On the Geometry of the Spin-Statistics Connection in Quantum Mechanics

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Zusammenfassung

Das Spin-Statistik-Theorem besagt, dass das statistische Verhalten eines Systems von identischen Teilchen durch deren Spin bestimmt ist: Teilchen mit ganzzahligem Spin sind Bosonen (gehorchen also der Bose-Einstein-Statistik), Teilchen mit halbzahligem Spin hingegen sind Fermionen (gehorchen also der Fermi-Dirac-Statistik). Seit dem ursprünglichen Beweis von Fierz und Pauli wissen wir, dass der Zusammenhang zwischen Spin und Statistik aus den allgemeinen Prinzipien der relativistischen Quantenfeldtheorie folgt.

Man kann nun die Frage stellen, ob das Theorem auch dann noch gültig bleibt, wenn man schwächere Annahmen macht als die allgemein üblichen (z.B. Lorentz-Kovarianz). Es gibt die verschiedensten Ansätze, die sich mit der Suche nach solchen schwächeren Annahmen beschäftigen. Neben dieser Suche wurden über viele Jahre hinweg Versuche unternommen einen geometrischen Beweis für den Zusammenhang zwischen Spin und Statistik zu finden. Solche Ansätze werden hauptsächlich, durch den tieferen Zusammenhang zwischen der Ununterscheidbarkeit von identischen Teilchen und der Geometrie des Konfigurationsraumes, wie man ihn beispielsweise an dem Gibbs'schen Paradoxon sehr deutlich sieht, motiviert. Ein Versuch der diesen tieferen Zusammenhang ausnutzt, um ein geometrisches Spin-Statistik-Theorem zu beweisen, ist die Konstruktion von Berry und Robbins (BR). Diese Konstruktion basiert auf einer Eindeutigkeitsbedingung der Wellenfunktion, die Ausgangspunkt erneuerten Interesses an diesem Thema war.

Die vorliegende Arbeit betrachtet das Problem identischer Teilchen in der Quantenmechanik von einem geometrisch-algebraischen Standpunkt. Man geht dabei von einem Konfigurationsraum $\mathcal{Q}$ mit einer endlichen Fundamentalgruppe $\pi_1(\mathcal{Q})$ aus. Diese hat eine Darstellung auf dem Raum $C(\tilde{\mathcal{Q}})$, wobei $\tilde{\mathcal{Q}}$ die universelle Überlagerung von $\mathcal{Q}$ bezeichnet. Die Wirkung von $\pi_1(\mathcal{Q})$ auf $\tilde{\mathcal{Q}}$ induziert nun eine Teilung von $C(\tilde{\mathcal{Q}})$ in disjunkte Moduln über $C(\mathcal{Q})$, die als Räume von Schnitten bestimmter flacher Vektorbündel über $\mathcal{Q}$ interpretiert werden können. Auf diese Weise lässt sich die geometrische Struktur des Konfigurationsraums $\mathcal{Q}$ in der Struktur des Funktionenraums $C(\tilde{\mathcal{Q}})$ kodieren. Durch diese Technik ist es nun möglich die verschiedensten Ergebnisse, die das Problem der Ununterscheidbarkeit betreffen, auf klare, systematische Weise zu reproduzieren. Ferner findet man mit dieser Methode eine globale Formulierung der BR- Konstruktion. Ein Ergebnis dieser globalen Betrachtungsweise ist, dass die Eindeutigkeitsbedingung der BR-Konstruktion zu Inkonsistenzen führt. Ein weiterführendes Proposal hat die Begründung der Fermi-Bose-Alternative innerhalb unseres Zugangs zum Gegenstand.

Contents

1 Introduction

1.1 Motivation

Being a consequence of the general principles of relativistic Quantum Field Theory, the Spin-Statistics theorem has found rigorous proofs in the context of axiomatic, as well as of algebraic Quantum Field Theory [Fie39, Pau40, LZ58, SW00, DHR71, DHR74, GL95]. In spite of many efforts, this has not been the case in non relativistic Quantum Mechanics. A proof of the Spin-Statistics theorem, which does not rely as heavily on concepts of relativistic Quantum Field Theory(QFT) as the established ones, is something desirable, for several reasons. There are many examples of phenomena taking place outside the relativistic realm (one could think of Bose-Einstein Condensation, Superconductivity or the Fractional Quantum Hall Effect as relevant ones, not to mention the striking consequences of Pauli's Exclusion Principle as the prediction of "exchange" interactions, or the explanation of the periodic table) which depend essentially on the Spin-Statistics relation for its description.

Furthermore, a proof based on assumptions which are different from the standard ones could also be of benefit for the understanding of QFT itself. For instance, a proof which does not make use of the full Lorentz group could provide hints towards the understanding of Spin-Statistics in more general situations, such as theories where a background gravitational field is present*, or theories on non-commutative space-times.

The idea that the observed correlation between Spin and Statistics may, perhaps, be derived without making use of relativistic QFT is not a new one. Much work has been devoted to the point of view that *quantum indistinguishability*, if correctly incorporated into quantum theory, might lead to a better understanding of Spin-Statistics. Most of the work in this direction is based on formulations where the quantum theory of a system of indistinguishable particles is obtained from a quantization procedure, whose starting point is a classical configuration space.

In one of the first works of this kind, Laidlaw and DeWitt found out[LD71] that when applying the path integral formalism to a system consisting of a finite number of non-relativistic, identical spinless particles in three spatial dimensions, the topology of the corresponding configuration space imposes certain restrictions on the propagator. From this, they were able to deduce that only particles obeying Fermi or Bose statistics

*Although it is possible to generalize the proofs on Minkowski space to curved background space-times[Ver01], here we are interested in alternative approaches, of geometric nature.

are allowed (this *Fermi-Bose alternative* is an input in the standard proofs of axiomatic QFT).

Leinaas and Myrheim considered a similar situation in [LM77], in an analysis that was motivated by the relevance of indistinguishability to Gibbs' paradox. They reproduced the results of [LD71] by obtaining the Fermi-Bose alternative in three dimensional space for spinless particles. But, in addition, they also found that in one and two dimensions the statistics parameter could, in principle, take infinitely many values. In that same work, they remarked that their results could provide a geometrical basis for a derivation of the Spin-Statistics theorem.

A lot of work based on this kind of "configuration space approach" has been done since then, in an effort to find a simpler proof of the Spin-Statistics theorem, in comparison to the relativistic, analytic ones.

Usually, such attempts are based on the point of view that, in non relativistic Quantum Mechanics, indistinguishability, together with the rotational properties of the wave function describing a system of identical particles, are the physical concepts lying at the core of the problem. There are several reasons that, from a theoretical perspective, really seem to justify such a standpoint. First of all, spin is an intrinsic property of a particle which is closely related to spatial rotations: For example, just because of the definition of spin in terms of $SU(2)$ representations, the wave function of a particle of spin S changes its phase by a factor $(-1)^{2S}$ under a 2π rotation. The factor $(-1)^{2S}$ is indeed a quantity that can be directly measured from the interference pattern (i.e. as a *relative* phase) of a two-slit-type experiment with neutrons where one of the beams is subject to a magnetic field [Ber67, WCOE75]. The change in the sign of the wave function is then due to precession of the neutrons in the region where the magnetic field is present and provides a clear experimental evidence of the rotational properties of the wave function (of a fermion, in this case). The fact that this phase is the same as that defining the statistics of a system of identical particles of spin S, could be considered as a hint pointing towards a simple physical explanation of Spin-Statistics. This point of view is partially substantiated by the work of Finkelstein and Rubinstein [FR68] on a kind of *topological* Spin-Statistics theorem. Here, exchange of two identical objects is really correlated with the 2π rotation of one of them, only that the objects under consideration are extended ones, known as *kinks*, or *topological solitons*. In the Finkelstein-Rubinstein approach, one considers configurations of non-linear classical fields. The fields are assumed to take values on a given, fixed manifold and to satisfy some boundary conditions (e.g., fields must take a constant value at spatial infinity). The space of all fields -the configuration space- can be given a suitable topology, thus allowing a classification of fields according to the connected component they belong to. A field differing from the constant one only in a bounded region of space is referred to as a *localized kink* and it can carry a "charge", according to the component it belongs to. Such kinks are objects that, because of the existence of a notion of localization, may be regarded as "extended-" (in contrast to "point-") particles. In a few words, the main result of Finkelstein-Rubinstein is that an exchange of two identical kinks is equivalent (in the sense of homotopy) to a 2π rotation of one of them. It must be emphasized that in their proof, the idea of pair creation/annihilation is introduced and used extensively in order to establish the mentioned equivalence.

Although the work of Finkelstein-Rubinstein appeared almost a decade before those of Leinaas-Myrheim or of Laidlaw-DeWitt, it has not been possible to obtain analogous results for systems of *point* particles, at the quantum level. A proposal, inspired in part by the results of Finkelstein-Rubinstein, has been put forward by Balachandran *et.al.* in a series of papers [BDG+90, BDG+93] where configuration spaces for point particles allowing for the possibility of pair creation and annihilation are considered. In this approach, spin is modelled (within these classical configuration spaces) by means of "frames" which are attached to the particles. Due to the possibility of pair creation/annihilation, the configuration space admits configurations of any number of particles and antiparticles, thus being highly non trivial from the mathematical point of view. The topology given to the configuration space allows for coincidences of particles and antiparticles, but in a restricted way. The restrictions for arbitrary coincidences are dictated by consistency conditions, and also by analogies with the topological approach of Finkelstein-Rubinstein for kinks. An independent approach, using a similar kind of configuration space (with a different coincidence condition and a different topology) has been worked out by Tscheuschner [Tsc89]. A drawback common to these approaches is that, although the idea of pair creation/annihilation is included at the level of the classical configuration space, there is no clear interpretation of the results in terms of Quantum Field Theory. The fact that these configuration spaces admit an arbitrary number of particles also makes an interpretation in terms of Quantum Mechanics a difficult task. Also, since the configuration space is not a differentiable manifold, it is not clear how questions about connections and parallel transport, which are used to define, for example, the momentum operator or the statistics parameter (as a holonomy), can be reformulated.

It has been suggested [Tsc89, BDG+93] that these configuration spaces may be obtained from the Hilbert space of the quantum theory by exploiting the projective nature of the space of pure states. Unfortunately, the argumentations -in this respect- remain at a heuristic level.

In any case, once the topology of the configuration space upon which quantization is to be carried out becomes non-trivial (as a consequence of quantum indistinguishability, or otherwise) a careful consideration of the mathematical aspects of the theory becomes necessary.

In order to understand this last point and to put it in perspective, let us recall that since the advent of Gauge Theories to Particle and High Energy Physics in the fifties and the "discovery" of the deep mathematical concepts underlying the Gauge Principle[WY75], the mathematical theory of fiber bundles and connections has played an essential role in the formulation and understanding of physical theories of the fundamental interactions. Modern Differential Geometry and Topology have played a crucial role for the understanding of fundamental issues as, for example, quantization of Gauge Theories, anomalies and renormalization, topological effects as solitons, instantons, monopoles, non-trivial vacua and many others.

In most cases, the use of these mathematical structures is mainly restricted to the *classical* part of the theory. Just to give an example, in the quantization of Gauge Theories, a thorough understanding of the geometry of constrained Hamiltonian systems is very important for the quantization of the theory. In contrast to this, the presence of a kind of "Gauge Principle" is inherent to Quantum Theory. Indeed, the original intention of H. Weyl when using his *"Eichprinzip"* was to obtain a unification of General Relativity and Electromagnetism by means of a re-scaling of the metric. The (conformal) exponential factor responsible for the change in scale was real, and as is well known, it was shown by Einstein that the proposal was not viable. Soon after that, it was recognized by London that Weyl's ideas really made sense, but not without substantial changes: The exponential factor had to be actually chosen as complex, a phase factor, related not to the classical theories but to the quantum mechanical wave function. The consequences of this fact -the gauge freedom of the wave function- are widely known, with many interesting applications. The geometric character of the quantum mechanical phase can be clearly recognized in situations like, for example, the Aharanov-Bohm effect. Also widely known are geometric phases [Kat50, Ber84], whose geometric nature was originally pointed out by Simon [Sim83]. It is in this context, that of the geometry of quantum states in non-relativistic Quantum Mechanics, that the Spin-Statistics problem fits in. When the topology of the configuration space is complicated enough, new possibilities for the realization of quantum states appear. Indeed, the whole scheme of quantization by means of a "Correspondence Principle" works so nicely for the theory of, say, one electron in three-dimensional Euclidean space, just because the topology and geometry of the configuration space are, in a certain sense, trivial. Then, as a consequence of the Stone von Neumann uniqueness theorem, it is unnecessary to look for realizations of quantum states apart from the usual ones, namely in the Hilbert space $L^2(\mathbb{R}^3)$. The fact that the standard canonical commutation relations can be represented as operators in this space can be traced back to the existence of a transitive abelian group of transformations [Ish84]. But as soon as one considers a more general type of configuration space, even the application of the Correspondence Principle becomes an issue by itself, because in this case, a suitable analog of the canonical commutation relations must be found. Since the representation of the resulting operators might take place in a space which is not a space of functions on the configuration space, one can in this way see why other possibilities for the realization of quantum states become possible and, in some cases, even necessary.

For the understanding of these situations, a thorough mathematical analysis becomes necessary and indeed can, in many cases, give very concrete answers. For example, in the case of Quantum Mechanics of spinless particles on a (possibly multiply-connected) configuration space, which has been discussed extensively in the literature, one has a general result, which can be stated as follows:

1.1.1 Theorem (cf.[LD71]). *Let $\mathcal{Q}$ be the manifold that corresponds to the classical configuration space of a spinless particle. Then, the inequivalent quantizations of this system are in bijective correspondence with the one dimensional unitary representations of the fundamental group of $\mathcal{Q}$.*

Let us make a few comments about this result.

- Since no spin is being considered, the wave function takes values on a one-dimensional vector space. This vector space might vary with the coordinates of the configuration space.
- We do not specify, for the moment being, the specific quantization approach (canonical, path integral, etc.). Since the only input is a configuration space, we can only face the kinematical part of the problem. Hence, inequivalent quantizations here refers to the different choices for the Hilbert space representation of the theory: There might be several ones, which fall in classes that could be termed "superselection sectors" and that can be put in bijective correspondence with topological invariants of the configuration space. A complete quantization must also include a classical dynamical principle as input.
- The results of Laidlaw-DeWitt and Leinaas-Myrheim can be regarded as a particular case of 1.1.1: Define, for a system of N particles in d spatial dimensions,

$$\widetilde{\mathcal{Q}} := \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{N-\text{times}} \backslash \triangle, \tag{1.1.1}$$

  where $\triangle = \{(x_1, \ldots, x_n) \,|\, x_i = x_j$, for at least some pair $(i,j), i \neq j\}$ denotes the set of configurations where two or more particles coincide. Since configurations differing only by a permutation of particles are considered the same, one has an equivalence relation $(x_1, \ldots, x_n) \sim (x_{\sigma(1)}, \ldots, x_{\sigma(N)})$ (with $\sigma \in S_N$ any permutation of the indices $\{1, \ldots, N\}$). The configuration spaces considered in references [LD71] and [LM77] are obtained from $\widetilde{\mathcal{Q}}$ by taking the quotient with respect to this equivalence relation:

$$\mathcal{Q} := \widetilde{\mathcal{Q}}/S_N. \tag{1.1.2}$$

  One can show that, for $d \geq 3$, $\pi_1(\mathcal{Q}) \cong S_N$, so that there are only two possibilities, corresponding to the two characters of S_N. For $d = 2$ it is well known that $\pi_1(\mathcal{Q})$ is isomorphic to Artin's braid group $\mathcal{B}_N$ [Art47]. This leads to new possibilities, *anyonic* statistics [Wil82]. For systems obeying this kind of statistics, the statistics parameter is not a sign, as in the Fermi or Bose cases, but a phase $e^{i\theta}$.

The application of theorem1.1.1 to the case of identical spinless particles is of particular relevance, since the imposition of a symmetrization postulate becomes, in this case, unnecessary: The two possible statistics, the fermionic and the bosonic one, arise as a direct consequence of the non-trivial topology of the configuration space and thus of indistinguishability. To prove the Spin-Statistics theorem in this particular case, it would be necessary to show that only bosonic statistics is allowed. An attempt to prove this, under some continuity assumptions for the wave function, has been carried out by Peshkin [Pes03b].

The possibility of using a result analogous to theorem 1.1.1 for non vanishing spin seems even more remote, mainly for two reasons. Firstly, whereas in the spinless case the quantum theory is obtained from a quantization procedure, for non-vanishing spin there is no classical theory to start with. One can adopt the point of view that the zero-spin case, for which quantization poses no problem, provides the justification for the use of vector bundles over the classical configuration space in order to define the wave function and then introduce spin by considering bundles with fibers with a dimension corresponding to the given value of the spin. But then, because the vector bundles involved in these cases are of higher rank, the restrictions imposed by the topology of the configuration space are not as strong as in the spin zero case. In fact, even a rigorous derivation of the Fermi-Bose alternative for nonzero spin is still lacking (some remarks and suggestions were already made in [LM77]). This is certainly a necessary step in order to, eventually, prove a Spin-Statistics theorem in this context.

The case of general spin has recently been considered by Berry and Robbins, who in [BR97] have provided an *explicit* construction, in which the quantum mechanics of two identical particles is formulated along the lines described in [LM77]. Let us explain in some detail the construction of [BR97] (henceforth referred to as the BR-construction), in order to point out some features that have served as the starting point for the present work and, at the same time, in order to motivate the approach that will be developed in the following chapters.

The BR approach is motivated, in part, by the idea that there is an unsatisfactory feature in the usual treatment of indistinguishability in non relativistic QM, namely, that the wave function of a system of identical fermions is not single-valued, in the sense that under the exchange of two particles it changes its sign. Since, because of indistinguishability, this exchange has no physical meaning, there should be no change at all. This amounts to the requirement that the wave function should be defined on a configuration space where indistinguishability is already incorporated, i.e., the space $\mathcal{Q}$ of eqn. (1.1.2). Although the idea of working with wave functions defined on this quotient space is implicit in their work, Berry and Robbins' construction is really carried out on $\widetilde{\mathcal{Q}}$ (as defined in eq. (1.1.1)), a configuration space of distinguishable particles. In this way, they avoid the use of mathematical concepts that would be necessary otherwise.

The construction was originally proposed for the special case of two particles of spin S. The basic idea consists in replacing the usual spin states $|s, m_1\rangle \otimes |s, m_2\rangle$, by position-dependent ones: $|s, m_1\rangle \otimes |s, m_2\rangle(r_1, r_2)$. This is the only way one can impose single-valuedness (in the sense explained below, see equation (1.1.5)) and still retain Pauli's exclusion principle.

Adopting the notation "M" for the quantum numbers $\{m_1, m_2\}$ (in that order) and

"$\overline{M}$" for $\{m_2, m_1\}$, we can express the exchange of spin states simply as $|M\rangle \to |\overline{M}\rangle$. Since the exchange of particles does not affect the center of mass position vector, one can reduce the dependence of the new spin vectors $|M(r_1, r_2)\rangle$ on the position vectors to only the relative position one, $r = r_1 - r_2$. Thus, under exchange of both *positions* and *spins* of the two particles, we have: $|M\rangle(r) \to |\overline{M}(-r)\rangle$.

The position-dependent spin basis is obtained by means of a unitary operator $U = U(r)$ that acts on a vector space which contains the usual (fixed) spin vectors. The dimension of this space is equal to $N_S = \frac{1}{6}(4S+1)(4S+2)(4S+3)$, for reasons that are specific to the construction: It is based on Schwinger's representation of spin, which uses raising and lowering operators. Leaving aside for the moment the technical details of the construction, let us concentrate on the basic properties of the spin basis resulting from it:

1.1.2 Definition. (Transported Spin Basis).

(i) The map
$$\begin{aligned} S^2 &\longrightarrow \mathbb{C}^{N_S} \\ r &\longmapsto |M(r)\rangle := U(r)|M\rangle \end{aligned}$$
is well defined and smooth for all M.

(ii) The following *"exchange"* rule holds:
$$|\overline{M}(-r)\rangle = (-1)^{2S}|M(r)\rangle. \tag{1.1.3}$$

(iii) The *"parallel transport"* condition $\langle M'(r(t))|\frac{d}{dt}M(r(t))\rangle = 0$ is satisfied for all M and M', and for every smooth curve $t \mapsto r(t)$.

The two-particle wave function is then written, in terms of the new spin basis, as follows:
$$|\Psi(r)\rangle = \sum_M \Psi_M(r)|M(r)\rangle. \tag{1.1.4}$$

1.1.3 Single-valuedness of the wave function. *Essential to the BR approach is the imposition of single-valuedness on the wave function, through the following condition:*
$$|\Psi(r)\rangle \stackrel{!}{=} |\Psi(-r)\rangle. \tag{1.1.5}$$

Let us point out some relevant features:

- The existence, for each S, of a basis satisfying the conditions stated in def.1.1.2 is not obvious at all. One advantage of its construction by means of Schwinger oscillators is that it works for all values of S. But its generalization to more than two particles poses technical difficulties, and leads to non-trivial mathematical questions; among them, a conjecture of Atiyah which has received considerable attention (see [AB02] and references therein).

- In the original proposal of BR, the sign in eq. (1.1.3) was assumed to be of the form $(-1)^K$ and the conjecture was made that any basis satisfying the three properties stated in definition 1.1.2 would have to satisfy $(-1)^K = (-1)^{2S}$. The assertion of this conjecture was later found not to be true. Counterexamples have been given in [BR00].

- In any case, for 2 particles the BR construction, using Schwinger's oscillators model, gives the correct sign, and this for every value of S. It is therefore an interesting matter to try to find the reason (in case there is one) for this.

- A direct consequence from eq. (1.1.5), when the spin vectors satisfy conditions (i)-(iii) from definition 1.1.2, is the relation $\Psi_{\overline{M}}(-r) = (-1)^{2S}\Psi_M(r)$, between the coefficient functions. This could, in principle, be interpreted as the usual form of the Spin-Statistics relation †, but only if the conjecture about the sign in eq. (1.1.3) had turned out to be true.

1.2 On the present work

Having discussed the basic ideas that serve as motivation for the present thesis, we now turn to a brief description of the method and techniques that have been used in our study of the Spin-Statistics problem.

In the last years, the BR proposal has given place to a renewed discussion of the Spin-Statistics problem, perhaps due to the fact that, in contrast to other approaches, what they present is a very *concrete* model which can be computed and also to the fact that the construction gave the correct connection between Spin and Statistics. Nevertheless, the existence of alternative constructions and the use of Schwinger's oscillators in the original one tend to obscure the real meaning of a "transported" spin basis. It would therefore be interesting to find a method which, while keeping the concreteness, also leads to a better understanding of the problem.

For instance, although the motivation for the imposition of single-valuedness as a way to include indistinguishability in the formalism is clear, its implementation by means of eq.(1.1.5) is not. Let us explain this in more detail.

In the BR construction, the wave function is implicitly considered as a section of a vector bundle, whose basis is supposed to be the physical configuration space $\mathcal{Q}$. As is well known, it is not possible in general to represent a section by means of a single function: The eventual non triviality of the bundle where it is defined makes the use of *local trivializations* necessary. A consequence of this is that, in order to recover a section as a whole, one needs several functions, one for each trivializing neighborhood. Thus, at every point on the basis manifold, the section will take (apparently) as many

†First one has to show that these functions satisfy the same differential equation as the usual ones, see [BR97]

values as there are trivializing neighborhoods covering that point. It should be emphasized that this "multiple-valuedness" is only something apparent. A section is -in the same way as a function- a map, which to each point on the basis manifold assigns a *unique* value on a vector space. The difference with a section is that the vector space might be different for every point. Now, as shown in this work, the equivalence between projective modules and vector bundles [Ser58, Swa62] can be used as a tool in order to obtain an algebraic characterization of certain vector bundles related to the configuration spaces $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$. More concretely, it will be shown that it is possible to represent a *section* defined on a bundle over $\mathcal{Q}$ as a (possibly vector-valued) *function* defined on $\widetilde{\mathcal{Q}}$. Since the resulting function is not anymore defined on the physically correct configuration space, its value now depends on the ordering of the particles and it is not necessarily invariant under exchange of them (it must, though, be equivariant). Thus, the imposition of strict invariance under exchange of particles for a wave function defined on $\tilde{\mathcal{Q}}$ as a vector-valued function seems not to be the most adequate form to incorporate indistinguishability into the theory. This point will be worked out in detail in chapter 5, where a comparison with the Berry-Robbins approach will be made.

In this work (some of which results can be found in [PPRS04]), we will therefore study the Spin-Statistics problem by making use of (some notions of) the theory of projective modules as an alternative, equivalent description of vector bundles. While from the mathematical point of view the descriptions of vector bundles in the usual differential-geometric language and in terms of projective modules are completely equivalent (as asserted by the well known Serre-Swan theorem[Ser58, Swa62]) from the point of view of our purposes, a description in terms of modules will be, at some points, much more convenient. The proposal to study the Spin-Statistics problem using projective modules was made in [Pas01]. In the present thesis, it will be further developed, so as to be in a position to study the case of an arbitrary number of particles. The consideration of the symmetries of the problem will also play an important role in this work. In this respect, an approach in the spirit of [HPRS87] will be followed, mainly by consideration of the different group actions involved.

In order to illustrate how these concepts can be used to study the Spin-Statistics problem, let us consider the spin basis of the BR-construction, in the special case of $S = 1/2$. There are, in this case, four physical, position dependent spin states, that are obtained by application of an unitary operator $U(r)$ to the fixed ones. The resulting spin states can be written as a linear combination of ten vectors $|e_1\rangle, \ldots, |e_{10}\rangle$. The first four coincide with the fixed spin states,

$$
\begin{aligned}
|+,+\rangle &= |e_1\rangle, \\
|-,-\rangle &= |e_2\rangle, \\
|+,-\rangle &= |e_3\rangle, \\
|-,+\rangle &= |e_4\rangle.
\end{aligned}
$$

For the transported spin states, one finds:

$$\begin{aligned}
|+,+(r)\rangle &:= U(r)|+,+\rangle = -e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}|e_7\rangle + \cos\theta|e_1\rangle + e^{i\phi}\frac{\sin\theta}{\sqrt{2}}|e_9\rangle \\
|-,-(r)\rangle &:= U(r)|-,-\rangle = -e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}|e_8\rangle + \cos\theta|e_2\rangle + e^{i\phi}\frac{\sin\theta}{\sqrt{2}}|e_{10}\rangle \\
|+,-(r)\rangle &:= U(r)|+,-\rangle = -e^{-i\phi}\frac{\sin\theta}{2}|e_5\rangle + \cos^2\frac{\theta}{2}|e_3\rangle - \sin^2\frac{\theta}{2}|e_4\rangle + e^{i\phi}\frac{\sin\theta}{2}|e_6\rangle \\
|-,+(r)\rangle &:= U(r)|-,+\rangle = -e^{-i\phi}\frac{\sin\theta}{2}|e_5\rangle + \cos^2\frac{\theta}{2}|e_4\rangle - \sin^2\frac{\theta}{2}|e_3\rangle + e^{i\phi}\frac{\sin\theta}{2}|e_6\rangle
\end{aligned} \tag{1.2.1}$$

Here, θ and ϕ are the polar angles of the relative position vector r. The fixed spin basis is chosen so that, at the north pole (r_0), the following holds: $|m_1, m_2(r_0)\rangle = |m_1, m_2\rangle$.

The spin basis vector $|m_1, m_2(r)\rangle$ at position r is obtained from the application of $U(r)$ to the vector $|m_1, m_2\rangle$, but an alternative point of view is that this same vector is obtained by means of an operator that, at each point r, projects the space spanned by all the $|e_i\rangle$ onto the subspace generated by $|m_1, m_2(r)\rangle$. Such an operator can be easily constructed using $U(r)$: Starting at the point r one applies $U(r)^{-1}$ to go to $r = r_0$, there one projects to $|m_1, m_2\rangle$ and finally returns back to r by applying $U(r)$.

The meaning of the basis vectors eq. (1.2.1), obtained in BR with the help of the Schwinger construction, apart from the fact that they satisfy the properties stated in definition 1.1.2, is not so clear. But, as shown in chapter 4, one can transform to the basis of total angular momentum and then define the four projection operators (corresponding to the singlet and the triplet states) as described above. One then obtains the following result: The singlet state does not depend on r and the projectors corresponding to each triplet state are all equal and can be written, in matrix form, as follows:

$$p(r) = \begin{pmatrix} \frac{1}{2}\sin^2\theta & -\frac{1}{\sqrt{2}}\sin\theta\cos\theta e^{-i\varphi} & -\frac{1}{2}\sin^2\theta e^{-2i\varphi} \\ -\frac{1}{\sqrt{2}}\sin\theta\cos\theta e^{i\varphi} & \cos^2\theta & \frac{1}{\sqrt{2}}\sin\theta\cos\theta e^{-i\varphi} \\ -\frac{1}{2}\sin^2\theta e^{2i\varphi} & \frac{1}{\sqrt{2}}\sin\theta\cos\theta e^{i\varphi} & \frac{1}{2}\sin^2\theta \end{pmatrix}. \tag{1.2.2}$$

The advantages of expressing the spin basis in terms of eq. (1.2.2) will be evident later, but for now let us only state the fact that, as a projective module over the ring of all continuous, even functions on the two-sphere, this projector gives place to a module which is isomorphic to the module of odd functions on the sphere, over the same ring. It has been shown in [Pas01] that this isomorphism is a direct consequence of the $SU(2)$ symmetry of the two-sphere. This shows that this is an alternative way to recover the BR-construction, but one which *(i)* clarifies the role played by the $SU(2)$ symmetry of the problem and *(ii)* uses this same symmetry to systematically construct a projective module, that turns out to reproduce exactly the formulae for the transported spin basis in the BR-construction.

Furthermore, such features of the spin basis as the parallel transport condition, can be deduced in this context very easily: The equivalent assertion is that the expression

$pdpdp$ vanishes or, in other words, the bundle corresponding to the projector p is *flat*. Additional to the $SU(2)$ symmetry, there is also an exchange symmetry, as a consequence of indistinguishability. In the case of two particles, this can be easily handled; but for a finite, arbitrary number of particles, the situation changes. As will be shown in this work, the approach using the language of projective modules allows for a very clear treatment of the general case.

We finish this section with a description of the content of the present thesis.

In Chapter 2 (Quantization on multiply-connected configuration spaces), a review of some of the basic facts about Quantum Mechanics of spinless particles on multiply-connected configuration spaces is made, based on the standard literature on the topic. The main purpose of this chapter is to present the theoretical background which is the starting point of the problem treated in this thesis. Although all of the results presented in this chapter are well known, they are of much relevance, since they provide well-founded explanations for many of the assumptions that are made in the study of the Spin-Statistics problem, specially the *physical* reasons why such mathematical structures as vector bundles and connections must be used.

Chapter 3 (G-spaces and projective modules) is devoted to the deduction of some mathematical results that will be needed in the following chapters. The case of a G-free space M (with G a finite group) and $G-$bundles over it is considered. The known equivalence of $G-$bundles over M and bundles over M/G is reformulated in a way that allows a useful interpretation when the same problem is formulated in terms of projective modules. It is shown that, given a vector bundle ξ over M/G, the module of sections $\Gamma(\xi)$ is isomorphic to the submodule of invariant sections of the pull-back of ξ. An alternative version of this result in terms of only the underlying algebras $C(M)$ and $C(M/G)$ is worked out in last part of the chapter, where a decomposition of $C(M)$ into $C(M/G)$-submodules is obtained, that allows to represent sections of bundles over M/G as functions on M.

Chapter 4 (The spin zero case) is devoted to the discussion of two examples (using the sphere and the projective space), where the methods of chapter 3 are illustrated. These examples are used in the last section of the chapter in order to discuss the case of two spin zero particles.

In Chapter 5 (Applications to the Berry-Robbins approach to Spin-Statistics), the methods developed in the previous chapters are applied in an analysis of the Berry-Robbins construction. This analysis leads to the conclusion that the single-valuedness condition, as stated in [BR97] is inconsistent.

Chapter 6 (Further developments) contains a proposal, based on the techniques developed in Chapter 3 and on the conclusions drawn from the analysis in Chapter 5, to study the Spin-Statistics problem.

2 Quantization on multiply-connected configuration spaces

Quantum indistinguishability forces us to consider configuration spaces that have non-trivial topologies. If the states of the quantum theory are represented as functions on these configuration spaces, the problem on quantization on multiply connected space immediately arises. In the present chapter, of motivational character, we will review one of the possible ways in which one can implement a quantization program in such spaces. For us, the relevance of the method reviewed in the present chapter is that it provides a firm conceptual basis that supports the assumptions we will make in the following part of the work.

2.1 Motivation

Having its origin in the fact that in Quantum Mechanics physical (pure) states are represented not by vectors in Hilbert space but by *rays* on it, the *gauge freedom* of the quantum mechanical wave function leads very naturally to questions of a topological nature. Consider a wave function $\psi : \mathcal{Q} \to \mathbb{C}$ defined on some classical configuration space $\mathcal{Q}$. Since it is only to $|\psi(q)|^2$ -the probability density- that a physical meaning is attached, we are always free to modify the wave function by adding a global phase factor, $\psi(q) \mapsto e^{i\theta}\psi(q)$, without changing the physical predictions. A more general kind of transformation is indeed allowed: a local one, of the form $\psi(q) \mapsto e^{i\varphi(q)}\psi(q)$.

The question, as to what extent is it possible to make a globally well defined choice of phase, i.e., of a function $\mathcal{Q} \to \mathbb{C} : q \mapsto e^{i\varphi(q)}$, was already considered by Pauli [Pau39]. He noticed that this is possible whenever $\mathcal{Q}$ is simply connected.

Phase ambiguities do also show up in the path integral formulation of Quantum Mechanics [FH65], where they arise as a consequence of the gauge freedom of the Lagrangian. Consider a classical system defined on a configuration space $\mathcal{Q}$ and described by a Lagrangian $L(q, \dot{q}; t)$. A gauge transformation of the Lagrangian, of the form

$$L(q, \dot{q}; t) \mapsto L'(q, \dot{q}; t) = L(q, \dot{q}; t) + \frac{d}{dt}M(q, t)$$

preserves, at the classical level, the equations of motion. Its effect on the amplitude

$$K(q^{(b)}, t^{(b)}; q^{(a)}, t^{(a)}) = \int \mathcal{D}[q] e^{\frac{i}{\hbar}S[q]}$$

is just a global phase change:

$$K(q^{(b)},t^{(b)};q^{(a)},t^{(a)}) \mapsto K'(q^{(b)},t^{(b)};q^{(a)},t^{(a)}) = e^{\frac{i}{\hbar}\phi(q^{(b)},t^{(b)};q^{(a)},t^{(a)})}K(q^{(b)},t^{(b)};q^{(a)},t^{(a)})$$

From $S[q] = \int_{t^{(a)}}^{t^{(b)}} dt\, L(q,\dot{q};t)$ one readily checks that $\phi(q^{(b)},t^{(b)};q^{(a)},t^{(a)}) = M(q^{(b)},t^{(b)}) - M(q^{(a)},t^{(a)})$, that is, the phase depends only on the extremal points, which are held fixed throughout. The situation changes drastically if $\mathcal{Q}$ happens to be non-simply connected [LD71] for, in that case, the phase change in the integrand, $\exp(i/\hbar S[q])$, depends on the *homotopy class* to which the given path $q(t)$ belongs. This means that a gauge transformation of the Lagrangian introduces relative phases between the different contributions to the amplitude $K(q^{(b)},t^{(b)};q^{(a)},t^{(a)})$, making it ill-defined.

The following example illustrates this point very clearly. Consider the motion of a free particle on a two dimensional plane with a "hole" on it:

$$\mathcal{Q} = \mathbb{R}^2 \setminus \{0\}; \quad L(q,\dot{q};t) = \frac{1}{2}\dot{q}^2.$$

Consider now the function

$$\begin{aligned} F : \mathcal{Q} \quad &\rightarrow \quad \mathbb{R}^2 \\ q = \quad &\mapsto \quad (F_1(q),F_2(q)) = \frac{1}{2}\left(\frac{-q_2}{q_1^2+q_2^2}, \frac{q_1}{q_1^2+q_2^2}\right). \end{aligned}$$

Since $\partial_1 F_2 = \partial_2 F_1$, it follows that

$$L(q,\dot{q};t) \mapsto L'(q,\dot{q};t) = L(q,\dot{q};t) + F(q)\cdot\dot{q}$$

is a gauge transformation. Consider now the following family of paths, labelled by $n \in \mathbb{Z}$:

$$\gamma_n(t) = (\cos(2n\pi t), \sin(2n\pi t))\,.$$

(The path γ_n is a representative of the class in the fundamental group of $\mathcal{Q}$ which is labelled by n. Thus, paths corresponding to different values of n are not deformable (by means of a homotopy) into each other). With $q^{(a)} = 0 = q^{(b)}$, $t^{(a)} = 0$ and $t^{(b)} = 1$, we obtain:

$$e^{\frac{i}{\hbar}S'[\gamma_n]} = e^{\frac{i}{\hbar}n\pi}e^{\frac{i}{\hbar}S[\gamma_n]}.$$

We therefore see how paths belonging to different homotopy classes induce different phase changes in the integrand of the path integral, under the *same* gauge transformation of the Lagrangian.

This new feature of the path integral for non-simply connected spaces was first pointed out by Schulman[Sch68]. A more general treatment was later on presented by Laidlaw and DeWitt[LD71], where they showed how the non-trivial topology of the configuration space leads to different, inequivalent quantizations of the same classical system.

Very roughly, the idea of their proof is as follows. Since, as we have just seen, paths belonging to different homotopy classes cannot be included in the path integral at the same time, only partial amplitudes K^α containing sums over paths belonging to the same homotopy class (labeled by the element α of the fundamental group $\pi_1(\mathcal{Q})$*) can be consistently defined. In order to include all paths one then adds the different amplitudes K^α, but there is no *a priori* reason why this sum might not be a weighted sum, of the form

$$K = \sum_{\alpha \in \pi_1(\mathcal{Q})} \chi(\alpha) K^\alpha. \tag{2.1.1}$$

The form of the weight factors $\chi(\alpha)$ is determined from certain -rather technical- properties that the partial propagators can be shown to have and that we will not discuss here (for details, see [LD71] and [Sch68]).

However, it is important to point out that physical considerations play an essential role in the determination of the weight factors. For example, the consistency requirement

$$|K(c, t_c; a, t_a)| = \left| \int K(c, t_c; b, t_b) K(b, t_b; a, t_a) db \right|,$$

imposed on the total propagator, has a clear physical meaning and is used in the derivation of the final result, which can be stated as follows:

2.1.1 Theorem (cf.[LD71])**.** *The weight factors $\chi(\alpha)$ in eq. (2.1.1) must form a one-dimensional unitary representation of the fundamental group.*

An immediate consequence of this result is that there are as a many possible quantizations of the classical system described by L on $\mathcal{Q}$, as there are characters of $\pi_1(\mathcal{Q})$. Although not clear at this point, these quantizations are inequivalent for different choices of the character χ.

A particularly relevant application of this theorem is obtained for a system of identical particles: Consider n identical particles moving in three dimensional Euclidean space. If the configuration space is, as described in the first chapter, taken to be

$$\mathcal{Q} = (\mathbb{R}^3 \times \cdots \times \mathbb{R}^3 \setminus \Delta)/S_N,$$

where S_N denotes the permutation group, then there are exactly two characters, corresponding to the completely symmetric/antisymmetric one dimensional representations, and hence to Bose/Fermi statistics.

In spite of its applicability being restricted to systems of *spinless* particles, this is indeed a very strong result: It says that, if we take indistinguishability properly into account, there is no need for a symmetrization postulate since, in three dimensions, the two physically observed statistics, fermionic or bosonic, are the only allowed ones.

*Here we are considering paths beginning at some point a and ending at some point b in $\mathcal{Q}$. In order to assign an element of $\pi_1(\mathcal{Q})$ to each such path, an arbitrary point $x_0 \in \mathcal{Q}$ is chosen, as well as an assignment of paths ("homotopy mesh"[LD71]) $C(y)$ going from y to x_0, for every y in $\mathcal{Q}$. In this way, a path γ going from a to b, can be assigned the map $C(a)\gamma C^{-1}(b)$, which belongs to a given element of $\pi_1(\mathcal{Q}, x_0)$

At the beginning of this section, we started our discussion with questions related to the problem of fixing the phase of the wave function globally. This is a question that can be more clearly formulated using the notion of fiber bundle. Since the phase is (at least locally) a $U(1)$-valued function, one can consider it as a section of a principal $U(1)$-bundle. Equivalently, one can consider a line bundle associated to this $U(1)$-bundle, and take the wave function to be a cross-section of this line bundle. In this context, different quantizations arise, among other things, from the choice of the bundle where the wave function is defined. The relation of a fiber bundle approach to the Feynman functional one we have just discussed is not clear at all, but we shall see, in the next sections, how the conclusion of theorem 2.1.1 might be obtained from a quantization scheme that is based on the action of a so-called *canonical group* on the classical phase space. Within this approach, the quantum theory is obtained from the representation theory of the canonical group, and this in turn leads naturally to a fiber bundle formulation.

The Feynman functional approach has the advantage of providing a very clear picture of why the topology of the configuration space of a classical system might be seen reflected in the corresponding quantized theory: In the sum over "histories" prescription, all paths between two given points give contributions to the probability amplitude, so that they provide information on the global structure of the configuration space. On the other hand, the approach presented in the next sections deals with the difficulties that arise when one attempts to apply a canonical quantization prescription to a system with a configuration space that has a non-trivial topology (in the sense of being, e.g., multiply connected). Then, the need to use fiber bundles and representation theory arises in a very natural way, providing a justification, from the point of view of physics, of the setting on which the present work is based.

2.2 Canonical Quantization from Group Actions

2.2.1 Preliminary remarks

We have discussed the effects that the global structure of a classical configuration space may have on the quantum theory obtained from a classical Lagrangian by means of Feynman's path integral formulation. As already pointed out, this approach is particularly useful in order to get a "feeling" of why -if at all- the topology of the configuration space should play a role in the quantization of the system. For us, the interest in these kind of "topological effects" lies in the fact that, as illustrated in the previous section, for a system of N identical spinless particles, the Fermi-Bose alternative follows directly as a restriction imposed by the topology of the configuration space. The topological non-triviality of this space, expressed by the fact that its fundamental group is isomorphic to the permutation group of N elements, determines the allowed statistics without the need to consider a symmetrization postulate.

Whether the fact that the topology of this configuration space is not trivial might be used to derive the Spin-Statistics relation, has been (and still is) a source of controversy. Many authors are of the opinion that there is no way at all of relating spin with statistics, unless one goes over to relativistic quantum field theory (see, for example, [Wig00]). Perhaps this is true: Much effort has been put in the search for a non-relativistic proof and, as time went by, it became clearer that there is some *essential* physical assumption that is still missing. This could be, for example, some kind of condition that must be imposed in the non-relativistic theory, arising as a "shadow" or "remanent" of properties inherent to the relativistic quantum theory. If such a condition could be found and consistently incorporated into a configuration space (topological) approach in order to obtain a non relativistic version of the Spin-Statistics Theorem, we could certainly learn more about the relativistic case. Another interesting possibility is that topological spaces such as the one defined in eq.(1.1.2) or generalizations thereof might appear in a natural way in a Quantum Field Theory. Ideas along this line have been put forward by Tscheuschner [Tsc89] and Balachandran et. al. [BDG+90, BDG+93]. A full mathematical understanding of the very interesting (but to a great extent heuristic) ideas presented by these authors would justify by itself further studies within a configuration space approach.

In spite of all this, we want to stick to the point of view that the configuration space defined in eq.(1.1.2) *does* have something to do with the relation between spin and statistics. The reasons that -to our opinion- justify our assuming this point of view, are the following:

- **Relevance of the structure of classical theories for the formulation and understanding of quantum ones:**

  Although nowadays there exist approaches to Quantum Theory which are not based on a quantization of some underlying classical theory, both the historical and conceptual relevance of the structure of the classical mechanics (of particles and fields) for the formulation of Quantum Theory cannot be overseen. Fundamental structures and concepts, like the symplectic structure of phase space, the Poisson bracket algebra or Noether's theorem have played (and still play) a primordial role as guidelines in the search for a Quantum Theory in many branches of physics. Additionally, there are many situations where all one has to start with is a classical theory that is believed to be a limit of some, more fundamental, quantum theory. Needless to say, two prominent examples of this situation are the relations between Maxwell's theory of electromagnetism and Quantum Electrodynamics on one hand and between General Relativity and its sought-after quantum version, on the other. In the same way, one expects some kind of relation between the classical mechanics of a system of identical particles and its quantum version.

- **The notion of space in quantum mechanics:** In most physical theories, space-time plays a passive role, in the sense that it is the "stage" where all physical phenomena take place. The space-time of every theory has a symmetry group. The dynamical rules of a theory must be formulated in such a way that they are compatible with the symmetry group, but space-time itself does not play a dynamical role†. In this sense, one could say that space-time is "what remains when one takes away all particles, fields, etc..". From this point of view, in quantum mechanics, when one considers one particle, space is what remains after removing this particle: $\mathbb{R}^3$. But, when considering two particles, the structure of the "space" that remains after taking away the particles should be expected to be very different from the one corresponding to a single particle. This because of indistinguishability. The configuration space of eq.(1.1.2) may be seen as the mathematical expression of this idea.

- **Gibbs' paradox:** This was one of the main motivations for Leinaas and Myrheim [LM77] in order to study the configuration space eq.(1.1.2). Gibbs' paradox shows that, *even if we remain at a classical level* ‡, when asking questions about the microscopic properties of a system of identical particles, the configuration space eq.(1.1.2) must be used, in order to avoid inconsistencies.

- **Relevance of the Spin-Statistics relation in the non-relativistic context:** As pointed out in the introduction, the observed relation between Spin and Statistics has many consequences and is crucial for the understanding of many physical phenomena that take place in a quantum, but non-relativistic, domain. Non-relativistic Quantum Mechanics is a theory that can be consistently formulated without the need of concepts of relativity. On the other hand, the only (generally accepted) proofs of the Spin-Statistics Theorem make use of Poincaré invariance in an essential way. There remains the question of why must one make use in non-relativistic Quantum Mechanics, a theory standing on its own, of a result that has been proved somewhere else?

- **Relation to relativistic QFT:** Even if at the end it turns out that there is no way of finding a non-relativistic proof of the Spin-Statistics Theorem, the techniques developed in such a search could eventually be used to have a different look at the problem such as an interpretation of the relativistic, analytical proofs in geometric/topological terms. It is in part because of this possibility that the configuration approach to indistinguishability proposed in the present work, which is of a geometric/topological nature, has been also formulated in a more algebraic language, one that lies closer to the one of Quantum Field Theory.

Having stated the reasons why we are interested in studying the effects that a configuration space as eq.(1.1.2) might have on the Quantum Theory, we now turn to the central theme of this chapter: Quantization on a multiply-connected configuration space. Assume we are given a classical configuration space $\mathcal{Q}$ with a non-trivial global structure (the specific example of interest for us is that of $\mathcal{Q}$ being multiply-connected).

†General Relativity being, of course, an exception.

‡With this we mean: Even if the Maxwell-Boltzmann distribution is being used, see [LM77].

We then try to construct a quantum theory, having as starting point a classical theory based on $\mathcal{Q}$. When such an attempt is made, one encounters very soon problems with the application of the usual "quantization rules" that work so well in the case of $\mathcal{Q} = \mathbb{R}^n$. The aim of the brief account to quantization that we present in this chapter is to provide well-founded arguments in favor of the idea that, when dealing with a globally-structured configuration space $\mathcal{Q}$, some of the usual assumptions that are made in the case of $\mathbb{R}^n$, have to be reconsidered, this leading to the need to consider a more general setting for the quantum theory. This more general setting will be the starting point for our considerations about the Spin-Statistics relation.

Among the assumptions we alluded to above, we have:

(1.) The Canonical Commutation Relations (CCR),

$$\left[\hat{q}^i, \hat{p}_j\right] = i\hbar\delta^i_j, \quad \left[\hat{q}^i, \hat{q}^j\right] = 0 = \left[\hat{p}_i, \hat{p}_j\right], \tag{2.2.1}$$

hold.

(2.) A (Pure) state ψ is a *function* defined on $\mathcal{Q}$ and taking values on a complex vector space:

$$\psi : \mathcal{Q} \to \mathbb{C}^r.$$

The fact that the CCR cannot hold unrestricted for any configuration space is something plausible because, whereas in the case of $\mathcal{Q} = \mathbb{R}^n$ the position and momentum operators are the quantized versions of the *global* coordinates $\{q^1, \ldots, q^n; p_1, \ldots, p_n\}$ (defined for the phase space $T^*\mathbb{R}^n$), in a general configuration space the use of more than one local coordinate chart might be necessary, and then the need to define *global* position and momentum operators, or suitable analogs of them, arises. The kind of permutation relations that these new operators obey will then clearly depend on the global structure of $\mathcal{Q}$. The "breakdown" of the usual CCR has consequences in the way that pure states are realized. This is so because the space where these states "live" is a representation space of an abstract algebra whose generators obey the CCR. The position and momentum operators can then be seen as the operators representing the generators of this algebra. But if the algebra admits different, inequivalent representations, it could well happen that the representation space takes a form different from the usual one (a space of complex functions defined on $\mathcal{Q}$). That this is not the case for the familiar example of quantum theory on $\mathbb{R}^n$ is a direct consequence of the Stone-von Neumann theorem.

In some cases, the representation space where the state vectors are defined takes the form of a space of sections of some vector bundle over $\mathcal{Q}$. For a given classical theory, different, inequivalent quantizations may be possible, and this could be reflected in the fact that the corresponding state functions will be sections defined on topologically inequivalent vector bundles over $\mathcal{Q}$. It could also happen that two (inequivalent) representations take place on the space of sections of the *same* bundle, but that there is some parameter that singles out the representations as being inequivalent. One way in which this may happen is in the choice of a certain *flat connection* that makes its appearance in the process of quantization.

The remarks above will be illustrated in the remaining sections of this chapter, following an approach to quantization where the relevance of group actions and its relation to topological effects in quantum theory are emphasized. The basic reference we have followed for this topic is the article by C.J. Isham, from the Les Houches session of 1983 [Ish84]. Other useful references are [Mor92, HMS89]. There are, of course, many approaches to quantization, from quite different points of view. One of them is, for example, geometric quantization [Sou69, Woo80]. More recent approaches include, for example, deformation quantization [BFF$^+$78, Fed94]. But, for our purposes, the very readable account of Isham will suffice.

2.2.2 From CCR to group actions and back

A very nice way to see that the usual CCR cannot hold for every configuration space, is provided by quantum theory based on the positive real line, $\mathbb{R}_+$. Assume that the usual definition for the position and momentum operators can also be used in this case:

$$\begin{aligned} (\hat{x}\psi)(x) &= x\psi(x) \\ (\hat{p}\psi)(x) &= -i\hbar\frac{d}{dx}\psi(x). \end{aligned} \tag{2.2.2}$$

Let us assume that we can take as Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+, dx)$. Now, assume for a moment that the operator defined by eq. (2.2.2) is self-adjoint (it is not). If this were the case, then we could affirm that this operator is the infinitesimal generator of translations, i.e., with $U(a) := e^{-ia\hat{p}}$ we would obtain a unitary operator in $\mathcal{H}$ whose action on wave functions would be

$$(U(a)\psi)(x) = \psi(x - \hbar a). \tag{2.2.3}$$

But clearly this cannot be possible, since we could always choose a in such a way that the support of $U(a)\psi$ ends up lying outside $\mathbb{R}_+$. Therefore, the usual definition of position and momentum operators does not work for this space. If the basic CCR cannot be imposed on this configuration space, what kind of operators could then be defined on it, and in that case, what kind of commutation relations would they obey? In order to answer this questions, it is necessary to answer first another one: What is the “origin” of the usual CCR in the familiar case of quantum theory on $\mathbb{R}$ (or $\mathbb{R}^n$)? This question is to be interpreted in the following sense. As the above example suggests, the fact that the CCR cannot hold in a given configuration space $\mathcal{Q}$, has something to do with the structure of $\mathcal{Q}$. Note that the main difference between $\mathbb{R}$ and $\mathbb{R}_+$ is that, whereas the former is a vector space, the latter is not. So, the question about the “origin” of the CCR on $\mathbb{R}$ could be reformulated by asking: What are the properties of $\mathbb{R}$ that allow the CCR to hold?

One way to answer the question posed above is by considering the exponentiated (Weyl) form of the CCR. Defining the unitary operators

$$U(a) := e^{-ia\hat{p}}, \quad V(b) := e^{-ib\hat{x}}, \tag{2.2.4}$$

acting on $L^2(\mathbb{R}, dx)$, we obtain the following relations:

$$\begin{aligned} U(a_1)U(a_2) &= U(a_1 + a_2), \\ V(b_1)V(b_2) &= V(b_1 + b_2), \\ U(a)V(b) &= e^{i\hbar ab}V(b)U(a). \end{aligned} \tag{2.2.5}$$

These relations follow readily from the action of $U(a)$ and $V(b)$ on wave functions:

$$\begin{aligned} (U(a)\psi)(x) &= \psi(x - \hbar a), \\ (V(b)\psi)(x) &= e^{-ibx}\psi(x). \end{aligned} \tag{2.2.6}$$

The commutation relations eq.(2.2.5) suggest that we are dealing with a representation of some group. This is in fact true, and the underlying group is called the *Heisenberg Group*. It can be defined not only for $\mathbb{R}$ but also for $\mathbb{R}^n$. As a set, it is given by $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, with the group law

$$(a_1, b_1; r_1) \cdot (a_2, b_2; r_2) := (a_1 + a_2, b_1 + b_2; r_1 + r_2 + \frac{1}{2}(b_1 \cdot a_2 - b_2 \cdot a_1)). \tag{2.2.7}$$

From $(a, 0; 0)^{-1} = (-a, 0; 0)$ and $(0, b; 0)^{-1} = (0, -b; 0)$, one checks:

$$\begin{aligned} (a, 0; 0) \cdot (0, b; 0) \cdot (a, 0; 0)^{-1} \cdot (0, b; 0)^{-1} &= \\ &= (a, 0; 0) \cdot (0, b; 0) \cdot (-a, 0; 0) \cdot (0, -b; 0) \\ &= (a, b; -\frac{1}{2}ab) \cdot (-a, -b; -\frac{1}{2}ab) \\ &= (0, 0; -ab). \end{aligned}$$

Thus we see that, if we define a representation $\mathcal{R}$ of the Heisenberg group on the space $L^2(\mathbb{R}, dx)$ by

$$\begin{aligned} \mathcal{R}((a, 0; 0)) &:= U(a), \\ \mathcal{R}((0, b; 0)) &:= V(b), \\ \mathcal{R}((0, 0; r)) &:= e^{i\hbar r}\mathrm{id}, \end{aligned} \tag{2.2.8}$$

we can get the commutation relations eq. (2.2.5) from this representation of the Heisenberg group. In order to see how the Heisenberg group is related to the symplectic structure of the phase space $T^*\mathbb{R}^n$, it is convenient to consider its action (through the operators U and V) on the position and momentum operators:

$$\begin{aligned} U(a)\hat{x}U(a)^{-1} &= \hat{x} - \hbar a, \\ V(b)\hat{p}V(b)^{-1} &= \hat{p} + \hbar b. \end{aligned} \tag{2.2.9}$$

These relations suggest that we consider the following transformation:

$$\begin{aligned}(\mathbb{R}^n \times \mathbb{R}^n) \times T^*\mathbb{R}^n &\rightarrow T^*\mathbb{R}^n \\ ((a,b),(q,p)) &\mapsto (q-a, p+b).\end{aligned} \tag{2.2.10}$$

If we see $\mathbb{R}^n \times \mathbb{R}^n$ as an additive group, then eq.(2.2.10) gives place to a group action on $T^*\mathbb{R}^n$. Although not clear at this point, starting from this group action one can arrive in a quite systematic way to the conclusion that the Heisenberg group provides the solution to the quantization problem on $\mathbb{R}^n$. For the moment, let us just mention that the action eq.(2.2.10) is symplectic, transitive and effective.

Summing up, what we have done until now is to realize that there is a group acting on the classical phase space, that this group is in a certain form related to the Heisenberg group and that the CCR (in their Weyl form) can be obtained from a representation of the Heisenberg group on $L^2(\mathbb{R}, dx)$.

The link between the classical and the quantum theory on R^n can be directly established at the infinitesimal level, by considering Dirac's quantization conditions, which include the replacement of classical observables f by operators $\hat{f}$ in such a way that the Poisson bracket of two observables is mapped to the commutator of the corresponding operators. In terms of the Heisenberg group, this can be formulated in the following way. The Lie algebra of the Heisenberg group, as a vector space, is given by $\mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R}$. The Lie bracket is

$$[(a_1, b_1; r_1), (a_2, b_2; r_2)] = (0, 0; b_1 \cdot a_2 - b_2 \cdot a_1). \tag{2.2.11}$$

This makes clear that the CCR are a representation of this Lie algebra. Consider now the subalgebra of the Poisson bracket algebra of T^*R^n given by the family of functions $\{P(a,b;c) \in C^\infty(T^*\mathbb{R}^n, \mathbb{R}) \mid a, b \in \mathbb{R}^n, c \in \mathbb{R}\}$, where

$$P(a,b;c) := \sum (a^i p_i + b_i q^i) + c. \tag{2.2.12}$$

This is a $(2n+1)$-dimensional subspace of $C^\infty(\mathbb{R}^n, \mathbb{R})$ and also a Lie subalgebra, because it is closed under the Poisson bracket:

$$\{P(a_1, b_1; r_1), P(a_2, b_2; r_2))\} = P(0, 0; b_1 \cdot a_2 - b_2 \cdot a_1). \tag{2.2.13}$$

This last relation allows one to formulate the above mentioned Dirac's quantization condition by means of the map

$$P(a,b;c) \mapsto \sum (a^i \hat{p}_i + b_i \hat{q}^i) + i\hbar c \,\mathrm{id}. \tag{2.2.14}$$

Note that from eqns. (2.2.11) and (2.2.13) it is clear that this map takes the Poisson bracket $\{q^i, p_j\} = \delta^i_j$ to $[\hat{q}^i, \hat{p}_j] = i\hbar\delta^i_j$.

This seems to make the consideration of the symplectic action of the additive group $\mathbb{R}^n \times \mathbb{R}^n$ on $T^*\mathbb{R}^n$ unnecessary. But, as explained below, in a generic case, the existence of a group action on phase space, fulfilling certain conditions, will be the starting point for the construction of the corresponding quantum theory.

With these remarks about the case of $T^*\mathbb{R}^n$ in mind, let us very briefly present the main ideas about the quantization scheme presented in [Ish84]. The starting point is a classical phase space $\mathcal{P}$, i.e., a symplectic manifold. In most examples this phase space is the one associated to a configuration space $\mathcal{Q}$: $\mathcal{P} = T^*\mathcal{Q}$.

There are two main stages into which the scheme can be divided.

(1) Find a finite dimensional Lie Group ($\mathcal{C}$) which is related (in a way to be specified) to a group ($\mathcal{G}$) of symplectic transformations of the phase space $\mathcal{P}$ and such that:

  (i) Its Lie algebra $\mathcal{L}(\mathcal{C})$ is a subalgebra of the Poisson bracket algebra $(C^\infty(\mathcal{P}, \mathbb{R}), \{\ ,\ \})$.

  (ii) $\mathcal{L}(\mathcal{C})$ is big enough to generate as many (classical) observables as possible.

(2) Study the unitary, irreducible representations of $\mathcal{C}$. The self-adjoint generators can be considered as the quantized versions of the corresponding classical observables.

As observed before, in the case of $\mathcal{Q} = \mathbb{R}^n$, the relation between the groups $\mathcal{C}$ (Heisenberg group) and $\mathcal{G}$ (the additive group $\mathbb{R}^n \times \mathbb{R}^n$) is not clear a priori. This is due to the existence of a certain obstruction (for details, see further below) that arises when one attempts to assign a classical observable (i.e. a function in $(C^\infty(\mathcal{P}, \mathbb{R}))$ to each element of $\mathcal{L}(\mathcal{G}))$ in such a way that the Lie bracket is preserved.

One of the reasons why the scheme consists in starting with a group that acts on $\mathcal{P}$ and then in trying to associate its Lie algebra with functions on $\mathcal{P}$ is that if this association can be constructed in the form of a Lie algebra isomorphism

$$P : \mathcal{L}(\mathcal{G}) \hookrightarrow C^\infty(\mathcal{P}, \mathbb{R}), \tag{2.2.15}$$

one can define a quantization map by fixing a representation U of the group and assigning to each function lying in the image of P the self-adjoint generator obtained from U by means of P^{-1}.

Whereas the existence of a map P having the desired properties is not something obvious, in the *opposite* direction, there is a natural way to implement such a procedure. Indeed, given a (finite dimensional) Lie subalgebra $\mathfrak{l}$ of $C^\infty(\mathcal{P}, \mathbb{R})$, consider the Hamiltonian vector field that a function $f \in \mathfrak{l}$ generates: ξ_f. This vector field gives place to a one-parameter subgroup, acting by symplectic transformations. If these vector fields are complete, their one-parameter subgroups will generate a group $\mathcal{G}$ of symplectic transformations and, if the mapping sending $\mathfrak{l}$ into the set of Hamiltonian vector fields is injective, we obtain a Lie algebra isomorphism $\mathfrak{l} \cong \mathcal{L}(\mathcal{G})$.

Thus, the idea is, starting with a group $\mathcal{G}$ of symplectic transformations, to find a kind of "inverse" to the map

$$\begin{aligned} \jmath : C^\infty(\mathcal{P}, \mathbb{R}) &\rightarrow \text{Ham VF}(\mathcal{P}) \\ f &\mapsto -\xi_f \end{aligned} \tag{2.2.16}$$

and then to pre-compose it with a map $\gamma : \mathcal{L}(\mathcal{G}) \rightarrow \text{Ham VF}(\mathcal{P})$. The map γ is naturally induced by the $\mathcal{G}$-action on $\mathcal{P}$, but in order that its image be restricted to only the set of *Hamiltonian* vector fields and that it be an isomorphism, some conditions must be imposed.

The situation can be summarized by saying that one looks for a map P that makes the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r] & C^\infty(\mathcal{P}, \mathbb{R}) \arrow[r, "\jmath"] & \text{HamVF}(\mathcal{P}) \arrow[r] & 0 \\ & & & \mathcal{L}(\mathcal{G}) \arrow[u, "\gamma"'] \arrow[ul, dashed, "P"] & \end{tikzcd}$$

The kernel of $\jmath$ consists of the set of constant functions on phase space, so the first row represents a short exact sequence. There are two main problems in relation to this diagram. The first one has to do with the conditions that should be imposed on the action in order that the map γ produces globally Hamiltonian vector fields. The second one comes from the requirement that the map P respects the Lie algebra structure. In order to understand this in some detail, let us consider, point by point, some of the more relevant features.

• <u>The map γ:</u>

Given a symplectic action of a Lie group on phase space, $\mathcal{G} \times \mathcal{P} \rightarrow \mathcal{P}$, one can construct a map $\gamma : \mathcal{L}(\mathcal{G}) \rightarrow \text{VF}(\mathcal{P})$ as follows. Given $A \in \mathcal{L}(\mathcal{G})$, consider the map sending $t \in \mathbb{R}$ to $e^{tA} \in \mathcal{G}$. Then, using the action, one can define a one-parameter subgroup of symplectic transformations on $\mathcal{P}$, given by

$$\phi_t^A(x) := e^{-tA} \cdot x \quad (x \in \mathcal{P}).$$

From this we obtain a vector field γ^A, defined through its action on functions:

$$\gamma_x^A(f) := \frac{d}{dt} f(\phi_t^A(x))|_{t=0}. \tag{2.2.17}$$

From this definition, it follows that the map $\gamma : A \mapsto \gamma^A$ is a Lie algebra homomorphism. It is clear that ϕ^A is the local flow of γ^A. But ϕ_t^A, being a symplectic transformation, preserves the symplectic form (ω), i.e. $\phi_t^{A*}\omega = \omega$. This, together with the relation $\frac{d}{dt}\phi_t^{A*}(\omega) = \phi_t^{A*}(L_{\gamma^A}\omega)$, implies that the Lie derivative $L_{\gamma^A}\omega$ vanishes. This is a necessary condition in order that γ^A be the Hamiltonian vector field of some function on phase space. But it is not sufficient. To see this, notice that from $L_{\gamma^A} = \imath_{\gamma^A} \circ d + d \circ \imath_{\gamma^A}$ and $d\omega = 0$, it follows that $\imath_{\gamma^A}\omega$ is a closed one-form. But the requirement that $\gamma^A = \xi_f$ for some function f is equivalent to the requirement that $\imath_{\gamma^A}\omega = df$. We thus arrive at the conclusion that a sufficient condition in order that the image of the map γ lies entirely in the set of Hamiltonian vector fields, Ham VF($\mathcal{P}$), is that every closed one-form on $\mathcal{P}$ be also exact. In other words, we must require that the first cohomology group of $\mathcal{P}$ vanishes. This imposes a restriction of a topological nature on the phase spaces $\mathcal{P}$ that can be quantized using the present approach. Finally, let us mention that a sufficient condition for the map γ to be injective is that the action be effective. (A less restrictive condition can be imposed, allowing one to replace the group $\mathcal{G}$ by a covering of it, see [Ish84]).

• The map P and the obstruction cocycle:

If the $\mathcal{G}$-action is effective, and if the first cohomology group of the phase space vanishes, we obtain an isomorphism γ from $\mathcal{L}(\mathcal{G})$ into the set of Hamiltonian vector fields. But in order to be able to assign an observable to each element of $\mathcal{L}(\mathcal{G})$, we need to find a suitable map $P : \mathcal{L}(\mathcal{G}) \to C^\infty(\mathcal{P})$. As indicated in the diagram above, the idea is that the diagram commutes, that is for $A \in \mathcal{L}(\mathcal{G})$ we want to have ($P^A \equiv P(A)$):

$$\gamma^A = -\xi_{P^A}. \tag{2.2.18}$$

This is a natural requirement, since, as we have observed, the map $\jmath$ assigns a Hamiltonian vector field to every function on phase space, in a natural way. Up to the isomorphism γ, we are now looking for an assignment in the opposite direction, and it is natural to expect that, if $\gamma(A) = \jmath(f)$, then $P(A) = f$ (at least up to a constant, since $\ker \jmath = \mathbb{R}$). The map P should also be a Lie algebra isomorphism. From the requirement eq.(2.2.18) and the fact that γ is a Lie algebra homomorphism, we obtain:

$$\xi_{P^{[A,B]}} = \xi_{\{P^A,P^B\}}. \tag{2.2.19}$$

But, since $\ker \jmath = \mathbb{R}$, the only thing we can conclude is that

$$z(A,B) := \{P^A, P^B\} - P^{[A,B]} \tag{2.2.20}$$

is a constant. There is some freedom in the choice of the map P: If P^A satisfies eq.(2.2.18), then the map P', defined by

$$P'^A := P^A + d(A), \tag{2.2.21}$$

where d belongs to the dual of $\mathcal{L}(\mathcal{G})$, is also linear and satisfies eq.(2.2.18). The problem is then reduced to find a suitable $d \in \mathcal{L}(\mathcal{G})^*$ in such a way that the constants $z(A,B)$ in eq.(2.2.20) vanish for all A, B.

Eq.(2.2.20) defines a real-valued 2-cocycle, that is, a map $z : \mathcal{L}(\mathcal{G}) \times \mathcal{L}(\mathcal{G}) \to \mathbb{R}$ satisfying

$$\begin{aligned} &z(A,B) = -z(B,A) \\ &z(A,[B,C]) + z(B,[C,A]) + z(C,[A,B]) = 0. \end{aligned}$$

If the cocycle z is of the form

$$z(A,B) = d([A,B]), \tag{2.2.22}$$

for some $d \in \mathcal{L}(\mathcal{G})^*$, it is called a 2-coboundary. In this case we obtain, for the map P' defined in eq.(2.2.21),

$$\begin{aligned}\{P'^A, P'^B\} - P'^{[A,B]} &= \{P^A, P^B\} - P^{[A,B]} - d([A,B]) \\ &= z(A,B) - d([A,B]) \\ &= 0.\end{aligned}$$

Note that every 2-coboundary is also a 2-cocycle. Defining an equivalence relation on the set of 2-cocycles by saying that two 2-cocycles are equivalent if they differ by a 2-coboundary, we obtain what is called the second cohomology group of the Lie algebra $\mathcal{L}(\mathcal{G})$ (with values in $\mathbb{R}$). We thus see that the group $\mathcal{G}$ and its action on $\mathcal{P}$ define a class in this cohomology group, and the map P with the desired properties can be defined if and only if this class vanishes. If the cocycle can be made to vanish, we obtain a Lie algebra isomorphism $A \mapsto P^A$. Choosing a representation of $\mathcal{G}$, we can quantize by assigning a self-adjoint generator to every observable $P^A \in C^\infty(\mathcal{P})$. But if the cocycle cannot be made to vanish, an additional step is necessary. The idea is to replace $\mathcal{L}(\mathcal{G})$ by a Lie algebra $\mathcal{E}$, in such a way that the resulting cocycle vanishes. The new algebra $\mathcal{E}$ will be what is known as a central extension (by $\mathbb{R}$) of $\mathcal{L}(\mathcal{G})$. $\mathcal{E}$ is defined on $\mathcal{L}(\mathcal{G}) \oplus \mathbb{R}$, with Lie bracket

$$[(A,r),(B,s)] = ([A,B], z(A,B)). \tag{2.2.23}$$

Using the natural homomorphism

$$\begin{aligned}\beta : \mathcal{E} &\to \mathcal{L}(\mathcal{G}) \\ (A,r) &\mapsto A,\end{aligned}$$

we can replace the old map $P : \mathcal{L}(\mathcal{G}) \to C^\infty(\mathcal{P},\mathbb{R})$ by a new one $\tilde{P} : \mathcal{E} \to C^\infty(\mathcal{P},\mathbb{R})$, defined by

$$\tilde{P}^{(A,r)} := P^A + r. \tag{2.2.24}$$

This new map provides a solution to our problem. Indeed, from eqns.(2.2.23) and (2.2.24), it follows that

$$\{\tilde{P}^{(A,r)}, \tilde{P}^{(B,s)}\} = \tilde{P}^{[(A,r),(B,s)]}.$$

From this new Lie algebra we obtain a Lie group, $\mathcal{C}$ (the *canonical group*, in the terminology of [Ish84]), whose unitary representations can be used to obtain a quantization of the system.

• Transitivity of the $\mathcal{G}$-action:

A transitive group action will guarantee that sufficiently many observables (functions on phase space) can be quantized. There are other requirements that can be imposed instead of transitivity, but this one is reasonable and works well in many known examples. A particular consequence of this condition is that the vector field γ^A will always span the tangent of phase space at every point (see [Ish84], section 4.3.4).

2.2.3 Representations of the canonical group

When the phase space $\mathcal{P}$ is the cotangent bundle of a configuration space $\mathcal{Q}$, there are two natural classes of transformations that one can consider. The first class is the one induced by the diffeomorphism group of $\mathcal{Q}$ on $T^*\mathcal{Q}$: If $\phi \in \mathrm{Diff}\mathcal{Q}$, then using the pull-back of ϕ we obtain a transformation ($l \in T^*_q\mathcal{Q}$ and $v \in T^*_q\mathcal{Q}$) defined by:

$$\langle \phi^* l, v\rangle_q = \langle l, \phi_* v\rangle_{\phi(q)}. \tag{2.2.25}$$

The map $l \mapsto \phi^* l$ is, in particular, a symplectic transformation on $T^*\mathcal{Q}$. Thus, subgroups of $\mathrm{Diff}\mathcal{Q}$ are possible candidates for the canonical group. The difficulty is that this action is not transitive. For this reason, this class of transformations has to be complemented by a second one. The second class of transformations is induced by functions on $\mathcal{Q}$, as follows. Given $h \in C^\infty(\mathcal{Q}, \mathbb{R})$ we can use its exterior differential to generate "translations" along the fibers of $T^*\mathcal{Q}$, mapping $l \in T^*_q\mathcal{Q}$ to $l - (dh)_q \in T^*_q\mathcal{Q}$. It turns out that this transformation is also a symplectic transformation. Notice that since h acts only through its differential dh, it is really the quotient group $C^\infty(\mathcal{Q}, \mathbb{R})/\mathbb{R}$ that is being considered: Two functions differing by a constant produce the same transformation.

We thus have two different group actions on $T^*\mathcal{Q}$:

$$\begin{array}{rcl} \mathrm{Diff}\mathcal{Q} \times T^*\mathcal{Q} & \to & T^*\mathcal{Q} \\ (\phi, l) & \mapsto & \phi^{-1*} l \end{array}$$

and

$$\begin{array}{rcl} C^\infty(\mathcal{Q}, \mathbb{R})/\mathbb{R} \times T^*\mathcal{Q} & \to & T^*\mathcal{Q} \\ (\alpha, l) & \mapsto & l - (d\alpha)_q \quad (l \in T^*_q\mathcal{Q}). \end{array}$$

It is not difficult to see that the set of $\mathrm{Diff}\mathcal{Q}$-induced transformations plus the set of the $C^\infty(\mathcal{Q}, \mathbb{R})$-induced ones is transitive on $T^*\mathcal{Q}$. It is therefore natural to try to express them as the action of a *single* group on $T^*\mathcal{Q}$, whose underlying set is $C^\infty(\mathcal{Q}, \mathbb{R})/\mathbb{R} \times \mathrm{Diff}\mathcal{Q}$. The group law is determined by the requirement that it must be compatible with the action. Denoting with τ the action, we have (for $\alpha \in C^\infty(\mathcal{Q}, \mathbb{R})/\mathbb{R}$, $\phi \in \mathrm{Diff}\mathcal{Q}$ and $l \in T^*_q\mathcal{Q}$):

$$\tau_{(\alpha,\phi)}(l) := \phi^{-1*}(l) - (d\alpha)_{\phi(q)}.$$

From the requirement $\tau_{g_2} \circ \tau_{g_1} \overset{!}{=} \tau_{g_2 g_1}$ we obtain:

$$\begin{array}{rl} \tau_{(\alpha_2,\phi_2)} \circ \tau_{(\alpha_1,\phi_2)}(l) = & \\ = & \tau_{(\alpha_2,\phi_2)}(\phi_1^{-1*}(l) - (d\alpha_1)_{\phi_1(q)}) \\ = & \phi_2^{-1*}(\phi_1^{-1*}(l) - (d\alpha)_{\phi_1(q)}) - (d\alpha_2)_{\phi_2 \circ \phi_1(q)} \\ = & ((\phi_2 \circ \phi_1)^{-1})^*(l) - d(\phi_2^{-1*}\alpha_1 - \alpha_2)_{\phi_2 \circ \phi_1(q)} \\ = & \tau_{(\phi_2 \circ \phi_1, \alpha_1 \circ \phi_2^{-1} + \alpha_2)}(l). \end{array}$$

Thus, the structure obtained is that of the semi-direct product $C^\infty(\mathcal{Q})/\mathbb{R} \ltimes \mathrm{Diff}\mathcal{Q}$. This group has several interesting properties. Among them, it fulfills all the requirements needed to implement the quantization procedure (for details, the reader is referred to [Ish84]). In particular, the map γ that it induces produces Hamiltonian vector fields. For the class of examples where this group can be used, the quantization problem reduces to finding a suitable finite dimensional subspace W of $C^\infty(\mathcal{Q})/\mathbb{R}$ and a finite dimensional subgroup G of Diff$\mathcal{Q}$ and then to study the representations of the group $W \ltimes G$.

The fact that for the scheme we have been discussing the basic structure is that of a group acting by symplectic transformations on phase space, provides the justification for a re-examination of the basic assumption, stated in the last section, that pure states are always expressible as *functions*. Indeed, since the quantum theory is obtained from the unitary representations of the canonical group, there is no *a priori* reason to discard one representation in favor of another one. Inequivalent representations might be associated with the same classical system, leading to different quantizations of it. Some of these representations will lead to states that are expressible in the form of functions, but there might be other representations for which the representation space is, for instance, some space of cross-sections of a vector bundle over the configuration space. These kind of representation arise naturally when one has a group G acting on a space M. If, for example, $\psi : M \to \mathbb{C}$ belong to some set of functions on M, the G-action induces a transformation on the space of functions, mapping ψ to $g \cdot \psi$, where

$$g \cdot \psi(m) := \psi(g^{-1} \cdot m). \tag{2.2.26}$$

Depending on the particular situation, the set of functions may be a space of (square) integrable, or continuous functions on M, and one is interested in G-actions that preserve this set, thus making it a representation space. But there is also the possibility of considering a vector bundle η over M, and of defining a G-representation on some space of sections of this bundle. The only difficulty is that eq.(2.2.26) cannot be applied directly, because, if ψ is now a cross-section, the vectors $\psi(m)$ and $\psi(g^{-1}\cdot m)$ will belong, in general, to different vector spaces (the fibers over m and $g^{-1} \cdot m$, respectively). A suitable generalization of eq.(2.2.26) can be obtained if it is possible to lift the G-action on M to one on the total space of the bundle.

The search for lifts of the action leads naturally to the consideration of the covering spaces of M, because of the existence of "lifting theorems" guarantee, under certain assumptions, the existence of lifts (to the covering space) of functions from a given space into M. Once a lift of the G-action on M to one of its covering spaces has been found, it can be passed to the vector bundles associated with the covering space. The universal covering space of M, $\widetilde{M}$, is an example of particular interest. In this case the structure group of the (principal) bundle $\widetilde{M} \to M$ is the fundamental group $\pi_1(M)$. Although we are neglecting a lot of technical points here, it should be at least plausible that in this case, if the representations are obtained from G-lifts to vector bundles over M and, in turn, these G-lifts are obtained from lifts to the $\pi_1(M)$-principal bundle $\widetilde{M}$, there will be some relation between the representations and the fundamental group $\pi_1(M)$. This is exactly what happens when one considers the quantum mechanics of a spinless particle in a multiply connected configuration space $\mathcal{Q}$. First one has to find the canonical group following the "steps" described previously. And then, to construct the representations of this canonical group, one looks for the corresponding G-lifts on the universal covering $\widetilde{\mathcal{Q}}$. As explained in detail in [Ish84], the final result is that the inequivalent quantizations of this system are classified by the one dimensional, unitary representations of $\pi_1(\mathcal{Q})$, exactly as stated in theorem 2.1.1.

3 G-spaces and Projective Modules

In this chapter we consider the free action of a finite group G on a manifold M and the relation between vector bundles over the quotient space M/G and G-equivariant vector bundles over M. This well-known relation is presented in such a way that a new formulation, in algebraic terms, becomes possible. In particular, it is shown that the algebra $C(M)$ of complex valued, continuous functions on M admits a decomposition into finitely generated, projective modules over the algebra $C(M/G)$. The relation between these projective modules and the irreducible representations of the group G is worked out in detail. The results obtained in this chapter provide the basis for the discussion of the Spin-Statistics relation in the next ones.

3.1 Equivariant bundles

Since the configuration space for a system of N indistinguishable particles is a quotient space, obtained from the action of the permutation group of N elements on the space $\mathbb{R}^3 \times \cdots \times \mathbb{R}^3 \setminus \Delta$, it is of relevance for us to consider topological spaces M that carry the structure of a G-space. When studying objects defined on the orbit space M/G, it turns out to be a good idea to consider them as arising from objects defined on the original space M. For instance, if we are interested in bundles over M/G, it is convenient to consider G-equivariant bundles over M. These are bundles that also carry a (special type) of G-action. Under some assumptions, it is then possible to describe a bundle over M/G as the quotient of an equivariant bundle over M.

The analog of a G-space at the level of vector bundles is a G-equivariant vector bundle, or just G-bundle, defined below (some definitions and theorems used in this section are collected in appendix A).

3.1.1 Definition. A vector bundle $\xi = (E(\xi), \pi, M)$ over the G-space M is called a G-bundle when the following conditions hold:

- The total space $E(\xi)$ is itself a G-space (the corresponding action will be denoted with τ).
- The projection π is G-equivariant, i.e. $\pi \circ \tau_g = \rho_g \circ \pi$ for all g in G or, equivalently, the diagram

$$\begin{tikzcd} E(\xi) \arrow[r, "\tau_g"] \arrow[d, "\pi"'] & E(\xi) \arrow[d, "\pi"] \\ M \arrow[r, "\rho_g"] & M. \end{tikzcd}$$

commutes.

- The restriction $\tau_g\big|_{\pi^{-1}(m)} : \pi^{-1}(m) \longrightarrow \pi^{-1}(g \cdot m)$ of the action to the fibers is a vector space isomorphism.

3.1.2 Definition (G-Bundle morphism)**.** A morphism between two G-bundles is a G-equivariant bundle morphism. The notation $\xi_1 \cong_G \xi_2$ will be used whenever ξ_1 and ξ_2 are equivalent as G-bundles.

Let M and N be two topological spaces and $\phi : M \to N$ a continuous map between them. Let ξ be a vector bundle over N, with projection map $\pi : E(\xi) \to N$. The *pull-back* $\phi^*\xi$ of ξ under ϕ is a bundle, defined over M, with total space

$$E(\phi^*\xi) := \{(x, y) \in (M \times E(\xi)) \,|\, \phi(x) = \pi(y))\}.$$

At first, the bundles ξ and $\phi^*\xi$ are only related by the fact that it is possible to lift ϕ to a bundle morphism $\hat{\phi} : \phi^*\xi \to \xi$. But if we have an action $\rho : G \times M \to M$ of, say, a finite group G, then an interesting relation between ξ and $\phi^*\xi$ emerges, that we proceed to explain. Let $q : M \to M/G$ be the canonical projection. If we put above $\phi = q$ and $N = M/G$, then the pull-back q^* induces an action τ_ξ of G on $E(q^*\xi)$, given by

$$\tau_\xi(g, (m, y)) = g \cdot (m, y) := (g \cdot m, y), \quad g \in G, \quad (m, y) \in E(q^*\xi) \tag{3.1.1}$$

This action is also free and so we see that, because of §A.9 and §A.11, the quotient $E(q^*\xi)/G$ is also a manifold. As we will see, one finds that $E(q^*\xi)/G$ is the total space of a vector bundle (denoted $q^*\xi/G$) over M/G, which is isomorphic to ξ:

$$q^*\xi/G \cong \xi.$$

On the other hand, let η be a G-bundle over M. The quotient $E(\eta)/G$ is, again, the total space of a bundle over M/G, whose pull-back is G-isomorphic to η:

$$q^*(\eta/G) \cong_G \eta.$$

For G finite, we have:

3.1.3 Theorem (cf.[Ati67])**.** *If M is G-free, there is a bijective correspondence between G-bundles over M and bundles over M/G by $\eta \to \eta/G$.*

Proof. Let M be a G-space with free action ρ and let η be a G-bundle over M with projection map π and action τ. The action τ must also be free, because suppose we have $y \in E(\eta)$ and $g \in G$ with $g \cdot y = y$. Since $\pi(g \cdot y) = \pi(y)$ and η is a G-bundle, it follows that $\pi(g \cdot y) = g \cdot \pi(y)$, so that $g = e$.

It then follows, from theorem A.9, that $E(\eta)/G$ has a differentiable structure. Let us denote with q and $\bar{q}$ the respective projections:

$$q : M \to M/G\,, \qquad \bar{q} : E(\eta) \to E(\eta)/G.$$

Choose now an open cover $\{\tilde{U}_{\alpha,i}\}_{\alpha\in\mathcal{I},i\in\mathcal{I}_G}$ for M, exactly as in theorem A.9 (here $\mathcal{I}_G = \{1,2,\ldots,|G|\}$) and choose local trivialisations of the form

$$\tilde{\psi}_{\alpha,i} : \pi^{-1}(\tilde{U}_{\alpha,i}) \longrightarrow \tilde{U}_{\alpha,i} \times \mathbb{C}^n \qquad (\, n = \mathrm{Rank}(E)\,).$$

These local trivialisations can serve, at the same time, as charts for $E(\eta)$, provided $\tilde{U}_{\alpha,i}$ is identified with a neighborhood in $\mathbb{R}^m$ and $\{\pi^{-1}(\tilde{U}_{\alpha,i})\}_{\alpha,i}$ chosen as cover for $E(\eta)$. Since π is equivariant, this cover has the same properties as $\{\tilde{U}_{\alpha,i}\}_{\alpha,i}$, namely (here we use the shorthand notation $\tau_i(\,\cdot\,) := \tau(g_i,\cdot)$):

$$\begin{aligned}
\pi^{-1}(\tilde{U}_{\alpha,i})) = \tau_i(\pi^{-1}(\tilde{U}_{\alpha,1})) &\qquad (\forall i),\\
\pi^{-1}(\tilde{U}_{\alpha,i}) \cap \pi^{-1}(\tilde{U}_{\alpha,j}) = \emptyset &\qquad (i \neq j),\\
\bar{q}(\pi^{-1}(\tilde{U}_{\alpha,i})) = \bar{q}(\pi^{-1}(\tilde{U}_{\alpha,j})) &\qquad (\forall i,j).
\end{aligned}$$

From theorem A.9 we then know that charts for $E(\eta)/G$ can be defined by

$$\tilde{\psi}_{\alpha,1}\circ\bar{q}_{\alpha,1}^{-1} :\ \bar{q}(\pi^{-1}(\tilde{U}_{\alpha,1})) \longrightarrow \tilde{U}_{\alpha,1} \times \mathbb{C}^n, \tag{3.1.2}$$

where $\bar{q}_{\alpha,i} := \bar{q}|_{\pi^{-1}(\tilde{U}_{\alpha,i})}$. The bundle projection $\bar{\pi}$ is defined in such a way that $\bar{q}$ becomes a bundle morphism:

$$\begin{aligned}
\bar{\pi} : E(\eta)/G &\longrightarrow M/G \\
\bar{q}(y) &\longmapsto q(\pi(y)).
\end{aligned} \tag{3.1.3}$$

$\bar{\pi}$ is well defined, since π is equivariant (see below). From $U_\alpha := q(\tilde{U}_{\alpha,i})$ and $\bar{\pi}\circ\bar{q} = q\circ\pi$ we see that the following holds (for all i):

$$\bar{\pi}^{-1}(U_\alpha) = \bar{q}(\pi^{-1}(\tilde{U}_{\alpha,i})). \tag{3.1.4}$$

Indeed, we have:

$$\begin{aligned}
[y] \in \bar{q}(\pi^{-1}(\tilde{U}_{\alpha,i})) &\Leftrightarrow \exists g \in G \text{ s.t. } y \in \pi^{-1}(g \cdot \tilde{U}_{\alpha,1})\\
&\Leftrightarrow \exists g \in G \text{ s.t. } \pi(y) \in g \cdot \tilde{U}_{\alpha,1}\\
&\Leftrightarrow [\pi(y)]\ (= \bar{\pi}([y])) \in U_\alpha\\
&\Leftrightarrow [y] \in \bar{\pi}^{-1}(U_\alpha).
\end{aligned}$$

It then follows, from (3.1.2) and (3.1.4), that $\psi_\alpha := (q \times \mathrm{id})\circ\tilde{\psi}_{\alpha,1}\circ\bar{q}_{\alpha,1}^{-1}$ gives place to a local trivialisation. That is, we get homeomorphisms:

$$\psi_\alpha : \bar{\pi}^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^n. \tag{3.1.5}$$

Let us check that $\bar{\pi}$ is well defined. We have:

$$\begin{aligned}[y_1]=[y_2] \quad &\Rightarrow \quad \exists g \in G \text{ so that } y_2 = g\cdot y_1\\ &\Rightarrow \quad \pi(y_2)=\pi(g\cdot y_1)=g\cdot\pi(y_1)\\ &\Rightarrow \quad [\pi(y_2)]=[\pi(y_1)].\end{aligned}$$

That $\bar{\pi}$ preserves fibers follows directly from the definition, eq. (3.1.3):

$$\begin{aligned}\pi(y_1)=\pi(y_2) \quad \Rightarrow \quad \bar{\pi}(\bar{q}(y_1)) &= q(\pi(y_1))\\ &= q(\pi(y_2))\\ &= \bar{\pi}(\bar{q}(y_2)).\end{aligned}$$

Now, what the theorem asserts is that for a G-bundle η the following holds:

- $\eta \simeq_G q^*(\eta/G)$:

$$\begin{tikzcd} E(\eta) \arrow[r, "\bar{q}"'] \arrow[d] & E(\eta)/G \arrow[d] \\ M \arrow[r, "q"'] & M/G, \end{tikzcd}$$

whereas for a bundle ξ over M/G one has:

- $q^*(\xi)/G \simeq \xi$:

$$\begin{tikzcd} q^*(\xi) \arrow[r] \arrow[d] & \xi \arrow[d] \\ M \arrow[r, "q"'] & M/G. \end{tikzcd}$$

These isomorphisms must be established. Transition functions for η/G are determined by the condition

$$\psi_\alpha \circ \psi_\beta^{-1}([x],t) = ([x], g_{\alpha,\beta}([x])t). \tag{3.1.6}$$

They can be related to the transition functions of η, $\tilde{g}_{\alpha_i,\beta_j}$, as follows. Consider $[x] \in U_\alpha \cap U_\beta$ and choose a representative $x \in [x]$. Then $x \in \tilde{U}_{\alpha_i} \cap \tilde{U}_{\beta_j}$ for some i, j. In particular, there are elements $g_i, g_j \in G$ such that $\tilde{U}_{\alpha_i} = g_i\tilde{U}_{\alpha_1}$ and $\tilde{U}_{\beta_j} = g_j\tilde{U}_{\beta_1}$. From the definition of ψ_α in terms of $\tilde{\psi}_{\alpha_1}$ it then follows that

$$g_{\alpha,\beta}([x]) = g_j^{-1}\, \tilde{g}_{\beta_j,\alpha_i}(x)\, g_i. \tag{3.1.7}$$

Here, g_i is a shorthand notation for the action of the group element g_i on $\mathbb{C}^n$ induced by τ through the local trivializations of the bundle. Eq. (3.1.7) is obtained in exactly the same way as eq. (A.3), see also diagram (A.4). On the other hand, the transition functions $g^{pb}_{\alpha,\beta}$ for the pull-back bundle $q^*(\eta/G)$ are, by definition,

$$g^{pb}_{\alpha,\beta}(x) = g_{\alpha,\beta}([x]). \tag{3.1.8}$$

This shows the equivalence of η with $q^*(\eta/G)$. Equations (3.1.7) and (3.1.8) give place to a bundle isomorphism $h : \eta \to q^*(\eta/G)$, and it is easy to check that, with respect to the G-action naturally defined on $q^*(\eta/G)$ (see eq. (3.1.1)), h is G-equivariant. This means that η and $q^*(\eta/G)$ are also equivalent as equivariant bundles. The isomorphism $\xi \cong q^*(\xi)/G$ is established in a similar way. $\square$

3.2 Equivariant trivial bundles

It is possible, based on the equivalence of theorem 3.1.3, to describe bundles over M/G by means of (corresponding, G-equivariant) ones over M. Of course it may happen that a given bundle η over M admits several, non-equivalent, G-bundle structures. In this case, the corresponding quotient bundles over M/G are inequivalent. This means: If we want to *regard* η as a bundle over M/G, then we must take its G-bundle structure into account. We are thus interested in a more compact formulation of this equivalence. As we will see, a reformulation of theorem 3.1.3 in algebraic terms will serve this purpose. A first step in this direction can already be given, using the proofs of theorems A.9 and 3.1.3, if we introduce the following definition.

3.2.1 Definition. Let $\xi = (E(\xi), \pi, M)$ be a vector bundle. Let $\{\phi_\alpha\}_\alpha$ a partition of unity for M ($\sum_\alpha |\phi_\alpha(x)|^2 = 1$) and $\{g_{\alpha,\beta}\}$ a choice of transition functions for ξ, both subordinated to a given open cover $\{U_\alpha\}_\alpha$ of M. The *projector* P_ξ corresponding to ξ will be defined as the $C(M)$-valued block matrix with entries

$$(P_\xi(x))_{\alpha,\beta} = |\phi_\alpha(x)|\, g_{\alpha,\beta}(x)\, |\phi_\beta(x)|.$$

We will see in the next section that the bundle ξ can equivalently be described by the projector P_ξ. The transition functions for the quotient bundle η/G from theorem 3.1.3 are given by eq. (3.1.7).

It is of interest for us to see if it is possible to use the data of an equivariant bundle (action, transition functions, etc..) in order to obtain the projector of the *quotient bundle*. We say this is of interest for us, because the data of the equivariant bundle will contain, apart from information about the group action, functions defined on M, whereas the entries of the projectors will be functions on M/G.

A particular case one can consider is that of a trivial equivariant vector bundle. This does not necessarily mean that the group action is also trivial. The map defining the action on the total space of the bundle must satisfy certain relations; these can be interpreted as cocycle conditions and can, in principle, be used to establish the equivalence classes of actions on that trivial bundle. Here we will consider the much simpler case of an equivariant trivial bundle with the action on the total space of the bundle given by a representation of the group. This seems to be enough for the analysis of indinstinguishability in Quantum Mechanics (see, however comments on chapter 6). In this particular case, one can easily express the projector describing the quotient bundle in terms of functions *on* M, as illustrated below.

Keeping the same notation as in §A.9 and §3.1.3, let $\{U_\alpha\}_{\alpha\in I}$ and $\{\tilde{U}_{\alpha,i}\}_{\alpha,i}$ be covers of M/G and M, respectively (in particular we assume that $U_\alpha = q(\tilde{U}_{\alpha,i})$ and that eqns. (A.1) and (A.2) hold). Consider an irreducible, unitary representation $R : G \to Gl(n_R, \mathbb{C})$. Then the bundle $\eta = (M \times \mathbb{C}^{n_R}, \pi_1, M)$ is equivariant with respect to the G-action

$$h \cdot (x,t) := (h \cdot x, R(h)t). \tag{3.2.1}$$

Since η is trivial, all $\tilde{g}_{\beta_j,\alpha_i}$ equal the identity of G and, therefore, the transition functions for η/G are given by (compare equation (3.1.7))

$$\begin{aligned} g_{\alpha,\beta} : U_\alpha \cup U_\beta &\to Gl(n_R, \mathbb{C}) \\ [x] &\mapsto g_{\alpha,\beta}([x]) = R(h_j^{-1})R(h_i), \end{aligned} \tag{3.2.2}$$

with i and j depending on some choice of $x \in [x]$ (the product $h_j^{-1}h_i$ is independent of this choice). Choosing a partition of unity $\{\phi_\alpha\}_\alpha$ for M/G (with Supp $\phi_\alpha \subset U_\alpha$ and $\sum_{\alpha\in I} |\phi_\alpha([x])|^2 = 1$, for $[x]$ in M/G) we define, for every α, the following map:

$$\Psi_{\alpha_1}(x) := \begin{cases} \phi_\alpha([x]), & \text{if } x \in \tilde{U}_{\alpha_1} \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.3}$$

This is in fact a *smooth* function $M \to \mathbb{R}$, because Supp $\phi_\alpha \subset U_\alpha$ and $\tilde{U}_{\alpha,1} \simeq U_\alpha$. Using the action, we can generate $|G|$ functions from $\Psi_{\alpha,1}$. These functions give a partition of unity for M, with respect to the open cover $\{U_{\alpha_i}\}_{\alpha,i}$. To show this, let x be any representative of $[x] \in U_\alpha$. Then there is exactly one open set $\tilde{U}_{\alpha,k}$, that contains x (the index $k \in \{1,\ldots,|G|\}$ obviously depends on x). It follows that there is only one non-vanishing term in the sum $\sum_{h\in G} |\Psi_{\alpha_1}(h^{-1}\cdot x)|^2$ and from this we conclude:

$$\sum_{h\in G} |\Psi_{\alpha,1}(h^{-1}\cdot x)|^2 = |\Psi_{\alpha,1}(h_k^{-1}\cdot x)|^2 = |\phi_\alpha([x])|^2 \quad (x \in \tilde{U}_{\alpha,k}).$$

That means,

$$\sum_{\alpha\in I}\sum_{h\in G} |\Psi_{\alpha_1}(h^{-1}\cdot x)|^2 = 1, \qquad (\forall x \in M), \tag{3.2.4}$$

as claimed. Now we can write down the projector P corresponding to η/G, in such a way, that only functions *on* M are used:

$$\begin{aligned} P_{\alpha\beta}([x]) &= \phi_\alpha([x])g_{\alpha\beta}([x])\phi_\beta([x]) \qquad \text{(from definition 3.2.1)} \\ &= \phi_\alpha([x])\big(R(h_i^{-1})R(h_j)\big)\phi_\beta([x]) \\ &= \big(\Psi_{\alpha_1}(h_i^{-1}\cdot x)R(h_i^{-1})\big)\,\big(R(h_j)\Psi_{\beta_1}(h_j^{-1}\cdot x)\big) \\ &= \left(\sum_{\tilde{h}\in G}\Psi_{\alpha_1}(\tilde{h}^{-1}\cdot x)R(\tilde{h}^{-1})\right)\left(\sum_{h\in G}\Psi_{\beta_1}(h^{-1}\cdot x)R(h)\right). \end{aligned}$$

The entries of (the block (α,β) of the) projector can be expressed in terms of the matrix components of the representation. One obtains:

$$\begin{aligned} \big(P_{\alpha\beta}([x])\big)_{ij} &= \sum_{\tilde{h}\in G}\sum_{h\in G}\Psi_{\alpha_1}(\tilde{h}^{-1}\cdot x)\Psi_{\beta_1}(h^{-1}\cdot x)\big(R(\tilde{h}^{-1})R(h)\big)_{ij} \\ &= \sum_{k=1}^{n_R}\left(\sum_{\tilde{h}\in G}R^*(\tilde{h})_{ki}\Psi_{\alpha_1}(\tilde{h}^{-1}\cdot x)\right)\left(\sum_{h\in G}R(h)_{kj}\Psi_{\beta_1}(h^{-1}\cdot x)\right). \end{aligned}$$

Introducing the functions ($\alpha \in I, i, j \in \{1, \ldots, n_R\}$)

$$\Psi^{\alpha}_{ij}(x) := \frac{n_R}{|G|} \sum_{h \in G} R^*(h)_{ij} \Psi_{\alpha_1}(h^{-1} \cdot x), \tag{3.2.5}$$

we obtain a more compact expression for the entries of the projector:

$$\left(P_{\alpha\beta}([x])\right)_{ij} = \frac{|G|^2}{n_R^2} \sum_{k=1}^{n_R} \Psi^{\alpha}_{ki}(x) \Psi^{\beta *}_{kj}(x). \tag{3.2.6}$$

The relevance of this formula, for us, is the following. The possibility of describing vector bundles by means of projectors (as the one from definition 3.2.1) comes from the Serre-Swan theorem, that establishes an equivalence between categories of vector bundles on one hand and of finitely generated, projective modules on the other (this equivalence will be discussed in some detail in the next section). The usefulness of eq. (3.2.6) is that we have a vector bundle η/G over the quotient space M/G but, using the Serre-Swan theorem, we know that this bundle can be equivalently described by the corresponding projector. In our example, the bundle in question is the quotient bundle of a trivial bundle η defined on M, which is G-equivariant. The G-action on the total space is determined by a representation of G. Since -as can be seen in eq. (3.2.6)- the components of this projector can be expressed completely in terms of functions defined on M (and not on M/G), we are led to the idea that certain vector bundles over M/G might be expressible in terms of functions defined on M. This is in fact true, and in section 3.3 we will see how the G-action on M naturally induces a decomposition of $C(M)$ into projective $C(M/G)$-modules. The projectors describing these modules will be shown to be precisely of the form we have just obtained in eq. (3.2.6).

3.3 Projective Modules

The purpose of this section is to obtain an alternative version of theorem 3.1.3 in terms of the spaces of sections associated with the bundles involved. There are mainly two motivations for doing this.

- As explained in the introduction, we are looking for a formulation of the quantum mechanics of a system of identical particles in which the wave function is considered to be a section of a suitable vector bundle defined on the configuration space $\mathcal{Q}$ (eq.(1.1.2)). We remarked, at the end of chapter 2, that when the configuration space is multiply-connected, as is $\mathcal{Q}$, it is a good strategy to consider its universal cover, $\widetilde{\mathcal{Q}}$. In the case of "spin zero" quantum mechanics, a standard procedure is to consider, at first, subsidiary functions $\tilde{\psi} : \widetilde{\mathcal{Q}} \to \mathbb{C}$. It is important to emphasize that such a function is not the physical wave function. The latter (call it ψ) should have $\mathcal{Q}$ as its domain of definition and it might be that it is really not a function but a section on some bundle over $\mathcal{Q}$. Although $\tilde{\psi}$ is not the physical wave function, some conditions can be imposed on it in order that it represents the physical one. In that case, one says that $\tilde{\psi}$ is "projectable" to $\mathcal{Q}$. In order that such a function be "projectable", it is required (for obvious physical reasons) that, for $x \in \mathcal{Q}$ and for any $\tilde{x}$ in the fiber $q^{-1}(x)$ (q denotes the projection $q : \widetilde{\mathcal{Q}} \to \mathcal{Q}$) $|\tilde{\psi}(\tilde{x})|^2$ depends only on the point x, independently of $\tilde{x}$. Restricting $\tilde{\psi}$ in this way, we force its domain of definition to be "almost" $\mathcal{Q}$, for if $\tilde{x}_1$ and $\tilde{x}_2$ belong to the same fiber $q^{-1}(x)$, $\tilde{\psi}(\tilde{x}_1)$ and $\tilde{\psi}(\tilde{x}_2)$ may still differ by a phase. These two points are related by an element σ of $\pi_1(\mathcal{Q})$* ($\tilde{x}_2 = \sigma \cdot \tilde{x}_1$) and one can show†, that the phase can only depend on σ. The restriction on $\tilde{\psi}$ takes the form

$$\tilde{\psi}(\sigma \cdot \tilde{x}) = \chi(\sigma)\tilde{\psi}(\tilde{x}), \tag{3.3.1}$$

where χ is, as in theorem 1.1.1, a one dimensional unitary representation of $\pi_1(\mathcal{Q})$. In other words, $\tilde{\psi}$ is required to be $\pi_1(Q)$-equivariant. It is in this sense that one should understand the statement, often found in the literature, that on multiply-connected spaces, "multiple-valued" functions are allowed. In this context, under multi-valuedness of ψ one should understand equivariance (as expressed by eq. (3.3.1)) of $\tilde{\psi}$. This somewhat confusing situation has, to our opinion, not been discussed in a clear and systematic way, and it leads to further confusion when spin degrees of freedom are taken into account (see discussion in section 5.3). The reformulation of theorem 3.1.3 we give in this section (see theorem 3.3.5 below) provides a suitable framework for handling the situation discussed above.

- The second reason is that this reformulation serves as a first step towards a formulation of the problem in only algebraic terms. Using the language of algebras and modules might prove to be useful in the search for a relation between the (mainly geometric) configuration-space approaches to the Spin-Statistics relation, and the Field theoretical ones, in which proofs of the Spin-Statistics theorem are available. It would be very interesting to eventually find an interpretation of the analytical proofs in geometrical terms. The relation of these proofs to, for example, the work of Finkelstein-Rubinstein[FR68] on topological solitons, is something that has not been clarified. One of the difficulties is certainly the difference of "languages" (the one algebraic/analytical, the other topological).

With the above remarks as motivation, let us recall two basic results that relate categories of topological spaces with categories of algebraic objects. For M a compact Hausdorff topological space, the space $C(M)$ is an algebra that, by the Gelfand-Naimark theorem, carries all the information about the (topological) space M. By the same theorem, this correspondence between topological spaces and complex algebras is bijective. Schematically, we have:

*Recall that $\pi_1(\mathcal{Q})$ acts on $\widetilde{\mathcal{Q}}$ by Deck transformations.

†See [Mor92] and references therein.

$$\begin{pmatrix} \text{Compact Hausdorff} \\ \text{topological spaces} \end{pmatrix} \xleftrightarrow[\text{Naimark}]{\text{Gelfand-}} \begin{pmatrix} \text{Unital, commutative} \\ C^*\text{-algebras} \end{pmatrix}.$$

An analogous result for vector bundles is also available: if η is a vector bundle over the compact space M, then it is well known that $\Gamma(\eta)$ carries the structure of a finitely generated and projective $C(M)$-module. The module multiplication

$$\begin{array}{ccc} C(M) \times \Gamma(\eta) & \longrightarrow & \Gamma(\eta) \\ (a, \psi) & \longrightarrow & a \cdot \psi \end{array} \tag{3.3.2}$$

is defined by pointwise multiplication: $(a \cdot \psi)(m) := a(m)\psi(m)$ (the easier notation $a\psi$ for the module product is usually used). The Serre-Swan theorem establishes the following bijective correspondence:

$$\begin{pmatrix} \text{Vector bundles} \\ \text{over M} \end{pmatrix} \xleftrightarrow[\text{Swan}]{\text{Serre-}} \begin{pmatrix} \text{Finitely generated,} \\ \text{projective } C(M)\text{-modules} \end{pmatrix}.$$

These two theorems (Gelfand-Neumark and Serre-Swan) provide complete equivalences between classes of topological objects on one side and of algebraic ones on the other. Let us, for the sake of concreteness, state the Serre-Swan theorem.

3.3.1 Theorem (Serre-Swan (cf.[Ser58],[Swa62])). *Let M be a compact, finite dimensional manifold and let $\mathcal{E}$ be a $C(M)$-module. Then there exists a vector bundle $\eta = (E(\eta), \pi, M)$ whose $C(M)$-module of sections $\Gamma(\eta) = \{\sigma : M \to E(\eta) \,|\, \sigma \text{ is continuous and } \pi\circ\sigma = id\}$ is isomorphic to $\mathcal{E}$ if and only if $\mathcal{E}$ is finitely generated and projective.*

In order to obtain an algebraic formulation of theorem 3.1.3, all that must be done, in view of theorem 3.3.1, is to replace all assertions about the relations between η and η/G by assertions about the corresponding spaces of sections, $\Gamma(\eta)$ and $\Gamma(\eta/G)$. This task can be easily performed once one realizes that the pull-back of a vector bundle, at the algebraic level, can be obtained from the tensor product of appropriate algebras and modules. The precise assertion is formulated in theorem 3.3.2, but before proving it we will make some remarks from which the proof follows quite naturally.

Let

$$\rho : G \times M \to M \tag{3.3.3}$$

be a G-action on the topological space M. It, in turn, induces an action $G \times C(M) \to C(M) : (g, a) \mapsto g \cdot a$, where

$$(g \cdot a)(m) := a(\rho_{g^{-1}}(m)) = a(g^{-1} \cdot m). \tag{3.3.4}$$

Analogously, for a G-bundle η with corresponding action $\tau : G \times E(\eta) \to E(\eta)$, an action on the space of sections, $\Gamma(\eta)$, will be induced, as follows: For $\psi \in \Gamma(\eta)$ and $g \in G$, $g \cdot \psi : M \to E(\eta)$ is defined through

$$(g \cdot \psi)(m) := \tau_g(\psi(g^{-1} \cdot m)). \tag{3.3.5}$$

The two actions (on $C(M)$ and on $\Gamma(\eta)$) are related, through the module multiplication, as follows:

$$g \cdot (a\psi) = (g \cdot a)(g \cdot \psi). \tag{3.3.6}$$

Consider now a continuous map $\phi : M \to N$ and a vector bundle ξ over N. We seek a characterization of the $C(M)$-Module $\Gamma(\phi^*\xi)$ where, instead of $\phi^*\xi$, only the $C(N)$-Module $\Gamma(\eta)$ is used. In other words, what we are looking for is a $C(M)$-module which is isomorphic to $\Gamma(\phi^*\xi)$, but constructed from the data ϕ, $C(N)$, $C(M)$ and $\Gamma(\xi)$ (no explicit mention of $\phi^*\xi$). This means that somehow we have to compare $\Gamma(\xi)$ with $\Gamma(\phi^*\xi)$. But $\Gamma(\xi)$ and $\Gamma(\phi^*\xi)$ are modules over different rings ($C(N)$ and $C(M)$, respectively), so it is only possible to compare them either by considering $\Gamma(\phi^*\xi)$ as a $C(N)$-module, or the other way around, that is, by regarding $\Gamma(\xi)$ as a $C(M)$-module. Both alternatives are possible, but only through the second one do we obtain the isomorphism we are looking for.

To see this, notice that ϕ induces a ring homomorphism

$$\begin{aligned} \phi^* : C(N) &\longrightarrow C(M) \\ f &\longmapsto \phi^* f := f \circ \phi. \end{aligned}$$

This homomorphism can be used to obtain a $C(N)$-module structure on $\Gamma(\phi^*\xi)$:

$$\begin{aligned} C(N) \times \Gamma(\phi^*\xi) &\longrightarrow \Gamma(\phi^*\xi) \\ (\, f \,,\, s \,) &\longmapsto f \cdot s := (\phi^* f)s. \end{aligned} \tag{3.3.7}$$

For any given section $\sigma \in \Gamma(\xi)$, we define a section

$$\begin{aligned} \phi^*\sigma : M &\longrightarrow E(\phi^*\xi) \\ x &\longmapsto (x, \sigma\circ\phi(x)). \end{aligned} \tag{3.3.8}$$

This gives place to a homomorphism of $C(N)$-modules

$$\begin{aligned} F^\phi : \Gamma(\xi) &\longrightarrow \Gamma(\phi^*\xi) \\ \sigma &\longmapsto F^\phi(\sigma) \equiv \phi^*\sigma. \end{aligned} \tag{3.3.9}$$

F^ϕ is clearly a $C(N)$-linear map:

$$F^\phi(f \cdot \sigma) = \phi^*(f \cdot \sigma) \overset{(3.3.8)}{=} (f\circ\phi)\phi^*\sigma \overset{(3.3.7)}{=} f \cdot \phi^*\sigma = f \cdot F^\phi(\sigma).$$

But it is also clear that F^ϕ is not, in general, an isomorphism. Indeed, although we may choose generators for $\Gamma(\phi^*\xi)$ of the form $\sigma_i' = F^\phi(\sigma_i)$, we see from $\mathrm{Im}(F^\phi) = \{\sum_i (f_i\circ\phi)\sigma_i' \,|\, f_i \in C(N)\}$ that F^ϕ is not surjective in general, because the elements in the image module are only linear combinations of the generators over the *subspace* $\phi^*(C(N))$ of $C(M)$. In other words: The generators σ_i' may be multiplied only by elements of $C(N)$. We thus see that a change of ring, performed with the help of the tensor product, is necessary. With the previous remark as motivation, we set

$$\begin{aligned} \Phi : C(M) \otimes_{C(N)} \Gamma(\xi) &\longrightarrow \Gamma(\phi^*\xi) \\ \sum_k a_k \otimes \sigma_k &\longmapsto \sum_k a_k F^\phi(\sigma_k). \end{aligned} \tag{3.3.10}$$

3.3.2 Theorem (cf. [GBVF01]). *The map defined through (3.3.10) is an isomorphism $C(M) \otimes_{C(N)} \Gamma(\xi) \cong \Gamma(\phi^*\xi)$ of $C(M)$-modules.*

Proof. (i) Φ is a $C(M)$-module homomorphism:

$\Phi(a_1 \cdot (a_2 \otimes \sigma)) = \Phi((a_1 a_2) \otimes \sigma) = a_1 a_2 F(\sigma) = a_1 \Phi(a_2 \otimes \sigma)$.

(ii) If $\Gamma(\xi)$ is generated by $\{\sigma_i\}_{i=1,\ldots,r}$, then $\Gamma(\phi^*\xi)$ is generated by $\{F(\sigma_i)\}_{i=1,\ldots,r}$:

The fiber ξ_y over $y \in N$ is generated by the vectors $\{\sigma_i(y)\}$. From $\phi^*\xi_x = \{x\} \times \xi_{\phi(x)}$ it is then clear that the vectors $\{(x, \sigma_i(\phi(x)))\}_i$ generate the fiber $\phi^*\xi_x$. The assertion follows from this fact, together with (3.3.9).
(iii) Φ is surjective:

Given $s \in \Gamma(\phi^*\xi)$, there exist -because of *(ii)*- functions $a_1, \ldots, a_r \in C(M)$ such that $s = \sum_i a_i F^\phi(\sigma_i)$.This implies $s = \Phi(\sum_i a_i \otimes \sigma_i)$.
(iv) Φ is injective: Clear if $\Gamma(\xi)$ is a free module, because in that case there exists a basis. In the general case, there exists some η such that $\Gamma(\xi \oplus \eta)$ is a free module, so that there is an isomorphism $\Phi' : C(M) \otimes_{C(N)} \Gamma(\xi \oplus \eta) \to \Gamma(\phi^*(\xi \oplus \eta))$. The inclusions $i : C(M) \otimes \Gamma(\xi) \hookrightarrow C(M) \otimes \Gamma(\xi \oplus \eta)$ and $j : \Gamma(\phi^*\xi) \hookrightarrow \Gamma(\phi^*\xi) \oplus \Gamma(\phi^*\eta)$ are injective and, moreover, $j \circ \Phi' = \Phi \circ i$: Φ must also be injective. □

If -for the situation in (3.3.3)- we set $N \equiv M/G$ and $\phi \equiv q : M \to M/G$ in thm. 3.3.2, then we can construct an injective $C(M/G)$-module homomorphism

$$\Phi^G : \Gamma(\xi) \hookrightarrow \Gamma(q^*\xi),$$

as follows. Recall that G acts on $C(M)$ by (3.3.4). In the next section it will be shown that this action induces a decomposition of $C(M)$ into $C(M/G)$-submodules (see eq. (3.4.27)). All that we need from this result is the fact that $C(M)$ may be decomposed in the form

$$C(M) = C(M)_+ \oplus \mathcal{E},$$

where $C(M)_+$ stands for the subalgebra $\{a \in C(M) \mid g \cdot a = a \ \forall\, g \in G\}$ of G-invariant functions and also that $\mathcal{E}$ has the structure of a projective $C(M/G)$-module. Noticing that the algebras $C(M)_+$ and $C(M/G)$ are isomorphic, we obtain:

$$C(M) \otimes_{C(M/G)} \Gamma(\xi) \cong \Gamma(\xi) \oplus \left(\mathcal{E} \otimes_{C(M/G)} \Gamma(\xi)\right). \tag{3.3.11}$$

Denote with $i : \Gamma(\xi) \hookrightarrow C(M) \otimes_{C(M/G)} G(\xi)$ the corresponding inclusion. Making use of thm. 3.3.2 we can put $\Phi^G := \Phi \circ i$ and in that way obtain the desired result.

3.3.3 Remark. It is important to notice that, although $\Phi^G(\Gamma(\xi))$ and $\Gamma(\xi)$ are isomorphic as $C(M/G)$-modules, $\Phi^G(\Gamma(\xi))$ is actually contained in $\Gamma(q^*\xi)$. This means that, although every section from $\Gamma(\xi)$ can be "replaced" by one from $\Phi^G(\Gamma(\xi))$, sections from $\Phi^G(\Gamma(\xi))$ may *only* be multiplied by functions in $C(M)_+$, if we want to identify $\Phi^G(\Gamma(\xi))$ and $\Gamma(\xi)$ as *modules*.

3.3.4 Remark. From Eqns. (3.3.10) and (3.3.11) we see that $\Phi^G(\sigma) = F^q(\sigma)$.

It is possible to give a better description of the image of Φ^G. In fact, one finds that $\Phi^G(\Gamma(\xi))$ equals the space of invariant sections of the pull-back bundle:

3.3.5 Theorem.

$$\Gamma(\xi) \cong \Phi^G(\Gamma(\xi)) = \Gamma^G(q^*\xi) := \{s \in \Gamma(q^*\xi) \mid g \cdot s = s \ \forall\, g \in \Gamma\}.$$

Proof. The first equality is clear, since Φ^G is an injective homomorphism. Every section from $\Phi^G(\Gamma(\xi))$ is of the form $q^*\sigma$, with $\sigma \in \Gamma(\xi)$. From the definition of $q^*\sigma$ (see Eq. (3.3.8)) and from the form of the action τ induced induced on $q^*\xi$ by q (see Eq.(3.1.1)) it follows that a section of the form $q^*\sigma$ is invariant:

$$\begin{aligned} (g \cdot q^*\sigma)(x) &= \tau_g(g^{-1} \cdot x, \sigma \circ q(g^{-1} \cdot x)) \\ &\overset{(3.1.1)}{=} (x, \sigma \circ q(g^{-1} \cdot x)) \\ &= (x, \sigma \circ q(x)) \\ &= q^*\sigma(x). \end{aligned}$$

Conversely, every invariant section must be of the form $q^*\sigma$: Given $s \in \Gamma(q^*\xi)$, there is a continuous map $y : M \to E(\xi)$ with $\pi \circ y = q$ and $s : x \mapsto (x, y(x))$. If s is invariant, then $y(g \cdot x) = y(x)$ holds for all g in G, so that one can define a section $\sigma \in \Gamma(\xi)$ through $\sigma([x]) := y(x)$, for which $s = q^*\sigma$ holds. □

3.4 Decomposition of $C(M)$

Let G be a finite group. Denote with $\vartheta_i : G \to \mathrm{Gl}(U^i) \quad (i = 1, \dots, N)$ its irreducible representations. A general representation $\vartheta : G \to \mathrm{Gl}(V)$ can always be reduced, i.e., V can be written in the form

$$V = V^1 \oplus \cdots \oplus V^N, \tag{3.4.1}$$

where every subspace V^j is invariant under G and can be reduced further,

$$V^j = V^j_1 \oplus \cdots \oplus V^j_{m_j}, \tag{3.4.2}$$

in such a way that $\dim(V^j_r) = n_j := \dim(U^j)$ for all $r = 1, \dots, m_j$ (m_j = multiplicity of ϑ_j in V). In particular, for a given j, all the representations $\vartheta|_{V^j_r}$ are equivalent to ϑ_j. The decomposition of V then takes the form

$$V \cong m_1 U^1 \oplus \cdots \oplus m_N U^N. \tag{3.4.3}$$

Since all subspaces V^i_r (for i fixed) are equivalent as representations, one is interested in finding a basis for every V^i_r with the property that the representing matrices are all the same, for $r = 1, \dots, m_i$. Obviously such basis do exist, so the question would rather be: How can one find such basis systematically?.

3.4.1 Remark. The problem of determining such "symmetry respecting" basis is closely related to the fact that, in contrast to eq. (3.4.1), the decomposition in eq. (3.4.2) is not unique. As we will see, after a concrete choice for the representing matrices of ϑ_j has been made, it is possible to define certain operators that allow one to find those symmetry-respecting basis. Of course, a choice of basis for the different subspaces amounts to writing down an explicit decomposition of V^j. It is clear from this point of view that the decomposition cannot be unique.

In the case of the *regular* representation, the construction of the operators mentioned in the previous remark is particularly clear. Since the formulae obtained in this case can be applied in general, we want to discuss this example in detail.

3.4.2 Definition. For G a finite group, let $\mathcal{F}(G)$ denote the vector space of complex functions on G (it dimension equals the order of the group, $|G|$). The *regular representation*, ρ^G, is the representation on $\mathcal{F}(G)$ induced by the group multiplication

$$\left(\rho^G(a)f\right)(b) := f(a^{-1}b) \qquad (f \in \mathcal{F}(G); a, b \in G).$$

3.4.3 Remark. It is a well-known fact that the regular representation contains all irreducible representations of G, each one with a multiplicity equal to its dimension:

$$\mathcal{F}(G) \cong \bigoplus_{i=1}^{N} n_i U^i \tag{3.4.4}$$

In particular, for the dimension of $\mathcal{F}(G)$, we have:

$$\dim(\mathcal{F}(G)) = \sum_{i=1}^{N} n_i \dim(U^i) = \sum_{i=1}^{N} (n_i)^2. \tag{3.4.5}$$

Although an additional decomposition of $n_i U^i$ is not canonically given, it is possible to, in a sense, "factor out" these subspaces, if one regards $\mathcal{F}(G)$ as a representation of $G \times G$. For this purpose we consider a $G \times G$-action on G, defined by

$$\begin{array}{rcl} (G \times G) \times G & \to & G \\ ((g_1, g_2), h) & \mapsto & g_1 h g_2^{-1}. \end{array} \tag{3.4.6}$$

The representation induced by this action on $\mathcal{F}(G)$ will be denoted with $\hat{r}^G$.

Another $G \times G$ representation can be defined as follows. Let $\tilde{\vartheta}_i$ $(i = 1, \ldots, N)$ be the representation induced by ϑ_i on the dual space U^{i*}: $(\tilde{\vartheta}_i(g)\varphi)(u) := \varphi(\vartheta_i(g^{-1})u) \quad (u \in U^i, \varphi \in U^{i*}, g \in G)$. It follows that $\vartheta_i \otimes \tilde{\vartheta}_i$ is an irreducible $G \times G$ representation on $U^i \otimes U^{i*}$. This representation can be realized inside $\mathcal{F}(G)$. This is a consequence of the following fact:

3.4.4 Theorem. *The map $S_i : U^i \otimes U^{i*} \to \mathcal{F}(G)$, defined by*

$$S_i(u \otimes \varphi)(g) := \varphi(\vartheta_i(g^{-1})u), \tag{3.4.7}$$

is $G \times G$-equivariant.

Proof. For $(a,b) \in G \times G$, $u \otimes \varphi \in U^i \otimes U^{i*}$ and $g \in G$, we have:

$$\begin{aligned} S_i\left((a,b)\cdot u\otimes\varphi\right)(g) &= S_i\left(\vartheta_i(a)u\otimes\tilde{\vartheta}_i(b)\varphi\right)(g) \\ &= \left(\tilde{\vartheta}_i(b)\varphi\right)\left(\vartheta_i(g^{-1})\vartheta_i(a)u\right) \\ &= \varphi\left(\vartheta_i((a^{-1}gb)^{-1})u\right) \\ &= S_i(u\otimes\varphi)((a,b)^{-1}\cdot g) \\ &= \left(\hat{r}^G(a,b)S_i(u\otimes\varphi)\right)(g) \\ &= \left((a,b)\cdot S_i(u\otimes\varphi)\right)(g). \end{aligned}$$

□

3.4.5 Theorem. *The decomposition of $\mathcal{F}(G)$ into irreducible $G\times G$-representations is*

$$\mathcal{F}(G) \cong \bigoplus_{i=1}^{N} U_i \otimes U_i^*. \tag{3.4.8}$$

Proof. It follows from theorem 3.4.4, together with Schur's lemma (since S_i is injective), that every representation $\vartheta_i \otimes \tilde{\vartheta}_i$ (i=1,…,N) is contained in $\mathcal{F}(G)$, that is, we have an equivalence of representations: $(U^i\otimes U^{i*}, \vartheta_i\otimes\tilde{\vartheta}_i) \sim (\mathcal{F}(G)|_{\mathrm{Im}(S_i)}, \hat{r}^G|_{\mathrm{Im}(S_i)})$. The direct sum of *all* $U^i\otimes U^{i*}$ must appear as a sub-representation in $\mathcal{F}(G)$, since $\vartheta_i\otimes\tilde{\vartheta}_i \sim \vartheta_j\otimes\tilde{\vartheta}_j$ precisely when $i=j$. Therefore, equation (3.4.8) must hold -as an isomorphism of $G\times G$ representations- since $\dim(\oplus_i U^i\otimes U^{i*}) = \sum_i n_i^2 = \dim(\mathcal{F}(G))$. □

From the previous theorem we learn that the two $G\times G$-representations, $(\mathcal{F}(G), \hat{r}^G)$ and $(\bigoplus_i(U^i\otimes U^{i*}), \bigoplus_i(\vartheta_i\otimes\tilde{\vartheta}_i))$, are equivalent. The isomorphism is given explicitly by the sum $S := \bigoplus_i S_i$ of all S_i from theorem 3.4.4, so that, if we choose a basis $\{e_r^{(i)}\}_r$ for U^i, it is now possible to write every element of $\mathcal{F}(G)$ as a sum over every of its components in the subspaces $U^i\otimes U^{i*}$.

Let $\{\tilde{e}_r^{(i)}\}_r$ be the dual basis of U^{i*}, induced by $\{e_r^{(i)}\}_r$. Denote with $R^{(i)}(g)$ the representing matrices of ϑ_i, with respect to $\{e_r^{(i)}\}_r$.

Every function $f\in\mathcal{F}(G)$ can be written in the form $f = \sum_{g\in G} f(g)\chi_g$, where χ_g stands for the characteristic function of g ($\chi_g(h)$ equals one if $g=h$, zero otherwise). We are seeking now those coefficients $\lambda^i_{r,r'}$ for which the following holds:

$$\chi_g = S\left(\sum_{i=1}^{N}\sum_{r,r'=1}^{n_i} \lambda^i_{r,r'} e_r^{(i)} \otimes \tilde{e}_{r'}^{(i)}\right). \tag{3.4.9}$$

Inserting (3.4.7) into (3.4.9), leads to:

$$\begin{aligned}
\chi_g(h) &\stackrel{!}{=} S\left(\sum_{i=1}^{N}\sum_{r,r'=1}^{n_i}\lambda^i_{r,r'}e^{(i)}_r\otimes\tilde{e}^{(i)}_{r'}\right)(h)\\
&= \sum_{i=1}^{N}\sum_{r,r'=1}^{n_i}\lambda^i_{r,r'}S_i(e^{(i)}_r\otimes\tilde{e}^{(i)}_{r'})(h)\\
&= \sum_{i=1}^{N}\sum_{r,r'=1}^{n_i}\lambda^i_{r,r'}\tilde{e}^{(i)}_{r'}(\vartheta_i(h^{-1})e^{(i)}_r)\\
&= \sum_{i=1}^{N}\sum_{r,r',r''=1}^{n_i}\lambda^i_{r,r'}R^{(i)}_{r'',r}(h^{-1})\tilde{e}^{(i)}_{r'}(e^{(i)}_{r''})\\
&= \sum_{i=1}^{N}\sum_{r,r'=1}^{n_i}\lambda^i_{r,r'}R^{(i)}_{r',r}(h^{-1}) \quad\Rightarrow
\end{aligned}$$

$$\sum_{h\in G}R^{(j)}_{k,l}(h)\chi_g(h) = \sum_{i=1}^{N}\sum_{r,r'=1}^{n_i}\lambda^i_{r,r'}\underbrace{\sum_{h\in G}R^{(j)}_{k,l}(h)R^{(i)}_{r',r}(h^{-1})}_{=\delta_{i,j}\delta_{k,r}\delta_{l,r'}\frac{|G|}{n_j}} \quad\Rightarrow$$

$$\lambda^i_{r,r'} = \frac{n_i}{|G|}R^{(i)}_{r,r'}(g) \tag{3.4.10}$$

It then follows, for f in $\mathcal{F}(G)$:

$$\begin{aligned}
f(g) &= \sum_{h\in G}f(h)\chi_h \qquad\qquad (3.4.11)\\
&= \sum_{h\in G}f(h)S\left(\sum_{i=1}^{N}\frac{n_i}{|G|}\sum_{r,r'=1}^{n_i}R^{(i)}_{r,r'}(h)e^{(i)}_r\otimes\tilde{e}^{(i)}_{r'}\right)(g)\\
&= \sum_{h\in G}f(h)\sum_{i=1}^{N}\frac{n_i}{|G|}\sum_{r,r'=1}^{n_i}R^{(i)}_{r,r'}(h)R^{(i)}_{r',r}(g^{-1})\\
&= \sum_{i=1}^{N}\sum_{r=1}^{n_i}\frac{n_i}{|G|}\sum_{h\in G}R^{(i)}_{r,r}(h^{-1})f(h^{-1}g).
\end{aligned}$$

From this formula we learn that the operator

$$\frac{n_i}{|G|}\sum_{r=1}^{n_i}\sum_{g\in G}R^{(i)}_{r,r}(g^{-1})g\cdot(\,-\,) \tag{3.4.12}$$

is the projection operator $\mathcal{F}(G)\to U^i\otimes U^{i*}$. Because of the (vector space) isomorphism $U^i\otimes U^{i*}\simeq \dim(U^i)U^i = n_iU^i$, one is led to consider, in (3.4.12), only the single terms in the sum over r:

$$\frac{n_i}{|G|}\sum_{g\in G} R^{(i)}_{r,r}(g^{-1})g\cdot(\,-\,). \tag{3.4.13}$$

We will now see that these are still projection operators, which turn out to be very useful for the algebraic description of G-bundles and of their quotients.

Before concluding this section, let us come back to remark 3.4.1, in order to further explain the role played by the operators defined through the formula (3.4.13) in the construction of a symmetry-preserving basis.

Consider, once again, a general representation, ϑ, of G. The representation space V is to be decomposed, as in equations (3.4.1) and (3.4.2), into irreducible subspaces. For a given j, let us assume that the representation ϑ_j appears in this decomposition and, moreover, that a symmetry-preserving basis $\{b_{\alpha,k}\}_{1\le\alpha\le m_j;1\le k\le n_j}$ for V^j is known.

That means: For α fixed, $\{b_{\alpha,k}\}_{1\le k\le n_j}$ is a basis for V^j_α and

$$\vartheta(g)b_{\alpha,l}=\sum_k d^{(j)}_{k,l}(g)b_{\alpha,k},$$

holds, i.e., the representing matrices in the space spanned by $\{b_{\alpha,k}\}_k$ are the same for all α. Defining now, in accordance with our previous considerations, the operators

$$P^{(j)}_{k,l}=\frac{n_j}{|G|}\sum_{g\in G} d^{(j)}_{k,l}(g^{-1})\vartheta(g), \tag{3.4.14}$$

we obtain, from the orthogonality relations for the matrix components of irreducible representations,

$$P^{(j)}_{k,l}b_{\alpha,k'}=\delta_{k,k'}b_{\alpha,k}. \tag{3.4.15}$$

The array $(b_{1,k},b_{2,k}\ldots,b_{m_j,k})$ is transformed by $P^{(j)}_{k,l}$ into $(b_{1,l},b_{2,l}\ldots,b_{m_j,l})$, componentwise. All other vectors $b_{\alpha,k'}$ $(k'\neq k)$ in the basis of V^j belong to the kernel of $P^{(j)}_{k,l}$. In this basis, $P^{(j)}_{k,l}$ is given by a matrix whose (l',k') component equals $\delta_{l,l'}\delta_{k,k'}$. From this, one concludes that the rank of $P^{(j)}_{k,l}$ is m_j.

Since ϑ_j appears in the decomposition of V, there exists $w_1\in V$ with $v_{1,1}:=P^{(j)}_{1,1}w_1\neq 0$. Moreover, $v_{1,1}$ must be a linear combination of the form

$$v_{1,1}=\lambda_1 b_{1,1}+\lambda_2 b_{2,1}+\cdots+\lambda_{m_j}b_{m_j,1}, \tag{3.4.16}$$

because in every V^j_α, $P^{(j)}_{k,k}$ projects on the k^{th} basis vector. Applying $P^{(j)}_{1,k}$ $(k=2,\ldots,n_j)$ to $v_{1,1}$ we obtain further vectors $v_{1,k}:=P^{(j)}_{1,k}v_{1,1}$ that, because of (3.4.15), are given by a linear combination of the vectors $b_{1,k},\ldots,b_{m_j,k}$, with the *same* coefficients λ_α as in (3.4.16):

$$v_{1,k}=\sum_{\alpha=1}^{m_j}\lambda_\alpha b_{\alpha,k}. \tag{3.4.17}$$

3.4.6 Theorem. *The vectors $v_{1,1}, \ldots, v_{1,n_j}$ form a basis for ϑ_j. When written in terms of this basis, the representing matrices are exactly the same as for $\{b_{\alpha,k}\}_k$.*

Proof.

$$\begin{aligned}
\vartheta(g)v_{1,k} &= \vartheta(g)\left(\sum_{\alpha=1}^{m_j} \lambda_\alpha b_{\alpha,k}\right) && (3.4.18)\\
&= \sum_{l=1}^{n_j} d^{(j)}_{l,k}(g) \sum_{\alpha=1}^{m_j} \lambda_\alpha b_{\alpha,l} && (3.4.19)\\
&= \sum_{l=1}^{n_j} d^{(j)}_{l,k}(g) v_{1,l}. && (3.4.20)
\end{aligned}$$

□

Since $P^{(j)}_{1,1}$ has rank m_j, we can still find other $m_j - 1$ independent vectors w_α ($\alpha = 2, \ldots, m_j$) such that $P^{(j)}_{1,1} w_\alpha \neq 0$. Defining now

$$v_{\alpha,k} := P^{(j)}_{1,k} w_\alpha \quad (k = 1, \ldots, n_j),$$

we obtain a symmetry-preserving basis for V^j, since the assertion of proposition 3.4.6 remains valid for $\alpha = 2, \ldots, n_j$:

$$\vartheta(g)v_{\alpha,k} = \sum_{l=1}^{n_j} d^{(j)}_{l,k}(g) v_{\alpha,l}.$$

The meaning of the operators $P^{(j)}_{k,l}$ becomes clearer by arranging the basis vectors $\{v_{\alpha,k}\}_{\alpha,k}$ in a "matrix", as follows:

$$\begin{tikzcd}
\tilde{V}^j_1 : & v_{1,1} & \cdots & v_{1,k} \arrow[loop above, "P^{(j)}_{k,k}"] \arrow[rr, bend left, "P^{(j)}_{k,l}"] & \cdots & v_{1,l} & \cdots & v_{1,n_j} \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\tilde{V}^j_\alpha : & v_{\alpha,1} & \cdots & v_{\alpha,k} & \cdots & v_{\alpha,l} & \cdots & v_{\alpha,n_j} \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\tilde{V}^j_{m_j} : & v_{m_j,1} & \cdots & v_{m_j,k} & \cdots & v_{m_j,l} & \cdots & v_{m_j,n_j}
\end{tikzcd}$$

Here, the space $\tilde{V}^j_\alpha = \text{span}\{v_{\alpha,k}\}_k$ bears the irreducible representation ϑ_j. Its representing matrices are, in the basis $\{v_{\alpha,k}\}_k$, the same as in the definition of $P^{(j)}_{k,l}$, namely $d^{(j)}_{k,l}$. Regarding the matrix M whose entries are (the vectors) $M_{\alpha,k} = v_{\alpha,k}$, we can then state the following:

- The operator $P^{(j)}_{k,k}$ is the projection in the subspace of V^j generated by the k^{th} column of M.
- The operator $P^{(j)}_{k,l}$ maps the k^{th} column of M into the l^{th} column, componentwise.

3.4.7 Remark. The non-uniqueness of the decomposition in eq. (3.4.2) is now obvious: the basis $\{v_{\alpha,k}\}_{\alpha,k}$ induces a decomposition

$$V^j = \tilde{V}^j_1 \oplus \cdots \oplus \tilde{V}^j_{m_j},$$

that depends both on the choice of the matrices in the definition of $P^{(j)}_{k,l}$, eq. (3.4.14), as in the choice of the vectors $w_1, \ldots, w_{m_j}$.

We have already mentioned that geometric objects as, for example, vector bundles, may be described from an algebraic point of view. In particular, we saw in the last section that in the case of a G-bundle η over a free G-space M it is possible to describe the quotient bundle η/G by means of the module $\Gamma^G(\eta)$. We now want to show that we can still go further and obtain a description of the quotient bundle as a subspace of the algebra $\mathcal{A} := C(M)$. In fact, we have already come close to this objective, in section 3.3: The entries of the projector in eq. (3.2.6) are elements of $\mathcal{A}$. Furthermore, by looking at eq. (3.2.5), we realize that the projection operators defined just above in eq. (3.4.14) also here come into play. Indeed, one can proceed analogously to the example of the regular representation, because an action $\rho : G \times M \to M$ gives place to a representation of G on $\mathcal{A}$ (as given in eq. (3.3.4)). The trivial representation

$$\mathcal{A}_1 := \{f \in \mathcal{A} |\, g \cdot f = f \;\; \forall g \in G\}, \tag{3.4.21}$$

is contained in $\mathcal{A}$ as a subalgebra. As we have already seen, there is an algebra isomorphism $C(M/G) \simeq A_1$ (in section 3.3 the notation $C(M)_+$ was used instead of $\mathcal{A}_1$). Let us consider eq.(3.2.5). There we obtained, from a single function (Ψ_{α_1}), $n_R \times n_R$ new functions, with the help of the representation R. The definition of these functions is, to a certain extent, singled out by the explicit form of the projector. This can be seen in eq. (3.2.6). In their form they are similar to the operators defined in eq. (3.4.14) and for this reason we will make the following ansatz:

Let R be an n_R-dimensional, irreducible, unitary representation of G. For $i, j \in \{1, \ldots, n_R\}$ set

$$\begin{aligned} E^R_{ij} : \mathcal{A} &\to \mathcal{A} \\ f &\mapsto E^R_{ij} f, \end{aligned}$$

where

$$E^R_{ij}f(x) := \frac{n_R}{|G|}\sum_{h\in G} R_{ij}(h^{-1})f(h^{-1}\cdot x) \qquad (x \in M). \tag{3.4.22}$$

The behavior of these functions under the G-action is easy to establish. For example, for $f \in \mathcal{A} = C(M)$ and $h \in G$, we have:

$$\begin{aligned}
\big[h\cdot(E^R_{ij}f)\big](x) &= (E^R_{ij}f)(h^{-1}\cdot x) \\
&= \frac{n_R}{|G|}\sum_{\tilde{h}\in G} R_{ij}(\tilde{h}^{-1})f((h\tilde{h})^{-1}\cdot x)) \\
&= \frac{n_R}{|G|}\sum_{h'\in G} R_{ij}(h'^{-1}h)f(h'^{-1}\cdot x) \\
&= \sum_{k=1}^{n_R} R_{kj}(h)\frac{n_R}{|G|}\sum_{h'\in G} R_{ik}(h'^{-1})f(h'^{-1}\cdot x) \\
&= \sum_{k=1}^{n_R} R_{kj}(h)E_{ik}f(x).
\end{aligned}$$

Summarizing,

$$h\cdot(E^R_{ij}f) = \sum_{k=1}^{n_R} R_{kj}(h)E^R_{ik}f. \tag{3.4.23}$$

If we keep i fixed, then the n_R functions $\{E^R_{i,1}f,\ldots,E^R_{i,n_R}f\}$ form (in case they do not vanish everywhere) a basis for the representation R, exactly as in thm. 3.4.6. On the other hand, letting first h act on f, we obtain

$$E^R_{ij}(h\cdot f) = \sum_{k=1}^{n_R} R_{ik}(h)(E^R_{kj}f). \tag{3.4.24}$$

The maps E^R_{ij} are, analogously to the $P^{(j)}_{kl}$, orthogonal to each other: For R_1 and R_2 two irreducible, unitary representations, one has

$$E^{R_1}_{ik}E^{R_2}_{mn} = \delta_{R_1,R_2}\delta_{km}E^{R_1}_{in}. \tag{3.4.25}$$

Let f be any function in $\mathcal{A}$. It is to be expected, from eqns. (3.4.11) and (3.4.12), that f may be written in the following form:

$$f = \sum_R\sum_{i=1}^{n_R} E^R_{ii}f, \tag{3.4.26}$$

where the first sum is performed over all inequivalent, irreducible representations. Let us check that this is indeed the case:

$$\begin{aligned}
\sum_R \sum_{i=1}^{n_R} E_{ii}^R f(x) &= \sum_R \sum_{i=1}^{n_R} \frac{n_R}{|G|} \sum_{h\in G} R_{ii}^*(h) f(h^{-1}\cdot x) \\
&= \sum_R \sum_{h\in G} \frac{n_R}{|G|} \underbrace{\sum_{i=1}^{n_R} R_{ii}^*(h)}_{\chi_R^*(h)} f(h^{-1}\cdot x) \qquad (n_R = \chi_R(e)) \\
&= \sum_{h\in G} \frac{1}{|G|} \underbrace{\sum_R \chi_R(e)\chi_R^*(h)}_{=|G|\delta_{e,h}} f(h^{-1}\cdot x) \\
&= f(x).
\end{aligned}$$

This shows that we have found the following decomposition of $\mathcal{A}$:

$$\mathcal{A} = \bigoplus_{R,i} \mathcal{A}_{R,i}\ , \tag{3.4.27}$$

with $\mathcal{A}_{R,i} := E_{ii}^R(A)$.

The trivial representation $\mathcal{A}_1$ appears exactly once in the decomposition. As stated above, it is an algebra isomorphic to $C(M/G)$. It is clear that the spaces $\mathcal{A}_{R,i}$ are $\mathcal{A}_1$-modules, since $h\cdot f = f$ for $h \in G$ and $f \in \mathcal{A}_1$, so it follows that $fg \in \mathcal{A}_{R,i}\ \forall f \in \mathcal{A}_1, g \in \mathcal{A}_{R,i}$. What is not yet so clear is whether these modules are finitely generated and projective.

3.4.8 Theorem. *The $\mathcal{A}_1$-module $\mathcal{A}_{R,i} := E_{ii}^R(\mathcal{A})$ is, for any unitary, irreducible representation R and $i \in \{1,\dots,n_R\}$, finitely generated.*

Proof. For $f \in \mathcal{A}_{R,i}$ we have $E_{ii}^R f = f$; that means:

$$\frac{n_R}{|G|}\sum_{h\in G} R_{ii}^*(h) f(h^{-1}\cdot x) = f(x) \quad \forall x \in M.$$

Let now $\{\phi_\alpha\}_{\alpha\in I}$ be, as in section 3.2, a partition of unity for M/G, subordinated to the cover $\{U_\alpha\}_{\alpha\in I}$. Because of the isomorphism $C(M/G) \cong \mathcal{A}_1$, we may regard ϕ_α as an element from $\mathcal{A}_1$. In particular, the notations $\phi_\alpha(x)$ ($\phi \in \mathcal{A}_1$) and $\phi_\alpha([x])$ ($\phi \in C(M/G)$) will be used interchangeably, according to convenience. It is clear, from eqns. (3.2.5) and (3.4.22), that

$$\Psi_{ij}^\alpha = E_{ij}^R \Psi_{\alpha_1} \tag{3.4.28}$$

holds, with Ψ_{α_1} as defined in the example of section 3.2. One also sees that (with the same conventions as in section 3.2)

$$\mathrm{Supp}(f\phi_\alpha) \subseteq q^{-1}(U_\alpha) = \bigcup_{j=1}^{n_R} \tilde{U}_{\alpha_j}. \tag{3.4.29}$$

Furthermore, $f\phi_\alpha$ can be "split" in $N \equiv |G|$ different functions f_{α_i}, in such a way that

$$\phi_\alpha f = \sum_{j=1}^{N} f_{\alpha_j}, \quad \text{mit} \quad \text{Supp}(f_{\alpha_j}) \subset \tilde{U}_{\alpha_j} \forall j \tag{3.4.30}$$

holds. Namely, defining

$$f_{\alpha_j}(x) := \begin{cases} (\phi_\alpha f)\big|_{\tilde{U}_{\alpha_j}}(x), & \text{in case } \; x \in \tilde{U}_{\alpha_j} \\ 0, & \text{otherwise,} \end{cases} \tag{3.4.31}$$

we see that (3.4.30) is satisfied, since for α fixed all $\tilde{U}_{\alpha_j}$ are disjoint. The maps f_{α_j} are well-defined, continuous functions, because $\text{Supp}(\phi_\alpha f) \subseteq \tilde{U}_{\alpha_j}$. By defining the (invariant) functions

$$f^s_{\alpha_j} := \sum_{h\in G} h \cdot f_{\alpha_j}, \tag{3.4.32}$$

the following identity is then obtained:

$$\sum_{h_j\in G} f^s_{\alpha_j}(x)\Psi_{\alpha_1}(h_j^{-1}\cdot x) = \phi_\alpha([x])^2 f(x). \tag{3.4.33}$$

It is enough, in order to check (3.4.33), to consider $x \in \tilde{U}_{\alpha_k}$, since both sides of the equation have support in $q^{-1}(U_\alpha) = \cup_{j=1}^N \tilde{U}_{\alpha_j}$. One then obtains:

$$\begin{aligned}
\sum_{h_j\in G} f^s_{\alpha_j}(x)\Psi_{\alpha_1}(h_j^{-1}\cdot x) &= f^s_{\alpha_k}(x)\Psi_{\alpha_1}(h_k^{-1}\cdot x) \\
&= f^s_{\alpha_k}(x)\phi_\alpha([x]) \\
&= \sum_{h_j\in G} f_{\alpha_k}(h_j^{-1}\cdot x)\phi_\alpha([x]) \\
&= f_{\alpha_k}(x)\phi_\alpha([x]) \\
&= \phi_\alpha([x])^2 f(x).
\end{aligned}$$

Summing over α,

$$\sum_\alpha \sum_{h_j\in G} f^s_{\alpha_j}(x)\Psi_{\alpha_1}(h_j^{-1}\cdot x) = \sum_\alpha \phi_\alpha([x])^2 f(x) = f(x).$$

That means

$$f = \sum_{\alpha\in I}\sum_{h_j\in G} f^s_{\alpha_j}(h_j \cdot \Psi_{\alpha_1}),$$

so that

$$\begin{aligned}
f(x) &= E^R_{ii} f(x) = \Big[E^R_{ii}\Big(\sum_{\alpha\in I}\sum_{h_j\in G} f^s_{\alpha_j}(h_j\cdot\Psi_{\alpha_1})\Big)\Big](x)\\
&= \Big[\sum_{\alpha\in I}\sum_{h_j\in G} E^R_{ii}\Big(f^s_{\alpha_j}(h_j\cdot\Psi_{\alpha_1})\Big)\Big](x)\\
&= \sum_{\alpha\in I}\sum_{h_j\in G} f^s_{\alpha_j}(x)\big[E^R_{ii}(h_j\cdot\Psi_{\alpha_1})\big](x)\\
&= \sum_{\alpha\in I}\sum_{h_j\in G} f^s_{\alpha_j}(x)\sum_{k=1}^{n_R} R_{ik}(h_j)\big[\underbrace{E^R_{ik}\Psi_{\alpha_1}}_{=\Psi^\alpha_{ik}}\big](x)\\
&= \sum_{\alpha\in I}\sum_{k=1}^{n_R}\underbrace{\Big(\sum_{h_j\in G} f^s_{\alpha_j}(x)R_{ik}(h_j)\Big)}_{\in\mathcal{A}_1}\Psi^\alpha_{ik}(x).
\end{aligned}$$

We have, therefore:

$$f\in\mathcal{A}_{R,i}\quad\Rightarrow\quad f=\sum_{\alpha\in I}\sum_{k=1}^{n_R}\Big(\sum_{h_j\in G} f^s_{\alpha_j}R_{ik}(h_j)\Big)\Psi^\alpha_{ik}. \tag{3.4.34}$$

The assertion follows, since $E^R_{ii}\Psi^\alpha_{ik}=\Psi^\alpha_{ik}$, so that $\{\Psi^\alpha_{ik}\}^{\alpha\in I}_{k\in\{1,\ldots,n_R\}}$ constitutes a set of generators of $\mathcal{A}_{R,i}$ over $\mathcal{A}_1$. □

In section 3.2 we have seen how to associate a projector P^R to every representation R. The explicit form of this projector is given by eq. (3.2.6). On the other hand, the decomposition -eq. (3.4.27)- of $\mathcal{A}$ shows how every irreducible, unitary representation R gives place to n_R $\mathcal{A}_1$-modules $\mathcal{A}_{R,1},\ldots,\mathcal{A}_{R,n_R}$ that turn out to be finitely generated. A natural question is whether these modules are also projective and, in that case, what is their relation to the above mentioned projectors P^R. The answer is given by the following proposition.

3.4.9 Proposition. *There is, for every unitary, irreducible representation R an integer N_R such that $P^R(\mathcal{A}_1^{N_R})\cong\mathcal{A}_{R,i}$ for every $i\in 1,\ldots,n_R$.*

Proof. Let us identify the sets of generators of both modules by sending the column of P^R that is labeled by the indexes (β,k) to the generator $\Psi^{\beta*}_{ik}$ of $\mathcal{A}_{R,i}$. The assertion now follows from the following identity:

$$\sum_{\alpha\in I}\sum_{l=1}^{n_R}\Psi^{\alpha*}_{k',l}\Psi^\alpha_{k,l}=\frac{n_R^2}{|G|^2}\delta_{k',k}. \tag{3.4.35}$$

In fact, from (3.2.6):

$$\sum_{\alpha\in I}\sum_{k=1}^{n_R}\big(P^R_{\alpha\beta}([x])\big)_{kj}\Psi^{\alpha*}_{ik}(x) \;=\; \sum_{\alpha\in I}\sum_{k=1}^{n_R}\frac{|G|^2}{n_R^2}\sum_{k'=1}^{n_R}\Psi^\alpha_{k'k}(x)\Psi^{\beta*}_{k'j}(x)\Psi^{\alpha*}_{ik}(x). \tag{3.4.36}$$

From the identity above, it follows that the right hand side of the last expression equals $\Psi_{ij}^{\beta *}(x)$. $\square$

4 The spin zero case

As already pointed out in the introduction, it was for spin zero particles that the Fermi-Bose alternative was deduced from considerations from the topology of the configuration space. Although this special case has been extensively studied (see, for example,[LD71, LM77, HMS89]), it is still controversial, in connection to the Spin-Statistics relation[Pes03b, Pes03a, AM03]. In fact, although any attempt to relate the spin of a system of identical particles with the statistics they obey should take, in principle, all values of the spin into account, there are some hints pointing to the idea that if it were possible to show that spin zero particles must be bosons, then the proof of the non-zero spin case could follow from it*. In the present chapter, however, we will use this example mainly in order to illustrate our approach, showing how well-known results can be recast in a very clear and compact form, using the techniques developed in the previous chapter. After recalling the standard construction of line bundles over the sphere, we consider their description by means of projective modules. The relevance of this example (bundles over the sphere), for us, is twofold. Firstly, the sphere is the covering space of the configuration space of a system of two identical particles (the projective space). It is of interest for us, therefore, to have a description of (equivariant) line bundles over the sphere, in particular in terms of projectors. In addition to this, the rotational symmetry of the sphere can be used to study the correct definition of angular momentum operators (when the configuration space is the sphere). This ideas will be used later as the starting point for the definition of spin operators, when spin is taken into account. In the last part of the chapter, the projective space, as configuration space for two spin-zero particles, is considered.

4.1 S^2 as Configuration Space

4.1.1 Line Bundles over S^2

Complex line bundles over the sphere S^2 can be described in a very compact manner by means of projective modules. Recall that the 2-sphere can be identified with the complex projective space $\mathbb{C}P^1$. Using homogeneous coordinates $[z_0 : z_1]$, $\mathbb{C}P^1$ can be described locally by means of two charts. Setting

$$U_0 := \{[z_0 : z_1] \mid z_0 \neq 0\} \quad \text{and} \quad U_1 := \{[z_0 : z_1] \mid z_1 \neq 0\},$$

*Some ideas in this direction are presented in chapter 6.

define local charts as follows:

$$\begin{aligned} z: \qquad U_0 &\longrightarrow \mathbb{C} \\ [z_0 : z_1] &\longmapsto z_1/z_0 \end{aligned}$$

and

$$\begin{aligned} \zeta: \qquad U_1 &\longrightarrow \mathbb{C} \\ [z_0 : z_1] &\longmapsto z_0/z_1. \end{aligned}$$

With this we obtain the following transition function: $\zeta \circ z^{-1}(w) = 1/w$. On the sphere, consider the stereographic projections

$$\begin{aligned} \psi_N: \quad S^2 \setminus \{N\} &\longrightarrow \quad \mathbb{C} \\ (x_1, x_2, x_3) &\longmapsto \tfrac{x_1 + ix_2}{1 - x_3} \end{aligned}$$

and

$$\begin{aligned} \psi_S: \quad S^2 \setminus \{S\} &\longrightarrow \quad \mathbb{C} \\ (x_1, x_2, x_3) &\longmapsto \tfrac{x_1 + ix_2}{1 + x_3}. \end{aligned}$$

Identifying $S^2 \setminus \{N\}$ with U_0 and $S^2 \setminus \{S\}$ with U_1 by means of the maps $(x_1, x_2, x_3) \mapsto [1 - x_3 : x_1 + ix_2]$ and $(x_1, x_2, x_3) \mapsto [x_1 - ix_2 : 1 + x_3]$, respectively, we obtain a diffeomorphism $S^2 \approx \mathbb{C}P^1$. In terms of spherical coordinates, this identification takes the form

$$z \equiv \frac{e^{i\phi} \sin\theta}{1 - \cos\theta}, \qquad \zeta \equiv \frac{e^{-i\phi} \sin\theta}{1 + \cos\theta}. \tag{4.1.1}$$

Now we construct a line bundles on the sphere, the *Hopf* bundle, as follows. Identifying S^2 with $\mathbb{C}P^1$ as explained above, consider the trivial bundle over $\mathbb{C}P^1$ with total space $\mathbb{C}P^1 \times \mathbb{C}^2$. Since in $\mathbb{C}P^1$ all scalar multiples of a point (z_0, z_1) in $\mathbb{C}^2$ belong to the same equivalence class $[z_0 : z_1]$, there is an obvious projection, in which the second component of a point $([z_0 : z_1], (\lambda_0, \lambda_1)) \in \mathbb{C}P^1 \times \mathbb{C}^2$ is mapped into the complex line generated by any representative $(z_0, z_1) \in \mathbb{C}^2$. Explicitly, we obtain a line bundle $\mathcal{L}$, as a sub-bundle of the trivial bundle $\mathbb{C}P^1 \times \mathbb{C}^2 \to \mathbb{C}P^1$, where the total space of $\mathcal{L}$ is the image of the projection map

$$\begin{aligned} \pi: \quad \mathbb{C}P^1 \times \mathbb{C}^2 \qquad &\longrightarrow \quad \mathbb{C}P^1 \times \mathbb{C}^2 \\ ([z_0 : z_1], (\lambda_0, \lambda_1)) \quad &\longmapsto \quad \left([z_0 : z_1], \frac{\lambda_0 \bar{z}_0 + \lambda_1 \bar{z}_1}{z_0 \bar{z}_0 + z_1 \bar{z}_1} (z_0, z_1) \right). \end{aligned} \tag{4.1.2}$$

There is also an inclusion map $\imath : \mathcal{L} \to \mathbb{C}P^1 \times \mathbb{C}^2$. The projection and the inclusion maps induce maps π_* and $\imath_*$ between the corresponding spaces of sections, so they can be composed with the trivial connection $\nabla^{(0)} = (d, d)$ on $\mathbb{C}P^1 \times \mathbb{C}^2 \to \mathbb{C}P^1$, in order to obtain a connection on $\mathcal{L}$ (the Grassmann connection) by setting $\nabla := (\mathrm{id} \otimes \pi_*) \circ \nabla^{(0)} \circ \imath_*$.

In terms of the local parametrisation $\mathbb{C} \to \mathbb{C}P^1 : w \mapsto [1 : w]$, and of the local section $\sigma(z) = ([1 : z], (1, z))$, the connection takes the following form:

$$\nabla(\sigma) = \sigma \otimes \left(\frac{\bar{w} dw}{1 + w\bar{w}} \right). \tag{4.1.3}$$

The curvature of this connection, written in terms of local coordinates on the sphere, is given by [MT97]:

$$F^{\nabla} = \frac{i}{2} \sin\theta d\theta \wedge d\phi. \tag{4.1.4}$$

We can use the map π to construct the projector that describes the line bundle $\mathcal{L}$. Let us write x for $[z_o : z_1]$, the second component of the map π is a map projecting (λ_0, λ_1) onto the complex line generated by (z_0, z_1). At each point x we can then compute the matrix $p(x)$ corresponding to this projection. From the definition of π we have:

$$\begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \frac{\lambda_0 \bar{z}_0 + \lambda_1 \bar{z}_1}{z_0 \bar{z}_0 + z_1 \bar{z}_1} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}. \tag{4.1.5}$$

From this we obtain the following matrix components for $p(x)$:

$$\begin{aligned} p_{11}(x) &= \frac{z_0 \bar{z}_0}{z_0 \bar{z}_0 + z_1 \bar{z}_1}, & p_{12}(x) &= \frac{z_0 \bar{z}_1}{z_0 \bar{z}_0 + z_1 \bar{z}_1}, \\ p_{21}(x) &= \frac{z_1 \bar{z}_0}{z_0 \bar{z}_0 + z_1 \bar{z}_1}, & p_{22}(x) &= \frac{z_1 \bar{z}_1}{z_0 \bar{z}_0 + z_1 \bar{z}_1}. \end{aligned}$$

Let us express $p(x)$ in terms of local coordinates on the sphere. On U_0 we have

$$z(x) = \frac{z_1}{z_0} \equiv \frac{e^{i\phi} \sin\theta}{1 - \cos\theta},$$

so that

$$\begin{aligned} p_{11}(x) &= \frac{z_0 \bar{z}_0}{z_0 \bar{z}_0 + z_1 \bar{z}_1} \\ &= \frac{1}{1 + \frac{\sin^2\theta}{(1-\cos\theta)^2}} \\ &= \frac{1 + \cos\theta}{2}, \end{aligned}$$

$$\begin{aligned} p_{12}(x) &= \frac{z_0 \bar{z}_1}{z_0 \bar{z}_0 + z_1 \bar{z}_1} \\ &= \frac{z_0}{z_1} \frac{1}{1 + |z_0/z_1|^2} \\ &= \frac{1 - \cos\theta}{e^{i\phi} \sin\theta} \left(1 + \frac{(1 - \cos\theta)^2}{\sin^2\theta} \right)^{-1} \\ &= \frac{e^{-i\phi}}{2} \sin\theta, \end{aligned}$$

and so on, leading to

$$p = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & 1-\cos\theta \end{pmatrix}. \tag{4.1.6}$$

Notice that this is a globally well-defined, matrix-valued function on the sphere. We obtain the same result if we work on the chart U_1. There, we have

$$\zeta(x) = \frac{z_0}{z_1} \equiv \frac{e^{-i\phi}\sin\theta}{1+\cos\theta}.$$

The matrix components obtained are the same. For example,

$$\begin{aligned} p_{12}(x) &= \frac{z_0\bar{z}_1}{z_0\bar{z}_0 + z_1\bar{z}_1} \\ &= \frac{z_0}{z_1}\frac{1}{1+|z_0/z_1|^2} \\ &= \frac{e^{i\phi}\sin\theta}{1+\cos\theta}\left(1+\frac{\sin^2\theta}{(1+\cos\theta)^2}\right)^{-1} \\ &= \frac{e^{-i\phi}}{2}\sin\theta. \end{aligned}$$

In terms of the projector p, the connection takes the form pd[GBVF01]. As an explicit calculation shows, the curvature form tr$(pdpdp)$ coincides with F^∇. It is a well-known fact that equivalence classes of line bundles over the sphere are in one to one correspondence with the integers. The integer labelling a specific line bundle ξ over the sphere may be obtained by choosing any connection on ξ and integrating its curvature two-form over the sphere. Up to a factor of $2\pi i$, this integral is the first Chern number of the bundle. A projector describing a line bundle on the sphere with Chern number equal to n is given by

$$p_n = \frac{1}{2}\begin{pmatrix} 1+\cos n\theta & \sin n\theta e^{-i\phi} \\ \sin n\theta e^{i\phi} & 1-\cos n\theta \end{pmatrix}. \tag{4.1.7}$$

The reader is suggested to look at references [GBVF01], [Pas01] and [Lan01] for further considerations on this example.

4.1.1 Remark. It is interesting to consider the example of the sphere as configuration space in connection to Dirac's magnetic monopole. It is well known that the presence of a magnetic monopole forces the configuration space of an electron to be (effectively) S^2. Dirac's quantization condition states that the electric and magnetic charges are related through $eg/\hbar c = n/2$, with n integer. In this case, the electron's wave function is a section of a line bundle on the sphere, with Chern number n. This fact has many consequences, one of them being that a correct definition of *angular momentum operators* requires a careful consideration of the bundle structure. In fact, although the electron's spin is being neglected, it turns out that the physically admissible angular momentum operator can produce a change in sign in the electron's wave function (when n is odd) or no change at all (when n is even) under a 2π rotation. The reader is advised to look at [BL81], p.203, for a physically-motivated discussion of this effect. The relation of these angular momentum operators to the lifting of the $SU(2)$ action on the sphere to the line bundles is made explicit in [GBVF01].

4.1.2 Angular Momentum coupled to a Magnetic Field

Let us now consider an example related to *geometric phases*. It has been discussed by Berry and Robbins in [RB94] and is relevant for us insofar as it seems to have provided the motivation for the idea that eventually led these authors to the construction of the spin basis proposed in [BR97]. We will see how the bundles underlying this example can also be described in terms of projectors as the one displayed in eq. (4.1.7).

Consider a quantum mechanical system described by the Hamiltonian

$$H = \vec{B} \cdot \vec{J} \tag{4.1.8}$$

describing the coupling of a magnetic field $\vec{B}$ with an angular momentum $\vec{J}$ (electron or nuclear spin, for instance). For a given value of $\vec{B}$, the eigenvectors of H can be expressed in terms of $|j, m\rangle$ (the eigenvectors of J_z and J^2), as follows:

$$|j, n(\vec{B})\rangle := D(\vec{B})|j, m\rangle, \tag{4.1.9}$$

where D is determined by the polar angles, φ and θ, of $\vec{B}$:

$$D(\vec{B}) := e^{-iJ_z\varphi}e^{-iJ_y\theta}. \tag{4.1.10}$$

Suppose that the magnitude of the magnetic field is kept fixed, but different possible orientations are allowed. In this case, the Hamiltonian above can be seen as depending on the parameters (φ, θ), the parameter space being the sphere. For such a system time evolution leads to what is known as geometric phase: If the parameters appearing in the Hamiltonian change† in time describing a loop on parameter space: $t \mapsto \vec{B}(t)$ ($\vec{B}(0) = \vec{B}(1)$), the original state $|j, m(\vec{B}(0))\rangle$ gains a phase (additional to the usual, dynamical one) which is given by the following expression:

$$\exp\left(-i\int_0^1 \langle j, m(\vec{B}(t))|\frac{d}{dt}|j, m(\vec{B}(t))\rangle\right) \tag{4.1.11}$$

As pointed out in [Sim83], this phase may be understood as the holonomy of a connection on a vector bundle which is naturally determined by the parameter-dependent Hamiltonian. The Chern number corresponding to this bundle can be shown to be equal to $2m$.

A description of the bundle, in terms of projectors, can be obtained as follows. For j fixed, the vectors $|j, m\rangle$ $(-|j| \leq m \leq |j|)$ form a basis for the representation space V^j.

†This change is usually assumed to be adiabatic, but more general situations are also allowed.

On V^j, consider the projection $P_m^{(0)}$ onto the subspace generated by $|j,m\rangle$. Using this, we can define a family of projections, parametrised by $\vec{B}$, namely the projections on the subspace generated by $|j,m(\vec{B})\rangle$:

$$P_m := D(\vec{B})P_m^{(0)}D(\vec{B})^\dagger. \tag{4.1.12}$$

This projector describes a bundle over the sphere (assuming $|\vec{B}|=1$), with total space $\{(\vec{B},\vec{v}) \in S^2 \times V^j : \ P_m(\vec{B})\vec{v} = \vec{v}\}$. In other words, we are considering a bundle over the parameter space of H, where the fibers are generated by the (parameter-dependent) eigenvectors of H. The curvature of the connection $P_m d$, integrated over S^2, gives us the Chern number associated to P_m:

$$c_1(P_m) = \frac{1}{2\pi i}\int_{S^2} \mathrm{tr}(P_m dP_m dP_m).$$

The integral can be evaluated as follows. From the definitions of D and P_m, we have:

$$dP_m = i[P_m, J_z]d\varphi + [P_m, DJ_yD^\dagger]d\theta.$$

Using $D^\dagger J_z D = \cos\theta J_z - \sin\theta J_x$ and taking the cyclicity of the trace into account, one then obtains $c_1(P_m) = 2m$. In the case of the sphere, equivalence classes of line bundles can be labelled by their Chern number, *i.e.* two line bundles over the sphere are isomorphic if and only if their Chern numbers coincide. Hence we see, from the previous computation, that $P_{n/2} \cong p_n$.

The case $m = 0$ is particularly interesting. The curvature of the connection it gives place to vanishes, implying that there is no geometrical phase arising from closed loops in the parameter space (sphere). However, it turns out that under a cycle going from some point on the sphere to its antipode, the eigenstates change only in a sign, which can also be interpreted as a geometrical phase (for a general value of m, the states $|j,m(-\vec{B})\rangle$ and $|j,-m(\vec{B})\rangle$ coincide, up to a phase factor. This implies that $|j,m=0(\vec{B})\rangle$ and $|j,m=0(-\vec{B})\rangle$ differ by a phase, and it can be shown that the value of this phase, $(-1)^j$, is independent of the path[‡] joining the points $\vec{B}$ and $-\vec{B}$, see [RB94]. When $m=0$ one can, therefore, identify antipodal points on the parameter space, giving place to an effective parameter space, obtained under the identification $\vec{B} \sim -\vec{B}$. This, and the fact that the phase depends on the value j of the angular momentum, provides a motivation to think that the structures appearing in this example could be related to the Spin-Statistics problem (see the remarks at the end of [RB94]).

4.2 $\mathbb{R}\mathrm{P}^2$ as Configuration Space

4.2.1 Line bundles over $\mathbb{R}\mathrm{P}^2$

As an application of the methods discussed in section 3.4 above we will work out the case of the projective space, because this is the relevant space to be considered when

[‡]This path, when projected to $\mathbb{R}P^2$, must belong to the non-trivial element of the fundamental group.

discussing the problem of two indistinguishable particles. Of particular importance is the consideration of line bundles on it, together with the connections and $SU(2)$-bundle structures naturally associated to them.

We begin by considering the situation of section 3.4, in the specific case where $\widetilde{M} = S^2$, $G = \mathbb{Z}_2$ and $M = \mathbb{R}P^2$. Here we have $\mathcal{A} = C(S^2)$. There are only two irreducible representations of $\mathbb{Z}_2$: the trivial one, denoted by R_+ and the "sign" representation $\sigma \mapsto (-1)^\sigma$, denoted by R_-. Equation (3.4.27) in this case thus reads, with $\mathcal{A}_+ = \mathcal{E}_1^{R_+}(\mathcal{A})$ and $\mathcal{A}_- := \mathcal{E}_1^{R_-}(\mathcal{A})$:

$$\mathcal{A} = C(S^2) = \mathcal{A}_+ \oplus \mathcal{A}_-. \tag{4.2.1}$$

$\mathcal{A}_+$ denotes the *subalgebra* of symmetric functions on the sphere, and $\mathcal{A}_-$ the $\mathcal{A}_+$*-module* of antisymmetric ones. With $S^2 = \{x = (x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3 = 1\}$ and $\mathbb{R}P^2 = S^2/\{x \sim -x\}$, we have

$$\begin{array}{rcl} q : S^2 & \to & \mathbb{R}P^2 \\ x & \mapsto & [x] = \{x, -x\} \end{array}$$

Since $\mathcal{A}_+$ is isomorphic to $C(\mathbb{R}P^2)$, which in turn may be identified with the module of sections of the trivial line bundle (L_+) over $\mathbb{R}P^2$, proposition 3.4.9 applied to R_+ gives no new information. Applied to R_- instead, it allows us to identify $\mathcal{A}_-$ as the module of sections of a line bundle (L_-) over $\mathbb{R}P^2$. Let us use the shorter notation p_- for the corresponding projector. Its matrix components can be obtained as follows. Consider the situation of section 3.2 for the case $M = S^2$, $G = \mathbb{Z}_2$ and $\eta = (S^2 \times \mathbb{C}, \pi_1, S^2)$. As representation we choose R_-. Using the same conventions as in the example, consider the following open cover of $S^2/\mathbb{Z}_2$ ($\alpha = 1, 2, 3$):

$$U_\alpha = \{[x] \equiv [(x_1, x_2, x_3)] \in S^2/\mathbb{Z}_2 \mid x_\alpha \neq 0\}.$$

It comes from the projection of a cover of S^2 given by open sets ($\alpha = 1, 2, 3;\ i = 1, 2$)

$$\tilde{U}_{\alpha_i} = \{(x_1, x_2, x_3) \in S^2 \mid (-1)^i x_\alpha < 0\}$$

and hence, according to equation (3.2.2), the transition functions for the bundle $\eta/\mathbb{Z}_2$ are

$$g_{\alpha,\beta}([x]) = \mathrm{sgn}(x_\alpha)\mathrm{sgn}(x_\beta). \tag{4.2.2}$$

Note that the right hand side is independent of the choice of representative $(x_1, x_2, x_3) \in [x]$. According to definition 3.2.1, we need a partition of unity, subordinated to the cover $\{U_\alpha\}_\alpha$. Choosing $\phi_\alpha([x]) = |x_\alpha|$, we obtain the matrix components of p_-:

$$\begin{array}{rcl} (p_-)_{\alpha\beta} : \mathbb{R}P^2 & \to & \mathbb{C} \\ {[x]} & \mapsto & x_\alpha x_\beta. \end{array} \tag{4.2.3}$$

Note that even though each matrix component of p_- is defined in terms of a representative $x \in [x]$, it is an even function on S^2 and therefore equivalent to a function on $\mathbb{R}P^2$.

Remark. We have thus obtained an isomorphism of $\mathcal{A}_+$-modules under which the *antisymmetric functions* $x_1, x_2, x_3 \in \mathcal{A}_-$ can be regarded as a set of generators for $\Gamma(L_-)$. Explicitly, we can make the following identification:

$$x_i \equiv \begin{pmatrix} x_i x_1 \\ x_i x_2 \\ x_i x_3 \end{pmatrix}.$$

Thus, under this identification, any section of L_- can be expressed as a linear combination of the form $a_1x_1 + a_2x_2 + a_3x_3$, with $a_i \in \mathcal{A}_+$. The (module) multiplication of the function a_i (regarded as element of $C(\mathbb{R}P^2)$) and x_i (regarded as a section of L_-) is implemented in the isomorphic image of $\Gamma(L_-)$ inside $\mathcal{A}$ by the usual multiplication of functions on S^2.

Some of the properties of p_- are:

- The line bundle L_- represented by the projector p_- is not trivial. This can be seen as follows: the isomorphisms $\Gamma(L_-) \simeq p_-(\mathcal{A}_+^3) \simeq \mathcal{A}_-$ allow us to represent every section as an antisymmetric function on the sphere. One can then use the fact that every such function vanishes at some point to show that we cannot find a nowhere vanishing section of L_-.

- The connection naturally associated to this bundle is defined as $\nabla = p_-\, d$. A direct calculation shows that it has vanishing curvature, i.e., it is a flat connection. Its action on an arbitrary section can be found, using the Leibniz rule, once we know its action on the set $\{x_i\}_i$ of generators. By direct computation we find:

$$\begin{aligned} p_- d \begin{pmatrix} x_i x_1 \\ x_i x_2 \\ x_i x_3 \end{pmatrix} &= \begin{pmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3^2 \end{pmatrix} \begin{pmatrix} d(x_i x_1) \\ d(x_i x_2) \\ d(x_i x_3) \end{pmatrix} \\ &= \sum_j \begin{pmatrix} x_j x_1 \\ x_j x_2 \\ x_j x_3 \end{pmatrix} d(x_i x_j), \end{aligned}$$

leading to the following formula for ∇:

$$\nabla x_i = \sum_{j=1}^{3} x_j \otimes d(x_i x_j). \tag{4.2.4}$$

Here the product $x_i x_j$ is to be understood as an element of $C(\mathbb{R}P^2)$, whereas $x_i \in \mathcal{A}_-$ is to be *interpreted* as an element in $p_-(\mathcal{A}_+^3)$, or a section on L_-. Note that the exterior differential d makes reference to $\mathbb{R}P^2$.

The formula above can also be obtained, taking advantage of the last remark, if we work exclusively on the sphere. Indeed, from $x_1^2+x_2^2+x_3^2=1$, we have:

$$\begin{aligned}
dx_1 &= \frac{x_1}{2}\cdot 0 + dx_1 \\
&= \frac{x_1}{2}d(x_1^2+x_2^2+x_3^2) + dx_1 \\
&= x_1(x_1dx_1+x_2dx_2+x_3dx_3)+dx_1 \\
&= (x_1^2+1)dx_1 + x_1x_2dx_2 + x_1x_3dx_3 \\
&= (2x_1^2+x_2^2+x_3^2)dx_1 + x_1x_2dx_2 + x_1x_3dx_3 \\
&= (2x_1^2dx_1) + (x_2^2dx_1+x_1x_2dx_2) + (x_3^2dx_1+x_1x_3dx_3) \\
&= x_1d(x_1x_1) + x_2d(x_1x_2) + x_3d(x_1x_3)
\end{aligned}$$

and similarly for x_2 and x_3, thus providing an alternative derivation of (4.2.4).

- Making use of (4.2.4), the holonomy group of ∇ can be computed. It depends only on the fundamental group of $\mathbb{R}P^2$ (because ∇ is flat) and it turns out to be isomorphic to $\mathbb{Z}_2$.

- Using standard results of topology, it can be shown that the only line bundles on $\mathbb{R}P^2$ are in fact L_+ and L_-. Moreover, a bundle ξ of rank k can always be written as the sum of a line bundle and the trivial bundle of rank $k-1$.

- The bundle L_- is $SU(2)$ equivariant, in the sense of definition 3.1.1.

- The $SU(2)$ action on L_- and parallel transport on it, by means of ∇ "coincide" in a certain sense, specified below (see eq. (4.2.25)).

The last two points deserve special attention. We have obtained the projector p_- using the method outlined in section 3.4. This construction is well adapted to deal with the quotient map $q : S^2 \to \mathbb{R}P^2$ and with the calculation of the holonomy of ∇, but there is an independent way of arriving at a projector describing the same bundle as p_- which takes the $SU(2)$ equivariance explicitly into account. This construction of the projector, based on its $SU(2)$ symmetry, was proposed in [Pas01] and will be presented in the next section. From that construction it is easy to see how to define an $SU(2)$ action on the corresponding bundle. We will then see how this $SU(2)$ action and parallel transport are closely related. But before that, it is convenient to close this section giving yet another description of the non-trivial line bundle L_- over the projective space, in terms of local trivialisations (this will be used when we compare parallel transport with the $SU(2)$ action, in the next section).

We will denote points on the sphere by $x=(x_1,x_2,x_3)$. Under the quotient map q, x goes to an equivalence class $[x]=\{x,-x\}$, representing a point in $\mathbb{R}P^2$. Using homogeneous coordinates, $[x]$ can equivalently be described as the line generated by x, regarded as a vector in $\mathbb{R}^3$:

$$[x]=\{\lambda x|\lambda\in\mathbb{R}\}.$$

An open cover of the projective plane is given by

$$U_i = \{[x] \mid x_i \neq 0\}, \quad (i = 1, 2, 3).$$

The corresponding local charts $\{(U_i, h_i)\}_i$ are given by

$$\begin{aligned} h_1 : U_1 &\longrightarrow \mathbb{R}^2 \\ [x] &\longmapsto \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right) \end{aligned} \tag{4.2.5}$$

and analogously for h_2 and h_3. The bundle L_- can be constructed as a sub-bundle of the trivial bundle $\mathbb{R}P^2 \times \mathbb{C}^3$. The explicit construction goes as follows. First, the total space $E(L_-)$ of the bundle is defined, as a <u>*set*</u>, in the following way:

$$E(L_-) := \{([x], \lambda|\phi(x)\rangle) \in \mathbb{R}P^2 \times \mathbb{C}^3 \mid \lambda \in \mathbb{C}, x \in [x]\}, \tag{4.2.6}$$

where

$$|\phi(x)\rangle = (x_1, x_2, x_3). \tag{4.2.7}$$

Note that there is some ambiguity in the description of the elements of $E(L_-)$, namely, the choice of λ is not unique, since we are using a representative x of $[x]$ in the argument of $|\phi(-)\rangle$ which of course is a well defined function only over the *sphere* , not over the projective plane. But in view of the relation $|\phi(-x)\rangle = -|\phi(x)\rangle$ (that is also satisfied by $|\psi\rangle$), we may describe an element $y \in E(L_-)$ in (exactly) two ways:

$$y = ([x], \lambda|\phi(x)\rangle) = ([x], (-\lambda)|\phi(-x)\rangle).$$

Notice that the same situation is faced in the construction of *tautological* bundles over projective spaces (see for example [MS74]) .

$E(L_-)$ is given the relative topology with respect to the product space $\mathbb{R}P^2 \times \mathbb{C}^3$. The projection map is obviously defined as

$$\pi([x], \lambda|\phi(x)\rangle) := [x].$$

Local trivialisations can then be defined through $(i = 1, 2, 3)$:

$$\begin{aligned} \varphi_i : \quad \pi^{-1}(U_i) &\longrightarrow U_i \times \mathbb{C} \\ ([x], \lambda|\phi(x)\rangle) &\longmapsto ([x], \mathrm{sgn}(x_i)\lambda). \end{aligned} \tag{4.2.8}$$

The map φ_i is well defined because, by definition, $[x] \in U_i$ if and only if $x_i \neq 0$, hence $\mathrm{sgn}(x_i)$ is defined throughout U_i. Also note that the term $\mathrm{sgn}(x_i)\lambda$ is not ambiguous. This follows from the remark on λ above: If y belongs to the fiber over $[x]$ then, assuming that we have chosen $x \in [x] = \{x, -x\}$, there exists a unique $\lambda \in \mathbb{C}$ such that $y = ([x], \lambda|\phi(x)\rangle)$. Had we made the other possible choice, namely $-x$, it would be rather $y = ([x], (-\lambda)|\phi(-x)\rangle)$. In any case we get

$$\varphi_i(y) = ([x], \mathrm{sgn}(x_i)\lambda) = ([x], \mathrm{sgn}(-x_i)(-\lambda)).$$

The map φ_i is surjective: Given $([x]\,, w) \in U_i \times \mathbb{C}$, we may put $\lambda \equiv \lambda(x) = \mathrm{sgn}(x_i)w$. Then,

$$\begin{aligned} \varphi_i([x]\,, \lambda|\phi(x)\rangle) &= ([x]\,, \mathrm{sgn}(x_i)\lambda) \\ &= ([x]\,, w). \end{aligned} \tag{4.2.9}$$

(Again, the choice of λ depends on a choice of representative x, but the product $\mathrm{sgn}(x_i)\lambda$ does not). Injectivity can also be easily checked. It follows from the relation $|\phi(-x)\rangle = -|\phi(x)\rangle$.

Having defined local trivialisations, we proceed to examine the corresponding transition functions. The result is

$$\varphi_i \circ \varphi_j^{-1}([x]\,, w) = ([x]\,, \mathrm{sgn}(x_i x_j)w),$$

that is, the transition functions describing the bundle are

$$\begin{aligned} g_{ij} : U_i \cap U_j &\longrightarrow \mathbb{C}^* \\ [x] &\longmapsto \mathrm{sgn}(x_i x_j). \end{aligned} \tag{4.2.10}$$

Recalling the Serre-Swan equivalence, and using the partition of unity given by the functions $f_i([x]) := (x_i)^2$ $(i = 1, 2, 3)$, we may construct the projector corresponding to this bundle. Recall that in general the entries of a projector constructed out of a bundle are $p_{ij} = \sqrt{f_i f_j}\, g_{ij}$. Using the result above, we find:

$$p_{ij}([x]) = x_i x_j.$$

We have thus recovered the expression eq. (4.2.3) for the projector, which had been obtained in the context of the more general construction outlined in chapter 3.

4.2.1 Remark. Let us remark that, although we have used $|\phi\rangle$ for the definition of the total space of the bundle L_-, it is not a section on it (indeed, the map $x \mapsto (x, |\phi(x)\rangle)$ defines a non vanishing section on the pull-back of L_-, a bundle over the *sphere*). The role of $|\phi(x)\rangle$ is to generate a complex line on $\mathbb{C}^3$ which gives the fiber over $[x]$. Of course this makes sense only if the complex lines (in $\mathbb{C}^3$) generated by $|\phi(-x)\rangle$ and $|\phi(x)\rangle$ coincide. But this is ensured by the relation $|\phi(-x)\rangle = -|\phi(x)\rangle$. This apparent ambiguity is, to our opinion, a source of confusion in the Berry-Robbins construction. If, instead, we choose to give a global description of the bundle by means of the corresponding projector, the problem does not even appear. This is so because the projection onto the fiber over $[x]$ is a concept which is independent of a choice of representative or, in other words, because the components of the projector $|\phi(x)\rangle\langle\phi(x)|$ are *even* functions.

Now we consider local frames and parallel transport. Let us define

$$\begin{aligned} e_i : \mathbb{R}P^2 &\longrightarrow E(L_-) \\ [x] &\longmapsto ([x]\,, x_i|\phi(x)\rangle). \end{aligned} \tag{4.2.11}$$

These global sections vanish at some points, so they can only give place to *local* frames, given by their restrictions to the respective neighborhoods U_i. Since the connection we consider on L_- is flat, the result of parallel transport can only depend on the homotopy type of the chosen path. For our purposes, it will be enough to consider parallel transport along the following path[§]:

$$\begin{array}{rcl} \gamma : (-\pi/2, \pi/2) & \longrightarrow & U_3 \subset \mathbb{R}P^2 \\ t & \longmapsto & [\,(\sin(t), 0, \cos(t)\,)\,] \end{array} \tag{4.2.12}$$

On U_3, any section can be written in terms of e_3. The action of the connection ∇ (eq.(4.2.4)) on it is

$$\nabla e_3 = e_1 \otimes d\alpha_1 + e_2 \otimes d\alpha_2 + e_3 \otimes d\alpha_3,$$

where $\alpha_i([x]) = x_i x_3$. From $(\alpha_1(\gamma(t))' = \cos^2 t - \sin^2 t$, $(\alpha_2(\gamma(t))' = 0$ and $(\alpha_3(\gamma(t))' = -2\cos t \sin t$ (note that in these expressions there is no dependence on the choice of representatives) we obtain, using the fact that $\cos t \neq 0$ on the interval considered:

$$\begin{array}{rcl} \nabla_{\dot\gamma(t)} e_3(t) & = & (\cos^2 t - \sin^2 t)\, e_1(t) - (2\cos t \sin t)\, e_2(t) \\ & = & -\tan t \; e_3(t). \end{array} \tag{4.2.13}$$

To find a parallel section along γ, we must solve the equation

$$\frac{d\omega}{dt} e_3(t) + \omega(t) \nabla_{\dot\gamma(t)} e_3(t) = 0.$$

The solution is given by $\omega(t) = (\cos t)^{-1}$, so that

$$\begin{array}{rrcl} s: & (-\pi/2, \pi/2) & \longrightarrow & E(L_-) \\ & t & \longmapsto & (\gamma(t), (\sin t, 0, \cos t)) \end{array} \tag{4.2.14}$$

is a parallel section *along* γ.

4.2.2 SU(2) equivariance

The sphere, being a homogeneous space for $SU(2)$ ($S^2 \cong SU(2)/U(1)$) carries a natural $SU(2)$-action

$$SU(2) \times S^2 \longrightarrow S^2.$$

The induced action on the space of functions makes $\mathcal{A} := C(S^2)$ a representation space, which contains all odd-integer dimensional irreducible $SU(2)$ representations -a well known fact- of which the spherical harmonics form suitable bases. If V^j denotes the $2j+1$ dimensional representation, we have

$$\mathcal{A} \simeq \bigoplus_{j \in N_0} V^j.$$

[§] The reason for this choice of path is that it can be obtained from the action of $SU(2)$ on $\mathbb{R}P^2$.

(Apart from the $SU(2)$ action, there is also the $\mathbb{Z}_2$ one given by the parity operator, with respect to which $\mathcal{A}$ can be decomposed, as before, into symmetric and antisymmetric parts: $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$).

Let us now form the tensor product of $\mathcal{A}$ with V^1:

$$\mathcal{A}^3 \simeq \mathcal{A} \otimes V^1.$$

The $SU(2)$ action induced through (3.3.4) on $\mathcal{A}$ is given, at the Lie algebra level, by the angular momentum operators L_i (i=1,2,3), which act as derivations, that is, for $\psi, \varphi \in \mathcal{A}$, the relation $L_i(\psi\varphi) = L_i(\psi)\varphi + \psi L_i(\varphi)$ holds. We can use the representation of the Lie algebra on V^1, given by (3×3) matrices τ_i, to form a product representation on $\mathcal{A} \otimes V^1$:

$$J_i := L_i \otimes 1_3 + \mathrm{id}_{\mathcal{A}} \otimes \tau_i. \tag{4.2.15}$$

The product $\mathcal{A} \otimes V^1$ can be decomposed into irreducibles with respect to this representation.

$$\begin{aligned} \mathcal{A} \otimes V^1 &\simeq \left(\bigoplus_{j \in N_0} V^j \right) \otimes V^1 \\ &\simeq V^1 \oplus (V^0 \oplus V^1 \oplus V^2) \oplus (V^1 \oplus V^2 \oplus V^3) \oplus \cdots \end{aligned} \tag{4.2.16}$$

Note that the trivial representation V^0 appears only once. Thus, there is a unique scalar element, up to normalization, with respect to this representation. From

$$|J, M\rangle = \sum_{m_1, m_2} (j_1\, m_1, j_2\, m_2 | J\, M) |j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

and

$$(j\, m j\ -m|0\, 0) = \frac{(-1)^{j-m}}{\sqrt{2j+1}},$$

we obtain, for the scalar element, the following expression:

$$\sqrt{4\pi/3}\, (Y_{1,1} \otimes |1, -1\rangle - Y_{1,0} \otimes |0, 0\rangle + Y_{1,-1} \otimes |1, 1\rangle)\,.$$

Identifying $\mathcal{A}^3$ with $\mathcal{A} \otimes V^1$, this is the same as

$$|\psi\rangle := \sqrt{\frac{4\pi}{3}} \begin{pmatrix} Y_{1,-1} \\ -Y_{1,0} \\ Y_{1,1} \end{pmatrix} = \begin{pmatrix} \frac{e^{-i\varphi}}{\sqrt{2}} \sin\theta \\ -\cos\theta \\ -\frac{e^{i\varphi}}{\sqrt{2}} \sin\theta \end{pmatrix}. \tag{4.2.17}$$

One checks readily that $|\psi\rangle$ is indeed a scalar element, that is that $J_i|\psi\rangle = 0$ holds. In terms of the group representation, this means that $\mathcal{D}^{(1)}(g)|\psi(g^{-1}x)\rangle = |\psi(x)\rangle$ or, equivalently, that

$$\mathcal{D}^{(1)}(g)|\psi(x)\rangle = |\psi(gx)\rangle. \tag{4.2.18}$$

This property of $|\psi\rangle$ will be used below in order to construct an $SU(2)$ bundle structure on the line bundle it gives place to. As an $\mathcal{A}$-valued vector, $|\psi\rangle$ has remarkable properties. One of them is that it provides the construction we are looking for, namely, we have the following result.

4.2.2 Proposition (cf.[Pas01]). *Define a projector on $\mathcal{A}^3$ by $p := |\psi\rangle\langle\psi|$. Then, the following isomorphism of $\mathcal{A}_+$ modules holds:*

$$p(\mathcal{A}_+^3) \simeq \mathcal{A}_-. \tag{4.2.19}$$

Proof. First of all, let us note that the spherical harmonics $\{Y_{1,1}, Y_{1,0}, Y_{1,-1}\}$ are a set of generators of $\mathcal{A}_-$ over $\mathcal{A}_+$. Written in matrix form, the projector is given by

$$p = \begin{pmatrix} \frac{1}{3} - \sqrt{\frac{4\pi}{45}} Y_{2,0} & -\sqrt{\frac{4\pi}{15}} Y_{2,-1} & -\sqrt{\frac{8\pi}{15}} Y_{2,-2} \\ \sqrt{\frac{4\pi}{15}} Y_{2,1} & \frac{1}{3} + \sqrt{\frac{16\pi}{45}} Y_{2,0} & \sqrt{\frac{4\pi}{15}} Y_{2,-1} \\ -\sqrt{\frac{8\pi}{15}} Y_{2,2} & -\sqrt{\frac{4\pi}{15}} Y_{2,1} & \frac{1}{3} - \sqrt{\frac{4\pi}{45}} Y_{2,0} \end{pmatrix}. \tag{4.2.20}$$

Denote with σ_+, σ_0 and σ_- the three columns of p in (4.2.20). They are a set of generators of the module $p(\mathcal{A}_+^3)$, since every element s on it is of the form

$$s = a\sigma_+ + b\sigma_0 + c\sigma_-,$$

with a, b and c in $\mathcal{A}_+$. Therefore, we define the map $\varrho : p\left(\mathcal{A}_+^3\right) \to \mathcal{A}_-$ as the $\mathcal{A}_+$-linear extension of

$$\begin{aligned} \varrho\left(\sigma_\pm\right) &:= Y_{1,\pm} \\ \varrho\left(\sigma_0\right) &:= Y_{1,0}. \end{aligned} \tag{4.2.21}$$

The generators $\{\sigma_i\}_i$ are not independent over $\mathcal{A}_+$, there are relations among them, of the form $\sum_j (\delta_{ij} - p_{ji})\sigma_j = 0$. They can be read off from (4.2.20):

$$\begin{aligned} \left(\frac{2}{3} + \sqrt{\frac{4\pi}{45}} Y_{2,0}\right) \sigma_+ - \sqrt{\frac{4\pi}{15}} Y_{2,1}\, \sigma_0 + \sqrt{\frac{8\pi}{15}} Y_{2,2}\, \sigma_- &= 0, \\ \sqrt{\frac{4\pi}{15}} Y_{2,-1}\, \sigma_+ + \left(\frac{2}{3} - \sqrt{\frac{16\pi}{45}} Y_{2,0}\right) \sigma_0 + \sqrt{\frac{4\pi}{15}} Y_{2,1}\, \sigma_- &= 0, \\ \sqrt{\frac{8\pi}{15}} Y_{2,-2}\, \sigma_+ - \sqrt{\frac{4\pi}{15}} Y_{2,-1}\, \sigma_0 + \left(\frac{2}{3} - \sqrt{\frac{4\pi}{45}} Y_{2,0}\right) \sigma_- &= 0. \end{aligned}$$

A direct calculation shows that these relations are also satisfied by the generators of $\mathcal{A}_-$. This completes the proof and establishes an equivalence of modules between p and p_-. $\square$

We have seen several ways in which L_- and its module of sections can be described. Using the method described in chapter 3, we have obtained $\Gamma(L_-) \cong p_-(\mathcal{A}_+^3)$, with $p_- = |\phi\rangle\langle\phi|$ and ϕ given by eq. (4.2.7). From 4.2.2 it follows that $\Gamma(L_-) \cong p_(\mathcal{A}_+^3)$, with $p_=|\psi\rangle\langle\psi|$ and ψ given by eq. (4.2.17). We have, therefore,

$$p_-(\mathcal{A}_+^3) \cong \mathcal{A}_- \cong p_(\mathcal{A}_+^3). \tag{4.2.22}$$

The description of the total space of L_- using eq. (4.2.6) has been used in order to find a parallel section along the section γ. From that calculation one sees that if γ is extended to a closed loop, the holonomy corresponding to that loop is -1. We will now consider the $SU(2)$ equivariance of L_- and we will compare it with parallel transport. For that purpose, let us consider an alternative description of the total space of L_-, obtained by using $|\psi\rangle$ in eq. (4.2.6) instead of $|\phi\rangle$. This implies that the open cover and transition functions will be different, but, because of eq. (4.2.22), they give place to isomorphic bundles. The advantage of this particular construction of the bundle lies in the fact that $|\psi\rangle$ is invariant under $SU(2)$. Equation (4.2.18), which we write again below, expresses this fact in a convenient way:

$$\mathcal{D}^{(1)}(g)|\psi(x)\rangle = |\psi(gx)\rangle. \tag{4.2.23}$$

Since $|\psi(x)\rangle$ ($= -|\psi(-x)\rangle$) spans the fiber over $[x]$, we see that the action of an element $g \in SU(2)$ on $y = ([x]\,, \lambda\psi(x)) \in \pi^{-1}([x])$ can be correctly defined by setting

$$\begin{aligned} \tau_g(y) &:= ([gx]\,, \lambda\mathcal{D}^{(1)}(g)|\psi(x)\rangle) \\ &= ([gx]\,, \lambda|\psi(gx)\rangle). \end{aligned} \tag{4.2.24}$$

In fact, from the last expression we see explicitly that y is mapped into the correct fiber: $\tau_g(y) \in \pi^{-1}(g \cdot [x])$. From this and the fact that we are using a representation of $SU(2)$ for its definition, the map τ fulfills the conditions stated in definition (3.1.1), giving place to an $SU(2)$-bundle structure on L_-.

In order to compare parallel transport with the $SU(2)$ action just defined, we use (4.2.24) to find the explicit form of the action on L_-, when described in terms of $|\phi\rangle$. First let us note that, for all x, we have $|\psi(x)\rangle = U|\phi(x)\rangle$, where U is a unitary matrix (with constant entries). Hence, we may replace ψ by ϕ in (4.2.24), provided we also replace $\mathcal{D}^{(1)}(g)$ by $U^\dagger\mathcal{D}^{(1)}(g)U$. For $g \in SU(2)$ written in matrix form as $g = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$ we have, for the 3-dimensional representation,

$$\mathcal{D}^{(1)}(g) = \begin{pmatrix} u^2 & \sqrt{2}uv & v^2 \\ -\sqrt{2}u\bar{v} & (|u|^2 - |v|^2) & \sqrt{2}\bar{u}v \\ \bar{v}^2 & -\sqrt{2}\bar{u}\bar{v} & \bar{u}^2 \end{pmatrix}.$$

This leads to

$$U^\dagger\, \mathcal{D}^{(1)}(g)\, U = \begin{pmatrix} \frac{1}{2}(u^2 - v^2 - \bar{v}^2 + \bar{u}^2) & -\frac{i}{2}(u^2 + v^2 - \bar{v}^2 - \bar{u}^2) & -(uv + \bar{u}\bar{v}) \\ \frac{i}{2}(u^2 + \bar{v}^2 - v^2 - \bar{u}^2) & \frac{1}{2}(u^2 + v^2 + \bar{v}^2 + \bar{u}^2) & -i(uv - \bar{u}\bar{v}) \\ \frac{1}{2}(u\bar{v} + \bar{u}v) & -\frac{i}{2}(u\bar{v} - \bar{u}v) & (|u|^2 - |v|^2) \end{pmatrix}.$$

The path $\gamma(t) = [(\sin t, 0, \cos t)]$ may be obtained from the $SU(2)$ action on $\mathbb{R}P^2$. In fact, with $x_0 = (0, 0, 1)$ and

$$g_t = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix},$$

one gets $\gamma(t) = g_t \cdot [x_0]$. Now, since $([x_0], (0,0,1))$ lies in the fiber over $[x_0]$, we may consider its image under the action of g_t:

$$\begin{aligned}\tau_{g_t}([x_0], (0,0,1)) &= \left(g_t \cdot [x_0]\,,\ U^\dagger\, \mathcal{D}^{(1)}(g_t) U \quad (0,0,1)^t\right) \\ &= (\gamma(t), (\sin t, 0, \cos t)) \\ &\overset{(4.2.14)}{=} s(t). \end{aligned} \tag{4.2.25}$$

4.2.3 Two Identical Particles of Spin Zero

As remarked in the introduction, the classical configuration space of a system of N identical particles moving in $\mathbb{R}^3$ is defined the quotient space, $\mathcal{Q}_N = \widetilde{\mathcal{Q}}_N/S_N$, obtained from the natural action of the permutation group S_N on the space

$$\widetilde{\mathcal{Q}}_N = \{(r_1, \ldots, r_N) \in \mathbb{R}^{3N} |\, r_i \neq r_j \text{ for all pairs}(i,j)\}.$$

This is a multiply-connected space and so, according to the general considerations presented in chapter 2, and following [LM77], we consider wave functions to be given by square integrable sections of some vector bundle on $\mathcal{Q}_N$.

Let us now use the results of the previous sections in order to draw some conclusions about the special case $N = 2$, for spin zero. Here, after performing a transformation to center of mass and relative coordinates, one sees that $\mathcal{Q}_2$ is of the same homotopy type as a two-sphere S^2, this latter representing the space of normalized relative coordinates of the two particles. Under exchange, the relative coordinate r goes to $-r$, so that after quotienting out by the action of $S_2 \cong \mathbb{Z}_2$, we obtain the projective space $\mathbb{R}P^2$. For our purposes it is therefore enough to consider $\widetilde{\mathcal{Q}}_2 = S^2$ and $\mathcal{Q}_2 = \mathbb{R}P^2$ for the configuration space.

Define now $\mathcal{A} := C(S^2)$. We have seen how the $\mathbb{Z}_2$-action on S^2 induces one on $\mathcal{A}$, leading to a decomposition into subspaces of even and odd functions: $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$. The sub-algebra $\mathcal{A}_+$ is isomorphic to $C(\mathbb{R}P^2)$.

As we mentioned before, there are -up to equivalence- only two complex line bundles on $\mathbb{R}P^2$. One of them is the trivial one, L_+, and the other is L_-. The corresponding modules of sections $\Gamma(L_\pm)$ are isomorphic, respectively, to $\mathcal{A}_\pm$. We have seen that the connection naturally associated to $p(\mathcal{A}_+^3)$ is flat and that its holonomy group is $\mathbb{Z}_2$. This corresponds to the fact that a pair of particles whose wave function is an element of $p(\mathcal{A}_+^3) \cong \Gamma(L_-)$ obey Fermi statistics. From this observations it follows that, for spin zero particles, the only possible statistics are the fermionic and the bosonic one and we see how the Fermi-Bose alternative for spinless particles is obtained as a direct consequence of the topology of the configuration space, a well known result. Note that this result is obtained from an intrinsic treatment of indistinguishability, where no use of a symmetrization postulate is made.

When higher values of the spin are considered, the Fermi-Bose alternative has no direct relation to the topology of the configuration space. In chapter 6 we propose to use the $SU(2)$-equivariance of bundles on the configuration space in order to deduce the Fermi-Bose alternative.

Although it will not be done here, let us mention that the recent discussion of the spin zero case presented in [Pes03b] can be clarified enormously using our approach. In particular, our approach allows us to show in detail why the proof presented there fails.

5 Applications to the Berry-Robbins approach to Spin-Statistics

We have remarked in the previous chapters that one of the motivations in the search for a geometric proof of the Spin-Statistics theorem is the fact that, for identical spinless particles, the Fermi-Bose alternative follows as a consequences of the non-trivial topology of the configuration space. The possibility of a geometric origin of the Spin-Statistics relation was already proposed by Leinaas and Myrheim [LM77] and has been a subject of interest for many years. But there is a big obstacle in the way to eventually achieving such a goal: The results in the motivational case (i.e. the example of quantum mechanics of identical spinless particles) are obtained within a framework in which *quantization* ideas play a key role. It is more than questionable that a method based on a quantization scheme may also be applied when electron spin (which has no classical analog*) is considered. A reasonable way out of this difficulty is to adopt a compromise position, by adding "by hand" the structures needed to describe spin to the ones that have already been obtained for the zero spin case by a well founded quantization algorithm. Such a strategy necessarily involves a certain amount of trial and error and it is for this reason that the implementation of different constructions and a comparison between them is welcome, since this might provide some insight into the problem.

Among these constructions are the one proposed by Berry and Robbins [BR97], as well as further developments of it [BR00, HR04]. The original construction attempts to tackle the issue of indistinguishability, making use of a certain single-valuedness requirement on the wave function, in the hope that a derivation of the Spin-Statistics theorem will be achieved. It has the virtue of being concrete and of reproducing the correct relation between Spin and Statistics for all values of the spin but, this one not being the only construction satisfying their assumptions (see [BR00]), the derived Spin-Statistics relation gets lost. Anyway, the Berry-Robbins approach leaves, in our opinion, many open questions, and the formalism used does not very easily allow to isolate technical aspects from physical ones. It is therefore of interest to have a new look at these constructions from a different point of view. This is what is done in this chapter, where the tools developed in chapter 3 are used to reproduce and, most importantly, to interpret the Berry-Robbins construction. Our method will allow us to show that the single-valuedness condition of Berry and Robbins is not consistent.

*See, however, [BH50].

5.1 Review of the construction

We begin with a review of the construction presented in [BR97], which is restricted to the two particle case. Recall that for two identical particles moving in three spatial dimensions the configuration space, $\mathcal{Q}$, is obtained from $\widetilde{\mathcal{Q}} = \{(r_1, r_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid r_1 \neq r_2\}$ by identifying exchanged configurations: $(r, r') \sim (r', r)$. The resulting space, $\mathcal{Q} = \widetilde{\mathcal{Q}}/\sim$, is homotopy equivalent to the projective plane $\mathbb{R}P^2$. This is easily seen by performing, in $\widetilde{\mathcal{Q}}$, the change of coordinates $(r_1, r_2) \mapsto (R, r)$, where R is the center of mass position vector, and r the relative position vector. From this we see that $\widetilde{\mathcal{Q}}$ splits as the product $\mathbb{R}^3 \times \mathbb{R}_+ \times S^2$. Under exchange, all points of the first two factors (that are contractible) remain unchanged, whereas points in the sphere are mapped to their antipodes. Identification of antipodes in the sphere gives place to a projective plane, so we might as well set $\widetilde{\mathcal{Q}} = S^2$ and $\mathcal{Q} = \mathbb{R}P^2$.

The spin degrees of freedom are introduced in the formalism in the following way. In a preliminary step, consider the usual description by means of states of the form $|s, m_1\rangle \otimes |s, m_2\rangle$. Following [BR97], let us introduce the notation $|M\rangle$ for these states, where the index "M" refers to the *ordered* pair of quantum numbers $\{m_1, m_2\}$ (the quantum number s will be implicitly understood from now on). For the pair $\{m_2, m_1\}$, which elements were permuted, the index "$\overline{M}$" will be used†. Now, the spin basis $\{|M\rangle\}_M$ will be replaced by a "transported" one, that depends on the positions: $\{|M(r)\rangle\}_M$, $r \in \widetilde{Q}$.

In order to obtain the transported basis from the fixed one, a position-dependent unitary operator $U(r)$ is introduced. The idea is to construct $U(r)$ in such a way that a continuous exchange of the particles' positions is accompanied by a (continuous, as well) exchange of spin labels. This will require the embedding of the fixed basis $\{|M\rangle\}_M$ in a larger vector space, that will be denoted with V. Schwinger's representation of spin in terms of creation and annihilation operators proves to be particularly useful for the construction of an operator $U(r)$ with the required properties. Let us explain why.

In the Schwinger representation, the spin operators (for one particle states) are constructed with the help of operators $a^{(\dagger)}, b^{(\dagger)}$ satisfying the commutation relations

$$[a, a^\dagger] = 1 \qquad [b, b^\dagger] = 1.$$

One then defines the spin operators as follows:

$$\begin{aligned} S_1 &:= \frac{1}{2}(a^\dagger b + b^\dagger a) \\ S_2 &:= \frac{i}{2}(b^\dagger a - a^\dagger b) \\ S_3 &:= \frac{1}{2}(a^\dagger a - b^\dagger b). \end{aligned} \tag{5.1.1}$$

†In other words: if $|M\rangle \equiv |s, m_1\rangle \otimes |s, m_2\rangle$, then $|\overline{M}\rangle \equiv |s, m_2\rangle \otimes |s, m_1\rangle$.

The commutation relations $[S_i, S_j] = i\epsilon_{ijk}S_k$ hold, so these operators provide a representation of $\mathfrak{su}(2)$. Common eigenvectors of $\mathbf{S}^2$ and S_3 are given by

$$|s, m_s\rangle := \frac{(a^\dagger)^{s+m_s}(b^\dagger)^{s-m_s}}{\sqrt{(s+m_s)!(s-m_s)!}}|0\rangle. \tag{5.1.2}$$

If we denote with $n_a(n_b)$ the exponent of $a^\dagger(b^\dagger)$ in equation (5.1.2), then we can write the quantum numbers s and m_s in the following form:

$$s = \frac{1}{2}(n_a + n_b) \qquad m_s = \frac{1}{2}(n_a - n_b). \tag{5.1.3}$$

From this expression, one sees that a change in the sign of the quantum number m_s, due to a rotation by an angle of π in which $\hat{e}_3$ goes to $-\hat{e}_3$, can also be effected by exchanging $a^\dagger$ with $b^\dagger$. Now, since in order to describe the spin states of two particles more operators are needed (say a_1, b_1, a_2, b_2, plus the corresponding creation operators), one might ask: What meaning do the different possible exchanges of operators have, in terms of rotations? For, whereas the relation between the exchange $a_i \leftrightarrow b_i$ and the change in sign of $m_s^{(i)}$ under a π rotation is clear from the expression $S_3^{(i)} = 1/2(a_i^\dagger a_i - b_i^\dagger b_i)$ for the third component of the spin operator of particle i, exchanges of the form $a_1^{(\dagger)} \leftrightarrow a_2^{(\dagger)}$ or $b_1^{(\dagger)} \leftrightarrow b_2^{(\dagger)}$ should be related, respectively, to operators of the form $E_{a,3} = 1/2(a_1^\dagger a_1 - a_2^\dagger a_2)$ or $E_{b,3} = 1/2(b_1^\dagger b_1 - b_2^\dagger b_2)$ and so on, which do not have an immediate interpretation in terms of rotations. The complete set of operators related to this type of exchange ($1 \leftrightarrow 2$) is, according to equation (5.1.1),

$$\begin{aligned}
E_{a,1} &:= \frac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1), & E_{b,1} &:= \frac{1}{2}(b_1^\dagger b_2 + b_2^\dagger b_1),\\
E_{a,2} &:= \frac{i}{2}(a_2^\dagger a_1 - a_1^\dagger a_2), & E_{b,2} &:= \frac{i}{2}(b_2^\dagger b_1 - b_1^\dagger b_2),\\
E_{a,3} &:= \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2), & E_{b,3} &:= \frac{1}{2}(b_1^\dagger b_1 - b_2^\dagger b_2).
\end{aligned} \tag{5.1.4}$$

One checks readily that they satisfy the same commutation relations as the spin operators ($[E_{a(b),i}, E_{a(b),j}] = i\epsilon_{ijk}E_{a(b),k}$) and hence provide yet another representation of $\mathfrak{su}(2)$. Since the a-operators commute with the b-ones, this is still true for $\mathbf{E} := \mathbf{E_a} + \mathbf{E_b}$ (where $\mathbf{E_{a(b)}} := (E_{a(b),1}, E_{a(b),2}, E_{a(b),3})$). Exponentiating it, we obtain a representation $\rho^E : SU(2) \to \mathrm{Gl}(V)$, where V is the space generated by the basis

$$|n_{1a}, n_{2a}, n_{1b}, n_{2b}\rangle = \frac{(a_1^\dagger)^{n_{1a}}(a_2^\dagger)^{n_{2a}}(b_1^\dagger)^{n_{1b}}(b_2^\dagger)^{n_{2b}}}{\sqrt{(n_{1a})!(n_{2a})!(n_{1b})!(n_{2b})!}}|0\rangle. \tag{5.1.5}$$

$SU(2)$ can be parametrized by elements of the form $g_\psi(n) = \exp(\frac{i}{2}\psi n \cdot \sigma)$, where $-\pi < \psi \leq \pi$, n is a unit vector and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ (Pauli matrices). Using this parametrization of SU, we can use the representation ρ^E to get a map

$$\begin{aligned}
U : S^2 &\to \mathrm{Gl}(V) &\quad (5.1.6)\\
r &\to U(r) := \rho^E(g_\theta(-n(r))), &\quad (5.1.7)
\end{aligned}$$

where $n(r) = e_3 \times r$. In terms of the generators E_i, we have:

$$U(r) = e^{-i\theta n(r)\cdot E}, \tag{5.1.8}$$

Doing this, the meaning of the exchanges of the form $(1 \leftrightarrow 2)$ becomes clear since, as we will now see, the map U provides a means to realize simultaneously the exchange of the particles' positions *and* spins, in a continuous way.

We can now define the following position-dependent vectors

$$|n_{1a}, n_{2a}, n_{1b}, n_{2b}(r)\rangle := U(r)|n_{1a}, n_{2a}, n_{1b}, n_{2b}\rangle, \tag{5.1.9}$$

for which the following relation can be established:

$$|n_{1a}, n_{2a}, n_{1b}, n_{2b}(r)\rangle = (-1)^{(n_{2a}+n_{2b})} e^{i\phi(n_{1a}+n_{1b}-n_{2a}-n_{2b})} |n_{2a}, n_{1a}, n_{2b}, n_{1b}(r)\rangle. \tag{5.1.10}$$

(ϕ denotes the azimuthal angle of r). According to eqs. (5.1.2) and (5.1.3), the 2-particle state with quantum numbers $M = \{m_1, m_2\}$ is obtained through the replacements

$$\begin{aligned} n_{1a} &= s + m_1, \qquad n_{2a} = s + m_2, \\ n_{1b} &= s - m_1, \qquad n_{2b} = s - m_2 \end{aligned}$$

in equation (5.1.5). In this case, equation (5.1.10) takes, with $|M(r)\rangle := U(r)|M\rangle$, the following form:

$$|M(r)\rangle = (-1)^{2s}|\overline{M}(-r)\rangle. \tag{5.1.11}$$

This is a relation of great relevance to the Berry-Robbins construction. Let us now check equation (5.1.10).

Setting $U_{a(b)}(r) = e^{-i\theta n(r)\cdot E_{a(b)}}$, it follows from $[a_\lambda^{(\dagger)}, b_{\lambda'}^{(\dagger)}] = 0$ that $[U_a(r), U_b(r)] = 0$ and hence that $U(r) = U_a(r)U_b(r)$. With the help of the Baker-Hausdorff identity the following relations are obtained:

$$\begin{aligned} U_a(r)a_1^\dagger U_a^\dagger(r) &= \cos\frac{\theta}{2}a_1^\dagger + e^{i\phi}\sin\frac{\theta}{2}a_2^\dagger, \\ U_a(r)a_2^\dagger U_a^\dagger(r) &= \cos\frac{\theta}{2}a_2^\dagger - e^{-i\phi}\sin\frac{\theta}{2}a_1^\dagger, \end{aligned}$$

and analogously for $b_\lambda^\dagger$. From this one then obtains:

$$U(r)(a_1^\dagger)^{n_{1a}}(a_2^\dagger)^{n_{2a}}(b_1^\dagger)^{n_{1b}}(b_2^\dagger)^{n_{2b}}|0\rangle =$$

$$\begin{aligned}
&= U(r)(a_1^\dagger)^{n_{1a}}U^\dagger(r)\cdots U(r)(b_2^\dagger)^{n_{2b}}U^\dagger(r)|0\rangle = \\
&= U_a(r)(a_1^\dagger)^{n_{1a}}(a_2^\dagger)^{n_{2a}}U_a^\dagger(r)U_b(r)(b_1^\dagger)^{n_{1b}}(b_2^\dagger)^{n_{2b}}U_b^\dagger(r)|0\rangle = \\
&= \Big(\cos\frac{\theta}{2}a_1^\dagger + e^{i\phi}\sin\frac{\theta}{2}a_2^\dagger\Big)^{n_{1a}}\Big(-e^{-i\phi}\sin\frac{\theta}{2}a_1^\dagger + \cos\frac{\theta}{2}a_2^\dagger\Big)^{n_{2a}} \times \\
&\quad \times\Big(\cos\frac{\theta}{2}b_1^\dagger + e^{i\phi}\sin\frac{\theta}{2}b_2^\dagger\Big)^{n_{1b}}\Big(-e^{-i\phi}\sin\frac{\theta}{2}b_1^\dagger + \cos\frac{\theta}{2}b_2^\dagger\Big)^{n_{2b}} = \\
&= \sigma\Big(e^{-i\phi}\cos\frac{\theta}{2}a_1^\dagger + \sin\frac{\theta}{2}a_2^\dagger\Big)^{n_{1a}}\Big(\sin\frac{\theta}{2}a_1^\dagger - e^{i\phi}\cos\frac{\theta}{2}a_2^\dagger\Big)^{n_{2a}} \times \\
&\quad \times\Big(e^{-i\phi}\cos\frac{\theta}{2}b_1^\dagger + \sin\frac{\theta}{2}b_2^\dagger\Big)^{n_{1b}}\Big(\sin\frac{\theta}{2}b_1^\dagger - e^{i\phi}\cos\frac{\theta}{2}b_2^\dagger\Big)^{n_{2b}} = \\
&= \sigma\Big(-e^{-i(\phi+\pi)}\sin(\frac{\theta+\pi}{2})a_1^\dagger + \cos(\frac{\theta+\pi}{2})a_2^\dagger\Big)^{n_{1a}}\Big(\cos(\frac{\theta+\pi}{2})a_1^\dagger + e^{i(\phi+\pi)}\sin(\frac{\theta+\pi}{2})a_2^\dagger\Big)^{n_{2a}} \times \\
&\quad \times\Big(-e^{-i(\phi+\pi)}\sin(\frac{\theta+\pi}{2})b_1^\dagger + \cos(\frac{\theta+\pi}{2})b_2^\dagger\Big)^{n_{1b}}\Big(\cos(\frac{\theta+\pi}{2})b_1^\dagger + e^{i(\phi+\pi)}\sin(\frac{\theta+\pi}{2})b_2^\dagger\Big)^{n_{2b}} = \\
&= \sigma U(-r)(a_1^\dagger)^{n_{2a}}(a_2^\dagger)^{n_{1a}}(b_1^\dagger)^{n_{2b}}(b_2^\dagger)^{n_{1b}}|0\rangle,
\end{aligned}$$

where $\sigma = (-1)^{(n_{2a}+n_{2b})}e^{i\phi(n_{1a}+n_{1b}-n_{2a}-n_{2b})}$.

The transported spin basis defined through equation (5.1.11) has certain properties, that we already mentioned in the introduction and that were originally conjectured to fully characterize it (as pointed out in [BR00], this is not the case). We list them here once again.

5.1.1 Definition (Transported Spin Basis).

(i) The map

$$\begin{aligned} S^2 &\longrightarrow \mathbb{C}^{N_S} \\ r &\longmapsto |M(r)\rangle := U(r)|M\rangle \end{aligned}$$

is well defined and smooth for all M.

(ii) The following *"exchange"* rule holds:

$$|\overline{M}(-r)\rangle = (-1)^{2S}|M(r)\rangle. \tag{5.1.12}$$

(iii) The *"parallel transport"* condition $\langle M'(r(t))|\frac{d}{dt}M(r(t))\rangle = 0$ is satisfied for all M and M', and for every smooth curve $t \mapsto r(t)$.

Having constructed the transported spin basis, the next step is the definition of the wave function in terms of it:

$$|\Psi(r)\rangle = \sum_M \Psi_M(r)|M(r)\rangle. \tag{5.1.13}$$

Since the usual spin states have been replaced by the transported ones, and the wave function now expressed in terms of them, it is also necessary to modify the momentum and spin operators. This can be done with the help of the operator $U(r)$ and it can be shown that, after the new operators are introduced, the coefficient functions $\Psi_M(r)$ satisfy the same differential equation as in the usual case (i.e. Schrödinger's equation).

The single-valuedness requirement, imposed on the wave function by the condition

$$\boxed{|\Psi(r)\rangle \stackrel{!}{=} |\Psi(-r)\rangle} \tag{5.1.14}$$

is essential to the Berry-Robbins approach. As a consequence of indistinguishability, the physical configuration space is, as explained before, not S^2 but $\mathbb{R}P^2$. But both the transported spin vectors $|M(r)\rangle$ as well as the wave function equation (5.1.13) are defined as maps whose domain of definition is S^2. As far as we can see, the motivation of Berry-Robbins for the imposition of the condition equation (5.1.14) on the wave function was precisely to "force" its domain of definition to be $\mathbb{R}P^2$, the physically correct configuration space. Although from the physical point of view such an assumption is fully justified (in view of indistinguishability), its implementation by means of equation (5.1.14) leaves several questions open. A detailed discussion of this single-valuedness condition, from the point of view of our approach will be presented in section 5.3.

Assuming the properties stated in 5.1.1, a direct consequence of the single-valuedness condition equation(5.1.14) is the relation $\Psi_{\overline{M}}(-r) = (-1)^{2S}\Psi_M(r)$ for the coefficient functions, which is equivalent to the (physically correct) relation between spin and statistics. The validity of the last assertion is due to the fat that, as shown in [BR97], these coefficient functions satisfy the same differential equation as the usual ones (of course, in order to regard this result as a derivation of the Spin-Statistics theorem, it would be necessary to justify the introduction of a transported spin basis satisfying precisely the properties stated above).

So far, these are, in our view, the most relevant aspects of the Berry-Robbins construction. We have already pointed out that a particular feature of the Berry-Robbins approach is its concreteness, since the use of Schwinger's representation of spin allows to make explicit computations. But, in spite of this, it posses several problems. The role of Schwinger's representation is not clear. It is known that there are alternative constructions which are not based on it and that give the wrong Spin-Statistics connection [BR00]. Also, its relation to other approaches of a more geometric nature cannot be seen clearly, although the authors do affirm that a geometric structure -in the spirit of that presented in chapter 2- underlies the construction. In their words, *"What we are doing is setting up quantum mechanics on a 'two-spin bundle', whose six-dimensional base is the configuration space r_1, r_2 with exchanged configurations identified and coincidences $r_1 = r_2$ excluded (...). The fibres are the two-spin Hilbert spaces spanned by the transported basis $|M(r)\rangle$. The full Hilbert space consists of global sections of the bundle, i.e. singlevalued wave functions"* [BR97].

In the next sections we will be concerned with a clear formulation of the Berry-Robbins construction in such geometric terms, thus giving a precise meaning to the assertion quoted above. The methods developed in chapter 3 are particularly well suited for this purpose. The re-formulation of the construction in terms of projective modules presented in the next section will lead us to the conclusion, presented in section 5.3, that the single-valuedness condition of Berry-Robbins is inconsistent.

5.2 The transported spin basis and projectors

Specializing the construction to the spin $1/2$ case, we note that the operators $U(r)$ leave a 10-dimensional space (which contains the 4 fixed spin states) invariant. A basis for this space is given by

$$\begin{aligned}
|e_1\rangle &:= a_1^\dagger a_2^\dagger|0\rangle = |+,+\rangle, & |e_6\rangle &:= a_2^\dagger b_2^\dagger|0\rangle,\\
|e_2\rangle &:= b_1^\dagger b_2^\dagger|0\rangle = |-,-\rangle, & |e_7\rangle &:= \frac{(a_1^\dagger)^2}{\sqrt{2}}|0\rangle,\\
|e_3\rangle &:= a_1^\dagger b_2^\dagger|0\rangle = |+,-\rangle, & |e_8\rangle &:= \frac{(b_1^\dagger)^2}{\sqrt{2}}|0\rangle,\\
|e_4\rangle &:= a_2^\dagger b_1^\dagger|0\rangle = |-,+\rangle, & |e_9\rangle &:= \frac{(a_2^\dagger)^2}{\sqrt{2}}|0\rangle,\\
|e_5\rangle &:= a_1^\dagger b_1^\dagger|0\rangle, & |e_{10}\rangle &:= \frac{(b_2^\dagger)^2}{\sqrt{2}}|0\rangle.
\end{aligned} \tag{5.2.1}$$

Applying $U(r)$ to the first four basis vectors, and with the help of equation (5.1.12), we obtain the explicit expressions for the transported spin vectors, as linear combinations of *all* ten vectors $|e_1\rangle, \ldots, |e_{10}\rangle$:

$$\begin{aligned}
|+,+(r)\rangle &:= U(r)|+,+\rangle = -e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}|e_7\rangle + \cos\theta|e_1\rangle + e^{i\phi}\frac{\sin\theta}{\sqrt{2}}|e_9\rangle\\
|-,-(r)\rangle &:= U(r)|-,-\rangle = -e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}|e_8\rangle + \cos\theta|e_2\rangle + e^{i\phi}\frac{\sin\theta}{\sqrt{2}}|e_{10}\rangle\\
|+,-(r)\rangle &:= U(r)|+,-\rangle = -e^{-i\phi}\frac{\sin\theta}{2}|e_5\rangle + \cos^2\frac{\theta}{2}|e_3\rangle - \sin^2\frac{\theta}{2}|e_4\rangle + e^{i\phi}\frac{\sin\theta}{2}|e_6\rangle\\
|-,+(r)\rangle &:= U(r)|-,+\rangle = -e^{-i\phi}\frac{\sin\theta}{2}|e_5\rangle + \cos^2\frac{\theta}{2}|e_4\rangle - \sin^2\frac{\theta}{2}|e_3\rangle + e^{i\phi}\frac{\sin\theta}{2}|e_6\rangle
\end{aligned} \tag{5.2.2}$$

The transported spin vectors are maps

$$\begin{array}{rcl} S^2 & \longrightarrow & V \\ r & \longmapsto & |M(r)\rangle \end{array}$$

($V = \text{span}\{|e_1\rangle, \ldots, |e_{10}\rangle\}$). Because of the relation $|\overline{M}(-r)\rangle = -|M(r)\rangle$, it is not possible to regard them as maps $\mathbb{R}P^2 \to V$ (because of equation (5.1.14), this is indeed possible for the whole wave function $|\Psi(r)\rangle$). But this situation changes if we make a change to a basis of total angular momentum. First, we define the following auxiliary vectors:

$$\begin{array}{ccc} |1,-1\rangle^{(-1)} := |e_8\rangle & |1,-1\rangle^{(0)} := |e_2\rangle & |1,-1\rangle^{(1)} := |e_{10}\rangle \\ |1,0\rangle^{(-1)} := |e_5\rangle & |1,0\rangle^{(0)} := \frac{1}{\sqrt{2}}\left(|e_3\rangle + |e_4\rangle\right) & |1,0\rangle^{(1)} := |e_6\rangle \\ |1,1\rangle^{(-1)} := |e_7\rangle & |1,1\rangle^{(0)} := |e_1\rangle & |1,1\rangle^{(1)} := |e_9\rangle \\ & |0,0\rangle := \frac{1}{\sqrt{2}}\left(|e_3\rangle - |e_4\rangle\right). & \end{array}$$

Applying $U(r)$ to them, we obtain:

$$\begin{aligned} U(r)|1,m\rangle^{(-1)} &= \cos^2\frac{\theta}{2}|1,m\rangle^{(-1)} + e^{i\phi}\frac{\sin\theta}{\sqrt{2}}|1,m\rangle^{(0)} + e^{2i\phi}\sin^2\frac{\theta}{2}|1,m\rangle^{(1)} \\ U(r)|1,m\rangle^{(0)} &= -e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}|1,m\rangle^{(-1)} + \cos\theta|1,m\rangle^{(0)} + e^{i\phi}\frac{\sin\theta}{\sqrt{2}}|1,m\rangle^{(1)} \\ U(r)|1,m\rangle^{(1)} &= e^{-2i\phi}\sin^2\frac{\theta}{2}|1,m\rangle^{(-1)} - e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}|1,m\rangle^{(0)} + \cos^2\frac{\theta}{2}|1,m\rangle^{(1)} \\ U(r)|0,0\rangle &= |0,0\rangle. \end{aligned} \tag{5.2.3}$$

For the physical spin vectors (with $|1,0(r)\rangle := (1/\sqrt{2})(|+,-(r)\rangle + |-,+(r)\rangle)$, etc..) one then gets

$$\begin{aligned} |1,1(r)\rangle &= -e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}|1,1\rangle^{(-1)} + \cos\theta|1,1\rangle^{(0)} + e^{i\phi}\frac{\sin\theta}{\sqrt{2}}|1,1\rangle^{(1)} \\ |1,0(r)\rangle &= -e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}|1,0\rangle^{(-1)} + \cos\theta|1,0\rangle^{(0)} + e^{i\phi}\frac{\sin\theta}{\sqrt{2}}|1,0\rangle^{(1)} \\ |1,-1(r)\rangle &= -e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}|1,-1\rangle^{(-1)} + \cos\theta|1,-1\rangle^{(0)} + e^{i\phi}\frac{\sin\theta}{\sqrt{2}}|1,-1\rangle^{(1)} \\ |0,0(r)\rangle &= |0,0\rangle. \end{aligned} \tag{5.2.4}$$

Note that the vectors $|j,m(r)\rangle$ (for $(j,m) = (1,\pm 1),(1,0)$ and $(0,0)$) are all non vanishing for all r. Thus, we see that $r \mapsto |j,m(r)\rangle$ is a section in a bundle with total space $S^2 \times V_m$, where V_m is the space spanned by $\{|1,m\rangle^{(-1)}, |1,m\rangle^{(0)}, |1,m\rangle^{(1)}\}$, when $j = 1$ and the space spanned by $|0,0\rangle$, when $j = 0$. All four line bundles are trivial, since the corresponding sections are non vanishing. The difference lies in the symmetry/antisymmetry of the sections with respect to r, when they are regarded as vector valued functions on the sphere. This symmetry/antisymmetry could be regarded as an equivariance property with respect to some Z_2 action on each line bundle, but this is something we do not have at our disposal, so we will consider these line bundles as bundles over the sphere. One can then construct the corresponding projectors. The interesting thing about constructing these projectors is that the three corresponding to $j = 1$ turn out to have matrix components which are even functions. One can then interpret these projectors as defining projective modules over $C(\mathbb{R}P^2)$ and, as we will see, the form of these projectors is exactly the same as the one displayed in equation(4.2.20). Since the first three basis vectors in equation (5.2.4) have the same form, let us consider any of them, written in the form

$$|\chi(r)\rangle = -e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}|e_1\rangle + \cos\theta|e_2\rangle + e^{i\phi}\frac{\sin\theta}{\sqrt{2}}|e_3\rangle. \tag{5.2.5}$$

The (r-dependent) operator $P(r)$ that projects onto the vector space spanned by $|\chi(r)\rangle$ is defined by $P(r)|e_i\rangle := \langle\chi(r)|e_i\rangle|\chi(r)\rangle$. Explicitly, we have:

$$\begin{aligned}
P(r)|e_1\rangle &= \frac{\sin^2\theta}{2}|e_1\rangle - \frac{\sin\theta\cos\theta}{\sqrt{2}}e^{i\varphi}|e_2\rangle - \frac{\sin^2\theta}{2}e^{2i\varphi}|e_3\rangle \\
P(r)|e_2\rangle &= -\frac{\sin\theta\cos\theta}{\sqrt{2}}e^{-i\varphi}|e_1\rangle + \cos^2\theta|e_2\rangle + \frac{\sin\theta\cos\theta}{\sqrt{2}}e^{i\varphi}|e_3\rangle \\
P(r)|e_3\rangle &= -\frac{\sin^2\theta}{2}e^{-2i\varphi}|e_1\rangle + \frac{\sin\theta\cos\theta}{\sqrt{2}}e^{-i\varphi}|e_2\rangle + \frac{\sin^2\theta}{2}|e_3\rangle
\end{aligned} \tag{5.2.6}$$

In the basis $\{|e_i\rangle\}_i$, $P(r)$ has a matrix form, with components $M_{ij}(r)$ defined by $P(r)|e_i\rangle = \sum_j M_{ji}(r)|e_j\rangle$, so that

$$M(r) = \begin{pmatrix} \frac{1}{2}\sin^2\theta & -\frac{1}{\sqrt{2}}\sin\theta\cos\theta e^{-i\varphi} & -\frac{1}{2}\sin^2\theta e^{-2i\varphi} \\ -\frac{1}{\sqrt{2}}\sin\theta\cos\theta e^{i\varphi} & \cos^2\theta & \frac{1}{\sqrt{2}}\sin\theta\cos\theta e^{-i\varphi} \\ -\frac{1}{2}\sin^2\theta e^{2i\varphi} & \frac{1}{\sqrt{2}}\sin\theta\cos\theta e^{i\varphi} & \frac{1}{2}\sin^2\theta \end{pmatrix}. \tag{5.2.7}$$

We have thus obtained, from the transported spin basis, exactly the same expression for the projector that was considered in proposition 4.2.2. Here we are considering it as defining a projective module over the algebra $C(S^2)$, but given that its components are even functions, we could in principle consider it as giving place to a projective module over $C(\mathbb{R}P^2)$. This point of view will be considered in more detail in the next section.

Let us now use the obtained expression for the projector to show that each spin basis vector is a section of a line bundle (determined by the projector $P(r)$) that is parallel with respect to the connection Pd. Writing $|\chi(r)\rangle = \sum_i \chi_i(r)|e_i\rangle$ and $\vec{\chi} = (\chi_1, \chi_2, \chi_3) =$

$(-e^{-i\varphi}\sin\theta/\sqrt{2}, \cos\theta, e^{i\varphi}\sin\theta/\sqrt{2})$, we have:

$$\begin{aligned} Pd|\chi(r)\rangle &= P\sum_i d\chi_i(r)|e_i\rangle \\ &= \sum_i d\chi_i(r)P|e_j\rangle \\ &= \sum_{i,j} d\chi_i(r)M_{ji}(r)|e_j\rangle \\ &= \sum_j \underbrace{(M(r)\cdot d\vec{\chi}(r))_j}_{=0}|e_j\rangle \\ &= 0. \end{aligned} \tag{5.2.8}$$

This also shows that the parallel transport condition (third condition in definition 5.1.1) holds if and only if $|\chi\rangle$ is parallel transported with respect to the connection $\nabla = Pd$, because $|\chi\rangle$ is nowhere vanishing, so that $M(r)\cdot d\vec{\chi}(r) = 0$ exactly when $\langle\chi(r)|d\chi(r)\rangle = 0$.

In this section we have shown how, for the spin $1/2$ case, the transported spin basis gives place to a projector which contains, as direct summands, one copy of the trivial rank one projector and three copies of the projector from proposition 4.2.2, i.e., a projective module isomorphic to $\mathcal{A}_-$. Although it will not be done here, let us remark that, in the case of general spin, the transported spin basis constructed with the help of Schwinger's oscillators also gives place to a projective module. In the same way as in the case $s = 1/2$, after passing to the total angular momentum basis and decomposing the spin states according to the Clebsch-Gordan decomposition, one obtains a projective module that contains different copies of $\mathcal{A}_+$ and $\mathcal{A}_-$. Hence, concerning the geometric properties of the transported spin basis, it is enough to consider the case $s = 1/2$.

5.3 Single-valuedness of the wave function

As we have already pointed out, the requirement of single-valuedness imposed on the wave function in the form of equation (5.1.14) is an essential ingredient of the Berry-Robbins construction. In this subsection we will discuss that requirement, using the tools developed so far. We shall arrive at the conclusion that the single-valuedness condition of Berry and Robbins leads to a contradiction.

There are two instances -not altogether independent- where the concept of a "multiple-valued function" comes into play in the present context. Notice that this concept is actually related to the transformation properties of the function under a certain symmetry group. Before describing them, let us remark that physicists do not necessarily use the term "multiple-valued" in the same way mathematicians do. So, for example, in the theory of one complex variable, the function $z \mapsto \sqrt{z}$ is said to be multiple-valued, because there are two possible solutions (for w) to the equation $w^2 = z$. In contrast to this, consider the wave function of a spin $1/2$ particle. As a "mathematical" function, it is a single-valued one. But it has the property that, under a 2π rotation, it is not left unchanged: its value after the rotation is (-1) times its original value; hence the usage of the term "double-valued" in the physics literature. The physical need to impose single-valuedness was examined already in the early days of Quantum Mechanics. In particular, Pauli studied the problem and showed that considering multiple-valued functions (in the "mathematical"sense described above) leads to problems with the self-adjointness of related operators (see [Pau33, Pau39]). For a further analysis of the status of single-valuedness in physics (and in particular in view of effects like the Aharanov-Bohm), the reader is referred to [Mer62] and references therein.

Thus, we will also adopt the point of view that, as a "mathematical function" defined on configuration space, the wave function should be single-valued. Now, in the particular case of N identical particles, depending on which of the spaces ($\widetilde{\mathcal{Q}} = \mathbb{R}^{3N} \setminus \Delta$ or $\mathcal{Q} = (\mathbb{R}^{3N} \setminus \Delta)/S_N$) is considered as the physical one, one will impose single-valuedness to the wave function, whose domain of definition will be the *physical* configuration space. As argued in [LM77], due to Gibbs' paradox, in our case the physical configuration space should be chosen as $\mathcal{Q}$, and not as $\widetilde{\mathcal{Q}}$. In the configuration space approach to Spin-Statistics, this is perhaps one of the few points on which most authors agree.

Once we have chosen $\mathcal{Q}$ as configuration space, we are faced with the fact that, due to the non-trivial global structure of $\mathcal{Q}$, different possibilities for the definition of the wave function arise: It will be, in general, a cross-section of some vector bundle over $\mathcal{Q}$. As already pointed out‡, a section of a vector bundle is sometimes regarded as a multiple-valued function, because of the different functions that are needed in order to describe it in terms of local trivialisations. But let us, once again, remark that a section of a vector bundle is a single-valued map, even if the bundle is not trivial. This is the first instance we alluded to above.

The advantages of working on $\widetilde{\mathcal{Q}}$ -the universal cover of $\mathcal{Q}$- are well known and have been used extensively (see e.g. [HMS89, Mor92]). In particular, every wave function ψ, being a section of some bundle over $\mathcal{Q}$, can be brought to the form of a map $\widetilde{\psi} : \widetilde{\mathcal{Q}} \to \mathbb{C}^k$. Although this map is a single-valued one, it is of common usage, in the physics literature, to call it a "multiple-valued function". From the discussion at the beginning of section 3.3, we know that $\widetilde{\psi}$ should rather be referred to as an equivariant function. Once we know the value of $\widetilde{\psi}$ at some point $\tilde{q} \in \widetilde{\mathcal{Q}}$, we can use the equivariance property to find its value at all points $\sigma \cdot \tilde{q}$, for $\sigma \in S_N$. But the different points $\sigma \cdot \tilde{q}$ do represent one and the same physical configuration; hence the tendency to call $\widetilde{\psi}$ a "multiple-valued function". This is the second instance we alluded to above.

It is our impression that, instead of talking about *single-valuedness*, one should consider the notion of *well-definiteness*. In fact, the point with many of the standard exam-

‡See the remark on the fourth paragraph of section 1.2

ples where single-valuedness comes into play, having so far-reaching consequences as, for example, the quantization of angular momentum, is not really single-valuedness but well-definiteness. Considering the example of angular momentum, what is being "forced" when a condition of the form $\psi(\varphi) \stackrel{!}{=} \psi(\varphi + 2\pi)$ is imposed, is that ψ, as a map $\psi : S^1 \to \mathbb{C}$, be well defined. The imposition of this condition is necessary because S^1 is being described using only one local chart. In a global formulation, this problem would not even arise. Saying that ψ belongs to $C(S^1)$ automatically implies well-definiteness and, for that matter, also single-valuedness. The origin of the quantization of angular momentum is thus not so much due to the fact that ψ be single-valued, but to the fact that S^1 is a compact manifold, from which it follows that the spectrum of L_z is discrete.

Before embarking on a discussion of the single-valuedness condition of Berry-Robbins, let us summarize the situation as follows:

We are given a configuration space $\mathcal{Q}$ with $\pi_1(\mathcal{Q}) \neq 0$. Physical wave functions will be, in general, of the form $\psi \in \Gamma(\mathcal{Q}, \xi)$, for ξ some vector bundle over $\mathcal{Q}$. Because of theorem 3.3.5 we know that, as a module over $C(\mathcal{Q})$, $\Gamma^G(q^*\xi)$ is isomorphic to $\Gamma(\xi)$, where $q : \tilde{\mathcal{Q}} \to \mathcal{Q}$ and $G = \pi_1(\mathcal{Q})$. Thus, $\psi \in \Gamma(\mathcal{Q}, \xi)$ is isomorphically represented by $\tilde{\psi} \in \Gamma^G(\widetilde{\mathcal{Q}}, q^*\xi)$. If, as in the case of our interest, the bundle ξ is flat, $q^*\xi$ will be a trivial bundle and we might as well consider $\widetilde{\psi}$ to be a usual function, taking values in some $\mathbb{C}^k$.

With these preliminary remarks in mind, we proceed now to interpret the Berry-Robbins single-valuedness condition, within our formalism. Consider a wave function for two spin s particles written in terms of a transported spin basis $\{|M(r)\rangle\}$,

$$|\Psi(r)\rangle = \sum_M \Psi_M(r)|M(r)\rangle. \tag{5.3.1}$$

The transported spin vectors will be assumed to satisfy the exchange rule $|\bar{M}(-r)\rangle = (-1)^K|M(r)\rangle$. The choice of K will be left open for the moment. Notice, however, that it is possible to construct a spin basis satisfying the three requirements originally imposed by Berry-Robbins on the spin basis, irrespectively of the value of K. Changing to the total angular momentum ($\{|j,m\rangle\}$) basis, we can re-write the exchange condition as follows:

$$|j,m(-r)\rangle = (-1)^{2S-j+K}|j,m(r)\rangle. \tag{5.3.2}$$

The vectors $|M(r)\rangle$ are all non-vanishing, by assumption. It follows that all the vectors $|j,m(r)\rangle$ are non-vanishing, so that they give place to a trivial vector bundle η_K^{BR} over the sphere, with total space

$$E(\eta_K^{\text{BR}}) = \{(r, \sum_{j,m} \lambda_{j,m}|j,m(r)\rangle\,) \mid r \in S^2,\ \lambda_{j,m} \in \mathbb{C}\}. \tag{5.3.3}$$

Let us remark that, for any choice of the spin basis (including the choice of K), the bundles η_K^{BR} are all isomorphic to the trivial bundle $S^2 \times \mathbb{C}^{2(2s+1)} \to \mathbb{C}^{2(2s+1)}$. But, given we are interested, in particular, in the transformation properties of the spin basis under exchange, we keep track of the sign $(-1)^K$ by means of a sub-index in η_K^{BR}. Hence, η_K^{BR} denotes the explicit realization of the bundle with total space given by eq. (5.3.3), and not the equivalence class of this bundle. Similarly, from now on we will also label the spin basis elements and functions constructed from them with a sub-index K: $|M(r)\rangle_K$, $|j,m(r)\rangle_K$ and $|\Psi(r)\rangle_K$.

$|\Psi\rangle_K$ is a map $r \mapsto |\Psi(r)\rangle_K$ from S^2 to some vector space, but it gives place to a section $\Psi_K \in \Gamma(S^2, \eta_K^{\text{BR}})$, defined by $\Psi_K(r) := (r, |\Psi(r)\rangle_K)$. In any case, $|\Psi(r)\rangle_K$ is supposed to represent the physical wave function, and this one should have $\mathbb{R}P^2$ as domain of definition. This is the reason for the imposition of the single-valuedness condition.

In order to make the arguments presented below as clear as possible, let us highlight the following three features of the Berry-Robbins approach:

(A) Exchange rule: $|\bar{M}(-r)\rangle_K = (-1)^K |M(r)\rangle_K$.

(B) Single-valuedness: $|\Psi(-r)\rangle_K \overset{!}{=} |\Psi(r)\rangle_K$.

(C) Berry and Robbins statement, quoted at the end of section 5.1, that $|\Psi(r)\rangle_K$ represents a section of a bundle over $\mathbb{R}P^2$.

According to our previous discussion, the physical wave function ψ is a section of some vector bundle ξ over $\mathbb{R}P^2$: $\psi \in \Gamma(\mathbb{R}P^2, \xi)$. In order to make contact with the Berry-Robbins approach, we will make use of the results of chapter 3. Recall that the projection $q : S^2 \to \mathbb{R}P^2$ induces a natural $\mathbb{Z}_2$-action τ_ξ on the pull-back bundle $q^*\xi$ and that, according to theorem 3.3.5, the space of invariant sections (with respect to τ_ξ) of $q^*\xi$ is isomorphic, as a $C(\mathbb{R}P^2)$-module, to the space of sections of ξ:

$$\Phi^{\mathbb{Z}_2} : \Gamma(\mathbb{R}P^2, \xi) \xrightarrow{\cong} \Gamma^{\tau_\xi}(S^2, q^*\xi).$$

This means, in particular, that we can represent the wave function ψ as a section $\Phi^{\mathbb{Z}_2}(\psi)$ of the bundle $q^*\xi$. But $q^*\xi$ is a trivial bundle and has the same rank as η_K^{BR}, so that we also have isomorphisms $q^*\xi \cong \eta_K^{\text{BR}}$, for any choice of K. Now recall that, in order to regard a section in η_K^{BR} as a section in ξ, we must require it to be $\mathbb{Z}_2$-invariant. The action in η_K^{BR} (which is not present within the Berry-Robbins approach) must therefore be defined in such a way that the isomorphism $q^*\xi \cong \eta_K^{\text{BR}}$ still holds as an isomorphism of $\mathbb{Z}_2$ bundles. In this way, we can realize ψ as a map of the form (5.3.1), if we require the section $r \mapsto (r, |\Psi(r)\rangle_K)$ to be invariant with respect to the $\mathbb{Z}_2$-action of η_K^{BR}.

The $\mathbb{Z}_2$-actions on η_K^{BR} that are compatible with the Fermi-Bose alternative are of the form ($\sigma \in \mathbb{Z}_2$):

$$\tau_{\tilde{K}}^{\text{BR}}(\sigma)(r, |j,m(r)\rangle_K) = (\sigma \cdot r, (\text{sgn}\sigma)^{2s-j+\tilde{K}} |j,m(\sigma \cdot r)\rangle_K), \tag{5.3.4}$$

for $\tilde{K}$ even or odd. Notice that the transformation property **(A)** does not enter into the definition of the action. In particular, $\tilde{K}$ and K can be chosen independently.

If we now fix the bundle ξ (assuming that it is a bundle compatible with the Fermi-Bose alternative), we can use the isomorphism $\Phi^{\mathbb{Z}_2}$ and the requirement that $q^*\xi$ and η_K^{BR} be isomorphic $\mathbb{Z}_2$-bundles, in order to determine the value of $\tilde{K}$. This value of $\tilde{K}$ depends only on ξ. In particular, it is the same for the two different choices of K (even and odd). If we want $\Psi_K(r) = (r, |\Psi(r)\rangle_K)$ to be an invariant section, then we must have $\sigma \cdot \Psi_K \equiv \Psi_K$. But

$$\Psi_K(r) = (r, \sum_{j,m} \Psi_{j,m}^{(K)}(r)|j, m(r)\rangle_K),$$

whereas

$$\begin{aligned}
\sigma \cdot \Psi_K(r) &= \tau_{\tilde{K}}^{\text{BR}}(\sigma)(\Psi_K(\sigma^{-1} \cdot r)) \\
&= \tau_{\tilde{K}}^{\text{BR}}(\sigma)(\sigma^{-1} \cdot r, \sum_{j,m} \Psi_{j,m}^{(K)}(\sigma^{-1} \cdot r)|j, m(\sigma^{-1} \cdot r)\rangle_K) \\
&= (r, \sum_{j,m} \Psi_{j,m}^{(K)}(\sigma^{-1} \cdot r)(\text{sgn}\sigma)^{2s-j+\tilde{K}}|j, m(r)\rangle_K),
\end{aligned}$$

implying

$$\Psi_{j,m}^{(K)}(-r) = (-1)^{2s-j+\tilde{K}}\Psi_{j,m}^{(K)}(r). \tag{5.3.5}$$

This result is independent of the value of K and is a consequence of assuming **(C)**.

Now, let us consider the two possible choices of K:

- $\underline{K = \tilde{K}}$:

$$\begin{aligned}
|\Psi(-r)\rangle_{\tilde{K}} &= \sum_{j,m} \Psi_{j,m}^{(\tilde{K})}(-r)|j, m(-r)\rangle_{\tilde{K}} \\
&\overset{(\mathbf{A}),\text{with } K\equiv\tilde{K}}{=} \sum_{j,m} \Psi_{j,m}^{(\tilde{K})}(-r)\left((-1)^{2s-j+\tilde{K}}|j, m(r)\rangle_{\tilde{K}}\right) \\
&\overset{(5.3.5)}{=} \sum_{j,m} \left((-1)^{2s-j+\tilde{K}}\Psi_{j,m}^{(\tilde{K})}(r)\right)\left((-1)^{2s-j+\tilde{K}}|j, m(r)\rangle_{\tilde{K}}\right) \\
&= |\Psi(r)\rangle_{\tilde{K}}.
\end{aligned}$$

- $\underline{K = \tilde{K} + 1}$:

$$\begin{aligned}
|\Psi(-r)\rangle_{\tilde{K}+1} &= \sum_{j,m} \Psi_{j,m}^{(\tilde{K}+1)}(-r)|j, m(-r)\rangle_{\tilde{K}+1} \\
&\overset{(\mathbf{A}),\text{with } K\equiv\tilde{K}+1}{=} \sum_{j,m} \Psi_{j,m}^{(\tilde{K}+1)}(-r)\left((-1)^{2s-j+\tilde{K}+1}|j, m(r)\rangle_{\tilde{K}+1}\right) \\
&\overset{(5.3.5)}{=} \sum_{j,m} \left((-1)^{2s-j+\tilde{K}}\Psi_{j,m}^{(\tilde{K}+1)}(r)\right)\left((-1)^{2s-j+\tilde{K}+1}|j, m(r)\rangle_{\tilde{K}+1}\right) \\
&= -|\Psi(r)\rangle_{\tilde{K}+1}.
\end{aligned}$$

Therefore, we see that:

- The specification of a transported spin basis with the exchange property **(A)**, together with the single-valuedness condition **(B)** can be interpreted in terms of the specification of a $\mathbb{Z}_2$-bundle structure on η_K^{BR} and, at the same time, of the requirement that the corresponding section Ψ be $\mathbb{Z}_2$-invariant (as it should be, according to **(C)**), *provided* that we choose $K = \tilde{K}$.
- The choice $K = \tilde{K} + 1$ leads to a contradiction with the assumption of single-valuedness **(C)**.

The need for a position-dependent spin in the Berry-Robbins approach stems from the fact that they try to obtain ψ, the physical wave function, by means of **(B)**: The role of the spin basis is to ensure that the fibers of η_K^{BR} at opposite points on the sphere *coincide*, given that the bundle has been constructed as a subbundle of a trivial bundle. It then makes sense to compare the values of $|\Psi\rangle_K$ at different points, as is tacitly assumed by **(B)**.

Let us note that,from the point of view of our approach, there is no need for a position-dependent spin basis. This is so because ψ is a section of ξ and, given that $q^*\xi$ is a trivial bundle, we can use the isomorphism $\Phi^{\mathbb{Z}_2}$ to represent ψ as a function on S^2.

But our analysis does not only show that the introduction of a position-dependent spin basis is unnecessary, it also shows that the single-valuedness condition **(B)** is wrong: We start with $\psi \in \Gamma(\mathbb{R}P^2, \xi)$, using the isomorphism $\Phi^{\mathbb{Z}_2}$ we obtain an isomorphic image of ψ inside $\Gamma^{\tau_\xi}(S^2, q^*\xi)$. Then, we require the bundle η_K^{BR} to be $\mathbb{Z}_2$-equivalent to $(q^*\xi, \tau_\xi)$. This requirement fixes the value of $\tilde{K}$. Finally, choosing $K = \tilde{K}$ gives us a section $r \mapsto (r, |\Psi(r)\rangle_{\tilde{K}})$ which is an isomorphic image of ψ and for which $|\Psi(-r)\rangle_{\tilde{K}} = |\Psi(r)\rangle_{\tilde{K}}$ holds. But choosing $K = \tilde{K} + 1$ gives us a section $r \mapsto (r, |\Psi(r)\rangle_{\tilde{K}+1})$ which is *also* an isomorphic image of ψ, but for which, instead, $|\Psi(-r)\rangle_{\tilde{K}+1} = -|\Psi(r)\rangle_{\tilde{K}+1}$ holds.

The subtlety of the argument lies in the fact that we are working with projective modules $\widetilde{P}, P$ over two different rings: $C(S^2)$ and $C(\mathbb{R}P^2)$, respectively. We have seen that $\widetilde{P}$ carries also the structure of a $C(\mathbb{R}P^2)$-module. This permits the construction of a $C(\mathbb{R}P^2)$-module homomorphism $\Phi^{\mathbb{Z}_2} : P \to \widetilde{P}$ (thm. 3.3.5) that maps P *isomorphically* onto a submodule of $\widetilde{P}$ (here considered as $C(\mathbb{R}P^2)$-module). The important point is, then, to recognize the correct condition characterizing those elements of $\widetilde{P}$ that lie in $\Phi^{\mathbb{Z}_2}(P)$. As we have seen, this condition is expressed in the form

$$\sigma \cdot \Psi = \Psi \quad (\sigma \in Z_2), \tag{5.3.6}$$

a form that (to our opinion), while retaining the original motivation behind the single-valuedness condition, is free from ambiguities and mathematically correct. Physically, because of indistinguishability, the wave function is defined on $\mathbb{R}P^2$. Working on S^2 thus demands the introduction of some criterion that allows one to retain the correct physical interpretation. Although this might be the purpose of the single-valuedness condition of Berry-Robbins, we have seen in detail why the condition itself is not tenable. Instead we propose the imposition of *invariance* on the wave function (when defined on S^2) as stated in equation (5.3.6) as a more clear and concise way to incorporate indistinguishability into the (usual) formalism of Quantum Mechanics.

Let us remark that, in [HR04] a generalization of the BR construction to N particles has been given. Being based on the same assumptions (in particular the single-valuedness condition), it has the same problems as the one for two particles. The extension of our reasoning to that -more general- case is nevertheless straightforward, since our approach is model-independent.

6 Further developments

In this work, almost all of our attention has been directed towards a reformulation of the Berry-Robbins (BR) proposal in geometric/algebraic terms. This effort, *per se*, does not seem to provide an advance regarding the Spin-Statistics theorem. But it must be said that the BR proposal has generated, in last few years a growing interest in the issue of Spin-Statistics in (non-relativistic) Quantum Mechanics. It is for this reason that we have emphasized so much the problems of their approach. We have seen that the problems with the BR proposal are of a conceptual nature. This means that now we have to turn our attention to other directions. Two points that, to our opinion, deserve to be investigated further are the correct incorporation of spin degrees of freedom into the configuration space approach and a convincent proof of the Fermi-Bose alternative, of a geometric nature. As we will see below, it seems that these two questions are closely related.

It is natural, for reasons of simplicity, to attack these and other questions first at the level of a two particle system, the hope being always that, once the two particle case has been understood, the general, many-particle case can be understood without essential modifications. But if one is also interested in the relation of the geometric/topological character of quantum indistinguishability to the (relativistic QFT) Spin-Statistics connection, then a formalism able to handle any number of particles will be welcome. In this work we have performed all explicit computations in the two particle case, but the motivation for the approach presented in chapter 3 has been, precisely, to develop a formalism that (1) allows to understand and clarify alternative approaches and (2) can, potentially, be applied to the case of an arbitrary number of particles. Also, inspired by the philosophy of Non-commutative Geometry, one could think of improving the approach in such a way that, while retaining all the geometric properties inherent to quantum indistinguishability, is also formulated in a language, of more algebraic nature, that could help to build a bridge to QFT. Although certain attempts more or less related to this idea have been made (see, e.g. [Tsc89]), there are, at the moment, no concrete results.

In this last chapter we will, therefore, consider in some detail some of the points on which we think it is possible to make some advance in the near future.

6.1 $\mathrm{SU}(2)$ and Spin

When quantum mechanics is studied in a more general setting, such as the one we have been discussing in the previous chapters where the wave function is considered to be a section in a vector bundle, a reconsideration of many physical concepts becomes necessary. A particularly important one is angular momentum and, closely related to it, the definition of spin.

Let us start by considering the simplest possible example: A spin $1/2$ particle in 3 dimensions. In this case, one considers the state space

$$\mathcal{H} = L^2(R^3) \otimes V^{(1/2)}.$$

The spin state space $V^{(1/2)}$ has a basis $\{|+\rangle, |-\rangle\}$ with respect to which the wave function can be written:

$$\psi = \psi_+ \otimes |+\rangle + \psi_- \otimes |-\rangle.$$

The spin operators $S_\pm, S_3$ are, by definition, operators representing $\mathfrak{su}(2)$, acting on $V^{(1/2)}$ by

$$\begin{aligned} S_\mp|\pm\rangle &= |\mp\rangle \\ S_\pm|\pm\rangle &= 0 \\ S_3|\pm\rangle &= \pm\frac{1}{2}|\mp\rangle. \end{aligned}$$

Together with the angular momentum operators L_i, they give place to infinitesimal generators of rotations, acting on $\mathcal{H}$:

$$J_i = L_i \otimes \mathrm{Id}_{V^{(1/2)}} + \mathrm{Id}_{L^2(R^3)} \otimes S_i. \tag{6.1.1}$$

The infinitesimal generators L_i and S_i give, by exponentiation, the respective representations of the whole group $SU(2)$ on $L^2(R^3)$ and $V^{(1/2)}$. Explicitly, one has, for $g \in SU(2)$:

$$\begin{aligned} (g \cdot \psi_\pm)(x) &= \psi_\pm(g^{-1}x) \\ g \cdot |\pm\rangle &= \mathcal{D}^{1/2}(g)|\pm\rangle. \end{aligned}$$

Consequently, the total wave function transforms under $SU(2)$ as $\psi \to \psi' = g \cdot \psi$, with $g \cdot \psi$ defined by:

$$(g \cdot \psi)(x) := \mathcal{D}^{(1/2)}(g)(\psi(g^{-1}x)). \tag{6.1.2}$$

Looking back at equation (3.3.5), we see that by regarding the wave function as a section of the trivial bundle $\mathbb{R}^3 \times V^{1/2} \to \mathbb{R}^3$, the transformation of the wave function under rotations is the one induced from the following $SU(2)$-action on the bundle:

$$\begin{aligned} \tau : SU(2) \times (\mathbb{R}^3 \times V^{1/2}) &\longrightarrow (\mathbb{R}^3 \times V^{1/2}) \\ (g, (x, v)) &\longmapsto (g \cdot x,\, \mathcal{D}^{(1/2)}(g)v) \end{aligned} \tag{6.1.3}$$

We thus see that, even if the bundle were not trivial, there would be no problem in defining what the transformation of the wave function under a finite rotation is: Everything we need is that the bundle on which the wave function is defined also carries the structure of a $SU(2)$ bundle. Spin operators can then in principle be defined by considering the one-parameter subgroups of transformations induced by $\mathfrak{su}(2)$ through the action τ. But here care most be taken, when defining spin operators in terms of these infinitesimal generators, because they include as well the generators of "orbital" angular momentum. Whether these two (orbital and spin) angular momentum operators can be consistently separated, when defined by means of the action τ, is something that is not clear *a priori*. The reason for this is that, if the vector bundle where the wave function is defined is not trivial, the existence and the properties of an $SU(2)$ action τ depend directly on the "twists" of the bundle, therefore, it is to be expected that the infinitesimal generators deduced from the action τ also carry information about the global properties of the bundle, making a more careful analysis necessary. An interesting example illustrating this feature, is that of a (spinless) electron in the presence of a magnetic monopole field, an example that we mentioned briefly in remark 4.1.1.

Leinaas and Myrheim [LM77] proposed to define *local* spin operators, $S_i(x)$, varying continuously with x and acting linearly on each fiber of the bundle ξ where the wave function is defined. Each fiber ξ_x is then required to carry a representation of $\mathfrak{su}(2)$, given by the local operators $S_i(x)$. In the Berry-Robbins approach, spin operators do also depend on the position, though they are not defined on the configuration space, but on its universal covering. As with the spin basis, the spin operators are defined making use of the map U (see eq. (5.1.8)):

$$S_i(r) := U(r) S_i U^\dagger(r). \tag{6.1.4}$$

In the same way as in the proposal of Leinaas and Myrheim, the spin operators are defined in such a way that they act linearly -as the physically correct representation- on each fiber. These operators give rise, through exponentiation, to an $SU(2)$ bundle structure.

In chapter 4 we considered the $SU(2)$ equivariance of line bundles over the projective space. In that case, we saw how the $SU(2)$ action was induced by the action on the (trivial) vector bundle on the sphere (from which we obtain the bundles over the projective space). This seems to be the case, also in the case of a finite number of particles (see remark 6.2.1).

The bundles relevant in the N-particle case will also be flat. It is therefore to be expected that the $SU(2)$ actions defined on them (as suggested in remark 6.2.1) are related to parallel transport, in the same way as we checked in the two particle case (see eq. (4.2.25)).

One could use this property in order to define spin operators, in the following way. Denote parallel transport from q to q' by $\mathrm{PT}(q, q')$, where $q, q' \in \mathcal{Q}$. Consider the one-parameter subgroup of homomorphisms $t \mapsto g_t$ corresponding to the infinitesimal generator σ_i of $SU(2)$. Given a vector y in the fiber over q, define the Spin Operator S_i as follows:

$$S_i(q) := \lim_{t\to 0} \frac{i}{t}\big(\mathrm{PT}(g_t \cdot q, q)(\tau_{g_t} y) - y\big). \tag{6.1.5}$$

Note the q-dependence of S_i.

It is not difficult to see that, in the two particle case, the operators $S_i(r)$ defined through equation (6.1.4) can be obtained in this way. For this, let us consider the (Berry-Robbins) transported spin basis $\{|j, m(r)\rangle\}_{j,m}$ for a given value s of the spin. The representation $\mathcal{D}^s \otimes \mathcal{D}^s$, acting on the *fixed* spin basis, decomposes in irreducible ones $\mathcal{D}^j$ acting on each of the subspaces labeled by j ($j = 0, \ldots, 2s$). The transported spin basis gives place to a bundle η^s over the sphere (whose fibers are generated by the transported spin vectors at each point). We can use the operator $U(r)$ and the representations $\mathcal{D}^j$ to give this bundle the structure of an $SU(2)$ bundle. Since η^s is a subbundle of a trivial bundle, we can define an $SU(2)$ action on the total space as follows ($g \in SU(2)$):

$$\begin{array}{rcl} \tau_g : E(\eta^s) & \longrightarrow & E(\eta^s) \\ \left(r, \sum_{j,m} \lambda_{j,m} |j, m(r)\rangle\right) & \longmapsto & \left(g\cdot r, \sum_{j,m} \lambda_{j,m} U(g\cdot r)\mathcal{D}^j(g)U(r)^\dagger |j, m(r)\rangle\right) \end{array} \tag{6.1.6}$$

Using this $SU(2)$ action and the definition of spin operators given above, one recovers the operators defined in equation (6.1.4).

Let us recall from the discussion of chapter 5 that the only reason that seems to make necessary the introduction of a transported spin basis is the single-valuedness condition imposed on the wave function. We have shown that this condition is not consistent with the point of view according to which the wave function is a section on the physically correct configuration space. Furthermore, the bundle η^s can be trivialized, because all the spin basis vectors are non-vanishing. If we dispense from adopting the single-valuedness condition, then the construction of the spin basis becomes irrelevant: In that case, all that we do is to map the spin vector space $V^s \otimes V^s$ at each point r, to a vector space isomorphic to it, constructed with the help of the operator $U(r)$. The "twists" introduced by this mapping force the introduction of a transported spin basis, as well as a redefinition of all operators, including the spin operators. But, the bundle being trivial, there is no topological obstruction of any kind to the existence of all these structures. This does not mean that the construction of these structures is a trivial task (as the generalization of the Berry-Robbins construction to more than two particles shows), but in any case, one should be aware that in this way, with most certainty, no new information or restriction that could help in the search for a Spin-Statistics theorem will be found. In fact, whereas some authors affirm that the introduction of structures as, for example, a transported spin basis, results in a theory *"quite different from standard physics"* [AM03], we can easily see, using our approach, that the spin operators of BR are equivalent to the usual ones*.

*All one has to do is to apply the isomorphism $\mathrm{Hom}_{C(M)}(\Gamma(\xi_1), \Gamma(\xi_2)) \cong \Gamma(\mathrm{Hom}(\xi_1, \xi_2))$, valid for ξ_1 and ξ_2 bundles over M (cf. [GBVF01])

As far as we can see, it remains possible to define spin operators in this way, for the cases where the (correct) Spin-Statistics relation is violated. It would be interesting, however, to compare a global construction of the spin operators using our approach to other works (in particular [Kuc04]) on the Spin-Statistics relation, where angular momentum comes into play.

6.2 Exchange

In the standard formulation of quantum mechanics, the state space of a system of N identical particles is built from the 1-particle Hilbert space $\mathcal{H}^{(1)}$ as a subspace of the (N-fold) product space $\mathcal{H}^{(1)} \otimes \cdots \otimes \mathcal{H}^{(1)}$. This space is the natural one to consider in the non-interacting case, where the Hamiltonian of the system consists of a sum of independent terms, one for each particle. Exchange/permutation operators acting on this state space can be considered as dynamical variables which, due to the fact that the Hamilton operator is symmetrical with respect to all particles' labels (because of indistinguishability), are, actually, constants of the motion. Exchange symmetry is a different kind of symmetry, in comparison with other symmetries like translational or rotational, in the sense that the degeneracy it gives rise to cannot be lifted by any physical means (as e.g. the application of an external field) but by the imposition of an additional postulate on the theory. After imposing the symmetrization postulate, the Spin-Statistics theorem further restricts the states: Bosonic for integer spin particles and Fermionic for half-integer spin ones. One feature of this way of constructing the physical states is that one starts with the state space of the one particle problem $\mathcal{H}^{(1)}$. The many-particle state space is then constructed in a two-step fashion, first "putting together" all particles (i.e. forming the tensor product space) and the "taking away" the labeling implicit in the tensor product space by projecting into the physical state space.

By contrast, the configuration space approach attempts to construct in a "single step" multi-particle wave functions, without the need for permutation operators. Indistinguishability is included intrinsically into the definition of the classical configuration space and therefore no labelings are used. If the two formulations are to be considered as equivalent, then it is natural to think that the latter results form the former by some sort of quotient operation. In fact, the projection from arbitrary product states to only symmetric or antisymmetric ones can be considered as a realization, at the Hilbert space label, of the quotient operation sending an equivariant S_N-bundle to its quotient bundle, as stated in theorem 3.1.3. The relevance of this idea is that, even if it is a physical reason that leads us to consider configuration spaces where permuted configurations have been identified,

1. It is the tensor product space the one that is used in every practical computation, i.e., it is in terms of this space that all predictions of many-particle quantum mechanics are performed.

2. There are situations where it is not necessary to consider indistinguishability, as when the particles involved are far apart.

3. However different this modified approach to quantum mechanics might look from a conceptual point of view, a well defined mathematical relationship between both formulations must be also given. In the end, it is quantum mechanics, in the way it is usually is formulated, the one that has been used to make successful predictions of physical phenomena.

Therefore, we want to propose some assumptions that will help in establishing the link between the usual formulation and the proposed one.

As a starting point, consider once again the configuration space in the case of two identical particles, $\mathbb{R}P^2$. It has been obtained from the quotient map $q : S^2 \to \mathbb{R}P^2$ because this is the natural way to eliminate the redundancy in labeling in the formalism. If the particles do also posses spin degrees of freedom, it is of course necessary to consider exchange of both positions and spins. Again, exchanged configurations should be identified. This can be done by noting that the simultaneous exchange of positions and spins is a transformation that can be regarded as a $\mathbb{Z}_2$-action on the trivial bundle

$$E^s := S^2 \times (V^s \otimes V^s).$$

By proposition 3.1.3, we know that this action induces a quotient map

$$\hat{q} : E^s \longrightarrow E^s/\mathbb{Z}_2$$

whose image is a bundle over $\mathbb{R}P^2$. In this way, we use a $\mathbb{Z}_2$-action on E^s to construct the bundle on which the wave function will be defined. The requirement that this action should be *directly related to exchange of spins and positions* seems to us to be a natural and physically well-motivated one. On the other hand, suppose we are given a bundle $\xi \xrightarrow{\pi} \mathbb{R}P^2$ (where the physical wave function is supposed to be defined). The action (3.1.1) induced on $q^*\xi$ is not *a priori* related to exchange and could in fact be incompatible with it. This gives us the motivation to impose the following condition.

Assumption I *Suppose ξ^s is the vector bundle on which the physical wave function is defined. Then, the $\mathbb{Z}_2$-action induced by q on E^s through $E^s \cong q^*\xi^s$ must coincide with the operation realizing the exchange of both positions and spins.*

In other words, since $q^*\xi^s$ is isomorphic to E^s as a vector bundle, we have two $\mathbb{Z}_2$-structures on the same bundle, one of which is intrinsically related to exchange. What we require is that both actions coincide, that is, that the two bundles be equivalent also as $\mathbb{Z}_2$-bundles. As a next step, we have to find, in principle, all $\mathbb{Z}_2$-actions that are compatible with exchange. An action $\tau : \mathbb{Z}_2 \times E^s \to E^s$ has the general form

$$\tau_\sigma(r, v) = (\sigma \cdot r, T_r(\sigma)v), \qquad (r \in S^2, v \in V^s \otimes V^s),$$

where $T_r(\sigma)$ is a linear map. Using the symbols e and t for the elements of $\mathbb{Z}_2$ with $t^2 = e$, we must have $T_r(e) = \mathrm{Id}$ and $T_{-r}(t)T_r(t) = \mathrm{Id}$. Note that the map $T_r(\sigma)$ depends on r and thus is not in general a representation of $\mathbb{Z}_2$. But since there are only two equivalence classes of line bundles over $\mathbb{R}P^2$, we know that there can be only two classes of $\mathbb{Z}_2$-equivariant line bundles over S^2. Notice, however, that when the number of particles increases, the problem becomes much more complex. For the general case, we are going to leave open the question of how many such actions exist. It would be interesting if one could show that all S_N-equivariant bundles over the respective configuration spaces $\widetilde{\mathcal{Q}}_N$ (eq.(1.1.1)) were equivalent to bundles with an S_N-action given by maps

$$\tau_\sigma(q,v) = (\sigma \cdot q, T_q(\sigma)v), \qquad (q \in \widetilde{\mathcal{Q}}_N, v \in V^s \otimes \cdots \otimes V^s, \sigma \in S_N),$$

with $T_q(\sigma)$ *independent* of q. Returning now to the two particle example, note that τ realizes the exchange of positions automatically. In order that it also realizes the exchange of spin vectors, according to our assumption, we must require that $T(t)$ acts as follows on basis elements:

$$T(t)(|m_1\rangle \otimes |m_2\rangle) = e^{i\alpha(m_1,m_2)}|m_2\rangle \otimes |m_1\rangle,$$

where

$$e^{i\alpha(m_1,m_2)} = e^{-i\alpha(m_2,m_1)} \tag{6.2.1}$$

must be satisfied.

We are used to regard the effect of exchange on spins as the particular case where $e^{i\alpha(m_1,m_2)} = 1$ for all m_1 and m_2, but there is nothing forbidding us to consider the general case. At this point we cannot say more about the coefficients $e^{i\alpha(m_1,m_2)}$. But the situation will change, after considering how rotations are to be implemented on ξ^s.

6.2.1 Remark. Recall that it is in part because of proposition 3.3.5 that we work on E^s. The wave function will be defined, ultimately, on $\xi^s = E^s/\mathbb{Z}_2$, once we find the correct $\mathbb{Z}_2$ action τ. In the same way, the effect of rotations on the wave function must be implemented on ξ^s, as explained in the last section, not on E^s. But since we have not yet found τ, we cannot say what bundle ξ^s is and consequently what $SU(2)$ actions it has. On the other hand, on E^s we have the standard $SU(2)$ representation. If it commutes with τ, we obtain an induced action on ξ^s and the corresponding spin operators will act irreducibly on the fibers. The question is therefore whether *any* $SU(2)$-action on ξ^s is of that form (comes from one on E^s). The answer to this question is positive, because suppose we have an action $\tau^{\mathrm{SU(2)}}$ on ξ^s. We then get an induced action on $q^*\xi^s$ given ($g \in SU(2)$, $(m,y) \in q^*\xi^s$) by

$$g \cdot (m,y) = (g \cdot m, \tau_g^{\mathrm{SU(2)}}(y)).$$

By Assumption I, the $\mathbb{Z}_2$-action τ on E^s is equivalent to the $\mathbb{Z}_2$-action induced on $q^*\xi^s$ by q, see eq. (3.1.1). But the two actions on $q^*\xi^s$, $\mathbb{Z}_2$ and $SU(2)$, commute, because ($g \in SU(2), \sigma \in \mathbb{Z}_2$):

$$\begin{aligned}
\sigma \cdot (g \cdot (m,y)) &= (\sigma \cdot (g \cdot m), \tau_g^{\mathrm{SU(2)}}(y)) \\
&= (g \cdot (\sigma \cdot m), \tau_g^{\mathrm{SU(2)}}(y)) \\
&= g \cdot (\sigma \cdot (m,y)).
\end{aligned} \tag{6.2.2}$$

This means that τ also commutes with the $SU(2)$ action on E^s induced by $\tau^{\mathrm{SU(2)}}$. But this last action is (we expect) induced by a representation of $SU(2)$. Since the spin operators induced by $\tau^{\mathrm{SU(2)}}$ act locally on the fibers as representations isomorphic to the usual ones on $V^s \otimes V^s$, we have arrived at the conclusion that τ must commute with the usual $SU(2)$ representation on E^s (i.e. on $V^s \otimes V^s$). Therefore, we make the following

Assumption II *The $\mathbb{Z}_2$ action τ on E^s must commute with $SU(2)$.*

6.2.2 Proposition. Assumption II implies that all terms $e^{i\alpha(m_1,m_2)}$ are equal to a factor $(-1)^K$.

Proof. Let $|j, m\rangle$ denote the vectors in the basis of total angular momentum. The requirement that the actions commute implies that the corresponding *representations* commute:

$$\left[T(\sigma), \mathcal{D}^{(j)}(g)\right] |j, m\rangle = 0 \qquad (\sigma \in \mathbb{Z}_2, g \in SU(2)).$$

This is equivalent to requiring

$$[T(t), J_i] \, |j, m\rangle = 0, \quad i = 1, 2, 3$$

for the infinitesimal generators. From (6.2.1), it follows that $e^{i\alpha(s,s)} = (-1)^K$ for some integer K. Using $|2s, 2s\rangle = |s\rangle \otimes |s\rangle$ we obtain, from $[T(t), J_-] \, |2s, 2s\rangle = 0$:

$$\begin{aligned} T(t)|2s, 2s-1\rangle &= \frac{1}{\sqrt{4s}} T(t) J_- |2s, 2s\rangle \\ &= \frac{1}{\sqrt{4s}} J_- T(t) |2s, 2s\rangle \\ &= (-1)^K |2s, 2s-1\rangle. \end{aligned}$$

Applying $T(t)$ to all vectors $|2s, \mu\rangle$, we obtain, in the same way as above,

$$T(t)|2s, \mu\rangle = (-1)^K |2s, \mu\rangle, \qquad -2s \leq \mu \leq 2s.$$

From this we obtain:

$$\begin{aligned} (-1)^K |2s, \mu\rangle &= T(t) \sum_{m_1, m_2} (s\, m_1, s\, m_2 | 2s\, \mu) |m_1\rangle \otimes |m_2\rangle \\ &= \sum_{m_1, m_2} (s\, m_1, s\, m_2 | 2s\, \mu) e^{i\alpha(m_1, m_2)} |m_2\rangle \otimes |m_1\rangle \\ &= \sum_{m_1, m_2} (s\, m_1, s\, m_2 | 2s\, \mu) e^{i\alpha(m_1, m_2)} |m_1\rangle \otimes |m_2\rangle. \end{aligned}$$

Putting $\mu = m_1 + m_2$ in the last equation, we obtain, using the linear independence of the basis, $e^{i\alpha(m_2, m_1)} = (-1)^K$. Note that in order to use linear independence it is necessary that the respective Clebsch-Gordan coefficient be non vanishing. But this is true. In fact one has:

$$
\begin{aligned}
&(s\,m_1, s\,m_2|2s\,m_1+m_2) = \\
&\quad = \frac{(-1)^{2m_1+2m_2}\Gamma(2s)\sqrt{s\Gamma(2s+1-m_1-m_2)\Gamma(2s+1+m_1+m_2)}}{\sqrt{\Gamma(4s)\Gamma(s+1-m_1)\Gamma(s+1+m_1)\Gamma(s+1-m_2)\Gamma(s+1+m_2)}}.
\end{aligned}
$$

□

The conclusion, up to this point, is that there are two possible bundles on which the wave function can be constructed. They correspond to Fermi and Bose statistics. Let us spell this out in detail. If we write the wave function representing the state of the two particle system in the usual form

$$
\begin{aligned}
|\Psi\rangle : S^2 &\longrightarrow V^s \otimes V^s \\
r &\longmapsto \sum_{m,m'} \Psi_{m,m'}(r)|m\rangle \otimes |m'\rangle,
\end{aligned}
\tag{6.2.3}
$$

then we can, using proposition 3.3.5, interpret it as the section of a bundle on $\mathbb{R}P^2$, if we require the section

$$
\begin{aligned}
\Psi : S^2 &\longrightarrow S^2 \times V^s \otimes V^s \\
r &\longmapsto (r, |\Psi(r)\rangle)
\end{aligned}
\tag{6.2.4}
$$

to be invariant with respect to the $\mathbb{Z}_2$ action. Now, if instead of $\{|m_1\rangle \otimes |m_2\rangle\}$ we use the basis $\{j, m\}$ of total angular momentum, we see that the map $T(t)$ takes the form

$$
|j,m\rangle \longmapsto (-1)^{2S-j+K}|j,m\rangle.
$$

This means that under the quotient map $\hat{q}$, the subspace corresponding to fixed values of j and m goes into a line bundle isomorphic to L_- (if $2s-j+K$ is odd) or L_+ (if $2s-j+K$ is even). Writing $|\Psi\rangle$ in the total spin basis $\{|j,m\rangle\}$, we have:

$$
|\Psi(r)\rangle = \sum_{j,m} \Psi_{j,m}(r)|j,m\rangle. \tag{6.2.5}
$$

For $|\Psi\rangle$ to represent the wave function we must require that the section $\Psi : r \mapsto (r, |\Psi(r)\rangle)$ be invariant (proposition 3.3.5). We have:

$$
\begin{aligned}
(t\cdot\Psi)(r) &\overset{!}{=} \Psi(r) \Longleftrightarrow \\
\tau_t(\Psi(t^{-1}\cdot r)) &\overset{!}{=} \Psi(r).
\end{aligned}
\tag{6.2.6}
$$

But from

$$
\begin{aligned}
\tau_t(\Psi(t^{-1}\cdot r)) &= \tau_t(\Psi(-r)) \\
&= \tau_t(-r, |\Psi(-r)\rangle) \\
&= (r, (-1)^{(2S-j+K)}|\Psi(-r)\rangle)
\end{aligned}
$$

then we obtain, for the coefficient functions:

$$(-1)^{(2S-j+K)}\Psi_{j,m}(-r) = \Psi_{j,m}(r). \tag{6.2.7}$$

Transforming back to the $\{|m_1\rangle \otimes |m_2\rangle\}$ basis, we get:

$$\Psi_{m_2,m_1}(-r) = (-1)^{2S+K}\Psi_{m_1,m_2}(r), \tag{6.2.8}$$

that is, the Fermi-Bose alternative. Let us finish this section with some remarks concerning this derivation.

- As remarked above, for the case $N > 2$ case, the S_N actions that we have considered are not the most general one can conceive. It would be interesting to find out how many possibilities there are, in order to be able to extend our arguments to that general case.
- The derivation of the Fermi-Bose alternative we have proposed is consistent with the point of view we have adopted of working in a configuration space of indistinguishable particles. The equivariance of the bundles over the covering of the configuration space play the double role of defining how to obtain the vector bundle where the physical wave function is defined and at the same time realizes exchange. Our first proposed assumption says that the action that realizes exchange must be precisely the one with respect to which the quotient is taken on the bundle.
- As presented here, this derivation is specific to the two particle case, which is a restriction. But on the other hand, it works for any value of the spin.

A G-Spaces

In this appendix we collect some standard facts about G-spaces. They are used in chapter 3.

A.1 Definition. A *topological group* is a group G which at the same time is a topological space, such that the map

$$\begin{array}{rcl} G \times G & \to & G \\ (g,h) & \mapsto & gh^{-1} \end{array}$$

is continuous.

A.2 Definition (G-space). Let G be a topological group and M a topological space. A map $\rho : G \times M \to M$ is called a (left-) group action when, for every $m \in M$, $\rho(gh,m) = \rho(g,\rho(h,m))$ $(\forall g,h \in G)$ and $\rho(e,m) = m$. If the map ρ is continuous, M is said to be a *left G-space*. A right G-space is defined similarly.

A.3 Remark. It is usual to denote an action ρ in the following way: $g \cdot m := \rho(g,m)$. For $g \in G$ fixed, one considers the "partial map" ρ_g, defined through

$$\begin{array}{rcl} \rho_g : M & \to & M \\ m & \mapsto & g \cdot m \equiv \rho(g,m). \end{array}$$

A.4 Definition (G-Orbit). Let M be a left G-space. Given $m \in M$, the set

$$G \cdot m = \{g \cdot m \,|\, g \in G\} \subset M$$

is called the *orbit* of m and will be denoted $\mathcal{O}_m$.

A.5 Definition (Free action). A group action is called free when $G_x \equiv \{e\}$ for all x in M, i.e. when for all $x \in M$, $g \cdot x = x \Rightarrow g = e$ holds.

A.6 Remark. Equivalently, one can say: A group action is free if and only if for every g in $G \setminus \{e\}$, the fixed-point set $\{x \in M | g \cdot x = x\}$ is the empty set.

Let G be a discrete group and M a topological space. If additionally G acts on M, one may ask under what conditions does the quotient space M/G retain certain topological properties that M might posses. If, for instance, M/G is a Hausdorff space, then every point $p \in M/G$ is a closed set. Since the canonical projection $q : M \to M/G$ is continuous (by definition of the quotient topology), the set $q^{-1}(p)$ must also be closed or, in other words, if $p = q(x)$, then the orbit $\mathcal{O}_x$ must be a closed set. It is clear that not every group action satisfies this condition.

An interesting case is the one where M is a (differentiable) manifold. Under the assumption of a free and properly discontinuous action, it is possible to show that M/G inherits the structure of a (differentiable) manifold. Since this well-known result [Boo02, BG88] will be needed several times in this work, a proof of it will be given below. First we begin by explaining the term "properly-discontinuous".

A.7 Definition (Properly discontinuous action). A *properly discontinuous action* of a discrete group G on a topological (resp. differentiable) manifold is a continuous (resp. differentiable) action such that:

(i) $\forall x, y \in M$:
$y \notin \mathcal{O}_x \Rightarrow \exists\, U, V$ neighborhoods such that $x \in U$, $y \in V$ and $(G \cdot U) \cap V = \emptyset$.

(ii) There is, for every $x \in M$, a neighborhood U such that the set $\{h \in G \,|\, h \cdot U \cap U = \emptyset\}$ is finite.

A.8 Remark. The second condition may be replaced by the following, equivalent one*

(ii)′ For every $x \in M$, the isotropy group $G_x := \{g \in G \,|\, g \cdot x = x\}$ is finite and there is a neighborhood U of x such that $(g \cdot U) \cap U = \emptyset$ for $g \notin G_x$ and $g \cdot U = U$ for $g \in G_x$.

A.9 Theorem. Let M be a topological (resp. differentiable) manifold and G a discrete group acting freely and properly-discontinuously on M. Then M/G inherits from M the structure of a topological (resp. differentiable) manifold.

Proof. Set $n = \dim(M)$. Give M/G the quotient topology, that is, the weakest topology such that the canonical projection

$$\begin{aligned} q : M &\to M/G \\ x &\mapsto [x] \end{aligned}$$

is continuous (or equivalently: W is open in $M/G \Leftrightarrow q^{-1}(W)$ is open in M). We first verify that M/G is a topological manifold. The following properties must be checked:

- M/G is Hausdorff.
- Every point in M/G has a neighborhood homeomorphic to $\mathbb{R}^n$.
- M/G admits a countable basis for its topology.

*See [Boo02]

The fact that M/G is Hausdorff is a direct consequence from (i) in definition A.7 †. If $W \subset M$ is open, then so is $g \cdot W$ $(g \in G)$. This follows from the continuity of the action $\rho : G \times M \to M$ since, for fixed $g \in G$, the partial map $\rho_g : M \to M$ is also continuous and $g \cdot W = \rho_{g^{-1}}^{-1}(W)$. It follows that the union $\bigcup_{g \in G} g \cdot W$ is also open. From $q^{-1}(q(W)) = \bigcup_{g \in G} g \cdot W$ and the definition of quotient topology it follows that $q(W)$ is open, i.e., q is an open map.

On the other hand, we know that M has a countable basis $\{U_i\}_{i \in I}$ (I a countable index set). Using the fact that q is open, it is easy to show that $\{q(U_i)\}_{i \in I}$ is a countable basis for the topology of M/G. Indeed, let $V \subset M/G$ be an open set. Then we have:

$$
\begin{aligned}
q \text{ is continuous} \quad &\Rightarrow \quad q^{-1}(V) \text{ open in } M \;\Rightarrow \\
\exists J \subset I \text{ with } q^{-1}(V) = \bigcup_{j \in J} U_j \quad &\Rightarrow \quad V = q\left(\bigcup_{j \in J} U_j\right) = \bigcup_{j \in J} q(U_j).
\end{aligned}
$$

The assertion follows since, for every j, $q(U_j)$ is an open set (q is an open map).

For every x in M we have $G_x = \{e\}$, because the action is free. From (ii)′ it follows that for every x there is a neighborhood $\tilde{U}_x \ni x$ with $(g \cdot \tilde{U}_x) \cap \tilde{U}_x = \emptyset$ for all g in $G \setminus \{e\}$. Putting $U_{[x]} := q(\tilde{U}_x)$, one obtains a bijective map $q|_{\tilde{U}_x} : \tilde{U}_x \to U_{[x]}$. This map is moreover a homeomorphism, since q is open and continuous. We can assume, without loss of generality, that $\tilde{U}_x$ belongs to a chart $\{\tilde{U}_x, \tilde{\varphi}\}$. Then

$$
\varphi := \tilde{\varphi} \circ (q|_{\tilde{U}_x})^{-1} : U_{[x]} \to \tilde{\varphi}(\tilde{U}_x) \subset \mathbb{R}^n
$$

is also a homeomorphism. This means (since q is surjective) that M/G is a topological manifold.

When M is, in addition, a differentiable manifold, it is possible to construct an atlas for M/G in the following way. Let us denote the elements of G with g_i, $i \in \mathcal{I}_G$, where $\mathcal{I}_G \subset \mathbb{N}$ is an index set such that $|\mathcal{I}_G| = |G|$, $1 \in \mathcal{I}_G$. Choose the indexing in such a way that $g_1 = e$ holds and set $\rho_i(\,\cdot\,) := \rho(g_i, \cdot\,)$.

Now, it is convenient to choose charts $\{(\tilde{U}_{\alpha,i}\,,\, \tilde{\varphi}_{\alpha,i})\}_{\alpha \in \mathcal{I}, i \in \mathcal{I}_G}$ for M with the following properties:

- For α fixed, $\tilde{U}_{\alpha,i} = \rho_i(\tilde{U}_{\alpha,1})$ $(i \in \mathcal{I}_G)$.
- $\rho_i(\tilde{U}_{\alpha,1}) \cap \tilde{U}_{\alpha,1} = \emptyset$ $(i \in \mathcal{I}_G \setminus \{1\})$.

For such charts we then have, for all α:

$$
\begin{aligned}
\tilde{U}_{\alpha,i} \cap \tilde{U}_{\alpha,j} &= \emptyset \qquad (i \neq j), \\
q(\tilde{U}_{\alpha,i}) &= q(\tilde{U}_{\alpha,j}) \quad (\forall i, j).
\end{aligned} \tag{A.1}
$$

† They are, in fact, equivalent assertions, see [Boo02].

For this reason we may set $U_\alpha := q(\tilde{U}_{\alpha,i})$, with $i \in \mathcal{I}_G$ arbitrarily chosen. It follows that

$$q^{-1}(U_\alpha) = \bigcup_{i\in\mathcal{I}_G} \tilde{U}_{\alpha,i} = \bigcup_{i\in\mathcal{I}_G} \rho_i(\tilde{U}_{\alpha,1}) \tag{A.2}$$

is a union of pairwise disjoint neighborhoods. With $q_{\alpha,i} := q|_{\tilde{U}_{\alpha,i}}$, we can now define charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{I}}$ (as before) by means of

$$\varphi := \tilde{\varphi}_{\alpha,1}\circ q_{\alpha,1}^{-1} : U_\alpha \longrightarrow \tilde{\varphi}_{\alpha,1}(\tilde{U}_{\alpha,1}) \subset \mathbb{R}^n.$$

Let $\alpha, \beta \in \mathcal{I}$ be such that $U_\alpha \cap U_\beta \neq \emptyset$. Let $[x] \in U_\alpha \cap U_\beta$ and choose a representative $x \in U_{\alpha,1}$. Then there is a unique $i \in \mathcal{I}_G$ with $x \in U_{\beta,i}$, from which

$$\varphi_\beta\circ\varphi_\alpha^{-1} = \tilde{\varphi}_{\beta,1}\circ\rho_i^{-1}\circ\tilde{\varphi}_{\alpha,1}^{-1}$$

follows. Since the three maps on the right hand side of this equation are diffeomorphisms, the transition function $\varphi_\beta\circ\varphi_\alpha^{-1}$ is also a diffeomorphism. □

A.10 Remark. Of course, from $U_\alpha\cap U_\beta \neq \emptyset$ it does not follow, in general, that $\tilde{U}_{\alpha,i}\cap\tilde{U}_{\beta,j} \neq \emptyset$ for arbitrary i and j. But, if one chooses a certain i, a unique $j = j(i)$ is singled out, in such a way that $\tilde{U}_{\alpha,i}\cap\tilde{U}_{\beta,j} \neq \emptyset$ holds. Putting $\tilde{\rho}_i := \tilde{\varphi}_{\alpha,i}\circ\rho_i\circ\tilde{\varphi}_{\alpha,1}^{-1}$, we obtain the following expression for the transition functions:

$$\begin{aligned}
\varphi_\beta\circ\varphi_\alpha^{-1} &= (\tilde{\varphi}_{\beta,1}\circ q_{\beta,1}^{-1})\circ(\tilde{\varphi}_{\alpha,1}\circ q_{\alpha,1}^{-1})^{-1} = \tilde{\varphi}_{\beta,1}\circ(q_{\beta,1}^{-1}\circ q_{\alpha,1})\circ\tilde{\varphi}_{\alpha,1}^{-1} = \\
&= \tilde{\varphi}_{\beta,1}\circ(\rho_j^{-1}\circ\rho_i)\circ\tilde{\varphi}_{\alpha,1}^{-1} = (\tilde{\varphi}_{\beta,1}\circ\rho_j^{-1})\circ(\rho_i\circ\tilde{\varphi}_{\alpha,1}^{-1}) = \\
&= (\tilde{\rho}_j^{-1}\circ\tilde{\varphi}_{\beta,j})\circ(\tilde{\varphi}_{\alpha,i}^{-1}\circ\tilde{\rho}_i) = \\
&= \tilde{\rho}_j^{-1}\circ(\tilde{\varphi}_{\beta,j}\circ\tilde{\varphi}_{\alpha,i}^{-1})\circ\tilde{\rho}_i.
\end{aligned} \tag{A.3}$$

The diagram below might help to keep track of the different maps involved in the construction:

$$\begin{tikzcd}
\mathbb{R}^n & \tilde{U}_{\alpha,i} \arrow[l, "\tilde{\varphi}_{\alpha,i}"'] \arrow[r] & \tilde{U}_{\alpha,i}\cap\tilde{U}_{\beta,j} \arrow[r] & \tilde{U}_{\beta,j} \arrow[r, "\tilde{\varphi}_{\beta,j}"] & \mathbb{R}^n \\
\mathbb{R}^n \arrow[u, "\tilde{\rho}_i"] & \tilde{U}_{\alpha,1} \arrow[l, "\tilde{\varphi}_{\alpha,1}"'] \arrow[u, "\rho_i"'] \arrow[rr, "\rho_j^{-1}\circ\rho_i=", "=q_{\beta,1}^{-1}\circ q_{\alpha,1}"'] \arrow[d, "q_{\alpha,1}"] & & \tilde{U}_{\beta,1} \arrow[u, "\rho_j"] \arrow[r, "\tilde{\varphi}_{\beta,1}"] \arrow[d, "q_{\beta,1}"'] & \mathbb{R}^n \arrow[u, "\tilde{\rho}_j"'] \\
& U_\alpha \arrow[ul, "\varphi_\alpha"] \arrow[r] & U_\alpha\cap U_\beta \arrow[r] & U_\beta \arrow[ur, "\varphi_\beta"'] &
\end{tikzcd} \tag{A.4}$$

A.11 Remark. When G is finite, it is enough to require that ρ be a *free* action in order to apply theorem A.9. In this case, the second condition in §A.7 is trivially satisfied. The first one is also satisfied. To see this, let us first write $G = \{g_1, \ldots, g_N\}$ $(N = |G|)$. The orbit of $x \in M$ is the given by $\mathcal{O}_x = \{g_i\cdot x\}_i$. Let y be in M but with $y \notin \mathcal{O}_x$. Then $y \neq g_i \cdot x \;\forall i \in \{1, \ldots, N\}$. Since M is Hausdorff, we know that there are open neighborhoods $U_i \ni g_i \cdot x$ and $V_i \ni y$, for which $U_i \cap V_i = \emptyset$ holds. Defining $U := \bigcap_{i=1}^N (g_i^{-1} \cdot U_i)$ and $V := \bigcap_{i=1}^N V_i$, we thus get: U and V are open and $(g_i\cdot U)\cap V \subseteq U_i\cap V_i = \emptyset$, $i \in \{1, \ldots, N\}$. From this it follows that $(G \cdot U) \cap V = \emptyset$, i.e. (i) from §A.7 holds.

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