Generalized soluble groups of finite co-central rank

Dissertation zur Erlangung des Grades

"Doktor der Naturwissenschaften"

am Fachbereich Mathematik
der Johannes-Gutenberg-Universität Mainz

von

Achim Tresch

Mainz, im März 2002

V, ein großer Vektorraum
wagte sich zu fragen kaum
ob es vielleicht gelänge,
dass er in vord're Ränge
der auflösbaren Gruppen dränge.

Die meinten: "Nein, für deine Länge
herrscht bei uns zu große Enge.
Doch noch ist nichts verloren,
wir haben nämlich geseh'n,
dass keine zentralen Faktoren
der Mitgliedschaft entgegensteh'n."

So wurde mit Beifall und Toben
V in den Co-Rang eins erhoben.

Contents

Introduction

This thesis introduces the notion of the finite co-central rank of a group, a weakened Prüfer rank condition. The structure theory of generalized soluble groups of finite co-central rank will be developed.

Historical background: Groups of finite Prüfer rank.

A group is said to have *finite Prüfer rank* r if every finitely generated subgroup can be generated by r elements, and r is the least such integer. Such groups were first considered for instance by Baer, Mal'cev and S.N. Černikov. The notion of finite Prüfer rank has its origin in the theory of abelian groups since for an elementary abelian p-group A, the Prüfer rank coincides with the dimension of A regarded as a vector space over the field with p elements. Thus the finite Prüfer rank condition has a strong influence on the abelian subgroups and the abelian sections of a group. Henceforth, the Prüfer rank will be simply called the *rank* of a group.

The structure of abelian groups A of finite rank r is well-known ([14], 1.9-1.12). Thus for each prime p, the p-component of A is a direct product of at most r groups which are either cyclic groups or Prüfer p-groups. The factor group of A modulo its torsion subgroup embeds into a direct product of at most r copies of the additive group of the rationals. As expected, classes of groups containing many abelian sections can be described particularly well if they have finite Prüfer rank. Appropriate classes of groups considered in the literature are generalized nilpotent and generalized soluble groups. The standard reference for this topic is Robinson's book [35], see also M.R. Dixon [14]. We mention some important results on groups of finite rank.

Theorem. A. Let G be a group of finite Prüfer rank.

a) G is locally nilpotent if and only if G is hypercentral (Mal'cev, see [35], p.50).
b) G is locally soluble if and only if G is hyperabelian (cf. Robinson [35], pp.80-82).
c) G is locally (soluble-by-finite) if and only if G is (locally soluble)-by-finite (N.S. Černikov [8]).

For locally nilpotent groups of finite rank, a very precise structure theorem is available.

Theorem B. Let G be a locally nilpotent group of finite rank r.

a) If G is torsion-free, then it is nilpotent of nilpotency class at most r (Čarin [6], Theorem 9).
b) If G is periodic, then it has Černikov p-components (see M.R. Dixon [14], Theorem 7.6).

The structural results on hyperabelian groups of finite rank are listed in the next theorem.

Theorem C. Let G be a hyperabelian group of finite rank r.

a) If G is torsion-free, then G is soluble of r-bounded derived length (see M.R. Dixon [14], Theorem 9.3).
b) If G is periodic, then for any prime p, the Sylow p-subgroups of G are conjugate (Baer and Heineken [3]).
c) Each chief factor of G is finite and every maximal subgroup of G has finite index in G (Robinson [37], Theorems C and D).
d) G is $\mathcal{F}$-perfect if and only if G is radicable nilpotent with central torsion group (see [35], Theorem 9.31).

Here, a group is *$\mathcal{F}$-perfect* if it has no non-trivial finite homomorphic images. The group G is called *radicable* if for every $n \in \mathbb{N}$ and every $g \in G$, the equation $x^n = g$ has a solution in G. A group is a *minimax* group if it has a finite series each of whose factors satisfies either the maximal or the minimal condition on subgroups. The soluble minimax groups form an important subclass of the hyperabelian groups of finite rank, as the next result reveals.

Theorem D.

a) A finitely generated hyperabelian group of finite rank is a soluble minimax group (Robinson [36]).
b) Let G be a soluble minimax group and let R be the subgroup generated by all quasicyclic subgroups of G. Then R is the direct product of finitely many quasicyclic subgropus of G and R is the finite residual of G (Robinson [35], Theorem 10.33).

Research activity has been focussing on rank restrictions on abelian subgroups. Kargapolov [20] proved that a soluble group whose abelian subgroups have finite rank likewise has finite rank. A famous result of Šunkov [41] states that a locally finite group whose abelian subgroups all have finite rank likewise has finite rank and is (locally soluble)-by-finite. Locally soluble groups in which all abelian subgroups are of bounded rank are likewise hyperabelian of finite rank (Merzljakov [30]). The paper of Robinson [34] gives a comprehensive survey on the results in this area. Recently, weakened rank conditions have been considered by several authors such as Maj, Langobardi, Smith [25] and Dixon, Evans, Smith [12].

Scope of this work

Results on groups with finite Püfer rank rely heavily on the restrictions imposed on the abelian subgroups of a group. We introduce a new, less restrictive rank condition which does not immediately affect the abelian subgroups of a group.

Definition. Let $s \in \mathbb{N}$. A group is said to have *finite co-central rank* s if every finitely generated subgroup can be generated by s elements *modulo its centre*, and s is the least such integer. Denote the class of groups of finite co-central rank by $\mathfrak{R}$.

It is obvious that the elements of the centre $Z(G)$ of a group G have no influence on the co-central rank of G. In particular, all abelian groups are $\mathfrak{R}$-groups of co-central rank 1. So groups of finite co-central rank form a much wider class than the groups of finite Prüfer rank. In the following, we determine the structure of groups of finite co-central rank. It turns out that the apparently weaker finite co-central rank condition implies results which were formerly known only for groups of finite Prüfer rank. In particular, if we raplace "finite rank" by "finite co-central rank", Theorems A, C and D hold literally and Theorem B remains valid if we replace G by $G/Z(G)$.

Although (locally) nilpotent groups generally have many properties in common with abelian groups, (locally) nilpotent $\mathfrak{R}$-groups have a surprisingly restricted structure. We derive an analogue of the already mentioned theorem of Čarin [6].

Theorem (2.7, 2.15) Let G be a locally nilpotent group with finite co-central rank s. Then G is hypercentral and

a) If G is torsion-free, then $r(G/Z(G)) = s$. In particular, G is nilpotent of class at most $s+1$.
b) If G is a p-group, then it is abelian-by-finite and the Prüfer rank of $G/Z(G)$ is bounded by a function $f(p, s)$ depending only on p and s.

The above theorem establishes the best possible relation between Prüfer rank and co-central rank, since we cannot hope to control the centre of a group of finite co-central rank. It is also impossible to choose the function f of part b) independently of p.

It is much harder to show that a group of finite co-central rank is locally soluble if and only if it is hyperabelian. In the finite rank case, the eqivalence of locally soluble and hyperabelian groups is essentially a consequence of Zassenhaus' Theorem on the derived length of a soluble linear group (see [45], Theorem 3.7). In the finite co-central rank case, this tool is usually not available, as the ranks of the abelian sections in $\mathfrak{R}$-groups are not bounded in general. The main obstacle is to establish the existence of abelian normal subgroups of finite rank. This requires a detailed analysis of the way an $\mathfrak{R}$-group operates on its abelian normal subgroups. We obtain

Theorems 3.14, 3.15 Let the hyperabelian group G have finite co-central rank.

a) The group G has an ascending series with abelian factors of finite rank.
b) Each chief factor of G is finite and each maximal subgroup of G has finite index.

The structure of hyperabelian groups of finite co-central rank is described in more detail by the next theorem. Let $\mathfrak{X}$ be the class of locally nilpotent groups with finite p-components for each prime p.

Theorem 3.19 Let G be a locally soluble group of finite co-central rank s.

a) There exists an integer d depending only on s such that $G^{(d)}$ has a periodic locally nilpotent characteristic subgroup N with $G^{(d)}/N \in \mathfrak{X}$.
b) If G is torsion-free, then G is soluble of s-bounded derived length.

An important theorem of Lubotzky and Mann [26] says that finitely generated residually finite groups of finite rank are soluble-by-finite. Employing an analogue of this (Proposition 3.26), we deduce

Theorem 3.31 Let G be a group with finite co-central rank. The following conditions are equivalent:

1) G is locally soluble,
2) G is hyperabelian,
3) G is radical.

Here, a group is called *radical* if it has an ascending normal series with locally nilpotent factors.

The theorem of Lubotzky and Mann [26] was generalized by Dixon, Evans and Smith in [11]. They proved that a group which is residually (finite of bounded rank) already is (locally soluble)-by-finite. This enables us to weaken the hypothesis of Theorem 3.31 in such a way that G is merely residually (of bounded co-central rank), see Theorem 3.34.

The theorem of Šunkov [41] on locally finite groups of finite Prüfer rank is extended to locally finite $\mathfrak{R}$-groups. The finite simple sections in a locally finite $\mathfrak{R}$-group play a crucial role. It is verified in Proposition 4.7 that a locally finite $\mathfrak{R}$-group has no infinite non-abelian chief factors. The next theorem is then derived from this fact.

Theorem 4.9 If G is a locally finite group with finite co-central rank, then G is hyperabelian-by-finite and hyperfinite.

N.S.Černikov shows in [8] that locally (soluble-by-finite) groups of finite rank are (locally soluble)-by-finite. We generalize his result to groups of finite co-central rank which have a *section cover* consisting of locally (soluble-by-finite) groups. The notion of a section cover of a group is used to investigate simultaneously groups having a local system as well as groups having a residual system.

Let G be a group and let M be a set consisting of sections of G. There is a natural partial ordering on M defined by $A_1/B_1 \leq A_2/B_2$ if and only if A_1/B_1 is a section of A_2/B_2. We call M a *section cover* of G if

1) $(M, \leq)$ is a directed set, i.e. to every $i, j \in M$ there exists a $k \in M$ such that $i \leq k$, $j \leq k$.
2) For every $g \in G\backslash\{1\}$, there exists a section $A/B \in M$ such that $g \in A \setminus B$.

With this terminology, the main theorem of the thesis can be stated. For groups of finite co-central rank, large classes of groups are shown to coincide with the class of hyperabelian-by-finite groups.

Theorem 4.19 Let G be a group with finite co-central rank. The following conditions are equivalent:

1) G has a section cover consisting of locally (soluble-by-finite) groups,
2) G is locally (soluble-by-finite),
3) G is (locally soluble)-by-finite,
4) G is residually ((locally soluble)-by-finite),
5) G is generalized radical,
6) G is hyperabelian-by-finite.

Thus the investigation of generalized soluble groups of finite co-central rank naturally leads to the class of hyperabelian groups. For hyperabelian groups in turn, it has been demonstrated that almost the whole structure theory of groups of finite Prüfer rank can be replaced by the theory of groups of finite co-central rank.

Acknowledgements

It is a pleasure to express my gratitude to my Ph.D. supervisor Prof. A. for his guidance through the theory of infinite groups, for his friendly support and helpful advice. Lots of thanks are also due to Prof. S. whose valuable suggestions were a large source of inspiration. I appreciate the many time I spent in discussion with Prof. L. and Prof. K. While writing my thesis, the patience and warm encouragement of Imke T. was a great help to me. Above all, I thank my parents to whom I dedicate this work. It is the least I can do in recognition of their generous support and their confidence, which made all this possible to me.

Mainz, March 2002

Chapter 1

Preliminaries and Examples

Groups with finite Prüfer rank are subject to a large number of investigations (see [3],[4],[6],[8],[20],[26],[36],[35],[48]). Recall that a group G has finite Prüfer rank r if and only if every finitely generated subgroup can be generated by r elements, and r is minimal with this property. We introduce a less restrictive rank condition which does not immediately affect the structure of the abelian subgroups of a group.

Definition 1.1 *A group G has* finite co-central rank $s \in \mathbb{N}$ *if for every finitely generated subgroup H of G, the group H can be generated by s elements modulo its centre $Z(H)$, and s is the least integer with this property. If no such integer exists, G has infinite co-central rank. Denote the Prüfer rank of a group G by $r(G)$ and the co-central rank of G by $r_c(G)$. The class of all groups with finite co-central rank is denoted by the letter $\mathfrak{R}$.*

Clearly always $r_c(G) \leq r(G)$, so the finite co-central rank condition is a weaker condition than the finite Prüfer rank condtion. We give some examples of groups having finite co-central rank.

Examples 1.2 a) The groups of co-central rank 1 are exactly the abelian groups. Indeed, an abelian group clearly has co-central rank 1. A group of co-central rank 1 is locally (central-by-cyclic) by definition, so it is locally abelian and hence abelian.

b) Let G be a Miller-Moreno group (see [31]), that is a finite group all of whose proper subgroups are abelian, but G is non-abelian. Such a group must be 2-generated, so its co-central rank is 2.

c) Ol'shanskii [32] constructs groups which are infinite simple, 2-generated and any proper subgroup is finite abelian. These groups also have co-central rank 2, so one cannot hope to characterize all groups of finite co-central rank without imposing further structural conditions.

d) A group G whose central factor group $G/Z(G)$ has finite Prüfer rank has finite co-central rank, and $r_c(G) \leq r(G/Z(G))$. This is seen by letting H be a finitely generated subgroup of G. Then

$$d(H/Z(H)) \leq d(H/(H \cap Z(G))) = d(HZ(G)/Z(G)) \leq r(G/Z(G)) \ .$$

It is obvious that the property of having finite co-central rank at most s is inherited by subgroups, and one verifies that it is also inherited by homomorphic images:

Lemma 1.3 *Let G be a group with finite co-central rank s. If N is a normal subgroup of G, then the co-central rank of G/N is bounded by s.*

Proof. If H/N is a finitely generated subgroup of G/N, then $H = KN$ for some finitely generated subgroup K of G. Let Z be the preimage of the centre $Z(H/N)$ of H/N in G. Then $Z(K)N \leq Z$ and $(H/N)/Z(H/N) \cong KN/Z$ is a homomorphic image of $KN/Z(K)N$, which is in turn a homomorphic image of $K/Z(K)$. As the latter group can be generated by at most s elements, so does $(H/N)/Z(H/N)$. □

In contrast to groups with finite Prüfer rank the property of having finite co-central rank is not closed with respect to extensions, not even if they split. For instance, the holomorph of the additive group of rationals is a metabelian group with infinite co-central rank. It is not even known whether the property of having finite co-central rank is inherited by direct products.

Question 1.4 *Does the direct product $A \times B$ of two groups A and B of finite co-central rank again have finite co-central rank? If the answer is positive, is there a function $f(r_c(A), r_c(B))$ which bounds the co-central rank of $A \times B$?*

Example 1.2 d) raises the question whether any group with finite co-central rank has its central factor group of finite rank. However, the next example is a group of finite co-central rank whose central factor group has infinite rank and so the relation between both ranks is more complicated than it seems. The example also shows that several natural assumptions concerning groups of finite co-central rank are wrong. It is preceded by a lemma.

Lemma 1.5

a) Let the group G have a central subgroup K which intersects G' trivially. Then $r_c(G) = r_c(G/K)$.

b) If $G = N \rtimes A$ is the semidirect product of the normal subgroup N with the abelian group A, then $r_c(G) = r_c(G/(\,A \cap Z(G)\,))$.

Proof. a) We only have to prove that $r_c(G) \le r_c(G/K)$. Let H be a finitely generated subgroup of G. For any $y(H\cap K) \in Z(H/(H\cap K))$ we have $[y,H] \in G' \cap K = 1$ and thus $Z(H/(H\cap K)) = Z(H)/(H\cap K)$. Consequently,

$$\begin{aligned} d(H/Z(H)) &= d(\ (H/(H\cap K))\ /\ Z(H)/(H\cap K)\ ) \\ &= d(\ H/(H\cap K)\ /\ Z(H/(H\cap K))\ ) \\ &\le r_c(H/(H\cap K)) = r_c(HK/K) \le r_c(G/K)\ . \end{aligned}$$

Thus $r_c(G) \le r_c(G/K)$.
b) This follows from a) because $(A \cap Z(G)) \cap G' \le A \cap N = 1$. □

Example 1.6 *There exists a metabelian group G with finite co-cental rank 2 which has the following properties:*

1) *G' and G/G' have infinite Prüfer rank, G/G' is even uncountable.*
2) *G' is the unique maximal abelian normal subgroup of G.*
3) *$C_G(G') = G'$ and $Z(G) = 1$.*

Proof. First of all we note that if C_p is a cyclic group of prime order p and V is an irreducible $\mathbb{F}_qC_p$-module over the field $\mathbb{F}_q$ of order $q \ne p$, then the set of all subgroups of the semidirect product $K = V \rtimes C_p$ of V with C_p is

$$\{K\} \cup \{U \,|\, U \le V\} \cup \{C_p^g \,|\, g \in V\}\ .$$

Therefore K is a Miller-Moreno group (see [31]) and its co-central rank is 2.

Now let $(p_k)_{k\in\mathbb{N}}, (q_k)_{k\in\mathbb{N}}$ be strictly increasing seqences of primes with $p_k \ne q_j$ for all $k, j \in \mathbb{N}$. If $\langle \alpha_k \rangle$ is a group of order p_k, then by Maschke's Theorem, there is an irreducible faithful α_k-module V_k over the field $\mathbb{F}_{q_k}$. Let $(p_k)_{k\in\mathbb{N}}$ and $(q_k)_{k\in\mathbb{N}}$ be chosen such that the ranks of the modules A_k are unbounded for $k \in \mathbb{N}$. Put $A = \mathrm{Dr}_{k\in\mathbb{N}} V_k$, a group of infinite rank. Now choose a mapping $\tau : \mathbb{N} \to \mathbb{N}$ such that the preimage $\tau^{-1}(n)$ of n under τ is an infinite set for every $n \in \mathbb{N}$. Consider the set of all integer sequences $H = \{\,(f_k)_{k\in\mathbb{N}} \,|\, f_k \in \mathbb{Z} \text{ for all } k \in \mathbb{N}\,\}$ as an abelian group via componentwise addition. Define the semidirect product $G = A \rtimes H$, the action of H on A being determined by the rules

$$v_k^f = v_k \alpha_k^{f_{\tau(k)}} \quad \text{for all} \ \ k \in \mathbb{N},\ v_k \in V_k,\ f = (f_j)_{j\in\mathbb{N}} \in H\ .$$

It is easy to see that $G' = A$, so G' and G/G' have infinite rank, $G/G' \cong H$ is uncountable, and that proves claim 1). Clearly $r_c(G) \ge 2$ since G is not abelian. We show that $r_c(G) \le 2$. Let F be a finitely generated subgroup of G. Then F lies in $D = \langle f_1, ..., f_r, V_1, ..., V_r \rangle$ for some $r \in \mathbb{N}$ and some $f_1, ..., f_r \in H$. In order to prove $d(F/Z(F)) \le 2$ it is sufficient to show $r_c(D) \le 2$. Note that D is the semidirect product of its normal subgroup $(D \cap A) = \langle V_1, ..., V_r \rangle$ with $(D \cap H) = \langle f_1, ..., f_r \rangle$ and that $H \cap Z(D) = C_{D\cap H}(D \cap A)$. So by Lemma 1.5 b),

$$r_c(D) = r_c(D/(H \cap Z(D))) = \ r_c(D/C_{D\cap H}(D \cap A))\ .$$

Let $K = \mathrm{Dr}_{k=1}^{r}(V_k \rtimes \langle \alpha_k \rangle)$. Check that the mapping from $\{f_1, ..., f_r\}$ to K, given by

$$f_j \mapsto \alpha_1^{f_{j\,\tau(1)}} \cdot ... \cdot \alpha_r^{f_{j\,\tau(r)}} \ , \quad j = 1, ..., r \ ,$$

together with the obvious injection of $D \cap A$ into K, uniquely extends to a homomorphism $\gamma : D \to K$. The kernel of γ is then $C_{D\cap H}(D \cap A)$, so $D/C_{D\cap H}(D \cap A)$ is isomorphic to a subgroup of K. As has been noted above, each single semidirect product $V_k \rtimes \langle \alpha_k \rangle$ has co-central rank 2. Their direct product also has co-central rank 2 because all semidirect products are coprime by construction. So $r_c(D) = r_c(D/(H \cap Z(D))) \leq r_c(K) \leq 2$ and consequently $r_c(G) = 2$.

Let $g \in G \setminus A$, write $g = af$ for some $a \in A$ and some $f \in H$ which is not the null sequence. Choose an $n \in \mathbb{N}$ for which $f_n \neq 0$. Since $\tau^{-1}(n)$ is infinite and $(p_k)_{k\in\mathbb{N}}$ is strictly increasing, there is a $k \in \tau^{-1}(n)$ such that $p_k > |f_n|$. The order of α_k is p_k, so $\alpha_k^{f_n} \neq 1$. Let $v_k \neq 1$ be an element of V_k. Necessarily, $\alpha_k^{f_n}$ operates without fixed-points on $V_k \setminus \{0\}$, which implies

$$w_k = [v_k, g] = [v_k, f] = v_k^{-1} v_k^f = v_k^{-1}(v_k \alpha_k^{f_{\tau(k)}}) = v_k^{-1}(v_k \alpha_k^{f_n}) \neq 1 \ .$$

By the same argument, $[w_k, g] \neq 1$. The normal closure g^G of g in G contains $\langle g, [g, v_k]\rangle = \langle g, w_k\rangle$, a non-abelian group. Thus g^G is non-abelian for every $g \in G \setminus A$. As a result, any abelian normal subgroup of G must be contained in A, and $G' = A$ is the unique maximal abelian normal subgroup of G. The centralizer $C_G(A)$ must equal A, because otherwise for $g \in C_G(A) \setminus A$, the group $A\langle g\rangle$ would be an abelian normal subgroup of G which is not contained in A. Furthermore, the centre of G is contained in A, so $Z(G) = C_A(G) = C_A(H) = 1$ by construction of G. This proves claims 2) and 3) and concludes the example. □

If G is a group with finite co-central rank, finding normal subgroups of G modulo which G has finite Prüfer rank turns out to be a difficult but important task. The following lemma indicates some cases in which $r(G/NC_G(N))$ is finite for a normal subgroup N of G. This draws the attention towards abelian normal subgroups N of G since there, $N \leq C_G(N)$. It has been shown by Example 1.6 that the periodicity condition for the factor group $G/C_G(N)$ in part b) of the next lemma is essential.

Lemma 1.7 *Let the group G have finite co-central rank s. Let N be a normal subgroup of G.*

- *a) If there exists a finite subset $F \subseteq N$ such that $C_G(F) = C_G(N)$, then $r(G/NC_G(N)) \leq s$. In particular, if N is abelian then $r(G/C_G(N)) \leq s$.*
- *b) If N is abelian and the factor group $G/C_G(N)$ is periodic, then $r(G/C_G(N)) \leq 2s$.*

Proof. a) Let $H/NC_G(N)$ be a finitely generated subgroup of $G/NC_G(N))$. Then there exists a finitely generated subgroup K of G such that $H = KNC_G(N)$. Let

$L = \langle K, F\rangle$. The group $L/Z(L)$ is s-generated by the hypothesis, and $Z(L) \leq C_G(F) = C_G(N)$. It follows that

$$\begin{aligned} s &\geq d(L/Z(L)) \geq d(NC_G(N)L/NC_G(N)Z(L)) \\ &= d(KNC_G(N)/NC_G(N)) = d(H/NC_G(N)) . \end{aligned}$$

This proves $r(G/NC_G(N)) \leq s$. The second statement is now obvious.

b) If $G/C_G(N)$ is finite, then there exists some finite subset $F \subseteq N$ such that $C_G(N) = C_G(F)$ and the result follows from part a). For the general case, let K be a finitely generated subgroup of G. Since $d(K/Z(K)) \leq s$, there exists an s-generated subgroup S of K and a finitely generated subgroup $T \leq Z(K)$ such that $K = ST$. The group $TC_G(N)/C_G(N)$ is finitely generated, periodic and abelian, hence finite. So $d(TC_G(N)/C_G(N)) \leq s$ by the finite case. Consequently,

$$d(KC_G(N)/C_G(N)) = d(STC_G(N)/C_G(N)) \leq s + d(TC_G(N)/C_G(N)) \leq 2s .$$

This proves $r(G/C_G(N)) \leq 2s$. □

Later, other cases will occur in which the rank of $G/C_G(N)$ is finite for some normal subgroup N (Lemmas 2.6, 3.6, 3.9). We finish this chapter with two more useful examples. They ensure that groups with finite co-central rank do not contain any section isomorphic to some "large" wreath product.

Example 1.8 *Let p be a prime and $G = C_p \wr C_{mn}$ with positive integers m and n. Then the co-central rank of G is at least $n(m-1)/m$.*

Proof. Let $B = \langle c_1\rangle \times ... \times \langle c_{mn}\rangle$ be the base group of G with cyclic subgroups $\langle c_i\rangle$ of order p. Take a generator a of the top group of G such that $c_i^a = c_{i+1}$ for $1 \leq i \leq mn-1$ and $c_{nm}^a = c_1$. Examine the subgroup $H = \langle a^n, B\rangle$ of G. Clearly

$$Z(H) = C_B(a^n) = \langle c_j c_{j+n} ... c_{j+n(m-1)} \,|\, j = 1, ..., n\rangle .$$

Thus, considering B as an $\mathbb{F}_p$-vector space, we have $\dim Z(H) = n$ and $\dim(B/Z(H)) = n(m-1)$. If $d(H/Z(H)) = k$, H is generated modulo $Z(H)$ by some elements $a_1b_1, ..., a_kb_k$ with $a_i \in \langle a^n\rangle$ and $b_i \in B$. The derived subgroup H' of H is then generated by the commutators $[b_i, a^{nl}]$ with $1 \leq i \leq k$ and $1 \leq l \leq m$. Since $H' = [B, a^n] \cong B/C_B(a^n)$, we have $n(m-1) = \dim(H') \leq km$ which means that $r_c(G) \geq k \geq n(m-1)/m$. □

Example 1.9 *For any prime p the (restricted) wreath product $C_p \wr C_\infty$ has infinite co-central rank.*

Proof. Let $B = \mathrm{Dr}_{i\in\mathbb{Z}}\langle c_i\rangle$ with subgroups $\langle c_i\rangle$ of order p be the base group of G and $C_\infty = \langle t\rangle$. Show that for any positive integer n the subgroup $K = \langle c_0, c_1, \dots, c_{n-1}, t^n\rangle$ has co-central rank $r_c(K) = n+1$. Since K is a soluble $(n+1)$-generated group with trivial centre, we have to prove that it is not n-generated. A short calculation gives us $K/K' \cong \langle c_0\rangle \times \dots \times \langle c_{n-1}\rangle \times C_\infty$. Hence $d(K/Z(K)) = d(K) \geq d(K/K') = n+1$, which means that $r_c(G) = \infty$. $\square$

Recall that a group G is *minimax* if it has finite series whose factors satisfy the minimum or maximum condition on subgroups. For an account of the main results on soluble minimax groups we refer to [35], Chapter 10.
By Lemma 1.9, a group of finite co-central rank does not contain any section isomorphic to the wreath product $C_p \wr C_\infty$ for any prime p. A theorem of Kropholler [24] states that a finitely generated soluble group is minimax if for any prime p it has no section isomorphic to the wreath product $C_p \wr C_\infty$. So we obtain the following frequently used proposition.

Proposition 1.10 *A finitely generated soluble group with finite co-central rank is minimax.*

Chapter 2

Nilpotent and locally nilpotent $\mathfrak{R}$-groups

2.1 The central factor group of locally nilpotent $\mathfrak{R}$-groups

Regarding Example 1.2 d), the most convenient relation between co-central rank and Prüfer rank would be that a group G has finite co-central rank if and only if its central factor group $G/Z(G)$ has finite Prüfer rank. Nilpotent and locally nilpotent groups are rather well-behaved in this sense. For locally nilpotent p-groups and locally nilpotent torsion-free groups, there exist even bounds for $r(G/Z(G))$ which depend only upon the given prime p and the co-central rank of G.

Lemma 2.1 *Let G be a finite p-group with finite co-central rank s. Then there is an integer-valued function $f(p,s)$ which bounds the Prüfer rank of the factor group $G/Z(G)$.*

Proof. Without loss we may assume $s > 1$ since $f(p,1) = 1$ satisfies the requirements of the lemma. Let A be a maximal abelian normal subgroup of G. Then $C_G(A) = A$ and $r(G/A) \leq s$ by Lemma 1.7, so $G = \langle g_1, ..., g_s, A\rangle$ for some $g_i \in G$. Since $Z(G) = C_A(G) = \bigcap_{i=1}^{s} C_A(g_i)$, the factor group $A/Z(G)$ is isomorphic to a subgroup of the direct product $\mathrm{Dr}_{i=1}^{s} A/C_A(g_i)$ of the factor groups $A/C_A(g_i)$. Show that for any $g \in G$ the Prüfer rank of $A/C_A(g)$ is bounded by a function of p and s.

Fix a $g \in G$ and let $H = \langle A, g\rangle$. As $d(H/Z(H)) \leq s$, the subgroup H is generated modulo $Z(H)$ by some elements $a_1g_1, ..., a_sg_s$ with $a_i \in A$ and $g_i \in \langle g\rangle$ for $1 \leq i \leq s$. If A_i is the normal closure of a_i in H, then $AZ(H) = A_1...A_sZ(H)$. Since $C_A(g) = A \cap Z(H)$ and $A/A \cap Z(H) \cong AZ(H)/Z(H)$, the factor group $A/C_A(g)$ is a homomorphic image of the group $A_1...A_s$. Fix $i \in \{1, ..., s\}$ and show that

$r = r(A_i) \leq p^3 s$. Putting $V = A_i/A_i^p$ and considering V as a vector space over the field $\mathbb{F}_p$, the dimension of V equals r.

Clearly V is a cyclic $\mathbb{F}_p\langle g\rangle$-module and g acts unipotently on V. By the Jordan decomposition theorem (e.g. [17], Corollary 4.4), there exists a basis $v_1, ..., v_r$ of V such that $v_r g = v_r$ and $v_j g = v_j + v_{j+1}$ for $1 \leq j \leq r-1$. Define the integer n by $p^n < r \leq p^{n+1}$. For $n \leq 2$, one immediately has $r \leq p^3 \leq p^3 s$, so let $n > 2$. By the equation

$$v_1 g^k = \sum_{j=0}^{k} \binom{k}{j} v_{j+1}$$

for $1 \leq k \leq r$ it follows that $v_1 g^{p^n} = v_1 + v_{p^n+1}$. Let $\bar{V} = V/(v_{p^n+1}\mathbb{F}_p\langle g\rangle)$ and consider the semidirect product $\bar{V} \rtimes \langle g\rangle$. Clearly it is isomorphic to a section of H. Since the elements $\{\bar{v_1} g^i \mid 0 \leq i \leq p^n - 1\}$ are linearly independent over $\mathbb{F}_p$ and $\bar{v_1} g^{p^n} = \bar{v_1}$, this semidirect product modulo $C_{<g>}(\bar{V})$ is isomorphic to $C_p \wr C_{p^n}$. Applying Lemma 1.8 (with $m = p$ and $n = p^{n-1}$), we obtain that its co-central rank is at least $p^{n-2}(p-1)$, a number which must be less or equal than s. But then $r \leq p^{n+1} \leq p^3 p^{n-2}(p-1) \leq p^3 s$. Hence $r(A/C_A(g)) \leq r(A_1 \dots A_s) \leq p^3 s^2$ and thus

$$r(A/Z(G)) \leq \sum_{i=1}^{s} r(A/C_A(g_i)) \leq p^3 s^3 \ .$$

The inequality $r(G/Z(G)) \leq r(G/A) + r(A/Z(G))$ finally leads to

$$r(G/Z(G)) \leq s + p^3 s^3 \ . \qquad \square$$

The bound for the rank of $G/Z(G)$ found in Lemma 2.1 cannot be chosen independently of p. For instance, it is not difficult to verify that the wreath product $C_p \wr C_p$ of two cyclic groups of prime order p has co-central rank 2. But the rank of its central factor group is $p-1$ and hence unbounded as p runs through the primes. In the light of this example the above result establishes the best possible relation between Prüfer rank and co-central rank.

However, let A be an elementary abelian p-group and let g be an automorphism of A acting unipotently on A. If $A = a^{<g>}$ for some $a \in A$, then the nilpotency class of $A \rtimes \langle g\rangle$ equals the rank of A. Hence in a periodic nilpotent group of nilpotency class c, the rank of such a group $A = a^{<g>}$ is bounded by c. Following the lines of the proof of Lemma 2.1, we can bound the ranks of the A_i by c and finally obtain the bound $r(G/Z(G)) \leq s + s^2 c$. This bound is independent of the prime p, so we have proved the next lemma.

Lemma 2.2 *Let G be a periodic nilpotent group of class c with finite co-central rank s. Then the factor group $G/Z(G)$ has finite Prüfer rank bounded by an integer-valued function $f(c, s)$ which depends on c and s only.*

For the special case that the periodic group is nilpotent of class 2, the relation between co-central rank and Prüfer rank is best possible because we can state

Lemma 2.3 *Let G be a group of nilpotency class ≤ 2. If $G/Z(G)$ is periodic, then $r_c(G) = r(G/Z(G))$.*

Proof. Let $r_c(G) = s$. Since $r(G/Z(G)) \geq s$ by Example 1.2 d), it suffices to prove the converse, i.e. to show that $d(HZ(G)/Z(G)) \leq s$ for every finitely generated subgroup H of G. The quotient $HZ(G)/Z(G) \cong H/(H \cap Z(G))$ is finite. Let T be a transversal of $H \cap Z(G)$ in H containing the element 1. For $1 \neq t \in T$, choose some $x_t \in G$ that does not commute with t. Define the two finitely generated groups $X = \langle x_t \,|\, t \in T \setminus \{1\}\rangle$ and $Y = \langle X, H\rangle$. Let $h \in H \cap Z(Y)$, $h = tz$ for some $t \in T$ and some $z \in H \cap Z(G)$. Then $1 = [h, X] = [t, X]$ and therefore $t = 1$ by construction. Hence $h \in H \cap Z(G)$ and $H \cap Z(Y) = H \cap Z(G)$. We conclude

$$\begin{aligned} d(HZ(G)/Z(G)) &= d(H/H \cap Z(G)) = d(H/H \cap Z(Y)) \\ &= d(HZ(Y)/Z(Y)) \leq d(Y/Z(Y)) \leq s \end{aligned}$$

because the factor group $Y/Z(Y)$ is abelian. This proves $s = r(G/Z(G))$. □

An analogous result to Lemma 2.3 can be obtained in the case of torsion-free nilpotent groups. The next preparatory Lemma is probably well-known.

Lemma 2.4 *Let G be a torsion-free nilpotent group and let H be a subgroup of finite index. Then $Z_n(H) \leq Z_n(G)$ for all $n \in \mathbb{N}_0$.*

Proof. Let $|G : H| = t$ and let c be the nilpotency class of G. If $n < c$, the hypothesis is trivial, so assume $c \geq n$. By induction on $n \in \mathbb{N}_0$, assume that we have already proved $Z_{n-1}(H) \leq Z_{n-1}(G)$, the case $n = 0$ being obvious. Assume that $[G, Z_n(H)] \leq Z_j(G)$ for some j, $c \geq j \geq n$ (the case $j = c$ being trivial). Then mod $Z_{j-1}(G)$, we have

$$[G, Z_n(H)]^{t!} = [G^{t!}, Z_n(H)] \leq [H, Z_n(H)] \leq Z_{n-1}(H) \leq Z_{n-1}(G) \leq Z_{j-1}(G) \ ,$$

so $[G, Z_n(H)]^{t!} = 1$ mod $Z_{j-1}(G)$. But $G/Z_{j-1}(G)$ is torsion-free by [35], Theorem 2.25, hence $[G, Z_n(H)] \leq Z_{j-1}(G)$. Repeating this process, we arrive at $j = n$ and $[G, Z_n(H)] \leq Z_{n-1}(G)$, which means that $Z_n(H) \leq Z_n(G)$. This finishes the induction. □

Lemma 2.5 *Let G be a finitely generated and torsion-free nilpotent group with finite co-central rank s. Then $r(G/Z(G)) = s$.*

Proof. By [1], Lemma 10, there is a subgroup H of finite index in G such that $r(G/Z(G)) = d(HZ(G)/Z(G))$. Since G is torsion-free, $Z(H) \leq Z(G)$ by Lemma 2.4 so that $Z(H) = H \cap Z(G)$. Therefore

$$s \geq d(H/Z(H)) = d(H/(H \cap Z(G)) = d(HZ(G)/Z(G)) = r(G/Z(G)) .$$

On the other hand, $s \leq r(G/Z(G)$ generally holds, thus $s = r(G/Z(G)$. □

Let M be an abelian normal subgroup of the group G. Remember that by Lemma 1.7, the finitness of the rank of $G/C_G(M)$ can be proved if M is finitely generated or if $G/C_G(M)$ is periodic. Example 1.6 shows that in general the rank of $G/C_G(M)$ needs not be finite even if $G/C_G(M)$ is abelian torsion-free. The following lemma however describes an important situation in which this assertion still holds.

We introduce the Hirsch length $r_0(G)$ of a group G. A group G is said to have Hirsch length $r_0(G)$ if it has a finite series with infinite cyclic or periodic factors, and $r_0(G)$ is the number of infinite cyclic factors in this series. The Hirsch length is an invariant of G by the Jordan-Hölder Theorem. It is immediate that $r_0(G) = r_0(G/N) + r_0(N)$ if N is a normal subgroup of G. A theorem of Gluškov ([16], Theorem 1) says that $r(G) = r_0(G)$ if G is a torsion-free nilpotent group.

Lemma 2.6 *Let H and M be subgroups of G such that $C_H(M) \trianglelefteq H$. Let $\mathcal{M}$ be a directed local system of M satisfying $C_H(N) \trianglelefteq H$ for all $N \in \mathcal{M}$ (directed means that to every $N_1, N_2 \in \mathcal{M}$ there is an $N \in \mathcal{M}$ with $N_1, N_2 \leq N$). If there exists an integer s such that $H/C_H(N)$ is torsion-free nilpotent of rank $\leq s$ for every $N \in \mathcal{M}$, then so is $H/C_H(M)$.*

Proof. Choose some $N \in \mathcal{M}$ such that the rank of $H/C_H(N)$ is maximal. The lemma is proved if we can show $C_H(M) = C_H(N)$. Suppose that this is not the case. There exists some $K \in \mathcal{M}$ with $K > N$ and $C_H(K) < C_H(N)$. Since the three groups $H/C_H(K)$, $H/C_H(N)$ and $C_H(N)/C_H(K)$ are torsion-free nilpotent, the rank formula

$$\begin{aligned} r(H/C_H(K)) &= r_0(H/C_H(K)) = r_0(H/C_H(N)) + r_0(C_H(N)/C_H(K)) \\ &= r(H/C_H(N)) + r(C_H(N)/C_H(K)) > r(H/C_H(N)) \end{aligned}$$

holds, contradicting the choice of N. □

It is possible to prove the results of Lemma 2.1 and Lemma 2.5 for locally nilpotent groups.

Theorem 2.7 *Let G be a locally nilpotent group with finite co-central rank s.*

a) If G is torsion-free, then $r(G/Z(G)) = s$. In particular, G is nilpotent of class at most $s+1$.

b) If G is a p-group, then it is hypercentral, abelian-by-finite and the Prüfer rank $r(G/Z(G))$ is bounded by a number $f(p,s)$ depending only on p and s.

Proof. a) By Lemma 2.5, $r(F/Z(F)) \leq s$ for every finitely generated subgroup F of G. Let H be a finitely generated subgroup of G. Then, for every finitely generated subgroup F of G containing H, the group $H/C_H(F) \cong HZ(F)/Z(F)$ is torsion-free (because $F/Z(F)$ is so by [35], Theorem 2.25) and its rank is bounded by s. Since the finitely generated subgroups of G containing H form a directed local system of H-invariant subgroups of G, we can apply Lemma 2.6 and conclude

$$s \;\geq\; r(H/C_H(G)) \;=\; r(H/(H \cap Z(G))) \;=\; r(HZ(G)/Z(G)) \;.$$

This proves $r(G/Z(G)) \leq s$. In particular, $G/Z(G)$ is a locally nilpotent torsion-free group of finite Prüfer rank s and therefore nilpotent of class at most s by a theorem of Čarin (see [35], Lemma 6.37). So G is nilpotent of class at most $s+1$.

b) Let G be a locally finite p-group with finite co-central rank s. If H is a finite subgroup of G, then there exists a finite subgroup K of G containing H such that $H \cap Z(K) = H \cap Z(G)$ (see the proof of Lemma 2.3). By Lemma 1.7, there exists an integer-valued function $f(p,s)$ with $r(K/Z(K)) \leq f(p,s)$. Since

$$\begin{aligned} d(HZ(G)/Z(G)) \;&=\; d(H/H \cap Z(G)) \;=\; d(H/H \cap Z(K)) \\ &=\; d(HZ(K)/Z(K)) \;\leq\; f(p,s) \;\;, \end{aligned}$$

this yields $r(G/Z(G)) \leq f(p,s)$. By Corollary 1 to Theorem 6.36 of [35], $G/Z(G)$ is hypercentral, so G is hypercentral. By Corollary 2 to the same Theorem, the factor group $G/Z(G)$ contains an abelian radicable subgroup $Q/Z(G)$ of finite index in $G/Z(G)$. Hence the subgroup Q is abelian and has finite index in G. Indeed, if $x \in Q$ and n is the order of x, then the subgroup $[Q,x] \cong Q/C_Q(x)$ is radicable and $[Q,x]^n = [Q,x^n] = 1$ so that $[Q,x] = 1$. Thus G is abelian-by-finite and the theorem is proved. $\square$

Considering abelian groups A and B as $\mathbb{Z}$-modules, one can form their tensor product $A \otimes_{\mathbb{Z}} B$ (for further details see [17], chapter IV.5). The property needed here is that $r_0(A \otimes_{\mathbb{Z}} B) = r_0(A) \cdot r_0(B)$ (this can be proved by elementary calculations).

Corollary 2.8 *Let G be a locally nilpotent group with finite co-central rank s.*

a) If G is torsion-free, then the rank of G' is bounded by $\frac{1}{2}s(s-1)$.

b) If G is a p-group, then the rank of G' is bounded by a function $g(p,s)$ depending only on p and s.

Proof. a) Let $Z = Z(G)$ and $G_r = \gamma_r(G)$, $r = 1,...,s+1$. By Theorem 2.7, $r(G/Z) = s$ and $G_{s+2} = 1$. Use the commutator formula $[G_i, G_j] \leq G_{i+j}$ (see [33], 5.1.11) to check that for $r = 1,...,s$ the mappings

$$\alpha_r : G/ZG_2 \times ZG_r/ZG_{r+1} \to G_{r+1}/G_{r+2} \;, \quad (gZG_2, hZG_{r+1}) \mapsto [g,h]G_{r+2}$$

are well-defined, $\mathbb{Z}$-bilinear and surjective. By the universal mapping property of tensor products, this induces epimorphisms

$$\bar{\alpha}_r : G/ZG_2 \otimes_{\mathbb{Z}} ZG_r/ZG_{r+1} \to G_{r+1}/G_{r+2} \quad (*)$$

for $r = 1, ..., s$. The group G/Z is torsion-free by [35], Theorem 2.25. So by the theorem of Gluškov [16], $r_0(G/Z) = r(G/Z) \leq s$ and $r_0(G/ZG_2) \leq s$. Let $r_0(G/ZG_2) = k$, so $r_0(ZG_2/Z) \leq s-k$. If F is a maximal free abelian subgroup of G/ZG_2 with free generators $g_1ZG_2, ..., g_kZG_2$, then $\langle [g_i, g_j]G_3 \mid 1 \leq i < j \leq k \rangle$ is a subgroup of G_2/G_3 with periodic factor group. Hence

$$r_0(G_2/G_3) \;\leq\; \frac{1}{2}k(k-1) \;.$$

The Hirsch length of the other terms of the lower central series can be bounded using $(*)$:

$$r_0(G_{r+1}/G_{r+2}) \;\leq\; r_0(G/ZG_2)\cdot r_0(ZG_r/ZG_{r+1}) \;=\; k\cdot r_0(ZG_r/ZG_{r+1}) \;,\;\; r = 2, ..., s \;.$$

Clearly $1 = G_{s+2} \leq G_{s+1} \leq ... \leq G_2 = G'$ is a normal series in G'. Note that $G_{s+1} \leq Z$ and $Z = ZG_{s+1} \leq ZG_s \leq ... \leq ZG_1 = G$ is a normal series in G joining Z and G. Again using the theorem of Gluškov [16], we calculate the rank of the torsion-free group G' as

$$\begin{aligned} r(G') \;&=\; r_0(G') \;=\; \sum_{r=1}^{s} r_0(G_{r+1}/G_{r+2}) \;\leq\; \frac{1}{2}k(k-1) + k\cdot\sum_{r=2}^{s} r_0(ZG_r/ZG_{r+1}) \\ &=\; \frac{1}{2}k(k-1) + k\cdot r_0(ZG_2/Z) \;\leq\; \frac{1}{2}k(k-1) + k(s-k) \;\;. \end{aligned}$$

It is elementary to verify that $\max\limits_{k\in\{1,...,s\}} \left(\frac{1}{2}k(k-1) + k(s-k)\right) = \frac{1}{2}s(s-1)$. Therefore, $r(G') \leq \frac{1}{2}s(s-1)$.

b) Since every finitely generated subgroup of G' is contained in the commutator subgroup of some finitely generated subgroup of G, we may assume that G is finitely generated and prove that $r(G') \leq g(p, s)$. Let $S \hookrightarrow F \stackrel{\pi}{\twoheadrightarrow} G$ be a presentation of G, i.e. let π be an epimorphism from the free group F onto G with $S = \ker\pi$. If ϵ is the canonical epimorphism from G onto G/Z, then $R \hookrightarrow F \stackrel{\pi\epsilon}{\twoheadrightarrow} G/Z$ is a presentation of G/Z with $R = \ker(\pi\epsilon)$. The Schur multiplicator $M(G/Z)$ of G/Z is isomorphic to $(F' \cap R)/[F, R]$ by Hopf's formula ([33], 11.4.15). Furthermore $R\pi \leq \ker\epsilon = Z$. We conclude that $[F, R]\pi \leq [G, Z] = 1$, thus $[F, R] \leq F' \cap S$. Now calculate $r(G')$ as

$$\begin{aligned} r(G') \;&=\; r((F/S)') \;=\; r(F'S/S) \;\leq\; r(F'S/(F'S\cap R)) + r((F'S\cap R)/S) \\ &=\; r(F'R/R) + r((F'\cap R)S/S) \;=\; r((F/R)') + r((F'\cap R)/(F'\cap S)) \\ &\leq\; r((G/Z)') + r((F'\cap R)/[F, R]) \;\leq\; r(G/Z) + r(M(G/Z)) \;. \end{aligned}$$

The group G/Z is a finite p-group whose rank is bounded by a function $f(p, s)$ depending only on p and s by Theorem 2.7. The results of Lubotzky and Mann on the Schur multiplicator of groups of finite rank ([27], Theorem 4.2.3) show that the rank of $M(G/Z)$ is bounded by a function which depends only on the rank of G/Z. So $r(G') \leq r(G/Z) + r(M(G/Z))$ is also bounded by a function $g(p, s)$ depending on p and s only. □

Let G be a locally nilpotent group of finite rank with its maximal torsion subgroup T. Then T is the direct product of its Černikov p-components and G/T is nilpotent of finite rank ([14], Corollary 7.6). Thus a locally nilpotent group of finite rank is countable. On the one hand, one cannot control the cardinality of the centre of an $\mathfrak{R}$-group. But in view of Theorem 2.7 and its next corollary, one might conjecture that the central factor group of a locally nilpotent $\mathfrak{R}$-group is countable.

Corollary 2.9 *Let G be a locally nilpotent group of finite co-central rank. If G is either torsion-free or periodic, then G/Z is countable.*

Proof. By either a) or b) of Theorem 2.7, the factor group $G/Z(G)$ has finite rank and is therefore countable by the above remark. □

The following example however demonstrates that even for a nilpotent group G of class two, $G/Z(G)$ need not be countable. In particular, $G/Z(G)$ cannot have finite rank, so Theorem 2.7 does not hold for arbitrary locally nilpotent groups. The construction in Example 2.10 is a modification of Example 1.2.

Example 2.10 *There exists a class two nilpotent group G of co-central rank* 2 *with uncountable central factor group $G/Z(G)$.*

Proof. Let $(p_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of primes and define

$$A = \mathrm{Dr}_{k\in\mathbb{N}}\langle c_k\rangle \ , \quad \langle c_k\rangle \cong C_{p_k^2} \quad \text{and} \quad \alpha_k \in \mathrm{Aut}\langle c_k\rangle \ , \ c_k\alpha_k = c_k^{1+p_k} \ , \ k \in \mathbb{N} \ .$$

Let $H = \{f = (f_k)_{k\in\mathbb{N}} \mid f_k \in \mathbb{Z} \text{ for all } k \in \mathbb{N}\}$ be the set of all integer sequences, considered as an abelian group via componentwise addition. Form the semidirect product $G = A \rtimes H$, the action of H on A being determined by the rules

$$c_k^f = c_k\alpha_k^{f_k} \ , \quad k \in \mathbb{N}, \ f \in H.$$

First, we show that G has co-central rank 2. Let F be a finitely generated subgroup of G. Then F lies in $D = \langle f_1, ..., f_r, c_1, ..., c_r\rangle$ for some $r \in \mathbb{N}$ and some $f_1, ..., f_r \in H$. In order to prove $d(F/Z(F)) \leq 2$ it is sufficient to show $r_c(D) \leq 2$. Note that D

is the semidirect product of its normal subgroup $\bar{A} = (D \cap A) = \langle c_1, ..., c_r \rangle$ with $\bar{H} = (D \cap H) = \langle f_1, ..., f_r \rangle$ and that $H \cap Z(D) = C_{\bar{H}}(\bar{A})$. So by Lemma 1.5 b),

$$\begin{aligned} r_c(D) &= r_c(D/(H \cap Z(D))) = r_c(\bar{A}\bar{H}/C_{\bar{H}}(\bar{A})) \leq r(\bar{A}\bar{H}/C_{\bar{H}}(\bar{A})) \\ &\leq r(\bar{A}\bar{H}/\bar{A}C_{\bar{H}}(\bar{A})) + r(\bar{A}C_{\bar{H}}(\bar{A})/C_{\bar{H}}(\bar{A})) \\ &= r(\bar{H}/(\bar{H} \cap \bar{A}C_{\bar{H}}(\bar{A}))) + r(\bar{A}/(\bar{A} \cap C_{\bar{H}}(\bar{A}))) \\ &= r(\bar{H}/C_{\bar{H}}(\bar{A})(\bar{H} \cap \bar{A})) + r(\bar{A}) \\ &= r(\bar{H}/C_{\bar{H}}(\bar{A})) + r(\bar{A}) \end{aligned}$$

The group $\bar{A} \cong \mathrm{Dr}_{j=1}^r C_{p_j^2}$ is cyclic, so $r(\bar{A}) = 1$. The group $\bar{H}/C_{\bar{H}}(\bar{A})$ embeds into the direct product $\mathrm{Dr}_{j=1}^r \bar{H}/C_{\bar{H}}(c_j)$. The restriction of the operation of $\bar{H}$ to $\langle c_j \rangle$ is a homomorphism from $\bar{H}$ to $\langle \alpha_j \rangle$. Thus $\bar{H}/C_{\bar{H}}(c_j)$ is isomorphic to a subgroup of $\langle \alpha_j \rangle \cong C_{p_j}$. Consequently, $\bar{H}/C_{\bar{H}}(\bar{A})$ is isomorphic to a subgroup of the cyclic group $\mathrm{Dr}_{j=1}^r C_{p_j}$. The rank of $\bar{H}/C_{\bar{H}}(\bar{A})$ is therefore 1, and $r_c(D) \leq 1 + 1 = 2$.

Now we claim that $G/C_G(A)$ is uncountable which implies that $G/Z(G)$ is also uncountable. Suppose for a contradiction that $G/C_G(A)$ is at most countable. The set $S = \{f = (f_k)_{k\in\mathbb{N}} \mid f_k \in \{0, 1\} \text{ for all } k \in \mathbb{N}\}$ is an uncountable subset of H. As there are at most countably many distinct cosets of $C_G(A)$ in G, there exist elements $f, g \in S$, $f \neq g$, such that $fC_G(A) = gC_G(A)$. There is some $k \in \mathbb{N}$ for which $f_k \neq g_k$, without loss $f_k = 0$ and $g_k = 1$. The element $f^{-1}g$ centralizes A. In particular,

$$c_k = c_k^{f^{-1}g} = (c_k\alpha_k^{-f_k})^g = c_k^g = c_k\alpha_k^{g_k} = c_k^{1+p_k} ,$$

which is false. □

In the above example, let $B = \mathrm{Dr}_{k\in\mathbb{N}}\langle c_k^{p_k} \rangle \leq Z(G)$. By the definition of the operation of H on A, it follows that $[H, G] = [H, A] \leq B$. Therefore

$$(BH)^G = BH^G = BH[H, G] \leq BH$$

and BH is an abelian normal subgroup of G. Moreover, $G/BH = AH/BH \cong A/B$ is countable. This means that there is at least an abelian normal subgroup of G modulo which G is countable. Whether this is true in general is not known yet.

Questions 2.11 *Let G be a nilpotent or locally nilpotent group of finite co-central rank.*

- *a) Does there always exist an abelian normal subgroup of G modulo which G is countable ?*
- *b) Let $\alpha = 2^{\aleph_0}$, where $\aleph_0$ is the first infinite cardinal. Does $|G/Z(G)| \leq \alpha$ hold ?*
- *c) Let the maximal torsion subgroup of G be a π-group for π a finite set of primes. Is $G/Z(G)$ countable? Does $G/Z(G)$ have finite rank? If yes, is there a bound on $r(G/Z(G))$ which depends only on π and $r_c(G)$?*

2.2 Hypercentrality of locally nilpotent $\mathfrak{R}$-groups

A locally nilpotent group of finite Prüfer rank is always hypercentral. In Theorem 2.15, this is proved to be still true if G has finite co-central rank instead of finite Prüfer rank. The required preparation for Theorem 2.15 will also be useful for Chapter 3. The aim of the next three lemmas is to establish the existence of abelian normal subgroups of finite rank.

Let k denote either a finite field or the ring $\mathbb{Z}$ of integers throughout. The group ring of the group G over k is denoted by kG. All occurring modules are right modules. A kG-module N is called cyclic if it is generated by one element, i.e. there exists an $n \in N$ such that $N = nkG = \{nx \mid x \in kG\}$. If M is a kG-module, its rank $r(M)$ is understood to be the Prüfer rank of M as an abelian group.

Let M be an abelian normal subgroup of g which is either an elementary abelian p-group for some prime p or a torsion-free group. The centralizer $C_G(M)$ of M in G is a normal subgroup of G. Let $\bar{G} = G/C_G(M)$ and bars denote images of elements of G in $\bar{G}$. The group M may be regarded as a $k\bar{G}$-module for the Galois field $k = \mathbb{F}_p$ or for $k = \mathbb{Z}$, respectively, if we define $m\bar{g}$ to be the conjugate m^g of m under g in G and extend this operation to $k\bar{G}$. When there is no danger of confusion, we use the expressions kG-module and G-module synonymously.

Lemma 2.12 *Let G be an abelian group, M a kG-module and N a cyclic kG-module of finite rank d. If $M = \sum_{j\in\mathcal{J}} M_j$ is a sum of cyclic submodules M_j each of which is a homomorphic image of N, then every cyclic submodule of M has rank at most d.*

Proof. If n is a generator of N, then the mapping $kG \ni 1 \mapsto n \in N$ induces a kG-module homomorphism whose kernel is $\mathrm{Ann}_{kG}(n) = \{r \,|\, r \in kG, nr = 0\}$, the annihilator of n in kG. So $nkG \cong kG/\mathrm{Ann}_{kG}(n)$ as kG-modules. Note that for the abelian group G, $\mathrm{Ann}_{kG}(n) = \mathrm{Ann}_{kG}(N)$. Indeed, let m be an arbitrary element of N, write $m = nx$ for some $x \in kG$. Then, for $y \in \mathrm{Ann}_{kG}(n)$, $my = nxy = nyx = 0x = 0$ and y lies in $\mathrm{Ann}_{kG}(N)$. Thus $N \cong kG/\mathrm{Ann}_{kG}(N)$. Since M_j is a homomorphic image of the kG-module N, the annihilator of M_j in kG contains $\mathrm{Ann}_{kG}(N)$ for every $j \in \mathcal{J}$. So $\mathrm{Ann}_{kG}(M)$ also contains $\mathrm{Ann}_{kG}(N)$, because M is the sum of the M_j, $j \in \mathcal{J}$. For every $m \in M$ we therefore have $\mathrm{Ann}_{kG}(m) \geq \mathrm{Ann}_{kG}(N)$. Now $mkG \cong kG/\mathrm{Ann}_{kG}(m)$, hence

$$r(mkG) = r(kG/\mathrm{Ann}_{kG}(m)) \leq r(kG/\mathrm{Ann}_{kG}(N)) = r(nkG) = r(N) = d\ ,$$

and every cyclic submodule of M has rank at most d. □

Lemma 2.13 *Let g be a non-trivial element of the centre of the group G. Let M be a kG-module. If there exists a non-zero element $m \in M$ and a positive integer s such that*

1) the $k\langle g\rangle$-module $mk\langle g\rangle$ has finite rank, say d, and

2) every finitely generated $k\langle g\rangle$-submodule of M can be generated by s elements,

then mkG is a non-trivial kG-submodule of rank at most sd.

Proof. Put $L = mk\langle g\rangle$ and let $x \in kG$. Since the element g is central in G, the mapping $l \mapsto lx$ with $l \in L$ is a $k\langle g\rangle$-module homomorphism from L onto Lx. Thus $K = mkG = \sum_{x\in kG} Lx$ is the sum of cyclic $k\langle g\rangle$-modules Lx, and each Lx is an epimorphic image of L. Hence every cyclic $k\langle g\rangle$-submodule of K has rank at most d by Lemma 2.12. Let N be any finitely generated $k\langle g\rangle$-submodule of K. By the assumption of the lemma, N can be generated as a $k\langle g\rangle$-module by some s elements $m_1, ..., m_s \in K$. Hence

$$r(N) \leq \sum_{j=1,\ldots,s} r(m_j k\langle g\rangle) \leq sd\ .$$

This implies $r(K) \leq sd$. □

The following lemma is crucial for finding abelian normal subgroups of finite rank.

Lemma 2.14 *Let G be a group with finite co-central rank s and let M be an abelian normal subgroup which is either elementary abelian or torsion-free. If Z is the full preimage in G of the centre of the factor group $G/C_G(M)$, then for any elements $m \in M$ and $g \in Z$ the subgroup $[m,g]^G \leq M$ has finite rank and $r([m,g]^G) \leq sr(m^{<g>})$.*

Proof. Denote $\bar{G} = G/C_G(M)$ and, for every $h \in G$, let $\bar{h}$ be the image of h in $\bar{G}$. Put $k = \mathbb{F}_p$, if M is an elementary abelian p-group, and $k = \mathbb{Z}$ if M is torsion-free. Then M can be considered as a $k\bar{G}$-module with $m\bar{h} = m^h$ for every $m \in M$. If $g \in Z$, then $\bar{g}$ is a central element of $\bar{G}$ and so the mapping $\varphi : m \mapsto m(1-\bar{g})$ with $m \in M$ is a $k\bar{G}$-homomorphism from M into M. Indeed, for $x \in k\bar{G}$, we have $x\bar{g} = \bar{g}x$ and therefore $\varphi(mx) = mx(1-\bar{g}) = m(1-\bar{g})x = \varphi(m)x$. Hence $C_M(g) = \ker\varphi$ and $N = M(1-\bar{g}) = \mathrm{Im}\varphi$ are $k\bar{G}$-modules.

If $m(1-\bar{g}) = 0$, then $[m,g] = 1$ in G and the lemma is proved. Let $m(1-\bar{g}) \neq 0$. We want to apply Lemma 2.13 to the $k\bar{G}$-module N with $m(1-\bar{g})$ and $\bar{g}$ in the role of the elements m and g respectively.

The subgroup $\langle m, g\rangle$ of G is finitely generated and soluble of finite co-central rank, hence minimax by Proposition 1.10. Consequently, the normal closure $m^{<g>}$ has finite rank. In other words, the $k\langle\bar{g}\rangle$-module $mk\langle\bar{g}\rangle$ has finite rank, d say.

Let $K = m_1k\langle\bar{g}\rangle + ... + m_rk\langle\bar{g}\rangle$ be a finitely generated $k\langle\bar{g}\rangle$-submodule of M, $m_1, ..., m_r \in M$. Then $H = \langle m_1, ..., m_r, g\rangle$ is a subgroup of G such that $H = K\langle g\rangle$. As G has finite co-central rank s, there exist some elements $h_1, ..., h_s \in H$ such that $H = \langle h_1, ..., h_s\rangle Z(H)$. Therefore there are elements $n_1, ..., n_s$ from K such that

$H = \langle n_1, ..., n_s, g\rangle Z(H)$. Let $L = n_1 k\langle \bar{g}\rangle + ... + n_s k\langle \bar{g}\rangle \leq K$. Then $H = LZ(H)\langle g\rangle$. Observe that $K(\langle g\rangle \cap M) = H \cap M$ and $Z(H) \leq C_M(g)\langle g\rangle$. We can write

$$\begin{aligned} KC_M(g) &= (H \cap M)C_M(g) = HC_M(g) \cap M \\ &= LZ(H)\langle g\rangle C_M(g) \cap M = LC_M(g)\langle g\rangle \cap M \\ &= LC_M(g)(\langle g\rangle \cap M) = LC_M(g)\ . \end{aligned}$$

This implies that $KC_M(g)/C_M(g) = LC_M(g)/C_M(g)$ is s-generated as a $k\langle \bar{g}\rangle$-module. Thus every finitely generated $k\langle \bar{g}\rangle$-submodule of $M/C_M(g)$ is s-generated. Since $M/C_M(g) = N/\ker\varphi$ is isomorphic to $N = \mathrm{Im}\varphi$, the $k\langle \bar{g}\rangle$-module N is s-generated.

Applying Lemma 2.13 to the $k\bar{G}$-module N (with $m(1-\bar{g})$ and $\bar{g}$ in the role of the elements m and g), we obtain that the $k\bar{G}$-submodule $m(1-\bar{g})k\bar{G}$ has finite rank. In other words, the group $[m, g]^G$ is a G-invariant subgroup of M which has finite rank. The first statement of the Lemma is proved.

If $r(m^{<g>}) = d$, then $r(m(1-\bar{g})k\langle \bar{g}\rangle) \leq r(mk\langle \bar{g}\rangle) = d$. The application of Lemma 2.13 in the above situation then gives the bound $r([m, g]^G) = r(m(1-\bar{g})k\bar{G}) \leq sd$, which proves the second statement. □

We are now able to accomplish the structure theory for locally nilpotent $\mathfrak{R}$-groups.

Theorem 2.15 *A locally nilpotent group of finite co-central rank is hypercentral.*

Proof. Let G be locally nilpotent. Denote its torsion subgroup by T. By Theorem 2.7, G/T is nilpotent. Since the hypothesis is inherited by quotients, it suffices to prove that $T \cap Z(G) \neq 1$. Again by Theorem 2.7, T is hypercentral. Let $m \neq 1$ be an element of the socle of some prime component of $Z(T)$. Without loss we may assume $m \notin Z(G)$. As $T \leq C_G(m^G)$, the quotient group $G/C_G(m^G)$ is nilpotent. Choose some $g \in G \setminus C_G(m^G)$ such that $gC_G(m^G) \in Z(G/C_G(m^G))$. Then $[m, g] \neq 1$ and the pair (m, g) satisfies the requirements of Lemma 2.14. Consequently, $[m, g]^G$ is a finite normal subgroup of G. Thus, there exists a minimal non-trivial G-invariant subgroup M in $[m, g]^G$. Since chief factors of locally nilpotent groups are central ([22], Corollary 1.B.8), $M \leq T \cap Z(G)$ which proves the theorem. □

The *Hirsch-Plotkin radical* $H(G)$ of a group G is known to be the unique maximal locally nilpotent normal subgroup of G. One can define the upper Hirsch-Plotkin series of a group in the obvious way by setting $H_0 = 1$, $H_{\alpha+1}/H_\alpha = H(G/H_\alpha)$ and $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$ for limit ordinals λ. The terminus of this series is characteristic in G and is called *the radical* of G and is denoted by $\mathrm{Rad}(G)$. A group G is called *radical* if $G = \mathrm{Rad}(G)$. Every factor $H_{\alpha+1}/H_\alpha$ is hypercentral by Theorem 2.15 and hence possesses an ascending characteristic series with abelian factors. As a consequence, the radical of a group G has an ascending characteristic series with abelian factors. We state this fact as a corollary.

Corollary 2.16

a) *The radical* $\mathrm{Rad}(G)$ *of a group* G *of finite co-central rank is hyperabelian and has an ascending characeristic series with abelian factors. In particular for groups of finite co-central rank, the class of radical groups is identical with the class of hyperabelian groups.*

b) *Within the class of* $\mathfrak{R}$*-groups, the class of hyperabelian groups is extension closed.*

Proof. Statement a) is clear from the remark above. For the second statement, note that hyperabelian groups are radical and that the class of radical groups is extension closed (this is a Corollary to the Hirsch-Plotkin Theorem ([35], Theorem 2.31)), so b) follows. □

Chapter 3

Generalized soluble $\mathfrak{R}$-groups

We show that in general the structure of locally soluble periodic $\mathfrak{R}$-groups with finite co-central rank is similar to that of such groups with finite Prüfer rank. In particular, we prove that for $\mathfrak{R}$-groups the properties of being locally soluble, hyperabelian and radical are identical.

3.1 Locally soluble $\mathfrak{R}$-groups

The essential point in this section is the construction of abelian normal subgroups of locally soluble $\mathfrak{R}$-groups. We start considering the periodic case.

Lemma 3.1 *Let G be a periodic and locally soluble group with finite co-central rank. Then G is hyperabelian.*

Proof. Let $r_c(G) = s$ and let F/H be a chief factor of G. Since H/F is abelian (see [22] Corollary 1.B.5), the factor group $G/C_G(F/H)$ has finite Prüfer rank at most $2s$ by Lemma 1.7. The intersection N of all the centralizers of all the chief factors of G is a locally nilpotent normal subgroup of G by [22], Proposition 1.B.10. Therefore N is hypercentral by Theorem 2.7. On the other hand, the factor group G/N is residually of finite Prüfer rank $2s$ and hence itself is of rank at most $2s$. Thus G/N is hyperabelian by [35], Lemma 10.39, and so G is hyperabelian. □

The following lemma is a slight generalization of a well-known fact (see e.g. [39], Chapter 2, proof of Proposition 3).

Lemma 3.2 *Let G be a group and H a normal subgroup of G having an ascending series of G-invariant subgroups with abelian factors. If N is a G-invariant subgroup* *of H which is maximal with respect to the property that its derived subgroup N' is contained in the centre $Z(N)$, then $C_H(N) = Z(N)$.*

Proof. Assume that $C_H(N) \neq Z(N)$. Then there exists a normal subgroup L of G such that $Z(N) < L \leq C_H(N)$ and the factor group $L/Z(N)$ is abelian. Clearly $Z(N) \leq Z(L)$. Therefore the factor group $L/Z(L)$ is abelian and so $L' \leq Z(L)$. Hence $(LN)' \leq L'N' \leq Z(L)Z(N) \leq Z(LN)$. The maximality of N implies that $LN = N$ and thus $L \leq C_H(N) \cap N = Z(N)$, a contradiction. The lemma is proved. □

The structure of periodic locally soluble $\mathfrak{R}$-groups can be determined even more precisely.

Lemma 3.3 *If G is a periodic hyperabelian group with finite co-central rank s, then G has an abelian characteristic subgroup A such that the factor group G/A has rank at most $5s$.*

Proof. A hyperabelian group of finite co-central rank has an ascending characteristic series with abelian factors by Corollary 2.16. Regard G as a normal subgroup of its holomorph $G \rtimes \mathrm{Aut}(G)$. Application of Lemma 3.2 yields that G has a nilpotent characteristic subgroup N of class ≤ 2 such that $C_G(N) = Z(N)$. By a theorem of Kalužnin (see for instance [22], Theorem 1.C.1), the intersection $C_G(Z(N)) \cap C_G(N/Z(N))$ is a nilpotent characteristic subgroup of class at most 2. This implies $N = C_G(Z(N)) \cap C_G(N/Z(N))$, and so G/N embeds into the group

$$G/C_G(Z(N)) \times G/C_G(N/Z(N)) \cong G/C_G(Z(N)) \times (G/Z(N))/C_{G/Z(N)}(N/Z(N)) \ .$$

Both $G/C_G(Z(N))$ and $(G/Z(N))/C_{G/Z(N)}(N/Z(N))$ have rank $\leq 2s$ by Lemma 1.7 so that G/N has rank $\leq 4s$. Setting $A = Z(N)$ and taking into account that $r(N/A) \leq s$ by Lemma 2.3, we obtain $r(G/A) \leq 5s$ with A being characteristic in G, as required. □

A group G is said to be *radicable* if for every $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ such that $h^n = g$. Before proving Theorem 3.5, we need to establish a lemma on the operation of radicable p-groups on p'-groups.

Lemma 3.4 *Let p be a prime and let the periodic group $G = A \rtimes B$ be a semidirect product of its normal abelian p'-group A with a radicable abelian p-subgroup B. If G has finite co-central rank, then G is abelian.*

Proof. Without loss A is a non-trivial q-group for some prime $q \neq p$. It is sufficient to prove that G is hypercentral, because this implies $G = A \times B$. If we can show $A \cap Z(G) \neq 1$, then, by transfinite induction, A is contained in the hypercentre of

G and we are done. So suppose $A \cap Z(G) = 1$ and pick some $a \in A$ of order q and some $b \in B$ such that $[a, b] \neq 1$. By Lemma 2.14, the rank of the elementary abelian q-group $S = [a, b]^G$ is finite, so S is finite. The index of $C_B(S)$ in the radicable group B is finite, which implies $C_B(S) = B$. Thus S is centralized by $AB = G$ and $1 \neq S \leq A \cap Z(G)$, a contradiction. □

Combining Lemma 3.1 with Lemma 3.3, we deduce that a periodic locally soluble group G with finite co-central rank has an abelian normal subgroup A such that $r(G/A) \leq 5s$. By the next theorem, A can be chosen such that A is characteristic and G/A is residually finite. Since we cannot control the centre of a group with finite co-central rank, Theorem 3.5 seems quite satisfactory.

Theorem 3.5 *Let G be a periodic locally soluble group with finite co-central rank s. Then G is hyperabelian and has an abelian characteristic subgroup A such that the factor group G/A is residually finite and has finite Prüfer rank at most $5s$.*

Proof. Let G be a locally soluble and periodic group with finite co-central rank s. By Lemma 3.1 and Lemma 3.3, G has an abelian characteristic subgroup A with $r(G/A) \leq 5s$. Without loss of generality we may assume that A is a maximal abelian characteristic subgroup of G with this property. It follows from [35], Theorem 9.31 and Theorem 9.23 that the finite residual J/A of G/A is an abelian radicable normal subgroup of G/A. Let p be a prime and Q/A be the Sylow p-subgroup of J/A. If A_p is the Sylow p-subgroup of A, then the factor group Q/A_p is decomposable in a semidirect product of its normal subgroup A/A_p with a radicable abelian p-subgroup B/A_p of Q/A_p. This is a consequence of the Schur-Zassenhaus theorem on split extensions for finite groups ([33], Theorem 9.1.2) because Q/A is countable and the orders of elements of Q/A and A/A_p are coprime. Since Q/A is radicable, the factor group Q/A_p is abelian by Lemma 3.4. If we can show that $Q/A_{p'}$ is abelian, then Q embeds into the abelian group $(Q/A_p) \times (Q/A_{p'})$ and is characteristic in G. By the choice of A it follows that $Q \leq A$. Therefore $J \leq A$ and the proof is complete.
Dividing through $A_{p'}$, it can be assumed that $A = A_p$, thus Q is a locally finite p-group. By Theorem 2.7 b), Q contains an abelian normal subgroup T of finite index. Let T be maximal among these subgroups, let $|Q : T| = n$. The factor group Q/TA is finite and at the same time radicable as an epimorphic image of Q/A, hence $Q = TA$. So $Q' \leq T \cap A \leq Z(Q)$ and Q is nilpotent of class at most 2. In a nilpotent group, a maximal abelian normal subgroup is self-centralizing, so $T = C_Q(T)$. As $A^n \leq T$, we have $[A, T]^n = [A^n, T] \leq [T, T] = 1$ and $[A, T]$ has exponent dividing n. The homomorphism $\alpha : T \to \mathrm{Cr}_{a \in A}[A, T]$, $t \mapsto \mathrm{Cr}_{a \in A}[a, t]$ has the kernel $C_T(A)$. The quotient $T/C_T(A)$ is isomorphic to a subgroup of $\mathrm{Cr}_{a \in A}[A, T]$ and therefore has finite exponent dividing n. But $T/C_T(A)$ is also radicable as an epimorphic image of $T/(T \cap A) \cong Q/A$. It follows that T centralizes A. So $T = C_Q(T) \geq TA = Q$, and Q is abelian. □

By Proposition 1.10, every locally soluble group G with finite co-central rank is locally minimax. The quasicyclic subgroups of finitely generated subgroups of G form an abelian normal subgroup A of G (see [35], Theorem 10.33). Passing to the factor group G/A, we obtain a locally minimax group whose finitely generated subgroups have no quasicyclic subgroups. These groups play an essential role in the forthcoming theorem.

Lemma 3.6 *Let G be a group with finite co-central rank s. If A is an abelian minimax normal subgroup of G which contains no quasicyclic subgroups, then $r(G/C_G(A)) \leq s$.*

Proof. Since A is minimax, there is a finitely generated subgroup K of A such that A/K is radicable. Let H be a finitely generated subgroup of G and $L = \langle H, K \rangle$. It follows that

$$HC_G(A)/C_G(A) \ \cong\ H/C_H(A) \ \cong\ HC_L(A)/C_L(A) \ =\ L/C_L(A)\ .$$

Show that $Z(L) \leq C_L(A)$. Take any $l \in Z(L)$. Since $[A, l] \cong A/C_A(l)$ and $K \leq A \cap L \leq C_A(l)$, the subgroup $[A, l]$ is radicable as a homomorphic image of the radicable group A/K. Hence $[A, l]$=1 and $l \in C_L(A)$. Now we have

$$s \ \geq\ d(L/Z(L)) \ \geq\ d(L/C_L(A)) \ =\ d(HC_G(A)/C_G(A))\ ,$$

which is the desired conclusion. □

Theorem 3.7 *A locally soluble group G with finite co-central rank is hyperabelian.*

Proof. The property of being hyperabelian is countably recognizeable by [35], Corollary 1 to Theorem 8.34. Thus, it is enough to consider the case where the group G is countable. It is sufficient to construct an abelian normal subgroup in G. Let G be the union $G = \bigcup_{i=1}^{\infty} G_i$ of an ascending chain

$$1 \leq G_1 \leq \cdots \leq G_i \leq G_{i+1} \leq \ldots,\ i \in \mathbb{N}\ ,$$

of finitely generated soluble subgroups each of which is minimax by Proposition 1.10. The join of all quasicyclic subgroups which are contained in some G_i is then an abelian normal subgroup of G (see [35], Theorem 10.33). Factoring out this normal subgroup, we may assume that no G_i contains a quasicyclic subgroup. Let B_i be the last non-trivial term of the derived series of G_i. Because B_i is abelian minimax without quasicyclic subgroups, its torsion subgroup is finite. There is some integer $n_i \in \mathbb{N}$ such that $A_i = B_i^{n_i}$ is torsion-free and normal in G_i. By Lemma 3.6, $r(G_i/A_i) \leq s$. Using Lemma 2 of [11], there exists an integer d depending only on s such that $(G_i/A_i)^{(d)}$ is periodic nilpotent with finite prime components. Since $(G_i/A_i)^{(d)}$ is also minimax, $(G_i/A_i)^{(d)}$ is finite. It is sufficient to show that $G^{(d)}$ is hyperabelian, because within the class of $\mathfrak{R}$-groups the property of being hyperabelian is extension closed by Corollary 2.16. Replacing G by $G^{(d)}$, we may therefore assume that every factor group G_i/A_i is finite.

For each $i \in \mathbb{N}$, let R_i be the Hirsch-Plotkin radical of G_i so that $A_i \leq R_i$. Clearly the intersection $R_{i+1} \cap G_i$ lies in R_i and R_j normalizes R_i for all $j \geq i$. If an intersection of infinitely many R_i contains some element $x \neq 1$ then the normal closure x^G is locally nilpotent and therefore hypercentral by Theorem 2.15, its centre being an abelian normal subgroup of G. So assume without loss of generality that such intersections are trivial. Let R be the subgroup of G generated by all R_i with $i \in \mathbb{N}$ and $P_i = \langle R_j \mid j > i \rangle$. Then P_i is a normal subgroup of R and the factor group R/P_i has finite Prüfer rank at most s because it is isomorphic to a section of the factor group G_{i+1}/R_{i+1}. Further, the intersection $P_i \cap (R_1 \ldots R_i)$ lies in R_{i+1}. Indeed, if x belongs to this intersection, then $x = r_1 \ldots r_i = r_{i+1} \ldots r_{i+n}$ with some $n \geq 1$ so that we can take among such products one with minimal n. However, if $n > 1$, then

$$r_{i+n} \in (R_1 \ldots R_{i+n-1}) \cap R_{i+n} \ \leq \ G_{i+n-1} \cap R_{i+n} \ \leq \ R_{i+n-1} \ ,$$

contrary to the choice of n. Let $g \in \bigcap_{i \in \mathbb{N}} P_i$. Then there is some $i \in \mathbb{N}$ for which $g \in R_1 ... R_i$ and hence $g \in R_1 .... R_j \cap P_j \leq R_{j+1}$ for every $j \geq i$. Thus $g = 1$, and the intersection $\bigcap_{i \in \mathbb{N}} P_i$ is trivial. The subgroup R is therefore residually (of rank $\leq s$). Since R is locally soluble, it must be hyperabelian by [2], Theorem B.

Let M be a non-trivial abelian normal subgroup of R and let $1 \neq x \in M$. Then, starting with some $i \in \mathbb{N}$, $x \in G_j \cap M$ for every $j \geq i$. Moreover, x centralizes A_j. To see this, let $a \in A_j$. The group A_j normalizes $G_j \cap M$, so $[a, x, x] = 1$ and thus $[a, x^k] = [a, x]^k$ for every positive integer k. Since A_j has finite index in G_j and so $x^k \in A_j$ for some k, we have $[a, x]^k = 1$ for all $a \in A_j^n$. Since A_j is torsion-free, this implies $[a, x] = 1$, as required. Therefore the centralizer $C_{G_j}(x)$ has finite index in G_j for every $j \geq i$ and hence the normal closure x^{G_j} of x in each G_j is central-by-finite. Thus $S = x^G$ is locally central-by-finite so that its derived subgroup is a periodic locally soluble normal subgroup of G by the Theorem of Schur ([35], Theorem 4.12). If S is abelian, we are done. Otherwise, $S' \neq 1$ is hyperabelian by Theorem 3.1 and therefore possesses a non-trivial characteristic abelian subgroup by Corollary 2.16. The theorem is proved. □

3.2 Chief factors and maximal subgroups in hyperabelian $\mathfrak{R}$-groups

The concern in this section is about abelian normal subgroups of finite rank in hyperabelian $\mathfrak{R}$-groups. As the co-central rank condition provides a priori no restriction on abelian subgroups, it is quite surprising that non-trivial abelian normal subgroups of finite rank always exist. The key to the proof of this result is the careful examination of the following situation: Let G be a hyperabelian group of finite

co-central rank with an abelian normal subgroup $M \neq 1$ which is either elementary abelian or torsion-free. Let $H/C_G(M) \neq 1$ be an abelian normal subgroup of $G/C_G(M)$.
A first step has already been taken in Lemma 2.14. For $H/C_G(M) \leq Z(G/C_G(M))$, the existence of an abelian G-invariant subgroup of M of finite rank has been established. Relying on this fact, Lemma 3.10 shows that there exists a *non-central* abelian H-invariant subgroup $K \leq M$ of finite rank. Later in Lemma 3.13, the finiteness of the rank of K^G is proved provided K is a non-central (rationally) irreducible H-module. Finally, Theorem 3.14 settles the general case. We start with an easy observation which we state as a lemma.

Lemma 3.8 *a) Let M and H be normal subgroups of the group G with $[H, M, M] = 1$. Let M be torsion-free and let F be a subgroup of M such that M/F is periodic. Then $C_H(M) = C_H(F)$.*
b) Let M be a torsion-free abelian normal subgroup of the group H. Let F be a subgroup of M such that M/F is periodic. Then $[M, H]/[F, H]$ is periodic. If H operates trivially on F, then H operates trivially on M.

Proof. a) Let $m \in M$ and $k \in \mathbb{N}$ such that $m^k \in F$. For any $h \in C_H(F)$, we have $[m, h]^k = [m^k, h] \leq [F, C_H(F)] = 1$. Since M is torsion-free, this implies $[m, h] = 1$. As m was arbitrary, h centralizes M, hence $C_H(F) \leq C_H(M)$. The reverse inclusion is trivial.
b) As M is abelian, the condition $[H, M, M] \leq [M, M] = 1$ holds. Let $m \in M$ and $k \in \mathbb{N}$ such that $m^k \in F$. The commutator relation $[m, h]^k = [m^k, h] \in [F, H]$ shows that each member of the generating set $\{[m, h] \,|\, m \in M, h \in H\}$ has some power in $[F, H]$. But $[M, H]$ is abelian, so $[M, H]/[F, H]$ is periodic. If we assume $[F, H] = 1$, then the group $[M, H] = [M, H]/[F, H]$ is periodic and torsion-free as a subgroup of M, hence it is trivial. The lemma is proved. □

Lemma 3.9 contains some information about the ranks of factor groups modulo centralizers in groups with finite co-central rank. As it may be of independent use, we state it in a slightly more general way than needed.

Lemma 3.9 *Let H be a group with finite co-central rank s and M an abelian normal subgroup of H. If one of the following conditions holds, then $r(H/C_H(M)) \leq s$.*

- *a) The subgroup M is torsion-free of finite rank.*
- *b) $[M, H] \leq Z(H)$ and M is finitely generated as a $\mathbb{Z}H$-module.*

Moreover, if b) holds, the subgroup M has finite rank.

Proof. a) Let L be a maximal free abelian subgroup of M and let F be minimal set of generators of L. Then F is finite and $M/\langle F \rangle$ is periodic. By Lemma 3.8 a), $C_H(F) = C_H(\langle F \rangle) = C_H(M)$ and the result follows from Lemma 1.7.

b) Let $m \in M$. The normal closure of m in H is $m^H = \langle\, m, [m,H]\,\rangle$. Let $g \in C_H(m)$ and $h \in H$. Since $[m,H] \leq Z(H)$, it follows that

$$[m^h, g] = [m[m,h], g] = [m, g] = 1 \ ,$$

proving $g \in C_H(m^H)$ and $C_H(m^H) = C_H(m)$. If M is finitely generated as a $\mathbb{Z}H$-module, there exists a finite set $F = \{m_1, ..., m_r\} \subseteq M$ such that $M = F^H$. Therefore $C_H(M) = C_H(F)$ and Lemma 1.7 yields the desired result.

Assume that b) holds. It is straightforward to check that for every $m \in M$ the mapping $\tau : h \mapsto [m,h]$ with $h \in H$ is a group homomorphism from H onto $[m,H]$ because of $[m,H] \leq Z(H)$. Since $r(H/C_H(m)) \leq r(H/C_H(M)) \leq s$, the group $[m,H] \cong H/\mathrm{ker}\,\tau = H/C_H(m)$ has finite rank at most s. We conclude that $r(m^H) = r(\langle\, m, [m,H]\,\rangle) \leq s+1$. So every cyclic H-submodule of M has finite rank. As M is finitely generated as an H-module, the rank of M has to be finite, too. □

Lemma 3.10 *Let H be a group with finite co-central rank and let M be an abelian normal subgroup of H which is either an elementary abelian p-group or torsion-free. Assume that the factor group $H/C_H(M)$ is non-trivial and abelian. Then M contains an H-invariant subgroup K of finite rank such that K is not central in H.*

Proof. If $[M,H] \not\leq Z(H)$, there exist some elements $m \in M$ and $h \in H$ such that $[m,h] \notin Z(H)$. Hence the normal closure $K = [m,h]^H$ of $[m,h]$ in H lies in M and is a non-central subgroup of H. By Lemma 2.14, the subgroup K has finite rank.

Now assume that $[M,H] \leq Z(H)$. There exists some element $m \in M$ which is not central in H. Apply Lemma 3.9 c) to the normal closure $K = m^H$ of m in H which is a cyclic $\mathbb{Z}H$-module. As a result, K is of finite rank and clearly not central in G. □

The next lemma is essentially Exercise 15.1.8 of [33]. For the sake of completeness, we provide a proof.

Lemma 3.11

a) An irreducible $\mathbb{Q}$-linear group G of finite co-central rank is residually finite.
b) A group G of finite rank acting rationally irreducibly on a torsion-free abelian group A of finite rank is finitely generated.
c) An irreducible $\mathbb{Q}$-linear group G cannot have finite rank.

Proof. a) Assume that G is a $\mathbb{Q}$-linear group of finite co-central rank. The Tits alternative ([45], Theorem 10.16 and Corollary 10.17) shows that G is soluble-by-finite because a group of finite co-central rank does not have non-cyclic free subgroups. An irreducible soluble linear group is abelian-by-finite by [45], Lemma 3.5. Let A be an abelian normal subgroup of G of finite index. By Clifford's Theorem ([45], Theorem

1.7), A is completely reducible. For this, it is enough to prove the lemma for G an abelian irreducible $\mathbb{Q}$-linear group. Let G act irreducibly on the finite-dimensional $\mathbb{Q}$-vector space V. Then $\mathrm{End}_{\mathbb{Q}G}(V)$ is a division ring by Schur's Lemma ([33], 8.1.4). The centre F of $\mathrm{End}_{\mathbb{Q}G}(V)$ is a finite-dimensional $\mathbb{Q}$-division algebra and hence an algebraic number field. The elements of G belong to F and they form a multiplicative group of units of F. A Theorem of Skolem ([15], Theorem 127.2) describes the structure of the group of units of an algebraic number field. It is the direct product of a finite cyclic group with a free abelian group (in countably many generators). In particular, such a group is residually finite. G is isomorphic to a subgroup of this group and thus also residually finite.
b) By the hypothesis, G is an irreducible group of $\mathbb{Q}$-automorphisms of $A \otimes_{\mathbb{Z}} \mathbb{Q}$. Continuing the argument of a), we see that G must be finitely generated.
c) By b) G must be finitely generated. But a finitely generated group cannot be an irreducible $\mathbb{Q}$-linear group by Lemma 9.39.1 of [35]. □

Lemma 3.12 is probably known, but there seems to be no appropriate reference.

Lemma 3.12

a) Let H_1,H_2 be normal subgroups of the group G. Let $H_1/C_{H_1}(H_2)$ and $H_2/C_{H_2}(H_1)$ be abelian and let $[H_1, H_2] \leq Z(H_1H_2)$. Then there exists a surjection from $H_1/C_{H_1}(H_2) \otimes_{\mathbb{Z}} H_2/C_{H_2}(H_1)$ onto $[H_1, H_2]$.

b) Let L be an abelian normal subgroup of the group H. Let $H/C_H(L)$ be abelian and $[L, H, H] = 1$. If $H/C_H(L)$ and $L/C_L(H)$ both have finite rank, then so does $[H, L]$.

Proof. a) Let $i, j \in \{1, 2\}$, $i \neq j$. Let $h_i, g_i \in H_i$, $h_j \in H_j$ be arbitrary. The identity $[h_iC_{H_i}(H_j), h_jC_{H_j}(H_i)] = [h_i, h_j]$ shows that the mapping

$$\alpha : H_1/C_{H_1}(H_2) \times H_2/C_{H_2}(H_1)\,,\ (\,h_iC_{H_i}(H_j), h_jC_{H_j}(H_i)\,) \mapsto [h_i, h_j]\ ,$$

is well-defined. Because of $[H_1, H_2] \leq Z(H_1H_2)$, we have $[h_1g_1, h_2] = [h_1, h_2][g_1, h_2]$ and $[h_1, h_2g_2] = [h_1, h_2][h_1, g_2]$ for any $h_j, g_j \in H_j$. This proves that α is bilinear. By the fundamental mapping property of tensor products, α induces a $\mathbb{Z}$-module-homomorphism $\overline{\alpha} : H_1/C_{H_1}(H_2) \otimes_{\mathbb{Z}} H_2/C_{H_2}(H_1) \to [H_1, H_2]$. Since the commutators $[h_1, h_2]$, which generate $[H_1, H_2]$, all lie within the image of $\overline{\alpha}$, the homomorphism $\overline{\alpha}$ is surjective.
b) The specialization $H_1 = H$, $H_2 = L$ in part a) yields the existence of a surjection from $H/C_H(L) \otimes_{\mathbb{Z}} L/C_L(H)$ onto $[H, L]$. Assertion b) is then a consequence of the fact that having finite rank is a property inherited by tensor products (see p.55 of [35], Vol.2). □

In the situation described in the introduction to this section, it is now possible to establish the existence of abelian normal subgroups of G, provided the abelian normal subgroup M contains (rationally) irreducible $H/C_G(M)$-modules.

Lemma 3.13 *Let G be a hyperabelian group with finite co-central rank. Let M be an abelian normal subgroup of G and let $H/C_G(M)$ be an abelian normal subgroup of $G/C_G(M)$.*

a) If M is elementary abelian, then it cannot contain an infinite direct sum of non-central H-modules of bounded finite rank.

b) If M is elementary abelian and $K \leq M$ is an irreducible non-central H-module of finite rank, then K^G has finite rank.

c) If M is torsion-free and $K \leq M$ is a rationally irreducible non-central H-module of finite rank, then K^G has finite rank.

Proof. a) Suppose $N = \oplus_{j\in\mathcal{J}} K_j \leq M$ is the direct sum of non-central H-invariant subgroups K_j, $j \in \mathcal{J}$, whose rank is bounded by some integer r. Since the order of K_j is bounded by p^r, the order of $\mathrm{Aut}(K_j)$ is also bounded by some integer, l say. Then $H^{l!} \leq C_H(N)$ and $H/C_H(N)$ is of finite exponent. By Lemma 1.7, $H/C_H(N)$ has finite rank and so is finite. As a consequence, the number of non-isomorphic H-modules K_j is finite. Thus N contains a direct sum $\oplus_{i\in\mathbb{N}} K_i$ of infinitely many isomorphic non-central H-modules K_i. Let $k \in K_1$ and $h \in H$ such that $[k,h] \neq 1$. For $i \in \mathbb{N}$, let $\varphi_i : K_1 \to K_i$ be some H-module isomorphism and define $k_i = \varphi_i k$. Let $t \in \mathbb{N}$ and let $F = \langle h, k_1, ..., k_t\rangle$. F has co-central rank less or equal than s, so $F = \langle f_1, ..., f_s\rangle Z(F)$. Hence there are $m_1, ..., m_s \in F \cap N$ such that $F \cap N = m_1^{<h>} ... m_s^{<h>}(Z(F) \cap N)$. By Lemma 2.12, $r(m_i^{<h>}) \leq r(k^{<h>}) \leq r(K)$. As $F \cap N$ is the direct sum of the $k_i^{<h>}$, the centralizer of F in N is $Z(F) \cap N = \oplus_{i=1}^{t} C_{k_i^{<h>}}(h)$. As $C_{k_i^{<h>}}(h)$ is a proper subgroup of $k_i^{<h>}$, the rank of $(F \cap N)/(Z(F) \cap N)$ is at least t. On the other hand,

$$r((F \cap N)/(Z(F) \cap N)) \leq r(m_1^{<h>} ... m_s^{<h>}) \leq s \cdot r(K) .$$

Since t was arbitrary, it can be chosen greater than $s \cdot r(K)$, a contradiction.

b) The group K^G is a sum of irreducible H-modules K^g, $g \in G$. Therefore, $K^G = \oplus_{j\in\mathcal{J}} K_j$ is a direct sum of some conjugates K_j of K. Note that for every $j \in \mathcal{J}$, $|K_j| = |K|$ and that K_j is a non-trivial H-module. By part a) of the lemma, $\mathcal{J}$ must be finite, hence K^G is a finite direct sum of modules of finite rank. It therefore must have finite rank itself.

c) As K is torsion-free of finite rank, $H/C_H(K)$ has finite rank by Lemma 3.9 a). Thus $H/C_H(K)$ is finitely generated by Lemma 3.11 b). It can therefore be considered as a subgroup of $GL(r, R)$ for some finitely generated subring R of the rationals. For such a ring, the intersection $\bigcap_p pR$ is trivial as p runs through all primes. Now choose some $1 \neq k \in K$ and set $L = k^H$. Then L can be considered as a submodule of the direct product $R^r = R \times ... \times R$ (written additively). It follows that $\bigcap_p L^p \lesssim \bigcap_p (pR)^r = 0$, where p runs through all primes. H does not operate trivially on L since otherwise it would operate trivially on K by Lemma 3.8. Therefore, there exists some prime p such that L/L^p is not central in H. Two conjugates of K in G either have the same rank as their intersection, or their intersection is trivial. It follows that $N = L^G = \oplus_{j\in\mathcal{J}} L_j$ is a direct sum of conjugates L_j of L. Note that also L_j/L_j^p is not central in H for each $j \in \mathcal{J}$. Considering the group H/N^p (with N/N^p in the role of M), we may apply part a) of the lemma and conclude that $\mathcal{J}$ is finite. Then, L^G is a finite sum of conjugates of L and has finite rank. Since K^G/L^G is periodic, the rank of K^G equals the rank of L^G. □

With the aid of Lemma 3.13, the general case can be solved.

Theorem 3.14 *A hyperabelian group with finite co-central rank has an ascending series with abelian factors of finite rank (which are either elementary abelian p or torsion-free).*

Proof. Let G be a hyperabelian group with finite co-central rank s. As the hypothesis of the theorem is inherited by quotient groups, it is enough to construct in G an abelian normal subgroup of finite rank. G possesses some abelian normal subgroup $M \neq 1$ which is either elementary abelian or torsion-free. If M is central in G, any non-trivial cyclic subgroup of M is normal in G and of rank 1. Therefore we may assume that $C_G(M) \neq G$ and choose some non-trivial abelian normal subgroup $H/C_G(M)$ of $G/C_G(M)$.

case 1) Asssume $C_M(H) = 1$. By Lemma 3.10, M contains H-invariant subgroups of finite rank. Among these choose some $L \neq 1$ of minimal rank. L is not central in H because of $C_L(H) \leq C_M(H) = 1$. If M is elementary abelian, L is an irreducible H-module. If M is torsion-free, L is a rationally irreducible H-module. Apply either part b) or c) of Lemma 3.13 in order to obtain that the abelian group L^G has finite rank.

case 2) Assume $C_1 = C_M(H) \neq 1$ and $C_{M/C_1}(H/C_1) \neq 1$. Define $L/C_1 = C_{M/C_1}(H/C_1)$, so L is a normal subgroup of G satisfying $[L, H, H] = 1$. Then $H/C_H(L)$ is clearly abelian as a quotient group of $H/C_H(M)$. First assume that L is an elementary abelian p-group. Then $H/C_H(L)$ is an elementary abelian p-group, too. For if $h \in H$ and $l \in L$, note that $[L, H] \leq C_1$ and hence $[h^p, l] = [h, l]^p = 1$. It follows by Lemma 1.7 that $r(H/C_H(L)) \leq 2s$, so the order of $H/C_H(L)$ is less or equal than p^{2s}. Let T be a transversal of $C_H(L)$ in H. Since L is contained in the second centre of H, the mapping $h \mapsto [l, h]$ is a homomorphism from H to L for every $l \in L$. Therefore, $[L, H] = [L, T] = \langle [L, t] \,|\, t \in T\rangle$. If we can show that $r([L, t])$ is finite for each t from the finite set T, the rank of the non-trivial group $[L, H]$ is also finite and we are done. So let $t \in T$ and consider the finitely generated group $K = L\langle t\rangle$. Clearly $L \leq Z_2(K)$ and $K/Z_2(K)$ is cyclic, which implies that $Z_2(K) = K$ and K is nilpotent of class at most two. By Theorem 2.7, the rank of $K/Z(K)$ is finite and by Corollary 2.8 b), $r([L, t]) = r(K')$ is finite, too.

Secondly, assume that L is torsion-free. By Lemma 3.10 there exists an H-invariant subgroup K of L which is of finite rank and which is not central in H. The set $\{K^S \,|\, S \subseteq G, |S| < \infty\}$ is a directed local system of $N = K^G$. Clearly $H/C_H(N)$ is abelian because $C_H(N) \geq C_G(M)$ Let S be a subset of G. If $h \in H \setminus C_H(K^S)$, there exists some $g \in K^S$ such that $[h,g]$ is non-trivial and hence of infinite order. The commutator identity $[h^j, g] = [h,g]^j \neq 1$, $j \in \mathbb{N}$, shows that $H/C_H(K^S)$ is torsion-free. If S is finite, the rank of K^S is finite and $r(H/C_H(K^S)) \leq s$ by Lemma 3.9 a). Thus we may apply Lemma 2.6 and obtain $r(H/C_H(N)) \leq s$.

Next, we show that the rank of the abelian group $N/C_N(H)$ is finite. The set $\{\langle F\rangle C_H(N) \,|\, F \subseteq H, |F| \leq s\}$ constitutes a local system of H because the rank of $H/C_H(N)$ is bounded by s. Let F be a finite subset of H containing at most s elements. Note that $C_N(\langle F\rangle C_H(N)) = C_N(F) = \bigcap_{f\in F} C_N(f)$. So the factor group $N/C_N(F)$ is isomorphic to a subgroup of $\mathrm{Dr}_{f\in F} N/C_N(f)$. Let $f \in F$. Consider the homomorphism $\alpha : N \to N$, $n \mapsto [n,f]$ to see that $N/C_N(f) = N/\ker\alpha$ is isomorphic to the torsion-free group $[N,f]$. The group $N_1 = N\langle f\rangle$ is torsion-free and nilpotent since

$$\gamma_4[N_1] = [N_1', N_1, N_1] \leq [N, N_1, N_1] \leq [L, H, H] = 1 \ .$$

By Corollary 2.8,

$$\frac{1}{2}s(s-1) \geq r(N_1') = r([N,f]) = r(N/C_N(f)) \ .$$

We conclude that $N/C_N(\langle F\rangle C_H(N)) = N/C_N(F)$ is torsion-free and

$$r(N/C_N(F)) \ \leq \ r(\operatorname*{Dr}_{f\in F} N/C_N(f)\,) \ \leq \ |F| \cdot \frac{1}{2}s(s-1) \ \leq \ \frac{1}{2}s^2(s-1) \ .$$

A second application of Lemma 2.6 yields $r(N/C_N(H)) \leq \frac{1}{2}s^2(s-1)$. Both groups $H/C_H(N)$ and $N/C_N(H)$ have finite rank. By Lemma 3.12 b) (with N in the role of L), the rank of $[N,H]$ is finite. Since $K \leq N$ and $[K,H] \neq 1$, the commutator $[N,H]$ is a non-trivial abelian normal subgroup of G of finite rank.

case 3) In the last step, we consider the case where $C_1 = C_M(H) \neq 1$, but $C_{M/C_1}(H/C_1) = 1$. Passing to the factor group G/C_1, case 1) proves the existence of some G-invariant subgroup L, $C_1 < L \leq M$, with L/C_1 of finite rank. Choose L to be of minimal rank with these properties. Define $H_1 = C_H(L/C_1)$, a normal subgroup of G. Then, $[L, H_1, H_1] \leq [C_1, H_1] \leq [C_1, H] = 1$. First assume that $[L, H_1] \neq 1$. Then $H_1/C_{H_1}(L) = H_1/C_H(L)$ is non-trivial and abelian as a section of $H/C_H(M)$. This fact allows us to apply case 2) with the role of H played by H_1 and the group L in the place of M. We obtain a normal non-trivial abelian G-invariant subgroup of finite rank, so we are done in this case. Hence we may assume that $[L, H_1] = 1$, in other words $H_1 \leq C_H(L)$. As $C_H(L)$ is contained in $H_1 = C_H(L/C_1)$, necessarily $C_H(L/C_1) = C_H(L)$. If M is elementary abelian, then L/C_1 is finite and so is $H/C_H(L/C_1)$. If M is torsion-free, L/C_1 is a rationally irreducible H-module of finite rank. Then $H/C_H(L/C_1)$ has finite rank by Lemma 3.9 a) and so is finitely generated by Lemma 3.11. In any case, $H/C_H(L) = H/C_H(L/C_1)$ is finitely generated, say $H/C_H(L) = \langle h_1, ..., h_m\rangle C_H(L)$. The group $L/C_L(H) = L/C_1$ is also of finite rank. Let $F = \langle f_1, ..., f_n\rangle$ be a finitely generated subgroup of L. By elementary commutator calculations (see [35], Corollary to Lemma 2.12),

$$\begin{aligned}[F,H] &= \langle [f_i,h_j] \mid i=1,...,n;\ j=1,...,m\rangle^{FH} \\ &= \prod \{[f_i,h_j]^H \mid i=1,...,n;\ j=1,...,m\}\ .\end{aligned}$$

But the group $[f_i,h_j]^H$ has finite rank by Lemma 2.14. Thus $[F,H] = [FC_1,H]$ has finite rank. If L is a p-group, L/C_1 is finite and F can be chosen such that $L = FC_1$. As a result, the group $[L,H]$ is non-trivial, G-invariant and of finite rank. If L is torsion-free, choose F such that L/FC_1 is periodic. Then $[L,H]/[FC_1,H]$ is periodic by Lemma 3.8 b). Consequently, $[L,H]$ is a non-trivial normal subgroup of G which has the same finite rank as $[FC_1,H]$. This finishes the proof of the theorem. □

The next theorem is a direct consequence of Theorem 3.14. It precisely extends Theorem 9.39 of [35] from groups of finite rank to groups of finite co-central rank.

Theorem 3.15 *Let the hyperabelian group G have finite co-central rank. Then each chief factor of G is finite and each maximal subgroup of G has finite index.*

Proof. For the first part of the theorem it is enough to show that a minimal normal subgroup M of the group G is finite. Clearly M is either elementary abelian or torsion-free and radicable. By Theorem 3.14, the group G has an ascending series with abelian factors of finite rank. Intersecting M with this series, we see that M must have finite rank. Suppose M is torsion-free. Clearly M is not central in G because then for any $1 \neq m \in M$, $\langle m^2\rangle$ would be a proper non-trivial G-invariant subgroup of M. So M is an irreducible $G/C_G(M)$-module. By Lemma 3.9, $G/C_G(M)$ is of finite rank which is impossible by Lemma 3.11 c). Thus M must be elementary abelian of finite rank and hence finite. The second part of the theorem now follows from Lemma 9.39.2 of [35], which states that in hyperabelian groups with all chief factors finite, all maximal subgroups have finite index. □

For the remainder of this section, let $\mathfrak{X}$ denote the class of periodic locally nilpotent groups with finite prime components. The class $\mathfrak{X}$ is obviously closed with respect to forming subgroups and homomorphic images. An extension of the very useful Lemma 10.39 of [35] is Lemma 2 of [11]. It shows that for a locally soluble group of finite Prüfer rank r, there exists an integer d depending only on r such that $G^{(d)} \in \mathfrak{X}$. This implies e.g. that a locally soluble group of finite rank is hyperabelian. Another consequence is that a torsion-free locally soluble group of finite rank is soluble, a fact which is due to Čarin (see [14], Theorem 9.3). In Theorem 3.19 we prove a co-central rank version of this lemma. As in the finite rank case, Theorem 3.19 implies that a locally soluble group of finite co-central rank is hyperabelian. It therefore provides a refinement of Theorem 3.7, however with recurrence to Theorem 3.14. Further, we generalize Čarin's theorem to torsion-free locally soluble groups $\mathfrak{R}$-groups.

We need three preparatory lemmas.

Lemma 3.16 *Let the group G have finite co-central rank s and let A be a torsion-free abelian normal subgroup of G. Then for any prime p, the exponent of a p-subgroup of $G/C_G(A)$ divides p^s.*

Proof. For an abelian group M, write $\overline{M}$ for its divisible hull $M \otimes_{\mathbb{Z}} \mathbb{Q}$. Assume $gC_G(A) \in G/C_G(A)$ is an element of order p^k for some prime p and some integer $k \geq 2$. Since the order of $gC_G(A)$ is finite, there is a finitely generated $\langle g\rangle$-invariant subgroup B of A such that $h = gC_H(B) \in H/C_H(B)$ has order p^k, where $H = \langle B, g\rangle$. By Maschke's theorem, $\overline{B}$ is a completely reducible $\mathbb{Q}\langle h\rangle$-module. Hence there must be an irreducible submodule $\overline{C}$, $C \leq B$, on which $\langle h\rangle$ acts faithfully. Indeed, if this were not the case, the unique minimal normal subgroup $\langle h^{p^{k-1}}\rangle$ of $\langle h\rangle$ would act trivially on every irreducible submodule of B. This would imply $h^{p^{k-1}} = 1$, a contradiction. Regard h as a $\mathbb{Q}$-vector space homomorphism of $\overline{C}$ and let $m_h \in \mathbb{Q}[x]$ be the minimal polynomial of h. Clearly m_h is irreducible and divides $x^{p^k} - 1$. By [17], Proposition 8.2, $x^{p^k} - 1 = \prod_{j=0}^{k} \phi_{p^j}(x)$, where ϕ_d is the d-th cyclotomic polynomial. The cyclotomic polynomials are irreducible in $\mathbb{Q}[x]$ (see [17], Proposition 8.3). Hence $m_h = \phi_{p^j}$ for some $j \in \{0, ..., k\}$. Since m_h does not divide $x^{p^{k-1}} - 1$, necessarily $m_h = \phi_{p^k}$, and by [17], Proposition 8.2 and Exercise 8.2, $\deg \phi_{p^k} = (p-1)p^{k-1}$. It follows that

$$r(C) \;=\; \dim_{\mathbb{Q}} \overline{C} \;=\; \deg m_h \;=\; (p-1)p^{k-1} \quad (*).$$

The element $z = h^{p^{k-1}}$ has order p. Again by Maschke's theorem, there exists an irreducible faithful $\mathbb{Q}\langle z\rangle$-submodule $\overline{D}$ of $\overline{C}$ for some $D \leq C$. By the same reasoning as for h, $r(D) = \deg \phi_p = p - 1$. Pick any $1 \neq m \in D$. Note that $r([m, z]^{<z>}) = r(D)$ and $r([m, z]^{<h>}) = r(C)$. Now we apply Lemma 2.14 and obtain $r(C) = r([m, z]^{<h>}) \leq s \cdot r(D) = s(p-1)$. Combining this result with $(*)$, we obtain $p^{k-1} \leq s$, and the very rough estimation $s \geq p^{k-1} \geq (1+1)^{k-1} \geq 1 + (k-1) \cdot 1 = k$ concludes the proof (it is only the fact that k is bounded in terms of s which is important to us). □

The next lemma deals with the structure of p-groups in general. It is not difficult to see that a p-group G of finite rank r and of finite exponent p^e is necessarily finite and its order is bounded in terms of r and p^e. So its derived length is also (r, p, e)-bounded. Here, (r, p, e)-bounded means bounded by a function $f = f(r, p, e)$ depending on r, p and e only. We give a bound for the derived length of G which does not depend on p but only on r and e.

Lemma 3.17 *Let $G \neq 1$ be a finite p-group for some prime p. If G has rank r and exponent p^e, then its derived length is at most $r + e - 1$.*

Proof. Choose a maximal abelian normal subgroup A of G. It is well-known that $A = C_G(A)$. Let $A_0 = 1$ and A_n/A_{n-1} be the socle of A/A_{n-1}, $n = 1, ..., e$. By the hypothesis, $A_e = A$. Define $C_n = C_G(A_n/A_{n-1})$, $n = 1, ..., e$, and $C = \bigcap_{n=1}^{e} C_n$. The p-group G/C_n is isomorphic to a subgroup of the automorphism group of the elementary abelian p-group A_n/A_{n-1} of rank at most r, hence G/C_n is isomophic to a unipotent subgroup of $\mathrm{GL}(r, p)$. Such a group is nilpotent of class at most $r - 1$ by [9], Lemma 0.8. Hence G/C is nilpotent of class at most $r - 1$. The group C/A can be regarded as a group of automorphisms of A which stabilizes the chain $1 = A_0 \leq A_1 \leq ... \leq A_e = A$, i.e. C/A acts trivially on every quotient A_n/A_{n-1}, $n = 1, ..., e$. Hence it is nilpotent of class at most $e - 1$ by [9], Lemma 0.7. In conclusion, we have $G^{(r-1)} \leq C$, $C^{(e-1)} \leq A$ and $A' = 1$. Together this yields $G^{(r+e-1)} = G^{((r-1)+(e-1)+1)} = 1$. □

The next lemma uses the two previous lemmas to "move" an $\mathfrak{X}$-factor of some group G from its top to its bottom.

Lemma 3.18 *Let the group G have a normal subgroup N which has a finite G-invariant series of length d with torsion-free abelian factors. Let further $G/N \in \mathfrak{X}$. If G has finite co-central rank s, then $G^{(3sd)} \in \mathfrak{X}$.*

Proof. First consider the case $d = 1$, i.e. N is torsion-free abelian. Since $G/C_G(N)$ is periodic, its rank is at most $2s$ by Lemma 1.7. Let P be the p-component of the $\mathfrak{X}$-group $G/C_G(N)$ for some prime p. Then $\exp(P)$ divides p^s by Lemma 3.16. We employ Lemma 3.17 and conclude that the derived length of P is at most $2s + s - 1 = 3s - 1$. As $G/C_G(N)$ is the direct product of its p-components, $G^{(3s-1)} \leq C_G(N)$.

The group $C_G(N)/N$ is also an $\mathfrak{X}$-group, so it is the direct product of its finite prime components C_p/N, where p runs through the primes. Since C_p/N is central-by-(finite p-group), Schur's theorem ([35], Theorem 4.12) implies that C_p' is a finite p-group, too. So $C_G(N)'$ is the product of its finite normal p-components C_p' and therefore lies in $\mathfrak{X}$. We have proved $G^{(3s)} = G^{((3s-1)+1)} \in \mathfrak{X}$.

The general statement now follows by a simple induction on $d \in \mathbb{N}$. □

A famous theorem of Zassenhaus ([45], Theorem 3.7) is of fundamental importance for the following. It asserts that a soluble linear group G of degree s has s-bounded derived length. As a corollary ([45], Corollary 3.8), a locally soluble linear group is soluble. These results will be quoted without further reference.

Theorem 3.19 *Let G be a locally soluble group of finite co-central rank s.*

a) There exists an integer d depending only on s such that $G^{(d)}$ has a periodic locally nilpotent characteristic subgroup N with $G^{(d)}/N \in \mathfrak{X}$.

b) If G is torsion-free, then G is soluble of s-bounded derived length.

Proof. a) Let H be the Hirsch-Plotkin radical of G. By Theorem 3.14, there exists an ascending series $(H_\alpha)_{\alpha<\gamma}$ of G-invariant subgroups in H with elementary abelian or torsion-free factors of finite rank. Thus $G/C_G(H_{\alpha+1}/H_\alpha)$ is soluble as a linear locally soluble group and by Lemma 3.9, $r(G/C_G(H_{\alpha+1}/H_\alpha)) \leq s$ for all $\alpha+1<\gamma$. Let $C = \bigcap_{\alpha+1<\gamma} C_G(H_{\alpha+1}/H_\alpha)$. Since C stabilizes an ascending chain of G-invariant subgroups in H, it is contained in H by Lemma 8.17 of [35]; in particular, C is locally nilpotent.

Assume $C = 1$ first. The group G is residually (soluble of rank $\leq s$). Theorem 2 of [11] then says that there is a normal series $1 \leq M \leq L \leq G$ such that G/L is soluble of s-bounded derived length, L/M is residually (linear of s-bounded degree), and M is locally nilpotent. An application of the Zassenhaus theorem shows that L/M is residually (soluble of s-bounded derived length) and hence L/M is soluble of s-bounded derived length. Let T be the torsion subgroup of M. Then $M^{(s+1)} \leq T$ by Theorem 2.7. Thus G/T is soluble of s-bounded derived length. The group T is locally finite and residually (of rank $\leq s$), so $r(T) \leq s$. Applying Lemma 2 of [11], $T^{(f)} \in \mathfrak{X}$ for some f depending only on s. Hence there is some h depending only on s such that $G^{(h)} \in \mathfrak{X}$.

Return to the general case and put $K = G^{(h)}$. We obtain $KC/C \cong K/(K\cap C) \in \mathfrak{X}$. Let T be the torsion subgroup of $K\cap C$ and note that T is a periodic locally nilpotent normal subgroup of G. The factor group $(K\cap C)/T$ is torsion-free nilpotent of class at most $s+1$ by Theorem 2.7, so its upper central series is a characteristic finite series of length at most $s+1$ with torsion-free abelian factors (see [35], Theorem 2.25). By Lemma 3.18, $K^{(3s(s+1))}T/T \in \mathfrak{X}$. If we let $d = h+3s(s+1)$, then $G^{(d)}/(G^{(d)}\cap T) \in \mathfrak{X}$. Since $(G^{(d)}\cap T)$ is normal in G, it is contained in N, the intersection of $G^{(d)}$ with the torsion subgroup of H. Note that N is characteristic in G with $G^{(d)}/N \in \mathfrak{X}$, and d depends only on s.

b) For the integer d defined above, $G^{(d)}$ is periodic by part a). If G is torsion-free, this implies $G^{(d)} = 1$. □

3.3 Residual properties of $\mathfrak{R}$-groups

Recall that the finite residual $J = J(G)$ of a group G is the intersection of all its subgroups of finite index. A subgroup U of G is called $\mathcal{F}$-perfect if $J(U) = U$. The $\mathcal{F}$-perfect radical of $R = R(G)$ of G is the join of all its $\mathcal{F}$-perfect subgroups. It is obvious that R is the unique largest $\mathcal{F}$-perfect subgroup of G.

Lemma 3.20 *Let the normal subgroup H of the group G possess a local system consisting of G-invariant finite subgroups. Then H is centralized by the finite residual J of G. In particular, any abelian normal torsion subgroup which is the direct product of its Černikov prime components is centralized by J.*

Proof. For any $h \in H$, there exists some finite G-invariant subgroup F containing h. Hence, $C_G(F)$ has finite index in G and so $J \leq C_G(F)$. This proves $J \leq C_G(H)$. Let H be an abelian normal torsion subgroup of G with Černikov p-components for every prime p. Every p-component is the union of its upper socle series (the members of which are finite and hence centralized by J). Thus each p-component of H is centralized by J, and so is H. □

Proposition 3.21 *Let G be a hyperabelian group with finite co-central rank. Then the finite residual J of G has an ascending G-invariant series with J-central factors of finite rank.*

Proof. It is sufficient to assume $J \neq 1$ and prove that J contains a non-trivial G-invariant central subgroup of finite rank. By Theorem 3.14, the group G possesses an ascending series of normal subgroups with abelian factors of finite rank which are either torsion-free or elementary abelian. Intersecting this series with J, we see that there exists a non-trivial abelian normal subgroup A of J which is of finite rank and either elementary abelian or torsion-free. If A is elementary abelian, it is finite and hence $J \leq C_G(A)$. So it remains to consider the case when A is torsion-free. Let A be chosen to be G-invariant of minimal rank. Then G acts rationally irreducibly on A. If B is a non-trivial G-invariant subgroup of A, then A/B has Černikov p-components. By Lemma 3.20, A/B is centralized by J. Let D be the intersection of all non-trivial G-invariant subgroups of A. We conclude that J centralizes the quotient A/D. If D is trivial, we are done. Assume $D \neq 1$. Then D is the unique smallest G-invariant subgroup of A. Necessarily, $D^n = D$ for all $n \in \mathbb{N}$ and, since D is abelian, it is a radicable irreducible G-module. Thus the factor group $G/C_G(D)$ is an irreducible $\mathbb{Q}$-linear group of finite co-central rank and is therefore residually finite by Lemma 3.11. So $J \leq C_G(D)$ and J centralizes D. □

Clearly epimorphic images of radicable groups are radicable, and a non-trivial finite group is not radicable. A radicable group is therefore necessarily $\mathcal{F}$-perfect. The converse is not always true. There are metabelian groups which are $\mathcal{F}$-perfect, but not radicable as the next example shows.

Example 3.22 *Let H and A be non-trivial radicable groups, let A contain an element a_1 of infinite order. Then the wreath product $G = H \wr A$ is $\mathcal{F}$-perfect, but not radicable. In particular, choosing H and A isomorphic to the additive group of the rationals, there exists a torsion-free metabelian group which is $\mathcal{F}$-perfect, but not radicable.*

Proof. Let $B = \mathrm{Dr}_{a \in A} H_a$, $H_a \cong H$, be the base group of G. Clearly B is $\mathcal{F}$-perfect as a direct product of $\mathcal{F}$-perfect groups. The property of being $\mathcal{F}$-perfect is extension closed, so $G = B \rtimes A$ is $\mathcal{F}$-perfect, too.

Let $1 \neq h \in H_1$. Suppose there exists an element $g \in G$ such that $g^2 = ha_1$. Write g as $g = bc^{-1}$ with $b = (b_a)_{a \in A} \in B$ and $c \in A$. Then $ha_1 = g^2 = (bc^{-1})^2 = bb^c c^{-2}$. Hence $a_1 = c^{-2}$ and $h = bb^c$. For $x = (x_a)_{a \in A} \in B$ define the *support* of x as $\operatorname{supp} x = \{a \in A \mid x_a \neq 1\}$. Note that the support of x is always a finite subset of A and that $|\operatorname{supp} b| = |\operatorname{supp} b^c|$. Then

$$\begin{aligned} 1 \;&=\; |\operatorname{supp} h| \;=\; |\operatorname{supp} bb^c| \;\geq\; |\operatorname{supp} b \setminus \operatorname{supp} b^c| + |\operatorname{supp} b^c \setminus \operatorname{supp} b| \\ &=\; |\operatorname{supp} b| + |\operatorname{supp} b^c| - 2\,|\operatorname{supp} b \cap \operatorname{supp} b^c| \;=\; 2\,(|\operatorname{supp} b| - |\operatorname{supp} b \cap \operatorname{supp} b^c|) \end{aligned}$$

Necessarily, the last term in the above equation is zero. This requires $\operatorname{supp} b = \operatorname{supp} b^c$. As $b \neq 1$, there is some $t \in \operatorname{supp} b$. Then also $tc^k \in \operatorname{supp} b^{(c^k)} = \operatorname{supp} b$, $k \in \mathbb{N}$. But $c \in A$ has infinite order, so the set $\{tc^k \mid k \in \mathbb{N}\} \subseteq \operatorname{supp} b$ is infinite, a contradiction. Thus G is not radicable. □

As a consequence of Proposition 3.21, the situation of Example 3.22 does not occur in hyperabelian groups with finite co-central rank. In these groups, the properties of being radicable and of being $\mathcal{F}$-perfect are equivalent and imply nilpotence by the next corollary.

Corollary 3.23 *An $\mathcal{F}$-perfect hyperabelian group G of finite co-central rank s is radicable nilpotent of class $\leq s+2$ and has central torsion group. A perfect hyperabelian group with finite co-central rank is trivial.*

Proof. By hypothesis, G equals its finite residual. Proposition 3.21 yields that G is hypercentral. Theorem 9.23 and its Corollary 1 in [35] imply that G is radicable with central torsion group T. As G has finite co-central rank, G/T is nilpotent of class at most $s+1$ by Theorem 2.7. Thus G is nilpotent of class at most $s+2$. The second statement of the Corollary follows at once, since a perfect hyperabelian group is necessarily $\mathcal{F}$-perfect. □

In a locally finite group G, we call a maximal p-group a Sylow p-subgroup of G. In contrast to finite groups, Sylow p-subgroups need not be conjugate, they need not even be isomorphic, as demonstrated in Example 3.3 of [22]. For periodic hyperabelian groups of finite rank, this is however true (see the theorem of Baer and Heineken [3] on radical groups of finite rank). This fact still holds for groups of finite co-central rank.

Theorem 3.24 *Let G be a periodic hyperabelian group of finite co-central rank. Then for any prime p, the Sylow p-subgroups are conjugate.*

Proof. Let $s = r_c(G)$ and fix some prime p. By Lemma 3.3, G has a normal subgroup A such that $r(G/A) \leq 5s$. Denote the p-component of A by A_p. If all Sylow

p-subgroups of G/A_p are conjugate, then so are those of G. Factoring by A_p, we may assume that the p-subgroups of G all have $5s$-bounded rank. The subgroup B generated by all subgroups of G which are isomorphic to a Prüfer p-group is $\mathcal{F}$-perfect, so it is abelian by Corollary 3.23. Hence B is a characteristic p-group in G, and factoring by B, we may additionally assume that G contains no Prüfer p-subgroups. Let P be a Sylow p-subgroup of G. As P is locally nilpotent of finite rank, it is hypercentral. Choose a maximal abelian normal subgroup N of P, so $N = C_P(N)$. Since N is an abelian p-group of finite rank which contains no Prüfer groups, it is finite. Consequently, $P/N = P/C_P(N)$ is finite as an automorphism group of N, and P must be finite. Therefore the subgroup generated by any two Sylow p-groups P and Q is finite. Hence P and Q are conjugate by the classical result of Sylow. This proves the theorem. □

A group has finite abelian section rank if all of its elementary abelian p-sections are finite. A group with finite Prüfer rank clearly has finite abelian section rank. In the structure theory of hyperabelian groups with finite abelian section rank, the $\mathcal{F}$-perfect radical and the finite residual of a group G coincide ([35], Theorem 9.31). If G is hyperabelian with finite co-central rank, this statement becomes false. The finite residual and the $\mathcal{F}$-perfect radical are distinct even in the class of abelian groups (which is the class of groups with co-central rank 1). For example, let p be a prime and let A be an abelian p-group with $A = \langle a_0, a_j \mid j \in \mathbb{N}; a_0^p = 1, a_j^{p^j} = a_0 \rangle$. Any non-trivial subgroup contains $\langle a_0 \rangle$, so $\langle a_0 \rangle$ is contained in the finite residual of A. On the other hand, $A/\langle a_0 \rangle$ is a direct product of cyclic groups and hence residually finite. It follows that the finite residual of A equals $\langle a_0 \rangle$, but $\langle a_0 \rangle$ is not radicable. However it can be shown that the finite residual and the $\mathcal{F}$-perfect radical are not far apart from each other.

Proposition 3.25 *Let G be a hyperabelian-by-finite group with finite co-central rank s. Denote by J the finite residual of G and by R the $\mathcal{F}$-perfect radical of G. Then J/R is (periodic abelian)-by-(nilpotent of class $\leq s+2$).*

Proof. Without loss $R = 1$. If N is a normal subgroup of G of finite index, then the finite residual of G and the finite residual of N coincide as well as their $\mathcal{F}$-perfect radicals do. Consequently, there is no loss in assuming that G is hyperabelian. By Proposition 3.21, J is hypercentral. Note that J contains no radicable subgroups. Let T be the torsion subgroup of J, and let T_p be a prime component of T. Then $r(T_p/Z(T_p))$ is finite by Theorem 2.7, so $T_p/Z(T_p)$ is a Černikov group. The finite residual $Q_p/Z(T_p)$ of $T_p/Z(T_p)$ is radicable abelian, and Q_p is characteristic of finite index in T_p (cf. [35], p.68). The group Q_p is nilpotent of class at most two, but it must even be abelian. For suppose there is some $q \in Q_p \setminus Z(Q_p)$, then by the homomorphism $x \mapsto [x, q]$ the subgroup $[Q_p, q] \cong Q_p/C_{Q_p}(q)$ would be non-trivial and radicable, a contradiction. Moreover, $J \leq C_G(T_p/Q_p)$ since T_p/Q_p is finite. It follows that the finite residual Q of T is the direct product of the abelian p-groups Q_p, where

p runs through the primes, and J centralizes T/Q. The factor group J/T is nilpotent of class at most $s+1$ by Theorem 2.7, so J/Q is nilpotent of class at most $s+2$. $\square$

The following proposition is essentially a consequence from Lemma 4.1 which will be proved in the next chapter, and a theorem of Lubotzky and Mann (see [26] or [9], Theorem 6.10) about the solubility of finitely generated groups which are residually of bounded finite Prüfer rank.

Proposition 3.26 *A finitely generated residually finite group G with finite co-central rank s is soluble-by-finite and minimax.*

Proof. Suppose N is a normal subgroup of G with finite index in G. The factor group G/N contains at most $2s$ non-abelian chief factors by Lemma 4.1. Consequently there must be a normal subgroup M of finite index in G which is residually (finite and soluble). Since M is finitely generated, without loss of generality $G = M$. Hence for each normal subgroup N of finite index in G, there exists a normal subgroup A_N of G containing N such that A_N/N is abelian and $r(G/A_N) \leq 5s$ by Lemma 3.3. The intersection A of all such subgroups A_N is an abelian normal subgroup of G and the factor group G/A has *finite upper rank* $\leq 5s$ in the sense of [9], Definition 6.7 (i.e. every finite quotient of G has rank $\leq 5s$). Therefore G/A and also G is soluble by [9], Theorem 6.10. Finally, G is minimax by Proposition 1.10. $\square$

D. Segal [38] characterised the structure of finitely generated groups that are residually (finite soluble of bounded rank). He proved that such a group is nilpotent-by-quasilinear. According to Wehrfritz [45] p.186, a group is termed *quasilinear* if it is a subdirect product of finitely many linear groups over fields. Segal's result can be applied to groups that are residually (soluble of bounded co-central rank).

Lemma 3.27 *Let s be an integer. Let G be a finitely generated group which is residually (soluble of co-central rank $\leq s$). Then G is soluble-by-quasilinear.*

Proof. Let $\mathcal{R}$ be a residual system of G with G/K soluble of co-central rank $\leq s$ for every $K \in \mathcal{R}$. Let $K \in \mathcal{R}$. By Proposition 1.10, G/K is minimax. By Lemma 3.33, there exists a nilpotent normal subgroup B_K/K of G/K of class at most two such that $r(G/B_K) \leq 2s$. By Theorem 10.33 of [35], the finite residual J_K/B_K of G/B_K is abelian, so J_K/K is soluble of derived length at most 3. Letting $J = \bigcap_{K\in\mathcal{R}} J_K$, this group is also soluble of derived length at most 3. the factor group G/J is residually (finite soluble of bounded rank $2s$) and hence nilpotent-by-quasilinear by the above mentioned theorem of Segal [38]. Thus G is soluble-by-quasilinear. $\square$

The next Lemma on quasilinear groups is a direct consequence of the Tits alternative for linear groups.

Lemma 3.28 *A quasilinear group of finite co-central rank is soluble-by-(locally finite) and hyperabelian-by-finite.*

Proof. Let the quasilinear group G be a subgroup of the direct product $D = D_1 \times ... \times D_s$ of linear groups D_k, $k = 1, ..., s$. Let π_k be the canonical projection from D onto D_k. Since G has finite co-central rank and therefore contains no non-cyclic free subgroup, $G\pi_k$ is soluble-by-(locally finite) by the Tits alternative ([44], see [45] Theorem 10.16 and Corollary 10.17). Since G is isomorphic to a subdirect product of $H\pi_1, ..., H\pi_s$, it is soluble-by-(locally finite), too. A locally finite group of finite co-central rank is hyperabelian-by-finite by Theorem 4.9, so G is hyperabelian-by-finite. □

Combining Lemma 3.27 and Lemma 3.28, we obtain the next Proposition.

Proposition 3.29 *Let G be a finitely generated group of finite co-central rank. If G is residually soluble, it is soluble-by-finite.*

Proof. By Lemma 3.27, G has a soluble normal subgroup N with quasilinear factor group. By Lemma 3.28, G/N is soluble-by-(locally finite), hence soluble-by-finite. This proves that G is soluble-by-finite. □

Theorem 3.30 *A hyperabelian group G with finite co-central rank is locally soluble.*

Proof. We may assume that G is finitely generated and have to show that it is soluble. Form the soluble residual S of G, that is the intersection of all normal subgroups of G modulo which G is soluble. By Proposition 3.29, G/S is soluble-by-finite, hence soluble. Thus G/S' is soluble, too. By the definition of S, this implies $S = S'$ or, in other words, S has to be perfect. But a perfect hyperabelian group with finite co-central rank is trivial by Corollary 3.23, so $S = 1$ and G is soluble. □

At this point, we gather some of the most important results of this chapter into one theorem.

Theorem 3.31 *Let G be a group with finite co-central rank. The following conditions are equivalent:*

1) *G is locally soluble,*
2) *G is hyperabelian,*
3) *G is radical.*

Proof. Condition 1) implies condition 2) by Theorem 3.7, and 1) follows from 2) by Theorem 3.30. The implication from 2) to 3) is trivial, and the reverse direction has been proved in Corollary 2.16 a). □

In general, the product of all locally soluble normal subgroups of a group G needs not be locally soluble itself. But as a consequence of Theorem 3.31, for groups G of finite co-central rank, the product of all locally soluble normal subgroups of G coincides with its radical $\mathrm{Rad}(G)$ and is locally soluble. Hence we have the corollary

Corollary 3.32 *Let G be a group of finite co-central rank. Then there exists a unique maximal locally soluble normal subgroup of G, the so-called locally soluble radical S of G. Furthermore, G/S contains no non-trivial locally soluble normal subgroups.*

Proof. This follows from the corresponding properties of the radical of a group. □

In their paper [11], Dixon, Evans and Smith prove the equivalence of conditions 1), 2) and 3) of Theorem 3.31 under the hypothesis that G is residually (of bounded rank). Combining their result with Theorem 3.31, it is possible to give a common generalization. A lemma on soluble minimax groups is envolved in its proof.

Lemma 3.33 *Let G be a soluble minimax group containing no quasicyclic subgroups. If G has finite co-central rank s, then there exists a nilpotent characteristic subgroup N of G with class at most 2 such that $r(G/N) \leq 2s$.*

Proof. Arguing exactly as in Lemma 3.3, G contains a characteristic subgroup N of nilpotency class ≤ 2 such that G/N embeds into the direct product

$$G/C_G(Z(N)) \times G/C_G(N/Z(N)) \cong G/C_G(Z(N)) \times (G/Z(N))/C_{G/Z(N)}(N/Z(N)) \ .$$

Note that both $Z(N)$ and $N/Z(N)$ do not contain quasicyclic subgroups. Indeed, this follows immediately from the definition in the case of $Z(N)$. If $N/Z(N)$ would contain a quasicyclic subgroup $R/Z(N)$, choose some $n \in N$ for which the homomorphism $\alpha : R/Z(N) \to Z(N)$ defined by $rZ(N) \mapsto [r, n]$ is non-trivial. As a homomorphic image of $R/Z(N)$, the image of α is a quasicyclic subgroup of $Z(N)$ which is impossible. Applying Lemma 3.6, we obtain that both $G/C_G(Z(N))$ and $(G/Z(N))/(C_G(N/Z(N))/Z(N))$ have finite Prüfer rank at most s so that $r(G/N) \leq 2s$. □

Theorem 3.34 *Let s be an integer and let G be a group which is residually (of co-central rank $\leq s$). The following conditions are equivalent:*

1) G is locally soluble,
2) G is hyperabelian,
3) G is radical.

Proof. Let $\mathcal{R}$ be a residual system of G such that $r_c(G/K) \leq s$ for every $K \in \mathcal{R}$. For $K \in \mathcal{R}$, the factor group G/K is soluble by Theorem 3.31 and minimax by Proposition 1.10 provided one of the conditions holds. The finite residual J_K/K of G/K is abelian by [35], Theorem 10.33. The factor group G/J_K, $K \in \mathcal{R}$, is a soluble minimax and does not contain any quasicyclic subgroup. Then by Lemma 3.33, there exists a nilpotent normal subgroup N_K/J_K of G/J_K of class at most two such that $r(G/H_K) \leq 2s$. Let $N = \bigcap_{K\in\mathcal{R}} N_K$. Note that N is soluble of derived length at most 3 since $N^{(3)} = \bigcap_{K\in\mathcal{R}} N_K^{(3)} \leq \bigcap_{K\in\mathcal{R}} K = 1$. Dividing through N, we may assume that G is residually (of rank $\leq 2s$) and have to prove that the three conditions above are equivalent. This however is exactly the content of Theorem 4 of Dixon, Evans and Smith [11]. $\square$

Chapter 4

Locally finite and locally (soluble-by-finite) $\mathfrak{R}$-groups

4.1 Finite sections in $\mathfrak{R}$-groups

This section is mainly concerned with finite simple groups. It turns out that only boundedly many of them can occur in a composition series of a finite $\mathfrak{R}$-group. Furthermore, simple groups of given co-central rank s cannot be arbitrarily "large" in the sense that they are linear of s-bounded degree. These results serve to prepare the study of locally finite $\mathfrak{R}$-groups in section 2.

Lemma 4.1 *Let G be a finite group with finite co-central rank s. Then any direct product of non-abelian simple subgroups of G contains at most s factors. Any chief series of G has at most $2s$ non-abelian factors and any composition series of G has at most $2s^2$ non-abelian factors.*

Proof. Let $M_1 \times ... \times M_n$ be a direct product of non-abelian simple subgroups of G. By the Feit-Thompson Theorem each M_i contains involutions, and not all of them commute pairwise. Since the group generated by two involutions is a dihedral group, it is clear that each M_i with $1 \leq i \leq n$ contains a non-abelian dihedral subgroup K_i. The factor group $K_i/Z(K_i)$ has a cyclic group of order 2 as a homomorphic image. Taking $K = K_1 \times ... \times K_n$, we see that $K/Z(K)$ has an elementary abelian 2-group of rank n as an homomorphic image. Hence $s \geq d(K/Z(K)) \geq n$ and the first statement ist proved. For the second statement, we may assume that the soluble radical of G is trivial. Then the socle $Soc(G)$ of G is a direct product of at most s non-abelian simple groups and any G-invariant series in $Soc(G)$ has at most s non-abelian factors. Since $C_G(Soc(G)) = 1$, Lemma 1.7 implies that $r(G/Soc(G)) \leq s$. According to [46], Lemma 8.4.3, the number of non-abelian factors of a G-invariant series running from $Soc(G)$ to G is bounded by s (this was implicitly proved by Lubotzky and Mann in [26]). Thus any chief series in G can contain at most $s+s=2s$ non-abelian factors. As every chief factor of G is the direct product of simple groups, only at most s of them can be non-abelian by the argument above. Thus in any series in G there are at most $2s \cdot s$ non-abelian composition factors. □

Lemma 4.2 *Let G be a group with finite co-central rank s. Then any normal series in G contains at most $2s$ non-soluble finite factors.*

Proof. Let $M_1 \leq N_1 \leq M_2 \leq N_2 \leq ... \leq M_k \leq N_k \leq G$ be a normal series in G with finite non-soluble factors N_j/M_j, $j=1,...,k$. Let S_j/M_j be the soluble radical of N_j/M_j and T_j/S_j be the socle of N_j/S_j, $j=1,...,k$. Then T_j/S_j is a direct product of non-abelian simple subgroups. In particular, any normal subgroup of T_j/S_j is perfect. Define $C=\bigcap\{C_G(T_j/S_j)\,|\,j=1,..,k\}$ to be the intersection of the centralizers of the factors T_j/S_j. Then C is a normal subgroup of G of finite index. For convenience let $T_0=1$. We claim that the factors of the series $T_0C \leq T_1C \leq T_2C \leq ... \leq T_kC$ are non-soluble. We have

$$T_jC/T_{j-1}C \cong T_j/(T_{j-1}C \cap T_j) = T_j/(T_{j-1}(T_j \cap C))\ , \quad j=1,...,k \quad (*)$$

Now $(T_j \cap C)' \leq [T_j, C] \leq S_j$, so $(T_j \cap C)S_j/S_j$ is an abelian normal subgroup of T_j/S_j and hence must be trivial. Thus $T_{j-1}(T_j \cap C) \leq S_j$ and $T_jC/T_{j-1}C$ has an epimorphic image isomorphic to T_j/S_j by $(*)$. Therefore $T_jC/T_{j-1}C$ cannot be soluble and the claim is proved.

Refining the series $T_0C/C \leq ... \leq T_kC/C \leq G/C$ in the finite group G/C, one obtains a chief series with at least k non-abelian factors. Lemma 4.1 implies that k cannot be greater than $2s$. □

Lemma 4.2 has a corollary which is convenient for later use. Recall that a *generalized radical* group is a group which has an ascending normal series with factors that are finite or locally nilpotent.

Corollary 4.3 *Let the group G have finite co-central rank.*

a) If G is generalized radical, then it is (locally soluble)-by-finite and hyperabelian-by-finite.
b) If G has an ascending normal series of hyperabelian-by-finite groups, then it is hyperabelian-by-finite.

Proof. a) Let $\{G_\alpha\,|\,\alpha<\gamma\}$ be a normal ascending series of G with finite of locally nilpotent factors. By Lemma 4.2, the number of the finite non-soluble factors in this series is bounded by $2s$. Denote by C the intersection of the centralizers of these finitely many factors. Then C has finite index in G, and C has an ascending series with locally nilpotent factors. It follows that C is radical of finite co-central rank, which implies by Theorem 3.31 that it is locally soluble and hyperabelian.

b) Let $M \leq N$ be normal subgroups of G. If N/M is hyperabelian-by-finite, $\mathrm{Rad}(N/M)$ is a characteristic subgroup of finite index in N/M. By Corollary 2.16, $\mathrm{Rad}(N/M)$ has an ascending series of characteristic subgroups with abelian factors. We conclude that G is generalized radical. It follows from part a) that G is hyperabelian-by-finite. □

The next lemma is a variation of a lemma of Dixon, Evans and Smith ([11], Lemma 2.2). A group is called semisimple if it has no non-trivial abelian normal subgroups.

Lemma 4.4 *Let G be a finite semisimple group of finite co-central rank s and N the socle of G. Then G contains a normal subgroup $E \geq N$ of index at most $s!$ such that E/N is soluble.*

Proof. The socle N is a direct product $M_1 \times ... \times M_r$ of the homogeneous components M_k of N. For each $k = 1, ..., r$, the group M_k is the product of all simple subgroups of N of one isomorphism type S_k, more specifically, M_k is the direct product of s_k such subgroups. Therefore we have $\mathrm{Aut}M_k \cong \mathrm{Aut}S_K \wr \mathrm{Sym}(s_k)$, where the wreath product is taken with respect to the natural permutation representation of the symmetric group $\mathrm{Sym}(s_k)$. Because the homogeneous components are characteristic in N, we have $\mathrm{Aut}N = \mathrm{Aut}M_1 \times ... \times \mathrm{Aut}M_k$. As $C_G(N) = 1$, we can embed G in $\mathrm{Aut}N$. Let B_k denote the base group of $\mathrm{Aut}M_k$ for each k and write $A = B_1 \times ... \times B_k$, $E = G \cap A$. Then $E/N \leq A/N$ which is a direct product of groups each isomorphic to the outer automorphism group $\mathrm{Out}\,S_j = \mathrm{Aut}S_j/\mathrm{Inn}S_j$ for some $j \in \{1, ..., k\}$. By the Schreier conjecture (which has been proved using the classification of finite simple groups), $\mathrm{Out}\,S_k$ is soluble. Thus E/N is soluble. Now $G/E \cong GA/A \leq \mathrm{Aut}N/A \cong \mathrm{Sym}(s_1) \times ... \times \mathrm{Sym}(s_k)$. Since the number of direct simple factors is bounded by s by Lemma 4.1, we have $s_1 + ... + s_k \leq s$ and $|G/E| \leq s_1! \cdot ... \cdot s_k! \leq s!$. The lemma is proved. □

Given any finite group G of co-central rank s, Lemma 4.4 allows to locate the non-abelian chief factors of G in a convenient way. For if S denotes the maximal soluble normal subgroup of G, then by Lemma 4.4 the group G has a normal series $1 \leq S \leq N \leq E \leq G$ in which the factors $S/1$ and E/N are soluble, the order of the (possibly non-soluble) quotient G/E is s-bounded and N/S is a direct product of at most s non-abelian simple groups.

Lemma 4.5 *To every $s \in \mathbb{N}_0$, there exists an integer f depending only on s such that every finite simple non-abelian group of co-central rank $\leq s$ is linear of f-bounded degree.*

Proof. We use the classification of the finite simple groups. Clearly we may disregard the sporadic groups, since there are only finitely many of them. There are also only finitely many alternating or symmetric groups of co-central rank $\leq s$. This is because an alternating group $\mathrm{Alt}(n)$ of degree $n \geq 5(s+1)$ has a direct product of $(s+1)$ alternating groups of degree 5 as a subgroup. Hence its co-central rank is greater than s by Lemma 4.1. It remains to consider the simple groups G of Lie type. The order of the Weyl group of G is s-bounded because there are only finitely many symmetric groups which can occur as sections of the Weyl group (cf. [47], Lemma 4.1). Hence the Lie rank of G is s-bounded, and G is therefore linear of s-bounded degree. The lemma is proved. □

4.2 Locally finite $\mathfrak{R}$-groups

It was proved by Šunkov [41] that locally finite groups with finite Prüfer rank are hyperabelian-by-finite. The aim here is to generalize this result to locally finite groups of finite co-central rank. If a group of finite co-central rank is locally finite-soluble, then this follows from Theorem 3.7. In this section, we investigate the non-abelian chief factors of locally finite groups.

In a first step, we ensure that a locally finite group with finite co-central rank cannot contain infinite simple sections. Since linear locally finite simple groups always contain subgroups isomorphic to some projective special linear group over some locally finite field (see [7], Theorem 6.3.1), our first concern is about projective special linear groups.

Lemma 4.6 *If $\mathbb{F}$ is an infinite locally finite field, then the co-central rank of* $\mathrm{PSL}(2,\mathbb{F})$ *is infinite.*

Proof. Let p be the characteristic of $\mathbb{F}$. There exists an ascending series of finite subfields $\mathbb{F}_{p^{n_t}}$ of orders p^{n_t} with $n_{t+1} = n_t q_t$ for some prime integer q_t and each $t \geq 1$. The subgroup $\mathrm{PSL}(2, p^{n_t})$ of $\mathrm{PSL}(2,\mathbb{F})$ contains a subgroup isomorphic to the semidirect product of the additive group A of the field $\mathbb{F}_{p^{n_t}}$ with its multiplicative group $(\mathbb{F}_{p^{n_t}}^{\times})^2$ acting on A by multiplication (see for instance [19], Hauptsatz 2.8.27). Let l be the greatest common divisor of $p-1$ and 2. The subgroup $(\mathbb{F}_{p^{n_t}}^{\times})^2$ is cyclic of order $(p^{n_t}-1)/l$ and so contains an element c of order k with $k = (p^{q_1}-1)/l$. Put $H = \langle A, c\rangle$. Since the multiplication by c on A has no fixed points in A except 0, the subgroup H has trivial centre. Let $x_1, ..., x_m$ be a minimal set of generators of H. Clearly $x_i = a_i c_i$ with $a_i \in A$ and $c_i \in \langle c\rangle$ so that

$$\langle x_i, c\rangle \cap A = \langle a_i\rangle^{\langle c\rangle}\langle c\rangle \cap A = \langle a_i\rangle^{\langle c\rangle} .$$

Now A is an $\mathbb{F}_p$-vector space of dimension n_{t+1} and the dimension of $\langle a_i \rangle^{\langle c \rangle}$ as a subspace of A is at most the order of c which is equal to k. Therefore,

$$\begin{aligned} n_t &= dim(\langle x_1, ..., x_m \rangle \cap A) \leq dim(\langle a_1, ..., a_m, c \rangle \cap A) \\ &= dim(\langle a_1, ..., a_m \rangle^{\langle c \rangle}) \leq dim(\langle a_1 \rangle^{\langle c \rangle}) + ... + dim(\langle a_m \rangle^{\langle c \rangle}) \leq mk \ . \end{aligned}$$

Thus $r_c(\mathrm{PSL}(n, \mathbb{F})) \geq d(H/Z(H)) = d(H) = m \geq n_t/k$, which means that $r_c(\mathrm{PSL}(n, \mathbb{F}))$ is infinite. □

In the proof of the following proposition, we make use of the classification of finite simple groups.

Proposition 4.7 *A locally finite simple group with finite co-central rank is finite.*

Proof. Assume there exists an infinite and locally finite simple group G with finite co-central rank s. Then G contains no finite subgroup isomorphic to the wreath product $C_p \wr C_{mn}$ with $m = 2$ and $n = 2(s+1)$ and any prime p by Lemma 1.8. Therefore G is a linear group of Lie type over a locally finite field $\mathbb{F}$ by [18], Theorem 2.6 (here the classification of finite simple groups is needed). It is well-known that such a group contains a section which is isomorphic to $\mathrm{PSL}(2, \mathbb{F})$ for some locally finite field $\mathbb{F}$ (see [7], Theorem 6.3.1). But then Lemma 4.6 implies that $r_c(G) \geq r_c(\mathrm{PSL}(2, \mathbb{F})) = \infty$, contrary to the choice of G. □

As infinite simple sections cannot occur in locally finite groups of finite co-central rank, we turn to the discussion of the non-abelian chief factors of these groups. It follows from Lemma 4.1 that for each locally finite group G with finite co-central rank we can define the *composition rank* $co(G)$ of G to be the maximal nonnegative integer n with respect to the condition that G has a finite subgroup whose composition series has n non-abelian composition factors.

Lemma 4.8 *Let G be a non-abelian locally finite group with finite co-central rank. Then every non-abelian chief factor of G is finite.*

Proof. Suppose the contrary and let n be the minimal positive integer for which there exists a counterexample G whose infinite and non-abelian chief factor U/V has composition rank n. Passing to the factor group G/V, we may assume that $V = 1$ so that U is a non-abelian minimal normal subgroup of G. Since U cannot be any simple group by Proposition 4.7, there exists a proper normal subgroup N of U. Show that $co(U/N) < n$.

Indeed, in the other case there exists a finite subgroup K of U with $co(KN/N) = n$. Note that locally soluble chief factors of locally finite groups are abelian by [22], Corollary 1.B.4. Obviously N cannot be locally soluble because otherwise $U = \langle N^g \mid g \in G\rangle$ would be locally soluble and thus abelian. Hence N contains a non-soluble finite subgroup L. Put $X = \langle K, L\rangle$. Then $X = K(N \cap X)$ and $L \leq N \cap X$. Since $KN/N = XN/N \cong X/(N \cap X)$, we have $co(X) > n$, contrary to the assumption. Therefore $co(U/N) < n$, as claimed.

It is also clear that the factor group U/N cannot be locally soluble since otherwise U would be residually locally soluble, hence locally soluble and so U would be abelian. Therefore U/N has a finite non-abelian chief factor and thus the centralizer C of this factor in U is a proper normal subgroup of finite index in U such that the factor group U/C is non-soluble. Let M be a maximal normal subgroup of U containing C. As stated above, the factor group U/M is a non-abelian finite simple group. Since $\bigcap_{g\in G} M^g = 1$ and $co(U) = n$, the subgroup U must be finite. This final contradiction completes the proof. □

Theorem 4.9 *If G is a locally finite group with finite co-central rank, then G is hyperabelian-by-finite and hyperfinite.*

Proof. The non-abelian chief factors of G are perfect and finite by Lemma 4.8, and there are only finitely many of them by Lemma 4.2. Denote by C the intersection of these non-abelian chief factors. Then G/C is finite, and C is locally soluble by [22], Proposition 1.B.11. By Theorem 3.7, C is hyperabelian and so G is hyperabelian-by-finite. By Theorem 3.14, G is hyperfinite-by-finite and therefore hyperfinite. □

By Example 1.2 c), periodic groups of finite co-central rank need not be locally finite. But under the weak additional condition that they are locally graded, they are in fact locally finite. Here, a group is called *locally graded* if every finitely generated subgroup of it has a non-trivial finite epimorphic image. The extensive class of locally graded groups contains all free groups, locally finite groups, residually finite groups, and locally soluble groups, but it does not contain the pathological examples constructed by Ol'shanskii and Rips [32].

Theorem 4.10 *A periodic locally graded group of finite co-central rank is locally finite.*

Proof. Let H be a finitely generated subgroup of the periodic locally graded group G with finite co-central rank. If R is the finite residual of H, then H/R is periodic and soluble-by-finite by Proposition 3.26. Therefore H/R is finite and so $R = 1$. Thus G is locally finite and the proof is complete. □

Finally we prove a sufficient condition for a group to be locally finite. It is the co-central rank analogon of Theorem 1 of Dixon, Evans and Smith in [11].

Theorem 4.11 *Let G be a periodic group and let s be an integer such that G is residually (locally finite of co-central rank $\leq s$). Then G is locally finite.*

Proof. It is enough to show that a finitely generated group G satisfying the hypothesis is finite. So it can be assumed that G is residually (finite of co-central rank $\leq s$). Let $\mathcal{R}$ be a residual system of G with G/K finite of co-central rank $\leq s$ for every $K \in \mathcal{R}$. For $K \in \mathcal{R}$, the factor group G/K contains at most $2s$ non-abelian chief factors by Lemma 4.1. Choosing $M \in \mathcal{R}$ with $co(G/M)$ maximal, we conclude that M is residually (finite soluble of co-central rank $\leq s$). Since M is finitely generated, Proposition 3.27 implies that the periodic group M has a soluble normal subgroup L with M/L quasilinear. Clearly L is locally finite. It is well known that periodic linear groups are locally finite ([45], 9.1 and 9.33). Hence periodic quasilinear groups are locally finite, too. In particular, M/L is locally finite, forcing M to be locally finite. As M is finitely generated, M and thus G is finite. $\square$

4.3 Locally (soluble-by-finite) $\mathfrak{R}$-groups

N.S.Černikov [8] studied locally (soluble-by-finite) groups of finite rank which he proved to be (locally soluble)-by-finite. The following theorem generalizes this result to groups which have merely finite co-central rank. Note that if G is a soluble-by-finite group, the product of all its normal soluble subgroups is again soluble and it is called the soluble radical of G.

Lemma 4.12 *Let G be a soluble-by-finite minimax group containing no non-trivial periodic normal subgroups. Let R be the Hirsch-Plotkin radical of G. If G has finite co-central rank s, then $r(G/R) \leq s(s+1)$.*

Proof. The Hirsch-Plotkin radical R of G is a torsion-free locally nilpotent group of finite co-central rank, so it is nilpotent of class at most $s+1$ by Theorem 2.7. If S is the soluble radical of G, then $R \leq S$. In particular, the Hirsch-Plotkin radical of S and that of G coincide. Moreover, $C_S(R) = Z(R)$ by Lemma 2.32 of [35]. The facor group $C_G(R)S/S \cong C_G(R)/Z(R)$ is finite, so $C_G(R)$ is central-by-finite. As the derived subgroup $C_G(R)'$ is a finite normal subgroup in G by Schur's theorem, it must be trivial. Hence $C_G(R)$ is abelian.

Let $R_k = Z_k(R)$, $k = 0, ..., s+1$, be the upper central series of R. For $k = 0, ..., s$, the abelian minimax factors R_{k+1}/R_k are torsion-free by [35], Theorem 2.25. So we may apply Lemma 1.7 and obtain $r(G/C_G(R_{k+1}/R_k)) \leq s$. Define

$N = \bigcap_{k=0}^{s} C_G(R_{k+1}/R_k)$, a normal subgroup of G. It follows that $r(G/N) \leq s(s+1)$. The group $N/C_N(R)$ is an automorphism group of R which stabilizes the finite normal series $1 = R_0 \leq R_1 \leq ... \leq R_{s+1} = R$, so $N/C_N(R)$ is nilpotent by [22], Theorem 1.C.1. Since $C_N(R) \leq C_G(R)$ is abelian, N is soluble and contained in S. As N stabilizes an ascending S-invariant chain in R, N is contained in R by [35], Lemma 8.17. This implies $r(G/R) \leq s(s+1)$. □

Theorem 4.13 *Let the group G have finite co-central rank s. If G is locally (soluble-by-finite), it is (locally soluble)-by-finite and hyperabelian-by-finite.*

Proof. The property of being (locally soluble)-by-finite is countably recognizeable by [13], Lemma 3.5. This means that if all countable subgroups of G are (locally soluble)-by-finite, then so is G. We may therefore assume that $G = \bigcup_{k\in\mathbb{N}} G_k$ is the union of an ascending chain of finitely generated soluble-by-finite groups $G_1 \leq G_2 \leq ...$ which are minimax by Proposition 1.10. Let T_k be the maximal periodic normal subgroup of G_k and let S_k/T_k be the soluble radical of G_k/T_k. By Lemma 4.12, the rank of G_k/S_k is bounded by $s(s+1)$. The group T_k is normalized by all T_j with $j \leq k$, so $T = \langle T_k \,|\, k \in \mathbb{N}\rangle$ is locally finite. So by Theorem 4.9, the subgroup T contains a hyperabelian normal subgroup N of finite index $d \in \mathbb{N}$. If t_k denotes the soluble radical of T_k, then $|T_k/t_k| \leq |T_k/(T_k \cap N)| \leq d$ and the order of $G_k/C_{G_k}(T_k/t_k)$ is bounded by some integer f depending only on d. The quotient group G/G^f is locally finite, so hyperabelian-by-finite by Theorem 4.9.

Let $H = G^f$ and $H_k = G_k^f$ for $k \in \mathbb{N}$. Then H is the ascending union of the subgroups H_k, $k \in \mathbb{N}$. The intersection $H_k \cap S_k$ is contained in $C_{G_k}(T_k/t_k)$ by construction of H_k, so $H_k \cap S_k$ is soluble. Its quotient group $H_k/(H_k \cap S_k)$ has rank bounded by $s(s+1)$. If we denote the soluble radical of H_k by R_k, the rank of the finite semisimple factor group $F_k = H_k/R_k$ is also bounded by $s(s+1)$. Clearly F_k is isomorphic to a section A_{k+1}/B_{k+1} of F_{k+1} for some soluble B_{k+1}. Suppose that the orders F_k, $k \in \mathbb{N}$, are unbounded. Proposition 2.1 of Dixon, Evans and Smith [13] shows then that there exists a certain sequence of finite simple groups, contradicting Proposition 2.4 of the same paper.

So there is some integer m which bounds the orders of F_k, $k \in \mathbb{N}$. Again, the group H/H^m is locally finite and hyperabelian-by-finite. We are left to consider H^m. But $H^m = \bigcup_{k\in\mathbb{N}}(H_k^m)$ is the ascending union of the soluble groups $H_k^m \leq R_k$. Hence H^m is a locally soluble group which is hyperabelian by Theorem 3.31. The normal series $1 \leq H^m \leq H \leq G$ has hyperabelian-by-finite factors, so G is hyperabelian-by-finite by Corollary 4.3 and (locally soluble)-by-finite by Theorem 3.31. □

As an application of Theorem 4.13, we prove a theorem on arbitrary groups with finite co-central rank. Define the descending finite residual series of G by $J_0 = G$, $J_{\alpha+1} = J(J_\alpha)$ and $J_\lambda = \bigcap_{\alpha<\lambda} J_\alpha$ for limit ordinals λ. The terminus R of this series is clearly the $\mathcal{F}$-perfect radical of G. The factor group G/R is rather well-behaved as the next theorem shows. It also provides an a posteriori justification for the detailed examination of hyperabelian-by-finite $\mathfrak{R}$-groups.

Theorem 4.14 *Let G be a group of finite co-central rank which has trivial $\mathcal{F}$-perfect radical. Then G is hyperabelian-by-finite and its finite residual is abelian-by-(nilpotent of class $\leq s+2$).*

Proof. By the assumption, the descending finite residual series $(J_\alpha)_{\alpha<\gamma}$ terminates with 1. Proceed by induction on γ, the case $\gamma = 0$ being trivial. Suppose γ is a regular ordinal. Then $G/J_{\gamma-1}$ is hyperabelian-by-finite by induction. The quotient $J_{\gamma-1}/J_\gamma$ is residually finite and therefore locally (soluble-by-finite) by Proposition 3.26. By Theorem 4.19, $J_{\gamma-1}/J_\gamma$ is hyperabelian-by-finite. Thus the radical $\mathrm{Rad}(J_{\gamma-1}/J_\gamma)$ has finite index in $J_{\gamma-1}/J_\gamma$ and G/J_γ turns out to be generalized radical. This implies by Theorem 4.13 that G/J_γ is hyperabelian-by-finite and J_1/J_γ is abelian-by-(nilpotent of class $\leq s+2$) by Proposition 3.25. Finally, let γ be a limit ordinal. For each $0 < \alpha < \gamma$, J_1/J_α is abelian-by-(nilpotent of class $\leq s+2$) by the induction hypothesis. So J_1/J_α is soluble of derived length at most $s+3$. Consequently, J_1/J_γ is also soluble of derived length at most $s+3$ and G/J_γ is hyperabelian-by-finite. Again by Proposition 3.25, J_1/J_γ is abelian-by-(nilpotent of class $\leq s+2$). The theorem is proved. □

Proposition 4.15 *Let the group G have finite co-central rank. If G is residually ((locally soluble)-by-finite), it is (locally soluble)-by-finite.*

Proof. Let H be a finitely generated subgroup of G and let G be residually ((locally soluble)-by-finite). So H is residually (soluble-by-finite). Denote the finite residual of H by J. Then H/J is locally (soluble-by-finite) by Proposition 3.26 and J is residually soluble. A residually soluble group is locally (soluble-by-finite) by Proposition 3.29. Hence both J and H/J are hyperabelian-by-finite by Theorem 4.13. So H is hyperabelian-by-finite by Lemma 4.3, it is (locally soluble)-by-finite by Theorem 3.31. Since H is finitely generated, it is soluble-by-finite. We conclude that G is locally (soluble-by-finite) and thus (locally soluble)-by-finite by Theorem 4.13. □

The notion of a section cover of a group defined below is used to investigate simultaneously groups having a local system as well as groups having a residual system.

Definition 4.16 *Let G be a group and let M be a set consisting of sections of G. There is a natural partial ordering on M defined by*

$$A_1/B_1 \leq A_2/B_2 \quad \textit{iff} \quad A_1/B_1 \textit{ is a section of } A_2/B_2$$

We call M a section cover *of G if*

1) *$(M, \leq)$ is a directed set, i.e. to every $i, j \in M$ there exists a $k \in M$ such that $i \leq k$, $j \leq k$.*
2) *For every $g \in G \backslash \{1\}$, there exists a section $A/B \in M$ such that $g \in A \setminus B$.*

To illustrate the use of this definition, let $\mathfrak{X}$ be a subgroup closed class of groups and let the group G have a section cover M consisting of $\mathfrak{X}$-groups. If we require that for all sections $A/B \in M$ we have $B = 1$, then G is locally an $\mathfrak{X}$-group. On the other hand, if we have $A = G$ for all sections $A/B \in M$, then G is residually an $\mathfrak{X}$-group. However, the class of groups having a section cover consisting of $\mathfrak{X}$-groups is usually larger than the union of the classes of locally $\mathfrak{X}$-groups and residually $\mathfrak{X}$-groups. For example, the additive group of the rationals $\mathbb{Q}^+$ has a section cover consisting of finite groups, e.g. $\{ \frac{1}{n}\mathbb{Z}/n\mathbb{Z} \mid n \in \mathbb{N}\}$. But $\mathbb{Q}^+$ is neither locally finite nor residually finite.

For classes $\mathfrak{X}$, $\mathfrak{Y}$ of groups write $\mathfrak{X} \leq \mathfrak{Y}$ if every group belonging to $\mathfrak{X}$ also belongs to $\mathfrak{Y}$. A (group theoretical) *operation* A in the sense of [35] is a function assigning to each class of groups $\mathfrak{X}$ a class of groups $A\mathfrak{X}$ subject to the following conditions:

1) $A\mathfrak{I} = \mathfrak{I}$, where $\mathfrak{I}$ is the class of all identity groups
2) $\mathfrak{X} \leq A\mathfrak{X} \leq A\mathfrak{Y}$ for all classes $\mathfrak{X} \leq \mathfrak{Y}$

A class of groups $\mathfrak{X}$ is called A-closed if $A\mathfrak{X} = \mathfrak{X}$. A partial ordering of operations is defined as follows: $A \leq B$ means that $A\mathfrak{X} \leq B\mathfrak{X}$ for every class of groups $\mathfrak{X}$. The product AB of two operations A,B is defined as $(AB)\mathfrak{X} = A(B\mathfrak{X})$. Note that the product of operations is associative. An operation A is called a *closure operation* if $A^2 = A$. If $\mathcal{S}$ is a set of operations, then closure operation generated by this set, $K = \langle \mathcal{S} \rangle$, is defined to be the unique smallest closure operation for which $A \leq K$ for every $A \in \mathcal{S}$ (it is easy to see that such a K exists).

For our purpose, we need to consider the closure operations S, L and R which denote for subgroup closure, local closure and residual closure respectively. Let C be the operation that assigns to the group theoretical class $\mathfrak{X}$ the class of all groups that have a section cover consisting of $\mathfrak{X}$-groups. For further details on group theoretical classes and closure operations we recommend Chapter 1.1 of Robinson's book [35].

Lemma 4.17 *Let S, L, R, C be the operations defined above. Then*

a) $L, R \leq C$,
b) $\langle L, R \rangle S = \langle C \rangle S$. *In other words, if $\mathfrak{X}$ is a subgroup closed class, then $\mathfrak{X}$ is C-closed if and only if it is both L- and R-closed.*

Proof. Part a) was proved in the remark above. As an immediate consequence, $\langle L, R \rangle \leq \langle C \rangle$, thus $\langle L, R \rangle S \leq \langle C \rangle S$. For the converse direction in part b), Let

$\mathfrak{Y}$ be a class of groups and let $\mathfrak{X} = S\mathfrak{Y}$ be its subgroup closure. It is sufficient to show $C\mathfrak{X} \leq LR\mathfrak{X}$ since then

$$\langle C\rangle S\mathfrak{Y} = \langle C\rangle\mathfrak{X} \leq \langle LR\rangle\mathfrak{X} \leq \langle L,R\rangle\mathfrak{X} = \langle L,R\rangle S\mathfrak{Y}\ .$$

Let $G \in C\mathfrak{X}$ and let $\{A_j/B_j \mid j \in \mathcal{J}\}$ be a section cover of G consisting of $\mathfrak{X}$-groups. Let F be a finitely generated subgroup of G. Then $F \leq A_j$ for some $j \in \mathcal{J}$. For $k \in \mathcal{J}$ with $A_j/B_j \leq A_k/B_k$, the group $F/(F \cap B_k)$ is an $\mathfrak{X}$-group because it is isomorphic to a subgroup of the $\mathfrak{X}$-group A_k/B_k and $\mathfrak{X}$ is subgroup closed. The set $\{F \cap B_k \mid A_j/B_j \leq A_k/B_k\}$ is a residual system of F whose members have their factor group $F/(F \cap B_k)$ in $\mathfrak{X}$. Hence $F \in R\mathfrak{X}$. This shows $G \in LR\mathfrak{X}$. □

After all these preparations, we are now able to state and prove the next result.

Theorem 4.18 *Let the group G have finite co-central rank. If G has a section cover consisting of locally (soluble-by-finite) groups, then G is (locally soluble)-by-finite.*

Proof. Let $\mathfrak{X}$ be the class of (locally soluble)-by-finite groups. The class $L\mathfrak{X}$ is the class of locally (soluble-by-finite) groups. To state the theorem in other words, we have to prove

$$\mathfrak{R} \cap CL\mathfrak{X} \ = \ \mathfrak{R} \cap \mathfrak{X}\ .$$

Obviously it is sufficient to prove $\mathfrak{R} \cap CL\mathfrak{X} \leq \mathfrak{R} \cap \mathfrak{X}$. The class $\mathfrak{X}$ is subgroup closed. Using Lemma 4.17, we obtain

$$CL\mathfrak{X} \ \leq\ \langle C\rangle\mathfrak{X} \ =\ \langle C\rangle S\mathfrak{X} \ =\ \langle L,R\rangle S\mathfrak{X} \ =\ \langle L,R\rangle\mathfrak{X}\ .$$

Therefore, it is sufficient to prove $\mathfrak{R} \cap L\mathfrak{X} \leq \mathfrak{R} \cap \mathfrak{X}$ and $\mathfrak{R} \cap R\mathfrak{X} \leq \mathfrak{R} \cap \mathfrak{X}$. The first inequality is the content of Proposition 4.15 and the second one follows from Theorem 4.13. □

Let us conclude with the main Theorem of this thesis. For groups of finite co-central rank, large classes of groups coincide with the class of hyperabelian-by-finite groups. Theorem 4.19 also comprises important results from the last chapters, namely Theorem 3.31 and Theorem 4.18.

Theorem 4.19 *Let G be a group with finite co-central rank. The following conditions are equivalent:*

1) *G has a section cover consisting of locally (soluble-by-finite) groups,*
2) *G is locally (soluble-by-finite),*
3) *G is (locally soluble)-by-finite,*
4) *G is residually ((locally soluble)-by-finite),*
5) *G is generalized radical,*
6) *G is hyperabelian-by-finite.*

Proof. Obviously 3) implies 2) which in turn implies 1). That 3) follows from 1) has been proved in Theorem 4.18. Trivially 4) is a consequence of 3), and the reverse direction has been proved in Proposition 4.15. The equivalence of conditions 3) and 5) follows from Theorem 3.31. It is clear that 5) follows from 6), and the converse of this has been shown in Corollary 4.3. □

Theorem 4.19 points out the prominent role hyperabelian groups play in the theory of generalized soluble $\mathfrak{R}$-groups. It motivates the extensive investigation of hyperabelian $\mathfrak{R}$-groups which has been carried out in Chapter 3.

Index of Notation

Binary relations

$X = Y$	X is equal to Y
$X \subseteq Y$	X is a subset of Y
$X \subset Y$	X is a proper subset of Y
$H \leq G$	H is a subgroup of G
$H < G$	H is a proper subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
$H \operatorname{char} G$	H is a characteristic subgroup of G

Elements and groups

$\langle x_i \mid i \in \mathcal{I} \rangle$	subgroup generated by the set $\{x_i \mid i \in \mathcal{I}\}$
$[x, y]$	$x^{-1}y^{-1}xy$, commutator of x and y
$[X, Y]$	$\langle\, [x, y] \mid x \in X, y \in Y \,\rangle$
x^g	$g^{-1}xg$
x^G	$\langle x^g \mid g \in G \rangle$
H^G	$\langle H^g \mid g \in G \rangle$, normal closure of H in G
H^n	$\langle h^n \mid h \in H \rangle$ (n an integer)
XY	$\{xy \mid x \in X, y \in Y\}$
$H \times K$	direct product of H and K
$\mathrm{Dr}_{i \in \mathcal{I}} G_i$	direct product of the groups G_i, $i \in \mathcal{I}$
$\mathrm{Cr}_{i \in \mathcal{I}} G_i$	Cartesian product of the groups G_i, $i \in \mathcal{I}$
$H \rtimes K$	semidirect product of H with K
$H \wr K$	(restricted) standard wreath product of H with K
C_n	cyclic group of order n
C_{p^∞}	Prüfer p-group
$\mathrm{Sym}(n)$	symmetric group on $\{1, ..., n\}$
$\mathrm{Alt}(n)$	alternating group on $\{1, ..., n\}$
$\mathrm{GL}(n, p)$	general linear group of degree n over $\mathbb{F}_p$

Special subgroups

$Z(G)$	centre of G
$Z_\alpha(G)$	α-th term of the ascending central series of G atarting with $Z_0 = 1$
$Z_\infty(G)$	hypercentre of G
$C(g), C_G(g)$	centralizer of g (in G)
$C_G(H/K)$	$\{g \mid g \in G, [H, g] \leq K\}$
G'	commutator/derived subgroup of G
$G^{(\alpha)}$	α-th term of the derived series of G
$\gamma_\alpha(G)$	α-th term of the lower central series of G starting with $\gamma_1 = G$
$\mathrm{Aut}(G)$	automorphism group of G
$\mathrm{Inn}(G)$	group of inner automorphisms of G
$\mathrm{Out}(G)$	$\mathrm{Aut}(G)/\mathrm{Inn}(G)$
$J(G)$	finite residual of G

Numbers associated with a group G

$\|G\|$	cardinality of G
$\|G:H\|$	index of H in G
$\exp(G)$	exponent of the periodic group G
$d(G)$	minimum number of generators of a group G
$r(G)$	Prüfer rank of a group G
$r_c(G)$	co-central rank of G
$r_0(G)$	Hirsch length of G
$co(G)$	composition rank of G
$cl(G)$	nilpotency class of G
$dl(G)$	derived length of G

Rings and modules

$\mathbb{N}$	ring of positive integers
$\mathbb{Z}$	ring of integers
$\mathbb{F}_q$	Galois field of order q
$\mathbb{Q}$	field of rationals
$\mathbb{F}[x]$	ring of polynomials in x over the field $\mathbb{F}$
$\deg p$	degree of the polynomial $p \in \mathbb{F}[x]$
RG	group ring of the group G over the ring R
$\bigoplus_{i\in\mathcal{I}} M_i$	direct sum of the modules M_i, $i \in \mathcal{I}$
$A \otimes_{\mathbb{Z}} B$	tensor product of the $\mathbb{Z}$-modules A and B
$\mathrm{Ann}_R(S)$	annihilator of S in R

Bibliography

[1] **Amberg, B., Dickenschied, O., Sysak, Ya.P**
Subgroups of the adjoint group of a radical ring. Can. J. Math. **50** No.1, 3-15 (1998)

[2] **Amberg, B., Sysak, Ya.P.**
Locally soluble products of minimax groups. Groups - Korea '94, de Gruyter-Verlag, Berlin, 18-25 (1994)

[3] **Baer, R., Heineken, H.**
Radical groups with finite abelian subgroup rank. Illinois J. Math. **16**, 533-580 (1972)

[4] **Baer, R.**
Polyminimaxgruppen. Math. Ann. **175**, 1-42 (1968)

[5] **Belyaev, V.V.**
Locally finite groups with Černikov Sylow p-subgroups. Algebra and Logic **20**, 393-402 (1981)

[6] **Čarin, V. S.**
On groups possessing solvable increasing invariant series. Mat. Sb. **41**, 297-316 (1957)

[7] **Carter, R.W.**
Simple Groups of Lie Type. Wiley, New York (1972)

[8] **Černikov, N.S.**
A theorem on groups of finite special rank. Ukrainian Math. J. **42**, 855-861 (1990)

[9] **Dixon, J.D., du Sautoy, M.P.F., Mann, A., Segal, D.**
Analytic pro-p Groups. London Math. Soc. Lecture Notes Series 157 (1991)

[10] **Dixon, M.R., Evans, M.J., Smith, H.**
On groups with rank restrictions on subgroups. Groups St. Andrews '97, Lond. Math. Soc. Lect. Note Ser. **260**, 237-247 (1999)

[11] ———
On groups that are residually of finite rank. Isr. J. Math. **107**, 1-16 (1998)

[12] **Dixon, M.R., Evans, M.J., Smith, H.**
Locally (soluble-by-finite) groups with all proper insoluble subgroups of finite rank. Arch. Math. **68**, 100-109 (1997)

[13] ———
Locally (soluble-by-finite) groups of finite rank. J. Algebra **182**, 756-769 (1996)

[14] **Dixon, M.R.**
Rank conditions in groups. Lecture Notes Series Universita degli Studi di Trento (1995)

[15] **Fuchs, L.**
Infinite Abelian Groups Vol.36-II. Academic Press, London (1973)

[16] **Gluškov, V.M.**
On some questions in the theory of nilpotent and locally nilpotent groups without torsion. Mat. Sb. **30**, 79-104 (1952)

[17] **Hungerford, T.W.**
Algebra, 5^{th} *edition.* Springer-Verlag, Berlin (1989)

[18] **Hartley, B.**
Simple locally finite groups. In: Finite and locally finite groups Kluwer Acad. Publ, 1-44 (1995)

[19] **Huppert, B.**
Endliche Gruppen I. Springer-Verlag, Berlin (1967)

[20] **Kargapolov, M.I.**
On soluble groups of finite rank. Algebra i Logika **1**, 37-44 (1962)

[21] **Finiteness conditions and factorizations in infinite groups**
Russian Math. Surveys **47***, 81-126.* (1992)

[22] **Kegel, O.H., Wehrfritz, B.A.F.**
Locally Finite Groups. North-Holland Publishing Company, Amsterdam (1973)

[23] **Kim, Y., Rhemtulla, A.H.**
On locally graded groups. Groups - Korea '94, de Gruyter-Verlag, Berlin, 189-197 (1995)

[24] **Kropholler, P.H.**
On finitely generated soluble groups with no large wreath products. Proc. London Math. Soc. **49** Ser.III, 155-169 (1984)

[25] **Langobardi, P., Maj, M., Smith, H.**
A finiteness condition on non-nilpotent subgroups. Comm. Algebra **24**, 3567-3588 (1996)

[26] **Lubotzky, A., Mann, A.**
Residually finite groups of finite rank. Math. Proc. Camb. Phil. Soc. **106**, 385-388 (1989)

[27] **Lubotzky, A., Mann, A.**
Powerful p-groups. I. Finite groups. J. Algebra **105**, 484-505 (1987)

[28] **Mal'cev, A.I.**
On certain classes of infinite soluble groups. Mat. Sb., **28**, 567-588 (1951); *translated in Trans. Amer. Math. Soc.* **2**, *1-21* (1956)

[29] ———
On groups of finite rank. Mat. Sb. **22**, 351-352 (1948)

[30] **Merzljakov, Yu. I.**
Locally soluble groups of finite rank. Algebra i Logika **3**, 5-16 (1964); erratum, **8**, 686-690 (1969)

[31] **Miller, G. A., Moreno, H.C.**
Non-abelian groups in which every subgroup is abelian. Trans. Amer. Math. Soc. **4**, 398-404 (1903)

[32] **Ol'shanskii, A. Yu.**
Geometry of Defining Relations in Groups. Mathematics and its Applications, Vol. 70 Kluwer Academic Publishers, Dordrecht (1989)

[33] **Robinson, D.J.S.**
A Course in the Theory of Groups. Springer-Verlag, Berlin (1982)

[34] ———
A new treatment of soluble groups with finiteness conditions on their abelian subgroups. Bull. London Math. Soc **8**, 113-129 (1976)

[35] ———
Finiteness Conditions and generalized soluble Groups, Vols.1 and 2. Springer-Verlag, Berlin (1972)

[36] ———
A note on groups of finite rank. Compositio Math. **31**, 240-246 (1969)

[37] ———
Residual properties of some classes of infinite soluble groups. Proc. London Math. Soc.(3) **18**, 495-520 (1968)

[38] **Segal, D.**
A footnote on residually finite groups. Israel J. Math. **94**, 1-5 (1996)

[39] ———
Polycylcic Groups. Cambridge University Press, Cambridge (1983)

[40] **Shalev, A.**
Characterization of p-adic analytic groups in terms of wreath products. J. Algebra **145**, 204-208 (1992)

[41] **Šunkov, V.P.**
On locally finite groups of finite rank. Alg. i Logika **10**, 199-225 (1971)

[42] **Sysak, Y.P., Tresch, A.**
Groups with finite co-central rank. J.Group Theory **4**, 325-340 (2001)

[43] **Thomas, S.**
The classification of the simple periodic linear groups. Arch. Math. **41**, 103-116 (1983)

[44] **Tits, J.**
Free subgroups in linear groups. J.Algebra **20**, 250-270 (1972)

[45] **Wehrfritz, B.A.F.**
Infinite Linear Groups. Queen Mary College Mathematics Notes (1969)

[46] **Wilson, J.S.**
Profinite Groups. Clarendon Press, Oxford (1998)

[47] ———
Two generator conditions for residually finite groups. Bull. Lundon Math. Soc. **23**, 239-248 (1981)

[48] **Zaičev, D.I.**
Solvable groups of finite rank. Naukova Dumka, Kiev, 115-130 (1971)

Curriculum vitae

March 30, 1971	Born as the first child of Gerd Klaus Tresch, Studiendirektur (teacher), and Ingrid Hanna Tresch, teacher, in Worms, Germany
1977-1981	Elementary school "Worms-Herrnsheim" in Worms
1982-1990	"Staatliches Gauß- Gymnasium Worms" in Worms. Final degree: *Abitur* with grade "*sehr gut*"
1990-1991	Military service in Daun and Mainz
since 1991	Study of Physics and Mathematics at the Johannes-Gutenberg-University, Mainz
1993	*Vordiplom* in Mathematics
1994	*Vordiplom* in Physics
1997	*Diplom* (*Master degree*) in Mathematics with final grade *"excellent"*
1997-1999	Study of Medicine at the Johannes-Gutenberg-University, Mainz
1998-1999	Ph.D. scholarship from the German Federal State Rheinland-Pfalz
1999	*Physikum* in Medicine
1999-2001	Wissenschaftlicher Angestellter (Scientific Assistant) at the Department of Mathematics, Mainz

Publications

Die Krulldimension in der Gruppentheorie.
Diploma thesis, Mainz (1997)

Krull dimension in subgroup lattices.
Preprint-Reihe der Universität Mainz **7**, Mainz (1998)

Groups with finite co-central rank (together with Y.P.Sysak).
Preprint-Reihe der Universität Mainz **22**, Mainz (1999)

Groups with finite co-central rank (together with Y.P.Sysak).
J. Group Theory **4**, 325-340 (2001)

Von allen, die bis jetzt nach Wahrheit forschten, haben die Mathematiker allein eine Anzahl Beweise finden können, woraus folgt, dass ihr Gegenstand der allerleichteste gewesen sein müsse.

Descartes