Ergodicity and Regularity of Invariant Measure for Branching Markov Processes with Immigration

Dissertation

zur Erlangung des Grades
"Doktor der Naturwissenschaften"
am Fachbereich Physik, Mathematik und Informatik
der Johannes Gutenberg-Universität
in Mainz

Matthias Hammer

geboren in Mainz

Mainz, den 1.5.2012

1. Berichterstatter:
2. Berichterstatter:
3. Berichterstatter:

Datum der mündlichen Prüfung: 26.10.2012

D77 (Dissertation, Johannes Gutenberg-Universität Mainz)

Abstract

In this thesis we consider systems of finitely many particles moving on paths given by a strong Markov process and undergoing branching and reproduction at random times. The branching rate of a particle, its number of offspring and their spatial distribution are allowed to depend on the particle's position and possibly on the configuration of coexisting particles. In addition there is immigration of new particles, with the rate of immigration and the distribution of immigrants possibly depending on the configuration of pre-existing particles as well.

In the first two chapters of this work, we concentrate on the case that the joint motion of particles is governed by a diffusion with interacting components. The resulting process of particle configurations was studied by E. Löcherbach (2002, 2004) and is known as a *branching diffusion with immigration* (BDI). Chapter 1 contains a detailed introduction of the basic model assumptions, in particular an assumption of ergodicity which guarantees that the BDI process is positive Harris recurrent with finite invariant measure on the configuration space. This object and a closely related quantity, namely the invariant occupation measure on the single-particle space, are investigated in Chapter 2 where we study the problem of the existence of Lebesgue-densities with nice regularity properties. For example, it turns out that the existence of a continuous density for the invariant measure depends on the mechanism by which newborn particles are distributed in space, namely whether branching particles reproduce *at* their death position or their offspring are distributed according to an absolutely continuous transition kernel.

In Chapter 3, we assume that the quantities defining the model depend only on the spatial position but not on the configuration of coexisting particles. In this framework (which was considered by Höpfner and Löcherbach (2005) in the special case that branching particles reproduce at their death position), the particle motions are independent, and we can allow for more general Markov processes instead of diffusions. The resulting configuration process is a branching Markov process in the sense introduced by Ikeda, Nagasawa and Watanabe (1968), complemented by an immigration mechanism. Generalizing results obtained by Höpfner and Löcherbach (2005), we give sufficient conditions for ergodicity in the sense of positive recurrence of the configuration process and finiteness of the invariant occupation measure in the case of general particle motions and offspring distributions.

Zusammenfassung

In dieser Dissertation betrachten wir Systeme endlich vieler Partikel, die sich entlang von Pfaden eines starken Markovprozesses bewegen und dabei zu zufälligen Zeiten verzweigen und Nachkommen erzeugen. Die Verzweigungsrate eines Partikels, die Anzahl seiner Nachkommen und deren Verteilung im Raum hängen von der räumlichen Position des Partikels sowie möglicherweise von der Konfiguration aller koexistierenden Partikel ab. Zusätzlich erfolgt Immigration, wobei die entsprechende Rate sowie die räumliche Verteilung der Immigranten ebenfalls von der Konfiguration aller bereits existierenden Partikel abhängen dürfen.

In den ersten beiden Kapiteln betrachten wir den Fall, dass die gemeinsame Partikelbewegung durch eine Diffusion mit interagierenden Komponenten beschrieben wird. Der resultierende Partikelprozess heißt *branching diffusion with immigration* (BDI) und wurde von E. Löcherbach (2002, 2004) eingehend untersucht. Kapitel 1 enthält eine detaillierte Einführung der zugrundeliegenden Modellannahmen, insbesondere eine Ergodizitätsannahme, welche die positive Harris-Rekurrenz des BDI-Prozesses mit endlichem invariantem Maß auf dem Konfigurationsraum sicherstellt. Dieses sowie ein nahe verwandtes Objekt, das sogenannte invariante Okkupationsmaß auf dem Einpartikelraum, werden in Kapitel 2 untersucht, wo wir das Problem der Existenz von Lebesgue-Dichten mit wünschenswerten Regularitätseigenschaften behandeln. Beispielsweise stellt sich heraus, dass die Existenz einer stetigen Dichte für das invariante Maß davon abhängt, ob die Reproduktion direkt am Ort des Verzweigens stattfindet oder ob die Nachkommen eines verzweigenden Partikels gemäß eines absolutstetigen Übergangskerns im Raum verteilt werden.

In Kapitel 3 setzen wir voraus, dass die eingangs erwähnten Modellparameter (Partikelbewegung, Verzweigungsrate, Reproduktionsmechanismus) nur von der räumlichen Position und nicht von der Konfiguration aller koexistierenden Partikel abhängen. In diesem Rahmen (der für den Spezialfall, dass die Reproduktion direkt am Verzweigungsort stattfindet, von Höpfner und Löcherbach (2005) untersucht wurde) bewegen sich die Partikel unabhängig voneinander, und wir können als Einpartikelbewegung allgemeinere Markovprozesse anstelle von Diffusionen zulassen. Der resultierende Prozess kann als ein verzweigender Markovprozess im Sinne von Ikeda, Nagasawa und Watanabe (1968) mit zusätzlicher Immigration aufgefasst werden. In Verallgemeinerung der Resultate von Höpfner und Löcherbach (2005) beweisen wir hinreichende Bedingungen für Ergodizität im Sinne von positiver Rekurrenz des Partikelprozesses auf dem Konfigurationsraum und Endlichkeit des invarianten Okkupationsmaßes für den Fall allgemeiner Partikelbewegungen und räumlicher Nachkommensverteilungen.

Contents

Introduction

This thesis deals with branching diffusions (or more general Markov processes) with immigration, a class of spatial stochastic processes the simplest version of which can be informally described as follows: Imagine a system of finitely many particles, each living in $\mathbb{R}^d$ ($d \geq 1$) and moving *independently* of each other on diffusion paths according to a stochastic differential equation

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \tag{1}$$

where $(W_t)_{t\geq 0}$ is an m-dimensional ($m \geq d$) standard Brownian motion. Each particle *branches*, which means that it "dies" and produces a random number of offspring, according to a position-dependent branching rate $\kappa(\cdot) : \mathbb{R}^d \to \mathbb{R}_+$. More precisely, a particle situated at position $x \in \mathbb{R}^d$ at time $t > 0$ dies during a small time intervall $(t, t+h]$ with probability $\kappa(x) \cdot h + o(h)$ as $h \downarrow 0$. At its death time, it is replaced by a random number $k \in \mathbb{N}_0$ of offspring particles with probability $p_k(x)$, the k newborn particles (if $k \geq 1$) being distributed in $(\mathbb{R}^d)^k$ according to the law $Q(x;\cdot)^{\otimes k}$, where

$$Q(\cdot\,;\cdot) : \mathbb{R}^d \times \mathcal{B}_{\mathbb{R}^d} \to [0,1]$$

is some transition probability kernel. In addition, there is immigration at constant rate $c > 0$: At each immigration event, one new particle is added to the configuration of preexisting particles in a position selected according to some probability law ν on $\mathbb{R}^d$.

The mechanism just described results in a stochastic process $\boldsymbol{\eta} = (\boldsymbol{\eta}_t)_{t\geq 0}$ of particle configurations where at each time $t > 0$, the number of particles is random but finite. It is called a *branching diffusion with immigration* (henceforth: BDI) and can be considered as a spatial version of "classical" Galton-Watson branching processes with immigration, for which we refer the reader e.g. to [Zub1972] or [Pak1975]. One can also conceive of it as being obtained by "adding immigration" to a spatial branching Markov process in the sense introduced by [INW1968a]. The latter class of processes, characterized by the famous branching property, has of course been extensively studied in the literature: We refer e.g. to [EHK2004] or [Shi2006] for recent approaches; see also [Saw1976] for possible applications to population genetics. Depending on whether one is interested in distinguishable or indistinguishable particles, the state space $\mathcal{S}$ for a BDI $\boldsymbol{\eta}$ may be chosen to consist of ordered or unordered particle configurations. In the latter case, identifying an unordered configuration with the corresponding finite point measure, a BDI can also be regarded as a special kind of measure-valued process, which is the usual point of view in the modern literature on spatial branching processes.

From a probabilistic point of view, branching diffusions with immigration in the sense outlined above have been extensively studied in [HL2005]; see also [HHL2002] which contains an application to statistical inference, namely the non-parametric estimation of the branching rate. A modification of the model, investigated in [Löc2002a]-[Löc2002b] and [Löc2004], allows for interactions between the particles in both their spatial motion and the branching, reproduction and immigration mechanisms. In this case, the quantities defining the model all depend in addition also on the configuration of coexisting particles, inducing a major complication in the analysis. Another possible generalization admits general strong Markov processes instead of diffusions for the particle motion.

As in the classical Galton-Watson case, the addition of immigration to a spatial branching process opens the door to the possibility of ergodic behavior, provided a suitable assumption of *subcritical reproduction* holds. The focus of this work is on the study of this ergodic case. In particular, we will address the following problems:

- As in [HL2005], we are interested in sufficient conditions for ergodicity of the BDI $\boldsymbol{\eta}$ in the sense of positive Harris recurrence, with the void configuration Δ (the state of no existing particle) as a recurrent atom. In this case, the process $\boldsymbol{\eta}$ will admit a finite invariant measure m on the configuration space $\mathcal{S}$ which turns out to be given (up to normalization) by the expected occupation time of a Borel set $F \in \mathcal{B}_{\mathcal{S}}$ during one life cycle of the process $\boldsymbol{\eta}$:
$$m(F) = \boldsymbol{E}_\Delta\left[\int_0^R \mathbf{1}_F(\boldsymbol{\eta}_s)\,ds\right],$$
where R denotes the first return time to the void configuration Δ. A closely related object is the *invariant occupation measure* $\overline{m}$ on the single-particle space $\mathbb{R}^d$, which is defined as the expected occupation time of a Borel subset $B \in \mathcal{B}_{\mathbb{R}^d}$ by all particles during one life cycle of $\boldsymbol{\eta}$: In the measure-valued interpretation of $\boldsymbol{\eta}$, this means
$$\overline{m}(B) := \boldsymbol{E}_\Delta\left[\int_0^R \boldsymbol{\eta}_s(B)\,ds\right],$$
and we are also interested in conditions ensuring finiteness of the measure $\overline{m}$.

  Essentially, this is the question for a suitable notion of "spatial subcriticality".

- Under the assumption of ergodicity and finiteness of $\overline{m}$, we are interested in the regularity of the invariant measure m on the configuration space $\mathcal{S}$ and of the invariant occupation measure $\overline{m}$ on $\mathbb{R}^d$. In particular, the problem of the existence of Lebesgue-densities with nice regularity properties naturally arises, which is also relevant for statistical applications as e.g. considered in [HHL2002].

In the first two chapters of this thesis, we will work in the above-mentioned "interactive" framework of [Löc2002a]-[Löc2002b] where the particle motion is governed by a diffusion but there are interactions present in the spatial motion component as well as in the branching, reproduction and immigration mechanisms. The precise model assumptions, including a condition of ergodicity, are stated in Chapter 1, which also contains some remarks on how ergodicity can sometimes be verified in the interactive framework.

Chapter 2 is mainly devoted to the study of the invariant measure m in the ergodic case. This work is motivated by the observation due to R. Höpfner in [Höp2004] that the invariant measure m, even if it is absolutely continuous with respect to Lebesgue measure on the configuration space, can be very irregular: In particular, the density will in general be neither continuous nor bounded as long as branching particles reproduce exactly at their death position, i.e. as long as we have

$$Q(x;\cdot) = \delta_x(\cdot) \tag{2}$$

for the kernel Q governing the spatial distribution of the offspring of a particle branching at position x. Using an example given in [Höp2004], we explain this phenomenon in detail in Section 2.1, illustrating the point that if one hopes for the existence of a "nice" Lebesgue-density for the invariant measure m, one has by necessity to consider more general spatial offspring distributions than (2). In fact, this observation can be considered the starting point of our whole work. In one of our main results (Theorem 2.2.8, which is proved in Section 2.2), we show that (under suitable additional assumptions) a continuous density for m can indeed be obtained provided the offspring particles are distributed in space according to an absolutely continuous kernel

$$Q(x; dy) = q(x; y)dy. \tag{3}$$

Section 2.3 is devoted to the study of the invariant occupation measure $\overline{m}$ on $\mathbb{R}^d$ in the interactive framework. Namely, we show that a result by E. Löcherbach in [Löc2004] on the existence of a continuous and bounded Lebesgue-density for $\overline{m}$ which was proved for the case (2) continues to hold in case (3).

Chapter 3 is motivated by the work [HL2005], where the authors considered "non-interactive" branching diffusions with immigration as outlined in the first paragraph of this introduction under the assumption (2) that branching particles reproduce at their death position. In this framework, the independence of the motions of the particles allows for a much more refined analysis than in the interactive case. Generalizing this approach, we allow for arbitrary strong Markov processes instead of diffusions as single-particle motions and for general offspring distribution kernels $Q(x; \cdot)$. We call the resulting process $\boldsymbol{\eta}$ a *branching Markov process with immigration* (BMPI); it can be constructed by "adding immigration" to a general spatial branching Markov process in the sense of [INW1968a]. Our main results on this class of processes, many of which are generalizations of those in [HL2005], are to be found in Section 3.3. They rely in particular on a spatial notion of subcriticality originally due to [HL2005] which can be formulated in terms of branching Markov processes without immigration. We generalize this condition to our framework in Section 3.2, which should be viewed as a complement to the classical approach to branching Markov processes in [INW1968a]-[INW1969].

In the appendix, we state a "Continuity and Differentiation Lemma" for integrals depending on a parameter which is just a simple reformulation of a version of the Dominated Convergence Theorem known as Pratt's Theorem and which will be used extensively in the proofs of Section 2.2.

Chapter 1

The Model: Branching Diffusions with Immigration

In the first two chapters of this work, we will consider systems of finitely many particles, each living in $\mathbb{R}^d$ ($d \geq 1$), with the following features: The joint motion of $\ell \in \mathbb{N}$ particles is governed by a diffusion in $(\mathbb{R}^d)^\ell$ with interacting components. Each particle branches according to a rate depending on its position and on the configuration of coexisting particles. When a particle branches, it dies and produces a random number of offspring according to a position- and configuration-dependent reproduction law. The newborn particles are distributed randomly in space, again depending on the position of the branching parent particle and on the configuration of coexisting particles. In addition, immigration occurs at a configuration-dependent rate. At each immigration event, exactly one new particle is added to the system in a position depending on the configuration of already existing particles. The resulting stochastic process of finite particle configurations is called a *branching diffusion with immigration*, henceforth: BDI.

1.1 Basic Assumptions and Notations

In this section, we introduce the precise model assumptions and notations with which we will work throughout the first two chapters of this thesis. The general framework is essentially as in [Löc2002a] or [Löc2002b] (see also [Löc1999]).

We write $E := \mathbb{R}^d$ for the "single-particle space". A BDI as outlined above is a strong Markov process $\boldsymbol{\eta} = (\boldsymbol{\eta}_t)_t$ taking values in the space

$$\mathcal{S} := \dot{\bigcup_{\ell \in \mathbb{N}_0}} E^\ell \tag{1.1.1}$$

of finite ordered particle configurations, where $E^0 := \{\Delta\}$ denotes the void configuration (i.e. the state of no existing particle). Elements of the single particle space E will be denoted by $x, y, z, \ldots$ and elements of the configuration space $\mathcal{S}$ by boldfaced letters $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \ldots$. The length of a configuration $\boldsymbol{x}$ will be denoted by $\ell(\boldsymbol{x})$. Thus for $\boldsymbol{x} \in E^\ell$, $\ell \in \mathbb{N}_0$ we have $\boldsymbol{x} = (x^1, \ldots, x^\ell)$ with $x^j \in E$, $j = 1, \ldots, \ell$ and $\ell(\boldsymbol{x}) = \ell$. A metric on $\mathcal{S}$ may be defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) := \begin{cases} \frac{d^{(\ell)}(\boldsymbol{x},\boldsymbol{y})}{1+d^{(\ell)}(\boldsymbol{x},\boldsymbol{y})} & \text{if } \boldsymbol{x}, \boldsymbol{y} \in E^\ell, \\ |k - \ell| & \text{if } \boldsymbol{x} \in E^k, \boldsymbol{y} \in E^\ell, k \neq \ell, \end{cases} \tag{1.1.2}$$

where $d^{(\ell)}(\boldsymbol{x},\boldsymbol{y})$ denotes the Euclidean distance (or any equivalent metric) on $E^\ell = \mathbb{R}^{d\ell}$. With the induced topology, $\mathcal{S}$ is a locally compact Polish space, and we endow it with its Borel σ-algebra $\mathcal{B}_{\mathcal{S}}$.

1.1.1 Remark

The choice of ordered particle configurations for the state space means that we can distinguish between individual particles belonging to a configuration. Sometimes, it may be more appropriate to regard the particles in the process $\boldsymbol{\eta}$ as indistinguishable with respect to permutation of coordinates, i.e. to consider *unordered* configurations. Formally, this point of view corresponds to *symmetrization* of the space $\mathcal{S}$ of (1.1.1) and was adopted in [INW1968a]-[INW1968b] where general branching Markov processes (without immigration) were introduced. On the other hand, in the modern literature on spatial branching processes it is more common to identify an unordered configuration with the corresponding finite point measure.[1]

In the first two chapters of this work, we will stick to the framework of ordered configurations in order to stay in line with the set-up of [Löc2002a]-[Löc2002b]. However, all of the results to be given in the sequel do also hold for the unordered case. In Chapter 3, we will expressly allow for the choice of unordered configurations resp. finite point measures as state space since one of our results in that chapter will be proved for the unordered case only. ⧫

The following Assumptions 1.1.2-1.1.5 are assumed to hold throughout the first two chapters of this work. The first assumption governs the motion of particles between branching or immigration events:

1.1.2 Assumption (Particle Motion)

For all $\ell \in \mathbb{N}$, the ℓ-particle motion $X^\ell = (X^{1,\ell},\dots,X^{\ell,\ell})$ on E^ℓ is given by a system of stochastic differential equations

$$dX_t^{j,\ell} = b^{(\ell)}(X_t^{j,\ell};X_t^\ell)\,dt + \sigma^{(\ell)}(X_t^{j,\ell};X_t^\ell)\,dW_t^j, \qquad j = 1,\dots,\ell, \tag{1.1.3}$$

with independent m-dimensional ($m \ge d$) standard Brownian motions $W^1,\dots,W^\ell$ driving the motion of every particle, and drift and diffusion coefficients

$$b^{(\ell)}(\cdot\,;\cdot) : E \times E^\ell \to E, \qquad \sigma^{(\ell)}(\cdot\,;\cdot) : E \times E^\ell \to \mathbb{R}^{d\times m}$$

which are assumed to be regular enough such that (1.1.3) *has a global unique strong solution.*

1.1.3 Remarks

- It is convenient to complement the above assumption for $\ell = 0$ by the convention $X^0 \equiv \Delta$.
- We do not specify the stochastic basis (the sample space) on which the ℓ-particle motion happens to be defined; it could be the canonical path space $\mathcal{C}\left(\mathbb{R}_+;E^\ell\right)$ or any other suitable space. Throughout, we will denote by $P_{\boldsymbol{x}}$ the probability measure corresponding to the diffusion X^ℓ started at $\boldsymbol{x} = (x^1,\dots,x^\ell) \in E^\ell$, and corresponding expectations will be denoted by $E_{\boldsymbol{x}}$.
- Fix $\ell \in \mathbb{N}$ and define $\tilde{b}^{(\ell)} : E^\ell \to E^\ell$, $\tilde{\sigma}^{(\ell)} : E^\ell \to \mathbb{R}^{d\ell\times m\ell}$ by

$$\tilde{b}^{(\ell)}(\boldsymbol{x}) \equiv \begin{pmatrix} \tilde{b}_1^{(\ell)}(\boldsymbol{x}) \\ \vdots \\ \tilde{b}_\ell^{(\ell)}(\boldsymbol{x}) \end{pmatrix} := \begin{pmatrix} b^{(\ell)}(x^1;\boldsymbol{x}) \\ \vdots \\ b^{(\ell)}(x^\ell;\boldsymbol{x}) \end{pmatrix} \in E^\ell,$$

[1] On this identification, see also Remark 3.1.2 in Chapter 3 below.

$$\tilde{\sigma}^{(\ell)}(\boldsymbol{x}) := \begin{pmatrix} \sigma^{(\ell)}(x^1;\boldsymbol{x}) & 0 & \cdots & \cdots & 0 \\ 0 & \sigma^{(\ell)}(x^2;\boldsymbol{x}) & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & & \vdots \\ 0 & \cdots & & & \sigma^{(\ell)}(x^\ell;\boldsymbol{x}) \end{pmatrix} \in \mathbb{R}^{d\ell\times m\ell}$$

for $\boldsymbol{x} = (x^1,\dots,x^\ell) \in E^\ell$. Then the ℓ-particle motion $X^\ell = (X^{1,\ell},\dots,X^{\ell,\ell})$ is a solution to the SDE

$$dX_t^\ell = \tilde{b}^{(\ell)}(X_t^\ell)\,dt + \tilde{\sigma}^{(\ell)}(X_t^\ell)\,dW_t \tag{1.1.4}$$

on E^ℓ, where W now denotes an $m\ell$-dimensional standard Brownian motion. Defining $a^{(\ell)}(\cdot;\cdot) : E\times E^\ell \to \mathbb{R}^{d\times d}$ by

$$a^{(\ell)}(x;\boldsymbol{x}) := \sigma^{(\ell)}(x;\boldsymbol{x})\left(\sigma^{(\ell)}(x;\boldsymbol{x})\right)^T, \qquad (x,\boldsymbol{x}) \in E\times E^\ell \tag{1.1.5}$$

and $\tilde{a}^{(\ell)} : E^\ell \to \mathbb{R}^{d\ell\times d\ell}$ by

$$\tilde{a}^{(\ell)}(\boldsymbol{x}) := \tilde{\sigma}^{(\ell)}(\boldsymbol{x})\left(\tilde{\sigma}^{(\ell)}(\boldsymbol{x})\right)^T = \begin{pmatrix} a^{(\ell)}(x^1;\boldsymbol{x}) & 0 & \cdots & \cdots & 0 \\ 0 & a^{(\ell)}(x^2;\boldsymbol{x}) & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & & \vdots \\ 0 & \cdots & & & a^{(\ell)}(x^\ell;\boldsymbol{x}) \end{pmatrix}, \tag{1.1.6}$$

the generator of X^ℓ is given by

$$\mathscr{A}^{(\ell)} f(\boldsymbol{x}) = \frac{1}{2}\sum_{i,j=1}^{d\ell} \tilde{a}_{ij}^{(\ell)}(\boldsymbol{x})\partial_{ij}^2 f(\boldsymbol{x}) + \sum_{i=1}^{d\ell} \tilde{b}_i^{(\ell)}(\boldsymbol{x})\partial_i f(\boldsymbol{x}), \qquad \boldsymbol{x} \in E^\ell \tag{1.1.7}$$

for twice continuously differentiable functions with compact support $f \in \mathcal{C}_c^2(E^\ell)$. ♦

The branching and reproduction mechanism is governed by the following assumption:

1.1.4 Assumption (Branching and Reproduction Mechanism)
For each $\ell \in \mathbb{N}$, we are given a nonnegative measurable function

$$\kappa^{(\ell)}(\cdot;\cdot) : E\times E^\ell \to \mathbb{R}_+ \tag{1.1.8}$$

such that for all $t > 0$, $j = 1,\dots,\ell$ and $\boldsymbol{x} \in E^\ell$ we have

$$\int_0^t \kappa^{(\ell)}(X_s^{j,\ell};X_s^\ell)\,ds < \infty \qquad P_{\boldsymbol{x}}\text{ a-s.} \tag{1.1.9}$$

Further, for each $\ell \in \mathbb{N}$ we are given measurable functions

$$p_k^{(\ell)}(\cdot;\cdot) : E\times E^\ell \to [0,1], \qquad k \in \mathbb{N}_0 \tag{1.1.10}$$

such that $\sum_{k\in\mathbb{N}_0} p_k^{(\ell)}(\cdot;\cdot) \equiv 1$, and transition probabilities

$$Q_k^{(\ell)}(\cdot;\cdot;\cdot) : E\times E^\ell\times\mathcal{B}_{E^k} \to [0,1], \qquad k \in \mathbb{N}. \tag{1.1.11}$$

We put

$$Q_0^{(\ell)}(x;\boldsymbol{x};\cdot) := \delta_\Delta(\cdot), \qquad x \in E,\ \boldsymbol{x} \in E^\ell. \tag{1.1.12}$$

A particle belonging to a configuration $\boldsymbol{x} = (x^1, \dots, x^\ell) \in E^\ell$ *and situated at position* $x^i \in E$ *at time* $t > 0$ *branches at position- and configuration-dependent rate* $\kappa^{(\ell)}(x^i; \boldsymbol{x})$*, i.e. it dies during a small time intervall* $(t, t+h]$ *with probability* $\kappa^{(\ell)}(x^i; \boldsymbol{x}) \cdot h + o(h)$ *as* $h \downarrow 0$*. At its death time, it is replaced by a random number* $k \in \mathbb{N}_0$ *of offspring particles with probability* $p_k^{(\ell)}(x^i; \boldsymbol{x})$*, again depending on its position and the configuration of coexisting particles. The* k *offspring particles are distributed in* E^k *according to the law*

$$Q_k^{(\ell)}(x^i; \boldsymbol{x}; dv^1 \cdots dv^k) \qquad \text{on } (E^k, \mathcal{B}_{E^k}). \tag{1.1.13}$$

We refer to $\kappa^{(\ell)}(\cdot\,;\cdot)$ *as the branching rate, to* $\left(p_k^{(\ell)}(\cdot\,;\cdot)\right)_{k \in \mathbb{N}_0}$ *as the reproduction law and to* $\left(Q_k^{(\ell)}(\cdot\,;\cdot\,;\cdot)\right)_{k \in \mathbb{N}}$ *as the spatial offspring distribution.*

1.1.5 Assumption (Immigration Mechanism)
For each $\ell \in \mathbb{N}_0$*, we are given a nonnegative measurable function*

$$c^{(\ell)}(\cdot) : E^\ell \to \mathbb{R}_+ \tag{1.1.14}$$

called the immigration rate such that for all $t > 0$ *and* $\boldsymbol{x} \in E^\ell$

$$\int_0^t c^{(\ell)}(X_s^\ell)\, ds < \infty \qquad P_{\boldsymbol{x}}\text{-a.s.} \tag{1.1.15}$$

Moreover, for each $\ell \in \mathbb{N}_0$ *we are given a transition probability*

$$\nu^{(\ell)}(\cdot\,;\cdot) : E^\ell \times \mathcal{B}_E \to [0,1] \tag{1.1.16}$$

to which we refer as the immigration law. For $\ell = 0$*, we assume that*

$$c^{(0)}(\Delta) > 0 \tag{1.1.17}$$

and write $\nu^{(0)}(dv) := \nu^{(0)}(\Delta; dv)$*. New particles immigrate at configuration-dependent rate* $c(\cdot)$*: If there are* $\ell \in \mathbb{N}_0$ *particles at positions* $\boldsymbol{x} = (x^1, \dots, x^\ell) \in E^\ell$ *at time* $t > 0$*, then one new particle immigrates during a small time interval* $(t, t+h]$ *with probability* $c^{(\ell)}(\boldsymbol{x})h + o(h)$ *as* $h \downarrow 0$*. The immigrating particle is distributed in* E *according to the law* $\nu(\boldsymbol{x}; dv)$*, depending on the configuration* $\boldsymbol{x}$ *of already existing particles.*

Omitting the superscript ℓ, we will consider all functions and kernels defined above on $E \times E^\ell$ or on E^ℓ also as functions and kernels on $E \times \mathcal{S}$ or on $\mathcal{S}$ in the obvious way, namely $\kappa(x; \boldsymbol{x}) := \kappa^{(\ell)}(x; \boldsymbol{x})$ for $x \in E$, $\boldsymbol{x} \in \mathcal{S}$ with $\ell(\boldsymbol{x}) = \ell \in \mathbb{N}$, $\kappa(x; \Delta) := \kappa^{(0)}(x; \Delta) := 0$, and the same for the other quantities introduced above.

1.1.6 Remarks

- Our assumptions correspond to Assumptions 2.1-2.3 in [Löc2002a] resp. Assumptions 5.1-5.3 in [Löc2002b]. We differ from Löcherbach's framework in some minor respects: For example, we do not rearrange the particles at random at every branching or immigration event (Assumption 2.4 in [Löc2002a] resp. 5.4 in [Löc2002b]). Also, we allow for the case $p_1^{(\ell)}(x; \boldsymbol{x}) > 0$, and we do not require the offspring particles to choose their spatial positions independently of each other, i.e. the kernel $Q_k^{(\ell)}(x; \boldsymbol{x}; \cdot)$ of (1.1.11) need not be of product type. Moreover, newborn particles are not inserted at the end of the preexisting configuration but in the place of the branching parent.

- The above assumptions include in particular the framework of [HL2005], where particles move independently of each other on diffusion paths and reproduce according to a branching rate and reproduction law which are allowed to depend on the spatial position of a particle but not on the whole configuration of coexisting particles, and immigration occurs at constant rate: In this case, the quantities b, σ, κ, p_k, Q_k, c and ν are all independent of the configuration variable $\boldsymbol{x}$; let us call this the "purely position-dependent framework". Moreover, in [HL2005] and in most of the branching process literature it is assumed that branching particles reproduce *at* their death position, i.e. the offspring particles start their spatial motion at their parent's death position. Note that this means
$$Q_k^{(\ell)}(x;\boldsymbol{x};\cdot) = \delta_x(\cdot)^{\otimes k} \quad \text{on } E^k \tag{1.1.18}$$
in (1.1.13) for all $k,\ell \in \mathbb{N}$ and $(x,\boldsymbol{x}) \in E \times E^\ell$. ⧫

1.1.7 Remarks

- Note that at present, we do not impose any continuity conditions (like Assumption 2.6 in [Löc2002a]) on the functions and kernels introduced in the assumptions above since this is not necessary for the existence and construction of the process $\boldsymbol{\eta}$. (We will have to require continuity later in this work when we deal e.g. with the regularity of the invariant measure for $\boldsymbol{\eta}$.) Also note that we do not require boundedness of the branching or immigration rate as in [HL2005]. Of course, boundedness of $\kappa^{(\ell)}(\cdot;\cdot)$ and $c^{(\ell)}(\cdot)$ is an easy way to ensure (1.1.9) and (1.1.15), but it is not necessary: For example, by the continuity of the paths of X^ℓ, (1.1.9) and (1.1.15) hold provided $\kappa^{(\ell)}(\cdot;\cdot)$ and $c^{(\ell)}(\cdot)$ are bounded on compacts.

- We comment on the significance of conditions (1.1.9) and (1.1.15): Fix $\ell \in \mathbb{N}_0$ and start the ℓ-particle motion from $\boldsymbol{x} = (x^1,\ldots,x^\ell) \in E^\ell$. We define additive functionals
$$A_t^j := \int_0^t \kappa^{(\ell)}(X_s^{j,\ell};X_s^\ell)\,ds, \quad j=1,\ldots,\ell, \qquad A_t^{\ell+1} := \int_0^t c^{(\ell)}(X_s^\ell)\,ds.$$
Due to (1.1.9) and (1.1.15), these are (nonnegative) *finite* and thus (by dominated convergence) also continuous additive functionals of X^ℓ. Let $\zeta^1,\ldots,\zeta^\ell,\zeta^{\ell+1}$ be i.i.d. random variables which are exponentially distributed with parameter 1 and independent of X^ℓ. For $j=1,\ldots,\ell+1$, define a random time τ^j as the right-continuous inverse of the additive functional A^j at ζ^j, i.e.
$$\tau^j := \inf\{t>0 : A_t^j > \zeta^j\}, \qquad j=1,\ldots,\ell+1.$$
Then it is easy to see that
$$P_{\boldsymbol{x}}\left[\tau^j > t \,\middle|\, X^\ell\right] = \exp(-A_t^j), \qquad j=1,\ldots,\ell+1. \tag{1.1.19}$$
Thus conditionally on the evolution of X^ℓ, the distribution of the random "clock" τ^j is of exponential nature, but its "rate" depends on the state of X^ℓ via the additive functional A^j. Indeed, (1.1.19) is a formalization of what is meant by saying that $\kappa^{(\ell)}(\cdot;\cdot)$ and $c^{(\ell)}(\cdot)$ are the branching and the immigration rate, respectively. Depending on which of the clocks τ^j rings first, the jth particle branches or an additional particle immigrates,

i.e. the time of the first branching or immigration event is given by $\tau := \min_{j=1,\dots,\ell+1} \tau^j$. Since $\zeta^1, \dots, \zeta^{\ell+1}$ are independent, it satisfies

$$P_{\boldsymbol{x}}\left[\tau > t \,\middle|\, X^\ell\right] = \prod_{j=1}^{\ell+1} \exp(-A_t^j) = \exp\left(-\int_0^t \alpha^{(\ell)}(X_s^\ell)\,ds\right), \tag{1.1.20}$$

where we define

$$\alpha^{(\ell)}(\boldsymbol{x}) := \sum_{i=1}^{\ell} \kappa^{(\ell)}(x^i;\boldsymbol{x}) + c^{(\ell)}(\boldsymbol{x}), \qquad \boldsymbol{x} = (x^1,\dots,x^\ell) \in E^\ell,\ \ell \in \mathbb{N}_0. \tag{1.1.21}$$

Thus the function $\alpha^{(\ell)}(\cdot)$ gives the rate at which a branching or immigration event happens, starting from an ℓ-particle configuration. ♦

The existence under Assumptions 1.1.2-1.1.5 of a corresponding BDI $\boldsymbol{\eta}$ with the desired properties follows from the "killing and restarting"-procedure for Markov processes developed by Ikeda, Nagasawa and Watanabe in [INW1968b] (see also [Nag1977]). More precisely, given the quantities in Assumptions 1.1.2-1.1.5, the process $\boldsymbol{\eta}$ can be constructed in the following way: First, let $\boldsymbol{X} = (\boldsymbol{X}_t)_{t\geq 0}$ denote the $\mathcal{S}$-valued process describing a finite system of particles such that for each $\ell \in \mathbb{N}$, starting from ℓ particles $\boldsymbol{X}$ evolves as the given process X^ℓ on E^ℓ, without any branching or immigration. In other words, $\boldsymbol{X}$ is the direct sum process of the given ℓ-particle motions X^ℓ, $\ell \in \mathbb{N}$. For $\ell = 0$ (starting from the void configuration Δ), by convention we have $\boldsymbol{X}_t = \Delta$ for all $t > 0$.

The process $\boldsymbol{X}$ is now stopped or "killed" at the random time τ from (1.1.20) above, i.e. with configuration-dependent rate $\alpha(\cdot) : \mathcal{S} \to \mathbb{R}_+$ defined layer-wise as in (1.1.21). At its death time, it is "revived" or "restarted" with a new initial configuration chosen by a jump kernel $K : \mathcal{S} \times \mathcal{B}_{\mathcal{S}} \to [0,1]$ which is defined as follows: For each $\ell \in \mathbb{N}$, $i \in \{1,2,\dots,\ell\}$ and $k \in \mathbb{N}_0$ define a mapping $\Pi_{\ell,k,i} : E^\ell \times E^k \to E^{\ell-1+k}$ by

$$\Pi_{\ell,k,i}(\boldsymbol{x};\boldsymbol{v}) := (x^1,\dots,x^{i-1},v^1,\dots,v^k,x^{i+1},\dots,x^\ell), \qquad k \in \mathbb{N}, \tag{1.1.22}$$

$$\Pi_{\ell,0,i}(\boldsymbol{x}) := \Pi_{\ell,0,i}(\boldsymbol{x};\Delta) := (x^1,\dots,x^{i-1},x^{i+1},\dots,x^\ell), \qquad k = 0. \tag{1.1.23}$$

It can be interpreted as a mapping which replaces the ith particle x^i of a given ℓ-particle configuration $\boldsymbol{x} = (x^1,\dots,x^\ell) \in E^\ell$ by $k \in \mathbb{N}_0$ particles at positions $v^1,\dots,v^k$. Also, for $\ell \in \mathbb{N}_0$, $\boldsymbol{x} = (x^1,\dots,x^\ell) \in E^\ell$ and $v \in E$ we write

$$(\boldsymbol{x},v) := (x^1,\dots,x^\ell,v) \in E^{\ell+1}$$

for the configuration obtained by concatenation (with the understanding that $(\Delta,v) := v$ if $\ell = 0$). The jump kernel is then defined as follows:

$$K(\boldsymbol{x};\cdot) := \frac{\mathbf{1}_{\alpha^{(\ell)}(\boldsymbol{x})>0}}{\alpha^{(\ell)}(\boldsymbol{x})} \cdot \Bigg[\sum_{k\in\mathbb{N}_0}\sum_{i=1}^{\ell} \kappa^{(\ell)}(x^i;\boldsymbol{x})p_k^{(\ell)}(x^i;\boldsymbol{x}) \int_{E^k} Q_k^{(\ell)}(x^i;\boldsymbol{x};dv^1\cdots dv^k)\,\delta_{\Pi_{\ell,k,i}(\boldsymbol{x};\boldsymbol{v})}(\cdot) + c^{(\ell)}(\boldsymbol{x})\int_E \nu^{(\ell)}(\boldsymbol{x};dv)\,\delta_{(\boldsymbol{x},v)}(\cdot)\Bigg] \tag{1.1.24}$$

for $\boldsymbol{x} = (x^1, \ldots, x^\ell) \in E^\ell$, $\ell \in \mathbb{N}_0$. For the term with $k = 0$ in (1.1.24), recall our convention (1.1.12). Also, we use the convention $\sum_{i=1}^{0} \cdots := 0$ throughout so that for $\ell = 0$, $\boldsymbol{x} = \Delta$, Definition (1.1.24) means

$$K(\Delta; \cdot) = \nu^{(0)}(\cdot), \tag{1.1.25}$$

i.e. the immigration measure when no particles are present, interpreted as a measure on $\mathcal{S}$ which is concentrated on the single-particle layer E.

The "killing and restarting"-procedure described above can be made rigorous using the so-called "Revival Theorem" for Markov processes, see [INW1968b], Thm. 2.2 or [Nag1977], Thm. 2. Applying it under our assumptions above, the resulting process of particle configurations can be constructed as a strong Markov process

$$\boldsymbol{\eta} = (\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, (\boldsymbol{\mathcal{F}}_t)_{t\geq 0}, (\boldsymbol{P}_{\boldsymbol{x}})_{\boldsymbol{x}\in\mathcal{S}}, (\boldsymbol{\eta}_t)_{t\geq 0}, (\boldsymbol{\theta}_t)_{t\geq 0}) \tag{1.1.26}$$

on some suitable stochastic basis with a right-continuous and complete[2] filtration. The process $\boldsymbol{\eta}$ takes values in $\mathcal{S}_\partial = \mathcal{S} \cup \{\partial\}$, where ∂ is an extra point adjoined to the space $\mathcal{S}$ as a "cemetary" in order to account for the possibility of "explosion" of the process (accumulation of branching or immigration events in finite time).[3] Writing

$$\boldsymbol{\tau}_\infty = \inf\{t > 0 : \boldsymbol{\eta}_t \notin \mathcal{S}\} \leq \infty$$

for the (possibly finite) "life-time" (in the sense of explosion time), $\boldsymbol{\eta}$ has càdlàg sample paths before time $\boldsymbol{\tau}_\infty$, and we have an increasing sequence

$$0 =: \boldsymbol{\tau}_0 \leq \boldsymbol{\tau}_1 \leq \boldsymbol{\tau}_2 \leq \cdots \uparrow \sup_{n\in\mathbb{N}} \boldsymbol{\tau}_n = \boldsymbol{\tau}_\infty \tag{1.1.27}$$

of $(\boldsymbol{\mathcal{F}}_t)_t$-stopping times given by

$$\boldsymbol{\tau}_n = \boldsymbol{\tau}_{n-1} + \boldsymbol{\tau}_1 \circ \boldsymbol{\theta}_{\boldsymbol{\tau}_{n-1}}, \qquad n \in \mathbb{N} \tag{1.1.28}$$

corresponding to branching or immigration events in the process $\boldsymbol{\eta}$.

1.1.8 Remarks

- An alternative construction of $\boldsymbol{\eta}$ on a canonical path space was given (under additional assumptions) in [Löc2002b].

- Under suitable regularity conditions (like Feller properties of the diffusion X^ℓ and the jump kernel K and continuity of α) the infinitesimal generator $\mathcal{A}$ of the process $\boldsymbol{\eta}$ is given as follows: Denote by $\mathscr{A}$ the generator of the direct sum process $\boldsymbol{X}$ on $\mathcal{S}$ mentioned above, corresponding to a system of particles evolving on each layer E^ℓ as the given ℓ-particle motion X^ℓ, without any branching or immigration. The operator $\mathscr{A}$ acts on functions $f = (f^{(\ell)})_{\ell\in\mathbb{N}_0}$ on $\mathcal{S}$ layer-wise as

$$(\mathscr{A} f)^{(\ell)}(\boldsymbol{x}) = \mathscr{A}^{(\ell)} f^{(\ell)}(\boldsymbol{x}), \qquad \boldsymbol{x} \in E^\ell,$$

[2] This means complete w.r.t. the family of measures $(\boldsymbol{P}_{\boldsymbol{x}})_{\boldsymbol{x}\in\mathcal{S}}$; see e.g. [RY1999], Def. I.4.13 for the terminology.

[3] The cemetary ∂ must be carefully distinguished from the void configuration Δ. Unfortunately, in [INW1968b] the notation is just reversed; our notation is chosen so as to be in line with [HL2005] and [Nag1977].

where $\mathscr{A}^{(\ell)}$ is the generator of X^ℓ from (1.1.7). Then the generator $\mathcal{A}$ of the BDI $\boldsymbol{\eta}$ takes the form

$$\mathcal{A}f(\boldsymbol{x}) = \mathscr{A}f(\boldsymbol{x}) + \alpha(\boldsymbol{x})\int_S K(\boldsymbol{x};d\boldsymbol{y})\,(f(\boldsymbol{y}) - f(\boldsymbol{x}))\,.^4 \tag{1.1.29}$$

In this case, the process $\boldsymbol{\eta}$ can also be constructed from the generator (1.1.29) by means of the Hille-Yosida Theorem. See e.g. [Saw1970] who also investigates the generator in the more general case that the above-mentioned regularity conditions are violated. The probabilistic construction in [INW1968b] on the other hand does not require any conditions beyond those in Assumptions 1.1.2-1.1.5.

- Note that our assumptions allow for the possibility of "explosion" $\boldsymbol{\tau}_\infty < \infty$, i.e. there may be an accumulation of branching / immigration events in finite time. On sufficient conditions for non-explosion of a BDI, see e.g. [Löc2002a], Sec. 4; see also Subsection 3.2.5 in Chapter 3 below concerning the explosion problem for branching Markov processes (without immgration).

  Also note that Assumptions 1.1.4-1.1.5 ensure that the branching / immigration times $\boldsymbol{\tau}_n$ ($n \in \mathbb{N}$) are strictly positive, but not that they are a.s. finite (this is the reason for "$\leq$" rather than "$<$" in (1.1.27)). Indeed, $\boldsymbol{\tau}_1 < \infty$ holds $\boldsymbol{P_x}$-a.s. if and only if

$$\int_0^\infty \alpha^{(\ell)}(X_t^{(\ell)})\,dt = \infty \qquad P_{\boldsymbol{x}}\text{-a.s.}, \qquad \boldsymbol{x} \in E^\ell. \tag{1.1.30}$$

  If this holds for all $\boldsymbol{x} \in \mathcal{S}$, all stopping times $\boldsymbol{\tau}_n$ are finite $\boldsymbol{P_x}$-a.s., and we have $0 = \boldsymbol{\tau}_0 < \boldsymbol{\tau}_1 < \boldsymbol{\tau}_2 \cdots \uparrow \boldsymbol{\tau}_\infty \leq \infty$.

  Our reason for not imposing conditions ensuring $\boldsymbol{\tau}_\infty = \infty$ or $\boldsymbol{\tau}_n < \infty$ at this point is the following: In the next section, we will work under the stronger assumption that the BDI $\boldsymbol{\eta}$ is positive Harris recurrent with the void configuration Δ as a recurrent atom, which implies at once infinite lifetime of $\boldsymbol{\eta}$ and finiteness of all branching / immigration times $\boldsymbol{\tau}_n$. ♦

We proceed with some notation and definitions. For $\ell \in \mathbb{N}$, consider the ℓ-particle motion X^ℓ (which is a $d\ell$-dimensional diffusion) killed at rate $\alpha^{(\ell)}(\cdot)$. Again writing τ for the corresponding killing time, from (1.1.20) we see that its semigroup is given by

$$P_t^\alpha f(\boldsymbol{x}) \coloneqq E_{\boldsymbol{x}}\left[f(X_t^\ell)\mathbf{1}_{t<\tau}\right] = E_{\boldsymbol{x}}\left[f(X_t^\ell)\exp\left(-\int_0^t \alpha^{(\ell)}(X_s^\ell)\,ds\right)\right] \tag{1.1.31}$$

for $\boldsymbol{x} \in E^\ell$ and bounded measurable functions $f \in \mathscr{B}(E^\ell)$. The corresponding generator acts on $\mathcal{C}_c^2$-functions as

$$\mathscr{A}^{(\ell)} f(\boldsymbol{x}) - \alpha^{(\ell)}(\boldsymbol{x})f(\boldsymbol{x}), \qquad \boldsymbol{x} \in E^\ell,\ f \in \mathcal{C}_c^2(E^\ell) \tag{1.1.32}$$

with $\mathscr{A}^{(\ell)}$ as in (1.1.7). The occupation times of the killed ℓ-particle motion are given by the generalized resolvent

$$E_{\boldsymbol{x}}\left[\int_0^\tau \mathbf{1}_B(X_t^\ell)\,dt\right] = E_{\boldsymbol{x}}\left[\int_0^\infty \mathbf{1}_B(X_t^{(\ell)})\,e^{-\int_0^t \alpha^{(\ell)}(X_s^{(\ell)})\,ds}dt\right] =: R_\alpha^{(\ell)}(\boldsymbol{x};B) \tag{1.1.33}$$

[4] This is the general form for the generator of a Markov process of the "killed and revived"-type.

for $\boldsymbol{x} \in E^\ell$ and $B \in \mathcal{B}_{E^\ell}$. Note that $R_\alpha^{(\ell)}$ is a transition kernel on $E^\ell \times \mathcal{B}_{E^\ell}$ for each ℓ, but in general not finite under our assumptions. The configuration of the killed ℓ-particle motion at the time of killing (provided it is finite) is given by

$$E_{\boldsymbol{x}}\left[\mathbf{1}_{\tau<\infty} \cdot \mathbf{1}_B(X_\tau^\ell)\right] = \int_{E^\ell} R_\alpha^{(\ell)}(\boldsymbol{x}; d\boldsymbol{y})\alpha^{(\ell)}(\boldsymbol{y})\mathbf{1}_B(\boldsymbol{y}) =: [R_\alpha^{(\ell)}\alpha](\boldsymbol{x}; B) \tag{1.1.34}$$

for $\boldsymbol{x} \in E^\ell$, $B \in \mathcal{B}_{E^\ell}$, which is a substochastic kernel on $E^\ell \times \mathcal{B}_{E^\ell}$. Under the conditions (1.1.9) and (1.1.15), the formulas (1.1.33) and (1.1.34) follow easily from (1.1.20).

Now we return to the $\mathcal{S}$-valued BDI process $\boldsymbol{\eta}$ from (1.1.26). Throughout, expectations w.r.t. the probability measures $\boldsymbol{P}_{\boldsymbol{x}}$ will be denoted by $\boldsymbol{E}_{\boldsymbol{x}}$, $\boldsymbol{x} \in \mathcal{S}$, and we will write

$$\boldsymbol{P}_t(\boldsymbol{x}; F) := \boldsymbol{E}_{\boldsymbol{x}}[\mathbf{1}_F(\boldsymbol{\eta}_t) \cdot \mathbf{1}_{t<\boldsymbol{\tau}_\infty}], \qquad F \in \mathcal{B}_{\mathcal{S}},\ \boldsymbol{x} \in \mathcal{S},\ t > 0 \tag{1.1.35}$$

for the transition semigroup of $\boldsymbol{\eta}$.[5] Putting

$$R_\alpha^{(0)}(\Delta; \cdot) := \frac{1}{c(\Delta)} \cdot \delta_\Delta(\cdot),$$

we interpret R_α from (1.1.33) and $[R_\alpha \alpha]$ from (1.1.34) as transition kernels on $\mathcal{S} \times \mathcal{B}_{\mathcal{S}}$ such that if $\boldsymbol{x} \in E^\ell$, $\ell \in \mathbb{N}_0$, the measure $R_\alpha(\boldsymbol{x}; \cdot)$ charges only the layer E^ℓ:

$$R_\alpha(\cdot\,;\cdot) : \mathcal{S} \times \mathcal{B}_{\mathcal{S}} \to [0, \infty], \qquad R_\alpha(\boldsymbol{x}; \cdot) := R_\alpha^{(\ell)}(\boldsymbol{x}; \cdot \cap E^\ell) \quad \text{if } \boldsymbol{x} \in E^\ell,\ \ell \in \mathbb{N}_0, \tag{1.1.36}$$

where $R_\alpha^{(\ell)}$ is defined as the r.h.s. of (1.1.33). Now observe that since by construction the BDI $\boldsymbol{\eta}$ started from an ℓ-particle configuration $\boldsymbol{x} \in E^\ell$ coincides up to the first branching / immigration time $\boldsymbol{\tau}_1$ with the given ℓ-particle motion X^ℓ killed at rate $\alpha^{(\ell)}$, from (1.1.33) and (1.1.34) we get

$$\boldsymbol{E}_{\boldsymbol{x}}\left[\int_0^{\boldsymbol{\tau}_1} \mathbf{1}_F(\boldsymbol{\eta}_s)\, ds\right] = R_\alpha(\boldsymbol{x}; F), \qquad \boldsymbol{x} \in \mathcal{S},\ F \in \mathcal{B}_{\mathcal{S}}, \tag{1.1.37}$$

$$\boldsymbol{E}_{\boldsymbol{x}}\left[\mathbf{1}_{\boldsymbol{\tau}_1<\infty} \cdot \mathbf{1}_F(\boldsymbol{\eta}_{\boldsymbol{\tau}_1-})\right] = [R_\alpha \alpha](\boldsymbol{x}; F), \qquad \boldsymbol{x} \in \mathcal{S},\ F \in \mathcal{B}_{\mathcal{S}}. \tag{1.1.38}$$

The transition from $\boldsymbol{\eta}_{\boldsymbol{\tau}_1-}$ to $\boldsymbol{\eta}_{\boldsymbol{\tau}_1}$ (if $\boldsymbol{\tau}_1 < \infty$) is governed by the jump kernel K of (1.1.24):

$$\boldsymbol{E}_{\boldsymbol{x}}\left[\mathbf{1}_{\boldsymbol{\tau}_1<\infty} \cdot \mathbf{1}_F(\boldsymbol{\eta}_{\boldsymbol{\tau}_1}) \,|\, \boldsymbol{\eta}_{T_1-} = \boldsymbol{x}\right] = K(\boldsymbol{x}; F), \qquad \boldsymbol{x} \in \mathcal{S},\ F \in \mathcal{B}_{\mathcal{S}}, \tag{1.1.39}$$

and thus

$$\boldsymbol{E}_{\boldsymbol{x}}\left[\mathbf{1}_{\boldsymbol{\tau}_1<\infty} \cdot \mathbf{1}_F(\boldsymbol{\eta}_{T_1})\right] = [R_\alpha \alpha K](\boldsymbol{x}; F), \qquad \boldsymbol{x} \in \mathcal{S},\ F \in \mathcal{B}_{\mathcal{S}}. \tag{1.1.40}$$

In view of the strong Markov property of $\boldsymbol{\eta}$, this describes the evolution of the process on any random time interval $[\boldsymbol{\tau}_n, \boldsymbol{\tau}_{n+1}]$, $n \in \mathbb{N}_0$.

We close this section with some further notations and a simple lemma which will be used in the sequel:

1.1.9 Notations

Deviating somewhat from usual mathematical parlance, throughout this work we will say that a property of a function, measure or other object defined on the configuration space $\mathcal{S}$ holds *locally* if it holds in restriction to each layer E^ℓ, $\ell \in \mathbb{N}_0$. For example, we say that a

[5] Adopting the convention that a function $g : \mathcal{S} \to \mathbb{R}$ is extended to $\mathcal{S}_\partial$ by putting $g(\partial) := 0$, we could also dispense with the indicator $\mathbf{1}_{t<\boldsymbol{\tau}_\infty}$ in (1.1.35).

function $f = (f^{(\ell)})_{\ell\in\mathbb{N}_0} : \mathcal{S} \to \mathbb{R}$ is locally bounded and write $f \in \mathscr{B}^{(loc)}(\mathcal{S})$ if $f^{(\ell)} \in \mathscr{B}(E^\ell)$ for all $\ell \in \mathbb{N}_0$, where $\mathscr{B}(E^\ell)$ denotes the space of bounded Borel measurable functions on E^ℓ. A similar convention applies e.g. for the spaces $\mathcal{C}_b^{(loc)}(\mathcal{S})$ resp. $\mathcal{C}_0^{(loc)}(\mathcal{S})$ of functions which are continuous and bounded resp. vanishing at infinity in restriction to each layer. A measure $\mu = (\mu^{(\ell)})_{\ell\in\mathbb{N}_0}$ on $(\mathcal{S}, \mathcal{B}_\mathcal{S})$ is called locally finite if $\mu^{(\ell)}(E^\ell) < \infty$ for all $\ell \in \mathbb{N}_0$, in which case we write $\mu \in \mathcal{M}_f^{(loc)}(\mathcal{S})$. The Lebesgue measure $\boldsymbol{\lambda}$ on $\mathcal{S}$ is defined layer-wise,

$$\boldsymbol{\lambda}(\cdot) := \sum_{\ell\geq 0} \lambda^\ell(\cdot \cap E^\ell),$$

where for $\ell \in \mathbb{N}$, λ^ℓ is the usual $d\ell$-dimensional Lebesgue measure on $E^\ell = (\mathbb{R}^d)^\ell$, and $\lambda^0 := \delta_\Delta$. Integrals of a function f w.r.t. a measure μ on $\mathcal{S}$ will be denoted by any of the expressions

$$\int_\mathcal{S} f(\boldsymbol{x})\,\mu(d\boldsymbol{x}), \qquad \int_\mathcal{S} \mu(d\boldsymbol{x})\,f(\boldsymbol{x}), \qquad \mu(f), \qquad \langle \mu, f\rangle.$$

For integrals w.r.t. Lebesgue measure $\boldsymbol{\lambda}$, we will usually write $\int_\mathcal{S} f(\boldsymbol{x})\,d\boldsymbol{x} := \int_\mathcal{S} f(\boldsymbol{x})\,\boldsymbol{\lambda}(d\boldsymbol{x})$. As usual, the Banach space of (equivalence classes of) p-integrable functions ($p \geq 1$) on $\mathcal{S}$ w.r.t. a measure μ is denoted by $L^p(\mathcal{S};\mu)$, and the space of locally (in the sense introduced above) p-integrable functions will be denoted by $L^p_{(loc)}(\mathcal{S};\mu)$. For $\mu = \boldsymbol{\lambda}$, we simply write $L^p(\mathcal{S}) := L^p(\mathcal{S};\boldsymbol{\lambda})$ and $L^p_{(loc)}(\mathcal{S}):=L^p_{(loc)}(\mathcal{S};\boldsymbol{\lambda})$.

If $N(\cdot;\cdot) : \mathcal{S} \times \mathcal{B}_\mathcal{S} \to [0,\infty]$ is a (positive) transition kernel on $\mathcal{S}$, we interpret it also as an "operator" acting on nonnegative measurable functions f on $\mathcal{S}$ and on (positive) measures μ on $\mathcal{B}_\mathcal{S}$ in the usual way via

$$Nf(\boldsymbol{x}) := N(\boldsymbol{x}; f) := \int_\mathcal{S} N(\boldsymbol{x}; d\boldsymbol{y})\,f(y), \qquad \boldsymbol{x} \in \mathcal{S},$$

$$N^*\mu(F) := \mu N(F) := \int_\mathcal{S} \mu(d\boldsymbol{x})\,N(\boldsymbol{x}; F), \qquad F \in \mathcal{B}_\mathcal{S}.$$

Usually, we will write $N^*\mu$ rather than the perhaps more common notation μN. Obviously, we have

$$\langle \mu, Nf\rangle = \langle N^*\mu, f\rangle \tag{1.1.41}$$

for all nonnegative f and (positive) measures μ. If the kernel N is bounded, the above definitions and the duality (1.1.41) can of course be extended to bounded measurable functions f and signed measures μ, and N^* coincides with the functional analytic adjoint operator of $N : \mathscr{B}(\mathcal{S}) \to \mathscr{B}(\mathcal{S})$ restricted to the subspace of $(\mathscr{B}(\mathcal{S}))^*$ consisting of signed measures. If the measure $N^*\mu$ happens to be absolutely continuous w.r.t. Lebesgue measure $\boldsymbol{\lambda}$ on $\mathcal{S}$, we use the same symbol $N^*\mu$ to denote also (the equivalence class of) its $\boldsymbol{\lambda}$-density. Moreover, if μ is $\boldsymbol{\lambda}$-absolutely continuous with density f,

$$\mu(d\boldsymbol{x}) = f(\boldsymbol{x})\,d\boldsymbol{x},$$

then we write $N^*f := N^*\mu$. Note that in this case, N^*f is in general a measure rather than a function. For a transition kernel N and a nonnegative measurable function g on $\mathcal{S}$, we define two new kernels $[gN]$ and $[Ng]$ by multiplying N with the function g in the first and second argument respectively:

$$[gN](\boldsymbol{x}; d\boldsymbol{y}) := g(\boldsymbol{x})N(\boldsymbol{x}; d\boldsymbol{y}), \qquad \boldsymbol{x} \in \mathcal{S},$$

$$[Ng](\boldsymbol{x};d\boldsymbol{y}) := N(\boldsymbol{x};d\boldsymbol{y})\,g(\boldsymbol{y}), \qquad \boldsymbol{x}\in\mathcal{S}$$

(cf. the definition of the kernel $[R_\alpha\alpha]$ in (1.1.34)).

Finally, for a measure $\mu\in\mathcal{M}_f^{(loc)}(\mathcal{S})$ we define its Fourier transform $\mathscr{F}[\mu]:\mathcal{S}\to\mathbb{C}$ layer-wise by the Fourier transforms of its restrictions to E^ℓ:

$$\mathscr{F}[\mu](\boldsymbol{\xi}) := \int_{E^\ell} e^{-i\langle\boldsymbol{x},\boldsymbol{\xi}\rangle_\ell}\,\mu^{(\ell)}(d\boldsymbol{x}), \qquad \boldsymbol{\xi}=(\xi^1,\dots,\xi^\ell)\in E^\ell, \tag{1.1.42}$$

where $\langle\cdot,\cdot\rangle_\ell$ denotes the euclidean scalar product in E^ℓ. For $\ell=0$, we put $\langle\Delta,\Delta\rangle_0:=0$, so that $\mathscr{F}[\mu](\Delta)=\mu(\Delta)$ for all measures μ. The inverse Fourier transform is denoted by

$$\mathscr{F}^{-1}[\mu](\boldsymbol{\xi}) := \int_{E^\ell} e^{i\langle\boldsymbol{x},\boldsymbol{\xi}\rangle_\ell}\,\mu^{(\ell)}(d\boldsymbol{x}) = \mathscr{F}[\mu](-\boldsymbol{\xi}), \qquad \boldsymbol{\xi}\in E^\ell.$$

Now consider again the jump kernel K from (1.1.24). The following lemma describes the action of the kernel $[\alpha K]$ on a measure μ on $(\mathcal{S},\mathcal{B}_\mathcal{S})$. We omit its proof since it is obvious.

1.1.10 Lemma
Let $\mu=\left(\mu^{(\ell)}\right)_{\ell\in\mathbb{N}_0}$ be a measure on $(\mathcal{S},\mathcal{B}_\mathcal{S})$. Then the measure $\mu[\alpha K]\equiv[\alpha K]^*\mu$ is given in restriction to each layer E^ℓ, $\ell\in\mathbb{N}$ by

$$\begin{aligned}\Big\langle([\alpha K]^*\mu)^{(\ell)}\,,\,h\Big\rangle = \sum_{k=0}^{\ell}\sum_{j=1}^{\ell-k+1}\int_{\boldsymbol{x}\in E^{\ell-k+1}} \mu^{(\ell-k+1)}(d\boldsymbol{x})\,\kappa^{(\ell-k+1)}(x^j;\boldsymbol{x})\,p_k^{(\ell-k+1)}(x^j;\boldsymbol{x})\\ \int_{\boldsymbol{v}\in E^k} Q_k^{(\ell-k+1)}(x^j;\boldsymbol{x};d\boldsymbol{v})\,h\left(\Pi_{\ell-k+1,k,j}(\boldsymbol{x};\boldsymbol{v})\right)\\ +\int_{\boldsymbol{x}\in E^{\ell-1}}\mu^{(\ell-1)}(d\boldsymbol{x})\,c^{(\ell-1)}(\boldsymbol{x})\int_E \nu^{(\ell-1)}(\boldsymbol{x};dv)\,h(\boldsymbol{x},v)\end{aligned} \tag{1.1.43}$$

for a bounded measurable function $h\in\mathscr{B}(E^\ell)$, $\ell\in\mathbb{N}$. For $\ell=0$, we have

$$\left([\alpha K]^*\mu\right)^{(0)}(\{\Delta\}) = \int_E \mu^{(1)}(dx)\,\kappa^{(1)}(x;x)\,p_0^{(1)}(x;x). \tag{1.1.44}$$

1.2 Ergodicity and Invariant Measure

Consider a BDI $\boldsymbol{\eta}$ as introduced in the previous section. As in the case of "classical" Galton-Watson branching processes with immigration in continuous time (without spatial motion), the presence of immigration leads to the possibility of ergodic behavior, provided the branching component of $\boldsymbol{\eta}$ is "subcritical" in a suitable sense. The focus of this work is on the study of the invariant measure[6] of the process $\boldsymbol{\eta}$ and its properties in this case. Thus throughout the first part of this thesis, in addition to Assumptions 1.1.2-1.1.5 we will work under the following assumption:

1.2.1 Assumption (Recurrence)
We assume that the process $\boldsymbol{\eta}$ admits the void configuration Δ as a recurrent atom with finite expected return time: Defining

$$R := \inf_{n\in\mathbb{N}}\left\{\boldsymbol{\tau}_n : \boldsymbol{\eta}_{\boldsymbol{\tau}_n}=\Delta\right\}, \tag{1.2.1}$$

we suppose that

$$\boldsymbol{E}_{\boldsymbol{x}}[R]<\infty, \qquad \boldsymbol{x}\in\mathcal{S}. \tag{1.2.2}$$

[6] We recall that a measure μ on $(\mathcal{S},\mathcal{B}_\mathcal{S})$ is called *invariant* for the process $\boldsymbol{\eta}$ if $\mu=\mu\boldsymbol{P}_t\equiv\boldsymbol{P}_t^*\mu$ for all $t>0$, where $(\boldsymbol{P}_t)_t$ denotes the transition semigroup (1.1.35).

1.2.2 Remark
It is clear that Assumption 1.2.1 entails non-explosion of the BDI $\boldsymbol{\eta}$ as well as finiteness of all branching / immigration times: Thus in (1.1.27) we have

$$0 = \boldsymbol{\tau}_0 < \boldsymbol{\tau}_1 < \boldsymbol{\tau}_2 < \cdots \uparrow \infty = \boldsymbol{\tau}_\infty \qquad \boldsymbol{P}_{\boldsymbol{x}}\text{-a.s., } \boldsymbol{x} \in \mathcal{S}.$$

♦

Under Assumption 1.2.1, let us define a finite measure m on $(\mathcal{S}, \mathcal{B}_\mathcal{S})$ by

$$m(F) := \boldsymbol{E}_\Delta\Big[\int_0^R \mathbf{1}_F(\boldsymbol{\eta}_s)\,ds\Big], \qquad F \in \mathcal{B}_\mathcal{S}. \tag{1.2.3}$$

Note that $m(F)$ gives the expected occupation time of a Borel set $F \in \mathcal{B}_\mathcal{S}$ during one life cycle of the BDI $\boldsymbol{\eta}$.

Condition (1.2.2) is sufficient to ensure recurrence of the BDI $\boldsymbol{\eta}$ in a strong sense, and the measure m defined in (1.2.3) above turns out to be the (essentially unique) invariant measure for $\boldsymbol{\eta}$ on $(\mathcal{S}, \mathcal{B}_\mathcal{S})$:

1.2.3 Proposition
Grant Assumption 1.2.1. Then the BDI $\boldsymbol{\eta}$ is positive recurrent in the sense of Harris[7]*, and its invariant measure (unique up to normalization) coincides with m defined in* (1.2.3)*.*

Proof Let

$$R_0 := 0, \qquad R_1 := R, \qquad R_{n+1} := R_n + R \circ \boldsymbol{\theta}_{R_n}, \quad n \in \mathbb{N}$$

denote successive reentry times into the void configuration Δ. By (1.2.2) and the strong Markov property, the R_n are all finite a.s., and we can decompose the path of $\boldsymbol{\eta}$ into life cycles $(\boldsymbol{\eta}_s)_{s \in [R_n, R_{n+1})}$ which are i.i.d. under $\boldsymbol{P}_\Delta$. Hence by the strong law of large numbers it holds

$$\frac{1}{N}\int_0^{R_N} \mathbf{1}_F(\boldsymbol{\eta}_s)\,ds = \frac{1}{N}\sum_{n=0}^{N-1}\int_{R_n}^{R_{n+1}} \mathbf{1}_F(\boldsymbol{\eta}_s)\,ds \xrightarrow{N\uparrow\infty} \boldsymbol{E}_\Delta\Big[\int_0^R \mathbf{1}_F(\boldsymbol{\eta}_s)\,ds\Big] = m(F) \tag{1.2.4}$$

$\boldsymbol{P}_\Delta$-a.s. for all $F \in \mathcal{B}_\mathcal{S}$. If $m(F) > 0$, this clearly implies

$$\int_0^\infty \mathbf{1}_F(\boldsymbol{\eta}_s)\,ds = \lim_{N\uparrow\infty}\int_0^{R_N} \mathbf{1}_F(\boldsymbol{\eta}_s)\,ds = \infty \qquad \boldsymbol{P}_\Delta\text{-a.s.}$$

Since $R < \infty$ $\boldsymbol{P}_{\boldsymbol{x}}$-a.s. for all $\boldsymbol{x} \in \mathcal{S}$ by (1.2.2), the foregoing extends to all starting points $\boldsymbol{x}$, i.e. we have

$$\int_0^\infty \mathbf{1}_F(\boldsymbol{\eta}_s)\,ds = \infty \qquad \boldsymbol{P}_{\boldsymbol{x}}\text{-a.s., } \boldsymbol{x} \in \mathcal{S},\ m(F) > 0, \tag{1.2.5}$$

which is (one version among several equivalent ones of) the definition of Harris recurrence (see e.g. [ADR1969], condition (H) on p. 24). Also note that by choosing $F = \mathcal{S}$ in (1.2.4) we obtain

$$R_N/N \xrightarrow{N\uparrow\infty} \boldsymbol{E}_\Delta[R] \in (0,\infty) \qquad \boldsymbol{P}_\Delta\text{-a.s.} \tag{1.2.6}$$

By a classical result for Harris recurrent processes (see [ADR1969]), there exists a σ-finite invariant measure μ for $\boldsymbol{\eta}$ which is unique up to normalization. Moreover, the ratio limit

[7] For the definition of Harris recurrence, see e.g. [ADR1969] or [MT1993].

theorem for additive functionals of Harris recurrent processes (see e.g. [ADR1969], Thm 3.1. on p. 30, or [RY1999], Thm. X.3.12 on p. 427) says that

$$\frac{\int_0^t f(\boldsymbol{\eta}_s)\,ds}{\int_0^t g(\boldsymbol{\eta}_s)\,ds} \xrightarrow{t\uparrow\infty} \frac{\mu(f)}{\mu(g)} \qquad \boldsymbol{P}_{\boldsymbol{x}}\text{-a.s.} \tag{1.2.7}$$

for all $\boldsymbol{x} \in \mathcal{S}$ and $f, g \in L^1(\mathcal{S};\mu)$ with $\mu(g) \neq 0$.

Let $(E_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{B}_{\mathcal{S}}$ such that $E_k \uparrow \mathcal{S}$ and $\mu(E_k) < \infty$ for all $k \in \mathbb{N}$. Then $\mathbf{1}_{E_k} \in L^1(\mu)$ for all $k \in \mathbb{N}$. If μ were not finite, we would have

$$\frac{1}{t}\int_0^t \mathbf{1}_{E_k}(\boldsymbol{\eta}_s)\,ds \xrightarrow{t\uparrow\infty} 0 \qquad \boldsymbol{P}_{\boldsymbol{x}}\text{-a.s.}, \qquad \boldsymbol{x} \in \mathcal{S} \tag{1.2.8}$$

(see [RY1999], Ex. X.3.15). On the other hand, note that by choosing $F = \mathcal{S}$ in (1.2.4) we obtain

$$R_N/N \xrightarrow{N\uparrow\infty} \boldsymbol{E}_\Delta[R] > 0 \qquad \boldsymbol{P}_\Delta\text{-a.s.} \tag{1.2.9}$$

Hence $R_N \xrightarrow{N\uparrow\infty} \infty$ $\boldsymbol{P}_\Delta$-a.s., and together with (1.2.4)

$$\frac{1}{R_N}\int_0^{R_N} \mathbf{1}_{E_k}(\boldsymbol{\eta}_s)\,ds = \frac{N}{R_N}\cdot\frac{1}{N}\int_0^{R_N} \mathbf{1}_{E_k}(\boldsymbol{\eta}_s)\,ds \xrightarrow{N\uparrow\infty} \frac{m(E_k)}{\boldsymbol{E}_\Delta[R]} \qquad \boldsymbol{P}_\Delta\text{-a.s.} \tag{1.2.10}$$

Comparison of (1.2.8) and (1.2.10) for $\boldsymbol{x} = \Delta$ gives $m(E_k) = 0$ for all $k \in \mathbb{N}$, which is a contradiction since $E_k \uparrow \mathcal{S}$. Consequently, μ must be a finite measure, and $\boldsymbol{\eta}$ is recurrent positive. Since bounded functions are now known to be μ-integrable, the ratio limit theorem (1.2.7) implies

$$\frac{1}{t}\int_0^t \mathbf{1}_F(\boldsymbol{\eta}_s)\,ds \xrightarrow{t\uparrow\infty} \frac{\mu(F)}{\mu(\mathcal{S})} \qquad \boldsymbol{P}_{\boldsymbol{x}}\text{-a.s.}, \ \boldsymbol{x} \in \mathcal{S},\ F \in \mathcal{B}_{\mathcal{S}},$$

and on the other hand we have again by (1.2.4)

$$\frac{1}{R_N}\int_0^{R_N} \mathbf{1}_F(\boldsymbol{\eta}_s)\,ds = \frac{N}{R_N}\cdot\frac{1}{N}\int_0^{R_N} \mathbf{1}_F(\boldsymbol{\eta}_s)\,ds \xrightarrow{N\uparrow\infty} \frac{m(F)}{\boldsymbol{E}_\Delta[R]} \qquad \boldsymbol{P}_\Delta\text{-a.s.}$$

Comparison of the last two displays for $\boldsymbol{x} = \Delta$ shows that μ coincides with m defined in (1.2.3) up to a multiplicative constant. ■

The invariant measure m of (1.2.3), its properties and related quantities are the primary objects of study in this thesis.[8] A related quantity in which we will also be interested is the *invariant occupation measure* of $\boldsymbol{\eta}$, which is a measure on the single particle space $(E, \mathcal{B}_E)$ and is defined as follows. First, we need some additional notation: For a function $f : E \times \mathcal{S} \to \mathbb{R}$, let $\bar{f} : \mathcal{S} \to \mathbb{R}$ denote the function on the configuration space defined by

$$\bar{f}(\boldsymbol{x}) := \sum_{i=1}^{\ell} f(x^i;\boldsymbol{x}) \quad \text{if } \boldsymbol{x} = (x^1,\dots,x^\ell) \in E^\ell,\ \ell \in \mathbb{N}, \qquad \bar{f}(\Delta) := 0. \tag{1.2.11}$$

If f does not depend on the configuration variable and is of the form $f(x;\boldsymbol{x}) = \mathbf{1}_B(x)$ for some Borel set $B \in \mathcal{B}_E$, we also write

$$\boldsymbol{x}(B) := \overline{\mathbf{1}_B}(\boldsymbol{x}) = \sum_{i=1}^{\ell(\boldsymbol{x})} \mathbf{1}_B(x^i), \qquad \boldsymbol{x} \in \mathcal{S} \tag{1.2.12}$$

[8] The symbol m will always denote the measure defined in (1.2.3) and not the invariant probability distribution which is obtained by normalization.

for the number of particles of the configuration $\boldsymbol{x}$ with position in B. This notation is motivated by the measure-valued point of view, where $\boldsymbol{x}(B)$ is just the total mass of the Borel set B under the finite point measure $\boldsymbol{x} = \sum_{i=1}^{\ell} \delta_{x^i}$. Now we can give the definition of the invariant occupation measure:

1.2.4 Definition
Under Assumption 1.2.1 and making use of the notation (1.2.12)*, we define a measure* $\overline{m}$ *on* $(E, \mathcal{B}_E)$ *by*

$$\overline{m}(B) := \int_{\mathcal{S}} \boldsymbol{x}(B)\, m(d\boldsymbol{x}) = \boldsymbol{E}_{\Delta}\left[\int_0^R \boldsymbol{\eta}_s(B)\, ds\right], \qquad B \in \mathcal{B}_E. \tag{1.2.13}$$

The measure $\overline{m}$ *is called the* invariant occupation measure *or intensity measure of* m.

Note that the measure $\overline{m}$ describes (up to normalization) the expected occupation time of a subset $B \in \mathcal{B}_E$ by all particles whose life span is contained in one life cycle of $\boldsymbol{\eta}$. We emphasize that under Assumption 1.2.1 alone, it is generally not assured that $\overline{m}$ is a finite measure on $(E, \mathcal{B}_E)$, i.e. finiteness of $\overline{m}$ is a strictly stronger condition than (1.2.2). In fact, this is already true for classical Galton-Watson branching processes with immigration, without any spatial behavior; for an example to this effect, see e.g. Examples 3.2.16 and 3.3.10 below. Also note that finiteness of $\overline{m}$ means

$$\overline{m}(E) = \int_{\mathcal{S}} \ell(\boldsymbol{x})\, m(d\boldsymbol{x}) = \sum_{\ell \in \mathbb{N}} \ell \cdot m(E^{\ell}) < \infty \tag{1.2.14}$$

and thus concerns the decay of $m(E^{\ell})$, the mass of the ℓ particle-layer under the invariant measure, as $\ell \uparrow \infty$. In particular, (1.2.14) is ensured if $m(E^{\ell})$ decays exponentially in ℓ; for a sufficient condition ensuring this in a very special case, see Theorem 1.2.9 below.

1.2.5 Remarks
It is a natural question to ask for sufficient conditions for Assumption 1.2.1, i.e. for positive Harris recurrence of the process $\boldsymbol{\eta}$, or for finiteness (1.2.14) of the invariant occupation measure $\overline{m}$. The following remarks summarize what is known in this regard:

- Define the configuration length process (or total mass process, in the measure-valued picture) $Z^{\boldsymbol{\eta}} = (Z_t^{\boldsymbol{\eta}})_{t \geq 0}$ by

$$Z_t^{\boldsymbol{\eta}} := \ell(\boldsymbol{\eta}_t) \equiv \boldsymbol{\eta}_t(E), \qquad t \geq 0.$$

  Suppose that the branching rate κ, the reproduction probabilities $(p_k)_{k \in \mathbb{N}_0}$ and the immigration rate c of Assumptions 1.1.4-1.1.5 are constants:

$$\kappa^{(\ell)}(\cdot\,;\cdot) \equiv \kappa > 0, \qquad p_k^{(\ell)}(\cdot\,;\cdot) \equiv p_k \in [0,1], \qquad c^{(\ell)}(\cdot) \equiv c > 0 \tag{1.2.15}$$

  for all $k, \ell \in \mathbb{N}_0$. Then it is easy to see[9] that the distribution of $Z^{\boldsymbol{\eta}}$ under $\boldsymbol{P}_{\boldsymbol{x}}$ depends on $\boldsymbol{x} \in \mathcal{S}$ only through its length $\ell(\boldsymbol{x})$, and if we take some arbitrary $x \in E$ and define

$$P_i := P_{\underbrace{(x,x,\ldots,x)}_{i \text{ times}}}, \qquad i \in \mathbb{N}_0,$$

  then the $\mathbb{N}_0$-valued process $Z^{\boldsymbol{\eta}}$ together with the system of probability measures $(P_i)_{i \in \mathbb{N}_0}$ is a "classical" Galton-Watson branching process with immigration in continuous time

[9] Cf. [Löc1999], Thm. 6.5 on p. 47.

(henceforth GWI process). Thus one can apply the well-developed theory available for this class of processes[10] in order to check finiteness of m or $\overline{m}$. For example, it is known that the expected return time to 0 in a GWI process is finite (regardless of the starting value) provided reproduction is *subcritical*, i.e.

$$\varrho := \sum_{k \in \mathbb{N}} k p_k < 1$$

(see [Zub1972], Thm. 1', p. 182). Subcriticality $\varrho < 1$ is also sufficient (and necessary) for (1.2.14) to hold (see the third remark below).

- If the branching or immigration rate or the reproduction law are not constants, the process $Z^{\boldsymbol{\eta}}$ will generally be no longer Markov. However, it is still sometimes possible to "compare" the BDI $\boldsymbol{\eta}$ to another one with constant rates: For the following, we refer in particular to [Löc1999], Section 6. Citing [Löc2002a], Ex. 3.3 (p. 75), the result can be summarized as follows: Writing $p(\cdot\,;\cdot) \equiv (p_k(\cdot\,;\cdot))_{k\in\mathbb{N}_0}$ for the position- and configuration-dependent reproduction law of Assumption 1.1.4, suppose that $p(\cdot\,;\cdot)$ is upper bounded by some fixed subcritical reproduction law in a convolution sense, i.e.: Denoting by $\mathcal{M}_1(\mathbb{N}_0)$ the space of all probability distributions on $\mathbb{N}_0$, there is some $\hat{p} = (\hat{p}_k)_{k\in\mathbb{N}_0} \in \mathcal{M}_1(\mathbb{N}_0)$ and for each $(x, \boldsymbol{x}) \in E \times \mathcal{S}$ some $\tilde{p}(x; \boldsymbol{x}) = (\tilde{p}_k(x; \boldsymbol{x}))_{k\in\mathbb{N}_0} \in \mathcal{M}_1(\mathbb{N}_0)$ such that

$$\hat{\varrho} := \sum_{k\in\mathbb{N}} k\hat{p}_k < 1, \tag{1.2.16}$$

$$p(x; \boldsymbol{x}) * \tilde{p}(x; \boldsymbol{x}) = \hat{p}, \qquad (x, \boldsymbol{x}) \in E \times \mathcal{S}. \tag{1.2.17}$$

Suppose moreover that the branching rate $\kappa(\cdot\,;\cdot)$ is bounded away from 0 and that the immigration rate $c(\cdot)$ is bounded. Using the notation (1.2.11), let

$$\tau_t := \inf\left\{ s \geq 0 : \int_0^s \frac{\bar{\kappa}(\boldsymbol{\eta}_r)}{\ell(\boldsymbol{\eta}_r)}\, dr > t \right\}. \tag{1.2.18}$$

Then Löcherbach gives (under additional continuity conditions on the quantities in Assumptions 1.1.4-1.1.5) a coupled construction of the time-changed process $(\tilde{\boldsymbol{\eta}}_t)_t := (\boldsymbol{\eta}_{\tau_t})_t$ together with another BDI $\hat{\boldsymbol{\eta}}$ in which particles branch and immigrate at constant rates and reproduce according to the subcritical law $\hat{p}$, such that $\tilde{\boldsymbol{\eta}}$ is a "subpopulation" (or "thinning") of $\hat{\boldsymbol{\eta}}$. For the details of this coupled construction, see the proof of Thm. 6.10 in [Löc1999]. Since $\hat{\varrho} < 1$, $\hat{\boldsymbol{\eta}}$ is positive recurrent, and the same then holds for $\tilde{\boldsymbol{\eta}}$ and $\boldsymbol{\eta}$. In fact, more is true: For every non-decreasing function $G : \mathbb{N}_0 \to \mathbb{R}_+$, it holds that

$$\boldsymbol{E}_\Delta\left[\int_0^R G(Z_s^{\boldsymbol{\eta}})\, ds\right] \leq \text{const}\, \boldsymbol{E}_\Delta\left[\int_0^{\tilde{R}} G(Z_s^{\tilde{\boldsymbol{\eta}}})\, ds\right] \leq \text{const}\, \boldsymbol{E}_\Delta\left[\int_0^{\hat{R}} G(Z_s^{\hat{\boldsymbol{\eta}}})\, ds\right], \tag{1.2.19}$$

where $\tilde{R}$ and $\hat{R}$ of course denote the first return times to Δ in $\tilde{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{\eta}}$, respectively. Using the time change (1.2.18), the first inequality in (1.2.19) holds because $\tilde{\boldsymbol{\eta}}$ corresponds to a "slowed down version" of $\boldsymbol{\eta}$, as follows from the assumption that κ be bounded away from 0 by a simple change of variables. The second inequality follows

[10] See e.g. [Pak1975] or [Zub1972].

from the fact that $\tilde{\boldsymbol{\eta}}$ is a thinning of $\hat{\boldsymbol{\eta}}$. This comparison result is stated in [HHL2002][11] for the purely position-dependent framework; for the general case it is implicit in the results of [Löc1999], Sec. 6. Note that by choosing $G := 1$ resp. $G := \mathrm{id}$ in (1.2.19), we get finiteness of m resp. $\overline{m}$ since $\hat{\boldsymbol{\eta}}$ is subcritical. Moreover, choosing $G := \mathbf{1}_{\{k \in \mathbb{N}:\, k \geq \ell\}}$ in (1.2.19), $\ell \in \mathbb{N}$ fixed, we obtain an estimate for $m\left(\cup_{k \geq \ell} E^k\right)$ in terms of the corresponding quantity for the process $\hat{\boldsymbol{\eta}}$, for which sometimes exponential decay may be proved (see Theorem 1.2.9 below).

In the general "interactive" framework, where the quantities in Assumptions 1.1.2-1.1.5 are allowed to depend on both the spatial position of a particle and the configuration of coexisting particles, the method described above seems to be the only known general way to verify finiteness of m or $\overline{m}$.

- On the other hand, in the purely position-dependent framework of [HL2005] it is possible to use the fact that particles move independently of each other in order to give sufficient conditions for finiteness of m or $\overline{m}$ which are not based on the comparison to a BDI with constant rates: Under the assumption that the spatial offspring distribution is of the form (1.1.18) (branching particles reproduce at their death position) and additional continuity and boundedness conditions on the branching rate $\kappa(\cdot)$ and the position-dependent reproduction mean $\varrho(\cdot)$, the authors in [HL2005] prove that finiteness of the invariant occupation measure $\overline{m}$ is equivalent to the following condition of "spatial subcriticality":

$$E_\nu\left[\int_0^\infty \exp\left(-\int_0^t \kappa(X_s)(1-\varrho(X_s))\,ds\right)dt\right] < \infty. \tag{1.2.20}$$

  Observe that the above condition is formulated completely in terms of the branching rate κ, reproduction mean ϱ, immigration measure ν and single particle motion X, all of which "live" on the single particle space E. Also note that for constant p_k and $\kappa > 0$, this result implies that finiteness of $\overline{m}$ is equivalent to $\varrho < 1$. In particular, as stated in the first remark, this holds for "classical" GWI processes. In the position-dependent case, (1.2.20) is obviously satisfied if $\kappa(\cdot)$ is bounded away from 0 and $\varrho(\cdot)$ is bounded away from 1. However, boundedness of $\varrho(\cdot)$ away from 1 (although weaker than the combination of (1.2.16) and (1.2.17)) is far from necessary, since (1.2.20) can hold even though $\varrho(\cdot) \geq 1$ locally (it may even be unbounded). These problems are taken up in Chapter 3 of this thesis, where we also show that the spatial subcriticality condition (1.2.20) and some of the results in [HL2005] can be generalized to the case of more general spatial offspring distributions and general Markov processes as single-particle motions. ♦

Whenever (1.2.2) or (1.2.14) is ensured, we are interested in the existence and regularity of Lebesgue-densities for m on $\mathcal{S}$ or $\overline{m}$ on E, respectively; this subject is taken up in the next chapter. In order to attack these problems, we will need to work with different representations of the invariant measure m aside from that in (1.2.3). Let us start by deriving a series representation and a fixed point equation for m, both of which are already contained in [Höp2004] (see p. 18, eqns. (33) and (34); cf. also [Löc2004], Prop. 3.1). We remind the reader of the definition of the generalized resolvent kernel R_α in (1.1.36), of the jump kernel K in (1.1.24) and of our notations in 1.1.9 concerning the action of a kernel on a measure.

[11] See [HHL2002], Lemma 3 and 4.

1.2.6 Proposition

Grant Assumption 1.2.1. Then the invariant measure m *on the configuration space* $\mathcal{S}$ *has the series representation*

$$m(F) = \sum_{n=0}^{\infty} \boldsymbol{E}_{\Delta}\left[\mathbf{1}_{\boldsymbol{\tau}_n<R} \cdot R_\alpha(\boldsymbol{\eta}_{\boldsymbol{\tau}_n}; F)\right], \qquad F \in \mathcal{B}_{\mathcal{S}}. \tag{1.2.21}$$

Moreover, m *is invariant for the kernel* $[\alpha K]R_\alpha$*, i.e. it satisfies the following fixed point equation:*

$$m = R_\alpha^*[\alpha K]^* m. \tag{1.2.22}$$

Proof Let $F \in \mathcal{B}_{\mathcal{S}}$. Decomposing the life cycle of the BDI $\boldsymbol{\eta}$ into random time intervals $[\boldsymbol{\tau}_n, \boldsymbol{\tau}_{n+1})$ between successive branching/immigration events, by an application of the strong Markov property at time $\boldsymbol{\tau}_n$ we obtain

$$\begin{aligned} m(F) = \boldsymbol{E}_{\Delta}\left[\int_0^R \mathbf{1}_F(\boldsymbol{\eta}_s)\, ds\right] &= \sum_{n=0}^{\infty} \boldsymbol{E}_{\Delta}\left[\mathbf{1}_{\boldsymbol{\tau}_n<R} \int_{\boldsymbol{\tau}_n}^{\boldsymbol{\tau}_{n+1}} \mathbf{1}_F(\boldsymbol{\eta}_s)\, ds\right] \\ &= \sum_{n=0}^{\infty} \boldsymbol{E}_{\Delta}\left[\mathbf{1}_{\boldsymbol{\tau}_n<R} \cdot \boldsymbol{E}_{\boldsymbol{\eta}_{\boldsymbol{\tau}_n}}\left[\int_0^{\boldsymbol{\tau}_1} \mathbf{1}_F(\boldsymbol{\eta}_s)\, ds\right]\right] \\ &= \sum_{n=0}^{\infty} \boldsymbol{E}_{\Delta}\left[\mathbf{1}_{\boldsymbol{\tau}_n<R} \cdot R_\alpha(\boldsymbol{\eta}_{\boldsymbol{\tau}_n}; F)\right], \end{aligned}$$

where in the last step we have used that expected occupation times of $\boldsymbol{\eta}$ between successive branching/immigration events are given by the generalized resolvent kernel R_α, see (1.1.37).

We turn to the proof of the fixed point equation (1.2.22). By (1.2.21), we have

$$\begin{aligned} R_\alpha^*[\alpha K]^* m(F) &\equiv \int_{\mathcal{S}} m(d\boldsymbol{x})\, [\alpha K R_\alpha](\boldsymbol{x}; F) \\ &= \sum_{n=0}^{\infty} \boldsymbol{E}_{\Delta}\left[\mathbf{1}_{\boldsymbol{\tau}_n<R} \int_{\mathcal{S}} R_\alpha(\boldsymbol{\eta}_{\boldsymbol{\tau}_n}; d\boldsymbol{x})[\alpha K R_\alpha](\boldsymbol{x}; F)\right] \\ &= \sum_{n=0}^{\infty} \boldsymbol{E}_{\Delta}\left[\mathbf{1}_{\boldsymbol{\tau}_n<R} \int_{\mathcal{S}} [R_\alpha \alpha K](\boldsymbol{\eta}_{\boldsymbol{\tau}_n}; d\boldsymbol{x}) R_\alpha(\boldsymbol{x}; F)\right] \\ &= \sum_{n=0}^{\infty} \boldsymbol{E}_{\Delta}\left[\mathbf{1}_{\boldsymbol{\tau}_n<R} \cdot R_\alpha(\boldsymbol{\eta}_{\boldsymbol{\tau}_n+1}; F)\right] \end{aligned}$$

since the kernel $[R_\alpha \alpha K]$ governs the transition from $\boldsymbol{\eta}_{\boldsymbol{\tau}_n}$ to $\boldsymbol{\eta}_{\boldsymbol{\tau}_{n+1}}$ (see (1.1.40)). Decomposing

$$\{\boldsymbol{\tau}_n < R\} = \{\boldsymbol{\tau}_{n+1} < R\} \cup \{\boldsymbol{\tau}_n < R,\, \boldsymbol{\tau}_{n+1} = R\},$$

we see that the preceding is equal to

$$\begin{aligned} &\sum_{n=0}^{\infty} \boldsymbol{E}_{\Delta}\left[\mathbf{1}_{\boldsymbol{\tau}_{n+1}<R} \cdot R_\alpha(\boldsymbol{\eta}_{\boldsymbol{\tau}_{n+1}}; F) + \mathbf{1}_{\boldsymbol{\tau}_n<R, \boldsymbol{\tau}_{n+1}=R} \cdot R_\alpha(\Delta; F)\right] \\ &= \left(\sum_{n=0}^{\infty} \boldsymbol{E}_{\Delta}\left[\mathbf{1}_{\boldsymbol{\tau}_{n+1}<R} \cdot R_\alpha(\boldsymbol{\eta}_{\boldsymbol{\tau}_{n+1}}; F)\right] + R_\alpha(\Delta; F) \cdot \underbrace{\sum_{n=0}^{\infty} \boldsymbol{P}_{\Delta}\left[\boldsymbol{\tau}_n < R,\, \boldsymbol{\tau}_{n+1} = R\right]}_{=\boldsymbol{P}_\Delta[R<\infty]=1}\right) \\ &= \sum_{n=0}^{\infty} \boldsymbol{E}_{\Delta}\left[\mathbf{1}_{\boldsymbol{\tau}_n<R} \cdot R_\alpha(\boldsymbol{\eta}_{\boldsymbol{\tau}_n}; F)\right] \\ &= m(F). \end{aligned}$$

■

As in [Höp2004] or [Löc2004], the series representation (1.2.21) will be crucial for proving the existence and regularity of Lebesgue densities for m and $\overline{m}$ in the next chapter. On the other hand, the fixed point equation (1.2.22) can be used to investigate the problem of the decay of $m(E^\ell)$ as $\ell \uparrow \infty$, which is relevant to the question of finiteness of $\overline{m}$ in (1.2.14): Write

$$m_\ell := m(E^\ell), \qquad \ell \in \mathbb{N}_0 \tag{1.2.23}$$

for the mass of the ℓ particle-layer under the invariant measure. In the case where the branching and immigration rates and the reproduction probabilities are constants, (1.2.22) can be used to derive a recursion formula for the numbers m_ℓ.

1.2.7 Proposition
Assume that

$$\kappa^{(\ell)}(\cdot\,;\cdot) \equiv \kappa > 0, \quad p_k^{(\ell)}(\cdot\,;\cdot) \equiv p_k \in [0,1], \qquad \ell \in \mathbb{N},\, k \in \mathbb{N}_0,$$

$$c^{(\ell)}(\cdot) \equiv c > 0, \qquad \ell \in \mathbb{N}_0.$$

Define m_ℓ as in (1.2.23) *for $\ell \in \mathbb{N}_0$, and put $m_\ell := 0$ for $\ell < 0$. Then we have $m_0 = \frac{1}{c}$ and*

$$m_\ell = \frac{1}{\kappa p_0 \ell}\Big(\Big(\kappa(1-p_1)(\ell-1)+c\Big)m_{\ell-1} - \Big(\kappa p_2(\ell-2)+c\Big)m_{\ell-2} - \sum_{k=3}^{\ell-1}\kappa p_k(\ell-k)m_{\ell-k}\Big) \tag{1.2.24}$$

for $\ell \in \mathbb{N}$.

Proof Starting from the void configuration Δ, the time $\boldsymbol{\tau}_1$ is exponentially distributed with parameter c. Hence it is clear that

$$m_0 = m(\{\Delta\}) = \boldsymbol{E}_\Delta\Big[\int_0^R \mathbf{1}_{\{\Delta\}}(\boldsymbol{\eta}_s)\,ds\Big] = \boldsymbol{E}_\Delta[\boldsymbol{\tau}_1] = \frac{1}{c}.$$

Let $\ell \in \mathbb{N}_0$. Recalling (1.1.21) and (1.1.33), due to the assumption of constant rates κ and c we have

$$R_\alpha(\boldsymbol{y}; E^\ell) = \begin{cases} R_\alpha^{(\ell)}(\boldsymbol{y}; E^\ell) = \int_0^\infty e^{-(\kappa\ell+c)t}\,dt = \frac{1}{\kappa\ell+c}, & \boldsymbol{y} \in E^\ell \\ 0, & \boldsymbol{y} \notin E^\ell. \end{cases}$$

The action of the kernel $[\alpha K]$ on the measure m is described by Lemma 1.1.10: Since also $(p_k)_{k\in\mathbb{N}_0}$ is constant, (1.1.43) gives

$$[\alpha K]^* m(E^\ell) = \sum_{k=0}^{\ell} \kappa p_k(\ell-k+1)m_{\ell-k+1} + c m_{\ell-1}, \qquad \ell \in \mathbb{N}. \tag{1.2.25}$$

For $\ell = 0$, by (1.1.44) we have $[\alpha K]^* m(\{\Delta\}) = \kappa p_0 m_1$. In view of our convention $m_{-1} = 0$, formula (1.2.25) is thus valid for all $\ell \in \mathbb{N}_0$.

Now the fixed point equation (1.2.22) implies that for all $\ell \in \mathbb{N}_0$

$$\begin{aligned} m_\ell &= R_\alpha^*[\alpha K]^* m(E^\ell) \\ &\equiv \int_{\mathcal{S}} m(d\boldsymbol{x}) \int_{\mathcal{S}} [\alpha K](\boldsymbol{x}; d\boldsymbol{y}) R_\alpha(\boldsymbol{y}; E^\ell) \\ &= \frac{1}{\kappa\ell + c} \int_{\mathcal{S}} m(d\boldsymbol{x})[\alpha K](\boldsymbol{x}; E^\ell) \\ &= \frac{1}{\kappa\ell + c}[\alpha K]^* m(E^\ell) \\ &= \frac{1}{\kappa\ell + c}\left(\sum_{k=0}^{\ell} \kappa p_k(\ell - k + 1) m_{\ell-k+1} + c m_{\ell-1}\right) \\ &= \frac{1}{\kappa\ell + c}\left(\kappa p_0(\ell+1) m_{\ell+1} + \kappa p_1 \ell m_\ell + (\kappa p_2(\ell-1) + c)\, m_{\ell-1} + \sum_{k=3}^{\ell} \kappa p_k(\ell - k + 1) m_{\ell-k+1}\right). \end{aligned} \tag{1.2.26}$$

Solving this for $m_{\ell+1}$ yields formula (1.2.24) for all $\ell \in \mathbb{N}$. ∎

For use in the proof of Theorem 1.2.9 below, we remark that (1.2.26) can also be written as follows:

$$(\kappa(1 - p_1)\ell + c)\, m_\ell = \kappa p_0(\ell+1) m_{\ell+1} + \sum_{k=2}^{\ell} \kappa p_k(\ell - k + 1) m_{\ell-k+1} + c m_{\ell-1}, \qquad \ell \in \mathbb{N}. \tag{1.2.27}$$

The usefulness of the above formulas seems rather limited since the recursion is not of finite order, i.e. m_ℓ is expressed by *all* $m_{\ell-1}, m_{\ell-2}, \ldots, m_0$. A recursion formula of finite order is obtained if we assume that there is some uniform upper bound for the possible number of offspring. In particular, for the important case of binary branching it is possible to solve the recursion and to obtain an explicit formula:

1.2.8 Corollary
In addition to the assumptions of Proposition 1.2.7, suppose that branching is binary, $p_2 = 1 - p_0$. Then the following explicit formula for m_ℓ holds:

$$m_\ell = \frac{1}{\ell\kappa p_0}\left(\frac{p_2}{p_0}\right)^{\ell-1} \cdot \prod_{j=1}^{\ell-1}\left(1 + \frac{c}{j\kappa p_2}\right), \qquad \ell \in \mathbb{N}. \tag{1.2.28}$$

It is easy to deduce (1.2.28) from (1.2.24) by induction; we omit the proof at this point since it follows even more easily from equation (1.2.33) below. For the case of subcritical binary branching

$$p_2 < 1/2,$$

formula (1.2.28) implies at once (e.g. by an application of the ratio test) that m_ℓ decays exponentially as $\ell \uparrow \infty$. This was already observed in [Löc1999], Cor. 6.14.

Now suppose that branching is not necessarily binary, but that there is some fixed upper bound k_0 for the possible number of offspring (i.e. the distribution $(p_k)_{k\in\mathbb{N}_0}$ has finite support $\{0, 1, \ldots, k_0\}$). In this case, although we have been unable to solve the recursion (1.2.24) explicitly, we can at least show exponential decay of m_ℓ:

1.2.9 Theorem
In addition to the assumptions of Proposition 1.2.7, suppose that $p_0 < 1$, that there is some $k_0 \geq 2$ such that

$$p_{k_0} \neq 0, \qquad p_k = 0 \quad \text{for all } k > k_0 \tag{1.2.29}$$

and that reproduction is subcritical:

$$\varrho := \sum_{k \in \mathbb{N}} k p_k < 1.$$

Then there exists $C < \infty$ such that

$$m_\ell \leq C \cdot \frac{q^\ell}{\ell}, \qquad \ell \in \mathbb{N}, \tag{1.2.30}$$

where

$$q := \varrho^{\frac{1}{k_0 - 1}} < 1.$$

Proof For the purposes of the proof, it turns out to be more convenient to consider instead of m_ℓ the numbers

$$\mu_\ell := \ell m_\ell = \ell \cdot m(E^\ell), \qquad \ell \in \mathbb{N} \tag{1.2.31}$$

and to show the existence of $C < \infty$ such that $\mu_\ell \leq C q^\ell$ for all $\ell \in \mathbb{N}$.

We start from the recursion formula already proved in the form (1.2.27): Since $p_k = 0$ for $k > k_0$, we have $1 - p_1 = p_0 + \sum_{k=2}^{k_0} p_k$, and for all $j \geq k_0$ formula (1.2.27) reads as follows:

$$\kappa \left(p_0 + \sum_{k=2}^{k_0} p_k \right) \mu_j + c \frac{\mu_j}{j} = \kappa p_0 \mu_{j+1} + \sum_{k=2}^{k_0} \kappa p_k \mu_{j-k+1} + c \frac{\mu_{j-1}}{j-1}, \qquad j \geq k_0.$$

This can be rewritten as

$$\kappa p_0 (\mu_j - \mu_{j+1}) = \sum_{k=2}^{k_0} \kappa p_k (\mu_{j-k+1} - \mu_j) + c \left(\frac{\mu_{j-1}}{j-1} - \frac{\mu_j}{j} \right), \qquad j \geq k_0. \tag{1.2.32}$$

Now let $\ell \geq k_0$. Since $\varrho < 1$, it is known that $\sum_{j \in \mathbb{N}} \mu_j < \infty$ (see the first remark in 1.2.5). Hence by summing $\sum_{j \geq \ell} \dots$ in (1.2.32), we obtain

$$\kappa p_0 \mu_\ell = \kappa p_0 \sum_{j \geq \ell} (\mu_j - \mu_{j+1}) = \sum_{k=2}^{k_0} \kappa p_k \sum_{j \geq \ell} (\mu_{j-k+1} - \mu_j) + c \frac{\mu_{\ell-1}}{\ell - 1} = \sum_{k=2}^{k_0} \kappa p_k \sum_{j=1}^{k-1} \mu_{\ell-k+j} + c \frac{\mu_{\ell-1}}{\ell-1}.$$

This gives the following recursion formula in which μ_ℓ is expressed by $\mu_{\ell-1}, \mu_{\ell-2}, \dots, \mu_{\ell-k_0+1}$ for $\ell \geq k_0$:

$$\mu_\ell = \sum_{k=2}^{k_0} \frac{p_k}{p_0} \sum_{j=1}^{k-1} \mu_{\ell-k+j} + \frac{c}{\kappa p_0 (\ell - 1)} \mu_{\ell-1}, \qquad \ell \geq k_0. \tag{1.2.33}$$

Now observe that since $\varrho < 1$ and $p_0 < 1$, we have

$$\sum_{k=2}^{k_0} \frac{p_k}{p_0} (k-1) = \frac{\varrho - (1 - p_0)}{p_0} < \varrho.$$

Consequently, choosing $\ell_0 \geq k_0$ large enough we get

$$\sum_{k=2}^{k_0} \frac{p_k}{p_0} (k-1) + \frac{c}{\kappa p_0 (\ell - 1)} \leq \varrho, \qquad \ell \geq \ell_0. \tag{1.2.34}$$

Evidently, there is some constant $C < \infty$ such that $\mu_\ell \le Cq^\ell$ for the finitely many indices $\ell = 1, 2, \ldots, \ell_0$. Now use induction on $\ell \ge \ell_0$: If it is already known for some $\ell \ge \ell_0$ that $\mu_j \le Cq^j$ for all $j = 1, \ldots, \ell$, then by (1.2.33) and (1.2.34) we get for $\ell + 1$

$$\begin{aligned}
\mu_{\ell+1} &= \sum_{k=2}^{k_0} \frac{p_k}{p_0} \sum_{j=1}^{k-1} \mu_{\ell+1-k+j} + \frac{c}{\kappa p_0 \ell} \mu_\ell \\
&\le C \left(\sum_{k=2}^{k_0} \frac{p_k}{k_0} \sum_{j=1}^{k-1} q^{\ell+1-k+j} + \frac{c}{\kappa p_0 \ell} q^\ell \right) \\
&\le C q^{\ell-k_0+2} \left(\sum_{k=2}^{k_0} \frac{p_k}{p_0} (k-1) + \frac{c}{\kappa p_0 \ell} \right) \\
&\le C \cdot \varrho \cdot q^{\ell-k_0+2} \\
&= C q^{\ell+1}
\end{aligned}$$

by definition of $q = \varrho^{\frac{1}{k_0-1}}$. Thus (1.2.30) is proved for $\ell + 1$. ■

1.2.10 Remarks

- Note that since Theorem 1.2.9 is stated for the case of constant rates, it is essentially a result on *classical* (nonspatial) branching processes with immigration. Although it is possibly well-known, we have not found it anywhere in the corresponding literature.

- In view of the second remark in 1.2.5, the assertion of Theorem 1.2.9 holds also in the case of non-constant rates provided (1.2.16) and (1.2.17) are satisfied and (1.2.29) holds for the upper bound $\hat{p}$: In this case, for each $\ell \in \mathbb{N}$ choose $G_\ell := \mathbf{1}_{\{k \in \mathbb{N}:\, k \ge \ell\}}$ in (1.2.19) to obtain an estimate for $m\left(\cup_{k \ge \ell} E^k\right)$ in terms of the corresponding quantity for the process $\hat{\boldsymbol{\eta}}$ in which particles branch and immigrate at constant rates and reproduce according to the law $\hat{p}$, and to which Theorem 1.2.9 can be applied. ⧫

- Together with the previous remark, Theorem 1.2.9 shows that Assumption 4 on p. 670 in [HHL2002] follows from the rest of their assumptions whenever there is a uniform upper bound for the possible number of offspring.

The reader will have noticed that in the proofs of the results in this section (namely Propositions 1.2.3 and 1.2.6 and Theorem 1.2.9), we did not make use of the fact that the motion of the particles in our BDI $\boldsymbol{\eta}$ is governed by a diffusion: Thus these results remain valid for the case of more general particle motions as will be considered in Chapter 3 of this thesis.

Chapter 2

Absolute Continuity of m and $\overline{m}$

Whenever a BDI $\boldsymbol{\eta}$ is positive recurrent with finite invariant measure m on the configuration space $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$ or finite intensity measure $\overline{m}$ on the single particle-space $(E, \mathcal{B}_E)$, the question of the existence of Lebesgue densities for m or $\overline{m}$ and of their regularity properties naturally arises. Also, this question is relevant for statistical applications, as e.g. in [HHL2002] which concerns the (non-parametric) estimation of the branching rate in a BDI.

In the purely position-dependent framework, for constant rates and under some strong conditions on the single particle motion, the problem of absolute continuity of m was considered in [Höp2004]. In that work, it was shown that a Lebesgue density for m exists but that in general it cannot be expected to have good regularity properties like (global) continuity. The reason for this phenomenon turns out to be the assumption (also supposed in [Höp2004]) that branching particles reproduce *at* their position. We explain this problem in some detail in Section 2.1 and indicate how it can be remedied by modifying the latter assumption. This program is then carried out in Section 2.2 which contains what can be considered the main result of this chapter: Namely, under suitable assumptions we will prove the existence of a Lebesgue density for m which is continuous and bounded on each layer E^{ℓ}.

On the other hand, the existence of a continuous and bounded Lebesgue density for $\overline{m}$ on $E = \mathbb{R}^d$ was proved in [Löc2004], also assuming that branching particles reproduce at their death position. Since this assumption has to be modified in order to get a "nice" Lebesgue density for m on $\mathcal{S}$, we are naturally interested in generalizing also the result in [Löc2004] concerning absolute continuity of $\overline{m}$ to this case, which will be done in Section 2.3.

We emphasize that *throughout this chapter, we will take Assumptions 1.1.2-1.1.5 and 1.2.1 for granted.*

2.1 The Problem

We begin our investigation of absolute continuity with the following simple observation: Recall the series representation (1.2.21) and the fixed point equation (1.2.22) for m. From either one of those two representations, we see that *whenever the generalized resolvent kernel* R_α *of* (1.1.36) *admits a Lebesgue density, so does the invariant measure* m. More precisely, suppose that $R_\alpha(\boldsymbol{x}; d\boldsymbol{y}) = r_\alpha(\boldsymbol{x}; \boldsymbol{y}) d\boldsymbol{y}$ for all $\boldsymbol{x} \in \mathcal{S}$. Then (1.2.21) and (1.2.22) imply that m is $\boldsymbol{\lambda}$-absolutely continuous with density

$$\gamma(\boldsymbol{y}) := \frac{dm}{d\boldsymbol{\lambda}}(\boldsymbol{y}) = \sum_{n=0}^{\infty} \boldsymbol{E}_\Delta \left[\mathbf{1}_{\boldsymbol{\tau}_n < R} \cdot r_\alpha(\boldsymbol{\eta}_{\boldsymbol{\tau}_n}; \boldsymbol{y}) \right] \tag{2.1.1}$$

and that γ satisfies the fixed point equation

$$\gamma(\boldsymbol{y}) = \int_{\mathcal{S}} m[\alpha K](d\boldsymbol{x})\, r_\alpha(\boldsymbol{x};\boldsymbol{y}) = \int_{\mathcal{S}} ([\alpha K]^*\gamma)(d\boldsymbol{x})\, r_\alpha(\boldsymbol{x};\boldsymbol{y}), \qquad \boldsymbol{\lambda}\text{-a.e. } \boldsymbol{y} \in \mathcal{S}. \tag{2.1.2}$$

In the above display, we have made use of our notation $N^* f$ for the action of a kernel N on an absolutely continuous measure $f(\boldsymbol{x})d\boldsymbol{x}$, see the Notations 1.1.9. Moreover, by the definition of the invariant occupation measure $\overline{m}$ in (1.2.13) we see that if m is absolutely continuous with density $\gamma = (\gamma^{(\ell)})_{\ell \in \mathbb{N}_0}$, then $\overline{m}$ is absolutely continuous with density

$$\frac{d\overline{m}}{d\lambda}(x) = \sum_{\ell \in \mathbb{N}} \sum_{i=1}^{\ell} \int_{E^{\ell-1}} dx^1 \cdots dx^{i-1} dx^{i+1} \cdots dx^\ell \, \gamma^{(\ell)}(x^1, \dots, x^{i-1}, x, x^{i+1}, \dots, x^\ell) \tag{2.1.3}$$

(cf. also [Löc2004], Prop. 3.1).

Thus the existence of Lebesgue densities for m and $\overline{m}$ is a direct consequence of absolute continuity of the kernel R_α, which in turn depends on the regularity properties of the killed ℓ-particle motion (more precisely: its semigroup $(P_t^\alpha)_{t \geq 0}$ of (1.1.31)). However, even if $(P_t^\alpha)_{t \geq 0}$ possesses very strong smoothness properties and the invariant occupation density on $\mathbb{R}^d$ is smooth, in general this does not ensure regularity properties (like continuity or local boundedness) for the invariant density γ on the big configuration space $\mathcal{S}$. This observation is due to R. Höpfner (see [Höp2004], Ex. 1) who gave an example demonstrating that even if all other "input parameters" are smooth, as long as branching particles reproduce at their death position the invariant density can be very "strange-shaped":

2.1.1 Example

Consider a (binary) branching Brownian motion with immigration in $\mathbb{R}^d$, $d \geq 2$: Particles move independently of each other on Brownian paths, branch at constant rate $\kappa > 0$ and leave $k = 0$ or $k = 2$ offspring at their death position with probability p_0 or $p_2 = 1 - p_0 < 1/2$, respectively. Note that in this case we have for all $k \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$

$$Q_k^{(\ell)}(x;\boldsymbol{x};\cdot) = \delta_x(\cdot)^{\otimes k} \quad \text{on } E^k, \qquad (x,\boldsymbol{x}) \in E \times E^\ell \tag{2.1.4}$$

in (1.1.11). Immigration occurs at constant rate $c > 0$ with an immigration distribution $\nu(dx) = p(x)\,dx$ on $\mathbb{R}^d$, where $p \in \mathcal{C}_b^\infty(\mathbb{R}^d)$, $p(\cdot) > 0$ is a strictly positive bounded function with bounded derivatives of all orders. Since the rates are constant and $\varrho = 2p_2 < 1$, the corresponding BDI $\boldsymbol{\eta}$ is positive Harris recurrent with finite invariant measure m and finite invariant occupation measure $\overline{m}$ (see the first and third remark in 1.2.5).

By [HL2005], Thms. 3.5, 3.9, the measure $\overline{m}$ on $\mathbb{R}^d$ in this example is absolutely continuous with a Lebesgue density of class $\mathcal{C}^\infty(\mathbb{R}^d)$. Since Brownian motion has a transition density and the rates are constant, the measure m on the configuration space $\mathcal{S}$ is also absolutely continuous with Lebesgue density γ as in (2.1.1). However, γ has singularities at the points of a specified $\boldsymbol{\lambda}$-null set in $\mathcal{S}$, as shown in [Höp2004], Ex. 1, Prop. 1; in particular, there can be no continuous and locally bounded version of the invariant density. ⧫

For the sake of completeness of our exposition, we will give a proof of the "strange shape" of the invariant density in Höpfner's example below. It will become clear that the source of the problem is indeed the assumption (2.1.4) that particles reproduce at their parent's death position. Let $p_t^{(\ell)}$ denote the transition density of Brownian motion in $E^\ell = \mathbb{R}^{d\ell}$:

$$p_t^{(\ell)}(\boldsymbol{x}) := (2\pi t)^{-d\ell/2} \exp\left(-\frac{\|\boldsymbol{x}\|^2}{2t}\right), \qquad \boldsymbol{x} \in E^\ell,\ t > 0. \tag{2.1.5}$$

Since the branching and immigration rate are constant, the kernel R_α of (1.1.36) is in this case just the ordinary resolvent kernel of Brownian motion, given on the layer E^ℓ by

$$R_\alpha^{(\ell)}(\boldsymbol{x}; d\boldsymbol{y}) \equiv r_\alpha^{(\ell)}(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} \equiv r_\alpha^{(\ell)}(\boldsymbol{y}-\boldsymbol{x})\, d\boldsymbol{y},$$

where $r_\alpha^{(\ell)}$ is the resolvent density

$$r_\alpha^{(\ell)}(\boldsymbol{x}) \coloneqq \int_0^\infty e^{-(\kappa\ell+c)t} p_t^{(\ell)}(\boldsymbol{x})\, dt.$$

Note that the function $r_\alpha^{(\ell)}(\cdot) : \mathbb{R}^{d\ell} \to \mathbb{R}_+$ is continuous and bounded (in fact, $\mathcal{C}_b^\infty$) on each compact subset of $E^\ell \setminus \{0\}$, but has a singularity at the origin whenever $d\ell > 1$. "Explicit" formulas for the resolvent density of Brownian motion in terms of modified Bessel functions may be found in [Sat1999] (see eq. (30.29) on p. 204).

If one is interested in proving the existence of a continuous and locally bounded version of γ, the obvious idea coming to mind is to employ the fixed point equation (2.1.2): Since for $\boldsymbol{x} \in E^\ell$, the measure $R_\alpha(\boldsymbol{x};\cdot) = R^{(\ell)}(\boldsymbol{x};\cdot \cap E^\ell)$ charges only the layer E^ℓ, (2.1.2) gives

$$\gamma^{(\ell)}(\boldsymbol{y}) = \int_{E^\ell} \left([\alpha K]^*\gamma\right)^{(\ell)} (d\boldsymbol{x})\, r_\alpha^{(\ell)}(\boldsymbol{y}-\boldsymbol{x}) \tag{2.1.6}$$

for $\boldsymbol{\lambda}$-a.e. $\boldsymbol{y} \in E^\ell$, $\ell \in \mathbb{N}$. Under our assumption of constant rates, the measure $[\alpha K]^*\gamma$ is finite in restriction to E^ℓ; moreover, $\boldsymbol{y} \mapsto r_\alpha^{(\ell)}(\boldsymbol{y}-\boldsymbol{x})$ is continuous at each point $\boldsymbol{y} \in E^\ell$ for $\boldsymbol{\lambda}$-a.e. $\boldsymbol{x} \in E^\ell$. But due to the singularity at $\boldsymbol{x} = \boldsymbol{y}$ we cannot employ dominated convergence in order to conclude that the r.h.s. of (2.1.6) is continuous at the point $\boldsymbol{y}$; this will depend on the properties of the kernel K. In fact, in the case (2.1.4) we can show that the r.h.s. of (2.1.6) is *not* continuous: We define the set

$$\mathcal{N} \coloneqq \left\{\boldsymbol{x} = (x^1,\dots,x^\ell) \in \mathcal{S} : \ell \ge 2 \text{ and } \exists i \neq j : x^i = x^j\right\}$$

of configurations where two or more particles occupy the same position. Clearly $\mathcal{N}$ is a closed $\boldsymbol{\lambda}$-null set in $\mathcal{S}$. We will now prove the assertion of Example 2.1.1 in the following form:

2.1.2 Proposition
In the set-up of Example 2.1.1, there is a nonnegative function $\underline{\gamma}$ on $\mathcal{S}$ which is continuous on $\mathcal{N}^c$ and has singularities at all points of a nonempty subset $\varnothing \neq \tilde{\mathcal{N}} \subseteq \mathcal{N}$, i.e.

$$\underline{\gamma}(\boldsymbol{y}) \xrightarrow{\boldsymbol{y}\to\boldsymbol{y}_0} \infty, \qquad \boldsymbol{y}_0 \in \tilde{\mathcal{N}},$$

such that the invariant density γ is minorized by $\underline{\gamma}$:

$$\gamma(\cdot) \ge \underline{\gamma}(\cdot) \qquad \boldsymbol{\lambda}\text{-a.s.}$$

In particular, there can be no version of γ which is continuous and locally bounded.

Proof For the purposes of this proof, we modify slightly the notation (1.1.22): For $\ell \in \mathbb{N}$ and $j \in \{1,\dots,\ell\}$, let $\Pi_{\ell,2,j} : E^\ell \to E^{\ell+1}$ denote the mapping

$$\Pi_{\ell,2,j}(\boldsymbol{x}) \coloneqq (x^1,\dots,x^{j-1},x^j,x^j,x^{j+1},\dots,x^\ell), \qquad \boldsymbol{x} = (x^1,\dots,x^\ell) \in E^\ell$$

which replaces the jth particle with two copies of itself, in accordance with the assumption in (2.1.4) that branching particles reproduce at their death position. Further, let $\Pi_{\ell,0,j}$ as in (1.1.23).

Fix $\ell \in \mathbb{N}$. By (2.1.6) and Lemma 1.1.10 applied to $\mu(d\boldsymbol{x}) := m(d\boldsymbol{x}) = \gamma(\boldsymbol{x})\,d\boldsymbol{x}$ and $h(\boldsymbol{x}) := r_\alpha^{(\ell)}(\boldsymbol{y}-\boldsymbol{x})$, we obtain

$$\begin{aligned}\gamma^{(\ell)}(\boldsymbol{y}) &= \int_{E^\ell} ([\alpha K]^*\gamma)^{(\ell)}(d\boldsymbol{x})\, r_\alpha^{(\ell)}(\boldsymbol{y}-\boldsymbol{x})\\ &= \kappa p_0 \sum_{j=1}^{\ell+1} \int_{E^{\ell+1}} d\boldsymbol{x}\, \gamma^{(\ell+1)}(\boldsymbol{x})\, r_\alpha^{(\ell)}\left(\boldsymbol{y}-\Pi_{\ell+1,0,j}(\boldsymbol{x})\right)\\ &\quad + \kappa p_2 \sum_{j=1}^{\ell-1} \int_{E^{\ell-1}} d\boldsymbol{x}\, \gamma^{(\ell-1)}(\boldsymbol{x})\, r_\alpha^{(\ell)}\left(\boldsymbol{y}-\Pi_{\ell-1,2,j}(\boldsymbol{x})\right)\\ &\quad + c\int_{E^{\ell-1}} d\boldsymbol{x}\, \gamma^{(\ell-1)}(\boldsymbol{x}) \int_E dv\, p(v)\, r_\alpha^{(\ell)}\left(\boldsymbol{y}-(\boldsymbol{x},v)\right)\end{aligned} \tag{2.1.7}$$

for $\boldsymbol{\lambda}$-a.e. $\boldsymbol{y} \in E^\ell$.

Let us show first by induction that for all $\ell \in \mathbb{N}$, $\gamma^{(\ell)}$ is minorized almost surely by a strictly positive, continuous and bounded function $u^{(\ell)}$: In fact, for $\ell = 1$ it follows from the series representation (1.2.21) by retaining only the term for $n = 1$ that

$$m^{(1)}(B) \geq \boldsymbol{E}_\Delta\left[R_\alpha(\boldsymbol{\eta}_{\boldsymbol{\tau}_1}; B)\right] = \int_E dx\, p(x) \int_B dy\, r_\alpha^{(1)}(y-x) = \int_B p * r_\alpha^{(1)}(y)\, dy,$$

where $p \in \mathcal{C}_b^\infty(E)$ is the strictly positive immigration density assumed in Example 2.1.1. Thus we have

$$\gamma^{(1)}(\cdot) \geq p * r_\alpha^{(1)}(\cdot) \qquad \lambda\text{-a.s.},$$

and it is clear that $p * r_\alpha^{(1)}$ is strictly positive, continuous and bounded on E. Assuming that it is already known for some $\ell \in \mathbb{N}$ that $\gamma^{(\ell)} \geq u^{(\ell)}$ $\boldsymbol{\lambda}$-a.s., where $u^{(\ell)} \in \mathcal{C}_b(E^\ell)$ is strictly positive, we get from (2.1.7) by retaining only the last term corresponding to the immigration the minorization

$$\begin{aligned}\gamma^{(\ell+1)}(\boldsymbol{y}) &\geq c\int_{E^\ell} d\boldsymbol{x}\, \gamma^{(\ell)}(\boldsymbol{x}) \int_E dv\, p(v)\, r_\alpha^{(\ell+1)}\left(\boldsymbol{y}-(\boldsymbol{x},v)\right)\\ &= (\gamma^{(\ell)} \otimes p) * r_\alpha^{(\ell+1)}(\boldsymbol{y})\\ &\geq (u^{(\ell)} \otimes p) * r_\alpha^{(\ell+1)}(\boldsymbol{y})\end{aligned}$$

for $\boldsymbol{\lambda}$-a.e. $\boldsymbol{y} \in E^{\ell+1}$, where again it is clear that the r.h.s. of the above estimate is strictly positive and in $\mathcal{C}_b(E^{\ell+1})$.

Now let $\ell \geq 2$. Using the minorization just proved, by retaining only the term with p_2 in (2.1.7) corresponding to two offspring we obtain

$$\begin{aligned}\gamma^{(\ell)}(\boldsymbol{y}) &\geq \kappa p_2 \sum_{j=1}^{\ell-1} \int_{E^{\ell-1}} d\boldsymbol{x}\, u^{(\ell-1)}(\boldsymbol{x})\, r_\alpha^{(\ell)}\left(\boldsymbol{y}-\Pi_{\ell-1,2,j}(\boldsymbol{x})\right)\\ &= \kappa p_2 \sum_{j=1}^{\ell-1} \int_0^\infty dt\, e^{-(\kappa\ell+c)t} \int_{E^{\ell-1}} d\boldsymbol{x}\, u^{(\ell-1)}(\boldsymbol{x}) p_t^{(\ell)}\left(\boldsymbol{y}-\Pi_{\ell-1,2,j}(\boldsymbol{x})\right)\\ &=: \underline{\gamma}^{(\ell)}(\boldsymbol{y})\end{aligned} \tag{2.1.8}$$

for $\boldsymbol{\lambda}$-a.e. $\boldsymbol{y} \in E^{\ell-1}$. Now fix $j \in \{1,\ldots,\ell-1\}$ and consider the corresponding term in the above sum. We define

$$H_{\ell-1,2,j} := \Pi_{\ell-1,2,j}(E^{\ell-1}) \subseteq \mathcal{N} \cap E^\ell;$$

clearly $H_{\ell-1,2,j}$ is a closed Lebesgue null set in E^ℓ. By dominated convergence, it is clear that the function

$$\boldsymbol{y} \mapsto e^{-(\kappa\ell+c)t} \int_{E^{\ell-1}} d\boldsymbol{x}\, u^{(\ell-1)}(\boldsymbol{x})\, p_t^{(\ell)}\left(\boldsymbol{y} - \Pi_{\ell-1,2,j}(\boldsymbol{x})\right) \tag{2.1.9}$$

is continuous on all of E^ℓ, for each $t > 0$ fixed. We will show that this continuity persists after integration $\int_0^\infty dt \ldots$ at all points in $\mathcal{N}^c \cap E^\ell$, but fails at all points in $H_{\ell-1,2,j}$.

Let $\boldsymbol{y} \in \mathcal{N}^c \cap E^\ell$, i.e. $\boldsymbol{y} = (y^1, \ldots, y^\ell)$ where the y^i are all distinct. Then

$$2\delta := \operatorname{dist}\left(\boldsymbol{y}, H_{\ell-1,2,j}\right) > 0,$$

and for all $\tilde{\boldsymbol{y}}$ in a sufficiently small neighborhood of $\boldsymbol{y}$ we have

$$p_t^{(\ell)}\left(\tilde{\boldsymbol{y}} - \Pi_{\ell-1,2,j}(\boldsymbol{x})\right) = (2\pi t)^{-d\ell/2} \exp\left(-\frac{\|\tilde{\boldsymbol{y}} - \Pi_{\ell-1,2,j}(\boldsymbol{x})\|^2}{2t}\right) \le (2\pi t)^{-d\ell/2} e^{-\delta^2/2t}.$$

Thus in a neighborhood of our fixed $\boldsymbol{y}$, the function (2.1.9) is continuous and dominated by

$$e^{-(\kappa\ell+c)t}(2\pi t)^{-d\ell/2} e^{-\delta^2/2t} \|u^{(\ell-1)}\|_1$$

which is integrable $\int_0^\infty dt \ldots$ since the $e^{-\delta^2/2t}$-term kills the $t^{-d\ell/2}$-term as $t \downarrow 0$. This proves that $\underline{\gamma}^{(\ell)}$, defined as the r.h.s. of (2.1.8), is continuous at each point $\boldsymbol{y} \in \mathcal{N}^c \cap E^\ell$.

Now consider a point $\tilde{\boldsymbol{y}} \in H_{\ell-1,2,j}$, i.e. $\tilde{\boldsymbol{y}}$ is of the form

$$\tilde{\boldsymbol{y}} = \Pi_{\ell-1,2,j}(\boldsymbol{y}) = (y^1, \ldots, y^{j-1}, y^j, y^j, y^{j+1}, \ldots, y^{\ell-1})$$

with $\boldsymbol{y} = (y^1, \ldots, y^{\ell-1}) \in E^{\ell-1}$. Then elementary manipulation shows

$$\begin{aligned} p_t^{(\ell)}\left(\tilde{\boldsymbol{y}} - \Pi_{\ell-1,2,j}(\boldsymbol{x})\right) &= \left(p_t(y^j - x^j)\right)^2 \prod_{i \in \{1,\ldots,\ell-1\} \setminus \{j\}} p_t(y^i - x^i) \\ &= (4\pi t)^{-d/2} \cdot p_{t/2}(y^j - x^j) \prod_{i \in \{1,\ldots,\ell-1\} \setminus \{j\}} p_t(y^i - x^i) \\ &=: (4\pi t)^{-d/2} \cdot \tilde{p}_t(\boldsymbol{y} - \boldsymbol{x}). \end{aligned}$$

Thus the r.h.s. of (2.1.9) can now be rewritten as

$$e^{-(\kappa\ell+c)t}(4\pi t)^{-d/2} \cdot (u^{(\ell-1)} * \tilde{p}_t)(\boldsymbol{y}). \tag{2.1.10}$$

Observe that the above, as a function of $t > 0$, is not integrable on $\mathbb{R}_+$: Since $(\tilde{p}_t)_t$ is a Dirac sequence for $t \downarrow 0$ and $u^{(\ell-1)} \in \mathcal{C}_b(E^{\ell-1})$ is strictly positive, we get

$$u^{(\ell-1)} * \tilde{p}_t(\boldsymbol{y}) \xrightarrow{t\downarrow 0} u^{(\ell-1)}(\boldsymbol{y}) > 0,$$

and since $d \ge 2$ the factor $t^{-d/2}$ ensures that (2.1.10) not integrable at $t = 0$. Now take a sequence $\boldsymbol{y}_n \in E^\ell$ with $\boldsymbol{y}_n \xrightarrow{n\to\infty} \tilde{\boldsymbol{y}} = \Pi_{\ell,2,j}(\boldsymbol{y})$. Then for each $\varepsilon > 0$

$$\begin{aligned} \liminf_{n\to\infty} \underline{\gamma}^{(\ell)}(\boldsymbol{y}_n) &\ge \kappa p_2 \liminf_{n\to\infty} \int_\varepsilon^{1/\varepsilon} dt\, e^{-(\kappa\ell+c)t} \int_{E^{\ell-1}} d\boldsymbol{x}\, u^{(\ell-1)}(\boldsymbol{x})\, p_t^{(\ell)}\left(\boldsymbol{y}_n - \Pi_{\ell-1,2,j}(\boldsymbol{x})\right) \\ &= \kappa p_2 \int_\varepsilon^{1/\varepsilon} dt\, e^{-(\kappa\ell+c)t} \int_{E^{\ell-1}} d\boldsymbol{x}\, u^{(\ell-1)}(\boldsymbol{x}) p_t^{(\ell)}\left(\tilde{\boldsymbol{y}} - \Pi_{\ell-1,2,j}(\boldsymbol{x})\right) \\ &= \kappa p_2 \int_\varepsilon^{1/\varepsilon} dt\, e^{-(\kappa\ell+c)t} (4\pi t)^{-d/2}\, (u^{(\ell-1)} * \tilde{p}_t)(\boldsymbol{y}), \end{aligned}$$

where the first equality holds by continuity of (2.1.9) and dominated convergence since we integrate over a compact time interval. Now letting $\varepsilon \downarrow 0$, we conclude by monotone convergence and the fact that (2.1.10) is not integrable at $t = 0$ that $\underline{\gamma}^{(\ell)}(\boldsymbol{y}_n) \xrightarrow{n\to\infty} \infty$. Thus we have proved that $\underline{\gamma}^{(\ell)}$ is continuous on $\mathcal{N}^c \cap E^\ell$ but unbounded, with singularities at each point belonging to one of the $\boldsymbol{\lambda}$-null sets $H_{\ell-1,2,j}$, $j = 1, \dots, \ell-1$. Taking

$$\tilde{\mathcal{N}} := \bigcup_{\ell\geq 2} \bigcup_{j\in\{1,\dots,\ell-1\}} H_{\ell-1,2,j} \subseteq \mathcal{N},$$

our proposition is proved. ■

Example 2.1.1 demonstrates that as long as branching particles reproduce at their death position (i.e. (2.1.4) holds), even strong regularity properties of the other "input parameters" are generally not sufficient to deduce basic regularity properties (like continuity and local boundedness) for the invariant density γ. We will not investigate this phenomenon any further[1]; instead, our focus in this work is on the problem to find conditions which ensure that there *is* a continuous and locally bounded version of the invariant density. In view of the above analysis, clearly (2.1.4) has to modified; namely, we will replace it with the assumption that the spatial offspring distribution is absolutely continuous w.r.t. Lebesgue measure.

We conclude this section with a heuristic outline of the approach to be taken in the next section. We continue within the framework of Höpfner's example 2.1.1, except that we replace (2.1.4) by the condition that the spatial distribution of offspring is of "product type" with an absolutely continuous convolution kernel:

$$Q_k^{(\ell)}(x; \boldsymbol{x}; dv^1 \cdots dv^k) = \prod_{i=1}^{k} q(x - v^i)\, dv^1 \cdots dv^k \quad \text{on } E^k, \qquad k, \ell \in \mathbb{N}, \tag{2.1.11}$$

where $q \in L^1(E)$. The probabilistic meaning of (2.1.11) is the following: If a particle branching at $x \in E$ produces $k \in \mathbb{N}$ offspring, the newborn particles are distributed randomly in space, independently of each other and of the configuration $\boldsymbol{x}$ of coexisting particles, according to the law $q(x - y)\, dy$; note that this entails that the distribution of the "jump size" $x - y$ does not depend on the position x.

In contrast to (2.1.4), under (2.1.11) the kernel $[\alpha K]$ preserves absolutely continuous measures (see Lemma 2.2.9 below); thus $[\alpha K]^*\gamma$ is now a function rather than a general measure (in fact, this is one of our reasons for adopting this notation). Returning to the fixed point equation for the invariant density on the layer E^ℓ, the r.h.s. of (2.1.6) can now be written as a classical convolution of two functions:

$$\gamma^{(\ell)} = r_\alpha^{(\ell)} * ([\alpha K]^*\gamma)^{(\ell)}. \tag{2.1.12}$$

As before, due to the singularity of $r_\alpha^{(\ell)}$ at the origin we cannot immediately deduce continuity or boundedness of $\gamma^{(\ell)}$ since a priori we know nothing more about $([\alpha K]^*\gamma)^{(\ell)}$ than integrability. On the other hand, since γ appears on the r.h.s. of (2.1.12), the obvious idea is to iterate this equation. But since the operator $[\alpha K]^*$ (unlike R_α) does not act "layer-wise" (e.g. in the case of binary branching, $([\alpha K]^*\gamma)^{(\ell)}$ will depend on $\gamma^{(\ell-1)}$ and on $\gamma^{(\ell+1)}$), the

[1] See [Höp2004], where it is shown that in the above example (and also for more general diffusions under smoothness and uniform ellipticity conditions) the invariant density is even $\mathcal{C}^\infty$ on $\mathcal{N}^c$.

expressions we get by iterating the convolution operation soon become hardly tractable. The idea is now to apply the Fourier transformation to equation (2.1.12) in order to obtain

$$\mathscr{F}[\gamma^{(\ell)}] = \mathscr{F}[r_\alpha^{(\ell)}] \cdot \mathscr{F}[(([\alpha K]^*\gamma)^{(\ell)}] \tag{2.1.13}$$

and to iterate the above equation. For this, we have to analyze the action of the Fourier transformation $\mathscr{F}$ on the operator $[\alpha K]^*$. Under (2.1.11), it turns out that $\mathscr{F}[(([\alpha K]^*\gamma)^{(\ell)}]$ can be expressed in terms of $\mathscr{F}^{-1}[q]$, $\mathscr{F}[p]$ and the Fourier transforms $\mathscr{F}[\gamma^{(\ell-1)}]$, $\mathscr{F}[\gamma^{(\ell+1)}]$ of γ on the "neighboring layers", so that we can indeed iterate equation (2.1.13). Moreover, by iterating sufficiently often the resulting expression is seen to be integrable on E^ℓ provided $\mathscr{F}[q]$ and $\mathscr{F}[p]$ are in $L^1(E)$. Then the Fourier inversion theorem implies that

$$\gamma^{(\ell)} = (2\pi)^{-d\ell}\mathscr{F}^{-1}\left[\mathscr{F}[\gamma^{(\ell)}]\right] \qquad \boldsymbol{\lambda}\text{-a.s.}; \tag{2.1.14}$$

in particular, $\gamma^{(\ell)}$ must coincide with a continuous function (in fact, with a $\mathcal{C}_0$-function) $\boldsymbol{\lambda}$-a.s. By this method, we can prove:

2.1.3 Example
Let everything as in Example 2.1.1 except that we replace (2.1.4) with (2.1.11). If

$$\mathscr{F}[q] \in L^1(E), \qquad \mathscr{F}[p] \in L^1(E),$$

then the invariant density has a version which is continuous and locally bounded (in fact, locally $\mathcal{C}_0$):

$$\gamma^{(\ell)} \in \mathcal{C}_0(E^\ell), \qquad \ell \in \mathbb{N}.$$

♦

For a general diffusion instead of Brownian motion, the fixed point equation (2.1.2) will not have the convolution structure (2.1.12), and the approach outlined above will not work in the same way. Indeed, in the general case we do not have a result expressing the invariant density as an inverse Fourier transform as in (2.1.14). Instead of the fixed point equation, we have to work with the series representation (2.1.1). Under appropriate conditions, each term in the series has a continuous and locally bounded density, and it remains to show uniform convergence of the series on each layer E^ℓ. This will be achieved by assuming that the resolvent density $r_\alpha(\boldsymbol{x};\boldsymbol{y})$ admits an upper bound which is of the same form as the resolvent density of Brownian motion, and that the spatial offspring distribution is absolutely continuous and upper bounded by a kernel of the form (2.1.11). Then the same Fourier-type arguments as outlined above can be employed on the level of the upper bound in order to prove uniform convergence of the series in (2.1.1).

2.2 Existence of a Continuous and Locally Bounded Invariant Density

This section is devoted to the implementation of the program sketched above: Under the assumption that the spatial offspring distribution is absolutely continuous w.r.t. Lebesgue measure and suitable additional conditions, we will prove the existence of a continuous and locally bounded invariant density.

We continue to assume throughout that Assumptions 1.1.2-1.1.5 and 1.2.1 are satisfied. Note that the recurrence condition (1.2.2) implies finiteness of the generalized resolvent kernel R_α of (1.1.36) since for each $\ell \in \mathbb{N}$ and $\boldsymbol{x} \in E^\ell$ we have

$$R_\alpha(\boldsymbol{x}; \mathcal{S}) = R_\alpha^{(\ell)}(\boldsymbol{x}; E^\ell) = \boldsymbol{E}_{\boldsymbol{x}}[\boldsymbol{\tau}_1] \leq \boldsymbol{E}_{\boldsymbol{x}}[R] < \infty. \tag{2.2.1}$$

We begin by strengthening Assumption 1.1.2: Recalling that the semigroup $(P_t^\alpha)_{t\geq 0}$ of the ℓ-particle motion X^ℓ killed at rate $\alpha^{(\ell)}$ is given by (1.1.31), we impose a condition of non-degeneracy for $(P_t^\alpha)_{t\geq 0}$. Namely, we assume that it possesses a continuous transition density which satisfies a heat kernel estimate (i.e. admits an upper bound of the same form as the transition density of Brownian motion) for small $t > 0$:

2.2.1 Assumption (Transition density, heat kernel estimate)
For all $\ell \in \mathbb{N}$, we have

$$P_t^\alpha f(\boldsymbol{x}) = \int_{E^\ell} p_t^\alpha(\boldsymbol{x}; \boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y}, \qquad f \in \mathscr{B}(E^\ell),\ \boldsymbol{x} \in E^\ell,$$

where for each $t > 0$ and $\boldsymbol{x} \in E^\ell$ fixed, $p_t^\alpha(\boldsymbol{x}; \cdot)$ is continuous in the "forward variable" $\boldsymbol{y} \in E^\ell$. Furthermore, we assume that there exist $\varepsilon > 0$ and $C_\ell \in (0, \infty)$ such that

$$p_t^\alpha(\boldsymbol{x}; \boldsymbol{y}) \leq C_\ell \cdot t^{-d\ell/2} \exp\left(-\frac{\|\boldsymbol{x} - \boldsymbol{y}\|^2}{2C_\ell t}\right), \qquad t \in [0, \varepsilon],\ \boldsymbol{x}, \boldsymbol{y} \in E^\ell. \tag{2.2.2}$$

2.2.2 Remarks

- Since the r.h.s. of the estimate (2.2.2) is increasing in C_ℓ, without loss of generality we can take the constants C_ℓ to be increasing in $\ell \in \mathbb{N}$.

- Under Assumption 2.2.1, an estimate like (2.2.2) holds in fact on every compact time interval $[0, T]$, $T < \infty$: Indeed, using the Chapman-Kolmogorov identity one sees immediately that (2.2.2) extends to all $t \in [0, 2\varepsilon]$ (with a different constant C_ℓ of course), and so on. Thus without loss of generality we could take $\varepsilon = 1$ in (2.2.2); we write ε just to emphasize that the condition must be verified for small t only. For the same reason, no generality is lost by taking ε independent of ℓ.

- In the purely position-dependent framework of [HL2005] where b, σ and κ do not depend on the configuration variable and c is constant, by the independence of the motion of the particles Assumption 2.2.1 reduces to a condition on the semigroup of the single particle motion killed at rate κ. If also κ is constant, it reduces further to a condition on the semigroup of the single-particle motion itself. ⧫

2.2.3 Remarks

- Assumption 2.2.1 is in particular satisfied if the drift and diffusion coefficients in (1.1.3) as well as the branching and immigration rates are bounded smooth functions and *uniform ellipticity* holds[2]: Suppose that for all $\ell \in \mathbb{N}$, we have

$$b^{(\ell)}(\cdot\,;\cdot) \in \mathcal{C}_b^\infty(E \times E^\ell; E), \qquad \sigma^{(\ell)}(\cdot\,;\cdot) \in \mathcal{C}_b^\infty(E \times E^\ell; \mathbb{R}^{d\times m}),$$

$$\kappa^{(\ell)}(\cdot\,;\cdot) \in \mathcal{C}_b^\infty(E \times E^\ell; \mathbb{R}_+), \qquad c^{(\ell)}(\cdot) \in \mathcal{C}_b^\infty(E^\ell; \mathbb{R}_+),$$

[2]This is the set of conditions adopted in [Höp2004] and [Löc2004].

where $\mathcal{C}_b^\infty$ denotes the class of $\mathcal{C}^\infty$-functions which are bounded with bounded derivatives of all orders. Further, assume that for all $\ell \in \mathbb{N}$

$$\inf_{(x,\boldsymbol{x})\in E\times E^\ell} \inf_{v\in E, \|v\|=1} \langle a^{(\ell)}(x;\boldsymbol{x})v,\, v\rangle > 0 \tag{2.2.3}$$

with $a^{(\ell)} := \sigma^{(\ell)}\left(\sigma^{(\ell)}\right)^T$ as in (1.1.5). Rewriting the given system (1.1.3) of SDEs as a $d\ell$-dimensional diffusion as in (1.1.4), it is easily seen that the generator (1.1.32) of the ℓ-particle motion killed at rate $\alpha^{(\ell)}$ is a second order differential operator with bounded smooth coefficients and that we have uniform ellipticity

$$\epsilon_\ell := \inf_{\boldsymbol{x}\in E^\ell} \inf_{\boldsymbol{v}\in E^\ell,\, \|\boldsymbol{v}\|=1} \langle \tilde{a}^{(\ell)}(\boldsymbol{x})\boldsymbol{v},\, \boldsymbol{v}\rangle > 0, \tag{2.2.4}$$

where $\tilde{a}^{(\ell)}$ is defined in (1.1.6). By a classical result[3], there exists a transition density $p_t^\alpha(\boldsymbol{x};\boldsymbol{y})$ for the semigroup $P_t^\alpha(\boldsymbol{x};d\boldsymbol{y})$ which is smooth jointly in $(t,\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}_+ \times E^\ell \times E^\ell$ and such that for partial derivatives of arbitrary order $n \in \mathbb{N}_0$ w.r.t. $\boldsymbol{y}$ it holds

$$|\partial_{\boldsymbol{y}}^\beta p_t^\alpha(\boldsymbol{x};\boldsymbol{y})| \le C_{\ell,n}^{(1)} \cdot t^{-(d\ell+n)/2} \exp\left(-C_{\ell,n}^{(2)} \frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{2t}\right), \qquad 0 < t \le 1, \tag{2.2.5}$$

where $\beta \in \mathbb{N}_0^{d\ell}$ denotes a multiindex with length $|\beta| = \sum_{j=1}^{d\ell} \beta_j = n$. From this, for $n = 0$ we obtain an estimate as in (2.2.2) by taking

$$C_\ell := \max\left(C_{\ell,0}^{(1)},\, \frac{1}{C_{\ell,0}^{(2)}}\right).$$

The constant $C_{\ell,n}^{(2)}$ in (2.2.5) can usually be controlled by the ϵ_ℓ in (2.2.4) plus the bounds on the drift and diffusion coefficients and their derivatives, while the constant $C_{\ell,n}^{(1)}$ will typically also depend in a non-explicit way on the dimension $d\ell$ and will grow exponentially as $\ell \uparrow \infty$. As in [Löc2004], this induces a certain difficulty when dealing with the invariant occupation measure $\overline{m}$ in the next section. For the purposes of the present section however, the precise form or growth of the constants does not matter, and in order to save notation, whenever (2.2.5) is satisfied we combine the two constants into one as in (2.2.2).

- The smoothness conditions mentioned in the previous remark are strong, and often Assumption 2.2.1 will hold under weaker conditions. For example, it is a classical result from PDE theory that for the existence of a *continuous* transition density (as opposed to a smooth one) satisfying the estimate (2.2.2), it suffices to require that the quantities $b^{(\ell)}$, $\sigma^{(\ell)}$, $\kappa^{(\ell)}$ and $c^{(\ell)}$ be bounded and Hölder-continuous (see e.g. [Dyn1965], Vol. II, Appendix, § 6, pp. 225ff.). For this reason, we have chosen not to adopt the framework of bounded smooth coefficients and rates plus uniform ellipticity outlined in the previous remark, but instead to take Assumption 2.2.1 as it is stated as the basic condition on which our approach is based. As we shall see, continuity of the transition density in the forward variable and the estimate (2.2.2), plus the assumptions on the branching, reproduction and immigration mechanisms to be introduced below, are all that we shall need in order to prove the existence of a continuous and locally bounded invariant density for the BDI $\boldsymbol{\eta}$. ♦

[3] See e.g. [KS1985], Thm. (3.18); [Str2008], Ch. 3, in particular Thm. 3.3.11.

2.2.4 Remarks

- Under Assumption 2.2.1, for each $\ell \in \mathbb{N}$ the generalized resolvent kernel $R_\alpha^{(\ell)}(\boldsymbol{x};\cdot)$ of (1.1.33) has the density

$$r_\alpha^{(\ell)}(\boldsymbol{x};\boldsymbol{y}) := \int_0^\infty p_t^\alpha(\boldsymbol{x};\boldsymbol{y})\,dt, \qquad \boldsymbol{x},\,\boldsymbol{y} \in E^\ell \tag{2.2.6}$$

which for each fixed $\boldsymbol{x} \in E^\ell$ is integrable in the forward variable $\boldsymbol{y}$ by (2.2.1). Moreover, it is easy to see that $r_\alpha^{(\ell)}(\boldsymbol{x};\cdot)$ is continuous in $\boldsymbol{y} \in E^\ell \setminus \{\boldsymbol{x}\}$: Use the Chapman-Kolmogorov identity to write

$$\begin{aligned} r_\alpha^{(\ell)}(\boldsymbol{x};\boldsymbol{y}) &= \int_0^\varepsilon p_t^\alpha(\boldsymbol{x};\boldsymbol{y})\,dt + \int_0^\infty p_{t+\varepsilon}^\alpha(\boldsymbol{x};\boldsymbol{y})\,dt \\ &= \int_0^\varepsilon p_t^\alpha(\boldsymbol{x};\boldsymbol{y})\,dt + \int_0^\infty \int_{E^\ell} p_t^\alpha(\boldsymbol{x};\boldsymbol{z}) p_\varepsilon^\alpha(\boldsymbol{z};\boldsymbol{y})\,d\boldsymbol{z}dt \\ &= r_{\alpha,\varepsilon}^{(\ell)}(\boldsymbol{x};\boldsymbol{y}) + \int_{E^\ell} r_\alpha^{(\ell)}(\boldsymbol{x};\boldsymbol{z}) p_\varepsilon^\alpha(\boldsymbol{z};\boldsymbol{y})\,d\boldsymbol{z} \end{aligned} \tag{2.2.7}$$

for all $\boldsymbol{x}$, $\boldsymbol{y} \in E^\ell$, where

$$r_{\alpha,\varepsilon}^{(\ell)}(\boldsymbol{x};\boldsymbol{y}) := \int_0^\varepsilon p_t^\alpha(\boldsymbol{x};\boldsymbol{y})dt. \tag{2.2.8}$$

The second term on the r.h.s. of (2.2.7), as a function of $\boldsymbol{y}$, is continuous and bounded on all of E^ℓ by dominated convergence, since $p_\varepsilon^\alpha(\boldsymbol{z};\cdot)$ is continuous for all $\boldsymbol{z}$ and bounded uniformly in $\boldsymbol{z}$ by (2.2.2), and $r_\alpha^{(\ell)}(\boldsymbol{x};\cdot) \in L^1(E^\ell)$. For the term $r_{\alpha,\varepsilon}^{(\ell)}(\boldsymbol{x};\boldsymbol{y})$, we can use continuity of $p_t^\alpha(\boldsymbol{x};\cdot)$ plus the heat kernel estimate (2.2.2) and dominated convergence to obtain continuity in $\boldsymbol{y} \in E^\ell \setminus \{\boldsymbol{x}\}$. In general, $r_{\alpha,\varepsilon}^{(\ell)}(\boldsymbol{x};\cdot)$ and thus also $r_\alpha^{(\ell)}(\boldsymbol{x};\cdot)$ will have a singularity at the point $\boldsymbol{y} = \boldsymbol{x}$; as already mentioned in the previous section, this is in particular true for Brownian motion in $\mathbb{R}^d$, $d \geq 2$.

- Defining

$$R_{\alpha,\varepsilon}(\boldsymbol{x};d\boldsymbol{y}) := r_{\alpha,\varepsilon}(\boldsymbol{x};\boldsymbol{y})\,d\boldsymbol{y}, \qquad \boldsymbol{x},\boldsymbol{y} \in \mathcal{S}, \tag{2.2.9}$$

we can rewrite (2.2.7) as an identity for kernels:

$$R_\alpha = R_{\alpha,\varepsilon} + R_\alpha P_\varepsilon^\alpha,$$

or in "operator form" (recall the Notations 1.1.9)

$$R_\alpha^* = R_{\alpha,\varepsilon}^* + (P_\varepsilon^\alpha)^* R_\alpha^*. \tag{2.2.10}$$

The identity (2.2.10) will be used repeatedly in the sequel. The action of the operator $R_{\alpha,\varepsilon}^*$ on a measure μ on $(\mathcal{S},\mathcal{B}_\mathcal{S})$ is given by

$$R_{\alpha,\varepsilon}^*\mu(F) \equiv \int_\mathcal{S} \mu(d\boldsymbol{x}) R_{\alpha,\varepsilon}(\boldsymbol{x};F) = \int_F d\boldsymbol{y} \int_\mathcal{S} \mu(d\boldsymbol{x})\, r_{\alpha,\varepsilon}(\boldsymbol{x};\boldsymbol{y}), \qquad F \in \mathcal{B}_\mathcal{S}. \tag{2.2.11}$$

In particular, $R_{\alpha,\varepsilon}^*$ maps arbitrary measures to $\boldsymbol{\lambda}$-absolutely continuous ones, with the density given by $R_{\alpha,\varepsilon}^*\mu(\cdot) = \int_\mathcal{S} \mu(d\boldsymbol{x})\, r_{\alpha,\varepsilon}(\boldsymbol{x};\cdot)$. Taking an absolutely continuous measure $\mu(d\boldsymbol{x}) = f(\boldsymbol{x})\,d\boldsymbol{x}$, we have on each layer E^ℓ

$$\left(R_{\alpha,\varepsilon}^* f\right)^{(\ell)}(\boldsymbol{y}) = \int_{E^\ell} d\boldsymbol{x}\, r_{\alpha,\varepsilon}^{(\ell)}(\boldsymbol{x};\boldsymbol{y}) f^{(\ell)}(\boldsymbol{x}), \qquad \boldsymbol{y} \in E^\ell$$

from which together with the heat kernel estimate (2.2.2) we see that $\left(R^*_{\alpha,\varepsilon} f\right)^{(\ell)}$ is again integrable provided $f^{(\ell)}$ is. In particular, $R^*_{\alpha,\varepsilon}$ is well-defined as an operator on $L^1_{(loc)}(\mathcal{S})$ (and not just for nonnegative functions).[4]

Note however that for arbitrary $f \in L^1_{(loc)}(\mathcal{S})$ we cannot expect any continuity properties of $R^*_{\alpha,\varepsilon} f$ because of the singularity of $r_{\alpha,\varepsilon}(\cdot;\cdot)$ at $\boldsymbol{x} = \boldsymbol{y}$. This introduces a certain technical difficulty that will have to be dealt with in the proof of our main result later in this section. ♦

Next, we have to strengthen our assumptions on the branching, reproduction and immigration mechanisms. Namely, we assume that the kernels Q and ν of Assumptions 1.1.4-1.1.5 are absolutely continuous and upper bounded by kernels of a certain "simple" structure. In particular, the upper bounds should depend on the configuration variable only through its length, and the upper bound for the spatial offspring distribution kernel Q should be the product of a convolution kernel as in (2.1.11):

2.2.5 Assumption (Absolute Continuity of Offspring and Immigration Laws) *Let all notations be as in Assumptions 1.1.4 and 1.1.5.*

1. *We assume that the branching rate $\kappa(\cdot;\cdot)$, the reproduction probabilities $p_k(\cdot;\cdot)$ and the immigration rate $c(\cdot;\cdot)$ are continuous in the configuration variable $\boldsymbol{x}$. Further, we assume for all $\ell \in \mathbb{N}$ that $\kappa^{(\ell)}(\cdot;\cdot)$ and $c^{(\ell)}(\cdot;\cdot)$ are bounded.*[5]

2. *For all $k \in \mathbb{N}$, $\ell \in \mathbb{N}$ we assume the following: We have*

$$Q_k^{(\ell)}(x;\boldsymbol{x};dv^1\cdots dv^k) = q_k^{(\ell)}(x;\boldsymbol{x};v^1,\dots,v^k)\,dv^1\cdots dv^k \quad \text{on } (E^k,\mathcal{B}_{E^k}) \tag{2.2.12}$$

for all $(x,\boldsymbol{x}) \in E \times E^\ell$, where $q_k^{(\ell)}(x;\cdot;\cdot) : E^\ell \times E^k \to \mathbb{R}_+$ is continuous for each fixed $x \in E$. Furthermore, there is a function $\widehat{q}_k^{(\ell)}(\cdot) \in \mathcal{C}_0(E) \cap L^1(E)$ such that for all $x \in E$, $\boldsymbol{x} \in E^\ell$ and $(v^1,\dots,v^k) \in E^k$ it holds

$$q_k^{(\ell)}(x;\boldsymbol{x};v^1,\dots,v^k) \le \prod_{j=1}^k \widehat{q}_k^{(\ell)}(x - v^j). \tag{2.2.13}$$

For $k = 0$, we write

$$q_0^{(\ell)}(x;\boldsymbol{x};\Delta) := \widehat{q}_0^{(\ell)}(x) := 1, \qquad x \in E,\ \boldsymbol{x} \in \mathcal{S},\ \ell \in \mathbb{N}. \tag{2.2.14}$$

In addition, for the Fourier transform of the upper bound function $\widehat{q}_k^{(\ell)}(\cdot)$ we require

$$\mathscr{F}[\widehat{q}_k^{(\ell)}] \in L^1(E), \qquad k,\ell \in \mathbb{N}. \tag{2.2.15}$$

3. *For all $\ell \in \mathbb{N}_0$ we assume the following: It holds*

$$\nu^{(\ell)}(\boldsymbol{x};dv) = \pi^{(\ell)}(\boldsymbol{x};v)\,dv \quad \text{on } (E,\mathcal{B}_E) \tag{2.2.16}$$

[4] The same is generally not true for R^*_α.

[5] The upper bound need not be uniform in ℓ.

for all $\boldsymbol{x} \in E^\ell$, where $\pi^{(\ell)}(\cdot\,;\cdot) : E^\ell \times E \to \mathbb{R}_+$ is continuous. For $\ell = 0$, we simply write

$$\pi^{(0)}(v) := \pi^{(0)}(\Delta; v). \tag{2.2.17}$$

Moreover, there is a function $\widehat{\pi}^{(\ell)}(\cdot) \in \mathcal{C}_0(E) \cap L^1(E)$ such that for all $\boldsymbol{x} \in E^\ell$ and $v \in E$ we have

$$\pi^{(\ell)}(\boldsymbol{x}; v) \le \widehat{\pi}^{(\ell)}(v). \tag{2.2.18}$$

Finally, we require

$$\mathscr{F}[\widehat{\pi}^{(\ell)}] \in L^1(E), \qquad \ell \in \mathbb{N}_0. \tag{2.2.19}$$

2.2.6 Remarks

- Note that under Assumption 2.2.5, the killing rate $\alpha(\cdot)$ is locally bounded on $\mathcal{S}$, i.e. $\alpha^{(\ell)} \in \mathscr{B}(E^\ell)$ for all $\ell \in \mathbb{N}$.
- Also note that (in contrast to [Löc2002a]), we require continuity of the branching rate κ and of the reproduction probabilites p_k only in the configuration variable $\boldsymbol{x}$ and not in the position variable x. In particular, if κ and p_k are are purely position-dependent (as in the framework of [HL2005]), no continuity is required. Similarly, we no not assume continuity of the offspring densities $q_k^{(\ell)}$ in the position variable x, but only continuity in the configuration variable $\boldsymbol{x}$ and the "forward variables" $v^1, \dots, v^k$.
- By condition (2.2.13), the transition probability $Q_k^{(\ell)}(x^i; \boldsymbol{x}; \cdot)$ which governs the distribution of k offspring particles of a parent branching at position x^i and belonging to a configuration $\boldsymbol{x} = (x^1, \dots, x^\ell)$, in addition to being absolutely continuous, is also bounded from above by a kernel which depends on the whole configuration $\boldsymbol{x}$ only through its length ℓ and which is the k-fold product of a convolution kernel on E. Note that w.l.o.g., we could allow for different marginals $\widehat{q}_{k,j}^{(\ell)}(\cdot)$ ($j = 1, \dots, k$) on the r.h.s. of (2.2.13) by choosing $\widehat{q}_k^{(\ell)}(\cdot) := \sum_{j=1}^k \widehat{q}_{k,j}^{(\ell)}(\cdot)$. ♦

Before stating the main result of this section, we consider what seems to be a "natural" example of an absolutely continuous offspring distribution of the form (2.2.12) (cf. [Löc1999], Ex. 3.12a)):

2.2.7 Example

Suppose that the descendants of a particle dying at a position $x^i \in E$ belonging to a configuration $\boldsymbol{x} = (x^1, \dots, x^\ell) \in E^\ell$ are distributed according to a sharply concentrated normal distribution around x^i with a small variance. Our framework is flexible enough to let this variance depend on the position of the dying particle, the whole configuration of coexisting particles and the number of offspring. That is, in (2.2.12) we have

$$Q_k^{(\ell)}(x^i; \boldsymbol{x}; dv^1 \cdots dv^k) = \mathcal{N}\Bigg(\underbrace{(x^i, \dots, x^i)}_{k \text{ times}};\ \Lambda_k^{(\ell)}(x^i; \boldsymbol{x})\Bigg)(dv^1 \cdots dv^k) \qquad \text{on } (E^k, \mathcal{B}_{E^k}),$$

where

$$\Lambda_k^{(\ell)}(\cdot\,;\cdot) : E \times E^\ell \to \mathrm{Sym}(dk)$$

is a mapping taking values in the class of symmetric and positive definite $dk \times dk$-matrices. Assume that for all $k, \ell \in \mathbb{N}$, the function $\boldsymbol{x} \mapsto \Lambda_k^{(\ell)}(x; \boldsymbol{x})$ is continuous on E^ℓ for each fixed

$x \in E$, and that $\Lambda_k^{(\ell)}(\cdot;\cdot)$ on $E \times E^\ell$ is bounded and bounded away from 0, i.e. there is a constant $0 < M_k^{(\ell)} < \infty$ such that

$$\frac{1}{M_k^{(\ell)}}\|v\|^2 \le \left\langle \Lambda_k^{(\ell)}(x;\boldsymbol{x})v, v\right\rangle_k \le M_k^{(\ell)}\|v\|^2, \qquad (x,\boldsymbol{x},v) \in E \times E^\ell \times E^k.$$

Then we clearly obtain for the offspring density

$$\begin{aligned}
&q_k^{(\ell)}(x^i;\boldsymbol{x};v^1,\dots,v^k)\\
&= \left((2\pi)^{dk}\det\Lambda_k^{(\ell)}(x^i;\boldsymbol{x})\right)^{-1/2}\cdot\\
&\quad\cdot\exp\left(-\frac{1}{2}\left\langle (v^1-x^i,\dots,v^k-x^i), \Lambda_k^{(\ell)}(x^i;\boldsymbol{x})^{-1}(v^1-x^i,\dots,v^k-x^i)\right\rangle_k\right)\\
&\le C_{k,\ell}\cdot\exp\left(-\frac{\|(v^1-x^i,\dots,v^k-x^i)\|^2}{C_{k,\ell}}\right)
\end{aligned}$$

for some constant $C_{k,\ell} < \infty$. Thus it is clear that the conditions (2.2.13) and (2.2.15) are satisfied. ♦

Now we are prepared to state the main theorem of this section and chapter:

2.2.8 Theorem
Under Assumptions 2.2.1 and 2.2.5, the invariant measure m on the configuration space $(\mathcal{S},\mathcal{B}_\mathcal{S})$ admits a Lebesgue density $\gamma = (\gamma^{(\ell)})_{\ell\in\mathbb{N}_0}$ which is locally $\mathcal{C}_0$, i.e.

$$\gamma^{(\ell)} \in \mathcal{C}_0(E^\ell), \qquad \ell \in \mathbb{N}.$$

The proof of Theorem 2.2.8 will be given by a series of lemmas. By the analysis of section 2.1, we know already that under Assumption 2.2.1 the invariant measure m is absolutely continuous, with density γ given by the series representation (2.1.1) which in restriction to the layer E^ℓ reads

$$\gamma^{(\ell)}(\cdot) = \sum_{n=0}^{\infty} \boldsymbol{E}_\Delta\left[\mathbf{1}_{\boldsymbol{\tau}_n<R}\cdot r_\alpha^{(\ell)}(\boldsymbol{\eta}_{\boldsymbol{\tau}_n};\cdot)\right] \qquad \boldsymbol{\lambda}\text{-a.s.} \tag{2.2.20}$$

Our first task is to show that each term in the above series is in $\mathcal{C}_0(E^\ell)$ for all $\ell \in \mathbb{N}$. In a second step, we then have to show uniform convergence of the series on each layer E^ℓ in order to extend the $\mathcal{C}_0$-property to the limit.

We begin by rewriting the series representation in terms of the quantities given in Assumptions 1.1.2-1.1.5. Returning to the form (1.2.21) of the series representation, in order to account for the indicator $\mathbf{1}_{\boldsymbol{\tau}_n<R}$ we modify the jump kernel K of (1.1.24) to make the void configuration Δ an absorbing state: Define

$$K_0(\boldsymbol{x};\cdot) := \begin{cases} K(\boldsymbol{x};\cdot) & \text{if } \boldsymbol{x} \in \mathcal{S}\smallsetminus\{\Delta\},\\ \delta_\Delta(\cdot) & \text{if } \boldsymbol{x} = \Delta.\end{cases} \tag{2.2.21}$$

Then it is easily checked that for sets $F \in \mathcal{B}_{\mathcal{S}\smallsetminus\{\Delta\}}$ we have

$$\boldsymbol{E}_\Delta\left[\mathbf{1}_{\boldsymbol{\tau}_n<R}\cdot R_\alpha(\boldsymbol{\eta}_{\boldsymbol{\tau}_n};F)\right] = \left(R_\alpha^*[\alpha K_0]^*\right)^{n-1} R_\alpha^*\,\nu^{(0)}(F), \qquad n \in \mathbb{N}.$$

Since $\boldsymbol{E}_\Delta\left[\mathbf{1}_{\boldsymbol{\tau}_0<R}\cdot R_\alpha(\boldsymbol{\eta}_{\boldsymbol{\tau}_0};F)\right] = R_\alpha(\Delta;F) = 0$ for $F\in\mathcal{B}_{\mathcal{S}\smallsetminus\{\Delta\}}$, we obtain thus for the invariant measure in restriction to $\mathcal{S}\smallsetminus\{\Delta\}$ the representation[6]

$$m(F) = \sum_{n=0}^{\infty}\left(R_\alpha^*[\alpha K_0]^*\right)^n R_\alpha^*\,\nu^{(0)}(F), \qquad F\in\mathcal{B}_{\mathcal{S}\smallsetminus\{\Delta\}}. \tag{2.2.22}$$

Here R_α and K_0 are defined completely in terms of the quantities introduced in Assumptions 1.1.2-1.1.5.

Now observe that under Assumption 2.2.5, we start from an "initial condition" $\nu^{(0)}(dv) = \pi^{(0)}(v)dv$ with $\pi^{(0)}\in\mathcal{C}_0(E)$. Rewriting (2.2.22) in terms of the invariant density γ, we obtain the representation

$$\gamma = \sum_{n=0}^{\infty}\left(R_\alpha^*[\alpha K_0]^*\right)^n R_\alpha^*\,\pi^{(0)} = \sum_{n=0}^{\infty}\gamma_n \qquad \boldsymbol{\lambda}\text{-a.s.} \tag{2.2.23}$$

with

$$\gamma_n := \left(R_\alpha^*[\alpha K_0]^*\right)^n R_\alpha^*\,\pi^{(0)}, \qquad n\in\mathbb{N}_0, \tag{2.2.24}$$

and our task is to show that the $\mathcal{C}_0$-property is preserved by the action of the kernels R_α and $[\alpha K_0]$ in (2.2.23).

Our first observation is that under Assumption 2.2.5, the operator $[\alpha K_0]^*$ preserves nonnegative measurable functions resp. absolutely continuous measures. We need the following notation: For each $\ell\in\mathbb{N}$, $k\in\{0,1,\dots,\ell\}$ and $j\in\{1,\dots,\ell-k+1\}$ define a mapping

$$\Pi^*_{\ell,k,j}: E^\ell\times E\to E^{\ell-k+1}, \qquad \Pi^*_{\ell,k,j}(\boldsymbol{y};v) := (y^1,\dots,y^{j-1},v,y^{j+k},\dots,y^\ell) \tag{2.2.25}$$

which, given an ℓ-particle configuration $\boldsymbol{y} = (y^1,\dots,y^\ell)\in E^\ell$ and $v\in E$, replaces the k positions $(y^j,\dots,y^{j+k-1})$ with the position $v\in E$.

2.2.9 Lemma

Grant Assumption 2.2.5. Then for every nonnegative measurable $g:\mathcal{S}\to\mathbb{R}_+$, the measure $[\alpha K_0]^*g$ is $\boldsymbol{\lambda}$-absolutely continuous. The density is given on each layer by

$$\begin{aligned}
&\Big([\alpha K_0]^*g\Big)^{(\ell)}(\boldsymbol{y})\\
&= \sum_{k=0}^{\ell}\sum_{j=1}^{\ell-k+1}\int_E dv\, g^{(\ell-k+1)}\left(\Pi^*_{\ell,k,j}(\boldsymbol{y};v)\right)\cdot\kappa^{(\ell-k+1)}\left(v;\Pi^*_{\ell,k,j}(\boldsymbol{y};v)\right)\cdot\\
&\qquad\cdot p_k^{(\ell-k+1)}\left(v;\Pi^*_{\ell,k,j}(\boldsymbol{y};v)\right)\cdot q_k^{(\ell-k+1)}\left(v;\Pi^*_{\ell,k,j}(\boldsymbol{y};v);\,(y^j,\dots,y^{j+k-1})\right)\\
&\quad+\mathbf{1}_{\ell\ge 2}\cdot g^{(\ell-1)}(y^1,\dots,y^{\ell-1})\,c^{(\ell-1)}(y^1,\dots,y^{\ell-1})\,\pi^{(\ell-1)}\left((y^1,\dots,y^{\ell-1});\,y^\ell\right)
\end{aligned}\tag{2.2.26}$$

for $\boldsymbol{y} = (y^1,\dots,y^\ell)\in E^\ell$, $\ell\in\mathbb{N}$.[7] Moreover, $[\alpha K_0]^*$ preserves the space $L^1_{(loc)}(\mathcal{S})$.

Proof Let $g:\mathcal{S}\to\mathbb{R}_+$ measurable. The action of $[\alpha K_0]^*$ on a measure μ on $\mathcal{S}$ is "almost" the same as that of $[\alpha K]^*$ described in Lemma 1.1.10, the only difference being that we have to add an indicator $\mathbf{1}_{\ell\ge 2}$ in front of the last term in (1.1.43) since we cannot escape from the

[6] Cf. [Höp2004], eqn. (35). For the representation (2.2.22), we of course do not need Assumption 2.2.1 or 2.2.5.

[7] Concerning the term with $k = 0$ in (2.2.26), we remind the reader of our convention (2.2.14).

void configuration by an immigration event. With this modification, by (1.1.43) applied to $\mu(d\boldsymbol{x}) := g(\boldsymbol{x})d\boldsymbol{x}$ we obtain

$$\begin{aligned}\Big\langle\big([\alpha K_0]^* g\big)^{(\ell)}, h\Big\rangle = \sum_{k=0}^{\ell}\sum_{j=1}^{\ell-k+1}\int_{\boldsymbol{x}\in E^{\ell-k+1}} d\boldsymbol{x}\, g^{(\ell-k+1)}(\boldsymbol{x})\,\kappa^{(\ell-k+1)}(x^j;\boldsymbol{x})\,p_k^{(\ell-k+1)}(x^j;\boldsymbol{x})\\ \int_{\boldsymbol{v}\in E^k} d\boldsymbol{v}\, q_k^{(\ell-k+1)}(x^j;\boldsymbol{x};\boldsymbol{v})\,h\left(\Pi_{\ell-k+1,k,j}(\boldsymbol{x};\boldsymbol{v})\right)\\ + \mathbf{1}_{\ell\geq 2}\int_{\boldsymbol{x}\in E^{\ell-1}} d\boldsymbol{x}\, g^{(\ell-1)}(\boldsymbol{x})\,c^{(\ell-1)}(\boldsymbol{x})\int_E dv\,\pi^{(\ell-1)}(\boldsymbol{x};v)\,h(\boldsymbol{x},v)\end{aligned} \tag{2.2.27}$$

for each nonnegative measurable $h : E^\ell \to \mathbb{R}_+$, $\ell \in \mathbb{N}$. (Of course, for $g(\boldsymbol{x})d\boldsymbol{x}$ replaced by $\mu(d\boldsymbol{x})$ this formula describes also the action of $[\alpha K_0]^*$ on a general, not necessarily absolutely continuous measure μ on $\mathcal{S}$.)

Now it is easily seen by renaming the integration variables in the last display and Fubini's theorem that (2.2.26) holds: Fix $k \in \{0,\dots,\ell\}$ and $j \in \{\ell-k+1\}$ and consider the corresponding integral in the (double) sum in (2.2.27): Given $\boldsymbol{x} = (x^1,\dots,x^{\ell-k+1}) \in E^{\ell-k+1}$ and $\boldsymbol{v} = (v^1,\dots,v^k) \in E^k$ (this is to be interpreted as $\boldsymbol{v} = \Delta$ in case $k = 0$, of course), we put

$$\boldsymbol{y} := \Pi_{\ell-k+1,k,j}(\boldsymbol{x};\boldsymbol{v}) \in E^\ell.$$

Note that since this means

$$y^i := \begin{cases} x^i & \text{for } i = 1,\dots,j-1,\\ v^{i-j+1} & \text{for } i = j,\dots,j+k-1,\\ x^{i-k+1} & \text{for } i = j+k,\dots,\ell,\end{cases}$$

in view of the definition (2.2.25) we have

$$\boldsymbol{x} = (y^1,\dots,y^{j-1},v,y^{j+k},\dots,y^\ell) = \Pi^*_{\ell,k,j}(\boldsymbol{y};v),$$

$$\boldsymbol{v} = (y^j,\dots,y^{j+k-1}).$$

Letting in addition

$$v := x^j,$$

we can rewrite the integral in the sum (2.2.27) corresponding to our fixed (k,j) as

$$\begin{aligned}\int_{\boldsymbol{y}\in E^\ell} d\boldsymbol{y}\,h(\boldsymbol{y})\int_E dv\, g^{(\ell-k+1)}\left(\Pi^*_{\ell,k,j}(\boldsymbol{y};v)\right)\kappa^{(\ell-k+1)}\left(v;\Pi^*_{\ell,k,j}(\boldsymbol{y};v)\right)p_k^{(\ell-k+1)}\left(v;\Pi^*_{\ell,k,j}(\boldsymbol{y};v)\right)\cdot\\ \cdot\, q_k^{(\ell-k+1)}\left(v;\Pi^*_{\ell,k,j}(\boldsymbol{y};v);\,(y^j,\dots,y^{j-k+1})\right).\end{aligned}$$

The last term in (2.2.27), coming from the immigration, is accounted for in a similar way. Thus (2.2.26) is proved.

Finally, by taking $h \equiv 1$ on E^ℓ in (2.2.27) together with the bounds for κ, c, q and π from Assumption 2.2.5 we see that if $0 \leq g \in L^1_{(loc)}(\mathcal{S})$, the same is true of $[\alpha K_0]^* g$. Consequently, $[\alpha K_0]^*$ can be extended to an operator on all of $L^1_{(loc)}(\mathcal{S})$ preserving this space. ■

2.2.10 Remark
Note that in the proof of Lemma 2.2.9, we did not use any continuity properties of the densities $q_k^{(\ell)}$ and $\pi^{(\ell)}$. Moreover, the kernel K_0 can be replaced with the original jump kernel K of (1.1.24), causing the indicator $\mathbf{1}_{\ell\geq 2}$ in front of the last term in (2.2.26) to disappear. Then the above arguments show that whenever the kernels $Q_k^{(\ell)}$ and $\nu^{(\ell)}$ are $\boldsymbol{\lambda}$-absolutely continuous (i.e. (2.2.12) and (2.2.16) hold), the kernel $[\alpha K]$ preserves absolutely continuous measures. If also the kernel R_α is absolutely continuous (as under Assumption 2.2.1), this implies that the distribution of $\boldsymbol{\eta}$ at a branching or immigration event is absolutely continuous, since

$$\mathcal{L}(\boldsymbol{\eta}_{\boldsymbol{\tau}_n} \,|\, \boldsymbol{P}_\Delta) = ([\alpha K]^* R_\alpha^*)^{n-1}\, \pi^{(0)}, \qquad n \in \mathbb{N}$$

(see (1.1.40)). This is in sharp contrast to the case (1.1.18) that branching particles reproduce at their death position. ♦

We return to the series representation (2.2.23) of the invariant density. Our goal is to show that $\gamma_n^{(\ell)} \in \mathcal{C}_0(E^\ell)$ for all $n, \ell \in \mathbb{N}$ and that the series (2.2.23) converges uniformly on each layer E^ℓ. In this approach, we encounter the following problem: Under Assumption 2.2.5, we start from an absolutely continuous "initial condition" $\pi^{(0)} \in L^1_{(loc)}(\mathcal{S}) \cap \mathcal{C}_0^{(loc)}(\mathcal{S})$. While we have seen in Lemma 2.2.9 that absolute continuity is preserved by the operator $[\alpha K_0]^*$, it is not clear that the same is true of the $\mathcal{C}_0$-property: In fact, $[\alpha K_0]^* g$ will generally *not* be in $\mathcal{C}_0^{(loc)}(\mathcal{S})$ for arbitrary $g \in L^1_{(loc)}(\mathcal{S}) \cap \mathcal{C}_0^{(loc)}(\mathcal{S})$. For example, assume that κ and p_k are spatially constant and consider some $g = \big(g^{(\ell)}\big)_{\ell\in\mathbb{N}_0} \in L^1_{(loc)}(\mathcal{S}) \cap \mathcal{C}_0^{(loc)}(\mathcal{S})$. Then $[\alpha K_0]^* g \in L^1_{(loc)}(\mathcal{S})$ by Lemma 2.2.9, but considering the term corresponding to $k = 0$ in the sum in (2.2.26), we see that for each $\ell \in \mathbb{N}$ the density $([\alpha K_0]^* g)^{(\ell)}$ on the layer E^ℓ contains expressions of the form

$$\kappa p_0 \int_E dv\, g^{(\ell+1)}(y^1, \ldots, y^{j-1}, v, y^j, \ldots, y^{\ell+1}) \tag{2.2.28}$$

for $j = 1, \ldots, \ell+1$. But the expression in (2.2.28), considered as a function of $(y^1, \ldots, y^\ell) \in E^\ell$, will in general be neither continuous nor bounded even though $g^{(\ell+1)} \in \mathcal{C}_0(E^{\ell+1})$. Moreover, even if $g \in L^1_{(loc)}(\mathcal{S}) \cap \mathcal{C}_0^{(loc)}(\mathcal{S})$ is such that (2.2.28) happens to be a $\mathcal{C}_0$-function of $(y^1, \ldots, y^\ell)$, there are no general estimates on its $\|\cdot\|_\infty$-norm in terms of $\|g^{(\ell+1)}\|_\infty$, creating a further problem in view of our goal to ensure uniform convergence of the series (2.2.23) on E^ℓ.

In order to solve the above-mentioned problems, we have to introduce a suitable subspace of $L^1_{(loc)}(\mathcal{S}) \cap \mathcal{C}_0^{(loc)}(\mathcal{S})$ which is preserved by the operator $[\alpha K_0]^*$. We introduce some notation:

2.2.11 Notations
Let $\ell \in \mathbb{N}$ and $J \subseteq \{1, \ldots, \ell\}$. Given $\boldsymbol{x} = (x^1, \ldots, x^\ell) \in E^\ell$, we write $\boldsymbol{x}^J$ for the "subconfiguration" of particles with index in J:

$$\boldsymbol{x}^J = (x^j : j \in J) \in E^J,$$

with the understanding that $\boldsymbol{x}^\varnothing = \Delta$. For each such subset J, we will also write $\boldsymbol{x}^{J^c}$ instead of $\boldsymbol{x}^{\{1,\ldots,\ell\}\smallsetminus J}$, the complement being always understood with respect to $\{1, \ldots, \ell\}$.

Conversely, the generic element of E^J will be denoted $\boldsymbol{x}^J$, and given $\boldsymbol{x}^J = (x^j : j \in J) \in E^J$ and $\boldsymbol{x}^{J^c} = (x^j : j \in J^c) \in E^{J^c}$, the configuration in E^ℓ obtained by "putting them together" is denoted by $\boldsymbol{x} = (x^j : j \in \{1, \ldots, \ell\})$.

2.2.12 Definition
Given a function $f = (f^{(\ell)})_{\ell\in\mathbb{N}_0} : \mathcal{S} \to \mathbb{R}$ which is in $L^1_{(loc)}(\mathcal{S})$ or nonnegative, we define for each $\ell \in \mathbb{N}$ and $J \subseteq \{1, \dots, \ell\}$ a function $\mathcal{I}^J_\ell f$ on E^J which returns the integral of $f^{(\ell)}$ w.r.t. to the $\boldsymbol{x}^{J^c}$-variables as a function of $\boldsymbol{x}^J$:

$$\mathcal{I}^J_\ell f : E^J \to \mathbb{R}, \qquad \boldsymbol{x}^J \mapsto \int_{E^{J^c}} d\boldsymbol{x}^{J^c}\, f^{(\ell)}(\boldsymbol{x}), \tag{2.2.29}$$

with the understanding that $\mathcal{I}^J_\ell f := f^{(\ell)}$ for $J = \{1, \dots, \ell\}$. (For $J = \varnothing$, $\mathcal{I}^J_\ell f$ is just the constant $\int_{E^\ell} f^{(\ell)}(\boldsymbol{x})\, d\boldsymbol{x}$.)

We denote by $\mathscr{U}$ the subspace of locally integrable functions on $\mathcal{S}$ such that for all $\ell \in \mathbb{N}$ and nonempty $J \subseteq \{1, \dots, \ell\}$, the function $\mathcal{I}^J_\ell f$ is $\mathcal{C}_0$ on E^J:

$$\mathscr{U} := \left\{ f \in L^1_{(loc)}(\mathcal{S}) : \forall\, \ell \in \mathbb{N}, \varnothing \neq J \subseteq \{1, \dots, \ell\} : \mathcal{I}^J_\ell f \in \mathcal{C}_0(E^J) \right\}. \tag{2.2.30}$$

2.2.13 Remarks

- For all $f \in L^1_{(loc)}(\mathcal{S})$, we have of course $\mathcal{I}^J_\ell f \in L^1(E^J)$ for all $\ell \in \mathbb{N}$, $\varnothing \neq J \subseteq \{1, \dots, \ell\}$ by Fubini-Tonelli's theorem, but there is no reason for $\mathcal{I}^J_\ell f$ to be continuous or bounded, even if $f^{(\ell)} \in \mathcal{C}_0(E^\ell)$. The space $\mathscr{U}$ is introduced precisely to force this property of $\mathcal{I}^J_\ell f$.
- For $f \in L^1_{(loc)}(\mathcal{S})$, strictly speaking (2.2.29) defines $\mathcal{I}^J_\ell f$ only as an equivalence class in $L^1(E^J)$, of course; the condition in (2.2.30) is to be interpreted in the obvious way that there be some version of $\mathcal{I}^J_\ell f$ (i.e. an element in the equivalence class) which is a function in $\mathcal{C}_0(E^J)$.
- Note that in the definition of $\mathscr{U}$, we admit $J = \{1, \dots, \ell\}$. Thus if $f \in \mathscr{U}$, we have $f^{(\ell)} = \mathcal{I}^{\{1,\dots,\ell\}}_\ell f \in \mathcal{C}_0(E^\ell)$ for all ℓ, in particular

$$\mathscr{U} \subseteq L^1_{(loc)}(\mathcal{S}) \cap \mathcal{C}^{(loc)}_0(\mathcal{S}).$$

♦

Remember that our goal is to show that action of the kernels R_α and $[\alpha K_0]$ in the series (2.2.23) preserves the $\mathcal{C}_0$-property of the "initial condition" $\pi^{(0)}$. In this regard, the next lemma is crucial. For its proof, we need a suitable "continuity lemma" for integrals depending on a parameter. In the textbook literature, such lemmas are typically presented with conditions under which they are simple consequences of Lebesgue's dominated convergence theorem; however, for our purposes these "standard versions" will not do. Instead, we will use a simple reformulation of the version of dominated convergence sometimes referred to as Pratt's theorem (see e.g. [Els2009], Thm. VI.5.1). Roughly speaking, in Pratt's theorem the existence of a single dominating integrable function assumed in Lebesgue's theorem is weakened to the existence of a sequence of dominating functions whose integrals are known to converge. For the convenience of the reader, we state and prove the corresponding "continuity lemma" in the form we need it in the appendix, along with a "differentiation lemma" which will be needed later-on.

We recall the definition of the kernel $R_{\alpha,\varepsilon}$ in (2.2.9) and the identity (2.2.10). Further, let

$$\widehat{p}^{(\ell)}_t(\boldsymbol{x}) := C_\ell \cdot t^{-d\ell/2} \exp\left(-\frac{\|\boldsymbol{x}\|^2}{2C_\ell t}\right), \qquad \boldsymbol{x} \in E^\ell,\ \ell \in \mathbb{N},\ t > 0 \tag{2.2.31}$$

denote the upper bound from the heat kernel estimate (2.2.2).

2.2.14 Lemma
Under Assumptions 2.2.1 and 2.2.5, the following holds:

- *The operators* $R^*_{\alpha,\varepsilon}$ *and* $[\alpha K_0]^*$ *preserve the subspace* $\mathscr{U}$ *of* (2.2.30).
- *The operator* $(P^\alpha_\varepsilon)^*$ *maps* $L^1_{(loc)}(\mathcal{S})$ *into* $\mathscr{U}$.
- *If* $f \in \mathscr{U}$ *is nonnegative and such that* $R^*_\alpha f \in L^1_{(loc)}(\mathcal{S})$, *then* $R^*_\alpha f \in \mathscr{U}$.[8]

Proof We know already that the operators $(P^\alpha_\varepsilon)^*$, $R^*_{\alpha,\varepsilon}$ and $[\alpha K_0]^*$ preserve the space $L^1_{(loc)}(\mathcal{S})$ of locally integrable functions (see the second remark in 2.2.4 and Lemma 2.2.9). Thus in order to prove the assertions of the lemma, we only have to check the continuity condition in the Definition of $\mathscr{U}$ in (2.2.30).

Throughout the proof, we may and do assume without loss of generality that $f \geq 0$. We fix $\ell \in \mathbb{N}$ and some subset $\varnothing \neq J \subseteq \{1, \ldots, \ell\}$.

Consider first the operator $R^*_{\alpha,\varepsilon}$: For nonnegative $f \in \mathscr{U}$, by the heat kernel estimate (2.2.2) we have for each $\boldsymbol{x}^J \in E^J$

$$\begin{aligned} \mathcal{I}^J_\ell(R^*_{\alpha,\varepsilon} f)(\boldsymbol{x}^J) &\equiv \int_{E^{J^c}} d\boldsymbol{x}^{J^c}\, (R^*_{\alpha,\varepsilon} f)^{(\ell)}(\boldsymbol{x}) \\ &= \int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_{E^\ell} d\boldsymbol{y} \int_0^\varepsilon dt\, p^\alpha_t(\boldsymbol{y};\boldsymbol{x}) f^{(\ell)}(\boldsymbol{y}) \\ &\leq \int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_{E^\ell} d\boldsymbol{y} \int_0^\varepsilon dt\, \widehat{p}^{(\ell)}_t(\boldsymbol{y}-\boldsymbol{x}) f^{(\ell)}(\boldsymbol{y}). \end{aligned} \tag{2.2.32}$$

By Assumption 2.2.1, the integrand in the second line of (2.2.32)

$$\boldsymbol{x}^J \mapsto p^\alpha_t(\boldsymbol{y};\boldsymbol{x}) f^{(\ell)}(\boldsymbol{y})$$

is continuous on E^J for all fixed $(\boldsymbol{x}^{J^c}, \boldsymbol{y}, t) \in E^{J^c} \times E^\ell \times \mathbb{R}_+$. The same is true for the upper bound

$$\boldsymbol{x}^J \mapsto \widehat{p}^{(\ell)}_t(\boldsymbol{y}-\boldsymbol{x}) f^{(\ell)}(\boldsymbol{y}).$$

To prove continuity of $\mathcal{I}^J_\ell(R^*_{\alpha,\varepsilon} f)$, in view of the "Continuity Lemma" A.1 (see the appendix) it remains to show that the integral in the last line of (2.2.32) is continuous as a function of $\boldsymbol{x}^J$. We observe that it is equal to

$$\begin{aligned} &\int_{E^\ell} d\boldsymbol{y}\, f^{(\ell)}(\boldsymbol{y}) \int_0^\varepsilon dt \int_{E^{J^c}} d\boldsymbol{x}^{J^c}\, \widehat{p}^{(\ell)}_t(\boldsymbol{x}-\boldsymbol{y}) \\ &= C_\ell \int_{E^\ell} d\boldsymbol{y}\, f^{(\ell)}(\boldsymbol{y}) \int_0^\varepsilon dt\, t^{-d|J|/2} \exp\left(-\frac{\|\boldsymbol{x}^J - \boldsymbol{y}^J\|^2}{2C_\ell t}\right) \underbrace{\int_{E^{J^c}} d\boldsymbol{x}^{J^c}\, t^{-d|J^c|/2} \exp\left(-\frac{\|\boldsymbol{x}^{J^c} - \boldsymbol{y}^{J^c}\|^2}{2C_\ell t}\right)}_{=(2\pi C_\ell)^{d|J^c|/2}} \\ &= C'_{\ell,J} \int_{E^\ell} d\boldsymbol{y}\, f^{(\ell)}(\boldsymbol{y}) \int_0^\varepsilon dt\, t^{-d|J|/2} \exp\left(-\frac{\|\boldsymbol{x}^J - \boldsymbol{y}^J\|^2}{2C_\ell t}\right) \\ &= C'_{\ell,J} \int_{E^J} d\boldsymbol{y}^J\, \tilde{r}^{(\ell,J)}(\boldsymbol{x}^J - \boldsymbol{y}^J) \int_{E^{J^c}} d\boldsymbol{y}^{J^c}\, f^{(\ell)}(\boldsymbol{y}) \\ &= C'_{\ell,J} \cdot \left(\hat{r}^{(\ell,J)}_\varepsilon * \mathcal{I}^J_\ell f\right)(\boldsymbol{x}^J), \end{aligned} \tag{2.2.33}$$

[8] The restriction to nonnegative f is necessary since $R^*_\alpha f$ need not be well-defined for general $f \in L^1_{(loc)}(\mathcal{S})$.

where we have written $C'_{\ell,J} := C_\ell \,(2\pi C_\ell)^{d|J^c|/2}$ and

$$\hat{r}_\varepsilon^{(\ell,J)}(\boldsymbol{y}^J) := \int_0^\varepsilon dt\, t^{-d|J|/2} \exp\left(-\frac{\|\boldsymbol{y}^J\|^2}{2C_\ell t}\right), \qquad \boldsymbol{y}^J \in E^J. \tag{2.2.34}$$

We have $\hat{r}_\varepsilon^{(\ell,J)} \in L^1(E^J)$, and also $\mathcal{I}_\ell^J f \in \mathcal{C}_0(E^J)$ since $f \in \mathscr{U}$. As a convolution of an integrable function with a $\mathcal{C}_0$-function, (2.2.33) is thus a $\mathcal{C}_0$-function of $\boldsymbol{x}^J \in E^J$. Consequently, Lemma A.1 applies and gives continuity of $\mathcal{I}_\ell^J(R^*_{\alpha,\varepsilon} f)$ at each point $\boldsymbol{x}^J \in E^J$, and it follows from the estimate (2.2.32) that $\mathcal{I}_\ell^J(R^*_{\alpha,\varepsilon} f)$ is vanishing at infinity.

Next, we show that $\mathcal{I}_\ell^J((P_\varepsilon^\alpha)^* f) \in \mathcal{C}_0(E^J)$ for arbitrary nonnegative $f \in L^1_{(loc)}(\mathcal{S})$. The argument is essentially the same as before: Instead of (2.2.32), we have

$$\begin{aligned} \mathcal{I}_\ell^J((P_\varepsilon^\alpha)^* f)(\boldsymbol{x}^J) &\equiv \int_{E^{J^c}} d\boldsymbol{x}^{J^c}\, ((P_\varepsilon^\alpha)^* f)^{(\ell)}(\boldsymbol{x}) \\ &= \int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_{E^\ell} d\boldsymbol{y}\, p_\varepsilon^\alpha(\boldsymbol{y};\boldsymbol{x}) f^{(\ell)}(\boldsymbol{y}) \\ &\le \int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_{E^\ell} d\boldsymbol{y}\, \widehat{p}_\varepsilon^{(\ell)}(\boldsymbol{y}-\boldsymbol{x}) f^{(\ell)}(\boldsymbol{y}). \end{aligned} \tag{2.2.35}$$

As before, the integrands in the above display, as a function of $\boldsymbol{x}^J \in E^J$, are continuous for each fixed $(\boldsymbol{x}^{J^c}, \boldsymbol{y}) \in E^{J^c} \times E^\ell$, and integrating the upper bound gives

$$\int_{E^\ell} d\boldsymbol{y}\, f^{(\ell)}(\boldsymbol{y}) \int_{E^{J^c}} d\boldsymbol{x}^{J^c}\, \widehat{p}_t^{(\ell)}(\boldsymbol{x}-\boldsymbol{y}) = C''_{\ell,J} \cdot \left(\hat{p}_\varepsilon^{(\ell,J)} * \mathcal{I}_\ell^J f\right)(\boldsymbol{x}^J) \tag{2.2.36}$$

with some constant $C''_{\ell,J}$ and $\hat{p}_\varepsilon^{(\ell,J)}(\boldsymbol{y}^J) := \varepsilon^{-d|J|/2} \exp\left(-\frac{\|\boldsymbol{y}^J\|^2}{2C_\ell \varepsilon}\right)$, $\boldsymbol{y}^J \in E^J$. Since $\hat{p}_\varepsilon^{(\ell,J)} \in \mathcal{C}_0(E^J)$ (it is even a Schwartz function, of course) we now need only integrability of $\mathcal{I}_\ell^J f$ in order to conclude that the convolution in (2.2.36) is in $\mathcal{C}_0(E^J)$.

We now turn to the operator $[\alpha K_0]^*$. Using (2.2.26), we get

$$\begin{aligned} &\mathcal{I}_\ell^J\left([\alpha K_0]^* f\right)(\boldsymbol{x}^J) \\ &= \sum_{k=0}^{\ell} \sum_{j=1}^{\ell-k+1} \int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_E dv\, f^{(\ell-k+1)}(\Pi^*_{\ell,k,j}(\boldsymbol{x};v))\, \kappa^{(\ell-k+1)}(v;\Pi^*_{\ell,k,j}(\boldsymbol{x};v)) \cdot \\ &\qquad \cdot p_k^{(\ell-k+1)}(v;\Pi^*_{\ell,k,j}(\boldsymbol{x};v))\, q_k^{(\ell-k+1)}\left(v;\Pi^*_{\ell,k,j}(\boldsymbol{x};v);(x^j,\dots,x^{j+k-1})\right) \\ &\quad + \mathbf{1}_{\ell\ge 2} \int_{E^{J^c}} d\boldsymbol{x}^{J^c}\, f^{(\ell-1)}(x^1,\dots,x^{\ell-1})\, c^{(\ell-1)}(x^1,\dots,x^{\ell-1})\, \pi^{(\ell-1)}(x^1,\dots,x^{\ell-1};x^\ell) \end{aligned} \tag{2.2.37}$$

with $\Pi^*_{\ell,k,j}(\boldsymbol{x};v)$ as in (2.2.25). By Assumption 2.2.5 and continuity of f, for each term in the above sum the integrand depends continuously on $\boldsymbol{x}^J \in E^J$, for every $(v, \boldsymbol{x}^{J^c}) \in E \times E^{J^c}$ fixed. It is precisely at this point that we use the requirement in Assumption 2.2.5 that $c(\cdot)$, $\pi(\cdot\,;\cdot)$ are continuous and that $\kappa(v\,;\cdot)$, $p_k(v\,;\cdot)$, $q(v\,;\cdot\,;\cdot)$ are continuous for every fixed $v \in E$.

Now fix some $k \in \{0,\dots,\ell\}$ and $j \in \{1,\dots,\ell-k+1\}$. For the corresponding term in the large sum in (2.2.37), we have by Assumption 2.2.5 the upper bound

$$\boldsymbol{x}^J \mapsto \|\kappa^{(\ell-k+1)}\|_\infty \int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_E dv\, f^{(\ell-k+1)}\left(\Pi^*_{\ell,k,j}(\boldsymbol{x};v)\right) \prod_{i=j}^{j+k-1} \widehat{q}_k^{(\ell-k+1)}(v - x^i). \tag{2.2.38}$$

Obviously, the integrand in the above display does also continuously depend on $\boldsymbol{x}^J \in E^J$, for every $(v, \boldsymbol{x}^{J^c}) \in E \times E^{J^c}$ fixed. In view of the "Continuity Lemma" A.1, to conclude continuity in $\boldsymbol{x}^J$ of the corresponding term in (2.2.37) it remains therefore only to show that the integral in (2.2.38) is continuous as a function of $\boldsymbol{x}^J$.

In order to show this, decompose $J = J_1 \uplus J_2$, $J^c = J_1^c \uplus J_2^c$, where

$$J_1 := J \cap \{1, \dots, j-1, j+k, \dots, \ell\}, \qquad J_2 := J \cap \{j, \dots, j+k-1\},$$
$$J_1^c := J^c \cap \{1, \dots, j-1, j+k, \dots, \ell\}, \qquad J_2^c := J^c \cap \{j, \dots, j+k-1\}.$$

Observing that $f^{(\ell-k+1)}\left(\Pi^*_{\ell,k,j}(\boldsymbol{x}; v)\right)$ does not depend on the variables $x^j, \dots, x^{j+k-1}$ (in particular, it is independent of $\boldsymbol{x}^{J_2^c}$), we can change the order of integration in (2.2.38) to obtain

$$\begin{aligned}
&\int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_E dv\, f^{(\ell-k+1)}\left(\Pi^*_{\ell,k,j}(\boldsymbol{x}; v)\right) \prod_{i=j}^{j+k-1} \widehat{q}_k^{(\ell-k+1)}(v - x^i) \\
&= \int_E dv \int_{E^{J_1^c}} d\boldsymbol{x}^{J_1^c} \prod_{i \in J_2} \widehat{q}_k^{(\ell-k+1)}(v - x^i)\, f^{(\ell-k+1)}\left(\Pi^*_{\ell,k,j}(\boldsymbol{x}; v)\right) \underbrace{\int_{E^{J_2^c}} d\boldsymbol{x}^{J_2^c} \prod_{i \in J_2^c} \widehat{q}_k^{(\ell-k+1)}(v - x^i)} \\
&= \overbrace{\left\|\widehat{q}_k^{(\ell-k+1)}\right\|_1^{|J_2^c|}} \cdot \int_E dv \int_{E^{J_1^c}} d\boldsymbol{x}^{J_1^c} \prod_{i \in J_2} \widehat{q}_k^{(\ell-k+1)}(v - x^i) \cdot f^{(\ell-k+1)}\left(\Pi^*_{\ell,k,j}(\boldsymbol{x}; v)\right) \\
&\leq \left\|\widehat{q}_k^{(\ell-k+1)}\right\|_1^{|J_2^c|} \cdot \left\|\widehat{q}_k^{(\ell-k+1)}\right\|_\infty^{|J_2|} \cdot \int_E dv \int_{E^{J_1^c}} d\boldsymbol{x}^{J_1^c} f^{(\ell-k+1)}\left(\Pi^*_{\ell,k,j}(\boldsymbol{x}; v)\right).
\end{aligned} \tag{2.2.39}$$

Consider the last two lines of the above display: Again, in each case the integrand depends continuously on $\boldsymbol{x}^J$ since $f^{(\ell-k+1)}$ and $\widehat{q}_k^{(\ell-k+1)}$ are continuous. By another application of Lemma A.1, it remains therefore to show that integral in the last line of (2.2.39)

$$\int_E dv \int_{E^{J_1^c}} d\boldsymbol{x}^{J_1^c} f^{(\ell-k+1)}\left(x^1, \dots, x^{j-1}, v, x^{j+k}, \dots, x^\ell\right) \tag{2.2.40}$$

is a continuous function of $\boldsymbol{x}^J$. But this follows from the assumption that $f \in \mathscr{U}$; in fact, (2.2.40) is a $\mathcal{C}_0$-function of $\boldsymbol{x}^{J_1}$ and does not depend on $\boldsymbol{x}^{J_2}$. Thus we have proved that the term in (2.2.37) corresponding to our fixed (k, j) is continuous in $\boldsymbol{x}^J \in E^J$. It is also vanishing at infinity, as follows readily from the estimate (2.2.39) (for the $\boldsymbol{x}^{J_2}$-variables, use additionally the fact that $\widehat{q}_k^{(\ell-k+1)} \in \mathcal{C}_0(E)$ by Assumption 2.2.5).

We have proved that each of the terms in the large (double) sum in (2.2.37) is a $\mathcal{C}_0$-function of $\boldsymbol{x}^J \in E^J$. The last term, stemming from the immigration, is treated accordingly.

Finally, let $f \in \mathscr{U}$ nonnegative such that $R^*_\alpha f \in L^1_{(loc)}(\mathcal{S})$. Then $R^*_{\alpha,\varepsilon} f \in \mathscr{U}$ by step 1 and $(P^\alpha_\varepsilon)^* R^*_\alpha f \in \mathscr{U}$ by step 2, whence it follows from the identity (2.2.10) that also $R^*_\alpha f = R^*_{\alpha,\varepsilon} f + (P^\alpha_\varepsilon)^* R^*_\alpha f \in \mathscr{U}$. ■

2.2.15 Corollary

Grant Assumptions 2.2.1 and 2.2.5. Then each term $\gamma_n = \left(R^*_\alpha[\alpha K_0]^*\right)^n R^*_\alpha \pi^{(0)}$ in the series representation (2.2.23) of the invariant density belongs to the subspace $\mathscr{U}$:

$$\gamma_n(\cdot) \in \mathscr{U}, \qquad n \in \mathbb{N}.$$

In particular, each γ_n is in $\mathcal{C}_0^{(loc)}(\mathcal{S})$.

Proof First observe that we know a priori that $0 \le \gamma_n \le \gamma \in L^1_{(loc)}(\mathcal{S})$ for all $n \in \mathbb{N}_0$. Now the assertion follows readily from Lemma 2.2.14 by induction: For $n = 0$, the immigration density $\pi^{(0)}$, considered as a function on $\mathcal{S}$ which is concentrated on the layer E, clearly belongs to the subspace $\mathscr{U}$ by Assumption 2.2.5. Thus also $\gamma_0 = R^*_\alpha \pi^{(0)} \in \mathscr{U}$ by the third assertion in Lemma 2.2.14. If it is already known for some $n \in \mathbb{N}_0$ that $\gamma_n \in \mathscr{U}$, by the first assertion of Lemma 2.2.14 we get $[\alpha K_0]^* \gamma_n \in \mathscr{U}$, and since $R^*_\alpha([\alpha K_0]^*\gamma_n) = \gamma_{n+1} \in L^1_{(loc)}(\mathcal{S})$ the last assertion of Lemma 2.2.14 implies that also $\gamma_{n+1} \in \mathscr{U}$. ∎

2.2.16 Remark

Note that in the proof of Lemma 2.2.14, we did not need the conditions (2.2.15) and (2.2.19) on the integrability of the Fourier transforms of $\widehat{q}^{(\ell)}_k$ and $\widehat{\pi}^{(\ell)}$. Thus also Corollary 2.2.15 remains valid without these conditions. Moreover, in all arguments in the proof of Lemma 2.2.14 the kernel K_0 can be replaced by the original jump kernel K of (1.1.24), showing that all laws $\mathcal{L}(\boldsymbol{\eta}_{\boldsymbol{\tau}_n} \,|\, \boldsymbol{P}_\Delta)$, $n \in \mathbb{N}$, admit a density of class $\mathcal{C}_0$. ♦

In our program, it remains to prove (locally) uniform convergence of the series (2.2.23). Thus for each $\ell \in \mathbb{N}$ we have to control the uniform norm $\|\gamma^{(\ell)}_n\|_\infty$ as $n \to \infty$. Here we encounter the following problem: In view of Lemma 2.2.9, we can express $\gamma^{(\ell)}_{n+1} = (R^*_\alpha[\alpha K_0]^*\gamma_n)^{(\ell)}$ "explicitly" in terms of $\gamma^{(k)}_n$ on the layers $k = 0, \dots, \ell{+}1$. However, due to the presence of terms like (2.2.28) in $[\alpha K_0]^*\gamma_n$ it is not at all clear how to estimate $\|\gamma^{(\ell)}_{n+1}\|_\infty$ in terms of the uniform norms $\|\gamma^{(k)}_n\|_\infty$, $k = 0, \dots, \ell{+}1$. In fact, we will take a different approach and estimate $\|\gamma^{(\ell)}_{n+1}\|_\infty$ in terms not of the uniform norms but of the L^1-norms $\|\gamma_{n+1}\|_1, \|\gamma_n\|_1, \dots, \|\gamma_{n-k}\|_1$, where the number k will depend on a fixed layer ℓ but not on n. Since $\sum_n \|\gamma_n\|_1 = \|\gamma\|_1 = m(\mathcal{S}) < \infty$, this will imply $\sum_n \|\gamma^{(\ell)}_n\|_\infty < \infty$ for fixed ℓ and thus locally uniform convergence of (2.2.23).

We proceed with some notations and definitions:

2.2.17 Definition

Let $\ell \in \mathbb{N}$. With C_ℓ and ε as in the heat kernel estimate (2.2.2)*, we define $\widehat{p}^{(\ell)}_t(\cdot)$ as in* (2.2.31) *and*

$$\widehat{r}^{(\ell)}_\varepsilon(\boldsymbol{x}) := \int_0^\varepsilon \widehat{p}^{(\ell)}_t(\boldsymbol{x})\,dt = C_\ell \int_0^\varepsilon t^{-d\ell/2} \exp\left(-\frac{\|\boldsymbol{x}\|^2}{2C_\ell t}\right) dt, \qquad \boldsymbol{x} \in E^\ell. \tag{2.2.41}$$

As integrable functions, $\widehat{p}^{(\ell)}_t$ and $\widehat{r}^{(\ell)}_\varepsilon$ induce bounded convolution kernels resp. operators on $E^\ell \times \mathcal{B}_{E^\ell}$ which will be denoted by $\widehat{P}^{(\ell)}_t$ and $\widehat{R}^{(\ell)}_\varepsilon$, respectively:

$$\widehat{P}^{(\ell)}_t f(\boldsymbol{x}) := \int_{E^\ell} \widehat{p}_t(\boldsymbol{x}-\boldsymbol{y}) f^{(\ell)}(\boldsymbol{y}), \qquad \boldsymbol{x} \in E^\ell, \tag{2.2.42}$$

$$\widehat{R}^{(\ell)}_\varepsilon f(\boldsymbol{x}) := \int_{E^\ell} \widehat{r}_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) f^{(\ell)}(\boldsymbol{y}), \qquad \boldsymbol{x} \in E^\ell. \tag{2.2.43}$$

Omitting the superscript ℓ, we will also interpret the functions and kernels introduced above as objects defined on the configuration space $\mathcal{S}$ in the usual way.

Next, we define a kernel which is of the same form as $[\alpha K_0]$ but with all quantities replaced by their upper bounds from Assumption 2.2.5:

2.2.18 Definition

Under Assumption 2.2.5, put

$$D_\ell := \|\kappa^{(\ell)}\|_\infty \vee \|c^{(\ell)}\|_\infty \quad \text{for } \ell \in \mathbb{N}, \qquad D_0 := c(\Delta) \tag{2.2.44}$$

and define for each $\boldsymbol{x} \in \mathcal{S}$

$$\widehat{K}_0(\boldsymbol{x};\cdot) := D_{\ell(\boldsymbol{x})}\Bigg[\sum_{i=1}^{\ell(\boldsymbol{x})} \delta_{(x^1,\dots,x^{i-1},x^{i+1},\dots,x^{\ell})}(\cdot) + \sum_{k\in\mathbb{N}}\sum_{i=1}^{\ell(\boldsymbol{x})}\int_{E^k} dv^1\cdots dv^k \prod_{j=1}^{k}\widehat{q}_k^{(\ell)}(x^i - v^j)\,\delta_{(x^1,\dots,x^{i-1},v^1,\dots,v^k,x^{i+1},\dots,x^{\ell(\boldsymbol{x})})}(\cdot) + \mathbf{1}_{\boldsymbol{x}\neq\Delta}\int_E dv\,\widehat{\pi}^{(\ell)}(v)\,\delta_{(x^1,\dots,x^{\ell(\boldsymbol{x})},v)}(\cdot) + \mathbf{1}_{\boldsymbol{x}=\Delta}\cdot\delta_\Delta(\cdot)\Bigg]. \tag{2.2.45}$$

2.2.19 Remarks

- Due to their structural similarity, the action of the kernel $\widehat{K}_0$ on a measure μ on $(\mathcal{S},\mathcal{B}_\mathcal{S})$ is essentially the same as that of $[\alpha K_0]$ described in the proof of Lemma 2.2.9: Namely, in (2.2.27) we just have to replace all quantities by their upper bounds. In particular, $\widehat{K}_0$ preserves the space $\mathcal{M}_f^{(loc)}(\mathcal{S})$ of locally finite measures. It is also true that $\widehat{K}_0$ preserves absolute continuity, the density of $\widehat{K}_0^* g$ for a measurable nonnegative function $g:\mathcal{S}\to\mathbb{R}_+$ being given by an expression analogous to (2.2.26); however we will not need this fact in what follows.

- Under Assumptions 2.2.1 and 2.2.5, we have for all nonnegative measurable functions $f:\mathcal{S}\to\mathbb{R}_+$
$$R^*_{\alpha,\varepsilon} f \le \widehat{R}_\varepsilon f, \qquad (P^\alpha_\varepsilon)^* f \le \widehat{P}_\varepsilon f, \tag{2.2.46}$$
$$[\alpha K_0]^* f \le \widehat{K}_0 f, \tag{2.2.47}$$
and in view of the identity (2.2.10) also
$$R^*_\alpha f = R^*_{\alpha,\varepsilon} f + (P^\alpha_\varepsilon)^* R^*_\alpha f \le \widehat{R}_\varepsilon f + \widehat{P}_\varepsilon R^*_\alpha f.$$

♦

In order to control the uniform norms of $\gamma_n^{(\ell)} = ((R^*_\alpha[\alpha K_0]^*)^n R^*_\alpha \pi^{(0)})^{(\ell)}$, we will use the estimates (2.2.46) and (2.2.47) and then control the uniform norms of the upper bounds by a Fourier inversion argument. Thus we will need the Fourier transforms of $\widehat{r}_\varepsilon^{(\ell)}$ and $\widehat{p}_\varepsilon^{(\ell)}$ on each layer, and we will need to know how the Fourier transformation acts on the operator $\widehat{K}_0$. The following lemmas deal with these questions.

The first observation is that as the Fourier transforms of radially symmetric integrable functions, $\mathscr{F}[\widehat{r}_\varepsilon^{(\ell)}]$ and $\mathscr{F}[\widehat{p}_\varepsilon^{(\ell)}]$ are radially symmetric $\mathcal{C}_0$-functions on E^ℓ. Moreover, an elementary calculation yields that no matter what the dimension $d\ell$, $\mathscr{F}[\widehat{r}_\varepsilon^{(\ell)}]$ decays as $\|\boldsymbol{\xi}\|^{-2}$ at infinity. More precisely, define a function $h:\mathbb{R}_+\to\mathbb{R}_+$ by

$$h(r) := \frac{1-e^{-r^2}}{r^2}, \qquad r>0. \tag{2.2.48}$$

It is easy to check by elementary calculus that h is monotone decreasing and continuous in $r=0$ with $h(0)=1$.

2.2.20 Lemma
Grant Assumption 2.2.1. Then for all $\ell \in \mathbb{N}$, *there is a constant* $C'_\ell < \infty$ *depending on* C_ℓ *and* ε *of* (2.2.2) *such that for all* $\boldsymbol{\xi} \in E^\ell$

$$|\mathscr{F}[\widehat{p}_\varepsilon^{(\ell)}](\boldsymbol{\xi})| \le C'_\ell \cdot e^{-\frac{1}{2C'_\ell}\|\boldsymbol{\xi}\|^2}, \qquad |\mathscr{F}[\widehat{r}_\varepsilon^{(\ell)}](\boldsymbol{\xi})| \le C'_\ell \cdot h(\|\boldsymbol{\xi}\|). \tag{2.2.49}$$

Proof For $t > 0$ and $\boldsymbol{\xi} \in E^\ell$ we have

$$\begin{aligned}\mathscr{F}\left[\widehat{p}_t^{(\ell)}\right](\boldsymbol{\xi}) &= C_\ell t^{-d\ell/2} \cdot \mathscr{F}\left[e^{-\frac{\|\cdot\|^2}{2C_\ell t}}\right](\boldsymbol{\xi}) \\ &= C_\ell \cdot (2\pi C_\ell)^{d\ell/2}\,\mathscr{F}\left[(2\pi C_\ell\cdot t)^{-d\ell/2}\, e^{-\frac{1}{2}\frac{\|\cdot\|^2}{C_\ell t}}\right](\boldsymbol{\xi}) \\ &= C_\ell^{1+d\ell/2}\cdot(2\pi)^{d\ell/2}\exp\left(-\frac{1}{2}C_\ell t\|\boldsymbol{\xi}\|^2\right)\end{aligned} \tag{2.2.50}$$

and consequently

$$|\mathscr{F}[\widehat{p}_\varepsilon^{(\ell)}](\boldsymbol{\xi})| = \mathscr{F}[\widehat{p}_\varepsilon^{(\ell)}](\boldsymbol{\xi}) \le C''_\ell \cdot e^{-\frac{1}{2C''_\ell}\|\boldsymbol{\xi}\|^2}$$

with $C''_\ell := \max\left\{C_\ell^{1+d\ell/2}\cdot(2\pi)^{d\ell/2}, \frac{1}{C_\ell\varepsilon}\right\}$. Integrating $\int_0^\varepsilon dt \ldots$ in (2.2.50) for $\boldsymbol{\xi} \ne 0$, we get

$$\mathscr{F}\left[\widehat{r}_\varepsilon^{(\ell)}\right](\boldsymbol{\xi}) = \int_0^\varepsilon \mathscr{F}\left[\widehat{p}_t^{(\ell)}\right](\boldsymbol{\xi})\,dt = 2\,(2\pi C_\ell)^{d\ell/2}\cdot\frac{1-\exp\left(-\frac{1}{2}C_\ell\varepsilon\|\boldsymbol{\xi}\|^2\right)}{\|\boldsymbol{\xi}\|^2}. \tag{2.2.51}$$

It is easily checked that the function

$$g(r) := \frac{1-\exp\left(-\frac{1}{2}C_\ell\varepsilon r^2\right)}{1-\exp\left(-r^2\right)}, \qquad r > 0 \tag{2.2.52}$$

is monotone increasing if $C_\ell\varepsilon/2 < 1$ and monotone decreasing if $C_\ell\varepsilon/2 > 1$, with $\lim_{r\downarrow 0} g(r) = C_\ell\varepsilon/2$ and $\lim_{r\uparrow\infty} g(r) = 1$. Consequently, by (2.2.51) we get

$$\left|\mathscr{F}\left[\widehat{r}_\varepsilon^{(\ell)}\right](\boldsymbol{\xi})\right| \le C'''_\ell\cdot\frac{1-\exp\left(-\|\boldsymbol{\xi}\|^2\right)}{\|\boldsymbol{\xi}\|^2}$$

with $C'''_\ell := 2\,(2\pi)^{d\ell/2}\cdot\left(1 \vee \frac{C_\ell\varepsilon}{2}\right)$. We thus obtain (2.2.49) upon taking $C'_\ell := C''_\ell \vee C'''_\ell$. ∎

We turn to the action of the Fourier transformation on the operator $\widehat{K}_0^*$. We will need the following notation: For every $\ell \in \mathbb{N}$, $k \in \{0, 1, \ldots, \ell\}$ and $j \in \{1, 2, \ldots, \ell - k + 1\}$ we define a mapping $\Sigma_{\ell,k,j}$ which for each configuration $\boldsymbol{\xi} = (\xi^1, \ldots, \xi^\ell) \in E^\ell$ replaces the coordinate ξ^j by the sum of the k coordinates $\xi^j, \ldots, \xi^{j+k-1}$ and leaves the other coordinates unchanged:

$$\Sigma_{\ell,k,j} : E^\ell \to E^{\ell-k+1}, \qquad \Sigma_{\ell,k,j}\boldsymbol{\xi} := (\xi^1, \ldots, \xi^{j-1}, \xi^j + \cdots + \xi^{j+k-1}, \xi^{j+k}, \ldots, \xi^\ell). \tag{2.2.53}$$

In particular, for $k = 0$ we have

$$\Sigma_{\ell,0,j} : E^\ell \to E^{\ell+1}, \qquad \Sigma_{\ell,0,j}\boldsymbol{\xi} := (\xi^1, \ldots, \xi^{j-1}, 0, \xi^j, \ldots, \xi^\ell),$$

where the symbol 0 in the above display denotes of course the zero vector in $E = \mathbb{R}^d$.

The next lemma states that for a locally finite measure $\mu \in \mathcal{M}_f^{(loc)}(\mathcal{S})$, the Fourier transform of $\widehat{K}_0^*\mu$ can be expressed in terms of the Fourier transform of μ:

2.2.21 Lemma
Let $\mu \in \mathcal{M}_f^{(loc)}(\mathcal{S})$. Then $\widehat{K}_0^* \mu \in \mathcal{M}_f^{(loc)}(\mathcal{S})$, and its Fourier transform on each layer E^ℓ, $\ell \in \mathbb{N}$ is given by

$$\mathscr{F}\Big[(\widehat{K}_0^*\mu)^{(\ell)}\Big](\boldsymbol{\xi}) = \sum_{k=0}^{\ell} D_{\ell-k+1} \sum_{j=1}^{\ell+1-k} \mathscr{F}[\mu^{(\ell+1-k)}]\big(\Sigma_{\ell,k,j}\boldsymbol{\xi}\big) \cdot \prod_{m=1}^{k} \mathscr{F}^{-1}[\widehat{q}_k^{(\ell-k+1)}](\xi^{j+m-1}) \\ + \mathbf{1}_{\ell\geq 2} \cdot D_{\ell-1} \mathscr{F}[\mu^{(\ell-1)}](\xi^1,\dots,\xi^{\ell-1}) \cdot \mathscr{F}[\widehat{\pi}^{(\ell-1)}](\xi^\ell) \tag{2.2.54}$$

for $\boldsymbol{\xi} = (\xi^1,\dots,\xi^\ell) \in E^\ell$.

Proof Let $\mu \in \mathcal{M}_f^{(loc)}(\mathcal{S})$. As remarked in 2.2.19, in order to describe the action of the kernel $\widehat{K}_0$ on μ we just have to replace all quantities by their upper bounds in (2.2.27). This gives for every $h \in \mathscr{B}(E^\ell)$

$$\begin{aligned}
&\Big\langle (\widehat{K}_0^*\mu)^{(\ell)},\, h\Big\rangle \\
&= \sum_{k=0}^{\ell} D_{\ell-k+1} \sum_{j=1}^{\ell-k+1} \int_{E^{\ell-k+1}} \mu^{(\ell-k+1)}(dx^1\cdots dx^{\ell-k+1}) \int_{E^k} dv^1\cdots dv^k \prod_{m=1}^{k} \widehat{q}_k^{(\ell-k+1)}(x^j - v^m)\cdot \\
&\qquad\qquad \cdot h(x^1,\dots,x^{j-1},v^1,\dots,v^k,x^{j+1},\dots,x^{\ell-k+1}) \\
&\quad + \mathbf{1}_{\ell\geq 2}\cdot D_{\ell-1} \int_{E^{\ell-1}} \mu^{(\ell-1)}(dx^1\cdots dx^{\ell-1}) \int_E dv\, \widehat{\pi}^{(\ell-1)}(v)\, h(x^1,\dots,x^{\ell-1},v) \qquad (2.2.55)\\
&= \sum_{k=0}^{\ell} D_{\ell-k+1} \sum_{j=1}^{\ell-k+1} \int_{E^{\ell-k+1}} \mu^{(\ell-k+1)}(dx^1\cdots dx^{\ell-k+1}) \int_{E^k} dv^1\cdots dv^k \prod_{m=1}^{k} \widehat{q}_k^{(\ell-k+1)}(-v^m)\cdot \\
&\qquad\qquad \cdot h(x^1,\dots,x^{j-1},x^j+v^1,\dots,x^j+v^k,x^{j+1},\dots,x^{\ell-k+1}) \\
&\quad + \mathbf{1}_{\ell\geq 2}\cdot D_{\ell-1} \int_{E^{\ell-1}} \mu^{(\ell-1)}(dx^1\cdots dx^{\ell-1}) \int_E dv\, \widehat{\pi}^{(\ell-1)}(v)\, h(x^1,\dots,x^{\ell-1},v),
\end{aligned}$$

where the second equality is obtained by a change of variables $v^m \rightsquigarrow x^j + v^m$, $m = 1,\dots,k$. From (2.2.55), we see in particular that $\widehat{K}_0^*\mu \in \mathcal{M}_f^{(loc)}(\mathcal{S})$.

Now let $\boldsymbol{\xi} = (\xi^1,\dots,\xi^\ell) \in E^\ell$. Fix $k \in \{0,\dots,\ell\}$ and $j \in \{1,\dots,\ell-k+1\}$ and observe that for each $(x^1,\dots,x^{\ell-k+1}) \in E^{\ell-k+1}$ and $(v^1,\dots,v^k) \in E^k$ we get by the bilinearity of the scalar product

$$\begin{aligned}
&\big\langle \boldsymbol{\xi},\, \big(x^1,\dots,x^{j-1},x^j+v^1,\dots,x^j+v^k,x^{j+1},\dots,x^{\ell-k+1}\big)\big\rangle_\ell \\
&= \sum_{m=1}^{j-1} \langle x^m,\, \xi^m\rangle + \sum_{m=1}^{k} \langle x^j + v^m,\, \xi^{j+m-1}\rangle + \sum_{m=j+1}^{\ell-k+1} \langle x^m,\, \xi^{m+k-1}\rangle \\
&= \sum_{m=1}^{j-1} \langle x^m,\, \xi^m\rangle + \langle x^j,\, \sum_{m=1}^{k} \xi^{j+m-1}\rangle + \sum_{m=1}^{k} \langle v^m,\, \xi^{j+m-1}\rangle + \sum_{m=j+1}^{\ell-k+1} \langle x^m,\, \xi^{m+k-1}\rangle \\
&= \Big\langle (x^1,\dots,x^{\ell-k+1}),\, \Sigma_{\ell,k,j}\boldsymbol{\xi}\Big\rangle_{\ell-k+1} + \big\langle (v^1,\dots,v^k),\, (\xi^j,\dots,\xi^{j+k-1})\big\rangle_k
\end{aligned}$$

(using our conventions, this holds including the case $k = 0$). Consequently, choosing

$$h_{\boldsymbol{\xi}}(\boldsymbol{y}) := e^{-i\langle \boldsymbol{y},\, \boldsymbol{\xi}\rangle}, \qquad \boldsymbol{y} \in E^\ell$$

in (2.2.55), we get separation of the x and v variables and thus

$$\begin{aligned}
&\mathscr{F}\Big[(\widehat{K}_0^*\mu)^{(\ell)}\Big](\boldsymbol{\xi}) \equiv \Big\langle(\widehat{K}_0^*\mu)^{(\ell)},\, h_{\boldsymbol{\xi}}\Big\rangle \\
&= \sum_{k=0}^{\ell} D_{\ell-k+1} \sum_{j=1}^{\ell+1-k} \int_{E^{\ell+1-k}} \mu^{(\ell+1-k)}(dx^1 \cdots dx^{\ell+1-k})\, e^{-i\langle (x^1,\dots,x^{\ell-k+1}),\, \Sigma_{\ell,k,j}\boldsymbol{\xi}\rangle} \cdot \\
&\qquad\qquad \cdot \prod_{m=1}^{k} \int_E dv^m\, \widehat{q}_k^{(\ell-k+1)}(-v^m)\, e^{-i\langle v^m,\, \xi^{j+m-1}\rangle} \\
&\quad + \mathbf{1}_{\ell\geq 2}\cdot D_{\ell-1}\int_{E^{\ell-1}} \mu^{(\ell-1)}(dx^1\cdots dx^{\ell-1})\, e^{-i\langle (x^1,\dots,x^{\ell-1}),\,(\xi^1,\dots,\xi^{\ell-1})\rangle} \int_E dv\, \widehat{\pi}^{(\ell-1)}(v)\, e^{-i\langle v,\xi^\ell\rangle} \\
&= \sum_{k=0}^{\ell} D_{\ell-k+1} \sum_{j=1}^{\ell+1-k} \mathscr{F}[\mu^{(\ell+1-k)}]\big(\Sigma_{\ell,k,j}\boldsymbol{\xi}\big)\cdot \prod_{m=1}^{k} \mathscr{F}^{-1}[\widehat{q}_k^{(\ell-k+1)}](\xi^{j+m-1}) \\
&\quad + \mathbf{1}_{\ell\geq 2}\cdot D_{\ell-1}\mathscr{F}[\mu^{(\ell-1)}](\xi^1,\dots,\xi^{\ell-1})\cdot \mathscr{F}[\widehat{\pi}^{(\ell-1)}](\xi^\ell)
\end{aligned}$$

which is the desired result. ∎

2.2.22 Remark

Note the importance in the proof of Lemma 2.2.21 of the assumption that the upper bound for the offspring kernel has a convolution structure (see (2.2.13)): Indeed, it is this property which allows expressing $\mathscr{F}[\widehat{K}_0^*\mu]$ in terms of $\mathscr{F}[\mu]$. On the other hand, the absolute continuity of the offspring kernel and the immigration law are not at all important in Lemma 2.2.21: In fact, inspection of the proof shows that an analogous assertion holds whenever the offspring kernel and the immigration law depend on the configuration variable only through its length and the offspring kernel has a convolution structure

$$Q_k^{(\ell)}(x;\boldsymbol{x};\cdot) = \int_{E^k} N_k^{(\ell)}(dv^1\cdots dv^k)\delta_{(x+v^1,\dots,x+v^k)}(\cdot) \quad \text{on } (E^k,\mathcal{B}_{E^k}) \tag{2.2.56}$$

with a probability measure $N_k^{(\ell)}$ on $(E^k,\mathcal{B}_{E^k})$. In particular, the assertion of Lemma 2.2.21 holds also for the case $Q_k^{(\ell)}(x;\boldsymbol{x};\cdot) = \delta_x(\cdot)^{\otimes k}$ that branching particles reproduce at their death position. However, for the proof of Theorem 2.2.8 we will also need that the Fourier transforms of (the upper bounds of) the offspring kernel and immigration law are integrable on each layer, where absolute continuity enters again. ⧫

The next lemma is key to the control of the uniform norms $\|\gamma_n^{(\ell)}\|_\infty$ in the series (2.2.23). Let us define the following classes of functions: For $\ell\in\mathbb{N}$, set

$$\mathcal{H}_\ell := \Big\{|\mathscr{F}[\widehat{\pi}^{(m)}]|,\, |\mathscr{F}^{-1}[\widehat{q}_k^{(m)}]| : k,m\in\{1,\dots,\ell\}\Big\} \tag{2.2.57}$$

and

$$K_\ell := 1\vee \max\{\|h\|_\infty : h\in\mathcal{H}_\ell\} = 1\vee \max_{k,m=1,\dots,\ell}\Big(\|\widehat{\pi}^{(m)}\|_1, \|\widehat{q}_k^{(m)}\|_1\Big) < \infty. \tag{2.2.58}$$

Further, we will use the notation

$$\mathcal{S}_{\leq\ell} := \bigcup_{k=0}^{\ell} E^k. \tag{2.2.59}$$

Now consider a locally integrable function $g\in L^1_{(loc)}(\mathcal{S})$. Then for each $n\in\mathbb{N}_0$ and $\rho\in\{0,1\}$, the function $(\widehat{R}_\varepsilon\widehat{K}_0^*)^n\widehat{P}_\varepsilon^\rho g$ is again locally integrable, and the next lemma basically states that its Fourier transform on each layer E^ℓ is is dominated by a finite sum of functions of the form

$$\boldsymbol{\xi} \mapsto C_{\ell,n} \, \|g|_{\mathcal{S}_{\le \ell+n}}\|_1 \cdot e^{-\rho \frac{\|\boldsymbol{\xi}^J\|^2}{2C_{\ell,n}}} h(\|\boldsymbol{\xi}^J\|)^n \prod_{j \in J^c} h_j(\xi^j), \qquad \boldsymbol{\xi} = (\xi^1, \dots, \xi^\ell) \in E^\ell, \tag{2.2.60}$$

where $J \subseteq \{1, \dots, \ell\}$, $h_j \in \mathcal{H}_{\ell+n}$ for $j \in J^c = \{1, \dots, \ell\} \setminus J$ and h is the function defined in (2.2.48) (see also the Notations 2.2.11 for our terminology regarding subconfigurations).The constant $C_{\ell,n}$, the length of the finite sum and the choice of the subsets J and functions h_j all depend only on ℓ and n, i.e. the above bound is uniform in $g \in L^1_{(loc)}(\mathcal{S})$, $\boldsymbol{\xi} \in E^\ell$ and $\rho \in \{0, 1\}$. Note that we allow for $J = \varnothing$; in this case $h(\|\boldsymbol{\xi}^J\|) = h(0) = 1$ by our conventions. Similarly, if $J = \{1, \dots, \ell\}$, we understand the product $\prod_{j \in J^c} \cdots$ to be equal to 1.

As we will see, under our assumptions (2.2.60) is integrable on E^ℓ provided n is large enough. Thus by the Fourier inversion theorem, we can estimate $\|((\widehat{R}_\varepsilon \widehat{K}_0^*)^n \widehat{P}_\varepsilon^\rho g)^{(\ell)}\|_\infty$ in terms of $\|g|_{\mathcal{S}_{\le \ell+n}}\|_1$. This will lead to a suitable estimate of the uniform norms $\|\gamma_n^{(\ell)}\|_\infty$, enabling us to show uniform convergence on E^ℓ of the series (2.2.23).

2.2.23 Lemma

Let $\ell \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then there exist a constant $C_{\ell,n} < \infty$, a natural number $M_{\ell,n} \in \mathbb{N}$, for each $m \in \{1, 2, \dots, M_{n,\ell}\}$ a subset $J_{\ell,n,m} \subseteq \{1, \dots, \ell\}$ and for $j \in J^c_{\ell,n,m} \equiv \{1, \dots, \ell\} \setminus J_{\ell,n,m}$ a function $h_{\ell,n,m,j} \in \mathcal{H}_{\ell+n}$ such that the following holds: For all $g \in L^1_{(loc)}(\mathcal{S})$, $\rho \in \{0, 1\}$ and $\boldsymbol{\xi} = (\xi^1, \dots, \xi^\ell) \in E^\ell$ we have

$$\left| \mathscr{F}\left[\left((\widehat{R}_\varepsilon \widehat{K}_0^*)^n \widehat{P}_\varepsilon^\rho g \right)^{(\ell)} \right](\boldsymbol{\xi}) \right| \le C_{\ell,n} \cdot \|g|_{\mathcal{S}_{\le \ell+n}}\|_1 \sum_{m=1}^{M_{\ell,n}} e^{-\rho \frac{\|\boldsymbol{\xi}^{J_{\ell,n,m}}\|^2}{2C_{\ell,n}}} h\left(\|\boldsymbol{\xi}^{J_{\ell,n,m}}\| \right)^n \prod_{j \in J^c_{\ell,n,m}} h_{\ell,n,m,j}(\xi^j). \tag{2.2.61}$$

Proof The assertion will be proved by induction on $n \in \mathbb{N}_0$.

Consider $n = 0$. Then $(\widehat{R}_\varepsilon \widehat{K}_0^*)^n = \mathrm{Id}$ and (2.2.61) is trivial: Indeed, let $\ell \in \mathbb{N}$, $g \in L^1_{(loc)}(\mathcal{S})$ and $\boldsymbol{\xi} \in E^\ell$. For $\rho = 0$, the l.h.s. of (2.2.61) is equal to

$$|\mathscr{F}[g^{(\ell)}](\boldsymbol{\xi})| \le \|g^{(\ell)}\|_1 \le \|g|_{\mathcal{S}_{\le \ell}}\|_1.$$

For $\rho = 1$, it is equal to

$$|\mathscr{F}[(\widehat{P}_\varepsilon g)^{(\ell)}](\boldsymbol{\xi})| = |\mathscr{F}[\widehat{p}_\varepsilon^{(\ell)} * g^{(\ell)}](\boldsymbol{\xi})| = |\mathscr{F}[\widehat{p}_\varepsilon^{(\ell)}](\boldsymbol{\xi})| \cdot |\mathscr{F}[g^{(\ell)}](\boldsymbol{\xi})| \le C'_\ell \, e^{-\frac{\|\boldsymbol{\xi}\|^2}{2C'_\ell}} \cdot \|g|_{\mathcal{S}_{\le \ell}}\|_1,$$

where we have used (2.2.49). Thus (2.2.61) holds with $M_{\ell,0} = 1$, $J_{\ell,0,1} = \{1, \dots, \ell\}$ and $C_{\ell,0} = 1 \vee C'_\ell$.

Now suppose that for some $n \in \mathbb{N}_0$, the assertion of the lemma holds for *all* $\ell \in \mathbb{N}$; we will show that it then also holds for $n + 1$ and all $\ell \in \mathbb{N}$.

To this end, consider any $\ell \in \mathbb{N}$ which will remain fixed throughout the rest of this proof. Then according to our induction hypothesis, for all $k \in \{0, \dots, \ell\}$ there are constants $C_{\ell-k+1,n} < \infty$ and $M_{\ell-k+1,n} \in \mathbb{N}$ as well as subsets $J_{\ell-k+1,n,m} \subseteq \{1, \dots, \ell-k+1\}$ and functions $h_{\ell-k+1,n,m,j} \in$

$\mathcal{H}_{\ell-k+1+n} \subseteq \mathcal{H}_{\ell+n+1}$ for $m \in \{1,\dots,M_{\ell-k+1,n}\}$ and $j \in J^c_{\ell-k+1,n,m} = \{1,\dots,\ell-k+1\} \smallsetminus J_{\ell-k+1,n,m}$ such that the following holds: Whenever $g \in L^1_{(loc)}(\mathcal{S})$, $\rho \in \{0,1\}$ and $\tilde{\boldsymbol{\xi}} \in E^{\ell-k+1}$, we have

$$\begin{aligned}
&\left|\mathscr{F}\left[\left((\widehat{R}_\varepsilon \widehat{K}_0^*)^n \widehat{P}_\varepsilon^\rho g\right)^{(\ell-k+1)}\right](\tilde{\boldsymbol{\xi}})\right| \\
&\le C_{\ell-k+1,n} \cdot \|g|_{\mathcal{S}_{\le \ell-k+1+n}}\|_1 \sum_{m=1}^{M_{\ell-k+1,n}} \exp\left(-\rho \frac{\|\tilde{\boldsymbol{\xi}}^{J_{\ell-k+1,n,m}}\|^2}{2C_{\ell-k+1,n}}\right) \cdot \\
&\qquad \cdot h\left(\left\|\tilde{\boldsymbol{\xi}}^{J_{\ell-k+1,n,m}}\right\|\right)^n \prod_{j \in J^c_{\ell-k+1,n,m}} h_{\ell-k+1,n,m,j}(\tilde{\xi}^j).
\end{aligned} \tag{2.2.62}$$

For each $k \in \{0,\dots,\ell\}$ and $i \in \{1,\dots,\ell-k+1\}$ we define a bijection

$$\sigma_{\ell,k,i} : \{1,\dots,\ell-k+1\} \smallsetminus \{i\} \to \{1,\dots,i-1,i+k,\dots,\ell\}$$

by

$$\begin{aligned}
\sigma_{\ell,k,i}(j) &:= j && \text{for } j \in \{1,\dots,i-1\}, \\
\sigma_{\ell,k,i}(j) &:= j+k-1 && \text{for } j \in \{i+1,\dots,\ell-k+1\},
\end{aligned}$$

i.e. $\sigma_{\ell,k,i}$ acts as the identity on $\{1,\dots,i-1\}$ and maps $\{i+1,\dots,\ell-k+1\}$ to $\{i+k,\dots,\ell\}$ by "shifting" each index by $k-1$ to the right. Note that with this definition and with $\Sigma_{\ell,k,i} : E^\ell \to E^{\ell-k+1}$ as in (2.2.53), we have for every subset $J \subseteq \{1,\dots,\ell-k+1\}$ and $\boldsymbol{\xi} = (\xi^1,\dots,\xi^\ell) \in E^\ell$

$$\left(\Sigma_{\ell,k,i}\boldsymbol{\xi}\right)^{J\smallsetminus\{i\}} = \left(\xi^1,\dots,\xi^{i-1},\xi^i+\dots+\xi^{i+k-1},\xi^{i+k},\dots,\xi^\ell\right)^{J\smallsetminus\{i\}} = \boldsymbol{\xi}^{\sigma_{\ell,k,i}(J\smallsetminus\{i\})}. \tag{2.2.63}$$

Now pick any $g \in L^1_{(loc)}(\mathcal{S})$, $\boldsymbol{\xi} = (\xi^1,\dots,\xi^\ell) \in E^\ell$ and $\rho \in \{0,1\}$. By (2.2.49) and Lemma 2.2.21 applied to the function $f := (\widehat{R}_\varepsilon \widehat{K}_0^*)^n \widehat{P}_\varepsilon^\rho g \in L^1_{(loc)}(\mathcal{S})$, we have

$$\begin{aligned}
&\left|\mathscr{F}\left[\left((\widehat{R}_\varepsilon \widehat{K}_0^*)^{n+1} \widehat{P}_\varepsilon^\rho g\right)^{(\ell)}\right](\boldsymbol{\xi})\right| = \left|\mathscr{F}\left[\widehat{r}_\varepsilon^{(\ell)} * (\widehat{K}_0^* f)^{(\ell)}\right](\boldsymbol{\xi})\right| \\
&= \left|\mathscr{F}\left[\widehat{r}_\varepsilon^{(\ell)}\right](\boldsymbol{\xi})\right| \cdot \left|\mathscr{F}\left[\left(\widehat{K}_0^* f\right)^{(\ell)}\right](\boldsymbol{\xi})\right| \\
&\le C'_\ell \cdot h(\|(\boldsymbol{\xi})\|)\Bigg[\sum_{k=0}^{\ell}\sum_{i=1}^{\ell+1-k} |\mathscr{F}[f^{(\ell+1-k)}](\Sigma_{\ell,k,i}\boldsymbol{\xi})| \cdot D_{\ell-k+1} \prod_{r=i}^{i+k-1} |\mathscr{F}^{-1}[\widehat{q}_k^{(\ell-k+1)}](\xi^r)| \\
&\qquad + \mathbf{1}_{\ell\ge 2} \cdot |\mathscr{F}[f^{(\ell-1)}](\xi^1,\dots,\xi^{\ell-1})| \cdot D_{\ell-1} |\mathscr{F}[\widehat{\pi}^{(\ell-1)}](\xi^\ell)|\Bigg] \\
&\le C'_{\ell,n+1} \cdot h(\|(\boldsymbol{\xi})\|)\Bigg[\sum_{k=0}^{\ell}\sum_{i=1}^{\ell+1-k} |\mathscr{F}[f^{(\ell+1-k)}](\Sigma_{\ell,k,i}\boldsymbol{\xi})| \prod_{r=i}^{i+k-1} |\mathscr{F}^{-1}[\widehat{q}_k^{(\ell-k+1)}](\xi^r)| \\
&\qquad + |\mathscr{F}[f^{(\ell-1)}](\xi^1,\dots,\xi^{\ell-1})| \cdot |\mathscr{F}[\widehat{\pi}^{(\ell-1)}](\xi^\ell)|\Bigg],
\end{aligned}$$

where $C'_{\ell,n+1} := C'_\ell \cdot \max_{k=1,\dots,\ell+1} D_k$. Since $f = (\widehat{R}_\varepsilon \widehat{K}_0^*)^n \widehat{P}_\varepsilon^\rho g$, we can estimate each of the terms in the above sum using the induction hypothesis (2.2.62): This gives

$$\begin{aligned}
&\left|\mathscr{F}\left[\left((\widehat{R}_\varepsilon \widehat{K}_0^*)^{n+1}\widehat{P}_\varepsilon^\rho g\right)^{(\ell)}\right](\boldsymbol{\xi})\right| \\
&\leq C'_{\ell,n+1}\cdot h(\|\boldsymbol{\xi}\|)\cdot \\
&\cdot\Bigg[\sum_{k=0}^{\ell}\sum_{i=1}^{\ell+1-k} C_{\ell-k+1,n}\,\|g|_{\mathcal{S}_{\leq \ell-k+1+n}}\|_1 \sum_{m=1}^{M_{\ell-k+1,n}} \exp\left(-\rho\frac{\|(\Sigma_{\ell,k,i}\boldsymbol{\xi})^{J_{\ell-k+1,n,m}}\|^2}{2C_{\ell-k+1,n}}\right) h\left(\left\|(\Sigma_{\ell,k,i}\boldsymbol{\xi})^{J_{\ell-k+1,n,m}}\right\|\right)^n \\
&\qquad\qquad \cdot\prod_{j\in J^c_{\ell-k+1,n,m}} h_{\ell-k+1,n,m,j}\left((\Sigma_{\ell,k,i}\boldsymbol{\xi})^j\right)\prod_{r=i}^{i+k-1}|\mathscr{F}^{-1}[\widehat{q}_k^{(\ell-k+1)}](\xi^r)| \\
&\quad + C_{\ell-1,n}\,\|g|_{\leq\mathcal{S}_{\ell-1+n}}\|_1\sum_{m=1}^{M_{\ell-1,n}}\exp\left(-\rho\frac{\|(\xi^1,\dots,\xi^{\ell-1})^{J_{\ell-1,n,m}}\|^2}{2C_{\ell-1,n}}\right) h\left(\left\|(\xi^1,\dots,\xi^{\ell-1})^{J_{\ell-1,n,m}}\right\|\right)^n \\
&\qquad\qquad \cdot\prod_{j\in J^c_{\ell-1,n,m}} h_{\ell-1,n,m,j}(\xi^j)\cdot|\mathscr{F}[\widehat{\pi}^{(\ell-1)}](\xi^\ell)|\Bigg] \\
&\leq C''_{\ell,n+1}\,\|g|_{\mathcal{S}_{\leq \ell+n+1}}\|_1 \\
&\quad\cdot\Bigg[\sum_{k=0}^{\ell}\sum_{i=1}^{\ell+1-k}\sum_{m=1}^{M_{\ell-k+1,n}}\exp\left(-\rho\frac{\|(\Sigma_{\ell,k,i}\boldsymbol{\xi})^{J_{\ell-k+1,n,m}}\|^2}{2C''_{\ell,n+1}}\right) h(\|\boldsymbol{\xi}\|)\,h\left(\left\|(\Sigma_{\ell,k,i}\boldsymbol{\xi})^{J_{\ell-k+1,n,m}}\right\|\right)^n\cdot \\
&\qquad\qquad \cdot\prod_{j\in J^c_{\ell-k+1,n,m}} h_{\ell-k+1,n,m,j}\left((\Sigma_{\ell,k,i}\boldsymbol{\xi})^j\right)\prod_{r=i}^{i+k-1}|\mathscr{F}^{-1}[\widehat{q}_k^{(\ell-k+1)}](\xi^r)| \\
&\quad + \sum_{m=1}^{M_{\ell-1,n}}\exp\left(-\rho\frac{\|(\xi^1,\dots,\xi^{\ell-1})^{J_{\ell-1,n,m}}\|^2}{2C''_{\ell,n+1}}\right) h(\|\boldsymbol{\xi}\|)\,h\left(\left\|(\xi^1,\dots,\xi^{\ell-1})^{J_{\ell-1,n,m}}\right\|\right)^n \\
&\qquad\qquad \cdot\prod_{j\in J^c_{\ell-1,n,m}} h_{\ell-1,n,m,j}(\xi^j)\cdot|\mathscr{F}[\widehat{\pi}^{(\ell-1)}](\xi^\ell)|\Bigg]
\end{aligned} \tag{2.2.64}$$

with $C''_{\ell,n+1} := (C'_{\ell,n+1}\vee 1)\cdot\max_{k=1,\dots,\ell+1} C_{k,n}$. Recall the identity (2.2.63): Since

$$\left(\Sigma_{\ell,k,i}\boldsymbol{\xi}\right)^{J_{\ell-k+1,n,m}\smallsetminus\{i\}} = \boldsymbol{\xi}^{\sigma_{\ell,k,i}(J_{\ell-k+1,n,m}\smallsetminus\{i\})}$$

is a subconfiguration of both $\left(\Sigma_{\ell,k,i}\boldsymbol{\xi}\right)^{J_{\ell-k+1,n,m}}$ and $\boldsymbol{\xi}$, we have

$$h(\|\boldsymbol{\xi}\|)\cdot h(\|(\Sigma_{\ell,k,i}\boldsymbol{\xi})^{J_{\ell-k+1,n,m}}\|)^n \leq h(\|\boldsymbol{\xi}^{\sigma_{\ell,k,i}(J_{\ell-k+1,n,m}\smallsetminus\{i\})}\|)^{n+1} \tag{2.2.65}$$

and

$$\exp\left(-\rho\frac{\|(\Sigma_{\ell,k,i}\boldsymbol{\xi})^{J_{\ell-k+1,n,m}}\|^2}{2C''_{\ell,n+1}}\right) \leq \exp\left(-\rho\frac{\|\boldsymbol{\xi}^{\sigma_{\ell,k,i}(J_{\ell-k+1,n,m}\smallsetminus\{i\})}\|^2}{2C''_{\ell,n+1}}\right), \tag{2.2.66}$$

as both $h(\cdot)$ and $e^{-\rho\frac{\|\cdot\|^2}{2C''_{\ell,n+1}}}$ are monotone decreasing. Also, we clearly have

$$\begin{aligned}
&\prod_{j\in J^c_{\ell-k+1,n,m}} h_{\ell-k+1,n,m,j}\left((\Sigma_{\ell,k,i}\boldsymbol{\xi})^j\right)\\
&\le K_{n+\ell+1}\prod_{j\in J^c_{\ell-k+1,n,m}\setminus\{i\}} h_{\ell-k+1,n,m,j}\left((\Sigma_{\ell,k,i}\boldsymbol{\xi})^j\right)\\
&= K_{n+\ell+1}\prod_{j\in J^c_{\ell-k+1,n,m}\setminus\{i\}} h_{\ell-k+1,n,m,j}\left(\boldsymbol{\xi}^{\sigma_{\ell,k,i}(j)}\right)\\
&= K_{n+\ell+1}\prod_{j\in\sigma_{\ell,k,i}(J^c_{\ell-k+1,n,m}\setminus\{i\})} h_{\ell-k+1,n,m,\sigma^{-1}_{\ell,k,i}(j)}(\xi^j),
\end{aligned} \tag{2.2.67}$$

where $K_{n+\ell+1}\ge 1$ is defined in (2.2.58). Thus at the cost of an additional constant, we can "delete" the index i from the set $J_{\ell-k+1,n,m}$. Putting

$$C_{\ell,n+1} := K_{\ell+n+1}C''_{\ell,n+1}$$

and substituting (2.2.65), (2.2.66) and (2.2.67) into (2.2.64), we obtain

$$\begin{aligned}
&\left|\mathscr{F}\left[\left((\widehat{R}_\varepsilon\widehat{K}_0^*)^{n+1}\widehat{P}^\rho_\varepsilon g\right)^{(\ell)}\right](\boldsymbol{\xi})\right|\\
&\le C_{\ell,n+1}\cdot\|g|_{\mathcal{S}_{\le\ell+n+1}}\|_1\cdot\\
&\left[\sum_{k=0}^{\ell}\sum_{i=1}^{\ell+1-k}\sum_{m=1}^{M_{\ell-k+1,n}}\exp\left(-\rho\frac{\|\boldsymbol{\xi}^{\sigma_{\ell,k,i}(J_{\ell-k+1,n,m}\setminus\{i\})}\|^2}{2C_{\ell,n+1}}\right)h\left(\left\|\boldsymbol{\xi}^{\sigma_{\ell,k,i}(J_{\ell-k+1,n,m}\setminus\{i\})}\right\|\right)^{n+1}\cdot\right.\\
&\qquad\cdot\prod_{j\in\sigma_{\ell,k,i}(J^c_{\ell-k+1,n,m}\setminus\{i\})} h_{\ell-k+1,n,m,\sigma^{-1}_{\ell,k,i}(j)}(\xi^j)\prod_{r=i}^{i+k-1}|\mathscr{F}^{-1}[\widehat{q}_k^{(\ell-k+1)}](\xi^r)|\\
&\left.+\sum_{m=1}^{M_{\ell-1,n}}\exp\left(-\rho\frac{\|\boldsymbol{\xi}^{J_{\ell-1,n,m}}\|^2}{2C_{\ell,n+1}}\right)h\left(\left\|\boldsymbol{\xi}^{J_{\ell-1,n,m}}\right\|\right)^{n+1}\prod_{j\in J^c_{\ell-1,n,m}} h_{\ell-1,n,m,j}(\xi^j)\cdot|\mathscr{F}[\widehat{\pi}^{(\ell-1)}](\xi^\ell)|\right].
\end{aligned}$$

Now observe that each term in the above sum is of the form (2.2.60) with n replaced by $n+1$: For each $k\in\{0,\dots,\ell\}$, $i\in\{1,\dots,\ell-k+1\}$ and $m\in\{1,\dots,M_{\ell-k+1,n}\}$ we have

$$\{1,\dots,\ell\} = \sigma_{\ell,k,i}(J_{\ell-k+1,n,m}\setminus\{i\})\cup\sigma_{\ell,k,i}(J^c_{\ell-k+1,n,m}\setminus\{i\})\cup\{i,\dots,i+k-1\},$$

and the functions $h_{\ell-k+1,n,m,\sigma^{-1}_{\ell,k,i}(j)}$ and $\mathscr{F}^{-1}[\widehat{q}_k^{(\ell-k+1)}]$ are all in $\mathcal{H}_{\ell+n+1}$. Similarly, for $m\in\{1,\dots,M_{\ell-1,n}\}$ we have

$$\{1,\dots,\ell\} = J_{\ell-1,n,m}\cup J^c_{\ell-1,n,m}\cup\{\ell\},$$

and $h_{\ell-1,n,m,j}$ and $\mathscr{F}[\widehat{\pi}^{(\ell-1)}]$ are in $\mathcal{H}_{\ell+n+1}$ as well. From this we see that (2.2.61) holds for $n+1$ with $C_{\ell,n+1}$ defined as above, with

$$M_{\ell,n+1} = \sum_{k=0}^{\ell}(\ell-k+1)M_{\ell-k+1,n} + M_{\ell-1,n},$$

with $J_{\ell,n+1,m}$ suitably defined from the subsets $\sigma_{\ell,k,i}(J_{\ell-k+1,n,m}\setminus\{i\})$ resp. $J_{\ell-1,n,m}$, and with functions $h_{\ell,n+1,m,j}\in\mathcal{H}_{\ell+n+1}$ defined accordingly, where the choice of all these quantities is independent of g, $\boldsymbol{\xi}$ and ρ. This concludes the proof. ∎

With Lemma 2.2.23 at hand, we can finally finish the proof of our main Theorem 2.2.8:
Proof [of Theorem 2.2.8] We know already that each term $\gamma_n = (R^*_\alpha[\alpha K_0]^*)^n R^*_\alpha \pi^{(0)}$ in the series representation (2.2.23) of the invariant density belongs to the space $\mathscr{U} \subseteq \mathcal{C}_0^{(loc)}(\mathcal{S})$ (see Corollary 2.2.15). All that remains to show is uniform convergence on each layer E^ℓ of the series $\sum_{n=0}^\infty \gamma_n$. We recall the identity $R^*_\alpha = R^*_{\alpha,\varepsilon} + (P^\alpha_\varepsilon)^* R^*_\alpha$ from (2.2.10): Expanding it, we get easily by induction that for all $n \in \mathbb{N}$

$$(R^*_\alpha[\alpha K_0]^*)^n = (R^*_{\alpha,\varepsilon}[\alpha K_0]^*)^n + \sum_{k=0}^{n-1} (R^*_{\alpha,\varepsilon}[\alpha K_0]^*)^k (P^\alpha_\varepsilon)^* (R^*_\alpha[\alpha K_0]^*)^{n-k}.$$

Since $\gamma_m \geq 0$ for all $m \in \mathbb{N}_0$, we can use the bounds (2.2.46) and (2.2.47) to obtain for each $n_0 \in \mathbb{N}$ and $n > n_0$

$$\begin{aligned}
\gamma_n &= (R^*_\alpha[\alpha K_0]^*)^{n_0} \gamma_{n-n_0} \\
&= (R^*_{\alpha,\varepsilon}[\alpha K_0]^*)^{n_0} \gamma_{n-n_0} + \sum_{k=0}^{n_0-1} (R^*_{\alpha,\varepsilon}[\alpha K_0]^*)^k (P^\alpha_\varepsilon)^* \underbrace{(R^*_\alpha[\alpha K_0]^*)^{n_0-k} \gamma_{n-n_0}}_{=\gamma_{n-k}} \\
&\leq (\widehat{R}_\varepsilon \widehat{K}_0^*)^{n_0} \gamma_{n-n_0} + \sum_{k=0}^{n_0-1} (\widehat{R}_\varepsilon \widehat{K}_0^*)^k \widehat{P}_\varepsilon \gamma_{n-k} \qquad (2.2.68) \\
&= (\widehat{R}_\varepsilon \widehat{K}_0^*)^{n_0} \gamma_{n-n_0} + \sum_{k=0}^{n_0-1} (\widehat{R}_\varepsilon \widehat{K}_0^*)^k \widehat{P}_\varepsilon \gamma_{n-k} \\
&=: \widehat{\gamma}_n.
\end{aligned}$$

Now fix $\ell \in \mathbb{N}$ for the rest of the proof. Choose $n_0 \in \mathbb{N}$ with $n_0 > d\ell/2$ and consider any $n > n_0$. Then it follows from Lemma 2.2.23 that the Fourier transform $\mathscr{F}[\widehat{\gamma}_n^{(\ell)}]$ is integrable on E^ℓ: Indeed, for the term $(\widehat{R}_\varepsilon \widehat{K}_0^*)^{n_0} \gamma_{n-n_0}$ apply Lemma 2.2.23 for n_0, $g := \gamma_{n-n_0}$ and $\rho = 0$. This gives an upper bound

$$\left| \mathscr{F}\left[\left((\widehat{R}_\varepsilon \widehat{K}_0^*)^{n_0} \gamma_{n-n_0} \right)^{(\ell)} \right](\boldsymbol{\xi}) \right| \leq C_{\ell,n_0} \cdot \| \gamma_{n-n_0}|_{\mathcal{S}_{\leq \ell+n_0}} \|_1 \cdot H_{\ell,n_0}(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in E^\ell, \qquad (2.2.69)$$

where the function H_{ℓ,n_0} is defined as the sum on the r.h.s. of (2.2.61). Each term of this sum is of the form

$$\boldsymbol{\xi} \mapsto h(\|\boldsymbol{\xi}^J\|)^{n_0} \cdot \prod_{j \in J^c} h_j(\xi^j) \qquad (2.2.70)$$

for some subset $J \subseteq \{1, \ldots, \ell\}$ and $h_j \in \mathcal{H}_{\ell+n_0}$ for $j \in J^c$. But for any $\varnothing \neq J \subseteq \{1, \ldots, \ell\}$ we have by choice of $n_0 > d\ell/2$

$$\int_{E^J} h(\|\boldsymbol{\xi}^J\|)^{n_0} \, d\boldsymbol{\xi}^J = \text{const} \int_0^\infty r^{d|J|-1} h(r)^{n_0} \, dr \leq \text{const} \int_0^\infty 1 \wedge r^{d|J|-1-2n_0} \, dr < \infty$$

since $h(\cdot) \leq 1$, $h(r) \sim r^{-2}$ for $r \to \infty$ (see the definition of h in (2.2.48)). Since moreover $h_j \in L^1(E)$ by our assumptions (2.2.15) and (2.2.19), we see that (2.2.70) and thus H_{ℓ,n_0} is integrable on E^ℓ.

As to the other terms on the r.h.s. of (2.2.68), fix $k \in \{0, \ldots, n_0 - 1\}$ and apply Lemma 2.2.23 for k, $g := \gamma_{n-k}$ and $\rho = 1$ to obtain upper bounds

$$\left| \mathscr{F}\left[\left((\widehat{R}_\varepsilon \widehat{K}_0^*)^k \widehat{P}_\varepsilon \gamma_{n-k} \right)^{(\ell)} \right](\boldsymbol{\xi}) \right| \leq C_{\ell,k} \cdot \| \gamma_{n-k}|_{\mathcal{S}_{\leq \ell+k}} \|_1 \cdot H_{\ell,k}(\boldsymbol{\xi}), \qquad (2.2.71)$$

where $H_{\ell,k}$ is again defined as the sum on the r.h.s. of (2.2.61) which now consists of terms of the form

$$\boldsymbol{\xi} \mapsto e^{-\frac{\|\boldsymbol{\xi}^J\|^2}{2C_{\ell,k}}} h(\|\boldsymbol{\xi}^J\|)^k \prod_{j\in J^c} h_j(\xi^j)$$

with $J \subseteq \{1,\ldots,\ell\}$ and $h_j \in \mathcal{H}_{\ell+k} \subseteq \mathcal{H}_{\ell+n_0}$. As before, $H_{\ell,k}$ is integrable on E^ℓ, the integrability w.r.t. the $\boldsymbol{\xi}^J$-variables for $J \neq \varnothing$ now being ensured by the exponential term.

We have proved that $\mathscr{F}[\widehat{\gamma}_n^{(\ell)}] \in L^1(E^\ell)$, with $\widehat{\gamma}_n$ defined as the r.h.s. of (2.2.68). Thus it follows from (2.2.68) together with the Fourier inversion theorem that

$$\gamma_n^{(\ell)}(\boldsymbol{x}) \le \widehat{\gamma}_n^{(\ell)}(\boldsymbol{x}) = (2\pi)^{-d\ell}\mathscr{F}^{-1}\left[\mathscr{F}[\widehat{\gamma}_n^{(\ell)}]\right](\boldsymbol{x}), \qquad \boldsymbol{\lambda}\text{-a.e. } \boldsymbol{x} \in E^\ell.$$

But since $\gamma_n^{(\ell)}$ is continuous, the above estimate holds for *all* $\boldsymbol{x} \in E^\ell$, and together with the bounds (2.2.69) and (2.2.71) we conclude that

$$\begin{aligned} \|\gamma_n^{(\ell)}\|_\infty &\le (2\pi)^{-d\ell}\|\mathscr{F}[\widehat{\gamma}_n^{(\ell)}]\|_1 \\ &\le (2\pi)^{-d\ell}\left(C_{\ell,n_0}\|\gamma_{n-n_0}|_{\mathcal{S}_{\le \ell+n_0}}\|_1 \cdot \|H_{\ell,n_0}\|_1 + \sum_{k=0}^{n_0-1} C_{\ell,k}\|\gamma_{n-k}|_{\mathcal{S}_{\le\ell+k}}\|_1 \cdot \|H_{\ell,k}\|_1\right) \\ &\le C'_{\ell,n_0}\sum_{k=0}^{n_0}\|\gamma_{n-k}|_{\mathcal{S}_{\le\ell+n_0}}\|_1 \end{aligned} \tag{2.2.72}$$

with a constant C'_{ℓ,n_0} depending only on ℓ and n_0. But the r.h.s. of (2.2.72) is of course summable in $n > n_0$, proving uniform convergence of the series $\sum_n \gamma_n^{(\ell)}$ on E^ℓ. This concludes our proof. ■

2.2.24 Remark

The proof of Theorem 2.2.8 above, together with the fact that $\gamma_n \in \mathscr{U}$ for all n by Corollary 2.2.15, shows in fact the stronger assertion that the invariant density γ belongs to the space $\mathscr{U}$ defined in (2.2.30): Let $\ell \in \mathbb{N}$, $\varnothing \neq I \subseteq \{1,\ldots,\ell\}$ and fix $n_0 > d\ell/2$ as in the proof. Then from (2.2.68), we get for all $n > n_0$

$$\mathcal{I}_\ell^I \gamma_n \le \mathcal{I}_\ell^I \widehat{\gamma}_n.$$

Given $\tilde{\boldsymbol{\xi}} = (\tilde{\xi}^j : j \in I) \in E^I$, denote by $\boldsymbol{\xi} = (\xi^1,\ldots,\xi^\ell) \in E^\ell$ the configuration obtained by "amending" $\tilde{\boldsymbol{\xi}}$ with 0 for $j \in I^c$:

$$\xi^j := \begin{cases} \tilde{\xi}^j & \text{if } j \in I, \\ 0 & \text{if } j \in I^c \equiv \{1,\ldots,\ell\} \smallsetminus I. \end{cases}$$

Then by (2.2.69) and (2.2.71) we get estimates

$$\begin{aligned} \mathscr{F}[\mathcal{I}_\ell^I\widehat{\gamma}_n](\tilde{\boldsymbol{\xi}}) &\equiv \int_{E^I} d\boldsymbol{x}^I\, e^{-i\langle \boldsymbol{x}^I, \tilde{\boldsymbol{\xi}}\rangle}\mathcal{I}_\ell^I\widehat{\gamma}_n(\boldsymbol{x}^I) \\ &= \int_{E^I} d\boldsymbol{x}^I\, e^{-i\langle \boldsymbol{x}^I, \tilde{\boldsymbol{\xi}}\rangle}\int_{E^{I^c}} d\boldsymbol{x}^{I^c}\widehat{\gamma}_n^{(\ell)}(\boldsymbol{x}) \\ &= \int_{E^\ell} d\boldsymbol{x}\, e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle}\widehat{\gamma}_n^{(\ell)}(\boldsymbol{x}) \\ &= \mathscr{F}[\widehat{\gamma}_n^{(\ell)}](\boldsymbol{\xi}) \\ &\le C_{\ell,n_0}\cdot\|\gamma_{n-n_0}|_{\mathcal{S}_{\le\ell+n_0}}\|_1 \cdot H_{\ell,n_0}(\boldsymbol{\xi}) + \sum_{k=0}^{n_0-1} C_{\ell,k}\cdot\|\gamma_{n-k}|_{\mathcal{S}_{\le\ell+k}}\|_1\cdot H_{\ell,k}(\boldsymbol{\xi}). \end{aligned}$$

Here H_{ℓ,n_0} is a finite sum of terms of the form (2.2.70), with suitable subsets $J \subseteq \{1,\dots,\ell\}$ and $h_j \in \mathcal{H}_{\ell+n_0}$. By definition of $\boldsymbol{\xi}$ and the form of h, the terms in (2.2.70) can be estimated by

$$\begin{aligned} h(\|\boldsymbol{\xi}^J\|)^{n_0} \cdot \prod_{j\in J^c} h_j(\xi^j) &= h(\|\tilde{\boldsymbol{\xi}}^{I\cap J}\|)^{n_0} \prod_{j\in I\cap J^c} h(\tilde{\xi}^j) \prod_{j\in I^c\cap J^c} h_j(0) \\ &\le K^\ell_{\ell+n_0} \cdot h(\|\tilde{\boldsymbol{\xi}}^{I\cap J}\|)^{n_0} \prod_{j\in I\cap J^c} h(\tilde{\xi}^j), \end{aligned}$$

where $K_{\ell+n_0}$ is defined in (2.2.58). Since $n_0 > d\ell/2$, it is clear that the r.h.s. of the previous display is integrable as a function of $\tilde{\boldsymbol{\xi}}$ on E^I, no matter what the subset $J \subseteq \{1,\dots,\ell\}$ is. Analogous remarks apply to the functions $H_{\ell,k}$, $k \in \{0,\dots,n_0-1\}$. Thus the same reasoning as before shows that $\mathscr{F}[\mathcal{I}^I_\ell \widehat{\gamma}_n]$ is integrable on E^I, and applying the Fourier inversion theorem we get an estimate like (2.2.72) with γ_n on the l.h.s. replaced by $\mathcal{I}^I_\ell \gamma_n$. Since $\mathcal{I}^I_\ell \gamma_n \in \mathcal{C}_0(E^I)$ for all n, uniform convergence of the series then gives $\mathcal{I}^I_\ell \gamma \in \mathcal{C}_0(E^I)$ and thus $\gamma \in \mathscr{U}$. ⧫

If the transition density of the killed ℓ-particle motion is continuously differentiable with respect to the forward variable (as e.g. in the $\mathcal{C}^\infty_b$-framework of Remark 2.2.3) and an estimate like (2.2.5) for the partial derivatives is available, we can strengthen the conclusion of Theorem 2.2.8 accordingly: In this case, the invariant density is locally $\mathcal{C}^1_0$, i.e. continuously differentiable on each layer with all partial derivatives in $\mathcal{C}_0(E^\ell)$. As in Remark 2.2.24, this extends to all functions $\mathcal{I}^J_\ell \gamma$, $\varnothing \neq J \subseteq \{1,\dots,\ell\}$: Instead of the space $\mathscr{U}$ of (2.2.30), consider the subspace $\mathscr{U}^1$ of all locally integrable functions such that all functions $\mathcal{I}^J_\ell f$ are in $\mathcal{C}^1_0(E^J)$ and all partial derivatives can be calculated by "differentiating under the integral sign":

$$\mathscr{U}^1 := \left\{ f \in L^1_{(loc)}(\mathcal{S}) : \forall\, \ell \in \mathbb{N},\, \varnothing \neq J \subseteq \{1,\dots,\ell\} : \mathcal{I}^J_\ell f \in \mathcal{C}^1_0(E^J),\ \partial \mathcal{I}^J_\ell f = \mathcal{I}^J_\ell \partial f \right\}. \qquad (2.2.73)$$

Here, the symbol ∂ stands for any of the partial derivatives w.r.t. the $\boldsymbol{x}^J$-variables. Then we have:

2.2.25 Theorem
Grant Assumptions 2.2.1 and 2.2.5. In addition, suppose that the transition density $p^\alpha_t(\boldsymbol{x};\cdot)$ is continuously differentiable in the forward variable $\boldsymbol{y}$ with

$$\partial_{\boldsymbol{y}} p^\alpha_t(\boldsymbol{x};\boldsymbol{y}) \le C'_\ell \cdot t^{-(1+d\ell)/2} \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{2C'_\ell t} \right), \qquad t \in [0,\varepsilon],\ \boldsymbol{x},\boldsymbol{y} \in E^\ell; \qquad (2.2.74)$$

here, $\partial_{\boldsymbol{y}}$ denotes any of the partial derivatives w.r.t. $\boldsymbol{y}$. Then the invariant density γ belongs to the space $\mathscr{U}^1$; in particular, it is locally $\mathcal{C}^1_0$:

$$\gamma^{(\ell)}(\cdot) \in \mathcal{C}^1_0(E^\ell), \qquad \ell \in \mathbb{N}.$$

Proof The proof is only a slight variant of the proof of Theorem 2.2.8. Starting from the series representation (2.2.23), first one has to prove that $\gamma_n \in \mathscr{U}^1$ for all $n \in \mathbb{N}_0$. With $\mathscr{U}^1$ defined as in (2.2.73), Lemma 2.2.14 can be complemented as follows: The operator $R^*_{\alpha,\varepsilon}$ is "smoothing" in the sense that it maps $\mathscr{U}$ into $\mathscr{U}^1$, and $(P^\alpha_\varepsilon)^*$ maps $L^1_{(loc)}(\mathcal{S})$ into $\mathscr{U}^1$. This is proved using a "Differentiation Lemma" for integrals depending on a parameter, see the second part of Lemma A.1 in the appendix: Let $f \in \mathscr{U}$, again taken to be nonnegative without loss of generality. Fix $\ell \in \mathbb{N}$ and $\varnothing \neq J \subseteq \{1, \dots, \ell\}$. Recall that for $\boldsymbol{x}^J \in E^J$, we have

$$\mathcal{I}_\ell^J R_{\alpha,\varepsilon}^* f(\boldsymbol{x}^J) = \int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_{E^\ell} d\boldsymbol{y} \int_0^\varepsilon dt\, p_t^\alpha(\boldsymbol{y}; \boldsymbol{x}) f^{(\ell)}(\boldsymbol{y}).$$

By assumption, for each fixed $(\boldsymbol{x}^{J^c}, \boldsymbol{y}, t) \in E^{J^c} \times E^\ell \times \mathbb{R}_+$ the function

$$\boldsymbol{x}^J \mapsto p_t^\alpha(\boldsymbol{y}; \boldsymbol{x})\, f^{(\ell)}(\boldsymbol{y}) \tag{2.2.75}$$

is in $\mathcal{C}^1(E^J)$. Denote by $\boldsymbol{e}$ any of the canonical unit vectors in E^J and by $\partial_{\boldsymbol{e}}$ the corresponding partial derivative. By the estimate (2.2.74), we have

$$\begin{aligned}&\int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_{E^\ell} d\boldsymbol{y} \int_0^\varepsilon dt\, |\partial_{\boldsymbol{e}} p_t^\alpha(\boldsymbol{y}; \boldsymbol{x}) f^{(\ell)}(\boldsymbol{y})| \\ &\leq C'_\ell \int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_{E^\ell} d\boldsymbol{y} \int_0^\varepsilon dt\, t^{-(1+d\ell)/2} \exp\left(-\frac{\|\boldsymbol{y} - \boldsymbol{x}\|^2}{2C'_\ell t}\right) f^{(\ell)}(\boldsymbol{y}).\end{aligned} \tag{2.2.76}$$

Clearly, the integrand on the r.h.s. of the above display, considered as a function of $\boldsymbol{x}^J$, is continuous for each $(\boldsymbol{x}^{J^c}, \boldsymbol{y}, t) \in E^{J^c} \times E^\ell \times \mathbb{R}_+$ fixed. Thus in view of the "Differentiation Lemma" A.1, it remains only to show that the integral on the r.h.s. of (2.2.76) is a continuous function of $\boldsymbol{x}^J \in E^J$. This follows by the same arguments as in the first part of the proof of Lemma 2.2.14 above, the only difference being that we have to add an additional factor $t^{-1/2}$ in the integral $\int_0^\varepsilon dt \dots$: Then essentially the same calculation as in (2.2.33) shows

$$\int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_{E^\ell} d\boldsymbol{y} \int_0^\varepsilon dt\, t^{-(1+d\ell)/2} \exp\left(-\frac{\|\boldsymbol{y} - \boldsymbol{x}\|^2}{2C'_\ell t}\right) f^{(\ell)}(\boldsymbol{y}) = C'_{\ell,J} \cdot \left(\tilde{r}_\varepsilon^{(\ell,J)} * \mathcal{I}_\ell^J f\right)(\boldsymbol{x}^J) \tag{2.2.77}$$

for some constant $C'_{\ell,J}$ and $\tilde{r}_\varepsilon^{(\ell,J)}$ given by

$$\tilde{r}_\varepsilon^{(\ell,J)}(\boldsymbol{y}^J) := \int_0^\varepsilon dt\, t^{-(1+d|J|)/2} \exp\left(-\frac{\|\boldsymbol{y}^J\|^2}{2C'_\ell t}\right), \qquad \boldsymbol{y}^J \in E^J.$$

Observe that compared to (2.2.34), we have an additional factor $t^{-1/2}$ in the definition of $\tilde{r}_\varepsilon^{(\ell,J)}$, which however does not destroy integrability at $t = 0$. In particular, we still have $\tilde{r}_\varepsilon^{(\ell,J)} \in L^1(E^J)$. Since $f \in U$ by assumption, we have $\mathcal{I}_\ell^J f \in \mathcal{C}_0(E^J)$ and conclude as before that also $\tilde{r}_\varepsilon^{(\ell,J)} * \mathcal{I}_\ell^J f \in \mathcal{C}_0(E^J)$. This shows that the integral on the r.h.s. of (2.2.76) is a continuous function of $\boldsymbol{x}^J \in E^J$, whence we conclude by Lemma A.1 that $\mathcal{I}_\ell^J R_{\alpha,\varepsilon}^* f \in \mathcal{C}^1(E^J)$ and that the partial derivatives are given by "differentiating under the integral sign" as

$$\partial_{\boldsymbol{e}} \mathcal{I}_\ell^J R_{\alpha,\varepsilon}^* f(\boldsymbol{x}^J) = \int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_{E^\ell} d\boldsymbol{y} \int_0^\varepsilon dt\, \partial_{\boldsymbol{e}} p_t^\alpha(\boldsymbol{y}; \boldsymbol{x}) f^{(\ell)}(\boldsymbol{y}), \qquad \boldsymbol{x}^J \in E^J.$$

In particular, for $J = \{1, \dots, \ell\}$ we obtain that $R_{\alpha,\varepsilon}^* f \in \mathcal{C}^1(E^\ell)$, and the argument above also shows that

$$\partial_{\boldsymbol{e}} \left(\mathcal{I}_\ell^J R_{\alpha,\varepsilon}^* f\right) = \mathcal{I}_\ell^J \left(\partial_{\boldsymbol{e}} R_{\alpha,\varepsilon}^* f\right).$$

Finally, from the estimate (2.2.76) and (2.2.77) we see that $\partial_{\boldsymbol{e}} \mathcal{I}_\ell^J R_{\alpha,\varepsilon}^* f$ is vanishing at infinity, thus $\mathcal{I}_\ell^J R_{\alpha,\varepsilon}^* f \in \mathcal{C}_0^1(E^J)$. Thus we have proved $R_{\alpha,\varepsilon}^* f \in \mathscr{U}^1$ for $f \in \mathscr{U}$.

As in Lemma 2.2.14, the proof that $(P_\varepsilon^\alpha)^* f \in \mathscr{U}^1$ for arbitrary nonnegative $f \in L^1_{(loc)}(\mathcal{S})$ is even easier: We have

$$\mathcal{I}_\ell^J((P_\varepsilon^\alpha)^* f)(\boldsymbol{x}^J) = \int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_{E^\ell} d\boldsymbol{y}\, p_\varepsilon^\alpha(\boldsymbol{y};\boldsymbol{x}) f^{(\ell)}(\boldsymbol{y}) \tag{2.2.78}$$

and in analogy with (2.2.36)

$$\begin{aligned}
&\int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_{E^\ell} d\boldsymbol{y}\, |\partial_{\boldsymbol{e}} p_\varepsilon^\alpha(\boldsymbol{y};\boldsymbol{x}) f^{(\ell)}(\boldsymbol{y})| \\
&\le C'_\ell \int_{E^{J^c}} d\boldsymbol{x}^{J^c} \int_{E^\ell} d\boldsymbol{y}\, \varepsilon^{-(1+d\ell)/2} \exp\left(-\frac{\|\boldsymbol{y}-\boldsymbol{y}\|^2}{2C'_\ell}\right) f^{(\ell)}(\boldsymbol{y}) \\
&= C''_{\ell,J} \cdot \left(\tilde{p}_\varepsilon^{(\ell,J)} * \mathcal{I}_\ell^J f\right)(\boldsymbol{x}^J)
\end{aligned} \tag{2.2.79}$$

with some constant $C''_{\ell,J}$ and $\tilde{p}_\varepsilon^{(\ell,J)}(\cdot) := \varepsilon^{-(1+d|J|)/2} \exp\left(-\frac{\|\cdot\|^2}{2C'_\ell \varepsilon}\right) \in \mathcal{C}_0(E^J)$. Again we conclude by the "Differentiation Lemma" that we may differentiate under the integral sign in (2.2.78) and that $\partial_{\boldsymbol{e}}\left(\mathcal{I}_\ell^J (P_\varepsilon^\alpha)^* f\right) \in \mathcal{C}_0(E^J)$.

We have proved our claim $R^*_{\alpha,\varepsilon}$ maps $\mathscr{U}$ into $\mathscr{U}^1$ and that $(P_\varepsilon^\alpha)^*$ maps $L^1_{(loc)}(\mathcal{S})$ into $\mathscr{U}^1$. Consequently, the third assertion of Lemma 2.2.14 can be strengthened as follows: Whenever $f \in \mathscr{U}$ is nonnegative and such that $R^*_\alpha f \in L^1_{(loc)}(\mathcal{S})$, we have

$$R^*_\alpha f = R^*_{\alpha,\varepsilon} f + (P_\varepsilon^\alpha)^* R^*_\alpha f \in \mathscr{U}^1,$$

which together with (2.2.10) implies that for all $n \in \mathbb{N}_0$

$$\gamma_n = (R^*_\alpha [\alpha K_0]^*)^n R^*_\alpha \pi^{(0)} \in \mathscr{U}^1. \tag{2.2.80}$$

Note that for this argument we do not have to assume differentiability of π^0 or preservation of the subspace $\mathscr{U}^1$ by the operator $[\alpha K_0]^*$, due to the "smoothing properties" of $R^*_{\alpha,\varepsilon}$ and $(P_\varepsilon^\alpha)^*$.

Since the rest of the proof is similar to that of Theorem 2.2.8 given above, we will only sketch it: Fix $\ell \in \mathbb{N}$. For $J = \{1,\ldots,\ell\}$, write simply $\tilde{r}_\varepsilon^{(\ell)}$ resp. $\tilde{p}_\varepsilon^{(\ell)}$ in place of $\tilde{r}_\varepsilon^{(\ell,J)}$ resp. $\tilde{p}_\varepsilon^{(\ell,J)}$. For all $n \in \mathbb{N}$, using the identity (2.2.10) we have

$$\partial_{\boldsymbol{e}} \gamma_n^{(\ell)} = \partial_{\boldsymbol{e}} \left(R^*_\alpha [\alpha K_0]^* \gamma_{n-1}\right)^{(\ell)} = \partial_{\boldsymbol{e}} \left(R^*_{\alpha,\varepsilon} [\alpha K_0]^* \gamma_{n-1}\right)^{(\ell)} + \partial_{\boldsymbol{e}} \left((P_\varepsilon^\alpha)^* \gamma_n\right)^{(\ell)}.$$

Using the above estimates on the derivatives, we get

$$\begin{aligned}
|\partial_{\boldsymbol{e}} \gamma_n^{(\ell)}| &\le \tilde{C}_\ell \cdot \left(\tilde{r}_\varepsilon^{(\ell)} * \left([\alpha K_0]^* \gamma_{n-1}\right)^{(\ell)} + \tilde{p}_\varepsilon^{(\ell)} * \gamma_n^{(\ell)}\right) \\
&\le \tilde{C}_\ell \cdot \left(\tilde{r}_\varepsilon^{(\ell)} * \left(\widehat{K}_0^* \widehat{\gamma}_{n-1}\right)^{(\ell)} + \tilde{p}_\varepsilon^{(\ell)} * \gamma_n^{(\ell)}\right)
\end{aligned} \tag{2.2.81}$$

with $\widehat{\gamma}_{n-1}$ as in (2.2.68). By Lemma 2.2.21, the Fourier transform of the first term in the above convolution can by estimated as

$$\begin{aligned}
&|\mathscr{F}[\tilde{r}_\varepsilon^{(\ell)} * (\widehat{K}_0^* \widehat{\gamma}_n)^{(\ell)}](\boldsymbol{\xi})| = \underbrace{|\mathscr{F}[\tilde{r}_\varepsilon^{(\ell)}](\boldsymbol{\xi})|}_{\le 1} \cdot |\mathscr{F}[(\widehat{K}_0^* \widehat{\gamma}_n)^{(\ell)}](\boldsymbol{\xi})| \\
&\le \sum_{k=0}^{\ell} D_{\ell-k+1} \sum_{j=1}^{\ell+1-k} \mathscr{F}[\widehat{\gamma}_n^{(\ell+1-k)}]\left(\Sigma_{\ell,k,j} \boldsymbol{\xi}\right) \cdot \prod_{m=1}^{k} \mathscr{F}^{-1}[\widehat{q}_k^{(\ell-k+1)}](\xi^{j+m-1}) \\
&\quad + \mathbf{1}_{\ell \ge 2} \cdot D_{\ell-1} \mathscr{F}[\widehat{\gamma}_n^{(\ell-1)}](\xi^1,\ldots,\xi^{\ell-1}) \cdot \mathscr{F}[\widehat{\pi}^{(\ell-1)}](\xi^\ell).
\end{aligned} \tag{2.2.82}$$

By the form of $\widehat{\gamma}_n$ in (2.2.68) and Lemma 2.2.23, one proves as before that the r.h.s. of the previous display is integrable on E^ℓ for $n > n_0 := d\ell/2$ and that its L^1-norm is bounded up to some constant by $\sum_{k=0}^{n_0} \|\gamma_{n-k}|_{\mathcal{S}_{\le \ell+n_0}}\|_1$. By Fourier inversion, the same same upper bounds then holds also for $\|\tilde{r}_\varepsilon^{(\ell)} * \big(\widehat{K}_0^* \widehat{\gamma}_{n-1}\big)^{(\ell)}\|_\infty$. The second convolution in (2.2.81) poses no problem because $\tilde{p}_\varepsilon^{(\ell)}$ is bounded. In this way, we obtain an estimate similar to (2.2.72) for the derivatives:

$$\|\partial_{\boldsymbol{e}} \gamma_n^{(\ell)}\|_\infty \le \mathrm{const} \sum_{k=0}^{n_0} \|\gamma_{n-k}|_{\mathcal{S}_{\le \ell+n_0}}\|_1, \qquad n > n_0.$$

Thus the series of the partial derivatives converges uniformly on E^ℓ, proving that $\gamma^{(\ell)} \in \mathcal{C}_0^1(E^\ell)$.

Finally, to prove that also $\mathcal{I}_\ell^J \gamma \in \mathcal{C}_0^1(E^J)$ for subsets $\varnothing \ne J \subseteq \{1, \dots, \ell\}$, one uses similar arguments as outlined in Remark 2.2.24. ∎

Unfortunately, the method of proof employed above does not extend to higher derivatives, not even under the strong smoothness and uniform ellipticity conditions of Remark 2.2.3: Although under those conditions we have the estimate (2.2.5) for derivatives of arbitrary order $n \in \mathbb{N}$, the corresponding upper bound is of the form

$$(t, \boldsymbol{x}) \mapsto C_{\ell,n} \cdot t^{-(n+d\ell)/2} \exp\left(-\frac{\|\boldsymbol{x}\|^2}{2C_{\ell,n} t}\right), \qquad (t, \boldsymbol{x}) \in [0, \varepsilon] \times E^\ell \tag{2.2.83}$$

which is not integrable w.r.t. $dt \otimes d\boldsymbol{x}$ on $[0, \varepsilon] \times E^\ell$ if $n \ge 2$. In particular, integrating (2.2.83) w.r.t. $t \in [0, \varepsilon]$ does not give a function in $L^1(E^\ell)$ so that the above Fourier arguments cannot be employed (at least not directly).

However, there are special cases in which the invariant density γ is smooth: This is evidenced by the following Proposition which was announced as Example 2.1.3 in section 2.1:

2.2.26 Proposition

Consider a (not necessarily binary) branching Brownian motion with immigration in $\mathbb{R}^d$, $d \ge 1$, i.e. particles move independently of each other on Brownian paths. Suppose that the branching rate $\kappa > 0$, the immigration rate c and the reproduction probabilities p_k, $k \in \mathbb{N}_0$ are constants such that $\varrho = \sum_{k \in \mathbb{N}} k p_k < 1$ and that for $k \ge 1$, offspring particles are distributed in E^k according to an absolutely continuous convolution kernel, i.e. in (2.2.12) *we have*

$$q_k^{(\ell)}(x; \boldsymbol{x}; v^1, \dots, v^k) = \prod_{j=1}^k q(x - v^j) \quad \text{on } E^k, \qquad (x, \boldsymbol{x}) \in E \times E^\ell,\ k, \ell \in \mathbb{N}$$

with $q \in \mathcal{C}_0(E)$. For the immigration distribution, suppose that $\nu^{(\ell)}(\boldsymbol{x}; dv) = p(v)\, dv$ in (2.2.16)*, where $p \in \mathcal{C}_0(E)$. Then the following holds: If for some $n \in \mathbb{N}_0$ we have*

$$\|\cdot\|^n \mathscr{F}[p](\cdot) \in L^1(E), \qquad \|\cdot\|^n \mathscr{F}[q](\cdot) \in L^1(E), \tag{2.2.84}$$

then the invariant density γ has a version which is locally $\mathcal{C}_0^n$:

$$\gamma^{(\ell)} \in \mathcal{C}_0^n(E^\ell), \qquad \ell \in \mathbb{N}.$$

In particular, if q and p are Schwartz functions, then $\gamma^{(\ell)} \in \mathcal{C}_0^\infty(E^\ell)$ for all $\ell \in \mathbb{N}$.

Proof The heuristic of the proof was already outlined at the end of Section 2.1. Under the assumptions above, we can work with the fixed point equation

$$\gamma^{(\ell)} = r_\alpha^{(\ell)} * ([\alpha K]^* \gamma)^{(\ell)} . \tag{2.2.85}$$

instead of the series representation (2.2.23). This is due to the fact that in this case the spatial offspring distribution itself has the convolution structure which in (2.2.13) was assumed for the upper bound only (cf. also Remark 2.2.22). For the Fourier transform of the invariant density on the layer E^ℓ, we obtain

$$\mathscr{F}[\gamma^{(\ell)}] = \mathscr{F}[r_\alpha^{(\ell)}] \cdot \mathscr{F}[([\alpha K]^* \gamma)^{(\ell)}]. \tag{2.2.86}$$

The Fourier transform of the resolvent density $r_\alpha^{(\ell)}$ of $d\ell$-dimensional Brownian motion is easily calculated as

$$\mathscr{F}[r_\alpha^{(\ell)}](\boldsymbol{\xi}) = \frac{1}{\kappa\ell + c + \|\boldsymbol{\xi}\|^2/2} \leq \frac{1}{c + \|\boldsymbol{\xi}\|^2/2} =: \tilde{h}(\|\boldsymbol{\xi}\|), \qquad \boldsymbol{\xi} \in E^\ell. \tag{2.2.87}$$

We observe that Lemma 2.2.21 does also hold for the kernel $[\alpha K]$ in place of $[\alpha K_0]$, the only difference being that the indicator $\mathbf{1}_{\ell \geq 2}$ in front of the last term in (2.2.54) disappears. This allows for iteration of the equation (2.2.86): Substituting $\tilde{h}$ for h in the proof of Lemma 2.2.23 and using that $\mathscr{F}[q]$ and $\mathscr{F}[p]$ are in $L^1(E)$ by assumption, one shows by iterating $n_0 > d\ell/2$ times that $\mathscr{F}[\gamma^{(\ell)}]$ is integrable on E^ℓ. Then the Fourier inversion theorem implies that

$$\gamma^{(\ell)} = (2\pi)^{-d\ell} \mathscr{F}^{-1}\left[\mathscr{F}[\gamma^{(\ell)}]\right] \qquad \boldsymbol{\lambda}\text{-a.s.} \tag{2.2.88}$$

Thus we have expressed the invariant density as an inverse Fourier transform (this does not work in the general case). By the properties of the (inverse) Fourier transform, if (2.2.84) holds for some $n \geq 1$ it follows that $\gamma^{(\ell)}$ is n times continuously differentiable with derivatives in $\mathcal{C}_0(E^\ell)$, and the assertion of the proposition is proved. ■

We conclude this section with the following observation: The reader will have noticed that in the above example, what is important is not the fact that the particle motion is given by a Brownian motion *as such*, but rather the induced convolution structure and the estimate (2.2.87) ensuring integrability for sufficiently high powers of the Fourier transform of the resolvent density, so that the above iteration can be performed. Similar remarks apply to our approach in the general case, where the heat kernel estimate (2.2.2) permits the same Fourier arguments to be employed on the level of an upper bound. Possibly, the results of the present section can be generalized to the case of particle motions other than diffusions, provided an upper bound for the transition density with analogous properties is available. We think e.g. of estimates for the transition densities of jump processes as studied e.g. in [CKK2008]; this is a possible line of future research.

2.3 Existence of a Continuous and Bounded Density for $\overline{m}$

In this section, we turn to the study of the invariant occupation measure $\overline{m}$ on $(E, \mathcal{B}_E)$ under the assumption that it is finite. We continue to work in the general framework of Assumptions 1.1.2 and 1.1.4 of Chapter 1 and assume throughout that the recurrence condition of

Assumption 1.2.1 holds. Thus the BDI $\boldsymbol{\eta}$ is positive Harris recurrent with finite invariant measure m, and the invariant occupation measure $\overline{m}$ is defined as in (1.2.13). In addition, we will require exponential decay of $m(E^\ell)$ as $\ell \uparrow \infty$ (see Assumption 2.3.5 below) so that in particular $\overline{m}$ is a finite measure on $(E, \mathcal{B}_E)$, i.e. (1.2.14) holds.

As for the invariant measure m on the "big" configuration space $(\mathcal{S}, \mathcal{B}_\mathcal{S})$, we are interested in conditions ensuring the existence of a Lebesgue-density for $\overline{m}$ with good properties. In the fully "interactive" framework of Assumptions 1.1.2 and 1.1.4 where all quantities are allowed to be configuration-dependent, this problem was taken up in [Löc2004]: Under uniform ellipticity and strong smoothness and boundedness conditions on the drift and diffusion coefficients in (1.1.3), using Malliavin calculus Löcherbach proved the existence of a continuous and bounded density for $\overline{m}$ on E. However, in [Löc2004] it was also assumed that branching particles reproduce at their parent's death position, i.e.

$$Q_k^{(\ell)}(x;\boldsymbol{x};\cdot) = \delta_x(\cdot)^{\otimes k} \quad \text{on } E^k, \qquad (x,\boldsymbol{x}) \in E \times E^\ell \tag{2.3.1}$$

in (1.1.11). As we have seen in the previous two sections, this assumption precludes the existence of a continuous and locally bounded density for m on the configuration space $\mathcal{S}$, and in order to obtain such a density for m it has to be replaced by the assumption that the spatial offspring distribution is absolutely continuous. Thus we are naturally interested in extending the result in [Löc2004] to the case of absolutely continuous spatial offspring distributions.

In the purely position-dependent framework and under (2.3.1), the problem of the existence of a density for $\overline{m}$ and its regularity properties were investigated in [HL2005]. Naturally, due to the independence of the motion of the particles, better results are available in this framework: In particular, one can prove existence of a continuous bounded Lebesgue density under much weaker assumptions than those in [Löc2004] or in the present section (see e.g. [HL2005], Thm. 3.5 and Lemma 3.6). It is possible to generalize this to the case of more general spatial offspring distribution than (2.3.1); this is however beyond the scope of this thesis and will be taken up in a future work. The first steps in this direction are taken in Chapter 3 below, where we will show that some of the other results in [HL2005] (namely a spatial subcriticality condition ensuring that Assumption 1.2.1 holds) do hold for general spatial offspring distributions and even general strong Markov processes instead of diffusions in the purely-position dependent framework. See also the concluding remarks at the end of Chapter 3 for an outlook.

We start our investigation of $\overline{m}$ by recalling the notation (1.2.11): For a function $f : E \to \mathbb{R}$ on the single particle space, $\bar{f} : \mathcal{S} \to \mathbb{R}$ denotes the function

$$\bar{f}(\boldsymbol{x}) := \sum_{i=1}^{\ell} f(x^i) \quad \text{if } \boldsymbol{x} = (x^1, \dots, x^\ell) \in E^\ell,\ \ell \in \mathbb{N}, \qquad \bar{f}(\Delta) := 0 \tag{2.3.2}$$

on the configuration space. By the definition (1.2.13) of $\overline{m}$, we clearly have for every measurable $f : E \to \mathbb{R}$ which is nonnegative or bounded

$$\begin{aligned} \overline{m}(f) = m(\bar{f}) &= \sum_{\ell \in \mathbb{N}} \int_{E^\ell} m^{(\ell)}(d\boldsymbol{x}) \bar{f}(\boldsymbol{x}) \\ &= \sum_{\ell \in \mathbb{N}} \sum_{i=1}^{\ell} \int_{E^\ell} m^{(\ell)}(dx^1 \cdots dx^\ell) f(x^i). \end{aligned} \tag{2.3.3}$$

Thus $\overline{m}$ is determined by the "marginals" of the invariant measure m on the configuration space $\mathcal{S}$. Morevover, if m is $\boldsymbol{\lambda}$-absolutely continuous with density $\gamma = (\gamma^{(\ell)})_{\ell\in\mathbb{N}_0}$ on $\mathcal{S}$, we see from the last display that $\overline{m}$ has density

$$\begin{aligned}\frac{d\overline{m}}{d\lambda}(x) &= \sum_{\ell\in\mathbb{N}}\sum_{i=1}^{\ell}\int_{E^{\ell-1}} dx^1\cdots dx^{i-1}dx^{i+1}\cdots dx^{\ell}\,\gamma^{(\ell)}(x^1,\ldots,x^{i-1},x,x^{i+1},\ldots,x^{\ell}) \\ &= \sum_{\ell\in\mathbb{N}}\sum_{i=1}^{\ell}\mathcal{I}_{\ell}^{\{i\}}\gamma(x),\end{aligned} \tag{2.3.4}$$

where we have used the notation introduced in (2.2.29) with $J = \{i\}$. Let us agree to the following:

2.3.1 Notation

Let $f : \mathcal{S} \to \mathbb{R}$ nonnegative or in $L^1_{(loc)}(\mathcal{S})$. Fix $\ell \in \mathbb{N}$ and $i \in \{1,\ldots,\ell\}$. For $J = \{i\}$ in Definition 2.2.12, we will simply write $\mathcal{I}_\ell^i f$ in place of $\mathcal{I}_\ell^{\{i\}} f$, and the integral in (2.2.29) will be written

$$\int_{E^{\ell-1}} d\check{\boldsymbol{x}}^i\, f^{(\ell)}(\boldsymbol{x}) := \int_{E^{\ell-1}} dx^1\cdots dx^{i-1}dx^{i+1}\cdots dx^{\ell}\, f^{(\ell)}(\boldsymbol{x}) \equiv \mathcal{I}_\ell^i f(x^i).$$

Finally, we put

$$\mathcal{I}_\ell f := \sum_{i=1}^{\ell}\mathcal{I}_\ell^i f \tag{2.3.5}$$

and

$$\mathcal{I}_0 f := 0.$$

Now suppose that Assumptions 2.2.1 and 2.2.5 of the previous section are satisfied. Then by Theorem 2.2.8 and Remark 2.2.24, we know that m admits an invariant density γ which belongs to the subspace $\mathscr{U}$ of (2.2.30). Using the notation introduced above and the series representation (2.2.23) for γ, from (2.3.4) we get for the invariant occupation density the representation

$$\frac{d\overline{m}}{d\lambda} = \sum_{\ell\in\mathbb{N}}\mathcal{I}_\ell\gamma = \sum_{\ell\in\mathbb{N}}\sum_{n\in\mathbb{N}_0}\mathcal{I}_\ell\gamma_n \tag{2.3.6}$$

with $\gamma_n := (R_\alpha^*[\alpha K_0]^*)^n R_\alpha^*\pi^{(0)} \in \mathscr{U}$ for all $n \in \mathbb{N}_0$ (see Corollary 2.2.15). Thus all terms in either one of the above (double) series belong to the space $\mathcal{C}_0(E)$. However, in order to extend the $\mathcal{C}_0$-property to the limit we need uniform convergence and thus bounds on $\|\mathcal{I}_\ell\gamma\|_\infty$ resp. $\|\mathcal{I}_\ell\gamma_n\|_\infty$ which are summable in $\ell \in \mathbb{N}$ resp. $(\ell,n) \in \mathbb{N}\times\mathbb{N}_0$. But by the method of the previous section, we only get bounds of the form (2.2.72) with the l.h.s. replaced by $\mathcal{I}_\ell^i\gamma_n$ (see Remark 2.2.24):

$$\|\mathcal{I}_\ell^i\gamma_n\|_\infty \le C'_{\ell,n_0}\sum_{k=0}^{n_0}\|\gamma_{n-k}|_{\mathcal{S}_{\le\ell+n_0}}\|_1, \qquad i = 1,\ldots,\ell, \tag{2.3.7}$$

which are summable in $n > n_0$ for fixed ℓ but have no reason to be summable in ℓ: First, recall that the number n_0 in the above estimate depends on ℓ. Even aside from this problem, without further assumptions one usually cannot control the growth of the constant C'_{ℓ,n_0}, and in addition the L^1-norm in (2.3.7) is taken over more and more layers as $\ell \uparrow \infty$. Roughly speaking, the approach of the present section is to strengthen our assumptions in such a way that precisely these flaws of the estimate (2.3.7) can be repaired. Assuming that there is some fixed upper bound k_0 for the number of possible offspring, it is not hard to see that (2.3.7) can be improved to

$$\|\mathcal{I}_\ell^i \gamma_n\|_\infty \le C'_{\ell,n_0} \sum_{k=0}^{n_0} \sum_{j=\ell-n_0(k_0-1)}^{\ell+n_0} \|\gamma_{n-k}^{(j)}\|_1.$$

Summing over $n > n_0$ in the above display gives

$$\sum_{n>n_0} \|\mathcal{I}_\ell^i \gamma_n\|_\infty \le C'_{\ell,n_0}(n_0+1) \sum_{j=\ell-n_0(k_0-1)}^{\ell+n_0} \|\gamma^{(j)}\|_1 = C'_{\ell,n_0}(n_0+1) \sum_{j=\ell-n_0(k_0-1)}^{\ell+n_0} m_j \tag{2.3.8}$$

with $m_j := m(E^j)$ as in (1.2.23). Now suppose that n_0 can be chosen independently of ℓ, and that we have exponential decay of the numbers m_j. Then the sum on the r.h.s. of the previous display will also decay exponentially fast as $\ell \uparrow \infty$, and we would be done if the constants C'_{ℓ,n_0} grow "slow enough" (e.g., at most polynomially). This motivates the conditions to be given in the sequel. Recall that exponential decay of $(m_\ell)_\ell$ sometimes follows from the assumption that the possible number of offspring is uniformly bounded (see Theorem 1.2.9 and the Remarks 1.2.10 in Chapter 1). On the other hand, for the polynomial growth of the constants we have to rely on the hard work done in [Löc2004], since the sufficient conditions given there seem to be the only general results which are known in this regard. We begin by strengthening Assumption 2.2.1:

2.3.2 Assumption

Grant Assumption 2.2.1 and assume in addition that there exists an increasing sequence of constants

$$1 \le K_1 \le K_2 \le \cdots \le K_\ell \le K_{\ell+1} \le \cdots < \infty$$

growing at most polynomially in ℓ such that the following holds: For all $\ell \in \mathbb{N}$, $i \in \{1,\dots,\ell\}$, $y^i \in E$ and $\boldsymbol{x} = (x^1,\dots,x^\ell) \in E^\ell$ we have

$$\begin{aligned} \mathcal{I}_\ell^i\left(p_t^\alpha(\boldsymbol{x};\cdot)\right)(y^i) &\equiv \int_{E^{\ell-1}} p_t^\alpha(\boldsymbol{x};\boldsymbol{y})\, dy^1\cdots dy^{i-1} dy^{i+1}\cdots dy^\ell \\ &\le K_\ell \cdot t^{-d/2} \exp\left(-\frac{\|x^i - y^i\|^2}{2K_\ell t}\right), \qquad t \in [0,\varepsilon]. \end{aligned} \tag{2.3.9}$$

2.3.3 Remarks

- We observe that (2.3.9) is a "heat kernel estimate" for the marginals (in the forward variable) of the transition density for the killed ℓ-particle motion. Of course, an estimate like (2.3.9) follows readily from Assumption 2.2.1 alone; the important point in Assumption 2.3.2 is the requirement that the constants K_ℓ grow at most polynomially in $\ell \in \mathbb{N}$: In fact, simply integrating the r.h.s. of (2.2.2) we would obtain an estimate as in (2.3.9) but with
$$K_\ell = 1 \vee \left(C_\ell \cdot (2\pi C_\ell)^{d(\ell-1)/2}\right)$$
where C_ℓ is the constant from (2.2.2). Note that the above K_ℓ does not grow polynomially in ℓ even if C_ℓ itself does; anyway it is generally not feasible to control the growth of C_ℓ since it depends on the dimension $d\ell$ in a non-explicit way (see Remark 2.2.3).

- It is natural to ask for sufficient conditions for Assumption 2.3.2. A general answer is given in [Löc2004]. Löcherbach works in the framework of smooth bounded coefficients together with uniform ellipticity for the particle motion (1.1.3) (see also Remark 2.2.3) and proves an estimate of the form (2.3.9) using Malliavin calculus methods.[9] However, in order to ensure polynomial growth of the arising constants K_ℓ, certain extra conditions on the coefficients b and σ are required; they are in particular satisfied for interactions of mean-field type.[10] These conditions are quite strong, but no other sufficient conditions of a general nature seem to be known in the fully interactive framework.

- In the set-up of Assumption 2.2.1, assume in addition that particles move independently of each other (i.e. b and σ in (1.1.3) depend only on the position) and that the single-particle motion admits a transition density $p_t(\cdot;\cdot) : E \times E \to \mathbb{R}_+$ for which a heat kernel estimate (2.2.2) (with ℓ replaced by d) is available. Then the transition density $p_t^{(\ell)}(\cdot;\cdot) : E^\ell \times E^\ell \to \mathbb{R}_+$ for the ℓ-particle motion has product structure, and marginals of $p_t^\alpha(\boldsymbol{x};\cdot)$ in (2.3.9) can be estimated by

$$\mathcal{I}_\ell^i\left(p_t^\alpha(\boldsymbol{x};\cdot)\right)(y^i) \leq \mathcal{I}_\ell^i\left(p_t^{(\ell)}(\boldsymbol{x};\cdot)\right)(y^i) \leq C_d \cdot t^{-d/2}\exp\left(-\frac{\|x^i - y^i\|^2}{2C_d t}\right), \qquad t \in [0,\varepsilon].$$

  Thus in (2.3.9) we can take $K_\ell \equiv C_d$ independent of ℓ. Note that we can allow for interactions in the branching rate and in the reproduction and immigration mechanism (i.e. the rates κ and c as well as p_k, Q_k and ν of Assumption 2.2.5 are allowed to depend on the configuration variable). This gives a special class of examples where at least the $\mathcal{C}_b^\infty$-conditions on the drift and diffusion coefficients imposed in [Löc2004] can be relaxed somewhat since a heat kernel estimate for the transition density is often available under weaker assumptions (see Remark 2.2.3). ⧫

Our assumptions on the branching, reproduction and immigration mechanisms will be modified as follows:

2.3.4 Assumption
We require that Assumption 2.2.5 holds except that we do not assume (2.2.15) *and* (2.2.19). *In addition, grant the following:*

1. *There exists a fixed upper bound $k_0 \in \mathbb{N}$ for the possible number of offspring, i.e. we have*
$$p_k^{(\ell)}(x;\boldsymbol{x}) = 0 \tag{2.3.10}$$
for all $k > k_0$, $\ell \in \mathbb{N}$ and $(x,\boldsymbol{x}) \in E \times E^\ell$.

2. *The quantities $\|\kappa^{(\ell)}\|_\infty$, $\|c^{(\ell)}\|_\infty$, $\|\widehat{\pi}^{(\ell)}\|_\infty$, $\|\widehat{\pi}^{(\ell)}\|_1$ and $\|\widehat{q}_k^{(\ell)}\|_1$, $k = 1,\dots k_0$, all grow at most polynomially in $\ell \in \mathbb{N}$.*

2.3.5 Assumption
As in (1.2.23), *write $m_\ell := m(E^\ell)$ for the mass of the ℓ-particle layer under the invariant measure. We assume exponential decay of $(m_\ell)_{\ell\in\mathbb{N}}$, i.e. there exist $C < \infty$ and $q < 1$ such that*
$$m_\ell \leq Cq^\ell, \qquad \ell \in \mathbb{N}. \tag{2.3.11}$$

[9] See [Löc2004], inequalities (13) or (24) which are essentially estimates for the marginals of the transition density of the killed particle motion as in (2.3.9).

[10] See Assumptions H3-H6 in [Löc2004] , pp. 138ff., and the example on p. 141. The extra conditions require that certain operator norms defined in terms of the coefficients b and σ are bounded uniformly in ℓ.

2.3.6 Remarks

- Note that although in Assumption 2.3.4 we do not require the integrability conditions (2.2.15) and (2.2.19) on the Fourier transforms of the upper bounds for the offspring resp. immigration distributions, the assertion of Corollary 2.2.15 remains valid since it was proved without using these conditions (see Remark 2.2.16). Thus we know that under the above assumptions, each term in the double series on the r.h.s. of (2.3.6) belongs to $\mathcal{C}_0(E)$.
- Of course, Assumption 2.3.5 implies finiteness of $\overline{m}$.
- By Theorem 1.2.9, we know that Assumption 2.3.5 is fulfilled if the branching rate and the reproduction probabilities are constants, reproduction is subcritical and (2.3.10) holds. By Remark 1.2.10, this can be generalized to the case that the branching rate is bounded away from 0 and the reproduction law admits as upper bound a space- and configuration-independent subcritical law in a convolution sense, see (1.2.17). Thus there are important classes of processes for which Assumption 2.3.5 can be readily verified. ⧫

The combination of Assumptions 2.3.2-2.3.5 may seem awkward, but they are the weakest conditions we have been able to obtain which allow for the program sketched above to be carried out and, at the same time, are satisfiable by general theorems such as those in [Löc2004] and Theorem 1.2.9. At the same time, we do not exclude special cases in which these results may not be applicable but the above assumptions are satisfied. We now state the main result of this section:

2.3.7 Theorem
Grant Assumptions 2.3.2, 2.3.4 and 2.3.5. Then the invariant occupation measure $\overline{m}$ on $(E, \mathcal{B}_E)$ admits a density of class $\mathcal{C}_0(E)$.

The program for the proof of Theorem 2.3.7 is analogous to that of the last section: We know already that under the above assumptions, each term in the double series on the r.h.s. of (2.3.6) is in $\mathcal{C}_0(E)$. Thus it remains only to show uniform convergence of the series, i.e. we have to control the uniform norms $\|\mathcal{I}_\ell\gamma_n\|_\infty$. This will be done by a series of lemmas. As in [Löc2004], the basic idea is to estimate the marginals $\mathcal{I}_\ell\gamma_n$ in terms of the marginals $\mathcal{I}_\ell\gamma_{n-1}$ and to perform an iteration.[11] However, in the details our method of proof differs from that of the result in [Löc2004]: Namely, we will not estimate the uniform norm $\|\mathcal{I}_\ell\gamma_n\|_\infty$ in terms of $\|\mathcal{I}_\ell\gamma_{n-1}\|_\infty$ but (corresponding to the approach of the previous section) in terms of certain L^1-norms as outlined in (2.3.8).

We begin with some notations and definitions:

2.3.8 Notation
Suppose $T : E \times \mathcal{B}_E \to [0, \infty]$ is a kernel. If $p, q \in [1, \infty]$ are such that T induces a bounded linear operator $T : L^p(E) \to L^q(E)$, we write

$$\|T\|_{p\to q} := \sup_{f \in L^p(E)\smallsetminus\{0\}} \frac{\|Tf\|_q}{\|f\|_p} < \infty \tag{2.3.12}$$

for its operator norm, where for the purposes of this notation we identify $L^\infty(E)$ with $\mathscr{B}(E)$, the space of bounded measurable functions.

[11] See the proof of Theorem 4.2 in [Löc2004].

Next, as in Definition 2.2.17 we define upper bound kernels in terms of the r.h.s. of the estimate (2.3.9). Now the upper bound kernels are defined on the single particle space E; nevertheless they carry a superscript ℓ since they depend on the constant K_ℓ in (2.3.9) which is allowed to grow polynomially in ℓ:

2.3.9 Definition
Under Assumption 2.3.2, with K_ℓ and ε as in (2.3.9) *let us define*

$$\tilde{p}_t^{(\ell)}(x) := K_\ell \cdot t^{-d/2} \exp\left(-\frac{\|x\|^2}{2K_\ell t}\right), \qquad x \in E,\ t > 0, \tag{2.3.13}$$

$$\tilde{r}_\varepsilon^{(\ell)}(x) := \int_0^\varepsilon \tilde{p}_t^{(\ell)}(x)\, dt, \qquad x \in E. \tag{2.3.14}$$

The corresponding convolution kernels induced by $\tilde{p}_t^{(\ell)}$ and $\tilde{r}_\varepsilon^{(\ell)}$ will be denoted by $\tilde{P}_t^{(\ell)}$ and $\tilde{R}_\varepsilon^{(\ell)}$, respectively.

2.3.10 Remarks
Since $\tilde{p}_t^{(\ell)}$ and $\tilde{r}_\varepsilon^{(\ell)}$ are integrable, $\tilde{P}_t^{(\ell)}$ and $\tilde{R}_\varepsilon^{(\ell)}$ induce bounded operators on both $L^1(E)$ and $\mathscr{B}(E)$:

$$\|\tilde{R}_\varepsilon^{(\ell)} f\|_1 = \|\tilde{r}_\varepsilon^{(\ell)} * f\|_1 \le \|\tilde{r}_\varepsilon^{(\ell)}\|_1 \cdot \|f\|_1, \qquad f \in L^1(E),$$

$$\|\tilde{R}_\varepsilon^{(\ell)} f\|_\infty = \|\tilde{r}_\varepsilon^{(\ell)} * f\|_1 \le \|\tilde{r}_\varepsilon^{(\ell)}\|_1 \cdot \|f\|_\infty, \qquad f \in \mathscr{B}(E),$$

and the same for every $\tilde{P}_t^{(\ell)}$, $t > 0$. In the notation introduced above, we have

$$\|\tilde{R}_\varepsilon^{(\ell)}\|_{1\to1} = \|\tilde{R}_\varepsilon^{(\ell)}\|_{\infty\to\infty} = \|\tilde{r}_\varepsilon^{(\ell)}\|_1, \qquad \|\tilde{P}_t^{(\ell)}\|_{1\to1} = \|\tilde{P}_t^{(\ell)}\|_{\infty\to\infty} = \|\tilde{p}_t^{(\ell)}\|_1, \quad t > 0.$$

By dominated convergence, it is clear that $\tilde{R}_\varepsilon^{(\ell)}$ preserves the subspaces $\mathcal{C}_b(E)$ and $\mathcal{C}_0(E)$ of $\mathscr{B}(E)$, so that it can also be considered as an operator on these spaces. For $\tilde{P}_t^{(\ell)}$, the situation is even better since $\tilde{p}_t^{(\ell)}$ is bounded for all $t > 0$: Thus we also have $\|\tilde{P}_t^{(\ell)} f\|_\infty = \|\tilde{p}_t^{(\ell)} * f\|_\infty \le \|\tilde{p}_t^{(\ell)}\|_\infty \|f\|_1$ for $f \in L^1(E)$ and consequently

$$\|\tilde{P}_t^{(\ell)}\|_{1\to\infty} = \|\tilde{p}_t^{(\ell)}\|_\infty = K_\ell \cdot t^{-d/2} < \infty, \qquad t > 0. \tag{2.3.15}$$

Moreover, since $\tilde{p}_t^{(\ell)}$ is a Schwartz function for each $t > 0$ fixed, $\tilde{P}_t^{(\ell)}$ maps $L^1(E)$ into $\mathcal{C}_0(E)$ (even into $\mathcal{C}_b^\infty(E)$). The same is generally not true for $\tilde{R}_\varepsilon^{(\ell)}$: As in the previous section, $\tilde{r}_\varepsilon^{(\ell)}$ will in general have a singularity at the origin, and we will have $\|\tilde{R}_\varepsilon^{(\ell)}\|_{1\to\infty} = \infty$. However, the composition of "sufficiently many" of the operators $\tilde{R}_\varepsilon^{(\ell)}$ *does* act as a bounded operator $L^1(E) \to \mathscr{B}(E)$, see Lemma 2.3.15 below. ♦

We recall the definition of the kernel $R_{\alpha,\varepsilon}$ in (2.2.9) and the identity (2.2.10). Consider a nonnegative measurable function $f : \mathcal{S} \to \mathbb{R}_+$. Fix $\ell \in \mathbb{N}$ and $i \in \{1, \ldots, \ell\}$. Under Assumption

2.3.2, by the estimate (2.3.9) we get for the marginals of $R^*_{\alpha,\varepsilon} f$

$$\begin{aligned}
\mathcal{I}^i_\ell R^*_{\alpha,\varepsilon} f(y^i) &\equiv \int_{E^{\ell-1}} d\check{\boldsymbol{y}}^i R^*_{\alpha,\varepsilon} f(\boldsymbol{y}) \\
&= \int_{E^{\ell-1}} d\check{\boldsymbol{y}}^i \int_{E^\ell} d\boldsymbol{x}\, r^{(\ell)}_{\alpha,\varepsilon}(\boldsymbol{x};\boldsymbol{y})\, f^{(\ell)}(\boldsymbol{x}) \\
&= \int_{E^\ell} d\boldsymbol{x}\, f^{(\ell)}(\boldsymbol{x}) \int_0^\varepsilon dt \int_{E^{\ell-1}} d\check{\boldsymbol{y}}^i\, p^\alpha_t(\boldsymbol{x};\boldsymbol{y}) \\
&\le \int_{E^\ell} d\boldsymbol{x}\, f^{(\ell)}(\boldsymbol{x}) \int_0^\varepsilon dt\, \tilde{p}^{(\ell)}_t(x^i - y^i) \\
&= \int_E dx^i\, \tilde{r}^{(\ell)}_\varepsilon(y^i - x^i) \int_{E^{\ell-1}} d\check{\boldsymbol{x}}^i f^{(\ell)}(\boldsymbol{x}) \\
&= \left(\tilde{r}^{(\ell)}_\varepsilon * \mathcal{I}^i_\ell f\right)(y^i) = \tilde{R}^{(\ell)}_\varepsilon \mathcal{I}^i_\ell f(y^i).
\end{aligned} \tag{2.3.16}$$

In a similar vein, for $(P^\alpha_\varepsilon)^*$ we have

$$\mathcal{I}^i_\ell (P^\alpha_\varepsilon)^* f \le \tilde{P}^{(\ell)}_\varepsilon \mathcal{I}^i_\ell f. \tag{2.3.17}$$

Thus we can estimate the marginals of $R^*_{\alpha,\varepsilon} f$ and $(P^\alpha_\varepsilon)^* f$ pointwise by the marginals of f. Moreover, in view of the identity (2.2.10) it follows that

$$\mathcal{I}^i_\ell R^*_\alpha f = \mathcal{I}^i_\ell R^*_{\alpha,\varepsilon} f + \mathcal{I}^i_\ell (P^\alpha_\varepsilon)^* R^*_\alpha f \le \left(\tilde{R}^{(\ell)}_\varepsilon \mathcal{I}^i_\ell f + \tilde{P}^{(\ell)}_\varepsilon \mathcal{I}^i_\ell R^*_\alpha f\right). \tag{2.3.18}$$

Next, we need to know how to estimate the marginals of $[\alpha K_0]^* f$ in terms of (the marginals of) f for $f \ge 0$. We need the following definition:

2.3.11 Definition
Grant Assumption 2.3.4.

1. *For all $\ell \in \mathbb{N}$ and $k \in \{1, \dots, k_0\}$, we define an operator $\tilde{Q}^{(\ell)}_k$ by convolution with the function $z \mapsto \widehat{q}^{(\ell)}_k(-z)$, where $\widehat{q}^{(\ell)}_k$ is the upper bound for the spatial offspring density from Assumption 2.2.5:*

$$\tilde{Q}^{(\ell)}_k g(x) := \int_E dv\, g(v) \widehat{q}^{(\ell)}_k(v - x), \qquad x \in E. \tag{2.3.19}$$

2. *For all $\ell, n \in \mathbb{N}$ we define the following finite sets of operators:*

$$\mathcal{R}_{\ell,n} := \left\{\tilde{R}^{(i)}_\varepsilon : 1 \vee (\ell - n(k_0 - 1)) \le i \le \ell + n - 1\right\}, \tag{2.3.20}$$

$$\mathcal{Q}_{\ell,n} := \left\{\mathrm{Id},\ \tilde{Q}^{(i)}_k : 1 \vee (\ell - n(k_0 - 1)) \le i \le \ell + n - 1,\ 1 \le k \le k_0\right\}. \tag{2.3.21}$$

Moreover, let

$$A_\ell := 1 \vee \max\left\{\|\tilde{r}^{(i)}_\varepsilon\|_1 : 1 \le i \le \ell\right\}, \tag{2.3.22}$$

$$B_\ell := \max\left\{\|\widehat{q}^{(i)}_k\|_1 : 1 \le i \le \ell,\ 1 \le k \le k_0\right\} \ge 1. \tag{2.3.23}$$

2.3.12 Remark
Since $\widehat{q}_k^{(\ell)}$ is integrable, as before we have

$$\|\tilde{Q}_k^{(\ell)}\|_{1\to 1} = \|\tilde{Q}_k^{(\ell)}\|_{\infty\to\infty} = \|\widehat{q}_k^{(\ell)}\|_1 < \infty$$

for the operator norm of $\tilde{Q}_k^{(\ell)}$ as an operator on $L^1(E)$ or $\mathscr{B}(E)$.

Under Assumptions 2.3.2 and 2.3.4, it is clear that A_ℓ and B_ℓ are increasing and grow at most polynomially in ℓ. Observe that whenever $S \in \mathcal{R}_{\ell,n}$ and $T \in \mathcal{Q}_{\ell,n}$, we have $\|S\| \le A_{\ell+n-1}$ and $\|T\| \le B_{\ell+n-1}$. Also note that for $n \in \mathbb{N}$ fixed, the number of elements in $\mathcal{R}_{\ell,n}$ resp. $\mathcal{Q}_{\ell,n}$ is bounded by a constant depending only on n, uniformly in ℓ. In fact, for all $\ell, n \in \mathbb{N}$ we have

$$|\mathcal{R}_{\ell,n}| \le nk_0, \qquad |\mathcal{Q}_{\ell,n}| \le nk_0^2 + 1.$$

♦

2.3.13 Lemma
Under Assumption 2.3.4 there is an increasing sequence $(M'_\ell)_\ell$ of constants growing at most polynomially in ℓ such that the following holds: For all nonnegative measurable functions $g : \mathcal{S} \to \mathbb{R}_+$ and all $\ell \in \mathbb{N}$ we have the pointwise estimate

$$\mathcal{I}_\ell[\alpha K_0]^* g \le M'_\ell \left(\|g^{(\ell-1)}\|_1 + \sum_{m=1\vee(\ell-k_0+1)}^{\ell+1} \mathcal{I}_m g + \sum_{m=1\vee(\ell-k_0+1)}^{\ell} \tilde{Q}_{\ell-m+1}^{(m)} \mathcal{I}_m g \right). \tag{2.3.24}$$

Proof Recalling from (2.2.44) the definition of the constants $D_\ell = \|\kappa^{(\ell)}\|_\infty \vee \|c^{(\ell)}\|_\infty$, we put

$$M'_\ell := (\ell+1) \cdot \left(1 \vee \max_{1\le k\le \ell+1} D_k\right) \cdot B_{\ell+1}^{k_0} \cdot \|\widehat{\pi}^{(\ell-1)}\|_1 \cdot (1 \vee \|\widehat{\pi}^{(\ell-1)}\|_\infty), \qquad \ell \in \mathbb{N}. \tag{2.3.25}$$

Clearly, under Assumption 2.3.4 the constants M'_ℓ grow only polynomially in ℓ.

Now let $g : \mathcal{S} \to \mathbb{R}_+$ measurable and fix $\ell \in \mathbb{N}$. By Lemma 2.2.9, $[\alpha K_0]^* g$ is absolutely continuous and we know its density on E^ℓ, from which we easily get estimates for the marginals. In fact, in the proof of Lemma 2.2.14 we have already computed "marginals" $\mathcal{I}_\ell^J([\alpha K_0]^* g)$ for arbitrary subsets $J \subseteq \{1, \ldots, \ell\}$, see formula (2.2.37). Taking $J = \{i\}$ in (2.2.37) and using the upper bounds of Assumption 2.2.5, we obtain for all $i \in \{1, \ldots, \ell\}$ and $y^i \in E$

$$\begin{aligned} \mathcal{I}_\ell^i([\alpha K_0]^* g)(y^i) &\equiv \int_{E^{\ell-1}} d\check{\boldsymbol{y}}^i \left([\alpha K_0]^* g\right)^{(\ell)}(\boldsymbol{y}) \\ &\le \sum_{k=0}^{k_0\wedge \ell} D_{\ell-k+1} \sum_{j=1}^{\ell-k+1} \int_{E^{\ell-1}} d\check{\boldsymbol{y}}^i \int_E dv\, g^{(\ell-k+1)}\left(\Pi^*_{\ell,k,j}(\boldsymbol{y}; v)\right) \prod_{r=j}^{j+k-1} \widehat{q}_k^{(\ell-k+1)}(v - y^r) \\ &\quad + \mathbf{1}_{\ell\ge 2} D_{\ell-1} \int_{E^{\ell-1}} d\check{\boldsymbol{y}}^i g^{(\ell-1)}(y^1, \ldots, y^{\ell-1})\, \widehat{\pi}^{(\ell-1)}(y^\ell), \end{aligned} \tag{2.3.26}$$

where

$$\Pi^*_{\ell,k,j}(\boldsymbol{y}; v) = (y^1, \ldots, y^{j-1}, v, y^{j+k}, \ldots, y^\ell)$$

is defined in (2.2.25). (Also recall that $\widehat{q}_0^{(\ell-k+1)} := 1$ by convention (2.2.14).)

Now fix $k \in \{1, \dots, k_0 \wedge \ell\}$ and $j \in \{1, \dots, \ell - k + 1\}$ and consider the corresponding term in the sum (2.3.26). We observe that

$$\int_{E^{\ell-1}} d\check{\boldsymbol{y}}^i \int_E dv\, g^{(\ell-k+1)}(y^1, \dots, y^{j-1}, v, y^{j+k}, \dots, y^\ell) \prod_{r=j}^{j+k-1} \widehat{q}_k^{(\ell-k+1)}(v - y^r)$$
$$= \begin{cases} \|\widehat{q}_k^{\ell-k+1}\|_1^k \cdot \left[\mathcal{I}^i_{\ell-k+1} g\right](y^i) & \text{if } i \in \{1, \dots, j-1\} \\ \|\widehat{q}_k^{\ell-k+1}\|_1^{k-1} \cdot \left[\tilde{Q}_k^{(\ell-k+1)} \mathcal{I}^{i-r}_{\ell-k+1} g\right](y^i) & \text{if } i = j + r,\ r \in \{0, \dots, k-1\} \\ \|\widehat{q}_k^{\ell-k+1}\|_1^k \cdot \left[\mathcal{I}^{i-k+1}_{\ell-k+1} g\right](y^i) & \text{if } i \in \{j+k, \dots, \ell\}, \end{cases} \tag{2.3.27}$$

where we have used the definition of $\tilde{Q}_k^{(\ell-k+1)}$ in (2.3.19). The preceding formula is also true for $k = 0$ with the understanding that the second case does not occur. Summing (2.3.27) over $i = 1, \dots, \ell$ for fixed j gives an upper bound

$$\|\widehat{q}_k^{\ell-k+1}\|_1^k \left(\mathcal{I}_{\ell-k+1} g + \mathbf{1}_{k \neq 0}\, \tilde{Q}_k^{(\ell-k+1)} \mathcal{I}_{\ell-k+1} g\right)$$

which does not depend on j.

The last term in (2.3.26), stemming from the immigration, equals

$$\int_{E^{\ell-1}} d\check{\boldsymbol{y}}^i g^{(\ell-1)}(y^1, \dots, y^{\ell-1})\, \widehat{\pi}^{(\ell-1)}(y^\ell)$$
$$= \begin{cases} \|\widehat{\pi}^{(\ell-1)}\|_1 \cdot \mathcal{I}^i_{\ell-1} g(y^i) & \text{if } i \in \{1, \dots, \ell-1\} \\ \|g^{(\ell-1)}\|_1 \cdot \widehat{\pi}^{(\ell-1)}(y^\ell) & \text{if } i = \ell. \end{cases} \tag{2.3.28}$$

Consequently, summing over $i = 1, \dots, \ell$ in (2.3.26) we get an upper bound

$$\begin{aligned} &\mathcal{I}_\ell([\alpha K_0]^* g) \equiv \sum_{i=1}^{\ell} \mathcal{I}^i_\ell([\alpha K_0]^* g) \\ &\le \sum_{k=0}^{k_0 \wedge \ell} D_{\ell-k+1}\,(\ell - k + 1)\, \|\widehat{q}_k^{(\ell-k+1)}\|_1^k \left(\mathcal{I}_{\ell-k+1} g + \mathbf{1}_{k \neq 0} \tilde{Q}_k^{(\ell-k+1)} \mathcal{I}_{\ell-k+1} g\right) \\ &\quad + \mathbf{1}_{\ell \ge 2} D_{\ell-1} \left(\|\widehat{\pi}^{(\ell-1)}\|_1\, \mathcal{I}_{\ell-1} g + \|g^{(\ell-1)}\|_1\, \|\widehat{\pi}^{(\ell-1)}\|_\infty\right) \\ &\le M'_\ell \left(\|g^{(\ell-1)}\|_1 + \sum_{k=0}^{k_0 \wedge \ell} \mathcal{I}_{\ell-k+1} g + \sum_{k=1}^{k_0 \wedge \ell} \tilde{Q}_k^{(\ell-k+1)} \mathcal{I}_{\ell-k+1} g\right) \\ &= M'_\ell \left(\|g^{(\ell-1)}\|_1 + \sum_{m=1 \vee (\ell-k_0+1)}^{\ell+1} \mathcal{I}_m g + \sum_{m=1 \vee (\ell-k_0+1)}^{\ell} \tilde{Q}^{(m)}_{\ell-m+1} \mathcal{I}_m g\right) \end{aligned} \tag{2.3.29}$$

with M'_ℓ as in (2.3.25). ∎

For $g : \mathcal{S} \to \mathbb{R}_+$ measurable, combining (2.3.16) and (2.3.24) we get a pointwise estimate of the marginals of $R^*_{\alpha,\varepsilon}[\alpha K_0]^* g$ in terms of $\|g^{(\ell-1)}\|_1$ and of the marginals of g:

$$\begin{aligned} &\mathcal{I}_\ell R^*_{\alpha,\varepsilon}[\alpha K_0]^* g \\ &\le M'_\ell \left(\|\tilde{r}_\varepsilon^{(\ell)}\|_1 \cdot \|g^{(\ell-1)}\|_1 + \sum_{m=1 \vee (\ell-k_0+1)}^{\ell+1} \tilde{R}_\varepsilon^{(\ell)} \mathcal{I}_m g + \sum_{m=1 \vee (\ell-k_0+1)}^{\ell} \tilde{R}_\varepsilon^{(\ell)} \tilde{Q}^{(m)}_{\ell-m+1} \mathcal{I}_m g\right). \end{aligned} \tag{2.3.30}$$

In the next lemma, we iterate the above estimate in order to obtain a bound for the marginals of powers $\left(R^*_{\alpha,\varepsilon}[\alpha K_0]^*\right)^n g$:

2.3.14 Lemma

Grant Assumptions 2.3.2 and 2.3.4. Then there is an increasing sequence $(M_\ell)_\ell$ of constants growing at most polynomially in ℓ and a constant $C < \infty$ such that the following holds: For all nonnegative measurable functions $f : E \to \mathbb{R}_+$ and all $\ell, n \in \mathbb{N}$ we have

$$\begin{aligned} &\mathcal{I}_\ell \left(R^*_{\alpha,\varepsilon}[\alpha K_0]^*\right)^n f \\ &\le C^{n-1} M_\ell M_{\ell+1} \cdots M_{\ell+n-1} \sum_{k=1\vee(\ell-n(k_0-1))}^{\ell+n} \left(\sum_{j=0}^{n-1} \left\| \left((R^*_\alpha[\alpha K_0]^*)^j f \right)^{(k)} \right\|_1 \right. \\ &\qquad \left. + \sum_{S_1,\dots,S_n \in \mathcal{R}_{\ell,n}} \sum_{T_1,\dots,T_n \in \mathcal{Q}_{\ell,n}} S_1 \cdots S_n T_1 \cdots T_n \mathcal{I}_k f \right) \end{aligned} \tag{2.3.31}$$

with $\mathcal{R}_{\ell,n}$ and $\mathcal{Q}_{\ell,n}$ from Definition 2.3.11.

Proof With M'_ℓ as in (2.3.25), we set

$$\begin{aligned} C &:= k_0 + 1, \\ M_\ell &:= M'_\ell \cdot A_\ell B_\ell. \end{aligned} \tag{2.3.32}$$

Then it is clear that $(M_\ell)_\ell$ is increasing and grows at most polynomially in ℓ.

Now let $f : \mathcal{S} \to \mathbb{R}_+$ nonnegative and measurable. The assertion will be proved inductively in $n \in \mathbb{N}$. For $n = 1$, the estimate (2.3.31) follows directly from (2.3.30).

Now suppose that (2.3.31) holds for some $n \in \mathbb{N}$ and all $\ell \in \mathbb{N}$. We will show that it then also holds for $n+1$ and all $\ell \in \mathbb{N}$. To this end, fix $\ell \in \mathbb{N}$. Letting $g := (R^*_{\alpha,\varepsilon}[\alpha K_0]^*)^n f \ge 0$, we have by (2.3.30)

$$\begin{aligned} &\mathcal{I}_\ell (R^*_{\alpha,\varepsilon}[\alpha K_0]^*)^{n+1} f \equiv \mathcal{I}_\ell R^*_{\alpha,\varepsilon}[\alpha K_0]^* g \\ &\le M'_\ell \left(\|\tilde{r}^{(\ell)}_\varepsilon\|_1 \|g^{(\ell-1)}\|_1 + \sum_{m=1\vee(\ell-k_0+1)}^{\ell+1} \tilde{R}^{(\ell)}_\varepsilon \mathcal{I}_m g + \sum_{m=1\vee(\ell-k_0+1)}^{\ell} \tilde{R}^{(\ell)}_\varepsilon \tilde{Q}^{(m)}_{\ell-m+1} \mathcal{I}_m g \right). \end{aligned} \tag{2.3.33}$$

Now fix some $m \in \{1 \vee (\ell - k_0 + 1), \dots, \ell + 1\}$. Since $g \ge 0$, the induction hypothesis gives

$$\begin{aligned} \mathcal{I}_m g \le C^{n-1} M_m \cdots M_{m+n-1} &\sum_{k=1\vee(m-n(k_0-1))}^{m+n} \left(\sum_{j=0}^{n-1} \left\| \left((R^*_\alpha[\alpha K_0]^*)^j f \right)^{(k)} \right\|_1 \right. \\ &\left. + \sum_{S_1,\dots,S_n \in \mathcal{R}_{m,n}} \sum_{T_1,\dots,T_n \in \mathcal{Q}_{m,n}} S_1 \cdots S_n T_1 \cdots T_n \mathcal{I}_k f \right). \end{aligned} \tag{2.3.34}$$

Observe that the definition of $\mathcal{R}_{m,n}$ and $\mathcal{Q}_{m,n}$ implies

$$\mathcal{R}_{m,n} \subseteq \mathcal{R}_{\ell,n+1}, \qquad \mathcal{Q}_{m,n} \subseteq \mathcal{Q}_{\ell,n+1}. \tag{2.3.35}$$

Thus we can continue the estimate (2.3.34) to obtain

$$\begin{aligned} \mathcal{I}_m g \le C^{n-1} M_{\ell+1} \cdots M_{\ell+n} &\sum_{k=1\vee(\ell-(n+1)(k_0-1))}^{\ell+(n+1)} \left(\sum_{j=0}^{n-1} \left\| \left((R^*_\alpha[\alpha K_0]^*)^j f \right)^{(k)} \right\|_1 \right. \\ &\left. + \sum_{S_1,\dots,S_n \in \mathcal{R}_{\ell,n+1}} \sum_{T_1,\dots,T_n \in \mathcal{Q}_{\ell,n+1}} S_1 \cdots S_n T_1 \cdots T_n \mathcal{I}_k f \right), \end{aligned} \tag{2.3.36}$$

where we have also used that the sequence $(M_k)_k$ is increasing. Observe that the estimate (2.3.36) is independent of m. Applying the (positive) operator $\tilde{R}_\varepsilon^{(\ell)}$ and summing over m gives

$$\begin{aligned}
&\sum_{m=1\vee(\ell-k_0+1)}^{\ell+1} \tilde{R}_\varepsilon^{(\ell)} \mathcal{I}_m g \\
&\le C^{n-1} M_{\ell+1} \cdots M_{\ell+n} \sum_{k=1\vee(\ell-(n+1)(k_0-1))}^{\ell+(n+1)} \Bigg((k_0+1) \|\tilde{r}_\varepsilon^{(\ell)}\|_1 \cdot \sum_{j=0}^{n-1} \left\| \big((R_\alpha^*[\alpha K_0]^*)^j f \big)^{(k)} \right\|_1 \\
&\qquad + \sum_{S_1,\dots,S_n \in \mathcal{R}_{\ell,n+1}} \sum_{T_1,\dots,T_n \in \mathcal{Q}_{\ell,n+1}} \underbrace{\tilde{R}_\varepsilon^{(\ell)}}_{\in \mathcal{R}_{\ell,n+1}} S_1 \cdots S_n T_1 \cdots T_n \mathcal{I}_k f \Bigg) \\
&\le C^n A_\ell \cdot M_{\ell+1} \cdots M_{\ell+n} \sum_{k=1\vee(\ell-(n+1)(k_0-1))}^{\ell+(n+1)} \Bigg(\sum_{j=0}^{n-1} \left\| \big((R_\alpha^*[\alpha K_0]^*)^j f \big)^{(k)} \right\|_1 \\
&\qquad + \sum_{S_1,\dots,S_{n+1} \in \mathcal{R}_{\ell,n+1}} \sum_{T_1,\dots,T_n \in \mathcal{Q}_{\ell,n+1}} S_1 \cdots S_{n+1} T_1 \cdots T_n \mathcal{I}_k f \Bigg).
\end{aligned} \tag{2.3.37}$$

Similarly, applying the positive operator $\tilde{R}_\varepsilon^{(\ell)} \tilde{Q}_{\ell-m+1}^{(m)}$ to (2.3.36) for $m \in \{1 \vee (\ell - k_0 + 1), \dots, \ell\}$ and summing over m we obtain an estimate

$$\begin{aligned}
&\sum_{m=1\vee(\ell-k_0+1)}^{\ell} \tilde{R}_\varepsilon^{(\ell)} \tilde{Q}_{\ell-m+1}^{(m)} \mathcal{I}_m g \\
&\le C^{n-1} M_{\ell+1} \cdots M_{\ell+n} \sum_{k=1\vee(\ell-(n+1)(k_0-1))}^{\ell+(n+1)} \Bigg(k_0 \|\tilde{r}_\varepsilon^{(\ell)}\|_1 \|\tilde{Q}_{\ell-m+1}^{(m)}\| \cdot \sum_{j=0}^{n-1} \left\| \big((R_\alpha^*[\alpha K_0]^*)^j f \big)^{(k)} \right\|_1 \\
&\qquad + \sum_{S_1,\dots,S_n \in \mathcal{R}_{\ell,n+1}} \sum_{T_1,\dots,T_n \in \mathcal{Q}_{\ell,n+1}} \tilde{R}_\varepsilon^{(\ell)} \underbrace{\tilde{Q}_{\ell-m+1}^{(m)}}_{\in \mathcal{Q}_{\ell,n+1} \setminus \{\mathrm{Id}\}} S_1 \cdots S_n T_1 \cdots T_n \mathcal{I}_k f \Bigg) \\
&\le C^n A_\ell B_\ell \cdot M_{\ell+1} \cdots M_{\ell+n} \sum_{k=1\vee(\ell-(n+1)(k_0-1))}^{\ell+(n+1)} \Bigg(\sum_{j=0}^{n-1} \left\| \big((R_\alpha^*[\alpha K_0]^*)^j f \big)^{(k)} \right\|_1 \\
&\qquad + \sum_{S_1,\dots,S_{n+1} \in \mathcal{R}_{\ell,n+1}} \sum_{T_1,\dots,T_n \in \mathcal{Q}_{\ell,n+1}} \sum_{T_{n+1} \in \mathcal{Q}_{\ell,n+1} \setminus \{\mathrm{Id}\}} S_1 \cdots S_{n+1} T_1 \cdots T_n T_{n+1} \mathcal{I}_k f \Bigg),
\end{aligned} \tag{2.3.38}$$

where we have also used that $\tilde{Q}_{\ell-m+1}^{(m)}$ commutes with every $S_1, \dots, S_n$ since they are all convolution operators. Taking together (2.3.37) and (2.3.38), we have shown that

$$\sum_{m=1\vee(\ell-k_0+1)}^{\ell+1} \tilde{R}_\varepsilon^{(\ell)}\mathcal{I}_m g + \sum_{m=1\vee(\ell-k_0+1)}^{\ell} \tilde{R}_\varepsilon^{(\ell)}\tilde{Q}_{\ell-m+1}^{(m)}\mathcal{I}_m g$$
$$\le C^n A_\ell B_\ell \cdot M_{\ell+1}\cdots M_{\ell+n} \sum_{k=1\vee(\ell-(n+1)(k_0-1))}^{\ell+(n+1)} \Bigg(\sum_{j=0}^{n-1} \left\| \left((R_\alpha^*[\alpha K_0]^*)^j f\right)^{(k)} \right\|_1 \tag{2.3.39}$$
$$+ \sum_{S_1,\dots,S_{n+1}\in\mathcal{R}_{\ell,n+1}} \sum_{T_1,\dots,T_{n+1}\in\mathcal{Q}_{\ell,n+1}} S_1\cdots S_{n+1}T_1\cdots T_{n+1}\mathcal{I}_k f \Bigg).$$

For the first term on the r.h.s. of (2.3.33), since $R_{\alpha,\varepsilon}^* \le R_\alpha^*$ we obviously have

$$\begin{aligned} &\|\tilde{r}_\varepsilon^{(\ell)}\|_1 \|g^{(\ell-1)}\|_1 \\ &\le A_\ell \left\| \left((R_\alpha^*[\alpha K_0]^*)^n f\right)^{(\ell-1)} \right\|_1 \\ &\le C^n A_\ell B_\ell M_{\ell+1}M_{\ell+2}\cdots M_{\ell+n} \sum_{k=1\vee(\ell-(n+1)(k_0-1))}^{\ell+(n+1)} \left\| \left((R_\alpha^*[\alpha K_0]^*)^n f\right)^{(k)} \right\|_1 . \end{aligned} \tag{2.3.40}$$

Combining (2.3.39) with (2.3.40) and observing (2.3.32), we have proved that (2.3.31) holds for ℓ and $n+1$, and the proof is complete. ■

2.3.15 Lemma
Under Assumption 2.3.2, there is an increasing sequence of constants $1 \le \tilde{K}_1 \le \tilde{K}_2 \le \cdots < \infty$ which is polynomial in ℓ such that the following holds: For any $n \in \mathbb{N}$ with $n > d/2$, there is a constant $\tilde{C}_n < \infty$ such that for any choice of $\ell \in \mathbb{N}$ and $k_1, \dots, k_n \in \{1, \dots, \ell\}$ we have

$$\|\tilde{R}_\varepsilon^{(k_1)}\tilde{R}_\varepsilon^{(k_2)}\cdots\tilde{R}_\varepsilon^{(k_n)}\|_{1\to\infty} \le \tilde{C}_n \tilde{K}_\ell^n < \infty, \tag{2.3.41}$$

i.e. $R_\varepsilon^{(k_1)}\tilde{R}_\varepsilon^{(k_2)}\cdots R_\varepsilon^{(k_n)}$ acts as a bounded operator $L^1(E) \to \mathscr{B}(E)$ and its operator norm is bounded by the r.h.s. of (2.3.41), uniformly in $k_i \in \{1, \dots, \ell\}$.

Proof The claim is again most easily seen by a Fourier inversion argument: By the same calculation as in the proof of Lemma 2.2.20, for each $\ell \in \mathbb{N}$ we have an estimate

$$\mathscr{F}[\tilde{r}_\varepsilon^{(\ell)}](\xi) = 2\,(2\pi K_\ell)^{d/2} \cdot \frac{1 - e^{-\frac{1}{2}K_\ell\varepsilon\|\xi\|^2}}{\|\xi\|^2} \le \tilde{K}_\ell \cdot h(\|\xi\|), \qquad \xi \in E, \tag{2.3.42}$$

where h is the function from (2.2.48) and

$$\tilde{K}_\ell := 2\,(2\pi K_\ell)^{d/2}) \cdot \left(1 \vee \frac{1}{2}K_\ell\varepsilon\right) \tag{2.3.43}$$

(see in particular the argument around (2.2.51) and (2.2.52)). Due to Assumption 2.3.2, $(\tilde{K}_\ell)_{\ell\in\mathbb{N}}$ is polynomial in ℓ (in contrast to the constants C'_ℓ in (2.2.49) which are generally not).

Now fix $\ell \in \mathbb{N}$ and choose any $k_1, \dots, k_n \in \{1, \dots, \ell\}$. Then by (2.3.42), for each $\xi \in E$ we have

$$\left|\mathscr{F}\left[\tilde{r}_\varepsilon^{(k_1)} * \tilde{r}_\varepsilon^{(k_2)} * \cdots * \tilde{r}_\varepsilon^{(k_n)}\right](\xi)\right| = \prod_{i=1}^n \left|\mathscr{F}\left[\tilde{r}_\varepsilon^{(k_i)}\right](\xi)\right| \le \tilde{K}_\ell^n \cdot h(\|\xi\|)^n$$

since $\tilde{K}_\ell$ is increasing in ℓ. Recall that the function $h(\|\xi\|)$ from (2.2.48) is continuous and decays as $\|\xi\|^{-2}$ on E. Thus from the above estimate, by choice of $n > d/2$ and integration in hyperspherical coordinates we see that $\mathscr{F}\left[\tilde{r}_\varepsilon^{(k_1)} * \tilde{r}_\varepsilon^{(k_2)} * \cdots * \tilde{r}_\varepsilon^{(k_n)}\right]$ is integrable on E. By the Fourier inversion theorem, we conclude that

$$\tilde{r}_\varepsilon^{(k_1)} * \tilde{r}_\varepsilon^{(k_2)} * \cdots * \tilde{r}_\varepsilon^{(k_n)} = (2\pi)^{-d}\mathscr{F}^{-1}\left[\mathscr{F}\left[\tilde{r}_\varepsilon^{(k_1)} * \tilde{r}_\varepsilon^{(k_2)} * \cdots * \tilde{r}_\varepsilon^{(k_n)}\right]\right] \qquad \lambda\text{-a.s.}$$

In particular, $\tilde{r}_\varepsilon^{(k_1)} * \tilde{r}_\varepsilon^{(k_2)} * \cdots * \tilde{r}_\varepsilon^{(k_n)}$ coincides λ-a.s. with a bounded function (even a $\mathcal{C}_0$-function) which of course induces the same convolution operator, namely our given composition $\tilde{R}_\varepsilon^{(k_1)}\cdots\tilde{R}_\varepsilon^{(k_n)}$. Thus we have proved that for all $f \in L^1(E)$

$$\begin{aligned}\|\tilde{R}_\varepsilon^{(k_1)}\cdots\tilde{R}_\varepsilon^{(k_n)} f\|_\infty &= \|\tilde{r}_\varepsilon^{(k_1)} * \cdots * \tilde{r}_\varepsilon^{(k_n)} * f\|_\infty \\ &\le (2\pi)^{-d}\left\|\mathscr{F}\left[\tilde{r}_\varepsilon^{(k_1)} * \tilde{r}_\varepsilon^{(k_2)} * \cdots * \tilde{r}_\varepsilon^{(k_n)}\right]\right\|_1 \cdot \|f\|_1 \\ &\le (2\pi)^{-d}\tilde{K}_\ell^n \int_E h(\|\xi\|)^n \, d\xi \cdot \|f\|_1.\end{aligned}$$

Thus (2.3.41) holds with $\tilde{C}_n := (2\pi)^{-d}\int_E h(\|\xi\|)^n \, d\xi < \infty$. ∎

Now we can give the proof Theorem 2.3.7:

Proof [of Theorem 2.3.7] It remains only to show that the double series $\sum_{\ell\in\mathbb{N}}\sum_{n\in\mathbb{N}_0}\mathcal{I}_\ell\gamma_n$ in (2.3.6) converges uniformly on E.

As in (2.2.68), by expanding the identity (2.2.10) we get for every $n_0 \in \mathbb{N}$, $n \ge n_0$ and $\ell \in \mathbb{N}$

$$\begin{aligned}\mathcal{I}_\ell\gamma_n &= \mathcal{I}_\ell(R_\alpha^*[\alpha K_0]^*)^{n_0}\gamma_{n-n_0} \\ &= \mathcal{I}_\ell(R_{\alpha,\varepsilon}^*[\alpha K_0]^*)^{n_0}\gamma_{n-n_0} + \sum_{m=0}^{n_0-1}\mathcal{I}_\ell(R_{\alpha,\varepsilon}^*[\alpha K_0]^*)^m(P_\varepsilon^\alpha)^*\gamma_{n-m}.\end{aligned} \tag{2.3.44}$$

Now fix some $n_0 \in \mathbb{N}$ with $n_0 > d/2$. For $n \ge n_0$ and $\ell \in \mathbb{N}$, consider the first term on the r.h.s. of (2.3.44): Observing that $(R_\alpha^*[\alpha K_0]^*)^j\gamma_{n-n_0} = \gamma_{n-n_0+j}$, we use Lemma 2.3.14 to obtain an upper bound

$$\begin{aligned}&\|\mathcal{I}_\ell(R_{\alpha,\varepsilon}^*[\alpha K_0]^*)^{n_0}\gamma_{n-n_0}\|_\infty \\ &\le C^{n_0-1}M_\ell\cdots M_{\ell+n_0-1}\sum_{k=(\ell-n_0(k_0-1))\vee 1}^{\ell+n_0}\Bigg(\sum_{j=0}^{n_0-1}\|\gamma_{n-n_0+j}^{(k)}\|_1 \\ &\qquad + \sum_{S_1,\ldots,S_{n_0}\in\mathcal{R}_{\ell,n_0}}\ \sum_{T_1,\ldots T_{n_0}\in\mathcal{Q}_{\ell,n_0}}\|S_1\cdots S_{n_0}T_1\cdots T_{n_0}\mathcal{I}_k\gamma_{n-n_0}\|_\infty\Bigg).\end{aligned} \tag{2.3.45}$$

Since every S_i, $i = 1,\ldots,n_0$ is of the form $S_i = \tilde{R}_\varepsilon^{(k_i)}$ for some $k_i \le \ell + n_0 - 1$ (cf. the definition of $\mathcal{R}_{\ell,n_0}$ (2.3.20)) and since $n_0 > d/2$, we know from Lemma 2.3.15 that

$$\|S_1\cdots S_{n_0}\|_{1\to\infty} \le \tilde{C}_{n_0}\tilde{K}_{\ell+n_0-1}^{n_0}$$

uniformly in $S_1,\ldots,S_{n_0} \in \mathcal{R}_{\ell,n_0}$, with the constants $\tilde{C}_{n_0}$, $(\tilde{K}_\ell)_\ell$ as in (2.3.41). Further, we note that $\|T\|_{1\to1} \le B_{\ell+n_0-1}$ for all $T \in \mathcal{Q}_{\ell,n_0}$, where $(B_\ell)_\ell$ is defined in (2.3.23), and that $\|\mathcal{I}_k f\|_1 \le k\cdot\|f^{(k)}\|_1$ by definition of $\mathcal{I}_k$ in (2.3.5). Consequently, we get

$$\begin{aligned}\|S_1\cdots S_{n_0}T_1\cdots T_{n_0}\mathcal{I}_k\gamma_{n-n_0}\|_\infty &\le \|S_1\cdots S_{n_0}\|_{1\to\infty}\cdot\|T_1\|_{1\to1}\cdots\|T_{n_0}\|_{1\to1}\cdot\|\mathcal{I}_k\gamma_{n-n_0}\|_1 \\ &\le \tilde{C}_{n_0}\tilde{K}_{\ell+n_0-1}^{n_0}\cdot B_{\ell+n_0-1}^{n_0}\cdot k\|\gamma_{n-n_0}^{(k)}\|_1\end{aligned}$$

uniformly in $S_1, \dots, S_{n_0} \in \mathcal{R}_{\ell,n_0}$ and $T_1, \dots, T_{n_0} \in \mathcal{Q}_{\ell,n_0}$. Continuing the estimate (2.3.45), we thus obtain

$$\begin{aligned}
&\|\mathcal{I}_\ell (R^*_{\alpha,\varepsilon}[\alpha K_0]^*)^{n_0} \gamma_{n-n_0}\|_\infty \\
&\le C^{n_0-1} M_\ell \cdots M_{\ell+n_0-1} \sum_{k=1\vee(\ell-n_0(k_0-1))}^{\ell+n_0} \Bigg(\sum_{j=0}^{n_0-1} \|\gamma^{(k)}_{n-n_0+j}\|_1 \\
&\qquad + |\mathcal{R}_{\ell,n_0}|^{n_0} \cdot |\mathcal{Q}_{\ell,n_0}|^{n_0} \cdot \tilde{C}_{n_0} \tilde{K}^{n_0}_{\ell+n_0-1} B^{n_0}_{\ell+n_0-1} \cdot k \|\gamma^{(k)}_{n-n_0}\|_1 \Bigg) \\
&\le C^{n_0-1} M_\ell \cdots M_{\ell+n_0-1} \cdot |\mathcal{R}_{\ell,n_0}|^{n_0} |\mathcal{Q}_{\ell,n_0}|^{n_0} \tilde{C}_{n_0} \tilde{K}^{n_0}_{\ell+n_0-1} B^{n_0}_{\ell+n_0-1} \cdot \\
&\qquad \cdot 2 \sum_{j=0}^{n_0-1} \sum_{k=1\vee(\ell-n_0(k_0-1))}^{\ell+n_0} k \|\gamma^{(k)}_{n-n_0+j}\|_1 \\
&= C'_{\ell,n_0} \sum_{j=0}^{n_0-1} \sum_{k=1\vee(\ell-n_0(k_0-1))}^{\ell+n_0} k \|\gamma^{(k)}_{n-n_0+j}\|_1 .
\end{aligned} \tag{2.3.46}$$

with

$$C'_{\ell,n_0} := 2C^{n_0-1} M_\ell \cdots M_{\ell+n_0-1} \cdot |\mathcal{R}_{\ell,n_0}|^{n_0} |\mathcal{Q}_{\ell,n_0}|^{n_0} \cdot (1 \vee \tilde{C}_{n_0}) \cdot \tilde{K}^{n_0}_{\ell+n_0-1} B^{n_0}_{\ell+n_0-1}. \tag{2.3.47}$$

We observe that since $|\mathcal{R}_{\ell,n_0}|$ and $|\mathcal{Q}_{\ell,n_0}|$ are bounded by a constant which does not depend on ℓ and since M_ℓ, $\tilde{K}_\ell$ and B_ℓ are all polynomial in ℓ, the constant C'_{ℓ,n_0} in (2.3.47) is also polynomial in $\ell \in \mathbb{N}$ (cf. Remark 2.3.12; also remember that in contrast to the previous section, the choice of n_0 does not depend on ℓ but only on d).

Returning to (2.3.44), we fix $m \in \{1, \dots, n_0 - 1\}$ and consider the corresponding term in the sum on the r.h.s. of (2.3.44): Using again Lemma 2.3.14, we choose $f := (P^\alpha_\varepsilon)^* \gamma_{n-m}$ in (2.3.31) and get an estimate

$$\begin{aligned}
&\|\mathcal{I}_\ell (R^*_{\alpha,\varepsilon}[\alpha K_0]^*)^m (P^\alpha_\varepsilon)^* \gamma_{n-m}\|_\infty \\
&\le C^{m-1} M_\ell \cdots M_{\ell+m-1} \sum_{k=1\vee(\ell-m(k_0-1))}^{\ell+m} \Bigg(\sum_{j=0}^{m-1} \left\| \left((R^*_\alpha[\alpha K_0]^*)^j (P^\alpha_\varepsilon)^* \gamma_{n-m} \right)^{(k)} \right\|_1 \\
&\qquad + \sum_{S_1,\dots,S_m \in \mathcal{R}_{\ell,m}} \sum_{T_1,\dots T_m \in \mathcal{Q}_{\ell,m}} \|S_1 \cdots S_m T_1 \cdots T_m \mathcal{I}_k (P^\alpha_\varepsilon)^* \gamma_{n-m}\|_\infty \Bigg).
\end{aligned} \tag{2.3.48}$$

From the identity (2.2.10), it is immediately clear that $(P^\alpha_\varepsilon)^* R^*_\alpha f \le R^*_\alpha f$ for $f \ge 0$. Thus by definition of γ_{n-m}, we have

$$(P^\alpha_\varepsilon)^* \gamma_{n-m} = (P^\alpha_\varepsilon)^* (R^*_\alpha[\alpha K_0]^*)^{n-m} R^*_\alpha \pi^{(0)} \le (R^*_\alpha[\alpha K_0]^*)^{n-m} R^*_\alpha \pi^{(0)} = \gamma_{n-m}$$

and consequently for each $j \in \{0, \dots, m-1\}$ and $k \in \{1 \vee (\ell - m(k_0 - 1)), \dots, \ell + m\}$

$$\left\| \left((R^*_\alpha[\alpha K_0]^*)^j (P^\alpha_\varepsilon)^* \gamma_{n-m} \right)^{(k)} \right\|_1 \le \left\| \left((R^*_\alpha[\alpha K_0]^*)^j \gamma_{n-m} \right)^{(k)} \right\|_1 = \|\gamma^{(k)}_{n-m+j}\|_1. \tag{2.3.49}$$

Moreover, by (2.3.17) we have for all $k \in \{1 \vee (\ell - m(k_0 - 1)), \dots, \ell + m\}$

$$\mathcal{I}_k (P^\alpha_\varepsilon)^* \gamma_{n-m} \le \tilde{P}^{(k)}_\varepsilon \mathcal{I}_k \gamma_{n-m},$$

and thus by (2.3.15)

$$\|\mathcal{I}_k(P^\alpha_\varepsilon)^*\gamma_{n-m}\|_\infty \le \|\tilde{P}^{(k)}_\varepsilon \mathcal{I}_k\gamma_{n-m}\|_\infty \le \|\tilde{P}^{(k)}_\varepsilon\|_{1\to\infty}\cdot\|\mathcal{I}_k\gamma_{n-m}\|_1 \le K_{\ell+m}\varepsilon^{-d/2}\cdot k\|\gamma^{(k)}_{n-m}\|_1$$

since $(K_\ell)_\ell$ is increasing. Consequently, each term in the second (double) sum on the r.h.s. of (2.3.48) is bounded by

$$\begin{aligned}&\|S_1\cdots S_m T_1\cdots T_m\mathcal{I}_k(P^\alpha_\varepsilon)^*\gamma_{n-m}\|_\infty\\ &\le \|S_1\|_{\infty\to\infty}\cdots\|S_m\|_{\infty\to\infty}\cdot\|T_1\|_{\infty\to\infty}\cdots\|T_m\|_{\infty\to\infty}\cdot\|\mathcal{I}_k(P^\alpha_\varepsilon)^*\gamma_{n-m}\|_\infty\\ &\le A^m_{\ell+m-1}B^m_{\ell+m-1}\cdot K_{\ell+m}\varepsilon^{-d/2}\cdot k\|\gamma^{(k)}_{n-m}\|_1\end{aligned}\tag{2.3.50}$$

uniformly in $S_1,\dots S_m\in\mathcal{R}_{\ell,m}$ and $T_1,\dots,T_m\in\mathcal{Q}_{\ell,m}$. Substituting (2.3.49) and (2.3.50) into (2.3.48) and using the fact that $m\le n_0-1$, we obtain

$$\begin{aligned}&\|\mathcal{I}_\ell(R^*_{\alpha,\varepsilon}[\alpha K_0]^*)^m(P^\alpha_\varepsilon)^*\gamma_{n-m}\|_\infty\\ &\le C^{m-1}M_\ell\cdots M_{\ell+m-1}\sum_{k=1\vee(\ell-m(k_0-1))}^{\ell+m}\Bigg(\sum_{j=0}^{m-1}\|\gamma^{(k)}_{n-m+j}\|_1\\ &\qquad\qquad+|\mathcal{R}_{\ell,m}|^m|\mathcal{Q}_{\ell,m}|^m\cdot A^m_{\ell+m-1}B^m_{\ell+m-1}\cdot K_{\ell+m}\varepsilon^{-d/2}\cdot k\|\gamma^{(k)}_{n-m}\|_1\Bigg)\\ &\le C^{m-1}M_\ell\cdots M_{\ell+m-1}\cdot|\mathcal{R}_{\ell,m}|^m|\mathcal{Q}_{\ell,m}|^m\cdot A^m_{\ell+m-1}B^m_{\ell+m-1}K_{\ell+m}(1\vee\varepsilon^{-d/2})\cdot\\ &\qquad\cdot 2\sum_{j=0}^{m-1}\sum_{k=1\vee(\ell-m(k_0-1))}^{\ell+m}k\|\gamma^{(k)}_{n-m+j}\|_1\\ &\le C''_{\ell,n_0}\sum_{j=0}^{n_0-1}\sum_{k=1\vee(\ell-n_0(k_0-1))}^{\ell+n_0}k\|\gamma^{(k)}_{n-n_0+j}\|_1,\end{aligned}\tag{2.3.51}$$

where

$$C''_{\ell,n_0}:=2C^{n_0-1}M_\ell\cdots M_{\ell+n_0-1}|\mathcal{R}_{\ell,n_0}|^{n_0}|\mathcal{Q}_{\ell,n_0}|^{n_0}A^{n_0}_{\ell+n_0-1}B^{n_0}_{\ell+n_0-1}K_{\ell+n_0}(1\vee\varepsilon^{-d/2}).\tag{2.3.52}$$

As before, we observe that C''_{ℓ,n_0} is polynomial in ℓ.

It remains to consider the term with $m=0$ in the sum on the r.h.s. of (2.3.44). For this term, we have by (2.3.15) and (2.3.17)

$$\|\mathcal{I}_\ell(P^\alpha_\varepsilon)^*\gamma_n\|_\infty\le\|\tilde{P}^{(\ell)}_\varepsilon\mathcal{I}_\ell\gamma_n\|_\infty\le\|\tilde{P}^{(\ell)}_\varepsilon\|_{1\to\infty}\cdot\|\mathcal{I}_\ell\gamma_n\|_1\le K_\ell\varepsilon^{d/2}\cdot\ell\|\gamma_n\|_1.\tag{2.3.53}$$

Summing over $m=0,1,\dots,n_0-1$, from the estimates (2.3.51) (which does not depend on m) and (2.3.53) we obtain

$$\left\|\sum_{m=0}^{n_0-1}\mathcal{I}_\ell(R^*_{\alpha,\varepsilon}[\alpha K_0]^*)^m(P^\alpha_\varepsilon)^*\gamma_{n-m}\right\|_\infty\le n_0\cdot C''_{\ell,n_0}\sum_{j=0}^{n_0}\sum_{k=1\vee(\ell-n_0(k_0-1))}^{\ell+n_0}k\|\gamma^{(k)}_{n-n_0+j}\|_1.\tag{2.3.54}$$

Now define

$$C_{\ell,n_0}:=C'_{\ell,n_0}\vee C''_{\ell,n_0}.$$

Then it is clear that the sequence $(C_{\ell,n_0})_\ell$ is polynomial in $\ell \in \mathbb{N}$, and substituting (2.3.46) and (2.3.54) into (2.3.44), we conclude that for all $\ell \in \mathbb{N}$ and $n \geq n_0$

$$\|\mathcal{I}_\ell \gamma_n\|_\infty \leq (n_0+1) \cdot C_{\ell,n_0} \sum_{j=0}^{n_0} \sum_{k=1\vee(\ell-n_0(k_0-1))}^{\ell+n_0} k\|\gamma_{n-n_0+j}^{(k)}\|_1.$$

Summing over $n \geq n_0$ gives

$$\begin{aligned}\sum_{n\geq n_0} \|\mathcal{I}_\ell \gamma_n\|_\infty &\leq (n_0+1) C_{\ell,n_0} \sum_{j=0}^{n_0} \sum_{n\geq n_0} \sum_{k=1\vee(\ell-n_0(k_0-1))}^{\ell+n_0} k\|\gamma_{n-n_0+j}^{(k)}\|_1 \\ &\leq (n_0+1)^2 C_{\ell,n_0} \sum_{k=1\vee(\ell-n_0(k_0-1))}^{\ell+n_0} k \sum_{n\in\mathbb{N}_0} \|\gamma_n^{(k)}\|_1 \qquad (2.3.55) \\ &= (n_0+1)^2 C_{\ell,n_0} \sum_{k=1\vee(\ell-n_0(k_0-1))}^{\ell+n_0} \mu_k,\end{aligned}$$

where as in (1.2.31) we put

$$\mu_k := k \cdot m(E^k) \equiv k\|\gamma^{(k)}\|_1 \equiv k \sum_{n\in\mathbb{N}_0} \|\gamma_n^{(k)}\|_1.$$

By Assumption 2.3.5, the sequence $(\mu_\ell)_{\ell\in\mathbb{N}}$ decays exponentially fast in $\ell \in \mathbb{N}$, and consequently the same holds for the sequence $(\tilde{\mu}_\ell)_{\ell\in\mathbb{N}}$ with

$$\tilde{\mu}_\ell := \sum_{k=1\vee(\ell-n_0(k_0-1))}^{\ell+n_0} \mu_k.$$

Together with (2.3.55), this gives

$$\sum_{\ell\in\mathbb{N}} \sum_{n\geq n_0} \|\mathcal{I}_\ell \gamma_n\|_\infty < \infty. \qquad (2.3.56)$$

since C_{ℓ,n_0} is polynomial in ℓ. For $n = 0, \ldots, n_0 - 1$ we use the fact that by Assumption 2.3.4, only a finite number k_0 of offspring are possible at each reproduction event: For all $\ell \in \mathbb{N}$ and $\boldsymbol{x} \in E^\ell$, $K_0(\boldsymbol{x};\cdot)$ charges only the layers $E^{\ell-1}, \ldots, E^{\ell+k_0-1}$. Since we start at $n = 0$ from the immigration density $\pi^{(0)}$ which is concentrated on the single-particle layer E, this clearly implies for all $n \in \mathbb{N}$ that $\gamma_n = (R_\alpha^*[\alpha K_0]^*)^n R_\alpha^* \pi^{(0)}$ is concentrated on the finitely many layers $E^{(0)}, E, \ldots, E^{1+n(k_0-1)}$, i.e.

$$\gamma_n^{(\ell)} \equiv 0 \qquad \text{for } \ell > 1 + n(k_0 - 1).$$

In particular, we have for each fixed $n \in \mathbb{N}$ that

$$\sum_{\ell\in\mathbb{N}} \|\mathcal{I}_\ell \gamma_n^{(\ell)}\|_\infty = \sum_{\ell=1}^{1+n(k_0-1)} \|\mathcal{I}_\ell \gamma_n^{(\ell)}\|_\infty < \infty.$$

Using this for $n = 0, \ldots, n_0 - 1$, together with (2.3.56) we have proved that

$$\sum_{\ell\in\mathbb{N}} \sum_{n\in\mathbb{N}_0} \|\mathcal{I}_\ell \gamma_n\|_\infty < \infty,$$

which completes our proof. ∎

2.3.16 Remark
As for the invariant density on the configuration space $\mathcal{S}$, under additional conditions the above reasoning can be adapted to show differentiability of the invariant occupation density $\frac{d\overline{m}}{d\lambda}$ on $E = \mathbb{R}^d$: In fact, under the assumptions of Theorem 2.2.25 we know by its proof that $\gamma_n \in \mathscr{U}^1$ for all $n \in \mathbb{N}_0$ (see (2.2.80)), where $\mathscr{U}^1$ is defined in (2.2.73). In particular, each term $\mathcal{I}_\ell \gamma_n$ in the double series on the r.h.s. of (2.3.6) is in $\mathcal{C}_0^1(E)$, with $\partial \mathcal{I}_\ell \gamma_n = \mathcal{I}_\ell \partial \gamma_n$, and it remains only to show uniform convergence of the series of derivatives. For this, we need estimates like (2.3.9) but with $p_t^\alpha(\boldsymbol{x};\cdot)$ replaced by its partial derivatives in the forward variable, giving an additional factor $t^{-1/2}$ on the r.h.s. of (2.3.9) which does not affect the basic argument. This extra condition is in particular fulfilled in the $\mathcal{C}_b^\infty$-framework of [Löc2004] (see step 7 in the proof of Thm. 4.2, pp. 154f.). Again however, the method of proof breaks down for higher derivatives. ♦

We have seen that the basic result in [Löc2004] continues to hold if the assumption that branching particles reproduce at their death position is replaced by the assumption that their offspring is distributed according to an absolutely continuous law. We emphasize again that the results in this section are of interest primarily for the “interactive” case: For the purely-position dependent framework, the approach of [HL2005] is available which can be adapted to general spatial offspring distributions. The first steps in this regard are taken in the next chapter.

Chapter 3

General Results for the Purely Position-Dependent Framework

The final chapter of this thesis is devoted to a study of the purely position-dependent framework - the case that the quantities in Assumptions 1.1.2, 1.1.4 and 1.1.5 are all independent of the configuration (or its length). In this set-up, the problem of finding sufficient conditions for finiteness of the invariant measure m on $\mathcal{S}$ and of the occupation measure $\overline{m}$ on E, and for the existence of "nice" Lebesgue-densities, was taken up in [HL2005]. In that work (as in most of the literature on spatial branching processes), the authors assume that branching particles reproduce at their parent's death position. Since we have seen that this assumption has to be modified in order to obtain "nice" Lebesgue densities for m, we are naturally interested in generalizing the results in [HL2005] to more general spatial offspring distributions. At the same time, we will without extra cost admit that the motion of the particles is governed by a general strong Markov process instead of a diffusion. The resulting stochastic process of particle configurations will be called a *branching Markov process with immigration* (or BMPI for short) and again denoted by $\boldsymbol{\eta} = (\boldsymbol{\eta}_t)_{t \geq 0}$. Its "branching component" is a *branching Markov process* (without immigration, henceforth: BMP) in the sense introduced by [INW1968a] and has the fundamental *branching property* (see (3.2.11) below) which permits many problems concerning the process on the "big" configuration space $\mathcal{S}$ to be translated into problems concerning quantities defined on the "small" and more feasible single-particle space E.

This chapter is organized as follows: The main results concerning the invariant measure m and occupation measure $\overline{m}$ for a branching Markov process with immigration are to be found in Section 3.3. Before, we give in Section 3.1 the precise assumptions under which we will work throughout the remainder of this work. Section 3.2 contains a relatively detailed and self-contained "excursion" into the field of branching Markov processes (without immigration) in the tradition of the classical treatment in [INW1968a]-[INW1969]. We use this approach to generalize some of the results in [HL2005], namely a condition of "spatial subcriticality" and its characterizations, to our more general framework.

3.1 Basic Assumptions

Our set-up for this chapter is as follows: Instead of $E = \mathbb{R}^d$, we consider more generally a locally compact Polish space E endowed with its Borel σ-algebra $\mathcal{B}_E$. The space $\mathcal{S}$ of ordered configurations is defined as in (1.1.1) before. If d is a metric on E such that (E, d) is complete and $d^{(\ell)}$ is the induced product metric on E^ℓ, the definition (1.1.2) again gives a complete metric on $\mathcal{S}$, thus $\mathcal{S}$ is Polish (and locally compact) also in this more general case. We will now expressly allow for the space of unordered configurations resp. finite point measures as state space; we can prove one of our results for the unordered case only (Theorem 3.3.7 below). Formally, the space of unordered configurations is obtained by symmetrization of the space $\mathcal{S}$ of (1.1.1) (see [INW1968a], § 0.2, pp. 246ff.): For two (ordered) configurations $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$, write $\boldsymbol{x} \sim \boldsymbol{y}$ if they have the same length and one is obtained from the other by a permutation of the coordinates, and denote by $[\boldsymbol{x}]$ the equivalence class of $\boldsymbol{x} \in \mathcal{S}$ under the equivalence relation $\sim$. The corresponding quotient space

$$\tilde{\mathcal{S}} := \mathcal{S}/_\sim \tag{3.1.1}$$

is called the space of unordered configurations. Endowed with the quotient topology, $\tilde{\mathcal{S}}$ is again Polish and locally compact. Since $\tilde{\mathcal{S}}$ is obtained from the unsymmetrized state space $\mathcal{S}$ by an equivalence relation, results for the process of ordered configurations can usually be "carried over" to the case of unordered configurations via the quotient mapping.[1]

On the other hand, especially when considering functions acting on configurations, it is sometimes more natural and helpful to identify an unordered configuration $[\boldsymbol{x}] \in \tilde{\mathcal{S}}$ with the corresponding finite point measure $\sum_{i=1}^{\ell(\boldsymbol{x})} \delta_{x^i}$, thus adopting a measure-valued point of view:

3.1.1 Notations
Write $\mathcal{M}_f(E)$ for the space of all finite (positive) measures on $(E, \mathcal{B}_E)$. The subspace of measures of total mass $\ell \in \mathbb{N}_0$ will be denoted by $\mathcal{M}_\ell(E)$; in particular $\mathcal{M}_1(E)$ is the space of all probability measures on $(E, \mathcal{B}_E)$. By a finite point measure on $(E, \mathcal{B}_E)$ we mean a measure μ which is a sum of Dirac measures

$$\mu = \sum_{i=1}^{\ell} \delta_{x^i}$$

with $\ell \in \mathbb{N}_0$ and $x^i \in E$, $i = 1, \dots, \ell$. We write $\mathcal{M}_f^p(E)$ resp. $\mathcal{M}_\ell^p(E)$ for the space of all finite point measures resp. all finite point measures of total mass $\ell \in \mathbb{N}_0$. The space $\mathcal{M}_1^p(E, \mathcal{B}_E)$ of Dirac measures will be identified with E, thus we have $E \subseteq \mathcal{M}_f^p(E) \subseteq \mathcal{M}_f(E)$.

3.1.2 Remark
It is clear that the mapping

$$[\boldsymbol{x}] \mapsto \mu_{[\boldsymbol{x}]} := \sum_{i=1}^{\ell(\boldsymbol{x})} \delta_{x^i}, \qquad \boldsymbol{x} \in \mathcal{S} \tag{3.1.2}$$

defines a bijective correspondence between $\tilde{\mathcal{S}}$ and $\mathcal{M}_f^p(E)$ as well as between every $E^\ell/_\sim$ and $\mathcal{M}_\ell^p(E)$, $\ell \in \mathbb{N}_0$. As is well known, $\mathcal{M}_f(E)$ endowed with the topology of weak convergence is itself a Polish space: Given a (complete) metric d on E, the corresponding Prokhorov metric on $\mathcal{M}_f(E)$ is a (complete) metric for the topology of weak convergence.[2] As noted in [Löc1999], Remark A.3, for the right choice of the metric d the correspondence (3.1.2) defines a homeomorphism between the spaces $\tilde{\mathcal{S}}$ and $\mathcal{M}_f^p(E)$. This may be "obvious", but the proof

[1] In particular, this holds for the construction of the process of unordered configurations itself. See e.g. [INW1968b], Thm. 3.1 on p. 384.

[2] See e.g. [Els2009], pp. 401f., in particular Thms. VIII.4.35 and VIII.4.38.

of it seems not to.[3] It is in fact possible to choose a metric d on E in such a way that the correspondence (3.1.2) becomes an isometry between $\tilde{\mathcal{S}}$ and $\mathcal{M}_f^p(E)$, thus we can isometrically embed $\tilde{\mathcal{S}}$ into $\mathcal{M}_f(E)$ by identifying it with $\mathcal{M}_f^p(E)$.[4] For the purposes of this thesis however, we will not give the proof of this result but (in line with the rest of the literature) take it for granted that $\tilde{\mathcal{S}}$ can be identified with $\mathcal{M}_f^p(E)$. Consequently, given a branching Markov process on $\tilde{\mathcal{S}}$ (constructed along the lines of [INW1968b], say), using the correspondence (3.1.2) we can regard it also as a process on $\mathcal{M}_f^p(E)$ and vice versa, and every property of the process involving limits (e.g., càdlàg property of sample paths, weak convergence) can be carried over from one space to the other. Moreover, one can sometimes prove results for the process using the structure and properties of the larger space $\mathcal{M}_f(E) \supseteq \mathcal{M}_f^p(E)$, like continuity theorems for Laplace functionals. This will be done in the proof of Theorem 3.3.7 below. ♦

We proceed with some more notations and definitions which are mainly taken from [Nag1977].

3.1.3 Notations

From now on, the symbol $\mathcal{S}$ will denote both the space of ordered configurations (1.1.1) or the space of finite point measures $\mathcal{M}_f^p(E)$, i.e.

$$\mathcal{S} \in \left\{ \dot{\bigcup_{\ell \in \mathbb{N}_0}} E^\ell,\ \mathcal{M}_f^p(E) \right\}.$$

For either choice of $\mathcal{S}$, we define a formal "multiplication" $\bullet : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ as follows: For ordered configurations $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S} = \dot\bigcup_{\ell \in \mathbb{N}_0} E^\ell$, we define their "product" $\boldsymbol{x} \bullet \boldsymbol{y}$ by concatenation, i.e.

$$\boldsymbol{x} \bullet \boldsymbol{y} := (x^1, \ldots, x^\ell, y^1, \ldots, y^k) \in E^{\ell + k}$$

if $\boldsymbol{x} = (x^1, \ldots, x^\ell) \in E^\ell$, $\boldsymbol{y} = (y^1, \ldots, y^k) \in E^k$. For finite point measures, our "product" is defined just as the ordinary sum of two measures:

$$\boldsymbol{x} \bullet \boldsymbol{y} := \boldsymbol{x} + \boldsymbol{y}, \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathcal{S} = \mathcal{M}_p(E, \mathcal{B}_E).^5$$

Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be σ-finite measures on $\mathcal{S}$. The *convolution* of $\boldsymbol{P}$ and $\boldsymbol{Q}$, denoted by $\boldsymbol{P} * \boldsymbol{Q}$, is defined as the image of the product measure $\boldsymbol{P} \otimes \boldsymbol{Q}$ on $\mathcal{S} \times \mathcal{S}$ under the multiplication $\bullet$, i.e.

$$\int_{\mathcal{S}} g(\boldsymbol{z})\, \boldsymbol{P} * \boldsymbol{Q}(d\boldsymbol{z}) := \int_{\mathcal{S} \times \mathcal{S}} \boldsymbol{P} \otimes \boldsymbol{Q}(d\boldsymbol{x}, d\boldsymbol{y}) g(\boldsymbol{x} \bullet \boldsymbol{y}) = \int_{\mathcal{S}} \boldsymbol{P}(d\boldsymbol{x}) \int_{\mathcal{S}} \boldsymbol{Q}(d\boldsymbol{y})\, g(\boldsymbol{x} \bullet \boldsymbol{y}) \tag{3.1.3}$$

for every nonnegative measurable function $g : \mathcal{S} \to \mathbb{R}_+$.

Given a function $f : E \to \mathbb{R}$, we define $\hat{f} : \mathcal{S} \to \mathbb{R}$ by

$$\hat{f}(\boldsymbol{x}) := \prod_{j=1}^{\ell(\boldsymbol{x})} f(x^j), \qquad \boldsymbol{x} \in \mathcal{S}, \tag{3.1.4}$$

[3] A rigorous proof was not given in [Löc1999] and we also have not found it elsewhere in the literature.

[4] For a proof that E can be isometrically identified with the space $\mathcal{M}_1^p(E)$ of Dirac measures, see [Els2009], Cor. VIII.4.33 on p. 404.

[5] In view of this definition, it might be more appropriate to speak of a formal addition and use an additive notation; however, we want to stay in line with the terminology in [Nag1977].

with the understanding that for $\boldsymbol{x} = \Delta$, this means $\hat{f}(\Delta) := 1$. Obviously, $\hat{f}$ is a *multiplicative* function in the sense that $\hat{f}(\boldsymbol{x} \bullet \boldsymbol{y}) = \hat{f}(\boldsymbol{x}) \cdot \hat{f}(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$. Conversely, if $g : \mathcal{S} \to \mathbb{R}$ is multiplicative we clearly have $g = \widehat{g|_E}$; in particular, a multiplicative function is uniquely determined by its restriction to the single-particle space E. Analogous remarks apply to functions $g : \mathcal{S} \to \mathbb{R}$ which are *additive* in the sense that $g(\boldsymbol{x} \bullet \boldsymbol{y}) = g(\boldsymbol{x}) + g(\boldsymbol{y})$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$ and for which we continue to use the notation $g = \bar{f}$ with $f = g|_E$, see e.g. (2.3.2). Also recall from (1.2.12) that we use the notation $\boldsymbol{x}(B) = \overline{\mathbf{1}_B}(\boldsymbol{x})$ for the number of particles in a Borel set $B \in \mathcal{B}_E$ in both the ordered and the unordered framework.

3.1.4 Remark
The formal multiplication $\bullet$ and the notation (3.1.3) are primarily introduced to allow for a concise statement of structural properties of the semigroups of our processes, such as (3.2.11) and (3.3.3) below. Clearly, $\bullet$ is associative and continuous in both the ordered and the unordered case, but it is commutative only if $\mathcal{S} = \mathcal{M}_f^p(E)$. Consequently, $(\mathcal{S}, \bullet)$ is a topological semigroup in either case but Abelian only in the latter, and the convolution is commutative only in the unordered case $\mathcal{S} = \mathcal{M}_f^p(E)$. ⧫

We now give the conditions on the single-particle motion and on the branching, reproduction and immigration mechanisms under which we will work throughout this chapter:

3.1.5 Assumption (Particle Motion)
On the locally compact Polish space E, we are given a strong Markov process

$$X = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, (P_x)_{x \in E}, (X_t)_{t \geq 0}, (\theta_t)_{t \geq 0})$$

taking values in E, with càdlàg sample paths and a right-continuous, complete[6] *filtration $(\mathscr{F}_t)_t$. The process $X = (X_t)_{t \geq 0}$ is called the single particle motion. For all $\ell \in \mathbb{N}$, the joint motion X^ℓ of ℓ particles on E^ℓ is given by*

$$X^\ell = (X^{1,\ell}, \dots, X^{\ell,\ell})$$

with independent copies $X^{i,\ell}$ of X, $i = 1, \dots, \ell$.

3.1.6 Example
Under Assumption 1.1.2 of Chapter 1, suppose that the drift and diffusion coefficients b and σ depend only on the position but not on the configuration variable or its length:

$$b^{(\ell)}(\cdot\,;\cdot) \equiv b(\cdot) : \mathbb{R}^d \to \mathbb{R}^d, \qquad \sigma^{(\ell)}(\cdot\,;\cdot) \equiv \sigma(\cdot) : \mathbb{R}^d \to \mathbb{R}^{d \times m}.$$

Then for each $\ell \in \mathbb{N}$, the motion of ℓ particles is given by

$$dX_t^{j,\ell} = b(X_t^{j,\ell})\,dt + \sigma(X_t^{j,\ell})\,dW_t^j, \qquad j = 1, \dots, \ell, \tag{3.1.5}$$

with independent m-dimensional ($m \geq d$) standard Brownian motions $W^1, \dots, W^\ell$. Thus Assumption 3.1.5 is satisfied. This is the framework of [HL2005]. ⧫

3.1.7 Assumption (Branching and Reproduction Mechanism)
We are given a nonnegative measurable function

$$\kappa(\cdot) : E \to \mathbb{R}_+, \tag{3.1.6}$$

[6] Again, completeness is understood w.r.t. the family of measures $(P_x)_{x \in E}$.

called the branching rate, such that for all $x \in E$

$$A_t^\kappa := \int_0^t \kappa(X_s)\, ds < \infty \qquad P_x\text{-a.s.} \tag{3.1.7}$$

Moreover, we are given measurable functions

$$p_k(\cdot) : E \to [0,1], \qquad k \in \mathbb{N}_0 \tag{3.1.8}$$

such that $\sum_{k \in \mathbb{N}_0} p_k(\cdot) \equiv 1$ and

$$\varrho(x) := \sum_{k \in \mathbb{N}_0} k p_k(x) < \infty, \qquad x \in E. \tag{3.1.9}$$

Finally, we are given transition probabilities

$$Q_k(\cdot\,;\cdot) : E \times \mathcal{B}_{E^k} \to [0,1], \qquad k \in \mathbb{N} \tag{3.1.10}$$

and put

$$Q_0(x;\cdot) := \delta_\Delta(\cdot), \qquad x \in E.$$

A particle situated at $x \in E$ branches (independently of the configuration of coexisting particles) at position-dependent rate $\kappa(x)$. At its death time, it is replaced by a random number $k \in \mathbb{N}_0$ of offspring particles with probability $p_k(x)$, the k offspring particles being distributed in E^k according to the law

$$Q_k(x; dv^1 \cdots dv^k) \quad \text{on } (E^k, \mathcal{B}_{E^k}).$$

3.1.8 Assumption (Immigration Mechanism)
We are given a nonnegative constant $c \geq 0$, called the immigration rate, and a probability measure ν on the single particle space $(E, \mathcal{B}_E)$ to which we refer as the immigration law. Immigration of new particles occurs at constant rate c. At each immigration event, exactly one new particle is added to the pre-existing configuration in a position selected according to the law ν.

3.1.9 Remarks

- In the case of ordered configurations, we need a rule where to insert newborn particles in a preexisting configuration: As in the set-up of Chapter 1, we adopt the convention that offspring particles are inserted in place of the branching parent. On the other hand, in the unordered resp. measure-valued case one has to assume that for each $k \in \mathbb{N}$ and $x \in E$ the probability measure $Q_k(x;\cdot)$ is symmetric w.r.t. the permutation of coordinates on E^k, so that the BMP $\boldsymbol{\beta}$ is well-defined on the space of unordered configurations resp. finite point measures.

- Condition (3.1.7) ensures that the additive functional $A_t^\kappa = \int_0^t \kappa(X_s)\, ds$ of X is a.s. finite and thus (by dominated convergence) also continuous.

- We allow for $c = 0$ in Assumption 3.1.8 since in the next section we will consider branching Markov processes *without* immigration, before adding immigration in a second step in Section 3.3.

- Note that we assume that the reproduction mean ϱ in (3.1.9) is a finite-valued function, but not in general that it is bounded or continuous.

- It is sometimes convenient to combine the reproduction law $(p_k)_k$ and the spatial offspring distribution $(Q_k)_k$ into a single transition probability $J : E \times \mathcal{B}_{\mathcal{S}} \to [0,1]$ defined layer-wise as $J(x;\cdot)|_{E^k} := p_k(x) Q_k(x;\cdot)$, i.e.

$$J(x;\cdot) := \sum_{k \in \mathbb{N}_0} p_k(x) Q_k(x;\cdot), \qquad x \in E. \tag{3.1.11}$$

  Equivalently, we could also start with a given transition probability J on $E \times \mathcal{B}_s t$ and define p_k and Q_k respectively by

$$p_k(x) := J(x; E^k), \qquad Q_k(x;B) := \mathbf{1}_{J(x;E^k) \neq 0} \cdot \frac{J(x;B)}{J(x;E^k)}, \quad B \in \mathcal{B}_{E^k},\ x \in E.$$

- If the sequence of kernels $(Q_k)_k$ is given by

$$Q_k(x;\cdot) = Q(x;\cdot)^{\otimes k}, \qquad x \in E,\ k \in \mathbb{N} \tag{3.1.12}$$

  for some fixed transition probability kernel $Q : E \times \mathcal{B}_E \to [0,1]$, we refer to $(Q_k)_k$ as being of product type. The case considered most often in the literature (eg. in [HL2005]) is that $(Q_k)_k$ is of product type with

$$Q(x;\cdot) = \delta_x(\cdot), \qquad x \in E \tag{3.1.13}$$

  where branching particles reproduce exactly at their death position. ⧫

As in Chapter 1, the existence under the above assumptions of a corresponding BMPI $\boldsymbol{\eta}$ with the desired properties follows from the "killing and restarting"-procedure due to Ikeda, Nagasawa and Watanabe since the "Revival Theorem" ([INW1968b], Thm. 2.2 or [Nag1977], Thm. 2) does require that the to-be-revived process is a diffusion, but holds for general Markov processes as in Assumption 3.1.5. Again, the resulting process $\boldsymbol{\eta}$ of particle configurations is strong Markov, with possibly finite lifetime (in the sense of explosion time) $\boldsymbol{\tau}_\infty \leq \infty$, and with càdlàg sample paths before time $\boldsymbol{\tau}_\infty$. We will retain all notations introduced in Section 1.1. In particular, the rate function $\alpha(\cdot) : \mathcal{S} \to \mathbb{R}_+$ in the configuration process, the corresponding killing time τ and the kernel R_α governing the occupation time between branching or immigration events are given as in (1.1.21), (1.1.20) and (1.1.36), respectively. With these definitions, the formulas (1.1.33) and (1.1.34) for the ℓ-particle motion killed at rate α remain true also if the single-particle motion X is not a diffusion. Since starting from an ℓ-particle configuration, the process $\boldsymbol{\eta}$ up to the first branching / immigration time $\boldsymbol{\tau}_1$ evolves as X^ℓ up to time τ, we have formula (1.1.37) as before, but not necessarily (1.1.38) if the paths of X are not continuous. Moreover, with the jump kernel $K : \mathcal{S} \times \mathcal{B}_{\mathcal{S}} \to [0,1]$ defined as in (1.1.24), the state of $\boldsymbol{\eta}$ at time $\boldsymbol{\tau}_1$ is described by formula (1.1.40). Note however that we do not necessarily have (1.1.39) since under Assumption 3.1.7, the distribution of $\boldsymbol{\eta}_{\boldsymbol{\tau}_1}$ depends on X_τ and not on the left-hand limit $X_{\tau-}$.

There is a second route to the construction of a BMPI $\boldsymbol{\eta}$: Instead of applying the "killing-and-reviving"-procedure directly to the quantities given in the above assumptions, one can construct at first a BMP *without* immigration in the sense of [INW1968a]-[INW1968b], and then (again by means of the "Revival Theorem") add the immigration in a second step. This is the approach to be adopted in the sequel. As will be seen in Section 3.3 below, it allows for certain insights into the structure of the semigroup of $\boldsymbol{\eta}$ which turn out to be useful when we return to the problem of sufficient conditions for finiteness of the invariant measure m and of the occupation measure $\overline{m}$.

3.2 Branching Markov Processes (Without Immigration)

This section contains a relatively self-contained "excursion" into the field of branching Markov processes. Some of the results, besides serving as ingredients for the proofs in Section 3.3 below, may also be of some interest in themselves since they complement the classical approach in [INW1968a]-[INW1969] where this class of processes was first introduced and studied in a general setting.

Thus throughout this section we consider a system of finitely many particles which independently of each other move, branch and reproduce according to Assumptions 3.1.5 and 3.1.7, without any immigration (put $c = 0$ in Assumption 3.1.8).

3.2.1 A Short Review of the Theory of Branching Markov Processes

In this subsection, we give a short review of those aspects of the theory of branching Markov processes developed in [INW1968a]-[INW1968b] which will be needed in the sequel.[7]

A rigorous probabilistic construction of a general branching Markov process on the space (3.1.1) of unordered configurations was first given in [INW1968b].[8] For a somewhat more accessible presentation of essentially the same construction, see [Nag1977]; in that work, the space (1.1.1) of ordered configurations was employed. Our set-up is essentially the same as in these references, with one minor difference: The authors in [INW1968b] and [Nag1977] start with a given E-valued strong Markov process X^0 with (finite or infinite) lifetime ζ which determines the motion of particles between branching events. This process is "restarted" at time ζ according to a jump kernel depending on the left-hand limit $X^0_{\zeta-}$ (which is assumed to exist). In our framework however, the "givens" are a strong Markov process X and a killing rate κ with corresponding killing time τ, and the process X^0 in [INW1968b] corresponds to the process obtained by killing the paths of X at time τ. In this context, we find it more natural to let the distribution of newborn particles depend on the position X_τ of the dying particle *at* the branching time, and not on the left-hand limit thereof as in [INW1968b] or [Nag1977]. In this modification, we follow [Saw1970].[9]

Under Assumptions 3.1.5 and 3.1.7, the resulting branching Markov process will be denoted by

$$\boldsymbol{\beta} = (\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, (\boldsymbol{\mathcal{F}}_t)_{t\geq 0}, (\boldsymbol{P}_{\boldsymbol{x}})_{\boldsymbol{x}\in\mathcal{S}}, (\boldsymbol{\beta}_t)_{t\geq 0}, (\boldsymbol{\theta}_t)_{t\geq 0})$$

The process $\boldsymbol{\beta}$ is strong Markov taking values in $\mathcal{S}_\partial = \mathcal{S} \cup \{\partial\}$, where again ∂ is a cemetary in order to account for possible "explosion" (which now means accumulation of branching events in finite time). The notation $\boldsymbol{\tau}_n$ from the previous chapters will now be used to denote the branching times in $\boldsymbol{\beta}$: Writing

$$\boldsymbol{\tau}_\infty = \inf\{t > 0 : \boldsymbol{\beta}_t \notin \mathcal{S}\} \leq \infty$$

for the (possibly finite) "life-time" (in the sense of explosion time) of the process, $\boldsymbol{\beta}$ has càdlàg sample paths before time $\boldsymbol{\tau}_\infty$, and we have an increasing sequence

$$0 = \boldsymbol{\tau}_0 \leq \boldsymbol{\tau}_1 \leq \boldsymbol{\tau}_2 \leq \cdots \uparrow \sup_{n\in\mathbb{N}} \boldsymbol{\tau}_n = \boldsymbol{\tau}_\infty$$

[7] If no specific reference is given for a particular result in this subsection, either it can be considered a well-known "classical" fact or else it is due to the above-mentioned authors.

[8] The authors of that work did not adopt a measure-valued point of view.

[9] Note that due to the continuity of the paths, this distinction is irrelevant in case that the single particle motion X is a diffusion as in Chapters 1 and 2.

of $(\boldsymbol{\mathcal{F}}_t)_t$-stopping times given by

$$\boldsymbol{\tau}_n = \boldsymbol{\tau}_{n-1} + \boldsymbol{\tau}_1 \circ \boldsymbol{\theta}_{\boldsymbol{\tau}_{n-1}}, \qquad n \in \mathbb{N} \tag{3.2.1}$$

corresponding to branching events in the process $\boldsymbol{\beta}$.

We proceed with some more notation and some formulas which will be needed in the sequel. Throughout the remainder of this work, we will denote by $(\boldsymbol{T}_t)_t$ the transition semigroup of the BMP $\boldsymbol{\beta}$ on the configuration space $\mathcal{S}$ and by $(T_t)_{t\geq 0}$ the transition semigroup of the given single-particle motion X on E:

$$\boldsymbol{T}_t(\boldsymbol{x}; g) \equiv \boldsymbol{T}_t g(\boldsymbol{x}) := \boldsymbol{E}_{\boldsymbol{x}}[g(\boldsymbol{\beta}_t)\mathbf{1}_{t<\boldsymbol{\tau}_\infty}], \qquad g \in \mathscr{B}(\mathcal{S}),\ \boldsymbol{x} \in \mathcal{S}.$$

$$T_t(x; f) \equiv T_t f(x) := E_x[f(X_t)], \qquad f \in \mathscr{B}(E),\ x \in E.$$

Also, from now on the symbol τ will denote the random time determined by killing the paths of X at rate κ. It is characterized by

$$P_x[\tau > t \,|\, X] = e^{-A_t^\kappa}, \qquad t \geq 0,\ x \in E \tag{3.2.2}$$

with $A_t^\kappa = \int_0^t \kappa(X_s)\, ds$ as in (3.1.7) (cf. formula (1.1.20) for the interactive case). Under (3.1.7), the joint distribution of τ and X_τ is given by

$$P_x\left[\tau \leq t, X_\tau \in B\right] = E_x\left[\int_0^t ds\, e^{-A_s^\kappa}\kappa(X_s)\mathbf{1}_B(X_s)\right], \qquad t > 0,\ B \in \mathcal{B}_E,\ x \in E. \tag{3.2.3}$$

By construction, starting from a single particle at $x \in E$ the BMP $\boldsymbol{\beta}$ up to the first branching time $\boldsymbol{\tau}_1$ evolves as the single particle motion X up to time τ. Moreover, the joint distribution of the path strictly before and the state of $\boldsymbol{\beta}$ at the first branching event is completely determined by X, τ and the kernel J of (3.1.11). In order make this precise, we need a notation for (the distribution of) the path of $\boldsymbol{\beta}$ strictly before the first branching time $\boldsymbol{\tau}_1$. Therefore let us introduce the *killed process*

$$\boldsymbol{\beta}_t^{\boldsymbol{\tau}_1} := \begin{cases} \boldsymbol{\beta}_t, & t < \boldsymbol{\tau}_1 \\ \partial, & t \geq \boldsymbol{\tau}_1. \end{cases} \tag{3.2.4}$$

For a proof that $\boldsymbol{\beta}^{\boldsymbol{\tau}_1}$ is again a strong Markov process under the assumptions above, see e.g. [BG1968], Section III.3. Note that although $\boldsymbol{\tau}_1$ is not necessarily a functional of the path of $\boldsymbol{\beta}$,[10] it is a functional of the path of $\boldsymbol{\beta}^{\boldsymbol{\tau}_1}$ since it coincides with the first hitting time of ∂ in the killed process. Similarly, we denote by X^τ the strong E_∂-valued Markov process obtained by killing the paths of X at time τ. The semigroup of the killed process X^τ will be denoted by $(T_t^\kappa)_{t\geq 0}$, and under (3.1.7) we have

$$T_t^\kappa(x; f) \equiv T_t^\kappa f(x) := E_x[f(X_t)\mathbf{1}_{t<\tau}] = E_x\left[f(X_t)e^{-A_t^\kappa}\right], \qquad f \in \mathscr{B}(E),\ x \in E. \tag{3.2.5}$$

The joint distribution of $\boldsymbol{\beta}^{\boldsymbol{\tau}_1}$ and $\boldsymbol{\beta}_{\boldsymbol{\tau}_1}$ under $\boldsymbol{P}_x$, $x \in E$, can be obtained as a special case of Thm. 1 (ii) in [Nag1977] or Thm. 2.2 (ii) in [INW1968b] and is given as follows: If $\phi : E_\partial^{\mathbb{R}_{\geq 0}} \to \mathbb{R}$ is a bounded measurable functional and $g \in \mathscr{B}(E)$, using the killed processes just introduced and the definition of J in (3.1.11) we have

$$\boldsymbol{E}_x\left[\mathbf{1}_{\boldsymbol{\tau}_1<\infty} \cdot \phi(\boldsymbol{\beta}^{\boldsymbol{\tau}_1})g(\boldsymbol{\beta}_{\boldsymbol{\tau}_1})\right] = E_x\left[\mathbf{1}_{\tau<\infty} \cdot \phi(X^\tau)J(X_\tau; g)\right]. \tag{3.2.6}$$

[10] For example, this may occur if the possibility of one offspring is allowed.

Taking the first marginal in (3.2.6) gives the equivalence of the killed processes $\boldsymbol{\beta}^{\boldsymbol{\tau}_1}$ and X^τ (starting from a single particle),

$$\mathcal{L}(\boldsymbol{\beta}^{\boldsymbol{\tau}_1} \,|\, \boldsymbol{P}_x) = \mathcal{L}(X^\tau \,|\, P_x) \qquad \text{on } (E_\partial)^{\mathbb{R}_+}, \qquad x \in E. \tag{3.2.7}$$

In particular, the evolution of $\boldsymbol{\beta}$ before time $\boldsymbol{\tau}_1$ is governed by the killed semigroup $(T_t^\kappa)_t$:

$$\boldsymbol{E}_x[f(\boldsymbol{\beta}_t)\mathbf{1}_{t<\boldsymbol{\tau}_1}] = E_x[f(X_t)\mathbf{1}_{t<\tau}], \qquad x \in E,\ t \geq 0. \tag{3.2.8}$$

Since $\boldsymbol{\tau}_1$ resp. τ is a functional of the path of the killed process $\boldsymbol{\beta}^{\boldsymbol{\tau}_1}$ resp. X^τ, with suitable choice of ϕ we obtain moreover from (3.2.6) the joint distribution of $\boldsymbol{\tau}_1$ and $\boldsymbol{\beta}_{\boldsymbol{\tau}_1}$: Taking also (3.2.3) into account, it is given by

$$\begin{aligned}\boldsymbol{E}_x\left[\mathbf{1}_{\boldsymbol{\tau}_1 \leq t} \cdot g(\boldsymbol{\beta}_{\boldsymbol{\tau}_1})\right] &= E_x[\mathbf{1}_{\tau \leq t} \cdot J(X_\tau; g)] \\ &= E_x\left[\int_0^t ds\, e^{-A_s^\kappa} \kappa(X_s) J(X_s; g)\right] \\ &= \int_0^t ds\, [T_s^\kappa \kappa J](x; g)\end{aligned} \tag{3.2.9}$$

for all $g \in \mathscr{B}(\mathcal{S})$. Letting $t \uparrow \infty$ in the previous display, we see that

$$\boldsymbol{E}_x\left[\mathbf{1}_{\boldsymbol{\tau}_1 < \infty} \cdot g(\boldsymbol{\beta}_{\boldsymbol{\tau}_1})\right] = E_x\left[\int_0^\infty ds\, e^{-A_s^\kappa} \kappa(X_s) J(X_s; g)\right] = [R_\kappa \kappa J](x; g), \qquad x \in E,$$

where again R_κ denotes the generalized resolvent of the process X corresponding to the function κ:

$$R_\kappa(x; B) := E_x\left[\int_0^\infty \mathbf{1}_B(X_t)\, e^{-A_s^\kappa\, ds} dt\right] = E_x\left[\int_0^\tau \mathbf{1}_B(X_t)\, dt\right] \tag{3.2.10}$$

for $x \in E$, $B \in \mathcal{B}_E$.

We now turn to the fundamental *branching property* of the BMP $\boldsymbol{\beta}$. Following [Nag1977], it is most conveniently formulated as a property of the semigroup $(\boldsymbol{T}_t)_t$ in terms of the formal multiplication $\bullet$ and the convolution introduced in the Notations 3.1.3:

$$\boldsymbol{T}_t(\boldsymbol{x} \bullet \boldsymbol{y}; \cdot) = \boldsymbol{T}_t(\boldsymbol{x}; \cdot) * \boldsymbol{T}_t(\boldsymbol{y}; \cdot), \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathcal{S} \tag{3.2.11}$$

(see [Nag1977], eqn. (22), p. 433). Note that (3.2.11) is just a formal statement of the fact that starting from a "concatenation" of two configurations $\boldsymbol{x}$ and $\boldsymbol{y}$ of particles, the branching Markov process $\boldsymbol{\beta}$ evolves as two independent copies of itself starting from $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively.

For a multiplicative function $g : \mathcal{S} \to \mathbb{R}$ which is bounded or nonnegative, the branching property (3.2.11) implies

$$\boldsymbol{E}_{\boldsymbol{x}\bullet\boldsymbol{y}}[g(\boldsymbol{\beta}_t)] = \boldsymbol{T}_t g(\boldsymbol{x} \bullet \boldsymbol{y}) = \boldsymbol{T}_t g(\boldsymbol{x}) \cdot \boldsymbol{T}_t g(\boldsymbol{y}) = \boldsymbol{E}_{\boldsymbol{x}}[g(\boldsymbol{\beta}_t)] \cdot \boldsymbol{E}_{\boldsymbol{y}}[g(\boldsymbol{\beta}_t)], \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathcal{S},$$

i.e. the semigroup $(\boldsymbol{T}_t)_t$ preserves multiplicative functions.[11] In particular, given $f : E \to \mathbb{R}$ with $f \geq 0$ or $\|f\|_\infty \leq 1$ (then $\hat{f} \in \mathscr{B}(\mathcal{S})$), we have

$$\boldsymbol{T}_t \hat{f} = \left((\boldsymbol{T}_t \hat{f})|_E\right)^\wedge. \tag{3.2.12}$$

[11] For the case of unordered configurations, this is in fact equivalent to (3.2.11) and was adopted as definition of the branching property in [INW1968a].

For additive functions $g : \mathcal{S} \to \mathbb{R}$, the branching property reads

$$\boldsymbol{T}_t g(\boldsymbol{x} \bullet \boldsymbol{y}) = \boldsymbol{T}_t g(\boldsymbol{x}) \cdot \boldsymbol{P}_{\boldsymbol{y}}[t < \boldsymbol{\tau}_\infty] + \boldsymbol{T}_t g(\boldsymbol{y}) \cdot \boldsymbol{P}_{\boldsymbol{x}}[t < \boldsymbol{\tau}_\infty].$$

Consequently, the semigroup $(\boldsymbol{T}_t)_t$ preserves additive functions provided $\boldsymbol{\beta}$ does not explode, i.e. $\boldsymbol{\tau}_\infty = \infty$ $\boldsymbol{P}_{\boldsymbol{x}}$-a.s. for all $\boldsymbol{x} \in \mathcal{S}$. In this case, we have

$$\boldsymbol{T}_t \bar{f} = \overline{(\boldsymbol{T}_t \bar{f})|_E} \tag{3.2.13}$$

for all nonnegative measurable functions $f : E \to \mathbb{R}_+$.

By virtue of identities such as (3.2.12) or (3.2.13), the branching property sometimes allows in a sense to reduce the behavior of the "big" (configuration-valued) branching process $\boldsymbol{\beta}$ to the behavior of a single particle on the "small" and more feasible space E. We give some classical examples:

3.2.1 Examples

- Choose $f \equiv 1$ on E. Then $\hat{f} \equiv 1$ on $\mathcal{S}$, and

$$\boldsymbol{T}_t \hat{f}(\boldsymbol{x}) = \boldsymbol{P}_{\boldsymbol{x}}[t < \boldsymbol{\tau}_\infty] = \prod_{j=1}^{\ell(\boldsymbol{x})} \boldsymbol{P}_{x^j}[t < \boldsymbol{\tau}_\infty], \qquad \boldsymbol{x} \in \mathcal{S} \tag{3.2.14}$$

  is the probability that the "lifetime" of the process $\boldsymbol{\beta}$ (in the sense of explosion time) is greater than t. Since nonexplosion of $\boldsymbol{\beta}$ means $\boldsymbol{P}_{\boldsymbol{x}}[t < \boldsymbol{\tau}_\infty] = 1$ for all $t > 0$, $\boldsymbol{x} \in \mathcal{S}$, it is equivalent to the formally weaker assertion that (3.2.14) holds for all $x \in E$. See also Subsection 3.2.5 below.

- Choose $f \equiv 0$ on E. Then $\hat{f} = \mathbf{1}_\Delta$ on $\mathcal{S}$, and $\boldsymbol{T}_t \hat{f}(\boldsymbol{x}) = \boldsymbol{P}_{\boldsymbol{x}}[\boldsymbol{\beta}_t = \Delta]$ is the probability that there are no particles (i.e. $\boldsymbol{\beta}$ has gone extinct) at time t. By the branching property,

$$\boldsymbol{P}_{\boldsymbol{x}}[\boldsymbol{\beta}_t = \Delta] = \prod_{j=1}^{\ell(\boldsymbol{x})} \boldsymbol{P}_{x^j}[\boldsymbol{\beta}_t = \Delta], \tag{3.2.15}$$

  and thus for extinction at time t to hold it suffices to check (3.2.15) for initial conditions $x \in E$.

- Set $f := \mathbf{1}_B$ for a Borel set $B \in \mathcal{B}_E$. Then

$$\boldsymbol{T}_t \bar{f}(\boldsymbol{x}) = \boldsymbol{E}_{\boldsymbol{x}}\left[\overline{\mathbf{1}_B}(\boldsymbol{\beta}_t)\mathbf{1}_{t<\boldsymbol{\tau}_\infty}\right] = \boldsymbol{E}_{\boldsymbol{x}}[\boldsymbol{\beta}_t(B)\mathbf{1}_{t<\boldsymbol{\tau}_\infty}]$$

  is the expected number of particles in B at time t. If $\boldsymbol{\beta}$ does not explode, this is an additive function, and by the "additive" branching property (3.2.13) we obtain

$$\boldsymbol{E}_{\boldsymbol{x}}[\boldsymbol{\beta}_t(B)] = \sum_{j=1}^{\ell(\boldsymbol{x})} \boldsymbol{E}_{x^j}[\boldsymbol{\beta}_t(B)], \qquad \boldsymbol{x} \in \mathcal{S}. \tag{3.2.16}$$

♦

Of special importance to our approach will be the expected number of particles introduced in the last example above; see in particular Subsection 3.2.4 below. Following [INW1969], we adopt the following definition:

3.2.2 Definition

Assume that the BMP $\boldsymbol{\beta}$ *does not explode. For each* $t > 0$*, we define a kernel* $M_t : E \times \mathcal{B}_E \to [0, \infty]$ *giving the expected number of particles of the BMP* $\boldsymbol{\beta}$ *at time* t*, starting from a single particle at* $x \in E$*:*

$$M_t(x; B) := \boldsymbol{E}_x [\boldsymbol{\beta}_t(B)] = \boldsymbol{T}_t(x; \overline{\mathbf{1}_B}), \qquad x \in E,\, B \in \mathcal{B}_E. \tag{3.2.17}$$

3.2.3 Remark

Note that M_t is nothing but the transition semigroup of $\boldsymbol{\beta}$ restricted to functions of the form $\overline{\mathbf{1}_B}$ and arguments $x \in E$. It is clear that M_t is in fact a kernel, i.e. $M_t(x; \cdot)$ is a measure on $\mathcal{B}_E$ for each $x \in E$, but without further assumptions there is no reason why it should be finite, much less bounded in x. Nevertheless, for every nonnegative measurable function $f : E \to \mathbb{R}_+$ we can define $M_t f(x) := \int_E M_t(x; dy) f(y)$, and it is clear that $M_t f = (\boldsymbol{T}_t \bar{f})|_E$. By the "additive branching property" (3.2.13), we obtain

$$\boldsymbol{T}_t \bar{f} = \overline{M_t f}, \qquad t \geq 0, \tag{3.2.18}$$

whence we get immediately that for each $s, t \geq 0$ and $f : E \to \mathbb{R}_+$

$$M_t(M_s f)(x) = \boldsymbol{T}_t(\overline{M_s f})(x) = \boldsymbol{T}_t(\boldsymbol{T}_s \bar{f})(x) = \boldsymbol{T}_{t+s} \bar{f}(x) = M_{t+s} f(x)$$

(cf. [INW1969], Thm. 4.12 on p. 138). Thus the kernels M_t fulfill the semigroup property for nonnegative functions f. Under suitable additional assumptions, each M_t will be a bounded kernel, and thus the family $(M_t)_t$ will induce a semigroup of bounded operators on $\mathscr{B}(E)$ (which in general is not a contraction semigroup, of course). See Section 3.2.4 below. ⧫

We conclude this subsection with several key equations which will be used in the sequel. Recall that under $\boldsymbol{P}_x$, $x \in E$, the evolution of $\boldsymbol{\beta}$ up to the first branching time $\boldsymbol{\tau}_1$ is governed by (3.2.8) and the joint distribution of $\boldsymbol{\tau}_1$ and $\boldsymbol{\beta}_{\boldsymbol{\tau}_1}$ is given by (3.2.9) above. As a consequence, by conditioning on $\boldsymbol{\tau}_1$ we obtain for every $g = (g^{(\ell)})_{\ell \in \mathbb{N}_0} \in \mathscr{B}(\mathcal{S})$ the integral equation

$$\begin{aligned} \boldsymbol{T}_t g(x) \equiv \boldsymbol{E}_x [g(\boldsymbol{\beta}_t) \mathbf{1}_{t < \boldsymbol{\tau}_\infty}] &= E_x \Big[g^{(1)}(X_t) \mathbf{1}_{t<\tau}\Big] + E_x \Big[\mathbf{1}_{\tau \leq t} \int_{\mathcal{S}} J(X_\tau; d\boldsymbol{y}) \boldsymbol{T}_{t-\tau} g(\boldsymbol{y})\Big] \\ &= T_t^\kappa g^{(1)}(x) + \int_0^t ds \int_{\mathcal{S}} [T_s^\kappa \kappa J](x; d\boldsymbol{y}) \boldsymbol{T}_{t-s} g(\boldsymbol{y}) \end{aligned} \tag{3.2.19}$$

for all $x \in E$ and $t \geq 0$. Since (3.2.19) is basically just the strong Markov property at time $\boldsymbol{\tau}_1$, an analogous equation holds of course for arbitrary starting values $\boldsymbol{x} \in \mathcal{S}$ provided T_t^κ and J in the above display are replaced by appropriate quantities. The resulting equation on $\mathcal{S}$ is called *M-equation* by the authors in [INW1968a] (see eqn. (1.39), p. 276). In order not to burden the reader with too much notation, we state it only for starting values $x \in E$ since this is all that will be needed in the sequel. Specifically, we will need equation (3.2.19) for multiplicative and additive functions: Consider $f \in \mathscr{B}(E)$ with $\|f\| \leq 1$ and write

$$u_t(x) := \boldsymbol{T}_t \hat{f}(x), \qquad x \in E,\ t \geq 0$$

for the restriction of $\boldsymbol{T}_t \hat{f}$ to E. Applying equation (3.2.19) for $g = \hat{f}$ and using the branching property in the form (3.2.12) gives

$$u_t(x) = T_t^\kappa f(x) + \int_0^t ds \int_{\mathcal{S}} [T_s^\kappa \kappa J](x; d\boldsymbol{y}) \hat{u}_{t-s}(\boldsymbol{y}), \qquad x \in E,\ t \geq 0. \tag{3.2.20}$$

Similarly, we consider additive functions $g = \bar{f}$, with $f : E \to \mathbb{R}_+$ nonnegative measurable: Assuming that $\boldsymbol{\beta}$ does not explode, write

$$v_t(x) := M_t f(x) \equiv \boldsymbol{T}_t \bar{f}(x), \qquad x \in E,\ t \geq 0.$$

Applying (3.2.19) for $g = \bar{f}$ and using the "additive" branching property (3.2.13), we obtain

$$v_t(x) = T_t^\kappa f(x) + \int_0^t ds \int_{\mathcal{S}} [T_s^\kappa \kappa J](x; d\boldsymbol{y}) \bar{v}_{t-s}(\boldsymbol{y}), \qquad x \in E,\ t \geq 0. \tag{3.2.21}$$

Although (3.2.20) and (3.2.21) are equations on the single-particle space E, on their r.h.s. they formally involve the functions $\hat{u}_{t-s}$ resp. $\bar{v}_{t-s}$ living on the "big" configuration space $\mathcal{S}$, which are however completely determined by u_{t-s} resp. v_{t-s}. In order to get an equation completely in terms of quantities defined on E, we need the following definitions:

3.2.4 Definition

1. *We write $\mathscr{B}^+(E)$ resp. $\mathscr{B}_1^+(E)$ for the class of nonnegative resp. $[0,1]$-valued measurable functions on E:*

$$\mathscr{B}^+(E) := \{f \in \mathscr{B}(E) : f \geq 0\}, \qquad \mathscr{B}_1^+(E) := \{f \in \mathscr{B}^+(E) : \|f\|_\infty \leq 1\}.$$

Using the kernel $J : E \times \mathcal{B}_{\mathcal{S}} \to [0,1]$ from (3.1.11), we define a nonlinear operator $F : \mathscr{B}_1^+(E) \to \mathscr{B}_1^+(E)$, $f(\cdot) \mapsto F(\cdot\,; f)$ by

$$F(y; f) := J(y; \hat{f}) \equiv \sum_{k \in \mathbb{N}_0} p_k(y) Q_k(y; \hat{f}|_{E^k}), \qquad y \in E. \tag{3.2.22}$$

2. *We define a kernel $\tilde{Q} : E \times \mathcal{B}_E \to [0,1]$ as follows: For $x \in E$ and $f : E \to \mathbb{R}$ measurable and bounded or nonnegative, let*

$$\tilde{Q}(x; f) := \begin{cases} \frac{1}{\varrho(x)} J(x; \bar{f}) \equiv \frac{1}{\varrho(x)} \sum_{k \in \mathbb{N}_0} p_k(x) Q_k(x; \bar{f}|_{E^k}), & \text{if } \varrho(x) > 0 \\ \tilde{\nu}(f), & \text{if } \varrho(x) = 0, \end{cases} \tag{3.2.23}$$

where $\varrho(\cdot)$ is the reproduction mean from (3.1.9) and $\tilde{\nu}$ is any fixed probability measure on $(E, \mathcal{B}_E)$.

3.2.5 Remarks

- The definition of F is taken from [INW1969] (see e.g. Def. 4.2, (4.10) on p. 99). Denoting by

$$\tilde{F}(y; s) := \sum_{k \in \mathbb{N}_0} p_k(y) s^k, \qquad s \in [0,1],\ y \in E$$

  the generating function of the space-dependent probability law $(p_k(y))_{k \in \mathbb{N}_0}$, $y \in E$, for each constant function $f(\cdot) \equiv s \in [0,1]$ on E we clearly have

$$F(y; f) = \tilde{F}(y; s).$$

  Observe moreover that if $(Q_k)_k$ is of product type (see (3.1.12)), we have

$$F(y; f) = \sum_{k \in \mathbb{N}_0} p_k(y) \left(Qf(y)\right)^k = \tilde{F}(y; Qf(y)), \qquad f \in \mathscr{B}_1^+(E),\ y \in E.$$

- Since $J(x;\bar{\mathbf{1}}) = \sum_{k\in\mathbb{N}_0} p_k(x)Q_k(x;\bar{\mathbf{1}}) = \sum_{k\in\mathbb{N}_0} kp_k(x) = \varrho(x)$, the kernel $\tilde{Q}$ is a transition probability. Also note that since $\varrho(x) = 0$ implies $p_0(x) = 1$, since $Q_0(x;\cdot) = \delta_\Delta(\cdot)$ and $\bar{f}(\Delta) = 0$ we clearly have

$$J(x;\bar{f}) = \varrho(x)\tilde{Q}(x;f), \qquad x \in E. \tag{3.2.24}$$

If $(Q_k)_k$ is of product type (3.1.12), the kernel $[\varrho\tilde{Q}]$ coincides with $[\varrho Q]$. Our kernel $[\varrho\tilde{Q}]$ corresponds to the kernel G defined in (4.80) in [INW1969], p. 140. ⧫

Using the operator F from (3.2.22) and the kernel $\tilde{Q}$ from (3.2.23), the equations (3.2.20) and (3.2.21) can be restated as follows:

$$\begin{aligned} u_t(x) &= T_t^\kappa f(x) + \int_0^t ds\,[T_s^\kappa \kappa F](x;u_{t-s}) \\ &= E_x\left[f(X_t)\mathbf{1}_{t<\tau}\right] + E_x\left[\mathbf{1}_{\tau\le t}\, F(X_\tau;u_{t-\tau})\right], \end{aligned} \tag{3.2.25}$$

$$\begin{aligned} v_t(x) &= T_t^\kappa f(x) + \int_0^t ds\,[T_s^\kappa \kappa\varrho\tilde{Q}](x;v_{t-s}) \\ &= E_x\left[f(X_t)\mathbf{1}_{t<\tau}\right] + E_x\left[\mathbf{1}_{\tau\le t}\, \varrho(X_\tau)\tilde{Q}(X_\tau;v_{t-\tau})\right]. \end{aligned} \tag{3.2.26}$$

Thus we have two equations completely in terms of quantities defined on the single-particle space E. Equation (3.2.25) is termed *S-equation* by the authors in [INW1969],[12] and from their work we have the following result (see Cor. 2 on p. 114 and Thm. 4.13 on pp. 139f., respectively):

3.2.6 Theorem

1. *For $f \in \mathscr{B}_1^+(E)$, the function*

$$u_t(x) := \boldsymbol{T}_t\hat{f}(x) \equiv \boldsymbol{E}_x\left[\hat{f}(\boldsymbol{\beta}_t)\mathbf{1}_{t<\boldsymbol{\tau}_\infty}\right]$$

is the minimal solution of the S-equation (3.2.25) *in the class $\mathscr{B}_1^+(E)$.*

2. *Assume that $\boldsymbol{\beta}$ does not explode. Then for $f \in \mathscr{B}^+(E)$, the function*

$$v_t(x) := M_t f(x) \equiv \boldsymbol{T}_t\bar{f}(x) \equiv \boldsymbol{E}_x\left[\bar{f}(\boldsymbol{\beta}_t)\right]$$

is the minimal nonnegative solution of the equation (3.2.26).

It is by way of equations such as (3.2.25) and (3.2.26) that the branching property often allows for the reduction of problems on the “big” configuration space $\mathcal{S}$ to problems on the single-particle space E, which distinguishes the purely position-dependent framework from the “interactive” case considered in the first two chapters.

3.2.7 Remark

Under suitable regularity conditions on (the semigroup of) X, κ and J, equations (3.2.25) and (3.2.26) can be restated in differential form: More precisely, suppose that X is a *Feller process*, i.e. its semigroup $(T_t)_t$ preserves the space $\mathcal{C}_0(E)$ of continuous functions vanishing

[12] See e.g. Def. 4.4 on p. 102.

at infinity. As is well known, together with the càdlàg property of the paths this implies the strong continuity of $(T_t)_t$ on $\mathcal{C}_0(E)$, i.e.

$$\|T_t f - f\|_\infty \xrightarrow{t\downarrow 0} 0, \qquad f \in \mathcal{C}_0(E).$$

Let $(A, D(A))$ denote the infinitesimal generator of $(T_t)_t$ defined by

$$D(A) := \left\{ f \in \mathcal{C}_0(E) : \exists Af := \lim_{t\downarrow 0} \frac{T_t f - f}{t} \in \mathcal{C}_0(E) \right\}.^{13} \tag{3.2.27}$$

Moreover, suppose that $\kappa(\cdot) \in \mathcal{C}_b(E)$ and that the nonlinear operator F from (3.2.22) maps $\mathcal{C}_0(E) \cap \mathscr{B}_1^+(E)$ into $\mathcal{C}_0(E)$. Then given $f \in D(A) \cap \mathscr{B}_1^+(E)$, the function $u_t := \boldsymbol{T}_t \hat{f}|_E$ is in $D(A)$ for all $t > 0$, $t \mapsto u_t$ is (strongly) differentiable and we have

$$\frac{\partial}{\partial t} u_t(x) = A u_t(x) + \kappa(x) \left(F(x; u_t) - u_t(x) \right), \qquad x \in E,\ t > 0 \tag{3.2.28}$$

(see [INW1969], Cor. on p. 127).[14]

If in addition $\varrho \in \mathcal{C}_b(E)$ and $\tilde{Q}$ from (3.2.23) (considered as an operator on $\mathscr{B}(E)$) preserves the subspace $\mathcal{C}_0(E)$, the family of kernels $(M_t)_t$ of (3.2.17) induces a strongly continuous semigroup of bounded operators on $\mathcal{C}_0(E)$ with generator

$$Lf := Af - \kappa f + \kappa \varrho \tilde{Q} f, \qquad f \in D(L) = D(A) \tag{3.2.29}$$

(see [INW1969], Thm. 4.14, p. 143). In particular, again writing $v_t \equiv M_t f$ for $f \in D(L) = D(A)$, equation (3.2.26) can be restated as

$$\frac{\partial}{\partial t} v_t(x) = A v_t(x) + \kappa(x) \left(\varrho(x) \tilde{Q}(x; v_t) - v_t(x) \right), \qquad x \in E,\ t > 0. \tag{3.2.30}$$

♦

We conclude this subsection by continuing the examples given in 3.2.1 above:

3.2.8 Examples

- A classical application of the S-equation concerns the question of *nonexplosion* of the branching Markov process $\boldsymbol{\beta}$ (see e.g. [Nag1977], pp. 443f.): Continuing the first example in 3.2.1, choosing $f \equiv 1$ on E we get $T_t^\kappa f(x) = T_t^\kappa(x; E) = P_x[t < \tau]$ and

  $$u_t(x) \equiv \boldsymbol{T}_t \hat{f}(x) = \boldsymbol{P}_x[t < \boldsymbol{\tau}_\infty], \qquad x \in E,\ t > 0.$$

  Thus by Theorem 3.2.6, $u_t(\cdot)$ is the minimal solution in $\mathscr{B}_1^+(E)$ of

  $$u_t(x) = T_t^\kappa(x; E) + \int_0^t ds\, [T_s^\kappa \kappa F](x; u_{t-s}) = P_x[t < \tau] + E_x\left[\mathbf{1}_{\tau \le t} F(X_\tau; u_{t-\tau})\right] \tag{3.2.31}$$

  on E. Since on the other hand $u_t(\cdot) \equiv 1$ is always a solution of (3.2.31), it follows that nonexplosion is equivalent to uniqueness of the solution in $\mathscr{B}_1^+(E)$ of (3.2.31). From this one can deduce as in [Nag1977], Thm. 8 on p. 443, that nonexplosion is ensured whenever κ and ϱ are bounded. On more concerning the explosion problem, see Subsection 3.2.5 below.

[13] The limit is understood in the strong sense, i.e. w.r.t. the uniform norm.

[14] The above regularity conditions on X, κ and J ensure regularity in the sense of [INW1969], Def. 4.7 on pp. 115f., thus their Corollary on p. 127 is applicable.

- We continue the second example in 3.2.1: Choosing $f \equiv 0$ on E gives $T^\kappa f \equiv 0$ and

$$u_t(x) \equiv \boldsymbol{T}_t \hat{f}(x) = \boldsymbol{P}_x[\boldsymbol{\beta}_t = \Delta], \tag{3.2.32}$$

the probability that $\boldsymbol{\beta}$ has gone extinct at time t. Then Theorem 3.2.6 says that $u_t(\cdot)$ is the minimal solution in $\mathscr{B}_1^+(E)$ of the equation

$$u_t(x) = \int_0^t ds\,[T_s^\kappa \kappa F](x; u_{t-s}) = E_x\left[\mathbf{1}_{\tau \le t} F(X_\tau; u_{t-\tau})\right], \qquad x \in E \tag{3.2.33}$$

which can be used to study the extinction problem for branching Markov processes. For example, assume that X is a one-dimensional Brownian motion, κ and $(p_k)_k$ are constants and branching particles reproduce at their death position, i.e. $(Q_k)_k$ is of product type with $Q(x;\cdot) = \delta_x(\cdot)$. Then the regularity conditions mentioned in Remark 3.2.7 are satisfied, and (3.2.33) can be restated as a nonlinear partial differential equation of Kolmogorov–Petrovskii–Piskounov-type

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \kappa\left(\sum_{k \in \mathbb{N}_0} p_k u^k - u\right), \qquad u_0(\cdot) \equiv 0, \tag{3.2.34}$$

which in different contexts has been extensively studied in the literature (see e.g. [Bra1978] or [HHK2006] and the references therein for probabilistic approaches to this equation).

- Concerning the expected number of particles considered in the third example in 3.2.1, the characterization of $M_t f$ as the minimal nonnegative solution to (3.2.26) for nonnegative f allows to deduce explicit representations of the kernels M_t in terms of the given quantities X, κ and J on the single particle space E, as will be seen in Subsections 3.2.3 and 3.2.4 below. ♦

3.2.2 The Extinction Problem for Branching Markov Processes

Remember that our ultimate interest is in adding immigration to the picture so that the resulting branching Markov process with immigration (BMPI) $\boldsymbol{\eta}$ exhibits ergodic behavior, admitting the void configuration Δ as a recurrent atom as in Assumption 1.2.1 of Chapter 1. Clearly, for this to hold the "branching component" of the process (which is a BMP $\boldsymbol{\beta}$ as in the present section) should go extinct with finite expected extinction time, and our goal is to give sufficient conditions to this effect. In essence, the problem is to find a "spatial substitute" for the classical notion of subcriticality that the reproduction mean be strictly smaller than 1. In a branching diffusions context and under (3.1.13), an answer was given in [HL2005]. We will present the spatial subcriticality condition proposed in [HL2005] and generalize it to our framework in the next subsection (see Condition 3.2.19). At this point, for the sake of completeness of our exposition we take the opportunity to give a brief discussion of the extinction problem for general branching Markov processes. For branching Brownian motions with absorbing boundaries, this problem has been considered in the classical papers [Sev1958] and [Wat1965]; for a recent generalization to branching symmetric α-stable processes, see [Shi2006].

We begin with the formal definition of extinction:

3.2.9 Definition
Let $\boldsymbol{\beta}$ be a branching Markov process as in the previous subsection. We write T_e for the extinction time*, i.e. the hitting time of the void configuration Δ:*

$$T_e := \inf\{t > 0 : \boldsymbol{\beta}_t = \Delta\} \in [0, \infty]. \tag{3.2.35}$$

We say that the BMP $\boldsymbol{\beta}$ goes extinct if $\boldsymbol{P}_{\boldsymbol{x}}[T_e < \infty] = 1$ for all $\boldsymbol{x} \in \mathcal{S}$.

3.2.10 Remark
Since Δ is an absorbing state for $\boldsymbol{\beta}$ and the configuration length does not change between branching events, we clearly have

$$T_e = \inf\{t > 0 : \boldsymbol{\beta}_s = \Delta \text{ for all } s \geq t\} = \inf\{T_n : n \in \mathbb{N}_0, \boldsymbol{\beta}_{T_n} = \Delta\}.$$

For the same reason, we have $\{T_e < \infty\} \subseteq \{T_\infty = \infty\}$, and thus extinction implies non-explosion. ♦

3.2.11 Remark
For a branching Markov process $\boldsymbol{\beta}$ as in the previous subsection, we may as in the first remark in 1.2.5 define the configuration length or total mass process

$$Z_t^{\boldsymbol{\beta}} := \ell(\boldsymbol{\beta}_t), \qquad t \geq 0 \tag{3.2.36}$$

for the number of particles of the BMP $\boldsymbol{\beta}$ "alive" at time t. Again it is easy to see that if the branching rate κ and the reproduction probabilities p_k are spatially constant, $Z^{\boldsymbol{\beta}}$ is a "classical" Galton-Watson branching process in continuous time, to which the corresponding classical theory can be applied (see e.g. [Har1963], Ch. V or [AN1972], Ch. III). Thus $Z^{\boldsymbol{\beta}}$ and consequently $\boldsymbol{\beta}$ goes extinct with probability one (starting from a single particle at time 0) iff $\varrho < 1$ or $\varrho = 1$ and $p_1 \neq 1$. More specifically, the extinction probability coincides with the smallest nonnegative root of the equation $s = \tilde{F}(s)$, where $\tilde{F}$ is the generating function of $(p_k)_k$. Further, in the subcritical case $\varrho < 1$ the expected time to extinction is finite, whereas it can be finite or infinite in the critical case $\varrho = 1$ (see also Example 3.2.16 below).

By the coupling method sketched in the second remark in 1.2.5, the above can be extended to processes with position-dependent rates and reproduction probabilities, provided the strong condition (1.2.17) holds which requires that the position-dependent reproduction law $p_k(\cdot)_k$ be upper bounded by a fixed subcritical law in a convolution sense. It is precisely one of the advantages of the purely position-dependent framework that it is possible to go beyond these results which are based on a comparison to a branching process with constant rates. As we are about to see, the reason for this is the branching property and its consequences such as the S-equation (3.2.25). ♦

Let us employ the S-equation (3.2.25) in order to derive a characterization of the probability of extinction: By (3.2.15) and the fact that Δ is an absorbing state for $\boldsymbol{\beta}$, we have

$$\boldsymbol{P}_{\boldsymbol{x}}[T_e \leq t] = \boldsymbol{P}_{\boldsymbol{x}}[\boldsymbol{\beta}_t = \Delta] = \prod_{j=1}^{\ell(\boldsymbol{x})} \boldsymbol{P}_{x^j}[\boldsymbol{\beta}_t = \Delta] = \prod_{j=1}^{\ell(\boldsymbol{x})} \boldsymbol{P}_{x^j}[T_e \leq t], \qquad \boldsymbol{x} \in \mathcal{S},\ t > 0. \tag{3.2.37}$$

Letting $t \uparrow \infty$ in the above display, we see that the probability of extinction $\boldsymbol{x} \mapsto \boldsymbol{P}_{\boldsymbol{x}}[T_e < \infty]$ is a multiplicative function of $\boldsymbol{x} \in \mathcal{S}$. Thus it is completely determined by its restriction

$$u_e(x) := \boldsymbol{P}_x[T_e < \infty] \tag{3.2.38}$$

to initial values $x \in E$. In particular, extinction is in fact equivalent to the formally weaker assertion that $u_e(\cdot) \equiv 1$ on E. Moreover, letting $t \uparrow \infty$ in (3.2.33) we obtain by dominated convergence that the extinction probability $u_e(\cdot)$ on E is a solution to

$$u_e(x) = [R_\kappa \kappa F](x; u_e) = E_x \left[\mathbf{1}_{\tau < \infty} \cdot F(X_\tau; u_e) \right], \qquad x \in E, \tag{3.2.39}$$

where R_κ is the generalized resolvent kernel from (3.2.10). As a consequence, a sufficient condition for extinction is uniqueness of the constant function $u(\cdot) \equiv 1$ as solution to the above equation in the class $\mathscr{B}_1^+(E)$. Observe that since $F(x; 1) = 1$ for all $x \in E$, $u(\cdot) \equiv 1$ is a solution to equation (3.2.39) iff the kernel $[R_\kappa \kappa]$ is a transition probability, which is equivalent to $P_x[\tau < \infty] = 1$ and clearly a necessary condition for extinction.

In fact, we have also the converse assertion that extinction implies uniqueness in equation (3.2.39), since one can prove the following result:

3.2.12 Theorem
The extinction probability $u_e(x) = \boldsymbol{P}_x(T_e < \infty)$, $x \in E$, is the minimal solution of the equation

$$u(x) = [R_\kappa \kappa F](x; u) = E_x \left[\mathbf{1}_{\tau < \infty} \cdot F(X_\tau; u) \right], \qquad x \in E \tag{3.2.40}$$

in the class $\mathscr{B}_1^+(E)$. Consequently, the BMP $\boldsymbol{\beta}$ goes extinct if and only if equation (3.2.40) *has the unique solution $u(\cdot) \equiv 1$.*

3.2.13 Remarks

- We will not give the proof of the minimality asserted in Theorem 3.2.12 since this result will not be needed in the sequel. The minimality of $u_e(\cdot)$ does not strictly follow from minimality of $u_t(\cdot)$ in (3.2.33) but may be proved by analogous arguments (namely those given in the proof of Thm. 4.7 on p. 112 in [INW1969]).

- Although Theorem 3.2.12 is certainly well known, we have not found it stated in the literature in the general form above. An analogous assertion for branching Brownian motions with absorbing boundary is classical (see e.g. [Sev1958], Sec. 6, in particular Thm. 1 on p. 117; [Wat1965], Thm. 2.1 on p. 390). For branching symmetric α-stable processes with absorbing boundaries, it is proved as Prop. 3.1 in [Shi2006].

- Theorem 3.2.12 says that the function $u_e(\cdot)$ is the minimal fixed point of the (nonlinear) operator $R_\kappa \kappa F$ on the space $\mathscr{B}_1^+(E)$, giving a nice analogy to the classical result for Galton-Watson processes that the extinction probability is the minimal fixed point of the generating function in the interval $[0,1]$ (see [AN1972], Thm. 1 on p. 108 or [Har1963], Thm. 10.1 on p. 108). See also the Remarks 3.2.5 for the connection between the operator F and the generating function. ⧫

In order to illustrate the usefulness of the above characterization of the extinction probability, let us consider the following simple question which arises naturally in view of the theory of classical Galton-Watson processes: Is it always true that $\boldsymbol{\beta}$ goes extinct if the reproduction mean is at most critical in a pointwise sense, i.e. $\varrho(\cdot) \leq 1$ on E? This may seem "obvious", but again we have not found a result (for general X and $(Q_k)_k$) in the literature. Note that since the condition $\varrho(\cdot) \leq 1$ is substantially weaker than the assumption (1.2.17) that $(p_k(\cdot))_k$ is upper bounded by a spatially constant, critical reproduction law in a convolution sense, the coupling and comparison results mentioned in Remark 3.2.11 do not apply. We will see that the answer to the question posed above is basically yes, although care must be taken in order to exclude some "pathological" cases. Even for classical GW processes, for the equivalence of extinction and the condition $\varrho \le 1$ to hold one has to exclude the trivial case $p_1 = 1$ from consideration. In the spatial framework, consider e.g. $E = \mathbb{R}^d$ and a position-dependent reproduction law $(p_k(\cdot))_k$ which is pointwise (strictly) subcritical, $\varrho(\cdot) < 1$ on E, but such that $p_1(x) \xrightarrow{\|x\|\to\infty} 1$. If the single particle motion X is "sufficiently transient" on $\mathbb{R}^d$, i.e. runs out to the realm where $p_1(\cdot)$ is nearly equal to 1 fast enough, it is conceivable that the BMP $\boldsymbol{\beta}$ survives with positive probability although "pointwise subcriticality" holds. What we need is a suitable substitute for the classical condition $p_1 \neq 1$ in order to exclude this "pathological" behavior. We have the following result:[15]

3.2.14 Proposition
Suppose that the kernel $[R_\kappa \kappa]$ is a transition probability, that $\varrho(\cdot) \le 1$ on E, and that

$$\inf_{x\in E} R_\kappa \kappa p_0(x) = \inf_{x\in E} E_x\left[p_0(X_\tau)\right] > 0. \tag{3.2.41}$$

Then $\boldsymbol{\beta}$ goes extinct.

3.2.15 Remark
The kernel $[R_\kappa \kappa]$ being a transition probability is equivalent to the condition that the random time τ of (3.2.2) is finite P_x-a.s., for all $x \in E$. Since τ is distributed under P_x as the first branching time in the process $\boldsymbol{\beta}$ under $\boldsymbol{P}_x$, this is evidently a necessary condition for extinction to hold. Further, note that (3.2.41) is trivially satisfied provided $p_0(\cdot)$ is bounded away from 0, or equivalently $\sup_{x\in E} p_1(x) < 1$. ♦

Proof [of Proposition 3.2.14] As already mentioned in the first remark in 3.2.5, for fixed $s \in [0,1]$ the nonlinear operator F from Definition 3.2.4 applied to the constant function $f \equiv s$ coincides with the generating function of the reproduction law $(p_k(y))_{k\in\mathbb{N}_0}$ at s:

$$F(y;s) = \sum_{k\in\mathbb{N}_0} p_k(y) \int_{E^k} Q_k(y; dv^1 \cdots dv^k) f(v^1)\cdots f(v^k) = \sum_{k\in\mathbb{N}_0} p_k(y) s^k, \qquad y \in E.$$

We observe that for $s \in [0,1]$ and $k \ge 2$ we have

$$s^k - s = (s-1) \sum_{i=1}^{k-1} s^i \ge (s-1)(k-1)s.$$

[15] For Proposition 3.2.14, we do not need the minimality assertion of Theorem 3.2.12 but only the fact that the extinction probability is a solution to (3.2.40).

Consequently, for every $y \in E$

$$\begin{aligned}
F(y;s) - s &= \sum_{k \in \mathbb{N}_0} p_k(y)\left(s^k - s\right) \\
&= p_0(y) \cdot (1-s) + p_1(y) \cdot 0 + \sum_{k \geq 2} p_k(y)\left(s^k - s\right) \\
&\geq p_0(y)(1-s) + \sum_{k \geq 2} p_k(y)(s-1)(k-1)s \\
&= (1-s)\left(p_0(y) - s\sum_{k\geq 2} p_k(y)(k-1)\right) \\
&= (1-s)\left(p_0(y) - s\left(\sum_{k\geq 2} p_k(y)k - \sum_{k\geq 2} p_k(y)\right)\right) \\
&= (1-s)\Big(p_0(y) - s\Big(\varrho(y) - p_1(y) - (1 - p_0(y) - p_1(y))\Big)\Big) \\
&\geq (1-s)\Big(p_0(y) - s\Big(1 - p_1(y) - (1 - p_0(y) - p_1(y))\Big)\Big) \\
&= (1-s)^2 p_0(y).
\end{aligned}$$

Now let $s := \inf_{x\in E} u_e(x) \in [0,1]$, with $u_e(\cdot)$ as in (3.2.38). Since as we have seen above $u_e(\cdot)$ is a fixed point of the (positive!) operator $R_\kappa \kappa F$ on $\mathscr{B}_1^+(E)$ and by assumption the kernel $[R_\kappa \kappa]$ is a transition probability, we obtain

$$\begin{aligned}
s = \inf_{x\in E} u_e(x) = \inf_{x\in E}[R_\kappa \kappa F](x;u_e) &\geq \inf_{x\in E}\int_E [R_\kappa\kappa](x;dy)F(y;s) \\
&= s + \inf_{x\in E}\int_E [R_\kappa\kappa](x;dy)\left(F(y;s) - s\right) \\
&\geq s + \inf_{x\in E}\int_E [R_\kappa\kappa](x;dy)(1-s)^2 p_0(y) \\
&= s + (1-s)^2 \cdot \inf_{x\in E} R_\kappa \kappa p_0(x).
\end{aligned}$$

From this estimate it follows at once that if (3.2.41) holds, we must have $s = 1$, i.e. $u_e(\cdot) \equiv 1$. ■

Proposition 3.2.14 is a clear illustration of the power of the branching property in the purely position-dependent framework. (Note that we did not need any assumptions on the precise nature of the single-particle motion X or the spatial offspring distribution $(Q_k)_k$.) Namely, we have seen that as far as sufficient conditions for extinction are concerned, it allows us to replace upper bounds for the reproduction law in a convolution sense with upper bounds for the reproduction mean in a pointwise sense. It seems that an analogous result is not known in the fully interactive framework of Chapters 1 and 2.

This concludes our brief discussion of the extinction probability and of equation (3.2.39). As remarked at the beginning of this subsection, our primary interest is in the finiteness of the expected extinction time

$$\boldsymbol{e}(\boldsymbol{x}) := \boldsymbol{E}_{\boldsymbol{x}}[T_e], \qquad \boldsymbol{x} \in \mathcal{S}.$$

Unfortunately, the function $\boldsymbol{e}(\cdot)$ is generally neither multiplicative nor additive on the configuration space $\mathcal{S}$, and thus it has no reason to be determined by its restriction to the single particle space E. In particular, we do not have an equation in terms of quantities defined on E which characterizes the expected extinction time, as was the case for the extinction probability above. However, we make the trivial observation that

$$\boldsymbol{e}(\boldsymbol{x}) = \int_0^\infty \boldsymbol{P}_{\boldsymbol{x}}\left[T_e > t\right] dt = \int_0^\infty (1 - \boldsymbol{u}_t(\boldsymbol{x}))\, dt, \qquad \boldsymbol{x} \in \mathcal{S} \tag{3.2.42}$$

where $\boldsymbol{u}_t(\boldsymbol{x}) := \boldsymbol{P}_{\boldsymbol{x}}\left[T_e \leq t\right] = \boldsymbol{P}_{\boldsymbol{x}}\left[\boldsymbol{\beta}_t = \Delta\right]$ is known to be multiplicative by (3.2.37). Consequently,

$$\begin{aligned} \boldsymbol{e}(\boldsymbol{x} \bullet \boldsymbol{y}) &= \int_0^\infty (1 - \boldsymbol{u}_t(\boldsymbol{x} \bullet \boldsymbol{y}))\, dt = \int_0^\infty (1 - \boldsymbol{u}_t(\boldsymbol{x}) \cdot \boldsymbol{u}_t(\boldsymbol{y}))\, dt \\ &= \int_0^\infty \left(1 - \boldsymbol{u}_t(\boldsymbol{x}) + \underbrace{\boldsymbol{u}_t(\boldsymbol{x})}_{\in[0,1]}(1 - \boldsymbol{u}_t(\boldsymbol{y})) \right) dt \\ &\leq \int_0^\infty (1 - \boldsymbol{u}_t(\boldsymbol{x}))\, dt + \int_0^\infty (1 - \boldsymbol{u}_t(\boldsymbol{y}))\, dt = \boldsymbol{e}(\boldsymbol{x}) + \boldsymbol{e}(\boldsymbol{y}) \end{aligned} \tag{3.2.43}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$. Thus $\boldsymbol{e}(\cdot)$ is *subadditive* on $\mathcal{S}$, and at least *finiteness* of the expected extinction time reduces to checking finiteness of (3.2.42) for starting values $x \in E$. In general, all that can be done without introducing further assumptions is to study the asymptotic behavior of $\boldsymbol{P}_x[T_e > t]$ as $t \to \infty$ in order to check (3.2.42) for $x \in E$.

We conclude this subsection with the following example showing that already for classical branching processes without spatial behavior, finiteness of the expected extinction time can hold even in the critical case $\varrho = 1$. We give this example[16] since it will also show that in the context of branching Markov processes *with* immigration to be considered in Section 3.3, finiteness of the invariant occupation measure $\overline{m}$ on E is strictly stronger than finiteness of the invariant measure m on $\mathcal{S}$ (see Example 3.3.10 below):

3.2.16 Example
Consider a classical (nonspatial) Galton-Watson branching process in continuous time with branching rate $\kappa > 0$ and reproduction law $(p_k)_{k \in \mathbb{N}_0}$ given by

$$p_0 := \frac{b}{\delta}, \qquad p_1 := 1 - b, \qquad p_k := \frac{b}{\delta} \cdot (-1)^k \binom{\delta}{k} \geq 0 \quad \text{for } k \geq 2 \tag{3.2.44}$$

with $b \in (0,1)$ and $\delta \in (1,2)$. Here, $\binom{\delta}{k} \equiv \frac{\delta(\delta-1)\cdots(\delta-k+1)}{k!}$ denote the generalized binomial coefficients. The generating function of $(p_k)_k$ is given by

$$\tilde{F}(s) = s + \frac{b}{\delta} \sum_{k \in \mathbb{N}_0} (-1)^k \binom{\delta}{k} s^k = s + \frac{b}{\delta}(1 - s)^\delta, \qquad s \in [0,1],$$

whence we obtain immediately that $(p_k)_k$ is critical:

$$\varrho = \tilde{F}'(1) = 1.$$

[16]The author learned of this example in a lecture by Prof. Hans-Jürgen Schuh in the context of discrete-time branching processes with immigration.

Setting $u_t := P_1[T_e \le t]$, equation (3.2.28) reads as follows:

$$\frac{\partial}{\partial t} u_t = \kappa \cdot (\tilde{F}(u_t) - u_t) = \kappa \frac{b}{\delta} \cdot (1 - u_t)^{\delta}.$$

Solving this ODE (e.g. by separation of variables) gives

$$1 - u_t = \left(1 + b\frac{\delta - 1}{\delta} \cdot t\right)^{\frac{1}{1-\delta}}, \qquad t > 0,$$

from which we get finiteness of the expected extinction time $E_1[T_e] = \int_0^\infty (1 - u_t)\, dt < \infty$ since $\delta \in (1, 2)$.

By applying this argument to the total mass process $Z^{\boldsymbol{\beta}}$, the same holds for any spatial BMP $\boldsymbol{\beta}$ with constant branching rate and reproduction law as above, but with arbitrary single particle motion X and spatial offspring distribution $(Q_k)_k$. ♦

3.2.3 The Expected Occupation Time and a Spatial Notion of Subcriticality

We return to the problem of defining a suitable notion of "subcriticality" for spatial branching Markov processes. Consider a BMP $\boldsymbol{\beta}$ and suppose that it does not explode, i.e. $\boldsymbol{\tau}_\infty = \infty$ $\boldsymbol{P_x}$-a.s. for all $\boldsymbol{x} \in \mathcal{S}$. In this case, an even stronger property than finiteness of the expected extinction time is the condition that the total expected lifetime of all particles in the process $\boldsymbol{\beta}$ is finite: In terms of the total mass process $Z^{\boldsymbol{\beta}}$ of (3.2.36), this means

$$\boldsymbol{E_x}\left[\int_0^\infty Z_t^{\boldsymbol{\beta}}\, dt\right] < \infty, \qquad \boldsymbol{x} \in \mathcal{S}. \tag{3.2.45}$$

Since due to nonexplosion we have $Z_t^{\boldsymbol{\beta}} \ge 1$ as long as $0 \le t < T_e$, (3.2.45) implies

$$\boldsymbol{E_x}[T_e] = \boldsymbol{E_x}\left[\int_0^{T_e} dt\right] \le \boldsymbol{E_x}\left[\int_0^\infty Z_t^{\boldsymbol{\beta}}\, dt\right] < \infty, \tag{3.2.46}$$

thus $\boldsymbol{\beta}$ goes extinct with finite expected extinction time.

For classical Galton-Watson processes without spatial behavior, 3.2.45 is equivalent to subcriticality $\varrho < 1$. (This equivalence is also a consequence of Proposition 3.2.21 or Corollary 3.2.30 below.) In [HL2005], working in a branching diffusion context and under the assumption (3.1.13) that branching particles reproduce at their death position, Höpfner and Löcherbach proposed to impose precisely the condition (3.2.45) as a notion of subcriticality for spatial branching processes and gave several equivalent characterizations of it. The rest of this subsection and the following one are devoted to generalizing some of their results to our framework of arbitrary single-particle motions X and spatial offspring distributions $(Q_k)_k$. The generalization is straightforward since the relevant results can be proved using only the strong Markov property and the branching property of the BMP $\boldsymbol{\beta}$. While the precise form of X and $(Q_k)_k$ is not important in the statement of the subcriticality condition and of its characterizations, it is of course important in checking these conditions in concrete cases (see the examples to be given below). But also for a diffusion as single-particle motion and under (3.1.13), our results are a generalization of those in [HL2005] since our assumptions on the branching rate κ and the reproduction mean ϱ are weaker; in particular, we do not explicitly require that these quantities be bounded or continuous. However, throughout the rest of this subsection we will work under the condition of nonexplosion:

3.2.17 Assumption
The BMP $\boldsymbol{\beta}$ does not explode:

$$\boldsymbol{P}_{\boldsymbol{x}}[\boldsymbol{\tau}_\infty = \infty] = 1, \qquad \boldsymbol{x} \in \mathcal{S}. \tag{3.2.47}$$

It was already mentioned in the first example in 3.2.8 that the easiest way to ensure nonexplosion is to require boundedness of the branching rate κ and of the reproduction mean ϱ; these boundedness conditions are not necessary, however. We will discuss the explosion problem briefly in Subsection 3.2.5 below; for this and the next subsection, we simply accept nonexplosion as a condition.

Next, we define a kernel giving the total expected occupation time of a borel set $B \in \mathcal{B}_E$ by all particles in the BMP $\boldsymbol{\beta}$, starting from a single particle:

3.2.18 Definition
Under Assumption 3.2.17, let

$$H(x;B) := \boldsymbol{E}_x\left[\int_0^{T_e} \boldsymbol{\beta}_t(B)\,dt\right] \equiv \int_0^\infty M_t(x;B)\,dt, \qquad x \in E,\, B \in \mathcal{B}_E, \tag{3.2.48}$$

where $(M_t)_t$ is the family of kernels from Definition 3.2.2 giving the expected number of particles at time t.

Observe that (3.2.48) defines a (potentially infinite) kernel $H(\cdot\,;\cdot) : E \times \mathcal{B}_E \to [0,\infty]$. The above definition of H is the same as that of the kernel V in [HHL2002] (see immediately before Assumption 6, p. 670) or in [HL2005], Prop. 2.2. As an immediate consequence of the "additive branching property" in the form (3.2.18), occupation times starting from configurations $\boldsymbol{x} \in \mathcal{S}$ of arbitrary length are given by

$$\boldsymbol{E}_{\boldsymbol{x}}\left[\int_0^{T_e} \boldsymbol{\beta}_t(B)\,dt\right] = \overline{H(\cdot\,;B)}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathcal{S},\, B \in \mathcal{B}_E. \tag{3.2.49}$$

We now state the spatial subcriticality condition originally due to [HL2005]:

3.2.19 Condition (Spatial Subcriticality)
Grant Assumption 3.2.17 (nonexplosion).

- *We say that the branching Markov process $\boldsymbol{\beta}$ fulfills the* Spatial Subcriticality Condition *or is spatially subcritical if the kernel H from Definition 3.2.18 is finite, i.e.*

$$H(x;E) < \infty, \qquad x \in E. \tag{3.2.50}$$

- *We say that the* Uniform Spatial Subcriticality Condition *is fulfilled or that $\boldsymbol{\beta}$ is uniformly spatially subcritical if H is a bounded kernel, i.e.*

$$\sup_{x \in E} H(x;E) < \infty. \tag{3.2.51}$$

The occupation times kernel H is a kernel on the single particle space $(E, \mathcal{B}_E)$, but it is defined in terms of the branching process $\boldsymbol{\beta}$ on the "big" configuration space $\mathcal{S}$. We would like to have representations of H in terms of the "given" quantities X, κ and J (recall (3.1.11)) on the single particle space, in order to be able to check the spatial subcriticality condition 3.2.19.

Using the second assertion of Theorem 3.2.6 due to [INW1969] above, we get almost immediately a series representation for the kernel M_t of (3.2.17), from which by integrating w.r.t. $t \in \mathbb{R}_+$ we can derive without too much effort a series representation for the occupation times kernel H as in [HHL2002] or [HL2005].[17] The representation of M_t below may look complicated but will be of use also in the next subsection:

3.2.20 Lemma
Grant Assumption 3.2.17 (nonexplosion). Then for each $f \in \mathscr{B}^+(E)$, we have

$$M_t f(x) = T_t^\kappa f(x) + \sum_{n\in\mathbb{N}} \int_0^t ds_1 \int_E [T_{s_1}^\kappa \kappa\varrho\tilde{Q}](x; dy_1) \int_0^{t-s_1} ds_2 \int_E [T_{s_2}^\kappa \kappa\varrho\tilde{Q}](y_1; dy_2) \cdots$$
$$\cdots \int_0^{t-s_1-\ldots-s_{n-1}} ds_n \int_E [T_{s_n}^\kappa \kappa\varrho\tilde{Q}](y_{n-1}; dy_n) T_{t-s_1-\ldots-s_n}^\kappa f(y_n). \tag{3.2.52}$$

Proof By the second assertion in Theorem 3.2.6, $v_t(x) = M_t f(x)$ is the minimal nonnegative solution of the equation

$$v_t(x) = T_t^\kappa f(x) + \int_0^t ds\, [T_s^\kappa \kappa\varrho\tilde{Q}](x; v_{t-s}), \qquad x \in E,\ t \geq 0 \tag{3.2.53}$$

on E. But it is a standard argument that the minimal solution can be explicitly constructed by successive approximation: Setting $v_0(\cdot) \equiv 0$ and

$$v_t^{(n)}(x) := T_t^\kappa f(x) + \int_0^t ds\, [T_s^\kappa \kappa\varrho\tilde{Q}](x; v_{t-s}^{(n-1)}), \qquad n \in \mathbb{N},$$

it is easy to check that $(v_t^{(n)}(x))_n$ is increasing in n and that $v_t^{(\infty)}(x) := \lim_{n\to\infty} v_t^{(n)}(x) \in [0,\infty]$ gives the minimal nonnegative solution to (3.2.53) with initial function f. In fact, since equation (3.2.53) is linear, the resulting recursion can be computed explicitly, giving (3.2.52). ■

3.2.21 Proposition
Under Assumption 3.2.17 (nonexplosion), the occupation times kernel $H(\cdot;\cdot)$ of (3.2.48) *has the series representation*

$$H(x; B) = \sum_{n\in\mathbb{N}_0} [(R_\kappa \kappa\varrho\tilde{Q})^n R_\kappa](x; B), \qquad x \in E,\ B \in \mathcal{B}_E. \tag{3.2.54}$$

Consequently, for fixed $B \in \mathcal{B}_E$ the function $H(\cdot; B) : E \to [0,\infty]$ is the minimal nonnegative solution to the equation

$$\begin{aligned} u(x) &= R_\kappa(x; B) + [R_\kappa \kappa\varrho\tilde{Q}](x; u) \\ &\equiv E_x\Big[\int_0^\tau \mathbf{1}_B(X_t)\, dt\Big] + E_x\big[\mathbf{1}_{\tau<\infty} \cdot \varrho(X_\tau)\tilde{Q}(X_\tau; u)\big] \end{aligned} \tag{3.2.55}$$

on E.

[17] See [HHL2002], eqn. (16) and [HL2005], Proof of Lemma 1.4 in combination with Prop. 2.2.

Proof Let $B \in \mathcal{B}_E$. By Definition of H and Lemma 3.2.20, we have

$$\begin{aligned} H(x;B) &= \int_0^\infty M_t(x;B)\,dt \\ &= R_\kappa(x;B) + \sum_{n\in\mathbb{N}} \int_0^\infty dt \int_0^t ds_1 \int_E [T^\kappa_{s_1}\kappa\varrho\tilde{Q}](x;dy_1) \int_0^{t-s_1} ds_2 \int_E [T^\kappa_{s_2}\kappa\varrho\tilde{Q}](y_1;dy_2)\cdots \\ &\qquad \cdots \int_0^{t-s_1-\ldots-s_{n-1}} ds_n \int_E [T^\kappa_{s_n}\kappa\varrho\tilde{Q}](y_{n-1};dy_n) T^\kappa_{t-s_1-\ldots-s_n}(y_n;B). \end{aligned} \tag{3.2.56}$$

Fix $n \in \mathbb{N}$ and consider the corresponding integral in the above display: We integrate on $[0,\infty)^{n+1}$ w.r.t. $dtds_1\cdots ds_n$, where the variables $(t,s_1,\ldots,s_n) \in [0,\infty)^{n+1}$ are subject to the additional condition $s_1 + s_2 + \cdots + s_n \leq t$. Thus by Fubini's theorem

$$\begin{aligned} &\int_0^\infty dt \int_0^t ds_1 \int_0^{t-s_1} ds_2 \cdots \int_0^{t-s_1-\ldots-s_{n-1}} ds_n \int_E [T^\kappa_{s_1}\kappa\varrho\tilde{Q}](x;dy_1) \int_E [T^\kappa_{s_2}\kappa\varrho\tilde{Q}](y_1;dy_2)\cdots \\ &\qquad \cdots \int_E [T^\kappa_{s_n}\kappa\varrho\tilde{Q}](y_{n-1};dy_n) T^\kappa_{t-s_1-\ldots-s_n}(y_n;B) \\ &= \int_{[0,\infty)^{n+1}} dtds_1\cdots ds_n\, \mathbf{1}_{s_1+\cdots+s_n\leq t} \int_E [T^\kappa_{s_1}\kappa\varrho\tilde{Q}](x;dy_1) \int_E [T^\kappa_{s_2}\kappa\varrho\tilde{Q}](y_1;dy_2)\cdots \\ &\qquad \cdots \int_E [T^\kappa_{s_n}\kappa\varrho\tilde{Q}](y_{n-1};dy_n) T^\kappa_{t-s_1-\ldots-s_n}(y_n;B) \\ &= \int_0^\infty ds_1 \int_E [T^\kappa_{s_1}\kappa\varrho\tilde{Q}](x;dy_1) \int_0^\infty ds_2 \int_E [T^\kappa_{s_2}\kappa\varrho\tilde{Q}](y_1;dy_2)\cdots \\ &\qquad \cdots \int_0^\infty ds_n \int_E [T^\kappa_{s_n}\kappa\varrho\tilde{Q}](y_{n-1};dy_n) \int_{s_1+\cdots+s_n}^\infty dt\, T^\kappa_{t-s_1-\ldots-s_n}(y_n;B) \\ &= [(R_\kappa\kappa\varrho\tilde{Q})^n R_\kappa](x;B). \end{aligned}$$

Summing over n proves formula (3.2.54).

As in the proof of Lemma 3.2.20, it is clear by successive approximation that the minimal nonnegative solution of (3.2.55) is given by the series on the r.h.s. of (3.2.54), thus it coincides with $H(\cdot;B)$. ■

The series representation (3.2.54) for the occupation times kernel H is only preliminary. In the next subsection, we will give another representation of H, namely as a generalized resolvent of some auxiliary process on E built from the "given" quantities X, κ and J. However, even the representation (3.2.54) suffices to illustrate the point that in the context of spatial branching processes, "subcriticality" in the sense of Condition 3.2.19 can hold without assuming "smallness" of the reproduction mean $\varrho(\cdot)$ in a pointwise or uniform sense. The rest of this subsection is devoted to some remarks and examples in this regard.

3.2.22 Remarks

- A certain drawback of Condition 3.2.19 and Proposition 3.2.21 is that they are stated under the assumption of nonexplosion of the process $\boldsymbol{\beta}$. However, in many cases verifying finiteness resp. boundedness of the series (3.2.54) will also verify the nonexplosion assumption.[18] In particular, this is true in all of the examples to be given below. Let us also remark again that nonexplosion is ensured whenever κ and ϱ are bounded. Concerning the explosion problem, see also Subsection 3.2.5 below.

[18] We do not investigate the question whether the finiteness resp. boundedness of the series (3.2.54) as such implies nonexplosion (we suspect that it does not).

- The "obvious" way to ensure boundedness in x of the series (3.2.54) is of course to require boundedness of the kernels R_κ and $R_\kappa \kappa \varrho \tilde{Q}$ with

$$\sup_{x \in E} [R_\kappa \kappa \varrho \tilde{Q}](x; E) = \|R_\kappa \kappa \varrho\|_\infty < 1, \tag{3.2.57}$$

since in this case R_κ resp. $R_\kappa \kappa \varrho \tilde{Q}$ induce bounded resp. contraction operators in $\mathscr{B}(E)$. Alternatively, rewriting the series in the form

$$\sum_{n \in \mathbb{N}_0} (R_\kappa \kappa \varrho \tilde{Q})^n R_\kappa = R_\kappa + \sum_{n=0}^{\infty} [R_\kappa \kappa \varrho] \left(\tilde{Q}[R_\kappa \kappa \varrho]\right)^n \tilde{Q} R_\kappa, \tag{3.2.58}$$

we see that it is also bounded if R_κ and $R_\kappa \kappa \varrho$ are bounded and

$$\sup_{x \in E} [\tilde{Q} R_\kappa \kappa \varrho](x; E) = \|\tilde{Q} R_\kappa \kappa \varrho\|_\infty < 1. \tag{3.2.59}$$

Boundedness of the generalized resolvent kernel R_κ is of course most easily ensured by requiring that κ be bounded away from 0. As to (3.2.57) and (3.2.59), since both $[R_\kappa \kappa]$ and $\tilde{Q}$ are (sub)stochastic kernels, the simplest way to satisfy either of those conditions is to require that the reproduction mean $\varrho(\cdot)$ be uniformly bounded away from 1,

$$\|\varrho\|_\infty < 1. \tag{3.2.60}$$

- Although (3.2.60) is considerably weaker than the condition (1.2.17) in Chapter 1 that the reproduction law be bounded by a spatially constant subcritical law in a convolution sense, it is far from necessary: Clearly, for subcriticality the reproduction mean ϱ must be "small" in some sense, but this smallness need not be expressed in terms of the uniform norm but may be replaced with smallness in an L^p-sense, for example. The examples given below are intended to illustrate this point.

- Finally, let us remark that from the standpoint of ensuring boundedness in x of the series (3.2.54), boundedness of κ away from infinity is not important at all. Although boundedness of κ together with boundedness of ϱ is one possible sufficient condition for nonexplosion, we will see in Subsection 3.2.5 below that both conditions (3.2.57) and (3.2.59) are sufficient for nonexplosion as well. ⧫

We give some examples illustrating how conditions (3.2.57) and (3.2.59) and thus boundedness of the kernel H may be verified without assuming (3.2.60):

3.2.23 Examples

- For the following, let μ be a σ-finite reference measure on $(E, \mathcal{B}_E)$ and denote by $L^p \equiv L^p(E; \mu)$ the usual Banach space of (μ-equivalence classes of) p-integrable functions with respect to μ on E, $1 \le p < \infty$. For the corresponding p-norm of $f \in L^p$, we write simply $\|f\|_p$, omitting the dependence on μ in the notation. Since the main purpose of the following examples is to illustrate the point that (3.2.60) is not necessary for the uniform spatial subcriticality condition (3.2.51) to hold, we will assume for simplicity that the branching rate is constant:

$$\kappa(\cdot) \equiv \kappa > 0 \qquad \text{on } E.$$

Then R_κ is just the ordinary κ-resolvent of the semigroup $(T_t)_t$ of the single particle motion X. Now suppose in addition that $(T_t)_t$ admits a transition density w.r.t. the measure μ:

$$T_t(x;f) = \int_E p_t(x;y)f(y)\,\mu(dy), \qquad x \in E,\ f \geq 0.$$

Then $R_\kappa(x;\cdot)$ is of course also absolutely continuous w.r.t. μ, with density given by

$$r_\kappa(x;y) := \int_0^\infty e^{-\kappa t} p_t(x;y)\,\mu(dy), \qquad x \in E.$$

Now let $1 \leq p < \infty$ and p^* the adjoint exponent, $1/p + 1/p^* = 1$. By Hölder's inequality, we have

$$\sup_{x\in E}[R_\kappa \kappa \varrho \tilde{Q}](x;E) = \kappa \|R_\kappa \varrho\|_\infty \leq \kappa\,\|\varrho\|_p \cdot \sup_{x\in E} \|r_\kappa(x;\cdot)\|_{p^*}. \tag{3.2.61}$$

Consequently, (3.2.57) is fulfilled provided the r.h.s. of the above display above is smaller than 1. The situation is particularly nice if $E = \mathbb{R}^d$, $\mu = \lambda$ is Lebesgue measure and $r_\kappa(x;y) = r_\kappa(x-y)$ is a convolution kernel: In this case, $\|r_\kappa(x;\cdot)\|_{p^*} = \|r_\kappa\|_{p^*}$ for all $x \in E$, and thus (3.2.57) is satisfied provided $r_\kappa \in L^{p^*}$ for some $p^* \in (1,\infty]$ and

$$\|\varrho\|_p < \frac{1}{\kappa \|r_\kappa\|_{p^*}}, \tag{3.2.62}$$

i.e. if ϱ is "small enough" in an L^p-sense for the adjoint exponent $p \in [1,\infty)$.

Note that the foregoing is true whatever the kernel $\tilde{Q}$ is. In particular, it holds for the important case $\tilde{Q}(x;\cdot) = \delta_x(\cdot)$ where branching particles reproduce at their death position. Moreover, the foregoing can be easily generalized to the case that the branching rate is not constant, but bounded away from 0, $\kappa(\cdot) \geq \underline{\kappa} > 0$ on E: In this case, replace R_κ by $R_{\underline{\kappa}}$ to obtain the sufficient condition

$$\|\kappa\varrho\|_p \cdot \sup_{x\in E} \|r_{\underline{\kappa}}(x;\cdot)\|_{p^*} < 1$$

for (3.2.57).

- (Branching Brownian Motion) For a concrete example along the above lines, let $E = \mathbb{R}^d$ and X be a d-dimensional Brownian motion, and consider a constant branching rate $\kappa > 0$. Then R_κ is the convolution kernel

$$R_\kappa(x;f) = (r_\kappa * f)(x), \qquad x \in E$$

where $r_\kappa(x) = \int_0^\infty e^{-\kappa t} p_t(x)dt$ is the κ-resolvent density and $p_t(x) = (2\pi t)^{-d/2} e^{-\|x\|^2/2t}$ the transition density of Brownian motion. As we have seen in Chapter 2, the integrability properties of the resolvent density depend heavily on the dimension:

  - For $d = 1$, r_κ can be easily calculated:

$$r_\kappa(x) = \frac{1}{\sqrt{2\kappa}} e^{-\sqrt{2\kappa}|x|}, \qquad x \in \mathbb{R};$$

  see e.g. [RY1999], Exercise III.2.23 on p. 98 or [Sat1999], formula (30.31) on p. 205. Thus $r_\kappa \in L^{p^*}$ for all $p^* \in [0,\infty]$; in particular, r_κ is bounded. We have

$$\|r_\kappa\|_{p^*} = \frac{1}{\sqrt{2\kappa}} \left(p^* \sqrt{\frac{\kappa}{2}}\right)^{-1/p^*},$$

where the r.h.s. side of the above display should be read as $\sqrt{2/\kappa}$ if $p^* = \infty$. Thus (3.2.62) is fulfilled if $\varrho \in L^p$ for some $1 \le p \le \infty$ and

$$\|\varrho\|_p < \frac{1}{\kappa \|r_\kappa\|_{p^*}} = \left(\frac{2}{\kappa}\right)^{1/(2p)} \cdot (p^*)^{1/p*}.$$

Note that this allows not only for large pointwise values of ϱ (in fact, ϱ may be unbounded), but also for large values of $\|\varrho\|_p$ provided the branching rate κ is sufficiently small.

– For $d \ge 2$, we have seen in Chapter 2 that r_κ has a singularity at the origin. In this case, we have

$$r_\kappa \in L^{p^*} \qquad \text{for } 1 \le p^* < \frac{d}{d-2}, \tag{3.2.63}$$

where $d/(d-2)$ is understood as ∞ for $d = 2$. This follows e.g. from the explicit representation of r_κ in terms of Bessel functions: By [Sat1999], formula (30.29) on p. 204, r_κ is given up to a normalization constant by

$$r_\kappa(x) = \text{const} \cdot \|x\|^{-(d-2)/2} K_{(d-2)/2}(\sqrt{2\kappa}\|x\|),$$

where $K_\nu(\cdot)$ is the modified Bessel function of the second kind. The asymptotics of $K_\nu(r)$ as $r \downarrow 0$ is given by $K_\nu(r) \sim C_\nu r^{-\nu}$ for $\nu > 0$ and $K_0(r) \sim C \log(r)$ for $\nu = 0$ (see e.g. [Fol1992], p.160). Thus we get for $d \ge 3$

$$r_\kappa(x) \sim \text{const} \cdot \|x\|^{2-d}, \qquad \|x\| \downarrow 0$$

and for $d = 2$

$$r_\kappa(x) \sim \text{const} \cdot \log \|x\|, \qquad \|x\| \downarrow 0.$$

From this it follows that r_κ is p^*-integrable at the origin iff $p^* < d/(d-2)$. On the other hand, p^*-integrability at infinity is ensured for all $p^* \in [1, \infty)$ in any dimension, since $K_\nu(r)$ decays exponentially as $r \uparrow \infty$ (again see e.g. [Fol1992], p. 160). This proves (3.2.63), and consequently (3.2.62) will be satisfied for a branching Brownian motion in $\mathbb{R}^d$ ($d \ge 2$) if ϱ is "small enough" in an L^p-sense for some $p > d/2$.

We remark that for odd $d \ge 3$, using the recurrence identities for the modified Bessel function one can show by induction that the resolvent density r_κ can be expressed in terms of elementary functions as

$$r_\kappa(x) = \text{const}\, e^{-\sqrt{2\kappa}\|x\|} \,(\sqrt{2\kappa}\|x\|)^{2-d} P(\sqrt{2\kappa}\|x\|),$$

where P is a polynomial of degree $(d-3)/2$ and the constant in front is known explicitly. For example, for $d = 3$ we have

$$r_\kappa(x) = \frac{1}{2\pi\|x\|} e^{-\sqrt{2\kappa}\|x\|}$$

(see e.g. [Sat1999], eqn. (30.31)). In particular, $r_\kappa \in L^2$ and (3.2.62) is satisfied if

$$\|\varrho\|_2 < \frac{1}{\kappa\|r_\kappa\|_2} = \frac{2^{3/2}\pi}{\sqrt{\kappa}}.$$

- Suppose that X is a diffusion on $E = \mathbb{R}^d$ with transition density $(p_t)_t$ which satisfies a heat kernel estimate as in Assumption 2.2.1 for small $t > 0$ (recall that this is ensured whenever the drift and diffusion coefficient are bounded Hölder continuous and uniform ellipticity holds, see Remark 2.2.3):

$$p_t(x;y) \leq C \cdot t^{-d/2} \cdot e^{-\frac{\|x-y\|^2}{2Ct}} =: \tilde{p}_t(x-y), \qquad t \in [0,\varepsilon],\, x,y \in \mathbb{R}^d. \tag{3.2.64}$$

By the same Chapman-Kolmogorov argument as in (2.2.7), we get for the resolvent density

$$\begin{aligned} r_\kappa(x;y) &= \int_0^\varepsilon dt\, e^{-\kappa t} p_t(x;y) + \int_\varepsilon^\infty dt\, e^{-\kappa t} p_t(x;y) \\ &= \int_0^\varepsilon dt\, e^{-\kappa t} p_t(x;y) + e^{-\kappa\varepsilon} \int_{\mathbb{R}^d} dz\, r_\kappa(x;z) p_\varepsilon(z;y) \\ &=: r_\kappa^{(1)}(x;y) + r_\kappa^{(2)}(x;y). \end{aligned}$$

In view of the heat kernel estimate (3.2.64), the L^q-properties of $r_\kappa^{(1)}(x;\cdot)$ are at least as good as those of the resolvent density of Brownian motion considered above. The second term $r_\kappa^{(2)}(x;\cdot)$ is in every L^q, $q \in [1,\infty)$, since by Jensen's inequality

$$\begin{aligned} \kappa \|r_\kappa^{(2)}(x;\cdot)\|_q &= e^{-\kappa\varepsilon} \left(\int_{\mathbb{R}^d} dy \left(\int_{\mathbb{R}^d} dz\, \kappa\, r_\kappa(x;z) p_\varepsilon(z;y) \right)^q \right)^{1/q} \\ &\leq e^{-\kappa\varepsilon} \left(\int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \kappa\, r_\kappa(x;z) p_\varepsilon(z;y)^q \right)^{1/q} \\ &\leq e^{-\kappa\varepsilon} \left(\int_{\mathbb{R}^d} dz\, \kappa\, r_\kappa(x;z) \int_{\mathbb{R}^d} dy\, \tilde{p}_\varepsilon(z-y)^q \right)^{1/q} \\ &= e^{-\kappa\varepsilon} \cdot \|\tilde{p}_\varepsilon\|_q < \infty \end{aligned}$$

for all $x \in \mathbb{R}^d$, where we have used that $\tilde{p}_\varepsilon \in L^q$ for all $q \in [1,\infty)$. Consequently, as in the case of Brownian motion, (3.2.62) is satisfied provided ϱ is L^p-small for some $p > d/2$.

3.2.24 Example

In the previous examples, there were no restrictions on the kernel $\tilde{Q}$, but we needed the existence of a transition density for the process together with strong integrability properties of the resolvent density $r_\kappa(x;\cdot)$. This condition can be relaxed by restricting the class of $\tilde{Q}$ under consideration as follows: As before, we assume a constant branching rate $\kappa > 0$ (or more generally, $\kappa(\cdot) \geq \underline{\kappa} > 0$). Again let μ be a σ-finite reference measure on $(E, \mathcal{B}_E)$, but now suppose that μ is invariant for the semigroup $(T_t)_t$ of X:

$$\mu T_t(x;B) \equiv \int_E \mu(dx) T_t(x;B) = \mu(B), \qquad t \geq 0,\, B \in \mathcal{B}_E.$$

There are of course many classes of processes X for which an invariant measure exists: For example, suppose that X is a Lévy process on $E = \mathbb{R}^d$ (then d-dimensional Lebesgue measure is invariant for X by the independence of the increments and the translation invariance of Lebesgue measure) or that X is an ergodic (positive Harris recurrent) diffusion with invariant probability measure μ.

It is well-known (and easy to show by Jensen's inequality and invariance of μ) that whenever an invariant measure exists, the kernels $(T_t)_t$ induce a contraction semigroup on each $L^p \equiv L^p(E;\mu)$, $1 \le p < \infty$; in particular, for the kernel R_κ we have

$$\|R_\kappa f\|_p \le \frac{1}{\kappa}\|f\|_p, \qquad f \in L^p. \tag{3.2.65}$$

Now suppose in addition that $\tilde{Q}$ is absolutely continuous w.r.t. the invariant measure μ:

$$\tilde{Q}(x;dy) = q(x;y)\,\mu(dy), \qquad x \in E.$$

Instead of (3.2.57), we consider (3.2.59): Again denoting by p^* the adjoint exponent, by Hölder's inequality and (3.2.65) we obtain

$$\tilde{Q}R_\kappa\kappa\varrho(x) = \kappa\int_E \mu(dy)\,q(x;y)R_\kappa\varrho(y) \le \kappa\,\|q(x;\cdot)\|_{p^*}\cdot\|R_\kappa\varrho\|_p \le \|q(x;\cdot)\|_{p^*}\cdot\|\varrho\|_p.$$

Thus we can impose the L^{p^*}-integrability condition on the density q instead of the resolvent density (which need not even exist in this case). Consequently, (3.2.59) will be satisfied if $C := \sup_{x\in E}\|q(x;\cdot)\|_{p^*} < \infty$ for some $p^* \in (1,\infty]$ and ϱ is small enough in an L^p-sense, $\|\varrho\|_p < C^{-1}$. ♦

Although the above examples and arguments are thoroughly elementary, they suffice to illustrate the point that in the context of spatial branching processes there is an infinity of (classes of) examples where subcriticality in the sense of Condition 3.2.19 does not require "smallness" of the reproduction mean in a pointwise or uniform sense: The spatial motion component and offspring distribution leave a wide latitude for possible models where the uniform spatial subcriticality condition (3.2.51) holds even though ϱ can take large values locally, provided it is "small enough" in an L^p-sense. In the next subsection, we will see a different representation of the occupation times kernel H which allows for other types of examples where Condition 3.2.19 can be satisfied without assuming (3.2.60).

3.2.4 An Auxiliary Process and the Expected Number of Particles

In this subsection, we will derive an equivalent representation of the occupation times kernel H of (3.2.48) in terms of an auxiliary process $\tilde{X}$ living on the single particle space E. The definition of this auxiliary process will involve only the given single particle motion X, branching rate κ and reproduction and offspring law J of (3.1.11). We will see that under suitable conditions, the kernels M_t of (3.2.17) giving the expected number of particles at time $t > 0$ can be expressed in terms of a semigroup involving the auxiliary process. From this result, which is also of some independent interest, we get expected occupation times simply by integrating with respect to time $t \in \mathbb{R}_+$. This gives a representation of H as a generalized resolvent of the auxiliary process. For the case $E = \mathbb{R}^d$, a diffusion as single particle motion X and (3.1.13), this result was proved in [HL2005].[19]

As in the previous subsection, we continue to assume throughout that $\boldsymbol{\beta}$ does not explode. In order to motivate what follows, we recall the result due to [INW1969] already reported in the Remarks 3.2.7:[20] Suppose that X is a Feller process, that κ and ϱ are continuous and

[19] See [HL2005], Prop. 2.2

[20] See [INW1969], Thm. 4.14; see also [Wat1967].

bounded and that the kernel $\tilde{Q}$ preserves the space $\mathcal{C}_0(E)$. Then the family of kernels $(M_t)_t$ induces a strongly continuous semigroup of bounded operators on $\mathcal{C}_0(E)$ with generator

$$Lf = Af - \kappa f + \kappa\varrho\tilde{Q}f, \qquad f \in D(L) = D(A). \tag{3.2.66}$$

Now suppose for the moment that branching particles reproduce at their death position, i.e. $\tilde{Q}(x;\cdot) = \delta_x(\cdot)$. In that case, the generator (3.2.66) reads

$$Lf = Af - \kappa(1-\varrho)f, \qquad f \in D(L) = D(A), \tag{3.2.67}$$

and the corresponding semigroup $(M_t)_t$ will be of "Kac type":

$$M_t f(x) = E_x\left[f(X_t)^{-\int_0^t \kappa(X_s)(1-\varrho(X_s))\,ds}\right], \qquad x \in E,\ f \in \mathscr{B}(E). \tag{3.2.68}$$

For the case that X is a diffusion, the above formula appears already in [Wat1967], eqn. (2.22). The function $\kappa(1-\varrho)$ appearing in (3.2.68) will play a prominent role in the sequel; therefore we define

$$\gamma : E \to \mathbb{R}, \qquad \gamma := \kappa(1-\varrho).^{21} \tag{3.2.69}$$

If offspring particles do not start their motion at their parent's death position, the representation (3.2.68) for the kernel M_t will not hold, at least not with the given single particle motion X. However, we observe that in any case (3.2.66) can be rewritten as

$$Lf = Af + \kappa\varrho(\tilde{Q}f - f) - \gamma f =: \tilde{A}f - \gamma f. \tag{3.2.70}$$

In the above display, the term $\tilde{A}f = Af + \kappa\varrho(\tilde{Q}f - f)$ has a clear probabilistic meaning; namely, it is again the generator of a Feller process $\tilde{X}$ on E: From an analytical point of view, $\tilde{A}$ is nothing but a perturbation of the Feller generator A by the bounded and dissipative operator $\kappa\varrho(\tilde{Q}f - f)$ on $\mathcal{C}_0(E)$. Thus by general semigroup theory (see e.g. [Paz1983]), $\tilde{A}$ generates a Feller semigroup $(\tilde{T}_t)_t$ on $\mathcal{C}_0(E)$, to which then a corresponding stochastic process $\tilde{X}$ can be associated. From a probabilistic point of view, this process $\tilde{X}$ is obtained by killing the given process X at rate $\kappa\varrho$ and "restarting" it with $\tilde{Q}$ as jump kernel. From (3.2.70) we can then again obtain a representation of "Kac type" for M_t, but with X replaced by $\tilde{X}$:

$$M_t f(x) = E_x\left[f(\tilde{X}_t)e^{-\int_0^t \gamma(\tilde{X}_s)\,ds}\right], \qquad x \in E,\ f \in \mathscr{B}(E). \tag{3.2.71}$$

This gives a probabilistic representation of the expected number of particles in terms of an "auxiliary process" $\tilde{X}$ defined on the single-particle space.

Although this (very straightforward) generalization of (3.2.68) seems like a nice result as far as it goes, the argument outlined above has of course the drawback that we need boundedness and continuity of κ and ϱ and Feller properties of $(T_t)_t$ and $\tilde{Q}$ to make it rigorous. On the other hand, for the probabilistic construction of the process $\tilde{X}$, these strong regularity conditions are not needed since we can employ the "killing and reviving"-procedure of [INW1968b] to this end. Without the above regularity conditions, this gives us a "candidate" $\tilde{X}$ for formula (3.2.71), but it will no longer be ensured that its generator (or the generator of $(M_t)_t$) is given by (3.2.66).

[21] We use the symbol γ to stay in line with the notation in [HL2005]. There should be no danger of confusion with the invariant density considered in the previous chapter.

The first goal of this subsection is to prove the representation (3.2.71) without resorting to generators and their domains, using basically only the strong Markov property and the branching property of the process $\boldsymbol{\beta}$. In particular, (3.2.71) can be shown without requiring boundedness or continuity of κ and ϱ or Feller properties of $(T_t)_t$ and $\tilde{Q}$. We will however need the following condition:

3.2.25 Assumption
We assume that for all $x \in E$ and $t > 0$

$$A_t^{\kappa\varrho} := \int_0^t \kappa(X_s)\varrho(X_s)\, ds < \infty \qquad P_x\text{-}a.s. \tag{3.2.72}$$

Under Assumption 3.2.25, $A^{\kappa\varrho}$ is a finite and continuous additive functional of X. The random time determined by killing the paths of X at rate $\kappa\varrho$ will be denoted by σ. It is characterized by

$$P_x[\sigma > t \,|\, X] = e^{-A_t^{\kappa\varrho}}, \qquad t > 0,\, x \in E, \tag{3.2.73}$$

and Assumption 3.2.25 ensures that we have the analogue of formula (3.2.3), namely

$$E_x\left[\mathbf{1}_{\sigma \le t} f(X_\sigma)\right] = E_x\left[\int_0^t ds\, e^{-A_s^{\kappa\varrho}} \kappa(X_s)\varrho(X_s) f(X_s)\right], \qquad f \in \mathscr{B}(E) \tag{3.2.74}$$

for the joint distribution of σ and X_σ.

We are now ready to introduce the auxiliary process $\tilde{X}$ announced above:

3.2.26 Definition
The auxiliary process $\tilde{X}$ is defined as the strong Markov process on E_∂ obtained by killing the paths of X at rate $\kappa\varrho$ (i.e., at time σ) and restarting it according to the jump kernel $\tilde{Q}$.

As for the BMP $\boldsymbol{\beta}$, the existence of $\tilde{X}$ follows rigorously from the "killing and reviving"-procedure of [INW1968b]. The stochastic basis on which $\tilde{X}$ is defined will be denoted by

$$(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, (\boldsymbol{\mathcal{F}}_t)_{t\ge 0}, (\boldsymbol{P}_x)_{x\in E}, (\boldsymbol{\theta}_t)_{t\ge 0})$$

in order to distinguish it from the stochastic basis for the original single particle motion X, and expectations w.r.t. $\boldsymbol{P}_x$ will be denoted $\boldsymbol{E}_x$, $x \in E$.[22] The revival times in the auxiliary process (which correspond to the branching times $\boldsymbol{\tau}_n$ in the construction of the BMP $\boldsymbol{\beta}$) will be denoted by σ_n, $n \in \mathbb{N}$: They form a sequence of $(\boldsymbol{\mathcal{F}}_t)_{t\ge 0}$-stopping times with

$$\sigma_n = \sigma_1 + \sigma_{n-1} \circ \boldsymbol{\theta}_{\sigma_1}, \qquad \sigma_n \uparrow \sigma_\infty \le \infty.$$

As before, we may have an accumulation of revival events in finite time and thus $\sigma_\infty < \infty$: In this case, $\tilde{X}_t = \partial$ for $t \ge \sigma_\infty$. In particular, σ_∞ coincides with the life-time of $\tilde{X}$:

$$\sigma_\infty = \inf\{t \ge 0 : \tilde{X}_t = \partial\},$$

which is allowed to be finite. The killed processes $\tilde{X}^{\sigma_1}$ and X^σ are defined as before (cf. (3.2.4)); namely, $\tilde{X}^{\sigma_1}$ describes the path strictly before the first revival event and coincides in distribution with X^σ. As for any killed-and-revived Markov process, the joint distribution of $\tilde{X}^{\sigma_1}$ and $\tilde{X}_{\sigma_1}$ (i.e. of the path strictly before the first revival and the state at the first revival

[22] In order to save notation, we use the same symbols as for the stochastic basis of the BMP $\boldsymbol{\beta}$. This should not cause any confusion: Of course, in general $\boldsymbol{\beta}$ and $\tilde{X}$ are defined on different sample spaces.

event) is given by the analogue of formula (3.2.6), where we have to replace $\boldsymbol{\tau}_1$ by σ_1, τ by σ and J by $\tilde{Q}$. Thus (3.2.6) now reads as follows:

$$\begin{aligned}\boldsymbol{E}_x\left[\mathbf{1}_{\sigma_1<\infty}\cdot\phi(\tilde{X}^{\sigma_1},\tilde{X}_{\sigma_1})\right] &= E_x\left[\mathbf{1}_{\sigma<\infty}\int_E \tilde{Q}(X_\sigma;dy)\phi(X^\sigma,y)\right]\\ &= E_x\left[\int_0^\infty ds\,\kappa(X_s)\varrho(X_s)e^{-A_s^{\kappa\varrho}}\int_E \tilde{Q}(X_s;dy)\phi(X^s,y)\right]\end{aligned}\tag{3.2.75}$$

for every measurable and bounded or nonnegative functional $\phi : E_\partial^{[0,\infty)}\times E\to\mathbb{R}$, where for the second equality we have also used (3.2.74). The semigroup of the auxiliary process will be denoted by

$$\tilde{T}_t f(x) := \boldsymbol{E}_x\left[f(\tilde{X}_t)\mathbf{1}_{t<\sigma_\infty}\right],\qquad f\in\mathscr{B}(E),\ x\in E.\tag{3.2.76}$$

In the proof of Lemma 3.2.27 below, we will use that by construction of the process $\tilde{X}$ as a killed-and-revived Markov process, it satisfies the following version of the strong Markov property for the first revival time σ_1: For any bounded or nonnegative $\boldsymbol{\mathcal{F}}$-measurable $G:\boldsymbol{\Omega}\to\mathbb{R}$ we have

$$\boldsymbol{E}_x\left[\mathbf{1}_{\sigma_1<\infty}\cdot G\circ\boldsymbol{\theta}_{\sigma_1}\,|\,\boldsymbol{\mathcal{F}}_{\sigma_1}\right] = \mathbf{1}_{\sigma_1<\infty}\cdot\boldsymbol{E}_{\tilde{X}_{\sigma_1}}[G],\qquad x\in E\tag{3.2.77}$$

(see e.g. [Nag1977], p. 429). Note that this is stronger than the usual formulation of the strong Markov property[23] in which the random variable G is supposed to be $\sigma(\tilde{X})$-measurable (i.e., supposed to be a functional of the paths of $\tilde{X}$), whereas the σ-algebra $\boldsymbol{\mathcal{F}}$ obtained from the killing-and-reviving construction can be significantly larger. We will in fact use the following variant of (3.2.77) for a functional G which depends additionally on time: If $G(\cdot,\cdot):\boldsymbol{\Omega}\times\mathbb{R}_+\to\mathbb{R}$ is $\boldsymbol{\mathcal{F}}\otimes\mathcal{B}_{\mathbb{R}_+}$-$\mathcal{B}_{\mathbb{R}}$-measurable and bounded or nonnegative, we have

$$\boldsymbol{E}_x\left[\mathbf{1}_{\sigma_1<\infty}\cdot G(\boldsymbol{\theta}_{\sigma_1},\sigma_1)\,|\,\boldsymbol{\mathcal{F}}_{\sigma_1}\right] = \mathbf{1}_{\sigma_1<\infty}\cdot\boldsymbol{E}_{\tilde{X}_{\sigma_1}}\left[G(\cdot,s)\right]|_{s=\sigma_1}.\text{[24]}\tag{3.2.78}$$

Having introduced the auxiliary process $\tilde{X}$ on E, we now relate it to the BMP $\boldsymbol{\beta}$ on $\mathcal{S}$ and to the expected number of particles. First, under (3.1.7) and (3.2.72) we can define another finite continuous additive functional of X by

$$A_t^\gamma := A_t^\kappa - A_t^{\kappa\varrho} = \int_0^t \gamma(X_s)\,ds,\qquad t\geq 0\tag{3.2.79}$$

with $\gamma=\kappa(1-\varrho)$ as in (3.2.69). Of course, A^γ is in general neither positive nor increasing but may take values in all of $\mathbb{R}$.

Next, in terms of the auxiliary process $\tilde{X}$ and the function γ of (3.2.69) we define a family of kernels $\tilde{T}_t^\gamma : E\times\mathcal{B}_E\to[0,\infty]$, $t\geq 0$, as follows:

$$\tilde{T}_t^\gamma(x;B) := \boldsymbol{E}_x\left[\mathbf{1}_{t<\sigma_\infty}\mathbf{1}_B(\tilde{X}_t)e^{-\int_0^t\gamma(\tilde{X}_s)\,ds}\right],\qquad x\in E,\ B\in\mathcal{B}_E.\tag{3.2.80}$$

By the Markov property of the process $\tilde{X}$, it follows easily that the family of kernels $(\tilde{T}_t^\gamma)_{t\geq 0}$ has the semigroup property for nonnegative measurable $f:E\to\mathbb{R}_+$:

$$\tilde{T}_{t+s}^\gamma f(x) = \tilde{T}_t^\gamma\left(\tilde{T}_s^\gamma f\right)(x),\qquad s,t>0,\ x\in E.\tag{3.2.81}$$

[23]See e.g. [BG1968], p. 38.

[24]This notation is used to emphasize that the expectation on the r.h.s. of (3.2.78) is to be taken w.r.t. the first component of G only. Cf. e.g. [INW1968a], p. 240 for this variant of the strong Markov property.

Nevertheless, it would amount to an abuse of language at this point to call $(\tilde{T}_t^\gamma)_{t\geq 0}$ a semigroup since without additional assumptions the kernels $\tilde{T}_t^\gamma$ have no reason to be bounded.

Our goal is to show that $\tilde{T}_t^\gamma$ coincides with the kernel M_t giving the expected number of particles. First we show that for nonnegative functions, $\tilde{T}_t^\gamma f$ coincides with the r.h.s. of (3.2.52):

3.2.27 Lemma
Under Assumption 3.2.25, define $\tilde{T}_t^\gamma$ as in (3.2.80). Let $f : E \to \mathbb{R}_+$ nonnegative and measurable. Then for each $x \in E$ and $t \geq 0$, we have

$$\tilde{T}_t^\gamma f(x) = T_t^\kappa f(x) + \sum_{n\in\mathbb{N}} \int_0^t ds_1 \int_E [T_{s_1}^\kappa \kappa\varrho\tilde{Q}](x; dy_1) \int_0^{t-s_1} ds_2 \int_E [T_{s_2}^\kappa \kappa\varrho\tilde{Q}](y_1; dy_2) \cdots$$
$$\cdots \int_0^{t-s_1-\ldots-s_{n-1}} ds_n \int_E [T_{s_n}^\kappa \kappa\varrho\tilde{Q}](y_{n-1}; dy_n) T_{t-s_1-\ldots-s_n}^\kappa f(y_n). \tag{3.2.82}$$

Proof Given $f : E \to \mathbb{R}_+$, define for each $N \in \mathbb{N}$

$$v_t^{(N)}(x) := \boldsymbol{E}_x\left[\mathbf{1}_{t<\sigma_N} f(\tilde{X}_t) e^{-\int_0^t \gamma(\tilde{X}_r)\,dr}\right], \qquad x \in E,\ t \geq 0.$$

Then clearly

$$v_t^{(N)}(x) \xrightarrow{N\uparrow\infty} \boldsymbol{E}_x\left[\mathbf{1}_{t<\sigma_\infty} f(\tilde{X}_t) e^{-\int_0^t \gamma(\tilde{X}_r)\,dr}\right] = \tilde{T}_t^\gamma f(x)$$

for all $x \in E$ and $t \geq 0$ by monotone convergence. We will now show by induction that for all $N \in \mathbb{N}$, $t \geq 0$ and $x \in E$

$$v_t^{(N)}(x) = T_t^\kappa f(x) + \sum_{n=1}^{N-1} \int_0^t ds_1 \int_E [T_{s_1}^\kappa \kappa\varrho\tilde{Q}](x; dy_1) \int_0^{t-s_1} ds_2 \int_E [T_{s_2}^\kappa \kappa\varrho\tilde{Q}](y_1; dy_2) \cdots$$
$$\cdots \int_0^{t-s_1-\ldots-s_{n-1}} ds_n \int_E [T_{s_n}^\kappa \kappa\varrho\tilde{Q}](y_{n-1}; dy_n) T_{t-s_1-\ldots-s_n}^\kappa f(y_n), \tag{3.2.83}$$

from which the assertion of the lemma follows.

Remember that by construction, the auxiliary process $\tilde{X}$ before the first revival time σ_1 is equivalent to the original process X killed at rate $\kappa\varrho$ (i.e. at time σ from (3.2.73)). More formally, by taking the first marginal distribution in (3.2.75) we have

$$\mathcal{L}\left(\tilde{X}^{\sigma_1} \,|\, \boldsymbol{P}_x\right) = \mathcal{L}\left(X^\sigma \,|\, P_x\right) \qquad \text{on } (E_\partial)^{\mathbb{R}_+} \tag{3.2.84}$$

for the distribution of the killed processes. We thus obtain for $N = 1$

$$\begin{aligned} v_t^{(1)}(x) = \boldsymbol{E}_x\left[\mathbf{1}_{t<\sigma_1} f(\tilde{X}_t) e^{-\int_0^t \gamma(\tilde{X}_s)\,ds}\right] &= E_x\left[\mathbf{1}_{t<\sigma} f(X_t) e^{-\int_0^t \gamma(X_s)\,ds}\right] \\ &= E_x\left[e^{-A_t^{\kappa\varrho}} \cdot f(X_t) e^{-A_t^\gamma}\right] \\ &= E_x\left[f(X_t) e^{-A_t^\kappa}\right] = T_t^\kappa f(x) \end{aligned} \tag{3.2.85}$$

by definition of $\gamma = \kappa(1-\varrho)$. Now suppose that (3.2.83) holds for some $N \in \mathbb{N}$ and all $t \geq 0$, $x \in E$; we will show that it then holds also for $N+1$ and all t, x. Fix some $t \geq 0$ and $x \in E$. We have

$$\begin{aligned} v_t^{(N+1)}(x) &\equiv \boldsymbol{E}_x\left[\mathbf{1}_{t<\sigma_{N+1}} f(\tilde{X}_t) e^{-\int_0^t \gamma(\tilde{X}_r)\,dr}\right] \\ &= \boldsymbol{E}_x\left[\mathbf{1}_{t<\sigma_1} f(\tilde{X}_t) e^{-\int_0^t \gamma(\tilde{X}_r)\,dr}\right] + \boldsymbol{E}_x\left[\mathbf{1}_{\sigma_1\leq t<\sigma_{N+1}} f(\tilde{X}_t) e^{-\int_0^t \gamma(\tilde{X}_r)\,dr}\right]. \end{aligned} \tag{3.2.86}$$

The first expectation in the previous display is equal to $T_t^\kappa f(x)$ by (3.2.85). Using the identity $\sigma_{N+1} = \sigma_N \circ \boldsymbol{\theta}_{\sigma_1} + \sigma_1$, the integrand in the second expectation can be rewritten as

$$\begin{aligned}&\mathbf{1}_{\sigma_1 \le t < \sigma_{N+1}} f(\tilde{X}_t) e^{-\int_0^t \gamma(\tilde{X}_r)\,dr} \\ &= \mathbf{1}_{\sigma_1 \le t < \sigma_N \circ \boldsymbol{\theta}_{\sigma_1} + \sigma_1} f(\tilde{X}_{t-\sigma_1} \circ \boldsymbol{\theta}_{\sigma_1}) e^{-\int_0^{\sigma_1} \gamma(\tilde{X}_r)\,dr} \cdot e^{-\int_0^{t-\sigma_1} \gamma(\tilde{X}_r \circ \boldsymbol{\theta}_{\sigma_1})\,dr}.\end{aligned} \tag{3.2.87}$$

Defining $G : \boldsymbol{\Omega} \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$G(\omega, s) := \mathbf{1}_{s \le t < \sigma_N(\omega) + s} \cdot f(\tilde{X}_{t-s}(\omega)) e^{-\int_0^{t-s} \gamma(\tilde{X}_r(\omega))\,dr},$$

we can rewrite (3.2.87) as

$$e^{-\int_0^{\sigma_1} \gamma(\tilde{X}_r)\,dr} \cdot G(\boldsymbol{\theta}_{\sigma_1}, \sigma_1).$$

Here the term $e^{-\int_0^{\sigma_1} \gamma(\tilde{X}_r)\,dr}$ is $\boldsymbol{\mathcal{F}}_{\sigma_1}$-measurable since the additive functional $t \mapsto \int_0^t \gamma(\tilde{X}_r)\,dr$ of $\tilde{X}$ is $(\boldsymbol{\mathcal{F}}_t)_t$-adapted and continuous and thus also $(\boldsymbol{\mathcal{F}}_t)_t$-progressive. The functional G on the other hand is clearly $\boldsymbol{\mathcal{F}} \otimes \mathcal{B}_{\mathbb{R}_+}$-$\mathcal{B}_{\mathbb{R}}$-measurable.[25] Hence we can apply the strong Markov property in the form (3.2.78) and obtain by conditioning on the first revival event

$$\begin{aligned}&\boldsymbol{E}_x\Big[\mathbf{1}_{\sigma_1 \le t < \sigma_{N+1}} f(\tilde{X}_t) e^{-\int_0^t \gamma(\tilde{X}_r)\,dr}\Big] \\ &= \boldsymbol{E}_x\Big[e^{-\int_0^{\sigma_1} \gamma(\tilde{X}_r)\,dr} \cdot \boldsymbol{E}_x\Big[G(\boldsymbol{\theta}_{\sigma_1}, \sigma_1) \Big| \boldsymbol{\mathcal{F}}_{\sigma_1}\Big]\Big] \\ &= \boldsymbol{E}_x\Big[e^{-\int_0^{\sigma_1} \gamma(\tilde{X}_r)\,dr} \cdot \mathbf{1}_{\sigma_1 \le t} \cdot \boldsymbol{E}_{\tilde{X}_{\sigma_1}}\Big[\mathbf{1}_{t-s<\sigma_N} \cdot f(\tilde{X}_{t-s}) e^{-\int_0^{t-s} \tilde{X}_r\,dr}\Big]\Big|_{s=\sigma_1}\Big] \\ &= \boldsymbol{E}_x\Big[e^{-\int_0^{\sigma_1} \gamma(\tilde{X}_r)\,dr} \cdot \mathbf{1}_{\sigma_1 \le t} \cdot v_{t-\sigma_1}^{(N)}(\tilde{X}_{\sigma_1})\Big].\end{aligned} \tag{3.2.88}$$

We observe that the integrand inside the expectation in the last display depends only on $\tilde{X}^{\sigma_1}$ (the path strictly before the first revival event[26]) and on $\tilde{X}_{\sigma_1}$. But we know that the joint distribution of $\tilde{X}^{\sigma_1}$ and $\tilde{X}_{\sigma_1}$ is given by (3.2.75). Thus we can continue (3.2.88) to obtain

$$\begin{aligned}&\boldsymbol{E}_x\Big[e^{-\int_0^{\sigma_1} \gamma(\tilde{X}_r)\,dr} \cdot \mathbf{1}_{\sigma_1 \le t} \cdot v_{t-\sigma_1}^{(N)}(\tilde{X}_{\sigma_1})\Big] \\ &= E_x\Big[\int_0^t ds\, \kappa(X_s) \varrho(X_s) e^{-A_s^{\kappa\varrho}} \int_E \tilde{Q}(X_s; dy)\, e^{-\int_0^s \gamma(X_r)\,dr} \cdot v_{t-s}^{(N)}(y)\Big] \\ &= E_x\Big[\int_0^t ds\, \kappa(X_s) \varrho(X_s) e^{-A_s^{\kappa}} \int_E \tilde{Q}(X_s; dy)\, v_{t-s}^{(N)}(y)\Big] \\ &= \int_0^t ds \int_E T_s^\kappa(x; dy) \kappa(y) \varrho(y) \tilde{Q}(y; dz) v_{t-s}^{(N)}(z).\end{aligned} \tag{3.2.89}$$

[25] Note however that σ_N and thus also $G(\cdot, s)$ for fixed $s \ge 0$ is not necessarily a functional of the paths of $\tilde{X}$. This is the reason that we need the strong Markov property in the form (3.2.78).

[26] Remember that σ_1 is a function of $\tilde{X}^{\sigma_1}$ since it is the lifetime of the killed process.

Since by induction hypothesis, (3.2.83) holds for *all* $t \geq 0$ and $x \in E$, the above is equal to

$$\begin{aligned}
&\int_0^t ds \int_E T_s^\kappa(x;dy)\kappa(y)\varrho(y) \int_E \tilde{Q}(y;dz)\Bigg(T_{t-s}^\kappa f(z) \\
&\quad + \sum_{n=1}^{N-1} \int_0^{t-s} ds_1 \int_E [T_{s_1}^\kappa \kappa\varrho\tilde{Q}](z;dy_1) \int_0^{t-s-s_1} ds_2 \int_E [T_{s_2}^\kappa \kappa\varrho\tilde{Q}](y_1;dy_2)\cdots \\
&\quad \cdots \int_0^{t-s-s_1-\ldots-s_{n-1}} ds_n \int_E [T_{s_n}^\kappa \kappa\varrho\tilde{Q}](y_{n-1};dy_n) T_{t-s-s_1-\ldots-s_n}^\kappa f(y_n)\Bigg) \qquad (3.2.90)\\
&= \sum_{n=1}^{N} \int_0^t ds_1 \int_E [T_{s_1}^\kappa \kappa\varrho\tilde{Q}](x;dy_1) \int_0^{t-s_1} ds_2 \int_E [T_{s_2}^\kappa \kappa\varrho\tilde{Q}](y_1;dy_2)\cdots \\
&\quad \cdots \int_0^{t-s_1-\ldots-s_{n-1}} ds_n \int_E [T_{s_n}^\kappa \kappa\varrho\tilde{Q}](y_{n-1};dy_n) T_{t-s_1-\ldots-s_n}^\kappa f(y_n).
\end{aligned}$$

Combining (3.2.86)-(3.2.90), we have shown (3.2.83) for $N+1$, and the proof is complete. ■

Taking together Lemma 3.2.20 and Lemma 3.2.27, we obtain as an immediate consequence that under suitable conditions the kernels M_t of Definition 3.2.2 giving the expected number of particles can be represented in terms of the auxiliary process; namely they coincide with the kernels $\tilde{T}_t^\gamma$ defined in (3.2.80):

3.2.28 Theorem
Let $\boldsymbol{\beta}$ be a branching Markov process which does not explode (Assumption 3.2.17) and suppose that Assumption 3.2.25 holds. Then, with the auxiliary process $\tilde{X}$ from Definition 3.2.26 and the kernels $\tilde{T}_t^\gamma$ defined as in (3.2.80)*, we have for every $f \in \mathscr{B}^+(E)$*

$$M_t f(x) = \tilde{T}_t^\gamma f(x), \qquad x \in E,\ t \geq 0, \qquad (3.2.91)$$

which reads in probabilistic terms

$$\boldsymbol{E}_x\left[\bar{f}(\boldsymbol{\beta}_t)\right] = \boldsymbol{E}_x\left[\mathbf{1}_{t<\sigma_\infty} f(\tilde{X}_t) e^{-\int_0^t \gamma(\tilde{X}_s)\,ds}\right], \qquad x \in E,\ t \geq 0.$$

In particular, if $f = \mathbf{1}_B$ for a Borel set $B \in \mathcal{B}_E$, we get the formula

$$\boldsymbol{E}_x\left[\boldsymbol{\beta}_t(B)\right] = \boldsymbol{E}_x\left[\mathbf{1}_{t<\sigma_\infty} \mathbf{1}_B(\tilde{X}_t) e^{-\int_0^t \gamma(\tilde{X}_r)\,dr}\right] \qquad (3.2.92)$$

for the expected number of particles in B at time t.

3.2.29 Remarks

- The assumptions of Theorem 3.2.28 are in particular satisfied if κ and ϱ are bounded.
- Consider the special case $\tilde{Q}(x;\cdot) = \delta_x(\cdot)$ that branching particles reproduce exactly at their death position. In this case, construction of the auxiliary process $\tilde{X}$ as above amounts to "continuously reviving" the given process X at the random times σ_n determined by the additive functional $A^{\kappa\varrho}$. Nevertheless, the reader may worry that it is not clear a priori whether $\tilde{X}$ and X coincide since there might be an accumulation of revival events in finite time. However under Assumption 3.2.25, using the strong

Markov property for the given process X it is easy to show by induction that for all $n \in \mathbb{N}$

$$P_x[\sigma_n > t] = E_x\left[e^{-A_t^{\kappa\varrho}} \cdot \sum_{k=0}^{n-1} \frac{(A_t^{\kappa\varrho})^k}{k!}\right], \qquad x \in E,\ t \geq 0.$$

Hence it follows that $\sigma_n \uparrow \infty$ P_x-a.s. (see also [Saw1970], Cor. 3.1); thus $\tilde{X}$ and X do in fact coincide. On the other hand, the auxiliary process resp. the revival times σ_n are not really necessary in this case: In fact, using the strong Markov property for the given process X it is easy to show by induction that for all $N \in \mathbb{N}$ the r.h.s. of (3.2.83) is equal to

$$\sum_{n=0}^{N-1} E_x\left[f(X_t)e^{-A_t^{\kappa}} \frac{(A_t^{\kappa\varrho})^n}{n!}\right].$$

From this, the assertion of Theorem 3.2.28 can be deduced without introducing the auxiliary process $\tilde{X}$ resp. the revival times σ_n, with $\tilde{X}$ in (3.2.80) replaced by X:

$$M_t f(x) \equiv \boldsymbol{E}_x\left[\bar{f}(\boldsymbol{\beta}_t)\right] = E_x\left[f(X_t)e^{-A_t^{\gamma}}\right], \qquad f \in \mathscr{B}^+(E),\ x \in E,\ t \geq 0. \tag{3.2.93}$$

For the case of a diffusion as single particle motion X, formula (3.2.93) is of course well known and can even be considered "classical": It appears already in [Wat1967], where it is however proved using the generator (3.2.67) under the regularity assumptions mentioned at the beginning of this subsection. Our proof of Theorem 3.2.28 on the other hand uses essentially only the strong Markov property and the branching property of $\boldsymbol{\beta}$ resp. the structural properties of the auxiliary process $\tilde{X}$ as a killed-and-revived Markov process, which permit the generalization of the result for arbitrary single-particle motions and offspring distributions.

- The assumptions needed for Theorem 3.2.28 are fairly weak; on the other hand, the result as such is quite abstract. In concrete applications, one needs to be able to verify the nonexplosion hypothesis assumed in Theorem 3.2.28. In addition, in order to use the representation (3.2.91) to study the expected number of particles (its growth as $t \uparrow \infty$, say), one clearly wants the family of kernels $(\tilde{T}_t^{\gamma})_t$ to form a semigroup of bounded operators on $\mathscr{B}(E)$, so that classical semigroup theory can be applied. We will see in the next subsection that these two problems are actually closely connected. More precisely, consider the following condition: Suppose that the auxiliary process $\tilde{X}$ does not "explode" in the sense that there is no accumulation of revival events in finite time (i.e. $\sigma_\infty = \infty$), and that there is some $t_0 > 0$ such that

$$C_{t_0} := \sup_{x \in E} \boldsymbol{E}_x\left[\exp\left(\int_0^{t_0} [\kappa(\varrho - 1)^+](\tilde{X}_s)\,ds\right)\right] < \infty, \tag{3.2.94}$$

where $(\varrho - 1)^+$ denotes the positive part of the function $\varrho - 1$.[27] Then since the additive functional $t \mapsto \int_0^t [\kappa(\varrho-1)^+](\tilde{X}_s)\,ds$ is increasing we obviously have

$$\sup_{x\in E} \tilde{T}_t^{\gamma}(x;E) = \sup_{x \in E} \boldsymbol{E}_x\left[e^{-\int_0^t \gamma(\tilde{X}_r)\,dr}\right] \leq \sup_{x\in E} \boldsymbol{E}_x\left[e^{\int_0^{t_0}[\kappa(\varrho-1)^+](\tilde{X}_r)\,dr}\right] = C_{t_0} < \infty,$$

[27] This condition is in particular satisfied if κ and ϱ are bounded.

i.e. $\tilde{T}_t^\gamma$ is a bounded kernel for $0 \le t \le t_0$. By the semigroup property of the kernels $\tilde{T}_t^\gamma$ (see (3.2.81) above), this extends immediately to all $t_0 \le t \le 2t_0$ via

$$\tilde{T}_t^\gamma(x;E) = \int_E \tilde{T}_{t_0}^\gamma(x;dy)\tilde{T}_{t-t_0}^\gamma(y;E) \le C_{t_0}^2, \qquad x \in E,\, t_0 \le t \le 2t_0,$$

and by induction to all $t > 0$. Thus $(\tilde{T}_t^\gamma)_{t\ge 0}$ induces a semigroup of bounded operators on $\mathscr{B}(E)$ with operator norm $\|\tilde{T}_t^\gamma\|_{\infty\to\infty} = \sup_{x\in E} \tilde{T}_t^\gamma(x;E) < \infty$ to which the results of general semigroup theory can be applied. We will see in the next subsection that the same condition (3.2.94) is also sufficient to ensure nonexplosion of $\boldsymbol{\beta}$.

Although Theorem 3.2.28 seems to be of independent interest, its importance for us is mainly in the fact that it allows for another representation of the occupation times kernel H from Definition 3.2.18: Namely, by integrating with respect to time $t \in \mathbb{R}_+$ in (3.2.91) we obtain a formula for H which is analogous to that in [HL2005] (see their Prop. 2.2, p. 1033):

3.2.30 Corollary

Under the assumptions of Theorem 3.2.28, the occupation times kernel of (3.2.48) *coincides with the generalized γ-resolvent kernel of the auxiliary process $\tilde{X}$:*

$$H(x;B) = \tilde{R}_\gamma(x;B) := \boldsymbol{E}_x\left[\int_0^{\sigma_\infty} \mathbf{1}_B(\tilde{X}_t)e^{-\int_0^t \gamma(\tilde{X}_s)\,ds}\,dt\right], \qquad B \in \mathcal{B}_E,\ x \in E. \tag{3.2.95}$$

3.2.31 Remarks

- Consider again the case $\tilde{Q}(x;\cdot) = \delta_x(\cdot)$: Then the auxiliary process $\tilde{X}$ coincides with the given single particle motion X and $\sigma_\infty = \infty$ (see the first remark in 3.2.29 above); thus (3.2.95) reads

  $$H(x;B) = \boldsymbol{E}_x\left[\int_0^{\infty} \mathbf{1}_B(X_t)e^{-\int_0^t \gamma(X_s)\,ds}\,dt\right], \qquad B \in \mathcal{B}_E,\ x \in E.$$

  If X is a diffusion on $E = \mathbb{R}^d$, this reproduces the result in [HL2005] which was mentioned above. Note however that our assumptions are weaker since we require neither boundedness nor continuity of κ and ϱ but only nonexplosion of $\boldsymbol{\beta}$ plus finiteness of the additive functionals $(A_t^\kappa)_t$ and $(A_t^{\kappa\varrho})_t$ (see (3.1.7) and (3.2.72)).

- In view of the representation (3.2.95), Condition 3.2.19 (spatial subcriticality) may be checked in terms of the generalized resolvent kernel $\tilde{R}_\gamma$. We see yet again that the easiest way to ensure subcriticality is to require boundedness of κ away from 0 and boundedness of ϱ away from 1, but possibly the representation (3.2.95) opens the door to other examples which are of a different type than those given in 3.2.23 and 3.2.24 above.

3.2.5 On Nonexplosion for Branching Markov Processes

The results in the two previous subsections were stated under the hypothesis that the branching Markov process $\boldsymbol{\beta}$ does not explode. Therefore, this subsection is devoted to a brief discussion of the explosion problem for branching Markov processes. We will see that this is also closely connected with boundedness of the kernels $\tilde{T}_t^\gamma$ from (3.2.80). In line with the approach of the previous subsections, we are interested in sufficient conditions for nonexplosion which permit general single-particle motions and offspring distributions, and we would like to avoid boundedness or continuity assumptions on κ and ϱ. For the following, we draw in particular on the approach in [Nag1977], pp. 443ff. In the context of branching diffusions and with $\tilde{Q}(x;\cdot) = \delta_x(\cdot)$, the explosion problem was also studied in [Löc2002a], Sec. 4, using techniques from the theory of backward SDEs.

In the first example in 3.2.8, we have already reported the classical result that nonexplosion is equivalent to uniqueness in the class $\mathscr{B}_1^+(E)$ of $u_t(\cdot) \equiv 1$ as the solution to

$$u_t(x) = T_t^\kappa(x;E) + \int_0^t ds \int_E T_s^\kappa(x;dy)\kappa(y)F(y;u_{t-s}) \tag{3.2.96}$$

which is just the S-equation (3.2.25) for $f \equiv 1$. From this characterization, several important sufficient conditions for nonexplosion can be obtained. In this regard, the following well-known observation is basic[28]: For any two functions $u(\cdot), v(\cdot) \in \mathscr{B}_1^+(E)$, we always have

$$|\hat{u}(\boldsymbol{x}) - \hat{v}(\boldsymbol{x})| = \left| \prod_{i=1}^{\ell(\boldsymbol{x})} u(x^i) - \prod_{i=1}^{\ell(\boldsymbol{x})} v(x^i) \right| \le \sum_{i=1}^{\ell(\boldsymbol{x})} |u(x^i) - v(x^i)| = \overline{|u-v|}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathcal{S}$$

and thus using the kernel J from (3.1.11) and formula (3.2.24)

$$|F(y;u) - F(y;v)| = |J(y;\hat{u} - \hat{v})| \le J(y;|\hat{u}-\hat{v}|) \le J(y;\overline{|u-v|}) = \varrho(y)\tilde{Q}(y;|u-v|) \tag{3.2.97}$$

for all $y \in E$. Now let $u_t(\cdot) \equiv 1$, $v_t(x) := \boldsymbol{P}_x[t < \boldsymbol{\tau}_\infty]$ and denote $w_t(x) := u_t(x) - v_t(x) = \boldsymbol{P}_x[\boldsymbol{\tau}_\infty \le t]$. Since both u_t and v_t are solutions to (3.2.96), from (3.2.97) we get

$$0 \le w_t(x) \le \int_0^t ds \int_E T_s^\kappa(x;dy)\kappa(y)\varrho(y)\tilde{Q}(y;w_{t-s}). \tag{3.2.98}$$

Letting $t \uparrow \infty$, we obtain $w_t(x) \uparrow \boldsymbol{P}_x[\boldsymbol{\tau}_\infty < \infty] =: w_\infty(x)$, and monotone convergence in (3.2.98) gives

$$0 \le w_\infty(x) \le [R_\kappa \kappa\varrho\tilde{Q}](x;w_\infty). \tag{3.2.99}$$

From this it follows immediately that the condition $\|R_\kappa\kappa\varrho\|_\infty < 1$ as in (3.2.57) implies $\|w_\infty\|_\infty = 0$, and thus nonexplosion. Alternatively, suppose as in (3.2.59) that $\|\tilde{Q}R_\kappa\kappa\varrho\|_\infty < 1$ and $\|R_\kappa\kappa\varrho\|_\infty < \infty$. Iterating (3.2.99), we have for all $n \in \mathbb{N}$

$$0 \le w_\infty(x) \le [R_\kappa\kappa\varrho][\tilde{Q}R_\kappa\kappa\varrho]^n\tilde{Q}(x;w_\infty) \le \|R_\kappa\kappa\varrho\|_\infty \cdot \|\tilde{Q}R_\kappa\kappa\varrho\|_\infty^n \xrightarrow{n\to\infty} 0.$$

This proves our claim in the last remark in 3.2.22 that the "obvious" conditions (3.2.57) and (3.2.59) for boundedness of the series (3.2.54) do also ensure nonexplosion of the process $\boldsymbol{\beta}$ (of course, they are much stronger since they even ensure extinction).[29]

We return to the estimate (3.2.98) for fixed $t > 0$. Since $w_t(x) = \boldsymbol{P}_x[\boldsymbol{\tau}_\infty \le t] \le 1$ for all x, iterating it gives

$$\begin{aligned} 0 \le w_t(x) \le \int_0^t ds_1 \int_E [T_{s_1}^\kappa\kappa\varrho\tilde{Q}](x;dy_1) \int_0^{t-s_1} ds_2 \int_E [T_{s_2}^\kappa\kappa\varrho\tilde{Q}](y_1;dy_2)\cdots \\ \cdots \int_0^{t-s_1-\ldots-s_{n-1}} ds_n\, [T_{s_n}^\kappa\kappa\varrho\tilde{Q}](y_{n-1};E) \end{aligned} \tag{3.2.100}$$

[28] See e.g. [Nag1977], proofs of Thms. 8 and 9, pp. 443f.

[29] We have not been able to determine whether boundedness in x of the series on the r.h.s. of (3.2.54) as such is already sufficient for nonexplosion.

for all $t > 0$, $x \in E$ and $n \in \mathbb{N}$. From this we see in particular that if κ and ϱ are bounded, then

$$\|w_t\|_\infty \leq \frac{(t \cdot \|\kappa\varrho\|_\infty)^n}{n!} \xrightarrow{n\to\infty} 0,$$

implying nonexplosion of $\boldsymbol{\beta}$ (see [Nag1977], Thm. 8, p. 443). For another proof of this result, see [HL2005], Prop. 2.3, where however κ and ϱ were supposed to be continuous as well as bounded.

A variation of the above argument assumes boundedness of ϱ, but replace boundedness of κ by a weaker condition[30]:

3.2.32 Proposition
Assume that ϱ is bounded and that the following condition holds:

$$\sup_{x\in E} P_x[\tau \leq t] \equiv \sup_{x\in E} E_x\left[\int_0^t e^{-A_s^\kappa}\kappa(X_s)\,ds\right] \xrightarrow{t\downarrow 0} 0. \tag{3.2.101}$$

Then $\boldsymbol{\beta}$ does not explode.

Proof Under (3.2.101), choose $t_0 > 0$ small enough such that

$$C_{t_0} := \sup_{x\in E} P_x\left[\tau \leq t_0\right] < \|\varrho\|_\infty^{-1}. \tag{3.2.102}$$

Let $t \in [0, t_0]$. Then for the "innermost" integral in (3.2.100) we have

$$\begin{aligned}
&\sup_{y_{n-1}\in E} \int_0^{t-s_1-\ldots-s_{n-1}} ds_n\,[T_{s_n}^\kappa \kappa\varrho\tilde{Q}](y_{n-1}; E) \\
&\leq \|\varrho\|_\infty \sup_{y_{n-1}\in E} \int_0^{t-s_1-\ldots-s_{n-1}} ds_n\,[T_{s_n}^\kappa \kappa](y_{n-1}; E) \\
&= \|\varrho\|_\infty \sup_{y_{n-1}\in E} P_{y_{n-1}}[\tau \leq t - s_1 - \cdots - s_{n-1}] \\
&\leq C_{t_0} \cdot \|\varrho\|_\infty.
\end{aligned}$$

In this way, we can recursively estimate all terms in (3.2.100) and obtain

$$\|w_t\|_\infty \leq (C_{t_0} \cdot \|\varrho\|_\infty)^n \xrightarrow{n\to\infty} 0$$

for all $t \in [0, t_0]$ by choice of t_0. Thus $w_t(x) = \boldsymbol{P}_x[\boldsymbol{\tau}_\infty \leq t] = 0$ for all $t \in [0, t_0]$ and $x \in E$. This implies (see (3.2.14))

$$\boldsymbol{P}_{\boldsymbol{x}}[t < \boldsymbol{\tau}_\infty] = \prod_{j=1}^{\ell(\boldsymbol{x})} \boldsymbol{P}_{x^j}[t < \boldsymbol{\tau}_\infty] = 1, \qquad \boldsymbol{x} \in \mathcal{S},\ t \in [0, t_0].$$

But since $\boldsymbol{P}_{\boldsymbol{x}}[t < \boldsymbol{\tau}_\infty] = \boldsymbol{T}_t(\boldsymbol{x}; \mathcal{S})$, using the semigroup property this extends to all $t > 0$: Indeed, we have

$$\boldsymbol{P}_{\boldsymbol{x}}[t + t_0 < \boldsymbol{\tau}_\infty] = \boldsymbol{T}_{t+t_0}(\boldsymbol{x}; \mathcal{S}) = \int_{\mathcal{S}} \boldsymbol{T}_t(\boldsymbol{x}; d\boldsymbol{y})\boldsymbol{T}_{t_0}(\boldsymbol{y}; \mathcal{S}) = 1, \qquad \boldsymbol{x} \in \mathcal{S},\ t \in [0, t_0].$$

Inductively, we get $\boldsymbol{P}_{\boldsymbol{x}}[t < \boldsymbol{\tau}_\infty]$ for all $t > 0$. ■

[30] According to [Saw1970], p. 6, the following is also a "standard argument"; however no proof or reference is given in [Saw1970].

3.2.33 Remarks

- An obvious sufficient condition for (3.2.101) is the following:

$$\sup_{x\in E} E_x\left[A_t^\kappa\right] \equiv \sup_{x\in E} E_x\left[\int_0^t \kappa(X_s)\,ds\right] \xrightarrow{t\downarrow 0} 0.$$

  This follows immediately by Jensen's inequality, since

$$E_x\left[\int_0^t ds\, e^{-A_s^\kappa}\kappa(X_s)\right] = 1 - E_x[e^{-A_t^\kappa}] \le 1 - \exp\left(-E_x[A_t^\kappa]\right) \le E_x[A_t^\kappa].$$

- The above proof shows also that under the assumptions of Proposition 3.2.32, each of the kernels $\tilde{T}_t^\gamma$ in (3.2.80) is bounded, i.e. $(\tilde{T}_t^\gamma)_t$ is in fact a semigroup of bounded operators on $\mathscr{B}(E)$: Indeed, for $f \equiv 1$ the r.h.s. of (3.2.82) is dominated by

$$\tilde{T}_t^\gamma(x;E) \le 1 + \sum_{n\in\mathbb{N}} (C_{t_0}\cdot\|\varrho\|_\infty)^n < \infty, \qquad t\in[0,t_0]$$

  by choice of t_0. Since we already know that the kernels $\tilde{T}_t^\gamma$ fulfill the semigroup property (see (3.2.81)), this implies $\|\tilde{T}_t^\gamma\|_{\infty\to\infty} < \infty$ for all $t > 0$.

- Conversely, suppose we already know that $(\tilde{T}_t^\gamma)_t$ is a semigroup of bounded operators, $\|\tilde{T}_t^\gamma\|_{\infty\to\infty} < \infty$ for all $t > 0$. Suppose moreover that for some t_0 we have

$$\tilde{C}_{t_0} := \inf_{x\in E}\inf_{t\in[0,t_0]} T_t^\kappa(x;E) = \inf_{x\in E}\inf_{t\in[0,t_0]} P_x[\tau > t] > 0. \tag{3.2.103}$$

  Then the estimate (3.2.100) implies

$$\begin{aligned} w_t(x) \le \tilde{C}_{t_0}^{-1}\int_0^t ds_1 \int_E [T_{s_1}^\kappa \kappa\varrho\tilde{Q}](x;dy_1)\int_0^{t-s_1} ds_2\int_E [T_{s_2}^\kappa\kappa\varrho\tilde{Q}](y_1;dy_2)\cdots \\ \cdots\int_0^{t-s_1-\ldots-s_{n-1}} ds_n\int_E [T_{s_n}^\kappa\kappa\varrho\tilde{Q}](y_{n-1};dy_n)T_{t-s_1-\cdots-s_n}^\kappa(y_n;E) \end{aligned}$$

  for all $t\in[0,t_0]$. Since $\tilde{T}_t^\gamma(x;E) < \infty$ is assumed, by choosing $f\equiv 1$ in (3.2.82) we see that the r.h.s. of the previous display tends to 0 as $n\to\infty$. Consequently $w_t(\cdot)\equiv 0$ for $t\in[0,t_0]$, and as in the proof of Proposition 3.2.32 this extends to all $t > 0$ by the semigroup property.

  Thus we see that nonexplosion is closely connected with boundedness of the kernels $\tilde{T}_t^\gamma$ from (3.2.80). ♦

We also have the following result connecting the nonexplosion problem with the auxiliary process $\tilde{X}$ of Definition 3.2.26: Basically, it says that the sufficient condition (3.2.94) ensuring boundedness of the kernels $\tilde{T}_t^\gamma$ guarantees also nonexplosion of $\boldsymbol{\beta}$.

3.2.34 Proposition

Assume that the auxiliary process $\tilde{X}$ has infinite "life-time" in the sense that there is no accumulation of revival events in finite time:

$$\forall x\in E:\ \sigma_\infty = \infty \qquad \boldsymbol{P}_x\text{-a.s.} \tag{3.2.104}$$

Furthermore, assume that there is some $t_0 > 0$ such that

$$\sup_{x\in E}\boldsymbol{E}_x\left[\exp\left(\int_0^{t_0} ds\,\kappa(\tilde{X}_s)(\varrho(\tilde{X}_s)-1)^+\right)\right] < \infty. \tag{3.2.105}$$

Then $\boldsymbol{\beta}$ does not explode.

Proof Again we consider the estimate (3.2.100) for each $n \in \mathbb{N}$. Recalling that the auxiliary process $\tilde{X}$ up to the first revival time σ_1 is given by the single-particle motion X killed at rate $\kappa\varrho$, the "innermost" integral in (3.2.100) can be rewritten as

$$\begin{aligned}\int_0^{t-s_1-\ldots-s_{n-1}} ds_n\,[T^\kappa_{s_n}\kappa\varrho\tilde{Q}](y_{n-1};E) &= E_{y_{n-1}}\left[\int_0^{t-s_1-\ldots-s_{n-1}} ds_n\, e^{-A^\kappa_{s_n}}\kappa(X_{s_n})\varrho(X_{s_n})\right]\\ &= E_{y_{n-1}}\left[\int_0^{t-s_1-\ldots-s_{n-1}} ds_n\, e^{-A^{\kappa\varrho}_{s_n}}\kappa(X_{s_n})\varrho(X_{s_n})\cdot e^{-A^\gamma_{s_n}}\right]\\ &= \boldsymbol{E}_{y_{n-1}}\left[\mathbf{1}_{\sigma_1\le t-s_1-\ldots-s_{n-1}}\cdot e^{-\int_0^{\sigma_1}\gamma(\tilde{X}_s)\,ds}\right].\end{aligned}$$

By a repeated application of the strong Markov property in the form (3.2.78) and using (3.2.75), it can be obtained in $n-1$ steps that the r.h.s. of (3.2.100) equals

$$\begin{aligned}&\int_0^t ds_1\int_E[T^\kappa_{s_1}\kappa\varrho\tilde{Q}](x;dy_1)\int_0^{t-s_1} ds_2\int_E[T^\kappa_{s_2}\kappa\varrho\tilde{Q}](y_1;dy_2)\cdots\\ &\cdots\int_0^{t-s_1-\ldots-s_{n-2}} ds_{n-1}\int_E[T^\kappa_{s_{n-1}}\kappa\varrho\tilde{Q}](y_{n-2};dy_{n-1})\int_0^{t-s_1-\ldots-s_{n-1}} ds_n\,[T^\kappa_{s_n}\kappa\varrho\tilde{Q}](y_{n-1};E)\\ &= \boldsymbol{E}_x\left[\mathbf{1}_{\sigma_n\le t}\cdot e^{-\int_0^{\sigma_n}\gamma(\tilde{X}_s)\,ds}\right].\end{aligned}$$

Since we assume $\sigma_n\uparrow\infty$ and (3.2.105), dominated convergence gives

$$\begin{aligned}0\le w_t(x) &\le \boldsymbol{E}_x\left[\mathbf{1}_{\sigma_n\le t}\cdot e^{-\int_0^{\sigma_n}\gamma(\tilde{X}_s)\,ds}\right]\\ &\le \boldsymbol{E}_x\left[\mathbf{1}_{\sigma_n\le t}\cdot e^{\int_0^{t_0}[\kappa(\varrho-1)^+](\tilde{X}_s)\,ds}\right]\xrightarrow{n\to\infty}0\end{aligned}$$

for all $x\in E$ and $t\in[0,t_0]$. As before, this implies $w_t(\cdot)\equiv 0$ for all $t>0$ by the semigroup property. ■

3.2.35 Remarks

- Condition (3.2.105) is essentially a generalized uniform version of a condition given in [Nag1977] for the case (3.1.13) that branching particles reproduce at their parent's death position.[31] Recall that in this special case, we automatically have $\sigma_n\uparrow\infty$, and $\tilde{X}$ coincides with X (see the first remark in 3.2.29). On the other hand, in this case it is not necessary to introduce the random times σ_n at all: Using the Markov property of the process X, it is easy to show that the r.h.s. of the estimate (3.2.100) equals

$$E_x\left[\int_0^t ds\, e^{-A^\kappa_s}\kappa(X_s)\varrho(X_s)\frac{(A^{\kappa\varrho}_s)^{n-1}}{(n-1)!}\right]$$

  for all $x\in E$, $t\ge 0$ and $n\in\mathbb{N}$. Using (3.2.105), the above expression is easily seen to vanish for $n\to\infty$.

- The condition (3.2.104) that there is no accumulation of "revival events" in the auxiliary process can be checked by analogous arguments as those in Proposition 3.2.32: Namely, it is satisfied provided

$$\sup_{x\in E}P_x[\sigma\le t]\equiv\sup_{x\in E}E_x\left[\int_0^t e^{-A^{\kappa\varrho}_s}\kappa(X_s)\varrho(X_s)\,ds\right]\xrightarrow{t\downarrow 0}0,$$

[31] See [Nag1977], formula (64) on p. 444.

which by Jensen's inequality is in turn implied by

$$\sup_{x\in E} E_x\left[A_t^{\kappa\varrho}\right] \equiv \sup_{x\in E} E_x\left[\int_0^t \kappa(X_s)\varrho(X_s)\,ds\right] \xrightarrow{t\downarrow 0} 0.$$

See the proof of Proposition 3.2.32 and the first remark in 3.2.33. ♦

3.3 The Addition of Immigration

In the final section of this chapter (and thesis), we return to the problem of sufficient conditions for ergodicity of a branching Markov process with immigration (BMPI). Given a branching Markov process $\boldsymbol{\beta}$ (without immigration) fulfilling Assumptions 3.1.5 and 3.1.7 as in the previous section, let us add immigration at a constant rate $c > 0$ and according to an immigration law ν as in Assumption 3.1.8. The resulting BMPI will be denoted by $\boldsymbol{\eta}$. Again, the rigorous construction of $\boldsymbol{\eta}$ from $\boldsymbol{\beta}$ and the immigration mechanism can be accomplished by means of the "Revival Theorem" of [INW1968b]: Take a BMP $\boldsymbol{\beta}$, kill it at constant rate $c > 0$ and restart it according to a jump kernel $\tilde{K} : \mathcal{S} \times \mathcal{B}_{\mathcal{S}} \to [0,1]$ acting on bounded measurable functions as

$$\tilde{K}(\boldsymbol{x}; g) := (\delta_{\boldsymbol{x}} * \nu)(g) \equiv \int_E \nu(dv)\, g(\boldsymbol{x} \bullet v), \qquad g \in \mathscr{B}(\mathcal{S}). \tag{3.3.1}$$

For a BMPI $\boldsymbol{\eta}$ constructed in this way from a BMP $\boldsymbol{\beta}$ and an immigration mechanism, we will say that $\boldsymbol{\beta}$ is the *branching component* of $\boldsymbol{\eta}$. Since it is a "killed and revived"-process, under suitable regularity assumptions the generator of $\boldsymbol{\eta}$ is given by

$$\mathcal{A}f = \mathcal{A}^0 f + c(\tilde{K}f - f), \qquad f \in D(\mathcal{A}) = D(\mathcal{A}^0),$$

where $\mathcal{A}^0$ denotes the generator of the branching component $\boldsymbol{\beta}$ (cf. the second remark in 1.1.8). However, as before we want to avoid arguments based on generators and their domains. In particular, the "killing and reviving"-procedure of [INW1968b] permits construction of $\boldsymbol{\eta}$ from $\boldsymbol{\beta}$ without any regularity assumptions on the semigroup of $\boldsymbol{\beta}$ (like Feller properties) and on the immigration measure ν.

We begin with a result on the structure of the semigroup $(\boldsymbol{P}_t)_t$ of the BMPI $\boldsymbol{\eta}$. Although due to the presence of immigration, it does no longer have the branching property (3.2.11), it still has a specific structure: More precisely, in the next proposition we will show that

$$\boldsymbol{P}_t(\boldsymbol{x};\cdot) = \boldsymbol{T}_t(\boldsymbol{x};\cdot) * \boldsymbol{\nu}_t, \qquad \boldsymbol{x} \in \mathcal{S},\ t > 0. \tag{3.3.2}$$

Here $(\boldsymbol{T}_t)_t$ denotes the semigroup of the branching component $\boldsymbol{\beta}$, which is a BMP without immigration as in the previous section, whereas $\boldsymbol{\nu}_t$ is a probability measure on $\mathcal{S}$ representing the offspring of all immigrants up to time $t > 0$. Semigroups of the form (3.3.2) are known as a *skew-convolution semigroups* and are extensively used in the study of superprocesses with immigration; on this topic see e.g. [Li2002] or the survey [Li2006], Sec. 2. In our model, the immigration mechanism is simple enough so that the measure $\boldsymbol{\nu}_t$ can actually be calculated:

3.3.1 Proposition
Consider a BMPI $\boldsymbol{\eta}$ with branching component $\boldsymbol{\beta}$, immigration rate $c > 0$ and immigration law ν. Assume that $\boldsymbol{\beta}$ does not explode. Then the semigroup of $\boldsymbol{\eta}$ has the form

$$\boldsymbol{P}_t(\boldsymbol{x};\cdot) = \boldsymbol{T}_t(\boldsymbol{x};\cdot) * \boldsymbol{\nu}_t, \qquad \boldsymbol{x} \in \mathcal{S},\ t > 0, \tag{3.3.3}$$

where $(\boldsymbol{T}_t)_t$ *is the semigroup of* $\boldsymbol{\beta}$ *and for each* $t > 0$, $\boldsymbol{\nu}_t$ *is a probability measure on* $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$ *which is explicitly given by*

$$\boldsymbol{\nu}_t = e^{-ct}\left(\delta_\Delta + \sum_{n=1}^{\infty} c^n \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \, (\nu \boldsymbol{T}_{s_1}) * (\nu \boldsymbol{T}_{s_2}) * \cdots * (\nu \boldsymbol{T}_{s_n})\right).^{32} \quad (3.3.4)$$

In other words, for all $g \in \mathscr{B}(\mathcal{S})$ *we have*

$$\boldsymbol{P}_t g(\boldsymbol{x}) = \int_{\mathcal{S}\times\mathcal{S}} \boldsymbol{T}_t(\boldsymbol{x}; d\boldsymbol{y}) \boldsymbol{\nu}_t(d\boldsymbol{z})\, g(\boldsymbol{y} \bullet \boldsymbol{z}), \qquad \boldsymbol{x} \in \mathcal{S},$$

where

$$\boldsymbol{\nu}_t(g) = e^{-ct}\Bigg(g(\Delta) + \sum_{n=1}^{\infty} c^n \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n \int_S (\nu \boldsymbol{T}_{s_1})(d\boldsymbol{z}_1) \cdots \\ \cdots \int_S (\nu \boldsymbol{T}_{s_n})(d\boldsymbol{z}_n)\, g(\boldsymbol{z}_1 \bullet \boldsymbol{z}_2 \bullet \cdots \bullet \boldsymbol{z}_n)\Bigg). \quad (3.3.5)$$

Proof The proof is a variation on the theme "conditioning on the first revival event in a killed-and-revived Markov process". In the present context, "first revival event" means the first immigration event. Let us write T_1 for the first immigration time, which is an exponential time with parameter c.

Let $g \in \mathscr{B}(\mathcal{S})$. Since the BMPI $\boldsymbol{\eta}$ up to time T_1 evolves as the BMP $\boldsymbol{\beta}$ killed at rate c and the transition at the first immigration event is governed by the kernel (3.3.1), by conditioning on T_1 we get for all $\boldsymbol{x} \in \mathcal{S}$, $t > 0$

$$\begin{aligned} \boldsymbol{P}_t g(\boldsymbol{x}) &= \boldsymbol{E}_{\boldsymbol{x}}\left[g(\boldsymbol{\eta}_t)\mathbf{1}_{t<T_1}\right] + \boldsymbol{E}_{\boldsymbol{x}}\left[g(\boldsymbol{\eta}_t)\mathbf{1}_{t\ge T_1}\right] \\ &= e^{-ct}\boldsymbol{E}_{\boldsymbol{x}}\left[g(\boldsymbol{\beta}_t)\right] + \boldsymbol{E}_{\boldsymbol{x}}\left[\int_0^t ds\, ce^{-cs} \int \nu(dv) \boldsymbol{E}_{\boldsymbol{\beta}_s \bullet v}\left[g(\boldsymbol{\eta}_{t-s})\right]\right] \\ &= e^{-ct}\boldsymbol{T}_t g(\boldsymbol{x}) + \int_0^t ds\, ce^{-cs} \int_{\mathcal{S}} \left[\boldsymbol{T}_s(\boldsymbol{x};\cdot) * \nu\right](d\boldsymbol{z}) \boldsymbol{E}_{\boldsymbol{z}}\left[g(\boldsymbol{\eta}_{t-s})\right] \\ &= e^{-ct}\boldsymbol{T}_t g(\boldsymbol{x}) + \int_0^t ds\, ce^{-cs} \left[\boldsymbol{T}_s(\boldsymbol{x};\cdot) * \nu\right](\boldsymbol{P}_{t-s} g), \end{aligned} \quad (3.3.6)$$

where in the notation $\boldsymbol{T}_s(\boldsymbol{x};\cdot) * \nu$ of course we regard ν as a measure on $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$ which is concentrated on the single-particle layer $E \subseteq \mathcal{S}$. We claim that for each $N \in \mathbb{N}$

$$\boldsymbol{P}_t g(\boldsymbol{x}) = \left[\boldsymbol{T}_t(\boldsymbol{x};\cdot) * \boldsymbol{\nu}_t^{(N-1)}\right](g) + R_t^{(N)}(\boldsymbol{x}; g), \qquad \boldsymbol{x} \in \mathcal{S},\ t > 0, \quad (3.3.7)$$

where for all $t > 0$ and $N \in \mathbb{N}$, $\boldsymbol{\nu}_t^{(N-1)}$ is a substochastic measure on $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$ defined by

$$\boldsymbol{\nu}_t^{(N-1)} := e^{-ct}\left(\delta_\Delta + \sum_{n=1}^{N-1} c^n \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \, (\nu \boldsymbol{T}_{s_1}) * (\nu \boldsymbol{T}_{s_2}) * \cdots * (\nu \boldsymbol{T}_{s_n})\right)$$

and

$$R_t^{(N)}(\boldsymbol{x}; g) := \int_0^t ds_1\, ce^{-cs_1} \int_{\mathcal{S}} \left[\boldsymbol{T}_{s_1}(\boldsymbol{x};\cdot) * \nu\right](d\boldsymbol{z}_1) \int_0^{t-s_1} ds_2\, ce^{-cs_2} \int_{\mathcal{S}} \left[\boldsymbol{T}_{s_2}(\boldsymbol{z}_1;\cdot) * \nu\right](d\boldsymbol{z}_2) \cdots \\ \cdots \int_0^{t-s_1-\cdots-s_{N-1}} ds_N\, ce^{-cs_N} \int_{\mathcal{S}} \left[\boldsymbol{T}_{s_N}(\boldsymbol{z}_{N-1};\cdot) * \nu\right](d\boldsymbol{z}_N) \boldsymbol{P}_{t-s_1-\cdots-s_N} g(\boldsymbol{z}_N).$$

[32] Here, $\nu \boldsymbol{T}_s$ denotes of course the measure given by $\nu \boldsymbol{T}_s(F) = \int_E \nu(dv) \boldsymbol{T}_s(v; F) = \int_E \nu(dv) \boldsymbol{E}_v[\mathbf{1}_F(\boldsymbol{\beta}_s)]$ for $F \in \mathcal{B}_{\mathcal{S}}$. We do not use the "operator-theoretic" notation $\boldsymbol{T}_s^* \nu$ at this point in order to avoid any possible confusion with the convolution operation.

We will prove (3.3.7) by induction: For $N = 1$, equation (3.3.7) is indeed just (3.3.6) above. Assuming that (3.3.7) is already known to hold for some $N \in \mathbb{N}$ and all $\boldsymbol{x} \in \mathcal{S}$, $t > 0$, from (3.3.6) and induction hypothesis we get

$$
\begin{aligned}
&\boldsymbol{P}_t g(\boldsymbol{x}) \\
&= e^{-ct}\boldsymbol{T}_t g(\boldsymbol{x}) + \int_0^t ds\, ce^{-cs} \int_{\mathcal{S}} [\boldsymbol{T}_s(\boldsymbol{x}\,;\cdot) * \nu](d\boldsymbol{z})\boldsymbol{P}_{t-s} g(\boldsymbol{z}) \\
&= e^{-ct}\boldsymbol{T}_t g(\boldsymbol{x}) + \int_0^t ds\, ce^{-cs} \int_{\mathcal{S}} [\boldsymbol{T}_s(\boldsymbol{x}\,;\cdot) * \nu](d\boldsymbol{z}) \Big([\boldsymbol{T}_{t-s}(\boldsymbol{z};\cdot) * \boldsymbol{\nu}_{t-s}^{(N-1)}](g) + R_{t-s}^{(N)}(\boldsymbol{z}; g) \Big) \\
&= e^{-ct}\boldsymbol{T}_t g(\boldsymbol{x}) + \int_0^t ds\, ce^{-cs} \int_{\mathcal{S}} [\boldsymbol{T}_s(\boldsymbol{x}\,;\cdot) * \nu](d\boldsymbol{z}) [\boldsymbol{T}_{t-s}(\boldsymbol{z};\cdot) * \boldsymbol{\nu}_{t-s}^{(N-1)}](g) \\
&\qquad\qquad + \int_0^t ds\, ce^{-cs} \int_{\mathcal{S}} [\boldsymbol{T}_s(\boldsymbol{x}\,;\cdot) * \nu](d\boldsymbol{z}) R_{t-s}^{(N)}(\boldsymbol{z}; g).
\end{aligned}
\tag{3.3.8}
$$

We consider the second term on the r.h.s. of the previous display and observe that for fixed $s \in [0, t]$

$$
\begin{aligned}
&\int_{\mathcal{S}} [\boldsymbol{T}_s(\boldsymbol{x}\,;\cdot) * \nu](d\boldsymbol{z}) [\boldsymbol{T}_{t-s}(\boldsymbol{z};\cdot) * \boldsymbol{\nu}_{t-s}^{(N-1)}](g) \\
&= \int_{\mathcal{S}} \boldsymbol{T}_s(\boldsymbol{x}; d\boldsymbol{y}_1) \int_{\mathcal{S}} \nu(d\boldsymbol{y}_2) [\boldsymbol{T}_{t-s}(\boldsymbol{y}_1 \bullet \boldsymbol{y}_2;\cdot) * \boldsymbol{\nu}_{t-s}^{(N-1)}](g) \\
&= \int_{\mathcal{S}} \boldsymbol{T}_s(\boldsymbol{x}; d\boldsymbol{y}_1) \int_{\mathcal{S}} \nu(d\boldsymbol{y}_2) [\boldsymbol{T}_{t-s}(\boldsymbol{y}_1;\cdot) * \boldsymbol{T}_{t-s}(\boldsymbol{y}_2;\cdot) * \boldsymbol{\nu}_{t-s}^{(N-1)}](g) \\
&= [\boldsymbol{T}_t(\boldsymbol{x};\cdot) * (\nu \boldsymbol{T}_{t-s}) * \boldsymbol{\nu}_{t-s}^{(N-1)}](g),
\end{aligned}
\tag{3.3.9}
$$

where we have used the branching property (3.2.11) of $(\boldsymbol{T}_t)_t$ in the second and the semigroup property in the third equality. Thus the sum of the first two terms on the r.h.s. of (3.3.8) is equal to

$$
\begin{aligned}
&e^{-ct}\boldsymbol{T}_t g(\boldsymbol{x}) + \int_0^t ds\, ce^{-cs} [\boldsymbol{T}_t(\boldsymbol{x};\cdot) * (\nu \boldsymbol{T}_{t-s}) * \boldsymbol{\nu}_{t-s}^{(N-1)}](g) \\
&= e^{-ct}\boldsymbol{T}_t g(\boldsymbol{x}) + \Big[\boldsymbol{T}_t(\boldsymbol{x};\cdot) * \int_0^t ds\, ce^{-cs} [(\nu \boldsymbol{T}_{t-s}) * \boldsymbol{\nu}_{t-s}^{(N-1)}] \Big](g).
\end{aligned}
\tag{3.3.10}
$$

Further, note that

$$
\begin{aligned}
&\int_0^t ds\, ce^{-cs} (\nu \boldsymbol{T}_{t-s}) * \boldsymbol{\nu}_{t-s}^{(N-1)} \\
&= e^{-ct} \int_0^t ds\, ce^{cs} (\nu \boldsymbol{T}_s) * \boldsymbol{\nu}_s^{(N-1)} \\
&= e^{-ct} \int_0^t ds\, c\, (\nu \boldsymbol{T}_s) * \Big(\delta_\Delta + \sum_{n=1}^{N-1} c^n \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n\, (\nu \boldsymbol{T}_{s_1}) * \cdots * (\nu \boldsymbol{T}_{s_n}) \Big) \\
&= e^{-ct} \sum_{n=1}^{N} c^n \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n\, (\nu \boldsymbol{T}_{s_1}) * (\nu \boldsymbol{T}_{s_2}) * \cdots * (\nu \boldsymbol{T}_{s_n}) \\
&= \boldsymbol{\nu}_t^{(N)} - e^{-ct}\delta_\Delta
\end{aligned}
\tag{3.3.11}
$$

by the definition of $\boldsymbol{\nu}_t^{(N)}$, where we have employed a suitable index shift in the penultimate equality. Substituting this into (3.3.10), we see that (3.3.10) equals

$$e^{-ct}\boldsymbol{T}_t g(\boldsymbol{x}) + \Big[\boldsymbol{T}_t(\boldsymbol{x};\cdot) * \Big(\boldsymbol{\nu}_t^{(N)} - e^{-ct}\delta_\Delta\Big)\Big](g) = \Big[\boldsymbol{T}_t(\boldsymbol{x};\cdot) * \boldsymbol{\nu}_t^{(N)}\Big](g). \tag{3.3.12}$$

It remains to consider the third term on the r.h.s. of (3.3.8): From the definition of $R_t^{(N)}$, we get immediately

$$\begin{aligned}
&\int_0^t ds\, ce^{-cs}\, [\boldsymbol{T}_s(\boldsymbol{x}\,;\cdot) * \nu](d\boldsymbol{z}) R_{t-s}^{(N)}(\boldsymbol{z}; g) \\
&= \int_0^t ds\, ce^{-cs}\, [\boldsymbol{T}_s(\boldsymbol{x}\,;\cdot) * \nu](d\boldsymbol{z}) \int_0^{t-s} ds_1\, ce^{-cs_1} \int_{\mathcal{S}} [\boldsymbol{T}_{s_1}(\boldsymbol{z};\cdot) * \nu](d\boldsymbol{z}_1)\cdots \\
&\qquad \cdots \int_0^{t-s-s_1-\cdots-s_{N-1}} ds_N\, ce^{-cs_N} \int_{\mathcal{S}} [\boldsymbol{T}_{s_N} * \nu](d\boldsymbol{z}_N) \boldsymbol{P}_{t-s-s_1-\cdots-s_N} g(\boldsymbol{z}_N) \\
&= R_t^{(N+1)}(\boldsymbol{x}; g).
\end{aligned}$$

Together with (3.3.12), we have thus proved that the r.h.s. of (3.3.8) is equal to

$$\Big[\boldsymbol{T}_t(\boldsymbol{x};\cdot) * \boldsymbol{\nu}_t^{(N)}\Big](g) + R_t^{(N+1)}(\boldsymbol{x}; g),$$

i.e. (3.3.7) for $N+1$.

Now choose $g \equiv 1$ on $\mathcal{S}$. Then by (3.3.7) and nonexplosion of $\boldsymbol{\beta}$, we have

$$1 \ge \boldsymbol{P}_t(\boldsymbol{x};\mathcal{S}) = \underbrace{\boldsymbol{T}_t(\boldsymbol{x};\mathcal{S})}_{=1} \cdot \boldsymbol{\nu}_t^{(N-1)}(\mathcal{S}) + R_t^{(N)}(\boldsymbol{x};1) = \boldsymbol{\nu}_t^{(N-1)}(\mathcal{S}) + R_t^{(N)}(\boldsymbol{x};1) \tag{3.3.13}$$

for all $N \in \mathbb{N}$. Note the inequality sign above: We do not know yet that $\boldsymbol{\eta}$ does not explode. But since for the total mass of $\boldsymbol{\nu}_t^{(N-1)}$ we clearly have

$$\begin{aligned}
\boldsymbol{\nu}_t^{(N-1)}(\mathcal{S}) &= e^{-ct}\left(\delta_\Delta + \sum_{n=1}^{N-1} c^n \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n\, (\nu\boldsymbol{T}_{s_1}) * (\nu\boldsymbol{T}_{s_2}) * \cdots * (\nu\boldsymbol{T}_{s_n})\right)(\mathcal{S}) \\
&= e^{-ct}\left(1 + \sum_{n=1}^{N-1} c^n \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n\right) \\
&= e^{-ct} \sum_{n=0}^{N-1} \frac{(ct)^n}{n!} \xrightarrow{N\uparrow\infty} 1,
\end{aligned}$$

(3.3.13) implies $R_t^{(N)}(\boldsymbol{x};1) \xrightarrow{N\uparrow\infty} 0$ for all $\boldsymbol{x} \in \mathcal{S}$, $t > 0$. Now it follows for arbitrary $g \in \mathscr{B}(\mathcal{S})$ that

$$|R_t^{(N)}(\boldsymbol{x}; g)| \le \|g\|_\infty \cdot R_t^{(N)}(\boldsymbol{x};1) \xrightarrow{N\uparrow\infty} 0, \qquad \boldsymbol{x} \in \mathcal{S},\, t > 0,$$

and (3.3.7) implies

$$\boldsymbol{P}_t g(\boldsymbol{x}) = [\boldsymbol{T}_t(\boldsymbol{x};\cdot) * \boldsymbol{\nu}_t^{(N-1)}](g) + R_t^{(N)}(\boldsymbol{x}; g) \xrightarrow{N\uparrow\infty} [\boldsymbol{T}_t(\boldsymbol{x};\cdot) * \boldsymbol{\nu}_t](g)$$

for all $\boldsymbol{x} \in \mathcal{S}$, $t > 0$. Thus (3.3.3) is proved. ∎

3.3.2 Remark
As an immediate (not very surprising) corollary of Proposition 3.3.1, we see that the BMPI $\boldsymbol{\eta}$ does not explode provided the same holds for its branching component $\boldsymbol{\beta}$. ♦

Several other important corollaries can be deduced with the help of Proposition 3.3.1. The first one gives the action of the semigroup $(\boldsymbol{P}_t)_t$ on multiplicative or additive functions, which is considerably more simple than (3.3.5):

3.3.3 Corollary
Let $\boldsymbol{\eta}$ be a BMPI with branching component $\boldsymbol{\beta}$, immigration rate $c > 0$ and immigration law ν. Assume that $\boldsymbol{\beta}$ does not explode.

1. *Let $f \in \mathscr{B}_1^+(E)$. Then*

$$\boldsymbol{P}_t\hat{f}(\boldsymbol{x}) = \boldsymbol{T}_t\hat{f}(\boldsymbol{x}) \cdot \boldsymbol{\nu}_t(\hat{f}) = \boldsymbol{T}_t\hat{f}(\boldsymbol{x}) \cdot \exp\left(-c\int_0^t \nu\boldsymbol{T}_s(1-\hat{f})\,ds\right) \tag{3.3.14}$$

   for all $\boldsymbol{x} \in \mathcal{S}$, $t > 0$.

2. *Let $f \in \mathscr{B}^+(E)$. Then*

$$\begin{aligned}\boldsymbol{P}_t\bar{f}(\boldsymbol{x}) = \boldsymbol{T}_t\bar{f}(\boldsymbol{x}) + \boldsymbol{\nu}_t(\bar{f}) &= \boldsymbol{T}_t\bar{f}(\boldsymbol{x}) + c\int_0^t ds\,\nu\boldsymbol{T}_s(\bar{f})\\ &= \sum_{i=1}^{\ell(\boldsymbol{x})} M_t f(x^i) + c\int_0^t ds\,\nu M_s(f)\end{aligned} \tag{3.3.15}$$

   for all $\boldsymbol{x} \in \mathcal{S}$, $t > 0$. Here M_t is the kernel from Definition 3.2.2 giving the expected number of particles in the process $\boldsymbol{\beta}$.

Proof The first equality in (3.3.14) resp. (3.3.15) follows at once from the skew convolution property (3.3.3) for $g = \hat{f}$ resp. $g = \bar{f}$, and it remains only to show that $\boldsymbol{\nu}_t(\hat{f})$ resp. $\boldsymbol{\nu}_t(\bar{f})$ have the form claimed in the second equality. For each $t > 0$, we define a measure

$$\tilde{\boldsymbol{\nu}}_t := e^{ct}\boldsymbol{\nu}_t = \delta_\Delta + \sum_{n=1}^{\infty} c^n \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n\,(\nu\boldsymbol{T}_{s_1}) * (\nu\boldsymbol{T}_{s_2}) * \cdots * (\nu\boldsymbol{T}_{s_n})$$

on $\mathcal{S}$ with total mass e^{ct}.

Let $f \in \mathscr{B}_1^+(E)$. Then $\hat{f}$ is bounded on $\mathcal{S}$, and using (3.3.5) with $g := \hat{f}$ it is easy to check that the mapping $t \mapsto \tilde{\boldsymbol{\nu}}_t(\hat{f})$ is differentiable on $\mathbb{R}_+$ with

$$\frac{d}{dt}\tilde{\boldsymbol{\nu}}_t(\hat{f}) = c \cdot \nu\boldsymbol{T}_t(\hat{f}) \cdot \tilde{\boldsymbol{\nu}}_t(\hat{f}).$$

Solving this ODE (e.g. by separation of variables) gives

$$\tilde{\boldsymbol{\nu}}_t(\hat{f}) = \text{const} \cdot \exp\left(c\int_0^t \nu\boldsymbol{T}_s(\hat{f})\,ds\right).$$

Since $\tilde{\boldsymbol{\nu}}_0(\hat{f}) = \hat{f}(\Delta) = 1$, the constant in front must be equal to 1. Consequently,

$$\boldsymbol{P}_t\hat{f}(\boldsymbol{x}) = [\boldsymbol{T}_t(\boldsymbol{x};\cdot) * \boldsymbol{\nu}_t](\hat{f}) = \boldsymbol{T}_t\hat{f}(\boldsymbol{x}) \cdot \boldsymbol{\nu}_t(\hat{f}) = \boldsymbol{T}_t\hat{f}(\boldsymbol{x}) \cdot e^{-ct}\exp\left(c\int_0^t \nu\boldsymbol{T}_s(\hat{f})\,ds\right)$$

which is equal to (3.3.14).

Now let $f \in \mathscr{B}^+(E)$. If it is known that the "semigroup" $(M_t)_t$ from Definition 3.2.2 is in fact a semigroup of bounded operators on $\mathscr{B}(E)$ (as e.g. under (3.2.94)), then the mapping $t \mapsto \tilde{\boldsymbol{\nu}}_t(\bar{f})$ is finite-valued, and as above one can check that it solves the ODE

$$\frac{d}{dt}\tilde{\boldsymbol{\nu}}_t(\bar{f}) = c \cdot \big(\nu \boldsymbol{T}_t(\bar{f}) \cdot \tilde{\boldsymbol{\nu}}_t(\mathcal{S}) + \tilde{\boldsymbol{\nu}}_t(\bar{f})\big) = c \cdot \big(\nu \boldsymbol{T}_t(\bar{f}) \cdot e^{ct} + \tilde{\boldsymbol{\nu}}_t(\bar{f})\big)$$

with initial value $\tilde{\boldsymbol{\nu}}_0(\bar{f}) = 0$, from which we obtain (3.3.15). However, we have not stated boundedness of the kernels M_t as a hypothesis since it is also possible to show (3.3.15) directly without this assumption: Since $\delta_\Delta(\bar{f}) = 0$, from the definition of $\tilde{\boldsymbol{\nu}}_t$ and the additive branching property (3.2.13) we obtain

$$\tilde{\boldsymbol{\nu}}_t(\bar{f}) = \sum_{n=1}^{\infty} c^n \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \sum_{i=1}^{n} \nu \boldsymbol{T}_{s_i}(\bar{f}).$$

Fixing $n \in \mathbb{N}$, the corresponding term in the above series is equal to

$$\begin{aligned}
&c^n \int_{[0,t]^n} ds_1 \cdots ds_n \sum_{i=1}^{n} \nu \boldsymbol{T}_{s_i}(\bar{f})\, \mathbf{1}_{0 \le s_n \le s_{n-1} \le \cdots \le s_{i+1} \le s_i \le s_{i-1} \le \cdots \le s_2 \le s_1 \le t} \\
&= c \int_0^t ds\, \nu \boldsymbol{T}_s(\bar{f}) \cdot \\
&\qquad \cdot \underbrace{c^{n-1} \sum_{i=1}^{n} \int_{[0,t]^{n-1}} ds_1 \cdots ds_{i-1} ds_{i+1} \cdots ds_n\, \mathbf{1}_{0 \le s_n \le s_{n-1} \le \cdots \le s_{i+1} \le s \le s_{i-1} \le \cdots \le s_2 \le s_1 \le t}}_{= t^{n-1}/(n-1)!} \\
&= c \int_0^t ds\, \nu \boldsymbol{T}_s(\bar{f}) \cdot \frac{(ct)^{n-1}}{(n-1)!},
\end{aligned}$$

where in the first equality we have used Fubini's theorem and relabeled s_i as s; the identity used in the second equality can e.g. be shown by induction. Consequently, we get

$$\tilde{\boldsymbol{\nu}}_t(\bar{f}) = c \int_0^t \nu \boldsymbol{T}_s(\bar{f})\, ds \cdot \sum_{n=1}^{\infty} \frac{(ct)^{n-1}}{(n-1)!} = c \int_0^t \nu \boldsymbol{T}_s(\bar{f})\, ds \cdot e^{ct}$$

which yields (3.3.15). ∎

Of course, in (3.3.15) we can replace $M_t f$ resp. $M_s f$ by the series representation (3.2.52) or, provided Assumption 3.2.25 holds, by the representation (3.2.91) in terms of the kernels $(\tilde{T}_t^\gamma)_t$ from (3.2.80). The latter gives a nice formula for the expected number of particles of a BMPI $\boldsymbol{\eta}$ in terms of the auxiliary process $\tilde{X}$ from Definition 3.2.26:

3.3.4 Corollary

Let $\boldsymbol{\eta}$ be a BMPI with branching component $\boldsymbol{\beta}$, immigration rate $c > 0$ and immigration law ν such that $\boldsymbol{\beta}$ does not explode, and grant Assumption 3.2.25. Then for all $\boldsymbol{x} \in \mathcal{S}$ and $B \in \mathcal{B}_E$ we have

$$\begin{aligned}
&\boldsymbol{E}_{\boldsymbol{x}}[\boldsymbol{\eta}_t(B)] \\
&= \sum_{i=1}^{\ell(\boldsymbol{x})} \tilde{T}_t^\gamma(x^i; B) + c \int_0^t \nu \tilde{T}_s^\gamma(x^i; B)\, ds \\
&= \sum_{i=1}^{\ell(\boldsymbol{x})} \boldsymbol{E}_{x^i}\Big[\mathbf{1}_{t<\sigma_\infty} \mathbf{1}_B(\tilde{X}_t) e^{-\int_0^t \gamma(\tilde{X}_s)\, ds}\Big] + c\, \boldsymbol{E}_\nu\Big[\int_0^{t\wedge\sigma_\infty} \mathbf{1}_B(\tilde{X}_s) e^{-\int_0^s \gamma(\tilde{X}_r)\, dr}\, ds\Big].
\end{aligned} \tag{3.3.16}$$

Proof Combine (3.3.15) for $f = \mathbf{1}_B$ with Theorem 3.2.28. ■

We now turn to the question of ergodicity of $\boldsymbol{\eta}$. The main result in this regard is contained in [HL2005], Prop. 2.5; roughly speaking, it says the following: If the branching component $\boldsymbol{\beta}$ goes extinct with finite expected extinction time, then the first return time to the void configuration Δ in the BMPI $\boldsymbol{\eta}$ has finite expectation, i.e. Assumption 1.2.1 is satisfied. In particular, in this case $\boldsymbol{\eta}$ is positive recurrent with finite invariant measure given by (1.2.3).

3.3.5 Theorem
Let $\boldsymbol{\eta}$ be a BMPI with branching component $\boldsymbol{\beta}$, immigration rate $c > 0$ and immigration law ν. Denote by T_e the extinction time in $\boldsymbol{\beta}$ and by R the first return time to Δ in $\boldsymbol{\eta}$ (see (3.2.35) *resp.* (1.2.1)*). If the conditions*

$$\boldsymbol{E}_x [T_e] < \infty, \qquad x \in E \tag{3.3.17}$$

and

$$\boldsymbol{E}_\nu [T_e] < \infty \tag{3.3.18}$$

are satisfied, then we have

$$\boldsymbol{E}_{\boldsymbol{x}} [R] < \infty, \qquad \boldsymbol{x} \in \mathcal{S}. \tag{3.3.19}$$

Proof As already remarked, this result is essentially proved as Prop. 2.5 in [HL2005]. The authors work in a branching diffusion framework and assume the spatial subcriticality condition (3.2.50); however their proof uses only the (weaker) condition (3.3.17). Also, the nature of the single-particle motion is irrelevant in this regard. We give a sketch of the proof, referring the reader to [HL2005] for the details.

Starting from some configuration $\boldsymbol{x} \in \mathcal{S}$, consider the process given by all descendants of the initial population: This process is a BMP without immigration; in fact, it coincides with the branching component $\boldsymbol{\beta}$, and T_e gives the extinction time of this process. Further, let $\tilde{R}$ denote the first return time to Δ in the process given by all particles which do *not* descend from the initial population (i.e., descend from some immigrant). This process is a BMPI which is independent of $\boldsymbol{\beta}$ and distributed as $\boldsymbol{\eta}$ started from the immigration measure ν; in particular, $\tilde{R}$ under $\boldsymbol{P}_{\boldsymbol{x}}$ is distributed as R under $\boldsymbol{P}_\nu$. Finally, as before T_1 denotes the first immigration time in $\boldsymbol{\eta}$.

Now let $\boldsymbol{x} \in \mathcal{S} \smallsetminus \{\Delta\}$ and fix $t > 0$. Then as in step 2 in the proof of Prop. 2.5 in [HL2005], by conditioning on T_1 we get

$$\begin{aligned} \boldsymbol{P}_{\boldsymbol{x}}[R > t] &= \boldsymbol{P}_{\boldsymbol{x}}[T_e > t] + \boldsymbol{P}_{\boldsymbol{x}}[T_1 < T_e \le t, \tilde{R} > t] \\ &\le \boldsymbol{P}_{\boldsymbol{x}}[T_e > t] + \boldsymbol{P}_{\boldsymbol{x}}[T_1 < T_e, \tilde{R} > t] \\ &= \boldsymbol{P}_{\boldsymbol{x}}[T_e > t] + \int_0^t ds\, ce^{-cs} \boldsymbol{P}_{\boldsymbol{x}}[s < T_e] \cdot \boldsymbol{P}_\nu[R > t - s]. \end{aligned} \tag{3.3.20}$$

Taking $x \in E$ and integrating w.r.t. $\nu(dx)$ in (3.3.20), we obtain

$$\boldsymbol{P}_\nu[R > t] \le \boldsymbol{P}_\nu[T_e > t] + \int_0^t \boldsymbol{P}_\nu[R > t - s]\, \mu(ds),$$

where $\mu(ds) := ce^{-cs} \boldsymbol{P}_\nu[s < T_e]\, ds$ is a measure on $\mathbb{R}_+$ with total mass strictly smaller than 1. Since (3.3.18) is assumed, from the above recursion inequality it can be deduced that

$$\boldsymbol{E}_\nu[R] = \int_0^\infty \boldsymbol{P}_\nu[R > t]\, dt < \infty; \tag{3.3.21}$$

at this point we refer to the proof in [HL2005], p. 1036, for the details. Once (3.3.21) is established, we get the same for arbitrary starting values $\boldsymbol{x} \in \mathcal{S} \smallsetminus \{\Delta\}$: Indeed, by assumption (3.3.17) and subadditivity (3.2.43) of $\boldsymbol{x} \mapsto \boldsymbol{E_x}[T_e]$ we get $\boldsymbol{E_x}[T_e] < \infty$ for all $\boldsymbol{x} \in \mathcal{S}$, and thus from (3.3.20)

$$\begin{aligned}\boldsymbol{E_x}[R] = \int_0^\infty \boldsymbol{P_x}[R > t]\,dt \le \int_0^\infty \boldsymbol{P_x}[T_e > t]\,dt + \int_0^\infty dt \int_0^t ds\, ce^{-cs}\boldsymbol{P}_\nu[R > t - s] \\ = \boldsymbol{E_x}[T_e] + \boldsymbol{E}_\nu[R] < \infty\end{aligned}$$

for $\boldsymbol{x} \in \mathcal{S} \smallsetminus \{\Delta\}$. On the other hand, for $\boldsymbol{x} = \Delta$ it is clear that

$$\boldsymbol{E}_\Delta[R] = \boldsymbol{E}_\Delta[T_1] + \boldsymbol{E}_\Delta[R \circ \boldsymbol{\theta}_{T_1}] = 1/c + \boldsymbol{E}_\nu[R] < \infty,$$

thus (3.3.19) is proved. ■

3.3.6 Remarks

- Assumption (3.3.17) of Theorem 3.3.5 is of course in particular satisfied if the branching component $\boldsymbol{\beta}$ of $\boldsymbol{\eta}$ is spatially subcritical in the sense of Condition 3.2.19 of the previous section. Moreover, if the uniform version (3.2.51) of the spatial subcriticality condition holds, then (3.3.18) is satisfied for any choice of immigration measure $\nu \in \mathcal{M}_1(E)$.

- Whenever the conditions (3.3.17) and (3.3.18) are satisfied, we conclude by Theorem 3.3.5 and Proposition 1.2.3 that the BMPI $\boldsymbol{\eta}$ is positive Harris recurrent with (finite) invariant measure m given by (1.2.3). In this case, putting $f \equiv 0$ on E we have $\hat{f} = \mathbf{1}_\Delta$ on $\mathcal{S}$, and from (3.3.14) we get

$$\begin{aligned}\boldsymbol{E_x}[\boldsymbol{\eta}_t = \Delta] \equiv \boldsymbol{P}_t\hat{f}(\boldsymbol{x}) &= \boldsymbol{T}_t\hat{f}(\boldsymbol{x}) \cdot \exp\left(-c\int_0^t \nu\boldsymbol{T}_s(1 - \hat{f})\,ds\right) \\ &= \boldsymbol{P_x}[\boldsymbol{\beta}_t = \Delta] \cdot \exp\left(-c\int_0^t \boldsymbol{P}_\nu[\boldsymbol{\beta}_s \neq \Delta]\,ds\right) \\ &= \boldsymbol{P_x}[T_e \le t] \cdot \exp\left(-c\int_0^t \boldsymbol{P}_\nu[T_e > s]\,ds\right)\end{aligned} \tag{3.3.22}$$

  for all $\boldsymbol{x} \in \mathcal{S}$ and $t > 0$. We remark that for $\boldsymbol{x} = \Delta$, the foregoing is a direct spatial analogue of a classical formula for (nonspatial) Galton-Watson processes with immigration, see the equation on top of p. 182 in [Zub1972]. By extinction and since (3.3.18) is assumed, from (3.3.22) we get

$$\boldsymbol{P}_t\hat{f}(\boldsymbol{x}) \xrightarrow{t\uparrow\infty} 1 \cdot \exp\left(-c\int_0^\infty \boldsymbol{P}_\nu[T_e > s]\,ds\right) = \exp\left(-c\,\boldsymbol{E}_\nu[T_e]\right)$$

  for all $\boldsymbol{x} \in \mathcal{S}$. On the other hand, by the ergodic theorem for Harris recurrent processes[33] we have

$$\frac{1}{t}\int_0^t \boldsymbol{P}_s\hat{f}(\boldsymbol{x})\,ds \equiv \frac{1}{t}\boldsymbol{E_x}\left[\int_0^t \hat{f}(\boldsymbol{\eta}_s)\,ds\right] \xrightarrow{t\uparrow\infty} \frac{m(\hat{f})}{m(\mathcal{S})} = \frac{m(\Delta)}{\boldsymbol{E}_\Delta[R]}, \qquad m\text{-a.e. } \boldsymbol{x} \in \mathcal{S}.$$

  Comparison of the previous two displays shows that $\exp(-c\boldsymbol{E}_\nu[T_e]) = m(\Delta)/\boldsymbol{E}_\nu[R]$, and since $m(\Delta) = \boldsymbol{E}_\Delta[T_1] = 1/c$ we obtain the following nice formula expressing the

[33] See e.g. [ADR1967], Thm. II.1 on p. 166 or [ADR1969], Thm. 3.1 on p. 30.

expected return time to the void configuration in the BMPI $\boldsymbol{\eta}$ in terms of the expected extinction of the BMP $\boldsymbol{\beta}$ and the immigration mechanism:

$$\boldsymbol{E}_\Delta[R] = \frac{1}{c} \cdot \exp\left(c\, \boldsymbol{E}_\nu[T_e]\right). \tag{3.3.23}$$

Again this equation can be interpreted as a spatial analogue of a well-known formula for the expected return time to 0 in "classical" Galton-Watson processes with immigration, see [Zub1972], Thm. 1' on p. 182.[34] ⧫

Under the assumptions of Theorem 3.3.5, the arguments given in the last remark work in fact for any $f \in \mathscr{B}_1^+(E)$, not just $f \equiv 0$. In particular, for all $f \in \mathscr{B}_1^+(E)$ we have $0 \le 1 - \hat{f} \le 1 - \mathbf{1}_\Delta = \mathbf{1}_{\{\Delta\}^c}$ (recall that $\hat{f}(\Delta) = 1$); thus we can use dominated convergence in (3.3.22) to deduce convergence of $\boldsymbol{P}_t \hat{f}(\boldsymbol{x})$ as $t \uparrow \infty$. If the state space of unordered configurations resp. finite point measures is employed (i.e. $\mathcal{S} = \mathcal{M}_f^p(E)$), this can be turned into a proof that the BMPI $\boldsymbol{\eta}$ converges to its (normalized) invariant measure in distribution. The argument uses the embedding of $\mathcal{M}_f^p(E)$ in the larger space $\mathcal{M}_f(E)$ of all finite measures on E; unfortunately, it does not work for the case of ordered configurations: Recall that for a probability measure $\boldsymbol{P}$ on $\mathcal{M}_f(E)$, its *Laplace functional* is defined by

$$\mathcal{L}_{\boldsymbol{P}}(f) := \int_{\mathcal{M}_f(E)} e^{-\langle \mu, f\rangle} \boldsymbol{P}(d\mu), \qquad f \in \mathscr{B}^+(E); \tag{3.3.24}$$

here $\langle \mu, f\rangle \equiv \mu(f)$ denotes the integral of f w.r.t. the measure $\mu \in \mathcal{M}_f(E)$. Note that if $\mu = \boldsymbol{x} \in \mathcal{M}_f^p(E) \subseteq \mathcal{M}_f(E)$ is a finite point measure, we have

$$\langle \boldsymbol{x}, f\rangle = \boldsymbol{x}(f) = \sum_{i=1}^{\ell(\boldsymbol{x})} f(x^i) = \bar{f}(\boldsymbol{x})$$

and thus

$$e^{-\langle \boldsymbol{x}, f\rangle} = e^{-\bar{f}(\boldsymbol{x})} = \widehat{e^{-f}}(\boldsymbol{x}), \qquad f \in \mathscr{B}^+(E).$$

Consequently, for a probability measure $\boldsymbol{P}$ on $\mathcal{M}_f(E)$ which is concentrated on the finite point measures $\mathcal{M}_f^p(E)$, the Laplace functional takes the form

$$\mathcal{L}_{\boldsymbol{P}}(f) = \int_{\mathcal{S}} \widehat{e^{-f}}(\boldsymbol{x}) \boldsymbol{P}(d\boldsymbol{x}). \tag{3.3.25}$$

In complete analogy with probability measures on $\mathbb{R}^d$, there is a "Continuity Theorem" for the Laplace functional (see e.g. [Li2002], Lemma 2.1): If $(\boldsymbol{P}_n)_n$ is a sequence of probability measures on $\mathcal{M}_f(E)$ such that $\lim_{n\to\infty} \mathcal{L}_{\boldsymbol{P}_n}(f) =: \mathcal{L}(f)$ exists for all $f \in \mathscr{B}^+(E)$ and $\mathcal{L}(f) \to 1$ as $f \to 0$, then the functional $\mathcal{L}$ is the Laplace functional of some probability measure $\boldsymbol{P}$ on $\mathcal{M}_f(E)$ such that $\boldsymbol{P}_n \xrightarrow{n\to\infty} \boldsymbol{P}$ weakly.

3.3.7 Theorem

For $\mathcal{S} = \mathcal{M}_f^p(E)$, under the assumptions of Theorem 3.3.5 the BMPI $\boldsymbol{\eta}$ converges in distribution to its (normalized) invariant measure, for every choice of a starting configuration $\boldsymbol{x} \in \mathcal{S}$:

$$\mathcal{L}(\boldsymbol{\eta}_t \,|\, \boldsymbol{P}_{\boldsymbol{x}}) \xrightarrow{t\uparrow\infty} \frac{1}{\boldsymbol{E}_\Delta[R]} \cdot m, \qquad \boldsymbol{x} \in \mathcal{S}, \tag{3.3.26}$$

where m is defined in (1.2.3).

[34] Zubkov's theorem concerns the expected hitting time of the state 0, starting from a positive number of particles, but modified accordingly his formula coincides with equation (3.3.23).

Proof Let $\boldsymbol{x} \in \mathcal{S}$ and $f \in \mathscr{B}^+(E)$. Then putting $h := e^{-f}$, the function $\hat{h} = \widehat{e^{-f}}$ is in $\mathscr{B}_1^+(\mathcal{S})$ and strictly positive. Choosing $\boldsymbol{P} = \boldsymbol{P}_t(\boldsymbol{x};\cdot)$ in (3.3.25) for $t > 0$ and using (3.3.14) gives

$$\mathcal{L}_{\boldsymbol{P}_t(\boldsymbol{x};\cdot)}(f) = \boldsymbol{P}_t\hat{h}(\boldsymbol{x}) = \boldsymbol{T}_t\hat{h}(\boldsymbol{x}) \cdot \boldsymbol{\nu}_t(\hat{h}) = \boldsymbol{T}_t\hat{h}(\boldsymbol{x}) \cdot \exp\left(-c\int_0^t \nu\boldsymbol{T}_s(1-\hat{h})\,ds\right). \tag{3.3.27}$$

In view of the above-mentioned Continuity Theorem for the Laplace functional, we need only prove that the above converges for arbitrary choice of $f \in \mathscr{B}^+(E)$ as $t \uparrow \infty$, and that the limit is continuous in $f \equiv 0$.

We remark in passing that formula (3.3.27) is a representation of the Laplace functional of $\boldsymbol{P}_t(\boldsymbol{x};\cdot)$ in terms of the Laplace functional of $\boldsymbol{T}_t(\boldsymbol{x};\cdot)$. In particular, for $\boldsymbol{x} = \Delta$ we have $\boldsymbol{T}_t\hat{h}(\Delta) = 1$ and (3.3.27) can be rewritten as

$$\mathcal{L}_{\boldsymbol{P}_t(\Delta;\cdot)}(f) = \exp\left(-c\int_0^t \nu\boldsymbol{T}_s(1-\hat{h})\,ds\right) = \exp\left(-c\int_0^t \left(1-\mathcal{L}_{\nu\boldsymbol{T}_s}(f)\right)\,ds\right),$$

which again is a spatial analogue of a well-known classical formula for the generating function of a (nonspatial) branching process with immigration (see e.g. [Har1963], eqn. (16.3) on p. 118 or [Sev1957], eqn. (12) on p. 323).

We return to (3.3.27). For all $f \in \mathscr{B}^+(E)$ and $h := e^{-f}$, we have $0 \le 1-\hat{h} \le 1-\mathbf{1}_\Delta = \mathbf{1}_{\{\Delta\}^c}$ (remember that $\hat{h}(\Delta) = 1$) and thus

$$\begin{aligned}\int_0^t \nu\boldsymbol{T}_s(1-\hat{h})\,ds \le \int_0^t \nu\boldsymbol{T}_s(\mathbf{1}_{\{\Delta\}^c})ds &= \int_0^t \boldsymbol{E}_\nu\left[\boldsymbol{\beta}_s \neq \Delta\right]\,ds \\ &= \int_0^t \boldsymbol{P}_\nu\left[T_e > s\right]\,ds \xrightarrow{t\uparrow\infty} \boldsymbol{E}_\nu\left[T_e\right] < \infty\end{aligned} \tag{3.3.28}$$

since (3.3.18) is assumed. Further, from (3.3.17) we get in particular extinction of $\boldsymbol{\beta}$, thus $\boldsymbol{\beta}_t \xrightarrow{t\uparrow\infty} \Delta$ $\boldsymbol{P}_{\boldsymbol{x}}$-a.s. and by dominated convergence

$$\boldsymbol{T}_t\hat{h}(\boldsymbol{x}) \xrightarrow{t\uparrow\infty} \hat{h}(\Delta) = 1, \qquad \boldsymbol{x} \in \mathcal{S}. \tag{3.3.29}$$

Taking together (3.3.27), (3.3.28) and (3.3.29), for all $\boldsymbol{x} \in \mathcal{S}$ and $f \in \mathscr{B}^+(E)$ we get

$$\mathcal{L}_{\boldsymbol{P}_t(\boldsymbol{x};\cdot)}(f) \equiv \boldsymbol{P}_t\hat{h}(\boldsymbol{x}) \xrightarrow{t\uparrow\infty} \exp\left(-c\int_0^\infty \nu\boldsymbol{T}_s(1-\hat{h})\,ds\right) =: \mathcal{L}(f),$$

with $h \equiv e^{-f}$. Now consider a sequence $f_n \in \mathscr{B}^+(E)$ with $f_n \xrightarrow{n\to\infty} 0$ pointwise. Put $h_n := e^{-f_n}$. Then $\hat{h}_n \xrightarrow{n\to\infty} 1$ pointwise on $\mathcal{S}$, $\nu\boldsymbol{T}_s(1-\hat{h}_n) \xrightarrow{n\to\infty} 0$ for all $s \in \mathbb{R}_+$, and

$$\int_0^\infty \nu\boldsymbol{T}_s(1-\hat{h}_n)\,ds \xrightarrow{n\to\infty} 0$$

by dominated convergence since $\nu\boldsymbol{T}_s(1-\hat{h}_n)$ is dominated for all $n \in \mathbb{N}$ by $\boldsymbol{P}_\nu[T_e > s]$ which is integrable in $s \in \mathbb{R}_+$ (see the argument leading to (3.3.28) above). Consequently

$$\mathcal{L}(f_n) = \exp\left(-c\int_0^\infty \nu\boldsymbol{T}_s(1-\hat{h}_n)\,ds\right) \xrightarrow{n\to\infty} 1,$$

and $\mathcal{L}$ is continuous in $f = 0$. By the Continuity Theorem, $\mathcal{L}$ is the Laplace functional of some probability measure $\tilde{m}$ on $\mathcal{M}_f(E)$ such that $\mathcal{L}(\boldsymbol{\eta}_t \,|\, \boldsymbol{P}_{\boldsymbol{x}}) \xrightarrow{t\uparrow\infty} \tilde{m}$ for all $\boldsymbol{x} \in \mathcal{S}$. As weak limit,

$\tilde{m}$ is necessarily the unique invariant probability measure for $(\boldsymbol{P}_t)_t$, hence it must coincide with the normalized version of m of (1.2.3). ∎

We now turn to the study of the invariant occupation measure $\overline{m}$ of (1.2.13). Again, the results in this regard are generalizations of those in [HL2005] (see in particular their Thms. 1.6, 1.7 on pp. 1029ff.). The first result states that under positive Harris recurrence of the BMPI $\boldsymbol{\eta}$, the measure $\overline{m}$ (whether finite or not) coincides up to a constant with the measure $\nu H(\cdot) \equiv \int_E \nu(dx)H(x;\cdot)$, where H is the occupation times kernel of (3.2.48):

3.3.8 Proposition
Let $\boldsymbol{\eta}$ be a BMPI with branching component $\boldsymbol{\beta}$, immigration rate $c > 0$ and immigration measure ν. Suppose that Assumption 1.2.1 holds for $\boldsymbol{\eta}$; thus in particular $\boldsymbol{\eta}$ is positive Harris recurrent with Δ as recurrent atom and finite invariant measure m on $(\mathcal{S}, \mathcal{B}_\mathcal{S})$ as in (1.2.3). *Then the invariant occupation measure $\overline{m}$ on $(E, \mathcal{B}_E)$ is given by*

$$\overline{m}(B) = c\,\boldsymbol{E}_\Delta[R]\cdot \nu H(B) \equiv c\,\boldsymbol{E}_\Delta[R]\int_E \nu(dx)H(x;B), \qquad B \in \mathcal{B}_E, \tag{3.3.30}$$

where $H(\cdot\,;\cdot) : E \times \mathcal{B}_E \to [0,\infty]$ denotes the occupation times kernel (3.2.48) *for the branching component $\boldsymbol{\beta}$.*

Proof This is essentially proved as Prop. 2.6 in [HL2005], and again the proof goes through without changes in our more general framework. Nevertheless, we give a slightly different proof using the skew convolution property of the semigroup $(\boldsymbol{P}_t)_t$.

Since the invariant measure m of (1.2.3) has total mass $m(\mathcal{S}) = \boldsymbol{E}_\Delta[R]$, by the ergodic theorem for Harris recurrent processes[35] we know that for all $g \in L^1(\mathcal{S}; m)$

$$\frac{1}{t}\int_0^t \boldsymbol{P}_s g(\boldsymbol{x})\,ds \xrightarrow{t\uparrow\infty} \frac{m(g)}{\boldsymbol{E}_\Delta[R]}, \qquad m\text{-a.e. } \boldsymbol{x} \in \mathcal{S}. \tag{3.3.31}$$

By a simple monotone convergence argument, (3.3.31) clearly extends to all nonnegative measurable $g : \mathcal{S} \to \mathbb{R}_+$ where the limit is equal to $+\infty$ if $g \notin L^1(\mathcal{S}; m)$. Moreover, (3.3.31) must in particular hold for $\boldsymbol{x} = \Delta$ since $m(\Delta) > 0$.

Now let $f \in \mathscr{B}^+(E)$. Then (3.3.31) for $g := \bar{f} \geq 0$ and $\boldsymbol{x} := \Delta$ yields

$$\frac{1}{t}\int_0^t \boldsymbol{P}_s \bar{f}(\Delta)\,ds \xrightarrow{t\uparrow\infty} \frac{m(\bar{f})}{\boldsymbol{E}_\Delta[R]} = \frac{\overline{m}(f)}{\boldsymbol{E}_\Delta[R]} \leq \infty. \tag{3.3.32}$$

On the other hand, the assumption of recurrence of $\boldsymbol{\eta}$ implies in particular that its branching component $\boldsymbol{\beta}$ cannot explode. Consequently, by (3.3.15) we know that

$$\boldsymbol{P}_t \bar{f}(\Delta) = c\int_0^t \nu \boldsymbol{T}_s(\bar{f})\,ds \xrightarrow{t\uparrow\infty} c\int_0^\infty \nu \boldsymbol{T}_s(\bar{f})\,ds = c\cdot \nu H(f)$$

and thus also

$$\frac{1}{t}\int_0^t \boldsymbol{P}_s g(\Delta)\,ds \xrightarrow{t\uparrow\infty} c\cdot \nu H(f). \tag{3.3.33}$$

Comparison of (3.3.32) and (3.3.33) yields the desired result. ∎

The main result concerning finiteness of the invariant occupation measure $\overline{m}$ (corresponding to Thm. 1.6 in [HL2005]) is the following:

[35] [ADR1967], Thm. II.1 on p. 166 or [ADR1969], Thm. 3.1 on p. 30.

3.3.9 Theorem

1. *For a BMPI* $\boldsymbol{\eta}$ *with branching component* $\boldsymbol{\beta}$ *and immigration law* ν*, the following are equivalent:*

   - *Condition* (3.3.17) *of Theorem 3.3.5 is fulfilled, and the measure* νH *is finite.*
   - *Assumption 1.2.1 holds and the invariant occupation measure* $\overline{m}$ *of* (1.2.13) *is finite.*

   In particular, this holds if $\boldsymbol{\beta}$ *is spatially subcritical in the sense of Condition 3.2.19 and* νH *is finite.*

2. *Let* $\boldsymbol{\beta}$ *be a BMP. Then the following are equivalent:*

   - $\boldsymbol{\beta}$ *is uniformly spatially subcriticality in the sense of Condition 3.2.19.*
   - *For all* $\nu \in \mathcal{M}_1(E)$*, the BMPI obtained by taking* $\boldsymbol{\beta}$ *as branching component and* ν *as immigration law satisfies Assumption 1.2.1 with finite invariant occupation measure* $\overline{m}$*.*

In all of the above cases, $\overline{m}$ *is given by* (3.3.30)*.*

Proof

1. Condition (3.3.17) entails extinction and thus also nonexplosion of $\boldsymbol{\beta}$. Together with finiteness of νH, this implies that also condition (3.3.18) is satisfied (see (3.2.46)). Thus Theorem 3.3.5 applies and says that Assumption 1.2.1 holds for the BMPI $\boldsymbol{\eta}$. Proposition 3.3.8 then implies that $\overline{m}$ is of the form (3.3.30), in particular it is finite.

   Conversely, suppose that Assumption 1.2.1 holds and that $\overline{m}$ is finite. Again by Proposition 3.3.8, $\overline{m}$ is necessarily of the form (3.3.30), thus νH must be a finite measure. Moreover, (3.3.17) follows directly from (1.2.2).

   In addition, observe that spatial subcriticality in the sense of Condition 3.2.19 means by definition that $\boldsymbol{\beta}$ does not explode and that the occupation times kernel $H(\cdot;\cdot)$ is finite. Together, this implies condition (3.3.17), again see (3.2.46).

2. If $\boldsymbol{\beta}$ is uniformly spatially subcritical, this means that it does not explode and that the kernel H is bounded (see (3.2.51)). Then obviously νH is finite for all $\nu \in \mathcal{M}_1(E)$, and the desired conclusion follows from the first part of the theorem.

   Conversely, if Assumption 1.2.1 holds, $\overline{m}$ is necessarily given by (3.3.30) in view of Proposition 3.3.8. But if the r.h.s. of (3.3.30) is finite for *all* $\nu \in \mathcal{M}_1(E)$, it is easy to see that the kernel H must be bounded (see the argument in [HL2005], proof of Prop. 2.7).

■

Continuing Example 3.2.16, we see that finiteness of the invariant occupation measure $\overline{m}$ on $(E, \mathcal{B}_E)$ is a strictly stronger condition than finiteness of the invariant measure m on $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$:

3.3.10 Example
Consider a BMP $\boldsymbol{\beta}$ with arbitrary spatial motion and offspring distribution but with spatially constant branching rate $\kappa > 0$ and reproduction law $(p_k)_k$ as in (3.2.44). Then the total mass

process $Z^{\boldsymbol{\beta}}$ is a classical Galton-Watson process, and the expected extinction time $\boldsymbol{E}_x[T_e]$ is a constant function of $x \in E$ and is finite by the analysis in Example 3.2.16. Taking any $\nu \in \mathcal{M}_1(E)$ as immigration measure, we can apply Theorem 3.3.5 to conclude that Assumption 1.2.1 is fulfilled, in particular the invariant measure m of (1.2.3) is finite.

On the other hand, since κ is constant and $(p_k)_k$ is critical, by the series representation (3.2.54) of the occupation times kernel H or by its representation as a generalized resolvent of the auxiliary process $\tilde{X}$ as in (3.2.95)[36] that $H(x;E) = \infty$ for all $x \in E$. By Proposition 3.3.8, $\overline{m}$ is given up to a constant by the measure νH, consequently it cannot be finite for any choice of an immigration measure $\nu \in \mathcal{M}_1(E)$. ⧫

We conclude with a characterization of the invariant occupation measure $\overline{m}$ as the unique solution to a "balance equation" as in [HL2005], eqn. (10). Theorem 3.3.11 below is the generalization of their Thm. 1.7 and Prop. 4.1 to our present context. The arguments given in the proof of Prop. 4.1 in [HL2005] rely on the single particle motion X being a diffusion which can be constructed as a stochastic flow of diffeomorphisms, and also on the assumption $\tilde{Q}(x;\cdot) = \delta_x(\cdot)$ that branching particles reproduce at their parent's death position. However, assuming that we are in the framework of Remark 3.2.7 where X, κ, ϱ and $\tilde{Q}$ satisfy certain regularity conditions, one can adapt their proof using arguments from general semigroup theory: Under these conditions, the result follows straightforwardly from the fact that the action of the semigroup $(\boldsymbol{P}_t)_t$ on additive functions is given by (3.3.15). The proof uses some general facts from the theory of (Feller) semigroups for which we refer the reader e.g. to [Kal2002], Ch. 19. In addition, let us recall the concept of a *core* from operator theory: If $(A, D(A))$ is a closed linear operator on the space $\mathcal{C}_0(E)$ of continuous functions vanishing at infinity, then a subspace $\mathcal{D} \subseteq D(A)$ is called a core for A if the closure of $A|_{\mathcal{D}}$ is equal to A, i.e. if for every $f \in D(A)$ there exists a sequence $f_n \in \mathcal{D}$ with $\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0$ and $\|Af_n - Af\|_\infty \xrightarrow{n\to\infty} 0$. Of course, $D(A)$ itself is always a core for A. For more information on this concept, we refer the reader e.g. to [EK1986], Ch. 1 Sec. 3; see also the Remarks 3.3.12 below.

3.3.11 Theorem

Let $\boldsymbol{\eta}$ be a BMPI with branching component $\boldsymbol{\beta}$ which satisfies the following regularity conditions: The single particle motion X is a Feller process, i.e. $T_t(\mathcal{C}_0(E)) \subseteq \mathcal{C}_0(E)$, with infinitesimal generator denoted by $(A, D(A))$.[37] Moreover, we have $\tilde{Q}(\mathcal{C}_0(E)) \subseteq \mathcal{C}_0(E)$ and $\kappa, \varrho \in \mathcal{C}_b(E)$.[38]

1. *If $\boldsymbol{\beta}$ is spatially subcritical in the sense of Condition 3.2.19 and*
$$\nu H(E) = \nu \tilde{R}_\gamma(E) < \infty, \tag{3.3.34}$$
then the measure $\overline{m}$ (which is finite by Theorem 3.3.9) fulfills the balance equation
$$\overline{m}\left(Af - \kappa f + \kappa\varrho\tilde{Q}f\right) = -c\,\boldsymbol{E}_\Delta[R]\cdot\nu(f), \qquad f \in D(A). \tag{3.3.35}$$

2. *Let $\mathcal{D}$ be a core for A. If $\boldsymbol{\beta}$ satisfies the uniform spatial subcriticality condition* (3.2.51), *then the measure $\overline{m}$ is the unique element $\mu \in \mathcal{M}_f(E)$ such that*
$$\mu\left(Af - \kappa f + \kappa\varrho\tilde{Q}f\right) = -c\,\boldsymbol{E}_\Delta[R]\cdot\nu(f), \qquad f \in \mathcal{D}. \tag{3.3.36}$$

[36] Note that $\sigma_\infty = \infty$ in (3.2.95) since κ and ϱ are constants.

[37] See (3.2.27). As generator of a Feller semigroup, A is necessarily closed.

[38] Note that the assumption that κ and ϱ are bounded implies $M_t = \tilde{T}_t^\gamma$ and $H = \tilde{R}_\gamma$, where $\tilde{T}_t^\gamma$ resp. $\tilde{R}_\gamma$ are the semigroup resp. generalized resolvent defined in terms of the auxiliary process $\tilde{X}$ in (3.2.80) resp. (3.2.95).

Proof First of all, we recall from Remark 3.2.7 that the regularity conditions on X, κ, ϱ and $\tilde{Q}$ imposed above imply that the family of kernels $(M_t)_t$ of Definition 3.2.2 induces a strongly continuous semigroup of bounded operators on $\mathcal{C}_0(E)$ with generator

$$Lf := Af - \kappa f + \kappa\varrho\tilde{Q}f, \qquad f \in D(L) = D(A),$$

and equation (3.3.35) reads

$$\overline{m}\,(Lf) = -c\,\boldsymbol{E}_\Delta[R] \cdot \nu(f), \qquad f \in D(L). \tag{3.3.37}$$

As the generator of a strongly continuous semigroup of bounded operators, L is closed. Moreover, if $\mathcal{D}$ is a core for A then it is also a core for L since the latter is a bounded perturbation of A on $\mathcal{C}_0(E)$. More precisely, for $f \in D(L) = D(A)$ there exists a sequence $f_n \in \mathcal{D}$ such that $f_n \xrightarrow{n\to\infty} f$ and $Af_n \xrightarrow{n\to\infty} Af$ uniformly on E, whence under our assumptions we get

$$Lf_n = Af_n - \kappa f_n + \kappa\varrho\tilde{Q}f_n \xrightarrow{n\to\infty} Af - \kappa f + \kappa\varrho\tilde{Q}f = Lf$$

uniformly on E. Consequently, if μ is a finite measure on $(E, \mathcal{B}_E)$ such that (3.3.36) holds for all $f \in \mathcal{D}$, it holds also for all $f \in D(L) = D(A)$ by dominated convergence.

1. We prove the balance equation (3.3.35) for all $f \in D(A)$.

   For $f \in \mathscr{B}^+(E)$, we know from (3.3.15) that $\boldsymbol{P}_t\bar{f}$ is a nonnegative *finite-valued* function on $\mathcal{S}$ given by

$$\boldsymbol{P}_t\bar{f} = \overline{M_t f} + c\int_0^t ds\,\nu M_s(f) = \overline{\tilde{T}_t^\gamma f} + c\int_0^t ds\,\nu\tilde{T}_s^\gamma(f). \tag{3.3.38}$$

   Now let $f \in D(A)$; in particular, f is bounded. Then, although $\bar{f}$ is in general neither nonnegative nor bounded on $\mathcal{S}$, writing $\bar{f} = \overline{f^+ - f^-} = \overline{f^+} - \overline{f^-}$ we see that $\boldsymbol{P}_t\bar{f} = \boldsymbol{P}_t\overline{f^+} - \boldsymbol{P}_t\overline{f^-}$ is well-defined as a finite-valued function on $\mathcal{S}$ which is given by (3.3.38). Moreover, since $\overline{m}$ is finite, $\boldsymbol{P}_t\bar{f}$ is integrable w.r.t. m, as we see from

$$m(\boldsymbol{P}_t\bar{f}) = m(\boldsymbol{P}_t\overline{f^+}) - m(\boldsymbol{P}_t\overline{f^-}) = m(\overline{f^+}) - m(\overline{f^-}) = m(\bar{f}) = \overline{m}(f),$$

   using the invariance of m. Consequently, combining the last two displays we get

$$\begin{aligned} 0 &\equiv \frac{m(\boldsymbol{P}_t\bar{f}) - m(\bar{f})}{t} = m\left(\frac{\boldsymbol{P}_t\bar{f} - \bar{f}}{t}\right) \\ &= m\left(\frac{\overline{M_t f} - \bar{f}}{t}\right) + m(\mathcal{S}) \cdot c \cdot \frac{1}{t}\int_0^t ds\,\nu M_s(f) \\ &= \overline{m}\left(\frac{M_t f - f}{t}\right) + c\,\boldsymbol{E}_\Delta[R] \cdot \frac{1}{t}\int_0^t ds\,\nu M_s(f). \end{aligned} \tag{3.3.39}$$

   Since $f \in D(A) = D(L)$ and L is the generator of $(M_t)_t$, we have $\dfrac{M_t f - f}{t} \to Lf = Af - \kappa f + \kappa\varrho\tilde{Q}f$ uniformly on E as $t \downarrow 0$. Thus the first term in (3.3.39) tends to $\overline{m}(Af - \kappa f + \kappa\varrho\tilde{Q}f)$ by dominated convergence. The second term converges to $c\,\boldsymbol{E}_\Delta[R] \cdot \nu(f)$ since $\nu M_t(f) \to \nu(f)$ as $t \downarrow 0$ by the strong continuity of the semigroup $(M_t)_t$. Thus equation (3.3.35) is proved.

2. Now grant the uniform version (3.2.51) of the spatial subcriticality condition. Let $\mathcal{D}$ be a core for A, and let $\mu \in \mathcal{M}_f(E)$ be any finite measure such that (3.3.36) holds for all $f \in \mathcal{D}$. As already stated above, by dominated convergence (3.3.36) then holds also for all $f \in D(L) = D(A)$. It remains to show that this implies that μ coincides with $\overline{m}$.

   Let $f \in D(L)$. Since L is the generator of the (strongly continuous) semigroup $(M_t)_t$ on $\mathcal{C}_0(E)$, we have

$$M_t f(x) - f(x) = \int_0^t L M_s f(x)\, ds = \int_0^t M_s L f(x)\, ds, \qquad x \in E,\ t > 0.$$

   Integrating w.r.t. μ and using (3.3.36), we obtain

$$\begin{aligned}\mu(f) - \mu(M_t f) = -\mu\left(\int_0^t L M_s f\, ds\right) &= -\int_0^t \mu(L M_s f)\, ds \\ &= c\,\boldsymbol{E}_\Delta[R] \cdot \int_0^t \nu M_s(f)\, ds\end{aligned} \tag{3.3.40}$$

   for all $f \in D(L)$. Note that the interchange of the order of integration in the above display is permissible since

$$\|L M_s f\|_\infty = \|M_s L f\|_\infty \le \|M_s\|_{\infty\to\infty} \cdot \|Lf\|_\infty \le \text{const}\, \|Lf\|_\infty, \qquad s \in [0,t].$$

   Now observe that the r.h.s. in (3.3.40) converges to

$$c\,\boldsymbol{E}_\Delta[R] \cdot \int_0^\infty \nu M_s(f)\, ds = c\,\boldsymbol{E}_\Delta[R] \cdot \nu H(f) = \overline{m}(f),$$

   since under our assumptions $\overline{m}$ is given by (3.3.30). The l.h.s. in (3.3.40) converges to $\mu(f)$ in Césaro-average since

$$\begin{aligned}\left|\frac{1}{T}\int_0^T \mu(M_t f)\, dt\right| &= \left|\frac{1}{T}\int_E \mu(dx) \int_0^T dt\, M_t f(x)\right| \\ &\le \frac{1}{T} \cdot \|f\|_\infty \int_E \mu(dx) \int_0^T M_t(x;E)\, dt \\ &\le \frac{1}{T} \cdot \|f\|_\infty \cdot \mu H(E) \xrightarrow{T\uparrow\infty} 0,\end{aligned}$$

   where we have used that $H = \int_0^\infty M_t\, dt$ is a bounded kernel by the uniform spatial subcriticality condition (3.2.51) and that μ is a finite measure by assumption. Hence taking $t \uparrow \infty$ in (3.3.40) shows that

$$\mu(f) = c\,\boldsymbol{E}_\Delta[R] \cdot \nu H(f) = \overline{m}(f), \qquad f \in D(L),$$

   whence we conclude by dominated or monotone convergence that the measures μ and $\overline{m}$ coincide since they are both finite and $D(L) = D(A)$ is dense in $\mathcal{C}_0(E)$.

■

3.3.12 Remarks

- Note that under the uniform spatial subcriticality condition, the balance equation (3.3.35) holds for all $f \in D(A)$ by the first part of Theorem 3.3.11, but for the uniqueness assertion in the second part we need it only to hold for all f in the possibly smaller space $\mathcal{D}$.

- If $E = \mathbb{R}^d$, the domain $D(A)$ of the generator of the Feller process X will usually (though not always) contain at least the space of test functions $\mathcal{C}_c^\infty(\mathbb{R}^d)$. This is in particular true if the single particle motion X is a diffusion. In fact, under suitable regularity conditions on the drift and diffusion coefficients b and σ, the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ will even be a core for A: This is for example the case if b and $a := \sigma\sigma^T$ are bounded Hölder-continuous and uniform ellipticity holds, see e.g. [EK1986], Ch. 8, Thm. 1.6. It is also true (without uniform ellipticity) provided a is twice continuously differentiable with bounded second-order derivatives and b is Lipschitz-continuous, see [EK1986], Ch. 8, Thm 2.5. Note that the latter conditions are weaker than those in Assumption 3.1 in [HL2005], so that Theorem 3.3.11 comprises Prop. 4.1 in [HL2005] as a special case. Finally, we mention that if the single particle motion X is a Lévy process on $\mathbb{R}^d$, the space $\mathcal{C}_0^\infty(\mathbb{R}^d)$ of infinitely differentiable functions with derivatives of all orders vanishing at infinity is a core for its generator, see e.g. [Kal2002], Thm. 19.10. In all these cases, given a finite measure μ on $(E, \mathcal{B}_E)$ one need only check that equation (3.3.36) holds for all f in the respective core in order to conclude that μ coincides with the invariant occupation measure $\overline{m}$.

We conclude this chapter and thesis with the following outlook: As in [HL2005], the balance equation (3.3.35) can be used as a starting point to investigate the problem of the existence and regularity of a Lebesgue density for $\overline{m}$ in the purely position-dependent framework: Formally taking adjoints in (3.3.35) gives the equation

$$\left(A^* - \kappa + \tilde{Q}^* \kappa \varrho\right) \overline{m} = -c\boldsymbol{E}_\Delta[R] \cdot \nu. \tag{3.3.41}$$

Now assume an absolutely continuous immigration law $\nu(dv) = p(v)dv$ and suppose that the adjoints A^* and $\tilde{Q}^*$ can be interpreted as operators acting on a suitable function space (e.g. on $\mathcal{C}_0(E) \cap L^1(E)$, if one is interested in a density of class $\mathcal{C}_0$). Then the problem of finding a (nice) density for $\overline{m}$ turns into finding a solution g to

$$\left(A^* - \kappa + \tilde{Q}^* \kappa \varrho\right) g = -c\boldsymbol{E}_\Delta[R] \cdot p \tag{3.3.42}$$

in the respective function space, which under appropriate conditions can be studied by analytical methods. For example, if the single particle motion X is a diffusion, then the formal adjoint A^* of the generator is a second-order differential operator (which however does not have a direct probabilistic interpretation), and together with the assumption $\tilde{Q}(x; \cdot) = \delta_x(\cdot)$ that branching particles reproduce at their death position we obtain the equation

$$\left(A^* - \kappa(1 - \varrho)\right) g = -c\boldsymbol{E}_\Delta[R] \cdot p$$

which was used in [HL2005]. In the more general case where X is a diffusion but $\tilde{Q}(x; \cdot) \neq \delta_x(\cdot)$, (3.3.42) will be an integro-differential equation, and the main challenge is to give a sense to the adjoint $\tilde{Q}^*$ as an operator on $\mathcal{C}_0(E) \cap L^1(E)$ (or another suitable function space). This will of course require additional assumptions on the form of the spatial offspring distribution. Sometimes (as in the absolutely continuous case considered in Chapter 2), the interpretation of $\tilde{Q}^*$ is clear, while in other cases it is not. Let us mention that it is possible to specify a general class of offspring distributions for which the existence of a $\mathcal{C}_0$-density for $\overline{m}$ can indeed be proved and which covers both the case that offspring particles are born at their parent's death position and the case that their distribution is absolutely continuous. However, the proof of this result (which builds on Theorem 3.3.11) is beyond the scope of this thesis and is reserved for some future work.

Appendix A

Continuity and Differentiation Lemma

This appendix contains a simple "Continuity and Differentiation Lemma" for integrals depending on a parameter. It is just a reformulation of the formally weaker version of dominated convergence known as Pratt's Theorem, see e.g. [Els2009], Thm. VI.5.1.

A.1 Lemma ("Continuity Lemma and Differentiation Lemma")
Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, (U, d) a metric space and $f : \Omega \times U \to \mathbb{R}$ such that $f(\cdot, x) \in L^1(\Omega, \mu)$ for all $x \in U$. Define

$$F : U \to \mathbb{R}, \qquad F(x) := \int_\Omega f(\omega, x)\, \mu(d\omega). \tag{A.1}$$

1. *"Continuity Lemma": Fix a point $x_0 \in U$. Suppose that there exists a function $g : \Omega \times U \to \mathbb{R}_+$ with $g(\cdot, x) \in L^1(\Omega, \mu)$ for all $x \in U$ such that the following holds:*

   - *For μ-a.e. $\omega \in \Omega$, the mappings $f(\omega, \cdot) : U \to \mathbb{R}$ and $g(\omega, \cdot) : U \to \mathbb{R}_+$ are continuous at x_0.*
   - *For all $x \in U$: $|f(\omega, x)| \le g(\omega, x)$ for μ-a.e. $\omega \in \Omega$.*
   - *The function*
   $$G : U \to \mathbb{R}_+, \qquad G(x) := \int_\Omega g(\omega, x)\, \mu(d\omega) \tag{A.2}$$
   is continuous at x_0.

   Then also the function $F : U \to \mathbb{R}$ is continuous at the point x_0.

2. *"Differentiation Lemma": Suppose more specifically that $U \subseteq \mathbb{R}^d$ is an open subset of Euclidean space. Assume that there exists a function $g : \Omega \times U \to \mathbb{R}_+$ with $g(\cdot, x) \in L^1(\Omega, \mu)$ for all $x \in U$ such that the following holds:*

   - *For μ-a.e. $\omega \in \Omega$ we have $f(\omega, \cdot) \in \mathcal{C}^1(U)$ and*
   $$|\partial_i f(\omega, x)| \le g(\omega, x), \qquad x \in U,\ i = 1, \ldots, d,$$
   where $\partial_i f(\omega, x)$ denotes the partial derivative w.r.t. x_i of the function $f(\omega, \cdot)$ at the point $x \in U$.

- *For μ-a.e. $\omega \in \Omega$, $g(\omega, \cdot) \in \mathcal{C}(U)$.*
- *$G \in \mathcal{C}(U)$, where G is defined as in* (A.2)*.*

Then with F defined as in (A.1)*, we have also $F \in \mathcal{C}^1(U)$, $\partial_i f(\cdot, x) \in L^1(\mu)$ for all $x \in U$, and the partial derivatives of F are given by*

$$\partial_i F(x) = \int_\Omega \partial_i f(\omega, x)\, \mu(d\omega), \qquad x \in U. \tag{A.3}$$

Proof

1. By decomposing $f = f^+ - f^-$, it suffices to consider $f \geq 0$.

   Let $(x_n)_n$ a sequence in U such that $x_n \xrightarrow{n\to\infty} x_0 \in U$. Defining $f_n(\omega) := f(\omega, x_n)$, $g_n(\omega) := g(\omega, x_n)$, we have

   $$f_n(\cdot) \xrightarrow{n\to\infty} f(\cdot, x_0)\ \mu\text{-a.s.}, \qquad g_n(\cdot) \xrightarrow{n\to\infty} g(\cdot, x_0)\ \mu\text{-a.s.},$$
   $$|f_n(\cdot)| \leq g_n(\cdot)\ \mu\text{-a.s.}, \qquad n \in \mathbb{N}$$

   and
   $$\int_\Omega g_n(\omega)\, \mu(d\omega) = G(x_n) \xrightarrow{n\to\infty} G(x_0) = \int_\Omega g(\omega, x_0)\, \mu(d\omega).$$

   Thus the assumptions of Pratt's Theorem ([Els2009], Thm. VI.5.1) are fulfilled, and we obtain
   $$F(x_n) = \int_\Omega f_n(\omega)\, \mu(d\omega) \xrightarrow{n\to\infty} \int_\Omega f(\omega, x_0)\, \mu(d\omega) = F(x_0),$$
   yielding the desired conclusion.

2. As to the second assertion, fix $x \in U$, $i \in \{1, \dots, d\}$ and denote by e_i the ith canonical unit vector in $\mathbb{R}^d$. Consider the difference quotient for the corresponding partial derivative: We have

   $$\begin{aligned}\frac{1}{h}\Big(F(x + he_i) - F(x)\Big) &= \int_\Omega \mu(d\omega)\, \frac{1}{h}\Big(f(\omega, x + he_i) - f(\omega, x)\Big) \\ &= \int_\Omega \mu(d\omega)\, \frac{1}{h}\int_0^h ds\, \partial_i f(\omega, x + se_i)\end{aligned}$$

   by the fundamental theorem of calculus, since by assumption $\partial_i f(\omega, \cdot)$ is continuous for μ-a.e. $\omega \in \Omega$. Moreover, for μ-a.e. $\omega \in \Omega$ the integrand in the preceding display converges to $\partial_i f(\omega, x)$ as $h \to 0$ and is dominated in absolute value by

   $$\frac{1}{h}\int_0^h ds\, |\partial_i f(\omega, x + se_i)| \leq \frac{1}{h}\int_0^h ds\, g(\omega, x + se_i) \xrightarrow{h\to 0} g(\omega, x)$$

   since $g(\omega, \cdot)$ is supposed to be continuous for μ-a.e. $\omega \in \Omega$. Integrating the upper bound, we have

   $$\int_\Omega \mu(d\omega)\, \frac{1}{h}\int_0^h ds\, g(\omega, x + se_i) = \frac{1}{h}\int_0^h ds\, G(x + se_i) \xrightarrow{h\to 0} G(x) = \int_\Omega g(\omega, x)\, \mu(d\omega)$$

   since also $G(\cdot)$ is supposed to be continuous on U. Again it follows by an application of Pratt's Theorem that $\partial_i f(\cdot, x) \in L^1(\Omega, \mu)$ and that

   $$\frac{1}{h}\Big(F(x + he_i) - F(x)\Big) \xrightarrow{h\to 0} \int_\Omega \partial_i f(\omega, x)\, \mu(d\omega),$$

proving that F is partially differentiable on U and that (A.3) holds. Now continuity of $\partial_i F$ on U follows from the first part ("Continuity Lemma") applied to $\partial_i f$ in place of f.

■

Bibliography

[ADR1967] Azéma, J.; Duflo, M.; Revuz, D.: *Mesure invariante sur les classes récurrentes des processus de Markov.* Z. Wahrscheinlichkeitstheorie verw. Geb. **8** (1967), 157-181.

[ADR1969] Azéma, J.; Duflo, M.; Revuz, D.: *Mesure invariante des processus de Markov récurrents.* Sem. Prob. Strasbourg **3** (1969), 24-33.

[AN1972] Athreya, K. B.; Ney, P. E.: *Branching Processes.* Grundlehren der mathematischen Wissenschaften **196**. Berlin: Springer 1972.

[BG1968] Blumenthal, R. M.; Getoor, R. K.: *Markov Processes and Potential Theory.* New York: Academic Press 1968.

[Bra1978] Bramson, M. D.: *Maximal Displacement of Branching Brownian Motion.* Comm. Pure Appl. Math. **31** (1978), 531-581.

[CKK2008] Chen, Z.-Q.; Kim, P.; Kumagai, T.: *Weighted Poincaré inequality and heat kernel estimates for finite range jump processes.* Math. Ann. **342** (2008), 833-883.

[Dyn1965] Dynkin, E. B.: *Markov Processes, Vols. I-II.* Grundlehren der mathematischen Wissenschaften **121-122**. Berlin: Springer 1965.

[Fol1992] Folland, G. B.: *Fourier Analysis and its Applications.* Pacific Grove, CA: Wadsworth & Brooks/Cole 1992.

[EHK2004] Engländer, J.; Harris, S. C.; Kyprianou, A. E.: *Local extinction versus local exponential growth for spatial branching processes.* Ann. Prob. **32** (2004), 78-99.

[EK1986] Ethier, S. N.; Kurtz, T. G.: *Markov Processes: Characterization and Convergence.* New York: Wiley 1986.

[Els2009] Elstrodt, J.: *Maß- und Integrationstheorie.* 6th edition. Berlin: Springer 2009.

[Har1963] Harris, T. E.: *The Theory of Branching Processes.* Grundlehren der mathematischen Wissenschaften **119**. Berlin: Springer 1963.

[HHK2006] Harris, J. W.; Harris, S. C.; Kyprianou, A. E.: *Further probabilistic analysis of the Fisher–Kolmogorov–Petrovskii–Piscounov equation: one sided travelling-waves.* Ann. I. H. Poincaré – PR **42** (2006), 125-145.

[HHL2002] Höpfner, R.; Hoffmann, M.; Löcherbach, E.: *Non-parametric estimation of the death rate in branching diffusions.* Scand. J. Statist. **29** (2002), 665-692.

[HL2005] Höpfner, R.; Löcherbach, E.: *Remarks on ergodicity and invariant occupation measure in branching diffusions with immigration.* Ann. I. H. Poincaré – PR **41** (2005), 1025-1047.

[Höp2004] Höpfner, R.: *Strange shape of invariant density in branching diffusions with immigration.* Unpublished preprint, Johannes Gutenberg-Universität Mainz 2004.

[INW1968a] Ikeda, N; Nagasawa, M.; Watanabe, S.: *Branching Markov Processes I.* J. Math. Kyoto Univ. **8** (1968), 233-278.

[INW1968b] Ikeda, N; Nagasawa, M.; Watanabe, S.: *Branching Markov Processes II.* J. Math. Kyoto Univ. **8** (1968), 365-410.

[INW1969] Ikeda, N; Nagasawa, M.; Watanabe, S.: *Branching Markov Processes III.* J. Math. Kyoto Univ. **9** (1969), 95-160.

[Kal2002] Kallenberg, O.: *Foundations of Modern Probability.* 2nd edition. New York: Springer 2002.

[KS1985] Kusuoka, S.; Stroock, D: *Applications of the Malliavin Calculus II.* J. Fac. Sci. Univ. Tokyo: Sect. IA, Math. **32** (1985), 1-76.

[Li2002] Li, Z. H.: *Skew convolution semigroups and related immigration processes.* Theory Probab. Appl. **46** (2002), 274-296.

[Li2006] Li, Z. H.: *Branching Processes with Immigration and Related Topics.* Front. Math. China **1** (2006), 73-97.

[Löc1999] Löcherbach, E.: *Statistical Models and Likelihood Ratio Processes for Interacting Particle Systems with Branching and Immigration.* Ph. D. Thesis, Universität Paderborn 1999.

[Löc2002a] Löcherbach, E.: *LAN and LAMN for Systems of Interacting Diffusions with Branching and Immigration.* Ann. I. H. Poincaré – PR **38** (2002), 59-90.

[Löc2002b] Löcherbach, E.: *Likelihood Ratio Processes for Markovian Particle Systems with Killing and Jumps.* Stat. Infer. Stoch. process. **5** (2002), 153-177.

[Löc2004] Löcherbach, E.: *Smoothness of the intensity measure density for interacting branching diffusions with Immigration.* J. Funct. Anal. 215 (2004), 130-177.

[MT1993] Meyn, S. P.; Tweedie, R. L.: *Stability of Markovian Processes II: Continuous-Time Processes and Sampled Chains.* Adv. Appl. Prob. **25** (1993), 487-517.

[Nag1977] Nagasawa, M.: *Basic Models of Branching Processes.* Bull. Int. Stat. Inst. XLVII **2** (1977), 423-445.

[Pak1975] Pakes, A. G.: *On Markov Branching Processes with Immigration.* Sankhya **37** (1977), 129-138.

[Paz1983] Pazy, A.: *Semigroups of linear operators and applications to partial differential equations.* New York: Springer 1983.

[RY1999] Revuz, D.; Yor, M.: *Continuous Martingales and Brownian Motion.* Grundlehren der mathematischen Wissenschaften **293**. 3rd edition. Berlin: Springer 1999.

[Sat1999] Sato, K.-I.: *Lévy Processes and Inifinitely Divisible Distributions.* Cambridge Studies in Advanced Mathematics **68**. Cambridge: Cambridge University Press 1999.

[Saw1970] Sawyer, S. A.: *A Formula for Semigroups, with an Application to Branching Diffusion Processes.* Trans. AMS **152** (1970), 1-38.

[Saw1976] Sawyer, S. A.: *Branching diffusion processes in population genetics.* Adv. Appl. Prob. **8** (1976), 659-689.

[Sev1957] Sevast'yanov, B. A.: *Limit theorems for branching stochastic processes of special form.* Theor. Prob. Appl. **2** (1957), 321-331.

[Sev1958] Sevast'yanov, B. A.: *Branching stochastic processes for particles diffusing in a bounded domain with absorbing boundaries.* Theor. Prob. Appl. **3** (1958), 111-126.

[Shi2006] Shiozawa, Y.: *Extinction of Branching Symmetric α-stable Processes.* J. Appl. Prob. **43** (2006), 1077-1090.

[Str2008] Stroock, D. W.: *Partial Differential Equations for Probabilists.* Cambridge Studies in Advanced Mathematics **112**. Cambridge: Cambridge University Press 2008.

[Wat1965] Watanabe, S.: *On the branching process for Brownian particles with an absorbing boundary.* J. Math. Kyoto Univ. **4** (1965), 385-398.

[Wat1967] Watanabe, S.: *Limit Theorem for a Class of Branching Processes.* In: J. Chover (ed.), Markov processes and potential theory. Proc. Sympos. Math. Res. Center Madison Wis. 1967, pp. 205-232. New York: Wiley 1967

[Zub1972] Zubkov, A. M.: *Life-Periods of a Branching Process with Immigration.* Theor. Probab. Appl. **17** (1972), 174-183.