Fontaine Modules and F-T-crystals, A Dévissage

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Contents

Conventions and Notations

k: a perfect field of characteristic $p > 0$
$W = W(k)$: the Witt ring with residue field k
$W_n = W_n(k) = W(k)/p^n W(k)$
$\mathbb{N} = \{0, 1, 2, \cdots\}$
log = logarithmic
Monoid: Always with a unit and commutative.

Introduction

Let X be a proper smooth variety over k, $H^n_{cris}(X/W)$ be its n-th crystalline cohomology, which is known to be a finitely generated $W(k)$-module. By functoriality, the Frobenius morphism $F_X : X \to X$ induces a semilinear endomorphism of $H^n_{cris}(X/W)$, or in other words, a $W(k)$-linear morphism

$$\Phi : F_W^* H^n_{cris}(X/W) \to H^n_{cris}(X/W), \tag{0.0.1}$$

where F_W is the Frobenius endomorphism of $W(k)$. This action on $H^n_{cris}(X/W)$ will be called Frobenius action in the sequel.

It is a fundamental discovery made by Mazur [15] that the Frobenius action (0.0.1) can be used to study the Hodge filtration of $H^n_{dR}(X/k)$, see [2, Theorem 8.5] for a precise statement of this result. Couples like $(H^*_{cris}(X/W), \Phi)$ are prototypes of a much more general object, namely F-crystal, which appeared later in [11]. After that, Fontaine and Laffaile in [6] introduced a category, whose objects, roughly speaking, are F-crystals (M, Φ) equipped with a filtration $Fil^\bullet$ such that $\Phi(Fil^i M) \subseteq p^i M$ and

$$\sum_i \frac{\Phi(Fil^i M)}{p^i} = M. \tag{0.0.2}$$

It should be noted that the objects of their category are defined over a local field or its valuation ring. In [5] Faltings gives a sheafified version of this category, whose objects live on more general smooth schemes or log schemes defined over a discrete valuation ring. Besides satisfying the condition (0.0.2), objects of Faltings' category are endowed with an integrable connection ∇ satisfying Griffiths transversality. The term "Fontaine module" is used probably for the first time in the monograph [16] to refer to objects of this category.

Fontaine modules arise naturally in the following way. Let X be a smooth proper S-scheme, given a proper smooth family $Y \to X$ over X, the n-th de Rham cohomology $R^n f_* \Omega^\bullet_{Y/X}$ is then endowed with a Hodge filtration $F^\bullet_{Hdg}$ and a canonical connection ∇, called Gauss-Manin connection. Moreover, the action of ∇ on $R^n f_* \Omega^\bullet_{Y/X}$ satisfies the Griffiths transversality with respect to the Hodge filtration, i.e.

$$\nabla(F^i_{Hdg} R^n f_* \Omega^\bullet_{Y/X}) \subseteq F^{i-1}_{Hdg} R^n f_* \Omega^\bullet_{Y/X} \otimes \Omega^1_{X/S}. \tag{0.0.3}$$

When $S = \operatorname{Spec} W(k)$, by a comparison theorem [2, 7.26.3] we have an isomorphism

$$R^n f_*(\Omega^\bullet_{Y/X}) \cong R^n f_{cris*}(Y_0/X),$$

where Y_0 is the special fiber of Y over the closed point of S. In particular, there is an induced Frobenius action on the de Rham cohomology $R^n f_* \Omega^\bullet_{Y/X}$. One can prove the quadruple $(R^n f_* \Omega^\bullet_{Y/X}, F^\bullet_{Hdg}, \nabla, \Phi)$ is a Fontaine module on X in the sense of Faltings.

In a recent work by Ogus and Vologodsky, a new way to define Fontaine modules by using inverse Cartier transform is presented [17, Definitino 4.16]. Unlike the preceding definitions, Fontaine modules in this context are p-torsion, or in other words, defined on schemes in characteristic p. It should be noted that to define a Fontaine module on a smooth scheme X over k by their means, a lifting $\tilde{X}'$ of $X' = X \times_{Fr_k} k$ to $W_2(k)$ is indispensable.

To explain this version of Fontaine modules, we need to recall briefly what the 'inverse Cartier transform' is. Firstly, we are given two categories, $\mathrm{MIC}_{p-1}(X/k)$ and $\mathrm{HIG}_{p-1}(X'/k)$. Objects of the former category consist of $\mathcal{O}_X$-modules with an integrable connection whose p-curvature are nilpotent of level less than p while the objects of the latter one are Higgs modules on X' whose Higgs fields are nilpotent of level less than p. Then the Cartier transform $C_{\mathscr{X}}$ together with its quasi-inverse $C_{\mathscr{X}}^{-1}$, i.e. the inverse Cartier transform is a pair of functors establishing an equivalence between the two categories above. Such an equivalence generalizes the classical Cartier-Katz descent [10, Theorem 5.1], which sets up an equivalence between the category of quasi-coherent $\mathcal{O}_X$-modules equipped with an integrable connection with vanishing p-curvature and that of quasi-coherent $\mathcal{O}_{X'}$-modules, whose objects can also be regarded as Higgs modules on X' with zero Higgs fields.

To define Fontaine modules one also needs the following elementary construction. Let (E, ∇) be an object of $\mathrm{MIC}_{p-1}(X/k)$ endowed with a filtration $F^\bullet$ satisfying Griffiths transversality, by taking gradings of the connection with respect to the filtration, we get an $\mathcal{O}_X$-linear morphism:

$$Gr_{F^\bullet}\nabla : Gr_{F^\bullet}E \to Gr_{F^\bullet}E \otimes \Omega^1_{X/k},$$

which is an object $(Gr_{F^\bullet}E, Gr_{F^\bullet}\nabla)$ of $\mathrm{HIG}_{p-1}(X/k)$. Let $\pi : X' = X \times_{Fr_k} k \to X$ be the projection to the first factor, then the pullback $\pi^*(Gr_{F^\bullet}E, Gr_{F^\bullet}\nabla)$ defines an object of $\mathrm{HIG}_{p-1}(X'/k)$.

After the above preparations, a Fontaine module in the setting of [17] can be roughly described as an object (E, ∇) of $\mathrm{MIC}_{p-1}(X/k)$ endowed with filtration $F^\bullet$ satisfying Griffiths transversality such that there is an isomorphism from (E, ∇) to the object $C_{\mathscr{X}}^{-1}\pi^*(Gr_{F^\bullet}E, Gr_{F^\bullet}\nabla)$.

The construction of Cartier transforms and its inverse can be generalized to log smooth schemes, which is the subject of Schepler's thesis [18]. In this work, we use his log inverse Cartier transform to give a definition of log Fontaine modules by mimicking Ogus and Vologodsky's.

As our title suggests, F-T-crystals make up another aspect this work. This class of objects are introduced by Ogus in his work [16] for the first time as a generalization of the more classical F-crystals. They can be thought of as families of F-crystals satisfying certain transversality conditions. Moreover, locally these objects live on formal schemes formally smooth over a p-adic base, hence one can consider their reductions modulo p. Since its definition is too complicated to recall here, we refer the readers to (3.3.1) for it.

Recall that the aforementioned Faltings' category also generalizes F-crystals to families, its relation to the F-T-crystals has been worked out by Ogus. To be more precise, he constructed a functor from the category of F-T-crystals to Faltings' category and proves [16, Proposition 5.3.9] this functor defines an equivalence of categories when the base scheme is p-torsion free .

The main motivation of this work is to investigate the relation between Ogus and Vologodsky's Fontaine modules and the reduction modulo p of F-T-crystals. More precisely, given an F-T-crystal over a base where p is nilpotent but nonzero, we will prove its reduction defines an Fontaine module in the sense of Ogus and Vologodsky. However, we will show the reduction of more general F-T-crystals in the log context are not log Fontaine modules as defined in (2.2.5), (2.2.6).

Next we give a brief overview of the contents of the following chapters.

In the first chapter, we review the construction of Cartier transform and its quasi-inverse. The guiding line in the original construction in [17] is the fact that the ring of PD-differential operators is an Azumaya algebra. Then the remaining task is to find a splitting module for this sheaf of rings. Since given a splitting module for an Azumaya algebra, it is a general formalism to define an equivalence between the categories of modules over this Azumaya algebra and modules over the coefficient rings (Morita Equivalence).

In our setting, the categories on two sides are nothing but de Rham objects and Higgs objects. However, these two classes of objects are not classical in the sense that the integrable connections and Higgs fields appear in their PD forms. The only classical objects involved in this equivalence are those nilpotent of level less than p.

In a recent work [12], the authors give a more explicit construction of the inverse Cartier transform. An obvious advantage of their construction is the disappearance of the splitting module $\mathscr{B}_{\mathscr{X}/\mathscr{S}}$ used in [17]. In the second section of this chapter, we will give a sketch of their construction and more importantly prove this construction is equivalent to the inverse Cartier transform given in [17].

In the second chapter we will first introduce log Cartier transform constructed by Schepler in his thesis [18]. The objects involved in the most general version of the log Cartier transform are more cumbersome compared to the smooth case, as an extra action by index algebras on these objects is added. Since our main concern are classical log objects, it is desirable to find conditions under which such action can be relieved. To this end, we will introduce residue condition in the first section. Then in the second section we will review log Cartier transforms, especially the version in which the residue condition is satisfied. In the third section, we will generalize Lan-Sheng-Zuo's construction to the log case. As an application of this construction of log inverse Cartier transform, we will give a definition of log Fontaine modules.

In the third chapter we will introduce F-T-crystals [16] and prove the main result of this work. In the first section we define F-spans, which is necessary for generalizing F-crystals to more general bases. The second section is on T-crystals, which can be regarded as a crystal together with a filtration by submodules (not necessarily subcrystals) satisfying a certain transversality condition analogous to Griffiths transversality. In the last section we will give the definition of an F-T-crystal and more importantly prove our main result, i.e. Theorem 7.

In the two appendices, we collect some facts on log structures and crystalline sites that are used in the previous chapters. Many of these materials can be found in standard references, though some new remarks and omitted proofs in the original references are added.

Chapter 1

Cartier Transform and Fontaine Modules

In [17], based on the fact that the sheaf of PD-differential operators is an Azumaya algebra, Cartier transform and its quasi-inverse are constructed as a Morita equivalence. A large part of this construction is devoted to finding a splitting module for the Azumaya algebra. On the other hand, in [12], the authors give an explicit construction of inverse Cartier functor firstly over affine open subsets by using Frobenius liftings then over the global base by gluing the local pieces by Taylor formula. We will prove their construction coincides with the more abstract one given by Ogus and Vologodsky. Moreover, we will reformulate the definition of Fontaine modules.

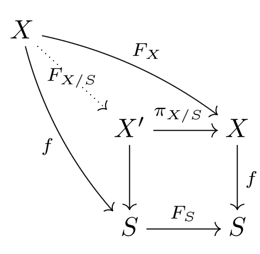
1.1 Cartier Transform and its Quasi-Inverse

In this section, we will study the structure of liftings of the relative Frobenius $F_{X/S}$. We will prove the sheaf of liftings of Frobenius is a crystal on $\mathrm{Cris}(X/S)$ and calculate the p-curvature of its associated connection. We will see these liftings provide splittings of the inverse Cartier operator, which can be used to construct a functor from Higgs objects to de Rham objects locally.

1.1.1 Cartier Isomorphism

First we recall the following well-known theorem due to Cartier.

Theorem 1. *Let $f : X \to S$ be a smooth morphism between schemes over k with relative dimension n, $F_S : S \to S$ (resp. $F_X : X \to X$) be the absolute Frobenius of S (resp. of X) and $X' = X \times_S S$ be the fiber product. The morphisms $\pi_{X/S} : X' \to X$ and $F_{X/S} : X \to X'$ are defined in a natural way as exhibited in the following commutative diagram.*



(1.1.1)

Then for any $0 \leq i \leq n$ we have a unique isomorphism

$$C^{-1}_{X/S} : \Omega^i_{X'/S} \xrightarrow{\cong} \mathscr{H}^i(F_{X/S*}\Omega^\bullet_{X/S}) \tag{1.1.2}$$

such that

1. $C^{-1}_{X/S}(1) = 1$

2. $C^{-1}_{X/S}(\omega \wedge \tau) = C^{-1}_{X/S}(\omega) \wedge C^{-1}_{X/S}(\tau)$

3. $C^{-1}_{X/S}(d\pi^*_{X/S}(a)) =$ *the class of* $a^{p-1}da$ *in* $\mathscr{H}^1(F_{X/S*}\Omega^\bullet_{X/S})$

where ω, τ *are relative differential forms and* a *is a local section of* $\mathcal{O}_X$.

Splitting of the Cartier Isomorphism

Let $Z^1_{X/S} := \ker(\Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S})$, then we have the following surjection

$$F_{X/S*}Z^1_{X/S} \twoheadrightarrow \mathscr{H}^1(F_{X/S*}\Omega^\bullet_{X/S}) \xrightarrow{C_{X/S}} \Omega^1_{X'/S}. \tag{1.1.3}$$

A result of Mazur [15] says that a lifting of the relative Frobenius $F_{X/S}$ over $W_2(k)$ provides a section of the composite of (1.1.3). More precisely, let $\tilde{F} : \tilde{X} \to \tilde{X}'$ be such a lifting (see Definition 1.1.2), then $\tilde{F}$ induces a morphism

$$d\tilde{F} : \Omega^1_{\tilde{X}'/\tilde{S}} \to \tilde{F}_*\Omega^1_{\tilde{X}/\tilde{S}}$$

Note that the above morphism is 0 after modulo p, therefore its image falls in $p\tilde{F}_*\Omega^1_{\tilde{X}/\tilde{S}}$ hence $d\tilde{F}$ induces a morphism from $\Omega^1_{X'/S}$ to $p\tilde{F}_*\Omega^1_{\tilde{X}/\tilde{S}}$. Moreover, the coefficient p in the image of $d\tilde{F}$ can be divided out, i.e. $d\tilde{F}$ can be written as $[p]\zeta_{\tilde{F}}$, where

$$\zeta_{\tilde{F}} : \Omega^1_{X'/S} \to F_{X/S*}\Omega^1_{X/S}. \tag{1.1.4}$$

and $[p]$ is given by the following

Lemma 1.1.1. *Let* M *be a flat* $\mathbb{Z}/p^2\mathbb{Z}$*-module, then the endomorphism of multiplication by* p

$$M \xrightarrow{p} M$$

defines an isomorphism

$$[p] : M/pM \cong pM.$$

Proof. Omitted. ∎

If $\tilde{F}(1 \otimes \tilde{a}) = \tilde{a}^p + p\tilde{b}$, where $\tilde{a}, \tilde{b} \in \mathcal{O}_{\tilde{X}}$, then it is easy to check

$$\zeta_{\tilde{F}}(1 \otimes a) = a^{p-1}da + db. \tag{1.1.5}$$

1.1.2 The Structure of Liftings of Frobenius

Definition 1.1.2. *Let* $f : X \to S$ *be a morphism between schemes in characteristic* $p > 0$*, a lifting of* f *modulo* p^n *is a morphism* $\tilde{f} : \tilde{X} \to \tilde{S}$ *between flat* $\mathbb{Z}/p^n\mathbb{Z}$*-schemes, fitting into the following Cartesian square*

$$\begin{array}{ccc} X & \longrightarrow & \tilde{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ S & \xrightarrow{i_S} & \tilde{S} \end{array}$$

where $i_S : S \to \tilde{S}$ *is a closed immersion.*

The liftings of the relative Frobenius $F_{X/S}$ is the main concern of this subsection. The following result in deformation theory will be used in the sequel.

Proposition 1.1.3. *[8, Theorem 5.9] Let* X *be a* S*-scheme and* $j : X_0 \to X$ *be a closed immersion defined by an ideal* J *such that* $J^2 = 0$*. Let* Y *be a smooth* S*-scheme and* $g : X_0 \to Y$ *be an* S*-morphism. There is an obstruction* $o(g, j) \in H^1(X_0, J \otimes_{\mathcal{O}_{X_0}} g^*T_{Y/S})$ *whose vanishing is necessary and sufficient for the existence of an* S*-morphism* $h : X \to Y$ *extending* g*, i.e. such that* $hj = g$*. When* $o(g, j) = 0$*, the set of extensions* h *of* g *is an affine space under* $H^0(X_0, J \otimes_{\mathcal{O}_{X_0}} g^*T_{Y/S})$.

Liftings of Frobenius as a Vector Group Torsor on Crystalline Site

Let $\tilde{S}$ be a flat lifting of S to $W_2(k)$ with its ideal $p\mathcal{O}_{\tilde{S}}$ endowed with the natural PD structure and $\mathrm{Cris}(X/\tilde{S})$ be the crystalline site. We will fix a flat lifting $\tilde{X}'$ of X' over $\tilde{S}$. As a convention, we occasionally use a locally free sheaf to denote the vector group scheme it defines. For instance, we use a locally free sheaf T on X to denote the affine group scheme $\mathrm{Spec}_X S^\bullet T^\vee$.

Let U be an open affine subscheme of X, $(U, \tilde{T}, i)$ be an object of $\mathrm{Cris}(X/\tilde{S})$ and $T = \tilde{T} \times_{\tilde{S}} S$, then the relative Frobenius $F_{T/S} : T \to T'$ factors through U'. Indeed, if we denote by I the ideal of $\mathcal{O}_T$ defining U, then for any element $a \in I$ we have $a^p = 0$ by the PD structure on I hence the claim follows. The induced morphism $T \to U'$ will be denoted by $f_{T/S}$.

Given a flat lifting $\tilde{T}$ of T to $\tilde{S}$, let $L_{\tilde{T}}$ be the set of liftings of $f_{T/S}$ to $\tilde{T}$. Then by proposition (1.1.3), if the set $L_{\tilde{T}}$ is nonempty, then it has a structure of torsor under the vector group $f^*_{T/S} T_{X'/S}$. Let $\mathrm{Cris}_f(X/\tilde{S})$ be the full subsite of $\mathrm{Cris}(X/\tilde{S})$ consisting of objects $(U, \tilde{T}, i)$ with $\tilde{T}$ flat over $\tilde{S}$, then the liftings of Frobenius form a sheaf of sets on $\mathrm{Cris}_f(X/\tilde{S})$. Firstly, let $\mathcal{L}_{\tilde{T}}$ be the sheaf on $\tilde{T}$ associate to the presheaf

$$\tilde{T}_1 \mapsto L_{\tilde{T}_1} := \{\text{Liftings of } f_{T_1/S} \text{ to } \tilde{T}_1\},$$

where $\tilde{T}_1$ is a open subscheme of $\tilde{T}$ and $T_1 = \tilde{T}_1 \times_{\tilde{S}} S$. If we are given a morphism

$$g : (U_1, \tilde{T}_1, i_1) \to (U_2, \tilde{T}_2, i_2),$$

between two objects of $\mathrm{Cris}_f(X/\tilde{S})$, then the transition map of the sheaf of lifting of Frobenius is given by

$$g^{-1}(\mathcal{L}_{\tilde{T}_2}) \to \mathcal{L}_{\tilde{T}_1}, \quad \tilde{F} \mapsto g \circ \tilde{F}.$$

If we denote this sheaf by $\mathcal{L}_{X/S}$, then for any object $(U, \tilde{T}, i)$ of $\mathrm{Cris}_f(X/S)$, by proposition 1.1.3 the restriction of $\mathcal{L}_{X/S}$ to $(U, \tilde{T}, i)$ is endowed with a $f^*_{T/S} T_{X'/S}$-torsor structure, which makes it an affine scheme over $\tilde{T}$. Let $\mathscr{A}_{(U,\tilde{T},i)}$ be the structure sheaf of this affine scheme, then by the transition map defined above one can see easily the sheaf of $\mathcal{O}_{X/\tilde{S}}$-module $\mathscr{A}$ defined by

$$(U, \tilde{T}, i) \mapsto \mathscr{A}_{(U,\tilde{T},i)}$$

form a crystal over $\mathrm{Cris}_f(X/S)$. The following lemma implies $\mathscr{A}$ is a crystal of $\mathcal{O}_{X/\tilde{S}}$-module over $\mathrm{Cris}(X/\tilde{S})$.

Lemma 1.1.4. *[17, Lemma 1.3] The categories of crystals of $\mathcal{O}_{X/\tilde{S}}$-modules over $\mathrm{Cris}_f(X/\tilde{S})$ and $\mathrm{Cris}(X/\tilde{S})$ are equivalent.*

Furthermore, as one can identify the category of crystals of $\mathcal{O}_{X/S}$-modules on $\mathrm{Cris}(X/S)$ and the category crystals of p-torsion $\mathcal{O}_{X/\tilde{S}}$-modules on $\mathrm{Cris}(X/\tilde{S})$, $\mathscr{A}$ is a crystal of $\mathcal{O}_{X/S}$-modules. The sheaf of liftings of Frobenius as a vector group torsor on $\mathrm{Cris}(X/S)$ will be denoted by $\mathscr{L}$. Then the $F^*_{X/S} T_{X'/S}$-torsor $\mathscr{L}_X$ is nothing but $\mathrm{Spec}\, \mathscr{A}_{\mathscr{X}/\mathscr{S}}$, where $\mathscr{A}_{\mathscr{X}/\mathscr{S}}$ is the value of $\mathscr{A}$ over X. In particular, let $\pi_{\mathscr{L}_X} : \mathscr{L}_X \to X$, then $\pi_{\mathscr{L}_X *} \mathcal{O}_{\mathscr{L}_X} \cong \mathscr{A}_{\mathscr{X}/\mathscr{S}}$ is endowed with an integrable connection. The objective of the rest of this subsection is to investigate this connection and its p-curvature.

A Natural Filtration on $\pi_{\mathscr{L}_X *} \mathcal{O}_{\mathscr{L}_X}$

The filtration we will construct on $\pi_{\mathscr{L}_X *} \mathcal{O}_{\mathscr{L}_X}$ is obtained by using Taylor expansions of its sections with respect to translation invariant differential operators (1.1.9). We will work in a little more general setting.

Let $\pi_{\mathbf{T}} : \mathbf{T} \to X$ be a vector group scheme over X and T be the sheaf of sections of $\mathbf{T}$ over X, then $\mathbf{T}$ can be written as $\mathrm{Spec}_X(S^\bullet \Omega)$, where $\Omega := \mathscr{H}om_{\mathcal{O}_X}(T, \mathcal{O}_X)$. Note that the natural pairing $T \times \Omega \to \mathcal{O}_X$ can be extended to an action of T on $S^\bullet \Omega$ by Leibnitz's rule. In other words, this action defines a map

$$T \to \pi_{\mathbf{T}*} T_{\mathbf{T}/X}, \qquad \xi \mapsto D_\xi, \tag{1.1.6}$$

where $T_{\mathbf{T}/X}$ is the tangent bundle of $\mathbf{T}$ relative to X. Moreover, the image of T under this map are exactly those translate invariant differential operators and we will identify T with its image in the sequel.

The action of T on $S^\bullet\Omega$ can be extended to higher order differential operators in the natural way. Since we are working in characteristic p, for any section ω of $S^\bullet\Omega$ and a differential operator D we have $D(\omega) = 0$ as long as the order of D is larger than pd, $d = \dim(X/S)$. Therefore the action

$$S^\bullet T \times S^\bullet\Omega \to S^\bullet\Omega$$

is discontinuous if we endow $S^\bullet T$ and $S^\bullet\Omega$ with the topology defined by the decreasing filtration $\{\bigcup_{i\geq n} S^i T\}_{n\geq 0}$ and discrete topology respectively.

Let $\Gamma_\bullet T := \sum_n \Gamma_n T$ ([2, A10]), then we still have a canonical paring

$$\Gamma_n T \times S^{n+m}\Omega \to S^m\Omega. \tag{1.1.7}$$

Moreover, by endowing the ring $\Gamma_\bullet T$ with the topology defined by PD-filtration $\{I^{[k]}\}_{k\in\mathbb{N}}$ [1] and $S^\bullet\Omega$ the discrete topology, the action (1.1.7) is continuous. This action can be extended to the completion $\hat{\Gamma}_\bullet T$ of $\Gamma_\bullet T$ with respect to the PD-filtration defined above.

Let ξ be a section of the morphism $\pi_{\mathbf{T}} : \mathbf{T} \to X$, and $t_\xi : \mathbf{T} \to \mathbf{T}$ be the induced translation map. Recall that D_ξ (1.1.6) is a translate invariant differential operator, then the exponential e^{D_ξ} now makes sense as an element of $\hat{\Gamma}_\bullet T$. Its action on $S^\bullet\Omega$ is given by Taylor's formula

$$t_\xi^*(f) = e^{D_\xi}(f). \tag{1.1.8}$$

Now given a $\mathbf{T}$-torsor $\mathbf{L}$ and $\pi_{\mathbf{L}} : \mathbf{L} \to X$, we can define a filtration on $\pi_{\mathbf{L}*}\mathcal{O}_{\mathbf{L}}$ as follows

$$N_n\pi_{\mathbf{L}*}\mathcal{O}_{\mathbf{L}} := \{x \in \pi_{\mathbf{L}*}\mathcal{O}_{\mathbf{L}} \mid D(x) = 0 \text{ for any } D \in I^{[n+1]}\Gamma_\bullet T\}. \tag{1.1.9}$$

The grading $Gr_N\pi_{\mathbf{L}*}\mathcal{O}_{\mathbf{L}}$ of $\pi_{\mathbf{L}*}\mathcal{O}_{\mathbf{L}}$ with respect to the above filtration is canonically isomorphic to $S^\bullet\Omega$. In particular, we have

$$0 \longrightarrow \mathcal{O}_X \longrightarrow N_1\pi_{\mathbf{L}*}\mathcal{O}_{\mathbf{L}} \longrightarrow \Omega \longrightarrow 0. \tag{1.1.10}$$

The element in $\mathrm{Ext}^1(\Omega, \mathcal{O}_X)$ corresponding to the above exact sequence is exactly the torsor $\mathbf{L}$.

If we are given a section l of $\mathbf{L}$, then l defines an isomorphism locally by

$$s_l : \mathbf{L} \to \mathbf{T}, \quad l' \mapsto l' - l,$$

hence locally an isomorphism

$$\sigma_l : S^\bullet\Omega \cong \pi_{\mathbf{L}*}\mathcal{O}_{\mathbf{L}}. \tag{1.1.11}$$

It is easy to see this isomorphism is the unique isomorphism of $\mathcal{O}_X$-algebras such that

$$\sigma_l(\omega)(l') = \langle \omega, l' - l\rangle$$

for any section l' of $\mathbf{L}$ and $\omega \in \Omega$.

Now we go back to the sheaf of liftings of Frobenius. If the we take $\mathbf{T}$ and $\mathbf{L}$ to be the vector group schemes associated to $F^*_{X/S}T_{X'/S}$ and $\mathscr{L}_X$, then from (1.1.11) we get

Proposition 1.1.5. *Let $\tilde{T}$ be an object of $\mathrm{Cris}_f(X/\tilde{S})$, $\tilde{F} : \tilde{T} \to \tilde{X}'$ be a lift of $f_{T/S}$. For any lifting $\tilde{a}' \in \mathcal{O}_{\tilde{X}'}$ of a section $a' \in \mathcal{O}_{X'}$ and lifting $\tilde{F}' : \tilde{T} \to \tilde{X}'$ of $f_{T/S}$, we have*

$$\begin{aligned}\sigma_{\tilde{F}}(f^*_{T/S}da')(\tilde{F}') &= \langle f^*_{T/S}da', \tilde{F}' - \tilde{F}\rangle \\ &= \frac{1}{[p]}(\tilde{F}'(\tilde{a}') - \tilde{F}(\tilde{a}')).\end{aligned}$$

Note the filtration $\{N_n\}_{n\geq 1}$ defined on $\mathscr{A}_{\mathscr{X}/\mathscr{S}} = \pi_{\mathscr{L}_X*}\mathcal{O}_{\mathscr{L}_X}$ can be extended to $\mathscr{A}$.

[1] A general member of $I^{[k]}$ is given by $x_1^{[i_1]}x_2^{[i_2]}\cdots x_n^{[i_n]}, i_1 + i_2 + \cdots + i_n \geq k$.

The Integrable Connection and p-Curvature

Next we will apply the results in the preceding paragraphs to the sheaf of liftings of Frobenius, which is an $F^*_{X/S}T_{X'/S}$-torsor. Now the exact sequence (1.1.10) becomes

$$0 \longrightarrow \mathcal{O}_X \longrightarrow N_1\pi_{\mathscr{L}_X *}\mathcal{O}_{\mathscr{L}_X} \longrightarrow F^*_{X/S}\Omega^1_{X'/S} \longrightarrow 0. \tag{1.1.12}$$

By construction, $N_1\pi_{\mathscr{L}_X *}\mathcal{O}_{\mathscr{L}_X}$ as a submodule of $\pi_{\mathscr{L}_X *}\mathcal{O}_{\mathscr{L}_X}$ is horizontal with respect to the connection on $\pi_{\mathscr{L}_X *}\mathcal{O}_{\mathscr{L}_X}$ associated to the crystal $\mathscr{A}$. Let $\tilde{U}$ be a flat lifting of some open subset U of X, $\tilde{F}$ be a lifting of $F_{U/S}$, and $\sigma_{\tilde{F}}$ be the splitting of (1.1.12) over U as defined in (1.1.11). Then for any local section ω' of $\Omega^1_{X'/S}$, we can prove

$$\nabla(\sigma_{\tilde{F}}(1\otimes\omega')) = -\zeta_{\tilde{F}}(\omega'). \tag{1.1.13}$$

Indeed, first observe that both sides of the formula are $\mathcal{O}_{X'}$-linear, hence we can assume $\omega' = da'$ for some section a' of $\mathcal{O}_{X'}$. Let $\tilde{T}$ be the diagonal of $\tilde{U}\times\tilde{U}$, $\tilde{h}_i : \tilde{T} \to \tilde{U}$ be the projection to the i-th factor and T be the reduction of $\tilde{T}$ modulo p. Let h_i be the reduction of $\tilde{h}_i$ modulo p, then the crystal structure gives an isomorphism

$$h_1^*N_1\pi_{\mathscr{L}_U *}\mathcal{O}_{\mathscr{L}_U} \cong N_1\pi_{\mathscr{L}_T *}\mathcal{O}_{\mathscr{L}_T} \cong h_2^*N_1\pi_{\mathscr{L}_U *}\mathcal{O}_{\mathscr{L}_U}.$$

By the definition of a connection associated to a crystal ([9, Theorem 6.2]) (note that here the PD-ideal has zero square), we have

$$\nabla(\sigma_{\tilde{F}}(1\otimes da')) = h_2^*(\sigma_{\tilde{F}}(1\otimes da')) - h_1^*(\sigma_{\tilde{F}}(1\otimes da')).$$

Since the right side is an affine function on $\mathscr{L}_T$, to prove (1.1.13) it suffices to evaluate its value at sections of $\mathscr{L}_T$. Note that $\tilde{F}_i = \tilde{F}\circ\tilde{h}_i \in \mathscr{L}_T$, for any lifting $\tilde{F}'$ of $f_{T/S}$, we have

$$\begin{aligned}
h_2^*(\sigma_{\tilde{F}}(1\otimes da'))(\tilde{F}') - h_1^*(\sigma_{\tilde{F}}(1\otimes da'))(\tilde{F}') &= h_2^*(\sigma_{\tilde{F}}(1\otimes da'))(\tilde{F}') - h_1^*(\sigma_{\tilde{F}}(1\otimes da'))(\tilde{F}')\\
&= \sigma_{\tilde{F}_2}(1\otimes da')(\tilde{F}') - \sigma_{\tilde{F}_1}(1\otimes da')(\tilde{F}')\\
&= \frac{1}{[p]}(\tilde{F}'(1\otimes\tilde{a}') - \tilde{F}_2(1\otimes\tilde{a}')) - \frac{1}{[p]}(\tilde{F}'(1\otimes\tilde{a}') - \tilde{F}_1(1\otimes\tilde{a}'))\\
&= \frac{1}{[p]}(\tilde{F}_1(1\otimes\tilde{a}') - \tilde{F}_2(1\otimes\tilde{a}'))\\
&= -\frac{1}{[p]}(d\tilde{F}(\tilde{a}'))\\
&= -\zeta_{\tilde{F}}(da').
\end{aligned}$$

The above computation implies in particular the image of the connection

$$\nabla : N_1\pi_{\mathscr{L}_X *}\mathcal{O}_{\mathscr{L}_X} \to N_1\pi_{\mathscr{L}_X *}\mathcal{O}_{\mathscr{L}_X}\otimes\Omega^1_{X/S}$$

falls in $\Omega^1_{X/S}$. Next we compute the p-curvature of this connection. Given a Frobenius lifting $\tilde{F}$ of $F_{X/S}$, let

$$N_1\pi_{\mathscr{L}_X *}\mathcal{O}_{\mathscr{L}_X} \cong \mathcal{O}_X \oplus F^*_{X/S}\Omega^1_{X'/S}$$

be the splitting of (1.1.12) determined by $\tilde{F}$. It is easy to see the sections of $N_1\pi_{\mathscr{L}_X *}\mathcal{O}_{\mathscr{L}_X}$ with nonzero images under the p-curvature map come from $F^*_{X/S}\Omega^1_{X/S}$. Following the notations of the previous computation, let $D \in T_{X/S}$ be a nonzero tangent vector, $a \in \mathcal{O}_X$ and $\zeta_{\tilde{F}}(d\pi^*_{X/S}a) = a^{p-1}da + db$, then we have

$$\begin{aligned}
&\nabla(D)^p(1\otimes d\pi^*_{X/S}a) - \nabla(D^{(p)})(1\otimes d\pi^*_{X/S}a)\\
=&\nabla(D)^{p-1}(-\langle D, a^{p-1}da + db\rangle) + \langle D^{(p)}, a^{p-1}da + db\rangle \qquad (1.1.14)\\
=&-\nabla(D)^{p-1}(a^{p-1}Da + Db) + a^{p-1}D^{(p)}a + D^{(p)}b\\
=&-\nabla(D)^{p-1}(a^{p-1}Da + Db) + a^{p-1}D^{(p)}a + D^{(p)}b\\
=&-\nabla(D)^{p-1}(a^{p-1}Da) + a^{p-1}D^{(p)}a\\
=&-(a^{p-1}D^{(p)}a - (Da)^p) + a^{p-1}D^{(p)}a \qquad (1.1.15)\\
=&(Da)^p
\end{aligned}$$

In the above computation, we apply the formula (1.1.13) in (1.1.14), and the Hochschild formula [19, Lemma 2] in (1.1.15). Therefore, the image of $(0, 1\otimes d\pi^*_{X/S}a)$ under the p-curvature map

$$\psi : \mathcal{O}_X \oplus F^*_{X/S}\Omega^1_{X'/S} \to (\mathcal{O}_X \oplus F^*_{X/S}\Omega^1_{X'/S}) \otimes_{\mathcal{O}_X} F^*_{X/S}\Omega^1_{X'/S}$$

is exactly $(1,0)\otimes(1\otimes d\pi^*_{X/S}a)$. It is easy to see the p-curvature map ψ coincides with the restriction of the usual differential $d_{\mathscr{L}_X/X} : \pi_{\mathscr{L}_X *}\mathcal{O}_{\mathscr{L}_X} \to \pi_{\mathscr{L}_X *}\Omega^1_{\mathscr{L}_X/X}$ to $N_1\pi_{\mathscr{L}_X *}\mathcal{O}_{\mathscr{L}_X}$.

The connection ∇ on $\mathscr{A}_{\mathscr{X}/\mathscr{S}}$ is nothing but the connection on $N_1\pi_{\mathscr{L}_X *}\mathcal{O}_{\mathscr{L}_X}$ extended by Leibnitz's rule.

1.1.3 Cartier Transform as Morita Equivalence

The construction of Cartier transform and its quasi-inverse is based on the fact that the sheaf of PD differential operators is an Azumaya algebra over $\mathbf{T}^*_{X'/S}$ (the affine scheme over X' associated to the relative cotangent bundle). As long as we can find a splitting module for it, the construction will follow immediately from the general formalism of Morita equivalence. Note that in the sequel we will use "Cartier transform" to mean "Cartier transform and its quasi-inverse" for short, which should be clear in the context.

Azumaya Nature of the Rings of PD Differential Operators

First we recall some facts on Azumaya algebra. Given a scheme Y, an Azumaya algebra on Y is a sheaf of associative algebra A, which is isomorphic to $\mathrm{End}_{\mathcal{O}_Y}(\mathcal{O}_Y^n)$ locally in the fppf topology. One can prove to give an Azumaya algebra A on Y is equivalent to give a sheaf of associative $\mathcal{O}_Y$-algebra A, which is locally free of finite rank as $\mathcal{O}_Y$-module and the canonical map

$$A \otimes A^{op} \to \mathrm{End}_{\mathcal{O}_Y}(A)$$

is an isomorphism.

Following notations as above, let $D_{X/S}$ be the sheaf of PD-differential operators on X. Then a first order PD-differential operator D is nothing but a derivation $D : \mathcal{O}_X \to \mathcal{O}_X$. Since we are working in characteristic p, the p-th iterative of D, denoted by $D^{(p)}$, is still a derivation. Let $c(D) := D^p - D^{(p)}$, then one can verify $c(gD) = g^p c(D)$ for any $g \in \mathcal{O}_X$, by adjointness, we have a $\mathcal{O}_{X'/S}$-linear morphism

$$c' : T_{X'/S} \to F_{X/S*}D_{X/S}, \qquad D' \mapsto (D'\otimes 1)^p - (D'\otimes 1)^{(p)}. \tag{1.1.16}$$

This morphism is nothing but the p-curvature map if we identify the category of $D_{X/S}$-modules with the category $\mathrm{MIC}(X/S)$ whose objects are quasi-coherent $\mathcal{O}_X$-modules endowed with an integrable connection [2, Theorem 4.8]. Moreover the image of c' falls in the center $\mathscr{Z}_{X/S}$ of $D_{X/S}$, hence $F_{X/S*}D_{X/S}$ can be regarded as a sheaf of algebra on $\mathbf{T}^*_{X'/S}$. We will denote it by $\mathscr{D}_{X/S}$.

The following proposition, firstly proved in [3], is the pillar stone of the construction of Cartier transform.

Proposition 1.1.6. *The above morphism* (1.1.16) *induces an isomorphism* $S^\bullet T_{X'/S} \cong F_{X/S*}\mathscr{Z}_{X/S}$. *This morphism makes* $\mathscr{D}_{X/S}$ *an Azumaya algebra over* $\mathbf{T}^*_{X'/S}$ *of rank* p^{2d}, *where* d *is relative dimension* $\dim(X/S)$.

Now we have an equivalent description of the category $\mathrm{MIC}(X/S)$ as follows. Firstly, the morphism $F_{X/S}$ is an homeomorphism of the underlying topological spaces, therefore the category $\mathrm{MIC}(X/S)$ is equivalent to the category of $(X', F_{X/S*}D_{X/S})$-modules, where $(X', F_{X/S*}D_{X/S})$ is a ringed space consisting of the topological space X' and the sheaf of rings $F_{X/S*}D_{X/S}$. Furthermore, by the definition of $\mathscr{D}_{X/S}$ the latter is easily seen to be equivalent to the category of $(\mathbf{T}^*_{X'/S}, \mathscr{D}_{X/S})$-modules.

The Splitting Module

In order to find a splitting module for the Azumaya algebra $\mathscr{D}_{X/S}$, it suffices to find an object in $\mathrm{MIC}(X/S)$ whose rank over the center of $\mathscr{D}_{X/S}$ is equal to p^d. Recall that we have obtained a sheaf of algebra

$$\mathscr{A}_{\mathscr{X}/\mathscr{S}} = \pi_{\mathscr{L}_X *}\mathcal{O}_{\mathscr{L}_X}$$

on X, which is endowed with an integrable connection. However it is not locally free as an $S^\bullet T_{X'/S}$-module via the p-curvature map. Indeed for any section a of $\mathscr{A}_{\mathscr{X}/\mathscr{S}}$ one can find $D \in S^\bullet T_{X'/S}$ of sufficiently high degree such that $c'(D)(a) = 0$. On the other hand, $\mathscr{B}_{\mathscr{X}/\mathscr{S}}$, the dual of $\mathscr{A}_{\mathscr{X}/\mathscr{S}}$ with respect to the pairing (1.1.7) is a $F^*_{X/S}\hat{\Gamma}_\bullet T_{X'/S}$-module locally free of rank 1. If we enlarge $D_{X/S}$ to

$$D^\gamma_{X/S} := D_{X/S} \otimes_{S^\bullet T_{X'/S}} \hat{\Gamma}_\bullet T_{X'/S},$$

then its center becomes $\hat{\Gamma}_\bullet T_{X'/S}$. In addition, $\mathscr{A}_{\mathscr{X}/\mathscr{S}}$ acquires a $D^\gamma_{X/S}$-module structure by (1.1.7), hence its dual $\mathscr{B}_{\mathscr{X}/\mathscr{S}}$ is also a $D^\gamma_{X/S}$-module. Moreover, since $\mathscr{B}_{\mathscr{X}/\mathscr{S}}$ is locally free of rank 1 as a $F^*_{X'/S}\hat{\Gamma}_\bullet T_{X'/S}$-module, it is locally free of rank p^d as a $\hat{\Gamma}_\bullet T_{X'/S}$-module. Thus it is a splitting module for the Azumaya algebra $F_{X/S*}D^\gamma_{X/S}$ over $\hat{\Gamma}_\bullet T_{X'/S}$. Note however that the enlarged Azumaya algebra no longer lives on the scheme $\mathbf{T}^*_{X'/S}$, but on the formal scheme $\hat{\mathbf{T}}^{*\gamma}_{X'/S}$, for details see [17, page 31].

Now let $\mathrm{MIC}_\gamma(X/S)$ (resp. $\mathrm{HIG}_\gamma(X'/S)$) be the category of quasi-coherent $\mathcal{O}_X$-modules endowed with an action of $D^\gamma_{X/S}$ (resp. the category of quasi-coherent $\mathcal{O}_{X'}$-modules endowed with an action of $\hat{\Gamma}_\bullet T_{X'/S}$). Based on the discussion above, the following theorem follows from Morita equivalence.

Theorem 2. *([17, Theorem 2.8]) Let $\mathscr{X}/\mathscr{S} = (X/S, \tilde{X}'/\tilde{S})$ be a pair consisting of a smooth S-scheme X together with a flat lift $\tilde{X}'/\tilde{S}$ of X'/S modulo p^2. The functor*

$$\begin{aligned} C_{\mathscr{X}/\mathscr{S}} : \mathrm{MIC}_\gamma(X/S) &\longrightarrow \mathrm{HIG}_\gamma(X'/S), \\ E &\longmapsto \iota^*\mathscr{H}om_{D^\gamma_{X/S}}(\mathscr{B}_{\mathscr{X}/\mathscr{S}}, E) \end{aligned}$$

defines an equivalence of categories, with quasi-inverse

$$\begin{aligned} C^{-1}_{\mathscr{X}/\mathscr{S}} : \mathrm{HIG}_\gamma(X'/S) &\longrightarrow \mathrm{MIC}_\gamma(X/S), \\ E' &\longmapsto \mathscr{B}_{\mathscr{X}/\mathscr{S}} \otimes_{\hat{\Gamma}_\bullet T_{X'/S}} \iota^*E', \end{aligned}$$

where ι is an endomorphism of $\hat{\mathbf{T}}^{*\gamma}_{X'/S}$ induced from the inverse operation of the group scheme $\mathbf{T}^*_{X'/S}$.

1.2 Inverse Cartier Transform and Fontaine Modules

The construction of the functors in Theorem 2 follows from the general formalism for Morita equivalence. The bewildering appearance of these functors makes it difficult for one to comprehend. In this subsection, we will introduce a construction of inverse Cartier functor in [12] based on Frobenius liftings rather than $\mathscr{B}_{\mathscr{X}/\mathscr{S}}$. Moreover, we will prove this construction is equivalent to $C^{-1}_{\mathscr{X}/\mathscr{S}}$ in Theorem 2. As an application, we reformulate the definition of Fontaine modules [17, Definition 4.16].

1.2.1 Lan-Sheng-Zuo's Construction of Inverse Cartier Transform

The construction of inverse Cartier transform given in [12] is first done locally by using Frobenius liftings then glued into a global object by using Taylor formula. We first review the local construction.

For any $\mathcal{O}_{X'}$-module E', it is well known that there is a canonical connection ∇_{can} on $F^*_{X/S}E'$ defined by $\nabla_{can}(a \otimes e') = da \otimes e'$. Note that E' can be regarded as a Higgs module endowed with zero Higgs field. Now if we are given a lifting of $F_{X/S}$ to $W_2(k)$, we can construct an integrable connection on $F^*_{X/S}E'$ for any Higgs module (E', θ) on X'. The construction is based on the elementary fact that the difference of two integrable connections on a fixed $\mathcal{O}_{X'}$-module is a Higgs field.

Let $\tilde{F}$ be the lifting of $F_{X/S}$, then the following composite defines a Higgs field on $F^*_{X/S}E'$

$$F^*_{X/S}E' \xrightarrow{F^*_{X/S}\theta} F^*_{X/S}E' \otimes F^*_{X/S}\Omega^1_{X'/S} \xrightarrow{1\otimes\zeta_{\tilde{F}}} F^*_{X/S}E' \otimes \Omega^1_{X/S}. \tag{1.2.1}$$

The new connection is defined to be

$$\nabla_{can} + (\mathrm{id} \otimes \zeta_{\tilde{F}}) \circ F_{X/S}^* \theta. \tag{1.2.2}$$

The pair consisting of $F_{X/S}^* E'$ together with the connection defined above will be denoted by

$$\Psi_{\tilde{F}}(E', \theta) \tag{1.2.3}$$

in the sequel. Note that this construction depends on choices of liftings of Frobenius.

Now let $S = \operatorname{Spec} k$ and suppose X is endowed with a $W_2(k)$-lifting $\tilde{X}$. Let (E, θ) be a Higgs bundle on X, then $\pi_{X/k}^*(E, \theta)$ is a Higgs bundle on X', where $\pi_{X/k} : X' \to X$ is the projection as shown in the diagram (1.1.1). Take a family of open affine subschemes $\{U_i\}_{i \in I}$ covering X, then locally they associate to $(E, \theta)_{U_i}$ a de Rham bundle $(F_X^* E_{U_i}, \nabla_{U_i})$, where ∇_{U_i} is the same way as the connection associated to $\pi_{X/k}^*(E, \theta)$ defined by (1.2.2). In other words, the locally associated de Rham bundle coincides with $\Psi_{\tilde{F}_{U_i}}(\pi_{X/k}^* E|_{U_i}, \pi_{X/k}^* \theta|_{U_i})$ (1.2.3), where $\tilde{F}_{U_i}$ is a lifting of F_{U_i} to $W_2(k)$.

In order to obtain a a global de Rham bundle, they choose a set of gluing matrices for the local de Rham bundles properly and prove these matrices satisfying gluing conditions for a flat bundle, i.e. a vector bundle endowed with an integrable connection.

Now we recall the definition of their gluing matrices. Let U_i, U_j be two open subschemes in the family $\{U_i\}_{i \in I}$, let e_{U_i} (resp. e_{U_j}) be a local basis for E_{U_i} (resp. E_{U_i}) and $M_{ij} \in GL_n(\mathcal{O}_{U_i \cap U_j})$ be the transition matrix, i.e. $e_{U_i} = M_{ij} e_{U_j}$. Let h_{ij} be the section of $\in F_X^* T_{U_i \cap U_j}$ determined by $\frac{dF_{U_i}}{[p]} - \frac{dF_{U_j}}{[p]} = dh_{ij}$ and

$$g_{ij} = \exp[h_{ij}(F_X^* \theta_{U_i})],$$

then the transition matrix of $\{\Psi_{\tilde{F}_{U_i}}(E, \theta)\}_{i \in I}$ is defined by

$$G_{ij} = g_{ij} F_X^* M_{ij}. \tag{1.2.4}$$

Then they show the matrices $\{G_{ij}\}_{i,j \in I}$ satisfies the following gluing conditions ([12, Theorem 3])

1. Over the open subset $U_i \cap U_j \cap U_k$, one has

$$G_{ij} G_{jk} = G_{ik}.$$

2. The connection gluing condition over $U_i \cap U_j$

$$\frac{dF_{U_i}}{[p]}(F_X^* \theta_{U_i}) = dG_{ij} G_{ij}^{-1} + G_{ij} \frac{dF_{U_j}}{[p]}(F_X^* \theta_{U_j}) G_{ij}^{-1}.$$

1.2.2 Equivalence of Lan-Sheng-Zuo and Ogus-Vologodsky

In this subsection, we will prove the inverse Cartier functor given by Lan-Sheng-Zuo is equivalent to the one given by Ogus and Vologodsky. In other words, given a Higgs bundle (E, θ) nilpotent of level than p, the associated de Rham bundle on X constructed above is nothing but $C_{\mathscr{X}/\mathscr{S}}^{-1}(\pi_{X/k}^*(E, \theta))$.

$C_{\mathscr{X}/\mathscr{S}}^{-1}$ Unscrewed

To prove the equivalence, we will replace $\mathscr{B}_{\mathscr{X}/\mathscr{S}}$ in definition of the functor $C_{\mathscr{X}/\mathscr{S}}^{-1}$ by some more down-to-earth terms.

Let $\{U_i\}_{i \in I}$ be a covering family of X, where U_i is an open subscheme of X admitting a lifting $\tilde{F}_i$ of the relative Frobenius $F_{U_i/S}$. Then $\tilde{F}_i$ provides an isomorphism (1.1.11) over U_i

$$\mathscr{A}_{\mathscr{U}_i/\mathscr{S}} \cong F_{U_i/S}^* S^\bullet \Omega_{X'/S}^1, \tag{1.2.5}$$

where $\mathscr{U}_i$ is the open subscheme of $\tilde{X}$ with the same underlying open subset as U_i. By taking duality we have an isomorphism

$$\mathscr{B}_{\mathscr{U}_i/\mathscr{S}} \cong \varinjlim \mathscr{H}om(\mathscr{A}_{\mathscr{U}_i/\mathscr{S}}/N_n\mathscr{A}_{\mathscr{U}_i/\mathscr{S}}, \mathcal{O}_X) \cong F^*_{U_i/S}\hat{\Gamma}_\bullet T_{X'/S}, \tag{1.2.6}$$

where $N_n\mathscr{A}_{\mathscr{U}_i/\mathscr{S}}$ is the filtration on $\mathscr{A}_{\mathscr{U}_i/\mathscr{S}}$ defined as in (1.1.9).

Now we analyze the $D^\gamma_{X/S}$-module structure of $\mathscr{B}_{\mathscr{U}_i/\mathscr{S}}$. Let $U_i' = U_i \times_X X'$, $\{\omega_i'\}_{1\leq i\leq d}$ be a local basis of $\Omega^1_{U_i'/S}$ and $\{\xi_i'\}_{1\leq i\leq d}$, $\xi_i' \in T_{U_i'/S}$ be its dual basis. Moreover, let $\xi_i = F^*_{U_i/S}(\xi_i')$ and $\omega_i = F^*_{U_i/S}(\omega_i')$ then $\{\xi_1^{[t_1]}\xi_2^{[t_2]}\cdots\xi_d^{[t_d]} | \sum_{1\leq k\leq d} t_k \leq n\}$ form a basis of $\mathscr{H}om(\mathscr{A}_{\mathscr{U}_i/\mathscr{S}}/N_{n+1}\pi_{\mathscr{A}*}\mathcal{O}_{\mathscr{A}}, \mathcal{O}_X)$. For any section D of $T_{X/S}$, we have

$$\begin{aligned}
&\langle \nabla_{\tilde{F}}(D)(\xi_1^{[t_1]}\xi_2^{[t_2]}\cdots\xi_d^{[t_d]}), \omega_1^{s_1}\omega_2^{s_2}\cdots\omega_d^{s_d}\rangle\\
=& -\langle \xi_1^{[t_1]}\xi_2^{[t_2]}\cdots\xi_d^{[t_d]}, \nabla_{\tilde{F}}(D)(\omega_1^{s_1}\omega_2^{s_2}\cdots\omega_d^{s_d})\rangle + \nabla_{\tilde{F}}(D)\langle \xi_1^{[t_1]}\xi_2^{[t_2]}\cdots\xi_d^{[t_d]}, \omega_1^{s_1}\omega_2^{s_2}\cdots\omega_d^{s_d}\rangle\\
=& -\langle \xi_1^{[t_1]}\xi_2^{[t_2]}\cdots\xi_d^{[t_d]}, \sum_{1\leq k\leq d}\nabla_{\tilde{F}}(D)(\omega_k)\omega_1^{s_1}\cdots\omega_k^{s_k-1}\cdots\omega_d^{s_d}\rangle + \nabla_{\tilde{F}}(D)\langle \xi_1^{[t_1]}\xi_2^{[t_2]}\cdots\xi_d^{[t_d]}, \omega_1^{s_1}\omega_2^{s_2}\cdots\omega_d^{s_d}\rangle
\end{aligned}$$

Since the second term is always zero, the above sum is nonzero only if the first term is, for which it is necessarily that $t_k = s_k + 1$ for some $1 \leq k \leq d$ and $s_l = t_l$ for $1 \leq l \leq d, l \neq k$. Let $\{D_k | 1 \leq k \leq d\}$ be a basis for $T_{U_i/S}$ and $\{d\alpha_k | 1 \leq k \leq d\}$ be its dual basis, then the action of $\nabla_{\tilde{F}}$ on $\xi_1^{[t_1]}\xi_2^{[t_2]}\cdots\xi_d^{[t_d]}$ is given by

$$\nabla_{\tilde{F}}(\xi_1^{[t_1]}\xi_2^{[t_2]}\cdots\xi_d^{[t_d]}) = -\sum_{1\leq k\leq d}\sum_{1\leq l\leq d}\nabla_{\tilde{F}}(D_k)(\omega_l)\xi_1^{[t_1]}\cdots\xi_l^{[t_l+1]}\cdots\xi_d^{[t_d]}\otimes d\alpha_k. \tag{1.2.7}$$

Next we can consider the $D^\gamma_{X/S}$-module structure of $C^{-1}_{\mathscr{X}/\mathscr{S}}(E')_{U_i}$. Firstly we have

$$\begin{aligned}
C^{-1}_{\mathscr{X}/\mathscr{S}}(E')_{U_i} = C^{-1}_{\mathscr{U}_i/\mathscr{S}}(E'_{U_i'}) &= \mathscr{B}_{\mathscr{U}_i/\mathscr{S}} \otimes_{\hat{\Gamma}_\bullet T_{U_i'/S}} \iota^* E'_{U_i'}\\
&\cong F^*_{U_i/S}\hat{\Gamma}_\bullet T_{X'/S} \otimes_{\hat{\Gamma}_\bullet T_{U_i'/S}} \iota^* E'_{U_i'}\\
&\cong F^*_{U_i/S}\iota^* E'_{U_i'}.
\end{aligned}$$

Take $1\otimes e \in C^{-1}_{\mathscr{X}/\mathscr{S}}(E')_{U_i}$ and let $t_i = 0, 1 \leq i \leq d$ in (1.2.7), then we get

$$\begin{aligned}
D_k(1\otimes e) &= -(\sum_{1\leq l\leq d}\nabla_{\tilde{F}}(D_k)(\omega_l)\xi_l)\otimes e\\
&= -\sum_{1\leq l\leq d}\nabla_{\tilde{F}}(D_k)(\omega_l)\otimes(-\xi_l'(e))\\
&= \sum_{1\leq l\leq d}\nabla_{\tilde{F}}(D_k)(\omega_l)\otimes\xi_l(e)
\end{aligned} \tag{1.2.8}$$

There is a minus sign in front of $\xi_l'(e)$ in (1.2.8) since the Higgs module $\iota^*E'_{U_i'}$ is nothing but $E'_{U_i'}$ with its Higgs field reversed. Now one can see the connection on $C^{-1}_{\mathscr{X}/\mathscr{S}}(E')_{U_i} \cong F^*_{U_i/S}E'_{U_i'/S}$ (as $\mathcal{O}_{U_i}$-modules) is nothing but

$$\nabla_{can} + \mathrm{id}\otimes\zeta_{\tilde{F}}\circ F^*_{U_i/S}\theta,$$

which coincides the construction (1.2.1), i.e. $\Psi_{\tilde{F}_i}(E',\theta)$.

One notices the above isomorphism depends on the choice of the lifting $\tilde{F}_i$. However, such ambiguity doesn't make an obstruction to obtain a global object in $\mathrm{MIC}_\gamma(X/S)$ by gluing the local objects $\{\Psi_{\tilde{F}_i}(E',\theta)\}_{i\in I}$. Indeed, let U_1, U_2 be two open subsets of X with nonempty intersection, $\tilde{F}_i$ be a lifting of $F_{U_i/S}$ to $\tilde{U}_i$ for $i = 1, 2$. Then the difference $\tilde{F}_2\tilde{j}_2 - \tilde{F}_1\tilde{j}_1$ defines an element ξ of $F^*_{U_1'\cap U_2'/S}T_{X/S}$, where $\tilde{j}_i : \tilde{U}_1 \times_{\tilde{S}} \tilde{U}_2 \to \tilde{U}_i$

is the i-th projection. Let $\sigma_{\tilde{F}_i}$ (see (1.1.11)) be the isomorphism (1.2.5) induced by $\tilde{F}_i$, then $\sigma_{\tilde{F}_1}\sigma_{\tilde{F}_2}^{-1}$ induces an automorphism of $F^*_{U'_1\cap U'_2/S}S^\bullet\Omega^1_{X'/S}$ given by the Taylor formula (1.1.8)

$$f \mapsto e^{\xi}(f).$$

By taking duality (see (1.1.7)) we obtain the following automorphism of $F^*_{U'_1\cap U'_2/S}\hat{\Gamma}_\bullet T_{X'/S}$

$$\alpha \mapsto \alpha e^{-\xi}.$$

For $i = 1, 2$, let $\varphi_i : C^{-1}_{\mathscr{X}/\mathscr{S}}(E')_{U_i} \cong F^*_{X/S}\iota^* E'_{U'_i}$, then we have the following isomorphism

$$\phi_{12} : \varphi_2(C^{-1}_{\mathscr{X}/\mathscr{S}}(E')_{U_2}) \to \varphi_1(C^{-1}_{\mathscr{X}/\mathscr{S}}(E')_{U_1}), \quad x \mapsto \exp(\theta_\xi)(x) = \sum_{k\geq 0} \frac{\theta^k_\xi(x)}{k!}, \quad x \in E'_{U_1\cap U_2}, \tag{1.2.9}$$

where $\theta_\xi = -(\mathrm{id}\otimes\xi)\circ F^*_{X/S}\theta$ and $\theta : E' \to E'\otimes\Omega^1_{X'/S}$ is the Higgs field.

It is easy to see the gluing morphisms $\{\phi_{ij}\}$ satisfies cocycle condition. So far we can give a more explicit though somehow circuitous way to describe $C^{-1}_{\mathscr{X}/\mathscr{S}}(E')$ for an object E' of $\mathrm{HIG}_\gamma(X'/S)$. This description applies in particular to any Higgs modules nilpotent of level less than or equal to $p-1$, as the terms in the sum on the right side of (1.2.9) make sense for all $k \leq p-1$.

The gluing formula (1.2.9) can be made more explicit in the following way. Let $\{\xi'_i\}_{1\leq i\leq d}$ be a local basis of $T_{X'/S}$ and $\{\omega'_i\}_{1\leq i\leq d}$ be its dual basis of $\Omega^1_{X'/S}$, then one can see easily $\theta^k_\xi(x)$ is given by

$$\sum_{\sum_i n_i = k} \langle\xi, 1\otimes\omega'_i\rangle^{n_i} \otimes \prod \theta^{n_i}_{\xi'_i}(x) \tag{1.2.10}$$

Theorem 3. *Let X be a smooth variety over k endowed with a flat lifting $\tilde{X}$ over $W_2(k)$ and (E,θ) be a Higgs bundle on X which is nilpotent of level less than p. Then the associated de Rham bundle on X constructed by Lan-Sheng-Zuo is isomorphic to the de Rham bundle $C^{-1}_{\mathscr{X}/\mathscr{S}}(\pi^*_{X/k}(E,-\theta))$, where $(\mathscr{X},\mathscr{S}) = (\tilde{X}\times_{W_2(k)} W_2(k), W_2(k))$.*

Proof. Firstly, the restriction of the de Rham bundle $C^{-1}_{\mathscr{X}/\mathscr{S}}(\pi^*_{X/k}(E,-\theta))$ coincides with the local de Rham constructed by Lan-Sheng-Zuo by using Frobenius liftings. Moreover, the gluing rule (1.2.4) coincides with (1.2.9).

□

Remark 1.2.1. *The advantage of the construction of inverse Cartier functor by Lan-Sheng-Zuo is obvious, as it simplifies the construction of Ogus and Vologodsky by removing $\mathscr{B}_{\mathscr{X}/\mathscr{S}}$. It should also be remarked that the gluing rule (1.2.4) used to obtain a global de Rham bundle is explicitly given in [12] for the first time. Moreover, based on such construction, one can give a reformulation of Fontaine modules.*

Fontaine Modules

The following definition of Fontaine module introduced in [17, 4.16] is based on the functor $C^{-1}_{\mathscr{X}/\mathscr{S}}$.

Definition 1.2.2. *([17, Definition 4.6]) A Fontaine module is a quadruple $(E, F^\bullet, \nabla, \phi)$, where (E,∇) is a coherent $\mathcal{O}_X$-module with an integrable connection ∇, $F^\bullet$ a decreasing filtration on E, i.e.*

$$E = F^k E \supseteq \cdots \supseteq F^l E \supseteq F^{l+1}E = 0, \qquad l-k \leq p-1,$$

satisfying Griffiths transversality

$$\nabla(F^i E) \subseteq F^{i-1}E\otimes\Omega^1_{X/S},$$

and ϕ is an isomorphism

$$(E,\nabla) \cong C^{-1}_{\mathscr{X}/\mathscr{S}}\pi^*_{X/S}(Gr_{F^\bullet}E, Gr_{F^\bullet}\nabla).$$

Examples for Fontaine modules are abundant by the following proposition.

Proposition 1.2.3. ([*17, Theorem* 4.17]) *Let $(E, F^\bullet, \nabla, \phi)$ be a Fontaine module on X, $h : X \to Y$ be a smooth proper morphism of relative dimension d. Suppose the length of the filtration on E is less than $p-d$, then the relative de Rham cohomology $(Rh_*^{DR}E, F^\bullet E)$ degenerates at E_1 and satisfies*

$$\phi : C^{-1}_{\mathscr{Y}/\mathscr{S}} \pi^*_{Y/S}(Gr_{F^\bullet} R^i h_*^{DR} E, Gr_{F^\bullet}\nabla) \cong (R^i h_*^{DR} E, \nabla),$$

which makes $(R^i h_*^{DR} E, \nabla, F^\bullet R^i h_*^{DR} E, \phi)$ a Fontaine module on Y. In particular, for $d < p$, $R^i h_*^{DR} \mathcal{O}_X$ is a Fontaine module.

In [17, page 98], the authors make a claim without proof that if the $W_2(k)$-lifting $\tilde{X}'$ of X' comes from a formal scheme $X_{W(k)}$ over $W(k)$, then the category of Fontaine modules is equivalent to the subcategory of p-torsion objects in Faltings' category $\mathscr{MF}^\nabla_{k,l}(X_{W(k)})$. This claim is not very clear to see directly from the abstract construction of inverse Cartier transform in [17]. However, by the construction of inverse Cartier transform by Lan-Sheng-Zuo, one can see easily the following equivalent reformulation of Fontaine modules just implies this claim.

Definition 1.2.4. *Let $\mathscr{X}/\mathscr{S} = (X/S, \tilde{X}'/\tilde{S})$ be a pair as described in Theorem 2, $\{U_i\}_{i\in I}$ be a family of open subschemes covering X such that each U_i is endowed with a lifting $\tilde{F}_i$ of $F_{U_i/S}$. A Fontaine module is a quadruple $(E, F^\bullet, \nabla, \{\phi_{\tilde{F}_i}\}_{i\in I})$, where (E, ∇) is a coherent $\mathcal{O}_X$-module with an integrable connection ∇, $F^\bullet$ is a decreasing filtration on E, i.e.*

$$E = F^k E \supseteq \cdots \supseteq F^l E \supseteq F^{l+1} E = 0, \qquad l - k \leq p-1,$$

satisfying Griffiths transversality

$$\nabla(F^i E) \subseteq F^{i-1} E \otimes \Omega^1_{X/S}, \quad \textit{and}$$

$$\phi_{\tilde{F}_i} : (E_{U_i}, \nabla) \to \Psi_{\tilde{F}_i}(\pi^*_{U_i/S}(Gr_{F^\bullet} E_{U_i}, Gr_{F^\bullet}\nabla))$$

is an isomorphism over U_i (for definition of $\Psi_{\tilde{F}_i}$ see (1.2.3)) such that the isomorphism

$$\phi_{\tilde{F}_j}\phi^{-1}_{\tilde{F}_i} : \Psi_{\tilde{F}_i}(\pi^*_{U_i/S}(Gr_{F^\bullet} E_{U_i}, Gr_{F^\bullet}\nabla))_{U_i\cap U_j} \to \Psi_{\tilde{F}_j}(\pi^*_{U_j/S}(Gr_{F^\bullet} E_{U_j}, Gr_{F^\bullet}\nabla))_{U_i\cap U_j} \tag{1.2.11}$$

is induced by the exponential $\exp(\xi_{ji})$, where $\xi_{ji} = \tilde{F}_j - \tilde{F}_i$ (1.2.9).

Remark 1.2.5. *The definition of a Fontaine module is not dependent on the choice of the covering family and the lifting of Frobenius. The proof can be done by using the gluing formula* (1.2.9).

Remark 1.2.6. *The Higgs and de Rham objects in the usual sense involved in the Cartier transforms are necessarily nilpotent. One might ask whether there exists a generalized Cartier transform which includes nonnilpotent objects in the usual sense as well. There is no general answer to this question, as can be seen from two examples below on a naive generalization of inverse Cartier transform to Higgs line bundles.*

Let X be a smooth variety over k. By definition a Higgs line bundle on X is uniquely determined by a line bundle L and a global 1-form $\omega \in H^0(X, \Omega_{X/k})$. We will use L_ω to denote his Higgs line bundle.

1. Let $X' = X \times_{Fr_k} k$, ω be a nonzero global 1-form, then we have a Higgs line bundle $(\mathcal{O}_{X'})_\omega$ on X' with a nonnilpotent Higgs field. Moreover, the 1-form ω determines a subsheaf $\ker\beta$ of $T_{X'/k}$, where β is the restriction of the paring $T_{X'/k} \times \Omega^1_{X'/k} \to \mathcal{O}_{X'}$ to $T_{X'/k} \times \mathcal{O}_{X'}.\omega$.

Recall that the sheaf of liftings of Frobenius defines an element of $H^1(X, F^*T_{X'/k})$. If we can prove this element falls in the image of $H^1(X, F^*\ker\beta) \to H^1(X, F^*T_{X'/k})$, then by our construction of inverse Cartier transform, the local de Rham objects $\Psi_{\tilde{F}_i}(\mathcal{O}_{X'}, \theta_\omega)$ can be glued into a global object. However, this condition is not easy to check.

2. On the other hand, if the Frobenius $F_{X/k}$ is liftable over $W_2(k)$, the naive generalization of the inverse Cartier transform yields a global object automatically. However, it might fail to be a faithful functor. For instance, let X be an ordinary abelian variety and $L_\omega = (L, \theta_\omega)$ be a Higgs line bundle on X', then under *the generalized inverse Cartier transform, the de Rham line bundle corresponding to L_ω is nothing but $(L^{\otimes p}, \nabla_{can} + \theta_{C^{-1}(\omega)})$, where ∇_{can} is the canonical connection on $L^{\otimes p}$ and C^{-1} is the inverse Cartier operator. Since X is ordinary, C^{-1} is bijective. In particular, if $L^{\otimes p} \cong \mathcal{O}_X$, there is a unique 1-form ω on X' such that the connection $\nabla_{can} + \theta_{C^{-1}(\omega)}$ is the usual differential $d : \mathcal{O}_X \to \Omega^1_{X/k}$. Let $n = \dim X$, now one can see easily there are p^n Higgs line bundle whose associated de Rham line bundle is $(\mathcal{O}_X, d)$.*

Chapter 2

Logarithmic Cartier Transforms

As its smooth counterpart, the construction of log Cartier transform is still based on Morita equivalence. Firstly one needs to prove the ring of (augmented) differential operators is an Azumaya algebra and then finds a splitting module for it. In this chapter, we will give a sketchy view of log Cartier transforms, which have been obtained by Schepler in his thesis [18]. In parallel with the previous chapter, we will reformulate the inverse log Cartier transform in terms of Frobenius liftings and then use it to define log Fontaine modules.

2.1 Logarithmic Cartier Descent and Residue Condition

As a special case of Cartier transform in the smooth setting, the classical Cartier descent defines an equivalence between the category of quasi-coherent $\mathcal{O}_X$-modules with canonical connections and the category of Higgs modules with vanishing p-curvatures on X'. However, the following example shows that its direct generalization to the logarithmic case fails to hold.

Example 2.1.1. *Let (X, M) be the affine line $\mathbb{A}^1_k$ endowed with the log structure defined by the closed point $0 \in \mathbb{A}^1_k$, then the log differential is given by*

$$d : \mathcal{O}_X \to \omega^1_{X/k}, \qquad f \mapsto tf'(t)\frac{dt}{t}.$$

Let $E_1 := (t) \subseteq k[t]$, it is easy to see E_1 is invariant under the above connection. However, the following morphism is not surjective

$$k[t] \otimes_{k[t]} E_1^{\nabla} \to E_1.$$

Let $E_2 := k[t]/E_1$, then the following morphism is not injective

$$k[t] \otimes_{k[t]} E_2^{\nabla} \to E_2.$$

The rest of this section is devoted to show that such new complexity can be circumvented by introducing residue condition.

Let $f : (X, \mathcal{M}) \to (S, \mathcal{N})$ be a morphism of log schemes, for convenience we denote by X, S and X^* the log schemes $(X, \mathcal{M}), (S, \mathcal{N})$ and $(X, f^*\mathcal{N})$ respectively. Then f factors as $X \to X^* \to S$. Recall that $\omega^1_{X^*/S} \cong \Omega^1_{X/S}$, hence we have an exact sequence of sheaves of log differentials on X

$$\omega^1_{X^*/S} \to \omega^1_{X/S} \to \mathcal{R}_{X/S}.$$

where $\mathcal{R}_{X/S} := \omega^1_{X/X^*}$. By definition (A.2.2), the morphism $d : \mathcal{O}_X \to \omega^1_{X/S}$ factors through $\omega^1_{X^*/S}$, in particular the above exact sequence induces a zero morphism $\mathcal{O}_X \to \mathcal{R}_{X/S}$.

Definition 2.1.2. *Let (E, ∇) be a sheaf of $\mathcal{O}_X$-module with integrable log connection, the residue map of this object is defined to be the composite of $E \to E \otimes \omega^1_{X/S}$ with the projection $\omega^1_{X/S} \to \mathcal{R}_{X/S}$.*

The following theorem establishes an equivalence between a subcategory of log Higgs objects with vanishing Higgs fields and log de Rham objects with integrable connections and vanishing p-curvatures.

Theorem 4. *([16, 1.3.4]) Assume that $X \to S$ is a log smooth morphism in characteristic p, $F : X \to X'$ be the exact relative Frobenius (see def. (A.3.7)) morphism of X/S. If (E, ∇) is a coherent sheaf with integrable connection, then the E^{∇} is a coherent $\mathcal{O}_{X'}$-module and there is a canonical horizontal map*

$$(F^* E^{\nabla}, \nabla') \to (E, \nabla), \quad a \otimes e \mapsto ae \tag{2.1.1}$$

If both the residue map and p-curvature vanish, then this map is surjective. If furthermore $Tor_1^{\mathcal{O}_X}(E, \mathcal{R}_{X/S})$ is 0, then it is bijective.

Remark 2.1.3. *E_1 in example 2.3.1 has nonzero residue map while E_2 has vanishing residue but doesn't satisfy the condition $Tor_1^{\mathcal{O}_X}(E_2, \mathcal{R}_{X/S}) = 0$.*

2.2 Logarithmic Cartier Transforms

2.2.1 Construction of Log Cartier Transform

As can be seen from the previous section, the classical Cartier descent cannot be generalized to log case directly. Though under suitable conditions (2.1.1) such generalization is still available, the de Rham and Higgs objects involved there are not easy to characterize as in the smooth case. A more thorough generalization of Cartier descent has been given by Lorenzon in [14] with the help of index algebra. His work is also the starting point of log Cartier transform obtained later by Schepler [18]. Though important, index algebra has a complicated definition and will not be used elsewhere in this work except the statement of Schepler's results. All we need to know about it is that an index algebra is a sheaf of $\mathcal{O}_X$-algebra on X.

Let $X \to S$ be a perfectly smooth morphism (see def. (A.3.7)) over a perfect field k of characteristic p, $\tilde{X}' \to \tilde{S}$ be a lifting of $X' \to S$ to $W_2(k)$. If we denote by $\overline{\mathcal{M}}_X^{gp}$ the quotient monoid $\mathcal{M}_X^{gp}/\mathcal{O}_X^*$, then we have a canonical $\overline{\mathcal{M}}_X^{gp}$-indexed algebra $\mathcal{A}_X^{gp}$, which is an $\mathcal{O}_X$-algebra equipped with an integrable connection and vanishing p-curvature. The flat section of $\mathcal{A}_X^{gp}$, denoted by $\mathcal{B}_{X/S}$ is a subring of $\mathcal{A}_X^{gp}$ and an $\mathcal{O}_{X'}$-algebra.

As mentioned at the beginning of this chapter, log Cartier transform is realized as a Morita equivalence. In this setting the Azumaya algebra is the ring of indexed PD-differential operators

$$\tilde{D}_{X/S} := \mathcal{A}_X^{gp} \otimes D_{X/S},$$

which should be considered as a counterpart of $D_{X/S}$ in the log setting. The center of $\tilde{D}_{X/S}$ is isomorphic to $\mathcal{B}_{X/S} \otimes_{\mathcal{O}_{X'}} S^{\bullet} T_{X'/S}$. The augmented ring of differential operators in log case is

$$\tilde{D}_{X/S}^{\gamma} := \tilde{D}_{X/S} \otimes_{S^{\bullet} T_{X'/S}} \hat{\Gamma} S^{\bullet} T_{X'/S}.$$

Let $\mathcal{X}/\mathcal{S} = (X/S, \tilde{X}'/\tilde{S})$ be a pair consisting of a perfectly smooth morphism $X \to S$ and a flat lifting $\tilde{X}'/\tilde{S}$ of X'/S over $W_2(k)$. Then one can prove by using log deformation theory (A.2.6) the sheaf of liftings of Frobenius is $F_{X/S}^* \omega_{X'/S}^1$-torsor. Let $\mathcal{K}_{\mathcal{X}/\mathcal{S}}$ be the sheaf of algebra associated this torsor over X, then its splitting module is

$$\check{\mathcal{K}}_{\mathcal{X}/\mathcal{S}}^{\mathcal{A}} := \check{\mathcal{K}}_{\mathcal{X}/\mathcal{S}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{A}_X^{gp},$$

where $\check{\mathcal{K}}_{\mathcal{X}/\mathcal{S}}$ is the $\mathcal{O}_X$-linear dual of $\mathcal{K}_{\mathcal{X}/\mathcal{S}}$. The construction of $\mathcal{K}_{\mathcal{X}/\mathcal{S}}$ and $\check{\mathcal{K}}_{\mathcal{X}/\mathcal{S}}$ is totally the same with $\mathscr{A}_{\mathscr{X}/\mathscr{S}}$ and $\mathscr{B}_{\mathscr{X}/\mathscr{S}}$, for more details see [18]. Consequently, we have the following

Theorem 5. *([18, Theorem 3.2]) Let $\mathrm{MIC}_{\gamma}^{\mathcal{A}}(X/S)$ be category of $\tilde{D}_{X/S}^{\gamma}$-modules, $\mathrm{HIG}_{\gamma}^{\mathcal{B}}(X'/S)$ be category of $\mathcal{B}_{X/S} \otimes_{\mathcal{O}_{X'}} \hat{\Gamma} S^{\bullet} T_{X'/S}$-modules, then the following functors define a quasi-inverse equivalence of categories*

$$\mathcal{C}_{\mathcal{X}/\mathcal{S}} : \mathrm{MIC}_{\gamma}^{\mathcal{A}}(X/S) \to \mathrm{HIG}_{\gamma}^{\mathcal{B}}(X'/S), \quad E \mapsto \iota_* \mathscr{H}om_{\tilde{D}_{X/S}^{\gamma}}(\check{\mathcal{K}}_{\mathcal{X}/\mathcal{S}}^{\mathcal{A}}, E),$$

$$\mathcal{C}_{\mathcal{X}/\mathcal{S}}^{-1} : \mathrm{HIG}_{\gamma}^{\mathcal{B}}(X'/S) \to \mathrm{MIC}_{\gamma}^{\mathcal{A}}(X/S), \quad E' \mapsto \check{\mathcal{K}}_{\mathcal{X}/\mathcal{S}}^{\mathcal{A}} \otimes_{\mathcal{O}_{\mathcal{G}}^{\mathcal{B}}} \iota_* E',$$

where $\mathcal{O}_{\mathcal{G}}^{\mathcal{B}} := \mathcal{B}_{X/S} \otimes_{\mathcal{O}_{X'}} \hat{\Gamma} S^{\bullet} T_{X'/S}$.

If we start with objects of $\mathrm{HIG}_\gamma(X'/S)$ and $\mathrm{MIC}_\gamma(X/S)$, then after tensored with suitable indexed algebras, they will fall in the categories in the above theorem. The following proposition compares the effects of "indexed log Cartier transform" and "non-indexed log Cartier transform" on these objects.

Proposition 2.2.1. *([18, Corollary 3.7])*

1. *Let E' be an object of $\mathrm{HIG}_\gamma(X'/S)$, then there is a natural isomorphism*

$$\mathcal{C}^{-1}_{\mathcal{X}/\mathcal{S}}(E'\otimes_{\mathcal{O}_{X'}}\mathcal{B}_{X/S})\cong(\check{\mathcal{K}}_{\mathcal{X}/\mathcal{S}}\otimes_{\mathcal{O}_\mathcal{G}}\iota_*E')\otimes_{\mathcal{O}_X}\mathcal{A}^{gp}_{\mathcal{X}}$$

   where $\mathcal{O}_\mathcal{G}=\hat{\Gamma}S^\bullet T_{X'/S}$.

2. *Let E be an object of $\mathrm{MIC}_\gamma(X/S)$. Then there is a natural map*

$$\iota_*\mathscr{H}om_{\mathcal{D}^\gamma_{X/S}}(\check{\mathcal{K}}_{\mathcal{X}/\mathcal{S}},E)\otimes_{\mathcal{O}_{X'}}\mathcal{B}_{X/S}\to\mathcal{C}_{\mathcal{X}/\mathcal{S}}(E\otimes_{\mathcal{O}_X}\mathcal{A}^{gp}_{\mathcal{X}}).$$

   Furthermore, the map is injective (resp. surjective) if and only if the natural map $F_X^*E_0^\nabla\to E_0$ is, where $E_0=\mathscr{H}om_{F^*\mathcal{O}_\mathcal{G}}(\check{\mathcal{K}}_{\mathcal{X}/\mathcal{S}},E)$ is equipped with the internal Hom connection.

Therefore, the indexed inverse log Cartier transform can by derived from the non-indexed inverse log Cartier transform.

Proposition 2.2.2. *[18, Corollary 3.10] Let $\mathrm{MIC}^0_\gamma(X/S)$ be the sub category of $\mathrm{MIC}_\gamma(X/S)$ consisting of objects (E,∇) whose residue map (see Definition 2.1.2) $\rho:E\to E\otimes\mathcal{R}_{X/S}$ satisfies $\rho_D^p=0$ for any $D\in T_{X/S}$. Then one can define the functors*

$$C^{-1}_{\mathcal{X}/\mathcal{S}}:\mathrm{HIG}_\gamma(X'/S)\to\mathrm{MIC}^0_\gamma(X/S),\quad C^{-1}_{\mathcal{X}/\mathcal{S}}(E'):=\check{\mathcal{K}}_{\mathcal{X}/\mathcal{S}}\otimes_{\mathcal{O}_\mathcal{B}}\iota_*E'$$

$$C_{\mathcal{X}/\mathcal{S}}:\mathrm{MIC}^0_\gamma(X'/S)\to\mathrm{HIG}_\gamma(X/S),\quad C_{\mathcal{X}/\mathcal{S}}(E):=\mathscr{H}om_{D^\gamma_{X/S}}(\check{\mathcal{K}}_{\mathcal{X}/\mathcal{S}},E).$$

Moreover, we have

1. *$C^{-1}_{\mathcal{X}/\mathcal{S}}$ is the left adjoint of $C_{\mathcal{X}/\mathcal{S}}$.*
2. *The unit $\eta:\mathrm{id}\to C_{\mathcal{X}/\mathcal{S}}\circ C^{-1}_{\mathcal{X}/\mathcal{S}}$ of this adjunction is an isomorphism and the counit $\epsilon:C^{-1}_{\mathcal{X}/\mathcal{S}}\circ C_{\mathcal{X}/\mathcal{S}}\to\mathrm{id}$ is an epimorphism. In particular, $C^{-1}_{\mathcal{X}/\mathcal{S}}$ is fully faithful and $C_{\mathcal{X}/\mathcal{S}}$ is faithful.*
3. *If $Tor_1^{\mathcal{O}_X}(E,\mathcal{R}_{X/S})=0$, $\epsilon_E:C^{-1}_{\mathcal{X}/\mathcal{S}}\circ C_{\mathcal{X}/\mathcal{S}}(E)\to E$ is an isomorphism.*

2.2.2 Lan-Sheng-Zuo's Construction in the Log Case

Following [12], we give below a more direct construction of inverse Cartier transform in the log case by using coordinate computation rather than Morita equivalence given by Schepler. First we fix some notations. Let $(X,\mathcal{M})$ be a log scheme smooth over Spec$k$ (equipped with trivial log structure) endowed with lifting $(\tilde{X},\tilde{\mathcal{M}})/W_2(k)$, ω^1_X be its sheaf of logarithmic differentials which is locally free of rank $d=\mathrm{dim}X$. By a log Higgs bundle on (X,M) we mean a locally free $\mathcal{O}_X$-module E together with an $\mathcal{O}_X$-linear map $\theta:E\to E\otimes\omega^1_X$ satisfying $\theta\wedge\theta=0$.

Since in terms of local basis, a log de Rham bundle is determined by a connection matrix whose entries are log differential forms, we will construct these connection matrices locally then glue them together. Let $\{U_i\}_{i\in I}$ be a covering family of X by affine open subschemes. For each U_i, we choose a lifting $\tilde{F}_{U_i}$ of the absolute Frobenius $F_{U_i}:U_i\to U_i$, then we have $\pi\tilde{F}_{U_i}(\tilde{m})=\pi\tilde{m}^p$, where $\pi:\tilde{\mathcal{M}}\to\mathcal{M}$ is the natural projection. Then we have $\tilde{F}_{U_i}(\tilde{m})=u\tilde{m}^p$ for some $u\in\ker\pi$ and $\tilde{\alpha}(u)=1+[p]\epsilon(m)$ for some $\epsilon(m)\in\mathcal{O}_X$, therefore

$$d\mathrm{log}(\tilde{F}_{U_i}(\tilde{m}))=d\mathrm{log}(u\tilde{m}^p)=d\mathrm{log}(u)+pd\mathrm{log}(\tilde{m})=[p]d\epsilon(m)+[p]d\mathrm{log}(m).$$

As in the smooth case (1.1.5), the morphism $d\mathrm{log}(m)\mapsto d\mathrm{log}(m)+d\epsilon(m)$ will be denoted by $\zeta_{\tilde{F}_{U_i}}$. Now we can associate to a log Higgs bundle (E,θ) locally the de Rham bundle $(F^*_{U_i}E_{U_i},d+(\mathrm{id}\otimes\zeta_{\tilde{F}_{U_i}})(F^*_{U_i}\theta_{U_i}))$ and we have the following

Proposition 2.2.3. *The connection* $\nabla_{U_i} = d + (\mathrm{id}\otimes\zeta_{\tilde{F}_{U_i}})(F_{U_i}^*\theta_{U_i})$ *is well defined and integrable.*

Proof. To say the connection ∇_{U_i} is well-defined is equivalent to say it is invariant under another choice of basis of E_{U_i}. Now given two set of basis $\{e_{U_i}\}$ and $\{e'_{U_i}\}$ of E_{U_i} satisfying $(e'_{U_i}) = A(e_{U_i})$, for some invertible matrix M with entries in $\mathcal{O}_{U_i}$. Let θ_{U_i} and θ'_{U_i} be the matrix representation of the Higgs field θ under the basis $\{e_{U_i}\}$ and $\{e'_{U_i}\}$ respectively, then we need to check

$$(\mathrm{id}\otimes\zeta_{\tilde{F}_{U_i}})(F_{U_i}^*\theta'_{U_i}) = d(F_{U_i}^*M)F_{U_i}^*M^{-1} + F_{U_i}^*M(\mathrm{id}\otimes\zeta_{\tilde{F}_{U_i}})(F_{U_i}^*\theta_{U_i})F_{U_i}^*M^{-1}.$$

Since $dF_{U_i}^*M = 0$ and $\theta'_{U_i} = M\theta_{U_i}M^{-1}$, the above equality holds. Next we prove the integrability of ∇_{U_i}. Since $\theta_{U_i}\wedge\theta_{U_i} = 0$, we have

$$(\mathrm{id}\otimes\zeta_{\tilde{F}_{U_i}})(F_{U_i}^*\theta_{U_i})\wedge(\mathrm{id}\otimes\zeta_{\tilde{F}_{U_i}})(F_{U_i}^*\theta_{U_i}) = (\mathrm{id}\otimes\bigwedge^2\zeta_{\tilde{F}_{U_i}})(F_{U_i}^*(\theta_{U_i}\wedge\theta_{U_i})) = 0.$$

We also have to show the composite $d(\mathrm{id}\otimes\zeta_{\tilde{F}_{U_i}})(F_{U_i}^*\theta_{U_i})$ vanishes. It suffices to show $d\zeta_{\tilde{F}_{U_i}}(F_{U_i}^*\omega) = 0$ for any one form ω, which follows easily from the definition of $\zeta_{\tilde{F}_{U_i}}$.

□

Next we will glue the locally associated de Rham bundles. For that purpose it suffices to construct a family of matrices A_{ij} where G_{ij} is defined over $U_i\cap U_j$ satisfying

1. (Bundle gluing condition) over $U_i\cap U_j\cap U_k$ we have $G_{ik} = G_{jk}G_{ij}$,
2. (Connection gluing condition) over U_{ij}, $(\mathrm{id}\otimes\zeta_{\tilde{F}_{U_i}})(F_{U_i}^*\theta_{U_i}) = dG_{ij}G_{ij}^{-1} + G_{ij}(\mathrm{id}\otimes\zeta_{\tilde{F}_{U_j}})(F_{U_j}^*\theta_{U_j})G_{ij}^{-1}$.

To construct G_{ij}, we need the following lemma which is an easy consequence of proposition (A.2.6)

Lemma 2.2.4. *For any two open subset* U_i *and* U_j*, we have* $\gamma_{ij} : F_{U_i\cap U_j}^*\omega_{X/k}^1 \to \mathcal{O}_X$ *such that over* $U_i\cap U_j$*,* $\zeta_{\tilde{F}_{U_i}} - \zeta_{\tilde{F}_{U_j}} = d\gamma_{ij}$ *and over* $U_i\cap U_j\cap U_k$*,* $\gamma_{ik} = \gamma_{ij}+\gamma_{jk}$*.*

Let $\{e_{U_i}\}$ be a basis of E_{U_i} over U_i, M_{ij} be a matrix with entries in $\mathcal{O}_{U_i\cap U_j}$ s.t. $(e_{U_i}) = M_{ij}(e_{U_j})$ over $U_i\cap U_j$ and $M_{ik} = M_{ij}M_{jk}$ over $U_i\cap U_j\cap U_k$. On the other hand, by the above lemma the nilpotence assumption on the Higgs field θ, we have locally well defined matrix

$$\exp[(\mathrm{id}\otimes\gamma_{ij})F^*\theta_{U_i}] = \sum_{1\le s\le n}\frac{[(\mathrm{id}\otimes\gamma_{ij})F^*\theta_{U_i}]^s}{s!}$$

Then the gluing matrix for $\{F_{U_i}^*E_{U_i}\}_{i\in I}$ is defined to be

$$G_{ij} = \exp[(\mathrm{id}\otimes\gamma_{ij})(F^*\theta_{U_i})]F^*M_{ij}.$$

Next we check the two gluing conditions above. First of all, we have

$$\begin{aligned}\exp[(\mathrm{id}\otimes\gamma_{jk})F^*\theta_{U_j}] &= \exp[(\mathrm{id}\otimes\gamma_{jk})(F^*(M_{ij}^{-1}\theta_{U_i}M_{ij}))]\\ &= \exp[F^*M_{ij}^{-1}(\mathrm{id}\otimes\gamma_{jk})(F^*\theta_{U_i})F^*M_{ij}]\\ &= F^*M_{ij}^{-1}\exp[(\mathrm{id}\otimes\gamma_{jk})(F^*\theta_{U_i})]F^*M_{ij}\end{aligned}$$

Then

$$\begin{aligned}G_{ij}G_{jk} &= \exp[(\mathrm{id}\otimes\gamma_{ij})(F^*\theta_{U_i})]F^*M_{ij}\exp[(\mathrm{id}\otimes\gamma_{jk})(F^*\theta_{U_j})]F^*M_{jk}\\ &= \exp[(\mathrm{id}\otimes(\gamma_{ij}+\gamma_{jk}))(F^*\theta_{U_i})]F^*M_{ij}M_{jk}\\ &= \exp[(\mathrm{id}\otimes\gamma_{ik})(F^*\theta_{U_i})]F^*M_{ik}\\ &= G_{ik}\end{aligned}$$

Then we prove the connection gluing condition. By definition we have

$$\begin{aligned}\nabla_{U_j}(F^*e_{U_i}) &= \nabla_{U_j}(G_{ij}F^*e_{U_j}) \\ &= dG_{ij}F^*e_{U_j} + G_{ij}(\mathrm{id}\otimes\zeta_{\tilde{F}_{U_j}})(F^*\theta_{U_j})(F^*e_{U_j}) \\ &= [dG_{ij}G_{ij}^{-1} + G_{ij}(\mathrm{id}\otimes\zeta_{\tilde{F}_{U_j}})(F^*\theta_{U_j})G_{ij}^{-1}](F^*e_{U_j})\end{aligned}$$

Since $dF^*M_{ij}=0$, $\theta_{U_j}=M_{ij}\theta_{U_i}M_{ij}^{-1}$, we have

$$\begin{aligned}dG_{ij}G_{ij}^{-1} &= d\exp[(\mathrm{id}\otimes\gamma_{ij})(F^*\theta_{U_i})]\exp[-(\mathrm{id}\otimes\gamma_{ij})(F^*\theta_{U_i})] \\ &= d[(\mathrm{id}\otimes\gamma_{ij})(F^*\theta_{U_i})] \qquad (2.2.1)\end{aligned}$$

and

$$\begin{aligned}G_{ij}(\mathrm{id}\otimes\zeta_{\tilde{F}_{U_j}})(F^*\theta_{U_j})G_{ij}^{-1} &= \exp[(\mathrm{id}\otimes\gamma_{ij})(F^*\theta_{U_i})](\mathrm{id}\otimes\zeta_{\tilde{F}_{U_j}})(F^*\theta_{U_j})\exp[-(\mathrm{id}\otimes\gamma_{ij})(F^*\theta_{U_i})] \\ &= (\mathrm{id}\otimes\zeta_{\tilde{F}_{U_i}})(F^*\theta_{U_i}) \qquad (2.2.2)\end{aligned}$$

By (2.2.1) and (2.2.2) we have

$$\begin{aligned}\nabla_{U_j}(F^*e_{U_i}) &= [d(\mathrm{id}\otimes\gamma_{ij})(F^*\theta_{U_i}) + (\mathrm{id}\otimes\zeta_{\tilde{F}_{U_i}})(F^*\theta_{U_i})](F^*e_{U_i}) \\ &= (\mathrm{id}\otimes\zeta_{\tilde{F}_{U_i}})(F^*\theta_{U_i})(F^*e_{U_i}) \\ &= \nabla_{U_i}(F^*e_{U_i}) \qquad (2.2.3)\end{aligned}$$

Therefore, we prove the following

Theorem 6. *The local de Rham bundles $\{(F^*_{U_i}E_{U_i},\nabla_{U_i})\}_{i\in I}$ can be glued into a de Rham bundle on X via the matrices $\{G_{ij}\}_{i,j\in I}$.*

For the log inverse Cartier transform $C^{-1}_{\mathcal{X}/\mathcal{S}}$ (Proposition 2.2.2), we still have the formulae (1.2.9), (1.2.10) and the proof for the smooth case still works in the log case, mutatis mutandis. Therefore, the generalization of Lan-Sheng-Zuo's construction given above coincides with the inverse Cartier transform in [18, Corollary 3.10].

2.2.3 Log Fontaine Modules

Next we give the definition of log Fontaine modules by imitating its smooth counterpart. Note that in the definition we employ the inverse Cartier functor $C^{-1}_{\mathcal{X}/\mathcal{S}}$ in Proposition (2.2.2).

Definition 2.2.5. *Let $X\to S$ be a perfectly smooth morphism of log schemes, a log Fontaine module on X is a quadruple $(E,F^\bullet,\nabla,\phi)$, where (E,∇) is a coherent $\mathcal{O}_X$-module E endowed with an integrable connection ∇, $F^\bullet$ a decreasing filtration on E, i.e.*

$$E=F^kE\supseteq\cdots\supseteq F^lE\supseteq F^{l+1}E=0,\qquad l-k\le p-1,$$

satisfying Griffiths transversality

$$\nabla(F^iE)\subseteq F^{i-1}E\otimes\omega^1_{X/S},$$

and ϕ is an isomorphism

$$(E,\nabla)\cong C^{-1}_{\mathcal{X}/\mathcal{S}}\pi^*_{X/S}(Gr_{F^\bullet}E,Gr_{F^\bullet}\nabla).$$

Note the elementary construction (1.2.1) can be carried out in the log case as well, for which we only need to replace the usual differentials by log differentials. Again, if we paraphrase the sheaf $\check{\mathcal{K}}_{\mathcal{X}/\mathcal{S}}$ in terms of local liftings of Frobenius, one has the following equivalent definition of log Fontaine modules.

Definition 2.2.6. *Let $\mathcal{X}/\mathcal{S} = (X/S, \tilde{X}'/\tilde{S})$ be a pair consisting of a perfectly smooth morphism $X \to S$ and a flat lifting $\tilde{X}'/\tilde{S}$ of X'/S over $W_2(k)$. Let $\{U_i\}_{i\in I}$ be a family of open subschemes of X such that each U_i is endowed with a lifting $\tilde{F}_i$ of $F_{U_i/S}$ to a flat thickening of U_i over $\tilde{S}$. A log Fontaine module is a quadruple $(E, F^\bullet, \nabla, \{\phi_{\tilde{F}_i}\}_{i\in I})$, where (E, ∇) is a coherent $\mathcal{O}_X$-module E with an integrable connection*

$$\nabla : E \to E \otimes \omega^1_{X/S},$$

$F^\bullet$ is a decreasing filtration on E, i.e.

$$E = F^k E \supseteq \cdots \supseteq F^l E \supseteq F^{l+1} E = 0, \qquad l - k \leq p - 1,$$

satisfying Griffiths transversality

$$\nabla(F^i E) \subseteq F^{i-1} E \otimes \omega^1_{X/S},$$

$$\phi_{\tilde{F}_i} : (E_{U_i}, \nabla) \to \Psi_{\tilde{F}_i}(\pi^*_{U_i/S}(Gr_{F^\bullet} E_{U_i}, Gr_{F^\bullet} \nabla))$$

is an isomorphism over U_i such that the isomorphism

$$\phi_{\tilde{F}_j} \phi^{-1}_{\tilde{F}_i} : \Psi_{\tilde{F}_i}(\pi^*_{U_i/S}(Gr_{F^\bullet} E_{U_i}, Gr_{F^\bullet} \nabla))_{U_i \cap U_j} \to \Psi_{\tilde{F}_j}(\pi^*_{U_j/S}(Gr_{F^\bullet} E_{U_j}, Gr_{F^\bullet} \nabla))_{U_i \cap U_j}$$

is induced by the exponential $\exp(\xi_{ji})$, where $\xi_{ji} = \tilde{F}_j - \tilde{F}_i$ (1.2.9).

Remark 2.2.7. *As in the smooth case, this definition is independent of choice of the covering family and the liftings of Frobenius. The proof can be obtained by using the formula* (1.2.9)*, in which the derivatives θ_k in the usual sense should be replaced by log derivatives.*

Chapter 3

F-T-Crystals and Fontaine Modules

In this chapter S will be a fixed formal log scheme endowed with a fine log structure flat over $W(k)$. For an integer $\nu \geq 0$, S_ν will be the reduction of S modulo $p^{\nu+1}$, hence a fine log scheme and flat over $W_{\nu+1}(k)$.

3.1 P-isogenies and F-spans

Let X/k be a proper smooth variety over a perfect field k of characteristic p, its crystalline cohomology $H^*_{cris}(X/k)$ is finitely generated $W(k)$-module and equipped with an action induced by the relative Frobenius

$$\Phi : F^*_{X/k} H^*_{cris}(X/k) \to H^*_{cris}(X/k).$$

The above morphism has the property that it is an isomorphism after tensored with $\mathbb{Q}$. This serves as the most fundamental example of an *F-crystal* in the classical setting, i.e. over a point. In this section we will define F-spans, which can be regarded as F-crystals over more general bases. To make our presentation more clear-cut, we will first do some abstraction work.

We fix an abelian category $\mathbf{Ab}$. An object E of $\mathbf{Ab}$ is called *p-torsion free* if the endomorphism $p.\mathrm{id}_E$ is injective. It is easy to see for a p-torsion free object E, the endomorphism $p^i.\mathrm{id}_E$ is injective for all $i \geq 0$, from which $P^iE := \mathrm{im}(p^i.\mathrm{id}_E)$ is isomorphic to E for all $i \geq 0$. Next we come to the following

Definition 3.1.1. *Let E'' and E be two p-torsion free objects, a morphism $\Phi : E'' \to E$ is called a p-isogeny of width n, if one can find a morphism $\Psi : E \to E''$ such that $\Psi \circ \Phi = p^n.\mathrm{id}_{E''}, \Phi \circ \Psi = p^n.\mathrm{id}_E$.*

Remark 3.1.2. *Let $\mathbf{S}$ be the arrows $E \xrightarrow{p^n} E$ in $\mathbf{Ab}$, where $n \geq 0$ and E is a p-torsion free object. Then $\mathbf{S}$ forms a multiplicative system [7, Chapter I, §3]. It is easy to check that to give a p-isogeny in $\mathbf{Ab}$ is equivalent to give an isomorphism in the category $\mathbf{Ab_S}$, the localization of $\mathbf{Ab}$ with respect to $\mathbf{S}$.*

Given a p-isogeny $\Phi : E'' \to E$, we can define a filtration $\{M^i\}_{i\geq 0}$ on E'' by

$$M^iE'' = \Phi^{-1}(P^iE).$$

By definition (3.1.1), one can find $\Psi : E \to E''$ such that $\Psi \circ \Phi = p^n.\mathrm{id}_{E''}, \Phi \circ \Psi = p^n.\mathrm{id}_E$. Then for any $k \geq n$ and $x \in M^kE''$, we have $\Phi(x) = p^ky$ for some $y \in E$, hence $p^nx = \Psi(\Phi(x)) = p^k\Psi(y)$, therefore $x = p^{k-n}\Psi(y)$ and $M^kE'' \subseteq P^{k-n}\Psi(E)$. Conversely, for any $y \in E$, we have $\Phi(p^{k-n}\Psi(y)) = p^{k-n}\Phi(\Psi(y)) = p^ky$, therefore $M^kE'' \supseteq P^{k-n}\Psi(E)$, hence we prove the following

Proposition 3.1.3. *For any $k \geq n$, $M^kE'' = P^{k-n}\Psi(E)$.*

The following fact is easy to verify and we omit its proof

$$PE'' \cap M^iE'' = PM^{i-1}E''. \tag{3.1.1}$$

From the above equation we can deduce, for any $i \geq j$,

$$P^jE'' \cap M^iE'' = P^jM^{i-j}E''. \tag{3.1.2}$$

Definition 3.1.4. *A filtration* $\{M^i\}$ *on* E'' *is said to be G-transversal to the ideal* (p) *and of level within* $[0,n]$*, if it satisfies the equation* (3.1.2) *and* $M^{i+1}E'' \subset PE''$ *if* $i \geq n$*,* $M^iE'' = E''$*, if* $i \leq 0$*.*

Proposition 3.1.5. *To give a p-isogeny* $\Phi : E'' \to E$ *of width n is equivalent to give a filtration* $\{M^i\}_{i\geq 0}$ *of* E'' *which is G-transversal to* (p) *and of level within* $[0,n]$*.*

Proof. The direction from p-isogeny to a filtration has been verified next we will prove the other direction. Let $E = M^nE''$, and the morphism $\Phi : E'' \to E$ is defined to be multiplication by p^n. This morphism is well defined since by definition (3.1.4), we have $P^nE'' \subseteq M^nE''$. The morphism $\Psi : E \to E''$ is defined to be the inclusion $i : M^nE'' \hookrightarrow E''$. It is easy to see Φ and Ψ satisfy the conditions in definition (3.1.1). □

Recall that the category of crystals over a smooth scheme is an abelian category [1], one can prove this is still true in the log smooth case. Therefore, we can make the following

Definition 3.1.6. *An F-span on X/S is a triple* (E', E, Φ)*, where* E' *is a crystal of p-torsion free* $\mathcal{O}_{X'/S}$*-module, E is a crystal of p-torsion free* $\mathcal{O}_{X/S}$*-module and* $\Phi : F^*_{X/S}E' \to E$ *is a p-isogeny. An F-span* (E', E, Φ) *is said to have width n if* Φ *has width n.*

Suppose we are given an F-span (E, E', Φ) of width n, let $E'' := F^*_{X/S}E'$, $A^\epsilon E'' := \sum_{0\leq k\leq n} p^{n-k}M^kE''$, then it is easy to see Φ induces an isomorphism

$$A^\epsilon E'' \to p^nE \cong E. \tag{3.1.3}$$

The above isomorphism will still denoted by Φ in the sequel.

Admissible F-spans

Let (E', E, Φ) be an F-span, $\{M^k\}$ be the filtration on E'' induced by Φ. Since the category of crystals on a log smooth scheme is an abelian category, the subobjects M^kE'' of $E'' := F^*_{X/S}E'$ are subcrystals.

Let Y be a LS thickening (see Definition (B.1.13)) of X. Then the value of E' over Y corresponds to an $\mathcal{O}_Y$-module E'_Y equipped with an integrable connection ∇. By functoriality, the value of the crystal $E'' = F^*_XE'$ over Y corresponds to the object $(F^*_{Y/S}E'_Y, F^*_{Y/S}\nabla)$. Finally, the objects corresponding to the values of the subcrystals M^kE'' on Y are nothing but the $\mathcal{O}_Y$-modules $M^k_YE''_Y$, where $E''_Y = F^*_{Y/S}E'_Y$.

Let $Y_0 = Y \times_S S_0$, then it is easy to see the p-curvature of the pullback of $(F^*_{Y/S}E'_Y, F^*_{Y/S}\nabla)$ to Y_0 is 0, in other words, we have isomorphism $F^*_{Y_0/S}E''^{\nabla}_{Y_0} \cong E''_{Y_0}$. A natural question is whether this is still true for the sub $\mathcal{O}_Y$-modules $M^k_YE''_Y$.

Definition 3.1.7. *([16, 5.2.9]) An F-span* $F^*_{X/S}E' \to E$ *is said to be admissible, iff for any LS thickening* Y*, the filtration* $\{M^k_{Y_0}\}$ *induced on* E''_{Y_0} *is descendable to* $E'_{Y'_0}$*, i.e. for all k we have an isomorphism*

$$F^*_{Y_0/S}(M^k_{Y_0}E''^{\nabla}_{Y_0}) \to M^k_{Y_0}E''_{Y_0},$$

where $E''_{Y_0}, M^k_{Y_0}E''_{Y_0}$ *are the restrictions of* $E''_Y, M^k_YE''_Y$ *to* Y_0 *respectively.*

By definition, admissibility can be checked locally over a family of LS thickenings covering X. For further development, we define a filtration ([16, 5.2.9.3]) $A^k_YE'_Y$ on E'_Y by assigning

$$A^k_YE'_Y = \eta^{-1}(F_{Y/S*}M^kF^*_{Y/S}E'_Y), \tag{3.1.4}$$

where $\eta : E'_Y \to F_{Y/S*}F^*_{Y/S}E'_Y$ is the natural morphism. Then one can prove

Proposition 3.1.8. *([16, 5.2.11]) For an admissible F-span, the filtration* A_Y *is G-transversal to* (p) *and compatible with* $F_{Y/S}$ *(see Definition* 3.2.7*) and the natural map*

$$F^*_{Y/S}A^k_YE'_Y \to M^kE''_Y$$

is an isomorphism. In particular, the the restriction of the filtration A^k_Y *to* Y_0 *coincides with* $M^k_{Y_0}E''^{\nabla}_{Y_0}$*.*

In the next section, we will see to give an admissible F-span is equivalent to give a filtered crystal satisfying certain transversality conditions. The filtration (3.1.4) defined above will play an important role there.

3.2 G-transversality and T-Crystals

In this section, we will introduce T-crystals, which are crystals endowed with a filtration satisfying certain transversality conditions. These objects are necessary to define F-T-crystals in the next section.

Definition 3.2.1. *([16, 2.1.1]) Let (E, A) be an $\mathcal{O}_T$-module endowed with a filtration $\{A^k E\}$ over a scheme T. Given a morphism $f : T' \to T$, we say (E, A) is normally transversal to f if the morphism $f^* A^k E \to f^* E$ is injective for all k.*

The image of the morphism $f^* A^k E \to f^* E$ will be denoted by A^k_f. If f is a closed immersion induced by an ideal J, then it is easy to see (A, E) normally transversal to f is equivalent to $JE \cap A^k E = JA^k E$.

Let $\mathcal{J} := \{\mathcal{J}^i\}$ be a filtration of $\mathcal{O}_T$ defined by sheaves of ideals $\mathcal{J}^i$, we say $\mathcal{J}$ is multiplicative if $\mathcal{J}^0 = \mathcal{O}_T$ and $\mathcal{J}^i \mathcal{J}^j \subset \mathcal{J}^{i+j}$ for all i, j. Note that we do not require the filtration is decreasing. An example of multiplicative filtration is given by the divided power ideal $\mathcal{J}^{[n]} = \gamma_n(\mathcal{J})$, where $\mathcal{J}$ is an ideal equipped with divided power γ.

Definition 3.2.2. *([16, 2.1.2, 2.1.3]) Let $\mathcal{J}$ be a quasi-coherent, multiplicative filtration of $\mathcal{O}_T$ and (E, A) be a filtered $\mathcal{O}_A$-module. We say (E, A) is G-transversal to $\mathcal{J}$ if for all k*

$$JE \cap A^k E = \mathcal{J}^1 A^{k-1} E + \mathcal{J}^2 A^{k-2} E + \cdots$$

where $J = \mathcal{J}^1$. If a pair (E, A) is G-transversal to $\mathcal{J}$ and $A^{n+1} E \subseteq JE, A^m E = E$, we say its $\mathcal{J}$-level is within $[m, n]$ and its width is less than or equal to $n - m$.

There is another weaker version of G-transversality, namely

Definition 3.2.3. *([16, 2.1.2]) Let $\mathcal{J}$ be a quasi-coherent, multiplicative filtration of $\mathcal{O}_T$ and (E, A) be a filtered $\mathcal{O}_A$-module. We say (E, A) is G'-transversal if for all k*

$$JE \cap A^k E \subseteq \mathcal{J}^1 A^{k-1} E + \mathcal{J}^2 A^{k-2} E + \cdots .$$

Remark 3.2.4.

1. *For a pair (E, A) G-transversal to $\mathcal{J}$ with $\mathcal{J}$-level within the interval $[m, n]$, $A^k E$ is determined by $A^i E, i < k$ when $k \geq n + 1$.*

2. *For a multiplicative filtration $\{\mathcal{J}^i\}$ satisfying $\mathcal{J}^i = J^i = (t^i)$ and a t-torsion free $\mathcal{O}_T$-module E, the G-transversality condition for (E, A) is equivalent to $JE \cap A^k E = \mathcal{J}^1 A^{k-1} E$, c.f. definition* (3.1.4).

One can pass from G'-transversality to G-transversality by the following *saturation* process.

Definition 3.2.5. *([16, 2.3.1]) Let (E, A) be a filtered sheaf of $\mathcal{O}_T$-module which is G'-transversal to $\mathcal{J}$. The following filtration*

$$A^k_{\mathcal{J}} E := A^k E + \mathcal{J}^1 A^{k-1} E + \mathcal{J}^2 A^{k-2} E + \cdots$$

is called the saturation of A with respect to $\mathcal{J}$.

Proposition 3.2.6. *([16, 2.3.1]) $A_{\mathcal{J}}$ has the following properties*

1. *$A_{\mathcal{J}}$ is coarser than A and G-transversal to $\mathcal{J}$. It is the finest filtration on E coarser than A and G-transversal to $\mathcal{J}$.*

2. *$A_{\mathcal{J}}$ is the coarsest filtration on E which is G'-transversal to $\mathcal{J}$, coarser than A and induces the same filtration on E/JE as A.*

3. *$A_{\mathcal{J}}$ is the unique filtration on E which is G-transversal to $\mathcal{J}$, coarser than A and induces the same filtration on E/JE as A.*

Definition 3.2.7. *([16, 2.3.3]) Suppose that (E, A) is a filtered sheaf of $\mathcal{O}_T$-module which is G-transversal to $\mathcal{J}$, let $i : X \to T$ be the closed immersion defined by J and let $g : X' \to X$ be a morphism. The pair (E, A) is said to be compatible with g if and only if (i^*E, A) is normally transversal to g. When g is a closed imbedding defined by an ideal I, we say (E, A) is compatible with I.*

Generally speaking, the above transversality properties do not behave well under base change, however we have the following

Proposition 3.2.8. *([16, 2.2.1]) Suppose $f : (T', \mathcal{J}') \to (T, \mathcal{J})$ is a morphism of schemes endowed with multiplicative filtration defined by quasi-coherent ideals. Let $i : X \hookrightarrow T$ and $i' : X \hookrightarrow T'$ be the inclusions defined by the ideals J and J', fitting into the following commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{i'} & T' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & T \end{array}$$

Let (E, A) be a filtered $\mathcal{O}_T$-module, and suppose (i^*E, A) is normally transversal to g. Then if (E, A) is normally transversal to i (resp. G'-transversal to $\mathcal{J}$), then (f^*E, A_f) is normally transversal to i' (resp. G'-transversal to $\mathcal{J}'$).

The above definition of G-transversality can be adapted to the crystalline setting in the following way. Let X be a log scheme smooth over S_ν for some $\nu \geq 0$, (E, A) be a filtered $\mathcal{O}_{X/S}$-module. (E, A) is said to be *G-transversal to the sheaf of PD-ideal $(\mathcal{J}_{X/S}, \gamma)$* if for any object (U, T, δ) of $\text{Cris}(X/S)$, (E_T, A_T) is G-transversal to $(\mathcal{J}_T, \gamma)$, where the multiplicative filtration is generated by the defining ideal of U in $\mathcal{O}_T$ with the divided power structure. Given a morphism $f : T' \to T$ in the crystalline topos $\text{Cris}(X/S)$, by proposition (3.2.8), we know the pullback filtration A_f on $E_{T'} \cong f^*E_T$ is G'-transversal to $(\mathcal{J}_{T'}, \gamma)$. Note that the filtration A_f is not necessarily G-transversal to $(\mathcal{J}_{T'}, \gamma)$. However, by proposition (3.2.6) the saturation $A_{f, \mathcal{J}_{T'}, \gamma}$ is G-transversal to $(\mathcal{J}_{T'}, \gamma)$.

Definition 3.2.9. *([16, 3.2.1]) A proto-T-crystal on X/S is a pair (E, A) where E is a crystal of $\mathcal{O}_{X/S}$-module and A is a filtration on E which is G-transversal to $(\mathcal{J}_{X/S}, \gamma)$, satisfying the following equivalent conditions ([16, 3.1.1])*

1. *For any morphism $f : T' \to T$ in $\text{Cris}(X/S)$,*

$$A^k E_{T'} = A_f^k E_{T'} + \mathcal{J}_{T'} A_f^{k-1} E_{T'} + \cdots + \mathcal{J}_{T'}^{[i]} A_f^{k-i} E_{T'} + \cdots$$

   where $A_f^k := \text{im}(f^*A^iE \to f^*E \cong E')$

2. *For any object (U, T, γ) of $\text{Cris}(X/S)$, the filtration induced by A_T on E_U is A_U.*

Definition 3.2.10. *A T-crystal is a proto-T-crystal (E, A) that is compatible with the closed subscheme of X defined by $p^i\mathcal{O}_X$ for all $i > 0$.*

Remark 3.2.11.
1. *When X/S is log smooth, the equivalent conditions in the definition of proto-T-crystals are automatically satisfied.*
2. *To give a crystal on $\text{Cris}(X/S)$ is equivalent to give a crystal on $\text{Cris}(X_0/S)$. However, it's no longer true for proto-T-crystals or T-crystals.*

Over local p-adic thickenings, we have the following description of proto-T-crystals and T-crystals.

Proposition 3.2.12. *([16, Theorem 3.2.3]) If X/S_ν is log smooth and Y is a log smooth lifting of X to S, then to give a proto-T-crystal is equivalent to give a triple (E, ∇, A), where (E, ∇) is a sheaf of $\mathcal{O}_Y$-module with an integrable connection ∇ and A is a filtration on E such that*

1. *$\nabla(A^kE) \subseteq A^{k-1}E \otimes \Omega^1_{Y/S}$,*

2. E is G-transversal to the defining ideal $(p^{\nu+1}, \gamma)$ of S_ν in S.

In particular, to give a T-crystal on X is equivalent to give a triple (E, ∇, A) on Y such that the conditions above are satisfied and (E_X, A_X) is compatible with $p^i\mathcal{O}_X$ for all $i \leq \nu + 1$.

Corollary 3.2.13. *Assuming the conditions in the above proposition, if (E, A, ∇) corresponds to a T-crystal then for any $1 \leq \delta \leq \nu + 1$, we have*

$$p^\delta E_Y \cap A^k E_Y = p^\delta A^k E_Y + p^{\nu+1} E_Y \cap A^k E_Y. \tag{3.2.1}$$

Proof. First observe that the right side of (3.2.1) is contained in the left side. By definition (3.2.7) and the equivalent description of T-crystal under proposition (3.2.12), we have

$$p^\delta E_X \cap A^k E_X = p^\delta A^k E_X. \tag{3.2.2}$$

Let $y \in E_Y$, $z \in A^k E_Y$, and $\bar{y}$, $\bar{z}$ be the images of y, z in E_X. If $p^\delta y$ and z have the same image in E_X, i.e. $p^\delta \bar{y} = \bar{z} \in p^\delta E_Y \cap A^k E_Y$, by (3.2.2) we have $\bar{z} = p^\delta \bar{z}_1$ where $\bar{z}_1$ is the image of some $z_1 \in A^k E_Y$ in $A^k E_X$. Therefore, if $z = p^\delta y$ for some $y \in E_Y$, then z can be written as $p^\delta z_1 + p^{\nu+1} z_2$ for some $z_1 \in A^k E_Y$, $z_2 \in E_Y$. Note that $p^{\nu+1} z_2 \in A^k E_Y$ hence $z_2 \in A^k E_Y$. In particular, we have the left side of (3.2.1) is contained in the right side hence the corollary is proved.

□

Let $D_Y(1)$ be the PD envelop of the diagonal in $Y \times_S Y$, and $p_i : D_Y(1) \to Y$ be the i-th projection. Then the connection ∇ on E_Y induces an isomorphism $p_1^* E_Y \cong p_2^* E_Y$. The following proposition will be used later.

Proposition 3.2.14. *([16, 3.1.3]) The isomorphism $p_1^* E_Y \cong p_2^* E_Y$ induces an isomorphism $A_1^k p_1^* E_Y \cong A_2^k p_2^* E_Y$, where the filtration $A_i^k p_i^* E_Y$ on $p_i^* E_Y$ is given by*

$$A_i^k := A^k_{p_i, J_Y} = A^k_{p_i} E_Y + J_Y A^{k-1}_{p_i} E_Y + \cdots + J_Y^{[i]} A^{k-i}_{p_i} E_Y + \cdots .$$

Admissible T-crystals and Admissible F-spans

Unlike crystals, the pullback of a T-crystal over X is not a T-crystal in general. Next we will consider the pullback of a T-crystal on X' under the Frobenius $F_{X/S} : X \to X'$.

Definition 3.2.15. *A T-crystal (E', A) on X' is called admissible if it is a p-torsion free crystal of $\mathcal{O}_{X'/S}$-module and compatible with $F_{X/S}$.*

Proposition 3.2.16. *([16, 5.2.5]) Let (E', A) be an admissible T-crystal on X'/S, then the pullback crystal $F^*_{X/S}(E', A)$ is horizontal, i.e. the filtration induced by A on $F^*_{X/S} E'$ are subcrystals. Moreover, this filtration on $F^*_{X/S} E'$ is G-transversal to the PD-ideal (p, γ). In particular, $F^*_{X/S}(E', A)$ is still a T-crystal.*

To any admissible T-crystal on X'/S, we can associate to it an F_γ-span on X/S, whose definition is given as follows.

Definition 3.2.17. *[16, Definition 5.2.3] An F_γ-span on X/S is a pair (E', M_γ) where E' is a p-torsion free crystal of $\mathcal{O}_{X'}$-module on X'/S and M_γ is a filtration of $F^*_{X/S} E'$ by subcrystals which is G-transversal to the PD-ideal (p, γ).*

We will follow the notation used in [16], i.e. the associated F_γ-span to an admissible T-crystal (E', A) is denoted by $\mu(E', A)$. One can prove [16, 5.2.8, 5.2.9], for an admissible F-span $\Phi : F^*_{Y/S} E' \to E$, there exists a T-crystal (E', A) such that $\mu(E', A) \cong (E', M_\gamma)$, where M_γ is the saturation of the filtration M with respect to the ideal (p, γ) (definition 3.2.5). The T-crystal will be denoted by $\alpha_{X/S}(\Phi)$.

Now given a T-crystal (E', A) on X' and suppose its width is less than p, then the same is true for $F^*_{X/S}(E', A)$. Under this condition, the property G-transversal to (p, γ) is equivalent to G-transversal to (p). Then we can prove the following

Proposition 3.2.18. *[16, Theorem 5.2.13] The functor μ establishes an equivalence between the category of admissible T-crystals on X'/S with width less than p and the category of admissible F-spans on X/S of width less than p.*

3.3 F-T-Crystals and Their Reductions

In this section S will be a formal $W(k)$-scheme equipped with a lifting F_S of the absolute Frobenius F_{S_0}. For a formal scheme $\mathcal{Y}$ over S_ν, we denote the fiber product $\mathcal{Y}_\nu \times_{S_\nu} S_\delta$ by $\mathcal{Y}_\delta$.

Definition 3.3.1. *([17, 5.3.1]) Suppose Y/S_ν is a perfectly smooth morphism of fine log schemes, and let X/S_0 be its reduction modulo p. An F-T-crystal on Y/S is a triple (E, Φ, B), in which E is a p-torsion free crystal of $\mathcal{O}_{X/S}$-module, $\Phi : F_X^* E \to E$ is a p-isogeny, and (E, B) is a T-crystal on Y/S_ν which is compatible with F_X and with $\pi_{X/S}$, such that $(E', A) := \pi^*_{X/S}((E, B)|_{X/S}) = \alpha((F_X^* E, E, \Phi))$.*

Given an F-T-crystal (E, B, Φ), the F-span and the T-crystal over X/S which it gives rise to are automatically admissible. Therefore, let $\{M^k\}$ be the filtration on $F_X^* E$ induced from the F-span structure, and $\{A^k\}$ be the filtration defined on E' (3.1.4), then we have a natural isomorphism

$$F^*_{X/S} A^k E' \cong M^k F_X^* E. \tag{3.3.1}$$

Indeed, by proposition (3.1.8), we can prove the above isomorphism over LS thickenings, then we can check these isomorphisms are compatible and can be written in the form in (3.3.1). Let

$$A^\epsilon E' = \sum_{0 \le k \le n} p^{n-k} A^k E',$$

by combining the isomorphism (3.1.3), we get

$$\Phi : F^*_{X/S} A^\epsilon E' \cong E. \tag{3.3.2}$$

Remark 3.3.2. *The subobject $A^\epsilon E'$ of E' is not invariant under the action of ∇'. In fact, given $p^{n-k} x_k \in A^\epsilon E'$ for some $x_k \in A^k E'$, $\nabla'(x_k)$ can be written as $\sum_i x^i_{k-1} \otimes \omega_i$ for some $x^i_{k-1} \in A^{k-1} E'$, from which one cannot deduce $p^{n-k} x^i_{k-1} \in A^\epsilon E'$.*

The proof of our main result, i.e. Theorem 7 is based on investigation of the reduction of F-T-crystals over LS thickenings locally. The following lemma will play a key role in the proof.

A Key Lemma [16, 4.5.3]

Let $\mathcal{Y}$ be a LS thickening of Y, (E', A) (by abuse of notation, short for $(E'_{\mathcal{Y}'}, A_{\mathcal{Y}'})$) be the value of (E', A) over $\mathcal{Y}' = \mathcal{Y} \times_{F_S} S$. We will consider the subobject $A^\epsilon E'$ of E'. Obviously, there is a natural surjective morphism

$$\alpha : \bigoplus_i A^i E' \twoheadrightarrow A^\epsilon E', \qquad (c_i) \mapsto \sum_i p^{n-i} c_i.$$

Let $j_i : A^i E \hookrightarrow A^{i-1} E$ be the natural inclusion, and ∂ be the morphism

$$\bigoplus_i A^i E' \to \bigoplus_i A^i E', \qquad (\cdots, 0, \overset{i}{c_i}, 0, \cdots) \mapsto (\cdots, 0, \overset{i}{-p c_i}, j_i(c_i), 0, \cdots) \tag{3.3.3}$$

Obviously, we have $\mathrm{im}(\partial) \subseteq \ker(\alpha)$. However, the inverse inclusion is generally not true, i.e. the following sequence is not necessarily exact at the middle term

$$\bigoplus_i A^i E' \xrightarrow{\partial} \bigoplus_i A^i E' \xrightarrow{\alpha} A^\epsilon E'.$$

However, if we consider the this sequence after modulo some power of p and assume the filtration satisfy suitable transversality condition, then things become different. The main point is that one can dig out a larger power of p from the transversality condition, which will eliminate the difference between $\ker(\alpha)$ and $\mathrm{im}(\partial)$ after modulo a suitable power of p.

Lemma 3.3.3. *Let δ be a positive integer satisfying $\nu \geq \delta$, using the notations in the above paragraphs, then we have the following commutative diagram with the lower row exact.*

$$\begin{array}{ccccc}
\oplus_i A^i E' & \xrightarrow{\quad\partial\quad} & \oplus_i A^i E' & \xrightarrow{\quad\alpha\quad} & A^\epsilon E' \\
\downarrow{\scriptstyle \pi_\delta} & & \downarrow{\scriptstyle \pi_\delta} & & \downarrow \\
\oplus_i A^i E' + p^\delta E'/p^\delta E' & \xrightarrow{\partial_\delta} & \oplus_i A^i E' + p^\delta E'/p^\delta E' & \xrightarrow{\alpha_\delta} & A^\epsilon E'/p^\delta A^\epsilon E'.
\end{array}$$

Proof. First of all, the filtration $\{A^i E'\}$ induces a filtration $\{A^i E' + p^\delta E'/p^\delta E'\}$ on $E'/p^\delta E'$ and it's easy to check the morphisms ∂ and α induce ∂_δ and α_δ as displayed in the above commutative diagram. Then by diagram chasing, the exactness of the lower row is equivalent to

$$\alpha^{-1}(p^\delta A^\epsilon E') = \ker\pi_\delta + \mathrm{im}\partial.$$

First we prove the direction $\supseteq$, obviously we have $\alpha^{-1}(p^\delta A^\epsilon E') \supseteq \mathrm{im}\partial$. Next we prove $\alpha^{-1}(p^\delta A^\epsilon E') \supseteq \ker\pi_\delta$. Suppose $x_i \in \ker\pi_\delta \cap A^i E'$, then since (E', A, ∇) is compatible with $p^\delta \mathcal{O}_\mathcal{Y}$, by corollary (3.2.13) we have

$$\begin{aligned}
x_i &\in p^\delta E' \cap A^i E' \\
&= p^\delta A^i E' + p^{h(1)} A^{i-1} E' + p^{h(2)} A^{i-2} E' + \cdots
\end{aligned}$$

where $h(k) = \mathrm{ord}_p\langle \frac{p^{k(\nu+1)}}{k!} \rangle$. Then since $\nu \geq \delta$, $h(k) \geq k + \delta$ hence

$$x_i \in p^\delta (A^i E' + p^1 A^{i-1} E' + p^2 A^{i-2} E' + \cdots)$$

and

$$\alpha(x_i) = p^{n-i} x_i \in p^\delta (p^{n-i} A^i E' + p^{n-i+1} A^{i-1} E' + \cdots) = p^\delta A^\epsilon E'.$$

Next we prove the direction $\alpha^{-1}(p^\delta A^\epsilon E') \subseteq \ker\pi_\delta + \mathrm{im}\partial$. Since α is surjective, it's enough to prove $\ker\alpha \subseteq \ker\pi_\delta + \mathrm{im}\partial$. Suppose $c = (c_i, \cdots) \in \ker\alpha, c_i \in A^i E'$, then we have

$$p^{n-i} c_i + p^{n-i+1} c_{i-1} + \cdots = 0. \tag{3.3.4}$$

Dividing both sides by p^{n-i}, and by (E', A) is compatible with $p\mathcal{O}_\mathcal{Y}$, we find

$$c_i \in pE' \cap A^i E' = pA^i E' + p^{h(1)} A^{i-1} E' + \cdots + p^{h(i-j)} A^j E' + \cdots.$$

Therefore c_i can be written as

$$c_i = py_i + p^{h(1)} y_{i-1} + \cdots + p^{h(i-j)} y_j + \cdots = py_i + p^{h(1)} y,$$

where $y_j \in A^j E', j \leq i$. If we use the right side of the above equality to substitute for c_i in (3.3.4), then we get

$$\begin{aligned}
0 &= p^{n-i}(py_i + p^{h(1)} y_{i-1} + \cdots + p^{h(i-j)} y_j + \cdots) + p^{n-i+1} c_{i-1} + \cdots \\
&= p^{n-i+1}(y_i + p^{h(1)-1} y_{i-1} + c_{i-1}) + p^{n-i+2}(p^{h(2)-2} y_{i-2} + c_{i-2}) + \cdots \\
&\qquad + p^{n-i+j}(p^{h(j)-j} y_{i-j} + c_{i-j}) + \cdots.
\end{aligned}$$

Let $z_j = p^{h(j)-j} y_{i-j}$, since $h(j) \geq j + \delta$, then $z_j \in p^\delta A^{i-j} E'$ hence $(\cdots, 0, z_j, 0, \cdots) \in \ker\pi_\delta$. Moreover, the above equality implies

$$(y_i + \overset{i-1}{z_{i-1}} + c_{i-1}, z_{i-2} + c_{i-2}, \cdots, z_{i-j} + c_{i-j}, \cdots) \in \ker\alpha.$$

By induction on the number of nonzero elements of (c_i), the above element falls in $\ker\pi_\delta + \mathrm{im}\partial$. Therefore, so does the element $(y_i \overset{i-1}{+} c_{i-1}, c_{i-2}, \cdots, c_{i-j}, \cdots)$. Note that our claim is proved since

$$\begin{aligned}
&(\overset{i}{0}, y_i + c_{i-1}, c_{i-2}, \cdots, c_{i-j}, \cdots) - (\overset{i}{c_i}, c_{i-1}, \cdots, c_{i-j}, \cdots) \\
=&(-c_i, y_i, 0, \cdots) \\
=&(-py_i - p^{h(1)}y, y_i, 0, \cdots) \\
=&\partial(\cdots, 0, \overset{i}{y_i}, 0, \cdots) - p^{h(1)}y,
\end{aligned}$$

and $p^{h(1)}y \in \ker\pi_\delta$. □

Reduction of F-T-Crystals

The main objective of this subsection is to show the reduction of an F-T-crystal modulo p can be endowed with a structure of a Fontaine module if X is smooth (in the classical sense) over S_0.

Let

$$L^\epsilon_\delta(E', A) := (\oplus_i A^i E' + p^{\delta+1}E')/p^{\delta+1}E' + \mathrm{im}\partial, \tag{3.3.5}$$

and $(E'_{\mathcal{Y}'}, A_{\mathcal{Y}'})$ be the value of (E', A) over an LS thickening $\mathcal{Y}'$ of some open subset of X'. Combining the key lemma above and (3.3.1), we obtain

$$M^\epsilon_{\mathcal{Y}} F^*_{\mathcal{Y}/S} E'_{\mathcal{Y}'} \otimes W_\delta \cong F^*_{\mathcal{Y}/S}(A^\epsilon_{\mathcal{Y}'} E'_{\mathcal{Y}'}/p^{\delta+1} A^\epsilon_{\mathcal{Y}'} E'_{\mathcal{Y}'}) \cong F^*_{\mathcal{Y}/S} L^\epsilon_\delta(E'_{\mathcal{Y}'}, A_{\mathcal{Y}'}), \tag{3.3.6}$$

where $M^\epsilon_{\mathcal{Y}} F^*_{\mathcal{Y}/S} E'_{\mathcal{Y}'} = \sum_i p^{n-i} M^i_{\mathcal{Y}} F^*_{\mathcal{Y}/S} E'_{\mathcal{Y}'}$ and $W_\delta = W(k)/p^{\delta+1}W(k)$.

On the other hand, recall the F-span structure Φ induces an isomorphism (3.1.3)

$$M^\epsilon F^*_X E \cong p^n E \cong E,$$

hence the restriction of Φ to $M^\epsilon_{\mathcal{Y}} F^*_{\mathcal{Y}/S} E'_{\mathcal{Y}'}$ is also an isomorphism, i.e. we have

$$M^\epsilon_{\mathcal{Y}} F^*_{\mathcal{Y}/S} E'_{\mathcal{Y}'} \cong E_{\mathcal{Y}}.$$

Combining this isomorphism with (3.3.6), we get

$$F^*_{\mathcal{Y}/S} L^\epsilon_\delta(E'_{\mathcal{Y}'}, A_{\mathcal{Y}'}) \cong E_{\mathcal{Y}} \otimes W_\delta. \tag{3.3.7}$$

Now we can state the main result of this work.

Theorem 7. *Let Y be a smooth S_ν-scheme, $\nu \geq 1$, X/S_0 be its reduction modulo p and (E, Φ, B) an F-T-crystal on Y/S of width less than p. Let ∇ be the integrable connection on E_Y corresponding to the crystal structure of E and Y_1 be the reduction of Y modulo p^2. Then the triple (E_X, B_X, ∇_X) together with the lifting $Y_1' = Y_1 \times_{F_{S_1}} S_1$ of X' to S_1 and the local isomorphisms* (3.3.7) *for $\delta = 0$ constitute a Fontaine module as defined in (1.2.4).*

Note that $L^\epsilon_0(E'_{\mathcal{Y}'}, A_{\mathcal{Y}'})$ is nothing but the restriction of $Gr_{A_{X'}} E'_{X'} \cong \pi^*_{X/S} Gr_{B_X} E_X$ to the reduction of $\mathcal{Y}'$ modulo p. Moreover, $L^\epsilon_0(E'_{\mathcal{Y}'}, A_{\mathcal{Y}'})$ endowed with a Higgs field $\pi^*_{X/S} Gr_{B_X} \nabla_X$. Besides, for any open affine subscheme U of X, one can find a local lifting $\mathcal{U}$ of U such that $\mathcal{U}_1 \cong Y_1 \times_Y U$. Therefore, by definition (1.2.4), we need only to verify the following two claims

1. The coherent $\mathcal{O}_X$-module $F^*_{\mathcal{Y}/S} L^\epsilon_0(E'_{\mathcal{Y}'}, A_{\mathcal{Y}'})$ together with connection induced by (3.3.7) coincides with $C^{-1}_{\mathcal{Y}_1'/S_1} \pi^*_{X/S}(Gr_{B_X} E_X, Gr_{B_X} \nabla_X)$, where $\mathcal{Y}_1'$ is the reduction of $\mathcal{Y}'$ modulo p^2.

2. The isomorphisms induced by (3.3.7) between $F^*_{\mathcal{Y}/S} L^\epsilon_0(E'_{\mathcal{Y}'}, A_{\mathcal{Y}'})$'s associated to different LS thickenings satisfy the condition in the definition of a Fontaine module (1.2.4).

Proof. The proofs of the two claims above will be given in the first and second step respectively.

Step 1. As the connection on $F^*_{\mathcal{Y}/S}L^\epsilon_0(E'_{\mathcal{Y}'}, A_{\mathcal{Y}'})$ induced from (3.3.7) is not so apparent to write down directly, we first consider the induced connection on $M^\epsilon_{\mathcal{Y}}F^*_{\mathcal{Y}/S}E'_{\mathcal{Y}'}$ as an intermediate. Let $\{t_i\}_{1\leq i\leq r}$ be a set of local coordinates for $\mathcal{Y}/S$, i.e. $\{dt_i\}_{1\leq i\leq r}$ form a basis of $\Omega^1_{\mathcal{Y}/S}$. If we denote by $\bar{t}_i$ be the image of t_i in $\mathcal{O}_{\mathcal{Y}_0}$, then $\{\bar{t}_i\}_{1\leq i\leq r}$ form a basis of $\Omega^1_{\mathcal{Y}_0/S_0}$. Let $\{\xi_i\}_{1\leq i\leq r}$ be the dual basis of $\{d\bar{t}_i\}_{1\leq i\leq r}$.

Let ∇' be the integrable connection on $E'_{\mathcal{Y}'}$ and $x_k\in A^k_{\mathcal{Y}'}E'_{\mathcal{Y}'}$, then by Griffiths transversality $\nabla'(x_k)$ can be written as $\sum_i x^i_{k-1}\otimes d(1\otimes t_i)$, where $x^i_{k-1}\in A^{k-1}_{\mathcal{Y}'}E'_{\mathcal{Y}'}$. Note that $1\otimes p^{n-k}x_k\in M^\epsilon_{\mathcal{Y}}F^*_{\mathcal{Y}/S}E'_{\mathcal{Y}'}$, if we let $F_{\mathcal{Y}/S}(1\otimes t_i)=t_i^p+p\tau_i$, the induced connection $F^*_{\mathcal{Y}/S}\nabla'$ on $M^\epsilon_{\mathcal{Y}}F^*_{\mathcal{Y}/S}E'_{\mathcal{Y}'}$ is given by

$$\begin{aligned}F^*_{\mathcal{Y}/S}\nabla'(1\otimes p^{n-k}x_k) &= (1\otimes\sum_i p^{n-k}x^i_{k-1})\otimes F^*_{\mathcal{Y}/S}(dt_i+p\tau_i)\\ &= (1\otimes\sum_i p^{n-k}x^i_{k-1})\otimes d(t_i^p+p\tau_i)\\ &= (1\otimes\sum_i p^{n-k+1}x^i_{k-1})\otimes (t_i^{p-1}dt_i+d\tau_i).\end{aligned}$$

Take $\delta=0$, by (3.3.6) one can see the induced connection on $F^*_{X/S_0}L^\epsilon_0(E'_{\mathcal{Y}'}, A_{\mathcal{Y}'})\cong F^*_{X/S_0}Gr_{A_X}E'_X$ is given by

$$(\bar{x}_k)\mapsto \theta_{\xi_i}(\bar{x}_k)\otimes\frac{dF_{\mathcal{Y}/S}(t_i)}{p} \tag{3.3.8}$$

where $\bar{x}_k$ is the image of x_k in $Gr_{A_{X'}}E_{X'}$ and $\theta=Gr_{A_{X'}}\nabla'_{X'}$. Hence this connection coincides with the one defined on $C^{-1}_{\mathcal{Y}'_1/S_1}\pi^*_{X/S}(Gr_{B_X}E_X, Gr_{B_X}\nabla_X)$ and the first claim is proved.

Step 2. Next we verify the isomorphisms (3.3.7) over different LS thickenings satisfy the transition condition in the definition of a Fontaine module (1.2.4). As a notification before the proof, one should bear in mind (see definition (B.1.13)) an LS thickening is not only a scheme but also endowed with a lifting of Frobenius.

For convenience, we write down the the isomorphism (3.3.7) in case $\mathcal{Y}$ is a LS thickening of an open subset of $U\subseteq X$ and $\delta=0$

$$\phi_{\mathcal{Y}}: F^*_{U/S}\pi^*_{U/S}Gr_{B_U}E_U\cong E_U. \tag{3.3.9}$$

Here we use the notation $\phi_{\mathcal{Y}}$ to indicate the definition of this isomorphism depends on the LS thickening $\mathcal{Y}$ of U. Let $\{U^i\}_{i\in I}$ be a family of open affine subschemes covering Y and $U^i_0\subseteq U^i$ be the closed subscheme defined by the ideal (p). We choose an LS thickening $\mathcal{Y}^i$ of U^i_0 for all i, on which (see definition (B.1.13)) there is a lifting $F_{\mathcal{Y}^i/S}$ of $F_{U^i_0/S_0}$.

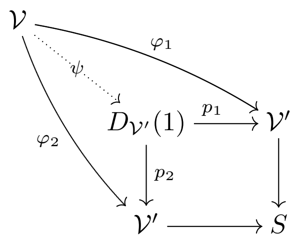
Given an open subset $V\subseteq U^i$, the LS thickening $\mathcal{Y}^i$ induces an LS thickening of V and we will denote it by $\mathcal{V}^i$. Moreover, it is easy to see the restriction of $\phi_{\mathcal{Y}^i}$ to V_0 coincides with $\phi_{\mathcal{V}^i}$. Therefore, to prove the second claim, we need only to check for any two open subsets $U^i, i=1,2$, in the covering family, the morphism $(\phi_{\mathcal{Y}^1}|_{V_0})^{-1}(\phi_{\mathcal{Y}^2}|_{V_0})$ satisfy the condition (1.2.11) in definition (1.2.4), where V_0 is the closed subscheme of $V=U^1\times_S U^2$ defined by the ideal (p).

If we denote by $\mathcal{V}^i$ the LS thickening of V_0 induced by $\mathcal{Y}^i$, $i=1,2$, then by smoothness criteria there is a unique isomorphism $\varepsilon:\mathcal{V}^1\xrightarrow{\cong}\mathcal{V}^2$. Thus the pair $(\mathcal{V}^1, \varepsilon^{-1}F_{\mathcal{V}^2/S}\varepsilon)$ provides another LS thickening of V_0 and we denote it by $\mathcal{W}^1$. It is easy to see the isomorphisms $\phi_{\mathcal{W}^1}$ and $\phi_{\mathcal{V}^2}$ coincide over V_0, in particular, $(\phi_{\mathcal{V}^2}|_{V_0})^{-1}(\phi_{\mathcal{W}^1}|_{V_0})$ satisfies the condition (1.2.11). If we could prove $(\phi_{\mathcal{W}^1}|_{V_0})^{-1}(\phi_{\mathcal{V}^1}|_{V_0})$ also satisfies the condition (1.2.11), then we are done. For saving notations, in the sequel the Frobenii $F_{\mathcal{V}^1/S}$ and $\varepsilon^{-1}F_{\mathcal{V}^2/S}\varepsilon$ on $\mathcal{V}^1$ will be denoted by φ_1 and φ_2 respectively and the LS thickening $\mathcal{V}^1$ will be denoted by $\mathcal{V}$.

Recall that the isomorphism (3.3.7) is induced from the morphism of crystals

$$\Phi: F^*_{X/S}E'\to E.$$

Thus to investigate $(\phi_{\mathcal{V}^1}|_{V_0})(\phi_{\mathcal{W}^1}|_{V_0})^{-1}$, one needs to investigate the difference between the pullbacks of E' via φ_1 and φ_2. Recall that fiber products exist in crystalline topos and the fiber product of a smooth object with itself is nothing but the PD-envelop of the diagonal in its self-product. Therefore the morphism (φ_1, φ_2) from $\mathcal{V}$ to $\mathcal{V}' = \mathcal{V} \times_{F_S} S$ factors through $D_{\mathcal{V}'}(1)$. In other words, if we denote by $p_i, i = 1, 2$, the natural projection from $D_{\mathcal{V}'}(1)$ to $\mathcal{V}'$, then there exists $\psi : \mathcal{V} \to D_{\mathcal{V}'}(1)$ such that $\varphi_i = p_i\psi$. To be more explicit, we have the following commutative diagram



In particular, the following isomorphism

$$\varphi_{21} : \varphi_1^* E'_{\mathcal{V}'} \cong \varphi_2^* E'_{\mathcal{V}'} \tag{3.3.10}$$

is nothing but the pullback of the the following isomorphism via ψ

$$\alpha_{21} : p_1^* E'_{\mathcal{V}'} \cong p_2^* E'_{\mathcal{V}'}. \tag{3.3.11}$$

On the other hand, by (3.2.14), (3.3.11) induces an isomorphism

$$\alpha_{21} : A_1^k p_1^* E'_{\mathcal{V}'} \cong A_2^k p_2^* E'_{\mathcal{V}'},$$

where the filtration A_i^k on $p_i^* E'_{\mathcal{V}'}$ is defined as

$$p_i^* A^k E'_{\mathcal{V}'} + J_{\mathcal{V}'} p_i^* A^{k-1} E'_{\mathcal{V}'} + \cdots + J_{\mathcal{V}'}^{[i]} p_i^* A^{k-i} E'_{\mathcal{V}'} + \cdots.$$

Now suppose $x \in A^k E'_{\mathcal{V}'}$, then the image of $p^{n-k} x \otimes 1$ under α_{21} is given by (see [16, Theorem 1.1.8])

$$\sum_I \eta_{12}^{[I]} \otimes \partial_I (p^{n-k} x). \tag{3.3.12}$$

In the above formula, $I = (n_1, \cdots, n_r)$, $\eta_{12}^{[I]} = \prod_{1 \le i \le r} \frac{(p_2^*(t_i') - p_1^*(t_i'))^{n_i}}{n_i!}$ and $\partial_I = \prod_{1 \le i \le r} \nabla_{t_i'}$, where $t_i' \in \mathcal{O}_{\mathcal{V}'}$ such that $\{dt_i'\}_{1 \le i \le r}$ is a local basis of $\Omega^1_{\mathcal{V}'/S}$.

Now apply the morphism ψ to $p^{n-k} x \otimes 1$ and $1 \otimes \partial_I (p^{n-k} x)$, then the image of $p^{n-k} x \otimes 1$ under φ_{21} is given by

$$\sum_I \prod_{1 \le i \le r} \frac{(\varphi_2(t_i') - \varphi_1(t_i'))^{n_i}}{n_i!} \otimes \partial_I (p^{n-k} x).$$

Note that $p | \varphi_2(t_i') - \varphi_1(t_i')$, hence the above result can be written as

$$\sum_I \prod_{1 \le i \le r} \frac{1}{n_i!} \left(\frac{\varphi_2(t_i') - \varphi_1(t_i')}{p} \right)^{n_i} \otimes p^{|I|+n-k} \partial_I (x). \tag{3.3.13}$$

By Griffiths transversality, $p^{|I|+n-k} \partial_I(x)$ also falls in $A^\epsilon E'_{\mathcal{V}'}$. Next we consider the images of $p^{n-k} x \otimes 1$ and $1 \otimes p^{|I|+n-k} \partial_I(x)$ under ρ_1 and ρ_2 respectively, where $\rho_i, i = 1, 2$ is the following projection

$$\rho_i : \varphi_i^* A^\epsilon E'_{\mathcal{V}'} \to \varphi_i^* A^\epsilon E'_{\mathcal{V}'} / p \varphi_i^* A^\epsilon E'_{\mathcal{V}'} \cong F^*_{V_0/S} \pi^*_{V_0/S} Gr_{B_{V_0}} E_{V_0} \tag{3.3.14}$$

Note the image of $\rho_1(p^{n-k} x \otimes 1)$ under $(\phi_{\mathcal{W}^1}|_{V_0})^{-1} (\phi_{\mathcal{V}^1}|_{V_0})$ is exactly $\rho_2(1 \otimes p^{|I|+n-k} \partial_I(x))$. Let y be the image of x under the composite

$$E'_{\mathcal{V}'} = \pi^*_{\mathcal{V}/S} E_{\mathcal{V}} \to \pi^*_{V_0/S} E_{V_0} \to \pi^*_{V_0/S} Gr_{B_{V_0}} E_{V_0}$$

then $\rho_1(p^{n-k}x\otimes 1)=y$ and $\rho_2(1\otimes p^{|I|+n-k}\partial_I(x))=\theta_I(y)$, where $\theta=Gr_{B_X}\nabla_X$ and

$$\theta_I=\prod_{1\le i\le r}\theta_{\xi'_i}^{n_i},$$

where $\{\xi'_i\}_{1\le i\le r}$ is a basis of $T_{V'_0/S}$ dual to the reduction of $\{dt'_i\}_{1\le i\le r}$ modulo p. On the other hand, note that after modulo p, $\frac{\varphi_2(t'_i)-\varphi_1(t'_i)}{p}$ is nothing but $\langle\xi,1\otimes dt'_i\rangle$, where ξ is the element of $F^*_{V_0/S}T_{V'_0/S}$ defined by the difference of the reductions of φ_1 and φ_2 modulo p^2. Therefore by (1.2.9), (1.2.10), $\rho_2(p^{|I|+n-k}\partial_I(x)\otimes 1)$ is exactly in the form (1.2.11) required by definition (1.2.4), hence the two claims are proved.

□

Remark on Log Fontaine modules and F-T-Crystals

Unlike the smooth case, now we will show the reduction of a log F-T-crystal is not a log Fontaine module as defined in (2.2.6).

Indeed, if we want to prove the theorem in the log smooth case, it suffices to verify the two claims as above in the smooth case. As can be seen easily, the first step of the above proof can be adapted to log smooth case just by replacing coordinates by log coordinates.

When it comes to the second claim, the argument in the former part of the second step above still works for the log smooth case. The difference starts to appear from the expression of $\eta_{12}^{[I]}$ in the formula (3.3.12). Let $\{m_i\}_{1\le i\le r}, m_i\in\mathcal{M}_{\mathcal{V}'}$ be a set of local log coordinates on $\mathcal{V}'$ and $t'_i\in\mathcal{O}_{\mathcal{V}'}$ be the image of m_i under the structure map $\mathcal{M}_{\mathcal{V}'}\to\mathcal{O}_{\mathcal{V}'}$. Then $\eta_{12}^{[I]}$ in the log setting is given by

$$\prod_{1\le i\le r}(u_i-1)^{[n_i]}\ ,$$

where u_i is locally defined by the unique element of $\ker(\mathcal{O}^*_{D_{\mathcal{V}'}(1)}\to\mathcal{O}^*_{\mathcal{V}'})$ satisfying $p_2^*(t'_i)=u_ip_1^*(t'_i)$. We can still prove the following formula

$$\frac{\psi(u_i-1)}{p}=\langle\xi,d\log\pi^*_{\mathcal{V}/S}m_i\rangle, \tag{3.3.15}$$

where ξ as before is the local section of $F^*_{V_0/S}T_{V'_0/S}$ defined by the difference of the reductions of φ_1 and φ_2 modulo p^2.

Let $i:\mathcal{V}'\to D(1)_{\mathcal{V}'}$ be the closed immersion of the diagonal, then we have the following exact sequence of monoid structures on $\mathcal{V}'$

$$0\to i^{-1}(1+J)\xrightarrow{\lambda}i^{-1}\mathcal{M}_{D_{\mathcal{V}'}(1)}\to\mathcal{M}_{\mathcal{V}'}\to 0,$$

where J is the defining ideal of $\mathcal{V}'$ in $D_{\mathcal{V}'}(1)$ and λ is induced by the natural inclusion $\mathcal{O}^*_{D_{\mathcal{V}'}(1)}\hookrightarrow\mathcal{M}_{D_{\mathcal{V}'}(1)}$.

Note that we have $1+J=\ker(\mathcal{O}^*_{D_{\mathcal{V}'}(1)}\to\mathcal{O}^*_{\mathcal{V}'})$, thus for each $1\le i\le r$ there exists a unique element $\chi(m_i)\in 1+J$ such that

$$p_2^\sharp(m_i)=p_1^\sharp(m_i)+\lambda(\chi(m_i)). \tag{3.3.16}$$

Applying $\alpha_{D_{\mathcal{V}'}(1)}:\mathcal{M}_{D_{\mathcal{V}'}(1)}\to\mathcal{O}_{D_{\mathcal{V}'}(1)}$ to this equality we find $\chi(m_i)$ is nothing but u_i defined above. Then we apply $\psi^\sharp$ to both sides of (3.3.16) we get

$$\psi^\sharp p_2^\sharp(m_i)=\psi^\sharp p_1^\sharp(m_i)+\psi^\sharp(\lambda(u_i)).$$

Since $\varphi_i=p_i\psi, i=1,2$, we get

$$\varphi_2^\sharp(m_i)=\varphi_1^\sharp(m_i)+\psi^\sharp(\lambda(u_i)).$$

Then apply the map $\alpha_{\mathcal{V}}:\mathcal{M}_{\mathcal{V}}\to\mathcal{O}_{\mathcal{V}}$ to the above equality, we obtain

$$\alpha_{\mathcal{V}}(\varphi_2^\sharp(m))=\alpha_{\mathcal{V}}(\varphi_1^\sharp(m))\alpha_{\mathcal{V}}(\psi^\sharp(\lambda(u_i))).$$

Since we have $\alpha_{\mathcal{V}}\psi^\sharp = \psi^\sharp \alpha_{D_{\mathcal{V}'}(1)}$, the above formula becomes

$$\alpha_{\mathcal{V}}(\varphi_2^\sharp(m_i)) = \alpha_{\mathcal{V}}(\varphi_1^\sharp(m_i))\psi(u_i).$$

By formula (A.2.1) to the two liftings φ_1 and φ_2, we have

$$\alpha_{\mathcal{V}}(\varphi_2^\sharp(m_i) - \varphi_1^\sharp(m_i)) = 1 + [p]\langle d\log \pi^*_{\mathcal{V}/S} m_i, \varphi_2 - \varphi_1\rangle,$$

hence (3.3.15) is proved. To prove the second claim, it remains to check

$$\rho_2(1 \otimes p^{|I|+n-k}\partial_I(x)) = \theta_I(y).$$

However, in the log setting, we have (see B.2.3)

$$\partial_I(x) = \prod_{1\le i\le r}\prod_{0\le j\le n_i-1}(\partial_i^{\log} - j),$$

which implies $\partial_I(x)$ is not even in $A^\epsilon E'_{\mathcal{V}'}$ unless $\partial_I(x) = 0$ for all $|I| \geq 1$. Therefore, the reduction of a log F-T-crystals is not a log Fontaine module in general.

Appendix A

Logarithmic Structures

A.1 Log Schemes

First we give some definitions on log schemes, which are necessary for further development.

Pre-log and Log structures on Schemes

Definition A.1.1. *Let X be a scheme, a pre-log structure on X is a sheaf of commutative monoid $\mathcal{M}$, endowed with a homomorphism $\alpha : \mathcal{M} \to \mathcal{O}_X$ with respect to the multiplicative monoid structure of $\mathcal{O}_X$. The pair $(\mathcal{M}, \alpha)$ is usually shortened as $\mathcal{M}$. When the restriction of α to $\alpha^{-1}(\mathcal{O}_X^*)$ is an isomorphism, we say $(\mathcal{M}, \alpha)$ defines a log structure on X.*

Let $(X_i, \mathcal{M}_i)$ be a scheme with pre-log structure $\alpha_i : \mathcal{M}_i \to \mathcal{O}_{X_i}$ for $i = 1, 2$. A morphism from $(X_1, \mathcal{M}_1)$ to $(X_2, \mathcal{M}_2)$ is a pair (f, h), where $f : X_1 \to X_2$ is a morphism of schemes and $h : f^{-1}(\mathcal{M}_2) \to \mathcal{M}_1$ is a homomorphism of monoids on X_1 making the following diagram commutative

$$\begin{array}{ccc} f^{-1}(\mathcal{M}_2) & \xrightarrow{h} & \mathcal{M}_1 \\ {\scriptstyle f^{-1}(\alpha_2)}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ f^{-1}(\mathcal{O}_{X_2}) & \longrightarrow & \mathcal{O}_{X_1} \end{array} .$$

Sometimes the notation (f, h) for a morphism between log schemes is abbreviated as a single f.

Definition A.1.2. *For a pre-log structure $(\mathcal{M}, \alpha)$ on X, the associated log structure, denoted by $\mathcal{M}^a$, is defined to be the push-out of the following diagram*

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^*) & \longrightarrow & \mathcal{M} \\ \downarrow & & \\ \mathcal{O}_X^* & & \end{array}$$

in the category of sheaves of monoids on X, endowed with

$$\mathcal{M}^a \to \mathcal{O}_X; \quad (m, a) \mapsto \alpha(m)a \quad (m \in \mathcal{M}, a \in \mathcal{O}_X^*).$$

The associated log structure defined above is universal in the sense that any morphism from the pre-log structure $\mathcal{M}$ to a log structure on X factors through $\mathcal{M}^a$.

Example A.1.3. 1. *The standard examples of log structures on a scheme are given by its divisors. Let X be a regular scheme, D be a a reduced divisor with normal crossings, then D defines a nontrivial log structure on X by*

$$\mathcal{M} = \{g \in \mathcal{O}_X \,|\, g \text{ is invertible outside } D\} \subset \mathcal{O}_X.$$

When the divisor is empty, $\mathcal{M} = \mathcal{O}_X^*$ is called the trivial log structure.

2. *Another typical example of log scheme is constructed from a monoid directly. Let P be a monoid, R be a commutative ring and $R[P]$ be the ring generated by P over R, then $\operatorname{Spec} R[P]$ has a canonical log structure defined by $P \to R[P]$. Such construction can be easily generalized to general schemes as follows. Given a scheme X, then any morphism of monoids $P \to \Gamma(X, \mathcal{O}_X)$ defines a pre-log structure on X by $P_X \to \mathcal{O}_X$, where P_X is the sheaf of constant monoid defined by P on X. Moreover, one can prove, if X is a R-scheme, P is integral (Definition A.1.8), and the induced morphism $X \to \operatorname{Spec} R[P]$ is flat, then the associated log structure of P_X can be identified with the submonoid of $\mathcal{O}_X$ generated by $\mathcal{O}_X^*$ and P_X. In particular, if the morphism $P \to \mathcal{O}_X$ factors through $\mathcal{O}_X^*$, then it defines a trivial log structure.*

Direct and Inverse Images, Fiber Products

Definition A.1.4. *Let $f : X \to Y$ be a morphism of schemes, $\mathcal{M}$ be a log structure on X, then the fiber product of the diagram*

$$f_*\mathcal{M} \to f_*\mathcal{O}_X \leftarrow \mathcal{O}_Y$$

in the category of sheaves of monoids is called the direct image of $\mathcal{M}$.

Definition A.1.5. *Let $f : X \to Y$ be a morphism of schemes, $\mathcal{N}$ be a log structure on Y, then the monoid $f^{-1}(\mathcal{N})$ together with the composite $f^{-1}(\mathcal{N}) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ defines a pre-log structure on X. The associated log structure, denoted by $f^*(\mathcal{N})$, is called the inverse image of $\mathcal{N}$ under f.*

Definition A.1.6. *A morphism $i : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is called a (resp. an exact) closed imbedding if*

1. *the underlying morphism of schemes is a closed imbedding,*
2. *morphism of log structure $i^*\mathcal{N} \to \mathcal{M}$ is surjective (resp. bijective).*

Given a finite inverse system of log schemes $\{X_\lambda, \mathcal{M}_\lambda\}_\lambda$, its inverse limit constructed as follows. First take the inverse limit X of $\{X_\lambda\}_\lambda$ in the category of schemes, then one has projections $p_\lambda : X \to X_\lambda$ and log structures $\{p_\lambda^*\mathcal{M}_\lambda\}_\lambda$ on X. Then let $\mathcal{M}$ be the inductive limit of of $\{p_\lambda^*\mathcal{M}_\lambda\}_\lambda$, one can see easily the log scheme $(X, \mathcal{M})$ is just the inverse limit as required. In particular, fiber products exist in the category of log schemes.

Fine Log Structures, Charts

Definition A.1.7. *A log structure on a scheme X is called quasi-coherent (coherent) if it is locally isomorphic to a log structure associated to a constant pre-log structure defined by a (resp. finitely generated) monoid P together with a homomorphism $P_X \to \mathcal{O}_X$.*

Definition A.1.8. *A monoid is called integral if the cancelation law holds, i.e. "$ab = ac \Rightarrow b = c$". A log structure is called integral if it is a sheaf of integral monoids.*

Definition A.1.9. *A fine log structure is one which is both coherent and integral.*

Definition A.1.10. *Let X be a scheme endowed with a fine log structure $\mathcal{M}$, a chart of $\mathcal{M}$ is a homomorphism $P_X \to \mathcal{M}$ where P is an finitely generated integral monoid such that the induced homomorphism $(P_X)^a \to \mathcal{M}$ is an isomorphism.*

Proposition A.1.11. *Given a morphism $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ between schemes with fine log structures, there exists locally charts $P_X \to \mathcal{M}$, $Q_Y \to \mathcal{N}$ and a homomorphism of monoids $Q \to P$ for which the diagram*

$$\begin{array}{ccc} Q_X & \longrightarrow & P_X \\ \downarrow & & \downarrow \\ f^{-1}(\mathcal{N}) & \longrightarrow & \mathcal{M} \end{array}$$

is commutative.

Given a monoid M, we use M^{gp} to denote the associated group $\{ab^{-1}|a,b\in M\}/\sim$, where the equivalence relation $\sim$ is defined by $ab^{-1}\sim cd^{-1}\Leftrightarrow sad=sbc$ for some $s\in M$. The following lemma provides a way to get a chart for a log scheme locally.

Lemma A.1.12. *Let X be a scheme with a fine log structure $\mathcal{M}$ and $x\in X$. Suppose we are given a finitely generated abelian group G and a homomorphism $h:G\to\mathcal{M}_x^{gp}$ such that the composite of h with the projection $\mathcal{M}_x^{gp}\to\mathcal{M}_x^{gp}/\mathcal{O}_{X,x}^*$ is surjective. Let $P=(h^{gp})^{-1}(\mathcal{M}_x)$, then $P\to\mathcal{M}_x$ can be extended to a chart $P_U\to\mathcal{M}|_U$ in an neighborhood U of x.*

From now on, all the log structures are assumed to be fine if there is no further specification.

A.2 Log Differentials

In the theory of schemes, the Kähler differential is fundamental concept based on which one can define s-moothness, étaleness, etc.. We will see in the following paragraphs, one can also define log Kähler differentials for log schemes.

Log Kähler Differentials

Definition A.2.1. *Let $(X,\mathcal{M})$ be a log scheme, $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-module, then a log derivation of $(X,\mathcal{M})$ with value in $\mathcal{E}$ is pair (D,δ) such that*

$$D(\alpha(m))=\alpha(m)\delta(m),\quad m\in\mathcal{M};$$

where $D:\mathcal{O}_X\to\mathcal{E}$ is $\mathcal{E}$-valued derivation in the usual sense and $\delta:\mathcal{M}\to\mathcal{E}$ is a homomorphism of monoids. Let $(f,h):(X,\mathcal{M})\to(Y,\mathcal{N})$ be a morphism of log schemes, then a log derivation with respect to $(Y,\mathcal{N})$ is a log derivation (D,δ) such that $D(f^*y)=\delta(h^*n)=0$ for any $y\in\mathcal{O}_X,n\in\mathcal{N}$.

A derivation of a log scheme is determined by δ, i.e. given two $\mathcal{E}$-valued derivations (D_1,δ_1), (D_2,δ_2), then $\delta_1=\delta_2\Rightarrow(D_1,\delta_1)=(D_2,\delta_2)$. The proof goes as follows. We may assume X is affine, then for any $g\in\mathcal{O}_X$, we have $V(g)\cup V(1+g)=X$, where $V(g)$ (resp. $V(1+g)$) denotes the open subset $\{x\in X|g(x)\neq 0\}$ (resp. $\{x\in X|1+g(x)\neq 0\}$) of X. In particular, one can find sections m_1,m_2 of $\mathcal{M}$ over $V(g)$ and $V(1+g)$ respectively such that $\alpha(m_1)=g$, $\alpha(m_2)=1+g$. Then one can see easily the claim follows from definition (A.2.1).

Now we come to the universal object for log derivatives of the log scheme $(X,\mathcal{M})$ with respect to $(Y,\mathcal{N})$.

Definition A.2.2. *Let $\alpha:\mathcal{M}\to\mathcal{O}_X$ and $\beta:\mathcal{N}\to\mathcal{O}_Y$ be pre-log structures, $f:(X,\mathcal{M})\to(Y,\mathcal{N})$ be a morphism. Then the relative log differential module $\Omega^1_{X/Y}(\mathcal{M}/\mathcal{N})$ or simply denoted by $\omega^1_{X/Y}$ is defined to be the quotient of*

$$\Omega^1_{X/Y}\oplus\mathcal{O}_X\otimes_{\mathbb{Z}}\mathcal{M}^{gp}$$

by the sub $\mathcal{O}_X$-modules locally generated by

1. $(d\alpha(m),0)-(0,\alpha(m)\otimes m)$, $m\in\mathcal{M}$;
2. $(0,1\otimes m)$, $m\in\operatorname{im}(f^{-1}(\mathcal{N})\to\mathcal{M})$.

The class of $(0,1\otimes m)$ in $\omega^1_{X/Y}$ is denoted by $d\log m$.

For a log derivative $\mathcal{O}_X\xrightarrow{(D,\delta)}\mathcal{E}$ with respect to $(Y,\mathcal{N})$, by the universal property of $\Omega^1_{X/Y}$, there exists an $\mathcal{O}_X$-linear morphism $\phi:\Omega^1_{X/Y}\to\mathcal{E}$. Now let

$$\varphi:\mathcal{O}_X\otimes_{\mathbb{Z}}\mathcal{M}^{gp}\to\mathcal{E},\quad a\otimes m\mapsto a\delta(m),$$

then one can see easily the morphism $\phi\oplus\varphi$ factors through $\omega^1_{X/Y}$. Moreover, since a log derivation is uniquely determined by δ, therefore the morphism from $\omega^1_{X/Y}$ to $\mathcal{E}$ defined above is unique.

Remark A.2.3. *Given two monoids P,Q and a homomorphism $Q\to P$, let $X=\operatorname{Spec}R[P]$, $Y=\operatorname{Spec}R[Q]$ be the log schemes endowed with the canonical log structures (A.1.3.2). Then there is an induced morphism of log schemes $X\to Y$ and $\omega^1_{X/Y}$ becomes $\mathcal{O}_X\otimes_{\mathbb{Z}}(P^{gp}/\operatorname{im}(Q^{gp}))$.*

Log Smooth Morphisms

One of the equivalent ways to to define smooth morphisms in the category of schemes is by using infinitesimal universal properties. This approach can be adopted to define smooth morphisms between log schemes as well. More precisely, we have

Definition A.2.4. *Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of log schemes with fine log structures fitting into the following commutative diagram*

$$\begin{array}{ccc} (T', \mathcal{L}') & \xrightarrow{s} & (X, \mathcal{M}) \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ (T, \mathcal{L}) & \xrightarrow{t} & (Y, \mathcal{N}) \end{array}$$

where i is a closed exact embedding with defining ideal I satisfying $I^2 = 0$. Then f is called log smooth (resp. étale) if for any commutative diagram as above, we can find a (resp. a unique) morphism $g : (T, \mathcal{L}) \to (X, \mathcal{M})$ such that $s = gi$ and $t = fg$.

Remark A.2.5. *Let $(Y, \mathcal{N})$ be the log scheme $\operatorname{Spec} W_2(k)$ endowed with the trivial log structure, $(X, \mathcal{M}) = (T, \mathcal{L})$ and i be the closed imbedding defined by the ideal p. If we take s (resp. t) to be the composite $i \circ F_{(T', \mathcal{L}')}$ (resp. $F_{(Y,\mathcal{N})} \circ f$), then the smoothness of f implies the local liftability of the Frobenius morphism of $(T', \mathcal{L}')$.*

Similarly, we also have the following result describing the set of morphisms solidifying the dashed arrow in the above diagram.

Proposition A.2.6. *([9, proposition 3.9]) Given a commutative diagram as in definition (A.2.4), there is a (noncanonical) bijection from the set S of liftings g of s satisfying the condition described in (A.2.4) and the set $\operatorname{Hom}_{\mathcal{O}_{T'}}(s^*\omega^1_{X/Y}, I)$.*

Proof. Let $g \in S$ be a fixed lifting of s, next we will associate each element $h \in S$ an $\mathcal{O}_{T'}$-linear homomorphism $\delta_g(h) : s^*\omega^1_{X/Y} \to I$. By definition (A.2.2), to define such a homomorphism it suffices to define an s-linear homomorphism from $\Omega^1_{X/Y} \oplus \mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{gp}$ to I which vanishes at the sections appearing in *1* and *2* of definition (A.2.2).

Given $a, b \in \mathcal{O}_X$, then we have

$$\begin{aligned} h(ab) - g(ab) &= h(a)(h(b) - g(b)) + g(b)(h(a) - g(a)) \\ &= s(a)(h - g)(b) + s(b)(h - g)(a) \end{aligned}$$

where the second equality follows from the fact that $\operatorname{im}(h - g) \in I$ and $I^2 = 0$. By the universal property of the sheaf of Kähler differential, $h - g$ defines a homomorphism $\Omega^1_{X/Y} \to s_*I$.

Next we define a homomorphism from $\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{gp}$ to I. Given $m \in \mathcal{M}$, by assumption we have $ig(m) = ih(m)$. We claim there is a unique unit $u(m) \in \ker i = \ker(\mathcal{L} \xrightarrow{i} \mathcal{L}')$ such that $h(m) = u(m)g(m)$, where $\mathcal{L}$ (resp. $\mathcal{L}'$) is the log structure on T (resp. T')

For that purpose it suffices to prove $\ker i$ is a group. If we denote the structure morphism of monoids $\mathcal{L} \to \mathcal{O}_T$, and $\mathcal{L}' \to \mathcal{O}_{T'}$ by β_T and $\beta_{T'}$ respectively, then we have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \ker i & \longrightarrow & i^{-1}\mathcal{L} & \xrightarrow{i} & \mathcal{L}' & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle \beta_T} & & \downarrow{\scriptstyle \beta_{T'}} & & \\ 0 & \longrightarrow & i^{-1}I & \longrightarrow & i^{-1}\mathcal{O}_T & \xrightarrow{p} & \mathcal{O}_{T'} & \longrightarrow & 0 \end{array}$$

Given $x \in \ker i$, then we have $p\beta_T(x) = \beta_{T'}i(x) = 1$ hence $\beta_T(x) = 1 + y$ for some $y \in I$. Note that $1 + y \in \mathcal{O}_T^*$ and $\beta_T|_{\beta_T^{-1}(\mathcal{O}_T^*)}$ is an isomorphism, we have $\beta^{-1}(1 - y) \in \ker i$ and $x\beta^{-1}(1 - y) = 1$ since $y^2 = 0$. Then $\ker i$ is a group and as we claimed. As a result, there is a unique element $u(m) \in \ker i$ such that $h(m) = u(m)g(m)$.

It is easy to see $u : \mathcal{M} \to \ker i$ defines a homomorphism of monoids and u can be extended to a homomorphism of groups $\mathcal{M}^{gp} \to \ker i$. Then we can define a homomorphism $\mathcal{M}^{gp} \to I$ by sending m to $\beta_T u(m) - 1$, and the equality $\beta_T u(m_1 m_2) - 1 = \beta_T u(m_1) - 1 + \beta_T u(m_2) - 1$ follows easily from the fact that $I^2 = 0$. Now one can get a homomorphism from $\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{gp}$ to I by extending $\beta_T u - 1$ $\mathcal{O}_X$-linearly. Moreover, let $\alpha : \mathcal{M} \to \mathcal{O}_X$ be the structure morphism on X, then we have

$$\begin{aligned} h(\alpha(m)) - g(\alpha(m)) &= \beta_T(h(m)) - \beta_T(g(m)) \\ &= \beta_T(u(m)g(m)) - \beta_T(g(m)) \\ &= \beta_T(g(m))(\beta_T u(m) - 1) \\ &= s(\alpha(m))(\beta_T u(m) - 1) \end{aligned}$$

from which the morphism $h - g \oplus \beta_T u - 1$ vanishes at the sections in *1* of definition (A.2.2). It is obvious $h(f(n)) = g(f(n)) = t(n)$ for $n \in \mathcal{N}$, therefore $h - g \oplus \beta_T u - 1$ also vanishes at the sections in *2* of definition (A.2.2) and we can define a morphism $\delta_g(h) \in \mathrm{Hom}_{\mathcal{O}_X}(\omega^1_{X/Y}, s_* I) \cong \mathrm{Hom}_{\mathcal{O}_{T'}}(s^*\omega^1_{X/Y}, I)$. The bijection of δ_g can be proved in the same way as in the smooth case and we omit it here. ☐

When I is a principal ideal, say a the generator p, then we have

$$\mathrm{Hom}_{\mathcal{O}_{T'}}(s^*\omega^1_{X/Y}, I) \cong \mathrm{Hom}_{\mathcal{O}_{T'}}(s^*\omega^1_{X/Y}, \mathcal{O}_{T'}) \cong s^* T_{X/Y}.$$

In particular, the morphism $\delta_g(h)$ can be written as $[p]\bar{\bar{\delta}}_g(h)$, where $\bar{\bar{\delta}}_g(h)$ is a section of $s^*T_{X/Y}$ and $[p]$ is defined in Lemma 1.1.1. If we use $h - g$ to denote the morphism $\bar{\bar{\delta}}_g(h)$, then we have the following

Corollary A.2.7. *When I is a principal ideal with a generator p, we have the following formula*

$$\beta_T(h^* m) = \beta_T(g^* m)(1 + p(h - g)(d \log m)) \tag{A.2.1}$$

Proof. The restriction of $\delta_g(h)$ to $\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{gp}$ is $\beta_T u - 1$ and $h(m) = u(m)g(m)$. ☐

The following proposition, which is specific to the log setting, gives an equivalent definition of smooth morphisms.

Proposition A.2.8. *Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism between schemes with fine log structures and $Q_Y \to \mathcal{N}$ be a chart on $(Y, \mathcal{N})$. Then f is smooth (étale) iff there exists locally on X a chart $P_X \to \mathcal{M}$ and a homomorphism $Q \to P$ satisfying*

(i) The kernel and torsion part of the cokernel (resp. The kernel and cokernel) of $Q^{gp} \to P^{gp}$ are finite groups of orders invertible in $\mathcal{O}_X$;

(ii) The induced morphism $X \to Y \times_{\mathrm{Spec}\,\mathbb{Z}[Q]} \mathrm{Spec}\,\mathbb{Z}[P]$ is étale (in the classical case).

Remark A.2.9.

1. *Condition (ii) of (A.2.8) may not be true for an arbitrary chart of f. For instance, let X, Y be schemes equipped with trivial log structures, f be a smooth morphism $X \to Y$, and the charts are taken to be the constant monoid consisting of the unit, then it is easy to see the condition (ii) is satisfied only if f as a morphism of schemes is étale.*

2. *In general, for a morphism between log schemes, neither the smoothness of this morphism itself nor the underlying morphism between schemes is implied by the other.*

3. *It can be proved that for a smooth morphism $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$, the relative differential sheaf is locally free. However, the converse is not true, which is different from what happens for usual schemes. This can be seen from the Proposition (A.2.8) combined with Remark (A.2.3), .*

4. *The smoothness of a morphism between log schemes doesn't imply the flatness of the morphism between the underlying schemes. A counter-example can be given as follows. Let Q be a monoid generated freely by two generators a_1, a_2, P be the monoid with generators a_1, a_2, b_1, b_2, modulo the relation $a_1 b_1 = a_2 b_2$, then we have log schemes $X = \mathrm{Spec}\,\mathbb{Z}[P]$, $Y = \mathrm{Spec}\,\mathbb{Z}[Q]$ equipped with the canonical log structures and an induced morphism $X \to Y$. By Proposition (A.2.8) this morphism is log smooth, however it is not flat on the underlying schemes.*

Logarithmic Coordinates

Let $(X, \mathcal{M}) \to (Y, \mathcal{N})$ be a log smooth morphism, then $\omega^1_{X/Y}$ is locally free and we can choose locally a set of sections $(m_1, \cdots, m_r)$ of M such that $(d \log m_1, \cdots, d \log m_r)$ constitutes a basis of $\omega^1_{X/Y}$. Any set of such sections will be called a log coordinate of X/Y. Given a log coordinate $(m_1, \cdots, m_r)$ of X/Y, then it defines a morphism from (X, M) to $\mathbb{A}^r_S = \mathbb{A}^r \times S$, where the fiber product is constructed in the category of log schemes and $\mathbb{A}^r = \operatorname{Spec} \mathbb{Z}[\mathbb{N}^r]$ is equipped with the canonical log structure. Moreover, by [9, 3.12], the above morphism is log étale.

A.3 Several Types of Log Morphisms

In the following paragraphs we will give a brief review of several types of morphisms exclusive for log schemes.

Integral and Exact Morphisms

Definition A.3.1. *A morphism $(X, \mathcal{M}) \to (Y, \mathcal{N})$ between schemes with fine log structures is said to be integral iff for any morphism $(Y', \mathcal{N}') \to (Y, \mathcal{N})$, the fiber product $(X, \mathcal{M}) \times_{(Y,\mathcal{N})} (Y', \mathcal{N}')$ is still an integral log scheme.*

The condition in the above definition is hard to check. There is an equivalent definition of integral morphism in terms of the morphisms of monoids. To introduce this definition we need the following lemma.

Lemma A.3.2. *Let $h : Q \to P$ be a homomorphism of integral monoids then the following conditions (i), (iv); (ii),(iii),(v) are equivalent respectively.*

(i) For any integral monoids Q' and for any homomorphism $g : Q \to Q'$, the push out of $P \leftarrow Q \to Q'$ in the category of monoids is integral.

(ii) The homomorphism $\mathbb{Z}[Q] \to \mathbb{Z}[P]$ induced by h is flat.

(iii) For any field k, the homomorphism $k[Q] \to k[P]$ induced by h is flat.

(iv) If $a_1, a_2 \in Q$, $b_1, b_2 \in P$ and $h(a_1)b_1 = h(a_2)b_2$, then there exists $a_3, a_4 \in Q$ and $b \in P$ such that $b_1 = h(a_3)b, b_2 = h(a_4)b$ and $a_1a_3 = a_2a_4$.

(v) The condition (iv) is satisfied and h is injective.

Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of schemes with integral log structures. Then for each $x \in X$, the conditions $(i) - (v)$ in lemma (A.3.2) are equivalent with $Q = f^*\mathcal{N}_x$ and $P = \mathcal{M}_x$. Moreover, they are equivalent to each of $(i) - (v)$ with Q and P replaced by $f^{-1}(\mathcal{N}/\mathcal{O}^*_Y)_x$ and $\mathcal{M}/\mathcal{O}^*_X$ respectively.

Remark A.3.3. *By remark A.2.9.4, a log smooth morphism is not necessarily integral.*

Definition A.3.1′. *A morphism $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ between schemes with fine log structures is called integral iff for any point $x \in X$, the induced morphism $f^{-1}(\mathcal{N}/\mathcal{O}^*_Y)_x \to (\mathcal{M}/\mathcal{O}^*_X)_x$ of monoids satisfy the equivalent conditions of lemma (A.3.2).*

Recall that we will consider only fine log structures, in particular we can find local charts for $(X, \mathcal{M})$ and $(Y, \mathcal{N})$. Let P and Q be finitely generated monoids such that we have isomorphisms $P_X \cong \mathcal{M}$ and $Q_Y \cong \mathcal{N}$ over some open neighborhoods of x and $f(x)$ respectively. Then by definitions of charts, we have surjective morphisms $Q \to f^{-1}(\mathcal{N}/\mathcal{O}^*_Y)_x$ and $P \to (\mathcal{M}/\mathcal{O}^*_X)_x$. Then it is easy to see if the morphism $Q_X \to P_X$ satisfies the condition (iv) above, so does the morphism $f^{-1}(\mathcal{N}/\mathcal{O}^*_Y)_x \to (\mathcal{M}/\mathcal{O}^*_X)_x$, i.e. h is integral.

Example A.3.4. *Using the same notations as above, if for any $y \in Y$, the monoid $(\mathcal{N}/\mathcal{O}^*_Y)_y$ is generated by one element, then the morphism is integral. This example also shows that log smoothness combined with integrality does not imply smoothness of the underlying schemes. However, one can prove the two conditions together imply the flatness of the morphism of the underlying schemes.*

Definition A.3.5. *A homomorphism of integral monoids $h : Q \to P$ is said to be exact if $Q = (h^{gp})^{-1}(P)$, where $h^{gp} : Q^{gp} \to P^{gp}$ is the homomorphism induced by h. Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism between schemes with integral log structures, then f is said to be exact if for any $x \in X$, the homomorphism $(f^*\mathcal{N})_x \to \mathcal{M}_x$ is exact.*

Remark A.3.6.

1. *It is easy to check an integral morphism is exact. On the other hand, a log smooth morphism is in general not exact.*
2. *Note that in the definition of exact morphism, we use the monoid $f^*\mathcal{N}$ instead of $f^{-1}\mathcal{N}$. However, one might still ask whether the morphism of monoids $f^{-1}\mathcal{N} \to \mathcal{M}$ is exact. The answer is generally no and one can give counterexamples easily when X is endowed with a pullback log structure $f^*\mathcal{N}$ from $(Y, \mathcal{N})$ via a morphism $f : X \to Y$.*

   On the other hand, if the morphism f is an exact closed imbedding defined by a nilpotent ideal, then the morphism mentioned above is exact. The proof goes as follows. Note that we have an surjection $\mathcal{O}_Y \to \mathcal{O}_X$, in particular we have a surjection $\mathcal{O}_Y^* \to \mathcal{O}_X^*$. Now if we denote the morphism $\mathcal{N} \xrightarrow{\beta} \mathcal{O}_Y \xrightarrow{p} \mathcal{O}_X$ by α, then $\alpha^{-1}(\mathcal{O}_X^*) = \beta^{-1}p^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_Y^*$. If we are given $n_1, n_2 \in \mathcal{N}$, such that

$$\frac{\overline{(n_1, 1)}}{\overline{(n_2, 1)}} = \overline{(n_3, g)}$$

   then we have $\overline{(n_1, 1)} = \overline{(n_2 n_3, g)}$ and by the remark under [9, 1.3], there exists $h_1, h_2 \in \mathcal{O}_Y^*$ such that $n_1\beta^{-1}(h_1) = n_2 n_3 \beta^{-1}(h_2)$, hence $n_1 = \beta^{-1}(h_1^{-1}h_2)n_2 n_3$ and $(f^{gp})^{-1}(f^*\mathcal{N}) = f^{-1}\mathcal{N}$.

Frobenius and Perfectly Smooth Morphisms

For a log scheme $(X, \mathcal{M})$ whose underlying scheme X is defined over $\mathbb{F}_p$, one can define a Frobenius endomorphism $F_{(X,\mathcal{M})}$ as follows. The morphism on X is the usual absolute Frobenius F_X and the homomorphism $F_X^{-1}(\mathcal{M}) \to \mathcal{M}$ is the multiplication by p on $\mathcal{M}$ under the identification of $F_X^{-1}(\mathcal{M})$ with $\mathcal{M}$.

With all preparations ready, we come to the definition of morphisms of Cartier type.

Definition A.3.7. *Let $f : (X, \mathcal{M}) \to (S, \mathcal{L})$ be a morphism of schemes with integral log structures over $\mathbb{F}_p$. The morphism f is said to be of Cartier type if it is integral and the morphism $(f, F_{(X,\mathcal{M})})$ from $(X, \mathcal{M})$ to the fiber product $(X, \mathcal{M}) \underset{(S,\mathcal{L})}{\times}{}_{F_{(S,\mathcal{L})}} (S, \mathcal{L})$ is exact. The morphism f is said to be perfectly smooth if it is smooth and of Cartier type.*

Example A.3.8.

1. *If the local chart of f is of the form $\mathbb{N} \to \mathbb{N}^r, n \mapsto (n, \cdots, n)$ with $r \geq 1$, then f is of Cartier type.*
2. *Let X be a smooth S-scheme over $\mathbb{F}_p$, D be a divisor of X relative to S with normal crossings. If we endow X and S with the log structure defined by D and trivial log structure, then the morphism of the log schemes $X \to S$ is of Cartier type.*

As hinted by definition (A.3.7), the 'relative Frobenius' in the log setting fails to be an exact morphism in general. Indeed, it is proved in [9, 4.10] that this morphism can be factored uniquely as a composite of an exact and a log étale morphism. We use the the term "exact relative Frobenius" or just relative Frobenius to mean the exact part of the aforementioned factorization. The preference for the exact part lies in the fact that it rather than the morphism $(f, F_{(X,\mathcal{M})})$ is involved in log Cartier descent.

Appendix B

Crystalline Topos and Crystals

B.1 Log Crystalline Sites

Grothendieck Topolgy

Definition B.1.1. *Let $\mathcal{C}$ be an category, a Grothendieck topology on $\mathcal{C}$ consists of a family of sets $Cov(X)$ whose elements are morphisms in $\mathcal{C}$ with target X such that*

1. *if $U \xrightarrow{f} X$ is an isomorphism in $\mathcal{C}$, then $\{U \xrightarrow{f} X\} \in Cov(X)$,*
2. *if $\{U_i \xrightarrow{f_i} X\}_{i \in I} \in Cov(X)$, and $\{V_j^i \xrightarrow{g_j^i} U_i\}_{j \in J_i} \in Cov(U_i)$, then $\{V_j^i \xrightarrow{f_i \circ g_j^i} X\}_{i \in I, j \in J_i} \in Cov(X)$,*
3. *if $\{U_i \xrightarrow{f_i} X\}_{i \in I} \in Cov(X)$, then for any morphism $X' \xrightarrow{g} X$ in $\mathcal{C}$, the fiber products $\{U_i \times_X X'\}_{i \in I}$ exist in $\mathcal{C}$, and $\{U_i \times_X X' \to X'\}_{i \in I} \in Cov(X')$.*

Definition B.1.2. *A site is a category equipped with a Grothendieck Topology.*

Example B.1.3. *Let X be a scheme, and X_{Zar} be the category whose objects are open subsets U of X and morphisms are open immersions $U \to V$. Then X_{Zar} has a natural Grothendieck topology on it, i.e. $Cov(U)$ are the morphisms in X_{Zar} with target U.*

The definition of a crystalline site for a scheme has been given in [2]. Roughly speaking, the objects of this site are PD-thickenings of open subsets of the scheme. This definition can be adapted to log schemes with only minor modifications. Let $(S, \mathcal{L})$ be a log scheme such that $m\mathcal{O}_S = 0$ for some integer m, I be a coherent ideal of $\mathcal{O}_S$ equipped with a PD-structure γ, then the quadruple $(S, \mathcal{L}, I, \gamma)$ (or $(S, \mathcal{L})$ for short) is a base over which the objects of log crystalline sites will be defined.

Definition B.1.4. *Let $(X, \mathcal{M})$ be a scheme endowed with a fine log structure over $(S, \mathcal{L})$ such that γ extends to X and $\mathrm{Cris}(X/S)$ be the log crystalline site. Then the objects of $\mathrm{Cris}(X/S)$ are quintuples $(U, T, \mathcal{M}_T, i, \delta)$, where U is an open subset of X, $i : (U, \mathcal{M}_U) \to (T, \mathcal{M}_T)$ is an exact closed imbedding over $(S, \mathcal{L})$ and δ is a PD-structure on the defining ideal of U in $\mathcal{O}_T$ compatible with γ. The morphisms of $\mathrm{Cris}(X/S)$ are defined in the natural way. A covering of an object $(U, T, \mathcal{M}_T, i, \delta)$ is a covering of (U, T, i, δ) under the Zariski topology forgetting the log structures.*

Recall that a presheaf of sets on a category is nothing but a contravariant functor from this category to the category of sets. When this category is a site, sheaves can be defined on it.

Definition B.1.5. *A sheaf of sets on a site $\mathcal{C}$ is presheaf $\mathcal{F}$ satisfying the following exact sequence for any object U and covering family $\{U_i\}_{i \in I} \in Cov(U)$*

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times U_j).$$

Definition B.1.6. *Let X/S be a log scheme, a log crystal of $\mathcal{O}_{X/S}$-module is a sheaf $\mathcal{F}$ of $\mathcal{O}_{X/S}$-module on the log crystalline site* $\mathrm{Cris}(X/S)$ *such that for any morphism $u : T \to T'$, the induced morphism $u^*\mathcal{F}(T') \to \mathcal{F}(T)$ is an isomorphism.*

Example B.1.7. *The structure sheaf $\mathcal{O}_{X/S}$ defined by $\mathcal{O}_{X/S}(U, T, M_T, i, \delta) = \Gamma(T, \mathcal{O}_T)$ is a crystal on* $\mathrm{Cris}(X/S)$.

Topoi, Inverse Images of Sheaves on Crystalline Sites

A topological space T can be made into a site $\mathbf{T}$ whose objects are open subsets, morphisms are inclusions and $Cov(U) = \{V | V \subseteq U \subseteq T \text{ is open}\}$. Let T_1, T_2 be two topological spaces, $\hat{T}_1$ and $\hat{T}_2$ be the category of sheaves on T_1 and T_2 respectively. Then a continuous morphism $f : T_1 \to T_2$ will induce functors $f_* : \hat{T}_1 \to \hat{T}_2$ and $f^* : \hat{T}_2 \to \hat{T}_1$. On the other hand, let $F : \mathbf{T}_2 \to \mathbf{T}_1$ be the functor induced by f, then one can see easily the functors f_* and f^* are determined by F.

Unlike topological spaces, given a S-morphism $X' \to X$ one cannot define a morphism from $\mathrm{Cris}(X/S)$ to $\mathrm{Cris}(X'/S)$ since one does not know how to define the pullback of an object of $\mathrm{Cris}(X/S)$ in $\mathrm{Cris}(X'/S)$, even X and X' are endowed with trivial log structures. In particular, the pullback of sheaves on crystalline sites cannot be defined in the standard way as above.

A topos, by definition, is the category of sheaves of sets over a site. Given a crystalline site $\mathrm{Cris}(X/S)$, we use $(X/S)^{\log}_{\mathrm{cris}}$ to denote the associated topos. Though a morphism from $\mathrm{Cris}(X/S)$ to $\mathrm{Cris}(X'/S)$ is not available, one can still define a pullback functor from $(X/S)^{\log}_{\mathrm{cris}}$ to $(X'/S)^{\log}_{\mathrm{cris}}$.

Let T an object of a site X, then T defines an object $\tilde{T}$ in $\hat{X}$, the category of presheaves on X by

$$T' \mapsto \tilde{T}(T') := \mathrm{Hom}_X[T', T].$$

If we are given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

where g is a PD-morphism, then the pullback of an object of $(X/S)^{\log}_{\mathrm{cris}}$ represented by an object of $\mathrm{Cris}(X/S)$ can be defined as follows.

Definition B.1.8. *Let $(U, T, \mathcal{M}_T, i, \delta)$ be an object of* $\mathrm{Cris}(X/S)$*, then the pullback $f^*(U, T, \mathcal{M}_T, i, \delta)$ is the sheaf on* $\mathrm{Cris}(X'/S')$ *defined by*

$$(U', T', \mathcal{M}'_{T'}, i', \delta') \mapsto I_f^{T'} := \text{PD-Hom}_f[(U', T', \mathcal{M}'_{T'}, i', \delta'), (U, T, \mathcal{M}_T, i, \delta)],$$

in which elements of $I_f^{T'}$ are PD*-morphisms $h : (T', \mathcal{M}'_{T'}) \to (T, \mathcal{M}_T)$ making the following diagram commutative*

$$\begin{array}{ccccc} (U', \mathcal{M}') & \xrightarrow{i'} & (T', \mathcal{M}'_{T'}) & \longrightarrow & (S', \mathcal{L}') \\ \downarrow f & & \downarrow h & & \downarrow g \\ (U, \mathcal{M}) & \xrightarrow{i} & (T, \mathcal{M}_T) & \longrightarrow & (S, \mathcal{L}) \end{array} . \tag{B.1.1}$$

Now for given any objects of $(X/S)^{\log}_{\mathrm{cris}}$, one can define its inverse image in the classical way.

Definition B.1.9. *Let $\mathcal{F}$ be an object of $(X/S)^{\log}_{\mathrm{cris}}$, the inverse image $f^{-1}_{\mathrm{cris}}\mathcal{F}$ is the sheaf associated to the presheaf*

$$T' = (U', T', \mathcal{M}'_{T'}, i', \delta') \mapsto \varinjlim_{I_f^{T'}} \mathcal{F}(T).$$

Similarly, one can define pullbacks of sheaf of $\mathcal{O}_{X'/S'}$-modules. Now let $h : (T', \mathcal{M}'_{T'}) \to (T, \mathcal{M}_T)$ be an element of $I_f^{T'}$, then we have the following canonical morphism

$$h^{-1}(\mathcal{F}_T) \to f_{\text{cris}}^{-1}(\mathcal{F})_{T'}.$$

Indeed, given an open subset V' of T', the sheaf $h^{-1}(\mathcal{F}_T)|_{V'}$ is the sheafification of the presheaf

$$V' \mapsto \varinjlim_{\substack{V \hookrightarrow T \\ h^{-1}(V) \supseteq V'}} \mathcal{F}(V).$$

The index set $\{V \mid V \hookrightarrow T, h^{-1}(V) \supseteq V'\}$ can be considered as an subcategory of $I_f^{T'}$ in the evident way, from which the morphism of presheaves and their sheafification follows. In particular, let $\mathcal{A}$ be a sheaf of rings on $\text{Cris}(X/S)$, then we have a homomorphism

$$h^{-1}(\mathcal{A}_T) \to f_{\text{cris}}^{-1}(\mathcal{A})_{T'},$$

hence for all $\mathcal{A}$-module $\mathcal{F}$ a homomorphism

$$h^{-1}(\mathcal{F}_T) \otimes_{h^{-1}(\mathcal{A}|_T)} f_{\text{cris}}^{-1}(\mathcal{A})_{T'} \to f_{\text{cris}}^{-1}(\mathcal{F})_{T'}. \tag{B.1.2}$$

Suppose we are given a sheaf of rings $\mathcal{A}'$ on $\text{Cris}(X'/S')$ and a homomorphism $f_{\text{cris}}^{-1}(\mathcal{A}) \to \mathcal{A}'$, then after tensoring $\mathcal{A}'$ on both sides of $(B.1.2)$, we have the following homomorphism of $\mathcal{A}'$-modules

$$h^*(\mathcal{F}_T) \to f_{\text{cris}}^*(\mathcal{F})_{T'}.$$

One can prove the following

Proposition B.1.10. *([1, IV, Corollaire 1.2.4]) When $\mathcal{F}$ is crystal of $\mathcal{A}$-module, the above morphism is an isomorphism.*

Fiber Products in Crystalline Topos

The following lemma is helpful for understanding the of pullback of a crystal.

Lemma B.1.11. *[2, 5.12] Let T_1, T_2 be two objects of $\text{Cris}(X/S)$, Y a S-scheme, and $q_i : T_i \to Y, i = 1, 2$, be two S-morphisms. Then there exists an object $T = (U, T, \delta)$ of $\text{Cris}(X/S)$, and two morphisms $p_i : T \to T_i$ in $\text{Cris}(X/S)$ such that for any object $T' = (U', T', \delta')$ of $\text{Cris}(X'/S')$ and two f-PD-morphisms $h_i : T' \to T_i$ satisfying $q_1 h_1 = q_2 h_2$, we have a unique f-PD-morphism $h : T' \to T$ such that $h_i = p_i h$.*

Next we consider the pullback of a crystal in a little more depth.

Let Y be a smooth S-scheme, and $i : X \hookrightarrow Y$ is a closed imbedding. By smoothness of Y/S, for any object T of $\text{Cris}(X/S)$, there exists locally on T a morphism to Y extending i. Moreover, if we denote by $D_{X,\gamma}(Y)$ the PD-envelop of X in Y, then the extended morphism from an open subset of T to Y factors through $D_{X,\gamma}(Y)$ [1, II Proposition 4.2.2(ii)]. Let $\mathcal{E}$ be a crystal on $\text{Cris}(X/S)$, then the value of the pullback of $\mathcal{E}$ on an object T' of $\text{Cris}(X'/S')$ obtained from an f-PD-morphism $h : T' \to T$ is essentially the pullback of $\mathcal{E}_{D_{X,\gamma}(Y)}$ via the morphism $T' \to T \to D_{X,\gamma}(Y)$. Now we are given two morphisms $h_1, h_2 : T' \to D_{X,\gamma}(Y)$, then by proposition (B.1.10) there exists an isomorphism

$$h_1^* \mathcal{E} \cong h_2^* \mathcal{E}. \tag{B.1.3}$$

With the help of lemma (B.1.11), now we can derive the isomorphism (B.1.3) in a more lucid way. Let $D(1)$ be the fiber product of $D_{X,\gamma}(Y)$ with itself over S in $\text{Cris}(X/S)$, then we have projections $p_i : D(1) \to D_{X,\gamma}(Y), i = 1, 2$, and the isomorphism (B.1.3) is nothing but the pullback of $p_1^* \mathcal{E}_{D_{X,\gamma}(Y)} \cong p_2^* \mathcal{E}_{D_{X,\gamma}(Y)}$ via a uniquely defined morphism $T' \to D_{X,\gamma}(Y)$.

Remark B.1.12. *One can prove the fiber product of $D_{X,\gamma}(Y)$ with itself over S in $\text{Cris}(X/S)$ is nothing but the* PD*-envelop of X in $Y \times_S Y$ via the diagonal imbedding $X \to Y \times_S Y$ [2, Lemma 5.12]. In particular, when X itself is smooth over S, the fiber product is nothing but the first order PD infinitesimal neighborhood of X in the diagonal embedding $X \to X \times_S X$.*

LS Thickening

Recall that in the definition of (log) crystalline site (B.1.4), we require the base scheme to be annihilated by some nonzero integer. We will see later this condition is necessary for one to translate the data given by a crystal into that given by a module with quasi-nilpotent integrable connection.

On the other hand, sometimes one needs to consider p-adic thickening of an open subset. In particular, we are interested in those thickenings which are endowed with a lifting of Frobenius. A standard thickening of this sort called "lifted situation" (LS) is introduced in [2, 8.2], [16, 1.2.6]. Now we recall its definition.

Definition B.1.13. *Let S be a formal W-scheme with the p-adic topology, equipped with a fine log structure. A lifted situation over S is a log scheme X smooth and integral over S_0, together with a lifting $F_{Y/S}: Y \to Y'$ of the exact relative Frobenius of X/S. A lifted situation is parallelizable if there exists system of log coordinates $(m_1, \cdots, m_n)$ for Y/S and $(m'_1, \cdots, m'_n)$ for Y'/S, such that $F^*_{Y/S}(m'_i) = pm_i$ for all i.*

By infinitesimal deformation theory, for any sufficiently small open subset of a log scheme smooth over k, one can always find LS thickenings for it.

B.2 Crystals and Connections

The Equivalence

As in the smooth case [2, Lemma 4.12], there is still an equivalence between category of crystals and modules with integrable connections in the log setting. More precisely, we have

Theorem 8. *[9, Theorem 6.2] Let $(S, \mathcal{L})$ be a log scheme such that $m\mathcal{O}_S = 0$ for some nonzero integer m, $(Y, \mathcal{N})$ be a log scheme smooth over $(S, \mathcal{L})$, $(X, \mathcal{M}) \to (Y, \mathcal{N})$ be a closed immersion and $(D, \mathcal{M}_D)$ be the* PD*-envelope of $(X, \mathcal{M})$ in $(Y, \mathcal{N})$. Then we have an equivalence between the following two categories:*

1. *The category of crystals on* $\mathrm{Cris}(X/S)$;
2. *The category of $\mathcal{O}_D$-modules $\mathfrak{M}$ on D with an integrable quasi-nilpotent connection*

$$\nabla : \mathfrak{M} \to \mathfrak{M} \otimes_{\mathcal{O}_Y} \omega^1_{Y/S}.$$

In [9], the condition "quasi-nilpotence" is defined as follows. Let $x \in X$ and $m_i \in \mathcal{M}_x, 1 \le i \le r$ such that $\{d \log m_i\}_{1 \le i \le r}$ forms a basis of $\omega^1_{Y/S,x}$. Then for any i and $a \in \mathfrak{M}_x$, there exists integers $j_1, \cdots, j_k; n_1, \cdots, n_k$ such that

$$\prod_{1 \le s \le k} (\nabla(\partial_i^{\log}) - j_s)^{n_s}(a) = 0, \tag{B.2.1}$$

where $\nabla(\partial_i^{\log})(a) = a_i$ if $\nabla(a) = \sum_{0 \le i \le r} a_i \otimes d \log m_i$.

Let $\alpha : \mathcal{N} \to \mathcal{O}_Y$ be the structure morphism, then for any i, we have $\nabla(\partial_i) = \alpha(m_i)\nabla(\partial_i^{\log})$, from which we can deduce easily

$$\nabla(\partial_i)^n = \alpha(m_i)^{-n} \prod_{0 \le j \le n-1} (\nabla(\partial_i^{\log}) - j). \tag{B.2.2}$$

It is easy to see the quasi-nilpotence condition (B.2.1) is equivalent to the quasi-nilpotence of ∇ defined in [2, Definition 4.10].

Instead of giving a proof of the theorem here, we only write down the correspondence. First we introduce some notations. Let $(D(1), \mathcal{M}_{D(1)})$ be the PD-envelop of the diagonal embedding $(X, \mathcal{M}) \to (Y, \mathcal{N}) \times_{(S, \mathcal{L})} (Y, \mathcal{N})$, and $p_i, i = 1, 2$, be the first and second projections. Let $x \in X$, $u_i \in \ker(\mathcal{O}^*_{D(1),x} \to \mathcal{O}^*_{X,x})$, satisfying $p_2^*(\alpha(m_i)) = u_i p_1^*(\alpha(m_i))$. Then we have an isomorphism

$$\mathcal{O}_{D,x}\langle T_1, \cdots, T_r\rangle \cong \mathcal{O}_{D(1),x}, \qquad T_i^{[n]} \mapsto (u_i - 1)^{[n]},$$

where $\mathcal{O}_{D,x}\langle T_1, \cdots, T_r\rangle$ denote the PD polynomial ring.

(2) $\Rightarrow$ (1). Let $(\mathfrak{M}, \nabla)$ be an object in the second category, then we have an isomorphism $\eta : p_2^*\mathfrak{M} \to p_1^*\mathfrak{M}$ given by

$$1 \otimes a \mapsto \sum_{n \in \mathbb{N}^r} \left(\prod_{1 \le i \le r} (u_i - 1)^{[n_i]} \right) \otimes \left(\prod_{1 \le i \le r, 0 \le j \le n_i - 1} (\nabla(\partial_i^{\log}) - j) \right) (a). \tag{B.2.3}$$

Moreover, the isomorphism above satisfy the transitivity condition, i.e. if we let $D(2)$ be the PD-envelop of the diagonal embedding of $(X, \mathcal{M})$ into $(Y, \mathcal{N}) \times_{(S,\mathcal{L})} (Y, \mathcal{N}) \times_{(S,\mathcal{L})} (Y, \mathcal{N})$, and p_{12}, p_{23}, p_{13} be the projections from $D(2)$ to $D(1)$, then we have

$$p_{13}^*(\eta) = p_{23}^*(\eta) p_{12}^*(\eta).$$

The associated crystal $\mathcal{F}$ on $\mathrm{Cris}(X/S)$ can be defined as follows. Let $(U, T, \mathcal{M}_T, i, \delta)$ be an object of $\mathrm{Cris}(X/S)$, we can find locally a morphism $h : (T, \mathcal{M}_T) \to (Y, \mathcal{N})$ extending the morphism $(X, \mathcal{M}) \to (D, \mathcal{M}_D) \hookrightarrow (Y, \mathcal{N})$ by the smoothness of $(Y, \mathcal{N})/(S, \mathcal{L})$ and [1, II Proposition 4.2.2(ii)]. As we mentioned earlier, the morphism h indeed factors through $(D, \mathcal{M}_D)$. We define $\mathcal{F}_T$ by $h^*\mathfrak{M}$, note that we have only defined $\mathcal{F}_T$ locally and need to prove they patch well. Indeed, by lemma (B.1.11) and remark (B.1.12), for any two extensions $h_i, i = 1, 2$, we can find a morphism $h : (T, \mathcal{M}_T) \to (D(1), \mathcal{M}_{D(1)})$ such that $h_i = p_i h$. Thus there is an isomorphism $h^*(\eta) : h_2^*\mathfrak{M} \cong h_1^*\mathfrak{M}$. Moreover, these isomorphisms satisfy the transition condition, hence now we can define $\mathcal{F}_T$ globally.

(1) $\Rightarrow$ (2). Given a crystal $\mathcal{F}$ on $\mathrm{Cris}(X/S)$, let $\mathfrak{M} = \mathcal{F}_D$, then by the definition of a crystal, we have

$$\eta : p_1^*\mathfrak{M} \cong p_2^*\mathfrak{M}.$$

The above isomorphism can be written in the following form:

$$1 \otimes a \mapsto \sum_{n \in \mathbb{N}^r} \left(\prod_{1 \le i \le r} (u_i - 1)^{[n_i]} \right) \otimes \eta_n(a).$$

Then the corresponding object in the second category is given by

$$\nabla(a) = \sum_{1 \le i \le r} \eta_{e_i}(a) \otimes d \log m_i,$$

where $\{e_i\}_{1 \le i \le r}$ is the natural basis of $\mathbb{N}^r$.

Quasi-nilpotence of Connection and Nilpotence of p-Curvature

As we have seen above, in order to reformulate an object $(\mathfrak{M}, \nabla)$ in the language of crystals, one needs the connection ∇ to be quasi-nilpotent. This is equivalent to say that the p-curvature of ∇ is nilpotent. This can be seen easily in the smooth setting. Let $t_i \in \mathcal{O}_X, 1 \le i \le r, x \in X$ such that $\{dt_i\}_{1 \le i \le r}$ form a basis of $\Omega^1_{X/k,x}$. Let $\{\partial_i\}_{1 \le i \le r}$ derivations such that $\partial_i t_j = \delta_{ij}$. Then we have the derivation $\partial_i^{(p)} = 0$. Thus the p-curvature $\psi_\nabla(\partial_i) = \nabla(\partial_i)^p - \nabla(\partial_i^{(p)}) = \nabla(\partial_i)^p$. Therefore, given an object $(\mathfrak{M}, \nabla)$ and $a \in \mathfrak{M}_x$, the condition $\nabla(\partial_i)^n(a) = 0$ for some n is equivalent to $\psi_\nabla(\partial_i)^n(a) = 0$ for some n.

In the log case, we can choose a local coordinate $(m_1, \cdots, m_n)$ and let $(\partial_1^{\log}, \cdots, \partial_n^{\log})$ be the dual basis. Then one can prove ([16, Remark 1.2.2]) $(\partial_i^{\log})^{(p)} = \partial_i^{\log}$ and

$$\psi_\nabla(\partial_i^{\log}) = \nabla(\partial_i^{\log})^p - \nabla(\partial_i^{\log}) = \prod_{0 \le j \le p-1} (\nabla(\partial_i^{\log}) - j).$$

In particular, the condition $[\psi_\nabla(\partial_i^{\log})]^n(a) = 0$ for some n is equivalent to (B.2.1).

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Zussamenfassung

Das Hauptziel der Arbeit ist es, die Beziehung zwischen Fontaine Modulen und F-T-Kristall zu studieren.

Im ersten Kapitel wird die Definition von Fontaine Modulen, die auf die inversen Cartier Transform setzt erinnern wir von Ogus und Vologodsky errichtet. Neben der Erinnerung an die ursprüngliche Konstruktion des inversen Cartier Transform, eine direktere Konstruktion, die wir auch vorstellen von G.T. Lan, M. Sheng und K. Zuo. Darüber hinaus beweisen wir die Gleichwertigkeit der beiden Konstruktion.

Im zweiten Kapitel werden wir uns daran erinnern, den Konstruktion von inversen Cartier Transform in der Log Einstellung von D. Schepler und verallgemeinern die Lan-Sheng-Zuo Konstruktion an dieser Einstellung. Darüber hinaus geben wir eine Definition von Log Fontaine Modulen.

Im dritten Kapitel werden wir erinnern an die Definition von F-T-Kristall und beweisen das wichtigste Ergebnis dieser Arbeit:

Theorem. Sei Y eine glatte S_ν-Schema, wobei S_ν ist eine flache $W_{\nu+1}(k)$-Schema, $\nu \geq 1$, und X/S_0 seine Reduction modulo p sein. Bei einem F-T-Kristall (E, Φ, B) auf Y der Breite von weniger als p und let (E_Y, B_Y, ∇_Y) die entsprechende gefilterte O_Y-modulen mit einer integrierbar Zusammenhang ausgestattet. Anschliesend wird die Reduktion dieses Objekt modulo p definiert eine Fontaine Modulen auf X/S_0 im dem Sinne der Ogus und Vologodsky.

Abstract

The main objective of this work is to study the relation between Fontaine modules and F-T-crystals.

In the first chapter we review the definition of Fontaine modules, which relies on the inverse Cartier transform constructed by Ogus and Vologodsky. Besides recalling the original construction of inverse Cartier transform, we also introduce a more direct construction by G.T. Lan, M. Sheng and K. Zuo. Moreover, we prove the equivalence of these two construction.

In the second chapter we review the construction of inverse Cartier transform in the log setting by D. Schepler and generalize Lan-Sheng-Zuo's construction to this setting. Moreover, we give a definition of log Fontaine modules.

In the third chapter we recall the definition of F-T-crystals and prove the main result of this work:

Theorem. Let Y be a smooth S_ν-scheme, where S_ν is a flat $W_{\nu+1}(k)$-scheme, $\nu \geq 1$, and X/S_0 be its reduction modulo p. Given an F-T-crystal (E, Φ, B) on Y of width less than p and let (E_Y, B_Y, ∇_Y) be the corresponding filtered O_Y-module endowed with an integrable connection. Then the reduction of this object modulo p defines a Fontaine module on X/S_0 in the sense of Ogus and Vologodsky.

Lebenslauf

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