Quantum Einstein Gravity

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Advancements of Heat Kernel-based Renormalization Group Studies

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Abstract

The asymptotic safety scenario allows to define a consistent theory of quantized gravity within the framework of quantum field theory. The central conjecture of this scenario is the existence of a non-Gaussian fixed point of the theory's renormalization group flow, that allows to formulate renormalization conditions that render the theory fully predictive. Investigations of this possibility use an exact functional renormalization group equation as a primary non-perturbative tool. This equation implements Wilsonian renormalization group transformations, and is demonstrated to represent a reformulation of the functional integral approach to quantum field theory.

As its main result, this thesis develops an algebraic algorithm which allows to systematically construct the renormalization group flow of gauge theories as well as gravity in arbitrary expansion schemes. In particular, it uses off-diagonal heat kernel techniques to efficiently handle the non-minimal differential operators which appear due to gauge symmetries. The central virtue of the algorithm is that no additional simplifications need to be employed, opening the possibility for more systematic investigations of the emergence of non-perturbative phenomena. As a by-product several novel results on the heat kernel expansion of the Laplace operator acting on general gauge bundles are obtained.

The constructed algorithm is used to re-derive the renormalization group flow of gravity in the Einstein-Hilbert truncation, showing the manifest background independence of the results. The well-studied Einstein-Hilbert case is further advanced by taking the effect of a running ghost field renormalization on the gravitational coupling constants into account. A detailed numerical analysis reveals a further stabilization of the found non-Gaussian fixed point.

Finally, the proposed algorithm is applied to the case of higher derivative gravity including all curvature squared interactions. This establishes an improvement of existing computations, taking the independent running of the Euler topological term into account. Known perturbative results are reproduced in this case from the renormalization group equation, identifying however a unique non-Gaussian fixed point.

Zusammenfassung

Das Konzept der Asymptotischen Sicherheit ermöglicht die Konstruktion einer konsistenten Theorie der Quantengravitation in Form einer Quantenfeldtheorie. Die zentrale Annahme ist hierbei die Existenz eines nicht-gaußschen Fixpunktes des Renormierungsgruppenflusses, der es erlaubt Renormierungsbedingungen zu formulieren. Zur Untersuchung dieses Szenarios findet eine nicht-störungstheoretische exakte Renormierungsgruppengleichung Anwendung. Diese beschreibt Wilsonsche Renormierungsgruppentransformationen und stellt eine äquivalente Formulierung einer Quantenfeldtheorie im Pfadintegralzugang dar.

Die vorliegende Arbeit entwickelt einen algebraischen Algorithmus zur systematischen Konstruktion der Renormierungsgruppenflüsse von Eichtheorien und Quantengravitation. Dieser verwendet die nicht-diagonale Wärmekernentwicklung, die eine Auswertung von Operatorspuren über nicht-minimale Differentialoperatoren erlaubt. Der Vorteil dieses Algorithmus ist, dass keine weiteren vereinfachenden Annahmen (z.B. über Hintergrundfelder) gebraucht werden und ermöglicht damit die systematische Analyse von nicht-störungstheoretischen Phänomenen. Im Zuge dieser Entwicklung enthält die vorliegende Arbeit auch neue Ergebnisse im Bezug auf die Wärmekernentwicklung des Laplaceoperators auf einem beliebigen Vektorbündel.

Der erarbeitete Algorithmus wird verwendet um den Renormierungsgruppenfluss der Quantengravitation in der Einstein-Hilbert Trunkierung abzuleiten. Dabei wird explizit die Hintergrundunabhängigkeit des Ergebnisses demonstriert. Weiterhin werden die Effekte der Wellenfunktionsrenormierung im Geistsektor auf das Laufen der Kopplungskonstanten der Gravitation untersucht. Eine detaillierte Analyse zeigt, dass die zusätzlichen Beiträge zu einer Stabilisierung des nicht-gaußschen Fixpunktes beitragen.

Abschließend wird der Algorithmus auf Gravitation mit allen marginalen Wechselwirkungsoperatoren angewendet. Dieser Ansatz bildet die Grundlage für die weitere Erforschung des Theorienraumes der Quantengravitation, unter Berücksichtigung der topologischen Euler-Invarianten. Diese Rechnung zeigt die Konsistenz des neuen Zugangs mit klassischen störungstheoretischen Ergebnissen, und kann die Zweideutigkeit der bisher gefundenen nicht-gaußschen Fixpunkte auflösen.

Contents

1. Overture

1.1. High Energy Metaphysics

It is the foundation of theoretical physics to pursue the logical structure of a formalized description of nature. Thereby, it is the phenomenon of uniformity in experimental physics which serves as the basis for any attempt to capture the spectrum of possible observations in such a theoretical paradigm and is what allows to speak about a structure of nature itself. Primarily, the result of any experiment is a collection of data. In terms of its information content, this data may be compressible in form of an algorithm, a prescription which reproduces the full set of data, provided that any remaining discrepancy is statistically accounted for as error of measurement. A theory can in this sense be thought of as a representation of an infinite amount of data in a compact way, since it encodes for implications on an infinite number of possible experiments. This means that a theory which is concerned with observable phenomena can only be acquired as an extrapolation of already established observations, and is therefore a hypothesis about the regularity in nature. In reverse, any phenomena are compatible with any number of models, all being distinct in logically unrelated aspects.

The concept of an effective theory can be defined as a description that is constructed under neglection of a class of effects, considered to be inessential for the investigation of a problem at hand, in order to achieve an artificial compression of informational content. For the purpose of a significant reduction of the mathematical complexity of a model, this reduction can help to make a formalization accessible, without being fully consistent beyond a reign of validity of the description. Due to the principal preliminary status of descriptions of nature, every theory must be seen as an effective one, acknowledging the impossibility to identify any as fundamentally true. On the other hand however, no future insight can shatter the validity of thoroughly confirmed theories, as long as they are restricted to their respective limiting reigns of validity. Therefore, progress in the field of theoretical physics is sought in form of a generalization of an established mathematical formalism to extend its reign of validity, while at the same time rendering the known to be a special case, which thus retains its validity in a limit case.

Although generalizations of existing concepts cannot be logically constrained, advances on purely theoretical grounds can be achieved through unification. The guiding principle for uniting models with different ranges of applicability into one is not just the resulting simplification, but is motivated by the need for consistency. Indeed, the effects found in distinct experimental situations must combine to produce a certain outcome for any constellation imaginable, even if never practically realized. Such thought experiments yield predictions that will depend on underlying theoretical assumptions, in as far as any empirical law requires to be generalized in a particular way. While in the paradigm of pure mathematics one has to rely on logic exclusively, theoretical physics often relies on the overall consistency of nature itself, the fact that every experiment has a unique outcome, corresponding to an exact algebra. This is what allows to trust the mathematical structures, as long as they have proven to accurately represent at least an aspect of nature, even if used only as calculation instructions rather than a complete theory. The importance of rigour in mathematical descriptions is essential, but in a way relaxed to the requirement that there has to exist only some way to unambiguously define all operations that are required for the representation of physical principles. Without preconceptions about what to consider as standard definition, there does not exist a notion of naturality in mathematical physics. If rigour is abandoned even in the sense of allowing for weaker versions of certain concepts, one leaves the ground of logic [1].

Although it is only rarely the case that a collection of theoretical insights on a class of phenomena are restrictive enough to lead to conclusive statements, investigations on the robust predictions associated to any involved assumption are useful. Robust are such consequences that are necessarily connected to an assumption, allowing no further choices that would manifest in observable features. Therefore, all non-unique ways to construct formulations should respect certain defining structural principles, so that all essential consequences do not depend strongly on the alternatives. Note that any equivalent formulation of a model points to alternative ways to generalize it, thus shifting the emphasize on which of its contents is to be considered as a more fundamental principle, and which merely appear in particular realizations. However all extensions introduce more freedom of choice. So at least key observables should be robust against small variations of inner parameters to encode physical information in a sensible way. This requirement applies to effective theories while abstaining from any statement about nature in itself.

For the phenomenology of high energy particle physics, quantum field theory plays the role of a meta-theory [2]. This is to say that it is proposed to represent the logical structure which captures the microscopic dynamics in nature. The elementary algebra encoded this way allows for a multitude of models, describing different possible systems, using probability amplitudes of elementary events as the primary observable. Within the domain of theoretical high energy physics today remain a number of conceptual questions concerning the mathematical representation of phenomenologically relevant features. Quantum field theory is in several toy models, usually in reduced dimensionality, successfully treated rigorously with the use of analytical continuations [3]. This serves as a justification for the generalized application of equivalent manipulations in more complicated realistic models, although there are no known proofs of well-definedness.

For example, a standard attempt to constructively formulate a local quantum field theory uses excitations based on a Hilbert space, classified in terms of representations of internal and external symmetry groups. Although being the basic objects of observation, an interacting particle becomes an excessively complicated object, being formally defined by the time evolution of a one particle state

$$\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}Ht}\, \Psi^\dagger(x)\, |0\rangle\,, \tag{1.1.1}$$

with the Hamiltonian H and an operator valued distribution $\Psi(x)$ as quantum field. Already in the case of pure quantum electrodynamics (QED), the result is a product state including any number of electron-positron pairs and photons, completely inaccessible by any known method of analysis. As another manifestation of the quasi-particle problem, consider the usual approach of quantizing free particle states by canonical commutation relations, which are in turn coupled in perturbation theory. Without a mass gap, spurious infrared divergences will appear in the resulting amplitudes. In this case, the concept of a finite detector resolution needs to be introduced, rendering all such states with position space field excitations sufficiently close to each other indistinguishable, since they would be observed as a single particle carrying the total momentum. This construction turns out to be sufficient to cancel the infrared divergence of virtual effects of infinitely low momenta [4]. That well defined observables are indeed found by such remarkable resummations of the perturbative expansion should hint at the principal soundness of the theoretical approach, with only a simple, compact formulation missing.

A further obstacle presents itself as the concept of renormalization. It is one of the major structural consequences of quantum field theory, reappearing in any of its formulations [5]. It was realized already early on in the development of quantum field theory, that expressions supposed to represent observable features were strictly divergent. The proper mathematical treatment remained an integral point of concern until the discovery of Yang-Mills theory, which culminated in the establishment of the standard model of particle physics. The initial problem of the UV divergence of functional representations of scattering amplitudes in momentum space was rigorously resolved within the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization scheme. It does however not provide a resummable renormalized perturbative series in general, as seen for example in QED where precision observables like the electron's magnetic moment can be computed, despite the theory being ill-defined by itself.[1] With the generalization of QED's unitary gauge symmetry to non-abelian symmetry groups in Yang-Mills theory, these remaining issues disappeared due to the Borel-summability of its perturbative series, related to the asymptotic freedom of the theory [6]. In the form of quantum chromodynamics (QCD), this gave the first example of a phenomenologically relevant theory without an obvious deficiency, describing non-trivial interactions in a renormalizable way. However even today, a fully rigorous construction of phenomenologically relevant quantum field theories is still missing, mainly due to an insufficient understanding of renormalization and its embedding in the structure of such theories. After proving to be successful at describing the physics of high energy scattering experiments, attempts have been started to elaborate the notion of curved spacetime into quantum field theory. The generalization of concepts based on Minkowski space to construct a quantized theory of dynamical gravity seems again to threaten the status of quantum field theory, due to its incompatibility with renormalization as it came to be understood in the foundation of the standard model.

Both the theory of General Relativity describing gravity classically and quantum field theory as it is used for the construction of models of high energy scattering experiments are remarkably successful in their predictions of observations. In a semi-classical estimate for a fully localized particle, general relativity would imply a Schwarzschild horizon which causally isolates the particle inside from its surrounding. This contradicts the phenomenology of high energy particle physics, with various distinct species of particles interacting according to specific rules. Arguing that the energy should be considered as

[1]This can be seen as the triviality problem, stating that QED does not describe interactions when treated in a self-consistent manner.

smeared in a region of the Compton wavelength still requires unconvincing extrapolations, since genuine gravitational interactions are with respect to a quantized metric degree of freedom. It is also inconsistent to have gravitation in a classical form when coupled to quantized degrees of freedom, since a quantum field theory of matter fields on curved spacetime induces running gravitational coupling constants via the renormalization of the matter fields alone. The central question to be addressed here is weather quantum field theory is a suitable formalism to include gravitation in a consistent way.

Since renormalization entails much of the phenomenology and reoccurs in any formalization of quantum field theory, it must be seen as a structurally essential component of the theory itself. The algebraic form of renormalization manifests itself in a scale dependence with immediate observable consequences. The scaling behaviour of coupling constants, leading to variable interaction strengths, as well as the emergence of quantum anomalies are in fact required to match experimentally confirmed phenomena. Furthermore, the Yang-Mills interactions appearing in the standard model are renormalizable in a predictive manner specifically by being asymptotically free. The vanishing of coupling constants in the high energy limit, known as asymptotic freedom, can be seen as the emergence of a fixed point of a renormalization group transformation which corresponds to only a few attractive directions given by the terms already present in the action. This means that any possible higher order operator would be rendered irrelevant and suppressed in the low energy limit. The structure of the renormalization group thus reveals a mechanism by which the perturbatively renormalizable interactions are selected from the complete set of possible operators in an effective field theory. By virtue of this insight, the main practical problem of non-renormalizability occurring in a perturbative attempt to quantize gravity may be avoided in a similar way. The conceptual idea of asymptotic safety [7, 8] is to formulate renormalization conditions in relation to a generalized fixed point. To have a renormalizable theory, it is sufficient for such a fixed point to possess only finitely many UV-attractive directions, which may but include interactions, in which case it would be called a non-Gaussian fixed point.

Since renormalization about an interacting fixed point action is not in general accessible by perturbative techniques, the investigation of the asymptotic safety scenario requires the use of non-perturbative methods. In principle theories entailing a strong coupling limit are compatible with the framework of quantum field theory, and several approaches to extract relevant information in such cases are known. Important examples for such techniques are Dyson-Schwinger equations, lattice simulations, perturbative approximations using strong-weak coupling dualities, and exact renormalization group equations. The latter presents a most convenient tool for a discussion on a conceptual level while being algebraically accessible, and will serve as the foundation of the discussions in this thesis. Renormalization group methods adopting Wilson's modern viewpoint on renormalization [9] are constructed on a functional formulation of quantum field theory and provide the means to extract information on the correlation functions without referring to a notion of asymptotic states at any time.

Another virtue of an analysis based on the scaling behaviour of coupling constants is that it allows to extrapolate critical phenomena. Determining the phase portrait of a given theory can reveal qualitative changes in its behaviour, since the fixed points do not only allow for renormalization but dictate the topology of possible renormalization group trajectories. Related studies are especially relevant to elicit the phase structure in infrared QCD and to understand phenomena in solid state physics like Bose-Einstein condensation and supra conduction [10]. Fixed points controlling the high energy limit are well known from, e. g., $O(N)$-sigma-models [11], the Thirring model [12], and the Wilson-Fisher fixed point in scalar field theories [13]. More recently, non-Gaussian fixed points have been considered in Yukawa-systems [14] and the Higgs sector [15] to realize mechanisms in the standard model beyond the ones allowed within perturbative renormalization. With such a fixed point, gravity can have a well-defined description within the framework of non-perturbatively renormalizable quantum field theories, as will be explained in more detail in the next section.

For its many demonstrated successes quantum field theory underwent a change in perspective, gaining more trust in the formalism despite the lack of mathematical rigour. Particularly the concept of renormalization, initially perceived as a sign for deficiency, received much clarification in the context of the structure of the renormalization group. These developments motivate to pursue an approach to quantum field theory that is much more centred around this structure. The renormalization group is a key algebraic structure that while not being appropriately appreciated in current formalisms may allow for an ambiguity free formulation of high energy physics when put on rigorous grounds. It is one of the main concerns of this thesis to contribute to an overall understanding of renormalization and its role played in quantum field theory, particularly combined with curved spacetime.

1.2. The Quest for Quantum Gravity

Formulating a consistent and predictive quantum theory for gravity is one of the prime challenges of theoretical high energy physics today. The construction of quantum gravity as a perturbative quantum field theory based on a metric degree of freedom has many parallels to the case of Yang-Mills theory. Despite the formal similarity, such a straightforward attempt to the quantized description of gravity fails, due to the presence of a dimensionful coupling constant spoiling renormalizability. In this respect, 4-dimensional gravity behaves like Yang-Mills theory in more than 4 spacetime dimensions, where its coupling constant acquires negative mass dimension. Newton's constant G_N is already present in the Einstein-Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{|g|} \Big(R - 2\Lambda \Big) \,, \tag{1.2.1}$$

describing the classical theory of general relativity. As one can see from power counting arguments, it implies that the gravitational interaction vertices come with positive powers of momenta, which make the contributions of UV modes diverge. The cancellation of UV divergence in the process of renormalization then requires the inclusion of an infinite number of higher derivative terms, each introducing a further coupling constant. In pure gravity, it is still possible to renormalise the action S_{EH} at the 1-loop level [16]. However the 2-loop contributions introduce the Goroff-Sagnotti term [17, 18], which corresponds to the first appearing interaction that cannot be absorbed into the Einstein-Hilbert action. All higher loop orders produce ever more such terms. Although being formally well defined, it is therefore not possible to formulate renormalization conditions in a meaningful way, as observables will in general depend on all the coupling constants introduced by renormalization.

The inclusion of higher derivative terms beyond the Einstein-Hilbert action can cure this issue. Adding the curvature squared terms

$$S_{\mathrm{HD}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{|g|} \Big(aR_{\mu\nu}R^{\mu\nu} + bR^2 + R - 2\Lambda \Big) \,, \tag{1.2.2}$$

which come with dimensionless coupling constants in 4 dimensions make the theory asymptotically free and thus renormalizable [19, 20]. The mechanism in such higher derivative gravity models can be understood as a modified propagator scaling, with a momentum dependence like

$$G(p^2) = \frac{m^2}{p^2(p^2 + m^2)} = \frac{1}{p^2} - \frac{1}{p^2 + m^2} \,. \tag{1.2.3}$$

For large momenta one has $G(p^2) \sim p^{-4}$, so that amplitudes become stronger suppressed compared to the usual p^{-2}-behaviour, thus fixing the power counting. This suppression can also be conceived as a Pauli-Villars regulated propagator, subtracting the contribution of a massive copy of the original degree of freedom. Unfortunately, this way negative energy or negative norm states are introduced, leading to a dynamical instability or loss of S-matrix unitarity. In total the above action describes massless spin-2, accompanied by massive spin-2, and massive spin-0 modes with negative norm. This raises a serious question about the possibility of a consistent formulation, that is both unitary and renormalizable within quantum field theory.

Many attempts have since been started to find a suitable UV completion for a quantum theory of spin-2 excitations. For one, modifications of the action of curvature squared gravity S_{HD} have been considered to change the number and nature of the propagating degrees of freedom. Constructions like the Pauli-Fierz mass term can be used to describe topologically massive gravity. Some related models of Hořava gravity [21] pursue a balancing of the unitarity violating degrees of freedom by the use of spacetime isotropy violating terms. This class of models may thus achieve formal consistency, but their quantum corrections accordingly do not respect Lorentz symmetry, so that compatibility with experimental observation is unclear.

A different line of research is stronger oriented on the conceptual implications of a quantized spacetime geometry. Loop Quantum Gravity is motivated by the attempt of canonically quantizing gravity, see [22] for a review. The singularity encountered in the Wheeler-deWitt equation is therein avoided with the introduction of a finite step length via the choice of representation of the Heisenberg algebra. The formalism makes use of holonomies as degrees of freedom which allows to write the Hamiltonian constraints in a simplified form. The resulting Gauss and Diffeomorphism constraints are solved on a Hilbert space of spin networks, but the remaining Hamiltonian constraint could not yet be fully taken into account. Due to this technical problem, the status of the theory is still unclear.

Initially considered as a way to reproduce the Regge trajectories of mesons, the approach of string theory generalizes the variables of quantum mechanics to one-dimensional objects. These basic objects are coordinates of an embedding geometry with the interpretation of spacetime subject to fixed (open strings) or periodic (closed strings) boundary condition [23]. An analysis of the resulting closed string excitation spectrum reveals an ubiquitous spin-2 mode. Furthermore, the conditions for anomaly cancellation in this theory resemble Einstein's equations for the target space geometry, so that the known dynamics of gravitation are reproduced in a limit. The elegance of the approach to describe any interactions geometrically however culminates practically in very involved mathematical problems. Also, the theory requires at least in its better studied forms supersymmetry as well as extra dimensions for consistency, which both lack experimental evidence.

A yet more radical conceptual paradigm shift is realized in the use of pregeometric variables. If the gravitational degrees of freedom defined only on an abstract group manifold are such that they form a metric field only as a condensate, spacetime becomes an emergent phenomenon. For example, a very general formulation of this sort is Group Field Theory [24]. Herein, the equivalents of Feynman graphs receive a reinterpretation, with each n-vertex replaced by an $(n-1)$-dimensional simplex connected to form a discretized spacetime manifold. A partition sum then corresponds to a weighted average over those quantum geometries, which allows it to describe quantum gravitational effects. Due to the abstract nature of such theories, many dynamical phenomena like the splitting of universes into topologically disconnected regions become possible. It is thus not known to what extent realism is maintained.

All the above mentioned proposals, among various others, may of course turn out to be accurate theories of quantum gravity. However for the lack of experimental signatures guiding modifications of established physics, a more conservative approach to the problem is favoured. Indeed, despite the apparent breakdown of quantum field theory due to gravitational effects, the standard model of particle physics works very well in perturbation theory. The standard model does by itself include a few unresolved issues, for example related to the magnitude of CP violation or the mechanism of confinement.[2] Nevertheless, the question about the consistency of gravity in this picture is, without any direct experimental signal, on a purely logical level. Since the structure of the renormalization group emerges directly within a field theoretic quantization, the implied possibility of non-perturbative renormalizability is worth consideration. The mere failure of the perturbative approach to quantum field theory as the origin of the conceptual problems is, from this perspective, much less speculative than alternative research programs. With the confirmation of a standard model-like Higgs particle, together with the lack of signatures

[2]One may also consider the gauge anomaly cancellation in the standard model, involving the conspiracy of several logically unrelated parameters, to expect a deeper underlying structure.

of new physics (e.g. supersymmetry) strongly hints at asymptotic safety as the mechanism uniting high energy physics with gravity realized in nature. In this proposal quantum gravity does not need any unobserved features, like extra dimensions or additional particles, to be consistent.

Based on the fixed point structure of gravity in $d = 2 + \epsilon$ dimensions [25, 26], it was conjectured by Weinberg [7], that the UV-behaviour of gravity in four dimensions is controlled by an interacting fixed point, so that the theory is non-perturbatively renormalizable or asymptotically safe. Similar to Yang-Mills theory in 4 dimensions, gravity in $d = 2$ dimensions is power counting renormalizable, and accordingly shows a free, Gaussian fixed point. For higher dimensionality, a non-Gaussian fixed point (NGFP) of the gravitational renormalization group flow emerges. The key idea of asymptotic safety is that this fixed point serves as a way to restrict the parametrical freedom of the theory, replacing the power counting argument of perturbation theory. For renormalization group trajectories attracted to it at high energies, the fixed point ensures that dimensionless coupling constants remain finite, so that physical quantities are safe from UV divergences. Importantly, this allows to generalize the perturbative way to formulate renormalization conditions, so that only finitely many free parameters remain to determine the renormalization group trajectory realized in nature. Estimating the critical exponents that control the nature of the fixed point by a mean field power counting suggests that at most the marginal coupling constants become relevant, while all higher ones become enslaved by the fixed point. In four dimensions, this implies that all interactions beyond curvature squared terms are irrelevant, assuming that quantum effects will not turn out to be too strong to invalidate the argument [27]. This would suffice to define a quantum field theory of gravity as a UV complete, renormalizable theory, rendering merely the perturbative technique inapplicable.

To arrive at a fully consistent theory, its unitarity has to be established in order to guarantee the conservation of probability. In contrast to the higher derivative actions in perturbation theory, the propagator in asymptotically safe gravity carries a non-trivial scale dependence, estimated as

$$G(p^2) \sim p^{-2+\eta} \,, \tag{1.2.4}$$

with the anomalous dimension η [28]. While at high energies close to the fixed point, one has $\eta \approx -2$ leading to a similar correction of power counting as described above, for lower energies the propagator remaines essentially unchanged. This way, a mechanism controlling the dynamical degrees of freedom can be envisioned.

The asymptotic safety mechanism is non-trivially improving the situation as compared to perturbative computations as demonstrated in [29–31] by the inclusion of matter fields. In a perturbative treatment, gravity is no longer 1-loop renormalizable when it becomes coupled to matter. Instead, the additional contributions to the gravitational β functions have no strong impact in a renormalization group study, and so do not endanger the non-perturbative renormalizability. Similarly, the inclusion of the Goroff-Sagnotti term, appearing as the first non-absorbable divergence at 2-loop perturbation theory, can be expected not to have any severe effect, as it would not have in higher derivative gravity.

More specifically, the absence of corrections at the order of the Planck scale as they would be expected from semi-classical estimates in the coupling of gravity to the standard model implies a non-relevance of this scale. This can be understood as a strong sign for the need to take renormalization group effects on gravity into account, which render all intrinsic scales relative to resolution. In fact, the application of the ideas of asymptotic safety to the standard model coupled to gravity leads to an accurate prediction of the Higgs mass [32]. Furthermore, the interplay of gravity with the gauge fields improves their renormalization behaviour so that the triviality problem of pure QED is circumvented [33]. The defining signature of asymptotically safe gravity is the dominance of the renormalization group behaviour at high energies by the NGFP, leading to a strong suppression of gravitation as compared to the expectation following perturbative arguments. Besides this phenomenon, the proposal that gravity is subject to substantial renormalization effects also leads to a comparably strong influence on the low energy limit, due to the dimensionality of its coupling constants. This way, some yet unexplained cosmological phenomena, e.g. dark matter, may be accounted for due to the quantum nature of the spacetime geometry itself [34].

In general it can be said that if a UV fixed point exists in the renormalization group flow, it will have an influence on the high energy behaviour, and might even provide the mechanism to make gravity renormalizable. The concept of asymptotic safety alone does thereby not determine the entire theory of quantum gravity. A specific fixed point action describing the precise phenomenology could be given in terms of different and possibly inequivalent degrees of freedom, and contain alternating interaction monomials. The main point to note is however that a quantized version of ordinary Einstein gravity along these lines is well imaginable to be a consistent theory, and to be in full accordance with all observed data. Although the true theory of quantum gravity may turn out to require much more severe changes in the mathematical framework than one could anticipate now, it still needs to contain classical general relativity in a limit. The asymptotic safety program is to show a viable pathway to contain gravity in standard quantum field theory as a mere consequence of the intrinsic structure of renormalization, and is in this sense in contrast to any proposed paradigm shift in high energy physics.

A key ingredient in the investigation of this possibility is the gravitational version of an exact renormalization group equation [35]. Generally, there does not exist a notion of an exact solution of such an equation, since the choice of initial conditions represents an input of information in its own right, and is lacking a strict prescription. This is why in practice, only certain interaction terms are taken into account in order to find non-self-consistent approximate solutions of renormalization group equations. This procedure of truncating the most general form of action is justified in the sense of an effective field theory, interpreting any neglected terms as secondarily generated in renormalization, which carry only inessential information for the considered couplings.

Especially for the case of gravity, to elucidate the fixed point structure of the renormalization group flow and to understand its properties is of central importance for the asymptotic safety program. For reviews and detailed references on this subject see [36–42]. In this class, the truncation ansatz is spanned by diffeomorphism invariant operators, which are build from the metric of the spacetime manifold.[3] The most studied case, the so called Einstein-Hilbert truncation, encompasses a scale-dependent Newton's constant G_k and cosmological constant Λ_k. This setup has been analysed in a number of works, for example [35, 44–50], studying the influence of different gauge fixing, cutoff dependence, and the effect of extra spacetime dimensions. These investigations led to an impressive body of evidence that gravity indeed possesses a suitable NGFP, located at positive Newton's constant and cosmological constant, being very robust against any change of parameters present. Notably, the essential features of this picture already emerge from the structurally significantly simpler renormalization group equations obtained by the reduction of the gravitational degrees of freedom to the scalar conformal factor [51–53].

Subsequently, the Einstein-Hilbert ansatz was extended in a series of works, to include further interactions. The first refining computations included a higher derivative R^2 interaction [54–56]. Higher order polynomials in the scalar curvature within the framework

[3] It is as well viable to realize this scenario in terms of different degrees of freedom. For example a spin connection formulation was investigated in [43] with promissing results.

of $f(R)$-gravity were studied in [27, 57–60] and with non-minimal coupling to scalar fields in [61]. Some non-local operators in [57,62], and the Weyl-squared interactions in [31,63,64]. All these computations have identified a NGFP of the gravitational renormalization group flow, providing substantial evidence for the asymptotic safety scenario. The latter started an important line of research, since the interactions encoded by contractions of higher powers of the curvature tensor are characteristic features of gravity. In comparison with the $f(R)$ approach, which can even allow for studies of a basis including infinite coupling constants [65, 66], computations involving such tensor structures are as of now still restricted to the curvature squared terms. The reason for this discrepancy is found in the fact that a simple spherical background geometry can be employed in the former case to reduce the computational involvement significantly. In contrast to this case, the most general graviton propagator including higher derivative terms can only be given on a generic background geometry.

All of the above mentioned computations hint at the proposition that it is a characteristic of actions describing graviation to give rise to the NGFP. If this indeed turns out to be the case, it should be possible to elicite the conditions of the emergence of this fixed point by analytical means. Specifically, the remarkable stability of the properties found in the comparably simple Einstein-Hilbert truncation, demands clarification. Although it seems very much to be a robust feature and not as an artifact of the used methods of computation, most of what has been done is a case-by-case study, being able to compare only the numerical values of fixed point position and its critical properties. Investigating the behaviour of solutions of the renormalization group equation under the inclusion or neglection of interaction terms in a systematic way should provide insights into the underlying mechanism and may thus allow to find analytical statements about the existence of fixed points. It is in this context especially important to learn to what extent the sign of critical exponents is conserved upon inclusion of further terms, since these are used to classify the corresponding interactions as relevant or irrelevant. So far, this was possible only in very limited ways, because the technicalities of the computations required simplifying assumptions, foremost concerning the choice of gauge fixing and background geometry, to be manageable. It is one of the main advances elaborated in this thesis to present an algorithm for the solution of renormalization group equations, capable of dealing with the related complications. As this algorithm can be fully automated, a lot of applications become possible, that would not be feasible by manual calculations.

1.3. Outline

The rest of this thesis is organized as follows.

In chapter 2 we lay the conceptual foundation of the renormalization group. The first section gives a recap of the euclidean functional integral approach to quantum field theory and its perturbative treatment. This serves as the basis for a detailed conceptual discussion of the emergence of the renormalization group in the following section. Herein we also elaborate on its relation to the perturbative approach to improve the common understanding of its place within the framework of quantum field theory, and to motivate the attempt to define quantum field theories via the renormalization group. The third section finally presents the derivation of the exact renormalization group equation in several functional forms. This way we introduce the renormalization group equation at the level of the quantum effective action, which is the central tool for computations in the remainder of this thesis. The derived equation is generally applicable to models with complicated non-linear dynamics such as Yang-Mills theory and gravity, which makes its use a formidable technique for the systematic extraction of relevant and non-perturbative quantum effects, that avoids the evaluation of many Feynman diagrams.

The concept of curved spacetime is introduced into the context of quantum field theory in the first section of chapter 3. We make use of the heat kernel expansion to define covariant propagators and exemplify the perturbative computation of quantum corrections in the presence of gravitational fields. The second section gives a detailed discussion of the off-diagonal heat kernel method. Here we employ the deWitt algorithm to compute all heat kernel coefficients on a general vector bundle with non-minimal operator insertions to third order in the curvature. In the third section, these results are embedded into a novel formalism which allows to evaluate very general operator traces including non-minimal derivatives, as they appear in any quantum field theory of gauge degrees of freedom.

Chapter 4 combines the results of chapters 2 & 3, to establish a general algorithm for the solution of the renormalization group equation in the context of gauge symmetries. The first section reviews the quantized description of gauge theories. Here we focus on the specific complications arising due to gauge symmetries in practical computations, and introduce suitable decompositions for vector and tensor fields with according projection operators. The second section presents the computation of heat kernel coefficients being constrained to the subspaces defined via these projectors. These newly derived coefficients show a singularity structure which is reminiscent of infrared divergencies in quantum field theory. The meaning of these findings for gauge theories and the practical treatment of gauge fields in the renormalization group equation are discussed. Finally, as one of the main accomplishments of this thesis, an algorithm for the solution of renormalization group equations using an arbitrary expansion scheme is developed in the third section. Its virtue lies with the applicability to general gauge theories, in principle not requiring any prerequisites and does in particular not rely on specific simplifying choices for background geometry or gauge fixing, intrinsic to previous methods. With the implementation in computer algebra, the foundation of an extension of the studies of the quantum gravitational renormalization group flow including higher tensorial interactions is laid.

The renormalization group equation is solved for the Einstein-Hilbert case in chapter 5. Using the general algorithm presented in the preceding chapter, we derive the β functions for Newton's and the cosmological constant, thus demonstrating their background independence explicitly. Moreover, a new method of analysis is proposed, focusing on the term structure of the β functions in an attempt to elicit the mechanism which gives rise to the remarkable stability of the non-Gaussian fixed point. The second section extends the previous setup by a non-trivial running field renormalization in the Faddeev-Popov ghost sector. Making contact with the standard Einstein-Hilbert computation in the literature, we engage in a detailed analysis of the properties of the persisting gravitational fixed point. Here we include a study of the stability of the flow and a discussion of the effect of extra spacetime dimensions.

In chapter 6 we engage in the completion of the curvature squared ansatz for the renormalization group equation. With the mathematical foundations for this endeavour established in form of the above mentioned algorithm, we are able to perform the full non-perturbative computation. The first section outlines this computation, deriving all operators appearing in the functional traces. The second section continues the evaluation of the final result in a perturbative limit, demonstrating the reproduction of the associated β functions in a closed and systematic way. Furthermore, the non-universal terms found in this way improve the known results by allowing for only one unique non-Gaussian fixed point, thereby identifying a second one as unphysical.

Chapter 7 gives a concluding summary of the work presented in this thesis, and comments on the meaning of the accomplishments for future research.

In three appendices, we cover some reoccurring topics, which are relevant in multiple chapters. Appendix A reviews the derivation of a general basis of curvature monomials up to third order, which is used to represent the results of heat kernel computations. In appendix B we collect numerous commutation relations of covariant derivatives, which are used in many computations throughout the thesis. Appendix C defines and discusses the threshold functions, used to conveniently capture the cutoff dependence appearing in the renormalization group equation, and gives a comparison of some commonly used choices of infrared cutoffs.

Parts of the content of this thesis is already published in [67–70].

2. Functional Renormalization

2.1. The Structure of Quantum Field Theory

There exist a number of ways to define a quantum field theory, which is a quantum theory of infinitely many degrees of freedom, represented by fields assuming independent values on each spacetime point. On an abstract account, the theory presents an arrow mapping a category with the interpretation of spacetime into an observable algebra of local measurement operators [71, 72]. One rather elegant way to realize this structure is the use of a functional formulation [73], which allows to represent the mapping by operator valued distributions, forming a basis for the quantum fields. In the euclidean version, a probability measure can be constructed which provides the correct distribution of observables in a manifestly stochastic formalism. This so called path integral approach will serve as the basis for the discussions in this thesis.

The basic observables of a quantum field theory are scattering cross sections σ, given by the matrix elements as

$$\sigma_{\mathrm{in}\to\mathrm{out}} \;\sim\; |\langle\Psi_{\mathrm{out}}|\mathcal{T}\mathrm{e}^{-\mathrm{i}\int\mathrm{d}^d x\, h[\mathcal{X}]}|\Psi_{\mathrm{in}}\rangle|^2\,, \tag{2.1.1}$$

with a Hamiltonian density operator $h[\mathcal{X}]$ defining the time evolution of the model at hand, acting on the interacting Hilbert space vectors Ψ. Here, $\mathcal{X}$ denotes the full collection of fields under consideration, abbreviated into a notation with any spacetime or internal group indices suppressed. Via the famous LSZ reduction formula [74], these matrix elements can be expressed as integrals over the n-point correlation functions

$$G_n(x_1,\ldots,x_n) = \langle 0|\mathcal{T}\mathcal{X}(x_1)\ldots\mathcal{X}(x_n)|0\rangle\,, \tag{2.1.2}$$

given as normalized time ordered vacuum expectation values of the quantum fields $\mathcal{X}$.[1] For a compact notation, the set of these functions can be formally collected into a single

[1] The G_n are also known as the Wightman functions, which are usually studied on a complex continuation of spacetime. For the purpose of this chapter it is more convenient to use their Wick-rotated counterpart, the Schwinger functions.

generating functional Z, defining

$$G_n(x_1,\dots,x_n) = \frac{\delta}{\delta J(x_1)} \dots \frac{\delta}{\delta J(x_n)} Z[J]\Big|_{J=0}\,, \tag{2.1.3}$$

as its n-th functional derivative. The explicit form of $Z[J]$ follows immediately in terms of a formal power series

$$Z[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^d x_1 \dots \int \mathrm{d}^d x_n \; G_n(x_1,\dots,x_n) J(x_1)\dots J(x_n)\,, \tag{2.1.4}$$

thus allowing the use of functional techniques to represent many essential identities in quantum field theories in a compact form. The infinite number of integrations appearing in this formula does not cause any ambiguities since the expression serves merely the purpose of introducing a notation which units the equations (2.1.2) into one. The functional Z preserves the identities (2.1.3) in its moments expansion, and thus convergence of the sum will not be required at any time. With the definitions (2.1.2), for simplicity written in the form $G_n(x_1,\dots,x_n) =< \mathcal{X}(x_1)\dots\mathcal{X}(x_n) >$, the generating functional $Z[J]$ becomes

$$\begin{aligned} Z[J] &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^d x_1 \dots \int \mathrm{d}^d x_n \;\; < \mathcal{X}(x_1)\dots\mathcal{X}(x_n) > J(x_1)\dots J(x_n) \\ &= < \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^d x_1 \; \mathcal{X}(x_1) J(x_1) \dots \int \mathrm{d}^d x_n \; \mathcal{X}(x_n) J(x_n) > \\ &= < \mathrm{e}^{\int \mathrm{d}^d x \; \mathcal{X}(x) J(x)} >\,, \end{aligned} \tag{2.1.5}$$

the (Wick-rotated) characteristic function of the associated distribution, in complete analogy to the usual definition in probability theory [75]. From this expression one can construct the expectation values of any function of the fields in terms of functional derivatives

$$< F[\mathcal{X}] >= F\left[\frac{\delta}{\delta J}\right] \; Z[J]\,\Big|_{J=0}\,. \tag{2.1.6}$$

In order to give explicit account for the expectation values, and thus determine the specific predictions of a model, the definition of a probability measure μ on the space of quantum fields is required. This can in general be written as

$$< F[\mathcal{X}] >= \mu(F[\mathcal{X}]) = \frac{1}{\mathcal{N}} \int \mathrm{d}\mu(\mathcal{X}) \; F[\mathcal{X}]\,, \tag{2.1.7}$$

including a normalization factor $\mathcal{N} = \int \mathrm{d}\mu(\mathcal{X})$. Establishing this definition grounds the formalism in a stochastic framework, where the fields $\mathcal{X}$ are no longer operator valued, but are now represented as random variables, subject to a distribution on the right hand side of (2.1.7). While the statistical nature of the theory is captured naturally, to preserve the canonical commutation relations it is sufficient to realize bosonic fields as ordinary complex valued, and fermionic fields as Grassmann valued random variables. The functional $Z[J]$ becomes

$$\begin{aligned} Z[J] =& \frac{1}{\mathcal{N}} \int \mathrm{d}\mu(\mathcal{X})\ \mathrm{e}^{\int \mathrm{d}^d x\ \mathcal{X}(x) J(x)} \\ =& \frac{1}{\mathcal{N}} \int D\mathcal{X}\ \tilde{Z}[\mathcal{X}]\ \mathrm{e}^{\int \mathrm{d}^d x\ \mathcal{X}(x) J(x)}\,, \end{aligned} \tag{2.1.8}$$

introducing the standard notation in quantum field theory as functional Laplace transformation in the second line. At this point it is sufficient to understand the integration over functions $D\mathcal{X}$ as a formal integral, for which substitution and partial integration rules hold by definition, without reference to the existence of the integral.[2]

To connect the stochastic formulation to a specific theory, we demand that its equations of motion are respected in the sense of expectation values

$$< \frac{\delta S[\mathcal{X}]}{\delta \mathcal{X}} - J >= 0\,, \tag{2.1.9}$$

giving the interpretation of external sources to the functions J. It is now an easy task to recover the Dyson-Schwinger equations in functional form by

$$\begin{aligned} 0 =& \int D\mathcal{X}\ \tilde{Z}[\mathcal{X}] \Big(\frac{\delta S[\mathcal{X}]}{\delta \mathcal{X}} - J \Big) \mathrm{e}^{\int \mathrm{d}^d x\ \mathcal{X}(x) J(x)} \\ =& \int D\mathcal{X}\ \Big(\frac{\delta S[\mathcal{X}]}{\delta \mathcal{X}} \tilde{Z}[\mathcal{X}] + \frac{\delta \tilde{Z}[\mathcal{X}]}{\delta \mathcal{X}} \Big) \mathrm{e}^{\int \mathrm{d}^d x\ \mathcal{X}(x) J(x)}\,, \\ \Rightarrow \quad & \frac{\delta S[\mathcal{X}]}{\delta \mathcal{X}} \tilde{Z}[\mathcal{X}] + \frac{\delta \tilde{Z}[\mathcal{X}]}{\delta \mathcal{X}} = 0\,, \end{aligned} \tag{2.1.10}$$

which determines the distribution function $\tilde{Z}[\mathcal{X}] = \mathrm{e}^{-S[\mathcal{X}]}$.

To summarize the above derivation of the euclidean functional integral formulation of quantum field theory, the observables are essentially determined by the generating functional

$$Z[J] = \frac{\int D\mathcal{X}\ \mathrm{e}^{-S[\mathcal{X}] + (\mathcal{X}, J)}}{\int D\mathcal{X}\ \mathrm{e}^{-S[\mathcal{X}]}}\,, \tag{2.1.11}$$

with a normalization such that $Z[0] = 1$ and the usual scalar product $(\mathcal{X}, J) = \int_x \mathcal{X}(x) J(x)$. One does also arrive at this expression as a field theoretical generalization of the path integral in quantum mechanics, which provides a formula for the matrix elements of the time evolution operator

$$< x' | \mathrm{e}^{-i \int \mathrm{d}t\ H} | x >= \prod_{n=0}^{\infty} \Big[\int \mathrm{d}x(t_n) \Big]\ \mathrm{e}^{\mathrm{i} S(x)} \Big|_{x(-\infty) = x}^{x(\infty) = x'}\,. \tag{2.1.12}$$

[2] This is a generalization guided by the intuition of an ordinary integral, analogous to formal sums, for which the operations of adding and multiplying are defined without requiring the convergence of the sum.

To achieve better analytical control and ensure the convergence, it is common practice to perform an analytical continuation to an imaginary time variable, thus eliminating the factor i in the exponent. This step of euclideanization preserves the structure of the theory, guaranteed by the Osterwalder-Schrader theorem, establishing explicitly the euclidean analogues of the Wightman axioms within the stochastic formulation [76].[3] Furthermore, the expression (2.1.11) can be understood, in statistical terms, as a partition function. This is due to the fact that the mathematical structure of measure theory provides a general formalism to describe probabilistic observables by the use of a weighted sum, given the probabilities of elementary events. The factors determining this distribution are generalized Boltzmann weights, as one has in statistical field theory

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\beta H} \,, \tag{2.1.13}$$

showing an inherent similarity to quantum physics. This comparison can also be exploited to find a finite temperature formulation of quantum field theory, since a Boltzmann factor is identical to a time evolution operator with the inverse temperature β appearing as an imaginary time variable.[4]

Although the statistical formulation is conceptually more transparent than alternative constructions of quantum field theories, it hides some mathematical difficulties in the explicit definition of the functional integral. It turns out to be quite a hard task to give mathematically precise meaning to an appropriate measure, as denoted $D\mathcal{X}$ above. The reason why a straightforward interpretation fails is that there does not exist a Lebesgue measure on infinite dimensional spaces. One way to make direct use of such an expression is by employing a lattice regularization, where the functional integration is replaced by a finite product $D\mathcal{X} = \prod_{i \in L} \mathrm{d}\mathcal{X}(x_i)$ over the set of lattice nodes L, with a certain discrete spacing $x_{i+1} = x_i + a$. This definition eliminates at the same time UV divergences, since no momenta bigger then $\frac{1}{a}$ can appear. However the approach forfeits spacetime symmetries, which can only be restored at the continuum limit $a \to 0$. A rigorous construction can be attempted using cylinder sets, which are the preimages of linear maps of a vector space into measurable spaces, thus inducing a (pre-)measure on the infinite dimensional vector

[3]In the case of gravity, the Lorentzian and Euclidean signature field equations are known to have in general inequivalent solution spaces. Nevertheless a statistical sampling of geometries still entails comparable features. See [77] for more details.

[4]Note that therefore the euclidean continuation becomes ambiguous for finite temperature calculations of non-equilibrium phenomena, as it mixes time and temperature in an indistinguishable way. This will however be of no concern for the purposes of this thesis.

space. It is most convenient in quantum field theory to employ the Wiener measure, which is a version of this procedure for probability distributions restricted to be of Gaussian form. This allows to reside with a formal treatment of the functional integrals, which is preferable as the main interest is to capture the underlying algebraic structure, and demonstrates the role of renormalization in quantum field theory. As a starting point to derive the perturbative expansion, we split the action functional into a quadratic kinetic part and an interaction part,

$$S[\mathcal{X}] = \tfrac{1}{2}\int_x \mathcal{X} D_0 \mathcal{X} + S_{\text{int}}[\mathcal{X}]\,, \tag{2.1.14}$$

where D_0 abbreviates any differential operator and $S_{\text{int}}[\mathcal{X}] = \mathcal{O}(\mathcal{X}^3)$.[5] It is then possible to extract the higher powers of the fields out of the functional integral to be left with a purely Gaussian integral, which is defined by decomposition into infinitely many one-dimensional integrals. This is a way to realize the Feynman-Kac formula to find

$$\begin{aligned} Z[J] =& \frac{1}{\mathcal{N}} \int D\mathcal{X}\ \mathrm{e}^{-S[\mathcal{X}]+\int_x \mathcal{X}J} \\ =& \frac{1}{\mathcal{N}}\ \mathrm{e}^{-S_{\text{int}}[\frac{\delta}{\delta J}]} \int D\mathcal{X}\ \mathrm{e}^{\int_x \left(-\frac{1}{2}\mathcal{X}D_0\mathcal{X}+\mathcal{X}J\right)} \\ =& \frac{1}{\mathcal{N}}\ \mathrm{e}^{-S_{\text{int}}[\frac{\delta}{\delta J}]}\ c\det(D_0^{-\frac{1}{2}})\ \mathrm{e}^{\int_x \frac{1}{2}JD_0^{-1}J} \\ =& \frac{\mathrm{e}^{-S_{\text{int}}[\frac{\delta}{\delta J}]}\ \mathrm{e}^{\int_x \frac{1}{2}JD_0^{-1}J}}{\mathrm{e}^{-S_{\text{int}}[\frac{\delta}{\delta J}]}\ \mathrm{e}^{\int_x \frac{1}{2}JD_0^{-1}J}\big|_{J=0}}\,, \end{aligned} \tag{2.1.15}$$

where the functional determinant of D_0 and an infinite constant c cancel with the normalization factor in the last step.[6] Recalling the formula for the n-point functions (2.1.3), we have

$$G_n(x_1,\ldots,x_n) \sim \tfrac{\delta}{\delta J(x_1)}\cdots\tfrac{\delta}{\delta J(x_n)}\mathrm{e}^{-S_{\text{int}}[\frac{\delta}{\delta J}]}\ \mathrm{e}^{\int_x \frac{1}{2}JD_0^{-1}J}\ \big|_{J=0}\,, \tag{2.1.16}$$

giving rise to the perturbative series of a given model, found from this expression by expanding in the coupling constants, parametrizing the interaction terms contained in S_{int}. The term structure following from this scheme has a diagrammatic representation in terms of Feynman graphs as follows. Any two derivatives with respect to J produce a free propagator $G(x-y) = D_0^{-1}\delta(x-y)$, determined by the quadratic part of the action, represented in a graph by a line connecting the points x and y. The n coordinates

[5]The integration over x will later be generalized for curved spacetimes of arbitrary dimension, defining $\int_x = \int \mathrm{d}^d x\sqrt{|g|}$.

[6]The exponent of D_0 in the determinant changes sign for integration over fermionic fields.

in the argument of G_n remain as external lines, while the integrations residing in S_{int} imply the connection of lines to one point, called an interaction vertex. It is exactly these integrations which lead to the UV divergence of quantum field theories.[7] Therefore one can see that the normalization $\mathcal{N}$ will cancel exactly the contribution of the so called vacuum graphs, which have no external lines. The perturbative approximation scheme thus relies on the coupling constants to be small parameters to expand in. However even if they are, the series converges poorly in many cases, and in practice only a finite number of terms is taken into account.

So far in this discussion, we used the so called disconnected n-point functions. These turn out to be less convenient, since even in a free theory there are purely combinatorial contributions to expectation values, as can be seen from the formula (2.1.16). Instead, to focus on genuine interaction processes, it is desirable to restrict the discussion to terms corresponding to connected graphs. This can be achieved fairly easy, since all disconnected graphs can be retrieved as a direct product of connected ones. Put in stochastic terms, the moments expansion provided by the generating functional $Z[J]$ should be replaced by a cumulant expansion, generated by the functional $W[J] = \log Z[J]$.[8] The connected n-point correlation functions $\frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} W[J]$ have the property to fall off rapidly for spacelike separation of any two of its arguments (because the propagators do). This makes their momentum space equivalents

$$\frac{\delta}{\delta J(p_1)} \cdots \frac{\delta}{\delta J(p_n)} W[J]\Big|_{J=0} = \frac{\delta}{\delta J(p_1)} \cdots \frac{\delta}{\delta J(p_n)} \log \mathrm{e}^{-S_{\text{int}}[\frac{\delta}{\delta J}]}\, \mathrm{e}^{\int_p \frac{1}{2} J(p) G(p) J(-p)}\Big|_{J=0}\,, \tag{2.1.17}$$

preferable to be used in scattering theory for practical phenomenological calculations.

Since the field expectation values are given as the first functional derivatives

$$\Phi(x) := < \mathcal{X}(x) > = \frac{\delta W}{\delta J(x)}\,, \tag{2.1.18}$$

it is possible to find a description of quantum field theory entirely in terms of these. The switch of variables is achieved by a generalized Legendre transformation, defining the functional

$$\Gamma[\Phi] = \sup_J \left(\int_x \Phi J - W[J] \right)\,, \tag{2.1.19}$$

[7] These integrals are usually written in momentum space. The notorious divergences are then created by the upper (UV) limit going to infinity.

[8] This is because the exponential map assigns exactly the correct combinatorial factors to generate all possible connected contributions to a full disconnected function.

in relation to W.[9] The functional Γ is called the quantum effective action, since its first derivative

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = J(x) \tag{2.1.20}$$

can be read as the equations of motion for the classical fields Φ with external source fields $J(x)$, and thus incorporates the quantum corrections effectively in a classical description. The Legendre transformation relates the Hessians $W^{(2)}$ and $\Gamma^{(2)}$ as inverse matrices in field space. Therefore, Γ generates the one-particle irreducible (1PI) graphs of a model, from which it is possible to reproduce all graphs, as illustrated by resumming the self-energy part of the full propagator (or connected 2-point function),

$$W^{(2)} = G\sum_{n=0}^{\infty}\Big((G^{-1}-\Gamma^{(2)})G\Big)^n = \Big(\Gamma^{(2)}\Big)^{-1}, \tag{2.1.21}$$

where products of functions are understood as convolution, and $(G^{-1}-\Gamma^{(2)})$ is introduced to contain by definition all non-trivial 1PI contributions of a given model. Thus in analogy to (2.1.4), one can write the expansion

$$\Gamma[\Phi] = \sum_{n=0}^{\infty}\frac{1}{n!}\int\mathrm{d}^d x_1\ldots\int\mathrm{d}^d x_n\ \Gamma_n(x_1,\ldots,x_n)\Phi(x_1)\ldots\Phi(x_n)\,, \tag{2.1.22}$$

defining the proper vertices $\Gamma_n(x_1,\ldots,x_n)$. From the definition (2.1.19) one can relate Γ to the generating functional Z by

$$\begin{aligned}\mathrm{e}^{-\Gamma[\Phi]} &= Z\ \mathrm{e}^{-\int_x \Phi\frac{\delta\Gamma}{\delta\Phi}}\\ &= \frac{1}{\mathcal{N}}\int D\mathcal{X}\ \mathrm{e}^{-S[\mathcal{X}]+\int_x(\mathcal{X}-\Phi)\frac{\delta\Gamma}{\delta\Phi}}\\ &= \frac{1}{\mathcal{N}}\int D\mathcal{X}\ \mathrm{e}^{-S[\mathcal{X}+\Phi]+\int_x \mathcal{X}\frac{\delta\Gamma}{\delta\Phi}}\,,\end{aligned} \tag{2.1.23}$$

shifting the integration variable in the second step. An expansion of the action in the exponent around the average field Φ yields the effective action organized in contributions of specific number of loop integrals

$$\Gamma[\Phi] = -\log\int D\mathcal{X}\ \mathrm{e}^{-S[\Phi]+\int_x\mathcal{X}(\frac{\delta\Gamma}{\delta\Phi}-S'[\Phi])-\frac{1}{2}\int_x\mathcal{X}\cdot S^{(2)}[\Phi]\cdot\mathcal{X}-\sum_{n=3}^{\infty}\frac{1}{n!}\int_x S^{(n)}_{\mathrm{int}}[\Phi]\cdot\mathcal{X}^n}\,. \tag{2.1.24}$$

The leading quantum corrections are found by neglecting higher powers of the fluctuation field $\mathcal{X}$. Tracking only the 1-loop contribution, one can perform the Gaussian integration to establish

$$\begin{aligned}\Gamma[\Phi] &= -\log\int D\mathcal{X}\ \mathrm{e}^{-S[\Phi]+\int_x\mathcal{X}(\frac{\delta\Gamma}{\delta\Phi}-\frac{\delta S}{\delta\Phi})-\frac{1}{2}\int_x\mathcal{X}\cdot S^{(2)}[\Phi]\cdot\mathcal{X}}\\ &= S[\Phi]+\tfrac{1}{2}\mathrm{Tr}\log S^{(2)}[\Phi]\,,\end{aligned} \tag{2.1.25}$$

[9] This transformation is unique and reversible in the case of convex functionals, otherwise the back transformation gives the convex hull.

with appropriate normalization of the logarithm.[10] Note that here the full Hessian of the action appears as a function of the background field, so that in contrast to the perturbative expansion (2.1.15), $\Gamma_{1-\text{loop}}$ contains already all orders of the coupling constants.

2.2. The Renormalization Group

The expressions discussed in the last section typically suffer from UV divergence. This can be seen by a counting of powers of momenta appearing in any specific n-point function, as a measure for the superficial degree of divergence. Since for a full correlation function, any number of vertices may appear in the corresponding Feynman graphs, one requires the degree of divergence not to increase with the number of vertices. This condition can be translated to the restriction of coupling constants to not carry negative mass dimension. If this condition is met, the procedure of perturbative renormalization can be applied. Algebraically, this is done by analytic continuation of the products of distributions occurring in the perturbative expansion (2.1.15).[11]

The perspective taken in this thesis is that of renormalization being an essential part in the phenomenology of quantum field theory. This is in contrast to the point of view that the appearance of divergence points at an inherent deficit of quantum field theory, which thus must be seen as an in itself incomplete theory, an approximation to some yet to be found completion, or at the very least requiring certain auxiliary features to serve as consistent description of nature microscopically. Taking on the other hand its acquired successes as hint for the structural soundness of renormalization and with it of quantum field theory as such, the study of related features attempted in the following may help on the way to find a simpler and more compact formulation and in fact will prove to allow insights that do not depend on perturbative methods.

Starting from the expression for a divergent n-point function (2.1.17), it is essential to preserve the dependence on the external momenta in all manipulations, since these encode the information of scattering processes. In a first step of regularization, a replacement of the form $W^{(n)}(p) \to W^{(n)}(p, a)$ provides a regular supplement expression, which is connected to the original one in a limit $a \to a'$. Since the integrand of the divergent integral itself is an analytic function, a continuation in the sense of a formal integral

[10]Here we used the operator identity $\log \det A = \mathrm{Tr} \log A$.

[11]Several schemes for doing so are known, most significantly the BPHZ subtraction method or the Epstein-Glaser construction, which gives an axiomatic definition at least for sufficiently simple cases.

measure is defined as the difference

$$W^{(n)}_{\mathrm{ren}}(p) = \lim_{a\to a'} W^{(n)}(p,a) - W^{(n)}(k,a)\,, \tag{2.2.1}$$

with respect to a reference momentum scale k, where the singularities cancel between the limits of the two terms. The so defined renormalized quantities $W^{(n)}_{\mathrm{ren}}$ are now finite, but carry a dependence on the arbitrary scale k, which should be understood analogously to a choice of units. As can be seen from the definition (2.2.1), the subtraction is formally equivalent to the introduction of another set of Feynman graphs with opposite sign. The corresponding new vertices required to achieve the above cancellation for a given theory are called its counter terms. The counter terms of the same form as those already present in the classical action can be absorbed into a redefinition of coupling constants. Thus renormalization simply replaces every occurrence of a divergent sub-graph by a renormalized version containing effective vertices, in a recursive manner, for counterterms stemming from some fixed loop order inserted as vertices contribute in turn only to higher orders.

To remain in the functional integral formulation manifestly, the equivalent prescription is to modify the integration measure to include the counter terms so that

$$\int (D\mathcal{X})'\, \mathrm{e}^{-S[\mathcal{X}]} = \int D\mathcal{X}\; \mathrm{e}^{-S_{\mathrm{c.t.}}[\mathcal{X}]}\mathrm{e}^{-S[\mathcal{X}]} = \int D\mathcal{X}\; \mathrm{e}^{-S_{\mathrm{ren}}[\mathcal{X}]}\,. \tag{2.2.2}$$

Notably, the required counter terms may not necessarily be selectable to satisfy all symmetries of the action, in which case a lost symmetry is said to be anomalous. In this context, renormalization can be thought of as introducing an anomaly in the classically scale invariant description and is therefore a genuine quantum effect. As a consequence of the counter terms, observables become subject to renormalization group flow, which can be seen as follows. Each counter term introduces a new parameter, which is fixed by imposing a renormalization condition, fitting a measurement at certain momentum scale k. Now, this renormalization scale can be chosen arbitrarily, yet measurements at different scales may not give equal results, but must only satisfy a consistency condition, set by the renormalization procedure. Specifically, since a model can be renormalized at any scale, there must be a systematic relation of the phenomenology at all such scales. This is the structure of the renormalization group. The notion of a perturbatively renormalizable theory is such that only finitely many counter terms are required, so that its action always contains the same terms with only the coupling constants changing with scale, which is to say the action is self-similar under renormalization group transformations.

A renormalization condition like $\Gamma^{(n)}(p=k) = g^n_{\text{ren,k}}$ defines renormalized coupling constants as proper vertices, and it is only by this step that the physical meaning of the coupling constants is fixed, because their meaning will be changed if different finite parts of the counter terms are chosen. To have all renormalization effects be grasped by the effective action, one assumes a parametrization of the form

$$\Gamma_k[\Phi] = \int_x \sum_i g_{i,k} I_i[\Phi]\,, \tag{2.2.3}$$

with the renormalized dimensionless coupling constants $g_{i,k}$. Because observables may not be influenced by the arbitrary scale parameter k, we can demand

$$k\tfrac{d}{dk}\Gamma_k^{(n)} = \left(k\frac{\partial}{\partial k} + \beta_i \frac{\partial}{\partial g_i} - \tfrac{1}{2}n\eta\right)\Gamma_k^{(n)} = 0\,, \tag{2.2.4}$$

known as the Callan-Symanzik equation. Its significance lies with the fact that herein is expressed that the explicit scale dependence of the counter terms is cancelled by the so called running of the renormalized coupling constants. This effect is captured by the β functions

$$\beta_i(g_1,\dots,g_n) = k\tfrac{\partial g_{i,k}}{\partial k}\,, \tag{2.2.5}$$

and the anomalous dimension

$$\eta(g_1,\dots,g_n) = -k\tfrac{\partial}{\partial k}\log Z_k\,, \tag{2.2.6}$$

where Z_k is a field renormalization, introduced as $\Phi_k = Z_k^{-1/2}\Phi$ in order to absorb a scale dependence of the fields and in turn to have the leading terms of the 2-point function $<\Phi(p)\Phi(-p)>_{\text{1PI}}$ become scale independent. An infinitesimal change in the scale parameter $k \to k + \mathrm{d}k$ thus becomes absorbed into a change of the coupling constants $g_i \to g_i + \beta_i\, \mathrm{d}k/k$. Both β and η do not explicitly depend on k, which is a manifestation of the group structure of renormalization. The crucial implication is that the relations between different finite scales and the unrenormalized theory are of the same form.

To formulate this insight at first in an abstract language, we start with an effective action of the form (2.2.3), viewing the coupling constants g_i as coordinates of a vector space spanned by the field monomials $\{\Gamma\} = \{I_1,\dots\}$, on which a renormalization map $\mathcal{R}$ is defined by

$$\Gamma_{\text{ren,k}} = \mathcal{R}_{(k,\infty)}(\Gamma_{\text{bare}})\,, \tag{2.2.7}$$

and represent the renormalization group by an operation

$$\mathcal{R}_{(k_1,k_3)} = \mathcal{R}_{(k_1,k_2)}\mathcal{R}_{(k_2,k_3)}\,. \tag{2.2.8}$$

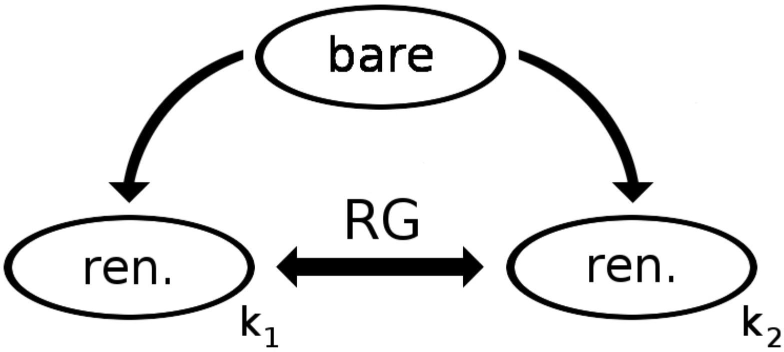

Figure 2.1.: Illustration of the relation induced by the renormalization of a bare action at two different scales k_1 and k_2, mediated by the renormalization group (RG).

Here the unrenormalized (bare) theory corresponds to an infinite scale, where the counter terms vanish. The emergence of the renormalization group as consistency requirement is illustrated in figure 2.1. In order not to remain dependent on the limitations of perturbation theory, this allows to give a generalized definition of renormalizability. An effective action Γ is renormalizable with respect to $\mathcal{R}$ if a continuum limit ($k \to \infty$) can be defined, such that only a finite number of coefficients remain undetermined. For this purpose the definition (2.2.7) should not simply be inverted, so instead one can study infinitesimal renormalization group transformations and integrate their trajectories. Thus renormalizability can be formulated as a condition on the existence of a fixed point action

$$\Gamma^* = \int_0^\infty \mathcal{R}_{(k+dk,k)}(\Gamma_k) = \lim_{N\to\infty} \sum_{n=0}^{N} \mathcal{R}_{(k+\frac{n+1}{N},k+\frac{n}{N})}(\Gamma_k)\,, \tag{2.2.9}$$

with $\{\Gamma^*\}$ being finite dimensional. Herein Γ^* is defined at infinite scale but in a still renormalized form in the sense that UV divergences remain absent. The condition on finite dimensionality is required to assure that even in the presence of infinitely many coupling constants, they are constrained to be functions of only a finite number among them. This is realized by having the fixed point of the renormalization map possess a basin of attraction which selects a finite dimensional subspace of $\{\Gamma\}$. As seen following the idea of (2.2.9), Γ^* defines a fixed point of the RG map since

$$\Gamma^* = \lim_{N\to\infty} \mathcal{R}^N \Gamma_k\,, \qquad \Rightarrow \quad \Gamma^* = \mathcal{R}\Gamma^*\,, \tag{2.2.10}$$

symbolically for a sequence of infinitesimal transformations $\mathcal{R}$. The phenomenon of asymptotic freedom in perturbation theory is thereby generalized to the mechanism

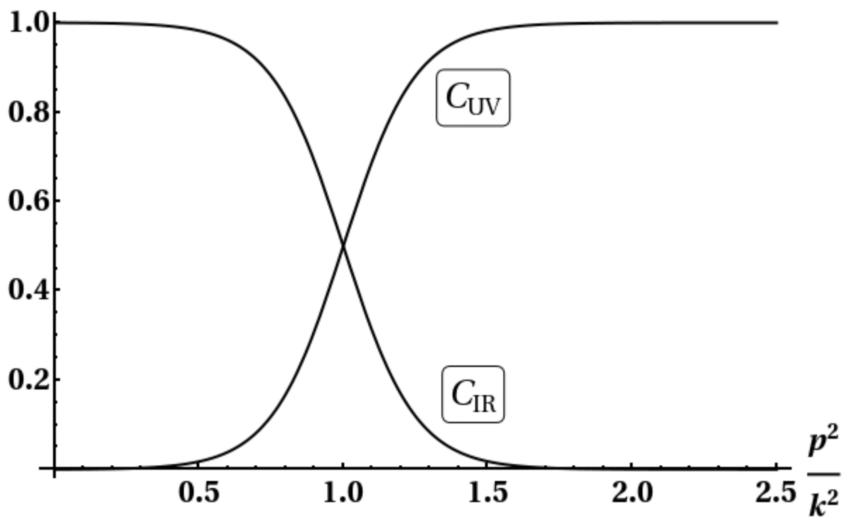

Figure 2.2.: Typical examples for the smooth cutoff functions C_{IR} and C_{UV} dressing the corresponding propagators to represent the separation of high and low momentum modes. To reproduce the full fields unambiguously, they are chosen to add to 1 at every point, while suppressing the respective momentum regions defined with respect to the scale parameter k.

called asymptotic safety [7]. The essential common property is that Γ^* does not itself carry a scale dependence, and therefore defines an end-point of renormalization group trajectories. A generalized notion of renormalization along these lines is essential for the proper realization of a quantized theory of gravity, which generates an infinite number of counter terms in a perturbative treatment [78].

A continuous realization of the renormalization group (2.2.7) with (2.2.8) is found with the Wilsonian idea of successive integration of momentum shells [9], ultimately allowing to study its structural consequences by means of a differential equation. To establish a split of the fields into low (IR) and high (UV) momentum modes according to

$$\mathcal{X} = \mathcal{X}_{\mathrm{IR}} + \mathcal{X}_{\mathrm{UV}} \,, \tag{2.2.11}$$

a reference scale k has to be given, in relation to which the separation is explained. For this purpose, we define the modified momentum space propagators

$$G_{\mathrm{IR/UV}}(p^2) = C_{\mathrm{IR/UV}}\Big(\frac{p^2}{k^2}\Big) G(p^2) \,, \tag{2.2.12}$$

dressed with the cutoff functions C_{IR} and C_{UV}, respectively. As illustrated in figure 2.2, these functions are chosen to extrapolate monotonously between 0 and 1, with the condition that $C_{\mathrm{IR}} + C_{\mathrm{UV}} = 1$ everywhere, and an inflection point at $p^2 = k^2$. With the propagators (2.2.12) that are accordingly suppressing the fields in the IR or UV region, the split (2.2.11) can be written with the use of formally independent fields $\mathcal{X}_{\mathrm{IR}}$ and $\mathcal{X}_{\mathrm{UV}}$, representing the two regions of momentum modes of each field $\mathcal{X}$. One can thus give an expression for the generating functional Z (2.1.11) with an action $S = S_{\mathrm{kin}} + S_{\mathrm{int}}$ (2.1.14) by

$$\begin{aligned} Z[J] &= \frac{1}{\mathcal{N}'} \int D\mathcal{X}_{\mathrm{IR}}\, D\mathcal{X}_{\mathrm{UV}}\ \mathrm{e}^{-\frac{1}{2}\int_x \mathcal{X}_{\mathrm{IR}} \frac{D_0}{C_{\mathrm{IR}}} \mathcal{X}_{\mathrm{IR}} - \frac{1}{2}\int_x \mathcal{X}_{\mathrm{UV}} \frac{D_0}{C_{\mathrm{UV}}} \mathcal{X}_{\mathrm{UV}} - S_{\mathrm{int}}[\mathcal{X}_{\mathrm{IR}} + \mathcal{X}_{\mathrm{UV}}] + \left(\mathcal{X}_{\mathrm{IR}} + \mathcal{X}_{\mathrm{UV}}, J\right)} \\ &= \frac{1}{\mathcal{N}'} \int D\mathcal{X}_{\mathrm{IR}}\, D\mathcal{X}\ \mathrm{e}^{-S[\mathcal{X}] + \left(\mathcal{X}, J\right)} \mathrm{e}^{-\frac{1}{2}\int_x \left(\mathcal{X}_{\mathrm{IR}} - C_{\mathrm{IR}}\mathcal{X}\right) \frac{D_0}{C_{\mathrm{IR}} C_{\mathrm{UV}}} \left(\mathcal{X}_{\mathrm{IR}} - C_{\mathrm{IR}}\mathcal{X}\right)} \\ &= \frac{1}{\mathcal{N}} \int D\mathcal{X}\ \mathrm{e}^{-S[\mathcal{X}] + (\mathcal{X}, J)}\,, \end{aligned} \tag{2.2.13}$$

which reduces to the original form after a shift in the $\mathcal{X}_{\mathrm{UV}}$ integration, and redefinition of the normalization constant to absorb the Gaussian $\mathcal{X}_{\mathrm{IR}}$ integral, following the ideas in [79]. The manifest separation of momentum modes implies a doubling of types of lines appearing in the generated Feynman graphs. While the full set of graphs will always reproduce the amplitudes independent of the scale k as seen from (2.2.13), individual graphs will depend on the exact shape of the cutoff functions. More specifically, any loop integrals become restricted to the corresponding momentum regions of the fields, due to the cutoff functions, smoothly suppressing IR or UV contributions. Since the UV divergence of the original full momentum integrals thus becomes restricted to $\mathcal{X}_{\mathrm{UV}}$ loops, the modified Feynman rules leave loop integrals over $\mathcal{X}_{\mathrm{IR}}$ finite, as they are constrained to momenta $p^2 \lesssim k^2$. This fact motivates the definition of an effective generating functional $Z_k[\mathcal{X}_{\mathrm{IR}}, J]$ for the low momentum fields by

$$Z[J] = \int D\mathcal{X}_{\mathrm{IR}}\ Z_k[\mathcal{X}_{\mathrm{IR}}, J]\ \mathrm{e}^{-\frac{1}{2}\int_x \mathcal{X}_{\mathrm{IR}} \frac{D_0}{C_{\mathrm{IR}}} \mathcal{X}_{\mathrm{IR}}}\,, \tag{2.2.14}$$

with the high momentum contributions integrated according to

$$\begin{aligned} Z_k[\mathcal{X}_{\mathrm{IR}}, J] &= \frac{1}{\mathcal{N}'} \int D\mathcal{X}_{\mathrm{UV}}\ \mathrm{e}^{-\frac{1}{2}\int_x \mathcal{X}_{\mathrm{UV}} \frac{D_0}{C_{\mathrm{UV}}} \mathcal{X}_{\mathrm{UV}} - S_{\mathrm{int}}[\mathcal{X}_{\mathrm{IR}} + \mathcal{X}_{\mathrm{UV}}] + \left(\mathcal{X}_{\mathrm{IR}} + \mathcal{X}_{\mathrm{UV}}, J\right)} \\ = \mathrm{e}^{-S_{\mathrm{int}}[\mathcal{X}_{\mathrm{IR}}]} &\frac{1}{\mathcal{N}'} \int D\mathcal{X}_{\mathrm{UV}}\ \mathrm{e}^{-\frac{1}{2}\int_x \mathcal{X}_{\mathrm{UV}} \frac{D_0}{C_{\mathrm{UV}}} \mathcal{X}_{\mathrm{UV}} - \sum_{n=1}^{\infty} \frac{1}{n!} S_{\mathrm{int}}^{(n)}[\mathcal{X}_{\mathrm{IR}}](\mathcal{X}_{\mathrm{UV}})^n + \left(\mathcal{X}_{\mathrm{IR}} + \mathcal{X}_{\mathrm{UV}}, J\right)}\,. \end{aligned} \tag{2.2.15}$$

Only the proper kinetic term for the low momentum fluctuations remains in the full functional $Z[J]$ explicitly, so that $Z_k[\mathcal{X}_{\mathrm{IR}}, J]$ accounts for their effective interactions. The interaction part of the action S_{int} appears in the second line of (2.2.15) with the modification by virtual high momentum modes, contributing to any order in $\mathcal{X}_{\mathrm{IR}}$. These correspond to effective vertices that are created from the bare vertices connected by the high momentum propagators, when replacing all their internal lines by a new point-like vertex. Thus the high momentum modes act like heavy particles whose interactions produce effective (non-renormalizable) interactions among the light particle modes. Herein, the IR field plays the role of a background field. As argued above, divergences are constrained to the UV integration, which thus requires regularization. The crucial point of the construction above is the finiteness of the remaining functional. Since the scale of separation k is left arbitrary, the means to study renormalization group transformations analytically is provided. Analogous to the definition of β functions (2.2.5) in perturbation theory, the unphysical dependence on k leads to the definition of a renormalization group equation.

To grasp the structure of the renormalization group in a differential equation and to give a precise definition to the fixed point action (2.2.9), we will now focus on infinitesimal changes of k. The low momentum generating functional Z_k produces loop integrals that are essentially covering the interval $[k, \infty]$, depending on the exact shape of the cutoff function C_{UV}. Therefore a small shift in k gives rise to the difference $Z_{k+\mathrm{d}k} - Z_k$ which will be dominated by loop integrals over the interval $[k, k+\mathrm{d}k]$, where $\frac{\partial C_{\mathrm{IR/UV}}}{\partial k}$ peaks. Likewise, an associated effective action $\Gamma_k[\mathcal{X}_{\mathrm{IR}}, \Phi]$ via generalization of (2.1.23) and parametrized as in (2.2.3) allows to write

$$\Gamma_{k+\mathrm{d}k} - \Gamma_k = \frac{\partial \Gamma_k}{\partial k}\,\mathrm{d}k = \int_x \sum_i \beta_i I_i[\mathcal{X}_{\mathrm{IR}}, \Phi]\,\frac{\mathrm{d}k}{k}\,, \tag{2.2.16}$$

which is independent of the chosen scheme of UV regularization. At the same time, a genuine generalization of the perturbative method of renormalization is achieved in this way, since all terms of an expansion of the functional integral are reproduced, and thus the scale dependence naturally captures the virtual particle contributions. The difference to the Wilsonian point of view is that here the effects of renormalization are translated from shifts in the renormalization conditions into changes of the scale k, corresponding to finite renormalization group transformations exclusively. It is a manifestation of the group structure of renormalization that running coupling constants can be found this way.

In the following section, we will derive a renormalization group equation in terms of the effective action Γ_k based on the concepts discussed so far, which is suitable for practical calculations. The functional Γ_k can be understood as a one parameter family of effective descriptions, valid at energy scales $p^2 \approx k^2$, because the lower momentum modes left out in the integration give only negligible contributions in that case. For this reason, it also accounts for a change in resolution of an observation, capturing the quantum effects of resolution scale dependent structure functions, as they famously appear in hadron physics.

2.3. Functional Renormalization Group Equations

A number of different renormalization group equations exist in the literature, mostly depending on the details of how cutoff functions are introduced, as well as on which generating functional is used in their derivation. For example, there is the Wegner-Houghton equation [80], the Polchinski equation [81], and the Wetterich equation [82] which will be used for practical computations in the later chapters of this thesis. Since our main concern lies on theories subject to gauge symmetries and especially on gravity, the split of an action into kinetic and interaction part (2.1.14) should be avoided, as non-abelian gauge transformations will require a cancellation between derivative and interaction terms to ensure invariance. For the purpose of representing the idea of Wilsonian renormalization, it is sufficient to introduce a damping of low momentum modes in the functional integral representation of the generating functional. To suppress the low momentum contributions and thus to effectively decouple them, we define

$$Z_k[J] = \int D\mathcal{X}\, \mathrm{e}^{-S[\mathcal{X}]-\Delta S_k[\mathcal{X}]+\left(J,\mathcal{X}\right)}\,, \tag{2.3.1}$$

which in comparison with (2.2.15) does not keep explicit track of the IR modes. The connection to the previous definition is realized by

$$Z_k[J] = Z_k[\mathcal{X}_{\mathrm{IR}} = 0, J]\,, \tag{2.3.2}$$

with a cutoff action quadratic in the quantum fields accounting for an analogue of the modification of propagators (2.2.12)

$$\Delta S_k[\mathcal{X}] = \tfrac{1}{2}\int_p \mathcal{X}\,\mathcal{R}_k(p^2)\,\mathcal{X}\,, \tag{2.3.3}$$

so that the full action S remains unmodified. In this implementation, the infrared cutoff function $\mathcal{R}_k(p^2)$ satisfies $\mathcal{R}_k \propto k^2$ for $p^2 \ll k^2$ and vanishes for high-momentum modes $\mathcal{R}_k \to 0$ as $p^2 \gg k^2$. In this way, it provides a k-dependent mass term for fluctuations with momenta $p^2 \lesssim k^2$, that thus freeze out.

From (2.3.1) one can immediately find the equation

$$\begin{aligned}\partial_k Z_k &= \int D\mathcal{X} \left(-\tfrac{1}{2} \int_p \mathcal{X}\, \partial_k \mathcal{R}_k\, \mathcal{X} \right) \mathrm{e}^{-S[\mathcal{X}]-\Delta S[\mathcal{X}]+\left(J,\mathcal{X}\right)} \\ &= -\tfrac{1}{2} \int D\mathcal{X} \left(\mathrm{Tr}\; \partial_k \mathcal{R}_k\, \mathcal{X}^2 \right) \mathrm{e}^{-S[\mathcal{X}]-\Delta S[\mathcal{X}]+\left(J,\mathcal{X}\right)} \\ &= -\tfrac{1}{2}\, \mathrm{Tr}\; \partial_k \mathcal{R}_k \int D\mathcal{X}\; \mathcal{X}^2\, \mathrm{e}^{-S[\mathcal{X}]-\Delta S[\mathcal{X}]+\left(J,\mathcal{X}\right)} \\ &= -\tfrac{1}{2}\, \mathrm{Tr}\; \partial_k \mathcal{R}_k \frac{\delta^2 Z_k}{\delta J^2}\,,\end{aligned} \tag{2.3.4}$$

by applying the k derivative and writing the momentum integration as functional trace. For the corresponding generating functional of connected correlation functions $W_k[J] = \log Z_k[J]$, this implies

$$\partial_k W_k = -\tfrac{1}{2}\, \mathrm{Tr}\; \partial_k \mathcal{R}_k \left[\frac{\delta^2 W_k}{\delta J^2} + \left(\frac{\delta W_k}{\delta J} \right)^2 \right]. \tag{2.3.5}$$

A renormalization group equation describing the running of proper vertices formulated in terms of a scale dependent generalization of the effective action Γ_k can be found using its relation to W_k according to the Legendre transformation (2.1.19). Taking the k dependence of $J_k = \frac{\delta \Gamma_k}{\delta \Phi}$ induced by the property (2.1.20) into account, the derivatives relate simply by

$$\partial_k W_k[J] = -\partial_k \Gamma_k[\Phi]\,, \tag{2.3.6}$$

wherein J is understood as scale independent. The r.h.s. of (2.3.5) can by translated using the definition of the average field (2.1.18), and the rule for the Hessian $\Gamma_k^{(2)} = \frac{\delta^2 \Gamma_k}{\delta \Phi^2}$ of a Legendre transform (2.1.21), yielding

$$\partial_k \Gamma_k = \tfrac{1}{2}\, \mathrm{Tr}\; \partial_k \mathcal{R}_k \left[(\Gamma_k^{(2)})^{-1} + \Phi^2 \right]. \tag{2.3.7}$$

The additive Φ^2 term produces the derivative of the cutoff action ΔS_k, which is absorbed into a shift in the definition of Γ_k. This redefinition implies

$$\begin{aligned} &\Gamma_k[\Phi] = \overline{\Gamma}_k[\Phi] + \Delta S_k[\Phi]\,, \\ \Rightarrow\; &\begin{cases} \partial_k \Gamma_k[\Phi] &= \partial_k \overline{\Gamma}_k[\Phi] + \tfrac{1}{2} \int_p \Phi\, \partial_k \mathcal{R}_k(p^2)\, \Phi \\ &= \partial_k \overline{\Gamma}_k[\Phi] + \tfrac{1}{2}\, \mathrm{Tr}\; \partial_k \mathcal{R}_k\; \Phi^2\,, \\ \Gamma_k^{(2)}[\Phi] &= \overline{\Gamma}_k^{(2)}[\Phi] + \mathcal{R}_k\,, \end{cases}\end{aligned} \tag{2.3.8}$$

which will be accounted for when discussing the proper initial conditions for the resulting differential equation. The final form of the 1PI functional renormalization group equation is now found as

$$\partial_k \overline{\Gamma}_k = \tfrac{1}{2}\, \mathrm{Tr}\; \partial_k \mathcal{R}_k \left(\overline{\Gamma}_k^{(2)} + \mathcal{R}_k \right)^{-1}. \tag{2.3.9}$$

This equation was first derived in [82], and is also known as the RG or flow equation. It has a conceptual similarity to the Callan-Symanzik equation (2.2.4), with the difference that the unphysical scale dependence originates from the cutoff action and the separation of momentum modes encoded by it, rather then counter terms. The renormalization group equation in this form provides the foundation of the computations in this thesis.

The factor $\partial_k \mathcal{R}_k$ on the r.h.s. ensures that the trace in the RG equation is finite and peaked at momenta $p^2 \approx k^2$, so that no UV regularisation is required. To find the appropriate initial conditions for this integro-differential equation, a relation of the so called effective average action $\overline{\Gamma}_k$ to scale independent quantities is needed. From (2.1.23) one can derive the expression

$$\begin{aligned} \mathrm{e}^{-\overline{\Gamma}_k[\Phi]} &= \mathrm{e}^{-\Gamma_k[\Phi]+\Delta S_k[\Phi]} \\ &= Z_k\ \mathrm{e}^{-\int_x \Phi \frac{\delta \Gamma_k}{\delta \Phi} + \Delta S_k[\Phi]} \\ &= \int D\mathcal{X}\ \mathrm{e}^{-S[\mathcal{X}]-\Delta S_k[\mathcal{X}]+\Delta S_k[\Phi]+\int_x (\mathcal{X}-\Phi)\frac{\delta \Gamma_k}{\delta \Phi}} \\ &= \int D\mathcal{X}\ \mathrm{e}^{-S[\mathcal{X}]-\frac{1}{2}\int_x (\mathcal{X}-\Phi)\mathcal{R}_k(\mathcal{X}-\Phi)+\int_x (\mathcal{X}-\Phi)\frac{\delta \overline{\Gamma}_k}{\delta \Phi}} \\ &= \int D\mathcal{X}\ \mathrm{e}^{-S[\mathcal{X}+\Phi]-\frac{1}{2}\int_x \mathcal{X}\mathcal{R}_k \mathcal{X}+\int_x \mathcal{X}\frac{\delta \overline{\Gamma}_k}{\delta \Phi}}\,, \end{aligned} \tag{2.3.10}$$

with the help of which the limits $k \to 0$ and $k \to \infty$ can be investigated. Since $\mathcal{R}_k$ vanishes at $k = 0$, the functional $\overline{\Gamma}_{k=0}$ can be identified with the ordinary effective action Γ. To simplify the opposite limit where

$$\lim_{k\to\infty} \mathcal{R}_k = \infty\,, \tag{2.3.11}$$

we use a field rescaling $\mathcal{X} \to \frac{\mathcal{X}}{\sqrt{\mathcal{R}_k}}$ to obtain

$$\begin{aligned} \mathrm{e}^{-\overline{\Gamma}_k[\Phi]} &= \int D\mathcal{X}\ \mathrm{e}^{-S[\mathcal{X}+\Phi]-\frac{1}{2}\int_x \mathcal{X}\mathcal{R}_k \mathcal{X}+\int_x \mathcal{X}\frac{\delta \overline{\Gamma}_k}{\delta \Phi}} \\ &= \int D\mathcal{X}\ \mathrm{e}^{-S[\frac{\mathcal{X}}{\sqrt{\mathcal{R}_k}}+\Phi]-\frac{1}{2}\int_x \mathcal{X}^2+\int_x \frac{\mathcal{X}}{\sqrt{\mathcal{R}_k}}\frac{\delta \overline{\Gamma}_k}{\delta \Phi}} \\ &\to \int D\mathcal{X}\ \mathrm{e}^{-S[\Phi]-\frac{1}{2}\int_x \mathcal{X}^2} \ = \ \mathrm{e}^{-S[\Phi]}\,. \end{aligned} \tag{2.3.12}$$

Thus $\overline{\Gamma}_k$ is in this limit given by the bare action S.[12] In the following the bar on the shifted functional $\overline{\Gamma}_k$ will be dropped, since the connection to the ordinary functionals is established via the boundary conditions for the RG equation, summarized as

$$\Gamma_{k=0}[\Phi] = \Gamma[\Phi]\,, \qquad \Gamma_{k=\infty}[\Phi] = S[\Phi]\,. \tag{2.3.13}$$

[12]See [83] for more details on this point.

For the practical use of the RG equation one starts with an ansatz for the effective action, parametrizing the field monomials $I_i[\Phi]$ with dimensionful coupling constants $u_i(k)$ that carry the full k dependence, like

$$\Gamma_k[\Phi] = \int_x \sum_i u_i(k) I_i[\Phi] \,. \tag{2.3.14}$$

The coupling constants describe the relative weight of the various terms. Observables however are always dimensionless ratios of quantities, which is to say that a system of units has to be chosen for dimensionful observables. For in the context of the renormalization group all scales are compared to the parameter k, the dimensionful coupling constants u_i can be translated to dimensionless couplings g_i by factors of the scale parameter k like

$$g_i = k^{-n_i} u_i \,, \qquad [u_i] = [k]^{n_i} \,, \tag{2.3.15}$$

without any assumption. It follows for the β functions (2.2.5) that

$$\beta_i = \partial_t g_i = \Big(- n_i g_i + k^{-n_i} \partial_t u_i \Big) \,, \tag{2.3.16}$$

where the dimensionless derivative $\partial_t = k \frac{\mathrm{d}}{\mathrm{d}k}$ with $t = \log k$ was defined. From (2.3.16) we see that in general the β_i vanish only for $g_i = 0$, due to the dimensional running induced by the first term for canonical dimensions $n_i \neq 0$. If the dimensional running dominates the β function, quantum effects are accounted for as small corrections, since no actual scaling behaviour would be expressed by a coupling constant with $\partial_t u_i = 0$. At the same time, it is this term that may allow for non-trivial fixed points $\beta_i = 0$ for coupling constants with negative mass dimension, that would spoil a perturbative renormalization.

From a parametrization of the effective action (2.3.14), the scale derivative

$$\partial_t \Gamma_k[\Phi] = \int_x \sum_i \partial_t u_i I_i[\Phi] = \int_x \sum_i \Big(\beta_i + n_i g_i\Big) k^{n_i} I_i[\Phi] \,, \tag{2.3.17}$$

becomes a generating functional for the (shifted) β functions. Therefore the RG equation (2.3.9), rewritten as

$$\partial_t \Gamma_k = \tfrac{1}{2} \mathrm{Tr} \left[\Big(\Gamma_k^{(2)} + \mathcal{R}_k \Big)^{-1} \, \partial_t \mathcal{R}_k \right] \,, \tag{2.3.18}$$

provides a definition for the vector field of β functions by

$$\begin{aligned} \beta_i =& - n_i g_i + k^{-n_i} \frac{\delta(\partial_t \Gamma_k)}{\delta I_i} \bigg|_{\Phi=0} \\ =& - n_i g_i + k^{-n_i} \frac{1}{2} \frac{\delta}{\delta I_i} \mathrm{Tr} \left[\Big(\Gamma_k^{(2)} + \mathcal{R}_k \Big)^{-1} \, \partial_t \mathcal{R}_k \right] \bigg|_{\Phi=0} \,, \end{aligned} \tag{2.3.19}$$

where (2.3.18) was used in the second step. This equation describes the renormalization group flow expressed in terms of a projection of the functional trace on the base monomials in the original ansatz. The RG equation (2.3.18) has the general form of a 1-loop amplitude, graphically represented as a single propagator $\Gamma_k^{(2)}$ dressed by the cutoff function $\mathcal{R}_k$, integrated with an insertion $\partial_t \mathcal{R}_k$. This can also be seen by

$$\partial_t \Gamma_k = \tfrac{1}{2}\mathrm{Tr}\left[\partial_t \mathcal{R}_k \,\frac{\delta}{\delta \mathcal{R}_k}\, \log\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)\right] = \tfrac{1}{2}\,\tilde{\partial}_t \,\mathrm{Tr}\,\log\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)\,, \tag{2.3.20}$$

where $\tilde{\partial}_t$ acts by definition only on the cutoff function. From this last expression one can see that if the full k dependence resides in $\mathcal{R}_k$, then Γ_k is given as the 1-loop contribution for an action $S_k[\Phi] = \Gamma[\Phi] + \Delta S_k[\Phi]$ according to (2.1.25). Employing an expansion $\Gamma_k = S + \Gamma_{1,k} + \Gamma_{2,k} + \ldots$ in loop orders, the RG equation can also reproduce higher loop orders by reinsertion of the $\Gamma_{n-1,k}$ under the trace [84].

For the example of a scalar field theory in the local potential approximation we have

$$\begin{aligned} \Gamma_k[\Phi] &= -\tfrac{1}{2}\int_x \partial_\mu \Phi \partial^\mu \Phi + V_k[\Phi]\,, \\ \text{with } V_k[\Phi] &= \sum_n \frac{u_n}{n!}\int_x \Phi^n\,. \end{aligned} \tag{2.3.21}$$

Plugging this expression into the RG equation allows to give the flow of the full potential $V[\Phi]$, yielding

$$\begin{aligned} \partial_t V_k[\Phi] &= \tfrac{1}{2}\mathrm{Tr}\left[\left(\Box + \mathcal{R}_k + V_k''[\Phi]\right)^{-1} \partial_t \mathcal{R}_k\right] \\ &= \tfrac{1}{2}\int \frac{d^d p}{(2\pi)^d}\, \frac{\partial_t \mathcal{R}_k(p^2)}{p^2 + \mathcal{R}_k(p^2) + V_k''[\Phi]}\,. \end{aligned} \tag{2.3.22}$$

Here, the trace is expressed as an ordinary integral in momentum space. The resulting integro-differential equation can be studied numerically, for instance to verify the existence of the Wilson-Fischer fixed point in $d = 3$ dimensions. The local potential approximation can in principle be refined by addition of higher derivative terms [85]. Similar and further complications arise however when dealing with gauge theories. Handling these will be a concern of the following chapters, with a special emphasize put on gravity.

Once the β functions are derived, they can be used to analyse the renormalization group behaviour of the underlying model. For this purpose all relevant information is encoded in a phase diagram, consisting of the trajectories of the coupling constants, generated by the β functions (2.2.5). The global structure of such diagrams is determined by the fixed points $\beta = 0$, since any trajectory can end only there or must escape to

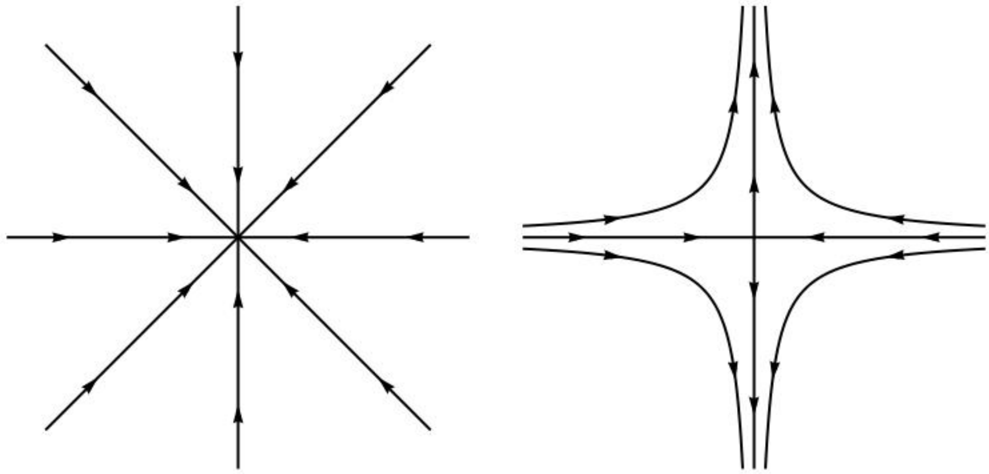

Figure 2.3.: Examples for 2-dimensional phase portraits around the vicinity of a fixed point with only attractive eigendirections (left) and with one attractive and one repelling eigendirection (right), respectively.

infinity, and curves of smooth vector fields cannot cross each other. Expanding around a fixed point g^*

$$\partial_t g_i = \beta_i(g) = \underbrace{\beta_i(g^*)}_{=0} + \frac{\mathrm{d}\beta_i}{\mathrm{d}g_j}(g_j - g_j^*) + \mathcal{O}\Big((g_j - g_j^*)^2\Big) \,, \tag{2.3.23}$$

the leading term is defined by the stability matrix $(D\beta)_{ij} = \frac{\mathrm{d}\beta_i}{\mathrm{d}g_j}$. The linearised flow around a fixed point in terms of $\bar{g}_i = g_i - g_i^*$ thus becomes

$$\partial_t \bar{g}_i = (D\beta)_{ij} \bar{g}_j \,, \tag{2.3.24}$$

which has an exact solution in terms of the eigenvalues and eigenvectors of $(D\beta)_{ij}$. This is given by

$$\begin{aligned} &(D\beta)_{ij} v_j = \lambda_i v_i \,, \\ \Rightarrow \quad &\bar{g}_i(k) = \sum_n c_{i,n} \mathrm{e}^{\lambda_n t} v_n = \sum_n c_{i,n} k^{-\theta_n} v_n \,, \end{aligned} \tag{2.3.25}$$

where the negative eigenvalues $\theta_n = -\lambda_n$ appear as critical exponents. Their sign determines whether the fixed point is UV-attractive ($\theta_n > 0$) or UV-repelling ($\theta_n < 0$) in the associated eigendirections v_n for increasing k. These cases are exemplified by typical pictures of phase portraits in figure 2.3, showing the implied local form of trajectories. As renormalization scheme independent quantities, the critical exponents present an important characterisation of quantum field theories. Drawing the analogy to the theory of critical phenomena, the scale parameter k can be seen as an order parameter, controlling the continuum limit as a second order phase transition of a formally regularized model. Interestingly, from this point of view the UV divergences can be expected to appear in the same way as the correlation length is known to diverge at a critical point, without spoiling the soundness of the mathematical description.

The functional renormalization group equations derived in this chapter constitute a rather universal and highly flexible tool for unlocking non-perturbative information, in the sense that all powers of the coupling constants appear in the inverse propagator $\Gamma_k^{(2)}$ in (2.3.18), and their magnitude is not required to be small. Furthermore, it even allows to renormalise theories with infinitely many coupling constants like (2.3.22), provided only finitely many are attracted to a UV fixed point, which is impossible by perturbative renormalization. For this reason, the RG equation is especially well suited to study effective field theories, which are constructed by including all leading terms into an action functional, that are allowed by the assumed degrees of freedom and symmetries (conservation laws), with only their coefficients remaining unknown. Assuming that quantum effects are not too big, we can estimate from (2.3.16) that $\beta_i \sim -n_i g_i$, neglecting the derivative term. This implies a suppression of exactly those coupling constants with negative canonical mass dimension n_i in the infrared limit, which would spoil perturbative renormalizability. Such coupling constants accordingly do not appear in the Standard Model of particle physics, rendering it likely to be understood as low energy limit of an effective field theory, with any non-renormalizable coupling constants effectively absent in the observable low energy regime. In turn, the solution of the RG equation determines the scale dependence of the coupling constants. Thus, it is one of the main practical virtues of functional renormalization group equations that non-perturbative features of a theory, like phase transitions or fixed points of the renormalization group flow can be accessed by comparably simple approximations. This method is regularly used in a plethora of settings ranging from condensed matter, over statistical and particle physics up to gravity [86–90].

Despite these advantages over other means to do calculations in quantum field theory, the renormalization group approach also has some practical limitations. Foremost, it relies on the fact that a single typical scale exists to which a renormalization parameter can be related. In multi-scale problems, the concept of running coupling constants is less clear, which also makes a definition of precision observables practically impossible. Asymptotic safety as a generalized form of renormalizability can be realized and studied within RG equations. However, assuming the existence of a fixed point action, it is still largely unclear how to obtain it in explicit form, since a choice of an action must always be given as initial condition. Instead, the β functions computed within a certain effective field theory ansatz can be interpreted irrespective of such an attempt. In this approach, all terms rejected in the projection (2.3.19) are read as corresponding to only effective couplings, that are inevitably generated in a Wilsonian RG step as explained before. In conclusion, the mechanism of asymptotic safety can be expected to control the behaviour of many models that effectively encode physical information, and should accordingly be taken into consideration.

3. The Heat Kernel Expansion

3.1. Quantum Field Theory on Curved Spacetime

The heat kernel is a mathematical method with a wide range of applicability both in mathematics and theoretical physics, providing analytical means to study the spectrum of operators [91–93]. The main virtue of its use within quantum field theory is grounded in the fact that the heat kernel offers a systematic way to generalize many of its central quantities in a covariant way to general curved spacetimes, which is required for the study of a quantized theory of gravity.

The functional renormalization group equation discussed in the last chapter allows to define a quantized version of general relativity by taking the metric field $g_{\mu\nu}$ of a Riemannian manifold with the interpretation of spacetime as dynamical degree of freedom. Since in this formulation the fields have a simple representation as random variables that capture their quantum nature, the metric will still play the role of a distance measure of spacetime by relating covariant and contravariant vectors

$$v_\mu = g_{\mu\nu} v^\nu \,, \quad v^\mu = g^{\mu\nu} v_\nu \,, \quad g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu \,, \tag{3.1.1}$$

like in the classical theory. Furthermore, the metric induces a unique torsion free connection

$$\Gamma^\alpha{}_{\mu\nu} = \tfrac{1}{2} g^{\alpha\beta} \Big(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu} \Big) \,, \tag{3.1.2}$$

the Levi-Civita connection, defining the covariant derivative

$$\begin{aligned} D_\mu v_\alpha &= \partial_\mu v_\alpha - \Gamma^\lambda{}_{\mu\alpha} v_\lambda \,, \\ D_\mu v^\alpha &= \partial_\mu v^\alpha + \Gamma^\alpha{}_{\mu\lambda} v^\lambda \,, \end{aligned} \tag{3.1.3}$$

that generalizes the ordinary directional derivative in flat spacetime, in a way that leaves the metric covariantly constant ($D_\mu g_{\rho\sigma} = 0$). Thus D_μ describes infinitesimal parallel transports of vectors along smooth curves on the curved manifold.[1] Considering a closed

[1] More correctly, an affine connection defines a mapping of vector fields between the tangent spaces of infinitesimally close points of the manifold, and thus induces a notion of transport on the manifold.

path of such transports, a vector will in general not be mapped into itself but instead show a relative rotation, the deficit angle. A universal measure of this effect is given by the Riemann curvature tensor, defined via the commutator

$$[D_\mu, D_\nu] v_\alpha = R_{\mu\nu\alpha}{}^{\lambda} v_\lambda \,. \tag{3.1.4}$$

More general and higher commutators which are useful for practical calculations on curved spacetime are collected in appendix B. Explicitly the curvature tensor is given in terms of the connection (3.1.2) as

$$R^\mu{}_{\nu\rho\sigma} = \partial_\nu \Gamma^\mu{}_{\rho\sigma} - \partial_\rho \Gamma^\mu{}_{\nu\sigma} + \Gamma^\mu{}_{\nu\gamma}\Gamma^\gamma{}_{\rho\sigma} - \Gamma^\mu{}_{\rho\gamma}\Gamma^\gamma{}_{\nu\sigma} \,. \tag{3.1.5}$$

It satisfies the symmetry properties (A.1.1) and the Bianchi identities (A.1.3) and (A.1.4), discussed in appendix A. The tensor $R_{\mu\nu\rho\sigma}$ and its non-trivial contractions, the Ricci tensor $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ and Ricci scalar $R = R^\mu{}_\mu$, are by construction covariant under reparametrisations of spacetime. This property allows to formulate an action principle, respecting the reparametrisation or diffeomorphism invariance as a symmetry in the description of the dynamics of gravity. The basis of monomials of the curvature tensor in terms of which a gravitational action can be given is discussed in appendix A.

The expressions discussed in the last chapter can now be generalized to arbitrary curved spacetimes using the replacement rules

$$\begin{aligned} \partial_\mu &\to D_\mu \,, \\ \mathrm{d}^d x &\to \mathrm{d}^d x \sqrt{g} \,, \end{aligned} \tag{3.1.6}$$

to covariantize the underlying action. The kinetic operator of a field will then typically be of general second order form

$$\Delta_0 = -g^{\mu\nu} D_\mu D_\nu + E \,, \tag{3.1.7}$$

including an endomorphism term E. Given such a differential operator of Laplace-type on a closed and torsionless Riemannian manifold, the associated heat kernel operator is defined as the exponential

$$H(s) = \mathrm{e}^{-s\Delta_0} \,, \tag{3.1.8}$$

which represents a formal solution to the generalized heat flow equation

$$\left(\partial_s + \Delta_0\right) H(s) = 0 \,. \tag{3.1.9}$$

This definition has immediate use in perturbative calculations at 1-loop order. In terms of the off-diagonal matrix elements of the heat kernel

$$\begin{aligned} H(x,y;s) &= \langle y|\mathrm{e}^{-s\Delta_0}|x\rangle \\ &= \mathrm{e}^{-s\Delta_{0,x}}\ \delta(x-y)\,, \end{aligned} \tag{3.1.10}$$

a definition of the propagator can be given as

$$\begin{aligned} G(x-y) &= \Delta_0^{-1}\delta(x-y) \\ &= \int_0^\infty ds\ \mathrm{e}^{-s\Delta_0}\ \delta(x-y) \\ &= \int_0^\infty ds\ H(x,y;s)\,, \end{aligned} \tag{3.1.11}$$

using its relation to the kinetic second order differential operator. In flat space, where $\Delta_0 = -\partial_\mu\partial^\mu$, this can be expressed easily in momentum space using the Schwinger-representation

$$G(p^2) = \frac{1}{p^2+m^2} = \int_0^\infty ds\ \mathrm{e}^{-s(p^2+m^2)}\,, \tag{3.1.12}$$

for the example of a massive scalar field. In this form, the integration over any loop momenta can always be performed, since the UV limit is exponentially suppressed. Therefore one can rewrite 1-loop integrals, as for example the contribution to the quantum correction to the 2-point function

$$\int d^dp'\ G(p'^2)G((p-p')^2) = \int_0^\infty ds_1 \int_0^\infty ds_2 \int d^dp'\ \mathrm{e}^{-s_1(p'^2+m^2)-s_2((p-p')^2+m^2)}\,, \tag{3.1.13}$$

shifting their divergence to the remaining integrations over s. While in general curved spaces there is no simple momentum space representation, the extension is straightforward in terms of the replacement

$$\int \frac{d^dp}{(2\pi)^d}\ \mathrm{e}^{-sp^2} \to \mathrm{Tr}\ \mathrm{e}^{-s\Delta}\,, \tag{3.1.14}$$

expressed as trace over the heat kernel.

Moreover, the heat kernel allows to compute counterterms and anomalies in an elegant way, by rewriting the 1-loop effective action (2.1.25) in the form

$$\Gamma_{1-\mathrm{loop}} = \tfrac{1}{2}\mathrm{Tr}\log\Delta_0 = \tfrac{1}{2}\int_0^\infty \frac{ds}{s}\ \mathrm{Tr}\ [\mathrm{e}^{-s\Delta_0}]\,. \tag{3.1.15}$$

In conclusion, the significance of the heat kernel in quantum field theory lies in its ability to generalize such expressions on flat space in a direct way to Riemannian manifolds as background geometry. After exponentiation of an operator like in (3.1.14) and (3.1.15), all computation is reduced to the functional trace

$$\mathrm{Tr}\ \mathrm{e}^{-s\Delta_0} = \int_x \langle x|\mathrm{e}^{-s\Delta_0}|x\rangle = \int_x H(x,x;s)\,, \tag{3.1.16}$$

which has a systematic expansion in terms of curvature monomials. In fact, a general trace involving a function of Δ_0 can formally be related to the traced heat kernel by a Laplace transform

$$\begin{aligned} f(x) &= \int_0^\infty ds\, \tilde{f}\,(s)\ \mathrm{e}^{-sx}\,, \\ \Rightarrow \quad \mathrm{Tr}\ f(\Delta_0) &= \int_0^\infty ds\, \tilde{f}\,(s)\ \mathrm{Tr}\ H(s)\,. \end{aligned} \tag{3.1.17}$$

Thus, a wide class of computations in quantum field theory involving the same differential operator Δ_0 are reduced to the calculation of the single object Tr $H(s)$, and can be kept in explicit covariance [94].

The trace of the heat kernel has two main such expansion schemes. For one, there is the local or early-time expansion in terms of powers of s

$$\mathrm{Tr}\ H(s) = (4\pi s)^{-d/2} \sum_{n=0}^{\infty} \int d^d x \sqrt{g} s^n\, \mathrm{tr}\, \overline{A_n}\,, \tag{3.1.18}$$

generally referred to as the Seeley-deWitt expansion [95, 96]. The quantities $\overline{A_n}[R](x)$ are the so called heat kernel coefficients and are local functions of the curvature invariants and their covariant derivatives. The other scheme is based on a non-local expansion in terms of curvature tensors and schematically represented by [97]

$$\mathrm{Tr}\ H(s) = (4\pi s)^{-d/2} \sum_n s^n \int \prod_{i=1}^{n} \left(d^d x_i \sqrt{g(x_i)}\right) F\left(s\Delta_{i_1}, \ldots, s\Delta_{i_n}\right) \mathcal{R}_{i_1} \ldots \mathcal{R}_{i_n}\,. \tag{3.1.19}$$

This expansion involves arbitrary powers of derivatives at every order in the curvature tensors $\mathcal{R}_i = \mathcal{R}_i(x_i)$, and thus realizes a derivative expansion. In (3.1.19) each operator Δ_i is acting only on the corresponding invariant $\mathcal{R}_i$ and the tensor structure of the invariants has been suppressed for brevity.

With the use of these expressions, it is quite easy to reproduce some general properties of perturbative quantum gravity. Considering a simple model of a scalar field on curved spacetime, the second variation of the action assumes the form

$$S^{(2)}[\Phi] = \Delta + V''[\Phi]\,, \tag{3.1.20}$$

where $\Delta = -\sqrt{g}^{-1}\partial_\mu \sqrt{g}\partial^\mu$ denotes the Laplace-Beltrami operator. The corresponding 1-loop quantum corrections to the effective action in terms of the average field and the background curvature tensor are found as

$$\begin{aligned}
\Gamma_{1-\text{loop}}[\Phi, R] &= \tfrac{1}{2}\int_0^\infty \frac{ds}{s}\ \mathrm{Tr}\ \mathrm{e}^{-s(\Delta+V''[\Phi])} \\
&= \tfrac{1}{2}\int_0^\infty \frac{ds}{s}\ (4\pi s)^{-d/2}\sum_{n=0}^{\infty}\int d^dx\sqrt{g}\ s^n\overline{A_n}[R]\ \mathrm{e}^{-sV''[\Phi]} \\
&= \tfrac{1}{2}(4\pi)^{-d/2}\sum_{n=0}^{\infty}\Gamma(n-\tfrac{d}{2})\int d^dx\sqrt{g}\ (V''[\Phi])^{d/2-n}\overline{A_n}[R]\,,
\end{aligned} \tag{3.1.21}$$

neglecting terms $\sim \mathcal{O}(\Delta V'')$, by applying the identities (3.1.15), (3.1.18) and identifying the integral representation of the Euler gamma function in the last step. Here, dimensional regularization was used to avoid the poles at $\Gamma(-\mathbb{N}_0)$. These are a direct manifestation of the UV divergence, stemming from the lower bound of the s integration. The integral may also diverge at the upper bound if V'' is not strictly positive. Such IR divergences originate from a zero eigenvalue of the Laplacian and should cancel in any observable with a fixed detector resolution taken into account. One can read off that the first three terms in the sum (3.1.21) are UV divergent for $d = 4$ spacetime dimensions, and thus induce counter terms of the same form. By dimensional analysis one can establish that the coefficients are $A_n \propto R^n$. Therefore we conclude that the renormalization of matter fields on curved spacetime already induces running coupling constants for dynamical gravity.[2] This is a sign for the need to quantize gravitational degrees of freedom in order to have a consistent theory. Specifically, there will be a contribution to the renormalization of the cosmological constant by $\sqrt{g}A_0$, to Newton's constant by $\sqrt{g}A_1$, and a term proportional to A_2.

Notably, the 1-loop effective action of pure gravity based on the Einstein-Hilbert action with appropriate gauge fixing will have a second variation of the same form (3.1.20). Making use of the Gauss-Bonnet theorem to rewrite $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 4R_{\mu\nu}R^{\mu\nu} - R^2 + E$ with the topological invariant E (see appendix A.3), and taking the equations of motion for the matter free case into account, one has the on-shell condition

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu} \quad \Rightarrow \quad R_{\mu\nu} = g_{\mu\nu}\frac{R}{d}\,, \quad R = \frac{2d\Lambda}{d-2}\,, \tag{3.1.22}$$

for $d \neq 2$, allowing to absorb the curvature squared terms residing in A_2 in a non-linear renormalization of the form $g_{\mu\nu} \to Z^{-1/2}(1 + cR)g_{\mu\nu}$, thus establishing the 1-loop renormalizability of pure gravity. However, as soon as any matter fields are included, this

[2]Similarly, the kinetic term of a Yang-Mills field would be induced as a counter term for a bare action that describes only matter fields on a corresponding gauge bundle.

property is lost and the counter terms $\sim R^2$ imply the perturbative non-renormalizability at 1-loop order.

The heat kernel also proves to be of essential importance in the solution of a renormalization group equation of the form (2.3.18), since it is in general exactly of 1-loop form. For the purpose of investigating the renormalization group behaviour of gravity, an expansion in terms of curvature invariants as it is provided by (3.1.18) is essential. Due to the implementation of the cutoff function (2.3.3) containing full dependence on the Laplacian, the operator under the trace becomes significantly more complicated, compared to the examples given above. Moreover, calculations based on gauge fields with general kinetic terms of non-Laplacian form can be reduced to the evaluation of

$$\mathrm{Tr}\left[D_{\mu_1}\ldots D_{\mu_k}\mathrm{e}^{-s\Delta_0}\right] = \int_x D_{\mu_1}\ldots D_{\mu_k} H(x,y;s)\Big|_{y=x}\,, \tag{3.1.23}$$

using the off-diagonal heat kernel. In the following, the required techniques are developed and expansion coefficients computed.

3.2. Off-diagonal Heat Kernel Coefficients

In order to establish a universal solution scheme for the exact renormalization group equation (2.3.18), the evaluation of operator traces must be handled systematically. This is achieved with the help of the heat kernel expansion, allowing for a recurrence relation for the off-diagonal heat kernel coefficients. The main virtue of the deWitt method used here is that it allows to compute operator traces of the general non-minimal form

$$\mathrm{Tr}\left[D_{\mu_1}D_{\mu_2}\ldots D_{\mu_n}\;H(s)\right], \tag{3.2.1}$$

by making use of the off-diagonal coefficients $A_n(x,y)$, generalizing the $\overline{A_n}(x)$. This is in contrast to a direct computation of only the diagonal ones appearing in

$$\mathrm{Tr}\,H(s) = \int_x H(x,x;s) = (4\pi s)^{-d/2}\int_x \sum_{n=0}^{\infty} s^n A_n(x,x)\,, \tag{3.2.2}$$

as seen from the minimal trace (3.1.18). In the following, a detailed description of the procedure is given.

3.2.1. Recurrence Relation for Off-diagonal Heat Kernel Coefficients

We assume that spacetime is a closed Riemannian manifold without boundary and of arbitrary dimension d. The Laplace operator Δ_0 in the heat kernel (3.1.8) is taken to be of general second order form

$$\begin{aligned}\Delta_0 =& -g^{\mu\nu}\partial_\mu\partial_\nu + a^\mu\partial_\mu + b \\ =& -g^{\mu\nu}D_\mu D_\nu + E \\ =& \Delta + E\,,\end{aligned} \tag{3.2.3}$$

where in the second line it is cast into standard notation [91], involving a covariant derivative operator $D_\mu = \nabla_\mu + A_\mu$ and an endomorphism E.[3] The Laplacian built from the covariant derivative only is denoted by $\Delta = -D^\mu D_\mu$, and ∇_μ is the covariant torsionless spacetime derivative compatible with the metric $g_{\mu\nu}$. We define A_μ to be a general connection on an internal bundle over the spacetime manifold and assume without loss of generality, that E is an endomorphism on the same bundle. Whenever this is not the case, it is sufficient to take it to be a direct product of the different respective bundles. Further, the sum of the curvatures of the connection A_μ and the Levi-Civita connection is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] + [\nabla_\mu, \nabla_\nu]\ . \tag{3.2.4}$$

For notational simplicity, all internal indices are suppressed, so that the quantities A_μ, $F_{\mu\nu}$ and E are understood as matrices on the internal space.

The heat kernel $H(s)$ owes its name to the fact that it solves a generalized heat equation with boundary condition $H(0) = \mathbf{1}$, where $\mathbf{1}$ denotes the identity on the internal space. To arrive at the heat equation in the form of an ordinary partial differential equation, it is convenient to express $H(s)$ in terms of its matrix elements,

$$H(x, y; s) \equiv \langle y|H(s)|x\rangle = \langle y|\mathrm{e}^{-s\Delta_0}|x\rangle\,, \tag{3.2.5}$$

called off-diagonal since their basis is formed at different points x and y of the manifold. The definition (3.2.5) is equivalently given as an initial value problem of the heat-equation

$$(\partial_s + \Delta_{0,x})\, H(x, y; s) = 0\,, \qquad H(x, y; 0) = \langle y|x\rangle = \delta_{x,y}\mathbf{1}\,. \tag{3.2.6}$$

The solution of this differential equation has the interpretation of heat propagating on the manifold, according to the operator $\Delta_{0,x}$, from a source located in y. Here s plays the role of a diffusion-time for the process.

The initial value problem (3.2.6) can be solved explicitly in the simple case of a flat manifold, where both the connection A_μ and the endomorphism E vanish. Here, the

[3] This can always be done as long as the manifold is torsionless [98].

matrix elements (3.1.10) are given by

$$\begin{aligned} H(x,y;s)\Big|_{\text{flat}} &= \mathrm{e}^{-s\Delta_{0,x}}\delta(x-y) \\ &= \int_k \frac{\mathrm{d}^d k}{(2\pi)^d}\mathrm{e}^{-sk^2}\mathrm{e}^{ikx-iky} \\ &= (4\pi s)^{-d/2}\mathrm{e}^{-\frac{(x-y)^2}{4s}}\,. \end{aligned} \tag{3.2.7}$$

This expression serves as an ansatz to find the general solution. Introducing a function $\Omega(x,y;s)$, it is parametrized by

$$H(x,y;s) = (4\pi s)^{-d/2}\ \mathrm{e}^{-\frac{\sigma(x,y)}{2s}}\ \Omega(x,y;s)\,. \tag{3.2.8}$$

Here $\sigma(x,y)$ is half the squared geodesic distance between x and y, called the "world function" [94]. It generalizes the flat space distance measure in (3.2.7) in a covariant way and satisfies

$$\frac{1}{2}\sigma_{;\mu}\sigma^{;\mu} = \sigma\,, \tag{3.2.9}$$

for arbitrary spacetime points x, y, and $\sigma(x,x) = 0$.[4] In the flat limit, (3.2.9) implies the identification $\sigma(x,y) = \frac{1}{2}(x-y)^2$, but will depend on the metric in the general case.

In order to find the partial differential equation satisfied by $\Omega(x,y;s)$, one substitutes the ansatz (3.2.8) into the heat-equation (3.2.6) to obtain

$$(\partial_s + \Delta_{0,x})\,H = \frac{1}{(4\pi s)^{d/2}}\mathrm{e}^{-\frac{\sigma}{2s}}\left(-\frac{d}{2s}\Omega + \frac{1}{2s}\sigma_{;\mu}{}^{\mu}\Omega + \frac{1}{s}\sigma_{;\mu}\Omega_{;}{}^{\mu} + \partial_s\Omega - \Omega_{;\mu}{}^{\mu} + E\Omega\right)\,. \tag{3.2.10}$$

The heat-equation is solved if the bracket on the right hand side vanishes.[5] Note that a solution of the heat equation is thus given for any pair of functions Ω and σ fulfilling this condition. The identification of σ with the world function is however crucial for the simplification of this problem. Inserting the early-time expansion of the heat kernel

$$\Omega(x,y;s) = \sum_{n\geq 0} s^n A_n(x,y)\,, \tag{3.2.11}$$

and requiring that the bracket in (3.2.10) vanishes at all orders of s independently, this yields the recursive equation

$$\left(n - \tfrac{d}{2} + \tfrac{1}{2}\sigma_{;\mu}{}^{\mu}\right)A_n + \sigma_{;}{}^{\mu}A_{n;\mu} - A_{n-1;\mu}{}^{\mu} + EA_{n-1} = 0\,,\quad n\geq 0\,, \tag{3.2.12}$$

[4] We use a semicolon to abbreviate any covariant derivative, $D_\mu a \equiv a_{;\mu}$. All derivatives are understood to be with respect to the coordinate x.

[5] This fact shows that the general solution can indeed be written as (3.2.8), provided an appropriate function $\Omega(x,y;s)$ exists. A van Vleck determinant does not explicitly appear in the heat kernel expansion this way.

	$\mathcal{R}^0$	$\mathcal{R}^{1/2}$	$\mathcal{R}^1$	$\mathcal{R}^{3/2}$	$\mathcal{R}^2$	$\mathcal{R}^{5/2}$	$\mathcal{R}^3$
	$D^2\sigma$	$D^3\sigma$	$D^4\sigma$	$D^5\sigma$	$D^6\sigma$	$D^7\sigma$	$D^8\sigma$
0	A_0	D^1A_0	D^2A_0	D^3A_0	D^4A_0	D^5A_0	D^6A_0
1			A_1	D^1A_1	D^2A_1	D^3A_1	D^4A_1
2					A_2	D^1A_2	D^2A_2
3							A_3

Table 3.1.: Analysis of the terms entering the recursion relation (3.2.12) at coincidence limit (we omit the overline for brevity). In order to compute an entry, one needs to compute every object that is above and on its left. Here $\mathcal{R}$ counts the number of curvature tensors and of square covariant derivatives. (For example $D^m R^n$ counts $\mathcal{R}^{n+m/2}$.)

subject to the initial conditions $A_{-1} = 0$ and $A_0 = \mathbf{1}$. On a non-trivial bundle, the additional indices carried by the covariant derivatives are inherited to the coefficients A_n.

Equation (3.2.12) still constitutes a complicate partial differential equation for the coefficients $A_n(x, y)$. In order to solve it, we exploit that the full off-diagonal heat kernel coefficients for non-coinciding points can be expressed as the geodesic expansion [99]

$$A_n(x,y) = \sum_{m\geq 0} \frac{(-1)^m}{m!} \overline{D_{\mu_1} \dots D_{\mu_m} A_n}\, \sigma^{;\mu_1} \dots \sigma^{;\mu_m} \,, \tag{3.2.13}$$

where the overline denotes the coincidence limit of any bi-tensor $C(x, y)$,

$$\overline{C}(x) := \lim_{y\to x} C(x,y) \,. \tag{3.2.14}$$

Notice that here, it is sufficient to know the quantities $\overline{D_{(\mu_1} \dots D_{\mu_m)} A_n}$ symmetrized in their indices.[6] Substituting (3.2.13) into (3.2.12) allows to recursively determine the expansion-coefficients (3.2.13) at coincident points $y \to x$. Since only covariant expressions are used in its derivation, the $A_n(x, y)$ are given as an expansion in curvature monomials.

To solve (3.2.12) for any $\overline{A_n}$, the coincidence limits of derivatives of A_{n-1} as well as σ are required. Comparing powers of the curvatures occurring in these objects, the systematics is easily found as summarized in Table 3.1. For example, to compute all coefficients up to the 6-derivative order ($\mathcal{R}^3$), one needs the coincidence limit of 8 derivatives acting on σ, 6 derivatives acting on A_0 and so on.

[6]It is important to note that in general $\overline{D_{(\mu_1} \dots D_{\mu_m)} A_n} \neq D_{(\mu_1} \dots D_{\mu_m)} \overline{A_n}$.

The coincidence limit of the derivatives of $\sigma(x,y)$ can be computed by making use of the property [100]

$$\overline{\sigma_{;\mu(\alpha_1\ldots\alpha_n)}} = 0\,, \quad n \geq 2\,, \tag{3.2.15}$$

implying that the only contribution beyond two derivatives will come from commutator terms. To exploit this, one can successively rewrite non-symmetric combinations of the derivatives as a sum of symmetric and antisymmetric pieces, and expressing the latter in terms of curvatures using the commutation rule (B.1.1). This method however proves to be inconvenient for the derivation of higher derivative terms without any symmetrization present. Instead, the quantities $\overline{\sigma_{;\mu_1\ldots\mu_n}}$ are computed much more efficiently with the help of the defining equation (3.2.9) [101]. With the initial condition $\overline{\sigma} = 0$, it is straightforward to find the first few expressions, by successively applying derivatives to the equation. After taking n derivatives of (3.2.9), terms appearing with $n+1$ derivatives vanish in the coincidence limit because $\overline{\sigma_{;\mu}}$ does. Inserting all lower order results and commuting the n indices in the remaining terms into a unique order immediately reveals the result. The method is conceptually straightforward and requires only simple algebra, yet the higher order terms increase in size quickly and render a manual computation unfeasible. With the help of computer algebra, up to the eighth derivative of σ was computed in the coincidence limit in the preparation of this thesis, to be readily accessible for heat kernel computations. However with the algorithm described above, it is only a matter of computation time to produce arbitrarily high derivatives in the same way. The coincidence limits up to fifth order in the derivatives are

$$\begin{aligned}
\overline{\sigma} &= 0\,, \qquad \overline{\sigma_{;\mu}} = 0\,, \qquad \overline{\sigma_{;\mu\nu}} = g_{\mu\nu}\,,\\
\overline{\sigma_{;\mu\nu\rho}} &= 0\,, \qquad \overline{\sigma_{;\mu\nu\rho\sigma}} = -\tfrac{1}{3}\left(R_{\mu\rho\nu\sigma} + R_{\mu\sigma\nu\rho}\right)\,,\\
\overline{\sigma_{;\mu\nu\rho\sigma\alpha}} &= -\tfrac{1}{4}\left(R_{\mu\nu\rho\sigma;\alpha} + R_{\mu\nu\rho\alpha;\sigma} + R_{\mu\sigma\rho\nu;\alpha} + R_{\mu\sigma\nu\alpha;\rho} + R_{\mu\alpha\nu\rho;\sigma} + R_{\mu\alpha\nu\sigma;\rho}\right)\,,
\end{aligned} \tag{3.2.16}$$

and to sixth order with symmetrization

$$\begin{aligned}
\overline{\sigma_{;\alpha\beta(\mu\nu\rho\sigma)}} = &-\tfrac{12}{5}R_{(\mu|\alpha|\nu|\beta|;\rho\sigma)} - \tfrac{4}{5}R_{(\mu\nu|\alpha|}{}^{\gamma}R_{\rho|\beta|\sigma)\gamma} - \tfrac{4}{5}R_{\gamma(\mu\nu|\alpha|}R_{\rho\sigma)\beta}{}^{\gamma} + \tfrac{8}{15}R_{\gamma(\mu\nu|\alpha|}R_{\rho|\beta|\sigma)}{}^{\gamma}\\
&+\tfrac{16}{45}R_{\gamma(\mu\nu\rho}R_{\sigma)\alpha\beta}{}^{\gamma} - \tfrac{8}{15}R_{\gamma(\mu\nu\rho}R_{\sigma)\beta\alpha}{}^{\gamma} + \tfrac{4}{9}R_{\gamma(\mu\nu\rho}R_{\sigma)}{}^{\gamma}{}_{\alpha\beta}\,.
\end{aligned} \tag{3.2.17}$$

The full unsymmetrized, as well as the expressions including seven and eight derivatives are too large to be presented here.[7]

[7] For the sake of demonstrating the fast increase in complexity, we state that $\overline{D^6\sigma}$ contains 92 terms, $\overline{D^7\sigma}$ contains 790 terms, and $\overline{D^8\sigma}$ close to 12000 terms.

With the derivatives of σ known to sufficient order, equation (3.2.12) is solved for the quantities $\overline{D_{\mu_1} \dots D_{\mu_m} A_n}$, for any $n, m \geq 0$, using the same method. It is because of the second order nature of the Laplacian operator Δ_0 that makes the recursion close, which is reflected by the fact that only positive powers of s are required in the ansatz (3.2.8) with (3.2.11) [100]. One can obtain an infinite set of algebraic equations by applying derivatives to (3.2.12) and taking the coincidence limit. These can then be solved recursively, substituting all lower order objects to find the next higher order in curvature quantities. Once all required derivatives are found for any n, one can proceed to $n+1$, until all ingredients to the heat kernel expansion up to a desired order are found. The results found by this procedure are listed in the next section.

Notably, the recursive relation for the heat kernel coefficients (3.2.12) becomes *independent* of the spacetime dimension, once the coincidence limit is taken, since $\overline{\sigma_{;\mu}{}^{\mu}} = d$ cancels the multiplicative d in (3.2.12). This is an important observation because it implies that the heat kernel coefficients of Laplace-type operators cannot depend on the dimension explicitly. We stress that this property does not hold for more general, non-minimal operators.

3.2.2. Heat Kernel Coefficients on a Vector Bundle

To solve traces of the general form (3.2.1), the deWitt-algorithm is used for determining their curvature expansion recursively. Following the discussion of the previous section, it is straightforward to implement the recursive equations (3.2.9) and (3.2.12) in a computer algebra software [102, 103] to find the $\overline{A_{n;\mu_1..\mu_m}}$ explicitly. With this algorithm realized in Mathematica, we are able to generalize the previously known results of [68, 99, 100, 104] to differential operators on a general gauge bundle including an arbitrary endomorphism. All coefficients contributing up to third order in the curvatures ($\mathcal{R}^3$) were computed that way.

Up to second order in the curvatures, the coincidence limit of the heat kernel coefficients and their derivatives are found as

$$\begin{aligned}
\overline{A_0} &= 1\,, \\
\overline{D_\mu A_0} &= 0\,, \\
\overline{D_{(\nu} D_{\mu)} A_0} &= \frac{1}{6} R_{\mu\nu}\,, \\
\overline{D_{(\alpha} D_\nu D_{\mu)} A_0} &= \frac{1}{4} R_{(\mu\nu;\alpha)}\,, \\
\overline{D_{(\beta} D_\alpha D_\nu D_{\mu)} A_0} &= \frac{3}{10} R_{(\mu\nu;\alpha\beta)} + \frac{1}{12} R_{(\beta\alpha} R_{\mu\nu)} + \frac{1}{15} R_{\gamma(\beta|\delta|\alpha} R^{\gamma}{}_{\nu}{}^{\delta}{}_{\mu)}\,, \\
\overline{A_1} &= -E + \frac{1}{6} R\,, \\
\overline{D_\mu A_1} &= -\frac{1}{2} E_{;\mu} - \frac{1}{6} F_{\nu\mu;}{}^{\nu} + \frac{1}{12} R_{;\mu}\,, \\
\overline{D_{(\nu} D_{\mu)} A_1} &= -\frac{1}{3} E_{;(\mu\nu)} - \frac{1}{6} R_{\mu\nu} E - \frac{1}{6} F_{\alpha(\mu;}{}^{\alpha}{}_{\nu)} + \frac{1}{6} F_{\alpha(\nu} F^{\alpha}{}_{\mu)} \\
&\quad + \frac{1}{20} R_{;(\mu\nu)} - \frac{1}{60} \Delta R_{\mu\nu} + \frac{1}{36} R R_{\mu\nu} \\
&\quad - \frac{1}{45} R_{\nu\alpha} R^{\alpha}{}_{\mu} + \frac{1}{90} R_{\alpha\beta} R^{\alpha}{}_{\nu}{}^{\beta}{}_{\mu} + \frac{1}{90} R^{\alpha\beta\gamma}{}_{\nu} R_{\alpha\beta\gamma\mu}\,, \\
\overline{A_2} &= \frac{1}{6} \Delta E + \frac{1}{2} E^2 - \frac{1}{6} R E + \frac{1}{12} F_{\mu\nu} F^{\mu\nu} \\
&\quad - \frac{1}{30} \Delta R + \frac{1}{72} R^2 - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{180} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}\,.
\end{aligned} \tag{3.2.18}$$

The terms containing five and six symmetrized derivatives of A_0 read

$$\overline{D_{(\gamma} D_\beta D_\alpha D_\nu D_{\mu)} A_0} = \frac{1}{3} R_{(\nu\mu;\alpha\beta\gamma)} + \frac{5}{12} R_{(\gamma\beta} R_{\nu\mu;\alpha)} + \frac{1}{3} R_{\rho(\gamma|\theta|\beta} R^{\rho}{}_{\nu}{}^{\theta}{}_{\mu;\alpha)}\,, \tag{3.2.19}$$

and

$$\begin{aligned}
\overline{D_{(\delta} D_\gamma D_\beta D_\alpha D_\nu D_{\mu)} A_0} &= \frac{5}{14} R_{(\nu\mu;\alpha\beta\gamma\delta)} + \frac{3}{4} R_{(\delta\gamma} R_{\nu\mu;\alpha\beta)} + \frac{4}{7} R_{\rho(\delta|\theta|\gamma} R^{\rho}{}_{\nu}{}^{\theta}{}_{\mu;\alpha\beta)} \\
&\quad + \frac{15}{28} R_{\rho(\gamma|\theta|\beta;\delta} R^{\rho}{}_{\nu}{}^{\theta}{}_{\mu;\alpha)} + \frac{5}{8} R_{(\gamma\beta;\gamma} R_{\nu\mu;\alpha)} \\
&\quad + \frac{5}{72} R_{(\delta\gamma} R_{\beta\alpha} R_{\nu\mu)} + \frac{1}{6} R_{(\delta\gamma} R_{|\rho|\beta|\theta|\alpha} R^{\rho}{}_{\nu}{}^{\theta}{}_{\mu)} \\
&\quad + \frac{8}{63} R^{\rho}{}_{(\delta|\theta|\gamma} R^{\theta}{}_{\beta|\lambda|\alpha} R^{\lambda}{}_{\nu|\rho|\mu)}\,.
\end{aligned} \tag{3.2.20}$$

The expressions for three symmetrized derivatives acting on A_1 and one derivative acting on A_2 contain both the endomorphism and curvature of the bundle. They are given by

$$\begin{aligned}
\overline{D_{(\alpha}D_\nu D_{\mu)}A_1} \;=\; & -\frac{1}{4}E_{;(\mu\nu\alpha)}+\frac{1}{30}R_{;(\mu\nu\alpha)}-\frac{1}{4}ER_{(\mu\nu;\alpha)}-\frac{1}{4}E_{;(\alpha}R_{\mu\nu)}\\
&-\frac{3}{20}F_{\rho(\mu;}{}^{\rho}{}_{\nu\alpha)}+\frac{1}{5}F_{\rho(\alpha}F^{\rho}{}_{\nu;\mu)}+\frac{3}{10}F_{\rho(\mu;\alpha}F^{\rho}{}_{\nu)}-\frac{1}{12}F_{\rho(\alpha;}{}^{\rho}R_{\mu\nu)}\\
&-\frac{1}{30}F_{\rho(\mu;\alpha}R^{\rho}{}_{\nu)}-\frac{1}{10}F_{\rho(\alpha}R_{\mu\nu);}{}^{\rho}+\frac{1}{10}F_{\rho(\alpha}R^{\rho}{}_{\nu;\mu)}\\
&-\frac{1}{15}F_{\rho(\alpha;|\sigma|}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu)}-\frac{1}{40}(\Delta R_{(\mu\nu})_{;\alpha)}+\frac{1}{24}RR_{(\mu\nu;\alpha)}\\
&+\frac{1}{24}R_{;(\alpha}R_{\mu\nu)}-\frac{1}{15}R_{\rho(\alpha}R^{\rho}{}_{\nu;\mu)}+\frac{1}{60}R_{\rho\sigma}R^{\rho}{}_{(\mu}{}^{\sigma}{}_{\nu;\alpha)}\\
&+\frac{1}{60}R_{\rho\sigma;(\alpha}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu)}+\frac{1}{30}R^{\rho\sigma\tau}{}_{(\alpha}R_{|\rho\sigma\tau|\nu;\mu)}\,,
\end{aligned} \tag{3.2.21}$$

and

$$\begin{aligned}
\overline{D_\mu A_2} \;=\; & \frac{1}{12}(\Delta E)_{;\mu}+\frac{1}{3}E_{;\mu}E+\frac{1}{6}EE_{;\mu}+\frac{1}{12}E_{;\rho}F^{\rho}{}_{\mu}+\frac{1}{12}F^{\rho}{}_{\mu}E_{;\rho}\\
&+\frac{1}{12}EF^{\rho}{}_{\mu;\rho}+\frac{1}{12}F^{\rho}{}_{\mu;\rho}E-\frac{1}{12}E_{;\mu}R-\frac{1}{12}ER_{;\mu}+\frac{1}{60}\Delta(F^{\rho}{}_{\mu;\rho})\\
&-\frac{1}{60}F_{\rho\mu}F^{\rho\sigma}{}_{;\sigma}+\frac{1}{45}F^{\rho\sigma}F_{\rho\mu;\sigma}+\frac{1}{30}F_{\rho\sigma;\mu}F^{\rho\sigma}+\frac{1}{30}F_{\rho\mu;\sigma}F^{\rho\sigma}\\
&+\frac{1}{45}F^{\rho\sigma}F_{\rho\sigma;\mu}-\frac{1}{60}F^{\rho}{}_{\sigma;\rho}F^{\sigma}{}_{\mu}-\frac{1}{36}F^{\rho}{}_{\mu;\rho}R-\frac{1}{30}F^{\rho}{}_{\mu}R_{;\rho}\\
&+\frac{1}{30}F^{\rho\sigma}R_{\rho\mu;\sigma}-\frac{1}{90}F_{\rho\mu;\sigma}R^{\rho\sigma}+\frac{1}{180}F^{\rho}{}_{\sigma;\rho}R^{\sigma}{}_{\mu}-\frac{1}{45}F_{\rho\sigma;\gamma}R_{\mu}{}^{\rho\sigma\gamma}\\
&-\frac{1}{60}(\Delta R)_{;a}+\frac{1}{72}RR_{;\mu}-\frac{1}{180}R^{\rho\sigma}R_{\rho\sigma;\mu}+\frac{1}{180}R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa;\mu}\,,
\end{aligned} \tag{3.2.22}$$

respectively. Recall that E and $F_{\mu\nu}$ may be matrix-valued with respect to the internal bundle and therefore, in general, do not commute. The coefficients $\overline{D_{(\mu}D_\nu D_\alpha D_{\beta)}A_1}$ and $\overline{D_{(\alpha}D_{\beta)}A_2}$ which, following table 3.1 also enter the recursive construction of $\overline{A_3(x)}$ are not given here, because they are very lengthy expressions and therefore of little practical value when written explicitly. For the purpose of automated heat kernel computations, all these objects of cubical order in the curvature are readily obtained. Lastly, we state the non-derivative coefficient at third order in the curvatures

$$
\begin{aligned}
\overline{A_3} \;=\; & -\frac{1}{6}E^3 - \frac{1}{12}(\Delta E)E + \frac{1}{12}E_{;\mu}E_{;}{}^{\mu} - \frac{1}{12}E(\Delta E) - \frac{1}{60}(\Delta\Delta E) \\
& + \frac{1}{60}E_{;\nu}F^{\mu\nu}{}_{;\mu} - \frac{1}{60}F^{\mu\nu}{}_{;\mu}E_{;\nu} - \frac{1}{20}EF_{\mu\nu}F^{\mu\nu} - \frac{1}{90}F_{\mu\nu}EF^{\mu\nu} - \frac{1}{45}F_{\mu\nu}F^{\mu\nu}E \\
& - \frac{1}{90}(\Delta F_{\mu\nu})F^{\mu\nu} + \frac{1}{45}F_{\mu\nu;\rho}F^{\mu\nu;\rho} - \frac{1}{180}F_{\mu\nu}(\Delta F^{\mu\nu}) \\
& + \frac{1}{180}F^{\mu}{}_{\nu;\mu}F^{\rho\nu}{}_{;\rho} + \frac{1}{45}F^{\mu\nu}F_{\mu}{}^{\rho}{}_{;\rho\nu} + \frac{1}{90}F^{\mu\rho}{}_{;\rho\nu}F_{\mu}{}^{\nu} - \frac{1}{90}F_{\mu\nu}F^{\nu\rho}F^{\mu}{}_{\rho} \\
& + \frac{1}{12}E^2R + \frac{1}{36}(\Delta E)R - \frac{1}{30}E_{;\mu}R_{;}{}^{\mu} + \frac{1}{30}E(\Delta R) - \frac{1}{90}E_{;\mu\nu}R^{\mu\nu} \\
& - \frac{1}{72}ER^2 + \frac{1}{180}ER_{\mu\nu}R^{\mu\nu} - \frac{1}{180}ER_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\
& + \frac{1}{72}F_{\mu\nu}F^{\mu\nu}R + \frac{1}{30}F_{\mu\nu}F^{\mu\rho}R^{\nu}{}_{\rho} - \frac{1}{180}F_{\mu\nu}F_{\rho\sigma}R^{\mu\nu\rho\sigma} \\
& + \frac{1}{280}(\Delta\Delta R) - \frac{1}{280}R(\Delta R) + \frac{17}{5040}R_{;\mu}R_{;}{}^{\mu} + \frac{1}{420}R_{;\mu\nu}R^{\mu\nu} + \frac{1}{630}R_{\mu\nu}(\Delta R^{\mu\nu}) \\
& - \frac{1}{2520}R_{\mu\nu;\rho}R^{\mu\nu;\rho} - \frac{1}{1260}R_{\mu\nu;\rho}R^{\nu\rho;\mu} - \frac{1}{420}R_{\mu\nu\rho\sigma}(\Delta R^{\mu\nu\rho\sigma}) + \frac{1}{560}R_{\mu\nu\rho\sigma;\lambda}R^{\mu\nu\rho\sigma;\lambda} \\
& + \frac{1}{1296}R^3 - \frac{1}{1080}RR_{\mu\nu}R^{\mu\nu} + \frac{1}{5670}R_{\mu\nu}R^{\mu\rho}R^{\nu}{}_{\rho} \\
& - \frac{1}{1890}R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma} + \frac{1}{1080}RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{1}{945}R_{\mu\nu}R^{\mu}{}_{\rho\sigma\tau}R^{\nu\rho\sigma\tau} \\
& + \frac{1}{567}R^{\mu}{}_{\nu}{}^{\rho}{}_{\sigma}R_{\mu\alpha\rho\beta}R^{\nu\alpha\sigma\beta} + \frac{11}{11340}R^{\mu\nu}{}_{\rho\sigma}R_{\mu\nu\alpha\beta}R^{\rho\sigma\alpha\beta}\,.
\end{aligned}
\tag{3.2.23}
$$

This formula generalizes previous computations of $\overline{A_3}$ with traced internal space indices [91], and in the limit of trivial connection ($F_{\mu\nu} = 0$) [99]. The reproduction of those results as special cases provides a strong cross-check for the Mathematica code. Note that in the above expressions, a full covariant derivative D is identical to the covariant spacetime derivative ∇ when it acts on the spacetime curvature tensors R.

Lastly, it remains to be shown how the heat kernel coefficients for the case of a Laplace operator acting on scalar-, vector- and symmetric tensor fields is obtained from the general formulas given above.[8] In the case of a pure tangent bundle, we set $E = 0$ and $A_\mu = 0$, so that the field strength becomes $F_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$. This commutator is then to be evaluated, acting on the various representations,

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu]\,\phi &= 0\,, \\
[\nabla_\mu, \nabla_\nu]\,v^\alpha &= R_{\mu\nu}{}^{\alpha}{}_{\beta}v^\beta\,, \\
[\nabla_\mu, \nabla_\nu]\,h^{\alpha\beta} &= 2R_{\mu\nu}{}^{(\alpha}{}_{(\gamma}\delta^{\beta)}_{\delta)}h^{\gamma\delta}\,,
\end{aligned}
\tag{3.2.24}
$$

[8]See [96] for some early work on the spin-dependence of the $A_n(x, y)$.

where ϕ, v^α and $h^{\alpha\beta}$ are test scalar-, vector- and symmetric tensor fields, respectively. Thus, the relations

$$\begin{aligned} F_{\mu\nu}\big|_{\text{scalar}} &= 0\,, \\ F_{\mu\nu}{}^{\alpha}{}_{\beta}\big|_{\text{vector}} &= R_{\mu\nu}{}^{\alpha}{}_{\beta}\,, \\ F_{\mu\nu}{}^{\alpha\beta}{}_{\gamma\delta}\Big|_{\text{tensor}} &= 2R_{\mu\nu}{}^{(\alpha}{}_{(\gamma}\delta^{\beta)}_{\delta)}\,, \end{aligned} \tag{3.2.25}$$

between field strengths and Riemann curvatures are implied. The index structure in (3.2.25) is to be understood as a soldering of the internal indices carried by the field strength, so that $F_{\mu\nu}{}^{\alpha}{}_{\beta} = F_{\mu\nu}{}^{a}{}_{b}\, e^\alpha_a\, e^b_\beta$ and $F_{\mu\nu}{}^{\alpha\beta}{}_{\gamma\delta} = F_{\mu\nu}{}^{a}{}_{b}\, e^{\alpha\beta}_a\, e^b_{\gamma\delta}$, with normalization and completeness conditions holding for the solder forms e:

$$e^\alpha_a\, e^a_\beta = \delta^\alpha_\beta\,, \qquad e^{\alpha\beta}_a\, e^a_{\gamma\delta} = \tfrac{1}{2}(\delta^\alpha_\gamma\delta^\beta_\delta + \delta^\alpha_\delta\delta^\beta_\gamma)\,. \tag{3.2.26}$$

Substituting the appropriate field strength (3.2.25) into the traced heat kernel coefficients $\mathrm{tr}_j\overline{D_{\mu_1}\dots D_{\mu_k}A_n}$, the specific expansion coefficients of the heat kernel, traced over the corresponding field space are obtained. Here the subscript $j = 0,1,2$ indicates that the trace is summing eigenvalues of its argument when acting on scalars, vectors and symmetric 2-tensors, respectively. The diagonal heat kernel expansion up to third order in the curvature takes the form

$$\begin{aligned} \mathrm{Tr}_j\left[\mathrm{e}^{-s\Delta}\right] =& (4\pi s)^{-d/2}\int d^dx\sqrt{g}\,\left\{\mathrm{tr}_j\,\overline{A_0} + s\,\mathrm{tr}_j\,\overline{A_1} + s^2\,\mathrm{tr}_j\,\overline{A_2}\right\} \\ =& (4\pi s)^{-d/2}\int d^dx\sqrt{g}\,\left\{c^0 + s\,c^1\,\mathcal{R}^1 + s^2\sum_{i=1}^{3} c^2_i\,\mathcal{R}^2_i + s^3\sum_{i=1}^{10} c^3_i\,\mathcal{R}^3_i\right\}, \end{aligned} \tag{3.2.27}$$

with the coefficients c^n_i listed in table 4.2 in the next chapter, and the basis monomials $\mathcal{R}^n_i$ defined in (A.1.2).

3.3. Evaluation of Non-minimal Traces

The derivatives of the heat kernel coefficients derived in the last section can be used to evaluate operator traces with non-minimal derivative insertions. The trace of the heat kernel defined in terms of its matrix elements (3.2.5) is

$$\begin{aligned} \mathrm{Tr}[\mathrm{e}^{-s\Delta_0}] &= \mathrm{tr}\int d^dx\sqrt{g}\;\langle x|\,\mathrm{e}^{-s\Delta_0}\,|x\rangle \\ &= \mathrm{tr}\int d^dx\sqrt{g}\;H(x,x;s)\,, \end{aligned} \tag{3.3.1}$$

where "tr" denotes the trace in the internal space. Putting the definitions (3.2.8) and (3.2.11) together, we have the heat kernel at non-coincident points

$$H(x,y;s) := \langle y|\mathrm{e}^{-s\Delta}|x\rangle = (4\pi s)^{-d/2}\,\mathrm{e}^{-\frac{\sigma(x,y)}{2s}}\sum_{n=0}^{\infty} s^n A_n(x,y)\,. \tag{3.3.2}$$

The generalization to the non-minimal case including arbitrary insertions of uncontracted covariant derivatives inside the trace becomes

$$\mathrm{Tr}[D_{\mu_1}\dots D_{\mu_n}e^{-s\Delta_0}] = \mathrm{tr}\int d^dx\sqrt{g}\;\langle x|\,D_{\mu_1}\dots D_{\mu_n}e^{-s\Delta_0}\,|x\rangle\;. \tag{3.3.3}$$

Operators of this form will appear in any kind of gauge theories and are thus of great importance. The relevance of these traces lies in the fact that very general expressions involving a Laplacian operator and uncontracted derivatives can be computed. For a general function general function of Δ_0, we have

$$\mathrm{Tr}[D_{\mu_1}\dots D_{\mu_n}f\,(\Delta_0)] = \mathrm{tr}\int d^dx\sqrt{g}\;\int ds\;\tilde{f}\,(s)\,\overline{D_{\mu_1}\dots D_{\mu_n}H(x,y;s)}\,, \tag{3.3.4}$$

where f is written as the Laplace-transform $f\,(\Delta_0) = \int ds\;\tilde{f}\,(s)\,\mathrm{e}^{-s\Delta_0}$. The use of this technique has been successfully applied to sophisticated traces appearing in functional renormalization group calculations [67–69, 105], where it allowed to generalize the computations to operator traces which are not accessible by standard heat kernel techniques.

For a general operator insertion $\mathcal{O}$, the matrix elements in (3.3.3) can be reduced to the off-diagonal heat kernel coefficients by insertion of a complete set of states, yielding

$$\begin{aligned}\langle x|\,\mathcal{O}\,\mathrm{e}^{-s\Delta_0}|x\rangle \;&=\; \int d^dx'\sqrt{g(x')}\;\langle x|\,\mathcal{O}_x\,|x'\rangle\langle x'|\mathrm{e}^{-s\Delta_0}|x\rangle\\ &= \int d^dx'\sqrt{g(x')}\;\langle x|x'\rangle\;\mathcal{O}_x\;H(x,x';s)\\ &= \int d^dx'\,\delta(x-x')\;\mathcal{O}_x\;H(x,x';s)\\ &= \overline{\mathcal{O}_x\;H(x,x';s)}\;\equiv\;\mathcal{O}_x\;H(x,x';s)\Big|_{x=x'}\,,\end{aligned} \tag{3.3.5}$$

by making use of the fact that $\mathcal{O}_x$ acts on the coordinate x only. This manipulation is the central building block for the computations in quantum gravity pursued in this thesis. For the important special case of uncontracted covariant derivative operators, (3.3.5) implies

$$\langle x|\,D_{\mu_1}\dots D_{\mu_n}e^{-s\Delta_0}\,|x\rangle \;=\; \overline{D_{\mu_1}\dots D_{\mu_n}H(x,y;s)}\;=:\;H_{\mu_1\dots\mu_n}(x,s)\,, \tag{3.3.6}$$

introducing the abbreviations $H_{\mu_1\dots\mu_n}$ for the coincidence limits, in terms of which the non-minimal traces

$$\mathrm{Tr}[D_{\mu_1}\dots D_{\mu_n}\mathrm{e}^{-s\Delta_0}] \;=\; \mathrm{tr}\int d^dx\sqrt{g}\;H_{\mu_1\dots\mu_n}(x,s) \tag{3.3.7}$$

are given.

The remaining task is to compute the quantities $H_{\mu_1\ldots\mu_n}(x,s)$. In order to express these in terms of the heat kernel coefficients (3.2.18), the derivatives acting on $H(x,y;s)$ can be applied explicitly to the definition (3.3.2), whereby they will act on both $\sigma(x,y)$ and the $A_n(x,y)$. With the help of (3.2.15) and $\overline{\sigma_\mu}=0$, totally symmetrized covariant derivatives acting on $\sigma(x,y)$ can only produce contributions when exactly two derivatives are present, yielding

$$\overline{\sigma_{;(\mu\nu)}} = g_{\mu\nu}\,. \tag{3.3.8}$$

Using this fact simplifies the form of the resulting expressions significantly.

We give here the results for the first six of the matrix elements with symmetrized derivatives:

$$\begin{aligned}
H(x,s) &= (4\pi s)^{-d/2}\sum_{n\geq 0} s^n\overline{A_n}\\
H_\mu(x,s) &= (4\pi s)^{-d/2}\sum_{n\geq 0} s^n\overline{D_\mu A_n}\\
H_{(\mu\nu)}(x,s) &= (4\pi s)^{-d/2}\sum_{n\geq 0} s^{n-1}\Big(-\frac{1}{2}g_{\mu\nu}\overline{A_n}+\overline{D_{(\mu}D_{\nu)}A_{n-1}}\Big)\\
H_{(\mu\nu\rho)}(x,s) &= (4\pi s)^{-d/2}\sum_{n\geq 0} s^{n-1}\Big(-\frac{3}{2}g_{(\rho\nu}\overline{D_{\mu)}A_n}+\overline{D_{(\rho}D_\nu D_{\mu)}A_{n-1}}\Big)\\
H_{(\mu\nu\rho\lambda)}(x,s) &= (4\pi s)^{-d/2}\sum_{n\geq 0} s^{n-2}\Big(\frac{3}{4}g_{(\lambda\rho}g_{\nu\mu)}\overline{A_n}-3g_{(\lambda\rho}\overline{D_\nu D_{\mu)}A_{n-1}}\\
&\quad +\overline{D_{(\lambda}D_\rho D_\nu D_{\mu)}A_{n-2}}\Big)\\
H_{(\mu\nu\rho\lambda\alpha)}(x,s) &= (4\pi s)^{-d/2}\sum_{n\geq 0} s^{n-2}\Big(\frac{15}{4}g_{(\alpha\lambda}g_{\rho\nu}\overline{D_{\mu)}A_n}\\
&\quad -5g_{(\alpha\lambda}\overline{D_\rho D_\nu D_{\mu)}A_{n-1}}+\overline{D_{(\alpha}D_\lambda D_\rho D_\nu D_{\mu)}A_{n-2}}\Big)\\
H_{(\mu\nu\rho\lambda\alpha\beta)}(x,s) &= (4\pi s)^{-d/2}\sum_{n\geq 0} s^{n-3}\Big(-\frac{15}{8}g_{(\beta\alpha}g_{\lambda\rho}g_{\nu\mu)}\overline{A_n}\\
&\quad +\frac{45}{4}g_{(\beta\alpha}g_{\lambda\rho}\overline{D_\nu D_{\mu)}A_{n-1}}-\frac{15}{2}g_{(\beta\alpha}\overline{D_\lambda D_\rho D_\nu D_{\mu)}A_{n-2}}\\
&\quad +\overline{D_{(\beta}D_\alpha D_\lambda D_\rho D_\nu D_{\mu)}A_{n-3}}\Big)\,.
\end{aligned} \tag{3.3.9}$$

In these expressions, the sums are understood with the boundary conditions $A_{-1}=A_{-2}=\cdots=0$. The general unsymmetrized formulas can always be recovered from the symmetrized ones by commutation of the derivatives, writing for each pair of indices a sum of symmetrized and anti-symmetrized terms. For example we have

$$\begin{aligned} H_{\mu\nu}(x,s) =& H_{(\mu\nu)}(x,s) + H_{[\mu\nu]}(x,s) \\ =& H_{(\mu\nu)}(x,s) + \overline{D_{[\mu}D_{\nu]}H(x,y,s)} \\ =& H_{(\mu\nu)}(x,s) + F_{\mu\nu}\,H(x,s) \\ =& \,(4\pi s)^{-d/2}\sum_{n\geq 0} s^{n-1}\Big(-\frac{1}{2}g_{\mu\nu}\overline{A_n} + F_{\mu\nu}\overline{A_{n-1}} + \overline{D_{(\mu}D_{\nu)}A_{n-1}}\Big)\,, \end{aligned} \tag{3.3.10}$$

where the commutator in the second term becomes the curvature tensor $F_{\mu\nu}(x)$. Thus, all traces of the form (3.3.7) are given with the explicit expressions for the expansion coefficients (3.2.18), (3.2.19), (3.2.20), (3.2.21), (3.2.22), and (3.2.23).

For the matrix elements with even number of derivatives, using the heat kernel coefficients on a trivial bundle

$$\overline{A_0} = 1\,, \qquad \overline{A_1} = \tfrac{1}{6}R\,, \qquad \overline{D_\mu D_\nu A_0} = \tfrac{1}{6}R_{\mu\nu}\,, \tag{3.3.11}$$

we can write in general the fully symmetric expression

$$\begin{aligned} H_{\mu_1\dots\mu_{2n}} = \; (4\pi s)^{-d/2}\Big\{&\,(-2s)^{-n}\,\tfrac{(2n)!}{2^n n!}\,g_{(\mu_1\mu_2}\cdots g_{\mu_{2n-1}\mu_{2n})}\Big(1+\tfrac{1}{6}sR\Big) \\ &+\tfrac{1}{6}\,(-2s)^{-(n-1)}\,\tfrac{(2n)!}{2^n(n-1)!}\,g_{(\mu_1\mu_2}\cdots g_{\mu_{2n-3}\mu_{2n-2}}R_{\mu_{2n-1}\mu_{2n})}\Big\}\,, \end{aligned} \tag{3.3.12}$$

valid up to terms of $\mathcal{O}(\mathcal{R}^2)$. For later reference, the first two of these read

$$\begin{aligned} H_{\alpha\beta} &= (4\pi s)^{-d/2}\Big\{-\frac{1}{2s}\,g_{\alpha\beta}\,\Big(1+\tfrac{1}{6}sR\Big) + \frac{1}{6}R_{\alpha\beta}\Big\}\,, \\ H_{\alpha\beta\mu\nu} &= (4\pi s)^{-d/2}\Big\{\frac{1}{4s^2}\,(g_{\alpha\mu}g_{\beta\nu}+g_{\alpha\nu}g_{\beta\mu}+g_{\alpha\beta}g_{\mu\nu})\,\Big(1+\tfrac{1}{6}sR\Big) \\ &\quad -\frac{1}{12s}\,(g_{\alpha\mu}R_{\beta\nu}+g_{\alpha\nu}R_{\beta\mu}+g_{\beta\mu}R_{\alpha\nu}+g_{\beta\nu}R_{\alpha\mu}+g_{\alpha\beta}R_{\mu\nu}+g_{\mu\nu}R_{\alpha\beta})\Big\}\,. \end{aligned} \tag{3.3.13}$$

Note that the H-tensors with any number of derivative indices give contributions to all orders of the curvature.

A generalization of (3.3.9) based on the non-local heat kernel expansion (3.1.19) is possible, as shown in [106], where an expansion including only the first order in the curvatures is studied. Notably, such a generalization would give many insights to important physical problems, and further increases the range of applicability of the methods discussed in this chapter.

4. Renormalization Under Gauge Symmetry

4.1. Gauge Theory & Field Decompositions

Quantum field theories which are known to describe realistic interaction processes are usually defined in terms of gauge fields, which entail unphysical off-shell components in order to maintain a manifestly covariant formalism. Theories with such internal gauge degrees of freedom are formulated on a fiber bundle, which is a manifold E equipped with a projection $\pi : E \to M$ into another lower-dimensional base manifold M. Locally, this projection induces a split of the bundle into $E = M \times F$, where F is called the fiber and is given the interpretation of internal spaces, attached to the base manifold at each point. A gauge field is the connection on a bundle over the spacetime manifold M and with the fiber equipped with the action of a structure or gauge group.[1] Invariance of the action functional under a gauge transformation, which is a local change of coordinates of the fiber, ensures the dynamical irrelevance of the associated components of the fields. Any field being a non-trivial representation of the gauge group will be involved in this transformation and thus describes particles charged with respect to an implied conserved current.

In the example of Yang-Mills theory on flat spacetime, one has a 1-form connection field A_μ which is in the adjoint representation of an underlying Lie group. Thus, it transforms like an element of the associated Lie algebra $[t^a, t^b] = f^{abc} t^c$, yielding

$$\delta_\xi A_\mu = \partial_\mu \xi + [A_\mu, \xi] = D_\mu \xi \,, \tag{4.1.1}$$

with a Lie algebra valued field ξ, and the covariant derivative D_μ as defined in the last chapter. Establishing the invariance of the action

$$\delta_\xi S[A_\mu] = 0 \,, \tag{4.1.2}$$

leads to the conclusion that not all of the components of A_μ will actually appear as on-shell degrees of freedom. Specifically, the gauge transformation is allowed to change

[1] In gravity, a metric connection can be used, as described in the last chapter.

only those components of A_μ on which the action does not depend. The Yang-Mills action

$$S_{\mathrm{YM}}[A_\mu] = \int d^d x \,\mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}\,, \tag{4.1.3}$$

with field strength $F_{\mu\nu} = [D_\mu, D_\nu]$ gives rise to the inverse propagator

$$\mathcal{G}^{-1}_{\mu\nu} = -g_{\mu\nu}\partial_\alpha\partial^\alpha + \partial_\mu\partial_\nu\,, \tag{4.1.4}$$

which has a zero mode $\mathcal{G}^{-1}_{\mu\nu}\partial^\nu\phi = 0$. Therefore, the propagator appearing in the perturbative expansion (2.1.15) and thus in the n-point functions (2.1.16) does not exist.

The consistent quantization of a model subject to gauge invariance requires to work around this issue, which is done by the implementation of gauge fixing. This procedure involves the removal of the zero eigenvalues from the propagator, formally making the unphysical gauge degrees of freedom propagate. To compensate for the resulting contributions, any non-abelian gauge theory has to include additional ghost fields. In order to be able to cancel corresponding modes of the gauge field, these ghosts are required to be subject to Fermi statistics despite being an integer spin representation. The formal breaking of gauge invariance is mediated via the BRST construction [107, 108], to preserve the physical content of the model. The central object of this construction is a BRST symmetry transformation δ_B, that replaces the original gauge symmetry by extension of the field space by the fermionic ghost degrees of freedom, thus being a form of (off-shell) supersymmetry [109]. Specifically, δ_B acts on the gauge field like a gauge transformation, parametrized by the ghost

$$\delta_B A_\mu = D_\mu c\,, \tag{4.1.5}$$

and on the ghost and anti-ghost fields such that the transformation becomes nilpotent

$$\delta_B^2 = 0\,. \tag{4.1.6}$$

Understood as a graded derivation with this property, δ_B induces a cohomology, providing the means to give a topological classification of observable algebras by the study of non-trivial solutions of $\delta_B F = 0$, with F being a functional in the space of quantum fields. A notion of gauge invariance is defined in this context in terms of the cohomology classes $[F] = \{F + \delta_B G\}$, expressing that two objects A, B are gauge equivalent if they differ only by a BRST-closed term, such that $A - B = \delta_B C$. By virtue of the nilpotency condition, the set of gauge invariant objects is identified with the cocycle $\{F, \delta_B F = 0\} = \mathrm{Ker}\,\delta_B$, which has the subset of trivial solutions given as the coboundary $\{F, F = \delta_B G\} = \mathrm{Im}\,\delta_B$.

Observables are finally explained by restriction of the set of all functionals to the quotient space

$$F_{\mathrm{obs}} \in \mathrm{Ker}\, \delta_B \Big/ \mathrm{Im}\, \delta_B \,. \tag{4.1.7}$$

The extension to BRST symmetry allows a shift of the action by an arbitrary coboundary element S_{gf}

$$S \to S + S_{\mathrm{gf}}\,, \qquad S_{\mathrm{gf}} = \delta_B K\,, \tag{4.1.8}$$

since the invariance $\delta_B S = 0$ remains preserved, without changing the physical content of the model. The gauge fixing term S_{gf} picks a coordinate system in the invariant supersymmetric configuration space, which remarkably allows to modify the term quadratic in the gauge field in the full action. Thus, the ambiguity to define inverse propagators in a gauge theory can be removed, establishing BRST as the genuine gauge symmetry of the quantum theory.

In turn, the n-point functions (2.1.3) are not gauge independent by themselves, but need to be restricted to the observable subspace (4.1.7). This is conveniently expressed in terms of the generating functional

$$\begin{aligned} &\int D\mathcal{X}\ \delta_B\ \mathrm{e}^{-S[\mathcal{X}]+\int_x J\mathcal{X}} \\ =\ &\int D\mathcal{X}\, \Big(\sum_i \int_x J_i\ \delta_B \mathcal{X}_i \Big)\ \mathrm{e}^{-S[\mathcal{X}]+\int_x J\mathcal{X}} \\ =\ &\sum_i \int_x J_i\ \Big[\delta_B \mathcal{X}_i\Big]\big(\frac{\delta}{\delta J}\big)\ Z[J]\ :=\ 0\,, \end{aligned} \tag{4.1.9}$$

since δ_B acts like a derivative, and invariance of the functional integral measure is assumed. The last expression together with all functional derivatives thereof are known as the Slavnov-Taylor identities. These encode relations among the n-point correlation functions, by virtue of which the scattering amplitudes become independent of the gauge fixing. The identities (4.1.9) can easily be expressed in terms of the effective action

$$\sum_i \int_x \Big[\delta_B \mathcal{X}_i\Big](\Phi)\ \frac{\delta \Gamma}{\delta \Phi_i}\ =\ \hat{W}\,\Gamma\ =\ 0\,, \tag{4.1.10}$$

using the relations (2.1.20) and (2.1.23). On a more general account, the operator $\hat{W}$ measures the magnitude by which invariance is violated. In the context of the RG equation (2.3.18), the action becomes modified by the cutoff term (2.3.3), which leads to the modified Slavnov-Taylor identities

$$\hat{W}\,\Gamma_k = \sum_i \int_x \Big[\delta_B \mathcal{X}_i\Big](\Phi)\ \mathcal{R}_{k,i}\ \Phi_i\,. \tag{4.1.11}$$

These recover the original form for $k = 0$ where $\mathcal{R}_k$ vanishes [35].

To reproduce the well known Faddeev-Popov Lagrangian as special case, the gauge fixing term S_{gf} is chosen with

$$
\begin{aligned}
K &= \int_x \bar{C}\; F[A]\,, \\
\delta_B \bar{C} &= \tfrac{1}{2\alpha} F[A]\,, \\
\delta_B F[A] &= \frac{\delta F[A]}{\delta A_\mu}\, \delta_B A_\mu\,,
\end{aligned}
\tag{4.1.12}
$$

defining a smeared gauge fixing condition $F[A] \approx 0$, linear in the gauge field, to yield

$$
S_{\mathrm{gf}} = \delta_B K = \frac{1}{2\alpha}\int_x F[A]^2 + \int_x \bar{C}\frac{\delta F[A]}{\delta A_\mu} D_\mu C\,, \tag{4.1.13}
$$

for Yang-Mills theory.[2] In the case of gravity, the gauge symmetry is given as the group of local diffeomorphisms. These define general coordinate transformations on the manifold, infinitesimally realized by the Lie derivative $\mathcal{L}_v$

$$
\mathcal{L}_v\, T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} = v^\alpha T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n;\alpha} - \sum_{k=1}^{m} v^{\mu_k}{}_{;\alpha} T^{\mu_1\ldots\alpha\ldots\mu_m}{}_{\nu_1\ldots\nu_n} + \sum_{k=1}^{n} v^\alpha{}_{;\nu_k} T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\alpha\ldots\nu_n}\,, \tag{4.1.14}
$$

acting on a tensor T as directional derivative along the vector field v^α. For the metric on a torsion-free manifold we find

$$
\mathcal{L}_v\, g_{\mu\nu} = v^\alpha g_{\mu\nu;\alpha} + v^\alpha{}_{;\mu} g_{\alpha\nu} + v^\alpha{}_{;\nu} g_{\mu\alpha} = v_{\nu;\mu} + v_{\mu;\nu}\,, \tag{4.1.15}
$$

so that a change of the local coordinate system described by x_μ by a vector field ξ_μ is given by

$$
\begin{aligned}
\delta_\xi x_\mu &= \xi_\mu(x)\,, \\
\delta_\xi g_{\mu\nu}(x) &= D_\mu \xi_\nu(x) + D_\nu \xi_\mu(x)\,.
\end{aligned}
\tag{4.1.16}
$$

For a theory of gravity with a metric degree of freedom, an appropriate gauge fixing term analogous to (4.1.13) is thus realized in the expression

$$
\begin{aligned}
S_{\mathrm{gf}} &= \delta_B \int_x \bar{C}^\mu F_\mu[g] \\
&= \frac{1}{2\alpha}\int_x F^\mu[g] F_\mu[g] + \int_x \bar{C}^\mu \frac{\delta F_\mu[g]}{\delta g_{\rho\sigma}} (D_\rho C_\sigma + D_\sigma C_\rho)\,,
\end{aligned}
\tag{4.1.17}
$$

[2]The BRST transformation defined here is only on-shell nilpotent. For our purpose it is sufficient to note that it can always be extended to hold off-shell by the inclusion of a Nakanishi-Lautrup field with on-shell condition $B = \frac{1}{2\alpha}F[A]$, if required.

with the vector valued ghost fields $\bar{C}_\mu$ and C_μ. The original Faddeev-Popov construction [110] attempts to exclude an integration over gauge equivalent configurations from the functional integral measure. This is done by insertion of unity in the form

$$\int Da\, \delta(F[A^a])\det(\tfrac{dF[A^a]}{da}) = 1 \tag{4.1.18}$$

inside the functional integral, that allows to isolate an infinite factor corresponding to the volume of gauge orbits, after a gauge transformation parametrized by a. In this perspective, the ghost term is understood as exponentiation of the determinant of a gauge transformation

$$\det(\mathcal{M}) = \int D\bar{C}\, DC\, \mathrm{e}^{-\int_x \bar{C}\mathcal{M}C}\,, \tag{4.1.19}$$

with the ghost kernel given by

$$\begin{aligned} \mathcal{M} &= \frac{\delta F[A]}{\delta A_\mu} D_\mu\,,\\ \mathcal{M}_\mu{}^\nu &= \frac{\delta F_\mu[g]}{\delta g_{\rho\sigma}}(D_\rho\delta^\nu_\sigma + D_\sigma\delta^\nu_\rho)\,, \end{aligned} \tag{4.1.20}$$

for Yang-Mills and gravity, respectively. Although it is the formal supersymmetrization of the gauge degrees of freedom by the ghost fields that guarantees the cancellation of gauge dependence in the observables of the quantized theory, the choice (4.1.12) suffers from the Gribov ambiguity, related to the appearance of zero modes in the ghost kernel $\mathcal{M}$ [111]. Since this affects the IR behaviour of a theory more then the UV, we can ignore this problem for the present purpose.

Finally, by virtue of the gauge condition F, smeared by a Gaussian weight with the width α, a non-ambiguous definition of a propagator can be given. The gauge fixed action with $F[A] = \partial^\mu A_\mu$ replaces the inverse Yang-Mills propagator (4.1.4) by

$$\mathcal{G}^{-1}_{\mu\nu} = -g_{\mu\nu}\partial_\alpha\partial^\alpha + \left(1-\alpha^{-1}\right)\partial_\mu\partial_\nu\,, \tag{4.1.21}$$

for $\alpha < \infty$. It is useful to formalize the decomposition of fields to isolate the gauge dependent part, as it can simplify the computation of functional traces for the case of gauge theories.

Decomposition of Vector Fields

In Yang-Mills theory it is a common practice to decompose the fluctuations of the vector field A_μ

$$A_\mu = A_{\mathrm{T},\mu} + A_{\mathrm{L},\mu}\,, \tag{4.1.22}$$

into its transversal and longitudinal part [74] by a generalization of Helmholtz's theorem. Following [112], this decomposition is readily given on a general Riemannian (background) manifold in the context of gravity or quantum field theory on curved spacetime. Specifically, the local decomposition of a generic vector field v^μ [113] is given by

$$v^\mu = v_{\rm T}^\mu + D^\mu \phi \,, \qquad D_\mu v_{\rm T}^\mu \equiv 0 \,, \tag{4.1.23}$$

describing the split into transversal part $v_{\rm T}^\mu$ and longitudinal part $v_{\rm L}^\mu = D^\mu\phi$, with the former subject to a differential transversality constraint. This split is uniquely defined under the assumption of a closed manifold, up to a constant shift in ϕ.[3] The transversal decomposition of a Yang-Mills field $A_\mu = A_\mu^{\rm T} + D_\mu\phi$ isolates the gauge invariant part according to the transformation (4.1.1), since

$$A'_\mu = A_\mu + D_\mu \xi = A_\mu^{\rm T} + D_\mu(\phi + \xi) \,, \tag{4.1.24}$$

leaving the transversal component invariant and constituting the longitudinal to be a gauge degree of freedom.[4]

The decomposition (4.1.23) can be implemented by the projection operators

$$\Pi_{\rm L}{}_\mu{}^\nu \equiv -D_\mu \frac{1}{\Delta} D^\nu \,, \qquad \Pi_{\rm T}{}_\mu{}^\nu \equiv \delta_\mu^\nu + D_\mu \frac{1}{\Delta} D^\nu \,, \tag{4.1.25}$$

acting on the space of unconstrained vector fields. These satisfy the projector properties of idempotency and orthogonality

$$\Pi_{\rm L} \cdot \Pi_{\rm L} = \Pi_{\rm L} \,, \qquad \Pi_{\rm T} \cdot \Pi_{\rm T} = \Pi_{\rm T} \,, \qquad \Pi_{\rm L} \cdot \Pi_{\rm T} = \Pi_{\rm T} \cdot \Pi_{\rm L} = 0 \,, \tag{4.1.26}$$

which ensure the orthogonality of the decomposition. Furthermore, one has the relations

$$\Pi_{\rm L}{}_\mu{}^\nu D_\nu = D_\mu \,, \qquad D^\mu \Pi_{\rm L}{}_\mu{}^\nu = D^\nu \,, \qquad [\Pi_{\rm T}]_\mu{}^\nu = [\mathbf{1}_1]_\mu{}^\nu - [\Pi_{\rm L}]_\mu{}^\nu \,, \tag{4.1.27}$$

with the identity operator $[\mathbf{1}_1]_\mu{}^\nu = \delta_\mu{}^\nu$, following directly from the definition. By virtue of these, a vector v^μ is projected onto its irreducible components

$$\Pi_{\rm T}{}_\mu{}^\nu v_\nu = v_{{\rm T},\mu} \,, \qquad \Pi_{\rm L}{}_\mu{}^\nu v_\nu = v_{{\rm L},\mu} = D_\mu \phi \,. \tag{4.1.28}$$

[3] The assumption of the background being closed could be relaxed if appropriate assumptions on the fall-off of the metric and fluctuation fields are fulfilled.

[4] The reduction to the two on-shell helicity states requires to also take the equations of motion into account.

For practical computations it is convenient to have an explicit expression for Π_{L}, where the Laplacian is commuted to either side. This can be achieved with the curvature expansion (B.2.4), to obtain in terms of the n-fold commutators (B.1), when commuting to the very left

$$\begin{aligned} \Pi_{\mathrm{L}\mu}{}^{\nu} v_\nu &= -\frac{1}{\Delta} \sum_{n=0}^{\infty} \left(\frac{1}{\Delta^n} (-1)^n \; [\, D_\mu \,, \, \Delta \,]_n \right) D^\nu v_\nu \\ &= \Big[-\Delta^{-1} D_\mu D^\nu + \Delta^{-2} R_{\mu\alpha} D^\alpha D^\nu + \mathcal{O}(\mathcal{R}^2) \Big] v_\nu \,, \end{aligned} \tag{4.1.29}$$

or the very right

$$\begin{aligned} \Pi_{\mathrm{L}\mu}{}^{\nu} v_\nu &= -\, D_\mu \sum_{n=0}^{\infty} \left([\, D^\nu \,, \, \Delta \,]_n \, \frac{1}{\Delta^n} \right) \frac{1}{\Delta} \, v_\nu \\ &= \Big[-D_\mu D^\nu \Delta^{-1} + D_\mu D_\alpha R^{\alpha\nu} \Delta^{-2} + \mathcal{O}(\mathcal{R}^2) \Big] v_\nu \,, \end{aligned} \tag{4.1.30}$$

respectively. The explicit expressions for the multi-commutators appearing in these expansions up to order $n = 4$ are given in appendix B, in (B.1.2) and (B.1.5).

The employment of a decomposition of vector fields (4.1.23) requires an adaptation of the functional integral measure by a Jacobian J_{vec} such that

$$D[v_\mu] = J_{\mathrm{vec}} \; D[v_\mu^{\mathrm{T}}] D[\phi] \,. \tag{4.1.31}$$

Following [114] the Jacobian can be found by explicit evaluation of a normalized Gaussian integral with the decomposition inserted

$$\begin{aligned} 1 &= \int D[v_\mu] \;\; \exp\Big[-\tfrac{1}{2} \int d^d x \sqrt{g} \; v^\mu v_\mu \Big] \\ &= J_{\mathrm{vec}} \int D[v_\mu^{\mathrm{T}}] D[\phi] \;\; \exp\Big[-\tfrac{1}{2} \int d^d x \sqrt{g} \; \left(v^{\mathrm{T}\mu} v_\mu^{\mathrm{T}} + \phi \Delta \phi \right) \,. \end{aligned} \tag{4.1.32}$$

Solving for J_{vec} we find

$$\begin{aligned} \Rightarrow \quad J_{\mathrm{vec}} &= \left\{ \int D[\phi] \;\; \exp\Big[-\tfrac{1}{2} \int d^d x \sqrt{g} \; \phi \Delta \phi \Big] \right\}^{-1} \\ &= \det(\Delta)^{\varepsilon} \,. \end{aligned} \tag{4.1.33}$$

The remaining functional integral evaluated according to the rules for bosonic fields implies the exponent $\varepsilon = \frac{1}{2}$, whereas for fermionic fields we have $\varepsilon = -1$.

Decomposition of Symmetric 2-Tensor Fields

The gravitational fluctuations of a metric degree of freedom $h_{\mu\nu}$ can be decomposed in a transverse-traceless part $h_{\mu\nu}^{\mathrm{T}}$, a vector ξ_μ, and a scalar h, representing the trace-part. The York-decomposition [113] reads explicitly

$$h_{\mu\nu} = h_{\mu\nu}^{\mathrm{T}} + D_\mu \xi_\nu + D_\nu \xi_\mu - \tfrac{2}{d} g_{\mu\nu} D^\alpha \xi_\alpha + \tfrac{1}{d} g_{\mu\nu} h \,, \tag{4.1.34}$$

with the component fields subject to the constraints

$$D^\mu h^{\mathrm{T}}_{\mu\nu} = 0\,, \qquad g^{\mu\nu} h^{\mathrm{T}}_{\mu\nu} = 0\,, \qquad g^{\mu\nu} h_{\mu\nu} = h\,. \tag{4.1.35}$$

The comparison with the coordinate transformation (4.1.16) reveals that here the vector ξ_μ absorbs the unphysical gauge dependence, however including also a shift of the trace part h [114]. The unconstrained vector ξ_μ can be further decomposed according to (4.1.23), so that with $\xi_\mu = \xi^{\mathrm{T}}_\mu + D_\mu \sigma$ the complete transverse-traceless (TT) decomposition is

$$h_{\mu\nu} = h^{\mathrm{T}}_{\mu\nu} + D_\mu \xi^{\mathrm{T}}_\nu + D_\nu \xi^{\mathrm{T}}_\mu + 2 D_\mu D_\nu \sigma - \tfrac{2}{d} g_{\mu\nu} D^2 \sigma + \tfrac{1}{d} g_{\mu\nu} h\,. \tag{4.1.36}$$

In this form, all components are in their irreducible representations. However, it can be more convenient to reside with the "minimal" TT-decomposition (4.1.34), if only projection on the $h^{\mathrm{T}}_{\mu\nu}$ component is required. Ambiguities in this decomposition arise if the background metric admits Killing vectors or conformal Killing vectors, since these constitute zero modes of the ξ_μ or ξ^{T}_μ parts.

To formalize the decomposition (4.1.34), covariant projection operators onto the transverse-traceless, vector and scalar subspaces can be constructed. Firstly, on has

$$\begin{aligned} [\,\mathbf{1}_2\,]_{\alpha\beta}{}^{\rho\sigma} &= \tfrac{1}{2} \left(\delta_\alpha{}^\rho \delta_\beta{}^\sigma + \delta_\alpha{}^\sigma \delta_\beta{}^\rho \right)\,, \\ [\,\Pi_{\mathrm{tr}}\,]_{\alpha\beta}{}^{\rho\sigma} &= \tfrac{1}{d} g_{\alpha\beta} g^{\rho\sigma}\,, \end{aligned} \tag{4.1.37}$$

the unit-operator $\mathbf{1}_2$ on the space of symmetric matrices, and Π_{tr} projecting on the trace part. After applying a projection on the trace-free part via $\mathbf{1}_2 - \Pi_{\mathrm{tr}}$, and acting with a covariant derivative on the decomposition, only the vector ξ_μ remains. To bring this remaining expression into standard form, we define

$$[\,\Pi_{2\mathrm{L}}\,]_{\alpha\beta}{}^{\rho\sigma} = \, [P_1]^\mu_{\alpha\beta} \left[P_2^{-1}\right]^\nu_\mu (-D^\gamma)\, [\,\mathbf{1}_2 - \Pi_{\mathrm{tr}}\,]_{\gamma\nu}{}^{\rho\sigma}\,, \tag{4.1.38}$$

with

$$\begin{aligned} [P_1]^\mu_{\alpha\beta} &= \, 2 D_{(\alpha} \delta^\mu_{\beta)} - \tfrac{2}{d} g_{\alpha\beta} D^\mu\,, \\ \left[P_2^{-1}\right]^\nu_\mu &= \left[\Delta \delta^\mu_\nu - R^\mu{}_\nu - \tfrac{d-2}{d} D^\mu D_\nu \right]^{-1}\,, \\ \Rightarrow \qquad & - D^\alpha\, [P_1]^\mu_{\alpha\beta} = [P_2]^\mu_\beta\,. \end{aligned} \tag{4.1.39}$$

These operators are chosen so that $\Pi_{2\mathrm{L}}$ applied to $h_{\mu\nu}$ reproduces the ξ-part as it appears in the decomposition (4.1.34). Applying the projectors (4.1.37) and (4.1.38) to the minimal TT-decomposition (4.1.34), thus gives

$$\begin{aligned} [\,\Pi_{\mathrm{tr}} h\,]_{\mu\nu} &= \tfrac{1}{d}\, g_{\mu\nu}\, h\,, \\ [\,\Pi_{2\mathrm{L}} h\,]_{\mu\nu} &= [P_1]^\alpha_{\mu\nu} \xi_\alpha = D_\mu \xi_\nu + D_\nu \xi_\mu - \tfrac{2}{d} g_{\mu\nu} D^\alpha \xi_\alpha\,, \end{aligned} \tag{4.1.40}$$

the vector- and trace-parts of $h_{\mu\nu}$, respectively. Finally the projector on the transverse-traceless part $h^{\mathrm{T}}_{\mu\nu}$ follows to be given as

$$\Pi_{2\mathrm{T}} = \mathbf{1}_2 - \Pi_{2\mathrm{L}} - \Pi_{\mathrm{tr}}\,, \qquad [\Pi_{2\mathrm{T}} h]_{\mu\nu} = h^{\mathrm{T}}_{\mu\nu}\,. \tag{4.1.41}$$

Since the complicated pseudo-differential operator $\left[P_2^{-1}\right]_\mu^{\ \nu}$ (4.1.39) severely complicates working with the projector $\Pi_{2\mathrm{T}}$, it is necessary to recast this operator into an expansion in the background curvature. Systematically, the inverse assumes the form

$$\left[P_2^{-1}\right]_\alpha^{\ \beta} \xi_\beta = \left(\Delta^{-1}\delta_\alpha{}^\beta + [P_{2,0}]_\alpha^{\ \beta} + [P_{2,R}]_\alpha^{\ \beta}\right)\xi_\beta + \mathcal{O}(\mathcal{R}^2, D\mathcal{R})\,, \tag{4.1.42}$$

where $P_{2,0}$ captures the resummed contribution of the $D^\mu D_\nu$-term at zeroth order, and $P_{2,R}$ the terms at first order in R. Explicitly, we compute as a geometric series in $q = (d-2)/d < 1$

$$\begin{aligned}
[P_{2,0}]_\alpha^{\ \beta}\xi_\beta &= \sum_{n=0}^{\infty} q^{n+1}\,\frac{1}{\Delta}D_\alpha\,\left(D^\gamma\frac{1}{\Delta}D_\gamma\right)^n\, D^\beta\,\frac{1}{\Delta}\,\xi_\beta \\
&= \sum_{n=0}^{\infty} q^{n+1}\,\frac{1}{\Delta}\,D_\alpha\,\left(-1 + \frac{1}{\Delta^2}R^{\mu\nu}D_\mu D_\nu\right)^n\, D^\beta\frac{1}{\Delta}\,\xi_\beta \\
&= \left[\frac{d-2}{2(d-1)}\,\frac{1}{\Delta}D_\alpha D^\beta\frac{1}{\Delta} + \frac{(d-2)^2}{4(d-1)^2}R^{\mu\nu}D_\mu D_\nu D_\alpha D^\beta\frac{1}{\Delta^4}\right]\xi_\beta\,,
\end{aligned} \tag{4.1.43}$$

and

$$\begin{aligned}
&[P_{2,R}]_\alpha^{\ \beta}\xi_\beta \\
&= \left[\frac{1}{\Delta}R_\alpha{}^\beta\frac{1}{\Delta} + \sum_{n=1}^{\infty}\frac{q^n}{\Delta^{n+2}}\left(2R_{(\alpha}{}^\mu D_\mu D^{\beta)}D^{2(n-1)} + (n-1)D^{2(n-2)}R^{\mu\nu}D_\mu D_\nu D_\alpha D^\beta\right)\right]\xi_\beta \\
&= \left[\frac{1}{\Delta}R_\alpha{}^\beta\frac{1}{\Delta} + \sum_{n=1}^{\infty}(-q)^n\left(-\frac{2}{\Delta^3}R_{(\alpha}{}^\mu D_\mu D^{\beta)} + \frac{n-1}{\Delta^4}R^{\mu\nu}D_\mu D_\nu D_\alpha D^\beta\right)\right]\xi_\beta \\
&= \left[\frac{1}{\Delta}R_\alpha{}^\beta\frac{1}{\Delta} + \frac{d-2}{d-1}\frac{1}{\Delta^3}R_{(\alpha}{}^\mu D_\mu D^{\beta)} + \frac{(d-2)^2}{4(d-1)^2}\frac{1}{\Delta^4}R^{\mu\nu}D_\mu D_\nu D_\alpha D^\beta\right]\xi_\beta\,.
\end{aligned} \tag{4.1.44}$$

The resummations (4.1.43) and (4.1.44) are exact up to terms $\mathcal{O}(\mathcal{R}^2, D\mathcal{R})$. Higher orders can be calculated in the same way, taking higher commutators into account. Substituting these results into (4.1.42) and applying the commutator expansion (B.2.4) to the first term in (4.1.43) finally reveals

$$\begin{aligned}
\left[P_2^{-1}\right]_\alpha^{\ \beta}\xi_\beta = &\Big[\tfrac{1}{\Delta}\delta_\alpha{}^\beta + \tfrac{1}{2}\tfrac{d-2}{d-1}D_\alpha D^\beta\tfrac{1}{\Delta^2} + R_\alpha{}^\beta\tfrac{1}{\Delta^2} + \tfrac{d-2}{d-1}R_\alpha{}^\mu D_\mu D^\beta\tfrac{1}{\Delta^3} \\
&\qquad + \tfrac{1}{2}\tfrac{(d-2)^2}{(d-1)^2}R^{\mu\nu}D_\mu D_\nu D_\alpha D^\beta\tfrac{1}{\Delta^4}\Big]\xi_\beta + \mathcal{O}(\mathcal{R}^2, D\mathcal{R})\,.
\end{aligned} \tag{4.1.45}$$

With the expansion of the inverse of P_2^{-1} at hand, it is now straightforward to give the projector Π_{2L} (4.1.38) in terms of a perturbative series in the background curvature. Again, we give this operator both with the Laplacian commuted through to the left and right, written in the form

$$\begin{aligned}[\,\Pi_{2L}\,]_{\mu\nu}{}^{\rho\sigma}\phi_{\rho\sigma} &= \left[\,\Pi^{0l}_{2L}+\Pi^{1l}_{2L}+\Pi^{1}_{2L}\right]_{\mu\nu}{}^{\rho\sigma}\phi_{\rho\sigma}+\mathcal{O}(\mathcal{R}^2,D\mathcal{R})\\ &= \left[\,\Pi^{0r}_{2L}+\Pi^{1r}_{2L}+\Pi^{1}_{2L}\right]_{\mu\nu}{}^{\rho\sigma}\phi_{\rho\sigma}+\mathcal{O}(\mathcal{R}^2,D\mathcal{R})\,,\end{aligned} \tag{4.1.46}$$

where, Π^{0l}_{2L} and Π^{0r}_{2L} denote the part with no curvature terms, Π^{1l}_{2L} and Π^{1r}_{2L} capture the corresponding commutator contributions, and Π^{1}_{2L} is the linear curvature contribution originating from the inversion formula (4.1.45). These parts are given by

$$\begin{aligned}[\,\Pi^{0r}_{2L}\,]_{\alpha\beta}{}^{\rho\sigma} =&\; -2D_{(\alpha}\delta^{(\rho}_{\beta)}D^{\sigma)}\tfrac{1}{\Delta}+\tfrac{1}{d-1}g^{\rho\sigma}D_{(\alpha}D_{\beta)}\tfrac{1}{\Delta}+\tfrac{1}{d-1}g_{\alpha\beta}D^{(\rho}D^{\sigma)}\tfrac{1}{\Delta}\\ &-\tfrac{d-2}{d-1}D_{(\alpha}D_{\beta)}D^{(\rho}D^{\sigma)}\tfrac{1}{\Delta^2}+\tfrac{1}{d(d-1)}g_{\alpha\beta}g^{\rho\sigma}\,,\\ [\,\Pi^{0l}_{2L}\,]_{\alpha\beta}{}^{\rho\sigma} =&\; -2\tfrac{1}{\Delta}D_{(\alpha}\delta^{(\rho}_{\beta)}D^{\sigma)}+\tfrac{1}{d-1}\tfrac{1}{\Delta}g^{\rho\sigma}D_{(\alpha}D_{\beta)}+\tfrac{1}{d-1}\tfrac{1}{\Delta}g_{\alpha\beta}D^{(\rho}D^{\sigma)}\\ &-\tfrac{d-2}{d-1}\tfrac{1}{\Delta^2}D_{(\alpha}D_{\beta)}D^{(\rho}D^{\sigma)}+\tfrac{1}{d(d-1)}g_{\alpha\beta}g^{\rho\sigma}\,,\end{aligned} \tag{4.1.47}$$

with the commutators corrections

$$\begin{aligned}[\,\Pi^{1r}_{2L}\,]_{\alpha\beta}{}^{\rho\sigma} =&\; \left(2D_{(\alpha}\delta^{\tau}_{\beta)}-\tfrac{1}{d-1}g_{\alpha\beta}D^{\tau}+\tfrac{2(d-2)}{(d-1)}D_{(\alpha}D_{\beta)}D^{\tau}\tfrac{1}{\Delta}\right)\times\\ &\times\left(R^{\lambda\mu}D_{\lambda}\delta^{\nu}_{\tau}-2R_{\tau}{}^{\mu\lambda\nu}D_{\lambda}\right)[\,\mathbf{1}_2-\Pi_{\rm tr}\,]_{\mu\nu}{}^{\rho\sigma}\tfrac{1}{\Delta^2}\\ &-\tfrac{d-2}{d(d-1)}\,g_{\alpha\beta}\,R^{\lambda\mu}\,D_{\lambda}D^{\nu}\,\tfrac{1}{\Delta^2}\,[\,\mathbf{1}_2-\Pi_{\rm tr}\,]_{\mu\nu}{}^{\rho\sigma}\,,\\ [\,\Pi^{1l}_{2L}\,]_{\alpha\beta}{}^{\rho\sigma} =&\; \Bigg[\tfrac{1}{\Delta^2}\left(2R_{(\alpha}{}^{\lambda}D_{\lambda}\delta^{\nu}_{\beta)}-4R^{\mu}{}_{(\alpha}{}^{\nu}{}_{\beta)}D_{\mu}+\tfrac{2}{d(d-1)}g_{\alpha\beta}R^{\mu\nu}D_{\mu}\right)\\ &+\tfrac{2(d-2)}{d-1}\tfrac{1}{\Delta^3}\left(2R_{(\alpha}{}^{\lambda}D_{\beta)}D_{\lambda}D^{\nu}-2R^{\lambda}{}_{\alpha}{}^{\sigma}{}_{\beta}D_{\lambda}D_{\sigma}D^{\nu}-D_{(\alpha}D_{\beta)}D_{\mu}R^{\mu\nu}\right)\Bigg]\times\\ &\times\;D^{\gamma}\;[\,\mathbf{1}_2-\Pi_{\rm tr}\,]_{\gamma\nu}{}^{\rho\sigma}\,,\end{aligned} \tag{4.1.48}$$

and the commutator free part appearing in both expressions

$$\begin{aligned}[\,\Pi^{1}_{2L}\,]_{\alpha\beta}{}^{\rho\sigma} =&\; \left(2D_{(\alpha}\delta^{\mu}_{\beta)}-\tfrac{2}{d}g_{\alpha\beta}D^{\mu}\right)\times\\ &\times\left(R_{\mu}{}^{\nu}\tfrac{1}{\Delta^2}+\tfrac{d-2}{d-1}R_{\mu}{}^{\gamma}D_{\gamma}D^{\nu}\tfrac{1}{\Delta^3}-\tfrac{(d-2)^2}{2(d-1)^2}R^{\gamma\delta}D_{\gamma}D_{\delta}D_{\mu}D^{\nu}\tfrac{1}{\Delta^4}\right)\times\\ &\times\left(D^{(\rho}\delta^{\sigma)}_{\nu}-\tfrac{1}{d}g^{\rho\sigma}D_{\nu}\right)\,.\end{aligned} \tag{4.1.49}$$

Note that at the level of the approximation (neglecting all terms of $\mathcal{O}(\mathcal{R}^2,D\mathcal{R})$), all covariant derivatives in (4.1.48) and (4.1.49) can be commuted freely.

The employment of the minimal TT-decomposition (4.1.34) of the metric tensor requires to insert an appropriate Jacobian J_{TT} in the functional integral. It contributes a compensating factor to the measure

$$D[h_{\mu\nu}] = J_{\mathrm{TT}}\; D[h^{\mathrm{T}}_{\mu\nu}]D[h]D[\xi_\mu]\,. \tag{4.1.50}$$

This Jacobian is again found by solving a Gaussian integral. We find the relation

$$\begin{aligned} 1 &= \int D[h_{\mu\nu}]\;\; \exp\Big[-\tfrac{1}{2}\int d^dx\sqrt{g}\; h^{\mu\nu}h_{\mu\nu}\Big] \\ &= J_{\mathrm{TT}}\; \int D[h^{\mathrm{T}}_{\mu\nu}]D[h]D[\xi_\mu]\;\; \exp\Big[-\tfrac{1}{2}\int d^dx\sqrt{g}\; \Big(h^{\mathrm{T}\mu\nu}h^{\mathrm{T}}_{\mu\nu} + \tfrac{1}{d}h^2 + 2\xi_\mu \mathcal{M}^\mu{}_\nu \xi^\nu\Big)\Big] \\ &\propto J_{\mathrm{TT}}\; \det(\mathcal{M})^{-\frac{1}{2}}\,, \end{aligned} \tag{4.1.51}$$

up to constant factors, with the operator

$$\mathcal{M}^\mu{}_\nu = \Delta\, \delta^\mu{}_\nu - (1-\tfrac{2}{d})D^\mu D_\nu - R^\mu{}_\nu\,. \tag{4.1.52}$$

Solving for J_{TT} yields

$$\begin{aligned} J_{\mathrm{TT}} &\propto \det(\mathcal{M})^{\frac{1}{2}} \;=\; \det(\mathcal{M})^{-\frac{1}{2}}\det(\mathcal{M})^1 \\ &= \int D[\Upsilon_\mu]D[\bar{b}_\mu]D[b_\mu]\;\; \exp\Big[-\tfrac{1}{2}\int d^dx\sqrt{g}\; \Big(\Upsilon_\mu\mathcal{M}^\mu{}_\nu\Upsilon^\nu + \bar{b}_\mu\mathcal{M}^\mu{}_\nu b^\nu\Big)\Big]\,, \end{aligned} \tag{4.1.53}$$

where in the last line $\mathcal{M}^\mu{}_\nu$ was exponentiated using the Faddeev-Popov trick (4.1.19) with the newly introduced auxiliary bosonic (Υ) and fermionic ($\bar{b}$, b) vector fields.

4.2. Projected Heat Kernel Expansions

In order to simplify the functional traces appearing in the RG equation (2.3.18) in the case of gauge and gravitational theories, it is often useful to employ a field decomposition like (4.1.23) or (4.1.34). The method of transversal field decomposition was first advocated in [45] and subsequently employed by a number of groups [57, 58, 115] to diagonalize the propagator. Defining heat kernel expansions on the subspaces of transversal vector and transverse-traceless tensor fields allows to evaluate operator traces constrained to according modes in the same way as for unconstrained fields (3.2.27). The corresponding heat kernel coefficients are known for the special classes of maximally symmetric backgrounds [45] or Lichnerowicz-Laplacians on Einstein spaces [31]. In this section, we engage in a detailed study of constrained heat traces, and compute their expansion coefficients for a general background, based on the off-diagonal heat kernel technique, discussed in chapter 3.

Heat Kernel Coefficients for Transverse Vector Fields

In this section we compute the early-time expansion of the heat kernel resulting from a projected Laplace operator acting on transverse vector fields, up to third order in the curvature. To simplify matters, we assume $D_\mu = \nabla_\mu$ to contain only the Levi-Civita connection and adopt the former notation throughout. In this computation we will make extensive use of the off-diagonal heat kernel coefficients derived in the previous chapter.

In order to construct a well-defined operator trace, it is crucial to observe that the eigenvalue equation for the standard Laplace operator is, in general, incompatible with the transverse condition. This can be seen as follows: Suppose that ξ_i^μ is an eigenfunction of Δ on the space of transverse vector fields

$$\Delta \xi_i^\mu = \lambda_i \xi_i^\mu \,, \qquad D_\mu \xi_i^\mu = 0 \,. \tag{4.2.1}$$

Applying D_μ to this equation, the left-hand-side becomes

$$D_\mu \Delta \xi_i^\mu = [D_\mu, \Delta]\, \xi_i^\mu = -D^\mu R_{\mu\nu} \xi_i^\nu \,, \tag{4.2.2}$$

where we used the transverse condition together with (B.1.5). Since the right hand side vanishes identically, we obtain the condition

$$D^\mu R_{\mu\nu} \xi_i^\nu = 0 \,. \tag{4.2.3}$$

On a general Riemannian manifold, this identity does not hold. Thus the eigenspace of Δ does not decompose into the direct sum of transverse and longitudinal vector fields in general. Notably, the special case of Einstein manifolds discussed in appendix A.2 are an exception, as they have the property that $R_{\mu\nu} \propto g_{\mu\nu}$, ensuring that (4.2.3) is satisfied. This implies that Δ does respect the split of vectors into transverse and longitudinal components expressed by $[\Delta, \Pi_{\mathrm{T}}{}^{\mu}{}_{\nu}]v^{\nu} = 0$, which can be verified by use of the equations (4.2.33).

To compute heat traces on the restricted space of transverse vector fields ξ^{μ} satisfying the constraint $D_{\mu}\xi^{\mu} = 0$, we define the projected Laplace operator

$$\tilde{\Delta}^{\mu}{}_{\nu} \equiv \Pi_{\mathrm{T}}{}^{\mu}{}_{\alpha}\Delta\Pi_{\mathrm{T}}{}^{\alpha}{}_{\nu}\,. \tag{4.2.4}$$

The projectors Π_{T} ensure, that $\tilde{\Delta}^{\mu}{}_{\nu}$ propagates only transversal modes. Moreover, even for the case of a general Riemannian manifold, all eigenfunctions of $\tilde{\Delta}^{\mu}{}_{\nu}$ are transversal vector fields by construction. However, it is due to the projectors entering the definition (4.2.4) that $\tilde{\Delta}^{\mu}{}_{\nu}$ is a non-local, pseudo-differential operator. Thus it is a priori unclear to which extend the standard results for the Seeley-deWitt expansion (3.1.18) carry over to the case of differentially constrained fields. This will be investigated in the following.

The heat-trace on the space of transverse vector fields as the object of interest can now be defined by a heat operator of the projected Laplacian $\tilde{\Delta}$, reading

$$S_{1\mathrm{T}} \equiv \mathrm{Tr}_1\, \Pi_{\mathrm{T}}\, \mathrm{e}^{-s\Pi_{\mathrm{T}}\Delta\Pi_{\mathrm{T}}}\,. \tag{4.2.5}$$

The additional projector in front of the exponential serves the purpose to remove the contribution of the longitudinal vector modes from the zeroth order of the exponential series. Since $\tilde{\Delta}$ is not of generalized Laplace-type, the results of the previous chapter cannot be applied straightforwardly to find the early-time expansion of $S_{1\mathrm{T}}$. Its explicit computation starts from expanding the exponential appearing in (4.2.5) in a power series

$$S_{1\mathrm{T}} = \sum_{n\geq 0} \frac{(-s)^n}{n!}\, \mathrm{Tr}_1\, \Pi_{\mathrm{T}}\, (\Delta\Pi_{\mathrm{T}})^n\ , \tag{4.2.6}$$

where the idempotency of Π_{T} was taken into account. Subsequently, the series is resummed by forming commutators of the projectors and the (unprojected) Laplacian, to collect all remaining powers of Δ into an exponential. This will provide a curvature expansion since multi-commutators and products of lower order commutators with a certain total number of Laplacians are bound to contain an increasing number of (derivatives of) curvatures. Working out the combinatorics in (4.2.6), the curvature expansion of $S_{1\mathrm{T}}$ up to order $\mathcal{R}^3$ is given by

$$S_{1\mathrm{T}} = \sum_{n=0}^{4} \frac{(-s)^n}{n!}\, S_{1\mathrm{T}}^{(n)} + \mathcal{O}(\mathcal{R}^4)\,, \tag{4.2.7}$$

where

$$
\begin{aligned}
S^{(0)}_{1\mathrm{T}} &= \mathrm{Tr}_1\, \Pi_{\mathrm{T}}\, \mathrm{e}^{-s\Delta}\,, \\
S^{(1)}_{1\mathrm{T}} &= \mathrm{Tr}_1\, \Pi_{\mathrm{T}}\, [\,\Delta\,,\,\Pi_{\mathrm{T}}\,]\; \mathrm{e}^{-s\Delta}\,, \\
S^{(2)}_{1\mathrm{T}} &= \mathrm{Tr}_1\, \Pi_{\mathrm{T}} \Big([\,\Delta\,,\,\Pi_{\mathrm{T}}\,]^2 + [\,\Delta\,,\,[\,\Delta\,,\,\Pi_{\mathrm{T}}\,]] \Big)\, \mathrm{e}^{-s\Delta}\,, \\
S^{(3)}_{1\mathrm{T}} &= \mathrm{Tr}_1\, \Pi_{\mathrm{T}} \Big(2\,[\,\Delta\,,\,\Pi_{\mathrm{T}}\,]\,[\,\Delta\,,\,[\,\Delta\,,\,\Pi_{\mathrm{T}}\,]] + [\,\Delta\,,\,[\,\Delta\,,\,\Pi_{\mathrm{T}}\,]]\,[\,\Delta\,,\,\Pi_{\mathrm{T}}\,] \\
&\qquad\qquad + [\,\Delta\,,\,[\,\Delta\,,\,[\,\Delta\,,\,\Pi_{\mathrm{T}}\,]]] + [\,\Delta\,,\,\Pi_{\mathrm{T}}\,]^3 \Big)\, \mathrm{e}^{-s\Delta}\,, \\
S^{(4)}_{1\mathrm{T}} &= \mathrm{Tr}_1\, \Pi_{\mathrm{T}} \Big(3\,[\,\Delta\,,\,\Pi_{\mathrm{T}}\,]\,[\,\Delta\,,\,[\,\Delta\,,\,[\,\Delta\,,\,\Pi_{\mathrm{T}}\,]]] + 3\,[\,\Delta\,,\,[\,\Delta\,,\,\Pi_{\mathrm{T}}\,]]^2 \\
&\qquad\qquad + [\,\Delta\,,\,[\,\Delta\,,\,[\,\Delta\,,\,\Pi_{\mathrm{T}}\,]]]\,[\,\Delta\,,\,\Pi_{\mathrm{T}}\,] + [\Delta\,,\,[\,\Delta\,,\,[\,\Delta\,,\,[\,\Delta\,,\,\Pi_{\mathrm{T}}\,]]]] \Big)\, \mathrm{e}^{-s\Delta}\,.
\end{aligned}
\tag{4.2.8}
$$

To find these expressions in terms of the basis monomials (A.1.2) defined in appendix A, it is sufficient to keep track of all operator insertions which contain up to three powers of the curvature or terms where two covariant derivatives act on two curvature tensors. Terms with more derivatives acting on a curvature tensor can only give rise to surface terms and may thus be discarded.

In order to apply the off-diagonal heat kernel technique to the curvature expansion (4.2.7), we have to deal with the inverse powers of the Laplacians appearing in the projectors and their commutators. To this end, we employ the identity

$$
\frac{1}{\Delta^n} = \int_0^\infty \frac{t^{n-1}}{(n-1)!} \mathrm{e}^{-t\Delta}\, dt\,. \tag{4.2.9}
$$

By combining the exponentials of the Laplace operator, the traces in (4.2.8) can be cast into the form

$$
\mathrm{Tr}_1 \left[\mathcal{O}\, \frac{1}{\Delta^n}\, \mathrm{e}^{-s\Delta} \right] = \int_0^\infty \frac{t^{n-1}}{(n-1)!}\, \mathrm{Tr}_1 \left[\mathcal{O}\, \mathrm{e}^{-(s+t)\Delta} \right] dt\,, \tag{4.2.10}
$$

where $\mathcal{O} = \mathcal{R} D_{\mu_1} \dots D_{\mu_n}$ denote any non-minimal operator insertions of the form (3.3.7). Thus the operator traces on the right hand side can be evaluated via the off-diagonal heat kernel techniques explained in section 3.3. The non-locality of the projection operators thus leads to auxiliary t-integrals under the trace. Upon inserting the H functions (3.3.9), these integrals assume the generic form

$$
I(n, k - \tfrac{d}{2}) \;:=\; \int_0^\infty \frac{t^{n-1}}{(n-1)!} (s+t)^{k-d/2} dt \;=\; \frac{\Gamma(\frac{d}{2} - k - n)}{\Gamma(\frac{d}{2} - k)}\, s^{n+k-\frac{d}{2}}\,. \tag{4.2.11}
$$

The gamma functions appearing here show the typical form of an IR divergence as seen within dimensional regularization. Indeed the r.h.s. of (4.2.11) becomes singular for $n + k \geq d/2$, that is, if the order of the heat kernel expansion $\mathcal{R}^{n+k}$ exceeds $d/2$.

One would thus expect that the coefficients in the Seeley-deWitt expansion of $S_{1\mathrm{T}}$ will develop divergences at $\mathcal{O}(\mathcal{R}^{d/2})$. For explicit computation in the next subsection, we will use (4.2.11) to regularize the Seeley-deWitt coefficients by analytically continuing in the spacetime dimension d, assuming that it is sufficiently large for all integrals to converge. In this context, we remark that it is necessary to keep the spacetime dimension unspecified, so that no terms are lost in the traced heat-equation

$$\mathrm{Tr}\Big[\Delta H(s)\Big] = -\tfrac{d}{ds}\,\mathrm{Tr}\Big[H(s)\Big] = (4\pi)^{-d/2}\sum_{n\geq 0}(\tfrac{d}{2}-n)s^{n-1-d/2}A_n\,. \tag{4.2.12}$$

Integrating this relation should reproduce the standard heat kernel

$$\mathrm{Tr}\Big[H(s)\Big] = \int_0^\infty \mathrm{Tr}\Big[\Delta H(s+t)\Big]dt\,, \tag{4.2.13}$$

which requires that none of the factors $(\frac{d}{2}-n)$ vanishes in the above formula.

We will now proceed with the evaluation of the expressions $S_{1\mathrm{T}}^{(n)}$ defined in (4.2.8). Substituting the explicit form of Π_T given in (4.1.25) into $S_{1\mathrm{T}}^{(0)}$ and using (4.2.9) the trace can be written as

$$S_{1\mathrm{T}}^{(0)} = \mathrm{Tr}_1\Big[\delta_\mu^\nu \mathrm{e}^{-s\Delta}\Big] + \int_0^\infty \mathrm{Tr}_1\Big[D_\mu \mathrm{e}^{-t\Delta}D^\nu \mathrm{e}^{-s\Delta}\Big]dt\,. \tag{4.2.14}$$

Here the first trace is a standard heat trace on the space of unconstrained vector fields and is readily evaluated by substituting the heat kernel coefficients given in the second column of table 4.2 according to (3.2.27). The second trace is evaluated via the off-diagonal heat kernel. We first combine the two exponentials using the Baker-Campbell-Hausdorff formula

$$\begin{aligned}\mathrm{Tr}_1\Big[D_\mu \mathrm{e}^{-t\Delta}D^\nu \mathrm{e}^{-s\Delta}\Big] &= \sum_{n=0}^\infty \frac{1}{n!}\,(-t)^n\,\mathrm{Tr}_1\Big[[D_\mu,\Delta]_n\,D^\nu \mathrm{e}^{-(s+t)\Delta}\Big]\\ &= \sum_{n=0}^\infty \frac{1}{n!}\,(-t)^n\,T_1^{(n)}\,,\end{aligned} \tag{4.2.15}$$

where $[D_\mu,\Delta]_n$ denote the n-fold commutators defined in (B.1). The commutators appearing on the r.h.s. ensure that (4.2.15) constitutes a curvature expansion. All terms contributing to the basis (A.1.2) are generated by the first five terms of this expansion, so that we truncate (4.2.15) at $n=4$. Substituting the commutators (B.1.2), it is a straightforward application of the off-diagonal heat kernel to evaluate the $T_1^{(n)}$, yielding

$$\begin{aligned}
T_1^{(0)} =& \frac{1}{(4\pi(s+t))^{d/2}} \int d^dx\sqrt{g}\,\Bigg[\tfrac{-d}{2(s+t)} - \tfrac{4+d}{12}\mathcal{R}^1 + (s+t)\Big(- \tfrac{d+8}{144}\mathcal{R}^2_1 + \tfrac{d-34}{360}\mathcal{R}^2_2 - \tfrac{d-4}{360}\mathcal{R}^2_3\Big)\\
&+ \frac{(s+t)^2}{24}\Big(- \tfrac{5d+12}{140}\mathcal{R}^3_1 - \tfrac{d+36}{70}\mathcal{R}^3_2 - \tfrac{d+12}{108}\mathcal{R}^3_3 + \tfrac{d-30}{90}\mathcal{R}^3_4 + \tfrac{16d+345}{945}\mathcal{R}^3_5\\
&\qquad - \tfrac{4d+207}{315}\mathcal{R}^3_6 - \tfrac{d}{90}\mathcal{R}^3_7 - \tfrac{d-174}{630}\mathcal{R}^3_8 - \tfrac{17d-102}{3780}\mathcal{R}^3_9 + \tfrac{d-6}{135}\mathcal{R}^3_{10}\Big)\Bigg]\,,\\
T_1^{(1)} =& -\frac{1}{(4\pi(s+t))^{d/2}} \int d^dx\sqrt{g}\,\Bigg[\tfrac{1}{2(s+t)}\mathcal{R}^1 + \tfrac{1}{12}\mathcal{R}^2_1 + \tfrac{1}{3}\mathcal{R}^2_2\\
&+ \frac{s+t}{24}\Big(- \tfrac{6}{5}\mathcal{R}^3_1 + \tfrac{8}{5}\mathcal{R}^3_2 + \tfrac{1}{6}\mathcal{R}^3_3 + \tfrac{19}{15}\mathcal{R}^3_4 - \tfrac{22}{15}\mathcal{R}^3_5 + \tfrac{56}{15}\mathcal{R}^3_6 + \tfrac{1}{15}\mathcal{R}^3_7 - \tfrac{4}{15}\mathcal{R}^3_8\Big)\Bigg]\,,\\
T_1^{(2)} =& \frac{1}{(4\pi(s+t))^{d/2}} \int d^dx\sqrt{g}\,\Bigg[- \tfrac{1}{2(s+t)}\mathcal{R}^2_2 + \frac{1}{12}\Big(3\mathcal{R}^3_1 - \mathcal{R}^3_4 + 2\mathcal{R}^3_5 - 6\mathcal{R}^3_6\Big)\Bigg]\,,\\
T_1^{(3)} =& \frac{1}{(4\pi)^{d/2}(s+t)^{d/2+1}} \int d^dx\sqrt{g}\,\Big[\tfrac{1}{2}\mathcal{R}^3_1 + \tfrac{1}{2}\mathcal{R}^3_2 + \tfrac{1}{2}\mathcal{R}^3_5 - \mathcal{R}^3_6\Big]\,,\\
T_1^{(4)} =& \frac{1}{(4\pi)^{d/2}(s+t)^{d/2+2}} \int d^dx\sqrt{g}\,\Big[\tfrac{1}{2}\mathcal{R}^3_1 + \mathcal{R}^3_2 + \mathcal{R}^3_5 - \mathcal{R}^3_6\Big]\,.
\end{aligned} \tag{4.2.16}$$

After computing the auxiliary t-integration and adding the vector-trace contribution in (4.2.14), the final result for $S_{1\mathrm{T}}^{(0)}$ is found. Together with the following partial results, this is written in the general form

$$S_{1\mathrm{T}}^{(n)} = \frac{1}{(4\pi s)^{d/2}} \int d^dx\sqrt{g}\,\Bigg[c^0 + s\,c^1\,R + s^2\sum_{i=1}^{3} c_i^2\,\mathcal{R}^2_i + s^3\sum_{i=1}^{10} c_i^3\,\mathcal{R}^3_i\Bigg]\,, \tag{4.2.17}$$

with the d-dependent coefficients explicitly given in the second column of table 4.1.

In $S_{1\mathrm{T}}^{(1)}$ appears the lowest order commutator, evaluating to

$$\Big[\Delta, \Pi_{\mathrm{T}\mu}{}^{\nu}\Big]v_\nu = \Big(\Pi_{\mathrm{L}\mu}{}^{\alpha}R_\alpha{}^{\nu} - R_\mu{}^{\alpha}\Pi_{\mathrm{L}\alpha}{}^{\nu}\Big)v_\nu\,. \tag{4.2.18}$$

With the orthogonality of the projectors $\Pi_\mathrm{T}\cdot\Pi_\mathrm{L} = 0$, this trace becomes

$$S_{1\mathrm{T}}^{(1)} = -\mathrm{Tr}_1\left[\Pi_{\mathrm{T}\mu}{}^{\alpha}\,R_\alpha{}^{\beta}\,\Pi_{\mathrm{L}\beta}{}^{\nu}\mathrm{e}^{-s\Delta}\right]\,. \tag{4.2.19}$$

In order to cast this expression into standard form, we express the projection operators via (4.1.29) and (4.1.30), where the inverse Laplacians appear to the very left and very right of the commutator insertions. This leads to

$$S_{1\mathrm{T}}^{(1)} \simeq \mathrm{Tr}_1\left[\left(\delta_\mu{}^{\alpha} + \frac{1}{\Delta}\sum_{n=0}^{3}\Big((-\Delta)^{-n}\,[D_\mu\,,\,\Delta]_n\Big)D^\alpha\right)R_\alpha{}^{\beta}\,D_\beta\left(\sum_{n=0}^{3}[D^\nu\,,\,\Delta]_n\,\frac{1}{\Delta^n}\right)\frac{1}{\Delta}\mathrm{e}^{-s\Delta}\right], \tag{4.2.20}$$

	$S^{(0)}_{1\mathrm{T}}$	$S^{(1)}_{1\mathrm{T}}$	$S^{(2)}_{1\mathrm{T}}$
c^0	$d-1$	0	0
c^1	$\frac{d}{6}-\frac{d+6}{6d}=\frac{(d-3)(d+2)}{6d}$	0	0
c^2_1	$\frac{d}{72}-\frac{1}{72}\frac{d+10}{d-2}=\frac{(d-5)(d+2)}{72(d-2)}$	$-\frac{1}{d(d+2)}$	$-\frac{1}{d(d+2)}$
c^2_2	$-\frac{d}{180}+\frac{d^2-32d+180}{180d(d-2)}=-\frac{d^3-3d^2+32d-180}{180d(d-2)}$	$\frac{1}{(d+2)}$	$\frac{1}{(d+2)}$
c^2_3	$\frac{d-1}{180}-\frac{1}{12}=\frac{d-16}{180}$	0	0
c^3_1	$\frac{d}{336}+\frac{1}{120}-\frac{5d^2+32d+464}{1680(d-4)(d+2)}$	$-\frac{d^3+4d^2+24d-24}{6(d-2)d(d+2)(d+4)}$	$\frac{d^2-14d+8}{2(d-2)d(d+2)(d+4)}$
c^3_2	$\frac{d}{840}-\frac{1}{30}-\frac{d^3+40d^2-64d-1120}{840(d-4)d(d+2)}$	$\frac{d^2+4d-24}{6(d-2)d(d+4)}$	$-\frac{4}{(d-2)(d+2)(d+4)}$
c^3_3	$\frac{d}{1296}-\frac{d+14}{1296(d-4)}$	$-\frac{1}{6(d-2)d}$	$-\frac{d^2+12d-4}{6(d-2)d(d+2)(d+4)}$
c^3_4	$-\frac{d}{1080}+\frac{d^2-30d+236}{1080(d-4)(d-2)}$	$\frac{d^2+d+10}{6(d-2)d(d+2)}$	$\frac{d^3+14d^2-4d+40}{6(d-2)d(d+2)(d+4)}$
c^3_5	$-\frac{4d}{2835}+\frac{1}{30}+\frac{16d^4+377d^3-1954d^2+2272d-30240}{11340(d-4)(d-2)d(d+2)}$	$-\frac{d^2+8d+32}{6d(d+2)(d+4)}$	$-\frac{d^2-2d+16}{3d(d+2)(d+4)}$
c^3_6	$\frac{d}{945}-\frac{1}{30}-\frac{4d^2+223d-1460}{3780(d-4)(d+2)}$	$\frac{d^3-12d-32}{3(d-2)d(d+2)(d+4)}$	$-\frac{4(d^2+8)}{3(d-2)d(d+2)(d+4)}$
c^3_7	$\frac{d}{1080}-\frac{1}{72}-\frac{d+2}{1080(d-4)}$	0	0
c^3_8	$\frac{d}{7560}-\frac{1}{90}-\frac{d-172}{7560(d-4)}$	0	0
c^3_9	$\frac{17(d-1)}{45360}-\frac{1}{180}$	0	0
c^3_{10}	$-\frac{d-1}{1620}+\frac{1}{90}$	0	0

Table 4.1.: Summary of the traced heat kernel coefficients of the partial traces $S^{(n)}_{1\mathrm{T}}$ entering the computation (4.2.7). These coefficients appear in the general form (4.2.17), with the basis monomials defined in (A.1.2).

where the series of higher-order commutators can be terminated at order $n = 3$, since (4.2.19) already contains one explicit power of the curvature. Substituting the explicit expressions for the commutators (B.1.2) and (B.1.5), the sub-traces are evaluated with Mathematica. The final result for $S^{(1)}_{1\mathrm{T}}$ in the form (4.2.17) is given with the coefficients in the third column of table 4.1.

The $S^{(2)}_{1\mathrm{T}}$ part (4.2.8) decomposes into two traces containing the product of two single commutators (4.2.18) and the double-commutator $[\Delta, [\Delta, \Pi_{\mathrm{T}}]]$. The latter can be constructed recursively from the single commutator (4.2.18). In order to collect all Laplace operators in a single function one has to take into account that the curvature tensors are not covariantly constant and thus do not commute with the (inverse) Laplacians. The corresponding commutation relation is given by

$$\Big[\Delta^{-1}, R_{\mu\nu}\Big] v^\nu = \Big(2R_{\mu\nu;\alpha}D^\alpha - (\Delta R_{\mu\nu}) + 4R_{\mu\nu;\alpha\beta}D^\beta D^\alpha \Delta^{-1}\Big)\,\Delta^{-2} v^\nu + \mathcal{O}(D^3\mathcal{R})\,, \quad (4.2.21)$$

which captures all terms contributing to the required order. With this formula, the explicit results are found via Mathematica and read

$$\begin{aligned}
&\mathrm{Tr}_1\,\Pi_\mathrm{T}\,[\,\Delta\,,\,\Pi_\mathrm{T}\,]^2\;\mathrm{e}^{-s\Delta} = \\
&\quad \tfrac{1}{(4\pi s)^{d/2}}\int d^dx\sqrt{g}\,\Big[\tfrac{1}{d(d+2)}\Big(\mathcal{R}_1^2 - d\mathcal{R}_2^2\Big) \\
&\qquad + \tfrac{s}{(d-2)d(d+2)}\Big[\tfrac{d^3+d^2+6d-8}{2(d+4)}\mathcal{R}_1^3 - \tfrac{2(d^2-8)}{d+4}\mathcal{R}_2^3 + \tfrac{d^2+20}{6(d+4)}\mathcal{R}_3^3 \\
&\qquad\quad - \tfrac{d^3-4d^2+32d+40}{6(d+4)}\mathcal{R}_4^3 + \tfrac{d^3+8d^2-4d-32}{3(d+4)}\mathcal{R}_5^3 - \tfrac{3d^3+2d^2-24d-32}{3(d+4)}\mathcal{R}_6^3\Big]\Big]\,,
\end{aligned} \quad (4.2.22)$$

and

$$\begin{aligned}
&\mathrm{Tr}_1\,\Pi_\mathrm{T}\,[\,\Delta\,,\,[\,\Delta\,,\,\Pi_\mathrm{T}\,]]\;\mathrm{e}^{-s\Delta} = \\
&\quad \tfrac{1}{(4\pi s)^{d/2}}\int d^dx\sqrt{g}\,\Big[-\tfrac{2}{d(d+2)}\Big(\mathcal{R}_1^2 - d\mathcal{R}_2^2\Big) \\
&\qquad - \tfrac{s}{6(d-2)d(d+2)}\Big[\tfrac{3(d^3+20d-16)}{d+4}\mathcal{R}_1^3 - \tfrac{12(d-4)(d+2)}{d+4}\mathcal{R}_2^3 + 2(d+2)\mathcal{R}_3^3 \\
&\qquad\quad - 2(d^2+d+10)\mathcal{R}_4^3 + \tfrac{4(d-2)(d^2+4d+16)}{d+4}\mathcal{R}_5^3 - \tfrac{2(d-4)(3d^2+10d+16)}{d+4}\mathcal{R}_6^3\Big]\Big]\,.
\end{aligned} \quad (4.2.23)$$

The final result for $S^{(2)}_{1\mathrm{T}}$ is the sum of (4.2.22) and (4.2.23) and is in terms of the curvature expansion (4.2.17) given by the coefficients in the fourth column of table 4.1.

The contributions of $S^{(3)}_{1\mathrm{T}}$ and $S^{(4)}_{1\mathrm{T}}$ can be computed along the same lines and are conveniently presented in terms of the abbreviations

$$\begin{aligned}
\mathcal{C}_1 \equiv &\frac{1}{(4\pi s)^{d/2}}\,\frac{1}{d(d+2)(d+4)}\int d^dx\sqrt{g}\,\Big[d(d-1)\mathcal{R}_1^3 + (d^2-8)\mathcal{R}_2^3 - 2\mathcal{R}_3^3 \\
&\qquad\qquad + 3d\mathcal{R}_4^3 + d(d+4)\mathcal{R}_5^3 - 2d(d+2)\mathcal{R}_6^3\Big]\,, \\
\mathcal{C}_2 \equiv &\frac{1}{(4\pi s)^{d/2}}\,\frac{1}{(d+2)(d+4)}\int d^dx\sqrt{g}\,\Big[(d-6)\mathcal{R}_1^3 + 2(d+2)\mathcal{R}_2^3 + 2d\mathcal{R}_5^3 - 2d\mathcal{R}_6^3\Big]\,.
\end{aligned} \quad (4.2.24)$$

The sub-traces appearing in $S^{(3)}_{1\mathrm{T}}$ are given by

$$
\begin{aligned}
2\,\mathrm{Tr}_1\Big[\Pi_{\mathrm{T}}\,[\Delta\,,\Pi_{\mathrm{T}}]\,[\Delta\,,[\Delta\,,\Pi_{\mathrm{T}}]]\;\mathrm{e}^{-s\Delta}\Big] &= 2\,\mathcal{C}_1\,,\\
\mathrm{Tr}_1\Big[\Pi_{\mathrm{T}}\,[\Delta\,,[\Delta\,,\Pi_{\mathrm{T}}]]\,[\Delta\,,\Pi_{\mathrm{T}}]\;\mathrm{e}^{-s\Delta}\Big] &= -\,\mathcal{C}_1\,,\\
\mathrm{Tr}_1\Big[\Pi_{\mathrm{T}}\,[\Delta\,,[\Delta\,,[\Delta\,,\Pi_{\mathrm{T}}]]]\;\mathrm{e}^{-s\Delta}\Big] &= -\,\mathcal{C}_2\,,\\
\mathrm{Tr}_1\Big[\Pi_{\mathrm{T}}\,[\Delta\,,\Pi_{\mathrm{T}}]^3\;\mathrm{e}^{-s\Delta}\Big] &= 0\,,
\end{aligned}
\tag{4.2.25}
$$

and for the traces contributing to $S^{(4)}_{1\mathrm{T}}$ we find

$$
\begin{aligned}
3\,\mathrm{Tr}_1\Big[\Pi_{\mathrm{T}}\,[\Delta\,,\Pi_{\mathrm{T}}]\,[\Delta\,,[\Delta\,,[\Delta\,,\Pi_{\mathrm{T}}]]]\;\mathrm{e}^{-s\Delta}\Big] &= 3\,s^{-1}\,\mathcal{C}_2\,,\\
3\,\mathrm{Tr}_1\Big[\Pi_{\mathrm{T}}\,[\Delta\,,[\Delta\,,\Pi_{\mathrm{T}}]]^2\;\mathrm{e}^{-s\Delta}\Big] &= -\,3\,s^{-1}\,\mathcal{C}_2\,,\\
\mathrm{Tr}_1\Big[\Pi_{\mathrm{T}}\,[\Delta\,,[\Delta\,,[\Delta\,,\Pi_{\mathrm{T}}]]]\,[\Delta\,,\Pi_{\mathrm{T}}]\;\mathrm{e}^{-s\Delta}\Big] &= s^{-1}\,\mathcal{C}_2\,,\\
\mathrm{Tr}_1\Big[\Pi_{\mathrm{T}}\,[\Delta\,,[\Delta\,,[\Delta\,,[\Delta\,,\Pi_{\mathrm{T}}]]]]\;\mathrm{e}^{-s\Delta}\Big] &= 2\,s^{-1}\,\mathcal{C}_2\,.
\end{aligned}
\tag{4.2.26}
$$

Substituting these results in (4.2.8) yields

$$
S^{(3)}_{1\mathrm{T}} = \mathcal{C}_1 - \mathcal{C}_2\,,
\tag{4.2.27}
$$

and

$$
S^{(4)}_{1\mathrm{T}} = 3\,s^{-1}\,\mathcal{C}_2\,.
\tag{4.2.28}
$$

The final result for $S_{1\mathrm{T}}$ defined in (4.2.5) is obtained by substituting the intermediate results given in table 4.1, (4.2.27) and (4.2.28) into the expansion (4.2.7). In terms of the basis (A.1.2) it is given as

$$
\begin{aligned}
&S_{1\mathrm{T}} \simeq \frac{1}{(4\pi s)^{d/2}}\int d^dx\sqrt{g}\left[c^0 + s\,c^1\,R + s^2\sum_{i=1}^{3}c^2_i\,\mathcal{R}^2_i + s^3\sum_{i=1}^{10}c^3_i\,\mathcal{R}^3_i\right]\,,\\
&\qquad c^0 = d-1\,, \qquad\qquad c^1 = \tfrac{(d-3)(d+2)}{6d}\,,\\
&\text{with}\quad c^2_1 = \tfrac{d^4-d^3-16d^2+16d-72}{72(d-2)d(d+2)}\,,\quad c^2_2 = -\tfrac{d^4-d^3+116d^2-296d-360}{180(d-2)d(d+2)}\,,\quad c^2_3 = \tfrac{d-16}{180}\,,\\
&\qquad c^3_{i=1,..,8} \sim \tfrac{1}{d-4}\,,\qquad c^3_9 = \tfrac{17d-269}{45360}\,,\qquad c^3_{10} = -\tfrac{d-19}{1620}\,,
\end{aligned}
\tag{4.2.29}
$$

with all coefficients c^n_i to curvature order $\mathcal{R}^3$ defined in the last column of table 4.2 at the end of this section. These coefficients for the transverse vector trace do not decompose into the difference of those of the vector and the scalar trace, as one might expect by counting the degrees of freedom. This is due to the interaction between transverse and longitudinal modes. Instead, it is possible to write any coefficient of the transverse trace as

$$c^n_{i,1\mathrm{T}} \quad = \quad c^n_{i,1} - c^n_{i,0} + Q_1\left(d\right)/Q_2\left(d\right)\,, \tag{4.2.30}$$

with the aforementioned difference, modulo a correction term where $Q_2\left(d\right)$ is a polynomial of one degree higher than $Q_1\left(d\right)$. This means that, in the limit $d \to \infty$, the partial decoupling of degrees of freedom becomes exact, a situation that is reminiscent of many mean field computations at large d.

As expected from the t-integrals (4.2.11), which are finite for sufficiently large d only, the $S_{1\mathrm{T}}$ trace has poles in even spacetime dimensions, with coefficients at order $\mathcal{R}^n$ diverging for $n > d/2$. Notably, all results can be safely continued to any odd dimensionality, since for odd d, the gamma functions in (4.2.11) never become singular.

In the context of these dimensional poles, it is interesting to consider Einstein spaces, reviewed in appendix A.2. These constitute a special class of Riemannian manifolds where the projection operators (4.1.25) commute with the unmodified Laplacian Δ, so that the replacement with $\tilde{\Delta}$ (4.2.4) is unnecessary. The corresponding heat kernel coefficients are readily obtained from the general result (4.2.29), taking the geometrical identities (A.2.4) into account. This defines the trace in terms of the Einstein space basis $\mathcal{E}^n_i$ as

$$\begin{aligned} S_{1\mathrm{T}}\Big|_{\mathrm{ES}} = \tfrac{1}{(4\pi s)^{d/2}} \int d^dx\sqrt{g}\Big\{&(d-1) + \tfrac{(d-3)(d+2)}{6d} s\,\mathcal{E}^1 \\ &+ s^2\left(\tfrac{5d^3-7d^2-58d-180}{360d^2}\,\mathcal{E}^2_1 + \tfrac{d-16}{180}\,\mathcal{E}^2_2\right) \\ &+ s^3\left(\tfrac{35d^4-77d^3-604d^2-3512d-7560}{45360d^3}\,\mathcal{E}^3_1 + \tfrac{7d^2-111d-127}{7560d}\,\mathcal{E}^3_2 + \tfrac{17d-269}{45360}\,\mathcal{E}^3_3 - \tfrac{d-19}{1620}\,\mathcal{E}^3_4\right)\Big\}\,. \end{aligned} \tag{4.2.31}$$

Remarkably, in the Einstein space limit, the heat kernel coefficients in (4.2.29) combine in such a way that all the singularities in even dimensionality d cancel, rendering all coefficients finite in any dimension $d > 0$. For this reason, it is illustrative to re-derive (4.2.31) by using the Einstein condition from the beginning. Since $[\Delta, \Pi_\mathrm{T}]|_{\mathrm{ES}} = 0$, all $S^{(n)}_{1\mathrm{T}}$ for $n \geq 1$ in the expansion (4.2.7) vanish and

$$S_{1\mathrm{T}}\Big|_{\mathrm{ES}} = S^{(0)}_{1\mathrm{T}} \tag{4.2.32}$$

holds exactly. Exploiting further that on an Einstein space

$$D_\alpha\, f\big(\Delta\big)\phi = f\big(\Delta + \tfrac{R}{d}\big)D_\alpha\phi\,, \qquad D_\alpha\, f\big(\Delta\big)v^\alpha = f\big(\Delta - \tfrac{R}{d}\big)D_\alpha v^\alpha\,, \tag{4.2.33}$$

for a general function of the Laplacian, the trace $S_{1\mathrm{T}}^{(0)}$ (4.2.14) can directly be cast into the form

$$S_{1\mathrm{T}}\Big|_{\mathrm{ES}} = \mathrm{Tr}_1\left[\delta^\nu_\mu\,\mathrm{e}^{-s\Delta}\right] + \int_0^\infty \mathrm{Tr}_1\left[D_\mu D^\nu \mathrm{e}^{-(s+t)\Delta}\right]\mathrm{e}^{-t\frac{R}{d}}\,dt\,. \tag{4.2.34}$$

The resulting traces are easily evaluated with the off-diagonal heat kernel (3.3.7), confirming the result (4.2.31). Here the Ricci scalar in the exponential corresponds to a summation of the multi-commutators in (4.2.15). Notably, this factor leads to an exponential suppression of the integrand for large values of t, rendering the auxiliary integrals finite. Thus it is this exponential that ensures that the heat kernel coefficients in $S_{1\mathrm{T}}\Big|_{\mathrm{ES}}$ are free from dimensional poles to all orders in the curvature. Moreover, the fact that the general computation restricted to the Einstein space limit and the Einstein space computation lead to the same finite result indicates that the origin of the dimensional poles is not due to the use of the early-time expansion of the off-diagonal heat kernel. We will elaborate this point further at the end of this section.

Heat Kernel Coefficients for Transverse-Traceless Tensor Fields

A similar computation of heat traces for 2-tensors constrained to the transverse-traceless subspace by the projector $\Pi_{2\mathrm{T}}$ (4.1.41) is significantly more involved. Using the insights we gained from the transverse vector case, we can estimate that dimensional poles will appear here because of the non-locality introduced by the projectors. These poles will however not disappear on Einstein spaces, because instead $[\Delta, {\Pi_{2\mathrm{T}}}^{\mu\nu}{}_{\rho\sigma}]h^{\rho\sigma} = 0$ holds only on a maximally symmetric manifold. Due to the increase in computational effort and the limited usefulness of such heat kernel coefficients, we will resort to the computation of the trace

$$S_{2\mathrm{T}} = \mathrm{Tr}_2\left[\Pi_{2\mathrm{T}}\,\mathrm{e}^{-s\Delta}\right]\,. \tag{4.2.35}$$

As an approximation to the fully projected heat kernel, the use of the ordinary Laplacian operator in the exponential, as opposed to $\Pi_{2\mathrm{T}}\Delta\Pi_{2\mathrm{T}}$, corresponds to the full propagator of unconstrained fields, with merely the trace being constrained to the modes of transverse-traceless field configurations. Following the estimate for the integrals (4.2.11), this expression will give finite results up to order $\mathcal{R}^2$ in $d = 4$ dimensions.

Using the definition of the projector (4.1.41) we find

$$\begin{aligned} S_{2\mathrm{T}} &= \mathrm{Tr}_2\left[\mathrm{e}^{-s\Delta}\right] - \mathrm{Tr}_2\left[\mathrm{e}^{-s\Delta}\Pi_{2\mathrm{L}}\right] - \mathrm{Tr}_2\left[\mathrm{e}^{-s\Delta}\Pi_{\mathrm{tr}}\right] \\ &=: S_{2\mathrm{T}}^{(2)} + S_{2\mathrm{T}}^{(1)} + S_{2\mathrm{T}}^{(0)}\,. \end{aligned} \tag{4.2.36}$$

Both $S^{(2)}_{2\mathrm{T}}$ and $S^{(0)}_{2\mathrm{T}}$ reduce to standard heat kernel formulas, noting that

$$S^{(0)}_{2\mathrm{T}} = -\mathrm{Tr}_2\left[\mathrm{e}^{-s\Delta}\,\Pi_{\mathrm{tr}}\right] = -\mathrm{Tr}_0\left[\mathrm{e}^{-s\Delta}\right]\,, \tag{4.2.37}$$

so their contribution is directly given by (3.2.27). In order to obtain $S^{(1)}_{2\mathrm{T}}$ we substitute $\Pi_{2\mathrm{L}}$, (4.1.38), and use the cyclicity of the trace to write

$$\begin{aligned}
S^{(1)}_{2\mathrm{T}} &= \mathrm{Tr}_1\left[(-D^\gamma)\,[\mathbf{1}_2-\Pi_{\mathrm{tr}}]_{\gamma\beta}{}^{\mu\nu}\mathrm{e}^{-s\Delta}\,[P_1]^\alpha_{\mu\nu}\left[P_2^{-1}\right]^{\beta'}_\alpha\right]\\
&= \mathrm{Tr}_1\left[\mathrm{e}^{-s\Delta}\right] + s\,\mathrm{Tr}_1\left[\mathrm{e}^{-s\Delta}\,[D^\gamma,\Delta]\;[P_1]^\alpha_{\gamma\beta}\left[P_2^{-1}\right]^{\beta'}_\alpha\right]\\
&\quad-\frac{s^2}{2}\,\mathrm{Tr}_1\left[\mathrm{e}^{-s\Delta}\;[[D^\gamma,\Delta],\Delta]\,[P_1]^\alpha_{\gamma\beta}\left[P_2^{-1}\right]^{\beta'}_\alpha\right]+\mathcal{O}(\mathcal{R}^3)\\
&=: T^{(0)}_2 + T^{(1)}_2 + T^{(2)}_2 + \mathcal{O}(\mathcal{R}^3)\,.
\end{aligned} \tag{4.2.38}$$

To arrive at the second line, we exploited the orthogonality of the projectors $\Pi_{\mathrm{tr}}\cdot P_1 = 0$, used the commutator expansion (B.2.2) and $-D^\gamma[P_1]^\alpha_{\gamma\beta}[P_2^{-1}]^{\beta'}_\alpha = \delta^{\beta'}_\beta$, following from (4.1.39). Subsequently, the commutators (B.1.7) and the perturbative inverse P_2^{-1} (4.1.45), can be substituted explicitly. After expanding the products, the full trace is built from linear combination of basis integrals, which can be evaluated with the off-diagonal heat kernel. These are summarized here for $d = 4$ to second order in the curvature:

$s\mathrm{Tr}_0\left[\frac{1}{\Delta}\mathrm{e}^{-s\Delta}R^2\right]$	$=$	$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}R^2$
$s\mathrm{Tr}_0\left[\frac{1}{\Delta}\mathrm{e}^{-s\Delta}R^{\mu\nu}D_\mu D_\nu\right]$	$=$	$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left[-\frac{1}{4s}R-\frac{1}{12}R^2+\frac{1}{6}R_{\mu\nu}R^{\mu\nu}\right]$
$s\mathrm{Tr}_0\left[\frac{1}{\Delta^2}\mathrm{e}^{-s\Delta}R^{\mu\alpha}R_\alpha{}^\nu D_\mu D_\nu\right]$	$=$	$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left[-\frac{1}{4}R^{\mu\nu}R_{\mu\nu}\right]$
$s\mathrm{Tr}_0\left[\frac{1}{\Delta^2}\mathrm{e}^{-s\Delta}RR^{\mu\nu}D_\mu D_\nu\right]$	$=$	$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left[-\frac{1}{4}R^2\right]$
$s\mathrm{Tr}_0\left[\frac{1}{\Delta^3}\mathrm{e}^{-s\Delta}R^{\mu\nu}R^{\alpha\beta}D_\mu D_\nu D_\alpha D_\beta\right]$	$=$	$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left[\frac{1}{24}R^2+\frac{1}{12}R^{\mu\nu}R_{\mu\nu}\right]$
$s\mathrm{Tr}_0\left[\frac{1}{\Delta^2}\mathrm{e}^{-s\Delta}R_{\alpha\beta}R^{\alpha\mu\nu\beta}D_\mu D_\nu\right]$	$=$	$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left[\frac{1}{4}R^{\mu\nu}R_{\mu\nu}\right]$

In the evaluation of these operator traces, it is crucial to evaluate any commutators before the open indices may be traced, since β' must thereby be treated as contracted with a vector $\phi_{\beta'}$ to the right of the expression. Following this route a lengthy but straightforward computation yields

$$\begin{aligned}
T^{(1)}_2 &= \frac{1}{(4\pi s)^2}\int d^4x\sqrt{g}\left[\tfrac{5}{3}sR+\tfrac{19}{27}s^2R^2-\tfrac{22}{27}s^2R_{\mu\nu}R^{\mu\nu}-\tfrac{4}{3}s^2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right]\,,\\
T^{(2)}_2 &= \frac{1}{(4\pi s)^2}\int d^4x\sqrt{g}\left[\tfrac{17}{18}s^2R_{\mu\nu}R^{\mu\nu}+\tfrac{2}{3}s^2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right]\,,
\end{aligned} \tag{4.2.39}$$

which together with the standard vector-trace $T_2^{(0)}$ combine to

$$S_{2\mathrm{T}}^{(1)} = \frac{1}{(4\pi s)^2}\int d^4x\sqrt{g}\left[4 + \tfrac{7}{3}sR + \tfrac{41}{54}s^2R^2 + \tfrac{29}{270}s^2R_{\mu\nu}R^{\mu\nu} - \tfrac{131}{180}s^2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right] . \tag{4.2.40}$$

With all partial results given, the heat kernel coefficients for the 2T-trace (4.2.36) in $d = 4$ are defined by the expansion

$$S_{2\mathrm{T}} = \frac{1}{(4\pi s)^2}\int d^4x\sqrt{g}\left[5 - \tfrac{5}{6}\,sR - \tfrac{137}{216}\,s^2R^2 - \tfrac{17}{108}\,s^2R_{\mu\nu}R^{\mu\nu} + \tfrac{5}{18}\,s^2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right] , \tag{4.2.41}$$

up to terms of order $\mathcal{R}^3$.

Zero Modes and Spherical Background Manifolds

The heat kernel coefficients for constrained vector and tensor fields (4.2.29) and (4.2.41) still contain zero modes of the corresponding decompositions. For the case of a transverse vector, the decomposition (4.1.23) involves the derivative of a scalar field, whose constant mode therefore does not contribute therein. In a split of the transverse vector trace into unconstrained vector and scalar trace

$$\begin{aligned}\mathrm{Tr}_{1\mathrm{T}}\,\mathrm{e}^{-s\Delta} &= \mathrm{Tr}_1\,\mathrm{e}^{-s\Delta} + \mathrm{Tr}_1'\,D_\mu\Delta^{-1}D^\nu\,\mathrm{e}^{-s\Delta} \\ &= \mathrm{Tr}_1\,\mathrm{e}^{-s\Delta} - \mathrm{Tr}_0'\,\mathrm{e}^{-s\Delta+s\frac{R}{d}} \\ &= \mathrm{Tr}_1\,\mathrm{e}^{-s\Delta} - \mathrm{Tr}_0\,\mathrm{e}^{-s\Delta+s\frac{R}{d}} + \mathrm{e}^{s\frac{R}{d}}\,,\end{aligned} \tag{4.2.42}$$

according to the projection operator (4.1.25), a prime is used to indicate summation over non-constant modes only. In the second step, use has been made of the identity (4.2.33) on Einstein spaces. If a more general geometry is required, the commutator $[D^\nu, \mathrm{e}^{-s\Delta}]$ can always be worked out by expansion. Finally in the last step, the contribution of a constant scalar mode was explicitly subtracted from the scalar trace Tr_0' following

$$\mathrm{Tr}_0'\,f(\Delta) = \mathrm{Tr}_0\,f(\Delta) - f(0)\,, \tag{4.2.43}$$

to compensate for the overcounting in Tr_0. In order to find the zero mode correction of the heat kernel coefficients stemming from this subtraction, it has to be written in the form of a spacetime integration. Since this correction is independent of the volume element, it has to contribute to a topological term, given as the Euler-character χ_E of the manifold, as reviewed in appendix A.3. Inserting unity in the form of $1 = \chi_\mathrm{E}/\chi_\mathrm{E}$, the zero mode contribution in $d = 4$ can be written as

$$\mathrm{e}^{s\frac{R}{d}} = \frac{1}{(4\pi)^2}\frac{1}{2\chi_\mathrm{E}}\int d^4x\sqrt{g}\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\mathrm{e}^{s\frac{R}{d}}\,. \tag{4.2.44}$$

The heat kernel coefficients of $S_{1\mathrm{T}}^{(0)}$ in table 4.1 therefore receive the corrections in $d = 4$ proportional to the inverse Euler character

$$\begin{aligned} &\mathrm{tr}_{1\mathrm{T}}\overline{A_0} = 3\,, \quad \mathrm{tr}_{1\mathrm{T}}\overline{A_1} = \tfrac{1}{4}R\,, \\ &\mathrm{tr}_{1\mathrm{T}}\overline{A_2} = \left(-\tfrac{1}{24} + \tfrac{1}{2\chi_\mathrm{E}}\right) R^2 + \left(\tfrac{1}{40} - \tfrac{2}{\chi_\mathrm{E}}\right) R_{\mu\nu}R^{\mu\nu} - \left(\tfrac{1}{15} - \tfrac{1}{2\chi_\mathrm{E}}\right) R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\,, \end{aligned} \tag{4.2.45}$$

starting at order $\mathcal{R}^2$.

The heat kernel coefficients for transverse vectors on a spherical manifold are reproduced form the Einstein space result (4.2.31), using the definitions of a sphere (A.2.6). Since S^4 is a maximally symmetric space, all its curvature tensors can be expressed in terms of the metric and the covariantly constant Ricci scalar. This way we obtain

$$\begin{aligned} S_{1\mathrm{T}}\Big|_{\mathrm{Sphere}} = \tfrac{1}{(4\pi s)^{d/2}} \int d^dx\sqrt{g}\Big\{&(d-1) + \tfrac{(d-3)(d+2)}{6d} sR + \tfrac{5d^4-12d^3-47d^2-186d+180}{360d^2(d-1)} s^2R^2 \\ &+ \tfrac{35d^6-147d^5-331d^4-3825d^3-676d^2+10992d-7560}{45360d^3(d-1)^2} s^3R^3\Big\}\,, \end{aligned} \tag{4.2.46}$$

which for $d = 4$ becomes

$$S_{1\mathrm{T}}\Big|_{\mathrm{Sphere},d=4} = \tfrac{1}{(4\pi s)^2} \int d^4x\sqrt{g}\Big\{3 + \tfrac{1}{4}sR - \tfrac{67}{1440}s^2R^2 - \tfrac{4321}{362880}s^3R^3\Big\}\,. \tag{4.2.47}$$

This trace still contains the contribution of the constant scalar mode, which does not contribute to the transverse decomposition (4.1.23). In order to obtain the final result, we subtract this part as indicated in (4.2.42) with (4.2.44) and $\chi_\mathrm{E}(S^4) = 2$, computed in (A.3.10). Taking the correction term into account, the heat kernel coefficients for transversal vectors on the 4-sphere become

$$\begin{aligned} &\mathrm{tr}_{1\mathrm{T}}\overline{A_0} = 3\,, &&\mathrm{tr}_{1\mathrm{T}}\overline{A_1} = \tfrac{1}{4}R\,, \\ &\mathrm{tr}_{1\mathrm{T}}\overline{A_2} = -\tfrac{7}{1440}R^2\,, &&\mathrm{tr}_{1\mathrm{T}}\overline{A_3} = -\tfrac{541}{362880}R^3\,. \end{aligned} \tag{4.2.48}$$

Notably, this result is in complete agreement with previous computations [45, 57, 58].

Similarly, the minimal transverse-traceless decomposition for tensor fields (4.1.34) admits zero modes which have to be subtracted from the heat kernel coefficients (4.2.41). In this case, vector fields satisfying the conformal Killing equation

$$D_\mu\xi_\nu + D_\nu\xi_\mu - \tfrac{1}{2}g_{\mu\nu}D^\alpha\xi_\alpha = 0\,, \tag{4.2.49}$$

will not contribute to the tensor field, and therefore have to be excluded from the operator traces. The number of modes to be excluded is given by the number of Killing vectors

$n_{\rm KV}$ (satisfying $D_\mu \xi_\nu + D_\nu \xi_\mu = 0$) plus the number of conformal Killing vectors $n_{\rm CKV}$ (which solve (4.2.49) with $D^\alpha \xi_\alpha^{\rm CKV} \neq 0$). Following the discussion of the zero modes for the transverse vector case (4.2.42), these modes give rise to corrections expressed as the Euler character $\chi_{\rm E}$. This extra contribution is absent in the generic case where the manifold does not posses particular symmetries. For the general case of a manifold with $N = n_{\rm KV} + n_{\rm CKV}$, the heat kernel coefficients on the space of transverse traceless tensors are accordingly given by

$$\begin{aligned} &{\rm tr}_{2{\rm T}} \overline{A_0} = 5\,, \quad {\rm tr}_{2{\rm T}} \overline{A_1} = -\tfrac{5}{6} R\,, \\ &{\rm tr}_{2{\rm T}} \overline{A_2} = \left(-\tfrac{137}{216} + \tfrac{N}{2\chi_{\rm E}}\right) R^2 - \left(\tfrac{17}{108} + \tfrac{2N}{\chi_{\rm E}}\right) R_{\mu\nu} R^{\mu\nu} + \left(\tfrac{5}{18} + \tfrac{N}{2\chi_{\rm E}}\right) R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\,. \end{aligned} \tag{4.2.50}$$

Finally, we reproduce the results for a spherical background geometry [45, 57, 58]. The symmetries of the 4-sphere admit $n_{\rm KV} = 10$ Killing- and $n_{\rm CKV} = 5$ conformal Killing vectors. Using the relations (A.2.5) defining the sphere, together with $\chi_{\rm E} = 2$ and $N = 15$, one obtains

$${\rm tr}_{2{\rm T}} \overline{A_0} = 5\,, \qquad {\rm tr}_{2{\rm T}} \overline{A_1} = -\tfrac{5}{6} R\,, \qquad {\rm tr}_{2{\rm T}} \overline{A_2} = -\tfrac{1}{432} R^2\,, \tag{4.2.51}$$

from the general result (4.2.50). Thus we establish that the projection method employed here recovers the heat kernel coefficients, found via other methods.

Regularization of the Poles

The infrared divergences of the auxiliary integrals (4.2.11) appear in the heat kernel expansion of $S_{1\rm T}$ in the form of poles in even dimensions, as shown in table 4.2. The first divergence in $d = 4$ spacetime dimensions appears at order $\mathcal{R}^3$, neglecting the boundary term ΔR. Notably, however, the coefficients of $\mathcal{R}^3_9$ and $\mathcal{R}^3_{10}$ are finite for any dimensionality.

Remarkably, the occurrence of the singularities is not related to the use of the early-time expansion of the heat kernel employed in (4.2.7). Based on power-counting arguments, we estimated that divergences are present also if a non-local expansion (3.1.19) of the projected trace is employed. This estimate leads to the same divergence structure in the form of dimensional poles, as that indicated by the results reported in table 4.2. We take this as a strong indication that our poles are not an artifact of the expansion, but rather a genuine feature of the projected traces.

Any attempt to regularize (4.2.29) has to be implemented with care in order not to affect the unambiguous result for Einstein spaces, since in this limit the poles cancel with a factor $(d-4)$ forming in the numerator. If no pole is present, these contributions would otherwise go to zero. In order to take the limit $d \to 4$, we expand in $\varepsilon = d-4$ and make use of the formula

$$\begin{aligned} s^{n-d/2}\frac{f(d)}{d-4} &= s^{n-2}\Big(f'(4) + f(4)\Big(\tfrac{1}{\varepsilon} - \tfrac{1}{2}\log\left(\tfrac{s}{s_0}\right)\Big)\Big) + \mathcal{O}(\varepsilon) \\ &= s^{n-2}\Big(f'(4) - \tfrac{1}{2}f(4)\log\left(\tfrac{s}{s_0'}\right)\Big) + \mathcal{O}(\varepsilon)\,, \end{aligned} \tag{4.2.52}$$

where the $\frac{1}{\varepsilon}$-pole is absorbed into a (infinite) redefinition of the scale $s_0' = s_0\, \mathrm{e}^{2/\varepsilon}$. In this way s_0' plays the role of an infrared cutoff and allows us to obtain the limit as

$$\begin{aligned} S_{1\mathrm{T}}\Big|_{d\to 4} = \frac{1}{(4\pi s)^2}\int d^{4+\epsilon}x\sqrt{g}\,\Big\{3 \,+ \tfrac{1}{4}\,sR \,+ s^2\Big(-\tfrac{1}{48}\,R^2 \,-\tfrac{7}{120}\,\mathcal{R}_2^2 \,-\tfrac{1}{15}\,\mathcal{R}_3^2\Big) \\ + s^3\Big(\Big(\tfrac{1363}{20160} + \tfrac{1}{30}\log\left(\tfrac{s}{s_0'}\right)\Big)\,\mathcal{R}_1^3 \,-\Big(\tfrac{67}{2016} + \tfrac{1}{60}\log\left(\tfrac{s}{s_0'}\right)\Big)\,\mathcal{R}_2^3 \\ + \Big(\tfrac{41}{3456} + \tfrac{1}{144}\log\left(\tfrac{s}{s_0'}\right)\Big)\,\mathcal{R}_3^3 \,-\Big(\tfrac{263}{2880} + \tfrac{11}{360}\log\left(\tfrac{s}{s_0'}\right)\Big)\,\mathcal{R}_4^3 \\ + \Big(\tfrac{233}{1512} + \tfrac{1}{45}\log\left(\tfrac{s}{s_0'}\right)\Big)\,\mathcal{R}_5^3 \,-\Big(\tfrac{397}{5040} + \tfrac{1}{90}\log\left(\tfrac{s}{s_0'}\right)\Big)\,\mathcal{R}_6^3 \\ -\Big(\tfrac{1}{90} - \tfrac{1}{360}\log\left(\tfrac{s}{s_0'}\right)\Big)\,\mathcal{R}_7^3 \,-\Big(\tfrac{3}{280} + \tfrac{1}{90}\log\left(\tfrac{s}{s_0'}\right)\Big)\,\mathcal{R}_8^3 \\ -\tfrac{67}{15120}\,\mathcal{R}_9^3 \,+\tfrac{1}{108}\,\mathcal{R}_{10}^3\Big)\Big\}\,. \end{aligned} \tag{4.2.53}$$

The appearance of the logarithmic terms is an immediate consequence of the dimensional regularization employed. The Seeley-deWitt expansion is still valid after dimensional regularization, in the sense that the singular $\log(s)$ terms are always multiplied by a power of s, and so maintain a continuous $s \to 0$ limit. Using this regularization scheme, in principle no information is lost with a finite choice of s_0' in (4.2.53). In fact, it is still possible to recover the correct Einstein space limit from (4.2.53), provided that the explicit dependence of s_0' on $\varepsilon = d-4$ is taken into account, so that it can combine with the d-dependence of the Ricci- and scalar curvature tensors in their Einstein space limit.

To demonstrate that the origin of the divergence lies in the low momentum limit of the spectrum of the inverse Laplacian in the projectors (4.1.25), we consider a one-parameter family of modified operators

$$\check{\Pi}^{\mathrm{T}}_{\mu\nu} = g_{\mu\nu} + aD_\mu\frac{1}{\Delta}D_\nu\,, \tag{4.2.54}$$

that are idempotent for $a=1$ only. With this definition the modified projected Laplacian can be expanded in the form

$$\check{\Delta}_{\mu\nu} = (\check{\Pi}^{\mathrm{T}}\,\Delta\,\check{\Pi}^{\mathrm{T}})_{\mu\nu} = \Delta g_{\mu\nu} + a(2-a)D_\mu D_\nu + \mathcal{O}\big(\mathcal{R},\Delta^{-1}\big)\,, \tag{4.2.55}$$

where all the pseudo-differential contributions due to the commutators of the inverse Laplacian with the covariant derivatives are schematically contained in $\mathcal{O}(\mathcal{R}, \Delta^{-1})$. At zeroth order in the curvature, the heat kernel of $\check{\Delta}$ is singular at $a = 1$, due to the degeneracy of the operator [92, 116]. Since for other values of a, $\check{\Pi}^{\mathrm{T}}$ is not a projector any more, we can exclude the degeneracy, that is specifically its infinite number of zero eigenvalues corresponding to the volume of the subspace of longitudinal modes, as source of the divergence. Tracking this modification in the traces (4.2.8) reveals a simple polynomial dependence of (4.2.29) on the parameter a. Therefore a discontinuity for the special value $a = 1$ is not present. An explicit calculation in fact shows that the dimensional poles vanish only for $a = 0$, reproducing the heat kernel of the standard Laplacian.

Alternatively, we can remove the zero eigenvalue from the spectrum of the Laplacian specifically by shifting it with an infrared scale $m^2 > 0$, defining the operator

$$\Pi^{\mathrm{T,m^2}}_{\mu\nu} = g_{\mu\nu} + D_\mu \Big(\Delta + m^2\Big)^{-1} D_\nu \, . \tag{4.2.56}$$

The scale m^2 can be chosen arbitrarily small, suppressing the low momentum modes while leaving the high momentum spectrum, essentially, unaltered. Using this modified operator in place of the projectors in (4.2.5), the exponentiation via (4.2.9) would be done with the regulated inverse Laplacian $(\Delta+m^2)^{-1}$. This accounts for an additional regularizing factor e^{-tm^2}, which renders the modification of the integrals (4.2.11) convergent at any order of the curvature. We conclude that the long range (infrared) modes of the pseudo-differential projection operators cause the breakdown of the heat kernel expansion (4.2.29). This situation is analogue to the case of a scalar field on curved spacetime, whose propagator can be defined via the heat kernel in a local expansion only if it is massive (or otherwise IR regulated).

For practical purposes, it is conceivable to use a modified operator like (4.2.56), if m^2 can be identified with an infrared scale already present in a particular problem. For example in the context of the renormalization group, the identification $m^2 = k^2$ can be used. In general however, such a procedure has the disadvantages that the limit $m^2 \to 0$ is discontinuous and arbitrary powers of the dimensionless combination (sm^2) will occur in heat kernel traces. Instead, we suggest to follow (4.2.52) and to introduce an infrared scale in a purely dimensionally regulated setup.

	scalar [0]	vector [1]	symmetric tensor [2]	transverse vector [1T]
c^0	1	d	d_2	$d_{1\mathrm{T}}$
c^1	$\frac{1}{6}$	$\frac{d}{6}$	$\frac{d_2}{6}$	$\frac{d_{1\mathrm{T}}}{6} - \frac{1}{d}$
c^2_1	$\frac{1}{72}$	$\frac{d}{72}$	$\frac{d_2}{72}$	$\frac{d_{1\mathrm{T}}}{72} - \frac{d^2-d+6}{6(d-2)d(2+d)}$
c^2_2	$-\frac{1}{180}$	$-\frac{d}{180}$	$-\frac{d_2}{180}$	$-\frac{d_{1\mathrm{T}}}{180} - \frac{2d^2-5d-6}{3(d-2)d(2+d)}$
c^2_3	$\frac{1}{180}$	$\frac{d}{180} - \frac{1}{12}$	$\frac{d_2}{180} - \frac{d+2}{12}$	$\frac{d_{1\mathrm{T}}}{180} - \frac{1}{12}$
c^3_1	$\frac{1}{336}$	$\frac{d}{336} + \frac{1}{120}$	$\frac{d_2}{336} + \frac{d+2}{120}$	$\frac{d_{1\mathrm{T}}}{336} + \frac{1}{120} + \frac{8d^4-73d^3+208d^2-428d+240}{30(d-4)(d-2)d(d+2)(d+4)}$
c^3_2	$\frac{1}{840}$	$\frac{d}{840} - \frac{1}{30}$	$\frac{d_2}{840} - \frac{d+2}{30}$	$\frac{d_{1\mathrm{T}}}{840} - \frac{1}{30} + \frac{2d^4-17d^3+42d^2+88d-320}{10(d-4)(d-2)d(d+2)(d+4)}$
c^3_3	$\frac{1}{1296}$	$\frac{d}{1296}$	$\frac{d_2}{1296}$	$\frac{d_{1\mathrm{T}}}{1296} - \frac{d^4-2d^3-4d^2+8d+288}{72(d-4)(d-2)d(d+2)(d+4)}$
c^3_4	$-\frac{1}{1080}$	$-\frac{d}{1080}$	$-\frac{d_2}{1080}$	$-\frac{d_{1\mathrm{T}}}{1080} - \frac{19d^3-82d^2+148d-1200}{180(d-4)(d-2)d(d+4)}$
c^3_5	$-\frac{4}{2835}$	$-\frac{4d}{2835} + \frac{1}{30}$	$-\frac{4d_2}{2835} + \frac{d+2}{30}$	$-\frac{4d_{1\mathrm{T}}}{2835} + \frac{1}{30} + \frac{41d^4-136d^3-44d^2-896d+960}{90(d-4)(d-2)d(d+2)(d+4)}$
c^3_6	$\frac{1}{945}$	$\frac{d}{945} - \frac{1}{30}$	$\frac{d_2}{945} - \frac{d+2}{30}$	$\frac{d_{1\mathrm{T}}}{945} - \frac{1}{30} - \frac{29d^4-139d^3-86d^2+376d+960}{45(d-4)(d-2)d(d+2)(d+4)}$
c^3_7	$\frac{1}{1080}$	$\frac{d}{1080} - \frac{1}{72}$	$\frac{d_2}{1080} - \frac{d+2}{72}$	$\frac{d_{1\mathrm{T}}}{1080} - \frac{1}{72} - \frac{1}{180(d-4)}$
c^3_8	$\frac{1}{7560}$	$\frac{d}{7560} - \frac{1}{90}$	$\frac{d_2}{7560} - \frac{d+2}{90}$	$\frac{d_{1\mathrm{T}}}{7560} - \frac{1}{90} + \frac{1}{45(d-4)}$
c^3_9	$\frac{17}{45360}$	$\frac{17d}{45360} - \frac{1}{180}$	$\frac{17d_2}{45360} - \frac{d+2}{180}$	$\frac{17d_{1\mathrm{T}}}{45360} - \frac{1}{180}$
c^3_{10}	$-\frac{1}{1620}$	$-\frac{d}{1620} + \frac{1}{90}$	$-\frac{d_2}{1620} + \frac{d+2}{90}$	$-\frac{d_{1\mathrm{T}}}{1620} + \frac{1}{90}$

Table 4.2.: The traced heat kernel coefficients in the early-time expansion (3.2.27) on the space of scalars, vectors, symmetric 2-tensors and transversal vectors ($1T$), respectively. The number of components in these field spaces in dependence of the spacetime dimension are given by $d_2 = \frac{1}{2}d(d+1)$ and $d_{1\mathrm{T}} = d - 1$, respectively.

4.3. Algorithmic Solution of the Renormalization Group Equation

In this section, we analyse the structure of Wetterich-type functional renormalization group equations (2.3.18), which provide an exact description of the Wilsonian renormalization group flow, in the presence of gauge symmetries. The prime computational challenge when extracting non-perturbative physical information from this equation is the evaluation of the operator traces appearing on its right hand side. This becomes particularly difficult in the context of curved spacetime, since in that case the spectra of the differential operators appearing inside the RG equation are generically unknown. To bypass the related obstacles, we present here an explicit algorithm to compute approximate solutions of the exact RG equation in a flexible and systematic way. Its key element is the algorithmic reduction of the (usually very complicated) operators appearing in the RG equation to become accessible by the off-diagonal heat kernel expansion, discussed in chapter 3. This procedure is very general and in particular does not rely on the choice of a particular background geometry, showing the background independence of the formalism manifestly. Thus, the algorithm provides access to the β functions, unlocking information that was previously restricted by the choice of the background as well as gauge fixing conditions. This procedure can be performed by purely algebraic methods, not requiring any numerics. The bookkeeping required in a practical computation can be easily handled by computer algebra software, which provides access to computations which have until now been out of reach due to technical limitations. Indeed, an automation of the solution of RG equations is required for increasingly involved problems to be approached in the future. In previous calculations a number of simplifying choices specifically tuned to a given problem were preventing the generalization. However when reaching a certain threshold complication which does not allow for many simplifying assumptions anyway, it will in fact become the easier route to code an algorithm. This point is reached in gravity with the attempt of a full curvature squared ansatz, for which a handling of the appearing tensor structures and huge heat kernel coefficients is manually unfeasible.

The RG equation (2.3.18) for the effective average action $\Gamma_k[\Phi, \bar{\Phi}]$ with background fields denoted as $\bar{\Phi}$, on which the algorithm is based, takes the form [82]

$$\partial_t \Gamma_k[\Phi, \bar{\Phi}] = \tfrac{1}{2} \mathrm{STr} \left[\left(\Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1} \partial_t \mathcal{R}_k \right] , \tag{4.3.1}$$

where "STr" was written to indicate a generalized functional trace which includes a minus sign for the ghost and fermion fields, and a factor two for complex fields. Schematically, the RG flow $\partial_t \Gamma_k$ is found in three steps: one has to compute the second variation $\Gamma_k^{(2)}$, invert the dressed operator $(\Gamma_k^{(2)} + \mathcal{R}_k)$, and finally evaluate the trace. In contrast to the treatment of purely scalar field theories as demonstrated in (2.3.22), complications arise for the case of gauge theories. These are mostly related to the fact that the functional Hessian $\Gamma_k^{(2)}$ inherits the index structure of the quantum fields and the presence of non-minimal differential operators therein.

The following presentation is based on the case of gravity, but we stress that the algorithm is applicable to other gauge theories like Yang-Mills theory as well.

The Background Field Method

The inclusion of a standard gauge fixing term like (4.1.13) or (4.1.17) would forfeit the ordinary gauge invariance of the effective action, with the consequence that no simple organizing principle for the construction of an effective field theory exists. This difficulty can be lifted with the background field method [117]. The essential idea is to employ a background field split, which for the example of a metric field reads

$$g_{\mu\nu} = h_{\mu\nu} + \bar{g}_{\mu\nu} \,, \tag{4.3.2}$$

defining a fluctuation field $h_{\mu\nu}$ shifted by a classical background field $\bar{g}_{\mu\nu}$, so that a gauge fixing (w.r.t. h) can be used while a remnant invariance (w.r.t. $\bar{g}$) is maintained. To this end, a full gauge transformation (4.1.15) is accordingly split as

$$\delta_\xi g_{\mu\nu} = \hat{\delta}_\xi h_{\mu\nu} + \bar{\delta}_\xi \bar{g}_{\mu\nu} \,, \tag{4.3.3}$$

where $\hat{\delta}_\xi$ and $\bar{\delta}_\xi$ act only on the fluctuation and background fields, respectively. With these definitions, the invariance of an action functional can be written in the form

$$\delta_\xi \; S_0[g_{\mu\nu}] = (\hat{\delta}_\xi + \bar{\delta}_\xi) \; S_0[h_{\mu\nu} + \bar{g}_{\mu\nu}] = 0 \,. \tag{4.3.4}$$

This relation is now to be replaced by a gauge fixed action $S[h_{\mu\nu}, \bar{g}_{\mu\nu}]$, with a priori arbitrary dependence on the background field. While herein the full gauge symmetry is lost, one can always retain the background invariance

$$\bar{\delta}_\xi S[h_{\mu\nu} = 0, \bar{g}_{\mu\nu}] = 0 \,, \tag{4.3.5}$$

by covariantization of the gauge fixing condition with respect to $\bar{g}$. This is easily achieved by replacing any partial derivatives in the non-invariant terms with a background covariant derivative, defined as

$$\bar{D}_\mu v_\nu = \partial_\mu v_\nu - \Gamma[\bar{g}]^\alpha{}_{\mu\nu} v_\alpha \,, \tag{4.3.6}$$

and $\Delta = -\bar{D}_\mu \bar{D}^\mu$ is accordingly understood in this context.

A corresponding effective action $\Gamma[\Phi, \bar{\Phi}]$ following (2.1.23) will assume the form

$$\begin{aligned} \mathrm{e}^{-\Gamma[\Phi,\bar{\Phi}]} &= \int D\mathcal{X} \; \mathrm{e}^{-S[\mathcal{X}+\bar{\Phi}] - S_{\mathrm{gf}}[\mathcal{X},\bar{\Phi}] + \int_x J(\mathcal{X}-\Phi)} \\ &= \int D\mathcal{X} \; \mathrm{e}^{-S[\mathcal{X}] - S_{\mathrm{gf}}[\mathcal{X}-\bar{\Phi},\bar{\Phi}] + \int_x J(\mathcal{X}-\Phi-\bar{\Phi})} \,, \end{aligned} \tag{4.3.7}$$

which implies that the background field dependence $\bar{\Phi}$ is sufficient to recover the ordinary effective action as

$$\Gamma[\Phi=0,\bar{\Phi}] \;=\; \Gamma[\Phi]\Big|_{\Phi=\bar{\Phi}} \,. \tag{4.3.8}$$

Note that the gauge fixing action $S_{\mathrm{gf}}[\mathcal{X}-\bar{\Phi},\bar{\Phi}]$ carries an unusual dependence on the average fields in order for this relation to hold [118], however its explicit form is up to this point left arbitrary.

The background field construction ensures that the gauge fixed functional $\Gamma_k[\Phi,\bar{\Phi}]$, although not being an observable, can be organized in (background) covariant terms exclusively. This way, the background independence of the β functions derived with the RG equation is explicit. Consequently, an ansatz in the form of

$$\Gamma_k[\Phi,\bar{\Phi}] = \int d^dx\sqrt{\bar{g}}\,\sum_i u_i(k) I_i[\Phi,\bar{\Phi}] \tag{4.3.9}$$

can be used to determine the the RG flow by projection on the base monomials I_i, in a straightforward generalization of (2.3.19). However, the most general form of Γ_k does now allow for separate dependence on its two arguments, studied for example for bi-metric gravity in [53, 119].

Construction of the Dressed Full Propagator in Field Space

After computing the second variation of an ansatz used for the effective average action Γ_k via

$$\left[\Gamma_k^{(2)}\right]_{ij}(x,y) = \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\,\frac{\delta^2\Gamma_k}{\delta\Phi_i(x)\delta\Phi_j(y)}\,, \tag{4.3.10}$$

to find the dressed full propagator $[\Gamma_k^{(2)}+\mathcal{R}_k]^{-1}$, a suitable cutoff $\mathcal{R}_k$ has to be constructed and the resulting operator has to be inverted as a matrix in field space. The main challenge in this procedure is the occurrence of non-minimal derivative terms in gauge theories, requiring either a decomposition of the fields or an involved resummation. The general operator structure appearing in the functional Hessian can be schematically written as

$$\left[\Gamma_k^{(2)}\right]_{ij} = \underbrace{\mathcal{K}_i(\Delta)\,\mathbf{1}_{ij}}_{\text{kinetic terms}} + \underbrace{\mathcal{D}_{ij}(\bar{D}_\mu)}_{\text{non-minimal derivatives}} + \underbrace{\mathcal{V}_{ij}(\bar{R},\bar{D}_\mu)}_{\text{background vertices}}\,, \tag{4.3.11}$$

where i,j abbreviate the indices on the underlying field space. Here, $\mathcal{K}_i(\Delta)$ is the diagonal part of the inverse full propagator of the i-th field, containing only Laplacian operators constructed from the background metric. The other two pieces group the off-diagonal operators, with $\mathcal{V}_{ij}$ including all vertices that include at least one power of a background quantity to expand the inversion in, and with $\mathcal{D}_{ij}$ containing all the remaining terms with derivatives but no such object. Such terms generically appear since the second variation of Γ_k inherits the index structure of the fields. The formalism allows to expand in any structure, here for single metric gravity taken as the curvature tensor of the background metric $\bar{R}$, enabling to retain manifestly covariant expressions.

In order for the cutoff term (2.3.3) to modify the inverse propagator $\Gamma_k^{(2)}$ of every field in a comparable manner, the cutoff insertion $\mathcal{R}_k$ has to be adapted to all kinetic terms individually. The specific way to adjust the cutoff admits some freedom concerning the prescription to do so. Following [58], the standard (Type I) regularization would dress any appearance of the Laplace operator with a cutoff term. In contrast, with the content of $\Gamma_k^{(2)}$ classified as in (4.3.11), we suggest to tailor the cutoff to the free kinetic term $\mathcal{K}$ only, leaving out Laplacians in the vertices $\mathcal{V}$. Hence the momentum dependent IR cutoff is introduced according to the replacement rule

$$\Delta \mapsto P_k = \Delta + R_k(\Delta)\,, \tag{4.3.12}$$

with a shape function R_k, applied to $\mathcal{K}$. This corresponds to the mapping $\Gamma_k^{(2)} \mapsto \Gamma_k^{(2)} + \mathcal{R}_k$, implying the definition of $\mathcal{R}_k$. As a consequence, the matrix-valued kinetic terms in the regulated functional Hessian take the form

$$\mathcal{K}(\Delta) \mapsto \mathcal{P}(\Delta) := \mathcal{K}(P_k) = \mathcal{K}(\Delta) + \mathcal{R}_k(\Delta)\,. \tag{4.3.13}$$

We thus have the definition $\mathcal{R}_k(\Delta) = \mathcal{K}(P_k) - \mathcal{K}(\Delta)$ from which the scale derivative $\partial_t \mathcal{R}_k$ can be determined. This construction guarantees that the IR cutoff $\mathcal{R}_k$ is diagonal in field space. Note that at no step in the derivation of β functions do we have to specify the explicit form of the shape function R_k.

It is in particular for the $\mathcal{D}$ terms that a direct expansion in $\mathcal{V}$ is prohibited, since these can contribute with any power to every order in the curvature. As an example for a typical non-minimal operator, the gauge fixed free Yang-Mills propagator (4.1.21) assumes the form

$$\mathcal{G}_{\mu\nu} = \left(g_{\mu\nu}\Delta + \left(1 - \tfrac{1}{\alpha}\right)\bar{D}_\mu \bar{D}_\nu\right)^{-1}\,, \tag{4.3.14}$$

with background covariant derivatives. One way to remove these terms which is commonly used is to choose an adapted gauge fixing, as the Feynman gauge $\alpha = 1$ in the above example. This practice does however suffer from technical limitations. For one, the gauge fixing condition must be restricted to a very particular choice, which does not allow for a comprehensive study of gauge dependence. Secondly, for effective field theories with higher derivative terms, there is not enough freedom in the choice of gauge fixing to remove all $\mathcal{D}$-type terms. For the example of higher derivative gravity, the leading four-derivative terms can be removed with conveniently chosen gauges, but two-derivative terms still remain [120].

A systematic prescription for the elimination of $\mathcal{D}$-type terms is the employment of a transverse decomposition of vector- or tensor-valued fields according to (4.1.23) and (4.1.34), respectively [68, 112, 121]. After inserting the projection operators as $\mathbf{1}_1 = \Pi_T + \Pi_L$ for a vector field (4.1.25), and $\mathbf{1}_2 = \Pi_{2\mathrm{T}} + \Pi_{2\mathrm{L}} + \Pi_{\mathrm{tr}}$ for a symmetric tensor field (4.1.41), all differential operators can be written as Laplacians acting on the various subspaces of the field spaces. This is a consequence of the transverse constraints of the fluctuation fields, by virtue of which derivatives in $\mathcal{D}$ contracting with them vanish. The remaining components of the unconstrained fields carry fewer spacetime indices and thus force derivatives to contract with each other. Using again the example (4.3.14), with a transverse decomposition we have

$$A^\mu\Big[g_{\mu\nu}\Delta + \Big(1-\tfrac{1}{\alpha}\Big)\,\bar{D}_\mu\bar{D}_\nu\Big]A^\nu = \; A^\mu_{\mathrm{T}}\Big[g_{\mu\nu}\Delta\Big]A^\nu_{\mathrm{T}} + \phi\Big[\bar{D}_\mu\Delta\bar{D}^\mu + \Big(1-\tfrac{1}{\alpha}\Big)\,\Delta^2\Big]\phi\,, \quad (4.3.15)$$

which organizes into diagonal terms for arbitrary α.

Following the steps described above, the regulated functional Hessian in field space assumes the form

$$\Big[\Gamma^{(2)} + \mathcal{R}_k\Big]_{ij} = \; \mathcal{P}_i(\Delta)\,\delta_{ij} \; + \; \mathcal{V}_{ij}\,. \quad (4.3.16)$$

Herein we defined the dressed inverse propagators $\mathcal{P}_i(\Delta) = \mathcal{K}_i(P_k) = \mathcal{K}_i(\Delta + R_k(\Delta))$, and the $\mathcal{V}_{ij}$ are understood as suitably projected onto the subspaces spanned by the fluctuation fields. The inversion of $\Big[\Gamma^{(2)}_k + \mathcal{R}_k\Big]$ is now well defined as an expansion, since $\mathcal{V}_{ij}$ contains the background quantity organizing the expansion scheme of the RG equation, ensuring that only a finite number of terms contribute to a specified order. This scheme resembles a perturbative expansion in the generalized interaction vertices $\mathcal{V}_{ij}$ with propagator insertions $\mathcal{P}^{-1}$, following from (4.3.16) as

$$\Big[\Gamma^{(2)} + \mathcal{R}_k\Big]^{-1}_{ij} = \; \frac{1}{\mathcal{P}_i}\,\delta_{ij} \; - \; \frac{1}{\mathcal{P}_i}\,\mathcal{V}_{ij}\,\frac{1}{\mathcal{P}_j} \; + \; \frac{1}{\mathcal{P}_i}\,\mathcal{V}_{ik}\,\frac{1}{\mathcal{P}_k}\,\mathcal{V}_{kj}\,\frac{1}{\mathcal{P}_j} \; \mp \; \ldots\,. \quad (4.3.17)$$

Note that here only diagonal elements of this block matrix need to be computed, owed to the fact that the cutoff operator and thus $\partial_t\mathcal{R}_k$ is diagonal in field space by construction, so no other parts contribute to the trace in the RG equation. The drawback of this method is that the projection operators implementing the field decompositions appear in the trace, so that a projected heat kernel expansion, as discussed in the last section, has to be used. The heat traces thus constrained to the corresponding subspaces have diverging coefficients on general backgrounds beyond a certain order in the curvature. For the computation of results in full generality, a regularization like (4.2.53) is therefore required.

An alternative method lifting this complication is the full resummation of the non-minimal operators appearing in $\Gamma_k^{(2)}$ [69, 92]. For this purpose, we write the form of the regulated Hessian as

$$\begin{aligned}\Big[\Gamma^{(2)} + \mathcal{R}_k\Big]_{ij} &= \mathcal{P}_i(\Delta)\,\mathbf{1}_{ij} \;+\; \mathcal{D}_{ij} \;+\; \mathcal{V}_{ij} \\ &= \mathcal{Q}_{ij} \;+\; \mathcal{V}_{ij}\,.\end{aligned} \tag{4.3.18}$$

In contrast to (4.3.16), this expression still contains the non-minimal $\mathcal{D}$ terms, and combines them with the diagonal derivative terms in $\mathcal{Q}$. In order to expand the inverse in the form of (4.3.17) with $\mathcal{Q}$ in place of $\mathcal{P}$, the inverse operator $\mathcal{Q}^{-1}_{ij}$ is required. It can be found systematically in a curvature expansion as follows. Since a full set of projectors provides a basis in field space, a general operator decomposes on a flat background into components acting on each subspace like

$$\begin{aligned}\mathcal{Q}_{ij} &= \Big(\sum_a \Pi^a_{ik}\Big)\,\mathcal{Q}_{kl}\,\Big(\sum_a \Pi^a_{lj}\Big) \\ &= \sum_a \,\Pi^a_{ik}\,\mathcal{Q}^a\,\Pi^a_{kj} + \mathcal{O}(\mathcal{R})\,.\end{aligned} \tag{4.3.19}$$

Neglecting all curvatures, such a decomposition is always diagonal, and thus immediately allows to give the inverse in flat space by

$$\mathcal{Q}^{-1}_{ij} = \sum_a \,\Pi^a_{ik}\,\frac{1}{\mathcal{Q}^a}\,\Pi^a_{kj} + \mathcal{O}(\mathcal{R})\,. \tag{4.3.20}$$

In the example of the Yang-Mills propagator (4.3.14), the transversal and longitudinal parts are easily found to yield

$$\begin{aligned}\mathcal{G}^{\rho\sigma} &= \bigg\{\Big({\Pi_{\mathrm{T}}}^{\rho\mu} + {\Pi_{\mathrm{L}}}^{\rho\mu}\Big)\Big[g_{\mu\nu}\Delta + \Big(1-\tfrac{1}{\alpha}\Big)\,\bar{D}_\mu\bar{D}_\nu\Big]\Big({\Pi_{\mathrm{T}}}^{\nu\sigma} + {\Pi_{\mathrm{L}}}^{\nu\sigma}\Big)\bigg\}^{-1} \\ &= {\Pi_{\mathrm{T}}}^{\rho}{}_{\mu}\,\frac{1}{\Delta}\,{\Pi_{\mathrm{T}}}^{\mu\sigma} \;+\; {\Pi_{\mathrm{L}}}^{\rho}{}_{\mu}\,\frac{\alpha}{\Delta}\,{\Pi_{\mathrm{L}}}^{\mu\sigma} \;+\; \mathcal{O}(\mathcal{R})\,,\end{aligned} \tag{4.3.21}$$

where the rewriting $\bar{D}_\mu\bar{D}_\nu = -\Delta\Pi_{\mathrm{L}\mu\nu} + \mathcal{O}(\mathcal{R})$ has been used. Note that it is by virtue of this last expression that the use of the projection operators is unproblematic on flat space, and no decomposition of the fluctuating fields is implied.

The full inverse of an operator $\mathcal{Q}$ is then defined as

$$\mathcal{Q}^{-1} = \mathcal{Q}_0^{-1} + \mathcal{B}\,, \tag{4.3.22}$$

with the curvature free part $\mathcal{Q}_0^{-1} = \sum_a \Pi^a_{ik} \frac{1}{\mathcal{Q}^a} \Pi^a_{kj}$, and $\mathcal{B}$ containing all orders in the curvatures. To find this correction term, we derive

$$\begin{aligned} & \mathcal{Q}\mathcal{Q}^{-1} = \mathcal{Q}(\mathcal{Q}_0^{-1} + \mathcal{B}) = 1 \\ \Leftrightarrow \quad & \mathcal{Q}\mathcal{B} = 1 - \mathcal{Q}\mathcal{Q}_0^{-1} \\ \Leftrightarrow \quad & \mathcal{B} = (\mathcal{Q}_0^{-1} + \mathcal{B})(1 - \mathcal{Q}\mathcal{Q}_0^{-1}) \\ \Leftrightarrow \quad & \mathcal{B} = \mathcal{Q}_0^{-1} \sum_{n=1}^{\infty} \left(1 - \mathcal{Q}\mathcal{Q}_0^{-1}\right)^n . \end{aligned} \tag{4.3.23}$$

Inserted in the definition (4.3.22), we find the resummation formula

$$\mathcal{Q}^{-1} \; = \; \mathcal{Q}_0^{-1} \sum_{n=0}^{\infty} \left(1 - \mathcal{Q}\mathcal{Q}_0^{-1}\right)^n , \tag{4.3.24}$$

which expresses the inverse of any operator as the operator itself and its inverse on flat space. Furthermore, since $\mathcal{B} = \mathcal{O}(R)$ by definition, we have

$$\mathcal{W} := \left(1 - \mathcal{Q}\mathcal{Q}_0^{-1}\right) \propto R \, , \tag{4.3.25}$$

establishing (4.3.24) as an expansion in the curvature. In conclusion, we arrive at the complete inverted operator, schematically written as

$$\left[\Gamma^{(2)} + \mathcal{R}_k\right]^{-1} = \mathcal{Q}^{-1} \sum_{n\geq 0} \left[\mathcal{V}\mathcal{Q}^{-1}\right]^n = \mathcal{Q}_0^{-1} \sum_{k\geq 0} \mathcal{W}^k \sum_{n\geq 0} \left[\mathcal{V}\mathcal{Q}_0^{-1} \sum_{k\geq 0} \mathcal{W}^k\right]^n , \tag{4.3.26}$$

wherein the expansion in $\mathcal{V}$ and $\mathcal{W}$ to any desired order is explained. Using this technique to invert the operator structure in (4.3.18) is not limited by any prerequisites. In practice however, it involves significantly more computational effort than the alternatives discussed before. The achieved software implementation of the algorithm will therefore remove any of the technical limitations of previously used solution schemes and make much more sophisticated computations available.

Decoupling of Physical Degrees of Freedom In Landau Gauge

The BRST construction of the Lagrangian of a gauge theory manages the degrees of freedom by employing a gauge fixing term, accompanied with the Faddeev-Popov ghost term (4.1.13). We will demonstrate here that it is possible to separate the contributions of these gauge variant terms to the running of coupling constants from those of the physical part of the action. To achieve this, a gauge field A is split into physical and gauge components $A = A_\mathrm{p} + A_\mathrm{g}$ such that the gauge invariant action depends on A_p only, and $A_{\rm g}$ carries the full gauge dependence.[5] For the important examples of Yang-Mills theory and gravity, realizations of such splits are given by the transverse (4.1.23) and transverse-traceless (4.1.34) decompositions, respectively.

A properly chosen gauge fixing condition contains besides background fields $\bar{A}$ only fluctuations of $A_{\rm g}$ linearly, here denoted as

$$F[A,\bar{A}] = F[A_{\rm g},\bar{A}] = \mathcal{F}[\bar{A}]\, A_{\rm g}\,. \tag{4.3.27}$$

Accordingly, the gauge fixing term can be written in the form

$$\begin{aligned} S_{\rm gf} &= \frac{1}{2\alpha}\int d^dx\sqrt{\bar{g}}\; F[A]^2 \\ &= \frac{1}{2\alpha}\int d^dx\sqrt{\bar{g}}\; A_{\rm g} G A_{\rm g}\,, \end{aligned} \tag{4.3.28}$$

where the operator $G = \mathcal{F}^\dagger \circ \mathcal{F}$ depends on background fields only. The choice $\alpha = 0$ of the gauge fixing parameter leads to a factorization of the operator traces into a physical sector, containing only the contribution of gauge invariant field components, and a gauge dependent sector. The latter captures the contributions of the gauge fixing, ghost and auxiliary terms, which arise in the application of the field decompositions. This part is universal and, once computed to sufficient order, it can be reused in many setups as it does not depend on the choice of the matter part of Γ_k.

To write the structure of the RG equation in terms of the decomposed gauge fields, they are arranged in a multiplet $\Phi = (\phi, A_{\rm g})$, where only the gauge dependent part $A_{\rm g}$ is separated and ϕ stands for all other fields occurring in the action, including $A_{\rm p}$ and the Faddeev-Popov ghosts. Any of these fields may be further decomposed into component fields later on. With the gauge part formally separated, the quadratic part of the effective average action can schematically be written as

$$\Gamma_k^{\rm quad} = \tfrac{1}{2}\int d^dx\sqrt{\bar{g}}\;\Phi \begin{pmatrix} L & Q \\ \widetilde{Q} & \frac{1}{\alpha}G + H \end{pmatrix} \Phi\,, \tag{4.3.29}$$

where the block structure is chosen such that the lower right is the quadratic part of the pure gauge component, L denotes the contributions from all remaining fields and Q and $\widetilde{Q}$ are mixed terms. The contribution of the gauge fixing term is written separately to track the α dependence explicitly. With appropriately chosen cutoff terms $\mathcal{R}_i$, the RG

[5] See [122] for a more careful discussion of this decomposition for the case of gravity.

equation becomes

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{STr}\begin{pmatrix} L+\mathcal{R}_L & Q+\mathcal{R}_Q \\ \widetilde{Q}+\mathcal{R}_{\widetilde{Q}} & \frac{1}{\alpha}(G+\mathcal{R}_G)+H+\mathcal{R}_H \end{pmatrix}^{-1} \partial_t \begin{pmatrix} \mathcal{R}_L & \mathcal{R}_Q \\ \mathcal{R}_{\widetilde{Q}} & \frac{1}{\alpha}\mathcal{R}_G+\mathcal{R}_H \end{pmatrix}. \tag{4.3.30}$$

The crucial thing to note here is that a suitably constructed cutoff for the gauge component $\mathcal{R}_G$ must come with the same pre-factor, where only the gauge parameter has been written explicitly. The Landau-deWitt gauge ($\alpha = 0$) presents a preferred choice for the gauge parameter since in this case there is no smearing of the gauge condition $F[A, \bar{A}] = 0$, and it corresponds to a fixed point value [123]. In contrast to perturbation theory, the form of the RG equation allows to implement this choice before any actual computations. However the occurrence of α^{-1} in the cutoff term does not allow to take the limit $\alpha \to 0$ right away. Rather, we must keep track of α in linear order to cancel the inverse. The block matrix inverted by an expansion around $\alpha = 0$ yields

$$\begin{pmatrix} L & Q \\ \widetilde{Q} & \frac{1}{\alpha}G+H \end{pmatrix}^{-1} = \begin{pmatrix} L^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \alpha \begin{pmatrix} L^{-1}QG^{-1}\widetilde{Q}L^{-1} & -L^{-1}QG^{-1} \\ -G^{-1}\widetilde{Q}L^{-1} & G^{-1} \end{pmatrix} + \mathcal{O}(\alpha^2)\,. \tag{4.3.31}$$

This formula requires the invertibility of L and G. We have to keep this in mind for the explicit construction of the gauge fixing term.

Multiplication with the cutoff matrix and taking the limit $\alpha \to 0$, we find

$$\begin{aligned} \partial_t \Gamma_k &= \frac{1}{2}\mathrm{STr}\begin{pmatrix} (L+\mathcal{R}_L)^{-1}\partial_t\mathcal{R}_L & 0 \\ 0 & (G+\mathcal{R}_G)^{-1}\partial_t\mathcal{R}_G \end{pmatrix} + \mathcal{O}(\alpha) \\ &= \frac{1}{2}\mathrm{STr}\left((L+\mathcal{R}_L)^{-1}\partial_t\mathcal{R}_L\right) + \frac{1}{2}\mathrm{STr}\left((G+\mathcal{R}_G)^{-1}\partial_t\mathcal{R}_G\right)\,. \end{aligned} \tag{4.3.32}$$

Thus we see that in the Landau gauge limit, the structure of the RG equation simplifies such that the mixed terms Q and $\widetilde{Q}$ as well as H drop out identically. Furthermore, the pure gauge components decouple from the remaining fields, and their contribution is determined by the gauge fixing term only.

Following (4.1.13), a standard ghost term of the form

$$\begin{aligned} \Gamma_k^{\mathrm{gh}} &= \int d^dx\sqrt{\bar{g}}\,\bar{C}\,\frac{\delta F}{\delta A}\,\delta_C A \\ &= \int d^dx\sqrt{\bar{g}}\,\bar{C}\,\mathcal{M}\,C \end{aligned} \tag{4.3.33}$$

is needed in the case of non-abelian gauge theories. Because of the simple quadratic form of (4.3.33), it is possible to examine the structure of L by separating the ghost fields from the remaining ones. Making use of the background independence of the RG equation, we can set the background ghost field to zero, eliminating the cross-terms stemming from the ghost-gauge field interaction. This choice does however not allow to identify contributions to coupling constants of ghost terms, which will be considered in the following chapter. The block structure becomes

$$L = \begin{pmatrix} K & 0 \\ 0 & \mathcal{M} \end{pmatrix}, \tag{4.3.34}$$

where $\mathcal{M}$ is the kernel of the ghost action (4.3.33) and K is the still undetermined block containing all other second variations of the action. Since the cutoff $\mathcal{R}_L$ must be of the same form, the resulting trace in the RG equation decomposes as

$$\frac{1}{2}\mathrm{STr}(L+\mathcal{R}_L)^{-1}\partial_t\mathcal{R}_L = \frac{1}{2}\mathrm{Tr}(K+\mathcal{R}_K)^{-1}\partial_t\mathcal{R}_K - \mathrm{Tr}(\mathcal{M}+\mathcal{R}_\mathcal{M})^{-1}\partial_t\mathcal{R}_\mathcal{M}\,, \tag{4.3.35}$$

and thus

$$\partial_t\Gamma_k = \frac{1}{2}\mathrm{Tr}(K+\mathcal{R}_K)^{-1}\partial_t\mathcal{R}_K + \frac{1}{2}\mathrm{Tr}(G+\mathcal{R}_G)^{-1}\partial_t\mathcal{R}_G - \mathrm{Tr}(\mathcal{M}+\mathcal{R}_\mathcal{M})^{-1}\partial_t\mathcal{R}_\mathcal{M}\,. \tag{4.3.36}$$

In order to arrive at an entirely self-contained treatment of the gauge degrees of freedom, it is necessary to include auxiliary fields that account for the correct Jacobian of the applied field decompositions. For Yang-Mills theory and gravity, these are given in (4.1.33) and (4.1.53), respectively. The resulting auxiliary contributions are found by exponentiating the Jacobians, analogous to (4.1.19). Since the projection operators implementing the transverse field decompositions act always linearly, the resulting terms will be strictly quadratic in the auxiliary fields and therefore decouple from the other fields like the ghost sector. Therefore the gauge and ghost as well as the auxiliary terms decouple from the physical field components in the Landau-deWitt gauge. This result allows to evaluate their contribution to the RG flow completely independently, leaving the matter field content of the model arbitrary, and to give universal expressions to be reused in a broad class of computations.

Evaluating the Traces via Off-diagonal Heat Kernel Expansion

Substituting the expanded regulated propagator $\left[\Gamma_k^{(2)} + \mathcal{R}_k\right]^{-1}$ in the form of (4.3.17) or (4.3.26) into the r.h.s. of the RG equation (4.3.1) results in a series of individual operator traces

$$\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}\partial_t\mathcal{R}_k\right] = \sum_n \mathrm{Tr}_{j_n}\left[f_n(\Delta)\,\hat{\mathcal{O}}_n\right]\,, \tag{4.3.37}$$

on field spaces distinguished by $j \in \{0, 1, 2, 1\mathrm{T}, 2\mathrm{T}\}$ for scalar, vector, symmetric 2-tensor, or transversal vector and tensor fields, respectively. In the absence of non-minimal operator insertions ($\hat{\mathcal{O}} = 1$), these traces are directly given in terms of the heat kernel coefficients according to (3.2.27). The evaluation of general traces that contain background insertions with derivative operators of non-Laplacian form is done with the use of the off-diagonal heat kernel method, following (3.3.4).

Employing the commutation relations collected in appendix B to arrange the trace arguments so that all Laplacians are collected in a single function f, the trace can be written as

$$\begin{aligned} \mathrm{Tr}_j \left[f(\Delta)\, \hat{\mathcal{O}} \right] &= \int_0^\infty ds\, \tilde{f}(s)\, \langle x_i |\, \mathrm{e}^{-s\Delta}\, \hat{\mathcal{O}}\, | x_i \rangle \\ &= \int_0^\infty ds\, \tilde{f}(s)\, \mathrm{tr}\, \overline{\hat{\mathcal{O}}_x\, H(x, x'; s)}\,, \end{aligned} \tag{4.3.38}$$

performing an (inverse-) Laplace transformation

$$f(\Delta) = \int_0^\infty ds\, \tilde{f}(s)\, \mathrm{e}^{-s\Delta}\,, \tag{4.3.39}$$

and using (3.3.5) for a basis of fields $|x_i\rangle$ in the representation indicated by j. More explicitly, representing the operator insertion in the form

$$\hat{\mathcal{O}} = \sum_{k=0}^{n} M^{\alpha_1 \ldots \alpha_k}\, \bar{D}_{(\alpha_1} \cdots \bar{D}_{\alpha_k)}\,, \tag{4.3.40}$$

with the $M^{\alpha_1 \ldots \alpha_k}$ denoting any tensors depending on the background fields, written symmetrically without loss of generality, the traces (4.3.38) assume the form

$$\begin{aligned} \mathrm{Tr}_j \left[f(\Delta)\, \hat{\mathcal{O}} \right] &= \int_0^\infty ds\ \tilde{f}(s)\, \mathrm{Tr}_j \left[\mathrm{e}^{-s\Delta} \sum_{k=0}^{n} M^{\alpha_1 \ldots \alpha_k}\, \bar{D}_{(\alpha_1} \cdots \bar{D}_{\alpha_k)} \right] \\ &= \int d^d x \sqrt{g} \int_0^\infty ds\ \tilde{f}(s)\, \mathrm{tr}_j \left[\sum_{k=0}^{n} M^{\alpha_1 \ldots \alpha_k}\, H_{\alpha_1 \ldots \alpha_k}(s) \right] \\ &= \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{g} \int_0^\infty ds\, \tilde{f}(s) \sum_i s^{k_i}\, I_i\,. \end{aligned} \tag{4.3.41}$$

In the last line, the base monomials I_i defined in (4.3.9) are used to indicate that the covariant expressions for the H-tensors, contracted with the tensor part of the insertion M, identifies the corresponding β functions for the couplings u_i. Notably, to any order in a curvature expansion, the maximum number of derivatives n in the above formulas is restricted to be twice the number of curvature tensors in the case of a scalar trace, since each power of the curvature contained in $\mathcal{V}$ can at most be contracted with two covariant derivatives. For traces over vector or 2-tensors fields, the number of required derivatives increases by 2 and 4 respectively, since here the $\mathcal{V}$ term carries according open indices. Having all required H-tensors (3.3.9) at our disposal, it is easy to systematically evaluate the traces (4.3.41).

The final form of the result is most conveniently written in terms of the Mellin-transforms

$$Q_n[f] = \int_0^\infty ds\, s^{-n}\, \tilde{f}(s)\,. \tag{4.3.42}$$

These functionals can be re-expressed in terms of the original function f as

$$\begin{aligned} Q_n[f] &= \tfrac{1}{\Gamma(n)} \textstyle\int_0^\infty dz\, z^{n-1}\, f(z)\,, && n > 0\,, \\ Q_{-n}[f] &= (-1)^n \tfrac{\mathrm{d}^n}{\mathrm{d}z^n}\, f(z)\Big|_{z=0}\,, && n \geq 0\,, \end{aligned} \tag{4.3.43}$$

for positive or negative powers of s, respectively. For additional bare powers of the Laplacian inside the argument of these functionals, it follows the property

$$Q_n[\Delta^p\, f(\Delta)] = \frac{\Gamma[n+p]}{\Gamma[n]}\, Q_{n+p}[f(\Delta)]\,, \qquad n > 0 \wedge p > 0\,, \tag{4.3.44}$$

which is useful for the simplification of the evaluated form of the functional traces.

The solution of the RG equation (4.3.1) can be schematically given with the use of (4.3.37), (4.3.41) and finally (4.3.42) as

$$\begin{aligned} \partial_t \Gamma_k &= \frac{1}{2} \mathrm{STr} \left[\left(\Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1}\, \partial_t \mathcal{R}_k \right] \\ &= \frac{1}{2} \sum_n\ \mathrm{Tr}_{j_n} \left[f_n(\Delta)\, \hat{\mathcal{O}}_n \right] \\ &= \frac{1}{2(4\pi)^{d/2}} \int d^dx \sqrt{g}\, \sum_n\, \int_0^\infty ds\, \tilde{f}_n(s)\, \sum_i\, s^{k_{i,n}}\, I_i \\ &= \frac{1}{2(4\pi)^{d/2}} \int d^dx \sqrt{g}\, \sum_{n,i}\, Q_{-k_{i,n}}[f_n]\, I_i\,. \end{aligned} \tag{4.3.45}$$

With the scale derivative on the l.h.s. assuming the form (2.3.17), the β functions for the coupling constants multiplying the interaction monomials I_i are found by projecting onto these according to (2.3.19). Comparing the last line of (4.3.45) with the initial form (4.3.9), the scale derivative of the coupling constants is determined and we find

$$\begin{aligned} \beta_i &= -n_i g_i + k^{-n_i}\, \frac{\delta}{\delta I_i}\, \partial_t \Gamma_k \Big|_{\Phi=0} \\ &= -n_i g_i + k^{-n_i}\, \frac{1}{2(4\pi)^{d/2}}\, \sum_n\, Q_{-k_{i,n}}[f_n]\,. \end{aligned} \tag{4.3.46}$$

The projection of the RG flow onto the subspace of all possible interaction monomials spanned by the $I_i[\Phi, \bar{\Phi}]$ presents a technique to find an approximate solution of the RG equation. This is so because in general no self-consistent flow is described in terms of the monomials present in an ansatz for Γ_k of the form (4.3.9), in which case any interactions left out in the projection are said to be truncated. Usually, symmetric truncations are studied, where the same terms serving as input on the r.h.s. of the RG equation and thus generating the scale dependence of coupling constants by their fluctuations, are projected on to find their scaling behaviour. In general though, it is possible to determine the individual contributions of each possible monomial to the β function associated to any other. To understand this approach, note that any renormalization group equation emerges in the general context of quantum field theory and therefore does not transport any information about a specific class of models, e.g. gravity. The ansatz employed as initial condition for the RG equation is what serves as a definition of the fluctuating degrees of freedom and is thus to be seen as an important input in its own respect. As already pointed out in chapter 2, assuming a specific ansatz as representing an effective field theory, one can read the resulting scaling behaviour under Wilsonian renormalization group flow as the generation of effective vertices stemming from the integration of high momentum modes. Since also those β functions for vertices left out in the projection (4.3.46) would depend on the coupling constants taken into consideration, the corresponding interaction is effectively switched on, even with its coupling constants set to zero initially. Any such higher order interaction being generated dynamically would be in principle given in terms of the couplings present in the renormalized action functional. However, when starting from an effective action formulation, this relation is lost, so that any of the proper vertices are treated as independently running. Back-reactions between the running of such effective vertices are encoded due to the dependence of all the β functions on all possible coupling constants. It is in this sense that a neglection of higher interactions presents an approximation of the full renormalization group behaviour.

Systematically motivated schemes commonly used as organizing principles for a truncated ansatz (4.3.9) constitute, foremost, the derivative or the vertex expansions. Herein, one includes all interactions containing up to a maximal number of derivatives, or up to a certain order of the fluctuation field Φ, respectively. The former approach allows to access information which is relevant for the UV limit of a model more easily using a basis of local invariants, while the latter keeps track of the full momentum dependence in structure functions for each vertex.[6] These approximations allow to extract non-perturbative information from the RG equation (4.3.1), without having to rely on a

[6]For reviews on the derivative expansion in the case of scalar field theory see [79, 124].

small parameter to expand in.

Depending on the complexity of the ansatz (4.3.9), inverting its second variation and projecting the operator traces onto these monomials can in practice become quite involved. In previous works, the task was significantly simplified by the choice of a particular class of background geometries. While the formalism guarantees that β functions will not depend on this choice, the major drawback of this method is that some of them will be rendered indistinguishable on such backgrounds, because generally different $I_i[\Phi, \bar{\Phi}]$ become proportional to each other. In this way potentially important structural information is hidden, for one is really only determining the running of a certain combination of otherwise independent coupling constants.[7] The algorithm for the approximate solution of the RG equation introduced in this section is able to retain an entirely unspecified background geometry, lifting these limitations. We stress that the possibility to study the disentangled contribution of arbitrary tensor structures is crucial for a comprehensive analysis of the asymptotic safety scenario, in particular to determine the relevant directions in the RG flow and to engage in an investigation of the underlying mechanism. This can be done in a structural analysis of the β functions, commencing in a study of the stability of fixed points to extent the results of [125] for increasingly sophisticated approximations. Incorporating the projection method (4.3.46), this algorithm can be used to keep track of the origin and effect of contributions to β functions in the RG equation, so that more precise conditions on the existence of fixed points can be elicited in the future.

Summary

In this section we developed a universal algorithm for the approximate solution of the RG equation (4.3.1). This procedure is applicable irrespective of the underlying degrees of freedom and is especially tailored to bypass the complications arising due to the presence of gauge symmetries. The algorithm is summarized by the following explicit steps:

[7] For the example of gravity, the background metric can be restricted to be that of a spherical manifold, in which case all curvature monomials reduce to powers of the Ricci scalar. See appendix A for more details.

1. expand Γ_k to quadratic order in the fluctuation fields to find the second variation (4.3.11);
2. implement the cutoff $\mathcal{R}_k$ via replacement rule (4.3.13);
3. invert the dressed propagator $[\Gamma_k^{(2)}+\mathcal{R}_k]$ as a series expansion (4.3.17) using field decompositions, or performing the resummation (4.3.26);
4. apply commutation rules to collect all derivatives in the trace arguments and write it in the form (4.3.37);
5. evaluate the traces as shown in (4.3.41), inserting the H-tensors via replacement rule;
6. project the resulting terms onto the monomials I_i to identify the β functions (4.3.46).

Each one of these steps is entirely algebraic and can thus be handled by computer algebra software. This is not only convenient but necessary for treating truncations beyond a certain complexity, since the size of intermediate results grows quickly over what can be managed manually. The method has already been successfully applied to handle sophisticated operator traces in [67–69, 105]. The algorithm presented here in liaison with its implementation on a computer brings a multitude of applications into computational reach.

The algorithm aids practical computations in any expansion scheme of the RG equation. Notably, the construction does not require any reference to a particular choice of background geometry, nor does it depend on a specific gauge fixing. This is one of its main improvements compared to previous computation methods. For the systematic investigation of the derivative expansion in gravity theories, relaxing this technical limitation is of central importance. It is due to this generality that makes the proposed method well suited for the automated processing of the RG equation.

5. Quantum Einstein Gravity

In this chapter, the renormalization group equation (2.3.18) derived in chapter 2, will be applied to the case of gravity in first order in the derivative expansion. This so called Einstein-Hilbert truncation is the most extensively studied approximation of the gravitational renormalization group flow, see for example [35, 45–47, 58]. Here we employ a metric degree of freedom, which is sufficient to define a quantization of the spacetime geometry in the absence of fermions. Making use of the algorithm developed in chapter 4, we re-derived the β functions of the single-metric Einstein-Hilbert truncation in a completely background invariant calculation. Subsequently, this computation will be extended to include non-trivial contributions stemming from the renormalization of the Faddeev-Popov ghost fields. The aim of this chapter is to demonstrate the existence of a non-Gaussian fixed point in a way that can be generalized to improved approximations. This allows for a more systematic investigation underlying the emergence of the fixed point in the future.

5.1. The Einstein-Hilbert truncation

Following the discussion of the technical difficulties arising in the solution of the RG equation for theories with gauge symmetries in the last chapter, we will now demonstrate the practical application of the algorithm suggested therein. This method allows to evaluate the RG equation for a completely generic background metric, showing that the resulting β functions depend on the gauge fixing and cutoff scheme, but are manifestly independent of the background geometry.[1] In this section, we consider the Einstein-Hilbert truncation in $d = 4$, treated within the universal algorithm constructed in section 4.3, to demonstrate the robustness of the gravitational fixed point. To find the β functions in a background independent way, our construction leaves the background metric $\bar{g}_{\mu\nu}$ unspecified. The only technical assumption is that $\bar{g}_{\mu\nu}$ is a (Euclidean) metric on a compact,

[1]See also [58] for a related discussion.

closed, and complete manifold, which guarantees that the minimal TT-decomposition and heat kernel expansion are well defined.[2]

The ansatz for the gravitational part of the effective average action for in the present case reads

$$\Gamma_k^{\mathrm{grav}}[g] = \frac{1}{16\pi G_k} \int d^4x \sqrt{g} \Big(- R + 2\Lambda_k \Big) \,, \tag{5.1.1}$$

with a scale dependent Newton's constant G_k and cosmological constant Λ_k. For convenience we introduce the couplings

$$u_0 = \frac{\Lambda_k}{8\pi G_k} \,, \qquad u_1 = -\frac{1}{16\pi G_k} \,, \tag{5.1.2}$$

together with their dimensionless counterparts

$$g_0 = u_0 k^{-4} \,, \qquad g_1 = u_1 k^{-2} \,. \tag{5.1.3}$$

Attributed with the gauge fixing and ghost actions defined in (4.1.17), the effective average action for single metric gravity can be written in the form

$$\Gamma_k[h, \bar{C}, C; \bar{g}] = \Gamma_k^{\mathrm{grav}}[\bar{g} + h] + \underbrace{S^{\mathrm{gf}}[h, \bar{g}] + S^{\mathrm{ghost}}[h, \bar{C}, C; \bar{g}] + S^{\mathrm{aux}}[h, \bar{g}]}_{S^{\mathrm{uni}}} \,. \tag{5.1.4}$$

As shown in section 4.3, the use of standard gauge fixing terms and field decompositions leads to recurring terms, which for geometric Landau gauge ($\alpha = 0$) decouple in the RG equation, and therefore can be evaluated independently of the specific model under consideration. Here, the RG equation decomposes in the form (4.3.36), separating the combined universal addition of the action S^{uni} from the gravitational part, so that we can write

$$\partial_t \Gamma_k = \mathcal{S}^{\mathrm{grav}} + \mathcal{S}^{\mathrm{univ}} = \mathcal{S}^{\mathrm{grav}} + \mathcal{S}^{\mathrm{gf}} + \mathcal{S}^{\mathrm{gh}} + \mathcal{S}^{\mathrm{aux}} \,. \tag{5.1.5}$$

Herein $\mathcal{S}^{\mathrm{grav}}$ captures the trace contributions of the physical degrees of freedom in the gravitational sector, represented as the transverse-traceless tensor $h^{\mathrm{T}}_{\mu\nu}$ and the trace h defined in (4.1.34). By definition $\mathcal{S}^{\mathrm{univ}}$ contains all terms that reoccur in any action for single metric gravity, including gauge fixing, ghost terms and Jacobians for the implemented decompositions.

[2]For a computation on a manifold with boundary, see [126].

5.1.1. Universal Contributions

The transverse traceless decomposition (4.1.34) employed in the computation of the gravitational contributions in the RG equation serves as a way to isolate the degrees of freedom that are subject to a gauge fixing condition in the vector component ξ_μ. In the following, we compute the corresponding re-occurring contributions $\mathcal{S}^{\text{univ}}$, arising from the gauge fixing-, ghost- and auxiliary actions, to leading orders in the background curvature. The final results are given in $d = 4$ dimensions for simplicity, but the generalization to arbitrary d is straightforward.

The Gauge Fixing Contribution

In the case of gravity, the general coordinate transformation invariance is broken with a standard harmonic gauge fixing condition of the form

$$\begin{aligned} F_\mu =& D^\nu h_{\mu\nu} - \beta D_\mu h \\ =& D^2 \xi_\mu + (1 - \tfrac{2}{d}) D_\mu D^\nu \xi_\nu + R_\mu{}^\nu \xi_\nu + (\tfrac{1}{d} - \beta) D_\mu h \,, \end{aligned} \tag{5.1.6}$$

retaining the background gauge invariance as an intact symmetry.[3] If we allow the trace part h to occur in the gauge fixing condition, the Hessian of S^{gf} will carry a zero eigenvalue. To ensure the invertibility of the operator G in (4.3.28) one has to choose $\beta = 1/d$, eliminating h from the gauge term. The explicit form of the gauge fixing term now reads

$$\begin{aligned} S^{\text{gf}} =& \frac{1}{2\alpha} \int d^d x \sqrt{g}\; g^{\mu\nu} F_\mu F_\nu \\ =& \frac{1}{2\alpha} \int d^d x \sqrt{g}\; \xi_\mu \left[\delta^\mu{}_\alpha \Delta - (1 - \tfrac{2}{d}) D^\mu D_\alpha - R^\mu{}_\alpha \right] \left[\delta^\alpha{}_\nu \Delta - (1 - \tfrac{2}{d}) D^\alpha D_\nu - R^\alpha{}_\nu \right] \xi^\nu \\ =& \frac{1}{2\alpha} \int d^d x \sqrt{g}\; \xi_\mu \Big[\delta^\mu{}_\nu \Delta^2 - \tfrac{3d^2 - 8d + 4}{d^2} \Delta D^\mu D_\nu - 2 R^\mu{}_\nu \Delta - \tfrac{d^2 - 4d + 4}{d^2} R^\mu{}_\alpha D^\alpha D_\nu \\ & \qquad + \tfrac{2d - 4}{d} R^\alpha{}_\nu D^\mu D_\alpha + R^{\mu\alpha} R_{\alpha\nu} \Big] \xi^\nu \,. \end{aligned} \tag{5.1.7}$$

The second term in the last line contains a non-minimal operator without a background curvature. Following the first strategy outlined in section 4.3, this $\mathcal{D}$-type contribution is eliminated with a transverse decomposition (4.1.23) of the vector $\xi_\mu = \xi^{\text{T}}_\mu + D_\mu \omega$ with

[3] To lighten the notation, we drop the bar on the background metric $\bar{g}_{\mu\nu}$, whenever it is the only metric field besides the fluctuations $h_{\mu\nu}$, appearing in the computations in this chapter.

$D^\mu \xi^{\mathrm{T}}_\mu = 0$. This finally leads to

$$
\begin{aligned}
S^{\mathrm{gf}} = \frac{1}{2\alpha}\int d^dx\sqrt{g}\Big\{ &\xi^{\mathrm{T}}_\mu \Big[\delta^\mu{}_\nu \Delta^2 - 2R^\mu{}_\nu\Delta + R^{\mu\alpha}R_{\alpha\nu}\Big]\xi^{\mathrm{T}\nu} \\
&+\xi^{\mathrm{T}}_\mu\Big[\tfrac{4-6d}{d}R^{\mu\nu}\Delta D_\nu + \tfrac{4-2d}{d}R^\mu{}_\alpha R^{\alpha\nu}D_\nu\Big]\omega \\
&+\omega\Big[\tfrac{2d+4}{d}R^\mu{}_\nu\Delta D_\mu + 2R^\mu{}_\nu R_{\mu\alpha}D^\alpha - 6R^{\alpha\beta}R_{\alpha\mu\beta\nu}D^\mu\Big]\xi^{\mathrm{T}\nu} \\
&+\omega\Big[\tfrac{4(d-1)^2}{d^2}\Delta^3 + \tfrac{8(d-1)}{d}R^{\mu\nu}\Delta D_\mu D_\nu + \tfrac{8-7d}{d}R^{\mu\alpha}R^\nu{}_\alpha D_\mu D_\nu \\
&\quad + \tfrac{6d-8}{d}R^{\alpha\beta}R_{\alpha\mu\beta\nu}D^\mu D^\nu\Big]\omega\Big\} \\
= \frac{1}{2\alpha}\int d^dx\sqrt{g}\,\Big\{ &\xi^{\mathrm{T}}_\mu\Big[\mathcal{K}^{\mathrm{gf}}_{1\mathrm{T}} + \mathcal{V}^{\mathrm{gf}}_{1\mathrm{T}}\Big]^\mu{}_\nu\xi^{\mathrm{T}\nu} + \omega\Big[\mathcal{K}^{\mathrm{gf}}_0 + \mathcal{V}^{\mathrm{gf}}_0\Big]\omega \\
&+ \xi^{\mathrm{T}}_\mu\Big[\mathcal{V}^{\mathrm{gf}}_\times\Big]^\mu\omega + \omega\Big[\mathcal{V}^{\mathrm{gf}}_\times\Big]^\dagger_\nu\xi^{\mathrm{T}\nu}\Big\}\,,
\end{aligned}
\tag{5.1.8}
$$

which allows to identify the kinetic terms

$$
\Big[\mathcal{K}^{\mathrm{gf}}_{1\mathrm{T}}\Big]^\mu{}_\nu = \Delta^2\delta^\mu{}_\nu\,, \qquad \Big[\mathcal{K}^{\mathrm{gf}}_0\Big] = \tfrac{4(d-1)^2}{d^2}\Delta^3\,,
\tag{5.1.9}
$$

and vertices

$$
\begin{aligned}
\Big[\mathcal{V}^{\mathrm{gf}}_{1\mathrm{T}}\Big]^\mu{}_\nu &= -R^\mu{}_\nu\Delta - \Delta R^\mu{}_\nu + R^{\mu\alpha}R_{\alpha\nu}\,, \\
\Big[\mathcal{V}^{\mathrm{gf}}_0\Big] &= 4(1-\tfrac{1}{d})\left(\Delta D_\mu R^{\mu\nu}D_\nu + D_\mu R^{\mu\nu}D_\nu\Delta\right) - 4D_\mu R^{\mu\alpha}R_\alpha{}^\nu D_\nu\,, \\
\Big[\mathcal{V}^{\mathrm{gf}}_\times\Big]^\mu &= -4(1-\tfrac{1}{d})R^{\mu\nu}D_\nu\Delta - 2\Delta R^{\mu\nu}D_\nu + 2R^{\mu\nu}R_\nu{}^\alpha D_\alpha\,, \\
\Big[\mathcal{V}^{\mathrm{gf}}_\times\Big]^\dagger_\nu &= 4(1-\tfrac{1}{d})\Delta D^\mu R_{\mu\nu} + 2D^\mu R_{\mu\nu}\Delta - 2D^\mu R_{\mu\alpha}R^\alpha{}_\nu\,.
\end{aligned}
\tag{5.1.10}
$$

Here we neglected terms involving covariant derivatives of the curvature tensor, since they would not contribute to the projection on the monomials in (5.1.1). The cutoff function is adapted following the rule (4.3.13) of regularizing the kinetic terms, yielding the corresponding cutoff operators

$$
\mathcal{R}^{\mathrm{gf},1\mathrm{T}}_k = (P_k^2 - \Delta^2)\delta^\mu{}_\nu\,, \qquad \mathcal{R}^{\mathrm{gf},0}_k = \tfrac{4(d-1)^2}{d^2}(P_k^3 - \Delta^3)\,.
\tag{5.1.11}
$$

The regularized inverse propagators are defined by $\mathcal{P}^{\mathrm{gf}}_j = \mathcal{K}^{\mathrm{gf}}_j + \mathcal{R}^{\mathrm{gf},j}_k$, where $j = 0, 1\mathrm{T}$.

Using the inversion formula (4.3.17), the final step consists of evaluating the individual resulting operator traces, following the prescription (4.3.45). Since the vertices (5.1.10) are defined on the transverse and longitudinal field subspaces, they have to be traced with the projection operators (4.1.25). As for the present purpose, the results are required only to linear order in $\mathcal{V}$ and in $d = 4$, we terminate the curvature expansion inversion at this order. Since $\mathcal{V}_\times$ and $\mathcal{V}^\dagger_\times$ are both linear in R, the cross-terms do not contribute here.

With the trace split into a transverse vector and scalar part, the 1T-sector evaluates to

$$\begin{aligned}\mathcal{S}^{\rm gf}_{\rm 1T} &= \frac{1}{2}{\rm Tr}_1\left[\frac{1}{\mathcal{P}^{\rm gf}_{\rm 1T}}\partial_t\mathcal{R}^{\rm gf,1T}_k\Pi_{\rm T}\right] - \frac{1}{2}{\rm Tr}_1\left[\frac{1}{\mathcal{P}^{\rm gf}_{\rm 1T}}\mathcal{V}^{\rm gf}_{\rm 1T}\cdot\Pi_{\rm T}\frac{1}{\mathcal{P}^{\rm gf}_{\rm 1T}}\partial_t\mathcal{R}^{\rm gf,1T}_k\right] \\ &= \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left[3\,Q_2[f^1] + R\left(\tfrac{1}{4}Q_1[f^1] + 3Q_3[f^3]\right)\right],\end{aligned} \tag{5.1.12}$$

while the scalar sector gives

$$\begin{aligned}\mathcal{S}^{\rm gf}_{0} &= \frac{1}{2}{\rm Tr}_0\left[\frac{1}{\mathcal{P}^{\rm gf}_{0}}\partial_t\mathcal{R}^{\rm gf,0}_k\right] - \frac{1}{2}{\rm Tr}_0\left[\frac{1}{\mathcal{P}^{\rm gf}_{0}}\mathcal{V}^{\rm gf}_{0}\frac{1}{\mathcal{P}^{\rm gf}_{0}}\partial_t\mathcal{R}^{\rm gf,1T}_k\right] \\ &= \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left[\tfrac{3}{2}\,Q_2[f^1] + R\left(\tfrac{1}{4}Q_1[f^1] + 6Q_4[f^4]\right)\right].\end{aligned} \tag{5.1.13}$$

Here the Q functionals are defined in (4.3.42), and their argument indicates the number of propagators in a term by

$$f^n(\Delta) = \frac{1}{(P_k)^n}\partial_t R_k\,. \tag{5.1.14}$$

The complete contribution of the gauge sector is then given by

$$\begin{aligned}\mathcal{S}^{\rm gf} &= \mathcal{S}^{\rm gf}_{\rm 1T} + \mathcal{S}^{\rm gf}_0 \\ &= \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left[\tfrac{9}{2}\,Q_2[f^1] + R\left(\tfrac{1}{2}Q_1[f^1] + 3Q_3[f^3] + 6Q_4[f^4]\right)\right] + \mathcal{O}(\mathcal{R}^2)\,.\end{aligned} \tag{5.1.15}$$

Ghost Contributions

The ghost term corresponding to the gauge fixing (5.1.6) with $\beta = 1/d$ is given by

$$\begin{aligned}S^{\rm ghost} &= \int d^dx\sqrt{\bar{g}}\,\bar{C}_\mu\,\bar{g}^{\mu\nu}\,\frac{\delta F_\nu}{\delta h_{\alpha\beta}}\,\mathcal{L}_C(\bar{g}+h)_{\alpha\beta} \\ &= \int d^dx\sqrt{\bar{g}}\,\bar{C}_\mu\,\mathcal{M}^\mu{}_\nu\,C^\nu\,,\end{aligned} \tag{5.1.16}$$

with the Faddeev-Popov operator

$$\mathcal{M}^\mu{}_\nu = \bar{g}^{\mu\alpha}\bar{g}^{\lambda\sigma}\left[\bar{D}_\lambda g_{\alpha\nu}D_\sigma + \bar{D}_\lambda g_{\sigma\nu}D_\alpha - 2\beta\bar{D}_\alpha g_{\lambda\nu}D_\sigma\right]. \tag{5.1.17}$$

Here we distinguish the full quantum metric $g_{\alpha\beta} = h_{\alpha\beta} + \bar{g}_{\alpha\beta}$ from the background metric $\bar{g}_{\alpha\beta}$, with respect to which derivatives with a bar are covariant. Since only variations with respect to the ghost fields are required to obtain the contribution to the gravitational coupling constants, we set $g_{\alpha\beta} = \bar{g}_{\alpha\beta}$ in the following. Thus the action (5.1.16), dropping an irrelevant overall sign, reduces to

$$S^{\rm ghost} = \int d^dx\sqrt{g}\,\bar{C}_\mu\left[\Delta\delta^\mu{}_\nu - (1-\tfrac{2}{d})D^\mu D_\nu - R^\mu{}_\nu\right]C^\nu\,, \tag{5.1.18}$$

which immediately reveals the quadratic variation in the ghost fields.

Again we carry out the transverse decomposition of the anti-ghost field $\bar{C}_\mu = \bar{C}^{\mathrm{T}}_\mu + D_\mu \bar{\eta}$ and likewise for the ghost, to have S^{ghost} assume a two-by-two block form without $\mathcal{D}$-type operators. In terms of the decomposed fields, with covariantly constant curvature, the action reads

$$\begin{aligned} S^{\mathrm{ghost}} &= \int d^dx\sqrt{g}\Big\{\bar{C}^{\mathrm{T}}_\mu \left[\Delta\delta^\mu{}_\nu - R^\mu{}_\nu\right] C^{\mathrm{T}\nu} + \bar{\eta}\left[(2-\tfrac{2}{d})\Delta^2 + 2R^{\mu\nu}D_\mu D_\nu\right]\eta \\ &\qquad + \bar{\eta}\left[2R_{\mu\nu}D^\mu\right]C^{\mathrm{T}\nu} - \bar{C}^{\mathrm{T}}_\mu\left[2R^{\mu\nu}D_\nu\right]\eta\Big\} \\ &= \int d^dx\sqrt{g}\Big\{\bar{C}^{\mathrm{T}}_\mu \left[\mathcal{K}^{\mathrm{gh}}_{1\mathrm{T}} + \mathcal{V}^{\mathrm{gh}}_{1\mathrm{T}}\right]^\mu{}_\nu C^{\mathrm{T}\nu} + \bar{\eta}\left[\mathcal{K}^{\mathrm{gh}}_0 + \mathcal{V}^{\mathrm{gh}}_0\right]\eta \\ &\qquad + \bar{\eta}\left[\mathcal{V}^{\mathrm{gh}}_\times\right]^\dagger_\nu C^{\mathrm{T}\nu} + \bar{C}^{\mathrm{T}}_\mu\left[\mathcal{V}^{\mathrm{gh}}_\times\right]^\mu \eta\Big\}\,. \end{aligned} \tag{5.1.19}$$

The operator structure found herein, classified according to (4.3.16) defines the entries

$$\begin{aligned} \left[\mathcal{P}^{\mathrm{gh}}_{1\mathrm{T}}\right]^\mu{}_\nu &= \delta^\mu{}_\nu P_k\,, & \left[\mathcal{P}^{\mathrm{gh}}_0\right] &= (2-\tfrac{2}{d})P_k^2\,, \\ \left[\mathcal{V}^{\mathrm{gh}}_{1\mathrm{T}}\right]^\mu_\nu &= -R^\mu_\nu\,, & \left[\mathcal{V}^{\mathrm{gh}}_0\right] &= 2D_\mu R^{\mu\nu}D_\nu\,, \\ \left[\mathcal{V}^{\mathrm{gh}}_\times\right]^\mu &= -2R^{\mu\nu}D_\nu\,, & \left[\mathcal{V}^{\mathrm{gh}}_\times\right]^\dagger_\nu &= 2D^\mu R_{\mu\nu}\,, \end{aligned} \tag{5.1.20}$$

where the cutoff operators are again introduced by application of the rule (4.3.13), with $P_k = \Delta + R_k$. Explicitly they read

$$\mathcal{R}^{\mathrm{gh},1\mathrm{T}}_k = \delta^\mu{}_\nu R_k\,, \qquad \mathcal{R}^{\mathrm{gh},0}_k = (2-\tfrac{2}{d})\left(P_k^2 - \Delta^2\right)\,. \tag{5.1.21}$$

The operators (5.1.20), restricted to their corresponding transverse subspaces with the projection operators (4.1.25), enter the inversion formula (4.3.17). The contributions of the ghost sector $\mathcal{S}^{\mathrm{gh}}$ entering into (5.1.5), split into transverse vector and scalar parts read

$$\begin{aligned} \mathcal{S}^{\mathrm{gh}}_{1\mathrm{T}} =&\; -\mathrm{Tr}_1\left[\frac{1}{\mathcal{P}^{\mathrm{gh}}_{1\mathrm{T}}}\,\partial_t\mathcal{R}^{\mathrm{gh},1\mathrm{T}}_k\,\Pi_{\mathrm{T}}\right] + \mathrm{Tr}_1\left[\frac{1}{(\mathcal{P}^{\mathrm{gh}}_{1\mathrm{T}})^2}\,\partial_t\mathcal{R}^{\mathrm{gh},1\mathrm{T}}_k\,\Pi_{\mathrm{T}}\,\mathcal{V}^{\mathrm{gh}}_{1\mathrm{T}}\,\Pi_{\mathrm{T}}\right] \\ &- \mathrm{Tr}_1\left[\frac{1}{(\mathcal{P}^{\mathrm{gh}}_{1\mathrm{T}})^3}\,\partial_t\mathcal{R}^{\mathrm{gh},1\mathrm{T}}_k\,\mathcal{V}^{\mathrm{gh}}_{1\mathrm{T}}\,\Pi_{\mathrm{T}}\,\mathcal{V}^{\mathrm{gh}}_{1\mathrm{T}}\,\Pi_{\mathrm{T}}\right] \\ &- \mathrm{Tr}_1\left[\frac{1}{(\mathcal{P}^{\mathrm{gh}}_{1\mathrm{T}})^2}\frac{1}{\mathcal{P}^{\mathrm{gh}}_0}\,\partial_t\mathcal{R}^{\mathrm{gh},1\mathrm{T}}_k\,(\mathcal{V}^{\mathrm{gh}}_\times)^\dagger\,\mathcal{V}^{\mathrm{gh}}_\times\,\Pi_{\mathrm{T}}\right] \\ &+ \mathrm{Tr}_1\left[\frac{1}{\mathcal{P}^{\mathrm{gh}}_{1\mathrm{T}}}\,\partial_t\mathcal{R}^{\mathrm{gh},1\mathrm{T}}_k\,\left[\Pi_{\mathrm{T}}\,,\,\frac{1}{\mathcal{P}^{\mathrm{gh}}_{1\mathrm{T}}}\right]\,\Pi_{\mathrm{T}}\,\mathcal{V}^{\mathrm{gh}}_{1\mathrm{T}}\,\Pi_{\mathrm{T}}\right] \\ =:&\; -\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,\left\{\mathcal{C}^1_{1\mathrm{T}} + \mathcal{C}^2_{1\mathrm{T}} + \mathcal{C}^3_{1\mathrm{T}} + \mathcal{C}^4_{1\mathrm{T}} + \mathcal{C}^5_{1\mathrm{T}}\right\} + \mathcal{O}(\mathcal{R}^3)\,, \end{aligned} \tag{5.1.22}$$

and

$$
\begin{aligned}
\mathcal{S}_0^{\text{gh}} = &- \text{Tr}_0 \left[\frac{1}{\mathcal{P}_0^{\text{gh}}} \, \partial_t \mathcal{R}_k^{\text{gh},0} \right] + \text{Tr}_0 \left[\frac{1}{(\mathcal{P}_0^{\text{gh}})^2} \, \partial_t \mathcal{R}_k^{\text{gh},0} \, \mathcal{V}_0^{\text{gh}} \right] \\
&- \text{Tr}_0 \left[\frac{1}{(\mathcal{P}_0^{\text{gh}})^3} \, \partial_t \mathcal{R}_k^{\text{gh},0} \, (\mathcal{V}_0^{\text{gh}})^2 \right] \\
&- \text{Tr}_0 \left[\frac{1}{(\mathcal{P}_0^{\text{gh}})^2} \frac{1}{\mathcal{P}_{1\text{T}}^{\text{gh}}} \, \partial_t \mathcal{R}_k^{\text{gh},0} \, \mathcal{V}_\times^{\text{gh}} \, \Pi_{\text{T}} \, (\mathcal{V}_\times^{\text{gh}})^\dagger \right] \\
=: &- \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \, \left\{ \mathcal{C}_0^1 + \mathcal{C}_0^2 + \mathcal{C}_0^3 + \mathcal{C}_0^4 \right\} + \mathcal{O}(\mathcal{R}^3) \, .
\end{aligned}
\tag{5.1.23}
$$

The overall negative sign here stems from the supertrace for fermionic fields. The individual partial traces are evaluated with the off-diagonal heat kernel expansion up to second order in the curvature. Setting $d = 4$ they this yields

$$
\begin{aligned}
\mathcal{C}_{1\text{T}}^1 &= 3Q_2[f_{1\text{T}}^1] + \tfrac{1}{4} R Q_1[f_{1\text{T}}^1] + \left[-\tfrac{1}{24} R^2 + \tfrac{1}{40} R_{\mu\nu} R^{\mu\nu} - \tfrac{1}{15} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right] Q_0[f_{1\text{T}}^1] \, , \\
\mathcal{C}_{1\text{T}}^2 &= \tfrac{3}{4} R Q_2[f_{1\text{T}}^2] + \left[\tfrac{1}{8} R^2 - \tfrac{1}{4} R_{\mu\nu} R^{\mu\nu} \right] Q_1[f_{1\text{T}}^2] \, , \\
\mathcal{C}_{1\text{T}}^3 &= \left[\tfrac{1}{24} R^2 + \tfrac{7}{12} R_{\mu\nu} R^{\mu\nu} \right] Q_2[f_{1\text{T}}^3] \, , \\
\mathcal{C}_{1\text{T}}^4 &= \left[-\tfrac{1}{12} R^2 + \tfrac{1}{3} R_{\mu\nu} R^{\mu\nu} \right] Q_3[\tilde{f}_\times] \, , \\
\mathcal{C}_{1\text{T}}^5 &= 0 \, , \\
\mathcal{C}_0^1 &= Q_2[f_0^1] + \tfrac{1}{6} R Q_1[f_0^1] + \left[\tfrac{1}{72} R^2 - \tfrac{1}{180} R_{\mu\nu} R^{\mu\nu} + \tfrac{1}{180} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right] Q_0[f_0^1] \, , \\
\mathcal{C}_0^2 &= R Q_3[f_0^2] + \left[\tfrac{1}{6} R^2 - \tfrac{1}{3} R_{\mu\nu} R^{\mu\nu} \right] Q_2[f_0^2] \, , \\
\mathcal{C}_0^3 &= \left[R^2 + 2 R_{\mu\nu} R^{\mu\nu} \right] Q_4[f_0^3] \, , \\
\mathcal{C}_0^4 &= \left[-\tfrac{1}{12} R^2 + \tfrac{1}{3} R_{\mu\nu} R^{\mu\nu} \right] Q_3[f_\times] \, .
\end{aligned}
\tag{5.1.24}
$$

Here the functions

$$
\begin{aligned}
f_{1\text{T}}^n(\Delta) &:= \frac{1}{(\mathcal{P}_{1\text{T}}^{\text{gh}})^n} \, \partial_t \mathcal{R}_k^{\text{gh},1\text{T}} = f^n(\Delta) \, , \\
f_0^n(\Delta) &:= \frac{1}{(\mathcal{P}_0^{\text{gh}})^n} \, \partial_t \mathcal{R}_k^{\text{gh},0} = \frac{2d^{n-1}}{(2d-2)^{n-1}} \, f^{2n-1}(\Delta) \, , \\
f_\times(\Delta) &:= \frac{1}{(\mathcal{P}_0^{\text{gh}})^2} \frac{1}{\mathcal{P}_{1\text{T}}^{\text{gh}}} \, \partial_t \mathcal{R}_k^{\text{gh},0} = \frac{d}{d-1} \, f^4(\Delta) \, , \\
\tilde{f}_\times(\Delta) &:= \frac{1}{(\mathcal{P}_{1\text{T}}^{\text{gh}})^2} \frac{1}{\mathcal{P}_0^{\text{gh}}} \, \partial_t \mathcal{R}_k^{\text{gh},1\text{T}} = \frac{d}{2d-2} \, f^4(\Delta) \, ,
\end{aligned}
\tag{5.1.25}
$$

capture the scalar parts of the trace arguments. Accounting for the relative factors herein, they reduce to the plain functions $f^n(\Delta)$ defined in (5.1.14). The full contribution of the ghost sector is obtained upon insertion to give

$$
\begin{aligned}
\mathcal{S}^{\mathrm{gh}} =& \mathcal{S}^{\mathrm{gh}}_{1\mathrm{T}} + \mathcal{S}^{\mathrm{gh}}_{0} \\
=& -\frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \Big\{ 5Q_2[f^1] + \Big(\tfrac{7}{12} Q_1[f^1] + \tfrac{3}{4} Q_2[f^2] + \tfrac{4}{3} Q_3[f^3] \Big) R \\
& + \Big(-\tfrac{1}{72} Q_0[f^1] + \tfrac{1}{8} Q_1[f^2] + \tfrac{19}{72} Q_2[f^3] - \tfrac{1}{6} Q_3[f^4] + \tfrac{8}{9} Q_4[f^5] \Big) R^2 \\
& + \Big(\tfrac{1}{72} Q_0[f^1] - \tfrac{1}{4} Q_1[f^2] + \tfrac{5}{36} Q_2[f^3] + \tfrac{2}{3} Q_3[f^4] + \tfrac{16}{9} Q_4[f^5] \Big) R_{\mu\nu} R^{\mu\nu} \\
& - \tfrac{1}{18} Q_0[f^1] \; R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \Big\} + \mathcal{O}(\mathcal{R}^3) \,.
\end{aligned}
\tag{5.1.26}
$$

Jacobians of the Transverse Decompositions

The final contribution to the universal sector comes from the Jacobians arising from the transverse decompositions of the fluctuation fields, employed to simplify the operator structure under the trace in the RG equation. The Jacobian from the minimal transverse-traceless decomposition of the metric fluctuation is given in (4.1.53) are accounted for with the help of auxiliary fields. Notably, the operator $\mathcal{M}_\mu{}^\nu$ defined in (4.1.52) coincides with the one arising in the ghost action (5.1.18). Therefore the contribution of the auxiliary fields will be proportional to the ghost contribution. With the correct factors for bosonic and fermionic vector fields, we find the result

$$
\mathcal{S}^{\mathrm{aux}}_{\mathrm{TT}} = \frac{1}{2} \mathcal{S}^{\mathrm{gh}} \,, \tag{5.1.27}
$$

where $\mathcal{S}^{\mathrm{gh}}$ is given by (5.1.26).

Furthermore, the decomposition of vector fields into their transversal and longitudinal components throughout the computation also leads to Jacobian factors. In total, we have decomposed the bosonic vector ξ_μ in the gauge fixing action (5.1.8), the fermionic vectors $\bar{C}_\mu$, C_μ in the ghost action (5.1.19), and another auxiliary set of one bosonic and one complex fermionic vector in (4.1.53) in order to arrive at (5.1.27). The Jacobians for two bosonic and two complex fermionic vectors given in (4.1.33) yield the complete correction of the functional measure

$$
J^{\mathrm{total}}_{\mathrm{vec}} = \det(\Delta)^{1/2} \det(\Delta)^{1/2} \det(\Delta)^{-1} \det(\Delta)^{-1} = \det(\Delta)^{-1} \,. \tag{5.1.28}
$$

This determinant can be exponentiated by introducing two real auxiliary bosonic scalar fields ω_1 and ω_2 like

$$
J^{\mathrm{total}}_{\mathrm{vec}} = \det(\Delta)^{-1} = \int D\omega_1 \, D\omega_2 \;\; \exp\Big[-\tfrac{1}{2} \int_x \omega_1 \Delta \omega_1 - \tfrac{1}{2} \int_x \omega_2 \Delta \omega_2 \Big] \,. \tag{5.1.29}
$$

The contribution of this term to the RG flow is consequently given by a simple scalar trace, evaluated with the standard heat kernel coefficients to give

$$
\begin{aligned}
\mathcal{S}^{\text{aux}}_{\text{vec}} &= \mathrm{Tr}_0\left[\frac{1}{P_k}\,\partial_t R_k\right]\\
&= \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,\Big\{Q_2[f^1] + \tfrac{1}{6}R\,Q_1[f^1]\\
&\qquad + \left(\tfrac{1}{72}R^2 - \tfrac{1}{180}R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)Q_0[f^1]\Big\}\\
&\qquad + \mathcal{O}(\mathcal{R}^3)\,.
\end{aligned}
\tag{5.1.30}
$$

Finally, the combined contribution of all auxiliary fields appearing due to field decompositions is given by

$$
\begin{aligned}
\mathcal{S}^{\text{aux}} &= \mathcal{S}^{\text{aux}}_{\text{TT}} + \mathcal{S}^{\text{aux}}_{\text{vec}}\\
&= \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\Big\{ -\tfrac{3}{2}Q_2[f^1] + \left(-\tfrac{1}{8}Q_1[f^1] - \tfrac{3}{8}Q_2[f^2] - \tfrac{2}{3}Q_3[f^3]\right)R\\
&\quad + \left(\tfrac{1}{48}Q_0[f^1] - \tfrac{1}{16}Q_1[f^2] - \tfrac{19}{144}Q_2[f^3] + \tfrac{1}{12}Q_3[f^4] - \tfrac{4}{9}Q_4[f^5]\right)R^2\\
&\quad + \left(-\tfrac{1}{80}Q_0[f^1] + \tfrac{1}{8}Q_1[f^2] - \tfrac{5}{72}Q_2[f^3] - \tfrac{1}{3}Q_3[f^4] - \tfrac{8}{9}Q_4[f^5]\right)R_{\mu\nu}R^{\mu\nu}\\
&\quad + \tfrac{1}{30}Q_0[f^1]\,R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Big\} + \mathcal{O}(\mathcal{R}^3)\,.
\end{aligned}
\tag{5.1.31}
$$

Combining the Universal Contributions

In total, the gauge fixing, ghost and auxiliary terms contribute with

$$
\begin{aligned}
\mathcal{S}^{\text{uni}} &= \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,\Big\{(\tfrac{9}{2}a_{\text{gf}} - 5a_{\text{gh}} - \tfrac{3}{2}a_{\text{aux}})Q_2[f^1]\\
&\qquad + R\Big((\tfrac{1}{2}a_{\text{gf}} - \tfrac{7}{12}a_{\text{gh}} - \tfrac{1}{8}a_{\text{aux}})Q_1[f^1] + (-\tfrac{3}{4}a_{\text{gh}} - \tfrac{3}{8}a_{\text{aux}})Q_2[f^2]\\
&\qquad + (3a_{\text{gf}} - \tfrac{4}{3}a_{\text{gh}} - \tfrac{2}{3}a_{\text{aux}})Q_3[f^3] + 6a_{\text{gf}}\,Q_4[f^4]\Big)\Big\}\\
&= \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,\Big\{k^4\left(\tfrac{9}{2}a_{\text{gf}} - 5a_{\text{gh}} - \tfrac{3}{2}a_{\text{aux}}\right) + R\,k^2\left(\tfrac{5}{2}a_{\text{gf}} - \tfrac{85}{36}a_{\text{gh}} - \tfrac{61}{72}a_{\text{aux}}\right)\Big\}\,.
\end{aligned}
\tag{5.1.32}
$$

Here we substituted the threshold functions (C.7) and evaluated them for the optimized cutoff (C.12) in the second line. The coefficients $a_{\text{gf}}, a_{\text{gh}}, a_{\text{aux}} \equiv 1$ are introduced here only for the purpose of tracing the origin of the full contribution back to the corresponding terms in the action. This allows for a more systematic analysis of the structure of the resulting β functions.

We can see here that the gauge fixing and ghost terms are almost cancelling each other, so that the complete result for the universal part is well approximated by the auxiliary contributions alone. We argue that this feature is not accidental, but a remnant of the BRST supersymmetry discussed in section 4.1. Indeed, the quadratic parts G and $\mathcal{M}$ of the gauge fixing action in (5.1.7) and the ghost action in (5.1.18) satisfy the relation $G = \mathcal{M}^2$. More generally, for a gauge fixing condition $F[A] = \mathcal{F}\, A_g = \mathcal{F}\,\Pi_g\, A$ with a projector on the gauge dependent components Π_g, the quadratic form of the gauge fixing action will be of the form $G = \Pi_g \mathcal{F}^\dagger \mathcal{F} \Pi_g$. Comparing with (5.1.16), a corresponding Faddeev-Popov operator can be written as $\mathcal{M} = \mathcal{F}\,\Pi_g$, because a gauge transformation must act as a projector on the pure gauge part. Therefore, the relation remains valid for other choices of gauge fixing conditions, and implies

$$\frac{1}{2}\mathrm{Tr}(G+\mathcal{R}_G)^{-1}\partial_t\mathcal{R}_G - \mathrm{Tr}(\mathcal{M}+\mathcal{R}_\mathcal{M})^{-1}\partial_t\mathcal{R}_\mathcal{M} \approx 0\,, \tag{5.1.33}$$

for the gauge and ghost traces in (4.3.36). The cancellation becomes exact only if the cutoffs are chosen so that $(G+\mathcal{R}_G) = (\mathcal{M}+\mathcal{R}_\mathcal{M})^2$, which is a consequence of the breaking of BRST invariance as witnessed in (4.1.11). This condition could always be implemented by setting $\mathcal{R}_G = \mathcal{R}_\mathcal{M}(2\mathcal{M}+\mathcal{R}_\mathcal{M})$. However, this strongly suggests that in general the combined contribution of the gauge fixing and ghost sector should never have strong influence on physically relevant features. The full contribution of all universal sectors, as it is found in the computation above reads

$$\mathcal{S}^{\mathrm{uni}} = \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,\Big\{ -2Q_2[f^1] + R\Big(-\tfrac{5}{24}Q_1[f^1] - \tfrac{9}{8}Q_2[f^2] + Q_3[f^3] + 6Q_4[f^4]\Big)\Big\}\,. \tag{5.1.34}$$

Note that this does not coincide with the result found in [27, 57] because of the difference in cutoff scheme implementation and gauge choice.

5.1.2. Analysis of the Gravitational Renormalization Group Flow

The contribution of the physical fields in the gravitational sector denoted as $\mathcal{S}^{\mathrm{grav}}$ in (5.1.5), with the ansatz (5.1.1) for Γ_k^{grav}, is now determined following the solution routine for the RG equation (4.3.1) laid out in section 4.3. In terms of the coupling constants (5.1.2), we will here consider the Einstein-Hilbert form

$$\Gamma_k^{\mathrm{grav}}[g] = \int d^4x\sqrt{g}\,\Big(u_0 + u_1 R\Big)\,. \tag{5.1.35}$$

After expanding to second order in the fluctuation field $h_{\mu\nu}$, and subsequently carrying out the minimal transverse-traceless decomposition (4.1.34), we find the quadratic part

$$\Gamma_k^{\mathrm{grav,quad}} = \frac{1}{2}\int d^4x\sqrt{g}\left\{h^{\mathrm{T}}_{\alpha\beta}\Big[\mathcal{K}_{2\mathrm{T}}^{\alpha\beta\mu\nu} + \mathcal{V}_{2\mathrm{T}}^{\alpha\beta\mu\nu}\Big]\,h^{\mathrm{T}}_{\mu\nu} + h\Big[\mathcal{K}_0 + \mathcal{V}_0\Big]h\right\}. \tag{5.1.36}$$

Here only the terms with the transverse-traceless tensor $h^{\mathrm{T}}_{\mu\nu}$ and the scalar trace h are retained, with the vector component ξ_μ dropping out due to the decoupling for $\alpha = 0$ as explained in section 4.3. Neglecting the bar on background quantities, the operators in (5.1.36) are given by

$$\begin{aligned}
\mathcal{K}_{2\mathrm{T}}^{\alpha\beta\mu\nu} &= -g^{\alpha\mu}g^{\beta\nu}\,(u_1\Delta + u_0)\;, \\
\mathcal{V}_{2\mathrm{T}}^{\alpha\beta\mu\nu} &= u_1(-g^{\alpha\mu}g^{\beta\nu}R + 2R^{\alpha\mu}g^{\beta\nu} + 2R^{\alpha\mu\beta\nu})\,, \\
\mathcal{K}_0 &= \tfrac{3}{8}u_1\Delta + \tfrac{1}{4}u_0\,, \\
\mathcal{V}_0 &= 0\,.
\end{aligned} \tag{5.1.37}$$

Notably, there are no cross-terms and a potential is only present in the transverse part. The operators $\mathcal{K}_{2\mathrm{T}}$ and $\mathcal{V}_{2\mathrm{T}}$ are understood as restricted to the transverse-traceless subspace via the projection operator (4.1.41). Accordingly, this projector has to be included when computing functional traces in the following.

For the gravitational sector, the matrix valued IR cutoff $\mathcal{R}_k$ introduced by a modification of the action corresponding to (2.3.3) assumes the form

$$\Delta S_k = \frac{1}{2}\int d^4x\sqrt{g}\left\{h^{\mathrm{T}}_{\alpha\beta}\,[\mathcal{R}_k^{2\mathrm{T}}]^{\alpha\beta\mu\nu}\,h^{\mathrm{T}}_{\mu\nu} + h\,\mathcal{R}_k^0\,h\right\}. \tag{5.1.38}$$

According to the scheme (4.3.13), only the kinetic terms $\mathcal{K}_i$ are regulated, so that there are no curvature terms entering $\mathcal{R}_k$. For the operators (5.1.37), this is achieved by

$$[\mathcal{R}_k^{2\mathrm{T}}]^{\alpha\beta\mu\nu} = -\,g^{\alpha\mu}g^{\beta\nu}\,u_1\,R_k\,, \qquad \mathcal{R}_k^0 = \tfrac{3}{8}u_1 R_k\,. \tag{5.1.39}$$

This way, the inclusion of $\mathcal{R}_k$ results in replacing

$$\mathcal{K}_{2\mathrm{T}} \to \mathcal{P}_{2\mathrm{T}} = \mathcal{K}_{2\mathrm{T}} + \mathcal{R}_k^{2\mathrm{T}}\,, \qquad \mathcal{K}_0 \to \mathcal{P}_0 = \mathcal{K}_0 + \mathcal{R}_k^0\,, \tag{5.1.40}$$

which, instead of Δ, now includes the regulated inverse propagators $P_k = \Delta + R_k$ as arguments, with $R_k(\Delta)$ denoting the yet undetermined scalar shape function. Note that the kinetic operators $\mathcal{K}_{2\mathrm{T}}$ and $\mathcal{K}_0$ have opposite signs, corresponding to the conformal instability in euclidean gravity. This circumstance does not pose any problem in the context of the RG equation, since it is insensitive to any constant pre-factors, which cancels between $(\Gamma_k^{(2)} + \mathcal{R}_k)$ and $\partial_t \mathcal{R}_k$. Thus it will have all fields contribute to the β functions with the sign determined by their bosonic or fermionic nature.

The full regulated propagator $\left[\Gamma_k^{(2)} + \mathcal{R}_k\right]^{-1}$ is found from the inversion formula (4.3.17). The contribution of the gravitational sector in the RG equation then assumes the form

$$\mathcal{S}^{\text{grav}} = \mathcal{T}_{2\text{T}} + \mathcal{T}_0 + \mathcal{O}(\mathcal{R}^2)\,, \tag{5.1.41}$$

decomposing into the transverse-traceless tensor and scalar parts. The traces are given by

$$\mathcal{T}_{2\text{T}} \;=\; \frac{1}{2}\text{Tr}_2 \left[\Pi_{2\text{T}} \left(\frac{1}{\mathcal{P}_{2\text{T}}} - \frac{1}{\mathcal{P}_{2\text{T}}}\Pi_{2\text{T}}\, \mathcal{V}_{2\text{T}}\, \Pi_{2\text{T}} \frac{1}{\mathcal{P}_{2\text{T}}}\right) \Pi_{2\text{T}}\; \partial_t \mathcal{R}_k^{2\text{T}}\right]\,, \tag{5.1.42}$$

and

$$\mathcal{T}_0 \;=\; \frac{1}{2}\text{Tr}_2 \left[\Pi_{\text{tr}}\, \frac{1}{\mathcal{P}_0}\, \Pi_{\text{tr}}\; \partial_t \mathcal{R}_k^0\right] \;=\; \frac{1}{2}\text{Tr}_0 \left[\frac{1}{\mathcal{P}_0}\; \partial_t \mathcal{R}_k^0\right]\,, \tag{5.1.43}$$

respectively. In order to identify the contributions to the running of the coupling constants u_0 and u_1 in (5.1.35), it is sufficient to expand only to linear order in the curvature, so all terms of the order $(\mathcal{V}_{2\text{T}})^2$ and higher do not contribute in the present computation. The projection operators Π_{tr} and $\Pi_{2\text{T}}$ are defined in (4.1.37) and (4.1.41) and ensure that all terms are restricted to their proper subspaces.

Since the scalar trace (5.1.43) contains only Laplacian operators, it is given in terms of the heat kernel coefficients summarized in table 4.2. The tensorial operator trace (5.1.42) consists of the two terms

$$\begin{aligned}
\mathcal{T}_{2\text{T}}^1 &= \frac{1}{2}\text{Tr}_2 \left[\, \Pi_{2\text{T}}\, \frac{1}{P_{2\text{T}}}\, \Pi_{2\text{T}}\; \partial_t R_k^{2\text{T}} \,\right]\,, \\
\mathcal{T}_{2\text{T}}^2 &= -\,\frac{1}{2}\text{Tr}_2 \left[\, \Pi_{2\text{T}}\, \frac{1}{P_{2\text{T}}}\, \Pi_{2\text{T}}\, \mathcal{V}_{2\text{T}}\, \Pi_{2\text{T}}\, \frac{1}{P_{2\text{T}}}\, \Pi_{2\text{T}}\; \partial_t R_k^{2\text{T}} \,\right]\,.
\end{aligned} \tag{5.1.44}$$

In the first of these traces the projectors can be moved together, creating a commutator term

$$\frac{1}{2}\text{Tr}_2 \left[\, \Pi_{2\text{T}}\, \left[\frac{1}{P_{2\text{T}}}, \Pi_{2\text{T}}\right]\; \partial_t R_k^{2\text{T}} \,\right] = \mathcal{O}(\mathcal{R}^2)\,, \tag{5.1.45}$$

which vanishes at first order in the curvature since it is trace-free. To see this, we compute

$$[\Pi_{2\text{T}}]_{\mu\nu}{}^{\rho\sigma} \left[\, [\Pi_{2\text{T}}]_{\rho\sigma}{}^{\alpha\beta}, f(\Delta)\right] h_{\alpha\beta} \;=\; f'(\Delta)\; [C_{2\text{T}}]_{\mu\nu}{}^{\alpha\beta}\, h_{\alpha\beta} + \mathcal{O}(\mathcal{R}^2)\,, \tag{5.1.46}$$

with use of the formula (B.2.4), yielding

$$[C_{2\text{T}}]_{\mu\nu}{}^{\alpha\beta} \;=\; [\Pi_{2\text{T}}]_{\mu\nu}{}^{\rho\sigma} \left(-2R_{(\rho}{}^{\lambda}\delta_{\sigma)}^{(\alpha}D^{\beta)}D_\lambda + 4R^{\lambda}{}_{(\rho}{}^{\tau}{}_{\sigma)}D_\lambda\delta_\tau^{(\alpha}D^{\beta)}\right) \frac{1}{\Delta}\,. \tag{5.1.47}$$

This expression is obtained from the first term in (4.1.47), with all other terms being proportional to either $g_{\alpha\beta}$, D_α, or D_β and are therefore annihilated when contracted with

$\Pi_{2\mathrm{T}}$. Finally we can verify

$$\mathrm{tr}_2 \left[C_{2\mathrm{T}}\right]_{\mu\nu}{}^{\alpha\beta} = \tfrac{1}{2}\left(\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\beta \delta^\nu_\alpha\right) \left[C_{2\mathrm{T}}\right]_{\mu\nu}{}^{\alpha\beta} = 0\,, \tag{5.1.48}$$

ensuring that the commutator does not contribute in the present computation, and the partial trace can be written as

$$\mathcal{T}^1_{2\mathrm{T}} = \frac{1}{2}\mathrm{Tr}_2\left[\frac{1}{P_{2\mathrm{T}}}\,\partial_t R_k^{2\mathrm{T}}\,\Pi_{2\mathrm{T}}\right]. \tag{5.1.49}$$

This expression is now easily evaluated with the heat kernel coefficients for constrained tensor fields, given in (4.2.41). In the remaining piece $\mathcal{T}^2_{2\mathrm{T}}$ it is sufficient for the present computation to replace the projector with its zero-order part (4.1.47) and to commute all covariant derivatives freely, since $\mathcal{V}_{2\mathrm{T}}$ already contains one power of the curvature. Thus we can write

$$\mathcal{T}^2_{2\mathrm{T}} = -\frac{1}{2}\mathrm{Tr}_2\left[\frac{1}{P^2_{2\mathrm{T}}}\,\partial_t R_k^{2\mathrm{T}}\,\mathcal{V}_{2\mathrm{T}}\,\Pi^{0r}_{2\mathrm{T}}\right] + \mathcal{O}(\mathcal{R}^2)\,, \tag{5.1.50}$$

which requires the off-diagonal heat kernel scheme (3.3.7) for its evaluation.

The resulting partial contributions from the gravitational action (5.1.35) are given as

$$\begin{aligned}
\mathcal{T}_0 &= \frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left\{Q_2[f^1_0] + \tfrac{1}{6}\,Q_1[f^1_0]\,R\right\},\\
\mathcal{T}^1_{2\mathrm{T}} &= \frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left\{5\,Q_2[f^1_{2\mathrm{T}}] - \tfrac{5}{6}\,Q_1[f^1_{2\mathrm{T}}]\,R\right\},\\
\mathcal{T}^2_{2\mathrm{T}} &= \frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left\{\tfrac{10}{3}u_1\,Q_2[f^2_{2\mathrm{T}}]\,R\right\},
\end{aligned} \tag{5.1.51}$$

with the scalar functions of the Laplacian under the operator traces being captured by

$$\begin{aligned}
f^n_{2\mathrm{T}}(\Delta) &:= \frac{1}{(\mathcal{P}_{2\mathrm{T}})^n}\partial_t\mathcal{R}_k^{2\mathrm{T}} = \frac{-\partial_t(u_1 R_k)}{(-u_1 P_k - u_0)^n}\,,\\
f^n_0(\Delta) &:= \frac{1}{(\mathcal{P}_0)^n}\partial_t\mathcal{R}_k^0 = \frac{\tfrac{3}{8}\,\partial_t(u_1 R_k)}{\left(\tfrac{3}{8}u_1 P_k + \tfrac{1}{4}u_0\right)^n}\,.
\end{aligned} \tag{5.1.52}$$

In terms of these functions, the total result for the gravitational contribution reads

$$\begin{aligned}
\mathcal{S}^{\mathrm{grav}} = \frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,\Big\{&\left(Q_2[f^1_0] + 5Q_2[f^1_{2\mathrm{T}}]\right)\\
&+ R\left(\tfrac{1}{6}Q_1[f^1_0] - \tfrac{5}{6}Q_1[f^1_{2\mathrm{T}}] + \tfrac{10}{3}u_1 Q_2[f^2_{2\mathrm{T}}]\right)\Big\}.
\end{aligned} \tag{5.1.53}$$

Notably, this result for the RG flow of the Einstein-Hilbert truncation coincides with that obtained in earlier computations [57], where the background metric was set to be the one of the four-sphere and only the standard heat kernel expansion was used. The direct comparison provides confirmation for the new techniques, whose virtue lies in their generalized applicability, not requiring for any such specific choice. The demonstrated derivation with the background metric $g_{\mu\nu}$ left completely unspecified demonstrates explicitly the background-independence of the result (5.1.53).

To arrive at the final expression for $\partial_t \Gamma_k$ in (5.1.5), we combine the result from the gravitational part (5.1.53) with the universal contributions given in (5.1.34). The β functions governing the running of the cosmological constant and Newton's constant are found by projection on the monomials $I_0 = \sqrt{g}$ and $I_1 = \sqrt{g}R$ according to (4.3.46). This way we find the scale derivatives

$$\begin{aligned}\partial_t u_0 &= \frac{1}{(4\pi)^2}\left(\tfrac{1}{2}Q_2[f_0^1] + \tfrac{5}{2}Q_2[f_{2\mathrm{T}}^1] - 2Q_2[f^1]\right),\\ \partial_t u_1 &= \frac{1}{(4\pi)^2}\left(\tfrac{1}{12}Q_1[f_0^1] - \tfrac{5}{12}Q_1[f_{2\mathrm{T}}^1] + \tfrac{5}{3}u_1 Q_2[f_{2\mathrm{T}}^2]\right.\\ &\qquad\left. - \tfrac{5}{24}Q_1[f^1] - \tfrac{9}{8}Q_2[f^2] + Q_3[f^3] + 6Q_4[f^4]\right).\end{aligned} \tag{5.1.54}$$

Switching to the dimensionless constants (5.1.3), and using the identities (C.7) to express the functionals $Q_n[f]$ in terms of the threshold functions $\Phi_n^p(\omega)$ defined in appendix C, the β functions assume the form

$$\begin{aligned}\partial_t g_0 &= -\,4g_0 + \tfrac{1}{(4\pi)^2}A_0(\tfrac{g_0}{g_1}) + \tfrac{1}{(4\pi)^2}\left(1 + \tfrac{1}{2}\tfrac{\partial_t g_1}{g_1}\right)B_0(\tfrac{g_0}{g_1}),\\ \partial_t g_1 &= -\,2g_1 + \tfrac{1}{(4\pi)^2}A_1(\tfrac{g_0}{g_1}) + \tfrac{1}{(4\pi)^2}\left(1 + \tfrac{1}{2}\tfrac{\partial_t g_1}{g_1}\right)B_1(\tfrac{g_0}{g_1}),\end{aligned} \tag{5.1.55}$$

with the cutoff shape dependent functions given by

$$\begin{aligned}A_0(\lambda) &= 5\Phi_2^1(-2\lambda) + \Phi_2^1(-\tfrac{4}{3}\lambda) - 4\Phi_2^1(0),\\ B_0(\lambda) &= 5\widetilde{\Phi}_2^1(-2\lambda) + \widetilde{\Phi}_2^1(-\tfrac{4}{3}\lambda),\\ A_1(\lambda) &= -\,\tfrac{5}{6}\Phi_1^1(-2\lambda) + \tfrac{1}{6}\Phi_1^1(-\tfrac{4}{3}\lambda) - \tfrac{10}{3}\Phi_2^2(-2\lambda)\\ &\qquad - \tfrac{5}{12}\Phi_1^1(0) - \tfrac{9}{4}\Phi_2^2(0) + 2\Phi_3^3(0) + 12\Phi_4^4(0),\\ B_1(\lambda) &= -\,\tfrac{5}{6}\widetilde{\Phi}_1^1(-2\lambda) + \tfrac{1}{6}\widetilde{\Phi}_1^1(-\tfrac{4}{3}\lambda) - \tfrac{10}{3}\widetilde{\Phi}_2^2(-2\lambda).\end{aligned} \tag{5.1.56}$$

Finally, the expressions (5.1.55) can be solved for the β functions for the dimensionless Newton's constant $g = k^2 G_k$ and cosmological constant $\lambda = k^{-2}\Lambda_k$. Introducing the anomalous dimension of Newton's constant, defined as

$$\eta_N = -\frac{\partial_t u_1}{u_1} = \frac{2gA_1(\lambda)}{2\pi + gB_1(\lambda)}, \tag{5.1.57}$$

we find the relations

$$\begin{aligned}\partial_t\lambda &= (\eta_N - 2)\lambda + \frac{g}{2\pi}\left(A_0(\lambda) - \tfrac{1}{2}\eta_N B_0(\lambda)\right)\,,\\ \partial_t g &= (\eta_N + 2)g\,.\end{aligned} \tag{5.1.58}$$

In such a system of β functions, the origin of every term can be traced to the field monomials present in the effective action (5.1.35) by their coupling constants. The simplest ansatz for a non-trivial gravitational action is realized with only a scalar curvature term. By neglecting all contributions of the cosmological constant (setting $\lambda = 0$), we can reproduce this case. With only the coupling constant g remaining, there is a unique gravitational non-Gaussian fixed point (NGFP) at

$$g^* = -\frac{2\pi}{A_1(0) + B_1(0)}\,. \tag{5.1.59}$$

Evaluation with a Fermi-type cutoff (C.14) with $T^{-1} = 100$, and with optimized cutoff (C.10) gives the fixed point values

$$g^*\Big|_{\text{Fermi}} = 1.15827\,, \qquad g^*\Big|_{\text{opt}} = 1.59855\,, \tag{5.1.60}$$

respectively. The corresponding critical exponent can also be found algebraically as

$$\theta = -\frac{\partial(\partial_t g)}{\partial g}(g^*) = 2\left(1 + \frac{B_1(0)}{A_1(0)}\right)\,, \tag{5.1.61}$$

which has a value of $\theta = 2.521646$ with the Fermi-cutoff, and $\theta = 2.58447$ with the optimized cutoff. Surprisingly, a fixed point with similar exponent can be found in much more complicated computations, where contributions of higher curvature terms mix with those present here, and do not allow for an algebraic solution. A more detailed study of the RG flow including a cosmological constant and a non-trivial ghost field renormalization is given in the next section.

Examining the full set of β functions (5.1.58), one verifies that with an anomalous dimension $\eta_N \neq -2$, the only fixed point is the Gaussian one at $(g = 0, \lambda = 0)$. Any non-trivial solution of the fixed point equations $\partial_t\lambda = 0$, $\partial_t g = 0$ thus has to satisfy $\eta_N = -2$, which allows to eliminate the coupling constant g. This way we find the condition

$$A_0(\lambda) + B_0(\lambda) + 4\lambda\Big(A_1(\lambda) + B_1(\lambda)\Big) = 0\,, \tag{5.1.62}$$

in terms of λ alone. For every solution of this equation λ^*, there is a value for $g^* = -\frac{2\pi}{A_1(\lambda^*)+B_1(\lambda^*)}$, such that the pair corresponds to a fixed point. For instructional purposes alone, we cast the condition (5.1.62) into numerical form, substituting (5.1.56) and using the optimized cutoff (C.12). Assuming ($\lambda \neq 1/2$) and ($\lambda \neq 3/4$) to get rid of the denominators, we obtain

$$\begin{aligned} &180a_{2\mathrm{T}} + 36a_0 + 243a_{\mathrm{gf}} - 270a_{\mathrm{gh}} - 81a_{\mathrm{aux}} \\ &+\lambda\Big(- 1350a_{2\mathrm{T}} - 90a_0 - 756a_{\mathrm{gf}} + 930a_{\mathrm{gh}} + 249a_{\mathrm{aux}}\Big) \\ &+\lambda^2\Big(+ 2020a_{2\mathrm{T}} - 72a_0 - 612a_{\mathrm{gf}} + 200a_{\mathrm{gh}} + 220a_{\mathrm{aux}}\Big) \\ &+\lambda^3\Big(- 720a_{2\mathrm{T}} + 216a_0 + 3744a_{\mathrm{gf}} - 3320a_{\mathrm{gh}} - 1276a_{\mathrm{aux}}\Big) \\ &+\lambda^4\Big(- 2880a_{\mathrm{gf}} + 2720a_{\mathrm{gh}} + 976a_{\mathrm{aux}}\Big) = 0\,, \end{aligned} \tag{5.1.63}$$

with the coefficients a_i indicating the origin of the corresponding contribution. As already observed in (5.1.32) the gauge and ghost contributions (multiplied by a_{gf} and a_{gh}) typically cancel otherwise significant factors mutually. For small λ the transverse-traceless sector ($a_{2\mathrm{T}}$) clearly dominates, as one would expect from a genuinely gravitational feature, while the conformal scalar mode (a_0) is mostly negligible.

The positions and critical exponents for all real valued fixed points found in the full system of β functions (5.1.58) or under neglection of specific contributions are respectively shown in table 5.1. The fixed point in the first line generalizes the NGFP (5.1.59). Note that here the critical exponents become a complex pair of numbers. Furthermore, a second fixed point appears with the inclusion of the scalar contributions. In general there may be more fixed point solutions in extended systems of β functions. Typically, these additional fixed points can however be identified as artifacts of the computation by studying their dependence on unphysical parameters as well as their critical exponents. It is in contrast to these spurious fixed points that a unique gravitational fixed point is to be considered a robust feature of gravitational quantum theories investigated this far.

Of course the genuine spin-2 degree of freedom in form of the transverse-traceless contribution is what gives rise to the fixed point. The remaining terms are important corrections that will have to be taken into account for any computation in the full theory. However, to understand the mechanism behind the stability of the fixed point, the situation becomes more transparent when systematically tracing the effect of individual terms. This technique of analysis may shed some light onto the mechanism underlying the emergence of the non-Gaussian fixed point, when applied to more sophisticated computations including higher orders of the curvature tensor. An NGFP was already found in a number of different computational schemes [35, 45–47, 58], but it is for the generalizability of the method used here that allows this new approach, going beyond a numerical comparison. Since gauge fixing and cutoff dependences will always change the

	g^*	λ^*	$g^*\lambda^*$	θ
all $a_i = 1$	1.0021	0.134414	0.134696	$2.37141 \pm 2.27954\,\mathrm{i}$
	2.31123	0.721226	1.66691	$3.55617\,,\ -289.16$
$a_0 = 0$	1.11987	0.0936343	0.104858	$2.42937 \pm 1.86549\,\mathrm{i}$
$a_{\mathrm{gf}} = a_{\mathrm{gh}} = 0$	0.918917	0.149409	0.137294	$2.49265 \pm 2.34403\,\mathrm{i}$
	2.18328	0.721161	1.5745	$3.50987\,,\ -277.625$
only $a_{2\mathrm{T}} = 1$	0.863188	0.177487	0.153205	$2.92881 \pm 2.61774\,\mathrm{i}$
	-39.0277	1.87807	-73.2967	$5.28929\,,\ 10.2049$

Table 5.1.: Fixed point values of the Einstein-Hilbert truncation for dimensionless Newton's constant g, and cosmological constant λ, as well as their product and associated critical exponents θ. In the full systems of β functions (first line), a second unphysical non-Gaussian fixed point appears as a result of the approximation. This fixed point vanishes if the contributions of the scalar gravitational mode is neglected (second line). The real parts of the critical exponents reveal that a fixed point with similar properties as in the one coupling approximation (5.1.59) persists in the full system, but also when taking only the transverse-traceless contributions into account (see last line). The neglection of gauge fixing and ghost terms (third line) changes very little.

results slightly, it is important to engage in a more systematic investigation of the validity of the asymptotic safety scenario.[4] The present analysis provides a further contribution in this direction.

5.2. Running Ghost Field Renormalization

In this section the purely gravitational renormalization group flow will be extended to include the quantum effects captured by the field renormalization of the Faddeev-Popov ghosts. Since a ghost term is essential for the standard quantization of non-abelian gauge fields, its back-reaction to the physical coupling constants needs to be taken into account for consistency. Specifically we will augment the Einstein-Hilbert ansatz discussed in the

[4] See [125] for a related study.

last section by the power-counting marginal field renormalization of the ghost fields.

A strong motivation to engage in this ghost-improved computation originates from the analogy to QCD where the interplay of the gluon and ghost scaling behaviour is essential for the IR physics of the theory [88, 127–129]. Specifically, the phenomenon of ghost enhancement is an integral part of the Gribov-Zwanziger confinement criterion and related to the anti-screening behaviour of QCD. While it is clear, that there is also a non-trivial interplay between the ghosts and the metric fluctuations in gravity, its conceptual role is much less clear. Here we compute the anomalous dimension of the ghost propagator and study its effect on the running of Newton's constant and the cosmological constant to contribute to an improved understanding.

Effects of the ghost sector in quantum gravity where first considered in [130], followed by a comparable computation of the ghost anomalous dimension using a spectrally adjusted cutoff and a flat-space projection technique in [131]. Using off-diagonal heat kernel methods to solve the RG equation as explained in section 4.3, we are able to derive our results under more general conditions, allowing for a systematic analysis.

5.2.1. Derivation of the β Functions

To compute the gravitational β functions in the ghost-improved Einstein-Hilbert truncation, we use an ansatz for the effective average action containing the three terms

$$\Gamma_k[g, C, \bar{C}; \bar{g}, c, \bar{c}] = \Gamma_k^{\text{grav}}[g] + S^{\text{gf}}[g; \bar{g}] + S_k^{\text{ghost}}[g, C, \bar{C}; \bar{g}, c, \bar{c}] , \tag{5.2.1}$$

depending on the metric $g_{\mu\nu}$ and the average ghost fields $C, \bar{C}$ and their corresponding background fields $\bar{g}$ and $c, \bar{c}$. They are related by

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} , \qquad C_\mu = c_\mu + f_\mu , \qquad \bar{C}_\mu = \bar{c}_\mu + \bar{f}_\mu , \tag{5.2.2}$$

where $h_{\mu\nu}$ and $f_\mu, \bar{f}_\mu$ denote the expectation value of the quantum fluctuations around the background. Note that these fluctuations are not required to be small in any sense. In a purely gravitational approximation as exemplified in the last section, computations are simplified by setting the background ghost fields to zero. This does however not allow to keep track of the ghost kinetic term, so that we must use a non-trivial ghost background in the following. Due to the presence of these fermionic fields the super-trace in the RG equation (4.3.1) is important to take the different statistics into account.

The gravitational part Γ_k^{grav} is again taken to be of the Einstein-Hilbert form

$$\Gamma_k^{\text{grav}}[g] = 2\,\kappa^2\, Z_k^N \int \mathrm{d}^d x \sqrt{g}\, \left(-R + 2\Lambda_k\right) , \tag{5.2.3}$$

where $\kappa^2 = (32\pi G_0)^{-1}$ with G_0 is a fixed reference scale and Z_k^N denotes the field renormalization for the graviton.[5] This way we express the running Newton's constant $G_k = G_0/Z_k^N$ in analogous terms to the ghost field renormalization Z_k^c. The action is supplemented by the gauge fixing term $S^{\text{gf}}[g;\bar{g}]$ according to (4.1.17) with the harmonic gauge (5.1.6). Since the decoupling of physical and gauge degrees of freedom is not as useful for the computation in this section, here we choose the gauge fixing parameters $\alpha = 1$ and $\beta = 1/2$ for direct comparability with older results [35, 46]. Thus we have

$$S^{\text{gf}}[h;\bar{g}] = \kappa^2\, Z_k^N \int\!\mathrm{d}^d x\sqrt{\bar{g}}\;\bar{g}^{\mu\nu}F_\mu F_\nu\,, \qquad F_\mu = \bar{D}^\rho h_{\rho\mu} - \tfrac{1}{2}\bar{D}_\mu \bar{g}^{\alpha\beta}h_{\alpha\beta}\,. \tag{5.2.4}$$

With this gauge fixing scaled by the pre-factor of the gravitational action, all non-minimal derivative terms in its second variation are cancelled. The corresponding Faddeev-Popov determinant is captured by the ghost term

$$S_k^{\text{ghost}}[g, C, \bar{C}; \bar{g}, c, \bar{c}] = -\sqrt{2}\, Z_k^c \int\!\mathrm{d}^d x\sqrt{\bar{g}}\;\bar{C}_\mu\,\mathcal{M}^\mu{}_\nu\, C^\nu\,, \tag{5.2.5}$$

containing the field renormalization of the ghosts Z_k^c and the operator

$$\mathcal{M}^\mu{}_\nu = \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda(g_{\rho\nu}D_\sigma + g_{\sigma\nu}D_\rho) - \bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda(g_{\sigma\nu}D_\rho)\,. \tag{5.2.6}$$

The gauge-choice (5.2.4) has the main virtue, that it allows for a straightforward comparison to earlier results obtained in the Einstein-Hilbert truncation without ghost-improvement, by setting $Z_k^c = 1$. Taking the ∂_t-derivative of the ansatz (5.2.1) and setting the fluctuation fields to zero afterwards yields

$$\partial_t \Gamma_k = 2\kappa^2 \int d^d x\sqrt{\bar{g}}\left[-(\partial_t Z_k^N)\bar{R} + 2\,\partial_t(Z_k^N \Lambda_k)\right] - \sqrt{2}(\partial_t Z_k^c)\int d^d x\sqrt{\bar{g}}\,\bar{c}^\mu \bar{D}^2 c_\mu\,. \tag{5.2.7}$$

In the ghost term we identify only the minimal kinetic part, which is sufficient for the present purpose. Thus to identify the interaction monomials whose coefficients encode the running of the coupling constants Z_k^N, Z_k^c and Λ_k, it suffices to keep the Einstein-Hilbert monomials and the ghost kinetic term. The projection of these terms as indicated in the schematic solution of the RG equation (4.3.45) yields the desired β functions.

Since non-minimal operators do not appear with the gauge choice (5.2.4), it is convenient to decompose the metric fluctuations $h_{\mu\nu}$ only into their traceless and trace part

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{d}\bar{g}_{\mu\nu}h\,, \qquad h = \bar{g}^{\mu\nu}h_{\mu\nu}\,, \qquad \bar{g}^{\mu\nu}\hat{h}_{\mu\nu} = 0\,. \tag{5.2.8}$$

[5] More precisely, the coupling constant Z_k^N encodes the running of the background Newton's constant in the present context.

A projector on the traceless part is given by

$$[\mathbf{1}_{2\mathrm{t}}]_{\mu\nu}{}^{\rho\sigma} \;=\; \tfrac{1}{2}\Big(\delta_\mu{}^\rho\delta_\nu{}^\sigma + \delta_\mu{}^\sigma\delta_\nu{}^\rho\Big) - \tfrac{1}{d}g_{\mu\nu}g^{\rho\sigma}\,, \tag{5.2.9}$$

satisfying $\hat{h}_{\mu\nu} \;=\; [\mathbf{1}_{2\mathrm{t}}]_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma}$, as it follows from (4.1.37). The second variation of Γ_k in (5.2.1) with respect to these fluctuations was already given in [35]. For $\Gamma_k^{\mathrm{grav}} + S^{\mathrm{gf}}$ the result is

$$\Gamma_k^{\mathrm{grav,quad}} = \tfrac{1}{2}\kappa^2 Z_k^N \int\!\mathrm{d}^d x\sqrt{\bar{g}}\;\Big\{\hat{h}_{\mu\nu}\Big[\Delta - 2\Lambda_k + C_T\bar{R}\Big]\hat{h}^{\mu\nu} - \tfrac{d-2}{2d}h\Big[\Delta - 2\Lambda_k + C_S\bar{R}\Big]h\Big\}\,, \tag{5.2.10}$$

with

$$C_T = \frac{d^2-3d+4}{d(d-1)}\,, \qquad C_S = \frac{d-4}{d}\,, \tag{5.2.11}$$

and $\Delta = -\bar{D}^2$ denoting the covariant Laplacian constructed from the background metric. This result is given with the background metric $\bar{g}$ chosen as the one of a d-dimensional sphere (A.2.5), which is general enough to distinguish the interaction monomials in the Einstein-Hilbert ansatz (5.2.1).

The new contributions originate from the ghost action and are of a more complicated form owed to the non-trivial background ghost field. Thus we first give the intermediate results obtained from expanding (5.2.6) in $h_{\mu\nu}$ when contracted with a dummy vector C^ν

$$\begin{aligned}
\overline{\mathcal{M}}^{\mu}{}_{\nu}\,C^\nu &= \Big[\bar{D}^2 + \tfrac{1}{d}\,\bar{R}\Big]\;C^\mu\,,\\
\overline{\delta_h\mathcal{M}}^{\mu}{}_{\nu}\,C^\nu &= \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda\Big[h_{\rho\alpha}\bar{D}_\sigma + h_{\sigma\alpha}\bar{D}_\rho + (\bar{D}_\alpha h_{\rho\sigma})\Big]\,C^\alpha\\
&\quad - \bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda\Big[h_{\sigma\alpha}\bar{D}_\rho + \tfrac{1}{2}(\bar{D}_\alpha h_{\rho\sigma})\Big]\,C^\alpha\,,\\
\overline{\delta_h^2\mathcal{M}}^{\mu}{}_{\nu}\,C^\nu &= 0\,.
\end{aligned} \tag{5.2.12}$$

Here the bar indicates background quantities, obtained by setting $g = \bar{g}$. Remarkably, the second variation in the last line vanishes independently of the choice of background or gauge. Based on these results, we obtain the quadratic form in the ghost sector

$$\begin{aligned}
S_k^{\mathrm{ghost,quad}} &= \sqrt{2}Z_k^c\int\!\mathrm{d}^d x\sqrt{\bar{g}}\;\Big\{\bar{f}_\mu\Big[\Delta - \tfrac{1}{d}\bar{R}\Big]f^\mu - hA^\alpha\bar{f}_\alpha - h\bar{A}_\alpha f^\alpha\\
&\qquad\qquad - \hat{h}^{\mu\nu}Q_{\mu\nu}{}^\alpha\bar{f}_\alpha - \hat{h}_{\mu\nu}\bar{Q}^{\mu\nu}{}_\alpha f^\alpha\Big\}\\
&= \sqrt{2}Z_k^c\int\!\mathrm{d}^d x\sqrt{\bar{g}}\;\Big\{\bar{f}_\mu\Big[\Delta - \tfrac{1}{d}\bar{R}\Big]f^\mu - \bar{f}_\alpha\widetilde{A}^\alpha h - f^\alpha\widetilde{\bar{A}}_\alpha h\\
&\qquad\qquad - \bar{f}_\alpha\widetilde{Q}^\alpha{}_{\mu\nu}\hat{h}^{\mu\nu} - f^\alpha\widetilde{\bar{Q}}_\alpha{}^{\mu\nu}\hat{h}_{\mu\nu}\Big\}\,.
\end{aligned} \tag{5.2.13}$$

Here the two lines are equivalent up to surface terms and define the Grassmann-valued operators $A, \bar{A}, Q, \bar{Q}$, and their adjoints, respectively. To keep track of the ghost kinetic term it is sufficient to take the background ghost field as transversal

$$\bar{D}_\mu c^\mu = 0\,, \qquad \bar{D}_\mu \bar{c}^\mu = 0\,. \tag{5.2.14}$$

In the sequel, we resort to this choice of background to simplify all expressions. The explicit expressions read

$$\begin{aligned}
A^\alpha =& \frac{1}{d}\left[\bar{D}_\sigma c^\alpha \bar{D}^\sigma + \bar{D}^\alpha c^\sigma \bar{D}_\sigma - (1-\tfrac{d}{2}) c^\sigma \bar{D}_\sigma \bar{D}^\alpha\right]\,,\\
\bar{A}_\alpha =& -\frac{1}{d}\left[\bar{D}^\sigma \bar{c}_\alpha \bar{D}_\sigma + \bar{D}_\alpha \bar{c}^\sigma \bar{D}_\sigma\right]\,,\\
\widetilde{A}^\alpha =& \frac{1}{d}\left[\bar{D}^2 c^\alpha + \bar{R}^{\alpha\sigma} c_\sigma + \bar{D}_\sigma c^\alpha \bar{D}^\sigma + (2-\tfrac{d}{2})\bar{D}^\alpha c^\sigma \bar{D}_\sigma + (1-\tfrac{d}{2}) c^\sigma \bar{D}^\alpha \bar{D}_\sigma\right]\,,\\
\widetilde{\bar{A}}_\alpha =& -\frac{1}{d}\left[\bar{D}^2 \bar{c}_\alpha + \bar{D}^\sigma \bar{c}_\alpha \bar{D}_\sigma + \bar{R}_{\alpha\sigma}\bar{c}^\sigma + \bar{D}_\alpha \bar{c}^\sigma \bar{D}_\sigma\right]\,,
\end{aligned} \tag{5.2.15}$$

and

$$\begin{aligned}
Q_{\mu\nu}{}^\alpha =& \delta^\alpha_{(\mu}\bar{D}^\sigma c_{\nu)}\bar{D}_\sigma - c_\sigma \bar{D}^\sigma \bar{D}_{(\nu}\delta^\alpha_{\mu)} + \bar{D}^\alpha c_{(\mu}\bar{D}_{\nu)} - \bar{D}_{(\mu}c_{\nu)}\bar{D}^\alpha\\
&- \frac{1}{d}\bar{g}_{\mu\nu}\left[-c_\sigma \bar{D}^\sigma \bar{D}^\alpha + \bar{D}^\sigma c^\alpha \bar{D}_\sigma + \bar{D}^\alpha c^\sigma \bar{D}_\sigma\right]\,,\\
\bar{Q}^{\mu\nu}{}_\alpha =& \bar{D}_\alpha \bar{D}^{(\nu}\bar{c}^{\mu)} - \delta^{(\nu}_\alpha \bar{D}_\sigma \bar{c}^{\mu)}\bar{D}^\sigma - \delta^{(\nu}_\alpha \bar{D}^{\mu)}\bar{c}_\sigma \bar{D}^\sigma + \bar{D}^{(\nu}\bar{c}^{\mu)}\bar{D}_\alpha\\
&+ \frac{1}{d}\bar{g}^{\mu\nu}\left[\bar{D}^\sigma \bar{c}_\alpha \bar{D}_\sigma + \bar{D}_\alpha \bar{c}_\sigma \bar{D}^\sigma\right],\\
\widetilde{Q}^\alpha{}_{\mu\nu} =& \delta^\alpha_{(\mu}\bar{D}^2 c_{\nu)} + R_{(\mu}{}^\alpha{}_{\nu)}{}^\sigma c_\sigma + \delta^\alpha_{(\mu}c^\sigma \bar{D}_{\nu)}\bar{D}_\sigma\\
&+ \delta^\alpha_{(\mu}\bar{D}^\sigma c_{\nu)}\bar{D}_\sigma + \bar{D}^\alpha c_{(\nu}\bar{D}_{\mu)} + \delta^\alpha_{(\mu}\bar{D}_{\nu)}c^\sigma \bar{D}_\sigma - \bar{D}_{(\mu}c_{\nu)}\bar{D}^\alpha\\
&- \frac{1}{d}\bar{g}_{\mu\nu}\left[\bar{D}^2 c^\alpha + \bar{R}^{\alpha\sigma}c_\sigma + \bar{D}^\sigma c^\alpha \bar{D}_\sigma + 2\bar{D}^\alpha c_\sigma \bar{D}^\sigma + c^\sigma \bar{D}^\alpha \bar{D}_\sigma\right],\\
\widetilde{\bar{Q}}_\alpha{}^{\mu\nu} =& -\delta^{(\mu}_\alpha\left[\bar{D}_\sigma \bar{c}^{\nu)}\bar{D}^\sigma + \bar{D}^2 \bar{c}^{\nu)} + \bar{D}^{\nu)}\bar{c}_\sigma \bar{D}^\sigma + \bar{R}^{\nu)\sigma}\bar{c}_\sigma\right] + \bar{D}^{(\mu}\bar{c}^{\nu)}\bar{D}_\alpha\\
&+ \frac{1}{d}\bar{g}^{\mu\nu}\left[\bar{D}^\sigma \bar{c}_\alpha \bar{D}_\sigma + \bar{D}^2 \bar{c}_\alpha + \bar{D}_\alpha \bar{c}_\sigma \bar{D}^\sigma + \bar{R}_{\alpha\sigma}\bar{c}^\sigma\right]\,,
\end{aligned} \tag{5.2.16}$$

where the covariant derivatives to the left of the ghost fields act on c or $\bar{c}$ only, and $(\mu\nu) = \frac{1}{2}(\mu\nu + \nu\mu)$ denotes symmetrization with unit strength. The respective last lines in the Q operators ensure that they are traceless in the indices $\mu\nu$.

Following the derivation of the RG equation, the action is supplemented by an IR regulating term (2.3.3) of the form

$$\Delta S_k = \int d^dx\sqrt{\bar{g}}\left\{\tfrac{1}{2}\,\hat{h}^{\mu\nu}\,[\mathcal{R}^2_k]_{\mu\nu}{}^{\rho\sigma}\,\hat{h}_{\alpha\beta} + \tfrac{1}{2}\,h\,\mathcal{R}^0_k\,h + \bar{f}^\mu\,[\mathcal{R}^{\rm c}_k]_\mu{}^\nu\,f_\nu\right\}, \tag{5.2.17}$$

dressing the Laplacians appearing in the kinetic terms. The insertions are adapted to the quadratic forms (5.2.10) and (5.2.13) according to the prescription (4.3.12). This determines

$$\begin{aligned}
[\mathcal{R}^2_k]_{\mu\nu}{}^{\rho\sigma} &= \kappa^2\,Z^N_k\,R_k\,[\mathbf{1}_{2\rm t}]_{\mu\nu}{}^{\rho\sigma}\,, \qquad \mathcal{R}^0_k = -\tfrac{d-2}{2d}\,\kappa^2\,Z^N_k\,R_k\,,\\
[\mathcal{R}^{\rm c}_k]_\mu{}^\nu &= \sqrt{2}\,Z^{\rm c}_k\,R_k\,\delta_\mu{}^\nu\,,
\end{aligned} \tag{5.2.18}$$

with the traceless projector given in (5.2.9). Observe that $\mathcal{R}_k$ inherits a non-trivial k-dependence via the field renormalization factors Z_k^N and Z_k^c.

To construct the inverse of the dressed Hessian $(\Gamma_k^{(2)} + \mathcal{R}_k)$, we define the multiplets

$$\Phi = \{\hat{h}_{\mu\nu}, h, f^\alpha, \bar{f}_\alpha\}\,, \qquad \bar{\Phi} = \{\hat{h}_{\mu\nu}, h, -\bar{f}_\alpha, f^\alpha\}\,, \tag{5.2.19}$$

adopting the convention of a skew-symmetric metric in field space [88, 132], to deal with the anti-commuting fields more conveniently. Without further modifications to the RG equation, instead of (4.3.10) the Hessian is thus equivalently defined as

$$\left[\Gamma_k^{(2)}\right]_{ij}(x,y) = \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\,\frac{\delta^2\Gamma_k}{\delta\bar{\Phi}_i(x)\delta\Phi_j(y)}\,, \tag{5.2.20}$$

with all variations acting from the left. With the cutoffs (5.2.18) in place, we have the full dressed inverse propagator of the block matrix form

$$\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)_{ij} = \begin{bmatrix} \mathbb{K} & \mathbb{Q} \\ \tilde{\mathbb{Q}} & \mathbb{M} \end{bmatrix}, \tag{5.2.21}$$

with the entries given from the quadratic forms (5.2.10) and (5.2.13). They read for the graviton sector

$$\mathbb{K} = \kappa^2 Z_k^N \begin{bmatrix} \left(P_k - 2\Lambda_k + C_T\bar{R}\right)[\mathbf{1}_{2\mathrm{t}}]_{\mu\nu}{}^{\rho\sigma} & 0 \\ 0 & -\frac{d-2}{2d}\left(P_k - 2\Lambda_k + C_S\bar{R}\right) \end{bmatrix}, \tag{5.2.22}$$

the ghost sector

$$\mathbb{M} = \sqrt{2}\,Z_k^c \begin{bmatrix} \left(P_k - \frac{1}{d}\bar{R}\right)\delta_\mu{}^\nu & 0 \\ 0 & \left(P_k - \frac{1}{d}\bar{R}\right)\delta^\mu{}_\nu \end{bmatrix}, \tag{5.2.23}$$

and for the mixed terms

$$\mathbb{Q} = \sqrt{2}Z_k^c \begin{bmatrix} \bar{Q}^{\mu\nu}{}_\alpha & Q^{\mu\nu,\alpha} \\ \bar{A}_\alpha & A^\alpha \end{bmatrix}, \qquad \tilde{\mathbb{Q}} = \sqrt{2}Z_k^c \begin{bmatrix} \tilde{Q}^{\alpha,\mu\nu} & \tilde{A}^\alpha \\ -\tilde{\bar{Q}}_\alpha{}^{\mu\nu} & -\tilde{\bar{A}}_\alpha \end{bmatrix}, \tag{5.2.24}$$

defined by the operators (5.2.15) and (5.2.16). In terms of these blocks, the inverse of (5.2.21) is found as

$$\begin{bmatrix} \mathbb{K} & \mathbb{Q} \\ \tilde{\mathbb{Q}} & \mathbb{M} \end{bmatrix}^{-1} = \begin{bmatrix} \left(\mathbb{K} - \mathbb{Q}\mathbb{M}^{-1}\tilde{\mathbb{Q}}\right)^{-1} & -\mathbb{K}^{-1}\mathbb{Q}\left(\mathbb{M} - \tilde{\mathbb{Q}}\mathbb{K}^{-1}\mathbb{Q}\right)^{-1} \\ -\mathbb{M}^{-1}\tilde{\mathbb{Q}}\left(\mathbb{K} - \mathbb{Q}\mathbb{M}^{-1}\tilde{\mathbb{Q}}\right)^{-1} & \left(\mathbb{M} - \tilde{\mathbb{Q}}\mathbb{K}^{-1}\mathbb{Q}\right)^{-1} \end{bmatrix}. \tag{5.2.25}$$

Expanding the inverse up to second order in the background ghost fields, and taking into account the minus sign originating from the supertrace, the RG equation (4.3.1) becomes

$$\begin{aligned}\partial_t \Gamma_k = &\tfrac{1}{2}\,\mathrm{Tr}_{(2,0)}\left[\left(\mathbb{K}^{-1}+\mathbb{K}^{-1}\mathbb{Q}\mathbb{M}^{-1}\tilde{\mathbb{Q}}\mathbb{K}^{-1}\right)\partial_t\mathcal{R}_k^{\mathrm{grav}}\right]\\ &-\tfrac{1}{2}\,\mathrm{Tr}_{(1,1)}\left[\left(\mathbb{M}^{-1}+\mathbb{M}^{-1}\tilde{\mathbb{Q}}\mathbb{K}^{-1}\mathbb{Q}\mathbb{M}^{-1}\right)\partial_t\mathcal{R}_k^{\mathrm{gh}}\right]\,,\\ =:&\ \mathcal{S}_2+\mathcal{S}_0+\mathcal{S}_{\mathrm{c}}+\mathcal{G}_2+\mathcal{G}_0+\mathcal{G}_{\mathrm{c}}\,,\end{aligned} \tag{5.2.26}$$

with the cutoff terms $\mathcal{R}_k^{\mathrm{grav}} = \mathrm{diag}[\mathcal{R}_k^2,\mathcal{R}_k^0]$, and $\mathcal{R}_k^{\mathrm{gh}} = \mathrm{diag}[\mathcal{R}_k^{\mathrm{c}},\mathcal{R}_k^{\mathrm{c}}]$, up to terms not contributing to the β functions of interest here. As indicated in the last line, the full trace decomposes into operator traces on the space of traceless symmetric tensors (2), scalars (0) and vectors (1), where the two separate vector traces from the ghost fields yield equal contributions, combining to $(\mathcal{S}_{\mathrm{c}}+\mathcal{G}_{\mathrm{c}})$. The $\mathcal{S}_i$ are defined as the terms not including Q and $\tilde{Q}$ and are thus independent of the background ghost fields, while the $\mathcal{G}_i$ denote the new contributions originating from these. Substituting the explicit expressions for the block matrices, we obtain

$$\begin{aligned}\mathcal{S}_2 =& \frac{1}{2}\,\mathrm{Tr}_2\left[\frac{[\mathbf{1}_{2\mathrm{T}}]^{\mu\nu}{}_{\rho\sigma}}{Z_k^N(P_k-2\Lambda_k+C_T\bar{R})}\,\partial_t(Z_k^N\,R_k)\right]\,,\\ \mathcal{S}_0 =& \frac{1}{2}\,\mathrm{Tr}_0\left[\frac{1}{Z_k^N(P_k-2\Lambda_k+C_S\bar{R})}\,\partial_t(Z_k^N\,R_k)\right]\,,\\ \mathcal{S}_{\mathrm{c}} =& -\mathrm{Tr}_1\left[\frac{\delta_\alpha{}^\beta}{Z_k^c(P_k-\frac{1}{d}\bar{R})}\,\partial_t(Z_k^c\,R_k)\right]\,.\end{aligned} \tag{5.2.27}$$

These partial traces give rise to the β functions for Newton's constant and the cosmological constant. The β function for Z_k^c is captured by the terms of second order in the background ghost fields. Neglecting the curvature terms and making use of the cyclicity of the trace, these are found as

$$\begin{aligned}\mathcal{G}_2 =& -\frac{Z_k^c}{\sqrt{2}\kappa^2(Z_k^N)^2}\,\mathrm{Tr}_2\left[\frac{\partial_t(Z_k^N R_k)}{(P_k-2\Lambda)^2}\left(Q^{\mu\nu}{}_\alpha\frac{1}{P_k}\tilde{\bar{Q}}^\alpha{}_{\rho\sigma}-\bar{Q}^{\mu\nu}{}_\alpha\frac{1}{P_k}\tilde{Q}^\alpha{}_{\rho\sigma}\right)\right]\,,\\ \mathcal{G}_0 =& \frac{2d}{d-2}\frac{Z_k^c}{\sqrt{2}\kappa^2(Z_k^N)^2}\,\mathrm{Tr}_0\left[\frac{\partial_t(Z_k^N R_k)}{(P_k-2\Lambda)^2}\left(A^\alpha\frac{1}{P_k}\tilde{\bar{A}}_\alpha-\bar{A}_\alpha\frac{1}{P_k}\tilde{A}^\alpha\right)\right]\,,\\ \mathcal{G}_{\mathrm{c}} =& -\frac{1}{\sqrt{2}\kappa^2 Z_k^N}\,\mathrm{Tr}_1\Bigg[\frac{\partial_t(Z_k^c R_k)}{P_k^2}\Bigg(\tilde{Q}_\alpha{}^{\mu\nu}\frac{1}{P_k-2\Lambda}\bar{Q}_{\mu\nu}{}^\beta-\tilde{\bar{Q}}^{\mu\nu}{}_\alpha\frac{1}{P_k-2\Lambda}Q_{\mu\nu}{}^\beta\\ &-\frac{2d}{d-2}\left(\tilde{A}_\alpha\frac{1}{P_k-2\Lambda}\bar{A}^\beta-\tilde{\bar{A}}_\alpha\frac{1}{P_k-2\Lambda}A^\beta\right)\Bigg)\Bigg]\,.\end{aligned} \tag{5.2.28}$$

Notably, the insertions appear here always in pairs, with each part giving exactly the same contribution to the $\mathcal{G}_i$. Evaluating both parts individually thus provides a highly non-trivial crosscheck for the evaluation of these traces.

The β functions for the coupling constants Z_k^N, Z_k^c, and Λ_k can now be computed from (5.2.27) and (5.2.28). It is useful to introduce the anomalous dimensions of Newton's constant and the ghost field renormalization

$$\eta_N = -\partial_t \ln(Z_k^N)\,, \qquad \eta_c = -\partial_t \ln(Z_k^c)\,, \tag{5.2.29}$$

together with the dimensionless coupling constants

$$g_k = G_k k^{d-2} = (Z_k^N)^{-1}\, G_0\, k^{d-2}\,, \qquad \lambda_k = \Lambda_k k^{-2}\,. \tag{5.2.30}$$

The β functions for g_k and λ_k are encoded in the monomials of the gravitational sector in (5.2.3). These are generated by the traces $\mathcal{S}_i$ (5.2.27), which are evaluated straightforwardly by applying the early-time heat kernel expansion (3.2.27). Equating the resulting coefficients with (5.2.7) provides the flow equations for Λ_k and $G_k = (32\pi Z_k^N \kappa^2)^{-1}$. Rewriting the functions of the Laplacians in the traces captured following the scheme (4.3.45), by inserting the identity (C.5) we obtain

$$\begin{aligned}
\partial_t \left(\tfrac{\Lambda_k}{8\pi G_k}\right) =& \tfrac{k^d}{(4\pi)^{d/2}} \left\{ \tfrac{d(d+1)}{2} \left[\Phi^{1,0}_{d/2}(-2\lambda_k) - \tfrac{1}{2}\eta_N \widetilde{\Phi}^{1,0}_{d/2}(-2\lambda_k)\right] - 2d \left[\Phi^{1,0}_{d/2}(0) - \tfrac{1}{2}\eta_c \widetilde{\Phi}^{1,0}_{d/2}(0)\right]\right\}\,, \\
-\partial_t \left(\tfrac{1}{16\pi G_k}\right) =& \tfrac{k^{d-2}}{(4\pi)^{d/2}} \left\{ \tfrac{d(d+1)}{12} \left[\Phi^{1,0}_{d/2-1}(-2\lambda_k) - \tfrac{1}{2}\eta_N \widetilde{\Phi}^{1,0}_{d/2-1}(-2\lambda_k)\right] \right. \\
& - \tfrac{d(d-1)}{2} \left[\Phi^{2,0}_{d/2}(-2\lambda_k) - \tfrac{1}{2}\eta_N \widetilde{\Phi}^{2,0}_{d/2}(-2\lambda_k)\right] \\
& \left. - \tfrac{d}{3} \left[\Phi^{1,0}_{d/2-1}(0) - \tfrac{1}{2}\eta_c \widetilde{\Phi}^{1,0}_{d/2-1}(0)\right] - 2 \left[\Phi^{2,0}_{d/2}(0) - \tfrac{1}{2}\eta_c \widetilde{\Phi}^{2,0}_{d/2}(0)\right] \right\}.
\end{aligned} \tag{5.2.31}$$

The threshold functions $\Phi^p_n(\omega)$ herein capture the cutoff dependence of the result as discussed in appendix C. In the limit $Z_k^c = 1$, $\eta_c = 0$, neglecting the quantum corrections from the ghost fields, this result agrees with earlier computations [35, 46]. The terms proportional to η_c are novel and capture the back-reaction of the quantum effects in the ghost sector on the running of the gravitational coupling constants.

The final step is the computation of η_c. This requires extracting the background ghost kinetic term from (5.2.28). The differential operators entering into these traces are not minimal, so that the early-time heat kernel expansion is no longer applicable and the more sophisticated off-diagonal heat kernel technique (3.3.5) is needed. This method allows the computation of the traces $\mathcal{G}_i$, containing operator insertions involving the background ghost fields. Retaining the ghost kinetic term only, we find

$$
\begin{aligned}
\mathcal{G}_{2\mathrm{T}} &= -\tfrac{\sqrt{2}}{(4\pi)^{d/2}}\tfrac{Z_k^c}{\kappa^2(Z_k^N)^2}\left[\tfrac{4d^2-d-8}{4d}Q_{d/2+1}[f_1^N] - \tfrac{d^2-2}{2d}Q_{d/2+2}[f_2^N]\right]\int d^dx\sqrt{\bar{g}}\,\bar{c}^\mu\bar{D}^2c_\mu\,,\\
\mathcal{G}_0 &= -\tfrac{\sqrt{2}}{(4\pi)^{d/2}}\tfrac{Z_k^c}{\kappa^2(Z_k^N)^2}\left[\tfrac{d-4}{d(d-2)}Q_{d/2+1}[f_1^N] - \tfrac{1}{d}Q_{d/2+2}[f_2^N]\right]\int d^dx\sqrt{\bar{g}}\,\bar{c}^\mu\bar{D}^2c_\mu, \qquad (5.2.32)\\
\mathcal{G}_1 &= -\tfrac{1}{(4\pi)^{d/2}}\tfrac{1}{\sqrt{2}\kappa^2 Z_k^N}\left[\tfrac{2d^2-5d-2}{2(d-2)}Q_{d/2+1}[f_1^c] + d\,Q_{d/2+2}[f_2^c]\right]\int d^dx\sqrt{\bar{g}}\,\bar{c}^\mu\bar{D}^2c_\mu\,.
\end{aligned}
$$

Here the Q functionals are defined in (4.3.42) and

$$
\begin{aligned}
f_1^N &\equiv \tfrac{\partial_t(Z_k^N R_k)}{(P_k-2\Lambda)^2}\,\tfrac{1}{P_k}\,, \qquad & f_2^N &\equiv -\tfrac{\partial_t(Z_k^N R_k)}{(P_k-2\Lambda)^2}\,\tfrac{\mathrm{d}}{\mathrm{d}x}\tfrac{1}{P_k(x)}\Big|_{x=\Delta}\,,\\
f_1^c &\equiv \tfrac{\partial_t(Z_k^c R_k)}{P_k^2}\,\tfrac{1}{P_k-2\Lambda}\,, \qquad & f_2^c &\equiv -\tfrac{\partial_t(Z_k^c R_k)}{P_k^2}\,\tfrac{\mathrm{d}}{\mathrm{d}x}\tfrac{1}{P_k(x)-2\Lambda}\Big|_{x=\Delta}\,,
\end{aligned}
\qquad (5.2.33)
$$

denote the functions of the Laplacian appearing in the trace arguments. Substituting the relations of the above functionals to the threshold functions (C.8) and equating the result with the ghost kinetic term in (5.2.7), we find

$$
\begin{aligned}
\partial_t Z_k^c = \frac{4\,Z_k^c\,g_k}{(4\pi)^{d/2-1}}\Big\{ & C_{\mathrm{gr}}\left(\Phi^{2,1}_{d/2+1}(-2\lambda_k) - \tfrac{1}{2}\eta_N\tilde{\Phi}^{2,1}_{d/2+1}(-2\lambda_k)\right)\\
& + C_{\mathrm{gh}}\left(\Phi^{1,2}_{d/2+1}(-2\lambda_k) - \tfrac{1}{2}\eta_c\tilde{\Phi}^{1,2}_{d/2+1}(-2\lambda_k)\right) \qquad (5.2.34)\\
& + d\,(\eta_N-\eta_c)\left(\check{\Phi}^{2,2}_{d/2+2}(-2\lambda_k) + \hat{\Phi}^{2,2}_{d/2+2}(-2\lambda_k)\right)\Big\}\,,
\end{aligned}
$$

with the constants

$$
C_{\mathrm{gr}} = \frac{4d^2-9d-2}{d-2}\,, \qquad C_{\mathrm{gh}} = \frac{2d^2-5d-2}{d-2}\,. \qquad (5.2.35)
$$

The terms containing $\Phi^{2,2}_{d/2+2}$ and $\check{\Phi}^{2,2}_{d/2+2}$ originating from the $Q_{d/2+2}[f_2^I]$ drop out, due to the cancellation of the corresponding coefficients.

The β functions are finally obtained by solving (5.2.31) and (5.2.34) for $\partial_t\lambda_k$, $\partial_t g_k$ and η_c. This yields

$$
\partial_t\lambda_k = \beta_\lambda\,, \qquad \partial_t g_k = \beta_g = (d-2+\eta_N)\,g_k\,, \qquad (5.2.36)
$$

with

$$
\begin{aligned}
\beta_\lambda = &-(2-\eta_N)\lambda_k + \tfrac{1}{2}g_k(4\pi)^{1-d/2}\\
&\Big[2d(d+1)\Phi^{1,0}_{d/2}(-2\lambda_k) - 8d\Phi^{1,0}_{d/2}(0) - d(d+1)\eta_N\widetilde{\Phi}^{1,0}_{d/2}(-2\lambda_k) + 4d\eta_c\widetilde{\Phi}^{1,0}_{d/2}(0)\Big]\,,
\end{aligned}
\qquad (5.2.37)
$$

and the anomalous dimensions

$$\begin{aligned}\eta_N &= \frac{gB_1(\lambda) + g^2\Big[C_3(\lambda)C_4(\lambda) - B_1(\lambda)C_2(\lambda)\Big]}{1 - g\Big[B_2(\lambda) + C_2(\lambda)\Big] + g^2\Big[B_2(\lambda)C_2(\lambda) - C_1(\lambda)C_3(\lambda)\Big]}\,,\\ \eta_c &= \frac{gC_4(\lambda) + g^2\Big[B_1(\lambda)C_1(\lambda) - B_2(\lambda)C_4(\lambda)\Big]}{1 - g\Big[B_2(\lambda) + C_2(\lambda)\Big] + g^2\Big[B_2(\lambda)C_2(\lambda) - C_1(\lambda)C_3(\lambda)\Big]}\,.\end{aligned} \tag{5.2.38}$$

The functions $B_i(\lambda)$ and $C_i(\lambda)$ are defined in terms of the threshold functions (see appendix C for details), which in the following will all be evaluated at the argument $\omega = -2\lambda$, which is suppress for notational simplicity. The B_i reproduce exactly the terms obtained in [35], neglecting the ghost corrections

$$\begin{aligned}B_1(\lambda) &= \tfrac{1}{3}\,(4\pi)^{1-d/2}\Big(d(d+1)\Phi^{1,0}_{d/2-1} - 6d(d-1)\Phi^{2,0}_{d/2} - 4d\Phi^{0,1}_{d/2-1} - 24\Phi^{0,2}_{d/2}\Big)\,,\\ B_2(\lambda) &= -\tfrac{1}{6}(4\pi)^{1-d/2}\Big(d(d+1)\tilde{\Phi}^{1,0}_{d/2-1} - 6d(d-1)\tilde{\Phi}^{2,0}_{d/2}\Big)\,.\end{aligned} \tag{5.2.39}$$

The quantum corrections from the field renormalization of the ghosts are encoded in

$$\begin{aligned}C_1(\lambda) &= (4\pi)^{1-d/2}\Big(2C_{\rm gr}\tilde{\Phi}^{2,1}_{d/2+1} - 4d(\tilde{\Phi}^{2,2}_{d/2+2} + \hat{\Phi}^{2,2}_{d/2+2})\Big)\,,\\ C_2(\lambda) &= (4\pi)^{1-d/2}\Big(2C_{\rm gh}\tilde{\Phi}^{1,2}_{d/2+1} + 4d(\tilde{\Phi}^{2,2}_{d/2+2} + \hat{\Phi}^{2,2}_{d/2+2})\Big)\,,\\ C_3(\lambda) &= \tfrac{1}{3}\,(4\pi)^{1-d/2}\Big(2d\tilde{\Phi}^{0,1}_{d/2-1} + 12\tilde{\Phi}^{0,2}_{d/2}\Big)\,,\\ C_4(\lambda) &= -\,(4\pi)^{1-d/2}\Big(4C_{\rm gr}\Phi^{2,1}_{d/2+1} + 4C_{\rm gh}\Phi^{1,2}_{d/2+1}\Big)\,,\end{aligned} \tag{5.2.40}$$

with the coefficients $C_{\rm gr}$ and $C_{\rm gh}$ defined in (5.2.35).

The expressions (5.2.37) and (5.2.38) in terms of which the desired β functions (5.2.34) and (5.2.36) are given constitute the central result of this section. As expected, the inclusion of the field renormalization for the ghosts gives non-trivial contributions to the β functions for g and λ. These encompass the terms proportional to η_c in β_λ and the qualitatively new g^2-terms in η_N. The leading contributions from the ghost sector are suppressed by one power of g, relative to the leading Einstein-Hilbert term

$$\eta_N = gB_1(\lambda) + g^2\Big(B_1(\lambda)B_2(\lambda) + C_3(\lambda)C_4(\lambda)\Big) + \mathcal{O}(g^3)\,. \tag{5.2.41}$$

Thus in the classical regime, $g \ll 1$, the ghost-improvement may be neglected. In the quantum regime close to the NGFP where $g \approx 1$, however, we expect that these corrections become important.

The results obtained within the standard Einstein-Hilbert truncation [35, 39] are recovered exactly by setting $C_i = 0$. Because of the direct comparability of the computations, these new terms allow for the effects of the ghost fields to be traced analytically. Although the β functions are not observables themselves and will thus in general carry unphysical information, the access to their term structure makes more detailed fixed point analyses possible, not restricted to numerical comparisons. A similar investigation with the inclusion of further interaction terms is proposed as providing the means to new structural insights in the emergence of the gravitational fixed point. This way, the mechanism establishing it not to be an artifact of the used approximations may be revealed.

5.2.2. The Ghost-improved Renormalization Group Flow

Investigating the influence of the new terms on the gravitational RG flow is the subject of the following discussion. In this context, it is useful to observe that Z_k^c enters into β_λ via η_c only and is in turn completely determined by g_k, λ_k. Thus, substituting the explicit formula for η_c into β_λ, the running of Z_k^c decouples and allows to analyse the gravitational RG flow in the two-dimensional g, λ-subsystem. Once an RG trajectory for g_k, λ_k is found, it can be plugged back into η_c to obtain the running of the ghost anomalous dimension. We will now exploit this decoupling and first discuss the fixed point structure of the ghost-improved Einstein-Hilbert truncation for $d = 4$, before focusing on the phase portrait and the fixed point structure including extra-dimensions.

The crucial requirement of the asymptotic safety scenario is the fixed point structure of the gravitational β functions. Thus, we start our investigation by looking for fixed points g^*, λ^* where $\beta_g = \beta_\lambda = 0$ simultaneously. In the vicinity of such a point, the linearised β functions are given by

$$\partial_t g_i = \mathbf{B}_{ij}(g_j - g_j^*)\,, \tag{5.2.42}$$

where $\mathbf{B}_{ij} = \partial_{g_j}\beta_{g_i}|_{g=g^*}$, and $g_i = \{g, \lambda\}$. The critical exponents θ_i defined in (2.3.25) as minus the eigenvalues of $\mathbf{B}_{ij}$, provide an important characterisation of the fixed point. In particular, eigendirections with a positive (negative) real part of θ are UV-attractive (UV-repulsive) for trajectories close to the fixed point.

Inspecting the β functions (5.2.36) immediately reveals the Gaussian fixed point (GFP)

$$g^* = 0\,, \qquad \lambda^* = 0\,, \qquad \eta_N^* = \eta_c^* = 0\,. \tag{5.2.43}$$

Truncation	λ^*	g^*	$g^*\lambda^*$	η_c^*	$\mathrm{Re}(\theta)$	$\mathrm{Im}(\theta)$	cutoff
EH + ghost	0.135	0.859	0.116	-1.774	1.935	2.012	opt
EH + ghost	0.260	0.355	0.092	-1.846	2.070	2.439	exp ($s = 1$)
EH	0.193	0.707	0.136	–	1.475	3.043	opt

Table 5.2.: Properties of the NGFP arising from the ghost-improved β functions (5.2.36). The first two lines show the position, the marginal product $g^*\lambda^*$, the ghost anomalous dimension η_c^*, and the critical exponents of the fixed point obtained with the optimized cutoff (C.10) and exponential cutoff (C.13) with $s = 1$, respectively. For comparison, the third line displays the characteristics of the NGFP found in the standard Einstein-Hilbert truncation [58].

This fixed point corresponds to the free theory, and constitutes a saddle point in the g-λ-plane. It has one attractive and one repulsive eigendirection with critical exponents given by the canonical mass dimensions of G and Λ, respectively. The numerical analysis of the ghost-improved β functions also shows a unique NGFP. Its position and properties are shown in the first two lines of table 5.2. Herein and in the remainder of the fixed point analysis, we will set $d = 4$, before commenting on general d at the end of the section.

The non-Gaussian fixed point is situated at $g^* > 0, \lambda^* > 0$ and UV-attractive in both g, λ. Substituting its position into η_c determines the ghost anomalous dimension $\eta_c^* \approx -1.8$.[6] For comparison, the third line of table 5.2 displays the properties of the NGFP obtained within the standard Einstein-Hilbert truncation without ghost-improvement. We observe that the actual numerical values of the product $g^*\lambda^*$ and the critical exponents are shifted by approximately 30%.[7] This is in the typical range for the cutoff scheme dependence observed in [58]. Most remarkably, both the standard and the ghost-improved Einstein-Hilbert truncation give rise to the same fixed point structure. This is highly non-trivial, as the new contributions to the gravitational β functions are of the same order of magnitude as the other previously known terms. We interpret this result as a striking confirmation of the gravitational fixed point structure disclosed by the standard

[6]When including the marginal Z_k^c in the set of coupling constants, η_c^* is also the critical exponent associated with the new (UV-irrelevant) eigendirection. However, since the running of η_c is completely determined by g_k, λ_k,the field renormalization Z_k^c is an inessential coupling.

[7]This product is expected to be less strongly dependent on the choice of cutoff, since it corresponding to a power-counting marginal combination with $g_k\lambda_k = G_k\Lambda_k$.

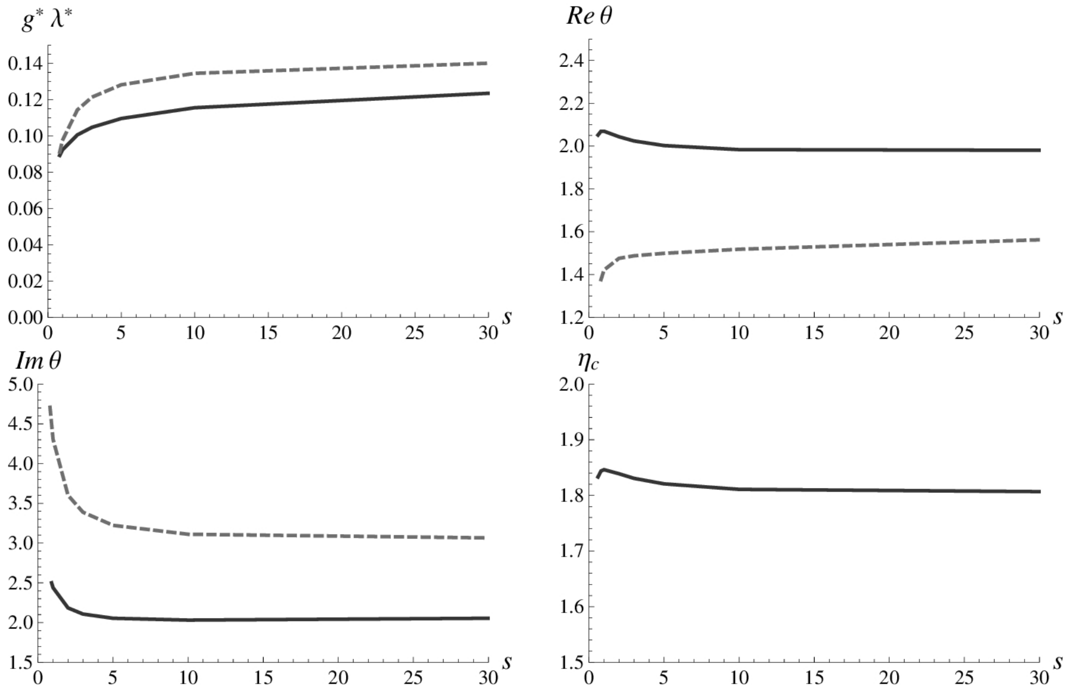

Figure 5.1.: Stability analysis for universal quantities in the Einstein-Hilbert truncation (dashed line), previously obtained in [46], and upon including the ghost field renormalization (solid line). The ghost-improvement significantly decreases the cutoff scheme dependence.

Einstein-Hilbert truncation [44–47, 58].[8]

The main virtue of the ghost-improvement becomes apparent, when investigating the stability of the physical quantities $g^*\lambda^*$, θ, η_c^* with respect to the variation of the IR cutoff R_k. To illustrate this point, we resort to the exponential cutoff (C.13), and determine the properties of the NGFP for varying shape parameter s. Figure 5.1 shows the resulting cutoff scheme dependence of the physical quantities for the standard and ghost-improved computation. Remarkably, the ghost-improvement reduces the unphysical cutoff scheme dependence by factors 1.4, 2.0 and 3.2 for the product $g^*\lambda^*$, Re θ and Im θ, respectively. The scheme-dependence of η_c^* can only be determined in the ghost-improved computation. Here the variation is approximately 2%.

The magnitude of cutoff scheme dependence of a result of the RG equation serves as a measure for its reliability, since such unphysical input should be inessential. Our

[8] Qualitatively, our picture is also confirmed by the very recent results [131], which study the ghost-improved Einstein-Hilbert truncation employing a spectrally adjusted cutoff. The numerical variations observed in the two computations are within the typical range expected from the different cutoff schemes.

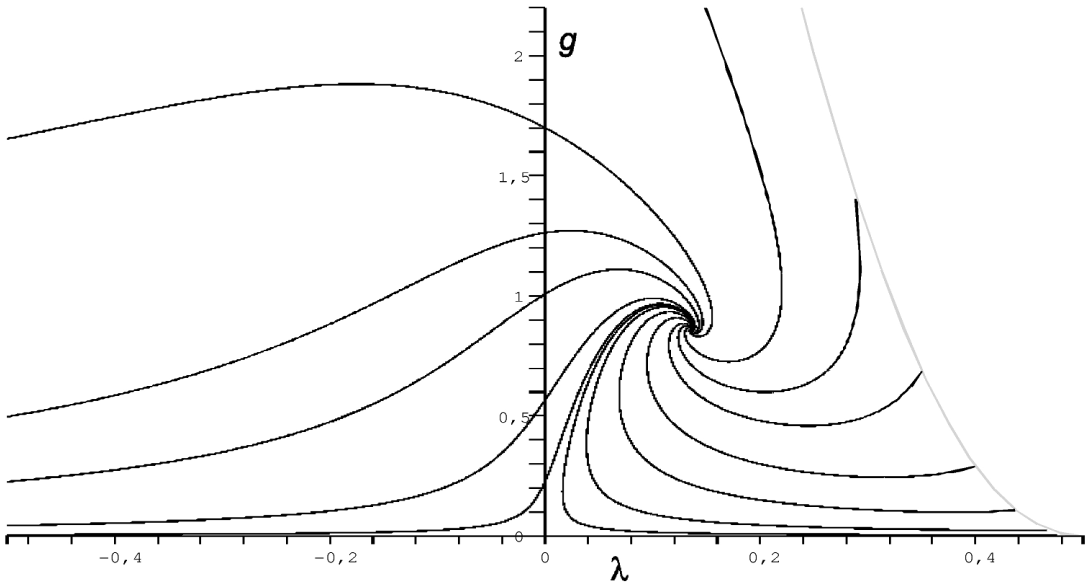

Figure 5.2.: The RG flow of g_k, λ_k obtained from the numerical integration of the ghost-improved β functions (5.2.36) using the optimized cutoff (C.10). The light-gray line indicates a boundary of the coupling constant space where the β functions diverge. The phase portrait is in complete agreement with the one obtained from the standard Einstein-Hilbert truncation.

findings indicate that the ghost-improvement substantially increases the quality of the Einstein-Hilbert approximation and therefore lends more credibility to the gravitational fixed point being a robust feature. In particular the fixed point properties given in table 5.2 are more stable than anticipated from earlier computations alone.

After analysing the fixed point structure, we determine the phase portrait resulting from the ghost-improved β functions. We start by investigating the gravitational RG flow on the g-λ−plane, before studying the behaviour of the anomalous dimensions along some typical sample trajectories.

The phase portrait resulting from the numerical integration of the β functions (5.2.36) is depicted in figure 5.2. We first observe that $g_k = 0$ is a fixed line, which cannot be crossed by the flow. For $g_k > 0$ the flow is dominated by the interplay of the NGFP and the GFP. In this regime, the UV behaviour of the RG trajectories is controlled by the NGFP, which acts as a UV attractor. Following the RG flow from this fixed point towards the IR, the RG trajectories undergo a crossover from the NGFP to the "classical regime" dominated by the GFP. Depending on whether the trajectory turns to the left (type Ia), right (type IIIa) or hits the GFP (type IIa), the classical theory has a negative, positive

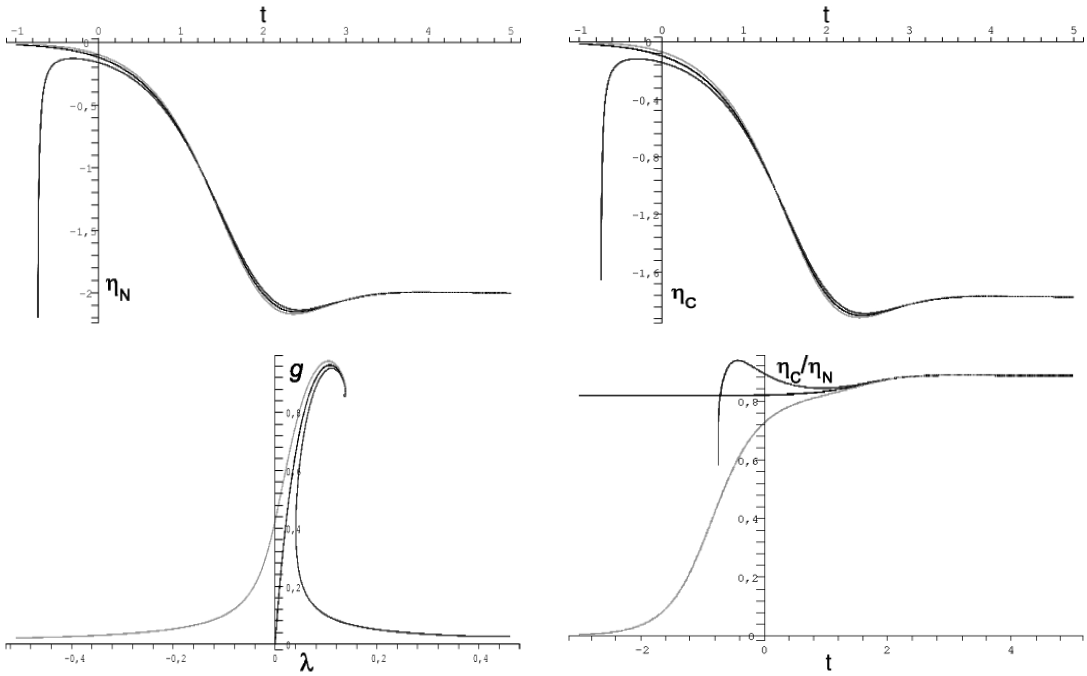

Figure 5.3.: RG flow of the anomalous dimensions η_N (upper left) and η_c (upper right) along the sample RG trajectories of type Ia (top line), type IIa (middle line) and type IIIa (bottom line) shown in the lower left diagram. The ratio η_c/η_N is depicted in the lower right diagram. In the IR, this ratio approaches zero, a finite constant, or rapidly decreases after a peak for trajectories of type Ia, type IIa and type IIIa, respectively.

or zero cosmological constant. The trajectories with positive cosmological constants, however, cannot be continued to $k = 0$, but terminate at a finite value of k when reaching the boundary of the phase space. The latter is indicated by the light-gray line, which constitutes a singularity in the β functions at finite g, λ. Observe that the ghost-improved phase portrait is in complete qualitative agreement with the standard Einstein-Hilbert truncation [40, 46, 58].

It is now illustrative to pick one sample trajectory for each of the classes distingushed above and study the k-dependence of the anomalous dimensions η_N and η_c along the flow. The resulting diagrams are shown in figure 5.3. In the UV limit for large values of $t = \ln(k/k_0)$, the anomalous dimensions are determined by the NGFP, so that $\eta_N^* = -2$ and $\eta_c^* = -1.77$ for the optimized cutoff. Lowering t and approaching the IR limit, the anomalous dimensions undergo a crossover towards the classical theory with $\eta_N \approx 0$, $\eta_c \approx 0$. The steep increase at the end of the type IIIa trajectory is caused by the singularity of the β functions (light-gray line in figure 5.2), and heralds the termination of the trajectory at a finite value k. The UV limit of the ratio η_c/η_N is governed by the NGFP and takes the value $\eta_c^*/\eta_N^* = 0.89$. Following the flow towards the IR, the ratio undergoes a crossover and asymptotes to 0, the cutoff scheme dependent value $54\Phi_3^{0,3}/(24\Phi_2^{0,2} - \Phi_1^{0,1})$, or a finite value at the termination point, for trajectories of type Ia, type IIa, and type IIIa, respectively.

It is worthwhile to have a closer look at the singularity causing the termination of the type IIIa trajectories. Figure 5.3 shows, that at this point in coupling constant space, the anomalous dimensions η_N and η_c diverge. This can be traced back to the vanishing of the denominators in (5.2.38). Here, the additional terms from the running ghost field renormalization has a very non-trivial effect. While the denominator arising from the standard Einstein-Hilbert truncation has a term linear in g only, the ghost-improvement adds an additional term quadratic in g. This may provide an elegant mechanism for lifting this singularity by shifting the zeros of the denominator to complex values g. However, for the B_i, C_i given in (5.2.39) and (5.2.40), this mechanism is not realized, and may require a further improvement of the truncation before becoming operational.

The β functions (5.2.37) and (5.2.38) are continuous in the spacetime dimension. In the sequel, we will exploit this feature and analyse the resulting gravitational fixed point structure for varying dimension d. Based on the standard Einstein-Hilbert truncation, a similar analysis has been carried out in [46, 48, 50]. Since some TeV-scale gravity models [133–136], hinge on the existence of the NGFP in the presence of extra-dimensions, it is worthwhile to complement the previous results by including the ghost-improvements. Remarkably, the d-dimensional fixed point structure is strikingly similar to the one obtained in four dimensions. This is in crucial contrast to the situation in perturbation theory, where a change of the dimension alters the power counting criterium, so that any given model is only renormalizable for sufficiently low dimension. Asymptotic safety is in this respect a direct generalization that continues the perturbative fixed points to higher dimensions.

Here, the GFP (5.2.43) exists for all d. Furthermore, there is a unique generalization of the NGFP for all dimensions $3 \leq d \leq 25$ considered here. Its properties are shown in figure 5.4. The fixed point is situated at positive $g^* > 0$, $\lambda^* > 0$ and UV attractive in both g and λ. Below $d < 24$ its critical exponents are given by a complex pair $\mathrm{Re}\theta \pm \mathrm{i}\mathrm{Im}\theta$. For $d \geq 24$ the imaginary part of the critical exponents vanishes and we have two real critical exponents, which are still UV attractive. All these results are in perfect agreement

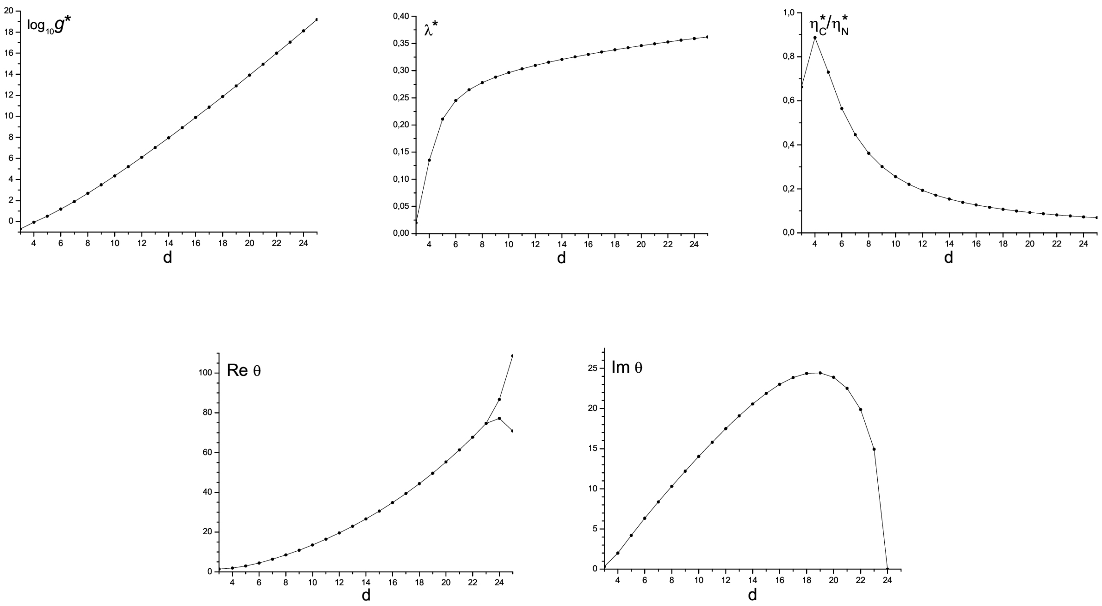

Figure 5.4.: Properties of the NGFP for general spacetime dimension d, obtained with the optimized cutoff (C.10). There is a unique NGFP for all $3 \leq d \leq 25$ considered.

with earlier findings based on the standard Einstein-Hilbert truncation. An interesting feature arises in the ratio η_c^*/η_N^*, with $\eta_N^* = 2 - d$, shown in the top right diagram of figure 5.4. This ratio is peaked at $d = 4$, where it reaches almost unity, and decreases for both $d > 4$ and $d < 4$. It is a rather curious observation that in $d = 4$ the graviton and ghost propagators have a similar anomalous dimension in the UV limit, highlighting this particular value of the spacetime dimension.[9] While it is clearly desirable to get a better understanding of this result, the underlying analysis is beyond the scope of the present discussion.

Notably, for $d = 4$ the fixed point value for the ghost anomalous dimension is very close to the anomalous dimension of Newton's constant $\eta_N^* = -2$. It can be anticipated that this is not accidental, but a consequence of spacetime becoming effectively two-dimensional at short distances, where the physics is controlled by the non-Gaussian fixed point [28, 137]. The argument is based on the observation that the propagator of a field

[9] A related argument, concluding that $d = 4$ is special, is based on the spectral dimension of spacetime computed from the running of Newton's constant and the cosmological constant and has been put forward in [28].

with anomalous dimension η is proportional to $p^{-2+\eta}$. Therefore, in the vicinity of the UV fixed point, both the graviton and ghost propagator behave approximately as p^{-4}, which translates into a logarithmic correlator in position space. In this case the spacetime seen by both fields is effectively two-dimensional, suggesting $\eta_c^* = -2$ in the full theory. A similar dynamical reduction of dimension has also been observed within the framework of Causal Dynamical Triangulations [138], and in a variety of other quantum gravity approaches [139]. For a more detailed discussion of the emergent properties of fractal spacetimes see [140].

6. Higher Derivative Gravity

The fixed point structure of a given theory can already be deduced from rather simple approximations of the exact RG flow, as demonstrated in the previous chapter for the case of gravity in the Einstein-Hilbert approximation. However, establishing the consistency of the approximation and answering questions concerning, e.g., the critical exponents of a given fixed point, and the number of relevant coupling constants appearing in a UV complete model requires much more sophistication. A natural organizing principle for the operators retained in the truncation ansatz for Γ_k best suited for an investigation that focuses on the UV limit of a model is provided by the derivative expansion. This scheme, ordering the interaction terms by the canonical mass-dimension of the corresponding coupling constants, corresponds to a curvature expansion for the case of gravity. In this chapter, we lay out the extension of existing results, aiming for the completion of the four-derivative truncation, initiated in [56, 63], by including the scale dependence of the Gauss-Bonnet term (see appendix A.3 for details). The first section demonstrates the computation of the non-perturbative β functions of a full curvature squared ansatz. In the second section, the results are presented for the computation in a perturbative limit in $d = 4$ dimensions.

6.1. The Renormalization Group Flow of Higher Derivative Gravity

The gravitational part of the effective average action Γ_k is chosen here to contain all interaction monomials build from the curvature tensor with four or less powers of momentum, neglecting only total derivative terms. This is commonly parametrized by

$$\Gamma_k^{\mathrm{grav}} = \int d^dx\sqrt{g}\left[2Z_k\Lambda_k - Z_kR + \frac{1}{2\sigma_k}C^2 - \frac{\omega_k}{3\sigma_k}R^2 + \frac{\theta_k}{\sigma_k}E\right], \tag{6.1.1}$$

where $Z_k \equiv 1/(16\pi G_k)$ contains the dimensionful Newton's constant G_k, and all coupling constants are allowed to depend on the RG scale k. In this action functional, we abbreviate

$$C^2 \equiv C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = \tfrac{2}{(d-2)(d-1)}R^2 - \tfrac{4}{d-2}R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\,, \tag{6.1.2}$$

the square of the Weyl tensor, and

$$\begin{aligned} E &= R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ &= \tfrac{d(d-3)}{(d-2)(d-1)}R^2 - \tfrac{4(d-3)}{(d-2)}R_{\mu\nu}R^{\mu\nu} + C^2\,, \end{aligned} \tag{6.1.3}$$

the integrand of the Euler topological invariant in $d = 4$ dimensions, given in (A.3.9). In order to give the second variation of this ansatz more conveniently, we reparametrize the action as

$$\Gamma_k^{\text{grav}} = \int d^dx\sqrt{g}\,\Big[2Z_k\Lambda_k - Z_kR + a_kR^2 + b_kR_{\mu\nu}R^{\mu\nu} + c_kE\Big]\,. \tag{6.1.4}$$

The coupling constants in this expression are related to the ones in (6.1.1) by

$$a_k = -\tfrac{1}{\sigma_k}\Big(\tfrac{d(d-3)}{2(d-2)(d-1)} + \tfrac{\omega_k}{3}\Big)\,,\quad b_k = \tfrac{2}{\sigma_k}\,\tfrac{d-3}{d-2}\,,\quad c_k = \tfrac{1}{\sigma_k}\Big(\theta_k + \tfrac{1}{2}\Big)\,. \tag{6.1.5}$$

Due to its topological character, the metric variations of the Euler term E will become relevant only in dimensions $d > 4$.

The gauge fixing action accompanying Γ_k^{grav} is here taken to be of the form

$$S^{\text{gf}} = \frac{1}{2}\int d^dx\sqrt{\bar{g}}\,F_\mu\,Y^{\mu\nu}\,F_\nu\,, \tag{6.1.6}$$

where $F_\mu = \bar{D}^\nu h_{\mu\nu} - \beta\bar{D}_\mu h^\nu{}_\nu$, and the bar denotes covariant derivatives with respect to the background metric. Since the gravitational part of the action contains terms with up to four derivatives of the fluctuation fields, we also allow for four derivatives in S^{gf}, employing the minimal gauge [141]

$$Y^{\mu\nu} = b_k\Big[\bar{g}^{\mu\nu}\,\Delta + \gamma^{\text{b}}\,\bar{D}^\mu\bar{D}^\nu + V_{\text{b}}^{\mu\nu}\Big]\,, \tag{6.1.7}$$

where $\Delta \equiv -\bar{D}^2$. To remove all non-minimal four derivative terms in the second variation of Γ_k, the gauge parameters are chosen as

$$\beta = 1 + \frac{b_k}{4a_k}\,,\qquad \gamma^{\text{b}} = 1 + \frac{2a_k}{b_k}\,,\qquad V_{\text{b}}^{\mu\nu} = \bar{R}^{\mu\nu}\,. \tag{6.1.8}$$

This type of higher derivative gauge fixing results in two ghost terms in the action which take the Faddeev-Popov determinant [120] into account. The operator F_μ defines a ghost action in the usual way, defined in (4.1.20). For the above choice, it becomes

$$S_{\text{c}}^{\text{gh}} = \int d^dx\sqrt{\bar{g}}\;\bar{C}_\mu\,\Big[\Delta\delta^\mu_\nu + (1 + \tfrac{b_k}{2a_k})\,\bar{D}^\mu\bar{D}_\nu - \bar{R}^\mu{}_\nu\Big]\,C^\nu\,. \tag{6.1.9}$$

The determinant of the additional operator $Y^{\mu\nu}$ in the gauge fixing action (6.1.6) is captured by the real third ghost field b_μ in the action

$$S^{\rm gh}_{\rm b} = \frac{1}{2}\int d^dx\sqrt{\bar g}\, b_\mu\, Y^{\mu\nu}\, b_\nu\,. \tag{6.1.10}$$

Putting the gauge fixing and the ghost terms together with the gravitational action (6.1.1), the complete effective average action assumes the form

$$\Gamma_k[h,\bar C, C, b;\bar g] = \Gamma^{\rm grav}_k[\bar g + h] + S^{\rm gf}[h,\bar g] + S^{\rm gh}_{\rm c}[h,\bar C, C;\bar g] + S^{\rm gh}_{\rm b}[h,b;\bar g]\,. \tag{6.1.11}$$

We now proceed with the solution of the RG equation (4.3.1) for this functional following the algorithm laid out in section 4.3.

Upon substituting the ansatz (6.1.11), the supertrace in (4.3.1) splits into a gravitational and the two ghost contributions, $\partial_t\Gamma_k = T^{\rm grav} + T^{\rm c} + T^{\rm b}$, defined by

$$\begin{aligned}
T^{\rm grav} &:= \tfrac{1}{2}{\rm Tr}\left[\left(\frac{\delta^2(\Gamma^{\rm grav}_k + S^{\rm gf})}{\delta h\,\delta h} + \mathcal{R}^{\rm grav}_k\right)^{-1}\partial_t\mathcal{R}^{\rm grav}_k\right],\\
T^{\rm gh}_{\rm c} &:= -\,{\rm Tr}\left[\left(\frac{\delta^2 S^{\rm gh}_{\rm c}}{\delta\bar c\,\delta c} + \mathcal{R}^{\rm c}_k\right)^{-1}\partial_t\mathcal{R}^{\rm c}_k\right],\\
T^{\rm gh}_{\rm b} &:= -\,\tfrac{1}{2}{\rm Tr}\left[\left(\frac{\delta^2 S^{\rm gh}_{\rm b}}{\delta b\,\delta b} + \mathcal{R}^{\rm b}_k\right)^{-1}\partial_t\mathcal{R}^{\rm b}_k\right].
\end{aligned} \tag{6.1.12}$$

We proceed with the evaluation of the ghost traces before engaging the more involved gravitational trace.

Computation of the Ghost Traces

The operators in the two ghost terms (6.1.9) and (6.1.10) are of the same form, and with a standard cutoff $\mathcal{R}_k$ (4.3.12) can both be written as

$$[S^{\rm gh,(2)} + \mathcal{R}_k]_\mu{}^\nu = P_k(\Delta)\,\delta_\mu{}^\nu + \gamma\bar D_\mu\bar D^\nu + V_\mu{}^\nu = \mathcal{Q}^{\rm gh}{}_\mu{}^\nu + V_\mu{}^\nu\,, \tag{6.1.13}$$

where γ is a fixed parameter depending on the coupling constant ω_k and $V_\mu{}^\nu$ is an endomorphism proportional to the Ricci tensor. Here we defined the operator

$$\begin{aligned}
\mathcal{Q}^{\rm gh}{}_\mu{}^\nu &= P_k\,\delta_\mu{}^\nu + \gamma\bar D_\mu\bar D^\nu\\
&= P_k\,\Pi_{\rm T}{}_\mu{}^\nu + \Big(P_k - \gamma\Delta\Big)\,\Pi_{\rm L}{}_\mu{}^\nu + \mathcal{O}(\mathcal{R})\,,
\end{aligned} \tag{6.1.14}$$

according to the scheme (4.3.18). With the use of the projection operators $\Pi_{\rm T}$ and $\Pi_{\rm L}$ given in (4.1.25), the inverse operator in flat space is easily found. Neglecting all curvature terms, we obtain

$$\begin{aligned}
\mathcal{Q}_0^{\mathrm{gh},-1}{}_\mu{}^\nu &= \frac{1}{P_k}\,\Pi_{\mathrm{T}\mu}{}^\nu + \frac{1}{P_k-\gamma\Delta}\Pi_{\mathrm{L}\mu}{}^\nu \\
&= \frac{1}{P_k}\,\delta_\mu{}^\nu - \gamma\bar{D}_\mu\bar{D}^\nu\frac{1}{P_k(P_k-\gamma\Delta)}\,.
\end{aligned} \tag{6.1.15}$$

Based on this expression, the inverse of $\mathcal{Q}^{\mathrm{gh}}$ is given as a curvature expansion with the formula (4.3.24). The curvature correction appearing therein is here found by

$$\begin{aligned}
\mathcal{W}^{\mathrm{gh}}{}_\mu{}^\nu &= \delta_\mu{}^\nu - \mathcal{Q}^{\mathrm{gh}}{}_\mu{}^\alpha\,\mathcal{Q}_0^{\mathrm{gh},-1}{}_\alpha{}^\nu \\
&= -\gamma\bar{D}_\mu\bar{D}^\nu\frac{1}{P_k} + \gamma P_k\bar{D}_\mu\bar{D}^\nu\frac{1}{P_k(P_k-\gamma\Delta)} - \gamma^2\bar{D}_\mu\Delta\bar{D}^\nu\frac{1}{P_k(P_k-\gamma\Delta)} \\
&= \gamma\Big([P_k,\bar{D}_\mu\bar{D}^\nu] + \gamma\bar{D}_\mu[\bar{D}^\nu,\Delta]\Big)\,\frac{1}{P_k(P_k-\gamma\Delta)}\,,
\end{aligned} \tag{6.1.16}$$

which is written in terms of commutators in the last step. The commutator involving the function $P_k(\Delta)$ can be evaluated with the formula (B.2.4), up to the required order in the curvature. Subsequently, the inverse of $\mathcal{Q}^{\mathrm{gh}}$ is acquired in the form

$$\begin{aligned}
\mathcal{Q}^{\mathrm{gh},-1}{}_\mu{}^\nu &= \mathcal{Q}_0^{\mathrm{gh},-1}{}_\mu{}^\alpha\Big(\delta_\alpha{}^\nu + \mathcal{W}^{\mathrm{gh}}{}_\alpha{}^\nu + \mathcal{W}^{\mathrm{gh}}{}_\alpha{}^\beta\mathcal{W}^{\mathrm{gh}}{}_\beta{}^\nu\Big) + \mathcal{O}(\mathcal{R}^3) \\
&= \mathcal{Q}_0^{\mathrm{gh},-1}{}_\mu{}^\alpha\Big(\delta_\alpha{}^\nu + \mathcal{W}_1^{\mathrm{gh}}{}_\alpha{}^\nu + \mathcal{W}_2^{\mathrm{gh}}{}_\alpha{}^\nu + \mathcal{W}_1^{\mathrm{gh}}{}_\alpha{}^\beta\mathcal{W}_1^{\mathrm{gh}}{}_\beta{}^\nu\Big) + \mathcal{O}(\mathcal{R}^3)\,,
\end{aligned} \tag{6.1.17}$$

introducing a separation $\mathcal{W}^{\mathrm{gh}} = \mathcal{W}_1^{\mathrm{gh}} + \mathcal{W}_2^{\mathrm{gh}}$ into terms including one or two curvature terms, respectively. Together with the inverse of the second variation (6.1.13) expanded in the endomorphism $V_\mu{}^\nu$ we find the full dressed propagator

$$\begin{aligned}
\Big[S^{\mathrm{gh},(2)} + R_k\Big]^{-1}{}_\mu{}^\nu &= \mathcal{Q}^{\mathrm{gh},-1}{}_\mu{}^\alpha - \mathcal{Q}^{\mathrm{gh},-1}{}_\mu{}^\alpha V_\alpha{}^\beta\mathcal{Q}^{\mathrm{gh},-1}{}_\beta{}^\nu \\
&\quad + \mathcal{Q}^{\mathrm{gh},-1}{}_\mu{}^\alpha V_\alpha{}^\beta\mathcal{Q}^{\mathrm{gh},-1}{}_\beta{}^\gamma V_\gamma{}^\delta\mathcal{Q}^{\mathrm{gh},-1}{}_\delta{}^\nu + \mathcal{O}(\mathcal{R}^3)\,.
\end{aligned} \tag{6.1.18}$$

The ghost traces T^{gh} in (6.1.12) can be obtained after substituting (6.1.18) and (6.1.17). To second order in the curvature, the result is given in terms of eight partial traces

$$\begin{aligned}
T^{\mathrm{gh}} &= \mathrm{Tr}\,\Big[S^{\mathrm{gh},(2)} + R_k\Big]^{-1}\partial_t R_k \\
&= \mathrm{Tr}\,\Big[\mathcal{Q}_0^{\mathrm{gh},-1} + \mathcal{Q}_0^{\mathrm{gh},-1}\mathcal{W}_1^{\mathrm{gh}} - \mathcal{Q}_0^{\mathrm{gh},-1}V\mathcal{Q}_0^{\mathrm{gh},-1} \\
&\qquad + \mathcal{Q}_0^{\mathrm{gh},-1}\mathcal{W}_2^{\mathrm{gh}} + \mathcal{Q}_0^{\mathrm{gh},-1}\mathcal{W}_1^{\mathrm{gh}}\mathcal{W}_1^{\mathrm{gh}} \\
&\qquad - \mathcal{Q}_0^{\mathrm{gh},-1}\mathcal{W}_1^{\mathrm{gh}}V\mathcal{Q}_0^{\mathrm{gh},-1} - \mathcal{Q}_0^{\mathrm{gh},-1}V\mathcal{Q}_0^{\mathrm{gh},-1}\mathcal{W}_1^{\mathrm{gh}} \\
&\qquad + \mathcal{Q}_0^{\mathrm{gh},-1}V\mathcal{Q}_0^{\mathrm{gh},-1}V\mathcal{Q}_0^{\mathrm{gh},-1}\Big]\,\partial_t R_k + \mathcal{O}(\mathcal{R}^3)\,,
\end{aligned} \tag{6.1.19}$$

which can now be evaluated via the off-diagonal heat kernel expansion. In $d = 4$ dimensions and using the optimized cutoff (C.10) the result for this generic ghost trace becomes

$$\begin{aligned} T^{\text{gh}} = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \Big[& k^4 \left(3 - \tfrac{2}{\gamma} - \tfrac{2}{\gamma^2} \ln(1-\gamma)\right) \\ & + k^2 \left(1 + \tfrac{1}{2}\psi - \tfrac{1}{3\gamma} \ln(1-\gamma)\right) R - k^2 \left(\tfrac{3}{4} + \tfrac{1}{2\gamma(1-\gamma)} + \tfrac{1}{2\gamma^2} \ln(1-\gamma)\right) V \\ & + \left(\tfrac{1}{24}\psi^2 + \tfrac{1}{6}\psi + \tfrac{1}{9}\right) R^2 + \left(\tfrac{1}{12}\psi^2 + \tfrac{1}{6}\psi - \tfrac{2}{45}\right) R_{\mu\nu}R^{\mu\nu} - \tfrac{11}{90} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ & - \left(\tfrac{1}{12}\psi^2 + \tfrac{1}{6}\psi + \tfrac{1}{3}\right) RV - \left(\tfrac{1}{6}\psi^2 + \tfrac{2}{3}\psi\right) R_{\mu\nu}V^{\mu\nu} \\ & + \tfrac{1}{24}\psi^2 V^2 + \left(1 + \tfrac{1}{2}\psi + \tfrac{1}{12}\psi^2\right) V_{\mu\nu}V^{\mu\nu} \Big] , \end{aligned} \tag{6.1.20}$$

where $\psi \equiv \gamma/(1-\gamma)$ and $V \equiv V_\mu{}^\mu$. Notably, the fourth order part is universal, and does not depend on the form of the cutoff. It agrees with earlier results [92] and thus provides an independent verification. Moreover, the γ dependence in these terms is such that the limit $\gamma \to 0$ is smooth and the result reduces to the trace for the minimal differential operator corresponding to (6.1.13). The explicit expressions for T^{gh}_{c} and T^{gh}_{b} are finally obtained from (6.1.20) by substituting

$$\begin{aligned} \gamma^{\text{b}} &= 1 + \tfrac{2a_k}{b_k} = \tfrac{1}{3}(1 - 2\omega_k) , \qquad & V_{\text{b}}^{\mu\nu} &= \bar{R}^{\mu\nu} , \\ \gamma^{\text{c}} &= 1 + \tfrac{b_k}{2a_k} = \tfrac{2\omega_k - 1}{2(\omega_k+1)} , \qquad & V_{\text{c}}^{\mu\nu} &= -\bar{R}^{\mu\nu} , \end{aligned} \tag{6.1.21}$$

and taking the corresponding prefactors into account.

Second Variation of the Gravitational Action

For the second variation of the gravitational part together with the gauge fixing action $(\Gamma_k^{\text{grav}} + S^{\text{gf}})$ we have

$$\Gamma_k^{\text{grav,quad}} = \frac{1}{2} \int d^dx \sqrt{\bar{g}} \left[h^{\mu\nu} \, \mathcal{H}_{\mu\nu\rho\sigma} \, h^{\rho\sigma} \right] , \tag{6.1.22}$$

with the insertion assuming the general form [141]

$$\mathcal{H}_{\mu\nu\rho\sigma} = K_{\mu\nu\rho\sigma} \Delta^2 + \hat{D}_{\mu\nu\rho\sigma}{}^{\alpha\beta} \bar{D}_{(\alpha} \bar{D}_{\beta)} + \hat{W}_{\mu\nu\rho\sigma} \, . \tag{6.1.23}$$

A non-minimal four derivative piece ($\sim D_{(\alpha} D_\beta D_\gamma D_{\delta)}$) does not appear here, because such terms cancel with the contributions of the gauge fixing term (6.1.6). Although the algorithm used here is capable to handle the operator in full generality, it is more convenient to set the computation up in this way in order to retain comparability with known results and to reduce the computational effort. Furthermore, a one derivative piece is irrelevant as it would only contribute to surface terms. In the following we drop the bar on background quantities, and neglect the variations of the Euler term E. The individual tensors in (6.1.23) are given by [120]

$$K_{\mu\nu\rho\sigma} = \frac{b_k}{2}\Big(\delta_{\mu\nu,\rho\sigma} - \Big(1 + \frac{b_k}{4a_k}\Big) g_{\mu\nu} g_{\rho\sigma}\Big) \,, \tag{6.1.24}$$

in the minimal Laplace-squared term,

$$\begin{aligned}\hat{D}_{\mu\nu\rho\sigma}{}^{(\alpha\beta)} = &\Big(a_k R - \frac{Z_k}{2}\Big)\Big(\delta_{\mu\nu,\rho\sigma} g^{\alpha\beta} - g_{\mu\nu} g_{\rho\sigma} g^{\alpha\beta} + 2 g_{\mu\nu} \delta_{\rho\sigma,}{}^{\alpha\beta} - 2 g_{\mu\rho} \delta_{\nu\sigma,}{}^{\alpha\beta}\Big) \\ &+ 4 a_k R_{\rho\sigma}\Big(g_{\mu\nu} g^{\alpha\beta} - \delta_{\mu\nu,}{}^{\alpha\beta}\Big) + b_k R^{\alpha\beta}\Big(\delta_{\mu\nu,\rho\sigma} - \tfrac{1}{2} g_{\mu\nu} g_{\rho\sigma}\Big) \\ &+ 2 b_k R_{\mu\rho\nu\sigma} g^{\alpha\beta} + 2 b_k g_{\mu\nu} R_\rho{}^{(\alpha} \delta_\sigma{}^{\beta)} - 4 b_k g_{\mu\rho} R_\nu{}^{(\alpha} \delta_\sigma{}^{\beta)} \,,\end{aligned} \tag{6.1.25}$$

in the non-minimal two derivative term, and

$$\begin{aligned}\hat{W}_{\mu\nu\rho\sigma} = &\Big(a_k R - \frac{Z_k}{2}\Big)\Big(R_{\mu\rho\nu\sigma} + 3 R_{\mu\rho} g_{\nu\sigma} - 2 g_{\mu\nu} R_{\rho\sigma}\Big) + 2 a_k R_{\mu\nu} R_{\rho\sigma} \\ &- \tfrac{1}{2}\Big(\delta_{\mu\nu,\rho\sigma} - \tfrac{1}{2} g_{\mu\nu} g_{\rho\sigma}\Big)\Big(2 Z_k \Lambda_k - Z_k R + a_k R^2 + b_k R_{\alpha\beta} R^{\alpha\beta}\Big) \\ &+ b_k \Big(R_{\mu\rho} R_{\nu\sigma} - g_{\mu\nu} R^{\alpha\beta} R_{\rho\alpha\sigma\beta} - g_{\mu\nu} R_\rho{}^\alpha R_{\alpha\sigma} \\ &\qquad + 3 g_{\mu\rho} R^{\alpha\beta} R_{\nu\alpha\sigma\beta} + 2 R_\mu{}^\alpha{}_\nu{}^\beta R_{\rho\alpha\sigma\beta}\Big) \,,\end{aligned} \tag{6.1.26}$$

where the symmetries ($\mu \leftrightarrow \nu$), ($\rho \leftrightarrow \sigma$) and (($\mu\nu$) $\leftrightarrow$ ($\rho\sigma$)) in the lower indices are understood implicitly in all of these expressions.[1] The unit operator on symmetric 2-tensors is here denoted by $\delta_{\mu\nu,\rho\sigma} = \frac{1}{2}(g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho})$. For the result in general spacetime dimensions the Euler term can easily be included, because its second variation $\delta^2 E \propto R$ includes at least one curvature in each term. Therefore, the following discussion applies entirely unchanged.

To find the inverse of the operator $\mathcal{H}$ given in (6.1.23) accompanied by the cutoff $\mathcal{R}_k$, following (4.3.18) it is cast in the form

$$\mathcal{H}_{\mu\nu\rho\sigma} + \mathcal{R}^{\text{grav}}_{k,\mu\nu\rho\sigma} = \mathcal{Q}_{\mu\nu\rho\sigma} + \mathcal{V}_{\mu\nu\rho\sigma} \,, \tag{6.1.27}$$

with $\mathcal{V}$ containing all terms with at least one curvature. The curvature free part is given

[1] At this stage it is consistent to drop all terms containing derivatives of curvatures, since these do not carry any information about the flow of the coupling constants contained in the ansatz (6.1.1).

by

$$\begin{aligned}\mathcal{Q}_{\mu\nu\rho\sigma} = \mathcal{P}(\Delta)\ \delta_{\mu\nu,\rho\sigma} + \widehat{\mathcal{P}}(\Delta)\ g_{\mu\nu}g_{\rho\sigma} \\ - \tfrac{1}{4}Z_k\Big(g_{\mu\nu}D_\rho D_\sigma + g_{\mu\nu}D_\sigma D_\rho + g_{\rho\sigma}D_\mu D_\nu + g_{\rho\sigma}D_\sigma D_\rho\Big) \\ + \tfrac{1}{8}Z_k\Big(g_{\mu\rho}D_\nu D_\sigma + g_{\mu\rho}D_\sigma D_\nu + g_{\mu\sigma}D_\nu D_\rho + g_{\mu\sigma}D_\rho D_\nu \\ + g_{\nu\rho}D_\mu D_\sigma + g_{\nu\rho}D_\sigma D_\mu + g_{\nu\sigma}D_\mu D_\rho + g_{\nu\sigma}D_\rho D_\mu\Big)\,,\end{aligned} \tag{6.1.28}$$

with all symmetries written explicitly, and the functions of the Laplacian defined as

$$\begin{aligned}\mathcal{P}(\Delta) &= \tfrac{1}{2}b_k\, P_k(\Delta)^2 + \tfrac{1}{2}Z_k\,\Delta - Z_k\Lambda_k\,, \\ \widehat{\mathcal{P}}(\Delta) &= -\tfrac{1}{2}b_k\Big(1 + \tfrac{b_k}{4a_k}\Big)\,P_k(\Delta)^2 - \tfrac{1}{2}Z_k\,\Delta + \tfrac{1}{2}Z_k\Lambda_k\,.\end{aligned} \tag{6.1.29}$$

For the construction of the cutoff, we chose $\mathcal{R}_k$ in such a way that it regulates the highest highest power of the Laplacians appearing in the kinetic terms, following the replacement rule $\Delta \to P_k(\Delta) \equiv \Delta + R_k(\Delta/k^2)$. Thus we have

$$\mathcal{R}^{\text{grav}}_{k,\mu\nu\rho\sigma}(\Delta) = K_{\mu\nu\rho\sigma}\Big(P_k(\Delta)^2 - \Delta^2\Big)\,, \tag{6.1.30}$$

for the cutoff operator in (6.1.27).

Inverting the Gravitational Operator

Using the projector technique (4.3.20) for the decomposition of 2-tensors (4.1.41), the inverse of $\mathcal{Q}$ (6.1.28) in flat space is found in the form

$$\begin{aligned}{\mathcal{Q}_0^{-1}}_{\alpha\beta\mu\nu} =& D_\alpha D_\beta D_\mu D_\nu\ f_1(\Delta) + \Big(g_{\alpha\beta}D_\mu D_\nu + g_{\mu\nu}D_\alpha D_\beta\Big)\ f_2(\Delta) \\ &+ \Big(g_{\alpha\mu}D_\beta D_\nu + g_{\alpha\nu}D_\beta D_\mu + g_{\beta\mu}D_\alpha D_\nu + g_{\beta\nu}D_\alpha D_\mu\Big)\ f_3(\Delta) \\ &+ g_{\alpha\beta}g_{\mu\nu}\ f_4(\Delta) + \delta_{\alpha\beta,\mu\nu}\ f_5(\Delta)\,.\end{aligned} \tag{6.1.31}$$

Here the functions of the Laplacian read

$$\begin{aligned}f_1(\Delta) &= \tfrac{1}{4}(d-2)Z_k^2\Big(\mathcal{P} + 2\widehat{\mathcal{P}} + \tfrac{1}{2}Z_k\Delta\Big)X^{-1}\,, \\ f_2(\Delta) &= \tfrac{1}{2}Z_k\Big(\mathcal{P} - \tfrac{1}{2}Z_k\Delta\Big)\Big(\mathcal{P} + 2\widehat{\mathcal{P}} + \tfrac{1}{2}Z_k\Delta\Big)X^{-1}\,, \\ f_3(\Delta) &= -\tfrac{1}{4}Z_k\Big(d\mathcal{P}\widehat{\mathcal{P}} + \mathcal{P}^2 - \tfrac{1}{2}(d-1)Z_k\Delta(2\widehat{\mathcal{P}} + \tfrac{1}{2}Z_k\Delta)\Big)X^{-1}\,, \\ f_4(\Delta) &= -\Big(\mathcal{P} - \tfrac{1}{2}Z_k\Delta\Big)\Big(\mathcal{P}\widehat{\mathcal{P}} - Z_k\widehat{\mathcal{P}}\Delta - \tfrac{1}{4}Z_k^2\Delta^2\Big)X^{-1}\,, \\ f_5(\Delta) &= \mathcal{P}^{-1}\,, \\ X(\Delta) &= \mathcal{P}\Big(\mathcal{P} - \tfrac{1}{2}Z_k\Delta\Big)\Big(d\mathcal{P}\widehat{\mathcal{P}} + \mathcal{P}^2 - \tfrac{1}{2}(d-1)Z_k\Delta(2\widehat{\mathcal{P}} + \tfrac{1}{2}Z_k\Delta)\Big)\,,\end{aligned} \tag{6.1.32}$$

given in terms of the functions (6.1.29) and the common denominator denoted as $X(\Delta)$. Before we can write the curvature expansion of the inverted operator $\mathcal{Q}$, the correction $\mathcal{W}^{\text{grav}} = 1 - \mathcal{Q}\mathcal{Q}_0^{-1}$ defined in (4.3.25) must be computed. Thus we evaluate this expression and rewrite it in terms of commutators to find

$$
\begin{aligned}
&\mathcal{Q}_{\rho\sigma}{}^{\alpha\beta}\mathcal{Q}_0^{-1}{}_{\alpha\beta\mu\nu} - \delta_{\rho\sigma,\mu\nu} = \\
&[\mathcal{P}, D_\rho D_\sigma D_\mu D_\nu]\, f_1 + \Bigg\{ - \tfrac{1}{2} g_{\rho\sigma}[\widehat{\mathcal{P}}\,\Delta, D_\mu D_\nu]\, f_1 \\
&\quad -\tfrac{1}{4} Z_k\, g_{\rho\sigma}\Big([\Delta^2, D_\mu D_\nu] - D^\alpha[\Delta, D_\alpha]D_\mu D_\nu\Big)\, f_1 - \tfrac{1}{4} Z_k\, D_\rho D_\sigma[\Delta, D_\mu D_\nu]\, f_1 \\
&\quad +\tfrac{1}{8} Z_k\Big(2[D_\beta, D_\rho]D_\sigma D^\beta + 3D_\rho[D_\beta, D_\sigma]D^\beta - D_\rho[\Delta, D_\sigma]\Big)D_\mu D_\nu\, f_1 \\
&\quad +\tfrac{1}{2}\, g_{\rho\sigma}[\mathcal{P}, D_\mu D_\nu]\, f_2 + \tfrac{d}{2} g_{\rho\sigma}[\widehat{\mathcal{P}}, D_\mu D_\nu]\, f_2 + \tfrac{1}{4} Z_k\, g_{\rho\sigma}[\Delta, D_\mu D_\nu]\, f_2 \\
&\quad +\tfrac{1}{4} Z_k\, g_{\mu\nu}\Big(- 2D_\rho[\Delta, D_\sigma] + [D_\beta, D_\rho]D^\beta D_\sigma\Big)\, f_2 \\
&\quad -\tfrac{1}{8} Z_k\, g_{\mu\nu}\, g_{\rho\sigma}\Big(D^\alpha[D_\beta, D_\alpha]D^\beta - D^\beta[\Delta, D_\beta]\Big)\, f_2 \\
&\quad +2g_{\mu\rho}[\mathcal{P}, D_\sigma D_\nu]\, f_3 + 2g_{\rho\sigma}[\widehat{\mathcal{P}}, D_\mu D_\nu]\, f_3 \\
&\quad -\tfrac{1}{4} Z_k\Big(- [D_\mu, D_\rho]D_\sigma D_\nu - 2D_\rho[D_\mu, D_\sigma]D_\nu\Big)\, f_3 \\
&\quad -\tfrac{1}{2} Z_k\, g_{\rho\mu}\Big(2D_\sigma[\Delta, D_\nu] - [D_\beta, D_\sigma]D^\beta D_\nu\Big)\, f_3 \\
&\quad -\tfrac{1}{2} Z_k\, g_{\rho\sigma}\Big(- 2D_\mu[\Delta, D_\nu] + [D_\beta, D_\mu]D^\beta D_\nu\Big)\, f_3 \\
&\quad +\tfrac{1}{4} Z_k\, g_{\rho\mu}[D_\nu, D_\sigma]\, f_5 + \Big(\rho \leftrightarrow \sigma\Big)\Bigg\}\,,
\end{aligned}
\tag{6.1.33}
$$

which is understood as acting on a symmetric tensor $h^{\mu\nu}$ to the right, and the argument for the functions $\mathcal{P}(\Delta)$ and $\widehat{\mathcal{P}}(\Delta)$ is suppressed. After evaluating the simple commutators, this expression becomes

$$
\begin{aligned}
&\Big(\mathcal{Q}_{\rho\sigma}{}^{\alpha\beta}\mathcal{Q}_0^{-1}{}_{\alpha\beta\mu\nu} - \delta_{\rho\sigma,\mu\nu}\Big)h^{\mu\nu} = \\
&\Bigg([\mathcal{P}, D_\rho D_\sigma D_\mu D_\nu]\, f_1 - g_{\rho\sigma}[\widehat{\mathcal{P}}\,\Delta, D_\mu D_\nu]\, f_1 + g_{\rho\sigma}[\mathcal{P}, D_\mu D_\nu]\, f_2 + d\, g_{\rho\sigma}[\widehat{\mathcal{P}}, D_\mu D_\nu]\, f_2 \\
&\quad +2\Big(g_{\mu\rho}[\mathcal{P}, D_\sigma D_\nu] + g_{\mu\sigma}[\mathcal{P}, D_\rho D_\nu]\Big)\, f_3 + 4g_{\rho\sigma}[\widehat{\mathcal{P}}, D_\mu D_\nu]\, f_3 \\
&\quad -\tfrac{1}{2} Z_k\, g_{\rho\sigma}[\Delta^2, D_\mu D_\nu]\, f_1 - \tfrac{1}{2} Z_k\, D_\rho D_\sigma[\Delta, D_\mu D_\nu]\, f_1 + \tfrac{1}{2} Z_k\, g_{\rho\sigma}[\Delta, D_\mu D_\nu]\, f_2\Bigg)h^{\mu\nu} \\
&-\tfrac{1}{4} Z_k\Big(2R_\rho{}^\beta{}_\sigma{}^\alpha D_\alpha D_\beta - 3R_\sigma{}^\alpha D_\rho D_\alpha - 3R_\rho{}^\alpha D_\sigma D_\alpha\Big)D_\mu D_\nu h^{\mu\nu}\, f_1 \\
&-\tfrac{1}{2} Z_k\, g_{\rho\sigma} R^{\alpha\beta} D_\alpha D_\beta D_\mu D_\nu h^{\mu\nu}\, f_1 - \tfrac{1}{2} Z_k\, g_{\rho\sigma} R^{\alpha\beta} D_\alpha D_\beta h_\mu{}^\mu\, f_2 \\
&-\tfrac{1}{4} Z_k\Big(2R_\rho{}^\alpha{}_\sigma{}^\beta D_\alpha D_\beta - 3R_\rho{}^\alpha D_\sigma D_\alpha - 3R_\sigma{}^\alpha D_\rho D_\alpha\Big)\, h_\mu{}^\mu\, f_2 \\
&-2Z_k\, g_{\rho\sigma} R^\mu{}_\alpha{}^\nu{}_\beta D_\mu D_\nu h^{\alpha\beta}\, f_3 - \tfrac{1}{4} Z_k\Big(R_{\rho\beta\sigma}{}^\alpha D_\alpha D_\nu h^{\beta\nu} + R_\rho{}^\alpha{}_{\sigma\beta} D_\alpha D_\nu h^{\beta\nu}
\end{aligned}
\tag{6.1.34}
$$

$$
\begin{aligned}
&+2R_\rho{}^\beta{}_{\sigma\alpha}D_\beta D_\nu h^{\alpha\nu} + 2R_\rho{}^\alpha{}_{\sigma\beta}D_\beta D_\nu h^{\alpha\nu} + 4R_{\rho\alpha\beta}{}^\gamma D_\sigma D_\gamma h^{\alpha\beta} + 4R_{\sigma\alpha\beta}{}^\gamma D_\rho D_\gamma h^{\alpha\beta} \\
&-3R_{\beta\sigma}D_\rho D_\nu h^{\beta\nu} - 3R_{\beta\rho}D_\sigma D_\nu h^{\beta\nu} - 2R^\beta{}_\sigma D_\beta D_\nu h_\rho{}^\nu - 2R^\beta{}_\rho D_\beta D_\nu h_\sigma{}^\nu\Big) f_3 \\
&-\tfrac{1}{2}Z_k\Big(2R_{\rho\beta\sigma\nu}h^{\beta\nu} - R_{\sigma\beta}h_\rho{}^\beta - R_{\rho\beta}h_\sigma{}^\beta\Big) f_5\,.
\end{aligned}
$$

The remaining, more complicated commutators are best evaluated using computer algebra software, since they increase in size significantly.

The gravitational contribution is then given in terms of partial traces, following from the curvature expansion formula (4.3.26) for the full dressed propagator. This yields

$$
\begin{aligned}
T^{\mathrm{grav}} &= \mathrm{Tr}\left[\Gamma_k^{\mathrm{grav},(2)} + S^{\mathrm{gf},(2)} + \mathcal{R}_k^{\mathrm{grav}}\right]^{-1}\partial_t\mathcal{R}_k^{\mathrm{grav}} \\
&= \mathrm{Tr}\Big[\mathcal{Q}_0^{-1} + \mathcal{Q}_0^{-1}\mathcal{W}_1^{\mathrm{grav}} - \mathcal{Q}_0^{-1}\mathcal{V}_1\mathcal{Q}_0^{-1} \\
&\qquad + \mathcal{Q}_0^{-1}\mathcal{W}_2^{\mathrm{grav}} + \mathcal{Q}_0^{-1}\mathcal{W}_1^{\mathrm{grav}}\mathcal{W}_1^{\mathrm{grav}} - \mathcal{Q}_0^{-1}\mathcal{V}_2\mathcal{Q}_0^{-1} \\
&\qquad - \mathcal{Q}_0^{-1}\mathcal{W}_1^{\mathrm{grav}}\mathcal{V}_1\mathcal{Q}_0^{-1} - \mathcal{Q}_0^{-1}\mathcal{V}_1\mathcal{Q}_0^{-1}\mathcal{W}_1^{\mathrm{grav}} \\
&\qquad + \mathcal{Q}_0^{-1}\mathcal{V}_1\mathcal{Q}_0^{-1}\mathcal{V}_1\mathcal{Q}_0^{-1}\Big]\,\partial_t\mathcal{R}_k^{\mathrm{grav}} + \mathcal{O}(\mathcal{R}^3)\,,
\end{aligned}
\tag{6.1.35}
$$

where the potential defined in (6.1.27) is split as $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$ into a piece containing only one and containing two occurences of the curvature, respectively. This step is convenient to reduce the computational expense in the automatized evaluation of these partial traces.

Following the algorithm presented in section 4.3, the traces in (6.1.35) can be solved by a replacement rule, making use of identities of the form (3.3.7). Since the resulting expressions are very large, the computation can only be handled reasonably by an implementation of this step on a computer. The qualitative discussion of the result is out of the scope of this thesis. In the next section, we will resort to the evaluation in the perturbative limit to demonstrate its relation to known results.

6.2. Perturbative β Functions and Their Fixed Points

In this section we will re-derive the perturbative β functions of higher derivative gravity [20, 141] from the general discussion in the last section. Since the RG equation introduces a dimensionful scale parameter, it keeps track of the quadratic and quartic divergences, which drop out when using dimensional regularization.[2] Similar studies have been carried

[2]More precisely, the dimensional regularization scheme allows to discard all but the power-counting marginal divergences via analytic continuation.

out before in [142,143], where it was observed that the contribution from terms with higher divergence has a drastic effect on the fixed point structure of the renormalization group flow. These contributions shift the fixed point for Newton's constant and cosmological constant to non-zero values, rendering the theory asymptotically safe instead of asymptotically free. The main purpose of the present section is to demonstration that the proposed algorithm used in the evaluation of the traces recovers these results. Surprisingly, the regularization scheme intrinsic to this algorithm unveils certain features in the fixed point structure of the theory, that have not been stressed before.

For the perturbative computation, it is sufficient to solve the RG equation in a 1-loop limit. Therefore we write the equation in the 1-loop form (2.3.20), from which it can be inferred that the scale dependence of all coupling constants in its r.h.s. can here be neglected. For the scale derivative of the cutoff operator (6.1.30), we can thus write

$$\partial_t \mathcal{R}^{\text{grav}}_{k,\mu\nu\rho\sigma}(\Delta) = 2K_{\mu\nu\rho\sigma} P_k(\Delta)\, \partial_t R_k(\Delta)\,, \tag{6.2.1}$$

with the tensor $K_{\mu\nu\rho\sigma}$ given in (6.1.24). Motivated by this form of the cutoff, it is convenient to define the operators

$$V_{\mu\nu\rho\sigma}{}^{(\alpha\beta)} := [K^{-1}]_{\mu\nu}{}^{\gamma\delta} \hat{D}_{\gamma\delta\rho\sigma}{}^{(\alpha\beta)}\,, \qquad U_{\mu\nu\rho\sigma} := [K^{-1}]_{\mu\nu}{}^{\gamma\delta} \hat{W}_{\gamma\delta\rho\sigma}\,, \tag{6.2.2}$$

in relation to the pieces $\hat{D}$ (6.1.25) and $\hat{W}$ (6.1.26). The inverse of K is easily found to be

$$[K^{-1}]_{\mu\nu}{}^{\gamma\delta} = \tfrac{2}{b_k}\, \delta_{\mu\nu,}{}^{\gamma\delta} - \tfrac{2(4a_k+b_k)}{b_k(-4a_k+4da_k+db_k)}\, g_{\mu\nu} g^{\gamma\delta}\,. \tag{6.2.3}$$

In terms of these operators, the second variation of the action (6.1.23), supplemented by the cutoff operator (6.1.30) assumes the form

$$\mathcal{H}_{\mu\nu\rho\sigma} + \mathcal{R}^{\text{grav}}_{k,\mu\nu\rho\sigma} = K_{\mu\nu}{}^{\gamma\delta}\Big(P_k(\Delta)^2\, \delta_{\gamma\delta,\rho\sigma} + V_{\gamma\delta\rho\sigma}{}^{(\alpha\beta)} \bar{D}_\alpha \bar{D}_\beta + U_{\gamma\delta\rho\sigma} \Big)\,. \tag{6.2.4}$$

To find the inverse of this expression in the perturbative limit, we will expand in V and U. In contrast to the full resummation (6.1.33) performed in the last section, this computation takes only finite orders of Z_k into account.

Thus treating V and U as interaction vertices of mass-dimension two and four, respectively, the inverse of the regulated propagator in the gravitational trace (6.1.12) is given by

$$\begin{aligned}
&\Big(P_k(\Delta)^2\, \delta_{\mu\nu,\rho\sigma} + V_{\mu\nu\rho\sigma}{}^{(\alpha\beta)} \bar{D}_\alpha \bar{D}_\beta + U_{\mu\nu\rho\sigma} \Big)^{-1} \\
&= \Big(\delta_{\mu\nu,\rho\sigma}\, P_k(\Delta)^{-2} + V_{\mu\nu\rho\sigma}{}^{(\alpha\beta)} \bar{D}_\alpha \bar{D}_\beta\, P_k(\Delta)^{-4} + U_{\mu\nu\rho\sigma}\, P_k(\Delta)^{-4} \\
&\quad + V_{\mu\nu}{}^{\lambda\tau(\alpha\beta)} V_{\lambda\tau\rho\sigma}{}^{(\gamma\delta)} \bar{D}_\alpha \bar{D}_\beta \bar{D}_\gamma \bar{D}_\delta\, P_k(\Delta)^{-6} \Big) + \mathcal{O}(\mathcal{R}^3)\,,
\end{aligned} \tag{6.2.5}$$

up to terms with higher powers of the curvature, which are outside the truncation considered here. The gravitational trace is in turn found as

$$
\begin{aligned}
T^{\text{grav}} &= \tfrac{1}{2}\mathrm{Tr}_2\left[\left(P_k(\Delta)^2\,\delta_{\mu\nu,\rho\sigma} + V_{\mu\nu\rho\sigma}{}^{(\alpha\beta)}\bar{D}_\alpha\bar{D}_\beta + U_{\mu\nu\rho\sigma}\right)^{-1} 2P_k(\Delta)\,\partial_t R_k(\Delta)\right] \\
&= \mathrm{Tr}_2\left[\tfrac{\partial_t R_k}{P_k}\right] - \mathrm{Tr}_2\left[U\;\tfrac{\partial_t R_k}{P_k^3}\right] - \mathrm{Tr}_2\left[V^{(\alpha\beta)}\bar{D}_\alpha\bar{D}_\beta\;\tfrac{\partial_t R_k}{P_k^3}\right] \\
&\quad + \mathrm{Tr}_2\left[V^{(\alpha\beta)}V^{(\gamma\delta)}\bar{D}_\alpha\bar{D}_\beta\bar{D}_\gamma\bar{D}_\delta\;\tfrac{\partial_t R_k}{P_k^5}\right] + \mathcal{O}(\mathcal{R}^3)\,,
\end{aligned}
\tag{6.2.6}
$$

where the tensor K cancelled between the second variation and the derivative of the cutoff term (6.2.1), and summation over internal indices is implicit in the last line.

The evaluation of these traces is remarkably easy when employing the off-diagonal heat kernel methods developed in chapter 3. Each of the four partial traces can be given without specifying the tensors V and U, since all non-minimal derivative operators are already written explicitly in the expansion (6.2.6). In $d = 4$ dimensions and using the optimized cutoff (C.10) as shape function, the automatized computation yields

$$
\begin{aligned}
T^{\text{grav}} = \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,\Big[&10k^4 + k^2\left(\tfrac{10}{3}R + \tfrac{1}{6}V_i{}^i{}_\mu{}^\mu\right) + \tfrac{5}{18}R^2 - \tfrac{1}{9}R_{\mu\nu}R^{\mu\nu} - \tfrac{8}{9}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \\
&- \tfrac{1}{6}R_{\mu\nu}V_i{}^{i\mu\nu} + \tfrac{1}{12}RV_i{}^i{}_\mu{}^\mu - U_i{}^i + \tfrac{1}{48}V_{ij\mu}{}^\mu V^{ji}{}_\nu{}^\nu + \tfrac{1}{24}V_{ij\mu\nu}V^{ji\mu\nu}\Big]\,.
\end{aligned}
\tag{6.2.7}
$$

This result encompasses the well-known expressions for the universal four-derivative terms. The explicit gravitational trace is found by substituting the rather lengthy expressions for V and U (6.2.2).

Finally, we can combine the three traces (6.1.12) to obtain the total result. The final form of the RG equation with the gravitational contribution (6.2.7) and the ghost contributions (6.1.20) with the parameters (6.1.21) substituted, yields

$$
\begin{aligned}
\partial_t\Gamma_k =& \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\bigg[\frac{133}{20}C^2 - \frac{196}{45}E + \frac{5}{36}\left(1 + 8\omega_k + 12\omega_k^2\right)R^2 \\
&- \left\{\tfrac{Z_k\sigma_k}{12\omega_k}p_7 + \tfrac{k^2}{72(1-2\omega_k)}p_4 - \tfrac{k^2}{12(1-2\omega_k)^2}\,p_2\ln\left(\tfrac{2}{3}(1+\omega_k)\right)\right\}R \\
&+ \tfrac{Z_k^2\sigma_k^2(1+20\omega_k)}{8\omega_k^2} + \tfrac{Z\sigma\left((4+112\omega_k)\Lambda_k + k^2p_5\right)}{6\omega_k} + \tfrac{k^4p_3}{36(1-2\omega_k)} + \tfrac{k^4\,p_1}{6(1-2\omega_k)^2}\ln\left(\tfrac{2}{3}(1+\omega_k)\right)\bigg]\,.
\end{aligned}
\tag{6.2.8}
$$

Herein and in the following, we abbreviated

$$
\begin{aligned}
&p_1 = 6 - 96\omega_k - 48\omega_k^2\,, && p_2 = 65 + 28\omega_k + 8\omega_k^2\,, && p_3 = 162 - 540\omega_k\,, \\
&p_4 = 35 - 218\omega_k - 352\omega_k^2\,, && p_5 = -2 - 20\omega_k\,, && p_6 = 1 + 86\omega_k + 40\omega_k^2\,, \\
&p_7 = 3 + 26\omega_k - 40\omega_k^2\,,
\end{aligned}
\tag{6.2.9}
$$

to express the non-universal terms.

Discussion of the Results

The β functions $\partial_t g_i = \beta_{g_i}$ governing the scale dependence of the coupling constants contained in (6.1.1) can finally be read off by comparison with the coefficients of the curvature polynomials appearing in (6.2.8). For the marginal couplings, the β functions are universal in the sense that they do not depend on the regularization scheme. This is due to the properties (C.9) of the threshold functions. Explicitly, we find

$$\begin{aligned}
\beta_\sigma = & -\tfrac{1}{(4\pi)^2}\,\frac{133}{10}\,\sigma^2\,,\\
\beta_\theta = & -\tfrac{1}{(4\pi)^2}\,\frac{7(56+171\theta)}{90}\sigma\,,\\
\beta_\omega = & -\tfrac{1}{(4\pi)^2}\,\frac{(25+1098\omega+200\omega^2)}{60}\sigma\,,
\end{aligned} \tag{6.2.10}$$

with the index k dropped. Despite our quite different computational approach, this result agrees with earlier computations [141, 142]. The β functions governing the running of Newton's constant and the cosmological constant are most conveniently written in terms of the dimensionless quantities $g = k^2/16\pi Z_k$ and $\lambda = k^{-2}\Lambda$. Here we obtain

$$\begin{aligned}
\beta_g =& \, 2g - \tfrac{\sigma g}{192\pi^2\omega}p_7 - \tfrac{g^2}{12\pi}\left[\tfrac{1}{6(1-2\omega)}\,p_4 - \tfrac{1}{(1-2\omega)^2}\,p_2 \ln\left(\tfrac{2}{3}(1+\omega)\right)\right]\,,\\
\beta_\lambda =& -2\lambda + \tfrac{\sigma^2(1+20\omega^2)}{4096\pi^3 g\omega^2} + \tfrac{g}{12\pi(1-2\omega)^2}(p_1+\lambda p_2)\ln\left(\tfrac{2}{3}(1+\omega)\right)\\
& + \tfrac{g}{72\pi(1-2\omega)}(p_3-\lambda p_4) + \tfrac{\sigma}{192\pi^2\omega}(p_5+\lambda p_6)\,.
\end{aligned} \tag{6.2.11}$$

These are non-universal results and will change in their explicit form with the applied regularization scheme. Therefore they are expected to differ from the derivation [142], which employs the RG equation with a Type III cutoff, adjusted to the second variation with $\mathcal{R}_k = \Gamma_k^{(2)} R_k$. In particular, the logarithmic terms in (6.2.11) are a novel feature in the present Type I cutoff computation. Their appearance can be traced back to the denominators $(P_k - a\Delta)$ appearing in the ghost sector (6.1.15), which are absent in the spectrally adjusted case.

We will close this section with a discussion of the fixed point structure of the β functions (6.2.10) and (6.2.11). The equation $\beta_\sigma(g_i^*) = 0$ has the sole solution $\sigma^* = 0$, indicating that σ_k vanishes logarithmically at high energies. Thus the coupling is asymptotically free. The remaining equations in (6.2.10) give rise to the known fixed point solutions

$$\mathrm{FP}_{1,2}: \qquad \sigma^* = 0\,, \qquad \theta^* = -171/56\,, \qquad \omega^*_{1,2} = \{-0.00228\,,\,-5.47\,\}\,, \tag{6.2.12}$$

with two possible values for ω. However, substituting this result into (6.2.11) we find that only FP_1 constitutes a fixed point of the full system, at the values

$$\text{NGFP}: \qquad \sigma^* = 0\,, \quad \theta^* = -171/56\,, \quad \omega^* = -0.00228\,, \quad \lambda^* = 0.39\,, \quad g^* = 2.39\,. \tag{6.2.13}$$

This non-Gaussian fixed point is UV-attractive in all five coupling constants. Similarly to the computations in [142],

the functional RG scheme employed here also takes into account quadratic and quartic divergences in the regularization procedure [142, 143]. The latter give rise to a fundamental contribution to the flow of Newton's constant and the cosmological constant, whose UV-behavior is then governed by a non-Gaussian fixed point λ^*, g^* instead of the Gaussian fixed point seen within dimensional regularization.

Concerning the second fixed point found before, we note that the non-universal β functions are well-defined in the region $\omega > -1$ only. This limit can be traced back to the requirement of positivity of the ghost operator (6.1.13), and manifests itself in the appearance of the logarithmic terms in (6.1.20). Since FP_2 is not within this bound, it cannot be completed to a FP on the full theory space. We take this as a strong indication that this fixed point is unphysical, establishing a unique non-Gaussian fixed point for the curvature squared case, at least in the perturbative limit analysed here.

7. Conclusions

Renormalization group techniques are capable to reveal many structurally non-perturbative features of a theory. In this thesis, we reviewed the derivation [35, 82] of an exact renormalization group equation and highlighted its relation to perturbative methods in quantum field theory. A main motivation to consider such an equation as a central object in quantum field theory is the ongoing search for a proper UV completion of gravity in a quantized description. Instead of pursuing to construct alternative models within a perturbative approach, the concept of asymptotic safety may explain the mechanism by which gravity can be quantized consistently. This scenario is presented as a direct generalization of the perturbative renormalization, as it is successfully applied in Yang-Mills theories. This renders Quantum Einstein Gravity a viable option for the realization of gravitation in nature.

The practical use of functional renormalization group equations is mathematically rather involved. This is especially so in the presence of gauge symmetries, which give rise to complicated non-minimal differential operators. Since computations on a non-trivial curved spacetime manifold furthermore require to express momentum integrals via operator traces, renormalization group studies of quantum gravity depend on heat kernel methods. The technique, being originally developed for the evaluation of amplitudes in quantum field theories on curved spacetimes, generalizes the loop integrals to a trace that captures the spectrum of the corresponding covariant Laplace operator. As a mathematical tool, the heat kernel expansion is very useful in many calculations. The expansion of such traces in terms of curvature monomials constitute reoccurring universal coefficients, which reappear in perturbation theory on a non-trivial background manifold as well as in the solution of renormalization group equations. Once these are computed, many applications become accessible, justifying an interest in the heat kernel in its own respect.

In this thesis, the deWitt-method was used to recursively determine the off-diagonal heat kernel expansion of a Laplace operator on a general bundle, including an arbitrary endomorphism, up to third order in the curvature tensors. The algorithm has been implemented on computer algebra systems [102, 103], making the manual handling of the huge expressions for higher orders unnecessary. Our results generalize previous calculations [68, 99, 100, 104] by leaving the internal space and spacetime dimension unspecified. All the results for the traced expansion of the heat kernel of the Laplace operator acting on scalar-, vector- and symmetric tensor fields are conveniently summarized in table 4.2. Our results allow the systematic evaluation of the traced heat kernel, including insertions of covariant derivative operators with open indices inside the trace. This is crucial for the treatment of gauge theories without simplifying gauge fixing or background choices.

Furthermore projected heat kernel traces over the subspace of transversally constrained fields were defined and computed, extending the applicability of heat kernel techniques in general. Notably, the Laplace operator projected on the transverse vector subspace becomes non-local due to the projection operators. Owed to this non-locality, the Seeley-deWitt coefficients of the heat trace, on a general d-dimensional manifold, diverge starting from order $\mathcal{R}^{d/2+1}$. The divergences are of infrared nature and appear as dimensional poles, when dimensional regularization is adopted. Remarkably, we showed that these singularities cancel for transversal vector fields if spacetime is an Einstein manifold, ensuring that the standard Laplacian commutes with the projector onto the transverse subspace.

The newly computed non-minimal heat-traces provide the basis for an algebraic algorithm to find approximate solutions of the functional renormalization group equation systematically. This algorithm, presented in chapter 4, allows to extent previous approximations of the renormalization group flow of gauge theories and gravity in numerous ways. It is capable of inverting any operator given as a polynomial of covariant derivatives in terms of an expansion in any background object. In combination with the background field method, computations can be done in a manifestly covariant way, and to arbitrary order in the curvature tensors. The occurring non-minimal kinetic operators can be handled based on the off-diagonal heat kernel techniques. Thereby, each step in the computation can be carried out while leaving the choice of gauge fixing and background geometry completely arbitrary.

Thus overcoming the mathematical limitations of previously known methods, the extension of effective gauge field theories to include higher derivative terms is now straightforward. In particular for the case of gravity, the most sophisticated renormalization group studies [27,31,57,61,63,64] to date are mostly restricted to a spherical manifold (for

$f(R)$-type truncations), or an Einstein manifold (including the Weyl-squared invariant) as background. In this way it is not possible to distinguish all invariants that would be independent on a generic manifold. The β functions obtained are accordingly describing the running of certain linear combinations of coupling constants only. While providing useful insights on the stability of the non-Gaussian fixed point, these computations are insufficient for example to properly answer questions about the number of relevant directions. Therefore, to extend the present computations of the gravitational renormalization group flow and to account for more involved interactions containing tensor structures, the background must be left arbitrary.

One of the next milestones in the research program focussing around asymptotically safe gravity is the inclusion of all curvature squared interactions. This computation is demonstrated in chapter 6, where the algorithm is applied to invert the full second variation in terms of a curvature expansion without imposing a specific background geometry. The resulting traces encoding the non-perturbative running of the five coupling constants involved can be evaluated with an implementation of the heat kernel rules in computer algebra. It is further shown that the known results of higher derivative gravity are reproduced in the perturbative limit from the renormalization group equation. The universal part of the β functions agrees exactly in any method of computation and admits two non-Gaussian fixed points. The present work improves upon this finding as we explicitly show that the existence of one of the fixed points depends on the choice of regularization scheme. This indicates that it is most probably unphysical, and implies a unique UV fixed point in this computational setup.

Demonstrating the systematic solution of the RG equation in the Einstein-Hilbert truncation we are able to establish the background independence of the resulting β functions explicitly. In the course of this computation, it is shown that standard gauge fixing and ghost terms create contributions to the renormalization group flow which almost cancel mutually and are thus negligible. This phenomenon can be traced back to the modified BRST symmetry of the generating functionals subject to the RG equation. Furthermore, contributions from gauge dependent terms and physical proper vertices decouple in the Landau gauge. In this context, a method of analysis is proposed, which allows to compare β functions by their full term structure.

The gravitational renormalization group flow in the Einstein-Hilbert truncation supplemented by a field renormalization of the ghosts is investigated in detail. The latter induces non-trivial corrections to the β functions for Newton's constant and the cosmological constant, giving rise to a non-Gaussian fixed point with strikingly similar properties compared to the standard Einstein-Hilbert truncation. The ghost-improvement leads to a significant decrease of the unphysical cutoff scheme dependence. This finding further substantiates the asymptotic safety scenario of quantum gravity.

The β functions found in this computation contain the standard result explicitly, revealing precisely the effect of the newly considered terms. This fact opens interesting possibilities for the investigation of the algebraic structure giving rise to the gravitational fixed point. This line of research may contribute to reveal more systematics behind the renormalization group flow and may point the way to construct a proof of the existence of an UV fixed point for gravity in the future.

With the technical limitations of earlier methods of computation out of the way, we understand how to extend the present approximations further by the inclusion of curvature cubed and higher terms. Thus the effect of the explicit tensor structure of higher derivative gravity actions can be investigated, providing a highly non-trivial test of the asymptotic safety scenario. Likewise, the developed techniques are applicable to Yang-Mills theory. Here, they could contribute to an improvement of the understanding of the phase structure of infrared QCD, allowing for higher derivative effective actions with arbitrary spacetime dimension and gauge parameter.

Since the neccessary computational effort grows quickly when taking such extensions into account, an automation of the mathematical steps involved in the evaluation of the exact RG equation is not only convenient but inevitable to handle increasingly complicated tensor structures. The universal algorithm provides this possibility and has already been implemented in important parts using Mathematica to perform many computations presented in this thesis. A full automation is clearly viable, from which the future developments of renormalization group studies will greatly profit.

A. Basis of Curvature Monomials on Riemannian Manifolds

As discussed in chapter 3, to write covariant gravitational action functionals, a basis of curvature momonials $\mathcal{R}_i^n$ is required. Such a basis is also conveniently used to represent the results for the early-time expansion of the heat kernel, so that it is given in terms of independent coefficients for a given order n. Because of the symmetries and Bianchi identities fulfilled by the Riemann curvature tensor, it is not entirely trivial to find a non-reducible set of such monomials. In this appendix, we construct such a basis of of fully contracted curvature tensors up to third order, following the results of [104, 144, 145]. Throughout this appendix, we will consider the case of general d-dimensional Riemannian manifolds without boundaries.

A.1. Curvature Basis for General Manifolds

The heat kernel expansions (3.2.27) and (4.2.29) require a basis for all curvature invariants build from six or less covariant derivatives. The Riemann curvature tensor defined in (3.1.4) has the symmetries

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}\,, \qquad R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}\,, \qquad R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}\,, \tag{A.1.1}$$

and is a suitable covariant object for this purpose. Following [144] the resulting basis contains 15 elements, which we choose as follows[1]

$$
\begin{aligned}
&\mathcal{R}^0 = 1\,, && && \\
&\mathcal{R}^1 = R\,, && && \\
&\mathcal{R}^2_1 = R^2\,, && \mathcal{R}^2_2 = R_{\mu\nu}R^{\mu\nu}\,, && \mathcal{R}^2_3 = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \\
&\mathcal{R}^3_1 = R\, D^2 R\,, && \mathcal{R}^3_2 = R_{\mu\nu}\, D^2 R^{\mu\nu}\,, && \mathcal{R}^3_3 = R^3\,, \\
&\mathcal{R}^3_4 = R R_{\mu\nu}R^{\mu\nu}\,, && \mathcal{R}^3_5 = R_\mu{}^\nu R_\nu{}^\alpha R_\alpha{}^\mu\,, && \mathcal{R}^3_6 = R_{\mu\nu}R_{\alpha\beta}R^{\mu\alpha\nu\beta}\,, \\
&\mathcal{R}^3_7 = R R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\,, && \mathcal{R}^3_8 = R_{\mu\nu}R^{\mu\alpha\beta\gamma}R^\nu{}_{\alpha\beta\gamma}\,, && \mathcal{R}^3_9 = R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu}\,, \\
&\mathcal{R}^3_{10} = R^\alpha{}_\mu{}^\beta{}_\nu R^\mu{}_\rho{}^\nu{}_\sigma R^\rho{}_\alpha{}^\sigma{}_\beta\,, && &&
\end{aligned}
\tag{A.1.2}
$$

with $D^2 \equiv g^{\mu\nu}D_\mu D_\nu = -\Delta$. Note that in $d = 4$ dimensions the integrand of (A.3.11) vanishes, so that only 9 independent $\mathcal{R}^3$ terms remain.

To arrive at this basis, one has to find relations which allow to express any additional curvature monomials in terms of the above by the use of the first and second Bianchi identities

$$R_{\mu[\nu\alpha\beta]} = 0\,, \tag{A.1.3}$$

and

$$R_{\mu\nu\alpha\beta;\rho} + R_{\mu\nu\beta\rho;\alpha} + R_{\mu\nu\rho\alpha;\beta} = 0\,, \quad R_{\mu\nu\alpha\beta}{}^{;\beta} = R_{\mu\alpha;\nu} - R_{\nu\alpha;\mu}\,, \quad R_{\mu\nu}{}^{;\nu} = \tfrac{1}{2}R_{;\mu}\,. \tag{A.1.4}$$

A very useful collection of curvature identities implied by these identities has been given in [145]. For completeness we summarize the ones important for our construction in the following.

At order $\mathcal{R}^2$, (A.1.3) implies

$$R^{\mu\nu\alpha\beta}R_{\mu\alpha\nu\beta} = \tfrac{1}{2}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\,, \tag{A.1.5}$$

while at order $\mathcal{R}^3$ there are three relations between different contractions of the Riemann tensor

$$
\begin{aligned}
&R^{\mu\alpha\nu\beta}R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma} = \tfrac{1}{2}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu}\,, \\
&R_{\alpha\beta}{}^{\rho\sigma}R^{\alpha\mu\beta\nu}R_{\rho\mu\sigma\nu} = \tfrac{1}{4}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu}\,, \\
&R_{\mu\alpha\nu\beta}R^{\mu\rho\nu\sigma}R^\alpha{}_\sigma{}^\beta{}_\rho = -\tfrac{1}{4}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} + R^\alpha{}_\mu{}^\beta{}_\nu R^\mu{}_\rho{}^\nu{}_\sigma R^\rho{}_\alpha{}^\sigma{}_\beta\,.
\end{aligned}
\tag{A.1.6}
$$

[1] Taking total derivatives into account, there is one more invariant at $\mathcal{O}(\mathcal{R}^2)$ and seven additional curvature monomials at $\mathcal{O}(\mathcal{R}^3)$, see eqs. (A.1.9) and (A.1.10) below.

In addition the combination of (A.1.3) and (A.1.4) allows to derive

$$\begin{aligned} R_{\mu\nu\alpha\beta}\Delta R^{\mu\nu\alpha\beta} = & -4R_{\mu\nu;\alpha\beta}R^{\mu\alpha\nu\beta} - 2R_{\mu\nu}R^{\mu}_{\alpha\beta\gamma}R^{\nu\alpha\beta\gamma} + R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} \\ & + 4R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}R^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta}\,, \end{aligned} \tag{A.1.7}$$

and

$$R_{\mu\nu}R^{\mu\alpha;\nu}{}_{\alpha} = \tfrac{1}{2}R_{;\mu\nu}R^{\mu\nu} + R_{\mu}{}^{\nu}R_{\nu}{}^{\alpha}R_{\alpha}{}^{\mu} - R_{\mu\nu}R_{\alpha\beta}R^{\mu\alpha\nu\beta}\,. \tag{A.1.8}$$

Once the curvature monomials appear under the volume integral, the condition of working on a manifold without boundary allows to integrate by parts freely. At order $\mathcal{R}^2$ this eliminates the surface term

$$\int d^dx\sqrt{g}\,\Delta R = 0\,, \tag{A.1.9}$$

while at order $\mathcal{R}^3$ we obtain the seven additional identities

$$\begin{aligned} \int d^dx\sqrt{g}\,\Delta\Delta R &= 0\,, \\ \int d^dx\sqrt{g}\,R^{\mu\nu}R_{;\mu\nu} &= \tfrac{1}{2}\int d^dx\sqrt{g}\,\mathcal{R}^3_1, \\ \int d^dx\sqrt{g}\,R_{\mu\nu;\alpha\beta}R^{\mu\alpha\nu\beta} &= \int d^dx\sqrt{g}\,\left[\mathcal{R}^3_2 - \tfrac{1}{4}\mathcal{R}^3_1 - \mathcal{R}^3_5 + \mathcal{R}^3_6\right]\,, \\ \int d^dx\sqrt{g}\,R_{;\mu}R^{;\mu} &= -\int d^dx\sqrt{g}\,\mathcal{R}^3_1\,, \\ \int d^dx\sqrt{g}\,R_{\alpha\beta;\mu}R^{\alpha\beta;\mu} &= -\int d^dx\sqrt{g}\,\mathcal{R}^3_2\,, \\ \int d^dx\sqrt{g}\,R_{\alpha\beta;\mu}R^{\alpha\mu;\beta} &= \int d^dx\sqrt{g}\,\left[-\tfrac{1}{4}\mathcal{R}^3_1 - \mathcal{R}^3_5 + \mathcal{R}^3_6\right]\,, \\ \int d^dx\sqrt{g}\,R_{\mu\nu\alpha\beta;\rho}R^{\mu\nu\alpha\beta;\rho} &= \int d^dx\sqrt{g}\left[\mathcal{R}^3_1 - 4\mathcal{R}^3_2 + 4\mathcal{R}^3_5 - 4\mathcal{R}^3_6 - 2\mathcal{R}^3_8 + \mathcal{R}^3_9 + 4\mathcal{R}^3_{10}\right]. \end{aligned} \tag{A.1.10}$$

In total, these establish the required relations to reduce any integrated combination of fully contracted curvatures to the basis (A.1.2).

A.2. Curvature Basis for Einstein Spaces

A special class of Riemannian manifolds are Einstein spaces, where the Ricci tensor is proportional to the metric

$$R_{\mu\nu} = \tfrac{1}{d}g_{\mu\nu}R\,. \tag{A.2.1}$$

In connection with the Bianchi-identities (A.1.4) this definition entails

$$R_{;\mu} = 0\,, \quad R_{\alpha\beta;\mu} = 0\,, \quad R_{\alpha\beta\gamma\mu;}{}^{\mu} = 0\,, \tag{A.2.2}$$

for any dimension $d \neq 2$.

As a direct consequence of these additional relations, the general basis of curvature monomials (A.1.2) degenerates, so that, up to $\mathcal{O}(\mathcal{R}^3)$, an Einstein space has only eight distinguished curvature monomials. Explicitly, these can be chosen as

$$
\begin{aligned}
&\mathcal{E}^0 = 1\,, \\
&\mathcal{E}^1 = R\,, \\
&\mathcal{E}^2_1 = R^2\,, && \mathcal{E}^2_2 = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\,, \\
&\mathcal{E}^3_1 = R^3\,, && \mathcal{E}^3_2 = RR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\,, \\
&\mathcal{E}^3_3 = R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu}\,, && \mathcal{E}^3_4 = R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}R^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta}\,.
\end{aligned}
\tag{A.2.3}
$$

Using the equations (A.2.1) and (A.2.2), the additional basis elements in (A.1.2) can be expressed in terms of the $\mathcal{E}^n_m$ by

$$
\begin{aligned}
&\mathcal{R}^2_2 = \tfrac{1}{d}\mathcal{E}^2_1\,, \\
&\mathcal{R}^3_1 = 0\,, && \mathcal{R}^3_2 = 0\,, && \mathcal{R}^3_4 = \tfrac{1}{d}\mathcal{E}^3_1\,, \\
&\mathcal{R}^3_5 = \tfrac{1}{d^2}\mathcal{E}^3_1\,, && \mathcal{R}^3_6 = \tfrac{1}{d^2}\mathcal{E}^3_1\,, && \mathcal{R}^3_8 = \tfrac{1}{d}\mathcal{E}^3_2\,.
\end{aligned}
\tag{A.2.4}
$$

Lastly, the d-spheres S^d pose a maximally symmetric special class of Einstein-manifolds. In their case, the relations

$$
\begin{aligned}
&R_{\mu\nu} = \tfrac{1}{d}g_{\mu\nu}R\,, && R_{\mu\nu\rho\sigma} = \tfrac{1}{d(d-1)}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R\,, \\
&R_{\mu\nu}R^{\mu\nu} = \tfrac{1}{d}R^2\,, && R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \tfrac{2}{d(d-1)}\,R^2\,,
\end{aligned}
\tag{A.2.5}
$$

determine all curvature invariants in terms of a constant Ricci scalar or, equivalently, the radius of the sphere. Consequently, the basis of curvature monomials contains only one element at each order $\mathcal{R}^n$, proportional to the corresponding power R^n. The additional basis elements in (A.2.3) then satisfy

$$
\begin{aligned}
&\mathcal{E}^2_2 = \tfrac{2}{d(d-1)}R^2\,, \\
&\mathcal{E}^3_2 = \tfrac{2}{d(d-1)}R^3\,, && \mathcal{E}^3_3 = \tfrac{4}{d^2(d-1)^2}R^3\,, && \mathcal{E}^3_4 = \tfrac{d-2}{d^2(d-1)^2}R^3\,.
\end{aligned}
\tag{A.2.6}
$$

A.3. The Gauss-Bonnet Theorem

Further important implications for a quantum gravitational action principle are due to topological identities. Here we give a brief discussion, which is conveniently adopting the language of differential forms.[2]

[2]For a derivation in the special case of $d = 4$, see [16]

An n-form A has a representation in local coordinates

$$A = \sum A_{\mu_1 \dots \mu_n} \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} \,, \tag{A.3.1}$$

with components $A_{\mu_1 \dots \mu_n}$ and the antisymmetric exterior wedge product. Integration of n-forms on n-dimensional manifolds is explained via the totally antisymmetric tensor $\varepsilon^{\mu_1 \dots \mu_n}$ by

$$\int_M A = \int_M d^n x \sqrt{g} \, \varepsilon^{\mu_1 \dots \mu_n} A_{\mu_1 \dots \mu_n} \,, \qquad \dim M = n \,. \tag{A.3.2}$$

The exterior derivative d maps n-forms on $n+1$-forms. An n-form in $d < n$ dimensions is identically zero, because at least two of the coordinates in (A.3.1) have to coincide. With the help of Stokes' theorem we can write

$$\int_M A - \int_N A = \int_{\partial K} A = \int_K dA = 0 \,, \tag{A.3.3}$$

for manifolds M, N being smooth deformations of one topology, so that their difference is a closed path ∂K. Therefore we have

$$\int_M A = \int_N A \,, \tag{A.3.4}$$

for any A, implying that the integral cannot depend on local differences, and is in particular independent of the metric of these manifolds.

The significance of this fact is that non-trivial identities emerge when considering integrands which themselves involve the metric tensor. Especially the covariant curvature 2-form

$$\mathcal{R}^{\mu\nu} = R^{\mu\nu}{}_{\alpha\beta} \, dx^\alpha \wedge dx^\beta \,, \tag{A.3.5}$$

is useful as it is used to construct gravitational action functionals. On even dimensional manifolds we can build a top dimensional form from the curvature, giving rise to a topological action. The Gauss-Bonnet theorem relates this integral to the Euler character χ_{E} of the manifold by

$$\chi_{\mathrm{E}}(M) \sim \int_M \varepsilon_{\mu_1 \nu_1 \cdots \mu_n \nu_n} \, \mathcal{R}^{\mu_1 \nu_1} \wedge \dots \wedge \mathcal{R}^{\mu_n \nu_n} \,. \tag{A.3.6}$$

Written in local coordinate form, the Euler character assumes the form

$$\chi_{\mathrm{E}} = \frac{1}{2(4\pi)^n} \int d^{2n} x \sqrt{g} \, \tfrac{(2n)!}{2n} \, \delta^{\alpha_1}_{[\mu_1} \delta^{\beta_1}_{\nu_1} \dots \delta^{\alpha_n}_{\mu_n} \delta^{\beta_n}_{\nu_n]} \, R^{\mu_1 \nu_1}{}_{\alpha_1 \beta_1} \dots R^{\mu_n \nu_n}{}_{\alpha_n \beta_n} \,, \tag{A.3.7}$$

in terms of the Riemann tensor on manifolds with even dimension $d = 2n$. This expression is by construction independent of the metric on M, and turns out to be an index only sensitive to the topology of the manifold.

For $d = 2$, the Euler character becomes

$$\chi_{\mathrm{E}}\Big|_{d=2} = \frac{1}{16\pi} \int d^2x \sqrt{g}\, R \,, \tag{A.3.8}$$

proportional to the Einstein-Hilbert action. In $d = 4$ dimensions, we have

$$\chi_{\mathrm{E}}\Big|_{d=4} = \frac{1}{2(4\pi)^2} \int d^4x \sqrt{g} \left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right), \tag{A.3.9}$$

which is frequently used to simplify actions of higher derivative gravity to eliminate one of the three curvature squared monomials. Since the Euler character appears in the computation of constrained heat kernel coefficients in section 4.2 to appropriately account for zero modes, we explicitly compute the expression on a sphere. Here the volume in terms of the constant scalar curvature is $\mathrm{Vol}(S^4) = 384\,\pi^2\, R^{-2}$, so that one obtains

$$\chi_{\mathrm{E}}(S^4) = \frac{1}{2(4\pi)^2}\, \mathrm{Vol}(S^4)\, \frac{1}{6}\, R^2 = 2 \,, \tag{A.3.10}$$

using the relations (A.2.5).

Finally, we give the Euler term for $d = 6$

$$\chi_{\mathrm{E}}\Big|_{d=6} = \frac{1}{6(4\pi)^3} \int d^6x \sqrt{g} \left(4\mathcal{R}^3_3 - 48\mathcal{R}^3_4 + 64\mathcal{R}^3_5 + 96\mathcal{R}^3_6 + 12\mathcal{R}^3_7 - 96\mathcal{R}^3_8 + 16\mathcal{R}^3_9 - 32\mathcal{R}^3_{10} \right). \tag{A.3.11}$$

Note that the integrand in this expression is exactly zero in $d = 4$ dimensions, because it corresponds to a 6-form. This fact can be used to eliminate one of the curvature cubed monomials in the general d-dimensional basis (A.1.2).

B. Commutator Identities

This appendix gives some explicit expressions for multi-commutators that appear in the evaluation of traces over non-minimal operators. In order to simplify notation, we introduce the n-fold commutators

$$\Big[\, D_\mu \,,\, \Delta \,\Big]_n \equiv \Big[\Big[\, D_\mu \,,\, \Delta \,\Big]_{n-1} \,,\, \Delta\Big]\,, \tag{B.1}$$

with $\Big[\, D_\mu \,,\, \Delta \,\Big]_0 = D_\mu$ and $\Big[\, D_\mu \,,\, \Delta \,\Big]_1 = \Big[\, D_\mu \,,\, \Delta \,\Big]$. Here we identify $D_\mu = \nabla_\mu$ excluding a connection on an internal space and define the Laplacian $\Delta = -g^{\mu\nu} D_\mu D_\nu$.

Throughout the thesis, symmetrization is with unit strength:

$$T_{(\alpha\beta)} = \tfrac{1}{2} \left(T_{\alpha\beta} + T_{\beta\alpha} \right) \,, \tag{B.2}$$

for any tensor T.

B.1. Commutators of Covariant Derivatives on Curved Spacetime

The commutator of two covariant derivatives acting on an arbitrary tensor $T_{\alpha_1 \ldots \alpha_n}$ is given by

$$[D_\mu \,, D_\nu]\, T_{\alpha_1 \ldots \alpha_n} = \sum_{k=1}^{n} R_{\mu\nu\alpha_k}{}^{\rho}\, T_{\alpha_1 \ldots \alpha_{k-1} \rho \alpha_{k+1} \ldots \alpha_n} \,. \tag{B.1.1}$$

The computation of operator traces as they appear in the expansion (4.2.8) employs an expansion in multi-commutators (B.1) involving the Laplacian acting on scalar and vector fields, and the evaluation of the RG equation for gravity (4.3.1) requires commutators on symmetric tensor fields. Here we summarize the corresponding expressions up to order

$\mathcal{R}^{7/2}$, whereby all surface terms in that order have also been neglected:

$$\begin{aligned}
\Big[D_\mu,\Delta\Big]\,\phi &= R_\mu{}^\alpha D_\alpha\phi\,,\\
\Big[D_\mu,\Delta\Big]_2\phi &= \Big(R_\mu{}^\alpha R_\alpha{}^\beta-(\Delta R_\mu{}^\beta)+2R_\mu{}^{\alpha;\beta}D_\alpha\Big)\,D_\beta\,\phi\,,\\
\Big[D_\mu,\Delta\Big]_3\phi &= \Big(R_\mu{}^\nu R_\nu{}^\alpha R_\alpha{}^\beta-(\Delta(R_\mu{}^\alpha R_\alpha{}^\beta))+2(R_\mu{}^\alpha R_\alpha{}^\beta)_{;\gamma}D^\gamma\\
&\qquad-(\Delta R_\mu{}^\alpha)R_\alpha{}^\beta+4R_{\mu\alpha}{}^{;\beta\gamma}D_\gamma D^\alpha\Big)D_\beta\phi+2R_\mu{}^{\alpha;\beta}\Big[D_\beta D_\alpha,\Delta\Big]\phi\,,\\
\Big[D_\mu,\Delta\Big]_4\phi &= 4R_\mu{}^{(\alpha;\beta)\gamma}\Big(2R^\nu{}_\alpha D_\gamma D_\beta D_\nu-2R_\alpha{}^\nu{}_\beta{}^\lambda D_\gamma D_\lambda D_\nu\\
&\qquad+R_\gamma{}^\sigma D_\sigma D_\alpha D_\beta-4R^\sigma{}_\gamma{}^\nu{}_\alpha D_\sigma D_\beta D_\nu\Big)\phi+\mathcal{O}(\mathcal{R}^{7/2})\,,
\end{aligned}\tag{B.1.2}$$

with the commutator appearing in the third term given by

$$\Big[D_\beta D_\alpha,\Delta\Big]\phi=\Big(2R^\mu{}_{(\beta;\alpha)}-R_{\alpha\beta}{}^{;\mu}+2R^\mu{}_{(\alpha}D_{\beta)}-2R_\alpha{}^\mu{}_\beta{}^\nu D_\nu\Big)D_\mu\phi\,.\tag{B.1.3}$$

Notably, the first three commutators are exact, while in the fourth one we only displayed terms up to $\mathcal{O}(\mathcal{R}^3)$.

For vector fields, we have the uncontracted commutator

$$[\,D_\mu\,,\,\Delta\,]\,\phi_\rho\ \ =R_\mu{}^\alpha D_\alpha\phi_\rho-2R^\alpha{}_\mu{}^\beta{}_\rho\,D_\alpha\phi_\beta-(D_\alpha R^\alpha{}_\mu{}^\beta{}_\rho)\phi_\beta\,,\tag{B.1.4}$$

together with those with contracted indices:

$$\begin{aligned}
\Big[D^\alpha,\Delta\Big]_1\phi_\alpha &= -\,D_\alpha\Big(R^{\alpha\beta}\phi_\beta\Big)\,,\\
\Big[D^\alpha,\Delta\Big]_2\phi_\alpha &= \Big(2R_{\alpha\beta}R^{\alpha\mu\beta\nu}-R^{\mu\alpha}R_\alpha{}^\nu-R^{;\mu\nu}+(\Delta R^{\mu\nu})-2R^{\alpha\nu;\mu}D_\alpha\Big)D_\mu\phi_\nu\\
&\qquad+(D_\mu(R^\mu{}_\alpha R^{\alpha\beta}))\phi_\beta+(D_\alpha\Delta R^{\alpha\beta})\phi_\beta\,,\\
\Big[D^\alpha,\Delta\Big]_3\phi_\alpha &= \Big(2R_{\alpha\beta}R^{\alpha\mu\beta\nu}-R^{\mu\alpha}R_\alpha{}^\nu-R^{;\mu\nu}+(\Delta R^{\mu\nu})\Big)\Big(R_\mu{}^\rho D_\rho\phi_\nu-2R^\rho{}_\mu{}^\sigma{}_\nu D_\rho\phi_\sigma\Big)\\
&\qquad-2R^{\alpha\nu;\mu}\,[D_\alpha D_\mu,\Delta]\,\phi_\nu-4R^{\alpha\nu;\mu\sigma}D_\sigma D_\alpha D_\mu\phi_\nu+\mathcal{O}(\mathcal{R}^{7/2})\,,\\
\Big[D^\alpha,\Delta\Big]_4\phi_\alpha &= -\,4R^{\alpha\nu;\mu\beta}\Big(2R_\mu{}^\lambda D_\lambda D_\beta D_\alpha\phi_\nu+R_\alpha{}^\lambda D_\lambda D_\beta D_\mu\phi_\nu-4R_\mu{}^\sigma{}_\nu{}^\lambda D_\beta D_\alpha D_\sigma\phi_\lambda\\
&\qquad-2R_\alpha{}^\sigma{}_\nu{}^\lambda D_\beta D_\mu D_\sigma\phi_\lambda-4R_\mu{}^\sigma{}_\alpha{}^\lambda D_\beta D_\sigma D_\lambda\phi_\nu-2R_\mu{}^\sigma{}_\beta{}^\lambda D_\sigma D_\alpha D_\lambda\phi_\nu\Big)+\mathcal{O}(\mathcal{R}^{7/2})\,.
\end{aligned}\tag{B.1.5}$$

These expressions are completed by the commutator

$$\begin{aligned}
\Big[D_\alpha D_\mu,\Delta\Big]\phi_\nu &= R_\mu{}^\rho D_\alpha D_\rho\phi_\nu+R_{\alpha\rho}D^\rho D_\mu\phi_\nu-2R_\mu{}^\rho{}_\nu{}^\sigma D_\alpha D_\rho\phi_\sigma-2R_\alpha{}^\rho{}_\mu{}^\sigma D_\rho D_\sigma\phi_\nu\\
&\quad-2R_\alpha{}^\rho{}_\nu{}^\sigma D_\rho D_\mu\phi_\sigma-R_{\alpha\mu;\rho}D^\rho\phi_\nu+R_{\alpha\rho;\mu}D^\rho\phi_\nu+R_{\mu\rho;\alpha}D^\rho\phi_\nu\\
&\quad-R_{\alpha\nu;}{}^\rho D_\mu\phi_\rho+R_\alpha{}^\rho{}_{;\nu}D_\mu\phi_\rho-R_{\mu\nu;}{}^\rho D_\alpha\phi_\rho+R_\mu{}^\rho{}_{;\nu}D_\alpha\phi_\rho\\
&\quad-2R_\mu{}^\rho{}_\nu{}^\sigma{}_{;\alpha}D_\rho\phi_\sigma-R_{\mu\nu;}{}^\rho{}_\alpha\phi_\rho+R_\mu{}^\rho{}_{;\nu\alpha}\phi_\rho\,.
\end{aligned}\tag{B.1.6}$$

For the case of symmetric 2-tensors, the expressions including only two orders in the curvature are

$$
\begin{aligned}
[\, D_\mu \,,\, \Delta\,]_1\, \phi_{\rho\sigma} &= R_\mu{}^\alpha D_\alpha \phi_{\rho\sigma} - 4R_{\alpha\mu}{}^\beta{}_{(\rho}\, D^\alpha \phi_{\sigma)\beta} - 2\left(D_\alpha R^\alpha{}_\mu{}^\beta{}_{(\rho)}\right) \phi_{\sigma)\beta}\,, \\
[\, D^\gamma \,,\, \Delta\,]_2\, \phi_{\gamma\beta} &= -\,R^{\gamma\nu}\left[R_\nu{}^\mu D_\mu \phi_{\gamma\beta} + 4R^\mu{}_{(\gamma\nu\alpha} D^\alpha \phi_{\beta)\mu}\right] \\
&\qquad + 2R^\lambda{}_\beta{}^{\gamma\nu}\left[R_\nu{}^\mu D_\mu \phi_{\gamma\lambda} + 4R^\mu{}_{(\gamma\nu\alpha} D^\alpha \phi_{\lambda)\mu}\right] \\
&\qquad + 4(D_\alpha R^\lambda{}_\beta{}^{\gamma\nu}) D^\alpha D_\nu a_{\gamma\lambda}\,.
\end{aligned}
\tag{B.1.7}
$$

B.2. Commutators Involving Composite Operators

For two continuous linear operators X, Y the Hadamard lemma states

$$
\mathrm{e}^X\, Y\, \mathrm{e}^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n \Big[Y, X\Big]_n\,,
\tag{B.2.1}
$$

where the n-fold commutator is defined recursively by $\Big[Y, X\Big]_n = \Big[\Big[Y, X\Big]_{n-1}, X\Big]$. With the identification $X = \pm s\Delta$, this formula provides an exact curvature expansion for the commutator of the exponentiated Laplacian and an arbitrary operator Y,

$$
\begin{aligned}
\Big[\, Y \,,\, \mathrm{e}^{-s\Delta}\Big] &= -\sum_{n=1}^{\infty} \frac{1}{n!}\, s^n \Big[\, Y \,,\, \Delta\,\Big]_n\, \mathrm{e}^{-s\Delta} \\
&= \mathrm{e}^{-s\Delta} \sum_{n=1}^{\infty} \frac{1}{n!}\, (-s)^n \Big[\, Y \,,\, \Delta\,\Big]_n\,.
\end{aligned}
\tag{B.2.2}
$$

Since arbitrary locally integrable functions can be represented as a formal Laplace-transformation

$$
f(s) = \int_t \tilde{f}(t)\; \mathrm{e}^{-st}\,,
\tag{B.2.3}
$$

the expansion (B.2.2) implies the more general commutator with a function of the Laplacian

$$
\begin{aligned}
[Y, f(\Delta)] &= \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{n-1} \Big[\, Y \,,\, \Delta\,\Big]_n\, f^{(n)}(\Delta) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!}\, f^{(n)}(\Delta) \Big[\, Y \,,\, \Delta\,\Big]_n\,,
\end{aligned}
\tag{B.2.4}
$$

where $f^{(n)}$ denotes the n-th derivative of f, and all Laplacians have been moved to the very right and very left, respectively.

C. Threshold Functions and Cutoff Scheme

The solution of the RG equation in the form (4.3.45) still contains the Q functionals (4.3.42), with their argument indicating the dependence of the functional traces on the Laplacian. In order to capture the cutoff scheme dependence and highlight the structure of the β functions it is convenient to write the expressions for the functionals $Q_n[f]$ (4.3.43) for $n > 0$ in terms of the dimensionless threshold functions [35]

$$\begin{aligned}
\Phi^p_n(\omega) &:= \frac{1}{\Gamma(n)} \int_0^\infty dz\ z^{n-1} \frac{R^{(0)}(z) - zR^{(0)\prime}(z)}{(z + R^{(0)}(z) + \omega)^p}\,, \\
\widetilde{\Phi}^p_n(\omega) &:= \frac{1}{\Gamma(n)} \int_0^\infty dz\ z^{n-1} \frac{R^{(0)}(z)}{(z + R^{(0)}(z) + \omega)^p}\,.
\end{aligned} \tag{C.1}$$

In the computation of the ghost field renormalization in section 5.2, we encounter structures that motivate the more general definitions

$$\begin{aligned}
\Phi^{p,q}_n(\omega) &:= \frac{1}{\Gamma(n)} \int_0^\infty dz\ z^{n-1} \frac{R^{(0)}(z) - zR^{(0)\prime}(z)}{(z + R^{(0)}(z) + \omega)^p\ (z + R^{(0)}(z))^q}\,, \\
\tilde{\Phi}^{p,q}_n(\omega) &:= \frac{1}{\Gamma(n)} \int_0^\infty dz\ z^{n-1} \frac{R^{(0)}(z)}{(z + R^{(0)}(z) + \omega)^p\ (z + R^{(0)}(z))^q}\,, \\
\check{\Phi}^{p,q}_n(\omega) &:= \frac{1}{\Gamma(n)} \int_0^\infty dz\ z^{n-1} \frac{R^{(0)\prime}(z)\, \Big(R^{(0)}(z) - zR^{(0)\prime}(z)\Big)}{(z + R^{(0)}(z) + \omega)^p\ (z + R^{(0)}(z))^q}\,, \\
\hat{\Phi}^{p,q}_n(\omega) &:= \frac{1}{\Gamma(n)} \int_0^\infty dz\ z^{n-1} \frac{R^{(0)}(z)\ R^{(0)\prime}(z)}{(z + R^{(0)}(z) + \omega)^p\ (z + R^{(0)}(z))^q}\,.
\end{aligned} \tag{C.2}$$

Herein appears the dimensionless shape function $R^{(0)}$, defined with respect to the IR cutoff function

$$R_k\Big(\Delta\Big) = k^2 R^{(0)}\Big(\frac{\Delta}{k^2}\Big)\,, \tag{C.3}$$

introduced in (4.3.12), and the prime denotes the derivative with respect to its argument. The shape function interpolates monotonically between $R^{(0)}(0) = 1$ and $\lim_{z\to\infty} R^{(0)}_k(z) = 0$ to define the separation of IR and UV modes of the fluctuation fields.

Observe that the threshold functions (C.2) with $p = 0$ do not depend on ω. These definitions naturally generalize the ones (C.1), which are recovered in the special cases

$$\Phi^{p,0}_n(\omega) = \Phi^p_n(\omega)\,, \quad \Phi^{0,q}_n(\omega) = \Phi^q_n(0)\,, \quad \tilde{\Phi}^{p,0}_n(\omega) = \tilde{\Phi}^p_n(\omega)\,, \quad \tilde{\Phi}^{0,q}_n(\omega) = \tilde{\Phi}^q_n(0)\,. \tag{C.4}$$

The relation to the $Q_n[f]$ is given by

$$Q_n\left[\frac{\partial_t(Z_k^I R_k)}{Z_k^I(P_k+\omega)^p}\right] = 2k^{2(n-p+1)}\left[\Phi_n^{p,0}(\tfrac{\omega}{k^2}) + \frac{1}{2}\,\frac{\partial_t Z_k^I}{Z_k^I}\,\tilde{\Phi}_n^{p,0}(\tfrac{\omega}{k^2})\right], \tag{C.5}$$

where Z_k^I denotes any dimensionless generalized field renormalization factor present in the kinetic terms of $\Gamma_k^{(2)}$. For the dimensionful coupling constant (5.1.2), we further have

$$Q_n\left[\frac{\partial_t(u_1 R_k)}{u_1(P_k+\omega)^p}\right] = 2Q_n\left[\frac{R_k}{(P_k+\omega)^p}\right] + Q_n\left[\frac{\partial_t(g_1 R_k)}{g_1(P_k+\omega)^p}\right], \tag{C.6}$$

Together with (C.5) this implies for the functions defined in (5.1.14) and (5.1.52)

$$\begin{aligned} Q_n[f^p] &= 2k^{2(n-p+1)}\,\Phi_n^p(0)\,,\\ Q_n[f^p_{2\mathrm{T}}] &= 2(-g_1)^{1-p}k^{2(n-2p+2)}\Big(\Phi_n^p(\tfrac{g_0}{g_1}) + \tfrac{1}{2}(\tfrac{\partial_t g_1}{g_1}+2)\widetilde{\Phi}_n^p(\tfrac{g_0}{g_1})\Big)\,,\\ Q_n[f^p_0] &= 2(\tfrac{3}{8}g_1)^{1-p}k^{2(n-2p+2)}\Big(\Phi_n^p(\tfrac{2}{3}\tfrac{g_0}{g_1}) + \tfrac{1}{2}(\tfrac{\partial_t g_1}{g_1}+2)\widetilde{\Phi}_n^p(\tfrac{2}{3}\tfrac{g_0}{g_1})\Big)\,. \end{aligned} \tag{C.7}$$

For the functions occurring in the computation of the ghost field renormalization defined in (5.2.33), we have

$$\begin{aligned} Q_n[f_1^N] &= 2\,Z_k^N\,k^{2n-4}\Big(\Phi_n^{2,1}(-2\lambda_k) - \tfrac{1}{2}\eta_N\tilde{\Phi}_n^{2,1}(-2\lambda_k)\Big)\,,\\ Q_n[f_1^c] &= 2\,Z_k^c\,k^{2n-4}\Big(\Phi_n^{1,2}(-2\lambda_k) - \tfrac{1}{2}\eta_c\tilde{\Phi}_n^{1,2}(-2\lambda_k)\Big)\,,\\ Q_n[f_2^I] &= 2\,Z_k^I\,k^{2n-6}\Big(\Phi_n^{2,2}(-2\lambda_k) + \check{\Phi}_n^{2,2}(-2\lambda_k) - \tfrac{1}{2}\eta_I\Big(\tilde{\Phi}_n^{2,2}(-2\lambda_k) + \hat{\Phi}_n^{2,2}(-2\lambda_k)\Big)\Big)\,. \end{aligned} \tag{C.8}$$

These equations allow us to express the β functions for Newton's and the cosmological constant, as well as for the ghost field renormalization in section 5.2 with the remaining dependence on the shape function residing in the threshold functions. Although the β functions are not themselves observables, they encode some universal, cutoff-independent information. In particular, for $p = n+1$ and $\omega = 0$ we can write

$$\begin{aligned} \Phi_n^{n+1}(0) &= \frac{1}{\Gamma(n)}\int_0^\infty dz\; z^{n-1}\frac{R^{(0)}(z) - zR^{(0)\prime}(z)}{(z+R^{(0)}(z))^{n+1}}\\ &= \frac{1}{\Gamma(n)}\int_0^\infty dz\;\frac{\partial}{\partial z}\left[\frac{1}{n}\,\frac{z^n}{(z+R^{(0)}(z))^n}\right]\\ &= \frac{1}{\Gamma(n+1)}\,. \end{aligned} \tag{C.9}$$

Such a term does therefore not depend on the cutoff which guarantees that the 1-loop contributions to the running of dimensionless coupling constants is universal.

In the analysis of fixed points and renormalization group trajectories, the shape function dependence requires to choose an explicit cutoff scheme. For the numerical

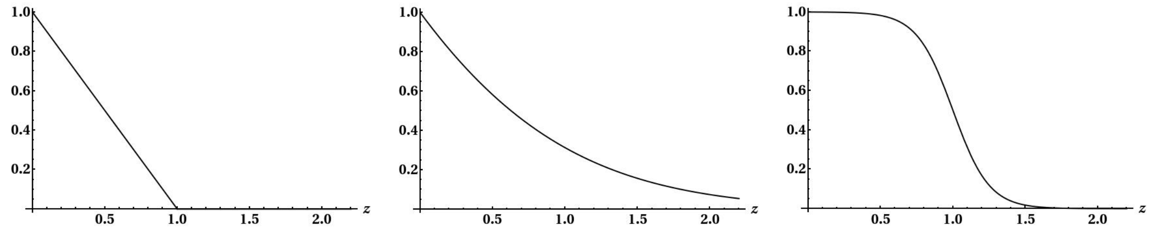

Figure C.1.: Typical examples for the shape function $R^{(0)}(z)$ appearing in the final results for β functions. From left to right the plots show the optimized cutoff (C.10), the exponential cutoff (C.13) for $s = 2$, and the Fermi-cutoff (C.14) for $T = 1/8$.

studies of β functions, one typically employs one of the shape functions introduced in the following. To compare these alternatives, plots are presented in figure C.1.

A particularly convenient choice is the optimized cutoff [146], defined by

$$R^{(0),\mathrm{opt}}(z) = (1 - z)\,\Theta(1 - z)\,, \tag{C.10}$$

with the Heaviside theta function Θ. With this choice we have

$$(R^{(0),\mathrm{opt}})'(z) = -\Theta(1 - z)\,, \tag{C.11}$$

so that the reign of integration in (C.2) is constrained to the interval $[0, 1]$. In this case the integrals can be carried out analytically to yield

$$\begin{aligned} \Phi^{p,q}_n(w) &= \frac{1}{\Gamma(n+1)}\frac{1}{(1+w)^p}\,, & \tilde{\Phi}^{p,q}_n(w) &= \frac{1}{\Gamma(n+2)}\frac{1}{(1+w)^p}\,, \\ \check{\Phi}^{p,q}_n(w) &= -\frac{1}{\Gamma(n+1)}\frac{1}{(1+w)^p}\,, & \hat{\Phi}^{p,q}_n(w) &= -\frac{1}{\Gamma(n+2)}\frac{1}{(1+w)^p}\,. \end{aligned} \tag{C.12}$$

Here, the threshold functions degenerate such that they become independent of the index q. Because of the simplicity of these expressions, the optimized cutoff is frequently used especially when the β functions become very complicated.

The non-analytic behaviour of $R^{(0),\mathrm{opt}}(z)$ at $z = 1$ can cause ambiguities,requiring an analytic continuation for Q_n with $n < 1$. This can be avoided by the use of a smooth shape function. The most commonly used example is the exponential cutoff

$$R^{(0),\exp}(z; s) = \frac{sz}{\mathrm{e}^{sz} - 1}\,, \tag{C.13}$$

The shape parameter s allows to smoothly vary the implementation of the IR cutoff. In contrast to the optimized cutoff, the integrals in the threshold functions cannot be carried out analytically for this class of cutoffs. Thus one has to resort to numerical integration when evaluating the threshold functions.

Finally, we advocate the use of a smooth shape function resembling a properly normalized Fermi-distribution

$$R^{(0),\mathrm{fermi}}(z;T) = \frac{\mathrm{e}^{-1/T} + 1}{\mathrm{e}^{(z-1)/T} + 1}\,, \tag{C.14}$$

with the temperature parameter T. In contrast to the above alternatives, its derivative is peaked at $z = 1$. Therefore this choice realizes the conceptual ideas laid out in chapter 2 most accurately. As for the exponential cutoff, numerical integration is required when applying this cutoff.

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