REPRESENTATION THEOREMS FOR INDEFINITE QUADRATIC FORMS AND APPLICATIONS

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Contents

Abstract

This thesis is devoted to the Representation Theorems for symmetric indefinite (that is non-semibounded) sesquilinear forms and their applications. In particular, we consider the case where the operator associated with the form does not have a spectral gap around zero.

Furthermore, the relation between reducing graph subspaces, solutions to operator Riccati equations, and block diagonalisation of diagonally dominant block operator matrices is investigated.

By means of the Representation Theorems, a corresponding relation is established for operators associated with indefinite forms and form Riccati equations. In this framework, an explicit block diagonalisation and a spectral decomposition of the Stokes operator as well as a representation for its kernel are obtained.

We apply the Representation Theorems to forms given by $\langle \operatorname{grad} u, h(\cdot) \operatorname{grad} v \rangle$, where the coefficient matrices $h(\cdot)$ are allowed to be sign-indefinite. As a result, indefinite self-adjoint differential operators $\operatorname{div} h(\cdot) \operatorname{grad}$ with homogeneous Dirichlet or Neumann boundary conditions are constructed. Examples of such kind are operators related to the modelling of optical metamaterials and left-indefinite Sturm-Liouville operators.

Zusammenfassung

Diese Arbeit widmet sich den Darstellungssätzen für symmetrische indefinite (das heißt nicht-halbbeschränkte) Sesquilinearformen und deren Anwendungen. Insbesondere betrachten wir den Fall, dass der zur Form assoziierte Operator keine Spektrallücke um Null besitzt.

Desweiteren untersuchen wir die Beziehung zwischen reduzierenden Graphräumen, Lösungen von Operator-Riccati-Gleichungen und der Block-Diagonalisierung für diagonaldominante Block-Operator-Matrizen.
Mit Hilfe der Darstellungssätze wird eine entsprechende Beziehung zwischen Operatoren, die zu indefiniten Formen assoziiert sind, und Form-Riccati-Gleichungen erreicht. In diesem Rahmen wird eine explizite Block-Diagonalisierung und eine Spektralzerlegung für den Stokes Operator sowie eine Darstellung für dessen Kern erreicht.

Wir wenden die Darstellungssätze auf durch $\langle \operatorname{grad} u, h(\cdot) \operatorname{grad} v \rangle$ gegebene Formen an, wobei Vorzeichen-indefinite Koeffizienten-Matrizen $h(\cdot)$ zugelassen sind. Als ein Resultat werden selbstadjungierte indefinite Differentialoperatoren $\operatorname{div} h(\cdot) \operatorname{grad}$ mit homogenen Dirichlet oder Neumann Randbedingungen konstruiert. Beispiele solcher Art sind Operatoren die in der Modellierung von optischen Metamaterialien auftauchen und links-indefinite Sturm-Liouville Operatoren.

Introduction

In this thesis, we investigate indefinite (that is non-semibounded) quadratic forms and their representation by self-adjoint operators. We focus on forms related to operators without spectral gap around zero.

In particular, we use these representations to investigate the Stokes operator together with its block diagonalisation, as well as self-adjoint differential operators $\operatorname{div} h(\cdot) \operatorname{grad}$ of second order with indefinite coefficient matrices $h(\cdot)$.

This thesis can be subdivided into five parts.

In the first Part, Chapters one to three, we prove new variants of the classical First and Second Representation Theorem for indefinite symmetric sesquilinear forms where the associated operator does not have a gap around zero. Furthermore, we determine the kernel of the associated operators.

The second Part, Chapter four, is an intermezzo in the operator framework, where we consider the relation between reducing graph subspaces, solutions to operator Riccati equations, and diagonalisations for block operator matrices. We obtain a new diagonalisation for diagonally dominant block operator matrices with unbounded off-diagonal entries.

In Part three, Chapters five to seven, we define the Stokes operator by a sesquilinear form and determine its kernel. We generalise the diagonalisation of block operator matrices of the preceding part to the diagonalisation of forms by means of solutions to the form Riccati equation. Furthermore, we apply this diagonalisation to the Stokes operator and obtain a decomposition of its spectrum.

In Part four, Chapters eight and nine, we apply the First Representation Theorem to define self-adjoint differential operators of second order, namely $\operatorname{div} h(\cdot) \operatorname{grad}$ with homogeneous Dirichlet or Neumann boundary values. We allow that the coefficient matrices $h(\cdot)$ are sign-indefinite. We also include the one-dimensional case of left-indefinite Sturm-Liouville operators and compute the spectrum for the special case where h is the sign function.

In the final Part, we give a conclusion of the obtained results, revisit open problems and give conjectures pointing to future research.

We now go into more detail on the different parts of this thesis.

In the first Part, we consider symmetric sesquilinear forms $\mathfrak{b}$ in a Hilbert space $\mathcal{H}$ and their representation by self-adjoint operators.

For any bounded sesquilinear form $\mathfrak{b}$, the Riesz Representation Theorem yields that there is a unique bounded operator B such that

$$\mathfrak{b}[x, y] = \langle x, By \rangle \tag{1}$$

for all $x, y \in \mathcal{H}$. Conversely, any bounded self-adjoint operator B defines a unique bounded sesquilinear form $\mathfrak{b}$ by (1). In combination, this is the one-to-one correspondence of bounded forms and bounded operators.

The representation (1) remains valid for closed symmetric semibounded sesquilinear forms, this classical result goes back to Friedrichs. More precisely, there exists a unique self-adjoint operator B with domain $\mathrm{Dom}(B) \subseteq \mathrm{Dom}[\mathfrak{b}]$ such that

$$\mathfrak{b}[x,y] = \langle x, By\rangle \quad \text{for all } x \in \mathrm{Dom}[\mathfrak{b}],\ y \in \mathrm{Dom}(B). \tag{2}$$

Furthermore, any semibounded self-adjoint operator B defines a unique closed symmetric semibounded sesquilinear form by

$$\mathfrak{b}[x,y] = \langle |B|^{1/2}x, \mathrm{sign}(B)|B|^{1/2}y\rangle \quad \text{for all } x,y \in \mathrm{Dom}[\mathfrak{b}] = \mathrm{Dom}(|B|^{1/2}) \tag{3}$$

which satisfies (2) again. Thus, the correspondence between closed symmetric semibounded forms and semibounded self-adjoint operators is one-to-one, see, e.g. [**43**, Section VI.2].

The validity of the representations (2) and (3) is usually called the First and Second Representation Theorem, respectively.

A classical example is the Dirichlet Laplacian $-\Delta_D$ defined by the sesquilinear form $\mathfrak{b}[f,g] := \langle \mathrm{grad}\, f, \mathrm{grad}\, g\rangle$ for $f,g \in H_0^1(\Omega)$. Further classical examples in quantum mechanics are the Laplacian with delta potential in dimension one, $-\frac{d^2}{dx^2} + \delta$ (see [**55**, Example X.2.3]), and the Laplacian with radial symmetric potential $-\Delta - r^{-\alpha}$ for $\alpha \in [\frac{3}{2}, 2)$ in dimension three, see [**55**, Section X.2]. Note that the last two operators cannot be considered as perturbations of the Laplacian in the sense of operators but can be defined by (2) as a perturbation in the sense of quadratic forms.

Further motivation to investigate the representation of forms by operators can also be derived from quantum mechanics. Namely, the physically measurable quantity is the energy, which is not the quantum mechanical Hamiltonian itself, but its expectation given by a quadratic form. Therefore, the definition of this operator by means of quadratic forms is physically the most natural one, c.f. the discussion in [**62**, Section II.1].

For indefinite (that is, non-semibounded) sesquilinear forms, the situation is more involved. In this case, the usual notion of closedness becomes meaningless. Moreover, there are examples of self-adjoint operators B associated with the form $\mathfrak{b}$ (in the sense that equation (2) holds) such that $\mathrm{Dom}[\mathfrak{b}] \neq \mathrm{Dom}(|B|^{1/2})$, see, e.g. [**22**] and [**36**, Example 2.11]. This implies that the form $\mathfrak{b}$ cannot be reconstructed from the operator B. Also, it may happen that two (or even infinitely many) forms define the same self-adjoint operator B but this operator defines only one of these forms by means of (3), see, e.g. [**36**, Example 2.11 and Proposition 4.2]. An example for this non-semibounded situation is the Dirac operator with Coulomb potential $-\mathrm{i}\alpha\cdot\mathrm{grad} + m\beta + \frac{v}{|x|}$ for $v \in (1/2, 1)$, which cannot be defined as an operator perturbation, see [**53**].

The indefinite situation has been studied by several authors. We point out only a few works on this topic.

- In [**49**] and [**50**], McIntosh introduced a new notion of closedness for arbitrary forms and obtained a new First Representation Theorem. These works are based on the consideration of different topologies on the underlying Hilbert space.
- In the papers [**23**] by Fleige and [**24**] by Fleige, Hassi, and de Snoo, the Representation Theorems are proved by Krein space methods. The idea of these papers is to consider the indefinite form on the Hilbert space as a positive form on a Krein space. In this sense, the indefiniteness of the form is translated into the indefinite inner product of the Krein space.

- In [**36**], Grubišić, Kostrykin, Makarov, and Veselić gave new proofs of the Representation Theorems by Hilbert space methods for forms $\mathfrak{b}$ of the type

$$\mathfrak{b}[x,y] := \langle A^{1/2}x, HA^{1/2}y\rangle, \quad x,y \in \operatorname{Dom}[\mathfrak{b}] = \operatorname{Dom}(A^{1/2}), \tag{4}$$

where A and H are self-adjoint operators such that A is strictly positive and H as well as its inverse are bounded.

In each of these approaches, it is essential that the associated operator B itself, or the shifted operator $B + \lambda I$ associated with $\mathfrak{b} + \lambda I$, will be boundedly invertible. For instance, in [**36**], the operator B is constructed as the inverse of some bounded operator. In [**23**] and [**24**], the corresponding spectral gap around zero allows to consider $\mathfrak{b} + \lambda I$ as a strictly positive form in a Krein space and, hence, to define the operator $B + \lambda I$ with a bounded inverse.

In contrast to these papers, we mainly investigate the setting, where the operator A in the form $\mathfrak{b}$ as in (4) is not strictly positive or even has a non-trivial kernel. In this case, we cannot expect that zero is in the resolvent set of the associated operator B. We do not shift by a multiple of the identity but by a bounded self-adjoint involution J such that the form $\mathfrak{b} + J$ fits into the framework of [**36**], where $B + J$ has a spectral gap around zero. The general idea of this approach is already contained in the papers [**53**] by Nenciu and [**65**] by Veselić. Our approach here is to assume that there is a self-adjoint involution J_A commuting with A which is also compatible with H, that is,

$$(I + J_A)H(I + J_A) \geq \alpha(I + J_A) \quad \text{and} \quad (I - J_A)H(I - J_A) \leq -\alpha(I - J_A)$$

for a suitable constant α. In this case, $\mathfrak{b} + J_A$ is in the framework of [**36**] and we obtain a generalisation of the First Representation Theorem, see Theorem 1.2.3. It turns out that this approach can also be used to consider symmetric off-diagonal perturbations of diagonal forms, see Theorem 1.5.3. We show that if the domain stability condition

$$\operatorname{Dom}(A^{1/2}) = \operatorname{Dom}(|B|^{1/2}) \tag{5}$$

holds, then the Second Representation Theorem can also be preserved from [**36**], see Theorem 2.1.1. We even obtain that the equivalent criteria, in particular the equivalence of the inclusions $\operatorname{Dom}(A^{1/2}) \subseteq \operatorname{Dom}(|B|^{1/2})$ and $\operatorname{Dom}(A^{1/2}) \supseteq \operatorname{Dom}(|B|^{1/2})$, as well as the sufficient conditions for the domain stability condition can be carried over, see Theorem 2.2.4 and Lemma 2.2.5. Furthermore, we give an explicit representation for the kernel of the associated operator B, see Lemma 1.6.2.

Also, we briefly investigate the setting of forms $\mathfrak{b}$ as in (4), where A is strictly positive, H is bounded, but the inverse of H may be unbounded. In this situation, the First Representation Theorem defines an essentially self-adjoint but not self-adjoint operator (Example 1.3.2) unless additional conditions like $\operatorname{Dom}(H^{-1}) \supseteq \operatorname{Dom}(A^{1/2})$ are imposed, see Theorem 3.1.1. For this case, we also provide a variant of the Second Representation Theorem (Theorem 3.2.1) but give an example where $\operatorname{Dom}(|B|^{1/2})$ is a proper subset of $\operatorname{Dom}(A^{1/2})$, see Example 3.2.3. This broken symmetry in the domain stability condition (5) distinguishes these two settings from each other.

Part two is based on the joint work [**48**] with K. A. Makarov and A. Seelmann and is independent of the first part.

In [**3**], Albeverio, Makarov, and Motovilov considered block operator matrices composed of a diagonal part A and a symmetric bounded off-diagonal part V. They provided the correspondence of reducing graph subspaces and solutions to the operator Riccati equation. As a consequence, they obtained a block diagonalisation of the operator $A+V$

by the diagonal operator $A + VY$. Namely

$$A + V = (I + Y)(A + VY)(I + Y)^{-1}, \tag{6}$$

where the skew-symmetric operator Y is a strong solution of the operator Riccati equation

$$AY - YA - YVY + V = 0.$$

Under mild regularity conditions on the unbounded perturbation V, we follow [**3**] in the discussion of reducing subspaces and their relation to the operator Riccati equation. However, we obtain the alternative diagonalisation

$$A + V = (I - Y)^{-1}(A - YV)(I - Y) \tag{7}$$

under mild regularity assumptions on the unbounded operator V, see Theorem 4.3.6. For these V, it turns out that the diagonalisation (7) even implies the diagonalisation (6), see Remark 4.4.2. Since the domain of $A - YV$ does not depend on Y, the diagonalisation (7) is more suitable than (6) for unbounded perturbations V.

Part three is based on the joint work [**38**] with L. Grubišić, V. Kostrykin, K. A. Makarov, and K. Veselić. We define the Stokes operator B_S on Lipschitz domains by application of the representation theory for indefinite forms from the first Part, see Theorem 5.1.2.

Recall that the Stokes operator, which is related to the stationary linearised Stokes on the domain Ω,

$$-\nu\Delta u + \operatorname{grad} p = f, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0,$$

appears in the investigation of fluid dynamics and is thus of interest in mathematical physics, see, e.g., [**61**]. The Stokes operator can be represented by the block operator matrix

$$\begin{pmatrix} -\boldsymbol{\Delta} & -\operatorname{grad} \\ \operatorname{div} & 0 \end{pmatrix}.$$

For other approaches to define the Stokes operator, we refer to [**61**], [**21**], and [**35**]. Furthermore, we obtain that the kernel of the Stokes operator is trivial for Lipschitz domains with infinite volume and is one-dimensional if the volume is finite, see Theorem 5.2.4.

We carry over the diagonalisation of diagonally dominant block operator matrices of the preceding part to the diagonalisation of operators associated with indefinite forms. This form technique allows to consider upper dominant block operator matrices including the Stokes operator. The first step in this technique is to show that the positive spectral part of the associated operator is a graph subspace. To obtain this, we prove that the corresponding difference of the spectral projectors is bounded by $\sqrt{2}/2$ which is an extension of the Tan 2Θ-Theorem of [**37**] by Grubišić, Kostrykin, Makarov, and Veselić to the case without spectral gap around zero, see Theorem 6.1.1. Furthermore, we extend this estimate in Theorem 6.1.6 to the case, where the kernel of the diagonal part admits a splitting and obtain the solvability of the operator Riccati equation in this case. This is in part a generalisation of [**1**, Theorem 6.3] by Adamjan, Langer, and Tretter to diagonally dominant block operator matrices, where the off-diagonal part is unbounded.

We observe that an orthogonal decomposition of the Hilbert space $\mathcal{H}$ reduces the form $\mathfrak{b}$ if and only if it reduces the representing operator B, see Lemma 6.2.4. By this observation, we carry out the diagonalisation of the operator B by finding a reducing subspace for its form $\mathfrak{b}$. Similarly to the operator case, we obtain that the graph subspace of X, for which X^*X respectively XX^* is sufficiently regular, reduces the form given by

$\mathfrak{b}[x,y] := \mathfrak{a}[x, J_A y] + \mathfrak{v}[x,y]$ if and only if the operator X is a solution of the corresponding *form Riccati equation*, see Theorem 6.3.1. To be more explicit, we say that X is a solution of the form Riccati equation if

$$\mathfrak{a}_+[-X^* y_-, x_+] - \mathfrak{a}_-[y_-, X x_+] + \mathfrak{v}[-X^* y_- \oplus 0, 0 \oplus X x_+] + \mathfrak{v}[0 \oplus y_-, x_+ \oplus 0] = 0 \tag{8}$$

holds for all $x_+ \in \mathrm{Dom}[\mathfrak{a}_+]$, $y_- \in \mathrm{Dom}[\mathfrak{a}_-]$, where $\mathfrak{a} = \mathfrak{a}_+ \oplus \mathfrak{a}_-$ is the orthogonal decomposition induced by the self-adjoint involution J_A and $\mathfrak{v}$ is off-diagonal. The regularity condition on X^*X respectively XX^* may be difficult to verify in applications but can be granted by sufficient conditions on the form $\mathfrak{b}$, see Theorem 6.3.6. We verify these sufficient conditions by application of interpolation theory to the results for the operator case of the preceding part. Furthermore, we obtain that the contractive solution X of the form Riccati equation related to the Stokes operator is unique, see Theorem 6.4.3. Using the form Riccati equation (8), we obtain an explicit diagonalisation of the Stokes operator B_S, see Theorem 7.1.6. As a consequence of the block diagonalisation, the positive spectrum of the Stokes operator is bounded below by the smallest eigenvalue of the Dirichlet Laplacian for sufficiently regular quasi-bounded domains, see Lemma 7.4.2. for these domains, the essential spectrum of the Stokes operator is the essential spectrum of the *Cosserat operator* $\overline{\operatorname{div} \mathbf{\Delta}^{-1} \operatorname{grad}}$, where $-\mathbf{\Delta}$ is the vector valued Dirichlet Laplacian. The Cosserat operator has been studied extensively which provides additional information on the spectrum of the Stokes operator. For properties of the Cosserat operator, we only point out the works [**28**] and [**14**] and the survey article [**58**] as well as the references therein.

Part four is based mainly on the joint work [**41**] with A. Hussein, V. Kostrykin, D. Krejčiřík, and K. A. Makarov. We define self-adjoint differential operators of second order given by the form

$$\mathfrak{b}[u,v] := \langle \operatorname{grad} u, h(\cdot) \operatorname{grad} v \rangle \tag{9}$$

on a bounded domain Ω with an indefinite coefficient matrix $h(\cdot)$, where $u, v \in H_0^1(\Omega)$ for Dirichlet boundary conditions and $u, v \in H^1(\Omega)$ for Neumann boundary conditions. The motivation for the consideration of forms of this type are recent results in Physics, more concretely in the investigation of so called optical metamaterials. See [**64**], [**59**], and [**60**] for information on these materials from the physics point of view. For a mathematical point of view on this topic, we refer to [**8**], [**9**], and [**32**].

Our interest lies in the interpretation of the differential expression $\operatorname{div} h(\cdot) \operatorname{grad}$ as a self-adjoint operator in the Hilbert space $L^2(\Omega)$. As a special case in dimension one, we get the self-adjointness of left-indefinite Sturm-Liouville operators. To define the indefinite differential operator of second order with Dirichlet and with Neumann boundary conditions in Theorem 8.2.2 and Theorem 9.1.6, respectively, we bring the form (9) into the structure of (3) using the polar decomposition of the gradient operator. In this sense, we represent the auxiliary form

$$\tilde{\mathfrak{b}}[x,y] := \langle (-\operatorname{grad} \operatorname{div})^{1/2} x, h(\cdot)(-\operatorname{grad} \operatorname{div})^{1/2} y \rangle. \tag{10}$$

by a self-adjoint operator. On the subspace complementary to $\mathrm{Ker}(\operatorname{div})$, this fits into the framework of [**36**] if we substitute the multiplication with $h(\cdot)$ by the application of the operator $Qh(\cdot)Q^*$, where Q^* is the imbedding of $\mathrm{Ran}(\operatorname{grad})$ into $L^2(\Omega)^n$.

It remains to show that $Qh(\cdot)Q^*$ is boundedly invertible. In dimension one, we have an explicit representation of the operator Q. As a consequence, we have a complete understanding of $Qh(\cdot)Q^*$ and derive the First Representation Theorem for left-indefinite Sturm-Liouville operators in the Dirichlet and Neumann case, see Corollaries 8.5.1 and 9.2.1. In higher dimension, we do not have an explicit representation of $Qh(\cdot)Q^*$ but provide examples in the Dirichlet case, where this operator is boundedly invertible, see Corollary 8.4.9 and Proposition 8.4.12, respectively. Finally, we show that the form (9) satisfies the Second Representation Theorem if the auxiliary form (10) does.

In the last part, we close this work by a conclusion on the results of the thesis and give conjectures on some of the open problems. We also indicate fields which are of interest for future investigations as well as ongoing research.

Notation

$\mathbb{R}_+ = (0, \infty)$
$\mathcal{H}$: Hilbert space with inner product $\langle \cdot\, , \,\cdot \rangle$ and norm $\| \cdot \|$
Dom, Ran, Ker: Domain, range and kernel of operators
$\sigma(\cdot)$, $\rho(\cdot)$: Spectrum and resolvent set of operators
$\mathfrak{b}[\,\cdot\, , \,\cdot\,]$, B: (indefinite) Form and associated operator
Ω: Domain in $\mathbb{R}^n$
$B_r(x)$: Open ball of radius r around x
J_A: Self-adjoint involution commuting with A, p. 18
$P_A = (I + J_A)/2$, $P_A^\perp = (I - J_A)/2$: orthogonal projectors onto $\mathcal{H}_+$, $\mathcal{H}_-$, p. 18
$A_+ \oplus A_-$: Orthogonal decomposition of A with respect to $\mathcal{H}_+ \oplus \mathcal{H}_-$, p. 18
$\ell^{2,p}$: Weighted Hilbert space of sequences, p. 21
$\tilde{\mathfrak{b}} = \mathfrak{b} + J_A$, $\widetilde{B}$: Auxiliary form and associated operator , p. 26
$\widetilde{H}$: Auxiliary operator, p. 26
$\mathfrak{L}_\pm$: Subspace related to $\operatorname{Ker} B$, p. 27
sgn: Unitary version of the signum, p. 33
$\mathcal{G}(\mathcal{H}_0, X)$: Graph subspace, p. 46
Y, T: Block operators related to X, X^*, p. 46
$\mathsf{E}_B(\mathbb{R}_+)$: Spectral projector related to the positive spectrum of B, p. 50
$H^1(\Omega), H^1_0(\Omega)$: Sobolev spaces, completion of $C^\infty(\Omega)$, $C_0^\infty(\Omega)$, p. 57
$B_S, \mathfrak{b}_S$: Stokes operator and its form, p. 58, p. 57
$-\Delta, -\boldsymbol{\Delta}$: Laplacian, p. 57
div , grad: Divergence and gradient operators, p. 57
$E^2(\Omega)$: natural domain of div , p. 58
$\operatorname{dist}(\,\cdot\, , \,\cdot\,)$: Distance between two sets or a point and a set, p. 62
$E(\lambda)$: Spectral family of a self-adjoint operator, p. 67
X_0: Operator given by $\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+)) = \mathcal{G}(\mathcal{H}_+, X_0)$, p. 70
$\mathcal{K}_+ \oplus \mathcal{K}_-$: Orthogonal decomposition induced by the positive part of B, p. 71
$\mathfrak{a}_\pm$: Decomposition of the form $\mathfrak{a}$, p. 73
$\widehat{B}_\pm$: Operator unitarily equivalent to $B_\pm$, p. 90
B_D, B_N: $\operatorname{div} H \operatorname{grad}$ with Dirichlet/Neumann boundary values, p. 103, p. 127
$L^2_\sigma(\Omega)$: Divergence free vector fields, p. 104
$H(\Omega)$: Orthogonal complement to $\operatorname{Ran} D \oplus L^2_\sigma(\Omega)$ in $L^2(\Omega)$, p. 104
Γ: Common boundary between Ω_+ and Ω_-, p. 111
$H^{1/2}(\cdot)$: Fractional order Sobolev space, p. 112
$H^{1/2}_{00}(\cdot)$: Functions that can be extended by zero to functions in $H^{1/2}$, p. 112
$H^{1/2}_0(\cdot)$: Fractional order Sobolev space with zero boundary trace, p. 112
$\mathscr{H}^{1/2}(\Gamma)$: Fractional order Sobolev space, p. 112
$\gamma_\pm, \tau_\pm$: Trace maps, p. 113
$\Lambda_\pm$: Dirichlet-to-Neumann map, p. 113
$H^1_{0,\partial\Omega\cap\partial\Omega_\pm}(\Omega_\pm)$: Functions with vanishing trace on $\partial\Omega \cap \partial\Omega_\pm$, p. 113

CHAPTER 1

The First Representation Theorem

1.1. Associated operators and represented forms

Let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ that is linear in the second argument.

If $\mathfrak{b}$ is a symmetric sesquilinear form, we will, as usual, denote the corresponding quadratic form by $\mathfrak{b}[x] := \mathfrak{b}[x,x]$. For brevity, we write just *form* for a symmetric sesquilinear form. By a non-negative or indefinite form we will always understand that the corresponding quadratic form is non-negative or *indefinite*, that is, bounded neither from above nor from below.

The correspondence between symmetric sesquilinear forms and operators can be expressed by the following two concepts.

Definition 1.1.1 (cf. [**43**, Section VI.2]). *Let* $\mathfrak{b}$ *be a symmetric sesquilinear form on* $\mathrm{Dom}[\mathfrak{b}]$ *and let* B *be a self-adjoint operator on* $\mathrm{Dom}(B)$.

(a) *The operator* B *is said to be associated with the form* $\mathfrak{b}$ *if*

$$\mathfrak{b}[x,y] = \langle x, By\rangle \quad \text{for all } x \in \mathrm{Dom}[\mathfrak{b}],\ y \in \mathrm{Dom}(B) \subseteq \mathrm{Dom}[\mathfrak{b}].$$

In this case, the form $\mathfrak{b}$ *is said to satisfy the* First Representation Theorem.

(b) *If* B *is associated with* $\mathfrak{b}$*, then the form* $\mathfrak{b}$ *is said to be represented by the operator* B *if*

$$\mathfrak{b}[x,y] = \langle |B|^{1/2}x, \mathrm{sign}(B)|B|^{1/2}y\rangle \quad \text{for all } x,y \in \mathrm{Dom}[\mathfrak{b}] = \mathrm{Dom}(|B|^{1/2}).$$

In this case, the form $\mathfrak{b}$ *is said to satisfy the* Second Representation Theorem*.*

In this sense, the form $\mathfrak{b}$ gives rise to the associated operator B but the form can only be recovered from the operator if the *domain stability condition* $\mathrm{Dom}[\mathfrak{b}] = \mathrm{Dom}(|B|^{1/2})$ is satisfied. Clearly, if $\mathrm{Dom}(|B|^{1/2})$ is too small, the form given by $\langle |B|^{1/2}x, \mathrm{sign}(B)|B|^{1/2}y\rangle$ defines a restriction of $\mathfrak{b}$ and an extension of $\mathfrak{b}$, if it is too large.

In the following, we generalise the Representation Theorems (and the corresponding ideas of the proofs) in the version of [**36**, Theorems 2.3 and 2.10] by Grubišić, Kostrykin, Makarov, and Veselić, for quadratic forms with a *gap around zero*, that is $A \geq cI > 0$, to the case of *forms without gap*, that is, where we allow $\min \sigma(A) \geq 0$. We collect these statements for the gap case in the following Theorem. Note that the associated operator has a spectral gap around zero in the gap case. For forms without gap, there is in general no such spectral gap.

Theorem 1.1.2 (The Representation Theorems: Gap case).
Let A, H *be self-adjoint operators. Suppose that* A *is strictly positive and that* H *is bounded, boundedly invertible. Define the symmetric sesquilinear form* $\mathfrak{b}$ *by*

$$\mathfrak{b}[x,y] := \langle A^{1/2}x, HA^{1/2}y\rangle, \quad x,y \in \mathrm{Dom}[\mathfrak{b}] = \mathrm{Dom}(A^{1/2}).$$

Then the operator

$$B := A^{1/2}HA^{1/2}, \quad \mathrm{Dom}(B) = \{x \in \mathrm{Dom}(A^{1/2}) \mid HA^{1/2}x \in \mathrm{Dom}(A^{1/2})\}$$

is the unique self-adjoint operator associated with the form $\mathfrak{b}$. *Furthermore,* $\mathrm{Dom}(B)$ *is a core for* $A^{1/2}$ *and* $(\alpha h_-, \alpha h_+) \subset \rho(B)$*, where* $A \geq \alpha I$ *and* (h_-, h_+) *is a maximal spectral gap of the operator* H *containing zero.*

If additionally the domain stability condition $\mathrm{Dom}(|B|^{1/2}) = \mathrm{Dom}(A^{1/2})$ *holds, then the form* $\mathfrak{b}$ *is represented by the operator* B.

1.2. The general case

To start the investigation of the Representation Theorems without gap, we fix the following assumptions.

Hypothesis 1.2.1. *Let* A *be a non-negative self-adjoint operator and let* H *be a bounded, boundedly invertible self-adjoint operator on the same Hilbert space* $\mathcal{H}$.

Furthermore, let J_A *be a non-trivial self-adjoint involution commuting with* A *in the sense that*

$$J_A x \in \mathrm{Dom}(A) \quad \text{and} \quad J_A A x = A J_A x \text{ for all } x \in \mathrm{Dom}(A).$$

Suppose that the orthogonal projectors $P_A := \frac{1}{2}(I + J_A)$ *and* $P_A^\perp := \frac{1}{2}(I - J_A)$ *satisfy*

$$P_A H P_A \geq \alpha P_A \quad \text{and} \quad P_A^\perp H P^\perp \leq -\alpha P_A^\perp \quad \text{for some } \alpha \in (0, 1], \tag{1.1}$$

that is, $\langle P_A x, H P_A x\rangle \geq \alpha \|P_A x\|^2$ *and* $\langle P_A^\perp x, H P_A^\perp x\rangle \leq -\alpha \|P_A^\perp x\|^2$ *for all* $x \in \mathcal{H}$.

Note that the condition (1.1) in the hypothesis above is not trivial, see Example 1.3.1 below.

Remark 1.2.2. *The following observations can be made under Hypothesis 1.2.1.*

(a) *The involution* J_A *induces an orthogonal decomposition*

$$\mathcal{H} = \mathrm{Ran}\, P_A \oplus \mathrm{Ran}\, P_A^\perp =: \mathcal{H}_+ \oplus \mathcal{H}_-$$

of the Hilbert space $\mathcal{H}$. *Since* P_A *and* $P_A^\perp$ *commute with* A*, the subspaces* $\mathcal{H}_+$ *and* $\mathcal{H}_-$ *are reducing subspaces for the operator* A*, see Definition 4.1.3 below or* [**66**, Section 2.5] *for this notion.*

With respect to this decomposition we have the block representations

$$J_A = \begin{pmatrix} I_{\mathcal{H}_+} & 0 \\ 0 & -I_{\mathcal{H}_-} \end{pmatrix}, \quad A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}$$

with $A_+ := P_A A P_A$, $A_- := P_A^\perp A P_A^\perp$*, where the operators* $A_\pm$ *are self-adjoint on the Hilbert spaces* $\mathcal{H}_\pm$ *with domains* $\mathrm{Dom}(A) \cap \mathcal{H}_\pm$*, respectively.*

In this sense, we always assume J_A *and* A *to be diagonal block operator matrices of this structure.*

(b) *Hypothesis 1.2.1 consists of finding a triple* (A, H, J_A) *of operators such that condition* (1.1) *is satisfied. If we start with a given pair* (A, J_A)*, any bounded self-adjoint operator* H *can be represented as a block operator with respect to the decomposition induced by* J_A*. Namely*

$$H = \begin{pmatrix} H_+ & R \\ R^* & H_- \end{pmatrix} \quad \text{with } H_+ := P_A H P_A,\ H_- := P_A^\perp H P_A^\perp,\ R := P_A H P_A^\perp,$$

where $H_\pm \colon \mathcal{H}_\pm \to \mathcal{H}_\pm$ *are bounded, self-adjoint and* $R \colon \mathcal{H}_- \to \mathcal{H}_+$ *is bounded. Thus, condition* (1.1) *can be rewritten as*

$$H_+ \geq \alpha I_{\mathcal{H}_+} \quad \text{and} \quad H_- \leq -\alpha I_{\mathcal{H}_-} \text{ for some } \alpha \in (0, 1]. \tag{1.2}$$

Note that any bounded self-adjoint operator H *satisfying* (1.2) *is automatically boundedly invertible, see* [**46**, Remark 2.8].

We now formulate the First Representation Theorem in the most general setting we consider in this chapter.

THEOREM 1.2.3 (The First Representation Theorem: General case). *Assume Hypothesis 1.2.1, and let $\mathfrak{b}$ be the symmetric sesquilinear form given by*

$$\mathfrak{b}[x,y] := \langle A^{1/2}x, HA^{1/2}y\rangle, \quad \mathrm{Dom}[\mathfrak{b}] = \mathrm{Dom}(A^{1/2}).$$

Then, there exists a unique self-adjoint operator B with $\mathrm{Dom}(B) \subseteq \mathrm{Dom}[\mathfrak{b}]$ and

$$\mathfrak{b}[x,y] = \langle x, By\rangle \quad \textit{for all } x \in \mathrm{Dom}[\mathfrak{b}],\ y \in \mathrm{Dom}(B).$$

Moreover, the operator B is given by

$$B = A^{1/2}HA^{1/2}$$

on the natural domain

$$\mathrm{Dom}(B) = \{x \in \mathrm{Dom}(A^{1/2}) \mid HA^{1/2}x \in \mathrm{Dom}(A^{1/2})\}.$$

Furthermore, $\mathrm{Dom}(B)$ is a core for the operators $(A+I)^{1/2}$ and $A^{1/2}$.

In the case where the operator A is strictly positive, the statement of Theorem 1.2.3 is already contained in [**36**, Theorem 2.3] and condition (1.1) is not needed.

The following observation allows to pull back the case of non-negative A to the case of strictly positive $A + I$ without changing the domain of the square roots of these operators.

REMARK 1.2.4. *Let $c \geq 0$ and let $A \geq -cI$ be self-adjoint. Then, the domain equality*

$$\mathrm{Dom}(|A|^{1/2}) = \mathrm{Dom}(A + (c+1)I)^{1/2}$$

holds since both operators

$$|A|^{1/2}(A + (c+1)I)^{-1/2} \quad \textit{and} \quad (A + (c+1)I)^{1/2}(|A|^{1/2} + I)^{-1}$$

are bounded by functional calculus.

We now turn to the proof of Theorem 1.2.3.

PROOF OF THEOREM 1.2.3. Consider the perturbed form

$$\tilde{\mathfrak{b}} := \tilde{\mathfrak{b}} + J_A \quad \text{on} \quad \mathrm{Dom}[\tilde{\mathfrak{b}}] := \mathrm{Dom}[\mathfrak{b}]$$

given by

$$\tilde{\mathfrak{b}}[x,y] = \mathfrak{b}[x,y] + \langle x, J_A y\rangle.$$

Using the domain equality in Remark 1.2.4 and the commutativity of J_A and P_A with functions of A, the perturbed form can be rewritten as

$$\tilde{\mathfrak{b}}[x,y] = \langle (A+I)^{1/2}x, \widetilde{H}(A+I)^{1/2}y\rangle,$$

where

$$\begin{aligned} \widetilde{H} &:= (A^{1/2}(A+I)^{-1/2})HA^{1/2}(A+I)^{-1/2} + (A+I)^{-1}J_A \\ &=: H_0 + (A+I)^{-1}J_A. \end{aligned} \tag{1.3}$$

In this case, the operator $\widetilde{H}$ is self-adjoint and bounded since $A^{1/2}(A+I)^{-1/2}$ is bounded and self-adjoint by functional calculus.

Let $x \in \mathcal{H}$, then we verify

$$\langle P_A x, H_0 P_A x\rangle = \langle P_A A^{1/2}(A+I)^{-1/2}x, HP_A A^{1/2}(A+I)^{-1/2}x\rangle.$$

By hypothesis (1.1), we get the estimate

$$\langle P_A x, H_0 P_A x\rangle \geq \alpha\langle P_A A^{1/2}(A+I)^{-1/2}x, P_A A^{1/2}(A+I)^{-1/2}x\rangle.$$

Using the commutativity of P_A with functions of A and the equality

$$(A^{1/2}(A+I)^{-1/2})^2 = A(A+I)^{-1} = I-(A+I)^{-1},$$

we can rewrite this estimate as

$$\langle P_A x, H_0 P_A x\rangle \geq \alpha\langle P_A x, P_A x\rangle - \alpha\langle P_A x, (A+I)^{-1}P_A x\rangle.$$

With definition (1.3), the equality $J_A P_A = P_A$, and $\alpha \leq 1$, it follows that

$$P_A \widetilde{H} P_A \geq \alpha P_A - \alpha P_A(A+I)^{-1}P_A + P_A(A+I)^{-1}J_A P_A \geq \alpha P_A.$$

In a similar way, noting that $J_A P_A^\perp = -P_A^\perp$, we obtain that $P_A^\perp \widetilde{H} P_A^\perp \leq -\alpha P_A^\perp$. As a consequence $\widetilde{H}$ is boundedly invertible, see [**46**, Remark 2.8].

Since $A+I$ is strictly positive and $\widetilde{H}$ bounded, boundedly invertible, we can apply the First Representation Theorem [**36**, Theorem 2.3] to the form $\tilde{\mathfrak{b}}$. We obtain that the operator

$$\widetilde{B} := (A+I)^{1/2}\widetilde{H}(A+I)^{1/2}$$

on its natural domain

$$\operatorname{Dom}(\widetilde{B}) = \bigl\{x \in \operatorname{Dom}(A+I)^{1/2} \bigm| \widetilde{H}(A+I)^{1/2}x \in \operatorname{Dom}(A+I)^{1/2}\bigr\} \subseteq \operatorname{Dom}(A^{1/2})$$

is the unique self-adjoint operator with $\operatorname{Dom}(\widetilde{B}) \subseteq \operatorname{Dom}[\tilde{\mathfrak{b}}]$ associated with the form $\tilde{\mathfrak{b}}$, that is,

$$\tilde{\mathfrak{b}}[x,y] = \langle x, \widetilde{B}y\rangle \quad \text{for all } x \in \operatorname{Dom}[\mathfrak{b}],\ y \in \operatorname{Dom}(\widetilde{B}).$$

Additionally, $\operatorname{Dom}(\widetilde{B})$ is a core for $(A+I)^{1/2}$.

Setting $B := \widetilde{B} - J_A$ on $\operatorname{Dom}(B) := \operatorname{Dom}(\widetilde{B})$, we obtain that

$$\mathfrak{b}[x,y] = \langle x, By\rangle \quad \text{for all } x \in \operatorname{Dom}[\mathfrak{b}],\ y \in \operatorname{Dom}(B).$$

Furthermore, we have

$$\begin{aligned}\operatorname{Dom}(B) &= \bigl\{x \in \operatorname{Dom}(A^{1/2}) \bigm| A^{1/2}(A+I)^{-1/2}HA^{1/2}x \in \operatorname{Dom}(A^{1/2})\bigr\}\\ &= \bigl\{x \in \operatorname{Dom}(A^{1/2}) \bigm| (A+I)^{-1/2}HA^{1/2}x \in \operatorname{Dom}(A)\bigr\}\\ &= \bigl\{x \in \operatorname{Dom}(A^{1/2}) \bigm| HA^{1/2}x \in \operatorname{Dom}(A^{1/2})\bigr\}\end{aligned}$$

and, hence,

$$B = (A+I)^{1/2}\widetilde{H}(A+I)^{1/2} - J_A = A^{1/2}HA^{1/2}.$$

The core property with respect to $A^{1/2}$ is a direct consequence of the equivalence of the corresponding graph norms, compare Remark 1.2.4. □

A closer look at the proof above shows that the operator $B+J_A$ is boundedly invertible since it can be written as $(A+I)^{1/2}\widetilde{H}(A+I)^{1/2}$. The operator B however may have an unbounded inverse or may even have a kernel if A has, see Example 1.2.8 below.

In principle, the idea in the proof above, that is to push open a spectral gap by a suitable bounded additive perturbation, is already present in the proof of [**65**, Theorem 2.4]. Namely, in the notation of that work, the sum of forms $h+\alpha_Q$ is associated with the operator product $(a+b|H|)^{1/2}C_\zeta(a+b|H|)^{1/2}$, where $(a+b|H|)$ is boundedly invertible and C_ζ is bounded, boundedly invertible.

Note that the assumptions in Theorem 1.2.3 respectively [**36**, Theorem 2.3] grant the closedness (see Section 1.4 for a suitable notion) of the corresponding form $\mathfrak{b} + J_A$ respectively $\mathfrak{b}$ itself, see the discussion in Section 1.4.

We now compare Theorem 1.2.3 for non-negative operators A respectively [**36**, Theorem 2.3] for strictly positive A, which each give a variant of the First Representation Theorem in different settings.

REMARK 1.2.5. *Theorem 1.2.3 is a supplement to* [**36**, Theorem 2.3] *in the sense that new pairs of operators* (A, H) *can be treated, where* A *is allowed to be a non-negative operator.*

Theorem 1.2.3 is not an extension of [**36**, Theorem 2.3] *since there are cases of strictly positive operators* A *which are not covered by Theorem 1.2.3 but can be treated by* [**36**, Theorem 2.3]. *A suitable* 2×2 *matrix example for this is given by*

$$A := \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In this case, every self-adjoint involution J_A *commuting with* A *is diagonal too. So up to the choice of a sign, we would have* $J_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *Since* $P_A H P_A = 0$ *in this case, the condition* (1.1) *in Hypothesis 1.2.1 cannot be satisfied for the pair* (A, H).

To consider further examples, we introduce the following notation.

DEFINITION 1.2.6. *Let* $\ell^{2,p}$ *be the space of complex sequences* $(a_k)_{k\in\mathbb{N}}$, *such that*

$$\sum_{k\in\mathbb{N}} k^p |a_k|^2 < \infty.$$

We abbreviate $\ell^2 := \ell^{2,0}$ *for the corresponding Hilbert space.*

To show that Theorem 1.2.3 indeed is a supplement to [**36**, Theorem 2.3], consider the following example.

EXAMPLE 1.2.7. *With the notation in Definition 1.2.6, consider the Hilbert space*

$$\mathcal{H} = (\ell^2 \oplus \ell^2) \oplus (\ell^2 \oplus \ell^2) =: \mathcal{H}_+ \oplus \mathcal{H}_-$$

and define the operators acting by multiplication as

$$A := \bigoplus_{k\in\mathbb{N}} \left(\begin{array}{cc|cc} k & 0 & 0 & 0 \\ 0 & k^{-1} & 0 & 0 \\ \hline 0 & 0 & k^{-1} & 0 \\ 0 & 0 & 0 & k \end{array}\right), \quad H := \bigoplus_{k\in\mathbb{N}} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right),$$

where $\mathrm{Dom}(A) := (\ell^{2,2} \oplus \ell) \oplus (\ell^2 \oplus \ell^{2,2})$ *and* A *does not have a bounded inverse. In this case, we can choose* $J_A := H = \widetilde{H}$ *to satisfy Hypothesis 1.2.1 but this case cannot be treated directly by* [**36**, Theorem 2.3].

We now show that B may have an unbounded inverse.

EXAMPLE 1.2.8. *In the same notation as above, the consideration of the bounded operators on* $\ell^2 \oplus \ell^2$,

$$A := \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 1 & 0 \\ 0 & k^{-1} \end{pmatrix}, \quad J_A = H := \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

shows that indeed $B + J_A$ *is boundedly invertible and* B *has an unbounded inverse. Here, we have*

$$B + J_A = \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 2 & 0 \\ 0 & -(1 + k^{-1}) \end{pmatrix}, \quad B = \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 1 & 0 \\ 0 & -k^{-1} \end{pmatrix}.$$

1.3. Counterexamples

We now illustrate the importance of Hypothesis 1.2.1 for the First Representation Theorem. The following example shows that hypothesis (1.1) ensures the self-adjointness of the symmetric operator $B = A^{1/2}HA^{1/2}$ associated with the form $\mathfrak{b}$ if $\min\sigma(A) = 0$.

EXAMPLE 1.3.1. *Using the notation in the Definition 1.2.6 above, we define the self-adjoint operators A and H by*

$$A := \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} k+1 & 0 \\ 0 & (k+1)^{-1} \end{pmatrix}, \quad H := \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\operatorname{Dom}(H) := \ell^2 \oplus \ell^2$ *and* $\operatorname{Dom}(A) := \ell^{2,2} \oplus \ell^2 \subset \mathcal{H}$. *Here, the operators A and H are self-adjoint,* $\min\sigma(A) = 0$ *and $H = H^{-1}$ is bounded.*

Hypothesis (1.1) *is not satisfied since there is no suitable involution J_A commuting with A. Indeed, assume that such a J_A exists, then, since A has a simple spectrum, J_A is a function of A, see* [**17**, Proposition VIII.3.6]. *By the diagonal block structure of A, the operators J_A and P_A also must be block diagonal. Since each block of A itself is diagonal, the corresponding blocks of P_A are also diagonal. Considering the block for $k = 1$, Remark 1.2.5 shows that it is not possible to find such a projector P_A since not even its first block can be constructed.*

To see that $A^{1/2}HA^{1/2}$ is not self-adjoint, let A_k and H_k denote the k-th block of A and H, respectively. Then, we have

$$A_k^{1/2}H_kA_k^{1/2} = H_k.$$

In this sense, the symmetric operator $A^{1/2}HA^{1/2}$ is associated with the form $\mathfrak{b}$ but is not closed on the natural domain

$$\{x \in \operatorname{Dom}(A^{1/2}) \mid HA^{1/2}x \in \operatorname{Dom}(A^{1/2})\} \subseteq \ell^{2,1} \oplus \ell^2 \subset \ell^2 \oplus \ell^2 = \operatorname{Dom}(H).$$

The closure of this operator is self-adjoint, so that the operator is only essentially self-adjoint.

The phenomenon appearing in the example above can be explained in the following way. The operator A has arbitrarily large and arbitrarily small spectral parts. The operator H maps the large spectral parts to the small ones and vice versa in such way, that the product $A^{1/2}HA^{1/2}$ remains bounded on its natural domain. The product is not closed then.

If A is strictly positive as in [**36**], the large spectral parts have no counterpart to be mapped to. Therefore, the closedness of the product $A^{1/2}HA^{1/2}$ on the natural domain is preserved.

This distinguishes the case of strictly positive A, where B is automatically self-adjoint, from the case of non-negative A, where additional conditions have to be imposed. Note that the explicit example above is a special case of a general example considered in [**29**, Remark 2.7]. There, the general example is used to show that the operator $\overline{A^{1/2}HA^{1/2}}$ can be bounded, even if $A^{1/2}$ is unbounded and the spectrum of H contains only the two points $\{-1, 1\}$. This general example is originally due to G. Teschl.

The following example shows that if we weaken the assumptions of [**36**, Theorem 2.3] in another way, that is, if A is strictly positive but H^{-1} is unbounded, the associated operator may also be only essentially self-adjoint.

EXAMPLE 1.3.2. *With the notation of Definition 1.2.6 consider*

$$A := \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 1 & 0 \\ 0 & k+1 \end{pmatrix}, \quad H := \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 1 & 0 \\ 0 & -(k+1)^{-1} \end{pmatrix},$$

where $\operatorname{Dom}(A) := \ell^{2,0} \oplus \ell^{2,2} \subset \ell^2 \oplus \ell^2 =: \mathcal{H}$. *Then*

$$A_k^{1/2} H_k A_k^{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $A^{1/2} H A^{1/2}$ *is not closed on the natural domain*

$$\{x \in \operatorname{Dom}(A^{1/2}) \mid HA^{1/2}x \in \operatorname{Dom}(A^{1/2})\} \subseteq \ell^2 \oplus \ell^{2,1} \subset \ell^2 \oplus \ell^2 = \mathcal{H}.$$

As in the example before the operator is only essentially self-adjoint.

In this example, the operator H scales the large spectral parts in such a way, that the product remains bounded on the natural domain. In this case, the associated operator is not closed on its natural domain.

If however $\operatorname{Dom}(H^{-1}) \supseteq \operatorname{Dom}(A^{1/2})$, then the operator $A^{1/2}HA^{1/2}$ is self-adjoint, see Theorem 3.1.1. Further investigation of the situation, where H^{-1} is unbounded is contained in [**41**]. We will revisit this situation briefly in Chapter 3.

1.4. Closedness of indefinite forms

The First Representation Theorem gives a self-adjoint operator associated with the form $\mathfrak{b}$. It is well known that closed semibounded symmetric forms always are associated with self-adjoint operators. For forms that are indefinite, the usual notion of closedness makes no sense anymore. Extending the usual notion of closedness to indefinite forms, McIntosh showed that this association is also true for indefinite forms, see [**49**, Theorem 4.2]. We will give a definition of this notion below, see Definition 1.4.1.

In this sense, the forms corresponding to the counterexamples Example 1.3.1 and 1.3.2 cannot be closed since the associated operators are only essentially self-adjoint.

Under Hypothesis 1.2.1, the form $\mathfrak{b} + J_A$ is closed (see Lemma 1.4.2 below) and thus associated with a self-adjoint operator $B + J_A$. We conclude that the form $\mathfrak{b}$ itself is associated with the self-adjoint operator B.

It is an open problem whether the form $\mathfrak{b}$ itself is closed under Hypothesis 1.2.1 or not. Also, we do not know whether the self-adjointness of the associated operator B can be obtained directly, that is, without considering the auxiliary form $\mathfrak{b} + J_A$, or not.

Definition 1.4.1 (cf. [**50**]). *Let* $\mathcal{H}_1, \mathcal{H}_2$ *be dense subspaces of a Hilbert space* $\mathcal{H}$ *and let* $\mathfrak{s}$ *be a sesquilinear form with* $\mathfrak{s}[x,y]$ *defined for* $x \in \mathcal{H}_1,\ y \in \mathcal{H}_2$. *Assume that the subspaces* $\mathcal{H}_1, \mathcal{H}_2$ *can be endowed with a Hilbert space structure. Furthermore, suppose that the form* $\mathfrak{s}$ *is bounded in the sense of*

$$|\mathfrak{s}[x,y]| \le c\|x\|_{\mathcal{H}_1} \cdot \|y\|_{\mathcal{H}_2} \quad \text{for some } c < \infty \text{ and all } x \in \mathcal{H}_1,\ y \in \mathcal{H}_2.$$

Then, by the Riesz Representation Theorem, there exists a bounded linear operator $S\colon \mathcal{H}_1 \to \mathcal{H}_2$ *with* $\mathfrak{s}[x,y] = \langle Sx, y\rangle_{\mathcal{H}_2}$. *If the operator* S *is an isomorphism, the form* $\mathfrak{s}$ *is called* regular. *If the Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are continuously imbedded in* $\mathcal{H}$ *and* $\mathfrak{s}$ *is regular, then the form* $\mathfrak{s}$ *is called* 0-closed. *If the form* $\mathfrak{s} + \lambda$ *is* 0*-closed for some* $\lambda \in \mathbb{C}$, *then* $\mathfrak{s}$ *is called* closed.

For our purposes, the explicit notion of closedness is not essential, it suffices to have the following statements that are due to McIntosh.

Lemma 1.4.2. (a) *Let* $\mathfrak{s}$ *be a non-negative symmetric form with* $\operatorname{Dom}[\mathfrak{s}] \subseteq \mathcal{H}$, *then* $\mathfrak{s}$ *is closed (in the notion of Definition 1.4.1) if and only if* $\operatorname{Dom}[\mathfrak{s}]$ *is complete with respect to the inner product*

$$(x,y)_{\operatorname{Dom}[\mathfrak{s}]} := \langle x,y\rangle_{\mathcal{H}} + \mathfrak{s}[x,y],$$

that is, if and only if it is closed in the usual sense of [**43**], *see* [**49**, Theorem 4.1].

(b) *Let T_1, T_2 be closed operators on a Hilbert space $\mathcal{H}$ with $0 \in \rho(T_1) \cap \rho(T_2)$ and let $\mathfrak{t}$ be the form given by*

$$\mathfrak{t}[x,y] := \langle T_1 x, T_2 y\rangle, \quad x \in \mathrm{Dom}(T_1),\ y \in \mathrm{Dom}(T_2).$$

Then the form $\mathfrak{t}$ is 0-closed, see [**49**, Example (b)].

(c) *The form $\mathfrak{s}$ is 0-closed if and only if zero belongs to the resolvent set of the associated operator, see* [**49**, Theorem 3.2].

(d) *Let $\mathfrak{t}$ be as in (b) and let $\mathfrak{s}$ be a form defined on the same (or larger) domain with*

$$|\mathfrak{s}[x,y]| \le c\,\|T_1 x\| \cdot \|T_2 y\| \quad \textit{for all}\, x \in \mathrm{Dom}(T_1),\ y \in \mathrm{Dom}(T_2)$$

and some constant $c < 1$. Then the form sum $\mathfrak{t}+\mathfrak{s}$ is 0-closed, see [**49**, Theorem 8.1].

1.5. The off-diagonal case

In this section, we consider the setting of an indefinite diagonal form with an off-diagonal additive perturbation. Under suitable assumptions, this setting can also be treated by the technique of Theorem 1.2.3.

Let $\mathfrak{a}$ be a non-negative, closed sesquilinear form. By the First Representation Theorem for non-negative forms [**43**, Theorem VI.2.1], the form $\mathfrak{a}$ is associated with a non-negative self-adjoint operator A. By the corresponding Second Representation Theorem [**43**, Theorem VI.2.23], the form $\mathfrak{a}$ is even represented by A.

In this setting, we impose the following assumptions.

Hypothesis 1.5.1. *Let J_A be a self-adjoint involution commuting with A and let*

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad \mathcal{H}_\pm := \mathrm{Ran}(I \pm J_A)$$

be the decomposition induced by J_A. Suppose that $\mathfrak{v}$ is a symmetric sesquilinear form on

$$\mathrm{Dom}[\mathfrak{v}] \supseteq \mathrm{Dom}[\mathfrak{a}] = \mathrm{Dom}(A^{1/2}),$$

and assume that $\mathfrak{v}$ is $(\mathfrak{a}+I)$-bounded, which means that there exists a finite constant β with

$$|\mathfrak{v}[x]| \le \beta \|(A+I)^{1/2} x\|^2 = \beta(\mathfrak{a}+I)[x],\ x \in \mathrm{Dom}(A^{1/2}). \tag{1.4}$$

Suppose furthermore, that $\mathfrak{v}$ is off-diagonal with respect to the decomposition induced by J_A, that is,

$$\mathfrak{v}[J_A x, y] = -\mathfrak{v}[x, J_A y] \quad \textit{for all}\ x,y \in \mathrm{Dom}[\mathfrak{a}]. \tag{1.5}$$

The forms $\mathfrak{v}$ satisfying (1.4) and (1.5) can be described explicitly in terms of the operator $(A+I)^{1/2}$.

Remark 1.5.2. *Let $\mathfrak{v}$ satisfy Hypothesis 1.5.1. Then $\mathfrak{v}$ can explicitly be rewritten on* $\mathrm{Dom}[\mathfrak{a}]$ *as*

$$\mathfrak{v}[x,y] = \langle \widetilde{R}(A+I)^{1/2}x, (A+I)^{1/2}y\rangle, \quad x,y \in \mathrm{Dom}[\mathfrak{a}] = \mathrm{Dom}(A^{1/2}), \tag{1.6}$$

where $\widetilde{R}$ is a bounded self-adjoint operator with $\|\widetilde{R}\| \le \beta$. The existence of the bounded operator $\widetilde{R}$ follows from [**43**, Lemma VI.3.1]. *Since the form $\mathfrak{v}$ is assumed to be symmetric and off-diagonal with respect to the decomposition induced by J_A, we have that the operator*

$$\widetilde{R} = \widetilde{R}^* = \begin{pmatrix} 0 & R \\ R^* & 0 \end{pmatrix}, \ \textit{with}\ R := P_A \widetilde{R} P_A^\perp$$

has to be self-adjoint and off-diagonal.

We are now ready to formulate the First Representation Theorem in this setting (cf. [**36**, Theorem 2.5]). This setting is a special case of the general case, where Hypothesis 1.2.1 is satisfied.

THEOREM 1.5.3 (The First Representation Theorem: Off-diagonal case). *Assume Hypothesis 1.5.1 and let $\mathfrak{b}$ be the symmetric sesquilinear form given by*

$$\mathfrak{b}[x,y] := \mathfrak{a}[x, J_A y] + \mathfrak{v}[x,y], \quad x,y \in \mathrm{Dom}[\mathfrak{b}] = \mathrm{Dom}[\mathfrak{a}].$$

Then, there exists a unique self-adjoint operator B on $\mathrm{Dom}(B) \subseteq \mathrm{Dom}[\mathfrak{b}]$ *with*

$$\mathfrak{b}[x,y] = \langle x, By\rangle \quad \text{for all } x \in \mathrm{Dom}[\mathfrak{a}],\ y \in \mathrm{Dom}(B).$$

Furthermore $\mathrm{Dom}(B)$ *is a* form core *for* $\mathfrak{a}$*, that is,* $\mathrm{Dom}(B)$ *is dense in* $\mathrm{Dom}[\mathfrak{a}]$ *with respect to the norm* $\sqrt{(\mathfrak{a}+I)[\,\cdot\,]}$.

PROOF. Consider the perturbed form $\tilde{\mathfrak{b}}$ on $\mathrm{Dom}[\tilde{\mathfrak{b}}] := \mathrm{Dom}[\mathfrak{b}]$ given by

$$\tilde{\mathfrak{b}}[x,y] := \mathfrak{b}[x,y] + \langle x, J_A y\rangle = (\mathfrak{a}+I)[x, J_A y] + \mathfrak{v}[x,y]. \tag{1.7}$$

Let $\mathfrak{h}$ be the bounded form given by

$$\mathfrak{h}[x,y] := \tilde{\mathfrak{b}}[(A+I)^{-1/2}x, (A+I)^{-1/2}y].$$

Then, by Remark 1.5.2, the form $\mathfrak{h}$ corresponds to the bounded operator

$$\widetilde{H} := J_A + \widetilde{R} = \begin{pmatrix} I_{\mathcal{H}_+} & R \\ R^* & -I_{\mathcal{H}_-} \end{pmatrix}, \tag{1.8}$$

where $R\colon \mathcal{H}_- \to \mathcal{H}_+$. By [**46**, Remark 2.8] the operator $\widetilde{H}$ is boundedly invertible.

We compute that

$$\tilde{\mathfrak{b}}[x,y] = \mathfrak{h}[(A+I)^{1/2}x, (A+I)^{1/2}y] = \langle (A+I)^{1/2}x, \widetilde{H}(A+I)^{1/2}y\rangle,\ x,y \in \mathrm{Dom}[\tilde{\mathfrak{b}}].$$

By the Representation Theorem in the gap case, [**36**, Theorem 2.3], the form $\tilde{\mathfrak{b}}$ is associated with a self-adjoint operator $\widetilde{B}$ that satisfies $\mathrm{Dom}(\widetilde{B}) \subset \mathrm{Dom}[\mathfrak{b}]$ and

$$\tilde{\mathfrak{b}}[x,y] = \langle x, \widetilde{B}y\rangle \quad \text{for all } x \in \mathrm{Dom}[\mathfrak{a}],\ y \in \mathrm{Dom}(\widetilde{B}).$$

The self-adjoint operator associated with the form $\mathfrak{b}$ is then $B := \widetilde{B} - J_A$. The core property is a direct consequence of the corresponding core property in Theorem 1.2.3. ☐

REMARK 1.5.4. *The First Representation Theorem 1.5.3 we give here is a special case of* [**53**, Theorem 2.1]. *To see this, consider $U := J_A$ as the unitary part of the polar decomposition of the self-adjoint operator $J_A A$. We then set*

$$\mathfrak{h}_{A_1}[x,y] := \langle |J_A A|^{1/2}x, J_A |J_A A|^{1/2}y\rangle + 1\langle x, J_A y\rangle = \langle |A|^{1/2}x, J_A|A|^{1/2}y\rangle + \langle x, J_A y\rangle$$

in equation (2.5) of [**53**]. *The off-diagonal form $\mathfrak{v}$ we investigate is, by* [**53**, Definition 2.1]*, then a form perturbation of $J_A A$. Indeed, the first two conditions in* [**53**, Definition 2.1] *can be seen directly, namely*

$$\mathrm{Dom}[\mathfrak{v}] \supseteq \mathrm{Dom}(A^{1/2}) = \mathrm{Dom}(|J_A A|^{1/2})$$

and

$$|\mathfrak{v}[x,y]| \le \beta \|(A+I)^{1/2}x\| \cdot \|(A+I)^{1/2}y\|.$$

It remains to note that the operator $\widetilde{H} = J_A + \widetilde{R}$ is boundedly invertible by the assumption that $\mathfrak{v}$ is off-diagonal. Since we can translate $J_A \equiv T$, $\widetilde{R} \equiv V_1$ into the notation of [**53**], *the sum $T + V_1$ has a bounded inverse and thus also the last condition in* [**53**] *is satisfied.*

REMARK 1.5.5. *In the general and the off-diagonal case, we have that the operator* $\widetilde{H}$ *as well as the operator* $\widetilde{B} = B + J_A$, *which is associated with* $\tilde{\mathfrak{b}} = \mathfrak{b} + J_A$, *in the corresponding proofs, are boundedly invertible, see* [**46**, Remark 2.8] *and* [**36**, Theorem 2.3].

More concretely, we even have the estimate $(-c, c) \subset \rho(B+J_A)$, *where* $c := \|\widetilde{H}^{-1}\|^{-1}$ *is a lower estimate on the spectral gap of* $\widetilde{H}$. *In the general case, we have* $c \leq \alpha \leq 1$ *and* $c \leq 1$ *in the off-diagonal case.*

Additionally, $\widetilde{H}$ *is bounded and* $\widetilde{B}$ *can be represented as*

$$\widetilde{B} = (A+I)^{1/2}\widetilde{H}(A+I)^{1/2}.$$

In contrast to the general case, Theorem 1.5.3 is an extension to the corresponding Theorem [**36**, Theorem 2.5] (cf. Remark 1.2.5 for the general case). This is contained in the following remark.

REMARK 1.5.6. *Theorem 1.5.3 is a generalisation of* [**36**, Theorem 2.5].

Indeed, for strictly positive $\mathfrak{a} \geq c > 0$, *the two sided estimate*

$$\mathfrak{a}[x] \leq (\mathfrak{a}+I)[x] = \mathfrak{a}[x] + c^{-1}c\,\|x\|^2 \leq \mathfrak{a}[x] + c^{-1}\mathfrak{a}[x] = (1+c^{-1})\mathfrak{a}[x]$$

implies the equivalence between $\mathfrak{a}$*-boundedness and* $(\mathfrak{a}+I)$*-boundedness in this case. This yields that forms satisfying the requirements of Theorem 1.5.3 also satisfy those of* [**36**, Theorem 2.5].

We now compare the two versions of the First Representation Theorem to each other.

REMARK 1.5.7.
(a) *The off-diagonal case in Theorem 1.5.3 is a special case of Theorem 1.2.3, where the Hypothesis* (1.1) *is automatically satisfied. In this case, the involution* J_A *creating the spectral gap is already given by the diagonal structure of the form* $\mathfrak{a}$. *Indeed, in this situation, equations* (1.7) *and* (1.8) *imply that the perturbed form* $\mathfrak{b} + J_A$ *and thus also* $\mathfrak{b}$ *satisfy the First Representation Theorem 1.5.3. In the general case however, finding a suitable perturbation* J_A *may be difficult or not possible at all as Example 1.3.1 illustrates.*
(b) *The difference between the results of Theorems 1.2.3 and 1.5.3 lies in the representation of the operator* B *associated to the form* $\mathfrak{b}$. *In the general case, we have the product formula*

$$B = A^{1/2}HA^{1/2}$$

involving only the operators A *and* H *defining the form* $\mathfrak{b}$. *As a consequence, an explicit representation*

$$\operatorname{Ker} B = \{x \in \operatorname{Dom}(A^{1/2}) \mid HA^{1/2}x \in \operatorname{Ker} A^{1/2}\}$$

can be directly deduced.

In the off-diagonal case, if the form $\mathfrak{v}$ *is only* $(\mathfrak{a}+I)$*-bounded but not bounded with respect to the form* $\mathfrak{a}$, *a corresponding operator* H *seems to be artificial.*

The best representation we have is

$$B = (A+I)^{1/2}\widetilde{H}(A+I)^{1/2} - J_A.$$

Indeed, the operator B *cannot be written as a product with respect to* $A^{1/2}$ *and a bounded operator* H *like in the first case, since in this case the operator* H *would formally be given by the block operator matrix*

$$\begin{pmatrix} I_{\mathcal{H}_+} & A_+^{-\frac{1}{2}}(A_+ + I_{\mathcal{H}_+})^{\frac{1}{2}}R(A_- + I_{\mathcal{H}_-})^{\frac{1}{2}}A_-^{-\frac{1}{2}} \\ A_-^{-\frac{1}{2}}(A_- + I_{\mathcal{H}_-})^{\frac{1}{2}}R^*(A_+ + I_{\mathcal{H}_+})^{\frac{1}{2}}A_+^{-\frac{1}{2}} & -I_{\mathcal{H}_-} \end{pmatrix}.$$

If $\min\sigma(A_\pm) = 0$*, the off-diagonal entries of this matrix are either unbounded or may not exist at all if* $A_\pm$ *has a non-trivial kernel. So if such an operator* H *exists, it would in general be unbounded. In this case, an explicit representation of the kernel is more difficult to obtain. This will be carried out in Lemma 1.6.2 below.*

For a strictly positive form $\mathfrak{a}$*, respectively operator* A*, however, the off-diagonal part* $\mathfrak{v}$ *is even* $\mathfrak{a}$*-bounded by Remark 1.5.6. Thus* $B = A^{1/2}HA^{1/2}$ *is still valid by direct application of* [**36**, Theorem 2.5 and Lemma 2.2] *with the operator*

$$H = \begin{pmatrix} I_{\mathcal{H}_+} & R \\ R^* & -I_{\mathcal{H}_-} \end{pmatrix}.$$

In this case, the kernel can be represented as in the general case.

Before we turn to the study of the kernel of B, we give a sufficient condition for its triviality by means of perturbation theory.

LEMMA 1.5.8. *Let* $\mathfrak{b}$ *be the form as in Theorem 1.2.3 or Theorem 1.5.3, respectively and let* B *be the associated operator. Suppose that the operator* $\widetilde{H}$ *such that*

$$B + J_A = (A+I)^{1/2}\widetilde{H}(A+I)^{1/2}$$

satisfies $\|\widetilde{H}^{-1}\| < 1$*, then zero belongs to the resolvent set of* B*.*

PROOF. The proof is a combination of the statements in Lemma 1.4.2. The form $\mathfrak{b} + J_A$ is 0-closed by the representation

$$(\mathfrak{b} + J_A)[x,y] = \langle (A+I)^{1/2}x, \widetilde{H}(A+I)^{1/2}y\rangle, \quad x,y \in \operatorname{Dom}(A^{1/2}).$$

If $\|\widetilde{H}^{-1}\| < 1$, we estimate for $x,y \in \operatorname{Dom}(A^{1/2})$

$$|\langle x, -J_A y\rangle| \le \|x\|\cdot\|y\| \le \|\widetilde{H}^{-1}\|\cdot\|(A+I)^{1/2}x\|\cdot\|\widetilde{H}(A+I)^{1/2}y\|.$$

Thus $\mathfrak{b} = (\mathfrak{b} + J_A) - J_A$ is 0-closed and B is boundedly invertible by Lemma 1.4.2. □

1.6. Computation of the kernel

For strictly positive forms $\mathfrak{a}$, the operator B associated with

$$\mathfrak{b}[x,y] = \mathfrak{a}[x, J_A y] + \mathfrak{v}[x,y]$$

in [**36**, Theorem 2.5] is boundedly invertible. If the form $\mathfrak{a}$ respectively the operator A is only non-negative, the operator B may have a kernel. In the following, we give a representation for the kernel in the off-diagonal case of Theorem 1.5.3. We will apply this representation to the Stokes operator, see Theorem 5.2.4 below.

Recall that by part (a) of Remark 1.2.2, the operator A decomposes as $A = A_+ \oplus A_-$ with self-adjoint $A_\pm$.

DEFINITION 1.6.1. *Under Hypothesis 1.5.1, we define the following subspaces*

$$\mathfrak{L}_\pm := \{x_\pm \in \operatorname{Dom}(A_\pm^{1/2}) \mid \mathfrak{v}[x_+ \oplus 0, 0 \oplus x_-] = 0 \quad \textit{for all } x_\mp \in \operatorname{Dom}(A_\mp^{1/2})\}. \tag{1.9}$$

Note that $\mathfrak{L}_+ \oplus \{0\}$ and $\{0\} \oplus \mathfrak{L}_-$ are not necessarily subsets of $\operatorname{Dom}(B)$ or closed. This fact has to be taken into account for the computation of the kernel of B.

We are now ready to give a representation for the kernel of B with respect to the kernels of the components $A_\pm$. This is a generalisation of [**46**, Theorem 2.2] by Kostrykin, Makarov, and Motovilov in the case of bounded operators respectively forms.

LEMMA 1.6.2. *Let B be the operator associated with the form $\mathfrak{b}$ in Theorem 1.5.3. Then we have that*

$$\operatorname{Ker} B = (\operatorname{Ker} A_+ \cap \mathfrak{L}_+) \oplus (\operatorname{Ker} A_- \cap \mathfrak{L}_-)\,.$$

PROOF. Suppose first that $x = x_+ \oplus x_- \in \operatorname{Ker} B \subseteq \operatorname{Dom}(A^{1/2})$ with respect to $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. By the First Representation Theorem 1.5.3, it follows that

$$0 = \langle y, Bx\rangle = \mathfrak{b}[y,x] = \mathfrak{a}[y, J_A x] + \mathfrak{v}[y,x] \text{ for all } y \in \operatorname{Dom}(A^{1/2}).$$

Writing $y = y_+ \oplus y_-$ with $y_\pm \in \operatorname{Dom}(A_\pm^{1/2})$, the Second Representation Theorem for the non-negative form $\mathfrak{a}$ (see [**43**, Theorem VI.2.23]) yields that

$$\langle A_+^{1/2} y_+, A_+^{1/2} x_+\rangle_{\mathcal{H}_+} - \langle A_-^{1/2} y_-, A_-^{1/2} x_-\rangle_{\mathcal{H}_-} + \mathfrak{v}[y_+ \oplus 0, 0 \oplus x_-] + \mathfrak{v}[0 \oplus y_-, x_+ \oplus 0] = 0.$$

Choosing $y_- = 0$, respectively $y_+ = 0$, we arrive at

$$\langle A_+^{1/2} y_+, A_+^{1/2} x_+\rangle_{\mathcal{H}_+} + \mathfrak{v}[y_+ \oplus 0, 0 \oplus x_-] = 0 \tag{1.10}$$

and

$$-\langle A_-^{1/2} y_-, A_-^{1/2} x_-\rangle_{\mathcal{H}_-} + \mathfrak{v}[0 \oplus y_-, x_+ \oplus 0] = 0, \tag{1.11}$$

respectively. In particular, if $y_+ = x_+$ and $y_- = x_-$, we have that

$$\big\|A_+^{1/2} x_+\big\|_{\mathcal{H}_+}^2 + \mathfrak{v}[x_+ \oplus 0, 0 \oplus x_-] = 0 \tag{1.12}$$

and

$$-\big\|A_-^{1/2} x_-\big\|_{\mathcal{H}_-}^2 + \overline{\mathfrak{v}[x_+ \oplus 0, 0 \oplus x_-]} = 0, \tag{1.13}$$

Suppose that $x_+ \notin \operatorname{Ker} A_+ = \operatorname{Ker} A_+^{1/2}$. Then, from (1.12) we get that

$$\mathfrak{v}[x_+ \oplus 0, 0 \oplus x_-] < 0$$

and from (1.13) follows that

$$\mathfrak{v}[x_+ \oplus 0, 0 \oplus x_-] = \overline{\mathfrak{v}[x_+ \oplus 0, 0 \oplus x_-]} \geq 0,$$

which yields a contradiction. Thus, $x_+ \in \operatorname{Ker} A_+$.

Using equation (1.10) again, we obtain that

$$\mathfrak{v}[y_+ \oplus 0, 0 \oplus x_-] = 0 \quad \text{ for all } y_+ \in \operatorname{Dom}(A_+^{1/2})$$

and, hence, $x_- \in \mathfrak{L}_-$. Similarly, one proves that $x_- \in \operatorname{Ker} A_-$ and $x_+ \in \mathfrak{L}_+$. This proves the inclusion

$$\operatorname{Ker} B \subseteq (\operatorname{Ker} A_+ \cap \mathfrak{L}_+) \oplus (\operatorname{Ker} A_- \cap \mathfrak{L}_-)\,.$$

We now turn to the converse inclusion. By the $(\mathfrak{a}+I)$-boundedness of $\mathfrak{v}$ in Hypothesis 1.5.1, the auxiliary form

$$\mathfrak{r}[x_+, y_-] := \mathfrak{v}[(A_+ + I_{\mathcal{H}_+})^{-1/2} x_+ \oplus 0, 0 \oplus (A_- + I_{\mathcal{H}_-})^{-1/2} y_-], \quad x_+ \in \mathcal{H}_+,\ y_- \in \mathcal{H}_-$$

is bounded. Hence, there exists a bounded operator $R : \mathcal{H}_- \to \mathcal{H}_+$ such that

$$\mathfrak{r}[x_+, y_-] = \langle x_+, R y_-\rangle_{\mathcal{H}_+}.$$

Noticing $\operatorname{Dom}(A_\pm^{1/2}) = \operatorname{Ran}\big((A_\pm + I_{\mathcal{H}_\pm})^{-1/2}\big)$, we get that

$$\mathfrak{L}_+ = \{(A_+ + I_{\mathcal{H}_+})^{-1/2} x \,|\, x \in \operatorname{Ker} R^*\} = (A_+ + I_{\mathcal{H}_+})^{-1/2} \operatorname{Ker} R^*, \tag{1.14}$$

In the same way we obtain that

$$\mathfrak{L}_- = (A_- + I_{\mathcal{H}_-})^{-1/2} \operatorname{Ker} R. \tag{1.15}$$

Let $x_+ \in \operatorname{Ker} A_+ \cap \mathfrak{L}_+$ and $x_- \in \operatorname{Ker} A_- \cap \mathfrak{L}_-$. Then, by (1.14) and (1.15), there exist $u_+ \in \operatorname{Ker} R^*$ and $u_- \in \operatorname{Ker} R$ such that

$$x_+ = (A_+ + I_{\mathcal{H}_+})^{-1/2} u_+ \quad \text{and} \quad x_- = (A_+ + I_{\mathcal{H}_+})^{-1/2} u_-.$$

Obviously, from $x_+ \in \operatorname{Ker} A_+ \subset \operatorname{Dom}(A_+)$, it follows that $u_+ \in \operatorname{Dom}(A_+^{1/2})$.

Similarly, we have $u_- \in \operatorname{Dom}(A_-^{1/2})$.

We claim that $u_+ \in \operatorname{Ker} A_+$ and $u_- \in \operatorname{Ker} A_-$. Indeed, we have that

$$x_+ = (A_+ + I_{\mathcal{H}_+}) x_+ = (A_+ + I_{\mathcal{H}_+})^{1/2} u_+,$$

which implies that $u_+ = (A_+ + I_{\mathcal{H}_+})^{-1/2} x_+$ and, thus, $u_+ \in \operatorname{Dom}(A_+^{3/2})$. Hence, we arrive at the conclusion that

$$A_+ u_+ = (A_+ + I_{\mathcal{H}_+})^{-1/2} A_+ x_+ = 0,$$

which proves that $u_+ \in \operatorname{Ker} A_+$. In the same way we also have $u_- \in \operatorname{Ker} A_-$. By Remark 1.5.5 and equation (1.8) we get the following representation for the operator B (cf. Remark 1.5.7):

$$B = (A + I)^{1/2} \widehat{H} (A + I)^{1/2} \tag{1.16}$$

with the operator

$$\widehat{H} := \begin{pmatrix} I_{\mathcal{H}_+} - (A_+ + I_{\mathcal{H}_+})^{-1} & R \\ R^* & -I_{\mathcal{H}_-} + (A_- + I_{\mathcal{H}_-})^{-1} \end{pmatrix}, \tag{1.17}$$

which follows from

$$B = (A + I)^{1/2} \widetilde{H} (A + I)^{1/2} - J_A$$

with

$$\widetilde{H} = \begin{pmatrix} I_{\mathcal{H}_+} & R \\ R^* & -I_{\mathcal{H}_-} \end{pmatrix}.$$

Identifying $x = x_+ \oplus x_-$ with the vector $\begin{pmatrix} x_+ \\ x_- \end{pmatrix}$, we compute

$$\begin{aligned}
&\widehat{H}(A + I)^{1/2} \begin{pmatrix} x_+ \\ x_- \end{pmatrix} \\
&= \begin{pmatrix} (A_+ + I_{\mathcal{H}_+})^{1/2} x_+ - (A_+ + I_{\mathcal{H}_-})^{-1/2} x_+ + R(A_- + I_{\mathcal{H}_-})^{1/2} x_- \\ R^*(A_+ + I_{\mathcal{H}_+})^{1/2} x_+ - (A_- + I_{\mathcal{H}_-})^{1/2} x_- + (A_- + I_{\mathcal{H}_-})^{-1/2} x_- \end{pmatrix} \\
&= \begin{pmatrix} u_+ - (A_+ + I_{\mathcal{H}_+})^{-1} u_+ + R u_- \\ R^* u_+ - u_- + (A_- + I_{\mathcal{H}_-})^{-1} u_- \end{pmatrix} \\
&= \begin{pmatrix} A_+(A_+ + I_{\mathcal{H}_+})^{-1} u_+ \\ -A_-(A_- + I_{\mathcal{H}_-})^{-1} u_- \end{pmatrix} = \begin{pmatrix} (A_+ + I_{\mathcal{H}_+})^{-1} A_+ u_+ \\ -(A_- + I_{\mathcal{H}_-})^{-1} A_- u_- \end{pmatrix} = 0.
\end{aligned}$$

From the representation (1.17), it follows that $x \in \operatorname{Ker} B$ which completes the proof. $\square$

CHAPTER 2

The Second Representation Theorem

In this chapter, we consider the Second Representation Theorem simultaneously in the situations, where either Hypothesis 1.2.1 or 1.5.1 is satisfied. These situations can be treated simultaneously since the operator B associated with the form $\mathfrak{b}$ can be represented in the same way, see Remark 1.5.5. Namely

$$B = (A+I)^{1/2}\widetilde{H}(A+I)^{1/2} - J_A,$$

and the operator $B + J_A$ is boundedly invertible in both cases.

2.1. The Second Representation Theorem

As a convention, we set $\operatorname{sign}(0) := 0$.

THEOREM 2.1.1 (The Second Representation Theorem). *Let $\mathfrak{b}$ be given as in Theorem 1.2.3 or Theorem 1.5.3, and let B be the associated operator. Furthermore, suppose that*

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(A^{1/2}). \tag{2.1}$$

Then, the operator B represents the form $\mathfrak{b}$, that is,

$$\mathfrak{b}[x,y] = \langle |B|^{1/2}x, \operatorname{sign}(B)|B|^{1/2}y\rangle \quad \textit{for all } x,y \in \operatorname{Dom}[\mathfrak{b}] = \operatorname{Dom}(|B|^{1/2})$$

holds.

Note that this theorem gives the one-to-one correspondence between the form $\mathfrak{b}$ and the operator B. However, it is not clear whether Hypothesis 1.2.1 already implies condition (2.1). For strictly positive A, an example where (2.1) is not satisfied is given by [**36**, Example 2.11]. In this example, condition (1.1) is not satisfied, so that this example cannot be considered as a counterexample here.

Before we turn to the proof, we need some preparations starting with the well known Heinz Inequality (see [**39**]) in the formulation of [**61**, Lemma 3.2.3].

LEMMA 2.1.2 (The Heinz Inequality). *Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Let $S\colon \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Assume that T_1, T_2 are strictly positive, self-adjoint operators with $\operatorname{Dom}(T_1) \subseteq \mathcal{H}_1$ and $\operatorname{Dom}(T_2) \subseteq \mathcal{H}_2$. Suppose that S maps $\operatorname{Dom}(T_1)$ into $\operatorname{Dom}(T_2)$ and that there is a constant c such that*

$$||T_2 S x||_2 \le c \cdot ||T_1 x||_1 \textit{ for all } x \in \operatorname{Dom}(T_1).$$

Then S maps $\operatorname{Dom}(T_1^\nu)$ into $\operatorname{Dom}(T_2^\nu)$ for all $0 \le \nu \le 1$.

COROLLARY 2.1.3. *Let T_1, T_2 be two strictly positive, self-adjoint operators in the Hilbert space $\mathcal{H}$. If the domain equality $\operatorname{Dom}(T_1) = \operatorname{Dom}(T_2)$ holds, then also the domain equality for the roots holds, that is,*

$$\operatorname{Dom}(T_1^\nu) = \operatorname{Dom}(T_2^\nu) \quad \textit{for all } \nu \in [0,1].$$

PROOF. By the domain equality, the operators $T_1T_2^{-1}$ and $T_2T_1^{-1}$ are both closed, defined on $\mathcal{H}$, and thus bounded by the Closed Graph Theorem.

For $x \in \mathrm{Dom}(T_1) = \mathrm{Dom}(T_2)$, we get the estimates

$$\|T_1x\| \le \left\|T_1T_2^{-1}\right\| \cdot \|T_2x\| \quad \text{and} \quad \|T_2x\| \le \left\|T_2T_1^{-1}\right\| \cdot \|T_1x\|.$$

By Lemma 2.1.2 for $S = I$, the equality $\mathrm{Dom}(T_1^\nu) = \mathrm{Dom}(T_2^\nu)$ holds for all $\nu \in [0,1]$. □

As a direct application, we get that the perturbation J_A does not change the domain of the square roots.

LEMMA 2.1.4. *Let B and J_A be the operators in either version of the First Representation Theorem 1.2.3 or 1.5.3. Then the domain equality*

$$\mathrm{Dom}(|B|^{1/2}) = \mathrm{Dom}(|B+J_A|^{1/2})$$

holds.

PROOF. By Remark 1.5.5, we have that $B+J_A$ is boundedly invertible. The operator $|B|+I$ is boundedly invertible by functional calculus. Clearly, one has

$$\mathrm{Dom}(|B+J_A|) = \mathrm{Dom}(B+J_A) = \mathrm{Dom}(B) = \mathrm{Dom}(|B|) = \mathrm{Dom}(|B|+I).$$

By Corollary 2.1.3 and Remark 1.2.4, we obtain the domain equality

$$\mathrm{Dom}(|B+J_A|^{1/2}) = \mathrm{Dom}((|B|+I)^{1/2}) = \mathrm{Dom}(|B|^{1/2}). \qquad \square$$

We are now ready to give a proof of the Second Representation Theorem.

PROOF OF THEOREM 2.1.1. Clearly, we have $\mathrm{Dom}(A^{1/2}) = \mathrm{Dom}((A+I)^{1/2})$ by functional calculus and $\mathrm{Dom}(|B|^{1/2}) = \mathrm{Dom}(|B+J_A|^{1/2})$ by Lemma 2.1.4. Recall that $B+J_A$ is boundedly invertible by Remark 1.5.5.

Taking into account (2.1), the First Representation Theorem 1.2.3, respectively 1.5.3, yields that

$$\mathfrak{b}[x,y] = \langle |B|^{1/2}x, \mathrm{sign}(B)|B|^{1/2}y\rangle \quad \text{for all } x \in \mathrm{Dom}(|B|^{1/2}), y \in \mathrm{Dom}(B).$$

We fix $x \in \mathrm{Dom}(|B|^{1/2})$ and define the functionals l_1 and l_2 on $\mathrm{Dom}(A^{1/2})$ by

$$l_1(y) := \mathfrak{b}[x,y] = \langle A^{1/2}x, HA^{1/2}y\rangle, \quad l_2(y) := \langle |B|^{1/2}x, \mathrm{sign}(B)|B|^{1/2}y\rangle.$$

These two functionals agree on $\mathrm{Dom}(B) \subseteq \mathrm{Dom}(A^{1/2})$ and we show that they agree on the whole of $\mathrm{Dom}(A^{1/2})$.

To do this, we prove that the shifted functionals

$$\tilde{l}_1(y) := l_1(y) + \langle x, J_Ay\rangle, \quad \tilde{l}_2(y) := l_2(y) + \langle x, J_Ay\rangle$$

agree on $\mathrm{Dom}((A+I)^{1/2}) = \mathrm{Dom}(|B+J_A|^{1/2})$, then the desired equality holds.

By Remark 1.5.5, we get the representation

$$\tilde{l}_1(y) = \langle (A+I)^{1/2}x, \widetilde{H}(A+I)^{1/2}y\rangle.$$

with the bounded, boundedly invertible operator $\widetilde{H}$.

Analogously, since $B+J_A$ has a bounded inverse, we get

$$\tilde{l}_2(y) = \langle |B+J_A|^{1/2}x, G|B+J_A|^{1/2}y\rangle$$

with the bounded, boundedly invertible operator

$$G := \mathrm{sign}(B+J_A).$$

By the boundedness of $\widetilde{H}$, we have that $\tilde{l}_1$ is continuous on the Hilbert space

$$\Big(\mathrm{Dom}((A+I)^{1/2}), \langle (A+1)^{1/2}\,\cdot\,, (A+1)^{1/2}\,\cdot\,\rangle\Big) =: \mathcal{H}_{A+I}.$$

In the same way, we have that $\tilde{l}_2$ is continuous on the Hilbert space

$$\Big(\operatorname{Dom}(|B+J_A|^{1/2}), \langle |B+J_A|^{1/2}\,\cdot\,, |B+J_A|^{1/2}\,\cdot\,\rangle\Big) =: \mathcal{H}_{B+J_A}.$$

The domain equality $\operatorname{Dom}((A+I)^{1/2}) = \operatorname{Dom}(|B+J_A|^{1/2})$ yields that the operator $(A+I)^{1/2}$ is $|B+J_A|^{1/2}$-bounded and vice versa. Since both operators $A+I$ and $|B+J_A|$ are strictly positive, the operators

$$(A+I)^{1/2}|B+J_A|^{-1/2} \quad \text{and} \quad |B+J_A|^{1/2}(A+I)^{-1/2}$$

are positive and bounded, so that the corresponding norms

$$\|x\|_{A+I} := \Big\|(A+I)^{1/2}x\Big\| \quad \text{and} \quad \|x\|_{B+J_A} := \Big\||B+J_A|^{1/2}x\Big\|$$

are equivalent on the Hilbert space $\mathcal{H}_{A+I}$.

Since $\operatorname{Dom}(B) = \operatorname{Dom}(B+J_A) = \operatorname{Dom}(|B+J_A|)$ is a core for $|B+J_A|^{1/2}$ (see [**43**, Theorem V.3.35]), it follows that $\operatorname{Dom}(B)$ is dense in $\mathcal{H}_{A+I}$.

The two functionals $\tilde{l}_1$ and $\tilde{l}_2$ are both closed since $\widetilde{H}$ and G are boundedly invertible. By the uniqueness of the closure, we have $\tilde{l}_1 = \tilde{l}_2$ on $\mathcal{H}_{A+I}$ and the claim follows. $\square$

2.2. The domain stability condition

The domain stability condition

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(A^{1/2})$$

in the hypothesis of Theorem 2.1.1 is in general hard to verify directly. We are thus interested in equivalent characterisations (Theorem 2.2.4 below) and in sufficient criteria (Lemma 2.2.5 below) for this condition. These characterisations and criteria are natural extensions to the ones in [**36**], where A is strictly positive. An additional characterisation in terms of reducing subspaces for the form $\mathfrak{b}$ is contained in Remark 6.2.3 below.

In order to start the investigation of the stability condition, we need the following tools. The first one is the Second Resolvent Identity (see, e.g., [**56**, Section 2.2]).

Remark 2.2.1 (The Second Resolvent Identity). *Let T_1, T_2 be closed linear operators on the same domain $\operatorname{Dom}(T_1) = \operatorname{Dom}(T_2)$. Assume that the resolvent sets $\rho(T_1)$ and $\rho(T_2)$ intersect. Denote the resolvents of T_i by $R_\lambda(T_i) := (\lambda I - T_i)^{-1}$. Then, for any $\lambda \in \rho(T_1) \cap \rho(T_2)$, the difference of the resolvents satisfies*

$$R_\lambda(T_1) - R_\lambda(T_2) = R_\lambda(T_1)(T_1 - T_2)R_\lambda(T_2) = R_\lambda(T_2)(T_1 - T_2)R_\lambda(T_1). \tag{2.2}$$

Another tool we use is the following.

Lemma 2.2.2 ([**36**, Lemma 3.1]). ,

Let $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle)$ and $(\mathcal{H}', \langle\,\cdot\,,\,\cdot\,\rangle')$ be Hilbert spaces. Assume that $\mathcal{H}'$ is continuously imbedded in $\mathcal{H}$.

If $S\colon \mathcal{H} \to \mathcal{H}$ is a bounded map leaving the set $\mathcal{H}'$ invariant, then the operator S' induced by S on $\mathcal{H}'$ is bounded in the topology of $\mathcal{H}'$.

In the following investigations, we want to consider the sign of the operator B as a unitary operator. Since B may have a kernel, we need to choose the sign of zero to be either $+1$ or -1. All the following statements are independent of this concrete choice, so we leave this choice open. However, in Lemma 2.2.5 below, it is convenient to have this freedom of choice. We define the unitary version of the sign by

$$\operatorname{sgn}(x) := \begin{cases} -1, & x < 0, \\ s, & x = 0, \quad s \in \{-1, 1\}. \\ +1, & x > 0, \end{cases} \tag{2.3}$$

Note that, by functional calculus, $\operatorname{sign}(B)f(B) = \operatorname{sgn}(B)f(B)$ for any function f with $f(0) = 0$. Furthermore, since the interval $(-1, 1)$ is not contained in the range of the function f defined by $f(x) := x+\operatorname{sgn}(x)$, the operator $B+\operatorname{sgn}(B)$ is boundedly invertible by functional calculus.

We now need the following observations.

Lemma 2.2.3. *Let the assumptions of the First Representation Theorem 1.2.3 or 1.5.3 be satisfied. Then, the operators*

$$(A + I)^{1/2}(B + \operatorname{sgn}(B))^{-1}(A + I)^{1/2} \text{ defined on } \operatorname{Dom}(A^{1/2}) \tag{2.4}$$

and

$$(A+I)^{-1/2}(B+\operatorname{sgn}(B))(A+I)^{-1/2} \text{ defined on the dense set } (A+I)^{1/2}\operatorname{Dom}(B) \tag{2.5}$$

can be extended, by closure, to bounded operators on $\mathcal{H}$. Since they are inverse to each other, they are boundedly invertible.

Proof. The first operator is obviously densely defined. For the second operator, note that $\operatorname{Dom}(B) = \operatorname{Dom}(B + J_A)$. With help of Remark 1.5.5, it follows that

$$(A + I)^{1/2}\operatorname{Dom}(B) = \widetilde{H}^{-1}\operatorname{Dom}(A^{1/2}),$$

so that this operator also is densely defined. Since $B + J_A = (A + I)^{1/2}\widetilde{H}(A + I)^{1/2}$ is boundedly invertible, we have that

$$\widehat{L} := (A + I)^{1/2}(B + J_A)^{-1}(A + I)^{1/2}$$

is bounded on $\operatorname{Dom}(A^{1/2})$.

Since both operators $B + \operatorname{sgn}(B)$ and $B + J_A$ are closed, boundedly invertible and defined on $\operatorname{Dom}(B)$, we have that $0 \in \rho(B + \operatorname{sgn}(B)) \cap \rho(B + J_A)$. We now apply the Second Resolvent Identity (2.2) in both variants. Setting for brevity $J := \operatorname{sgn}(B)$ and $S := J - J_A$, we obtain that

$$\begin{aligned}(B + \operatorname{sgn}(B))^{-1} &= (B + J_A)^{-1} + (B + J_A)^{-1}S(B + J)^{-1}\\ &= (B + J_A)^{-1} + (B + J_A)^{-1}S\big((B + J_A)^{-1} + (B + J)^{-1}S(B + J_A)^{-1}\big).\end{aligned}$$

Thus, we get that

$$\begin{aligned}&(A + I)^{1/2}(B + \operatorname{sgn}(B))^{-1}(A + I)^{1/2}\\ &= \widehat{L} + \widehat{L}(A + I)^{-1/2}S(A + I)^{-1/2}\widehat{L} + \widehat{L}(A + I)^{-1/2}S(B + J)^{-1}S(A + I)^{-1/2}\widehat{L}\end{aligned}$$

is bounded. By the identity $B + J_A = (A + I)^{1/2}\widetilde{H}(A + I)^{1/2}$, the operator

$$\widehat{M} := (A + I)^{-1/2}(B + J_A)(A + I)^{-1/2}$$

is a bounded operator on its natural domain $(A + I)^{1/2}\operatorname{Dom}(B)$. Thus

$$(A + I)^{-1/2}(B + \operatorname{sgn}(B))(A + I)^{-1/2} = \widehat{M} + (A + I)^{-1/2}(J - J_A)(A + I)^{-1/2}$$

is bounded. □

If we consider the same operators as in (2.4) and (2.5), only with the absolute value $|B + \operatorname{sgn}(B)|$ instead of $B + \operatorname{sgn}(B)$, this extension to bounded operators on $\mathcal{H}$ is equivalent to the domain stability condition $\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(A^{1/2})$, see the theorem below.

Theorem 2.2.4 (cf. [**36**, Theorem 3.2]). *Let B be the operator associated with the form $\mathfrak{b}$ in either Theorem 1.2.3 or 1.5.3. Then the following statements are equivalent.*

(i) $\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(A^{1/2})$,
(ii) $\operatorname{Dom}(|B|^{1/2}) \supseteq \operatorname{Dom}(A^{1/2})$,
(ii') $\operatorname{Dom}(|B|^{1/2}) \subseteq \operatorname{Dom}(A^{1/2})$,
(iii) $L := (A+I)^{-1/2}|B+\operatorname{sgn}(B)|(A+I)^{-1/2}$ *is a bounded symmetric operator on*

$$\operatorname{Dom}(L) := (A+I)^{1/2}\operatorname{Dom}(B),$$

(iii') $M := (A+I)^{1/2}|B+\operatorname{sgn}(B)|^{-1}(A+I)^{1/2}$ *is a bounded symmetric operator on*

$$\operatorname{Dom}(M) := \operatorname{Dom}(A^{1/2}),$$

(iv) $K := (A+I)^{1/2}\operatorname{sgn}(B)(A+I)^{-1/2}$ *is a bounded involution on $\mathcal{H}$,*
(v) $\operatorname{sgn}(B)\operatorname{Dom}(A^{1/2}) \subseteq \operatorname{Dom}(A^{1/2})$.

Proof. For brevity, we set $J := \operatorname{sgn}(B)$. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (ii') are obvious.

(ii) $\Rightarrow$ (iii): Since $\operatorname{Dom}(A^{1/2}) \subseteq \operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(|B+J|^{1/2})$, we have that the operator $|B+J|^{1/2}(A+I)^{-1/2}$ is closed on $\mathcal{H}$ and is thus bounded. Define the positive form

$$\mathfrak{l}[x,y] := \langle x, Ly\rangle \quad \text{on} \quad \operatorname{Dom}[\mathfrak{l}] := \operatorname{Dom}(L) = (A+I)^{1/2}\operatorname{Dom}(B).$$

This form can be represented as a bounded form

$$\mathfrak{l}[x,y] = \langle |B+J|^{1/2}(A+I)^{-1/2}x, |B+J|^{1/2}(A+I)^{-1/2}y\rangle.$$

Thus, the associated operator L is bounded. Note that $\operatorname{Dom}(L) = \operatorname{Dom}[\mathfrak{l}]$ is dense in $\mathcal{H}$ by Lemma 2.2.3, so that the closure of L is a bounded operator on $\mathcal{H}$.

(ii') $\Rightarrow$ (iii'): Similarly to the implication before, we have that the operator product $(A+I)^{1/2}|B+J|^{-1/2}$ is bounded and the densely defined positive form

$$\mathfrak{m}[x,y] := \langle x, My\rangle \quad \text{on} \quad \operatorname{Dom}[\mathfrak{m}] := \operatorname{Dom}(M) = \operatorname{Dom}(A)^{1/2}$$

can be represented as a bounded form

$$\mathfrak{m}[x,y] = \langle (A+I)^{1/2}|B+J|^{-1/2}x, (A+I)^{1/2}|B+J|^{-1/2}y\rangle.$$

Thus, the closure of M is a bounded operator on $\mathcal{H}$.

(iii) $\Rightarrow$ (iv): The operator K is closed on its natural domain

$$\operatorname{Dom}(K) = \{x \in \mathcal{H} \mid \operatorname{sgn}(B)(A+I)^{-1/2}x \in \operatorname{Dom}((A+I)^{1/2})\}.$$

Furthermore, since $\operatorname{Dom}(B) \subseteq \operatorname{Dom}(A^{1/2})$ and $\operatorname{sgn}(B)$ leaves $\operatorname{Dom}(B)$ invariant, we have that

$$\operatorname{Dom}(L) = (A+I)^{1/2}\operatorname{Dom}(B) \subseteq \operatorname{Dom}(K).$$

Let $x \in \operatorname{Dom}(X)$, then, taking into account $\operatorname{sgn}(B+J) = \operatorname{sgn}(B)$, it follows that

$$\begin{aligned} Kx &= (A+I)^{1/2}\operatorname{sgn}(B)(A+I)^{-1/2}x = (A+I)^{1/2}\operatorname{sgn}(B+J)(A+I)^{-1/2}x \\ &= (A+I)^{1/2}(B+J)^{-1}|B+J|(A+I)^{-1/2}x = \big((A+I)^{1/2}(B+J)^{-1}(A+I)^{1/2}\big)Lx. \end{aligned}$$

By hypothesis and Lemma 2.2.3, respectively, both operators in the product can be extended to bounded operators on $\mathcal{H}$. Thus $K|_{\operatorname{Dom}(L)}$ can be boundedly extended to $\mathcal{H}$. By the closedness of K, it follows that K is bounded with $\operatorname{Dom}(K) = \mathcal{H}$. The operator K is an involution since $K^2 = I$.

(iii') $\Rightarrow$ (iv): As in the implication before, $(A+I)^{1/2}\operatorname{Dom}(B) \subseteq \operatorname{Dom}(K)$ is dense.

Let $x \in (A+I)^{1/2}\operatorname{Dom}(B)$, then, in the same way,

$$\begin{aligned} Kx &= (A+I)^{1/2}\operatorname{sgn}(B+J)(A+I)^{-1/2}x = (A+I)^{1/2}|B+J|^{-1}(B+J)(A+I)^{-1/2}x \\ &= \big((A+I)^{1/2}|B+J|^{-1}(A+I)^{1/2}\big)\cdot\big((A+I)^{-1/2}(B+J)(A+I)^{-1/2}\big)x \\ &= M\big((A+I)^{-1/2}(B+J)(A+I)^{-1/2}\big)x, \end{aligned}$$

where both operators in the product can be boundedly extended to $\mathcal{H}$ by Lemma 2.2.3. As before, K is a bounded involution on $\operatorname{Dom}(K) = \mathcal{H}$.

(iv) $\Rightarrow$ (v): Since $\operatorname{Dom}(K) = \mathcal{H}$ by assumption, $\operatorname{sgn}(B)$ leaves $\operatorname{Dom}(A^{1/2})$ invariant.

(v) $\Rightarrow$ (i): We first consider the case of strictly positive $\widetilde{H}$.

Then, the positive-definite form

$$\tilde{\mathfrak{b}}[x,y] = \langle (A+I)^{1/2}x, \widetilde{H}(A+I)^{1/2}y\rangle, \quad x,y \in \operatorname{Dom}[\tilde{\mathfrak{b}}] = \operatorname{Dom}(A^{1/2})$$

can be represented as

$$\tilde{\mathfrak{b}}[x,y] = \langle \widetilde{H}^{1/2}(A+I)^{1/2}x, \widetilde{H}^{1/2}(A+I)^{1/2}y\rangle,$$

so that $\tilde{\mathfrak{b}}$ is closed by the closedness of the operator $\widetilde{H}^{1/2}(A+I)^{1/2}$.

The First Representation Theorem for non-negative forms (see, e.g., [**43**, Theorem VI.2.6]) implies the existence of a non-negative operator $\widetilde{B}$ associated with the form $\tilde{\mathfrak{b}}$.

By construction, we have $\widetilde{B} = B + J_A$.

The Second Representation Theorem for positive-semidefinite sesquilinear forms [**43**, Theorem VI.2.23] yields the equality $\operatorname{Dom}[\tilde{\mathfrak{b}}] = \operatorname{Dom}(\widetilde{B}^{1/2})$.

Using Lemma 2.1.4, the claim follows by observing

$$\operatorname{Dom}[\tilde{\mathfrak{b}}] = \operatorname{Dom}[\mathfrak{b}] \quad \text{and} \quad \operatorname{Dom}(\widetilde{B}^{1/2}) = \operatorname{Dom}(B^{1/2}).$$

We now consider the case, where $\widetilde{H}$ is not necessarily positive. Define the Hilbert space

$$\mathcal{H}_{A+I} := \Big(\operatorname{Dom}(A^{1/2}), \langle (A+I)^{1/2}\,\cdot\,, (A+I)^{1/2}\,\cdot\,\rangle\Big).$$

Let J_{A+I} be the operator induced by $\operatorname{sgn}(B)$ on $\mathcal{H}_{A+I}$. The space $\mathcal{H}_{A+I}$ is continuously imbedded in $\mathcal{H}$ and, by part (v), the operator $\operatorname{sgn}(B)$ leaves $\mathcal{H}_{A+I}$ invariant (as a set).

Then, by Lemma 2.2.2, the operator J_{A+I} is continuous on $\mathcal{H}_{A+I}$. Since $J^2 = I$, we even have that J_{A+I} is a bounded involution, not necessarily unitary.

It follows that $K = (A+I)^{1/2}J(A+I)^{-1/2}$ is a bounded involution on $\mathcal{H}$. Observing $\operatorname{sgn}(B+J) = \operatorname{sgn}(B)$ and $\operatorname{Dom}(B) \subseteq \operatorname{Dom}(A^{1/2})$, we have that

$$\begin{aligned} |B+J| &= (B+J)\operatorname{sgn}(B+J) = (B+J)J \\ &= (A+I)^{1/2}(A+I)^{-1/2}(B+J)(A+I)^{-1/2}(A+I)^{1/2}J(A+I)^{-1/2}(A+I)^{1/2} \\ &= (A+I)^{1/2}\widetilde{M}K(A+I)^{1/2}, \end{aligned}$$

with the abbreviation $\widetilde{M} := (A+I)^{-1/2}(B+J)(A+I)^{-1/2}$ for the operator in Lemma 2.2.3. Since $|B+J|$ is non-negative, $\widetilde{M}K$ also has to be non-negative. Both $\widetilde{M}$ and K are Hilbert space isomorphisms.

Hence, the self-adjoint operator $\widetilde{M}K$ has a bounded inverse and is thus strictly positive. Considering the positive form

$$\hat{\mathfrak{b}}[x,y] = \langle (A+I)^{1/2}x, (\widetilde{M}K)(A+I)^{1/2}y\rangle, \quad x,y \in \operatorname{Dom}(A^{1/2})$$

associated with the operator $|B+J|$, we get from the first case and functional calculus that

$$\operatorname{Dom}(A^{1/2}) = \operatorname{Dom}(|B+J|^{1/2}) = \operatorname{Dom}(|B|^{1/2})$$

holds. This completes the proof. $\square$

We now give sufficient, but in general not necessary, criteria for the domain stability condition. These criteria were introduced in [**36**, Lemma 3.6] for the case of strictly positive A. The following lemma shows that they can be extended to the case, where A is only non-negative.

LEMMA 2.2.5 (cf. [**36**, Lemma 3.6]). *Let the assumptions of Theorem* 1.2.3 *be satisfied, and let B be the operator associated with the form $\mathfrak{b}$. If one of the following conditions*

(a) *the operator H maps* $\mathrm{Dom}(A^{1/2})$ *onto itself;*
(b) *the operator H is strictly positive or strictly negative;*
(c) *the operator B is semibounded*

holds, then the domain stability condition

$$\mathrm{Dom}(|B|^{1/2}) = \mathrm{Dom}(A^{1/2}) \tag{2.6}$$

is satisfied.

PROOF. (a) Since H is bijective as a map on $\mathrm{Dom}(A^{1/2})$ by assumption, the natural domain of the operator $B = A^{1/2}HA^{1/2}$ coincides with $\mathrm{Dom}(A)$. Thus, the domains of the positive, boundedly invertible operators $A+I$ and $|B|+I$ coincide. The domain stability condition (2.6) now follows from Corollary 2.1.3 and Remark 1.2.4.

(b) If the operator H is strictly positive, then B is non-negative, but may still have a kernel. Choosing $\mathrm{sgn}(0) = 1$, we have $\mathrm{sgn}(B) = I$. In this case the condition (v) in Theorem 2.2.4 is trivially satisfied. If H is strictly negative, choose $\mathrm{sgn}(0) = -1$ so that in this case $\mathrm{sgn}(B) = -I$ and condition (v) is again satisfied.

(c) Without loss of generality, assume B to be bounded from below. By Remark 1.5.5, the operator $B + J_A$ is boundedly invertible. Since B is bounded from below and J_A is bounded, the operator $B + J_A + cI$ is strictly positive for all sufficiently large constants $c > 0$. For such constants, we have the representation

$$B + J_A + cI = (A+I)^{1/2}\big(\widetilde{H} + c(A+I)^{-1}\big)(A+I)^{1/2} \tag{2.7}$$

on $\mathrm{Dom}(B) \subseteq \mathrm{Dom}(A^{1/2})$, where $\widetilde{H}$ is bounded and boundedly invertible. Since the left-hand side of (2.7) is a non-negative operator, we have that

$$\widetilde{H} + c(A+I)^{-1} \geq 0$$

holds. We show that this operator is even strictly positive. To see this, it suffices to verify that $\widetilde{H} + c(A+I)^{-1}$ is boundedly invertible. Note that

$$\widetilde{H} + c(A+I)^{-1} = c\widetilde{H}\left(c^{-1}I + \widetilde{H}^{-1}(A+I)^{-1}\right),$$

so that this operator is invertible if

$$-c^{-1} \notin \sigma(\widetilde{H}^{-1}(A+I)^{-1}).$$

Recall that for bounded self-adjoint operators T_1, T_2 the well known spectral identity

$$\sigma(T_1T_2) \setminus \{0\} = \sigma(T_2T_1) \setminus \{0\}$$

holds, see, e.g., [**56**, Exercise 2.4.11]. Thus, the operator has a bounded inverse if

$$-c^{-1} \in \rho((A+I)^{-1/2}\widetilde{H}^{-1}(A+I)^{-1/2}) = \rho((B+J_A)^{-1}).$$

Since $B + J_A$ is, by assumption, bounded from below and boundedly invertible, the negative spectrum of $B + J_A$ is contained in a bounded interval away from zero, so that $-c^{-1}$ does not belong to the spectrum if c is sufficiently large. For these constants c, the operator $\widetilde{H} + c(A+I)^{-1}$ is then strictly positive. From part (b) and (2.7), we deduce that

$$\mathrm{Dom}((B + J_A + cI)^{1/2}) = \mathrm{Dom}((A+I)^{1/2}).$$

By Lemma 2.1.4 and Remark 1.2.4, the equality of the domains of $A^{1/2}$ and $|B|^{1/2}$ holds. □

In Theorem 1.5.3 we investigated off-diagonal form perturbations $\mathfrak{v}$ of indefinite forms $\mathfrak{a}[\,\cdot\, J_A \,\cdot\,]$. In the following, we consider the special case of forms

$$\mathfrak{v} = \langle |V|^{1/2}\,\cdot\,, \mathrm{sign}(V)|V|^{1/2}\,\cdot\,\rangle, \quad \mathrm{Dom}[\mathfrak{v}] = \mathrm{Dom}(|V|^{1/2})$$

represented by a self-adjoint operator V and give a sufficient condition for the domain stability condition (2.1) in terms of the operator V. First, we fix the following standard notation.

Definition 2.2.6. *The following statements can be derived e.g. from* [**56**, Definitions 8.1, 10.7, and Proposition 10.4].

(a) *Let T_1, T_2 be closed operators with* $\mathrm{Dom}(T_2) \supseteq \mathrm{Dom}(T_1)$ *and*

$$\|T_2 x\| \le a\,\|T_1 x\| + b\,\|x\|\,, \quad x \in \mathrm{Dom}(T_1)$$

for some constants $a, b \ge 0$. Denote by a_0 the infimum of such a. Then T_2 is called relatively T_1-bounded *and a_0 is the* relative bound *of T_2 with respect to T_1. T_2 is called* infinitesimally small *or* infinitesimal *with respect to T_1 if $a_0 = 0$.*

(b) *Let $\mathfrak{t}_1, \mathfrak{t}_2$ be symmetric forms on $\mathcal{H}$ with* $\mathrm{Dom}[\mathfrak{t}_1] \subseteq \mathrm{Dom}[\mathfrak{t}_2]$. *Suppose that $\mathfrak{t}_1$ is semibounded from below and that*

$$|\mathfrak{t}_2[x,x]| \le a|\mathfrak{t}_1[x]| + b\,\|x\|^2\,, \quad x \in \mathrm{Dom}[\mathfrak{t}_1]$$

for some constants $a, b \ge 0$. Denote by a_0 the infimum of all those a. Then $\mathfrak{t}_2$ is called relatively $\mathfrak{t}_1$-bounded *and a_0 is the* relative bound *of $\mathfrak{t}_2$ with respect to $\mathfrak{t}_1$.*

(c) *Let T_1, T_2 be self-adjoint operators, where T_1 is semibounded from below, and let $j \in \{1, 2\}$ and*

$$\mathfrak{t}_j[x,y] := \langle |T_j|^{1/2} x, \mathrm{sign}(T_j)|T_j|^{1/2} y\rangle, \quad \mathrm{Dom}[\mathfrak{t}_j] = \mathrm{Dom}(|T_j|^{1/2}),$$

be the related forms. Then the operator T_2 is called relatively T_1-form bounded *if the form $\mathfrak{t}_2$ is relatively $\mathfrak{t}_1$-bounded. The* T_1-form bound *of T_2 is defined as the $\mathfrak{t}_1$-bound of $\mathfrak{t}_2$.*

Lemma 2.2.7. *Let A be the self-adjoint operator associated with the non-negative form $\mathfrak{a}$ by the First Representation Theorem* [**43**, Theorem VI.2.6]. *Suppose, that J_A is a self-adjoint involution commuting with A and $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is the orthogonal decomposition induced by J_A. Furthermore, let V be an off-diagonal, self-adjoint operator that has relative bound $a_0 < 1$ with respect to A. Let $\mathfrak{v}$ be represented by V, that is,*

$$\mathfrak{v}[x,y] = \langle |V|^{1/2} x, \mathrm{sign}(V)|V|^{1/2} y\rangle, \quad x, y \in \mathrm{Dom}[\mathfrak{v}] = \mathrm{Dom}(|V|)^{1/2}. \tag{2.8}$$

Then the operator sum $J_A A + V$ coincides with the operator B associated with the form sum $\mathfrak{b} = \mathfrak{a}[\,\cdot\,, J_A\,\cdot\,] + \mathfrak{v}$ defined by Theorem 1.5.3. Furthermore, the domain stability condition $\mathrm{Dom}(A^{1/2}) = \mathrm{Dom}(|B|^{1/2})$ *holds.*

PROOF. First, note that the non-negative form $\mathfrak{a}$ satisfies the Second Representation Theorem [**43**, Theorem VI.2.23], so that

$$\mathfrak{a}[x,y] = \langle A^{1/2}x, A^{1/2}y\rangle, \quad x,y \in \operatorname{Dom}[\mathfrak{a}] = \operatorname{Dom}(A^{1/2}).$$

The form-domain of the operator V is $\operatorname{Dom}(|V|^{1/2})$, see, e.g., [**56**, Definition 10.3]. It follows from [**55**, Theorem X.18], that the operator V is also relatively A-form bounded with the relative bound $0 \le a_0 < 1$. In this sense, we have that the form-domain of the operator V satisfies

$$\operatorname{Dom}(|V|^{1/2}) = \operatorname{Dom}[\mathfrak{v}] \supseteq \operatorname{Dom}[\mathfrak{a}] = \operatorname{Dom}(A^{1/2})$$

and

$$|\mathfrak{v}[x]| \le a\mathfrak{a}[x] + b\,\|x\|^2 \le (a+b)(\mathfrak{a}+I)[x], \quad x \in \operatorname{Dom}(A^{1/2})$$

for suitable constants $a_0 \le a < 1$ and b. In this case, the forms $\mathfrak{a}$ and $\mathfrak{v}$ satisfy Hypothesis 1.5.1 and Theorem 1.5.3 is applicable. Thus, there is an operator B associated with the form $\mathfrak{a}[\,\cdot\,, J_A\,\cdot\,] + \mathfrak{v}$. It remains to note that the form $\mathfrak{v}$ is off-diagonal since the operator V is off-diagonal by assumption.

Since V has A-bound $a_0 < 1$ and J_A is unitary, we also have the same bound with respect to $J_A A$. The Kato-Rellich Theorem, see, e.g., [**55**, Theorem X.12] allows to define the self-adjoint operator

$$J_A A + V \text{ on } \operatorname{Dom}(J_A A + V) = \operatorname{Dom}(A) \subseteq \operatorname{Dom}(A^{1/2}) = \operatorname{Dom}[\mathfrak{b}].$$

In order to prove that $J_A A + V$ equals B, we show that $J_A A + V$ is associated with $\mathfrak{b}$ and the claim then follows from the uniqueness in Theorem 1.5.3. To see this, let

$$x \in \operatorname{Dom}[\mathfrak{b}] = \operatorname{Dom}[\mathfrak{a}] = \operatorname{Dom}(A^{1/2}), \quad y \in \operatorname{Dom}(J_A A + V) = \operatorname{Dom}(A) \subseteq \operatorname{Dom}(A^{1/2}),$$

then

$$\langle x, (J_A A + V)y\rangle = \langle x, AJ_A y\rangle + \langle |V|^{1/2}x, \operatorname{sign}(V)|V|^{1/2}y\rangle = \mathfrak{a}[x, J_A y] + \mathfrak{v}[x,y] = \mathfrak{b}[x,y].$$

Thus, we have $B = J_A A + V$ on $\operatorname{Dom}(B) = \operatorname{Dom}(A)$. Since $|B| + I$ and $A + I$ are strictly positive and have the same domain, the desired domain equality for the roots holds by Corollary 2.1.3 and Remark 1.2.4. $\square$

CHAPTER 3

Representation Theorems for H with unbounded inverse

In this chapter, we briefly investigate the First and Second Representation Theorem in the setting of Theorem 1.2.3 under different assumptions. Here, we consider A to have a bounded inverse and H to have a (possibly) unbounded inverse. In Example 1.3.2, we already saw that additional conditions will be needed for the First Representation Theorem to hold with a self-adjoint associated operator.

3.1. The First Representation Theorem

The following First Representation Theorem was obtained in the joint work [**41**] with A. Hussein, V. Kostrykin, D. Krejčiřík, and K. A. Makarov. For completeness sake, we restate the statement and the corresponding proof.

THEOREM 3.1.1 ([**41**]). *Let A and H be self-adjoint operators, where* $\inf \sigma(A) > 0$ *and H is bounded with a trivial kernel and*

$$\operatorname{Dom}(H^{-1}) \supseteq \operatorname{Dom}(A^{1/2}).$$

Then the sesquilinear form

$$\mathfrak{b}[x,y] := \langle A^{1/2}x, HA^{1/2}y\rangle, \quad x,y \in \operatorname{Dom}(A^{1/2})$$

admits the representation

$$\mathfrak{b}[x,y] = \langle x, By\rangle, \quad x \in \operatorname{Dom}(A^{1/2}),\ y \in \operatorname{Dom}(B) \subseteq \operatorname{Dom}(A^{1/2})$$

for a unique self-adjoint operator B. The operator B is explicitly given by $A^{1/2}HA^{1/2}$ on its natural domain and has a bounded inverse.

PROOF. Since $\operatorname{Dom}(H^{-1}A^{-1/2}) = \mathcal{H}$ by assumption and $H^{-1}A^{-1/2}$ is closed as the product of the closed operator H^{-1} and the bounded operator $A^{-1/2}$, it follows that $H^{-1}A^{-1/2}$ is a bounded operator by the Closed Graph Theorem. As a consequence, the operator $S := A^{-1/2}H^{-1}A^{-1/2}$ is bounded and self-adjoint with a trivial kernel. Thus its inverse is self-adjoint and it suffices to show the identity $S^{-1} = A^{1/2}HA^{1/2} = B$.

To see this, observe that

$$\operatorname{Dom}(S^{-1}) = \operatorname{Ran} S = A^{-1/2}\operatorname{Ran}(H^{-1}A^{-1/2}).$$

Hence, if $x \in \operatorname{Dom}(S^{-1})$, then $x \in \operatorname{Dom}(A^{1/2})$ and $A^{1/2}x \in \operatorname{Ran}(H^{-1}A^{-1/2})$. In this case $HA^{1/2}x \in \operatorname{Ran}(A^{-1/2}) = \operatorname{Dom}(A^{1/2})$, so that $\operatorname{Dom}(S^{-1/2}) \subseteq \operatorname{Dom}(B)$.

For the converse inclusion, let $x \in \operatorname{Dom}(B)$. Then we have that $x \in \operatorname{Dom}(A^{1/2})$ and $HA^{1/2}x \in \operatorname{Dom}(A^{1/2})$. As a consequence, it follows that $x = A^{1/2}y$ for some $y \in \mathcal{H}$ with $Hy \in \operatorname{Dom}(A^{1/2})$. Consequently $y = H^{-1}A^{-1/2}z \in H^{-1}\operatorname{Ran}(A^{-1/2})$ for some $z \in \mathcal{H}$, which in turn implies $x = A^{-1/2}H^{-1}A^{-1}z \in \operatorname{Ran} S = \operatorname{Dom}(S^{-1})$. □

Some further variants of the First Representation Theorem in the situation where H is not boundedly invertible can be found in [**41**].

3.2. The Second Representation Theorem

We now investigate the corresponding Second Representation Theorem for this setting.

THEOREM 3.2.1. *Assume that the hypotheses of the First Representation Theorem 3.1.1 are satisfied. Suppose that additionally the domain stability condition*

$$\mathrm{Dom}(|B|^{1/2}) = \mathrm{Dom}(A^{1/2})$$

is satisfied. Then the form $\mathfrak{b}$ *also satisfies*

$$\mathfrak{b}[x,y] = \langle |B|^{1/2}x, \mathrm{sign}(B)|B|^{1/2}y\rangle, \quad x,y \in \mathrm{Dom}(A^{1/2}) = \mathrm{Dom}(|B|^{1/2}).$$

PROOF. Fix $x \in \mathrm{Dom}(A^{1/2})$, then by the First Representation Theorem 3.1.1, the functionals l_1 and l_2, defined on their natural domains $\mathrm{Dom}(A^{1/2})$, $\mathrm{Dom}(|B|^{1/2})$, respectively, given by

$$l_1(y) := \langle A^{1/2}x, HA^{1/2}y\rangle, \quad l_2(y) := \langle |B|^{1/2}x, \mathrm{sign}(B)|B|^{1/2}y\rangle$$

agree on $\mathrm{Dom}(B) \subseteq \mathrm{Dom}(|B|^{1/2}) = \mathrm{Dom}(A^{1/2})$. We show that this equality extends to $\mathrm{Dom}(A^{1/2})$. Note that $A^{1/2}|B|^{-1/2}$ and $|B|^{1/2}A^{-1/2}$ are bounded by the domain stability condition and the strict positivity of A and $|B|$. Let

$$\delta := \frac{1}{2}\big\|A^{1/2}|B|^{-1/2}\big\|^{-2}.$$

Consider now the two shifted functionals on $\mathrm{Dom}(A^{1/2})$ given by

$$\tilde{l}_1(y) := l_1(y) + \langle A^{1/2}x, \delta\,\mathrm{sign}(H)A^{1/2}y\rangle = \langle A^{1/2}x, (H + \delta\,\mathrm{sign}(H))A^{1/2}y\rangle$$

and

$$\begin{aligned}\tilde{l}_2(y) &:= l_2(y) + \langle A^{1/2}x, \delta\,\mathrm{sign}(H)A^{1/2}y\rangle \\ &= \langle |B|^{1/2}x, \Big(\mathrm{sign}(B) + \delta(A^{1/2}|B|^{-1/2})^*\,\mathrm{sign}(H)A^{1/2}|B|^{-1/2}\Big)|B|^{1/2}y\rangle.\end{aligned}$$

Note that by functional calculus $H + \delta\,\mathrm{sign}(H)$ is bounded and boundedly invertible. Furthermore, the operator

$$G := \mathrm{sign}(B) + \delta(A^{1/2}|B|^{-1/2})^*\,\mathrm{sign}(H)A^{1/2}|B|^{-1/2}$$

is bounded and boundedly invertible since $(-\frac{3}{4}, \frac{3}{4}) \subset \rho(\mathrm{sign}(B))$ and

$$\Big\|\delta(A^{1/2}|B|^{-1/2})^*\,\mathrm{sign}(H)A^{1/2}|B|^{-1/2}\Big\| \le \frac{1}{2}.$$

Consequently, the functionals $\tilde{l}_2$ and $\tilde{l}_2$ coincide on the set $\mathrm{Dom}(B)$, which is dense in $\mathrm{Dom}(|B|^{1/2})$, and are closed on $\mathrm{Dom}(A^{1/2})$. By the uniqueness of the closure, the equality of $\tilde{l}_1$ and $\tilde{l}_2$ extends to $\mathrm{Dom}(|B|^{1/2})$ in the same way as in the proof of Theorem 2.1.1. In this case, also $l_1 = l_2$ holds on $\mathrm{Dom}(|B|^{1/2})$ completing the proof. □

REMARK 3.2.2. *We do not have an appropriate example showing that the requirements of Theorem 3.2.1 can simultaneously be satisfied in full extend, namely*

$$\mathrm{Dom}(|B|^{1/2}) = \mathrm{Dom}(A^{1/2}) \quad \textit{and} \quad \mathcal{H} \supset \mathrm{Dom}(H^{-1}) \supsetneq \mathrm{Dom}(A^{1/2}),$$

so that the inverse of H is unbounded. By the lack of an appropriate example, Theorem 3.2.1 is only of theoretical value in its full extend so far. If instead $\mathrm{Dom}(H^{-1}) = \mathcal{H}$*, then H is bounded, boundedly invertible and the Second Representation Theorem for the form* $\mathfrak{b}$ *is already contained in* [**36**, Theorem 2.10]. *In this case, we obtain a new variant of its proof.*

If H has a bounded inverse, the inclusions between $\operatorname{Dom}(A^{1/2})$ and $\operatorname{Dom}(|B|^{1/2})$ are equivalent (see Theorem 2.2.4). In the situation that H has an unbounded inverse, this equivalence does not hold anymore. A simple example for this is the following.

EXAMPLE 3.2.3. *With the notation as in Definition 1.2.6, introduce*

$$A := \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}, \quad H := \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 1 & 0 \\ 0 & k^{-1/2} \end{pmatrix} = A^{-1/2},$$

where

$$\operatorname{Dom}(A) = \ell^2 \oplus \ell^{2,2}, \quad \operatorname{Dom}(H^{-1}) = \ell^2 \oplus \ell^{2,2} = \operatorname{Dom}(A^{1/2}).$$

The form defined by

$$\mathfrak{b}[x,y] := \langle A^{1/2}x, HA^{1/2}y\rangle, \quad \operatorname{Dom}[\mathfrak{b}] = \operatorname{Dom}(A^{1/2})$$

satisfies the First Representation Theorem 3.1.1 with $B = A^{1/2}HA^{1/2} = A^{1/2}$. Since B is unbounded, we only have the strict inclusion

$$\operatorname{Dom}(|B|^{1/2}) \supset \operatorname{Dom}(B) = \operatorname{Dom}[\mathfrak{b}] = \operatorname{Dom}(A^{1/2}). \tag{3.1}$$

Recall that this behaviour is not possible if H^{-1} is bounded, see Theorem 2.2.4.

In this case, direct computation shows that we have that

$$\mathfrak{b}[x,y] = \langle |B|^{1/2}x, \operatorname{sign}(B)|B|^{1/2}y\rangle, \quad x,y \in \operatorname{Dom}[\mathfrak{b}] \subset \operatorname{Dom}(|B|^{1/2}).$$

However, we can define the form $\tilde{\mathfrak{b}}$ with

$$\tilde{\mathfrak{b}}[x,y] := \langle |B|^{1/2}x, \operatorname{sign}(B)|B|^{1/2}y\rangle, \quad x,y \in \operatorname{Dom}(|B|^{1/2})$$

on a larger domain and we can not only reconstruct the form $\mathfrak{b}$ but even the extension $\tilde{\mathfrak{b}}$ from the associated operator B.

CHAPTER 4

Graph subspaces, Riccati equations and block diagonalisation

This chapter is based on the joint work [**48**] with A. Seelmann and K. A. Makarov.

Here, we investigate the block diagonalisation of unbounded symmetric, diagonally dominant block operator matrices on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$,

$$\begin{pmatrix} A_0 & W^* \\ W & A_1 \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix} + \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix} =: A + V.$$

The considerations of this chapter will be taken over to operators associated with indefinite forms instead of block operator matrices in Chapters 6 and 7 below. Nevertheless, the considerations presented in the current chapter are of interest on their own.

The block diagonalisation of operator matrices, where the off-diagonal part V is additionally assumed to be bounded, has already been investigated by Albeverio, Makarov, and Motovilov in [**3**]. There, it turned out that the decomposition

$$\mathcal{H} = \mathcal{G}(\mathcal{H}_0, X) \oplus \mathcal{G}(\mathcal{H}_0, X)^\perp$$

given by the graph of the operator X reduces the operator $A+V$ if and only if the block operator matrix

$$Y = \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}$$

is a strong solution of the operator Riccati equation

$$AY - YA - YVY + V = 0.$$

As a consequence, the explicit block diagonalisation of $A + V$,

$$(I + Y)^{-1}(A + V)(I + Y) = A + VY = \begin{pmatrix} A_0 + W^*X & 0 \\ 0 & A_1 - WX^* \end{pmatrix}, \tag{4.1}$$

has been obtained in a natural way by the invariance of the graph subspaces. Due to the boundedness of V, the natural domain of $A + VY$ is just $\mathrm{Dom}(A)$.

However, the corresponding proof in [**3**] has a gap in reasoning. Namely, in [**3**, Lemma 5.3 and Theorem 5.5] it has been taken for granted that $I + Y$ maps $\mathrm{Dom}(A)$ onto itself. Discussions with K. A. Makarov confirmed that this indeed requires a detailed proof. This proof has been provided by A. Seelmann for bounded V and will be part of his Ph.D. thesis [**57**].

The present author adapted the technique used by A. Seelmann to show the correspondence between reducing graph subspaces and solutions to the operator Riccati equation in the case of *unbounded* perturbations V under certain regularity conditions on the perturbation.

However, the natural diagonalisation (4.1) is difficult to show if V is unbounded since in this case the domain of $A + VY$ depends explicitly on the operator Y and, a priori, does not have to coincide with $\mathrm{Dom}(A)$. The author noted that this difficulty can be circumvented by considering the alternative block diagonalisation

$$(I - Y)(A + V)(I - Y)^{-1} = A - YV \tag{4.2}$$

of $A+V$ with $\operatorname{Dom}(A-YV)=\operatorname{Dom}(A)$. In contrast to (4.1), the operator $A-YV$ has a domain which is independent of Y even if the perturbation is unbounded.

Sufficient conditions for the correspondence between reducing graph subspaces and solutions to the operator Riccati equation and, hence, for the diagonalisation (4.2) can be given in terms of relative bounds on V with respect to A which can easily be verified.

These results on bounded and unbounded perturbations were published in the joint work [**48**].

A further result of the collaboration [**48**] is that the diagonalisation (4.2) already implies the diagonalisation (4.1), so that (4.2) is stronger in this sense. In Chapter 6 the diagonalisation (4.2) will be used for the correspondence of reducing graph subspaces for the form $\mathfrak{b}$ and solutions to the form Riccati equation.

The content of this chapter does not depend on the preceding chapters.

4.1. Invariant and reducing graph subspaces for operators

Before we can turn to the diagonalisation, we have to fix the following notation and assumptions.

HYPOTHESIS 4.1.1. *Let $\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_1$ be an orthogonal decomposition of the Hilbert space $\mathcal{H}$. Denote by A the self-adjoint block diagonal operator*

$$A:=\begin{pmatrix}A_0 & 0\\ 0 & A_1\end{pmatrix},\quad \operatorname{Dom}(A)=\operatorname{Dom}(A_0)\oplus\operatorname{Dom}(A_1)$$

and by V the symmetric off-diagonal block operator

$$V:=\begin{pmatrix}0 & W^*\\ W & 0\end{pmatrix},\quad \operatorname{Dom}(V)=\operatorname{Dom}(W)\oplus\operatorname{Dom}(W^*),$$

where $W\colon\mathcal{H}_0\supseteq\operatorname{Dom}(W)\to\mathcal{H}_1$ and $W^*\colon\mathcal{H}_1\supseteq\operatorname{Dom}(W^*)\to\mathcal{H}_0$. Furthermore, assume that

$$\operatorname{Dom}(V)\supseteq\operatorname{Dom}(A),$$

so that the operator sum $A+V$ is defined on $\operatorname{Dom}(A+V)=\operatorname{Dom}(A)$.

Note that we changed the roles of W and W^* in comparison to [**48**] for sake of notational accordance with Chapters 5 to 7.

In the following, we introduce the notions of graph subspaces, invariant subspaces, and reducing subspaces of the Hilbert space $\mathcal{H}$.

DEFINITION 4.1.2. *Let $\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_1$ be an orthogonal splitting of the Hilbert space $\mathcal{H}$ and let $X\colon\mathcal{H}_0\to\mathcal{H}_1$ be a bounded operator. Then the subspace*

$$\mathcal{G}(\mathcal{H}_0,X):=\{x\oplus Xx\mid x\in\mathcal{H}_0\}$$

is called the graph subspace *of $\mathcal{H}$ with respect to X and $\mathcal{H}_0$, or shorter, the* graph *of X. In a similar way, we denote the graph of $-X^*$ by*

$$\mathcal{G}(\mathcal{H}_1,-X^*):=\{-X^*x\oplus x\mid x\in\mathcal{H}_1\}.$$

It is well known that

$$\mathcal{G}(\mathcal{H}_0,X)\oplus\mathcal{G}(\mathcal{H}_1,-X^*)=\mathcal{H}$$

gives another orthogonal decomposition of the Hilbert space $\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_1$.

We introduce the bounded block operators Y and T by

$$Y:=\begin{pmatrix}0 & -X^*\\ X & 0\end{pmatrix},\quad T:=I+Y=\begin{pmatrix}I_{\mathcal{H}_0} & -X^*\\ X & I_{\mathcal{H}_1}\end{pmatrix}. \tag{4.3}$$

Then we have the alternative representations

$$\mathcal{G}(\mathcal{H}_0, X) = \operatorname{Ran}\left(T|_{\mathcal{H}_0}\right) \quad \text{and} \quad \mathcal{G}(\mathcal{H}_1, -X^*) = \operatorname{Ran}\left(T|_{\mathcal{H}_1}\right).$$

Definition 4.1.3 ([**56**], Section 1.4). *Let S be a linear operator on the Hilbert space $\mathcal{H}$. Then a closed subspace $\mathcal{G} \subseteq \mathcal{H}$ is called* invariant *for the operator S if*

$$\operatorname{Ran}\left(S|_{\operatorname{Dom}(S)\cap\mathcal{G}}\right) \subseteq \mathcal{G}.$$

If $\mathcal{G}$ and its orthogonal complement $\mathcal{G}^\perp$ are invariant for the operator S and the domain of the operator S splits as

$$\operatorname{Dom}(S) = (\operatorname{Dom}(S) \cap \mathcal{G}) \oplus (\operatorname{Dom}(S) \cap \mathcal{G}^\perp), \tag{4.4}$$

then the subspace $\mathcal{G}$ is called reducing *for the operator S.*

Note that $\mathcal{G}$ is reducing for S if and only if $\mathcal{G}^\perp$ is reducing for S.

Note also that even for self-adjoint operators S the splitting condition (4.4) of the domain is not automatically satisfied if both $\mathcal{G}$ and $\mathcal{G}^\perp$ are invariant, see [**56**, Example 1.8].

An equivalent characterisation for reducing subspaces in terms of projectors is given in Remark 6.2.1 below. However, in this chapter, this alternative characterisation is not used.

The analysis of the mapping properties of the operators Y, T and T^* is crucial in the considerations below. We introduce the following notation for this analysis.

Definition 4.1.4. *Let $X\colon \mathcal{H}_0 \to \mathcal{H}_1$ be a bounded operator and Y as in* (4.3)*. Define the subspace $\mathcal{D} \subseteq \mathcal{H}$ by*

$$\mathcal{D} := \{x \in \operatorname{Dom}(A) \mid Yx \in \operatorname{Dom}(A)\} = \mathcal{D}_0 \oplus \mathcal{D}_1,$$

where

$$\mathcal{D}_0 := \{f \in \operatorname{Dom}(A_0) \mid Xf \in \operatorname{Dom}(A_1)\}, \quad \mathcal{D}_1 := \{g \in \operatorname{Dom}(A_1) \mid X^*g \in \operatorname{Dom}(A_0)\}.$$

By definition, the set $\mathcal{D}$ is the maximal linear subset of $\operatorname{Dom}(A)$ that Y maps into $\operatorname{Dom}(A)$.

We restate the following result of [**48**].

Lemma 4.1.5 ([**48**, Lemma 2.5 and Lemma 2.6]). *Assume Hypothesis 4.1.1. Let $\mathcal{C} \subseteq \operatorname{Dom}(A)$ be a linear subset and let T, Y be the operators as in* (4.3)*. Then:*

(a) *If one of the operators T, T^* and Y maps $\mathcal{C}$ into $\operatorname{Dom}(A)$, then so do the others.*
(b) *The operators T and T^* are both boundedly invertible and T^{-1} maps $\operatorname{Dom}(A)$ into $\operatorname{Dom}(A)$ if and only if $(T^*)^{-1}$ does. In this case, the set $\mathcal{D}$ can alternatively be written as*

$$\mathcal{D} = \operatorname{Ran}\left(T^{-1}|_{\operatorname{Dom}(A)}\right) = \operatorname{Ran}\left((T^*)^{-1}|_{\operatorname{Dom}(A)}\right). \tag{4.5}$$

In view of the lemma above, $\mathcal{D}$ is the maximal linear subset of $\operatorname{Dom}(A)$ that is mapped into $\operatorname{Dom}(A)$ by the operator T and $\mathcal{D}$ is also maximal in this sense for the operator T^*.

We now give a characterisation for invariant graph subspaces.

Lemma 4.1.6 ([**48**, Lemma 2.3]). *Assume Hypothesis 4.1.1. Then the following statements are equivalent.*

(i) *The graph subspaces $\mathcal{G}(\mathcal{H}_0, X)$ and $\mathcal{G}(\mathcal{H}_1, -X^*)$ are both invariant for the operator sum $A + V$;*
(ii) *The operator Y satisfies*

$$AYx - YAx - YVYx + Vx = 0 \quad \text{for } x \in \mathcal{D}; \tag{4.6}$$

(iii) *The operator T^* satisfies*

$$T^*(A+V)x = (A - YV)T^*x \quad \text{for } x \in \mathcal{D}. \tag{4.7}$$

Proof. Noting that

$$\operatorname{Dom}(A) \cap \mathcal{G}(\mathcal{H}_0, X) = \{f \oplus Xf \mid f \in \mathcal{D}_0\}$$

and that

$$(A+V)(f \oplus Xf) = (A_0 f + W^* X f) \oplus (Wf + A_1 X f),$$

we see directly that $\mathcal{G}(\mathcal{H}_0, X)$ is invariant for $A+V$ if and only if

$$(W + A_1 X)f = X(A_0 + W^* X)f \quad \text{for } f \in \mathcal{D}_0,$$

that is

$$A_1 X f - X A_0 f - X W^* X f + W f = 0 \quad \text{for } f \in \mathcal{D}_0. \tag{4.8}$$

In the same way, we see that $\mathcal{G}(\mathcal{H}_1, -X^*)$ is invariant if and only if

$$A_0 X^* g - X^* A_1 g + X^* W X^* g - W^* g = 0 \quad \text{for } g \in \mathcal{D}_1. \tag{4.9}$$

The equivalence of (i) and (ii) then follows by combining (4.8) and (4.9) to the single equation (4.6). Taking into account the identity $T^* = I - Y$, equation (4.7) is just a reformulation of (4.6). This completes the proof. □

Note that condition (ii) states that Y is a strong solution of the Riccati equation $AY - YA - YVY + V = 0$ restricted to $\mathcal{D}$. This equation is also considered in [**47**, Corollary 3.2] by Langer and Tretter under the condition that the spectra of A_0 and A_1 are subordinated, that is $(A_0 x, x) \geq 0$ for $x \in \operatorname{Dom}(A_0)$ and $(A_1 x, x) \leq 0$ for $x \in \operatorname{Dom}(A_1)$ and that the kernel of $A+V$ is trivial.

We are now ready to present the first results on the relation between invariant and reducing subspaces.

Lemma 4.1.7 ([**48**, Lemma 2.7]). *Assume Hypothesis 4.1.1. Then the following statements are equivalent.*

(i) *The graph subspace $\mathcal{G}(\mathcal{H}_0, X)$ is reducing for $A+V$;*
(ii) *The graph subspaces $\mathcal{G}(\mathcal{H}_0, X)$ and $\mathcal{G}(\mathcal{H}_1, -X^*)$ are both invariant for $A+V$ and the operator T^{-1} (respectively $(T^*)^{-1}$) maps $\operatorname{Dom}(A)$ into itself;*
(iii) *The operator inclusion*

$$T^*(A+V) \supseteq (A - YV)T^* \tag{4.10}$$

holds.

Proof. Note that we have the following intersections of the domain with the graph subspaces:

$$\operatorname{Dom}(A+V) \cap \mathcal{G}(\mathcal{H}_0, X) = \{f \oplus Xf \mid f \in \mathcal{D}_0\} = \operatorname{Ran}(T|_{\mathcal{D}_0}),$$
$$\operatorname{Dom}(A+V) \cap \mathcal{G}(\mathcal{H}_1, -X^*) = \{-X^* g \oplus g \mid f \in \mathcal{D}_1\} = \operatorname{Ran}(T|_{\mathcal{D}_1}).$$

This yields that

$$\big(\operatorname{Dom}(A+V) \cap \mathcal{G}(\mathcal{H}_0, X)\big) \oplus \big(\operatorname{Dom}(A+V) \cap \mathcal{G}(\mathcal{H}_1, -X^*)\big) = \operatorname{Ran}\left(T|_{\mathcal{D}}\right).$$

Therefore, $\mathcal{G} := \mathcal{G}(\mathcal{H}_0, X)$ is reducing for $A+V$ if and only if both $\mathcal{G}$ and $\mathcal{G}^\perp$ are invariant and

$$\operatorname{Dom}(A) = \operatorname{Dom}(A+V) = \operatorname{Ran}\left(T|_{\mathcal{D}}\right). \tag{4.11}$$

With Lemma 4.1.5, it follows that (4.11) holds if and only if T^{-1}, respectively $(T^*)^{-1}$, maps $\operatorname{Dom}(A)$ into itself. This gives the equivalence of (i) and (ii).

Suppose that (i) holds. Then by the invariance of the graph and Lemma 4.1.6, we have that

$$T^*(A+V)x = (A-YV)T^*x \quad \text{for } x \in \mathcal{D}.$$

Together with (4.11) and the alternative representation (4.5) of $\mathcal{D}$ in terms of T^* in Lemma 4.1.5, we also have that

$$\operatorname{Dom}(A) = \operatorname{Ran}\left(T^*|_{\mathcal{D}}\right). \tag{4.12}$$

With $\operatorname{Dom}(A-YV) = \operatorname{Dom}(A)$ and (4.12) it follows that

$$\operatorname{Dom}\big((A-YV)T^*\big) = \mathcal{D} \subseteq \operatorname{Dom}(A+V) = \operatorname{Dom}\big(T^*(A+V)\big).$$

Hence, (iii) is shown.

Conversely, assume that (iii) holds. The right-hand side of (4.10) has natural domain $\mathcal{D}$, so that

$$T^*(A+V)x = (A-YV)T^*x \quad \text{for } x \in \mathcal{D}.$$

By Lemma 4.1.6 both graph subspaces $\mathcal{G}(\mathcal{H}_0, X)$ and $\mathcal{G}(\mathcal{H}_1, -X^*)$ are invariant for $A+V$. Moreover, we have that

$$\operatorname{Ran}\Big((T^*)^{-1}|_{\operatorname{Dom}(A)}\Big) = \operatorname{Dom}((A-YV)T^*) \subseteq \operatorname{Dom}(T^*(A+V)) = \operatorname{Dom}(A).$$

Thus (ii) holds and the proof is completed. □

4.2. Operator Riccati equation and block diagonalisation

We now investigate the inclusion $T^*(A+V) \subseteq (A-YV)T^*$ which is converse to the one in (4.10). This inclusion is related to the operator Riccati equation.

LEMMA 4.2.1 ([**48**, Lemma 2.9]). *Assume Hypothesis 4.1.1. Then the inclusion*

$$T^*(A+V) \subseteq (A-YV)T^*$$

holds if and only if Y is a strong solution of the operator Riccati equation

$$AY - YA - YVY + V = 0, \tag{4.13}$$

that is $\operatorname{Ran}\left(Y|_{\operatorname{Dom}(A)}\right) \subseteq \operatorname{Dom}(A)$ *and*

$$AYx - YAx - YVYx + Vx = 0 \quad \textit{for } x \in \operatorname{Dom}(A).$$

In this case the domain equality $\operatorname{Dom}(A) = \mathcal{D}$ *holds.*

PROOF. Note that by Lemma 4.1.5 the operator Y maps $\operatorname{Dom}(A)$ into itself if and only if T^* does. In this case, by the maximality of $\mathcal{D} \subseteq \operatorname{Dom}(A)$ as a set being mapped by Y into $\operatorname{Dom}(A)$, the equality $\operatorname{Dom}(A) = \mathcal{D}$ follows. The equivalence of the statements then follows from the observation that

$$\operatorname{Dom}(A) = \operatorname{Dom}(A+V) = \operatorname{Dom}(A-YV).$$ □

A direct combination of Lemma 4.1.7 and Lemma 4.2.1 gives the following result.

THEOREM 4.2.2 ([**48**, Theorem 2.10]). *Assume Hypothesis 4.1.1. Then, the operator sum $A+V$ admits the block diagonalisation*

$$T^*(A+V)(T^*)^{-1} = A - YV = \begin{pmatrix} A_0 + X^*W & 0 \\ 0 & A_1 - XW^* \end{pmatrix} \tag{4.14}$$

if and only if

(i) *the graph subspace $\mathcal{G}(\mathcal{H}_0, X)$ is reducing for $A+V$, and*

(ii) *the operator Y is a strong solution of the operator Riccati equation*

$$AY - YA - YVY + V = 0.$$

4.3. Operator Riccati equation and reducing graph subspaces

We now address the question when the conditions (i) and (ii) in Theorem 4.2.2 are equivalent. In this case, one inclusion between the operators $T^*(A+V)$ and $(A-YV)T^*$ implies even the equality. Furthermore, the reducing subspace $\mathcal{G} = \mathcal{G}(\mathcal{H}_0, X)$ is then associated with the strong solution X of the operator Riccati equation.

In the special case that the graph subspace $\mathcal{G} := \mathcal{G}(\mathcal{H}_0, X)$ is also a spectral subspace of $A+V$, that is,

$$\mathcal{G} = \operatorname{Ran}(\mathsf{E}_{A+V}(M))$$

for some Borel set $M \subset \mathbb{R}$, it is clear that $\mathcal{G}$ is reducing for $A+V$. In this case, if the equivalence of (i) and (ii) in Theorem 4.2.2 takes place, we have that the operator $A+V$ can be block diagonalised with respect to the spectral decomposition into M and $\mathbb{R} \setminus M$. This diagonalisation can then be given explicitly in terms of the solution X of the Riccati equation.

The main tool to investigate under which assumptions the equivalence of (i) and (ii) in Theorem 4.2.2 holds is the following simple observation.

Lemma 4.3.1 ([**56**, Lemma 1.3]). *Let T_1 and T_2 be two linear operators such that $T_2 \subseteq T_1$. If T_1 is injective and T_2 surjective, then $T_1 = T_2$.*

Proof. For sake of completeness, we reproduce the proof here.

Let $x \in \operatorname{Dom}(T_1)$. Since T_2 is surjective by hypothesis, there is $y \in \operatorname{Dom}(T_2)$ such that $T_1x = T_2y$. By the inclusion $T_2 \subseteq T_1$, we obtain that $T_1x = T_1y$, which, by the injectivity of T_1, implies that $x = y \in \operatorname{Dom}(T_2)$. This yields that $\operatorname{Dom}(T_1) \subseteq \operatorname{Dom}(T_2)$ and thus $T_1 = T_2$. $\square$

For our purposes, we give the following variant of Lemma 4.3.1.

Corollary 4.3.2 ([**48**, Corollary 3.2 and Remark 3.3]). *Let T_1 and T_2 be linear operators on Hilbert spaces $\mathcal{H}_1$, and $\mathcal{H}_2$, respectively, and let $S\colon \mathcal{H}_1 \to \mathcal{H}_2$ be an isomorphism. Suppose that*

$$ST_1 \subseteq T_2S.$$

If there exists some constant $\lambda \in \mathbb{C}$ such that $T_1 - \lambda$ is surjective and $T_2 - \lambda$ is injective, then

$$ST_1 = T_2S \tag{4.15}$$

holds as an operator equality. In this case, the operators $T_1 - \lambda$ and $T_2 - \lambda$ are both bijective.

In particular, the equality (4.15) holds if T_1 and T_2 are closed operators with intersecting resolvent sets, that is, $\rho(T_1) \cap \rho(T_2) \neq \varnothing$.

As a consequence, we get the following conditions for the implications between (i) and (ii) in Theorem 4.2.2. These conditions are of an a posteriori type since they involve the operator Y.

Theorem 4.3.3 ([**48**, Theorem 3.4]). *Assume Hypothesis 4.1.1.*

(a) *Let $\mathcal{G}(\mathcal{H}_0, X)$ be a reducing graph subspace for $A+V$. Then Y is a strong solution of the operator Riccati equation for $A+V$,*

$$AY - YA - YVY + V = 0$$

if there is some constant $\lambda \in \mathbb{C}$ such that

$$A + V - \lambda \text{ is injective} \quad \text{and} \quad A - YV - \lambda \text{ is surjective.} \tag{4.16}$$

(b) *Let Y be a strong solution of the operator Riccati equation* (4.13)*. Then the graph subspace $\mathcal{G}(\mathcal{H}_0, X)$ is reducing for $A+V$ if there is some constant $\lambda \in \mathbb{C}$ such that*

$$A+V-\lambda \text{ is surjective} \quad \text{and} \quad A-YV-\lambda \text{ is injective.} \tag{4.17}$$

PROOF. (a) By Lemma 4.1.7, the property of $\mathcal{G}(\mathcal{H}_0, X)$ to be reducing for $A+V$ can be written as

$$(A-YV)T^* \subseteq T^*(A+V).$$

In this case, we also have the reformulation

$$(T^*)^{-1}(A-YV) \subseteq (A+V)(T^*)^{-1}.$$

Considering the isomorphism $S := (T^*)^{-1}$, it follows from Corollary 4.3.2 that even equality in the inclusion above holds. Thus, we have that

$$(T^*)^{-1}(A-YV) = (A+V)(T^*)^{-1}$$

giving the block diagonalisation and the claim follows by Theorem 4.2.2.

(b) By Lemma 4.2.1 the operator Y is a strong solution of the Riccati equation if and only if

$$T^*(A+V) \subseteq (A-YV)T^*.$$

With Corollary 4.3.2 for $S := T^*$, the operator equality

$$T^*(A+V) = (A-YV)T^*$$

holds. Again, this gives the block diagonalisation as in Theorem 4.2.2 and the claim follows. ☐

In view of Corollary 4.3.2, we get in both conditions (4.16) and (4.17) that the operators $A+V-\lambda$ and $A-YV-\lambda$ are even bijective. However, in order to apply the theorem, it suffices to check the injectivity, respectively surjectivity, for the corresponding operators.

Recall that we imposed the condition $\operatorname{Dom}(V) \supseteq \operatorname{Dom}(A)$ as a general assumption for this chapter. However, this assumption was not essential in the considerations above.

REMARK 4.3.4 (cf. [**48**, Remark 3.7]). *The condition*

$$\operatorname{Dom}(V) \supseteq \operatorname{Dom}(A)$$

in Hypothesis 4.1.1 can be dropped in the considerations above. Namely, the statements remain valid if $\operatorname{Dom}(A)$ *is substituted everywhere by* $\operatorname{Dom}(A) \cap \operatorname{Dom}(V)$*. In this case, Y is said to be a strong solution of the operator Riccati equation $AY - YA - YVY + V = 0$ if*

$$\operatorname{Ran}\left(Y|_{\operatorname{Dom}(A)\cap\operatorname{Dom}(V)}\right) \subseteq \operatorname{Dom}(A) \cap \operatorname{Dom}(V)$$

and

$$AYx - YAx - YVYx + Vx = 0 \quad \text{for } x \in \operatorname{Dom}(A) \cap \operatorname{Dom}(V).$$

In turn, the set $\mathcal{D}$ has to be redefined as

$$\mathcal{D} := \{x \in \operatorname{Dom}(A) \cap \operatorname{Dom}(V) \mid Yx \in \operatorname{Dom}(A) \cap \operatorname{Dom}(V)\}.$$

Note that in general the intersection $\operatorname{Dom}(A) \cap \operatorname{Dom}(V)$ can be trivial. If this is the case, it is clear that investigation of $A+V$ makes no sense. It makes sense to consider this operator sum if it is at least densely defined. We revisit the case, where $\operatorname{Dom}(A)$ is in general not a subspace of $\operatorname{Dom}(V)$ in Lemma 6.3.8 below. There, the density of the intersection $\operatorname{Dom}(A) \cap \operatorname{Dom}(V)$ can be granted.

In Theorem 4.3.3, the conditions imposed on $A+V$ do not involve Y, so that these conditions are of *a priori* type. The conditions imposed on $A-YV$ however are of *a posteriori* type since they explicitly depend on the operator Y.

It is thus natural to seek for stronger a priori conditions that imply the a posteriori conditions in (4.16) and (4.17), respectively. Classical perturbation results in [**43**] can be applied to grant these conditions.

LEMMA 4.3.5 (cf. [**48**, Lemma 4.3]). *Assume Hypothesis 4.1.1. If the operators V and YV are both A-bounded with A-bound less than one, then $A+V$ is self-adjoint, $A-YV$ is closed and the intersection of the resolvent sets is not empty, that is,*

$$\rho(A+V)\cap\rho(A-YV)\neq\varnothing.$$

In particular, this is the case if V is infinitesimal with respect to A.

PROOF. Since the perturbation V is symmetric and A is self-adjoint, it follows from [**43**, Theorem V.4.3] that the operator $A+V$ also is self-adjoint on $\operatorname{Dom}(A)$. To prove the claim, it therefore suffices to show that $A-YV$ is closed and that there is a constant $k\geq 0$ such that

$$\mathrm{i}\lambda\in\rho(A-YV)\quad\text{for } \lambda\in\mathbb{R},\ |\lambda|>k.$$

In order to prove this, choose $a\geq 0$ and $0\leq b<1$ such that

$$\|YVx\|\leq a\,\|x\|+b\,\|Ax\|\quad\text{for } x\in\operatorname{Dom}(A).$$

For $\lambda\neq 0$, it follows from [**43**, Theorem V.3.16] that

$$\big\|(A-\mathrm{i}\lambda)^{-1}\big\|\leq|\lambda|^{-1}\quad\text{and}\quad\big\|A(A-\mathrm{i}\lambda)^{-1}\big\|<1. \tag{4.18}$$

Define

$$k:=\frac{a}{1-b}\geq 0,$$

and let $\lambda\in\mathbb{R}$ with $|\lambda|>k$. Then, the estimates in (4.18) imply that

$$a\,\big\|(A-\mathrm{i}\lambda)^{-1}\big\|+b\,\big\|A(A-\mathrm{i}\lambda)^{-1}\big\|<\frac{a}{|\lambda|}+b<1. \tag{4.19}$$

Hence, by [**43**, Theorem IV.3.17], the operator $A-YV$ is closed, and $\mathrm{i}\lambda$ belongs to its resolvent set. □

We now combine Lemma 4.3.5 and the Theorems 4.3.3 and 4.2.2 to obtain the following block diagonalisation for certain diagonally dominant block operator matrices.

THEOREM 4.3.6. *Assume Hypothesis 4.1.1. Suppose furthermore that V is infinitesimal with respect to A and that $X\colon\mathcal{H}_0\to\mathcal{H}_1$ is bounded. Then the graph space $\mathcal{G}(\mathcal{H}_0,X)$ is reducing for the operator $A+V$ if and only if*

$$Y=\begin{pmatrix}0&-X^*\\X&0\end{pmatrix}$$

is a strong solution of the operator Riccati equation

$$AY-YA-YVY+V=0, \tag{4.20}$$

that is

$$\operatorname{Ran}\big(Y|_{\operatorname{Dom}(A)}\big)\subseteq\operatorname{Dom}(A)$$

and

$$AYx-YAx-YVYx+Vx=0\quad\textit{for } x\in\operatorname{Dom}(A).$$

In this case, the operators $T=I+Y$ and $T^*=I+Y$ are bounded, boundedly invertible on $\mathcal{H}$ and bijective on $\operatorname{Dom}(A)$. *Furthermore, the equality*

$$T^*(A+V)=(A-YV)T^* \tag{4.21}$$

holds.

In Theorem 6.1.9 below, we will obtain a criterion for the spectral subspace $\mathsf{E}_{A+V}(\mathbb{R}_+)$ to be a graph subspace with respect to the first component in the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. In view of Theorem 4.3.6, this yields the solvability of the Riccati equation since the spectral graph subspace clearly is reducing.

4.4. Sylvester equations, alternative and unitary diagonalisation

The results we obtained in this chapter so far relate $A+V$ to the operator $A-YV$. However, it is also possible to relate $A+V$ to the operator $A+VY$ instead. Recall that for bounded perturbations V, the latter has been investigated already in [**3**, Lemma 5.3, Theorem 5.5], see also [**57**]. It is natural to try to extend these results on the block diagonalisation of $A+V$ with respect to $A+VY$ as well as their relation to solutions to the Riccati equation to unbounded perturbations V. If the perturbation V is unbounded, the operator $A+VY$ is more difficult to investigate than $A-YV$ since its domain depends on Y. For the operator $A+VY$ it is in general not clear that there exists a point $\lambda \in \rho(A+VY) \cap \rho(A+V)$ or that this operator is even closed on its natural domain $\operatorname{Dom}(A) \cap \operatorname{Dom}(VY)$. Without further assumptions we cannot apply the crucial Lemma 4.3.1 on $A+VY$ instead of $A-YV$.

We reproduce those results related to $A+VY$ in [**57**], respectively [**48**], that still hold for unbounded perturbations V.

Furthermore, we give a comparison to the results presented here and combine the results to a unitary block diagonalisation.

THEOREM 4.4.1 ([**48**, Theorem 2.13], cf. [**3**, Lemma 5.3, Theorem 5.5]). *Assume Hypothesis 4.1.1. Then the operator $A+V$ admits the diagonalisation*

$$T^{-1}(A+V)T = A+VY = \begin{pmatrix} A_0 + W^*X & 0 \\ 0 & A_1 - WX^* \end{pmatrix} \tag{4.22}$$

if and only if

(i) *the graph subspace $\mathcal{G}(\mathcal{H}_0, X)$ is reducing for $A+V$, and*

(ii) *the operator Y satisfies* $\operatorname{Ran}\left(Y|_{\operatorname{Dom}(A+VY)}\right) \subseteq \operatorname{Dom}(A)$ *and*

$$Ax - YAx - YVYx + Vx = 0 \quad \textit{for } x \in \operatorname{Dom}(A+VY).$$

Note that a diagonalisation similar to (4.22) also appears in [**47**]. For the case considered there ([**47**, Corollary 3.2]), it is required that the spectra of A_0 and A_1 are subordinated in the sense that A_0 is non-negative and A_1 is non-positive and that the kernel of $A+V$ is trivial. Furthermore, it is supposed that the graph $\mathcal{G}(\mathcal{H}, X)$ is a spectral subspace which is automatically reducing. The considerations we present here do not require additional information of this type.

We now compare the diagonalisation that can be obtained from (4.21) to the one given by (4.22).

REMARK 4.4.2 (cf. [**48**, Theorem 2.10]). *Since* $\operatorname{Dom}(A+VY)$ *is a subset of* $\operatorname{Dom}(A)$*, it is clear that the conditions in (ii) of Theorem 4.2.2 are stronger than the corresponding conditions in Theorem 4.4.1. In this sense, also the related diagonalisation is stronger. Namely, if the diagonalisation*

$$T^*(A+V)(T^*)^{-1} = A - YV \tag{4.23}$$

holds, then so does

$$T^{-1}(A+V)T = A+VY. \tag{4.24}$$

Remark 4.4.3 (cf. [**48**, Remark 2.11]).
The two block diagonalisations (4.23) *and* (4.24) *can be combined to*

$$T^*T(A+VY)(T^*T)^{-1} = A - VY.$$

In particular, the operators $A+V$, $A+VY$ and $A-YV$ are similar to one another. Note that

$$T^*T = \begin{pmatrix} I_{\mathcal{H}_0} + X^*X & 0 \\ 0 & I_{\mathcal{H}_1} + XX^* \end{pmatrix} = TT^*$$

is normal and block diagonal. With the polar decomposition (see, e.g., [**43**, Section VI.2.7]*) of $T = U|T|$ as well as $T^* = |T|U^* = |T^*|U$, where U is a unitary transformation, we get that $A+V$ is unitary equivalent to a block diagonal matrix (cf.* [**3**, Theorem 5.5(iii)]*). Namely,*

$$U^*(A+V)U = |T|(A+VY)|T|^{-1} = |T|^{-1}(A-YV)|T| = \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix},$$

where

$$B_0 := (I_{\mathcal{H}_0} + X^*X)^{1/2}(A_0 + W^*X)(I_{\mathcal{H}_0} + X^*X)^{-1/2}$$

and

$$B_1 := (I_{\mathcal{H}_1} + XX^*)^{1/2}(A_1 - WX^*)(I_{\mathcal{H}_1} + XX^*)^{-1/2}$$

on their natural domains

$$\operatorname{Dom}(B_0) = \operatorname{Ran}\left((I_{\mathcal{H}_0} + X^*X)^{1/2}\Big|_{\operatorname{Dom}(A_0)} \right),$$

$$\operatorname{Dom}(B_1) = \operatorname{Ran}\left((I_{\mathcal{H}_1} + XX^*)^{1/2}\Big|_{\operatorname{Dom}(A_1)} \right).$$

In this sense, both bock diagonalisations (4.23) *and* (4.24) *lead to the same unitary block diagonalisation.*

In Chapter 7 below, we give a unitary diagonalisation of the Stokes operator B_S introduced in Chapter 5 by a form sum.

If both diagonalisations hold, then $A+VY$ and $A-YV$ are similar to each other. We give a simple explanation for the appearance of both operators in our considerations. Recall that the Riccati equation is essential for the investigation of reducing graph subspaces. We have two possibilities to simplify the Riccati equation to a Sylvester equation. Namely, we have

$$0 = (A - YV)Y - YA + V = AY - YA - YVY + V = AY - Y(A+VY) + V = 0.$$

In the left-hand equation, Y is a solution of a Sylvester equation with the coefficients $A - YV$ and A. In the right-hand equation, Y is a strong solution of the corresponding equation with the coefficients A and $A+VY$. These two equations clearly carry the same information but only in the left-hand equation we can determine the domain of the coefficients without solving the equation. This freedom of bracketing explains why there are two diagonalisations of $A+V$, the usual related to $A+VY$ and the new one related to $A-YV$.

As it turns out the operators $A+VY$ and $A-YV$ are not only similar to $A+V$ but also to $A-V$. This symmetry yields a sign-symmetry of the spectrum of $A+V$ with respect to the perturbation V.

Remark 4.4.4 (cf. [**48**, Remark 2.12]). *Introducing the symmetry*

$$J = J^* = \begin{pmatrix} I_{\mathcal{H}_0} & 0 \\ 0 & -I_{\mathcal{H}_1} \end{pmatrix},$$

it is easy to see that we have the unitary transformation

$$J(A+V)J = A + JVJ = A - V.$$

Also, a short computation shows that $\mathcal{G}(\mathcal{H}_0, X)$ *is a reducing graph subspace for* $A+V$ *if and only if* $\mathcal{G}(\mathcal{H}_0, -X)$ *is reducing for* $A-V$*. In a similar way, we have that* Y *is a strong solution of the corresponding operator Riccati equation* (4.13) *for* $A+V$ *if and only if* $JYJ = -Y$ *is a strong solution of the Riccati equation for* $A-V$*. As a consequence, the spectra of* $A+V$ *and* $A-V$ *agree and the operators can be diagonalised at the same time. For sums of forms, a similar observation can be made, see Lemma 6.3.9.*

For a more detailed study of the relation between $A+V$ and $A+YV$ for unbounded V, see [**48**]. An extensive study of the case of bounded perturbations V is contained in the Ph. D. thesis [**57**] of A. Seelmann.

CHAPTER 5

The Stokes operator

5.1. Definition of the Stokes operator

This chapter is based on the joint work [**38**] with L. Grubišić, V. Kostrykin, K. A. Makarov, and K. Veselić.

We apply the results of Chapter 1 to define the Stokes operator B_S by the First Representation Theorem and compute its kernel. The Stokes operator is of interest in fluid dynamics and is a simple example for the approach of Chapter 1 since its form is bounded from below.

Recall that the Stokes operator is related to the stationary linearised Stokes system

$$-\nu\Delta u + \operatorname{grad} p = f, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0,$$

where $\nu > 0$ is the viscosity, see, e.g., [**61**].

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$ with Lipschitz boundary. We allow Ω to be bounded or unbounded. We introduce the following Hilbert spaces

$$\mathcal{H}_+ = L^2(\Omega)^n := L^2(\Omega) \otimes \mathbb{C}^n, \quad \mathcal{H}_- := L^2(\Omega) \quad \text{and} \quad \mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-.$$

We consider the Sobolev space $H_0^1(\Omega)^n \subset L^2(\Omega)^n$ of vector-valued functions on Ω given by the closure of $C_0^\infty(\Omega)^n$ with respect to the Sobolev norm

$$\|u\|^2_{H^1(\Omega)^n} := \int_\Omega \langle u(x), u(x)\rangle_{\mathbb{C}^n} dx + \sum_{i=1}^n \int_\Omega \langle D_i u(x), D_i u(x)\rangle_{\mathbb{C}^n} dx,$$

where $D_i u$ denotes the partial derivative of u with respect to the i-th component of the vector x.

Definition 5.1.1 ([**38**]). *For arbitrary $\varphi, \psi \in H_0^1(\Omega)^n \oplus L^2(\Omega)$ define the form $\mathfrak{b}_S$ by*

$$\mathfrak{b}_S[\varphi,\psi] := \sum_{j=1}^n \int_\Omega \langle D_j u(x), D_j v(x)\rangle dx + \int_\Omega \Big(q(x)\, \overline{\operatorname{div} u(x)} + \overline{p(x)} \operatorname{div} v(x) \Big)\, dx, \tag{5.1}$$

where $\varphi = u \oplus p$ and $\psi = v \oplus q$.

This form is associated with the differential expression appearing on the left-hand side of the Stokes system

$$\begin{pmatrix} -\boldsymbol{\Delta} & -\operatorname{grad} \\ \operatorname{div} & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \tag{5.2}$$

where f is an external forcing and $-\boldsymbol{\Delta}$ is the componentwise application of the Laplacian $-\Delta$. In this sense, we have $\boldsymbol{\Delta}(u_1,\dots,u_n) = (\Delta u_1,\dots,\Delta u_n)$ and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

We consider the form $\mathfrak{b}_S$ in (5.1) as a perturbation of the closed non-negative sesquilinear form $\mathfrak{a}$ on $H_0^1(\Omega)^n \oplus L^2(\Omega)$ given by

$$\mathfrak{a}[\varphi,\psi] := \mathfrak{a}_+[u,v] - \mathfrak{a}_-[p,q] \quad \text{with} \quad \mathfrak{a}_- := 0$$

and

$$\mathfrak{a}_+[u,v] := \sum_{j=1}^{n} \int_\Omega \langle D_j u(x), D_j v(x)\rangle_{\mathbb{C}^n} dx.$$

The domain of the self-adjoint operator $A_+ = -\boldsymbol{\Delta}$ associated with the form $\mathfrak{a}_+$ is precisely $(H_0^1(\Omega) \cap \{u \in L^2(\Omega) \mid \Delta u \in L^2(\Omega)\})^n$. This follows from the consideration of the components $u_1, \dots, u_n$ of $u \in L^2(\Omega)^n$ and application of [**43**, Example VI.2.13], to the operators

$$S := \mathrm{grad}, \quad \mathrm{Dom}(S) = H_0^1(\Omega)$$

and

$$S^* = -\mathrm{div}\,, \quad \mathrm{Dom}(S^*) = E^2(\Omega) := \{u \in L^2(\Omega)^n \mid \mathrm{div}\, u \in L^2(\Omega)\}.$$

In the special case that Ω is bounded and either convex or with Hölder $C^{1,1}$-boundary, we have the explicit representation

$$\mathrm{Dom}(A_+) = (H_0^1(\Omega) \cap H^2(\Omega))^n$$

by [**34**, Theorems 2.2.2.3 and 3.2.1.2].

We now define the Stokes operator B_S associated with the form $\mathfrak{b}_S$.

Theorem 5.1.2 (The Stokes Operator [**38**]). *The form $\mathfrak{b}_S$ given by* (5.1) *admits the representation*

$$\mathfrak{b}_S[\varphi,\psi] = \langle \varphi, B_S\psi\rangle,\ \varphi \in H_0^1(\Omega)^n \oplus L^2(\Omega),\ \psi \in \mathrm{Dom}(B_S) \subset H_0^1(\Omega)^n \oplus L^2(\Omega) \tag{5.3}$$

for a unique self-adjoint operator B_S. This operator will be called the Stokes operator.

Proof. We introduce the diagonal part of $\mathfrak{b}_S$ in the following way.

$$\mathfrak{a}[\varphi,\psi] := \sum_{j=1}^{n} \int_\Omega \langle D_j u(x), D_j v(x)\rangle_{\mathbb{C}^n} dx,$$

then the non-negative form $\mathfrak{a}$ is represented by the block operator matrix $\begin{pmatrix} -\boldsymbol{\Delta} & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the decomposition given by the involution J_A with $J_A(u \oplus p) = u \oplus (-p)$. The remaining off-diagonal part $\mathfrak{v}$ of $\mathfrak{b}_S$ is given by

$$\mathfrak{v}[\varphi,\psi] := \int_\Omega q(x)\,\overline{\mathrm{div}\, u(x)}dx + \int_\Omega \overline{p(x)}\,\mathrm{div}\, v(x)dx.$$

Using Theorem 1.5.3, it remains to show the $(\mathfrak{a}+I)$-boundedness of the off-diagonal part $\mathfrak{v}$ to define the Stokes operator B_S.

Let $\varphi := u \oplus p$, then it suffices to estimate

$$\begin{aligned} 2\left|\int_\Omega p\overline{\mathrm{div}\, u}dx\right| &\le 2\|p\|_{L^2(\Omega)}\|\mathrm{div}\, u\|_{L^2(\Omega)} \\ &\le \|\mathrm{div}\, u\|_{L^2(\Omega)}^2 + \|p\|_{L^2(\Omega)}^2 \\ &\le \mathfrak{a}_+[u] + \|p\|_{L^2(\Omega)}^2 \\ &\le (\mathfrak{a}+I)[u \oplus p], \qquad p \in L^2(\Omega), \quad u \in H_0^1(\Omega)^n. \end{aligned} \tag{5.4}$$

The estimate $\|\operatorname{div} u\|^2_{L^2(\Omega)} \leq \mathfrak{a}_+[u]$ holds since for $u = (u_1, \dots, u_n) \in C_0^\infty(\Omega)^n$ we have the identity

$$\begin{aligned}\mathfrak{a}_+[u] - \|\operatorname{div} u\|^2 &= \sum_{i,j=1}^n \int_\Omega \left|\frac{\partial u_i}{\partial x_j}\right|^2 dx - \int_\Omega \left|\sum_{i=1}^n \frac{\partial u_i}{\partial x_i}\right|^2 dx \\ &= \sum_{i,j=1}^n \int_\Omega \left|\frac{\partial u_i}{\partial x_j}\right|^2 dx - \sum_{i,j=1}^n \int_\Omega \overline{\frac{\partial u_i}{\partial x_i}} \frac{\partial u_j}{\partial x_j} dx.\end{aligned}$$

We thus obtain that

$$\begin{aligned}\mathfrak{a}_+[u] - \|\operatorname{div} u\|^2 &= \sum_{i,j=1, i\neq j}^n \int_\Omega \left(\left|\frac{\partial u_i}{\partial x_j}\right|^2 - \overline{\frac{\partial u_i}{\partial x_i}} \frac{\partial u_j}{\partial x_j}\right) dx \\ &= \sum_{\substack{i,j=1\\ i<j}}^n \int_\Omega \left(\left|\frac{\partial u_i}{\partial x_j}\right|^2 + \left|\frac{\partial u_j}{\partial x_i}\right|^2 - 2\mathrm{Re}\, \overline{\frac{\partial u_i}{\partial x_i}} \frac{\partial u_j}{\partial x_j}\right) dx \\ &= \sum_{\substack{i,j=1\\ i<j}}^n \int_\Omega \left|\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right|^2 dx + 2\mathrm{Re} \sum_{\substack{i,j=1\\ i<j}}^n \int_\Omega \left(\overline{\frac{\partial u_i}{\partial x_j}} \frac{\partial u_j}{\partial x_i} - \overline{\frac{\partial u_i}{\partial x_i}} \frac{\partial u_j}{\partial x_j}\right) dx.\end{aligned}$$

Suppose that $\operatorname{supp}(u_i) \subseteq B_r(0)$ for some $r > 0$ and all $i \in \{1, \dots, n\}$. Extending each u_i by zero to $\tilde{u}_i \in C_0^\infty(\mathbb{R}^n)$, integration by parts yields

$$\int_\Omega \left(\overline{\frac{\partial u_i}{\partial x_j}} \frac{\partial u_j}{\partial x_i} - \overline{\frac{\partial u_i}{\partial x_i}} \frac{\partial u_j}{\partial x_j}\right) dx = \int_{B_{2r}(0)} \left(\overline{\frac{\partial \tilde{u}_i}{\partial x_j}} \frac{\partial \tilde{u}_j}{\partial x_i} - \overline{\frac{\partial \tilde{u}_i}{\partial x_i}} \frac{\partial \tilde{u}_j}{\partial x_j}\right) dx = 0.$$

By density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$, the claim follows. □

Note that there are also alternative approaches to define the Stokes operator B_S.

REMARK 5.1.3. (a) *An approach to define the Stokes operator is to consider the Helmholtz projector P and to define the operator product $-\nu P\Delta$ on the Sobolev space of divergence free vectorfields, see, e.g., the book of Sohr, in particular* [**61**, Section III.2].

(b) *The Stokes operator can also be defined as the block operator matrix*

$$\begin{pmatrix} -\boldsymbol{\Delta} & -\operatorname{grad} \\ \operatorname{div} & 0 \end{pmatrix}.$$

Its properties can be investigated directly by studying the Schur complement as in [**21**] *by Faierman, Fries, Mennicken, and Möller.*

(c) *An approach based on pseudo-differential operators is contained in* [**35**] *by Grubb and Greymonat.*

(d) *Another form based approach to define the Stokes operator is closely related to the one we presented and uses the KLMN Theorem. Using the Young Inequality, we obtain that the form $\mathfrak{v}$ is infinitesimal with respect to the non-negative form $\mathfrak{a}$. Namely, for arbitrary $\varepsilon > 0$ we have*

$$2\left|\int_\Omega p\overline{\operatorname{div} u} dx\right| \leq \varepsilon \mathfrak{a}[u \oplus p] + \frac{1}{\varepsilon} \|u \oplus p\|^2. \tag{5.5}$$

Thus, by the KLMN Theorem (see, e.g., [**55**, Theorem X.17]*), there is a unique self-adjoint operator B associated with the form $\mathfrak{b}_S$ such that*

$$\operatorname{Dom}(B) \subseteq H_0^1(\Omega)^n \oplus L^2(\Omega)$$

allowing the representation

$$\mathfrak{b}_S[\varphi,\psi] = \langle \varphi, B\psi\rangle, \quad \varphi \in \mathrm{Dom}[\mathfrak{b}_S],\ \psi \in \mathrm{Dom}(B).$$

By the uniqueness in Theorem 5.1.2, this operator agrees with B_S.

An important difference between the form $\mathfrak{b}_S$ and the operator B_S is the following.

REMARK 5.1.4. *The block operator matrix on the left-hand side of* (5.2) *is upper dominant, see* [**63**, Definition 2.2.1] *for a definition of this notion. This follows from* [**63**, Corollary 2.1.20], *noting that* $\mathrm{Dom}(\mathrm{div}\,) \supset \mathrm{Dom}((-\boldsymbol{\Delta})^{1/2})$ *and that the zero operator is automatically infinitesimal with respect to the gradient operator.*

However, the form $\mathfrak{b}_S$ in (5.1) *is diagonally dominant, that is, the domain of $\mathfrak{b}_S$ is just the domain of the diagonal part $\mathfrak{a}$.*

This difference in the dominance can occur since we have the freedom to push the gradient operator to the other side in the scalar product of the weak formulation of the corresponding form. In this sense, we have $-\mathrm{div}$ *on the left-hand side instead of* grad *on the right-hand side of the scalar product defining the form $\mathfrak{b}_S$, so that the divergence operator appears twice in* (5.1).

The Stokes operator and its form satisfy the Second Representation Theorem as well.

REMARK 5.1.5. *The form $\mathfrak{b}_S$ of the Stokes operator satisfies also the Second Representation Theorem. To see this, note that by Remark 5.1.3 and by the KLMN Theorem* [**55**, Theorem X.17]*, the operator B_S is semibounded from below. Thus, there is a constant c such that the form $\mathfrak{b}_S+cI$ is non-negative and satisfies the Second Representation Theorem* [**43**, Theorem VI.2.23]. *From this, using Remark 1.2.4, we get that*

$$\mathrm{Dom}(A^{1/2}) = \mathrm{Dom}[\mathfrak{a}] = \mathrm{Dom}[\mathfrak{b}_S] = \mathrm{Dom}[\mathfrak{b}_S{+}cI] = \mathrm{Dom}((B_S{+}cI)^{1/2}) = \mathrm{Dom}(|B_S|^{1/2}),$$

which yields that the Second Representation Theorem holds.

5.2. The kernel of the Stokes operator

We will now compute the kernel of the Stokes operator. We first show that the kernel of the Laplacian with homogeneous Dirichlet boundary values is trivial. Although this statement seems to be known, we found no reference for this statement in the case of unbounded domains.

For bounded domains, [**30**, Corollary 8.2] shows that the corresponding weak solution vanishes identically. This Corollary is based on the chain rule for $H^1(\Omega)$ functions on bounded domains Ω.

We show that the argumentation in [**30**, Corollary 8.2] extends to arbitrary domains. An important step in the proof of this claim is the following chain rule.

LEMMA 5.2.1 ([**19**, Satz 5.19 and Satz 5.20]).
Let $f \in C^1(\mathbb{R})$ with bounded derivative, $|f'| \le c$ for some constant c, and $f(0) = 0$. Furthermore, let Ω be a domain, then for $u \in H^1(\Omega)$, the composition $f \circ u$ satisfies

$$f \circ u \in H^1(\Omega), \quad D(f \circ u) = (f' \circ u) \cdot Du,$$

where D is the gradient operator on $H^1(\Omega)$. Additionally, $u^+ := \max\{u, 0\}$ satisfies

$$u^+ \in H^1(\Omega) \text{ and } Du^+(x) = Du(x) \cdot \chi_{\{u>0\}}(x) \text{ almost everywhere,}$$

where $\chi_{\{u>0\}}$ is defined almost everywhere and given by $\chi_{\{u>0\}}(x) = 1$ if $u(x) > 0$ and $\chi_{\{u>0\}}(x) = 0$ if $u(x) \le 0$.

We modify this result to hold for the class $H_0^1(\Omega)$, so that the zero boundary values are preserved.

LEMMA 5.2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain and let $u \in H_0^1(\Omega)$ be arbitrary. Then the positive part u^+ of u satisfies*

$$u^+ \in H_0^1(\Omega).$$

PROOF. By Lemma 5.2.1, we have $u^+ \in H^1(\Omega)$ and need only to verify that u^+ can be approximated by $C_0^\infty(\Omega)$ functions. This approximation with respect to the Sobolev norm of $H^1(\Omega)$ will be carried out in three steps.

Step 1: u^+ is approximated by $(\sqrt{u^2+\varepsilon^2}-\varepsilon)\chi_{\{u>0\}}$ for $\varepsilon \to 0$. To see this, we compute

$$\begin{aligned} &\left\| u^+ - (\sqrt{u^2+\varepsilon^2}-\varepsilon)\chi_{\{u>0\}} \right\|_{H^1(\Omega)}^2 \\ &= \int_\Omega \left| u^+ - (\sqrt{u^2+\varepsilon^2}-\varepsilon)\chi_{\{u>0\}} \right|^2 dx + \int_{\{u>0\}} \left| Du - \frac{u}{\sqrt{u^2+\varepsilon^2}} Du \right|^2 dx. \end{aligned} \tag{5.6}$$

We have $(\sqrt{u^2+\varepsilon^2}-\varepsilon)\chi_{\{u>0\}} \leq u^+ \in L^2(\Omega)$ and the pointwise convergence almost everywhere

$$(\sqrt{u^2+\varepsilon^2}-\varepsilon)\chi_{\{u>0\}} \to u^+, \; \varepsilon \to 0.$$

Analogously, we get the estimate

$$\left| \frac{u \cdot \chi_{\{u>0\}}}{\sqrt{u^2+\varepsilon^2}} Du \right| \leq |Du| \chi_{\{u>0\}} \in L^2(\Omega)$$

and the convergence pointwise almost everywhere

$$\frac{u \cdot \chi_{\{u>0\}}}{\sqrt{u^2+\varepsilon^2}} Du \to Du \cdot \chi_{\{u>0\}}, \; \varepsilon \to 0.$$

For both summands in (5.6), the convergence to zero now follows from the Dominated Convergence Theorem.

Step 2: Let $\varepsilon > 0$ be fixed and let $(u_m)_{m\in\mathbb{N}} \subset C_0^\infty(\Omega)$ be an approximating sequence for u in the $H^1(\Omega)$ norm for $m \to \infty$, then (up to taking a subsequence), we have that $(\sqrt{u^2+\varepsilon^2}-\varepsilon)\chi_{\{u>0\}}$ is approximated by

$$(\sqrt{u_m^2+\varepsilon^2}-\varepsilon)\chi_{\{u_m>0\}} =: u_{\varepsilon,m} \in C_0^1(\Omega).$$

For a proof of this convergence consider the map $f_\varepsilon \in C^1(\mathbb{R})$ given by

$$f_\varepsilon(s) := (\sqrt{s^2+\varepsilon^2}-\varepsilon)\chi_{\{s>0\}}(s).$$

Using the chain rule in Lemma 5.2.1 above, we get

$$\|f_\varepsilon(u) - f_\varepsilon(u_m)\|_{H^1(\Omega)}^2 = \int_\Omega |f_\varepsilon(u_m) - f_\varepsilon(u)|^2 dx + \int_\Omega |f_\varepsilon'(u)Du - f_\varepsilon'(u_m)Du_m|^2 dx. \tag{5.7}$$

By the estimates $|f_\varepsilon(u_m) - f_\varepsilon(u)| \leq \sup_{s\in\mathbb{R}} |f_\varepsilon'(s)| \cdot |u - u_m|$ and $\sup_{s\in\mathbb{R}} |f_\varepsilon'(s)| \leq 1$, we get the convergence to zero of the first summand in (5.7) since we have

$$\int_\Omega |f_\varepsilon(u_m) - f_\varepsilon(u)|^2 dx \leq \int_\Omega |u_m - u|^2 dx \to 0, m \to \infty.$$

The second summand in (5.7) can be estimated by a sum of three integrals, namely

$$\begin{aligned}
&\int_\Omega |f'_\varepsilon(u)Du - f'_\varepsilon(u_m)Du_m|^2 dx \\
&\le \int_\Omega (|Du - Du_m| + |Du| \cdot |f'_\varepsilon(u_m) - f'_\varepsilon(u)|)^2 dx \\
&\le \int_\Omega |Du_m - Du|^2 dx + \int_\Omega |Du|^2 |f'_\varepsilon(u_m) - f'_\varepsilon(u)|^2 dx \\
&\quad + 2\int_\Omega |Du| \cdot |Du_m - Du| \cdot |f'_\varepsilon(u_m) - f'_\varepsilon(u)| dx
\end{aligned}$$

The first integral on the right-hand side converges by definition of the $H^1(\Omega)$ norm. Since $u_m \to u,\ m \to \infty$ in $L^2(\Omega)$, there is a subsequence, which we name again by (u_m), such that $u_m \to u$ pointwise almost everywhere. The continuity of the derivative f'_ε implies $f'_\varepsilon(u_m) \to f'_\varepsilon(u),\ m \to \infty$ pointwise almost everywhere. The Dominated Convergence Theorem implies the convergence of the second Integral. For the remaining third integral, use $|f'_\varepsilon(s)| \le 1$ and estimate

$$2\int_\Omega |Du| \cdot |Du_m - Du| \cdot |f'_\varepsilon(u_m) - f'_\varepsilon(u)| dx \le 4\int_\Omega |Du| \cdot |Du_m - Du| dx.$$

We then use the Cauchy-Schwarz Inequality to obtain the convergence to zero of (5.7).

Step 3: Let ε, m be fixed. Denote by j_δ the standard mollifier with support contained in the ball $B_\delta(0)$ (see e.g. [**19**, Section 4.3]), where $\delta < \operatorname{dist}(\operatorname{supp}(u_{\varepsilon,m}), \partial\Omega)$. Then $u_{\varepsilon,m}$ is approximated by the convolution $j_\delta * u_{\varepsilon,m},\ \delta \to 0$. This approximation follows from [**19**, Lemma 4.22], noting that $u_{\varepsilon,m} \in C_0^1(\Omega)$, and that the derivatives satisfy

$$\frac{\partial}{\partial x_i}(j_\delta * u_{\varepsilon,m}) = j_\delta * \frac{\partial}{\partial x_i} u_{\varepsilon,m},\ i \in \{1, \dots n\}.$$

Combining the three steps, we have a sequence $\big(j_\delta * u_{\varepsilon,m}\big)$ in $C_0^\infty(\Omega)$ converging to u^+ completing the proof. □

A simplified version of the proof of the weak maximum principle in [**30**, Theorem 8.1] now shows the triviality of the kernel of the Dirichlet Laplacian $-\Delta$ in arbitrary domains. For bounded domains, this is a direct consequence of the weak maximum principle, see, e.g. [**30**, Corollary 8.2].

THEOREM 5.2.3. *Let Ω be a domain and let $u \in H_0^1(\Omega)$ with $\Delta u = 0$ in the weak sense, that is,*

(5.8)
$$L(u,v) := \int_\Omega \langle Du, Dv\rangle dx = 0 \quad \text{for all } v \in H_0^1(\Omega),\ \text{with } v \ge 0 \text{ almost everywhere.}$$

Then u equals zero almost everywhere.

PROOF. Without loss of generality, assume u to be real-valued. Let $\Delta u \ge 0$ in the weak sense, that is,

$$L(u,v) \le 0 \quad \text{for all } v \in H_0^1(\Omega) \text{ with } v \ge 0.$$

By Lemma 5.2.2 we can choose

$$v := u^+ \in H_0^1(\Omega) \quad \text{with} \quad Dv = Du \cdot \chi_{\{u>0\}}.$$

Thus, we have

$$\int_\Omega \langle Du, Dv\rangle dx \le 0.$$

This yields the equality

$$\int_{\{u>0\}} |Du|^2 dx = 0.$$

So either the Lebesgue measure of the set $\{u > 0\}$ is zero or $Du = 0$ almost everywhere on $\{u > 0\}$. In the first case, we immediately have $u \leq 0$ almost everywhere. If $Du = 0$ almost everywhere on $\{u > 0\}$, we have by definition $Dv \equiv 0$ and thus v is constant and equal to zero, so that also in this case $u \leq 0$ almost everywhere.

Analogously, we get $u \geq 0$ almost everywhere if we assume $\Delta u \leq 0$. In combination we obtain $u = 0$ almost everywhere. □

We now give an explicit representation for the kernel of the Stokes operator.

THEOREM 5.2.4 ([**38**]). *Let Ω be a domain with Lipschitz boundary. Then the kernel of the Stokes operator B_S is trivial for domains with infinite volume and consists of constant functions $0 \oplus c$ for domains with finite volume.*

PROOF. By Theorem 5.2.3 (respectively [**30**, Corollary 8.2] for bounded domains), the kernel of the Laplacian $-\Delta$ in $H_0^1(\Omega)$ is trivial for any domain Ω with Lipschitz boundary. Thus the kernel of $-\boldsymbol{\Delta} = A_+$ is trivial too.

Let W denote the closure of div defined on $C_0^\infty(\Omega)^n$ with respect to the graph norm $\sqrt{\|\cdot\|^2 + \|\mathrm{div}\,\cdot\|^2}$. Obviously, $H_0^1(\Omega)^n$ is a dense subset of $\mathrm{Dom}(W)$. It is straightforward to verify that the domain of the adjoint operator $W^* = -\,\mathrm{grad}$ is $H^1(\Omega)$. We then get the representation

$$\mathfrak{b}_S[\varphi,\psi] = \langle A_+^{1/2}u, A_+^{1/2}v\rangle_{L^2(\Omega)^n} + \langle Wu, q\rangle_{L^2(\Omega)} + \langle p, Wv\rangle_{L^2(\Omega)}, \tag{5.9}$$

for $\varphi = u \oplus p$ and $\psi = v \oplus q \in H_0^1(\Omega)^n \oplus L^2(\Omega)$.

The kernel $\mathrm{Ker}\, W^*$ has dimension one and is spanned by constant functions on Ω. For domains with infinite volume, the only integrable constant is zero, so the set $\mathfrak{L}_-$ given in Definition 1.6.1 is the subspace of constant functions or only the function vanishing identically.

The claim follows now from the representation in Lemma 1.6.2,

$$\mathrm{Ker}\, B_S = (\mathrm{Ker}\, A_+ \cap \mathfrak{L}_+) \oplus (\mathrm{Ker}\, A_- \cap \mathfrak{L}_-)\,. \qquad \square$$

REMARK 5.2.5. *For strictly positive forms $\mathfrak{a}$, the kernel of B given by Theorem 1.5.3 is always trivial. The Stokes operator B_S on bounded domains shows that indeed the kernel may be non-trivial if the operator A associated with the diagonal part $\mathfrak{a}$ of $\mathfrak{b}$ is only non-negative.*

CHAPTER 6

Subspace perturbation and solutions to the form Riccati equation

This chapter is in part based on the joint work [**38**] with L. Grubišić, V. Kostrykin, K. A. Makarov, and K. Veselić.

We consider graph subspaces which reduce an indefinite diagonal form perturbed by an off-diagonal form, namely

$$\mathfrak{b}[x,y] =: \mathfrak{a}[x, J_A y] + \mathfrak{v}[x,y], \quad x,y \in \operatorname{Dom}[\mathfrak{a}]$$

as in Theorem 1.5.3. This is a preparation for an explicit block diagonalisation of the associated operator B. The considerations here are closely related to the preceding Chapter 4, where we investigated graph subspaces that reduce the operator sum of the diagonal operator $J_A A$ and the off-diagonal perturbation V.

6.1. Estimate of the subspace perturbation

Let B be the operator associated with the form $\mathfrak{b}$. To begin this chapter, we investigate when $\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+))$, the positive spectral subspace of the operator B, is a graph space with respect to $\mathcal{H}_+$, the positive spectral subspace of $J_A A$. Recall that $J_A A$ is self-adjoint and diagonal with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ induced by the involution J_A.

The first step for us is to review the situation where $\mathfrak{a} \geq cI > 0$ is strictly positive. The following Tan 2Θ Theorem on the subspace perturbation problem, which we reproduce in the notation we used so far, has been obtained in this case.

Theorem 6.1.1 (The Tan 2Θ Theorem, cf. [**37**, Theorem 3.1]).
Assume Hypothesis 1.5.1 and let

$$\mathfrak{a} \geq cI > 0.$$

For $\mu \in (-c,c)$ set

$$v_\mu := \sup_{0 \neq x \in \operatorname{Dom}[a]} \frac{|\mathfrak{v}[x]|}{(\mathfrak{a} - \mu J_A)[x]} < \infty$$

and

$$v := \inf_{\mu \in (-c,c)} v_\mu .$$

Then the operator B associated with the form $\mathfrak{b} := \mathfrak{a}[\,\cdot\,, J_A\,\cdot\,] + \mathfrak{v}$ satisfies $(-c,c) \subseteq \rho(B)$. In this case, the difference of the spectral projectors of $J_A A$ and B on $\mathbb{R}_+$ satisfies the estimate

$$\|\mathsf{E}_{J_A A}(\mathbb{R}_+) - \mathsf{E}_B(\mathbb{R}_+)\| \leq \sin\left(\frac{1}{2}\arctan v\right) < \frac{\sqrt{2}}{2} < 1. \tag{6.1}$$

An immediate consequence of (6.1) is that by [**45**, Corollary 3.4], the subspace $\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+))$ is a graph subspace with respect to $\mathcal{H}_+$. Namely, there is a bounded operator $X_0 \colon \mathcal{H}_+ \to \mathcal{H}_-$ with $\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+)) = \mathcal{G}(\mathcal{H}_+, X_0) = \{x \oplus X_0 x \mid x \in \mathcal{H}_+\}$, see Definition 4.1.2.

The question that arises here is whether the spectral subspace $\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+))$ is still a graph subspace with respect to $\mathcal{H}_+$ in the case where $\mathfrak{a}$ is only non-negative. Recall that in this case it is possible that the operator B has no spectral gap around zero or has even a non-trivial kernel. In general, the following example shows that the bound $\sqrt{2}/2$ can be attained if $\mathfrak{a}$ is only non-negative.

EXAMPLE 6.1.2. *Let $\mathcal{H} := \ell^2 \oplus \ell^2$ be the Hilbert space as in Definition 1.2.6. Define the bounded operators*

$$A := \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 0 \end{pmatrix}, \quad J_A := \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V := \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding bounded forms clearly satisfy Hypothesis 1.5.1 and the associated operator B coincides with the operator sum $J_A A + V$. For brevity, we set $\varepsilon := \frac{1}{k}$ and obtain that

$$B = \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} \varepsilon & 1 \\ 1 & 0 \end{pmatrix} =: \bigoplus_{k\in\mathbb{N}} B_k.$$

The characteristic polynomial of B_k is given by

$$\chi_{B_k}(\lambda) = \lambda^2 - \varepsilon\lambda - 1.$$

The eigenspace related to the positive eigenvalue of B_k is generated by $\begin{pmatrix} \frac{\varepsilon}{2} + \frac{\sqrt{\varepsilon^2+4}}{2} \\ 1 \end{pmatrix}$, thus we have the dyadic product

$$\mathsf{E}_{B_k}(\mathbb{R}_+) = \frac{1}{\frac{1}{2}\varepsilon^2 + \frac{\varepsilon}{2}\sqrt{\varepsilon^2+4}} \cdot \begin{pmatrix} \frac{\varepsilon^2}{2} + 1 + \frac{\varepsilon}{2}\sqrt{\varepsilon^2+4} & \frac{\varepsilon}{2} + \frac{1}{2}\sqrt{\varepsilon^2+4} \\ \frac{\varepsilon}{2} + \frac{1}{2}\sqrt{\varepsilon^2+4} & 1 \end{pmatrix}$$

and $\mathsf{E}_B(\mathbb{R}_+) = \bigoplus_{k\in\mathbb{N}} \mathsf{E}_{B_k}(\mathbb{R}_+)$. With

$$\mathsf{E}_{J_A A}(\mathbb{R}_+) = \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

we get

$$\begin{aligned} &\mathsf{E}_{J_A A}(\mathbb{R}_+) - \mathsf{E}_B(\mathbb{R}_+) \\ &= \bigoplus_{k\in\mathbb{N}} \frac{1}{2\varepsilon^2 + 4 + \varepsilon\sqrt{\varepsilon^2+4}} \cdot \begin{pmatrix} 2 & -\varepsilon - \sqrt{\varepsilon^2+4} \\ -\varepsilon - \sqrt{\varepsilon^2+4} & -2 \end{pmatrix} =: \bigoplus_{k\in\mathbb{N}} S_k. \end{aligned}$$

The eigenvalues of the 2×2 matrix S_k are

$$\pm\frac{\sqrt{2}}{2} \cdot \sqrt{1 - \frac{\varepsilon}{\sqrt{\varepsilon^2+4}}} \longrightarrow \pm\frac{\sqrt{2}}{2} \ \text{as } k \to \infty$$

and thus the bound $\frac{\sqrt{2}}{2}$ on the difference of the spectral projectors can be attained.

We now show sufficient conditions under which the projector difference has norm less or equal than $\sqrt{2}/2$. The main idea there is to create a spectral gap by adding the perturbation $\frac{1}{n}J_A$ and to verify that the estimate remains valid in the limit $n \to \infty$, when the spectral gap of $B + \frac{1}{n}J_A$ around zero closes. In the following, we present the necessary tools for these considerations.

LEMMA 6.1.3 ([**43**, Corollary VIII.1.6 and Theorem VIII.1.15]).
Let S_n and S be self-adjoint operators in a Hilbert space $\mathcal{H}$ for all $n \in \mathbb{N}$. Furthermore,

let $\mathcal{D} \subseteq \bigcap_{n\in\mathbb{N}} \operatorname{Dom}(S_n)$ be a core for the operator S such that $S_n u \to Su$ for all $u \in \mathcal{D}$ and $n \to \infty$. Furthermore, let

$$S_n = \int \lambda dE_n(\lambda), \quad S = \int \lambda dE(\lambda)$$

be the spectral representations, where E_n and E respectively, are the corresponding spectral families. Then, for fixed λ, the strong limit satisfies

$$\operatorname*{s-lim}_{n\to\infty} E_n(\lambda) = E(\lambda)$$

provided that the limit from below $E(\lambda_-)$ coincides with $E(\lambda)$, that is $\mathsf{E}_S(\{\lambda\}) = 0$.

Note that $E(\lambda_-) = E(\lambda)$ whenever λ is not an eigenvalue of the corresponding operator, see [**43**, Section X.1.1].

LEMMA 6.1.4. *Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of bounded self-adjoint operators converging weakly to a bounded self-adjoint operator S,*

$$\operatorname*{w-lim}_{n\to\infty} S_n = S. \tag{6.2}$$

Suppose additionally that $\limsup_{n\to\infty} \|S_n\| \le \|S\|$. Then $\lim_{n\to\infty} \|S_n\| = \|S\|$.

PROOF. Note that (6.2) implies that $||S|| \le \liminf ||S_n||$, see [**43**, Equation (3.2) in Chapter III.]. Together with the estimate on the lim sup, this directly yields the claim. $\square$

In the following, we give a sufficient criterion for $\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+))$ to be a graph subspace with respect to $\mathcal{H}_+$.

The corresponding estimate is a generalisation of the Tan 2Θ Theorem 6.1.1 to the case of non-negative forms $\mathfrak{a}$.

THEOREM 6.1.5. *Assume Hypothesis 1.5.1 and let B be the operator given by the First Representation Theorem 1.5.3 in the off-diagonal case. Furthermore, suppose that the operator A associated with the form $\mathfrak{a}$ satisfies*

$$A|_{\mathcal{H}_+} > 0, \quad A|_{\mathcal{H}_-} \ge 0. \tag{6.3}$$

Additionally, suppose that the kernel of B is trivial, which in view of Lemma 1.6.2 and $A_+ > 0$ can be rewritten as

$$\operatorname{Ker} A_- \cap \mathfrak{L}_- = \{0\}.$$

Then the difference of the spectral projectors satisfies the estimate

$$\|\mathsf{E}_{J_A A}(\mathbb{R}_+) - \mathsf{E}_B(\mathbb{R}_+)\| \le \frac{\sqrt{2}}{2} < 1. \tag{6.4}$$

PROOF. By hypothesis (1.4) we have the estimate

$$|\mathfrak{v}[x]| \le \beta(\mathfrak{a} + I) \le (n\beta)\left(\mathfrak{a} + \frac{1}{n} I\right)[x] \quad \text{for } n \in \mathbb{N}.$$

Thus, the form $\mathfrak{v}$ is $(\mathfrak{a} + \frac{1}{n} I)$-bounded for each $n \in \mathbb{N}$.

Following the proof of Theorem 1.5.3, we have that the form $\mathfrak{b}_n$ given by

$$\mathfrak{b}_n[x, y] := \mathfrak{b}[x, y] + \frac{1}{n}\langle x, J_A y\rangle$$

defines a self-adjoint operator B_n with $\left(-\frac{1}{n}, \frac{1}{n}\right)$ in the resolvent set and the operator B can be rewritten as

$$B = B_n - \frac{1}{n} J_A.$$

In this case $\mathrm{Dom}(B)$ is a core for each B_n and B and $B_n\varphi \to B\varphi$, $n \to \infty$ for all $\varphi \in \mathrm{Dom}(B)$. Since zero is not an eigenvalue of B by assumption, the limit from below of the spectral family of B satisfies $E_B(0-) = E_B(0)$ (see e.g. [**43**, Section X.1.1]). By Lemma 6.1.3 and the relation between spectral family and spectral projector, it follows that

$$\operatorname*{s-lim}_{n\to\infty} \mathsf{E}_{B_n}(-\infty, 0] = \operatorname*{s-lim}_{n\to\infty} E_{B_n}(0) = E_B(0) = \mathsf{E}_B(-\infty, 0].$$

The complementary projectors then satisfy

$$\operatorname*{s-lim}_{n\to\infty} \mathsf{E}_{B_n}(\mathbb{R}_+) = \mathsf{E}_B(\mathbb{R}_+).$$

The projectors then also converge in the weak sense. For brevity, define

$$J_A A_n := J_A\left(A + \frac{1}{n} I\right).$$

Then, by (6.3), we have the following representation with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$:

$$\mathsf{E}_{J_A A_n}(\mathbb{R}_+) = \mathsf{E}_{J_A A}(\mathbb{R}_+) = \begin{pmatrix} I_{\mathcal{H}_+} & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } n \in \mathbb{N}.$$

Consider now the bounded auxiliary operators

$$S_n := \mathsf{E}_{J_A A_n}(\mathbb{R}_+) - \mathsf{E}_{B_n}(\mathbb{R}_+), \quad S := \mathsf{E}_{J_A A}(\mathbb{R}_+) - \mathsf{E}_B(\mathbb{R}_+).$$

Set

$$v_n := \sup_{0 \neq x \in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{(\mathfrak{a} + \frac{1}{n} I)[x]} \leq n\beta < \infty.$$

We then have that either

$$\|S\| \leq \limsup_{n\to\infty} \|S_n\| \quad \text{or} \quad \limsup_{n\to\infty} \|S_n\| \leq \|S\|$$

has to hold. In the first case, we apply Theorem 6.1.1 with $c = \frac{1}{n}$ and $\mu = 0$ and have that

$$\|S\| \leq \limsup_{n\to\infty} \|S_n\| \leq \limsup_{n\to\infty} \sin\left(\frac{1}{2}\arctan(v_n)\right) \leq \frac{\sqrt{2}}{2}.$$

In the second case, we have with Theorems 6.1.4 and 6.1.1 that

$$\|S\| = \limsup_{n\to\infty} \|S_n\| \leq \frac{\sqrt{2}}{2}$$

Thus, $\|S\| \leq \frac{\sqrt{2}}{2}$ holds in both cases. This completes the proof. □

Note that due to Remark 1.5.6, the theorem above is indeed a generalisation of the Tan 2Θ Theorem. Clearly, the estimate we give here is not optimal for strictly positive forms $\mathfrak{a}$ but suffices to obtain that the spectral subspace is a graph subspace for non-negative $\mathfrak{a}$.

Furthermore, note that the condition (6.3) is reasonable if we want to consider the positive spectral part of B as a graph space with respect to $\mathcal{H}_+$. Namely, in this case, we have $\mathrm{Ran}(\mathsf{E}_{J_A A}(\mathbb{R}_+)) = \mathcal{H}_+$ and we compare the two spectral subspaces of $J_A A$ and of B for the open interval $\mathbb{R}_+$.

With slight modifications, we can extend the result above to allow a splitting of the kernel of $J_A A$.

THEOREM 6.1.6. *Assume Hypothesis 1.5.1 and let B be the operator given by the First Representation Theorem 1.5.3 in the off-diagonal case. Furthermore, suppose that the operator $A = A_+ - A_-$ on $\mathcal{H}_+ \oplus \mathcal{H}_-$ satisfies $A_+ \geq 0$ and $A_- \geq 0$. Additionally, suppose that the kernel of B is trivial, that is, by Lemma 1.6.2,*

$$(\operatorname{Ker} A_+ \cap \mathfrak{L}_+) \oplus (\operatorname{Ker} A_- \cap \mathfrak{L}_-) = \{0\}.$$

Then, the estimate

$$\|P_A - \mathsf{E}_B(\mathbb{R}_+)\| \leq \frac{\sqrt{2}}{2} < 1$$

holds, where P_A is the projector onto $\mathcal{H}_+$ and $\mathsf{E}_B(\mathbb{R}_+) = \mathsf{E}_B(\overline{\mathbb{R}_+})$ is the spectral projector of B for the interval $[0, \infty)$.

PROOF. In the same way as before, consider the bounded auxiliary operators

$$S_n := P_A - \mathsf{E}_{B_n}(\mathbb{R}_+), \quad S := P_A - \mathsf{E}_B(\mathbb{R}_+).$$

Note that P_A is in general not a spectral projector for $J_A A$ since the kernel splits as $\operatorname{Ker}(J_A A) =: \mathcal{N}_+ \oplus \mathcal{N}_- \subseteq \mathcal{H}_+ \oplus \mathcal{H}_-$. However, we have that

$$P_A = E_{J_A A_n}(\mathbb{R}_+) \quad \text{for all } n \in \mathbb{N}.$$

Thus, it is a spectral projector for the perturbed operator. The claim now follows as before by Theorems 6.1.1 and 6.1.4. ☐

It is only sufficient but not necessary that B has a trivial kernel for the estimate (6.4) to hold.

EXAMPLE 6.1.7. *Let $\mathcal{H} = (\ell^2 \oplus \ell^2) \oplus (\ell^2 \oplus \ell^2)$. Define the following operators acting by multiplication with 2×2 block matrices*

$$A = \bigoplus_{k \in \mathbb{N}} \left(\begin{array}{cc|cc} k^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), J_A = \bigoplus_{k \in \mathbb{N}} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right), V = \bigoplus_{k \in \mathbb{N}} \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right).$$

In a similar way as in Example 6.1.2, the difference $\mathsf{E}_{J_A A}(\mathbb{R}_+) - \mathsf{E}_B(\mathbb{R}_+)$ is again a matrix with 2×2 blocks with eigenvalue zero of multiplicity 2 and non-zero eigenvalues of modulus less than $\frac{\sqrt{2}}{2}$ converging to $\pm\frac{\sqrt{2}}{2}$ but the kernel of B is not trivial.

If the projector difference is small enough then the spectral subspace is indeed a graph subspace.

REMARK 6.1.8. *Assume the hypothesis of Corollary 6.1.5. Then, setting for brevity*

$$P := \mathsf{E}_{J_A A}(\mathbb{R}_+), \quad Q := \mathsf{E}_B(\mathbb{R}_+),$$

we have that

$$\|P - Q\| \leq \frac{\sqrt{2}}{2} < 1.$$

*By this estimate, we get from [**45**, Corollary 3.4] that $\operatorname{Ran} Q$ is a graph subspace associated with the subspace $\operatorname{Ran} P = \mathcal{H}_+$. This means that there exists a bounded operator $X_0 \colon \mathcal{H}_+ \to \mathcal{H}_-$ with $\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+)) = \mathcal{G}(\mathcal{H}_+, X_0)$ and*

$$\|X_0\| = \frac{\|P - Q\|}{\sqrt{1 - \|P - Q\|^2}} \leq 1.$$

Example 6.1.2 shows that the bound 1 on the norm of X_0 can be attained when the spectral gap around zero closes.

As a convention, we will always write X_0 for the operator related to

$$\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+)) =: \mathcal{G}(\mathcal{H}_+, X_0) = \{x \oplus X_0 x \mid x \in \mathcal{H}_+\}. \tag{6.5}$$

The following Theorem combines Theorems 4.3.6 and 6.1.6 to obtain the solvability of the Riccati equation for diagonally dominant block operator matrices. This result is in part a generalisation of [**1**, Theorem 6.3] by Adamyan, Langer, and Tretter.

THEOREM 6.1.9. *Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be the orthogonal decomposition induced by the self-adjoint involution J_A and let*

$$J_A A + V := \begin{pmatrix} A_+ & W^* \\ W & -A_- \end{pmatrix}$$

be a block operator matrix, where A and V are self-adjoint, $A \geq 0$ and V has $J_A A$-bound less than one. Furthermore, let

$$\mathfrak{v}[x,y] := \langle |V|^{1/2} x, \operatorname{sign}(V)|V|^{1/2} y\rangle, \quad x, y \in \operatorname{Dom}(|V|^{1/2})$$

be the form represented by V. Additionally, suppose that

$$(\operatorname{Ker} A_+ \cap \mathfrak{L}_+) \oplus (\operatorname{Ker} A_- \cap \mathfrak{L}_-) = \{0\}$$

in the notation of Lemma 1.6.2.

Then the operator Riccati equation for $J_A A + V$ is solvable in the strong sense, that is,

$$AYx - YAx - YVYx + Vx = 0 \quad \text{for } x \in \operatorname{Dom}(A)$$

for some bounded operator Y.

One solution of the Riccati equation is a skew-symmetric contraction and satisfies

$$Y_0 = \begin{pmatrix} 0 & -X_0^* \\ X_0^* & 0 \end{pmatrix},$$

where $\operatorname{Ran}(\mathsf{E}_{J_A A+V}(\mathbb{R}_+)) = \mathcal{G}(\mathcal{H}_+, X_0)$.

PROOF. Since J_A is an isometry, the operator V is also relatively bounded with respect to A with the same bound, see, e.g., [**56**, Lemma 7.1]. By [**55**, Theorem X.18] and Definition 2.2.6, the form $\mathfrak{v}$ is relatively bounded with respect to the form $\mathfrak{a}$ with the same bound. Thus, we have that

$$|\mathfrak{v}[x]| \leq \beta(\mathfrak{a} + I)[x]$$

for some constant β. By Theorem 6.1.6, we have that

$$\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+)) = \mathcal{G}(\mathcal{H}_+, X_0)$$

for the contraction $X_0 \colon \mathcal{H}_+ \to \mathcal{H}_-$ and the operator B associated with the form

$$\mathfrak{b} := \mathfrak{a}[\,\cdot\,, J_A\,\cdot\,] + \mathfrak{v}.$$

By the Kato-Rellich Theorem (see, e.g., [**55**, Theorem X.12]), the operator sum $J_A A + V$ is self-adjoint on $\operatorname{Dom}(A) \subseteq \operatorname{Dom}(A^{1/2})$. This operator sum, which is the block operator matrix, has to coincide with B by the uniqueness in the First Representation Theorem 1.5.3 and $\operatorname{Dom}[\mathfrak{b}] = \operatorname{Dom}(A^{1/2})$ of the corresponding form.

Since $\operatorname{Ran}(E_B(\mathbb{R}_+))$ is a reducing graph subspace for the block operator matrix B, Theorem 4.3.6 implies that Y_0 is a strong solution of the Riccati equation. □

Note that we do not claim that contractive solutions of the operator Riccati equation in the theorem above are unique. In this sense, we do not obtain a complete extension of [**1**, Theorem 6.3] since we cannot grant the uniqueness of that theorem. Clearly, if A_- is bounded, then, by the $J_A A$ boundedness of the symmetric operator V, this operator is also bounded. In this case, the theorem above uses the same assumptions as [**1**, Theorem 6.3] and the uniqueness can be provided by [**1**, Theorem 6.3] directly.

We return to the problem of uniqueness of solutions in Section 6.4 again. There, it will be clear why the case of non-semibounded $J_A A$ is more complicated in view of uniqueness.

6.2. Reducing subspaces for forms

By the preceding section, we have a criterion for $\operatorname{Ran}(E_B(\mathbb{R}_+))$ to be a graph subspace. We now turn to the investigation of reducing subspaces. For brevity, we set

$$\mathcal{K}_+ := \operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+)), \quad \mathcal{K}_- := \operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+))^{\perp},$$

then we get an additional decomposition $\mathcal{K}_+ \oplus \mathcal{K}_-$ of the Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

In order to generalise the concept of reducing subspaces (see Definition 4.1.3) from operators to forms, we give an alternative characterisation of reducing subspaces.

Remark 6.2.1 (cf. [**66**, Satz 2.60]). *Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ be an orthogonal decomposition of the Hilbert space $\mathcal{H}$. Denote the orthogonal projector onto $\mathcal{H}_0$ by P. Then the decomposition reduces a symmetric operator S if and only if*

$$PS \subseteq SP,$$

that is,

(1) $Pu \in \operatorname{Dom}(S)$ *for all* $u \in \operatorname{Dom}(S)$ *and*
(2) $PSu = SPu$ *for all* $u \in \operatorname{Dom}(S)$.

We do not know a suitable generalisation of the weaker notion of invariant subspaces (see Chapter 4) from operators to forms. For the notion of reducing subspace, the natural generalisation to forms is the following.

Definition 6.2.2 ([**38**]). *Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ be an orthogonal decomposition of the Hilbert space $\mathcal{H}$ and let P be the orthogonal projector onto $\mathcal{H}_0$. The decomposition is called* reducing *for the symmetric densely defined sesquilinear form $\mathfrak{s}$ with domain* $\operatorname{Dom}[\mathfrak{s}] \subseteq \mathcal{H}$ *if and only if*

(1) $Pu \in \operatorname{Dom}[\mathfrak{s}]$ *for all* $u \in \operatorname{Dom}[\mathfrak{s}]$ *and*
(2) $\mathfrak{s}[Pu, v] = \mathfrak{s}[u, Pv]$ *for all* $u, v \in \operatorname{Dom}[\mathfrak{s}]$.

For sake of simplicity, we will use "reducing subspace" and "reducing decomposition" synonymously since the orthogonal complement of a reducing subspace is reducing as well.

We give a condition in terms of reducing subspaces that is equivalent to the domain stability condition $\operatorname{Dom}(A^{1/2}) = \operatorname{Dom}(|B|^{1/2})$. This extends the conditions presented in Theorem 2.2.4.

Remark 6.2.3. *Let $\mathfrak{b}$ satisfy the First Representation Theorem 1.2.3 or 1.5.3 and let B be the associated operator. Then the domain stability condition*

$$\operatorname{Dom}(A^{1/2}) = \operatorname{Dom}(|B|^{1/2})$$

holds if and only if the decomposition induced by the unitary operator $\operatorname{sgn}(B)$ *(where* $\operatorname{sgn}(0)$ *can be chosen as $+1$ or -1) reduces the form $\mathfrak{b}$.*

To see this, recall that $\operatorname{Dom}[\mathfrak{b}] = \operatorname{Dom}(A^{1/2})$ *and that the domain stability condition* $\operatorname{Dom}(A^{1/2}) = \operatorname{Dom}(|B|^{1/2})$ *is equivalent to condition (v) in Theorem 2.2.4, that is,* $\operatorname{sgn}(B)\operatorname{Dom}(A^{1/2}) \subseteq \operatorname{Dom}(A^{1/2})$. *Set $P := \frac{1}{2}(\operatorname{sgn}(B) + I)$. The claim now follows directly, noticing that* $\operatorname{sgn}(B)$ *maps* $\operatorname{Dom}[\mathfrak{b}] = \operatorname{Dom}(A^{1/2})$ *into itself if and only if P does, and that P as well as* $\operatorname{sgn}(B)$ *commutes with functions of B. Furthermore, if* *the stability condition holds, the Second Representation Theorem 2.1.1 can be applied to rewrite* $\mathfrak{b}$ *as*

$$\mathfrak{b}[x,y] = \langle |B|^{1/2}x, \operatorname{sgn}(B)|B|^{1/2}y\rangle = \langle |B|^{1/2}x, \operatorname{sign}(B)|B|^{1/2}y\rangle, \quad x,y \in \operatorname{Dom}[\mathfrak{b}],$$

where sign *is the usual sign with* $\operatorname{sign}(0) = 0$*.*

In the following, we investigate the relation between reducing subspaces for forms and reducing subspaces for the representing operator.

LEMMA 6.2.4. *Let* $\mathfrak{b}$ *be a sesquilinear form satisfying the hypothesis of the Second Representation Theorem 2.1.1 and let* B *be the representing operator.*

Then any decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ *reduces the form* $\mathfrak{b}$ *if and only if it reduces the operator* B*.*

The proof of this statement is based on an observation made by Albeverio and Motovilov. Namely, weak solutions of Sylvester equations are even strong solutions.

LEMMA 6.2.5 (cf. [**4**, Lemma 4.2]). *Let* T_1, T_2 *be closed, densely defined linear operators on Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*, respectively. Suppose that* S *is a linear map from* $\operatorname{Dom}(T_1)$ *to* $\mathcal{H}_2$*.*

Furthermore, let the bounded operator Z *be a weak solution of the Sylvester equation*

$$ZT_1 - T_2 Z = S,$$

that is,

$$\langle ZT_1 f, g\rangle - \langle Zf, T_2^* g\rangle = \langle Sf, g\rangle \quad \textit{for all } f \in \operatorname{Dom}(T_1),\ g \in \operatorname{Dom}(T_2^*).$$

Then Z *is even a strong solution of the Sylvester equation* $ZT_1 - T_2Z = S$*, that is, the inclusion*

$$\operatorname{Ran}(Z|_{\operatorname{Dom}(T_1)}) \subseteq \operatorname{Dom}(T_2)$$

and also

$$ZT_1 f - T_2 Z f = Sf \quad \textit{for all } f \in \operatorname{Dom}(T_1)$$

hold.

PROOF. The proof of this lemma is the same as for [**4**, Lemma 4.2] in the case of bounded S. In order to verify that S can be allowed to be unbounded, we reproduce this proof here. Let Z be a weak solution of the Riccati equation. Fix $f \in \operatorname{Dom}(T_1)$ and introduce the linear functional $\mathfrak{l}_f$ on $\operatorname{Dom}(\mathfrak{l}_f) = \operatorname{Dom}(T_2^*)$ by

$$\mathfrak{l}_f(g) := \langle T_2^* g, Zf\rangle.$$

Since Z is a weak solution, we can rewrite

$$\mathfrak{l}_f(g) = \langle g, ZT_1 f\rangle - \langle g, Sf\rangle,$$

where we only use that S can be defined on $\operatorname{Dom}(T_1)$. We obtain that the functional $\mathfrak{l}_f$ is bounded with

$$|\mathfrak{l}_f(g)| \le |\langle g, ZT_1 f - Sf\rangle| \le c_f \|g\|, \quad g \in \operatorname{Dom}(T_2^*),$$

where $c_f = \|ZT_1 f - Sf\|$. Furthermore, the functional $\mathfrak{l}_f$ is densely defined since its domain $\operatorname{Dom}(\mathfrak{l}_f) = \operatorname{Dom}(T_2^*)$ is dense in the Hilbert space $\mathcal{H}$ as the domain of the adjoint of a closed, densely defined operator.

As a consequence, we obtain that $Zf \in \operatorname{Dom}\big((T_2^*)^*\big) = \operatorname{Dom}(T_2)$, which implies $\operatorname{Ran}(Z|_{\operatorname{Dom}(T_1)}) \subseteq \operatorname{Dom}(T_2)$. In this case

$$\langle g, ZT_1 f - T_2 Z f - Sf\rangle = 0 \quad \text{for all } g \in \mathcal{H}_2,$$

so that $ZT_1 f - T_2 Z f - Sf = 0$ for all $f \in \operatorname{Dom}(T_1)$. □

We are now ready to give a proof of Lemma 6.2.4.

Proof of Lemma 6.2.4. Denote by P the orthogonal projector onto $\mathcal{H}_0$. Recall, that by assumption

$$\mathrm{Dom}[\mathfrak{b}] = \mathrm{Dom}(A^{1/2}) = \mathrm{Dom}(|B|^{1/2}).$$

For the first implication, let $\mathcal{H}_0 \oplus \mathcal{H}_1$ reduce the form $\mathfrak{b}$. Then, by the Second Representation Theorem 2.1.1, we have that the inclusion

$$P\,\mathrm{Dom}(|B|^{1/2}) \subseteq \mathrm{Dom}(|B|^{1/2})$$

holds and that

$$\langle |B|^{1/2}Px, \mathrm{sign}(B)|B|^{1/2}y\rangle = \langle |B|^{1/2}x, \mathrm{sign}(B)|B|^{1/2}Py\rangle \quad \text{for all } x, y \in \mathrm{Dom}(|B|^{1/2})$$

hold. Restricting this last equality to $\mathrm{Dom}(B)$ gives that

$$\langle Px, By\rangle = \langle Bx, Py\rangle = \langle PBx, y\rangle \text{ for all } x, y \in \mathrm{Dom}(B).$$

Since B is self-adjoint, the bounded operator P is a weak solution of the Sylvester equation

$$ZB - BZ = 0.$$

By Lemma 6.2.5, the operator P also is a strong solution of this equation, that is,

$$\mathrm{Ran}(P|_{\mathrm{Dom}(B)}) \subseteq \mathrm{Dom}(B) \quad \text{and} \quad PBx = BPx \text{ for all } x \in \mathrm{Dom}(B).$$

Thus, the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ reduces the operator B by Remark 6.2.1.

For the converse implication, let $\mathcal{H}_0 \oplus \mathcal{H}_1$ reduce the operator B. By [**66**, Satz 8.23], the decomposition also reduces both operators $|B|^{1/2}$ and $\mathrm{sign}(B)$. Together with

$$\mathrm{Dom}[\mathfrak{b}] = \mathrm{Dom}(A^{1/2}) = \mathrm{Dom}(|B|^{1/2})$$

this yields $P\,\mathrm{Dom}[\mathfrak{b}] \subseteq \mathrm{Dom}[\mathfrak{b}]$ and P commutes with $\mathrm{sign}(B)$ and $|B|^{1/2}$. Thus

$$\mathfrak{b}[Pu, v] = \langle |B|^{1/2}Pu, \mathrm{sign}(B)|B|^{1/2}v\rangle = \langle |B|^{1/2}u, \mathrm{sign}(B)|B|^{1/2}Pv\rangle = \mathfrak{b}[u, Pv]$$

and $\mathcal{H}_0 \oplus \mathcal{H}_1$ reduces the form $\mathfrak{b}$. □

The following observation is an immediate consequence of Lemma 6.2.4.

Remark 6.2.6. (a) *(cf.* [**56**, Exercise 10.8.2] *for the case $A_\pm > 0$)*
Since the operator J_A commutes with A, by Hypothesis 1.5.1, also $I \pm J_A$ commute with A and thus $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ reduces A by Remark 6.2.1.
We then have the splitting $A = A_+ + A_-$, where $A_\pm$ is the restriction of A to $\mathcal{H}_\pm$. This yields $J_A A = A_+ - A_-$.
Since the form $\mathfrak{a}$ is non-negative, the Second Representation Theorem [**43**, Theorem VI.2.23] *and* $\mathrm{Dom}[\mathfrak{a}] = \mathrm{Dom}(A^{1/2})$ *hold. By Lemma 6.2.4 also the form $\mathfrak{a}$ is reduced by this decomposition.*
Thus, $\mathfrak{a}$ is decomposed as a sum of two non-negative forms

$$\mathfrak{a}[x_+ \oplus x_-, y_+ \oplus y_-] = \mathfrak{a}_+[x_+, y_+] + \mathfrak{a}_-[x_-, y_-],$$

where $\mathfrak{a}_\pm = \mathfrak{a}|_{\mathcal{H}_\pm}$, $\mathrm{Dom}[\mathfrak{a}_\pm] = \mathrm{Dom}[\mathfrak{a}] \cap \mathcal{H}_\pm$ *and*

$$x = x_+ \oplus x_-, y = y_+ \oplus y_- \in \mathrm{Dom}[\mathfrak{a}_+] \oplus \mathrm{Dom}[\mathfrak{a}_-] = \mathrm{Dom}[\mathfrak{a}].$$

From this, we get the decomposition

$$\mathfrak{a}[x_+ \oplus x_-, J_A(y_+ \oplus y_-)] = \mathfrak{a}_+[x_+, y_+] - \mathfrak{a}_-[x_-, y_-].$$

The restricted forms $\mathfrak{a}_\pm$ *are each non-negative and thus satisfy the Second Representation Theorem* [**43**, Theorem VI.2.23], *so that the alternative representation*

$$\mathfrak{a}[x_+ \oplus x_-, J_A(y_+ \oplus y_-)] = \langle A_+^{1/2} x_+, A_+^{1/2} y_+ \rangle - \langle A_-^{1/2} x_-, A_-^{1/2} y_- \rangle,$$

with respect to $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ *and* $\mathrm{Dom}[\mathfrak{a}_\pm] = \mathrm{Dom}(A_\pm^{1/2})$ *holds.*

(b) *The orthogonal decomposition*

$$\mathcal{H} = \mathcal{K}_+ \oplus \mathcal{K}_- = \mathrm{Ran}(\mathsf{E}_B(\mathbb{R}_+)) \oplus \mathrm{Ran}(\mathsf{E}_B(\mathbb{R}_+))^\perp$$

clearly reduces the operator B *by functional calculus. If in addition* B *is associated with the form* $\mathfrak{b}$ *by the First Representation Theorem 1.2.3 or 1.5.3 and the domain stability condition* $\mathrm{Dom}(|B|^{1/2}) = \mathrm{Dom}(A^{1/2})$ *holds, then this decomposition also reduces the form* $\mathfrak{b}$ *by Lemma 6.2.4.*

If the form $\mathfrak{b}$ *additionally satisfies the assumptions of Corollary 6.1.5, then* $\mathcal{K}_+ = \mathrm{Ran}(\mathsf{E}_B(\mathbb{R}_+)) = \mathcal{G}(\mathcal{H}_+, X_0)$ *is a graph subspace with respect to a contraction* X_0 *with* $||X_0|| \le 1$*, see Remark 6.1.8.*

In this case, the kernel of B *is trivial and we have two natural decompositions of* $\mathcal{H}$.

Namely, we have $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ *induced by the isometry* J_A *and we also have the decomposition* $\mathcal{H} = \mathcal{K}_+ \oplus \mathcal{K}_-$*, where* $\mathcal{K}_- = \mathcal{G}(\mathcal{H}_-, -X_0^*)$*. The latter decomposition is induced by the isometry* $J = \mathrm{sgn}(B) = \mathrm{sign}(B)$ *and reduces the operator* B.

Simple calculation gives an alternative characterisation of reducing subspaces for forms.

LEMMA 6.2.7 ([**38**]). *Any orthogonal decomposition* $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ *reduces a symmetric densely defined sesquilinear form* $\mathfrak{s}$ *with domain* $\mathrm{Dom}[\mathfrak{s}] \subseteq \mathcal{H}$ *if and only if*

$$Pu \in \mathrm{Dom}[\mathfrak{s}] \text{ for all } u \in \mathrm{Dom}[\mathfrak{s}] \quad \text{and} \quad \mathfrak{s}[P^\perp u, Pv] = 0 \text{ for all } u, v \in \mathrm{Dom}[\mathfrak{s}],$$

where P *is the orthogonal projector onto* $\mathcal{H}_0$ *and* $P^\perp$ *is the orthogonal projector onto* $\mathcal{H}_1$.

PROOF. Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ reduce $\mathfrak{s}$. Then, we have that

$$\mathfrak{s}[P^\perp u, Pv] = \mathfrak{s}[u, Pv] - \mathfrak{s}[Pu, Pv] = \mathfrak{s}[u, Pv] - \mathfrak{s}[u, P^2 v] = 0.$$

Conversely, assume that $Pu \in \mathrm{Dom}[\mathfrak{s}]$ and $\mathfrak{s}[P^\perp u, Pv] = 0$ holds for $u, v \in \mathrm{Dom}[\mathfrak{s}]$. Since $\mathfrak{s}$ is symmetric, it follows $\mathfrak{s}[Pv, P^\perp u] = 0$. By the identity

$$\mathfrak{s}[P^\perp v, Pu] = \mathfrak{s}[v, Pu] - \mathfrak{s}[Pv, u] + \mathfrak{s}[Pv, P^\perp u],$$

we get that $\mathfrak{s}[v, Pu] = \mathfrak{s}[Pv, u]$. □

6.3. Graph subspaces and solutions to form Riccati equations

Recall that in Chapter 4 we related reducing graph subspaces for the operator sum $B = J_A A + V$ to solutions to the operator Riccati equation.

In this section, we consider reducing graph subspaces for the form given by

$$\mathfrak{b}[x, y] := \mathfrak{a}[x, J_A y] + \mathfrak{v}[x, y]$$

and relate these to solutions to the form Riccati equation. The following theorem for forms corresponds to Theorem 4.3.3 for operators.

THEOREM 6.3.1 ([**38**]). *Assume the hypothesis of the First Representation Theorem 1.5.3 in the off-diagonal case. Furthermore, let*

$$\mathcal{H}_0 \oplus \mathcal{H}_1 := \mathcal{G}(\mathcal{H}_+, X) \oplus \mathcal{G}(\mathcal{H}_-, -X^*) = \mathcal{H} \tag{6.6}$$

be a decomposition of $\mathcal{H}$ for a bounded operator X. Assume additionally that at least one of the equalities

$$\operatorname{Ran}\left((I_{\mathcal{H}_+} + X^*X)^{-1}|_{\operatorname{Dom}(A_+^{1/2})} \right) = \operatorname{Dom}(A_+^{1/2}) \tag{6.7}$$

or

$$\operatorname{Ran}\left((I_{\mathcal{H}_-} + XX^*)^{-1}|_{\operatorname{Dom}(A_-^{1/2})} \right) = \operatorname{Dom}(A_-^{1/2}) \tag{6.8}$$

holds.

Then the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ reduces the form $\mathfrak{b}$ if and only if X satisfies the following conditions

(i) $\operatorname{Ran}\left(X^*|_{\operatorname{Dom}(A_-^{1/2})} \right) \subseteq \operatorname{Dom}(A_+^{1/2})$,

(ii) $\operatorname{Ran}\left(X|_{\operatorname{Dom}(A_+^{1/2})} \right) \subseteq \operatorname{Dom}(A_-^{1/2})$,

(iii) *X is a solution of the form Riccati equation corresponding to $\mathfrak{b}$, that is*

$$\mathfrak{a}_+[-X^*y_-, x_+] - \mathfrak{a}_-[y_-, Xx_+] + \mathfrak{v}[-X^*y_- \oplus 0, 0 \oplus Xx_+] + \mathfrak{v}[0 \oplus y_-, x_+ \oplus 0] = 0 \tag{6.9}$$

holds for all $x_+ \in \operatorname{Dom}(A_+^{1/2}) = \operatorname{Dom}[\mathfrak{a}_+]$, $y_- \in \operatorname{Dom}(A_-^{1/2}) = \operatorname{Dom}[\mathfrak{a}_-]$.

PROOF. By the definition of the form $\mathfrak{b}$ and Lemma 6.2.6, it follows that

$$\operatorname{Dom}[\mathfrak{b}] = \operatorname{Dom}[\mathfrak{a}] = \operatorname{Dom}(A^{1/2}) = \operatorname{Dom}(A_+^{1/2}) \oplus \operatorname{Dom}(A_-^{1/2}).$$

It is straightforward to verify that P, the orthogonal projector onto $\mathcal{G}(\mathcal{H}_+, X)$, is given by the block operator matrix (see e.g. [**45**, Remark 3.6])

$$P = \begin{pmatrix} (I_{\mathcal{H}_+} + X^*X)^{-1} & (I_{\mathcal{H}_+} + X^*X)^{-1}X^* \\ X(I_{\mathcal{H}_+} + X^*X)^{-1} & X(I_{\mathcal{H}_+} + X^*X)^{-1}X^* \end{pmatrix} \tag{6.10}$$

written with respect to the original orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ induced by the involution J_A.

Remark that the operator $(I_{\mathcal{H}_+} + X^*X)\colon \mathcal{H}_+ \to \mathcal{H}_+$ is bijective. In a similar way, one gets that $P^\perp$, the orthogonal projector onto $\mathcal{G}(\mathcal{H}_-, -X^*)$, is given by

$$P^\perp = \begin{pmatrix} X^*(I_{\mathcal{H}_-} + XX^*)^{-1}X & -X^*(I_{\mathcal{H}_-} + XX^*)^{-1} \\ -(I_{\mathcal{H}_-} + XX^*)^{-1}X & (I_{\mathcal{H}_-} + XX^*)^{-1} \end{pmatrix} \tag{6.11}$$

and $(I_{\mathcal{H}_-} + XX^*)\colon \mathcal{H}_- \to \mathcal{H}_-$ is bijective.

Note that P maps $\operatorname{Dom}[\mathfrak{b}]$ into itself if and only if the complementary projector $P^\perp$ does.

The "only if" part: Assume first that

$$\operatorname{Ran}\left((I_{\mathcal{H}_+} + X^*X)^{-1}|_{\operatorname{Dom}(A_+^{1/2})} \right) \supseteq \operatorname{Dom}(A_+^{1/2}) \tag{6.12}$$

is valid.

Let $X\colon \mathcal{H}_+ \to \mathcal{H}_-$ be a bounded operator and assume that the orthogonal decomposition

$$\mathcal{H} = \mathcal{G}(\mathcal{H}_+, X) \oplus \mathcal{G}(\mathcal{H}_-, -X^*)$$

reduces the sesquilinear form $\mathfrak{b}$. By the first condition in Definition 6.2.2, we get that

$$(I_{\mathcal{H}_+} + X^*X)^{-1}x_+ + (I_{\mathcal{H}_+} + X^*X)^{-1}X^*y_- \in \mathrm{Dom}(A_+^{1/2}), \tag{6.13}$$

$$X(I_{\mathcal{H}_+} + X^*X)^{-1}x_+ + X(I_{\mathcal{H}_+} + X^*X)^{-1}X^*y_- \in \mathrm{Dom}(A_-^{1/2}) \tag{6.14}$$

holds for all $x_+ \in \mathrm{Dom}(A_+^{1/2}),\ y_- \in \mathrm{Dom}(A_-^{1/2})$.

Considering $y_- = 0$ in (6.13), we obtain that

$$(I_{\mathcal{H}_+} + X^*X)^{-1}\,\mathrm{Dom}(A_+^{1/2}) \subseteq \mathrm{Dom}(A^{1/2}).$$

Together with assumption (6.12), it follows that $(I_{\mathcal{H}_+} + X^*X)^{-1}$ is a bijective map from $\mathrm{Dom}(A_+^{1/2})$ into itself. Using this bijectivity, the consideration of $x_+ = 0$ in (6.13) and $y_- = 0$ in (6.14) yields that (i) and (ii) are satisfied.

To see that X is a solution of the form Riccati equation, note that we have

$$P^{\perp}(-X^*y_- \oplus y_-) = -X^*y_- \oplus y_- \quad \text{and} \quad P(x_+ \oplus Xx_+) = x_+ \oplus Xx_+$$

for all $x_+ \in \mathcal{H}_+,\ y_- \in \mathcal{H}_-$. With Lemma 6.2.7, we have in particular that

$$\mathfrak{b}[-X^*y_- \oplus y_-, x_+ \oplus Xx_+] = 0 \quad \text{for all } x_+ \in \mathrm{Dom}(A_+^{1/2}),\ y_- \in \mathrm{Dom}(A_-^{1/2}).$$

By the decomposition of

$$\mathfrak{b}[x,y] = \mathfrak{a}[x, J_A y] + \mathfrak{v}[x,y]$$

with respect to the involution J_A, where $\mathfrak{a}$ is a non-negative diagonal form and $\mathfrak{v}$ is an off-diagonal form and Remark 6.2.6, the claim (iii) follows.

In the case of $\mathrm{Ran}\left((I_{\mathcal{H}_-} + XX^*)^{-1}|_{\mathrm{Dom}(A_-^{1/2})}\right) \supseteq \mathrm{Dom}(A_-^{1/2})$, the projector $P^{\perp}$ maps $\mathrm{Dom}[\mathfrak{b}]$ into itself. Thus $(I_{\mathcal{H}_-} + XX^*)^{-1}$ is bijective on $\mathrm{Dom}(A_-^{1/2})$.

The conditions (i)-(iii) now follow in a similar way.

The "if" part: Assume first that

$$\mathrm{Ran}\left((I_{\mathcal{H}_+} + X^*X)^{-1}|_{\mathrm{Dom}(A_+^{1/2})}\right) \subseteq \mathrm{Dom}(A_+^{1/2}) \tag{6.15}$$

holds. Assume further that equation (6.9) has a bounded solution X satisfying (i) and (ii). The Riccati equation (6.9) can be rewritten as

$$\mathfrak{b}[-X^*y_- \oplus y_-, x_+ \oplus Xx_+] = 0 \quad \text{for all } x_+ \in \mathrm{Dom}(A_+^{1/2}),\ y_- \in \mathrm{Dom}(A_-^{1/2}).$$

By the assumptions (i), (ii) and equation (6.15), the projector P maps $\mathrm{Dom}[\mathfrak{b}]$ into itself.

Since $P^{\perp} = I_{\mathcal{H}} - P+$, the projector $P^{\perp}$ also maps $\mathrm{Dom}[\mathfrak{b}]$ into itself. As a consequence $(I_{\mathcal{H}_-} + XX^*)^{-1}$ maps $\mathrm{Dom}(A_-^{1/2})$ into itself.

Let $y = y_+ \oplus y_- \in \mathrm{Dom}[\mathfrak{b}] = \mathrm{Dom}(A_+^{1/2}) \oplus \mathrm{Dom}(A_-^{1/2})$. Then, by (6.15), there exists $\tilde{y}_+ \in \mathrm{Dom}(A_+^{1/2})$ with

$$Py = \begin{pmatrix} (I_{\mathcal{H}_+} + X^*X)^{-1}(y_+ + X^*y_-) \\ X(I_{\mathcal{H}_+} + X^*X)^{-1}(y_+ + X^*y_-) \end{pmatrix} = \begin{pmatrix} \tilde{y}_+ \\ X\tilde{y}_+ \end{pmatrix}.$$

In the same way, for $x = x_- \oplus x_- \in \mathrm{Dom}[\mathfrak{b}]$ there exists $\tilde{x}_- \in \mathrm{Dom}(A_-^{1/2})$ such that $P^{\perp}x = -X^*\tilde{x}_- \oplus \tilde{x}_-$. This implies

$$\mathfrak{b}[P^{\perp}x, Py] = 0 \quad \text{for all } x, y \in \mathrm{Dom}[\mathfrak{b}].$$

Thus, by Lemma 6.2.7, the orthogonal decomposition

$$\mathcal{H} = \mathcal{K}_+ \oplus \mathcal{K}_- \text{ with } \mathcal{K}_+ = \mathcal{G}(\mathcal{H}_+, X),\ \mathcal{K}_- = \mathcal{G}(\mathcal{H}_-, -X^*)$$

reduces the form $\mathfrak{b}$.

If $\operatorname{Ran}\left((I_{\mathcal{H}_-}+XX^*)^{-1}|_{\operatorname{Dom}(A_-^{1/2})}\right) \subseteq \operatorname{Dom}(A_-^{1/2})$ holds, then also $P^\perp$ and P map $\operatorname{Dom}[\mathfrak{b}]$ into itself. □

Remark that in the same way as in Chapter 4, an operator X is a solution of the form Riccati equation if X and its adjoint X^* map accordingly (that is (i) and (ii) of Theorem 6.3.1 are satisfied) and (6.9) holds.

At the first glance, the conditions (6.7) and (6.8) in the theorem above seem to be unexpected since the corresponding operators neither appear in the reducing decomposition nor in the form Riccati equation of that theorem. However, these conditions appear in a natural way by the representation of the projectors P and $P^\perp$.

Additionally, we make the following observations on (6.7) and (6.8) in Theorem 6.3.1.

Remark 6.3.2. *Note that in each implication of the equivalence in Theorem 6.3.1, only one of the two inclusions $\subseteq$ and $\supseteq$ in the assumptions* (6.7) *and* (6.8)*, respectively, is used. The other inclusion is automatically satisfied in this implication.*

Namely, if the decomposition reduces $\mathfrak{b}$, then the projector P leaves $\operatorname{Dom}[\mathfrak{a}] = \operatorname{Dom}[\mathfrak{b}]$ *invariant. The operator $(I_{\mathcal{H}_+}+X^*X)^{-1}$ thus has to map* $\operatorname{Dom}(A^{1/2})$ *into itself.*

Conversely, if X is a solution of the form Riccati equation (6.9)*, such that (i) and (ii) of Theorem 6.3.1 are satisfied, then $(I_{\mathcal{H}_+}+X^*X)$ maps* $\operatorname{Dom}(A^{1/2})$ *into itself. Since the operator $(I_{\mathcal{H}_+}+X^*X)$ is bijective on $\mathcal{H}_+$, it follows that*

$$\operatorname{Ran}\left((I_{\mathcal{H}_+}+X^*X)^{-1}|_{\operatorname{Dom}(A_+^{1/2})}\right) \supseteq \operatorname{Dom}(A_+^{1/2}).$$

Lemma 6.3.3. *Assume the hypotheses of Corollary 6.1.5 and let $\mathfrak{b}$ satisfy the Second Representation Theorem. Furthermore, let*

$$\mathcal{H} = \operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+)) \oplus \operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+))^\perp$$

be an orthogonal decomposition of the Hilbert space $\mathcal{H}$.

Then the conditions (6.7) *and* (6.8) *are equivalent.*

Proof. By Remark 6.2.6, the decomposition reduces the operator B and the form $\mathfrak{b}$ as well. Furthermore, by Remark 6.1.8, the spectral subspaces are graph subspaces,

$$\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+)) = \mathcal{G}(\mathcal{H}_+, X_0), \quad \operatorname{Ran}(\mathsf{E}_B(-\infty,0]) = \mathcal{G}(\mathcal{H}_-, -X_0^*)$$

for the bounded operator X_0. Assume that condition (6.7),

$$\operatorname{Ran}\left((I_{\mathcal{H}_+}+X_0^*X_0)^{-1}|_{\operatorname{Dom}(A_+^{1/2})}\right) = \operatorname{Dom}(A_+^{1/2})$$

is valid. Thus, by Theorem 6.3.1, X_0 satisfies the form Riccati equation (6.9) and

$$X_0\colon \operatorname{Dom}(A_+^{1/2}) \to \operatorname{Dom}(A_-^{1/2}), \quad X_0^*\colon \operatorname{Dom}(A_-^{1/2}) \to \operatorname{Dom}(A_+^{1/2})$$

holds. Thus the bijective map $(I_{\mathcal{H}_-}+X_0X_0^*)\colon \mathcal{H}_- \to \mathcal{H}_-$ maps $\operatorname{Dom}(A_-^{1/2})$ into itself. Since the projector $P^\perp$ given by (6.11) maps $\operatorname{Dom}[\mathfrak{b}] = \operatorname{Dom}(A_+^{1/2}) \oplus \operatorname{Dom}(A_-^{1/2})$ into itself, it follows that $(I_{\mathcal{H}_-}+X_0X_0^*)^{-1}$ also maps $\operatorname{Dom}(A_-^{1/2})$ into itself and the claim (6.8) now follows from the bijectivity of $(I_{\mathcal{H}_-}+X_0X_0^*)^{-1}$ on $\mathcal{H}_-$.

Conversely, if (6.8) holds, the implication follows in a similar way. □

In applications, it may be difficult to check the conditions (6.7) or (6.8) since explicit knowledge on the operator X and its adjoint X^* is needed. We are thus interested in sufficient assumptions on the form $\mathfrak{b}$, such that these conditions are satisfied. Note that the problem to find sufficient conditions in the form framework is similar to the problem in the operator framework in Chapter 4, see in particular Theorem 4.3.3.

The whole situation is simpler if one of the forms $\mathfrak{a}_+$ or $\mathfrak{a}_-$ or their corresponding operators are bounded.

LEMMA 6.3.4. *Assume Hypothesis 1.5.1.*

(a) *If $\mathfrak{a}_+$ respectively $\mathfrak{a}_-$ is bounded, then the assumption* (6.7) *respectively* (6.8) *in Theorem 6.3.1 is satisfied.*
(b) *If additionally $\beta < 1$ in Hypothesis 1.5.1 and $\mathfrak{a}_-$ is bounded, then the form $\mathfrak{b}$ is bounded from below. If instead $\mathfrak{a}_+$ is bounded, then $\mathfrak{b}$ is bounded from above.*

PROOF. It suffices to consider the case of bounded $\mathfrak{a}_-$.

(a) The statement is obvious noting that the associated operator A_- satisfies

$$\operatorname{Dom}(A_-^{1/2}) = \mathcal{H}_-$$

if $\mathfrak{a}_-$ is bounded.

(b) Let $x = x_+ \oplus x_- \in \operatorname{Dom}[\mathfrak{a}] = \operatorname{Dom}[\mathfrak{a}_+] \oplus \operatorname{Dom}[\mathfrak{a}_-]$, then we have that

$$\mathfrak{a}[x, J_A x] = \mathfrak{a}[x, x] - 2\mathfrak{a}_-[x_-].$$

We now rewrite $\mathfrak{b}$ as the difference

$$\mathfrak{b}[x] = \big((\mathfrak{a} + I)[x] + \mathfrak{v}[x]\big) - \big(2\mathfrak{a}_-[x_-] + ||x||^2\big),$$

where the second form is bounded by assumption. The first form is non-negative since (1.4) with $\beta < 1$ allows to write

$$(\mathfrak{a} + I)[x] + \mathfrak{v}[x] \geq (\mathfrak{a} + I)[x] - |\mathfrak{v}[x]| \geq (1 - \beta)(\mathfrak{a} + I)[x] \geq 0.$$

As a consequence, $\mathfrak{b}$ is bounded from below. $\square$

An example where $A_- = 0$ and thus (6.8) holds, is the form $\mathfrak{b}_S$ represented by the Stokes operator B_S introduced in Theorem 5.1.2.

If both A_+ and A_- are unbounded, we show that for A-infinitesimal operators V and forms $\mathfrak{v}$ defined by (2.8), the additional assumptions (6.7) respectively (6.8) in Theorem 6.3.1 are satisfied.

The principal idea for this is to use the results on reducing subspaces for operator sums in Chapter 4. The main tools in this process are interpolation by the Heinz Inequality (Lemma 2.1.2) and the coincidence of weak and strong solutions to operator Sylvester equations (Lemma 6.2.5).

Under the assumptions of Theorem 4.3.6 of the preceding Chapter 4, we get the following interpolation result.

LEMMA 6.3.5. *Let $A \geq 0$ be a self-adjoint operator and let J_A be a self-adjoint involution commuting with A. Assume furthermore that V is symmetric, infinitesimal with respect to A and off-diagonal with respect to J_A. Suppose that the graph space of the bounded operator X is a spectral subspace for the operator $B = J_A A + V$, that is*

$$\mathcal{G} := \mathcal{G}(\mathcal{H}_+, X) = \operatorname{Ran}(\mathsf{E}_B(M))$$

for some Borel set $M \subset \mathbb{R}$. Denote

$$Y := \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}$$

and assume that the domain stability condition

$$\operatorname{Dom}(A^{1/2}) = \operatorname{Dom}(|B|^{1/2})$$

is satisfied. Then $T^* := I - Y$ and its inverse are bijective on $\operatorname{Dom}(A^{1/2})$.

PROOF. Clearly, the spectral subspace $\mathcal{G}$ is reducing for B and T^* is bijective on $\mathrm{Dom}(A)$ by Theorem 4.3.6. By the relative boundedness of V, the product $V(A+I)^{-1}$ is bounded. Since T^* is bijective on $\mathrm{Dom}(A)$ by hypothesis, we can rewrite (4.21) as

$$(A+V)(T^*)^{-1} = (T^*)^{-1}(A - YV). \tag{6.16}$$

For brevity, denote

$$\mathcal{T} := (T^*)^{-1}.$$

Let $x \in \mathrm{Dom}(A) = \mathrm{Dom}(B)$, then, since the operators J_A and $\mathrm{sgn}(B)$ are unitary, we can use (6.16) to obtain

$$\begin{aligned}
\|(|B|+I)\mathcal{T}x\| &\leq \|\mathrm{sgn}(B)B\mathcal{T}x\| + \|\mathcal{T}x\| \\
&\leq \|B\mathcal{T}x\| + \left\|\mathcal{T}(A+I)^{-1}\right\| \cdot \|(A+I)x\| \\
&\leq \|\mathcal{T}(J_A A - YV)x\| + \|\mathcal{T}\| \cdot \|(A+I)x\| \\
&\leq \|\mathcal{T}\| \cdot \left\|(A - J_A YV)(A+I)^{-1}(A+I)x\right\| + \|\mathcal{T}\| \cdot \|(A+I)x\|.
\end{aligned}$$

With $A = A + I - I$, we can estimate further

$$\begin{aligned}
&\|(|B|+I)\mathcal{T}x\| \\
&\leq \|\mathcal{T}\|\big(2 + \|(I + J_A YV)(A+I)^{-1}\|\big) \cdot \|(A+I)x\| \\
&\leq \|\mathcal{T}\| \left(3 + \|Y\| \cdot \left\|V(A+I)^{-1}\right\|\right) \cdot \|(A+I)x\| \\
&=: c\,\|(A+I)x\|.
\end{aligned}$$

From the Heinz Inequality (Lemma 2.1.2), it follows that

$$(T^*)^{-1} \colon \mathrm{Dom}((A+I)^{1/2}) \to \mathrm{Dom}((|B|+I)^{1/2}).$$

By Remark 1.2.4, $(T^*)^{-1}$ maps $\mathrm{Dom}(A^{1/2})$ into itself.

Similar arguments below show that also T^* maps $\mathrm{Dom}(A^{1/2})$ into itself. To see this, note that V is A-bounded with bound less than 1. It follows from [**63**, Lemma 2.1.6] that V is also B-bounded with bound less than 1. In this case, $V(|B|+I)^{-1}$ is bounded. Remark that

$$S := (|B|+I)T^*(|B|+I)^{-1}$$

is defined on $\mathcal{H}$ since T^* maps $\mathrm{Dom}(B)$ into itself. Furthermore, S is closed since $(|B|+I)$ is closed and $T^*(|B|+I)^{-1}$ is bounded. Consequently, S is bounded.

Let $x \in \mathrm{Dom}(A)$, then with help of equation (4.21), we have the estimate

$$\begin{aligned}
\|(A+I)T^*x\| &= \|(J_A A + J_A)T^*x\| \\
&= \|(J_A A - YV + J_A + YV)T^*x\| \\
&\leq \|T^*Bx\| + \|(J_A + YV)T^*x\| \\
&\leq \left\|T^*\,\mathrm{sgn}(B)|B|(|B|+I)^{-1}(|B|+I)x\right\| + \|(J_A+YV)T^*x\|.
\end{aligned}$$

Noting that $\left\||B|(|B|+I)^{-1}\right\| \leq 1$, we can estimate further

$$\begin{aligned}
\|(A+I)T^*x\| &\leq \|T^*\| \cdot \left\||B|(|B|+I)^{-1}\right\| \cdot \|(|B|+I)x\| + \|(J_A+YV)T^*x\| \\
&\leq \|T^*\| \cdot \|(|B|+I)x\| + \left\|(J_A+YV)T^*(|B|+I)^{-1}(|B|+I)x\right\| \\
&\leq \left(2\,\|T^*\| + \left\|YVT^*(|B|+I)^{-1}\right\|\right)\|(|B|+I)x\| \\
&\leq \left(2\,\|T^*\| + \|Y\| \cdot \left\|V(|B|+I)^{-1}\right\| \cdot \|S\|\right)\|(|B|+I)x\| \\
&=: c\,\|(|B|+I)x\|.
\end{aligned}$$

From the Heinz Inequality, Lemma 2.1.2, it follows

$$T^*\colon \operatorname{Dom}((|B|+I)^{1/2}) \to \operatorname{Dom}((A+I)^{1/2}).$$

By Remark 1.2.4, T^* maps $\operatorname{Dom}(A^{1/2})$ into itself completing the proof. □

The following theorem shows that the assumptions (6.7) respectively (6.8) in Theorem 6.3.1 are satisfied under strong conditions on the off-diagonal perturbation $\mathfrak{v}$. Namely, we require B to coincide with the operator sum $J_A A + V$ on $\operatorname{Dom}(A)$.

To show that a solution to the form Riccati equation gives a reducing graph space in the theorem below, the domain inclusion $\operatorname{Dom}(V) \supseteq \operatorname{Dom}(A^{1/2})$ is used. For the converse implication, it suffices to have that V is infinitesimal with respect to A. By [**63**, Proposition 2.1.19], the domain inclusion $\operatorname{Dom}(V) \supseteq \operatorname{Dom}(A^{1/2})$ already implies that V is infinitesimal with respect to A and is thus the stronger condition.

THEOREM 6.3.6. *Let $\mathfrak{a}$ be a non-negative form and $\mathfrak{v}$ be the form associated with the self-adjoint operator V by Lemma 2.2.7. Furthermore, suppose that V is infinitesimal with respect to A. Let*

$$\mathfrak{b} := \mathfrak{a}[\,\cdot\,, J_A\,\cdot\,] + \mathfrak{v}$$

and let $X\colon \mathcal{H}_+ \to \mathcal{H}_-$ be a bounded operator. Suppose that

$$\mathcal{G}(\mathcal{H}_+, X) \oplus \mathcal{G}(\mathcal{H}_-, -X^*) = \mathcal{H}$$

is a decomposition of $\mathcal{H}$. Then the following statements hold.

(a) *If $\mathcal{G}(\mathcal{H}_+, X) = \operatorname{Ran}(\mathsf{E}_B(M))$ is a spectral subspace of the associated operator B for some Borel set M, then X is a solution of the form Riccati equation, that is, the conditions (i)-(iii) in Theorem 6.3.1 are satisfied.*

(b) *If X satisfies (i)-(iii) in Theorem 6.3.1 and additionally*

$$\operatorname{Dom}(V) \supseteq \operatorname{Dom}(A^{1/2})$$

holds, then the decomposition

$$\mathcal{H} = \mathcal{G}(\mathcal{H}_+, X) \oplus \mathcal{G}(\mathcal{H}_-, -X^*)$$

reduces the form $\mathfrak{b}$.

PROOF. By Lemma 2.2.7, the operator B associated with the form $\mathfrak{b}$ satisfies the domain stability condition

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(A^{1/2}).$$

To show part (a), note that the spectral decomposition $\mathcal{H} = \mathcal{G}(\mathcal{H}_+, X) \oplus \mathcal{G}(\mathcal{H}_-, -X^*)$ reduces the form $\mathfrak{b}$ as well as the operator B, see Lemma 6.2.4. From Theorem 4.3.6 and Lemma 6.3.5, it follows that

$$T^* = I - Y = \begin{pmatrix} I_{\mathcal{H}_+} & X^* \\ -X & I_{\mathcal{H}_-} \end{pmatrix}$$

is bijective on $\operatorname{Dom}(A^{1/2}) = \operatorname{Dom}(A_+^{1/2}) \oplus \operatorname{Dom}(A_-^{1/2})$ thus the conditions (i) and (ii) are satisfied. By Lemma 6.2.7, we have in particular

$$\mathfrak{b}[-X^* y_- \oplus y_-, x_+ \oplus X x_+] = 0 \quad \text{for all } x_+ \in \operatorname{Dom}(A_+^{1/2}),\ y_- \in \operatorname{Dom}(A_-^{1/2})$$

since

$$P^{\perp}(-X^* y_- \oplus y_-) = -X^* y_- \oplus y_- \quad \text{and} \quad P(x_+ \oplus X x_+) = x_+ \oplus X x_+$$

remain fixed under the corresponding projectors for any $x_+ \in \mathcal{H}_+, y_- \in \mathcal{H}_-$.

By the decomposition of

$$\mathfrak{b}[x, y] = \mathfrak{a}[x, J_A y] + \mathfrak{v}[x, y]$$

with respect to J_A, where $\mathfrak{a}$ is a non-negative diagonal form and $\mathfrak{v}$ is off-diagonal, the claim (iii) follows from Remark 6.2.6 part (b).

For part (b) of the theorem, let X satisfy the conditions (i)-(iii) and suppose that $\mathrm{Dom}(V) \supseteq \mathrm{Dom}(A^{1/2})$. Then V is automatically infinitesimal with respect to A (see [**63**, Corollary 2.1.20]) and

$$\mathfrak{v}[x, y] = \langle x, Vy\rangle$$

holds for $x, y \in \mathrm{Dom}(A^{1/2})$. Since Y maps $\mathrm{Dom}(A^{1/2})$ into $\mathrm{Dom}(A^{1/2})$, the operator $YVY - V$ can be defined on $\mathrm{Dom}(A^{1/2})$.

By part (b) of Remark 6.2.6, condition (iii) can be rewritten as

$$\langle A_+^{\frac{1}{2}} X^* y_-, A_+^{\frac{1}{2}} x_+\rangle + \langle A_-^{\frac{1}{2}} y_-, A_-^{\frac{1}{2}} X x_+\rangle + \mathfrak{v}[X^* y_- \oplus 0, 0 \oplus X x_+] - \overline{\mathfrak{v}[x_+ \oplus 0, 0 \oplus y_-]} = 0.$$

For simplicity, we write this equation in a short way as

$$\langle A_+^{\frac{1}{2}} X^* y_-, A_+^{\frac{1}{2}} x_+\rangle + \langle A_-^{\frac{1}{2}} y_-, A_-^{\frac{1}{2}} X x_+\rangle + \mathfrak{v}[X^* y_-, X x_+] - \overline{\mathfrak{v}[x_+, y_-]} = 0. \tag{6.17}$$

We now restrict (6.17) to $x_+ \in \mathrm{Dom}(A_+)$, $y_- \in \mathrm{Dom}(A_-)$ and get that

$$\langle y_-, XA_+ x_+\rangle + \langle A_- y_-, X x_+\rangle + \langle y_-, XW^* X x_+\rangle - \langle y_-, W x_+\rangle = 0.$$

Considering the complex conjugate of (6.17), it follows in the same way that

$$\langle A_+ x_+, X^* y_-\rangle + \langle x_+, X^* A_- y_-\rangle + \langle x_+, X^* W X^* y_-\rangle - \langle x_+, W^* y_-\rangle = 0.$$

Combining these two equations, we obtain that

$$\langle J_A A y, Y x\rangle - \langle y, Y J_A A x\rangle - \langle y, YVYx\rangle + \langle y, Vx\rangle = 0, \quad x, y \in \mathrm{Dom}(A)$$

This equation can be rewritten as

$$\langle J_A A y, Y x\rangle - \langle y, Y J_A A x\rangle = \langle y, (YVY - V)y\rangle$$

Thus, Y is a weak solution of the Sylvester equation

$$J_A A Z - Z J_A A = YVY - V,$$

for the unknown Z and, by Lemma 6.2.5, also a strong solution of this equation. In this case, Y is a strong solution of the Riccati equation

$$J_A A Z - Z J_A A - ZVZ + V = 0.$$

By Theorem 4.3.6, the decomposition reduces the operator B and by Lemma 6.2.4 also the form $\mathfrak{b}$. □

If the operator V has a large domain, we can rewrite the form Riccati equation in a special way.

Remark 6.3.7. *If V satisfies* $\mathrm{Dom}(V) \supseteq \mathrm{Dom}(A^{1/2})$*, then V is infinitesimal with respect to A (see* [**63**, Corollary 2.1.20]*) and the Riccati equation* (6.17) *can be rewritten as*

$$-\langle A_+^{1/2} X^* y_-, A_+^{1/2} x_+\rangle - \langle A_-^{1/2} y_-, A_-^{1/2} X x_+\rangle - \langle X^* y_-, W X x_+\rangle + \langle W y_-, x_+\rangle = 0 \tag{6.18}$$

for all $x_+ \in \mathrm{Dom}(A_+^{1/2})$, $y_- \in \mathrm{Dom}(A_-^{1/2})$ where $V = \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix}$.

In the following lemma we will give a compromise between Theorem 6.3.1, where additional assumptions on the mapping properties of the solution X of the form Riccati equation have to be made and Theorem 6.3.6, where $B = J_A A + V$ has to be defined as an operator sum on $\mathrm{Dom}(A)$. Below we require conditions on the form $\mathfrak{b}$ and on the operator $J_A A - YV - \lambda$ for some λ on the imaginary axis instead of (6.7) or (6.8). For applications, it may be easier to verify these conditions. In the following lemma, we have a special case, where $\mathrm{Dom}(V) \supseteq \mathrm{Dom}(A)$ is in general not satisfied but $J_A A + V$ is densely defined, compare Remark 4.3.4.

LEMMA 6.3.8. *Let the form $\mathfrak{b}$ be defined by the First Representation Theorem 1.5.3 in the off-diagonal case. Additionally, suppose that $\mathfrak{b}$ satisfies the Second Representation Theorem 2.1.1. Let*

$$V := (A+I)^{1/2}\widetilde{R}(A+I)^{1/2}$$

in the notation of Remark 1.5.2. Assume that there exists a point $0 \neq \lambda \in \mathrm{i}\mathbb{R}$ such that $J_A A - YV - \lambda$ has full range and a bounded inverse. Then the form $\mathfrak{b}$ is reduced by the decomposition $\mathcal{H} = \mathcal{G}(\mathcal{H}_+, X) \oplus \mathcal{G}(\mathcal{H}_-, -X^*)$ if and only if $Y := \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}$ satisfies the Riccati equation

$$J_A AY - YJ_A A - YVY + V = 0 \quad \textit{strongly on } \mathrm{Dom}(A) \cap \mathrm{Dom}(V),$$

that is

$$J_A AYx - YJ_A Ax - YVYx + Vx = 0 \textit{ for all } x \in \mathrm{Dom}(B) = \mathrm{Dom}(A) \cap \mathrm{Dom}(V). \tag{6.19}$$

PROOF. By Lemma 6.2.4, the form $\mathfrak{b}$ is reduced by the decomposition if and only if the associated operator B is reduced. Note that the self-adjoint operator B defined in Theorem 1.5.3 coincides with $(A+I)^{1/2}\widetilde{H}(A+I)^{1/2} - J_A$ on the corresponding natural domain. A short computation shows that this domain is in fact $\mathrm{Dom}(A) \cap \mathrm{Dom}(V)$ and thus B is given by the operator sum $J_A A + V$. By the self-adjointness of B, the operator $B - \lambda$ has full range and a bounded inverse for any non-trivial $\lambda \in \mathrm{i}\mathbb{R}$ and the domain $\mathrm{Dom}(B) = \mathrm{Dom}(A) \cap \mathrm{Dom}(V)$ is dense in $\mathcal{H}$. By the assumption on $J_A A + V - \lambda$ and $J_A A - YV - \lambda$, these operators have full range and bounded inverses. The claim then follows from Theorem 4.3.6 and Remark 4.3.4. □

It remains to note that as in the operator case, there is a sign-symmetry in the off-diagonal perturbation $\mathfrak{v}$ and the graph spaces $\mathcal{G}(\mathcal{H}_+, X)$, compare Remark 4.4.4.

LEMMA 6.3.9. *Assume Hypothesis 1.5.1 and let*

$$\mathfrak{b}[x,y] := \mathfrak{a}[x, J_A y] + \mathfrak{v}[x,y] \quad \textit{and} \quad \tilde{\mathfrak{b}}[x,y] := \mathfrak{a}[x, J_A y] - \mathfrak{v}[x,y]$$

be two forms. Then the following statements hold.

(a) *The graph space $\mathcal{G}(\mathcal{H}_+, X)$ is reducing for the form $\mathfrak{b}$ if and only if $\mathcal{G}(\mathcal{H}_+, -X)$ is reducing for the form $\tilde{\mathfrak{b}}$.*
(b) *The operator X is a solution of the form Riccati equation* (6.9) *for $\mathfrak{b}$ if and only if $-X$ is a solution of the corresponding equation for $\tilde{\mathfrak{b}}$.*

PROOF. For part (a), note that the orthogonal projector P onto $\mathcal{G}(\mathcal{H}_+, X)$ is given by

$$P = \begin{pmatrix} (I_{\mathcal{H}_+} + X^*X)^{-1} & (I_{\mathcal{H}_+} + X^*X)^{-1}X^* \\ X(I_{\mathcal{H}_+} + X^*X)^{-1} & X(I_{\mathcal{H}_+} + X^*X)^{-1}X^* \end{pmatrix}.$$

Thus, the orthogonal projector Q onto $\mathcal{G}(\mathcal{H}_+, -X)$ is given by

$$Q = \begin{pmatrix} (I_{\mathcal{H}_+} + X^*X)^{-1} & -(I_{\mathcal{H}_+} + X^*X)^{-1}X^* \\ -X(I_{\mathcal{H}_+} + X^*X)^{-1} & X(I_{\mathcal{H}_+} + X^*X)^{-1}X^* \end{pmatrix}.$$

This yields $Q = P - 2\begin{pmatrix} 0 & -(I_{\mathcal{H}_+} + X^*X)^{-1}X^* \\ -X(I_{\mathcal{H}_+} + X^*X)^{-1} & 0 \end{pmatrix}$.

By the sign-symmetry of the statement, it suffices to show one of the implications. Let $\mathcal{G}(\mathcal{H}_+, X)$ be reducing for the form $\mathfrak{b}$, then

$$P\,\mathrm{Dom}[\mathfrak{b}] \subseteq \mathrm{Dom}[\mathfrak{b}] = \mathrm{Dom}(A_+^{1/2}) \oplus \mathrm{Dom}(A_-^{1/2}) = \mathrm{Dom}[\tilde{\mathfrak{b}}].$$

Thus, we also have $Q\,\mathrm{Dom}[\tilde{\mathfrak{b}}] \subseteq \mathrm{Dom}[\tilde{\mathfrak{b}}]$. It remains to show that Q can commute from one side to the other in the form $\tilde{\mathfrak{b}}$.

Let $x_+, y_+ \in \mathrm{Dom}[\mathfrak{a}_+] = \mathrm{Dom}(A_+^{1/2}),\ x_-, y_- \in \mathrm{Dom}[\mathfrak{a}_-] = \mathrm{Dom}(A_-^{1/2})$. Then

$$\begin{aligned}
&\tilde{\mathfrak{b}}[Q(x_+ \oplus 0), y_+ \oplus y_-] \\
&= \tilde{\mathfrak{b}}[(I_{\mathcal{H}_+} + X^*X)^{-1}x_+ \oplus -X(I_{\mathcal{H}_+} + X^*X)^{-1}x_-, y_+ \oplus y_-] \\
&= \mathfrak{a}_+[(I_{\mathcal{H}_+} + X^*X)^{-1}x_+, y_+] - \mathfrak{a}_-[X(I_{\mathcal{H}_+} + X^*X)^{-1}x_+, -y_-] \\
&\qquad + \mathfrak{v}[0 \oplus X(I_{\mathcal{H}_+} + X^*X)^{-1}x_+, y_+ \oplus 0] + \mathfrak{v}[(I_{\mathcal{H}_+} + X^*X)^{-1}x_+ \oplus 0, 0 \oplus -y_-] \\
&= \mathfrak{b}[(I_{\mathcal{H}_+} + X^*X)^{-1}x_+ \oplus X(I_{\mathcal{H}_+} + X^*X)^{-1}x_+, y_+ \oplus y_-] \\
&= \mathfrak{b}[P(x_+ \oplus 0), y_+ \oplus -y_-] = \mathfrak{b}[x_+ \oplus 0, P(y_+ \oplus -y_-)] \\
&= \mathfrak{a}_+[x_+ \oplus 0, (I_{\mathcal{H}_+} + X^*X)^{-1}(y_+ - X^*y_-)] \\
&\qquad - \mathfrak{v}[x_+ \oplus 0, 0 \oplus X(I_{\mathcal{H}_+} + X^*X)^{-1}(-Xy_+ + X^*y_-)] \\
&= \tilde{\mathfrak{b}}[x_+ \oplus 0, Q(y_+ \oplus y_-)].
\end{aligned}$$

In a similar way, we have that $\tilde{\mathfrak{b}}[Q(0 \oplus x_-), y_+ \oplus y_-] = \tilde{\mathfrak{b}}[0 \oplus x_-, Q(y_+ \oplus y_-)]$ and thus $\tilde{\mathfrak{b}}[x, Qy] = \tilde{\mathfrak{b}}[Qx, y]$ for all $x,\ y \in \mathrm{Dom}[\tilde{\mathfrak{b}}]$.

The claim of part (b) follows immediately by substituting X and $-X$ in the form Riccati equation (6.9). $\square$

The statement of Lemma 6.3.9 is a generalisation of Remark 4.4.4. In this sense, the forms $\mathfrak{a}[\,\cdot\,, J_A\,\cdot\,] \pm \mathfrak{v}$ can be diagonalised at the same time as in the case of the operator sums $J_AA \pm V$. As a consequence, the spectra of the operators associated with the forms $\mathfrak{a}[\,\cdot\,, J_A\,\cdot\,] \pm \mathfrak{v}$ coincide.

6.4. Uniqueness of solutions

We now investigate the uniqueness of solutions to the form Riccati equation in special semibounded cases. We need the following lemma as preparation.

Lemma 6.4.1 ([**45**, Lemma 6.1]). *Let $\mathcal{H}$ be a separable Hilbert space with decomposition into the orthogonal sum of two subspaces $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Furthermore, let $X, Z\colon \mathcal{H}_0 \to \mathcal{H}_1$ with $\|X\|, \|Z\| \le 1$ be two contractions such that the orthogonal projectors in $\mathcal{H}$ onto their graphs $\mathcal{G}(\mathcal{H}_0, X)$ and $\mathcal{G}(\mathcal{H}_0, Z)$ commute. Then*

$$Z\big|_{\mathfrak{L}} = -X\big|_{\mathfrak{L}}, \qquad Z\big|_{\mathfrak{L}^\perp} = X\big|_{\mathfrak{L}^\perp},$$

where

$$\mathfrak{L} := \mathrm{Ker}(I_{\mathcal{H}_0} + Z^*X), \qquad \mathfrak{L}^\perp := \mathcal{H}_0 \ominus \mathfrak{L}.$$

Moreover, $\mathfrak{L}$ is a subspace of $\mathrm{Ker}(I_{\mathcal{H}_0} - X^*X) \cap \mathrm{Ker}(I_{\mathcal{H}_0} - Z^*Z)$.

Lemma 6.4.2. *Let $\mathfrak{b}$ be a form satisfying the assumption of Corollary 6.1.5. Additionally, suppose that $\mathfrak{a} = \mathfrak{a}_+ \oplus \mathfrak{a}_-$, where $\mathfrak{a}_+$ or $\mathfrak{a}_-$ is a bounded form.*

Then the operator X_0 given by $\mathcal{G}(\mathcal{H}_+, X_0) = \mathrm{Ran}(\mathsf{E}_B(\mathbb{R}_+))$ satisfies $1 \notin \sigma_p(|X_0|)$ and is a contraction, that is $\|X_0\| \le 1$ and one is not an eigenvalue of $|X_0|$.

PROOF. The fact that X_0 is a contraction follows directly from Remark 6.1.8. For the eigenvalue consideration, we modify the idea of the proof of [**46**, Theorem 2.4]. Let $X_0 = U|X_0|$ be the polar decomposition for X_0, where $U\colon \mathcal{H}_+ \to \mathcal{H}_-$ is a partial isometry with initial subspace $(\operatorname{Ker} X_0)^\perp$ and final subspace $\overline{\operatorname{Ran} X_0}$ and $|X_0| = (X_0^* X_0)^{1/2}$ is the absolute value of X_0.

Let f be an eigenvector of $|X_0|$ corresponding to the eigenvalue one, that is,

$$|X_0|f = f, \quad 0 \neq f \in \mathcal{H}_+.$$

By $\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+)) = \mathcal{G}(\mathcal{H}_+, X_0)$ we can consider

$$F := f \oplus X_0 f = f \oplus Uf \in \operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+)) \tag{6.20}$$

and

$$G := (-X_0^* Uf) \oplus Uf = -f \oplus Uf \in \operatorname{Ran}(\mathsf{E}_B(-\infty, 0]). \tag{6.21}$$

We now show $F \in \operatorname{Dom}(A_+^{1/2})$ and $G \in \operatorname{Dom}(A_-^{1/2})$.

Consider first the case that $\mathfrak{a}_+$, respectively A_+, is bounded. In this case, we have that

$$f \in \operatorname{Dom}(A_+^{1/2}) \quad \text{and} \quad Uf = X_0 f \in \operatorname{Dom}(A_-^{1/2})$$

by condition (ii) in Theorem 6.3.1.

In a similar way, if $\mathfrak{a}_-$ is bounded, we have that

$$Uf \in \operatorname{Dom}(A_-^{1/2}) \quad \text{and} \quad f = X_0^* Uf \in \operatorname{Dom}(A_+^{1/2})$$

by condition (i) in Theorem 6.3.1. Since F, G are in $\operatorname{Dom}[\mathfrak{b}]$ and in different parts of the decomposition $\mathcal{H} = \operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+)) \oplus \operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+))^\perp$, we have that

$$\mathfrak{b}[F, G] = 0.$$

Using the symmetry of the off-diagonal part $\mathfrak{v}$, we conclude that

$$\mathfrak{b}[F, G] = -\mathfrak{a}_+[f, f] - \mathfrak{a}_-[Uf, Uf].$$

These two equations on $\mathfrak{b}[F, G]$ however give a contradiction to the positivity assumption of the operator A_+ associated with the form $\mathfrak{a}_+$. $\square$

We now show that the form Riccati equation (6.9) for the Stokes operator B_S has a unique contractive solution.

THEOREM 6.4.3. *The form Riccati equation for the Stokes Operator B_S can be written as*

$$\sum_{j=1}^{n} \int_\Omega \langle -D_j(X^* q)(x), D_j u(x)\rangle - \int_\Omega (Xu)(x)\overline{\operatorname{div}\,(X^* q)(x)}dx + \int_\Omega \overline{q(x)}\operatorname{div} u(x)dx \tag{6.22}$$

and has a unique contractive solution.

PROOF. Equation (6.22) is just a reformulation of (6.9) for the form $\mathfrak{b}_S$ of the Stokes operator in Definition 5.1.1.

By Remark 5.1.5, the form $\mathfrak{b}_S$ satisfies the Second Representation Theorem 2.1.1. Thus, the form $\mathfrak{b}_S$ is reduced by the subspace $\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+))$, see Remark 6.2.6.

In the case, that the domain Ω has infinite volume, the kernel of the Stokes operator B_S is trivial by Theorem 5.2.4 and by Remark 6.1.8, the subspace

$$\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+)) = \mathcal{G}(\mathcal{H}_+, X_0)$$

is a graph space for a contraction X_0. By Lemma 6.4.2, we have that $1 \notin \sigma_p(|X_0|)$.

If Ω has a finite volume, the smallest Dirichlet-eigenvalue δ_0 of the operator $-\Delta$ on $H_2(\Omega) \cap H_0^1(\Omega)$ is strictly positive. Let $\mu \in (0, \delta_0)$, then $\mathfrak{a}_+ - \mu I$ is strictly positive and $\mathfrak{a}_- - \mu I_{\mathcal{H}_-} = -\mu I_{\mathcal{H}_-}$ is strictly negative. A computation similar to (5.4) shows that the off-diagonal part $\mathfrak{v}$ is $(\mathfrak{a}_+ + \mu I_{\mathcal{H}_+})$-bounded in the sense of (1.4). It follows from [**37**, Theorem 2.4(ii)] that $(-\mu, \delta_0 - \mu)$ belongs to the resolvent set of $B_S - \mu I$. In this case, [**37**, Theorem 3.1] implies that

$$\|\mathsf{E}_{A-\mu I}(\mathbb{R}_+) - \mathsf{E}_{B_S-\mu I}(\mathbb{R}_+)\| < \frac{\sqrt{2}}{2}.$$

By the assumption on μ, we have that

$$\mathsf{E}_{A-\mu I}(\mathbb{R}_+) = \mathsf{E}_A(\mathbb{R}_+) \quad \text{and} \quad \mathsf{E}_{B_S-\mu I}(\mathbb{R}_+) = \mathsf{E}_{B_S}(\mathbb{R}_+).$$

Together with [**45**, Corollary 3.4] this implies that $\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+))$ is related even to a strict contraction X_0 with $\|X_0\| < 1$.

So, in any case $1 \notin \sigma_p(|X_0|)$. It remains now to show the uniqueness of the contractive solution. Let X be a contractive solution of (6.22) that satisfies (i) and (ii) of Theorem 6.3.1, then by Theorem 6.3.6, the graph subspace $\mathcal{G}(\mathcal{H}, X)$ also is reducing for both $\mathfrak{b}$ and B.

Since the graph $\mathcal{G}(\mathcal{H}_+, X_0)$ of X_0 is a spectral subspace of B_S, the orthogonal projectors onto the graphs of X_0 and X commute. To see this, let P be the projector onto $\mathcal{G}(\mathcal{H}_+, X_0)$ and let Q be the projector onto $\mathcal{G}(\mathcal{H}_+, X)$. Then Q reduces any measurable function of B_S including the characteristic function $P = \mathsf{E}_{B_S}(\mathbb{R}_+)$ (see [**66**, Satz 8.23]), thus $\mathcal{G}(\mathcal{H}_+, X)$ reduces the operator P. Since P is bounded, it follows that $PQ = QP$ by Remark 6.2.1.

By Lemma 6.4.1, we have $\operatorname{Ker}(I_{\mathcal{H}_+} + X^* X_0) = \{0\}$ since it is a subspace of

$$\operatorname{Ker}(I_{\mathcal{H}_+} - X_0^* X_0) = \{0\}.$$

The uniqueness of the solution now follows from Lemma 6.4.1. □

It remains an open problem whether the uniqueness of contractive solutions to the form Riccati equation can be preserved in a more general, non-semibounded setting. The main problem is to extend Lemma 6.4.2 to non-semibounded situations. To obtain the uniqueness, we only have to show that $1 \notin \sigma_p(|X_0|)$. However, we can only exclude corresponding eigenfunctions in $\operatorname{Dom}(A_+^{1/2})$. Since $|X_0|$ is a bounded operator it is possible that corresponding eigenfunctions are not in this domain.

Recall that a similar problem in the investigation of the uniqueness of contractive solutions to the operator Riccati equation appears already in Theorem 6.1.9. As in the form case, we would have to exclude that one is an eigenvalue to obtain the uniqueness, but only eigenvalues in a corresponding domain can be excluded.

CHAPTER 7

Diagonalisation of representing operators

This chapter is based on the joint work [**38**] with L. Grubišić, V. Kostrykin, K. A. Makarov, and K. Veselić.

The aim is to give an explicit block diagonalisation for operators defined by the First Representation Theorem in the off-diagonal case. We will use the correspondence of reducing graph subspaces and solutions to the form Riccati equation obtained in the preceding chapter. The assumptions we impose for this technique are all satisfied for the Stokes operator B_S, respectively the form $\mathfrak{b}_S$.

7.1. Preliminaries

We collect the assumptions of this chapter in the following hypothesis.

Hypothesis 7.1.1. *Assume that $\mathcal{H}_+ \oplus \mathcal{H}_-$ is an orthogonal decomposition of $\mathcal{H}$. Furthermore, let*

$$\begin{aligned}\mathfrak{b}[x_+ \oplus x_-, y_+ \oplus y_-] &:= \langle A_+^{1/2} x_+, A_+^{1/2} y_+\rangle - \langle A_-^{1/2} x_-, A_-^{1/2} y_-\rangle + \langle W x_+, y_-\rangle + \langle x_-, W y_+\rangle \\ &=: \mathfrak{a}_+[x_+, y_+] - \mathfrak{a}_-[x_-, y_-] + \mathfrak{v}[x_+ \oplus x_-, y_+ \oplus y_-],\end{aligned}$$

where $\operatorname{Dom}[\mathfrak{b}] = \operatorname{Dom}(A_+^{1/2}) \oplus \operatorname{Dom}(A_-^{1/2})$*,* $A_\pm \geq 0$ *is self-adjoint and*

$$\operatorname{Dom}(W) \supseteq \operatorname{Dom}(A_+^{1/2}).$$

Suppose that $\mathfrak{b}$ satisfies the First and Second Representation Theorem with the associated self-adjoint operator B. Additionally, let $\operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+)) = \mathcal{G}(\mathcal{H}_+, X_0)$ be a graph space for some contraction X_0 and assume that this operator is a solution of the form Riccati equation (6.9)*.*

Indeed, all the hypotheses above are satisfied for the form $\mathfrak{b}_S$ of the Stokes operator, see Theorem 5.1.2 and Remarks 5.1.5 as well as 6.4.3.

We now give some preparatory observations for the block diagonalisation.

Lemma 7.1.2 ([**38**]). *Assume Hypothesis 7.1.1 and let X_0 be the contraction given by $\mathcal{G}(\mathcal{H}_+, X_0) = \operatorname{Ran}(\mathsf{E}_B(\mathbb{R}_+))$.*

(a) *Let* $\operatorname{Ran}\left((I_{\mathcal{H}_+} + X_0^* X_0)^{\pm 1}|_{\operatorname{Dom}(A_+^{1/2})}\right) = \operatorname{Dom}(A_+^{1/2})$*, then*

$$\operatorname{Ran}\left((I_{\mathcal{H}_+} + X_0^* X_0)^{s}|_{\operatorname{Dom}(A_+^{1/2})}\right) = \operatorname{Dom}(A_+^{1/2}) \text{ for all } s \in [-1, 1].$$

(b) *If* $\operatorname{Ran}\left((I_{\mathcal{H}_-} + X_0 X_0^*)^{\pm 1}|_{\operatorname{Dom}(A_-^{1/2})}\right) = \operatorname{Dom}(A_-^{1/2})$*, then*

$$\operatorname{Ran}\left((I_{\mathcal{H}_-} + X_0 X_0^*)^{s}|_{\operatorname{Dom}(A_-^{1/2})}\right) = \operatorname{Dom}(A_-^{1/2}) \text{ for all } s \in [-1, 1].$$

Proof. It suffices to consider part (a), the proof of part (b) is analogous.

If A_+ is bounded, the statement is trivially satisfied since the operator $(I + X_0^* X_0)$ and its inverse are bounded. Also, the statement is obvious in the case of $s \in \{0, \pm 1\}$.

By assumption, the operators $(A_+ + I)^{1/2}(I + X_0^* X_0)^{\pm 1}(A_+ + I)^{-1/2}$ are closed and defined on $\mathcal{H}_+$. By the Closed Graph Theorem, they are also bounded. Furthermore, the operator $(A_+ + I)^{1/2} X_0^* X_0 (A_+ + I)^{-1/2}$ is bounded and similar to the non-negative operator $X_0^* X_0$.

Assume now that $s \in (0,1)$, then the bounded operator $(I + X_0^* X_0)^s$ admits the integral representation

$$(I + X_0^* X_0)^s x = \frac{\sin(s\pi)}{\pi} \int_0^\infty w^{s-1}(I + X_0^* X_0)\big((1+w)I + X_0^* X_0\big)^{-1} x dw$$

for all $x \in \mathcal{H}_+$ (see, e.g., [**56**, Proposition 5.16]).

In order to get the inclusion $\operatorname{Ran}\left((I + X^*X)^s|_{\operatorname{Dom}(A_+^{1/2})}\right) \subseteq \operatorname{Dom}(A_+^{1/2})$ for $s > 0$, it suffices to show that the Hilbert space valued function $f\colon [0,\infty) \to \mathcal{H}_+$ with

$$f(w) := w^{s-1}(A_+ + I)^{1/2}(I + X_0^* X_0)\big((1+w)I + X_0^* X_0\big)^{-1}(A_+ + I)^{-1/2} x$$

is integrable on $\mathbb{R}_+$ for any $x \in \mathcal{H}_+$. One can easily verify the identity

$$\begin{aligned}
&(A_+ + I)^{1/2}\big((1+w)I + X_0^* X_0\big)^{-1}(A_+ + I)^{-1/2} \\
&= \big((1+w)I + (A_+ + I)^{1/2} X_0^* X_0 (A_+ + I)^{-1/2}\big)^{-1} \\
&= (1+w)^{-1}\big(I + (1+w)^{-1}(A_+ + I)^{1/2} X_0^* X_0 (A_+ + I)^{-1/2}\big)^{-1} \quad \text{for all } w \geq 0.
\end{aligned}$$

Note that the operator $(A_+ + I)^{1/2}\big((1+w)I + X_0^* X_0\big)^{-1}(A_+ + I)^{-1/2}$ is similar to the bounded operator $\big((1+w)I + X_0^* X_0\big)^{-1}$ and thus bounded for all $w \geq 0$.

Furthermore, for all sufficiently large $w > 0$ we have that

$$\big\|\big(I + (1+w)^{-1}(A_+ + I)^{1/2} X_0^* X_0 (A_+ + I)^{-1/2}\big)^{-1}\big\| \leq c$$

for some constant c. As a consequence, the estimate

$$\left\|(A_+ + I)^{1/2}((1+w)I + X_0^* X_0)^{-1}(A_+ + I)^{-1/2}\right\| \leq \frac{c}{w}$$

holds for some constant c and sufficiently large w. We now decompose f as a product

$$\begin{aligned}
f(w) &= w^{s-1}(A_+ + I)^{1/2}\big((1+w)I + X_0^* X_0\big)^{-1}(A_+ + I)^{-1/2} \\
&\qquad \cdot (A_+ + I)^{1/2}(I + X_0^* X_0)^{-1}(A_+ + I)^{-1/2} x \\
&=: w^{s-1}(A_+ + I)^{1/2}\big((1+w)I + X_0^* X_0\big)^{-1}(A_+ + I)^{-1/2} z,
\end{aligned}$$

where

$$z := (A_+ + I)^{1/2}(I + X_0^* X_0)^{-1}(A_+ + I)^{-1/2} x \in \mathcal{H}_+,$$

so that the function f is bounded and decays sufficiently fast to grant the integrability.

The inclusion

$$\operatorname{Ran}\left((I + X_0^* X_0)^s|_{\operatorname{Dom}(A_+^{1/2})}\right) \subseteq \operatorname{Dom}(A_+^{1/2}) \quad \text{for } s \in (-1, 0)$$

follows from the case of positive $1+s$ and the decomposition as a product of bounded operators

$$\begin{aligned}&(A_+ + I)^{1/2}(I + X_0^*X_0)^s(A_+ + I)^{-1/2}\\&= (A_+ + I)^{1/2}(I + X_0^*X_0)^{-1}(A_+ + I)^{-1/2}\\&\quad\cdot (A_+ + I)^{1/2}(I + X_0^*X_0)^{1+s}(A_+ + I)^{-1/2}.\end{aligned}$$

Since now both $(I + X_0X_0^*)^{\pm s}$ map $\mathrm{Dom}(A_+^{1/2})$ into itself for $s \in (-1, 1)$, the desired mapping property follows. □

Recall that the conditions in part (a) and (b) of Lemma 7.1.2 are satisfied under Hypothesis 7.1.1, see Remark 6.3.2 and Lemma 6.3.3. We now define the rotation that maps $(\mathcal{H}_+, \mathcal{H}_-)$ to $(\mathrm{Ran}(\mathsf{E}_B(\mathbb{R}_+)), \mathrm{Ran}(\mathsf{E}_B(\mathbb{R}_+))^\perp)$.

Definition 7.1.3 ([**38**]). *Let $U\colon \mathcal{H} \to \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be the linear map given by the block operator matrix representation*

$$U = \begin{pmatrix} (I + X_0^*X_0)^{-1/2} & -X_0^*(I + X_0X_0^*)^{-1/2} \\ X_0(I + X_0^*X_0)^{-1/2} & (I + X_0X_0^*)^{-1/2} \end{pmatrix},$$

where X_0 is the operator given by $\mathrm{Ran}(\mathsf{E}_B(\mathbb{R}_+)) = \mathcal{G}(\mathcal{H}_+, X_0)$.

Simple calculation shows that U is unitary. By the bijectivity of $(I + X_0^*X_0)^{-1/2}$ *on* $\mathcal{H}_+$ *and* $(I + X_0X_0^*)^{-1/2}$ *on* $\mathcal{H}_-$*, respectively, we have*

$$\begin{aligned}\mathrm{Ran}(U|_{\mathcal{H}_+}) &= \mathcal{G}(\mathcal{H}_+, X_0) = \mathrm{Ran}(\mathsf{E}_B(\mathbb{R}_+)) =: \mathcal{K}_+,\\ \mathrm{Ran}(U|_{\mathcal{H}_-}) &= \mathcal{G}(\mathcal{H}_-, -X_0^*) = \mathrm{Ran}(\mathsf{E}_B(\mathbb{R}_+))^\perp =: \mathcal{K}_-.\end{aligned}$$

By Lemma 7.1.2 and conditions (i) and (ii) in Theorem 6.3.1, we also have that

$$\mathrm{Ran}\left(U|_{\mathrm{Dom}(A_+^{1/2})\oplus\mathrm{Dom}(A_-^{1/2})}\right) \subseteq \mathrm{Dom}(A_+^{1/2}) \oplus \mathrm{Dom}(A_-^{1/2}).$$

Decomposing U into two parts, we obtain two unitary operators U_+, U_- with

$$U_+\colon \mathcal{H}_+ \to \mathcal{K}_+, \quad U_+x_+ := (I + X_0^*X_0)^{-1/2}x_+ \oplus X_0(I + X_0^*X_0)^{-1/2}x_+$$

and

$$U_-\colon \mathcal{H}_- \to \mathcal{K}_-, \quad U_+x_- := -X_0^*(I + X_0X_0^*)^{-1/2}x_- \oplus (I + X_0X_0^*)^{-1/2}x_-.$$

These operators map the positive respectively negative spectral subspace of $A_+ - A_-$ to the corresponding subspace for B.

Let $B = B_+ - B_-$ be the decomposition into positive and negative part of B with $B_+ > 0$ and $B_- \geq 0$. Then, by the domain stability condition $\mathrm{Dom}(A^{1/2}) = \mathrm{Dom}(|B|^{1/2})$ and $\mathrm{Ran}(\mathsf{E}_B(\mathbb{R}_+)) = \mathcal{G}(\mathcal{H}_+, X_0)$, we have that

$$\mathrm{Ran}\left(U_+|_{\mathrm{Dom}(A_+^{1/2})}\right) \subseteq \mathrm{Dom}(B_+^{1/2}) \quad \text{and} \quad \mathrm{Ran}\left(U_-|_{\mathrm{Dom}(A_-^{1/2})}\right) \subseteq \mathrm{Dom}(B_-^{1/2}).$$

We are now ready to give a diagonalisation of the form $\mathfrak{b}$ satisfying Hypothesis 7.1.1.

Lemma 7.1.4 ([**38**]). *Let $\mathfrak{b}$ satisfy the Hypothesis 7.1.1. Then the sesquilinear forms given by*

$$\begin{aligned}\hat{\mathfrak{b}}_+[x_+, y_+] &:= \mathfrak{a}_+[(I + X_0X_0^*)^{1/2}x_+, (I + X_0^*X_0)^{-1/2}y_+]\\&\qquad + \mathfrak{v}[(I + X_0^*X_0)^{1/2}x_+ \oplus 0, 0 \oplus X_0(I + X_0^*X_0)^{-1/2}y_+],\\ \hat{\mathfrak{b}}_-[x_-, y_-] &:= \mathfrak{a}_-[(I + X_0X_0^*)^{-1/2}x_-, (I + X_0X_0^*)^{1/2}y_-]\\&\qquad + \mathfrak{v}[X_0^*(I + X_0^*X_0)^{-1/2}x_- \oplus 0, 0 \oplus (I + X_0X_0^*)^{1/2}y_-],\end{aligned} \tag{7.1}$$

where $x_\pm, y_\pm \in \mathrm{Dom}[\mathfrak{a}_\pm]$*, are closed and non-negative. Furthermore, these forms can be diagonalised with respect to the decomposition* $B = B_+ - B_-$ *as*

$$\hat{\mathfrak{b}}_+[x_+, y_+] = \langle B_+^{1/2} U_+ x_+, B_+^{1/2} U_+ y_+ \rangle_{\mathcal{K}_+}, \tag{7.2}$$

$$\hat{\mathfrak{b}}_-[x_-, y_-] = \langle B_-^{1/2} U_- x_-, B_-^{1/2} U_- y_- \rangle_{\mathcal{K}_-}. \tag{7.3}$$

Proof. Note that U maps $\mathrm{Dom}[\mathfrak{b}]$ into itself. Let $\hat{\mathfrak{b}}_+$ be the symmetric sesquilinear form on $\mathrm{Dom}[\hat{\mathfrak{b}}_+] = \mathrm{Dom}[\mathfrak{a}_+]$ given by

$$\hat{\mathfrak{b}}_+[x_+, y_+] := \mathfrak{b}[U(x_+ \oplus 0), U(y_+ \oplus 0)],$$

then, by the Second Representation Theorem for non-negative forms [**43**, Theorem VI.2.23] and the mapping properties of U and U_+, the form

$$\hat{\mathfrak{b}}_+[x_+, y_+] = \langle B_+^{1/2} U_+ x_+, B_+^{1/2} U_+ y_+ \rangle_{\mathcal{H}_+}$$

is closed and non-negative. On the other hand, we can compute $U(x_+ \oplus 0)$ explicitly and get that

$$\hat{\mathfrak{b}}_+[x_+, x_+] = \mathfrak{b}[(I + X_0^* X_0^*)^{-1/2} x_+ \oplus X_0 (I + X^* X)^{-1/2} x_+].$$

Using the form Riccati equation (6.9), we can rewrite $\hat{\mathfrak{b}}_+$ as

$$\begin{aligned} \hat{\mathfrak{b}}_+[x_+] = {} & \mathfrak{a}_+[(I + X_0^* X_0)^{-1/2} x_+] - \mathfrak{a}_-[X_0 (I + X_0^* X_0)^{-1/2} x_+] \\ & + \mathfrak{v}[(I + X_0^* X_0)^{-1/2} x_+ \oplus 0, 0 \oplus X_0 (I + X_0^* X_0)^{-1/2} x_+] \\ & + \mathfrak{v}[0 \oplus X_0 (I + X_0^* X_0)^{-1/2} x_+, (I + X_0^* X_0)^{-1/2} x_+ \oplus 0]. \end{aligned} \tag{7.4}$$

By Lemma 7.1.2 and condition (ii) in Theorem 6.3.1, we can insert the two elements $(I + X_0^* X_0)^{-1/2} x_0 \in \mathrm{Dom}[\mathfrak{a}_+]$ and $X_0 (I + X_0^* X_0)^{-1/2} x_+ \in \mathrm{Dom}[\mathfrak{a}_-]$ into the form Riccati equation (6.9) to obtain that

$$\begin{aligned} 0 = {} & - \mathfrak{a}_+[X_0^* X_0 (I + X_0^* X_0)^{-1/2} x_+, (I + X_0^* X_0)^{-1/2} x_+] \\ & - \mathfrak{a}_-[X_0 (I + X_0^* X_0)^{-1/2} x_+, X_0 (I + X_0^* X_0)^{-1/2} x_+] \\ & + \mathfrak{v}[X_0^* X_0 (I + X_0^* X_0)^{-1/2} x_+ \oplus 0, 0 \oplus X_0 (I + X_0^* X_0)^{-1/2} x_+] \\ & + \mathfrak{v}[0 \oplus X_0 (I + X_0^* X_0)^{-1/2} x_+, (I + X_0^* X_0)^{-1/2} x_+ \oplus 0]. \end{aligned} \tag{7.5}$$

Combining equations (7.4) and (7.5), we get for all $x_+ \in \mathrm{Dom}[\mathfrak{a}_+]$ that

$$\begin{aligned} \hat{\mathfrak{b}}_+[x_+, x_+] = {} & \mathfrak{a}_+[(I + X_0 X_0^*)^{1/2} x_+, (I + X_0^* X_0)^{-1/2} x_+] \\ & + \mathfrak{v}[(I + X_0^* X_0)^{1/2} x_+ \oplus 0, 0 \oplus X_0 (I + X_0^* X_0)^{-1/2} x_+]. \end{aligned}$$

The claim now follows by polarisation. The form $\hat{\mathfrak{b}}_-$ can be considered in the same way. □

Note that for the block diagonalisation of the form $\mathfrak{b}$ we only needed that the off-diagonal part $\mathfrak{v}$ is defined on $\mathrm{Dom}[\mathfrak{a}]$ and did not use the representation by an operator W. However, for the diagonalisation of the operator B, we have to relay on this condition.

By the First Representation Theorem for non-negative forms [**43**, Theorem VI.2.1], the forms $\hat{\mathfrak{b}}_+$ respectively $\hat{\mathfrak{b}}_-$ are associated with self-adjoint operators $\widehat{B}_+$ respectively $\widehat{B}_-$. These operators contain all information on the positive and negative parts of B, B_+ and B_-, respectively.

LEMMA 7.1.5. *The operators $B_\pm$ and $\widehat{B}_\pm$ are unitary equivalent.*

PROOF. Note that $U_\pm\colon \mathcal{H}_\pm \to \mathcal{K}_\pm$ is unitary. By equations (7.2) and (7.3), the claim follows from [**43**, Example VI.2.13] since

$$\widehat{B}_\pm = (B_\pm^{1/2}U_\pm)^* B_\pm^{1/2} U_\pm = U_\pm^* B_\pm U_\pm. \qquad \square$$

We now carry over the diagonalisation of the forms $\hat{\mathfrak{b}}_\pm$ in Lemma 7.1.4 to the diagonalisation of the associated operators.

LEMMA 7.1.6. *Assume Hypothesis 7.1.1. Then the positive part $\widehat{B}_+$ can be rewritten as*

$$\widehat{B}_+ = (I + X_0^* X_0)^{1/2}(A_+ + X_0^* W)(I + X_0^* X_0)^{-1/2}. \tag{7.6}$$

Furthermore, the negative part $\widehat{B}_-$ satisfies the inclusion

$$\widehat{B}_- \subseteq (I + X_0 X_0^*)^{-1/2}\big(A_- + (W X_0^*)^*\big)(I + X_0 X_0^*)^{1/2}. \tag{7.7}$$

If additionally A_- and B_- are bounded, then the inclusion above is an equality.

PROOF. We consider first the positive part. By the Second Representation Theorem for the non-negative form $\mathfrak{a}_+$ ([**43**, Theorem VI.2.23]) and the definition of $\mathfrak{v}$ and $\mathrm{Dom}(W) \supseteq \mathrm{Dom}(A_+^{1/2})$, we can rewrite the form $\hat{\mathfrak{b}}_+$ as

$$\begin{aligned}\hat{\mathfrak{b}}_+[x_+, y_+] = {}& \langle A_+^{1/2}(I + X_0^* X_0)^{1/2} x_+, A_+^{1/2}(I + X_0^* X_0)^{-1/2} y_+\rangle \\ &+ \langle W(I + X_0^* X_0)^{1/2} x_+, X_0(I + X_0^* X_0)^{-1/2} y_+\rangle,\ x_+, y_+ \in \mathrm{Dom}(A_+^{1/2}).\end{aligned}$$

We restrict this equation to x_+ in the dense set

$$\mathrm{Dom}(A_+(I + X_0^* X_0)^{1/2}) \subseteq \mathrm{Dom}(A_+^{1/2})$$

and obtain that

$$\hat{\mathfrak{b}}_+[x_+, y_+] = \big\langle (I + X_0^* X_0)^{-1/2}(A_+ + X_0^* W)(I + X_0^* X_0)^{1/2} x_+, y_+\big\rangle$$

for all $x_+ \in \mathrm{Dom}(A_+(I + X_0^* X_0)^{1/2})$, $y_+ \in \mathrm{Dom}(A_+^{1/2})$. From [**43**, Corollary VI.2.4], it follows that

$$(I + X_0^* X_0)^{-1/2}(A_+ + X_0^* W)(I + X_0^* X_0)^{1/2} \subseteq \widehat{B}_+.$$

To show that even equality holds, note that the operators $\widehat{B}_+$ and A_+ are self-adjoint and non-negative. Following the lines of the proof of Lemma 4.3.5, the perturbation results [**43**, Theorems V.3.16 and VI.3.17] yield that $A_+ + X_0^* W$ is closed and $\mathrm{i}\lambda \in \rho(A_+ + X_0^* W)$ for all real λ with sufficiently large absolute value. Thus,

$$(I + X_0^* X_0)^{-1/2}(A_+ + X_0^* W)(I + X_0^* X_0)^{1/2} \quad \text{and} \quad \widehat{B}_+$$

have a common point on the imaginary axis in their resolvent sets. The claim now follows from Corollary 4.3.2.

In the same way, restricting $\hat{\mathfrak{b}}_-[x_-, y_-]$ to $x_- \in \mathrm{Dom}(A_-(I + X_0 X_0^*)^{-1/2})$, we have that

$$\hat{\mathfrak{b}}_-[x_-, y_-] = \langle (I + X_0 X_0^*)^{1/2}(A_- + W X_0^*)(I + X_0 X_0^*)^{-1/2} x_-, y_-\rangle.$$

In the same way as before, it follows that

$$\widehat{B}_- \supseteq (I + X_0 X_0^*)^{1/2}(A_- + W X_0^*)(I + X_0 X_0^*)^{-1/2}. \tag{7.8}$$

Remark that in general we cannot grant, by means of perturbation theory, that $\widehat{B}$ and $(A_- + W X_0^*)$ have a common point in their resolvent sets, see also the discussion at the beginning of Section 4.4.

The inclusion (7.7) follows then by taking the adjoint on both sides of (7.8).

If we additionally suppose that A_- and B_- are bounded, then by taking the adjoint, the bounded operator $\widehat{B}_-$ defined on $\mathcal{H}$ is extended by

$$\widehat{B}_- \subseteq (I + X_0X_0^*)^{-1/2}\big(A_- + (WX_0^*)^*\big)(I + X_0X_0^*)^{1/2},$$

so that equality must hold. □

Recall that if $\mathfrak{a}_-$ is bounded, the boundedness of B_- can be granted by Lemma 6.3.4 if $\beta < 1$.

Under stronger assumptions, we can reproduce the diagonalisation of Chapter 4 by means of forms.

Remark 7.1.7. *If, additionally to Hypothesis 7.1.1, we impose that*

$$\mathrm{Dom}(W^*) \supseteq \mathrm{Dom}(A_-)^{1/2},$$

then we alternatively get

$$\widehat{B}_- = (I + X_0X_0^*)^{1/2}(A_- + X_0^*W^*)(I + X_0X_0^*)^{-1/2},$$

using the same arguments as in the case of $\widehat{B}_+$.

In this case, the operator B *given by the form* $\mathfrak{b}[x,y] := \mathfrak{a}[J_Ax,y] + \mathfrak{v}[x,y]$ *can also be represented as the operator* $B = J_AA + V$*, where*

$$J_AA = \begin{pmatrix} A_+ & 0 \\ 0 & -A_- \end{pmatrix}, \quad V = \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix}.$$

To see this, note that $\mathrm{Dom}(V) \supseteq \mathrm{Dom}(J_AA)$*, so that* V *is infinitesimal with respect to* J_AA *(see* [**63**, Corollary 2.1.20]*) and the operator is defined by the Kato-Rellich Theorem* [**55**, Theorem X.12]. *For operators of this type, the diagonalisation is already contained in Theorem 4.3.6.*

We now apply the diagonalisation in Lemma 7.1.6 to the Stokes operator B_S. This allows a closer inspection of the positive and negative part of its spectrum. It turns out that these parts are related to the Dirichlet Laplacian and the Cosserat operator, respectively, see Sections 7.4 and 7.5 below. We start with the investigation of the Laplacian.

7.2. Properties of the Laplacian

Recall that in Theorem 5.2.3, we established that the kernel of the Laplacian $-\Delta$ with homogeneous Dirichlet boundary values is trivial for arbitrary Lipschitz domains.

To obtain further spectral information, we use the following classification of unbounded domains which is due to Glazman, see [**31**, Section IV.49], see also [**26**, Section 4.2].

Let $\Omega \subseteq \mathbb{R}^n$ be a domain, then Ω is called *quasi-conical* if it contains a sphere of arbitrarily large radius. If Ω is not quasi-conical but contains infinitely many pairwise disjoint spheres of the same radius, then Ω is called *quasi-cylindrical.* Finally, if Ω is none of the above, then Ω is called *quasi-bounded.* The most prominent representants of these types of domains are half-spaces, cylinders, and bounded domains, respectively.

Recall that a domain is quasi-bounded if and only if, for any radius $r > 0$, only a finite number of pairwise disjoint spheres of radius r are contained. In this sense, an unbounded domain Ω is quasi-bounded if and only if it is *narrow at infinity* in the sense of [**2**, 6.9],

$$\lim_{x\in\Omega,\ |x|\to\infty} \mathrm{dist}(x, \partial\Omega) = 0.$$

We now collect facts on the Dirichlet Laplacian for domains in the classification above.

REMARK 7.2.1. (a) *For quasi-conical domains the spectrum of* $-\Delta$ *is continuous and consists of the whole half-line* $[0, \infty)$. *This is a direct consequence of the non-negativity of* $-\Delta$ *and the inclusion* $[0, \infty) \subseteq \sigma_c(-\Delta)$ *for the continuous spectrum in* [**31**, Theorem IV.8].

(b) *For quasi-bounded, sufficiently regular domains, the operator* $-\Delta$ *has a compact resolvent and thus the spectrum is purely discrete. Sufficient regularity conditions can be given in terms of the capacity of the domain and are optimal in the sense of compact imbeddings of* $H_0^1(\Omega)$ *into* $L^2(\Omega)$, *see* [**2**, Theorem 6.19].
To be more precise on the regularity, let $K \subset \mathbb{R}^n$ *be a cube of length* k. *Then, for* $u \in C^\infty(K)$ *we define*

$$I_K(u) := k^2 \sum_{|\alpha|=1} \int_K |D^\alpha u(x)|^2 dx.$$

Denote by $C^\infty(K, E)$ *the set of all non-trivial smooth functions vanishing in a neighbourhood of* E, *then the capacity* $Q(K, E)$ *of a closed subset* E *in* K *is denoted by*

$$Q(K, E) := \inf \left\{ \frac{I_K(u)}{\|u\|^2_{L^2(K)}} \middle| u \in C^\infty(K, E) \right\}.$$

The imbedding of $H_0^1(\Omega)$ *into* $L^2(\Omega)$ *is compact if and only if for every* $\varepsilon > 0$ *there exists* $k < 1$ *and* $r \geq 0$ *such that*

$$Q(K, K \setminus \Omega) \geq k^2/\varepsilon$$

for every cube K *of length* k *with* $K \cap (\Omega \setminus B_r(0)) \neq \varnothing$, *that is,* K *intersects with the parts in* Ω *outside of the ball of radius* r *around zero.*

Note that $\langle -\Delta\varphi, \psi\rangle = \langle \operatorname{grad}\varphi, \operatorname{grad}\psi\rangle$ *for all* $\varphi, \psi \in C_0^\infty(\Omega)$ *and that the imbedding of* $H_0^1(\Omega)$ *into* $L^2(\Omega)$ *is compact. The compactness of the operator* $(-\Delta)^{-1}$ *and the discreteness of its spectrum follow now by the same argumentation as in* [**11**, Theorem 5].

Remark that for bounded domains satisfying the cone condition, the compactness of the imbedding follows directly from the Rellich-Kondrachov Theorem, see, e.g., [**2**, Theorem 6.3].

(c) *For quasi-cylindrical domains, localisation of the spectrum and its type is more complicated. We will only illustrate the behaviour in three special cases.*

The first case are limit-cylindrical domains in $\mathbb{R}^3$ *as in* [**31**, Theorem IV.16]. *Let* Ω_0 *be a bounded domain in the two-dimensional plane* $\{x_3 = 0\} \subset \mathbb{R}^3$.

Suppose that the boundary of Ω_0 *is described by the radius*

$$r = r(\varphi), \ \varphi \in [0, 2\pi)$$

in polar coordinates. Let δ_0 *denote the smallest Dirichlet eigenvalue of the two dimensional Laplacian* $-\Delta$ *on* Ω_0. *Furthermore, let* $\Omega \subset \mathbb{R}^3$ *be given by the surface*

$$r = r(\varphi)(1 + f(x_3)),$$

where $f(x_3) > -1$ *with* $\lim_{x_3 \to \infty} f(x_3) = 0$. *Then the continuous spectrum of the Dirichlet Laplacian consists of the interval* $[\delta_0, \infty)$.

If $\limsup_{x_3 \to \infty} x_3^2 f(x_3) < \frac{1}{8\delta_0}$, *then there are only finitely many eigenvalues below the continuous spectrum.*

If instead $\limsup_{x_3 \to \infty} x_3^2 f(x_3) > \frac{1}{8\delta_0}$, *then there exist infinitely many eigenvalues below the continuous spectrum that accumulate only at* δ_0.

The second case are tubular neighbourhoods around infinite, sufficiently smooth curves in $\mathbb{R}^n$ *that are asymptotically straight, see* [**10**] *and the discussion in* [**26**, Section 4.2]*. In this case, the essential spectrum is the interval* $[\delta_0, \infty)$*, where* δ_0 *is the smallest Dirichlet eigenvalue of the* $(n-1)$*-dimensional cross-section. However, if the domain is not a tube itself, there are eigenvalues below the essential spectrum. We do not know whether the number of these eigenvalues is finite or convergence to* δ_0 *holds as in the first case.*

The third case are half-cylinders in $\mathbb{R}^n$ *that are bounded in* m *directions, that is,*

$$\Omega = \big\{x = (x_1, \ldots, x_n) \mid |x_i| < \rho,\ i \in \{1, \ldots, m\},\ x_i \in (0, \infty),\ i \in \{m+1, \ldots, n\}\big\},$$

then the essential spectrum of $-\Delta$ *is* $[\pi^2 m \rho^{-2}, \infty)$*, see* [**20**, Theorem X.6.6].

Note that the Laplacian is invariant under translations and rotations, so that the statements remain valid for domains that can be transformed into the structures above, see the discussion at the end of [**20**, Section X.6.1].

By a *sufficiently regular domain*, we always mean regularity for the boundary in the suitable case of the remark above.

The most important property for us is that only in sufficiently regular quasi-bounded domains, we can grant the compactness of the resolvent of the Dirichlet Laplacian.

7.3. Numerical ranges of forms

In this section, we investigate the numerical range and the quadratic numerical range for forms. This allows to give an estimate on the lower bound of the positive spectrum of the Stokes operator. The quadratic numerical range for forms is a natural generalisation of the corresponding range for operators. For a discussion of the numerical range and the quadratic numerical range of block operators see the book [**63**] by Tretter.

The numerical range of an operator B is denoted as

$$W(B) := \{\langle x, Bx\rangle \mid x \in \mathrm{Dom}(B),\ \|x\| = 1\}.$$

Definition 7.3.1 ([**38**]). *Let* $\mathfrak{b}$ *be a form as in Theorem 1.5.3, then we denote the numerical range of* $\mathfrak{b}$ *by*

$$W[\mathfrak{b}] := \{\mathfrak{b}[x] \mid x \in \mathrm{Dom}[\mathfrak{b}],\ \|x\| = 1\}.$$

The quadratic numerical range of $\mathfrak{b}$ *is given by the union of spectra of* 2×2 *matrices,*

$$W^2[\mathfrak{b}] := \bigcup_{\substack{x_+ \oplus x_- \in \mathrm{Dom}[\mathfrak{a}_+] \oplus \mathrm{Dom}[\mathfrak{a}_-], \\ \|x_+\| = \|x_-\| = 1}} \sigma \begin{pmatrix} \mathfrak{a}_+[x_+] & \mathfrak{v}[x_+ \oplus 0, 0 \oplus x_-] \\ \overline{\mathfrak{v}[x_+ \oplus 0, 0 \oplus x_-]} & -\mathfrak{a}_-[x_-] \end{pmatrix}.$$

The following properties are natural generalisations of the corresponding statements for operators and are shown in a similar way, see, e.g., [**63**] for the operator case.

Lemma 7.3.2 ([**38**]). *Let* $\mathfrak{b}$ *be a form as in The First Representation Theorem 1.5.3 in the off-diagonal case and let* B *be the self-adjoint associated operator. Furthermore, let* $W(B)$ *be the numerical range of the operator* B*. Then*

(a) $\sigma(B) \subseteq \overline{W^2[\mathfrak{b}]}$;
(b) $W(B) \subseteq W[\mathfrak{b}] \subseteq \overline{W(B)} \subseteq \overline{(\inf \sigma(B), \sup \sigma(B))}$*. The same inclusions hold for the non-negative form* $\mathfrak{a} = \mathfrak{a}_+ \oplus \mathfrak{a}_-$ *and its associated operator* A*;*
(c) $W^2[\mathfrak{b}] \subseteq W[\mathfrak{b}]$;
(d) $\inf \sigma(B) = \inf \overline{W^2[\mathfrak{b}]}, \quad \sup \sigma(B) = \sup \overline{W^2[\mathfrak{b}]}$;
(e) $W[\mathfrak{a}_\pm] \subseteq W^2[\mathfrak{b}]$ *if* $\dim \mathcal{H}_\pm > 1$*;*

(f) *if* $\mathfrak{a}_\pm \geq \alpha_\pm I$ *for some* $\alpha_\pm \geq 0$, *then*

$$\sigma(B) \subseteq (-\infty, -\alpha_-] \cup [\alpha_+, \infty).$$

Proof. (a) First, let $\lambda \in \mathbb{R}$ be an eigenvalue of B with corresponding eigenfunction $u \in \mathrm{Dom}(B)$. Since $\mathrm{Dom}(B) \subseteq \mathrm{Dom}(A^{1/2}) = \mathrm{Dom}(A_+^{1/2}) \oplus \mathrm{Dom}(A_-^{1/2})$, we have the unique decomposition $u = u_+ \oplus u_-$ with $u_\pm \in \mathrm{Dom}(A_\pm^{1/2})$. We set $\hat{u}_\pm := \|u_\pm\|^{-1} u_\pm$ if $u_\pm \neq 0$ and choose $\hat{u}_\pm$ in $\mathrm{Dom}(A_\pm^{1/2})$ arbitrary with $\|\hat{u}_\pm\| = 1$ if $u_\pm = 0$. From the eigenvalue equation, we obtain that

$$\langle \hat{u}_+ \oplus 0, Bu \rangle = \lambda \langle \hat{u}_+, u_+ \rangle, \quad \langle 0 \oplus \hat{u}_-, Bu \rangle = \lambda \langle \hat{u}_-, u_- \rangle.$$

By the First Representation Theorem, we can rewrite these equations in a 2×2 matrix form

$$\begin{pmatrix} \mathfrak{a}_+[\hat{u}_+] & \mathfrak{v}[\hat{u}_+ \oplus 0, 0 \oplus \hat{u}_-] \\ \overline{\mathfrak{v}[\hat{u}_+ \oplus 0, 0 \oplus \hat{u}_-]} & -\mathfrak{a}_-[\hat{u}_-] \end{pmatrix} \begin{pmatrix} \|u_+\| \\ \|u_-\| \end{pmatrix} = \lambda \begin{pmatrix} \|u_+\| \\ \|u_-\| \end{pmatrix}.$$

As a consequence $\lambda \in W^2[\mathfrak{b}]$.

If $\lambda \in \sigma(B)$ is not an eigenvalue, the Weyl criterion [**54**, Theorem VII.12] implies that there exists a sequence $(u^{(n)})_{n\in\mathbb{N}} \subset \mathrm{Dom}(B)$ with $\|u^{(n)}\| = 1$ and $(B-\lambda)u^{(n)} \to 0$, $n \to \infty$.

In the same way as before, we write $u^{(n)} = u_+^{(n)} \oplus u_-^{(n)} \in \mathrm{Dom}(A^{1/2})$ and introduce $\hat{u}_\pm^{(n)}$ for the normalised components. Then, we have that

$$\langle (B-\lambda)u^{(n)}, \hat{u}_+^{(n)} \oplus 0 \rangle =: v_+^{(n)}, \quad \langle (B-\lambda)u^{(n)}, 0 \oplus \hat{u}_-^{(n)} \rangle =: v_-^{(n)},$$

both converge to zero. By the First Representation Theorem 1.5.3, these equations can be rewritten as

$$\begin{pmatrix} (\mathfrak{a}_+ - \lambda)[\hat{u}_+^{(n)}] & \mathfrak{v}[\hat{u}_+^{(n)} \oplus 0, 0 \oplus \hat{u}_+^{(n)}] \\ \overline{\mathfrak{v}[\hat{u}_+^{(n)} \oplus 0, 0 \oplus \hat{u}_-^{(n)}]} & -(\mathfrak{a}_- + \lambda)[\hat{u}_-^{(n)}] \end{pmatrix} \begin{pmatrix} \|u_+^{(n)}\| \\ \|u_-^{(n)}\| \end{pmatrix} = \begin{pmatrix} v_+^{(n)} \\ v_-^{(n)} \end{pmatrix}. \tag{7.9}$$

Let $\mathcal{B}_n - \lambda$ denote the matrix in (7.9), then, by the definition of the Euclid norm on $\mathbb{R}^2$, we obtain the estimate

$$\begin{aligned} 1 &= \sqrt{\|u_+^{(n)}\|^2 + \|u_-^{(n)}\|^2} \\ &\leq \|(\mathcal{B}_n - \lambda)^{-1}\| \cdot \sqrt{(v_+^{(n)})^2 + (v_-^{(n)})^2} = \frac{\sqrt{(v_+^{(n)})^2 + (v_-^{(n)})^2}}{\mathrm{dist}(\lambda, \sigma(\mathcal{B}_n))}. \end{aligned}$$

This yields that

$$\mathrm{dist}(\lambda, \sigma(\mathcal{B}_n)) \leq \sqrt{(v_+^{(n)})^2 + (v_-^{(n)})^2} \to 0, \ n \to \infty$$

and consequently $\lambda \in \overline{W^2[\mathfrak{b}]}$.

(b) The first inclusion $W(B) \subseteq W[\mathfrak{b}]$ follows directly from the First Representation Theorem 1.5.3 noting that $\mathrm{Dom}(B) \subseteq \mathrm{Dom}[\mathfrak{b}]$.

For the second inclusion, $W[\mathfrak{b}] \subseteq \overline{W(B)}$, recall that $\mathrm{Dom}(B)$ is a core for the operator $(A+I)^{1/2}$. By Remark 1.5.5, we have that

$$\mathfrak{b}[x,x] = \langle (A+I)^{1/2}x, \widetilde{H}(A+I)^{1/2}x \rangle - \langle x, J_A x \rangle$$

holds for $x \in \mathrm{Dom}((A+I)^{1/2})$. With $B = (A+I)^{1/2}\widetilde{H}(A+I)^{1/2} - J_A$, the claim follows from the core property.

The last inclusion, $\overline{W(B)} \subseteq \overline{(\inf \sigma(B), \sup \sigma(B))}$, follows directly from the well known convexity of the numerical range and [**67**, Aufgabe VII.5.24(c)].

In the same way, the inclusions for the non-negative $\mathfrak{a}$ respectively A follow since $\mathrm{Dom}(A)$ is a core for $(A+I)^{1/2}$.

(c) Let $\lambda \in W^2[\mathfrak{b}]$, then, there exist $x_\pm \in \mathrm{Dom}(A^{1/2})$ with $\|x_\pm\| = 1$ and a point $c = (c_1, c_2) \in \mathbb{R}^2$ with $\|c\| = 1$ such that

$$\begin{pmatrix} \mathfrak{a}_+[x_+] & \mathfrak{v}[x_+ \oplus 0, 0 \oplus x_-] \\ \overline{\mathfrak{v}[x_+ \oplus 0, 0 \oplus x_-]} & -\mathfrak{a}_-[x_-] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Taking the scalar product with c yields

$$\mathfrak{b}[c_1 x_+ \oplus c_2 x_-] = \lambda,$$

where $\|c_1 x_+ \oplus c_2 x_-\| = 1$ and the claim follows.

(d) Note that by parts (b) and (c), we have $\overline{W^2[\mathfrak{b}]} \subseteq \overline{(\inf\sigma(B), \sup\sigma(B))}$. The claim now follows from part (a) since $\inf\sigma(B), \sup\sigma(B) \in \overline{W^2[\mathfrak{b}]}$.

(e) Assume that $\dim \mathcal{H}_- > 1$, then for each $x_+ \in \mathrm{Dom}[\mathfrak{a}_+]$, $\|x_+\| = 1$, there is an element $x_- \in \mathrm{Dom}[\mathfrak{a}_-]$, $\|x_-\| = 1$ with $\mathfrak{v}[x_+ \oplus 0, 0 \oplus x_-] = 0$. To see this, note that by Remark 1.5.2

$$\mathfrak{v}[x_+ \oplus 0, 0 \oplus x_-] = \langle R^*(A_+ + I)^{1/2} x_+, (A_- + I)^{1/2} x_- \rangle.$$

Let $f \in \mathcal{H}_+$, then, by $\dim \mathcal{H}_- > 1$, there exists an element $g \in \mathcal{H}_-$ such that $\langle R^* f, g \rangle_{\mathcal{H}_-} = 0$.

By the bijectivity of $(A+I)^{1/2} \colon \mathrm{Dom}((A+I)^{1/2}) \to \mathcal{H}$, a suitable x_- with $\mathfrak{v}[x_+ \oplus 0, 0 \oplus x_-] = 0$ then exists. In this case, we have that

$$\mathfrak{a}_+[x_+] \in \sigma \begin{pmatrix} \mathfrak{a}_+[x_+] & 0 \\ 0 & \mathfrak{a}_-[x_-] \end{pmatrix} \subseteq W^2[\mathfrak{b}].$$

(f) The claim follows directly, noting that the spectrum of the 2×2 matrix

$$\begin{pmatrix} a_+ & v \\ \overline{v} & -a_- \end{pmatrix}, \quad 0 \le a_\pm < \infty,\ v \text{ bounded}$$

is located outside of the interval $(-a_-, a_+)$. As a consequence $W^2[\mathfrak{b}]$ is outside of $(-\alpha_-, \alpha_+)$ and the claim follows.

$\square$

7.4. The positive part of the Stokes operator

Combining the results of the lemma above with the results of Section 7.2, we obtain that the positive part of the Stokes operator B_S defined by the form (5.9) is bounded from below by the smallest spectral value of the Dirichlet Laplacian. This bound is given by the smallest Dirichlet eigenvalue of the Laplacian if Ω is a (sufficiently regular) quasi-bounded or a limit-cylindrical domain. In these cases, the positive part of B_S is strictly positive. It turns out that for (sufficiently regular) quasi-bounded domains the positive part of B_S has a compact resolvent, and thus the positive spectrum is discrete, see the following two statements.

Lemma 7.4.1. *Assume Hypothesis 7.1.1 and let $\widehat{B}_+$ be the operator associated with the form $\hat{\mathfrak{b}}_+$ in Lemma 7.1.4. Furthermore, suppose that $\mathfrak{a}_+$ is strictly positive.*

Then the operator $\widehat{B}_+^{1/2} A_+^{-1/2}$ and its inverse are bounded. If $\mathfrak{a}_-$ is strictly positive then $\widehat{B}_-^{1/2} A_-^{-1/2}$ and its inverse are bounded.

Proof. Let A_+ respectively $\mathfrak{a}_+$ be strictly positive, then the mapping properties of X_0, namely

$$\mathrm{Ran}\left(X_0|_{\mathrm{Dom}(A_+^{1/2})} \right) \subseteq \mathrm{Dom}(A_+^{1/2}),$$

and those of $(I+X_0^*X_0)^{\pm 1/2}$ in Lemma 7.1.2 grant that the non-negative quadratic form

$$\begin{aligned}\hat{\mathfrak{b}}[A_+^{-1/2}x_+] &= \mathfrak{a}_+[(I+X_0^*X_0)^{1/2}A_+^{-1/2}x_+,(I+X_0^*X_0)^{-1/2}A_+^{-1/2}x_+] \\ &\quad + \mathfrak{v}[(I+X_0^*X_0)^{1/2}A_+^{-1/2}x_+\oplus 0, 0\oplus X_0(I+X_0^*X_0)^{-1/2}A_+^{-1/2}x_+]\end{aligned}$$

is bounded. By the Second Representation Theorem [**43**, Theorem VI.2.23] for the form $\hat{\mathfrak{b}}_+$, we have that $\hat{\mathfrak{b}}[A_+^{-1/2}x_+] = \langle \widehat{B}_+^{1/2}A_+^{-1/2}x_+, \widehat{B}_+^{1/2}A_+^{-1/2}x_+\rangle$ is bounded for all $x_+ \in \mathcal{H}_+$. Thus $\widehat{B}_+^{1/2}A_+^{-1/2}$ is bounded.

For the inverse, note that $\widehat{B}_+ = U_+^*B_+U_+$ is strictly positive since B_+ is strictly positive, see Lemma 7.3.2. Note that also $\mathrm{Dom}(\widehat{B}_+^{1/2}) = \mathrm{Dom}[\hat{\mathfrak{b}}_+] = \mathrm{Dom}(A_+^{1/2})$ by the Second Representation Theorem. Then $A_+^{1/2}\widehat{B}_+^{-1/2}$ is closed and by

$$\mathrm{Ran}(\widehat{B}_+^{-1/2}) = \mathrm{Dom}(\widehat{B}_+^{1/2}) = \mathrm{Dom}(A_+^{1/2})$$

also defined on the whole of $\mathcal{H}_+$. Consequently $A_+^{1/2}\widehat{B}_+^{-1/2}$ is bounded. □

Using the observation above, we get further spectral information on the Stokes operator B_S in quasi-bounded domains.

LEMMA 7.4.2. *For sufficiently regular quasi-bounded domains Ω, the positive part of B_S has a compact resolvent so that the positive spectrum is discrete.*

PROOF. The discreteness follows directly from the compactness of the resolvent. Note that the vector valued Dirichlet Laplacian $A_+ = -\boldsymbol{\Delta}$ and its square root have a compact resolvent. Thus, the positive part of B_S is equivalent to the operator $\widehat{B}_+$ which satisfies

$$\widehat{B}_+^{-1} = A_+^{-1} - A_+^{-1/2}\big(I + A_+^{1/2}\widehat{B}_+^{-1/2}\cdot \widehat{B}_+^{-1/2}A_+^{1/2}\big)A_+^{-1/2}.$$

This operator is compact by Lemma 7.4.1. Indeed, the two operators

$$\widehat{B}_+^{-1/2}A_+^{1/2} = (A_+^{1/2}\widehat{B}_+^{-1/2})^* \quad \text{and} \quad A_+^{1/2}\widehat{B}_+^{-1/2}$$

are bounded and the compactness follows. □

7.5. The negative part of the Stokes operator

We now turn to the investigation of the negative spectral part of the Stokes operator B_S. We obtain that for sufficiently regular quasi-bounded domains Ω, the essential spectrum of the operator B_S coincides with the essential spectrum of the Cosserat operator $\overline{\operatorname{div}(-\boldsymbol{\Delta})^{-1}\operatorname{grad}}$ defined on $L^2(\Omega)$. This generalises the result of [**21**, Theorem 3.11] by Faierman, Fries, Mennicken, and Möller for bounded C^2-domains in $\mathbb{R}^3$, see also [**35**] and [**51**] for this result for C^∞-domains.

THEOREM 7.5.1. *Let Ω be a sufficiently regular quasi-bounded domain, then the essential spectra of B_S and $\overline{\operatorname{div}(-\boldsymbol{\Delta})^{-1}\operatorname{grad}}$ coincide.*

Before we can turn to the proof of this statement, we need some preparation.

Recall that for sufficiently regular domains, the resolvent of the Dirichlet Laplacian is compact. It turns out that in this case also the solution X_0 of the form Riccati equation with $\mathrm{Ran}(\mathsf{E}_{B_S}(\mathbb{R}_+))$ is compact.

LEMMA 7.5.2. *Let Ω be a sufficiently regular quasi-bounded domain, then X_0 is compact.*

PROOF. Using (6.17), we can rewrite the form Riccati equation for the Stokes operator as

$$\big\langle (A_+^{1/2} + A_+^{-1/2} X_0^* W) X^* q, A_+^{1/2} v \big\rangle_{L^2(\Omega)^n} = \Big\langle \overline{A_+^{-1/2} W^*} q, A_+^{1/2} v \Big\rangle_{L^2(\Omega)^n} \tag{7.10}$$

for all $q \in L^2(\Omega)$ and $v \in H_0^1(\Omega)^n$, where the closure $\overline{A_+^{-1/2} W^*}$ is a bounded operator.

Note that $A_+ = -\boldsymbol{\Delta}$ as well as $A_+^{1/2}$ are surjective as maps from $H_0^1(\Omega)$ to $L^2(\Omega)^n$, so that (7.10) turns into the operator identity

$$(A_+^{1/2} + A_+^{-1/2} X_0^* W) X_0^* = \overline{A_+^{-1/2} W^*}. \tag{7.11}$$

Recall that the operator $\widehat{B}_+$ associated with the form $\hat{\mathfrak{b}}_+$ in (7.1) is strictly positive by part (f) of Lemma 7.3.2 and (7.6) yields that

$$A_+ + X_0^* W = (I + X_0 X_0^*)^{1/2} \widehat{B}_+ (I + X_0 X_0^*)^{-1/2}.$$

In this case $A_+ + X_0^* W = A_+^{1/2}(I + A_+^{-1/2} X_0^* W A_+^{-1/2}) A_+^{1/2}$ is boundedly invertible.

We claim that also

$$I + A_+^{-1/2} X_0^* W A_+^{-1/2} =: I + K$$

is boundedly invertible.

To see this, note that $A_+^{-1/2}$ is compact and that $W A_+^{-1/2}$ is bounded. In this case K is compact and thus the spectrum of $I + A_+^{-1/2} X_0^* W A_+^{-1/2}$ can accumulate only at the point 1. This operator is then boundedly invertible if and only if its kernel is trivial. Assume that the kernel is not trivial, then the surjectivity of $A_+^{1/2}$ implies that $A_+ + X_0^* W$ has a non trivial kernel which is not possible since this operator is related to the strictly positive operator $\widehat{B}_+$. To obtain the claim that X_0^* (and thus also X_0) is compact, we rewrite (7.11) as

$$X_0^* = A_+^{-1/2}(I + A_+^{-1/2} X_0^* W A_+^{-1/2})^{-1} \overline{A_+^{-1/2} W^*}. \tag{7.12}$$

□

We also use the following operator identity which brings the Cosserat operator into play.

LEMMA 7.5.3. *Let X_0 be a solution of the form Riccati equation* (6.9)*, then the identity for bounded operators*

$$(I + X_0 X_0^*) W X_0^* - W A_+^{-1/2} (W A_+^{-1/2})^* = -(W X_0^*)^* X_0 A_+^{-1} X_0^* (W X_0^*)$$

holds.

PROOF. We set $x_+ = X_0^* y_-$ into the form Riccati equation (6.9) to obtain that the equation

$$\begin{aligned} \langle A_+^{1/2} X_0^* y_-, A_+^{1/2} X_0^* y_- \rangle_{L^2(\Omega)^n} &+ \langle W X_0^* y_-, (I + X_0 X_0^*) y_- \rangle_{L^2(\Omega)} \\ &= \langle y_-, W X_0^* y_- \rangle_{L^2(\Omega)} + \langle W X_0^* y_-, y_- \rangle_{L^2(\Omega)} \end{aligned} \tag{7.13}$$

holds for all $y_- \in L^2(\Omega)$. Since all the operator products above are bounded by the mapping property $\operatorname{Ran} X_0^* \subseteq \operatorname{Dom}(A_+^{1/2}) \subseteq \operatorname{Dom}(W)$ of X_0^* in condition (ii) of Theorem 6.3.1, equation (7.13) can be turned into the identity for bounded quadratic forms

$$\langle (I + X_0 X_0^*) W X_0^* y_-, y_- \rangle = \Big\langle \big(-(A_+^{1/2} X_0^*)^* (A_+^{1/2} X_0^*) + W X_0^* + (W X_0^*)^* \big) y_-, y_- \Big\rangle.$$

By polarisation, this gives the identity for bounded self-adjoint operators

$$(I + X_0 X_0^*) W X_0^* = -(A_+^{1/2} X_0^*)^* (A_+^{1/2} X_0^*) + W X_0^* + (W X_0^*)^*. \tag{7.14}$$

Note that $WA_+^{-1/2}$ is a bounded operator, so that

$$K := A_+^{-1/2} X_0^* W A_+^{-1/2}$$

is a bounded operator. Using the representation (7.12) of X_0^*, we derive the following representations

$$WX_0^* = (WA_+^{1/2})(I+K)^{-1}\overline{A_-^{-1/2}W^*},$$

$$(A_+^{1/2}X_0^*)^*(A_+^{1/2}X_0^*) = (W^*A_+^{-1/2})(I+K^*)^{-1}(I+K)^{-1}\overline{A_-^{-1/2}W^*}.$$

We now introduce the bounded operator

$$Z := (I+K)^{-1} + (I+K^*)^{-1} - (I+K^*)^{-1}(I+K)^{-1}.$$

Together with (7.14), this yields

$$(I + X_0X_0^*)WX_0^* = WA_+^{-1/2}Z(WA_+^{-1/2})^*$$

It is a straightforward computation to verify the identity

$$(I+K^*)(Z-I)(I+K) = -KK^*.$$

Combining the last two equations, we obtain the claimed identity

$$\begin{aligned}
&(I+XX^*)WX_0^* - WA_+^{-1/2}(WA_+^{-1/2})^* \\
&= WA_+^{-1/2}(Z+I)(WA_+^{-1/2})^* \\
&= -(WA_+^{1/2})(I+K)^{-1}KK^*(I+K)^{-1}(WA_+^{-1/2})^* \\
&= -(WA_+^{1/2})(I+K^*)^{-1}(WA_+^{-1/2})^*X_0A_+^{-1/2}A_+^{-1/2}X_0^*(WA_+^{-1/2})(I+K)^{-1}(WA_+^{-1/2})^* \\
&= -(WX_0^*)^*X_0A_+^{-1}X_0^*(WX_0^*).
\end{aligned}$$

□

Note that the self-adjoint operator $WA_+^{-1/2}(WA_+^{-1/2})^*$ is equal to the Cosserat operator $\overline{WA_+^{-1}W^*}$ since it is a bounded closed extension of the densely defined operator $WA_+^{-1}W^*$.

We now turn to the proof of Theorem 7.5.1

Proof of Theorem 7.5.1. From Lemma 7.1.5 and Lemma 7.1.6, it follows that the negative part of B_S is similar to the operator

$$\widehat{B}_- = (I+X_0X_0^*)^{-1/2}(WX_0^*)^*(I+X_0X_0^*)^{1/2}.$$

By Lemma 7.5.3, we have that the difference $(I+X_0X_0^*)WX_0^* - \overline{WA_+^{-1}W^*}$ is a compact operator which we denote by L. We then obtain the representation

$$\widehat{B}_- - (I+X_0X_0^*)^{-1/2}\overline{WA_+^{-1}W^*}(I+X_0X_0^*)^{-1/2} = (I+X_0X_0^*)^{-1/2}L^*(I+X_0X_0^*)^{-1/2},$$

where the right-hand side is also compact. We now show that the difference

$$(I+X_0^*)^{-1/2}\overline{WA_+^{-1}W^*}(I+X_0^*)^{-1/2} - \overline{WA_+^{-1}W^*}$$

is a compact operator. To see this, recall that X_0 is compact by Lemma 7.5.2, so that the operator $(I+X_0X_0^*)^{-1/2} - I$ is also compact. In this case, we have that

$$\begin{aligned}
&(I+X_0^*)^{-1/2}\overline{WA_+^{-1}W^*}(I+X_0^*)^{-1/2} - \overline{WA_+^{-1}W^*} \\
&= \big((I+X_0X_0^*)^{-1/2}+I\big)\overline{WA_+^{-1}W^*}(I+X_0X_0^*)^{-1/2} + \overline{WA_+^{-1}W^*}\big((I+X_0X_0^*)^{-1/2}+I\big).
\end{aligned}$$

Thus $\widehat{B}_- - \overline{WA_+^{-1}W^*}$ is compact and the claim follows from the unitary equivalence of $\widehat{B}_-$ and the negative part of B_S. $\square$

We call the operator $\overline{WA_+^{-1}W^*}$ the Cosserat operator since it is directly related to the Cosserat eigenvalue problem of finding $\lambda \in \mathbb{R}$ such that

$$\mathbf{\Delta} v = \operatorname{grad} p \text{ in the domain } \Omega, \quad v|_{\partial\Omega} = 0, \quad p = \lambda \operatorname{div} v,$$

that is,

$$p = \lambda \operatorname{div} \mathbf{\Delta}^{-1} \operatorname{grad} p.$$

Note that for $\lambda \neq 0$ this eigenvalue problem can be rewritten as

$$\frac{1}{\lambda} \mathbf{\Delta} v = \operatorname{grad} \operatorname{div} v.$$

In [**28**] Gaultier and Lezaun showed that for bounded domains the spectrum of the operator $\overline{WA_+^{-1}W^*}$ on $L^2(\Omega)/\mathbb{R}$ is contained in the interval $[c, 1]$, where c is the optimal constant in the coercivity estimate $||u||_{L^2(\Omega)}^2 \leq c|| \operatorname{grad} u||_{H^{-1}(\Omega)^n}$. This constant is known for several domains like discs in $\mathbb{R}^2$ ($c = 1/2$ and the spectrum consists only of $1/2$ and 1) or spheres in $\mathbb{R}^3$ ($c = 1/3$ and the eigenvalues are 1 and $\frac{m}{2m+1}$ for $m \in \mathbb{N}$, only the point $1/2$ belongs to the continuous spectrum), see [**28**].

These results have been generalised by Crouzeix in [**14**] to bounded domains with at least C^3-boundary. Namely, in this case, the essential spectrum consists only of $1/2$ and 1, see [**14**, Corollary 4]. For exterior domains, the same result has been obtained in [**68**, Theorem 1.1] by Weyers.

For domains with corners the situation is more complicated. The best result we know of is the following. Namely, for $\Omega \subset \mathbb{R}^2$ piecewise smooth with opening angles ω_j, we have

$$\sigma_{\mathrm{ess}}\left(\overline{WA_+^{-1}W^*}\right) = \bigcup_{\text{corners } j} \left[\frac{1}{2} - \frac{|\sin\omega_j|}{2\omega_j}, \frac{1}{2} + \frac{|\sin\omega_j|}{2\omega_j}\right] \cup \{1\}.$$

This result is contained in [**13**, Theorem 3.3] by Costabel, Crouzeix, Dauge, and Lafranche, see also the slide talks [**12**] and [**16**] by Costabel and Dauge, respectively.

CHAPTER 8

The indefinite operator $\operatorname{div} h(\cdot) \operatorname{grad}$ in the Dirichlet case

This chapter is based on the joint work [**41**] with A. Hussein, V. Kostrykin, D. Krejčiřík, and K. A. Makarov.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary for $n \geq 2$ or a bounded open interval if $n = 1$. Here, we investigate forms of the type

$$\mathfrak{b}_D[u, v] := \langle \operatorname{grad} u, H \operatorname{grad} v \rangle, \ u, v \in H_0^1(\Omega), \tag{8.1}$$

where H acts by multiplication with a sign-indefinite, bounded $n \times n$ matrix $h(x)$ which is Hermitian for almost all x in Ω and has a bounded inverse. It is natural to relate these forms to sign-indefinite differential expressions of the type

$$(B_D u)(x) = -\operatorname{div} h(x) \operatorname{grad} u(x). \tag{8.2}$$

In dimension $n = 1$ these expressions are related to left indefinite Sturm-Liouville operators and in dimension $n = 2$ or $n = 3$, these expressions appear in the investigation of so called metamaterials, see Section 8.1 below.

The aim of this chapter is to define B_D as a self-adjoint operator associated with the form $\mathfrak{b}_D$ above.

8.1. Motivation

In the sixties, Veselago studied hypothetical materials in [**64**], where the electric permittivity ε and the magnetic permeability μ may change sign, so called *metamaterials*. In principle, these materials have a sign-indefinite refraction index leading to strange effects like refraction to the other side one would usually expect and an inverse Doppler effect. These materials do not exist in nature, so that these considerations remained a toy model for some time. However this model has recently come to interest by the experimental construction of these materials by electro-magnetic fields. In this sense, electro-magnetic fields can simulate the same behaviour in a region as a metamaterial in that region would have, see the survey article [**59**] and the article [**60**], both by Smith *et al.*, and the references therein. By a combination of materials and metamaterials one hopes to create a 'cloaking effect' hiding objects from detection.

For the mathematical modelling of metamaterials and their cloaking properties, see e.g. the papers [**8**] by Bonnet-Ben Dhia, Ciarlet, and Zwölf, [**9**] by Bouchitté and Schweizer, or [**32**] by Greenleaf, Kurylev, Lassas, and Uhlmann.

The particular motivation for this chapter is the abstract setting of the paper [**9**]. This setting can, in a simplified version, be understood as the consideration of differential operators of the type

$$(B_\varepsilon u)(x) := -\operatorname{div} h_\varepsilon(x) \operatorname{grad} u(x)$$

with Dirichlet boundary conditions, where

$$h_\varepsilon(x) = \begin{cases} (-1 + \mathrm{i}\varepsilon) I & x \in \Omega_- \\ +I & x \in \Omega_+ \end{cases}$$

on a domain Ω that is split into $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$, $\Omega_1 \cap \Omega_2 = \varnothing$. In this setting the limiting operator $B_0 := \lim_{\varepsilon \to 0} B_\varepsilon$ is investigated. In this case, the operator B_ε can be constructed by classical form methods. Namely, the form given by

$$\langle \operatorname{grad} u, h_\varepsilon(x) \operatorname{grad} v \rangle$$

has numerical range

$$\{\langle \operatorname{grad} u, h_\varepsilon(\cdot) \operatorname{grad} u \rangle_{L^2(\Omega)^n} \mid u \in H_0^1(\Omega),\ \|u\|_{\mathcal{H}^1(\Omega)} = 1\}$$

contained in a sector of the complex plane, where the opening angle approaches $\pi/2$ as $\varepsilon \to 0$. Up to a rotation into the right halfplane, the form is then sectorial in the usual sense of [**43**, Section VI.1.2] and thus associated with an operator, see [**43**, Theorem VI.2.1].

Our aim is to define the limiting operator directly by means of indefinite forms. Indeed, Theorem 8.2.6 shows that $B_D^{-1} = \lim_{\varepsilon \to 0} B_\varepsilon^{-1}$, so that the limiting process gives the operator we construct.

From the physical point of view, the additive perturbation $\mathrm{i}\varepsilon$ in h corresponds to a small absorption in the metamaterial. However, one is interested in non-absorbing materials and their physical properties. Thus, it is reasonable to eliminate the limiting process and to define the self-adjoint operator $\operatorname{div} h(\cdot) \operatorname{grad}$ with $h(x) = \pm 1$ for $x \in \Omega_\pm$ directly.

8.2. The general case

We begin this section introducing the following notation. Let $L^2(\Omega)^n$ denote the Hilbert space of square integrable vector valued functions on Ω. By D, we denote the gradient operator with Dirichlet boundary conditions, that is

$$Du = \operatorname{grad} u, \quad u \in H_0^1(\Omega).$$

We now collect several properties of D.

LEMMA 8.2.1 ([**41**]). *The gradient operator D is closed, densely defined with trivial kernel. The range of D, denoted by $\mathcal{L}$, is a closed subspace of $L^2(\Omega)^n$. The explicit representation*

$$\mathcal{L} = \operatorname{Ran} D = \{v \in L^2(\Omega)^n \mid v = \operatorname{grad} \varphi, \quad \varphi \in H_0^1(\Omega)\} \subset L_0^2(\Omega)^n$$

holds, where

$$L_0^2(\Omega)^n := \left\{ v \in L^2(\Omega)^n \;\middle|\; \int_\Omega v(x) dx = 0. \right\} \tag{8.3}$$

is the subset of functions in $L^2(\Omega)^n$ with vanishing mean value in each component.

PROOF. For the closedness of $\operatorname{Ran} D$, consider a sequence $(v_j)_{j \in \mathbb{N}} \subset \operatorname{Ran} D$ with $v_j \to u \in L^2(\Omega)^n$, $j \to \infty$.

By definition, we have $v_j = \operatorname{grad} \varphi_j$ for some $\varphi_j \in H_0^1(\Omega)$ and, thus,

$$\|\operatorname{grad} \varphi_j - \operatorname{grad} \varphi_k\|_{L^2(\Omega)^n} \to 0,\ j, k \to \infty.$$

The Poincaré Inequality yields that

$$\begin{aligned} \|\varphi_j - \varphi_k\|_{H^1(\Omega)}^2 &= \|\varphi_j - \varphi_k\|_{L^2(\Omega)} + \|\operatorname{grad} \varphi_j - \operatorname{grad} \varphi_k\|_{L^2(\Omega)^n}^2 \\ &\le (c^2 + 1) \|\operatorname{grad} \varphi_j - \operatorname{grad} \varphi_k\|_{L^2(\Omega)^n}^2 \to 0,\ j, k \to \infty, \end{aligned}$$

where c is the Poincaré constant of Ω. Thus, $(\varphi_j)_{j \in \mathbb{N}}$ is a Cauchy sequence in the complete space $H^1(\Omega)$. Since $H_0^1(\Omega)$ is a closed subset of $H^1(\Omega)$, we have

$$\varphi_j \to \varphi \in H_0^1(\Omega),\ j \to \infty.$$

Consequently, the sequence $(v_j)_{j\in\mathbb{N}}$ converges to $v := \operatorname{grad}\varphi \in \operatorname{Ran} D \subseteq L^2(\Omega)^n$ showing the closedness of $\operatorname{Ran} D$.

The inclusion $\operatorname{Ran} D \subseteq L^2_0(\Omega)^n$ follows from the Gauß-Green Formula, see [**52**, Theorem 3.1.1]. Let $e \in \mathbb{R}^n$ be a fixed vector and $\varphi \in H^1_0(\Omega)$, then

$$\int_\Omega \langle \operatorname{grad}\varphi(x), e\rangle dx = \int_{\partial\Omega} \varphi(x)\langle \nu(x), e\rangle d\sigma(x) = 0,$$

where $\nu(x)$ denotes the exterior unit normal at the boundary point $x \in \partial\Omega$ and σ is the surface measure on $\partial\Omega$. Thus $\operatorname{Ran} D$ is orthogonal to all constant vectorfields in $L^2(\Omega)^n$ and the claim follows. The triviality of $\operatorname{Ker} D$ follows directly from the Poincaré Inequality. □

Remark that in dimension $n = 1$, where $\Omega = (a, b)$ is an open interval, we have the equality $\operatorname{Ran} D = L^2_0(\Omega)$. To see this, note that for arbitrary $f \in L^2_0(\Omega)$, the function g given by

$$g(x) =: \int_a^x f(s)ds \tag{8.4}$$

is in $H^1_0(\Omega)$ and satisfies $Dg = f$ almost everywhere.

In dimension $n \geq 2$ we have that $\operatorname{Ran} D$ may be smaller than $L^2_0(\Omega)^n$.

Since the subspace $\mathcal{L} = \operatorname{Ran} D$ is closed by the lemma above, this subspace itself can be considered as a Hilbert space imbedded in $L^2(\Omega)^n$.

We imbedd $\operatorname{Ran} D$ into $L^2(\Omega)^n$ in the following way. Let $Q\colon L^2(\Omega)^n \to \mathcal{L}$ denote the partial isometry given by

$$Qv := \begin{cases} v, & v \in \operatorname{Ran} D, \\ 0, & v \perp \operatorname{Ran} D. \end{cases}$$

The adjoint Q^* of Q is just the imbedding of $\mathcal{L}$ into $L^2(\Omega)$ and $I - Q^*Q$ is the Leray projector onto the divergence free vectorfields of $L^2(\Omega)^n$. See [**25**, Section II.3] for further investigation of the Helmholtz-Leray decomposition and the Leray projector. Remark that we do not consider Q as a projector since we treat $L^2(\Omega)^n$ and $\operatorname{Ran} D$ as different Hilbert spaces.

It is well known, that the adjoint D^* of the gradient operator D is the divergence operator, namely

$$D^*v = -\operatorname{div} v, \quad v \in E^2(\Omega) := \{v \in L^2(\Omega)^n \mid \operatorname{div} v \in L^2(\Omega)\}.$$

Note that $E^2(\Omega)$ is the closure of $H^1(\Omega)^n$, the space of vector valued Sobolev functions, with respect to the graph norm $(\|\cdot\|^2 + \|\operatorname{div}\cdot\|^2)^{1/2}$. In dimension $n = 1$, we even have $E^2(\Omega) = H^1(\Omega)$.

We are now ready to state the main Theorem of this chapter.

THEOREM 8.2.2 ([**41**]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain for $n \geq 2$ or a bounded open interval for $n = 1$. Furthermore, let $h \in L^\infty(\Omega, \mathbb{C})^{n\times n}$ be such that*

(i) *$h(x)$ is Hermitian for almost all $x \in \Omega$,*
(ii) *the operator $QHQ^*\colon \mathcal{L} \to \mathcal{L}$ is boundedly invertible, where H is the operator on $L^2(\Omega)^n$ acting by multiplication with $h(x)$.*

Then

(a) *there exists a unique self-adjoint operator B_D with $\operatorname{Dom}(B_D) \subseteq H^1_0(\Omega)$ such that the First Representation Theorem*

$$\langle u, B_D v\rangle_{L^2(\Omega)} = \mathfrak{b}_D[u, v] = \langle \operatorname{grad} u, H \operatorname{grad} v\rangle_{L^2(\Omega)}, \quad u \in H^1_0(\Omega),\ v \in \operatorname{Dom}(B_D)$$

holds;

(b) *the operator B_D is explicitly given by $B_D = D^*HD$ on the natural domain*

$$\operatorname{Dom}(B_D) = \{u \in H_0^1(\Omega) \mid HDu \in E^2(\Omega)\}.$$

The domain $\operatorname{Dom}(B_D)$ is a core for the gradient operator D with Dirichlet boundary conditions;

(c) *the operator B_D is semibounded if and only if QHQ^* is sign-definite;*

(d) *the operator B_D has a spectral gap around zero. Namely, if $\delta_0 > 0$ denotes the smallest eigenvalue of the Dirichlet Laplacian $-\Delta_D$ on $L^2(\Omega)$ and*

$$\alpha := \left\|(QHQ^*)^{-1}\right\|^{-1} \tag{8.5}$$

denotes the width of the spectral gap of QHQ^* around zero, then

$$(-\alpha\delta_0, \alpha\delta_0) \subseteq \rho(B_D).$$

In particular, B_D is boundedly invertible with $\left\|B_D^{-1}\right\| \leq \frac{1}{\alpha\delta_0}$;

(e) *the inverse B_D^{-1} is compact and the spectrum of B_D is purely discrete.*

Note that we introduce the subscript D in B_D to put emphasis on the Dirichlet boundary condition and to distinguish from the Neumann case that will be treated in the following chapter.

Before we start the proof of Theorem 8.2.2, further investigation of the operators D, D^* and their polar decomposition is needed.

Recall that, by the Lipschitz boundary of Ω, there exists a normal trace operator $\gamma_\nu \colon E^2(\Omega) \to H^{-1/2}(\partial\Omega)$ assigning the normal component at the boundary to any function in $E^2(\Omega)$, see [**61**, Lemma II.1.2.2]. The kernel of D^* splits into the orthogonal sum of the two closed subspaces

$$\begin{aligned} L^2_\sigma(\Omega) &:= \{v \in L^2(\Omega)^n \mid \operatorname{div} v = 0,\ \gamma_\nu v = 0\}, \\ H(\Omega) &:= \{v \in L^2(\Omega)^n \mid v = \operatorname{grad}\varphi,\ \varphi \in H^1(\Omega),\ \Delta\varphi = 0\}, \end{aligned} \tag{8.6}$$

so that we have the orthogonal decomposition

$$L^2(\Omega)^n = \operatorname{Ran} D \oplus L^2_\sigma(\Omega) \oplus H(\Omega),$$

see [**17**, Proposition IX.1.1].

The range of D^* is the whole of $L^2(\Omega)$. In the case of dimension $n = 1$, this can be obtained directly by integration as in (8.4). For the higher dimensional case $n \geq 2$, split $L^2(\Omega) = L^2_0(\Omega) \oplus C$, where C is the subspace of constant functions and

$$L^2_0(\Omega) := L^2_0(\Omega)^1 = \left\{ u \in L^2(\Omega) \;\middle|\; \int_\Omega u(x)dx = 0 \right\}$$

as in (8.3). Let $f = f_1 \oplus f_2 \in L^2_0(\Omega) \oplus C = L^2(\Omega)$, then by [**61**, Lemma II.2.1.1], we have $f_1 = \operatorname{div}\varphi_1$ for some $\varphi_1 \in H^1_0(\Omega)$. Since the equation $\operatorname{div}\varphi_2 = f_2 \in C$ is solvable with some φ_2 in $H^1(\Omega)$, we have $f = \operatorname{div}\varphi$ with $\varphi = \varphi_1 + \varphi_2 \in H^1(\Omega) \subset E^2(\Omega)$ and the claim follows.

Since both operators D and D^* are closed, they admit the polar decompositions, see [**43**, Section VI.2.7],

$$D = U|D| = |D^*|U, \quad D^* = U^*|D^*| = |D|U^*, \tag{8.7}$$

where $U \colon L^2(\Omega) \to L^2(\Omega)^n$ is a partial isometry with initial subspace

$$(\operatorname{Ker} D)^\perp = \{0\}^\perp = L^2(\Omega)$$

and final subspace $\overline{\operatorname{Ran} D} = \operatorname{Ran} D = \mathcal{L} \subseteq L^2_0(\Omega)^n$.

An immediate consequence of the polar decomposition is the following.

LEMMA 8.2.3 ([**41**]). *The partial isometry U maps $\operatorname{Dom}(D)$ onto $\operatorname{Dom}(D^*) \cap \operatorname{Ran} D$.*

Proof. The operator D can be considered as a surjective map between the Banach spaces

$$\Big(H_0^1(\Omega), \|\cdot\|_{H^1(\Omega)}\Big) \quad \text{and} \quad \Big(\operatorname{Ran} D, \|\cdot\|_{L^2(\Omega)}\Big).$$

By the Bounded Inverse Theorem, there exists a bounded inverse of D as a map

$$D^{-1}\colon \operatorname{Ran} D \to H_0^1(\Omega) \subset L^2(\Omega).$$

Let $v \in \operatorname{Dom}(D^*) \cap \operatorname{Ran} D$ be arbitrary and set $u := D^{-1}|D^*|v$. By construction, we have $u \in \operatorname{Dom}(D)$ and the polar decomposition (8.7) yields that

$$|D^*|Uu = Du = |D^*|v.$$

As a consequence, $v - Uu \in \operatorname{Ker}|D^*| = \operatorname{Ker} D^*$. By assumption, we have that both v and Uu belong to $\operatorname{Ran} D$. By the closedness of $\operatorname{Ran} D$, we have that

$$v - Uu \in \operatorname{Ran} D = (\operatorname{Ker} D^*)^\perp.$$

This implies $v - Uu = 0$, so that $v = Uu$ and the proof is completed. □

The investigation of the operators D, D^* and U above allows to examine the operators DD^* and D^*D more closely.

We define the Dirichlet-Laplace operator $-\Delta_D = D^*D$ and the grad-div operator DD^* on their natural domains

$$\begin{aligned}\operatorname{Dom}(-\Delta_D) = \operatorname{Dom}(D^*D) &= \{\varphi \in H_0^1(\Omega) \mid \operatorname{grad}\varphi \in E^2(\Omega)\}\\ &= \{\varphi \in H_0^1(\Omega) \mid \Delta\varphi \in L^2(\Omega)\}\end{aligned}$$

and

$$\operatorname{Dom}(DD^*) = \{v \in E^2(\Omega) \mid \operatorname{div} v \in H_0^1(\Omega)\}.$$

The operator $-\Delta_D$ has a trivial kernel since $\operatorname{Ran} D \perp \operatorname{Ker} D^*$ and $\operatorname{Ker} D = \{0\}$, whereas the kernel of DD^* is non-trivial, namely with the notation of (8.6), we have that

$$\operatorname{Ker}(DD^*) = \operatorname{Ker} D^* = L^2_\sigma(\Omega) \oplus H(\Omega).$$

From the polar decomposition (8.7), we obtain that

$$DD^* = UD^*DU^* = U(-\Delta_D)U^*, \tag{8.8}$$

so that the spectra $\sigma(DD^*) \setminus \{0\} = \sigma(-\Delta_D)$ agree up to the point zero. We even have that the spectral multiplicities agree outside zero. Namely for all $\lambda > 0$

$$\dim(\operatorname{Ker}(DD^* - \lambda)) = \dim(\operatorname{Ker}(D^*D - \lambda)).$$

Lemma 8.2.4. *The decomposition* $L^2(\Omega)^n = \operatorname{Ran} D \oplus \operatorname{Ker} D^*$ *reduces the* grad-div *operator* DD^*.

Proof. Let $P\colon L^2(\Omega)^n \to L^2(\Omega)^n$ denote the orthogonal projector onto $\operatorname{Ran} D$, so that $I - P$ is the orthogonal projector onto $\operatorname{Ker} D^*$. It is well known that in this situation $P = UU^*$ holds, see, e.g., [**20**, Section IV.3]. Let $v \in \operatorname{Dom}(DD^*)$, then we have that $Pv = v - (I-P)v \in \operatorname{Dom}(DD^*)$ since $\operatorname{Ker} D^*$ is a proper subset of $\operatorname{Dom}(DD^*)$. Furthermore, using the polar decomposition again, it follows that

$$PDD^*v = UU^*D|D|U^*v = UU^*DD^*UU^*v = DD^*Pv.$$

The claim now follows from Remark 6.2.1, respectively [**66**, Satz 2.60]. □

Since $\operatorname{Ran} D$ is a reducing subspace for DD^* by the lemma above, we can split of the kernel of DD^*, $\operatorname{Ker}(DD^*) = \operatorname{Ker} D^*$, without loosing any information on the action of DD^*. This splitting will be crucial for the proof of Theorem 8.2.2.

Definition 8.2.5 ([**41**]). *Let*

$$A := DD^*|_{\operatorname{Ran}(D)}$$

denote the restriction of DD^* *into the reducing subspace*

$$\operatorname{Ran} D = (\operatorname{Ker} D^*)^\perp = (\operatorname{Ker}(DD^*))^\perp.$$

The operator A clearly is a self-adjoint. Since $\varphi \in H_0^1(\Omega)$ together with $\Delta\varphi \in H_0^1(\Omega)$ already implies that $\operatorname{grad}\varphi \in E^2(\Omega)$, the domain of A can be written as

$$\begin{aligned}\operatorname{Dom}(A) &= \operatorname{Ran} D \cap \operatorname{Dom}(DD^*)\\ &= \{v \in L^2(\Omega)^n \mid v = \operatorname{grad}\varphi,\ \varphi \in H_0^1(\Omega),\ \Delta\varphi \in H_0^1(\Omega)\}.\end{aligned}$$

Since $\operatorname{Ran} D^* = L^2(\Omega)$, we have that $\operatorname{Ran} A = \operatorname{Ran}(DD^*) = \operatorname{Ran} D$. Furthermore, from (8.8), we have that A and $-\Delta_D$ are unitary equivalent. As a consequence, A has a compact inverse with $\|A^{-1}\| \le \frac{1}{\delta_0}$, where δ_0 is the smallest eigenvalue of the Dirichlet Laplacian $-\Delta_D$.

We now turn to the proof of Theorem 8.2.2.

Proof of Theorem 8.2.2. We consider first the auxiliary sesquilinear form

$$\mathfrak{b}[u,v] := \langle A^{1/2}u, QHQ^*A^{1/2}v\rangle_{\mathcal{L}}, \quad u,v \in \operatorname{Dom}(A^{1/2}) \subseteq \mathcal{L}, \tag{8.9}$$

where $\mathcal{L} = \operatorname{Ran} D$ is a Hilbert space and $A = DD^*|_{\operatorname{Ran}(D)}$ is a self-adjoint operator with

$$A^{1/2} = |D^*||_{\operatorname{Ran} D}, \quad \operatorname{Dom}(A^{1/2}) = \operatorname{Dom}(|D^*|) \cap \mathcal{L} = \operatorname{Dom}(D^*) \cap \mathcal{L}. \tag{8.10}$$

Since A is strictly positive and QHQ^* is bounded, boundedly invertible by assumption, we can apply the First Representation Theorem in the gap case [**36**, Theorem 2.3]. Thus, there exists a unique self-adjoint, boundedly invertible operator B in the Hilbert space $\mathcal{L}$ with

$$\operatorname{Dom}(B) = \{v \in \operatorname{Dom}(A^{1/2}) \mid QHQ^*A^{1/2}v \in \operatorname{Dom}(A^{1/2})\} \subseteq \operatorname{Dom}(A^{1/2}) \tag{8.11}$$

which is associated with the form $\mathfrak{b}$, that is,

$$\langle u, Bv\rangle_{\mathcal{L}} = \mathfrak{b}[u,v],\ u \in \operatorname{Dom}(A^{1/2}),\ v \in \operatorname{Dom}(B).$$

We have $0 < \delta_0 = \min\sigma(A)$ for the smallest Dirichlet eigenvalue δ_0 and by assumption $(-\alpha,\alpha) \subset \rho(QHQ^*)$, so that $(-\alpha\delta_0, \alpha\delta_0) \subset \rho(B)$.

Let $\widehat{B}$ denote the trivial extension of B to the whole of $L^2(\Omega)^n$, that is,

$$\widehat{B}v = \begin{cases} Bv, & v \in \operatorname{Dom}(B)\\ 0, & v \in \mathcal{L}^\perp = L^2_\sigma(\Omega) \oplus H(\Omega)\end{cases}$$

with

$$\begin{aligned}\operatorname{Dom}(\widehat{B}) &= \{u \oplus v \mid u \in \operatorname{Dom}(B),\ v \in L^2_\sigma(\Omega) \oplus H(\Omega)\}\\ &\subseteq (\operatorname{Dom}(D^*) \cap \operatorname{Ran} D) \oplus L^2_\sigma(\Omega) \oplus H(\Omega).\end{aligned} \tag{8.12}$$

Set

$$B_D := U^*\widehat{B}U$$

on the natural domain

$$\operatorname{Dom}(B_D) = \{u \in L^2(\Omega) \mid Uu \in \operatorname{Dom}(\widehat{B})\}. \tag{8.13}$$

By Lemma 8.2.3, the partial isometry $U\colon L^2(\Omega) \to L^2(\Omega)^n$ maps $H_0^1(\Omega) = \operatorname{Dom}(D)$ onto $\operatorname{Dom}(D^*) \cap \operatorname{Ran} D$, so that U does not map into the kernel of $\widehat{B}$.

As a consequence $\mathrm{Dom}(B_D) \subseteq H_0^1(\Omega)$ and B_D is a self-adjoint operator with $(-\alpha\delta_0, \alpha\delta_0) \subset \rho(B_D)$. With the equations (8.13), (8.12) and (8.11) as well as the identities $D = |D^*|U$ and $\mathrm{Dom}(D^*) = E^2(\Omega)$, we obtain that

$$\mathrm{Dom}(B_D) = \{u \in H_0^1(\Omega) \mid HDu \in E^2(\Omega)\}.$$

Furthermore, since A^{-1} is compact, also $B^{-1} = A^{-1/2}(QHQ^*)^{-1}A^{-1/2}$ and B_D^{-1} are compact. By [**36**, Theorem 2.3], $\mathrm{Dom}(B)$ is a core for $A^{1/2}$, so that by trivial extension $\mathrm{Dom}(\widehat{B})$ is a core for $(DD^*)^{1/2}$. It follows that $\mathrm{Dom}(B_D) = U^* \,\mathrm{Dom}(\widehat{B})$ is a core for

$$(DD^*)^{1/2}U = |D^*|U = D.$$

Finally, for $u \in H_0^1(\Omega)$ and $v \in \mathrm{Dom}(B_D)$, straightforward computation shows that

$$\begin{aligned} \langle u, B_D v\rangle_{L^2(\Omega)} &= \langle Uu, \widehat{B}Uv\rangle_{L^2(\Omega)^n} = \langle QUu, BQUv\rangle_{\mathcal{L}} \\ &= \mathfrak{b}[QUu, QUv] = \langle A^{1/2}QUu, QHQ^*A^{1/2}QUv\rangle_{\mathcal{L}} \\ &= \langle Q^*A^{1/2}QUu, HQ^*A^{1/2}QUv\rangle_{L^2(\Omega)^n}. \end{aligned}$$

With $Q^*HQ = |D^*|$, we get the representation

$$\begin{aligned} \langle u, B_D v\rangle_{L^2(\Omega)} &= \langle |D^*|Uu, H|D^*|Uv\rangle_{L^2(\Omega)^n} \\ &= \langle Du, HDv\rangle_{L^2(\Omega)^n}. \end{aligned}$$

This completes the proof of Theorem 8.2.2. □

The operator B_D constructed above is indeed the limit of the operators B_ε of Section 8.1 as we intended. We only restate this result explicitly in the following lemma.

LEMMA 8.2.6 ([**40**, Corollary 5.8, Remark 5.9]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$, $\Omega_1 \cap \Omega_2 = \varnothing$ be a non-trivial splitting of Ω. Furthermore, let*

$$h_\varepsilon(x) := \begin{cases} (-1+\mathrm{i}\varepsilon)I & x \in \Omega_- \\ +I & x \in \Omega_+ \end{cases}, \quad h(x) := \begin{cases} -I & x \in \Omega_- \\ +I & x \in \Omega_+ \end{cases}.$$

Denote by $\mathfrak{b}_\varepsilon$ the sesquilinear form given by

$$\mathfrak{b}_\varepsilon[u,v] := \langle \operatorname{grad} u, h_\varepsilon(x) \operatorname{grad} v\rangle, \quad u, v \in H_0^1(\Omega)$$

and let B_ε be the associated operator. Let H be the operator acting by multiplication with $h(x)$ and suppose that QHQ^* is boundedly invertible. Then we have the operator convergence

$$B_\varepsilon^{-1} \to B_D^{-1},\ \varepsilon \to 0.$$

A closer look at the proof of Theorem 8.2.2 leads to the following remark.

REMARK 8.2.7. *Investigations of the indefinite form given by*

$$\langle Du, HDv\rangle_{L^2(\Omega)^n}, \quad u, v \in H_0^1(\Omega)$$

can be simplified by the polar decomposition to investigations of the more accessible indefinite form

$$\tilde{\mathfrak{b}}[\tilde{u}, \tilde{v}] := \langle |D^*|\tilde{u}, H|D^*|\tilde{v}\rangle_{L^2(\Omega)^n},$$

see the proof of Theorem 8.2.2.

The operator $|D^*| \geq 0$ is self-adjoint and H is bounded and boundedly invertible by assumption. Thus, the form $\tilde{\mathfrak{b}}$ can be viewed in the context of Chapter 1. However, in general, the spectrum of $A^{1/2} := |D^*|$ cannot be separated from zero and it is not clear that a non-trivial involution $J_{|D^*|^2}$ commuting with $|D^*|^2$ exists such that (1.1) *is satisfied.*

In this sense, it is not clear if Hypothesis 1.2.1 can be satisfied in this setting of indefinite differential operators of second order.

For this reason, we used another approach here. Namely, since the spectrum of $|D^*|$ *has zero as an isolated eigenvalue, splitting off the reducing subspace* $\operatorname{Ker}|D^*|$ *gives a reduced sesquilinear form in the framework of* [**36**] *in the gap case. The mapping properties of* U *(see Lemma 8.2.3) grant that no information is lost during this splitting.*

8.3. The operator QHQ^* in dimension $n = 1$

Recall that it is essential for Theorem 8.2.2 to have that the inverse of QHQ^* is bounded on the Hilbert space $\mathcal{L}$. This property is investigated here.

The difficulty in the analysis of QHQ^* lies in the non-localness of this operator. Namely, H acts by pointwise multiplication with $h(x)$ and is thus a local operator. However, the operator Q does not act pointwise, see the one dimensional case below.

In dimension $n = 1$, we can give a complete description of the spectral properties of QHQ^*. In higher dimensions, we give sufficient criteria for the boundedness of $(QHQ^*)^{-1}$ in special cases.

THEOREM 8.3.1 ([**41**]). *Let* $n = 1, h \in L^\infty(\Omega)$. *Then*

(a) *the point spectrum* $\sigma_p(QHQ^*)$ *of* QHQ^* *is given by the disjoint union*

$$\sigma_p(QHQ^*) = \{\lambda \in \mathbb{R} \mid |h^{-1}(\{\lambda\})| > 0\} \dot\cup \{\lambda \in \mathbb{R} \mid (h(\cdot) - \lambda)^{-1} \in L^2_0(\Omega)\},$$

where $|\cdot|$ *denotes the Lebesgue measure on* $\mathbb{R}$. *Moreover, all eigenvalues in the set* $\{\lambda \in \mathbb{R} \mid |h^{-1}(\{\lambda\})| > 0\}$ *are of infinite multiplicity and the eigenvalues in* $\{\lambda \in \mathbb{R} \mid (h(\cdot) - \lambda)^{-1} \in L^2_0(\Omega)\}$ *are simple;*

(b) *the essential spectrum* $\sigma_{\mathrm{ess}}(QHQ^*)$ *agrees with the essential spectrum of* H. *This spectrum is the essential range of the function* $h \colon \Omega \to \mathbb{R}$,

$$\sigma_{\mathrm{ess}}(QHQ^*) = \{\lambda \in \mathbb{R} \mid |h^{-1}(B_\varepsilon(\lambda))| > 0 \quad \textit{for all} \quad \varepsilon > 0\},$$

where $B_\varepsilon(\lambda)$ *is the open ball of radius* ε *around* λ.

(c) λ *is in the resolvent set of* QHQ^* *if* $\int_\Omega \frac{dx}{h(x)-\lambda} \neq 0$ *and* $|h(x) - \lambda| > \delta$ *for some* $\delta > 0$ *and almost all* $x \in \Omega$.

PROOF. (a) Since $\mathcal{L} = \operatorname{Ran} D = L^2_0(\Omega)$, the partial isometry Q is explicitly given by

$$Qf = f - \frac{1}{|\Omega|}\int_\Omega f(x)dx, \quad f \in L^2(\Omega).$$

The adjoint Q^* is the imbedding $\operatorname{Ran} D \to L^2(\Omega)$.

Let $\lambda \in \mathbb{R}$ be an eigenvalue of QHQ^* and let $f \in \mathcal{L} = L^2_0(\Omega)$ be a corresponding eigenfunction. Then

$$h(x)f(x) - \frac{1}{|\Omega|}\int_\Omega h(x)f(x)dx = \lambda f(x) \quad \text{for almost all } x \in \Omega.$$

Since f has mean value zero by assumption, we can rewrite this equation as

$$(h(x) - \lambda)f(x) = \frac{1}{|\Omega|}\int_\Omega (h(x) - \lambda)f(x)dx.$$

Thus, $(h - \lambda)f$ has to be constant almost everywhere, so that we have the following two cases.

If $|h^{-1}(\{\lambda\})| = 0$, the eigenfunction f has to coincide with $\frac{c}{h-\lambda} \neq 0$ for some constant c. This function then has to belong to the space $L^2_0(\Omega)$. Thus, the eigenspace is of dimension one.

Otherwise, if instead $|h^{-1}(\{\lambda\})| > 0$, we have that the function $h(x) - \lambda$ vanishes for all $x \in h^{-1}(\{\lambda\})$. Therefore f is an eigenfunction if it vanishes on $\Omega \setminus h^{-1}(\{\lambda\})$ and satisfies

$$\int_{h^{-1}(\{\lambda\})} f(x)dx = 0.$$

This yields that the eigenvalue λ is of infinite multiplicity.

(b) Let P be the orthogonal projector onto the one dimensional space of constant functions in $L^2(\Omega)$. Then $P = I - Q^*Q$ and we have the representation

$$Q^*QHQ^*Q = H - PH - HP + PHP,$$

so that Q^*QHQ^*Q is a finite rank perturbation of the multiplication operator H and $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(Q^*QHQ^*Q)$. It remains to note that Q^*QHQ^*Q and QHQ^* are unitary equivalent.

(c) Let $\lambda \in \mathbb{R}$ be such that $|h(x) - \lambda| > \delta$ almost everywhere and

$$\int_\Omega (h(x) - \lambda)^{-1} dx \neq 0.$$

To show that λ belongs to the resolvent set of QHQ^*, let $g \in L^2_0(\Omega)$ and consider the equation

$$(QHQ^* - \lambda)f = g, \; f \in L^2_0(\Omega).$$

As in part (a), this equation can be written as

$$(h(x) - \lambda)f(x) - g(x) = \frac{1}{|\Omega|}\int_\Omega \big((h(x) - \lambda)f(x) - g(x)\big)dx,$$

so that $(h(x) - \lambda)f(x) - g(x)$ is constant almost everywhere. By assumption $(h-\lambda)^{-1}$ is essentially bounded on Ω. As a consequence, the function f is given by

$$f(x) := \frac{g(x)}{h(x) - \lambda} + \frac{c}{h(x) - \lambda}$$

for some constant c is square integrable. Since $\int_\Omega \frac{dx}{h(x)-\lambda} \neq 0$, we can choose c in such way that $\int_\Omega f(x)dx = 0$. Hence $(QHQ^* - \lambda)\colon \mathcal{L} \to \mathcal{L}$ is surjective and consequently we have $\lambda \in \rho(QHQ^*)$. □

We now give a sharp estimate on the width of the spectral gap of the operator QHQ^* around zero. Let $h_\pm$ denote the positive respectively negative part of h with

$$h_\pm(x) := \pm\frac{h(x) + |h(x)|}{2} \geq 0.$$

COROLLARY 8.3.2 ([**41**]). *Let $n = 1$ and Ω be a bounded interval. Furthermore, let $h, h^{-1} \in L^\infty(\Omega)$. Then the operator QHQ^* is invertible if and only if $\int_\Omega h(x)^{-1}dx \neq 0$. The inverse is bounded if and only if $\int_\Omega h(x)^{-1}dx \neq 0$ and zero is not in the essential range of h. In this case, α, the width of the spectral gap of QHQ^* as in* (8.5)*, satisfies the lower bound $\alpha \geq r$, where*

$$r := \min\left(\{\text{ess inf}\, h_+, \text{ess inf}\, h_-\} \cup \left\{\lambda \in \mathbb{R} \;\middle|\; \int_\Omega \frac{1}{h(x) - \lambda}dx = 0\right\}\right).$$

In this case, we have $(-r, r) \subset \rho(QHQ^*)$.

PROOF. Note that for any $\lambda \in (-\text{ess inf } h_-, \text{ess inf } h_+)$, the function $x \mapsto \frac{1}{h(x)-\lambda}$ belongs to $L^2(\Omega)$. If $\int_\Omega \frac{1}{h(x)-\lambda} dx \neq 0$, then $\lambda \in \rho(QHQ^*)$ by Theorem 8.3.1 part (c).

If there is a number $\lambda_0 \in (-\text{ess inf } h_-, \text{ess inf } h_+)$ such that $\int_\Omega \frac{1}{h(x)-\lambda_0} dx = 0$, then λ_0 is an eigenvalue by Theorem 8.3.1 part (a). By the assumption on h, $\int_\Omega h(x)^{-1} dx \neq 0$, we have that λ_0 is not zero. Since the map $\lambda \mapsto \int_\Omega \frac{1}{h(x)-\lambda} dx$ is strictly increasing on the interval $(-\text{ess inf } h_-, \text{ess inf } h_+)$, there exists at most one such λ_0. $\square$

The following examples show the sharpness of the estimate for the width of the spectral gap of QHQ^* around zero in Corollary 8.3.2.

REMARK 8.3.3 ([**41**]). (a) *Let $\Omega := (-1, 2)$ and let $h(x) := \operatorname{sign}(x)$ on Ω, then the operator QHQ^* has the infinitely degenerate eigenvalues -1, 1 and the simple eigenvalue $-1/3$. To see this, note that*

$$\int_{(-1,2)} \frac{1}{\operatorname{sign}(x) + \frac{1}{3}} dx = 0.$$

In this case $(-1, 1) \setminus \{-1/3\}$ is contained in the resolvent set and $r = 1/3$.

(b) *Let $\Omega := (-4, 1)$ and let h be given by*

$$h(x) := -1,\ x < 0, \quad h(x) := 1 + x^{1/4},\ x \geq 0,$$

then $(h(x) - \lambda)^{-1} \in L^2(\Omega)$ for all $\lambda \in (-1, 1]$. Furthermore, the equation

$$\int_\Omega (h(x) - \lambda)^{-1} dx = 0$$

has no solution λ in the interval $(-1, 1]$. Hence, the only eigenvalue -1 of QHQ^* is of infinite multiplicity and $r = 1$.

In dimension $n = 1$, we have an explicit representation of the operator Q, namely

$$Qf = f - \frac{1}{|\Omega|} \int_\Omega f(x) dx, \quad f \in L^2(\Omega)^n.$$

In dimension $n \geq 2$, we do not know a comparable representation of the operator Q. This makes the higher dimensional case more difficult to investigate.

8.4. The operator QHQ^* in dimension $n \geq 2$

In this section we give examples of higher dimension where QHQ^* is boundedly invertible.

Before we can turn to the first example, we need to collect some well known general facts on Sobolev spaces, traces and boundary value problems.

LEMMA 8.4.1. *Let $\Sigma \subset \mathbb{R}^n, n \geq 2$, be a bounded domain with Lipschitz boundary.*

(a) *There is a bounded trace map $\gamma\colon H^1(\Sigma) \to H^{1/2}(\partial\Sigma)$. The trace map has a bounded right inverse.*

(b) *The inhomogeneous boundary value problem for $f \in H^{-1}(\Sigma)$*

$$-\Delta u = f, \quad u|_{\partial\Sigma} = 0$$

has a unique solution $u \in H_0^1(\Sigma)$. This solution satisfies the estimate

$$||u||_{H^1(\Sigma)} \leq c||f||_{H^{-1}(\Sigma)}$$

for some constant c independent of f.

(c) *The divergence operator*

$$\operatorname{div} : L^2(\Sigma)^n \to H^{-1}(\Sigma)$$

is a bounded map.

(d) *The homogeneous boundary value problem for* $g \in H^{1/2}(\partial\Sigma)$

$$-\Delta u = 0, \quad u|_{\partial\Sigma} = g \tag{8.14}$$

has a unique solution $u \in H_0^1(\Omega)$. *This solution satisfies the estimate*

$$||u||_{H^1(\Sigma)} \leq c||g||_{H^{1/2}(\partial\Sigma)}$$

for some constant c *independent of* g.

Proof. (a) This statement is shown in [**27**, Theorema 1.I]. A more modern formulation of this statement is contained in [**34**, Theorem 1.5.1.3]

(b) This result is a combination of [**42**, Theorems 1.3 (a) and 1.1 (a)] in the case of dimension $n = 2$ and $n \geq 3$, respectively.

(c) For arbitrary $f \in L^2(\Omega_\pm)^n$ and $\varphi \in C_0^\infty(\Omega_\pm)$, we have in the distributional sense

$$(\operatorname{div} f)(\varphi) = -f(\operatorname{grad}\varphi) = -\int_{\Omega_\pm} \langle f, \operatorname{grad}\varphi\rangle dx,$$

so that the Cauchy-Schwarz Inequality yields

$$|(\operatorname{div} f)(\varphi)| \leq \|f\|_{L^2(\Omega_\pm)^n} \cdot \|\varphi\|_{H^1(\Omega_\pm)},$$

which implies the desired continuity.

(d) For sake of completeness we reproduce the proof of this classical statement. Let $g \in H^{1/2}(\partial\Sigma)$ be arbitrary. By part (a), we can define

$$G := \gamma^{-1} g \in H^1(\Sigma) \quad \text{with} \quad ||G||_{H^1(\Sigma)} \leq c||g||_{H^{1/2}(\partial\Sigma)}.$$

We now claim that $\Delta G \in H^{-1}(\Sigma)$. To see this, note that the gradient $\operatorname{grad}\colon H^1(\Sigma) \to L^2(\Sigma)^n$ is bounded and that by part (c) also the divergence $\operatorname{div} : L^2(\Sigma)^n \to H^{-1}(\Sigma)$ is bounded. Consequently also $\Delta = \operatorname{div}\operatorname{grad}$ is a bounded map and the claim follows.

Consider now the boundary value problem

$$-\Delta v = \Delta G, \quad v|_{\partial\Sigma} = 0.$$

By part (b) this boundary value problem has a unique solution v that satisfies $||v||_{H^1(\Sigma)} \leq c||\Delta G||_{H^{-1}(\Sigma)}$ together with the continuity of the trace, we have

$$||v||_{H^1(\Sigma)} \leq c||g||_{H^{1/2}(\Sigma)}.$$

As a consequence $u := v + G$ solves the boundary value problem (8.14) and satisfies the estimate $||u||_{H^1(\Sigma)} \leq c||g||_{H^{1/2}(\Sigma)}$.

□

We now turn to the examples. For the first example, assume that $\Omega_\pm \subset \Omega \subset \mathbb{R}^n$, $\Omega_- \cap \Omega_+ = \varnothing$, $\overline{\Omega_+ \cup \Omega_-} = \overline{\Omega}$, are open bounded sets with Lipschitz boundaries such that

$$\Gamma := \partial\Omega_+ \cap \partial\Omega_- \cap \Omega,$$

the *interior common boundary*, respectively the interface between Ω_+ and Ω_-, consists of a finite number of disjoint components Γ_i, $i = 1, \ldots, N$. Furthermore, let

$$h(x) = \begin{cases} h_+ I & \text{if } x \in \Omega_+, \\ -h_- I & \text{if } x \in \Omega_-, \end{cases} \qquad a_\pm > 0. \tag{8.15}$$

Without loss of generality we additionally assume that $\partial\Omega \cap \partial\Omega_+$, has a strictly positive surface measure on $\partial\Omega$. In this case each component Γ_i of Γ is a Lipschitz manifold without boundary, either closed (if it is a boundary of a subdomain of Ω) or open (if $\partial(\partial\Omega_+ \cap \partial\Omega_-) \subseteq \partial\Omega$).

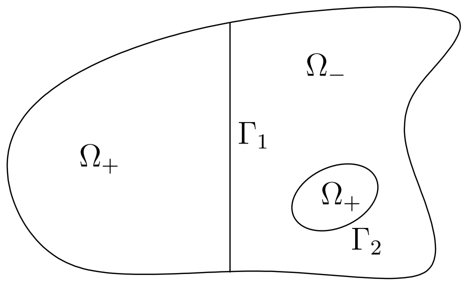

FIG. 1. Domains with interior common boundary Γ

As preparation we collect the following facts on Sobolev spaces of fractional order in this setting.

Let Σ be an arbitrary bounded Lipschitz domain in $\mathbb{R}^n$, then we define the Sobolev space $H^{1/2}_{00}(\Sigma) \equiv \widetilde{W}^{1/2}_2(\Sigma)$ as the subspace of the fractional order Sobolev space $H^{1/2}(\Sigma)$ such that the trivial continuation

$$\widetilde{u}(x) := \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

of each $u \in H^{1/2}_{00}(\Sigma)$ lies in $H^{1/2}(\mathbb{R}^n)$ (see, e.g., [**34**, Definition 1.3.2.5]). This subspace is in general strictly smaller than $H^{1/2}(\Sigma)$.

Furthermore, we have that $H^{1/2}(\Sigma) = H^{1/2}_0(\Sigma)$, the space of $H^{1/2}(\Sigma)$ functions with vanishing trace, agrees with the closure of $C^\infty_0(\Sigma)$ with respect to the $H^{1/2}$-norm, see [**34**, Theorem 1.4.2.4] and the discussion thereafter. By [**34**, Corollary 1.4.4.10] we have the explicit representation

$$H^{1/2}_{00}(\Sigma) = \left\{ u \in H^{1/2}(\Sigma) \,\middle|\, \frac{u}{\sqrt{\operatorname{dist}(\cdot, \partial\Sigma)}} \in L^2(\Sigma) \right\}.$$

It is a Hilbert space with the norm

$$\|u\|^2_{H^{1/2}_{00}(\Sigma)} := \|u\|^2_{H^{1/2}(\Sigma)} + \int_\Sigma \frac{|u(x)|^2}{\operatorname{dist}(\,\cdot\,, \partial\Sigma)} dx.$$

The set $C^\infty_0(\Sigma)$ is dense in $H^{1/2}_{00}(\Sigma)$ with respect to this norm, this is a direct combination of [**34**, Definition 1.3.2.5, Lemma 1.3.2.6 and Theorem 1.4.2.2].

Note that the dual $H^{1/2}_{00}(\Sigma)'$ is strictly bigger than $H^{-1/2}(\Sigma)$, more explicitly

$$H^{1/2}_{00}(\Sigma)' = \{u_1 + u_2 \,|\, u_1 \in H^{-1/2}(\Sigma),\ \sqrt{\operatorname{dist}(\,\cdot\,, \partial\Sigma)}u_2 \in L^2(\Sigma)\}.$$

We define the Hilbert space

$$\begin{aligned} \mathscr{H}^{1/2}(\Gamma) := \big\{ g : \Gamma \to \mathbb{C} \,|\, g|_{\Gamma_i} \in H^{1/2}(\Gamma_i) \quad &\text{if } \Gamma_i \text{ is a closed manifold and} \\ g|_{\Gamma_i} \in H^{1/2}_{00}(\Gamma_i) \quad &\text{if } \Gamma_i \text{ is an open manifold} \big\} \end{aligned}$$

with the norm

$$\|g\|^2_{\mathscr{H}^{1/2}(\Gamma)} := \sum_{\Gamma_i \text{ closed}} \|g_i\|^2_{H^{1/2}(\Gamma_i)} + \sum_{\Gamma_i \text{ open}} \|g_i\|^2_{H^{1/2}_{00}(\Gamma_i)}.$$

The properties of the fractional order Sobolev spaces mentioned above immediately imply the following results.

LEMMA 8.4.2 ([**38**]). *The functions $k_\pm$ given by*

$$k_\pm(x) = \begin{cases} 0, & x \in \partial\Omega_\pm \setminus \Gamma, \\ g(x), & x \in \Gamma \end{cases} \tag{8.16}$$

are in $H^{1/2}(\partial\Omega_\pm)$, respectively, if and only if $g \in \mathscr{H}^{1/2}(\Gamma)$.

Denote by $H^1_{0,\partial\Omega\cap\partial\Omega_\pm}(\Omega_\pm)$ the Sobolev subspace of $H^1(\Omega_\pm)$ consisting of functions with vanishing boundary traces on the *outer boundary* $\partial\Omega \cap \partial\Omega_\pm$, respectively. Note that $\partial\Omega \cap \partial\Omega_\pm = \partial\Omega_\pm \setminus \Gamma$.

Let $\gamma_\pm : H^1(\Omega_\pm) \to H^{1/2}(\Gamma)$ be the operator assigning each $u_\pm \in H^1(\Omega_\pm)$ to its trace on the interior common boundary $\Gamma = \partial\Omega_+ \cap \partial\Omega_- \cap \Omega$. Observe that for any $u \in H^1_0(\Omega)$, $\gamma_+ u_+ = \gamma_- u_-$ holds almost everywhere on Γ, where $u_\pm$ denotes the restriction of u onto $\Omega_\pm$, $u_\pm = u|_{\Omega_\pm}$. Indeed, considering the operator

$$\widetilde{\gamma} u := \gamma_+ u_+ - \gamma_- u_-, \qquad \widetilde{\gamma} : H^1_0(\Omega) \to H^{1/2}(\Gamma),$$

we have $\widetilde{\gamma} u = 0$ for all $u \in C_0^\infty(\Omega)$. The claim now follows from a standard density argument.

LEMMA 8.4.3. *(i) The mappings $\gamma_\pm$ are continuous and surjective considered as maps*

$$\gamma_\pm \colon H^1_{0,\partial\Omega\cap\partial\Omega_\pm}(\Omega_\pm) \to \mathscr{H}^{1/2}(\Gamma).$$

In particular, they have continuous right inverses $\gamma_\pm^{-1}$.

(ii) The mappings

$$\tau_\pm : H^1_0(\Omega) \to \mathscr{H}^{1/2}(\Gamma), \qquad \tau_\pm u := \gamma_\pm \left(u\big|_{\Omega_\pm} \right),$$

are continuous and surjective. In particular, for any $g \in \mathscr{H}^{1/2}(\Gamma)$ there exists an $u \in H^1_0(\Omega)$ such that $u|_{\Omega_\pm}$ are harmonic and $\tau_\pm u = g$.

PROOF. Claim (i) follows immediately from Lemma 8.4.2 since the usual boundary trace operators on $\Omega_\pm$ have a continuous right inverse, see Lemma 8.4.1, part (a).

To prove claim (ii) for arbitrary $g \in \mathscr{H}^{1/2}(\Gamma)$, we consider the boundary value problems

$$\Delta u_\pm = 0 \quad \text{in} \quad \Omega_\pm$$

with the boundary data

$$u_\pm|_{\partial\Omega\cap\partial\Omega_\pm} = 0 \quad \text{and} \quad u_\pm|_\Gamma = g.$$

By Lemma 8.4.2 the boundary data on $\partial\Omega_\pm$ are in $H^{1/2}(\partial\Omega_\pm)$. Thus, the boundary value problems have unique weak solutions $u_\pm \in H^1_{0,\partial\Omega\cap\partial\Omega_\pm}(\Omega_\pm)$. Now we consider the function u on the whole of Ω given by

$$u(x) := \begin{cases} u_+(x), & \text{if} \quad x \in \Omega_+, \\ u_-(x), & \text{if} \quad x \in \Omega_-. \end{cases}$$

Since $\gamma_+ u_+ = \gamma_- u_- = g$ by construction, it is straightforward to verify that $u \in H^1_0(\Omega)$ with $\tau_\pm u = g$. This proves the surjectivity of the operators $\tau_\pm$. □

For every $u \in \mathscr{H}^{1/2}(\Gamma)$ we define the *Dirichlet-to-Neumann maps*

$$\Lambda_\pm : \mathscr{H}^{1/2}(\Gamma) \to \mathscr{H}^{1/2}(\Gamma)$$

in the following way. Let $g \in \mathscr{H}^{1/2}(\Gamma)$ be given, then we denote by $u_\pm \in H^1(\Omega_\pm)$ the unique weak solutions to the Dirichlet boundary value problems

$$\Delta u_\pm = 0 \quad \text{in} \quad \Omega_\pm \tag{8.17}$$

with

$$u_\pm|_{\partial\Omega\cap\partial\Omega_\pm} = 0 \quad \text{and} \quad u_\pm|_\Gamma = g. \tag{8.18}$$

On the Hilbert space $\mathscr{H}^{1/2}(\Gamma)$ we consider the quadratic forms given by

$$\mathfrak{q}_\pm[g] := \int_{\Omega_\pm} |\operatorname{grad} u_\pm(x)|^2 dx.$$

From the continuity of the right inverse of the boundary trace operators $\gamma_\pm$ (Lemma 8.4.3) and the estimate $\|v_\pm\|_{H^1(\Omega_\pm)} \leq C\|\Delta\gamma_\pm^{-1}g\|_{H^{-1}(\Omega_\pm)}$ on the norm of the weak solution to the inhomogeneous boundary value problem

$$\Delta v_\pm = -\Delta\gamma_\pm^{-1}g, \quad v_\pm \in H_0^1(\Omega_\pm),$$

see Lemma 8.4.1 part (d), it follows that

$$\|u_\pm\|_{H^1(\Omega_\pm)} \leq C\|g\|_{H^{1/2}(\Gamma)} \leq C\|g\|_{\mathscr{H}^{1/2}(\Gamma)},$$

which in turn implies that the forms $\mathfrak{q}_\pm$ are bounded. Hence, they uniquely define self-adjoint bounded non-negative operators $\Lambda_\pm$ such that

$$\mathfrak{q}_\pm[g] = \langle g, \Lambda_\pm g\rangle_{\mathscr{H}^{1/2}(\Gamma)} \quad \text{for any } g \in \mathscr{H}^{1/2}(\Gamma),$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathscr{H}^{1/2}(\Gamma)}$ denotes the inner product in $\mathscr{H}^{1/2}(\Gamma)$.

Due to the assumption $|\partial\Omega\cap\partial\Omega_+| > 0$, the form $\mathfrak{q}_+$ is coercive. Indeed, by a version of the Poincaré Inequality $\|u_+\|^2 \leq c_1\|\operatorname{grad} u_+\|^2$ in [**69**, Section 4.5] and by the continuity of the boundary trace operator γ_+,

$$\|g\|^2_{\mathscr{H}^{1/2}(\Gamma)} \leq c_2\|u_+\|^2_{H^1(\Omega_+)}$$

in Lemma 8.4.3, we have the estimate

$$\begin{aligned}\|g\|^2_{\mathscr{H}^{1/2}(\Gamma)} &\leq c_2\|u_+\|^2_{H^1(\Omega_+)} = c_2\|u_+\|^2_{L^2(\Omega_+)} + c_2\|\operatorname{grad} u_+\|^2_{L^2(\Omega_+)}\\ &\leq c_2(c_1+1)\|\operatorname{grad} u_+\|^2_{L^2(\Omega_+)} = c_2(c_1+1)\mathfrak{q}_+[g].\end{aligned}$$

Hence, the operator itself Λ_+ is boundedly invertible.

For further analysis of the operator, we introduce the canonical isometric isomorphism between $\mathscr{H}^{1/2}(\Gamma)$ and its dual $\mathscr{H}^{1/2}(\Gamma)'$.

DEFINITION 8.4.4. *Let $J\colon \mathscr{H}^{1/2}(\Gamma) \to \mathscr{H}^{1/2}(\Gamma)'$ be the canonical isometric isomorphism, that is, the linear mapping with the property that*

$$\langle f, g\rangle_{\mathscr{H}^{1/2}(\Gamma)} = (Jg)\,(\overline{f})$$

holds for all $f, g \in \mathscr{H}^{1/2}(\Gamma)$. In particular,

$$\langle g, \Lambda_\pm g\rangle_{\mathscr{H}^{1/2}(\Gamma)} = (J\Lambda_\pm g)\,(\overline{g}).$$

Based on this isometry and the quadratic forms $\mathfrak{q}_\pm$, we can interpret $\Lambda_\pm$ as a Dirichlet-to-Neumann map.

REMARK 8.4.5. *Since the solutions $u_\pm \in H^1(\Omega)$ of* (8.17),(8.18) *above are harmonic in $\Omega_\pm$, we have that* $\operatorname{grad} u_\pm \in E^2(\Omega_\pm)$*, the natural domain of* div *. In the sense of distributions, the Gauß-Green formula (in the version of* [**61**, Lemma II.1.2.3]*) yields*

$$(J\Lambda_\pm g)\,(\overline{g}) = \int_{\Omega_\pm} |\operatorname{grad} u_\pm(x)|^2 dx = (\partial_\nu u_\pm)\,(\overline{k_\pm}),$$

where $\partial_\nu u_\pm \in H^{-1/2}(\partial\Omega_\pm)$ is the derivative of $u_\pm$ in the direction of the unit normal ν pointing out of $\Omega_\pm$ and $k_\pm$ are defined by (8.16)*. Since $k_\pm$ are supported on the interior common boundary Γ, we have*

$$(J\Lambda_\pm g)\,(\overline{g}) = (\,\partial_\nu u_\pm|_\Gamma)\,(\overline{k_\pm})$$

for any $g \in \mathscr{H}^{1/2}(\Gamma)$. The restriction $\partial_\nu u_\pm|_\Gamma$ is in $\mathscr{H}^{1/2}(\Gamma)'$ by [**34**, Proposition 1.4.2.3]*. Thus, the operator $J\Lambda_\pm$ indeed maps the Dirichlet data g to the Neumann values $\partial_\nu u_\pm$ on Γ.*

We generalise the remark above on the quadratic forms $\mathfrak{q}_\pm$ to the corresponding sesquilinear forms.

Remark 8.4.6. *For any $f, g \in \mathscr{H}^{1/2}(\Gamma)$ we use polarisation to obtain that*

$$\langle f, \Lambda_\pm g\rangle_{\mathscr{H}^{1/2}(\Gamma)} = \int_{\Omega_\pm} \langle \overline{\operatorname{grad} u_\pm(x)}, \operatorname{grad} w_\pm(x)\rangle_{\mathbb{C}^n}\, dx,$$

where $u_\pm$ are the unique weak solutions of the boundary value problems (8.17), (8.18) *and $w_\pm$ those of $\Delta w_\pm = 0$ in $\Omega_\pm$ with the boundary conditions*

$$w_\pm|_{\partial\Omega\cap\partial\Omega_\pm} = 0 \qquad \textit{and} \qquad w_\pm|_\Gamma = f.$$

From Remark 8.4.5 it follows that

$$(J\Lambda_\pm g)\,(\overline{f}) = (\,\partial_\nu u_\pm|_\Gamma)\,(\overline{f})$$

for any $f \in \mathscr{H}^{1/2}(\Gamma)$.

Lemma 8.4.7. *Let $u_\pm \in H^1(\Omega_\pm)$ be the unique weak solution of the boundary value problem* (8.17), (8.18)*. Then*

$$\int_{\Omega_\pm} \langle \operatorname{grad} v(x), \operatorname{grad} u_\pm(x)\rangle_{\mathbb{C}^n}\, dx = \langle \tau_\pm v, \Lambda_\pm g\rangle_{\mathscr{H}^{1/2}(\Gamma)} \tag{8.19}$$

holds for all $v \in H^1_0(\Omega)$.

Proof. Since $\operatorname{grad} u_\pm \in E^2(\Omega_\pm)$, by the Gauß-Green formula (in the version of [**61**, Lemma II.1.2.3]) and Remark 8.4.6, we obtain

$$\begin{aligned}\int_{\Omega_\pm} \langle \operatorname{grad} v(x), \operatorname{grad} u_\pm(x)\rangle_{\mathbb{C}^n}\, dx &= (\partial_\nu u_\pm)\left(\overline{v|_{\partial\Omega_\pm}}\right) = (\,\partial_\nu u_\pm|_\Gamma)\,(\overline{\tau_\pm v})\\ &= (J\Lambda_\pm g)(\overline{\tau_\pm v}) = \langle \tau_\pm v, \Lambda_\pm g\rangle_{\mathscr{H}^{1/2}(\Gamma)},\end{aligned}$$

where $v|_{\partial\Omega_\pm}$ denote the boundary traces of $v|_{\Omega_\pm}$ on the whole boundary $\partial\Omega_\pm$.

□

We now introduce the operator $\Lambda_+^{-1}\Lambda_- : \mathscr{H}^{1/2}(\Gamma) \to \mathscr{H}^{1/2}(\Gamma)$. Recall that Λ_+ has a bounded inverse by the assumption $|\partial\Omega \cap \partial\Omega_+| > 0$. Since

$$\Lambda_+^{-1}\Lambda_- = \Lambda_+^{-1/2}\left(\Lambda_+^{-1/2}\Lambda_-\Lambda_+^{-1/2}\right)\Lambda_+^{1/2},$$

the operator $\Lambda_+^{-1}\Lambda_-$ is similar to a self-adjoint non-negative operator. Hence, the spectrum of the operator $\Lambda_+^{-1}\Lambda_-$ is real and non-negative.

Note that the spectral analysis of the operator $\Lambda_+^{-1}\Lambda_-$ defined by Dirichlet-to-Neumann maps can be regarded as a weak form of the following problem: Find the values of the parameter μ for which the system

$$\begin{aligned}
&\Delta u_\pm = 0 \quad \text{in} \quad \Omega_\pm,\\
&u_- = u_+ \quad \text{on} \quad \partial\Omega_- \cap \partial\Omega_+,\\
&\mu\partial_\nu u_- + \partial_\nu u_+ = 0,\\
&u_\pm|_{\partial\Omega\cap\partial\Omega_\pm} = 0
\end{aligned}$$

has non-trivial solutions $u_\pm$, see [**33**] for this interpretation.

We now obtain a characterisation of the spectrum of $QHQ*$ in terms of the operator $\Lambda_+^{-1}\Lambda_-$.

THEOREM 8.4.8 ([**38**]). *The spectrum of the operator QHQ^* equals*

$$\{h_+\} \cup \{-h_-\} \cup \left\{ \frac{h_+ - \mu h_-}{1+\mu} \;\Big|\; \mu \in \sigma(\Lambda_+^{-1}\Lambda_-),\, \mu \neq 0 \right\}.$$

Furthermore, the points $\pm h_\pm$ are eigenvalues of infinite multiplicity, the eigenvalues λ of the operator QHQ^* and $\mu = \frac{h_+-\lambda}{h_-+\lambda}$ of the operator $\Lambda_+^{-1}\Lambda_-$ have the same multiplicity. In particular, the operator QHQ^* is boundedly invertible if and only if h_+/h_- is in the resolvent set of the operator $\Lambda_+^{-1}\Lambda_-$. In this case, we have

$$\big\|(QHQ^*)^{-1}\big\| = \max_{\mu\in\sigma(\Lambda_+^{-1}\Lambda_-)} \frac{1+\mu}{|h_+ - \mu h_-|}. \tag{8.20}$$

PROOF. Observe that $QHQ^* - \lambda = Q(H-\lambda)Q^*$, for all $\lambda \in \mathbb{R}$.

The equation $Q(H-\lambda)Q^* \operatorname{grad}\varphi = 0$ holds for some $\varphi \in H_0^1(\Omega)$ if and only if

$$\langle \operatorname{grad}\psi, (H-\lambda)\operatorname{grad}\varphi\rangle_{L^2(\Omega)} = 0$$

for all $\psi \in H_0^1(\Omega)$. Equivalently, we have that

$$(h_+ - \lambda)\int_{\Omega_+} \overline{\operatorname{grad}\psi(x)}\operatorname{grad}\varphi(x)dx - (h_- + \lambda)\int_{\Omega_-} \overline{\operatorname{grad}\psi(x)}\operatorname{grad}\varphi(x)dx = 0 \tag{8.21}$$

for any $\psi \in H_0^1(\Omega)$.

Considering $\lambda = h_+$, we have that

$$\int_{\Omega_-} \overline{\operatorname{grad}\psi(x)}\operatorname{grad}\varphi(x)dx = 0$$

for any $\psi \in H_0^1(\Omega)$. This implies that $\varphi = 0$ almost everywhere in Ω_-, so that we have $\varphi|_{\Omega_+} \in H_0^1(\Omega_+)$. Hence, h_+ is an eigenvalue of the operator QHQ^* of infinite multiplicity. Similarly, we conclude that $-h_-$ is an eigenvalue of the operator QHQ^* of infinite multiplicity.

Assume now that $\lambda \neq \pm h_\pm$ is in the spectrum of the operator QM_AQ^*.

Then, according to the Weyl criterion [**54**, Theorem VII.12], there is a sequence $(\varphi_j)_{j\in\mathbb{N}}$ in $H_0^1(\Omega)$ such that $\|\operatorname{grad}\varphi_j\|_{L^2(\Omega)^n} = 1$ and

$$\lim_{j\to\infty} \|Q(H-\lambda)Q^* \operatorname{grad}\varphi_j\|_{L^2(\Omega)} = 0. \tag{8.22}$$

By the Poincaré Inequality, the limit (8.22) is equivalent to the convergence

$$\begin{aligned}
&\langle \operatorname{grad}\psi, (H-\lambda)\operatorname{grad}\varphi_j\rangle_{L^2(\Omega)}\\
&= (h_+ - \lambda)\int_{\Omega_+} \overline{\operatorname{grad}\psi(x)}\operatorname{grad}\varphi_j(x)dx - (h_- + \lambda)\int_{\Omega_-} \overline{\operatorname{grad}\psi(x)}\operatorname{grad}\varphi_j(x)dx\\
&\xrightarrow[j\to\infty]{} 0 \text{ uniformly in } \psi \in H_0^1(\Omega).
\end{aligned} \tag{8.23}$$

We define

$$\operatorname{grad} \rho_j := Q(H-\lambda)Q^* \operatorname{grad} \varphi_j \quad \text{with } \rho_j \in H_0^1(\Omega).$$

Introducing $g_j := \tau_\pm \varphi_j$, we obtain from (8.23) that for any $\psi \in H_0^1(\Omega_\pm)$

$$(h_\pm \mp \lambda)\langle \operatorname{grad} \psi, \operatorname{grad} \varphi_j \rangle_{L^2(\Omega)^n} = \langle \operatorname{grad} \psi, \operatorname{grad} \rho_j \rangle_{L^2(\Omega)^n}.$$

Thus, the functions $\varphi_j^\pm := \varphi_j|_{\Omega_\pm} \in H^1_{0,\partial\Omega_\pm \setminus \Gamma}(\Omega_\pm)$ are the unique weak solutions of the boundary value problem

$$\Delta \varphi_j^\pm = \frac{\Delta \rho_j^\pm}{h \pm \mp \lambda} \in H^{-1}(\Omega_\pm) \quad \text{in} \quad \Omega_\pm, \tag{8.24}$$

where $\rho_j^\pm = \rho_j|_{\Omega_\pm}$, with

$$\varphi_j^\pm \Big|_{\partial\Omega_\pm \setminus \Gamma} = 0 \quad \text{and} \quad \varphi_j^\pm \Big|_\Gamma = g_j. \tag{8.25}$$

Let $\widetilde{\varphi}_j^\pm \in H^1_{0,\partial\Omega_\pm \setminus \Gamma}(\Omega_\pm)$ be the unique weak solutions of the boundary value problem

$$\Delta \widetilde{\varphi}_j^\pm = 0 \quad \text{in} \quad \Omega_\pm,$$

with the boundary values

$$\widetilde{\varphi}_j^\pm \Big|_{\partial\Omega_\pm \setminus \Gamma} = 0 \quad \text{and} \quad \widetilde{\varphi}_j^\pm \Big|_\Gamma = g_j.$$

Then, the difference $\varphi_j - \widetilde{\varphi}_j \in H_0^1(\Omega_\pm)$ is the unique weak solution of the boundary value problem

$$\Delta(\varphi_j^\pm - \widetilde{\varphi}_j^\pm) = \frac{\Delta \rho_j^\pm}{a_\pm \mp \lambda} \quad \text{in} \quad \Omega_\pm$$

with the homogeneous Dirichlet boundary conditions on $\partial\Omega_\pm$. Hence, this difference satisfies the estimate

$$\begin{aligned} \| \operatorname{grad} \varphi_j^\pm - \operatorname{grad} \widetilde{\varphi}_j^\pm \|_{L^2(\Omega_\pm)} \leq \| \varphi_j^\pm - \widetilde{\varphi}_j^\pm \|_{H^1(\Omega_\pm)} &\leq c \| \Delta \rho_j^\pm \|_{H^{-1}(\Omega_\pm)} \\ &\leq c \| \operatorname{grad} \rho_j^\pm \|_{L^2(\Omega_\pm)} \to 0, \ j \to \infty, \end{aligned} \tag{8.26}$$

where the last inequality follows from the continuity of the divergence operator

$$\operatorname{div} : L^2(\Omega_\pm)^n \to H^{-1}(\Omega_\pm),$$

in Lemma 8.4.1.

Combining this continuity with (8.23), we arrive at the conclusion that

$$(h_+ - \lambda) \int_{\Omega_+} \overline{\operatorname{grad} \psi(x)} \operatorname{grad} \widetilde{\varphi}_j(x) dx - (h_- + \lambda) \int_{\Omega_-} \overline{\operatorname{grad} \psi(x)} \operatorname{grad} \widetilde{\varphi}_j(x) dx \xrightarrow[j\to\infty]{} 0$$

uniformly in $\psi \in H_0^1(\Omega)$. Since $\widetilde{\varphi}_j$ is harmonic in both Ω_+ and Ω_-, by Lemma 8.4.7 we obtain that

$$(h_+ - \lambda)\langle \tau_+ \psi, \Lambda_+ g_j \rangle_{\mathscr{H}^{1/2}(\Gamma)} - (h_- + \lambda)\langle \tau_- \psi, \Lambda_- g_j \rangle_{\mathscr{H}^{1/2}(\Gamma)} \xrightarrow[j\to\infty]{} 0 \tag{8.27}$$

uniformly in $\psi \in H_0^1(\Omega_\pm)$. By the surjectivity in Lemma 8.4.3, we have that

$$(h_+ - \lambda)\langle f, \Lambda_+ g_j \rangle_{\mathscr{H}^{1/2}(\Gamma)} - (h_- + \lambda)\langle f, \Lambda_- g_j \rangle_{\mathscr{H}^{1/2}(\Gamma)} \xrightarrow[j\to\infty]{} 0 \tag{8.28}$$

uniformly in $f \in \mathscr{H}^{1/2}(\Gamma)$. Therefore,

$$\|(h_+ - \lambda)\Lambda_+ g_j - (h_- + \lambda)\Lambda_- g_j\|_{\mathscr{H}^{1/2}(\Gamma)} \to 0, \ j \to \infty. \tag{8.29}$$

By the continuity in Lemma 8.4.3, we have that $\|g_j\|_{\mathscr{H}^{1/2}(\Gamma)} \leq c\|\varphi_j\|_{H^1(\Omega)}$.

We now claim that

$$\|g_j\|_{\mathscr{H}^{1/2}(\Gamma)} \geq c' \|\varphi_j\|_{H^1(\Omega)} \geq c' \| \operatorname{grad} \varphi_j \|_{L^2(\Omega)^n} = c'$$

for some constant $c' > 0$ independent of j. Indeed, assume that there is a subsequence $(g_{j_k})_{k\in\mathbb{N}}$ converging to zero. Then $\varphi^{\pm}_{j_k}$, defined as solutions of the boundary value problems (8.24), (8.25) satisfy the estimates

$$\|\varphi^{\pm}_{j_k}\|_{H^1(\Omega)} \le c\left(\|\Delta\rho^{\pm}_{j_k}\|_{H^{-1}(\Omega)} + \|g_{j_k}\|_{\mathscr{H}^{1/2}(\Gamma)}\right).$$

By (8.26), the right-hand side converges to zero, which contradicts the assumption $\|\operatorname{grad}\varphi_{j_k}\|_{L^2(\Omega)} = 1$ with

$$\varphi_{j_k}(x) := \begin{cases}\varphi^{+}_{j_k}(x), & x\in\Omega_+,\\ \varphi^{-}_{j_k}(x), & x\in\Omega_-.\end{cases}$$

The Weyl criterion applied to (8.29) then implies that $\mu = \frac{h_+-\lambda}{h_-+\lambda}$ is in the spectrum of the operator $\Lambda_+^{-1}\Lambda_-$.

Conversely, assume that μ is in the spectrum of the operator $\Lambda_+^{-1}\Lambda_-$. Again by the Weyl criterion there is a sequence $g_j \in \mathscr{H}^{1/2}(\Gamma)$ with $\|g_j\|_{\mathscr{H}^{1/2}(\Gamma)} = 1$, such that (8.28) holds uniformly in $h \in \mathscr{H}^{1/2}(\Gamma)$. By Lemma 8.4.3 we obtain that (8.27) holds uniformly in $\psi \in H^1_0(\Omega)$. Let now $\varphi_j \in H^1_0(\Omega)$ be the sequence constructed from g_j as in the proof of Lemma 8.4.3 (ii), that is,

$$\varphi_j(x) = \begin{cases}\varphi^{+}_{j}(x), & x\in\Omega_+,\\ \varphi^{-}_{j}(x), & x\in\Omega_-,\end{cases}$$

where $\varphi_j^{\pm}$ are the unique weak solutions of the boundary value problems $\Delta\varphi_j^{\pm} = 0$ in $\Omega_\pm$ with boundary values $\varphi_j^{\pm}\Big|_{\partial\Omega\cap\partial\Omega_\pm} = 0$ and $\varphi_j^{\pm}\Big|_{\Gamma} = g_j$.

Hence, (8.23) holds uniformly in $\psi \in H^1_0(\Omega)$, which implies (8.22).

Thus, λ with $\mu = (h_+ - \lambda)/(h_- + \lambda)$ is in the spectrum of the operator QHQ^*.

From the considerations above, it follows that $\lambda \in \mathbb{R}$ is an eigenvalue of QHQ^* of multiplicity $1 \le m \le \infty$ if and only if $\mu = (h_+ - \lambda)/(h_- + \lambda)$ is an eigenvalue of the operator $\Lambda_+^{-1}\Lambda_-$ of the same multiplicity.

Observe that

$$\|\,(QHQ^*)^{-1}\,\| = \frac{1}{|\lambda_0|},$$

where λ_0 is the point of the spectrum of QHQ^* with the smallest absolute value,

$$|\lambda_0| = \min_{\lambda\in\sigma(QM_AQ^*)} |\lambda|.$$

Since $\lambda \in \sigma(QHQ^*)$ if and only if it has the representation $\lambda = \frac{h_+-\mu h_-}{1+\mu}$ for some $\mu \in \sigma(\Lambda_+^{-1}\Lambda_-)$, we arrive at the representation (8.20). □

We can apply the Theorem above to the special case of a symmetric splitting.

Corollary 8.4.9 ([**38**]). *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain symmetric with respect to the hyperplane $x_1 = 0$, that is,*

$$x = (x_1, x_2, \dots, x_n) \in \Omega \quad \textit{if and only if} \quad (-x_1, x_2, \dots, x_n) \in \Omega.$$

Assume furthermore that

$$\Omega_+ = \{x\in\Omega \mid x_1 > 0\} \quad \textit{and} \quad \Omega_- = \{x\in\Omega \mid x_1 < 0\}.$$

Then the operator QHQ^* is boundedly invertible if and only if $h_- \neq h_+$. The norm of its inverse is $2/|h_+ - h_-|$.

In the case $h_- = h_+$ zero is an eigenvalue of QHQ^* of infinite multiplicity.

PROOF. Due to the symmetry we have $\Lambda_+ = \Lambda_-$, so that $\Lambda_+^{-1}\Lambda_- = I$. Therefore, the resolvent set of $\Lambda_+^{-1}\Lambda_-$ is $\mathbb{C} \setminus \{1\}$. As a consequence, the spectrum of QHQ^* consists of three eigenvalues h_+, $(h_+ - h_-)/2$, $-h_-$, each of infinite multiplicity. □

Note that the corollary above is a generalisation of [**7**, Remark 3.2] by Bonnet-Ben Dhia, Chesnel, and Ciarlet to dimensions $n \geq 2$. The following observation connects this work with [**7**].

LEMMA 8.4.10 (cf. [**41**]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $h, h^{-1} \in L^\infty(\Omega)$.*
Then the operator QHQ^* is boundedly invertible if and only if the self-adjoint operator $S : H_0^1(\Omega) \to H_0^1(\Omega)$ defined by the sesquilinear form

$$\mathfrak{s}[\varphi, \psi] := \langle \operatorname{grad} \varphi, h(\,\cdot\,) \operatorname{grad} \psi \rangle_{L^2(\Omega)}, \quad \varphi, \psi \in H_0^1(\Omega) \tag{8.30}$$

is boundedly invertible.

PROOF. We endow the Hilbert space $H_0^1(\Omega)$ with the inner product

$$\langle \varphi, \psi \rangle_{H_0^1(\Omega)} := \langle \operatorname{grad} \varphi, \operatorname{grad} \psi \rangle_{L^2(\Omega)},$$

which, by the Poincaré Inequality, is equivalent to the standard inner product in $H^1(\Omega)$. The sesquilinear form $\mathfrak{s}$ is bounded. Hence, by the First Representation Theorem there exists a bounded operator $S : H_0^1(\Omega) \to H_0^1(\Omega)$ associated with this form. Recall that the operator grad : $H_0^1(\Omega) \to L^2(\Omega)$ is an isometrical isomorphism of Hilbert spaces. Since

$$\langle \varphi, F\psi \rangle_{H_0^1(\Omega)} = \langle \operatorname{grad} \varphi, \operatorname{grad}(F\psi) \rangle_{L^2(\Omega)}$$

for any bounded operator F on $H_0^1(\Omega)$, we obtain $S = \operatorname{grad}^{-1} QHQ^* \operatorname{grad}$ by comparison. The claimed equivalence now follows directly since grad is an isomorphism. □

Since the Dirichlet Laplacian $-\Delta_D$ is an isomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$, we immediately arrive at the following conclusion.

COROLLARY 8.4.11 ([**41**]). *The operator QHQ^* is boundedly invertible if and only if the operator*

$$\operatorname{div} QHQ^* \operatorname{grad} : H_0^1(\Omega) \to H^{-1}(\Omega)$$

is an isomorphism.

In [**7**] the notion of T-coercivity was introduced. In this sense, a form $\mathfrak{b}$ is T-coercive if the form given by $\mathfrak{b}[x, Ty]$ is coercive. Using the concept of T-coercivity, a number of domains and coefficient functions h have been presented in [**7**], for which the operator

$$\operatorname{div} QHQ^* \operatorname{grad} \colon H_0^1(\Omega) \to H^{-1}(\Omega)$$

is an isomorphism and thus QHQ^* is boundedly invertible. We present only the following statement of that paper to give an additional example, where QHQ^* is boundedly invertible.

PROPOSITION 8.4.12 ([**7**, Theorem 3.3]). *Let $\Omega \subset \mathbb{R}^2$ be a disc of radius $R > 0$ and let $\theta_0 \in (0, 2\pi)$. Set*

$$\Omega_+ := \{(r\cos\theta, r\sin\theta) \mid 0 < r < R,\ 0 < \theta < \theta_0\},$$
$$\Omega_- := \{(r\cos\theta, r\sin\theta) \mid 0 < r < R,\ \theta_0 < \theta < 2\pi\}.$$

Furthermore, let

$$h(x) := \begin{cases} h_+(x) I_2, & x \in \Omega_+, \\ -h_-(x) I_2, & x \in \Omega_-, \end{cases}$$

with $h_\pm(x) \geq c > 0, i = 1,2$ almost everywhere. Then QHQ^* is boundedly invertible whenever

$$\max\left\{\frac{\operatorname{ess\,inf}_{x\in\Omega_+} h_+(x)}{\operatorname{ess\,sup}_{x\in\Omega_-} h_-(x)}, \frac{\operatorname{ess\,inf}_{x\in\Omega_-} h_-(x)}{\operatorname{ess\,sup}_{x\in\Omega_+} h_+(x)}\right\} > \max\left\{\frac{2\pi-\theta_0}{\theta_0}, \frac{\theta_0}{2\pi-\theta_0}\right\}.$$

8.5. Left-indefinite Sturm Liouville operators

Let $n = 1$ and $\Omega \subset \mathbb{R}$ be a bounded open interval, then we can combine Theorem 8.2.2 and Corollary 8.3.2 to get the following statements on the (in general) indefinite Sturm-Liouville operator B_D.

COROLLARY 8.5.1 ([**41**]). *Let $h \in L^\infty(\Omega)$ be a real valued function with*

$$h^{-1} \in L^2(\Omega), \int_\Omega h(x)^{-1}dx \neq 0$$

and let H be the operator on $L^2(\Omega)$ acting by multiplication with the function h. Then

(a) *there exists a unique self-adjoint operator B_D with $\operatorname{Dom}(B_D) \subseteq H_0^1(\Omega)$ such that*

$$\langle u, B_D v\rangle_{L^2(\Omega)^2} = \langle u', Hu'\rangle_{L^2(\Omega)},\ u \in H_0^1(\Omega),\ v \in \operatorname{Dom}(B_D),$$

where the domain of B_D is given by $\operatorname{Dom}(B_D) = \{u \in H_0^1(\Omega) \mid hu' \in H^1(\Omega)\}$ and

$$(B_D u)(x) = \frac{d}{dx}\left(h(x)\frac{d}{dx}u(x)\right) = (hu')'(x);$$

(b) *the operator B_D is semibounded if and only if h is sign-definite almost everywhere;*

(c) *the inverse B_D^{-1} is compact and the spectrum of B_D is purely discrete.*

We have a closer look at the special case of $h(x) := \operatorname{sign}(x)$, which is exemplary for the setting of left-indefinite Sturm-Liouville operators.

LEMMA 8.5.2. *Let $\Omega := (-a, b)$ be a bounded interval, where $a \neq b$ are positive constants and let $h(x) := \operatorname{sign}(x)$. Then the natural domain of B_D is*

$$M := \{f \in C_0(\overline{\Omega}) | f' \in L^2(\Omega),\ \operatorname{sign}(\,\cdot\,)f' \in C(\overline{\Omega}),\ \big(\operatorname{sign}(\,\cdot\,)f'\big)' \in L^2(\Omega)\},$$

where $C_0(\overline{\Omega})$ is the set of functions which are continuous up to the boundary with Dirichlet boundary values, and f' denotes the weak derivative of the function f.

PROOF. By the assumption $a \neq b$, we have $\int_\Omega \frac{1}{h(x)}dx \neq 0$, so that Corollary 8.5.1 can be applied. Here, $\operatorname{sign}(\,\cdot\,)f'$ is continuous if and only if f' is continuous on $(-a, 0)$ and on $(0, b)$ and the jump-condition on the derivative in zero, $f'(0_-) = -f'(0_+)$, is satisfied. Let $f \in \operatorname{Dom}(B_D)$, then

$$f \in \operatorname{Dom}(D) = H_0^1(\Omega) \text{ and } h(\,\cdot\,)f' \in \operatorname{Dom}(D^*) = H^1(\Omega).$$

By the Sobolev Imbedding Theorem, [**2**, Theorem 6.3], we get

$$f \in C_0(\overline{\Omega}),\ f' \in L^2(\Omega) \quad \text{and} \quad \operatorname{sign}(\,\cdot\,)f' \in C(\overline{\Omega}),\ (\operatorname{sign}(\,\cdot\,)f')' \in L^2(\Omega),$$

thus $f \in M$. Conversely, let $f \in M$, then

$$f \in C_0(\overline{\Omega}),\ f' \in L^2(\Omega) \quad \text{and} \quad h(\,\cdot\,)f' \in C(\overline{\Omega}),\ (h(\,\cdot\,)f')' \in L^2(\Omega),$$

thus $f \in H_0^1(\Omega),\ h(\,\cdot\,)f' \in H^1(\Omega)$ and, consequently, $f \in \operatorname{Dom}(B_D)$. □

We now construct the eigenvalues and eigenfunctions for the left-indefinite Sturm-Liouville operator for $h = \operatorname{sign}$.

LEMMA 8.5.3. *Let* $\Omega := (-a, b)$ *and* $h(x) := \operatorname{sign}(x)$, *then positive eigenvalues* λ *of the left-indefinite Sturm-Liouville operator* $D^*h(\,\cdot\,)D$ *are solutions of the implicit equation*

$$\cosh(\sqrt{\lambda}b)\sin(\sqrt{\lambda}a) - \cos(\sqrt{\lambda}a)\sinh(\sqrt{\lambda}b) = 0.$$

and satisfy the asymptotics $\lambda_j^{(+)} \sim \frac{\pi^2(j+1/4)^2}{a^2}$ *for* $j \to \infty$. *The negative eigenvalues satisfy*

$$\cos(\sqrt{|\lambda|}b)\sinh(\sqrt{|\lambda|}a) - \cosh(\sqrt{|\lambda|}a)\sin(\sqrt{|\lambda|}b) = 0$$

and the asymptotics $\lambda_j^{(-)} \sim -\frac{\pi^2(j+1/4)^2}{d^2}$ *for* $j \to \infty$.

PROOF. To compute the eigenfunctions and eigenvalues, we make the ansatz

$$f := f_1\chi_{(-a,0)} + f_2\chi_{[0,b)}.$$

If $\lambda > 0$, we choose

$$f_1(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x), \quad f_2(x) = c_1\cosh(\sqrt{\lambda}x) - c_2\sinh(\sqrt{\lambda}x)$$

for constants c_1, c_2. Then f automatically satisfies the continuity and the jump condition of its derivative in 0. In order that f is non-trivial and satisfies the boundary conditions at a and b, the determinant of $\begin{pmatrix} \cos(\sqrt{\lambda}a) & -\sin(\sqrt{\lambda}a) \\ \cosh(\sqrt{\lambda}b) & -\sinh(\sqrt{\lambda}b) \end{pmatrix}$ has to be zero. Thus λ has to be a solution of the equation

$$\cosh(\sqrt{\lambda}b)\sin(\sqrt{\lambda}a) - \cos(\sqrt{\lambda}a)\sinh(\sqrt{\lambda}b) = 0.$$

If $\cos(\sqrt{\lambda}a)$ is not zero, this yields $\tan(\sqrt{\lambda}a) = \tanh(\sqrt{\lambda}b)$. By the asymptotics

$$\lim_{\lambda\to\infty} \tanh(\sqrt{\lambda}b) = 1,$$

it follows that

$$\lambda_j^{(+)} \sim \frac{\pi^2(j+1/4)^2}{a^2} \quad \text{for} \quad j \to \infty.$$

Negative eigenvalues λ can be computed in the same way making the ansatz

$$f_1(x) = c_1\cosh(\sqrt{|\lambda|}x) + c_2\sinh(\sqrt{|\lambda|}x), \quad f_2(x) = c_1\cos(\sqrt{|\lambda|}x) - c_2\sin(\sqrt{|\lambda|}x)$$

for constants c_1, c_2, where λ is a solution of the equation

$$\cos(\sqrt{|\lambda|}b)\sinh(\sqrt{|\lambda|}a) - \cosh(\sqrt{|\lambda|}a)\sin(\sqrt{|\lambda|}b) = 0.$$

In this case, we obtain that

$$\lambda_j^{(-)} \sim -\frac{\pi^2(j+1/4)^2}{b^2} \quad \text{for} \quad j \to \infty. \qquad \square$$

In this case, the lemma above shows that for $h = \operatorname{sign}$, the modulus of the positive respectively negative eigenvalues of the indefinite operator $\frac{d}{dx}h(x)\frac{d}{dx}$ on $(-a, b)$ satisfies the well known Weyl asymptotics of the eigenvalues for the Dirichlet Laplacian Δ_D on the domain $(-a, 0)$ respectively $(0, b)$, see the survey article [**5**].

Note that in dimension $n = 1$ the Weyl asymptotics for the left-indefinite Sturm-Liouville problem is already contained e.g. in [**6**]. However, there seems to be no direct generalisation of the technique used there to higher dimensional cases.

Based on the construction here, this asymptotic can be carried over to other functions h and to certain special cases in higher dimensions. These results are contained in the Ph. D. thesis of A. Hussein [**40**] and also appear in the joint work [**41**].

8.6. The Second Representation Theorem

In this section, we briefly investigate the Second Representation Theorem for the sesquilinear form

$$\mathfrak{b}_D[u,v] = \langle Du, HDv\rangle_{L^2(\Omega)^n}, \quad \operatorname{Dom}[\mathfrak{b}] = H_0^1(\Omega).$$

This form satisfies the Second Representation Theorem if the auxiliary form

$$\mathfrak{b}[u,v] := \langle A^{1/2}u, QHQ^*A^{1/2}v\rangle_{\operatorname{Ran} D}, \quad \operatorname{Dom}[\mathfrak{b}] = \operatorname{Dom}(A^{1/2}) = \operatorname{Dom}(D^*) \cap \operatorname{Ran} D$$

as in (8.9) does.

THEOREM 8.6.1. *Assume that the domain stability condition*

$$\operatorname{Dom}[\mathfrak{b}] = \operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(D^*) \cap \operatorname{Ran}(D)$$

holds, then also

$$\operatorname{Dom}[\mathfrak{b}_D] = \operatorname{Dom}(D) = H_0^1(\Omega) = \operatorname{Dom}(|B_D|^{1/2})$$

is satisfied. In this case, we have

$$\mathfrak{b}_D[u,v] = \langle Du, HDv\rangle_{L^2(\Omega)^n} = \langle |B_D|^{1/2}u, \operatorname{sign}(B_D)|B_D|^{1/2}v\rangle_{L^2(\Omega)} \tag{8.31}$$

for $u,v \in \operatorname{Dom}(|B_D|^{1/2})$.

PROOF. Note that since $UU^*\colon L^2(\Omega)^n \to L^2(\Omega)^n$ is the orthogonal projector onto $\operatorname{Ran} D$ (see [**20**, Section IV.3]), we have $QUU^*Q^* = I_{\mathcal{L}}$, where $I_{\mathcal{L}}$ is the identity on $\operatorname{Ran} D$. Furthermore, the operator identities

$$|B_D|^{1/2} = U^*Q^*|B|^{1/2}QU$$

and

$$\operatorname{sign}(B_D) = U^*Q^*\operatorname{sign}(B)QU$$

hold. To see these identities, recall that we have the representation

$$B_D = U^*Q^*BQU$$

and $\operatorname{Dom}(B) \subseteq \operatorname{Dom}(D^*) \cap \operatorname{Ran} D \subset \operatorname{Ran} D$ as well as $\operatorname{Ran} B \subseteq \operatorname{Ran} D$ by the construction of B_D in the proof of Theorem 8.2.2. Furthermore, $\operatorname{Dom}(B_D) \subseteq \operatorname{Dom}(D)$ and $QU|_{\operatorname{Dom}(D)}$ is isometric. In this case the spectral families of B and B_D satisfy the identity

$$U^*QE_B(\lambda)QU = E_{B_D}(\lambda)$$

from which the identities for the sign and the square root follow by functional calculus. It can now be seen from

$$|B_D|^{1/2} = U^*Q^*|B|^{1/2}QU,$$

where

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(A^{1/2}) = \operatorname{Dom}(D^*) \cap \operatorname{Ran} D = U\operatorname{Dom}(D),$$

that $\operatorname{Dom}(|B_D|^{1/2}) = \operatorname{Dom}(D)$.

Based on these operator identities, (8.31) follows by direct computation

$$
\begin{aligned}
&\langle |B_D|^{1/2}u, \operatorname{sign}(B_D)|B_D|^{1/2}v\rangle_{L^2(\Omega)} \\
&= \langle U^*Q^*|B|^{1/2}QUu, U^*Q^* \operatorname{sign}(B)QUU^*Q^*|B|^{1/2}QUv\rangle_{L^2(\Omega)} \\
&= \langle QUU^*Q^*|B|^{1/2}QUu, \operatorname{sign}(B)QUU^*Q^*|B|^{1/2}QUv\rangle_{\operatorname{Ran} D} \\
&= \langle |B|^{1/2}QUu, \operatorname{sign}(B)|B|^{1/2}PUv\rangle_{\operatorname{Ran} D} \\
&= \mathfrak{b}[QUp, QUq] = \langle A^{1/2}QUu, QHQ^*A^{1/2}QUv\rangle_{\operatorname{Ran} D} \\
&= \langle |D^*|Uu, H|D^*|Uu\rangle_{L^2(\Omega)^n} \\
&= \langle Du, HDv\rangle_{L^2(\Omega)^n}
\end{aligned}
$$

□

The difficulty in the application of Theorem 8.6.1 lies in showing the domain stability condition for the auxiliary form $\mathfrak{b}$.

In applications, the sufficient criteria introduced in Lemma 2.2.5 will in general not be satisfied, as e.g. in the one dimensional case of $\Omega := (-1, 2)$ and $h(x) := \operatorname{sign}(x)$. In this case, the operator B is non-semibounded, QHQ^* is not sign-definite, and QHQ^* does not map $\operatorname{Dom}(D^*) \cap \operatorname{Ran}(D)$ into itself.

It is not clear whether one (and thus all) of the equivalent conditions in Theorem 2.2.4 can be satisfied in the setting of $\operatorname{div} h(\,\cdot\,) \operatorname{grad}$ considered here.

CHAPTER 9

The indefinite operator $\operatorname{div} h(\cdot) \operatorname{grad}$ in the Neumann case

9.1. The general case

In this chapter, we modify considerations of Chapter 8 (respectively [**41**]) from the case of Dirichlet boundary values to Neumann boundary values.

The following well known definitions and results can be taken from the proof of [**61**, Lemma II.2.4.1] for $n \geq 2$ and, modified accordingly, also hold in the case of $n = 1$.

DEFINITION 9.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain or a bounded interval for $n = 1$ and let $L_0^2(\Omega)$ be the space of $L^2(\Omega)$ functions f with mean value zero, that is, $\int_\Omega f dx = 0$.*

Let D_N denote the closed operator $D_N = \operatorname{grad}$ on the domain

$$\operatorname{Dom}(D_N) = \{u \in L_0^2(\Omega) \mid \operatorname{grad} u \in L^2(\Omega)^n\} \subset H^1(\Omega)^n.$$

Then the adjoint D_N^* of D_N is given by $-\operatorname{div}$ on

$$\operatorname{Dom}(D_N^*) = \{v \in L^2(\Omega)^n \mid \operatorname{div} v \in L^2(\Omega),\ \nu.v|_{\partial\Omega} = 0\}, \tag{9.1}$$

where $\nu.v$ is the scalar product of the outer normal ν and v in $\mathbb{R}^n$. In dimension $n = 1$, $\nu.v|_{\partial\Omega} = 0$ can be substituted by $v|_{\partial\Omega} = 0$. The domain of D_N^* is dense in

$$E^2(\Omega) := \{v \in L^2(\Omega)^n \mid \operatorname{div} v \in L^2(\Omega)\}$$

since $C_0^\infty(\Omega)$ is contained. The kernel of D_N^* is denoted as

$$KerD_N^* = L_\sigma^2(\Omega) := \{v \in L^2(\Omega)^n \mid \operatorname{div} v = 0,\ \nu.v|_{\partial\Omega} = 0\},$$

the kernel of D_N is trivial. In a similar way to Lemma 8.2.1, the range of D_N is closed in $L^2(\Omega)^n$ by the Poincaré Inequality.

From [**17**, Proposition IX.1.1], we get the orthogonal decomposition

$$L^2(\Omega)^n = \operatorname{Ran} D_N \oplus L_\sigma^2(\Omega).$$

We have the polar decomposition

$$D_N = U|D_N| = |D_N^*|, \quad D_N^* = U^*|D_N^*| = |D_N|U^*, \tag{9.2}$$

where U is the partial isometry with initial space $L_0^2(\Omega) = (\operatorname{Ker} D_N)^\perp$ and final space $\operatorname{Ran} D_N \subset L^2(\Omega)^n$.

We define the grad–div operator and the Neumann Laplacian on their natural domains as follows.

DEFINITION 9.1.2. *Let $D_N D_N^*$ be the operator with $D_N D_N v = -\operatorname{grad}\operatorname{div} v$ with*

$$\operatorname{Ker}(D_N D_N^*) = \operatorname{Ker}(D_N^*)$$

on

$$\begin{aligned}\operatorname{Dom}(D_N D_N^*) &= \{v \in \operatorname{Dom}(D_N^*) \mid D_N^* v \in \operatorname{Dom}(D_N)\} \\ &= \{v \in L^2(\Omega)^n \mid \operatorname{div} v \in L^2(\Omega),\ \nu.v|_{\partial\Omega} = 0,\ \operatorname{div} v \in \operatorname{Dom}(D_N)\} \\ &= \{v \in L^2(\Omega)^n \mid \nu.v|_{\partial\Omega} = 0,\ \operatorname{div} v \in L_0^2(\Omega),\ \operatorname{grad}\operatorname{div} v \in L^2(\Omega)^n\}.\end{aligned}$$

The adjoint of this operator is the Neumann Laplacian $-\Delta_N$ with

$$-\Delta_N u = D_N^* D_N u$$

and

$$\begin{aligned}\operatorname{Dom}(-\Delta_N) &= \operatorname{Dom}(D_N^* D_N) = \{u \in \operatorname{Dom}(D_N) \mid \operatorname{grad} u \in \operatorname{Dom}(D_N^*)\}\\ &= \{u \in L_0^2(\Omega) \mid \operatorname{grad} u \in L^2(\Omega)^n,\ \operatorname{div}\operatorname{grad} u \in L^2(\Omega),\ \nu.\operatorname{grad} u|_{\partial\Omega} = 0\}\\ &= \{u \in L_0^2(\Omega) \mid \operatorname{grad} u \in L^2(\Omega)^n,\ \Delta u \in L^2(\Omega),\ \nu.\operatorname{grad} u|_{\partial\Omega} = 0\}\end{aligned}$$

REMARK 9.1.3. *The same argumentation as in the preceding chapter shows that*

(a) *the partial isometry U maps $\operatorname{Dom}(D_N)$ onto $\operatorname{Dom}(D_N^*) \cap \operatorname{Ran} D_N$;*
(b) *the decomposition $L^2(\Omega)^n = \operatorname{Ran} D_N \oplus \operatorname{Ker}(D_N^*)$ reduces the* grad–div *operator $D_N D_N^*$;*
(c) *the spectra $\sigma(D_N D_N^*) \setminus \{0\}$ and $\sigma(-\Delta_N)$ agree, and for all $\lambda > 0$, the spaces $\operatorname{Ker}(D_N D_N^* - \lambda)$ and $\operatorname{Ker}(D_N^* D_N - \lambda)$ have the same dimension.*

In the same way as in the preceding chapter, we consider the reduction of $D_N D_N^*$ on its reducing subspace.

LEMMA 9.1.4. *The operator*

$$A := D_N D_N^*|_{(\operatorname{Ker} D_N^*)^\perp}$$

is self-adjoint and unitary equivalent to $-\Delta_N$. If Ω has a smooth boundary, A has a compact resolvent.

PROOF. Since $(\operatorname{Ker} D_N^*)^\perp = \operatorname{Ran} D_N$ is a reducing subspace for $D_N D_N^*$, the operator A is self-adjoint. The domain is given by

$$\begin{aligned}&\operatorname{Dom}(A) = \operatorname{Ran}(D_N) \cap \operatorname{Dom}(D_N D_N^*)\\ &= \{v \in L^2(\Omega)^n \mid v = \operatorname{grad} u,\ u \in L_0^2(\Omega),\ \nu.\operatorname{grad} u|\partial\Omega = 0,\ \Delta u \in L_0^2(\Omega) \cap H^1(\Omega)^n\}.\end{aligned}$$

We have that the ranges of A and D_N agree since $\operatorname{Ran} D_N^* = L_0^2(\Omega)$. This identity follows since the equation $\operatorname{div} v = f$ has a solution $v \in H_0^1(\Omega)^n \subset \operatorname{Dom}(D_N^*)$ for any $f \in L_0^2(\Omega)$, see [**61**, Lemma II.2.1.1].

By the polar decomposition, we see that A is unitary equivalent to $-\Delta_N$. Thus, $\|A^{-1}\| \le \mu_0^{-1}$, where μ_0 is the smallest Neumann eigenvalue. If Ω has a smooth boundary, $-\Delta_N$ has a compact resolvent by [**18**, Theorem 7.2.2] and thus also A has a compact resolvent. ☐

We introduce the following analogon to the operator Q.

DEFINITION 9.1.5. *Let $R\colon L^2(\Omega)^n \to \operatorname{Ran} D_N$ be the map given by*

$$Ru = \begin{cases} u, & u \in \operatorname{Ran} D_N \\ 0, & u \in \operatorname{Ran} D_N^\perp \end{cases}$$

By the closedness of the subspace $\operatorname{Ran} D_N$*, we can consider this space again as a Hilbert space which we denote by $\mathcal{M}$.*

In a similar way to the Dirichlet case of Theorem 8.2.2, we obtain the following result in the Neumann case.

THEOREM 9.1.6. *Let $\Omega \subset \mathbb{R}^n$, be a bounded Lipschitz domain for $n \ge 2$ or a bounded open interval for $n = 1$. Let $h \in L^\infty(\Omega; \mathbb{C})^{n\times n}$ and let H be the operator that acts by multiplication with $h(\,\cdot\,)$ such that*

(a) *$h(x)$ is Hermitian for almost all $x \in \Omega$,*
(b) *the operator $RHR^* : \operatorname{Ran} D_N \to \operatorname{Ran} D_N$ is boundedly invertible.*

Then

(i) *there exists a unique self-adjoint operator* B_N *with* $\mathrm{Dom}(B_N) \subset (H^1(\Omega) \cap L_0^2(\Omega))$ *such that*

$$\langle v, B_N u\rangle_{L^2(\Omega)} = \langle \operatorname{grad} v, H \operatorname{grad} u\rangle_{L^2(\Omega)^n}$$

holds for all $v \in (H^1(\Omega) \cap L_0^2(\Omega))$ *and all* $u \in \mathrm{Dom}(B_N)$*, where the domain is given by*

$$\mathrm{Dom}(B_N) = \{u \in (H^1(\Omega) \cap L_0^2(\Omega)) \mid H D_N u \in \mathrm{Dom}(D_N^*)\}$$

with $\mathrm{Dom}(D_N^*)$ *as in* (9.1). *For any* $u \in \mathrm{Dom}(B_N)$ *and almost all* $x \in \Omega$*, one has*

$$(B_N u)(x) = \operatorname{div} h(x) \operatorname{grad} u(x),$$

the domain $\mathrm{Dom}(B_N)$ *is a core for the gradient operator* D_N*;*

(ii) *the operator* B_N *is semibounded if and only if* RHR^* *is sign-definite;*

(iii) *the open interval* $(-\alpha\mu_0, \alpha\mu_0)$ *with*

$$\alpha := \|(RHR^*)^{-1}\|^{-1} \tag{9.3}$$

and $\mu_0 > 0$ *the smallest eigenvalue of the Neumann Laplacian* $-\Delta_N$ *in* $L_0^2(\Omega)$*, belongs to the resolvent set of* B_N*. In particular,* B_N *is boundedly invertible with* $\|B_N^{-1}\| \leq \frac{1}{\alpha\mu_0}$*;*

(iv) *the inverse* B_N^{-1} *is compact if* Ω *has a smooth boundary. In particular, the spectrum of* B_N *is purely discrete.*

Proof. Consider the sesquilinear form given by

$$\mathfrak{b}[u, v] = \langle A^{1/2} u, RHR^* A^{1/2} v\rangle_{\operatorname{Ran} D_N}, \quad u, v \in \mathrm{Dom}(A^{1/2}) \subseteq \operatorname{Ran} D_N,$$

where

$$\mathrm{Dom}(A^{1/2}) = \mathrm{Dom}(|D_N^*|) \cap \operatorname{Ran} D_N = \mathrm{Dom}(D_N^*) \cap \operatorname{Ran} D_N = U(\mathrm{Dom}(D_N))$$

for the partial isometry U in the polar decomposition (9.2).

By the First Representation Theorem in the gap case, [**36**, Theorem 2.3], there exists a unique self-adjoint, boundedly invertible operator B in the Hilbert space $\mathcal{M}$, such that

$$\mathfrak{b}[u, v] = \langle u, Bv\rangle, \quad u \in \mathrm{Dom}(A^{1/2}),\ v \in \mathrm{Dom}(B) \subset \mathrm{Dom}(A^{1/2}).$$

We now extend B to an operator on $L_0^2(\Omega)^n$ by

$$\widehat{B}u = \begin{cases} Bu, & u \in \mathrm{Dom}(B) \\ 0, & u \in (\operatorname{Ran} D_N)^\perp = \operatorname{Ker} D_N^* \end{cases}$$

with $\mathrm{Dom}(\widehat{B}) = \mathrm{Dom}(B) \oplus \operatorname{Ker}(D_N^*)$. We define the operator

$$B_N := U^* \widehat{B} U$$

on the natural domain

$$\mathrm{Dom}(B_N) = \{u \in L_0^2(\Omega) \mid Uu \in \mathrm{Dom}(\widehat{B})\} \subset \mathrm{Dom}(D_N) \oplus \{u \in L_0^2(\Omega) \mid Uu \in \operatorname{Ker}(D_N^*)\}.$$

Since $\operatorname{Ker} D_N = \{0\}$ and $D_N = |D_N^*|U$, it follows, that U does not map to $\operatorname{Ker}(D_N^*)$. Furthermore, we have $U\,\mathrm{Dom}(B_N) = \mathrm{Dom}(\widehat{B})$ so that B_N is a self-adjoint operator and the open interval $(-\alpha\mu_0, \alpha\mu_0)$ is in its resolvent set. Combining the results above on the domains $\mathrm{Dom}(B), \mathrm{Dom}(\widehat{B})$ and $\mathrm{Dom}(B_N)$, we obtain that

$$\mathrm{Dom}(B_N) = \{u \in H^1(\Omega) \cap L_0^2(\Omega) \mid H D_N u \in \mathrm{Dom}(D_N^*)\}.$$

If the domain Ω has a smooth boundary, the operator A^{-1} is compact. In this case also the inverse

$$B_N^{-1} = A^{-1/2}(RHR^*)^{-1}A^{-1/2}$$

is compact. As a consequence, the spectrum of B_N is discrete.

By the First Representation Theorem, $\operatorname{Dom}(\widehat{B})$ is a core for $(D_N D_N^*)^{1/2}$. Hence,

$$\operatorname{Dom}(B_N) = U^* \operatorname{Dom}(\widehat{B})$$

is a core for

$$(D_N D_N^*)^{1/2}U = |D_N^*|U = D_N.$$

For any $u \in \operatorname{Dom}(B_N)$ and $v \in H^1(\Omega) \cap L_0^2(\Omega)$ we have that

$$\begin{aligned}
\langle u, B_N v\rangle_{L^2(\Omega)} &= \langle Uu, \widehat{B}Uv\rangle_{L^2(\Omega)^n} = \langle RUu, BRUv\rangle_{\mathcal{M}} \\
&= \mathfrak{b}[RUu, RUv] = \langle A^{1/2}RUu, RHR^*A^{1/2}RUv\rangle_{\mathcal{M}} \\
&= \langle R^*A^{1/2}RUu, R^*A^{1/2}RUv\rangle_{L^2(\Omega)^n}.
\end{aligned}$$

Since $R^*A^{1/2}R = |D_N^*|$, we have

$$\begin{aligned}
\langle u, B_N v\rangle_{L^2(\Omega)} &= \langle |D_N^*|Uu, H|D_N^*|Uv\rangle_{L^2(\Omega)^n} \\
&= \langle D_N u, HD_N v\rangle_{L^2(\Omega)^n}.
\end{aligned}$$

□

We now extend the operator $D_N^* H D_N$ from the space $L_0^2(\Omega)$ of mean value zero to the whole $L^2(\Omega)$.

REMARK 9.1.7. *The self-adjoint operator $(B_N, \operatorname{Dom}(B_N))$ on $L_0^2(\Omega)$ can be extended by zero to a self-adjoint operator on $L^2(\Omega)$ with kernel consisting of constant functions.*

Since $\dim\big(L^2(\Omega)/L_0^2(\Omega)\big) = 1$, we can set $L^2(\Omega) = L_0^2(\Omega) \oplus C$, where C is the set of constant functions on Ω. Consider now

$$\widehat{B_N} := \begin{pmatrix} B_N & 0 \\ 0 & 0 \end{pmatrix}, \quad \operatorname{Dom}(\widehat{B_N}) = \operatorname{Dom}(B_N) \oplus C.$$

This block operator matrix defines a self-adjoint operator by [**63**, Corollary 2.29].

The natural domain of $\widehat{B_N}$ is

$$\operatorname{Dom}(\widehat{B_N}) = \{u \in H^1(\Omega) \mid \operatorname{div} H \operatorname{grad} u \in L^2(\Omega),\ \nu.(H \operatorname{grad} u)|_{\partial\Omega} = 0\}.$$

In the statements we obtained so far, the Dirichlet case and the Neumann case are very close. The only difference is that one has to consider $H^1(\Omega) \cap L_0^2(\Omega)$ instead of $H_0^1(\Omega)$ as the domain of the gradient and additional regularity of the boundary is needed for the compactness of the resolvent in the Neumann case.

However the difference lies in the properties of the operators QHQ^* and RHR^*.

We first investigate the Neumann case in dimension $n = 1$.

9.2. The operator RHR^*

In dimension $n = 1$, we have $\operatorname{Ran} D = L_0^2(\Omega)$ and $\operatorname{Ran} D_N = L^2(\Omega)$, so that Q is a non-trivial map and R is a trivial map.

To see the identity $\operatorname{Ran} D_N = L^2(\Omega)$, let $\Omega = (a, b)$, $f \in L^2(\Omega)$ be a bounded interval, then one easily verifies that g defined by

$$g(x) := \int_a^x f(y)dy - \frac{1}{|\Omega|}\int_\Omega \int_a^z f(y)dy\, dz$$

is in $H^1(\Omega) \cap L_0^2(\Omega)$ and satisfies $g' = f$ almost everywhere.

An immediate consequence of R being trivial is the following.

COROLLARY 9.2.1. *Let $n = 1, \Omega = (a, b)$ be a bounded interval. Then the operator*

$$B_N := \operatorname{div} H \operatorname{grad}$$

with natural domain

$$\{u \in H^1(\Omega) \cap L^2_0(\Omega) \mid (hu')' \in L^2(\Omega),\ u'(a) = u'(b) = 0\}$$

is boundedly invertible if $h, h^{-1} \in L^\infty(\Omega)$. Furthermore, B_N has a spectral gap in the interval $(-\mu_0\alpha, \mu_0\alpha)$, where μ_0 is the smallest eigenvalue of the Neumann Laplacian $-\Delta_N$ and

$$\alpha^{-1} := \operatorname*{ess\,sup}_{x\in\Omega} |h^{-1}(x)|.$$

We now compare QHQ^* and RHR^*.

REMARK 9.2.2. *In dimension $n = 1$, we have that QHQ^* has a bounded inverse if $h, h^{-1} \in L^\infty(\Omega)$ and $\int_\Omega h(x)^{-1}dx \neq 0$. For the boundedness of $(RHR)^{-1}$, the last condition is not needed. In this case, boundedness of $(QHQ^*)^{-1}$ is the stronger condition.*

Note that $\Omega = (-1, 1)$, $h(x) = \operatorname{sign}(x)$, with

$$\sigma(RHR) = \{\pm 1\} \quad \text{and} \quad \sigma(QHQ^*) = \{0, \pm 1\}$$

shows that this condition is indeed stronger.

It is an open problem whether the condition on QHQ^* is still the stronger one in arbitrary dimensions. On this problem, we note that

$$(\operatorname{Ran} D_N)^\perp = L^2_\sigma(\Omega) \quad \text{and} \quad (\operatorname{Ran} D)^\perp = L^2_\sigma(\Omega) \oplus H(\Omega)$$

(see [**17**, Proposition IX.1.1]*), so that the space of functions that have to be mapped to zero by Q is larger than the corresponding space for R. A natural modification of Corollary 8.4.9 and Proposition 8.4.12 to the Neumann case is not known to us.*

Like in the Dirichlet case, we compute the eigenfunctions and eigenvalues for the interval $\Omega := (-c, d)$ and $h(x) := \operatorname{sign}(x)$ in the Neumann case.

LEMMA 9.2.3. *The positive eigenvalues λ of $D_N^* H D_N$ are solutions of the implicit equation*

$$\cosh(\sqrt{\lambda}d)\sin(\sqrt{\lambda}c) + \cos(\sqrt{\lambda}c)\sinh(\sqrt{\lambda}d) = 0.$$

and satisfy $\lambda_j^{(+)} \sim \frac{\pi^2(j-1/4)^2}{c^2}$ for $j \to \infty$. The negative eigenvalues satisfy

$$\cos(\sqrt{|\lambda|}d)\sinh(\sqrt{|\lambda|}c) + \cosh(\sqrt{|\lambda|}c)\sin(\sqrt{|\lambda|}d) = 0$$

and the asymptotics $\lambda_j^{(-)} \sim -\frac{\pi^2(j-1/4)^2}{d^2}$ for $j \to \infty$.

PROOF. The proof is similar to the one of Lemma 8.5.3, the same ansatz functions satisfying the conditions in 0 are used. It remains to verify the Neumann condition in the endpoints $-c$ and d for those functions to get a condition on λ. For positive λ the determinant of $\begin{pmatrix} \sin(\sqrt{\lambda}c) & \cos(\sqrt{\lambda}c) \\ \sinh(\sqrt{\lambda}d) & -\cosh(\sqrt{\lambda}d) \end{pmatrix}$. Thus λ has to be a solution of the equation

$$\cosh(\sqrt{\lambda}d)\sin(\sqrt{\lambda}c) + \cos(\sqrt{\lambda}c)\sinh(\sqrt{\lambda}d) = 0.$$

In the same way as in Lemma 8.5.3, this yields the asymptotics

$$\lambda_j^{(+)} \sim \frac{\pi^2(j-1/4)^2}{c^2} \quad \text{for} \quad j \to \infty.$$

A short computation shows that the mean-value condition $\int_\Omega f(x)dx = 0$ follows from the Neumann condition in $-c$ and d.

The statement for negative eigenvalues follows in the same way. $\square$

As in the preceding chapter, direct computation shows that a Second Representation Theorem holds in the Neumann case provided that the domain stability condition is satisfied.

LEMMA 9.2.4 (cf. Theorem 8.6.1). *Let the domain stability condition*

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(A^{1/2})$$

be satisfied, where $\operatorname{Dom}(A^{1/2}) = \operatorname{Dom}(D_N^*) \cap \operatorname{Ran}(D_N) = U \operatorname{Dom}(D_N)$. *Then we have that* $\operatorname{Dom}(D_N) = \operatorname{Dom}(|B_N|^{1/2})$ *and we obtain the representation*

$$\langle D_N u, H D_N v\rangle_{L^2(\Omega)^n} = \langle |B_N|^{1/2} u, \operatorname{sign}(B_N)|B_N|^{1/2} v\rangle, \quad u, v \in \operatorname{Dom}(D_N).$$

However, as in the Dirichlet case, it is an open problem to verify the domain stability condition $\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(A^{1/2})$ for the corresponding auxiliary form.

Conclusion, open problems and future research

We close this work with an informal survey on the results and the open problems. We speculate on the corresponding conjectures and point out future research.

In this thesis, we provided strategies for the problem of finding a self-adjoint operator associated with an indefinite form as well as strategies for the problem of the block diagonalisation of an operator with respect to a graph subspace.

The strategy to consider indefinite forms $\mathfrak{b}$ and to find the associated operator is as follows. First check whether the form fits into the off-diagonal setting $\mathfrak{b} = \mathfrak{a}[\cdot, J_A \cdot] + \mathfrak{v}$, where $\mathfrak{a}$ is strictly positive or non-negative, and verify the corresponding regularity conditions.

If the form $\mathfrak{b}$ is not in the off-diagonal setting, put it into the framework of

$$\tilde{\mathfrak{b}}[x, y] = \langle B_1 x, B_2 y \rangle,$$

where $0 \in \rho(B_1) \cap \rho(B_2)$ and $\tilde{\mathfrak{b}} = \mathfrak{b} + J$ for some bounded operator J. In this sense, create a spectral gap for the associated operator $\widetilde{B}$ by a bounded perturbation J. The case, where J is a multiple of the identity, so that the whole form is shifted in the complex plane, is the approach by McIntosh.

Alternatively, push open the spectral gap around zero by adding a more general bounded perturbation J. If there is a guess on the associated operator B, for instance if

$$\mathfrak{b}[x, y] = \left\langle A^{1/2} x, H A^{1/2} y \right\rangle, \tag{$*$}$$

where $B = A^{1/2} H A^{1/2}$ is expected, look for an operator J close to the sign of this operator.

If the form $\mathfrak{b}$ can be written as $(*)$, where A has a kernel, but otherwise is separated form zero, split off the kernel (as for the div H grad operator) and reconsider the reduced form. This splitting comes at the price that one has to deal with some operator QHQ^* on a reducing subspace instead of the simpler operator H.

For these cases, we obtain an operator B associated with $\mathfrak{b}$. To check that the operator also represents the form, verify the sufficient criteria for the domain stability condition $\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(A^{1/2})$ or the equivalent conditions.

For the block diagonalisation of block operator matrices with respect to some graph spaces, the strategy is as follows.

First, classify the operator into diagonal dominant, upper (or lower) dominant, and off-diagonal dominant.

For the diagonal dominant case, stay in the operator framework and check that the reducing subspace corresponding to the diagonalisation is a graph subspace. To do this, verify that the corresponding projector difference satisfies $||P - Q|| \le \sqrt{2}/2$. Now consider the diagonalisation with respect to $A - YV$ and the operator Riccati equation as in Chapter 4.

For the upper dominant case, for instance the Stokes operator, apply form methods. Namely, rewrite the corresponding form in a diagonal dominant way, verify the conditions for the generalisation of the Tan 2Θ Theorem (Theorem 6.1.5) and diagonalise this form by means of the form Riccati equation. Use then the fact that reducing subspaces for operators and reducing subspaces for forms coincide to obtain an explicit block diagonalisation of the operator.

The diagonalisation of off-diagonal dominant matrices was not considered here. A corresponding strategy can be derived from [**15**], where Cuenin gave an explicit block diagonalisation of the Dirac operator with Coulomb potential which is exemplary for this case.

We now point to the open problems, conjectures and the ongoing research.

Chapter 1: By Hypothesis 1.2.1, we could grant that the form $\mathfrak{b} + J_A$ is closed (even 0-closed) in the sense of McIntosh and that the operator B associated with $\mathfrak{b}$ is self-adjoint. We conjecture that in this case also the form $\mathfrak{b}$ is closed. Numerical computations for infinite block matrices with 2×2 blocks as in Example 1.2.8 confirm this conjecture. However, thinking of a proof by perturbation theory, we meet the following problem:

If the operator A has a kernel, the form $\mathfrak{b}$ with $\mathfrak{b}[x, y] := \langle A^{1/2}x, HA^{1/2}y\rangle$ clearly is degenerate since $\mathfrak{b}[x, y] = 0$ for all x does not imply $y = 0$. Thus, the form $\mathfrak{b}$ cannot be 0-closed (see [**49**]) and the perturbation result in part (d) of Lemma 1.4.2 cannot be applied since this result directly implies the 0-closedness of $\mathfrak{b} = (\mathfrak{b} + J_A) - J_A$.

We do not know any suitable variant of the perturbation result granting only the closedness.

Chapter 2: If A is strictly positive, there is an example, where only the First but not the Second Representation Theorem is valid. Namely

$$A := \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 1 & 0 \\ 0 & k^2 \end{pmatrix}, \quad H := \bigoplus_{k\in\mathbb{N}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

see [**36**, Example 2.11]. Note that in this example, the operator H is purely off-diagonal. We do not have an appropriate example of this type for the case of only non-negative A since purely off-diagonal H are excluded by Hypothesis (1.1). It is possible that the condition (1.1) is strong enough to even grant the Second Representation Theorem.

Chapter 3: If A is bad, that is only non-negative, most of the statements in [**36**], including the Second Representation Theorem, can be preserved if there is a suitable perturbation J_A creating a spectral gap.

If instead H is bad in the sense of H^{-1} being unbounded, numerical examples of infinite block matrices with 2×2 blocks as in Example 3.2.3 suggest that

$$\mathrm{Dom}(A^{1/2}) \subset \mathrm{Dom}(|B|^{1/2})$$

always holds in this situation. In this case, the form one can reconstruct is even an extension of the original form. In this sense, the case of bad H seems to be of a different character as the case of bad A. The case of bad H and its applications will be investigated further in [**41**].

Chapter 4: The statements of this chapter are formulated for symmetric diagonally dominant operator matrices but in fact also off-diagonally dominant matrices can be considered. An important example in this class that one would like to diagonalise is the Dirac operator with Coulomb potential on $\mathbb{R}^3$,

$$H_\Phi = \begin{pmatrix} I + \Phi & 0 \\ 0 & -I + \Phi \end{pmatrix} + \begin{pmatrix} 0 & -i\sigma \cdot \mathrm{grad} \\ -i\sigma \cdot \mathrm{grad} & 0 \end{pmatrix} =: A + V,$$

where σ is the vector containing the Pauli matrices and Φ is a potential of Coulomb type.

A diagonalisation for this operator by means of forms has been investigated recently by Cuenin in [**15**] for Coulomb potentials $\Phi(x) := \frac{Z\alpha}{|x|}$, where $\alpha \approx 1/137$ is the fine structure constant and $Z \leq 124$.

For $Z \leq 87$, this operator can be defined in the operator sense, see [**43**, Section VI.5.4]. However, to obtain a diagonalisation of this operator by the technique of Chapter 4, the major issues are how to grant that the spectral subspaces are graph subspaces and that the operator inclusion (4.10),

$$T^*(A+V) \supseteq (A-YV)T^*,$$

can be turned into an identity. If the diagonal part A is dominant, the identity can be established by perturbation theory. However, if the off-diagonal part V is dominant, as for the Dirac operator, we do not know any corresponding result.

Another field for future investigation could be the block diagonalisation of random block operators appearing e.g. in the physical modelling of mesoscopic disordered systems like dirty superconductors as considered in [**44**].

Chapter 6: Here, we meet the following problems.

For the correspondence between reducing graph subspaces and solutions to the form Riccati equation we required the conditions (6.7) respectively (6.8) in Theorem 6.3.1. By stronger assumptions, we could grant these conditions in Theorem 6.3.6. However, these assumptions are related to the operator B and not to the form $\mathfrak{b}$. It would be more natural, dealing with form Riccati equations, to expect a condition in terms of forms. In Chapter 4, we had a similar problem in the operator setting and could solve it by the operator extension result in Corollary 4.3.2. A similar form extension result, which is unknown to us, could be used to grant the conditions in Theorem 6.3.1 in a similar way.

In Section 6.4, we established that the form Riccati equation (6.22) for the Stokes operator B_S has a unique contractive solution. We expect that the uniqueness still holds in the general situation of Theorem 6.3.1. Considering the form Riccati equation for $\mathfrak{b} + \frac{1}{n} J_A$, the proof of Corollary 6.1.5 implies that this equation has a strictly contractive solution X_n for each $n \in \mathbb{N}$. Since $1 \notin \sigma_p(X_n)$, these solutions are the unique contractive solutions, compare Theorem 6.4.3. In this case, the perturbed equation is uniquely solvable in the set of contractive operators, but we cannot exclude bifurcation of solutions when n approaches infinity and thus the gap around zero closes.

Chapter 7: In Chapter 4, we obtained a block diagonalisation of diagonally dominant, indefinite block operator matrices defined as a sum of operators. In this chapter, we obtain a block diagonalisation for upper-dominant, semibounded operators defined by sesquilinear forms. The extension to non-semibounded operators that are not in the framework of Chapter 4 remains an open problem.

Chapter 8: In dimension $n = 1$, we have an explicit representation of the operator QHQ^* and can thus grant the boundedness of its inverse. In higher dimensions, we do not have a suitable representation and are thus limited in our considerations to a few special cases.

An open problem is the construction of the operator div $h(x)$ grad on unbounded domains, even in the well understood case of $n = 1$. The issue here is that for unbounded domains the spectrum of the Dirichlet Laplacian $-\Delta_D$ can (in general) not be separated from zero even though the kernel is trivial. In this case, the First Representation Theorem in [**36**] cannot be applied. Also, it is not clear how to verify the Hypothesis 1.2.1 for the First Representation Theorem 1.2.3 for unbounded domains.

Chapter 9: In dimension $n = 1$, the Neumann case is simpler than the Dirichlet case, since the spectrum of RHR^* in this case is just the essential range of the function h. In higher dimension however the Neumann case seems to be more complicated than the Dirichlet case. The major issue here is to obtain a result corresponding to Theorem 8.4.8. We could not obtain a suitable substitute for the Dirichlet-to-Neumann maps $\Lambda_\pm$ for the common boundary $\partial\Omega_+ \cap \partial\Omega_-$. The problem in this case is to solve the Laplace equation with mixed boundary values on Lipschitz domains and to obtain a suitable estimate on the solution.

"Imagination is more important than knowledge.
Knowledge is limited.
Imagination encircles the world."

Albert Einstein

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