The Euler number of O'Grady's 10-dimensional symplectic manifold

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Abstract

We compute the Euler number of the 10-dimensional exceptional irreducible symplectic manifold constructed by O'Grady. The idea is to construct its Lagrangian fibration and to compute the Euler numbers of the fibers. It turns out that almost all of the fibers have Euler number 0 and therefore the problem is reduced to the computation of the Euler numbers of the rest of the fibers. Those fibers are the moduli spaces of semistable sheaves on the singular curves and the principle part of the dissertation is devoted to the calculation of the Euler numbers of those moduli spaces. These results are of independent interest.

Keywords:
Euler number, Irreducible symplectic manifold, Lagrangian fibration, Moduli space

Zusammenfassung

Wir berechnen die Eulerzahl der 10-dimensionalen exzeptionellen irreduziblen symplektischen Mannigfaltigkeit, die von O'Grady konstruiert wurde. Die Idee besteht darin, zunächst eine Lagrangefaserung zu konstruieren und dann die Eulerzahlen der Fasern zu berechnen. Es stellt sich heraus, dass fast alle Fasern die Eulerzahl 0 haben, und deswegen reduziert sich das Problem auf die Berechnung der Eulerzahlen der übrigen Fasern. Diese Fasern sind Modulräume von halbstabilen Garben auf singulären Kurven. Der Hauptteil dieser Dissertation ist der Berechnung der Eulerzahlen dieser Modulräume gewidmet. Diese Resultate sind von unabhängigem Interesse.

Schlagwörter:
Euler Zahl, Irreduzible symplektische Mannigfaltigkeit, Lagrangefaserung, Modulraum

Contents

Introduction

The irreducible holomorphic symplectic manifolds, along with complex tori and Calabi-Yau manifolds, are the building stones of the compact Kähler manifolds with trivial first Chern class. Recall that a compact Kähler manifold X is called an irreducible symplectic manifold if it is simply connected and has a holomorphic symplectic form generating $H^0(X, \Omega_X^2)$. Only a few such manifolds, up to deformation, are known, namely:

1. The K3-surfaces. The second Betti number here is $b_2 = 22$.
2. The Hilbert schemes $X^{[n]}$, where X is a K3-surface and $n \geq 2$. Here $b_2 = 23$.
3. Generalized Kummer varieties $K^n X$, where X is a two-dimensional torus and $n > 2$. Here $b_2 = 7$.
4. 10-dimensional example introduced by O'Grady [36]. Here $b_2 = 24$.
5. 6-dimensional example introduced by O'Grady [37]. Here $b_2 = 8$.

It is an intriguing problem to prove that this list exhausts all the irreducible symplectic manifolds up to deformation. Another problem worth special attention is the study of the manifolds listed above and in particular finding their diverse invariants. The Betti numbers of Hilbert schemes and generalized Kummer varieties were computed by Göttsche [18, 19] using the Weil conjectures. For example, the Euler numbers of the Hilbert schemes of a K3-surface X are determined by the formula

$$\sum_{n \geq 0} e(n) q^n = \prod_{m \geq 1} (1 - q^m)^{-24},$$

where $e(n) := e(X^{[n]})$; e.g. $e(2) = 324$, $e(5) = 176256$. Another computation of the Betti numbers together with the Hodge numbers was made by Göttsche and Soergel [20], using the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber for perverse sheaves [5] and its refinement for mixed Hodge modules due to Saito [40]. The invariants of the 6-dimensional example were computed by O'Grady [37] and Rapagnetta [39]. Our aim is to compute the Euler number of O'Grady's 10-dimensional example.

To explain the idea of the computation let us recall briefly the construction of this example. We consider a projective K3-surface X with a sufficiently general polarization H and construct a (singular) moduli space $M_X(2,0,4)$ of rank 2 semistable sheaves on X with first Chern class 0 and second Chern class 4. This moduli space can be equipped with a symplectic form (over the nonsingular part) and then one shows that it has a resolution of singularities such that the symplectic form can be extended to the whole resolution and remains non-degenerate. Such a resolution is called a symplectic resolution. It is the required 10-dimensional irreducible symplectic manifold.

Deforming the polarized K3-surfaces, one can assume that $\operatorname{Pic} X = \mathbb{Z}[H]$ and $H^2 = 2$, where H is an ample divisor (see [35]). Then $M_X(2,0,4) = M_X(2n^2)$, where in the last moduli space we consider the Hilbert polynomial with respect to the polarization H. It was proved by O'Grady in [35, Part 2.2] that there is a birational map between $M_X(2n^2)$ and $M_X(4n+2) = M_X(0,2H,2)$. The last moduli space can also be equipped with a symplectic form (over the nonsingular part) and it also has a symplectic resolution, so both symplectic resolutions are birational and hence diffeomorphic by the result of Huybrechts [25]. It follows that both symplectic resolutions have the same Euler number.

The symplectic resolution of the moduli space $M_X(4n+2)$ is obtained by certain blow-ups. Computation of the Euler numbers of the strata of $M_X(4n+2)$ occurring in these blow-ups allows us to compute the Euler number of the symplectic resolution, our ultimate goal. In order to find the Euler number of $M_X(0,2H,2) = M_X(4n+2)$ (and also the Euler numbers of its strata) we construct the fibration $M_X(4n+2) \longrightarrow |2H|$ that maps a sheaf in the moduli space to its support. The fibre of this fibration over a curve $C \in |2H|$ has the same Euler number as a moduli space $M_C(4n+2)$, where the Hilbert polynomial in the last moduli space is considered with respect to the induced polarization on C. It turns out that the Euler number of the fibre can be nonzero only for a finite number of curves in $|2H|$. To compute the Euler numbers of the strata of $M_X(4n+2)$, we determine all such curves, compute the Euler numbers of the corresponding fibers and sum them up according to their membership with respect to strata.

Theorem 1. *The Euler number of O'grady's 10-dimensional symplectic manifold is equal to* $e(5) + 2e(2) = 176904$.

Just for comparison, the Euler number of $M_X(4n+1)$ equals $e(X^{[5]}) = e(5)$. In the course of our computations we will encounter quite exotic curves, some of them non reduced and some of them reducible. The core of the thesis consists of the computation of the Euler numbers of the moduli spaces of semistable sheaves on such curves.

Let us now describe the structure of the thesis in more detail. The first chapter is devoted to the definition of the O'Grady example and describing its properties. First, we recall the basic results concerning the moduli spaces of semistable sheaves on the projective varieties. Then we introduce the framework of (rational) Lagrangian fibrations for a more conceptual exposition of the further material of the chapter. We recall the construction of O'Grady's 10-dimensional symplectic variety (the symplectic resolution of $M_X(2,0,4)$, where X is a K3-surface) and also give its birational version (the symplectic resolution of $M_X(0,2H,2)$), which is more suitable for our computations as it possesses the Lagrangian fibration mentioned above. We construct this fibration in the final section. Intuitively, it is a map $M_X(4n+2) \longrightarrow |2H|$ sending a sheaf to it support. We have tried to be quite accurate proving that it is a regular morphism of schemes.

The fiber of $M_X(4n+2)$ over a curve $C \in |2H|$ is related to the moduli space $M_C(4n+2)$. All sheaves in the last moduli space have ranks 1 or 2. The leitmotif of the second and third chapters is the calculation of the Euler numbers of the moduli spaces of rank 2 semistable sheaves on various curves.

The second chapter is devoted to the study of the moduli spaces of semistable sheaves on reduced (possibly reducible) curves. We study the moduli spaces of those sheaves that have the same nonzero rank on all the components of the curve. First of all we will prove that whenever a curve has a nonrational component, the Euler number of the moduli space equals zero. This result reduces greatly the number of curves we have to consider. After that we study the moduli spaces over different types of reduced curves with rational components that we will encounter later in the course of the computation of the Euler number of $M_X(4n+2)$. The first curve that we study is a rational curve with one node. Here we do actually much more than we need, namely, we classify all indecomposable semistable sheaves. The approach we use goes back to the work of Drozd and Greuel [14], where they classified the indecomposable sheaves on such a curve in terms of certain combinatorial data. Our result can be compared with the work of Burban [7], who classified the stable bundles and with the work of Burban and Kreußler [9] who classified indecomposable semistable sheaves of rank zero. Using the Fourier-Mukai transform, they reduced the problem to the classification of finite dimensional modules over the ring $k[\![x,y]\!]/(xy)$, the problem solved by Gelfand and Ponomarev [17]. Our approach is certainly more elementary and gives a classification for arbitrary degree.

Another curve we have to investigate is a rational curve with $n > 1$ nodes. The moduli spaces that we actually need are the moduli spaces of rank 2 semistable sheaves having a fixed even degree. The Euler number of the moduli space of rank 2 semistable sheaves having a fixed odd degree was computed by Wu [42]. He has shown that it is equal to n. In the case of the even degree we will show that the Euler number equals 1 for any n. In our calculation we follow the approach of Drozd and Greuel [14], to classify the torsion free sheaves on a singular curve by means of torsion free sheaves on its normalization together with some additional data. All required results can be found in Appendix B. The cases of even and odd degree are quite analogous under this approach, so we will also reprove the result of Wu. The last curve considered in this chapter is a curve consisting of two (possibly singular) components intersecting with each other transversally in two nonsingular points. The only moduli spaces we will be interested in here are the moduli spaces of semistable sheaves having a fixed characteristic and having rank one on each component. The answer here depends again on the parity of the characteristic.

The third chapter is devoted to the analysis of the moduli spaces of semistable sheaves on double curves. These are the curves C such that the ideal sheaf I of C_{red} in C satisfies $I^2 = 0$ and I is a line bundle over C_{red}. Here we are interested in certain moduli spaces of semistable sheaves over C having rank 2 (see Lemma 3.1.1 for the definition of the rank). We show that a sheaf in the moduli space is either an $\mathcal{O}_{C_{\mathrm{red}}}$-module or can be represented as an extension of sheaves over C_{red}. The part corresponding to the sheaves of the second type is fibered over the Jacobian of C_{red} and, using this fibration, we compute the Euler number of the moduli space.

The fourth chapter is devoted to the calculation of the Euler number of $M_X(4n+2)$ and of its symplectic resolution. Here is the place where all calculations of the second and third chapters finally find their application. To get the trace of the singularities of the curves from $|2H|$, we choose special K3-surfaces. Namely, for any sextic $B \in |\mathcal{O}_{\mathbb{P}^2}(2)|$, we consider a $2:1$ map $\pi : X \longrightarrow \mathbb{P}^2$ branched along B. The surface X is a K3-surface and for a sufficiently general B it satisfies $\operatorname{Pic} X = \mathbb{Z}[H]$, where H is a preimage of a line in $\mathbb{P}^2$. For a sufficiently general B the curve in $|2H|$ is rational only if its image in $\mathbb{P}^2$ is a quadric having very specific intersection multiplicities with B. To compute the number of such quadrics for the occurring intersection multiplicities we use the result of Gathmann [16]. Knowing the intersection multiplicities of a conic C_0 with B, we also know the types of singularities of its preimage C in X. Using the results of the previous chapters we obtain the Euler number of the corresponding moduli space on C. Summing everything up we get the Euler number of the symplectic resolution of $M_X(4n+2)$.

Chapter 1

Basic constructions

1.1 Moduli spaces of semistable sheaves

In this section, we introduce moduli spaces of semistable sheaves on a projective scheme. We refer to [41] and [26] for a full treatment of the subject. Let X be a projective scheme over an algebraically closed field k of characteristic zero and fix a polarization (i.e., an ample line bundle) on X. All sheaves on X will be assumed to be coherent, unless stated otherwise. Given a sheaf F on X, one defines its Hilbert polynomial by

$$P(F, n) = \chi(F(n)).$$

There is a connection between the degree of the Hilbert polynomial of F and the dimension of F. Recall first, that the dimension $\dim F$ of F is defined to be the dimension of $\operatorname{supp} F$. The sheaf F is called pure dimensional if any subsheaf $G \subset F$ with $\dim G < \dim F$ is zero.

Lemma 1.1.1. *Let F be a sheaf of dimension d on X. Then the degree of $P(F, n)$ equals d, i.e., the Hilbert polynomial of F can be written in the form*

$$P(F, n) = \sum_{i=0}^{d} a_i(F) \frac{n^i}{i!},$$

with $a_d(F) \neq 0$. All the coefficients $a_i(F)$ are integers. If X is an integral scheme and F has dimension equal to $\dim X$, then the rank of F (i.e., the dimension of the stalk of F at the generic point) equals $a_d(F)/a_d(\mathcal{O}_X)$.

The quotient $p(F, n) = P(F, n)/a_d(F)$ is called the normalized Hilbert polynomial of F.

Definition 1.1.2. The sheaf F on X is called semistable if it is pure dimensional and for any proper nontrivial subsheaf $G \subset F$,

$$p(G,n) \leq p(F,n), \qquad \forall n \gg 0.$$

If a strict inequality holds for all G as above, the sheaf F is called stable.

Lemma 1.1.3. *The sheaf F is semistable if and only if for any (nonzero, non-bijective) surjection $F \twoheadrightarrow H$,*

$$p(F,n) \leq p(H,n), \qquad \forall n \gg 0.$$

The sheaf F is stable if and only if strict inequality holds.

Lemma 1.1.4. *The category of semistable sheaves with a given normalized Hilbert polynomial is abelian.*

Lemma 1.1.5 (Jordan–Hölder filtration)**.** *Let F be a semistable sheaf. Then there exists a filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l = F$$

such that F_{i+1}/F_i, $0 \leq i < l$, are stable sheaves having normalized Hilbert polynomial $p(F_{i+1}/F_i) = p(F)$. The factors of the filtration do not depend on the filtration up to permutation.

Definition 1.1.6. Two semistable sheaves F_1 and F_2 with the same normalized Hilbert polynomial p are called S-equivalent if the factors of their Jordan–Hölder filtration coincide up to permutation. A semistable sheaf is called polystable if it is a direct sum of stable sheaves. Any semistable sheaf is S-equivalent to a unique polystable sheaf.

We now recall the basic definitions and results concerning the moduli space of semistable sheaves. Denote by **Sch** the category of algebraic schemes over k (i.e., schemes of finite type over $\operatorname{Spec} k$); this category is svelte (i.e., equivalent to a small category). Given a polynomial $P \in \mathbb{Q}[x]$, we define a functor $\mathcal{M}_X(P) : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$. First define an auxiliary functor $\mathcal{M}'_X(P) : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ in the following way: for any $S \in \mathbf{Sch}$, $\mathcal{M}'_X(P)(S)$ is the set of all flat S-families of sheaves on X (i.e., coherent sheaves on $S \times X$ flat over S) with all fibers being semistable and having Hilbert polynomial P; for any morphism of schemes $f : T \to S$ define $\mathcal{M}'_X(P)(f) : \mathcal{M}'_X(P)(S) \to \mathcal{M}'_X(P)(T)$ by pullbacks. Then define the functor $\mathcal{M}_X(P)$ as a quotient functor of $\mathcal{M}'_X(P)$ with respect to an equivalence relation: two families $F_1, F_2 \in \mathcal{M}'_X(P)(S)$ are equivalent if there exists an invertible sheaf L on S such that $F_2 \simeq F_1 \otimes p_S^* L$. Analogously we define a functor $\mathcal{M}^s_X(P) : \mathbf{Sch}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ corresponding to families of stable sheaves on X with Hilbert polynomial P.

Given a svelte category $\mathcal{A}$, we denote by $\widehat{\mathcal{A}} = \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ the category of presheaves on $\mathcal{A}$, i.e., the category of functors from $\mathcal{A}^{\mathrm{op}}$ to $\mathbf{Set}$. By the Yoneda lemma we may consider $\mathcal{A}$ as a full subcategory of $\widehat{\mathcal{A}}$. Given an object $X \in \mathcal{A}$, the corresponding object in $\widehat{\mathcal{A}}$ will also be denoted by X.

Definition 1.1.7. Let $\mathcal{A}$ be a svelte category and $F : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ a functor. We say that a pair (X, u), where $X \in \mathcal{A}$ and $u : F \longrightarrow X$ is a morphism in $\widehat{\mathcal{A}}$, corepresents F if any morphism $v : F \longrightarrow Y$, where $Y \in \mathcal{A}$, factors uniquely as $F \xrightarrow{u} X \longrightarrow Y$. We say that the pair (X, u) universally corepresents F if for any morphism $X' \longrightarrow X$ in $\mathcal{A}$, the fiber product $X' \times_X F$ (fiber products exist in $\widehat{\mathcal{A}}$) is corepresented by X'.

Theorem 1.1.8. *There exists a projective scheme $M_X(P)$ that universally corepresents the functor $\mathcal{M}_X(P)$. Closed points in $M_X(P)$ correspond to S-equivalence classes of semistable sheaves with Hilbert polynomial P. There exists an open subscheme $M^S_X(P)$ of $M_X(P)$ that universally corepresents $\mathcal{M}^s_X(P)$.*

Definition 1.1.9. The scheme $M_X(P)$ corepresenting the functor $\mathcal{M}_X(P)$ is called the moduli space of semistable sheaves on X with Hilbert polynomial P.

1.2 Fibrations of irreducible symplectic manifolds

We first recall the definition of an irreducible symplectic manifold from the introduction.

Definition 1.2.1. A compact Kähler manifold X is called an irreducible symplectic manifold if it is simply connected and $H^0(X, \Omega^2_X)$ is generated by a symplectic form.

Of particular interest are the irreducible symplectic manifolds possessing fibrations.

Definition 1.2.2. A fibration of a normal algebraic variety X is a proper surjective morphism $f : X \longrightarrow B$ such that B is normal, $0 < \dim B < \dim X$ and a general fiber of f is connected.

The importance of fibrations can be explained by the following result of Matsushita [31, 32, 33].

Theorem 1.2.3. *Let X be a projective irreducible symplectic manifold of dimension $2n$ possessing a fibration $f : X \longrightarrow B$ with smooth and projective base B. Then the following hold.*

1. *f is a lagrangian fibration, i.e., a general fiber of f is a lagrangian submanifold. A general fiber of f is an abelian variety.*
2. *$\dim B = n$.*
3. *B has ample anticanonical bundle and has Picard number one.*
4. *$h^{p,q}(B) = \delta_{p,q}$.*

There is a conjecture that the base variety in the above theorem is always isomorphic to $\mathbb{P}^n$. In fact, one cannot hope that any irreducible symplectic manifold admits a fibration. Nevertheless, a fibration usually exists for some irreducible symplectic manifold birational to the given one.

Definition 1.2.4. Let X be an irreducible symplectic manifold. A rational fibration of X is a rational map $X \dashrightarrow B$ such that there exists an irreducible symplectic manifold X' with a birational map $g : X' \dashrightarrow X$ such that $fg : X' \dashrightarrow B$ is regular and is a fibration.

An example of the above situation is the moduli space $M_X(2, 0, 4)$ (more precisely, its symplectic resolution), where X is a K3-surface equipped with a sufficiently general polarization. Under certain conditions on X, this moduli space is birational to $M_X(0, 2, 2)$ and the latter possesses a fibration with base $\mathbb{P}^5$. This will be discussed below.

1.3 O'Grady's example and its birational version

In this section we recall the construction of the 10-dimensional example of O'Grady [36]. It is a symplectic resolution of the moduli space $M_X(2, 0, 4)$, where X is a K3-surface equipped with a sufficiently general polarization.

For simplicity, we impose some restriction on X and its polarization from the very beginning. Let X be a K3-surface with $\operatorname{Pic} X = \mathbb{Z}[H]$ and $H^2 = 2$, where H is an ample divisor. For any coherent sheaf F on X, its first Chern class is a multiple of H, so we may write it just as the corresponding integer.

Let the triple (r, c_1, c_2) denote the rank and Chern classes of F. Using the Riemann–Roch formula, we can express the Hilbert polynomial of F with respect to the polarization H in the form

$$P(F, n) = rn^2 + c_1H \cdot n + (c_1^2/2 - c_2 + 2r),$$

where c_1H and c_1^2 denote the intersection numbers, so if $c_1 = mH$, then $c_1H = 2m$ and $c_1^2 = 2m^2$. One sees that conversely, the Hilbert polynomial determines uniquely the triple (r, c_1, c_2). If the polynomial P and the triple (r, c_1, c_2) are related as above, we will also denote the moduli space $M_X(P)$ of semistable sheaves with Hilbert polynomial P by $M_X(r, c_1, c_2)$. In particular, under this identification, the moduli space $M_X(2, 0, 4)$ equals $M_X(2n^2)$. The expected dimension of $M_X(r, c_1, c_2)$ (see e.g., [26, 4.5]) equals the dimension of $\mathrm{Ext}^1(F, F)$, where F is a sheaf of type (r, c_1, c_2). Using the Riemann–Roch formula for $\chi(F, F) = \sum_{i\geq 0}(-1)^i \dim \mathrm{Ext}^i(F, F)$ one can show that this dimension equals

$$c_1^2 - 2r(c_1^2/2 - c_2 + r) + 2.$$

In particular, the expected dimension of $M_X(2, 0, 4)$ equals 10.

The moduli space $M = M_X(2, 0, 4)$ has a stratification $M \supset \Sigma \supset \Omega$, where $M \backslash \Sigma = M^s$ consists of the stable sheaves, $\Sigma = M^{\mathrm{sing}}$ consists of the sheaves $I_Z \oplus I_W$ with $Z, W \in X^{[2]}$ and $\Omega = \Sigma^{\mathrm{sing}}$ consists of the sheaves $I_Z \oplus I_Z$ with $Z \in X^{[2]}$. O'Grady considers the Kirwan desingularization [27] $\widehat{M} \longrightarrow M$, which is obtained by first blowing up along Ω and then blowing up along the proper transform of Σ. The Mukai symplectic form on the smooth locus of M can be extended to the whole of $\widehat{M}$, but it is degenerate along $\widehat{\Omega}$, the proper transform of Ω. It is shown in [36] that $\widehat{M}$ can be contracted along $\widehat{\Omega}$, the obtained variety $\widetilde{M}$ is smooth and the induced symplectic form on $\widetilde{M}$ is nondegenerate. This gives the symplectic resolution of M.

Consider a different moduli space $M_X(0, 2H, 2) = M_X(4n+2)$. It also has dimension 10 and can be equipped with a symplectic form over the smooth locus. It is proved by Lehn and Sorger [29] that it has a symplectic resolution obtained by just one blow-up (the same is proved for $M_X(2, 0, 4)$ there). The varieties $M_X(2, 0, 4)$ and $M_X(0, 2H, 2)$ are birational, so the same holds for their symplectic resolutions and by the result of Huybrechts [25] these resolutions have the same Euler number. We recall the construction of a birational equivalence between $M_X(2n^2)$ and $M_X(4n+2)$ (see e.g., [35, Subsection 2.2]).

Note that any locally free sheaf $F \in M_X(2n^2)$ is necessarily a stable sheaf. Indeed, otherwise it would be polystable, hence it would be a sum of two line bundles with Hilbert polynomial n^2, but $\mathrm{Pic}\, X = \mathbb{Z}[H]$ and $P(\mathcal{O}_X(k), n) = (k + n)^2 + 2$ for any integers k and n. Thus, the Hilbert polynomial of a line bundle cannot equal n^2. Given a locally free sheaf $F \in M_X(2n^2)$ such that $h^0(F^\vee(1)) = 2$, we consider the canonical morphism $H^0(F^\vee(1)) \otimes \mathcal{O}_X \longrightarrow F^\vee(1)$ of rank 2 vector bundles, take its determinant and define $C(F)$ to be the induced divisor. We note that $c_1(F^\vee(1)) = 2H$ and therefore $C(F) \in |2H|$. Define an open dense subscheme of $M_X(2n^2)$

$$M^0 = \{F \in M_X(2n^2) \mid h^0(F^\vee(1)) = 2,\ F \text{ is loc. free and } C(F) \text{ is smooth}\}.$$

Given a sheaf $L \in M_X(4n+2)$, we will denote its support by $C(L)$. We have that $C(L) \in |2H|$. The sheaves on $C(L)$ will be often identified with the corresponding sheaves on$X$. Define an open dense subscheme of $M_X(4n+2)$

$$M^1 = \{L \in M_X(4n+2) \mid h^0(L) = 2,\ L \text{ is glob. gener. and } C(L) \text{ is smooth}\}.$$

Our goal is to prove the following proposition.

Proposition 1.3.1 (cf. [35, Proposition 2.2.5])**.** *M^0 is isomorphic to M^1.*

Given a coherent sheaf E on X of codimension c, one defines its dual sheaf to be $E^D = \mathcal{E}xt^c_{\mathcal{O}_X}(E, \omega_X)$ (see e.g., [26, Definition 1.1.7]). There exists a canonical morphism $\theta_E : E \longrightarrow E^{DD}$ and E is called reflexive if θ_E is an isomorphism. The nice feature of this definition of the dual is that it does not depend on the smooth ambient space (X in our case). In particular, if E is a vector bundle over a smooth curve $C \subset X$ then it is reflexive. We mention also that θ_E is injective if and only if E is pure (see [26, Proposition 1.1.10]).

Lemma 1.3.2. *Given a sheaf $L \in M^1$, consider an exact sequence*

$$0 \longrightarrow E \xrightarrow{f} H^0(L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0.$$

Then E is a stable rank 2 line bundle on X having Hilbert polynomial $2(n-1)^2$. We have $h^0(E^\vee) = 2$.

Proof. This construction is a so-called elementary transformation of $H^0(L) \otimes \mathcal{O}_X$ along L (see e.g., [26, Section 5.2]). To prove that E is locally free, we note that $\mathcal{O}_C$ has homological dimension 1 over $\mathcal{O}_X$, hence the same holds for L and the above exact sequence must be a locally free resolution over $\mathcal{O}_X$, so E is locally free over $\mathcal{O}_X$. The statement about the Hilbert polynomial is straightforward.

If E is not stable then it contains some destabilizing line bundle. Let $g : \mathcal{O}_X(k) \hookrightarrow E$ be such a line bundle. From the inequality of polynomials

$(n+k)^2+2 \geq (n-1)^2$, we deduce $k \geq -1$. Let us show that the map $fg : \mathcal{O}_X(k) \longrightarrow \mathcal{O}_X^{\oplus 2}$ vanishes on $C = C(L)$. It will then follow that

$$fg \in \operatorname{Hom}(\mathcal{O}_X(k), \mathcal{O}_X^{\oplus 2}(-C)) = H^0(\mathcal{O}_X^{\oplus 2}(-k-2)) = 0,$$

contradicting the injectivity of f and g.

The sheaf L is pure dimensional, hence it is a line bundle over C. The kernel of the surjective map $H^0(L) \otimes \mathcal{O}_C \longrightarrow L$ is a line bundle on C and comparing the determinants we see that it is isomorphic to L^{-1}. Hence there is an exact sequence

$$0 \longrightarrow L^{-1} \longrightarrow H^0(L) \otimes \mathcal{O}_C \longrightarrow L \longrightarrow 0.$$

It follows that the map $f|_C : E|_C \longrightarrow H^0(L) \otimes \mathcal{O}_C$ maps $E|_C$ to L^{-1}, hence

$$(fg)|_C \in \operatorname{Hom}(\mathcal{O}_C(k), L^{-1}) = H^0(L^{-1}(-k)) = 0,$$

where the last equality follows from the fact that degree of $L^{-1}(-k)$ is negative. Indeed, $g_C = \frac{1}{2}C^2 + 1 = 5$ (see e.g., [3, Proposition 8.13]), hence

$$\deg L = \chi(L) + g_C - 1 = 2 + 5 - 1 = 6$$

and $\deg(L^{-1}(-k)) = -6 - k < 0$.

To show that $h^0(E^\vee) = 2$ we have to prove by Serre duality that $h^2(E) = 2$. By stability, we have $h^0(E) = 0$, so we just have to show that $h^1(E) = 0$. This follows from the long exact sequence

$$H^0(L) \otimes H^0(\mathcal{O}_X) \longrightarrow H^0(L) \longrightarrow H^1(E) \longrightarrow H^0(L) \otimes H^1(\mathcal{O}_X).$$

□

The correspondence $L \longmapsto F := E(1)$ defines a map $M^1 \longrightarrow M^0$. We just have to show that $C(F)$ is smooth. In fact, $C(F) = C(L)$. Applying the functor $\mathcal{H}om(-, \mathcal{O}_X)$ to

$$0 \longrightarrow E \longrightarrow \mathcal{O}_X^{\oplus 2} \longrightarrow L \longrightarrow 0$$

we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus 2} \longrightarrow E^\vee \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_X}(L, \mathcal{O}_X) = L^D \longrightarrow 0.$$

The divisor of the determinant of $\mathcal{O}_X^{\oplus 2} \longrightarrow E$ is equal the support of L^D. But L^D is just a line bundle on $C(L)$ by the above remarks.

This construction also gives an idea of the inverse map. Namely, for a sheaf $F \in M^0$, the corresponding sheaf $L \in M^1$ must be a dual of the cokernel of a certain monomorphism $\mathcal{O}_X^{\oplus 2} \longrightarrow E^\vee$ with $E := F(-1)$ (there is a unique such map up to isomorphism of $\mathcal{O}_X^{\oplus 2}$ in view of $h^0(E^\vee) = 2$). The Chern character of F equals $\operatorname{ch}(F) = (2, 0, -4)$, therefore also $\operatorname{ch}(F^\vee) = (2, 0, -4)$, $P(F^\vee, n) = 2n^2$ and $P(E^\vee, n) = 2(n+1)^2$.

Lemma 1.3.3. *$F^\vee$ is stable.*

Proof. If it is not then there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_X(k) \longrightarrow F^\vee \longrightarrow T \longrightarrow 0$$

with $(k+n)^2 + 2 \geq n^2, n \gg 0$, i.e., $k \geq 0$ and pure T. Taking duals, we get a nonzero map $F \longrightarrow \mathcal{O}_X(-k)$ and it follows from the stability of F that $n^2 < (n-k)^2 + 2, n \gg 0$, i.e., $k \leq 0$. So, we may assume that $k = 0$ and therefore $P(T,n) = n^2 - 2$. As T is pure, there is an embedding $T \hookrightarrow T^{\vee\vee}$ and the quotient $T^{\vee\vee}/T$ is zero-dimensional (see e.g., [26, Example 1.1.16]). Therefore the Hilbert polynomial of $T^{\vee\vee}$ equals $n^2 + c$ for some constant c. Using the fact that $T^{\vee\vee}$ is a line bundle, we obtain $T^{\vee\vee} \simeq \mathcal{O}_X$ and hence there exists a nonzero morphism $F^\vee \longrightarrow \mathcal{O}_X$. Taking duals, we get a nonzero map $\mathcal{O}_X \longrightarrow F$ contradicting to the stability of F. □

Lemma 1.3.4. *The map $H^0(E^\vee) \otimes \mathcal{O}_X \longrightarrow E^\vee$ is a monomorphism.*

Proof. Consider an exact sequence

$$0 \longrightarrow T \longrightarrow H^0(E^\vee) \otimes \mathcal{O}_X \longrightarrow E^\vee.$$

Assume that T is nonzero. Then T is pure dimensional, hence it is embedded in its reflexive closure $T \hookrightarrow T^{\vee\vee}$ and the quotient $T^{\vee\vee}/T$ is zero-dimensional. The map $T \longrightarrow H^0(E^\vee) \otimes \mathcal{O}_X$ can be factored through $T \longrightarrow T^{\vee\vee}$, hence there is an embedding $T^{\vee\vee}/T \hookrightarrow E^\vee$. As $E^\vee$ is pure dimensional, we deduce that $T^{\vee\vee}/T = 0$ and T is locally free. Its rank cannot be equal to 2 as otherwise $E^\vee$ would contain a subsheaf of rank 0, which would contradict to the fact that $F^\vee$ is stable. Hence T is a line bundle and there exists $k \in \mathbb{Z}$ with $T \simeq \mathcal{O}_X(k)$. The Hilbert polynomial of $(H^0(E^\vee) \otimes \mathcal{O}_X)/T$ equals $2(n^2+2) - (n+k)^2 - 2$. The stability of $E^\vee$ implies

$$2(n^2+2) - (n+k)^2 - 2 < (n+1)^2, \qquad n \gg 0$$

or, equivalently, $k \geq 0$. The existence of the embedding $T \hookrightarrow \mathcal{O}_X^{\oplus 2}$ implies that $k \leq 0$ and hence $T \simeq \mathcal{O}_X$. It is clear that the map

$$\mathrm{Hom}(\mathcal{O}_X, H^0(E^\vee) \otimes \mathcal{O}_X) \longrightarrow \mathrm{Hom}(\mathcal{O}_X, E^\vee)$$

is an isomorphism. Hence the embedding $\mathcal{O}_X \longrightarrow H^0(E^\vee) \otimes \mathcal{O}_X$ cannot be mapped to zero in $\mathrm{Hom}(\mathcal{O}_X, E^\vee)$, which is a contradiction. □

Lemma 1.3.5. *Let F and E be as above and consider an exact sequence*

$$0 \longrightarrow H^0(E^\vee) \otimes \mathcal{O}_X \longrightarrow E^\vee \longrightarrow N \longrightarrow 0. \tag{1.1}$$

Then the support of N equals $C(F)$ and N is a line bundle over $C(F)$.

Proof. We will write $\mathcal{O}_X^{\oplus 2}$ for $H^0(E^\vee)\otimes\mathcal{O}_X$. The curve $C = C(F)$ is defined as a set as

$$\{x \in X \mid \mathcal{O}_X^{\oplus 2}(x) \longrightarrow E^\vee(x) \text{ is not an isomorphism}\}.$$

N is supported on $C(F)$ and a local computation shows that actually the ideal of C in X annihilates N, i.e., N is a module over $\mathcal{O}_C$. Indeed, let locally $X = \operatorname{Spec} A$ and let the map $H^0(E^\vee)\otimes\mathcal{O}_X \longrightarrow E^\vee$ be given by the square matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : A^2 \longrightarrow A^2$. Then locally the ideal of C is generated by $s = ad - bc$ and N is defined as a cokernel of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. To show that $sN = 0$, one has to prove that $sA^2 \subset \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) A^2$, which easily follows from the linear algebra.

Applying the functor $(- \otimes_{\mathcal{O}_X} \mathcal{O}_C)$ to (1.1) we get

$$0 \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_X}(N,\mathcal{O}_C) \longrightarrow H^0(E^\vee)\otimes\mathcal{O}_C \longrightarrow E^\vee|_C \longrightarrow N \longrightarrow 0. \qquad (1.2)$$

It follows that $\operatorname{Tor}_1^{\mathcal{O}_X}(N,\mathcal{O}_C)$ is torsion free on C and is actually locally free as C is smooth. Applying $(L \otimes_{\mathcal{O}_X} -)$ to the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

we get

$$0 \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_X}(N,\mathcal{O}_C) \longrightarrow N(-C) \longrightarrow N \longrightarrow N \longrightarrow 0$$

and hence $\operatorname{Tor}_1^{\mathcal{O}_X}(N,\mathcal{O}_C) \simeq N(-C)$, which implies that N is locally free over C. The Hilbert polynomial of N equals $2(n+1)^2 - 2(n^2+2) = 4n-2$ and so N has rank 1 over C. □

Lemma 1.3.6. *The sheaf $L = N^D$ is contained in M_1.*

Proof. It was mentioned already that

$$N^D = \mathcal{E}xt^1_{\mathcal{O}_X}(N,\mathcal{O}_X) = \mathcal{H}om(N,\omega_C) = N^{-1}\otimes\omega_C.$$

We note that by the adjunction formula $\omega_C = \mathcal{O}_C(C)$ and hence we can write also $L = N^{-1}(C)$. By Serre duality $\chi(L) = -\chi(N) = 2$ and therefore $P(L,n) = 4n+2$. Moreover, applying the functor $H^0(-)$ to the exact sequence (1.1), we get $h^0(N) = 0$, which implies $h^1(L) = 0$ and therefore $h^0(L) = 2$. It remains to prove that L is globally generated. The exact sequence (1.2) can be rewritten in the form

$$0 \longrightarrow N(-C) \longrightarrow \mathcal{O}_C^2 \longrightarrow E^\vee|_C \longrightarrow N \longrightarrow 0.$$

The cokernel of $N(-C) \longrightarrow \mathcal{O}_C^2$ is a subsheaf of $E^\vee|_C$, hence it is torsion free and therefore locally free. Taking the determinants, we obtain that this cokernel is isomorphic to $N^{-1}(C) = L$. It follows from the existence of the surjective map $\mathcal{O}_C^2 \longrightarrow L$ that L is globally generated. □

The correspondence $F \longmapsto L = N^D$ defines a map $M^0 \longrightarrow M^1$. The fact that two constructed maps are inverse to each other follows easily from the above remarks. This completes the proof of the proposition.

Remark 1.3.7. As we have mentioned in the Introduction, the thesis is devoted to the study of the moduli space $M_X(4n+2)$ and its symplectic resolution. It is sufficiently interesting to compare it with a smooth symplectic manifold $M_X(4n+1)$. The latter is birational to the Hilbert scheme $X^{[5]}$. Indeed, the general point of $M_X(4n+1)$ can be written in the form (C, L), where $C \in |2H|$ is smooth (hence $g_C = \frac{1}{2}C^2 + 1 = 5$) and L is a line bundle on C having Euler characteristic 1 or, equivalently, degree 5. Moreover, it follows from the upper semi-continuity that for a general (C, L) as above one has $h^1(X, L) = 0$ (or, equivalently, $h^0(X, L) = 1$), therefore L defines a unique effective divisor on C of degree 5 and this gives a point in $X^{[5]}$. One can easily show that this provides a birational map from $M_X(4n+1)$ to $X^{[5]}$ (see e.g. [4, Proposition 1.3]). It follows by a result of Huybrechts [25] that $M_X(4n+1)$ and $X^{[5]}$ are diffeomorphic and

$$e(M_X(4n+1)) = e(X^{[5]}) = 176256,$$

which is the coefficient at q^5 in $\prod_{k\geq 1}(1-q^k)^{-24}$ by the Göttsche formula [18]. We can compare this number with the Euler number of the symplectic resolution of $M_X(4n+2)$ which equals 176904.

1.4 Fibration $M_X(4n+2) \longrightarrow |2H|$

In this section X is, as before, a K3-surface with $\mathrm{Pic}(X) = \mathbb{Z}[H]$, where H is an ample divisor with $H^2 = 2$. Our goal is to construct a map $M = M_X(4n+2) \longrightarrow |2H|$ that sends a sheaf to its supporting curve. In view of the universal property of M, in order to construct a map $M \longrightarrow |2H|$ we have to construct a map from the functor $\mathcal{M}_X(4n+2)$ to $|2H|$ (see Definition 1.1.9). Given a family $\mathcal{F} \in \mathcal{M}'_X(4n+2)(S)$, we will construct a map $S \longrightarrow |2H|$ sending a point $s \in S$ to the support of the fiber $\mathcal{F}_s$.

Let us first recall the definition of the Fitting ideals (see e.g., [15, Section 20.2]). Given a finite module M over a noetherian ring R, consider its free presentation

$$P_1 \xrightarrow{\varphi} P_0 \longrightarrow M \longrightarrow 0$$

with $\mathrm{rk}\, P_0 = r$. Then the Fitting ideal $\mathrm{Fitt}_i\, M$ is defined to be the image of the map

$$\wedge^{r-i} P_1 \otimes (\wedge^{r-i} P_0)^\vee \longrightarrow R$$

induced by $\wedge^{r-i} P_1 \longrightarrow \wedge^{r-i} P_0$. In other words, $\operatorname{Fitt}_i M$ is generated by $(r-i)$-th order minors of φ (if the bases of P_0 and P_1 are chosen). This definition does not depend on the presentation of M and one has

$$\operatorname{Fitt}_0 M \subset \operatorname{Fitt}_1 M \subset \dots$$

and $(\operatorname{ann} M)^r \subset \operatorname{Fitt}_0 M \subset \operatorname{ann} M$ (see [15, Proposition 20.7]). The first nonzero Fitting ideal is denoted by $I(M)$. An important property of Fitting ideals is that they commute with base change (see [15, Corollary 20.5]). Analogous definitions can be made for coherent sheaves over noetherian schemes.

Let us return to our setup. If F is a semistable sheaf over X with a Hilbert polynomial $4n+2$ then $\operatorname{ann} F \neq 0$ so that already the zeroth Fitting ideal is nonzero and $I(F) = \operatorname{Fitt}_0 F$. Let us show that $I(F)$ is an ideal of the curve in $|2H|$. Consider an exact sequence

$$0 \longrightarrow E \longrightarrow P \longrightarrow F \longrightarrow 0$$

of coherent sheaves over X with P locally free. There is a factorization

$$E \hookrightarrow E^{\vee\vee} \hookrightarrow P$$

and an embedding $E^{\vee\vee}/E \hookrightarrow F$. Using the facts that $E^{\vee\vee}/E$ is zero-dimensional and F is pure, we obtain that $E = E^{\vee\vee}$, so E is locally free. It follows that $I(F)$ is equal to the image of the map $\det E \otimes (\det P)^{-1} \longrightarrow \mathcal{O}_X$ induced by $\det E \longrightarrow \det P$. The corresponding curve is a divisor of the line bundle $(\det E)^{-1} \otimes \det P$ whose first Chern class equals $c_1(F) = 2H$ and so $I(F)$ defines a curve in $|2H|$.

We can do the same relatively. Let $\mathcal{F}$ be a flat family over $S \times X \longrightarrow S$ with fibers being semistable sheaves with the Hilbert polynomial $4n+2$.

Lemma 1.4.1. *There exists a locally free resolution*

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

of sheaves over $S \times X$.

Proof. There exists an exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow \mathcal{F} \longrightarrow 0$$

with P locally free. It is easy to see that K is flat over S. By Lemma C.1, for proving that K is locally free it is enough to show that, for any $s \in S$, the sheaf K_s is locally free over X. As $\mathcal{F}$ is flat there is an exact sequence

$$0 \longrightarrow K_s \longrightarrow P_s \longrightarrow \mathcal{F}_s \longrightarrow 0$$

and we have seen already that K_s is then necessarily locally free. □

The map $\det P_1 \longrightarrow \det P_0$ is a monomorphism on the fibers. Therefore by Proposition C.2 it is itself a monomorphism and its cokernel is flat over S. We can identify $I(\mathcal{F})$ with $\det P_1 \otimes (\det P_0)^{-1} \hookrightarrow \mathcal{O}_X$. The quotient $\mathcal{O}_{S\times X}/I(\mathcal{F})$ is flat over S and has the Hilbert polynomial

$$P(\mathcal{O}_X, n) - P(\mathcal{O}_X(-2H), n) = (n^2+2) - ((n-2)^2+2) = 4n-4$$

on the fibers. It induces a map $S \longrightarrow \mathrm{Quot}_X(\mathcal{O}_X, 4n-4)$ by the universal property of Quot-schemes. Notice that $\mathrm{Quot}_X(\mathcal{O}_X, 4n-4)$ is isomorphic to $|2H|$. Indeed, it is clear that any curve in $|2H|$ defines a point in $\mathrm{Quot}_X(\mathcal{O}_X, 4n-4)$. To show the converse, consider a subscheme $Y \subset X$ with a Hilbert polynomial $P(\mathcal{O}_Y, n) = 4n-4$. Let I be an ideal of Y. Then $P(I, n) = (n-2)^2+2$ and therefore $P(I^{\vee\vee}, n) = (n-2)^2+c$ for some integer c. As $I^{\vee\vee}$ is a line bundle and $\mathrm{Pic}\, X = \mathbb{Z}[H]$, we deduce that $I^{\vee\vee} \simeq \mathcal{O}_X(-2H)$ and $P(I^{\vee\vee}, n) = (n-2)^2+2$. This implies that $\chi(I^{\vee\vee}/I) = 0$ and hence $I = I^{\vee\vee} \simeq \mathcal{O}_X(-2H)$. Altogether implies that Y is a divisor in $|2H|$. This completes the construction of the map $S \longrightarrow |2H|$ and consequently the construction of the map $M_X(4n-2) \longrightarrow |2H|$.

Let us now briefly explain our further strategy. There is a $2:1$ map $X \xrightarrow{\pi} |H| = \mathbb{P}^2$ branched along a sextic in $\mathbb{P}^2$ (see e.g., [4, Proposition 8.13]). Moreover, the natural morphisms $\mathcal{O}_{\mathbb{P}^2}(m) \longrightarrow \pi_*\pi^*\mathcal{O}_{\mathbb{P}^2}(m) = \pi_*\mathcal{O}_X(mH)$ induce the maps $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) \longrightarrow H^0(X, \mathcal{O}_X(mH))$, which are isomorphisms for $m = 1, 2$ (the maps are injective and the spaces have the same dimension). It follows that we can identify $|H|$ with $|\mathcal{O}_{\mathbb{P}^2}(1)|$ and $|2H|$ with $|\mathcal{O}_{\mathbb{P}^2}(2)|$. For any curve in $|2H|$ there are three possibilities:

1. The curve is the preimage of a smooth quadric in $\mathbb{P}^2$ and is integral.

2. The curve is the preimage of two intersecting lines in $\mathbb{P}^2$ and hence is a union of two intersecting curves from $|H|$.

3. The curve is the preimage of a double line in $\mathbb{P}^2$ and hence is nonreduced and of the form $2C$, where $C \in |H|$.

Denote the corresponding subsets of $|2H|$ by Y_1, Y_2, Y_3 and their preimages under $M_X(4n+2) \longrightarrow |2H|$ by M_1, M_2, M_3, respectively. To find the Euler number of $M_X(4n+2)$ we will find the Euler numbers of these strata. The fiber over any curve C from $|2H|$ can be identified with the moduli space $M_C(4n+2)$ of semistable sheaves on C (with respect to the polarization $\mathcal{O}_C(H)$) and we will study more closely these moduli spaces in all three cases.

Chapter 2

Moduli spaces of sheaves over reduced curves

Throughout this section C will be a reduced projective curve over $\mathbb{C}$ with a fixed polarization, unless otherwise explicitly stated. Our goal is to compute the Euler number of some moduli spaces of semistable sheaves of nonzero rank on C. First of all, we will show that the Euler number is zero whenever C has a non-rational component. In the case of rational curves we consider just the curves having nodes as singularities and study the moduli spaces of semistable sheaves having rank at most 2. In the simplest case of a rational curve with one node we are able to provide a complete classification of semistable sheaves of arbitrary rank. This result is used later to compute the Euler number of the moduli space of rank 2 semistable sheaves on the rational curve with arbitrary many nodes.

2.1 Tensor products with line bundles

We begin with definitions. Let $\pi : \widetilde{C} \longrightarrow C$ be a normalization of C. We will say that a sheaf F on C has a constant rank if it has equal ranks on all irreducible components of C. If C is connected and F is locally free then F has a constant rank. Even in the case of disconnected curves, we will assume that locally free sheaves have constant ranks. Given a linear polynomial P, we will denote by $M_C(P)$ the moduli space of semistable sheaves on C having the Hilbert polynomial P and having a constant rank. We will denote by $C_1, \dots, C_s$ the integral components of C.

Lemma 2.1.1. *Let F be a sheaf on C and let L be a line bundle. Then*

$$\chi(\pi^*(F \otimes L)) - \chi(F \otimes L) = \chi(\pi^*F) - \chi(F).$$

Proof. There is an exact sequence

$$0 \longrightarrow A \longrightarrow F \longrightarrow \pi_*\pi^*F \longrightarrow B \longrightarrow 0,$$

where A and B are zero-dimensional sheaves. We deduce $\chi(\pi^*F) - \chi(F) = l(B) - l(A)$. Tensoring the sequence with L we obtain analogously $\chi(\pi^*(F \otimes L)) - \chi(F \otimes L) = l(B \otimes L) - l(A \otimes L) = l(B) - l(A)$ and the claim follows. □

Lemma 2.1.2. *Let F be a sheaf on C with ranks $r_1, \ldots, r_s$ on the components and L be a line bundle with degrees $d_1, \ldots, d_s$ on the components. Then $\chi(F \otimes L) - \chi(F) = \sum r_i d_i$.*

Proof. We get from the previous lemma and the Riemann-Roch theorem

$$\chi(F \otimes L) - \chi(F) = \chi(\pi^*(F \otimes L)) - \chi(\pi^*F) = \sum r_i \deg(\pi^*L)_i = \sum r_i d_i.$$

□

Corollary 2.1.3. *With the notations of the lemma, let L be a polarization of C. Then the Hilbert polynomial of F has the following form $P(F, n) = (\sum r_i d_i)n + \chi(F)$. In particular, if F has constant rank r, then $P(F, n) = r \deg C \cdot n + \chi(F)$.*

Remark 2.1.4. Let C be an integral curve. Then for any coherent sheaf F on C (not necessarily of finite Tor-dimension) we can formally define $\deg F := \chi(F) - \operatorname{rk} F \chi(\mathcal{O}_C)$. For a locally free sheaf F this definition coincides with the usual one by the Riemann-Roch formula (for singular curves). It follows from Corollary 2.1.3 that the notion of semistability with respect to any polarization of C coincides with the notion of semistability with respect to the usual slope function $\mu(F) = \deg F / \operatorname{rk} F$. We will denote by $M_C(r, d)$ the moduli space of semistable sheaves on C having rank r and degree d.

Corollary 2.1.5. *Let F be a semistable sheaf and L be a line bundle having degree zero on all components. Then $F \otimes L$ is semistable.*

Proof. It follows from Lemma 2.1.2 that $\chi(F \otimes L) = \chi(F)$ and analogously $P(F \otimes L, n) = P(F, n)$. The same holds for any subsheaf of F. □

Lemma 2.1.6. *Let F be a sheaf on C having a constant rank r and let L be a line bundle such that $F \otimes L \simeq F$. Then π^*L^r is trivial.*

Proof. We deduce $\pi^*F \otimes \pi^*L \simeq \pi^*F$ and taking G to be a torsion free part of π^*F we get $G \otimes \pi^*L \simeq G$. Therefore $\det G \otimes \pi^*L^r \simeq \det G$ and the claim follows. □

2.2 Moduli spaces over non-rational curves

The aim of this section is to prove the following

Proposition 2.2.1. *Let C be a reduced curve having at least one nonrational component and let P be a nonconstant linear polynomial. Then $M_C(P)$ has Euler number zero.*

To prove the proposition, we will use the arguments of [4] (there it was proved an analogous assertion for the Euler number of a Jacobian). Namely, we will show that $M_C(P)$ possesses a free action of a finite group of arbitrary large order, so that the Euler number should be a multiple of these orders and hence is zero. Let us denote by JC the group of line bundles (modulo isomorphisms) having degree zero on all components.

Lemma 2.2.2 (cf. [4])**.** *The natural map $JC \twoheadrightarrow J\widetilde{C}$ is surjective and its kernel is a divisible group.*

Proof. We will use the description of invertible sheaves from the Appendix B. It is shown there (see Remark B.5), that $JC \twoheadrightarrow J\widetilde{C}$ is surjective. The elements of the kernel come from invertible elements of $\widetilde{\mathcal{O}}/J$. To be more precise, there is an exact sequence of groups

$$(\widetilde{\mathcal{O}}/J)^* \twoheadrightarrow JC \twoheadrightarrow J\widetilde{C} \twoheadrightarrow 0.$$

It is enough to show that the group $(\widetilde{\mathcal{O}}/J)^*$ is divisible. Note that $\widetilde{\mathcal{O}}/J$ is a product of rings of the form $\mathbb{C}[t]/t^n$, so we must prove that $(\mathbb{C}[t]/t^n)^*$ is divisible. This follows from the next lemma. □

Lemma 2.2.3. *The group $k[\![t]\!]^*$ is divisible if k is an algebraically closed field of characteristic 0.*

Proof. Given $f = \sum_{i\geq 0} a_i t^i \in k[\![t]\!]^*$ and $n > 0$, we will show that there exists $g \in k[\![t]\!]^*$ s.t. $g^n = f$. We construct inductively $g_m = \sum_{i=0}^{m-1} b_i t^i$ s.t. $g_m^n \equiv f \pmod{t^m}$. As k is algebraically closed, we can find $b_0 \in k$ s.t. $b_0^n = a_0$. Assume that g_m is already constructed. Then the condition

$$(g_m + b_m t^m)^n \equiv g_m^n + n b_0 b_m t^m \equiv f \pmod{t^{m+1}}$$

uniquely determines b_m. □

Divisible groups are injective objects in the category of abelian groups and this implies that the morphism $JC \twoheadrightarrow J\widetilde{C}$ has a (set theoretical) section $s : J\widetilde{C} \twoheadrightarrow JC$. Using this section we can equip $M_C(P)$ with an action of the group $J\widetilde{C}$ (see Lemma 2.1.5). Next, $J\widetilde{C}$ is an abelian variety of positive dimension, therefore it contains a subgroup G of order p for any prime p (see [34, p. 63]).

Lemma 2.2.4. *If p does not divide r then the action of G on $M_C(P)$ is free.*

Proof. Let $L \in G$ and $F \in M_C(P)$ be such that $s(L) \otimes F \simeq F$. Then it follows from Lemma 2.1.6 that L^r is trivial. But L^p is also trivial, hence already L is trivial. □

It follows that the Euler number of $M_C(P)$ is a multiple of p, whenever p and r are coprime. Therefore $e(M_C(P))$ is zero.

2.3 Semistable sheaves on a rational curve with one node

The aim of this section is to classify the semistable sheaves on a rational curve with one node. The classification of indecomposable vector bundles on it was done by Drozd and Greuel in [14] in terms of certain combinatorial objects, which are easy to handle. Using their technique one can also classify all indecomposable torsion-free sheaves. Partial results on the classification of the semistable sheaves can be found in [7, 8, 9].

For our classification we introduce and analyze certain combinatorial objects - chains and cycles of integers, which are used for the classification of indecomposable torsion-free sheaves. With any aperiodic cycle $\mathbf{a}$ one associates an indecomposable locally free sheaf $\mathcal{B}(\mathbf{a})$ (see [14] or Section 2.3.1) and with any chain $\mathbf{b}$ one associates an indecomposable non-locally free sheaf $\mathcal{S}(\mathbf{b})$. The conditions of (semi)stability of $\mathcal{B}(\mathbf{a})$ and $\mathcal{S}(\mathbf{b})$ imply certain conditions on the cycle $\mathbf{a}$ and the chain $\mathbf{b}$. We will call these conditions the conditions of (semi)stability of cycles and chains, respectively. Our main result is

Proposition 2.3.1. *Given an aperiodic cycle $\mathbf{a}$, the sheaf $\mathcal{B}(\mathbf{a})$ is (semi)stable if and only if the cycle $\mathbf{a}$ is (semi)stable. Given a chain $\mathbf{b} = (b_1, \ldots, b_r)$, the sheaf $\mathcal{S}(\mathbf{b})$ is (semi)stable if and only if the chain $(b_1+1, b_2, \ldots, b_{r-1}, b_r+1)$ is (semi)stable.*

This means that we only need to classify the (semi)stable chains and cycles. This is a purely combinatorial problem, and the main result here can be described as follows. Let $M_{\mathrm{cyc}}(r,d)$ be the set of all aperiodic semistable cycles ($M^s_{\mathrm{cyc}}(r,d)$ for stable cycles) of rank r and degree :

Proposition 2.3.2. *There is a natural bijection between $M_{\mathrm{cyc}}(r,d)$ and $M_{\mathrm{cyc}}(r,d+r)$ and if $0 < d < r$ then there is a natural bijection between $M_{\mathrm{cyc}}(r,d)$ and $M_{\mathrm{cyc}}(d,d-r)$. As a corollary there is a bijection between $M_{\mathrm{cyc}}(r,d)$ and $M_{\mathrm{cyc}}(h,0)$, where $h = \gcd(r,d)$. Analogously for $M^s_{\mathrm{cyc}}(r,d)$ and for chains.*

Thus, the description of $M_{\mathrm{cyc}}(r,d)$ is reduced to the description of $M_{\mathrm{cyc}}(h,0)$, $h = \gcd(r,d)$, and the later is given in Lemma 2.3.26. Among other things, we prove that for $h > 1$ there are no stable cycles (chains) and for $h = 1$ there is just one semistable cycle (chain) which is actually stable. Using these results, we can compute the Euler number of the moduli spaces

Corollary 2.3.3. *The Euler number of $M^s_C(r,d)$ equals 1 if $\gcd(r,d) = 1$ and equals 0 otherwise.*

2.3.1 Semistable sheaves

Let C be a rational curve with one node over an algebraically closed field k of characteristic 0. Let $\pi : \widetilde{C} \to C$ be its normalization ($\widetilde{C} \simeq \mathbb{P}^1$). Given a torsion-free sheaf F over C of rank r and degree d (recall that $\deg F = \chi(F) - r\chi(\mathcal{O}_C) = \chi(F)$), we will say that F is of type (r,d). We will denote the set of torsion-free, indecomposable sheaves of type (r,d) by $\mathcal{E}(r,d)$.

Let us give a description of sheaves in $\mathcal{E}(r,d)$ according to [14]. Let p, p^* be preimages of a singular point in C under π. For any line bundle $L \simeq \mathcal{O}(n)$ over $\widetilde{C}$ we fix once and for all the bases of the fibers $L(p)$ and $L(p^*)$. To make possibly few choices we do this in the following way. Fix some section s of $\mathcal{O}(1)$ having zero in some point different from p and p^*. Then s^n will induce nonzero elements of the fibers of $\mathcal{O}(n)$ over p and p^*, giving the necessary bases.

Given a finite sequence of integers $\mathbf{a} = (a_1, \dots, a_r)$ (throughout the paper we call it just chain, cf. Definition 2.3.14), a natural number m and an element $\lambda \in k^*$, construct the vector bundle $\mathcal{B}(\mathbf{a}, m, \lambda)$ over C in the following way (see [14], [9] or Appendix B for more formal description). We consider the sheaves $B_i = \mathcal{O}(a_i)^{\oplus m}$ over $\widetilde{C}$, then take the direct image π_* of their sum and make the following identifications over the singular point. Glue $B_1(p^*)$ with $B_2(p)$, $B_2(p^*)$ with $B_3(p)$ and so on up to identification of $B_r(p^*)$ with $B_1(p)$. The gluing matrices (with respect to the chosen above bases) are all unit matrices except the matrix gluing $B_r(p^*)$ with $B_1(p)$ which is a Jordan block of size m with an eigenvalue λ.

Remark 2.3.4. Note that if $\mathbf{a} = (a_1, \dots, a_r)$ and $\mathbf{a}' = (a_2, \dots, a_r, a_1)$ is its cyclic shift, then $\mathcal{B}(\mathbf{a}, m, \lambda) \simeq \mathcal{B}(\mathbf{a}', m, \lambda)$. We define a cycle to be an equivalence class of chains modulo cyclic shifts (cf. Definition 2.3.14).

Proposition 2.3.5. *The sheaves $\mathcal{B}(\mathbf{a}, m, \lambda)$ with aperiodic (see Definition 2.3.17) cycles $\mathbf{a}$ describe all indecomposable locally free sheaves over C. Different cycles induce non-isomorphic sheaves.*

In particular, consider the cycle $\mathbf{0} = (0)$ and define $F_m := \mathcal{B}(\mathbf{0}, m, 1)$. One can show that $F_1 \simeq \mathcal{O}_C$ and there is an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow 0.$$

Lemma 2.3.6 (see [43])**.** *There are isomorphisms*

$$\mathcal{B}(\mathbf{a}, m, \lambda) \simeq \mathcal{B}(\mathbf{a}, 1, \lambda) \otimes F_m, \qquad \mathcal{B}(\mathbf{a}, 1, \lambda^r) \simeq \mathcal{B}(\mathbf{a}, 1, 1) \otimes \mathcal{B}(\mathbf{0}, 1, \lambda),$$

where r is the length of $\mathbf{a}$.

We will denote $\mathcal{B}(\mathbf{a}, 1, 1)$ by $\mathcal{B}(\mathbf{a})$.

Corollary 2.3.7. *The sheaf $\mathcal{B}(\mathbf{a}, m, \lambda)$ is semistable if and only if $\mathcal{B}(\mathbf{a})$ is semistable. The sheaf $\mathcal{B}(\mathbf{a}, m, \lambda)$ is stable if and only if $m = 1$ and $\mathcal{B}(\mathbf{a})$ is stable.*

Proof. We know that the sheaf F_m has a filtration with factors isomorphic to $\mathcal{O}_C$. Therefore the sheaf $\mathcal{B}(\mathbf{a}, m, \lambda)$ has a filtration with factors isomorphic to $\mathcal{B}(\mathbf{a}, 1, \lambda)$. Hence $\mathcal{B}(\mathbf{a}, m, \lambda)$ is semistable if and only if $\mathcal{B}(\mathbf{a}, 1, \lambda)$ is semistable and $\mathcal{B}(\mathbf{a}, m, \lambda)$ can be stable just if $m = 1$. It is clear that $\mathcal{B}(\mathbf{a}, 1, \lambda)$ is (semi)stable if and only if $\mathcal{B}(\mathbf{a}, 1, 1)$ is. $\square$

Let us now describe the non-locally free indecomposable sheaves. Given a chain of integers $\mathbf{a} = (a_1, \ldots, a_r)$, define a torsion-free sheaf $\mathcal{S}(\mathbf{a})$ as follows. Take the direct image π_* of the sum of $B_i = \mathcal{O}(a_i)$ and make the following identifications over the singular point. Glue $B_1(p^*)$ with $B_2(p)$, $B_2(p^*)$ with $B_3(p)$ and so on just identifying their bases. The fibers $B_1(p)$ and $B_r(p^*)$ are not identified.

Proposition 2.3.8. *The sheaves $\mathcal{S}(\mathbf{a})$ describe all indecomposable torsion-free non-locally free sheaves over C. Different chains induce non-isomorphic sheaves.*

Our goal is to determine which of the sheaves $\mathcal{B}(\mathbf{a}, m, \lambda)$ and $\mathcal{S}(\mathbf{a})$ are (semi)stable. It follows from Corollary 2.3.7 that in the case of locally free sheaves we can restrict ourselves just to $\mathcal{B}(\mathbf{a})$.

Remark 2.3.9. Given a chain $\mathbf{a} = (a_1, \ldots, a_r)$, one can easily show that $\deg \mathcal{B}(\mathbf{a}) = \sum a_i$, $\deg \mathcal{S}(\mathbf{a}) = \sum a_i + 1$ and $\operatorname{rk} \mathcal{B}(\mathbf{a}) = \operatorname{rk} \mathcal{S}(\mathbf{a}) = r$.

Proposition 2.3.10. *Let $\mathbf{a} = (a_1, \ldots, a_r)$ be a cycle and $\mathbf{b} = (b_1, \ldots, b_k)$ be its subchain (see Definition 2.3.15). Then there is an exact sequence*

$$0 \longrightarrow \mathcal{S}((b_1 - 1, b_2, \ldots, b_{k-1}, b_k - 1)) \longrightarrow \mathcal{B}(\mathbf{a}) \longrightarrow \mathcal{S}(\mathbf{b}') \longrightarrow 0,$$

where for $k = 1$ we consider just $\mathcal{S}((b_1 - 2))$ and $\mathbf{b}'$ is the complement of $\mathbf{b}$ in $\mathbf{a}$.

Proof. Without lost of generality we may assume $\mathbf{b} = (a_1, \dots, a_k)$. Let us denote $B_i = \mathcal{O}_{\widetilde{C}}(a_i)$. We consider the direct image π_* of the sum

$$(B_1 \otimes \mathcal{O}_{\widetilde{C}}(-p)) \oplus B_2 \oplus \cdots \oplus B_{k-1} \oplus (B_k \otimes \mathcal{O}_{\widetilde{C}}(-p^*))$$

and identify their fibers precisely like in the construction of $\mathcal{S}$. The module obtained in this way is isomorphic to $\mathcal{S}((b_1-1, b_2, \dots, b_{k-1}, b_k-1))$ and there is a natural embedding of this module to $\mathcal{B}(\mathbf{a})$ (the fiber of $(B_1 \otimes \mathcal{O}_{\widetilde{C}}(-p))$ in point p goes to zero both in fibers $B_r(p^*)$ and $B_1(p)$, and analogously for the fiber of $(B_k \otimes \mathcal{O}_{\widetilde{C}}(-p^*))$ in point p^*). It is clear that the factor-module will be isomorphic to the direct image of $B_{k+1} \oplus \cdots \oplus B_r$ with identifications $B_{k+1}(p^*) \simeq B_{k+2}(p), \dots B_{r-1}(p^*) \simeq B_r(p)$. But such a module is precisely $\mathcal{S}((a_{k+1}, \dots, a_r))$. □

Corollary 2.3.11. *Let* $\mathbf{a} = (a_1, \dots, a_r)$ *be a cycle such that* $\mathcal{B}(\mathbf{a})$ *is a semistable sheaf. Then for any proper subchain* $\mathbf{b} = (b_1, \dots, b_k)$ *of* $\mathbf{a}$ *it holds*

$$\frac{\sum_{i=1}^{k} b_i - 1}{k} \leq \frac{\sum_{i=1}^{r} a_i}{r}.$$

If $\mathcal{B}(\mathbf{a})$ *is stable then the inequalities are strict.*

Proposition 2.3.12. *Let* $\mathbf{a} = (a_1, \dots, a_r)$ *be a chain and* $\mathbf{b} = (b_1, \dots, b_k)$ *be its subchain that does not contain* a_1 *and* a_r*. Then there is an embedding*

$$\mathcal{S}((b_1 - 1, b_2, \dots, b_{k-1}, b_k - 1)) \hookrightarrow \mathcal{S}(\mathbf{a}).$$

If $\mathbf{b}$ *is a subchain containing* a_1 *or* a_r *(say,* $\mathbf{b} = (a_1, \dots, a_k)$*) then there is an exact sequence*

$$0 \longrightarrow \mathcal{S}((a_1, a_2, \dots, a_{k-1}, a_k - 1)) \longrightarrow \mathcal{S}(\mathbf{a}) \longrightarrow \mathcal{S}((a_{k+1}, \dots, a_r)) \longrightarrow 0.$$

Proof. The proof goes through the same lines as in Proposition 2.3.10. □

Corollary 2.3.13. *Let* $\mathbf{a} = (a_1, \dots, a_r)$ *be a chain such that* $\mathcal{S}(\mathbf{a})$ *is a semistable sheaf and let*

$$\mathbf{a}' := (a_1 + 1, a_2, \dots, a_{r-1}, a_r + 1).$$

Then for any proper subchain $\mathbf{b} = (b_1, \dots, b_k)$ *of* $\mathbf{a}'$ *it holds*

$$\frac{\sum_{i=1}^{k} b_i - 1}{k} \leq \frac{\sum_{i=1}^{r} a'_i - 1}{r}.$$

If $\mathcal{S}(\mathbf{a})$ *is stable then the inequalities are strict.*

Proof. If $\mathcal{S}(\mathbf{a})$ is semistable then for any subchain $\mathbf{b} = (b_1, \ldots, b_k)$ of $\mathbf{a}'$ that does not contain a'_1 and a'_r we have (see Proposition 2.3.12 and Remark 2.3.9)

$$\frac{\sum_{i=1}^k b_i - 2 + 1}{k} \leq \frac{\sum_{i=1}^r a_i + 1}{r} = \frac{\sum_{i=1}^r a'_i - 1}{r}.$$

If $\mathbf{b}$ is a subchain of $\mathbf{a}'$ containing a'_1 or a'_r (say, $\mathbf{b} = (a'_1, \ldots, a'_k)$) then according to Proposition 2.3.12 there is an embedding $\mathcal{S}((b_1 - 1, b_2, \ldots, b_{k-1}, b_k - 1)) \hookrightarrow \mathcal{S}(\mathbf{a})$ and this implies

$$\frac{\sum_{i=1}^k b_i - 2 + 1}{k} \leq \frac{\sum_{i=1}^r a_i + 1}{r} = \frac{\sum_{i=1}^r a'_i - 1}{r}.$$

The claim about stability is analogous. □

Corollaries 2.3.11 and 2.3.13 suggest that one can define stability conditions for chains and cycles. We will do this in the next section. After the classification of (semi)stable chains and cycles we will be able to prove that the stability of chains and cycles is not only necessary for the stability of the corresponding sheaves (as proved in Corollaries 2.3.11 and 2.3.13) but is also sufficient.

2.3.2 Semistable chains and cycles

Recall the definition of chains and cycles from the previous section.

Definition 2.3.14. Define a chain to be a finite sequence of integers. Define a cycle to be an equivalence class of finite sequences of integers, where the equivalence is generated by relations

$$(a_1, a_2, \ldots, a_r) \sim (a_2, a_3, \ldots, a_r, a_1).$$

We will usually write representing sequences instead of the corresponding cycles.

Definition 2.3.15. Given a chain $(a_1, \ldots, a_r)$, define its subchain as any chain of the form $(a_i, a_{i+1}, \ldots, a_j)$, where $1 \leq i \leq j \leq r$. Given a cycle $(a_1, \ldots, a_r)$, define its subchain as any chain of the form $(a_i, a_{i+1}, \ldots, a_{i+k})$, where $1 \leq i \leq r$, $0 \leq k < r$ and we identify a_{r+1} with a_1, a_{r+2} with a_2 and so on.

For example, the cycle $(1, 2, 3, 1, 2, 3)$ contains the subchain $(3, 1, 2, 3, 1)$ but the chain $(1, 2, 3, 1, 2, 3)$ does not.

Definition 2.3.16. Given a chain $\mathbf{a} = (a_1, \dots, a_r)$, we call any its subchain containing a_1 or a_r an extreme subchain of $\mathbf{a}$.

Definition 2.3.17. A cycle is called aperiodic if its sequence cannot be written as a concatenation of equal proper subsequences.

Definition 2.3.18. Given a chain $\mathbf{a} = (a_1, \dots, a_r)$, define its degree, rank, and slope by

$$\deg \mathbf{a} = \sum_{i=1}^{r} a_i - 1, \qquad \operatorname{rk} \mathbf{a} = r, \qquad \mu(\mathbf{a}) = \frac{\deg \mathbf{a}}{\operatorname{rk} \mathbf{a}}.$$

Given a cycle $\mathbf{a} = (a_1, \dots, a_r)$, define its degree, rank, and slope by

$$\deg \mathbf{a} = \sum_{i=1}^{r} a_i, \qquad \operatorname{rk} \mathbf{a} = r, \qquad \mu(\mathbf{a}) = \frac{\deg \mathbf{a}}{\operatorname{rk} \mathbf{a}}.$$

For example, the slope of the chain $(1, 2, 3, 1, 2, 3)$ equals $\frac{11}{6}$ and the slope of the cycle $(1, 2, 3, 1, 2, 3)$ equals 2.

Definition 2.3.19. The chain (cycle) $\mathbf{a} = (a_1, \dots, a_r)$ is called semistable if for any its subchain $\mathbf{b}$ it holds

$$\mu(\mathbf{b}) \leq \mu(\mathbf{a}).$$

If the inequality is strict for any proper subchain then $\mathbf{a}$ is called stable. A proper subchain $\mathbf{b}$ of a chain (cycle) $\mathbf{a}$ is called a destabilizing subchain of $\mathbf{a}$ if $\mu(\mathbf{b}) \geq \mu(\mathbf{a})$.

For example, the chain $(1, 0, 0, 1)$ is stable and the cycle $(1, 0, 0, 1)$ is not stable, because it has slope $1/2$ and contains a destabilizing subchain $(1, 1)$ having the same slope. In what follows, we will classify (semi)stable chains and cycles. For example, the only stable chain of rank 7 and degree 4 is $(1, 1, 0, 1, 0, 1, 1)$ and the only stable (aperiodic) cycle of rank 7 and degree 4 is $(1, 0, 1, 0, 1, 0, 1)$.

A chain (cycle) of rank r and degree d will be said to be of type (r, d). We want to classify all (semi)stable chains and aperiodic cycles of a fixed type (r, d). The set of semistable (stable) chains of type (r, d) will be denoted by $M_{\mathrm{ch}}(r, d)$ ($M^s_{\mathrm{ch}}(r, d)$). The set of aperiodic semistable (stable) cycles of type (r, d) will be denoted by $M_{\mathrm{cyc}}(r, d)$ ($M^s_{\mathrm{cyc}}(r, d)$). We are going to prove

Proposition 2.3.20. *For any pair (r, d) ($r \in \mathbb{Z}_{>0}, d \in \mathbb{Z}$) the set $M_{\mathrm{ch}}(r, d)$ is finite and non-empty. If $\gcd(r, d) > 1$ then $M^s_{\mathrm{ch}}(r, d) = \emptyset$. If $\gcd(r, d) = 1$ then $M_{\mathrm{ch}}(r, d) = M^s_{\mathrm{ch}}(r, d)$ consists of one element. The same assertions hold for aperiodic cycles.*

The study of semistable chains and semistable cycles is quite analogous but we will deal with them separately for simplicity.

Lemma 2.3.21. *A chain* $\mathbf{a} = (a_1, a_2, \dots a_r)$ *is (semi)stable if and only if the chain* $(a_1+1, a_2+1 \dots, a_r+1)$ *is (semi)stable. In particular, there is a natural bijection* $M_{\mathrm{ch}}(r,d) \simeq M_{\mathrm{ch}}(r, d+r)$ *and we can always assume* $0 \le \deg \mathbf{a} < r$.

Lemma 2.3.22. *For any subchain* $\mathbf{b}$ *of a semistable chain* $\mathbf{a} = (a_1, \dots, a_r)$ *one has*

$$\mu(\mathbf{a}) \le \frac{\sum b_i + 1}{\operatorname{rk} \mathbf{b}}.$$

Moreover, if $\mathbf{b}$ *is an extreme subchain then*

$$\mu(\mathbf{a}) \le \frac{\sum b_i}{\operatorname{rk} \mathbf{b}}.$$

Proof. Let us first prove the assertion for extreme subchains. We may assume that $\mathbf{b} = (a_1, \dots, a_{k_1})$. Denote $k_2 = r - k_1$, $x_1 = \sum_{i=1}^{k_1} a_i$, $x_2 = \sum_{i=k_1+1}^{r} a_i$. Then the semistability of $\mathbf{a}$ implies

$$\frac{x_2 - 1}{k_2} \le \frac{x_1 + x_2 - 1}{k_1 + k_2},$$

hence

$$\frac{x_1 + x_2 - 1}{k_1 + k_2} \le \frac{x_1}{k_1}.$$

Let us now assume that $\mathbf{b} = (a_{k_1+1}, \dots, a_{k_1+k_2})$ is not extreme, i.e. $k_1 \ge 1$ and $k_3 := r - k_1 - k_2 \ge 1$. We denote $x_1 = \sum_{i=1}^{k_1} a_i$, $x_2 = \sum_{i=k_1+1}^{k_1+k_2} a_i$, and $x_3 = \sum_{i=k_1+k_2+1}^{r} a_i$. It holds by our assumptions

$$\frac{x_1 - 1}{k_1} \le \frac{x_1 + x_2 + x_3 - 1}{k_1 + k_2 + k_3}, \qquad \frac{x_3 - 1}{k_3} \le \frac{x_1 + x_2 + x_3 - 1}{k_1 + k_2 + k_3},$$

hence

$$\frac{x_1 + x_3 - 2}{k_1 + k_3} \le \frac{x_1 + x_2 + x_3 - 1}{k_1 + k_2 + k_3}$$

and therefore

$$\frac{x_1 + x_2 + x_3 - 1}{k_1 + k_2 + k_3} \le \frac{x_2 + 1}{k_2}.$$

□

Corollary 2.3.23. *If a chain* $\mathbf{a} = (a_1, \dots, a_r)$ *is semistable then* $\mu(\mathbf{a}) \le a_1$, $\mu(\mathbf{a}) \le a_r$ *and for any* k *one has* $\mu(\mathbf{a}) \le a_k + 1$.

It follows that in a semistable chain $(a_1, \dots, a_r)$ for any indices i, j one has $a_i - 1 \leq \mu(a) \leq a_j + 1$ and therefore the difference between any a_i and a_j is not greater than 2. Hence, the elements of $\mathbf{a}$ can take at most 3 consecutive values.

Lemma 2.3.24. *If a semistable chain* $\mathbf{a} = (a_1, \dots, a_r)$ *of type* (r, d) *contains elements with difference* 2 *then* d *is a multiple of* r *(hence* $d = 0$ *under the assumption* $0 \leq d < r$*).*

Proof. Assume there are elements in $\mathbf{a}$ equal to $m - 1$ and $m + 1$. Then we have $(m + 1) - 1 \leq \mu(\mathbf{a}) \leq (m - 1) + 1$ and therefore $d/r = m$ is an integer. □

We come now to the main fact allowing us to prove Proposition 2.3.20 and to classify all semistable chains. It serves as a basis of our reduction of chains.

Proposition 2.3.25. *Let* $0 < d < r$. *Then there is a natural bijection between* $M_{\mathrm{ch}}(r, d)$ *and* $M_{\mathrm{ch}}(d, d - r)$. *Analogously for stable chains.*

Proof. Let $\mathbf{a} = (a_1, \dots, a_r)$ be a semistable chain of type (r, d). It follows from Lemma 2.3.24 that its elements can take at most two consecutive values. Obviously, they can be only 0 or 1. From the inequality $a_1 \geq \mu(\mathbf{a}) > 0$ one gets $a_1 = 1$. Analogously $a_r = 1$. From the condition $\sum_{i=1}^{r} a_i - 1 = d$ we obtain that there are $d + 1$ 1's among the elements of $\mathbf{a}$. Let $b_1, \dots, b_d$ be the lengths of consecutive zero-blocks between the 1's. We have $\sum_{i=1}^{d} b_i = r - d - 1$. Now, the chain $\mathbf{a}$ consisting of 0's and 1's is semistable if and only if the inequality from the definition 2.3.19 holds for any subchain starting and ending with a one. This can be written as follows. Any subchain $(b_{j+1}, \dots, b_{j+k})$ of the chain $(b_1, \dots, b_d)$ should satisfy

$$\frac{(k+1) - 1}{\sum_{i=j+1}^{j+k} b_i + k + 1} \leq \frac{d}{\sum_{i=1}^{d} b_i + d + 1},$$

or, equivalently,

$$\frac{\sum_{i=j+1}^{j+k} b_i + 1}{k} \geq \frac{\sum_{i=1}^{d} b_i + 1}{d},$$

which can be written in the form

$$\frac{\sum_{i=j+1}^{j+k}(-b_i) - 1}{k} \leq \frac{\sum_{i=1}^{d}(-b_i) - 1}{d}.$$

But this says precisely that the chain $(-b_1, -b_2, \dots, -b_d)$ is semistable. Its degree is $\sum_{i=1}^{d}(-b_i) - 1 = -(r - d - 1) - 1 = d - r$. The last thing to prove is that, conversely, any such semistable chain will give nonnegative numbers b_i so that we can reconstruct the chain $\mathbf{a}$. But the semistability condition for $(-b_1, \dots, -b_d)$ implies $-b_i - 1 \le (d-r)/d < 0$ and therefore $b_i \ge 0$. Altogether implies that there is a bijection between $M_{\mathrm{ch}}(r,d)$ and $M_{\mathrm{ch}}(d, d-r)$. The proof for stable chains goes through the same lines. □

This proposition shows that we can reduce the classification of (semi)stable chains of type (r,d) to the classification of (semi)stable chains of type $(d, d-r)$, i.e., of those with a smaller rank. The later can be reduced to $M_{\mathrm{ch}}(d, r_0)$ (respectively, to $M^s_{\mathrm{ch}}(d, r_0)$), where $0 \le r_0 < d$ by Lemma 2.3.21. Repeating these reductions we will finally end up with $M_{\mathrm{ch}}(h, 0)$ (respectively, $M^s_{\mathrm{ch}}(h,0)$), where $h = \gcd(r,d)$. So, Proposition 2.3.20 should be proved (and classification should be done) only for the type $(h,0)$.

For example, let us describe $M_{\mathrm{ch}}(7,4)$. We write our reductions as follows

$$M_{\mathrm{ch}}(7,4) \simeq M_{\mathrm{ch}}(4, 4-7) \simeq M_{\mathrm{ch}}(4,1) \simeq M_{\mathrm{ch}}(1, 1-4) \simeq M_{\mathrm{ch}}(1,0).$$

Thus, we take the unique element $(1) \in M_{\mathrm{ch}}(1,0)$ and reconstruct the element from $M_{\mathrm{ch}}(7,4)$ going from the right to the left in our sequence of isomorphisms. We get $(-2) \in M_{\mathrm{ch}}(1,-3)$ and therefore the element of $M_{\mathrm{ch}}(4,1)$ consists of two 1's with a zero-block of length 2 between them, so we get $(1,0,0,1) \in M_{\mathrm{ch}}(4,1)$. Then $(0,-1,-1,0) \in M_{\mathrm{ch}}(4,-3)$ and the element of $M_{\mathrm{ch}}(7,4)$ consists of five 1's with zero-blocks of lengths $(0,1,1,0)$ between them, so we get $(1,1,0,1,0,1,1) \in M_{\mathrm{ch}}(7,4)$.

Lemma 2.3.26. *The semistable chains of type $(r,0)$ are of the form*

$$(0,\dots,0,1,0,\dots,0,-1,0,\dots,0,1,\dots\dots,-1,0,\dots,0,1,0,\dots,0),$$

where 1 and -1 alternate and the zero-blocks are of arbitrary lengths (the whole sequence must be, of course, of length r). If $r > 1$, none of these chains is stable. If $r = 1$ there is precisely one semistable chain (1) and it is stable.

Proof. Let $\mathbf{a} = (a_1, \dots, a_r)$ be semistable of type $(r,0)$. Then we have $a_i - 1 \le \mu(\mathbf{a}) = 0$, $a_i + 1 \ge \mu(\mathbf{a}) = 0$ and therefore $-1 \le a_i \le 1$. Let there be k elements in $\mathbf{a}$ which are equal to 1 and l elements which are equal to -1. We have then $\deg \mathbf{a} = l - k - 1 = 0$ and so $l = k+1$. If there exists a subchain $\mathbf{b}$ containing only zeros and ones with at least two ones then $\deg \mathbf{b} \ge 2 - 1 > 0$ and therefore $\mu(\mathbf{b}) > \mu(\mathbf{a}) = 0$, which is impossible. This together with $l = k+1$ imply that 1 and -1 alternate in $\mathbf{a}$ and therefore $\mathbf{a}$ has a required form. Conversely, if a chain $\mathbf{a}$ has the form like in the condition of the lemma then, first of all, its degree equals 0. For any subchain $\mathbf{b}$ the difference between the numbers of 1's and -1's is not greater than 1 and therefore $\deg \mathbf{b} \leq 0$, which implies $\mu(\mathbf{b}) \leq \mu(\mathbf{a}) = 0$. To prove that $\mathbf{a}$ is not stable if $r > 1$ we notice that for a proper subchain (1) of $\mathbf{a}$ one has $\mu((1)) = 0 = \mu(\mathbf{a})$. The last assertion of the lemma is trivial. $\square$

This lemma implies immediately Proposition 2.3.20 (for the case of chains). The further considerations are of independent interest.

Lemma 2.3.27. *A chain $\mathbf{a} = (a_1, \ldots, a_r)$ is semistable (stable) if and only if for any its extreme subchain $\mathbf{b}$ it holds $\mu(\mathbf{b}) \leq \mu(\mathbf{a})$ ($\mu(\mathbf{b}) < \mu(\mathbf{a})$). In particular, if a chain $\mathbf{a}$ is non-stable then it contains an extreme destabilizing subchain.*

Proof. Assuming that for any extreme subchain $\mathbf{b}$ of $\mathbf{a}$ it holds $\mu(\mathbf{b}) \leq \mu(\mathbf{a})$, we will show that $\mathbf{a}$ is semistable. Let $\mathbf{c} = (a_{k_1+1}, a_{k_1+2}, \ldots a_{k_1+k_2})$ be a subchain of $\mathbf{a}$. We denote $k_3 = r - k_1 - k_2$, $x_1 = \sum_{i=1}^{k_1} a_i$, $x_2 = \sum_{i=k_1+1}^{k_1+k_2} a_i$, and $x_3 = \sum_{i=k_1+k_2+1}^{r} a_i$. We want to show that $\mu(\mathbf{c}) \leq \mu(\mathbf{a})$, so we may suppose that $\mathbf{c}$ is not extreme, hence $k_1 \geq 1$ and $k_3 \geq 1$. The same proof as in Lemma 2.3.22 shows

$$\frac{x_1 + x_2 + x_3 - 1}{k_1 + k_2 + k_3} \leq \frac{x_3}{k_3}, \quad \frac{x_1 + x_2 + x_3 - 1}{k_1 + k_2 + k_3} \leq \frac{x_1}{k_1}$$

and therefore

$$\frac{x_1 + x_2 + x_3 - 1}{k_1 + k_2 + k_3} \leq \frac{x_1 + x_3}{k_1 + k_3}.$$

This implies

$$\frac{x_2 - 1}{k_2} \leq \frac{x_1 + x_2 + x_3 - 1}{k_1 + k_2 + k_3},$$

i.e. $\mu(\mathbf{c}) \leq \mu(\mathbf{a})$. The proof for stable chains is analogous. $\square$

Lemma 2.3.28. *Let $\mathbf{a} = (a_1, \ldots, a_r)$ be a semistable chain and $\mathbf{b} = (a_1, \ldots, a_k)$ be its extreme destabilizing subchain. Then the chains $\mathbf{b}$ and $\mathbf{b}' = (a_{k+1} + 1, a_{k+2}, \ldots, a_r)$ are semistable chains with slope $\mu(\mathbf{a})$.*

Proof. It follows from the condition

$$\frac{\sum_{i=1}^{k} a_i - 1}{k} = \frac{(\sum_{i=1}^{k} a_i - 1) + \sum_{i=k+1}^{r} a_i}{r}$$

that

$$\mu(\mathbf{b}') = \frac{\sum_{i=k+1}^{r} a_i}{r - k} = \frac{(\sum_{i=1}^{k} a_i - 1) + \sum_{i=k+1}^{r} a_i}{r} = \mu(\mathbf{a}).$$

The semistability of $\mathbf{b}$ is trivial. To prove the semistability of $\mathbf{b}'$ we note that if $\mathbf{c}$ is a subchain of $\mathbf{b}'$ not containing the element $a_{k+1}+1$ then $\mu(\mathbf{c}) \leq \mu(\mathbf{a}) =$

$\mu(\mathbf{b}')$. If $\mathbf{c}$ contains $a_{k+1}+1$ then it is of the form $(a_{k+1}+1, a_{k+2}, \dots, a_{k+l})$ and therefore it would follow from

$$\mu(\mathbf{c}) = \frac{\sum_{i=k+1}^{k+l} a_i}{l} > \mu(\mathbf{a}), \qquad \frac{\sum_{i=1}^{k} a_i - 1}{k} = \mu(\mathbf{a})$$

that

$$\frac{\sum_{i=1}^{k+l} a_i - 1}{k+l} > \mu(\mathbf{a}),$$

which is impossible as $\mathbf{a}$ is semistable. □

We return to (semi)stable cycles.

Lemma 2.3.21′. The cycle $\mathbf{a} = (a_1, a_2, \dots a_r)$ is semistable (stable) if and only if the cycle $(a_1+1, a_2+1 \dots, a_r+1)$ is semistable (stable). In particular, there is a bijection $M_{\mathrm{cyc}}(r,d) \simeq M_{\mathrm{cyc}}(r, d+r)$ and we may always assume $0 \le \deg a < r$.

Lemma 2.3.23′. If the chain $\mathbf{a} = (a_1, \dots, a_r)$ is semistable then for any index k it holds $\mu(\mathbf{a}) \le a_k + 1$.

Proof. Without loss of generality we may assume $k = r$. Semistability of $\mathbf{a}$ implies

$$(\sum_{i=1}^{r-1} a_i - 1)/(r-1) \le (\sum_{i=1}^{r} a_i)/r,$$

hence

$$\sum_{i=1}^{r} a_i \le r a_r + r$$

and the claim follows. □

Hence for any semistable cycle $\mathbf{a} = (a_1, \dots, a_r)$ one has $a_i - 1 \le \mu(\mathbf{a}) \le a_j - 1$ for any indices i, j. As above, we deduce that the elements of $\mathbf{a}$ can take at most three consecutive values.

Lemma 2.3.24′. If a semistable chain of type (r, d) contains elements with difference 2 then d is a multiple of r (hence $d = 0$ under the assumption $0 \le d < r$).

Proposition 2.3.25′. Let $0 < d < r$. Then there is a bijection between $M_{\mathrm{cyc}}(r,d)$ and $M_{\mathrm{cyc}}(d, d-r)$. Analogously for stable aperiodic cycles.

Proof. Let $\mathbf{a} = (a_1, \ldots, a_r)$ be a semistable cycle of type (r, d). We know that its elements can take at most two consecutive values. Obviously, they can be only 0 and 1. From the condition $\sum_{i=1}^{r} a_i = d$ we get that there are d ones among the elements of $\mathbf{a}$. Let $b_1, \ldots, b_d$ be the lengths of consecutive zero-blocks between the ones. We have $\sum_{i=1}^{d} b_i = r - d$. Now, the cycle $\mathbf{a}$ consisting of zeros and ones is semistable if and only if the inequality from Definition 2.3.19 holds for any subchain starting and ending with a one. This can be written as follows. For any subchain $(b_{j+1}, \ldots, b_{j+k})$ of the cycle $(b_1, \ldots, b_d)$ one should have

$$\frac{(k+1)-1}{\sum_{i=j+1}^{j+k} b_i + k + 1} \le \frac{d}{\sum_{i=1}^{d} b_i + d},$$

or, equivalently,

$$\frac{\sum_{i=j+1}^{j+k} b_i + 1}{k} \ge \frac{\sum_{i=1}^{d} b_i}{d},$$

which can be written in the form

$$\frac{\sum_{i=j+1}^{j+k}(-b_i) - 1}{k} \le \frac{\sum_{i=1}^{d}(-b_i)}{d}.$$

But this says precisely that the cycle $(-b_1, -b_2, \ldots, -b_d)$ is semistable. Its degree equals $\sum_{i=1}^{d}(-b_i) = d - r$. It remains to prove that, conversely, any such semistable cycle will produce nonnegative numbers b_i so that we can reconstruct the cycle $\mathbf{a}$. But the semistability condition implies $-b_i - 1 \le (d-r)/d < 0$, therefore $b_i \ge 0$. It is clear that the cycle $\mathbf{a}$ is aperiodic if and only if $\mathbf{b}$ is aperiodic. Altogether implies that there is a bijection between $M_{\mathrm{cyc}}(r, d)$ and $M_{\mathrm{cyc}}(d, d - r)$. The proof for stable cycles goes through the same lines. □

Using this proposition, precisely as it was done for chains, we can reduce the study of $M_{\mathrm{cyc}}(r, d)$ to the study of $M_{\mathrm{cyc}}(h, 0)$, where $h = \gcd(r, d)$.

For example, let us describe $M_{\mathrm{cyc}}(7, 4)$. We write our reductions as follows

$$M_{\mathrm{cyc}}(7,4) \simeq M_{\mathrm{cyc}}(4, 4-7) \simeq M_{\mathrm{cyc}}(4,1) \simeq M_{\mathrm{cyc}}(1, 1-4) \simeq M_{\mathrm{cyc}}(1,0).$$

Thus, we take the unique element $(0) \in M_{\mathrm{cyc}}(1, 0)$ and reconstruct the element from $M_{\mathrm{cyc}}(7, 4)$ going from the right to the left in our sequence of isomorphisms. We get $(-3) \in M_{\mathrm{cyc}}(1, -3)$ and therefore the element of $M_{\mathrm{cyc}}(4, 1)$ equals $(1, 0, 0, 0)$ (the length of zero-block equals 3). Then $(0, -1, -1, -1) \in M_{\mathrm{cyc}}(4, -3)$ and the element of $M_{\mathrm{cyc}}(7, 4)$ has zero-blocks of lengthes $(0, 1, 1, 1)$, so it looks like $(1, 1, 0, 1, 0, 1, 0)$. Clearly, it is equivalent to $(1, 0, 1, 0, 1, 0, 1) \in M_{\mathrm{cyc}}(7, 4)$.

Lemma 2.3.26′. The semistable cycles of type $(r, 0)$ are of the form

$$(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0, 1, \ldots\ldots, -1, \ldots, 0),$$

where 1 and -1 alternate and zero-blocks are arbitrary (the sequence should be of course of length r). If $r > 1$, none of these cycles is stable aperiodic. If $r = 1$ there is just one semistable cycle (1) and it is stable.

Proof. Let $\mathbf{a} = (a_1, \ldots, a_r)$ be semistable of type $(r, 0)$. Then we have $a_i - 1 \leq \mu(\mathbf{a}) = 0$, $a_i + 1 \geq \mu(\mathbf{a}) = 0$ and therefore $-1 \leq a_i \leq 1$. Let there be k elements in $\mathbf{a}$ which equal 1 and l elements which equal -1. We have then $\deg \mathbf{a} = l - k = 0$, so $l = k$. If there exists a subchain $\mathbf{b}$ containing only zeros and ones with at least two ones then $\deg \mathbf{b} \geq 2 - 1 > 0$ and therefore $\mu(\mathbf{b}) > \mu(\mathbf{a}) = 0$, which is impossible. This, with $l = k$ imply that 1 and -1 alternate in $\mathbf{a}$ and therefore $\mathbf{a}$ has the required form. Conversely, if a chain $\mathbf{a}$ has the form like in the condition of the lemma then, first of all, its degree equals 0. For any subchain $\mathbf{b}$ the difference between the numbers of 1's and -1's is no greater than 1 and therefore $\deg \mathbf{b} \leq 0$, which implies $\mu(\mathbf{b}) \leq \mu(\mathbf{a}) = 0$. To prove that any aperiodic $\mathbf{a}$ is non-stable if $r > 1$ we notice that it contains nonzero elements, because otherwise it would be periodic. But for a proper subchain (1) of $\mathbf{a}$ one has $\mu((1)) = 0 = \mu(\mathbf{a})$, so $\mathbf{a}$ is non-stable. The last assertion of the lemma is trivial. □

Lemma 2.3.28′. Let $\mathbf{a} = (a_1, \ldots, a_r)$ be a semistable cycle and $\mathbf{b} = (a_1, \ldots, a_k)$ be its destabilizing subchain. Then the chains $\mathbf{b}$ and $\mathbf{b}' = (a_{k+1} + 1, a_{k+2}, \ldots, a_{r-1}, a_r + 1)$ are semistable chains with slope $\mu(\mathbf{a})$.

Proof. It follows from the condition

$$\frac{\sum_{i=1}^{k} a_i - 1}{k} = \frac{(\sum_{i=1}^{k} a_i - 1) + (\sum_{i=k+1}^{r} a_i + 1)}{r}$$

that

$$\mu(\mathbf{b}') = \frac{\sum_{i=k+1}^{r} a_i + 1}{r - k} = \frac{(\sum_{i=1}^{k} a_i - 1) + (\sum_{i=k+1}^{r} a_i + 1)}{r} = \mu(\mathbf{a}).$$

The semistability of $\mathbf{b}$ is trivial. To prove the semistability of $\mathbf{b}'$ we note that if $\mathbf{c}$ is a subchain of $\mathbf{b}'$ not containing elements $a_{k+1} + 1$ and $a_r + 1$ then $\mu(\mathbf{c}) \leq \mu(\mathbf{a}) = \mu(\mathbf{b}')$. If $\mathbf{c}$ is a proper subchain of $\mathbf{b}'$ containing, say, $a_{k+1} + 1$ then it is of the form $(a_{k+1} + 1, a_{k+2}, \ldots, a_{k+l})$ and therefore it would follow from

$$\mu(\mathbf{c}) = \frac{\sum_{i=k+1}^{k+l} a_i}{l} > \mu(\mathbf{a}), \qquad \frac{\sum_{i=1}^{k} a_i - 1}{k} = \mu(\mathbf{a})$$

that

$$\frac{\sum_{i=1}^{k+l} a_i - 1}{k+l} > \mu(\mathbf{a}),$$

which is impossible as $\mathbf{a}$ is semistable. $\square$

2.3.3 Classification of semistable sheaves

We know how to classify the (semi)stable chains and cycles, so the classification of (semi)stable sheaves will be complete if we will prove that it holds the converse of Corollaries 2.3.11 and 2.3.13. We do this in four steps.

Lemma 2.3.29. *The sheaf $\mathcal{B}(\mathbf{a})$ is stable if and only if the cycle $\mathbf{a}$ is stable. In this case degree and rank are coprime.*

Proof. The "only if" part is already proved. Let $\mathbf{a}$ be a stable cycle of type (r, d). We know that necessarily r and d are coprime and $\mathbf{a}$ is the unique stable cycle of type (r, d). There exist stable locally free sheaves of type (r, d) (see e.g. [7]). Let $\mathcal{B}(\mathbf{b}, m, \lambda)$ be any of them. Then $m = 1$ and $\mathbf{b}$ is stable of type (r, d), hence $\mathbf{b} = \mathbf{a}$. It follows that $\mathcal{B}(\mathbf{a})$ is stable. $\square$

Lemma 2.3.30. *The sheaf $\mathcal{S}(\mathbf{a})$ ($\mathbf{a} = (a_1, \dots, a_r)$) is stable if and only if the chain $\mathbf{a}' = (a_1 + 1, a_2, \dots, a_{r-1}, a_r + 1)$ is stable. In this case degree and rank are coprime.*

Proof. The "only if" part is already proved. Let $\mathbf{a}'$ be a stable chain of type (r, d). Then r and d are coprime and $\mathbf{a}'$ is the unique stable chain of type (r, d). Let $M_C(r, d)$ denote the moduli space of stable sheaves of type (r, d) over C. The subspace of $M_C(r, d)$ consisting of the locally free sheaves $\mathcal{B}(\mathbf{b}, 1, \lambda)$ (where $\mathbf{b}$ is a unique stable cycle of type (r, d)) is isomorphic to k^*. It follows from the projectivity of $M_C(r, d)$ that it cannot coincide with k^* and therefore it contains some $\mathcal{S}(\mathbf{c})$, so that the corresponding chain $\mathbf{c}'$ of type (r, d) is stable and we deduce from the uniqueness of stable chains of type (r, d) that $\mathbf{c}' = \mathbf{a}'$ hence $\mathbf{c} = \mathbf{a}$ and $\mathcal{S}(\mathbf{a})$ is stable. $\square$

Lemma 2.3.31. *The sheaf $\mathcal{S}(\mathbf{a})$ ($\mathbf{a} = (a_1, \dots, a_r)$) is semistable if and only if the chain $\mathbf{a}' = (a_1 + 1, a_2, \dots, a_{r-1}, a_r + 1)$ is semistable.*

Proof. The "only if" part is already proved. Conversely, if the chain $\mathbf{a}'$ is stable, then we are done. So, let us assume that $\mathbf{a}'$ is semistable but not stable. Then it contains an extreme destabilizing subchain, which without lost of generality we will assume to be of the form $(a_1 + 1, a_2, \dots, a_k)$. By Lemma 2.3.28, we know that the chains $(a_1 + 1, a_2, \dots, a_k)$ and $(a_{k+1} +$

$1, a_{k+2}, \ldots, a_{r-1}, a_r+1)$ are semistable with the same slope $\mu(\mathbf{a}')$, so by induction on rank we deduce that $\mathcal{S}((a_1, a_2, \ldots, a_{k-1}, a_k - 1))$ and $\mathcal{S}((a_{k+1}, a_{k+2}, \ldots, a_{r-1}, a_r))$ are semistable with the slope $\mu(\mathbf{a}')$. Now, it follows from the exact sequence of Proposition 2.3.12 that $\mathcal{S}(\mathbf{a})$ is also semistable. □

Lemma 2.3.32. *The sheaf $\mathcal{B}(\mathbf{a})$ is semistable if and only if the cycle $\mathbf{a}$ is semistable.*

Proof. The "only if" part is already proved. Conversely, if the cycle $\mathbf{a}$ is stable, then we are done. So, let us assume that $\mathbf{a}$ is semistable but not stable. Then it contains a destabilizing subchain which, without lost of generality, we will assume to be of the form $(a_1, \ldots, a_k)$. By Lemma 2.3.28′, we know that the chains $(a_1, a_2, \ldots, a_k)$ and $(a_{k+1}+1, a_{k+2}, \ldots, a_{r-1}, a_r+1)$ are semistable with the same slope $\mu(\mathbf{a})$, therefore the sheaves $\mathcal{S}((a_1-1, a_2, \ldots, a_{k-1}, a_k-1))$ and $\mathcal{S}((a_{k+1}, a_{k+2}, \ldots, a_{r-1}, a_r))$ are semistable with the slope $\mu(\mathbf{a}')$. Now, it follows from the exact sequence of Proposition 2.3.10 that $\mathcal{B}(\mathbf{a})$ is also semistable. □

Altogether gives

Proposition 2.3.1. Given an aperiodic cycle $\mathbf{a}$, the sheaf $\mathcal{B}(\mathbf{a})$ is (semi)stable if and only if the cycle $\mathbf{a}$ is (semi)stable. Given a chain $\mathbf{b} = (b_1, \ldots, b_r)$, the sheaf $\mathcal{S}(\mathbf{b})$ is (semi)stable if and only if the chain $(b_1+1, b_2, \ldots, b_{r-1}, b_r+1)$ is (semi)stable.

We formulate now some corollaries.

Corollary 2.3.33. *If* $\gcd(r, d) > 1$ *then there are no stable sheaves in $\mathcal{E}(r, d)$. The number of non-locally free semistable sheaves is finite and non-zero. The family of semistable locally free sheaves in $\mathcal{E}(r, d)$ is parameterized by a finite (non-empty) union of copies of k^*.*

Corollary 2.3.34. *If* $\gcd(r, d) = 1$ *then all semistable sheaves in $\mathcal{E}(r, d)$ are stable. There is precisely one non-locally free semistable sheaf. The family of semistable locally free sheaves in $\mathcal{E}(r, d)$ is parameterized by k^**

2.4 Moduli spaces over a rational curve with many nodes

The aim of this subsection is to compute the Euler number of the moduli space of rank 2 semistable sheaves on a rational singular curve having just nodes as singularities. In the case of odd degree it was computed in [42].

Let C be a rational projective curve over $\mathbb{C}$ only with nodes as singularities, which we denote by $q_1, \ldots, q_n$. In order to determine the Euler number of $M_C(2,d)$ we will study the action of the Jacobian JC on it. One knows that $JC \simeq (\mathbb{C}^*)^n$ (see Remark B.5 and the discussion before it).

Let F be a torsion free sheaf on C, q be one of double points of C and p, p^* be its preimages under the normalization $\pi : \widetilde{C} \longrightarrow C$. We may associate with F a triple (F', V, i) (see Appendix B). One has $V_q = F(q)$ (by definition $F(q) = F_q/m_qF_q$, where $m_q \subset \mathcal{O}_q$ is a maximal ideal) and $i_q = (i_p, i_p^*)^t : V_q \hookrightarrow F'(p) \oplus F'(p^*)$ is s.t. $i_p : V_q \longrightarrow F'(p)$ and $i_{p^*} : V_q \longrightarrow F'(p^*)$ are surjective. The sheaf F is locally free at q if and only if $\dim F(q) = \operatorname{rk} F$.

Definition 2.4.1. Let C be a nodal (integral) curve and F be a torsion free sheaf on it. We define an nlf-index (a non locally free index) of F at q by $i(F,q) = \dim F(q) - \operatorname{rk} F$. Clearly $0 \le i(F,q) \le \operatorname{rk} F$ and F is locally free at q if and only if $i(F,q) = 0$.

Lemma 2.4.2. *Let $\pi : C' \longrightarrow C$ be a normalization of C at node q. Let F be a torsion free sheaf over C and F' be a torsion free part of π^*F. Then* $\deg F = \deg F' + i(F,q)$.

Proof. One see that $\chi(F') - \chi(F) = 2\operatorname{rk}(F) - \dim F(q)$ (see the cartesian diagram in Lemma B.1). Therefore

$$\begin{aligned}\deg F' - \deg F &= \chi(F') - \chi(F) - \operatorname{rk} F(\chi(\mathcal{O}_{C'}) - \chi(\mathcal{O}_C)) \\ &= \chi(F') - \chi(F) - \operatorname{rk} F = \operatorname{rk}(F) - \dim F(q).\end{aligned}$$

□

We are interested only in sheaves with $\operatorname{rk} F = 2$ and for them one has $0 \le i(F,q) \le 2$. By Lemma D.9, we may study just those sheaves of the moduli space which are fixed under the JC-action.

Lemma 2.4.3. *A torsion free sheaf F which is locally free at some double point q (i.e. $i(F,q) = 0$) is not fixed under the action of JC.*

It follows that we should study only the sheaves with $1 \le i(F,q) \le 2$ for any double point q . The main ingredient of this study is the following Proposition (cf. [42])

Proposition 2.4.4. *Let F be a torsion free sheaf of rank 2. If there are at least two double points q_1, q_2 with $i(F,q_1) = i(F,q_2) = 1$, then F is not fixed under the action of JC.*

Proof. First of all, we may suppose that C has just two double points q_1 and q_2. Indeed, if $C' \longrightarrow C$ is a normalization of C outside q_1 and q_2, then the canonical map $JC \longrightarrow JC'$ is surjective (see Appendix B) so, if the group JC acts trivially on the sheaf F, then JC' acts trivially on the induced sheaf F'. So, it is enough to prove the proposition for C', which has just two nodes. We will need an auxiliary

Lemma 2.4.5. *Let $\pi : C' \longrightarrow C$ be a normalization outside a double point q. If $i(F, q) = 1$ and F is JC-fixed then a torsion free part F' of $\pi^*(F)$ is indecomposable.*

Proof. The map $\pi^* : JC \longrightarrow JC'$ is surjective. Therefore if JC acts trivially on F , then JC' acts trivially on F'. Suppose that there exists a decomposition $F' = F_1 \oplus F_2$. As $i(F', q) = 1$, a nlf-index of one of the summands, say F_1, equals 0 and it follows that F_1 is locally free at q. We deduce that the action of JC' on F_1 (and therefore on F') is nontrivial, hence a contradiction. □

Let $\widetilde{\pi} : \widetilde{C} \longrightarrow C$ be a (full) normalization, so that $\widetilde{C}$ is a rational curve. Let $(\widetilde{F}, V, i)$ be a triple corresponding to the sheaf F. We may decompose $\widetilde{F} = \mathcal{O}(n_1) \oplus \mathcal{O}(n_2)$ with $n_1 \leq n_2$. In order to classify the triples with a given sheaf $\widetilde{F}$ we should describe the automorphisms of $\widetilde{F}$. We will distinguish the cases $n_1 = n_2$ and $n_1 < n_2$.

case $n_1 = n_2$:

Let us suppose that F is fixed under JC-action. Also, we may suppose without loss of generality that $n_1 = n_2 = 0$. Let $\pi : C' \longrightarrow C$ be a normalization in the point q_1 (i.e. outside the point q_2) and F' be a torsion free part of $\pi^*(F)$. The point $\pi^{-1}(q_2)$ will be denoted by q_2. Then the degree $\deg F' = \deg \widetilde{F} + i(F', q_2) = n_1 + n_2 + 1 = 1$. Moreover, as $i(F', q_2) = 1$ it follows from Lemma 2.4.5 that F' is indecomposable. In the notation of Section 2.3 it is isomorphic to $\mathcal{S}((0, 0))$ and we know that this sheaf is stable and in particular simple. As it was explained in Appendix B the isomorphism classes of the triples (F', V, i) (with a fixed F') may be identified with the subspaces $V \subset F'(p_1) \oplus F'(p_1^*)$ such that the induced maps $i_1 : V \longrightarrow F'(p_1)$ and $i_1^* : V \longrightarrow F'(p_1^*)$ are surjective. The dimension $\dim V = \operatorname{rk} F + i(F, q_1) = 3$. The action of $\ker(JC \longrightarrow JC') \simeq \mathbb{C}^*$ on the isomorphism class of (F', V, i), where $i = (i_1, i_1^*)^t : V \hookrightarrow F'(p_1) \oplus F'(p_1^*)$ is given by changing $i = (i_1, i_1^*)^t$ to $i' = (i_1, \lambda i_1^*)^t$, where $\lambda \in \mathbb{C}^*$. By our assumption (F', V, i) and (F', V, i') are isomorphic. This means that $\operatorname{im} i = \operatorname{im} i'$, hence the sum of these images is a 3-dimensional subspace of $F'(p_1) \oplus F'(p_1^*)$. For $\lambda \neq 1$ the sum of $\operatorname{im} i$ and $\operatorname{im} i'$ contains the images of $(i_1, 0)^t$ and $(0, i_1^*)^t$ which generate the whole space $F'(p_1) \oplus F'(p_1^*)$. Hence $\operatorname{im} i + \operatorname{im} i'$ is 4-dimensional giving a contradiction.

case $n_1 < n_2$**:**
We again suppose that F is fixed under JC-action. As before, we consider the triple $(\widetilde{F}, V, i) \in \mathrm{Coh}^{\widetilde{\pi}}_{tf}\, C$ corresponding to F. As it was explained in Appendix B, we may write the matrices of i_1, i_1^*, i_2, i_2^* after choosing the bases of the corresponding fibers. We choose everything in such a way that the first rows of the matrices correspond to $\mathcal{O}(n_1)$. Let us now analyze the automorphisms of $\widetilde{F} \simeq \mathcal{O}(n_1) \oplus \mathcal{O}(n_2)$. They are given by the matrices $\left(\begin{smallmatrix} a_1 & 0 \\ \varphi & a_2 \end{smallmatrix}\right)$, where a_1, a_2 are two nonzero scalars corresponding to automorphisms of $\mathcal{O}(n_1)$ and $\mathcal{O}(n_2)$ and $\varphi : \mathcal{O}(n_1) \longrightarrow \mathcal{O}(n_2)$. Scalars a_1, a_2 will induce the simultaneous multiplications of the rows of all 4 matrices by these scalars. Morphism φ will induce the operation of adding the first row multiplied by a certain scalar to the second row for any of 4 matrices (possibly with different scalars for different matrices). We can always choose φ in such a way that it will be 0 in one prescribed point among p_1, p_1^*, p_2, p_2^* and nonzero in other 3 points, so we can always make any of 4 scalars to be equal 0. The automorphisms of V will induce arbitrary simultaneous operations on the columns of i_1 and i_1^* and simultaneous operations on the columns of i_2 and i_2^*.

Doing all these operations we may reduce the matrix (i_1, i_1^*) to one of the following forms

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & a & 0 \end{array}\right), \tag{2.1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right), \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & a & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right), \tag{2.2}$$

where a is some nonzero scalar. We are going to show that under our assumption (that F is JC-fixed) the second and fourth cases cannot occur. Let $\pi : C' \longrightarrow C$ be a normalization outside the point q_1 and F' be a torsion free part of $\pi^*(F)$. Then it follows from Lemma 2.4.5 that F' is indecomposable. Considering the canonical map $\widetilde{C} \longrightarrow C'$ (normalization in q_1), we get a triple $(\widetilde{F}, V_{q_1}, (i_1, i_1^*)^t)$ corresponding to the sheaf F'. Assuming that (i_1, i_1^*) has a form of 2-nd or 4-th cases we deduce that the triple $(\widetilde{F}, V_{q_1}, (i_1, i_1^*)^t)$ is decomposable and hence F' is decomposable, a contradiction. Note that in the third case, after certain column permutations (simultaneous for i_1 and i_1^*) one can get the interchanged pair of matrices (i_1^*, i_1) of the first case. It will be therefore enough for our purposes to consider just the first case. The same considerations hold also for the matrix (i_2, i_2^*).

As it was explained in Appendix B, the action of $(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2 \simeq JC$ on the triple $(\widetilde{F}, V, i)$ is given by changing i to i', where $i_1' = i_1$, $i_1'^* = \lambda_1 i_1^*$

$i'_2 = i_2$, $i'^*_2 = \lambda_2 i^*_2$. Under assumption that F is JC-fixed, we should have $(\widetilde{F}, V, i) \simeq (\widetilde{F}, V, i')$, i.e. the matrix i can be reduced to the matrix i' by described earlier operations. In order to show that this is false we will even enlarge the set of possible operations. Namely, for any of the matrices i_1, i^*_1, i_2, i^*_2 we allow an addition of the first row multiplied by an arbitrary scalar to the second row, with no any operations on the other 3 matrices. Assuming that (i_1, i^*_1) has a form of the first case, we may reduce the matrix (i_1, i^*_1, i_2, i^*_2) to one of the following forms

$$\left(\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & b & 0 & 0 \end{array}\right), \quad \left(\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & b & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right),$$

where b is some nonzero scalar (we use the fact that the matrix (i_2, i^*_2) can be reduced to the 1-st or 3-rd cases of (2.1)). The matrices corresponding to different b cannot be reduced to each other by the allowed operations and therefore these matrices give non isomorphic triples $(\widetilde{F}, V, i)$. Now we note that the action of $(1, \lambda) \in (\mathbb{C}^*)^2 \simeq JC$ with $\lambda \neq 1$ on the triple $(\widetilde{F}, V, i)$ will necessarily change the scalar b and therefore, the action is non trivial. $\square$

Now we are able to prove the main fact of this subsection

Proposition 2.4.6. *Let C be a rational curve with n nodes. Then we have*

$$e(M_C(2,d)) = \begin{cases} 1, & \text{if } d \text{ is even;} \\ n, & \text{if } d \text{ is odd.} \end{cases}$$

The proposition is obvious for $n = 0$. Let us assume now that C has one node. In this case we have a complete description of semistable sheaves and we can prove the following

Lemma 2.4.7. *Let C be a rational curve with one node. Then the Euler number of $M_C(r, d)$ equals 1 for any degree d and (positive) rank r.*

Proof. Let $h = \gcd(r, d)$, $r' = r/h$, $d' = d/h$. Then we know that $M_C(r, d) = S^h M_C(r', d')$. Also one knows that $e(M_C(r', d')) = 1$. $\square$

We turn now to the general case. As we have shown, in order to find the Euler number of $M_C(2, d)$ we should study just the subspace of $M_C(2, d)$ consisting of such sheaves F that for all (except possibly one) nodes q_i one has $i(F, q_i) = 2$ and for an exceptional node (if any) one has $i(F, q_i) = 1$.

Lemma 2.4.8. *Let $\pi : C' \longrightarrow C$ be a normalization of C at a double point q. Then the map π_* induces an isomorphism between the moduli space $M_{C'}(r, d)$ and the subspace of the moduli space $M_C(r, d + r)$ consisting of such sheaves F that $i(F, q) = r$.*

Let now $M_i(2,d)$ denote the subspace of $M_C(2,d)$ consisting of sheaves having a nlf-index 1 in point q_i and equal 2 in all the other nodes. Let $\pi_i : C_i \longrightarrow C$ denote a normalization of C outside the point q_i. Denote the subspace of $M_{C_i}(2,d)$ consisting of sheaves having a nlf-index 1 in point q_i by $N_i(2,d)$. It follows from the lemma that there is an isomorphism between $M_i(2,d)$ and $N_i(2,d-2(n-1))$. Therefore

$$e(M_i(2,d)) = e(N_i(2,d)) = \begin{cases} 0, & \text{if } d \text{ is even;} \\ 1, & \text{if } d \text{ is odd.} \end{cases}$$

The last equality follows from the results of Section 2.3.3. Indeed, given a rational curve B with one node q, there is just one stable sheaf F in $M_B(2,1)$ with $i(F,q) = 1$ and there are no stable sheaves in $M_B(2,0)$, so any sheaf $F \in M_B(2,0)$ is decomposable and therefore if $i(F,q) = 1$ then F is not fixed by JB (cf. Lemma 2.4.5). We deduce that the Euler number of the subspace of $M_C(2,d)$ consisting of sheaves having a nlf-index 1 in precisely one node, equals 0 if d is even and equals n if d is odd.

Let now $\pi : \widetilde{C} \longrightarrow C$ be a normalization of C, so that $\widetilde{C} \simeq \mathbb{P}^1$. Then, as before, we can show that the Euler number of the subspace of $M(2,d)$ consisting of sheaves having a nlf-index 2 in all nodes equals the Euler number of $M_{\widetilde{C}}(2,d)$. The last one equals 1 if d is even and equals 0 if d is odd. We deduce that $e(M_C(2,d))$ equals n if d is even and equals 1 if d is odd.

2.5 One special case

In this subsection we will consider the moduli space of semistable sheaves over a curve consisting of two (possibly singular) components intersecting in two nodes. This very specific situation will be needed later. As we have seen in Corollary 2.1.3, the notion of semistability depends on the degrees of components of C. One natural choice, is when all the components have the same degree. So, let us assume from now on that a polarization of a reduced curve C is chosen in such a way that all the components $C_1, \ldots, C_s$ of C have the same degree d.

Lemma 2.5.1. *Let F be a sheaf on C having ranks $r_1, \ldots r_s$ on the components. Then*

$$P(F,n) = (\sum r_i)dn + \chi(F).$$

Remark 2.5.2. It follows that the notion of semistability with respect to a chosen polarization coincides with the notion of semistability with respect to a slope function $\mu(F) := \chi(F)/(\sum r_i)$. The Hilbert polynomial of a sheaf F

having a constant rank r and Euler number χ equals $P(F,n) = rsdn+\chi$. We will denote the moduli space $M_C(P)$ by $M_C(r,\chi)$ in this subsection (earlier we wrote a degree as a second parameter and it is a more standard notation).

Assume now that $s = 2$ and that the components C_1 and C_2 intersect transversally in two nodes q_1 and q_2. We will study the moduli space $M_C(1,\chi)$. Clearly, there is an isomorphism $M_C(1,\chi) \simeq M_C(1,\chi+2)$, so one should distinguish just the parity of χ.

Proposition 2.5.3. *Let C be as before. Then the Euler number of $M_C(1,\chi)$ equals $e(\overline{J}C_1)\cdot e(\overline{J}C_2)$ if χ is even and equals $2e(\overline{J}C_1)\cdot e(\overline{J}C_2)$ if χ is odd.*

Proof. Given a torsion free sheaf F of constant rank on C we define as before a non locally free index $i(F,q_1) = \dim F(q_1) - \operatorname{rk} F$. Then $0 \leq i(F,q_1) \leq 1$ and F is locally free at q_1 if and only if $i(F,q_1) = 0$. Analogously for q_2. Given a sheaf F from $M_C(1,\chi)$, we distinguish four cases

1. $i(F,q_1) = i(F,q_2) = 1$. This implies that $F = F_1 \oplus F_2$, where F_j is a sheaf on C_j. It follows that $\chi(F_1) = \chi(F_2)$, so this case is possible only if χ is even. The corresponding part of $M_C(1,\chi)$ is isomorphic then to $\overline{J}C_1 \times \overline{J}C_2$. The Euler number of this part equals $e(\overline{J}C_1)\cdot e(\overline{J}C_2)$.

2. $i(F,q_1) = 0$ and $i(F,q_2) = 1$. Let us denote the corresponding torsion free components of F on C_1 and C_2 by F_1 and F_2 respectively. Then $\chi(F) = \chi(F_1)+\chi(F_2)-1$. The sheaf F must be necessarily stable and we get from the existence of an exact sequence

$$0 \longrightarrow F_1(-q_1) \longrightarrow F \longrightarrow F_2 \longrightarrow 0$$

   that $\chi(F_1) - 1 < \chi(F_2)$, i.e. $\chi(F_1) \leq \chi(F_2)$. Analogously $\chi(F_2) \leq \chi(F_1)$ and therefore $\chi(F_1) = \chi(F_2)$. It follows that χ must be odd. Conversely, given torsion free sheaves F_1 and F_2 of rank 1 on C_1 and C_2, we can glue them together in a unique way, to get a stable sheaf F with $i(F,q_1) = 0$, $i(F,q_2) = 1$. The corresponding part of $M_C(1,\chi)$ is isomorphic then to $\overline{J}C_1 \times \overline{J}C_2$. The Euler number of this part equals $e(\overline{J}C_1)\cdot e(\overline{J}C_2)$.

3. $i(F,q_1) = 1$, $i(F,q_2) = 0$. This case can be treated analogously to the previous one.

4. $i(F,q_1) = 0$, $i(F,q_2) = 0$. We again denote the corresponding torsion free components of F on C_1 and C_2 by F_1 and F_2. Then sheaf F must

be again stable and it holds $\chi(F) = \chi(F_1) + \chi(F_2) - 2$. We deduce from the existence of an exact sequence

$$0 \longrightarrow F_1(-q_1 - q_2) \longrightarrow F \longrightarrow F_2 \longrightarrow 0$$

that $\chi(F_1) - 2 < \chi(F_2)$, i.e. $\chi(F_1) \leq \chi(F_2) + 1$. Analogously $\chi(F_2) \leq \chi(F_1) + 1$. If χ is even then it holds $\chi(F_1) = \chi(F_2)$ and if χ is odd then it holds $\chi(F_1) = \chi(F_2) \pm 1$. Conversely, given such sheaves F_1 and F_2 we can glue them together (up to a scalar from $\mathbb{C}^*$) to get a stable sheaf F. The choice of this scalar defines a free action of $\mathbb{C}^*$ on the corresponding part of $M_C(1, \chi)$ and therefore the contribution of this part to the Euler number of $M_C(1, \chi)$ equals 0.

Corollary 2.5.4. *Let C be as before. Then the Euler number of $M^s_C(1, \chi)$ equals 0 if χ is even and equals $2e(\overline{J}C_1) \cdot e(\overline{J}C_2)$ if χ is odd.*

$\square$

Chapter 3

Moduli spaces of sheaves over double curves

In the last section we have studied the moduli spaces of rank 2 semistable sheaves on the reduced curves. In this section we will occupy ourselves with the nonreduced curves. More specifically, we will study the moduli spaces of rank 2 semistable sheaves on the double curves, which we define below.

3.1 Sheaves over double curves

Let C be an irreducible (but possibly nonreduced) projective curve, C_{red} be the associated reduced curve and $I \subset \mathcal{O}_C$ be the ideal sheaf of C_{red}. There is an integer N such that $I^N = 0$.

Lemma 3.1.1. *For any coherent sheaf F on C define its rank and degree by*

$$\operatorname{rk} F := \sum_k \operatorname{rk}_{C_{\mathrm{red}}}(I^k F / I^{k+1} F), \quad \deg F := \chi(F) - \operatorname{rk} F \chi(\mathcal{O}_{C_{\mathrm{red}}}).$$

Then both these invariants are compatible with the corresponding invariants for sheaves on C_{red} and moreover, they are additive with respect to short exact sequences of sheaves on C. If a polarization of C is given, then it holds

$$P(F, n) = \operatorname{rk} F \deg C_{\mathrm{red}} \cdot n + \chi(F).$$

Proof. Assertions about additivity follow easily from the last assertion. Given any coherent sheaf F on C, we consider a decreasing sequence of sheaves $F \supset IF \supset I^2 F \supset \ldots \supset I^N F = 0$. Then any sheaf $I^k F / I^{k+1} F$ can be considered as an $\mathcal{O}_{C_{\mathrm{red}}}$ module and we get (see Corollary 2.1.3)

$$P(F,n) = \sum_k P(I^kF/I^{k+1}F,n)$$
$$= \sum_k (\operatorname{rk}(I^kF/I^{k+1}F)\deg C_{\mathrm{red}}\cdot n + \chi(I^kF/I^{k+1}F)) = \operatorname{rk}F\deg C_{\mathrm{red}}\cdot n + \chi(F).$$

□

Remark 3.1.2. We see that also for a nonreduced curve the notion of semistability with respect to any polarization coincides with the notion of semistability with respect to the slope function $\mu(F) = \deg F/\operatorname{rk}F$.

Definition 3.1.3. Let C be an irreducible curve and $I \subset \mathcal{O}_C$ be an ideal sheaf of C_{red}. We say that C is a double curve if $I^2 = 0$ and I is an invertible sheaf considered as an $\mathcal{O}_{C_{\mathrm{red}}}$-module.

The aim of the chapter is to investigate certain moduli spaces on the double curves. From now on the curve C will be assumed to be a double curve. We are especially interested in the situation when it can be embedded into a smooth projective surface.

Lemma 3.1.4. *Let $C \subset X$ be a double curve embedded into a smooth projective surface. Let $J \subset \mathcal{O}_X$ be the ideal sheaf of C_{red} as a subscheme of X. Then the ideal sheaf of C equals J^2.*

Proof. Let $J_1 \subset \mathcal{O}_X$ be the ideal sheaf of C. Then $J^2 \subset J_1 \subset J$ and J/J_1 is a locally free sheaf of rank 1 over $\mathcal{O}_X/J$. However J/J^2 is also a locally free sheaf of rank 1 over $\mathcal{O}_X/J$ (because J is an invertible sheaf on X). This implies that an epimorphism $J/J^2 \longrightarrow J/J_1$ is an isomorphism, hence $J_1 = J^2$. □

Let F be a pure coherent sheaf of rank 2 on C that is not an $\mathcal{O}_{C_{\mathrm{red}}}$-module (i.e. $IF \neq 0$). Then F/IF can be considered as an $\mathcal{O}_{C_{\mathrm{red}}}$-module and its torsion and torsion free parts constitute an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow F/IF \longrightarrow E \longrightarrow 0.$$

Lemma 3.1.5. *Let F be a pure coherent sheaf of rank 2 on C, that is not an $\mathcal{O}_{C_{\mathrm{red}}}$-module. Then there is a canonical exact sequence (with the above notations)*

$$0 \longrightarrow E \otimes I \longrightarrow F \longrightarrow F/IF \longrightarrow 0.$$

Proof. The condition $I^2 = 0$ implies that the multiplication map $F \otimes I \longrightarrow F$ factorizes as $F \otimes I \longrightarrow F/IF \otimes I \longrightarrow F$. The composition

$$\mathcal{T} \otimes I \longrightarrow F/IF \otimes I \longrightarrow F$$

is zero because F is pure-dimensional. This implies that $F/IF \otimes I \longrightarrow F$ factorizes as $F/IF \otimes I \longrightarrow E \otimes I \longrightarrow F$. The exactness of

$$E \otimes I \longrightarrow F \longrightarrow F/IF \longrightarrow 0$$

is now obvious. Let us prove that the first map is a monomorphism. The sheaves E and F/IF have the same rank, hence we get from the last exact sequence that it is either 1 or 2. But if $\operatorname{rk} F/IF = 2 = \operatorname{rk} F$ then $\operatorname{rk} IF = 0$ and it follows from the purity of F that $IF = 0$ contradicting to the condition that F is not an $\mathcal{O}_{C_{\mathrm{red}}}$-module. It follows that $\operatorname{rk} E = \operatorname{rk} F/IF = 1$ and that the map $E \otimes I \longrightarrow F$ has image of rank 1. Noting that $E \otimes I$ is torsion-free of rank 1 as E is torsion free of rank 1 and I is invertible, we deduce that the kernel must be zero. □

The condition on the degree in the next lemma is motivated by the problem we will get in Subsection 4.3.3.

Lemma 3.1.6. *Let F be a stable sheaf of rank 2 and of even degree on C that is not an $\mathcal{O}_{C_{\mathrm{red}}}$-module. Also assume that $\deg I = -2$. Then F/IF is torsion free, i.e. $E \simeq F/IF$ and there is a canonical exact sequence*

$$0 \longrightarrow E \otimes I \longrightarrow F \longrightarrow E \longrightarrow 0.$$

Proof. We know that

$$\chi(F) = \chi(E \otimes I) + \chi(F/IF) = (\chi(E) + \deg I) + (\chi(E) + \chi(\mathcal{T})).$$

Then we get from the epimorphism $F \longrightarrow F/IF \longrightarrow E$ that

$$\frac{2\chi(E) + \deg I + \chi(\mathcal{T})}{2} < \chi(E),$$

that is $\deg I + \chi(\mathcal{T}) < 0$. The condition that $\deg F$ is even implies that also $\chi(F)$ is even and therefore also $\deg I + \chi(\mathcal{T})$ is even. It follows that $\deg I + \chi(\mathcal{T}) \leq -2$, hence $\chi(\mathcal{T}) \leq 0$. But $\chi(\mathcal{T})$ is the length of $\mathcal{T}$. Therefore $\mathcal{T} = 0$. □

We can prove a "converse" statement:

Lemma 3.1.7. *Let E_1, E_2 be torsion free sheaves on C_{red} of rank 1, such that $\mu(E_2) < \mu(E_1)$. Let*

$$0 \longrightarrow E_2 \longrightarrow F \longrightarrow E_1 \longrightarrow 0$$

be an exact sequence of coherent sheaves on C such that F is not an $\mathcal{O}_{C_{\mathrm{red}}}$-module. Then F is stable.

Proof. First of all, the module F is pure-dimensional. Indeed, let $G \subset F$ be a submodule having a zero-dimensional support. Then $E_2 \cap G \subset E_2$ has a zero-dimensional support and as E_2 is torsion-free, we deduce that $E_2 \cap G = 0$. It follows that G can be embedded in E_1 and this again implies that $G = 0$.

To prove the stability, consider any proper nonzero submodule $G \subset F$. Assume that $E_2 \cap G = 0$. Then G can be considered as a submodule of E_1, hence G is torsion free of rank 1. Consider the exact sequence

$$0 \longrightarrow E_2 \oplus G \longrightarrow F \longrightarrow E_1/G \longrightarrow 0$$

over X. We get $\operatorname{rk} E_1/G = 0$ and therefore $\operatorname{Hom}((E_1/G) \otimes_{\mathcal{O}_C} I, E_2 \oplus G) = 0$. This implies (see Corollary 3.2.2) that the exact sequence comes actually from an exact sequence over C_{red}, contradicting the condition that F is not an $\mathcal{O}_{C_{\mathrm{red}}}$-module. So, we assume that $E_2 \cap G \neq 0$, and therefore $\operatorname{rk}(E_2 \cap G) = 1$. Assuming that $E_2 \cap G \neq G$, we get $0 \neq G/(E_2 \cap G) \simeq (G + E_2)/E_2 \subset E_1$, therefore $\operatorname{rk} G/(E_2 \cap G) = 1$. We deduce that $\operatorname{rk} G = 2$ and $\mu(G) = \deg G/2 < \deg F/2 = \mu(F)$. Assuming that $E_2 \cap G = G$, we get $G \subset E_2$ and therefore $\operatorname{rk} G = 1$ and $\mu(G) = \deg G \leq \deg E_2 < (\deg E_1 + \deg E_2)/2 = \mu(F)$. □

Under the assumption that $\deg I = -2$, we can prove the semistability for an arbitrary F:

Lemma 3.1.8. *Let E be a torsion free sheaf on C_{red} of rank 1 and assume that $\deg I = -2$. Then, given any non-split extension*

$$0 \longrightarrow E \otimes I \longrightarrow F \longrightarrow E \longrightarrow 0$$

over C, the sheaf F is semistable.

Proof. One proves that F is pure as before. To prove the stability, consider any proper nonzero submodule $G \subset F$. Assume that $(E \otimes I) \cap G = 0$. Then G can be considered as a proper submodule of E (if $G = E$ then the sequence splits), hence

$$\mu(G) = \deg G \leq \deg E - 1 = \frac{2 \deg E + \deg I}{2} = \mu(F).$$

If $(E \otimes I) \cap G \neq 0$ we can just repeat the arguments of the previous lemma. □

We are going to prove that whenever we deal with curves embedded into a smooth projective surface, the sheaf E as in Lemma 3.1.6 must be locally free. To do this we will compare the extensions groups of sheaves on C and on the ambient space.

3.2 Extensions in ambient space

Proposition 3.2.1 (Ambient space spectral sequence)**.** *Let X be a projective scheme and let $i : Y \hookrightarrow X$ be a closed imbedding. Then for arbitrary quasi-coherent sheaves E_1 on X and E_2 on Y there exists a spectral sequence*

$$\mathrm{Ext}^p_{\mathcal{O}_Y}(L^q i^* E_1, E_2) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(E_1, i_* E_2).$$

Proof. Consider two functors $F = i^* : (\mathrm{QCoh}\, X)^{\mathrm{op}} \longrightarrow (\mathrm{QCoh}\, Y)^{\mathrm{op}}$ (in this notation F is left exact) and $G = \mathrm{Hom}_{\mathcal{O}_Y}(-, E_2) : (\mathrm{QCoh}\, Y)^{\mathrm{op}} \longrightarrow \mathbf{Ab}$. It is well-known that there are enough F-acyclic and G-acyclic objects. We want to construct the Grothendieck spectral sequence

$$(R^p G)(R^q F)(E_1) \Rightarrow (R^{p+q} GF)(E_1)$$

using Proposition A.9. We define a subcategory $\mathcal{I} \subset \mathrm{QCoh}$ to be the category of all locally free sheaves I (possibly of infinite rank) on X such that FI is G-acyclic. It is easy to see that $\mathcal{I}$ is closed under extensions and for any exact pair $I_1 \longrightarrow I_2 \longrightarrow I_3$ of quasi-coherent sheaves on X if I_2, I_3 are in $\mathcal{I}$ then so is I_1 (if we consider $\mathcal{I}^{\mathrm{op}}$ as embedded into $(\mathrm{QCoh}\, X)^{\mathrm{op}}$ then the last condition means that the embedding reflects monomorphisms). To apply Proposition A.9 we just have to show that any object in $\mathrm{QCoh}\, X$ is a quotient of an object in $\mathcal{I}$. One knows that any quasi-coherent sheaf on X is a union of its coherent subsheaves (see e.g. [22, Theorem 9.4.7]). If we could represent any of these coherent sheaves as a quotient of an object in $\mathcal{I}$, then the whole quasi-coherent sheaf would be also represented as a quotient of a sheaf in $\mathcal{I}$, because $\mathcal{I}$ is closed under (arbitrary) coproducts (this follows from the facts that coproducts of locally free sheaves are locally free and coproducts of G-acyclic sheaves are G-acyclic). So, we consider a coherent sheaf E on X. Choosing a polarization on X we can find $n \geq 0$ such that $E(n)$ is globally generated and $H^i(Y, E_2(n)) = 0$, $i \geq 1$. It follows that there exists a surjective morphism $\mathcal{O}_X^{\oplus r} \longrightarrow E(n)$ and therefore a surjective morphism $I := \mathcal{O}_X(-n)^{\oplus r} \longrightarrow E$. Moreover, it follows from the condition $H^i(Y, E_2(n)) = 0$ that $\mathrm{Ext}^i_{\mathcal{O}_Y}(\mathcal{O}_Y(-n), E_2) = 0$ and therefore $FI = i^* I = \mathcal{O}_Y(-n)$ is G-acyclic, hence $I \in \mathcal{I}$. $\square$

Corollary 3.2.2 (Ambient space exact sequence). *Let X be a projective scheme, let $i : Y \hookrightarrow X$ be a closed embedding and let $I \subset \mathcal{O}_X$ be the ideal sheaf of Y. Then for any quasi-coherent sheaves E_1, E_2 on Y there is a canonical exact sequence*

$$\begin{aligned} 0 \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_Y}(E_1, E_2) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(E_1, E_2) \\ \xrightarrow{\partial} \operatorname{Hom}_{\mathcal{O}_Y}(E_1 \otimes_{\mathcal{O}_X} I, E_2) \longrightarrow \operatorname{Ext}^2_{\mathcal{O}_Y}(E_1, E_2) \longrightarrow \operatorname{Ext}^2_{\mathcal{O}_X}(E_1, E_2). \end{aligned}$$

Proof. We get from Proposition 3.2.1 the following exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_Y}(E_1, E_2) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(E_1, E_2) \\ \xrightarrow{\partial} \operatorname{Hom}_{\mathcal{O}_Y}(L^1 i^* E_1, E_2) \longrightarrow \operatorname{Ext}^2_{\mathcal{O}_Y}(E_1, E_2) \longrightarrow \operatorname{Ext}^2_{\mathcal{O}_X}(E_1, E_2). \end{aligned}$$

So we just have to show that $L^1 i^* E_1 \simeq E_1 \otimes_{\mathcal{O}_X} I$. Applying $(E_1 \otimes -)$ to the short exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

we get the exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_X}(E_1, \mathcal{O}_Y) \longrightarrow E_1 \otimes_{\mathcal{O}_X} I \longrightarrow E_1 \otimes_{\mathcal{O}_X} \mathcal{O}_X \\ \longrightarrow E_1 \otimes_{\mathcal{O}_X} \mathcal{O}_Y \longrightarrow 0. \end{aligned}$$

The map $E_1 \otimes_{\mathcal{O}_X} I \longrightarrow E_1$ is zero because E_1 is an $\mathcal{O}_Y$-module. Therefore $L^1 i^* E_1 \simeq \operatorname{Tor}_1^{\mathcal{O}_X}(E_1, \mathcal{O}_Y) \simeq E_1 \otimes_{\mathcal{O}_X} I$. □

Let us describe the boundary map

$$\partial : \operatorname{Ext}^1_{\mathcal{O}_X}(E_1, E_2) \longrightarrow \operatorname{Hom}_{\mathcal{O}_Y}(L^1 i^* E_1, E_2)$$

in terms of extensions (here E_1 is an $\mathcal{O}_X$-module and E_2 is an $\mathcal{O}_Y$-module). Given an extension

$$\varepsilon : \; 0 \longrightarrow E_2 \longrightarrow F \longrightarrow E_1 \longrightarrow 0$$

over X we consider the following diagram with exact rows and columns

$$\begin{tikzcd} & E_2 \otimes I \arrow[r] \arrow[d, "m_{E_2}"] & F \otimes I \arrow[r] \arrow[d, "m_F"] & E_1 \otimes I \arrow[r] \arrow[d, "m_{E_1}"] & 0 \\ 0 \arrow[r] & E_2 \arrow[r] & F \arrow[r] & E_1 \arrow[r] & 0. \end{tikzcd} \tag{3.1}$$

Clearly $\ker m_{E_1} = L^1 i^* E_1$ and $\operatorname{coker} m_{E_2} = E_2$.

Lemma 3.2.3. *The boundary* $\partial(\varepsilon) \in \mathrm{Hom}_{\mathcal{O}_Y}(L^1 i^* E_1, E_2)$ *equals the connecting morphism*

$$\ker m_{E_1} \longrightarrow \operatorname{coker} m_{E_2}$$

of the Snake lemma applied to the above diagram.

Proof. Let $0 \to K \to P \to E_1 \to 0$ be an exact sequence of quasi-coherent sheaves on X, where $P \to E_1$ is a first component of the resolution used to construct the Grothendieck spectral sequence as in Proposition 3.2.1. Then

$$\mathrm{Ext}^1_{\mathcal{O}_X}(E_1, i_* E_2) = \operatorname{coker}(\mathrm{Hom}_{\mathcal{O}_Y}(i^*P, E_2) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(i^*K, E_2))$$

and $L^1 i^* E_1 = \ker(i^*K \to i^*P)$. The lengthy but straightforward analysis shows that the boundary map

$$\partial : \mathrm{Ext}^1_{\mathcal{O}_X}(E_1, i_* E_2) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(L^1 i^* E_1, E_2)$$

is given on the representative $f \in \mathrm{Hom}_{\mathcal{O}_Y}(i^*K, E_2)$ of $\varepsilon \in \mathrm{Ext}^1_{\mathcal{O}_X}(E_1, E_2)$ by the composition

$$L^1 i^* E_1 \hookrightarrow i^*K \xrightarrow{f} E_2.$$

The map $L^1 i^* E_1 \hookrightarrow i^*K$ occurs actually as a connecting morphism of the Snake lemma applied to the diagram

$$\begin{tikzcd} & K \otimes I \arrow[r] \arrow[d, "m_K"] & P \otimes I \arrow[r] \arrow[d, "m_P"] & E_1 \otimes I \arrow[r] \arrow[d, "m_{E_1}"] & 0 \\ 0 \arrow[r] & K \arrow[r] & P \arrow[r] & E_1 \arrow[r] & 0. \end{tikzcd} \tag{3.2}$$

The map $\mathrm{Hom}_{\mathcal{O}_Y}(i^*K, E_2) \to \mathrm{Ext}^1_{\mathcal{O}_X}(E_1, E_2)$ associates with any $f : i^*K \to E_2$ an exact sequence ε on the bottom of the following diagram with a left cocartesian square

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "f"] & P \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E_2 \arrow[r] & F \arrow[r] & E_1 \arrow[r] & 0. \end{tikzcd}$$

The map between these exact sequences induces the map between the diagrams (3.2) and (3.1) and we get the map between the exact sequences of the Snake lemma applied to both diagrams (3.2) and (3.1)

$$\begin{tikzcd} \ker m_P \arrow[r] \arrow[d] & \ker m_{E_1} \arrow[r] \arrow[d, equal] & \operatorname{coker} m_K \arrow[r] \arrow[d] & \operatorname{coker} m_P \arrow[d] \\ \ker m_F \arrow[r] & \ker m_{E_1} \arrow[r] & \operatorname{coker} m_{E_2} \arrow[r] & \operatorname{coker} m_F. \end{tikzcd}$$

We can rewrite the middle square in the form

$$\begin{array}{ccc} L^1 i^* E_1 & \longrightarrow & i^* K \\ \| & & \downarrow f \\ L^1 i^* E_1 & \longrightarrow & E_2. \end{array}$$

This implies that the map $\partial(\varepsilon)$, which equals the composition

$$L^1 i^* E_1 \hookrightarrow i^* K \xrightarrow{f} E_2$$

coincides with a connecting morphism $L^1 i^* E_1 \to E_2$ of the diagram (3.1). $\square$

From now on we will assume that both the sheaves E_1 and E_2 are $\mathcal{O}_Y$-modules.

Lemma 3.2.4. *With the same notations as above, the sequence*

$$\varepsilon: \ 0 \to E_2 \to F \to E_1 \to 0$$

has the property that $F/IF \to E_1$ is an isomorphism (or equivalently $E_2 = IF$) if and only if the boundary $\partial(\varepsilon) \in \operatorname{Hom}(E_1 \otimes I, E_2)$ is surjective. If $E_2 = E_1 \otimes I$ this is equivalent to the requirement that $\partial(\varepsilon) \in \operatorname{End}(E_1 \otimes I)$ is an isomorphism.

Proof. As we have proved, the boundary $\partial(\varepsilon) : E_1 \otimes I \to E_2$ is defined in such a way that the composition

$$F \otimes I \to E_1 \otimes I \to E_2 \to F$$

is given by the multiplication in F. This implies that $E_2 = IF$ if and only $E_1 \otimes I \to E_2$ is surjective. Concerning the last assertion, one knows that an endomorphism of a noetherian module is surjective if and only if it is bijective (see e.g. [6, Lemma 2.2.2]). $\square$

From now on we will assume that $E_1 = E$ and $E_2 = E \otimes_{\mathcal{O}_X} I$ for some sheaf E on Y. We will investigate extensions $\varepsilon \in \operatorname{Ext}^1_{\mathcal{O}_X}(E, E \otimes_{\mathcal{O}_X} I)$ as in the lemma above, i.e. such that $F/IF \to E$ is an isomorphism.

Lemma 3.2.5. *Assume that E is a simple sheaf on $Y \subset X$ (i.e. $\operatorname{End}_{\mathcal{O}_Y}(E) = \mathbb{C}$). Then there is a bijection between the set of isomorphism classes of sheaves F on X occurring in extensions*

$$\varepsilon: \ 0 \to E \otimes I \to F \to E \to 0$$

such that $F/IF \to E$ is an isomorphism and the set of extensions $\varepsilon \in \operatorname{Ext}^1_{\mathcal{O}_X}(E, E \otimes I)$ such that $\partial(\varepsilon) = 1_{E \otimes I}$.

Proof. We construct a map from the second set to the first in an obvious way: with any exact sequence

$$\varepsilon:\ 0 \to E\otimes I \to F \to E \to 0,$$

such that $\partial(\varepsilon) = 1_{E\otimes I}$, we associate the sheaf $F(\varepsilon) := F$. To prove that the map is surjective, consider any exact sequence

$$\varepsilon:\ 0 \to E\otimes I \to F \to E \to 0$$

such that $f := \partial(\varepsilon) \in \mathrm{End}(E\otimes I)$ is an isomorphism. Then $\partial(f_*^{-1}(\varepsilon)) = f^{-1}\partial(\varepsilon) = 1_{E\otimes I}$ and obviously $F(f_*^{-1}(\varepsilon)) \simeq F$, so F is contained in the image of the map. Assume that there are extensions

$$\varepsilon_1:\ 0 \to E\otimes I \xrightarrow{j_1} F_1 \xrightarrow{p_1} E \to 0,$$

$$\varepsilon_2:\ 0 \to E\otimes I \xrightarrow{j_2} F_2 \xrightarrow{p_2} E \to 0$$

such that $\partial(\varepsilon_1) = \partial(\varepsilon_2) = 1$ and $F_1 \simeq F_2$. We fix some isomorphism $f : F_1 \to F_2$ and get the following commutative diagram

$$\begin{tikzcd}
F_1\otimes I \arrow[r, "p_1\otimes 1"] \arrow[d, "f\otimes 1"'] & E\otimes I \arrow[r, "\partial(\varepsilon_1)"] \arrow[d, "\lambda\otimes 1"] & E\otimes I \arrow[r, "j_1"] \arrow[d, "\mu"] & F_1 \arrow[d, "f"] \\
F_2\otimes I \arrow[r, "p_2\otimes 1"] & E\otimes I \arrow[r, "\partial(\varepsilon_2)"] & E\otimes I \arrow[r, "j_2"] & F_1,
\end{tikzcd}$$

where λ and μ are defined as the compositions of isomorphisms

$$\lambda : E \xrightarrow{p_1^{-1}} F_1/IF_1 \xrightarrow{f} F_2/IF_2 \xrightarrow{p_2} E,$$

$$\mu : E\otimes I \xrightarrow{j_1} IF_1 \xrightarrow{f} IF_2 \xrightarrow{j_2^{-1}} E\otimes I.$$

It follows that $\mu = \lambda\otimes 1_I$ as $\partial(\varepsilon_1) = \partial(\varepsilon_2) = 1_{E\otimes I}$. Moreover, there is an isomorphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & E\otimes I \arrow[r, "j_1"] \arrow[d, "\mu"] & F_1 \arrow[r, "p_1"] \arrow[d, "f"] & E \arrow[r] \arrow[d, "\lambda"] & 0 \\
0 \arrow[r] & E\otimes I \arrow[r, "j_2"] & F_2 \arrow[r, "p_2"] & E \arrow[r] & 0
\end{tikzcd}$$

that implies $\varepsilon_1 = \mu_*^{-1}\lambda^*(\varepsilon_2)$. As E is simple, we may consider $\lambda \in \mathrm{End}_{\mathcal{O}_Y}(E)$ as a scalar, hence

$$\varepsilon_1 = \mu_*^{-1}\lambda^*\varepsilon_2 = \lambda\mu_*^{-1}(\varepsilon_2) = (\lambda\otimes 1)_*\mu_*^{-1}(\varepsilon_2) = \varepsilon_2.$$

This proves injectivity. $\square$

Corollary 3.2.6. *Let E be a simple coherent sheaf on $Y \subset X$ and M be the set of isomorphism classes of sheaves F on X occurring in extensions $\varepsilon : \ 0 \to E \otimes I \to F \to E \to 0$ such that $F/IF \to E$ is an isomorphism. Then only the following possibilities for M can occur:*

1. *If $\partial : \mathrm{Ext}^1_{\mathcal{O}_X}(E, E\otimes I) \to \mathrm{End}(E\otimes I)$ is surjective then $M \simeq \partial^{-1}(1_{E\otimes I})$.*
2. *If $\partial : \mathrm{Ext}^1_{\mathcal{O}_X}(E, E\otimes I) \to \mathrm{End}(E\otimes I)$ is not surjective then M is empty .*

Proof. The only thing we have to prove is that if there exists some ε such that $\partial(\varepsilon)$ is an isomorphism then $\partial : \mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes I) \to \mathrm{End}(E \otimes I)$ is surjective. Let $f = \partial(\varepsilon)$ and let $g \in \mathrm{End}(E \otimes I)$ be any endomorphism. Then

$$\partial((gf^{-1})_*\varepsilon) = gf^{-1}\partial(\varepsilon) = g.$$

□

Our aim now is to investigate, when for a given simple sheaf E on $Y \subset X$ there exist extensions as above, i.e. when the boundary map $\partial : \mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes I) \to \mathrm{End}(E \otimes I)$ is surjective.

Proposition 3.2.7. *Let Y be an integral curve on a smooth projective surface X and E be a coherent torsion-free sheaf on Y. Then the boundary map $\partial : \mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes I) \to \mathrm{End}(E \otimes I)$ is surjective if and only if E is locally free.*

Proof. One direction is clear: if E is locally free, then we deduce from the exact sequence

$$\mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes_{\mathcal{O}_X} I) \xrightarrow{\partial} \mathrm{End}(E \otimes_{\mathcal{O}_X} I) \to \mathrm{Ext}^2_{\mathcal{O}_Y}(E, E \otimes I)$$

and the fact that $\mathrm{Ext}^2_{\mathcal{O}_Y}(E, E \otimes_{\mathcal{O}_X} I) \simeq H^2(Y, E^\vee \otimes_{\mathcal{O}_Y} E \otimes_{\mathcal{O}_X} I) = 0$ that ∂ is surjective.

To prove the other direction, we will construct explicitly the first terms of the Grothendieck spectral sequence as in Proposition 3.2.1. Considering E as a sheaf on X, we can find a surjection $P_0 \to E$ such that P_0 is locally free and $P_0|_Y$ is acyclic with respect to the functor $\mathrm{Hom}_{\mathcal{O}_Y}(-, E \otimes I)$ (i.e. $\mathrm{Ext}^i_{\mathcal{O}_Y}(P_0|_Y, E\otimes I) = 0$, $i \geq 1$, recall the proof of Proposition 3.2.1). As E is torsion free, we deduce from the Auslander-Buchsbaum formula, that for all points $x \in X$, E_x has a global projective dimension smaller or equal than 1. It follows that $P_1 = \ker(P_0 \to E)$ is locally free. Applying to an exact sequence

$0 \to P_1 \to P_0 \to E \to 0$ the functors $\mathrm{Hom}_{\mathcal{O}_X}(-, E\otimes I)$ and $(-\otimes \mathcal{O}_Y)$ we get two exact sequences (recall from 3.2.2 that $\mathrm{Tor}_1^{\mathcal{O}_X}(E, \mathcal{O}_Y) \simeq E\otimes I$)

$$\mathrm{Hom}_{\mathcal{O}_X}(P_0, E\otimes I) \to \mathrm{Hom}_{\mathcal{O}_X}(P_1, E\otimes I) \to \mathrm{Ext}^1_{\mathcal{O}_X}(E, E\otimes I) \\ \to \mathrm{Ext}^1_{\mathcal{O}_X}(P_0, E\otimes I) = 0,$$

$$0 \to E\otimes I \xrightarrow{\varphi} P_1|_Y \to P_0|_Y \to E \to 0.$$

It follows that there exists a unique factorization (the second row is not exact in general)

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_X}(P_0, E\otimes I) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{O}_X}(P_1, E\otimes I) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{\mathcal{O}_X}(E, E\otimes I) \arrow[r] \arrow[d, dotted, "\partial"] & 0 \\
\mathrm{Hom}_{\mathcal{O}_Y}(P_0|_Y, E\otimes I) \arrow[r] & \mathrm{Hom}_{\mathcal{O}_Y}(P_1|_Y, E\otimes I) \arrow[r, "\varphi^*"] & \mathrm{Hom}_{\mathcal{O}_Y}(E\otimes I, E\otimes I) &
\end{tikzcd}$$

Assuming that ∂ is surjective, we deduce that

$$\varphi^* : \mathrm{Hom}_{\mathcal{O}_Y}(P_1|_Y, E\otimes I) \to \mathrm{Hom}_{\mathcal{O}_Y}(E\otimes I, E\otimes I)$$

is surjective, therefore $\varphi : E\otimes I \to P_1|_Y$ is a splitting monomorphism. This implies that $E\otimes I$ is locally free. As I is invertible (one can apply, for example, the Auslander-Buchsbaum formula to $\mathcal{O}_Y$), we deduce that E is locally free. $\square$

3.3 Euler number of a moduli space

So far, we have investigated how one can describe a stable sheaf on a double curve C in terms of certain sheaves on a reduced curve C_{red}. In order to use the previous results we need certain restrictive assumptions which we list below:

1. C can be embedded into a smooth projective surface (in order to use Proposition 3.2.7).
2. The ideal sheaf of $C_{\mathrm{red}} \subset C$ has degree -2 (in order to use Lemma 3.1.6).

With these assumptions we have the following

Lemma 3.3.1. *Let F be a stable sheaf from $M_C(2, 2k)$ that is not an $\mathcal{O}_{C_{\mathrm{red}}}$-module. Then $E = F/IF$ is an invertible sheaf on C_{red} and there is an exact sequence*

$$0 \to E\otimes I \to F \to E \to 0.$$

Proof. In view of Lemma 3.1.6, the only thing we have to prove is that E is locally free. Take any embedding $C \subset X$ into a smooth projective surface and let $J \subset \mathcal{O}_X$ be an ideal sheaf of C_{red} as a subscheme of X. Consider the above extension as an extension over X and denote it by ε. As $F/JF = F/IF \simeq E$, we get from Corollary 3.2.6 that ∂ is surjective and using Proposition 3.2.7 we deduce that E is locally free. □

Remark 3.3.2. One can construct in an obvious way the map $M_{C_{\mathrm{red}}}(2,2k) \longrightarrow M_C(2,2k)$. Its image is closed because $M_{C_{\mathrm{red}}}(2,2k)$ is projective. The complement M_0 consists precisely of those stable sheaves that are not $\mathcal{O}_{C_{\mathrm{red}}}$-modules.

Lemma 3.3.3. *Let $M_0 \subset M_C(2,2k)$ be as before. There is a map $M_0 \longrightarrow J_{k+1}C_{\mathrm{red}}$ that to any sheaf F from M_0 associates an invertible sheaf F/JF on C_{red}. The fiber over an invertible sheaf $E \in J_{k+1}C_{\mathrm{red}}$ is isomorphic (non-canonically) to $\mathrm{Ext}^1_{C_{\mathrm{red}}}(E, E\otimes I) \simeq H^1(C_{\mathrm{red}}, I)$.*

Proof. To construct the map $M_0 \longrightarrow J_{k+1}C_{\mathrm{red}}$ as described, we have to show, that given a flat family $\mathcal{F}$ of sheaves over $S \times C$ with fibers over S being stable sheaves of rank 2 and degree $2k$ over C that are not $\mathcal{O}_{C_{\mathrm{red}}}$-modules, the family $\mathcal{F} \otimes \mathcal{O}_{C_{\mathrm{red}}}$ is again flat.

There is an exact sequence

$$\mathcal{F}\otimes I \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}\otimes \mathcal{O}_{C_{\mathrm{red}}} \longrightarrow 0,$$

that induces short exact sequences on the fibers in view of Lemma 3.3.1. Using Proposition C.2 we deduce that $\mathcal{F}\otimes \mathcal{O}_{C_{\mathrm{red}}}$ is flat. □

Corollary 3.3.4. *It holds*

$$e(M_C(2,2k)) = e(M_{C_{\mathrm{red}}}(2,2k)) + e(JC_{\mathrm{red}}),$$
$$e(M^s_C(2,2k)) = e(M^s_{C_{\mathrm{red}}}(2,2k)) + e(JC_{\mathrm{red}}),$$

where JC_{red} is a Jacobian consisting of line bundles of degree 0.

Proof. It is clear that $e(M_C(2,2k)) = e(M_{C_{\mathrm{red}}}(2,2k)) + e(M_0)$. On the other hand we deduce from the previous Lemma that $e(M_0) = e(JC_{\mathrm{red}}) * e(H^1(C_{\mathrm{red}}, I)) = e(JC_{\mathrm{red}})$ (we use $e(\mathbb{C}^n) = 1$, see Appendix D). □

Corollary 3.3.5. *If C_{red} is non-rational, then*

$$e(M_C(2,2k)) = 0, \quad e(M^s_C(2,2k)) = 0.$$

Corollary 3.3.6. *If C_{red} is a singular nodal rational curve then*

$$e(M_C(2,2k)) = 1, \quad e(M^s_C(2,2k)) = 0.$$

If $C_{\mathrm{red}} \simeq \mathbb{P}^1$ then

$$e(M_C(2,2k)) = 2, \quad e(M^s_C(2,2k)) = 1.$$

Chapter 4

Computation of the Euler number

4.1 General sextics

Let X be a $K3$ surface with a divisor H such that $H^2 = 2$ and $\operatorname{Pic} X = \mathbb{Z}[H]$. A complete linear system $|H|$ defines a map $X \longrightarrow \mathbb{P}^2$, which is a $2:1$ map branched along some sextic $B \in |\mathcal{O}_{\mathbb{P}^2}(6)|$ (see e.g. [3, Proposition VIII.13 (iii)]). Conversely, given a smooth sextic B in $\mathbb{P}^2$ we can construct a $K3$ surface $X = X_B$ together with a $2:1$ map $p_B : X_B \longrightarrow \mathbb{P}^2$ branched along B. Namely, one defines $\mathcal{O}_{X_B} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$ with a multiplication given locally by

$$(f_1, g_1) * (f_2, g_2) \longmapsto (f_1 f_2 + g_1 g_2 s, f_1 g_2 + f_2 g_1),$$

where $s \in \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$ is an equation of B. It can be shown that X is a projective K3 surface (see e.g. [28, Example 1.6]). Let $H \subset X$ be an inverse image of some line h in $\mathbb{P}^2$. Then for a general choice of B it holds $\operatorname{Pic} X \simeq \mathbb{Z}[H]$ (see e.g. [3, Exercise VIII.22 (16)]). One can easily show that the natural maps $|h| \longrightarrow |H|$ and $|2h| \longrightarrow |2H|$ are bijective.

As we have shown already, computing the Euler number of the moduli space of semistable sheaves over a reduced curve $C \in |2H|$ (or $C \in |H|$) we may assume that the components of the curve C are rational. We will show that such curves can be easily described if the sextic B is sufficiently general.

Definition 4.1.1. Let C be a reduced curve, $\pi : \widetilde{C} \longrightarrow C$ be its normalization and $\widetilde{\mathcal{O}}_C := \pi_* \mathcal{O}_{\widetilde{C}}$ be the integral closure of the sheaf of algebras $\mathcal{O}_C$. For any $x \in C$ we define the number δ_x to be the length $l(\widetilde{\mathcal{O}}_{C,x}/\mathcal{O}_{C,x})$.

Lemma 4.1.2. *Let C be a reduced curve and $x \in C$ be an A_k singularity. Then $\delta_x = [(k+1)/2]$.*

Proof. The completion of the ring of the A_k singularity is isomorphic to $R = \mathbb{C}[\![x, y]\!]/(x^{k+1} - y^2)$. If k is even, say $k = 2m$, then there is an inclusion

$R \hookrightarrow \mathbb{C}[\![t]\!]$, $x \longmapsto t^2$, $y \longmapsto t^{2m+1}$. Both rings have the same field of fractions, hence $\mathbb{C}[\![t]\!]$ is an integral closure of R. The dimension of the quotient equals $m = [(k+1)/2]$. If k is odd, say $k = 2m-1$, then there is an inclusion $R \hookrightarrow \mathbb{C}[\![t]\!] \oplus \mathbb{C}[\![t]\!]$, $x \longmapsto (t,t)$, $y \longmapsto (t^m, -t^m)$. Both rings have the same ring of fractions, hence $\mathbb{C}[\![t]\!] \oplus \mathbb{C}[\![t]\!]$ is an integral closure of R. The dimension of the quotient equals $m = [(k+1)/2]$. $\square$

Let C_0 be a smooth curve in $\mathbb{P}^2$ having s intersection points of multiplicities $\lambda_1, \dots, \lambda_s$ with a smooth sextic B. Then the preimage C of C_0 in X_B has singularities $A_{\lambda_1 - 1}, \dots, A_{\lambda_s - 1}$.

Remark 4.1.3. To see the last claim, let p be an intersection point of C_0 and B having the multiplicity m. The completion of $\mathcal{O}_{C_0,p}$ is isomorphic to $\mathbb{C}[\![t]\!]$ and the divisor induced by B is given locally at p by the ideal (t^m). The preimage of C_0 is given locally at p by the same construction as X_B. That is, the completion of the corresponding ring is isomorphic to $k[\![t]\!] \oplus k[\![t]\!]$ as a vector space and multiplication is given by

$$(f_1, g_1) * (f_2, g_2) \longmapsto (f_1 f_2 + g_1 g_2 t^m, f_1 g_2 + f_2 g_1).$$

This ring is isomorphic to $k[\![x,y]\!]/(x^m - y^2)$, where the map is given by $(f,g) \longmapsto f(x) + g(x)y$.

It follows that the length $l(\widetilde{\mathcal{O}}_C/\mathcal{O}_C) = \sum [\lambda_i/2]$ by the previous lemma. This implies that

$$\chi(\mathcal{O}_{\widetilde{C}}) = \chi(\mathcal{O}_C) + \sum [\lambda_i/2]$$

and therefore the arithmetic genus

$$g_{\widetilde{C}} = g_C - \sum [\lambda_i/2].$$

So, the curve C is rational if and only if $g_C = \sum [\lambda_i/2]$. Assuming that C is indeed a rational curve, we consider two cases

- $C_0 \in |h|$, i.e. C_0 is a line. Then $g_C = 2$ and we have

$$\sum \lambda_i = 6, \qquad \sum [\lambda_i/2] = 2.$$

- $C_0 \in |2h|$, i.e. C_0 is a quadric. Then $g_C = 5$ and we have

$$\sum \lambda_i = 12, \qquad \sum [\lambda_i/2] = 5.$$

The equality in the second formula holds in both cases if and only if there are precisely two odd numbers among $\lambda_1, \dots, \lambda_s$.

Remark 4.1.4. In what follows we will assume that $\lambda_1 \geq \cdots \geq \lambda_s$, so that $\lambda = (\lambda_1, \ldots, \lambda_s)$ is a partition. As usual, we write the partition λ also in the form

$$(1^{k_1} 2^{k_2} \ldots r^{k_r} \ldots),$$

where k_i denotes the number of elements in λ that are equal to i.

Our goal is to prove the following proposition

Proposition 4.1.5. *For a general sextic B it holds*

1. *The curve in $|H|$ is rational iff its image in $|h|$ has intersection multiplicities with B equal $(1^2 2^2)$.*
2. *The integral curve in $|2H|$ is rational iff its image in $|2h|$ has intersection multiplicities with B equal $(1^2 2^5)$.*

There is a finite number of such curves and they have just A_1 singularities. Two such curves from $|H|$ (or their images in $|h|$) intersect outside of B.

We start with some technical results.

Proposition 4.1.6. *Let $k < n$ be two positive integers and let $\mathcal{D} \subset |\mathcal{O}_{\mathbb{P}^2}(k)|$ be the open family of smooth curves. Then for a general curve from $|\mathcal{O}_{\mathbb{P}^2}(n)|$ the number of intersection points (without multiplicities) with any curve from $\mathcal{D}$ is greater or equal than $kn - k^2 - 1$ and an equality can hold just for a finite number of elements from $\mathcal{D}$.*

Proof. Let $\mathcal{D}' \subset |\mathcal{O}_{\mathbb{P}^2}(n)|$ be the open subset consisting of integral curves. Then any $C_0 \in \mathcal{D}$ and $B \in \mathcal{D}'$ don't have a common irreducible component and therefore their intersection $C_0 \cap B$ (with multiplicities) is well defined. It can be considered as a divisor of degree kn on C_0 or on B if B is smooth. Define $\mathcal{A} := \mathcal{D} \times \mathcal{D}'$ and

$$\mathcal{B} := \{(C_0, D) \mid C_0 \in \mathcal{D},\ D \in |\mathcal{O}_{C_0}(n)|\}.$$

Then there is a map $\mathcal{A} \to \mathcal{B}$ that associates with any $(C_0, B) \in \mathcal{A}$ a pair $(C_0, C_0 \cap B) \in \mathcal{B}$. Let us estimate the dimension of its fiber over some $(C_0, D) \in \mathcal{B}$. We get from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(n-k) \to \mathcal{O}_{\mathbb{P}^2}(n) \to \mathcal{O}_{C_0}(n) \to 0$$

that the sequence

$$0 \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-k)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)) \to H^0(\mathbb{P}^2, \mathcal{O}_{C_0}(n)) \to 0$$

is also exact ($H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) = 0$ for any $m \in \mathbb{Z}$). Therefore the dimensions of the fibres of $\mathcal{A} \longrightarrow \mathcal{B}$ are equal to $h^0(\mathcal{O}_{\mathbb{P}^2}(n-k))$.

Now we stratify the space $\mathcal{A}$ in the following way. Let $\lambda = (\lambda_1, \dots, \lambda_s)$ be a partition of kn (we assume $\lambda_s \geq 1$). Then we define $\mathcal{A}(\lambda) \subset \mathcal{C}$ to be the set of pairs (C_0, B) such that $C_0 \cap B$ can be written in the form $\sum_{i=1}^s \lambda_i x_i$, where $x_i \in C_0$ are pairwise distinct. We stratify analogously the space $\mathcal{B}$. The fiber of the canonical map $\mathcal{B}(\lambda) \longrightarrow \mathcal{D}$ over $C_0 \in \mathcal{D}$ is contained in a certain s-dimensional space (corresponding to the choices of $x_i \in C_0$, $1 \leq i \leq s$), therefore

$$\dim \mathcal{B}(\lambda) \leq \dim |\mathcal{O}_{\mathbb{P}^2}(k)| + s$$

and this implies that

$$\dim \mathcal{A}(\lambda) \leq \dim |\mathcal{O}_{\mathbb{P}^2}(k)| + h^0 \mathcal{O}_{\mathbb{P}^2}(n-k) + s.$$

We claim that the condition $s < kn - k^2 - 1$ implies that $\dim \mathcal{A}(\lambda) < \dim |\mathcal{O}_{\mathbb{P}^2}(n)|$ and therefore the canonical map $\mathcal{A}(\lambda) \longrightarrow |\mathcal{O}_{\mathbb{P}^2}(n)|$ has an image of codimension at least 1. Indeed, we just have to show that

$$\binom{k+2}{2} + \binom{n-k+2}{2} + s < \binom{n+2}{2}$$

and this is equivalent to the condition $s < kn - k^2 - 1$.

This implies that a general curve from $|\mathcal{O}_{\mathbb{P}^2}(n)|$ is not contained in the images of $\mathcal{A}(\lambda) \longrightarrow |\mathcal{O}_{\mathbb{P}^2}(n)|$ with $\lambda = (\lambda_1, \dots, \lambda_s)$ such that $s < kn - k^2 - 1$ and therefore a general curve from $|\mathcal{O}_{\mathbb{P}^2}(n)|$ has at least $kn - k^2 - 1$ intersection points with any curve from $\mathcal{D}$. Analogously, if $s = kn - k^2 - 1$ then $\dim \mathcal{A}(\lambda) \leq \dim |\mathcal{O}_{\mathbb{P}^2}(n)|$ and therefore the map $\mathcal{A}(\lambda) \longrightarrow |\mathcal{O}_{\mathbb{P}^2}(n)|$ is generically finite. This implies that a general curve from $|\mathcal{O}_{\mathbb{P}^2}(n)|$ can intersect just a finite number of curves from $\mathcal{D}$ in exactly $kn - k^2 - 1$ points. □

Corollary 4.1.7. *For a general sextic in $\mathbb{P}^2$ the number of intersection points with any line is greater or equal than 4 and the number of intersection points with any smooth quadric is greater or equal than 7. An equality holds (for a general sextic) just for a finite number of lines (smooth quadrics).*

Definition 4.1.8. Let $k < n$ be positive integers and $B \in |\mathcal{O}_{\mathbb{P}^2}(n)|$ be a general curve as in Proposition 4.1.6. We call a smooth curve $C_0 \in |\mathcal{O}_{\mathbb{P}^2}(k)|$ an exceptional curve with respect to B if the number s of intersection points of B with C_0 equals $k(n-k)-1$. If the multiplicities of intersection points are $\lambda_1 \geq \dots \geq \lambda_s$, then we call C_0 an exceptional curve of type $\lambda = (\lambda_1, \dots, \lambda_s)$ with respect to B.

Lemma 4.1.9. *For a general curve $B \in |\mathcal{O}_{\mathbb{P}^2}(n)|$ any two lines that are exceptional with respect to B, intersect outside of B.*

Proof. For a general curve from $|\mathcal{O}_{\mathbb{P}^2}(n)|$, the number of intersection points of it with an exceptional line equals by definition $n-2$. We proceed similarly to the proof of Proposition 4.1.6. Let $\mathcal{D} = |\mathcal{O}_{\mathbb{P}^2}(1)|$ and $\mathcal{D}' \subset |\mathcal{O}_{\mathbb{P}^2}(n)|$ be an (open) subset consisting of integral curves. Define $\mathcal{A} := \mathcal{D} \times \mathcal{D} \times \mathcal{D}'$ and

$$\mathcal{B} := \{(C_0, D) \mid C_0 \in \mathcal{D},\ D \in |\mathcal{O}_{C_0}(n)|\}.$$

There is then a map $\mathcal{A} \longrightarrow \mathcal{B} \times \mathcal{B}$ defined by

$$(C_0, C_1, B) \longmapsto (C_0, C_0 \cap B) \times (C_1, C_1 \cap B).$$

One can prove as in 4.1.6, that the dimensions of the fibers of $\mathcal{A} \longrightarrow \mathcal{B} \times \mathcal{B}$ are less or equal than $h^0(\mathcal{O}_{\mathbb{P}^2}(n-2))$.

Given any two partitions of n

$$\lambda^0 = (\lambda^0_1, \ldots, \lambda^0_s), \quad \lambda^1 = (\lambda^1_1, \ldots, \lambda^1_s),$$

where $s = n-2$, $\lambda^0_s \geq 1$ and $\lambda^1_s \geq 1$, we define $\mathcal{A}(\lambda^0, \lambda^1) \subset \mathcal{A}$ to consist of those triples (C_0, C_1, B) that $C_0 \neq C_1$, the intersection of C_0 and C_1 is contained in B and the multiplicities of the intersection points of C_i with B are given by λ^i, $i = 0, 1$. In a similar way we define $\mathcal{B}'(\lambda^0, \lambda^1) \subset \mathcal{B} \times \mathcal{B}$ to consist of those collections $(C_0, D_0) \times (C_1, D_1)$ that $C_0 \neq C_1$, D_0 and D_1 contain the intersection point $C_0 \cap C_1$ and their multiplicities are given by λ^0 and λ^1 respectively. The fibers of the canonical map $\mathcal{B}'(\lambda^0, \lambda^1) \longrightarrow |\mathcal{O}_{\mathbb{P}^2}(1)| \times |\mathcal{O}_{\mathbb{P}^2}(1)|$ have dimension less or equal $2s-1$, therefore

$$\dim \mathcal{B}'(\lambda^0, \lambda^1) \leq 2 + 2 + 2s - 1 = 2n - 1$$

and

$$\dim \mathcal{A}(\lambda^0, \lambda^1) \leq \dim \mathcal{B}'(\lambda^0, \lambda^1) + h^0 \mathcal{O}_{\mathbb{P}^2}(n-2) \leq \binom{n}{2} + 2n - 1.$$

It follows that

$$\dim \mathcal{A}(\lambda^0, \lambda^1) < \dim |\mathcal{O}_{\mathbb{P}^2}(n)| = \binom{n+2}{2} - 1$$

and therefore the canonical map $\mathcal{A}(\lambda^0, \lambda^1) \longrightarrow |\mathcal{O}_{\mathbb{P}^2}(n)|$ has the image of codimension at least 1. Arguing as before we get the statement of the lemma. $\square$

Lemma 4.1.10. *Assume that B is a general sextic in $\mathbb{P}^2$ like in Corollary 4.1.7 and C_0 is a line in $\mathbb{P}^2$ such that its preimage in X_B is a rational curve. Then C_0 is an exceptional curve of type $(1^2 2^2)$ with respect to B. In particular, there are only finitely many such lines.*

Proof. Let C_0 has intersections with B of multiplicities $\lambda_1, \dots, \lambda_s$. As we have mentioned already there are precisely two odd numbers among $\lambda_1, \dots, \lambda_s$. If C_0 is a non exceptional line then $s \geq 5$, hence $\sum \lambda_i \geq 2 \cdot 1 + 3 \cdot 2 > 6$ that is impossible. If C_0 is an exceptional line then $s = 4$ and therefore $\sum \lambda_i \geq 2 \cdot 1 + 2 \cdot 2 = 6$. An equality holds if and only if the multiplicities are $(1^2 2^2)$. □

Remark 4.1.11. One may call an exceptional line of type $(1^2 2^2)$ with respect to a sextic B a bitangent of B as it is tangent to B in two points. The number of bitangents of a general curve from $|\mathcal{O}_{\mathbb{P}^2}(n)|$ equals (see [24, Exercise IV.2.3])

$$\frac{1}{2}n(n-2)(n-3)(n+3)$$

In particular, for a general sextic one has 324 bitangents. We will compute this number later using a different technique (see Proposition 4.1.13).

Lemma 4.1.12. *Assume that B is a general sextic in $\mathbb{P}^2$ as in Corollary 4.1.7 and C_0 is a smooth quadric in $\mathbb{P}^2$ such that its preimage in X_B is a rational curve. Then C_0 is an exceptional curve of type $(1^2 2^5)$ with respect to B. In particular, there are only finitely many such quadrics.*

Proof. Let C_0 has intersections with B of multiplicities $\lambda_1, \dots, \lambda_s$. As we have mentioned already there are precisely two odd numbers among $\lambda_1, \dots, \lambda_s$. If C_0 is a non exceptional quadric then $s \geq 8$, hence $\sum \lambda_i \geq 2 \cdot 1 + 6 \cdot 2 > 12$ that is impossible. If C_0 is an exceptional quadric then $s = 7$ and therefore $\sum \lambda_i \geq 2 \cdot 1 + 5 \cdot 2 = 12$. An equality holds if and only if the multiplicities are $(1^2 2^5)$. □

To find the number of the described exceptional lines one can use the following classical result.

Proposition 4.1.13 (De Jonquières Formula, see [1, §8.5])**.** *Let B be a smooth curve of genus g, $\mathcal{D}$ be a linear system of dimension r and degree d. Let $(a_1^{n_1} \dots a_k^{n_k})$ be a partition of d such that $\sum_i n_i = d - r$. Then the virtual number of divisors in $\mathcal{D}$ having n_i points of multiplicity a_i for all i equals the coefficient of $\prod t_i^{n_i}$ in*

$$(1 + \sum a_i^2 t_i)^g (1 + \sum a_i t_i)^{d-r-g}.$$

Corollary 4.1.14. *Let B be a general sextic in $\mathbb{P}^2$ as in Corollary 4.1.7 and Lemma 4.1.9. Then the number of exceptional lines of type* $(1^2 2^2)$ *with respect to B equals* 324.

Proof. Writing down a long exact sequence for cohomologies of a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1-6) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow \mathcal{O}_B(1) \longrightarrow 0,$$

we get an isomorphism $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \longrightarrow H^0(B, \mathcal{O}_B(1))$. This implies that instead of studying exceptional curves of type $(1^2 2^2)$ in $|\mathcal{O}_{\mathbb{P}^2}(1)|$ we can study divisors in $|\mathcal{O}_B(1)|$ having multiplicities $(1^2 2^2)$. The curve B has genus $g_B = \binom{6-1}{2} = 10$. A linear system $\mathcal{D} = |\mathcal{O}_B(1)|$ has dimension $h^0(\mathcal{O}_{\mathbb{P}^2}(1)) - 1 = 2$ and degree 6. The numbers a_i and n_i as in Proposition are $a_1 = 1$, $a_2 = 2$, $n_1 = 2$, $n_2 = 2$. It follows that the number of exceptional lines of type $(1^2 2^2)$ equals the coefficient of $t_1^2 t_2^2$ in

$$(1 + t_1 + 4t_2)^{10}(1 + t_1 + 2t_2)^{-6}$$

and one can easily check that this number equals 324. $\square$

Remark 4.1.15. In a similar way we could apply the De Jonquières Formula to find the virtual number of exceptional quadrics of type $(1^2 2^5)$ with respect to B. This number equals the coefficient of $t_1^2 t_2^5$ in

$$(1 + t_1 + 4t_2)^{10}(1 + t_1 + 2t_2)^{-3}$$

which is 71136. However this virtual number does not equal the actual number we are looking for because of the correction term caused by the fact that not all divisors in $\mathcal{O}_{\mathbb{P}^2}(2)$ are primitive.

Fortunately enough, the problem of finding the number of smooth quadrics 5-tangent to a general sextic is quite old and was solved by Gathmann [16] (as we have shown, there are at least 7 different intersection points and therefore the intersection multiplicities are $(1^2 2^5)$, as required). There he used the Gromov-Witten invariants to compute the number of smooth quadrics 5-tangent to general curves in $\mathcal{O}_{\mathbb{P}^2}(n)$ for $n \geq 5$. In particular, for a general sextic there are

$$70956 = e(5) - 2\binom{e(2)}{2} - 2e(2)$$

such quadrics (see [16, Corollary 3.6]), where $e(n)$ was defined in Introduction. This is the number we need.

Remark 4.1.16. It follows from the above discussion that for a general sextic the number of integral rational curves in the linear system $|2H|$ equals 70956. This result can be compared with a famous formula of Yau and Zaslow (see e.g. [4]) that computes the number of rational curves in the complete linear system on the $K3$-surface, assuming that all the curves in the linear system are integral.

4.2 Stringy E-function of $M_X(4n+2)$

The goal of this section is to represent the stringy E-function (see Appendix D) of $M_X(4n+2)$ as a sum of the E-polynomial $E(M_X^s(4n+2))$ of the smooth locus and of some additional term. Then we can do the same for the stringy Poincaré polynomial and stringy Euler number. Analogous calculations can be found in [10], where they used the Kirwan desingularization of $M_X(4n+2)$ in order to find the stringy E-function. Our calculation is much easier as we are using the desingularization of Lehn and Sorger [29] consisting of just one blow-up along the singular locus.

Let us recall that the moduli space $M = M_X(4n+2)$ can be stratified $M \supset \Sigma \supset \Omega$, where $M \backslash \Sigma = M^s$ consists of the stable sheaves, $\Sigma = M^{\mathrm{sing}}$ consists of the sheaves $F \oplus G$ with $F, G \in M_X(2n+1)$ and $\Omega = \Sigma^{\mathrm{sing}}$ consists of the sheaves $F \oplus F$ with $F \in M_X(2n+1)$. The moduli space $M_X(2n+1)$ is birational to the Hilbert scheme $X^{[2]}$ (see e.g. Remark 1.3.7), hence they have the same Hodge numbers and $E(M_X(2n+1)) = E(X^{[2]})$. It was shown by Lehn and Sorger [29] that the blow-up $\widetilde{M}$ of M along Σ is nonsingular and is a crepant resolution of M. Moreover, the map $\widetilde{M} \longrightarrow M$ is a locally trivial fibration over $\Sigma - \Omega$ with fibers isomorphic to $\mathrm{Gr}^\omega(1,2)$ (see Lemma D.6 for the definition of symplectic Grassmanians) and it is a locally trivial fibration over Ω with fibers isomorphic to $\mathrm{Gr}^\omega(2,4)$. It follows that

$$
\begin{aligned}
E_{st}(M) =& E(\widetilde{M}) \\
=& E(M^s) + E(\mathrm{Gr}^\omega(1,2))E(\Sigma - \Omega) + E(\mathrm{Gr}^\omega(2,4))E(\Omega) \\
=& E(M^s) + (1+uv)E(\Sigma - \Omega) + (1+uv)(1+u^2v^2)E(\Omega).
\end{aligned}
$$

The additional term for the Euler number can be computed now as follows. We know that $e(\Omega) = e(X^{[2]}) = e(2) = 324$ and $e(\Sigma - \Omega) = e(S^2\Omega - \Omega) = \binom{e(2)}{2}$, where $S^2\Omega$ means the symmetric product of Ω. It follows

Proposition 4.2.1. *It holds*

$$
e_{st}(M) = e(\widetilde{M}) = e(M^s) + 2\binom{e(2)}{2} + 4e(2) = e(M^s) + 105948.
$$

Remark 4.2.2. We can compute also the additional term for the Poincaré polynomial. We know that $P(X) = 1 + 22t^2 + t^4$ and using the Göttsche formula for Hilbert schemes we get

$$P(\Omega) = P(X^{[2]}) = 1 + 23t^2 + 276t^4 + 23t^6 + t^8.$$

Using the Macdonald formula for the Betti numbers of the symmetric product one can show that

$$P(\Sigma; t) = P(S^2\Omega; t) = \frac{1}{2}(P(\Omega; t)^2 + P(\Omega; t^2)).$$

Doing the substitutions in the formula for the stringy E-function, one can obtain an explicit expression for the additional term, but it is to big to write it down here.

Remark 4.2.3. It was shown in [10, p. 4] that

$$e_{st}(M) = e(M^s) + e(X^{[2]})^2 + 3e(X^{[2]}).$$

Predictably enough, it coincides with our formula.

We see that in order to compute the stringy Euler number of M, we have to compute the Euler number of M^s. This is the content of the next section.

4.3 The Euler numbers of strata of $M_X(4n + 2)$

Let B be a general sextic in $\mathbb{P}^2$ as in Corollary 4.1.7 and Lemma 4.1.9. Let $p : X \longrightarrow \mathbb{P}^2$ be a $2 : 1$ covering branched along B. Recall that we have stratified the moduli space

$$M = M_X(4n + 2) = M_1 \cup M_2 \cup M_3$$

in accordance with the type of supporting divisors of semistable sheaves from M (see Section 1.4). Analogously we stratify the stable part

$$M^s = M_1^s \cup M_2^s \cup M_3^s.$$

4.3.1 The Euler number of M_1^s

The subset $M_1 \subset M$ corresponds to those sheaves in M whose supporting divisor is an integral divisor in $|2H|$ (or equivalently, is a preimage of a smooth quadric in $\mathbb{P}^2$). Let C be some integral curve from $|2H|$. The fiber of $M_1 \longrightarrow |2H|$ over $C \in |2H|$ is isomorphic to $M_C(4n + 2)$. As C has degree 4, all the sheaves in $M_C(4n+2)$ have rank 1 over C and in particular they all are stable. We know that the Euler number of $M_C(4n+2)$ is zero, whenever the curve C is non-rational. Therefore we may assume that C is rational. Let C be a preimage of a quadric C_0 on $\mathbb{P}^2$. It follows from Lemma 4.1.12 that C_0 is an exceptional quadric of type $(1^2 2^5)$. In particular C has just A_1 singularities and therefore $e(M_C(4n+2)) = 1$ (see e.g. [4]). Summing over all exceptional quadrics of type $(1^2 2^5)$ (their number was computed in Section 4.1) we get

$$e(M_1) = e(M_1^s) = e(5) - 2\binom{e(2)}{2} - 2e(2) = 70956.$$

4.3.2 The Euler number of M_2^s

The subset $M_2 \subset M$ corresponds to those sheaves in M whose supporting divisor consists of two different divisors from $|H|$. Let $C = C' \cup C''$ be a curve from $2H$, where $C', C'' \in |H|$. Denote by C_0' and C_0'' the lines in $\mathbb{P}^2$ corresponding to C' and C'' respectively. Again, the fiber of $M_2 \longrightarrow |2H|$ over C is isomorphic to $M_C(4n+2)$ (recall that $M_C(4n+2)$ means the moduli space of semistable sheaves of constant rank, in our case of rank 1) and its Euler number is zero, whenever any of its components C' or C'' is non-rational. So assume that C' and C'' are rational curves. It follows then from Lemma 4.1.10 that C_0' and C_0'' are exceptional lines of type $(1^2 2^2)$. Their intersection point lies outside of B by Lemma 4.1.9, therefore the curves C' and C'' intersect transversally at two nonsingular points (preimages of the intersection point of C_0' and C_0'') and each of them has two A_1 singularities. Using Corollary 2.5.4 we get $e(M_2^s) = 0$.

4.3.3 The Euler number of M_3^s

The subset $M_3 \subset M$ corresponds to those sheaves in M whose supporting divisor is a double curve $C = 2C_{\mathrm{red}}$, where $C_{\mathrm{red}} \in |H|$. Denote by C_0 the line in $\mathbb{P}^2$ corresponding to C_{red}. Again, the fiber of $M_3 \longrightarrow |2H|$ over C is isomorphic to $M_C(4n+2)$. Recall that the curve $C_{\mathrm{red}} \in |H|$ has arithmetic genus 2 and degree 2. It follows from Lemma 3.1.1 that for any coherent sheaf F on C it holds

$$P(F, n) = 2\,\mathrm{rk}\,F \cdot n + \deg F - \mathrm{rk}\,F.$$

In particular, $M_C(4n+2) = M_C(2,4)$. We deduce from Corollary 3.3.5 that $e(M_C(2,4)) = e(M_C^s(2,4)) = 0$, whenever C_{red} is non-rational. So, assume that C_{red} is rational. Then C_0 is an exceptional line of type $(1^2 2^2)$ with respect to B. In particular, C_{red} has just A_1 singularities and we deduce from Corollary 3.3.6 that $e(M^s_C(2,4)) = 0$. Therefore $e(M^s_3) = 0$.

Now we are ready to prove the main result of the dissertation

Theorem 1. The Euler number of O'grady's 10-dimensional symplectic manifold is equal to $e(5) + 2e(2) = 176904$.

Proof. It follows from the above discussion that

$$e(M^s) = e(5) - 2\binom{e(2)}{2} - 2e(2) = 70956.$$

We obtain from Proposition 4.2.1

$$e_{st}(M) = e(\widetilde{M}) = e(M^s) + 2\binom{e(2)}{2} + 4e(2) = e(5) + 2e(2) = 176904.$$

$\square$

Appendix A

Grothendieck spectral sequences

In this Appendix we give a slight modification of the construction of the Grothendieck spectral sequences. Recall, that the usual construction (see e.g. [21, Theorem 2.4.1]) requires for a pair $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ of left exact functors between abelian categories that there are enough injectives in $\mathcal{A}$ and $\mathcal{B}$ and F maps injectives from $\mathcal{A}$ to G-acyclic objects in $\mathcal{B}$. We will weaken a little these requirements in order to be able to construct a Grothendieck spectral sequence in our specific situation in Section 3.2.

Definition A.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. We call an object $A \in \mathcal{A}$ weakly F-acyclic if for any short exact sequence $0 \to A \to A' \to A'' \to 0$ in $\mathcal{A}$ the induced map $FA' \to FA''$ is an epimorphism. If there exists a right derived functor $RF : \mathcal{D}^+\mathcal{A} \to \mathcal{D}^+\mathcal{B}$ then an object $A \in \mathcal{A}$ is called F-acyclic if $R^iF(A) = 0$, $i \geq 1$. Clearly, F-acyclic objects are weakly F-acyclic.

Lemma A.2. *The family of all weakly F-acyclic objects in $\mathcal{A}$ is closed under extensions (i.e. is an exact subcategory in $\mathcal{A}$, see e.g. [38]).*

Proof. Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be a short exact sequence with A_1 and A_3 being weakly F-acyclic. Let $0 \to A_2 \to A_2' \to A_2'' \to 0$ be any short exact sequence. Then we may construct the following commutative diagram

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, equal] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[d] \\ A_1 \arrow[r] & A_2' \arrow[r] \arrow[d] & A_3' \arrow[d] \\ & A_2'' \arrow[r, equal] & A_3'', \end{tikzcd}$$

where all long rows and columns are exact pairs (i.e. are parts of short exact sequences). It follows then from the assumptions on A_1 and A_3 that $FA_2' \twoheadrightarrow FA_3'$ and $FA_3' \twoheadrightarrow FA_3''$ are epimorphisms. Hence $FA_2' \twoheadrightarrow FA_2''$ is also an epimorphism. $\square$

Lemma A.3. *Assume that the left exact functor $F : \mathcal{A} \to \mathcal{B}$ has a right derived functor. Then the category of all F-acyclic objects is closed under extensions (i.e. is an exact subcategory) and its embedding in $\mathcal{A}$ reflects monomorphisms or equivalently, for any exact pair $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{A}$, if X, Y are F-acyclic then so is Z).*

We are going to give a handy criterion for an existence of right derived functors.

Definition A.4. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. An exact subcategory $\mathcal{I} \subset \mathcal{A}$ (i.e. a full subcategory that is closed under extensions) of weakly F-acyclic objects is called a *dense subcategory of F-acyclic objects* if the embedding $\mathcal{I} \subset \mathcal{A}$ reflects monomorphisms (or equivalently, for any exact pair $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{A}$ if $X, Y \in \mathcal{I}$ then $Z \in \mathcal{I}$) and any object in $\mathcal{A}$ can be embedded into an object in $\mathcal{I}$. We say that there are enough F-acyclic objects in $\mathcal{A}$ if there exists a dense subcategory of F-acyclic objects.

Remark A.5. Obviously, if there are enough injective objects in $\mathcal{A}$, then the subcategory of all injective objects in $\mathcal{A}$ is a dense subcategory of F-acyclic objects.

Proposition A.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories and assume that there are enough F-acyclic objects. Then there exists a right derived functor $RF : \mathcal{D}^+\mathcal{A} \to \mathcal{D}^+\mathcal{B}$.*

Proof. Let $\mathcal{I}$ be a dense subcategory of F-acyclic objects and denote by $\mathcal{H}^+\mathcal{I}$ the homotopy category of bounded below complexes of objects of $\mathcal{I}$. Then any bounded below complex in $\mathcal{A}$ admits a quasi-isomorphism into a complex from $\mathcal{H}^+\mathcal{I}$ (see e.g. [23, Lemma I.4.6] or take the total complex of the Cartan-Eilenberg $\mathcal{I}$-resolution constructed in Lemma A.8). Furthermore, for any acyclic complex $I \in \mathcal{H}^+\mathcal{I}$ the complex FI is again acyclic (here we use the condition that the embedding $\mathcal{I} \subset \mathcal{A}$ reflects monomorphisms). These two properties of the triangulated subcategory $\mathcal{H}^+\mathcal{I} \subset \mathcal{H}^+\mathcal{A}$ allow us to use [23, Theorem I.5.1] to deduce that there exists a derived functor $RF : \mathcal{D}^+\mathcal{A} \to \mathcal{D}^+\mathcal{B}$. $\square$

Corollary A.7. *Let $\mathcal{I}$ be a dense subcategory of F-acyclic objects. Then all the objects in $\mathcal{I}$ are F-acyclic. So the terminology of the Definition A.4 is adequate.*

Proof. The derived functor $RF : \mathcal{D}^+\mathcal{A} \to \mathcal{D}^+\mathcal{B}$ is constructed in such a way that any $I \in \mathcal{H}^+\mathcal{I}$ is mapped to FI. In particular for any $I \in \mathcal{I}$ it holds $R^iF(I) = 0$, $i \geq 1$. □

Lemma A.8. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{I}$ be a family of objects in $\mathcal{A}$ closed under direct sums and such that any object in $\mathcal{A}$ can be embedded into an object in $\mathcal{I}$. Then any complex in $\mathcal{A}$ has a Cartan-Eilenberg $\mathcal{I}$-resolution (this means that any complex A in $\mathcal{A}$ has a resolution by a double complex I in $\mathcal{I}$ and for any $p \in \mathbb{Z}$ the cocycles (coboundaries, cohomologies) in A^p are resolved by the complex of cocycles (coboundaries, cohomologies) in $I^{p,*}$ that we require to be again a complex in $\mathcal{I}$) .*

Proof. The result is classical whenever all the objects in $\mathcal{I}$ are injective. In a general situation we need just minor modifications. Let A be a complex in $\mathcal{A}$ and denote by Z^n (respectively B^n, H^n) its cycles (respectively boundaries, cochains) in degree n. There are obvious exact pairs

$$Z^n \to A^n \to B^{n+1}, \quad B^n \to Z^n \to H^n$$

and we are going to construct the $\mathcal{I}$-resolutions of these exact pairs and then glue them together. Consider some embedding $A^n \hookrightarrow I_0^n$, where I_0^n is an $\mathcal{I}$-object. We get then also embeddings $Z^n \to I_0^n$ and $B^n \to I_0^n$. Besides that, we consider embeddings $H^n \sqcup_{Z^n} A^n \hookrightarrow I_1^n$, where I_1^n is an $\mathcal{I}$-object. Note that $H^n \to H^n \sqcup_{Z^n} A^n$ is a push-forward of a monomorphism $Z^n \to A^n$ and is therefore itself a monomorphism. So, we get a monomorphism $H^n \to I_1^n$. We construct now the following commutative diagrams with exact pairs of morphisms in rows and monomorphisms in columns

$$\begin{tikzcd}
B^n \arrow[r] \arrow[d] & Z^n \arrow[r] \arrow[d] & H^n \arrow[d] \\
I_0^n \arrow[r] & I_0^n \oplus I_1^n \arrow[r] & I_1^n
\end{tikzcd}$$

and

$$\begin{tikzcd}
Z^n \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & B^{n+1} \arrow[d] \\
I_0^n \oplus I_1^n \arrow[r] & I_0^n \oplus I_1^n \oplus I_0^{n+1} \arrow[r] & I_0^{n+1}.
\end{tikzcd}$$

We can consider the short exact sequences of the cokernels of vertical morphisms and repeat the whole process. After all we get compatible $\mathcal{I}$-resolutions of A^n, Z^n, B^n and H^n. Gluing these resolutions together one gets a Cartan-Eilenberg $\mathcal{I}$-resolution. □

Proposition A.9 (Grothendieck spectral sequence)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors between abelian categories and assume that there are enough F-acyclic and G-acyclic objects. Moreover, assume that there exists a dense subcategory $\mathcal{I} \subset \mathcal{A}$ of F-acyclic objects such that its objects are mapped by F to G-acyclic objects. Then for any $A \in \mathcal{A}$ there exists a spectral sequence*

$$(R^pG)(R^qF)(A) \Rightarrow R^{p+q}(GF)(A).$$

Proof. First of all, there are enough GF-acyclic objects and therefore there exists a right derived functor of GF. Indeed, the category $\mathcal{I}$ is exact, consists of weakly GF-acyclic objects, the embedding $\mathcal{I} \subset \mathcal{A}$ reflects monomorphisms and any object in $\mathcal{A}$ can be embedded into an object in $\mathcal{I}$. In view of the existence of Cartan-Eilenberg $\mathcal{I}$-resolutions the construction of the spectral sequence repeats literally the classical one. □

Remark A.10. Assume that there are enough F-acyclic, G acyclic and GF acyclic objects. To show the existence of a Grothendieck spectral sequence for a given object $A \in \mathcal{A}$ it is enough to find a resolution

$$A \to I^0 \to I^1 \to \dots$$

such that all I^n are F-acyclic and all FI^n are G-acyclic.

Remark A.11. All the results of this subsection can be easily reformulated in the case of contravariant or right exact functors.

Appendix B

Description of torsion free sheaves

Let C be a reduced projective curve and $\pi : C' \longrightarrow C$ be a (partial) normalization. We say that a coherent sheaf over C is torsion free if it is of pure dimension 1, see e.g. [26, Defintion 1.1.2]. In this appendix we will describe the torsion free sheaves over C in terms of torsion free sheaves over C' and certain gluing data. Usually we will write just $\mathcal{O}$ for $\mathcal{O}_C$ and $\mathcal{O}'$ for $\pi_*\mathcal{O}_{C'}$. All the coherent sheaves over C' will be considered as $\mathcal{O}'$-modules. The category of torsion-free $\mathcal{O}_C$-modules will be denoted by $\operatorname{Coh}_{tf} C$, analogously for C'. Given a module $F \in \operatorname{Coh}_{tf} C$, define a module $F' \in \operatorname{Coh}_{tf} C'$ to be the torsion free part of $\mathcal{O}' \otimes_{\mathcal{O}} F$, i.e. $F' = (\mathcal{O}' \otimes_{\mathcal{O}} F)/\operatorname{tors}(\mathcal{O}' \otimes_{\mathcal{O}} \mathrm{F})$. Define a conductor ideal to be $J := \operatorname{ann}_{\mathcal{O}} \mathcal{O}'/\mathcal{O}$. One sees easily that J is both an $\mathcal{O}$ and an $\mathcal{O}'$ ideal and the support of $\mathcal{O}/J$ is contained in the singular locus of C.

Lemma B.1 (cf. [14])**.** *Let $F \in \operatorname{Coh}_{tf} C$. Then the canonical morphisms $F \longrightarrow F'$ and $F/JF \longrightarrow F'/JF'$ are monomorphisms and there is a cartesian diagram in* $\operatorname{Coh} C$

$$\begin{array}{ccc} F & \longrightarrow & F/JF \\ \big\downarrow & & \big\downarrow{\scriptstyle i} \\ F' & \longrightarrow & F'/JF'. \end{array}$$

The sheaf F is locally free (of constant rank r) if and only if F' is locally free of constant rank r and F/JF is a free $\mathcal{O}/J$-module of rank r.

Proof. Morphism $F \longrightarrow F'$ is a monomorphism as F is torsion free. In particular $JF \longrightarrow JF'$ is a monomorphism and actually an isomorphism, as $J \otimes_{\mathcal{O}} F \longrightarrow JF'$ is surjective (this is because $\mathcal{O}' \otimes_{\mathcal{O}} F \longrightarrow F'$ is surjective).

It follows that $F/JF \longrightarrow F'/JF'$ is a monomorphism. The fact that the diagram is cartesian follows from the surjectivity of $JF \longrightarrow JF'$ and an easy diagram chasing.

The "only if" part of the last claim of the lemma is trivial, so let us prove the converse. It is enough to prove that F is free in any singular point, so we fix such a point and work just with stalks of sheaves in this point (using the same notation as for sheaves). As F/JF is a free $\mathcal{O}/J$ module of rank r, there exists a morphism $\mathcal{O}^r \longrightarrow F$ inducing an isomorphism $(\mathcal{O}/J)^r \longrightarrow F/JF$. It follows that $\mathcal{O}^r \longrightarrow F/JF$ is surjective, and hence, by Nakayama lemma, that $\mathcal{O}^r \longrightarrow F$ is surjective. This implies, that the induced morphism $\mathcal{O}'^r \longrightarrow F'$ is surjective and therefore it is an isomorphism, as the modules are free and have the same rank. It follows, that $\mathcal{O}^r \longrightarrow F$ induces isomorphisms in three points of the cartesian diagram of the first part, therefore it is itself an isomorphism. □

Remark B.2. For A_1 and A_2 singularities the condition that F/JF is a free $\mathcal{O}/J$ module is empty because $\mathcal{O}/J$ is a field. So one has just certain rank conditions.

In view of the proved lemma we give the following definition.

Definition B.3. Define the category $\mathrm{Coh}^{\pi}_{tf}\, C$ to be the category whose objects are triples (G, V, i), where $G \in \mathrm{Coh}_{tf}\, C'$, V is a finite module over $\mathcal{O}/J$ and $i : V \longrightarrow G/JG$ is an $\mathcal{O}/J$-monomorphism, inducing an $\mathcal{O}'/J$ epimorphism $\mathcal{O}'/J \otimes_{\mathcal{O}/J} V \longrightarrow G/JG$. Morphism between two object (G_1, V_1, i_1) and (G_2, V_2, i_2) is a pair of morphisms $f_G : G_1 \longrightarrow G_2$ and $f_V : V_1 \longrightarrow V_2$ such that the following diagram commutes

$$\begin{tikzcd} V_1 \arrow[r, "f_V"] \arrow[d, "i_1"'] & V_2 \arrow[d, "i_2"] \\ G_1 \arrow[r, "f_G"] & G_2 \end{tikzcd}$$

We have shown that there is a functor

$$\varphi : \mathrm{Coh}_{tf}\, C \longrightarrow \mathrm{Coh}^{\pi}_{tf}\, C$$

sending a torsion free sheaf F to $(F', F/JF, i : F/JF \longrightarrow F'/JF')$.

Lemma B.4 (cf. [14])**.** *The functor $\varphi : \mathrm{Coh}_{tf}\, C \longrightarrow \mathrm{Coh}^{\pi}_{tf}\, C$ is an equivalence of the categories. The inverse functor ψ is given on an arbitrary object*

(G, V, i) by a cartesian product F

$$\begin{tikzcd} F \arrow[r] \arrow[d] & V \arrow[d, "i"] \\ G \arrow[r] & G/JG. \end{tikzcd}$$

Proof. If F is constructed from the triple (G, V, i) then it is clear that $F \to G$ is a monomorphism and therefore F is a torsion free $\mathcal{O}$ module, so the functor ψ is correctly defined. It follows from the previous lemma that $\psi\varphi \simeq \mathrm{Id}$. Let us show that $\varphi\psi \simeq \mathrm{Id}$. So, let there be given a triple (G, V, i) and F be constructed as earlier. We have to show that there is a canonical isomorphism between $(F', F/JF, F/JF \to F'/JF')$ and (G, V, i). From the condition that $\mathcal{O}'/J \otimes_{\mathcal{O}/J} V \to G/JG$ is surjective it follows that $\mathcal{O}' \otimes_{\mathcal{O}} F \to G/JG$ is surjective and therefore (by Nakayama lemma) $\mathcal{O}' \otimes_{\mathcal{O}} F \to G$ is surjective. The torsion of $\mathcal{O}' \otimes_{\mathcal{O}} F$ goes to zero under the last morphism, so one has surjective $F' \to G$. Both modules correspond to torsion free sheaves over C' and their ranks coincide on all components (consider nonsingular points). It follows that the surjective morphism $F' \to G$ is actually an isomorphism. We still have to show the existence of a canonical isomorphism $F/JF \to V$. Note that $F \to V \to G/JG$ maps JF to zero and as $V \to G/JG$ is a monomorphism, already $F \to V$ should map JF to zero, hence one has a (canonical) morphism $F/JF \to V$. It is surjective because $F \to V$ is. Next, we have shown already that $\mathcal{O}' \otimes_{\mathcal{O}} F \to G$ is surjective, therefore $J \otimes_{\mathcal{O}} F \to J \otimes_{\mathcal{O}'} G$ is surjective, hence $JF \to JG$ is. One easily gets now by diagram chasing, that $F/JF \to V$ is a monomorphism. $\square$

Let us look in more detail on the case of C having just nodes as singularities. Let $\pi : C' \to C$ be a normalization of nodes $q_1, \ldots, q_n$ of C (C' can still be singular). Looking on the local picture one sees that $\mathcal{O}/J$ is a sum of skyscrapers $\oplus_{k=1}^{n} \mathcal{O}(q_k)$. Let us denote by p_k, p_k^* the preimages of q_k under π. Given a triple (G, V, i), we see that $G/JG = \oplus_{k=1}^{n}(G(p_k) \oplus G(p_k^*))$, $V = \oplus_{k=1}^{n} V_{q_k}$ and i can be given by its components

$$i_k : V_{q_k} \to G(p_k) \qquad i_k^* : V_{q_k} \to G(p_k^*), \qquad k = 1, \ldots, n. \tag{B.1}$$

The surjectivity of $\mathcal{O}'/J \otimes_{\mathcal{O}/J} V \to G/JG$ is equivalent to the surjectivity of i_k and i_k^*. The injectivity of $i : V \to G/JG$ is equivalent to the injectivity of $(i_k, i_k^*)^t : V_{q_k} \to G(p_k) \oplus G(p_k^*)$. An isomorphism between two triples $(\varphi, \psi) : (G, V, i) \to (G, V, i')$ is given by isomorphisms $\varphi : G \to G$ and $\psi_k : V_{q_k} \to V_{q_k}$

satisfying certain commutativity conditions. Namely, there should commute

$$\begin{tikzcd}
V_{q_k} \arrow[rr, "\psi_k"] \arrow[d, "(i_k, i_k^*)^t"'] & & V_{q_k} \arrow[d, "(i'_k, i'^*_k)^t"] \\
G(p_k) \oplus G(p_k^*) \arrow[rr, "\varphi(p_k) \oplus \varphi(p_k^*)"'] & & G(p_k) \oplus G(p_k^*).
\end{tikzcd}$$

In particular, the isomorphism classes of triples with a simple G are given just by subspaces V_{q_k} of $G(p_k) \oplus G(p_k^*)$ for all k, with surjective morphisms in (B.1).

Given a triple (G, V, i) and an isomorphism $\varphi : G \longrightarrow G$ we may construct a new embedding $i' : V \hookrightarrow G/JG$ as a composition $V \xhookrightarrow{i} G/JG \xrightarrow{\overline{\varphi}} G/JG$. There is an isomorphism of triples $(\varphi, \mathrm{Id}_V) : (G, V, i) \longrightarrow (G, V, i')$. Analogously, given an isomorphism $\psi : V \longrightarrow V$ of $\mathcal{O}/J$ modules, we may construct a new embedding i' as a composition $V \xrightarrow{\psi} V \xhookrightarrow{i} G/JG$ and there will be an isomorphism of triples $(\mathrm{Id}_G, \psi) : (G, V, i') \longrightarrow (G, V, i)$. In this way me can look for a canonical form of embeddings i in order to classify the isomorphism classes of triples (G, V, i) with fixed G and V. Namely, we fix the bases of $V_{q_k}, G(p_k), G(p_k^*)$, $k = 1, \ldots, n$ and write the corresponding matrices of i_k and i_k^*. Then, the isomorphism $\varphi : G \longrightarrow G$ will induce certain operations on the rows of the matrices of i_k, i_k^* $\forall k = 1, \ldots, n$ and the isomorphisms $\psi_k : V_{q_k} \longrightarrow V_{q_k}$ will induce simultaneous operations on the columns of i_k and i_k^*.

If C is a rational curve and C' is its normalization, then G can be decomposed as a sum of line bundles and one can easily describe all automorphisms of G and get all possible row transformations of the matrices of i_k and i_k^*. In particular, if (G, V, i) corresponds to an element from JC, then G is an invertible sheaf of degree 0 and therefore $G \simeq \mathcal{O}_{C'}$. The isomorphisms $i_k : V_{q_k} \longrightarrow G(p_k)$ and $i_k^* : V_{q_k} \longrightarrow G(p_k^*)$ induce an isomorphism $G(p_k) \longrightarrow G(p_k^*)$ and fixing the (standard) bases of these fibers, we get certain scalars λ_k for any double point q_k. The n-tuple $(\lambda_1, \ldots, \lambda_n)$ uniquely determines the isomorphism class of (G, V, i) and this shows that $JC \simeq (C^*)^n$. Given such an n-tuple $(\lambda_1, \ldots, \lambda_n)$ corresponding to an invertible sheaf $L \in JC$ and a triple (G, V, i) corresponding to a torsion free sheaf F, we may describe a triple representing an isomorphism class of the sheaf $L \otimes F$. This is a triple (G, V, i') (G and V are the same as earlier) such that $i'_k = i_k$ and $i'^*_k = \lambda_k i_k^*$ $\forall k = 1, \ldots, n$.

Remark B.5. Let C be any reduced curve and $\pi : C' \longrightarrow C$ be its normalization. If (G, V, i) corresponds to an invertible sheaf on C, then V is a free

$\mathcal{O}/J$ module of rank 1. Therefore an $\mathcal{O}/J$ morphism $V \longrightarrow G/JG$ is given just by an element of G/JG. This element should generate G/JG as an $\mathcal{O}'/J$ module (see the definition of $\operatorname{Coh}^{\pi}_{tf} C$). In particular, one sees that for any invertible sheaf G on C' there exists a triple (G, V, i) of this kind, therefore $\operatorname{Pic} C \longrightarrow \operatorname{Pic} C'$ is surjective. The kernel of this map consists of the triples $(\mathcal{O}', V, i)$ (modulo isomorphisms) that are given by generators of $\mathcal{O}'/J$, i.e. by invertible elements of $\mathcal{O}'/J$.

Appendix C

Some facts about flat families

Lemma C.1 (cf. [26, Lemma 2.1.7]). *Let $f : X \longrightarrow Y$ be a morphism of noetherian schemes and F be a coherent sheaf on X flat over Y. If, for any $y \in Y$, the sheaf F_y is locally free over X_y then F is locally free over X.*

Proof. Let $x \in X$ and $y := f(x)$. We will prove that F_x is free over $\mathcal{O}_{X,x}$. Let r be the rank of $F(x)$ over $k(x)$. By Nakayama lemma there exists an exact sequence of modules over $\mathcal{O}_{X,x}$

$$0 \longrightarrow T \longrightarrow \mathcal{O}_{X,x}^r \longrightarrow F_x \longrightarrow 0.$$

As F_x is flat over $\mathcal{O}_{Y,y}$, we get after applying $- \otimes_{\mathcal{O}_{Y,y}} k(y)$

$$0 \longrightarrow T \otimes_{\mathcal{O}_{Y,y}} k(y) \longrightarrow \mathcal{O}_{X_y,x}^r \longrightarrow (F_y)_x \longrightarrow 0.$$

Using the condition that F_y is locally free, we deduce that the map $\mathcal{O}_{X_y,x}^r \longrightarrow (F_y)_x$ is an isomorphism and hence $T \otimes_{\mathcal{O}_{Y,y}} k(y) = 0$. This implies that $T \otimes_{\mathcal{O}_{X,x}} k(x) = 0$ and it follows now from Nakayama lemma that $T = 0$. This means that F_x is free over $\mathcal{O}_{X,x}$. $\square$

Proposition C.2. *Let $X \longrightarrow Y$ be a morphism of locally noetherian schemes and $f : F \longrightarrow G$ be a morphism of coherent sheaves on X such that G is flat over Y. Then the following conditions are equivalent:*

1. *f is a monomorphism and* $\operatorname{coker} f$ *is flat over Y.*
2. *For any $y \in Y$, the map $f_y : F_y \longrightarrow G_y$ of sheaves over X_y is a monomorphism.*

Proof. The statement is local with respect to X and Y, so we can assume that X and Y are affine. Then the implication $1 \Rightarrow 2$ is obvious. Let us prove the other one. Let $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$ and $A \longrightarrow B$ be the ring homomorphism corresponding to the morphism $X \to Y$ of schemes. Also denote by $M \to N$ the homomorphism of B-modules corresponding to $f : F \to G$. By our assumption, for any prime ideal $p \subset A$ the map $M/pM \to N/pN$ is injective. Given a prime ideal $q \subset B$, denote by p its preimage in A. We claim that $M_q \to N_q$ is injective and its cokernel is flat over A_p. The map $A_p \to B_q$ is a local map of noetherian local rings and the claim will follow from [30, Proposition 20.E] if we show that the map $M_q \to N_q$ induces a monomorphism

$$M_q \otimes_{A_p} A_p/pA_p \to N_q \otimes_{A_p} A_p/pA_p. \tag{C.1}$$

Notice that

$$M_q \otimes_{A_p} A_p/pA_p \simeq M_q/pM_q \simeq M_q \otimes_A A/p \simeq B_q \otimes_B M \otimes_A A/p$$

and so it is a localization of M/pM with respect to a multiplicative system $B\backslash q$. The same holds for N and we deduce from the injectivity of $M/pM \to N/pN$ and the exactness of localization that the map in (C.1) is also injective, as desired. Altogether implies that $M \to N$ is injective and its cokernel is flat over A. $\square$

Remark C.3. If X and Y are schemes locally of finite type over some field k then the second condition of the theorem should be checked just for closed points of Y. Indeed, it is enough to check the first condition of the theorem for closed points of X and under our assumptions the images of such points are necessarily closed in X, which allows us to repeat the argument of the proposition.

Corollary C.4. *Let X be a locally noetherian scheme and $f : F \to G$ be a morphism of sheaves on X with G being locally free. Assume that, for any $x \in X$, the induced map of fibers $f(x) : F(x) \to G(x)$ is a monomorphism. Then f is a monomorphism and* coker f *is locally free.*

Appendix D

Preliminaries on the Euler number

Given a complex algebraic variety X, we denote by $H_*(X,\mathbb{Q})$ its Borel-Moore homology groups and denote by $H^c_*(X,\mathbb{Q})$ its usual (singular) homology groups. There are isomorphisms $H^c_k(X,\mathbb{Q}) \simeq H^k(X,\mathbb{Q})^\vee$ and $H_k(X,\mathbb{Q}) \simeq H^k_c(X,\mathbb{Q})^\vee$, where $H^k_c(X,\mathbb{Q})$ denote the k-th cohomology group with compact support. Moreover, Poincaré duality says that for a smooth X of dimension n there are isomorphisms $H_k(X,\mathbb{C}) \simeq H^{2n-k}(X,\mathbb{C})$ and $H^c_k(X,\mathbb{C}) \simeq H^{2n-k}_c(X,\mathbb{C})$.

Definition D.1. The Euler number of an algebraic variety X is defined to be

$$e(X) = \sum_k (-1)^k \dim H_k(X,\mathbb{Q}).$$

It is clear from the above discussion that for a smooth or proper X one can use also the usual homologies or any of two types of cohomologies to define its Euler number. We use Borel-Moore homologies (or equivalently, cohomologies with compact support) to define the Euler number because of the following fact. If $i : Z \hookrightarrow X$ is a closed embedding and $j : U \hookrightarrow X$ is its open complement then for any sheaf F over X there is an exact triangle in the derived category of abelian sheaves over X (see e.g. [5, Subsection 1.4.3.4])

$$j_!j^*F \longrightarrow F \longrightarrow i_*i^*F \longrightarrow$$

and applying the functor $R\Gamma_c(X,-)$ we get an exact triangle

$$R\Gamma_c(U,F|_U) \longrightarrow R\Gamma_c(X,F) \longrightarrow R\Gamma_c(Z,F|_Z) \longrightarrow .$$

Taking $F = \mathbb{Q}_X$, we deduce that $e(X) = e(U) + e(Z)$.

Remark D.2. There is also a triangle

$$i_* i^! F \longrightarrow F \longrightarrow j_* j^* F \longrightarrow$$

and applying to it the functor $R\Gamma(X, -)$ we get a triangle

$$R\Gamma(Z, i^! F) \longrightarrow R\Gamma(X, F) \longrightarrow R\Gamma(U, F|_U) \longrightarrow .$$

However, we cannot deduce from this the existence of the triangle

$$R\Gamma(Z, \mathbb{Q}_Z) \longrightarrow R\Gamma(X, \mathbb{Q}_X) \longrightarrow R\Gamma(U, \mathbb{Q}_U) \longrightarrow$$

because in general $i^! \mathbb{Q}_X \not\simeq \mathbb{Q}_Z$.

Given a smooth projective variety X, one defines its Hodge polynomial to be

$$E(X; u, v) = \sum_{p,q} (-1)^{p+q} \dim H^{p,q}(X, \mathbb{C}) u^p v^q.$$

This definition can be extended to all algebraic varieties so that the polynomial will be additive with respect to the complements [12]. There exists a natural mixed Hodge structure $(W_., F^{\cdot})$ on the cohomologies with compact support $H^k_c(X, \mathbb{Q})$ (see [12] or see [13] for the mixed Hodge structure on $H^k(X, \mathbb{Q})$). Define then $h^{p,q}(H^k_c(X, \mathbb{C}))$ to be the dimension of the (p,q)-th component of $H^k_c(X, \mathbb{C})$ i.e.

$$h^{p,q}(H^k_c(X, \mathbb{C})) = \dim(F^p \mathbf{Gr}^W_{p+q} H^k_c(X, \mathbb{C}) \cap \overline{F^q \mathbf{Gr}^W_{p+q} H^k_c(X, \mathbb{C})}).$$

Definition D.3. The E-polynomial (also called Hodge-Deligne polynomial or virtual Hodge polynomial) of a complex algebraic variety X is defined to be

$$E(X; u, v) = \sum_{p,q} \left(\sum_k (-1)^k h^{p,q}(H^k(X, \mathbb{C})) \right) u^p v^q.$$

Remark D.4. The Euler number of an algebraic variety X equals $e(X) = E(X; 1, 1)$. One defines the Poincaré polynomial of a smooth projective variety X by

$$P(X; t) = \sum_k (-1)^k \dim H^k(X, \mathbb{Q}) t^k.$$

Then $P(X; t) = E(X; t, t)$ and we can extend the Poincaré polynomial to all algebraic varieties taking $P(X; t) = E(X; t, t)$.

Let us formulate the basic properties of the E-polynomial

1. If $X = \coprod X_i$ is a disjoint union of locally closed subvarieties then $E(X) = \sum E(X_i)$.
2. If $f : X \longrightarrow Y$ is a locally trivial fibration with fiber F with respect to the Zarisky topology then $E(X) = E(Y) \cdot E(F)$.

For example $E(\mathbf{pt}) = 1$, $E(\mathbb{P}^1) = 1 + uv$. Let us denote $\mathbb{L} = E(\mathbb{C}) = uv$. Then it holds $E(\mathbb{C}^n) = \mathbb{L}^n$ and

$$E(\mathbb{P}^n) = 1 + \mathbb{L} + \cdots + \mathbb{L}^n = \frac{1 - \mathbb{L}^{n+1}}{1 - \mathbb{L}}.$$

More generally

Lemma D.5. *It holds*

$$E(\mathrm{Gr}(k, n)) = \prod_{i=1}^{k} \frac{1 - \mathbb{L}^{n-i+1}}{1 - \mathbb{L}^i}.$$

Proof. We prove the claim by induction on k. For $k = 1$, $\mathrm{Gr}(k, n) = \mathbb{P}^{n-1}$ and the statement is clear. For $k > 1$ consider the incidence variety

$$Z = \{(V, W) \in \mathrm{Gr}(k-1, n) \times \mathrm{Gr}(k, n) \mid V \subset W\}.$$

Then the projection $Z \longrightarrow \mathrm{Gr}(k-1, n)$ is a bundle with fiber $\mathrm{Gr}(1, n-k+1)$ and $Z \longrightarrow \mathrm{Gr}(k, n)$ is a bundle with fiber $\mathrm{Gr}(1, k)$. It holds therefore

$$E(\mathrm{Gr}(k, n)) = E(\mathrm{Gr}(k-1, n)) \frac{E(\mathrm{Gr}(1, n-k+1))}{E(\mathrm{Gr}(1, k))}$$

and we use the induction assumption. $\square$

We will need later also the E-polynomials of symplectic Grassmanians. Given a symplectic vector space $(\mathbb{C}^{2n}, \omega)$, we denote by $\mathrm{Gr}^{\omega}(k, 2n)$ the set of k-dimensional subspaces of $\mathbb{C}^n$ isotropic with respect to ω (i.e. the restriction of ω to the subspace is zero). In the same way as above one can prove

Lemma D.6 (cf. [10, Lemma 3.1])**.** *It holds*

$$E(\mathrm{Gr}^{\omega}(k, 2n)) = \prod_{i=1}^{k} \frac{1 - \mathbb{L}^{2n-2i+2}}{1 - \mathbb{L}^i}.$$

In particular we deduce for the Grassmanian of Lagrangian subspaces

$$E(\mathrm{Gr}^{\omega}(n, 2n)) = \prod_{i=1}^{n} (1 + \mathbb{L}^i).$$

Remark D.7. It is clear that one can compute in the same way the E-polynomial of any flag variety of a semi-simple algebraic group using the Bruhat decomposition.

Remark D.8. An important property of the Euler number is that it is multiplicative also with respect to the locally trivial fibrations in etale topology, i.e. if $f : X \longrightarrow Y$ is a locally trivial fibration with fiber F with respect to the etale topology then $e(X) = e(Y) \cdot e(F)$. This implies for example that if $f : X \longrightarrow Y$ is a morphism with almost all fibers (i.e. except the finite number) having zeroth Euler number, then $e(X)$ equals the sum of Euler numbers of the fibers. Indeed, one can stratify $Y = \coprod Y_i$ in such a way that f is locally trivial over Y_i with respect to the etale topology and now our statement easily follows.

Lemma D.9. *Let T be a torus (i.e. the product of $\mathbb{C}^*$) acting on a quasi-projective variety X. Then $e(X) = e(X^T)$, where X^T is subspace of X consisting of fixed points under the action of T.*

Proof. This fact is a folklore but we give an elementary proof for the convenience of the reader. Doing induction on the dimension of T we can reduce the problem to $T \simeq \mathbb{C}^*$. Let p be any prime number and μ_{p^n} be a subgroup of $\mathbb{C}^*$ consisting of p^n-th roots of unit. The sequence

$$X \supset X^{\mu_{p^1}} \supset \dots \supset X^{\mu_{p^n}} \supset \dots$$

stabilizes, say from index N, as X is noetherian. Also one see that $\bigcap_n X^{\mu_{p^n}} = X^{\mathbb{C}^*}$ as $\bigcup_n \mu_{p^n}$ is dense in $\mathbb{C}^*$ (in the Zariski topology). It follows that $X^{\mathbb{C}^*} = X^{\mu_{p^N}}$. Therefore the action of μ_{p^N} on $X - X^{\mathbb{C}^*}$ does not have fixed points. As $X - X^{\mathbb{C}^*}$ is quasi-projective, there exists a geometric quotient

$$X - X^{\mathbb{C}^*} \longrightarrow (X - X^{\mathbb{C}^*})/\mu_{p^N}$$

and all of its fibers are isomorphic to orbits. As they all are nontrivial it follows that the numbers of their elements are multiples of p. Hence, the Euler number of $X - X^{\mathbb{C}^*}$ is a multiple of p. As it holds for any prime p, we deduce that $e(X - X^{\mathbb{C}^*}) = 0$. $\square$

Under certain conditions, given a singular algebraic variety X, one can predict the E-polynomial (and the Euler number) of its crepant resolution without knowing the later. This is the essence of the definition of the stringy E-function.

Definition D.10. A resolution of singularities $f : Y \longrightarrow X$ of a normal irreducible algebraic variety X is called a log resolution of X if the exceptional locus D of f consists of smooth irreducible components $D_1, \ldots, D_m$ with only normal crossings. A normal irreducible variety X is called log terminal if K_X is a $\mathbb{Q}$-Cartier divisor and for some (hence any) log resolution $f : Y \longrightarrow X$, writing $K_Y = f^*K_X + \sum_{i=1}^m a_i D_i$ with $a_i \in \mathbb{Q}$, we have $a_i > -1$ for all i.

Definition D.11. Let X be a log terminal algebraic variety and $f : Y \longrightarrow X$ be a log resolution with $K_Y = f^*K_X + \sum_{i=1}^m a_i D_i$. Define for any $J \subset I = \{1, \ldots, m\}$, $D_J = \cap_{i \in J} D_i$ and $D_J^0 = D_J - \cup_{i \in I-J} D_i$. Then the stringy E-function of X is defined by

$$E_{st}(X; u, v) = \sum_{J \subset I} E(D_J^0; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1}.$$

The stringy Poincaré function is defined by $P_{st}(X; t) = E_{st}(X; t, t)$. The stringy Euler number $e_{st}(X)$ is defined by $\lim_{u,v \to 1} E_{st}(X; u, v)$. In other words

$$e_{st}(X) = \sum_{J \subset I} e(D_J^0) \prod_{j \in J} \frac{1}{a_j + 1}.$$

Theorem D.12 (see [2, Theorem 3.8] or [11, Theorem 3.6])**.** *Let X be a log terminal algebraic variety. Then the stringy E-function of X does not depend on the choice of a log resolution.*

In particular it follows that if there exists a crepant resolution $f : Y \longrightarrow X$ then $E_{st}(X) = E(Y)$.

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Selbständigkeitserklärung

Hiermit versichere ich, daß ich die vorliegende Dissertation selbständig und ohne unerlaubte Hilfe angefertigt habe.

Mainz, den 20. Oktober 2006

Sergiy Mozgovyy