4 - 27c_2^2)x^2 - 4c_3^6 \ . \end{aligned}$$

As the equation depends only on X^2 the zeros are the square roots of the zeros of the equation for $z := x^2$:

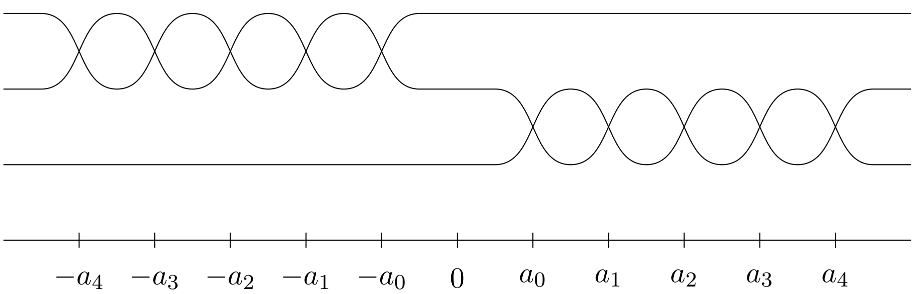
$$0 = (9c_1^2 - 108c_2)z^5 - 18c_1c_3^2z^4 + (9c_3^4 + 4c_1^3 - 54c_1c_2)z^3 \qquad (5.3)$$
$$+ (54c_2c_3^2 - 12c_1^2c_3^2)z^2 + (12c_1c_3^4 - 27c_2^2)z - 4c_3^6 \ . \qquad (5.4)$$

The geometric genus of $\mathcal{C}_+$ respectively $\mathcal{C}_-$ is 3 as:

$$2 - 2g(\mathcal{C}_+) = \chi(\mathcal{C}_+) = 3\chi(\mathbb{P}^1) - \#R = 6 - 10 = -4 \ .$$

□

The ramification of the $3:1$ cover of the Painlevé–curves of $\mathbb{P}^1$ is indicated below:



,

where a_i are the square roots of the zeros of equation (5.3). From the Laurent series solution of the flow of χ_H we obtain a basis of $\mathcal{P}(\mathcal{C}_+ + \mathcal{C}_-)$:

$$\begin{aligned}
\varphi_1 &:= Q \\
\varphi_2 &:= U \\
\varphi_3 &:= PU - 2QW \\
\varphi_4 &:= 2Q^4 + P^2 + Q^2U \\
\varphi_5 &:= 2Q^5 + QP^2 + 2Q^3U + \frac{1}{2}QU^2 + PW \\
\varphi_6 &:= \frac{1}{4}P^2U - QPW + Q^2V \\
\varphi_7 &:= -\frac{1}{2}Q^2PU + 2Q^3W + \frac{1}{2}QUW + PV \ .
\end{aligned}$$

Proposition 36 *The image of the general fibre under the morphism $\varphi_{\mathcal{L}(\mathcal{C}_+ + \mathcal{C}_-)}$ in $\mathbb{P}^7$ is the smooth degree 16 surface generated by the 7 quadric and 2 quartic equations:*

$$\begin{aligned}
0 &= (4h^2 - g)z_0z_2 - 4l^2z_0z_4 + 8hz_1z_5 - 4z_5^2 - 4z_4z_6 + 2z_3z_7 \\
0 &= (16h^3 - 4hg)z_0^2 - 16h^2z_0z_4 + l^2z_2z_4 + 4hz_4^2 - 4l^2z_1z_5 - 4l^2z_0z_6 + (8h)z_2z_6 \\
0 &= (8h^2 - 2g)z_0^2 + z_3^2 - 8hz_0z_4 + 2z_4^2 \\
0 &= (4h^2 - g)z_2^2 - 8gz_0z_4 + 16hz_2z_6 + 16z_6^2 + 8z_7^2 \\
0 &= 4gz_1^2 + (-4h^2 + g)z_0z_2 + 2hz_2z_4 - 8hz_0z_6 + 4z_4z_6 + 2z_3z_7 \\
0 &= (32h^4 - 16h^2g + 2g^2)z_0^2 + (4h^2l^2 - l^2g)z_0z_2 + (-32h^3 + 8hg)z_0z_4 \\
&\quad + (8h^2 - 2g)z_4^2 - 8hl^2z_1z_5 + (16h^2 - 4g)z_2z_6 - 4l^2z_4z_6 - 2l^2z_3z_7 \\
0 &= (-8h^3 + 2hg)z_0^2z_1z_3 + (4h^2 - g)z_0z_1z_3z_4 + (4h^2 - g)z_0^2z_3z_5 \\
&\quad - 2hz_0z_3z_4z_5 + 2l^2z_0z_1z_3z_6 + z_2z_3z_5z_6 + 4z_1z_3z_6^2 \\
&\quad + 2l^2z_0z_1z_4z_7 + 4z_1z_4z_6z_7 - 4z_0z_5z_6z_7 \\
0 &= (32h^4 - 16h^2g + 2g^2)z_0^2z_1z_3 + (4h^2l^2 - l^2g)z_0z_1z_2z_3 \\
&\quad + (-16h^3 + 4hg)z_0z_1z_3z_4 + (-16h^3 + 4hg)z_0^2z_3z_5 + (8h^2 + 2g)z_0z_3z_4z_5 \\
&\quad + (8h^2 - 2g)z_1z_2z_3z_6 - 8hz_2z_3z_5z_6 - 8z_3z_5z_6^2 \\
&\quad + (16h^2l^2 - 4l^2g)z_0^2z_1z_7 + (-32h^3 + 8hg)z_0z_1z_2z_7 \\
&\quad + (8h^2 - 2g)z_1z_2z_4z_7 + (8h^2 - 2g)z_0z_2z_5z_7 - 4z_3z_5z_7^2 \ .
\end{aligned}$$

Proof. This can be done by a direct computation using SINGULAR. □

We also computed a basis for $\mathcal{P}(2\mathcal{C}_+ + 2\mathcal{C}_-)$. As this vector space should be isomorphic to the global sections of a line bundle inducing a $(4, 8)$ polarisation $\dim(\mathcal{P}(2\mathcal{C}_+ + 2\mathcal{C}_-)) = 4.8 = 32$. The basis is written down in section 6.6.

5.4 Algebraic integrability

With the use of the global sections of the $(4,8)$–polarisation we are able to prove the algebraic complete integrability of the system $(\mathcal{X}, \mathbb{F}_{\mathcal{X}})$. The global sections of the $(4,8)$–polarisation defines an embedding of the generic fibre in $\mathbb{P}^{31}$. Its image has the degree:

$$(4\mathcal{C}).(4\mathcal{C}) = 4.4.\mathcal{C}.\mathcal{C} = 16.(2p_a - 2) = 64 \ .$$

This point can be verified using SINGULAR for the special choice $H = 1$, $G = 1$, and $F = 1$. Furthermore we obtain that the ideal is generated by 400 quadratic equations and the intersection with $z_0 = 0$ defines twice the two curves $\mathcal{C}_+$ and $\mathcal{C}_-$ which yields that the degree of $\mathcal{C}_+$ and $\mathcal{C}_-$ in $\mathbb{P}^{31}$ equals 16. Using this we obtain:

$$h^0(\mathcal{L}(2\mathcal{C}_+ + 2\mathcal{C}_-)|_{\mathcal{C}_\pm}) = \deg(\mathcal{L}(2\mathcal{C}_+ + 2\mathcal{C}_-)|_{\mathcal{C}_\pm}) + p_a - 1 = 16 + 3 - 1 = 18 \ .$$

We expect that $\mathcal{C}_+$ respectively $\mathcal{C}_-$ are contained in a 17 dimensional linear subspace. At least both curves are contained in the linear subspace defined by $z_0 = 0$ $z_1 = 0$, $z_4 = 0$, $z_7 = 0$, and $z_9 = 0$.
If we intersect the image of the general fibre in $\mathbb{P}^7$ with the hyperplane $z_0 = 0$ we obtain two smooth curves of genus 3 intersecting each other in 4 different points, as to be expected since:

$$\frac{1}{2}(\mathcal{C}_+ + \mathcal{C}_-).(\mathcal{C}_+ + \mathcal{C}_-) = \frac{1}{2}\left(\mathcal{C}_+^2 + \mathcal{C}_-^2 + 2\mathcal{C}_+.\mathcal{C}_-\right) = \frac{1}{2}\left(4 + 4 + 2\mathcal{C}_+.\mathcal{C}_-\right) = 2.4 = 8 \ ,$$

and hence:

$$\mathcal{C}_+.\mathcal{C}_- = 4 \ .$$

We expect that the general fibre is the affine part of a $(1,2)$–polarised abelian surface so it could be the Prym variety of a two to one cover of a genus 3 curve with an elliptic curve. Therefore we try to find the -1 action on the surface. Searching for a -1 action on the parameters induced by the action of mapping t to $-t$ is always a good point to start. We obtain the following $\mathbb{Z}/2\mathbb{Z}$ action on the parameter of the Laurent series solution:

$$\tau : (t, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) \mapsto (-t, -\gamma_1, -\gamma_2, \gamma_3, \gamma_4, -\gamma_5) \ .$$

The quotients of the affine Painlevé curves $\mathcal{C}_\pm$ are elliptic curves. The action of τ extends to the action on the variables:

$$\tau : (Q, P, U, W, L, V) \mapsto (-Q, P, U, -W, L, V) \ .$$

As τ maps χ_H and χ_G to $-\chi_H$ and $-\chi_G$ it has to be the action of -1 if the general fibre completes in to an abelian surface. The action of τ lifts to the embedding in $\mathbb{P}^7$:

$$\tau : (z_0 : z_1 : z_2 : z_3 : z_4 : z_5 : z_6 : z_7) \mapsto (z_0 : -z_1 : -z_2 : z_3 : z_4 : -z_5 : z_6 : z_7)$$

and we observe, that the quotient of the curves $\mathcal{C}_+$ and $\mathcal{C}_-$ by the τ action in $\mathbb{P}^7$ are smooth elliptic curves.

Theorem 14 *The integrable system* $(\mathcal{X}, \mathbb{F}_{\mathcal{X}})$ *is algebraic completely integrable. The general fibre is isomorphic to* $Prym(\mathcal{C}_+/\mathcal{C}'_+)\setminus(\mathcal{C}_+ + \mathcal{C}_-)$. *Where* $\mathcal{C}_+$ *and* $\mathcal{C}_-$ *are translates of each other and* $\mathcal{C}'_+$ *denotes the elliptic curve that is obtained from the quotient by the* -1*–action on the abelian surface.*

Proof. The $(4, 8)$–polarisation defined by the divisor $2\mathcal{C}_+ + 2\mathcal{C}_-$ defines a projectively normal embedding in $\mathbb{P}^{31}$. A Singular computation shows that χ_H and χ_G extend to the completion of the fibre in $\mathbb{P}^{31}$ and that χ_H is transversal at the general point of the divisor that compactifies the generic fibre. Furthermore χ_H vanishes nowhere on the completion and thus by Proposition 4 and Theorem 5 the completion of the generic fibre is an abelian surface. τ defines a $\mathbb{Z}/2\mathbb{Z}$–action on the abelian surface and since τ maps the vector fields $(\chi_H, \chi_G) \mapsto (-\chi_H, -\chi_G)$ it defines the -1 action of the abelian surface for a proper choice of the neutral element 0. From theorem 2 we obtain that the abelian surface is isomorphic to the Prym of the double cover $\mathcal{C} \to \mathcal{C}'$. □

As a consequence we obtain the following theorem.

Theorem 15 *The integrable system* $(\mathbb{C}^6, \mathbb{F}_3)$ *is generalised algebraic integrable. Its generic fibre is the affine part of an abelian algebraic group that is a* $\mathbb{C}^*$ *extension of the abelian surface of theorem 14.*

Proof. The embedding of the $\mathbb{C}^*$ quotient of the general fibre in $\mathbb{P}^{31}$ via the $(4, 8)$–polarisation can be extended to an embedding of the general fibre in $\mathbb{P}^{31} \times \mathbb{P}^6$ by adding the linear coordinates $q_1, q_2, q_3, p_1, p_2, p_3$ to the global sections $\varphi_1, \ldots, \varphi_{31}$

$$\begin{array}{ccc} \mathcal{F}_c & \hookrightarrow & \mathbb{P}^{31} \times \mathbb{P}^6 \\ \downarrow & & \downarrow \\ \mathcal{F}_c/\mathbb{C}^* & \hookrightarrow & \mathbb{P}^{31} \end{array} .$$

Using Singular the proof can be completed by verifying the conditions of Theorem 4. □

Again like in the Henon-Heilés $\varepsilon = 16$ example we started with an almost algebraic integrable system where the generic fibre completes in a $\mathbb{Z}/2\mathbb{Z}$ cyclic cover of an abelian surface. We obtain a generalised algebraic integrable system from an embedding in a higher dimensional integrable system which is not an embedding of integrable systems. After quoting out the $\mathbb{C}^*$–action we ended up with an integrable system having an extra parameter. This system can be thought of as a deformation of the original integrable system. The casimir $F : \mathcal{X} \to \mathbb{C}$ defines a family of integrable systems and $(X, \mathbb{F}_X)$ is obtained as $F^{-1}(0)$.

6 Appendix

6.1 Non algebraic integrable systems

In this section we want to illustrate which cases of integrable systems may occur beside the four nice cases we studied before. A class of very simple integrable systems is obtained from polynomials on $\mathbb{C}^2$ together with the standard Poisson–structure. Here we deal with systems having only one degree of freedom and as the rank of the Poisson–bracket is 2 every non–constant polynomial defines an integrable system. We consider a Hamiltonian function of the type:

$$H = \frac{1}{2}p^2 + F(q) \ ,$$

where F is a square free polynomial of degree d. If $d < 3$ the general fibre of H defines a rational curve and the flow of the Hamiltonian vector field χ_H induces a linear action on the fibre. This is the simplest case of a generalised algebraic completely integrable system. For $d = 3$ and $d = 4$ the generic fibre of H is the affine part of an elliptic curve and χ_H extends to a global vector field on the completion. If $F(q)$ has only even exponents of q and is of degree $d = 6$ or $d = 8$ the generic fibre completes in to a $\mathbb{Z}/2\mathbb{Z}$–cyclic cover of an elliptic curve and we obtain an almost algebraic integrable system. An almost generalised algebraic integrable system is obtained if F is of degree 4 and obtains only even exponents of q.

In general if the degree of F is greater than 4 the general fibre is a curve of genus 2 or higher. If this curve yields no cyclic cover of an elliptic curve or $\mathbb{P}^1$ we have an example that fits in none of the four classes of algebraic integrable systems.

In this case the vector field do not extend to a vector field of the completion. The next example is a analogue to this behaviour for an integrable system on $\mathbb{C}^4$.

Example 2 We take the degree 5 case of the Hénon-Heiles hierarchy which is given by the two polynomials:

$$H := \frac{1}{2}(p_1^2 + p_2^2) + 3q_1^4 q_2 + 16q_1^2 q_2^3 + 16q_2^5$$
$$G := \frac{1}{2}q_1^6 + 6q_1^4 q_2^2 + 8q_1^2 q_2^4 + q_1 p_1 p_2 - q_2 p_1^2.$$

Pol Vanhaecke [Van96] showed, that the generic fibre completes in to the Theta divisor of the Jacobian of a hyperelliptic genus 3 curve.

An other possibility of not being algebraic integrable is, that the vector fields behave in the wrong way. To illustrate this we write down a family of abelian surfaces which looks like an aci system but with meromorphic vector fields:

Example 3 Let $f : X \to B$ be a family of abelian surfaces. We denote by $\mathcal{A}_c$ the general member. Let D_c be an ample divisor on $\mathcal{A}_c$, that is a family of divisors over B which is generically ample and denote by $\mathcal{V}_1$ and $\mathcal{V}_2$ a basis of the holomorphic vector fields. Let $s \in H^0(\mathcal{O}_{\mathcal{A}_c}(D_c))$ be a meromorphic section. We set $s_1 = \mathcal{V}_1 s$ and $s_2 = \mathcal{V}_2 s$, and $\tilde{\mathcal{V}}_1 = s_1 \mathcal{V}_1$ and $\tilde{\mathcal{V}}_2 = s_2 \mathcal{V}_2$. Now:

$$[\tilde{V}_1, \tilde{\mathcal{V}}_2] = \mathcal{V}_1 s_2 - \mathcal{V}_2 s_1 = \mathcal{V}_1 \mathcal{V}_2 s - \mathcal{V}_2 \mathcal{V}_1 s = [\mathcal{V}_1, \mathcal{V}_2] s = 0$$

and

$$\tilde{\mathcal{V}}_1 \wedge \tilde{\mathcal{V}}_2 = s_1 s_2 \mathcal{V}_1 \wedge \mathcal{V}_2$$

hence $\tilde{\mathcal{V}}_1$ and $\tilde{\mathcal{V}}_2$ commute and their exterior product vanishes nowhere outside of $(s_1) + (s_2)$. Furthermore (s) is contained in $(s_1) + (s_2) =: D'_c$ hence D'_c is ample and there exists an very ample divisor $\tilde{D}_c$ supported on D'_c containing all components of D'_c. The image of X under the map:

$$\varphi_{\tilde{D}_c} : X \to \mathbb{P}^N$$

has the property, that on its affine piece $f : X \to B$ induces a fibration, such that the push out of the vector fields $\tilde{\mathcal{V}}_i$ is holomorphic, commutes, and their wedge product is nowhere zero.

6.2 Complex Abelian Varieties

Complex abelian varieties play a central role in the analyse of the geometry of integrable systems. The results of this section can be found in [BL04].

6.2.1 Complex Tori and Line Bundles

Definition 16 Complex Torus
A g dimensional *complex torus* is the quotient $X = V/\Lambda$ of a g dimensional vector space V by a rank $2g$ $\mathbb{Z}$ sub–module $\Lambda \subset V$ with the property $\mathbb{R} \otimes \Lambda = V$.

Since $\Lambda \simeq \mathbb{Z}^{2g}$, we obtain from any isomorphism $V \simeq \mathbb{C}^g$ a matrix $\Pi \in M(g \times 2g, \mathbb{C})$ defining the isomorphism from $\mathbb{Z}^{2g}$ to $\Lambda \subset V \simeq \mathbb{C}^g$. This matrix is called the *period matrix* of X. We can always find a basis $\{\lambda_1, \dots, \lambda_{2g}\}$ of Λ such that in the dual coordinates $\{x_1, \dots, x_{2g}\}$ the period matrix is of the form (Δ_δ, Z), where Δ_δ is a diagonal matrix having only integers on the diagonal. A g–dimensional complex torus is homeomorphic to $(\mathbb{S}^1)^{2g}$ and we must pay attention not to mix up with the g–dimensional algebraic torus $(\mathbb{C}^*)^g$ which is homotopy equivalent to $(\mathbb{S}^1)^g$.
Not all complex tori are complex varieties. Indeed, they are if and only if they carry a positive line bundle L, which means that $c_1(L)$ is non–degenerate with positive eigenvalues. Let L denote a positive line bundle on X. By a Theorem of Kronecker we can find a basis $(dx_1, \dots, dx_g, dy_1, \dots, dy_g)$ of $\Omega^1_{dR}(X)$ dual to a basis of Λ, that yields:

$$c_1(L) = \sum_{i=1}^{g} \delta_i dx_i \wedge dy_i$$

and all $\delta_i \in \mathbb{N}$, with $\delta_i|\delta_j$ for $i < j$. We call a positive line bundle with the first chern class $c_i(L)$ above on X a *polarisation of type* $(\delta_1, \ldots, \delta_g)$. A complex torus with a fixed polarisation is called an *abelian variety*, since by the Riemann relations the polarised complex tori are exactly the complex tori that are projective. We denote $NS(X)$ the Néron-Severi group of X which is the group of hermitian forms $H : V \times V \to \mathbb{C}$, such that $Im(H(\Lambda, \Lambda)) \subset \mathbb{Z}$. A *semi character* for an hermitian form $H \in NS(X)$ is a map $\chi : \Lambda \to \mathbb{S}^1 \subset \mathbb{C}$ that yields for all $\lambda, \mu \in \Lambda$ we obtain:

$$\chi(\lambda\mu) = \chi(\lambda)\chi(\mu)e^{\pi i Im(H(\lambda,\mu))} .$$

We define the group $\mathcal{P}(\Lambda)$ to be the set of pairs (H, χ), where $H \in NS(X)$ and χ is a semi character for H. The group action is defined by $(H_1, \chi_1) \circ (H_2, \chi_2) := (H_1H_2, \chi_1\chi_2)$.

Theorem 16 ***(Appell–Humbert)*** *Let $X = V/\Lambda$. There is a canonical isomorphism of exact sequences:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Hom(X) & \longrightarrow & \mathcal{P}(X) & \longrightarrow & NS(X) & \longrightarrow & 0 \\ & & \simeq\downarrow & & \simeq\downarrow & & =\downarrow & & \\ 1 & \longrightarrow & Pic^0(X) & \longrightarrow & Pic(X) & \longrightarrow & NS(X) & \longrightarrow & 0 \end{array} .$$

A direct consequence of the Appell-Humbert Theorem is $\mathcal{P}(X) \simeq Pic(X)$. For a pair $(H, \chi) \in \mathcal{P}(X)$ we denote by $L(H, \chi)$ the corresponding line bundle. As a part of the proof of the previous theorem one shows that $c_1(L(H, \chi)) = H$. Now as a consequence of the Appell-Humbert Theorem we find that for each positive line bundle L on X of type $(\delta_1, \ldots, \delta_g)$ there exists a positive line bundle M of type $(1, {}^{\delta_2}/_{\delta_1}, \ldots, {}^{\delta_g}/_{\delta_1})$ such that $L = M^{\delta_1}$. We can see this directly if we assume that $L = L(H, \chi)$, we set $M := L({}^1/_{\delta_1}H, \chi^{1/\delta_1})$ and check that $({}^1/_{\delta_1}H, \chi^{1/\delta_1}) \in \mathcal{P}(X)$, which is true since all δ_i are divisible by δ_1.
Let $f : X \to X$ be a homomorphism of a complex torus $X = V/\Lambda$. Then f is induced by a linear map $F : V \to V$, which has the property $F(\Lambda) \subset \Lambda$, and a translation $t_f : V \to V,\ v \mapsto v + f(0)$, namely $f(\overline{v}) = \overline{(F(v) + f(0))}$. Together with the Appell-Humbert-Theorem, we find:

$$f^*L(H, \chi) = L(f^*H, F^*\chi).$$

For a translation t_x by x on X we obtain the following result:

$$t_x^*L(H, \chi) = L\Big(H, \chi e^{\left(2\pi i Im(H(x,.))\right)}\Big). \tag{6.1}$$

The elements $x \in X$ for which L is translation invariant form a group:

$$K(L) := \left\{x \in X \mid t_x^*L = L\right\} .$$

Which is

$$K(L) \simeq \mathbb{Z}/\delta_1\mathbb{Z} \times \cdots \times \mathbb{Z}/\delta_g\mathbb{Z} \tag{6.2}$$

for a line bundle $L(H, \chi)$ of type $(\delta_1, \ldots, \delta_g)$, since $H = c_1(L) = \sum_{i=1}^{g} \delta_i dx_i \wedge dy_i$. Another consequence of the equation (6.1) is the following theorem:

Theorem 17 ***Theorem of the square***
Let $X = V/\Lambda$ be a complex torus, L a line bundle on X and $x, y \in X$. Then we have:

$$t^*_{x+y}L = t^*_x L \otimes t^*_y L \otimes L^{-1}.$$

Let $X = V/\Lambda$ and $X' = V'/\Lambda'$ be two complex tori. Since $X \times X'$ is again a complex torus, we can consider a holomorphic morphism $f : X \to X'$ as a holomorphic morphism $\tilde{f} : X \times X' \to X \times X'$ by $\tilde{f}(x, y) = (0, f(x))$. Such a morphism is again given by a composition of a translation and a morphism induced by a linear map $F : V \to V'$. If the map F is an isomorphism we call $f : X \to X'$ an isogeny. In general, an isogeny is not invertible. But we find for each isogeny $f : X \to X'$ there is an isogeny $g : X' \to X$ and an integer $n \in \mathbb{N}$ such that $f \circ g = e_n$, where e_n is the morphism induced by the multiplication with n on V. If we apply the Theorem of the square recursively and tensorise with L^{n-1} on both sides, for $x_1, \ldots, x_n \in X$ with $\sum_{i=1}^n x_1 = 0$ we obtain:

$$\bigoplus_{i=1}^{n} t^*_{x_i} L = L^n. \tag{6.3}$$

For a positive line bundle L on a complex torus V/Λ it is possible to write down a global section $\theta \in H^0(L)$ by giving a quasi–periodic meromorphic function on V. Via the action of $K(L)$ we are able to produce all sections in $H^0(L)$ an hence we find that $h^0(L) = ord(K(L)) = \Pi_{i=1}^g \delta_i$. A study of the L-valued harmonic forms on a complex torus proves that for positive line bundles L one has $H^i(L) = 0$ for $i > 0$. For a positive line bundle the Euler characteristic is $\chi(L) = h^0(L) = \Pi_{i=1}^g \delta_i$. Since $c_1(L) = \sum_{i=1}^g \delta_i dx_i \wedge dy_i$ one computes $L^g = \int_X c_1(L) g = g! \Pi_{i=1}^g \delta_i$, which leads to the following Theorem:

Theorem 18 ***Riemann-Roch Theorem*** *Let L be a positive line bundle on a complex torus of dimension g, then:*

$$\chi(L) = h^0(L) = \frac{1}{g!} L^g = \Pi_{i=1}^g \delta_i \ .$$

The Riemann-Roch Theorem $\chi(L) = \frac{1}{g!} L^g$ even holds for any line bundle on a complex torus, but for our purpose we need only a statement for positive line bundles and this one allows us to compute the dimension of $H^0(L)$. Let L be a positive line bundle on $X = V/\Lambda$. Then L induces a morphism $\varphi_L : X \to \mathbb{P}^{h^0(L)-1}$ by:

$$X \ni x \mapsto (\sigma_0(x) : \cdots : \sigma_n(x)) \in \mathbb{P}^n$$

where $n = h^0(L) - 1$ and $\sigma_0, \ldots, \sigma_n$ is a basis of $H^0(L)$. Property (6.3) is the main ingredient to prove the following Theorem:

Theorem 19 *Let L be a positive line bundle of type $(\delta_1, \ldots, \delta_g)$ on X. Then:*

(i) $\varphi_L : X \to \mathbb{P}^{h^0(L)-1}$ is a holomorphic map for $\delta_1 \geq 2$.

(ii) $\varphi_L : X \to \mathbb{P}^{h^0(L)-1}$ is an embedding for $\delta_1 \geq 3$.

The idea of the proof is to write:

$$L = M^{\otimes \delta_1} = t^*_{\left(\sum_{i=1}^{\delta_1 - 1} -x_i\right)} M \otimes \left(\bigotimes_{i=1}^{\delta_1 - 1} t^*_{x_i} M \right).$$

If D is a divisor such that $M = \mathcal{L}(D)$, we find:

$$\delta_1 D \sim \sum_{i=1}^{\delta_1 - 1} (D + x_i) - \left(D - \sum_{i=1}^{\delta_1 - 1} x_i \right)$$

and by arranging the divisors appropriately on X and by using some extra properties of divisors corresponding to positive line bundles one can proof the previous theorem.
Let L, L' denote two line bundles on an abelian variety X. Then the multiplication of meromorphic functions on X induces a multiplication map:

$$\mu : H^0(L) \times H^0(L') \to H^0(L \otimes L').$$

Theorem 20 *Let L denote a positive line bundle on the abelian variety X. The multiplication map*

$$\mu : H^0(L^n) \times H^0(L^m) \to H^0(L^{m+n})$$

is surjective if $n \geq 2$ and $m \geq 3$.

As a direct consequence for a positive line bundle L the embedding

$$\varphi_{L^m} : X \to \mathbb{P}^{h^0(L^m)-1}$$

is projectively normal for $m \geq 3$. Another consequence is that we get some extra information about the image of φ_{L^m}.

Theorem 21 *Let L be an ample line bundle on X, then the ideal of the image $\varphi_{L^m}(X) \in \mathbb{P}^{h^0(L^m)-1}$*

1. *is generated by quadratic equations if $m \geq 4$*
2. *is generated by quadratic and cubic equations if $m = 3$*
3. *is generated by equations of degree 2, 3 and 4 if $m = 2$.*

For the further analysis of Abelian varieties we need some properties of maps from smooth varieties into abelian varieties and sub varieties of Abelian varieties.

Theorem 22 *Let Y be a smooth variety, X an abelian variety and $f : Y \to X$ a rational map. Then f extends to all of Y.*

Using the next theorem one can give a first answer to the question, which varieties cannot be subvarieties of an abelian variety.

Proposition 37 *Any rational map $f : \mathbb{P}^n \to X$ to an abelian variety is constant.*

A direct consequence of the last proposition is, that there is no non trivial rational action on an abelian variety X, since the orbit of one point would induce a non constant rational map $\mathbb{P}^1 \to X$. If an abelian variety has an abelian variety as subvariety, there is the following result:

Theorem 23 ***Poincaré's Reducibility Theorem*** *Let (X, L) be a polarised abelian variety, $Y \subset X$ an abelian subvariety. Then X has an abelian subvariety Z such that X is isogenous to $Y \times Z$.*

The subvariety Z is unique up to the chosen polarisation L.

6.2.2 Jacobi and Prym Varieties

Let $\mathcal{C}$ be a smooth projective curve over $\mathbb{C}$ of genus g. There is a natural map $H_1(\mathcal{C}, \mathbb{Z}) \to H^0(\Omega_{\mathcal{C}})^*$ given by:

$$\gamma : H^0(\Omega_{\mathcal{C}}) \to \mathbb{C} \ , \quad \omega \mapsto \int_\gamma \omega.$$

Now we can embed $H_1(\mathcal{C}, Z)$ to $H_1(\mathcal{C}, \mathbb{C}) = H^1_{dR}(\mathcal{C})^* = H^0(\Omega_{\mathcal{C}}) \oplus \overline{H^0(\Omega_{\mathcal{C}})}$. Since the integral along the path γ is invariant under complex conjugation, the map $H_1(\mathcal{C}, \mathbb{Z}) \to H^0(\Omega_{\mathcal{C}})$ is injective. Thus $H_1(\mathcal{C}, \mathbb{Z})$ is a rank $2g$ free $\mathbb{Z}$ submodule of $H^0(\Omega_{\mathcal{C}})^*$, which is a g–dimensional vector space. We define the *Jacobian* of $\mathcal{C}$ as:

$$J(\mathcal{C}) := H^0(\Omega_{\mathcal{C}})/H_1(\mathcal{C}, \mathbb{Z}) \ ,$$

which is a complex torus. To show that $J(\mathcal{C})$ is an abelian variety we fix a basis $\lambda_1, \ldots, \lambda_{2g}$ of $H_1(\mathcal{C}, \mathbb{Z})$ with intersection matrix $\begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix}$. Now $\lambda_1, \ldots, \lambda_{2g}$ is a basis of $H^0(\Omega_{\mathcal{C}})^*$ as an $\mathbb{R}$–vector space. We denote by E the alternating form induced by $\begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix}$. Then $H(u, v) := E(iu, v) + iE(u, v)$ is an element of $NS(J(\mathcal{C}))$, which is positive. Hence the Jacobian is a polarised abelian variety of typ $(1, \ldots, 1)$. It is not obvious that $H \in NS(J(\mathcal{C}))$. One option to prove this is using the Riemann–Relations [GH94] or [BL04]. A divisor Θ that induces this polarisation, i.e. $c_1(\mathcal{L}(\Theta)) = H$ is called *theta divisor.* If we choose a divisor $D = \sum_{i=1}^N (p_i - q_i) \in Div^0(\mathcal{C})$ we obtain a map $Div^0(\mathcal{C}) \to J(\mathcal{C})$, called the *Abel–Jacobi map*:

$$D = \sum_{i=1}^N (p_i - q_i) \quad \longmapsto \quad \{\omega \mapsto \sum_{i=1}^N \int_{q_i}^{p_i} \omega\} \quad mod \quad H_1(\mathcal{C}, \mathbb{Z}).$$

The kernel of the Abel–Jacobi map consists exactly of the principal divisors in $Div^0(\mathcal{C})$, hence the Abel Jacobi map induces an isomorphism between $Pic^0(\mathcal{C})$ and $J(\mathcal{C})$. If we fix a line bundle $L_n \in Pic^n(\mathcal{C})$, we get a map:

$$\alpha_{L_n} : Pic^n(\mathcal{C}) \to J(\mathcal{C}) \ , \quad L \mapsto L \otimes L_n^{-1}$$

which is a bijection. Let $\mathcal{C}^{(n)}$ denote the n-fold symmetric product of $\mathcal{C}$. An element in $\mathcal{C}^{(n)}$ can be identified with a divisor of degree n on $\mathcal{C}$ which defines an element in

$Pic^n(\mathcal{C})$. If we map this line bundle with α_{L^n} to $J(\mathcal{C})$, we get a map from $\mathcal{C}^{(\backslash)}$ to $J(\mathcal{C})$. Let $D_n \in Div^n(\mathcal{C})$ be a divisor corresponding to the line bundle L_n we write $\alpha_{D_n} : \mathcal{C}^{(n)} \to J(\mathcal{C})$ for this map. If n equals g the genus of $\mathcal{C}$, this map is an isomorphism of abelian varieties. For $n = 1$ we get an embedding of C in $J(\mathcal{C})$. Let $c \in \mathcal{C}$, we denote by $\mathcal{L}(c)$ the corresponding line bundle on $\mathcal{C}$ and write:

$$\alpha_c : \mathcal{C} \to J(\mathcal{C}) \quad \mathcal{C} \ni x \mapsto \alpha_{\mathcal{L}(c)}(x).$$

In the following theorem we see that all maps from $\mathcal{C}$ to an abelian variety factor via the last map over $J(\mathcal{C})$:

Theorem 24 ***Universal Property of the Jacobian*** *Suppose that $\varphi : \mathcal{C} \to X$ is a morphism from a smooth curve $\mathcal{C}$ to an abelian variety X. Then there exists a homomorphism of abelian varieties $\tilde{\varphi} : J(\mathcal{C}) \to X$ such that for all $c \in \mathcal{C}$ the diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\varphi} & X \\ \downarrow{\scriptstyle \alpha_c} & & \downarrow{\scriptstyle t_{-\varphi(c)}} \\ J(\mathcal{C}) & \xrightarrow{\tilde{\varphi}} & X \end{array}$$

commutes.

If a smooth curve $\mathcal{C}$ has an $\mathbb{Z}/2\mathbb{Z}$ involution $\sigma : \mathcal{C} \to \mathcal{C}$ we get a $2:1$ cover

$$\pi : \mathcal{C} \to \mathcal{C}' := \mathcal{C}/\langle\sigma\rangle.$$

If the genus of $\mathcal{C}'$ is greater than 0 the Jacobian $J(\mathcal{C})$ is isogenous to the product of $J(\mathcal{C}')$ and an other abelian variety which is called the Prym variety of the double cover $\mathcal{C} \to \mathcal{C}'$, $Prym(\mathcal{C}/\mathcal{C}')$. The involution σ on $\mathcal{C}$ induces an involution on $H^0(\Omega_{\mathcal{C}})^\vee$, $H_1(\mathcal{C})$ and hence on the jacobian $J(\mathcal{C}) = H^0(\Omega_{\mathcal{C}})^\vee / H_1(\mathcal{C})$ which we denote by $\tilde{\sigma} : J(\mathcal{C}) \to J(\mathcal{C})$.
Using the involution $\tilde{\sigma}$ we define two morphisms:

$$\begin{aligned} N_Y : J(\mathcal{C}) \to J(\mathcal{C}) \quad & x \mapsto x + \tilde{\sigma}(x) \\ N_Z : J(\mathcal{C}) \to J(\mathcal{C}) \quad & x \mapsto x - \tilde{\sigma}(x)\ . \end{aligned}$$

The image $Y := N_Y(J(\mathcal{C}))$ is isomorphic to $J(\mathcal{C}')$, $Z := N_Z(J(\mathcal{C}))$ is $Prym(\mathcal{C}/\mathcal{C}')$ and $J(\mathcal{C})$ is isogenous to $Y \times Z$. The Abel-Jacobi map induces a morphism $\mathcal{C}$ to $Prym(\mathcal{C}/\mathcal{C}')$:

$$\pi_c : \qquad \mathcal{C} \xrightarrow{\alpha_c} J(\mathcal{C}) \xrightarrow{N_Z} Prym(\mathcal{C}/\mathcal{C}')$$

The Universal Property of the Jacobian induces the following Theorem:

Theorem 25 ***Universal Property of the Prym variety*** *Suppose $\varphi : \mathcal{C} \to X$ is a morphism from a smooth curve $\mathcal{C}$ into an abelian variety X as above. If $\varphi \circ \sigma = -1 \circ \varphi$*

there exist a unique homomorphism $\psi : Prym(\mathcal{C}/\mathcal{C}') \to X$ such that for all $c \in \mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\varphi} & X \\ \downarrow{\scriptstyle \pi_c} & & \downarrow{\scriptstyle t_{-\varphi(c)}} \\ Prym(\mathcal{C}/\mathcal{C}') & \xrightarrow{\psi} & X \end{array} .$$

In the next step we want to fix the induced polarisation of $Prym(\mathcal{C}/\mathcal{C}')$. Therefore we consider the case where a principally polarised abelian variety is isogenous to the product of two abelian subvarieties which are complementary i.e. their intersection is finite.

Proposition 38 *Let (X, L) be a principally polarised abelian variety and Y, Z complementary abelian subvarieties. Denote with ι_Y and ι_Z the embedding of Y and Z in X we get:*

$$K(\iota_Y^* L) \simeq Y \cap Z.$$

By the symmetry of the situation we find $K(\iota_Y^* L) \simeq K(\iota_Z^* L)$. Since the embedding of $J(\mathcal{C}')$ is given by the σ invariant of $J(\mathcal{C})$ that is $\left(H^0(\Omega_{\mathcal{C}})^*\right)^\sigma / H_1(\Omega_{\mathcal{C}})^\sigma$ the induced polarisation $\iota_{J(\mathcal{C}')}\Theta \simeq 2\Theta$ is equivalent to twice the principal polarisation of $J(\mathcal{C}')$. We denote by g the genus of $\mathcal{C}$ and with g' the genus of $\mathcal{C}'$ then Hurewitzś Theorem implies that $g \geq 2g'$. By equation (6.2) and Proposition 38 we get that the induced polarisation on $Prym(\mathcal{C}\ \mathcal{C}')$ is of type $(1, \ldots, 1, 2, \ldots, 2)$, where the last g' entries are equal to 2.

Remark 2 The universal property of the Prym yields the interesting question when is a given abelian variety X with a -1_X invariant possibly singular curve $\mathcal{C}$ isomorphic to the Prym of the normalisation $\tilde{\mathcal{C}}$ of this curve and its -1 quotient. If $\mathcal{C}$ is not contained in an abelian subvariety of X, from the dimension of the Prym we have the following condition on the curves:

$$dim(X) = g(\mathcal{C}) - g(\mathcal{C}') = g(\mathcal{C}') + \deg(R) - 1\ .$$

Here $\mathcal{C}' := \mathcal{C}/\langle -1_X \rangle$ and g denotes the geometric genus X is isogenous to the Prym variety of the normalisations $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}'$ of $\mathcal{C}$ and $\mathcal{C}'$.

If $\mathcal{C}$ is smooth we obtain an even stronger result:

Theorem 26 *Let X be an abelian variety and $\mathcal{C} \subset X$ a smooth curve not contained in an abelian subvariety that is invariant under the -1_X–action. We denote $\mathcal{C}'$ the quotient of $\mathcal{C}$ by -1_X. If $dim(X) = g(\mathcal{C}) - g(\mathcal{C}')$ the Abel-Jacobi map induces an isomorphism between X and $Prym(\tilde{\mathcal{C}}/\tilde{\mathcal{C}}')$.*

Proof. The morphism $\tilde{\mathcal{C}} \to X$ factors through $\psi : Prym(\tilde{\mathcal{C}}/\tilde{\mathcal{C}}') \to X$. As $\mathcal{C}$ is not contained in an abelian subvariety of X, ψ has to be surjective and as the dimension of the Prym equals the dimension of X this has to be an isogeny. We will assume that $\psi(0) = 0$ for simplicity, if this is not the case we can write ψ as a composition of such a morphism and a translation. Then ψ is an isomorphism if $\psi_* : H_1(Prym(\tilde{\mathcal{C}}/\tilde{\mathcal{C}}'), \mathbb{Z}) \to H_1(X, \mathbb{Z})$ is

an isomorphism. Since ψ is an isogeny, ψ_* is injective and it remains to show that it is surjective.
From the construction of the Prym we know that $H_1(\mathcal{C},\mathbb{Z}) \to H_1(Prym(\mathcal{C}/\mathcal{C}'),\mathbb{Z})$ is surjective. If we can show that $H_1(\mathcal{C},\mathbb{Z}) \to H_1(X,\mathbb{Z})$ is surjective we are done, since we obtain the triangle from the universal property of the Prym:

$$\begin{array}{ccc} H_1(\mathcal{C},\mathbb{Z}) & & \\ \downarrow & \searrow & \\ H_1(Prym(\mathcal{C}/\mathcal{C}'),\mathbb{Z}) & \longrightarrow & H_1(X,\mathbb{Z}) \end{array} .$$

Bertinis theorem [GH94] implies, that we can find a filtration of hyperplane sections:

$$\mathcal{C} = X_1 \subset X_2 \subset \cdots \subset X_{dim(X)-1} \subset X_{dim(X)} = X$$

where all X_i are smooth. Lefschetz hyperplane section theorem implies that $H_1(X_{i-1},\mathbb{Z}) \to H_1(X_i,\mathbb{Z})$ is surjective and so is $H_1(\mathcal{C},\mathbb{Z}) \to H_1(X,\mathbb{Z})$. $\square$

For singular curves this is definitely false.

6.2.3 A $(2,2)$ polarised Prym variety

Let (A,Θ) denote a principally polarised abelian surface. We want to put a $(2,2)$ polarisation on A by a smooth genus 4 curve $\tilde{\mathcal{C}}$:

$$\iota_{\mathcal{C}} : \tilde{\mathcal{C}} \to A$$

and denote by $\mathcal{C} := \iota_{\mathcal{C}}(\tilde{\mathcal{C}})$ the image on A. We consider the non degenerate case where A is not isogenous to the product of two elliptic curves. The geometric Riemann-Roch Theorem 18 implies:

$$\chi(\mathcal{L}(\mathcal{C})) = h^0(\mathcal{L}(\mathcal{C})) = \frac{1}{2}\mathcal{C}.\mathcal{C} = \frac{1}{2}(2p_a - 2) = \delta_1.\delta_2 = 4$$

and hence $p_a = 5$. The line bundle $\mathcal{L}(\mathcal{C})$ defines a holomorphic map:

$$\varphi_{\mathcal{L}(\mathcal{C})} : A \longrightarrow \mathbb{P}^3.$$

Up to a translation $\varphi_{\mathcal{L}(\mathcal{C})}$ gives an embedding of the Kummer surface $K = A/(-1)$ in $\mathbb{P}^3$. The equation definig the image of K in $\mathbb{P}^3$ is a quartic. K has exactly 16 A_1 singular points and 16 planes that intersect with K in a conic. The 16 planes and singular points form a 16_6 configuration. On each of the 16 planes are 6 singular points and each of the 16 singular points lies on 6 planes. By Theorem 19

$$\varphi_{\mathcal{L}(2\mathcal{C})} : A \longrightarrow \mathbb{P}^{15}$$

is an embedding and by Theorem 20 it is even projectively normal. The ideal of the image is generated by quadratic equations via Theorem 21. If $\mathcal{C}$ is invariant under the action of -1_A the line bundle $\mathcal{L}(\mathcal{C})$ is symmetric. Let $\mathcal{C}' := \mathcal{C}/\langle -1_A \rangle$ denote the quotient. Since $\mathcal{C}$ has only one singular point and -1_A induces a $\mathbb{Z}/2\mathbb{Z}$ action on $\mathcal{C}$ the singular point has to be a fixed point of the symmetry action. Since $H^0(\mathcal{L}(\mathcal{C}))$ is a sub vector space of $H^0(\mathcal{L}(2\mathcal{C}))$ we find a projection $p : \mathbb{P}^{15} \dashrightarrow \mathbb{P}^3$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \hookrightarrow & \mathbb{P}^{15} & \hookleftarrow & \mathcal{C} = \varphi_{\mathcal{L}(2\mathcal{C})}(A) \cap \{z_0 = 0\} \\ \downarrow{\scriptstyle 2:1} & & \vdots{\scriptstyle p} & & \downarrow{\scriptstyle 2:1} \\ K & \hookrightarrow & \mathbb{P}^3 & \hookleftarrow & \mathcal{C}' = \varphi_{\mathcal{L}(\mathcal{C})}(A) \cap \{z_0 = 0\} \end{array} .$$

The interesting case for us is the one where the geometric genus of $\mathcal{C}'$ is 2. By the embedding of K in $\mathbb{P}^3$ we get an embedding of $\mathcal{C}$ in $\mathbb{P}^2$ of degree 4. By the Plücker formula $\mathcal{C}'$ has to have one singular point that lowers the genus by 1. By Hurwitz's theorem the $2:1$ cover $\mathcal{C} \to \mathcal{C}'$ has to be ramified along a divisor of degree 2. Since $\mathcal{C}$ has only one singular point it has to be a ramification point. The $\mathbb{Z}/2\mathbb{Z}$ action on $\mathcal{C}$ defines an $\mathbb{Z}/2\mathbb{Z}$ action on its normalisation $\tilde{\mathcal{C}}$ and the quotient defines a normalisation $\tilde{\mathcal{C}}' \to \mathcal{C}'$. The universal property of the Prym variety induces the next statement:

Theorem 27 *Let A be an abelian surface $\mathcal{C} \subset A$ a A_1-singular curve on A of genus* 4 *defining a* $(2,2)$ *polarisation. If -1_A induces a $\mathbb{Z}/2\mathbb{Z}$ action on $\mathcal{C}$ and $\mathcal{C}' := \mathcal{C}/\langle -1_A \rangle$ is of geometric genus* 2*, A is isogenous to the prym variety $Prym(\tilde{\mathcal{C}}/\tilde{\mathcal{C}}')$, where $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}'$ denote the normalisations of $\mathcal{C}$ and $\mathcal{C}'$ and we get:*

1. *$\mathcal{L}(\mathcal{C})$ is an symmetric line bundle and induces an embedding of $K := A/\langle -1_A \rangle$ in $\mathbb{P}^3$.*
2. *$\mathcal{L}(2\mathcal{C})$ is very ample and $\varphi_{\mathcal{L}(2\mathcal{C})} : A \to \mathbb{P}^{15}$ is an projectively normal embedding.*
3. *$\varphi_{\mathcal{L}(2\mathcal{C})}(A)$ is a degree* 32 *surface in $\mathbb{P}^{15}$ and its ideal is generated by quadratic equations.*
4. *The line bundle $\mathcal{L}(2\mathcal{C})|_{\mathcal{C}}$ defines an embedding of $\mathcal{C}$ in $\mathbb{P}^{11}$ the degree of the image is* 16*.*

Proof. Since $\mathcal{C}$ is an ample divisor on A it is not contained in an abelian subvariety of A. Therefore the image of the prym map is all of A since the dimension of the prym equals the dimension of A the prym map is an isogeny. Since $\mathcal{C}$ is invariant under the action of $\langle -1_A \rangle$ the line bundle $\mathcal{L}(\mathcal{C})$ is symmetric. By Theorem 19 and 20 we get the second point. Now

$$(\mathcal{L}(2\mathcal{C}))^2 = (2\mathcal{C}).(2\mathcal{C}) = 4.(2p_a - 2) = 32$$

hence we get with Theorem 21 the third point. From the Riemann-Roch Theorem for curves it follows:

$$l((2\mathcal{C})|_{\mathcal{C}}) = deg((2\mathcal{C})|_{\mathcal{C}}) + 1 - p_a = 16 + 1 - 5 = 12 .$$

This implies the last point. □

A principally polarised abelian surface $\mathcal{A}$, that is not isogenous to the product of two elliptic curves, yields always a curve of geometric genus 4 in $|\mathcal{L}(2\Theta)|$. As $\varphi_{\mathcal{L}(2\Theta)}$ embeds the Kummer surface $K = \mathcal{A}/\langle -1_{\mathcal{A}} \rangle$ in $\mathbb{P}^3$ as hypersurface of degree 4 the intersection of a hyperplane of $\mathbb{P}^3$ with the Kummer surface yields a curve of degree 4. The image of the zero in $\mathcal{A}$ is one of the 16 nodes on the Kummer surface, as it is invariant under the $-1_{\mathcal{A}}$ action. If we choose a hyperplane in $\mathbb{P}^3$ containing the image of the zero and no other singular point of the Kummer the intersection of the Kummer with this hyperplane is a degree 4 curve in $\mathbb{P}^2$ with a node, so of genus 2. Its preimage then is of geometric genus 4 with a node.

6.3 $\varepsilon = 16$ Hénon-Heiles

Basis of $H^0(\mathcal{L}(2D))$:

$$
\begin{aligned}
z_0 &= 1 \\
z_1 &= Q \\
z_2 &= U \\
z_3 &= W \\
z_4 &= \frac{32}{3}Q^3 + P^2 + \frac{4}{3}QU \\
z_5 &= PU - 4QW \\
z_6 &= \frac{64}{3}Q^4 + 2QP^2 + \frac{16}{3}Q^2U + PW \\
z_7 &= U^2 \\
z_8 &= -2QPU + 16Q^2W + UW + 3PV \\
z_9 &= \frac{32}{3}Q^3U + P^2U + \frac{4}{3}QU^2 \\
z_{10} &= \frac{1}{3}PU^2 + \frac{32}{3}Q^3W + P^2W \\
z_{11} &= \frac{1024}{9}Q^6 + \frac{64}{3}Q^3P^2 + P^4 + \frac{256}{9}Q^4U + \frac{8}{3}QP^2U + \frac{16}{9}Q^2U^2 \\
z_{12} &= -4Q^3PU - \frac{3}{8}P^3U - \frac{5}{6}QPU^2 + 16Q^4W + \frac{3}{2}QP^2W + 4Q^2UW \\
&\quad + \frac{1}{12}U^2W - \frac{1}{4}PL^2 + QWV \\
z_{13} &= -\frac{2048}{3}Q^7 - 128Q^4P^2 - 6QP^4 - \frac{512}{3}Q^5U - 16Q^2P^2U - \frac{32}{3}Q^3U^2 \\
&\quad + P^2U^2 + 64Q^3PW + 6P^3W - 16Q^2L^2 - 256Q^4V - 24QP^2V - 16Q^2UV \\
z_{14} &= 64Q^4PU + 6QP^3U + 16Q^2PU^2 + PU^3 - 512Q^5W - 48Q^2P^2W - 128Q^3UW \\
&\quad + 3P^2UW - 8QU^2W + 24QPL^2 - 96Q^3PV - 9P^3V - 48QPUV + 48Q^2WV \\
z_{15} &= -\frac{1792}{3}Q^6U - 112Q^3P^2U - \frac{21}{4}P^4U - \frac{544}{3}Q^4U^2 - 18QP^2U^2 \\
&\quad - \frac{40}{3}Q^2U^3 + 16Q^2PUW + PU^2W + 64Q^3L^2 + 6QUL^2 - 96Q^3UV \\
&\quad - 10QU^2V - 24QPWV + 24Q^2V^2
\end{aligned}
$$

Basis of $H^0 = (\mathcal{L}(3D))$:

$$
\begin{aligned}
z_0 &= 1\\
z_1 &= Q\\
z_2 &= P\\
z_3 &= U\\
z_4 &= W\\
z_5 &= QU\\
z_6 &= \frac{32}{3}Q^3 + P^2\\
z_7 &= PU - 4QW\\
z_8 &= 8Q^4 + \frac{3}{4}QP^2 + Q^2U\\
z_9 &= -\frac{64}{3}Q^4 - 2QP^2 + PW\\
z_{10} &= U^2\\
z_{11} &= -2Q^3P - \frac{3}{16}P^3 - \frac{1}{2}QPU + Q^2W\\
z_{12} &= UW\\
z_{13} &= \frac{32}{3}Q^3U + P^2U + \frac{4}{3}QU^2\\
z_{14} &= -\frac{2}{3}Q^3U - \frac{1}{12}QU^2 - \frac{1}{2}QPW + Q^2V\\
z_{15} &= -\frac{1}{4}PU^2 + QUW\\
z_{16} &= \frac{1}{3}PU^2 + \frac{32}{3}Q^3W + P^2W\\
z_{17} &= \frac{1024}{9}Q^6 + \frac{64}{3}Q^3P^2 + P^4 + \frac{256}{9}Q^4U\\
&\quad + \frac{8}{3}QP^2U + \frac{16}{9}Q^2U^2\\
z_{18} &= U^3\\
z_{19} &= 8Q^3PU + \frac{3}{4}P^3U + QPU^2 - 32Q^4W - 3QP^2W - 4Q^2UW\\
z_{20} &= \frac{8}{3}Q^3PU + \frac{1}{4}P^3U - \frac{32}{3}Q^4W - QP^2W\\
&\quad + \frac{2}{3}Q^2UW + \frac{1}{12}U^2W + QWV
\end{aligned}
$$

$$
\begin{aligned}
z_{21} &= \frac{512}{3}Q^7 + 32Q^4P^2 + \frac{3}{2}QP^4 + 64Q^5U + 6Q^2P^2U \\
&\quad + \frac{16}{3}Q^3U^2 + 8Q^3PW + \frac{3}{4}P^3W + QPUW \\
z_{22} &= \frac{2048}{3}Q^7 + 128Q^4P^2 + 6QP^4 + 256Q^5U + 24Q^2P^2U \\
&\quad + \frac{64}{3}Q^3U^2 + P^2U^2 + 64Q^3PW + 6P^3W - 16Q^2L^2 - 128Q^4V - 12QP^2V \\
z_{23} &= -128Q^6P - 24Q^3P^3 - \frac{9}{8}P^5 - 64Q^4PU - 6QP^3U - 8Q^2PU^2 \\
&\quad + 128Q^5W + 12Q^2P^2W + 32Q^3UW + QU^2W - 3QPL^2 + 3QPUV \\
z_{24} &= \frac{512}{3}Q^6P + 32Q^3P^3 + \frac{3}{2}P^5 + 64Q^4PU + 6QP^3U \\
&\quad + 8Q^2PU^2 - \frac{32}{3}Q^3UW + P^2UW + 4QPL^2 + 32Q^3PV + 3P^3V \\
z_{25} &= \frac{1024}{9}Q^6U + \frac{64}{3}Q^3P^2U + P^4U + \frac{256}{9}Q^4U^2 \\
&\quad + \frac{8}{3}QP^2U^2 + \frac{16}{9}Q^2U^3 \\
z_{26} &= \frac{512}{3}Q^6U + 16Q^3P^2U + \frac{64}{3}Q^4U^2 - 2QP^2U^2 \\
&\quad + 128Q^4PW + 12QP^3W + 40Q^2PUW + PU^2W + 64Q^3L^2 - 256Q^5V \\
&\quad - 24Q^2P^2V - 128Q^3UV - 8QU^2V - 12QPWV \\
z_{27} &= \frac{32}{9}Q^3PU^2 + \frac{1}{3}P^3U^2 + \frac{4}{9}QPU^3 \\
&\quad + \frac{1024}{9}Q^6W + \frac{64}{3}Q^3P^2W + P^4W + \frac{128}{9}Q^4UW \\
&\quad + \frac{4}{3}QP^2UW \\
z_{28} &= \frac{8}{3}Q^3PU^2 + \frac{1}{8}P^3U^2 + \frac{2}{3}QPU^3 - \frac{16}{3}Q^4UW \\
&\quad + \frac{3}{2}QP^2UW - 2Q^2U^2W + 8Q^2PL^2 + QWL^2 - 12Q^2PUV \\
&\quad + \frac{1}{4}PU^2V + 32Q^3WV + 3QPV^2 \\
z_{29} &= \frac{32768}{27}Q^9 + \frac{1024}{3}Q^6P^2 + 32Q^3P^4 + P^6 \\
&\quad + \frac{4096}{9}Q^7U + \frac{256}{3}Q^4P^2U + 4QP^4U + \frac{512}{9}Q^5U^2 \\
&\quad + \frac{16}{3}Q^2P^2U^2 + \frac{64}{27}Q^3U^3
\end{aligned}
$$

$$z_{30} = -32Q^6PU - 6Q^3P^3U - \frac{9}{32}P^5U - \frac{32}{3}Q^4PU^2 - QP^3U^2 - \frac{5}{6}Q^2PU^3 + 128Q^7W + 24Q^4P^2W + \frac{9}{8}QP^4W + 48Q^5UW + \frac{9}{2}Q^2P^2UW + \frac{14}{3}Q^3U^2W + \frac{1}{16}P^2U^2W + \frac{1}{12}QU^3W - 2Q^3PL^2 - \frac{3}{16}P^3L^2 - \frac{1}{4}QPUL^2 + 8Q^4WV + \frac{3}{4}QP^2WV + Q^2UWV$$

$$z_{31} = -\frac{65536}{9}Q^10 - 2048Q^7P^2 - 192Q^4P^4 - 6QP^6 - \frac{8192}{3}Q^8U - 512Q^5P^2U - 24Q^2P^4U - \frac{1024}{3}Q^6U^2 - \frac{64}{3}Q^3P^2U^2 + P^4U^2 - \frac{128}{9}Q^4U^3 + \frac{4}{3}QP^2U^3 + \frac{2048}{3}Q^6PW + 128Q^3P^3W + 6P^5W + \frac{256}{3}Q^4PUW + 8QP^3UW - \frac{512}{3}Q^5L^2 - 16Q^2P^2L^2 - \frac{64}{3}Q^3UL^2 - \frac{8192}{3}Q^7V - 512Q^4P^2V - 24QP^4V - 512Q^5UV - 48Q^2P^2UV - \frac{64}{3}Q^3U^2V$$

$$z_{32} = -\frac{16384}{3}Q^9P - 1536Q^6P^3 - 144Q^3P^5 - \frac{9}{2}P^7 - 2048Q^7PU - 384Q^4P^3U - 18QP^5U - 256Q^5PU^2 - 24Q^2P^3U^2 - \frac{32}{3}Q^3PU^3 + P^3U^3 - 8192Q^8W - 1536Q^5P^2W - 72Q^2P^4W - 3072Q^6UW - 288Q^3P^2UW - 384Q^4U^2W - 24QP^2U^2W - 16Q^2U^3W + 384Q^4PL^2 + 36QP^3L^2 + 64Q^3WL^2 - 3072Q^6PV - 576Q^3P^3V - 27P^5V - 1536Q^4PUV - 144QP^3UV - 96Q^2PU^2V + 1536Q^5WV + 144Q^2P^2WV + 128Q^3UWV$$

$$z_{33} = -\frac{14336}{3}Q^9U - 1344Q^6P^2U - 126Q^3P^4U - \frac{63}{1}6P^6U - 2048Q^7U^2 - 392Q^4P^2U^2 - \frac{75}{4}QP^4U^2 - 288Q^5U^3 - 28Q^2P^2U^3 - \frac{40}{3}Q^3U^4 + 128Q^5PUW + 12Q^2P^3UW + 24Q^3PU^2W + \frac{3}{4}P^3U^2W + QPU^3W + 512Q^6L^2 + 48Q^3P^2L^2 + 112Q^4UL^2 + \frac{9}{2}QP^2UL^2 + 6Q^2U^2L^2 - 768Q^6UV - 72Q^3P^2UV - 176Q^4U^2V - \frac{15}{2}QP^2U^2V - 10Q^2U^3V - 192Q^4PWV - 18QP^3WV - 24Q^2PUWV + 192Q^5V^2 + 18Q^2P^2V^2 + 24Q^3UV^2$$

$$
\begin{aligned}
z_{34} = & -384Q^6PU^2 - 66Q^3P^3U^2 - \frac{45}{16}P^5U^2 - 120Q^4PU^3 \\
& - \frac{39}{4}QP^3U^3 - 9Q^2PU^4 - 10240Q^9W - 2880Q^6P^2W - 270Q^3P^4W \\
& - \frac{135}{16}P^6W - 2560Q^7UW - 576Q^4P^2UW - \frac{63}{2}QP^4UW \\
& - 27Q^2P^2U^2W + 28Q^3U^3W + QU^4W - 384Q^5PL^2 - 36Q^2P^3L^2 \\
& - 84Q^3PUL^2 + \frac{45}{8}P^3UL^2 - 192Q^4WL^2 - 36QP^2WL^2 \\
& - 30Q^2UWL^2 + 576Q^5PUV + 54Q^2P^3UV + 180Q^3PU^2V \\
& - \frac{9}{8}P^3U^2V + 9QPU^3V - 1536Q^6WV - 144Q^3P^2WV \\
& - 384Q^4UWV + 27QP^2UWV - 18Q^2U^2WV + 72Q^2PL^2V \\
& - 144Q^4PV^2 - \frac{27}{2}QP^3V^2 - 108Q^2PUV^2 + 72Q^3WV^2 \\
z_{35} = & -\frac{5373952}{27}Q^{12} - \frac{671744}{9}Q^9P^2 - 10496Q^6P^4 \\
& - 656Q^3P^6 - \frac{123}{8}P^8 - \frac{753664}{9}Q^10U - 23552Q^7P^2U \\
& - 2208Q^4P^4U - 69QP^6U - \frac{57344}{9}Q^8U^2 - \frac{3584}{3}Q^5P^2U^2 \\
& - 56Q^2P^4U^2 + \frac{43264}{27}Q^6U^3 + \frac{1712}{9}Q^3P^2U^3 \\
& + \frac{23}{6}P^4U^3 + 288Q^4U^4 + \frac{40}{3}QP^2U^4 + \frac{116}{9}Q^2U^5 \\
& + 2560Q^6PUW + 480Q^3P^3UW + \frac{45}{2}P^5UW + \frac{2560}{3}Q^4PU^2W \\
& + 78QP^3U^2W + \frac{232}{3}Q^2PU^3W + 2048Q^7L^2 + 384Q^4P^2L^2 \\
& + 18QP^4L^2 + 768Q^5UL^2 + 56Q^2P^2UL^2 + \frac{256}{3}Q^3U^2L^2 \\
& + \frac{160}{3}Q^3PWL^2 + P^3WL^2 - 8Q^2L^4 - \frac{2048}{3}Q^7UV \\
& - 128Q^4P^2UV - 6QP^4UV - \frac{640}{3}Q^5U^2V - \frac{32}{3}Q^3U^3V \\
& + P^2U^3V + \frac{10}{3}QU^4V - \frac{160}{3}Q^3PUWV + 5P^3UWV \\
& + 16QPU^2WV - 384Q^4L^2V - 12QP^2L^2V + 320Q^4UV^2 \\
& + 48Q^2PWV^2 - 32Q^3V^3
\end{aligned}
$$

6.4 The degree 3 Grammaticos potential

We computed a basis for $H^0(\mathcal{L}(2\mathcal{C}_P))$ as:

$$
\begin{aligned}
\varphi_1 &:= q_1\\
\varphi_2 &:= q_2\\
\varphi_3 &:= p_2\\
\varphi_4 &:= q_2^2\\
\varphi_5 &:= q_2p_1 + \tfrac{2}{3}q_1p_2 + \tfrac{1}{3}q_2p_2\\
\varphi_6 &:= q_1^2q_2 - \tfrac{1}{2}q_1q_2^2 - \tfrac{1}{2}q_2^3 - \tfrac{1}{3}p_1p_2\\
\varphi_7 &:= -\tfrac{3}{2}q_1q_2^2 + \tfrac{3}{2}q_2^3 - \tfrac{1}{3}p_2^2\\
\varphi_8 &:= q_2^2p_1 + \tfrac{2}{3}q_1q_2p_2 + \tfrac{1}{3}q_2^2p_2\\
\varphi_9 &:= q_1^3q_2 - \tfrac{3}{4}q_1q_2^3 - \tfrac{1}{4}q_2^4 + \tfrac{1}{2}q_2p_1^2\\
\varphi_{10} &:= 2q_1^4 - \tfrac{9}{2}q_1^2q_2^2 + q_1q_2^3 + \tfrac{3}{2}q_2^4 + q_1p_1^2 + q_2p_1p_2\\
\varphi_{11} &:= \tfrac{3}{2}q_1q_2^2p_1 - \tfrac{3}{2}q_2^3p_1 + 2q_1^2q_2p_2 - q_1q_2^2p_2 - q_2^3p_2 - \tfrac{1}{3}p_1p_2^2\\
\varphi_{12} &:= \tfrac{3}{2}q_2^3p_1 + \tfrac{3}{2}q_1q_2^2p_2 + \tfrac{1}{9}p_2^3\\
\varphi_{13} &:= -9q_1^2q_2^3 + \tfrac{9}{2}q_1q_2^4 + \tfrac{9}{2}q_2^5 + 6q_2^2p_1p_2 + 2q_1q_2p_2^2 + q_2^2p_2^2\\
\varphi_{14} &:= \tfrac{3}{2}q_1^3q_2^2 - 3q_1^2q_2^3 - \tfrac{3}{4}q_1q_2^4 + \tfrac{9}{4}q_2^5 - q_1q_2p_1p_2 + \tfrac{3}{2}q_2^2p_1p_2 - \tfrac{1}{3}q_1^2p_2^2 + \tfrac{1}{3}q_1q_2p_2^2\\
\varphi_{15} &:= -\tfrac{9}{2}q_1q_2^3p_1 + \tfrac{9}{4}q_2^4p_1 - \tfrac{9}{2}q_1^2q_2^2p_2 + \tfrac{9}{4}q_2^4p_2 - \tfrac{1}{3}q_1p_2^3 - \tfrac{1}{3}q_2p_2^3\\
\varphi_{16} &:= \tfrac{3}{4}q_2^4p_1 + \tfrac{1}{2}q_1^2q_2^2p_2 + \tfrac{1}{2}q_1q_2^3p_2 - \tfrac{1}{4}q_2^4p_2 - \tfrac{1}{3}q_2p_1p_2^2 - \tfrac{1}{9}q_1p_2^3\\
\varphi_{17} &:= 2q_1^5q_2 - q_1^4q_2^2 - \tfrac{5}{2}q_1^3q_2^3 + \tfrac{1}{4}q_1^2q_2^4 + q_1q_2^5 + \tfrac{1}{4}q_2^6 + q_1^2q_2p_1^2 - \tfrac{1}{2}q_1q_2^2p_1^2 + q_2^3p_1^2\\
&\quad - \tfrac{2}{3}q_1^3p_1p_2 + \tfrac{5}{2}q_1q_2^2p_1p_2 + \tfrac{7}{6}q_2^3p_1p_2 - \tfrac{1}{3}p_1^3p_2 + \tfrac{2}{3}q_1^2q_2p_2^2 + \tfrac{2}{3}q_1q_2^2p_2^2 + \tfrac{1}{6}q_2^3p_2^2\\
\varphi_{18} &:= 3q_1^3q_2^2p_1 + \tfrac{3}{4}q_1^2q_2^3p_1 - \tfrac{39}{8}q_1q_2^4p_1 + \tfrac{9}{8}q_2^5p_1 + \tfrac{3}{2}q_2^2p_1^3 + 2q_1^4q_2p_2 + \tfrac{3}{2}q_1^3q_2^2p_2 - 3q_1^2q_2^3p_2\\
&\quad - \tfrac{7}{8}q_1q_2^4p_2 + \tfrac{3}{8}q_2^5p_2 + q_1q_2p_1^2p_2 - \tfrac{1}{2}q_1q_2p_1p_2^2 - \tfrac{3}{4}q_2^2p_1p_2^2 - \tfrac{1}{9}q_1^2p_2^3 - \tfrac{4}{9}q_1q_2p_2^3 - \tfrac{7}{36}q_2^2p_2^3\\
\varphi_{19} &:= 2q_1^7 - 4q_1^6q_2 - 6q_1^5q_2^2 + 8q_1^4q_2^3 + \tfrac{43}{8}q_1^3q_2^4 - 3q_1^2q_2^5 - \tfrac{17}{8}q_1q_2^6 - \tfrac{1}{4}q_2^7 + 2q_1^4p_1^2 - 4q_1^3q_2p_1^2\\
&\quad - 3q_1^2q_2^2p_1^2 + 4q_1q_2^3p_1^2 - \tfrac{5}{4}q_2^4p_1^2 + 1/2q_1p_1^4 - q_2p_1^4 - 3q_1q_2^3p_1p_2 - \tfrac{3}{2}q_2^4p_1p_2 - \tfrac{2}{3}q_1^4p_2^2\\
&\quad - \tfrac{1}{2}q_1^2q_2^2p_2^2 - \tfrac{5}{6}q_1q_2^3p_2^2 - \tfrac{1}{4}q_2^4p_2^2 - \tfrac{1}{3}q_1p_1^2p_2^2
\end{aligned}
$$

$$\begin{aligned}
\varphi_{20} := {}& 3q_1^4q_2^2p_1 - 3q_1^3q_2^3p_1 + \tfrac{9}{4}q_1^2q_2^4p_1 - \tfrac{3}{4}q_1q_2^5p_1 - \tfrac{3}{2}q_2^6p_1 + \tfrac{3}{2}q_1q_2^2p_1^3 - \tfrac{3}{2}q_2^3p_1^3 + 4q_1^5q_2p_2 \\
& - 2q_1^4q_2^2p_2 - 2q_1^3q_2^3p_2 + \tfrac{1}{2}q_1^2q_2^4p_2 - \tfrac{1}{4}q_1q_2^5p_2 - \tfrac{1}{4}q_2^6p_2 + 2q_1^2q_2p_1^2p_2 - q_1q_2^2p_1^2p_2 \\
& - 4q_2^3p_1^2p_2 - \tfrac{2}{3}q_1^3p_1p_2^2 - \tfrac{5}{2}q_1q_2^2p_1p_2^2 - \tfrac{4}{3}q_2^3p_1p_2^2 - \tfrac{1}{3}p_1^3p_2^2 - \tfrac{2}{3}q_1^2q_2p_2^3 - \tfrac{2}{3}q_1q_2^2p_2^3 - \tfrac{1}{6}q_2^3p_2^3 \\
\varphi_{21} := {}& 3q_1^5q_2^3 - \tfrac{9}{8}q_1^4q_2^4 - 8q_1^3q_2^5 + \tfrac{75}{16}q_1^2q_2^6 + \tfrac{33}{8}q_1q_2^7 - \tfrac{43}{16}q_2^8 + \tfrac{3}{2}q_1^2q_2^3p_1^2 - \tfrac{3}{4}q_1q_2^4p_1^2 - \tfrac{19}{8}q_2^5p_1^2 \\
& - 2q_1^3q_2^2p_1p_2 - \tfrac{1}{2}q_1^2q_2^3p_1p_2 + \tfrac{13}{8}q_1q_2^4p_1p_2 - \tfrac{19}{8}q_2^5p_1p_2 - q_2^2p_1^3p_2 - \tfrac{2}{3}q_1^4q_2p_2^2 - \tfrac{1}{2}q_1^3q_2^2p_2^2 \\
& - \tfrac{7}{12}q_1^2q_2^3p_2^2 - \tfrac{13}{24}q_1q_2^4p_2^2 + \tfrac{2}{3}q_2^5p_2^2 - \tfrac{1}{3}q_1q_2p_1^2p_2^2 + \tfrac{1}{9}q_1q_2p_1p_2^3 + \tfrac{55}{108}q_2^2p_1p_2^3 \\
& + \tfrac{1}{54}q_1^2p_2^4 + \tfrac{7}{54}q_1q_2p_2^4 \\
\varphi_{22} := {}& -3q_1^4q_2^3p_1 - \tfrac{21}{4}q_1^3q_2^4p_1 + 9q_1^2q_2^5p_1 + \tfrac{33}{8}q_1q_2^6p_1 - \tfrac{39}{8}q_2^7p_1 - \tfrac{3}{2}q_1q_2^3p_1^3 - \tfrac{15}{4}q_2^4p_1^3 \\
& - 3q_1^5q_2^2p_2 - 6q_1^4q_2^3p_2 + \tfrac{15}{2}q_1^3q_2^4p_2 + \tfrac{15}{2}q_1^2q_2^5p_2 - \tfrac{33}{8}q_1q_2^6p_2 - \tfrac{15}{8}q_2^7p_2 - \tfrac{3}{2}q_1^2q_2^2p_1^2p_2 \\
& - 6q_1q_2^3p_1^2p_2 - \tfrac{21}{4}q_2^4p_1^2p_2 - 3q_1^2q_2^2p_1p_2^2 - \tfrac{13}{2}q_1q_2^3p_1p_2^2 - \tfrac{3}{2}q_2^4p_1p_2^2 - \tfrac{2}{9}q_1^4p_2^3 \\
& - \tfrac{14}{9}q_1^3q_2p_2^3 - \tfrac{11}{6}q_1^2q_2^2p_2^3 + \tfrac{1}{18}q_1q_2^3p_2^3 + \tfrac{1}{18}q_2^4p_2^3 - \tfrac{1}{9}q_1p_1^2p_2^3 - \tfrac{1}{9}q_2p_1^2p_2^3 \\
& + \tfrac{1}{9}q_1p_1p_2^4 - \tfrac{1}{27}q_1p_2^5 \\
\varphi_{23} := {}& \tfrac{27}{4}q_1^5q_2^4 - \tfrac{27}{4}q_1^4q_2^5 - \tfrac{27}{2}q_1^3q_2^6 + \tfrac{27}{2}q_1^2q_2^7 + \tfrac{27}{4}q_1q_2^8 - \tfrac{27}{4}q_2^9 + \tfrac{9}{2}q_1^2q_2^4p_1^2 - \tfrac{9}{4}q_1q_2^5p_1^2 \\
& - \tfrac{45}{8}q_2^6p_1^2 - 3q_1^3q_2^3p_1p_2 + \tfrac{9}{2}q_1^2q_2^4p_1p_2 - \tfrac{33}{4}q_2^6p_1p_2 - 3q_2^3p_1^3p_2 - q_1^4q_2^2p_2^2 + q_1^3q_2^3p_2^2 \\
& - \tfrac{3}{2}q_1^2q_2^4p_2^2 - \tfrac{11}{4}q_1q_2^5p_2^2 + \tfrac{7}{8}q_2^6p_2^2 - 2q_1q_2^2p_1^2p_2^2 - \tfrac{5}{2}q_2^3p_1^2p_2^2 - \tfrac{2}{3}q_1^2q_2p_1p_2^3 \\
& - \tfrac{5}{3}q_1q_2^2p_1p_2^3 + \tfrac{1}{3}q_2^3p_1p_2^3 - \tfrac{1}{9}q_1^3p_2^4 - \tfrac{1}{3}q_1^2q_2p_2^4 - \tfrac{1}{18}q_2^3p_2^4\,.
\end{aligned}$$

6.5 The Duistermaat System

The following SINGULAR script commutes the Laurent Series solutions to χ_F of the deformed Duistermaat System $(\mathbb{C}^8, \mathbb{F}_4)$. We may pay attention, that the variable t denotes the square root $\sqrt{t}$. The integer MaxOrd defines the maximal order of the series and the integer MaxDeg the maximal degree of a generators of the ideal of the indicial locus, which are used to solve the linear equations, which are given over $\mathcal{O}_\mathcal{I}$ where $\mathcal{I}$ denotes the inditial locus. If MaxDeg is chosen to small the equations can not be solved and the dimension of I is -1 which is written in the document ":a␣Fehlermeldungen".

Listing 6.1: Laurent Series

```
LIB"primdec.lib";
LIB"normal.lib";
LIB"qhmoduli.lib";
int MaxDeg = 24;
int MaxOrd = 16;
//==========RING==========/
ring r = 0, (q(1..4),p(1..4),z(1..3),l),(dp(8),lp);
poly H = q(1)^2 + p(1)^2 + q(2)^2 + q(4)^2
+ p(2)^2 + p(4)^2 + 2*q(3)^2 + 2*p(3)^2;
poly F = q(3)*(2*(q(1)^2 - p(1)^2)
+ (q(2)^2 + q(4)^2 - p(2)^2 - p(4)^2))
```

```
+ 2*p(3)*(2*q(1)*p(1) + q(2)*p(2) + q(4)*p(4));
poly G1 = (p(1)^2*(q(2)^2 + q(4)^2)
- 2*p(1)*q(1)*(p(2)*q(2) + q(4)*p(4))
+ q(1)^2*(p(2)^2 + p(4)^2))*(p(2)^2 + p(4)^2 + q(2)^2 + q(4)^2)
+ 2*(1/2*q(3)*(q(2)^2 + q(4)^2 - p(2)^2 - p(4)^2)
+ p(3)*(q(2)*p(2) + q(4)*p(4)))^2;
poly L = -q(4)*p(2)+q(2)*p(4);
poly G = G1 - (q(1)^2 + p(1)^2)*L^2;
matrix X[8][1];
for (int i = 1; i <= 8; i++)
{
        X[i,1] = PBk(F,var(i),4);
}
matrix E = diag(1,8);
matrix Nu = E;
matrix k[8][8];
for (int i = 1; i <= 8; i++)
{
        for (int j = 1; j <= 8; j++)
        {
                k[i,j] = diff(X[i,1],var(j));
        }
}
matrix K = k + Nu;
ideal il;
for (int i = 1; i <= 8; i++)
{
        il = il, var(i) + X[i,1];
}
poly c = det(K - l*E);
list Lil = primdecGTZ(il);
//for (int i = 1; i <= 3; i++)
//{
//      ideal I(i) = Lil[i][2];
//      ideal J(i) = std(I(i));
//      poly C(i) = reduce(c,J(i));
//}
ideal J(2);
J(2)[1]=p(4);
J(2)[2]=p(2);
J(2)[3]=q(4);
J(2)[4]=q(2);
J(2)[5]=16*p(1)^2+4*p(3)+1;
J(2)[6]=4*q(3)*p(1)-4*q(1)*p(3)-q(1);
```

```
J(2)[7]=4*q(1)*p(1)+q(3);
J(2)[8]=16*q(3)^2+16*p(3)^2-1;
J(2)[9]=4*q(1)*q(3)+4*p(1)*p(3)-p(1);
J(2)[10]=16*q(1)^2-4*p(3)+1;
ideal IL = std(J(2));
K = RedComp(K,IL);
poly C = reduce(c,IL);
matrix V(0)[8][1];
for (int i = 1; i <= 8; i++)
{
        V(0)[i,1] = reduce(var(i),IL);
}
//==========RING==========/
ring r1 = 0, (q(2..4),p(2..4),q(1),p(1)),(dp(6),dp);
ideal il = imap(r,IL);
ideal IL = std(il);
ideal B = 1;
for (int i = 1; i <= MaxDeg; i++)
{
        B = B, kbase(IL,i);
}
int N = size(B);
//==========RING==========/
ring R = (0,c(1..6)), (q(2..4),p(2..4),q(1),p(1),z(1..8*N),t),
(dp(6),dp(2),lp);
ideal il = imap(r,IL);
ideal IL = std(il);
ideal B = imap(r1,B);
matrix K = imap(r,K);
matrix E = diag(1,8);
matrix Nu = E;
matrix V[8][1];
for (int i = 1; i <= size(B); i++)
{
for (int j = 1; j <= 8; j++)
{
V[j,1] = V[j,1] + B[i]*z(8*(i-1) + j);
}
}
matrix V(0) = RedComp(imap(r,V(0)),IL);
//===EV = 1/2
matrix V(1)[8][1];
V(1)[2,1]=-p(3)+1/4;
V(1)[6,1]=q(3);
```

```
matrix V(2)[8][1];
V(2)[4,1]=q(3);
V(2)[8,1]=p(3)+1/4;
//===EV = 3/2
matrix V(3)[8][1];
V(3)[2,1]=-p(3)-1/4;
V(3)[6,1]=q(3);
matrix V(4)[8][1];
V(4)[4,1]=-p(3)-1/4;
V(4)[8,1]=q(3);
//===EV = 2
matrix V(5)[8][1];
V(5)[1,1]=q(1);
V(5)[3,1]=-2*q(3);
V(5)[5,1]=p(1);
V(5)[7,1]=-2*p(3);
//===EV = 3
matrix V(6)[8][1];
V(6)[1,1]=-p(1);
V(6)[3,1]=p(3);
V(6)[5,1]=q(1);
V(6)[7,1]=-q(3);
V(1) = RedComp(V(1),IL);
V(2) = RedComp(V(2),IL);
V(3) = RedComp(V(3),IL);
V(4) = RedComp(V(4),IL);
V(5) = RedComp(V(5),IL);
V(6) = RedComp(V(6),IL);
matrix W = V(0) + t*(c(1)*V(1) + c(2)*V(2)) + t^3*(c(3)*V(3)
+ c(4)*V(4)) + t^4*c(5)*V(5) + t^6*c(6)*V(6);
for (int i = 1; i <= MaxOrd; i++)
{
        matrix wW = W + t^i*V;
        map iW = r, wW[1,1], wW[2,1], wW[3,1], wW[4,1],
        wW[5,1], wW[6,1], wW[7,1], wW[8,1];
        matrix D = iW(X) + Nu*wW - t/2*diff(wW,t);
        matrix rD = D/(t^(i+1));
        matrix Di = RedComp((D/(t^i) - t*rD),IL);
        ideal iI;
        for (int j = 1; j <= 8; j++)
        {
                matrix M = coef(Di[j,1],q(1)*q(3)*p(1)*p(3));
                for (int n = 1; n <= ncols(M); n++)
                {
```

```
                    iI = iI, M[2,n];
            }
    }
    ideal iJ = std(iI);
    ideal I(i) = iI;
    int N(i) = dim(iJ);
    print("step " + string(i));
    print("dim(I) = "+string(N(i)));
    write(":a Fehlermeldungen", "step " + string(i)
    + ":" + "dim(I) = " + string(N(i)));
    matrix We = RedComp(V,iJ);
    for (int j = 1; j <= 8*N; j++)
    {
            We = subst(We,z(j),0);
    }
    W = W + t^i*We;
}
write(":w Poly1",W[1,1]);
write(":w Poly2",W[2,1]);
write(":w Poly3",W[3,1]);
write(":w Poly4",W[4,1]);
write(":w Poly5",W[5,1]);
write(":w Poly6",W[6,1]);
write(":w Poly7",W[7,1]);
write(":w Poly8",W[8,1]);
exit;
```

6.6 The degree 4 Grammaticos Potential

A basis of $H^0(\mathcal{L}(\mathcal{C}_+ + \mathcal{C}_-)^2)$:

$$
\begin{aligned}
\varphi_1 &= Q\\
\varphi_2 &= Q^2\\
\varphi_3 &= P\\
\varphi_4 &= U\\
\varphi_5 &= QU\\
\varphi_6 &= W\\
\varphi_7 &= PU - 2QW\\
\varphi_8 &= U^2\\
\varphi_9 &= 2Q^4 + P^2 + Q^2U\\
\varphi_{10} &= 2Q^5 + QP^2 + Q^3U\\
\varphi_{11} &= -\frac{1}{2}QPU + Q^2W\\
\varphi_{12} &= Q^3U + \frac{1}{2}QU^2 + PW\\
\varphi_{13} &= 2Q^4P + P^3 + 2Q^2PU - 2Q^3W\\
\varphi_{14} &= 2Q^4U + P^2U + Q^2U^2\\
\varphi_{15} &= -\frac{1}{2}PU^2 + QUW\\
\varphi_{16} &= 2Q^6 + Q^2P^2 + 2Q^4U + \frac{1}{2}Q^2U^2 + QPW\\
\varphi_{17} &= QP^2U - 4Q^2PW + 4Q^3V\\
\varphi_{18} &= 2Q^5U + 2Q^3U^2 + \frac{1}{2}QU^3 + 4Q^2PW + PUW - 4Q^3V\\
\varphi_{19} &= \frac{1}{2}QPU^2 + 2Q^4W + P^2W\\
\varphi_{20} &= -\frac{1}{2}Q^3PU + 2Q^4W + \frac{1}{2}Q^2UW + QPV\\
\varphi_{21} &= 2Q^4PU + P^3U + Q^2PU^2 - 4Q^5W - 2QP^2W - 2Q^3UW\\
\varphi_{22} &= P^2U^2 - 4QPUW + 4Q^2UV\\
\varphi_{23} &= -\frac{1}{2}Q^2PU^2 + 2Q^3UW + \frac{1}{2}QU^2W + PW^2\\
\varphi_{24} &= 6Q^9 + 6Q^5P^2 + \frac{3}{2}QP^4 + 8Q^7U + 4Q^3P^2U\\
&\quad + \frac{7}{2}Q^5U^2 + \frac{3}{4}QP^2U^2 + \frac{1}{2}Q^3U^3 + 2Q^4PW + P^3W + Q^3W^2\\
\varphi_{25} &= -Q^5PU - \frac{1}{2}QP^3U - Q^3PU^2 - \frac{1}{4}QPU^3 + 2Q^6W\\
&\quad + Q^2P^2W + 2Q^4UW - \frac{1}{2}P^2UW + \frac{1}{2}Q^2U^2W + QPW^2
\end{aligned}
$$

$$\begin{aligned}
\varphi_{26} &= 4Q^8P + 4Q^4P^3 + P^5 + 8Q^6PU + 4Q^2P^3U + 4Q^4PU^2 \\
&+ \frac{1}{2}Q^2PU^3 - 8Q^7W - 4Q^3P^2W - 6Q^5UW + QP^2UW - Q^3U^2W - 2Q^2PW^2 \\
\varphi_{27} &= 2Q^10 + 2Q^6P^2 + \frac{1}{2}Q^2P^4 + 3Q^8U + Q^4P^2U - \frac{1}{4}P^4U \\
&+ \frac{3}{2}Q^6U^2 - \frac{1}{4}Q^2P^2U^2 + \frac{1}{4}Q^4U^3 + 2Q^5PW + QP^3W + 2Q^3PUW - Q^4W^2 \\
\varphi_{28} &= \frac{3}{2}Q^9U + \frac{5}{2}Q^5P^2U + \frac{7}{8}QP^4U + \frac{9}{4}Q^7U^2 \\
&+ 2Q^3P^2U^2 + \frac{9}{8}Q^5U^3 + \frac{3}{16}QP^2U^3 + \frac{3}{16}Q^3U^4 - 4Q^6PW \\
&- 2Q^2P^3W - \frac{7}{2}Q^4PUW + \frac{1}{4}P^3UW + 2Q^5W^2 + \frac{1}{4}Q^3UW^2 + 4Q^7V \\
&+ 2Q^3P^2V + 3Q^5UV + \frac{1}{2}Q^3U^2V + Q^3V^2 \\
\varphi_{29} &= -Q^7PU - \frac{1}{2}Q^3P^3U + \frac{1}{2}QP^3U^2 + \frac{1}{2}Q^3PU^3 + \frac{1}{8}QPU^4 \\
&+ 4Q^8W + 2Q^4P^2W + 2Q^6UW - \frac{3}{2}Q^2P^2UW - \frac{1}{2}Q^4U^2W + \frac{1}{4}P^2U^2W \\
&- \frac{1}{4}Q^2U^3W + 2Q^3PW^2 - \frac{1}{2}QPUW^2 + 2Q^5PV + QP^3V + 2Q^3PUV - 2Q^4WV \\
\varphi_{30} &= P^4U^2 - 8QP^3UW + 16Q^2P^2W^2 + 8Q^2P^2UV - 32Q^3PWV + 16Q^4V^2 \\
\varphi_{31} &= -Q^8PU - Q^4P^3U - \frac{1}{4}P^5U - \frac{3}{2}Q^6PU^2 - \frac{7}{8}Q^2P^3U^2 \\
&- \frac{3}{4}Q^4PU^3 - \frac{1}{8}Q^2PU^4 + 2Q^9W + 2Q^5P^2W + \frac{1}{2}QP^4W + 3Q^7UW \\
&+ \frac{5}{2}Q^3P^2UW + \frac{3}{2}Q^5U^2W - \frac{3}{8}QP^2U^2W + \frac{1}{4}Q^3U^3W \\
&- \frac{5}{2}Q^4PW^2 - \frac{1}{4}P^3W^2 + \frac{1}{2}Q^2PUW^2 - \frac{1}{2}Q^3W^3 - Q^4PUV \\
&+ 4Q^5WV + Q^3UWV + Q^2PV^2 \ .
\end{aligned}$$

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