SHIMURA SUBVARIETIES AND THE TORELLI LOCUS

Dissertation zur Erlangung des Grades

Doktor der Naturwissenschaften

am Fachbereich Physik, Mathematik und Informatik

der Johannes Gutenberg-Universität Mainz

von

Abolfazl Mohajernaser

geboren in Teheran

2014

Abstract

We investigate the Shimura subvarieties in the locus of abelian and cyclic covers of $\mathbb{P}^1$. In particular show that for $s = 4, 5$, the family of abelian covers branched along s points and satisfying an irreducibility condition is Shimura if and only if it is equal to the largest irreducible subvariety of A_g over which the action of abelian group extends to the whole universal abelian scheme. For $s = 5$ there are exceptional cases that we study in some detail. We also show that for s large enough, there is no Shimura subvariety in the locus of abelian covers of $\mathbb{P}^1$ with $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. We achieve these by using an obstruction of Dwork and Ogus and also by studying the generic Mumford-Tate and monodromy group of such families.

Zusammenfassung

Wir untersuchen A_g, den Modulraum der abelschen Varietäten auf Existenz der Shimura Untervarietäten und insbesondere den Locus der abelschen und zyklischen Überlagernugen von $\mathbb{P}^1$. Wir betrachten alle solche Familien die eine Irreduzibilität Bedingung erfüllen. Für $s = 4, 5$ zeigen wir -bis auf zwei Ausnahmefälle für $s = 5$- dass die Familie ist genau dann eine Shimura Familie, wenn die Wirkung der abelschen Gruppe auf das ganze universelle abelsche Schema erweitert werden kann und es gibt keine größere Untervarietät mit dieser Eigenschaft. Wir beweisen auch, dass für s genügend groß, es gibt keine Shimura Varietät die sich aus einer Familie der abelschen Überlagerungen von $\mathbb{P}^1$ mit $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ ergibt. Wir erreichen das, durch verwendung einer Obstruktion von Dwork und Ogus sowie durch Berechnung der generischen Mumford-Tate Gruppe dieser Familien.

Contents

Introduction

In this thesis we deal with the occurrence of Shimura varieties in the locus of Jacobians of abelian (and cyclic) covers of $\mathbb{P}^1$, occasionally we also deal with totally geodesic subvarieties. Roughly speaking, we are going to show that under some conditions there is no Shimura subvarieties with in this locus for suitable genera.

Let $f : Y \to T$ be a family of abelian covers of $\mathbb{P}^1$. We will explicitly construct and describe such families and show the important features of their structures. By varying the brnch points, this family gives rise naturally to a subvariety Z of A_g. There exists a Shimura subvariety of PEL type which we denote by $S(\mu_G)$. This Shimura subvariety is constructed using the Hodge classes which are endomorphisms coming from the group action of the abelian Galois group G. Therefore $S(\mu_G)$ contains Z and if these two are equal, then the family is a Shimura family, i.e. a family which gives rise to a Shimura subvariety of A_g. Using this and a simple computer program, we can check for which families this equality holds. In this way, we obtain a table (table 1) which appears also in [MO]. We show that this table is exhaustive for $s = 4$ and that there are no more examples satisfying the above equality for approximately big monodromy data. But if $Z \neq S(\mu_G)$, it does not mean that Z is not a Shimura subvariety as there might be Hodge classes in our family that are not endomorphisms coming from the linear group action. In the case where G is cyclic, however it is shown by Moonen that Z is not actually a Shimura subvariety in this case. When G is abelian (and non-cyclic) this is more complicated to achieve. Let s be the number of branch points of the family. When $s = 4$ and the family satisfies an irreducibility condition, we show the following:

Theorem. If $s = 4$, then Z is Shimura if and only if $Z = S(\mu_G)$.

For $s = 5$, the above theorem also holds, with only two exceptions. We will show that these two families are totally geodesic families. Therefore by a result of Moonen, the investigation of whether or not these families are Shimura boils down to verify whether they have a CM point, i.e. there exists a fiber whose Jacobian is a CM abelian variety.

For $s > 5$, the problem is more complicated because of some technical difficulties which we will make clear in section 3.10. In particular, we can not use the Dwork-Ogus obstruction to get a contradiction. We will instead study the (generic) monodromy group and the generic Mumford-Tate group of the families and make use of a result of André that shows that for most cases the generic Mumford-Tate group can not be that of a Shimura family. These computations show that for s large enough there does not exist any Shimura subvariety in the locus of abelian covers provided that the Galois group is $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ and there exists an element $n \in G$ with $d_n \geq 2$ and $d_{-n} \geq 2$. Where d_n is the dimension of the eigenspace of the action of G on the variation of Hodge structures of the family. We also give some examples of families of cyclic coverings of $\mathbb{P}^1$ that are not Shimura families but they do contain a positive dimensional Shimura subvariety. This answers a question of Oort which asks if such cases exist at all.

Chapter 1. Hodge structures and their variations

1.1. Hodge structures

In this chapter we present the basic results and notions from Hodge theory that we will need in the later chapters.

Definition 1.1.1. Let R be any ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$. Let V be an R-module. An R-Hodge structure of weight k on V is given by a decomposition

$$V \otimes_R \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$

such that $V^{q,p} = \overline{V^{p,q}}$

Example 1.1.2. Let X be a compact Kähler manifold. Then the $\mathbb{Z}$-module $H^k(X, \mathbb{Z})$ carries a Hodge structure. Indeed, we have:

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X) \text{ where } H^{p,q}(X) = H^q(X, \Omega_X^p)$$

One can define the Hodge structures in terms of *Hodge filtration*. A Hodge filtration is a decreasing filtration of V by complex subspaces F^pV that satisfy:

$$F^pV \cap \overline{F^qV} = 0 \text{ for } p + q = k + 1$$

One can recover the Hodge structure from the Hodge filtration and vice versa:

$$V^{p,q} = F^p \cap \overline{F^qV}$$

$$F^pV = \bigoplus_{i \geq p} V^{i,k-i}$$

Definition 1.1.3. Let R be a ring with $\mathbb{Z} \subseteq R \subset \mathbb{R}$. A *polarized R-*Hodge structure on an of weight k on an R-module V is an R-Hodge structure on V together with a bilinear form $Q : V \times V \to R$ which is symmetric if k is even and antisymmetric otherwise, and whose extension to $V \otimes_R \mathbb{C}$ satisfies:

i) The Hodge decomposition is orthogonal for Q, i.e. $Q(v, v') = 0$ for $v \in V^{p,q}, v' \in V^{p',q'}$ with $p \neq q'$.

ii) $i^{p-q}Q(v, \overline{v}) > 0$ for $v \in V^{p,q} \setminus \{0\}$.

Example 1.1.4. Let X be a compact Kähler manifold. The Hodge structure on $H^k(X, \mathbb{Z})_{prim}$ is polarized. In fact a polarization Q is given by

$Q(\alpha, \beta) = \int_X \wedge^{n-k}(\omega) \wedge \alpha \wedge \beta$

Here ω is the Kähler form of X.

Let $\mathbb{S} = Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$. A Hodge structure on V according to Deligne can be understood as a certain homomorphism from $\mathbb{S}$ to $GL(V)$:

Proposition 1.1.5. Let V be an $\mathbb{R}$-vector space. A real Hodge structure on V defines an action of $\mathbb{S}$ on $V \otimes \mathbb{C}$ by

$z.v^{p,q} = z^p\overline{z}^q v^{p,q}$ for every $v^{p,q} \in V^{p,q}$ and $z \in \mathbb{S}(\mathbb{R})$

In this way one gets a homomorphism $h : \mathbb{S} \to GL(V)$. Conversely, any such representation corresponds to a Hodge structure on V.

Proof. See [D1], 1.1.1.

Example 1.1.6. Let V be an $\mathbb{R}$-vector space. A *complex structure* on V is a linear morphism $J : V \to V$ such that $J^2 = -1$. Any complex structure on V gives a Hodge structure of type $(-1, 0) + (0, -1)$ by:

$h : \mathbb{C}^\times \to GL(V_\mathbb{R})$
$a + bi \mapsto a + bJ$

Conversely, any such Hodge structure of the above type defines a complex structure on V by proposition 1.1.5.

Hodge structures of the above type are the main Hodge structures that we encounter in this thesis and in fact we have the following key theorem.

Theorem 1.1.7. There is a correspondence between the set of polarized abelian varieties of dimension g and polarized Hodge structures (L, h, Q) of type $(-1, 0) + (0, -1)$ on a torsion-free lattice L of rank $2g$. This correspondence gives a bijection between sets of polarized abelian varieties $A = V/L$ and polarized Hodge structures on $L \otimes \mathbb{R}$ of the above type.

Proof. Let us sketch a proof of this important statement. if (L, h, Q) carries a Hodge structure of the above type, then by example, the Hodge structure gives a complex structure on $L_{\mathbb{R}}$ and makes it to a $\mathbb{C}$-vector space. On the other hand, we have an isomorphism

$$L \otimes \mathbb{R} \hookrightarrow L \otimes \mathbb{C} \to V^{0,-1},$$

Therefore, we can consider Q as an alternating form on $V^{0,-1}$. Since $Q(iv, v) < 0$ for $v \in V^{0,-1}$, we can set $E = -Q$ and it will be a positive definite Hermitian form making the complex torus $V^{0,-1}/L$ an abelian variety.

Conversely, if (A, E) is an abelian variety with $A = V/L$, set $Q = -E$. according to example we have a Hodge structure of type $(-1, 0) + (0, -1)$ on L corresponding to $J = -i$. □

Let us note that if A is an abelian variety over $\mathbb{C}$, then by the above theorem $H_1(A, \mathbb{Q})$ carries a Hodge structure of type $(-1, 0) + (0, -1)$, see [Mi].

Let V be a $\mathbb{Q}$-vector space equipped with a Hodge structure of weight k. This Hodge structure by proposition 1.1.5 corresponds to a representation

$$h : \mathbb{S} \to GL(V_{\mathbb{R}})$$

The following definition is very important for us in these notes:

Definition 1.1.8. The *Mumford-Tate group* of a $\mathbb{Q}$-Hodge structure (V, h) is the smallest $\mathbb{Q}$-algebraic subgroup of $GL(V)$ that contains the image of h. That is, the smallest subgroup G such that

$$h(\mathbb{S}) \subseteq G \times_{\mathbb{Q}} \mathbb{R}.$$

We denote the Mumford-Tate group of the hodge structure (V, h) by $MT(V, h)$ or simply $MT(V)$.

1.2. Variations of Hodge structures

Let $f : X \to Y$ be a smooth family of algebraic manifolds. In this section we are going to study the behavior of Hodge structures in such families. We will use these results in the subsequent chapters.

Definition 1.2.1. Let X be a topological space. By a *local system* on X we mean a locally constant sheaf with stalk G, where G is an abelian group.

Let $\mathbb{V}$ be a local system on X and let $\gamma : [0, 1] \to X$ be an arc in X. The pull-back $\gamma^*\mathbb{V}$ is a local system on $[0, 1]$. In particular, $(\gamma^*\mathbb{V})_0 = \mathbb{V}_{\gamma(0)}$ and $(\gamma^*\mathbb{V})_1 = \mathbb{V}_{\gamma(1)}$. Consider a point x_0 in X as a base point. The above observation shows that the local system gives rise to a representation

$$\rho : \pi_1(X, x_0) \to Aut(\mathbb{V}_{x_0})$$

We call the image of ρ, the *monodromy group* of the local system.

Example 1.2.2. Let $f : X \to Y$ be a smooth map of complex manifolds. Then the sheaf $R^k f_*\mathbb{C}$ is a local system on Y for $k \in \mathbb{N}$.

Let $\mathcal{V} = R^k f_*\mathbb{C} \otimes \mathcal{O}_X$. Let U be a simply connected open subset of Y. Over U, $\mathcal{V}$ is a constant sheaf. Now set:

$\sigma = \sum \alpha_i \otimes \sigma_i \in \mathcal{V}(U)$,

Where $\alpha_i \in R^k f_* \mathbb{C}(U)$ and $\sigma_i \in \mathcal{O}(U)$. We define a connection

$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_Y$ by:

$\sigma = \sum \alpha_i \otimes \sigma_i \mapsto \sum \alpha_i \otimes d\sigma_i$.

The above connection is called the *Gauss-Manin* connection.

Proposition 1.2.3. Let $f : X \to Y$ be a smooth morphism of algebraic manifolds and consider $\mathcal{V} = R^k f_* \mathbb{C} \otimes \mathcal{O}_X$. Then $\mathcal{V}$ admits a filtration $F^\bullet$ by holomorphic subbundles such that $F^p H^k(X_y, \mathbb{C}) = F^p_y \mathcal{V}$.

Proof. See for example [Vo].

The Gauss-Manin connection satisfies a very important property:

$\nabla(F^p \mathcal{V}) \subseteq F^{p-1} \mathcal{V} \otimes \Omega^1$ for every p.

We call this property the *Griffiths transversality.*

We are now ready to define the main object of this chapter:

Definition 1.2.4. Let X be complex manifold and R be a ring such that $\mathbb{Z} \subseteq R \subset \mathbb{R}$. A *variation of R-Hodge structures* of weight k, is a local system $\mathbb{V}_R$ of R-modules of finite rank and a filtration $\mathcal{F}^\bullet$ of $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_X$ by holomorphic subbundles such that:

i) Griffiths transversality holds

ii) For every $x \in X$, $(\mathcal{V}_x, \mathcal{F}^\bullet_x)$ is a Hodge structure of weight k.

A variation $\mathcal{V}$ of Hodge structures is called *polarized*, if there exists a locally constant bilinear form Q on $\mathcal{V}$ such that $(\mathcal{V}_x, \mathcal{F}^\bullet_x, Q_x)$ is a polarized Hodge structure of weight k for all $x \in X$

1.3. The generic Mumford-Tate group

A smooth family of algebraic manifolds gives rise to a family of Hodge structures. The notion of Mumford-Tate groups also can be generalized to families:

Proposition 1.3.1. Let S be a connected complex manifold. Let $\mathcal{V}$ be a variation of $\mathbb{Q}$-Hodge structures of weight k over S. Then there exists a countable union $\Sigma \subset S$ of submanifolds such that all $MT(\mathcal{V}_s)$ coincide for $s \in S \setminus \Sigma$. For all $s' \in S$ and $s \in S \setminus \Sigma$ it holds that $MT(\mathcal{V}_{s'}) \subset MT(\mathcal{V}_s)$.

Proof. This was originally proved in [CDK]. See also [M2].

The group $M = MT(\mathcal{V}_s)$ for $s \in S \setminus \Sigma$ is called the *generic Mumford-Tate group.*

Like the Mumford-Tate group, the monodromy group can also be defined for a variation of Hodge structures.

Definition 1.3.2. Let $(\mathcal{V}, \mathcal{F}^\bullet, Q)$ be a polarized variation of Hodge structures over a connected complex manifold S. For $s \in S$, we define $Mon^0(\mathcal{V}_s)$ to be the identity component of the Zariski closure of the monodromy group in $GL(\mathcal{V}_s)$.

There is a close relation between the monodromy group and the Mumford-Tate group of a variation of Hodge structures. We will describe this relation below and will use it in the next chapters extensively:

Proposition 1.3.3. With the assumptions and notations as in the previous definition and proposition 1.3.1, $Mon^0(\mathcal{V}_s)$ is a normal subgroup of $MT^{der}(\mathcal{V}_s)$ for all $s \in S \setminus \Sigma$. If there exists a point $s \in S$ such that $MT(\mathcal{V}_s)$ is abelian (hence a torus), then we have :

$$Mon^0(\mathcal{V}_s) = MT^{der}(\mathcal{V}_s).$$

for all $s \in S \setminus \Sigma$.

Proof. The first statement is proved by Deligne in [D2]. The second one even holds in a more general setting of mixed Hodge structures and is proved by André in [A].

According to the last proposition, the groups $Mon^0(\mathcal{V}_s)$ coincide for all $s \in S \setminus \Sigma$. We therefore omit s and simply write $Mon^0(\mathcal{V})$ as the monodromy group of the variation of Hodge structures.

Recall that a point that satisfies the last condition of the above proposition, namely a point s with $MT(\mathcal{V}_s)$ abelian, is called a *CM point*. The study of CM points is one of our main goals in these notes and we will come back to them in chapter 3.

Chapter 2. Shimura Varieties

2.1. Premilinaries

In this chapter we are going to review the basic concepts and constructions of Shimura varieties. In the future chapters we will be working with these constructions and their applications. Since one of the main objects in the definition of a Shimura variety is a reductive algebraic group, we first recall some definitions and well-known properties of algebraic groups.

Definition 2.1.1. Let G be a $\mathbb{Q}$-algebraic group. Let $R(G)$ be the radical of G. That is, $R(G)$ is the maximal connected normal solvable algebraic subgroup of G. Let $R_u(G)$ be the set of unipotent elements inside $R(G)$. We say that:

• G is reductive if $R_u(G) = \{1\}$.

• G is semisimple if $R(G) = \{1\}$

• G is simple is $\{1\}$ and $\{G\}$ are the only connected normal algebraic subgroups of G.

By [D3], 1.1, there exists exact sequences

$1 \to G^{der} \to G \to T \to 1$

$1 \to Z(G) \to G \to G^{ad} \to 1$

$1 \to Z(G^{der}) \to Z(G) \to T \to 1$

Where T is the maximal commutative quotient of G and G^{der} is the derived group of G i.e. the group generated by the commutators of G.

For our future use, the following proposition is of key importance:

Proposition 2.1.2. Let G be a reductive $\mathbb{Q}$-algebraic group. Then G^{ad} is a semisimple group with trivial center.

Proof. By [D3] 1.1, the above exact sequences induce an isogeny $G^{der} \to G^{ad}$ with kernel $Z(G^{der})$. The proposition then follows from a more general fact that if G is a connected algebraic group, then $Z(G)^0$ and G^{der} are respectively a torus and a semisimple group and moreover G is the almost direct product of these two subgroups.

The next important notion for the definition of a Shimura variety is the notion of Cartan involution which we define here:

Definition 2.1.3. Let G be a connected $\mathbb{R}$-algebraic group and θ be an involution of G. We say that θ is a Cartan involution if the following Lie group:

$$G^{\theta}(\mathbb{R}) = \{g \in G(\mathbb{C}) | g = \theta(\overline{g})\}$$

is a compact subgroup of $G(\mathbb{C})$.

The following theorem reveals the importance of Cartan involutions in characterising the reductive groups.

Theorem 2.1.4. An $\mathbb{R}$-algebraic group G is a reductive group if and only if it has a Cartan involution in which case any two Cartan involutions are conjugate to each other by an inner automorphism.

Proof. See [S].

Remark 2.1.5. Let G be a connected algebraic group over $\mathbb{R}$ such that $G(\mathbb{R})$ is compact. Then it follows from the definition that the identity map id is a Cartan involution because $G^{id}(\mathbb{R}) = G(\mathbb{R})$ and this Lie group is compact by assumption. It follows from the previous theorem that G is a reductive algebraic group. Moreover, this argument shows that any compact linear

$\mathbb{R}$-algebraic group is a reductive group having the identity map as a Cartan involution. Since by theorem 2.1.4, any two Cartan involutions are conjugate, this shows that id is the unique Cartan involution of G.

Now we are going to define the Shimura datum which is a buliding block of Shimura varieties.

Definition 2.1.6. A *Shimura datum* is a pair (G, X) where G is a reductive $\mathbb{Q}$-algebraic group and X is a conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the following conditions:

1) The morphism induced by h on the Lie group $Lie(G)_{\mathbb{C}}$ acts only by the characters $\frac{z}{\bar{z}}$, 1 and $\frac{\bar{z}}{z}$.

2) The map $adoh(i)$ is a Cartan involution on $G^{ad}_{\mathbb{C}}$.

3) G^{ad} has no $\mathbb{Q}$-factor H such that $H(\mathbb{R})$ is compact.

Remark 2.1.7. The above conditions for a Shimura datum can be rephrased as follows:

1′) The representation induced by h on $Lie(G)_{\mathbb{C}}$ corresponds to a Hodge structure of type $(1, -1) \oplus (0, 0) \oplus (-1, 1)$

2′) The restriction of the inner automorphism of $h(i)$ is a Cartan involution on G^{der}. This is because of the fact that G^{der} and G^{ad} are isogenous and hence their compact subgroups correspond.

3′) G^{ad} has no $\mathbb{Q}$-factor H to which $ad(h(i))$ projects trivially. The equivalence of this and 3) above is just the content of remark 2.1.5.

2.1.8. Examples

• The simplest example of a Shimura datum is given by a torus. Let T be a torus over $\mathbb{Q}$. Consider a homomorphism $h : \mathbb{C}^\times \to T(\mathbb{R})$. As T is a commutative algebraic group, h is fixed by the conjugation action and hence it makes sense to consider the class $\{h\}$ consisting only of h. All of the above conditions are then trivially satisfied and hence we get a Shimura datum by definition. This example, although simple, turns out to be very useful for our study of CM abelian varieties and special points.

- Set $G = GL_{2,\mathbb{Q}}$ and consider the set X of conjugacy classes of homomorphisms $h_0 : \mathbb{S} \to GL_{2,\mathbb{R}}$, $h_0(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Then (G, X) has the structure of a Shimura datum and there is a bijection $X \cong \mathbb{C} \setminus \mathbb{R}$ given by $h_0 \mapsto i$.

Definition 2.1.9. Let (G, X) and (M, Y) be two Shimura data. A *morphism of Shimura data* between (M, Y) and (G, X) is a morphism $f : M \to G$ of algebraic groups that respects the classes X and Y. In other words, if $y : \mathbb{S} \to M_{\mathbb{R}}$ is in Y, then $x = f \circ y : \mathbb{S} \to G_{\mathbb{R}}$ lies in X. f will be called a *closed immersion* of Shimura data, if map $f : M \to G$ is a closed immersion of algebraic varieties.

Hermitian symmetric domains

One of the most important properties of Shimura data is that they give rise to Hermitian symmetric domains. These will be used also in the construction of Shimura varieties which are the main ingredients of this chapter. Regarding this, we will first need to define the Hermitian symmetric domains and recall their basic properties. Throughout this section, M will always be a $\mathcal{C}^\infty$ manifold.

Definition 2.1.10. An *almost complex structure* on a $\mathcal{C}^\infty$ manifold M is a smooth family $(J_p)_{p\in M} : T_pM \to T_pM$ of automorphisms of tangent spaces T_pM, such that $J_p^2 = -1$ for every $p \in M$.

From the definition, it follows that each J_p induces a complex structure on each T_pM. We call such a pair (M, J) an *almost complex manifold.*

Definition 2.1.11. A smooth 2-*tensor field* on M is a family of bilinear maps $g_p : T_pM \times T_pM$ such that for all smooth vector fields X and Y the map $p \to g_p(X,Y)$ is smooth. If in addition, each g_p is a is symmetric and positive-definite for $p \in M$, then the 2-tensor field g is called a *Riemannian metric* on M.

Definition 2.1.12. If (M, J) is a connected almost complex manifold, a *Hermitian metric* on M is a Riemannian metric g such that $g(JX, JY) = g(X, Y)$ for every vector field X and Y. M equipped with such a Hermitian metric is called a *Hermitian manifold.*

From this definition it follows that each g_p is the real part of a unique Hermitian form h_p on each T_pM.

We are now ready to define the Hermitian symmetric spaces:

Definition 2.1.13. A Hermitian symmetric space is a Hermitian manifold such that for each $p \in M$, there is an involution s_p such that p is an isolated fixed point of s_p.

The relation between Hermitian symmetric spaces and Shimura data is given as follows: for $h : \mathbb{S} \to G_{\mathbb{R}}$ a Shimura datum, let K_h be the stabilizer of h under conjugation. Then the space $D = G(\mathbb{R})/K_h(\mathbb{R})$ parametrizes the elements of the conjugacy classes of h. It turns out ([H], II. §4) that D is a $\mathcal{C}^\infty$ manifold and the elements of $G(\mathbb{R})$ are diffeomorphisms of D. Now from condition 1 of the definition of a Shimura datum, it follows that the action of $h(\mathbb{S})$ on $Lie(G)_{\mathbb{C}}$ gives a decomposition into eigenspaces of the types $(-1,1) \oplus (0,0) \oplus (1,-1)$. The intersection of the $(0,0)$-piece with $Lie(G_{\mathbb{R}})$ coincides with $Lie(K_h(\mathbb{R}))$. Therefore the real vector space $T_h(D) = Lie(G(\mathbb{R}))/Lie(K_h(\mathbb{R}))$ inherits a complex structure J_h. Now the relation between Shimura data and Hermitian symmetric spaces is stated in the following proposition:

Proposition 2.1.14. Each connected component of D is a Hermitian symmetric space.

Proof. [R], proposition 1.4.8.

2.2. Shimura varieties

Let (G, X) be a Shimura datum and let $\mathbb{A}^f$ be the ring of finite adeles. If K is a compact open subgroup of $G(\mathbb{A}^f)$, consider the double coset:

$$Sh_K(G, X) = G(\mathbb{Q}) \setminus X \times (G(\mathbb{A}^f)/K)$$

Where the action is given by $q(x, a)k = (qx, qak)$.

To understand the structure of $Sh_K(G, X)$ better, we need the following definition:

Definition 2.2.1. Let G be a $\mathbb{Q}$-algebraic group. A subgroup Γ of $G(\mathbb{Q})$ is called *arithmetic*, if Γ is commensurable with $G(\mathbb{Z})$. That is, if $\Gamma \cap G(\mathbb{Z})$ is of finite index in both Γ and $G(\mathbb{Z})$.

Lemma 2.2.2. Let $K \subseteq G(\mathbb{A}^f)$ be a compact open subgroup. Then $\Gamma = K \cap G(\mathbb{Q})$ is an arithmetic subgroup.

Proof. See [Mi] Proposition 4.1.

Proposition 2.2.3. Let $K \subseteq G(\mathbb{A}^f)$ be a compact open subgroup and $C = G(\mathbb{Q}) \setminus G(\mathbb{A}^f)/K$. Set $\Gamma_{[g]} := gKg^{-1} \cap G(\mathbb{Q})^+$ for $[g] \in C$. Then:

$$Sh_K(G, X) = \sqcup_{[g]\in C} \Gamma_{[g]} \setminus D^+.$$

Proof. [Mi], Lemma 5.13.

Now if $K' \subseteq K$ are compact open subgroups, there is a natural morphism $Sh_{K'}(G,X) \to Sh_K(G,X)$.

Definition 2.2.4. The Shimura variety $Sh(G,X)$ associated to the Shimura datum (G,X) is defined to be the limit:

$Sh(G,X) = \varprojlim_K Sh_K(G,X)$.

Where the limit is taken over all compact open subgroups K of $G(\mathbb{A}^f)$.

In order to determine the algebraicity of $Sh_K(G,X)$ (and hence $Sh(G,X)$) we need the following famous theorem due to Baily and Borel :

Theorem 2.2.5. Let D be a bounded symmetric domain and Γ an arithmetic subgroup of $Aut(D)^+$. The quotient $\Gamma \setminus D$ has the structure of a complex quasi-projective algebraic variety. If Γ is moreover torsion-free, then this structure is unique.

Proof. [BB].

Corollary 2.2.6. The Shimura variety $Sh_K(G,X)$ as constructed above has the structure of a complex algebraic variety.

Proof. Combine proposition 2.2.3 with the Baily-Borel theorem above.

Remark 2.2.7. Note that a morphism $f : (M,Y) \to (G,X)$ of Shimura data as defined in 2.1.9 gives rise naturally to a morphism $Sh(f) : Sh(M,Y) \to Sh(G,X)$ of Shimura varieties. Also, if $K' \subseteq K$ are compact open subgroups, then by definition of a Shimura variety, there is a natural morphism $Sh_{K',K} : Sh_{K'}(G,X) \to Sh_K(G,X)$.

Let $\gamma \in G(\mathbb{A}^f)$. Given compact open subgroups K, K' with $K' \subseteq \gamma K \gamma^{-1}$, we define another important morphism of Shimura varieties as follows:

Definition 2.2.8. Let $T_\gamma = [.\gamma] : Sh_{K'}(G,X) \to Sh_K(G,X)$ be given on $\mathbb{C}$-vlaued points by $T_\gamma[x, aK'] = [x, a\gamma K]$. We call T_γ, the *Hecke correspondence* associated with γ.

2.3. Shimura subvarieties of A_g

Let $\mathcal{A}_g$ be the moduli stack of complex principally polarized abelian varieties. In this section we are going to show that $\mathcal{A}_g$ has the structure of a Shimura variety. At least in this thesis, this Shimura variety is the most important one as we will be mainly working with this Shimura variety and it's Shimura subvarieties. We will see that Shimura subvarieties of this Shimura variety have imoprtant and rich arithmetic and geometric peoperties. Let us first prove the following theorem:

Theorem 2.3.1. $\mathcal{A}_g$ has the structure of a Shimura variety.

Proof. The proof will be sketchy. For more details see [MO] or [Mi]. We have seen in chapter 1 that every polarized Hodge structure on a $\mathbb{Q}$-vector space of dimension $2g$, gives rise to a homomorphism $h : \mathbb{S} \to Gsp_{2g,\mathbb{R}}$. The space of conjugacy classes of h, can be identified with $\mathcal{H}_g$, the Siegel upper half space. The pair $(Gsp_{2g,\mathbb{Q}}, \mathcal{H}_g)$ is a Shimura datum. Indeed, for $G = Gsp_{2g,\mathbb{Q}}$, G^{ad} is $\mathbb{Q}$-simple and $G^{ad}(\mathbb{R})$ is not compact so condition 3 of definition 2.1.6 is satisfied. To see that adJ induces a Cartan involution on G^{ad}, note that $J^2 = -1$ lies in the center of $Sp_{2g}(\mathbb{R})$ and ψ is a J-polarization for $Sp_{2g,\mathbb{R}}$. Then [Mi], proposition 1.20 shows that adJ is a Cartan involution, which is condition 2 of definition 2.1.6 . Finally, the Lie algebra of G is as follows:

$$Lie(G) = \{f \in End(V) | \psi(f(u), v) + \psi(u, f(v)) = 0\}$$

The decomposition of $V(\mathbb{C}) = V^{-1,0} + V^{0,-1}$ gives a corresponding decomposition on $End(V)$ on which the action of f satisfies condition 1 of definition. These observations show that in fact (G, X) is a Shimura datum. We show that $\mathcal{A}_g$ is the associated Shimura variety. To this end, define a compact open subgroup $K_m \subseteq G(\mathbb{A}^f)$ as:

$K_m = \{\gamma \in G(V \otimes \widehat{\mathbb{Z}}, \phi) | \gamma \equiv 1(mod m)\}$

Denoting by $\mathcal{A}_{g,m,\mathbb{Q}}$ as the moduli of abelian varieties equipped with a level $m \geq 3$ structure, we get as isomorphism $\beta : \mathcal{A}_{g,m,\mathbb{Q}} \to Sh_{K_m}(G_{2g,\mathbb{Q}}, \mathcal{H}_g)$ as follows:

If (A, λ, α) is a $\mathbb{C}$-valued point of $\mathcal{A}_{g,m,\mathbb{Q}}$ corresponding to an abelian variety A with peincipal polarization λ and level structure α, the polarization λ gives a polarization $\psi : H \times H \to \mathbb{Z}(1)$ for $H = H_1(A, \mathbb{Z})$. The Hodge structure on H, corresponds to a point $x \in \mathcal{H}_g$. A level structure corresponds to an isomorphism $A[m] \cong H/mH$ commuting with the Weil pairing (i.e. such that $e^{\lambda}(P, Q) = exp(\frac{\psi(y,z)}{m})$ where $y, z \in H/mH$ are representatives of $P, Q \in A[m](\mathbb{C})$). The level structure α then corresponds to an element $\gamma K_m \in G(V \otimes \mathbb{Z}/m\mathbb{Z}, \psi) = G(V \otimes \widehat{\mathbb{Z}}, \psi)/K_m$. The isormophism β send (A, λ, α) to $[x, \gamma K_m] \in Sh_{K_m}(G_{2g,\mathbb{Q}}, \mathcal{H}_g)$.

Note that the above proof shows that not only $\mathcal{A}_g$, but also A_g is a Shimura variety.

We have seen that A_g can be realized as a Shimura subvariety. It therefore makes sense to define the Shimura subvarieties of A_g. Although as already mentioned, we will be mainly working with this Shimura variety and it's subvarieties, we will define the notion of a Shimura subvariety in the general setting.

Definition 2.3.2. A closed subvariety $Z \subseteq Sh_K(G, X)$ is called a *Shimura subvariety* (or special subvariety), if there exist a Shimura datum (M, Y) and a morphism of Shimura data $f : (M, Y) \to (G, X)$ and an element $\gamma \in G(\mathbb{A}^f)$, such that Z is an irreducible component of the image of the map:

$$Sh(M, Y) \xrightarrow{Sh(f)} Sh(G, X) \xrightarrow{T_\gamma} Sh(G, X) \to Sh_K(G, X)$$

Where T_γ is the Hecke transform associated with γ defined in definition 2.2.8.

An alternative definition of a Shimura subvariety is as follows: Let M be a subgroup of G. Define $Y_M \subseteq X$ as:

$$Y_M = \{x : \mathbb{S} \to G_{\mathbb{R}} | x \in X, x(\mathbb{S}) \subseteq M_{\mathbb{R}}\}$$

Definition 2.3.3. A closed subvariety $Z \subseteq Sh_K(G, X)$ is called a *Shimura subvariety* if there exist a subgroup M of G, an irreducible component Y^+ of Y_M and an element $\gamma \in G(\mathbb{A}^f)$, such that Z is the image of $Y^+ \times \{\gamma K\}$ in $Sh_K(G, X)$.

For the equivalence of the above two definitions, see [M2], remark 2.6.

Example 2.3.4. Let $G = Gsp_{2g}(\mathbb{Q})$. As we have seen in 2.3.1, the Shimura variety associated to G is $\mathcal{A}_g$. Let (A, λ, α) be a principally polarized abelian variety with level $m \geq 3$ structure. Set $D = End^0(A)$. Using the definitions above we get a Shimura subvariety of A_g as follows: There is a Hodge structure on $H = H_1(A, \mathbb{Z})$. This Hodge structure corresponds to a point h_0 in $\mathcal{H}_g$. Let $M = Gsp_{2g,\mathbb{Q}} \cap GL_D(H_{\mathbb{Q}})$ and define Y_M as in 2.3.3. Of course the homomorphism $h_0 : \mathbb{S} \to G_{\mathbb{R}}$ factors through $M_{\mathbb{R}}$. Now there is a connected component Y^+ of $\mathcal{H}_g$ containing h_0. As explained earlier, the level structure on A corresponds to a class $\gamma K_m \in G(V \otimes \mathbb{Z}/m\mathbb{Z}, \psi) = G(V \otimes \widehat{\mathbb{Z}}, \psi)/K_m$. Define the subvariety S as the image of $Y^+ \times \{\gamma K_m\}$ in $Sh_{K_m}(Gsp_{2g,\mathbb{Q}}, \psi)$. By definition, this subvariety is a Shimura subvariety. It is in fact the largest irreducible subvariety in A_g which contains A and over which the action of D extends to the whole universal family of abelian varieties. See [MO], Example 3.12.

2.4. Totally geodesic subvarieties

In this section we define the totally geodesic subvarieties of Shimura subvarieties. We will see some examples of them arising from families of abelian covers of $\mathbb{P}^1$ later. We will also see that there is a close relation between totally geodesic and Shimura subvarieties.

Definition 2.4.1. Let $Z \subseteq Sh_K(G,X)$ be an irreducible subvariety. There is a connected component X^+ of X and an element γK such that Z is contained in the image of $X^+ \times \{\gamma K\}$ in $Sh_K(G,X)$. Z is called a *totally geodesic subvariety* if there exists a totally geodesic subvariety $Y \subseteq X^+$ (as defined in [H], 1 §14) such that Z is the image of $Y \times \{\gamma K\}$ in $Sh_K(G,X)$.

To understand the relation between Shimura subvarieties and totally geodesic subvarieties we first need the notion of a CM abelian variety. This is defined as:

Definition 2.4.2. An abelian variety A of dimension d is called a *CM abelian variety* or an abelian variety *of CM type* if there are number fields K_i with $\oplus K_i \subseteq End^0(A)$ such that $[K_i : \mathbb{Q}] = 2d$.

Let $x \in A_g$. The point x corresponds to a Hodge structure of type $(-1,0) \oplus (0,-1)$ and hence to a point of $\mathcal{H}_g$ given by a homomorphism $h : \mathbb{S} \to Gsp_{2g,\mathbb{R}}$ which gives the Hodge structure. Let MT_x be the Mumford-Tate group of this Hodge structure (definition 1.1.8).

Proposition 2.4.3. An abelian variety A corresponding to $x \in A_g$ is a CM abelian variety if and only if x is an irreducible Shimura subvariety of dimension zero.

Proof. It is shown in [Mu] that a CM abelian variety corresponds to a point x as above such that MT_x is a torus. On the other hand, we have already seen (2.1.8) that a torus gives rise naturally to a Shimura datum (and hence to a Shimura variety of dimension zero). Conversely, if x is a Shimura subvariety of dimension zero then $M = MT_x$ where M is a subgroup in definition 2.3.3 and there exists a zero-dimensional irreducible component $Y^+ \subseteq Y_M$ (i.e. a point) such that x is the image of Y^+ in A_g. Since Y^+ is zero dimensional, it follows that the homomorphism $x : \mathbb{S} \to M_{\mathbb{R}}$ is fixed by conjugation. Now let C be the centralizer of $x(\mathbb{S})$ which is connected and reductive, see [Sp], lemma 15.3.2. If T is a maximal torus of C, we have that $x(\mathbb{S}) \subseteq T$. Since any other Torus containing T will also centralize $x(\mathbb{S})$, it follows that T is actually a maximal torus of $G_{\mathbb{R}}$. By a general fact from group theory, there is an element $g \in G(\mathbb{R})$ and a maximal torus S over $\mathbb{Q}$ such that $gTg^{-1} = S_{\mathbb{R}}$. But then $x = g.x : \mathbb{S} \to G_{\mathbb{R}}$ factors through $S_{\mathbb{R}}$ which shows that x is a CM point.

The following theorem due to Moonen explains the relation between Shimura and totally geodesic subvarieties.

Theorem 2.4.4. A totally geodesic subvariety of A_g is a Shimura subvariety if and only if it contains a CM point.

Proof. See [M2], theorem 4.3.

Remark 2.4.5. The above theorem shows that if a totally geodesic subvariety contains a CM point, then it is algebraic and hence a Shimura variety. If $Z \subseteq A_g$ is a Shimura subvariety and if M is the group in definition 2.3.3, then the set of CM points in Y_M are stable under the action of $M(\mathbb{R})$, as M is an abelian subgroup. Since $M(\mathbb{Q})$ is dense in $M(\mathbb{R})$, the Shimura subvariety Z will not just contain one, but a dense subset of CM points.

Chapter 3. Shimura subvarieties in the Torelli locus

3.1. Preliminaries

Let M_g be the coarse moduli scheme of smooth complex curves of genus g and let A_g be the moduli space of principally polarized complex abelian varieties of dimension g. There exists a well-known morphism, the so called *Torelli morphism* $j : M_g \to A_g$ which assigns to a curve C, it's Jacobian $(J(C), \lambda)$ as a principally polarized abelian variety (ppav) with polarization λ. As we will be always working with A_g and abelin varieties with principal polarization, we omit the polarization λ and only write $J(C)$ as the Jacobian of C. Torelli's theorem asserts that this morphism is injective, see [We]. It is shown (see [OS]) that the Torelli morphism is in fact an immersion. Note that instead of A_g and M_g we could have worked with moduli stacks $\mathcal{M}_g$ and $\mathcal{A}_g$ and we again have a Torelli morphism $j : \mathcal{M}_g \to \mathcal{A}_g$. In this case, it is no longer true that j is an immersion as it is ramified over the hyperelliptic locus. However, away from the hyperelliptic locus and restricted to the hyperelliptic locus again the Torelli morphism is an immersion. We call the image of the Torelli map $j(M_g)$, the *open Torelli locus* and denote it by T_g° . By it's definition, it consists of the Jacobians of smooth curves of genus g. The closure of the open Torelli locus inside A_g is called the *closed Torelli locus*, which we denote by T_g. It can be precisely shown what the boundary of $T_g \backslash T_g^\circ$ is. Before showing this we show how the Torelli morphism can be extended to the locus of singular curves. The Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ of $\mathcal{M}_g$ parametrizes stable curves of genus g. The boundary $\Delta = \overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ is a normal crossing divisor with components Δ_i for $0 \leq i \leq \lfloor \frac{g}{2} \rfloor$. Δ is the locus of stable curves with exactly one node (and in fact the locus of curves with δ nodes is of pure codimension δ in $\overline{\mathcal{M}_g}$). The components of Δ parametrize certain curves as follows (see [HM], §2): Δ_0 is the closure of the locus of irreducible curves with a single node. For $i \neq 0$, Δ_i is the closure of the locus of curves that have exactly two connected components of genera i and $g - i$ meeting at an ordinary double point. The complement of the locus Δ_0 corresponds to the locus $\mathcal{M}_g^{ct}$ of curves of *compact type*. That is, the locus of curves whose connected components of Picard scheme are proper which is equivalent (over $\mathbb{C}$) to say that the irreducible components of the curve are non-singular, the graph of components is a tree and the sum of the genera is equal to g. Now, if C is a curve of compact type in the above sense, the identity component $J(C) = Pic^0(C)$ is an abelian variety and hence the Torelli morphism can be extended to a morphism

$$j : \mathcal{M}_g^{ct} \to \mathcal{A}_g$$

Note that similarly we have a Torelli morphism $j : M_g^{ct} \to A_g$ on the level of coarse moduli schemes and by the above description, in fact $T_g = j(M_g^{ct})$. Now, we have the following theorem:

Theorem 3.1.1. The boundary $T_g \setminus T_g^{\circ}$ consists precisely of the locus of decomposable Jacobians. That is, the locus of Jacobians that are isomorphic to a product of lower dimensional abelian varieties as ppav.

Proof. In fact by the above discussion, a point lies in $T_g \setminus T_g^{\circ}$ if and only if the corresponding abelian variety is that of a reducible curve of compact type. i.e. curves C with two connected components meeting in one ordinary double point. In this case, $J(C)$ is isomorphic to the product of Jacobians of the connected components. Conversely, if the Jacobian of a curve C decomposes as a product of ppav, then it's theta divisor is reducible and hence it can not be the Jacobian of a non-singular irreducible curve.

The main purpose of these notes is to investigate the existence of Shimura subvarieties that are contained in the Torelli locus in the sense that will be made precise below:

Definition 3.1.2. Let Z be a Shimura subvariety of A_g. We say that Z *lies* in the Torelli locus, if $Z \subset T_g$ and $Z \cap T_g^{\circ} \neq \emptyset$.

So in particular, the Shimura subvariety should not lie entirely in the boundary of the Torelli locus. This condition is crucial, because one can easily construct a lot of Shimura subvarieties lying entirely in the boundary for every $g \geq 2$ (see [MO], remark 4.3).

The quest of Shimura subvarieties (or *special* subvarieties or subvarieties of Hodge type) contained generically in the Torelli locus T_g and not fully contained in the boundary, is a longstanding problem which can be traced back at least to Shimura (see [Sh]). Although not realized then, more than two decades later, Coleman formulated his famous conjecture suggesting that for $g \geq 4$, there are only finitely many curves C of genus g over the complex numbers whose Jacobian $J(C)$ is an abelian variety of CM type. See [C]. Shortly after that, de Jong and Noot disproved that conjecture by finding examples of families of curves which give rise to Shimura subvarieties in A_g lying generically in the (open) Torelli locus and intersecting the (closed) Torelli locus non-trivially. The families that they found -which were also found in the aforementioned article of Shimura- were all families of cyclic coverings of $\mathbb{P}^1$. Their examples were of fiber genus 4 and 6 and they did explain the relation between their examples and the Coleman conjecture. Later, examples of higher genera ($g = 5$ and $g = 7$), again for families of cyclic coverings of $\mathbb{P}^1$, were found by Rohde [R]. Note that there are also several other results in this direction. Viehweg and Zuo for example studied in [VZ1] the occurrence of Shimura curves in the moduli stack of principally polarized abelian varieties relating it to Areklov equalities. See also [MVZ1] and [MVZ2]. Moonen completed the list of Shimura varieties arising from families of cyclic coverings of $\mathbb{P}^1$ in [M1]. Note that in [M1] Moonen did not find any new example of Shimura families and all of the examples there were already found by Rohde in [R]. But Moonen Showed that there are no further examples in the locus of cyclic covers of $\mathbb{P}^1$. The fiber genus of these examples are also bounded by 8. This suggests a correction of the Coleman conjecture in the following way:

The (corrected) Coleman conjecture. For $g \geq 8$, there are only finitely many smooth projective curves C over $\mathbb{C}$ of genus g such that $J(C)$ is an abelian variety of CM type.

Note that according to remark 2.4.5, every Shimura subvariety of A_g contains a dense subset of CM points. So we see that the Coleman conjecture is related to the following conjecture and in fact disproving this conjecture will disprove the Coleman conjecture:

Conjecture (C). For $g \geq 8$, there is no positive-dimensional Shimrua subvariety contained in T_g such that $Z \cap T_g^{\circ}$ is nonempty.

Another famous conjecture which links the above conjectures is the *André-Oort* conjecture:

André-Oort conjecture. Let $Z \subseteq A_g$ be a closed irreducible algebraic subvariety. If the set of CM points on Z is dense (in the Zariski topology), then Z is a Shimura subvariety.

We remark that if one assumes the André-Oort conjecture to be true, then the Coleman conjecture is even equivalent to conjecture (C). This can be shown easily: if there are infinitely many CM points in T_g° for some $g \geq 8$, consider the closure $\overline{CM(T_g^\circ)}$ in T_g of the infinite set of CM points on T_g°. Since $CM(T_g^\circ)$ is infinite by assumption, it's closure contains a subvariety of positive dimension, which by Anré-Oort should be a Shimura subvariety, contradicting conjecture (C).

3.2. Families of cyclic coverings of $\mathbb{P}^1$ and some examples

In this section we will explain shortly the results of [M1] about Shimura subvarieties in A_g arising from families of cyclic covers of $\mathbb{P}^1$ and construct some examples of them which contain Shimura subvarieties of positive dimension . In the next section, we describe more generally the construction of abelian coverings of $\mathbb{P}^1$ and their families in detail, see section 3.3. For the moment, we remark that a cyclic cover of $\mathbb{P}^1$ of degree N is given by an (affine) equation :

$w^N = \prod_{j=1}^{s}(z - z_j)^{r_j}$

The r_j give the monodromy data of the cover arounf the branch point z_j. By varying the branch points in $(\mathbb{P}^1)^s \setminus \widetilde{\Delta} = \{(z_1, ..., z_s) \in (\mathbb{P}^1)^s | z_i \neq z_j, i \neq j\}$, we obtain a smooth family of cyclic covers of $\mathbb{P}^1$. We will refer to the pair $(N, (r_1, ..., r_s))$ as the *ramification data* of the family or to the family as *being given* by this ramification data. Such a family gives rise to a subvariety $Z = Z(N, s, r)$ of A_g such that $dimZ = s - 3$. The cases where Z is a Shimura subvariety of A_g are completely classified in [M1]. His classification however, does not rule out that the variety Z contains a Shimura subvariety. In fact, in [O], Oort asks the following question:

Question (Question 7.8 in [O]). Let the situation be as above such that $Z = Z(N, s, r)$ is *not* a special (Shimura) subvariety. Can we find a situation where Z *contains* a positive dimensional Shimura subvariety?

In this section we will give some examples of non-Shimura varieties Z containing positive dimensional Shimura subvarieties, so we are going to give some answers to the above question.

Example 3.2.1. Consider the family $f : C \to T$ of cyclic covers of $\mathbb{P}^1$ given by the ramification data $(12, (4, 6, 7, 7))$. This family is a Shimura family as is shown in the paper of Moonen [M1]. Take a $t \in T$. The fiber C_t is a cyclic Galois cover of $\mathbb{P}^1$. The Galois group of $C_t \to \mathbb{P}^1$ is $\mathbb{Z}/12\mathbb{Z}$ which contains a normal subgroup $H \cong \mathbb{Z}/6\mathbb{Z}$. Now this cover factors as

$$C_t \to X_t \to \mathbb{P}^1$$

The quotient cover $C_t \to X_t$ is a Galois cover with Galois group $H \cong \mathbb{Z}/6\mathbb{Z}$. We have $X_t \cong \mathbb{P}^1$ as the quotient is corresponding to the equation $y^2 = (x - x_1)(x - x_2)$. The cover $X_t \to \mathbb{P}^1$ has Galois group $\mathbb{Z}/2\mathbb{Z}$ and so has degree 2 and is branched above 2 points.

The cover $C_t \to X_t(\cong \mathbb{P}^1)$ is ramified at 6 points with ramification indices $1, 1, 2, 2, 3, 3,$. This shows that our original family $f : C \to T$ is a 1-dimensional subfamily of the 3-dimensional family of cyclic covers given by the ramification data $(6, (1, 1, 2, 2, 3, 3))$. It is quite easy to see from the Moonen's list that this family is not a Shimura family. Indeed, for this family $dimS(\mu_m) = 5 \neq 3$, so by the results of Moonen, this family is not Shimura.

Note that the subfamily here is given over the closed subset (of $(\mathbb{P}^1)^6 \setminus \widetilde{\Delta}$) given by a set Γ of orbits of the $\mathbb{Z}/2\mathbb{Z}$-action on $\mathbb{P}^1$.

Therefore we have found a 3-dimensional non-Shimura family of cyclic covers which contains a 1-dimensional Shimura subvariety.

Similarly one finds out that:

• The family given by the ramification data $(2, (1, 1, 1, 1, 1, 1, 1, 1))$ (the universal family of hyperelliptic curves of genus $g = 3$) which is *not* a Shimura family (cf. [M1]) has a subfamily isomorphic to the family $(4, (1, 1, 2, 2, 2))$. So in this example Z contains a shimura surface.

• The family $(4, (1, 1, 1, 1, 2, 2))$ (a family of cyclic covers of fiber genus $g = 5$) has a subfamily isomorphic to the family $(8, (5, 5, 4, 2))$. In this case $dim Z = 3$ and Z contains a Shimura curve.

3.3. Families of abelian coverings of $\mathbb{P}^1$

As explained earlier, Moonen gave a full classification of Shimura subvarieties arising from families of cyclic coverings of $\mathbb{P}^1$ in [M1]. In [MO] Oort and Moonen asked whether one can obtain further Shimura subvarieties in the Torelli locus by taking families of abelian coverings of $\mathbb{P}^1$ with a non-cyclic Galois group. They also gave some examples of families of abelian covers of $\mathbb{P}^1$ which give rise to Shimura subvarieties in T_g. In this article we try to generalize their methods and classify Shimura subvarieties arising from families of abelian covers of the projective line. We fix integers $N \geq 2$ and $s \geq 4$ and an s-tuple $(z_1, ..., z_s)$ and consider a family of abelian covers $Y_t \to \mathbb{P}^1$ with an abelian Galois group G which is isomorphic to the column span of the matrix A and hence is a subgroup of the group $(\mathbb{Z}/N\mathbb{Z})^m$. By varying the branch points we obtain a subvariety whose closure Z (inside A_g) lies in T_g. Of course Z will be of dimension $s-3$ where s is the number of branch points of the covering and it lies in the Torelli locus cutting the open Torelli locus non-trivially. We then try to classify the cases where Z is a Shimura subvariety. Our method here is a generalization of that of Moonen-Oort [MO] and Moonen [M1]. Let us first recall the basic notions and constructions of abelian coverings of $\mathbb{P}^1$ and their families.

Construction of abelian covers of $\mathbb{P}^1$ and their families

An abelian cover is determined by a collection of equations in the following way: Consider an $m \times s$ matrix $A = (r_{ij})$ whose entries r_{ij} are in $\mathbb{Z}/N\mathbb{Z}$ for some $N \geq 2$. Set

$$w_i^N = \prod_{j=1}^{s} (z - z_j)^{\widetilde{r}_{ij}} \text{ for } i = 1, \cdots, m$$

Where $\widetilde{r}_{ij}$ is a lift of r_{ij} to $\mathbb{Z} \cap [0, N)$. Denote by $\widetilde{A} = (\widetilde{r}_{ij})$ and call it the *lifted matrix* of A. We impose the condition that the sum of the columns are zero. This implies that the cover is not ramified over infinity. The matrix A will be called the matrix of the covering. Note that our notations here are mostly that of [W]. Also we consider the row and column spans of the matrix A as modules over the ring $\mathbb{Z}/N\mathbb{Z}$ and so all of the operations with rows and columns will be carried out in the ring $\mathbb{Z}/N\mathbb{Z}$, i.e. it will be considered modulo

N. By the way, sometimes for preventing confusions we use the symbol $[]_N$ to show in which ring we are working and for example write $[r_{ij}]_N$ instead of r_{ij}. For each $1 \leq i \leq m$, the function w_i is an element of $\overline{\mathbb{C}(z)}$. The abelian cover, is then the Riemann surface with function field $\mathbb{C}(z)[w_1, ..., w_m]$. It can easily be seen that any Galois cover of $\mathbb{P}^1$ with abelian Galois (or deck) group is obtained in this way from a certain matrix A. The local monodromy at the branch point z_j is given by the column vector $(r_{1j}, ..., r_{mj})^t$ and thus the order of ramification at z_j is $\frac{N}{gcd(N,\widetilde{r}_{1j},..,\widetilde{r}_{mj})}$. Using this and the Riemann-Hurwitz formula, the genus g of the cover is then given by:

$$g = 1 + d(\tfrac{s-2}{2} - \tfrac{1}{2N} \textstyle\sum_{j=1}^{s} gcd(N, \widetilde{r}_{1j}, ..., \widetilde{r}_{mj}))$$

Where d is the degree of the covering. Note that the degree d of the covering (or equivalently the order of the Galois group) can be realized as the size of the row span (equivalently column span) of the matrix A, see [W].

Next we turn to define families of curves which are abelian coverings of the projective line.

Let $U \subset (\mathbb{A}^1)^s$ be the complement of the big diagonals. i.e. $U = \mathcal{P}_s = \{(z_1, ..., z_s) \in (\mathbb{A}^1)^s \mid z_i \neq z_j \forall i \neq j\}$. Over this open affine set, we define a family of abelian covers of $\mathbb{P}^1$ to have the equation:

$$w_i^N = \textstyle\prod_{j=1}^{s} (z - z_j)^{\widetilde{r}_{ij}} \text{ for } i = 1, \cdots, m.$$

Where the tuple $(z_1, ..., z_s)$ varies in U defined above and $\widetilde{r}_{ij}$ is a lift of r_{ij} to $\mathbb{Z} \cap [0, N)$ as before. In this way each w_i defines a cyclic cover of $\mathbb{P}^1$. If X is the total space of the above family of abelian covers and X_t is a fiber of this family, there exists an open subset $T \subset U$ and a smooth proper curve $f : C \to T$ with an action of the Galois group G and a G-equivariant morphism $g : C \to X_T$ such that for every $t \in T$, the morphism on fibers $g_t : C_t \to X_t$ is a normalization.

If $f : C \to T$ is a family of abelian covers constructed as above, we write $J \to T$ for the relative Jacobian of C over T. This family gives a natural map $\phi : T \to A_g$. Let $Z = Z(N, s, r)$ be the closure $\overline{\phi(T)}$ in A_g. Such a family therefore gives rise to a closed subvariety $Z = Z(N, s, r)$ in the moduli space A_g and we have $dim Z = s - 3$. This is because any three points on $\mathbb{P}^1$ can be moved to the points $0, 1, \infty$. We call the subvariety Z, the *moduli variety* associated to the family $f : C \to T$.

If $f_1 : C_1 \to \mathbb{P}^1$ and $f_2 : C_2 \to \mathbb{P}^1$ are two abelian coverings of $\mathbb{P}^1$, we say that they are *equivalent* if there exists an isomorphism $g : C_1 \to C_2$ respecting the coverings i.e. $f_1 = f_2 \circ g$. Two abelian covers (with the same N) are equivalent if and only if A and A' have the same row span. Based on the notion of equivalence we can state the following description of abelian Galois covers of $\mathbb{P}^1$:

Remark 3.3.1. There is a correspondence between the equivalence classes of non-ramified Galois coverings $f : R \to \mathbb{P}^1 \setminus S$ with $S = \{z_1, ..., z_s\}$ and normal subgroups of $\pi_1(\mathbb{P}^1 \setminus S)$ with quotient isomorphic to G, See [V], Theorem 5.14. In particular a Galois cover $f : C \to \mathbb{P}^1$ branched above the points $S = \{z_1, ..., z_s\}$ corresponds to a surjection $\phi : \pi_1(\mathbb{P}^1 \setminus S) \twoheadrightarrow G$. The fundamental group $\pi_1(\mathbb{P}^1 \setminus S)$ is generated by the loops γ_j around z_j with only the condition that $\gamma_1 ... \gamma_s = 1$ and the local monodromy around z_j is given by $\phi(\gamma_j)$.

3.4. The local system associated to an abelian cover

In this section we are going to represent an alternative construction of abelian coverings using line bundles and local systems. This construction resembles that of [EV] in the case of cyclic coverings of algebraic varieties. Let G be a finite abelian group. We denote by μ_G the group of the characters of G. i.e. $\mu_G = Hom(G, \mathbb{C}^*)$. Consider a Galois covering $\pi : X \to \mathbb{P}^1$ with Galois group G. The group G acts naturally on the sheaves $\pi_*(\mathcal{O})$ and $\pi_*(\mathbb{C})$ vis it's characters i.e. $f(gx) = \chi(g)f(x)$ for $\chi \in \mu_G$. Under this action the sheaf decomposes as direct sum of the eigensheaves corresponding to the characters of G. Let $L_\chi^{-1} = \pi_*(\mathcal{O}_X)_\chi$ and $\mathbb{C}_\chi = \pi_*(\mathbb{C})_\chi$ denote the eigensheaves corresponding to the character χ. L_χ is a line bundle and outside of the branch locus of π, $\mathbb{C}_\chi$ is a local system of rank 1. We will look more closely on these sheaves and describe them in detail. Consider an abelian cover given by the equation above which is branched along s points. For $j = 1, 2, .., s$, let G_j be the corresponding inertia subgroup of z_j. It is the subgroup of G consisting of elements that pointwise fix the elements of the inverse image $\pi^{-1}(z_j)$. It is a cyclic subgroup of G and it's order is equal to the ramification order of z_j which we have seen is equal to $N/gcd(\widetilde{r}_{1j},..,\widetilde{r}_{mj})$. Let g_j be the generator of G_j. If we identify G with μ_G, we can consider each element of G as a root of unity and therefore we get that g_j can be identified with $\alpha_j = e^{2\pi i \mu_j}$ where $\mu_j = \frac{gcd(\widetilde{r}_{1j},..,\widetilde{r}_{mj})}{N}$. We call the α_j the *local monodromy data* around z_j. Now we can describe the eigensheaves $\mathbb{C}_\chi$ and L_χ. Consider a character χ of G. If $\chi(g_j) = 1$, then $\mathbb{C}_\chi$ is a trivial local system at s. Otherwise the fiber of $\mathbb{C}_\chi$ is zero at s and the monodromy around s is given by the root of unity $\chi(g_j)$. Note that the fundamental group $\pi_1(\mathbb{P}^1 \setminus S)$ of the punctured Riemann sphere is generated by loops γ_k around each z_k satisfying the relation $\gamma_1 ... \gamma_s = 1$. This gives us the following description of the local system $\mathbb{C}_\chi$:

Theorem 3.4.1. The monodromy representation of the local system $\pi_*\mathbb{C}|_{\mathbb{P}^1\setminus S}$ is given by :

$$\rho : \pi_1(\mathbb{P}^1 \setminus S) \to GL_N(\mathbb{C})$$
$$\gamma_k \mapsto diag(e^{2\pi i j \mu_k} | j = 0, 1, ..., N-1).$$

Following [P1], we call the bundles L_χ and $z_j, j = 1,..,s$ considered as divisors in $\mathbb{P}^1$ the *building data* of the cover. The reason is that these data determine the cover completely. Let us now give a more explicit description of the line bundles L_χ for an abelian cover given by the equations above.

Lemma 3.4.2. Let $\widetilde{A} = (\widetilde{r}_{ij})$ be the lifted matrix of A. Let $a = (a_1, ..., a_m) \in G \subseteq \mathbb{Z}_N^m$ and consider $a.\widetilde{A} = (\alpha_1, .., \alpha_s)$. The sheaf $\pi_*(\omega)$ also decomposes with respect to the Galois group action. For the line bundles L_χ corresponding to the character χ associated to the element $a \in G$ and $\pi_*(\omega)_\chi$ we have:

$L_\chi = \mathcal{O}_{\mathbb{P}^1}(\sum_1^s < \frac{\alpha_i}{N} >)$, Where $< x >$ denotes the fractional part of the real number x.

and $\pi_*(\omega)_\chi = \omega_{\mathbb{P}^1} \otimes L_{\chi^{-1}} = \mathcal{O}_{\mathbb{P}^1}(-2 + \sum_1^s < -\frac{\alpha_i}{N} >)$

Proof. Note that since the sum of the columns of the matrix A is zero, the above sum is an integer. One can easily see that each section of the line bundle $\mathcal{O}_{\mathbb{P}^1}(\sum_1^s < \frac{\alpha_i}{N} >)$ is a function on which the Galois group acts as χ and conversely any such section must be a function of the above form. The rest of the lemma is [P2], Propostion 1.2.

Remark 3.4.3. Note that in the case of a cyclic cover of $\mathbb{P}^1$, the bundles L_χ and local systems $\mathbb{C}_\chi$ coincide with the bundles $\mathcal{L}^{(j)}$ and the local systems $\mathbb{L}_j$ of [R], as one expects naturally.

3.5. Shimura subvarieties of PEL type

One of the main kind of Shimura subvarieties that we encounter in our investigation of families of abelian coverings of $\mathbb{P}^1$, are Shmura subvarieties of *PEL type*. The name is an abbreviation of *polarization*, *endomosphims* and *level strcuture*. This is justified by the fact that they are constructed with the aid of certain endomorphisms of abelian varieties and in fact they are moduli spaces of abelian varieties together with level structure and these endomorphisms.

Let $(B, *)$ be a semisimple $\mathbb{Q}$-algebra with an involution $*$ and let (V, ψ) be a *symplectic* $(B, *)$-module. That is, a B-module V with an alternating $\mathbb{Q}$-bilinear form $\psi : V \times V \to \mathbb{Q}$ such that:

$\psi(bu, v) = \psi(u, b^* v)$ for all $b \in B$ and $u, v \in V$.

Let $G = Gsp(V, \psi)$ and set $M = G \cap GL_D(V)$. So M is the group of B-linear symplectic similitudes of V. Consider the set Y of $M(\mathbb{R})$-conjugacy classes of homomorphisms $h : \mathbb{S} \to M_{\mathbb{R}}$ defining a Hodge structure of type $(-1, 0) \oplus (0, -1)$ on V. This gives a Shimura datum (M, Y). The inclusion $M \hookrightarrow G$, gives rise to a morphism $(M, Y) \to (G, \mathcal{H}_g)$ of Shimura data and hence a morphism $Sh(M, Y) \to Sh_{K_m}(G, \mathcal{H}_g) = \mathcal{A}_{g,m}$ of Shimura varieties, defining a Shimura subvariety $S \subseteq A_g$.

We will then see that there is a Shimura subvariety of PEL type that contains our moduli subvariety Z of a family of abelian covers of $\mathbb{P}^1$.

3.6. Two Shimura subvarieties containing Z

In this section we describe two naturally constructed Shimura subvarieties of A_g associated to a family $f : C \to T$ of abelian covers that contain the moduli variety Z.

Let μ_G be the group of characters of the abelian group G. The Jacobians in the family $J \to T$ admit naturally an action of the group ring $\mathbb{Z}[\mu_G]$. This action defines a Shimura subvariety of PEL type $S(\mu_G)$ in A_g that contains Z. More precisely, by fixing a base point $t \in T$, there is a Hodge structure on $V = H_1(C_t, \mathbb{Z})$ which correspondes to a point $y \in \mathcal{H}_g$ in the Siegel space. On the vectore space $V_{\mathbb{Q}}$ there is a natural action of $F = \mathbb{Q}[\mu_G]$ and so $V_{\mathbb{Q}}$ has also the structure of an F-module. F is equipped with a natural involution $*$. The polarization ϕ on $V_{\mathbb{Q}}$ satisfies.

$\phi(bu, v) = \phi(u, b^*v)$ for all $b \in F$ and $u, v \in V$.

Define the subgroup M as in 2.3.3:

$M = Gsp(V_{\mathbb{Q}}, \phi) \cap GL_F(V_{\mathbb{Q}})$

If $h_0 : \mathbb{S} \to Gsp_{2g,\mathbb{R}}$ is the Hodge structure on $V = H_1(C_t, \mathbb{Z})$ corresponding to the point $y \in \mathcal{H}_g$, then by the above F-action, this homomorphism factors through the subvariety $M_{\mathbb{R}}$. Define

$Y_M = \{x : \mathbb{S} \to Gsp_{\mathbb{R}} | x \text{ factores through } M_{\mathbb{R}}\}$

Where each homomorphism $x : \mathbb{S} \to Gsp_{\mathbb{R}}$ is in the conjugacy class of the homomorphism h_0. By the above, the point y lies in Y_M and there is a connected component $Y^+ \subseteq Y_M$ containing y and $S(\mu_G)$ is equal to the image of the quotient map $\mathcal{H}_g \to Gsp(V, \phi)\backslash\mathcal{H}_g \cong A_g$.

Since $Z \subseteq S(\mu_G)$, we have that $s-3 \leq dimS(\mu_G)$. Therefore if $dimS(\mu_G) = s-3$, it follows that $Z = S(\mu_G)$ and hence Z will be a Shimura subvariety of A_g. We can find the dimension of the variety $S(\mu_G)$ by finding the tangent space to it at an arbitrary point. To do this we have to consider the eigenspaces of the action of the group G on cohomology. The group G acts naturally on the cohomology $H^1(C_t, \mathbb{C})$. There is also a natural action on

$H^{1,0} = H^0(C_t, \Omega_{C_t})$. By the action of G, for every $n \in G$, there is an eigenspace $H^0(C_t, \Omega_{C_t})_{(n)}$. Put $d_n = dim_{\mathbb{C}} H^0(C_t, \Omega_{C_t})_{(n)}$. We have:

Lemma 3.6.1. $dimS(\mu_G) = \sum_{2n \neq 0} d_n d_{-n} + \frac{1}{2} \sum_{2n=0} d_n(d_n + 1)$.

Note that $2.0 = 0$ in G and $d_0 = 0$, so in fact the second sum in the right hand side of the above equality is always meaningful and if $|G|$ is an odd number it will be zero.

Proof. We calculate $dimT_y(Y_M)$ at the point $y \in \mathcal{H}_g$. The dimension of tangent space of $S(\mu_G)$ at the point y will be equal to this number. to compute $dimT_y(\mathcal{H}_g)$, we note that:

$$T_y(\mathcal{H}_g) = Hom^{sym}(F^{1,0}, V_{\mathbb{C}}/F^{1,0}) :=$$

$$\{\beta : F^{1,0} \to V_{\mathbb{C}}/F^{1,0} \mid \overline{\phi}(v, \beta(v')) = \overline{\phi}(v', \beta(v)) \forall v, v' \in F^{1,0}\}$$

i.e. the elements of $T_y(\mathcal{H}_g)$ are given by the symmetric homomorphisms with respect to β from $F^{1,0}$ to $V_{\mathbb{C}}/F^{1,0}$, which means that each β is it's own dual via the isomorphisms induced by $\overline{\phi}$. The subspace $T_y(Y_M) \subset T_y(\mathcal{H}_g)$ is therefore given by the elements $\beta \in Hom^{sym}(F^{1,0}, V_{\mathbb{C}}/F^{1,0})$, that respect the F-action on V, that is, are $F_{\mathbb{C}}$-linear. Any such β can be written as the sum $\sum \beta_n$, where $\beta_n : F^{1,0}_{\mathbb{C},n} \to F^{0,1}_{\mathbb{C},n}$ is the induced action on the eigenspaces. These β_n should satisfy the relation

$$\overline{\phi}_n(v, \beta_{-n}(v')) = \overline{\phi}_{-n}(v', \beta_n(v)).$$

Note that the map $\overline{\phi}_n$ induced by the polarization ϕ, gives a duality between $F^{1,0}_{\mathbb{C},n}$ and $F^{0,1}_{\mathbb{C},(-n)}$. So we have a duality between β_n and β_{-n} if $n \neq -n$ in G. If $n = -n$ in G, i.e. if $2n = 0$ in G, this gives a self duality for β_k. Therefore $dimT_y(Y_M)$ is equal to:

$$\sum_{2n \neq 0} d_n d_{-n} + \frac{1}{2} \sum_{2n=0} d_n(d_n + 1) \quad \square$$

Construction 3.6.2. The construction of the second Shimura subvariety that contains Z, is in fact Mumford's construction of "variety of Hodge type". Namely, let M be the generic Mumford-Tate group of the family $f : C \to T$. For the definition and construction of generic Mumford-Tate group, look at [M2] or [R]. Note that M is a reductive $\mathbb{Q}$-algebraic group and let S_f be the natural Shimura variety associated to M. S_f is in fact the *smallest* Shimura subvariety that contains Z and it's dimension depends on the real adjoint group $M_{\mathbb{R}}^{ad}$. Namely, if $M_{\mathbb{R}}^{ad} = Q_1 \times ... \times Q_r$ is the decomposition of $M_{\mathbb{R}}^{ad}$ to $\mathbb{R}$-simple groups, then $dimS_f = \sum \delta(Q_i)$. Where $\delta(Q_i)$ is the dimension of the real group Q_i which can be read from from table V of [H]. We just remark that for $Q = PSU(p,q)$, $\delta(Q) = pq$ and for $Q = Psp_{2p}$, $\delta(Q) = \frac{p(p+1)}{2}$. In particular, Z is a Shimura subvariety if and only if $\sum \delta(Q_i) = s - 3$, i.e. if and only if $dimZ = dimS_f = s - 3$.

Computation of d_n. We have seen that the dimension of the Shimura variety $S(\mu_G)$ can be expressed in terms of the dimension of the eigensapaces of Galois action on the cohomology of the fibers. We will now try to compute these dimensions. Let $n = (a_1, .., a_m) \in G$ be an element. Since G is a finite abelian group, the groups G and μ_G are isomorphic and hence their elements correspond. Let $\chi \in \mu_G$ be the character corresponding to n and $\widetilde{A}$ be the lifted matrix of A, the matrix of the abelian cover, and the α_i as in lemma 3.4.2 We have the following lemma:

Proposition 3.6.3. For an abelian cover C, we have:

$$d_n = h^{1,0}_{\chi}(C) = -1 + \sum_1^s < -\frac{\alpha_j}{N} >$$

Proof. By Proposition 1.3, we have that:

$$\pi_*(\omega)_\chi = \omega_{\mathbb{P}^1} \otimes L_{\chi^{-1}} = \mathcal{O}_{\mathbb{P}^1}(-2 + \sum_1^s < \frac{\alpha_j}{N} >)$$

It follows that:

$$h^{1,0}_{\chi}(C) = h^0(\pi_*(\omega)_\chi) = -1 + \sum_1^s < \frac{\alpha_j}{N} >. \ \square$$

Note that it naturally follows that

$$d_{-n} = h^{1,0}_{\chi^{-1}}(C) = h^{0,1}_{\chi}(C) = -1 + \sum_1^s < -\frac{\alpha_j}{N} >.$$

Remark 3.6.4. There are other methods to compute the dimension of the eigenspaces d_n. Note that the abelian Galois group G of the covering is a (possibly proper) subgroup of $\mathbb{Z}_N^m$ and therefore we can show an element of G as an m-tuple $n = (a_1, ..., a_m)$. The space of differential forms with respect to the character n, is generated over $\mathbb{C}(z)$ by the form $\prod(z - z_j)^{-t_j(-n)}dz$, where $t_j(n) =< \frac{\sum_{i=1}^{m} a_i \widetilde{r}_{ij}}{N} >$ and $< . >$ denotes the fractional part of a real number. It is then straightforward to check that a meromorphic form $p(z)\prod(z - z_j)^{-t_j(-n)}dz$ is holomorphic if and only if $p(z)$ is a polynomial of degree at most $t(n) = \sum t_j(-n) - 1$. So that the dimension of H_n^1 is equal to $t(n)$(see [W], Lemma 2.6). Alternatively, one can use the Chevalley-Weil fromula to compute the the dimension of the eigenspaces. See [CW].

3.7. Examples of Shimura varieties arising from abelian covers

In [M1], Moonen completed the list of Shimura subvarieties generated by families of cyclic covers of $\mathbb{P}^1$ and proved that in the locus of cyclic covers of $\mathbb{P}^1$, there is no more Shimura varieties. The fiber genus of the families that he constructs is bounded by 8, confirming the bound given by the corrected version of Coleman conjecture, see section 3.1 . In [MO], Oort and Moonen give a table of 7 examples of abelian non-cylic Galois covers of $\mathbb{P}^1$, that generate Shimura subvarieties in A_g. All of these examples satisfy the equality $dimS(\mu_G) = s - 3$. Instead of computing the dimension of $S(\mu_G)$ with the formula above, their method to obtain these examples is based on analyzing the decomposition of Jacobians up to the isogeny under the action of group ring $\mathbb{Q}[\mu_G]$. Our argument here is more systematic and has the advantage that can be checked numerically on a computer. Checking whether this equality holds is something that can be checked by a computer and by using a computer program we have checked the examples which satisfy this equality. However, our computer search for did not provide a further example satisfying $dimS(\mu_G) = s - 3$. We are able, however to prove that for $s = 4$, the table contains all examples with $dimS(\mu_G) = s - 3 = 1$ (see theorem 3.7.1 below). We moreover study the families that *do not* appear in the table above i.e. those that do not satisfy the equality $dimS(\mu_G) = s - 3$. In this case, $Z \neq S(\mu_G)$ but it does not imply that Z is not a Shimura subvariety: it could still be a smaller Shimura subvariety (inside $S(\mu_G)$) or in other words, there might be Hodge classes, that are *not* given by the action of $\mathbb{Z}[\mu_G]$. We are able to show that some large classes of families, including all families with $s = 4$, do not give rise to Shimura subvarieties in A_g provided that the

Table 1: Monodromy data of families of abelian coverings that generate Shimura subvarieties.

genus	Galois group	N	monodromy data
1	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2	$\{(1,0)(1,0)(0,1)(0,1)\}$
2	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2	$\{(1,0)(1,0)(1,0)(1,1)(0,1)\}$
3	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	4	{(2,0)(2,1)(0,1)(0,2)}
3	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	4	{(2,0)(2,2)(0,1)(0,1)}
3	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2	{(1,0)(1,0)(1,1)(1,1)(0,1)(0,1)}
4	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	6	{(3,0)(3,1)(0,2)(0,3)}
4	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	3	{(1,0)(1,0)(1,2)(0,1)}

family satisfies an irreducibility condition.

Theorem 3.7.1. The families in table 1, give rise to Shimura subavarieties in A_g. For s=4 this table contains all examples for which $dimS(\mu_G) = s - 3(= 1)$ and contains all such examples for $4 \leq s \leq 7$, $2 \leq m \leq 5$ and $N \leq 20$.

Proof. One can compute the dimensions d_n of eigenspaces with the aid of the formula in proposition 3.6.3. It is straightforward to check that in all of these cases $dimS(\mu_G) = dimZ = s - 3$ and therefore $Z = S(\mu_G)$ is a Shimura subvariety of A_g. For $s = 4$, Note that if $dimS(\mu_G) = 1$, then the family satisfies the equality of lemma 1.3 and in fact the is a unique $a \in G$, such that $d_a = d_{-a} = 1$ and for all other $n \in G$, $d_n d_{-n} = 0$. We may therefore assume that the first row of the matrix A satisfies this equality. By results of [R] we know that there are only finitely many of these with $N \leq 12$ and by the aforementioned computer program we may check that the above examples are the only ones which satisfy $dimS(\mu_G) = 1$ and $N \leq 12$. This means that table 1 contains all examples with $dimS(\mu_G) = 1$. By using the same computer program we see that this table contains all examples with $dimS(\mu_G) = s - 3$ and with conditions as above. □

3.8. The Dwork-Ogus obstruction

As we remarked earlier, we are going to exclude further examples of Shimura subvarieties arising from families of abelian covers. To do this we will need an obstruction introduced by Dwrok and Ogus in [DO]. Although we will encounter cases that we can not use this obstruction, This obstruction remains crucial tool in proving that a certain variety is not a Shimura subvariety. The construction of the obstruction is as follows:

Let the $f : C \to T$ be a family of smooth projective curves with an irreducible base scheme T. We denote the sheaf of relative differentials with $\omega_{C/T}$ and the Hodge bundle $\mathbb{E} = \mathbb{E}(C/T) = f_*\omega_{C/T}$. Consider the Kodaira-Spencer map $\kappa : Sym^2(\mathbb{E}) \to \Omega^1_T$ (usually the dual of this map is defined to be the Kodaira-spencer map, but as we will mainly work with this map, rather than the original Kodaira-Spencer, we name it as such). The multiplication map $mult : Sym^2(\mathbb{E}) \to f_*(\omega^{\otimes 2}_{C/T})$ induces the following sheaves:

$$\mathcal{K} = Ker(mult) = Ker(Sym^2(\mathbb{E}) \to f_*(\omega^{\otimes 2}_{C/T}))$$

$$\mathcal{L} = Coker(mult^\vee) = Coker((f_*\omega^{\otimes 2}_{C/T})^\vee \to Sym^2(\mathbb{E})^\vee),$$

If the fibers are not hyperelliptic, by a famous result of Max Noether (see [OS]), $mult$ is surjective and $\mathcal{K}$ is dual to $\mathcal{L}$.

The Dwork-Ogus obstruction can be constructed only for families of curves which satisfy a further condition that we define explicitly below:

Definition 3.8.1. An abelian variety A of dimension g over an algebraically closed field k of characteristic $p > 0$ is said to be *ordinary* if it satisfies one of the following equivalent conditions:

i) The p-rank of A is equal to g, i.e. $A_p(k) \cong (\mathbb{Z}/p\mathbb{Z})^g$.

ii) The p-linear endomorphism on $H^1(A, \mathcal{O}_A)$ induced by the absolute Frobenius endomorphism F_A of A is isomorphism.

iii) The crystalline cohomology $H^1_{cris}(A/W)$ splits into a direct sum decomposition $H^1_{cris}(A/W) = U \oplus T$ invariant under the map F_A^* and such that $F_A^* \mid_U : U \to U$ is isomorphism and $F_A^* \mid_T : T \to T$ is p times an isomorphism.

A smooth projective curve C over k is said to be *ordinary* if it's Jacobian $J(C)$ is an ordinary abelian variety.

In the definition above, $W = W(k)$ is the *Witt ring* of the field k and $H^1_{cris}(A/W)$ can be realized as a lift of the de Rham cohomology $H^1_{dR}(A/k)$ over W. This means that $H^1_{cris}(A/W) \otimes k \cong H^1_{dR}(A/k)$. The decomposition in condition iii) above is a lift of the Hodge decomposition $H^1_{dR}(A/k) = F^1_{Hodge} \oplus F^1_{con}$. In particular U is a lift of conjugate fitration F^1_{con} and T is a lift of the Hodge filtration F^1_{Hodge}. Condition iii) above means that U is stable under the action of the absolute Frobenius F_A on the cohomology.

Now, if C is an ordinary smooth projective curve over a field k of positive characteristic and a principal polarization λ, Serre-Tate theory (see [K1] or [DO]) guaranties that there exists a canonical lifting J^{can} of J to the Witt ring $W(k)$. The question of whether the canonical lifting of a Jacobian J is again a Jacobian has been of main interest and Dwok and Ogus have shown in [DO] that even over the Witt ring of length 2, this is a very restrictive condition and in general is not true. Their method consists of constructing an obstruction β, such that $\beta = 0$ if and only if the canonical lifting J^{can} is a Jacobian. They then show that this obstruction is generically non-zero. We recall the construction of β in short. The curve C is called pre-W_2-*canonical* if the canonical lifting (J^{can}, λ^{can}) over $W_2(k)$ is isomorphic to the Jacobian of a smooth projective curve Y as a principally polarized abelian variety. According to Dwork-Ogus theory, the obstruction β_C to the existence of such Y, is the restriction of an element $\beta_{\mathbf{C}} \in Sym^2(F^{1,0})^\vee$ to the kernel $ker(\mu_C)$ of the multiplication map. This obstruction can be generalized to an obstruction for families $f : C \to T$ of ordinary curves to give an obstruction $\widetilde{\beta}_{C/T}$ which is a global section of $F_T^*\mathcal{L}(C/T)$ where $F_T : T \to T$ denotes the absolute Frobenius map and the value of $\widetilde{\beta}_{C/T}$ at $t \in T$ is equal to $F_k^*(\beta_{C_t/k})$. Note that since the family is assumed to have ordinary fibers, the inverse Cartier operator $\gamma : F_T^*\mathbb{E} \to \mathbb{E}$ is an $\mathcal{O}_T$-linear map and in fact it is the inverse transpose of the Frobenius action on $R^1f_*\mathcal{O}_C$. By a result of Katz in [K2], the pull-back $F_T^*\mathcal{L}(C/T)$ comes equipped with a natural flat connection:

$$\nabla : F_T^*\mathcal{L} \to F_T^*\mathcal{L} \otimes \Omega^1_{T/k}.$$

For the Dwork-Ogus obstruction $\widetilde{\beta}_{C/T}$, it holds that $-\nabla\widetilde{\beta}_{C/T} : F_T^*\mathcal{K} \to \Omega^1_{T/k}$ is equal to the composition

$$F_T^*\mathcal{K} \hookrightarrow F_T^*Sym^2(\mathbb{E}) \xrightarrow{S^2(\gamma)} Sym^2(\mathbb{E}) \xrightarrow{\kappa} \Omega^1_{T/k} \ (*)$$

In the above sequence $\gamma : F_T^*\mathbb{E} \to \mathbb{E}$ is the inverse Cartier operator i.e. the inverse transpose of the Frobenius action on $R^1f_*\mathcal{O}_C$. The matrix of this map will be called the *Hasse-Witt matrix* of the family. The map $\kappa : Sym^2(\mathbb{E}) \to \Omega^1_{T/k}$ is the *Kodaira-Spencer* map associated to the family $f : C \to T$.

Using the above description of $\nabla\widetilde{\beta}_{C/T}$, this gives us something computable which we will use later to show that the obstruction is not zero for our families.

Let $f : C \to T$ be as usual a family of abelian covers as in section 3.3. We can choose a prime number $p \equiv 1 \pmod N$ and an open subset U of $T \otimes \mathbb{F}_p$ such that for all $t \in U$, the fibers are ordinary curves in characteristic p. This is possible for example by results of [B]. For such p and U, consider the restricted family $C_U \to U$. The abelian group G also acts on the sheaves $\mathcal{L}(C_U/U)$ and gives the eigensheaf decomposition $\mathcal{L}(C_U/U) = \oplus_{n\in G}\mathcal{L}_{(n)}$. The same is true for $\mathbb{E}_U = \mathbb{E}(C_U/U)$ and $\mathcal{K}_U = \mathcal{K}(C_U/U)$. This in turn, gives us the decomposition $\widetilde{\beta}_{C_U/U} = \sum_n \widetilde{\beta}_n$. Here $\widetilde{\beta}_n$ is considered as a section of $F_U^*\mathcal{L}_n$.

The main observation here is that if the family gives rise to a Shimura subavariety in A_g, then the Dwrok-Ogus obstruction vanishes:

Lemma 3.8.2. For prime number p and open subset U as above, if the family gives rise to a Shimura subvariety $Z \subseteq A_g$, then for any $t \in U$ we have that the Jacobian J_t is pre-W_2-canonical and in particular $\widetilde{\beta}_{C_U/U} = 0$.

Proof. This follows from [M3] or [N]. In fact if the moduli variety Z is a Shimura subvariety, and $t \in T$ is an ordinary point (i.e. it's pre-image is an ordinary curve), then the canonical lifting J_t^{can} of J_t is a $W(k)$-valued point of Z. This means in particular that it is a Jacobian and hence J_t is pre-W_2-canonical. By Dwork-Ogus theory, this forces $\widetilde{\beta}_{C_U/U}$ to be zero. □

Now assuming that the fibers of the family are ordinary over U and the family gives rise to a Shimura subvariety in A_g, it follows from the lemma that $\widetilde{\beta}_{C_U/U} = 0$ and hence $\nabla\widetilde{\beta}_{C_U/U} = 0$. This shows that the composition map (*) should vanish identically.

From now on we just work with the restricted family C_U/U whose fibers are all ordinary instead of C/T and denote it simply as C/U. Next we remark that the sequence (*) factors through the map

$$Sym^2(\mathbb{E}) \to Sym^2(\mathbb{E})_{(0)} \xrightarrow{mult_{(0)}} f_*(\omega_{C/U}^{\otimes 2})_{(0)}$$

Where by the index (0) we mean the subspace of invariant elements under the action of G, i.e. the subspace on which G acts with the trivial character. This factorization follows from the general fact that the fiber of $Sym^2(\mathbb{E})_{(0)}$ at t can be identified with (dual of) the space of G-equivariant deformations of C_t, i.e. the deformations for which the G-action also deforms along. Since there is a G-action on our whole family, the Kodaira-Spencer map should factor through the above map. The last map in the above sequence is just multiplication of forms.

Proposition 3.8.3. With notations as above, the map

$$F_U^*\mathcal{K}_{(0)} \hookrightarrow F_U^*Sym^2(\mathbb{E}_U)_{(0)} \xrightarrow{S^2(\gamma)} Sym^2(\mathbb{E}_U)_{(0)} \xrightarrow{mult_{(0)}} f_*(\omega_{C/U}^{\otimes 2})_{(0)}$$

vanishes identically, provided that the family gives rise to a Shimura subvariety in A_g.

Proof. Let us first note that the induced Kodaira-Spencer map $\kappa_{(0)}$: $f_*(\omega_{C/T}^{\otimes 2})_{(0)} \to \Omega_T^1$ is injective. We remark that this map is in fact the dual of the usual Kodaira-Spencer map $\kappa : \Theta_T \to H^1(\Theta_{C/T})$. Therefore it's injectivity means the surjectivity of the Kodaira-Spencer map i.e. the versality (or completeness)of our family. Now if $D/k[\epsilon]$ is a G-equivariant first order deformation of the fiber C_t, the versality means that $D/k[\epsilon]$ can be obtained by pull-back from our family. But this is true, because in this case D/G is isomorphic to $\mathbb{P}^1_{k[\epsilon]}$ and so as an abelian cover of $\mathbb{P}^1$, it can be obtained by pull-back from our family. If the fibers are non-hyperelliptic, the vanishing of the above map follows directly from the theory of Dwork-Ogus, see [DO], together with the injectvity of $\kappa_{(0)}$ discussed above. In fact, according to [DO], the exact sequence (*) being equal to $-\nabla\widetilde{\beta}_{C/U}$ vanishes identically (lemma 3.1 above). Injectivity of $\kappa_{(0)}$ then gives the vanishing of the claimed map. So we may assume that the fibers are hyperelliptic curves. From this point on, everything goes like [M1], proposition 5.8. Namely, with $\iota \in Aut(C/U)$ being the hyperelliptic isomorphism we conclude from results of [OS] that although the multiplication map $Sym^2(\mathbb{E}_U) \to f_*(\omega^{\otimes 2})$ is no longer surjective, the induced map $mult_\iota : Sym^2(\mathbb{E}_U)_\iota \to f_*(\omega^{\otimes 2})_\iota$ on the the sheaves of invariants of ι is again surjective. Since our family is contained in the hyperelliptic locus, this implies that the map $mult_{(0)}$ is also surjective and this forces $\widetilde{\beta}_{(0)}$ to be an $\mathcal{O}_U$-linear map. If $\widetilde{\beta}_{(0)}$ is zero, of course $\nabla\widetilde{\beta}_{(0)}$ will be also zero. To complete the proof one can check that 0-component analogue of the exact sequence (*) also holds true for $\nabla\widetilde{\beta_{(0)}}$. □

3.9. The generalization of a lemma

For our classification purposes, we will need the generalization to the abelian case of a lemma in [B] that concerns only with cyclic coverings, see [B], lemma 5.1.i. This lemma allows us to compute explicitly the Hasse-Witt matrix of an abelian covering which considering the above constructions will be needed to compute the obstruction $\widetilde{\beta}_{C/U}$. Let $a = (a_1, ..., a_m) \in G \subseteq \mathbb{Z}_N^m$ be an element in the Galois group of the abelian covering. Let $\widetilde{A} = (\widetilde{r}_{ij})$ be the matrix whose entries $\widetilde{r}_{ij}$ are lifts of r_{ij} to $\mathbb{Z} \cap [0, N)$ of the covering. So the entries of $\widetilde{A}$ are lifts of the entries of A, the matrix of the covering, to $\mathbb{Z} \cap [0, N)$ and these two matrices determine each other uniquely. We denote by $a.\widetilde{A}$ the product of these as the product of $1 \times m$ and $m \times s$ matrices, thereby obtaining a $1 \times s$ matrix. Therefore we have that :

$$a.\widetilde{A} = (\textstyle\sum_1^m a_j\widetilde{r}_{j1}, ..., \sum_1^m a_j\widetilde{r}_{js}) = (\alpha_1, ..., \alpha_s)$$

Next take a prime number p such that $p \equiv 1 \pmod{N}$ and let $q = \frac{p-1}{N}$.

Lemma 3.9.1. With the notations as above, the $(h_{\nu\iota})$ entry of the Hasse-Witt matrix of the abelian covering Y is given by the formula:

$$\sum_{\sum l_i=\Sigma} \binom{q.[-\alpha_1]_N}{l_1} ... \binom{q.[-\alpha_s]_N}{l_s} z_1^{l_1} ... z_s^{l_s}$$

Where $\Sigma = (d_n - \iota)(p-1) + (\nu - \iota)$ and $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ and $[\alpha]_N$ denotes the representative of the integer α modulo N in $\{0, 1, .., N-1\}$.

Proof. Like in [B] take $U_1 = \mathbb{P}^1 - \{\infty\}$ and $U_2 = \mathbb{P}^1 - \{0\}$ and $V_i = \pi^{-1} U_i$ for $i = 1, 2$. Let

$$v_a = w_1^{a_1} ... w_m^{a_m} (z - z_1)^{-[\alpha_1]} ... (z - z_s)^{-[\alpha_s]}$$

Then we have:

$$\Gamma(V_1, \mathcal{O}_{V_1}) = \bigoplus_{a \in G} k[z] v_a$$

$$\Gamma(V_2, \mathcal{O}_{V_2}) = \bigoplus_{a \in G} k[z^{-1}] z^{-|a|-1} v_a$$

$$\Gamma(V_1 \cap V_2) = \bigoplus_{a \in G} k[z, z^{-1}] v_a$$

Defining $\xi_j = z^{-j} v_a$ for $j = 1, ..., |a|$, with $|a| = dim H^1(Y, \mathcal{O}_Y)_a$, we see that the ξ_j form a basis for $H^1(Y, \mathcal{O}_Y)_a = \frac{\Gamma(V_1 \cap V_2)_a}{\Gamma(V_1)_a + \Gamma(V_2)_a}$.

If B_a is the matrix of the Hasse-Witt map $F : H^1_a \to H^1_a$, then the (i, j) entry of B_a is given by the coefficient of ξ_i in ξ_j^p. This follows from the fact that $\xi_j \otimes 1$ determines a local basis for the bundle $F_T^*(R^1 f_* \mathcal{O}_C)_a$ and the Hasse-Witt operator $\gamma : F_T^*(R^1 f_* \mathcal{O}_C)_a \to (R^1 f_* \mathcal{O}_C)_a$ with respect to these bases is given by the p-th power endomorphism of $\mathcal{O}_C$. This is because of the fact that the p-linear composite map

$$R^1 f_* \mathcal{O}_C \to F_T^*(R^1 f_* \mathcal{O}_C) \to (R^1 f_* \mathcal{O}_C)$$

is induced by the p-th power endomorphism of $\mathcal{O}_C$ (cf.[K], 2.3.4.1.4). Now one sees that the coefficient of ξ_i in this polynomial is as claimed above:

$\sum_{\sum l_i=\Sigma} \binom{q.[-\alpha_1]_N}{l_1} \dots \binom{q.[-\alpha_s]_N}{l_s} z_1^{l_1} \dots z_s^{l_s}.$

Where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ and where $[\alpha]_N$ denotes the residue of an integer α modulo N. Occasionally, we just drop $[]_N$ and write only α. □

3.10. Excluding non-Shimura examples

At this point we are going to exculde the families of abelian covers of the projective line that do not give rise to Shimura subvarieties in A_g. For some technical reasons, working with families with 4 branch points is different from families with more branch points and it should be noted that this is in some sense the most important case, as most of the examples of Shimura families that have been found in [M1] or [MO] are obtained from families with 4 branch points. We therefore distinguish between this case and other cases. Before we state our results, we give an "irreducibility condition" that we assume all of the families till end of these notes satisfy:

Condition (*). We say that the family satisfies the condition (*) if the rows of the associated matrix are linearly independent over $\mathbb{Z}/N\mathbb{Z}$. For families of cyclic covers, this implies that the family is irreducible. For families of abelian covers it implies that all of the intermediate cyclic covers are irreducible.

The case of four branch points

Proposition 3.10.1. Let $Y \to T$ be a family of abelian covers with $s = 4$, i.e. with 4 branch points. Then the associated subvariety $Z \subseteq A_g$ is a Shimura subvariety if and only if $Z = S(\mu_G)$. i.e. if and only if it appears in table 1.

Proof. Clearly the statement holds if $Z = S(\mu_G)$ as $S(\mu_G)$ is a Shimura variety of PEL type. Now assume on the contrary that $Z \neq S(\mu_G)$ but Z is a Shimura subvariety and we will derive a contradiction. The assumption

$Z \neq S(\mu_G)$ implies that $dimS(\mu_G) > 1$. Note that we have already classified all cases where $dimZ = dimS(\mu_G) = 1$, i.e. the cases for which $Z = S(\mu_G)$. The fact that $dimS(\mu_G) > 1$ shows that there are pairs $a, a' \in G$ with $a' \neq \pm a$ such that $d_a = d_{-a} = 1$ and $d_{a'} = d_{-a'} = 1$. Therefore for every $l \in \{\pm a, \pm a'\}$, the Hasse-witt matrix A_l is a polynomial in $\mathbb{F}_p[z_1, .., z_4]$. Note that according to the discussion just before lemma 3.8.2, there is an open subset U of T and a suitable prime number p, such that all fibers above U are ordinary after reduction mod p. Now since the fibers are all ordinary curves in U, we conclude that the Hasse-Witt operator is an isomorphism and so A_l is invertible as a section of $\mathcal{O}_U$. Note that $\omega_a.\omega_{-a} = \omega_{a'}.\omega_{-a'}$ is a non-zero section of the bundle $f_*(\omega^{\otimes 2})$ and so we must have :

$$A_a.A_{-a} = A_{a'}.A_{-a'}$$

as polynomials.

We will show that this identity can not happen with the above conditions. The polynomials A_l are given by the above lemma and we could set $B_l = A_l \mid_{z_1=0}$. It means that we have :

$$B_l = \sum_{j_1+j_2+j_3=N-1} C(q.[-\alpha_2]_N, j_2)C(q.[-\alpha_3]_N, j_3)C(q.[-\alpha_4]_N, j_4)z_2^{j_2}z_3^{j_3}z_4^{j_4}.$$

Let $r_a(l)$ be the largest integer r such that B_l is divisible by t_l^r. We have that

$$r_a(l) = max\{0, q.\alpha_1 + q.\alpha_2 - (N-1)\}.$$

Similarly let $r_{\pm a}(l)$ be the largest integer r such that $B_a.B_{-a}$ is divisible by t_l^r. We have :

$$r_{\pm a}(l) = q.max\{\alpha_1 + \alpha_l, \alpha_k + \alpha_\lambda\} - (N-1).$$

Now the equality $A_a.A_{-a} = A_{a'}.A_{-a'}$ implies that $r_{\pm a}(l) = r_{\pm a'}$ and so we get the following equality:

$$\{\alpha_1 + \alpha_l, \alpha_k + \alpha_\lambda\} = \{\alpha'_1 + \alpha'_l, \alpha'_k + \alpha'_\lambda\}.$$

By an easy lemma in [M1] (lemma 6.3) we conclude that there exists an even permutation $\sigma \in A_4$ of order 2, such that $\alpha_i = \alpha'_{\sigma(i)}$. We first claim that $\sigma \neq 1$. This in fact follows from the above technical condition (*) which ensures that σ is not trivial i.e. that α_i and α'_i are not all the same. Furthermore, without loss of generality we can assume that $\alpha_i = r_{1i}$ for all $i = 1, .., 4$. That is, we may consider $(\alpha_1, ..., \alpha_4)$ as the first row of the matrix A of the abelian covering . We set $a_i = \alpha_i$ instead of r_{1i} for simplicity. Now since α_i and α'_i are different by the above argument, we may again without loss of generality suppose that:

$\alpha'_1 = a_2$, $\alpha'_2 = a_1$

$\alpha'_3 = a_4$, $\alpha'_4 = a_3$

by our assumptions on a_i and a'_i, we have that

$\sum[a_i]_N = \sum[a'_i]_N = 2N$

Suppose that $[a_1]_N + [a_2]_N = [a_3]_N + [a_4]_N = N$, or in other words, $[a_2]_N = -[a_1]_N$ and $[a_4]_N = -[a_3]_N$ in $\mathbb{Z}/N\mathbb{Z}$. This means that the two rows $n = (a_1, .., a_4)$ and $n' = (a'_1, .., a'_4)$ are linearly dependent and this contradicts condition (*). So the above equality does not hold and we may assume that $a_1 + a_2 < N$ and $a_3 + a_4 > N$. Now consider the row vector

$$n + n' = (a_1, .., a_4) + (a'_1, .., a'_4) = (a_1 + a_2, a_1 + a_2, a_3 + a_4, a_3 + a_4)$$

Note that condition (*) assures that $n + n' \neq \pm n$ and one can easily verify that this row vector also satisfies the conditions for a_i and a'_i (in fact $2([a_1 + a_2] + [a_3 + a_4]) = 2([(N-1)(a_1 + a_2)] + [(N-1)(a_3 + a_4)]) = 2N$) and so we may replace the second row $(a'_1, .., a'_4) = (a_2, a_1, a_4, a_3)$ by this row vector and the equality $A_n.A_{-n} = A_{n'}.A_{-n'}$ should hold for this row vector as n' and $(a_1, .., a_4)$ as n. We show that this is impossible. In fact, if this equality holds, it is easy to see that the left hand side must contain a monomial of the form $z_2^{\alpha} z_3^{\beta}$ and also a monomial of the form $z_1^{\gamma} z_4^{\delta}$. This means that $a_2 + a_3 = a_1 + a_4 = a_1 + a_3 = a_2 + a_4 = N$ which is exactly to say that $n = n' = (a_1, a_1, -a_1, -a_1)$. This is against our assumptions and this contradiction completes the proof. □

The cases where $s \geq 5$

As the families giving rise to Shimura subvarieties in table 1 all have the Galois group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, this is in a sense the most important case of the families. We therefore restrict our attention to families with such Galois groups and form now on we assume that the family has Galois group of the form $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ i.e. that the matrix A is a $2 \times s$ matrix.

In [M1] Moonen uses the Dwork-Ogus obstruction also in order to prove that further examples of Shimura families of cyclic covers of $\mathbb{P}^1$ do not exist for $s > 4$. The core observation in his proof is then that for a minimal Shimura family not existing in his table, there exists two integers n and n', such that $d_n = d_{-n'} = 1$ and $d_{-n} = d_{-n'} = s - 3$. To deduce the existence of these two elements, he argues that for $n \in (\mathbb{Z}/m\mathbb{Z})^*$, $d_n + d_{-n} = s - 2$. This point, as we will see explicitly later, does not remain necessarily true for families of abelian coverings. It could very well be that for all $n \in G$, $d_n + d_{-n} < s - 2$. Example III^* below (see theorem 3.10.6) is the simplest example of such a family. In fact by what we have seen so far, if this equality holds and the family is Shimura, then $N = 2, 3, 4, 5, 6$ and $s \leq 6$. So in order to exclude further examples first we will use another method based on the monodromy of a family of curves. We first need a definition :

Definition 3.10.2. Let $f : Y \to T$ be a family of abelian Galois covers of $\mathbb{P}^1$ as constructed in section 1. Then $\mathcal{L} = R^1 f_* \mathbb{C}$ is a polarized variation of Hodge structures (PVHS) of weight 1. This PVHS decomposes according to the action of the abelian Galois group G and the eigenspaces $\mathcal{L}_i$ (or $\mathcal{L}_\chi$ where $i \in G$ corresponds to character $\chi \in \mu_G$) are again variations of Hodge structures and we are mainly interested in these. Take a $t \in T$ and assume that $h^{1,0}((\mathcal{L}_i)_t) = a$ and $h^{0,1}((\mathcal{L}_i)_t) = b$. Then the polarization equips $(\mathcal{L}_i)_t$ with a Hermitian form with singature (a, b) (see [DM], 2.21 and 2.23). This implies that $Mon^0(\mathcal{L}_i) \subseteq U(a, b)$. In this case, we say that $\mathcal{L}_i$ is of type (a, b). The above observations are key to our further analysis. Let us first prove a lemma:

Lemma 3.10.3. Let $\mathcal{L}_i$ be an eigenspace as discussed above of type (a, b) with $ab \neq 0$. Then $Mon^0(\mathcal{L}_i) = SU(a, b)$, unless when $|G| = 2l$ is even and i is of order 2 in G, in which case there is a surjection from $Mon^0(\mathcal{L}_i)$ to $SU(n, n) = Sp_{2n}$. Where $n = d_i$.

Proof. Let $t \in T$ and let $\chi \in Hom(G, \mathbb{C}^*)$ be the character corresponding to i. Consider the cover $f_{\chi,t} : Y_{\chi,t} \to \mathbb{P}^1$ with group $\chi(G)$ branched only above the points z_j with local monodromy $\chi(\phi(\gamma_j))$ about z_j. Where ϕ is the surjection in remark 3.3.1. Note that $\chi(G)$ is a cyclic group and so $f_{\chi,t}$ is in fact a cyclic cover with group $\chi(G)$. Varying $t \in T$, we get a family of cyclic covers of $\mathbb{P}^1$. The eigenspace $\mathcal{L}_i$ is exactly the eigenspace corresponding to this family (or in other words, it is the $\mathcal{L}_1$ of this family of cyclic covers). Unless when $|G| = 2l$ is even and i is of order 2 in G, theorem 5.1.1 of [R] applies and we get that $Mon^0(\mathcal{L}_i) = SU(a, b)$. If $|G| = 2l$ is even and i is of order 2 in G by taking quotient of the family $f_{\chi,t}$, we obtain a family of hyperelliptic curves of the form $w^2 = (z - z_1)...(z - z_{2n+2})$. Note that in this case it follows from the formulas of proposition 3.6.3 that there are $2n + 2$ odd powers in the equation of $f_{\chi,t}$ for $n = d_i$. Now it is well-known that Mon^0 of a family of hyperelliptic curves is the full symplectic group and so the proof is completed.

Remark 3.10.4. Assume that $Y \to T$ is a family of curves and let M be the generic Mumford-Tate group of this family. Recall from construction 3.6.2, that there is a natural Shimura variety $S_f = Sh(M, Y)$ associated to M (which is a reductive group) and the dimension of S_f only depends on $M^{ad}_{\mathbb{R}}$. The Shimura datum comes from the Hodge structures of the fibers in the family. This Shimura variety is the smallest Shimura subvariety in A_g which contains Z. Our purpose is to show that for families of abelian covers with a big s, $M^{ad}_{\mathbb{R}} = \prod Q_i$ such that $\sum \delta(Q_i) > s-3$ and therefore the family is not a Shimura family. Here $\delta(Q_i)$ is as in construction 3.6.2. The following remark is well-knownn but very important for our goals:

Remark 3.10.5. If the family $f : Y \to T$ gives rise to a Shimura subvariety in A_g, then the connected monodromy group Mon^0 is a normal subgroup of the generic Mumford-Tate group M (in fact in this case, $Mon^0 = M^{der}$, as we have seen in section 1.3). Cosequently, if $M^{ad}_{\mathbb{R}} = \prod_1^l Q_i$ as a product of simple Lie groups, then then exists a subset $K \subseteq \{1, ..., l\}$, such that $Mon^{0,ad}_{\mathbb{R}} = \prod_{i\in K} Q_i$.

Our strategy is to show that for large s, there are eigenspaces $\mathcal{L}_i$ of types (a_i, b_i) with $\{a_i, b_i\} \neq \{a_j, b_j\}$ for $i \neq j$ and such that $\sum \delta(\mathcal{L}_i) > s - 3$. Then by the above remark , we conclude that $dimS_f > s - 3$. Note that this property is far from being true for the families of cyclic covers of $\mathbb{P}^1$. For those families it can happen that all of the eigenspaces are either unitary (i.e. $a_i = 0$ or $b_i = 0$) or of the same type. Take for example the family $(11, (1, 1, 1, 1, 7))$. In this case all of eigenspaces are either of type $(3, 0)$ (or $(0, 3)$) and hence unitary, or of type $(1, 2)$. Another important observation is that if in the family, one row, say the first row, does not have any 0 entry, then cyclic covering arising from this row is either a Shimura family (of cyclic covers) or the whole family will not be a Shimura family. This is true because if this family is not a Shimura family then by the above notations and observations, there are $\mathbb{R}$-simple factors Q_i, in the decomposition of $M^{ad}_{1,\mathbb{R}}$ such that $\sum \delta(Q_i) > s - 3$. Where M_1 is the Mumford-Tate group associated to this family of cyclic covers. As this is a sub-Hodge structure, we know from [VZ2], that M_1 is a quotient of M and therefore the factors Q_i also occur in decomposition of M^{ad} (note that M^{ad} is semi-simple group with trivial center) and so $dimS_f > s - 3$ i.e. the family is not a Shimura family. On the other hand, if this cyclic family is a Shimura family it must be one of the families in [R] (or [M1]) and therefore, N is one the 10 numbers in table 1 of [M1] or [R]. Of course this leaves only finitely many possibilities to investigate. So, if in one of the rows all of the entries are non-zero, according to the table in [M1], $N = 3, 4, 5, 6$ and we will exclude these in what follows, but if there are 0 entries in the rows, there are only three possibilities: $I) \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & 0 & b_2 & b_3 \end{pmatrix}$, $II) \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & b_1 & b_2 & b_3 \end{pmatrix}$, $III) \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & b_1 & b_2 & b_3 \end{pmatrix}$ or $IV) \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & 0 & b_1 & b_2 \end{pmatrix}$.

Also consider the families $III^{*}) \begin{pmatrix} 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$ with $N = 3$ and $III^{**}) \begin{pmatrix} 2 & 2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$ with $N = 4$, which are special cases respectively of families III. Then the following theorem holds:

The following theorem serves as an example for the methods and arguments based on the above remark which we will use later in the proof of proposition :

Theorem 3.10.6. A family $Y \to T$ of abelian covers of $\mathbb{P}^1$ with $s = 5$ branch points does not give rise to a Shimura variety in A_g except possibly the families III^* and III^{**}.

Proof. Following the discussions after remark 3.10.5, we see that the families $\begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & 0 & b_2 & b_3 \end{pmatrix}$ are either not Shimura or can only have $N = 3, 4, 5, 6$ because for these families the eigenspace $\mathcal{L}_{(1,1)}$ is that of a Shimura family of cyclic covers with 5 branch points which according to [M1] (or [R]), table 1 leaves only these 4 possibilities for N. An argument similar to the argument below shows that none of these families can give rise to a Shimura subvariety. Therefore we need only to consider families of the form $\begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & b_1 & b_2 & b_3 \end{pmatrix}$ or $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & b_1 & b_2 & b_3 \end{pmatrix}$ or $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & 0 & b_1 & b_2 \end{pmatrix}$. We first exclude the first family which is easier and our method can more easily be elucidated with this example. In this case, the eigenspace associated to the element $(1,1)$ is given by $(a_1, a_2, a_3+b_1, b_2, b_3)$ so we must have $a_3+b_1 = 0$, otherwise $N = 3, 4, 5, 6$, which can be excluded in each case using the same argument as follows: Likewise $a_3 - b_1 = 0$ and so we have that $a_3 = b_1 = \frac{N}{2}$. Consider the eigenspace associated to the element $(2,1)$ given by the cyclic cover $(2a_1, 2a_2, \frac{N}{2}, b_2, b_3)$ since non of a_1 and a_2 is zero, we have also that $2a_1 \neq 0$ and $2a_2 \neq 0$ (note that we assume that $\sum a_i = N$, otherwise we can replace a_i with $-a_i$ and we get an isomorphic cover for which $\sum a_i = N$). By what we said earlier, this implies that $N = 4, 6$ which we have to exclude now. For $N = 4$, the only possible families are $\begin{pmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$. All of the 3 families are not Shimura as we will show here. Take for example the first family. The eigenspace $\mathcal{L}_{(1,2)}$ has type $(1,2)$ and the eigenspace $\mathcal{L}_{(1,3)}$ has type $(1,1)$. This shows by the above remarks that $dimS_f = \sum \delta(Q_i) \geq 3$ Therefore $Z \neq S_f$ and the family is not Shimura. In the same way one sees easily that the other two families are not Shimura too. Also by the same method one can conclude that for $N = 6$, there does not exist any Shimura family. For $N = 3$ there is only one family, namely the family $\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$ which is not Shimura again because there is an eigenspace of type $(1,2)$ and another one of type $(1,1)$

which forces $dimS_f \geq 3$. Now the only exceptions that do not follow the regulation above are families III^*, III^{**}.$\square$

At this point, we don't know whether the families III^* and III^{**} give rise to Shimura subvarieties or not.

As we explained earlier, if one row of the family has no non-zero entries, then N can be one of the numbers $3, 4, 5, 6$. Therefore if we have a family with N other than the numbers above, the rows must always contain zero. Let the number of these zeros in the first row be l, i.e. the family has the form $\begin{pmatrix} a_{11} & \cdots & a_{1l} & a_{1l+1} & \cdots & a_{1r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{1l+1} & \cdots & b_{1r} & b_{1r+1} & \cdots & b_s \end{pmatrix}$. Furthermore suppose that $\sum a_i > 2N$ and $\sum(-a_i) = (\sum[-a_i]_N) > 2N$. For $s > 19$, we prove that:

Proposition 3.10.7. With the above condition and if $s > 19$, then the family

$$\begin{pmatrix} a_{11} & \cdots & a_{1l} & a_{1l+1} & \cdots & a_{1r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{1l+1} & \cdots & b_{1r} & b_{1r+1} & \cdots & b_s \end{pmatrix}$$

does not give rise to a Shimura subvariety in A_g.

Proof. Assume that the associated eigenspaces are of types $(k_1, r-k_1-2)$ and $(k_2, s-l-k_2-2)$. If these two types are different, we will have:

$$dimS_f \geq k_1(r-k_1-2) + k_2(s-l-k_2-2) \geq 2(r-4) + 2(s-l-4)$$

Now for $s \geq 19$, one sees that $2(r-4) + 2(s-l-4) > s-3$ and hence $dimS_f > s-3$. It remains to treat the case where the two types are the same which implies that $s = r + l$. We have:

$$dimS_f \geq k_2(s-l-k_2-2) \geq 2(s-l-4)$$

The right hand side is strictly greater than $s-3$ if and only if $r-l>5$. We may therefore assume that $r-l\leq 5$. We have

$$t(-1,1)=\sum_1^l(-a_i)+\sum_{l+1}^r(b_j-a_j)+\sum_r^s b_k\geq(l-1)N.$$

That is, for the eigenspace associated to the element $(-1,1)$, we have that $t(-1,1)\geq(l-1)$ and consequently $d_{(-1,1)}\geq l-2$. Similarly one sees that $d_{(1,-1)}\geq l-2$. Note that this eigenspace (i.e. the eigenspace associated with the element $(-1,1)$) can not be of the type $(k_2, s-l-k_2-2)$ (equivalently of the type $(k_1, r-k_1-2)$). For otherwise we will have $k_2\geq l-2$ and $s-l-k_2-2\geq l-2$. The first inequality says that $k_2>5$ (because $s>19$ and hence $l>7$) and the second one implies that $k_2\leq s-2l\leq 5$. This contradiction shows that we have a new eigenspace. Since $d_{(-1,1)}d_{(1,-1)}\geq(l-2)^2>l$, we conclude that:

$$dimS_f>s-3$$

and the claim follows. □

References

[A] Y. André, *Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part.* Compositio Math. 82(1992), pp. 1-24.

[B] I. Bouw, *The p-rank of ramified covers of curves.* Compositio Math. 126 (2001), 295-322.

[BB] W. Baily, A. Borel, *Compactification of arithemtic quotients of bounded symmetric domains.* Ann. of Math. 84 (1966) 442-528.

[C] R. Coleman, *Torsion points on curves.* In: Galois representations and arithmetic algebraic geometry; pp. 235-247. (Y. Ihara, ed.) Adv. Studies Pure Math. 12, North-Holland, Amsterdam, 1987.

[CDK] E. Cattani, P. Deligne, A. Kaplan, *On the locus of Hodge classes.* J.A.M.S. 8 (1995), pp.483-506

[CFG] E. Colombo, P. Frediani, A. Ghigi, *On totally geodesic submanifolds in the Jacobian locus.* arXiv:1309.1022.

[CW] C. Chevalley, A. Weil, *Über das Verhalten der Integrale 1. Gattung bei Automorphismen des Funktionenkörpers.* Abh. Math. Semin. Univ. Hambg. 10 (1934), 358-261.

[D1] P. Deligne, *Variétés de Shimura: interprétation modulaire, et tehcnique de construction de modèles canoniques.*. In: Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvalis, Ore., 1977)), Part 2, Proc. Sympos. Pure Math., XXXIII, AMS, Providence, R.I. (1979) 247-289.

[D2] P. Deligne, *La conjecture de Weil les sruface K3.* Invent. Math 15 (1972), pp.206-226.

[D3] P. Deligne, *Travaux de Shimura.* In: Seminaire Bourbaki, 389 (1970/71), Lecture notes in Mathematics 244, Springer-Verlag, Berlin (1977) 123-165.

[DM] P. Deligne, G.Mostow, *Monodromy of hypergeometric functions and non-lattice integral monodromy.* Inst. Hautes Études Sci. Publ. Math. 63 (1986), 5-89.

[DO] B. Dwork, A.Ogus, *Canonical liftings of jacobians.* Compositio Math. 58 (1986), 111-131.

[EV] H. Esnault, E. Viehweg, *Lectures on Vanishing Theorems.* DMV Seminar, Band 20, Birkhäuser 1992.

[H] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces.* Pure and Applied Mathematics 80, Academic Press, Inc., New York-London, 1978.

[HM] J. Harris, I. Morrison, *Moduli of curves.* Graduate Texts in Mathematics 187, Springer-Verlag (1998) Berlin Heidelberg New-York.

[K] N. Katz, *Algebraic solutions of differential equations.* Inventions math 18, 1-118. (1972)

[K1] N. Katz, *Travaux de Dwork.* Seminaire de Bourbaki, 1971-1972, exp.n 409, p.167-200.

[K2] N.Katz, *Nilpotent connections and the monodromy theorem: applications of a result of Turrittin.* Publ. Math. I.E.H.S. 39 (1971) 175-232.

[Mi] J. Milne, *Introduction to Shimura varieties.* (2004) preprint.

[M1] B. Moonen, *Special subavarieties arising from families of cyclic covers of the projective line.* Documenta Mathematica 15 (2010) 793-819.

[M2] B. Moonen, *Linearity of Shimura varieties. I.* J. Alg. Geom. 7 (1998), 539-567

[M3] B. Moonen, *Linearity of Shimura varieties. II.* Compositio Math. 114 (1998), 3-35.

[MO] B. Moonen, F. Oort, *The Torelli locus and special subvarieties.* 2010, preprint. Submitted to the Handbook of moduli (G.Farkas, I.Morrisson, eds).

[Mu] D. Mumford, *A note on Shimura's paper "Discontinuous groups and Abelian Varieties".* Math. Ann. 181 (1969) 345-351.

[MVZ1] M. Möller, E. Viehweg, K. Zuo, *Special families of curves, of abelian varieties, and of certain minimal manifolds over curves.* In: Global aspects of complex geometry; pp. 417-450. (F. Catanese et al., eds.) Springer, Berlin, 2006.

[MVZ2] M. Möller, E. Viehweg, K. Zuo, *Stability of Hodge bundles and a numerical characterization of Shimura varieties.*, 2007, arXiv:math.AG/0706.3462, to appear in J. Diff. Geom.

[N] R. Noot, *Hodge classes, Tate classes and local moduli of abelian varieties.* PhD thesis, University of Utrecht, 1992.

[O] F. Oort, *Special subvarieties in the Torelli locus.* UPenn Algebra seminar 2010 (informal manuscript).

[OS] F. Oort, J.Steenbrink, *The local Torelli problem for algebraic curves.* In: Journées de Géometrie Algébrique d'Angres, Sijthoff Noordhoff, 1980, pp.157-204.

[P1] R. Pardini, *Abelian covers of algebraic varieties.* J. reine angew. Math., 417 (1991), 191-213.

[P2] R. Pardini, *On the period map for abelian covers of projective varieties.* Annali della Scuola Normale Superiore di Pisa - Classe di Scienze 26.4 (1998): 719-735.

[R] J. Rohde, *Cyclic coverings, Calabi-Yau manifolds and complex multiplication.* Lecture Notes in Math. 1975, Springer, 2009.

[S] I. Satake, *Algebraic structures of symmetric domains.* Kano Memorial Letures 4, Iwanami Shoten, Princeton University Press (1980) USA.

[Sh] G. Shimura, *On purely transcendental fields of automorphic functions.* Ann. of Math. (2) 78 (1963), 149-192.

[Sp] T. Springer, *Linear algebraic groups. Second edition.* Progr. Math. 9, Birkhäuser Boston, Boston, MA, 1998.

[V] H. Völklein, *Groups as Galois groups: An introduction.* Cambridge studies in Advanced Mathematics 53, Cambridge University Press (1996) Cambridge.

[Vo] C. Voisin, *Hodge theory and complex algebraic geometry.* I. Cambridge Studies in Advanced Mathematics, 76. Cambrdige university Press, Cambrdige (2002).

[VZ1] E. Viehweg, K. Zuo, *A characterization of certain Shimura curves in the moduli stack of abelian varieties.* J. Diff Geom. 66 (2004), 233-287.

[VZ2] E. Viehweg, K. Zuo, *Complex multiplication, Griffiths-Yukawa couplings, and rigidity for families of hypersurfaces.* J. Alg. Geom. 14 (2005) 481-528.

[W] A. Wright, *Shwarz triangle mappings and Teichmüller curves: abelian square-tiled surfaces.* arXiv:1203.2683.

[We] A. Weil, *Zum Beweis des Torellischen Satzes.* Nachr. Akad. Wissensch. Göttingen. 1957, 33-53.