Picard–Fuchs Equations of Dimensionally Regulated Feynman Integrals

Dissertation zur Erlangung des Grades
"Doktor der Naturwissenschaften"
vorgelegt von
Raphael Zayadeh

Fachbereich Physik, Mathematik und Informatik
der Johannes Gutenberg Universität Mainz.

Mainz, Dezember 2013

Das Thema stellte Herr Prof. Stefan Müller–Stach

Contents

Chapter 1

Introduction

Calculating Feynman integrals is a problem that is originally known in physics. In perturbative quantum field theory one considers sums of Feynman integrals indexed by graphs that belong to the physical theory. This set of graphs is ordered by their number of loops (independent cycles). In any physical application this series, which is generally believed to be only an asymptotic series, is truncated at a certain loop order and then the coefficients are computed order by order. The difficulty of evaluating such a coefficient increases rapidly with the loop order. This is due to two facts. Firstly, the number of Feynman integrals that need to be evaluated increase rapidly with the given loop order and secondly, the Feynman integrals themselves become more complicated as the loop order grows. One, therefore, uses relations between the Feynman integrals to reduce the number of integrals involved in such a calculation. These are the integration by parts identities. A smaller number of non–reducible integrals remain, which may then be evaluated individually. It is this second step that we are interested in – evaluating a single Feynman integral.
The subject has been drawing intensive attention of mathematicians since the discovery of a systematic appearance of zeta and multi zeta values in the calculation of Feynman integrals, made by Broadhurst and Kreimer ([BK95], [BK97]). A Feynman integral can be written as a projective integral via the Feynman (or alternatively Schwinger) trick. With this Feynman (or Schwinger) parameter description of a Feynman integral, the number theoretic content of such an integral is captured by two homogeneous polynomials and their projective hypersurfaces. A Feynman integral may depend on additional parameters, such as masses and momenta. Only one of the two polynomials, called the second graph polynomial (or second Symanzik polynomial), depends on these parameters. Also the Feynman integral depends on the space–time dimension, which is usually four in a physical application.
A major complication in calculating Feynman integrals arises from the fact, that for some combinations of parameters, including the dimension, a Feynman integral may be divergent. If the Feynman integral is (absolutely) convergent and, moreover, all parameters are chosen to be rational numbers, the Feynman integral is a period.
A divergent Feynman integral is treated by various procedures, such as regularization and renormalization. We consider dimensional regularization replacing a divergent Feynman integral by a Laurent series in a regularization parameter. The coefficients in this series are then the objects of interest. It is known, that for rational parameters all coefficients in such a Laurent series are periods. We will be mainly interested in computing the terms of negative order, as well as the zero order term in such a Laurent series. Poles in the regularization parameter are caused by the singularities of the integrand. In a second step one removes the singularities from the final result. One distinguishes between ultraviolet and infrared singularities and usually handles them

separately. Working in dimensional regularization one may perform a renormalization procedure such as minimal subtraction or modified minimal subtraction to handle ultraviolet singularities. To remove infrared singularities there are various tools. Let us emphasize at this point, that we will not discuss this second step – the removal of singularities – in this dissertation. In particular, we will not discuss renormalization. Instead we will be interested in properties of the coefficients of a Laurent series of a Feynman integral and how these may be computed. For the topic of renormalization we have to refer to the literature (e.g. [IZ87]).
A particularly nice class of Feynman graphs has been studied extensively by mathematicians in the past decade. These are the so–called primitive (massless) graphs of ϕ^4 theory. In dimension four they give rise to an absolutely convergent integral. Therefore, the step of dimensional regularization can be omitted – the zero order term in the Laurent series in the regularization parameter is just the Feynman integral, while the polar part is absent. The property of being primitive ensures that, putting all external momenta to zero, in Feynman parameters we obtain a simplified formula:

$$I = \int_{\sigma} \frac{1}{\mathcal{U}^2} \Omega,$$

where the integration is over some semi–algebraic set and Ω is the standard n–form in the n–dimensional projective space. Here, only the simpler one of the two graph polynomials $\mathcal{U}$ appears. Therefore, the polynomial $\mathcal{U}$ and its projective hypersurface $X_{\mathcal{U}}$ has been the object of study. As the above integrals do not depend on any parameters, they are complex numbers. Inspired by the above–mentioned discovery of Broadhurst and Kreimer, Kontsevich speculated that the periods of $X_{\mathcal{U}}$ as above are multiple zeta values stemming from a mixed Tate motive. He conjectured that all such hypersurfaces $X_{\mathcal{U}}$ have polynomial point count functions. This was, however, disproven by Belkale and Brosnan ([BB03a]). They showed that the first graph hypersurfaces $\mathcal{U}_\Gamma$ of all graphs are sufficiently general to generate the Grothendieck ring $K_{0,mot}$. The question arose under what conditions would the (first) graph motive be mixed Tate. For two infinite families of graphs a positive answer was found [BEK06, Dor08]. Only recently the exact formula for the Feynman integrals of one of these families, the so–called zig–zag graphs, was proven by Brown and Schnetz [BS12]. They evaluate to single zeta values (up to rational numbers).
In this dissertation we are, complementary to that, interested in Feynman integrals that depend on momenta and masses. We, therefore, interpret Feynman integrals as functions in some parameters and we are interested in the functions that occur when evaluating Feynman integrals. The latter problem is, however, very difficult. A more humble goal is to ask for differential equations that Feynman integrals satisfy when varying parameters. The functions appearing in evaluating Feynman integrals as well as the differential equations satisfied by Feynman integrals, are the main focus of this dissertation. In particular, we would like to study variations of such objects over a one–dimensional base, which arise when we vary one parameter and fix the others. The variational parameter will always be a squared momentum or a mass (hence a scalar) in this dissertation. We chose to vary only one parameter for simplicity – we will obtain ordinary differential equations in this parameter. What we do can be generalized to variations with respect to more than one parameter, which will result in partial differential equations. As only the second graph polynomial depends on the parameters, it is apparent, that studying such a variation amounts to studying the second graph polynomial and its projective hypersurface. Therefore, these will be the main objects of interest in this dissertation. To be more precise, let us give an overview of the individual chapters.

The **first chapter** is this introduction.
In the **second chapter** we introduce Feynman graphs and Feynman integrals associated with

such a graph. We will discuss a procedure called dimensional regularization and reduction methods to reduce large sets of integrals to a few basic ones which are commonly called master integrals. A major complication arises from the fact that these can be divergent. This issue is resolved within dimensional regularization by replacing the ill–defined quantity by a Laurent series in a regularization parameter. The coefficients of this series are periods and the objects of interest in this dissertation. Convergent Feynman integrals can be interpreted as projective integrals using the Feynman parameter technique, which is also covered in this chapter. We would like to obtain information about the periods appearing as coefficients in the above–mentioned Laurent series by studying convergent Feynman integrals and applying Tarasov's generalized dimensional recurrence relations. We will therefore take a close look at these relations between Feynman integrals with shifted space–time dimensions.
The **third chapter** covers blowups of linear spaces associated with the two graph hypersurfaces entering the Feynman parameter description of a Feynman integral. These are needed to separate the polar locus of the projective form and the domain of integration. Divergencies can be detected by this method. These divergencies are avoided by shifting dimensions. We will discuss how this leads to an inhomogeneous Picard–Fuchs equation whose inhomogeneous part can be related to Feynman integrals associated with minors of the graph. Also main results from the theory of Picard–Fuchs equations and ways to compute these in the projective setting are recalled.
In the **fourth chapter**, we introduce a more formal approach to obtain differential equations directly for arbitrary dimension without first shifting dimensions and shifting back via Tarasov's recurrence relations. It is more flexible than our approach in chapter three, but it produces larger systems of linear equations.
Finally, in the **fifth chapter**, we will carry through our program for a finite family of two–loop graphs. Our procedure leads to new equations that have previously not been obtained by the usual method of integration by parts. Furthermore, by a result of Tarasov, the two–loop graphs considered here are sufficient to describe all two–loop two–point functions. We also give a three–loop example.

Chapter 2

Feynman Integrals

Feynman integrals arise in perturbative calculations in quantum field theories. In such a theory Feynman graphs are constructed according to Feynman rules and one formally assigns integrals to these graphs. These may depend on parameters, such as masses or momenta and a major complication arises from the fact, that these integrals often are divergent. These ill–defined quantities need to be replaced by finite values. We will discuss dimensional regularization, which is a successful and widely used regularization procedure. This leads to a Laurent series in a regularization parameter, whose polar part as well as zero order term will be the main interest in later chapters.

We discuss the Feynman parametrization, which is the bridge to the algebraic world. It allows us to assign periods to the Feynman integrals. If a Feynman integral is convergent, it is itself a period and the Feynman parametrization conveniently gives us a representation of the integral as a projective integral. The number theoretic content is governed by two homogeneous polynomials, that appear in the Feynman parametrization. Here, these will be called the first and the second graph polynomial. In the divergent case, the coefficients in their Laurent series are periods, but we have to work a little harder to detect their origin. We discuss Tarasov's dimensional recurrence relations which are helpful in this context. Finally, we introduce integration by parts identitites, which connect Feynman integrals of a graph to those of the minors of the graph and which, therefore, provide a useful mechanism of reduction.

2.1 Feynman Graphs

A Feynman graph is a pictorial representation of a physical process involving elementary particles moving in space–time. Particles can be created and deleted, represented by vertices of the graph joining a certain number of lines. We will assume an intuitive understanding of what is meant by graph, let us, however, briefly explain what we mean, when we say "graph". A Feynman graph differs a little from the notion of a graph in algebraic graph theory – it is a multigraph which may have external half–edges.

To be more precise, a (Feynman) *graph* Γ consists of a finite set $V_\Gamma = \{v_1, \ldots, v_k\}$ of *vertices*, a finite set of *edges* $E_\Gamma = \{e_1, \ldots, e_N\}$ (also called *lines*), and a finite set of *half–edges*. Formally, E_Γ is a finite set of oriented pairs (v_i, v_j) of vertices which may have a multiplicity, i.e. more than one edge may join the same two vertices. An edge of the form (v_i, v_i) is called a *tadpole* (which would be called *self–loop* in algebraic graph theory). A half–edge is an edge of the form (v_i) and will be called *external edge* henceforth. The edges of the graph will sometimes be called *internal edges* to distinguish them from the external ones, but we will usually drop the word

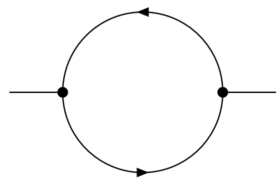

Figure 2.1: A Feynman graph.

"internal".
The *interior* of a graph is obtained from a graph by removing the external edges. A *subgraph* is obtained by removing internal edges. There are a lot of operations on graphs that can be useful for Feynman integral computations (see e.g. [Bro09b]). For our purposes the most important operation is the contraction of an edge, which means to delete the edge and identify its endpoints. A graph γ, that is obtained from another graph Γ by an iterated application of the deletion operation (remove an edge) or the contraction operation, we just described, is called a *minor* of the graph Γ.
Whenever we are interested in a graph, we are also interested in its minors. Later, we will be interested in differential equations of integrals that are associated with Feynman graphs. These will be inhomogeneous and we will see, that information about the inhomogeneous term of the differential equation of a Feynman integral associated with Γ will be contained in the minors of that graph.
A graph is called *connected*, if its interior is connected. The interior of a graph is naturally a simplicial complex. We define the loop number ℓ of a graph to be the number of edges minus the number of vertices plus one. For a connected graph this equals the first Betti number of its interior, which is just the formula for the Euler characteristic of the (interior of) the graph.
A connected graph is called *one–particle irreducible* (1PI), if it cannot be rendered disconnected by removing a single edge. We will restrict to graphs that are connected and 1PI. In the case when the graph is connected, the property of being one–particle irreducible is equivalent to the property of being core. A graph is called *core*, if for any edge e we have $h_1(\Gamma\backslash e) < h_1(\Gamma)$. Sometimes the notion of being core is prefered (see e.g. [BK08]), we will, however, use the notion of one–particle–irreducibility, which is standard in physics literature.
The motivation to restrict to graphs, that are connected and 1PI, comes from physics (see e.g. [IZ87]). Starting from a graph, we will be interested in the graph itself, as well as in its minors, that are obtained by allowing the contraction operation only. Note that the contraction operation respects these two properties, so we will never leave the class of connected 1PI graphs. Therefore, let us assume throughout this dissertation, that every graph is connected and 1PI. The Feynman graphs needed in a calculation are constructed from so–called Feynman rules, which depend on the physical theory. In this dissertation we will not discuss Feynman rules of different physical theories, we will instead start with a graph and its Feynman integrals, which we will define in the present chapter.
The reader interested in the physical background and how graphs are constructed from Feynman rules, is invited to consider one of the many textbooks on the subject [IZ87, PS95] or the lecture notes [Wei10].
Let us conclude this section with one final remark. In the physics literature one often encounters

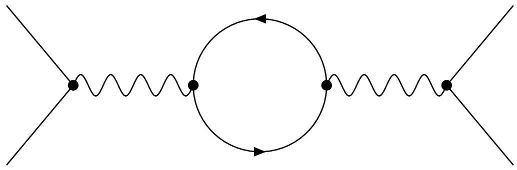

Figure 2.2: A graph with one type of vertex and two types of edges. It represents a process that can be described by Quantum Electrodynamics (QED), which is not a scalar theory.

Feynman graphs with more than one type of vertex or edge. One example is depicted in figure 2.2. Here, we will not consider such graphs, for us, there is always only one type of vertex and one type of edge. This is always the case in a so–called scalar theory, so let us for the moment restrict to such a theory. In section 2.8 we will discuss a reduction due to Tarasov, that will allow more flexibility.
We have mentioned, that a graph is constructed from Feynman rules, so not every graph one may draw necessarily belongs to the given theory. One may ask which graphs are interesting. From our perspective the answer is simple, every graph is interesting. In general, we will work bottom–up, whenever we encounter a graph, we will first try to understand the minors of the graph in question. In [Bro09b], F. Brown discusses a special class of graphs occuring in a scalar theory called (massless) ϕ^4 theory. These are the so–called primitive–divergent graphs of (massless) ϕ^4 theory, and Brown shows that every graph occurs as a minor of such a graph. This means, that in the sense of minors there is no unphysical graph. In conclusion, for us, every graph is interesting, but we have a useful ordering principle and that is by loop order. In the physics literature all one–loop graphs (resp. their Feynman integrals) are known analytically. On the other hand, even at the two–loop level, there are still open questions. Less is known, the higher the loop order of the graph. Therefore, a natural starting point for us will be two–loop graphs.

2.2 The Momentum Space Representation of a Feynman Integral

In a quantum field theory in D dimensions, space–time points x and momenta p are D–dimensional vectors, denoted by $x = (x^0, x^1, \dots, x^{D-1})$ and $p = (p^0, p^1, \dots, p^{D-1})$ respectively. Here D can be any positive integer. In a physical application the dimension usually equals four. A Feynman integral can be represented in *position space*, using the variables x above, or in *momentum space*, using the variables p (we will also use the letter k). Each representation has its own advantages. In this dissertation we choose to work with momentum space. There are two metrics on D–dimensional momentum space that are used, the *Minkowski* metric and the *Euclidean* metric. The Minkowski metric is given by the D by D diagonal matrix

$$\mathrm{diag}(1, -1, \dots, -1),$$

the Euclidean metric is given by the D by D unit matrix. When we write p^2 (or k^2), we mean the scalar product with respect to one of these metrics. When not explicitly stated otherwise we will work with the Minkowski metric.

Let us now define the Feynman integral associated with a graph Γ in its momentum space representation. Starting from a graph, we do the following:

- To each external edge, assign an (external) momentum p_i.
- To each internal edge, assign a momentum k_i and a mass m_i.
- Orient all external edges inwards.[1]

Note, that all masses are scalars (real numbers), whereas all momenta are D–vectors (with real entries).
A Feynman integral in its momentum space representation associated to a graph Γ with ℓ loops and N edges is of the form

$$I_{MS}(D,\Lambda) = C \cdot \int\limits_{\mathbb{R}^{D\cdot\ell}} \frac{d^D k_1 \dots d^D k_\ell}{\prod_{j=1}^{N} P_j}. \tag{2.1}$$

Here, the P_j are quadrics and C is a prefactor that we will discuss later. The Feynman quadrics P_j are obtained by setting

$$P_j = k_j^2 - m_j^2 + i\delta,$$

and applying *momentum conservation* to all vertices, which means at each vertex of the graph to put to zero the sum of the ingoing momenta minus the sum of the outgoing momenta. These terms are called *propagators* in physics literature and we will adopt this terminology. Applying momentum conservation we can eliminate all but ℓ of the momenta k_i. To see this, recall, that we have defined

$$\ell = N - V + 1,$$

where V is the number of vertices of the graph. Clearly N is the total number of internal momenta and $V - 1$ the number of conditions on the internal momenta implied by momentum conservation. The last condition assures conservation of the external momenta and with our convention to orient the external edges inwards, reads

$$\sum_{\text{external momenta}} p_i = 0.$$

Therefore, momentum conservation also fixes the kinematical invariants stemming from the external edges of the graph.[2] Finally, the vector Λ contains all the masses assigned to the edges of the graph and all the independend kinematical invariants. The extra term $i\delta$ is thought of as small and is added to avoid the pole on the real axis. We will usually omit this term in our notation.

Example Let us consider as an example the graph given in fig. 2.3. We have assigned momenta to the edges of the graph. Applying momentum conservation to the vertex on the left gives

$$k_1 - k_4 - p_1 = 0,$$

and we can eliminate k_4. Continuing with the vertex at the top, we obtain

$$-k_1 + k_2 + k_5 = 0,$$

[1] We choose this convention here, but other conventions are also possible. We need to orient the external edges in order to apply momentum conservation.

[2] For a graph with E external edges and external momenta $p_1, \dots, p_E$ we can use momentum conservation to eliminate p_E and use the variables $s_{jk} = (p_j + p_k)^2$ for $1 \le j,k \le E-1$ as kinematical invariants.

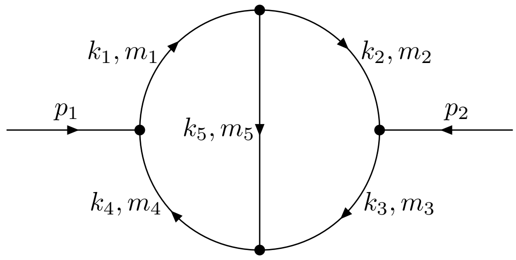

Figure 2.3: The master two–loop two–point graph. We have decorated the edges and external edges, but we have not yet applied momentum conservation.

eliminating k_5. The vertex at the bottom gives

$$-k_3 + k_4 - k_5 = 0,$$

which allows us to eliminate k_3. Finally the vertex on the right gives

$$-k_2 + k_3 - p_2 = 0.$$

Putting everything together, we obtain

$$\begin{aligned} k_4 &= k_1 - p_1, \\ k_5 &= k_1 - k_2, \\ k_3 &= k_2 - p_1, \\ p_2 &= -p_1. \end{aligned}$$

Denoting $p = p_1$ we obtain the Feynman integral

$$\int \frac{d^D k_1 d^D k_2}{(k_1^2 - m_1^2)\,(k_2^2 - m_2^2)\,[(k_2 - p)^2 - m_3^2]\,[(k_1 - p)^2 - m_4^2]\,[(k_1 - k_2)^2 - m_5^2]}.$$

Furthermore, we have seen that momentum conversation at each vertex implies overall momentum conservation. A graph with only two external edges, therefore, only depends on a single kinematical invariant.

Let us now introduce a procedure called *Wick rotation.* In the definition of the propagators we had to include a term $i\delta$ to shift the domain of integration away from the real axis, where we could have poles, due to the fact, that we have worked with the Minkowski metric. Recall, that in Minkowski space, any squared momentum is of the form

$$k^2 = (k^0)^2 - (k^1)^2 - \cdots - (k^{D-1})^2.$$

We want to consider functions that depend only on the square of momenta. Let f be a function, that depends on a single squared momentum k^2. The idea is to rotate from the real axis to the imaginary axis in the complex k_0 plane. We can find a closed contour in the complex k_0–plane, that includes the real axis (shifted by $i\delta$ to avoid the poles) and the imaginary axis and does not contain poles. Therefore, the integration over the whole contour is zero. If the arcs at infinity give a vanishing contribution, we obtain

$$\int\limits_{-\infty}^{\infty} dk_0 f(k_0) = -\int\limits_{i\infty}^{-i\infty} dk_0 f(k_0).$$

Let us now perform the simple change of variables

$$\begin{aligned} k_0 &= iK_0 \\ k_j &= K_j, \text{ for } 1 \le j \le D-1. \end{aligned}$$

This implies

$$k^2 = -K^2,$$

and

$$d^D k = i\, d^D K.$$

Together, we obtain

$$\int d^D k f(-k^2) = i \int d^D K f(K^2). \tag{2.2}$$

Here, the right hand side is an integral over Euclidean space. Using rotational invariance, we can express our integral as the volume of the unit sphere times a one–dimensional integral. The volume of the $(D-1)$–dimensional unit sphere is given by

$$\mathrm{vol}\left(S^{D-1}\right) = \frac{2\pi^{D/2}}{\Gamma(D/2)},$$

and the expression for our integral becomes

$$\int d^D K f(K^2) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dK f(K^2)\, K^{D-1}. \tag{2.3}$$

Here Γ denotes the gamma function, which for $\Re(z) > 0$ is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

It satisfies the functional equation

$$\Gamma(z+1) = z\,\Gamma(z),$$

and, for any positive integer n, the equation

$$\Gamma(n+1) = n!\,.$$

The gamma function can be extended meromorphically to the complex plane. It has simple poles at zero and all negative integers.

Note, that the right hand side of equation (2.3) makes sense for complex values of D. This leads to the idea of dimensional regularization, that we will discuss in section 2.4.

2.3 Divergencies and Power Counting

In the preceeding we have taken a graph and formally assigned an integral to it. The procedure is not sensitive to the matter of convergence and the integral we obtain may converge or diverge depending on the values of the masses and the kinematical invariants, as well as the dimension D. In this chapter we will take a close look under what circumstances we find a convergent integral. To this end we will define the superficial degree of divergence and introduce the method of power counting. How to replace a divergent Feynman integral by a meaningful quantity will then be the subject of the next section.

Let us begin with an example.

Example The Feynman integral corresponding to the one–loop graph depicted in fig. 2.4 reads

$$\int \frac{d^D k}{(k^2 - m_1^2)((k-p)^2 - m_2^2)}.$$

Let us put the external momentum p to zero (which amounts to removing the external edges, such that we have a graph without external edges) and the space–time dimension equal to four. In the case $m_1 = m_2 = 0$ we can apply Wick rotation and rotational invariance and formally write

$$\int \frac{d^4 k}{(k^2)^2} = i \int \frac{d^4 K}{(K^2)^2} = i\,(2\pi^2) \int_0^\infty \frac{dK}{K},$$

which has a divergency as K goes to infinity and a divergency as K goes to zero. The first kind of divergency, coming from a large momentum region is called an *ultraviolet divergency* (UV–divergency). The second kind of divergency, stemming from a small momentum region is called an *infrared divergency* (IR–divergency). Let us further observe, that for $m_1 > 0$ and $m_2 > 0$ we do not obtain a divergency from the lower boundary $K = 0$.

Let us now write the same integral in dimension two. We have

$$\int \frac{d^2 k}{(k^2)^2} = i \int \frac{d^2 K}{(K^2)^2} = i\,(2\pi) \int_0^\infty \frac{dK}{K^3},$$

and we do not obtain an ultraviolet divergency. Furthermore, if the masses are positive, we have a convergent integral.

Everything we have seen in this example remains true for general values of p, we have chosen $p = 0$ just for convenience.

We have seen, that Feynman integrals may acquire singularities from regions of large and small momenta, depending on how the parameters, including the dimension, are chosen. Fortunately, there is a simple method to determine whether a Feynmal integral with given parameters Λ and D is (absolutely) convergent[3] or not. Let us treat UV– and IR–divergencies separately. Beginning with UV–divergencies, we define for a Feynman integral as in equation (2.1) the *superficial degree of divergence*

$$\mathrm{sdd}(I) = D\ell - 2N,$$

[3]A physisist would say finite instead of absolutely convergent, we will just say convergent. If a Feynman integral is convergent, it is automatically absolutely convergent in the Euclidean region by a simple positivity argument. This will be obvious once we introduce Feynman parameters and define the Euclidean region, which will be the region we are interested in.

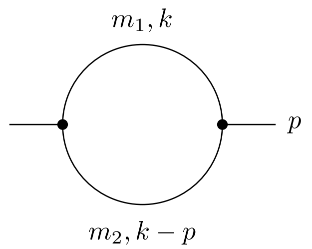

Figure 2.4: A one–loop self–energy graph. We have applied momentum conservation and omitted the orientation.

which counts the powers in the numerator and the denominator of equation (2.1). The semi–algebraic set, given by the inequations

$$m_i \geq 0,$$
$$q_i^2 \leq 0$$

for all masses m_i and all kinematical invariants q_i^2, is called the *Euclidean region.* We have

Theorem 2.3.1 *(i) If all masses assigned to the edges of* Γ *are strictly positive, the corresponding Feynman integral has no infrared divergencies in the Euclidean region.*

(ii) Assume, that there are no IR–divergencies. If $sdd(I_\gamma) < 0$ *for all connected and one–particle irreducible subdiagrams of* Γ *(including* Γ *itself), that consist of some vertices as well as all edges joining these vertices in* Γ*, then the Feynman integral corresponding to* Γ *is absolutely convergent in the Euclidean region.*

(iii) Conversely, if at least one subdiagram as in (ii) has a non–negative superficial degree of divergence, the corresponding Feynman integral is divergent.

The above Theorem is a collection of well–known results. A proof of part *(ii)* and *(iii)* can be found in chapter 8 of [IZ87]. Part *(ii)* is general, while part *(iii)* is true, because we have restricted ourselves to a scalar theory. Finally part *(i)* follows from the discussion in chapter 3. Similarly to what we have done for UV–divergencies, we can define a superficial degree of divergence for the IR–divergencies. We have, however, seen that IR–divergencies are absent, if the corresponding graph has no massless edges (i.e. no mass is zero). In the following we will handle UV–divergencies by shifting the space–time dimension, while we avoid IR–divergencies by either restricting to graphs with no massless edges or by putting a small mass on the massless edges. The convergent integral can then be evaluated as a function in the masses.

2.4 Dimensional Regularization

Let us assume in this chapter, that we are given a Feynman integral without IR–divergencies. Then we see by power counting that for some values of D the integral is convergent, while for others it is (UV–) divergent. The idea of dimensional regularization is to add to D a (complex) regularization parameter ε, such that for $\varepsilon = 0$ we have the original divergent integral. The Feynman integral is then interpreted as a function in ε, which extends meromorphically to the complex plane and the ill–defined quantity is replaced by a Laurent series in ε. One is then usually mostly interested in the polar part of this series as well as the zero order term. From a mathematician's point of view these coefficients are very interesting – they are periods in the sense of Kontsevitch and Zagier [KZ01].

The idea to replace the dimension by a complex variable was introduced to quantum field theory by 't Hooft and Veltman [HV72], Bollini and Giambiagi [BG72], Ashmore [Ash72], and Cicuta and Montaldi [CM72]. The method was also discovered by Speer and Westwater [SW71] in a more abstract setting. In this section we follow [Col84].

We would like to construct a function of a complex variable D, that coincides with the usual integration for positive integer values of D. We have to show the existence of such a function. This will be done by constructing an explicit definition. We furthermore have to show its uniqueness and in order for the definiton to be useful, we have to demonstrate, that the function constructed still has properties associated with ordinary integration.

Let there be given a finite set of vectors $\{k, q_1, \dots, q_r\}$ and let $f(k, q_1, \dots, q_r)$ be a *scalar* function, which means, that it only depends on scalar products of the given vectors. Here the vectors q_j can be thought of as momenta of external particles (associated with the external edges of the Feynman graph). For complex D we would like to define an operation

$$\int d^D k f(k, q_1, \dots, q_r),$$

that can be regarded as integration over a D–dimensional space. Since the dimension of a vector space is either a positive integer or infinity, this cannot be taken completely literally. We work in an infinite–dimensional Euclidean vector space $\mathbb{E}$. Let us assume, that k and all q_j are elements of this vector space. We can find a finite–dimensional subspace of $\mathbb{E}$, that contains all the q_j. This subspace will be denoted $\mathbb{E}_{||}$ and its orthogonal complement will be denoted $\mathbb{E}_\perp$. We can now decompose

$$\begin{aligned} k &= k_{||} + k_\perp \\ &= \sum_{j=1}^{J} k^j e_j + k_\perp, \end{aligned}$$

where $k_{||} \in \mathbb{E}_{||}$ and $k_\perp \in \mathbb{E}_\perp$, and $\{e_1, \dots, e_J\}$ is an orthonormal basis of $\mathbb{E}_{||}$.

We now define

$$\int d^D k f(k, q_1, \dots, q_r) = \int dk^1 \dots dk^J \int d^{D-J} k_\perp f(k, q_1, \dots, q_r), \tag{2.4}$$

and

$$\int d^{D-J} k_\perp f(k, q_1, \dots, q_r) = \frac{2\pi^{(D-J)/2}}{\Gamma((D-J)/2)} \int\limits_0^\infty dk_\perp k_\perp^{D-J-1} f(k, q_1, \dots, q_r), \tag{2.5}$$

and call $\int d^D k f(k, q_1, \dots, q_r)$ a D–dimensional integral. If all q_j are zero, this is nothing but equation (2.3). Definition (2.5) is motivated by the fact that f depends only on scalar products of the vectors, but not on the direction of $k_\perp$. We have mimicked the procedure of integrating over a sphere.

It remains to show, that our definition is independend of the choice of the subspace $\mathbb{E}_{||}$. Also one has to be careful at the lower boundary $k_\perp = 0$ if D is small. For a discussion of these issues, as well as a proof of the uniqueness of the definition, we refer to [Col84].

Let us now come to the properties of the D–dimensional integral. We have

- *Linearity*: For any complex numbers a and b

$$\int d^D k[a\, f(k) + b\, g(k)] = a \int d^D k f(k) + b \int d^D k g(k).$$

- *Scaling*: For any complex number s

$$\int d^D k f(sk) = s^{-D} \int d^D k f(k).$$

- *Translation invariance*: For any vector q

$$\int d^D k f(k+q) = \int d^D k f(k).$$

- *Fubini*:

$$\int d^D k_1 \int d^D k_2 f(k_1, k_2) = \int d^D k_2 \int d^D k_1 f(k_1, k_2).$$

  This can be used to define

$$\begin{aligned}\int d^D k_1 d^D k_2 f(k_1, k_2; q_1, \ldots, q_r) &= \int d^D k_1 \int d^D k_2 f(k_1, k_2; q_1, \ldots, q_r)\\ &= \int d^D k_2 \int d^D k_1 f(k_1, k_2; q_1, \ldots, q_r).\end{aligned}$$

- *Commutativity of integration and differentiation*:

$$\frac{\partial}{\partial q} \int d^D k f(k, q, \ldots) = \int d^D k \frac{\partial}{\partial q} f(k, q, \ldots).$$

- *Integration by parts*: For any vector q

$$\int d^D k \frac{\partial}{\partial k^\mu} q^\mu f(k) = 0.$$

- *Analyticity and recovery of ordinary integration*: Consider an integral

$$\int d^D k f(k, q_1, \ldots, q_r),$$

  which is convergent at $D = 4$. Then the integral is analytic in D and the parameters q_j, when D is close to four, if the integrand is analytic. If the q_j lie in the first four dimensions, then the integral at $D = 4$ has the same value as the ordinary four–dimensional integral of f. The same is true for convergent multiple integrals

$$\int d^D k_1 \ldots d^D k_\ell f(k_1, \ldots, k_\ell, q_1, \ldots, q_r),$$

  which might represent Feynman integrals of graphs with ℓ loops and r external edges.

- *Beta function formula*: For k–independent terms u and v, we have

$$\int d^D k \frac{(k^2)^a}{(uk^2+v)^b} = \frac{\pi^{D/2}}{\Gamma(D/2)} \frac{u^{-a-D/2}}{v^{b-a-D/2}} \frac{\Gamma(a+D/2)\,\Gamma(b-a-D/2)}{\Gamma(b)}.$$

Again, proofs for all these properties can be found in [Col84]. Let us comment on the name of the last property. Euler's beta function can be defined as

$$B(x,y) = \int_0^\infty dt \frac{t^{x-1}}{(t+1)^{x+y}}$$

for $\Re(x) > 0$ and $\Re(y) > 0$, and satisfies

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

For $u, v \neq 0$, we obtain the identity

$$\int_0^\infty dt \frac{t^a}{(ut+v)^b} = \frac{u^{-a-1}}{v^{b-a-1}} \frac{\Gamma(a+1)\,\Gamma(b-a-1)}{\Gamma(b)}, \tag{2.6}$$

which was used in the reduction algorithm of [Bro09a].
For any positive integer D, we have

$$\frac{\Gamma(D/2)}{\pi^{D/2}} \int d^D K \frac{(K^2)^a}{(uK^2+v)^b} = \int_0^\infty dK \frac{K^{a+D/2-1}}{(uK+v)^b},$$

and the right hand side is just a beta function via equation (2.6). Furthermore, we see, that the beta function formula contains equation (2.6) as a special case.

In this section we have described the D–dimensional integral of a scalar function f. We have denoted the momenta k_j and q_j and worked in an Euclidean space. Making the transition to Feynman integrals, let us remind the reader, that we have denoted Euclidean momenta by capital letters and that we will usually denote external momenta by p_j.

In [Eti00] and also in [Mey02] a different approach is proposed. There, an operator (the D–dimensional integral) is defined on a space of Schwartz functions. This may be applied to Feynman integrals with parameters. The essence of this approach is the following

Theorem 2.4.1 *A Feynman integral $I(D,\Lambda,\nu)$ in its momentum space representation as in equation (2.1) extends meromorphically to the whole complex plane. At any integer D we have a Laurent series*

$$I(D-2\varepsilon,\Lambda,\nu) = \sum_{j=-2\ell}^{\infty} c_j\,\varepsilon^j.$$

The fact that we acquire at most poles of order 2ℓ is well known and becomes evident when analyzing the sequence of blowups we discuss in chapter 3.

2.5 Integration by Parts Identities

One of the most powerful tools for evaluating Feynman integrals are the *integration by parts identities* (IBP–identities), that we will now introduce. In a physical application typically not only a single Feynman integral has to be computed. Instead one has to consider sums of Feynman integrals. The number of integrals involved in such a computation increases rapidly with the desired loop–order. On the other hand the evaluation of a single two or three–loop integral is already very difficult. Therefore, efficient reduction procedures are essential and the method of IBP–identities provides such a procedure.

2.5.1 Integrals with Irreducible Numerator

Let $\mathcal{S} = \{k_1, \dots, k_\ell, p_1, \dots, p_r\}$ be a set of momenta. We have called a function $f(\mathcal{S})$, that depends on these momenta only through their scalar products, *scalar*. The integrand of a Feynman integral as in equation (2.1) is a scalar function. We will now define Feynman integrals with irreducible numerators and arbitrary powers of the propagators associated to a Feynman graph, where until now we have spoken of the unique Feynman integral assigned to a Feynman graph. These more general objects will appear naturally in the reduction algorithm we will discuss in the next section.

Let us assign to a graph Γ with N internal edges, E external edges, dimension D and parameters Λ the following integrals

$$I(D, \Lambda, \nu) = \int d^D k_1 \dots d^D k_\ell \frac{\prod_{i=1}^{B} S_i^{\nu_{N+i}}}{\prod_{j=1}^{N} P_j^{\nu_j}}. \tag{2.7}$$

Here S_i can be a product of loop momenta or the product of a loop momentum with an external momentum. The exponents ν_i of the propagators are arbitrary non–negative integers[4]. Due to momentum conservation, (for $E \geq 1$) we have $E - 1$ independent external momenta, hence we obtain

$$B = \ell(E-1) + \frac{1}{2}\ell(\ell+1).$$

This set of integrals can be reduced to a set of integrals where the product in the numerator runs only from one to $B - N$. This is achieved by completing squares and we will give an example in a moment. In this way, for a given graph, the integrals of equation (2.7) reduce to integrals

$$\int d^D k_1 \dots d^D k_\ell \frac{1}{\prod_{j=1}^{N} (P_j)^{\nu_j}}, \tag{2.8}$$

for $B \leq N$, and to

$$\int d^D k_1 \dots d^D k_\ell \frac{\prod_{i=1}^{B-N} (S_i)^{\nu_{N+i}}}{\prod_{j=1}^{N} (P_j)^{\nu_j}}, \tag{2.9}$$

for $B > N$.
We call integrals as in (2.8) *scalar* Feynman integrals. If for an integral as in (2.9) at least one of the exponents in the numerator is positive, we call it a Feynman integral with *irreducible numerator*.

If an exponent in the denominator of an integral as in (2.9) is zero, the integral is considered simpler. In such a situation, it in fact belongs to a minor of the graph[5] Γ, namely the one with the corresponding edge contracted. Typically, we will work bottom–up, i.e. we consider all

[4] In some applications propagators with complex exponents are considered. We will, however, not consider this situation in this dissertation.

[5] In the literature one finds the terms Feynman graph or diagram. These are in principle used as synonyms. One also frequently finds the term Feynman amplitude instead of Feynman integral. We will consequently use the terms graph and integral.

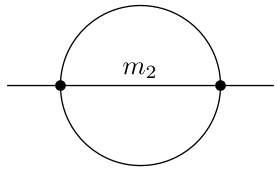

Figure 2.5: The two-loop sunrise graph. We will usually draw Feynman graphs in this way, omitting everything, that is not strictly necessary. While the propagators depend on the orientation of the graph, the Feynman integrals do not. This can be seen in the Feynman parameter description which we will discuss in section 2.6. The missing label for the external edge is obtained by momentum conservation.

integrals that belong to minors of Γ known and then gather information about the integrals of Γ by these known simpler integrals and additional information, that later will be provided by the second graph hypersurface, which we will define in section 2.6. Relations between integrals with shifted exponents are available from two sources, firstly, the IBP–identities, to be discussed in the following section and secondly, we have generalized dimensional recurrence relations, that will be discussed in section 2.8. In this sense, as we mentioned earlier, every graph becomes interesting, not only those of a certain theory.

Example Consider the graph depicted in fig. 2.5. According to the discussion in this chapter, it has associated the Feynman integrals

$$I(D,\Lambda,\nu) = \int d^Dk_1 d^Dk_2 \frac{(k_1\cdot p)^{\nu_4}(k_2\cdot p)^{\nu_5}(k_1\cdot k_2)^{\nu_6}}{(k_1^2-m_1^2)^{\nu_1}(k_2^2-m_2^2)^{\nu_2}\left[(p-k_1-k_2)^2-m_3^2\right]^{\nu_3}}, \tag{2.10}$$

where the entries of $\nu=(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6)$ are non–negative integers. We have $N=3$ and $B=5$, such that we can eliminate one term in the numerator. For notational convenience, let us denote the propagator, that is raised to the power ν_i in equation (2.10), by P_i. We would like to eliminate $P_6 = k_1k_2$. We have

$$k_1k_2 = \frac{1}{2}\left[2k_1p + 2k_2p - p^2 - k_1^2 - k_2^2 + (p-k_1-k_2)^2\right],$$

and therefore

$$\begin{aligned}
I(D,\Lambda,\nu) &= \int d^Dk_1 d^Dk_2 \frac{P_4^{\nu_4+1}P_5^{\nu_5}P_6^{\nu_6-1}}{P_1^{\nu_1}P_2^{\nu_2}P_3^{\nu_3}} + \int d^Dk_1 d^Dk_2 \frac{P_4^{\nu_4}P_5^{\nu_5+1}P_6^{\nu_6-1}}{P_1^{\nu_1}P_2^{\nu_2}P_3^{\nu_3}} \\
&\quad - \frac{1}{2}\int d^Dk_1 d^Dk_2 \frac{P_4^{\nu_4}P_5^{\nu_5}P_6^{\nu_6-1}\left(p^2+k_1^2+k_2^2-(p-k_1-k_2)^2\right)}{P_1^{\nu_1}P_2^{\nu_2}P_3^{\nu_3}} \\
&= \int d^Dk_1 d^Dk_2 \frac{P_4^{\nu_4+1}P_5^{\nu_5}P_6^{\nu_6-1}}{P_1^{\nu_1}P_2^{\nu_2}P_3^{\nu_3}} + \int d^Dk_1 d^Dk_2 \frac{P_4^{\nu_4}P_5^{\nu_5+1}P_6^{\nu_6-1}}{P_1^{\nu_1}P_2^{\nu_2}P_3^{\nu_3}} \\
&\quad - \frac{1}{2}\left(p^2+m_1^2+m_2^2-m_3^2\right)\int d^Dk_1 d^Dk_2 \frac{P_4^{\nu_4}P_5^{\nu_5}P_6^{\nu_6-1}}{P_1^{\nu_1}P_2^{\nu_2}P_3^{\nu_3}} \\
&\quad - \frac{1}{2}\int d^Dk_1 d^Dk_2 \frac{P_4^{\nu_4}P_5^{\nu_5}P_6^{\nu_6-1}}{P_1^{\nu_1-1}P_2^{\nu_2}P_3^{\nu_3}} - \frac{1}{2}\int d^Dk_1 d^Dk_2 \frac{P_4^{\nu_4}P_5^{\nu_5}P_6^{\nu_6-1}}{P_1^{\nu_1}P_2^{\nu_2-1}P_3^{\nu_3}} \\
&\quad + \frac{1}{2}\int d^Dk_1 d^Dk_2 \frac{P_4^{\nu_4}P_5^{\nu_5}P_6^{\nu_6-1}}{P_1^{\nu_1}P_2^{\nu_2}P_3^{\nu_3-1}}.
\end{aligned}$$

On the right–hand side of this equation, the power of the propagator P_6 is reduced. Iterating this procedure, we can express $I(D,\Lambda,\nu)$ as a sum of integrals, that do not depend on P_6. It is therefore enough to consider integrals

$$I(D,\Lambda,\nu) = \int d^Dk_1 d^Dk_2 \frac{(k_1\cdot p)^{\nu_4}(k_2\cdot p)^{\nu_5}}{(k_1^2-m_1^2)^{\nu_1}(k_2^2-m_2^2)^{\nu_2}\left[(p-k_1-k_2)^2-m_3^2\right]^{\nu_3}},$$

where the entries of $\nu = (\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$ are non–negative integers.

Note, that power counting as discussed earlier still works for integrals as in equation (2.8). For such an integral, we define

$$\mathrm{sdd}(I) = \ell D - 2\overline{\nu},$$

where $\overline{\nu} = \nu_1 + \cdots + \nu_N$.

2.5.2 Reduction via IBP–Identities

The reduction procedure, we would like to introduce in this section is based on the property of the D–dimensional integral, that we have called integration by parts. For a scalar function f, a momentum k and any D–vector q we have

$$\int d^Dk \frac{\partial}{\partial k^\mu} q^\mu f(k) = 0.$$

For Feynman integrals as in equation (2.9) we obtain the relations

$$\int d^Dk_1 \dots d^Dk_\ell \frac{\partial}{\partial k_h^\mu} q_l^\mu \frac{\prod_{i=1}^{B-N}(S_i)^{\nu_{N+i}}}{\prod_{j=1}^{N}(P_j)^{\nu_j}} = 0,$$

where k_h can be any internal momentum and q_l can be any internal or external momentum. This gives $\ell(\ell+E-1)$ identities, that we call *basic IBP–identities*. From these, all IBP–identities can be obtained by linearity. They were introduced in [CT81, Tka81]. Evaluating the differentiation leads to a sum of integrals with exponents, that are shifted by integers. To simplify our notation, let us introduce the operators

$$\begin{aligned}\mathbf{i}^+ I(D,\Lambda,\nu_1,\dots,\nu_B) &= I(D,\Lambda,\nu_1,\dots,\nu_i+1,\dots,\nu_B),\\ \mathbf{i}^- I(D,\Lambda,\nu_1,\dots,\nu_B) &= I(D,\Lambda,\nu_1,\dots,\nu_i-1,\dots,\nu_B),\end{aligned}$$

which shift one exponent by one. Applying a single IBP–identity leads to an equation where integrals with at most two shifts appear. This can be seen easily using the chain rule. The initial differentiation may increase an exponent in the denominator or reduce an exponent in the numerator by one. The inner derivative may then increase an exponent in the numerator, decrease an exponent in the denominator, or leave all exponents unchanged. Therefore an IBP–identity is of the form

$$\left(c + \sum_{i=1}^{N} c_i^+ \mathbf{i}^+ + \sum_{i=N+1}^{B} c_i^- \mathbf{i}^- + \sum_{i\neq j} c_{ij}^{\pm} \mathbf{i}^+\mathbf{j}^- + \sum_{\substack{0\le i\le N\\ N<j\le B}} c_{ij}^+ \mathbf{i}^+\mathbf{j}^+ + \sum_{\substack{0\le i\le N\\ N<j\le B}} c_{ij}^- \mathbf{i}^-\mathbf{j}^- \right) I(D,\Lambda,\nu) = 0,$$

with coefficients, that are rational functions.

We have associated with a graph an infinite number of integrals and the IBP–identities provide an infinite number of relations among them. One can choose an ordering on the integrals and try to reduce them according to this ordering. The integrals that cannot be reduced are called *master integrals*.

Let us be more precise. The Feynman integrals of a graph can be considered as elements of the field of functions $\mathscr{F}$ of B integer arguments $\nu_1,\dots,\nu_B$. A basis of this infinite–dimensional vector space is

$$H_{\nu_1,\dots,\nu_B}(\widetilde{\nu}_1,\dots,\widetilde{\nu}_B) = \delta_{\nu_1,\widetilde{\nu}_1}\dots\delta_{\nu_B,\widetilde{\nu}_B}.$$

A Feynman integral as a function in ν represents a point in $\mathscr{F}$ and an IBP–identity can be interpreted as a linear functional on this vector space. The set of relations is fixed by considering all IBP–identities with all possible values of ν substituted. Each such relation defines an element in $\mathscr{F}^*$ and all relations generate an infinite–dimensional vector subspace $\mathcal{R}\subset\mathscr{F}^*$. Let us consider the vector subspace of $\mathscr{F}$ given by

$$\mathcal{S} = \{ f\in\mathscr{F} \mid \langle r,f\rangle = 0 \quad \forall\, r\in\mathcal{R}\}.$$

Then the Feynman integral $F(\nu)$ is an element of $\mathcal{S}$. Now, fix an ordering on the Feynman integrals (i.e. an ordering on $\mathbb{Z}^B$). A master integral $F(\nu_1,\dots,\nu_B)$ is an integral, such that there is no element $r\in\mathcal{R}$ acting on F, such that all the points $(\widetilde{\nu}_1,\dots,\widetilde{\nu}_B)$ are lower than $(\nu_1,\dots,\nu_B)$ with respect to the ordering. The set of master integrals depends on the chosen ordering.

We see, that the number of master integrals is finite, if and only if, the vector space $\mathcal{S}$ is finite–dimensional. In this way, the following is shown in [SP10].

Theorem 2.5.1 *For a given graph Γ the number of master integrals is finite.*

This means, that the Feynman integrals associated with a graph can be reduced to a finite number of integrals using IBP–identities. To obtain these, an automated procedure is desireable. The widely used *Laporta algorithm* [Lap00] is such a procedure. It imposes an ordering on the Feynman integrals and uses Gauß–elimination to obtain a set of master integrals for the graph. Let us mention, that there are publicly available implementations of the Laporta algorithm, such as AIR [AL04], FIRE [Smi08] and Reduze [Stu10]. To summarize, we have seen in this section, that IBP–identities provide an efficient and successful reduction procedure to express (sums of) complicated integrals in terms of simpler ones. To any given graph we only have to consider finitely many master integrals.

Let us conclude this section with an example.

Example Let us derive the basic IBP–identities for the two–loop sunrise graph of fig. 2.5. We have $\ell\,(\ell + E - 1) = 6$ such identities. The Feynman integrals of the sunrise graph are

$$I(D, \Lambda, \nu) = \int d^D k_1 d^D k_2 \frac{(k_1 \cdot p)^{\nu_4} (k_2 \cdot p)^{\nu_5}}{(k_1^2 - m_1^2)^{\nu_1} (k_2^2 - m_2^2)^{\nu_2} \left[(p - k_1 - k_2)^2 - m_3^2\right]^{\nu_3}}. \tag{2.11}$$

Let us denote the integrand of the right–hand side of equation (2.11) by f. We have

$$\begin{aligned} \frac{\partial}{\partial k_1^\mu} k_1^\mu f &= D f + k_1^\mu \frac{\partial}{\partial k_1^\mu} f \\ &= \left(D + \nu_4 - 2\nu_1 - 2\nu_1 m_1^2 \mathbf{1}^+ - 2\nu_3 (k_1^2 + k_1 k_2 - k_1 p) \mathbf{3}^+\right) f. \end{aligned}$$

At this point the product $k_1 k_2$ appears and we can complete the square as in the previous example to obtain

$$\begin{aligned} 2\nu_3 (k_1^2 + k_1 k_2 - k_1 p) \mathbf{3}^+ f &= \nu_3 \left(2k_2 p - p^2 + k_1^2 - k_2^2 + (p - k_1 - k_2)^2\right) \mathbf{3}^+ f \\ &= \nu_3 \left(1 - (p^2 - m_1^2 + m_2^2 - m_3^2) \mathbf{3}^+ + \mathbf{1}^- \mathbf{3}^+ - \mathbf{2}^- \mathbf{3}^+ + 2 \cdot \mathbf{3}^+ \mathbf{5}^+\right) f. \end{aligned}$$

In the same way, we obtain all six basic IBP–identities. They read

$$\begin{aligned}
\frac{\partial}{\partial k_1^\mu} k_1^\mu I(D, \Lambda, \nu) &= \big(D - \nu_3 + \nu_4 - 2\nu_1 - 2\nu_1 m_1^2 \mathbf{1}^+ + \nu_3 (p^2 - m_1^2 + m_2^2 - m_3^2) \mathbf{3}^+ \\
&\quad - \nu_3 \mathbf{1}^- \mathbf{3}^+ + \nu_3 \mathbf{2}^- \mathbf{3}^+ - 2\nu_3 \mathbf{3}^+ \mathbf{5}^+\big) I(D, \Lambda, \nu) = 0, \\
\frac{\partial}{\partial k_2^\mu} k_2^\mu I(D, \Lambda, \nu) &= \big(D - \nu_3 + \nu_5 - 2\nu_2 - 2\nu_2 m_2^2 \mathbf{2}^+ + \nu_3 (p^2 + m_1^2 - m_2^2 - m_3^2) \mathbf{3}^+ \\
&\quad + \nu_3 \mathbf{1}^- \mathbf{3}^+ - \nu_3 \mathbf{2}^- \mathbf{3}^+ - 2\nu_3 \mathbf{3}^+ \mathbf{4}^+\big) I(D, \Lambda, \nu) = 0, \\
\frac{\partial}{\partial k_1^\mu} k_2^\mu I(D, \Lambda, \nu) &= \big(\nu_1 - \nu_3 + \nu_1 (p^2 + m_1^2 + m_2^2 - m_3^2) \mathbf{1}^+ + \nu_3 (p^2 + m_1^2 - m_2^2 - m_3^2) \mathbf{3}^+ \\
&\quad + \nu_1 \mathbf{1}^+ \mathbf{2}^- + \nu_3 \mathbf{1}^- \mathbf{3}^+ - \nu_1 \mathbf{1}^+ \mathbf{3}^- - \nu_3 \mathbf{2}^- \mathbf{3}^+ - 2\nu_1 \mathbf{1}^+ \mathbf{4}^+ \\
&\quad - 2\nu_1 \mathbf{1}^+ \mathbf{5}^+ - 2\nu_3 \mathbf{3}^+ \mathbf{4}^+ + \nu_4 \mathbf{4}^- \mathbf{5}^+\big) I(D, \Lambda, \nu) = 0, \\
\frac{\partial}{\partial k_2^\mu} k_1^\mu I(D, \Lambda, \nu) &= \big(\nu_2 - \nu_3 + \nu_2 (p^2 + m_1^2 + m_2^2 - m_3^2) \mathbf{2}^+ + \nu_3 (p^2 - m_1^2 + m_2^2 - m_3^2) \mathbf{3}^+ \\
&\quad + \nu_2 \mathbf{1}^- \mathbf{2}^+ + \nu_3 \mathbf{2}^- \mathbf{3}^+ - \nu_2 \mathbf{2}^+ \mathbf{3}^- - \nu_3 \mathbf{1}^- \mathbf{3}^+ - 2\nu_2 \mathbf{2}^+ \mathbf{5}^+ \\
&\quad - 2\nu_2 \mathbf{2}^+ \mathbf{4}^+ - 2\nu_3 \mathbf{3}^+ \mathbf{5}^+ + \nu_5 \mathbf{4}^+ \mathbf{5}^-\big) I(D, \Lambda, \nu) = 0, \\
\frac{\partial}{\partial k_1^\mu} p^\mu I(D, \Lambda, \nu) &= \big(\nu_4 p^2 \mathbf{4}^- - 2\nu_1 \mathbf{1}^+ \mathbf{4}^+ - 2\nu_3 \mathbf{3}^+ \mathbf{4}^+ - 2\nu_3 \mathbf{3}^+ \mathbf{5}^+ + 2\nu_3 p^2 \mathbf{3}^+\big) I(D, \Lambda, \nu) = 0, \\
\frac{\partial}{\partial k_2^\mu} p^\mu I(D, \Lambda, \nu) &= \big(\nu_5 p^2 \mathbf{5}^- - 2\nu_2 \mathbf{2}^+ \mathbf{5}^+ - 2\nu_3 \mathbf{3}^+ \mathbf{4}^+ - 2\nu_3 \mathbf{3}^+ \mathbf{5}^+ + 2\nu_3 p^2 \mathbf{3}^+\big) I(D, \Lambda, \nu) = 0.
\end{aligned}$$

Now let us assume we would like to evaluate the integral $I = I(4; p^2, m_1, m_2, 0; 2, 1, 1, 1, 0)$ as a function in p^2, where we view the two masses as arbitrary, but fixed. Define the integrals

$$\begin{aligned} I_0 &= I(D, p^2, m_1^2, m_2^2, 0; 1, 1, 1, 0, 0), \\ I_1 &= I(D, p^2, m_1^2, m_2^2, 0; 2, 1, 1, 0, 0), \\ I_2 &= I(D, p^2, m_1^2, m_2^2, 0; 1, 2, 1, 0, 0). \end{aligned}$$

We know by power counting, that these integrals are divergent in dimension four. We are therefore not interested in the integrals themselves, but in the coefficients of their ε–expansions around $D = 4$, viewed as functions in p^2. It is known, that these evaluate to polylogarithms (as functions in p^2). Let us now come to I in general dimension D. We apply the IBP–identity

$$\left(\frac{\partial}{\partial k_1^\mu} k_1^\mu + \frac{\partial}{\partial k_2^\mu} k_2^\mu - \frac{\partial}{\partial k_1^\mu} p^\mu \right) I_0 = 2(D-3) I_0 - 2m_1^2 I_1 - 2m_2^2 I_2 + 2I = 0,$$

to express I as a combination of I_0, I_1 and I_2. Indeed, the integrals I_0, I_1, I_2 are master integrals for the two–loop sunrise graph when $m_3 = 0$. Therefore, in this case, everything evaluates to polylogarithms. This fails to be true, when all masses are non–zero. The general case of arbitrary masses will be of interest in later chapters.

2.6 Feynman Parameters

The Feynman parameters are the bridge to the world of (complex) algebraic geometry. They allow it to write a Feynman integral as a projective integral. The Feynman parameter technique is widely known and can be found in many places in the literature (see e.g. [Kak93, IZ87]). In this chapter we will discuss the Feynman parameter prescription of a scalar Feynman integral. The restriction to scalar integrals will be justified in section 2.8, where we introduce a reduction procedure to scalar integrals.

2.6.1 The Feynman Trick

The Feynman trick in its simplest form is the equality

$$\frac{1}{P_1 P_2} = \int_0^1 dx \frac{1}{(xP_1 + (1-x)P_2)^2},$$

eliminating a product at the cost of introducing an integration. Equivalently, we may use the Schwinger trick which, in its simplest form, reads

$$\frac{1}{P} = \int_0^\infty dx \exp(-xP).$$

We decide to use the Feynman trick in the following. In its most general form the Feynman trick becomes

$$\prod_{i=1}^N \frac{1}{P_i^{\nu_i}} = \frac{\Gamma(\overline{\nu})}{\prod_{i=1}^N \Gamma(\nu_i)} \int_{x_i \geq 0} d^N x \; \delta\left(1 - \sum_{i=1}^N x_i\right) \left(\prod_{i=1}^N x_i^{\nu_i - 1}\right) \left(\sum_{i=1}^N x_i P_i\right)^{-\overline{\nu}}, \tag{2.12}$$

with $\overline{\nu} = \sum_{i=1}^{N} \nu_i$. The newly introduced integration variables x_i are called *Feynman parameters*. Let us use this equality to re–express scalar Feynman integrals. Starting from the momentum space representation of a scalar Feynman integral as in equation (2.8), we apply Wick rotation and the Feynman trick of equation (2.12). Then we may use translational invariance of the D–dimensional integral to complete the square, such that the integral depends only on the square of a momentum k_i. At this point we can use the beta function formula to integrate over the momenta k_i:

$$\int d^D k \frac{(k^2)^a}{(uk^2+v)^b} = \frac{\pi^{D/2}}{\Gamma(D/2)} \frac{u^{-a-D/2}}{v^{b-a-D/2}} \frac{\Gamma(a+D/2)\,\Gamma(b-a-D/2)}{\Gamma(b)}.$$

Iterating this procedure we can evaluate the integration over the momenta k_i, such that only the integration over the Feynman parameters x_i remains. For a scalar Feynman integral this procedure leads to the formula

$$I(D,\Lambda) = (-1)^{\overline{\nu}} i^{\ell} \pi^{\ell D/2} \frac{\Gamma(\overline{\nu}-\ell D/2)}{\prod\limits_{i=1}^{N}\Gamma(\nu_i)} \int\limits_{x_i \geq 0} d^N x\, \delta\left(1-\sum_{i=1}^{N} x_i\right)\left(\prod_{i=1}^{N} x_i^{\nu_i-1}\right) \frac{\mathcal{U}^{\overline{\nu}-(\ell+1)D/2}}{\mathcal{F}^{\overline{\nu}-\ell D/2}}. \quad (2.13)$$

Here, $\mathcal{U}$ and $\mathcal{F}$ are polynomials in the Feynman parameters. The polynomial $\mathcal{F}$ depends furthermore on the kinematical invariants and the masses. To obtain equation (2.13) one does not have to perform all the steps we have just described, the polynomials $\mathcal{U}$ and $\mathcal{F}$ can be read off the underlying Feynman graph. This will be the subject of the next subsection. Let us, however, already mention, that both polynomials $\mathcal{U}$ and $\mathcal{F}$ are homogeneous in the Feynman parameters (of degree ℓ and $\ell+1$ respectively).

The domain of integration is the real simplex given by the equation $\sum_{i=1}^{N} x_i = 1$ and inequations $x_i \geq 0$. In the literature the Feynman parameter description usually includes this choice of an affine open. On the other hand, we usually prefer affine opens of the kind $x_i = 1$ for a single variable x_i. We can choose any such open because the integral of equation (2.13) is a projective integral. In physics literature this is refered to as the Cheng–Wu Theorem (see e.g. [Smi04]).

We would like to alter equation (2.13) a little more. The integral of equation (2.13) comes with prefactors that do not affect the integration and we would like to drop these terms. There is, however, one subtlety. Within dimensional regularization, the term $\Gamma(\nu - \ell D/2)$ can lead to a single pole in ε. If a pole occurs, this is called an overall UV–divergence. We can, nevertheless, treat the gamma function and the remaining integral separately. The Laurent series of the gamma function is well known and the integral as well as the gamma function can be expanded as a Laurent series separately.

So dropping the prefactors including the gamma functions and defining

$$\Omega = \sum_{i=1}^{N} (-1)^{i-1}\, x_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_N,$$

where the hat means that the corresponding term is omitted and defining

$$\sigma = \left\{ x = (x_1 : \cdots : x_N) \in \mathbb{P}^{N-1} \mid x_i \in \mathbb{R},\ x_i \geq 0 \right\},$$

we refer to

$$I(D,\Lambda,\nu) = \int\limits_{\sigma} \left(\prod_{i=1}^{N} x_i^{\nu_i-1}\right) \frac{\mathcal{U}^{\overline{\nu}-(\ell+1)D/2}}{\mathcal{F}^{\overline{\nu}-\ell D/2}}\, \Omega \quad (2.14)$$

as the Feynman parameter description of $I(D, \Lambda, \nu)$, bearing in mind that when integrating we have to choose an affine open.
Let us now define the prefactor C entering in the momentum space representation of a Feynman integral. Everything we have stated so far remains correct, if we choose

$$C = (-1)^{\overline{\nu}} \frac{1}{i^{\ell} \pi^{\ell D/2}}.$$

This simplifies equation (2.13) but leaves the gamma factors. To obtain equation (2.14) we have to choose

$$\widetilde{C} = (-1)^{\overline{\nu}} \frac{1}{i^{\ell} \pi^{\ell D/2}} \frac{\prod_{i=1}^{N} \Gamma(\nu_i)}{\Gamma(\overline{\nu} - \ell D/2)}.$$

Each choice of the constant has advantages. In this chapter we use C. Later we will use $\widetilde{C}$ and only deal with the projective integral as in equation (2.14). It is of course straightforeward to change from one convention to the other. For a discussion on how our definitions are related to the usual definition of a Feynman integral in the physics literature, we invite the reader to have a look at appendix B of [BW09]. Especially one will find a physical constant μ entering the definition of a Feynman integral. For us it is of no relevance and we have set it equal to one.

2.6.2 Graph Polynomials

The two homogenous polynomials entering in the Feynman parameter description can be read off the underlying Feynman graph. They do not depend on the orientation of the graph, so we can drop it in the following. Let us begin with some auxiliary definitions. A *tree* is a connected and simply connected graph, i.e. a connected graph without loops. A disjoint union of k trees is called a *k–forest*, such that a tree is a 1–forest. Now let Γ be a connected one–particle irreducible graph. A *spanning tree* of Γ is a connected subgraph of Γ which is a tree and which contains all vertices of Γ. The set of all spanning trees of Γ is denoted by $\mathcal{T}_\Gamma$. With any edge e_i, we have associated a mass m_i, and from now on we would like to associate in addition the Feynman parameter x_i to e_i.

With the terminology just introduced define the polynomial

$$\mathcal{U} = \sum_{T \in \mathcal{T}_\Gamma} \prod_{e_i \notin T} x_i. \tag{2.15}$$

Observe that this definition is independent of the orientation of the graph and its external edges, also it depends neither on masses nor on kinematical invariants. Furthermore, a spanning tree T has $h_0(T) = 0$, $h_1(T) = 1$ and contains all vertices of the graph, so every spanning tree has exactly $N - \ell$ edges. This means that $\mathcal{U}$ is homogeneous of degree ℓ. It is a classic result that this polynomial is exactly the polynomial $\mathcal{U}$ in equation (2.14). It is called the *first graph polynomial* of the underlying graph, its zero set in $\mathbb{P}^{N-1}$ is called the *first graph hypersurface* of the graph. The second polynomial entering in the Feynman parameter description is a little more complicated. A *spanning 2–forest* of a graph Γ is a subgraph G which is a 2–forest that contains all vertices of Γ. It can be obtained from a spanning tree by removing exactly one edge. Let $\mathcal{T}_2$ denote the set of all spanning 2–forests of a graph and let us write an element of $\mathcal{T}_2$ in the form $\{T_1, T_2\}$, where each T_i denotes one tree of the 2–forest. We define

$$\mathcal{F}_0 = \sum_{\{T_1, T_2\} \in \mathcal{T}_2} \left(\prod_{e_i \notin \{T_1, T_2\}} x_i \right) \left(\sum_{p_j \in P_{T_1}} \sum_{p_k \in P_{T_2}} p_j \cdot p_k \right), \tag{2.16}$$

where P_T is the set of external momenta attached to a tree T, and

$$\mathcal{F}(\Lambda) = \mathcal{F}_0 + \left(\sum_{i=1}^{N} m_i^2 x_i \right) \mathcal{U}. \tag{2.17}$$

It follows that $\mathcal{F}$ is homogeneous of degree $\ell + 1$. Again it is a classic result that this polynomial is the polynomial $\mathcal{F}$ in equation (2.14). We call $\mathcal{F}$ the *second graph polynomial* of the graph Γ. Let us at this point emphasize, that we do not call $\mathcal{F}_0$ the second graph polynomial, both are only equal, if all masses are zero. The polynomials $\mathcal{U}$ and $\mathcal{F}_0$ are frequently called first and second Symanzik polynomial in literature. The polynomial $\mathcal{F}_0$, however, is of no relevance to us except when it coincides with $\mathcal{F}$.
The second graph polynomial depends on additional (complex) parameters given by the vector Λ. For general (but fixed) values of these parameters we call the hypersurface in $\mathbb{P}^{N-1}$ defined by the vanishing of $\mathcal{F}$ the *second graph hypersurface* of the graph. We will ultimately be interested in a variation of this hypersurface as one of the parameters varies.
Let us take a closer look at the kinematical invariants. We have agreed to orient external edges inwards, such that overall momentum conservation gives

$$\sum_{i=1}^{E} p_i = 0.$$

Each spanning 2–tree $\{T_1, T_2\}$ gives a partition $P_{T_1} \dot\cup P_{T_2} = \{p_1, \ldots, p_E\}$ of the set of all external momenta, and we can write

$$\sum_{p_j \in P_{T_1}} \sum_{p_k \in P_{T_2}} p_j \cdot p_k = \left(\sum_{p_j \in P_{T_1}} p_j \right) \left(\sum_{p_k \in P_{T_2}} p_k \right) = - \left(\sum_{p_j \in P_{T_1}} p_j \right)^2 .$$

The expressions on the right hand side of this equation are called the kinematical invariants of the graph. These, together with the masses define the set of parameters Λ. We see that $\mathcal{F}$ has positive coefficients as a polynomial in the Feynman parameters, if the entries of Λ are fixed in the Euclidean region.
The graph polynomials have many remarkable properties. Obviously $\mathcal{U}$ and $\mathcal{F}_0$ are linear in each variable. The second graph polynomial is linear in the variable x_i if, and only if, $m_i = 0$. Otherwise it is quadratic in x_i. Furthermore, every coefficient of the polynomial $\mathcal{U}$ equals $+1$.
Both graph polynomials can be related to graph polynomials of subgraphs and minors of the graph. For a graph Γ let us define

$$\Phi_i = \Phi_{\Gamma/e_i},$$

and

$$\Phi^{(i)} = \Phi_{\Gamma \setminus e_i},$$

where Φ can be any of the polynomials $\mathcal{U}$, $\mathcal{F}_0$ or $\mathcal{F}$. For any such choice of Φ, the polynomials Φ_i and $\Phi^{(i)}$ are independent of the variable x_i. Immediately from the definitions we obtain

$$\mathcal{U} = \mathcal{U}_i + x_i \mathcal{U}^{(i)}, \tag{2.18}$$

$$\mathcal{F}_0 = (\mathcal{F}_0)_i + x_i \mathcal{F}_0^{(i)}. \tag{2.19}$$

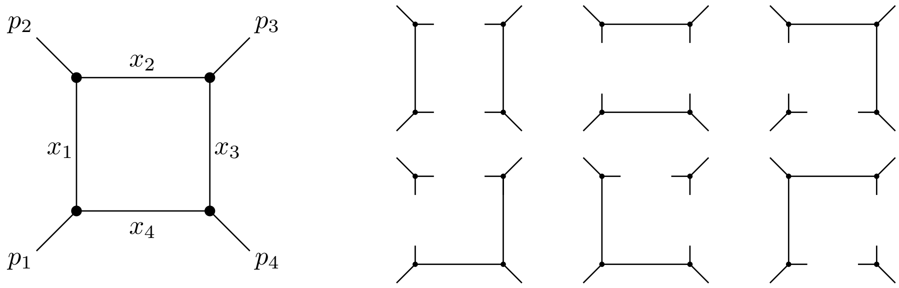

Figure 2.6: An example of a Feynman graph together with its spanning 2-forests.

This is called the deletion–contraction property and it is at the heart of many calculations involving Feynman integrals. For the second graph polynomial we obtain

$$\mathcal{F} = \mathcal{F}_i + x_i \, \mathcal{F}^{(i)} + m_i^2 x_i \, \mathcal{U}. \tag{2.20}$$

It follows, that

$$\Phi \mid_{x_i=0} = \Phi_i = \Phi_{\Gamma/e_i} \tag{2.21}$$

where Φ can be any of the polynomials $\mathcal{U}$, $\mathcal{F}_0$ or $\mathcal{F}$.
We will exploit these properties of the graph polynomials in chapter 3, where we investigate intersections of graph hypersurfaces with the coordinate divisor of the ambient projective space.

Example Let us consider the graph depicted in figure 2.6. It has four spanning trees and we immediately obtain

$$\mathcal{U}_\Gamma = x_1 + x_2 + x_3 + x_4.$$

The six spanning 2–trees of the graph are depicted in figure 2.6. Defining

$$t_1 = p_1^2, \; t_2 = p_2^2, \; t_3 = p_3^2, \; t_4 = p_4^2,$$

and

$$t_5 = (p_1 + p_2)^2 \,, \; t_6 = (p_2 + p_3)^2 \,,$$

we obtain

$$(\mathcal{F}_0)_\Gamma = -t_1 x_1 x_4 - t_2 x_1 x_2 - t_3 x_2 x_3 - t_4 x_3 x_4 - t_5 x_2 x_4 - t_6 x_1 x_3.$$

Let us remark, that in literature one may find other equivalent definitions of the graph polynomials. For a more detailed survey on graph polynomials one may consider [BW10].
Finally, in the remainder of this section, let us discuss the tadpole integrals that appear frequently as minors of graphs. The one–loop tadpole graph consists of one vertex and one self–looping edge that we assign a mass m. It has associated the integrals

$$T(D, m^2, \nu_1) = C_T \int \frac{d^D k}{(k^2 - m^2)^{\nu_1}}.$$

For $\nu_1 = 1$, we have

$$T(D, m^2, 1) = -\frac{1}{i\pi^{D/2}} \int \frac{d^D k}{(k^2 - m^2)} = \frac{1}{\pi^{D/2}} \int \frac{d^D k}{(K^2 + m^2)}$$
$$= \left(m^2\right)^{D/2-1} \Gamma(1 - D/2).$$

Here we have applied Wick rotation and evaluated the Euclidean integral using the beta function formula.
The two–loop tadpole graph consists of one vertex and two self–looping edges that we assign the masses m_1 and m_2, respectively. We evaluate the corresponding integrals using Feynman parameters and obtain for unit exponents

$$T_2(D, m_1^2, m_2^2, 1, 1) = \Gamma(2 - D) \int_0^\infty \frac{\mathcal{U}^{2-3D/2}}{\mathcal{F}^{2-D}} \Omega = \Gamma(2 - D) \int_0^\infty \frac{x_1^{-D/2}}{(x_1 m_1^2 + m_2^2)^{2-D}} dx_1$$
$$= \left(m_1^2\right)^{D/2-1} \left(m_2^2\right)^{D/2-1} \Gamma(1 - D/2)^2.$$

Here we have applied equation (2.6), which is a variant of the beta function formula. We see, that the double tadpole integral is a product of two tadpole integrals. If we repeat the same calculation with the constant $\widetilde{C}_T$, we find that this is no longer true. T_2 will differ from the product of two one–loop tadpole integrals by a gamma factor. When dealing with graphs that are one–vertex joins[6] of smaller graphs one might, therefore, want to use C_T. We are, however, most interested in the projective integral and do not make use of product structures.

2.7 Periods

Periods, as defined in [KZ01], are complex numbers, whose real and imaginary part are absolutely convergent integrals of rational forms with rational coefficients over domains in $\mathbb{R}^n$ given by polynomial equations and inequations. The set $\mathcal{P}$ of periods is countable and forms an algebra

$$\mathbb{Q} \subset \overline{\mathbb{Q}} \subset \mathcal{P} \subset \mathbb{C}.$$

It contains many interesting numbers, such as π, logarithms at rational arguments, and multi zeta values. Let us give a more formal definition of a period. Let X be a smooth variety over $\mathbb{Q}$ of dimension d, $D \subset X$ a normal crossing divisor, $\omega \in \Omega^d(x)$ an algebraic differential form on X of top degree, and $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$ a relative cycle. This data defines a complex number $\int_\gamma \omega$, the period of (X, D, ω, γ). Kontsevich defines in [Kon99]

Definition 2.7.1 *The space of effective periods $\mathcal{P}_+$ is defined as the vector space over $\mathbb{Q}$ generated by symbols $[(X, D, \omega, \gamma)]$ as above, and subject to the relations*

(i) Linearity: $[(X, D, \omega, \gamma)]$ is linear in ω and γ.

(ii) Substitution law: If $f : (X_1, D_1) \longrightarrow (X_2, D_2)$ is a morphism of pairs defined over $\mathbb{Q}$, $\gamma \in H_d(X_1(\mathbb{C}), D_1(\mathbb{C}); \mathbb{Q})$, and $\omega \in \Omega^d(X_2)$, then

$$[(X_1, D_1, f^*\omega, \gamma)] = [(X_2, D_2, \omega, f_*\gamma)].$$

[6] A one–vertex join of two graphs G_1 and G_2 is obtained by identifying one vertex of G_1 with one vertex of G_2.

(iii) Stokes formula: Let $\widetilde{D}$ denote the normalization of D, containing a normal crossing divisor $\widetilde{D}_1$ coming from double points of D. If $\beta \in \Omega^{d-1}(X)$ and $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$, then

$$[(X, D, d\beta, \gamma)] = [(\widetilde{X}, \widetilde{D}, \omega\mid_{\widetilde{D}_1}, \partial\gamma)],$$

where $\partial : H_d(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q}) \longrightarrow H_{d-1}(\widetilde{D}(\mathbb{C}), \widetilde{D}_1(\mathbb{C}); \mathbb{Q})$ is the boundary operator.

It is conjectured that the evaluation map $\mathcal{P}_+ \longrightarrow \mathbb{C}$ is injective.
Clearly, $2\pi i$ is a period. It is presently, however, not know whether π^{-1} is a period. One, therefore, defines the algebra $\mathcal{P}$ of periods to be $\mathcal{P} = \mathcal{P}_+[(2\pi i)^{-1}]$. Instead of considering forms of top degree one could also consider cohomology classes as in [HM12].

Example Let us consider the relative homology group

$$H_1(\mathbb{C}^*, \{1, r\}),$$

where r is some rational number $\neq 0, 1$. It is generated by a small circle around zero and a suitable path from one to r (e.g. an arc from one to r). Integrating against the differential form $\frac{dx}{x}$, we obtain the periods $2\pi i$ and $\log(r)$. We see that logarithms at rational arguments are periods.

Let us now discuss how periods appear in the context of Feynman integrals. The following Theorem (Theorem 2.8 in [BB03b]) allows for a lot of flexibility regarding the domain of integration, but is restrictive with respect to the integrand.

Theorem 2.7.2 (Belkale, Brosnan) *Let P be a smooth variety defined over $k \subset \mathbb{R} \cap \overline{\mathbb{Q}}$ and let $f \in \mathcal{O}(P)$ be a function. Let C be a compact preoriented semiarithmetic subset of $X_{f\geq 0}(\mathbb{R})$ defined over k. Then, if $\omega \in \Omega^n(P)$ is a differential form, the function*

$$I(s) = \int\limits_C f^s \omega$$

extends meromorphically to all of $\mathbb{C}$ with poles occurring only at negative integers. Moreover, for any $s_0 \in \mathbb{Z}$, the coefficients a_i in the Laurent expansion

$$I(s) = \sum_{i \geq n} a_i (s - s_0)^i$$

are periods.

Let us give some examples where this Theorem can be applied in our context. An interesting class of graphs are the primitive graphs of massless (four dimensional) φ^4 theory. Mathematicians became interested in these graphs, because of a systematic appearance of multi zeta values, that were discovered by Broadhurst and Kreimer in evaluating the Feynman integrals of these graphs [BK95, BK97]. They were further investigated in various papers (see e.g. [BEK06, Bro09a, Bro09b]).
A (connected) graph is called *primitive* (divergent), if it has $N = 2\ell$ edges and, furthermore, the number of edges of every proper subgraph is strictly greater than twice its number of loops. This last condition assures that every proper subgraph is convergent. If the powers of all exponets equal one and all external momenta are zero, the Feynman parametric representation reads

$$\int\limits_\sigma \frac{1}{\mathcal{U}_\Gamma^2} \Omega.$$

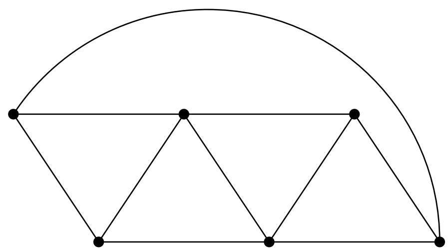

Figure 2.7: The zig–zag graph with five loops. All external momenta have been put to zero, therefore we do not draw external edges. Zig–zag graphs of higher loop order are obtained by adding more triangles.

Here, the Theorem can be applied with $s = -2$. Such an integral is convergent in dimension four (compare Proposion 5.2 in [BEK06]), hence its Laurent series at $D = 4$ has no polar part. Furthermore, the zero order term is just the integral itself. This period in this special situation is often a multi zeta value or a rational combination of multi zeta values.
Prominent examples are the graph of figure 2.3, which is known to evaluate to multi zeta values to all orders ([BW03]), the wheel and spokes graphs considered in [BEK06] and the zig–zag graphs. A zig–zag graph is depicted in figure 2.7. Recently an exact formula, previously conjectured by Broadhurst and Kreimer, has been proven for the zig–zag graphs by Brown and Schnetz [BS12]. Denoting by I_{Z_ℓ} the Feynman integral with unit exponents, zero masses and zero external momenta, associated with the zig–zag graph with ℓ loops, it reads

Theorem 2.7.3 (Brown, Schnetz) *The Feynman integral of the graph Z_ℓ at zero momenta and masses, in dimension four, is given by*

$$I_{Z_\ell} = 4\frac{(2\ell-2)!}{\ell!(\ell-1)!}\left(1 - \frac{1-(-1)^\ell}{2^{2\ell-3}}\right)\zeta(2\ell-3).$$

Another example, that we will discuss later are the banana–graphs. These consist of two vertices connected by N edges. For $N = 2, 3$ they are depicted in figure 5.3 in chapter 5. In dimension two, putting all exponents equal to one, we obtain

$$\int_\sigma \frac{1}{\mathcal{F}_\Gamma(\Lambda)}\Omega.$$

Again, these integrals are convergent, hence periods. We will be interested in a variation of these periods with respect to a kinematical invariant or a mass.

While multi zeta values are ubiquitous in the world of Feynman integrals, Belkale and Brosnan showed in [BB03a], that the graph hypersurfaces of graphs of massless four–dimensional ϕ^4 theory are general enough to generate the Grothendieck ring K_0. From that one expects to find Feynman integrals that do not evaluate to multi zeta values even in the rather simple massless four–dimensional ϕ^4 theory. Still, every number that will appear is a period. We have the following generalization of the above Theorem of Belkale and Brosnan ([BW09])

Theorem 2.7.4 (Bogner, Weinzierl) *Let $I(D, \Lambda, \nu)$ be a Feynman integral as in equation (2.9). In the case where* *(i) all kinematical invariants are negative or zero,*

(ii) all masses m_i are positive or zero,

(iii) all ratios of invariants and masses are rational,

the coefficients c_j of the Laurent expansion

$$I(D-2\varepsilon,\Lambda,\nu)=\sum_{j=-2\ell}^{\infty} c_j\,\varepsilon^j$$

are periods.

The first two conditions define the Euclidean region. This assures that the second graph polynomial has positive coefficients, which will be important later. In a stronger version, also proven in [BW09], even complex exponents are allowed. For us, however, the above is sufficient.

In this chapter, we have fixed all parameters and regarded a Feynman integral as a complex number. One can also vary some of the parameters. In this case one interprets a Feynman integral as a function in these parameters. It is then an interesting question which functions will appear. Following this viewpoint one discovers that (multiple) polylogarithms are omnipresent in the world of Feynman integrals. There are, however, also examples of graphs where multiple polylogarithms are not sufficient. The functions that appear when evaluating Feynman integrals as well as their differential equations will be the main focus of this dissertation.

2.8 Dimensional Recurrence Relations

In a physical theory that is not scalar, so–called *tensor integrals* may appear. These differ from scalar integrals by tensor structures that may appear in the numerator of the integrand. A dimensionally regularized tensor integral can be reduced to sums of scalar integrals.
In [Tar96], Tarasov defines an operator

$$T\left(\{q_i\},\left\{\frac{\partial}{\partial m_i^2}\right\},\mathbf{D}^+\right),$$

that depends on kinematical invariants, and that may contain mass derivatives as well as the operator $\mathbf{D}^+$, that is defined to increase the dimension by two. The operator T gives a reduction

$$I^{\text{tensor}}(D,\Lambda,\nu)=T\left(\{q_i\},\left\{\frac{\partial}{\partial m_i^2}\right\},\mathbf{D}^+\right)I^{\text{scalar}}(D,\Lambda,\nu),$$

of dimensionally regularized tensor integrals of arbitrary loop order to sums of scalar integrals. For one–loop integrals a reduction method had already been established in [PV79].
We will not make explicit use of such a reduction and refer to [Tar96, Tar97] for a precise definition of the operator T and detailed examples. We would, however, like to mention, that the operator T comes with two complications. Firstly, as the notation suggests, we apply mass derivatives to Feynman integrals. This may shift some of the exponents. Secondly, we shift the dimension by an integer. Therefore, the most complicated integrals we want to consider (and in principle need to consider) are scalar integrals with arbitrary integer exponents and dimension.

Sometimes it may be necessary to shift the dimension of a Feynman integral by an integer. This problem may arise when we cannot compute a Feynman integral directly and want to do manipulations in a dimension that is not the correct one (as given by the physical theory). It would then be important to shift back the dimension. Furthermore, integer shifts in dimension arise naturally in the reduction procedure we just described.
By [Tar96], for a scalar Feynman integral $I(D,\Lambda,\nu)$ as in equation (2.8), we have the relation

$$I(D,\Lambda,\nu) = \mathcal{U}_\Gamma(\nu_i\,\mathbf{i}^+)\,I(D+2,\Lambda,\nu). \tag{2.22}$$

Here $\mathcal{U}_\Gamma$ is the first graph polynomial of the graph Γ. The argument means that we have to replace the variable x_i by the operator $\nu_i\,\mathbf{i}^+$. The left–hand side of equation (2.22) contains a Feynman integral in dimension D, whereas the right–hand side contains sums of Feynman integrals in dimension $D+2$ (with increased powers of the propagators). It is, therefore, natural to call this relation a raising one.
For a similar relation lowering the dimension, let us introduce the auxiliary function

$$V\left(q_1,\ldots,q_r\right) = \begin{vmatrix} q_1^2 & \ldots & q_1\cdot q_r \\ \vdots & \ddots & \vdots \\ q_1\cdot q_r & \ldots & q_r^2 \end{vmatrix} \tag{2.23}$$

which assigns to a set of vectors $q_1,\ldots,q_r$ the above determinant. Now let

$$I(D,\Lambda,\nu) = \int d^Dk_1\ldots d^Dk_\ell \frac{1}{P_1^{\nu_1}\ldots P_N^{\nu_N}}$$

be a scalar Feynman integral depending on independent loop momenta $k_1,\ldots,k_\ell$ and external momenta $p_1,\ldots,p_{E-1}$. The lowering relation works, if the momenta form a complete basis in the sense that all products of an internal momentum with an internal or external momentum can be expressed uniquely in terms of the propagators P_j. According to the discussion in section 2.5, this is (for non–zero external momenta) the case, when $B\le N$. In this situation $V(k_1,\ldots,k_\ell,p_1,\ldots,p_{E-1})$ can be written as a polynomial Q in the propagators

$$V(k_1,\ldots,k_\ell,p_1,\ldots,p_{E-1}) = Q(P_1,\ldots,P_N)$$

of degree $\ell+E-1$. The relation lowering the dimension then reads ([Lee10])

$$I(D,\Lambda,\nu) = \frac{2^\ell\left[V(p_1,\ldots,p_{E-1})\right]^{-1}}{(D-E-\ell)_\ell}\;Q\left(\mathbf{1}^-,\ldots,\mathbf{N}^-\right)I(D-2,\Lambda,\nu), \tag{2.24}$$

where we have used the Pochhammer symbol

$$(\alpha)_n = \alpha\,(\alpha+1)\ldots(\alpha+n-1).$$

Let us observe that when expanding equation (2.24) in ε, on the right–hand side a pole in ε may appear. In dimensional regularization this poses no problems, in fact, it allows to connect convergent and divergent Feynman integrals. For graphs where the propagators do not form a complete basis in the above sense, we cannot apply the lowering relation. In this situation one may instead invert equation (2.22). An example will be given in chapter 5.

Finally, let us remark, that integrals with irreducible numerators can be reduced to scalar ones as described above. Depending on the situation it may, however, be convenient to keep the integrals with irreducible numerators instead of shifting the dimension via the operator T. Therefore, we will sometimes consider integrals with irreducible numerators as in equation (2.9), keeping in mind, that these can be reduced to scalar ones.

Example Let us reconsider the sunrise graph of figure (2.5) and put $m_1 = m_2 = 1$ and $m_3 = m$. Let us assume we are intersted in the special value at $p = 0$ of the associated integral where all exponents are equal to one. We have

$$I(D, m^2) = C_I \int d^D k_1 d^D k_2 \frac{1}{(k_1^2 - 1)(k_2^2 - 1)\left[(k_1 - k_2)^2 - m^2\right]}$$
$$= \int d^D K_1 d^D K_2 \frac{1}{\pi^D (K_1^2 + 1)(K_2^2 + 1)\left[(K_1 - K_2)^2 + m^2\right]}.$$

In dimension two this integral is convergent by power counting, hence a period by our previous discussion. Let us assume, that by some method we have obtained this value and we were interested in the Laurent expansion at $D = 4$. Even though the graph does not fulfill the condition $B \leq N$, the propagators form a complete basis, because we have put $p = 0$. We can apply the dimensional raising relation. Denoting $P_1 = K_1 + 1$, $P_2 = K_2 + 1$, and $P_3 = (K_1 - K_2) + m^2$, we have

$$V(K_1, K_2) = \begin{vmatrix} K_1^2 & K_1 \cdot K_2 \\ K_1 \cdot K_2 & K_2^2 \end{vmatrix} = (P_1 - 1)^2 (P_2 - 1)^2 - \frac{1}{4}\left(P_3 - P_1 - P_2 + 2 - m^2\right)^2.$$

We have to compute

$$\left[\left(\mathbf{1}^- - 1\right)^2 \left(\mathbf{2}^- - 1\right)^2 - \frac{1}{4}\left(\mathbf{3}^- - \mathbf{1}^- - \mathbf{2}^- + 2 - m^2\right)^2\right] I(D, m^2).$$

If one of the exponents of $I(D, m^2)$ drops to zero we have a double tadpole integral. We have computed these in a previous section. Putting everything together, we obtain

$$I(D, m^2) = \frac{m^2(4 - m^2)}{(D-3)(D-2)}\, I(D-2, m^2) + \frac{m^2 - 2 - 2m^{D-2}}{(D-3)(D-2)}\, \Gamma^2(2 - D/2).$$

Chapter 3

Picard–Fuchs Equations of Feynman Integrals in Integer Dimensions

The goal of this section is to interpret a Feynman integral $I = \int_\sigma \omega_\Gamma$ as a period of a (mixed) Hodge structure. If I is divergent in dimension D, this cannot be achieved directly. Our strategy is then to shift the dimension to a positive even integer $\widetilde{D}$, such that I converges in this dimension, evaluate the integral there and shift back to dimension D using Tarasov's generalized dimensional recurrence relations.
We have divided the singularities of I into two classes, the UV and the IR singularities[1]. We think of the UV singularities as those caused by the vanishing of $\mathcal{U}$ and the IR singularities as those caused by the vanishing of $\mathcal{F}$. In chapter 2 we have discussed a criterion for UV–finiteness and this will dictate how to choose the dimension $\widetilde{D}$. Before we can proceed we have to solve a related problem – in any dimension the polar locus of the Feynman form will intersect the domain of integration. It is clear by resolution of singularities in characteristic zero [Hir64], that the polar locus of ω_Γ and the domain of integration can be separated by blowups, except in the rare case when the intersection of the polar locus and the domain of integration contains a divisor in $\mathbb{P}^{N-1}$ (we will exclude such cases). The procedure of separating the polar locus and the domain of integration is commonly referred to as sector decomposition in physics literature. A sector decomposition algorithm for Feynman integrals of arbitrary graphs has been proposed by Binoth and Heinrich [BH00, BH04] and a termination problem of the algorithm has been solved by Bogner and Weinzierl [BW08]. In [BEK06], Bloch, Esnault and Kreimer construct an explicit sequence of blowups that can be read off the graph and that is valid whenever the second graph polynomial is not present in the denominator of the Feynman form. It is minimal in the sense that only the intersection of the domain of integration and the polar locus gets blown up, but no additional blowups on the exceptional divisors are required. As we have mentioned an algorithm for the blowup that terminates after finitely many steps is known, it is, however, not clear how to construct a minimal sequence of blowups and how to relate it to the graph and its subgraphs. Problems are caused by massless edges, if the second graph polynomial is contained in the denominator of the Feynman form. The massless edges may lead to more complicated combinatorics of

[1] Recall that we have agreed to ignore the gamma function which may occur in equation 2.13. If this term leads to a pole it is called an overall UV–divergence. Furthermore, we generally restrict to the Euclidean region, which affects the presence of IR–singularities.

the blowups and even to IR–singularities. The latter have to be avoided within our method. We regularize by adding small masses to massless edges and we will see, that the blowup constructed by Bloch, Esnault and Kreimer is sufficient, if we add enough masses.
Once this is achieved we briefly recall the notion of a Picard–Fuchs equation and how it can be computed in the projective setting. We will then see how a convergent Feynman integral that depends on additonal parameters such as masses and kinematical invariants leads to an inhomogeneous Picard–Fuchs equation, where the homogeneous part is determined by the second graph hypersurface of the graph itself and the inhomogeneous term can be related to the minors of the graph and their second graph hypersurfaces.

3.1 The Domain of Integration

The domain of integration σ of a Feynman integral depends solely on the number of edges of the corresponding graph. For a graph with N edges and any associated Feynman integral the domain of integration is the semi–algebraic set given by the equations and inequations

$$x_i \geq 0, \text{ for } 1 \leq i \leq N,$$
$$\sum_{i=1}^{N} x_i = 1$$

and can be described equivalently as the set

$$\left\{x = (x_1 : \cdots : x_N) \in \mathbb{P}^{N-1} \mid x_i \in \mathbb{R},\ x_i \geq 0\right\}$$

in projective space.
It is apparent that σ has a boundary that is contained in the coordinate divisor

$$B_0 := \left\{\prod_{i=1}^{N} x_i = 0\right\}.$$

Therefore, it represents a relative cycle with respect to B_0. Before we can pass to relative homology there is an additional problem, that has to be solved. The polar locus of the Feynman form ω intersects the domain of integration. Luckily, as long as we stay in the Euclidean region, this intersection is well behaved, as we are going to see.
Let us first observe that in the Euclidean region both graph polynomials have positive coefficients. In this case it is, therefore, clear that a Feynman form as in equation 2.14 cannot acquire poles at inner points of σ, but only on its boundary. Moreover, the polar locus along the boundary of σ can be located easily. It is given by a union of coordinate linear spaces, that can be read off the graph. Here, by linear coordinate space we mean a subspace of $\mathbb{P}^{N-1}$ given by the vanishing of a set of variables. To the i–th edge e_i of a graph Γ we have assigned the variable x_i. Identifying a graph with its set of edges and this set of edges with a subset S_Γ of $\{1, \ldots, N\}$, there is a 1–1 correspondance between the subgraphs of Γ and the linear coordinate spaces of $\mathbb{P}^{N-1}$, given by

$$\text{Subgraphs } G \subset \Gamma \longleftrightarrow \text{linear coordinate spaces } L \subset \mathbb{P}^{N-1}$$
$$G \mapsto L(G) = \left\{x = (x_1 : \cdots : x_N) \in \mathbb{P}^{N-1} \mid x_i = 0 \text{ for } i \in S_G\right\}$$
$$L = \left\{x \in \mathbb{P}^{N-1} \mid x_i = 0 \text{ for } i \in S\right\} \mapsto G(L) = \bigcup_{i \in S} e_i \subset \Gamma.$$

If a coordinate linear space corresponds to a graph with set of edges $S = \{i, j, k\}$ we will simply write L_{ijk} instead of $L(\{i, j, k\})$.

Lemma 3.1.1 *Let Γ be a graph and let X be either the zero set of its first graph polynomial $\mathcal{U}$ or the zero set of its second graph polynomial $\mathcal{F}$. In the latter case we assume that all masses and kinematical invariants have been fixed somewhere in the Euclidean region, such that $\mathcal{F}$ is a polynomial in $\mathbb{Q}[x_1, \ldots, x_N]$ with positive coefficients. In any case, X is a (possibly singular) hypersurface in $\mathbb{P}^{N-1}$. Then*

$$X(\mathbb{C}) \cap \sigma(\mathbb{R}) = \bigcup_{L \subset X} L(\mathbb{R}_{\geq 0}),$$

where the union is over all linear coordinate spaces $L \subset X$.

This is clear for any polynomial with positive coefficients. The next Lemma establishes the link with the subgraphs of Γ. First let us introduce some terminology.

Definition 3.1.2 *Let Γ be a graph. A subgraph G is called core, if removing any edge lowers $h_1(G)$. A core subgraph is called a cycle, if it has $h_1(G) = 1$. For a graph with at least two external edges, N edges and ℓ loops define new graphs by glueing two external edges without introducing a new vertex and dropping the other external edges. Each graph obtained in this way has $N + 1$ edges and $\ell + 1$ loops. These are called the closures of Γ. There is a unique closure when Γ has two external edges. In this case it is denoted $\overline{\Gamma}$. The graph obtained by contracting the new edge in $\overline{\Gamma}$ is called residue of Γ and is denoted $\Gamma^\bullet$. A subgraph (or minor) G of Γ naturally gives rise to a subgraph (or minor) in $\Gamma^\bullet$ which is denoted $G^\bullet$.*

Obviously, core equals 1PI. In the rest of the chapter we make the general assumption that Γ has two external edges.

Lemma 3.1.3 *(i) Let $X_{\mathcal{U}}$ be the zero set of the first graph polynomial $\mathcal{U}$ of Γ. A linear coordinate space L is contained in $X_{\mathcal{U}}$ if, and only if, its corresponding subgraph has at least one loop. The union of these spaces is stratified by the linear coordinate spaces that belong to core subgraphs of X.*

(ii) Fix all parameters in the Euclidean region, such that the unique kinematical invariant is negative and at least one mass is positive. Let $X_{\mathcal{F}}$ be the zero set of the second graph polynomial $\mathcal{F}$ of Γ. A linear coordinate space L is contained in $X_{\mathcal{F}}$ if, and only if, at least one of the following two conditions are met

(a) the subgraph that corresponds to L has at least one loop

(b) the subgraph that corresponds to L has at least one loop after passing to $\Gamma^\bullet$ and contains all massive edges of Γ.

Proof Let us begin with the case that X is the zero set of $\mathcal{U}$. In this case a proof can be found in [BEK06]. We sketch the proof. A coordinate linear space L_S is contained in X, if, and only if, every monomial of $\mathcal{U}$ contains at least one variable which belongs to S, which is the case if, and only if, no spanning tree of Γ contains the graph $G(L)$. If $G(L)$ has at least one loop the latter is the case. It is left to prove, that if $G(L)$ has no loop it is contained in a spanning tree of Γ. This is Lemma 3.2 in [BEK06]. Now the maximal linear coordinate spaces contained in X belong to cycles of Γ, their intersections correspond to unions of these. Hence we obtain a natural stratification on the union of linear coordinate spaces contained in X by the core subgraphs of Γ.

Let us now prove *(ii)*. First observe that a linear coordinate space L that is contained in $X_{\mathcal{U}}$ is automatically contained in $X_{\mathcal{F}}$. Let now L be a linear coordinate space that is not contained in $X_{\mathcal{U}}$. It is contained in $X_{\mathcal{F}}$ if, and only if, both $\mathcal{F}_0$ and $\sum_{i=1}^N m_i^2 x_i$ vanish on L. Therefore, $G(L)$ must contain all massive edges of Γ and furthermore $\mathcal{F}_0$ must vanish on L. Now observe that the intersection behavior of $\mathcal{F}_0$ does only depend on its monomials, but not on their coefficients. We, therefore, set all coefficients of $\mathcal{F}_0$ equal to one. With this normalization it is the first graph polynomial of the graph $\Gamma^\bullet$. This follows from the identity $\mathcal{U}_{\overline{\Gamma}} = x_{N+1}\mathcal{U}_\Gamma + \mathcal{U}_{\Gamma^\bullet}$ and the definition of $\mathcal{F}_0$. Now the statement follows from *(i)*. □

To describe the procedure of blowing up these linear spaces let us begin with an important special case. The following is Proposition 7.3 in [BEK06]. We have slightly modified it by adding an argument made in [BEK06] in the proof of the Proposition (see also the discussion in [BK08], chapter 3 and 5). In case $D = 4$ a graph without external edges is called *primitive*, if it has $N = 2\ell$ edges and every proper subgraph G has $N_G > 2\ell_G$.

Proposition 3.1.4 (Bloch, Esnault, Kreimer) *Let $D = 4$ and Γ be a primitive graph. The integrand of the Feynman integral of this graph reads*

$$\omega = \frac{\Omega}{\mathcal{U}^2}.$$

There exists a tower

$$P = P_r \xrightarrow{\pi_{r,r-1}} P_{r-1} \xrightarrow{\pi_{r-1,r-2}} \cdots \xrightarrow{\pi_{2,1}} P_1 \xrightarrow{\pi_{1,0}} \mathbb{P}^{N-1}$$
$$\pi = \pi_{1,0} \circ \cdots \circ \pi_{r,r-1}$$

where P_i is obtained from P_{i-1} by blowing up disjoint unions of strict transforms of coordinate linear spaces $L_i \subset X_\Gamma$ and such that

(i) *$\pi^*\omega$ has no poles along the exceptional divisors associated to the blowups.*

(ii) *let $B \subset P$ be the total transform in P of $B_0 \subset \mathbb{P}^N$. Then B is a normal crossings divisor in P. No face of B is contained in the strict transform Y of X in P.*

(iii) *the strict transform of σ in P does not meet Y.*

Such a tower is explicitly given by collecting all linear coordinate spaces that correspond to cycles of Γ

$$\mathcal{C} = \{L \subset X \mid G(L) \text{ is a cycle}\},$$

and forming the set

$$\mathscr{F} = \{L \subset X \mid L = \bigcap L^{(i)},\ L^{(i)} \in \mathcal{C}\}.$$

The blowup is then given step by step by blowing up at once all linear spaces in $\mathscr{F}$ with the same dimension in increasing order, beginning with these of minimal dimension. In this way in every step disjoint unions of linear coordinate spaces (resp. their strict transforms) are blown up.

Remark 3.1.5 *(i) A full proof of this Proposition can be found in [BEK06]. The fact that it is sufficient to blow up core subgraphs is explained in detail in chapters 3 and 5 of [BK08]. The set of linear coordinate spaces is semi–ordered by inclusion. We have chosen to blow up by dimension in every step which bounds the length of the tower by $N - 2 = 2\ell - 2$. Instead, like in [BEK06], one can choose to blow up flags of core subgraphs of Γ, which possibly shortens the length of the tower. It is then bounded by $\ell - 1$. Of course, the same linear coordinate spaces get blown up, but in the latter case all disjoint coordinate spaces (regardless of dimension) are blown up at the same time.*

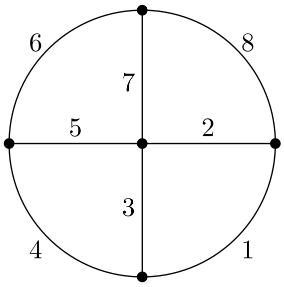

Figure 3.1: The wheel and spokes graph with four loops.

(ii) The form $\pi^*\omega$ vanishes on many of the exceptional divisors. If L_S is blown up the zero order of $\pi^*\omega$ on the exceptional divisor of L_S is $|S|-1$, the pole order on the other hand is clearly even. It follows from the Proposition that whenever $|S|$ is even the form $\pi^*\omega$ vanishes along L_S. In general, we expect the vanishing of $\pi^*\omega$ on many of the exceptional divisors.

Example Consider the wheel and spokes graph with four loops depicted in fig. 3.1. It has the following linear coordinate spaces associated to cycles

$$\begin{aligned}\mathcal{C} = \{&L_{123},\ L_{345},\ L_{567},\ L_{782},\\ &L_{1468},\ L_{1378},\ L_{1452},\ L_{3467},\ L_{5682},\\ &L_{14578},\ L_{12764},\ L_{23468},\ L_{13568}\}.\end{aligned}$$

The set $\mathscr{F}$ which contains all possible intersections of these linear spaces has 41 elements. We see, however, that $\pi^*\omega$ vanishes on all exceptional divisors of linear spaces that correspond to core subgraphs, except

$$L_{123},\ L_{345},\ L_{567},\ L_{782}.$$

One easily verifies that $\pi^*\omega$ vanishes on all exceptional divisors of the blowup constructed in Proposition 3.1.4, except the ones that correspond to linear spaces contained in

$$\begin{aligned}\mathscr{L} = \{&L_{123},\ L_{345},\ L_{567},\ L_{782},\ L_{12345},\ L_{12378},\ L_{34567},\ L_{25678},\\ &L_{1234567},\ L_{1234568},\ L_{1234578}, L_{1234678}, L_{1235678}, L_{1245678}, L_{1345678}, L_{2345678}\}.\end{aligned}$$

These are only 16 of the 41 exceptional divisors.

In light of Lemma 3.1.1 and 3.1.3 one expects that one has to at least blow up as in Proposition 3.1.4. One of the achievements of Proposition 3.1.4 is, that this blowup is indeed sufficient and no further blowups are required. This is far from general, simply blowing up the intersection as given by the above Lemmas is, in general, not sufficient to separate the strict transform of a divisor from the real non–negative points, even if the polynomial is at most quadratic in each variable.

Example Consider the polynomial $f = x_1x_2 + x_3^2$. It is homogeneous and at most quadratic in each variable. Let us denote its zero set in $\mathbb{P}^2$ by X and the real non–negative points in $\mathbb{P}^2$ as usual by σ. The intersection of X and σ are the two points L_{13}, L_{23}. We blow up the point L_{13}, which in the affine open $x_2 = 1$ is the origin. The blowup is given by the equation

$$x_1z_3 = x_3z_1,$$

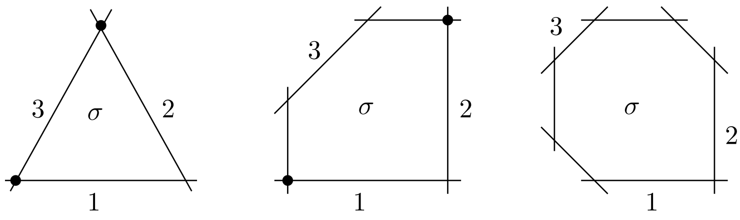

Figure 3.2: *left*: σ intersects X in the two marked points, the i–th line represents $x_i = 0$, *middle*: σ after blowing up the intersection points with X, σ still intersects X in two points, *right*: polar locus and domain of integration are separated after an additional blowup of two points.

where the exceptional divisor is a $\mathbb{P}^1$ in the variables z_1 and z_3. In the affine open $z_3 = 1$ we have

$$f = x_3 z_1 + x_3^2 = x_3(z_1 + x_3). \tag{3.1}$$

We see that the strict transform of f meets the strict transform of σ in the point $(z_1 : z_3) = (0 : 1)$ on the exceptional divisor of the blowup of $(x_1 : x_2 : x_3) = (0 : 1 : 0)$. To separate X and σ we, therefore, have to blow up this point. In the other affine open $z_1 = 1$ we obtain

$$f = x_1 + (x_1 z_3)^2 = x_1(1 + x_1 z_3^2), \tag{3.2}$$

and we see that the strict transform does not intersect σ. Blowing up the other point L_{23} leads to the same result, the strict transform intersects σ in one point on the exceptional divisor. The procedure is depicted in fig. 3.2. The point is that the strict transform of X which can be read off equations 3.1 and 3.2, did not contain all variables (the variable z_3 is missing) and, therefore, could vanish on the boundary of σ (even though it is linear).

It is all the more remarkable that in the situation of Feynman integrals it is in many interesting special cases indeed sufficient to blow up the intersection of the polar locus and the domain of integration as in Proposition 3.1.4. If the second graph polynomial is not present in the denominator this is the case. If the second graph polynomial is present in the denominator of the Feynman form the blowup may need to be modified. Cases with no masses are of no interest to us, the massless integrals depend on a single scale – the momentum – and we observe that it factors out of the integrand. Let us mention for completeness that, if there are no masses, one can proceed as in Proposition 3.1.4 with Γ replaced by $\Gamma^\bullet$. However, if $\Gamma^\bullet$ has a tadpole the second graph polynomial contains a coordinate divisor and if this term is not canceled by the numerator the domain of integration and the polar locus cannot be separated by blowups.
Let us reconsider Proposition 3.1.4. Here primitive graphs in dimension four are considered. The gamma function contributes an overall UV–divergence which can be handled separately and is ignored here. Furthermore, ω does not contain $\mathcal{F}$, so IR–singularities are automatically avoided. Properties *(ii)* and *(iii)* of the blowup constructed there can always be achieved by resolution of singularities over a field with characteristic zero. Property *(i)* shows that there is no UV–divergence for this class of graphs in dimension four. What is remarkable is the fact that the blowup can be related to the graph and its core subgraphs.
We do not restrict to primitive graphs, instead our strategy will be the following. Given any

Feynman integral associated to a graph and a dimension D. If the integral $I(D)$ is not UV–convergent, change the dimension by an integer, such that the shifted dimension $\widetilde{D}$ is still positive and even and in addition, $I(\widetilde{D})$ converges[2]. The denominator of the latter integrand can either be a power of $\mathcal{U}$ or a power of $\mathcal{F}$ or a product of both. In any case we have to check convergence by counting zero order and pole order of ω along the exceptional divisors of the blowup constructed above.

Lemma 3.1.6 *Let Γ be a graph and G a subgraph with connected components $G = G_1 \cup \cdots \cup G_k$. Then*

$$\mathcal{U}_\Gamma = \mathcal{U}_{\Gamma/G} \prod_{i=1}^{k} \mathcal{U}_{G_i} + R,$$

where R is a polynomial of degree at least $h_1(G)+1$ in the variables belonging to the edges of G.

The above Lemma is well–known and the proof is not repeated here. The following is well–known in physics literature.

Proposition 3.1.7 *Let Γ be a graph without tadpoles and $\widetilde{D}$ a dimension, such that $N_\Gamma \geq \ell\frac{\widetilde{D}}{2}$ and every proper subgraph G satisfies $N_G > \ell_G \frac{\widetilde{D}}{2}$. Then the associated scalar Feynman integral is UV–finite. Furthermore, it can be made finite by adding masses to massless lines.*

Proof Let us begin with the case $N_\Gamma = \ell\frac{\widetilde{D}}{2}$. We obtain

$$\omega_\Gamma = \frac{1}{\mathcal{U}^{\frac{\widetilde{D}}{2}}}\Omega.$$

Now the blowup of Proposition 3.1.4 is sufficient and we count pole order and zero order along the exceptional divisors. It is enough to consider the blowup of a single coordinate linear space $L(G)$ associated to a core subgraph G. According to Lemma 3.1.6 ω has pole order $\ell_G\frac{\widetilde{D}}{2}$ along the exceptional divisor. The zero order along the exceptional divisor clearly is $N_G - 1$. By assumption we have $N_G - 1 \geq \ell_G\frac{\widetilde{D}}{2}$. We do not have to add masses in this case. Next assume $N_\Gamma > \ell\frac{\widetilde{D}}{2}$. Let L be a coordinate linear space and assume $\mathcal{U}$ vanishes on L. Then clearly also $\mathcal{F}$ vanishes on L. If we add masses to all massless lines $\mathcal{U}$ and $\mathcal{F}$ vanish on the exact same linear spaces and to the same order. Again the blowup as above is sufficient. We write

$$\omega = \frac{\mathcal{U}^{N-(\ell+1)\widetilde{D}/2}}{\mathcal{F}^{N-\ell\widetilde{D}/2}}\Omega = \left(\frac{\mathcal{U}}{\mathcal{F}}\right)^{N-\ell\widetilde{D}/2} \frac{1}{\mathcal{U}^{\frac{\widetilde{D}}{2}}}\Omega$$

and conclude as above. □

Remark 3.1.8 *In the previous Proposition $\widetilde{D} = 2$ can be chosen for any graph without tadpole. If a graph has a tadpole the Feynman integral is a product and it is enough to consider only the graph which arises when removing all tadpoles. We have already discussed the tadpole integrals in chapter 2.*

In this chapter we have restricted to the Euclidean region to ensure that the second graph polynomial has positive coefficients. This bound on the kinematical invariants of the graph can

[2] As we mentioned earlier such a dimension might not exist, if the graph has many massless lines. In this case we can at least put small masses on some of the lines to obtain a convergent integral. In a physical application we would then be interested in the limit, as these masses go to zero. This limit would be taken as the last step.

sometimes be improved. For an example see Lemma 5.2.1.
We have constructed an explicit blowup $\pi : P \longrightarrow \mathbb{P}^{N-1}$ such, that the strict transform Y of the polar locus X of ω does not intersect the strict transform of σ (which we denote again by σ). If $\pi^*\omega$ does not acquire poles along the exceptional divisors of the blowup, the Feynman integral is a period. This is controlled by Proposition 3.1.7. In this situation we interpret σ as a cycle in

$$H_{N-1}(P\backslash Y, B\backslash Y)$$

and ω as a form in

$$H^{N-1}(P\backslash Y, B\backslash Y).$$

3.2 Picard–Fuchs Equations

The rest of this chapter is devoted to Picard–Fuchs equations and how to compute them in the projective setting. Our standard reference is [Del70]. Further useful literature on the topic of Picard–Fuchs equations includes [Hae87, Mal87, BP02, PS03, PS08]. In the section on the Griffiths–Dwork reduction procedure we follow [Gri69, CK00].
Let us recall some basics about differential operators and differential modules.
Let R be a commutative ring containing $\mathbb{Q}$. A *derivation* on R is an additive map $\partial : R \longrightarrow R$, that satisfies the Leibniz rule

$$\partial\,(ab) = \partial(a)b + a\partial(b),$$

for all $a, b \in R$. A ring equipped with a derivation is called a *differential ring*, a field equipped with a derivation is called a *differential field*. Usually we will denote the derivation of a differential ring by $a \longrightarrow a'$ and the k–fold application of the derivation by $(\,\cdot\,)^{(k)}$.

Definition 3.2.1 *Let R be a differential ring with derivation $(\,\cdot\,)'$, $R\{\partial\}$ be the free R–algebra generated by ∂, and $J \subset R\{\partial\}$ be the ideal generated by the elements $\{a\partial - \partial a - a' \mid a \in R\}$. We call*

$$R[\partial] = R\{\partial\}/J$$

the ring of differential operators over R.

An element $L \in R[\partial]$ is of the form $L = \sum_{i=0}^{n} a_i \partial^i$. The largest natural number, such that a_k is non–zero is called the *degree* of L.
Differential operators are closely related to differential modules.

Definition 3.2.2 *A differential module (M, ∂) is a finite–dimensional k–vector space together with an additive map $\partial : M \longrightarrow M$ that satisfies*

$$\partial(fm) = f'm + f\partial m$$

for all $f \in k$ and $m \in M$.

Instead of differential module we will usually write D–module. For a differential operator $L \in k[\partial]$, the quotient $M := k[\partial]/k[\partial]L$ is a D–module. Conversely, we can assign a differential operator to a given D–module. An element $e \in M$ is called a *cyclic vector*, if the elements $e, \partial e, \partial^2 e, \ldots$ generate M as a k–vector space. The proof of the following can be found in many places in the literature (see e.g. Proposition 2.9 in [PS03] and the references given there)

Proposition 3.2.3 *Every D–module M has a cyclic vector. In particular, there is a differential operator $L \in k[\partial]$, such that $M \cong k[\partial]/k[\partial]L$.*

The next Proposition shows when two D–modules are isomorphic. Let us denote the greatest common right divisor of two elements $L_1, L_2 \in k[\partial]$ by $\mathrm{GCRD}(L_1, L_2)$. We have

Proposition 3.2.4 *For $L_1, L_2 \in k[\partial]$, the D–modules $k[\partial]/k[\partial]L_1$ and $k[\partial]/k[\partial]L_2$ are isomorphic, if, and only if, L_1 and L_2 have the same degree and there exist elements $P, Q \in k[\partial]$ of smaller degree, such that $L_1 P = Q L_2$ and $\mathrm{GCRD}(P, L_2) = 1$.*

One of the key features of Picard–Fuchs differential operators is that the singularities of Picard–Fuchs operators are very mild. Let us introduce the notion of a Fuchsian differential operator.

Definition 3.2.5 *A point $s \in \mathbb{C}$ is called a regular singular point of the linear differential operator $L = \sum_{i=0}^{n} a_i \partial^i$, if the pole order of $a_{n-j}(s)/a_n(s)$ is at most j for $j = 1, \ldots, n$. A point $s \in \mathbb{C}$ is called regular, if none of the coefficients $a_j/a_n(s)$ has a pole in s. The point at infinity is defined to be regular or regular singular if the point 0 is a regular or a regular singular point, respectively, after applying the transformation $z \mapsto 1/z$. L is called Fuchsian if all points are regular or regular singular points of L.*

There are many equivalent formulations of a Fuchsian differential operator. According to a Theorem of Fuchs, the condition on the pole order of the coefficients of a Fuchsian differential operator L is equivalent to a growth condition on the solutions u of the differential equation $Lu = 0$. For a discussion of the Theorem and other equivalent reformulations of the property Fuchsian, see e.g. [Hae87, PS03].

By Lemma 6.11 in [PS03] we have a useful description of the coefficients of a Fuchsian differential operator.

Lemma 3.2.6 *Let $L = \sum_{i=0}^{n} a_i \partial^i$ with $\partial = \frac{d}{dz}$, $a_j \in \mathbb{C}(z)$, $a_n = 1$ and such that the only poles of the coefficients a_j are in a finite set $S = \{s_1, \ldots, s_{m-1}, \infty\} \subset \mathbb{P}^1$. L is Fuchsian with singular locus in S if, and only if, the a_j have the form*

$$a_{n-j} = \frac{b_j}{(z - s_1)^j \ldots (z - s_{m-1})^j}$$

with b_j polynomials of degrees less or equal to $j(m-1) - j$.

For differential modules the property of being Fuchsian usually is defined in a more complicated way. For our purposes it is convenient to use Proposition 3.16 in [PS03] as a definition.

Definition 3.2.7 *Let M be a differential $\mathbb{C}(z)$–module with cyclic vector e and let L be the minimal monic polynomial with $Le = 0$. M is called Fuchsian if L is Fuchsian.*

Let us now turn to Picard–Fuchs differential operators. We consider the following geometric situation. Suppose that we have a one–parameter family of smooth projective hypersurfaces given by a smooth projective morphism $\pi : \mathcal{X} \longrightarrow S$, where S is $\mathbb{P}^1$ minus finitely many points. The middle cohomology $H^k(X_t; \mathbb{C})$ of a fibre X_t carries a pure Hodge structure. Moreover, the $\mathbb{C}$–vector spaces $H^k(X_t; \mathbb{C})$ fit together to form a *local system* (which we define to be a locally constant sheaf of $\mathbb{C}$–vector spaces) on S. To be more precise, the sheaf $\mathbb{V} = R^k \pi_* \mathbb{C}$ is a local system on S which satisfies $\mathbb{V}_t = H^k(X_t; \mathbb{C})$. Local systems on S are closely related to holomorphic connections on S.

We define

Definition 3.2.8 *A holomorphic connection $(\mathcal{V}, \nabla)$ on S is a holomorphic vector bundle $\mathcal{V}$ equipped with an additive $\mathbb{C}_S$–linear map*

$$\nabla : \mathcal{V} \longrightarrow \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{V}$$

satisfying the Leibniz rule

$$\nabla(f \otimes v) = df \otimes v + f\nabla(v)$$

for f a local section of $\mathcal{O}_S$ and v a local section of $\mathcal{V}$. A local section v of $\mathcal{V}$ is called flat (or horizontal) if $\nabla(v) = 0$. The system of flat sections is denoted $\mathcal{V}^\nabla = \mathrm{Ker}(\nabla)$.

A connection extends for all $p \geq 1$ to a unique map

$$\nabla : \Omega^p_S \otimes \mathcal{V} \longrightarrow \Omega^{p+1}_S \otimes \mathcal{V}$$

satisfying

$$\nabla(f \otimes v) = df \otimes v + (-1)^p f \wedge \nabla(v).$$

The *curvature* of ∇ is the $\mathcal{O}_S$–linear map

$$\nabla^2 : \mathcal{V} \longrightarrow \Omega^2_S \otimes \mathcal{V}.$$

A connection is called *flat* (or *integrable*) if its curvature vanishes. By [Del70] we have

Theorem 3.2.9 *The functor*

$$(\mathcal{V}, \nabla) \mapsto \mathcal{V}^\nabla$$

is an equivalence between flat connections on S and local systems on S.

A quasi–inverse is given by

$$\mathbb{V} \mapsto (\mathbb{V} \otimes \mathcal{O}_S, 1 \otimes d).$$

Returning to our geometric situation, it follows that there is a unique flat connection $(R^k\pi_*\mathbb{C} \otimes \mathcal{O}_S, \nabla_{\mathrm{GM}})$ on S which has $R^k\pi_*\mathbb{C}$ as its local system of flat sections. It is called *Gauß–Manin connection.*

Locally, choosing an open set $U \subset S$ and a coordinate t on U we have the covariant derivative in the direction $\frac{d}{dt}$

$$\nabla_{\frac{d}{dt}} : \mathcal{V}(U) \longrightarrow \mathcal{V}(U),$$

$$\nabla_{\frac{d}{dt}} = \left(\frac{d}{dt} \otimes \mathrm{id}\right) \circ \nabla_{\mathrm{GM}}$$

which gives $\mathcal{V}(U)$ the structure of a differential $\mathcal{O}_S(U)$–module.

Let us now write $S = \mathbb{P}^1 \backslash D$, where D is a finite set of points. We study connections which may acquire poles of order one on D. The sheaf of one–forms on $\mathbb{P}^1$ which admit logarithmic poles at the points of D is denoted $\Omega^1_{\mathbb{P}^1}(\log D)$.

Definition 3.2.10 *A regular singular connection $(\mathcal{V}, \nabla)$ on $\mathbb{P}^1$ with singular locus in D is a pair consisting of a holomorphic vector bundle $\mathcal{V}$ on $\mathbb{P}^1$ together with a morphism*

$$\nabla : \mathcal{V} \longrightarrow \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathcal{V}$$

of sheaves of groups which satisfies for every open set U, $f \in \mathcal{O}_{\mathbb{P}^1}(U)$ and $v \in \mathcal{V}(U)$ the Leibniz rule

$$\nabla(f \otimes v) = df \otimes v + f\nabla(v).$$

Holomorphic vector bundles on $\mathbb{P}^1$ (with the usual analytic topology) can be compared to algebraic vector bundles on $\mathbb{P}^1$ (with the Zariski topology) using the GAGA principle ([Ser56]). A regular singular connection on $\mathbb{P}^1$ (Zariski topology) is defined as above, replacing holomorphic vector bundles by algebraic vector bundles. Then the GAGA principle assures that the categories of regular singular connections in the algebraic setting and in the analytic setting are equivalent. In the algebraic setting we have a natural link to (i.e. a functor to the category of) Fuchsian differential $\mathbb{C}(t)$–modules. This can be seen as follows. Let (M, ∇) be a regular singular connection on $\mathbb{P}^1$ with singular locus in D. The fibre M_η of M at the generic point η is a finite–dimensional $\mathbb{C}(t)$–vector space and $\Omega_{\mathbb{P}^1}(\log D)_\eta$ can be identified with the universal differential $\mathbb{C}(t)dt$ on $\mathbb{C}(t)$ over $\mathbb{C}$. The induced map $\nabla_\eta : M_\eta \longrightarrow \mathbb{C}(t)dt \otimes M_\eta$ determines a regular singular connection (M_η, ∇_η) which can be identified with a Fuchsian differential $\mathbb{C}(t)$–module (M_η, ∂) by setting $\partial = (\nabla_\eta)_{\frac{d}{dt}}$. For more details on the functor form the algebraic to the analytic setting and the second functor we just described see chapter 6 of [PS03].
Returning to the geometric situation, the Gauß–Manin connection $(\mathcal{V}, \nabla)$ can be extended to a regular singular connection $(\overline{\mathcal{V}}, \overline{\nabla})$ on $\mathbb{P}^1$ with singular locus in D (this follows from Theorem II.7.9 in [Del70]). By our previous discussion this gives a Fuchsian differential $\mathbb{C}(t)$–module (M_η, ∂). Choosing a cyclic vector we obtain a Fuchsian differential operator. Such an operator will be called *Picard–Fuchs operator*.
Classically, a Picard–Fuchs operator arises as follows. The middle cohomoloy groups $H^k(X_t)$ we have considered in this section underlie Poincaré duality. The Poincaré pairing can be extended to $\overline{\mathcal{V}}$ on all of $\mathbb{P}^1$ such that

$$\frac{d}{dt}\langle v, w\rangle = \left\langle \overline{\nabla}_{\frac{d}{dt}} v, w\right\rangle + \left\langle v, \overline{\nabla}_{\frac{d}{dt}} w\right\rangle$$

for a local coordinate t and sections v, w of $\overline{\mathcal{V}}$. In particular for a flat section v and an operator L which annihilates w we find that $\langle v, w\rangle$ is a solution of L. In this situation the pairing $\langle \cdot, \cdot\rangle$ is given by integrating a k–form w against a locally constant k–cycle. In conclusion, the periods of $H^k(X_t)$ are the solutions of the Picard–Fuchs operator L.
What we have discussed so far is the prototypical example of a *variation of Hodge structure*. In [SZ85] (see also [BZ97]), *variations of mixed Hodge structure* are defined. These covers families with open or singular fibres.
Let us make the transition to Feynman integrals. We fix a dimension D and consider a convergent Feynman integral (in dimension D), where we fix all parameters but one, which we consider variable. Let us assume, furthermore, that the second graph polynomial is present in the denominator of the Feynman form. Then the Feynman integral gives rise to a variation of (mixed) Hodge structure. From the Feynman form we obtain a (homogeneous) Picard–Fuchs operator. The boundary of the domain of integration then contributes an inhomogeneous term. Due to the deletion–contraction properties of the graph polynomials the inhomogeneous term is a sum of simpler Feynman integrals corresponding to minors of the graph.
In chapter 4 we will investigate the inhomogeneous Picard–Fuchs equation in dimensional regularization and in chapter 5 we will compute examples of the situation described presently in a suitably chosen fixed dimension D.

3.3 Griffiths–Dwork Reduction

In this section we discuss a method for calculating the Picard–Fuchs equation due to Griffiths and Dwork. Let $V \subset \mathbb{P}^{N-1}$ be a smooth projective hypersurface of degree d. By Griffiths [Gri69], elements of $H^{N-1}(\mathbb{P}^{N-1}\backslash V)$ can be represented by forms

$$\frac{P}{f^k}\,\Omega,$$

where f is the homogeneous polynomial defining V and P is homogeneous of degree $kd-N$. To analyze the middle cohomology of the complement $\mathbb{P}^{N-1}\backslash V$, we consider the residue map

$$\mathrm{Res}: H^{N-1}(\mathbb{P}^{N-1}\backslash V) \longrightarrow H^{N-2}(V).$$

To define it we need the tube–over–cycle map. For any topological $(N-2)$–cycle γ in V, let $T(\gamma)$ be the tube over γ, which is a circle bundle over γ contained in $\mathbb{P}^{N-1}\backslash V$. The residue is defined by

$$\int_\gamma \mathrm{Res}\,\frac{P}{f^k}\Omega = \int_{T(\gamma)} \frac{P}{f^k}\Omega.$$

Clearly, Res is well–defined on cohomology classes. Let now H be the hyperplane class. The primitive cohomology can be defined as

$$H^n_{\mathrm{prim}}(V) = \{\eta \in H^n(V) \mid \eta\cdot H = 0\}.$$

We have $dH \sim V$ and, therefore, $\mathrm{Res}(P\Omega/f^k)\cdot H = 0$. It follows, that the image of Res is contained in the primitive cohomology of V. Furthermore, the map $\mathrm{Res}: H^{N-1}(\mathbb{P}^{N-1}\backslash V) \longrightarrow H^n_{\mathrm{prim}}(V)$ is surjective. By Griffiths [Gri69], $P\Omega/f^k$ lies in $F^{N-k-1}H^{N-2}_{\mathrm{prim}}(V)$. For a family induced by $f(t)$, where f now depends also on the parameter t, we therefore find

$$\nabla_{\frac{d}{dt}}\left(\frac{P\Omega}{f^k}\right) = \frac{(-kPf' + fP')\Omega}{f^{k+1}} \in \mathcal{F}^{N-k-2},$$

where the prime denotes differentiation with respect to t. This shows Griffiths transversality for smooth projective hypersurfaces.

To find the Picard–Fuchs equation we have to calculate modulo exact forms. An effective way to do this was developed in [Gri69]. It is called Griffiths–Dwork reduction. Let A_j be homogeneous polynomials of degree $(k-1)d-(N-1)$. A projective $(N-2)$–form on $\mathbb{P}^{N-1}\backslash V$ can be written

$$\eta = \frac{1}{f^{k-1}}\sum_{j_1<j_2}\left[x_{j_1}A_{j_2} - x_{j_2}A_{j_1}\right] dx_1\wedge ...\wedge \widehat{dx_{j_1}}\wedge ...\wedge \widehat{dx_{j_2}}\wedge ...\wedge dx_N.$$

We have

$$d\eta = \frac{[(k-1)\sum_j A_j\partial_j f]\Omega}{f^k} - \frac{(\sum_j \partial_j A_j)\Omega}{f^{k-1}}. \tag{3.3}$$

It follows, that any form $P\Omega/f^k$ where P lies in the Jacobian ideal J of f can be reduced modulo an exact form to a form with a lower pole order. Conversely, if $P\Omega/f^k$ reduces modulo exact forms to an element of F^{N-k-2}, then P lies in the Jacobian ideal of f. Therefore, the map $P \longmapsto P\Omega/f^k$ defines an isomorphism

$$(R/J)_{kd-N} \cong H^{N-k-1,k-1}_{\mathrm{prim}}(V),$$

where $R = \mathbb{C}[x_1, \ldots, x_N]$ and the subscript denotes graded pieces. To illustrate the method we now assume that V is a Calabi–Yau hypersurface in $\mathbb{P}^{N-1}$, such that we have a unique holomorphic $(N-2)$–form $\omega = \mathrm{Res}(\Omega/f)$ on V. We assume that V is defined by a single equation f depending on a parameter $t \in S$, where S is one–dimensional. Then ω is a holomorphic $(N-2)$–form on the family of hypersurfaces $\mathcal{V}$ defined by f as t varies. We assume that f is a polynomial in t. To find the Picard–Fuchs equation of ω the Griffiths–Dwork method proceeds as follows:

- Choose a basis of the primitive cohomology of $\mathcal{V}$ represented by a collection of forms $\omega_i = P_i\Omega/f^{k_i}$ for $1 \le i \le r$.
- Repeatedly differentiate ω with respect to t to get sections

$$\omega, \nabla_{d/dt}(\omega), \ldots, \nabla^r_{d/dt}(\omega)$$

  of $\mathcal{F}^0$. Each of these can be expressed in terms of the basis modulo exact forms. Starting from the highest pole order, we can express any $\nabla^i_{d/dt}(\omega)$ in terms of the basis modulo J, most conveniently using Gröbner basis techniques in the ring $\mathbb{C}(t)[x_1, \ldots, x_N]$. Then we use equation (3.3) to express $\nabla^i_{d/dt}(\omega)$ modulo an exakt form as a linear combination of basis elements and a form η of strictly lower pole order. The coefficients in the linear combination will be rational functions of t. This process can be repeated until all forms are expressed in terms of the basis modulo exact forms.
- Since we have $r + 1$ sections and $\mathcal{F}^0$ has rank r, there has to be a relation between $\omega, \nabla_{d/dt}(\omega), \ldots, \nabla^r_{d/dt}(\omega)$ with coefficients in $\mathbb{C}(s)$, the Picard–Fuchs equation.

If the hypersurface V is singular, not everything we have stated in this section remains true but the basic procedure we just described can still be used. We will carry this through more generally in dimensional regularization in chapter 4.

Chapter 4

Picard–Fuchs Equations of Feynman Integrals in Complex Dimension

In this chapter we present a method to compute differential equations for Feynman integrals within dimensional regularization where the dimension D is left as a parameter. The differential equation is obtained by solving (possibly large) systems of linear equations. We follow closely [MWZ13] where the results have been presented for the first time.

Let us recall our notation. To a graph Γ with ℓ loops, E external edges and N (internal) edges, we have assigned loop momenta $k_1, \dots, k_\ell$, external momenta $p_1, \dots, p_E$ and masses $m_1, \dots, m_N$. The momenta flowing through the internal lines of the graph are

$$q_i = \sum_{j=1}^{\ell} \rho_{ij} k_j + \sum_{j=1}^{E} \sigma_{ij} p_j, \qquad \rho_{ij}, \sigma_{ij} \in \{-1, 0, 1\}.$$

This representation is obtained by using momentum conservation. The kinematical invariants are

$$s_{jk} = (p_j + p_k)^2, \quad 1 \leq j, k \leq E,$$

such that the Feynman integrals depend on the parameters

$$\Lambda = (s_{jk}, m_1, \dots, m_N).$$

In momentum space a Feynman integral of Γ is given by

$$I_{MS}(D, \Lambda, \nu) = C \cdot \int\limits_{\mathbb{R}^{D\cdot\ell}} \frac{d^D k_1 \dots d^D k_\ell}{\prod_{j=1}^{N} \left(q_j^2 - m_j^2\right)^{\nu_j}}, \tag{4.1}$$

and we now fix

$$C = (-1)^{\overline{\nu}} \left(\frac{1}{i\pi^{D/2}}\right) \frac{\prod_{j=1}^{N} \Gamma(\nu_j)}{\Gamma(\overline{\nu} - \ell D/2)},$$

according to our previous discussion, such that the Feynman parameter prescription reads

$$I(D, \Lambda, \nu) = \int\limits_{\sigma} \left(\prod_{j=1}^{N} x_j^{\nu_j - 1}\right) \frac{\mathcal{U}^{\overline{\nu} - (\ell+1)D/2}}{\mathcal{F}(\Lambda)^{\overline{\nu} - \ell D/2}} \ \Omega,$$

where $\overline{\nu} = \nu_1 + \cdots + \nu_N$. In this chapter we allow integer powers of the propagators and arbitrary parameters. We fix all but one parameter and consider the remaining parameter, as well as the dimension D variable.
Denoting the variable parameter by t, let us define

$$f = \left(\prod_{j=1}^{N} x_j^{\nu_j - 1} \right) \frac{\mathcal{U}^{\overline{\nu} - (\ell+1)D/2}}{\mathcal{F}(\Lambda)^{\overline{\nu} - \ell D/2}},$$

and

$$\omega_t = f\,\Omega.$$

As before we would like to find a differential equation of the form

$$L^{(r)}(\omega_t) = d\beta, \tag{4.2}$$

where r denotes the order of the differential operator and β is a $(N-2)$–form on projective space. Both r and β, as well as the coefficients of L are considered unknown. The results from previous sections suggest to look for a differential operator

$$L^{(r)} = \sum_{i=0}^{r} c_i \left(\frac{d}{dt} \right)^i,$$

where the coefficients c_i are rational functions in the kinematical invariants, the powers of the propagators and here also the dimension. They are not allowed to contain the Feynman parameters x_i and we would like to normalize the operator by demanding $c_r = 1$.
If we find an equation of the form (4.2), integration gives

$$L^{(r)}\left(I_\Gamma(D)\right) = \int_\sigma d\beta.$$

Within dimensional regularization there are no obstructions in using Stokes' Theorem (see e.g. [Eti00]) and we obtain

$$L^{(r)}\left(I_\Gamma(D)\right) = \int_{\partial\sigma} \beta. \tag{4.3}$$

This is the sought–after differential equation for $I_\Gamma(D)$ within dimensional regularization. The right–hand side of equation (4.3) is a sum of Feynman integrals in one less Feynman parameter (corresponding to minos of the graph). It can therefore be regarded as a sum of simpler integrals. This is a similar situation as in chapter 3. Here, however, the sum of integrals need not be convergent. Instead we regard the individual pieces as dimensionally regularized Feynman integrals. Before treating an integral $I_\Gamma(D)$ one should therefore obtain analytical solutions for the simpler integrals first. If one is interested in a solution to the differential equation, one additionally needs a boundary value. Here the value of $I_\Gamma(D)$ at $t = 0$ can often be obtained. Then equation (4.3) can be used to obtain $I_\Gamma(D)$. We will not address this different problem in this dissertation. Instead we will focus on obtaining the desired differential equation.

Let us take a closer look at β. Differentiating ω_t with respect to a kinematical invariant or a squared mass increases the power of the second graph polynomial in the denominator by one. In order to find a differential operator of order r we therefore make the ansatz

$$\beta = \frac{f}{\mathcal{F}^{r-1}}\,\alpha \tag{4.4}$$

for the $(N-2)$–form β of equation (4.2). Here α is an $(N-2)$–form without singularities. As in chapter 3, we try to find a form α whose coefficients are homogeneous of degree $(r-1)(\ell+1)+2$, such that the coefficients of β are of degree $-(N-2)$. As before we make the ansatz

$$\alpha = \sum_{j_1<j_2} (-1)^{j_1+j_2} \left[-x_{j_1}a_{j_2} + x_{j_2}a_{j_1}\right] dx_1 \wedge ... \wedge \widehat{dx_{j_1}} \wedge ... \wedge \widehat{dx_{j_2}} \wedge ... \wedge dx_N,$$

where the a_i are homogeneous polynomials of degree $h = [(r-1)(l+1)+1]$ in the variables x_i. For the polynomials a_i we assume the most general form. For example, if $N=3$ and $h=2$ we have

$$a_i = a^{(i)}_{200}x_1^2 + a^{(i)}_{020}x_2^2 + a^{(i)}_{002}x_3^2 + a^{(i)}_{110}x_1x_2 + a^{(i)}_{011}x_2x_3 + a^{(i)}_{101}x_1x_3.$$

The variables $a^{(i)}_{jkl}$ are independent of the Feynman parameters. The most general homogeneous polynomial of degree h in N variables has

$$\begin{pmatrix} N+h-1 \\ h \end{pmatrix}$$

monomials. This is the number of possibilities to pick h elements out of a set of N elements by not taking the order into account and with repetitions. For a given r we therefore have within our ansatz as unknown variables the variables c_0, c_1, ..., c_{r-1}, which appear in the Picard-Fuchs operator, as well as all coefficients appearing in the expansion into monomials of the polynomials a_i. This gives

$$N_{\text{unknowns}} = r + N \begin{pmatrix} (r-1)(\ell+1)+N \\ (r-1)(\ell+1)+1 \end{pmatrix} \tag{4.5}$$

unknown variables.
Our ansatz gives

$$d\beta = \sum_{j=1}^{N} (-1)^{j-1} \sum_{i=1}^{N} \partial_i \left[\frac{f}{\mathcal{F}^{r-1}}\left(-x_ia_j + x_ja_i\right)\right] dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_N.$$

Plugging this expression into equation (4.2) and comparing the coefficients of $dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_N$ we obtain for each j

$$x_j L^{(r)} f = \sum_{i=1}^{n} \partial_i \left[\frac{f}{\mathcal{F}^{r-1}}\left(-x_ia_j + x_ja_i\right)\right]. \tag{4.6}$$

To obtain a polynomial equation we need to clear the denominator. Therefore, we multiply both sides of equation (4.6) by

$$\mathcal{F}^{\nu-lD/2+r}\mathcal{U}^{-\nu+(\ell+1)D/2+1}\left(\prod_{j=1}^{N} x_j^{-\nu_j+2}\right).$$

We obtain

$$
\begin{aligned}
0 = x_j & \left(\prod_{k=1}^{N} x_k \right) \mathcal{U} \sum_{s=0}^{r} c_s \left(\ell \frac{D}{2} - \nu - s + 1 \right)_s \mathcal{F}^{r-s} \dot{\mathcal{F}}^s \\
& - \sum_{i=1}^{N} \left\{ \left(\prod_{k=1}^{N} x_k \right) \mathcal{U} \mathcal{F} \partial_i \left(-x_i a_j + x_j a_i \right) + \left(\nu_i - 1 \right) \left(\prod_{k=1, k \neq i}^{N} x_k \right) \mathcal{U} \mathcal{F} \left(-x_i a_j + x_j a_i \right) \right. \\
& \quad + \left(\nu - (\ell + 1) \frac{D}{2} \right) \left(\prod_{k=1}^{N} x_k \right) \left(\partial_i \mathcal{U} \right) \mathcal{F} \left(-x_i a_j + x_j a_i \right) \\
& \quad \left. + \left(\ell \frac{D}{2} - \nu - r + 1 \right) \left(\prod_{k=1}^{N} x_k \right) \mathcal{U} \left(\partial_i \mathcal{F} \right) \left(-x_i a_j + x_j a_i \right) \right\}. \qquad (4.7)
\end{aligned}
$$

Here we have denoted the derivative with respect to t by

$$\dot{\mathcal{F}} = \frac{d}{dt} \mathcal{F}.$$

Equation (4.7) is our master equation and we should pause a moment to contemplate the main features of this equation. First of all, each term of this equation is of degree one or zero in the unknown variables (the coefficients c_j and the coefficients appearing in the expansion of the polynomials a_i into monomials). Secondly, equation (4.7) is homogeneous of degree $[N + (\ell + 1)(r + 1)]$ in the variables x_i. Since equation (4.7) has to hold for all values of the variables x_i, the coefficient c of each monomial in the variables x_i has to vanish. But each coefficient c of a monomial in the variables x_i yields a linear equation $c = 0$ in the unknown variables. We thus obtain a (possibly large) system of linear equations for the unknown variables. In total we obtain by this method

$$N_{\text{equations}} = N \begin{pmatrix} (\ell+1)(r+1) + 2N - 1 \\ (\ell+1)(r+1) + N \end{pmatrix} \qquad (4.8)$$

equations for N_{unknowns}. The number N_{unknowns} has been given in equation (4.5). Of course, not all equations will be independent. With methods from linear algebra we may attempt to solve this system. If the system admits a solution, we have found a differential equation for the Feynman integral under consideration. In the case where a solution exists, there will be in general more than one solution. This is related to the fact that one can always add a closed $(N-2)$-form to β in equation (4.2). This does not affect our method. In order to find the differential equation one can pick any solution.
If the system does not admit a solution, we repeat the procedure by increasing the order r of the differential operator $L^{(r)}$ by one. In general this will result in a linear system with more unknowns and more equations. The practical limitation of this method is the ability to solve large systems of linear equations.

Let us now have a closer look at the inhomogeneous term

$$\int\limits_{\partial\sigma} \beta$$

in equation (4.3). Within dimensional regularisation the integration is over the N faces of the simplex σ. Let us consider one particular face. Without loss of generality we can consider the

N-th face. We consider the restriction of β to $x_N = 0$. If $\nu_N > 1$ we have

$$\beta|_{x_N=0} = 0.$$

Otherwise, if $\nu_N = 1$ we find

$$\beta|_{x_N=0} = (-1)^N a_N \left(\prod_{i=1}^{N-1} x_i^{\nu_i-1} \right) \frac{\mathcal{U}^{\nu-(\ell+1)D/2}}{\mathcal{F}^{\nu-\ell D/2+r-1}} \Bigg|_{x_N=0}$$
$$\times \sum_{j=1}^{N-1} (-1)^{j-1} x_j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{N-1}.$$

Now let

$$a_N = \sum_{\substack{m_1 \geq 0, \ldots, m_N \geq 0 \\ m_1+\cdots+m_N=(r-1)(\ell+1)+1}} a^{(N)}_{m_1 \ldots m_N} x^{m_1} \ldots x^{m_N}$$

be the expansion of the polynomial a_N into monomials. We recall that a_N is homogeneous of degree $(r-1)(\ell+1)+1$. On the N-th face only the monomials with $m_N = 0$ are relevant. We thus have

$$\beta|_{x_N=0} = (-1)^N \sum_{\substack{m_1 \geq 0, \ldots, m_{N-1} \geq 0 \\ m_1+\cdots+m_{N-1}=(r-1)(\ell+1)+1}} a^{(N)}_{m_1 \ldots m_{N-1} 0} \left(f^{(N)}_{m_1 \ldots m_{N-1}} \omega^{(N)} \right),$$

with

$$f^{(N)}_{m_1 \ldots m_{N-1}} = \left(\prod_{i=1}^{N-1} x_i^{\nu_i+m_i-1} \right) \frac{\mathcal{U}^{\nu-(\ell+1)D/2}}{\mathcal{F}^{\nu-\ell D/2+r-1}} \Bigg|_{x_N=0},$$
$$\omega^{(N)} = \sum_{j=1}^{N-1} (-1)^{j-1} x_j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{N-1}.$$

As we have seen in chapter 2, the polynomials

$$\mathcal{U}|_{x_N=0} \text{ and } \mathcal{F}|_{x_N=0}$$

are the graph polynomials of a graph, obtained from the original one by contracting the edge e_N. Therefore

$$f^{(N)}_{m_1 \ldots m_{N-1}} \omega^{(N)}$$

is the integrand of a Feynman integral in $(D + 2r - 2)$ space-time dimensions, where the edge e_N has been contracted. Thus the inhomogeneous term in equation (4.3) is given as a linear combination of Feynman integrals in $(D + 2r - 2)$ space–time dimensions, where one of the N edges of the original integral has been contracted. We remark that the representation of the inhomogeneous term as a linear combination of Feynman integrals is not necessarily unique. As previously already mentioned, we can always add a linear combination of Feynman integrals corresponding to a closed form γ.

Chapter 5

Differential Equations of Some Graphs of Small Loop Order

In the present chapter, we will discuss Feynman integrals of two and three–loop graphs with two external edges and arbitrary masses. Due to momentum conservation these depend on one momentum only, and we will derive ordinary differential equations for the corresponding Feynman Integrals with respect to the momentum parameter. New equations are derived in sections 5.2 through 5.4. These will be obtained by use of the methods introduced in chapters 3 and 4. The Feynman integrals will be interpreted as periods of variations of Hodge Structures (VHS).
Integrals corresponding to graphs with two loops and two external edges will be called two–loop two–point functions. To describe these, according to a result of Tarasov that we recall in section 5.1, it is enough to know three integrals. One will then know all two–loop two–point functions by use of IBP–Identities, differentiation and permutation of masses. The corresponding graphs are shown in figures 5.1 and 5.2.
In section 5.2 we will follow joint work with Stefan Müller–Stach and Stefan Weinzierl [MWZ12], where the new equation was first presented. We consider the simplest of the three aforementioned integrals, the so-called two–loop sunrise graph, which we have already encountered in chapter 2. In the literature it is sometimes called London transport or two–loop banana graph. It is the simplest non–trivial member in an infinite family of banana graphs. The ℓ-loop banana graph consists of two vertices, connected by $\ell + 1$ edges. We will consider the three–loop banana graph in section 5.6. The first three banana graphs are depicted in fig. 5.3. In section 5.2 we will recall known facts about the two–loop sunrise graph and derive a new differential equation in dimension two for one of its master integrals. This is precisely the integral needed by the aforementioned result of Tarasov. One distinguished fact about the sunrise integral is that the generic fibre, when viewing it as a VHS, is smooth. In fact it is a smooth elliptic curve.
In sections 5.3 and 5.4 we discuss the remaining two two–loop graphs and compute differential equations in an integer dimension for their master integrals. Here the generic fibre will be singular. The most complicated one, the master two–loop two–point function is considered in section 5.4.
In section 5.5 we discuss the relations between the sunrise integrals in dimensions $D = 2$ and $D = 4 - 2\varepsilon$. In particular we will explain why knowing the differential equation in dimension two is enough to compute the desired coefficients in the Laurent expansion around $D = 4$. We use the dimensional shift operators we have discussed in chapter 2. In section 5.6 we treat the three–loop banana graph. Finally in section 5.7 we will reconsider the two–loop sunrise graph, but this time we will derive an equation in general dimension D using the methods of chapter 4.

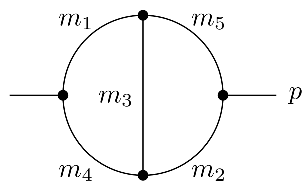

Figure 5.1: The master two–loop two–point graph.

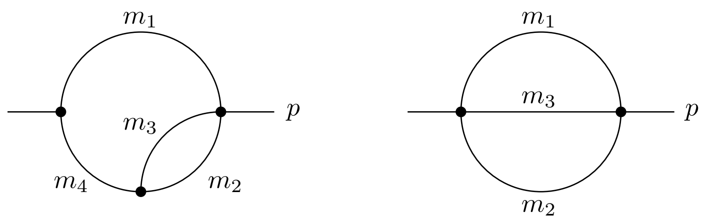

Figure 5.2: *left*: An intermediate graph. It is obtained from the master two–loop two–point graph by contracting edge e_5 which we have assigned the mass m_5 in fig. 5.1, *right*: The two–loop sunrise graph. It is obtained from the graph on the left by contracting edge e_4 which we have assigned the mass m_4.

5.1 A Basis for Two–Loop Two–Point Functions

In [Tar97], Tarasov finds a set of 30 integrals, from which all two–loop integrals can be obtained by generalized recurrence relations [Tar96]. Furthermore, he explains that these integrals are either trivial (reduce to one–loop integrals) or can be obtained by differentiation or by permutation of masses from only three integrals. For this reason we call the set of these three integrals a basis for all two–loop two–point functions. The integrals are:

$$M_{11111}(D,\Lambda) = C_M \cdot \int \frac{d^Dk_1 d^Dk_2}{(k_1^2-m_1^2)(k_2^2-m_5^2)((k_1-p)^2-m_4^2)((k_2-p)^2-m_2^2)((k_1-k_2)^2-m_3^2)},$$

$$H_{1111}(D,\Lambda) = C_H \cdot \int \frac{d^Dk_1 d^Dk_2}{(k_1^2-m_1^2)(k_2^2-m_2^2)((k_1-p)^2-m_3^2)((k_1+k_2-p)^2-m_4^2)},$$

$$S_{111}(D,\Lambda) = C_S \cdot \int \frac{d^Dk_1 d^Dk_2}{(k_1^2-m_1^2)(k_2^2-m_2^2)((k_1+k_2-p)^2-m_3^2)}.$$

Here we have changed the notation from [Tar97] a little. The i-th subscript represents the power of the i-th propagator. We can see by power counting, assuming that all masses are positive, that all three integrals are convergent in dimension two. The first integral M_{11111} is also convergent

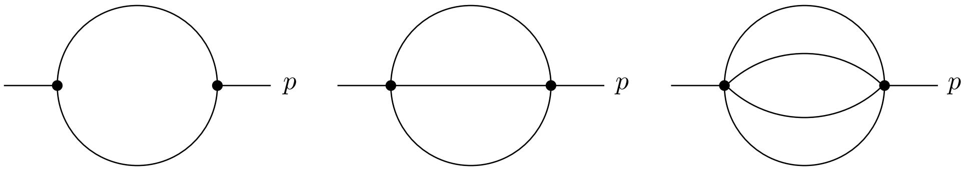

Figure 5.3: *left*: the one–loop banana (or self–energy) graph is completely known and not considered interesting in this context, *middle*: the two–loop sunrise graph which is gouverned by a family of elliptic curves, *right*: the three–loop banana graph which is gouverned by a family of K3–surfaces.

in dimension four. The corresponding Feynman graphs are shown in figures 5.1 and 5.2. To interpret the integrals as periods we switch to their representations in Feynman parameters, which are given by

$$\begin{aligned} M(4,\Lambda) &= \int_\sigma \frac{1}{\mathcal{U}_M \mathcal{F}_M} \cdot \Omega, \\ M(2,\Lambda) &= \int_\sigma \frac{\mathcal{U}_M^2}{\mathcal{F}_M^3} \cdot \Omega, \\ H(2,\Lambda) &= \int_\sigma \frac{\mathcal{U}_H}{\mathcal{F}_H^2} \cdot \Omega, \\ S(2,\Lambda) &= \int_\sigma \frac{1}{\mathcal{F}_S} \cdot \Omega, \end{aligned}$$

with

$$\begin{aligned} \mathcal{U}_M &= (x_1+x_4)(x_2+x_5) + (x_1+x_2+x_4+x_5)x_3, \\ \mathcal{F}_M &= -t\left((x_1+x_5)(x_2+x_4)x_3 + x_1x_4(x_2+x_5) + x_2x_5(x_1+x_4)\right) \\ &\quad + (m_1^2x_1 + m_2^2x_2 + m_3^2x_3 + m_4^2x_4 + m_5^2x_5)\cdot \mathcal{U}_M, \\ \mathcal{U}_H &= (\mathcal{U}_M)_5, \\ \mathcal{F}_H &= (\mathcal{F}_M)_5, \\ \mathcal{U}_S &= (\mathcal{U}_H)_4, \\ \mathcal{F}_S &= (\mathcal{F}_H)_4. \end{aligned}$$

Here we have dropped the subscript to distinguish between the two representations. We have also adopted the notation $t := p^2$, which we will use throughout this chapter. In the subsequent sections we will find differential equations of Picard–Fuchs type for these integrals.

5.2 The Two–Loop Sunrise Integral

In the present section we will deal with the two–loop sunrise integral, which we called S in the previous section. It has received significant attention in the literature. Despite this effort, an analytical answer in the general case of unequal masses could not be achieved before.
The two–loop sunrise graph is known to have four master integrals and in the general situation of unequal masses the IBP–approach finds only a coupled system of four first–order differential equations for the master integrals. We will recall from [CCLR98], how these are obtained. We will then see that in the special case of equal masses the IBP–approach leads to a second–order differential equation for the master integral we call S. This equation appeared first in literature in [BFT93].
In the case of equal masses by solving this differential equation, an analytical solution has been found for S in [LR04]. Following joint work with Stefan Müller–Stach and Stefan Weinzierl [MWZ12], we will show that also in the general case of unequal masses the integral S has to solve a second–order differential equation. To achieve this we will not use IBP–identities. Instead we interpret S as a period of a VHS and derive its Picard–Fuchs equation.

5.2.1 The Master Differential Equations for the Two–Loop Sunrise Graph

The master integrals of the two–loop sunrise graph are

$$\begin{aligned} S_0(D,\Lambda) &:= S(D,\Lambda), \\ S_1(D,\Lambda) &:= -\frac{\partial}{\partial m_1^2}\, S(D,\Lambda), \\ S_2(D,\Lambda) &:= -\frac{\partial}{\partial m_2^2}\, S(D,\Lambda), \quad \text{and} \\ S_3(D,\Lambda) &:= -\frac{\partial}{\partial m_3^2}\, S(D,\Lambda). \end{aligned}$$

By direct computation one finds the equations

$$\begin{aligned} &\left(t\frac{\partial}{\partial t} + m_1^2\frac{\partial}{\partial m_1^2} + m_2^2\frac{\partial}{\partial m_2^2} + m_3^2\frac{\partial}{\partial m_3^2} - (D-3)\right) S_0 = 0, \\ &\left(t\frac{\partial}{\partial t} + m_1^2\frac{\partial}{\partial m_1^2} + m_2^2\frac{\partial}{\partial m_2^2} + m_3^2\frac{\partial}{\partial m_3^2} - (D-4)\right) S_i = 0, \quad \text{for } i=1,2,3. \end{aligned}$$

Here new integrals appear – the mass derivatives of the S_i – which can be expressed in terms of the four master integrals using integration–by–parts identities. One thus obtains a system of first–order differential equations. It reads [CCLR98]

$$t\frac{d}{dt}\, y = Ay + c,$$

where

$$A = \begin{pmatrix} D-3 & m_1^2 & m_2^2 & m_3^2 \\ \frac{(D-3)(3D-8)}{2\cdot L} \cdot P_{1,0} & \frac{D-3}{L} \cdot P_{1,1} + \frac{D-4}{2} & \frac{D-3}{L} \cdot P_{1,2} & \frac{D-3}{L} \cdot P_{1,3} \\ \frac{(D-3)(3D-8)}{2\cdot L} \cdot P_{2,0} & \frac{D-3}{L} \cdot P_{2,1} & \frac{D-3}{L} \cdot P_{2,2} + \frac{D-4}{2} & \frac{D-3}{L} \cdot P_{2,3} \\ \frac{(D-3)(3D-8)}{2\cdot L} \cdot P_{3,0} & \frac{D-3}{L} \cdot P_{3,1} & \frac{D-3}{L} \cdot P_{3,2} & \frac{D-3}{L} \cdot P_{3,3} + \frac{D-4}{2} \end{pmatrix},$$

$$y = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Here we have

$$L\left(t, m_1^2, m_2^2, m_3^2,\right) = \left(t - (m_1+m_2+m_3)^2\right)\left(t - (m_1+m_2-m_3)^2\right)$$
$$\left(t - (m_1-m_2+m_3)^2\right)\left(t - (-m_1+m_2+m_3)^2\right).$$

Although we have written L in a factorised form, it is actually a polynomial in the variables t, m_1^2, m_2^2 and m_3^2. The functions $P_{i,j}$ are polynomials in t, m_1^2, m_2^2 and m_3^2 and can be obtained from the expressions given in [CCLR98] by setting $p^2 = -t$. The sign accounts for the change from the Euclidean to the Minkowski case. The c_i are combinations of tadpole integrals. Their coefficients, again, can be obtained from [CCLR98] by setting $p^2 = -t$.
In the general case of arbitrary masses this system is irreducible. In the case of equal masses $m_1 = m_2 = m_3 = m$, however, we find the relation

$$S_1 = S_2 = S_3$$

and the system reduces to a system of two first–order differential equations for the two remaining master integrals. One can extract from that a single second–order equation for $S = S_0$. With the conventions and the notation adopted here and setting $m = 1$, it reads (compare [LR04])

$$\left(\partial_t^2 + \frac{(12-3D)t^2 + 10(D-6)t + 9D}{2t(t-1)(t-9)}\,\partial_t + \frac{(D-3)((D-4)t+D+4)}{2t(t-1)(t-9)}\right) S(D,t,1,1,1)$$
$$= -\frac{24}{t(t-1)(t-9)}\frac{1}{(D-4)^2}. \tag{5.1}$$

Setting $D = 2$ this reduces to

$$\left(\partial_t^2 + \frac{3t^2-20t+9}{t(t-1)(t-9)}\,\partial_t + \frac{t-3}{t(t-1)(t-9)}\right) S(2,t,1,1,1) = -\frac{6}{t(t-1)(t-9)}. \tag{5.2}$$

This is an inhomogeneous Picard–Fuchs equation of a familiy of elliptic curves.

5.2.2 The Picard–Fuchs Equation for the Sunrise Integral with Arbitrary Masses

In this section we derive the Picard–Fuchs equation for the integral

$$S\,(2,\Lambda) = \int_\sigma \frac{1}{\mathcal{F}}\Omega.$$

Here

$$\mathcal{F} = -tx_1x_2x_3 + (m_1^2x_1 + m_2^2x_2 + m_3^2x_3)\cdot(x_1x_2 + x_1x_3 + x_2x_3)$$

like in the previous sections. We keep the masses fixed and denote by $\mathcal{X}$ the set of points $([x_1 : x_2 : x_3], t) \in \mathbb{P}^2 \times S$, for which $\mathcal{F} = 0$. Here S is a Zariski open subset of $\mathbb{P}^1$. We have the natural projection map $p : \mathcal{X} \to \mathbb{P}^2$ and we denote the fibre over t by X_t. This defines a family of elliptic curves. The generic fibre is smooth if all masses are positive, it is singular if at least one mass is zero. The most interesting case, that we will discuss first, is the one of positive masses. According to the results in section 3 we have to blow up the linear spaces L_{12}, L_{13} and L_{23}. Now let $P \xrightarrow{\pi} \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ in the three points L_{12}, L_{13} and L_{23}. We use the notation from chapter 3, denoting the strict transform of X_t by Y_t and the strict transform of σ again by σ. Here X_t is a smooth elliptic curve, so in this particular case X_t is isomorphic to Y_t for generic t. Furthermore, in P we have $\sigma \cap Y_t = \emptyset$. Now let $B_0 := \{x_1x_2x_3 = 0\} \subset \mathbb{P}^2$ and B its total transform. Clearly, the boundary of σ is contained in B. Define

$$H_t := H^2\,(P \setminus Y_t,\ B \setminus B \cap Y_t)\,.$$

The convergent Feynman–Integral $S(2,\Lambda)$ is a period of H_t and we compute its Picard–Fuchs equation.
In the part of the Euclidean region given by the inequality $t < 0$, we have

$$\begin{aligned} \omega_t &\in H^2(P \setminus Y_t,\ B \setminus B \cap Y_t), \quad \text{and} \\ \sigma &\in H_2(P \setminus Y_t,\ B \setminus B \cap Y_t). \end{aligned}$$

In this example the bound on the kinematical invariant can be improved.

Lemma 5.2.1 *Let $t_0 := (m_1 + m_2 + m_3)^2$ and $\mathbb{C}_{<t_0}$ be the complex numbers with the line $\{x \in \mathbb{R} \mid x \geq t_0\}$ removed. For any $t \in \mathbb{C}_{<t_0}$ the chain of integration σ intersects the graph hypersurface X_t precisely in the three points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$.*

Proof We have $\mathcal{F}(t) = -t\ x_1x_2x_3 + (m_1^2x_1 + m_2^2x_2 + m_3^2x_3)(x_1x_2 + x_1x_3 + x_2x_3)$. First observe that the boundary of σ intersects X_t precisely in the three points stated. We have to show that the intersection of X_t with the inner points of σ is the empty set for $t \in \mathbb{C}_{<t_0}$. This is obvious for $t \in \mathbb{C}\setminus \mathbb{R}$. Now let $\widetilde{t_0} = (m_1 + m_2 + m_3)^2 - \epsilon$, with $\epsilon \in \mathbb{R}_{>0}$. We restrict to the affine open $x_1 = 1$ and obtain the function

$$\mathcal{F} = -((m_1 + m_2 + m_3)^2 - \epsilon)x_2x_3 + (m_1^2 + m_2^2x_2 + m_3^2x_3)(x_2 + x_3 + x_2x_3).$$

We have to show that the equation

$$(m_1 + m_2 + m_3)^2 - \epsilon = (m_1^2 + m_2^2x_2 + m_3^2x_3)\left(\frac{1}{x_2} + \frac{1}{x_3} + 1\right)$$

has no positive real solution. Now $\varphi(x_2, x_3) := (m_1^2 + m_2^2x_2 + m_3^2x_3)(\frac{1}{x_2} + \frac{1}{x_3} + 1)$ is a continuous function from $U := \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ to $\mathbb{R}_{>0}$ which tends to infinity, when x_2 or x_3 tend to zero or infinity. Hence the set $\{x \in U \mid \varphi(x) \leq C\} =: K \subset U$ is compact and φ has its global minimum on K. We easily find the global minimum to be unique, namely the point $(x_2, x_3) = (\frac{m_1}{m_2}, \frac{m_1}{m_3})$. Now we find $\varphi(\frac{m_1}{m_2}, \frac{m_1}{m_3}) = (m_1 + m_2 + m_3)^2$ which proves the Lemma. $\square$

In the following we will assume $t \in \mathbb{C}_{<t_0}$. The differential equation which we derive will be valid in the region $\mathbb{C}_{<t_0}$. Note that $p^2 = t_0 = (m_1+m_2+m_3)^2$ is the physical threshold. The two–loop sunrise integral for values of p^2 above the threshold can then be obtained from the solution of the differential equation by analytic continuation.

In the following we will denote a generic fibre by X, resp. Y, dropping the subscript t. For any closed cycle ξ, that spans a cohomology class in $H_2(P \setminus Y)$, the integral

$$\int_\xi \omega_t$$

has a homogeneous Picard–Fuchs equation, that can be computed using the Griffiths–Dwork reduction.

We find the Picard Fuchs operator

$$L^{(2)} = \frac{d^2}{dt^2} + a(t)\frac{d}{dt} + b(t), \tag{5.3}$$

where

$$\begin{aligned} a(t) &= \frac{p_1(t)}{p_0(t)}, \\ b(t) &= \frac{p_2(t)}{p_0(t)}, \end{aligned}$$

with

$$\begin{aligned} p_1(t) = {} & 9t^6 - 32M_{100}t^5 + (37M_{200} + 70M_{110})\,t^4 - (8M_{300} + 56M_{210} + 144M_{111})\,t^3 \\ & - (13M_{400} - 36M_{310} + 46M_{220} - 124M_{211})\,t^2 \\ & - (-8M_{500} + 24M_{410} - 16M_{320} - 96M_{311} + 144M_{221})\,t \\ & - (M_{600} - 6M_{510} + 15M_{420} - 20M_{330} + 18M_{411} - 12M_{321} - 6M_{222})\,, \\ p_2(t) = {} & 3t^5 - 7M_{100}t^4 + (2M_{200} + 16M_{110})\,t^3 + (6M_{300} - 14M_{210})\,t^2 \\ & - (5M_{400} - 8M_{310} + 6M_{220} - 8M_{211})\,t + (M_{500} - 3M_{410} + 2M_{320} + 8M_{311} - 10M_{221})\,, \\ p_0(t) = {} & t\,L(t, m_1^2, m_2^2, m_3^2)\left(3t^2 - 2M_{100}t - M_{200} + 2M_{110}\right). \end{aligned}$$

Here L is the polynomial introduced in section 5.2.1. In order to present the result in a compact form we have introduced the monomial symmetric polynomials $M_{\lambda_1\lambda_2\lambda_3}$ in the variables m_1^2, m_2^2 and m_3^2. These are defined by

$$M_{\lambda_1\lambda_2\lambda_3} = \sum_\sigma \left(m_1^2\right)^{\sigma(\lambda_1)} \left(m_2^2\right)^{\sigma(\lambda_2)} \left(m_3^2\right)^{\sigma(\lambda_3)},$$

where the sum is over all distinct permutations of $(\lambda_1, \lambda_2, \lambda_3)$. A few examples are

$$\begin{aligned} M_{100} &= m_1^2 + m_2^2 + m_3^2, \\ M_{111} &= m_1^2 m_2^2 m_3^2, \\ M_{210} &= m_1^4 m_2^2 + m_2^4 m_3^2 + m_3^4 m_1^2 + m_2^4 m_1^2 + m_3^4 m_2^2 + m_1^4 m_3^2. \end{aligned}$$

So, for any cycle ξ in $H_2(P \setminus Y)$ we have

$$L^{(2)}\left(\int_\xi \omega_t\right) = 0.$$

Since the domain of integration σ is not a cycle we instead obtain

$$L^{(2)}\left(\int_\sigma \omega_t\right) = \int_\sigma d\beta_t = \int_{\partial\sigma} \beta_t =: g(t). \tag{5.4}$$

The integral over the boundary of σ corresponds to Feynman integrals of minors of the sunrise graph, so the function $g(t)$ can be considered simpler. We compute it explicitely and find

$$g(t) = \frac{p_3(t)}{p_0(t)},$$

with

$$\begin{aligned} p_3(t) = &-18t^4 + 24M_{100}t^3 + (4M_{200} - 40M_{110})\,t^2 + (-8M_{300} + 8M_{210} + 48M_{111})\,t \\ &+ (-2M_{400} + 8M_{310} - 12M_{220} - 8M_{211}) + 2c\,(t, m_1, m_2, m_3) \ln \frac{m_1^2}{\mu^2} \\ &+ 2c\,(t, m_2, m_3, m_1) \ln \frac{m_2^2}{\mu^2} + 2c\,(t, m_3, m_1, m_2) \ln \frac{m_3^2}{\mu^2} \end{aligned}$$

and

$$\begin{aligned} &c\,(t, m_1, m_2, m_3) = \\ &\quad \left(-2m_1^2 + m_2^2 + m_3^2\right) t^3 + \left(6m_1^4 - 3m_2^4 - 3m_3^4 - 7m_1^2m_2^2 - 7m_1^2m_3^2 + 14m_2^2m_3^2\right) t^2 \\ &\quad + \left(-6m_1^6 + 3m_2^6 + 3m_3^6 + 11m_1^4m_2^2 + 11m_1^4m_3^2 - 8m_1^2m_2^4 - 8m_1^2m_3^4 - 3m_2^4m_3^2 - 3m_2^2m_3^4\right) t \\ &\quad + \left(2m_1^8 - m_2^8 - m_3^8 - 5m_1^6m_2^2 - 5m_1^6m_3^2 + m_1^2m_2^6 + m_1^2m_3^6 + 4m_2^6m_3^2 + 4m_2^2m_3^6\right. \\ &\qquad \left. + 3m_1^4m_2^4 + 3m_1^4m_3^4 - 6m_2^4m_3^4 + 2m_1^4m_2^2m_3^2 - m_1^2m_2^4m_3^2 - m_1^2m_2^2m_3^4\right). \end{aligned}$$

The coefficients $c(t, m_i, m_j, m_k)$ of the logarithms of the masses vanish for equal masses. At this point one might wonder why the logarithms of the masses appear in the inhomogeneous term. Let us indicate where these come from.

As we have explained, the inhomogeneous term of the differential equation is a sum of Feynman integrals of minors of the graph. The double tadpole integral, that is associated with the graph that is obtained by contracting one of the edges of the sunrise graph, reads

$$T_2(D, m_1, m_2, \nu_1, \nu_2) = \int_\sigma x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} \frac{(x_1x_2)^{\nu_1+\nu_2-3/2D}}{(m_1^2x_1 + m_2^2x_2)\,(x_1x_2)^{\nu_1+\nu_2-D}}\,\Omega.$$

Especially, we have

$$T_2(4, m_1, m_2, 3, 3) = \int_\sigma \frac{1}{(m_1^2x_1 + m_2^2x_2)}\,\Omega = \int_0^\infty \frac{1}{m_1^2x_1 + m_2^2}\,dx_1.$$

The integrand on the right–hand side of the above equation is a logarithmic divergent function, which is holomorphic on the strip $(0,\infty)$.
For a class of functions $f(z)$, called hyperlogarithms, that are holomorphic on the real interval $(0,\infty)$ and have at most logarithmic singularities at $z=0,\infty$, Brown defines in [Bro09a] the regularized integral of $f(z)dz$ along $(0,\infty)$ as

$$\int_0^\infty f(z)dz = \operatorname{Reg}_{z=\infty} F(z) - \operatorname{Reg}_{z=0} F(z),$$

where $F(z)$ is a primitive of $f(z)$, and the regularized values $\operatorname{Reg}_{z=\infty} F(z)$ and $\operatorname{Reg}_{z=0} F(z)$ are obtained in the following way. Hyperlogarithms can be uniquely written in the form

$$F(z) = \sum_{i=0}^{m} F_i(z)\log^i(z),$$

where $F_i(z)$ is holomorphic at $z=\infty$, for $0\le i\le m$. Define

$$\operatorname{Reg}_{z=\infty} F(z) = F_0(\infty).$$

The regularized value at zero is defined analogously. The point now is, that in computing a convergent Feynman integral (like the inhomogenous part of our differential equation) one is allowed to break it into logarithmically divergent components, compute the regularized integral of each piece and add the contributions together. Again, we refer the reader to chapter 5 of [Bro09a] and the references given there for details. This can be applied to the above tadpole integral, so for a convergent integral I, that can be broken into a sum of the form

$$I = T_2(4, m_1, m_2, 3, 3) + \widetilde{I},$$

we obtain

$$I = \ln(m_1^2) - \ln(m_2^2) + \widetilde{I}.$$

In this way logarithms of masses appear.
Another way to see this is by analyzing the Gysin sequence together with the long exact sequence of relative cohomology. The latter reads

$$0 \longrightarrow H^1(B\backslash B\cap Y) \longrightarrow H^2(P\backslash Y,\ B\backslash B\cap Y) \longrightarrow H^2(P\backslash Y) \longrightarrow H^2(B\backslash B\cap Y),$$

where we have used $H^1(P\backslash Y)=0$. We find

$$B\cap Y = \left\{\left[0 : -\frac{m_3}{m_2} : 1\right], \left[-\frac{m_3}{m_1} : 0 : 1\right], \left[1 : -\frac{m_1}{m_2} : 0\right]\right\} \cup \{p_1, p_2, p_3\},$$

where p_i is a point on the exceptional divisor E_i. For $B\backslash B\cap Y$ we get the picture of figure 5.4.
To analyze $H^k\left(B\backslash B\cap Y\right)$ there is the spectral sequence of Hodge structures

$$E_1^{p,q} = H^q\left(B^{p+1}\backslash B^{p+1}\cap Y\right) \Rightarrow H^{p+q}\left(B\backslash B\cap Y\right),$$

where we take the obvious cover $B=\bigcup B_i$ and write $B^k = \coprod B_{i_1}\cap\cdots\cap B_{i_k}$.
We obtain

$$H^0(B\backslash B\cap Y) = \mathbb{Z},$$
$$H^1(B\backslash B\cap Y) = \mathbb{Z}, \text{ and}$$
$$H^k(B\backslash B\cap Y) = 0, \text{ for } k\neq 0,1.$$

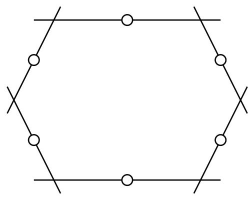

Figure 5.4: $B\backslash\ B\cap Y$. A closed chain of six copies of $\mathbb{P}^1$ with a point removed.

Summarizing, we get the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H^2(P\backslash Y,\ B\backslash B\cap Y) \longrightarrow H^2(P\backslash Y) \longrightarrow 0 \tag{5.5}$$

of mixed Hodge structures. For a smooth hypersurface inside some projective manifold such as the curve $Y \hookrightarrow P$ we have the Gysin sequence to analyze the complement $P\backslash Y$. It is an exact sequence of mixed Hodge structures and in our case reads

$$0 \longrightarrow H^2(P) \longrightarrow H^2(P\backslash Y) \longrightarrow H^1(Y)(-1) \longrightarrow 0.$$

We see that H_t has weights zero, two and three. In the case of equal masses the Gysin sequence can be shown to split and one finds, that the period $S(2,\Lambda)$ is supported on a subquotient of H_t which is an extension of $H^1(Y)(-1)$ by $\mathbb{Q}(0)$. From this one can predict a constant inhomogeneous term (the numerator of the inhomogenous term in equation 5.2). In the case of general masses we cannot split the weight two part of H_t and we must deal with more complicated extensions. Therefore, we can no longer deduce that the inhomogenous term will be constant.

In principle, equation (5.4) is the inhomogeneous differential equation we have been searching for. It can be written in a homogeneous form

$$\left(\partial_t - \frac{g'(t)}{g(t)}\right)\ L^{(2)}\left(S(2,\Lambda)\right) = 0, \tag{5.6}$$

where the prime denotes differentiation with respect to t. This equation annihilates the Feynman integral and has regular singular points. It is, however, not defined over the differential field $\mathbb{Q}(t)$ but only over $k(t)$ with $k = \mathbb{Q}(\ln m_1, \ln m_2, \ln m_3)$. To obtain an operator that is defined over $\mathbb{Q}(t)$, we can find a differential operator that is defined over $\mathbb{Q}(t)$ and annihilates the inhomogenous part of the differential equation.
As an example, let us discuss the case $m_1 = 2$ and $m_2 = m_3 = 1$. In this case our differential equation reads

$$L^{(2)}\left(S(2,t)\right) = \frac{-6t - 8\ \ln(2)}{t^2(t-4)(t-16)} =: g(t).$$

Define

$$L_g := \partial_t - \frac{g'(t)}{g(t)}.$$

The operator $L_g \cdot L^{(2)}$ has regular singular points and annihilates $S(2,t)$ but its coefficients are not in $\mathbb{Q}(t)$.

We look for a second order differential operator with coefficients in $\mathbb{Q}(t)$ that kills $g(t)$ and find

$$\widetilde{L} = \left(\partial_t + \left(\frac{2}{t} + \frac{1}{t-4} + \frac{1}{t-16}\right)\right)^2.$$

Now observe

$$L_g = \partial_t + \left(\frac{2}{t} + \frac{1}{t-4} + \frac{1}{t-16} - \frac{1}{t + \frac{4}{3}\ln(2)}\right),$$

and define

$$L_{\overline{g}} = \partial_t + \left(\frac{2}{t} + \frac{1}{t-4} + \frac{1}{t-16} + \frac{1}{t + \frac{4}{3}\ln(2)}\right).$$

We have

$$L_{\overline{g}} \cdot L_g = \widetilde{L},$$

which is just the identity

$$(\partial_t + b)^2 = \left(\partial_t + b + \frac{1}{t+a}\right) \cdot \left(\partial_t + b - \frac{1}{t+a}\right),$$

for $\partial_t a = 0$ and $b \in \mathbb{Q}(t)$.
Now the Picard–Fuchs operator

$$\widetilde{L} \cdot L^{(2)}$$

is of order four and the relation to the operator $L_g \cdot L^{(2)}$ is illustrated above.
Recall that the ring of differential operators over R, denoted $R[\partial]$, is defined as the free R–algebra generated by ∂, modulo the ideal J generated by elements $a\partial - \partial a - a'$ for $a \in R$. So in conclusion we have taken an element in $\mathbb{Q}(t)[\partial]$ and found a different factorization in $\mathbb{Q}(\ln(2))(t)[\partial]$. It turned out that the smaller operator $L_g \cdot L^{(2)}$ already killed the period.

Remark 5.2.2 *Note that in this way we cannot find a differential operator in dimension four without shifting dimensions. For $D = 4$ the Feynman form becomes $\omega = \frac{\mathcal{F}}{\mathcal{U}^3}$ where $\mathcal{U}$ is a smooth quadric. Its first cohomology group is thus trivial and without applying any differential operator we can find a form β, such that $\omega = d\beta$. To separate its polar locus from σ we still have to blow up the same three points. Now property (i) in Proposition (3.1.4) fails, because the integral diverges in dimension four. We have thus obtained a useless equation, namely $S(4, z) = \infty$.*

So far, we have discussed the sunrise integral in the most interesting case of positive masses. In the remainder of this section we will complete the discussion on the two–loop sunrise graph by covering the remaining cases shortly. This means that at least one mass has to be zero. Let us begin with the case of a single zero mass and two positive masses. According to the results of chapter 3 we still only have to blow up the three points L_{12}, L_{13}, L_{23}. Let $m_3 = 0$. Now we see that ω_t vanishes along L_{12} to order two, but the blowup of L_{12} only reduces the pole order by one. Therefore we hit an IR–singularity.
The integral is not a period and we cannot assign a Picard–Fuchs operator. We can, however, compute the homogeneous Picard–Fuchs operator L of the Feynman form and formally assign the equation

$$L\left(\int_\sigma \omega_t\right) = \int_\sigma d\beta_t = \int_{\partial\sigma} \beta_t = \infty. \tag{5.7}$$

This might appear to be of little use, we can, however, use it to compare results with a more formal computation we do in chapter 5.7. There, we will compute an equation valid in general dimension D. It should then specialize to the equation computed here and thus the inhomogeneous part should have a pole as D approaches two.
To compute the differential operator L in equation (5.7), observe that in cases with a massless line X_t is a singular cubic. From that we expect a linear differential operator. More precisely, when m_i is zero, the point L_{jk} with $i \neq j \neq k \neq i$ is a singular point of X_t and these are the only singular points of X_t. Its strict transform Y_t will then be smooth in the blowup P we considered earlier. We can then, as before, apply the Gysin sequence. This time we obtain a trivial sequence.
With that in mind let us look at our formula for the homogeneous term of the Picard–Fuchs equation (eq. 5.3). Setting one mass equal to zero and naming the others m and n, we obtain

$$\begin{aligned} L^{(2)} &= \partial_t^2 + \frac{p_4(t)}{p_7(t)}\,\partial_t + \frac{p_5(t)}{p_7(t)\,(t-(m+n)^2)\,(t-(m-n)^2)} \\ &= L_2 \cdot L_1, \end{aligned}$$

with

$$\begin{aligned} L_1 &= \left(\partial_t + \frac{t-m^2-n^2}{(t-(m+n)^2)\,(t-(m-n)^2)}\right), \\ L_2 &= \left(\partial_t + \frac{p_6(t)}{p_7(t)}\right), \end{aligned}$$

and

$$\begin{aligned} p_4(t) &= 9t^4 - 14(m^2+n^2)t^3 + 32m^2n^2t^2 + 6(m^6+n^6-m^4n^2-m^2n^4)t \\ &\quad - m^8 - n^8 + 4m^6n^2 + 4m^2n^6 - 6m^4n^4, \\ p_5(t) &= 3t^5 - 7(m^2+n^2)t^4 + 2(m^4+n^4+8m^2n^2)t^3 + 2(3m^6+3n^6-7m^4n^2-7m^2n^4)t^2 \\ &\quad - (5m^8+5n^8-8m^6n^2-6m^2n^6+6m^4n^4)t \\ &\quad + m^{10} + n^{10} - 3m^8n^2 - 3m^2n^8 + 2m^6n^4 + 2m^4n^6, \\ p_6(t) &= 6t^4 - 9(m^2+n^2)t^3 - (m^4+n^4-26m^2n^2)t^2 + 5(m^6+n^6-m^4n^2-m^2n^4)t \\ &\quad - m^8 - n^8 + 4m^6n^2 + 4m^2n^6 - 6m^4n^4, \\ p_7(t) &= t\,\left[3t^2 - 2(m^2+n^2)t - (m^2-n^2)^2\right]\,(t-(m+n)^2)\,(t-(m-n)^2). \end{aligned}$$

Indeed one easily verifies that the second operator L_1 is the sought–after homogeneous Picard–Fuchs operator of equation (5.7). We have found a linear operator as expected.
The case with two massless lines is also IR–singular and can be treated as above. We obtain

$$L_1 = \partial_t + \frac{1}{t-m^2}.$$

Remark 5.2.3 *We have seen that our method works best if all internal lines are massive. This is motivically and physically the most interesting case. If there is a single massless line the integral can be reduced to a one–loop Feynman integral and can therefore be considered known (see e.g. [Smi04]).*

This concludes our discussion of the differential equation of the sunrise integral in dimension two. We will treat the relation between dimension $D = 2$ and $D = 4 - 2\varepsilon$ in section 5.5.

5.3 An Intermediate Graph

The integral

$$H(2,\Lambda) = \int\limits_\sigma \frac{\mathcal{U}_H}{\mathcal{F}_H^2}\cdot\Omega,$$

of the graph on the left–hand side of figure 5.2 is convergent in dimension two, if all masses are positive. The middle cohomology of the desingularization of its second graph hypersurface $X_t \subset \mathbb{P}^3$ is

$$H^2(X_t) = \bigoplus^7 \mathbb{Z}(-1).$$

To separate the polar locus from the domain of integration we have to blow up the linear spaces L_{23}, L_{124}, and L_{134}. We find the differential equation

$$L^{(1)}\left(H(2,\Lambda)\right) = g(t),$$

with

$$\begin{aligned}
L^{(1)} &= \frac{d}{dt} + \frac{t-(m_1^2+m_4^2)}{(t-(m_1+m_4)^2)(t-(m_1-m_4)^2)},\\
g(t) &= \frac{1}{(t-(m_1+m_4)^2)(t-(m_1-m_4)^2)}\,\widetilde{I},\\
\widetilde{I} &= \int\limits_\sigma \frac{(t+3m_1^2-m_4^2)x_1x_2x_3 + m_2^2x_2^2(2x_1+4x_3) - 2m_1^2x_1^2x_3 - 2m_3^2x_3^2(x_1-x_2)}{(\mathcal{F}_H)_4^2}\,\Omega.
\end{aligned}$$

Here we see, that only the massive sunrise graph gives a contribution to the inhomogeneous term of the differential equation. The other minors of the graph give vanishing contributions. Furthermore, also the integrals over the exceptional divisors give vanishing contributions. The remaining integral $\widetilde{I}$ can be expressed in terms of the two sunrise integrals $S(2,\Lambda) = S(2,t,m_1,m_2,m_3)$ and $S^{'}(2,\Lambda) = \partial_t\, S(2,t,m_1,m_2,m_3)$ because their integrands span the cohomology group

$$H^2(\mathbb{P}^2\backslash X_S),$$

where X_S denotes the second graph hypersurface of the sunrise graph. We obtain

$$\begin{aligned}
\widetilde{I} &= \frac{a(t)}{N(t)}S^{'}(2,\Lambda) + \frac{b(t)}{N(t)}S(2,\Lambda)\\
a(t) &= -5t^3 + (6m_2^2 - 5m_1^2 + 6m_3^2 + 3m_4^2)t^2\\
&\quad + (-m_2^4 - m_3^4 + 9m_1^4 + 2m_1^2m_2^2 - 4m_2^2m_3^2 - 4m_1^2m_3^2 - 2m_2^2m_4^2 - 2m_1^2m_4^2 - 2m_3^2m_4^2)t\\
&\quad + (m_3-m_4)(m_3+m_4)\cdot R\\
b(t) &= -2(t^2 - t(m_2^2+m_3^2) + m_1^2(-m_1^2+m_2^2+m_3^2))\\
N(t) &= 3t^2 - 2(m_1^2+m_2^2+m_3^2)t + R,\\
R &= (m_1+m_2+m_3)(m_1+m_2-m_3)(m_1-m_2+m_3)(-m_1+m_2+m_3).
\end{aligned}$$

In the case $m_1 = m_2 = m_3 = m_4 = 1$ this simplifies to

$$\begin{aligned}
L^{(1)} &= \frac{d}{dt} + \frac{t-2}{t(t-4)},\\
g(t) &= -\frac{1}{t(t-4)}\left(\frac{5}{3}t\; S^{'}(2,t,1,1,1) - \frac{2}{3}S(2,t,1,1,1)\right).
\end{aligned}$$

The inhomogeneous Picard–Fuchs operator of $H(2,\Lambda)$ now reads

$$\left(\partial_t - \frac{g'(t)}{g(t)}\right) L^{(1)}.$$

In order to obtain an operator with coefficients in $\mathbb{Q}(t)$ one can simply multiply the Picard–Fuchs operator of the sunrise integral $g(t)$ to $L^{(1)}$ from the left.

5.4 The Two–Loop Master Graph

The two–loop master graph has very good convergence properties. It is UV–finite in dimension less or equal to four. Therefore, if we consider a four–dimensional theory we do not have to shift dimensions within our method. Let us check for IR–singularities. The Feynman form in dimension four is

$$\omega = \frac{1}{\mathcal{U}\mathcal{F}},$$

so IR–singularities could be possible. From counting pole and zero order on the exceptional divisors, we see that the only critical linear coordinate spaces for $\mathcal{F}_0$ are L_{12} and L_{34}. The polynomial $\mathcal{U}$ does, however, not vanish on these, so regardless of the masses we never have IR–singularites in dimension four.

Our method therefore applies to all possible combinations of masses. The period is most interesting when the graph contains a massive sunrise diagram as a minor, i.e. if masses m_1, m_2, m_3 or m_3, m_4, m_5 are positive. Let us discuss these cases, beginning with the homogeneous part of the differential equation. Observe that

$$\partial_t M(4,t) = -\int_\sigma \frac{\partial_t \mathcal{F}}{\mathcal{U}\mathcal{F}^2}\,\Omega,$$

where $\partial_t \mathcal{F} = \frac{1}{t}\mathcal{F}_0$. It follows that

$$\partial_t M(4,t) + \frac{1}{t} M(4,t) = \frac{1}{t}\int_\sigma \frac{\sum_{i=1}^5 m_i^2 x_i}{\mathcal{F}^2}\,\Omega. \tag{5.8}$$

If all masses are zero we have already found the differential equation, it is however not very helpful. From it we get a solution $M(4,t) = \frac{c}{t}$ involving a constant c. Is is known in the literature and given by $c = 6\,\zeta(3)$.

In all other cases equation (5.8) is not homogeneous and more useful. The right–hand side of equation (5.8) is again a period and has a Picard–Fuchs equation. We find that without regard to the masses the numerator of the right–hand side of (5.8) is contained in the Jacobian ideal of $\mathcal{F}$. This leads to the following equation

$$\left(\partial_t + \frac{1}{t}\right) M(4,t) = \frac{1}{t}\int_\sigma \frac{\sum_{i=1}^5 m_i^2 x_i}{\mathcal{F}^2}\,\Omega = \int_\sigma d\beta = \int_{\partial\sigma} \beta. \tag{5.9}$$

The right hand side of the above equation can be expressed in terms of minors of the graph, i.e. in terms of the double tadpole integrals and the integrals of the graph we have discussed in the previous section. Putting everything together, we find, that the Feynman integrals of the two–loop master graph reduce to sunrise integrals after applying two linear differential operators.

5.5 From $D = 2$ to $D = 4 - 2\varepsilon$

In the preceeding chapters we have obtained differential equations for Feynman integrals in dimension two and four. We have seen, that the sunrise integral is of particular importance and we only have an equation in dimension two available. We want to obtain from that information about the sunrise integral in dimension four. It is convergent in dimension two and UV–divergent in dimension four. It also has an overall UV–divergence in dimension four which we will not drop in this chapter. We can write

$$S(2-2\varepsilon,\Lambda) = \sum_{i=0}^{\infty} S^{(i)}(2,\Lambda)\varepsilon^i,$$

and

$$S(4-2\varepsilon,\Lambda) = \sum_{i=-2}^{\infty} S^{(i)}(4,\Lambda)\varepsilon^i.$$

Our ultimate goal is to compute the polar part of the latter series and its zero–order term. What we have is a differential equation for $S(2,\Lambda) = S^{(0)}(2,\Lambda)$. From that an analytic solution for $S^{(0)}(2,\Lambda)$ has been obtained in the recent paper [ABW13]. Let us recall that we have assigned to a graph a family of integrals by allowing integer exponents of the propagators. The scalar integrals assigned to the sunrise graph are of the form

$$S(D,\Lambda,\nu) = \int_\sigma \frac{x_1^{\nu_1-1}x_2^{\nu_2-1}x_3^{\nu_3-1}\mathcal{U}^{\overline{\nu}-(\ell+1)\frac{D}{2}}}{\mathcal{F}^{\overline{\nu}-\ell\frac{D}{2}}}\Omega,$$

where $\Lambda = (t, m_1^2, m_2^2, m_3^2)$, $\nu = (\nu_1,\nu_2,\nu_3)$ and $\overline{\nu} = \nu_1+\nu_2+\nu_3$.
Recall that the two–loop sunrise graph has four master integrals, which can be chosen

$$\begin{aligned}
S(D) &= S(D,\Lambda,1,1,1),\\
S_1(D) &= S(D,\Lambda,2,1,1),\\
S_2(D) &= S(D,\Lambda,1,2,1),\\
S_3(D) &= S(D,\Lambda,1,1,2).
\end{aligned}$$

Observe that the latter three integrals are just mass derivatives of S, we have for $i = 1,2,3$

$$S_i(D) = \mathbf{i}^+ S(D,\Lambda,1,1,1) = -\frac{\partial}{\partial m_i^2} S(D,\Lambda,1,1,1).$$

In conclusion, we know all the master integrals analytically in dimension two. They all converge, such that for $i = 1,2,3$, we have

$$S_i(2) = S_i^{(0)}(2,\Lambda).$$

From that we would like to know $S^{(0)}(4,\Lambda)$ and the polar part of $S(4,\Lambda)$ analytically.
In chapter 2 we have discussed two types of dimensional recurrence relations. The type of relation, that shifts the dimension down by two cannot be applied here. We have to use the type of relation, that shifts the dimension up by two. These read

$$S(D-2,\Lambda,\nu) = \mathcal{U}(\nu_1\mathbf{1}^+,\nu_2\mathbf{2}^+,\nu_3\mathbf{3}^+)\,S(D,\Lambda,\nu). \tag{5.10}$$

When applying this to the master integrals, the integrals on the right–hand side can be expressed in terms of the four master integrals and simpler integrals which correspond to minors of the graph. From the six basic IBP–identities we have discussed in chapter 2, Tarasov derives in [Tar97] the following relations

$$\begin{aligned}2\nu_1\nu_2 P\,\mathbf{1}^+\mathbf{2}^+ S(D,\Lambda,\nu) = &\Big[2\nu_1 h_{123}\mathbf{1}^+ + 2\nu_2 h_{213}\mathbf{2}^+ + 4\nu_3 m_3^2\sigma_{123}\mathbf{3}^+ \\ &+ \nu_2\nu_3 m_3^2\phi_2\mathbf{1}^-\mathbf{2}^+\mathbf{3}^+ + \nu_1\nu_3 m_3^2\phi_1\mathbf{1}^+\mathbf{2}^-\mathbf{3}^+ - 2\nu_1\nu_2\rho\mathbf{1}^+\mathbf{2}^+\mathbf{3}^- \\ &+ \frac{1}{2}(3D-2\nu_1-2\nu_2-2\nu_3-2)(D-\nu_1-\nu_2-\nu_3)\phi_3\Big] S(D,\Lambda,\nu), \qquad (5.11)\end{aligned}$$

where

$$\begin{aligned}P &= (t-(m_1+m_2+m_3)^2)(t-(m_1+m_2-m_3)^2)(t-(m_1-m_2+m_3)^2)(t-(m_1-m_2-m_3)^2),\\ \rho &= -\frac{1}{4}\frac{\partial P}{\partial t},\\ \phi_i &= \frac{1}{2}\frac{\partial}{\partial m_i^2}\left(\frac{\partial}{\partial m_1^2}+\frac{\partial}{\partial m_2^2}+\frac{\partial}{\partial m_3^2}+\frac{\partial}{\partial t}\right)P,\\ \sigma_{ijk} &= -\frac{1}{4}(D-\nu_i-2\nu_j)\phi_i - \frac{1}{4}(D-2\nu_i-\nu_j)\phi_j - \frac{1}{4}(2D-2\nu_i-2\nu_j-\nu_k-1)\phi_k,\\ h_{ijk} &= -\frac{1}{2}(D-2\nu_j-\nu_k)m_k^2\phi_i - \frac{1}{2}(2D-\nu_i-2\nu_j-2\nu_k-1)m_i^2\phi_k + (D-\nu_j-2\nu_k)\rho.\end{aligned}$$

Two additional relations follow from (5.11) by the interchanges

$$\begin{aligned}\mathbf{1}^+\mathbf{3}^+ S(D,\Lambda,\nu): &\quad 2,\mathbf{2}^\pm \longleftrightarrow 3,\mathbf{3}^\pm,\\ \mathbf{2}^+\mathbf{3}^+ S(D,\Lambda,\nu): &\quad 1,\mathbf{1}^\pm \longleftrightarrow 3,\mathbf{3}^\pm. \qquad (5.12)\end{aligned}$$

These relations cannot be applied when two edges have an exponent equal to one (i.e. two entries of ν are equal to one). In order to give a complete reduction Tarasov gives

$$\begin{aligned}2\nu_1(\nu_1+1)m_1^2 P\,\mathbf{1}^+\mathbf{1}^+ S(D,\Lambda,\nu) = &\Big[-(3D-2\nu_1-2\nu_2-2\nu_3-2)(D-\nu_1-\nu_2-\nu_3)\rho \\ &+ \nu_2\nu_3 m_2^2 m_3^2\phi_1\mathbf{1}^-\mathbf{2}^+\mathbf{3}^+ + \nu_1\nu_3 m_1^2 m_3^2\phi_2\mathbf{1}^+\mathbf{2}^-\mathbf{3}^+ + \nu_1\nu_2 m_1^2 m_2^2\phi_3\mathbf{1}^+\mathbf{2}^+\mathbf{3}^- \\ &+ \nu_1(D-2-2\nu_1)P\,\mathbf{1}^+ + \nu_1 m_1^2 S_{123}\mathbf{1}^+ + \nu_2 m_2^2 S_{213}\mathbf{2}^+ + \nu_3 m_3^2 S_{312}\mathbf{3}^+\Big] S(D,\Lambda,\nu) \qquad (5.13)\end{aligned}$$

with

$$S_{ijk} = -(D-2\nu_j-\nu_k)m_k^2\phi_j - (D-\nu_j-2\nu_k)m_j^2\phi_k + 2(2D-\nu_i-2\nu_j-2\nu_k-1)\rho.$$

Again, we have the two variants

$$\begin{aligned}\mathbf{2}^+\mathbf{2}^+ S(D,\Lambda,\nu): &\quad 1,\mathbf{1}^\pm \longleftrightarrow 2,\mathbf{2}^\pm,\\ \mathbf{3}^+\mathbf{3}^+ S(D,\Lambda,\nu): &\quad 1,\mathbf{1}^\pm \longleftrightarrow 3,\mathbf{3}^\pm. \qquad (5.14)\end{aligned}$$

Note, that the relations (5.11),(5.12),(5.13) and (5.14) show, that every $S(D,\Lambda,\nu)$ can indeed be reduced to combinations of the integrals $S(D), S_1(D), S_2(D), S_3(D)$ and simpler integrals.

With our choice of master integrals we obtain the linear system of equations

$$A\begin{pmatrix}S(D)\\S_1(D)\\S_2(D)\\S_3(D)\end{pmatrix} + R = \begin{pmatrix}S(D-2)\\S_1(D-2)\\S_2(D-2)\\S_3(D-2)\end{pmatrix}, \qquad (5.15)$$

where A is a matrix with entries that are rational functions in D, t and the masses, and R is a vector that contains rational combinations of simpler integrals. Inverting A we obtain the desired equation. This has been done in [Tar97] and the equation reads

$$\begin{aligned} S(D) = & \frac{1}{3t(D-3)(D-4)(3D-8)(3D-10)}\Big[c_0 S(D-2) \\ & + f(m_1^2, m_2^2, m_3^2) S_1(D-2) + f(m_2^2, m_1^2, m_3^2) S_2(D-2) + f(m_3^2, m_2^2, m_1^2) S_3(D-2) \\ & + g(m_1^2, m_2^2, m_3^2) S(D-2, \Lambda, 2, 2, 0) + g(m_1^2, m_3^2, m_2^2) S(D-2, \Lambda, 2, 0, 2) \\ & + g(m_2^2, m_3^2, m_1^2) S(D-2, \Lambda, 0, 2, 2)\Big], \end{aligned} \tag{5.16}$$

with

$$\begin{aligned} c_0 = & (D-4)^2 t^3 - 2u_1(D-4)(6D-23)t^2 \\ & + (5u_1^2(15D^2 - 117D + 224) - u_2(42D^2 - 331D + 640))t \\ & - \frac{1}{4}(D-5)(u_3(27D-90) - u_1 u_2(3D-2) - 2u_1^3(5D-26)), \\ u_1 = & m_1^2 + m_2^2 + m_3^2 \\ u_2 = & 3(m_1^4 + m_2^4 + m_3^4) + 2(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2) \\ u_3 = & m_1^2(m_1^4 - m_2^4 - m_3^4) + m_2^2(m_2^4 - m_1^4 - m_3^4) + m_3^2(m_3^4 - m_1^4 - m_2^4) + 10 m_1^2 m_2^2 m_3^2 \end{aligned}$$

and

$$\begin{aligned} f(m_1^2, m_2^2, m_3^2) = & m_1^2(t - m_1^2)[-2(D-4)t^2 + (4u_1(5D-18) - 24m_1^2(2D-7))t \\ & - 2(4D-13)u_1^2 + 2(9D-31)u_2 - 24m_2^2 m_3^2(4D-13) - 24m_1^4(2D-7)] \\ g(m_1^2, m_2^2, m_3^2) = & \frac{m_1^2 m_2^2}{(D-4)}(4(D-4)t^2 - 4(7D-24)(3m_3^2 - 2u_1)t \\ & - u_1^2(23D-80) + u_2(9D-32) - 12m_3^4(D-4) + 12m_1^2 m_2^2(7D-24)). \end{aligned}$$

The simpler integrals occuring on the right–hand side of equation (5.16) are know to all orders. Furthermore, we know the zero–order term of the master integrals in dimension two. Now the prefactor on the right–hand side contains a term $(D-4)^{-1}$ and therefore contributes a simple pole at $D = 4$. This is called a spurious pole. It causes us to lose information about the zero order term of $S(4, \Lambda)$. Therefore, equation (5.16) is not quite what we want. The occurrent phenomenon can be explained by looking at the Feynman forms involved in equation (5.15), setting $D = 4$. On the right–hand side we find

$$\begin{aligned} S_0(2) &= \int_\sigma \frac{1}{\mathcal{F}} \Omega \\ S_1(2) &= \int_\sigma \frac{x_1 \mathcal{U}}{\mathcal{F}^2} \Omega \\ S_2(2) &= \int_\sigma \frac{x_2 \mathcal{U}}{\mathcal{F}^2} \Omega \\ S_3(2) &= \int_\sigma \frac{x_3 \mathcal{U}}{\mathcal{F}^2} \Omega. \end{aligned}$$

The four Feynman forms lie in $H^1(X_{\mathcal{F}})$ and generate it. From equation (5.10) we see that in order to describe $S^{(0)}(4,\Lambda)$ we additionally need forms that lie in the bigger cohomology group $H^1(X_{\mathcal{UF}})$ but not in $H^1(X_{\mathcal{F}})$. Let us replace $S_2(D-2)$ and $S_3(D-2)$ by

$$\widetilde{S_2}(D) = S(D, \Lambda, 2, 1, 2)$$

and

$$\widetilde{S_3}(D) = S(D, \Lambda, 1, 2, 2).$$

We have

$$\widetilde{S_2}(4) = \int_\sigma \frac{x_2 x_3}{\mathcal{UF}} \Omega$$

and

$$\widetilde{S_3}(4) = \int_\sigma \frac{x_1 x_3}{\mathcal{UF}} \Omega.$$

Instead of the system (5.15) we compute the system

$$\widetilde{A} \begin{pmatrix} S(D) \\ S_1(D) \\ S_2(D) \\ S_3(D) \end{pmatrix} + \widetilde{R} = \begin{pmatrix} S(D-2) \\ S_1(D-2) \\ \widetilde{S_2}(D) \\ \widetilde{S_3}(D) \end{pmatrix}, \tag{5.17}$$

using relations (5.11) and (5.13) and their variants (5.12) and (5.14). Let us simplify the notation a little. Firstly, observe that σ_{ijk} and S_{ijk} are symmetric in two of their indices, so that we may drop these. Secondly, in the definition of h_{ijk}, σ_{ijk} and S_{ijk} the values of the exponents are not visible. We only need two cases. In the case $\nu = (1,1,1)$ let us write abusively for the rest of the chapter

$$\begin{aligned} h_{ijk} &= -\frac{1}{2}(D-3)m_k^2\phi_i - (D-3)m_i^2\phi_k + (D-3)\rho \\ \sigma_k &= -\frac{1}{4}(D-3)\phi_i - \frac{1}{4}(D-3)\phi_j - \frac{1}{2}(D-3)\phi_k \\ S_i &= -(D-3)m_k^2\phi_j - (D-3)m_j^2\phi_k + 4(D-3)\rho \end{aligned}$$

and in the case $\nu = (2,1,1)$ let us use the notation $\widetilde{h}_{ijk}$ and $\widetilde{\sigma}_k$.
We obtain

$$\widetilde{A} = \frac{1}{P} \begin{pmatrix} \frac{1}{4}(D-3)(3D-8)(\phi_1+\phi_2+\phi_3) & \tau_1 & \tau_2 & \tau_3 \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ \frac{1}{4}(D-3)(3D-8)\phi_2 & h_{132} & 2m_2^2\sigma_2 & h_{312} \\ \frac{1}{4}(D-3)(3D-8)\phi_1 & 2m_1^2\sigma_1 & h_{231} & h_{321} \end{pmatrix},$$

where

$$
\begin{aligned}
\tau_i &= h_{ijk} + h_{ikj} + 2m_i^2 \sigma_i \\
\widetilde{\tau}_i &= \widetilde{h}_{ijk} + \widetilde{h}_{ikj} + 2m_i^2 \widetilde{\sigma}_i \\
\partial_i P &= \frac{\partial}{\partial m_i^2} P \\
a_{2,1} &= \frac{(D-3)(3D-8)}{4P} \left(-\frac{2}{m_1^2}\, \widetilde{\tau}_1 \rho + \widetilde{\tau}_2 \phi_3 + \widetilde{\tau}_3 \phi_2 - \frac{1}{4} (\partial_1 P) \phi_1 \right) \\
a_{2,2} &= \frac{1}{2P} \left(\frac{(D-3)}{m_1^2} \widetilde{\tau}_1 P + \widetilde{\tau}_1 S_1 + 2\widetilde{\tau}_2 h_{123} + \widetilde{\tau}_3 h_{132} - m_1^2 \sigma_1 (\partial_1 P) \right) \\
&\quad + \frac{1}{4} (D-4)(3D-10)(\phi_1 + \phi_2 + \phi_3) \\
a_{2,3} &= \frac{1}{4P} \left(2 \frac{m_2^2}{m_1^2} \widetilde{\tau}_1 S_2 + 4\widetilde{\tau}_2 h_{213} + 8m_2^2 \widetilde{\tau}_3 \sigma_2 - (\partial_1 P) h_{231} \right) \\
a_{2,4} &= \frac{1}{4P} \left(2 \frac{m_3^2}{m_1^2} \widetilde{\tau}_1 S_3 + 4\widetilde{\tau}_3 h_{312} + 8m_3^2 \widetilde{\tau}_2 \sigma_3 - (\partial_1 P) h_{321} \right)
\end{aligned}
$$

and

$$
\widetilde{R} = \frac{1}{2P} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix}
$$

where

$$
\begin{aligned}
R_1 &= -\frac{1}{2} \Big((\partial_1 P)\, S(D,0,2,2) + (\partial_2 P)\, S(D,2,0,2) + (\partial_3 P)\, S(D,2,2,0) \Big), \\
R_2 &= \frac{1}{P} \left(\frac{m_2^2 m_3^2}{m_1^2} \widetilde{\tau}_1 \phi_1 + m_3^2 \widetilde{\tau}_2 \phi_2 + m_2^2 \widetilde{\tau}_3 \phi_3 + \frac{1}{2} (\partial_1 P) \rho \right) S(D,0,2,2) \\
&\quad + \frac{1}{P} \left(m_3^2 \widetilde{\tau}_1 \phi_2 + m_3^2 \widetilde{\tau}_2 \phi_1 - \frac{m_1^2}{4} (\partial_1 P) \phi_3 - 2\widetilde{\tau}_3 \rho \right) S(D,2,0,2) \\
&\quad + \frac{1}{P} \left(m_2^2 \widetilde{\tau}_1 \phi_3 - 2\rho \widetilde{\tau}_2 + m_2^2 \widetilde{\tau}_3 \phi_1 - \frac{m_1^2}{4} (\partial_1 P) \phi_2 \right) S(D,2,2,0) \\
&\quad - (\partial_2 P)\, S(D,3,0,2) - (\partial_3 P)\, S(D,3,2,0), \\
R_3 &= m_2^2 \phi_3\, S(D,0,2,2) - 2\rho\, S(D,2,0,2) + m_2 \phi_1\, S(D,2,2,0), \\
R_4 &= -2\rho\, S(D,0,2,2) + m_1^2 \phi_3\, S(D,2,0,2) + m_1 \phi_2\, S(D,2,2,0).
\end{aligned}
$$

Inverting the matrix $\widetilde{A}$, we obtain the desired equation for $S(D,\Lambda)$. The explicit expressions are rather lenghty and not given here. Setting $D = 4$, we have expressed $S(4,\Lambda)$ as a rational combination of the integrals $S(2,\Lambda)$, $S_1(2,\Lambda)$, $\widetilde{S_2}(4,\Lambda)$ and $\widetilde{S_3}(4,\Lambda)$ and simpler integrals. One checks, that here no spurious poles appear. The simpler integrals are known to all orders. An analytic solution for $S(2,\Lambda)$ (the zero–order term in the Laurent expansion around two) has been obtained in [ABW13] by solving the differential equation we have discussed in the previous section and which was first published in [MWZ12]. Therefore, also its mass derivative $S_1(2,\Lambda)$ is known. To obtain a complete answer we need in addition $\widetilde{S_2}(4,\Lambda)$ analytically. The last integral $\widetilde{S_3}(4,\Lambda)$ can then be obtained by interchanging masses. To fill this last gap one simply has to compute the differential equation of the convergent integral $\widetilde{S_2}(4,\Lambda)$ with the methods described in the previous sections and solve this equation like in [ABW13]. From that $S(4-2\varepsilon,\Lambda)$ would be known up to and including the zero–order term in the Laurent expansion.

5.6 A Three–Loop Example

The method we presented in the previous sections is not restricted to loop–order two. In principle it can be used in any loop–order. Let us give a three–loop example and discuss the three–loop banana graph depicted in fig. 5.3. We assume as before that all masses are positive so that we do not have to deal with IR–singularites.
In Feynman parameters the integrals associated to the three–loop banana graph read

$$B(D,\Lambda,\nu) = \int_\sigma \frac{x_1^{\nu_1-1}x_2^{\nu_2-1}x_3^{\nu_3-1}\mathcal{U}_B^{\overline{\nu}-(\ell+1)\frac{D}{2}}}{\mathcal{F}_B^{\overline{\nu}-\ell\frac{D}{2}}}\Omega,$$

with

$$\begin{aligned}
\mathcal{U} &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4,\\
\mathcal{F} &= -tx_1x_2x_3x_4 + (m_1^2x_1 + m_2^2x_2 + m_3^2x_3 + m_4^2x_4)\mathcal{U}.
\end{aligned}$$

We derive a differential equation for $B(D,\Lambda) := B(D,\Lambda,1,1,1,1)$. We see from power counting, that it is divergent in dimension four and convergent in dimension two. We have

$$B(2,\Lambda) = \int_\sigma \frac{1}{\mathcal{F}_B}\Omega.$$

For simplicity let us assume that all masses are equal to one. We have given by $\mathcal{F}_B$ a family of K3–surfaces and we compute its homogeneous Picard–Fuchs equation. We obtain the differential operator

$$L = \partial^3 + \left(\frac{6(t^2-15t+32)}{t(t-4)(t-16)}\right)\partial^2 + \left(\frac{7t^2-68t+64}{t^2(t-4)(t-16)}\right)\partial + \frac{1}{t^2(t-16)}$$

and a differential form β, such that

$$L(B(2,\Lambda)) = \int_\sigma d\beta = \int_{\partial\sigma}\beta.$$

It is straightforward to integrate the right–hand side. Calling the result $g(t)$ one obtains as before the inhomogeneous equation

$$\left(\partial - \frac{g'(t)}{g(t)}\right)L(B(2,\Lambda)) = 0.$$

At this point one can repeat the procedure we have carried out for the two–loop sunrise graph in the previous sections.

5.7 The Sunrise Integral in General Dimension

In this section we reconsider the sunrise graph in general dimension D. Here, we can no longer expect a homogeneous differential equation of order two. This is due to the fact, that both graph hypersurfaces play a role in general dimension. Carrying out the procedure we have described in chapter 4, we observe something interesting. For general masses and dimension we find an operator $L^{(4)}$ of order four, such that

$$L^{(4)}\left(S(D,\Lambda,\nu)\right) = I,$$

where I is a sum of simpler integrals. Setting $D = 4 - 2\varepsilon$, we find a decomposition

$$L^{(4)} = L^{(2)}\left(\partial_t + \frac{2}{t}\right)\partial_t + \varepsilon L_\varepsilon^{(3)}, \tag{5.18}$$

where $L^{(2)}$ is a differential operator of order two that is independent of ε and $L_\varepsilon^{(3)}$ is a differential operator of order three. The coefficients of the operators $L^{(2)}$ and $L_\varepsilon^{(3)}$ are rather long and not given here explicitly.

Let us conclude this section by pointing out that the decomposition given by equation (5.18) is very useful for solving the differential equation. The operator $L_\varepsilon^{(3)}$ appears with an extra factor of ε in equation (5.18). When solving the equation order by order in ε one can regard $\varepsilon L_\varepsilon^{(3)}$ as part of the inhomogeneous term and one is reduced to solving a second order differential equation and two additional integrations coming from the linear differential operators.

Bibliography

[ABW13] L. Adams, C. Bogner, S. Weinzierl, *The two-loop sunrise graph with arbitrary masses*, Journal of Mathematical Physics 54 (2013), 052303.

[AL04] C. Anastasiou and A. Lazopoulos, *Automatic integral reduction for higher order perturbative calculations*, JHEP, 2004(07), 46.

[Ash72] J. F. Ashmore, *A method of gauge–invariant regularization*, Nuovo Cimento Lett. 4.8 (1972), 289-290.

[BB03a] P. Belkale and P. Brosnan, *Matroids, motives and a conjecture of Kontsevich*, Duke Math. Journal, Vol. 116 (2003), 147-188

[BB03b] P. Belkale and P. Brosnan, *Periods and Igusa local zeta functions*, International Mathematics Research Notices 2003.49, (2003), 2655-2670

[BEK06] S. Bloch, H. Esnault, D. Kreimer: *On motives associated to graph polynomials*, Comm. Math. Phys. 267, (2006), no. 1, 181-225

[BFT93] D. J. Broadhurst, J. Fleischer, O. V. Tarasov, *Two-loop two-point functions with masses: Asymptotic expansions and Taylor series, in any dimension*, Zeitschrift für Physik C Particles and Fields 60.2, (1993), 287-301

[BG72] C.G. Bollini and J.J. Giambiagi, *Lowest order divergent graphs in ν–dimensional space*, Physics Letters B 40.5 (1972), 566-568.

[BH00] T. Binoth and G. Heinrich, *An automatized algorithm to compute infrared divergent multi-loop integrals*, Nuclear Physics B 585.3, (2000), 741-759

[BH04] T. Binoth and G. Heinrich, *Numerical evaluation of multi-loop integrals by sector decomposition*, Nuclear Physics B 680.1, (2004), 375-388

[BK95] D. J. Broadhurst and D. Kreimer, *Knots and numbers in ϕ^4 theory to 7 loops and beyond*, Int. J. Mod. Phys. C 6, (1995), 519

[BK97] D. J. Broadhurst and D. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, Phys. Lett. B 393 (3-4), (1997), 403-412

[BK08] S. Bloch and D. Kreimer, *Mixed Hodge Structures and Renormalization in Physics*, Comm. Num. Theor. Phys. 2.4, (2008), 637-718

[BP02] J. Bertin and C. Peters, *Variations of Hodge Structures, Calabi-Yau Manifolds, and Mirror Symmetry*, Introduction to Hodge Theory, (2002).

[Bro09a] F. Brown, *The massless higher–loop two–point function*, Comm. in Math. Physics 287, no. 3, (2009), 925-958.

[Bro09b] F. Brown, *On the periods of some Feynman integrals*, arXiv:0910.0114v1, (2009), 1-69

[BS12] F. Brown and O. Schnetz, *Proof of the zig-zag conjecture*, arXiv preprint arXiv:1208.1890, (2012)

[BW03] I. Bierenbaum and S. Weinzierl, *The massless two–loop two–point function*, Eur. Phys. J. C32, (2003), 67

[BW08] C. Bogner and S. Weinzierl, *Resolution of singularities for multi-loop integrals*, Computer Physics Communications 178.8, (2008), 596-610

[BW09] C. Bogner and S. Weinzierl, *Periods and Feynman integrals*, J. Math. Phys. 50, (2009), 042302

[BW10] C. Bogner and S. Weinzierl, *Feynman graph polynomials*, International Journal of Modern Physics A 25.13, (2010), 2585-2618

[BZ97] J.-L. Brylinski and S. Zucker, *An overview of recent advances in Hodge theory*, Several complex variables, VI 69, (1997), 39-142

[CCLR98] M. Caffo, H. Czyz, S. Laporta, E. Remiddi, *The Master Differential Equations for the 2–loop Sunrise Selfmass Amplitudes*, Nuovo Cim. A 111, (1998), 365

[CK00] D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry (mathematical surveys and monographs)*, AMS, 2000.

[CM72] G. M. Cicuta and E. Montaldi, *Analytic renormalization via continuous space dimension*, Nuovo Cimento Lett. 4.9 (1972), 329-332.

[Col84] J. Collins, *Renormalization*, Cambridge University Press, 1984.

[CT81] K. G. Chetyrkin and F. V. Tkachov, *Integration by parts: the algorithm to calculate β-functions in 4 loops* Nuclear Physics B 192.1, (1981), 159-204

[Del70] P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*, Lecture notes in Mathematics, vol. 163, Springer–Verlag, Heidelberg, 1970.

[Dor08] D. Doryn, *Cohomology of graph hypersurfaces associated to certain Feynman graphs*, arXiv preprint, arXiv:0811.0402 (2008).

[Eti00] P. Etingof, *Note on dimensional regularization*, Quantum fields and strings: a course for mathematicians 1, (2000), 597-607

[Gri69] P. A. Griffiths, *On the periods of certain rational integrals: I*, The Annals of Mathematics 90.3, (1969), 460-495

[Hae87] A. Haefliger, *Local theory of meromorphic connections in dimension one (Fuchs theory)*, Borel et al., Algebraic D-Modules, (1987), 129-149.

[Hir64] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Annals of Math. 79.2 (1964), 205-326

[HM12] A. Huber and S. Müller–Stach, *On the relation between nori motives and Kontsevich periods*, arXiv preprint, arXiv:1105.0865 (2011).

[HV72] G. 't Hooft and M. Veltman, *Regularization and renormalization of gauge fields*, Nuclear Physics B 44.1, (1972), 189-213.

[IZ87] C. Itzykson and J.-B. Zuber, *Quantum field theory*, Dover Publications, 2006.

[Kak93] M. Kaku, *Quantum field theory: a modern introduction*, Vol. 5., Oxford, UK: Oxford University Press, 1993.

[Kon99] M. Kontsevich, *Operads and motives in deformation quantization*, Letters in Mathematical Physics 48.1 (1999), 35-72.

[KZ01] M. Kontsevich and D. Zagier, *Periods*, Mathematics unlimited2001 and beyond, (2001), 771808

[Lap00] S. Laporta, *High-precision calculation of multiloop Feynman integrals by difference equations*, International Journal of Modern Physics A 15.32, (2000), 5087-5159

[Lee10] R. N. Lee, *Space–time dimensionality D as complex variable: Calculating loop integrals using dimensional recurrence relation and analytical properties with respect to D*, Nuclear Physics B 830.3, (2010), 474-492

[LR04] S. Laporta and E. Remiddi, *Analytic treatment of the two loop equal mass sunrise graph*, Nucl. Phys. B 704, (2005), 349

[Mal87] B. Malgrange, *Regular connections, after Deligne*, Algebraic D-modules, (1987), 151-172.

[Mey02] R. Meyer, *Dimensional Regularization*, Lecture at the workshop Theory of Renormalization and Regularization, Hesselberg, (2002), 1-9

[MWZ12] S. Müller–Stach, S. Weinzierl, R. Zayadeh, *A second–order differential equation for the two–loop sunrise graph with arbitrary masses*, Comm. Num. Theor. Phys. 6, (2012), 203

[MWZ13] S. Müller–Stach, S. Weinzierl, R. Zayadeh, *Picard–Fuchs equations for Feynman integrals*, to appear in Comm. Math. Phys., (2013), arXiv:1212.4389

[PS95] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Addison–Wesley, 1995.

[PS03] M. van der Put and M. F. Singer, *Galois Theory of Linear Differential Equations*, Grundlagen der Mathematischen Wissenschaften, vol. 328, Springer, Berlin, 2003.

[PS08] C. Peters and J. Steenbrink, *Mixed Hodge Structures*, Springer, 2008.

[PV79] G. Passarino and M. Veltman, *One-loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg model*, Nuclear Physics B 160.1, (1979), 151-207

[Ser56] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier 6, (1956), 1-42

[Smi04] V. A. Smirnov, *Evaluating Feynman Integrals*, (Springer Tracts Mod. Phys. 211) Springer, Berlin, Heidelberg, 2004.

[Smi08] A. V. Smirnov, *Algorithm FIRE – Feynman integral reduction*, JHEP, 2008(10), 107.

[SP10] A. V. Smirnov and A. V. Petukhov, *The number of master integrals is finite*, Letters in Math. Phys. 97.1, (2011), 37-44

[Stu10] C. Studerus, *Reduze - Feynman integral reduction in C++*, Computer Phys. Comm. 181.7, (2010), 1293-1300

[SW71] E.R. Speer and M.J. Westwater, *Generic Feynman amplitudes*, Ann. Inst. Henri Poincaré A14, No.1, (1971), 1-55

[SZ85] J. Steenbrink and S. Zucker, *Variation of mixed Hodge structure. I.*, Inventiones Mathematicae 80.3, (1985), 489-542

[Tar96] O. V. Tarasov, *Connection between Feynman integrals having different values of the space-time dimension*, Physical Review D 54.10, (1996), 6479

[Tar97] O. V. Tarasov, *Generalized recurrence relations for two-loop propagator integrals with arbitrary masses*, Nuclear Physics B 502.1, (1997), 455-482

[Tka81] F. V. Tkachov, *A theorem on analytical calculability of 4-loop renormalization group functions*, Physics Letters B 100.1, (1981), 65-68

[Wei10] S. Weinzierl, *Introduction to Feynman Integrals*, arXiv:hep-ph/1005.1855

Abstract

This thesis is devoted to studying differential equations of Feynman integrals. A Feynman integral depends on a dimension D. For integer values of D it can be written as a projective integral, which is called the Feynman parameter prescription. A major complication arises from the fact that for some values of D the integral can diverge. This problem is solved within dimensional regularization by continuing the integral as a meromorphic function on the complex plane and replacing the ill–defined quantity by a Laurent series in a dimensional regularization parameter. All terms in such a Laurent expansion are periods in the sense of Kontsevich and Zagier.
We describe a new method to compute differential equations of Feynman integrals. So far, the standard has been to use integration–by–parts (IBP) identities to obtain coupled systems of linear differential equations for the master integrals. Our method is based on the theory of Picard–Fuchs equations. In the case we are interested in, that of projective and quasiprojective families, a Picard–Fuchs equation can be computed by means of the Griffiths–Dwork reduction. We describe a method that is designed for fixed integer dimension. After a suitable integer shift of dimension we obtain a period of a family of hypersurfaces, hence a Picard–Fuchs equation. This equation is inhomogeneous because the domain of integration has a boundary and we only obtain a relative cycle. As a second step we shift back the dimension using Tarasov's generalized dimensional recurrence relations.
Furthermore, we describe a method to directly compute the differential equation for general D without shifting the dimension. This is based on the Griffiths–Dwork reduction. The success of this method depends on the ability to solve large systems of linear equations.
We give examples of two and three–loop graphs. Tarasov classifies two–loop two–point functions and we give differential equations for these. For us the most interesting example is the two–loop sunrise integral with arbitrary fixed masses and varying momentum. It was previously known not to evaluate to multiple polylogarithms, but an analytic answer could not be obtained. Its geometric and number theoretic content is governed by a family of elliptic curves. We provide an inhomogeneous Picard–Fuchs equation which in the meantime lead to an analytic answer of the two–loop sunrise integral. We give a three–loop example where we find a family of K3–surfaces.

Zusammenfassung

In der vorliegenden Arbeit beschäftige ich mich mit Differentialgleichungen von Feynman–Integralen. Ein Feynman–Integral hängt von einem Dimensionsparameter D ab und kann für ganzzahlige Dimension als projektives Integral dargestellt werden. Dies ist die sogenannte Feynman–Parameter Darstellung. In Abhängigkeit der Dimension kann ein solches Integral divergieren. Als Funktion in D erhält man eine meromorphe Funktion auf ganz $\mathbb{C}$. Ein divergentes Integral kann also durch eine Laurent–Reihe ersetzt werden und dessen Koeffizienten rücken in das Zentrum des Interesses. Diese Vorgehensweise wird als dimensionale Regularisierung bezeichnet. Alle Terme einer solchen Laurent–Reihe eines Feynman–Integrals sind Perioden im Sinne von Kontsevich und Zagier.
Ich beschreibe eine neue Methode zur Berechnung von Differentialgleichungen von Feynman–Integralen. Üblicherweise verwendet man hierzu die sogenannten "integration by parts" (IBP)–Identitäten. Die neue Methode verwendet die Theorie der Picard–Fuchs–Differentialgleichungen. Im Falle projektiver oder quasi–projektiver Varietäten basiert die Berechnung einer solchen Differentialgleichung auf der sogenannten Griffiths–Dwork–Reduktion.
Zunächst beschreibe ich die Methode für feste, ganzzahlige Dimension. Nach geeigneter Verschiebung der Dimension erhält man direkt eine Periode und somit eine Picard–Fuchs–Differentialgleichung. Diese ist inhomogen, da das Integrationsgebiet einen Rand besitzt und daher nur einen relativen Zykel darstellt. Mit Hilfe von dimensionalen Rekurrenzrelationen, die auf Tarasov zurückgehen, kann in einem zweiten Schritt die Lösung in der ursprünglichen Dimension bestimmt werden.
Ich beschreibe außerdem eine Methode, die auf der Griffiths–Dwork–Reduktion basiert, um die Differentialgleichung direkt für beliebige Dimension zu berechnen. Diese Methode ist allgemein gültig und erspart Dimensionswechsel. Ein Erfolg der Methode hängt von der Möglichkeit ab, große Systeme von linearen Gleichungen zu lösen.
Ich gebe Beispiele von Integralen von Graphen mit zwei und drei Schleifen. Tarasov gibt eine Basis von Integralen an, die Graphen mit zwei Schleifen und zwei externen Kanten bestimmen. Ich bestimme Differentialgleichungen der Integrale dieser Basis. Als wichtigstes Beispiel berechne ich die Differentialgleichung des sogenannten Sunrise–Graphen mit zwei Schleifen im allgemeinen Fall beliebiger Massen. Diese ist für spezielle Werte von D eine inhomogene Picard–Fuchs–Gleichung einer Familie elliptischer Kurven. Der Sunrise–Graph ist besonders interessant, weil eine analytische Lösung erst mit dieser Methode gefunden werden konnte, und weil dies der einfachste Graph ist, dessen Master–Integrale nicht durch Polylogarithmen gegeben sind. Ich gebe außerdem ein Beispiel eines Graphen mit drei Schleifen. Hier taucht die Picard–Fuchs–Gleichung einer Familie von K3–Flächen auf.