An Infinite Level Atom coupled to a Heat Bath

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Abstract

We study the mathematics of a finite particle system coupled to a heat bath.

The Standard Model of Quantum Electrodynamics at temperature zero yields a Hamiltonian H describing the energy of particles interacting with photons. In the Heisenberg picture the time evolution of the physical system is the action of a one-parameter-group $(\tau_t)_{t\in\mathbb{R}}$ on a set of observables $\mathcal{A}$:

$$\tau_t \,:\, A \mapsto \tau_t(A),\; t \in \mathbb{R},\; A \in \mathcal{A}$$

Note, that τ is related with solutions of the Schrödinger equation for H.
To consider states of $\mathcal{A}$ describing the physical system near its thermal equilibrium at temperature $T > 0$, we follow the ansatz of Jaksic and Pillet to construct a representation of $\mathcal{A}$.

Now, states are unit vectors in this representation and the time evolution, is described with the aid of the Standard Liouvillean $\mathcal{L}$.

The following results are derived or proved, respectively, in this thesis:

- the construction of the representation

- the self-adjointness of the Standard Liouvillean

- the existence of an equilibrium state in the representation

- the limit of large times for the physical system.

Zusammenfassung

Wir untersuchen die Mathematik endlicher, an ein Wärmebad gekoppelter Teilchensysteme.

Das Standard-Modell der Quantenelektrodynamik für Temperatur Null liefert einen Hamilton-Operator H, der die Energie von Teilchen beschreibt, welche mit Photonen wechselwirken. Im Heisenbergbild ist die Zeitevolution des physikalischen Systems durch die Wirkung einer Ein-Parameter-Gruppe $(\tau_t)_{t\in\mathbb{R}}$ auf eine Menge von Observablen $\mathcal{A}$ gegeben:

$$\tau_t \,:\, A \mapsto \tau_t(A),\ t \in \mathbb{R},\ A \in \mathcal{A}.$$

Diese steht im Zusammenhang mit der Lösung der Schrödinger-Gleichung für H.
Um Zustände von $\mathcal{A}$, welche das physikalische System in der Nähe des thermischen Gleichgewichts zur Temperatur T darstellen, zu beschreiben, folgen wir dem Ansatz von Jaksic und Pillet, eine Darstellung von $\mathcal{A}$ zu konstruieren.
Die Vektoren in dieser Darstellung definieren die Zustände, die Zeitentwicklung wird mit Hilfe des Standard Liouville-Operators $\mathcal{L}$ beschrieben.

In dieser Doktorarbeit werden folgende Resultate bewiesen bzw. hergeleitet:

- die Konstuktion einer Darstellung
- die Selbstadjungiertheit des Standard Liouville-Operators
- die Existenz eines Gleichgewichtszustandes in dieser Darstellung
- der Limes des physikalischen Systems für große Zeiten.

Danksagung

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Chapter 1

Introduction

In this work a system of n particles is considered with the aid of quantum mechanics. The particles are described in a non relativistic setting, so the kinetic energy is $T = \sum_{j=1}^{n} \frac{1}{2m_j}(\vec{p}_j)^2$. In this context is $\vec{p}_j$ the momentum of the j-th particle and m_j is its mass, the position in $\mathbb{R}^3$ is $\vec{x}_j$. The potential energy is $V(\underline{x}) = V(\vec{x}_1, \ldots, \vec{x}_n)$. Thus, the classical expression for the total energy is

$$H = T + V(\underline{x}) = \sum_{j=1}^{n} \frac{1}{2m_j}(\vec{p}_j)^2 + V(\vec{x}_1, \ldots, \vec{x}_n). \tag{1.1}$$

To take effects like uncertainty of the momentum $\vec{p}_j$ and the position $\vec{x}_j$ into account, one quantizes the system, i.e. H, $\vec{p}_j$ and $V(\underline{x})$ are replaced by operators acting on complex-valued wave functions ψ. The momentum is $\vec{p}_j := -\imath\vec{\nabla}_{x_j}$ and $V(\underline{x})$ means now, multiplication of a wave function with the potential energy $V(\underline{x})$. The Hamiltonian H is defined to be the operator on the right hand side of (1.1). The wave function ψ is the state of the system, the probability density to find the particles $1, \ldots, n$ at time t in the positions $x_1, \ldots, x_n$ is

$$(\vec{x}_1, \ldots, \vec{x}_n) \mapsto N^{-1} \cdot \left|\psi_t(\vec{x}_1, \ldots, \vec{x}_n)\right|^2,$$

N is the normalization constant. The Hamiltonian H determines the time evolution of a state, in the sense that ψ obeys the Schrödinger equation,

$$-\frac{\imath}{\hbar}(\partial_t\psi)(\vec{x}_1, \ldots, \vec{x}_n) = (H\psi_t)(\vec{x}_1, \ldots, \vec{x}_n). \tag{1.2}$$

$\hbar$ is the Planck constant divided by 2π. Usual we write states time independent, the connection is

$$\psi_t(\vec{x}_1, \ldots, \vec{x}_n) = (e^{\imath\frac{t}{\hbar}H}\psi)(\vec{x}_1, \ldots, \vec{x}_n). \tag{1.3}$$

A measurement of a quantity, called observable, is modeled as follows, one expresses the observable by an operator A, for example H for the energy, x_j for the position of the j-th particle. The expectation value of an observable corresponding to A in a state ψ is

$$N^{-1} \cdot \int \overline{\psi(\vec{x}_1, \ldots, \vec{x}_n)} (A\psi)(\vec{x}_1, \ldots, \vec{x}_n) d^3x_1 \cdots d^3x_n. \tag{1.4}$$

The second ingredient is a gas of infinitely many bosons that surrounds the particles. The bosons are for example photons of an electromagnetic field. The states for such a gas is a sequence of n particle states, $(\psi_m^{(ph)})_{m=0}^{\infty}$. The m-particle state is $\psi_m(\underline{x}) = \psi_m^{(ph)}(x_1, \ldots, x_m)$, we assume that ψ is symmetric, i.e. invariant under a permutation of x_i and x_j. The Hamiltonian $\check{H}$ for the gas is

$$(\check{H}\psi_m^{(ph)})(x_1, \ldots, x_m) = \sum_{j=1}^{m} \big(|p_j|\, \psi_m^{(ph)}\big)(x_1, \ldots, x_m). \tag{1.5}$$

It corresponds to the relativistic kinetic energy of m photons.

The states of the coupled system are now vectors in the tensor product $\mathcal{H}$ of the particle states and the photon states. The Hamiltonian H_λ is sum of the particle and the photon Hamiltonian and an interaction operator W,

$$H_\lambda = H \otimes 1 + 1 \otimes \check{H} + W. \tag{1.6}$$

We remark again, that we now consider time independent states. The time evolution is expressed by a group action on the observables

$$\tau_t^\lambda \,:\, A \mapsto e^{\imath \frac{t}{\hbar} H_\lambda} A e^{-\imath \frac{t}{\hbar} H_\lambda}. \tag{1.7}$$

A is in this context an observable for the coupled system. The expectation of A in ψ is

$$\omega(A) := N^{-1} \cdot \langle \psi | A\psi \rangle, \quad N := \|\psi\|^2. \tag{1.8}$$

In this work an algebra $\mathcal{A}$ of observables is specified, such as the $*$-automorphism group τ^λ. Our aim is to study the physical system at an inverse temperature β, therefore a representation π of $\mathcal{A}$ in a Hilbert space $\mathcal{K}$ is introduced, so that the vectors of $\mathcal{K}$ shall describe state over $\mathcal{A}$. In this representation the time evolution is defined by a unitary group $e^{\imath t \mathcal{L}_\lambda}$, generated by the so-called Standard Liouvillean $\mathcal{L}_\lambda$ being a self-adjoint operator that does not represent the energy of the physical system. The parameter $\lambda \in \mathbb{R}$ is the coupling constant, describing the coupling strength between particles and bosons. If the particle system is confined, i.e. the

Gibbs condition $Z := \text{Tr}\{e^{-\beta H}\} < \infty$ is fulfilled, we investigate the existence of an equilibrium state. Moreover we study the property, that states represented as vectors in $\mathcal{K}$ return to the equilibrium state in the large time limit.

At this point we give a short overview of related results in mathematical physics on this topic. The representations of the canonical commutation relations (CCR) describing an infinite free Bose gas of finite density are considered in a work of Araki and Woods [3]. An other fundamental work is due to Haag, Hugenholtz and Winnink [15], therein the representations of a C^*-algebra corresponding to thermal equilibrium of a system at given temperature T is studied. Moreover, properties of these representation are discussed in relation with the KMS-condition. A coupled system, consisting of a small system (finite level atom) and a heat bath is considered by Jaksic and Pillet in [16, 17]. They give a characterization of "Return to Equilibrium" in terms of spectral properties of the Liouvillean. However, their proof of return to equilibrium needs the assumption $|\lambda\beta| \ll 1$, i.e. the product of inverse temperature β and coupling constant λ is small. In a work of Bach, Fröhlich and Sigal [5] the "Return to Equilibrium" for the same model is shown for small coupling constant and a less restrictive infrared condition on the coupling function. To our knowledge the most general existence theorem for a KMS-state for the coupled system is formulated by Derezinski, Jaksic and Pillet in [9, 10].

Thermal Ionization is an other related topic, Fröhlich, Merkli and Sigal have shown in [12, 13] an atom coupled to the heat bath will be ionized in the limit of large times, if the atom is not confined.

The dipole approximation of a harmonic oscillator plays an important role in our work, it was treated rigorously by Arai [1, 2] at temperature zero. In those papers asymptotic completeness is shown the spectrum of the Hamiltonian is completely determined. We apply these results to the case of positive temperature. The analysis of an anharmonic oscillator in context of the Langevin equation is studied by Maassen in [18]. The same anharmonic oscillator is considered by Spohn [25] to prove asymptotic completeness of photon scattering.

To the contribution of this thesis we count

- The definition of a W^*-dynamical system $(\mathfrak{M}, \tau^\lambda)$, that represents a particle system in a heat bath (Bose gas). Models with minimally coupled interaction are included. Self-adjointness of the Liouvilleans is proved in Theorem 4.1.2, the existence of a $*$-automorphism is proved in 4.2.8. In Theorem 4.1.2 one do not need a commutator condition, that is for example assumed in [19], hence Liouvilleans describing thermal ionization are included.

- Assuming the existence $\mathrm{Tr}\{e^{-(\beta-\epsilon)H}\} < \infty$, of the partition function of the small system at inverse temperature $\beta \gg \epsilon > 0$ we formulate conditions on the coupling constant λ, and on the interaction W, that ensure the existence of a thermal equilibrium state ω_λ^β, see Theorem 4.2.1. It is also shown, that the KMS-boundary conditions are fulfilled and the vector representative of this state is cyclic and separating. The proof draws on ideas of the one given in [6] for a simpler model. In general the condition for existence of a KMS-state, $\|e^{\beta/2Q}\Omega_0^\beta\| < \infty$ formulated in [9, 10] includes not the same class of models.

- In Theorem 4.3.1 a specific variant of Theorem 4.2.1 geared to a harmonic oscillator with a dipole interaction is applied. In this case we show existence for small $|\lambda|$ independent of β. This result can not be obtained by applying abstract results of [9, 10], confer Remark 4.3.4.

- In the case of a harmonic oscillator with a dipole interaction we derive a second approach to establish existence of a KMS-state essentially using results stated in [1, 2]. In this case Return to Equilibrium is shown and a time-decay rate is given. Its positivity is due to Fermi Golden rule for the Hamiltonian, not for the Liouvillean. This is treated in Theorem 5.2.6. Moreover, both approaches are compared in Theorem 5.3.4. It turns out, that both models are mathematically equivalent and a complete spectral analysis of the Standard Liouvillean is obtained.

- The property of "Return to Equilibrium" in the harmonic oscillator case is preserved if a potential (depending on the coupling constant) is added. The rate of decay decreases if the potential increases, confer Theorem 5.4.1. The proof uses ideas formulated by Maassen in [18].

The work is subdivided into five chapters.

The first chapter is the introduction.

In the second chapter a few fundamental definitions and theorems are recalled to be self-contained. Some theorems are proved if they deal with W^*-algebras. For the remaining proofs the reader is referred to textbooks. The examples given in this chapter are mostly relevant for the later discussion. W^*-dynamical systems, the formalism of second quantization and the Weyl algebras are the main topics. In the most important part deals with a perturbation-theory for W^*-dynamical system, it uses ideas we found in [10]. The other statements and proofs have there origin in the textbooks of Bratteli and Robinson [7, 8] and the Lecture Notes on open quantum systems [4] by Attal et.al. In chapter three an equilibrium state for an Ideal Bose Gas, such as Gibbs states, are considered. Hamiltonians, generating the dynamics in the temperature zero case are presented. The Gibbs condition for several Hamiltonians describing confined particles is discussed. Furthermore, a representation of the algebra of observables and corresponding Liouvilleans are defined.

Here, statements concerning self-adjointness of the Liouvilleans , existence of KMS-states are formulated and proved. Theorem 4.1.2, Theorem 4.2.1, and Theorem 4.3.1 are found there. In the last chapter the dipole approximation of a harmonically bounded, respectively anharmonically bounded, particle is considered. This includes Theorem 5.2.6, Theorem 5.3.4, and Theorem 5.4.1.

Finally, we give a list of related problems or questions that seem to be interesting, but (far) out of the scope of this work.

1. The condition $\mathrm{Tr}\{e^{-\beta H_{el}}\} < \infty$ encodes that the particle is confined, which should be necessary for the existence of a thermal equilibrium state. But the condition might be too strong. Confer Section 3.3.

2. Is there a way to get rid of the smallness condition for $|\lambda\beta|$ in the Standard Model, which does not occur if H_{el} is just a finite level atom or if H_{el} is the harmonic oscillator ? Is $|\lambda| \ll 1$ sufficient ?

3. Is there a way to generalize Theorem 4.3.1 to the model considered by Arai independent of the UV-cutoff ? This would imply that ω_λ^β is in generally ω_0^β normal.

4. What happens if the inverse temperature tends to infinity ? Why does the Nelson Model have a thermal equilibrium state, even without restrictions to λ and β, but in the temperature zero case, there is no ground state in the Fock space?

5. It is possible to obtain spectral informations for the Liouvillean $\mathcal{L}_\lambda$ in the general case, although the techniques and methods for the finite level atom do not solve the problem

alone ? What is $\dim \ker(\mathcal{L}_\lambda)$, $\sigma_c(\mathcal{L}_\lambda)$ or $\sigma_{ac}(\mathcal{L}_\lambda)$?

Chapter 2

Mathematical Theory

2.1 C^*-Algebras, States and Representations

First, we give a short introduction in the theory of C^*-algebras and their states. The elements of a C^*-algebra are also called observables, they describe the physical system. The states stand for our information of a physical system.

Definition 2.1.1 (C^*-Algebra). *1. An algebra $\mathfrak{A}$ is a vector space with coefficient field $\mathbb{C}$ equipped with a multiplication, that is associative and distributive, i.e. for $A, B, C \in \mathfrak{A}$ and $a, b \in \mathbb{C}$ is $AB \in \mathfrak{A}$ and*

$$(AB)C = A(BC) \tag{2.1}$$

$$(A + B)C = AC + BC \tag{2.2}$$

$$a(bA) = (ab)A. \tag{2.3}$$

2. An algebra $\mathfrak{A}$ is called normed with norm $\|\cdot\|$, if $(\mathfrak{A}, \|\cdot\|)$ is a normed vector space and the norm is compatible with the multiplication, i.e.

$$\|AB\| \leq \|A\| \cdot \|B\|, \quad A, B \in \mathfrak{A} \tag{2.4}$$

3. A $*$-algebra $\mathfrak{A}$ is an algebra, such that for all $A \in \mathfrak{A}$ exists $A^* \in \mathfrak{A}$ and

$$(A^*)^* = A, \tag{2.5}$$

$$(aA + bB)^* = \bar{a}A^* + \bar{b}B^* \tag{2.6}$$

$$(AB)^* = B^*A^*. \tag{2.7}$$

4. *An identity* $\mathbb{1}$ *is an element of* $\mathfrak{A}$*, uniquely determined by*

$$A = \mathbb{1}A = A\mathbb{1}, \;\; A \in \mathfrak{A}. \tag{2.8}$$

An algebra with identity is called unital.

5. *A C^*-algebra* $\mathfrak{A}$ *is a normed, complete $*$-algebra, such that*

$$\|A^*A\| = \|A\|^2. \tag{2.9}$$

6. $\mathfrak{B} \subset \mathfrak{A}$ *is a $*$-subalgebra of a $*$-algebra* $\mathfrak{A}$*, if* $A^* \in \mathfrak{B}$ *for* $A \in \mathfrak{B}$.
If additionally, $\mathfrak{A}$ *is a C^*-algebra and* $\mathfrak{B}$ *is closed, then* $\mathfrak{B}$ *is a C^*-subalgebra.*

We give a few examples for C^*-algebras.

Example 2.1.2.
- $\mathbb{C}$ is a unital C^*-algebra, $a^* = \bar{a}$.
- The unital C^*-algebra $\mathcal{B}(\mathfrak{h})$ of bounded operator on a Hilbert space $\mathfrak{h}$. $\|\cdot\|_{\mathcal{B}(\mathfrak{h})}$ is the operator norm and A^* is the adjoint operator of A.
- $\mathrm{Com}(\mathfrak{h}) \subset \mathcal{B}(\mathfrak{h})$ the compact operators on a Hilbert space $\mathfrak{h}$ define a C^*-subalgebra of $\mathcal{B}(\mathfrak{h})$. $\mathrm{Com}(\mathfrak{h})$ is not unital in the case of an infinite dimensional $\mathfrak{h}$.
- Let Ξ be a topological space and $\mathcal{C}_b(\Xi;\mathbb{C})$ the continuous bounded functions with values in $\mathbb{C}$. $\mathcal{C}_b(\Xi;\mathbb{C})$ equipped with the uniform norm $\|\cdot\|_\infty$ is a commutative, unital C^*-algebra. For $f \in \mathcal{C}_b(\Xi;\mathbb{C})$, one defines f^* by pointwise complex conjugation.
- Let (X,μ) be a measure space. $L^\infty(X,\mu;\mathbb{C})$ is the commutative, unital C^*-algebra of μ-essentially bounded, complex-valued functions. Again, f^* is obtained by pointwise complex conjugation, the norm is the μ-essential supremum of f.

The next definition deals with maps that preserve the structure of C^*-algebras. An important role play the maps into the bounded operators of a Hilbert space.

Definition 2.1.3 (Representation)**.** *Let* $\mathfrak{A}, \mathfrak{B}$ *be C^*-algebras and* $\pi : \mathfrak{A} \to \mathfrak{B}$ *be a linear map.*

1. π *is a $*$-morphism, iff*

$$\pi[AB] = \pi[A]\pi[B] \tag{2.10}$$

$$\pi[A^*] = \pi[A]^*. \tag{2.11}$$

2. *A $*$-morphism π is isometric, if $\|\pi[A]\| = \|A\|$ for all $A \in \mathfrak{A}$. π is a $*$-isomorphism, if it is bijective. If additionally $\mathfrak{A} = \mathfrak{B}$, then an isometric $*$-isomorphism is called an $*$-automorphism.*

3. *If π is a $*$-morphism and $\mathfrak{B} = \mathcal{B}(\mathfrak{h})$, then π is a representation map of $\mathfrak{A}$ in $\mathfrak{h}$.*

Remark 2.1.4. Let $\mathfrak{A}$ be unital.

- The range $\pi[\mathfrak{A}]$ of π is a C^*-subalgebra of $\mathfrak{B}$.

- For $A \in \mathfrak{A}$ we have $\|\pi[A]\| \leq \|A\|$.

Confer ([7], Prop. 2.3.1).

The next example shows, in which way an isometric isomorphism between two Hilbert spaces yields an isometric $*$-isomorphism between the C^*-algebras of bounded operators.

Example 2.1.5.

- Let $U : \mathfrak{h} \to \mathfrak{h}'$ be an isometric isomorphism between two Hilbert spaces. $\alpha_U : \mathcal{B}(\mathfrak{h}) \to \mathcal{B}(\mathfrak{h}'), \quad A \mapsto UAU^{-1}$ is an isometric $*$-isomorphism.

- For Hilbert spaces $\mathfrak{h}$, $\mathfrak{h}'$ one can define a representation map $\pi : \mathcal{B}(\mathfrak{h}) \to \mathcal{B}(\mathfrak{h} \otimes \mathfrak{h}')$ by means of $\pi[A]\phi \otimes \psi := (A\phi) \otimes \psi$.

Definition 2.1.6 (States)**.** *Let $\mathfrak{A}$ be a unital C^*-algebra. A linear map $\omega : \mathfrak{A} \to \mathbb{C}$ is a state, iff*

$$\omega(A^*A) \geq 0, \quad A \in \mathfrak{A} \tag{2.12}$$

$$\omega(\mathbb{1}) = 1. \tag{2.13}$$

It can be shown, that states are always continuous maps with operator norm $\|\omega\| = 1$. For every $A \in \mathfrak{A}$ one has $\omega(A^*) = \overline{\omega(A)}$. The set of states is obviously closed and convex.

Example 2.1.7.

- Let $\phi \in \mathfrak{h}$ and $\|\phi\| = 1$. The vector state over $\mathcal{B}(\mathfrak{h})$ associated with ϕ is $\omega_\phi(A) = \langle\phi|A\phi\rangle$. A vector state is the finest preparation of a physical system, it encodes full information and is therefore called pure, as well.

- Let $1 \geq p_1 \geq p_2 \geq \ldots > 0$, such that $\sum_\nu p_\nu = 1$. Moreover, let $(\phi_n)_n$ be an orthonormal system of vectors in $\mathfrak{h}$. The operator $\rho \in \mathcal{B}(\mathfrak{h})$ defined by

$$\rho(\psi) = \sum_\nu p_\nu \langle \phi_\nu | \psi \rangle \phi_\nu, \ \psi \in \mathfrak{h} \tag{2.14}$$

is referred to as density operator. The corresponding state is

$$\omega_\rho(A) := \mathrm{Tr}\{\rho A\} = \sum_\nu p_\nu \langle \phi_\nu | A \phi_\nu \rangle, \ A \in \mathcal{B}(\mathfrak{h}). \tag{2.15}$$

ω_ρ is called a normal state. One can interpret that the physical system is with probability p_i in the state ω_{ϕ_i}, ω_ρ is a mixed state.

Definition 2.1.8 (Cyclic and Separating Vectors)**.** *Let $\phi \in \mathfrak{h}$ and $\mathfrak{A} \subset \mathcal{B}(\mathfrak{h})$ a subalgebra.*

1. *ϕ is cyclic for $\mathfrak{A}$, iff $\mathrm{c}\ell\, \mathfrak{A}\phi = \mathfrak{h}$.*

2. *ϕ is separating for $\mathfrak{A}$, iff $A\phi = 0$ implies $A = 0$, for all $A \in \mathfrak{A}$.*

Theorem 2.1.9 (GNS-Representation)**.** *Let $\mathfrak{A}$ be a unital C^*-algebra and ω a state.*

1. *There exists a Hilbert space $\mathfrak{h}_\omega$, a normed vector Ω_ω, and a representation map $\pi_\omega : \mathfrak{A} \to \mathcal{B}(\mathfrak{h}_\omega)$, such that $\pi_\omega[\mathbb{1}] = \mathbb{1}$. Furthermore Ω_ω is cyclic for $\pi_\omega[\mathfrak{A}]$ and*

$$\omega(A) = \langle \Omega_\omega | \pi_\omega[A] \Omega_\omega \rangle, \quad A \in \mathfrak{A} \tag{2.16}$$

$(\mathfrak{h}_\omega, \pi_\omega, \Omega_\omega)$ is a GNS-Triple corresponding to ω.

2. *Let $(\mathfrak{h}_\omega, \pi_\omega, \Omega_\omega)$ and $(\mathfrak{h}, \pi, \Omega)$ be GNS-triple. There is a natural isometric isomorphism $U : \mathfrak{h}_\omega \to \mathfrak{h}$, such that $U\Omega_\omega = \Omega$ and $\alpha_U \circ \pi_\omega = \pi$.*

Proof of 2.1.9: We only prove the second statement, for the first see ([7], Thm. 2.3.16). For $A \in \mathfrak{A}$ we define

$$U(\pi_\omega[A]\Omega_\omega) := \pi[A]\Omega. \tag{2.17}$$

First, we check that $U : \pi_\omega[\mathfrak{A}]\Omega_\omega \to \pi[\mathfrak{A}]\Omega$ is a well-defined map. Assume $\pi_\omega[A]\Omega_\omega = \pi_\omega[A']\Omega_\omega$, it follows directly from the definition of the scalar product.

$$\begin{aligned} \|U(\pi_\omega[A]\Omega_\omega) - U(\pi_\omega[A']\Omega_\omega)\|^2 &= \langle \pi[A - A']\Omega | \pi[A - A']\Omega \rangle \\ &= \langle \Omega | \pi[(A - A')^*(A - A')]\Omega \rangle = \omega((A - A')^*(A - A')) \\ &= \langle \Omega_\omega | \pi_\omega[(A - A')^*(A - A')]\Omega_\omega \rangle = \|\pi_\omega[A]\Omega_\omega - \pi_\omega[A']\Omega_\omega\|^2 = 0. \end{aligned} \tag{2.18}$$

To show linearity, pick $a, b \in \mathbb{C}$ and $A, B \in \mathfrak{A}$, the linearity of π and π_ω yields

$$\begin{aligned} U(a\pi_\omega[A]\Omega_\omega + b\pi_\omega[A]\Omega_\omega) &= U(\pi_\omega[aA+bB]\Omega_\omega) = \pi[aA+bB]\Omega \\ &= a\pi[A]\Omega + b\pi[B]\Omega = aU(\pi_\omega[A]\Omega_\omega) + bU(\pi_\omega[B]\Omega_\omega). \end{aligned} \tag{2.19}$$

U is isometric,

$$\begin{aligned} &\langle (U\pi_\omega[A]\Omega_\omega)|U(\pi_\omega[A]\Omega_\omega)\rangle = \langle \pi[A]\Omega|\pi[B]\Omega\rangle = \langle \Omega|\pi[A^*B]\Omega\rangle \\ &= \omega(A^*B) = \langle \Omega_\omega|\pi_\omega[A^*B]\Omega_\omega\rangle = \langle \pi_\omega[A]\Omega_\omega|\pi_\omega[B]\Omega_\omega\rangle. \end{aligned} \tag{2.20}$$

Since Ω_ω is cyclic for $\pi[\mathfrak{A}]$, the subspace $\pi_\omega[\mathfrak{A}]\Omega_\omega$ is dense in $\mathfrak{h}_\omega$. Hence U extends isometrically to a map $\mathfrak{h}_\omega \to \mathfrak{h}$.

U is surjective. Let $\psi \in \mathfrak{h}$. Since Ω is cyclic for $\pi_\omega[\mathfrak{A}]$, there is a sequence $(\phi_n)_n \subset h_\omega$ such that $\lim_{n\to\infty} U\phi_n = \psi$. Since U is isometric,

$$\|\phi_n - \phi_m\| = \|U\phi_n - U\phi_m\|. \tag{2.21}$$

Hence $(\phi_n)_n$ is a Cauchy-sequence in $\mathfrak{h}_\omega$ and therefore convergent. Thus

$$\psi = \lim_{n\to\infty} U\phi_n = U \lim_{n\to\infty} \phi_n \in \operatorname{ran} U. \tag{2.22}$$

since U is bounded. For α_U defined in (2.1.5)

$$\begin{aligned} &\alpha_U(\pi_\omega[A])(\pi[B]\Omega) = UAU^{-1}\pi[B]\Omega = U\pi_\omega[A]U^{-1}(U\pi_\omega[B]\Omega_\omega) \\ &= U\pi_\omega[A]\pi_\omega[B]\Omega_\omega = U\pi_\omega[AB]\Omega_\omega = \pi[AB]\Omega = \pi[A](\pi[B]\Omega), \end{aligned} \tag{2.23}$$

hence $\alpha_U \circ \pi_\omega = \pi$, since $\pi[\mathfrak{A}]\Omega$ is dense in $\mathfrak{h}$. □

We remark, that the assumption for $\mathfrak{A}$ to be unital is not necessary.

Definition 2.1.10 (ω-normal States). *Let ω be a state over $\mathfrak{A}$ and $(\mathfrak{h}_\omega, \pi_\omega, \Omega_\omega)$ the GNS-triple. A state μ is ω-normal, if there exists a density operator $\rho \in \mathcal{B}(\mathfrak{h}_\omega)$, such that $\mu(A) = \mathrm{Tr}\{\rho\,\pi_\omega[A]\}$, $A \in \mathfrak{A}$.*

The GNS-representation gives us a tool to define states, that are closely related to a given state ω. We remark, that ρ is in general not uniquely determined by μ. In the following example we give an explicit construction of a GNS-Triple, if ω_ρ is a normal state over $\mathcal{B}(\mathfrak{h})$.

Example 2.1.11 (GNS-Representation for a Normal State)**.** Let $\rho \in \mathcal{B}(\mathfrak{h})$ be a density operator, defined in Example 2.1.7 and $\mathfrak{g} := (\ker\,\rho)^{\perp} \subset \mathfrak{h}$. Choose π as defined in Example 2.1.5 for $\mathfrak{h}' = \mathfrak{g}$. We define

$$\Omega_\rho := \sum_\nu p_\nu^{1/2} \phi_\nu \otimes \phi_\nu \in \mathfrak{h}_\rho := \mathfrak{h} \otimes \mathfrak{g}. \tag{2.24}$$

The vector state corresponding to Ω_ρ is

$$\begin{aligned} \langle \Omega_\rho | \pi_\rho[A] \Omega_\rho \rangle_{\mathfrak{h}\otimes\mathfrak{g}} &= \Big\langle \sum_\nu p_\nu^{1/2} \phi_\nu \otimes \phi_\nu \Big| \sum_\mu p_\mu^{1/2} (A\phi_\mu) \otimes \phi_\mu \Big\rangle_{\mathfrak{h}\otimes\mathfrak{g}} \qquad (2.25)\\ &= \sum_{\nu,\mu} p_\nu^{1/2} p_\mu^{1/2} \langle \phi_\nu | A\phi_\mu \rangle_{\mathfrak{h}} \cdot \langle \phi_\nu | \phi_\mu \rangle_{\mathfrak{g}} = \sum_\nu p_\nu \langle \phi_\nu | A\phi_\nu \rangle_{\mathfrak{h}} = \mathrm{Tr}\{\rho A\} = \omega_\rho(A). \end{aligned}$$

Let $\psi \in \mathfrak{h}$ and $\phi_\nu \in \mathfrak{g}$ fixed and $A\phi := p_\nu^{-1/2} \langle \phi_\nu | \phi \rangle \psi,\ \phi \in \mathfrak{h}$. We observe that $\pi_\rho[A]\Omega_\rho = \psi \otimes \phi_\nu$. Therefore

$$c\ell\, \pi_\rho[\mathfrak{A}]\Omega_\rho \supseteq c\ell\, \mathrm{LH}\{\psi \otimes \phi_\nu \in \mathfrak{h}_\rho \,:\, \psi \in \mathfrak{h},\ \nu = 1, 2 \ldots\} = \mathfrak{h}_\rho. \tag{2.26}$$

Moreover, Ω_ρ is separating for $\pi_\rho[\mathcal{B}(\mathfrak{h})]$, iff $\ker\,\rho = \{0\}$.

2.2 W^*-Algebras

In this section we consider unital C^*-algebras $\mathfrak{A}$ embedded in $\mathcal{B}(\mathfrak{h})$. We are interested in algebras, that are closed in weaker topologies, and therefore it is possible to apply more technics and theorems.

Definition 2.2.1 (Commutant)**.** *Let $\mathfrak{S}$ be a non-empty subset of $\mathcal{B}(\mathfrak{h})$, such that $S^* \in \mathfrak{S}$ for $S \in \mathfrak{S}$. We denote by $\mathfrak{S}' := \{A \in \mathcal{B}(\mathfrak{h}) \,:\, AS = SA,\ \forall S \in \mathfrak{S}\}$ the <u>commutant</u> of $\mathfrak{S}$.*

Theorem 2.2.2 (Properties of the Commutant)**.**

1. *$\mathfrak{S}'$ is a unital C^*-subalgebra of $\mathcal{B}(\mathfrak{h})$.*

2. *$\mathfrak{S}'$ is weakly closed, i.e. if for a $C \in \mathcal{B}(\mathfrak{h})$ and for all $\epsilon > 0$ and $\phi_1, \ldots, \phi_n \in \mathfrak{h}$, $\psi_1, \ldots, \psi_n \in \mathfrak{h}$ exists $A \in \mathfrak{S}'$, such that*

$$\big| \langle \psi_i | (C - A) \phi_i \rangle \big| < \epsilon,\ i = 1, 2, \ldots, n, \tag{2.27}$$

then $C \in \mathfrak{S}'$.

Proof of 2.2.2. Obviously, $\mathbb{1} \in \mathfrak{S}'$. To show that $\mathfrak{S}'$ is a subalgebra, we pick $S \in \mathfrak{S}$ and $A, B \in \mathfrak{S}'$, $a, b \in \mathbb{C}$. Since

$$\begin{aligned} S(aA + bB) &= aSA + bSB = aAS + bBS = (aA + bB)S \\ S(AB) &= (AS)B = A(BS) = (AB)S, \end{aligned} \tag{2.28}$$

we obtain $(aA + bB) \in \mathfrak{S}'$ and $AB \in \mathfrak{S}'$. Since $S^* \in \mathfrak{S}$ we have

$$SA^* = (AS^*)^* = (S^*A)^* = A^*S. \tag{2.29}$$

Thus $A^* \in \mathfrak{S}'$. It suffices to show now the second statement. Assume $C \notin \mathfrak{S}'$, then exists an $\epsilon > 0$, and ϕ, $\psi \in \mathfrak{h}$ such as $S \in \mathfrak{S}$, for which

$$\big|\langle\phi|(SC - CS)\psi\rangle\big| \geq 2\epsilon. \tag{2.30}$$

Hence for all $A \in \mathfrak{S}'$,

$$\begin{aligned} 2\epsilon &\leq \big|\langle\phi|(SC - AS)\psi\rangle\big| + \big|\langle\phi|(AS - CS)\psi\rangle\big| \\ &= \big|\langle S^*\phi|(C - A)\psi\rangle\big| + \big|\langle\phi|(A - C)S\psi\rangle\big|. \end{aligned} \tag{2.31}$$

Hence C does not belong to the weak closure of $\mathfrak{S}'$. □

Theorem 2.2.3 (Bicommutant Theorem)**.** *Let $\mathfrak{A}$ a unital $*$-subalgebra.*

1. *$\mathfrak{A}''$ is a weakly closed, unital C^*-algebra containing $\mathfrak{A}$.*

2. *For all $A'' \in \mathfrak{A}''$, $\epsilon > 0$ and every sequence $(\psi_n)_n \subset \mathfrak{h}$ with $\sum_{n=1}^{\infty} \|\psi_n\|^2 < \infty$ exists $A \in \mathfrak{A}$, such that*

$$\sum_{n=1}^{\infty} \|A\psi_n - A''\psi_n\|^2 < \epsilon^2, \tag{2.32}$$

i.e. $\mathfrak{A}''$ is the σ-strong closure of $\mathfrak{A}$.

Lemma 2.2.4. *Let $A'' \in \mathfrak{A}''$. For all $\epsilon > 0$ and $\psi \in \mathfrak{h}$ exists an $A \in \mathfrak{A}$, such that*

$$\|A\psi - A''\psi\| < \epsilon. \tag{2.33}$$

Proof of 2.2.4. Let $B \in \mathcal{B}(\mathfrak{h})$ be fixed and assume for all $A \in \mathfrak{A}$

$$\|A\psi - B\psi\| \geq \epsilon. \tag{2.34}$$

To prove 2.2.4 we will show $B \notin \mathfrak{A}''$. First, define $\mathfrak{h}_0 := c\ell\,\mathfrak{A}\psi$ and P_0 to be the orthogonal projection onto $\mathfrak{h}_0$. For $A_1, A_2 \in \mathfrak{A}$ we obtain

$$A_1 P_0 A_2 \psi = A_1 A_2 \psi = P_0 A_1 A_2 \psi = P_0 A_1 P_0 A_2 \psi. \tag{2.35}$$

We conclude $A_1 P_0 = P_0 A_1 P_0$, since $\mathfrak{A}\psi$ is dense in $\mathfrak{h}_0$. Analogously, we conclude $A_1^* P_0 = P_0 A_1^* P_0$. Furthermore, since $P^* = P$ we have

$$A_1 P_0 = P_0 A_1 P_0 = (P_0 A_1^* P_0)^* = (A_1^* P_0)^* = P_0 A_1. \tag{2.36}$$

Hence $P_0 \in \mathfrak{A}'$. Now we will show that P_0 and B do not commute.
$P_0\psi = \psi$ because $\mathfrak{A}$ is unital. From Equation (2.34) follows

$$\begin{aligned} \|BP_0\psi - P_0 B\psi\| &\geq \|BP_0\psi - AP_0\psi\| - \|AP_0\psi - P_0 B\psi\| \\ &\geq \epsilon - \|AP_0\psi - P_0 B\psi\|. \end{aligned} \tag{2.37}$$

Since $P_0 B\psi \in \mathfrak{h}_0$ and $\mathfrak{A}\psi$ is dense in $\mathfrak{h}_0$ we obtain

$$\|BP_0\psi - P_0 B\psi\| \geq \epsilon - \inf_{A\in\mathfrak{A}} \|A\psi - P_0 B\psi\| = \epsilon > 0. \tag{2.38}$$

It follow $[B, P_0] \neq 0$ and therefore $B \notin \mathfrak{A}''$. □

Proof of 2.2.3. The first statement is a consequence of Theorem 2.2.2.
Let $\mathfrak{g} = \bigoplus_{n=1}^{\infty} \mathfrak{h}$ and $\|\phi\|_{\mathfrak{g}}^2 = \sum_{n=1}^{\infty} \|\phi_n\|_{\mathfrak{h}}^2$ for $\phi = (\phi_n)_n \in \mathfrak{g}$ and

$$\pi : \mathcal{B}(\mathfrak{h}) \to \mathcal{B}(\mathfrak{g}), \quad \pi[A]\phi = (A\phi_1, A\phi_2, \ldots). \tag{2.39}$$

The reader can easily check, that π is a representation map. $\pi[\mathfrak{A}]$ is a unital $*$-subalgebra of $\mathcal{B}(\mathfrak{g})$, By Lemma 2.2.4 we find for $\psi = (\psi_n)_n \in \mathfrak{g}$ and $\epsilon > 0$ a $C \in \pi[\mathfrak{A}]''$, such that

$$\|\pi[A]\psi - C\psi\|_{\mathfrak{g}} \leq \epsilon. \tag{2.40}$$

Next, we shall show that $\pi[\mathfrak{A}''] \supseteq \pi[\mathfrak{A}]''$. We write operators as infinite matrices, the entries are bounded operators on $\mathcal{B}(\mathfrak{h})$ that map one subspace in the infinite direct sum to another. Hence $\pi[A] = (\delta_{ij}A)_{i,j\in\mathbb{N}}$. Let $B \in \mathcal{B}(\mathfrak{g})$, we write

$$B = (B_{ij})_{i,j\in\mathbb{N}}, \quad C = (C_{ij})_{i,j\in\mathbb{N}}, \quad B_{ij},\, C_{ij} \in \mathcal{B}(\mathfrak{h}). \tag{2.41}$$

To the commutator $\pi[A]B - B\pi[A]$ belongs the matrix

$$(AB_{ij} - B_{ij}A)_{i,j\in\mathbb{N}}. \tag{2.42}$$

We conclude $B \in \pi[\mathfrak{A}]'$ iff $B_{ij} \in \mathfrak{A}'$ for all $i, j \in \mathbb{N}$. Let $E_{kl} := (\delta_{k,i} \cdot \delta_{l,j}\mathbb{1})_{i,j\in\mathbb{N}}$. Clearly, $E_{kl} \in \pi[\mathfrak{A}]'$. Otherwise

$$E_{kl}C = (C_{kj} \cdot \delta_{li})_{i,j\in\mathbb{N}}, \quad CE_{kl} = (C_{il} \cdot \delta_{jk})_{i,j\in\mathbb{N}} \tag{2.43}$$

coincide, hence all matrix entries are equal and we obtain

$$C = (C_{11} \cdot \delta_{ij})_{i,j\in\mathbb{N}}. \tag{2.44}$$

Direct calculation yields $C_{11} \in \mathfrak{A}''$ and therefore $C = \pi[C_{11}] \in \pi[\mathfrak{A}'']$. Equation (2.40) now implies $\left(\sum_{n=1}^{\infty} \|A\psi_n - C_{11}\psi_n\|^2\right)^{1/2} \leq \epsilon$. □

Lemma 2.2.5 (Cyclic and Separating Vectors for $\mathfrak{A}$ and $\mathfrak{A}'$). *Let $\mathfrak{A}$ be a unital subalgebra of $\mathcal{B}(\mathfrak{h})$ and $\phi \in \mathfrak{h}$. We have:*

$$\phi \textit{ is cyclic for } \mathfrak{A} \Leftrightarrow \phi \textit{ is separating for } \mathfrak{A}'.$$

Proof of 2.2.5. Let be $\phi \in \mathfrak{h}$ a cyclic vector for $\mathfrak{A}$, hence $\mathfrak{h} = c\ell\, \mathfrak{A}\phi$. Assume now $B \in \mathfrak{A}'$ with $B\phi = 0$. We need show that $B = 0$. Taking $A \in \mathfrak{A}$ one has $BA\phi = AB\phi = 0$. Thus $B(\mathfrak{A}\phi) = \{0\}$. Since B is a bounded operator, B is zero on $\mathfrak{h}$.
Conversely, let $\phi \in \mathfrak{h}$ be a separating vector for $\mathfrak{A}'$ and $\mathfrak{h}_0 := c\ell\, \mathfrak{A}\phi$. Our goal is to show that $\mathfrak{h}_0 = \mathfrak{h}$. Let P_0 be the orthogonal projection $\mathfrak{h}_0$. As in the proof of 2.2.3, we conclude $P \in \mathfrak{A}'$. From $\phi \in \mathfrak{h}_0$ we obtain $(\mathbb{1} - P_0)\phi = 0$, but $\mathbb{1} - P_0$ in $\mathfrak{A}'$. It follows $P = \mathbb{1}$ and $\mathfrak{h}_0 = \mathfrak{h}$. □

Lemma 2.2.6. $\mathfrak{S}' = \mathfrak{S}'''$, $\mathfrak{S}'' = \mathfrak{S}''''$ *for every non-empty subset $\mathfrak{S}$ of $\mathcal{B}(\mathfrak{h})$, for which $\mathfrak{S} = \{S^* : S \in \mathfrak{S}\}$.*

Proof of 2.2.6. Obviously, $A \subset B \Rightarrow B' \subset A'$. Since $\mathfrak{S} \subset \mathfrak{S}''$, it follows $\mathfrak{S}''' \subset \mathfrak{S}'$. But the inverse inclusion $\mathfrak{S}' \subset \mathfrak{S}'''$ holds, as well.$\mathfrak{S}' = \mathfrak{S}'''$ implies $\mathfrak{S}'' = \mathfrak{S}''''$. □

Example 2.2.7.

- $(\mathbb{C}\mathbb{1})' = \mathcal{B}(\mathfrak{h})$

- $\mathcal{B}(\mathfrak{h})' = \mathbb{C}\mathbb{1}$
- For a separable Hilbert space $\mathfrak{h}$ it is easy to show that $\mathrm{Com}(\mathfrak{h})'' = \mathcal{B}(\mathfrak{h})$. We consider the case $\dim \mathfrak{h} = \infty$. Let $(\phi_n)_{n=1}^{\infty}$ be an ONB of $\mathcal{B}(\mathfrak{h})$ and P_n the orthogonal projection onto $\mathrm{LH}\{\phi_1, \dots, \phi_n\}$. For $A \in \mathcal{B}(\mathfrak{h})$ the operator $P_n A$ is compact, since P_n has finite range. But obviously $A = \text{w-}\lim_{n\to\infty} P_n A$. We remark that Theorem 2.2.3 holds even for algebras without unit.

Definition 2.2.8 (W^*-Algebras). *A unital C^*-subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathfrak{h})$ is called a (concrete) W^*-algebra, iff $\mathfrak{A} = \mathfrak{A}''$.*

Remark 2.2.9 (Further Topologies). There a several other local convex topologies on $\mathcal{B}(\mathfrak{h})$, for which $\mathfrak{A}''$ is the closure of a unital $*$-subalgebra $\mathfrak{A}$. We only have introduced two: the weak topology and the σ-strong topology, they belong to the systems of semi-norms

- $\left\{\rho^1_{\phi,\psi} \,:\, \phi,\psi \in \mathfrak{h}, \quad \rho^1_{\phi,\psi}(A) = |\langle\psi|A\phi\rangle|\right\}$
- $\left\{\rho^2_{\phi} \,:\, \phi \in \oplus_{n=1}^{\infty}\mathfrak{h}, \quad \rho^2_{\phi}(A) = \left(\sum_{n=1}^{\infty} \|A\phi_n\|^2_{\mathfrak{h}}\right)^{1/2}\right\}$

We use in the following to more topologies

- The σ-weak topology : $\left\{\rho^3_{\phi,\psi} \,:\, \phi,\psi \in \oplus_{n=1}^{\infty}\mathfrak{h}, \quad \rho^3_{\phi,\psi}(A) = \sum_{n=1}^{\infty} |\langle\psi_n|A\phi_n\rangle_{\mathfrak{h}}|\right\}$
- The σ-strong* topology : $\left\{\rho^4_{\phi} \,:\, \phi \in \oplus_{n=1}^{\infty}\mathfrak{h}, \quad \rho^4_{\phi}(A) = \left(\sum_{n=1}^{\infty}(\|A\phi_n\|^2_{\mathfrak{h}} + \|A^*\phi_n\|^2_{\mathfrak{h}})\right)^{1/2}\right\}$

The reader is referred to ([7], Chap. 2.4) for the proofs and a more comprehensive representation of this topic.

Remark 2.2.10 ((Abstract) W^*-Algebras). In literature the concept of W^*-algebras is not restricted to subalgebras of $\mathcal{B}(\mathfrak{h})$. There is an intrinsic definition, but we do not need it.

2.3 Tomita-Takesaki-Theory

Let $\mathfrak{A}$ be a W^*-algebra and Ω a cyclic and separating vector for $\mathfrak{A}$. One can define

$$SA\Omega = A^*\Omega, \quad A \in \mathfrak{A}, \quad \mathrm{dom}(S) = \mathfrak{A}\Omega. \tag{2.45}$$

S is an anti-linear, closeable operator on $\mathfrak{h}$. Its closure $\overline{S}$ has a unique polar decomposition

$$\overline{S} = J\Delta^{1/2}, \tag{2.46}$$

where J is an anti-unitary operator and Δ is a positive self-adjoint operator. J is called the modular conjugation and Δ is the modular operator associated with $\mathfrak{A}$ and Ω, since $J = J^2$.

Theorem 2.3.1 (Tomita-Takesaki theorem)**.** *Under the above assumptions one has*

$$J\mathfrak{A}J = \mathfrak{A}' \tag{2.47}$$

$$\Delta^{\imath t}\mathfrak{A}\Delta^{-\imath t} = \mathfrak{A},\ t \in \mathbb{R}. \tag{2.48}$$

$(\Delta^{\imath t})_{t\in\mathbb{R}}$ *is the modular group associated with* $\mathfrak{A}$ *and* Ω*.*

Proof of 2.3.1. See ([7], Thm 2.5.14). □

Definition 2.3.2 (Natural Positive Cone)**.** *The natural positive cone associated with* $\mathfrak{A}$ *and* Ω *is*

$$\mathcal{C} := c\ell\{AJA\Omega\ :\ A \in \mathfrak{A}\}. \tag{2.49}$$

The following theorem holds:

Theorem 2.3.3 (Representation of Normal States)**.** *1.* $\mathcal{C}$ *is convex.*

2. $\mathcal{C}$ *is self-dual, i.e.*

$$\mathcal{C} = \{\phi \in \mathfrak{h}\ :\ \langle\psi|\phi\rangle \geq 0,\ \forall\psi \in \mathcal{C}\}. \tag{2.50}$$

3. For all $\phi \in \mathcal{C}$ *one has*

$$\phi \text{ is separating for } \mathfrak{A} \Leftrightarrow \phi \text{ is cyclic for } \mathfrak{A}.$$

4. For an arbitrary normal state ω *of* $\mathfrak{A}$ *exists a uniquely determined* $\phi \in \mathcal{C}$*, for which*

$$\omega(A) = \langle\phi|A\phi\rangle, \quad A \in \mathfrak{A}. \tag{2.51}$$

Proof of 2.3.3. See ([7], Prop 2.5.28, Prop 2.5.30, Thm 2.5.31). □

Remark 2.3.4. The Tomita-Takesaki-theory yields a characterization of the commutant of a W^*-algebra $\mathfrak{A}$, if a cyclic and separating vector exists. Moreover a $*$-automorphism group can be defined, such as a simple description of normal states. It turns out, confer Theorem 2.4.11, that there is a strong connection between KMS-states and the modular structure. In our case we can explicitly calculate J and Δ.

Example 2.3.5 (Modular Structure).

1. Let $\mathfrak{h}$ be a separable Hilbert space. The two-sided ideal $\mathfrak{K} := \mathcal{L}^2(\mathfrak{h}) \subset \mathcal{B}(\mathfrak{h})$ of Hilbert-Schmidt operators is a Hilbert space equipped with the scalar product $\langle \tau | \sigma \rangle_{\mathfrak{K}} := \mathrm{Tr}\{\tau^* \sigma\}$.
   We define a representation map $\mathcal{B}(\mathfrak{h}) \to \mathcal{B}(\mathfrak{K})$ by $\pi[A]\sigma = A\sigma$. Now,
   $$\langle \tau | \pi[A]\sigma \rangle_{\mathfrak{K}} = \mathrm{Tr}\{\tau^* A \sigma\} = \mathrm{Tr}\{(A^*\tau)^*\sigma\} = \langle \pi[A^*]\tau | \sigma \rangle_{\mathfrak{K}} \tag{2.52}$$
   yields that π is a $*$-morphism. One can show, that $\mathfrak{A} := \pi[\mathcal{B}(\mathfrak{h})]$ is a W^*-algebra. Let $\varrho \in \mathfrak{K}$ be a positive operator with $\ker \varrho = \{0\}$ and $\|\varrho\|_{\mathfrak{K}} = 1$. It follows directly from $\mathrm{ran}\,\varrho = (\ker \varrho)^{\perp} = \mathfrak{K}$, that ϱ is separating for $\mathfrak{A}$.
   Let $(\phi_n)_n \subset \mathfrak{h}$ be an ONB of eigenvectors for ϱ and $\varrho\phi_n = \epsilon_n \phi_n$. We define for $\phi \in \mathfrak{h}$ the operator $A\phi := \epsilon_j^{-1} \langle \phi_j | \phi \rangle\, \phi_i$.
   Obviously, $\pi[A]\varrho = E_{ij}$, where $E_{ij}\phi := \langle \phi_j | \phi \rangle\, \phi_i$. From
   $$c\ell\, \pi[\mathfrak{A}]\varrho \supseteq c\ell\, \mathrm{LH}\{E_{ij} \in \mathfrak{K} \,:\, i, j \in \mathbb{N}\} = \mathfrak{K}$$
   follows cyclicity of ϱ for $\mathfrak{A}$. Let us now consider the modular conjugation and the modular operator. For $\sigma \in \mathfrak{K},\ A \in \mathcal{B}(\mathfrak{h})$ and $\tau \in \mathrm{dom}(\Delta^{1/2}) := \{\tau \in \mathfrak{K} : \sum_{i=1}^{\infty} \|\varrho\tau\varrho^{-1}\phi_i\|_{\mathfrak{h}}^2 < \infty\}$ we define
   $$J\sigma = \sigma^*, \quad \Delta^{1/2}\tau = \varrho\tau\varrho^{-1}. \tag{2.53}$$
   We check $J\pi[A]^*\varrho = \varrho A = \Delta^{1/2}\pi[A]\varrho$. J is obviously anti-linear $J^2 = \mathbb{1}$, moreover
   $$\langle J\sigma | J\tau \rangle_{\mathfrak{K}} = \mathrm{Tr}\{(\sigma^*)^*\tau^*\} = \mathrm{Tr}\{\sigma\tau^*\} = \mathrm{Tr}\{\tau^*\sigma\} = \langle \tau | \sigma \rangle_{\mathfrak{K}}. \tag{2.54}$$

2. $\Delta^{1/2}$ is self-adjoint: Let $P_m = \mathbb{1}[\varrho \geq m^{-1}]$. For τ, $\tau' \in \mathrm{dom}(\Delta^{1/2})$ we have:
   $$\begin{aligned} \langle \Delta^{1/2}\tau | \tau' \rangle_{\mathfrak{K}} &= \lim_{m\to\infty} \langle \Delta^{1/2}\tau P_m | \tau' P_m \rangle_{\mathfrak{K}} = \lim_{m\to\infty} \mathrm{Tr}\{\varrho^{-1}\tau^*\varrho\tau' P_m\} \\ &= \lim_{m\to\infty} \mathrm{Tr}\{\tau^*\varrho\tau'\varrho^{-1}P_m\} = \lim_{m\to\infty} \langle \tau P_m | \Delta^{1/2}\tau' P_m \rangle_{\mathfrak{K}} = \langle \tau | \Delta^{1/2}\tau' \rangle_{\mathfrak{K}}. \end{aligned} \tag{2.55}$$

Assume $|\langle \Delta^{1/2}\tau|\sigma\rangle_{\mathfrak{K}}| \leq C\|\tau\|_{\mathfrak{K}}$ for all $\tau \in \mathrm{dom}(\Delta^{1/2})$. Choose $\tau_m = \varrho\sigma P_m \varrho^{-1}$.

$$\begin{aligned} C\|\tau_m\|_{\mathfrak{K}} &\geq |\langle \Delta^{1/2}\tau_m|\sigma\rangle_{\mathfrak{K}}| = |\operatorname{Tr}\{\varrho^{-2}P_m\sigma^*\varrho^2\sigma\}| \\ &= \operatorname{Tr}\{(\varrho\sigma P_m\varrho^{-1})^*(\varrho\sigma P_m\varrho^{-1})\} = \|\tau_m\|^2_{\mathfrak{K}}. \end{aligned} \tag{2.56}$$

We conclude that $\|\tau_m\|^2_{\mathfrak{K}} = \sum_{i=1}^\infty \|\varrho\sigma P_m\varrho^{-1}\phi_i\|^2_{\mathfrak{h}} \leq C^2$. Hence $\sum_{i=1}^\infty \|\varrho\sigma\varrho^{-1}\phi_i\|^2_{\mathfrak{h}} \leq C^2$ and $\sigma \in \mathrm{dom}(\Delta^{1/2})$.

3. $\Delta^{1/2}$ is positive: Since

$$\operatorname{Tr}\{P_m\sigma^*\varrho\sigma\varrho^{-1}P_m\} = \operatorname{Tr}\{(\varrho^{1/2}\sigma P_m\varrho^{-1/2})^*(\varrho^{1/2}\sigma P_m\varrho^{-1/2})\} \geq 0. \tag{2.57}$$

We obtain for $\sigma \in \mathrm{dom}(\Delta^{1/2})$ that

$$\langle\sigma|\Delta^{1/2}\sigma\rangle_{\mathfrak{K}} = \lim_{m\to\infty} \operatorname{Tr}\{P_m\sigma^*\varrho\sigma\varrho^{-1}P_m\} \geq 0. \tag{2.58}$$

4. $\mathfrak{A}' := J\mathfrak{A}J = \{B \in \mathcal{B}(\mathfrak{K}) \,:\, \exists A \in \mathcal{B}(\mathfrak{h}) \,:\, \forall \sigma \in \mathfrak{K} \quad B\sigma = \sigma A^*\}$.

5. $\mathcal{C} = c\ell\{AJA\varrho \,:\, A \in \mathcal{B}(\mathfrak{h})\}$. Since $AJA\varrho = A\varrho A^*$ all elements of $\mathcal{C}$ are positive operators. Let $\eta \in \mathfrak{K}$ be a positive operator. We define $A = \eta^{1/2}\varrho^{-1/2}P_m$. We conclude $\eta_m := A\varrho A^* = \eta^{1/2}P_m\eta^{1/2} \in \mathcal{C}$.

$$\begin{aligned} \|\eta - \eta_m\|^2_{\mathfrak{K}} &= \operatorname{Tr}\{(\eta^{1/2}(\mathbb{1}-P_m)\eta^{1/2})^2\} = \operatorname{Tr}\{(\mathbb{1}-P_m)\eta(\mathbb{1}-P_m)\eta(\mathbb{1}-P_m)\} \\ &\leq \operatorname{Tr}\{(\mathbb{1}-P_m)\eta^2(\mathbb{1}-P_m)\} \leq \|\eta(\mathbb{1}-P_m)\|^2_{\mathfrak{K}} \to 0 \end{aligned} \tag{2.59}$$

as $m \to \infty$. Hence $\eta \in \mathcal{C}$ and $\mathcal{C} = \{\sigma \in \mathfrak{K} \,:\, \sigma \geq 0\}$.

6. Let $\rho \in \mathcal{B}(\mathfrak{K})$ be a density operator and $(\sigma_n)_n \subset \mathfrak{K}$. an ONB of eigenvectors, such that $\rho\sigma_n = p_n\sigma_n$. The corresponding normal state ω_ρ over $\mathfrak{A}$ is

$$\begin{aligned} \omega_\rho(A) &:= \sum_{n=1}^\infty p_n\langle\sigma_n|A\sigma_n\rangle_{\mathfrak{K}} = \sum_{n=1}^\infty p_n \operatorname{Tr}\{\sigma_n^*A\sigma_n\} \\ &= \sum_{n=1}^\infty p_n \operatorname{Tr}\{\sigma_n\sigma_n^*A\} = \operatorname{Tr}\Big\{\sum_{n=1}^\infty p_n\sigma_n\sigma_n^*A\Big\}. \end{aligned} \tag{2.60}$$

Let $\sigma := \big(\sum_{n=1}^\infty p_n\sigma_n\sigma_n^*\big)^{1/2}$. Since $\operatorname{Tr}\{\sigma_n\sigma_n^*\} = 1$, we obtain that σ is a positive Hilbert-Schmidt operator with $\|\sigma\|_{\mathfrak{K}} = 1$ and

$$\omega_\rho(A) = \langle\sigma|A\sigma\rangle_{\mathfrak{K}}. \tag{2.61}$$

2.4 Dynamical-Systems

Definition 2.4.1 (W^*-dynamical System)**.** *Let $\mathfrak{A}$ be a C^*-algebra.*

1. *A $*$-automorphism group $\tau = (\tau_t)_{t\in\mathbb{R}}$ is a group*

$$\tau_{t+s}(A) = \tau_t(\tau_s(A)) \in \mathfrak{A}, \quad \tau_0(A) = A, \quad \forall A \in \mathfrak{A},\ \forall t, s \in \mathbb{R} \tag{2.62}$$

and for fixed $t \in \mathbb{R}$ one has

$$\tau_t(AB) = \tau_t(A)\tau_t(B), \quad \tau_t(aA + bB) = a\tau_t(A) + b\tau_t(B), \quad \tau_t(A^*) = (\tau_t(A))^*. \tag{2.63}$$

2. *For a W^*-algebra $\mathfrak{A} \subset \mathcal{B}(\mathfrak{h})$ a $*$-automorphism group τ is σ-weakly continuous, if for every $A \in \mathfrak{A}$*

$$\mathbb{R} \ni t \mapsto \tau_t(A) \in \mathcal{B}(\mathfrak{h}) \tag{2.64}$$

is continuous and $\mathcal{B}(\mathfrak{h})$ carries the σ-weak topology.

3. *$(\mathfrak{A}, \tau)$ is a W^*-dynamical system, if $\mathfrak{A}$ is a W^*-algebra and τ is a σ-weakly continuous automorphism group.*

4. *Let τ a $*$-automorphism group and ω a state over $\mathfrak{A}$. ω is τ-invariant, if*

$$\omega(\tau_t(A)) = \omega(A) \tag{2.65}$$

for all $A \in \mathfrak{A}$ and $t \in \mathbb{R}$.

Example 2.4.2 (A W^*-dynamical System)**.** Let H be a self-adjoint (not necessarily bounded) operator on $\mathfrak{h}$. For $A \in \mathcal{B}(\mathfrak{h})$ we define $\tau_t(A) = e^{\imath tH} A e^{-\imath tH},\ t \in \mathbb{R}$. $(\mathcal{B}(\mathfrak{h}), \tau)$ is a W^*-dynamical system and $\omega_\phi(A) := \langle \phi | A\phi \rangle$ is a τ-invariant state, whenever ϕ is a normed eigenvector of H. We sketch the proof:
Choose $\psi = (\psi_n)_n,\ \phi = (\phi_n)_n \in \bigoplus_{n=1}^{\infty} \mathfrak{h}$. For fixed $s \in \mathbb{R}$ and $A \in \mathcal{B}(h)$ we define $f_n(t) = |\langle \psi_n | \big(\tau_t(A) - \tau_s(A)\big)\phi_n \rangle|$. By Stone's theorem $\mathbb{R} \ni t \mapsto e^{\imath tH}\chi$ for $\chi \in \mathfrak{h}$ is continuous. It follows that f_n is continuous, moreover $|f_n(t)| \le \|\psi_n\| \cdot \|\phi_n\| \cdot \|A\|$. Hence $f(t) := \sum_{n=1}^{n} f_n(t) = \rho^3_{\psi,\phi}(\tau_t(A) - \tau_s(A))$ is the uniform limit of continuous functions and therefore continuous. We obtain $\lim_{t\to s} \rho^3_{\psi,\phi}(\tau_t(A) - \tau_s(A)) = f(s) = 0$. $\rho^3_{\psi,\phi}$ is an arbitrary semi-norm that generates the σ-weakly topology.

Lemma 2.4.3 (Analytic Elements of a W^*-Algebra)**.** *Let $(\mathfrak{A}, \tau)$ a W^*-dynamical system. There exists a $*$-subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ of* <u>*τ-analytic elements*</u>*, such that $\mathfrak{A}_0'' = \mathfrak{A}$.*
An element is τ-analytic, if

$$\mathbb{R} \ni t \mapsto \tau_t(A) \in \mathfrak{A}_0 \tag{2.66}$$

has an analytic extension to

$$\mathbb{C} \ni z \mapsto \tau_z(A) \in \mathfrak{A}_0. \tag{2.67}$$

Proof of 2.4.3. The proof can be found in ([4], page 170) □

Lemma 2.4.4. *Let $\mathfrak{A}$ be a W^*-algebra. Every $*$-automorphism is σ-weakly continuous.*

Proof of 2.4.4. The proof can be found in ([4], page 115, Cor 2.12) □

Let $S_\beta := \{z \in \mathbb{C} \, : \, 0 < \Im z < \beta\}$.

Definition 2.4.5 (KMS-States)**.** *Let $(\mathfrak{A}, \tau)$ be a W^*-dynamical system and $\beta > 0$. A normal state ω is (τ, β)-KMS-state, if for $A, B \in \mathfrak{A}$ there is a function $F_\beta(A, B, \cdot)$, that is analytic on S_β and continuous on its closure and taking the boundary conditions*

$$\begin{aligned} F_\beta(A, B, t) &= \omega(A\tau_t(B)) \\ F_\beta(A, B, t + \imath\beta) &= \omega(\tau_t(B)A) \end{aligned} \tag{2.68}$$

for $t \in \mathbb{R}$.

Example 2.4.6 (Gibbs state)**.** Let $\mathfrak{h}$ be a separable Hilbert space. Assume furthermore, that H is a self-adjoint operator, such that

$$Z_\beta := \mathrm{Tr}\{e^{-\beta H}\} < \infty \tag{2.69}$$

for some $\beta > 0$. For $A \in \mathcal{B}(\mathfrak{h})$ we define $\tau_t(A) = e^{\imath tH} A e^{-\imath tH}$, $t \in \mathbb{R}$ and $\omega(A) := Z_\beta^{-1} \cdot \mathrm{Tr}\{e^{-\beta H} A\}$. ω is positive by cyclicity of the trace, and normed since $\omega(\mathbb{1}) = 1$. We show, that ω is a (τ, β)-KMS state over $\mathcal{B}(\mathfrak{h})$. Let $F(A, B, z) = Z_\beta^{-1} \cdot \mathrm{Tr}\{e^{-(\beta + \imath z)H} A e^{\imath z H} B\}$ for $z \in S_\beta$. Let $0 \le \tau_1 := \Re(\beta + \imath z)$ and $0 \le \tau_2 = \Re(-\imath z)$. Obviously $\tau_1 + \tau_2 = \beta$. Since $e^{-(\beta+\imath z)H} \in \mathcal{L}^{\beta/\tau_1}(\mathfrak{h})$ and $e^{\imath z H} \in \mathcal{L}^{\beta/\tau_2}(\mathfrak{h})$ we conclude that $e^{-(\beta+\imath z)H} A e^{\imath z H} B$ is an operator of trace class. ($\mathcal{L}^\alpha(\mathfrak{h})$

are the operators of Schatten α-class). The analyticity properties for $F(A, B, \cdot)$ follow directly. Moreover, the boundary conditions in Equation (2.68) are derived from the cyclicity of the trace.

Lemma 2.4.7. *Let ω be a (τ, β)-KMS-state and $B \in \mathfrak{A}_0$ an analytic element. We have for $z \in \mathbb{C}$ and $A \in \mathfrak{A}$*

$$\omega(A\tau_{z+\imath\beta}(B)) = \omega(\tau_z(B)A). \tag{2.70}$$

In particular,

$$\omega(A\tau_{\imath\beta}(B)) = \omega(BA). \tag{2.71}$$

Proof of 2.4.7. By assumption

$$\begin{aligned} G &: \mathbb{C} \to \mathbb{C}, \quad z \mapsto \omega(A\tau_z(B)) \\ H &: \mathbb{C} \to \mathbb{C}, \quad z \mapsto \omega(\tau_z(B)A) \end{aligned} \tag{2.72}$$

are entire analytic. $F_\beta(A, B, z)$ and $G(z)$ are equal on $\imath\mathbb{R}$. By Schwarz reflection principle $F_\beta(A, B, z)$ and $G(z)$ are identical on $\mathrm{c}\ell\, S_\beta$ and

$$G(t + \imath\beta) = F_\beta(A, B, t + \imath\beta) = H(t),\ t \in \mathbb{R}. \tag{2.73}$$

Since $G(z + \imath\beta)$ and $H(z)$ coincide for $z \in \mathbb{R}$, they are identical. □

Theorem 2.4.8 (Time-Invariance of KMS-states)**.** *Let ω be a (τ, β)-KMS state. It follows that ω is τ-invariant.*

Proof of 2.4.8. Let A be an analytic element and $M := \max_{s\in[0,\beta]} \|\tau_z(A)\|$. Then for $s \in [0, \beta]$ and $t \in \mathbb{R}$

$$|\omega(\tau_{t+\imath s}(A))| = |\omega(\tau_t(\tau_{\imath s}(A)))| \le \|\tau_{\imath s}(A)\| \le M, \tag{2.74}$$

since $\|\omega\| = 1$ and since τ_t is an isometric $*$-automorphism for $t \in \mathbb{R}$. Hence $|\omega(\tau_z(A))| \le M$ for $z \in \mathrm{c}\ell\, S_\beta$. By Lemma 2.4.7 $\omega(\tau_z(A)) = \omega(\tau_{z+\imath\beta}(A))$, hence is entire analytic and bounded. By Liouville's theorem it is constant. In particular,

$$\omega(\tau_t(A)) = \omega(A), \quad t \in \mathbb{R} \tag{2.75}$$

for $A \in \mathfrak{A}_0$. By Lemmata 2.4.3 and 2.4.4 and since ω is normal, Equation (2.75) holds for $A \in \mathfrak{A}$. □

Theorem 2.4.9 (Unitary Implementation of Automorphism Groups)**.** *Let $(\mathfrak{A}, \tau)$ be a W^*-dynamical system and ω a normal τ-invariant state. Let $(\mathfrak{h}, \pi, \Omega)$ be the GNS-representation corresponding to ω. There is a uniquely determined strongly continuous group $(U_t)_{t\in\mathbb{R}}$, such that for $A \in \mathfrak{A}$*

$$U_t \pi(A) U_t^* = \pi(\tau_t(A)), \quad U_t\Omega = \Omega. \tag{2.76}$$

The infinitesimal generator $\mathcal{L}_\omega$ is the ω-Liouvillean.

Proof of 2.4.9. Let $\mathfrak{h}_0 := \pi[\mathfrak{A}]\Omega$. One has to define $U_t\pi[A]\Omega = \pi[\tau_t(A)]\Omega$. First we check well-definedness

$$\begin{aligned} \|\pi(\tau_t(A))\Omega - \pi(\tau_t(B))\Omega\|_\omega^2 &= \omega((\tau_t(A-B))^*\tau_t(A-B)) \quad (2.77)\\ = \quad \omega\big(\tau_t\big((A-B)^*(A-B)\big)\big) &= \omega\big((A-B)^*(A-B)\big) = \|\pi(A)\Omega - \pi(B)\Omega\|^2. \end{aligned}$$

Hence $\pi[\tau_t(A)]\Omega$ is determined by t and $\pi[A]\Omega$. By time-invariance of KMS-state

$$\|U_t\pi[A]\Omega\|^2 = \omega(\tau_t(A)^*\tau_t(A)) = \omega(\tau_t(A^*A)) = \omega(A^*A) = \|\pi[A]\Omega\|^2. \tag{2.78}$$

U_t is isometric on $\mathfrak{h}_0$. As in the proof of Theorem 2.1.9 one obtains that U_t is unitary. Since τ_t is a σ-weakly continuous group and ω is normal, one has

$$\lim_{t\to 0} \|U_t\pi(A)\Omega - \pi(A)\Omega\|^2 = 2\omega(A^*A) - 2\lim_{t\to 0} \Re\omega(A^*\tau_t(A)) = 0. \tag{2.79}$$

The continuity at zero follows from Lemma 2.4.3 and Lemma 2.4.4. □

Lemma 2.4.10. *Let ω be a (τ, β)-KMS-state. For a $A, B \in \mathfrak{A}_0$ one has*

$$\omega(\tau_{-\imath\beta/2}(A)\tau_{\imath\beta/2}(B)) = \omega(BA). \tag{2.80}$$

Proof of 2.4.10. Let

$$G(z) := \omega(\tau_z(A)\tau_{z+\imath\beta}(B)). \tag{2.81}$$

G is entire analytic, for $z = t \in \mathbb{R}$ we obtain

$$G(t) := \omega(\tau_t(A)\tau_{t+\imath\beta}(B)) = \omega(\tau_t(A\tau_{\imath\beta}(B))) = \omega(A\tau_{\imath\beta}(B)) = \omega(BA). \tag{2.82}$$

Hence $G(z) = \omega(BA)$ for all $z \in \mathbb{C}$. The statement follows by setting $z = -\imath\beta/2$. □

The next theorem clarifies the connection between KMS-states and the modular structure.

Theorem 2.4.11 (KMS-State/ Modular Structure)**.** *Let* $\Omega \in \mathfrak{h}$ *a normed vector, cyclic and separating for* $\mathfrak{A}$. *Assume* $\omega(A) := \langle \Omega | A\Omega \rangle$ *is a* (τ, β)*-KMS-state. Then the modular conjugation can be defined as*

$$J A\Omega := \tau_{\imath\beta/2}(A^*)\Omega, \ A \in \mathfrak{A}_0. \tag{2.83}$$

The modular operator is $\Delta = e^{-\beta\mathcal{L}_\omega}$*, where* $\mathcal{L}_\omega$ *is the* ω*-Liouvillean.*

Proof of 2.4.11. Since Ω is separating for $\mathfrak{A}$, one can define for $A \in \mathfrak{A}_0$

$$J A\Omega := \tau_{\imath\beta/2}(A^*)\Omega. \tag{2.84}$$

First we remark, that $\tau_z(cC + dD)\Omega = c\tau_z(C)\Omega + d\tau_z(D)\Omega$ and $\tau_z(C)^*\Omega = \tau_{\overline{z}}(C^*)\Omega$ for C, $D \in \mathfrak{A}_0$ and c, $d \in \mathbb{C}$. Hence

$$\begin{aligned} J(aA\Omega + bB\Omega) = J(aA + bB)\Omega = \tau_{\imath\beta/2}\big((aA + bB)^*\big)\Omega \\ = \ \overline{a}\tau_{\imath\beta/2}(A^*)\Omega + \overline{b}\tau_{\imath\beta/2}(B^*)\Omega = \overline{a}JA\Omega + \overline{b}JB\Omega. \end{aligned} \tag{2.85}$$

For $A, B \in \mathfrak{A}_0$

$$\begin{aligned} \langle JA\Omega | JB\Omega \rangle &= \langle \tau_{\imath\beta/2}(A^*)\Omega | \tau_{\imath\beta/2}(B^*)\Omega \rangle = \langle \Omega | \tau_{-\imath\beta/2}(A)\tau_{\imath\beta/2}(B^*)\Omega \rangle \\ &= \omega(\tau_{-\imath\beta/2}(A)\tau_{\imath\beta/2}(B^*)) = \omega(B^*A) = \langle B\Omega | A\Omega \rangle. \end{aligned} \tag{2.86}$$

Let $\mathfrak{h}_0 := \mathfrak{A}_0\Omega$. Since Ω is cyclic for $\mathfrak{A} = \mathfrak{A}_0''$, we have $\mathfrak{h} = c\ell\, \mathfrak{h}_0$. Because J is anti-linear and isometric on $\mathfrak{h}_0$, it can be uniquely extended to a conjugation on $\mathfrak{h}$. Let $\mathcal{L}_\omega$ be the ω-Liouvillean. Choose $\phi \in \bigcup_{n=1}^\infty \operatorname{ran}\{\mathbb{1}[|\mathcal{L}_\omega| \leq n]\}$. For $A \in \mathfrak{A}_0$ we define

$$f_\phi(z) = \langle \phi | \tau_z(A)\Omega \rangle, \quad g_\phi(z) = \langle e^{-\imath\overline{z}\mathcal{L}_\omega}\phi | A\Omega \rangle. \tag{2.87}$$

Both f_ϕ and g_ϕ are entire analytic, for $z = t$ we have

$$f_\phi(t) = \langle \phi | \tau_t(A)\Omega \rangle = \langle \phi | (e^{\imath t\mathcal{L}_\omega} A e^{-\imath t\mathcal{L}_\omega})\Omega \rangle = g_\phi(t). \tag{2.88}$$

Hence $f_\phi = g_\phi$. For $z = \imath\beta/2$ one has

$$\langle e^{-\beta/2\mathcal{L}_\omega}\phi | A\Omega \rangle = \langle \phi | JA^*\Omega \rangle = \langle \Omega | AJ\phi \rangle. \tag{2.89}$$

Equation (2.89) extends to $\mathfrak{A}_0'' = \mathfrak{A}$, hence for $A \in \mathfrak{A}$ $\langle e^{-\beta/2\mathcal{L}_\omega}\phi | A\Omega \rangle = \langle \phi | JA^*\Omega \rangle$. Since $\bigcup_{n=1}^\infty \operatorname{ran}\{\mathbb{1}[|\mathcal{L}_\omega| \leq n]\}$ is a core of $e^{-\beta/2\mathcal{L}_\omega}$, we obtain $A\Omega \in \operatorname{dom}(e^{-\beta/2\mathcal{L}_\omega})$ and $e^{-\beta/2\mathcal{L}_\omega}A\Omega = JA^*\Omega$. □

Example 2.4.12. Assume now $\mathfrak{h} \subseteq L^2(X, d\mu)$ is a separable, closed subspace. By $\overline{}$ we denote the complex conjugation. Assume also, that $\overline{f} \in \mathfrak{h}$ whenever $f \in \mathfrak{h}$. For an operator A on $\mathfrak{h}$ we define

$$\overline{A}f = \overline{A\overline{f}}, \quad \mathrm{dom}(\overline{A}) := \overline{\mathrm{dom}(A)}. \tag{2.90}$$

For a Hamiltonian H with $\mathrm{Tr}\{e^{-\beta H}\} < \infty$ as in Example 2.4.6 we have a (τ, β)-KMS-state ω. In Example 2.3.5 the modular structure on $\mathfrak{K} = L^2(\mathfrak{h})$ for a cyclic and separating vector is defined. The link between both is the following:
Let $(\phi_j)_{j=1}^\infty$ be a ONB of eigenvectors of H. We denote by E_{ij} the operator

$$E_{ij}\phi = \langle \phi_j | \phi \rangle \phi_i, \ \phi \in \mathfrak{h}. \tag{2.91}$$

The set $\{E_{ij} \, : \, i, j \in \mathbb{N}\}$ is an ONB of $\mathfrak{K}$. We identify $\mathfrak{K}$ with $\mathfrak{h} \otimes \mathfrak{h} \subset L^2(X \times X, d\mu \otimes d\mu)$ via the unitary p, that is defined by $p \, : \, E_{ij} \mapsto \phi_i(x)\overline{\phi_j(y)}$. We understand elements of $\mathfrak{h} \otimes \mathfrak{h}$ as L^2-functions of $(x, y) \in X \times X$. The modular conjugation becomes to $J\kappa(x, y) = \overline{\kappa(y, x)}$. The Liouvillean reads $L = H_x - \overline{H}_y$, where the subscript x means that H acts for fixed y, and similarly, the subscript y means that $\overline{H}$ acts for fixed x. Instead of $\pi[A]$ we only write A_x. The vector $\Omega(x, y) := Z_\beta^{-1/2} \sum_{j=1}^\infty e^{-\beta/2E_j} \phi_j(x)\overline{\phi_j(y)}$ is the cyclic vector, so that

$$\omega(A) = \langle \Omega | A_x \Omega \rangle_{\mathfrak{h}\otimes\mathfrak{h}}. \tag{2.92}$$

2.5 Perturbation of W^*-dynamical systems

Definition 2.5.1. *A closed operator A is affiliated with $\mathfrak{A}$, if*

$$\mathfrak{A}' \,\mathrm{dom}(A) \subset \mathrm{dom}(A) \quad AA' \supset A'A, \ \forall \, A' \in \mathfrak{A}'. \tag{2.93}$$

Lemma 2.5.2. *If A is self-adjoint and affiliated with $\mathfrak{A}$, then all bounded Borel functions of A belong to $\mathfrak{A}$.*

Proof of 2.5.2. The proof follows from ([7], Lemma 2.58) and the spectral theorem, confer also ([10], Thm 2.1) □

Theorem 2.5.3. *Let $\alpha_t(A) = e^{\imath t\mathcal{L}} A e^{-\imath t\mathcal{L}}$, $t \in \mathbb{R}$ be a $*$-automorphism group acting on $\mathfrak{A}$, i.e. $\mathcal{L}$ is a self-adjoint operator and $e^{\imath t\mathcal{L}} A e^{-\imath t\mathcal{L}} \in \mathfrak{A}$ for $A \in \mathfrak{A}$. Let Q be a self-adjoint operator*

affiliated with $\mathfrak{A}$*, such that* $\mathrm{dom}(\mathcal{L}) \cap \mathrm{dom}(Q)$ *is a core for* $\mathcal{L}$*,* Q *and* $\mathcal{L}+Q$*. (In particular,* $\mathcal{L}+Q$ *is assumed to be essentially self-adjoint.)*
Then

$$\alpha_t^Q(A) = e^{\imath t(\mathcal{L}+Q)} A e^{-\imath t(\mathcal{L}+Q)}, \; t \in \mathbb{R}, \; A \in \mathfrak{A} \tag{2.94}$$

defines a $*$*-automorphism group acting on* $\mathfrak{A}$*.*

Proof of 2.5.3. In the sense of weak convergence in $\mathcal{B}(\mathfrak{h})$ we obtain using Trotter's product formula

$$\begin{aligned} \alpha_t^Q(A) &= e^{\imath t(\mathcal{L}+Q)} A e^{-\imath t(\mathcal{L}+Q)} = \text{w-lim}_{n\to\infty} \left(e^{\imath \frac{t}{n}\mathcal{L}} e^{\imath \frac{t}{n} Q}\right)^n A \left(e^{-\imath\frac{t}{n}Q} e^{-\imath \frac{t}{n}\mathcal{L}}\right)^n \\ &= \text{w-lim}_{n\to\infty} \alpha_{\frac{t}{n}}\left(e^{\imath \frac{t}{n} Q} \cdots \alpha_{\frac{t}{n}}(e^{\imath\frac{t}{n}Q} A e^{-\imath \frac{t}{n}Q}) \cdots e^{-\imath\frac{t}{n}Q}\right). \end{aligned} \tag{2.95}$$

Since Q is affiliated with $\mathfrak{A}$, one has $e^{\imath\frac{t}{n}Q} \in \mathfrak{A}$ and $e^{\imath \frac{t}{n}Q} A e^{-\imath\frac{t}{n}Q} \in \mathfrak{A}$, whenever $A \in \mathfrak{A}$. Moreover, α leaves $\mathfrak{A}$ invariant. Hence $\alpha_t^Q(A)$ is the weak limit of operators in $\mathfrak{A}$, and therefore $\alpha_t^Q(A) \in \mathfrak{A}$. □

Theorem 2.5.4. *Let* $\Omega \in \mathfrak{h}$ *be a cyclic and separating vector and* J *be the modular conjugation associated with* Ω *and* $\mathfrak{A}$*. Let* $\alpha_t(A) = e^{\imath t\mathcal{L}} A e^{-\imath t \mathcal{L}}$ *be a* $*$*-automorphism, where* $\mathcal{L}$ *is a self-adjoint operator. Furthermore let* Q *be a self-adjoint operator affiliated with* $\mathfrak{A}$*, such that* $\mathrm{dom}(\mathcal{L}) \cap \mathrm{dom}(JQJ)$ *is a core for* $\mathcal{L}$*,* JQJ *and* $\mathcal{L} - JQJ$*. (In particular* $\mathcal{L}-JQJ$ *is assumed to be essentially self-adjoint.)*
Then

$$\alpha_t(A) = e^{\imath t(\mathcal{L}-JQJ)} A e^{-\imath t(\mathcal{L}-JQJ)}, \; t\in\mathbb{R}, \; A \in \mathfrak{A}. \tag{2.96}$$

Proof of 2.5.4. Obviously, $e^{\imath t JQJ} = J e^{-\imath t Q} J \in \mathfrak{A}'$. As in the proof of Theorem 2.5.3, we obtain for $A \in \mathfrak{A}$

$$\begin{aligned} & e^{\imath t(\mathcal{L}-JQJ)} A e^{-\imath t(\mathcal{L}-JQJ)} = \lim_{n\to\infty} \left(e^{\imath\frac{t}{n}\mathcal{L}} e^{-\imath\frac{t}{n}JQJ}\right)^n A \left(e^{\imath \frac{t}{n}JQJ} e^{-\imath\frac{t}{n}\mathcal{L}}\right)^n \\ &= \lim_{n\to\infty} \alpha_{\frac{t}{n}}\left(e^{-\imath\frac{t}{n}JQJ} \cdots \alpha_{\frac{t}{n}}(e^{-\imath\frac{t}{n}JQJ} \alpha_{\frac{t}{n}}(e^{-\imath\frac{t}{n}JQJ} A e^{\imath\frac{t}{n}JQJ}) e^{\imath\frac{t}{n}JQJ}) \cdots e^{\imath\frac{t}{n}JQJ}\right) \\ &= \lim_{n\to\infty} \alpha_{\frac{t}{n}}\left(e^{-\imath\frac{t}{n}JQJ} \cdots \alpha_{\frac{t}{n}}(e^{-\imath\frac{t}{n}JQJ} \alpha_{\frac{t}{n}}(A) e^{\imath\frac{t}{n}JQJ}) \cdots e^{\imath\frac{t}{n}JQJ}\right) \\ &= \alpha_t(A), \end{aligned} \tag{2.97}$$

since $\alpha_t(A) \in \mathfrak{A}$ for all $t \in \mathbb{R}$. □

We prove a simple statements, that will be needed:

Lemma 2.5.5. *Let H be a self-adjoint operator on a separable Hilbert space. $\beta > 0$ and $\phi \in \mathrm{dom}(e^{-\beta/2H})$. Then $\phi \in \mathrm{dom}(e^{-zH})$, whenever $0 \leq \Re z \leq \beta/2$.*

Proof. Proof of 2.5.5 We assume that ϕ is normed. Let μ be the probability measure corresponding to μ and H. By the spectral theorem we obtain $\int_{\mathbb{R}} e^{-\beta s}\mu(ds) < \infty$. But for $z \in S_{\beta/2}$ we have

$$\int_{\mathbb{R}} |e^{-zs}|^2 \mu(ds) = \int_{]-\infty,0]} e^{2\Re zs}\mu(ds) + \int_{]0,\infty[} e^{2\Re zs}\mu(ds) \leq 1 + \int_{\mathbb{R}} e^{-\beta s}\mu(ds) < \infty. \tag{2.98}$$

The spectral theorem yields Lemma 2.5.5. □

Theorem 2.5.6. *Assume $\Omega \in \mathrm{dom}(e^{-\beta/2(\mathcal{L}+Q)})$ be a cyclic and separating vector for a W^*-algebra $\mathfrak{A}$. $\mathcal{L}$ and J are the Standard Liouvillean and the modular conjugation corresponding to $\mathfrak{A}$ and J. Let $\Omega_Q := e^{-\beta/2(\mathcal{L}+Q)}\Omega \in \mathfrak{h}$*

The following it true:

1. $J\Omega_Q = \Omega_Q$
2. $\Omega_Q = e^{\beta/2(\mathcal{L}-JQJ)}\Omega$
3. $\mathcal{L}_Q\Omega_Q = 0$
4. $JA^*\Omega_Q = e^{-\beta/2\mathcal{L}_Q}A\Omega_Q$ *for* $A \in \mathfrak{A}$
5. Ω_Q *is separating for* $\mathfrak{A}$
6. Ω_Q *is cyclic for* $\mathfrak{A}$
7. *Let* $\omega_Q(A) := \|\Omega_Q\|^{-2} \cdot \langle \Omega_Q | A\Omega_Q \rangle$. ω_Q *is a* (α^Q, β)*-KMS-state.*

Proof of 2.5.6. First, we define $\Omega(z) = e^{-z(\mathcal{L}+Q)}\Omega$ for $z \in \mathcal{S}_{\beta/2}$ and $E(t) := e^{\imath t(\mathcal{L}+Q)}e^{-\imath t\mathcal{L}} \in \mathcal{B}(\mathfrak{h})$.

$$\begin{aligned} e^{\imath t(\mathcal{L}+Q)}e^{-\imath t\mathcal{L}} &= \text{s-lim}_{n\to\infty} \left(e^{\imath \frac{t}{n}\mathcal{L}}e^{\imath\frac{t}{n}Q}\right)^n e^{-\imath t\mathcal{L}} \\ &= \text{s-lim}_{n\to\infty} \alpha_{\frac{t}{n}}(e^{\imath\frac{t}{n}Q})\alpha_{\frac{2t}{n}}(e^{\imath\frac{t}{n}Q})\cdots\alpha_{\frac{nt}{n}}(e^{\imath\frac{t}{n}Q}) \\ &= \text{s-lim}_{n\to\infty} \left(e^{\imath\frac{t}{n}(\mathcal{L}-JQJ)}e^{\imath\frac{t}{n}Q}\right)^n e^{-\imath t(\mathcal{L}-JQJ)} \\ &= e^{\imath t(\mathcal{L}+Q-JQJ)}e^{-\imath t(\mathcal{L}-JQJ)}. \end{aligned} \tag{2.99}$$

From the second line in (2.99) follows that $E(t)$ is the strong limit of operators in $\mathfrak{A}$, therefore $E(t) = e^{\imath t(\mathcal{L}+Q-JQJ)}e^{-\imath t(\mathcal{L}-JQJ)} \in \mathfrak{A}$.

1. We choose $\phi \in \bigcup_{n\in\mathbb{N}} \operatorname{ran} \mathbb{1}[|\mathcal{L}| \leq n]$. Let

$$f(z) := \langle \phi | J\Omega(\overline{z})\rangle \text{ and } g(z) := \langle e^{-(\beta/2-\overline{z})\mathcal{L}}\phi | e^{-z(\mathcal{L}+Q)}\Omega\rangle.$$

Both f and g are analytic on $\mathcal{S}_{\beta/2}$ and continuous on its closure. Since

$$f(\imath t) = \langle \phi | JE(t)\Omega\rangle = \langle \phi | e^{-\beta\mathcal{L}/2}E(t)^*\Omega\rangle = g(\imath t),\ t\in\mathbb{R}. \tag{2.100}$$

By Schwarz theorem f and g are equal, particularly in $z=\beta/2$. It follows since $e^{-z(\mathcal{L}+Q)}$ is self-adjoint, and since ϕ is an arbitrary element of a core, that $J\Omega(\beta/2) = \Omega(\beta/2)$.

2. Similarly, we choose $\phi \in \bigcup_{n\in\mathbb{N}} \operatorname{ran} \mathbb{1}[|\mathcal{L}-JQJ| \leq n]$. Now we define $g(z) = \langle e^{\overline{z}(\mathcal{L}-JQJ)}\phi | e^{-z\mathcal{L}}\Omega\rangle$. Since $JE(t)J = e^{\imath t(\mathcal{L}-JQJ)}e^{-\imath t\mathcal{L}}$, g coincides for real $z = \imath t$ with $f(z) := \langle \phi | J\Omega(\overline{z})\rangle$. Hence they are equal in $z=\beta/2$ and therefore $\Omega \in \operatorname{dom}(e^{\beta/2(\mathcal{L}-JQJ)})$ and $\Omega(\beta/2) = e^{\beta/2(\mathcal{L}-JQJ)}\Omega$.

3. Choose $\phi \in \bigcup_{n\in\mathbb{N}} \operatorname{ran} \mathbb{1}[|\mathcal{L}_Q| \leq n]$. We define $g(z) := \langle e^{-\overline{z}\mathcal{L}_Q}\phi | e^{z(\mathcal{L}-JQJ)}\Omega\rangle$ and $f(z) := \langle \phi | \Omega(z)\rangle$ for z in the closure of $\mathcal{S}_{\beta/2}$. Again both functions are equal on the line $z = \imath t$, $t\in\mathbb{R}$. Hence f and g are identical, and therefore $\Omega(\beta/2) \in \operatorname{dom}(e^{-\beta/2\mathcal{L}_Q})$ and $e^{-\beta/2\mathcal{L}_Q}\Omega(\beta/2) = \Omega(\beta/2)$. We conclude that $\mathcal{L}_Q\Omega(\beta/2) = 0$.

4. Let $A\in\mathfrak{A}_0$ be an α_Q-analytic element of $\mathfrak{A}$. Hence

$$\begin{aligned} JA^*\Omega(-\imath t) &= JA^*E(t)\Omega = e^{-\beta/2\mathcal{L}}E(t)^*A\Omega_0^\beta \\ &= e^{-(\beta/2-\imath t)\mathcal{L}}e^{-\imath t(\mathcal{L}+Q)}A\Omega_0^\beta = e^{-(\beta/2-\imath t)\mathcal{L}}\alpha_{-t}^Q(A)e^{-\imath t(\mathcal{L}+Q)}\Omega. \end{aligned} \tag{2.101}$$

Let $\phi \in \bigcup_{n\in\mathbb{N}} \operatorname{ran} \mathbb{1}[|\mathcal{L}| \leq n]$. We define

$$f(z) = \langle \phi | JA^*\Omega(\overline{z})\rangle \quad\text{and}\quad g(z) = \langle e^{-(\beta/2-\overline{z})\mathcal{L}}\phi | \alpha_{\imath z}(A)\Omega(z)\rangle.$$

Since f and g are analytic and equal for $z=\imath t$, we have $JA^*\Omega(\beta/2) = \alpha_{\imath\beta/2}(A)\Omega(\beta/2)$. To finish the proof we choose $\phi \in \bigcup_{n\in\mathbb{N}} \operatorname{ran} \mathbb{1}[|\mathcal{L}_Q| \leq n]$. Let

$$f(z) := \langle \phi | \alpha_{\imath z}^Q(A)\Omega(\beta/2)\rangle \text{ and } g(z) := \langle e^{-\overline{z}\mathcal{L}_Q}\phi | A\Omega(\beta/2)\rangle.$$

For $z = \imath t$ we see

$$g(\imath t) = \langle \phi | e^{-\imath t \mathcal{L}_Q} A e^{\imath t \mathcal{L}_\lambda} \Omega(\beta/2) \rangle = \langle \phi | \alpha^Q_{-t}(A) \Omega(\beta/2) \rangle = f(\imath t). \quad (2.102)$$

Hence $A\Omega(\beta/2) \in \mathrm{dom}(e^{-\beta/2\mathcal{L}_Q})$ and $JA^*\Omega(\beta/2) = e^{-\beta/2\mathcal{L}_Q} A\Omega(\beta/2)$.

As in Equation (2.89) one can extend the result to all $A \in \mathfrak{A}$.

5. Let $A \in \mathfrak{A}_0$. We choose $\phi \in \bigcup_{n \in \mathbb{N}} \mathrm{ran}\, 1[|(\mathcal{L} + Q)| \le n]$. First, we have

$$JA^*\Omega(\beta/2) = \alpha_{\imath\beta/2}(A)\Omega(\beta/2). \quad (2.103)$$

Let $f_\phi(z) = \langle \phi | \alpha_z(A)\Omega(\beta/2) \rangle$ and $g_\phi(z) = \langle e^{\overline{z}(\mathcal{L}+Q)}\phi | A e^{-(\beta/2+z)(\mathcal{L}+Q)}\Omega \rangle$ for $-\beta/2 \le \Re z \le 0$. Both functions are continuous and analytic if $-\beta/2 < \Re z < 0$. Furthermore , $f_\phi(\imath t) = g_\phi(\imath t)$ for $t \in \mathbb{R}$. Hence $f_\phi = g_\phi$ and for $z = -\beta/2$

$$\langle \phi | JA^*\Omega(\beta/2) \rangle = \langle e^{-\beta/2(\mathcal{L}+Q)}\phi | A\Omega \rangle. \quad (2.104)$$

This equation extends to all $A \in \mathfrak{A}$, we obtain $A\Omega \in \mathrm{dom}(e^{-\beta/2(\mathcal{L}+Q)})$ and $e^{-\beta/2(\mathcal{L}+Q)}A\Omega = JA^*\Omega(\beta/2)$ for $A \in \mathfrak{A}$. Assume $A^*\Omega(\beta/2) = 0$, then $e^{-\beta/2(\mathcal{L}+Q)}A\Omega = 0$ and $A\Omega = 0$. Since Ω is separating, it follows that $A = 0$ and therefore $A^* = 0$.

6. Let $\mathcal{C}$ be the natural positive cone associated with J and Ω. To prove that $\phi \in \mathcal{C}$ it is sufficient to check that $\langle \phi | A\mathcal{J}A\Omega_0^\beta \rangle \ge 0$ for all $A \in \mathfrak{M}$. We have

$$\langle \Omega(\beta/2) | AJA\Omega_0^\beta \rangle = \overline{\langle JA^*\Omega(\beta/2) | A\Omega_0^\beta \rangle} = \overline{\langle e^{-\beta/2(\mathcal{L}+Q)} A\Omega_0^\beta | A\Omega_0^\beta \rangle} \ge 0.$$

7. For $A, B \in \mathfrak{A}$ and $z \in S_\beta$ we define

$$F_\beta(A, B, z) = c\langle e^{-\imath\overline{z}/2\mathcal{L}_Q} A^*\Omega_Q | e^{\imath z/2\mathcal{L}_Q} B\Omega_Q \rangle, \quad (2.105)$$

where $c := \|\Omega_Q\|^{-2}$. First, we observe

$$\begin{aligned} F_\beta(A, B, t) &= c\langle e^{-\imath t/2\mathcal{L}_Q} A^*\Omega_Q | e^{\imath t/2\mathcal{L}_Q} B\Omega_Q \rangle = c\langle \Omega_Q | A\alpha_t^Q(B)\Omega_Q \rangle \quad (2.106) \\ &= \omega_Q(A\alpha_t^Q(B)) \end{aligned}$$

and

$$\begin{aligned} &\omega_Q(\alpha_t^Q(B)A) = c\langle \alpha_t^Q(B^*)\Omega_Q | A\Omega_Q \rangle = c\langle JA\Omega_Q | J\alpha_t^Q(B^*)\Omega_Q \rangle \quad (2.107) \\ &= c\langle e^{-\beta/2\mathcal{L}_Q} A^*\Omega_Q | e^{-\beta/2\mathcal{L}_Q} \alpha_t^Q(B)\Omega_Q \rangle = c\langle e^{-\imath\overline{(\imath\beta+t)}/2\mathcal{L}_Q} A^*\Omega_Q | e^{\imath(\imath\beta+t)/2\mathcal{L}_Q} B\Omega_Q \rangle \\ &= F_\beta(A, B, t + \imath\beta). \end{aligned}$$

The requirements on the analyticity of $F_\beta(A,B,\cdot)$ are directly fulfilled, hence ω_Q is a (α_Q,β)-KMS-state. □

Lemma 2.5.7. *Let $Q\in\mathcal{B}(\mathfrak{h})$ self-adjoint and $\alpha_t^Q(A)=e^{\imath t(\mathcal{L}+Q)}Ae^{-\imath t(\mathcal{L}+Q)}$ one has for $t>0$*

$$\alpha_t^Q(A)=\alpha_t(A)+\sum_{n=1}^{\infty}\imath^n\int_{0\le t_n\le\ldots\le t_1\le t}[\alpha_{t-t_1}(Q),[\alpha_{t-t_2}(Q),\cdots[\alpha_{t-t_n}(Q),\alpha_t(A)]\cdots]]\,d\underline{t}. \quad (2.108)$$

The convergence is in the norm on $\mathcal{B}(\mathfrak{h})$. The integral is defined in a weak sense. For $t<0$ the statement is analog.

Proof of 2.5.7. First, we define for all $t\in\mathbb{R}$

$$Y_t=e^{-\imath t\mathcal{L}}e^{\imath t(\mathcal{L}+Q)}Ae^{-\imath t(\mathcal{L}+Q)}e^{\imath t\mathcal{L}},\quad Y_0:=A. \quad (2.109)$$

It is well known, that $\mathrm{dom}(\mathcal{L})=\mathrm{dom}(\mathcal{L}+Q)$ and that $e^{\imath t\mathcal{L}}$ and $e^{\imath t(\mathcal{L}+Q)}$ leave $\mathrm{dom}(\mathcal{L})$ invariant. For $\phi,\ \psi\in\mathrm{dom}(\mathcal{L})$ we obtain directly

$$\begin{aligned}\frac{d}{dt}\langle\phi|Y_t\psi\rangle &= \frac{d}{dt}\langle e^{-\imath t(\mathcal{L}+Q)}e^{\imath t\mathcal{L}}\phi|Ae^{-\imath t(\mathcal{L}+Q)}e^{\imath t\mathcal{L}}\psi\rangle \quad (2.110)\\ &= \langle e^{-\imath t(\mathcal{L}+Q)}(-\imath)Qe^{\imath t\mathcal{L}}\phi|Ae^{-\imath t(\mathcal{L}+Q)}e^{\imath t\mathcal{L}}\psi\rangle+\langle e^{-\imath t(\mathcal{L}+Q)}e^{\imath t\mathcal{L}}\phi|Ae^{-\imath t(\mathcal{L}+Q)}(-\imath)Qe^{\imath t\mathcal{L}}\psi\rangle\\ &= \langle\phi|\imath[\alpha_{-t}(Q),Y_t]\psi\rangle.\end{aligned}$$

It is a simple calculation to check that for all bounded operator $C,D\in\mathcal{B}(\mathfrak{h})$ and $\chi\in\mathfrak{h}$ the function $\mathbb{R}\ni Ce^{\imath t(\mathcal{L}+Q)}De^{\imath t\mathcal{L}}\chi$ is in $\mathcal{C}(\mathbb{R};\mathfrak{h})$. Since $\|Y_t\|\le\|A\|$ the fundamental theorem of calculus yields

$$|\langle\phi|(Y_t-Y_s)\psi\rangle|\le\left|\int_s^t\langle\phi|\imath[\alpha_r(Q),Y_r]\psi\rangle\,dr\right|\le 2\|\phi\|\,\|\psi\|\,|t-s|\,\|Q\|\,\|A\|. \quad (2.111)$$

The density of $\mathrm{dom}(\mathcal{L})$ in $\mathfrak{h}$ implies $Y_\cdot\in\mathcal{C}(\mathbb{R};\mathcal{B}(\mathfrak{h}))$.

For fixed $t_0\in\mathbb{R}$ and $T>0$ we define

$$\begin{gathered}F\ :\ \mathcal{C}([t_0-T,t_0+T];\mathcal{B}(\mathfrak{h}))\to\mathcal{C}([t_0-T,t_0+T];\mathcal{B}(\mathfrak{h})) \quad (2.112)\\ F[Z](t):=A_{t_0}+\imath\int_{t_0}^t[\alpha_{-r}(Q),Z_r]\,dr.\end{gathered}$$

The integral can be defined in a weak sense, since the integrand is weakly continuous. This means $\imath\int_{t_0}^t[\alpha_{-r}(Q),Z_r]\,dr$ is the bounded operator corresponding to the form

$$(\phi,\psi)\mapsto\imath\int_{t_0}^t\langle\phi|[\alpha_{-r}(Q),Z_r]\psi\rangle\,dr,\ \ \phi,\psi\in\mathfrak{h}.$$

For T small enough is F a contraction:

$$\|F[Z]-F[Z']\|_{[t_0-T,t_0+T]} \leq 2\,T\,\|Q\|\,\|Z-Z'\|_{[t_0-T,t_0+T]}, \tag{2.113}$$

where $\|Z\|_{[t_0-T,t_0+T]} = \sup_{t\in[t_0-T,t_0+T]}\|Z(t)\|$ is the norm of $\mathcal{C}([t_0-T,t_0+T];\mathcal{B}(\mathfrak{h}))$. Choosing $A_{t_0}=Y_{t_0}$ implies that Y is a fixed point of F.

We construct a solution by the Picard-Lindelöf iteration, formally $\tilde{Y}=\lim_{n\to\infty}F^{(n)}(A)$, and define

$$\tilde{Y}(t) = A + \sum_{n=1}^{\infty} \imath^n \int_{0\leq t_n\leq\ldots\leq t_1\leq t} [\alpha_{-t_1}(Q),[\alpha_{-t_2}(Q),\cdots[\alpha_{-t_n}(Q),A]\cdots]]d\underline{t}. \tag{2.114}$$

since $\left|\int_{0\leq t_n\leq\ldots\leq t_1\leq t} 1 d\underline{t}\right| \leq \frac{|t|^n}{n!}$ and

$$\|[\alpha_{-t_1}(Q),[\alpha_{-t_2}(Q),\cdots[\alpha_{-t_n}(Q),A]\cdots]]\| \leq 2^n\|Q\|^n\|A\|,$$

the series is norm-convergent resp.$\|\cdot\|$ and all $t\in\mathbb{R}$. Moreover, $\tilde{Y}$ is continuous and weakly differentiable, and

$$\begin{aligned}\tilde{Y}(t) &= A+\imath\int_0^t[\alpha_{-r}(Q),\tilde{Y}(r)]dr = \tilde{Y}(t_0)+\int_{t_0}^t\frac{d}{dr}\tilde{Y}(r)dr \\ &= \tilde{Y}(t_0)+\imath\int_{t_0}^t[\alpha_{-r}(Q),\tilde{Y}(r)]dr.\end{aligned} \tag{2.115}$$

Hence $\tilde{Y}\in\mathcal{C}([t_0-T,t_0+T];\mathcal{B}(\mathfrak{h}))$ is a fixed point for F. Since F is a contraction we obtain $\tilde{Y}=Y$ on $[t_0-T,t_0+T]$ and hence for $t\in\mathbb{R}$. But $\alpha_t^Q(A)=\alpha_t(Y(t))=\alpha_t(\tilde{Y}(t))$. □

2.6 Ergodic Properties

Let $(\mathfrak{A},\alpha)$ be a W^*-dynamical system. Ω is a normed, cyclic and separating vector and $\omega(A)=\langle\Omega|A\Omega\rangle$ is the vector state associated with Ω.

Definition 2.6.1. *1. $(\mathfrak{A},\alpha,\omega)$ is ergodic, if for all ω-normal states μ, one has*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mu(\alpha_t(A))dt=\omega(A), \quad \forall A\in\mathfrak{A}. \tag{2.116}$$

2. $(\mathfrak{A},\alpha,\omega)$ is mixing, if for all ω-normal states μ

$$\lim_{t\to\infty}\mu(\alpha_t(A))=\omega(A), \quad \forall A\in\mathfrak{A}. \tag{2.117}$$

An immediate consequence is

Corollary 2.6.2. *If* $(\mathfrak{A}, \alpha, \omega)$ *is mixing, then* $(\mathfrak{A}, \alpha, \omega)$ *is ergodic. If* $(\mathfrak{A}, \alpha, \omega)$ *is ergodic, then* ω *is time-invariant.*

Lemma 2.6.3 (Mean Ergodic Theorem)**.** *Let* $\mathcal{L}$ *be self-adjoint, then for* $\psi \in \mathfrak{h}$*.*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{\imath t\mathcal{L}} \psi dt = \mathbb{1}[\mathcal{L} = 0]\psi. \tag{2.118}$$

Proof of 2.6.3. For the proof see ([4],Thm. 3.13). □

Theorem 2.6.4 (Koopman Ergodic Theorem)**.** *Let* $\mathcal{L}_\omega$ *be the* ω*-Liouvillean of* α*.* $(\mathfrak{A}, \alpha, \omega)$ *is ergodic, iff* $\ker\{\mathcal{L}_\omega\} = \{\mathbb{C}\Omega\}$*.*

Proof of 2.6.4. First we consider the states μ, that can be written as $\mu(A) = \langle R\Omega | AR\Omega\rangle$ for $A \in \mathfrak{A},\ R \in \mathfrak{A}'$. From the Mean Ergodic Theorem follows

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mu(\alpha_t(A))dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \langle R\Omega|\alpha_t(A)R\Omega\rangle dt &(2.119)\\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \langle R^*R\Omega|\alpha_t(A)\Omega\rangle dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \langle R^*R\Omega|e^{\imath t\mathcal{L}_\omega}(A)\Omega\rangle dt\\
&= \langle R^*R\Omega|\mathbb{1}[\mathcal{L}_\omega = 0]A\Omega\rangle.
\end{aligned}$$

Let P be the orthogonal projection onto Ω. Since Ω is separating for $\mathfrak{A}$, it is cyclic for $\mathfrak{A}'$, furthermore

$$\mathfrak{h} = c\ell\,\mathrm{LH}\{R^*R\Omega\ :\ R \in \mathfrak{A}',\ \|R\Omega\| = 1\} = c\ell\,\mathrm{LH}\{A\Omega\ :\ A \in \mathfrak{A}\}. \tag{2.120}$$

Since $\omega(A) = \langle R^*R\Omega|\mathbb{1}[\mathcal{L}_\omega = 0]A\Omega\rangle$ we have for all $A \in \mathfrak{A}$ and all μ as specified above

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mu(\alpha_t(A))dt = \omega(A) \Leftrightarrow \mathbb{1}[\mathcal{L}_\omega = 0] = P. \tag{2.121}$$

Let μ be an arbitrary ω normal state. From the Tomita-Takesaki-theory follows, that there is a $\phi \in \mathcal{C}$ with $\mu(A) = \langle\phi|A\phi\rangle$ for $A \in \mathfrak{A}$. Assume $\mathbb{1}[\mathcal{L}_\omega = 0] = P$ and $R \in \mathfrak{A}',\ \|R\Omega\| = 1$.

$$\begin{aligned}
&\limsup_{T\to\infty} \Big|\frac{1}{2T} \int_{-T}^{T} \mu(\alpha_t(A))dt - \omega(A)\Big| &(2.122)\\
&= \limsup_{T\to\infty} \Big|\frac{1}{2T} \int_{-T}^{T} \big(\langle\phi|\alpha_t(A)\phi\rangle - \langle R\Omega|\alpha_t(A)R\Omega\rangle\big)dt\Big|\\
&\leq 2\|A\|\,\|R\Omega - \phi\|. &(2.123)
\end{aligned}$$

Since Ω is cyclic for $\mathfrak{A}'$, one has $\inf_{R\in\mathfrak{A}',\ \|R\Omega\|=1} \|R\Omega - \phi\| = 0$. □

Lemma 2.6.5. *Assume* $\sigma_{ac}(\mathcal{L}_\omega) = \sigma(\mathcal{L}_\omega) \setminus \{0\}$ *and* $\dim\ker\mathcal{L}_\omega = 1$, *then* $(\mathfrak{A}, \alpha, \omega)$ *is mixing.*

Proof of 2.6.5. First, we consider states $\mu(A) = \langle R\Omega | AR\Omega\rangle$ for $R \in \mathfrak{A}'$ with $1 = \langle R\Omega|R\Omega\rangle$. We obtain

$$\mu(\alpha_t(A)) - \omega(A) = \langle R\Omega|\alpha_t(A)R\Omega\rangle - \langle\Omega|A\Omega\rangle \tag{2.124}$$
$$= \langle R^*R\Omega|e^{\imath t\mathcal{L}_\omega}A\Omega\rangle - \langle R^*R\Omega|\Omega\rangle\langle\Omega|A\Omega\rangle = \langle R^*R\Omega|(\mathbb{1} - \mathbb{1}[\mathcal{L}_\omega = 0])e^{\imath t\mathcal{L}_\omega}A\Omega\rangle.$$

By the spectral theorem we can find a Borel measure $d\nu$ on $\sigma(\mathcal{L}_\omega) \setminus \{0\}$ such that

$$\langle R^*R\Omega|(\mathbb{1} - \mathbb{1}[\mathcal{L}_\omega = 0])e^{\imath t\mathcal{L}_\omega}A\Omega\rangle = \int_{\sigma(\mathcal{L}_\omega)\setminus\{0\}} e^{\imath t\lambda}\nu(d\lambda). \tag{2.125}$$

Since $\sigma(\mathcal{L}_\omega) \setminus \{0\} = \sigma_{ac}(\mathcal{L}_\omega)$, there is an absolutely integrable function ρ with respect to the Lebesgue-measure $d\lambda$, such that $d\nu = \rho d\lambda$. Hence we obtain

$$\lim_{t\to\infty} \mu(\alpha_t(A)) - \omega(A) = \lim_{t\to\infty} \int_{\sigma(\mathcal{L}_\omega)\setminus\{0\}} e^{\imath t\lambda}\rho(\lambda)d\lambda = 0, \tag{2.126}$$

by the Riemann-Lebesgue Lemma. An approximation argument extends Equation (2.126) to all normal state μ. □

2.7 Infinite Particle Space

2.7.1 Symmetrization

Let $(\mathfrak{h}, \langle\cdot,\cdot\rangle_1)$ be a separable Hilbert space. We remark that for the n-fold tensor product $\mathfrak{h}^n := (\bigotimes_{i=1}^n \mathfrak{h}, \langle\cdot,\cdot\rangle_n)$ one has

$$\bigotimes_{i=1}^n \mathfrak{h} = \mathrm{c}\ell\, \mathfrak{g}_n, \quad \mathfrak{g}_n := \mathrm{LH}\{f_1 \otimes \cdots \otimes f_n \;:\; f_i \in \mathfrak{h}\} \tag{2.127}$$

and

$$\langle f_1 \otimes \cdots \otimes f_n | g_1 \otimes \cdots \otimes g_n\rangle_n = \prod_{i=1}^n \langle f_i|g_i\rangle_1. \tag{2.128}$$

The *symmetric projection* $\mathcal{S}_n$ is defined by

$$\mathcal{S}_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\pi\in\mathfrak{S}(n)} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}. \tag{2.129}$$

In this context $\mathfrak{S}(n)$ is the symmetric group of permutations acting on$\{1, 2, \ldots, n\}$.

Lemma 2.7.1. *$\mathcal{S}_n$ is an orthogonal projection.*

Proof of 2.7.1. For any $\tau \in \mathfrak{S}(n)$ one has $\tau\mathfrak{S}(n) = \{\tau \circ \pi : \pi \in \mathfrak{S}(n)\} = \mathfrak{S}(n)$. That yields

$$\begin{aligned}\mathcal{S}_n^2 f_1 \otimes \cdots \otimes f_n &= \frac{1}{(n!)^2} \sum_{\tau,\pi\in\mathfrak{S}(n)} f_{\tau(\pi(1))} \otimes \cdots \otimes f_{\tau(\pi(n))} \\ &= \frac{1}{(n!)^2} \sum_{\tau\in\mathfrak{S}(n)} \sum_{\pi'\in\tau\mathfrak{S}(n)} f_{\pi'(1)} \otimes \cdots \otimes f_{\pi'(n)} \\ &= \Big(\frac{1}{n!} \sum_{\tau\in\mathfrak{S}(n)} 1\Big)\Big(\frac{1}{n!} \sum_{\pi'\in\mathfrak{S}(n)} f_{\pi'(1)} \otimes \cdots \otimes f_{\pi'(n)}\Big) \\ &= \mathcal{S}_n f_1 \otimes \cdots \otimes f_n.\end{aligned} \tag{2.130}$$

Since $\mathfrak{S}(n) = \{\pi^{-1} : \pi \in \mathfrak{S}(n)\}$ we have

$$\begin{aligned}&\langle g_1 \otimes \cdots \otimes g_n | \mathcal{S}_n f_1 \otimes \cdots \otimes f_n \rangle_n \\ &= \frac{1}{n!} \sum_{\pi\in\mathfrak{S}(n)} \langle g_1 \otimes \cdots \otimes g_n | f_{\pi(1)} 1 \otimes \cdots \otimes f_{\pi(n)} \rangle_n \\ &= \frac{1}{n!} \sum_{\pi\in\mathfrak{S}(n)} \prod_{i=1}^{n} \langle g_i | f_{\pi(i)} \rangle_1 = \frac{1}{n!} \sum_{\pi\in\mathfrak{S}(n)} \prod_{i=1}^{n} \langle g_{\pi^{-1}(i)} | f_i \rangle_1 \\ &= \frac{1}{n!} \sum_{\pi\in\mathfrak{S}(n)} \langle g_{\pi^{-1}(1)} \otimes \cdots \otimes g_{\pi^{-1}(n)} | f_1 \otimes \cdots \otimes f_n \rangle_n \\ &= \langle \mathcal{S}_n g_1 \otimes \cdots \otimes g_n | f_1 \otimes \cdots \otimes f_n \rangle_n.\end{aligned} \tag{2.131}$$

It follows, that $\mathcal{S}_n$ extends linearly to $f \in \mathfrak{g}_n$. By definition of the norm, we have

$$\|\mathcal{S}_n f\|_n^2 \;\leq\; \|\mathcal{S}_n f\|_n^2 + \|f - \mathcal{S}_n f\|_n^2 = \|f\|_n^2. \tag{2.132}$$

$\mathcal{S}_n$ can therefore be extended to a bounded operator on $\mathfrak{h}^n$ by continuity. Since $\mathcal{S}_n^2 = \mathcal{S}_n$ and $\mathcal{S}_n^* = \mathcal{S}_n$ on $\mathfrak{g}_n$, it is true on $\mathfrak{h}^n$. □

2.7.2 Fock Space

The Fock space is the vector space

$$\mathcal{F}[\mathfrak{h}] := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathfrak{h}^n \tag{2.133}$$

equipped with the norm

$$\|\cdot\|_{\mathcal{F}} := \Big(|\cdot|^2 + \sum_{n=1}^{\infty} \|\cdot\|_n^2\Big)^{1/2}. \tag{2.134}$$

The elements of $\mathcal{F}[\mathfrak{h}]$ are sequences $f = (f_n)_{n=0}^{\infty}$, such that $f_0 \in \mathbb{C}$ and $f_n \in \mathfrak{h}^n$, $n \in \mathbb{N}$ and $\|f\|_{\mathcal{F}} < \infty$.

$\mathcal{F}[\mathfrak{h}]$ is a separable Hilbert space. In this context $\mathfrak{h}^n$ is the n-particle subspace. $\Omega := (1, 0, \ldots)$ is the vacuum vector. A short calculation yields that

$$\mathcal{F}^0[\mathfrak{h}] := \{(f_n)_{n=0}^{\infty} \in \mathcal{F}[\mathfrak{h}] \,:\, \forall_{n\in\mathbb{N}}\, f_n \in \mathfrak{g}_n,\ \exists_{n_0\in\mathbb{N}}\forall_{n\geq n_0}\, f_n = 0\} \tag{2.135}$$

is a dense subspace of $\mathcal{F}[\mathfrak{h}]$. On $\mathcal{F}^0[\mathfrak{h}]$ creation- and annihilation-operators are defined for $h \in \mathfrak{h}$ by,

$$\begin{aligned} b^*(h) f_1 \otimes \cdots \otimes f_n &:= (n+1)^{1/2} \cdot h \otimes f_1 \otimes \cdots \otimes f_n, \quad b^*(h)\Omega := h \\ b(h) f_1 \otimes \cdots \otimes f_n &:= n^{1/2} \langle h | f_1 \rangle_1 \cdot f_2 \otimes \cdots \otimes f_n, \quad b(h)\Omega := 0. \end{aligned} \tag{2.136}$$

We obtain

$$\begin{aligned} &\langle b^*(h) f_1 \otimes \cdots \otimes f_n | g_1 \otimes \cdots \otimes g_{n+1} \rangle_{n+1} \\ &= (n+1)^{1/2} \langle h | g_1 \rangle_1 \prod_{j=1}^{n} \langle f_j | g_{j+1} \rangle_1 = \langle f_1 \otimes \cdots \otimes f_n | b(h) g_1 \otimes \cdots \otimes g_{n+1} \rangle_n \end{aligned} \tag{2.137}$$

for $n \in \mathbb{N}_0$. It follows $\langle b^*(h) f | g \rangle_{\mathcal{F}} = \langle f | b(h) g \rangle_{\mathcal{F}}$ for $f, g \in \mathcal{F}^0[\mathfrak{h}]$.

2.7.3 Bosonic Fock-Space

Not all vectors in $\mathcal{F}[\mathfrak{h}]$ describe bosons, one has to restrict to the subspace

$$\mathcal{F}_b[\mathfrak{h}] := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{S}_n \mathfrak{h}^n \tag{2.138}$$

called the Bosonic Fock Space. $\mathcal{F}_b^0[\mathfrak{h}]$ is defined analogously. $\mathcal{S}_n \mathfrak{h}^n$ is the symmetric n-particle subspace. We define (bosonic) creation operators by

$$\begin{aligned} a^*(h) f_1 \otimes \cdots \otimes f_n &:= \mathcal{S}_{n+1} b^*(h) \mathcal{S}_n f_1 \otimes \cdots \otimes f_n \\ &= (n+1)^{1/2} \mathcal{S}_{n+1} h \otimes f_1 \otimes \cdots \otimes f_n. \end{aligned} \tag{2.139}$$

and $a^*(h)\Omega = h$. The (bosonic) annihilation operators are

$$\begin{aligned} a(h) f_1 \otimes \cdots \otimes f_n &:= \mathcal{S}_{n-1} b(h) \mathcal{S}_n f_1 \otimes \cdots \otimes f_n \\ &= n^{-1/2} \sum_{m=1}^{n} \langle h | f_m \rangle\ \mathcal{S}_{n-1} f_1 \otimes \cdots \otimes \widehat{f_m} \otimes \cdots \otimes f_n \end{aligned} \tag{2.140}$$

and $a(h)\Omega = 0$. The symbol $\widehat{f_m}$ means, that the factor f_m is omitted. A straightforward calculation yields the so called Canonical Commutator Relations (CCR).

$$[a(h_1), a(h_2)] = 0, \quad [a^*(h_1), a^*(h_2)] = 0, \quad [a(h_1), a^*(h_2)] = \langle h_1 | h_2 \rangle_1 \tag{2.141}$$

That are identities of operators on $\mathcal{F}_b^0[\mathfrak{h}]$. From

$$\langle a^*(h) f | g \rangle_{\mathcal{F}} = \langle f | a(h) g \rangle_{\mathcal{F}}, \quad f, g \in \mathcal{F}_b^0[\mathfrak{h}] \tag{2.142}$$

follows that $a^*(h) \subset (a(h))^*$ and $a(h) \subset (a^*(h))^*$, i.e. $a^*(h)$ is a restriction of the adjoint operator of $a(h)$ and vice versa. From now on we identify $a^*(h)$ and $a(h)$ with their closures. The field operators are defined on $\mathcal{F}_b^0[\mathfrak{h}]$ as

$$\Phi(h) := \frac{a^*(h) + a(h)}{\sqrt{2}}. \tag{2.143}$$

$\Phi(h)$ is obviously symmetric and

$$a^*(h) = \frac{\Phi(h) - \imath\Phi(\imath h)}{\sqrt{2}}, \quad a(h) = \frac{\Phi(h) + \imath\Phi(\imath h)}{\sqrt{2}}. \tag{2.144}$$

Furthermore, it follows from (2.139) and 4.2 that

$$\begin{aligned} \mathcal{F}_b^0[\mathfrak{h}] &= \mathrm{LH}\{a^*(f_1)\cdots a^*(f_n)\Omega \,:\, n \in \mathbb{N}_0, \quad f_1, \ldots, f_n \in \mathfrak{h}\} \\ &= \mathrm{LH}\{\Phi(f_1)\cdots \Phi(f_n)\Omega \,:\, n \in \mathbb{N}_0, \quad f_1, \ldots, f_n \in \mathfrak{h}\}. \end{aligned} \tag{2.145}$$

Let $\mathrm{dom}(N) := \{f = (f_n)_{n=0}^\infty \in \mathcal{F}_b[\mathfrak{h}] \,:\, \sum_{n=1}^\infty n^2 \|f_n\|_n^2 < \infty\}$ and

$$Nf := (nf_n)_{n=0}^\infty, \quad \text{for } f = (f_n)_{n=0}^\infty. \tag{2.146}$$

N is the number operator, it is self-adjoint. One easily, that $\mathcal{F}_b^0[\mathfrak{h}]$ is a core of N. Applied to $\sum_{(i_1,\ldots,i_n)} f_{i_1} \otimes \cdots \otimes f_{i_n} \in \mathcal{S}_n \mathfrak{h}^n$ we obtain

$$\begin{aligned} \Big\| a^*(h) \sum_{(i_1,\ldots,i_n)} f_{i_1} \otimes \cdots \otimes f_{i_n} \Big\|_{n+1}^2 &\leq (n+1) \Big\| h \otimes \sum_{(i_1,\ldots,i_n)} f_{i_1} \otimes \cdots \otimes f_{i_n} \Big\|_{n+1}^2 \\ = (n+1)\|h\|_1^2 \cdot \Big\| \sum_{(i_1,\ldots,i_n)} f_{i_1} \otimes \cdots \otimes f_{i_n} \Big\|_n^2 &= \|h\|_1^2 \cdot \Big\| (N+1)^{1/2} \sum_{(i_1,\ldots,i_n)} f_{i_1} \otimes \cdots \otimes f_{i_n} \Big\|_n^2. \end{aligned} \tag{2.147}$$

The yields the relative bound

$$\|a^*(h) f\|_{\mathcal{F}} \leq \|h\|_1 \cdot \|(N+1)^{1/2} f\|_{\mathcal{F}}. \tag{2.148}$$

Otherwise,

$$\|a(h)f\|_{\mathcal{F}} \leq \|h\|_1 \cdot \|N^{1/2}f\|_{\mathcal{F}} \tag{2.149}$$

follows if additionally the CCR are used,

$$\begin{aligned}\|h\|_1^2 \cdot \|(N+1)^{1/2}f\|_{\mathcal{F}}^2 &\geq \langle a^*(h)f|a^*(h)f\rangle_{\mathcal{F}} = \langle f|a(h)a^*(h)f\rangle_{\mathcal{F}} \\ &= \langle f|a^*(h)a(h)f\rangle_{\mathcal{F}} + \langle h|h\rangle_1 \cdot \langle f|f\rangle_{\mathcal{F}} = \|a(h)f\|_{\mathcal{F}}^2 + \|h\|_1^2 \cdot \|f\|_{\mathcal{F}}^2.\end{aligned}$$

Consequently, we obtain for the field operator the relative bound

$$\|\Phi(h)f\|_{\mathcal{F}} \leq \sqrt{2} \cdot \|h\|_1 \cdot \|(N+1)^{1/2}f\|_{\mathcal{F}}. \tag{2.150}$$

2.7.4 Second Quantization

Let A be an essentially self-adjoint operator with core $D \subseteq \mathfrak{h}$. We define the second quantization of A by

$$d\Gamma(A)\Big|_{\mathcal{S}_n \bigotimes_{m=1}^n D} := \sum_{m=1}^n \big(\mathbb{1} \otimes \cdots \otimes \underbrace{A}_{m} \otimes \cdots \otimes \mathbb{1}\big), \quad d\Gamma(A)\Omega := 0. \tag{2.151}$$

$d\Gamma(A)$ is essentially self-adjoint on

$$\mathcal{F}_b^0[D] = \mathrm{LH}\{a^*(f_1)\cdots a^*(f_n)\Omega \,:\, f_i \in D,\ n \in \mathbb{N}_0\}. \tag{2.152}$$

In the following we do not distinguish between $d\Gamma(A)$ and its closure. Given A the spectrum and the pure point spectrum of $d\Gamma(A)$ is well known:

$$\sigma(d\Gamma(A)) = c\ell\Big(\{0\} \cup \bigcup_{n=1}^\infty \Big\{\sum_{m=1}^n \lambda_m \in \mathbb{R} : \lambda_m \in \sigma(A)\Big\}\Big) \tag{2.153}$$

$$\sigma_{pp}(d\Gamma(A)) = \{0\} \cup \bigcup_{n=1}^\infty \Big\{\sum_{m=1}^n \lambda_m \in \mathbb{R} : \lambda_m \in \sigma_{pp}(A)\Big\}. \tag{2.154}$$

2.7.5 The Natural Isomorphism $\mathcal{F}_b[\mathfrak{g}_1 \oplus \mathfrak{g}_2] = \mathcal{F}_b[\mathfrak{g}_1] \otimes \mathcal{F}_b[\mathfrak{g}_2]$

Let $\mathfrak{h} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{g}_1 \perp \mathfrak{g}_2$, such as $f_i \in \mathfrak{g}_1$ and $g_i \in \mathfrak{g}_2$. There is a unitary map $U : \mathcal{F}_b[\mathfrak{h}] \to \mathcal{F}_b[\mathfrak{g}_1] \otimes \mathcal{F}_b[\mathfrak{g}_2]$ determined by

$$\prod_{j=1}^n a^*(f_j) \prod_{j=1}^m a^*(g_j)\Omega \mapsto \big(\prod_{j=1}^n a^*(f_j)\Omega\big) \otimes \big(\prod_{j=1}^m a^*(g_j)\Omega\big) \tag{2.155}$$

for all $m, n \in \mathbb{N}_0$.

For an essentially self-adjoint operator A_i on $\mathfrak{g}_i$ with core D_i for $i = 1, 2$, one has that $A_1 \oplus A_2$ is essentially self-adjoint on $\mathfrak{h}$ with core $D_1 \oplus D_2$ and for $f_j \in D_1$ and $g_j \in D_2$

$$\begin{aligned} & U d\Gamma(A_1 \oplus A_2) \prod_{j=1}^{n} a^*(f_j) \prod_{j=1}^{m} a^*(g_j)\Omega \qquad (2.156) \\ = \;& U\big(\sum_{k=1}^{n} a^*(f_1)\cdots a^*(A_1 f_k)\cdots a^*(f_n)\big) \prod_{j=1}^{m} a^*(g_j)\Omega \\ & + U \prod_{j=1}^{n} a^*(f_j)\big(\sum_{k=1}^{m} a^*(g_1)\cdots a^*(A_2 g_k)\cdots a(g_m)\big)\Omega \\ = \;& \big(d\Gamma(A_1) a^*(f_1)\cdots a^*(f_n)\Omega\big) \otimes \prod_{j=1}^{m} a^*(g_j)\Omega + \prod_{j=1}^{n} a^*(f_j)\Omega \otimes \big(d\Gamma(A_2) a^*(g_1)\cdots a(g_m)\Omega\big). \end{aligned}$$

Hence $U d\Gamma(A_1 \oplus A_2) = \big(d\Gamma(A_1) \otimes 1 + 1 \otimes d\Gamma(A_2)\big)U$. For the field operators one can show

$$\Phi(f \oplus g)U = U\Phi(f) \otimes 1 + U1 \otimes \Phi(g). \qquad (2.157)$$

2.7.6 The Fock Space $\mathcal{F}_b[L^2(\mathbb{R}^3, d\mu)]$

We consider now the case, where $\mathfrak{h} = L^2(\mathbb{R}^3, d\mu)$ for some Borel measure μ. In the following we will identify

$$\begin{aligned} & \mathcal{S}_n\big(\otimes_{j=1}^{n} L^2(\mathbb{R}^3, d\mu)\big) \qquad (2.158) \\ = \;& \{f \in L^2((\mathbb{R}^3)^n, \otimes_{j=1}^{n} d\mu) \,:\, f(k_1, \dots, k_n) = f(k_{\pi 1}, \dots, k_{\pi n}),\ \pi \in \mathfrak{S}(n), a.e.\}. \end{aligned}$$

In this case creation and annihilation operators read for $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}_b[L^2(\mathbb{R}^3, d\mu)]$

$$\begin{aligned} (a(h) f_{n+1})(k_1, \dots, k_n) &= (n+1)^{1/2} \int d\mu(k_{n+1}) \overline{h(k_{n+1})} f_{n+1}(k_1, \dots, k_n, k_{n+1}), \\ (a^*(h) f_n)(k_1, \dots, k_{n+1}) &= (n+1)^{-1/2} \sum_{i=1}^{n} h(k_i) f_{n+1}(k_1, \dots, \hat{k}_i \dots, k_{n+1}). \end{aligned}$$

Let α be a real valued measurable function, we observe

$$(d\Gamma(\alpha) f_n)(k_1, \dots, k_n) = \sum_{i=1}^{n} \alpha(k_i) f_n(k_1, \dots, k_n) \qquad (2.159)$$

for $f \in \operatorname{dom}(d\Gamma(\alpha)) = \{(f_n)_{n=0}^\infty : \sum_{n=1}^\infty |\sum_{i=1}^n \alpha(k_i)|^2 |f_n(k_1,\dots,k_n)|^2 < \infty\}$.

For $h \in L^2(\Omega)$ and if $\mu(\operatorname{supp}(\alpha) \setminus \operatorname{supp}(h)) = 0$ we have

$$\begin{aligned} &\int |a(h) f_{n+1}(k_1,\dots,k_n)|^2 d\mu(k_1)\dots d\mu(k_n) && (2.160)\\ &= (n+1) \int d\mu(\underline{k}) \Big| \int d\mu(k_{n+1}) \overline{h(k_{n+1})} f_{n+1}(k_1,\dots,k_n,k_{n+1}) \Big|^2 \\ &\leq (n+1) \int d\mu(\underline{k}) \Big(\int_{\operatorname{supp}(h)} d\mu(k_{n+1}) \frac{|h(k_{n+1})|^2}{|\alpha(k_{n+1}))|} \Big) \\ &\quad \Big(\int d\mu(k_{n+1}) |\alpha(k_{n+1})| |f_{n+1}(k_1,\dots,k_n,k_{n+1})|^2 \Big) \\ &= \Big(\int_{\operatorname{supp}(h)} \frac{|h(k)|^2 d\mu(k)}{|\alpha(k))|} \Big) \| d\Gamma(|\alpha|)^{1/2} f_{n+1} \|^2_{L^2(\mathbb{R}^{3(n+1)})}. \end{aligned}$$

Hence one obtains the relative bounds

$$\begin{aligned} \|a(h)f\|^2 &\leq C^2 \|d\Gamma(|\alpha|)f\|^2 && (2.161)\\ \|a^*(h)f\|^2 &= \|a(h)f\|^2 + \|h\|_1^2 \|f\|^2 \leq C^2 \|d\Gamma(|\alpha|)f\|^2 + \|h\|_1^2\|f\|^2 \end{aligned}$$

for $f \in \operatorname{dom}(d\Gamma(|\alpha|))$and $C = \Big(\int_{\operatorname{supp}(h)} \frac{|h(k)|^2 d\mu(k)}{|\alpha(k))|} \Big)^{1/2}$.

2.7.7 Weyl Algebra in the Fock Representation

Theorem 2.7.2. *Let $h, g \in \mathfrak{h}$.*

1. *$\Phi(h)$ is essentially self-adjoint on $\mathcal{F}_b^0[\mathfrak{h}]$. After identifying $\Phi(h)$ with its closure, we define the Weyl operator by $W_F(h) := \exp(\imath \Phi(h))$.*

2. *If $\psi \in \operatorname{dom}(\Phi(h))$ then*

$$W_F(g)\psi \in \operatorname{dom}(\Phi(h)), \quad W_F(g)\Phi(h)\psi - \Phi(h)W_F(g)\psi = \Im\langle h|g\rangle\, W_F(g)\psi.$$

3. *$W_F(h)W_F(g) = e^{-\imath/2 \Im\langle h|g\rangle} W_F(h+g)$*

Proof of 2.7.2. Let $e_m(t) := \frac{(\imath t \Phi(h))^m}{m!} \Phi(f_1)\cdots\Phi(f_n)\Omega$. An iterated application of estimate (2.150) yields

$$\|e_m(t)\|_{\mathcal{F}} \leq \frac{2^{(m+n)/2}[(m+n+1)!]^{1/2}}{m!} (|t|\|h\|_1)^m \prod_{j=1}^n \|f_j\|_1. \tag{2.162}$$

Since $\frac{(m+n+1)!}{m!(n+1)!} \leq 2^{m+n+1}$ we derive

$$\|e_m(t)\|_{\mathcal{F}} \leq \frac{2^{(2m+2n+1)/2}[(n+1)!]^{1/2}}{[m!]^{1/2}} (|t|\|h\|_1)^m \prod_{j=1}^n \|f_j\|_1. \tag{2.163}$$

From the quotient criterion follows, that $\sum_{m=0}^{\infty} e_m(t)$ converges normally for $|t| \leq R,\ R > 0$. Nelson's analytic vector theorem completes the proof of the first part.

Let $\psi \in \mathrm{dom}(\Phi(h))$ and $\eta \in \mathcal{F}_b^0[\mathfrak{h}]$. Since η, $\Phi(g)^m\eta$, $\Phi(h)\eta \in \mathcal{F}_b^0[\mathfrak{h}]$ they are analytic vectors and

$$
\begin{aligned}
&\big\langle \eta \big| W_F(g)\Phi(h)\psi \big\rangle_{\mathcal{F}} = \big\langle W_F(-g)\eta \big| \Phi(h)\psi \big\rangle_{\mathcal{F}} \qquad (2.164)\\
&= \lim_{m\to\infty} \sum_{n=0}^{m} \frac{1}{n!} \big\langle \big((-\imath)\Phi(g)\big)^n \eta \big| \Phi(h)\psi \big\rangle_{\mathcal{F}} = \lim_{m\to\infty} \sum_{n=0}^{m} \frac{1}{n!} \big\langle \big(\Phi(h)(-\imath)\Phi(g)\big)^n \eta \big| \psi \big\rangle_{\mathcal{F}}\\
&= \lim_{m\to\infty} \sum_{n=0}^{m} \frac{1}{n!} \big\langle \big([\Phi(h), (-\imath)\Phi(g)\big)^n]\eta \big| \psi \big\rangle_{\mathcal{F}} + \lim_{m\to\infty} \sum_{n=0}^{m} \frac{1}{n!} \big\langle \big((-\imath)\Phi(g)\big)^n \Phi(h)\eta \big| \psi \big\rangle_{\mathcal{F}}\\
&= \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{(n-1)!} \big\langle \Im\langle h|g\rangle \big((-\imath)\Phi(g)\big)^{n-1} \eta \big| \psi \big\rangle_{\mathcal{F}} + \big\langle W_F(-g)\Phi(h)\eta \big| \psi \big\rangle_{\mathcal{F}}\\
&= \overline{\Im\langle h|g\rangle_1} \big\langle W_F(-g)\eta \big| \psi \big\rangle_{\mathcal{F}} + \big\langle W_F(-g)\Phi(h)\eta \big| \psi \big\rangle_{\mathcal{F}}.
\end{aligned}
$$

That is equivalent to

$$
\big\langle \Phi(h)\eta \big| W_F(g)\psi \big\rangle_{\mathcal{F}} = \langle \eta | \big(W_F(g)\Phi(h) + \Im\big\langle h\big|g\big\rangle_1 W_F(g)\big)\psi\rangle_{\mathcal{F}}. \qquad (2.165)
$$

Since $\mathcal{F}_b^0[\mathfrak{h}]$ is a core of $\Phi(h)$ the second part follows by the definition of self-adjointness.

Let $\phi, \psi \in \mathcal{F}_b^0[\mathfrak{h}]$ and $\rho(t) = \big\langle \phi \big| W_F(th)W_F(tg)W_F(-t(h+g))\psi\rangle_{\mathcal{F}}$. It is $\rho(0) = \langle\phi|\psi\rangle_{\mathcal{F}}$, furthermore

$$
\begin{aligned}
&\rho(t+\delta) - \rho(t) \qquad (2.166)\\
&= \big\langle (W_F(-\delta h) - \mathbb{1})W_F(-th)\phi \big| W_F((t+\delta)g)W_F(-(t+\delta)(h+g))\psi \big\rangle_{\mathcal{F}}\\
&\quad + \big\langle (W_F(-\delta g) - \mathbb{1})W_F(-tg)W_F(-th)\phi \big| W_F(-(t+\delta)(h+g))\psi \big\rangle_{\mathcal{F}}\\
&\quad + \big\langle W_F(th)W_F(tg)\phi \big| (W_F(-\delta(h+g)) - \mathbb{1})W_F(-t(h+g)\psi \big\rangle_{\mathcal{F}}.
\end{aligned}
$$

Since

$$
W_F(-th)\phi,\ W_F(-tg)W_F(-th)\phi \in \mathrm{dom}(\Phi(th)) \cap \mathrm{dom}(\Phi(tg)
$$

and $\|W(f)\| = 1$ for all $f \in \mathfrak{h}$, and $\text{s-lim}_{\delta\to 0}\, \delta^{-1}(W_F(-\delta k) - \mathbb{1}) = -\imath\Phi(k)$ for $k = h,\ g,\ h+g$, we obtain that ρ is differentiable and

$$\begin{aligned}\dot{\rho}(t) &= \big\langle \imath\Phi(-h)W_F(-th)\phi\big|W_F(tg)W_F(-t(h+g))\psi\big\rangle_{\mathcal{F}} \\ &\quad+\big\langle \imath\Phi(-g)W_F(-tg)W_F(-th)\phi\big|W_F(-t(h+g))\psi\big\rangle_{\mathcal{F}} \\ &\quad+\big\langle W_F(th)W_F(tg)\phi\big|\imath\Phi(-(h+g)W_F(-t(h+g)\psi\big\rangle_{\mathcal{F}} \\ &= -\imath\big\langle W_F(-th)\phi\big|[\Phi(-h),W_F(tg)]W_F(-t(h+g))\psi\big\rangle_{\mathcal{F}} \\ &= -\imath\Im\big\langle h\big|tg\big\rangle\rho(t).\end{aligned} \tag{2.167}$$

Solving this initial value problem for ρ yields

$$\rho(t)=\exp(-\imath t^2/2\Im\langle h|g\rangle)\langle\phi|\psi\rangle_{\mathcal{F}}. \tag{2.168}$$

Hence $W_F(h)W_F(g)W_F(-(h+g))=e^{-\imath/2\Im\langle h|g\rangle}$ for $t=1$. □

Theorem 2.7.3. *Let* $W_F(\mathfrak{h}):=\mathrm{c}\ell\,\mathrm{LH}\big\{W_F(f)\in\mathcal{B}(\mathcal{F}_b[\mathfrak{h}])\ :\ f\in\mathfrak{h}\big\}$. *Then*

$$W_F(\mathfrak{h})'=\{\mathbb{C}\mathbb{1}\}. \tag{2.169}$$

Proof of 2.7.3. Let $T\in W_F(\mathfrak{h})'$ for $f_1,\dots,f_n\in\mathfrak{h}$ and $g_1,\dots,g_m$ we have

$$\begin{aligned}&\langle\Phi(f_1)\cdots\Phi(f_n)\Omega|T\Phi(g_1)\cdots\Phi(g_m)\Omega\rangle_{\mathcal{F}} \\ &= \lim_{t\to0}\Big\langle\frac{W(tf_1)-1}{\imath t}\Phi(f_2)\cdots\Phi(f_n)\Omega\Big|T\Phi(g_1)\cdots\Phi(g_m)\Omega\Big\rangle_{\mathcal{F}} \\ &= \lim_{t\to0}\Big\langle\Phi(f_2)\cdots\Phi(f_n)\Omega\Big|T\frac{W(-tf_1)-1}{-\imath t}\Phi(g_1)\cdots\Phi(g_m)\Omega\Big\rangle_{\mathcal{F}} \\ &= \big\langle\Phi(f_2)\cdots\Phi(f_n)\Omega\big|T\Phi(f_1)\Phi(g_1)\cdots\Phi(g_m)\Omega\big\rangle_{\mathcal{F}} \\ &= \big\langle\Omega\big|T\Phi(f_n)\cdots\Phi(f_1)\Phi(g_1)\cdots\Phi(g_m)\Omega\big\rangle_{\mathcal{F}}.\end{aligned} \tag{2.170}$$

Using equation (2.144), we obtain

$$\begin{aligned}&\langle a^*(f_1)\cdots a^*(f_n)\Omega|Ta^*(g_1)\cdots a^*(g_m)\Omega\rangle_{\mathcal{F}} \\ &= \big\langle\Omega\big|Ta(f_n)\cdots a(f_1)a^*(g_1)\cdots a^*(g_m)\Omega\big\rangle_{\mathcal{F}} \\ &= \langle T^*\Omega|\Omega\rangle\cdot\big\langle\Omega\big|a(f_n)\cdots a(f_1)a^*(g_1)\cdots a^*(g_m)\Omega\big\rangle_{\mathcal{F}}.\end{aligned} \tag{2.171}$$

The last equation is to show. First we assume $n>m$, then $a(f_n)\cdots a(f_1)a^*(g_1)\cdots a^*(g_m)\Omega=0$ and the equality holds. Assume $n<m$, we obtain since

$T^* \in W_F(\mathfrak{h})'$

$$\begin{aligned} & \langle a^*(f_1)\cdots a^*(f_n)\Omega | T a^*(g_1)\cdots a^*(g_m)\Omega\rangle_{\mathcal{F}} \\ = \;& \overline{\langle a^*(g_1)\cdots a^*(g_m)\Omega | T^* a^*(f_1)\cdots a^*(f_m))\Omega\rangle_{\mathcal{F}}} \\ = \;& \overline{\langle T\Omega|\Omega\rangle} \cdot \overline{\langle \Omega | a(g_m)\cdots a(g_1) a^*(f_1)\cdots a^*(f_n)\Omega\rangle_{\mathcal{F}}} \\ = \;& \langle T^*\Omega|\Omega\rangle \cdot \big\langle \Omega \big| a(f_n)\cdots a(f_1)a^*(g_1)\cdots a^*(g_m)\Omega\rangle_{\mathcal{F}}. \end{aligned} \tag{2.172}$$

If $n = m$ one has $a(f_n)\cdots a(f_1)a^*(g_1)\cdots a^*(g_m)\Omega = c\cdot\Omega$ for $c \in \mathbb{R}$ and

$$\begin{aligned} & \langle a^*(f_1)\cdots a^*(f_n)\Omega | T a^*(g_1)\cdots a^*(g_m)\Omega\rangle_{\mathcal{F}} \\ = \;& \langle \Omega | Tc\Omega\rangle_{\mathcal{F}} = \langle T^*\Omega|\Omega\rangle_{\mathcal{F}} \cdot \langle \Omega | a(f_n)\cdots a(f_1)a^*(g_1)\cdots a^*(g_m)\Omega\rangle_{\mathcal{F}}. \end{aligned} \tag{2.173}$$

By linearity follows for $\phi, \psi \in \mathcal{F}_b^0[\mathfrak{h}]$ that

$$\langle \phi | T\psi\rangle = \langle T^*\Omega|\Omega\rangle_{\mathcal{F}} \cdot \langle \phi|\psi\rangle_{\mathcal{F}} \tag{2.174}$$

and $T = \langle T^*\Omega|\Omega\rangle_{\mathcal{F}} \cdot \mathbb{1}$. □

Corollary 2.7.4. *For all $A \in \mathcal{B}(\mathcal{F}_b[\mathfrak{h}])$, $\epsilon > 0$ and $(\psi_n)_{n=1}^\infty \in \bigoplus_{j=1}^n \mathcal{F}_b[\mathfrak{h}]$ exists $W \in W_F(\mathfrak{h})$, such that $\left(\sum_{n=1}^\infty \|W\psi_n - A\psi_n\|^2\right)^{1/2} < \epsilon$,*

Proof of 2.7.4. Theorem 2.2.3 and $\mathcal{B}(\mathcal{F}_b[\mathfrak{h}]) = W_F(\mathfrak{h})''$. □

2.8 The Abstract Weyl Algebra

Let $\mathfrak{h}$ be a Hilbert space and $\mathfrak{f} \subset \mathfrak{h}$ a (not necessarily closed) subspace.

Definition 2.8.1. *A C^*-algebra $\mathfrak{A}$ is a Weyl algebra over $\mathfrak{f}$, if $\mathfrak{A} = c\ell\,\mathrm{LH}\{W(f) \in \mathfrak{A} : f \in \mathfrak{f}\}$ The operators $W(f)$ have to fulfill the Canonical Commutation Relations (CCR), i.e.*

$$W(f)W(g) = e^{-\imath/2\cdot\Im\langle f|g\rangle} W(f+g) \quad f, g \in \mathfrak{f}. \tag{2.175}$$

Furthermore $W(f)^* = W(-f)$, $f \in \mathfrak{f}$. $W(f)$ is called a Weyl operator.

Obviously, $W(0) = \mathbb{1}$ and $W(f)$ is a unitary of $\mathfrak{A}$.

Theorem 2.8.2 (Uniqueness)**.** *Assume $\mathfrak{A}_1, \mathfrak{A}_2$ are Weyl algebras over $\mathfrak{f}$*

$$\mathfrak{A}_i = c\ell\,\mathrm{LH}\{W_i(f) \in \mathfrak{A}_i \,:\, f \in \mathfrak{f}\}, \quad i = 1, 2, \tag{2.176}$$

Then exists a unique $*$-isomorphism $\alpha : \mathfrak{A}_1 \to \mathfrak{A}_2$, such that

$$\alpha(W_1(f)) = W_2(f), \quad f \in \mathfrak{f}. \tag{2.177}$$

Proof of 2.8.2. Confer ([8], Thm. 5.2.8). □

We write in the following $\mathcal{W}(\mathfrak{f})$ for the Weyl algebra over $\mathfrak{f}$.

Theorem 2.8.3 (Bogoliubov-transform)**.** *Let $\mathfrak{f}_i \subset \mathfrak{h}_i$, $i = 1, 2$ and $v : \mathfrak{f}_1 \to \mathfrak{f}_2$. v is real linear and*

$$\Im\langle v(f)|v(g)\rangle_2 = \Im\langle f|g\rangle_1. \tag{2.178}$$

Furthermore, let $\mathfrak{A}_i$ be Weyl-algebras over $\mathfrak{f}_i$, then exists an unique, injective $*$-morphism $\alpha : \mathfrak{A}_1 \to \mathfrak{A}_2$ with

$$\alpha(W_1(f)) = W_2(v(f)). \tag{2.179}$$

α is called a Bogoliubov-transform.

Proof of 2.8.3. Let $W_3(f) := W_2(v(f))$, $f \in \mathfrak{f}_1$.

$$\mathfrak{A}_3 := c\ell\,\mathrm{LH}\{W_3(f) \in \mathfrak{A}_2 \,:\, f \in \mathfrak{f}_1\} \subset \mathfrak{A}_2 \tag{2.180}$$

Since

$$\begin{aligned} W_3(f)W_3(g) &= W_2(v(f))W_2(v(g)) = e^{-\imath/2\cdot\Im\langle v(f)|v(g)\rangle_2} W_2(v(f) + v(g)) \\ &= e^{-\imath/2\,\Im\langle f|g\rangle_1} W_2(v(f+g)) = e^{-\imath/2\,\Im\langle f|g\rangle_1} W_3(f+g), \end{aligned}$$

and

$$W_3(f)^* = W_2(v(f))^* = W_2(-v(f)) = W_2(v(-f)) = W_3(-f). \tag{2.181}$$

We conclude that $\mathfrak{A}_3$ is a Weyl-algebra over $\mathfrak{f}_1$. $\alpha : \mathfrak{A}_1 \to \mathfrak{A}_3$ is the $*$-isomorphism of 2.8.2 defined by $W_1(f) \mapsto W_3(f)$. □

Example 2.8.4.

- The main example is the Weyl algebra in the Fock representation defined in Theorem 2.7.2.
- The range of a Weyl algebra under a $*$-morphism is a Weyl-algebra.

Definition 2.8.5 (States over the Weyl algebra)**.** *Let ω be a state over $W(\mathfrak{f})$.*

1. *ω is regular, if $\mathbb{R} \ni t \mapsto \omega(W(tf)) \in \mathbb{C}$ is continuous for all $f \in \mathfrak{f}$.*
2. *ω is quasi-free, if there is a real bilinear form $q : \mathfrak{f}\times\mathfrak{f} \to \mathbb{C}$, and $\omega(W(f)) = e^{-q(f,f)}$, $f \in \mathfrak{f}$.*

Example 2.8.6. Let $\omega(W) = \langle\Omega|W\Omega\rangle$ for $W \in W_F(\mathfrak{h}) \subset \mathcal{B}(\mathcal{F}_b[\mathfrak{h}])$ and let Ω be the vacuum vector of $\mathcal{F}_b[\mathfrak{h}]$. We define $x(t) = \omega(W_F(tf))$. Obviously $x(0) = 1$ and

$$\dot{x}(t) = \imath\langle\Omega|W_F(tf)\Phi(f)\Omega\rangle = \imath 2^{-1/2}\langle\Omega|W_F(tf)a^*(f)\Omega\rangle = \imath 2^{-1/2}\langle\Omega|a_t^*(f)W_F(tf)\Omega\rangle, \quad (2.182)$$

where we have

$$a_t^*(f) = W_F(tf)a^*(f)W_F(-tf) = a^*(f) + \imath 2^{-1/2}t\|f\|_{\mathfrak{h}}^2. \quad (2.183)$$

It follows $\dot{x}(t) = -(1/2)t\|f\|_{\mathfrak{h}}^2 \cdot x(t)$ and solving the initial value problem for x yields

$$x(t) = \exp(-(1/4)t^2\|f\|_{\mathfrak{h}}^2) \Longrightarrow \omega(W_F(f)) = \exp(-(1/4)\|f\|_{\mathfrak{h}}^2). \quad (2.184)$$

Hence ω is a quasi-free state of $W_F(h)$.

Theorem 2.8.7 (Regular States)**.** *Let ω be a regular state over $W(\mathfrak{f})$ and $(\mathfrak{h}_\omega, \pi_\omega, \Omega_\omega)$ its GNS-representation. For all $f \in \mathfrak{f}$ exists a self-adjoint operator $\Phi_\omega(f)$ on $\mathfrak{h}_\omega$, such that*

$$W(tf) = e^{\imath t\Phi_\omega(f)}. \quad (2.185)$$

Proof of 2.8.7. $\mathbb{R} \ni t \mapsto \pi_\omega(W(tf)) \in \mathcal{B}(\mathfrak{h}_\omega)$ is a group of unitaries. The statement of the theorem follows from Stone's theorem, whenever the group is strongly continuous.
Let $\mathfrak{h}_0 = \pi_\omega(W(\mathfrak{f}))\Omega_\omega$.

$$\begin{aligned} &\|(\pi_\omega(W(tf)) - \mathbb{1})\pi_\omega(W(g))\Omega_\omega\|^2 \quad (2.186)\\ =\ &\big\langle\Omega_\omega|\pi_\omega(W(g))^*(\pi_\omega(W(tf)) - \mathbb{1})^*(\pi_\omega(W(tf)) - \mathbb{1})\pi_\omega(W(g))\Omega_\omega\big\rangle \\ =\ &\omega\Big(W(g)^*(W(tf) - \mathbb{1})^*(W(tf) - \mathbb{1})W(g)\Big) \\ =\ &2 - e^{\imath t\Im\langle f|g\rangle}\omega(W(-tf)) - e^{-\imath t\Im\langle f,g\rangle}\omega(W(tf)) \longrightarrow 0, \end{aligned}$$

if t tends to zero. Hence it is strongly continuous on $\mathfrak{h}_0$, but $\mathfrak{h}_0$ is dense in $\mathfrak{h}_\omega$. □

Theorem 2.8.8 (Wick's Theorem). *Let ω be a regular state over $W(\mathfrak{f})$.*

1. *Assume*

$$\mathbb{R}^n \ni (t_1, \dots, t_n) \mapsto \omega\big(W(t_n f_n) \cdots W(t_1 f_1)\big) \tag{2.187}$$

is smooth, then for $k = 1, \dots, n$

$$\Omega_\omega \in \operatorname{dom}(\Phi_\omega(f_1)), \quad \Phi_\omega(f_{k-1}) \cdots \Phi_\omega(f_1)\Omega_\omega \in \operatorname{dom}(\Phi_\omega(f_k)).$$

2. *If ω is quasi-free, then*

$$\langle \Omega_\omega | \Phi_\omega(f_1) \cdots \Phi_\omega(f_{2n-1})\Omega_\omega \rangle = 0$$
$$\langle \Omega_\omega | \Phi_\omega(f_1) \cdots \Phi_\omega(f_{2n})\Omega_\omega \rangle = \sum_{P \in \mathcal{P}_{2n}} \prod_{\{k<l\} \in P} \langle \Omega_\omega | \Phi_\omega(f_k)\Phi_\omega(f_l)\Omega_\omega \rangle.$$

$\mathcal{P}_n$ is the set of pairings of $\{1, \dots, 2n\}$. That is, $\mathcal{P}_n$ contains sets P of the power set of $\{1, \dots, 2n\}$. Each $P \in \mathcal{P}_n$ is a decomposition of $\{1, \dots, 2n\}$ into set with exactly two element.

Proof of 2.8.8. Theorem 2.8.8 is proved in the Appendix. □

Lemma 2.8.9. *If H is a self-adjoint operator on a separable Hilbert space $\mathfrak{h}$ and $\beta, \mu \in \mathbb{R}$, $\beta > 0$, such that $H - \mu \geq \epsilon > 0$ and $\operatorname{Tr}\{e^{-\beta H}\} < \infty$, then $\operatorname{Tr}\{e^{-\beta d\Gamma(H-\mu)}\} < \infty$.*

The parameter μ is often called the chemical potential.

Proof of 2.8.9. Since $\operatorname{Tr}\{e^{-\beta H}\} < \infty$, H has only discrete spectrum and we may write the eigenvalues in increasing order $E_1 \leq E_2 \dots$, repeated according to multiplicity. Let P_m be the projection onto the m-particle space $\mathcal{S}_m \bigotimes_{i=1}^m \mathfrak{h}$ of the bosonic Fock space. We obtain

$$e^{-\beta d\Gamma(H-\mu)} P_m = \Big(\bigotimes_{i=1}^m e^{-\beta(H-\mu)} \Big) P_m. \tag{2.188}$$

In $\bigotimes_{i=1}^m \mathfrak{h}$ the eigenstates of $\bigotimes_{i=1}^m e^{-\beta(H-\mu)}$ can be characterized by $(E_{i_1}, \dots, E_{i_m})$. But the eigenstates in $\mathcal{S}_m \bigotimes_{i=1}^m \mathfrak{h}$ can be indexed by

$$\big((E_{i_1}, n_1), \dots, (E_{i_m}, n_m)\big)\big), \quad i_1 < \dots < i_m, \quad n_1 + \dots + n_m = n, \quad n_i \in \mathbb{N}_0 \tag{2.189}$$

due to symmetry. That yields for the trace

$$\operatorname{Tr}\{e^{-\beta d\Gamma(H-\mu)} P_m\} = \sum_{i_1 < i_2 < \dots < i_m} \sum_{\substack{(n_1, \dots, n_m) \in \mathbb{N}_0^m \\ n_1 + \dots + n_m = m}} e^{-\beta \sum_{k=1}^m n_k (E_{i_k} - \mu)}. \tag{2.190}$$

For the full trace we obtain using the Neumann series

$$\begin{aligned}\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\} &= \sum_{m=0}^{\infty}\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}P_m\} \leq \prod_{k=1}^{\infty}\Big(1+\sum_{n=1}^{\infty}e^{-\beta n(E_k-\mu)}\Big) \\ &= \prod_{k=1}^{\infty}\frac{1}{1-e^{-\beta(E_k-\mu)}} \leq \exp\Big(\sum_{k=1}^{\infty}\ln\Big(1-e^{-\beta(E_k-\mu)}\Big)\Big).\end{aligned} \tag{2.191}$$

Using concavity of ln we obtain

$$\ln\Big(\frac{1}{1-e^{-\beta(E_k-\mu)}}\Big) \leq \frac{1}{1-e^{-\beta(E_k-\mu)}}-1 = \frac{e^{-\beta(E_k-\mu)}}{1-e^{-\beta(E_k-\mu)}} \leq \frac{e^{-\beta(E_k-\mu)}}{1-e^{-\beta\epsilon}}. \tag{2.192}$$

Inserting in Equation (2.191) yields

$$\begin{aligned}\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\} &\leq \exp\Big(\mathrm{Tr}\{e^{-\beta(H-\mu)}\}(1-e^{-\beta\epsilon})^{-1}\Big) \\ &\leq \exp\Big(\mathrm{Tr}\{e^{-\beta H}\}\frac{e^{\beta\mu}}{1-e^{-\beta\epsilon}}\Big) < \infty.\end{aligned} \tag{2.193}$$

We have shown the trace class property of $e^{-\beta d\Gamma(H-\mu)}$. □

Theorem 2.8.10 (Gibbs States of Second Quantized Operators)**.** *For $f \in \mathfrak{h}$ and $W(\mathfrak{h}) \subset \mathcal{B}(\mathcal{F}_b[\mathfrak{h}])$ we have*

$$\begin{aligned}\omega(W(f)) &:= Z^{-1}\cdot\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}W_F(f)\} \\ &= \exp\Big(-(1/4)\langle f|\coth((\beta/2)(H-\mu))f\rangle\Big),\end{aligned} \tag{2.194}$$

where $Z := \mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\}$ and H, μ, β as in Lemma 2.8.9.

Proof of 2.8.10. Pick $\epsilon > 0$ such that $H-\mu \geq 2\epsilon > 0$. For every $m \in \mathbb{N}$ we have

$$\begin{aligned}&\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}(N+1)^m\} = \mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu-\epsilon)}e^{-\beta\epsilon N}(N+1)^m\} \\ &\leq \mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu-\epsilon)}\}\|e^{-\beta\epsilon N}(N+1)^m\| < \infty,\end{aligned} \tag{2.195}$$

since $\|e^{-\beta\epsilon N}(N+1)^m\| < \infty$ because of the spectral theorem, $\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu-\epsilon)}\} < \infty$ by Lemma 2.8.9. For any operator A, such that $A(N+1)^{-m}$ is bounded be obtain that $e^{-\beta d\Gamma(H-\mu-\epsilon)}A$ extends to operator of trace class. It is simple to show, that one can choose any polynomial of creation- and annihilation operators for A. First, we remark that for $n \in \mathbb{N}_0$

$$\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n+1})\} = 0, \tag{2.196}$$

since for the orthogonal projection P_m onto the m-particle space we have $e^{-\beta d\Gamma(H-\mu)}P_m = P_m e^{-\beta d\Gamma(H-\mu)}$ and $P_m\Phi(f)^{2n+1})P_m = 0$. Moreover,

$$\begin{aligned} & \mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n-1}a(g)\} = \mathrm{Tr}\{a(g)e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n-1}\} \\ = \; & \mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}a(e^{-\beta(H-\mu)}g)\Phi(f)^{2n-1}\} \\ = \; & \frac{2n-1}{2}\langle e^{-\beta(H-\mu)}g|f\rangle \,\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n-2}\} \\ & + \mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n-1}a(e^{-\beta(H-\mu)}g)\}. \end{aligned} \tag{2.197}$$

For $h := (1-e^{-\beta(H-\mu)})g$ we obtain

$$\begin{aligned} & \mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n-1}a(h)\} \\ = \; & \frac{2n-1}{\sqrt{2}}\langle (e^{\beta(H-\mu)}-1)^{-1}h|f\rangle \;\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n-2}\}. \end{aligned} \tag{2.198}$$

Furthermore,

$$\begin{aligned} & \mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n-1}a^*(h)\} = \overline{\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}a(h)\Phi(f)^{2n-1}\}} \\ = \; & \overline{\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n-1}a(h)\}} + \frac{2n-1}{\sqrt{2}}\overline{\langle h|f\rangle}\cdot\overline{\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n-2}\}} \\ = \; & \frac{2n-1}{\sqrt{2}}\langle f|e^{-\beta(H-\mu)}(e^{\beta(H-\mu)}-1)^{-1}h\rangle \;\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n-2}\}. \end{aligned} \tag{2.199}$$

That yields

$$\begin{aligned} & \mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n}\} \\ = \; & \frac{2n-1}{\sqrt{2}}\langle f|\coth(\beta/2(H-\mu))f\rangle\cdot\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n-2}\} \\ = \; & \frac{(2n-1)(2n-3)\cdots 1}{2^n}\langle f|\coth(\beta/2(H-\mu))f\rangle^n\cdot\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\} \\ = \; & \frac{(2n)!}{4^n n!}\langle f|\coth(\beta/2(H-\mu))f\rangle^n\cdot\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\}. \end{aligned} \tag{2.200}$$

For a Weyl operator $W(f) = e^{\imath\Phi(f)}$ we have

$$\omega(W(f)) \;:=\; Z^{-1}\cdot\sum_{n=0}\frac{\imath^{2n}}{(2n)!}\,\mathrm{Tr}\{e^{-\beta d\Gamma(H-\mu)}\Phi(f)^{2n}\} = \exp\Big(-(1/4)\langle f|\coth(\beta/2(H-\mu))f\rangle\Big).$$

□

Chapter 3

Models in Nonrelativistic QED

3.1 Particle-Photon Interaction

In the Standard Model of QED, the particles are described with spin and with a potential. We are working in units, where the Planck-constant $\hbar$ and the speed of light c are equal to 1. The Hilbert space $\mathcal{H}_{el}$ is a closed subspace of $L^2(\mathbb{R}^{3N};\mathbb{C}^2)$. If all particles are identical fermions, then to $\mathcal{H}_{el}$ belong only antisymmetric wave functions, i.e for any permutation of $\{1,\ldots,N\}$, we have

$$\psi(k_{\pi 1},\ldots,k_{\pi N}) = \mathrm{sgn}(\pi)\,\psi(k_1,\ldots,k_N) \tag{3.1}$$

almost everywhere, for every $\psi \in \mathcal{H}_{el}$. The Hamiltonian for the particles is

$$H_{el} = \sum_{i=1}^{N}\left[\left(\vec{\sigma_i}\cdot(-\imath\vec{\nabla}_{x_i})\right)^2\right] + V, \tag{3.2}$$

where $\vec{\sigma_i} = (\sigma_i^x,\, \sigma_i^y,\, \sigma_i^z)$ is the vector of spin-matrices. V is a potential, i.e. a measurable function $V : \mathbb{R}^{3N} \to \mathbb{R}$, that acts by multiplication and leaves $\mathcal{H}_{el}$ invariant.
The Hilbert space for the bosons is

$$\mathcal{H}_f := \mathcal{F}_b[\mathfrak{h}], \quad \mathfrak{h}_{ph} := L^2(\mathbb{R}^3 \times \{\pm\};\mathbb{C}). \tag{3.3}$$

$\mathfrak{h}_{ph}$ is a Hilbert space equipped with the scalar product $\langle f|g\rangle_{\mathfrak{h}_{ph}} = \sum_{\mu=\pm}\int \overline{f_\mu(k)}g_\mu(k)d^3k$. The field Hamiltonian is the second quantization of the one-particle energy,

$$\check{H} = d\Gamma(|k|) \tag{3.4}$$

$|k|$ denotes the function that acts by multiplication with $|k|$ in each component.

The quantized radiation field is defined by

$$\vec{A}(x) = (2\pi)^{-3/2} \sum_{\mu=\pm} \int \frac{\hat{\rho}(k)}{(2|k|)^{1/2}} e^{-\imath k \cdot x} \vec{\epsilon}(k,\mu) a^*(k,\mu) d^3k + h.c. \tag{3.5}$$

$\vec{\epsilon}(k,\lambda)$ is a measurable functions over $\mathbb{R}^3 \times \{\pm\}$, for almost all $k \in \mathbb{R}^3$, the triple $(\vec{k},\ \vec{\epsilon}(k,+),\ \vec{\epsilon}(k,-))$ defines an ONB in $\mathbb{C}^3$. Formally, one can check that $\vec{A}(x)$ is in Coulomb gauge, i.e. $\operatorname{div}_x \vec{A}(x) = 0$. ρ is an artificial charge distribution of the particle, its Fourier transform $\hat{\rho}$ yields an UV-cutoff, in particular $\hat{\rho}(-k) = \overline{\hat{\rho}(k)}$. $a^*(k,\lambda)$, $a(k,\lambda)$ are creation- and annihilation- operators, for a definition of these objects, see ([22], Sec X.7). In our notation we have

$$A_i(x) = \Phi(G_{i,x}), \quad G_{i,x}(k) = (2\pi)^{-3/2} \frac{\hat{\rho}(k)}{|k|^{1/2}} e^{-\imath k \cdot x} \epsilon_i(k,\mu). \tag{3.6}$$

$G_{i,x}(k)$ is considered as an operator on $\mathcal{H}_{el}$ depending on $(k,\mu) \in \mathbb{R}^3 \times \{\pm\}$. We also use the notation $\Phi(\vec{G}_x) = \big(\Phi(G_{1,x}),\ \Phi(G_{2,x}),\ \Phi(G_{3,x})\big)$.

The interaction between both particle system and photons is obtained by minimal coupling, one replaces $-\imath\vec{\nabla}_{x_i} \to -\imath\vec{\nabla}_{x_i} - \lambda^{3/2}\Phi(\vec{G}_{\lambda^{1/2}x_i})$. λ is the coupling constant its physical value is $\frac{1}{137}$. we allow in our model λ to take any value not equal to zero. The full Hamiltonian now reads

$$H_\lambda = \sum_{i=1}^{N} \Big[\big(\vec{\sigma}_i \cdot (-\imath\vec{\nabla}_{x_i} - \lambda^{3/2}\Phi(\vec{G}_{\lambda^{1/2}x_i})) \big)^2 \Big] + V \otimes \mathbb{1} + \mathbb{1} \otimes \check{H}. \tag{3.7}$$

$$\tag{3.8}$$

The units in this model are $\hbar = c = 1$, the positions of the electrons are measured in units of $(1/2)r_{Bohr} = (2m_{el}e^2)^{-1}\hbar^2$, r_{Bohr} is the Bohr radius, e is the charge of the electron, m_{el} is the mass of the electron. The wave length is measured in units of $(1/2)\lambda r_{Bohr}$, the energy is chosen in unit of 4 Rydberg, with $4Ry = 2e^2 r_{Bohr}^{-1}$.

Other models describing the particle-photon interaction can be derived from the standard model. For example one neglects the spin of the particles and the polarization of the photons. The Hilbert space $\mathcal{H}_{el}$ is now a subspace of $L^2(\mathbb{R}^{3N})$ and the Fock space corresponds to $L^2(\mathbb{R}^3)$. As Hamiltonian one would consider

$$H_\lambda = \sum_{i=1}^{N} (-\imath\vec{\nabla}_{x_i} \otimes \mathbb{1} - \lambda^{3/2}\Phi(\vec{G}_{\lambda^{1/2}x_i}))^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes \check{H}. \tag{3.9}$$

To obtain other models, that are simpler to study, we introduce

$$U := \exp\Big(- \imath\lambda^{3/2} \sum_{i=1}^{N} x_i \otimes \Phi(\vec{G}_0)\Big)$$

U is an operator-valued gauge transformaton, known as the Pauli-Fierz transformation. We will not prove that U is a unitary, since the proof is simple. One obtains

$$\begin{aligned} U(-\imath\vec{\nabla}_{x_i})U^* &= -\imath\vec{\nabla}_{x_i} \otimes 1 + \lambda^{3/2} \sum_{i=1}^{N} x_i \otimes \Phi(\vec{G}_0) \qquad (3.10) \\ U\,\Phi(G_{j,\lambda^{1/2}x_i})\,U^* &= \Phi(G_{j,\lambda^{1/2}x_i}) + \lambda^{3/2} \sum_{m=1}^{N} \Im\langle x_m \vec{G}_0 | G_{j,\lambda^{1/2}x_i}\rangle_{\mathfrak{h}_{ph}} = \Phi(G_{j,\lambda^{1/2}x_i}) \\ U\check{H}U^* &= \mathbb{1} \otimes \check{H} + \lambda^{3/2} \sum_{i=1}^{N} x_i \otimes \Phi(\imath|k|\vec{G}_0) + \lambda^3 \Big(\sum_{i=1}^{N} x_i \otimes \mathbb{1}\Big)^2 \||k|^{1/2}\vec{G}_0\|^2_{\mathfrak{h}_{ph}}. \end{aligned}$$

The unitary transformed Hamiltonian is

$$\begin{aligned} H_1 &:= UH_\lambda U^* = \sum_{i=1}^{N} (-\imath\vec{\nabla}_{x_i} \otimes \mathbb{1} - \lambda^{3/2}\Phi(\vec{G}_{\lambda^{1/2}x_i} - \vec{G}_0))^2 + V \otimes \mathbb{1} + \check{H} \\ &\quad + \lambda^{3/2} \sum_{i=1}^{N} x_i \otimes \Phi(\imath|k|\vec{G}_0) + \lambda^3 \||k|^{1/2}\vec{G}_0\|^2_{\mathfrak{h}_{ph}} \Big(\sum_{i=1}^{N} x_i\Big)^2 \otimes 1. \qquad (3.11) \end{aligned}$$

In the following we call a model Dipole Approximation, if we consider have $\vec{G}_0$ in the interaction, i.e. the transformed Hamiltonian is

$$H_1^{DP} := \sum_{i=1}^{N} \big(-\Delta_{x_i} \otimes \mathbb{1} + \lambda^{3/2} x_i \otimes \Phi(\imath|k|\vec{G}_0)\big) + V \otimes \mathbb{1} + \mathbb{1} \otimes \check{H} + \lambda^3 \Big(\sum_{m=1}^{N} x_m\Big)^2 \||k|^{1/2}\vec{G}_0\|^2_{\mathfrak{h}_{ph}}. \quad (3.12)$$

This Approximation seems to be justified if one considers the bounded states of atoms, where x_j is small. The Pauli-Fierz Transformation has the property to improve the infrared singularity in the interaction by the factor $|k|$.

An alternative model, introduced by Nelson, uses $G_x := \frac{e^{\imath kx}\hat{\rho}(k)}{|k|^{1/2}}$. The particles have no spin and the photons have no polarization, i.e. the Fock space $\mathcal{F}_b[L^2(\mathbb{R}^3)]$ is considered. The Nelson Hamiltonian is

$$H_\lambda^{NM} := H_{el} \otimes 1 + 1 \otimes \check{H} + \lambda \sum_{i=1}^{n} \Phi(G_{x_i}). \qquad (3.13)$$

This model is used to study the behavior of H_λ^{NM} as $\hat{\rho}(k)$ tends to 1, i.e. the ultraviolet cutoff is removed. Confer [20].

3.2 The concrete Model

In the mathematical analysis we consider operators of the form

$$H_\lambda = H_{el} + W + \check{H}, \tag{3.14}$$

acting on $\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{F}_b[\mathfrak{h}]$, $\mathfrak{h} := L^2(\mathbb{R}^3)$. $\mathcal{H}_{el}$ is a Hilbert space of wave functions, i.e. a separable subspace of $L^2(X;\mathbb{C})$. $X = (X, \mathfrak{A}, \mu)$ is measure space and ν is a σ-finite, regular Borel measure. In principle, one could take an arbitrary Hilbert space, if one introduce an abstract conjugation map on it, we use in our case the pointwise complex conjugation. Its also possible to consider particles with spin and photons with polarizations, but to keep things simple we will not do.

The operator W denotes the interaction. We set

$$\begin{aligned} W &= \sum_{i=1}^{3} \lambda_i W_i \qquad (3.15) \\ W_1 &:= \lambda_1 \sum_{i=1}^{r} \big(\Phi(G_i)\Phi(H_i) + \Phi(H_i)\Phi(G_i)\big) \\ W_2 &:= \lambda_2 \Phi(F) \\ W_3 &:= \lambda_3 V, \end{aligned}$$

which can be specified to operators mentioned above. First, we remark, that we can write

$$\mathcal{H} \subset \{(f_n)_{n=0}^{\infty} \,:\, f_n \in L^2(X \times \mathbb{R}^{3n}, d\mu \otimes d^{3n}k),\ f(x,\cdot) \in L^2_{sym}(\mathbb{R}^{3n})\}. \tag{3.16}$$

The free Hamiltonian is

$$H_0 := H_{el} \otimes 1 + 1 \otimes \check{H}, \tag{3.17}$$

where H_{el} is self-adjoint and bounded below. We will see, that one obtains the same Liouvillean for H_{el} and $H_{el} + E$. Therefore, we may assume that

$$H_{el} \geq 1. \tag{3.18}$$

Partly, we need the assumption

Hypothesis 1 (Gibbs Condition)**.**

$$\mathrm{Tr}\{e^{-(\beta-\epsilon)H_{el}}\} < \infty \text{ for some } 0 < \epsilon \ll \beta. \tag{3.19}$$

The value β is of course the inverse temperature, the positive number $\epsilon >$ will yield a kind of regularization in the proof of Theorem 4.2.1. We will discuss this condition in section 3.3.

Let $G = \{G_k\}_{k\in\mathbb{R}^3},\ H = \{H_k\}_{k\in\mathbb{R}^3},\ F = \{F_k\}_{k\in\mathbb{R}^3}$ be families of closed operators (eventually depending on further indices). We assume, that $\mathrm{dom}(F_k^{\#}) \supset \mathrm{dom}(H_{el}^{1/2})$ and

$$\mathbb{R}^3 \ni k \mapsto G_k^{\#},\ H_k^{\#},\ H_{el}^{-1/2}F_k^{\#} \in \mathcal{B}(\mathcal{H}_{el}) \tag{3.20}$$

belong to $L^2(\mathbb{R}^3;\mathcal{B}(\mathcal{H}_{el}))$. Furthermore we assume

$$\begin{gathered}\int(|k|+|k|^{-1})\|G_k^{\#}\|^2_{\mathcal{B}(H_{el})}d^3k<\infty,\ \int(|k|+|k|^{-1})\|H_k^{\#}\|^2_{\mathcal{B}(H_{el})}d^3k<\infty \\ \int(|k|+|k|^{-1})\|H_{el}^{-1/2}F_k^{\#}\|^2_{\mathcal{B}(H_{el})}d^3k<\infty,\ \|H_{el}^{-\delta}V\|_{\mathcal{B}(H_{el})}<\infty,\end{gathered} \tag{3.21}$$

where $0 \le \delta < 1$ is fixed and $G_k^{\#},\ H_k^{\#},\ F_k^{\#}$ is either $G_k,\ H_k,\ F_k$ or $G_k^*,\ H_k^*,\ F_k^*$.
The critical singularity for $G_k^{\#}, H_k^{\#}$ or $F_k^{\#}$ at the origin is $|k|^{-1}$, that includes the physical case $|k|^{-1/2}$ resp. $|k|^{1/2}$.

We define for $f = (f_n)_{n=0}^{\infty}$,

$$\begin{aligned}&(a^*(F)f_n)(x,k_1,\dots,k_{n+1}) \\ &= (n+1)^{-1/2}\sum_{i=1}^{n+1}(F_{k_i}f_n)(x,k_1,\dots,\hat{k}_i,\dots,k_{n+1}),\end{aligned} \tag{3.22}$$

and $a(F)f_0(x) = 0$, resp.

$$\begin{aligned}&(a(F)f_{n+1})(x,k_1,\dots,k_n) \\ &= (n+1)^{1/2}\int dk_{n+1}(F^*_{k_{n+1}}f_{n+1})(x,k_1,\dots,k_n,k_{n+1}).\end{aligned} \tag{3.23}$$

The following bounds are obtained directly from Equation 5.10.

$$\begin{aligned}\|H_{el}^{-1/2}a(F)f\|^2_{\mathcal{H}_g} &\le \int|\alpha(k)|^{-1}\|H_{el}^{-1/2}F_k^*\|^2_{\mathcal{B}(\mathcal{H}_{el})}dk\cdot\|d\Gamma(|\alpha|)^{1/2}f\|^2_{\mathcal{H}_g} \\ \|H_{el}^{-1/2}a^*(F)f\|^2_{\mathcal{H}_g} &\le \int|\alpha(k)|^{-1}\|H_{el}^{-1/2}F_k\|^2_{\mathcal{B}(\mathcal{H}_{el})}dk\cdot\|d\Gamma(|\alpha|)^{1/2}f\|^2_{\mathcal{H}_g} \\ &\quad+\int\|H_{el}^{-1/2}F_k\|^2_{\mathcal{B}(\mathcal{H}_{el})}dk\cdot\|f\|^2_{\mathcal{H}_g}.\end{aligned} \tag{3.24}$$

For $G_k,\ H_k \in \mathcal{B}(\mathcal{H}_{el})$, the regularization $H_{el}^{-1/2}$ can be omitted. At this moment we say nothing about self-adjointness of H_λ. In most parts of this work (except in Chapter 5) need not this fact, nevertheless it can deduced by 4.1.2 and 4.1.3.

3.3 Gibbs Condition

In this section we discuss in detail for same relevant operators, which conditions imply that H_{el} as only discrete pure point spectrum and $\mathrm{Tr}\{e^{-\beta H_{el}}\} < \infty$. Only in this section we will write H for H_{el}.

In the first case we consider particles in a bounded region. Let $\Omega \subset \mathbb{R}^3$ be open and bounded. We set

$$\Omega_N = \underbrace{\Omega \times \cdots \times \Omega}_{N\text{-times}}.$$

The kinetic energy is H_0 is defined on $\mathcal{C}_0^\infty(\Omega_N)$ by $-\Delta_x$. H_0 is defined by Friedrichs extension on $L^2(\Omega_N)$. For the potential one can choose

$$W = \sum_{1 \le i < j \le N} V(x_i - x_j),$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is a potential that is an infinitesimally form bounded perturbation of H_0. Hence $H = H_0 + W$ can be defined as sum of quadratic forms in the sense of ([22], Thm. X.17). It follows for all $\beta > 0$ that $\mathrm{Tr}\{e^{-\beta H}\} < \infty$. For a proof see ([23], Thm. XIII.78).

In the second case we consider a particle system in $\mathbb{R}^3$ with confining potential. First assume $H = -\Delta_x + V$ and $V = V_1 + V_2$ is potential, where $V_1 \in L^1_{loc}(\mathbb{R}^n)$ is positive and for all $N > 0$ exists $R > 0$, so that the $V_1(x) \ge N$ for almost all x with $|x| \ge R$. V_2 is a form bounded perturbation of $-\Delta_x$ with relative bound $a < 1$. Then H defined as sum of quadratic forms has a compact resolvent and hence only discrete pure point spectrum. For the proof of self-adjointness see ([22], Thm. X.32 and Thm. X.17) for the rest see ([23], Thm. XIII.68).
To check the Gibbs condition we assume $V_1 \in L^2_{loc}(\mathbb{R}^n)$ and V_2 is perturbation of $-\Delta_x$ with relative bound $a < 1$. By ([22], Thm. X.29) H is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^n)$. Since in the quadratic form sense $V_2 \le a(-\Delta_x) + b$ we have

$$H \ge (1-a)(-\Delta_x) + V_1 - b. \tag{3.25}$$

By the Golden-Thompson inequality B.0.9 we have

$$\mathrm{Tr}\{e^{-\beta H}\} \le e^{\beta b}\,\mathrm{Tr}\{e^{-\beta/2(1-a)(-\Delta_x)} e^{-\beta V_1} e^{-\beta/2(1-a)(-\Delta_x)}\}. \tag{3.26}$$

We define now $A = e^{-(\beta/2)V_1}e^{-(\beta/2)(1-a)(-\Delta_x)}$. Using Fourier Transform $\mathcal{F}$ one calculate the integral kernel k_A of A,

$$\begin{aligned}(A\phi)(x) &= \int k_A(x,y)\phi(y)d^ny = e^{-(\beta/2)V_1(x)}\mathcal{F}^{-1}[e^{-(\beta/2)(1-a)(p^2)}\hat{\phi}](x) \\ &= (2\pi)^{-n/2}\int e^{-(\beta/2)V_1(x)}e^{-(\beta/2)(1-a)((x-y)^2)}\phi(y)d^ny\end{aligned} \tag{3.27}$$

for almost every $x \in \mathbb{R}^n$. From Lemma 4.2.2 follows

$$\|A\|_2^2 = \int |k_A(x,y)|^2 d^nxd^ny = (2\pi)^{-n}\int e^{-\beta V_1(x)}d^nx \cdot \int e^{-\beta(1-a)(y^2)}d^ny.$$

Since $\mathrm{Tr}\{e^{-\beta H}\} \leq e^{\beta b}\,\mathrm{Tr}\{A^*A\} = e^{\beta b}\|A\|_2^2$, it follows

$$\int e^{-\beta V_1(x)}d^nx \Longrightarrow \mathrm{Tr}\{e^{-\beta H}\} < \infty. \tag{3.28}$$

This means, at least a logarithmical growth at infinity is needed to fulfill the Gibbs condition.

The Gibbs condition indicates that the particle is confined to finite region. This is believed to be sufficient for the existence of an equilibrium state, since otherwise an escape to infinity is expected.

Remark 3.3.1 (The Representation of the Dynamical System for the Particles)**.** In our context the Gibbs condition enables us to define

$$\omega_{el}^{\beta}(A) = Z_\beta^{-1}\cdot \mathrm{Tr}\{e^{-\beta H_{el}}A\}, \quad Z_\beta = \mathrm{Tr}\{e^{-\beta H_{el}}\} \tag{3.29}$$

for $A \in \mathcal{B}(\mathcal{H}_{el})$. Whether the Gibbs condition is satisfied or not, one can define a representation of $\mathcal{B}(\mathcal{H}_{el})$ into $\mathcal{B}(\mathcal{K}_{el})$, $\mathcal{K}_{el} := \mathcal{H}_{el}\otimes\mathcal{H}_{el}$ by $\pi^{el} : A \mapsto A\otimes 1$. We use also the notation $\pi^{el}[A] = A_x$ introduced in Example 2.4.12.
In our case is $\mathfrak{M}_{el} := \pi^{el}[\mathcal{B}(\mathcal{H}_{el})] = \mathcal{B}(\mathcal{H}_{el}) \otimes \mathbb{1}$ a W^*-algebra. Its the commutant $\mathfrak{M}_{el}'$ is $\mathbb{1} \otimes \mathcal{B}(\mathcal{H}_{el})$. We have a modular conjugation $\mathcal{J}_{el}$ defined in 2.4.12 and in the case where the Gibbs condition is fulfilled, we have a cyclic and separating vector Ω_{el}^{β} and $(\mathcal{K}_{el}, \pi^{el}, \omega_{el}^{\beta})$ is a GNS-triple corresponding to ω_{el}^{β}, confer 2.4.12. In any case the Liouvillean is $\mathcal{L}_{el} := H_{el,x} - \overline{H}_{el,y}$ and $\tau_t^{el}(A) = e^{\imath\mathcal{L}_{el}}Ae^{-\imath\mathcal{L}_{el}}$ for $A \in \mathfrak{M}_{el}$.

3.4 Thermodynamic Limit

In this section we are interested in equilibrium states over $(\tau^f, W(\mathfrak{f}))$, which describe a Bose gas, infinitely extended, without condensation with an particle density due to Planck's law. In a *grand canonical setting* the equilibrium state is the so called Gibbs state

$$\omega(A) = \mathrm{Tr}\{\rho_\beta A\}, \quad \rho_\beta = \mathrm{Tr}\{e^{-\beta H}\}^{-1} \cdot e^{-\beta H} \tag{3.30}$$

It is not only a (τ, β)-KMS-state, ρ_β is the maximum of the entropy

$$\mathrm{Ent}(\rho) = -\mathrm{Tr}\{\rho \ln \rho\}, \quad \rho \in \big\{\sigma \in \mathcal{L}^1(\mathfrak{h}) \,:\, 0 \le \sigma,\ \mathrm{Tr}\{\sigma\} = 1,\ \mathrm{Tr}\{\sigma H\} = E_\beta\big\}. \tag{3.31}$$

E_β is a fixed energy depending on H and β. In a *grand canonical setting*, that belongs to an infinitely extended region, there is no explicit unique formula for an equilibrium state, in general.

We start with a restriction to a box Λ of length $L > 0$, and a Hilbert space $\mathcal{H}_\Lambda = L^2(\Lambda, d^3x)$. We use the Laplacian $-\Delta_x$ with periodic boundary conditions. The Hamiltonian we consider to describe a single boson in a box is $h^\Lambda := (-\Delta_x)^{1/2}$

We describe the Bose gas in the configuration space, while before we considered the momentum space, the one-particle Hamiltonian was the Fourier-transform of $(-\Delta_x)^{1/2}$ in the whole $\mathbb{R}^3$. For fixed L and $n \in \mathbb{Z}^3$ we define $e_n(x) = L^{-3/2} e^{\imath \frac{2\pi}{L} n \cdot x}$, $x \in \Lambda$. $\{e_n \,:\, n \in \mathbb{Z}^3\}$ is an ONB of eigenvectors of h^Λ, explicitly we have

$$h^\Lambda e_n = \frac{2\pi}{L} |n| e_n, \quad |n| = (n_1^2 + n_2^2 + n_3^2)^{1/2} \tag{3.32}$$

For every chemical potential $\mu < 0$ we check, that

$$Z := \mathrm{Tr}\{e^{-\beta(\mathfrak{h}^\Lambda - \mu)}\} = \sum_{n \in \mathbb{Z}^3} e^{-\beta(\frac{2\pi}{L}|n| - \mu)} \le 8 e^{\beta\mu} \Big(\sum_{n=0}^{\infty} e^{-\beta \frac{2\pi}{3L} n} \Big)^3 \le 8 e^{\beta\mu} \Big(1 - e^{-\beta \frac{2\pi}{3L}} \Big)^3 < \infty, \tag{3.33}$$

in the calculation we used $\frac{1}{3}(|n_1| + |n_2| + |n_3|) \le |n|$ and the Neumann series. By Lemma 2.8.9 we deduce from $e^{-\beta(h^\Lambda - \mu)} \in \mathcal{L}^1(\mathcal{H}_\Lambda)$ that $e^{-\beta(d\Gamma(\mathfrak{h}_\Lambda - \mu))} \in \mathcal{L}^1\big(\mathcal{F}_b[\mathcal{H}_\Lambda]\big)$. Let $f \in \mathcal{H}_\Lambda$. By Theorem 2.8.10 we obtain for the Gibbs state over the Weyl operators:

$$\omega^\Lambda_{\beta,\mu}(W(f)) = (Z^\Lambda_{\beta,\mu})^{-1} \mathrm{Tr}\{e^{-\beta(d\Gamma(\mathfrak{h}_\Lambda - \mu))} W(f)\} = \exp\Big(\frac{1}{4} \langle f | \coth((\beta/2)(H - \mu)) f \rangle_{\mathcal{H}_\Lambda} \Big). \tag{3.34}$$

For a f with compact support and all $\Lambda \supset \operatorname{supp} f$ we obtain for the Fourier coefficient

$$\langle e_n | f \rangle_{\mathcal{H}_\Lambda} = \hat{f}\big(\frac{2\pi}{L} n\big) \cdot \Big(\frac{2\pi}{L}\Big)^{3/2}. \tag{3.35}$$

$\hat{f}$ is the Fourier transform of f in $L^2(\mathbb{R}^3)$. Since $f = \sum_{n \in \mathbb{Z}^3} \langle e_n | f \rangle_{\mathcal{H}_\Lambda} e_n$, we obtain

$$\begin{aligned} \omega^\Lambda_{\beta,\mu}(W(f)) \quad &= \quad \exp\Big(\frac{1}{4} \sum_{n \in \mathbb{Z}^3} \coth\Big((\beta/2)\big(\frac{2\pi}{L}|n| - \mu\big) \Big) \big(\frac{2\pi}{L}\big)^3 \Big| \hat{f}\Big(\frac{2\pi}{L}|n|\Big) \Big|^2 \Big) \\ &\overset{L \to \infty}{\longrightarrow} \quad \exp\big(-\frac{1}{4} \langle \hat{f} | \coth((\beta/2)(|k| - \mu)) \hat{f} \rangle \big). \end{aligned} \tag{3.36}$$

using a Riemann approximation. To calculate the particle density, we use the formula for the photon number operator N_Λ on $\mathcal{F}_b[L^2(\Lambda, d^3x)]$:

$$\mathrm{N}_\Lambda = \sum_{n\in\mathbb{Z}^3} a^*(e_n)a(e_n). \tag{3.37}$$

The convergence is in a weak sense. We remark that

$$\omega^\Lambda_{\beta,\mu}(\Phi(e_n)^2) := -\partial_t^2\omega^\Lambda_{\beta,\mu}\big(W(te_n)\big)\Big|_{t=0} = \frac{1}{2}\coth((\beta/2)(\frac{2\pi}{L}|n|-\mu)), \tag{3.38}$$

confer Theorem 2.8.8. From $\Phi(e_n)^2 = (1/2)a(e_n)a(e_n) + (1/2)a^*(e_n)a^*(e_n) + a^*(e_n)a(e_n) + (1/2)$ and from $\omega^\Lambda_{\beta,\mu}(a(e_n)a(e_n)) = 0 = \omega^\Lambda_{\beta,\mu}(a^*(e_n)a^*(e_n))$ we conclude

$$\omega^\Lambda_{\beta,\mu}(a^*(e_n)a(e_n)) = \Big(\exp\big(\beta(\frac{2\pi}{L}|n|-\mu)\big)-1\Big)^{-1}. \tag{3.39}$$

For the particle number density one obtains

$$\begin{aligned}\frac{\omega^\Lambda_{\beta,\mu}(\mathrm{N}_\Lambda)}{|\Lambda|} &= L^{-3}\sum_{n\in\mathbb{Z}^3}\Big(\exp\big(\beta(\frac{2\pi}{L}|n|-\mu)\big)-1\Big)^{-1} \\ &\overset{L\to\infty}{\longrightarrow} (2\pi)^{-3}\int\Big(\exp\big(\beta(|k|-\mu)\big)-1\Big)^{-1}d^3k.\end{aligned} \tag{3.40}$$

Since $\omega^\Lambda_{\beta,\mu}$ is a (β,τ^Λ_t)-KMS state, where $\tau^\Lambda_t(A) = e^{\imath t d\Gamma(h_\lambda-\mu)}Ae^{-\imath t d\Gamma(h_\lambda-\mu)}$, and, formally,

$$d\Gamma(h_\Lambda-\mu) \overset{L\to\infty}{\longrightarrow} \check{H} - \mu\,\mathrm{N}, \tag{3.41}$$

we define the dynamical system, as follows:

Definition 3.4.1. *Let*

$$\mathcal{A}_f := W(\mathfrak{f}), \quad \mathfrak{f} = \{f\in L^2(\mathbb{R}^3) \,:\, \int(1+|k|^{-1})|f(k)|^2d^3k<\infty\}.$$

We define a linear functional by

$$\omega^\beta_f : \mathcal{A}_f \to \mathbb{C}, \quad W(f)\mapsto \exp\Big(-\frac{1}{4}\big\langle f|\coth\big((\beta/2)|k|\big)f\big\rangle\Big).$$

The $*$-automorphism group is $\tau^f_t(A) = e^{\imath t\check{H}}Ae^{-\imath t\check{H}}$ for $A\in\mathcal{A}_f$. $\check{H}$ is defined in 3.4. We also introduce the Planck density $\varrho(k) = (e^{\beta|k|}-1)^{-1}$. In terms of this function, we have $\omega^\beta_f(W(f)) = \exp\big(-\frac{1}{4}\langle f|(1+2\varrho)f\rangle\big)$.

3.5 A Derivation for the concrete Model at inverse Temperature β

Lemma 3.5.1. *There is an injective $*$-morphism*

$$\pi_{AW} : \mathcal{A}_f \to W[\mathfrak{h} \oplus \mathfrak{h}], \quad W(f) \mapsto W((1+\varrho)^{1/2} f \oplus \varrho^{1/2} \overline{f}), \tag{3.42}$$

such that for the vacuum vector Ω_f^β

$$\omega_f^\beta(W) = \langle \Omega_f^\beta | \pi_{AW}[W] \Omega_f^\beta \rangle_{\mathcal{F}_b[\mathfrak{h}\oplus\mathfrak{h}]}. \tag{3.43}$$

π_{AW} is the so called Araki-Woods-Isomorphism.

It follows directly, that ω_f^β is a state over $\mathcal{A}_f$.

Proof of 3.5.1. Let $v : \mathfrak{f} \subset \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$ be defined by $v(f) = (1+\varrho)^{1/2} f \oplus \varrho^{1/2} \overline{f}$. v is a real linear map and

$$\begin{aligned} \Im\langle v(f) | v(g) \rangle_{\mathfrak{h}\oplus\mathfrak{h}} &= \Im\langle (1+\varrho)^{1/2} f \oplus \varrho^{1/2} \overline{f} | (1+\varrho)^{1/2} g \oplus \varrho^{1/2} \overline{g} \rangle_{\mathfrak{h}\oplus\mathfrak{h}} \\ &= \Im\langle f | (1+\varrho) g \rangle_{\mathfrak{h}} + \Im\langle \overline{f} | \varrho \overline{g} \rangle_{\mathfrak{h}} = \Im\langle f | g \rangle_{\mathfrak{h}}. \end{aligned} \tag{3.44}$$

Theorem 2.8.3 and Example 2.8.6 yield the morphism π_{AW}. □

Lemma 3.5.2. *$(\mathcal{F}_b[\mathfrak{h}\oplus\mathfrak{h}], \pi_{AW}, \Omega_f^\beta)$ is a GNS-triple for ω_f^β. That is, Ω_f^β is cyclic for $\pi_{AW}[\mathcal{A}_f]$.*

Proof of 3.5.2. We have to show, that

$$\mathcal{F}_b[\mathfrak{h} \oplus \mathfrak{h}] = c\ell\{\pi_{AW}[W]\Omega_f^\beta : W \in \mathcal{A}_f\}. \tag{3.45}$$

Let $U := c\ell\{\pi_{AW}[W]\Omega_f^\beta : W \in \mathcal{A}_f\}$. We will show $\Phi_\beta(f_n)\cdots\Phi_\beta(f_1)\Omega_f^\beta \in U$, $f_i \in \mathfrak{f}$, where $\Phi_\beta(f) = \Phi\big((1+\varrho)^{1/2} f \oplus \varrho^{1/2}\overline{f}\big)$ Note that $\pi_{AW}[W(f)] = e^{\imath\Phi_\beta(f)}$. We proceed by mathematical induction. For $n = 0$ it is clear. By induction hypothesis $\Phi_\beta(f_n)\cdots\Phi_\beta(f_1)\Omega_f^\beta \in U$ and hence for all $t \in \mathbb{R}$.

$$\frac{\pi_{AW}[W(tf_{n+1})] - 1}{\imath t} \Phi_\beta(f_n)\cdots\Phi_\beta(f_1)\Omega_f^\beta \overset{t\to 0+}{\longrightarrow} \Phi_\beta(f_{n+1})\cdots\Phi_\beta(f_1)\Omega_f^\beta \in U. \tag{3.46}$$

because $\pi_{AW}[W(f_i)]$ leaves U invariant. Hence using CCR we have

$$a^*\big((1+\varrho)^{1/2} f_n \oplus \varrho^{1/2}\overline{f_n}\big)\cdots a^*\big((1+\varrho)^{1/2} f_1 \oplus \varrho^{1/2}\overline{f_1}\big)\Omega_f^\beta \in U. \tag{3.47}$$

Since $f \mapsto a^*(f)$ is linear, one obtains

$$\begin{aligned} a^*(\big((1+\varrho)^{1/2} f \oplus 0\big) &= (1/2) a^*\big((1+\varrho)^{1/2} f \oplus \varrho^{1/2}\overline{f}\big) - (1/2)\imath a^*\big((1+\varrho)^{1/2}\imath f \oplus \varrho^{1/2}\overline{(\imath f)}\big) \\ a^*(\big(0 \oplus \varrho^{1/2} f\big) &= (1/2) a^*\big((1+\varrho)^{1/2} f \oplus \varrho^{1/2}\overline{f}\big) + (1/2)\imath a^*\big((1+\varrho)^{1/2}\imath f \oplus \varrho^{1/2}\overline{(\imath f)}\big). \end{aligned}$$

Therefore for $g_1, \dots, g_n \in \mathfrak{f}$,

$$a^*\big((1+\varrho)^{1/2} f_n \oplus \varrho^{1/2} g_n\big) \cdots a^*\big((1+\varrho)^{1/2} f_1 \oplus \varrho^{1/2} g_1\big)\Omega_f^\beta \in U. \tag{3.48}$$

Since $c\ell_{\mathfrak{h}\oplus\mathfrak{h}}\{(1+\varrho)^{1/2} f \oplus \varrho^{1/2} g \,:\, f, g \in \mathfrak{f}\} = \mathfrak{h} \oplus \mathfrak{h}$, the statement follows with Equation (2.158) and Equation 2.145. □

We define $\mathfrak{M}_f := \pi_{AW}[\mathcal{A}_f]''$. At this point we remark, that $\mathfrak{M}_f$ is not $\mathcal{B}(\mathcal{F}_b[\mathfrak{h} \oplus \mathfrak{h}])$. Indeed, $\pi_{AW}[\mathcal{A}_f] \cong W(\mathfrak{f})$ up to a $*$-isomorphism, but it is not the Fock-representation of the CCR.

Lemma 3.5.3. *Ω_f^β is separating for $\mathfrak{M}_f$.*

Proof of 3.5.3. For $f, g \in \mathfrak{f}$ we have

$$\begin{aligned} & W((1+\varrho)^{1/2} f \oplus \varrho^{1/2}\overline{f})\, W(\varrho^{1/2} g \oplus (1+\varrho)^{1/2}\overline{g}) \\ = \;& \exp\Big(-\imath/2\Im\langle (1+\varrho)^{1/2} f \oplus \varrho^{1/2}\overline{f} | \varrho^{1/2} g \oplus (1+\varrho)^{1/2}\overline{g}\rangle_{\mathfrak{h}\oplus h}\Big) \\ & \cdot W(\varrho^{1/2} g \oplus (1+\varrho)^{1/2}\overline{g})\, W((1+\varrho)^{1/2} f \oplus \varrho^{1/2}\overline{f}) \\ = \;& W(\varrho^{1/2} g \oplus (1+\varrho)^{1/2}\overline{g})\, W((1+\varrho)^{1/2} f \oplus \varrho^{1/2}\overline{f}). \end{aligned} \tag{3.49}$$

Hence $W(\varrho^{1/2} g \oplus (1+\varrho)^{1/2}\overline{g}) \in \pi_{AW}[\mathcal{A}_f]' = \mathfrak{M}_f'$. As in Lemma 3.5.2, we obtain that Ω_f^β is cyclic for $\mathfrak{M}_f'$ and therefore separating for $\mathfrak{M}_f$. □

Next we observe, that

$$\pi_{AW}(\tau_t^f(W(f))) = W((1+\varrho)^{1/2} e^{\imath t|k|} f \oplus \varrho^{1/2} e^{-\imath t|k|}\overline{f}) = e^{\imath\mathcal{L}_f}\pi_{AW}(W(f)) e^{-\imath\mathcal{L}_f}, \tag{3.50}$$

where $\mathcal{L}_f := d\Gamma(|k| \oplus -|k|)$.

Lemma 3.5.4. *$\omega_f^\beta(A) := \langle \Omega_f^\beta | A \Omega_f^\beta\rangle$ is a (τ^f, β)-KMS-state over $\mathfrak{M}_f$, where $\tau_t^f(A) = e^{\imath t\mathcal{L}_f} A e^{-\imath t\mathcal{L}_f}$ for $A \in \mathfrak{M}_f$.*

Proof of 3.5.4. First, we observe

$$\begin{aligned}\omega_f^\beta(W(f)\,\tau_t^f(W(g))) &= \omega_f^\beta\big(\exp(-(\imath/2)\Im\langle f|e^{\imath t|k|}g\rangle)W(f+e^{\imath t|k|}g)\big) \\ &= \exp\big(-(1/2)\langle f|(1+\varrho)e^{\imath t|k|}g\rangle-(1/2)\langle g|\varrho e^{-\imath t|k|}f\rangle\big) \\ &\quad\cdot\exp\big(-(1/4)\|(1+\varrho)^{1/2}f\|^2-(1/4)\|(1+\varrho)^{1/2}g\|^2\big),\end{aligned} \tag{3.51}$$

and analogously,

$$\begin{aligned}&\omega_f^\beta(\tau_t^f(W(g))\,W(f)) \\ &= \exp\big(-(1/2)\langle g|(1+\varrho)e^{-\imath t|k|}f\rangle-(1/2)\langle f|\varrho e^{\imath t|k|}g\rangle\big) \\ &\quad\cdot\exp\big(-(1/4)\|(1+\varrho)^{1/2}f\|^2-(1/4)\|(1+\varrho)^{1/2}g\|^2\big).\end{aligned} \tag{3.52}$$

Using $\varrho(k)=(e^{\beta|k|}-1)^{-1}$, we define

$$\begin{aligned}&F(\pi_{AW}[W(f)],\pi_{AW}[W(g)],z) \\ &= \exp\Big(-(1/2)\Big\langle f\Big|\frac{e^{(\beta+\imath z)|k|}}{e^{\beta|k|}-1}g\Big\rangle-(1/2)\Big\langle g\Big|\frac{e^{-\imath z|k|}}{e^{\beta|k|}-1}f\Big\rangle\Big) \\ &\quad\cdot\exp\Big(-(1/4)\Big\|(1+\varrho)^{1/2}f\Big\|^2-(1/4)\Big\|(1+\varrho)^{1/2}g\Big\|^2\Big).\end{aligned} \tag{3.53}$$

A short calculation yields, that F is analytic on S_β, continuous on $c\ell\, S_\beta$ and takes the necessary boundary conditions defined in 2.4.5. By linearity we can define $F(A,B,\cdot)$ for $A,B\in U:=\mathrm{LH}\{\pi_{AW}[W(f)]\ :\ f\in\mathfrak{f}\}$. since $\pi_{AW}[W(\mathfrak{f})]=c\ell_{\|\cdot\|}U$. and $\mathfrak{M}_f$ is the σ-strong* closure of $\pi_{AW}[W(\mathfrak{f})]$, we can pick for every given $B\in\mathfrak{M}_f$ a $B_n\in U$, such that

$$\|(B^*-B_n^*)\Omega_f^\beta\|\le\frac{1}{n}\text{ and }\|(B-B_n)\Omega_f^\beta\|\le\frac{1}{n}. \tag{3.54}$$

Immediately, for $A\in U$

$$\begin{aligned}\big|F_\beta(A,B_n,t)-\omega_f^\beta(A\tau_t^f(B))\big| &= \big|\langle\Omega_f^\beta|Ae^{\imath t\mathcal{L}_f}B_n\Omega_f^\beta\rangle-\langle\Omega_f^\beta|Ae^{\imath t\mathcal{L}_f}B\Omega_f^\beta\rangle\big| \\ &\le \|A\|\,\|(B_n-B)\Omega_f^\beta\|\le n^{-1}\|A\|,\end{aligned} \tag{3.55}$$

and

$$\begin{aligned}\big|F_\beta(A,B_n,t+\imath\beta)-\omega_f^\beta(\tau_t^f(B)A)\big| &= \big|\langle\Omega_0^\beta|B_ne^{-\imath t\mathcal{L}_f}A\Omega_f^\beta\rangle-\langle\Omega_f^\beta|Be^{-\imath t\mathcal{L}_f}A\Omega_f^\beta\rangle\big| \\ &\le \|A\|\,\|(B_n^*-B^*)\Omega_f^\beta\|\le n^{-1}\|A\|.\end{aligned} \tag{3.56}$$

Hence $F_\beta(A,B_n,\cdot)$ and $F_\beta(A,B_n,\cdot+\imath\beta)$ converge uniformly against $\omega_f^\beta(A\tau_{(\cdot)}^f(B))$, resp. $\omega_f^\beta(\tau_{(\cdot)}^f(B)A)$. The Phragmen Lindelöf theorem states that (under further weak assumptions) an analytic functions defined on the strip S_β take their maximal modulus at the boundary of S_β. This means in our case

$$\begin{aligned} &\sup_{z\in c\ell\, S_\beta} |F_\beta(A,B_n,z)-F_\beta(A,B_m,z)| \\ &\leq \max\{\sup_{t\in\mathbb{R}}|F_\beta(A,B_n,t)-F_\beta(A,B_m,t)|,\ \sup_{t\in\mathbb{R}}|F_\beta(A,B_n,t+\imath\beta)-F_\beta(A,B_m,t+\imath\beta)|\}. \end{aligned} \tag{3.57}$$

We conclude, that $F_\beta(A,B_n,\cdot)$ is a Cauchy-sequence the uniform norm. Let $F_\beta(A,B,\cdot)$ be the limit, hence F_β is the function that yields the (τ^f,β)-KMS property of ω_f^β.

Now let $A,\ B\in\mathfrak{M}_f$ and $A_n\in U$, such that

$$\|(A_n-A)\Omega_f^\beta\|\leq n^{-1} \text{ and } \|(A_n^*-A^*)\Omega_f^\beta\|\leq n^{-1}. \tag{3.58}$$

As before, it follows $F_\beta(A,B,z):=\lim_{n\to\infty}F_\beta(A_n,B,z)$ is the uniform limit on $c\ell\, S_\beta$. □

Remark 3.5.5. We identify $\mathcal{K}_f := \mathcal{F}_b[\mathfrak{h}\oplus\mathfrak{h}]\cong\mathcal{F}_b[\mathfrak{h}]\otimes\mathcal{F}_b[\mathfrak{h}]$. and $\mathcal{L}_f := d\Gamma(|k|\oplus -|k|)\cong \check{H}\otimes 1 - 1\otimes\check{H} = \check{H}_l-\check{H}_r$, using the notation $A_l := A\otimes 1$ and $A_r = 1\otimes A$.

A summary:

Theorem 3.5.6. *$(\mathfrak{M}_f,\tau^f,\omega_f^\beta)$ is a W^*-dynamical system $(\mathfrak{M}_f,\tau^f,\omega_f^\beta)$. ω_f^β is a (τ^f,β)-KMS-state. $\Omega_f^\beta\in\mathcal{F}_b[\mathfrak{h}]\otimes\mathcal{F}_b[\mathfrak{h}]$ is a cyclic and separating vector, such that $\omega_f^\beta(A):=\langle\Omega_f^\beta|A\Omega_f^\beta\rangle$, $A\in\mathfrak{M}_f$. $\mathcal{L}_f$ is the ω_f^β-Liouvillean for τ^f in the sense of* (2.4.9) *and also the Standard Liouvillean for $(\mathfrak{M}_f,\Omega_f^\beta)$. The modular conjugation $\mathcal{J}_f$ is defined by*

$$\mathcal{J}_f a_l^*(f_1)\cdots a_l^*(f_n)a_r^*(g_1)\cdots a_r^*(g_m)\Omega_f^\beta = a_l^*(\overline{g_1})\cdots a_l^*(\overline{g_m})a_r^*(\overline{f_1})\cdots a_r^*(\overline{f_n})\Omega_f^\beta. \tag{3.59}$$

Remark 3.5.7. One easily obtains a representation π of $\mathcal{A}:=\mathcal{A}_{el}\otimes\mathcal{A}_f$ in $\mathcal{B}(\mathcal{K})$, where $\mathcal{K}:=\mathcal{K}_{el}\otimes\mathcal{K}_f$, by setting $\pi:=\pi^{el}\otimes\pi_f$. The W^*-dynamical system shall be $\mathfrak{M}:=\pi[\mathfrak{A}]''$. Furthermore we have

$$\pi[\tau_t^0(A)] = e^{\imath t\mathcal{L}_0}\pi[A]e^{-\imath t\mathcal{L}_0},\ t\in\mathbb{R}, \tag{3.60}$$

where $\mathcal{L}_0=\mathcal{L}_{el}\otimes 1+1\otimes\mathcal{L}_f = H_{el,x}-\overline{H}_{el,y}+\check{H}_l-\check{H}_r$, confer Remark 3.5.5. The modular conjugation is $\mathcal{J}:=\mathcal{J}_{el}\otimes\mathcal{J}_f$. If $\mathcal{H}_{el}$ fulfills the Gibbs condition, there is a cyclic and separating vector $\Omega_0^\beta=\Omega_{el}^\beta\otimes\Omega_f^\beta$. For $A\in\mathfrak{A}$ we have

$$\omega_0^\beta(A)=\langle\Omega_0^\beta|\pi[A]\Omega_0^\beta\rangle. \tag{3.61}$$

For $A \in \mathfrak{M}$ we take the right side of (3.61) as definition and choose

$$\tau_t^0(A) := e^{\imath t\mathcal{L}_0} A e^{-\imath t\mathcal{L}_0} \tag{3.62}$$

as the $*$-automorphism group.

Although, an $*$-automorphism group $\tau_t^\lambda(A) = e^{\imath t H_\lambda} A e^{-\imath t H_\lambda}$ on $\mathcal{A}$ may not exist, one can define an $*$-automorphism group τ^λ on $\mathfrak{M}$ that describes the dynamics of an interacting system. Therefore one replaces for example H_0 in the definition of H_λ by $\mathcal{L}_0$ and for $\Phi(G_i)$, $\Phi(H_i)$, $\Phi(F)$ by $\Phi_\beta(G_i)$, $\Phi_\beta(H_i)$, $\Phi_\beta(F)$. The interaction term reads now

$$Q := \lambda_1 \sum_{j=1}^{r} \big(\Phi_\beta(G_j)\Phi_\beta(H_j) + \Phi_\beta(H_j)\Phi_\beta(G_j)\big) + \lambda_2 \Phi_\beta(F) + \lambda_3 V_x. \tag{3.63}$$

For families of operators defined in Equation 3.20 we define $\Phi_\beta(F) = \Phi((1+\varrho)^{1/2}F \oplus \varrho^{1/2}\overline{F})$ and $\Phi((1+\varrho)^{1/2}F \oplus \varrho^{1/2}\overline{F}) = 2^{-1/2}a((1+\varrho)^{1/2}F \oplus \varrho^{1/2}\overline{F}) + h.c.$, confer Equation 3.22. To study the dynamics we introduce a Liouvillean that anti-commutes with J,

$$\mathcal{L}_\lambda = \mathcal{L}_0 + Q - JQJ \tag{3.64}$$

and define

$$\tau_t^\lambda(A) = e^{\imath t\mathcal{L}_\lambda} A e^{-\imath t\mathcal{L}_\lambda}, \;\; A \in \mathfrak{M}. \tag{3.65}$$

In the following we prove that $\mathcal{L}_\lambda$ is self-adjoint. That allows to define τ^λ and apply the theory of perturbations of W^*-dynamics, confer Theorem 2.5.6.

Chapter 4

Existence of Thermal Equilibrium States

4.1 The Liouvillean $\mathcal{L}_\lambda$

First, we define four auxiliary operators, which we use in Nelson's commutator theorem for $\mathcal{L}_\lambda$. Let fix notation:

$$\begin{aligned} \mathcal{L}_{aux}^{(1)} &:= (H_{el,x} + \overline{H}_{el,y}) + (\check{H}_{aux,l} + \check{H}_{aux,r} + 1) \qquad (4.1)\\ \mathcal{L}_{aux}^{(2)} &:= \big(H_{el,x} + Q\big) + \big(\overline{H}_{el,y} + \mathcal{J}Q\mathcal{J}\big) + c_1(\check{H}_{aux,l} + \check{H}_{aux,r} + 1) + c_2\\ \mathcal{L}_{aux}^{(3)} &:= \big(H_{el,x} + Q\big) + \overline{H}_{el,y} + c_1(\check{H}_{aux,l} + \check{H}_{aux,r} + 1) + c_2\\ \mathcal{L}_{aux}^{(4)} &:= H_{el,x} + \big(\overline{H}_{el,y} + \mathcal{J}Q\mathcal{J}\big) + c_1(\check{H}_{aux,l} + \check{H}_{aux,r} + 1) + c_2 \end{aligned}$$

be defined on $\mathrm{dom}(\mathcal{L}_{aux}^{(1)}) := \mathrm{dom}(\mathcal{L}_{aux}^{(2)}) := \mathrm{dom}(H_{el}) \otimes \overline{\mathrm{dom}(H_{el})} \otimes \mathrm{dom}(\check{H}_{aux}) \otimes \mathrm{dom}(\check{H}_{aux})$, where $\check{H}_{aux} := d\Gamma(1 + |k|)$. Recall, that $\mathcal{J} = \mathcal{J}_{el} \otimes \mathcal{J}_f$. In Example 2.4.12 we defined

$$H_{el,x} := H_{el} \otimes 1 \otimes 1 \otimes 1 \quad \text{and} \quad \overline{H}_{el,y} := 1 \otimes \overline{H_{el}} \otimes 1 \otimes 1.$$

Furthermore, following Remark 3.5.5 we have

$$\check{H}_{aux,l} = 1 \otimes 1 \otimes \check{H}_{aux} \otimes 1 \quad \text{and} \quad \check{H}_{aux,r} = 1 \otimes 1 \otimes 1 \otimes \check{H}_{aux}.$$

Lemma 4.1.1. *For sufficiently large values of $c_1, c_2 \geq 0$ is $\mathcal{L}_{aux}^{(i)}$, $i = 1, 2, 3, 4$ self-adjoint and positive. Moreover, there is a constant $c_3 > 0$ such that*

$$c_3^{-1} \|\mathcal{L}_{aux}^{(1)}\phi\| \leq \|\mathcal{L}_{aux}^{(i)}\phi\| \leq c_3 \|\mathcal{L}_{aux}^{(1)}\phi\|, \quad \phi \in \mathrm{dom}(\mathcal{L}_{aux}^{(1)}). \qquad (4.2)$$

Proof of 4.1.1. Let $N = d\Gamma(1)$ be the number operator.

$$\begin{aligned}
&\Phi_a(\eta G_z)\Phi_{a'}(\eta' H_z)\big(N_r + N_l + 1\big)^{-1} \\
&= \Phi_a(\eta G_z)\big(N_r + N_l + 1\big)^{-1}\Phi_{a'}(\eta' H_z) \\
&\quad -\imath\Phi_a(\eta G_z)\big(N_r + N_l + 1\big)^{-1}\Phi_{a'}(\imath\eta' H_z)\big(N_r + N_l + 1\big)^{-1},
\end{aligned} \tag{4.3}$$

where $a, a' \in \{l, r\},\ z \in \{x, y\}$ and $\eta, \eta' \in \{\varrho^{1/2}, (1+\varrho)^{1/2}\}$.
Equation (4.3) together with Equation (2.148) and (2.149) yields

$$\|\Phi_a(\eta G)\Phi_{a'}(\eta' H)\big(N_r + N_l + 1\big)^{-1}\| \le \text{const } \|\eta G\|_f \cdot \|\eta' H\|_f, \tag{4.4}$$

where $\|K\|_f^2 = \int \big(\|K^*(k)\|^2_{\mathcal{B}(\mathcal{H}_{el})} + \|K(k)\|^2_{\mathcal{B}(\mathcal{H}_{el})}\big) d^3k$. Furthermore, we have

$$\|\Phi_a(\eta F_z) H_{el,z}^{-1/2}(N_a + 1)^{-1/2}\| \le \text{const } \|\eta H_{el}^{-1/2} F\|_f. \tag{4.5}$$

Note, that

$$\|H_{el,z}^{1/2}(N_a + 1)^{1/2}(qH_{el,z} + q^{-1}N_a + 1)^{-1}\| \le 1 \tag{4.6}$$

for all $0 < q$. From $\|N\phi\| \le \|d\Gamma(|k| + 1)\phi\|$ follows that

$$\|Q\phi\| + \|\mathcal{J}Q\mathcal{J}\phi\| \le q'\|(H_{el,x} + \overline{H}_{el,y})\phi + c_1(\check{H}_{aux,l} + \check{H}_{aux,r} + 1)\phi\|. \tag{4.7}$$

for all $0 < q' < 1$ and for $c_1 \gg 0$ depending on q'. By the Kato-Rellich theorem ([22], Thm X.12) follows self-adjointness, boundedness from below of $\mathcal{L}_{aux}^{(i)}$ and that $\mathcal{L}_{aux}^{(1)}$ is $\mathcal{L}_{aux}^{(i)}$-bounded for every $c_2 \ge 0$ and $i = 2, 3, 4$. □

Theorem 4.1.2. *The operators*

$$\mathcal{L}_0, \quad \mathcal{L}_\lambda := \mathcal{L}_0 + Q - \mathcal{J}Q\mathcal{J}, \quad \mathcal{L}_0 + Q, \quad \mathcal{L}_0 - \mathcal{J}Q\mathcal{J} \tag{4.8}$$

are self-adjoint with domains including $\mathrm{dom}(\mathcal{L}_{aux}^{(1)})$. *Every core of* $\mathcal{L}_{aux}^{(1)}$ *is a core of the operators in Equation* (4.1).

Recall, that $\mathcal{L}_0$ was defined in Remark 3.5.7 and $\mathcal{J} = \mathcal{J}_{el} \otimes \mathcal{J}_f$.

Proof of 4.1.2. We restrict ourselves to the case of $\mathcal{L}_\lambda$. We check the assumptions of Nelson's commutator theorem ([22], Thm X.37). Using Lemma 4.2.7 it suffices to show that for $\phi \in$

$\mathrm{dom}(\mathcal{L}_{aux}^{(2)})$,

$$\|\mathcal{L}_\lambda\phi\| \le C\|\mathcal{L}_{aux}^{(1)}\phi\| \tag{4.9}$$

$$\left|\left\langle\mathcal{L}_\lambda\phi\middle|\mathcal{L}_{aux}^{(2)}\phi\right\rangle - \left\langle\mathcal{L}_{aux}^{(2)}\phi\middle|\mathcal{L}_\lambda\phi\right\rangle\right| \le C\left\|\mathcal{L}_{aux}^{(1)}\phi\right\|^2. \tag{4.10}$$

The first inequality follows from Equation (4.7). For the second inequality we observe

$$\begin{aligned}
&\left|\left\langle\mathcal{L}_\lambda\phi\middle|\mathcal{L}_{aux}^{(2)}\phi\right\rangle - \left\langle\mathcal{L}_{aux}^{(2)}\phi\middle|\mathcal{L}_\lambda\phi\right\rangle\right| \qquad (4.11)\\
&\le c_1\left|\left\langle Q\phi\middle|(\check{H}_{aux,l}+\check{H}_{aux,r})\phi\right\rangle - \left\langle(\check{H}_{aux,l}+\check{H}_{aux,r})\phi\middle|Q\phi\right\rangle\right|\\
&+c_1\left|\left\langle \mathcal{J}Q\mathcal{J}\phi\middle|(\check{H}_{aux,l}+\check{H}_{aux,r})\phi\right\rangle - \left\langle(\check{H}_{aux,l}+\check{H}_{aux,r})\phi\middle|\mathcal{J}Q\mathcal{J}\phi\right\rangle\right|\\
&+\left|\left\langle\mathcal{L}_f\phi\middle|Q\phi\right\rangle - \langle Q\phi|\mathcal{L}_f\phi\rangle\right| + \left|\left\langle\mathcal{L}_f\phi\middle|\mathcal{J}Q\mathcal{J}\phi\right\rangle - \left\langle\mathcal{J}Q\mathcal{J}\phi\middle|\mathcal{L}_f\phi\right\rangle\right|,
\end{aligned}$$

where we used, that in a strong sense

$$\left[H_{el,x}+Q,\ \overline{H}_{el,y}+\mathcal{J}Q\mathcal{J}\right]=0. \tag{4.12}$$

We remark that

$$\begin{aligned}
&[\Phi_a(\eta G)\Phi_a(\eta' H),\ d\Gamma_a(1+|k|)] \qquad (4.13)\\
&= \imath\Phi_a(\imath(1+|k|)\eta G)\Phi_a(\eta' H) + \imath\Phi_a(\eta G)\Phi_a(\imath(1+|k|)\eta' H)
\end{aligned}$$

and

$$[\Phi_a(\eta F),\ d\Gamma_a(1+|k|)] = \imath\Phi_a(\imath(1+|k|)\eta F), \tag{4.14}$$

hence the terms on the right side of (4.11) are infinitesimal perturbations of $\mathcal{L}_{aux}^{(1)}$. For $\mathcal{L}_0+Q$ and $\mathcal{L}_0-\mathcal{J}Q\mathcal{J}$ one has to consider the commutator with $\mathcal{L}_{aux}^{(3)}$, resp. $\mathcal{L}_{aux}^{(4)}$ in Equation 4.10. □

Remark 4.1.3. In the same way one can show, that H_λ is essentially self-adjoint on any core of $H_{el}+\check{H}_{aux}$, even if H_λ is not bounded below and is defined $\check{H}_{aux}=d\Gamma(1+|k|)$.

4.2 Equilibrium States

The goal of the following theorem is to give explicit conditions for H_{el} and W, which ensure $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2(\mathcal{L}_0+Q)})$. Recall, that Q is defined in 3.63 and Ω_0^β in 3.5.7.

Let $\gamma,\delta\ge 0$, so that

$$\int(|k|+|k|^{-1})\|H_{el}^{-\gamma}F_k^{\#}\|_{\mathcal{B}(H_{el})}^2 d^3k<\infty, \quad \|H_{el}^{-\delta}V\|_{\mathcal{B}(H_{el})}<\infty. \tag{4.15}$$

We are interested in small values for γ and δ.

Theorem 4.2.1. *There are two cases,*

1. *If* $0 \leq \gamma < 1/2$, *then for* $|\lambda_1| < C\beta^{-1}$, $\lambda_2, \lambda_3 \in \mathbb{R}$ *and all* $\beta > 0$ *we have* $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2(\mathcal{L}_0+Q)})$

2. *If* $\gamma = 1/2$ *then there is a constant* $C < \infty$, *such that for* $|\lambda_1|, |\lambda_2| < C\beta^{-1}$ *and* $\lambda_3 \in \mathbb{R}$ *we have* $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2(\mathcal{L}_0+Q)})$.

Furthermore we assume that the Gibbs condition is fulfilled for a small, but fixed $\epsilon > 0$:

$$\mathrm{Tr}\{e^{-(\beta-\epsilon)H_{el}}\} < \infty, \qquad \text{if } \gamma + \delta > 0. \tag{4.16}$$

When $\gamma = 0$ *and* $\delta = 0$ *only have to assume* $\mathrm{Tr}\{e^{-\beta H_{el}}\} < \infty$. *In this case the constant* C *depends on* $\epsilon > 0$.

First we introduce a regularized version of Q.

$$Q_N := \lambda_1 \sum_{j=1}^{r} \big(\Phi_\beta(G_{j,N})\Phi_\beta(H_{j,N}) + \Phi_\beta(H_{j,N})\Phi_\beta(G_{j,N})\big) + \lambda_2 \Phi_\beta(F_N) + \lambda_3 V_{x,N}, \tag{4.17}$$

where $G_{j,N} := P_N G_j P_N$, $H_{j,N} := P_N H_j P_N$, $F_N := P_N F P_N$ and $V_{x,N} := P_N V_x P_N$. $P_N := \mathbb{1}[H_{el} \leq N]$ is a spectral projection of H_{el}.

Lemma 4.2.2. Q_N *is self-adjoint with* $\mathrm{dom}(Q_N) \supset \mathrm{dom}(1 \otimes d\Gamma(\mathbb{1} \oplus \mathbb{1}))$. Q_N *is affiliated with* $\mathfrak{M}$, *i.e.* Q_N *is closed and commutes with all elements in* $\mathfrak{M}'$, *confer Definition 2.5.1.*

Proof of 4.2.2. By assumption, $V_{x,N}$ is bounded and $V_{x,N} \in \mathfrak{M}$, hence it suffices to consider the case $V_{x,N} = 0$. Let $\mathcal{K}_0 := \bigcup_{n=1}^{\infty} \mathrm{ran}\, \mathbb{1}[1 \otimes (\mathrm{N}_l + \mathrm{N}_r) \leq n]$. Obviously, $Q_N : \mathcal{K}_0 \to \mathcal{K}_0$. On $\mathcal{K}_0$ we have

$$-\imath[\Phi_\beta(F_N), 1 \otimes (\mathrm{N}_l + \mathrm{N}_r)] = \Phi_\beta(\imath F_N), \tag{4.18}$$

the same is true for $\Phi_\beta(G_{j,N})$ and $\Phi_\beta(H_{j,N})$. Therefore

$$\begin{aligned} -\imath[Q_N, 1 \otimes (\mathrm{N}_l + \mathrm{N}_r)] &= \lambda_1 \sum_{j=1}^{r} \big(\Phi_\beta(\imath G_{j,N})\Phi_\beta(H_{j,N}) + \Phi_\beta(G_{j,N})\Phi_\beta(\imath H_{j,N}) + h.c.\big) \\ &\quad + \lambda_2 \Phi_\beta(\imath F_N). \end{aligned} \tag{4.19}$$

Since Equation (4.4) holds, we have

$$\|Q_N\phi\| \leq \text{const}\,\|(\mathrm{N}_l + \mathrm{N}_r + 1)\phi\| \tag{4.20}$$

$$\|[Q_N, \mathrm{N}_l + \mathrm{N}_r]\phi\| \leq \text{const}\,\|(\mathrm{N}_l + \mathrm{N}_r + 1)\phi\| \tag{4.21}$$

for $\phi \in \mathcal{K}_0$, we can apply Nelson's commutator theorem ([22], Thm X.37) to show that Q_N is essentially self-adjoint on $\mathcal{K}_0$. Next, we remark that $A_y \cdot \mathcal{J}_f \pi_{AW}[\mathcal{W}(f)]\mathcal{J}_f : \mathcal{K}_0 \to \mathrm{dom}(Q_N)$, and

$$Q_N\big(A_y \cdot \mathcal{J}_f \pi_{AW}[\mathcal{W}(f)]\mathcal{J}_f\big)\phi = \big(A_y \cdot \mathcal{J}_f \pi_{AW}[\mathcal{W}(f)]\mathcal{J}_f\big)Q_N\phi, \quad \phi \in \mathcal{K}_0. \tag{4.22}$$

By closedness of Q_N the above equation extends to all $\phi \in \mathrm{dom}(Q_N)$. Since the strong closure of $U := \mathrm{LH}\{A_y \cdot \mathcal{J}_f \pi_{AW}[\mathcal{W}(f)]\mathcal{J}_f \,:\, A \in \mathcal{B}(h),\ f \in \mathfrak{f}\}$ is $\pi[\mathfrak{A}]' = \mathfrak{M}' = \mathcal{J}\mathfrak{M}\mathcal{J}$, we have for every $Y \in \mathfrak{M}'$, a sequence $Y_n \in U$, such that $Y_n \to Y$ strongly. Hence

$$Q_N Y_n \phi = Y_n Q_N \phi \to Y Q_N \phi, \quad Y_n\phi \to Y\phi, \quad n \to \infty. \tag{4.23}$$

From the closedness of Q_N follows $Y\phi \in \mathrm{dom}(Q_N)$ and $Q_N Y \supset Y Q_N$. Hence Q_N is affiliated with $\mathfrak{M}$. □

We observe, that $(e^{-s\mathcal{L}_0} Q_N e^{s\mathcal{L}_0})$ leaves $\mathrm{ran}\{P_N \otimes P_N \otimes 1 \otimes 1\}$ invariant and $\mathrm{ran}\{P_N \otimes P_N \otimes 1 \otimes 1\}^\perp \subset \ker(e^{-s\mathcal{L}_0} Q_N e^{s\mathcal{L}_0})$. Therefore, we can define

$$\psi(t, \underline{s}, n, N) := e^{t\mathcal{L}_0}(e^{-s_n\mathcal{L}_0} Q_N e^{s_n\mathcal{L}_0}) \cdots (e^{-s_1\mathcal{L}_0} Q_N e^{s_1\mathcal{L}_0})\Omega_0^\beta. \tag{4.24}$$

for $0 \leq s_n \leq s_{n-1} \leq \ldots \leq s_1 \leq \beta/2,\ n \in \mathbb{N}_0$

Lemma 4.2.3. *With the assumptions of Theorem 4.2.1 we have*

$$\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2(\mathcal{L}_0 + Q_N)}), \quad \sup_N \|e^{-\beta/2(\mathcal{L}_0+Q_N)}\Omega_0^\beta\| < \infty, \tag{4.25}$$

and

$$e^{-\beta/2(\mathcal{L}_0+Q_N)}\Omega_0^\beta = \sum_{n=0}^\infty (-1)^n \int_{\Delta_\beta^n} d\underline{s}\, \psi(0, \underline{s}, n, N) \tag{4.26}$$

for ψ defined in Equation (4.24). Here, $\Delta_\beta^n = \{(s_1, \ldots, s_n) \in \mathbb{R}^n \,:\, 0 \leq s_n \leq \ldots \leq s_1 \leq \beta\}$ is a simplex of dimension n.

Proof of 4.2.3. For $\phi \in \operatorname{ran} \mathbb{1}[|\mathcal{L}_0 + Q_N| \leq k]$ we have

$$\langle e^{-x(\mathcal{L}_0+Q_N)}\phi|\, e^{x\mathcal{L}_0}\Omega_0^\beta\rangle = \sum_{n=0}^{m}(-1)^n \int_{\Delta^n_{\beta/2}} d\underline{s}\, \big\langle \phi|\psi(0,\underline{s},n,N)\big\rangle$$
$$+(-1)^{m+1}\int_{\Delta^{m+1}_{\beta/2}} d\underline{s}\big\langle e^{-s_{m+1}(\mathcal{L}_0+Q_N)}\phi|\psi(s_{m+1},\underline{s},n,N)\big\rangle, \tag{4.27}$$

From Lemma 4.2.4 and Lemma 4.2.5 we obtain

$$\sum_{n=1}^{\infty}\int_{\Delta^n_{\beta/2}} \|\psi(0,\underline{s},n,N)\| d\underline{s} < \infty. \tag{4.28}$$

Since $\|e^{-s_{m+1}(\mathcal{L}_0+Q_N)}\phi\| \leq e^{\beta k}\|\phi\|$, the limit $m\to\infty$ for (4.27) exists and equals

$$\langle e^{-\beta/2(\mathcal{L}_0+Q_N)}\phi|\Omega_0^\beta\rangle = \big\langle \phi|\sum_{n=0}^{\infty}(-1)^n\int_{\Delta^n_{\beta/2}} d\underline{s}\,\psi(0,\underline{s},n,N)\big\rangle. \tag{4.29}$$

Since $\phi \in \bigcup_{k\in\mathbb{N}} \operatorname{ran}\{\mathbb{1}[|\mathcal{L}_0+Q_N| \leq k]\}$ is a core of $e^{-\beta/2(\mathcal{L}_0+Q_N)}$, we get $\Omega_0^\beta \in \operatorname{dom}(e^{-\beta/2(\mathcal{L}_0+Q_N)})$ and

$$e^{-\beta/2(\mathcal{L}_0+Q_N)}\Omega_0^\beta = \sum_{n=0}^{\infty}(-1)^n\int_{\Delta^n_{\beta/2}} d\underline{s}\,\psi(0,\underline{s},n,N). \tag{4.30}$$

Furthermore, Lemmata 4.2.4 and 4.2.5 allow to choose λ and $\beta > 0$ so small, that

$$\sup_N \|e^{-\beta/2(\mathcal{L}_0+Q_N)}\Omega_0^\beta\|^2 \leq \sup_N \langle\Omega_0^\beta|e^{-\beta(\mathcal{L}_0+Q_N)}\Omega_0^\beta\rangle \tag{4.31}$$
$$= \sup_N \sum_{n=0}^{\infty}\int_{\Delta^{2n}_\beta} d\underline{s}\,\langle\Omega|\psi(0,\underline{s},2n,N)\rangle < \infty.$$

□

Lemma 4.2.4. *Let $\psi(t,\underline{s},n,N)$ be defined as in Equation (4.24).*

1. *For all $m,n,N\in\mathbb{N}$ one has*

$$\int_{\Delta^n_{\beta/2}} d\underline{s}\int_{\Delta^m_{\beta/2}} d\underline{r}\,\big\langle\psi(0,\underline{r},m,N)|\psi(0,\underline{s},n,N)\big\rangle \tag{4.32}$$
$$= \int_{\Delta^{n+m}_\beta} d\underline{z}\,\mathbb{1}[z_m \geq \beta/2 \geq z_{m+1}]\big\langle\Omega_0^\beta\big|\psi(0,\underline{z},n+m,N)\big\rangle,$$

2. *and*

$$\Big\|\int_{\Delta^n_{\beta/2}}\psi(0,\underline{s},n,N)d\underline{s}\Big\|^2 \leq \int_{\Delta^{2n}_\beta}\Big|\big\langle\Omega_0^\beta|\psi(0,\underline{s},2n,N)\big\rangle\Big|d\underline{s}. \tag{4.33}$$

Proof of 4.2.4. First, we introduce the short cut $\psi(\underline{s},n) := \psi(0,\underline{s},n,N)$. Since $(e^{-s\mathcal{L}_0}Q_N e^{s\mathcal{L}_0}),\ s \in \mathbb{R}$ is affiliated with $\mathfrak{M}$, one has

$$\begin{aligned}
&\int_{\Delta^n_{\beta/2}} d\underline{s} \int_{\Delta^m_{\beta/2}} d\underline{r}\, \langle \psi(\underline{r},m)|\psi(\underline{s},n)\rangle \\
&= \int_{\Delta^n_{\beta/2}} d\underline{s} \int_{\Delta^m_{\beta/2}} d\underline{r}\, \langle (e^{-r_m\mathcal{L}_0}Q_N e^{r_m\mathcal{L}_0})\cdots(e^{-r_1\mathcal{L}_0}Q_N e^{r_1\mathcal{L}_0})\Omega_0^\beta \\
&\quad |(e^{-s_n\mathcal{L}_0}Q_N e^{s_n\mathcal{L}_0})\cdots(e^{-s_1\mathcal{L}_0}Q_N e^{s_1\mathcal{L}_0})\Omega_0^\beta\rangle \\
&= \int_{\Delta^n_{\beta/2}} d\underline{s} \int_{\Delta^m_{\beta/2}} d\underline{r}\, \overline{\langle \mathcal{J}(e^{-r_m\mathcal{L}_0}Q_N e^{r_m\mathcal{L}_0})\cdots(e^{-r_1\mathcal{L}_0}Q_N e^{r_1\mathcal{L}_0})\Omega_0^\beta} \\
&\quad \overline{|\mathcal{J}(e^{-s_n\mathcal{L}_0}Q_N e^{s_n\mathcal{L}_0})\cdots(e^{-s_1\mathcal{L}_0}Q_N e^{s_1\mathcal{L}_0})\Omega_0^\beta\rangle} \\
&= \int_{\Delta^n_{\beta/2}} d\underline{s} \int_{\Delta^m_{\beta/2}} d\underline{r}\, \Big\langle e^{-\beta/2\mathcal{L}_0}(e^{s_1\mathcal{L}_0}Q_N e^{-s_1\mathcal{L}_0})\cdots(e^{s_n\mathcal{L}_0}Q_N e^{-s_n\mathcal{L}_0})\Omega_0^\beta \Big| \\
&\quad e^{-\beta/2\mathcal{L}_0}(e^{r_1\mathcal{L}_0}Q_N e^{-r_1\mathcal{L}_0})\cdots(e^{r_m\mathcal{L}_0}Q_N e^{-r_m\mathcal{L}_0})\Omega_0^\beta\Big\rangle.
\end{aligned}$$

We used that $\mathcal{J}$ is a the modular conjugation with respect to Ω_0^β. Next, we introduce new variables for $\underline{r}$, namely $x_i := \beta - r_{m-i+1}$. Let $D^m_{\beta/2} := \{\underline{x} \in \mathbb{R}^m : \beta/2 \le x_m \le \ldots \le x_1 \le \beta\}$. Hence we get, using $e^{\beta\mathcal{L}_0}\Omega_0^\beta = \Omega_0^\beta$, that

$$\begin{aligned}
&\int_{\Delta^n_{\beta/2}} d\underline{s} \int_{\Delta^m_{\beta/2}} d\underline{r}\, \langle \psi(\underline{r},m)|\psi(\underline{s},n)\rangle \\
&= \int_{\Delta^n_{\beta/2}} d\underline{s} \int_{D^m_{\beta/2}} d\underline{x}\, \Big\langle (e^{s_1\mathcal{L}_0}Q_N e^{-s_1\mathcal{L}_0})\cdots(e^{s_n\mathcal{L}_0}Q_N e^{-s_n\mathcal{L}_0})\Omega_0^\beta \Big| \\
&\quad (e^{-x_m\mathcal{L}_0}Q_N e^{x_m\mathcal{L}_0})\cdots(e^{-x_1\mathcal{L}_0}Q_N e^{x_1\mathcal{L}_0})\Omega_0^\beta\Big\rangle \\
&= \int_{\Delta^{n+m}_\beta} d\underline{z}\, \mathbb{1}[z_m \ge \beta/2 \ge z_{m+1}]\Big\langle \Omega_0^\beta \Big| (e^{-z_{n+m}\mathcal{L}_0}Q_N e^{z_{n+m}\mathcal{L}_0})\cdots(e^{-z_1\mathcal{L}_0}Q_N e^{z_1\mathcal{L}_0})\Omega_0^\beta\Big\rangle.
\end{aligned}$$

Choosing $n = m$ yields the second part of the statement. □

For this section we write $Z_{el}^{\beta-\epsilon} = \mathrm{Tr}\{e^{-(\beta-\epsilon)H_{el}}\}$.

Lemma 4.2.5. *Let $n > 0$. There are constants $C_0, C_1, C_2, C_3 > 0$ independent of β, λ, N, such that*

$$\begin{aligned}
\int_{\Delta^n_\beta} \Big|\langle \Omega_0^\beta|\psi(0,\underline{s},n,N)\rangle\Big| d\underline{s} \;\le\; & (Z_{el}^{\beta-\epsilon}/Z_{el}^\beta) C_0 \sum_{n_1+2n_2+n_3=n} (C_1(1+\beta)|\lambda_1|)^{n_1} \\
& \Big(\frac{C_2\epsilon^{-2\gamma}(1+\beta)\lambda_2^2}{(1+2(1-\gamma)n_2)^{1-2\gamma}}\Big)^{n_2}\Big(\frac{C_3\epsilon^{-\delta}\beta|\lambda_3|}{((1-\delta)n_3)^{1-\delta}}\Big)^{n_3}.
\end{aligned} \tag{4.34}$$

for some $\epsilon > 0$. If $\gamma = \delta = 0$ one can choose $\epsilon = 0$. γ and δ are defined in Equation 4.15. ϵ is a small real, defined in Theorem 4.2.1.

Proof of 4.2.5. We observe that

$$\int_{\Delta^n_\beta} \left| \langle \Omega^\beta_0 | \psi(0, \underline{s}, n, N) \rangle \right| d\underline{s} = \int_{\Delta^n_1} d\underline{s}\, I_n(\beta, \underline{s}, N), \tag{4.35}$$

where

$$\begin{aligned} I_n(\beta, \underline{s}, N) &= \beta^n \omega^\beta_0 \Big(\big(e^{-\beta s_n H_0} W_{\underline{\lambda}} e^{\beta s_n H_0} \big) \cdots \big(e^{-\beta s_1 H_0} W_{\underline{\lambda}} e^{\beta s_1 H_0} \big) \Big) \\ &= (Z^\beta_{el})^{-1} \beta^n \sum_{\kappa \in \{1,2,3\}^n} \omega^\beta_f \Big(\mathrm{Tr}_{\mathcal{H}_{el}} \big\{ e^{-\beta H_{el}} \big(e^{-\beta s_n H_0} \lambda_{\kappa(n)} W_{\kappa(n)} e^{\beta s_n H_0} \big) \cdots \big(e^{-\beta s_1 H_0} \lambda_{\kappa(1)} W_{\kappa(1)} e^{\beta s_1 H_0} \big) \big\} \Big). \end{aligned} \tag{4.36}$$

Next, we give a short summary how to evaluate the quasi-free state in this case. From Wick's theorem about quasi-free states we conclude for $(1 + |k|^{-1/2}) f_1, \ldots, (1 + |k|^{-1/2}) f_{2m} \in L^2(\mathbb{R}^3)$ and $\sigma \in \{+, -\}^{2m}$

$$\begin{aligned} &\omega^\beta_f (a^{\sigma_{2m}} (e^{-\sigma_{2m} s_{2m} |k|} f_{2m}) \cdots a^{\sigma_1} (e^{-\sigma_1 s_1 |k|} f_1)) \\ &= \sum_{P \in \mathcal{Z}_2} \prod_{\substack{\{i,j\} \in P \\ i > j}} \omega^\beta_\lambda (a^{\sigma_i} (e^{-\sigma_i s_i |k|} f_i) a^{\sigma_j} (e^{-\sigma_j s_j |k|} f_j)). \end{aligned} \tag{4.37}$$

$\mathcal{Z}_2$ are the pairings, that is

$$P \in \mathcal{Z}_2 \text{ iff } P = \{Q_1, \ldots, Q_m\},\ \#Q_i = 2 \text{ and } \textstyle\bigcup_{i=1}^m Q_i = \{1, \ldots, 2m\}.$$

Of course $a^+ = a^*$ and $a^- = a$. For the expectation of the so called two-point functions, we obtain:

$$\begin{aligned} \omega^\beta_f (a^+ (e^{\beta s_i |k|} f_i) a^+ (e^{\beta s_j |k|} f_j)) &= 0 = \omega^\beta_f (a (e^{-\beta s_i |k|} f_i) a (e^{-\beta s_j |k|} f_j)) \\ \omega^\beta_f \big(a^+ (e^{\beta s_i |k|} f_i) a^- (e^{-\beta s_j |k|} f_j)\big) &= \int d^3k f_i(k) \overline{f_j(k)} \frac{e^{\beta (s_i - s_j)|k|}}{e^{\beta |k|} - 1} \\ \omega^\beta_f \big(a^- (e^{\beta s_i |k|} f_i) a^+ (e^{-\beta s_j |k|} f_j)\big) &= \int d^3k f_j(k) \overline{f_i(k)} \frac{e^{(\beta + \beta s_j - \beta s_i)|k|}}{e^{\beta |k|} - 1}. \end{aligned} \tag{4.38}$$

That implies for the expectation value of $2m$ creation- or annihilation operators:

$$\begin{aligned} &\omega^\beta_f (a^{\sigma_{2m}} (e^{-\sigma_{2m} \beta s_{2m} |k|} f_{2m}) \cdots a^{\sigma_1} (e^{-\sigma_1 \beta s_1 |k|} f_1)) \\ &= \int \nu (d^{3(2m)} k \otimes d^{2m} \underline{\tau})\ f^{\sigma_{2m}}_{2m} (k_{2m}, \tau_{2m}) \cdots f^{\sigma_1}_1 (k_1, \tau_1), \end{aligned} \tag{4.39}$$

where $f^+(k,\tau) := f(k)\mathbb{1}[\tau = +]$ and $f^-(k,\tau) := \overline{f(k)}\mathbb{1}[\tau = -]$.

ν is a measure on $(\mathbb{R}^3)^{2m} \times \{+,-\}^{2m}$, which expresses the pairings by means of Dirac-measures

$$\begin{aligned}&\nu(d^{3(2m)}\underline{k} \otimes d^{2m}\underline{\tau}) \\ &= \sum_{P\in\mathcal{Z}_{2m}} \sum_{\underline{\tau}\in\{+,-\}^{2m}} \prod_{\{i>j\}\in P} \delta_{\tau_i,-\tau_j}\delta_{k_i,k_j}\Big(\delta_{\tau_i,+}\frac{e^{\beta(s_i-s_j)|k_i|}}{e^{\beta|k_i|}-1} + \delta_{\tau_i,-}\frac{e^{(\beta-\beta(s_i-s_j))|k_i|}}{e^{\beta|k_i|}-1}\Big)d^{3(2m)}k.\end{aligned} \tag{4.40}$$

Let $M(m_1,m_2,m_3) = \{\kappa \in \{1,2,3\}^n \ : \#\kappa^{-1}(\{i\}) = m_i,\ i = 1,2,3\}$, it is the set vectors that have m_1 times the value 1 in a component, m_2 times the value 2 and m_3 times the value 3. We obtain

$$\begin{aligned}I_n(x,\underline{s},N) \ &= \sum_{\substack{(n_1,n_2,n_3)\in\mathbb{N}_0^3 \\ n_1+2n_2+n_3=n}} \sum_{\substack{\kappa\in M(n_1,2n_2,n_3) \\ m:=n_1+n_2}} \lambda_1^{n_1}\lambda_2^{2n_2}\lambda_3^{n_3} \int \nu(d^{3(2m)}k \otimes d^{2m}\tau) \\ &\quad (Z_{el}^{\beta})^{-1}\beta^n \operatorname{Tr}_{\mathcal{H}_{el}}\{e^{-(\beta-\beta(s_1-s_{2m}))H_{el}} I_{2m} e^{-\beta(s_{2m-1}-s_{2m})H_{el}} \cdots e^{-\beta(s_1-s_2)H_{el}} I_1\},\end{aligned} \tag{4.41}$$

where we have for $\kappa(j) = 1,2,3$

$$\begin{aligned}I_j \ &= \begin{cases} I_j(m,\tau,m',\tau'), & \kappa(j)=1 \\ I_j(m,\tau), & \kappa(j)=2 \\ I_j, & \kappa(j)=3 \end{cases} \\ &= \begin{cases} \sum_{\underline{\sigma}\in\{+,-\}^{2r}} \sum_{i=1}^{r} \mathbb{1}[(\tau,\tau') = (\sigma_{2i-1},\sigma_{2i})] G_i^{\sigma_{2i-1}}(m)\, H_i^{\sigma_{2i}}(m') + h.c., & \kappa(j)=1 \\ F^+(m)\mathbb{1}[\tau=+] + F^-(m)\mathbb{1}[\tau=-], & \kappa(j)=2 \\ V & \kappa(j)=3, \end{cases}\end{aligned} \tag{4.42}$$

where we have suppressed the index N. In the integral (4.41) we insert for (m,τ) and (m',τ') from left to right $k_{2m}, \tau_{2m}, k_{2m-1}, \tau_{2m-1}, \dots, k_1, \tau_1$.

In the next step we want to take the norm within the integral of (4.41). Additionally we use

$$\nu(d^3 2m \otimes d^{2m}\underline{\tau}) \le \sum_{P\in\mathcal{Z}_{2m}} \sum_{\underline{\tau}\in\{+,-\}^{2m}} \prod_{\{i>j\}\in P} \Big(\delta_{k_i,k_j}\coth(\beta|k_i|/2)\Big)d^{3(2m)}k. \tag{4.43}$$

Next, we apply Hölder's-Inequality for the trace, i.e.

$$|\operatorname{Tr}\{A_{2m}B_{2m}\cdots A_1B_1\}| \le \prod_{j=1}^{2m}\|B_j\| \cdot \prod_{j=1}^{2m}\operatorname{Tr}\{A_i^{p_j}\}^{p_j^{-1}}, \tag{4.44}$$

if $p_i \ge 1$ and $\sum_{i=1}^{2m} p_i^{-1} = 1$, such as $A_i \ge 0$. Confer Theorem B.0.8. Let

$$p_1 := (1 - s_1 + s_{2m})^{-1}, \quad p_i := (s_{i-1} - s_i)^{-1}, \quad i = 2,\dots,2m \tag{4.45}$$

We define

$$(A_j, B_j) := \begin{cases} \left(e^{-\beta p_j^{-1} H_{el}},\ I_j(m,\tau,m',\tau')\right), & \kappa(j) = 1 \\ \left(e^{-\beta p_j^{-1} H_{el}} H_{el}^{\gamma},\ H_{el}^{-\gamma} I_j(m,\tau,)\right), & \kappa(j) = 2 \\ \left(e^{-\beta p_j^{-1} H_{el}} H_{el}^{\delta},\ H_{el}^{-\delta} I_j\right), & \kappa(j) = 3 \end{cases}. \tag{4.46}$$

Let

$$\eta_1(k) = \max_{\sigma=\pm,\, i=1,\dots,r} \{\|G_i^\sigma(k)\|, \|H_i^\sigma(k)\|\}, \quad \eta_2(k) = \max_{\sigma=\pm} \|H_{el}^{-\gamma} F^\sigma(k)\|. \tag{4.47}$$

We obtain after applying Hölder's inequality and integrating over ν, using Equation (4.43), factors like

$$\begin{aligned} &\int \eta_i(k)\eta_j(k)\coth(\beta/2|k|)d^3k \qquad (4.48)\\ \le\ & \Big(\int \eta_i^2(k)\coth(\beta/2|k|)d^3k\Big)^{1/2}\Big(\int \eta_j^2(k)\coth(\beta/2|k|)d^3k\Big)^{1/2} \\ \le\ & \Big(\int \eta_i^2(k)(2+4(\beta|k|)^{-1})d^3k\Big)^{1/2}\Big(\int \eta_j^2(k)(2+4(\beta|k|)^{-1})d^3k\Big)^{1/2}. \end{aligned}$$

Note that the number of pairings is $\#\mathcal{Z}_2 = \frac{(2m)!}{2^m m!}$. That yields for the evaluation of I_n

$$\begin{aligned} |I_n(\beta\underline{s},N)| \ \le\ & (Z_{el}^\beta)^{-1}\beta^n \sum_{\substack{(n_1,n_2,n_3)\in\mathbb{N}\\ n_1+2n_2+n_3=n}} \sum_{\substack{\kappa\in M(n_1,2n_2,n_3)\\ m:=n_1+n_2}} \lambda_1^{n_1}\lambda_2^{2n_2}\lambda_3^{n_3}\frac{(2m)!}{2^m m!}2^{2n_1 r+2n_2} \\ & \prod_{i=1}^{2}\Big(\int \eta_i^2(k)(2+4(\beta|k|)^{-1})d^3k\Big)^{n_i} \prod_{i:\kappa(i)=1} \mathrm{Tr}_{\mathcal{H}_{el}}\left\{e^{-\beta H_{el}}\right\}^{p_i^{-1}} \\ & \prod_{i:\kappa(i)=2} \mathrm{Tr}_{\mathcal{H}_{el}}\left\{e^{-\beta H_{el}}H_{el}^{p_i\gamma}\right\}^{p_i^{-1}} \prod_{i:\kappa(i)=3} \mathrm{Tr}_{\mathcal{H}_{el}}\left\{e^{-\beta H_{el}}H_{el}^{p_i\delta}\right\}^{p_i^{-1}} \|H_{el}^{-\delta}V\|^{n_3} \end{aligned} \tag{4.49}$$

Furthermore, for $\epsilon > 0$

$$\mathrm{Tr}_{\mathcal{H}_{el}}\left\{e^{-\beta H_{el}}H_{el}^{p_i\gamma}\right\}^{p_i^{-1}} \le \|e^{-\epsilon H_{el}}H_{el}^{p_i\gamma}\|^{p_i^{-1}}\, \mathrm{Tr}_{\mathcal{H}_{el}}\left\{e^{-(\beta-\epsilon)H_{el}}\right\}^{p_i^{-1}}. \tag{4.50}$$

The spectral theorem for $\mathcal{H}_{el}$ states, that the norm in Equation (4.50) is less than the maximum of the function $f(r) = e^{-\epsilon r}r^{p_i\gamma}, \quad r \ge 0$. For $0 \le \gamma \le 1/2$ we obtain

$$\mathrm{Tr}_{\mathcal{H}_{el}}\left\{e^{-\beta H_{el}}H_{el}^{p_i\gamma}\right\}^{p_i^{-1}} \le \epsilon^{-\gamma}p_i^{\gamma}\, \mathrm{Tr}_{\mathcal{H}_{el}}\left\{e^{-(\beta-\epsilon)H_{el}}\right\}^{p_i^{-1}}. \tag{4.51}$$

The same inequality holds for δ instead of γ. We obtain

$$\begin{aligned} \int_{\Delta_1^n} I(x,\underline{s},N)d\underline{s} \ \le\ & \beta^n(Z_{el}^{\beta-\epsilon}/Z_{el}^\beta) \sum_{\substack{(n_1,n_2,n_3)\in\mathbb{N}\\ n_1+2n_2+n_3=n}} \sum_{\substack{\kappa\in M(n_1,2n_2,n_3)\\ m:=n_1+n_2}} \lambda_1^{n_1}\lambda_2^{2n_2}\lambda_3^{n_3}\frac{(2m)!}{2^m m!} \\ & \prod_{i=1}^{2}\Big(\int \eta_i^2(k)(2+4(\beta|k|)^{-1})d^3k\Big)^{n_i}\epsilon^{-2n_2\gamma-n_3\delta}\int_{\Delta_1^n} C_\kappa(\underline{s})d\underline{s}, \end{aligned} \tag{4.52}$$

where

$$C_\kappa(\underline{s}) = (1 - s_1 + s_n)^{-\alpha_1} \prod_{i=1}^{n-1} (s_i - s_{i+1})^{-\alpha_i}, \tag{4.53}$$

and

$$\alpha_j = \begin{cases} 0, & \kappa(j) = 1 \\ \gamma, & \kappa(j) = 2 \\ \delta, & \kappa(j) = 3 \end{cases} \cdot \tag{4.54}$$

We remark, that $\sum_{\kappa \in M(n_1, 2n_2, n_3)} 1 \leq 3^{n_1 + 2n_2 + n_3}$ and $\frac{(2m)!}{2^m m!} \leq 2^{n_1+n_2}(n_1+n_2)!$. Thus Lemma 4.2.6 yields the assertion. □

Lemma 4.2.6. *For the function C_κ we have*

$$\begin{aligned} &\Gamma(n_1 + n_2 + 1) \int_{\Delta_n} C_\kappa(\underline{s}) d\underline{s} \\ \leq\ & (n+1)^2 C_0' C_1'^n \big(n_1 + 2(1-\gamma)n_2\big)^{-(1-2\gamma)n_2} \big((1-\delta)n_3\big)^{-(1-\delta)n_3}. \end{aligned} \tag{4.55}$$

Proof of 4.2.6. We turn now to the integral

$$\int_{\Delta_n} C_\kappa(\underline{s}) d\underline{s} = \int_{\Delta_1^{2n}} d\underline{s} (1 - s_1 + s_n)^{-\alpha_1} \prod_{i=1}^{n-1} (s_i - s_{i+1})^{-\alpha_i}. \tag{4.56}$$

We define for $k = 1, \ldots, 2n$, a change of coordinates by $s_k = r_1 - \sum_{j=2}^k r_j$, the integral transforms to

$$\begin{aligned} \int_{S^n} (1 - (r_2 + \cdots + r_n))^{-\alpha_1} \prod_{i=2}^n r_i^{-\alpha_i} d^n \underline{r} &= \int_{T^{n-1}} (1 - (r_2 + \cdots + r_n))^{1-\alpha_1} \prod_{i=2}^n r_i^{-\alpha_i} d^{n-1} \underline{r} \\ &= \frac{\Gamma(1-\alpha_1)^{-1} \Gamma\big(1-\gamma\big)^{2n_2} \Gamma\big(1-\delta\big)^{n_3}}{\Gamma\big(n_1 + 2n_2(1-\gamma) + n_3(1-\delta)\big)}, \end{aligned} \tag{4.57}$$

where $S^{2n} := \{\underline{r} \in \mathbb{R}^{2n} : 0 \leq r_i \leq 1,\ r_2 + \cdots + r_{2n} \leq r_1\}$ and $T^{2n-1} := \{\underline{r} \in \mathbb{R}^{2n-1} : 0 \leq r_i \leq 1,\ r_2 + \cdots + r_{2n} \leq 1\}$. From the first to the second formula we integrate over dr_1. The last equality follows from a formula in [14] Formula 4.635 (4).

From Stirling's formula we obtain

$$(2\pi)^{1/2} x^{x-1/2} e^{-x} \leq \Gamma(x) \leq (2\pi)^{1/2} x^{x-1/2} e^{-x+1},\ x \geq 1. \tag{4.58}$$

Since $m = n_1 + n_2$ and $n = n_1 + 2n_2 + n_3$ we have for $n_1 + n_2 \geq 1$

$$\frac{\Gamma(n_1+n_2+1)}{\Gamma(n_1+2(1-\gamma)n_2+(1-\delta)n_3)} = \frac{(n_1+n_2+1)\Gamma(n_1+n_2)}{\Gamma(n_1+2(1-\gamma)n_2+(1-\delta)n_3)} \tag{4.59}$$

$$\leq \; (n+1)^2\Big(\frac{n_1+2(1-\gamma)n_2}{e}\Big)^{-(1-2\gamma)n_2}\Big(\frac{(1-\delta)n_3}{e}\Big)^{-(1-\delta)n_3}. \tag{4.60}$$

□

Lemma 4.2.7. *If* $\sup_{N\in\mathbb{N}} \|e^{-\beta/2(\mathcal{L}_0+Q_N)}\Omega_0^\beta\| < \infty$ *then* $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2(\mathcal{L}_0+Q)})$.

Proof of 4.2.7. For $f \in \mathcal{C}_0^\infty(\mathbb{R})$ and $\phi \in \mathcal{K}$ we define $\psi_N := f(\mathcal{L}_0+Q_N)\phi$. Obviously, for $g(r) = e^{-\beta/2r}f(r) \in \mathcal{C}_0^\infty(\mathbb{R})$ we have $e^{-\beta/2(\mathcal{L}_0+Q_N)}\psi_N = g(\mathcal{L}_0+Q_N)\phi$. Since $\mathcal{L}_0+Q_N \longrightarrow \mathcal{L}_0+Q$ for $N\to\infty$ in the strong resolvent sense, we know from [21] that

$$e^{-\beta/2(\mathcal{L}_0+Q_N)}\psi_N = g(\mathcal{L}_0+Q_N)\phi \longrightarrow g(\mathcal{L}_0+Q_N)\phi,\; N\to\infty, \tag{4.61}$$

and

$$\psi_N \longrightarrow \psi := f(\mathcal{L}_0+Q_N)\phi,\; N\to\infty. \tag{4.62}$$

Hence for $0 \leq x \leq \epsilon$

$$|\langle e^{-\beta/2(\mathcal{L}_0+Q_N)}\psi|\Omega_0^\beta\rangle| = \limsup_{N\to\infty} |\langle e^{-\beta/2(\mathcal{L}_0+Q_N)}\psi_N|\Omega_0^\beta\rangle| \leq a\|\psi\|, \tag{4.63}$$

Since $\{f(\mathcal{L}_0+Q)\phi \in \mathcal{K} : \phi\in\mathcal{K},\, f\in\mathcal{C}_0^\infty(\mathbb{R})\}$ is a core of $e^{-\beta/2(\mathcal{L}_0+Q)}$, we obtain $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2(\mathcal{L}_0+Q)})$. □

Corollary 4.2.8. *Let* $\tau_t^\lambda(A) := e^{\imath t\mathcal{L}_\lambda}Ae^{-\imath t\mathcal{L}_\lambda}$, $A\in\mathfrak{M}$. τ^λ *is a* $*$*-automorphism group for* $\mathfrak{M}$. *By the assumptions of Theorem 4.2.1 ,* $\omega_\lambda^\beta(A) := \langle\Omega_\lambda^\beta|A\Omega_\lambda^\beta\rangle$ *is a* (β,τ^λ)*-KMS-state.*

Proof of 4.2.8. Since $Q_N \to Q$ and $\mathcal{J}Q_N\mathcal{J} \to \mathcal{J}Q\mathcal{J}$ strongly for $N\to\infty$, we have $e^{\imath t(\mathcal{L}_0+Q_N)} \to e^{\imath t(\mathcal{L}_0+Q_N)}$, $e^{\imath t(\mathcal{L}_0-\mathcal{J}Q_N\mathcal{J})} \to e^{\imath t(\mathcal{L}_0-\mathcal{J}Q\mathcal{J})}$ and $e^{\imath t\mathcal{L}_{Q_N}} \to e^{\imath\mathcal{L}_\lambda}$. Therefore, for $A\in\mathfrak{M}$

$$\text{w-lim}_{N\to\infty} e^{\imath t\mathcal{L}_{Q_N}}Ae^{-\imath t\mathcal{L}_{Q_N}} = \tau_t^\lambda(A) \in \mathfrak{M} \tag{4.64}$$

$$\text{w-lim}_{N\to\infty} e^{\imath t\mathcal{L}_{Q_N}}e^{-\imath t(\mathcal{L}_0-\mathcal{J}Q_N\mathcal{J})} = \text{w-lim}_{N\to\infty} e^{\imath t(\mathcal{L}_0+Q_N)}e^{\imath t\mathcal{L}_0} = E(t)\in\mathfrak{M}, \tag{4.65}$$

where $E(t) := e^{\imath t\mathcal{L}_\lambda}e^{-\imath t(\mathcal{L}_0-\mathcal{J}Q\mathcal{J})} = e^{\imath t(\mathcal{L}_0+Q)}e^{\imath t\mathcal{L}_0}$ defined in Theorem 2.5.3. The rest follows by mimicking the proof of Theorem 2.5.3 using $E(t)\in\mathfrak{M}$. □

4.3 The Harmonic Oscillator

In this section we consider a concrete model to check, if one can get rid of the assumption on $|\lambda\beta| \ll 1$ in Theorem 4.2.1. Let

$$H_\lambda = H_{el} + \lambda\Phi(F) + \check{H}, \tag{4.66}$$

where $F = x \cdot f(k)$, with $(|k|^{-1/2} + |k|^{1/2})f \in L^2(\mathbb{R}^3)$. The form-factor is obtained by the dipole approximation. Furthermore, $\mathcal{H}_{el} = L^2(\mathbb{R})$ and $H_{el} = (1/2)(-\Delta_x + \alpha^2 x^2)$.
H_{el} is the harmonic oscillator with friction constant $\alpha > 0$. The Liouvillean for this model is denoted by

$$\mathcal{L}_{osc} = H_{el,x} - H_{el,y} + \mathcal{L}_f + Q - \mathcal{J}Q\mathcal{J}. \tag{4.67}$$

We can show.

Theorem 4.3.1. *Ω_0^β is in the domain of* $\mathrm{dom}(e^{-\beta/2(\mathcal{L}_0+Q)})$ *for all $\beta \in (0,\infty)$, whenever*

$$|2\alpha^{-1}\lambda| \| |k|^{-1/2} f \| < 1.$$

Proof of 4.3.1. We define the ladder operators for the harmonic oscillator

$$A^* = \frac{\alpha^{1/2}x - \imath\alpha^{-1/2}p}{\sqrt{2}}, \quad A = \frac{\alpha^{1/2}x + \imath\alpha^{-1/2}p}{\sqrt{2}}, \quad p = -\imath\partial_x, \tag{4.68}$$

$$\Phi(c) = 2^{-1/2}(cA^* + \overline{c}A), \quad \text{for } c \in \mathbb{C}.$$

These operators fulfill the CCR-relations and the harmonic oscillator is the number operator up to constants.

$$[A, A^*] = 1, \quad [A^*, A^*] = [A, A] = 0, \quad H_{el} = \alpha A^* A + \alpha/2, \tag{4.69}$$

$$[H_{el}, A] = -\alpha A, \quad [H_{el}, A^*] = \alpha A^*.$$

The vector $\Omega = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$ is called the vacuum vector. Note, that one can identify $\mathcal{F}_b[\mathbb{C}]$ with $L^2(\mathbb{R})$, since $\mathrm{LH}\{(A^*)^n\Omega \,:\, n \in \mathbb{N}_0\}$ is dense in $L^2(\mathbb{R})$. By Theorem 2.8.10 follows

$$\omega_\beta^{osc}\big(W(c)\big) = Z_\beta^{-1}\,\mathrm{Tr}\left\{e^{-\beta H_{el}} W(c)\right\} = \exp\big(-1/4\coth(\beta\alpha/2)|c|^2\big),$$

where $\mathcal{Z}_\beta := \mathrm{Tr}\{e^{-\beta H_{el}}\}$. First, we remark, that Equation 4.24 is defined for this model without regularization by $P_N := \mathbb{1}[H_{el} \leq N]$. Moreover we obtain from Lemma 4.2.4, that

$$\Big\| \int_{\Delta_{\beta/2}^n} \psi(0, \underline{s}, n, \infty) d\underline{s} \Big\|^2 \leq \int_{\Delta_\beta^{2n}} \Big| \big\langle \Omega_0^\beta | \psi(0, \underline{s}, 2n, \infty) \big\rangle . \Big| d\underline{s} =: h_{2n}(\beta, \lambda). \tag{4.70}$$

To show that $\Omega_\lambda^\beta \in \mathrm{dom}(e^{-\beta/2(\mathcal{L}_0+Q)})$ is suffices to prove, that $\sum_{n=0}^\infty h_{2n}(\beta,\lambda)^{1/2} < \infty$. Using that ω_β^{osc} is quasi-free, Wick's Theorem 2.8.8 yields

$$h_{2n}(\beta,\lambda) := \frac{(-\beta\lambda)^{2n}}{Z_{el}^\beta}\int_{\Delta_1^{2n}} d\underline{s}\ \mathrm{Tr}\left\{e^{-\beta H_{el}}\left(e^{-\beta s_{2n}H_{el}}xe^{\beta s_{2n}H_{el}}\right)\dots\left(e^{-\beta s_1 H_{el}}xe^{\beta s_1 H_{el}}\right)\right\}$$
$$\cdot\omega_f^\beta\big((e^{-\beta s_{2n}\check{H}}\Phi(f)e^{\beta s_{2n}\check{H}})\dots(e^{-\beta s_1\check{H}}\Phi(f)e^{\beta s_1\check{H}})\big).$$

Moreover, we have

$$\begin{aligned} e^{-\beta s_i H_{el}}xe^{\beta s_i H_{el}} &= (2\alpha)^{-1/2}\big(e^{-\beta\alpha s_i}A^* + e^{\beta\alpha s_i}A\big) \\ e^{-\beta s_i\check{H}}\Phi(f)e^{\beta s_i\check{H}} &= 2^{-1/2}\Big(a^*(e^{-\beta s_i|k|}f) + a(e^{\beta s_i|k|}f)\Big). \end{aligned} \tag{4.71}$$

Inserting the identities of Equation (4.71) in Equation (4.3) yields

$$\begin{aligned} h_{2n}(\beta,\lambda) &:= (\beta\lambda)^{2n}\int_{\Delta_1^{2n}} d\underline{s}\sum_{P\in\mathcal{Z}_2}\prod_{\{i,j\}\in P}K_{osc}(|s_i-s_j|,\beta)\sum_{P'\in\mathcal{Z}_2}\prod_{\{k,l\}\in P'}K_f(|s_k-s_l|,\beta) \\ &= \frac{(\beta\lambda)^{2n}}{(2n)!}\int_{[0,1]^{2n}} d\underline{s}\sum_{P,P'\in\mathcal{Z}_2}\prod_{\substack{\{i,j\}\in P\\ \{k,l\}\in P'}}K_{osc}(|s_i-s_j|,\beta)K_f(|s_k-s_l|,\beta), \end{aligned} \tag{4.72}$$

where for $k<l$ and $i<j$

$$\begin{aligned} K_f(|s_k-s_l|,\beta) &:= \omega_f^\beta((e^{-\beta s_k\check{H}}\Phi(f)e^{\beta s_k\check{H}})(e^{-\beta s_l\check{H}}\Phi(f)e^{\beta s_l\check{H}})) \\ &= 2^{-1}\int\frac{\cosh(\beta|s_k-s_l||k|-\beta|k|/2)}{\sinh(\beta|k|/2)}|f(k)|^2d^3k, \end{aligned} \tag{4.73}$$

and

$$\begin{aligned} K_{osc}(|s_i-s_j|,\beta) &:= \omega_\beta^{osc}(e^{-\beta s_i H_{el}}xe^{\beta s_i H_{el}}e^{-\beta s_j H_{el}}xe^{\beta s_j H_{el}}) \\ &= (2\alpha)^{-1}\frac{\cosh(\beta\alpha|s_i-s_j|-\beta\alpha/2)}{\sinh(\beta\alpha/2)}. \end{aligned} \tag{4.74}$$

The last equality in (4.72) holds, since the integrand is invariant with respect to a change of the axis of coordinates.

We interpret two pairings P and $P'\in\mathcal{Z}_2$ as an indirected graph $G=G(P,P')$, where $M_{2n}=\{1,\dots,2n\}$ is the set of points. Any graph in G has two kinds of lines, namely lines in $L_{osc}(G)$, which belong to elements of P and lines in $L_f(G)$, which belong to elements of P'.

Let $\mathcal{G}(A)$ be the set of undirected graphs with points in $A\subset M_{2n}$, such that for each point "i" in A, there is exact one line in $L_f(G)$, which begins in "i", and exact one line in $L_{osc}(G)$, which begins with "i". $\mathcal{G}_c(A)$ is the set of connected graphs. We do not distinguish, if points are connected by lines in $L_f(G)$ or by lines in $L_{osc}(G)$.

Let

$$\mathcal{P}_k := \{P \,:\, P = \{A_1, \ldots, A_k\},\ \emptyset \neq A_i \subset M_{2n},\ A_i \cap A_j = \emptyset \text{ for } i \neq j,\ \bigcup_{i=1}^{k} A_i = M_{2n}\} \tag{4.75}$$

be the family of decompositions of M_{2n} in k disjoint set. It follows

$$\begin{aligned} h_{2n}(\beta,\lambda) &= \frac{(\beta\lambda)^{2n}}{(2n)!} \sum_{G\in\mathcal{G}(M_{2n})} \int_{M_{2n}} d\underline{s} \prod_{\substack{\{i,j\}\in L_{osc}(G)\\ \{k,l\}\in L_f(G)}} K_{osc}(|s_i - s_j|,\beta) K_f(|s_k - s_l|,\beta) \\ &= \frac{(\beta\lambda)^{2n}}{(2n)!} \sum_{k=1}^{2n} \sum_{\{A_1,\ldots,A_k\}\in\mathcal{P}_k} \sum_{\substack{(G_1,\ldots,G_k)\\ G_a\in\mathcal{G}_c(A_a)}} \prod_{a=1}^{k} I(G_a, A_a, \beta) \\ &= \frac{(\beta\lambda)^{2n}}{(2n)!} \sum_{k=1}^{2n} \frac{1}{k!} \sum_{\substack{A_1,\ldots,A_k\subset M_{2n},\\ \{A_1,\ldots,A_k\}\in\mathcal{P}_k}} \sum_{\substack{(G_1,\ldots,G_k)\\ G_a\in\mathcal{G}_c(A_a)}} \prod_{a=1}^{k} I(G_a, A_a, \beta), \end{aligned} \tag{4.76}$$

where

$$I(G_a, A_a, \beta) := \int_{A_a} d\underline{s} \prod_{\substack{\{i,j\}\in L_{osc}(G_a)\\ \{k,l\}\in L_f(G_a)}} K_{osc}(|s_i - s_j|,\beta) K_f(|s_k - s_l|,\beta). \tag{4.77}$$

$\int_{A_a} d\underline{s}$ means, $\int_{-1}^{1} ds_{j_1} \int_{-1}^{1} ds_{j_2} \ldots \int_{-1}^{1} ds_{j_m}$, where $A_a = \{j_1, \ldots, j_m\}$ and $\#A_a = m$.

From the first to the second line we summarize terms with graphs, having connected components containing the same set of points. From the second to the third line the order of the components is respected, hence the correction factor $\frac{1}{k!}$ is introduced. Due to Lemma 4.3.3 the integral depends only on the number of points in the connected graph, i. e. $I(G, A, \beta) = I(\#A, \beta)$. Moreover, Lemma 4.3.3 states that $\beta^{\#A} \cdot I(\#A, \beta) \leq (2\||k|^{-1/2} f\|_{\mathfrak{h}} (\alpha\beta)^{-1})^{\#A} \cdot (C\beta + 1)$. To ensure that $\mathcal{G}_c(A_a)$ is not empty, $\#A_a$ must be even. For $(m_1, \ldots, m_k) \in \mathbb{N}^k$ with $m_1 + \ldots + m_k = n$ we obtain

$$\sum_{\substack{A_1,\ldots,A_k\subset M_{2n}, \#A_i = 2m_i\\ \{A_1,\ldots,A_k\}\in\mathcal{P}_k}} 1 = \frac{(2n)!}{(2m_1)!\cdots(2m_k)!}. \tag{4.78}$$

Let now be $A_a \subset M_{2n}$ with $\#A_a = 2m_a > 2$ fixed. In G_a are $\#A_a$ lines in $L_{osc}(G_a)$, since such lines have no points in common, we have $\frac{(2m_a)!}{m_a! 2^{m_a}}$ choices. Let now be the lines in $L_{osc}(G_a)$ fixed. We have now $\big((2m_a - 2)(2m_a - 4)\cdots 1\big)$ choices for m_a lines in $L_f(G_a)$, which yield a connected graph. Thus

$$\sum_{G_a\in\mathcal{G}_c(A_a),} 1 = \frac{(2m_a)!}{m_a! 2^{m_a}} \big((2m_a - 2)(2m_a - 4)\cdots 1\big) = \frac{(2m_a)!}{2m_a}. \tag{4.79}$$

For $\#A_a = 2$ exists only one connected graph. We obtain for h_{2n}

$$\begin{aligned} h_{2n}(\beta,\lambda) &= (\lambda)^{2n}\sum_{k=1}^{2n}\frac{1}{k!}\sum_{\substack{(m_1,\dots,m_k)\in\mathbb{N}^k\\ m_1+\dots+m_k=n}}\prod_{a=1}^{k}\frac{I(2m_a,\beta)(\beta^2)^{m_a}}{2m_a} \\ &\leq (2\alpha^{-1}\||k|^{-1/2}f\|\lambda)^{2n}\sum_{k=1}^{2n}\frac{1}{k!}\sum_{\substack{(m_1,\dots,m_k)\in\mathbb{N}^k\\ m_1+\dots+m_k=n}}\prod_{a=1}^{k}\frac{(C\beta+1)}{2m_a} \\ &\leq (2\alpha^{-1}\||k|^{-1/2}f\|\lambda)^{2n}\sum_{k=1}^{2n}\frac{\left((C\beta+1)/2\sum_{m=1}^{n}\frac{1}{m}\right)^k}{k!}. \end{aligned} \tag{4.80}$$

Since the $\sum_{m=1}^{n}\frac{1}{m}$ can be considered as a lower Riemann sum we have $\sum_{m=1}^{n}\frac{1}{m}\leq\ln(n+1)$. Thus,

$$\begin{aligned} h_{2n}(\beta,\lambda) &\leq (2\alpha^{-1}\||k|^{-1/2}f\|\lambda)^{2n}\sum_{k=1}^{2n}\frac{\left((C\beta+1)/2\ln(n+1)\right)^k}{k!} \\ &\leq (2\alpha^{-1}\||k|^{-1/2}f\|\lambda)^{2n}(n+1)^{(C\beta+1)/2}. \end{aligned} \tag{4.81}$$

Since $2|\lambda|\cdot\||k|^{1/2}f\| < \alpha$ the series $\sum_{n=0}^{\infty}h_{2n}(\beta,\lambda)^{1/2}$ converges absolutely for all $\beta > 0$. It follows, that

$$e^{-\beta/2(\mathcal{L}_0+Q)}\Omega_0^\beta = \sum_{n=0}^{\infty}\int_{\Delta^n_{\beta/2}}\psi(0,\underline{s},n,N)d\underline{s}$$

exists. □

Conversely, Equation (4.80) and Lemma 4.3.3 imply

$$h_{2n}(\beta,\lambda)\geq(\lambda/2)^{2n}\frac{I(2n,\beta)\beta^{2n}}{2n} = \frac{\left(\alpha^{-1}\int\frac{\beta^2\lambda^2/4|f(k)|^2}{\sinh(|k|\beta/2)\sinh(\beta\alpha/2)}dk\right)^n}{2n}. \tag{4.82}$$

Hence for every $\beta > 0$ exists a $\lambda\in\mathbb{R}$, such that $h_{2n}(\beta,\lambda)\geq\frac{1}{2n}$. Thus $\sum_{n=1}^{\infty}h_{2n}(\beta,\lambda)^{1/2}=\infty$

Remark 4.3.2. We can therefore not extended Theorem 4.3.1 to an existence proof for all $\lambda > 0$.

Lemma 4.3.3. *Following statements are true.*

$$I(G,A,\beta) = I(\#A,\beta),\ G\in\mathcal{G}_c(A) \tag{4.83}$$

$$I(\#A,\beta)\leq(2\||k|^{-1/2}f\|_{\mathfrak{h}}(\alpha\beta)^{-1})^{\#A}\cdot(C\beta+1) \tag{4.84}$$

$$I(\#A,\beta)\geq\left(\alpha^{-1}\int\frac{|f(k)|^2}{\sinh(|k|\beta/2)\sinh(\alpha\beta/2)}dk\right)^{\#A/2}, \tag{4.85}$$

where $\#A = 2m$ *and* $C = (1/2)\frac{\|f\|^2}{\||k|^{1/2}f\|^2}$.

Proof of 4.3.3. A relabeling of the integration variables yields

$$\begin{aligned} I(G,A,\beta) = \int_{[0,1]^{2m}} & d\underline{t}\, K_{osc}(|t_1-t_2|,\beta)K_f(|t_2-t_3|,\beta)\cdots \\ & K_{osc}(|t_{2m-1}-t_{2m}|,\beta)K_f(|t_{2m}-t_1|,\beta) \\ \leq \ & \int_{[0,1]^{2m}} d\underline{t}\, K_{osc}(|t_1-t_2|,\beta)K_f(|t_2-t_3|,\beta)\cdots K_{osc}(|t_{2m-1}-t_{2m}|,\beta) \\ & \sup_{s\in[0,1]} K_f(s,\beta). \end{aligned} \tag{4.86}$$

We transform due to $s_i := t_i - t_{i+1},\ i \leq 2m-1$ and $s_{2m} = t_{2m}$, hence $-1 \leq s_i \leq 1,\ i = 1,\ldots,2m$, since integrating a positive function we obtain

$$\begin{aligned} I(G,A,\beta) &\leq \int_{[-1,1]^{2m}} K_{osc}(|s_1|,\beta)\, K_f(|s_2|,\beta)\cdots K_{osc}(|s_{2m-1}|,\beta) d\underline{s} \sup_{s\in[0,1]} K_f(s,\beta) \\ &= \Big(\int_{-1}^{1} K_{osc}(|s|,\beta)\, ds\Big)^m \Big(\int_{-1}^{1} K_f(|s|,\beta)\, ds\Big)^{m-1} \cdot \sup_{s\in[0,1]} K_f(s,\beta). \end{aligned} \tag{4.87}$$

We recall that

$$\int_{-1}^{1} K_{osc}(|s|,\beta)\, ds = (2\alpha)^{-1} \int_{-1}^{1} \frac{\cosh(\beta\alpha|s| - \alpha\beta/2)}{\sinh(\alpha\beta/2)}\, ds = 2(\alpha^2\beta)^{-1}$$

and

$$\int_{-1}^{1} K_f(|s|,\beta)\, ds = \int_{-1}^{1}\int \frac{\cosh(\beta|s||k| - \beta|k|/2)|f(k)|^2}{2\sinh(\beta|k|/2)} dk\, ds = 2\int \frac{|f(k)|^2}{\beta|k|} dk.$$

Using $\coth(x) \leq 1 + 1/x$, we obtain

$$\sup_{s\in[0,1]} K_f(s,\beta) \leq 2^{-1}\int \frac{\cosh(\beta|k|/2)|f(k)|^2}{\sinh(\beta|k|/2)} dk \leq (1/2)\int |f(k)|^2 dk + \frac{1}{\beta}\int \frac{|f(k)|^2}{|k|} dk. \tag{4.88}$$

Due to the fact, that $t \mapsto K_f(t,\beta)$ and $t \mapsto K_{osc}(t,\beta)$ attain their minima at $t = 1/2$, we obtain the lower bound for $I(\#A,\beta)$. □

Remark 4.3.4. In the literature there is one criterion for $\Omega_0^\beta \in \mathrm{dom}(e^{-\beta/2(\mathcal{L}_0+Q)})$, to our knowledge, that can be applied in this situation [10]. One has to show that $\|e^{-\beta/2Q}\Omega_0^\beta\| < \infty$.

If we consider the case, where the criterion holds for $\pm\lambda$, then the expansion in λ must converge,

$$\|e^{-\beta/2Q}\Omega_0^\beta\|^2 = \sum_{n=0}^{\infty} \frac{(\lambda\beta)^{2n}}{(2n)!}\omega_{el}^\beta(x^{2n})\omega_f^\beta(\Phi(f)^{2n}) \tag{4.89}$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda\beta)^{2n}}{(2n)!}\Big(\frac{(2n)!}{n!2^n}\Big)^2 K_{osc}(0,\beta)^n K_f(0,\beta)^n$$

$$= \sum_{n=0}^{\infty}(\lambda\beta)^{2n}\alpha^{-n}\binom{2n}{n}2^{-2n}\Big(\coth(\alpha\beta/2)\int|f(k)|^2\coth(\beta|k|/2)dk\Big)^n$$

$$\geq \sum_{n=0}^{\infty}(\lambda\beta)^{2n}(4\alpha)^{-n}\Big(\int|f(k)|^2 dk\Big)^n. \tag{4.90}$$

Obviously, for any value of $\lambda \neq 0$, there is a $\beta > 0$, for which $\|e^{-\beta/2Q}\Omega_0^\beta\| < \infty$ is not fulfilled.

Chapter 5

Return to Thermal Equilibrium

In this chapter a slightly modified model is considered, it originates from a from an one dimensional harmonic oscillator coupled to the quantized radiation field by means of the dipole approximation.

This model was studied by Arai in [1, 2] for temperature zero. Therein the existence of a unique ground state is shown, such as asymptotic completeness of H_λ and $\check{H}$. We introduce the ladder operators for the harmonic oscillator,

$$A^* = \frac{x - \imath p}{\sqrt{2}}, \quad A^* = \frac{x + \imath p}{\sqrt{2}}. \tag{5.1}$$

defined as operators on the Schwartz space $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$. The operator x is the position operator defined by $(x\psi)(x) = x\psi(x)$ and p is the momentum operator for the particle, it is defined by $(p\psi)(x) = -\imath \frac{\partial \psi(x)}{\partial x}$ on $\mathcal{S}(\mathbb{R})$. It is an easy calculation to check that, A and A^* fulfill the Canonical Commutator Relations (CCR),

$$[A, A] = 0 = [A^*, A^*], \quad [A, A^*] = \mathbb{1}. \tag{5.2}$$

Moreover, there is a (up to a complex phase) unique normed vector Ω_0 in $L^2(\mathbb{R})$ in the kernel of A,

$$\Omega_0(x) = \pi^{-1/4} e^{-x^2/2}. \tag{5.3}$$

It is well known, that

$$c\ell \, \mathrm{LH}\{(A^*)^n \Omega_0 \in L^2(\mathbb{R}) | \, n \in \mathbb{N}_0\} = L^2(\mathbb{R}) =: \mathcal{H}_{el}.$$

Hence we can identify $L^2(\mathbb{R}) \cong \mathcal{F}_b[\mathbb{C}]$. Ω_0 is the vacuum vector. The position- and momentum-operator are the field operators $\Phi(1)$ and $\Phi(\imath)$.

We turn now to the model. The Hamiltonian for the particle is $H_{el} = \frac{1}{2}(p^2 + x^2)$ of the bosons the Hamiltonian is $\check{H} = \int |k| a(k)^* a(k) d^3k$. Hence the non-interacting Hamiltonian is $H_0 := H_{el} + \check{H}$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathcal{F}_b[L^2(\mathbb{R}^3)] \cong \mathcal{F}_b[\mathbb{C} \oplus L^2(\mathbb{R}^3)]$. The full Hamiltonian is

$$H_\lambda = H_{el} \otimes 1 + 1 \otimes \check{H} + \lambda W + \lambda^2/2 \, \| |k|^{-1} \hat{\rho} \|_{\mathfrak{h}}^2 \cdot x^2 \otimes 1, \tag{5.4}$$

W is given by $x \cdot \Phi(|k|^{-1/2} \hat{\rho})$. Confer with the derivation in Section 3.1.
We assume

- $(1 + |k|^{-1}) \hat{\rho} \in L^2(\mathbb{R}^3)$, $\hat{\rho}(k) > 0$, $k \in \mathbb{R}^3$.
- $\hat{\rho}$ is rotation invariant.
- In polar coordinates: $[0, \infty) \ni r \mapsto \hat{\rho}(r)$ has an anal. cont. to $\{z \in \mathbb{C} : |\Im z| \leq 2\pi\beta^{-1}\}$
- $\sup_{|s| \leq 2\pi\beta^{-1}} \int |(r + \imath s) \hat{\rho}(r + \imath s)|^2 dr < \infty$.
- $\hat{\rho}(r) = \hat{\rho}(-r)$, $r \in \mathbb{R}$

Note, that $\hat{\rho} > 0$ is a relict of the analysis of Arai.
The main advantage of this model is that H_λ is a quadratic operator in creation- and annihilation operators. Hence one expects, that

$$e^{\imath t H_\lambda} \Phi(c \oplus f) e^{-\imath t H_\lambda} = \Phi(w_t(c \oplus f))$$

for a real linear operator w_t on $\mathbb{C} \oplus \mathfrak{h}$, such that

$$\Im \langle w_t(c \oplus f) | w_t(c' \oplus f') \rangle_{\mathbb{C} \oplus \mathfrak{h}} = \Im \langle c \oplus f | c' \oplus f' \rangle_{\mathbb{C} \oplus \mathfrak{h}}.$$

This fact allows us to define a C^*-algebra of observables

$$\begin{aligned} \mathcal{A} &= c\ell \, \mathrm{LH} \Big\{ W(c \oplus f) \in \mathcal{B}(\mathcal{H}) \, : \, c \in \mathbb{C}, \, f \in \mathfrak{f} \Big\}, \\ \mathfrak{f} &= \{ f \in \mathfrak{h} \, : \, \int (1 + |k|^{-1}) |f(k)|^2 d^3k < \infty \}, \end{aligned}$$

where the closure is taken in the operator norm of $\mathcal{B}(\mathcal{H})$, and a $*$-automorphism group $\tau_t^\lambda(A) = e^{\imath t H_\lambda} A e^{-\imath t H_\lambda}$, $t \in \mathbb{R}$ for $A \in \mathcal{A}$. The operator $W(c \oplus f) := \exp(\imath \Phi(c \oplus f))$ is called a Weyl operator, $c \oplus f$ is a form factor and $\mathcal{A}$ a Weyl algebra.

The Equilibrium of the dynamical system $(\mathcal{A}, \tau^\lambda)$ is a state ω_λ^β that suffices the so called

(β, τ^λ)-KMS-condition, i.e. for all $A, B \in \mathcal{A}$ exists a complex function $F_\beta(A, B, \cdot)$ that is analytic in the strip $\{z \in \mathbb{C} : 0 < \Im z < \beta\}$, continuous on its closure and taking the boundary conditions

$$F_\beta(A, B, t) = \omega_\lambda^\beta(A\tau_t^\lambda(B)), \quad F_\beta(A, B, t + \imath\beta) = \omega_\lambda^\beta(\tau_t^\lambda(B)A). \tag{5.5}$$

The positive parameter β is the inverse temperature.

5.1 A Summary of Results due to Arai

In this section we recall definition and statements, that Arai made in [1, 2]. Therein a explicit unitary transform of H_λ into $\check{H}$ is defined. Therefore time dependent field operators $\Phi_k(t)$ for a sharp momentum $k \in \mathbb{R}^3$ are considered, i.e.

$$\Phi_k(t) := e^{\imath t H_\lambda} \Phi_k e^{-\imath t H_\lambda}, \quad x(t) := e^{\imath t H_\lambda} x e^{-\imath t H_\lambda}, \tag{5.6}$$

where $\Phi_k := \frac{1}{\sqrt{2|k|}}\big(a_k + a_k^*\big)$ is a operator valued distribution. These objects obey Heisenberg's equations of motion:

$$\frac{d^2}{dt^2}\begin{pmatrix} x(t) \\ \Phi_k(t) \end{pmatrix} = \begin{pmatrix} -1 - \lambda^2 \||k|^{-1}\hat{\rho}\|_{\mathfrak{h}}^2 & -\lambda \int d^3k\, \hat{\rho}(k)\, [\cdot] \\ -\lambda\hat{\rho} & -k^2 \end{pmatrix} \begin{pmatrix} x(t) \\ \Phi_k(t) \end{pmatrix}. \tag{5.7}$$

One can explicitly solve them for the Laplace transform with respect to t of $x(\cdot)$ and $\Phi_k(\cdot)$. The explicit formulas for it, are formulated with the help of the functions

Definition 5.1.1.

$$D(z) := -z + 1 + \lambda^2 \||k|^{-1}\hat{\rho}\|_{\mathfrak{h}}^2 + \lambda^2 \int d^3k\, \frac{\hat{\rho}(k)^2}{z - k^2}, \; z \in \mathbb{C} \setminus [0, \infty) \tag{5.8}$$

$$D_\pm(r) := \lim_{\epsilon \to 0+} D(r \pm \imath\epsilon), \; r \in [0, \infty) \tag{5.9}$$

$$Q(k) := -\lambda \frac{\hat{\rho}(k)}{D_+(k^2)} \tag{5.10}$$

$$Q_\pm(k) := (1/2)(|k|^{1/2} \pm |k|^{-1/2})Q(k). \tag{5.11}$$

and the operators on $\mathfrak{h}$

Definition 5.1.2.

$$(G_\epsilon g)(k) := \int \frac{g(k')}{(|k|\cdot|k'|)^{1/2}(k^2 - k'^2 + \imath\epsilon)} dk'. \tag{5.12}$$

$$G := \lim_{\epsilon\to 0+} G_\epsilon \tag{5.13}$$

$$Tg := g + \lambda|k|^{1/2} QG|k|^{1/2}\hat{\rho} g \tag{5.14}$$

$$T^* g := g - \lambda|k|^{1/2}\hat{\rho} G|k|^{1/2}\overline{Q} g \tag{5.15}$$

$$W_+ g := (1/2)\big\{|k|^{-1/2}T^*|k|^{1/2} + |k|^{1/2}T^*|k|^{-1/2}\big\} g \tag{5.16}$$

$$W_- g := (1/2)\big\{|k|^{-1/2}T^*|k|^{1/2} - |k|^{1/2}T^*|k|^{-1/2}\big\} g. \tag{5.17}$$

Using the Laplace transforms one can find explicit formulas for the asymptotic incoming creation- and annihilation operators. Since we have to calculate with the objects in Definition 5.1.1 and 5.1.2, we recall some results of Arai for these objects. The reader may skip the rest of this section and consult the Lemmata later.

The hypothesis on the analyticity and positivity of $\hat{\rho}$ implies directly

Lemma 5.1.3. *1. D is analytic in $\mathbb{C}\setminus[0,\infty)$,*

2. $D_\pm(s) := \lim_{\epsilon\to 0+} D(s+\imath\epsilon)$ exists and is continuous for $s\in[0,\infty)$,

3. $\inf_{s\in[0,\infty)} |D_\pm(s)| > 0$,

4. $|D(z)+z| < c_1$ and $|D(z)| > c_2$ for all $z\in\mathbb{C}\setminus[0,\infty)$ and $c_1, c_2 < \infty$.

Let $M_\alpha(\mathbb{R}^3) = \{f : \|f\|_\alpha = \||k|^\alpha f\|_{\mathfrak{h}} < \infty\}$, for $\alpha\in\mathbb{R}$. For the operators introduced in Equation (5.11) and (5.12) we have:

Lemma 5.1.4. *1. G_ϵ is bounded on $\mathfrak{h}$, uniformly for $\epsilon > 0$.*

2. $G := s - \lim_{\epsilon\to 0+} G_\epsilon$ exists as an operator on $\mathfrak{h}$.

3. G is bounded on $\mathfrak{h}$ and $M_{-1/2}(\mathbb{R}^3)$.

4. $G^* = -G$, i.e G is skew-symmetric on $\mathfrak{h}$.

Given a bounded operator A on $\mathfrak{h}$ we denote by $\overline{A}$ an operator acting on $g\in\mathfrak{h}$ by means of $(\overline{A}g)(k) := \overline{(A\overline{g})(k)}$. The bar is of course the complex conjugation.

Lemma 5.1.5. *1. T and T^* (see Equation (5.13)) are bounded on $M_\alpha(\mathbb{R}^3)$ for $\alpha = 1/2, 0, -1$.*

2. T^* *is the adjoint of* T.

3. *For a rotation invariant function* h *on* $\mathbb{R}^3$, *we have* $T^*hT = \overline{T}^*h\overline{T}$

4. *Furthermore, if* $hQ \in \mathfrak{h}$, *then* $T^*hQ = \overline{T}^*h\overline{Q}$.

5. $T^*Q = 0$

The next algebraic relations ensure that the incoming creation- and annihilation operators fulfill the CCR.

Lemma 5.1.6. *The operators* W_+ *and* W_- *defined in* (5.15) *and* (5.16) *are bounded on* $M_\alpha(\mathbb{R}^3)$ *for* $\alpha = -1/2, 0$ *and fulfill*

$$W_+^*W_+ - W_-^*W_- + P_+ - P_- = \mathbb{1}, \quad W_+W_+^* - \overline{W}_-\overline{W}_-^* = \mathbb{1}, \tag{5.18}$$

$$\overline{W}_+^*W_- - \overline{W}_-^*W_+ + P_{+-} - P_{-+} = 0, \quad W_-W_+^* - \overline{W}_+\overline{W}_-^* = 0, \tag{5.19}$$

where

$$P_\pm f = \langle Q_\pm | f\rangle_{\mathfrak{h}} \cdot Q_\pm, \quad P_{+-}f = \langle Q_- | f\rangle_{\mathfrak{h}} \cdot \overline{Q}_+, \quad P_{-+}f = \langle Q_+ | f\rangle_{\mathfrak{h}} \cdot \overline{Q}_-. \tag{5.20}$$

Furthermore W_- *is a Hilbert-Schmidt operator with integral kernel*

$$W_-(k,k') = \frac{\lambda\hat{\rho}(k)\overline{Q(k')}}{2(|k||k'|)^{1/2}(|k|+|k'|)}. \tag{5.21}$$

The starting point of our work is the following result:

Lemma 5.1.7. *The asymptotic creation- and annihilation- operators* $a_{in}^{\#}(f)$ *exist for* $f \in M_0(\mathbb{R}^3) \cap M_{-1/2}(\mathbb{R}^3)$,

$$a_{in}^{\#}(f) = \text{s-lim}_{t\to-\infty} e^{\imath tH_\lambda}e^{-\imath tH_0}a^{\#}(f)e^{\imath tH_0}e^{-\imath tH_\lambda}, \tag{5.22}$$

$\mathrm{dom}(a_{in}^{\#}(f)) \supset \mathrm{dom}(H_\lambda)$ *and*

$$a_{in}(f) = \langle Q_- | \overline{f}\rangle_{\mathfrak{h}} A^* + \langle Q_+ | \overline{f}\rangle_{\mathfrak{h}} A + a^*(W_-\overline{f}) + a(\overline{W}_+ f). \tag{5.23}$$

5.2 Return to Equilibrium for the Harmonic Oscillator

In this section we define a C^*-algebra of observables, a time-evolution group and an equilibrium state for the isolated bosonic system. A comparison with the dynamical system $(\mathcal{A}, \tau^\lambda)$ yields the existence of an equilibrium state of $(\mathcal{A}, \tau^\lambda)$.

In Lemma 5.2.1 the analyticity and the zeros of $D_+(\cdot)$ in the complex plane depending on the coupling parameter $\lambda \neq 0$ are studied. A dense set of form-factors is specified, which define a $*$-subalgebra $\mathcal{A}_{anal}$ of observables.

In Theorem 5.2.6 is proved Return to Equilibrium for a set of states. The exponential rate of decay is specified by the imaginary part of zeros of $D_+(\cdot)$, it follows Fermi's Golden Rule for H_λ.

5.2.1 Existence of an Equilibrium State

The free bosonic system is defined by a C^*-algebra $\mathcal{W}(\mathfrak{f})$ of observables and a $*$-automorphism group $\tau_t^f(A) = e^{\imath t \check{H}} A e^{-\imath t \check{H}},\ A \in \mathcal{W}(\mathfrak{f})$. The corresponding thermal Equilibrium State ω_f^β is

$$\omega_f^\beta(\mathcal{W}(h)) = \exp\big(-1/4\langle h|(1+2\varrho)h\rangle_{\mathfrak{h}}\big), \tag{5.24}$$

where $\varrho(k) = \frac{1}{e^{\beta|k|}-1}$ is the density due to Planck's law. It is well known that ω_f^β is a (β, τ^f)-KMS-state. For the coupled system without interaction ($\lambda = 0$) the Equilibrium state over $\mathcal{W}(\mathbb{C} \oplus \mathfrak{f})$ is

$$\omega_0^\beta(\mathcal{W}(c \oplus h)) = \exp\big(-1/4\langle c \oplus h|(1+2\varrho_0)c \oplus h\rangle_{\mathbb{C}\oplus\mathfrak{h}}\big), \tag{5.25}$$

for $\varrho_0(k) = \frac{1}{e^{\beta}-1} \oplus \frac{1}{e^{\beta|k|}-1}$. We remark that

$$\tau_t^0(\mathcal{W}(c \oplus h)) = e^{\imath t H_0}\mathcal{W}(c \oplus h)e^{-\imath t H_0} = \mathcal{W}(e^{\imath t}c \oplus e^{\imath t|k|}h). \tag{5.26}$$

To define a thermal Equilibrium State $\mathfrak{A} = W(\mathbb{C} \oplus \mathfrak{f})$ for $\lambda \neq 0$ we start from a result due to Arai [1, 2], where an explicit formula for the incoming creation- and annihilation- operators is given for H_λ at temperature zero.

Lemma 5.1.7 implies directly for $\Phi_{in}(f) = \frac{1}{\sqrt{2}}\big(a_{in}(f) + a_{in}^*(f)\big)$, that

$$\Phi_{in}(f) = \Phi\big((\langle \overline{Q}_+|f\rangle_{\mathfrak{h}} + \langle f|\overline{Q}_-\rangle_{\mathfrak{h}}) \oplus (\overline{W}_+ f + W_- \overline{f})\big). \tag{5.27}$$

We remark that $e^{\imath t H_\lambda}\Phi_{in}(f)e^{-\imath t H_\lambda} = \Phi_{in}(e^{\imath t|k|}f)$. A simple but lengthly calculation using Lemmata 5.1.6 and 5.1.7 yields $\Phi(c \oplus h) = \Phi_{in}(v(c \oplus h))$, where v is a real linear operator from

$\mathbb{C} \oplus \mathfrak{h}$ to $\mathfrak{h}$ defined by

$$v(c \oplus h) := \overline{W}_{+}^{*} h + c \cdot \overline{Q}_{+} - \overline{W}_{-}^{*} \overline{h} - \overline{c} \cdot \overline{Q}_{-}. \tag{5.28}$$

Note that v is surjective since

$$h = v\big((\langle \overline{Q_{+}} | h \rangle_{\mathfrak{h}} + \langle Q_{-} | \overline{h} \rangle_{\mathfrak{h}}) \oplus (\overline{W}_{+} h + W_{-} \overline{h})\big). \tag{5.29}$$

Hence it follows, that

$$\tau_t^{\lambda}(\Phi(c \oplus h)) = e^{\imath t H_{\lambda}} \Phi_{in}(v(c \oplus h)) e^{-\imath t H_{\lambda}} = \Phi_{in}(e^{\imath t |k|} v(c \oplus h)) = \Phi(w_t(c \oplus h)). \tag{5.30}$$

for the real linear, time dependent operator w_t defined by

$$\begin{aligned} w_t(c \oplus h) \;\; = \;\; & \big(\langle \overline{Q}_{+} | e^{\imath t |k|} v(c \oplus h) \rangle_{\mathfrak{h}} + \langle e^{\imath t |k|} v(c \oplus h) | \overline{Q}_{-} \rangle_{\mathfrak{h}}\big) \\ & \oplus \big(\overline{W}_{+} e^{\imath t |k|} v(c \oplus h) + W_{-} e^{-\imath t |k|} \overline{v(c \oplus h)}\big). \end{aligned} \tag{5.31}$$

Since $\Phi_{in}(e^{\imath t|k|} v(c \oplus h)) = \Phi_{in}(v(w_t(c \oplus h)))$ we have

$$e^{\imath t |k|} v(c \oplus h) = v(w_t(c \oplus h)). \tag{5.32}$$

Furthermore we have

$$\begin{aligned} & \Im \langle c \oplus h | c' \oplus h' \rangle_{\mathbb{C} \oplus \mathfrak{h}} = -\imath [\Phi(c \oplus h), \Phi(c' \oplus h')] \\ = \;\; & -\imath [\Phi_{in}(v(c \oplus h)), \Phi_{in}(v(c' \oplus h'))] = \Im \langle v(c \oplus h) | v(c' \oplus h') \rangle_{\mathfrak{h}}. \end{aligned} \tag{5.33}$$

Moreover, v is injective:

Assume $v(c \oplus h) = 0$, hence for all $f \in \mathfrak{f}$ and $c' \in \mathbb{C}$ we have $0 = \Im \langle v(c' \oplus f) | v(c \oplus h) \rangle_{\mathfrak{h}} = \Im \langle c' \oplus f | c \oplus h \rangle_{\mathbb{C} \oplus \mathfrak{h}}$. Since $\Im \langle (-\imath)(c' \oplus f) | c \oplus h \rangle_{\mathbb{C} \oplus \mathfrak{h}} = \Re \langle c' \oplus f | c \oplus h \rangle_{\mathbb{C} \oplus \mathfrak{h}} = 0$, it is $\langle c' \oplus f | c \oplus h \rangle_{\mathbb{C} \oplus \mathfrak{h}} = 0$. Hence $c \oplus h = 0$ follows from the density of $\mathfrak{f}$ in $\mathfrak{h}$.

Let

$$i \, : \, \mathcal{A} \to W(\mathfrak{f}), \; W(c \oplus h) \mapsto W(v(c \oplus h)). \tag{5.34}$$

Hence i defines a $*$-isomorphism, in particular a Bogoliubov-transform. It follows that $\omega_{\lambda}^{\beta} := \omega_f^{\beta} \circ i$, i.e

$$\omega_{\lambda}^{\beta}(W(c \oplus h)) = \exp(-(1/4) \| (1 + 2\varrho)^{1/2} v(c \oplus h) \|_{\mathfrak{h}}^2) \tag{5.35}$$

is a (β, τ^{λ})-KMS state over $\mathcal{W}(\mathbb{C} \oplus \mathfrak{f})$. We have

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\tau_t^{\lambda}} & \mathcal{A} & \xrightarrow{\omega_{\lambda}^{\beta}} & \mathbb{C} \\ \Big\downarrow i & & \Big\downarrow i & \nearrow \omega_f^{\beta} & \\ W(\mathfrak{f}) & \xrightarrow{\tau_t^{f}} & W(\mathfrak{f}). & & \end{array}$$

5.2.2 Return to Equilibrium

Lemma 5.2.1. *There is a function $G_\lambda : S_\lambda \to \mathbb{C}$, such that G_λ is an analytic continuation of $\mathbb{R} \ni r \mapsto D_+(r^2) \in \mathbb{C}$ in $S_\lambda = \{z \in \mathbb{C} : |\Im z| < |\Im \kappa_+(\lambda)|\}$ that has no zeros. Furthermore, $\kappa_+(\cdot)$ is even, analytic and*

$$\kappa_+(\lambda) = 1 + \kappa_2\lambda^2 + \dots, \tag{5.36}$$

where $\Im\kappa_2 = -2\pi^2\hat{\rho}^2(1)$. The result holds for small values of λ.

We remark that the same is true for $r \mapsto \overline{D_+(r^2)}$.

Definition 5.2.2. *Let κ be a real number, depending on $\lambda \neq 0$, , so that $0 < \kappa < |\Im\kappa_+(\lambda)|$ and $\kappa < 2\pi\beta^{-1}$. κ is the so-called decay rate.*

Proof of 5.2.1. Let G be defined for $|\Im z| < \eta$ and $\lambda \in \mathbb{C}$ by

$$G(z,\lambda) = -z^2 + 1 + \||k|^{-1}\hat{\rho}\|_{\mathfrak{h}}^2\lambda^2 + 2\pi\lambda^2 \int_{-\infty}^{\infty} \frac{\hat{\rho}^2(r+\imath\eta)(r+\imath\eta)}{z-(r+\imath\eta)}dr + 4\pi^2\imath\lambda^2\hat{\rho}^2(z)z. \tag{5.37}$$

Since $\hat{\rho}(r)$ is an even function, we may write

$$D(z^2) = -z^2 + 1 + \||k|^{-1}\hat{\rho}\|_{\mathfrak{h}}^2\lambda^2 + 2\pi\lambda^2 \int_{-\infty}^{\infty} .\frac{\hat{\rho}^2(r)r}{z-r}dr,\ \Im z > 0. \tag{5.38}$$

The residue theorem yields that $G_\lambda(\cdot) := G(\cdot,\lambda)$ is an analytic continuation of $D(z^2)$ into the lower half plane and hence

$$G(z,\lambda) = G(-z,\lambda) = G(z,-\lambda). \tag{5.39}$$

Let $s \geq 0$. We can choose $p_\epsilon(s)$, such that $s^2 + \imath\epsilon = p_\epsilon(s)^2$, $\Re z \geq 0$ and $\Im z > 0$, then

$$G(s,\lambda) = \lim_{\epsilon\to 0+} G(p_\epsilon(s),\lambda) = \lim_{\epsilon\to 0+} D(s^2+\imath\epsilon) = D_+(s^2). \tag{5.40}$$

Next, we define $P(z) = -z^2 + 1$. For $0 < \eta' < \eta$ we have

$$\sup_{\{z : |\Im z| < \eta'\}} |P(z) - G(z,\lambda)| \leq C_{\eta'}|\lambda|^2. \tag{5.41}$$

Since $\partial_z G(\pm 1, 0) = \mp 2$ the implicit function theorem states that we can find $\lambda_0' > 0$ and $0 < \epsilon < \eta'$ such that for $(z,\lambda) \in B_\epsilon(\pm 1) \times B_{\lambda'}(0)$

$$G(z,\lambda) = 0 \Leftrightarrow z = \kappa_\pm(\lambda) \tag{5.42}$$

for two analytic functions $\kappa_\pm : B_{\lambda_0'}(0) \to B_\epsilon(\pm 1)$ with $\kappa_\pm(0) = \pm 1$. Using (5.41) we can choose $0 < \lambda_0 < \lambda_0'$ that ensures $G(z,\lambda) \neq 0$ for $|z-1| \geq \epsilon$ and $|z+1| \geq \epsilon$ for all $|\lambda| \leq \lambda_0$.

By symmetry of G and uniqueness of $\kappa_\pm$ we have $\kappa_-(\lambda) = -\kappa_+(\lambda)$ and $\kappa_+(\lambda) = \kappa_+(-\lambda)$, in particular $\partial_\lambda^{(2n+1)}\kappa_+(0) = 0$. For the second derivative we have

$$\partial_\lambda^2\kappa_+(0) = -\frac{(\partial_\lambda^2 G)(1,0)}{(\partial_z G)(1,0)} = \||k|^{-1}\hat{\rho}\|_{\mathfrak{h}}^2 + 2\pi\mathcal{P}\int_{-\infty}^{\infty}\frac{\hat{\rho}^2(r)r}{1-r}dr - 4\pi^2\hat{\rho}^2(1)\imath, \tag{5.43}$$

where $\mathcal{P}\int_{-\infty}^{\infty}$ means the Cauchy Principal value. □

For $f \in \mathfrak{h}$ we define

$$\tilde{f}(r,\Theta) = \left\{ \begin{array}{ll} f(r,\Theta), & r \geq 0 \\ \overline{f(-r,\Theta)}, & r < 0 \end{array} \right\}, \tag{5.44}$$

where $(r,\Theta) \in [0,\infty) \times S^2$ and f is written in polar coordinates.

Definition 5.2.3. *Let*

$$\begin{aligned} H^2(\kappa) &:= \Big\{ f \in \mathfrak{h} \,:\, \mathbb{R} \ni r \mapsto \tilde{f}(r,\cdot) \in L^2(S^2) \text{ has an anal. contin. to } S_\kappa, \\ &\quad \sup_{|s|\leq\kappa}\int_{\mathbb{R}} |r+\imath s|^2 \|\tilde{f}(r+\imath s,\cdot)\|_{L^2(S^2)}^2 dr < \infty \Big\} \\ \mathcal{G}_{anal} &:= \Big\{ a\imath|k|^{1/2}\frac{f}{\overline{D_+(k^2)}} + b|k|^{-1/2}\frac{f}{\overline{D_+(k^2)}} \in \mathfrak{h} \,:\, f \in H^2(\kappa),\ a,b \in \mathbb{R} \Big\} \\ \mathcal{H}_{anal} &:= \{ c \oplus f \in \mathbb{C} \oplus \mathfrak{h} \,:\, v(c \oplus f) \in \mathcal{G}_{anal} \}. \end{aligned} \tag{5.45}$$

Lemma 5.2.4. *$\mathcal{G}_{anal}$ is dense in $M_0(\mathbb{R}^3) \cap M_{-1/2}(\mathbb{R}^3)$. $\mathcal{H}_{anal}$ is dense in $\mathbb{C} \oplus (M_0(\mathbb{R}^3) \cap M_{-1/2}(\mathbb{R}^3))$.*

Proof of 5.2.4. Clearly, $\mathcal{G}_{anal}$ is a dense real subspace of $\mathfrak{f}$, respectively the norms $\|\cdot\|_j,\ j = 0, -1/2$. To show that $\mathcal{H}_{anal}$ is dense in $\mathbb{C} \oplus (M_0(\mathbb{R}^3) \cap M_{-1/2}(\mathbb{R}^3))$, we observe, that $c \oplus 0 \in \mathcal{H}_{anal}$, since for $c = a + \imath b$

$$v(c \oplus 0) = a|k|^{-1/2}\overline{Q} - \imath b|k|^{1/2}\overline{Q}, \tag{5.46}$$

and Equation (5.10).

Let $f \in \mathfrak{h}$. We observe $f = \overline{W}_+ g + W_-\overline{g}$ for $g = \overline{W}_+^* f - \overline{W}_-^*\overline{f} \in \mathfrak{h}$. We choose now $(g_\nu)_\nu \subset \mathcal{G}_{anal}$ with $g_\nu \to g,\ \nu \to \infty$. Therefore we have

$$f_\nu := \overline{W}_+ g_\nu + W_-\overline{g}_\nu \to f,\ \nu \to \infty. \tag{5.47}$$

Moreover, for $c_\nu := \langle\overline{Q_+}|g_\nu\rangle_{\mathfrak{h}} - \langle Q_-|\overline{g}_\nu\rangle_{\mathfrak{h}}$ we obtain

$$g_\nu = v(c_\nu \oplus f_\nu), \tag{5.48}$$

hence $c_\nu \oplus f_\nu \in \mathcal{H}_{anal}$ and $0 \oplus f \in \mathrm{cl}\,\mathcal{H}_{anal}$. □

Lemma 5.2.5. *For $f, g \in \mathcal{G}_{anal}$ one has*

$$|\Re\langle f|(1+2\varrho)e^{\imath t|k|}g\rangle_{\mathfrak{h}}| \leq \mathrm{const}\, e^{-\kappa t} \tag{5.49}$$

$$|\Im\langle f|e^{\imath t|k|}g\rangle_{\mathfrak{h}}| \leq \mathrm{const}\, e^{-\kappa t}. \tag{5.50}$$

Proof of 5.2.5. We choose $f = \frac{\imath a|k|^{1/2}f'+b|k|^{-1/2}f'}{\overline{D_+(k^2)}}$ and $g = \frac{\imath a'|k|^{1/2}g'+b'|k|^{-1/2}g'}{\overline{D_+(k^2)}}$ for $a, b, a', b' \in \mathbb{R}$ and $f', g' \in H^2(\kappa)$. For (5.49) we obtain

$$\begin{aligned}
&\Re\langle f|(1+2\varrho)e^{\imath t|k|}g\rangle_{\mathfrak{h}} \\
&= 1/2\int \big(|k|aa' + bb'|k|^{-1}\big)(1+2\varrho(k))\frac{\overline{f'(k)}g'(k)e^{\imath t|k|} + h.c.}{|D_+(k^2)|^2}d^3k \\
&\quad +1/2\int \big(\imath a'b - \imath ab'\big)(1+2\varrho(k))\frac{\overline{f'(k)}g'(k)e^{\imath t|k|} - h.c.}{|D_+(k^2)|^2}d^3k.
\end{aligned} \tag{5.51}$$

Let $(P_{rad}f)(r) = (4\pi)^{-1}\int_{S^2} f(r,\Theta)d\Theta$, where f is written in polar-coordinates. We have

$$\begin{aligned}
&\Re\langle f|(1+2\varrho)e^{\imath t|k|}g\rangle_{\mathfrak{h}} \\
&= 2\pi\int_0^\infty r^2\big(raa' + bb'r^{-1}\big)\coth(\beta r/2)\frac{(P_{rad}\overline{f'})(r)P_{rad}(g')(r)e^{\imath tr} + h.c.}{|D_+(r^2)|^2}dr \\
&\quad +2\pi\int_0^\infty r^2\big(\imath a'b - \imath ab'\big)\coth(\beta r/2)\frac{(P_{rad}\overline{f'})(r)P_{rad}(g')(r)e^{\imath tr} - h.c.}{|D_+(r^2)|^2}dr \\
&= 2\pi\int_{\mathbb{R}} r^2\big(raa' + bb'r^{-1}\big)\coth(\beta r/2)\frac{\widetilde{(P_{rad}\overline{f'})}(r)\widetilde{P_{rad}(g')}(r)e^{\imath tr}}{|D_+(r^2)|^2}dr \\
&\quad +2\pi\int_{\mathbb{R}} r^2\big(\imath a'b - \imath ab'\big)\coth(\beta r/2)\frac{\widetilde{(P_{rad}\overline{f'})}(r)\widetilde{P_{rad}(g')}(r)e^{\imath tr}}{|D_+(r^2)|^2}dr.
\end{aligned} \tag{5.52}$$

The last line follows if we split the integrals in (5.51) in a part with $e^{\imath tr}$ and $e^{-\imath tr}$. In the second we substitute $r \to -r$ and then we integrate over $\mathbb{R}$. $\sim$ is defined in Equation (5.44). By Lemma 5.2.1 we obtain that the integrand is analytic on the strip $S_\kappa = \{z \in \mathbb{C} : |\Im z| \leq \kappa\}$. By the Cauchy's integral theorem, we can shift the contour to $\mathbb{R} \ni r \mapsto r + \imath\kappa$. That yields the decay rate for $t \to \infty$.

The proof of inequality (5.50) is analog. □

Let

$$\mathcal{A}_{anal} := \mathrm{LH}\{W(c\oplus f) \in \mathcal{A} \,:\, c \oplus f \in \mathcal{H}_{anal}\}. \tag{5.53}$$

Theorem 5.2.6. *For $A, B, C \in \mathcal{A}$ we have*

- ω_λ^β *is strongly clustering, i.e.* $\lim_{t\to\infty} \omega_\lambda^\beta(A\tau_t^\lambda(B)C) = \omega_\lambda^\beta(AC)\omega_\lambda^\beta(B)$.

Moreover, if $A, B, C \in \mathcal{A}_{anal}$ one has

- $|\omega_\lambda^\beta(A\tau_t^\lambda(B)C) - \omega_\lambda^\beta(AC)\omega_\lambda^\beta(B)| \leq \text{const}\, e^{-\kappa t}$.

Proof of 5.2.6. We prove the second part of the statement. Let $v_i := v(c_i \oplus f_i)$ for $c_i \oplus f_i \in \mathcal{H}_{anal}$ and $i = 1, 2, 3$.

$$\begin{aligned} &\omega_\lambda^\beta(W(c_1 \oplus f_1)\tau_t^\lambda(W(c_2 \oplus f_2))W(c_3 \oplus f_3)) \\ &= \omega_\lambda^\beta(W(c_1 \oplus f_1 + w_t(c_2 \oplus f_2) + c_3 \oplus f_3)) \\ &\quad \cdot \exp(-(\imath/2)\Im\{\langle c_1 \oplus f_1 | w_t(c_2 \oplus f_2)\rangle_{\mathbb{C}\oplus\mathfrak{h}} + \langle c_1 \oplus f_1 + w_t(c_2 \oplus f_2) | c_3 \oplus f_3\rangle_{\mathbb{C}\oplus\mathfrak{h}}\}) \end{aligned} \tag{5.54}$$

In the above calculation we used $\tau_t^\lambda(W(c_2 \oplus f_2)) = W(w_t(c_2 \oplus f_2))$ and twice the CCR relation, defined in Definition 2.8.1. Using now Equations (5.32) and (5.33) we obtain

$$\begin{aligned} &\omega_\lambda^\beta(W(c_1 \oplus f_1)\tau_t^\lambda(W(c_2 \oplus f_2))W(c_3 \oplus f_3)) \\ &= \exp(-1/4\|(1+2\varrho)^{1/2}(v_1 + e^{\imath t|k|}v_2 + v_3)\|_{\mathfrak{h}}^2) \\ &\quad \cdot \exp\big(-(\imath/2)\Im\{\langle v_1 | e^{\imath t|k|}v_2\rangle_{\mathfrak{h}} + \langle v_1 + e^{\imath t|k|}v_2 | v_3\rangle_{\mathfrak{h}}\}\big). \end{aligned} \tag{5.55}$$

From

$$\begin{aligned} &\omega_\lambda^\beta(W(c_1 \oplus f_1)W(c_3 \oplus f_3))\,\omega_\lambda^\beta(W(c_2 \oplus f_2)) \\ &= \exp(-\imath/2\Im\langle v_1|v_3\rangle_{\mathfrak{h}})\exp(-1/4\|(1+2\varrho)^{1/2}(v_1+v_3)\|_{\mathfrak{h}}^2)\exp(-1/4\|(1+2\varrho)^{1/2}e^{\imath t|k|}v_2\|_{\mathfrak{h}}^2) \end{aligned} \tag{5.56}$$

we conclude

$$\begin{aligned} &\omega_\lambda^\beta(W(c_1 \oplus f_1)\tau_t^\lambda(W(c_2 \oplus f_2))W(c_3 \oplus f_3)) \\ &= \omega_\lambda^\beta(W(c_1 \oplus f_1)W(c_3 \oplus f_3))\,\omega_\lambda^\beta(W(c_2 \oplus f_2)) \\ &\quad \cdot \exp(-(1/2)\Re\langle v_1 + v_3|(1+2\varrho)e^{\imath t|k|}v_2\rangle_{\mathfrak{h}})\exp(-(\imath/2)\Im\langle v_1 - v_3|e^{\imath t|k|}v_2\rangle_{\mathfrak{h}}). \end{aligned} \tag{5.57}$$

The rest follows by Lemma 5.2.5 and linearity. To prove the first statement, we assume $f_1, f_2, f_3 \in M_{-1/2}(\mathbb{R}^3) \cap M_0(\mathbb{R}^3)$. As before we obtain Equation 5.57, but only $v_1, v_2, v_3 \in M_{-1/2}(\mathbb{R}^3) \cap M_0(\mathbb{R}^3)$ is satisfied. The Riemann-Lebesgue Lemma yields

$$\lim_{t\to\infty} \omega_\lambda^\beta(W(c_1 \oplus f_1)\tau_t^\lambda(W(c_2 \oplus f_2))W(c_3 \oplus f_3)) = \omega_\lambda^\beta(W(c_1 \oplus f_1)W(c_3 \oplus f_3))\,\omega_\lambda^\beta(W(c_2 \oplus f_2)). \tag{5.58}$$

By linearity we obtain

$$\lim_{t\to\infty} \omega_\lambda^\beta(W_1\tau_t^\lambda(W_2)W(W_3)) = \omega_\lambda^\beta(W_1W_3)\,\omega_\lambda^\beta(W_2). \tag{5.59}$$

for $W_1,\, W_2,\, W_3 \in \mathcal{A}_0 := \mathrm{LH}\{W(c\oplus f) \in W(\mathbb{C}\oplus\mathfrak{f}) \,:\, c\in\mathbb{C},\, f\in\mathfrak{f}\}$. Since $\mathcal{A}_0$ is dense in $\mathcal{A}$ respectively the operator norm, we have

$$\lim_{t\to\infty} \omega_\lambda^\beta(A\tau_t^\lambda(B)C) = \omega_\lambda^\beta(AC)\omega_\lambda^\beta(B)$$

for $A,\, B,\, C \in \mathcal{A}$. □

5.3 Comparison with the Liouvillean Approach

At this point we summarize, what is discussed in Chapter 3 and Chapter 4:

In the Liouvillean approach the algebra $\tilde{\mathcal{A}} := \mathcal{B}(\mathcal{H}_{el})\otimes\mathcal{W}(\mathfrak{f})$ is considered. One starts from the (τ^0,β)-KMS-state

$$\omega_0^\beta(A\otimes W(f)) := Z_\beta^{-1}\,\mathrm{Tr}\{Ae^{-\beta H_{el}}\}\cdot\exp(-(1/4)\|(1+2\varrho)^{1/2}f\|_{\mathfrak{h}}^2), \tag{5.60}$$

where $Z_\beta := \mathrm{Tr}\{e^{-\beta H_{el}}\} < \infty$ and $A\in\mathcal{B}(\mathcal{H}_{el})$. Next, one makes an explicit GNS-construction $(\tilde{\mathcal{A}},\mathcal{K},\Omega_0^\beta)$ with $\omega_0^\beta(B) = \langle\Omega_0^\beta|\tilde{\pi}[B]\Omega_0^\beta\rangle_{\mathcal{K}},\ B\in\tilde{\mathcal{A}}$. One may choose $\mathcal{K} := L^2(\mathbb{R})\otimes L^2(\mathbb{R})\otimes\mathcal{F}_b[\mathfrak{h}]\otimes\mathcal{F}_b[\mathfrak{h}]$ and $\Omega_0^\beta = Z_\beta^{-1/2}\cdot k_{\beta/2}\otimes\Omega_{\mathfrak{h}}\otimes\Omega_{\mathfrak{h}}$, where $k_{\beta/2}$ is the Hilbert-Schmidt kernel of $e^{-\beta/2H_{el}}$ in $L^2(\mathbb{R}^2)\cong L^2(\mathbb{R})\otimes L^2(\mathbb{R})$.

The $*$-isomorphism $\tilde{\pi}$ is given by

$$\tilde{\pi}[A\otimes W(f)] = A\otimes 1\otimes W((1+\varrho)^{1/2}f)\otimes W(\varrho^{1/2}\overline{f}). \tag{5.61}$$

The time evolution is

$$\tilde{\pi}[\tau_0^t(A\otimes W(f)] = e^{\imath t\mathcal{L}_0}\tilde{\pi}[A\otimes W(f)]e^{-\imath t\mathcal{L}_0}, \tag{5.62}$$

where $\mathcal{L}_0 = \mathcal{L}_{el}\otimes 1 + 1\otimes\mathcal{L}_f,\ \mathcal{L}_{el} := H_{el}\otimes 1 - 1\otimes H_{el}$ and $\mathcal{L}_f = \check{H}\otimes 1 - 1\otimes\check{H}$. We define

$$H_{el} := 1/2\big(-\Delta_x + (1+\lambda^2\||k|^{-1}\hat{\rho}\|_{\mathfrak{h}}^2)x^2\big). \tag{5.63}$$

Let $\mathfrak{M} := \tilde{\pi}[\tilde{\mathcal{A}}]''$. Furthermore, the so-called modular conjugation $\mathcal{J}$ on $\mathcal{K}$ is defined by

$$\mathcal{J}A^*\Omega_0^\beta = e^{-\beta/2\mathcal{L}_0}A\Omega_0^\beta,\ A\in\mathfrak{M}. \tag{5.64}$$

Using $Q := \lambda\tilde{\pi}[W]$ one can define the standard Liouvillean

$$\mathcal{L}_\lambda := \mathcal{L}_0 + Q - \mathcal{J}Q\mathcal{J}. \tag{5.65}$$

Note, that the definition of $\tilde{\pi}$ must be extended from Weyl operators to field operators, as done in Remark 3.5.7. The time evolution of the interacting system is

$$\widetilde{\tau^\lambda}_t(A) = e^{\imath t\mathcal{L}_\lambda} A e^{-\imath t\mathcal{L}_\lambda}, \; A \in \mathfrak{M}. \tag{5.66}$$

In our context one has to verify, that the canonical imbedding j of $\mathcal{A}$ into $\tilde{\mathcal{A}}$ fulfills:

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{j} & \tilde{\mathcal{A}} & \xrightarrow{\tilde{\pi}} & \mathfrak{M} \\ \downarrow{\scriptstyle \tau_t^\lambda} & & & & \downarrow{\scriptstyle \widetilde{\tau^\lambda}_t} \\ \mathcal{A} & \xrightarrow{j} & \tilde{\mathcal{A}} & \xrightarrow{\tilde{\pi}} & \mathfrak{M}. \end{array}$$

Moreover,

$$\Omega_\lambda^\beta \;=\; c \cdot e^{-(\beta/2)(\mathcal{L}_0 + \lambda Q)} \Omega_0^\beta \tag{5.67}$$

is cyclic for $\tilde{\pi}[\tilde{\mathcal{A}}]$ and separating for $\mathfrak{M}$, and normed for some $c > 0$. Furthermore, Ω_λ^β is in the kernel of $\mathcal{L}_\lambda$ and

$$\widetilde{\omega_\lambda^\beta}(A) := \langle \Omega_\lambda^\beta | A \Omega_\lambda^\beta \rangle_{\mathcal{H}_\beta} \tag{5.68}$$

is a $(\widetilde{\tau^\lambda}, \beta)$-KMS-state over $\mathfrak{M}$, if $\lambda \in \mathbb{R}$ is small. The main theorem in this context is Theorem 4.3.1 with $\alpha^2 := 1 + \lambda^2 \| |k|^{-1} \hat{\rho} \|_{\mathfrak{h}}^2$.

Remark 5.3.1. In Theorem 2.5.6 is a fatal factor 2, that inhibits to prove existence of a $(\beta, \widetilde{\tau^\lambda})$-KMS-state with the Liouvillean approach for all $\lambda \neq 0$. We think, that Theorem 2.5.6 can be improved using a better estimate in Equation (4.87). Therein we used $-1 \leq s_i \leq 1$, but $-1 \leq \sum_{i=1}^m s_i \leq 1$ for $m = 1, \ldots, n-1$ is true, as well.

Lemma 5.3.2. $(\tilde{\pi} \circ j)[\mathcal{A}]'' = \mathfrak{M}$.

Proof of 5.3.2. By the Bicommutant-Theorem it suffices to show, that for every $X \in \tilde{\pi}[\tilde{\mathcal{A}}]$, $\phi \in \mathcal{K}$ and $\epsilon > 0$ exists a $Y \in (\tilde{\pi} \circ j)[\mathcal{A}]$ such that

$$\|X\phi - Y\phi\|_{\mathcal{K}} \leq \epsilon. \tag{5.69}$$

Using density and linearity arguments one can assume $X = \tilde{\pi}[A \otimes W(f)] = A \otimes 1 \otimes W((1 + \varrho)^{1/2} f) \otimes W(\varrho^{1/2}\overline{f})$, where $A \in \mathcal{B}(\mathcal{H}_{el})$, $\phi = \phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4$, where $\phi_i \in \mathcal{H}_{el}$ for $i = 1, 2$ and

$\phi_i \in \mathcal{F}_b[\mathfrak{h}]$ for $i = 3, 4$. Since $W(\mathbb{C})'' = \mathcal{B}(\mathcal{H}_{el})$ the Bicommutant-Theorem yields, that there is a $W \in W(\mathbb{C})$ such that $\|A\phi_1 - W\phi_1\|_{\mathcal{H}_{el}} \leq \epsilon$. Hence $Y := \tilde{\pi}[W \otimes W(f)]$ is an appropriate choice. □

Theorem 5.3.3. *If a cyclic KMS-state $\Omega_\lambda^\beta \in \mathcal{K}$ exists, one has*

$$\omega_\lambda^\beta(A) = \tilde{\omega_\lambda^\beta}((\tilde{\pi} \circ j)[A]), \;\; A \in \mathcal{A}. \tag{5.70}$$

Hence,

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\tilde{\pi} \circ j} & \mathfrak{M} \\ & \searrow^{\omega_\lambda^\beta} & \downarrow \tilde{\omega_\lambda^\beta} \\ & & \mathbb{C}. \end{array}$$

Proof of 5.3.3. Let $\phi := (\tilde{\pi} \circ j)(B)\Omega_\lambda^\beta \in \mathcal{K}$ with $\|\phi\|_{\mathcal{K}}^2 = \omega_0^\beta(B^*B) = 1$

$$\begin{aligned} & |\langle \Omega_\lambda^\beta | \widetilde{\pi} \circ j(A) \Omega_\lambda^\beta \rangle_{\mathcal{K}} - \langle \phi | e^{\imath t \mathcal{L}_\lambda} \widetilde{\pi} \circ j(A) e^{-\imath t \mathcal{L}_\lambda} \phi \rangle_{\mathcal{K}}| \\ = \;\; & |\langle \Omega_\lambda^\beta | e^{\imath t \mathcal{L}_\lambda} \widetilde{\pi} \circ j(A) e^{-\imath t \mathcal{L}_\lambda} \Omega_\lambda^\beta \rangle_{\mathcal{K}} - \langle \phi | e^{\imath t \mathcal{L}_\lambda} \widetilde{\pi} \circ j(A) e^{-\imath t \mathcal{L}_\lambda} \phi \rangle_{\mathcal{K}}| \\ \leq \;\; & 2\|\phi - \Omega_\lambda^\beta\|_{\mathcal{K}} \cdot \|A\|_{\mathcal{B}(\mathcal{K})}. \end{aligned} \tag{5.71}$$

Next, because of Theorem 5.2.6

$$\lim_{t \to \infty} \langle \phi | e^{\imath t \mathcal{L}_\lambda} \widetilde{\pi} \circ j(A) e^{-\imath t \mathcal{L}_\lambda} \phi \rangle_{\mathcal{K}} = \lim_{t \to \infty} \omega_\lambda^\beta(B^* \tau_t^\lambda(A) B) = \omega_\lambda^\beta(A). \tag{5.72}$$

Hence

$$|\langle \Omega_\lambda^\beta | \widetilde{\pi} \circ j(A) \Omega_\lambda^\beta \rangle_{\mathcal{K}} - \omega_\lambda^\beta(A)| \leq 2\|\phi - \Omega_\lambda^\beta\|_{\mathcal{K}} \cdot \|A\|_{\mathcal{B}(\mathcal{K})}. \tag{5.73}$$

On the other hand

$$\mathrm{c}\ell\, \tilde{\pi} \circ j[\mathcal{A}]\Omega_\lambda^\beta = \mathrm{c}\ell\, \tilde{\pi} \circ j[\mathcal{A}]''\Omega_\lambda^\beta = \mathrm{c}\ell\, \mathfrak{M}\Omega_\lambda^\beta = \mathcal{K}. \tag{5.74}$$

Therefore $\|\phi - \Omega_\lambda^\beta\|_{\mathcal{K}}$ can be chosen arbitrarily small, so that $\tilde{\omega_\lambda^\beta}(\widetilde{\pi} \circ j(A)) = \omega_\lambda^\beta(A)$ follows. □

Let $\mathcal{H}_f = \mathcal{F}_b[\mathfrak{h}] \otimes \mathcal{F}_b[\mathfrak{h}]$ and $\Omega_f^\beta := \Omega \otimes \Omega$. The Araki-Woods-Representation is

$$\pi_{AW}(W(g)) = W((1 + \varrho)^{1/2} g) \otimes W(\varrho^{1/2} \overline{g}). \tag{5.75}$$

We define $\mathfrak{M}_f := \pi_{AW}[W(\mathfrak{f})]'' \in \mathcal{B}(\mathcal{H}_f)$ and $\widetilde{\omega_f^\beta}(W(g)) = \langle \Omega_f^\beta | \pi_{AW}[W(g)]\Omega_f^\beta \rangle_{\mathcal{H}_f}$, such as $\widetilde{\tau^f}_t(A) = e^{\imath t\mathcal{L}_f} A e^{-\imath t\mathcal{L}_f}$, $A \in \mathcal{A}$. It is well known, that Ω_f^β is cyclic for $\pi_{AW}[W(\mathfrak{f})]$ and separating for $\mathfrak{M}_f$. Moreover, the following diagram is commutative

$$\begin{array}{ccccc} W(\mathfrak{f}) & \xrightarrow{\tau_t^f} & W(\mathfrak{f}) & \xrightarrow{\omega_f^\beta} & \mathbb{C} \\ \downarrow \pi_{AW} & & \downarrow \pi_{AW} & \nearrow \widetilde{\omega_f^\beta} & \\ \mathfrak{M}_f & \xrightarrow[\widetilde{\tau^f}_t]{} & \mathfrak{M}_f. & & \end{array}$$

In the following we identify ω_f^β and $\widetilde{\omega_f^\beta}$, such as $\widetilde{\tau^f}_t$ and τ_t^f.

Theorem 5.3.4. *There is an isometric isomorphism* $U : \mathcal{H}_f \to \mathcal{K}$, *such that* $Ue^{\imath t\mathcal{L}_f} = e^{\imath t\mathcal{L}_\lambda} U$ *and* $U\Omega_f^\beta = \Omega_\lambda^\beta$. *Let* $\gamma : \mathcal{B}(\mathcal{H}_f) \to \mathcal{B}(\mathcal{K}), \quad \gamma(A) = UAU^{-1}$. *Then* $\gamma \circ \pi_{AW} = \widetilde{\pi} \circ j \circ i^{-1}$.

$$\begin{array}{ccc} (\mathcal{H}_f, \Omega_f^\beta) & \xrightarrow{e^{\imath t\mathcal{L}_f}} & (\mathcal{H}_f, \Omega_f^\beta) \\ U \downarrow & & U \downarrow \\ (\mathcal{K}, \Omega_\lambda^\beta) & \xrightarrow{e^{\imath t\mathcal{L}_\lambda}} & (\mathcal{K}, \Omega_\lambda^\beta) \end{array} \qquad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\tilde{\pi} \circ j} & \mathcal{B}(\mathcal{K}) \\ i \downarrow & & \uparrow \gamma \\ W(\mathfrak{f}) & \xrightarrow[\pi_{AW}]{} & \mathcal{B}(\mathcal{H}_f). \end{array}$$

Proof of 5.3.4. Let $\mathcal{H}_1 := (\pi_{AW} \circ i)[\mathcal{A}]\Omega_f^\beta$ and $\mathcal{H}_2 := (\tilde{\pi} \circ j)[\mathcal{A}]\Omega_\lambda^\beta$. Since Ω_f^β is separating for $(\pi_{AW} \circ i)[\mathcal{A}]$ one can define

$$U : \mathcal{H}_1 \to \mathcal{H}_2,\ (\pi_{AW} \circ i)[A]\Omega_f^\beta \mapsto (\tilde{\pi} \circ j)[A]\Omega_\lambda^\beta. \tag{5.76}$$

We observe that

$$\begin{aligned} &\langle (\tilde{\pi} \circ j)[A]\Omega_\lambda^\beta | (\tilde{\pi} \circ j)[B]\Omega_\lambda^\beta \rangle_{\mathcal{K}} = \tilde{\omega}_\lambda^\beta((\tilde{\pi} \circ j)[A^*B]) = \omega_\lambda^\beta(A^*B) \\ &= \ \omega_f^\beta((\pi_{AW} \circ i)[A^*B]) = \langle (\pi_{AW} \circ i)[A]\Omega_f^\beta | (\pi_{AW} \circ i)[B]\Omega_f^\beta \rangle_{\mathcal{H}_f}. \end{aligned} \tag{5.77}$$

Therefore, U is an isometric isomorphism. Moreover,

$$\begin{aligned} &e^{\imath t\mathcal{L}_\lambda} U(\pi_{AW} \circ i)[A]\Omega_f^\beta = e^{\imath t\mathcal{L}_\lambda}(\tilde{\pi} \circ j)[A]\Omega_\lambda^\beta \\ &= \ e^{\imath t\mathcal{L}_\lambda}(\tilde{\pi} \circ j)[A]e^{-\imath t\mathcal{L}_\lambda}\Omega_\lambda^\beta = (\tilde{\pi} \circ j)[\tau_t^\lambda(A)]\Omega_\lambda^\beta = U(\pi_{AW} \circ i)[\tau_t^\lambda(A)]\Omega_f^\beta \\ &= \ Ue^{\imath t\mathcal{L}_f}(\pi_{AW} \circ i)[A]e^{\imath t\mathcal{L}_f}\Omega_f^\beta = Ue^{\imath t\mathcal{L}_f}(\pi_{AW} \circ i)[A]\Omega_f^\beta. \end{aligned} \tag{5.78}$$

That is $e^{\imath t\mathcal{L}_\lambda} U = Ue^{\imath t\mathcal{L}_f}$. Now, we extend U to an isometric map from $\mathrm{c}\ell\,\mathcal{H}_1$ onto $\mathrm{c}\ell\,\mathcal{H}_2$. The proof is complete, since $\mathrm{c}\ell\,\mathcal{H}_1 = \mathrm{c}\ell(\pi_{AW} \circ i)[\mathcal{A}]\Omega_f^\beta = \mathrm{c}\ell\,\pi_{AW}[W(\mathfrak{h})]\Omega_f^\beta = \mathcal{H}_f$ and $\mathrm{c}\ell\,\mathcal{H}_2 = \mathcal{K}$. □

Corollary 5.3.5. *1. The W^*-dynamical systems $(\mathfrak{M}, \tau^\lambda, \Omega_\lambda^\beta)$ and $(\mathfrak{M}_f, \tau^f, \Omega_f^\beta)$ are unitarily equivalent.*

2. $\mathcal{L}_\lambda$ is unitarily equivalent to $\mathcal{L}_f$ and $\mathrm{dom}(\mathcal{L}_\lambda) = U\,\mathrm{dom}(\mathcal{L}_f)$.

3. $\sigma(\mathcal{L}_\lambda) = \mathbb{R}$, $\sigma_{sc}(\mathcal{L}_\lambda) = \emptyset$, $\sigma_{ac}(\mathcal{L}_\lambda) = \mathbb{R} \setminus \{0\}$, $\sigma_{pp}(\mathcal{L}_\lambda) = \{0\}$ and Ω_λ^β is up to scalar multiples the only vector in the kernel of $\mathcal{L}_\lambda$. $(\mathfrak{M}, \tau^\lambda, \Omega_\lambda^\beta)$ is mixing.

Proof of 5.3.5. 1. Since $i(\mathcal{A}) = W(\mathfrak{f})$ and $(\widetilde{\pi} \circ j)[\mathcal{A}] \subset \mathfrak{M}$, we have

$$\gamma\,:\,\pi_{AW}[W(\mathfrak{f})] \to \mathfrak{M}. \tag{5.79}$$

Furthermore, since $\mathfrak{M}_f$ (resp. $\mathfrak{M}$) is the σ-weak closure of $\pi_{AW}[W(\mathfrak{f})]$ (resp. $(\widetilde{\pi} \circ j)[\mathcal{A}]$), and γ, γ^{-1} are σ-weakly continuous, we conclude that $\gamma\,:\,\mathfrak{M}_f \to \mathfrak{M}$ is a spatial $*$-isomorphism.

2. follows from $e^{\imath t\mathcal{L}_\lambda} = Ue^{\imath t\mathcal{L}_f}U^{-1}$.

3. The spectral of $\mathcal{L}_f$ well known. By Lemma 2.6.5 implies, that $(\mathfrak{M}, \tau^\lambda, \Omega_\lambda^\beta)$ is mixing.

□

5.4 Anharmonic Oscillator

In this section we consider an anharmonic oscillator in the dipole approximation, i.e. we replace in our model H_{el} by $H_{aosc} := H_{el} + V(x)$. The potential V is defined by

$$V(x) = \int_{\mathbb{R}} \nu(d\mu)e^{\imath x\mu}. \tag{5.80}$$

ν is a complex-valued Borel-measure, such that $\nu(A) = \overline{\nu(-A)}$ for any Borel set A, where $-A := \{-a \in \mathbb{R}\,:\,a \in A\}$. V is therefore real valued. Moreover, the conditions

$$a_i = \int_{\mathbb{R}} |\nu|(d\mu)|\mu|^i < \infty,\ i = 0, 1, 2, \tag{5.81}$$

where $|\nu|$ is the absolute value of ν and

$$\kappa > 2(a_0 + C_3a_2) \tag{5.82}$$

for $C_3 := 2\pi \int_{\mathbb{R}} \frac{\lambda^2 \left|\hat{\rho}(r+\imath\kappa)^2 (r+\imath\kappa)^2\right|}{\left|G_\lambda(r+\imath\kappa)\overline{G_\lambda(r-\imath\kappa)}\right|} dr$ have to be satisfied. This choice for V is due to Maassen [18], who studied the Langevin equation. V is also used by Spohn in a paper [25], in which a three dimensional harmonic oscillator, coupled to a field with polarization at temperature zero. For our reasons we define $P := \int_{\mathbb{R}} \nu(d\mu) W(\nu(\mu \oplus 0)) \in \mathfrak{M}_f$ and the corresponding standard Liouvillean

$$\mathcal{L}_P = \mathcal{L}_f + P - JPJ. \tag{5.83}$$

J is the modular conjugation corresponding to Ω_f^β. It is well known, that $\mathcal{L}_P$ is self-adjoint and $\tau_t^{\lambda,P}(A) := e^{\imath t \mathcal{L}_P} A e^{-\imath t \mathcal{L}_P} \in \mathfrak{M}_f$ for $A \in \mathfrak{M}_f$.

Moreover a $(\beta, \tau_t^{\lambda,P})$-KMS-state is given by $\omega_{\lambda,P}^\beta$, where $\omega_{\lambda,P}^\beta(A) := \langle \Omega_{\lambda,P}^\beta | A \Omega_{\lambda,P}^\beta \rangle_{\mathcal{H}_f}$ and $\Omega_{\lambda,P}^\beta := \| e^{-\beta/2(\mathcal{L}_f + P)} \Omega_f^\beta \|^{-1} e^{-\beta/2(\mathcal{L}_f+P)} \Omega_f^\beta$. It is also textbook knowledge, confer ([8] Thm. 5.4.4) and Theorem 2.5.6, that $\Omega_{\lambda,P}^\beta$ is cyclic and separating for $\mathfrak{M}_f$. We remark that

$$\tau_t^{\lambda,P}(A) = \tau_t^\lambda(A) + \sum_{n=1}^\infty \imath^n \int_{0 \le t_n \le \ldots t_1 \le t} d\underline{t}\, [\tau_{t-t_1}^\lambda(P), [\ldots [\tau_{t-t_n}^\lambda(P), \tau_t^\lambda(A)] \ldots]]. \tag{5.84}$$

Theorem 5.4.1. *For $A, B, C \in i(\mathcal{A}_{anal})$ we have*

$$\left| \omega_\lambda^\beta(A \tau_t^{\lambda,P}(B) C) - \omega_\lambda^\beta(AC) \widetilde{\omega_{\lambda,P}^\beta}(B) \right| \le \text{const} \exp(-(\kappa - 2(a_0 + C_3 a_2))t), \tag{5.85}$$

where $C_3 := 2\pi \int_{\mathbb{R}} \frac{\lambda^2 \left|\hat{\rho}(r+\imath\kappa)^2 (r+\imath\kappa)^2\right|}{\left|G_\lambda(r+\imath\kappa)\overline{G_\lambda(r-\imath\kappa)}\right|} dr$ and $\widetilde{\omega_{\lambda,P}^\beta}(B) = \lim_{t\to\infty} \omega_\lambda^\beta(\tau_t^{\lambda,P}(B))$.

Proof of 5.4.1. It suffices to assume $A = W(c_1 \oplus f_1)$, $B = W(c_2 \oplus f_2)$ and $C = W(c_3 \oplus f_3)$.

$$\begin{aligned} \omega_\lambda^\beta(A \tau_t^{\lambda,P}(B) C) &= \omega_\lambda^\beta(A \tau_t^\lambda(B) C) \\ &+ \sum_{n=1}^\infty \imath^n \int_{0 \le t_n \le \ldots t_1 \le t} d\underline{t}\, \omega_\lambda^\beta \Big(W(v(c_1 \oplus f_1)) [W(e^{\imath(t-t_1)|k|} v(\mu_1 \oplus 0)), [\ldots \\ &[W(e^{\imath(t-t_n)|k|} v(\mu_n \oplus 0)), W(e^{\imath t|k|} v(c_2 \oplus f_2))] \ldots]] W(v(c_3 \oplus f_3)) \Big). \end{aligned} \tag{5.86}$$

For the commutator of Weyl operators we obtain

$$[W(f), W(g)] = 2\imath \sin\big((1/2) \Im \langle f | g \rangle_{\mathfrak{h}}\big) \cdot W(f+g). \tag{5.87}$$

Let

$$\begin{aligned} C(\underline{t}, n, \underline{\mu}) \;\; := \;\; & (2\imath)^n \prod_{k=1}^n \sin\Big(\sum_{m=1}^{k-1} \Im \langle e^{\imath t_k |k|} v(\mu_k \oplus 0) | e^{\imath t_m |k|} v(\mu_m \oplus 0) \rangle_{\mathfrak{h}} / 2 \\ & + \Im \langle e^{-\imath t_k |k|} v(\mu_k \oplus 0) | v(c_2 \oplus f_2) \rangle_{\mathfrak{h}} / 2 \Big) \end{aligned} \tag{5.88}$$

and

$$g(\underline{t}, n, \underline{\mu}) := \sum_{m=1}^{n} e^{-\imath t_m |k|} v(\mu_m \oplus 0) + v(c_2 \oplus f_2). \tag{5.89}$$

We obtain

$$\begin{aligned} & C(\underline{t}, n, \underline{\mu}) \cdot W(e^{\imath t|k|} g(\underline{t}, n, \underline{\mu})) \\ = \; & [W(e^{\imath(t-t_1)|k|} v(\mu_1 \oplus 0)), [\ldots [W(e^{\imath(t-t_1)|k|} v(\mu_1 \oplus 0)), W(e^{\imath t|k|} v(c_2 \oplus f_2))] \ldots]]. \end{aligned} \tag{5.90}$$

Therefore,

$$\begin{aligned} \omega_\lambda^\beta(A\tau_t^{\lambda,P}(B)C) &= \omega_\lambda^\beta(A\tau_t^\lambda(B)C) \\ & + \sum_{n=1}^{\infty} \imath^n \int_{0 \le t_n \le \ldots t_1 \le t} d\underline{t} \int \nu(d\underline{\mu}) C(\underline{t}, n, \underline{\mu}) \\ & \omega_\lambda^\beta\Big(W(v(c_1 \oplus f_1)) W(e^{\imath t|k|} g(\underline{t}, n, \underline{\mu})) W(v(c_3 \oplus f_3))\Big) \\ = \; & \omega_\lambda^\beta(A\tau_t^\lambda(B)C) + \sum_{n=1}^{\infty} \imath^n \int_{0 \le t_n \le \ldots t_1 \le t} d\underline{t} \int \nu(d\underline{\mu}) \\ & \omega_\lambda^\beta\big(W(v(c_1 \oplus f_1)) W(v(c_3 \oplus f_3))\big) \omega_\lambda^\beta\big(W(g(t, n, \underline{\mu}))\big) \\ & C(\underline{t}, n, \underline{\mu}) \cdot \exp(\Delta(t, \underline{t}, n, \underline{\mu})), \end{aligned} \tag{5.91}$$

where

$$\begin{aligned} \Delta(t, \underline{t}, n, \underline{\mu}) \; := \; & -(1/2) \Re \langle v(c_1 \oplus f_1) + v(c_3 \oplus f_3) | (1 + 2\varrho) e^{\imath t|k|} g(\underline{t}, n, \underline{\mu}) \rangle_{\mathfrak{h}} \\ & -(\imath/2) \Im \langle v(c_1 \oplus f_1) - v(c_3 \oplus f_3) | e^{\imath t|k|} g(t, n, \underline{\mu}) \rangle_{\mathfrak{h}}. \end{aligned} \tag{5.92}$$

Due to Theorem (5.2.6) we have

$$|\omega_\lambda^\beta(A\tau_t^\lambda(B)C) - \omega_\lambda^\beta(AC)\omega_\lambda^\beta(\tau_t^\lambda(B))| \le C_0 e^{-\kappa t}. \tag{5.93}$$

From $|e^{\Delta(t,\underline{t},n,\underline{\mu})} - 1| \le |\Delta(t, \underline{t}, n, \underline{\mu})| e^{|\Re\Delta(t,\underline{t},n,\underline{\mu})|}$ and

$$\Big| \omega_\lambda^\beta\big(W(v(c_1 \oplus f_1)) W(v(c_3 \oplus f_3))\big) \omega_\lambda^\beta(W(g(\underline{t}, n, \underline{\mu}))) \exp(|\Re\Delta(t, \underline{t}, n, \underline{\mu})|) \Big| \le 1$$

follows that

$$\begin{aligned} & |\omega_\lambda^\beta(A\tau_t^{\lambda,P}(B)C) - \omega_\lambda^\beta(AC)\omega_\lambda^\beta(\tau_t^{\lambda,P}(B))| \\ \le \; & C_0 e^{-\kappa t} + \sum_{n=1}^{\infty} \int_{0 \le t_n \le \ldots t_1 \le t} d\underline{t} \int |\nu|(d\underline{\mu}) \, \big| C(\underline{t}, n, \underline{\mu}) \Delta(t, \underline{t}, n, \underline{\mu}) \big|. \end{aligned} \tag{5.94}$$

We observe

$$|\Delta(t,\underline{t},n,\underline{\mu})| \leq C_1(\sum_{i=1}^{n} e^{-(t-t_i)\kappa}|\mu_i| + e^{-t\kappa}) \tag{5.95}$$

for some $C_1 > 0$. Furthermore, Maassen's estimate (A.0.4) yields

$$|C(\underline{t},n,\underline{\mu})| \leq C_2 e^{-\kappa t_i}|\mu_i| \prod_{k=1}^{i-1}(1+C_3\mu_k^2), \tag{5.96}$$

where $C_2 > 0$ and $C_3 \geq 1$, C_3 is independent of $v(c_i \oplus f_i)$, $i = 1,2,3$.

$$\begin{aligned} & |\omega_\lambda^\beta(A\tau_t^{\lambda,P}(B)C) - \omega_\lambda^\beta(AC)\omega_\lambda^\beta(\tau_t^{\lambda,P}(B))| \\ \leq\ & C_0 e^{-\kappa t} + e^{-\kappa t} C_1 C_2 \sum_{n=1}^{\infty} \int_{0\leq t_n \leq \ldots t_1 \leq t} d\underline{t} \int |\nu|(d\underline{\mu}) \big(\sum_{i=2}^{n} \mu_i^2 \prod_{k=1}^{i-1}(1+C_3\mu_k^2) + 1 \big) \\ \leq\ & C_0 e^{-\kappa t} + e^{-\kappa t} C_1 C_2 \sum_{n=1}^{\infty} \int_{0\leq t_n \leq \ldots t_1 \leq t} d\underline{t} \int |\nu|(d\underline{\mu}) \prod_{k=1}^{n}(1+C_3\mu_k^2) \\ \leq\ & C_0 e^{-\kappa t} + C_1 C_2 \exp(-t(\kappa - a_0 - C_3 a_2)). \end{aligned} \tag{5.97}$$

Furthermore for $0 < t < s$

$$\begin{aligned} & |\omega_\lambda^\beta(\tau_s^{\lambda,P}(B)) - \omega_\lambda^\beta(\tau_t^{\lambda,P}(B))| \\ \leq\ & \sum_{n=1}^{\infty} \int_{0\leq t_n \leq \ldots \leq t_1 \leq s} d\underline{t}\ \mathbb{1}[t_1 \geq t] \int |\nu|(d\underline{\mu})\, |C(\underline{t},n,\underline{\mu})|. \end{aligned}$$

From Lemma A.0.4 follows

$$\begin{aligned} & \big|\omega_\lambda^\beta(\tau_s^{\lambda,P}(B)) - \omega_\lambda^\beta(\tau_t^{\lambda,P}(B))\big| \\ \leq\ & C_2 a_0 \sum_{n=1}^{\infty} 2^n \int_{0\leq t_n \leq \ldots \leq t_1 \leq s} d\underline{t}\ \mathbb{1}[t_1 \geq t](a_0 + C_3 a_2)^{n-1} e^{-\kappa t_1} \\ =\ & C_2 a_0 \sum_{n=1}^{\infty} 2^n \int_t^s \frac{(r(a_0+C_3a_2))^{n-1}}{(n-1)!} e^{-\kappa r} dr \\ \leq\ & 2C_2 a_0 \int_t^\infty \exp(-(\kappa - 2(a_0+C_3a_2))r)dr \\ \leq\ & \frac{2C_2a_0}{\kappa - 2(a_0+C_3a_2)} \exp(-(\kappa - 2(a_0+C_3a_2))t). \end{aligned} \tag{5.98}$$

We define $\widetilde{\omega_{\lambda,P}^\beta}(B) := \lim_{s\to\infty} \omega_\lambda^\beta(\tau_s^{\lambda,P}(B))$ using the Cauchy criterion. □

Lemma 5.4.2. $(\pi_{AW} \circ i)[\mathcal{A}_{anal}]'' = \mathfrak{M}_f$.

Proof. First, we observe ran $i = \mathrm{LH}\{W(f) \in W(\mathfrak{f}) : f \in \mathcal{H}_{anal}\}$. Since H_{anal} is dense in $M_0(\mathbb{R}^3) \cap M_{-1/2}(\mathbb{R}^3)$, for every $f \in M_0(\mathbb{R}^3) \cap M_{-1/2}(\mathbb{R}^3)$ exists a sequence $(f_n)_n \subset \mathcal{H}_{anal}$, so that

$$\sqrt{1+\varrho} f_n \oplus \sqrt{\varrho} \overline{f}_n \longrightarrow \sqrt{1+\varrho} f \oplus \sqrt{\varrho} \overline{f}, \quad n \to \infty,$$

where the limit is in the norm of $\mathfrak{h} \oplus \mathfrak{h}$. It follows, that

$$\text{s-lim}_{n\to\infty} W\big(\sqrt{1+\varrho} f_n \oplus \sqrt{\varrho} \overline{f}_n\big) = W(\sqrt{1+\varrho} f \oplus \sqrt{\varrho} \overline{f}).$$

We conclude, that $(\pi_{AW} \circ i)[\mathcal{A}_{anal}]'' = \mathrm{LH}\{W(\sqrt{1+\varrho} f \oplus \sqrt{\varrho} \overline{f}) : f \in \mathfrak{f}\}'' = \mathfrak{M}_f$. ☐

Corollary 5.4.3. *For every $\omega^\beta_{\lambda,P}$-normal state μ over $\mathfrak{M}_f$ and every $C \in \mathfrak{M}_f$ we obtain*

$$\lim_{t\to\infty} \mu(\tau^{\lambda,P}_t(C)) = \omega^\beta_{\lambda,P}(C).$$

Hence $\omega^\beta_{\lambda,P}$ is mixing.

Proof of 5.4.3. Let now $\phi \in \mathcal{H}_f$, $\|\phi\| = 1$ and A, $B \in (\pi_{AW} \circ i)[\mathcal{A}_{anal}]$, so that $\|A\Omega^\beta_\lambda\| = 1$.

$$\begin{aligned}
&|\langle\phi|\tau^{\lambda,P}_t(B)\phi\rangle_{\mathcal{H}_f} - \widetilde{\omega^\beta_{\lambda,P}}(B)| \qquad (5.99)\\
\leq\ & |\langle\phi|\tau^{\lambda,P}_t(B)\phi\rangle_{\mathcal{H}_f} - \langle A\Omega^\beta_f|\tau^{\lambda,P}_t(B)A\Omega^\beta_f\rangle| + |\omega^\beta_\lambda(A^*\tau^{\lambda,P}_t(B)A) - \widetilde{\omega^\beta_{\lambda,P}}(B)|\\
\leq\ & 2\|A\Omega^\beta_f - \phi\| \cdot \|B\| + |\omega^\beta_\lambda(A^*\tau^{\lambda,P}_t(B)A) - \widetilde{\omega^\beta_{\lambda,P}}(B)|
\end{aligned}$$

We obtain directly

$$\begin{aligned}
&\limsup_{t\to\infty} |\langle\phi|\tau^{\lambda,P}_t(B)\phi\rangle_{\mathcal{H}_f} - \widetilde{\omega^\beta_{\lambda,P}}(B)| \qquad (5.100)\\
\leq\ & 2\|B\| \inf\{\|A\Omega^\beta_f - \phi\| \mid A \in (\pi_{AW} \circ i)[\mathcal{A}_{anal}],\ \|A\Omega^\beta_f\| = 1\} = 0
\end{aligned}$$

Choosing $\phi = \Omega^\beta_{\lambda,P}$ we obtain by time-invariance of KMS-states, that $\omega^\beta_{\lambda,P} = \widetilde{\omega^\beta_{\lambda,P}}$ over $(\pi_{AW} \circ i)[\mathcal{A}_{anal}]$. Assume now $C \in \mathfrak{M}_f, (C_n)_n \in (\pi_{AW} \circ i)[\mathcal{A}_{anal}]$ and $D \in \mathfrak{M}'_f$, so that $C = \text{s-lim}_{n\to\infty} C_n$. We have

$$\langle D\Omega^\beta_{\lambda,P}|\tau^{\lambda,P}_t(C - C_n)D\Omega^\beta_{\lambda,P}\rangle = \langle D^*D\Omega^\beta_{\lambda,P}|\tau^{\lambda,P}_t(C - C_n)\Omega^\beta_{\lambda,P}\rangle = \langle D^*D\Omega^\beta_{\lambda,P}|e^{\imath t\mathcal{L}_P}(C - C_n)\Omega^\beta_{\lambda,P}\rangle \qquad (5.101)$$

Hence,

$$\begin{aligned}
&\limsup_{t\to\infty} |\omega^\beta_{\lambda,P}(C) - \langle D\Omega^\beta_{\lambda,P}|\tau_t^{\lambda,P}(C)D\Omega^\beta_{\lambda,P}\rangle| \\
&\leq \limsup_{t\to\infty} |\omega^\beta_{\lambda,P}(C_n) - \langle D\Omega^\beta_{\lambda,P}|\tau_t^{\lambda,P}(C_n)D\Omega^\beta_{\lambda,P}\rangle| \\
&\quad + \limsup_{t\to\infty} |\omega^\beta_{\lambda,P}(C-C_n) - \langle D\Omega^\beta_{\lambda,P}|\tau_t^{\lambda,P}(C-C_n)D\Omega^\beta_{\lambda,P}\rangle| \\
&\leq (1+\|D^*D\|)\cdot\|(C-C_n)\Omega^\beta_{\lambda,P}\|.
\end{aligned} \tag{5.102}$$

For $n \to \infty$ we obtain $\omega^\beta_{\lambda,P}(C) = \lim_{t\to\infty}\langle D\Omega^\beta_{\lambda,P}|\tau_t^{\lambda,P}(C)D\Omega^\beta_{\lambda,P}\rangle$ Since $\Omega^\beta_{\lambda,P}$ is separating for $\mathfrak{M}_f$, it is cyclic for $(\mathfrak{M}_f)'$. A simple approximation argument yields now

$$\omega^\beta_{\lambda,P}(C) = \lim_{t\to\infty}\langle\phi|\tau_t^{\lambda,P}(C)\phi\rangle \tag{5.103}$$

for all $\phi \in \mathcal{H}_f$. Apply now Theorem 2.3.3. □

Appendix A

Two Additional Proofs

Lemma A.0.4 (Maassen,[18]). *Let* $f_0, \ldots, f_n$ *be vectors in a Hilbert space* $\mathfrak{h}$*, and real numbers* $0 \le t_1 \le t_2 \le \ldots t_n$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ *and* $\alpha, \beta, \gamma > 0$*, such that*

$$|\Im\langle f_k|f_0\rangle| \le \alpha|\lambda_k|e^{-t_k\gamma}, \quad k > 0 \tag{A.1}$$

$$|\Im\langle f_k|f_j\rangle| \le \beta|\lambda_k\lambda_j| \cdot e^{-(t_k-t_j)\gamma}, \quad k > j > 0. \tag{A.2}$$

Then,

$$\Big|\prod_{k=1}^{n} \sin\big(\sum_{j=0}^{k-1} \Im\langle f_k|f_j\rangle\big)\Big| \le e^{-\gamma t_n}|\lambda_n|\alpha \prod_{k=1}^{n-1}(1+\beta\lambda_k^2). \tag{A.3}$$

Proof of A.0.4. First we remark that for $x, y \in \mathbb{R}$

$$|\sin(x+y)| \le |\sin(x)| + |\sin(y)|, \quad |\sin(x)| \le |x|, \quad |\sin(x)| \le 1. \tag{A.4}$$

We proceed by induction for n. Assume the statement of Lemma A.0.4 holds for $j = 1, \ldots, n-1$.

$$\begin{aligned}
&\Big|\prod_{k=1}^{n} \sin\big(\sum_{j=0}^{k-1} \Im\langle f_k|f_j\rangle\big)\Big| \qquad\qquad (A.5)\\
&= \Big|\sin\big(\sum_{j=0}^{n-1} \Im\langle f_n|f_j\rangle\big)\Big| \cdot \Big|\prod_{k=1}^{n-1} \sin\big(\sum_{j=0}^{k-1} \Im\langle f_k|f_j\rangle\big)\Big|\\
&\le \sum_{j=1}^{n-1} |\sin(\Im\langle f_n|f_j\rangle)| \cdot \prod_{k=1}^{j-1} \Big|\sin\big(\sum_{j=0}^{k-1} \Im\langle f_k|f_j\rangle\big)\Big| + \big|\sin\big(\Im\langle f_n|f_0\rangle\big)\big|\\
&\le \sum_{j=1}^{n-1} \Big(\beta|\lambda_n\lambda_j| \cdot e^{-(t_n-t_j)\gamma}\Big) \cdot \Big(e^{-\gamma t_j}|\lambda_j|\alpha \prod_{k=1}^{j-1}(1+\beta\lambda_k^2)\Big) + \alpha|\lambda_n|e^{-t_n\gamma}.
\end{aligned}$$

Since the right-hand side (r.h.s) of Equation (A.5) is less than the (r.h.s) of Equation (A.3). □

Theorem A.0.5 (Wick's Theorem). *Let ω be a regular state over $W(\mathfrak{f})$.*

1. *Assume*

$$\mathbb{R}^n \ni (t_1,\dots,t_n) \mapsto \omega\big(W(t_n f_n)\cdots W(t_1 f_1)\big) \tag{A.6}$$

is smooth, then for $k = 1,\dots,n$

$$\Omega_\omega \in \mathrm{dom}(\Phi_\omega(f_1)), \quad \Phi_\omega(f_{k-1})\cdots\Phi_\omega(f_1)\Omega_\omega \in \mathrm{dom}(\Phi_\omega(f_k))$$

2. *If ω is quasi-free, then*

$$\langle\Omega_\omega|\Phi_\omega(f_1)\cdots\Phi_\omega(f_{2n-1})\Omega_\omega\rangle = 0$$
$$\langle\Omega_\omega|\Phi_\omega(f_1)\cdots\Phi_\omega(f_{2n})\Omega_\omega\rangle = \sum_{P\in\mathcal{P}_{2n}}\prod_{\{k<l\}\in P}\langle\Omega_\omega|\Phi_\omega(f_k)\Phi_\omega(f_l)\Omega_\omega\rangle$$

Proof of A.0.5. Let

$$\begin{aligned} & s_n(t_{2n},\dots,t_1) \\ := \; & \omega\big(W(-t_{n+1}f_{n+1})\cdots W(-t_{2n}f_{2n})W(t_n f_n)\cdots W(t_1 f_1)\big) \end{aligned} \tag{A.7}$$

and

$$S_n(t_n,\dots,t_1) := \pi_\omega(W(t_n f_n))\cdots\pi_\omega(W(t_1 f_1))\Omega_\omega. \tag{A.8}$$

We want to show that $\partial_{t_n}\cdots\partial_{t_1}S_n(t_n,\dots,t_1)$ exists if s_n is smooth.
We proceed by induction for n. For $n=0$ is nothing to show. Let

$$r_{n+1}(t_{n+1}) = \partial_{t_{2n+1}}\cdots\partial_{t_{n+2}}\partial_{t_n}\cdots\partial_{t_1}s_{n+1}(t_{n+1}\dots,t_1,t_{n+1}\dots,t_1) \tag{A.9}$$

and

$$R_{n+1}(t_{n+1}) = \partial_{t_n}\cdots\partial_{t_1}S_{n+1}(t_{n+1},\dots,t_1). \tag{A.10}$$

We want to show that $R_{n+1}(t_{n+1})$ is differentiable in 0. Cauchy's criterion for the differential quotient reads

$$\begin{aligned} & \|x^{-1}\big(R_{n+1}(x)-R_{n+1}(0)\big) - y^{-1}\big(R_{n+1}(y)-R_{n+1}(0)\big)\|^2 \\ = \; & x^{-2}\|R_{n+1}(x)\|^2 + y^{-2}\|R_{n+1}(y)\|^2 + (x^{-1}-y^{-1})^2\|R_{n+1}(0)\|^2 \\ & -2x^{-1}y^{-1}\Re\langle R_{n+1}(x)|R_{n+1}(y)\rangle - 2x^{-1}(x^{-1}-y^{-1})\Re\langle R_{n+1}(x)|R_{n+1}(0)\rangle \\ & +2y^{-1}(x^{-1}-y^{-1})\Re\langle R_{n+1}(y)|R_{n+1}(0)\rangle \end{aligned} \tag{A.11}$$

per induction we obtain $\langle R_{n+1}(t_{n+1})|R_{n+1}(t_{n+1})\rangle = s_{n+1}(t_{n+1})$.

$$\begin{aligned}&\|x^{-1}\big(R_{n+1}(x)-R_{n+1}(0)\big)-y^{-1}\big(R_{n+1}(y)-R_{n+1}(0)\big)\|^2 \\ =\; & (x^{-2}+y^{-2}+(x^{-1}-y^{-1})^2)s_{k+1}(0)-x^{-1}y^{-1}(r_{n+1}(x-y)+r_{n+1}(y-x)) \\ & -x^{-1}(x^{-1}-y^{-1})(r_{n+1}(x)+r_{n+1}(-x)) \\ & +y^{-1}(x^{-1}-y^{-1})(r_{n+1}(y)+r_{n+1}(-y)).\end{aligned} \tag{A.12}$$

We define $\Delta(x)=x^{-2}(r_{n+1}(x)+r_{n+1}(-x)-2r_{n+1}(0)),\ x\neq 0$ and get

$$\begin{aligned}&\|x^{-1}\big(R_{n+1}(x)-R_{n+1}(0)\big)-y^{-1}\big(R_{n+1}(y)-R_{n+1}(0)\big)\|^2 \\ =\; & -x^{-1}y^{-1}(x-y)^2\Delta(x-y)-x(x^{-1}-y^{-1})\Delta(x)+y(x^{-1}-y^{-1})\Delta(y) \\ =\; & xy^{-1}(\Delta(x)-\Delta(x-y))+x^{-1}y(\Delta(y)-\Delta(x-y)).\end{aligned} \tag{A.13}$$

Since r_{n+1} is smooth, Δ extends to a function in $\mathcal{C}^1(\mathbb{R};\mathbb{C})$. From $\Delta(x)=\Delta(-x)$ follows, that the (r.h.s) of (A.13) tends to zero if $x,y\to 0$.
Because $[0,\infty)\ni r\mapsto\sqrt{r}$ is uniformly continuous, $\partial_{t_{n+1}}R_{n+1}(0)$ exists.
As a result of the differentiability of $S_n(t)$ in zero, we obtain

$$\Phi_\omega(f_{n-1})\cdots\Phi_\omega(f_1)\Omega_\omega\in\mathrm{dom}(\Phi_\omega(f_n)). \tag{A.14}$$

Let now ω be quasi-free. Hence

$$\begin{aligned}\omega(s_1,\ldots,s_n) &:= \omega(W(s_1f_1)\cdots W(s_nf_n)) \\ &= \prod_{k=1}^{n-1}\exp\Big((-(\imath/2)\sum_{l=k+1}^{n}\Im\langle f_k,f_l\rangle s_ks_l\Big)\omega\Big(W(\sum_{l=1}^{n}s_lf_l)\Big) \\ &= \prod_{k=1}^{n-1}\exp\Big((-(\imath/2)\sum_{l=k+1}^{n}\Im\langle f_k,f_l\rangle s_ks_l\Big)\prod_{k,l=1}^{n}\exp(-q(f_k,f_l)s_ks_l) \\ &= \exp\Big(-\sum_{1\le k<l\le n}\omega_{kl}s_ks_l-\sum_{k=1}^{n}\omega_{kk}s_k^2\Big).\end{aligned} \tag{A.15}$$

where $\omega_{kl}:=\big(\imath\Im\langle f_k,f_l\rangle/2+q(f_k,f_l)+q(f_l,f_k)\big)$, if $k<l$, and $\omega_{kk}:=q(f_k,f_k)$.
Let Pow_m be the power set of $\{1,\ldots,m\}$ and

$$\mathcal{P}_{2n}=\big\{P\subset\mathrm{Pow}_{2n}\ :\ P:=\{A_1,\ldots,A_n\} \tag{A.16}$$

$$A_i\cap A_j=\emptyset\,\text{for}\ i\neq j,\ \#A_i=2\big\} \tag{A.17}$$

the set of pairings of $\{1,\ldots,2n\}$. We obtain

$$\partial_{s_{2n-1}}\cdots\partial_{s_1}\omega(s_1,\ldots,s_{2n-1})\Big|_{\underline{s}=0} = 0 \tag{A.18}$$

$$\partial_{s_{2n}}\cdots\partial_{s_1}\omega(s_1,\ldots,s_{2n})\Big|_{\underline{s}=0} = (-1)^n \sum_{P\in\mathcal{P}_{2n}}\prod_{\{k<l\}\in P}\omega_{kl} \tag{A.19}$$

Proof by induction for n.

$$\partial_{s_1}\omega(s_1)\big|_{s_1=0} = -2\omega_{11}s_1\exp(-\omega_{11}s_1^2)\big|_{s_1=0} = 0 \tag{A.20}$$

$$\partial_{s_1}\partial_{s_2}\omega(s_1,s_2)\big|_{s_1,s_2=0} = \partial_{s_1}(-\omega_{12}s_1)\exp(-\omega_{11}s_1^2)\Big|_{s_1=0} = -\omega_{12}$$

Assume it is true for n.

$$\begin{aligned}
&\partial_{s_{2(n+1)-1}}\cdots\partial_{s_1}\omega(s_1,\ldots,s_{2(n+1)-1})\Big|_{\underline{s}=0} \\
&= \partial_{s_{2n}}\cdots\partial_{s_1}\Big(-\sum_{k=1}^{2n}\omega_{k,2n+1}s_k - 2\omega_{2n+1,2n+1}s_{2n+1}\Big) \\
&\quad \exp\Big(-\sum_{1\le k<l\le 2n+1}\omega_{kl}s_ks_l - \sum_{k=1}^{2n+1}\omega_{kk}s_k^2\Big)\Big|_{\underline{s}=0} \\
&= \partial_{s_{2n}}\cdots\partial_{s_1}\Big(-\sum_{m=1}^{2n}\omega_{m,2n+1}s_m\Big)\exp\Big(-\sum_{1\le k<l\le 2n}\omega_{kl}s_ks_l - \sum_{k=1}^{2n}\omega_{kk}s_k^2\Big)\Big|_{\underline{s}=0} \\
&= -\sum_{m=1}^{2n}\omega_{m,2n+1}s_m\partial_{s_{2n}}\cdots\partial_{s_1}\exp\Big(-\sum_{1\le k<l\le 2n}\omega_{kl}s_ks_l - \sum_{k=1}^{2n}\omega_{kk}s_k^2\Big)\Big|_{\underline{s}=0} \\
&\quad -\sum_{m=1}^{2n}\omega_{m,2n+1}\partial_{s_{2n}}\cdots\widehat{\partial_{s_m}}\cdots\partial_{s_1}\exp\Big(-\sum_{1\le k<l\le 2n}\omega_{kl}s_ks_l - \sum_{k=1}^{2n}\omega_{kk}s_k^2\Big)\Big|_{\underline{s}=0}
\end{aligned} \tag{A.21}$$

The first term of the (r.h.s) of Equation (A.21) is zero, since $\underline{s}=0$ The second term is zero by induction since only $2n-1$ derivatives are taken into account. Analogously, we obtain

$$\begin{aligned}
&\partial_{s_{2(n+1)}}\cdots\partial_{s_1}\omega(s_1,\ldots,s_{2(n+1)})\Big|_{\underline{s}=0} \\
&= -\sum_{m=1}^{2n+1}\omega_{m,2n+2}\partial_{s_{2n+1}}\cdots\widehat{\partial_{s_m}}\cdots\partial_{s_1}\exp\Big(-\sum_{\substack{1\le k<l\le 2n+1\\ k,l\neq m}}\omega_{kl}s_ks_l - \sum_{\substack{k=1\\ k\neq m}}^{2n+1}\omega_{kk}s_k^2\Big)\Big|_{\underline{s}=0}.
\end{aligned} \tag{A.22}$$

Let

$$\mathcal{P}^m_{2n+1} = \big\{P\subset \mathrm{Pow}_{2n+1}\ :\ P := \{A_1,\ldots,A_n\} \tag{A.23}$$

$$A_i\cap A_j=\emptyset \text{ for } i\neq j,\ \#A_i = 2,\ \forall_{i=1}^n\, m\in A_i\big\} \tag{A.24}$$

the set of pairings of $\{1,\ldots,2n+1\}\setminus\{m\}$. By induction we obtain for the (r.h.s.) of Equation (A.22) that

$$\partial_{s_{2(n+1)}}\cdots\partial_{s_1}\omega(s_1,\ldots,s_{2(n+1)})\Big|_{\underline{s}=0} \tag{A.25}$$

$$= -\sum_{m=1}^{2n+1}\omega_{m,2n+2}(-1)^n\sum_{P\in\mathcal{P}^m_{2n+1}}\prod_{\{k<l\}\in P}\omega_{kl}. \tag{A.26}$$

Since

$$(-\imath)^n\partial_{s_n}\cdots\partial_{s_1}\omega(s_1,\ldots,s_n)\Big|_{\underline{s}=0} = \langle\Omega_\omega|\Phi_\omega(f_1)\cdots\Phi_\omega(f_n)\Omega_\omega\rangle \tag{A.27}$$

and $\omega_{kl}=\langle\Omega_\omega|\Phi_\omega(f_k)\Phi_\omega(f_l)\Omega_\omega\rangle$ the proof is complete. □

Appendix B

Operator Theory

For a bounded self-adjoint operator on a Hilbert space $\mathfrak{h}$ we denote by $|A| := (A^*A)^{1/2}$ absolute value of A. $|A|$ is obviously positive and $A = |A|$, whenever A is positive. From now on we assume that $\mathfrak{h}$ is separable.

Definition B.0.6 (Trace of positive Operators)**.** *Let A be a positive, bounded operator on $\mathfrak{h}$ and $(\phi_n)_{n=1}^{\infty}$ any ONB of $\mathfrak{h}$.*

$$\mathrm{Tr}\{A\} := \sum_{n=1}^{\infty} \langle \phi_n | A\phi_n \rangle \tag{B.1}$$

is the trace of A, it is independent of the choice of $(\phi_n)_{n=1}^{\infty}$. If A has pure point spectrum, then $\mathrm{Tr}\{A\}$ *is the sum of the eigenvalues of A counted with multiplicity*

Definition B.0.7 (Operators of Schatten p-Class)**.** *For $1 \leq p < \infty$ we define for $A \in \mathcal{B}(h)$*

$$\|A\|_p = \mathrm{Tr}\{|A|^p\}^{1/p} \in [0, \infty]$$

and $\|A\|_\infty$ is the operator norm. Furthermore, for $1 \leq p < \infty$

$$\mathcal{L}^p(\mathfrak{h}) := \{A \in \mathcal{B}(\mathfrak{h}) \,:\, \|A\|_p < \infty\}. \tag{B.2}$$

Lemma B.0.8. *Let $1 \leq p < \infty$.*

1. *$(\mathcal{L}^p(\mathfrak{h}), \|\cdot\|_p)$ is a Banach space. $A \in \mathcal{L}^p(\mathfrak{h})$ implies $A^* \in \mathcal{L}^p(\mathfrak{h})$. Moreover $\|A^*\|_p = \|A\|_p$.*

2. *Assume $p, q, r \in [1, \infty]$ and $p^{-1} + q^{-1} = r^{-1}$. For $A \in \mathcal{L}^p(\mathfrak{h})$ and $B \in \mathcal{L}^q(\mathfrak{h})$ we have $AB \in \mathcal{L}^r(\mathfrak{h})$ and $\|AB\|_r \leq \|A\|_p \cdot \|B\|_q$. Only for this definition we set $\mathcal{L}^\infty(\mathfrak{h}) := \mathcal{B}(\mathfrak{h})$.*

3. *For $A \in \mathcal{L}^1(\mathfrak{h})$ one can define a bounded linear functional, the trace,* $\mathrm{Tr} : \mathcal{L}^1(\mathfrak{h}) \to \mathbb{C}$, *by*

$$\mathrm{Tr}\{A\} := \sum_{n=1}^{\infty} \langle \phi_n | A\phi_n \rangle_{\mathfrak{h}}.$$

where $(\phi_n)_{n=1}^{\infty}$ is an arbitrary ONB of $\mathfrak{h}$. The definition of Tr *is independent of $(\phi_n)_{n=1}^{\infty}$. $\mathcal{L}^1(\mathfrak{h})$ are the operators of trace class. For $A, B \in \mathcal{B}(\mathfrak{h})$ the trace is cyclic, that means*

$$AB,\, BA \in \mathcal{L}^1(\mathfrak{h}) \Rightarrow \mathrm{Tr}\{AB\} = \mathrm{Tr}\{BA\}. \tag{B.3}$$

4. *$(\mathcal{L}^2(\mathfrak{h}), \|\cdot\|_2)$ is a Hilbert space equipped with the scalar product $\langle A|B\rangle_2 := \mathrm{Tr}\{A^*B\}$. We have*

$$\|A\|_2 = \mathrm{Tr}\{A^*A\}^{1/2} = \Big(\sum_{n=1}^{\infty} \|A\phi_n\|_{\mathfrak{h}}^2\Big)^{1/2}.$$

for every ONB $(\phi_n)_{n=1}^{\infty}$ of $\mathfrak{h}$. $(\mathcal{L}^2(\mathfrak{h}), \|\cdot\|_2)$ are the Hilbert-Schmidt operators. If $\mathfrak{h} = L^2(X, \mu)$ and $A \in \mathcal{L}^2(\mathfrak{h})$ then exists an uniquely determined $k \in L^2(X \times X, \mu \otimes \mu)$ with $(A\phi)(x) = \int k(x,y)\phi(y)\mu(dy)$ for μ-almost every $x \in X$ and all $\phi \in \mathfrak{h}$. Moreover $\|A\|_2 = \|k\|_{L^2(X\times X, \mu\otimes\mu)}$.

Proof of B.0.8. The statements are textbook knowledge. The proof is partly in ([21],Chapter, VII.6) and in ([24], Thm. 2.7 and Thm. 2.8) □

Theorem B.0.9 (Golden-Thompson-Inequality)**.** *Let A, B be self-adjoint operators, bounded below. Assume, that $A + B$ is essentially self-adjoint on* $\mathrm{dom}(A) \cap \mathrm{dom}(B)$. *Then*

$$\|e^{-(A+B)}\|_p \leq \|e^{-A/2}e^{-B}e^{-A/2}\|_p, \quad 1 \leq p < \infty. \tag{B.4}$$

Proof of B.0.9. For the proof see ([24], Thm 8.5). Note, that one can prove the theorem under weaker assumptions. □

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