Convergence results for stochastic particle systems with social interaction

Dissertation zur Erlangung des Grades
Doktor der Naturwissenschaften

am Fachbereich Physik, Mathematik und Informatik
der Johannes Gutenberg-Universität in Mainz

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Mainz 2013

Datum der mündlichen Prüfung: 21. Januar 2014

D77 - Mainzer Dissertation

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Abstract

We consider stochastic individual-based models for social behavior of groups of N animals. In these models the trajectory of each animal is given by a stochastic differential equation with interaction. The social interaction is contained in the drift term of the SDE. We consider a global aggregation force and a short-range repulsion force. The repulsion range and strength gets rescaled with the number of animals N. We show that for N tending to infinity stochastic fluctuations disappear and a smoothed version of the empirical process converges uniformly in $L^2(\mathbb{R}^d)$ towards the solution of a nonlinear, nonlocal partial differential equation of advection-reaction-diffusion type. The rescaling of the repulsion in the individual-based model implies that the corresponding term in the limit equation is local while the aggregation term is non-local. Moreover, we discuss the effect of a predator on the system and derive an analogous convergence result. The predator acts as an repulsive force. Different laws of motion for the predator are considered. Finally, some simulations of individual-based systems with predator are shown.

Zusammenfassung

Wir betrachten stochastische Modelle für das Sozialverhalten von Gruppen von N Tieren, die auf einzelnen Individuen basieren. Die Trajektorie jedes einzelnen Tieres wird durch eine stochastische Differentialgleichung mit Interaktion beschrieben. Die soziale Interaktion ist im Driftterm der SDGL enthalten. Wir betrachten eine globale Aggregationskraft und eine Repulsionskraft mit kurzer Reichweite. Die Stärke und die Reichweite der Repulsion werden mit der Anzahl N der Tiere in der Gruppe reskaliert. Wir zeigen, dass für N gegen unendlich die stochastischen Fluktuationen verschwinden und eine geglättete Version des empirischen Prozesses gleichmäßig in $L^2(\mathbb{R}^d)$ gegen die Lösung einer nicht-linearen, nicht-lokalen partiellen Differentialgleichung vom Typ einer Advektions-Reaktions-Diffusionsgleichung konvergiert. Wegen der Reskalierung der Repulsion im individuenbasierenden Modell ist der entsprechende Term in der Differentialgleichung für das Kontinuummodell lokal, während der Aggregationsterm nicht lokal bleibt. Darüber hinaus untersuchen wir den Effekt eines Räubers auf das System und leiten ein entsprechendes Konvergenzresultat her. Der Räuber wirkt als abstoßende Kraft. Wir betrachten verschiedene Bewegungsgesetze für den Räuber. Zum Schluss zeigen wir einige Simulationsergebnisse für die individuenbasierenden Modelle mit Räuber.

Introduction

In the present thesis we discuss an individual-based (Lagrangian) model for social behavior of groups of animals. In [18] and [19] Oelschläger, Morale and Capasso proposed such an individual-based model describing the movement of animals in a social group. The movement of the animals is given by a family of stochastic differential equations with interaction. The equations are driven by independent Brownian motions. On the one hand, animals in a group tend to aggregate. On the other hand, there is a local repulsion between animals when they get too close. The drift term of the equations is used to model social behavior in the group such as these aggregation and repulsion effects. Aggregation is given by an interaction kernel G of McKean-Vlasov-Type and repulsion is given by some rescaled kernel function. The range of the repulsion kernel V_N decreases by a factor $N^{\beta/d}$ as the number of animals N in the group increases. Here, d denotes the dimension of the underlying space and $\beta \in (0,1)$ is a suitable constant.

Due to large computational demands individual-based models are not very useful to study the behavior of the system for large numbers of animals N. Therefore, our interest is to show that for N tending to infinity the stochastic fluctuations disappear and the particle density converges towards the solution ρ of a non-linear and non-local partial differential equation of advection-reaction-diffusion type

$$\begin{aligned} \partial_t \rho(x,t) = \frac{\sigma_\infty^2}{2}\Delta\rho(x,t) &+ \nabla\cdot(\rho(x,t)\nabla\rho(x,t)) \\ &- \nabla\cdot(\rho(x,t)(\nabla G * \rho(\cdot,t))(x)), \\ \rho(x,0) = \rho_0(x). & \end{aligned} \tag{1}$$

As a result of the rescaling of the repulsion kernel the coefficient corresponding to the repulsive force is local while the coefficient corresponding to the aggregation force remains non-local in the limit. Continuous models of non-linear and non-local advection-reaction-diffusion type for social groups of animals are considered in mathematics and biology by various authors, see for example Grünbaum [9], chapter 7 of Okubo and Levin [24], Mogilner and

Edelstein-Keshet [17] and the references specified on p.311 of [10].

In this thesis we give a L^2-convergence result for a smoothed version h_N of the empirical process of the individual-based system. To be more precise, we will show that under certain technical assumptions, see chapter 1.5, the following theorem holds.

Main theorem.

$$\lim_{N\to\infty} \mathbb{E} \sup_{t\leq T} \|h_N(\cdot,t) - \rho(\cdot,t)\|_{L^2}^2 = 0. \tag{2}$$

The precise statement of our main result can be found in theorem 2.1.7. This result is obtained by applying Gronwall's lemma to the time evolution of $\|h_N(t) - \rho(t)\|^2$. This method was already used by Oelschläger et al in [18], see also [22], [21], [23]. In [18] the existence of a sufficiently regular solution of (1) was posed as an assumption. Using weaker assumptions on our model, we give, in chapter 2, a proof of our main theorem. Furthermore, in case a diffusion remains in the limit ($\sigma_\infty > 0$), we give a rigorous proof for the existence of the suitable solution ρ of the continuum model. Convergence results of individual-based models towards continuum models for N to infinity can also be obtained by different means. For another approach for limit results of individual-based models see Sznitman [28] or Benachour, Roynette, Talay and Vallois [2].

As further generalization the influence of a predator on the animal swarm is discussed in this thesis. The predator acts as repulsive non-local force on all other animals in the group. The effect of this repulsion is that the animals try to get to a safe distance from the predator. We consider deterministic and stochastic laws of motion for the predator, see chapter 2.2. Analogous convergence results (see theorem 2.2.5 and 2.2.9) are derived in these cases.

As already mentioned, in case $\sigma_\infty > 0$ we prove the existence of a solution of (1). The existence proof is based on a classical semigroup approach and a fixed point iteration, see Kato [16]. The major mathematical difficulty in this framework is to show the stability of the corresponding evolution system. This method can easily be generalized and applied to the predator system, see chapter 3.4.

In chapter 1 we give a detailed discussion of our individual-based model and its continuum limit equation. Furthermore, we discuss the influence of a predator on the behavior of the system. In 1.3 a short heuristic derivation of the limit equation (1) is given (cf. Morale, Capasso and Oelschläger [19]). Moreover, we present some elementary results and lemmas used in the subsequent chapters. The proofs of our convergence results can be found in chapter 2. In chapter 3 we give the existence proofs for the solutions of the continuum advection-reaction-diffusion equations for the model with and without predator. Finally, in chapter 4 we discuss some simulation results for individual-based models with predator.

Chapter 1

A detailed description of the model

1.1 Lagrangian and Eulerian models for animal swarming

There are two essentially different approaches to modeling social groups of animals. One class of models is individual-based. The movement of every animal is calculated separately, according to a stochastic differential equation

$$dX_N^k = F[X_N^1, \dots, X_N^N]dt + \sigma_k d\mathbb{W}_t, \quad k = 1, \dots, N,$$

where the function $F[X_N^1, \dots, X_N^N]$ may contain social behavior and environmental constraints. Models of this type are called *Lagrangian* models, *individual-based* models or *stochastic* models.

A different approach is to describe the density distribution of animals using a partial differential equation. This type of model is called *Eulerian* model or *continuum* model. A typical partial differential equation for modeling social behavior is the advection-reaction-diffusion equation:

$$\partial_t \rho = \frac{\partial}{\partial x}\left(D\frac{\partial}{\partial x}\rho\right) - \frac{\partial}{\partial x}(u\rho) + R, \tag{1.1}$$

where D is a diffusivity tensor, u is the advection velocity and R is a vector which is used to model local effects on the particle density (see [10, p. 311]). The terms D, u and R may depend on ρ and thus equation (1.1) may be non-linear and non-local. In fact, non-linear terms are essential for modeling reasonable social behavior.

Lagrangian models are well-suited to describe small groups of animals over short periods of time. Due to computational requirements these models are not useful for larger groups. However, Eulerian models can effectively be applied to large groups of animals, but it is not possible to keep track of the movement of individual animals. A detailed discussion of Eulerian and Lagrangian models can be found in [10].

In this chapter we give an introduction to an individual-based model for animal swarming and all technical assumptions required to derive an Eulerian limit equation. This Eulerian model fits into the class (1.1). Furthermore, in chapter 2.2 we discuss the influence of a predator on the individual-based system and its continuum limit equation.

1.2 Particle Model

We give a short introduction to a slightly generalized version of the individual based model that was proposed and discussed in [18] and [19].

In this model we observe the spatial movement of a system of N particles (animals) in the space $\mathbb{R}^d, d \in \mathbb{N}$. Let $X_N^k(t), k = 1, \ldots, N$, denote the position of the k-th particle at time t. The particles are subject to a stochastic movement and a drift that is caused by mutual interaction (social behavior). Thus every X_N^k defines a $\mathbb{R}^d$-valued stochastic process.

1.2.1 Definition. The measure valued process X_N given by

$$X_N : t \mapsto \frac{1}{N} \sum_{k=1}^{N} \delta_{X_N^k(t)} \tag{1.2}$$

is called the *empirical process* of the N-particle system $(X_N^k)_{k=1,\ldots,N}$.

The movement of the individual particles is given via the following stochastic differential equations

$$dX_N^k(t) = F_N[X_N(t)](X_N^k(t))dt + \sigma_N d\mathbb{W}^k(t), \quad k = 1, \ldots N. \tag{1.3}$$

Here, $\mathbb{W}^k, k \in \mathbb{N}$, is a family of independent standard Brownian motions and σ_N is a non-negative sequence of constants depending only on the number of particles N such that

$$\lim_{N \to \infty} \sigma_N = \sigma_\infty. \tag{1.4}$$

The cases $\sigma_\infty > 0$ and $\sigma_\infty = 0$ lead to essentially different behavior in the limit N to ∞. If $\sigma_\infty = 0$, there is no diffusion term in the corresponding continuum model of the animal swarm.

The mutual interaction of the animals in the swarm is completely contained in the drift part F_N. The drift $F_N[X_N(t)](x)$ acting on a particle at position $x \in \mathbb{R}^d$ and time t is a function of the empirical process $X_N(t)$, i.e. the drift depends on the position of all particles in the system at time t. It can be splitted into multiple components. In the simplest case of interest we have a decomposition into two parts

$$F_N = F^A + F_N^R. \tag{1.5}$$

The first one F^A denotes the aggregation part and the second one denotes the repulsion part F_N^R. As the notation indicates only the repulsion part depends directly on the number of particles N. The aggregation depends only through the empirical process X_N on N. The aggregation is given by a function

$$G : \mathbb{R}^d \longrightarrow \mathbb{R}. \tag{1.6}$$

The function G is called the *potential of the aggregation force*. Later on, we will make some additional technical assumptions on G, see assumption (A6) in section 1.5. F^A can now be defined by

$$\begin{aligned} F^A[X_N(t)](X_N^k(t)) &= (\nabla G * X_N(t))(X_N^k(t)) \\ &= \frac{1}{N}\sum_{j=1}^{N} \nabla G(\cdot - X_N^j(t))(X_N^k(t)) \\ &= \frac{1}{N}\sum_{j=1}^{N} \nabla G(X_N^k(t) - X_N^j(t)),\ k = 1, \dots, N. \end{aligned} \tag{1.7}$$

In a very similar fashion the repulsion force is given by a kernel function

$$V_N : \mathbb{R}^d \longrightarrow [0, \infty). \tag{1.8}$$

The difference here is that the *potential of the repulsion force* V_N gets rescaled in the following way

$$V_N(x) = \chi_N^d V_1(\chi_N x) \tag{1.9}$$

with a scaling parameter

$$\chi_N = N^{\beta/d}, \quad N \in \mathbb{N}, \tag{1.10}$$

for fixed $\beta \in (0, 1)$ and V_1 is a probability density. Again, later on, we will require some additional assumptions on the function V_1 and the rescaling parameter β, see (A2), (A4) and (A5).

1.2.2 Remark. Observe that for every $N \in \mathbb{N}$ the rescaled density V_N is still a probability density. This is a direct consequence of the substitution rule

$$\int_{\mathbb{R}^d} \chi_N^d V_1(\chi_N x)\, dx = \int_{\mathbb{R}^d} V_1(y)\, dy = 1.$$

Now, we can define the repulsion term

$$F_N^R[X_N(t)](X_N^k(t)) = -(\nabla V_N * X_N(t))(X_N^k(t)). \tag{1.11}$$

The rescaling of the potential implies that the range of the repulsive force gets smaller as the number of particles N grows. Thus in the limit N to infinity there remains only a local repulsion force, while the aggregation force is a non-local term.

The names aggregation and repulsion are purely motivated by the biological interpretation of the model. In fact, the aggregation potential G may act repulsive and the repulsion potential may act aggregative on animals in the group.

In the next chapter we give an exact statement and a proof for our main convergence result, see theorem 2.1.7.

Observe that the solution of the limit partial differential equation (1) does not depend on the explicit choice of the function V_1. The limit equation (1) describes a multidimensional continuum model for animal swarming of type (1.1) with $D = \frac{\sigma_\infty^2}{2} + \rho$ and $u = \nabla G * \rho$. Furthermore, in the diffusion case ($\sigma_\infty \neq 0$) we can give a sufficient condition for the the existence of a solution of this cauchy problem (1). This is done in chapter 3 using a fixed point iteration and general semigroup theory.

In the case $\sigma_\infty = 0$ the same convergence theorem holds. Since we have no diffusion part in the Eulerian limit equation, equation (1) reads

$$\begin{aligned} \partial_t \rho(x,t) &= \nabla \cdot (\rho(x,t)\nabla\rho(x,t)) - \nabla \cdot (\rho(x,t)(\nabla G * \rho(\cdot,t))(x)) \\ \rho(x,0) &= \rho_0(x). \end{aligned} \tag{1.12}$$

All other parts of the main theorem remain unchanged. In (1.12) we have no uniform ellipticity. Since ρ is in $L^2(\mathbb{R}^d)$ and nonnegative the equation is degenerated elliptic. Therefore, we can not apply our proof from chapter 3 to show the existence of a solution in this case.

A further extension of these results will be presented in section 2.2 where we add a predator who acts as an additional repulsive force on all other animals in the swarm, i.e., we add a repulsive potential H to the interaction F_N. We consider different laws of motion for the predator particle. The existence result for the solution of the continuum partial differential equation can easily be generalized to this setting.

The following table summarizes the parts in the particle equation and the corresponding parts in the limit equations for the various parts of the particle model.

	particle equation	limit equation
local repulsion	$-(\nabla V_N * X_N(t))(X_N^k(t))$	$\nabla \cdot (\rho(t)\nabla\rho(t))$
global aggregation	$(\nabla G * X_N(t))(X_N^k(t))$	$-\nabla \cdot (\rho(t)(\nabla G * \rho(t)))$
predator action	$-(\nabla H * \delta_{P_N(t)})$	$(\nabla H * \delta_{P_\infty(t)})$
diffusion	$\sigma_\infty \neq 0$	$\frac{\sigma_\infty}{2}\Delta\rho(t)$

1.3 A heuristic derivation of the continuum limit

Let us consider a system without a predator and derive the continuum limit equation in a heuristic way. This heuristic derivation of the limit was given by Morale, Capasso and Oelschläger in [19]. In chapter 2 we give a rigorous proof for the convergence of the empirical process against this limit and thus, legitimize this calculation.

Let $f \in C_b^2(\mathbb{R}^d \times \mathbb{R})$. Using Itô's formula, equation (1.3) and the independence of the components of the Brownian motions $\mathbb{W}^k, k \in \mathbb{N}$, we get

$$\begin{aligned} f(X_N^k(t),t) = f(X_N^k(0),0) &+ \int_0^t F_N[X_N(s)](X_N^k(s)) \cdot \nabla f(X_N^k(s),s)\, ds \\ &+ \int_0^t (\partial_s f)(X_N^k(s),s)\, ds \\ &+ \int_0^t \frac{\sigma_N^2}{2}\Delta f(X_N^k(s),s)\, ds \\ &+ \sigma_N \int_0^t (\nabla f)(X_N^k(s),s)\, d\mathbb{W}^k(s). \end{aligned}$$

Therefore, taking the sum over k and dividing by N gives us

$$\begin{aligned} \langle X_N(t), f\rangle = \langle X_N(0), f(\cdot,0)\rangle &+ \int_0^t \langle X_N(s), F_N[X_N(s)](\cdot)\cdot\nabla f(\cdot,s)\rangle\, ds \\ &+ \int_0^t \langle X_N(s), (\partial_s f)(\cdot,s)\rangle\, ds \\ &+ \int_0^t \frac{\sigma_N^2}{2}\langle X_N(s), \Delta f(\cdot,s)\rangle\, ds \\ &+ \int_0^t \frac{\sigma_N}{N}\sum_{k=1}^N (\nabla f)(X_N^k(s),s)\, d\mathbb{W}^k(s). \end{aligned} \tag{1.13}$$

Now, let us for the moment assume that the martingale part

$$\int_0^t \frac{\sigma_N}{N} \sum_{k=1}^{N} (\nabla f)(X_N^k(s), s)\, d\mathbb{W}^k(s)$$

vanishes for N to infinity. A rigorous argument using Burkholder-Davis-Gundy inequality is given in lemma 2.1.4 (e). Furthermore, we assume that the empirical process X_N converges to a process, whose distribution has density ρ with respect to the Lebesgue measure such that

$$\lim_{N\to\infty} \langle X_N(t), f(\cdot,t)\rangle = \int_{\mathbb{R}^d} \rho(x,t) f(x,t)\, dx.$$

Because of $V_N(x) = \chi_N^d V_1(\chi_N x)$ we formally have $V_N \to \delta_0$. This formally implies

$$\lim_{N\to\infty} (\nabla V_N * X_N(t))(x) = \nabla\rho(x,t),$$
$$\lim_{N\to\infty} (\nabla G * X_N(t))(x) = \nabla G * \rho(x,t).$$

Since we consider the case without predator, we have

$$F_N[X_N(s)](\cdot) = \nabla G * X_N(s) - \nabla V_N * X_N(s)$$

and this formally converges against $\nabla G * \rho(\cdot,s) - \nabla\rho(\cdot,s)$. Thus, letting $N \to \infty$ in equation (1.13), we obtain

$$\begin{aligned}\langle \rho(\cdot,t), f(\cdot,t)\rangle = \langle \rho_0, f(\cdot,0)\rangle & \\ &+ \int_0^t \langle \rho(\cdot,s), (\nabla G * \rho(\cdot,s) - \nabla\rho(\cdot,s)) \cdot \nabla f(\cdot,s)\rangle\, ds \\ &+ \int_0^t \langle \rho(\cdot,s), (\partial_s f)(\cdot,s)\rangle\, ds \\ &+ \int_0^t \frac{\sigma_\infty^2}{2} \langle \rho(\cdot,s), \Delta f(\cdot,s)\rangle\, ds.\end{aligned} \tag{1.14}$$

Partial integration gives us

$$\int_0^t \langle \rho(\cdot,s), (\partial_s f)(\cdot,s)\rangle\, ds = \langle \rho(\cdot,t), f(\cdot,t)\rangle - \langle \rho_0, f(\cdot,0)\rangle - \int_0^t \langle \partial_s \rho(\cdot,s), f(\cdot,s)\rangle\, ds.$$

Therefore, we get from (1.14)

$$\int_0^t \langle \partial_s \rho(\cdot, s), f(\cdot, s)\rangle \, ds = \int_0^t \langle \nabla \cdot (\rho(\cdot, s)(\nabla \rho(\cdot, s) - \nabla G * \rho(\cdot, s)), f(\cdot, s)\rangle \, ds + \int_0^t \frac{\sigma_\infty^2}{2} \langle \Delta \rho(\cdot, s), f(\cdot, s)\rangle \, ds.$$

This a weak version of the continuum limit equation (1).

1.4 Notations and results from Fourier-Analysis

During this section, we introduce our notations and present a few elementary results from basic fourier analysis. Proofs can be found in most standard textbooks on partial differential equations.

1.4.1 Definition. For all functions $f \in L^1(\mathbb{R}^d)$ we define the *Fourier transform of f* by

$$\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} \, dx. \tag{1.15}$$

1.4.2 Remark. As usually, using Parseval's identity, the Fourier transform can be extended to a linear isomorphism on $L^2(\mathbb{R}^d)$ and on the space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions.

During this thesis, we make frequently use of this and many other properties of the Fourier transform.

1.4.3 Definition. For $m \in \mathbb{R}$ the space

$$H^m(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d) \mid \|(1 + |\xi|^2)^{m/2} \hat{f}(\xi)\|_{L^2(\mathbb{R}^d)} < \infty\}$$

is called (classical) *sobolev space* of order m. $H^m(\mathbb{R}^d)$ is a banach space with norm

$$\|f\|_{H^m} := \|f\|_{H^m(\mathbb{R}^d)} := \|(1 + |\xi|^2)^{m/2} \hat{f}(\xi)\|_{L^2(\mathbb{R}^d)}.$$

1.4.4 Proposition. *(a) We have* $[\widehat{(-i\partial_j)f}](\xi) = \xi_j \hat{f}(\xi)$ *and* $[\widehat{\xi_j f}](\xi) = i\partial_j \hat{f}(\xi)$ *for all* $f \in \mathcal{S}(\mathbb{R}^d)$ *and all* $\xi \in \mathbb{R}^d$.

(b) The partial derivatives

$$\partial_k : H^{n+1}(\mathbb{R}^d) \to H^n(\mathbb{R}^d), \quad k = 1, \ldots, d; \ n \in \mathbb{N},$$

are bounded operators $\|\partial_k f\|_{H^n} \le \|f\|_{H^{n+1}}, f \in H^{n+1}(\mathbb{R}^d)$.

(c) *Pointwise multiplication almost everywhere induces a continuous mapping*

$$H^n(\mathbb{R}^d) \times H^n(\mathbb{R}^d) \to H^n(\mathbb{R}^d), \quad n > \frac{d}{2}.$$

1.4.5 Definition. For a *multiindex* $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ one defines:

(a) $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}, \quad x \in \mathbb{R}^d$

(b) $\partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_d^{\alpha_d}$

1.4.6 Remark. Let $f \in H^{n+1}(\mathbb{R}^d)$. Then $\|\nabla f\|_{H^n} \leq \|f\|_{H^{n+1}}$, where

$$\|\nabla f\|_{H^n} := \max_{i=1,\dots,d} \|\partial_i f\|_{H^n}.$$

During this chapter and chapter 3 we make frequently use of the following well-known lemma, for a proof see for example [13, Satz 42.9].

1.4.7 Lemma (Sobolev Embedding Theorem)**.** *Let $m > \frac{d}{2} + k$ and $f \in H^m(\mathbb{R}^d)$ then $f \in C_b^k(\mathbb{R}^d)$ and there exists a constant $C_{k,m} > 0$, depending only on k and m, such that for all multiindices $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ we have $\|\partial^\alpha f\|_\infty \leq C_{k,m} \|f\|_{H^m}$.*

1.4.8 Definition. For all $f \in H^2(\mathbb{R}^d)$ we define $\|\nabla^2 \cdot f\|_{L^2(\mathbb{R}^d)}$ by:

$$\|\nabla^2 \cdot f\|_{L^2(\mathbb{R}^d)} := \left(\sum_{|\alpha|=2} \|\partial^\alpha f\|^2_{L^2(\mathbb{R}^d)} \right)^{1/2}.$$

1.5 Assumptions

In this section we present and discuss all technical assumptions (A1)-(A6) required to give a proof of the convergence of the particle model to its limit equation in theorem 2.1.7.

(A1) For some fixed $T > 0$ and some fixed constant $L \in \mathbb{N}, L > \frac{d}{2} + 2$, the cauchy problem (1) has a nonnegative solution

$$\rho \in C([0,T], H^{L+1+\frac{d}{2}}(\mathbb{R}^d)).$$

in the sense that $\partial_t \rho(t)$ exists for almost every t and equation (1) holds for almost every t.

1.5.1 Remark. (a) During this thesis we frequently consider ρ as a function on $\mathbb{R}^d \times [0,T]$ and write $\rho(x,t)$ instead of $[\rho(t)](x)$.

(b) Sobolev's Lemma 1.4.7 implies that $\rho(t) \in C_b^{L+1}(\mathbb{R}^d)$ for all $t \in [0,T]$.

(c) The continuity of ρ implies the existence of $R > 0$ such that for all $t \in [0,T]$

$$\rho(t) \in B_{H^{L+1+\frac{d}{2}}(\mathbb{R}^d)}(\rho_0, R),$$

where $B_{H^{L+1+\frac{d}{2}}(\mathbb{R}^d)}(\rho_0, R)$ denotes the ball with radius R and center ρ_0 in the sobolev space $H^{L+1+\frac{d}{2}}(\mathbb{R}^d)$.

(d) Uniqueness of the solution ρ is a simple consequence of our main result, theorem 2.1.7.

(A2) The repulsion potential satisfies

$$V_1 = W_1 * W_1 \tag{1.16}$$

for a symmetric probability density function $W_1 \in H^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} |z| W_1(z)dz < \infty$ holds.

As in equation (1.9) we can define a rescaled density W_N by

$$W_N(x) = \chi_N^d W_1(\chi_N x), \quad N \in \mathbb{N}. \tag{1.17}$$

And, exactly as in remark 1.2.2, we see that the rescaled repulsion kernels W_N are still probability density functions on $\mathbb{R}^d$.

1.5.2 Definition. Let

$$h_N(x,t) := (X_N(t) * W_N)(x) \tag{1.18}$$

define a *smoothed version of the empirical process.*

For notational simplicity we occasionally use another smoothed version of the empirical process g_N that is defined by replacing W_N with V_N in equation (1.18), i.e., we have

$$g_N(x,t) := (X_N(t) * V_N)(x) = (h_N(t) * W_N)(x).$$

1.5.3 Remark. Please observe that the densities V_N are used in two different ways. First, they are used to define the repulsive force between the particles of the system. And second, the functions V_N and its convolution roots W_N are used as mollifiers for the empirical process of the N-particle system $(X_N^k)_{k=1,\dots,N}$.

(A3) We have

$$\lim_{N\to\infty} \mathbb{E}\|h_N(\cdot,0)-\rho_0(\cdot)\|_{L^2}^2 = 0. \tag{1.19}$$

(A4) For all $k=1,\dots,d$, the partial derivatives $\partial_k \hat{W}_1$ are bounded and the equations

$$|\widehat{[y^\alpha \partial_k W_N]}(\xi)| \le C|\hat{W}_N(\xi)|, \quad 0<|\alpha|\le L, \xi\in\mathbb{R}^d \tag{1.20}$$

$$\|y^\alpha \partial_k W_1\|_{L^2} \le C, \quad |\alpha| = L+1 \tag{1.21}$$

hold for all $N\in\mathbb{N}$ and all $k=1,\dots,d$. Here, $L\in\mathbb{N}$ is the same constant as in assumption (A1).

(A5) The rescaling factor $\beta\in(0,1)$ satisfies the following condition:

$$\lim_{N\to\infty} \sigma_N^2 N^{(d+2)\beta/d-1} = 0. \tag{1.22}$$

(A6) The aggregation potential satisfies $\nabla G \in C_b^L(\mathbb{R}^d)$.

1.5.4 Remarks. (a) Clearly, in case $\beta < \frac{d}{d+2}$ condition (A5) is always fulfilled. If $\beta \ge \frac{d}{d+2}$, this condition implies $\sigma_\infty = 0$.

(b) The assumptions on the densities V_1 are very restrictive. For example, the standard normal density $f(s) = (2\pi)^{-d/2}e^{-\frac{1}{2}x^2}$ does not satisfy the conditions from assumption (A4). In 1.5.5 and 1.5.6 we give two examples of suitable probability density functions W_1.

(c) If ρ_0 is a probability density function we can construct a sequence of iid random variables $X_N^k(0)$ such that (A2) holds, see lemma 1.6.7.

(d) Please observe that the aggregation potential G may be unbounded. Assumption (A6) is condition on ∇G.

1.5.5 Example. Let $d \geq 1$ arbitrary and $r > \frac{d}{2} + \frac{1}{2}, b > 0$ then

$$\hat{W}_1(\xi) := (2\pi)^{-d/2} \left(1 + \left|\frac{\xi}{b}\right|^2\right)^{-r} \tag{1.23}$$

is the Fourier transform of a probability density function satisfying the conditions in assumption (A2) and assumption (A4).

In fact, in dimension $d = 1$ the density W_1 defined by equation (1.23) is the r-fold convolution of a bilateral exponential distribution (see [7, p. 503]). I.e., let X denote a random variable with $X = Y^+ * (-Y^-)$, where Y^+, Y^- are iid $\Gamma(r, b)$-distributed random variables. Then $\hat{W}_1$ is the Fourier transform of the density of X.

Proof. We prove that $\hat{W}_1$ satisfies (A2) and (A4). In example A.7 it is shown that W_1 is the Fourier transform of a probability density function. $r > \frac{d}{2} + \frac{1}{2}$ implies that $\hat{W}_1(\xi)$ and $(1 + (|\xi|/b)^2)^{1/2}\hat{W}_1(\xi), j = 1, \ldots, d$, are in $L^2(\mathbb{R}^d)$. Hence, W_1 is in $H^1(\mathbb{R}^d)$. Moreover, since $\hat{W}_1$ is a smooth function, the integral $\int_{\mathbb{R}^d} |z|W_1(z)\ dz$ is finite. Using the chain rule, we obtain for the derivative

$$\partial_k \hat{W}_1(\xi) = -2r(2\pi)^{-d/2}\xi_k(1 + (|\xi|/b)^2)^{-r-1}, \quad k = 1, \ldots, d.$$

This shows the boundedness of all first order partial derivatives of $\hat{W}_1(\xi)$. Clearly, $\xi_j W_1(\xi), j = 1, \ldots, d$, are smooth functions. Therefore, using standard properties of the Fourier transform, we see that W_1 satisfies equation (1.21) from assumption (A4). It remains to show that

$$|\partial_\xi \xi \hat{W}_N(\xi)| \leq C|\hat{W}_N(\xi)| \tag{1.24}$$

which is an equivalent condition to (1.20). Using $\hat{W}_N(\xi) = \hat{W}_1(\chi_N^{-1}\xi)$ and the product rule, one can easily verify that all partial derivatives of $\xi_k \hat{W}_1(\xi)$ can be written in the form

$$\partial^\alpha \xi_k \hat{W}_N(\xi) = \sum_{j=1}^{|\alpha|} P_j(\chi_N^{-1}\xi)(1 + |\chi_N^{-1}\xi b^{-1}|)^2)^{-j}\hat{W}_N(\xi), 0 < |\alpha| \leq L,$$

where P_j is a polynomial of degree less or equal $2j$. Hence, there exists a constant $C > 0$ such that for all $j = 1, \ldots, |\alpha|$

$$P_j(\chi_N^{-1}\xi)(1 + |\chi_N^{-1}\xi b^{-1}|)^2)^{-j} \leq C.$$

This yields equation (1.24). □

1.5.6 Example. In the case $d = 1$ ($L = 1$) we consider the density function

$$W_1(y) := \frac{b^r}{2\Gamma(r)} |y|^{r-1} \exp(-b|y|). \tag{1.25}$$

W_1 satisfies for $r \geq 1, b > 0$ the conditions from (A2) and (A4). And thus, $V_N := W_N * W_N, N \in \mathbb{N}$, with $W_N(x) := \chi_N W_1(\chi_N x)$ is a valid family of repulsion kernels.

W_1 is a symmetrized gamma density function. Observe that in case $r > 1$ this is not the same example as 1.5.5. This can easily be seen by comparing the Fourier transforms in equations (1.23) and (1.29).

Proof. Clearly, W_1 is smooth and symmetric function on $\mathbb{R} \setminus \{0\}$. Furthermore, W_1, $\partial_y W_1(y)$ and $\partial_y^2 W_1(y)$ are of exponential decay. Thus, we have $W_1 \in H^2(\mathbb{R})$, $\int_{\mathbb{R}} |z| W_1(z)\, dz < \infty$ and there exists a constant $C > 0$ such that $\|y^\alpha \partial_y W_1(y)\|_{L^2} \leq C$ holds for $\alpha = L + 1 = 2$. Hence, it remains to show that $\partial_\xi \hat{W}_1(\xi)$ is bounded and that

$$|[y\widehat{\partial_y W_N(y)}](\xi)| \leq C |\hat{W}_N(\xi)|.$$

Using the properties form proposition 1.4.4, we see that this condition is equivalent to

$$|\partial_\xi \xi \hat{W}_N(\xi)| \leq C |\hat{W}_N(\xi)|. \tag{1.26}$$

Let us compute the Fourier transform $\hat{W}_1$. Applying the substitution $z = (b + i\xi)y$ and the defintion of the gamma function, we obtain

$$\begin{aligned} \int_0^\infty & \frac{b^r}{2\Gamma(r)} y^{r-1} \exp(-by) \exp(-i\xi y)\, dy \\ &= \int_0^\infty \frac{b^r}{2\Gamma(r)} y^{r-1} \exp(-(b + i\xi)y)\, dy \\ &= \frac{b^r}{2\Gamma(r)} (b + i\xi)^{-r} \int_0^\infty z^{r-1} \exp(-z)\, dz \\ &= \frac{1}{2} \left(1 + i\frac{\xi}{b}\right)^{-r}. \end{aligned} \tag{1.27}$$

And analogously for negative y, we get

$$\int_{-\infty}^0 \frac{b^r}{2\Gamma(r)} (-y)^{r-1} \exp(by) \exp(-i\xi y)\, dy = \frac{1}{2} \left(1 - i\frac{\xi}{b}\right)^{-r}. \tag{1.28}$$

Thus, it follows that the Fourier transform $\hat{W}_1$ of W_1 is given by

$$\begin{aligned}\hat{W}_1(\xi) &= \frac{1}{2}(2\pi)^{-1/2}\left[\left(1+i\frac{\xi}{b}\right)^{-r} + \left(1-i\frac{\xi}{b}\right)^{-r}\right] \\ &= \frac{1}{2}(2\pi)^{-1/2}\frac{\left(1+i\frac{\xi}{b}\right)^{r} + \left(1-i\frac{\xi}{b}\right)^{r}}{\left(1+\frac{\xi^2}{b^2}\right)^{r}}.\end{aligned} \tag{1.29}$$

Note that $\hat{W}_N(\xi) = \hat{W}_1(\chi_N^{-1}\xi)$. Therefore, the remaining condition (1.26) is fulfilled. □

1.6 Lemmas

We close this chapter with a collection of elementary lemmas that are used in the second and third chapter. In all equations during this chapter C denotes a non-negative constant that may vary from line to line. The constant C may depend on the solution ρ, the aggregation and repulsion potentials, G and V_1 (and thus on W_1), and the terminal time T but C does not depend on the number of particles N nor on the time t.

The following lemma is a simple consequence of assumption (A4) and the rescaling of the probability density functions $W_N, N \in \mathbb{N}$ (see equation (1.17)).

1.6.1 Lemma. *There exists a constant $C > 0$ such that for all $\alpha \in \mathbb{N}^d$ with $|\alpha| = L+1$, L as in (A1), and all $k = 1, \dots, d$ we have*

$$\|y^\alpha \partial_k W_N(y)\|_{L^2} \le C\chi_N^{-1} \underset{N\to\infty}{\longrightarrow} 0. \tag{1.30}$$

Proof. Using the substitution $z = \chi_N y$, we get

$$\begin{aligned}\|y^\alpha \partial_k W_N(y)\|_{L^2} &= \left[\int_{\mathbb{R}^d} \left|y^\alpha \chi_N^{d+1}(\partial_k W_1)(\chi_N y)\right|^2 \, dy\right]^{1/2} \\ &= \chi_N^{\frac{d}{2}+1-|\alpha|}\left[\int_{\mathbb{R}^d} |z^\alpha(\partial_k W_1)(z)|^2 \, dz\right]^{1/2} \\ &\le C\chi_N^{-1},\end{aligned}$$

where the constant C is given by

$$C := \max\{\|z^\alpha(\partial_k W_1)(z)\|_{L^2(\mathbb{R}^d)} \mid k = 1, \dots, d;\ |\alpha| = L+1\}. \tag{1.31}$$

□

The following lemma lists a few elementary results, that will be used frequently during the calculations in the next section.

1.6.2 Lemma. *For all $N \in \mathbb{N}$ we have:*

(a) V_N is a symmetric probability density function on $\mathbb{R}^d$ such that

$$V_N = W_N * W_N. \tag{1.32}$$

(b) $W_N, \nabla W_N \in L^2(\mathbb{R}^d)$ and

$$\|W_N\|_{L^2} = \chi_N^{d/2} \|W_1\|_{L^2}, \tag{1.33}$$

$$\|\nabla W_N\|_{L^2} = \chi_N^{(d+2)/2} \|\nabla W_1\|_{L^2}. \tag{1.34}$$

(c) $\|h_N(s)\|_{L^2} \leq C\chi_N^{d/2}$ and $\|\nabla h_N(s)\|_{L^2} \leq C\chi_N^{(d+2)/2}$ for all $s \in [0,T]$.

(d) For any function $f \in L^2(\mathbb{R}^d)$ we have

$$\langle X_N(t), f * W_N \rangle = \langle X_N * W_N, f \rangle, \tag{1.35}$$

$$\langle X_N(t), f * \partial_i W_N \rangle = -\langle X_N * \partial_i W_N, f \rangle, \quad i = 1, \ldots, d, \tag{1.36}$$

$$\langle X_N(t), f * \nabla W_N \rangle = -\langle X_N * \nabla W_N, f \rangle. \tag{1.37}$$

Clearly, the same equations with $\rho(\cdot, t)$ instead of $X_N(t)$ hold.

Proof. (a) In remark 1.2.2 we have already seen that the rescaled functions V_N are probability density functions. Substituting $y = \chi_N z$ we get

$$\begin{aligned}
V_N(x) &= \chi_N^d V_1(\chi_N x) \\
&= \chi_N^d (W_1 * W_1)(\chi_N x) \\
&= \chi_N^d \int_{\mathbb{R}^d} W_1(\chi_N x - y) W_1(y)\, dy \\
&= \chi_N^{2d} \int_{\mathbb{R}^d} W_1(\chi_N(x - z)) W_1(\chi_N z)\, dz \\
&= (W_N * W_N)(x).
\end{aligned} \tag{1.38}$$

The symmetry of the repulsion potentials $V_N, N \in \mathbb{N}$, follows easily from the symmetry of W_N and the substitution $z = -y$:

$$\begin{aligned}
V_N(-x) &= \int_{\mathbb{R}^d} W_N(-x - y) W_N(y) dy \\
&= \int_{\mathbb{R}^d} W_N(-(x - z)) W_N(-z) dz \\
&= \int_{\mathbb{R}^d} W_N(x - z) W_N(z) dz \\
&= V_N(x).
\end{aligned} \tag{1.39}$$

(b) The substitution $y = \chi_N x$ gives

$$\|W_N\|_{L^2}^2 = \chi_N^{2d} \int_{\mathbb{R}^d} |W_1(\chi_N x)|^2 \, dx = \chi_N^d \int_{\mathbb{R}^d} |W_1(y)|^2 \, dy.$$

Since $W_1 \in L^2(\mathbb{R}^d)$ due to assumption (A2), the claimed result follows. Analogously, we get

$$\|\nabla W_N\|_{L^2}^2 = \chi_N^{2d+2} \int_{\mathbb{R}^d} |(\nabla W_1)(\chi_N x)|^2 \, dx = \chi_N^{d+2} \|\nabla W_1\|_{L^2}^2.$$

(c) Using Jensen's inequality, we obtain

$$\begin{aligned}
\|h_N(\cdot, s)\|_{L^2}^2 &= \int_{\mathbb{R}^d} |(X_N(s) * W_N)(x)|^2 \, dx \\
&= \int_{\mathbb{R}^d} \Big| \frac{1}{N} \sum_{k=1}^N W_N(x - X_N^k(s)) \Big|^2 \, dx \\
&\le \frac{1}{N} \sum_{k=1}^N \int_{\mathbb{R}^d} |W_N(x - X_N^k(s))|^2 \, dx \\
&= \|W_N(x)\|_{L^2}^2.
\end{aligned}$$

Thus, part (b) gives us the claimed result $\|h_N(\cdot, s)\|_{L^2} \le C\chi_N^d$. The second inequality $\|\nabla h_N(\cdot, s)\|_{L^2} \le C\chi_N^{(d+2)/2}$ follows exactly the same way.
(d) A straight forward calculation using the symmetry of the probability density functions W_N, gives

$$\begin{aligned}
\langle X_N, f * W_N \rangle &= \frac{1}{N} \sum_{k=1}^d (f * W_N)(X_N^k) \\
&= \frac{1}{N} \sum_{k=1}^d \int_{\mathbb{R}^d} f(y) W_N(X_N^k - y) dy \\
&= \int_{\mathbb{R}^d} f(y) \frac{1}{N} \sum_{k=1}^d W_N(y - X_N^k) dy \\
&= \int_{\mathbb{R}^d} f(y) (X_N * W_N)(y) dy \\
&= \langle X_N * W_N, f \rangle.
\end{aligned} \tag{1.40}$$

The same calculation with $\partial_i W_N$ instead of W_N shows

$$\langle X_N, f * \partial_i W_N \rangle = -\langle X_N * \partial_i W_N, f \rangle. \tag{1.41}$$

The minus sign here is due to the antisymmetry of the functions $\partial_i W_N, i = 1, \dots, d$. Finally, equation (1.37) follows directly from equation (1.36). □

1.6.3 Lemma. *For any $s \in [0,T]$ and $N \in \mathbb{N}$ we have*

$$\langle X_N(s), \Delta V_N * X_N(s)\rangle = -\|\nabla h_N(\cdot,s)\|_{L^2}^2. \tag{1.42}$$

Proof. A direct calculation shows

$$\begin{aligned}\langle X_N(s), \Delta V_N * X_N(s)\rangle &= \sum_{i=1}^{d} \langle X_N(s), \partial_i\partial_i(W_N * W_N * X_N(s))\rangle \\ &= \sum_{i=1}^{d} \langle X_N(s), (\partial_i W_N * \partial_i W_N * X_N(s))\rangle.\end{aligned}$$

Now, we can apply lemma 1.6.2 (d). This gives us

$$\begin{aligned}\langle X_N(s), \Delta V_N * X_N(s)\rangle &= -\sum_{i=1}^{d} \langle X_N(s) * \partial_i W_N, X_N(s) * \partial_i W_N\rangle \\ &= -\langle X_N(s) * \nabla W_N, X_N(s) * \nabla W_N\rangle \\ &= -\|\nabla h_N(\cdot,s)\|_{L^2}^2.\end{aligned}$$

□

For the definition of $\|\nabla \cdot f\|_{L^2}$ see 1.4.8.

1.6.4 Lemma. *There exists a constant $C > 0$ such that*

$$\|\nabla^2 \cdot \rho(t)\nabla\rho(t)\|_{L^2} \leq C \tag{1.43}$$

$$\|\nabla^2 \cdot \rho(t)(\rho(t) * \nabla G)(\cdot)\|_{L^2} \leq C \tag{1.44}$$

$$\|\Delta\rho(\cdot,s)\|_{L^2} \leq C \tag{1.45}$$

hold for all $t \in [0,T]$.

Proof. The inequalities are direct consequences of assumption (A1) and the properties of the Sobolev spaces. From proposition 1.4.4 (b) we get that $t \mapsto \nabla\rho(t)$ is continuous as a map from $[0,T]$ to $H^{L+\frac{d}{2}}(\mathbb{R}^d;\mathbb{R}^d)$. Furthermore, since $L > 1$ it follows from 1.4.4 (c) that $t \mapsto \rho(t)\nabla\rho(t)$ is continuous as a map from $[0,T]$ to $H^{L+\frac{d}{2}}(\mathbb{R}^d;\mathbb{R}^d)$. And finally, again by part (b) of remark 1.4.4, we obtain

$$\phi : [0,T] \to H^{L+\frac{d}{2}-2}(\mathbb{R}^d;\mathbb{R}^d); \quad t \mapsto \nabla^2 \cdot \rho(t)\nabla\rho(t)$$

is continuous. Because of $L+\frac{d}{2}-2>0$ the embedding $H^{L+\frac{d}{2}-2}(\mathbb{R}^d)\hookrightarrow L^2(\mathbb{R}^d)$ is continuous. Hence, ϕ induces a continuous map

$$\tilde{\phi}:[0,T]\to L^2(\mathbb{R}^d;\mathbb{R}^d);\quad t\mapsto \nabla^2\cdot\rho(t)\nabla\rho(t).$$

And now the compactness of the interval $[0,T]$ implies the existence of a constant $C>0$ such that equation (1.43) holds.

Clearly, we have for all $n\leq L+\frac{d}{2}+1$

$$\|\rho(t)*\nabla G\|_{H^n(\mathbb{R}^d)}\leq\|\nabla G\|_\infty\|\rho(t)\|_{H^n(\mathbb{R}^d)}, \tag{1.46}$$

i.e. $t\mapsto\rho(t)*\nabla G$ is continuous from $[0,T]$ to $H^{L+\frac{d}{2}+1}(\mathbb{R}^d;\mathbb{R}^d)$. And it follows as in the proof of the first inequality that

$$\tilde{\psi}:[0,T]\to L^2(\mathbb{R}^d;\mathbb{R}^d);\quad t\mapsto\nabla^2\cdot\rho(t)(\rho(t)*\nabla G)$$

is continuous. Again, the compactness of the interval $[0,T]$ implies the existence of a constant $C>0$ such that equation (1.44) holds.

Finally, inequality (1.45) follows directly from proposition 1.4.4 (b). □

In the next two lemmas we give several estimates for the error that is caused by convolution with W_N, resp. ∇W_N. This lemmas are used in chapter two to give an upper bound for $\|h_N(t)-\rho(t)\|_{L^2}^2$. The first lemma is from [23].

1.6.5 Lemma. *(a) Let $f\in C_b^1(\mathbb{R}^d)$ then for all $x\in\mathbb{R}$*

$$|f(x)-(f*W_N)(x)|\leq C\chi_N^{-1}\|\nabla f\|_\infty.$$

(b) Let $f\in H^1(\mathbb{R}^d)$ then

$$\|f-f*W_N\|_{L^2}^2\leq C\chi_N^{-2}\|\nabla f\|_{L^2}^2.$$

Proof. (a) Since W_N is a probability density, we have for all $x\in\mathbb{R}^d$

$$f(x)=\int_{\mathbb{R}^d}f(x)W_N(y)\,dy.$$

Now, using the substitution $z=\chi_n y$ and the mean value inequality, we obtain

$$\begin{aligned}
&|f(x) - (f * W_N)(x)| \\
&\quad = \left| \int_{\mathbb{R}^d} (f(x) - f(x-y)) \, W_N(y) \, dy \right| \\
&\quad \leq \|\nabla f\|_\infty \int_{\mathbb{R}^d} |y| W_N(y) \, dy \\
&\quad = \chi_N^{-1} \|\nabla f\|_\infty \int_{\mathbb{R}^d} \chi_N |y| W_1(\chi_N y) \chi_N^d \, dy \\
&\quad = \chi_N^{-1} \|\nabla f\|_\infty \int_{\mathbb{R}^d} |z| W_1(z) \, dz.
\end{aligned}$$

Since the integral $\int_{\mathbb{R}^d} |z| W_1(z) \, dz$ is finite due to assumption (A2), the result follows immediately.

(b) By Taylor's Theorem we have

$$\hat{W}_1(\chi_N^{-1}\xi) = \hat{W}_1(0) + \frac{1}{\chi_N} \sum_{i=1}^{d} \xi_i \partial_i \hat{W}_1(\theta_i \chi_N^{-1} \xi) \tag{1.47}$$

for some $\theta_i \in [0,1], i = 1, \ldots, d$. Since W_1 is a probability density function we have $\hat{W}_1(0) = (2\pi)^{-d/2}$. Furthermore, by (A4) the functions $\partial_i \hat{W}_1$ are bounded. This yields

$$|1 - (2\pi)^{d/2} \hat{W}_1(\chi_N^{-1}\xi)|^2 \leq C|\xi|^2 \chi_N^{-2}. \tag{1.48}$$

Using Parseval's identity and the well-known fourier multiplication formula $\widehat{f * W_N}(\xi) = (2\pi)^{d/2} \hat{f}(\xi) \hat{W}_N(\xi)$, we can now compute

$$\begin{aligned}
&\|f - f * W_N\|_{L^2}^2 \\
&\quad = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |1 - (2\pi)^{d/2} \hat{W}_N(\xi)|^2 \, d\xi \\
&\quad = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |1 - (2\pi)^{d/2} \hat{W}_1(\chi_N^{-1}\xi)|^2 \, d\xi \\
&\quad \leq C\chi_N^{-2} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |\xi|^2 \, d\xi \\
&\quad = C\chi_N^{-2} \|\nabla f\|_{L^2}^2.
\end{aligned}$$

Here, we used $\hat{W}_N(\xi) = \hat{W}_1(\chi_N^{-1}\xi)$. □

The following lemma is crucial for the second step of the convergence proof in chapter 2.

1.6.6 Lemma. *Let* $v \in C_b^{L+1}(\mathbb{R}^d, \mathbb{R}^d), f \in C_b^{L+1}(\mathbb{R}^d)$ *where* L *is as in assumption (A1), then:*

(a) For all $t \in [0, T]$ *we have*

$$|\langle X_N(t) - \rho(t)), (\nabla[h_N(t) - \rho(t)] * W_N) \cdot v\rangle| \\ \leq C(\|h_N(t) - \rho(t)\|_{L^2}^2 + \chi_N^{-2}). \quad (1.49)$$

(b) For $F \in C_b(\mathbb{R}^d)$ *and all* $t \in [0, T]$ *we have*

$$|\langle (X_N(t) - \rho(t)) * F, (\nabla[h_N(t) - \rho(t)] * W_N) \cdot v\rangle| \\ \leq C(\|h_N(t) - \rho(t)\|_{L^2}^2 + \chi_N^{-2}). \quad (1.50)$$

(c) For $U \in C_b(\mathbb{R}^d, \mathbb{R}^d)$ *and all* $t \in [0, T]$ *we have*

$$|\langle (X_N(t) - \rho(t)) * U, (\nabla[h_N(t) - \rho(t)] * W_N) f\rangle| \\ \leq C(\|h_N(t) - \rho(t)\|_{L^2}^2 + \chi_N^{-2}). \quad (1.51)$$

Proof. Let $v_k, k = 1, \dots, d$, denote the k-th component of the function v. Taylor's theorem gives

$$v_k(z) = \sum_{|\alpha| \leq L} \frac{y^\alpha}{\alpha!} \partial^\alpha v_k(z - y) + \sum_{|\alpha| = L+1} \frac{y^\alpha}{\alpha!} \partial^\alpha v_k(z - \theta_{k,\alpha}(y, z) y) \quad (1.52)$$

with $\theta_{k,\alpha}(y, z) \in [0, 1]$ for $z, y \in \mathbb{R}^d$. To keep the notation short we write X_N for $X_N(t)$, ρ for $\rho(\cdot, t)$ and h_N for $h_N(\cdot, t)$ during the following calculation. This should cause no confusion since all functions are evaluated at time t. Because of

$$|\langle X_N - \rho, \nabla(h_N - \rho) * W_N \cdot v\rangle| \leq \sum_{k=1}^{d} |\langle X_N - \rho, [(h_N - \rho) * \partial_k W_N]\, v_k\rangle| \quad (1.53)$$

it is sufficient to give a proof for fixed $k = 1, \dots, d$. Using equation (1.52), we get

$$
\begin{aligned}
&|\langle X_N-\rho,[(h_N-\rho)*\partial_k W_N]\, v_k\rangle| \\
&\quad=\Bigg|\Big\langle (X_N-\rho)(dz), \\
&\qquad\int_{\mathbb{R}^d}(h_N-\rho)(z-y)\partial_k W_N(y)\sum_{|\alpha|\le L}\frac{y^\alpha}{\alpha!}\partial^\alpha v_k(z-y)\,dy \\
&\qquad+\int_{\mathbb{R}^d}(h_N-\rho)(z-y)\partial_k W_N(y) \\
&\qquad\qquad\sum_{|\alpha|=L+1}\frac{y^\alpha}{\alpha!}\partial^\alpha v_k(z-\theta_{k,\alpha}(y,z)y)dy\Big\rangle\Bigg| \\
&\quad\le\sum_{|\alpha|\le L}\frac{1}{\alpha!}|\langle X_N-\rho,[(h_N-\rho)\partial^\alpha v_k]*[y^\alpha\partial_k W_N(y)]\rangle| \\
&\qquad+\sum_{|\alpha|=L+1}\frac{1}{\alpha!}\Bigg|\Big\langle (X_N-\rho)(dz), \\
&\qquad\qquad\int_{\mathbb{R}^d}(h_N-\rho)(z-y)y^\alpha\partial_k W_N(y)\;\partial^\alpha v_k(z-\theta_{k,\alpha}(y,z)y)\;dy\Big\rangle\Bigg| \\
&\quad=\sum_{|\alpha|\le L}\frac{1}{\alpha!}|\langle (X_N-\rho)*[y^\alpha\partial_k W_N(y)],(h_N-\rho)\partial^\alpha v_k\rangle| \\
&\qquad+\sum_{|\alpha|=L+1}\frac{1}{\alpha!}\Bigg|\Big\langle (X_N-\rho)(dz), \\
&\qquad\qquad\int_{\mathbb{R}^d}(h_N-\rho)(z-y)y^\alpha\partial_k W_N(y)\;\partial^\alpha v_k(z-\theta_{k,\alpha}(y,z)y)\;dy\Big\rangle\Bigg| \\
&\quad=:\sum_{|\alpha|\le L+1}\frac{1}{\alpha!}H_\alpha.
\end{aligned}
\tag{1.54}
$$

Let us start estimating the summand $\alpha=0$:

$$
\begin{aligned}
H_0&=|\langle (X_N-\rho)*\partial_k W_N,(h_N-\rho)v_k\rangle| \\
&=|\langle (X_N-\rho)*W_N,\partial_k[(h_N-\rho)v_k]\rangle| \\
&\le|\langle h_N-\rho,\partial_k[(h_N-\rho)v_k]\rangle|+|\langle\rho-\rho*W_N,\partial_k[(h_N-\rho)v_k]\rangle|.
\end{aligned}
\tag{1.55}
$$

For the first summand in the last line of this inequality we get by partial integration and the boundedness of v_k and $\partial_k v_k$

$$\begin{aligned}|\langle h_N - \rho, \partial_k[(h_N - \rho)v_k]\rangle| &= \frac{1}{2}|\langle \partial_k (h_N - \rho)^2, v_k\rangle| \\ &= \frac{1}{2}|\langle (h_N - \rho)^2, \partial_k v_k\rangle| \\ &\leq C\|h_N - \rho\|^2.\end{aligned} \tag{1.56}$$

Using partial integration, Schwarz inequality and lemma 1.6.5 (b) with $f = \partial_k \rho$, we obtain for the second summand of (1.55)

$$\begin{aligned}|\langle \rho - \rho * W_N, \partial_k[(h_N - \rho)v_k]\rangle| &= |\langle \partial_k \rho - \partial_k \rho * W_N, (h_N - \rho)v_k\rangle| \\ &\leq C\chi_N^{-1}\|h_N - \rho\|\end{aligned} \tag{1.57}$$

where we used that $\|\partial_k \rho\|_{L^2}$ is bounded due to assumption (A3). Summarizing the inequalities (1.55), (1.56) and (1.57) we obtain

$$H_0 \leq C(\|h_N - \rho\|^2 + \chi_N^{-2}). \tag{1.58}$$

Now, we consider the summands with $0 < |\alpha| \leq L$. First, note that equation (1.20) from assumption (A4) implies

$$\begin{aligned}&\|(X_N - \rho) * [y^\alpha \partial_k W_N]\|_{L^2}^2 \\ &\quad = \int_{\mathbb{R}^d} |\widehat{X_N - \rho}|^2(\xi)|\widehat{y^\alpha \partial_k W_N}|^2(\xi) d\xi \\ &\quad \leq \int_{\mathbb{R}^d} |\widehat{X_N - \rho}|^2(\xi)|\hat{W}_N|^2(\xi) d\xi \\ &\quad = \|(X_N - \rho) * W_N\|_{L^2}^2, \quad 0 < |\alpha| \leq L.\end{aligned} \tag{1.59}$$

Due to lemma 1.6.5 (b) with $f = \rho$ we get

$$\begin{aligned}\|(X_N - \rho) * W_N\|_{L^2} &\leq \|h_n - \rho\|_{L^2} + \|\rho - \rho * W_N\|_{L^2} \\ &\leq \|h_n - \rho\|_{L^2} + C\chi_N^{-1}.\end{aligned}$$

This together with equation 1.59 shows

$$\|(X_N - \rho) * [y^\alpha \partial_k W_N]\|_{L^2} \leq \|h_n - \rho\|_{L^2} + C\chi_N^{-1}, \quad 0 < |\alpha| \leq L. \tag{1.60}$$

Taking again the boundedness of the derivatives $\partial^\alpha v_k$ into account this gives us

$$\begin{aligned}H_\alpha &= |\langle (X_N - \rho) * [y^\alpha \partial_k W_N], [(h_N - \rho)\partial^\alpha v_k]\rangle| \\ &\leq C\|(X_N - \rho) * W_N\|_{L^2} \, \|h_N - \rho\|_{L^2} \\ &\leq C\|h_N - \rho\|_{L^2}^2 + \chi_N^{-2}, \quad 0 < |\alpha| \leq L.\end{aligned} \tag{1.61}$$

In a similar way, one obtains the inequalities for $|\alpha| = L+1$:

$$\begin{aligned}
H_\alpha &= |\langle X_N - \rho, \int_{\mathbb{R}^d} (h_N - \rho)(\cdot - y) y^\alpha \partial_k W_N(y) \partial^\alpha v_k(\cdot - y + \theta_{k,\alpha}(y,\cdot))\, dy\rangle| \\
&\leq C\langle |X_N - \rho|, \int_{\mathbb{R}^d} |h_N - \rho|(\cdot - y)|y^\alpha|\, |\partial_k W_N(y)|\, dy\rangle \\
&\leq C\langle |X_N - \rho| * \left[|y^\alpha|\, |\partial_k W_N(y)|\right], |h_N - \rho|\rangle \\
&\leq C\langle X_N * \left[|y^\alpha|\, |\partial_k W_N(y)|\right], |h_N - \rho|\rangle \\
&\qquad + C\langle \rho * \left[|y^\alpha||\partial_k W_N(y)|\right], |h_N - \rho|\rangle.
\end{aligned} \tag{1.62}$$

The last inequality follows from the fact that ρ and X_N are nonnegative. Now, Schwarz inequality and lemma 1.6.1 imply

$$\begin{aligned}
|\langle \rho * \left[|y^\alpha|\, |\partial_k W_N(y)|\right], |h_N - \rho|\rangle| &\leq \|\rho\|_\infty \|y^\alpha \partial_k W_N(y)\|_{L^2} \|h_N - \rho\|_{L^2} \\
&\leq C\chi_N^{-1} \|h_N - \rho\|_{L^2}.
\end{aligned} \tag{1.63}$$

On the other hand, Jensen's inequality gives us

$$\|X_N * [|y^\alpha|\, |\partial_k W_N(y)|]\|_{L^2} \leq \|y^\alpha \partial_k W_N(y)\|_{L^2}$$

(cf. proof of lemma 1.6.2 (b)). And it follows that

$$\begin{aligned}
|\langle X_N * \left[|y^\alpha|\, |\partial_k W_N(y)|\right], |h_N - \rho|\rangle| &\leq \|y^\alpha \partial_k W_N(y)\|_{L^2} \|h_N - \rho\|_{L^2} \\
&\leq C\chi_N^{-1} \|h_N - \rho\|_{L^2}.
\end{aligned} \tag{1.64}$$

Hence, for all $|\alpha| = L+1$ we have

$$|H_\alpha| \leq C\|h_N - \rho\|_{L^2} + \chi_N^{-2}.$$

This completes the proof of (a).

Part (b) of this lemma is shown in a similar way. Using Taylor's formula, as in equation (1.54), we obtain

$$\begin{aligned}
&|\langle (X_N - \rho) * U, [(h_N - \rho) * \partial_k W_N]\, v_k\rangle| \\
&\quad = \sum_{|\alpha| \leq L} \frac{1}{\alpha!} |\langle (X_N - \rho) * U * [y^\alpha \partial_k W_N], [(h_N - \rho)\partial^\alpha v_k]\rangle| \\
&\qquad + \sum_{|\alpha| = L+1} \frac{1}{\alpha!} \Big| \Big\langle [(X_N - \rho) * U](\cdot), \\
&\qquad\qquad \int_{\mathbb{R}^d} (h_N - \rho)(\cdot - y) \partial_k W_N(y)\, y^\alpha \partial^\alpha v_k(\cdot - \theta_{k,\alpha}(y,\cdot) y)\, dy \Big\rangle \Big| \\
&\quad =: \sum_{|\alpha| \leq L+1} \frac{1}{\alpha!} \tilde{H}_\alpha.
\end{aligned} \tag{1.65}$$

For $|\alpha| \le L$:

$$\begin{aligned}|\tilde{H}_\alpha| &\le C\|U\|_\infty \|(X_N - \rho) * [y^\alpha \partial_k W_N]\|_{L^2}\|h_N - \rho\|_{L^2}\\ &\le C\|(X_N - \rho) * W_N\|_{L^2}\|h_N - \rho\|_{L^2}\\ &\le C\|h_N - \rho\|_{L^2}^2 + \chi_N^{-2}.\end{aligned}$$

For $|\alpha| = L + 1$:

$$\begin{aligned}|\tilde{H}_\alpha| \le\ & C|\langle X_N * |U| * \big[|y^\alpha|\ |\partial_k W_N(y)|\big], |h_N - \rho|\rangle|\\ &+ C|\langle \rho * |U| * \big[|y^\alpha|\ |\partial_k W_N(y)|\big], |h_N - \rho|\rangle|.\end{aligned}$$

Direct calculation shows

$$\begin{aligned}&|\langle \rho * |U| * \big[|y^\alpha|\ |\partial_k W_N(y)||\big], |h_N - \rho|\rangle|\\ &\quad\le \|U\|_\infty \|\rho\|_\infty \|y^\alpha \partial_k W_N(y)\|_{L^2}\|h_N - \rho\|_{L^2}\\ &\quad\le C\chi_N^{-1}\|h_N - \rho\|_{L^2}.\end{aligned}\tag{1.66}$$

Using again Jensen's inequality (as in the proof of (a)), we obtain

$$\begin{aligned}&|\langle X_N * |U| * \big[|y^\alpha|\ |\partial_k W_N(y)|\big], |h_N - \rho|\rangle|\\ &\quad\le \|U\|_\infty \|y^\alpha \partial_k W_N(y)\|_{L^2}\|h_N - \rho\|_{L^2}\\ &\quad\le C\chi_N^{-1}\|h_N - \rho\|_{L^2}.\end{aligned}\tag{1.67}$$

Therefore, we get $|\tilde{H}_\alpha| \le C\|h_N - \rho\|_{L^2}$. This completes the proof of (b). Part (c) follows easily from part (b). □

1.6.7 Lemma. *Assume ρ_0 is a probability density function. Let $X_N^k(0), k = 1, \dots, N$, be families of iid random variables, whose distribution has density ρ_0. Then assumption (A2) is satisfied.*

Proof. For the sake of completeness we give the short proof from [23]. We have

$$\mathbb{E}\|h_N(0) - \rho_0\|_{L^2}^2 = \mathbb{E}\|h_N(0)\|_{L^2}^2 - 2\mathbb{E}\langle h_N(0), \rho_0\rangle + \|\rho_0\|_{L^2}^2.$$

Using $V_N = W_N * W_N$ and lemma 1.6.2 (d), we obtain

$$\begin{aligned}
\mathbb{E}\|h_N(0)\|_{L^2}^2 &= \frac{1}{N^2}\sum_{k,l=1}^{N} \mathbb{E}[V_N(X_N^k(0) - X_N^l(0))] \\
&= \frac{1}{N^2}\sum_{\substack{k,l=1\\k\neq l}}^{N} \int_{\mathbb{R}^d\times\mathbb{R}^d} V_N(x-y)\rho_0(x)\rho_0(y)\,dxdy + \frac{1}{N}V_N(0) \\
&= \frac{1}{N^2}(N^2 - N)\langle \rho_0 * V_N, \rho_0\rangle + \frac{1}{N}V_N(0) \\
&= \|\rho_0 * W_N\|_{L^2}^2 - \frac{1}{N}\|\rho_0 * W_N\|_{L^2}^2 + \frac{1}{N}V_N(0).
\end{aligned}$$

A similar calculation for the second summand shows

$$\begin{aligned}
\mathbb{E}\langle h_N(0), \rho_0\rangle &= \frac{1}{N}\sum_{k=1}^{N}\int_{\mathbb{R}^d} (\rho_0 * W_N)(x)\rho_0(x)\,dx \\
&= \langle \rho_0 * W_N, \rho_0\rangle.
\end{aligned}$$

Using $\|\rho_0 - \rho_0 * W_N\|_{L^2}^2 = \|\rho_0\|_{L^2}^2 - \|\rho_0 * W_N\|_{L^2}^2 - 2\langle \rho_0 * W_N, \rho_0\rangle$, we finally arrive at

$$\mathbb{E}\|h_N(0) - \rho_0\|_{L^2}^2 = \|\rho_0 - \rho_0 * W_N\|_{L^2}^2 - \frac{1}{N}\|\rho_0 * W_N\|_{L^2}^2 + \frac{1}{N}V_N(0).$$

Applying lemma 1.6.5 (b), $\|\rho_0 * W_N\|_{L^2} \le \|\rho_0\|_{L^2}$ and $V_N(0) \le C\chi_N^d$, we obtain

$$\mathbb{E}\|h_N(0) - \rho_0\|_{L^2}^2 \underset{N\to\infty}{\longrightarrow} 0.$$

□

Chapter 2

Proof of the convergence

2.1 A Convergence Result

In this section we give a proof of our main convergence result, theorem 2.1.7. We show that the smoothed version of the empirical process $(h_N(t))_t$ converges to a deterministic family of densities $(\rho(\cdot,t))_t$ that is given as a solution of the nonlinear partial differential equation (1) of advection-reaction-diffusion type. If assumption (A1) holds, i.e., we assume the existence of a sufficiently regular solution of equation (1), this proof works in both cases $\sigma_\infty > 0$ and $\sigma_\infty = 0$. We start by giving a short outline of the proof of the convergence of $(h_N(t))_t$ to its deterministic limit. The proof consists of two major steps.

- First, we write down the expression $\|h_N(t) - \rho(t)\|_{L^2}^2$ as integrals and stochastic integrals from 0 to t using Ito's formula and the dynamics of the solution ρ given by equation (1). This expansion is derived in theorem 2.1.3.

- In the second step, we give upper bounds for all the terms occurring during the first step such that we are able to apply Gronwall's lemma, see lemma 2.1.4.

In the next section we present and prove a generalization of this result for a model with a predator. Two different laws of motion for the predator are considered. The proof uses the results from this section.

In the following lemma we separately compute the time evolution of the terms on the right-hand side of

$$\|h_N(t) - \rho(t)\|_{L^2}^2 = \|h_N(t)\|_{L^2}^2 - 2\langle h_N(t), \rho(t)\rangle + \|\rho(t)\|_{L^2}^2. \tag{2.1}$$

2.1.1 Lemma. *For all $t \in [0,T]$ we have:*

(a) $$\|\rho(t)\|_{L^2}^2 = \|\rho(0)\|_{L^2}^2 - 2\int_0^t \langle \rho(s), |\nabla\rho(s)|^2\rangle_{L^2}\, ds + 2\int_0^t \langle \rho(s), (\nabla G * \rho(s))(\cdot)\cdot\nabla\rho(s)\rangle_{L^2}\, ds - \sigma_\infty^2\int_0^t \|\nabla\rho(s)\|_{L^2}^2\, ds.$$

(b) $$\|h_N(t)\|_{L^2}^2 = \|h_N(0)\|_{L^2}^2 + 2\int_0^t \langle X_N(s), \nabla g_N(s)\cdot(\nabla G * X_N(s))\rangle\, ds - 2\int_0^t \langle X_N(s), |\nabla g_N(s)|^2\rangle\, ds + \frac{2}{N}\sigma_N\sum_{k=1}^N\int_0^t \nabla g_N(X_N^k(s))\cdot d\mathbb{W}^k(s) - \sigma_N^2\int_0^t \|\nabla h_N(s)\|_{L^2}^2\, ds - \sigma_N^2\frac{1}{N}t\Delta V_N(0)$$

(c) $$\langle h_N(t), \rho(t)\rangle = \langle h_N(0), \rho(0)\rangle + \int_0^t \langle X_N(s), (\nabla G * X_N(s) - \nabla g_N(s))\cdot(\nabla\rho(s) * W_N)(\cdot)\rangle\, ds + \int_0^t \langle \rho(s), (\rho(s) * \nabla G - \nabla\rho(s))\cdot\nabla h_N(s)\rangle ds + \frac{1}{N}\sigma_N\sum_k\int_0^t \nabla(\rho(s) * W_N)(X_N^k(s))d\mathbb{W}^k(s) - \frac{1}{2}\left(\sigma_N^2 + \sigma_\infty^2\right)\int_0^t \langle\nabla h_N(s), \nabla\rho(s)\rangle\, ds.$$

Proof. (a) Using the partial differential equation (1) and partial integration for the last equality, we obtain

$$\begin{aligned}\|\rho(t)\|_{L^2}^2 &= \|\rho(0)\|_{L^2}^2 + \int_0^t \frac{d}{ds}\|\rho(s)\|_{L^2}^2\, ds\\ &= \|\rho(0)\|_{L^2}^2 + 2\int_0^t \langle \rho(s), \frac{d}{ds}\rho(s)\rangle_{L^2}\, ds\end{aligned}$$

$$
\begin{aligned}
&= \|\rho(0)\|_{L^2}^2 + \sigma_\infty^2 \int_0^t \langle \rho(s), \Delta\rho(s)\rangle \, ds \\
&\quad + 2\int_0^t \langle \rho(s), \nabla\cdot(\rho(s)\nabla\rho(s))\rangle_{L^2} \, ds \\
&\quad - 2\int_0^t \langle \rho(s), \nabla\cdot(\rho(s)(\nabla G * \rho(s))(\cdot))\rangle_{L^2} \, ds \\
&= \|\rho(0)\|_{L^2}^2 - \sigma_\infty^2 \int_0^t \|\nabla\rho(s)\|_{L^2}^2 \, ds \\
&\quad - 2\int_0^t \langle \rho(s), |\nabla\rho(s)|^2\rangle_{L^2} \, ds \\
&\quad + 2\int_0^t \langle \rho(s), (\nabla G * \rho(s))(\cdot)\cdot\nabla\rho(s)\rangle_{L^2} \, ds.
\end{aligned}
$$

(b) Applying lemma 1.6.2 (a) and (d), we get

$$
\begin{aligned}
\|h_N(t)\|_{L^2}^2 &= \langle X_N(t) * W_N, X_N(t) * W_N\rangle \\
&= \langle X_N(t), X_N(t) * V_N\rangle \\
&= \frac{1}{N^2}\sum_{k,l=1}^N V_N(X_N^k(t) - X_N^l(t)).
\end{aligned} \tag{2.2}
$$

Let us write $X_N^{k,(i)}$ for the i-th component of the d-dimensional stochastic process X_N^k. Now, applying Itô's formula to the expression $V_N(X_N^k(t) - X_N^l(t))$, we obtain

$$
\begin{aligned}
\|h_N(t)\|_{L^2}^2 = &\frac{1}{N^2}\sum_{k,l=1}^N V_N(X_N^k(0) - X_N^l(0)) \\
&+ \frac{1}{N^2}\sum_{k,l=1}^N \int_0^t \nabla V_N(X_N^k(s) - X_N^l(s))\cdot d(X_N^k - X_N^l)(s) \\
&+ \frac{1}{2N^2}\sum_{k,l=1}^N \int_0^t \sum_{i,j=1}^d \partial_i\partial_j V_N(X_N^k(s) - X_N^l(s)) \\
&\qquad\qquad \times d\langle X_N^{k,(i)} - X_N^{l,(i)}, X_N^{k,(j)} - X_N^{l,(j)}\rangle(s) \\
=: &\, T_1 + T_2 + T_3.
\end{aligned}
$$

Here, we used

$$\sum_{i=1}^{d} \int_0^t \partial_i V_N(X_N^{k,(i)}(s) - X_N^{l,(i)}(s)) \cdot d(X_N^k - X_N^l)(s)$$
$$= \int_0^t \nabla V_N(X_N^k(s) - X_N^l(s)) \cdot d(X_N^k - X_N^l)(s).$$

For the first term T_1 we get (cf. equation (2.2))

$$T_1 = \frac{1}{N^2} \sum_{k,l=1}^{N} V_N(X_N^k(0) - X_N^l(0)) = \|h_N(0)\|_{L^2}^2.$$

Note that for any bounded antisymmetric measurable function f from $\mathbb{R}^d$ to $\mathbb{R}^d$ we have

$$\sum_{k,l=1}^{N} \int_0^t f(X_N^k(s) - X_N^l(s)) \cdot d(X_N^k - X_N^l)(s)$$
$$= 2 \sum_{k,l=1}^{N} \int_0^t f(X_N^k(s) - X_N^l(s)) \cdot dX_N^k(s).$$

This and the stochastic differential equation (1.3) give us

$$\begin{aligned}
T_2 &= \frac{2}{N^2} \sum_{k,l=1}^{N} \int_0^t \nabla V_N(X_N^k(s) - X_N^l(s)) \cdot dX_N^k(s) \\
&= \frac{2}{N^2} \sum_{k,l=1}^{N} \int_0^t \nabla V_N(X_N^k(s) - X_N^l(s)) \cdot (\nabla G * X_N(s))(X_N^k(s))\, ds \\
&\quad - \frac{2}{N^2} \sum_{k,l=1}^{N} \int_0^t \nabla V_N(X_N^k(s) - X_N^l(s)) \cdot (\nabla V_N * X_N(s))(X_N^k(s))\, ds \\
&\quad + \frac{2}{N^2}\sigma_N \sum_{k,l=1}^{N} \int_0^t \nabla V_N(X_N^k(s) - X_N^l(s)) \cdot d\mathbb{W}^k(s) \\
&= 2 \int_0^t \langle X_N(s), (\nabla V_N * X_N(s)) \cdot (\nabla G * X_N(s)) \rangle\, ds \\
&\quad - 2 \int_0^t \langle X_N(s), (\nabla V_N * X_N(s)) \cdot (\nabla V_N * X_N(s)) \rangle\, ds \\
&\quad + \frac{2}{N}\sigma_N \sum_{k=1}^{N} \int_0^t (\nabla V_N * X_N(s))(X_N^k(s)) \cdot d\mathbb{W}^k(s).
\end{aligned}$$

Let us consider T_3. Please observe that the stochastic differential equation (1.3) directly implies that for $k, l \in \{1, \ldots, N\}$ and $i, j \in \{1, \ldots, d\}$ we have

$$d\langle X_N^{k,(i)} - X_N^{l,(i)}, X_N^{k,(j)} - X_N^{l,(j)}\rangle(s) = \begin{cases} 2\sigma_N^2 ds & \text{for } k \neq l,\ i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Thus, we obtain

$$\begin{aligned} T_3 &= \frac{\sigma_N^2}{N^2} \sum_{\substack{k,l=1 \\ k \neq l}}^N \int_0^t \Delta V_N(X_N^k(s) - X_N^l(s)) ds \\ &= \frac{\sigma_N^2}{N^2} \sum_{k,l=1}^N \int_0^t \Delta V_N(X_N^k(s) - X_N^l(s)) ds - \frac{\sigma_N^2}{N} t \Delta V_N(0) \\ &= -\sigma_N^2 \int_0^t \|h_N(s)\|_{L^2}^2 \, ds - \frac{\sigma_N^2}{N} t \Delta V_N(0). \end{aligned}$$

For the last equation we used lemma 1.6.3. This completes the proof of part (b).

(c) Since the mixed term $\langle h_N(\cdot, t), \rho(\cdot, t)\rangle$ contains both the deterministic density ρ and the stochastic smoothed version of the empirical process h_N we have to use equation (1) and Itô's formula to derive the claimed result. First, note that

$$d\langle X_N^{k,(i)}, X_N^{k,(j)}\rangle(s) = \delta_{ij}\sigma_N^2 ds \tag{2.4}$$

holds for any $k \in \{1, \ldots, N\}$ and any $i, j \in \{1, \ldots, d\}$. Now, we apply Itô's formula to the function $\rho * W_N$ and obtain

$$\begin{aligned} &\langle h_N(t), \rho(t)\rangle \\ &\quad = \langle X_N(t), \rho(t) * W_N\rangle \\ &\quad = \frac{1}{N} \sum_{k=1}^N (\rho(t) * W_N)(X_N^k(t)) \\ &\quad = \langle X_N(0), \rho(0) * W_N\rangle + \frac{1}{N} \sum_{k=1}^N \int_0^t (\nabla \rho(s) * W_N)(X_N^k(s)) \, dX_N^k(s) \\ &\qquad + \frac{1}{N} \sum_{k=1}^N \int_0^t (\partial_s \rho(s) * W_N)(X_N^k(s)) ds \\ &\qquad + \frac{1}{2N} \sum_{k=1}^N \sum_{i,j=1}^d \int_0^t (\partial_i \partial_j \rho(s) * W_N)(X_N^k(s)) \, d\langle X_N^{k,(i)}, X_N^{k,(j)}\rangle(s) \end{aligned}$$

$$
\begin{aligned}
&= \langle h_N(0), \rho(0)\rangle + \sum_{k=1}^{N} \int_0^t (\nabla\rho(s) * W_N)(X_N^k(s))\, dX_N^k(s) \\
&\quad + \int_0^t \langle X_N(s), \partial_s\rho(s) * W_N\rangle ds. \\
&\quad + \frac{1}{2}\sigma_N^2 \int_0^t \langle X_N(s), (\Delta\rho(s) * W_N)(\cdot)\rangle\, ds \\
&=: \langle h_N(0), \rho(0)\rangle + U_1 + U_2 + U_3.
\end{aligned}
$$

Taking the stochastic differential equation (1.3) into account, we get for the first term

$$
\begin{aligned}
U_1 = & \int_0^t \langle X_N(s), (\nabla G * X_N(s) - \nabla g_N(s)) \cdot (\nabla\rho(s) * W_N)(\cdot)\rangle\, ds \\
& + \frac{\sigma_N}{N} \sum_{k=1}^{N} \int_0^t \nabla(\rho(s) * W_N)(X_N^k(s)) d\mathbb{W}^k(s).
\end{aligned}
$$

Using Lemma 1.6.2 (d), the differential equation (1) and partial integration in the last step, we obtain

$$
\begin{aligned}
U_2 &= \int_0^t \langle X_N(s), \partial_s\rho(s) * W_N\rangle\, ds \\
&= \int_0^t \langle h_N(s), \partial_s\rho(s)\rangle\, ds \\
&= \int_0^t \langle h_N(s), \frac{\sigma_\infty^2}{2}\Delta\rho(s)\rangle\, ds \\
&\quad + \int_0^t \langle h_N(s), \nabla\cdot(\rho(s)\nabla\rho(\cdot, s)) - \nabla\cdot(\rho(s)(\rho(s) * \nabla G)(\cdot))\rangle ds \\
&= -\frac{\sigma_\infty^2}{2} \int_0^t \langle \nabla h_N(s), \nabla\rho(s)\rangle\, ds \\
&\quad + \int_0^t \langle \rho(s), (\rho(s) * \nabla G - \nabla\rho(s)) \cdot \nabla h_N(s)\rangle ds.
\end{aligned}
$$

Again, by lemma 1.6.2 (d) we get

$$
\begin{aligned}
U_3 &= \frac{\sigma_N^2}{2} \int_0^t \langle X_N(s), \Delta\rho(s) * W_N\rangle\, ds \\
&= -\frac{\sigma_N^2}{2} \int_0^t \langle \nabla h_N(s), \nabla\rho(s)\rangle\, ds.
\end{aligned}
$$

This completes the proof of part (c). □

The following theorem gives a decomposition of the expression $\|h_N(\cdot,t)-\rho(\cdot,t)\|_{L^2}^2$ into an aggregation part, a repulsion part, a diffusion part, a martingale part and a minor term. Let us introduce the following abbreviations to keep the notation short. We define

$$\begin{aligned} R(s) := &- \langle X_N(s), \nabla g_N(s) \cdot (\nabla g_N(s) - \nabla \rho(s) * W_N)\rangle \\ &- \langle \rho(s), \nabla \rho(s) \cdot (\nabla \rho(s) - \nabla h_N(s))\rangle\,, \end{aligned} \tag{2.5}$$

$$\begin{aligned} A(s) := &\langle X_N(s), (X_N(s) * \nabla G) \cdot (\nabla g_N(s) - \nabla \rho(s) * W_N)\rangle \\ &+ \langle \rho(s), (\rho(s) * \nabla G) \cdot (\nabla \rho(s) - \nabla h_N(s))\rangle \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} D(s) := &-\sigma_N^2 \left[\|\nabla h_N(s)\|_{L^2}^2 - \langle \nabla h_N(s), \nabla \rho(s)\rangle\right] \\ &- \sigma_\infty^2 \left[\|\nabla \rho(s)\|_{L^2}^2 - \langle \nabla h_N(s), \nabla \rho(s)\rangle\right]. \end{aligned} \tag{2.7}$$

Finally, let M_N denote the stochastic integral part of equation (2.9), i.e.

$$M_N(t) := \frac{2\sigma_N}{N} \int_0^t \sum_k \left(\nabla g_N(\cdot,s) - \nabla \rho(\cdot,s) * W_N\right)(X_N^k(s))\, d\mathbb{W}^k(s). \tag{2.8}$$

2.1.2 Remark. Because of lemma 1.6.2 (b), we have $\nabla W_N, W_N \in L^2(\mathbb{R})$. Therefore, $\nabla V_N = W_N * \nabla W_N$ is a bounded and continuous function. Hence, $\left(\nabla g_N(\cdot,s) - \nabla \rho(\cdot,s) * W_N\right)(X_N^k(s))$ is a bounded progressively measurable process and M_N is a martingale.

Observe that $A(s)$ contains the terms corresponding to the aggregation forces between the particles, $R(s)$ contains the terms corresponding to the repulsion forces and $D(s)$ the terms corresponding to the diffusion part. Using the notation from (2.5)-(2.8), we can state the main result of the first step.

2.1.3 Theorem. *We have*

$$\begin{aligned} \|h_N(t) - \rho(t)\|_{L^2}^2 = &\|h_N(0) - \rho(0)\|_{L^2}^2 + 2\int_0^t R(s)\, ds + 2\int_0^t A(s)\, ds \\ &+ 2\int_0^t D(s)\, ds - t\sigma_N^2 \frac{1}{N}\Delta V_N(0) + M_N(t), \qquad t \in [0,T]. \end{aligned} \tag{2.9}$$

Proof. Combining equation (2.1) with lemma 2.1.1 and sorting the terms on the right hand side we get equation (2.9). □

Our main task now is to find suitable estimates for all terms of this decomposition. In order to give a proof for these inequalities we will make frequently use of the lemmas 1.6.5 and 1.6.6.

2.1.4 Lemma. *For all $s \in [0,T]$ and all $N \in \mathbb{N}$ we have:*

(a) $R(s) \le C\|h_N(s)-\rho(s)\|_{L^2}^2 + C\chi_N^{-2}$,

(b) $A(s) \le C\|h_N(s)-\rho(s)\|_{L^2}^2 + C\chi_N^{-2}$,

(c) $|\Delta V_N(0)| \le C\chi_N^{d+2}$,

(d) $-\sigma_N^2 \left[\|\nabla h_N(s)\|_{L^2}^2 - \langle \nabla h_N(s), \nabla\rho(s)\rangle\right] \le C\sigma_N^2$,

(e) $\mathbb{E}\sup_{t\le T}|M_N(t)| \le \frac{CT\sigma_N^2}{N}\chi_N^{d+2} + \mathbb{E}\sup_{t\le T}\|h_N(t)-\rho(t)\|_{L^2}^2$,

(f) $D(s) \le C|\sigma_\infty^2 - \sigma_N^2|\|h_N(s)-\rho(s)\|_{L^2}$.

Proof. We start with the proof of part (a). Observe that

$$\begin{aligned}
R(s) = &- \langle X_N(s), \nabla g_N(s)\,(\nabla g_N(s) - \nabla\rho(s) * W_N)\rangle \\
&- \langle \rho(s), \nabla\rho(s)\cdot(\nabla\rho(s) - \nabla h_N(s))\rangle \\
= &- \left\langle X_N(s), |\nabla g_N(s) - \nabla\rho(s) * W_N|^2\right\rangle \\
&- \langle X_N(s), (\nabla g_N(s) - \nabla\rho(s) * W_N)\cdot(\nabla\rho(s) * W_N)\rangle \\
&- \langle \rho(s), \nabla\rho(s)\cdot(\nabla\rho(s) - \nabla h_N(s))\rangle\,.
\end{aligned}$$

Since the first term on the right hand side of this equation is non-positive, it is sufficient to show that

$$\begin{aligned}
\tilde R(s) := &-\langle X_N(s), (\nabla g_N(s) - \nabla\rho(s) * W_N)\cdot(\nabla\rho(s) * W_N)\rangle \\
&- \langle \rho(s), \nabla\rho(s)\cdot(\nabla\rho(s) - \nabla h_N(s))\rangle \\
\le &\; C\|h_N(s)-\rho(s)\|_{L^2}^2 + \chi_N^{-2}.
\end{aligned} \tag{2.10}$$

An elementary calculation gives us

$$\begin{aligned}
\tilde R(s) = &-\langle X_N(s) - \rho(s), (\nabla g_N(s) - \nabla\rho(s) * W_N)\cdot(\nabla\rho(s) * W_N)\rangle \\
&- \langle \rho(s), (\nabla g_N(s) - \nabla\rho(s) * W_N)\cdot(\nabla\rho(s) * W_N)\rangle \\
&- \langle \rho(s), \nabla\rho(s)\cdot(\nabla\rho(s) - \nabla h_N(s))\rangle \\
= &-\langle X_N(s) - \rho(s), (\nabla g_N(s) - \nabla\rho(s) * W_N)\cdot(\nabla\rho(s) * W_N)\rangle \\
&- \langle \rho(s)\nabla\rho(s) * W_N, \nabla g_N(s) - \nabla\rho(s) * W_N\rangle \\
&- \langle \rho(s)\nabla\rho(s), \nabla\rho(s) - \nabla h_N(s)\rangle\,.
\end{aligned}$$

Using partial integration for the last two summands, we obtain

$$\begin{aligned}\tilde{R}(s) &= -\langle X_N(s)-\rho(s), (\nabla g_N(s)-\nabla\rho(s)*W_N)\cdot(\nabla\rho(s)*W_N)\rangle \\ &\quad + \langle \nabla\cdot\rho(s)\nabla\rho(s)*W_N, g_N(s)-\rho(s)*W_N\rangle \\ &\quad + \langle \nabla\cdot\rho(s)\nabla\rho(s), \rho(s)-h_N(s)\rangle \\ &= -\langle X_N(s)-\rho(s), (\nabla g_N(s)-\nabla\rho(s)*W_N)\cdot\nabla\rho(s)*W_N\rangle \\ &\quad + \langle h_N(s)-\rho(s), \\ &\qquad (\nabla\cdot\rho(s)\nabla\rho(s)*W_N)*W_N - \nabla\cdot\rho(s)\nabla\rho(s)\rangle \\ &=: \tilde{R}_1(s)+\tilde{R}_2(s).\end{aligned}$$

From lemma 1.6.6 with $v=\nabla\rho*W_N=\rho*\nabla W_N\in C_b^{L+1}(\mathbb{R}^d)$ it follows that

$$|\tilde{R}_1(s)| \le C\|h_N(s)-\rho(s)\|_{L^2}^2+\chi_N^{-2} \tag{2.11}$$

and for $\tilde{R}_2$ Schwarz inequality gives

$$\begin{aligned}|\tilde{R}_2(s)| &= |\langle h_N(s)-\rho(s), \\ &\qquad (\nabla\cdot\rho(s)\nabla\rho(s)*W_N)*W_N-\nabla\cdot\rho(s)\nabla\rho(s)\rangle| \\ &\le \|h_N(s)-\rho(s)\|_{L^2}\times \\ &\qquad \|(\nabla\cdot\rho(s)\nabla\rho(s)*W_N)*W_N-\nabla\cdot\rho(s)\nabla\rho(s)\|_{L^2}.\end{aligned}$$

Using lemma 1.6.5 and equation (1.43) twice, we obtain for the second term

$$\begin{aligned}&\|(\nabla\cdot\rho(s)\nabla\rho(s)*W_N)*W_N-\nabla\cdot\rho(s)\nabla\rho(s)\|_{L^2} \\ &\quad\le \|(\nabla\cdot\rho(s)\nabla\rho(s)*W_N)*W_N-[\nabla\cdot\rho(s)\nabla\rho(s)]*W_N\|_{L^2} \\ &\qquad + \|[\nabla\cdot\rho(s)\nabla\rho(s)]*W_N-\nabla\cdot\rho(s)\nabla\rho(s)\|_{L^2} \\ &\quad\le C\chi_N^{-1}\|[\nabla\cdot\rho(s)\nabla\rho(s)]*W_N-\nabla\cdot\rho(s)\nabla\rho(s)\|_{L^2} \\ &\qquad + \|[\nabla\cdot\rho(s)\nabla\rho(s)]*W_N-\nabla\cdot\rho(s)\nabla\rho(s)\|_{L^2} \\ &\quad\le C(\chi_N^{-2}+\chi_N^{-1})\le C\chi_N^{-1}.\end{aligned}$$

Applying the elementary $ab\le a^2+b^2$, we get $\tilde{R}_2(s)\le C\|h_N(s)-\rho(s)\|_{L^2}^2+\chi_N^{-2}$. Together with (2.11) this shows $\tilde{R}(s)\le C\|h_N(s)-\rho(s)\|_{L^2}^2+\chi_N^{-2}$ and (a) follows. Next, we consider the aggregation term $A(s)$

$$\begin{aligned}A(s) &= \langle X_N(s), (X_N(s)*\nabla G)\,(\nabla g_N(s)-\nabla\rho(s)*W_N)\rangle \\ &\quad + \langle \rho(s), (\rho(s)*\nabla G)\,(\nabla\rho(s)-\nabla h_N(s))\rangle \\ &= \langle X_N(s)-\rho(s), (X_N(s)*\nabla G)\,(\nabla g_N(s)-\nabla\rho(s)*W_N)\rangle \\ &\quad + \langle \rho(s), (X_N(s)*\nabla G)\,(\nabla h_N(s)-\nabla\rho(s))*W_N\rangle \\ &\quad - \langle \rho(s), (\rho(s)*\nabla G)\,(\nabla h_N(s)-\nabla\rho(s))\rangle \\ &=: A_1(s)+A_2(s)+A_3(s).\end{aligned} \tag{2.12}$$

For A_1 we get by lemma 1.6.6 (a) with $v = X_N(s) * \nabla G$

$$|A_1(s)| \le C\|h_N(s) - \rho(s)\|_{L^2}^2 + \chi_N^{-2}. \tag{2.13}$$

Let us consider the terms A_2 and A_3:

$$\begin{aligned} A_2(s) + A_3(s) &= \langle [\rho(s)(X_N(s) * \nabla G)] * W_N, \nabla h_N(s) - \nabla\rho(s)\rangle \\ &\quad - \langle \rho(s)(\rho(s) * \nabla G), \nabla h_N(s) - \nabla\rho(s)\rangle \\ &= \langle [\rho(s)(X_N(s) * \nabla G)] * W_N - [\rho(s)(\rho(s) * \nabla G)] * W_N, \\ &\qquad \nabla h_N(s) - \nabla\rho(s)\rangle \\ &\quad + \langle [\rho(s)(\rho(s) * \nabla G)] * W_N - \rho(s)(\rho(s) * \nabla G), \\ &\qquad \nabla h_N(s) - \nabla\rho(s)\rangle \\ &=: \tilde{A}_2(s) + \tilde{A}_3(s). \end{aligned} \tag{2.14}$$

Using lemma 1.6.6 (c) with $U = \nabla G$, $f = \rho$ again, we obtain

$$\begin{aligned} |\tilde{A}_2(s)| &= |\langle (X_N(s) - \rho(s)) * \nabla G, \nabla(h_N(s) - \rho(s))\rho(s)\rangle| \\ &\le C\|h_N(s) - \rho(s)\|_{L^2}^2 + \chi_N^{-2}. \end{aligned} \tag{2.15}$$

Finally, lemma 1.6.5 together with equation (1.44) gives

$$\begin{aligned} |\tilde{A}_3(s)| &= \langle \nabla\cdot\rho(s)(\rho(s) * \nabla G) - \nabla\cdot\rho(s)(\rho(s) * \nabla G) * W_N, \\ &\qquad h_N(s) - \rho(s)\rangle \\ &\le C\chi_N^{-1}\|h_N(s) - \rho(s)\|_{L^2} \\ &\le C\chi_N^{-2} + C\|h_N(s) - \rho(s)\|_{L^2}^2. \end{aligned} \tag{2.16}$$

Summarizing inequalities (2.14) - (2.16) we get

$$A(s) = A_1(s) + \tilde{A}_2(s) + \tilde{A}_3(s) \le C\chi_N^{-2} + C\|h_N(s) - \rho(s)\|_{L^2}^2. \tag{2.17}$$

This shows (b).

(c) follows immediately from

$$\left|\Delta V_N(0)\right| = \left|\chi_N^d (\Delta V_1)(\chi_N x)\big|_{x=0}\right| \le C\chi_N^{d+2} \tag{2.18}$$

with the constant $C := \Delta V_1(0)$ and part (d) is implied by

$$\begin{aligned} &-\sigma_N^2\left[\|\nabla h_N(s)\|_{L^2}^2 - \langle \nabla h_N(s), \nabla\rho(s)\rangle\right] \\ &\quad\le -\frac{\sigma_N^2}{2}\left[\|\nabla h_N(s)\|_{L^2}^2 - 2\langle \nabla h_N(s), \nabla\rho(s)\rangle\right] \\ &\quad= \frac{\sigma_N^2}{2}\|\nabla\rho(s)\|_{L^2}^2 - \frac{\sigma_N^2}{2}\|\nabla(h_N(s) - \rho(s))\|_{L^2}^2 \end{aligned} \tag{2.19}$$

since $\|\nabla\rho(s)\|_{L^2}^2 \le C$, $s \in [0,T]$, by assumption (A3) and $-\frac{\sigma_N^2}{2}\|\nabla(h_N(s) - \rho(s))\|_{L^2}^2$ is negative.

(e) For any $T \in \mathbb{R}$ we have by Burkholder-Davis-Gundy inequality (see [14, chapter 3, thm 3.28])

$$\begin{aligned}\mathbb{E}\sup_{t\le T}|M_N(t)| &\le C\,\mathbb{E}\langle M_N(t)\rangle_T^{1/2}\\ &= \frac{2\sigma_N C}{N^{1/2}}\,\mathbb{E}\left[\int_0^T \langle X_N(s), |\nabla(h_N(s)-\rho(s)) * W_N|^2\rangle\, ds\right]^{1/2}.\end{aligned} \tag{2.20}$$

Furthermore, using Jensen inequality and Schwarz inequality in the last line, we obtain

$$\begin{aligned}\langle X_N(s),&|\nabla(h_N(\cdot,s)-\rho(\cdot,s)) * W_N|^2\rangle\\ &\le |\nabla(h_N(\cdot,s)-\rho(\cdot,s)) * W_N|_\infty^2\\ &= \left|\int_{\mathbb{R}_d}(h_N-\rho)(\cdot - y, s)\nabla W_N(y)\, dy\right|_\infty^2\\ &\le \|h_N(\cdot,s)-\rho(\cdot,s)\|_{L^2}^2\|\nabla W_N\|_{L^2}^2.\end{aligned} \tag{2.21}$$

In lemma 1.6.2 (b) we have already seen that $\|\nabla W_N\|_{L^2}^2 \le C\chi_N^{d+2}$ holds. Thus, we obtain

$$\int_0^T \langle X_N(s), |\nabla(h_N(s)-\rho(s)) * W_N|^2\rangle\, ds \le CT\chi_N^{d+2}\sup_{t\le T}\|h_N(t)-\rho(t)\|_{L^2}^2.$$

Therefore, we get from (2.20)

$$\mathbb{E}\sup_{t\le T}|M_N(t)| \le \frac{CT\sigma_N^2}{N}\chi_N^{d+2} + \mathbb{E}\sup_{t\le T}\|h_N(t)-\rho(t)\|_{L^2}^2. \tag{2.22}$$

The last inequality here was derived using the elementary $ab \le a^2 + b^2$ and Jensen's inequality. This completes the proof of (e). Finally, regrouping the terms on the right hand side of equation (2.7), we get

$$\begin{aligned}D(s) &= \sigma_\infty^2\langle\nabla\rho(s), \nabla(h_N(s)-\rho(s))\rangle\\ &\qquad - \sigma_N^2\langle\nabla h_N(s), \nabla(h_N(s)-\rho(s))\rangle\\ &= (\sigma_\infty^2 - \sigma_N^2)\langle\nabla\rho(s), \nabla(h_N(s)-\rho(s))\rangle\\ &\qquad - \sigma_N^2\|\nabla(h_N(s)-\rho(s))\|_{L^2}^2.\end{aligned}$$

Since $-\sigma_N^2 \|\nabla h_N(s) - \nabla \rho(s)\|_{L^2}^2$ is non-positive we can omit this term and obtain by partial integration

$$\begin{aligned} D(s) &\leq (\sigma_\infty^2 - \sigma_N^2)\langle \nabla \rho(s), \nabla h_N(s) - \nabla \rho(s)\rangle \\ &\leq |\sigma_\infty^2 - \sigma_N^2| \|\Delta \rho(s)\|_{L^2} \cdot \|h_N(s) - \rho(s)\|_{L^2}. \end{aligned}$$

Observe that $\|\Delta\rho(s)\|_{L^2}$ is bounded on the interval $[0, T]$ due to equation (1.45). □

2.1.5 Corollary. *For any $N \in \mathbb{N}$ we have*

$$\begin{aligned} \mathbb{E} \sup_{t \leq T} \|h_N(t) - \rho(t)\|_{L^2}^2 \leq \mathbb{E}\|h_N(0) - \rho(0)\|_{L^2}^2 + CT\gamma_N \\ + C \int_0^T \mathbb{E}\|h_N(s) - \rho(s)\|_{L^2}^2 \, ds \end{aligned} \tag{2.23}$$

where γ_N is given by

$$\gamma_N := \chi_N^{-2} + |\sigma_\infty^2 - \sigma_N^2| + \frac{\sigma_N^2}{N} \chi_N^{d+2}. \tag{2.24}$$

Proof. Applying the estimates from lemma 2.1.4 (a)-(d) and (f) to the terms on the right hand side of equation (2.9), gives

$$\begin{aligned} \|h_N(t) - \rho(t)\|_{L^2}^2 \leq & \|h_N(0) - \rho(0)\|_{L^2}^2 \\ & + Ct \left(\chi_N^{-2} + |\sigma_\infty^2 - \sigma_N^2| + \frac{\sigma_N^2}{N} \chi_N^{d+2} \right) \\ & + C \int_0^t \|h_N(s) - \rho(s)\|_{L^2}^2 \, ds \\ & + |M_N(t)|. \end{aligned} \tag{2.25}$$

Taking supremum over the time interval $[0, T]$ and expectation we arrive at

$$\begin{aligned} \mathbb{E} \sup_{t \leq T} \|h_N(t) - \rho(t)\|_{L^2}^2 \leq \mathbb{E} \sup_{t \leq T} & \|h_N(0) - \rho(0)\|_{L^2}^2 \\ & + CT \left(\chi_N^{-2} + |\sigma_\infty^2 - \sigma_N^2| + \frac{\sigma_N^2}{N} \chi_N^{d+2} \right) \\ & + C \int_0^T \mathbb{E}\|h_N(s) - \rho(s)\|_{L^2}^2 \, ds \\ & + \mathbb{E} \sup_{t \leq T} |M_N(t)|. \end{aligned} \tag{2.26}$$

Now, by taking lemma 2.1.4 (e) into account we get (2.23). □

Next, we need the following well-known lemma.

2.1.6 Lemma. *(Gronwall's Lemma) Let $u, \alpha : [0,T] \to \mathbb{R}$ be continuous functions with*

$$u(s) \leq \alpha(s) + K \int_0^s u(t)dt, \; s \in [0,T], \tag{2.27}$$

for a constant $K \geq 0$. Then

$$u(s) \leq \alpha(s) + K \int_0^s \alpha(t) e^{K(s-t)} dt \tag{2.28}$$

for all $s \in [0,T]$.

Proof. For the sake of completeness, we give the short proof. Let

$$f(t) := e^{-Kt} \int_0^t u(r)\, dr.$$

Using the product rule and the assumed integral inequality (2.27), we obtain for the derivative

$$\begin{aligned} f'(t) &= e^{-Kt} \left(u(t) - K \int_0^t u(r)\, dr \right) \\ &\leq e^{-Kt} \alpha(t). \end{aligned}$$

Now, integrating f' from 0 to s gives

$$f(s) \leq \int_0^s e^{-Kt} \alpha(t)\, dt.$$

Using this inequality and the definition of f, we obtain

$$\int_0^s u(r)\, dr \leq \int_0^s \alpha(t) e^{K(s-t)}\, dt.$$

Substituting this into inequality (2.27) completes the proof of Gronwall's Lemma. □

Now, we can give a proof of our main theorem.

2.1.7 Theorem. *Suppose assumptions (A1) to (A6) hold. Then*

$$\lim_{N\to\infty} \mathbb{E} \sup_{t \leq T} \|h_N(t) - \rho(t)\|_{L^2}^2 = 0. \tag{2.29}$$

Proof. Let

$$u_N(s) := \mathbb{E} \sup_{t \le s} \|h_N(t) - \rho(t)\|_{L^2}^2, \quad s \in [0, T].$$

Then it follows from equation (2.23) that

$$u_N(T) \le \mathbb{E}\|h_N(0) - \rho(0)\|_{L^2}^2 + CT\gamma_N + C\int_0^T u_N(s)\, ds \tag{2.30}$$

where γ_N is given by equation (2.24). Using Gronwall's Inequality 2.1.6, we obtain:

$$\begin{aligned} u_N(T) &\le \left[\mathbb{E}\|h_N(0) - \rho(0)\|_{L^2}^2 + CT\gamma_N\right] \left(1 + C\int_0^T \exp(C(T-s)\, ds\right) \\ &= \left[\mathbb{E}\|h_N(0) - \rho(0)\|_{L^2}^2 + CT\gamma_N\right] \exp(CT). \end{aligned} \tag{2.31}$$

Thus, in the limit $N \to \infty$ we get $u_N(T) \to 0$ since equation (1.22) from assumption (A5) gives us $\gamma_N \to 0$. □

2.1.8 Corollary. *In the situation of theorem 2.1.7, let $f \in C_b^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then*

$$\lim_{N\to\infty} \langle X_N(t), f\rangle = \langle \rho(t), f\rangle \tag{2.32}$$

uniformly in $t \in [0, T]$. The limit in equation (2.32) is with respect to convergence in probability.

Proof. First note that lemma 1.6.5 (a) gives us for any $t \in [0, T]$

$$\begin{aligned} |\langle X_N(t) - h_N(t), f\rangle| &= |\langle X_N(t), f - f * W_N\rangle| \\ &\le |f - f * W_N|_\infty \\ &\le C\chi_N^{-1}. \end{aligned}$$

This yields

$$\begin{aligned} |\langle X_N(t) - \rho(t), f\rangle| &\le |\langle X_N(t) - h_N(t), f\rangle| + |\langle h_N(t) - \rho(t), f\rangle| \\ &\le |\langle X_N(t), f - f * W_N\rangle| + C\|h_N(t) - \rho(t)\|_{L^2} \\ &\le C\chi_N^{-1} + C \sup_{t\le T} \|h_N(t) - \rho(t)\|_{L^2}. \end{aligned} \tag{2.33}$$

And now Theorem 2.1.7 implies the uniform convergence in probability. □

2.2 Model with Predator

In this section we observe the influence of a predator on the behavior of the animals in the swarm. The predator in the individual-based system is modeled as an additional particle that acts repulsive on all other particles in the system. In section 2.2.1 we consider a deterministic law of motion for the predator. Clearly, this is not a realistic model for the behavior of a predator. However, we can already see some of the essential features of a predator model. A predator who is attracted by the other particles in the swarm leads to a much more interesting model. A Lagrangian model of this type will be discussed in section 2.2.2.

Let $P_N(t)$ denote location of the predator in the space $\mathbb{R}^d$ at time t for the N-particle system. The predator acts in a way that adds an repulsive force on all other particles in the system. This is modeled by adding a force F^P to the drift term F_N in equation (1.5). This force is given by

$$\begin{aligned} F^P[P_N(t)](X_N^k(t)) &= -(\nabla H * \delta_{P_N(t)})(X_N^k(t)) \\ &= -(\nabla H)(X_N^k(t) - P_N(t)) \end{aligned}$$

where

$$H : \mathbb{R}^d \longrightarrow \mathbb{R} \tag{2.34}$$

is a sufficiently smooth function. More precisely, in addition to the assumptions from chapter 1 we require:

(AP) $\nabla H \in H^{L+1+\frac{d}{2}}(\mathbb{R}^d)$ where the constant L is as in assumption (A1), i.e., we have $L \in \mathbb{N}, L > \frac{d}{2} + 2$.

The function H is called the *predator potential*. Clearly, Sobolev's Lemma gives us $\nabla H \in C_b^{L+1}(\mathbb{R}^d)$. The repulsion given by H works in a very similar fashion as the aggregation potential G, i.e., there is no rescaling of the kernel.

In the limit N to infinity we derive the following deterministic limit equation

$$\begin{aligned} \partial_t \rho(x,t) = \frac{\sigma_\infty^2}{2}\Delta\rho(x,t) &+ \nabla\cdot(\rho(x,t)\nabla\rho(x,t)) \\ &- \nabla\cdot(\rho(x,t)(\nabla G * \rho(\cdot,t))(x)) \\ &+ \nabla\cdot(\rho(x,t)(\nabla H * \delta_{P_\infty(t)})(x)) \end{aligned} \tag{2.35}$$

$$\rho(x,0) = \rho_0(x). \tag{2.36}$$

for some continuous function $P_\infty : [0,T] \to \mathbb{R}^d$, depending on the law of motion of the predator particle.

2.2.1 Lemma. *There exists a constant $C > 0$ such that for all $t \in [0,T]$*

$$\|\nabla^2 \cdot [\rho(t)(\nabla H * \delta_{P_\infty(t)})]\|_{L^2} \leq C. \tag{2.37}$$

Proof. From assumption (AP) it follows that $\nabla H * \delta_{P_\infty(t)}$ is in $H^s(\mathbb{R}^d)$ for some $s > \frac{d}{2} + 2$. This together with $\rho \in C([0,T], H^{L+1+\frac{d}{2}}(\mathbb{R}^d))$ and proposition 1.4.4 (b) and (c) implies $\nabla^2 \cdot \rho(t)(\nabla H * \delta_{P_\infty(t)}) \in L^2(\mathbb{R}^d)$. Furthermore, the function

$$t \mapsto \|\nabla^2 \cdot [\rho(t)(\nabla H * \delta_{P_\infty(t)})]\|_{L^2}$$

is continuous. This completes the proof. □

We proceed now as in the previous section by splitting the expression $\|h_N - \rho\|^2_{L^2}$ into its parts. Let us define

$$\begin{aligned} RT(s) := &\langle \rho(s), (\nabla H * \delta_{P_\infty(s)}) \cdot (\nabla h_N(s) - \nabla \rho(s))\rangle \\ &- \langle X_N, (\nabla H * \delta_{P_N(s)}) \cdot (\nabla g_N(s) - \nabla \rho(s) * W_N)\rangle \end{aligned} \tag{2.38}$$

We have to repeat the calculations from lemma 2.1.1 and theorem 2.1.3. Let $\|\tilde{\rho}(t)\|^2_{L^2}, \|\tilde{h}_N(t)\|^2_{L^2}$, $\langle \tilde{h}_N(t), \tilde{\rho}(t)\rangle$ and $\|\tilde{h}_N(t) - \tilde{\rho}(t)\|^2_{L^2}$ be given by the right-hand sides of the equations in lemma 2.1.1, resp., the right-hand side of equation (2.9).

2.2.2 Theorem. *For all $N \in \mathbb{N}$ we have*

$$\|h_N(t) - \rho(t)\|^2_{L^2} = \|\tilde{h}_N(t) - \tilde{\rho}(t)\|^2_{L^2} + 2\int_0^t RT(s)ds \tag{2.39}$$

Proof. The same computation as in lemma 2.1.1 gives us

$$\begin{aligned} \|\rho(t)\|^2_{L^2} &= \|\tilde{\rho}(t)\|^2_{L^2} + 2\int_0^t \langle \rho(s), \nabla \cdot (\rho(s)(\nabla H * \delta_{P_\infty(s)})(\cdot))\rangle \, ds \\ &= \|\tilde{\rho}(t)\|^2_{L^2} - 2\int_0^t \langle \rho(s), (\nabla \rho(s)) \cdot (\nabla H * \delta_{P_\infty(s)})(\cdot)\rangle \, ds \end{aligned}$$

and

$$\begin{aligned} \|h_N(t)\|^2_{L^2} = \|\tilde{h}_N(t)\|^2_{L^2} - \frac{2}{N^2}\sum_{k,l=1}^N \int_0^t &(\nabla V_N)(X_N^k(s) - X_N^l(s)) \\ &\cdot (\nabla H * \delta_{P_N(s)})(X_N^k(s)) \, ds \\ = \|\tilde{h}_N(t)\|^2_{L^2} - 2\int_0^t &\langle X_N(s), (\nabla g_N)(s) \cdot (\nabla H * \delta_{P_N(s)})\rangle \, ds. \end{aligned}$$

Furthermore, we have

$$\begin{aligned}\langle h_N(t), \rho(t)\rangle_{L^2} &= \langle \tilde{h}_N(t), \tilde{\rho}(t)\rangle_{L^2} \\ &\qquad - \int_0^t \langle X_N(s), (\nabla H * \delta_{P_N(s)}) \cdot (\nabla \rho(s) * W_N)(\cdot)\rangle \, ds \\ &\qquad + \int_0^t \langle h_N(s), \nabla(\rho(s)(\nabla H * \delta_{P_\infty(s)}))\rangle \, ds \\ &= \langle \tilde{h}_N(t), \tilde{\rho}(t)\rangle_{L^2} \\ &\qquad - \int_0^t \langle X_N(s), (\nabla H * \delta_{P_N(s)}) \cdot (\nabla \rho(s) * W_N)(\cdot)\rangle \, ds \\ &\qquad + \int_0^t \langle \rho(s), \nabla h_N(s) \cdot (\nabla H * \delta_{P_\infty(s)}))\rangle \, ds.\end{aligned}$$

Applying equations (2.1) and (2.38), one obtains (2.39). □

2.2.1 Deterministic Predator

In this chapter we consider $P_N(t) := P_\infty(t) := P(t)$ for some deterministic function

$$P : [0, T] \to \mathbb{R}^d. \tag{2.40}$$

This implies

$$\begin{aligned}RT(s) = \langle \rho(s), (\nabla H * \delta_{P(s)}) \cdot (\nabla h_N(s) - \nabla \rho(s))\rangle \\ - \langle X_N, (\nabla H * \delta_{P(s)}) \cdot (\nabla g_N(s) - \nabla \rho(s) * W_N)\rangle.\end{aligned} \tag{2.41}$$

The following condition replaces assumption (A1) from section 1.5.

($\tilde{A}1$) For some fixed $T > 0$ and L as in (A1) the cauchy problem (2.35) has a nonnegative solution

$$\rho \in C([0, T], H^{L+1+\frac{d}{2}}(\mathbb{R}^d)).$$

in the sense that $\partial_t \rho(t)$ exists for almost every t and equation (2.35) holds for almost every t.

It remains to find an estimate for $RT(s)$ in terms of $\|h_N(s) - \rho(s)\|^2_{L^2}$.

2.2.3 Lemma. *For all $N \in \mathbb{N}$ we have*

$$RT(s) \le C\|h_N(s) - \rho(s)\|_{L^2}^2 + C\chi_N^{-2}, \quad s \in [0,T]. \tag{2.42}$$

Proof. We start by splitting $RT(s)$ into two parts

$$\begin{aligned} RT(s) = \langle X_N(s) - \rho(s), (\nabla H * \delta_{P(s)}) \cdot (\nabla h_N(s) - \nabla\rho(s)) * W_N \rangle \\ + \langle \rho(s), (\nabla H * \delta_{P(s)}) \\ \cdot [(\nabla h_N(s) - \nabla\rho(s)) * W_N - (\nabla h_N(s) - \nabla\rho(s))] \rangle \\ =: RT_1(s) + RT_2(s). \end{aligned}$$

For the RT_1 lemma 1.6.6 (a) with $v = \nabla G * \delta_{P(s)} \in C_b^{L+1}(\mathbb{R}^d)$ gives

$$RT_1(s) \le \|h_N(s) - \rho(s)\|_{L^2}^2 + C\chi_N^{-2} \tag{2.43}$$

and for second term $RT_2(s)$ we get by lemma 1.6.2 (d) and partial integration

$$\begin{aligned} RT_2(s) &= \langle (\nabla H * \delta_{P(s)})\rho(s), (\nabla h_N(s) - \nabla\rho(s)) * W_N - \nabla h_N(s) - \nabla\rho(s) \rangle \\ &= \langle \nabla \cdot [\rho(s)(\nabla H * \delta_{P(s)})] * W_N \\ &\qquad - \nabla \cdot [\rho(s)(\nabla H * \delta_{P(s)})], h_N(s) - \rho(s) \rangle. \end{aligned}$$

Using lemma (2.2.1), Cauchy's inequality and lemma 1.6.5 (b), we get

$$RT_2(s) \le C\chi_N^{-1}\|h_N(s) - \rho(s)\|_{L^2}^2 \le C\|h_N(s) - \rho(s)\|_{L^2}^2 + C\chi_N^{-2}. \tag{2.44}$$

This completes the proof of this lemma. □

Now, we get exactly as in the previous section.

2.2.4 Corollary. *For any $N \in \mathbb{N}$ we have*

$$\begin{aligned} \mathbb{E}\sup_{t\le T} \|h_N(t) - \rho(t)\|_{L^2}^2 \le \mathbb{E}\|h_N(0) - \rho(0)\|_{L^2}^2 + CT\gamma_N \\ + C\int_0^T \mathbb{E}\|h_N(s) - \rho(s)\|_{L^2}^2 \, ds \end{aligned} \tag{2.45}$$

where

$$\gamma_N := \chi_N^{-2} + |\sigma_\infty^2 - \sigma_N^2| + \frac{\sigma_N^2}{N}\chi_N^{d+2} \tag{2.46}$$

2.2.5 Theorem. *Suppose assumptions ($\tilde{A}1$), (A2) to (A6) and (AP) hold. Then*

$$\lim_{N\to\infty} \mathbb{E}\sup_{t\le T} \|h_N(t) - \rho(t)\|_{L^2}^2 = 0. \tag{2.47}$$

2.2.2 Stochastic Predator

Let us consider a predator whose movement is determined by a stochastic differential equation instead of a deterministic movement.

$$dP_N(t) = \int_{\mathbb{R}^d} h_N(x,t)w(x - P_N(t))dxdt + \sigma_{P,N} d\mathbb{W}_P(t), \tag{2.48}$$

$$P_N(0) = 0, \tag{2.49}$$

where $\mathbb{W}_P$ is a Brownian Motion independent of $\mathbb{W}^k, k \in \mathbb{N}$, and $\sigma_{P,N} > 0, N \in \mathbb{N}$, is a sequence with

$$\lim_{N\to\infty} \sigma_{P,N} = 0, \tag{2.50}$$

This condition assures that the movement of the predator is completely deterministic in the limit N to ∞. w is a weight function such that:

(AW) $w \in L^2(\mathbb{R}^d)$ and

$$\|w(\cdot - x) - w(\cdot - y)\|_{L^2} \leq C|x - y| \text{ for all } x, y \in \mathbb{R}^d \tag{2.51}$$

2.2.6 Remark. (Interpretation of equation (2.48))
If we take a weight functions of the form $w(x) = x\tilde{w}(x)$ with $\tilde{w} \geq 0$. Then equation (2.48) reads

$$dP_N(t) = \int_{\mathbb{R}^d} (x - P_N(t))h_N(x,t)\tilde{w}(x - P_N(t))dxdt + \sigma_{P,N} d\mathbb{W}_P(t), \tag{2.52}$$

$$P_N(0) = 0, \tag{2.53}$$

Note that $\int_{\mathbb{R}^d}(x - P_N(t))h_N(x,t)\tilde{w}(x - P_N(t))dx$ is a smoothed version of the barycenter of the particles $X_N, N \in \mathbb{N}$ relative to the position of the predator $P_N(t)$ at time t with some weight function $\tilde{w}$. Therefore, the term $\int_{\mathbb{R}^d}(x - P_N(t))h_N(x,t)w(x - P_N(t))dx$ implies that the predator is attracted (for $w > 0$) by the barycenter. Clearly, for $w \leq 0$ this would define a repulsive force. The function $\tilde{w}$ may be asymmetric and thus contain preferred directions of the predator.

Again, we can guess the limit of this system. The equation for the density of the swarming animals remains unchanged. But the movement of the predator now depends on ρ, which gives much more realistic ways for modeling animal behavior. For N to infinity we derive the following system of non-linear limit equations

$$\begin{aligned}
\partial_t \rho(x,t) &= \nabla \cdot (\rho(x,t)\nabla\rho(x,t)) - \nabla \cdot (\rho(x,t)(\nabla G * \rho(\cdot,t))(x)) \\
&\qquad + \nabla \cdot (\rho(x,t)(\nabla H * \delta_{P_\infty(t)})(x)), \quad x \in \mathbb{R}^d, \\
\rho(x,0) &= \rho_0(x), \quad x \in \mathbb{R}^d, \\
\partial_t P_\infty(t) &= \int_{\mathbb{R}^d} \rho(x,t) w(x - P_\infty(t))\, dx \\
P_\infty(0) &= 0.
\end{aligned} \tag{2.54}$$

A sufficient condition for the existence of a solution of this system will be given later on in chapter 3. For the moment, let us assume:

(AS) For some fixed $T > 0$ and L as in (A1) the cauchy problem (2.35) has a solution (ρ, P_∞) such that

$$\rho \in C([0,T], H^{L+1+\frac{d}{2}}(\mathbb{R}^d)), \quad P_\infty \in C([0,T], \mathbb{R}^d),$$

with non-negative ρ and $\partial_t \rho(t)$ exists for almost every t and equation (2.35) holds for almost every t.

This assumption replaces assumption (A1) in chapter 2.1, resp., assumption $(\tilde{A}1)$ from chapter 2.2.1.

2.2.7 Lemma. *We have*

$$|RT(s)| \le C(|P_N(s) - P_\infty(s)|^2 + \|h_N(\cdot,s) - \rho(\cdot,s)\|_{L^2}^2 + \chi_N^{-2}). \tag{2.55}$$

Proof. We start by splitting the interaction term corresponding to the predator action $RT(s)$ such that we can apply lemma 1.6.6.

$$\begin{aligned}
RT(s) &= \langle X_N(s) - \rho(s), (\nabla H * \delta_{P_N(s)}) \cdot (\nabla g_N(s) - \nabla\rho(s) * W_N)\rangle \\
&\quad + \langle \rho(s), (\nabla H * \delta_{P_N(s)}) \cdot (\nabla h_N(s) - \nabla\rho(s)) * W_N\rangle \\
&\quad - \langle \rho(s), (\nabla H * \delta_{P_\infty(s)}) \cdot (\nabla h_N(s) - \nabla\rho(s))\rangle \\
&= \langle X_N(s) - \rho(s), (\nabla H * \delta_{P_N(s)}) \cdot (\nabla g_N(s) - \nabla\rho(s) * W_N)\rangle \\
&\quad + \langle [\rho(s)(\nabla H * \delta_{P_N(s)})] * W_N, \nabla h_N(s) - \nabla\rho(s)\rangle \\
&\quad - \langle \rho(s)(\nabla H * \delta_{P_\infty(s)}), \nabla h_N(s) - \nabla\rho(s)\rangle \\
&= \langle X_N(s) - \rho(s), (\nabla H * \delta_{P_N(s)}) \cdot (\nabla g_N(s) - \nabla\rho(s) * W_N)\rangle \\
&\quad + \langle [\rho(s)(\nabla H * \delta_{P_N(s)})] * W_N - [\rho(s)(\nabla H * \delta_{P_\infty(s)})] * W_N, \\
&\qquad\qquad \nabla h_N(s) - \nabla\rho(s)\rangle \\
&\quad + \langle [\rho(s)(\nabla H * \delta_{P_\infty(s)})] * W_N - [\rho(s)(\nabla H * \delta_{P_\infty(s)})], \\
&\qquad\qquad \nabla h_N(s) - \nabla\rho(s)\rangle \\
&=: RT_1(s) + RT_2(s) + RT_3(s)
\end{aligned} \tag{2.56}$$

For RT_1 we obtain by lemma 1.6.6

$$|RT_1(s)| \le C(\|h_N(s) - \rho(s)\|_{L^2} + \chi_N^{-2}). \tag{2.57}$$

Using partial integration, lemma 2.2.1 and lemma 1.6.5, we obtain

$$|RT_3(s)| \le C(\|h_N(s) - \rho(s)\|_{L^2} + \chi_N^{-2}) \tag{2.58}$$

It remains to find an estimate for RT_2. Since assumption (AP) together with Sobolev's Imbedding Theorem 1.4.7 implies $\nabla H \in C_b^3(\mathbb{R}^d)$. the mean value inequality gives

$$|\Delta H(x - P_N(s)) - \Delta H(x - P_\infty(s))| \le C|P_N(s) - P_\infty(s)|, \tag{2.59}$$
$$|\nabla H(x - P_N(s)) - \nabla H(x - P_\infty(s))| \le C|P_N(s) - P_\infty(s)| \tag{2.60}$$

for all $x \in \mathbb{R}^d$. Hence, it follows that

$$\begin{aligned}
&\|\rho(s)(\Delta H * (\delta_{P_N(s)} - \delta_{P_\infty(s)}))\|_{L^2}^2 \\
&\qquad = \int_{\mathbb{R}^d} |\rho(x,s)|^2 |\Delta H(x - P_N(s)) - \Delta H(x - P_\infty(s))|^2\, dx \\
&\qquad \le C\|\rho(s)\|_{L^2}^2 |P_N(s) - P_\infty(s)|^2 \le C|P_N(s) - P_\infty(s)|^2
\end{aligned} \tag{2.61}$$

and

$$\begin{aligned}
&\|\nabla\rho(s)\cdot(\nabla H * (\delta_{P_N(s)} - \delta_{P_\infty(s)}))\|_{L^2}^2 \\
&\qquad \le \int_{\mathbb{R}^d} |\nabla\rho(x,s)|^2 |\nabla H(x - P_N(s)) - \nabla H(x - P_\infty(s))|^2\, dx \\
&\qquad \le C\|\nabla\rho(s)\|_{L^2}^2 |P_N(s) - P_\infty(s)|^2 \le C|P_N(s) - P_\infty(s)|^2.
\end{aligned} \tag{2.62}$$

Therefore, we get

$$
\begin{aligned}
RT_2(s) &\le \|\nabla \cdot [\rho(s)\nabla H * (\delta_{P_N(s)} - \delta_{P_\infty(s)})] * W_N\|_{L^2}\|h_N(s) - \rho(s)\|_{L^2} \\
&\le \|W_N\|_{L^1}\|\nabla \cdot [\rho(s)\nabla H * (\delta_{P_N(s)} - \delta_{P_\infty(s)})]\|_{L^2}\|h_N(s) - \rho(s)\|_{L^2} \\
&\le C|P_N(s) - P_\infty(s)|\|h_N(s) - \rho(s)\|_{L^2} \\
&\le C(|P_N(s) - P_\infty(s)|^2 + \|h_N(s) - \rho(s)\|_{L^2}^2).
\end{aligned} \tag{2.63}
$$

Now we have shown (see equations (2.56), (2.57), (2.58) and (2.63))

$$
|RT(s)| \le C(|P_N(s) - P_\infty(s)|^2 + \|h_N(\cdot, s) - \rho(\cdot, s)\|_{L^2}^2 + \chi_N^{-2}). \tag{2.64}
$$

□

2.2.8 Lemma. *We have*

$$
\begin{aligned}
|P_N(t) - P_\infty(t)|^2 \le C&|P_N(0) - P_\infty(0)|^2 \\
&+ C\int_0^t (\|h_N(s) - \rho(s)\|_{L^2}^2 + |P_N(s) - P_\infty(s)|^2)\, ds \\
&+ C\sigma_{P,N}^2|\mathbb{W}_P(t)|^2.
\end{aligned}
$$

Proof. For $|P_N(t) - P_\infty(t)|$ we get the following estimate from equation (2.48) and (2.54)

$$
\begin{aligned}
|P_N(t) - P_\infty(t)| &\le |P_N(0) - P_\infty(0)| + \left|\int_0^t dP_N - \int_0^t \frac{d}{ds}P_\infty(s)\, ds\right| \\
&\le |P_N(0) - P_\infty(0)| \\
&\quad + \int_0^t \left| \int_{\mathbb{R}^d} h_N(x,s)w(x - P_N(t))\, dx \right. \\
&\qquad\qquad \left. - \int_{\mathbb{R}^d} \rho(x,s)w(x - P_\infty(t))dx \right| ds \\
&\quad + \sigma_{P,N}|\mathbb{W}_P(t)| \\
&\le |P_N(0) - P_\infty(0)| \\
&\quad + \int_0^t \left| \int_{\mathbb{R}^d} [h_N(x,s) - \rho(x,s)]w(x - P_N(t))\, dx \right| ds \\
&\quad + \int_0^t \left| \int_{\mathbb{R}^d} \rho(x,s)[w(x - P_N(t)) - w(x - P_\infty(t))]dx \right| ds \\
&\quad + \sigma_{P,N}|\mathbb{W}_P(t)|.
\end{aligned}
$$

Taking the square of this inequality and estimating the mixed terms on the right hand side with $ab \leq a^2 + b^2$, we get

$$
\begin{aligned}
&|P_N(t) - P_\infty(t)|^2 \\
&\qquad \leq C|P_N(0) - P_\infty(0)|^2 \\
&\qquad + C\left(\int_0^t \left|\int_{\mathbb{R}^d} [h_N(x,s) - \rho(x,s)] w(x - P_N(t))\, dx\right| ds\right)^2 \\
&\qquad + C\left(\int_0^t \left|\int_{\mathbb{R}^d} \rho(x,s)[w(x - P_N(t)) - w(x - P_\infty(t))] dx\right| ds\right)^2 \\
&\qquad + C\sigma_{P,N}^2 |\mathbb{W}_P(t)|^2.
\end{aligned} \tag{2.65}
$$

Jensen's inequality gives us

$$
\begin{aligned}
&\left(\int_0^t \left|\int_{\mathbb{R}^d} [h_N(x,s) - \rho(x,s)] w(x - P_N(t))\, dx\right| ds\right)^2 \\
&\qquad\qquad \leq T \int_0^t \left|\int_{\mathbb{R}^d} [h_N(x,s) - \rho(x,s)] w(x - P_N(t))\, dx\right|^2 ds
\end{aligned} \tag{2.66}
$$

and

$$
\begin{aligned}
&\left(\int_0^t \left|\int_{\mathbb{R}^d} \rho(x,s)[w(x - P_N(t)) - w(x - P_\infty(t))] dx\right| ds\right)^2 \\
&\qquad \leq T \int_0^t \left|\int_{\mathbb{R}^d} \rho(x,s)[w(x - P_N(t)) - w(x - P_\infty(t))] dx\right|^2 ds.
\end{aligned} \tag{2.67}
$$

Using Schwarz inequality, we obtain

$$
\begin{aligned}
\left|\int_{\mathbb{R}^d} [h_N(x,s) - \rho(x,s)] w(x - P_N(s))\, dx\right| &\leq \|w\|_{L^2} \|h_N(s) - \rho(s)\|_{L^2} \\
&\leq C\|h_N(s) - \rho(s)\|_{L^2}
\end{aligned}
$$

for all $N \in \mathbb{N}$ since $w \in L^2(\mathbb{R}^d)$. For the inner integral in inequality (2.67) we get from assumption (AW)

$$
\left|\int_{\mathbb{R}^d} \rho(x,s)[w(x - P_N(s) - w(x - P_\infty(t))]\, dx\right| \leq C\|\rho(s)\|_{L^2} |P_N(s) - P_\infty(s)|.
$$

Since $\|\rho(s)\|_{L^2}$ is bounded and T is constant, the claimed result follows. $\square$

Taking supremum and expectation, we get

$$\begin{aligned}
&\mathbb{E}\sup_{t\le T}\|h_N(t)-\rho(t)\|_{L^2}^2+\mathbb{E}\sup_{t\le T}|P_N(t)-P_\infty(t)|^2\\
&\qquad\le \mathbb{E}\sup_{t\le T}\|\tilde{h}_N(s)-\tilde{\rho}(s)\|_{L^2}+2\int_0^t \mathbb{E}\sup_{t\le T}|RT(s)|\,ds\\
&\qquad\quad+\mathbb{E}\sup_{t\le T}|P_N(t)-P_\infty(t)|^2\\
&\qquad\le \mathbb{E}\|h_N(0)-\rho(s)\|_{L^2}^2+C\mathbb{E}|P_N(0)-P_\infty(0)|^2\\
&\qquad\quad+CT\left(\chi_N^{-2}+\sigma_N^2+\frac{\sigma_N^2}{N}\chi_N^{d+2}\right)\\
&\qquad\quad+C\int_0^T\left[\sup_{t\le T}\mathbb{E}\|h_N(s)-\rho(s)\|_{L^2}^2+\sup_{t\le T}\mathbb{E}|P_N(t)-P_\infty(t)|^2\right]\,ds\\
&\qquad\quad+C\sigma_{P,N}^2\sup_{t\le T}\mathbb{E}|\mathbb{W}_P(t)|^2.
\end{aligned} \tag{2.68}$$

2.2.9 Theorem. *Suppose assumptions (AS),(AW),(A3)-(A6) and (AP) hold. Furthermore, suppose that*

$$\lim_{N\to\infty}\mathbb{E}\|h_N(0)-\rho_0\|_{L^2}^2=0, \tag{2.69}$$

$$\lim_{N\to\infty}\mathbb{E}|P_N(0)-P_\infty(0)|^2=0. \tag{2.70}$$

Then

$$\lim_{N\to\infty}\mathbb{E}\sup_{t\le T}\left[\|h_N(t)-\rho(t)\|_{L^2}^2+|P_N(t)-P_\infty(t)|^2\right]=0 \tag{2.71}$$

Proof. This is the analogous result to theorem 2.1.7. Applying Gronwall's lemma 2.1.6 to equation (2.68) and letting $N\to\infty$, the result follows. Observe that $\sup_{t\le T}\mathbb{E}|\mathbb{W}_P(t)|^2\le\mathbb{E}\sup_{t\le T}|\mathbb{W}_P(t)|^2$ is finite due to Doob's L^p-inequality. □

Chapter 3

Existence of Solutions for the Limit PDE

In this chapter we give a for the existence of a local solution $\rho(x,t)$ of the nonlinear Cauchy problem:

$$
\begin{aligned}
\partial_t \rho(x,t) &= \nabla \cdot [(\frac{\sigma_\infty^2}{2} + \rho(x,t))\nabla \rho(x,t)] - \nabla \cdot ((\nabla G * \rho)\rho(x,t)), \quad t \in [0,T] \\
\rho(x,0) &= \rho_0(x)
\end{aligned}
\tag{3.1}
$$

This is the divergence form of the partial differential equation describing the Eulerian limit of the particle system in the diffusion case ($\sigma_\infty > 0$). In assumption (A1) the existence of this solution was posed as an assumption. We now give a sufficient condition for (A1). Furthermore, we show that the solution $\rho(\cdot,t)$ satisfies some regularity properties required in order to prove the convergence of the particle system to its limit equation, see theorem 2.1.7. For all s let $H^s_+(\mathbb{R}^d) := \{v \in H^s(\mathbb{R}^d) \mid v \geq 0\}$ be the set of all positive H^s functions on $\mathbb{R}^d$. For all $v \in H^{2n+3}_+(\mathbb{R}^d)$ we consider the linear operator $A(v) : H^{2n}(\mathbb{R}^d) \supseteq D(A) \to H^{2n}(\mathbb{R}^d)$ given by $D(A) := H^{2n+2}(\mathbb{R}^d)$ and

$$
A(v)\rho := \nabla \cdot [(\frac{\sigma_\infty^2}{2} + v)\nabla \rho] - \nabla \cdot ((\nabla G * v)\rho)
$$

for all functions $\rho \in D(A)$. Then the Cauchy problem (3.1) can be restated in the following form:

$$
\begin{aligned}
\partial_t \rho(x,t) &= A(\rho(\cdot,t))\rho(x,t), \quad t \in [0,T] \\
\rho(x,0) &= \rho_0(x)
\end{aligned}
$$

To show the existence of a solution we first consider a linearized version of the initial nonlinear Cauchy problem (3.1). We choose a fixed function $u_1 \in C([0,T], H^{2n}_+(\mathbb{R}^d))$ and replace the operator $A(\rho(\cdot,t))$ by $A(J_1 u(t))$, where $(J_j)_j$ is a sequence of mollifiers such that $J_j u(t) \in H^{2n+3}(\mathbb{R}^d)$ for all $u \in H^{2n}$:

$$\begin{aligned} \partial_t \rho(x,t) &= A(J_1 u_1(t))\rho(x,t), \quad t \in [0,T], \\ \rho(x,0) &= \rho_0(x). \end{aligned}$$

The operator $A(J_1 u_1(t))$ is uniformly elliptic. Thus this Cauchy problem has a unique positive solution u_2 which is again in $H^{2n}_+(\mathbb{R}^d)$. This solution can be constructed by semigroup methods. Therefore, we can inductively define a sequence $(u_j)_j$ such that u_{j+1} is the solution of

$$\begin{aligned} \partial_t \rho(x,t) &= A(J_j u_j(t))\rho(x,t), \quad t \in [0,T], \\ \rho(x,0) &= \rho_0(x). \end{aligned}$$

It remains to show that $(u_j)_j$ converges against a solution of the nonlinear Cauchy problem (3.1). This method is described in [16],[5] and [25]. Our main results are summarized in theorem 3.3.7. In section 3.4 we apply this iteration method to obtain a solution of the Eulerian stochastic predator system which was introduced in section 2.2.2 of the previous chapter.

Throughout this chapter we consider only the case $\sigma_\infty > 0$ (model with diffusion). In case $\sigma_\infty = 0$ the operator $A(v)$ is not uniformly elliptic and this method can not be applied.

3.1 C_0-semigroups and elliptic operators

In this section we fix our notation and collect some general results about C_0-semigroups and elliptic operators. A more detailed discussion of these topics can be found in [11], [12], [25] and [6].

3.1.1 Notation. During this chapter X denotes an arbitrary Banach space and A is always a linear operator $A : X \supseteq D(A) \to X$ on X with domain $D(A)$.

3.1.2 Definition. We say a linear operator $A : X \supseteq D(A) \to X$ is *dissipative on X* if

$$\|(\lambda - A)u\|_X \geq \lambda \|u\|_X$$

holds for all $u \in D(A)$ and all $\lambda > 0$.

3.1.3 Lemma. *Let A be a densely defined linear dissipative operator on X. Then A is closable.*

Proof. We take a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} Ax_n = y$ in X. In order to prove the stated result, it is sufficient to show that $y = 0$. Choose a second sequence $(y_n)_{n\in\mathbb{N}}$ with $(y_n)_{n\in\mathbb{N}} \subset D(A)$ such that $\lim_{n\to\infty} y_n = y$. Using the dissipativity of A, we obtain for all $\lambda > 0$ and all $n \in \mathbb{N}$:

$$\begin{aligned}\lambda\|y_n\|_X &= \lim_{m\to\infty} \lambda\|y_n - \lambda x_m\|_X \\ &\le \lim_{m\to\infty} \|(\lambda - A)(y_n - \lambda x_m)\|_X \\ &= \|(\lambda - A)y_n - \lambda y\|_X.\end{aligned}$$

This implies

$$\begin{aligned}\|y_n\|_X &\le \|\frac{1}{\lambda}(\lambda - A)y_n - y\|_X \\ &= \|y_n - \frac{1}{\lambda}Ay_n - y\|_X \underset{\lambda\to\infty}{\longrightarrow} \|y_n - y\|_X.\end{aligned}$$

Letting $n \to \infty$, we get $y = 0$. □

3.1.4 Definition. Let $L(x)$ be a second order differential operator given by

$$L(x) = \sum_{k,l=1}^{d} a_{kl}(x)\partial_k\partial_l + \sum_{j=1}^{d} b_j(x)\partial_j + c(x) \tag{3.2}$$

with coefficients $a_{kl}, b_j, c \in C_b(\mathbb{R}^d, \mathbb{R})$. We say the operator $L(x)$ is *uniformly elliptic* if there exists a constant $C_0 > 0$ such that

$$\sum_{k,l=1}^{d} a_{kl}(x)\xi_k\xi_l \ge C_0|\xi|^2$$

holds for all $\xi \in \mathbb{R}^d$ and all $x \in \mathbb{R}^d$. The constant C_0 is called *constant of ellipticity.*

For a proof of the following theorem see theorem 2.1.42 in the book of Jacob [12].

3.1.5 Theorem. *Let $L(x)$ be a uniformly elliptic second order differential operator as in (3.2) with coefficients $a_{kl} = a_{lk} \in C_b^2(\mathbb{R}^d)$, $b_j \in C_b^1(\mathbb{R}^d)$ and $c \in C_b(\mathbb{R}^d, \mathbb{R})$. Further, let $\lambda \ge \lambda_0$ sufficiently large. Then the operator $L(x, D) - \lambda$ is a bounded bijective operator from $H^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ with bounded inverse.*

3.1.6 Definition. Let X be a subspace of the space of all measurable functions on $\mathbb{R}^d$ and let $A : X \supseteq D(A) \to X$ be a linear operator. We say that A satisfies the *positive maximum principle* if $C_c^\infty(\mathbb{R}^d) \subseteq D(A)$ and for any $f \in C_c^\infty(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$ with $f(x_0) = \sup_{x\in\mathbb{R}^d} f(x) \geq 0$ we have $Af(x_0) \leq 0$.

3.1.7 Definition. The set $\rho(A)$ of all $\lambda \in \mathbb{C}$ for which $\lambda - A$ is invertible, i.e.,

$$\rho(A) := \{\lambda \in \mathbb{C} \mid (\lambda - A)^{-1} \in L(X)\},$$

is called the *resolvent set of* A. The family of operators

$$R_\lambda^A := (\lambda - A)^{-1}, \quad \lambda \in \rho(A)$$

is called the *resolvent* of A.

3.1.8 Definition. A C_0-semigroup $(T(t))_{t\geq 0}$ on a space of functions X with $C_c^\infty(\mathbb{R}^d) \subseteq X$ dense is called *positivity preserving* if $T(t)f \geq 0$ holds for every $t \geq 0$ and every positive continuous function $f \in C_c^\infty(\mathbb{R}^d) \subseteq D(A)$ where A is the generator of the semigroup $(T(t))_{t\geq 0}$.

Next, we state the well-known Lumer-Phillips-Theorem. For a proof see, for example, Theorem 4.3 in Pazy [25].

3.1.9 Theorem. *(Lumer-Phillips) Let A be a densely defined linear operator on X. If A is dissipative and there exists a $\lambda > 0$ such that*

$$R(\lambda - A) = X,$$

where $R(\lambda - A)$ is the range of $\lambda - A$, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.

3.1.10 Definition. The bounded operator $A_\lambda := \lambda^2 R_\lambda^A - \lambda \,\mathrm{id}$ is called *Yosida approximation of A*

3.1.11 Lemma. *Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ of contractions on X and A_λ its Yosida approximation. Then*

$$\lim_{\lambda\to\infty} e^{tA_\lambda} u = T(t)u \text{ for all } u \in X.$$

3.1.12 Lemma. *Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ of contractions on X. Further, let $M \subset X$ and assume that*

$$\lambda R_\lambda^A : M \to M$$

for all $\lambda > 0$. Then for every $t \geq 0$ the operator $T(t)$ leaves M invariant.

Proof. For $\lambda > 0$ let A_λ denote the Yosida approximation (see definition 3.1.10) of A. Furthermore, we have for all $t > 0$

$$e^{tA_\lambda} = e^{-t\lambda}\, e^{t\lambda^2 R_\lambda^A} = e^{-t\lambda} \sum_{j=0}^{\infty} \frac{(t\lambda)^j}{j!} (\lambda R_\lambda^A)^j. \tag{3.3}$$

This implies that $\exp(tA_\lambda)$ leaves M invariant. From lemma 3.1.11 we get

$$\lim_{\lambda\to\infty} e^{tA_\lambda} u = T(t)u.$$

Therefore, $T(t)$ leaves M invariant. □

3.1.13 Lemma. *Let $A : D(A) \to X$ be a linear operator satisfying the positive maximum principle. Furthermore let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Then the semigroup $(T(t))_{t\geq 0}$ is positivity preserving.*

Proof. Let $\lambda > 0$ and $f \in C_c^\infty(X)$. Assume f is not a non-negative function, i.e., we have $\inf_{x\in\mathbb{R}^d} f(x) < 0$. Choose $x_0 \in \mathbb{R}^d$ such that $f(x_0) = \inf_{x\in\mathbb{R}^d} f(x)$. The positive maximum principle implies

$$-[Af](x_0) = [A(-f)](x_0) \leq 0.$$

Therefore, we have

$$\inf_{x\in\mathbb{R}^d} [(\lambda - A)f](x) \leq [(\lambda - A)f](x_0) \leq \lambda f(x_0) < 0.$$

This shows that $(\lambda - A)f \geq 0$ implies $f \geq 0$. Hence, the set $M := \{f \in C_c^\infty(X) \mid f \geq 0\}$ is invariant under λR_λ^A for all $\lambda > 0$. Now, lemma 3.1.12 implies the semigroup $(T(t))_{t\geq 0}$ is positivity preserving. □

The following result is a special case of Browder [4], p.44.

3.1.14 Theorem. *Let $A(x)$ be a uniformly elliptic second order differential operator on $L^2(\mathbb{R}^d)$ as in equation (3.2) with $D(A) = H^2(\mathbb{R}^d)$ and such that*

(i) a_{ij}, b_j, c are bounded by a constant $M > 0$;

(ii) a_{ij}, b_j, c are Lipschitz continuous with constant $L > 0$;

(iii) the constant of ellipticity is $C_0 > 0$.

Then there exists a constant $K > 0$ such that for all $u \in D(A) = H^2(\mathbb{R}^d)$

$$\|u\|_{H^2}^2 \leq K\left(\|Au\|_{L^2}^2 + \|u\|_{L^2}^2\right) \tag{3.4}$$

Moreover, the constant K can be chosen such that it depends only on the constants M, L and C_0.

3.1.15 Definition. A family $A(t), t \in [0,T]$, of generators of C_0-semigroups on X is called *stable* if there are constants $M \geq 1$ and ω such that for all $t \in [0,T]$ the resolvent set of $A(t)$ satisfies

$$\rho(A(t)) \supset (\omega, \infty)$$

and for every finite sequence $0 \leq t_1 \leq \cdots \leq t_m \leq T, m \in \mathbb{N}$ and all $\lambda > \omega$

$$\left\| \prod_{i=1}^{m} R_\lambda^{A(t_i)} \right\|_{L(X)} \leq M(\lambda - \omega)^{-m}.$$

The product here is time-ordered, i.e. a factor with a larger t_i stands to the left of ones with smaller t_i. The constants M and ω are called *constants of stability.*

3.1.16 Remarks. (a) The stability of a family of generators on X is preserved when the norm in X is replaced by an equivalent norm. But the constants of stability may depend on the choice of the norm.

(b) It is clear from the definition of stability that a family of generators of contractions is stable.

To show the stability of a given family of operators directly is usually a difficult task. The next proposition gives a useful criterion for stability.

3.1.17 Proposition. *Let $\|\cdot\|_t, t \in [0,T]$, be a continuous family of norms on X, in the sense that*

$$\frac{\|x\|_t}{\|x\|_s} \leq e^{C|t-s|}, \quad x \in X \setminus \{0\},$$

holds for all $t, s \in [0,T]$. Furthermore let $A(t), t \in [0,T]$, be a family of operators such that every $A(t)$ is the generator of a contraction semigroup with respect to the norm $\|\cdot\|_t$. Then $A(t), t \in [0,T]$, is stable with respect to $\|\cdot\|_t$ for all $t \in [0,T]$.

Proof. See proposition 3.4 of [15]. □

3.1.18 Definition. A family of bounded operators $U(t,s), 0 \leq s \leq t \leq T$, on the Banach space X is called an *evolution system* if:

(a) $U(s,s) = \mathrm{id}$ for all $0 \leq s \leq T$.

(b) $U(t,r)U(r,s) = U(t,s)$ for all $0 \leq s \leq r \leq t \leq T$.

(c) The mapping $(t,s) \mapsto U(t,s)$ is strongly continuous for all $0 \le s \le t \le T$.

Part (a) of the following theorem adopts theorem 5.3.1 in the book of A.Pazy [25] to our setting.

3.1.19 Theorem. *Let $\{A(t), t \in [0,T]\}$ be a family of operators on a Banach space X and $Y \subset X$ densely and continuously embedded subspace satisfying the conditions*

(i) $\{A(t), t \in [0,T]\}$ is a stable family of generators of C_0-semigroups with constants of stability ω, M.

(ii) For all $t \in [0,T]$ the subspace $Y \subseteq X$ is $A(t)$-admissible.

(iii) The family of parts $\tilde{A}(t)$ of $A(t)$ in Y is a stable family of generators of C_0-semigroups in Y with constants of stability $\tilde{\omega}, \tilde{M}$.

(iv) For all $t \in [0,T]$ the domain $D(A)$ of A contains Y and $A(t)$ is a bounded operator from Y to X, i.e., $A(t) \in L(Y,X)$. Furthermore, $t \mapsto A(t)$ is continuous with respect to $\|\cdot\|_{L(Y,X)}$.

Then

(a) There exists a unique evolution system $\{U(t,s),\, 0 \le s \le t \le T\}$ in X satisfying

$$\begin{aligned} \|U(t,s)\|_{L(X)} &\le M e^{\omega(t-s)} \\ \partial_t^+ U(t,s)y\Big|_{t=s} &= A(s)y \\ \partial_s U(t,s)y &= -U(t,s)A(s)y \end{aligned}$$

for all $0 \le s \le t \le T$ and all $y \in Y$. Furthermore, $U(t,s)$ is positivity preserving if all semigroups generated by the operators $A(t)$ are positivity preserving. The derivatives ∂_t^+ and ∂_s are taken with respect to X.

Suppose X and Y are reflexive Banach spaces, then:

(b) $U(t,s)Y \subseteq Y$ and $\|U(t,s)\|_Y \le \tilde{M} e^{\tilde{\beta}(t-s)}$ for some constants $\tilde{M} \ge 1, \tilde{\beta} \ge 0$. Furthermore, $U(t,s)$ is weakly continuous in t,s with respect to Y.

(c) $\partial_t^+ U(t,s)y = A(s)U(t,s)y$ for $y \in Y$ and $t \ge s$.

(d) $\partial_t U(t,s)y$ exists for almost every t and

$$\partial_t U(t,s)y = A(t)U(t,s)y, \qquad y \in Y,$$

holds for almost every t.

Proof. All statements of (a) except the last one are shown in theorem 5.3.1 in [25]. If all $A(t)$ are generators of positivity preserving C_0-semigroups then the families $U_n(t,s)$ in the proof theorem 5.3.1 in [25] are clearly positivity preserving. Hence, we get $U(t,s)$ is positivity preserving.

Parts (b),(c) and (d) are shown in theorem 5.1 of [15]. □

The following result is used during the proof of our main result 3.3.7.

3.1.20 Lemma. *Let $q \in [0,1)$ and let $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subseteq [0,\infty)$ be non-negative sequences such that*

$$\alpha_{n+1} \le q\alpha_n + \beta_n \text{ for all } n \in \mathbb{N} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \beta_n < \infty.$$

Then

$$\sum_{n=1}^{\infty} \alpha_n < \infty.$$

Proof. We have

$$\sum_{n=1}^{\infty} \alpha_n = \alpha_1 + \sum_{n=1}^{\infty} \alpha_{n+1} \le \alpha_1 + q\sum_{n=1}^{\infty} \alpha_n + \sum_{n=1}^{\infty} \beta_n.$$

Therefore,

$$\sum_{n=1}^{\infty} \alpha_n \le \frac{1}{1-q}\left(\alpha_1 + \sum_{n=1}^{\infty} \beta_n\right) < \infty.$$

□

3.2 C_0-semigroups generated by A(v)

To give a proof for the existence of a solution of the Eulerian limit partial differential equation (1) we require the following additional convexity assumption on the aggregation potential G:

(AG) $\Delta G \geq 0$.

3.2.1 Definition. Let $L \in \mathbb{N}, L > \frac{d}{2} + 2$. For any function $v \in H^L_+(\mathbb{R}^d)$ we define the operator $A(v) : L^2(\mathbb{R}^d) \supseteq D(A(v)) \to L^2(\mathbb{R}^d)$ by

$$A(v)\rho := \nabla \cdot [(\frac{\sigma_\infty^2}{2} + v)\nabla\rho] - \nabla \cdot ((\nabla G * v)\rho) \tag{3.5}$$

for all functions ρ in the domain $D(A(v)) := H^2(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$ of $A(v)$.

3.2.2 Remark. Using Sobolev's Lemma 1.4.7, we get $v \in C_b^2(\mathbb{R}^d)$. Therefore, the operator $A(v)$ is well-defined. Replacing v by $\rho(t)$, we see that $A(\rho(t))$ is the divergence form of the operator on the right hand side of the partial differential equation describing the continuum limit system (1).

3.2.3 Lemma. *For all $v \in H^L_+(\mathbb{R}^d)$, $u \in H^2(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d} u\, A(v)u \; dx \leq 0.$$

Proof. Partial integration and $u(x)\nabla u(x) = \frac{1}{2}\nabla u^2(x)$ imply

$$\begin{aligned}\int_{\mathbb{R}^d} u(\nabla G * v) \cdot \nabla u \; dx &= \frac{1}{2}\int_{\mathbb{R}^d} (\nabla G * v) \cdot \nabla u^2 \; dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^d} (\Delta G * v)u^2 \; dx.\end{aligned}$$

Note that $\left(\frac{\sigma_\infty}{2} + v\right) \geq 0$ due to $\sigma_\infty > 0$ and the positivity of v. Furthermore, assumption (AG) implies $(\Delta G * v) \geq 0$. Hence, using partial integration, we get

$$\begin{aligned}\int_{\mathbb{R}^d} u\, A(v)u \; dx &= -\int_{\mathbb{R}^d} \left(\frac{\sigma_\infty}{2} + v\right) (\nabla u)^2 \; dx + \int_{\mathbb{R}^d} u \; (\nabla G * v) \cdot \nabla u \; dx \\ &\leq -\frac{1}{2}\int_{\mathbb{R}^d} (\Delta G * v)u^2 \; dx \leq 0.\end{aligned}$$

□

3.2.4 Lemma. *For all $v \in H^L_+(\mathbb{R}^d)$ the linear operator $A(v)$ is dissipative on $L^2(\mathbb{R}^d)$.*

Proof. For $u \in H^2(\mathbb{R}^d)$, $v \in H_+^L(\mathbb{R}^d)$ and $\lambda > 0$ it follows from lemma 3.2.3 and Schwarz inequality that

$$\begin{aligned}\lambda \|u\|_{L^2(\mathbb{R}^d)}^2 &\leq \lambda \int_{\mathbb{R}^d} u^2 \, dx - \int_{\mathbb{R}^d} u \, A(v) u \, dx \\ &= \int_{\mathbb{R}^d} u \, (\lambda - A(v)) u \, dx \\ &\leq \|(\lambda - A(v))u\|_{L^2(\mathbb{R}^d)} \|u\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Therefore, we have

$$\lambda \|u\|_{L^2(\mathbb{R}^d)} \leq \|(\lambda - A(v))u\|_{L^2(\mathbb{R}^d)}$$

for all $\lambda > 0$. This yields $A(v)$ is a dissipative operator. □

3.2.5 Corollary. *For all $v \in H_+^L(\mathbb{R}^d)$ the linear operator $A(v)$ is closable.*

Proof. This follows immediately from lemma 3.2.4 and lemma 3.1.3. □

In fact, in Corollary 3.2.9 we will see that for all non-negative functions $v \in H_+^L(\mathbb{R}^d)$ the linear operator $A(v)$ is already a closed operator.

3.2.6 Lemma. *For all $v \in H^L(\mathbb{R}^d)$ there exists a $\lambda > 0$ such that the range, $R(\lambda - A(v))$, of $\lambda - A(v)$ is $L^2(\mathbb{R}^d)$.*

Proof. Writing the operator $A(v)$ in the form (3.2) we get

$$\begin{aligned}a_{kl} &= \delta_{kl} \left(\frac{\sigma_\infty^2}{2} + v \right) \in C_b^2(\mathbb{R}^d), \\ b_j &= \partial_j v + (\partial_j G * v) \in C_b^1(\mathbb{R}^d), \\ c &= \Delta G * v \in C_b^2(\mathbb{R}^d).\end{aligned}$$

Therefore, the positivity of v gives

$$\sum_{k,l=1}^d a_{kl}(x) \xi_k \xi_l = \sum_{k=1}^d \left(\frac{\sigma_\infty^2}{2} + v \right) |\xi_k|^2 \geq \frac{\sigma_\infty^2}{2} |\xi|^2.$$

Because of $\sigma_\infty > 0$, we see that $A(v)$ is a uniformly elliptic second order differential operator and the result follows directly from Theorem 3.1.5. □

3.2.7 Lemma. *For all $v \in H_+^L(\mathbb{R}^d)$ the operator $A(v)$ satisfies the positive maximum principle.*

Proof. Choose $\rho \in C_c^\infty(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$ such that $\rho(x_0) = \sup_{x\in\mathbb{R}^d} \rho(x) \geq 0$. We write $A(v)\rho(x_0)$ in the form

$$\begin{aligned} A(v)\rho(x_0) = \left(\frac{\sigma_\infty^2}{2} + v\right) \Delta\rho(x_0) + \left[(\nabla v)\cdot(\nabla\rho)\right](x_0) \\ - \left[(\nabla G * v)\cdot(\nabla\rho)\right](x_0) - (\Delta G * v)\rho(x_0) \end{aligned} \tag{3.6}$$

and show that all terms on the right hand side of this equation are less or equal zero. Assumption (AG) directly implies $-(\Delta G * v)\rho(x_0) \leq 0$. Since $\rho(x_0)$ is an extremum of ρ we have $\nabla\rho(x_0) = 0$ and therefore we get for the second and third part of equation (3.6):

$$\begin{aligned} &\left[(\nabla G * v)\cdot(\nabla\rho)\right](x_0) = 0 \text{ and} \\ &\left[(\nabla v)\cdot(\nabla\rho)\right](x_0) = 0. \end{aligned}$$

Furthermore, the fact that ρ attains its maximum at x_0 implies that the Hessian $H_\rho(x_0)$ of ρ in x_0 is negative semidefinite. Thus, we have

$$\Delta\rho(x_0) = \operatorname{trace} H_\rho(x_0) \leq 0.$$

This completes the proof since $\sigma_\infty > 0$ and v is a non-negative function. □

3.2.8 Theorem. *For all $v \in H_+^L(\mathbb{R}^d)$ the operator $A(v)$ generates a positivity preserving C_0-semigroup of contractions on $L^2(\mathbb{R}^d)$.*

Proof. In lemma 3.2.4 we have shown that $A(v)$ is a dissipative linear operator on $L^2(\mathbb{R}^d)$. Furthermore, lemma 3.2.6 says that $A(v)$ satisfies the range condition of the Lumer-Phillips-Theorem 3.1.9. Hence, $A(v)$ is the generator of a C_0-semigroup of contractions on $L^2(\mathbb{R}^d)$. From lemma 3.2.7 together with lemma 3.1.13 it follows that this semigroup is positivity preserving. □

3.2.9 Corollary. *For all $v \in H_+^L(\mathbb{R}^d)$ the linear operator $A(v)$ is closed.*

Proof. Theorem 3.2.8 implies that the operator $A(v)$ is the generator of a C_0-semigroup. Hence, $A(v)$ is a closed operator. □

3.2.10 Definition. Let $m \in \mathbb{N}$ with $L \leq 2m$. For every $v \in H_+^{2m}(\mathbb{R}^d)$ we inductively define a scale of abstract Sobolev spaces $(H^n_{A(v)}, \|\cdot\|_n), n \in \mathbb{N}, n \leq m$, associated to the linear operator $A(v)$ by $H^1_{A(v)} := H^2(\mathbb{R}^d)$ and

$$H^n_{A(v)} := D((A(v))^n) := \{f \in D((A(v))^{n-1}) | A(v)f \in D((A(v))^{n-1})\}.$$

The family of norms is given by

$$\|\cdot\|_{0,v} := \|\cdot\|_{L^2} \text{ and } \|\cdot\|_{n,v} := \|\cdot\|_{n-1,v} + \|A(v)\cdot\|_{n-1,v}$$

Our aim now is to show that the space $H^n_{A(v)}$ equals the classical Sobolev space of order $2n$ and thus is independent of the choice of the function v. But the norms $\|\cdot\|_n$ clearly depend on v.

3.2.11 Corollary. *For $L \leq 2n \leq 2m$ and all $R > 0$ we have $H^{2n}(\mathbb{R}^d) = H^n_{A(v)}$ and there exist constants $C_{1;n}, C_{2;n} > 0$ such that for all $v \in H^{2m+1}_+(\mathbb{R}^d)$ with $\|v\|_{H^{2m+1}} \leq R$ we have*

$$C_{1;n}\|\cdot\|_{H^{2n}} \leq \|\cdot\|_{n,v} \leq C_{2;n}\|\cdot\|_{H^{2n}} \tag{3.7}$$

Proof. We first show that $H^{2n}(\mathbb{R}^d) = H^n_{A(v)}$. For $n = 1$ this is clear from the definition of the space $H^1_{A(v)}$. Now, assume we have already shown that the equality holds for $n-1$. Let $u \in H^n_{A(v)}$, this is, by definition, equivalent to $u \in H^{n-1}_{A(v)}$ and $A(v)u \in H^{n-1}_{A(v)}$. Furthermore, due to our induction assumption this is equivalent to $u \in H^{2(n-1)}(\mathbb{R}^d)$ and $A(v)u \in H^{2(n-1)}(\mathbb{R}^d)$. For any $\lambda \in \mathbb{R}$ this gives $\lambda u - A(v)u \in H^{2(n-1)}(\mathbb{R}^d)$. Choosing λ large enough we get from 3.1.5 that $u \in H^{2n}(\mathbb{R}^d)$. This shows $H^n_{A(v)} \subseteq H^{2n}(\mathbb{R}^d)$.

On the other hand, $u \in H^{2n}(\mathbb{R}^d)$ clearly implies $u \in H^{2(n-1)}(\mathbb{R}^d)$ and $A(v)u \in H^{2(n-1)}(\mathbb{R}^d)$ and thus we obtain $u \in H^{n-1}_{A(v)}$ and $A(v)u \in H^{n-1}_{A(v)}$. This yields $u \in H^n_{A(v)}$ and completes the proof of $H^n_{A(v)} = H^{2n}(\mathbb{R}^d)$

We prove the result in case $n = 1$. If $n > 1$ the result follows by induction. Writing the linear differential operator $A(v)$ in the form (3.2) we get

$$\begin{aligned} a_{kl} &= \delta_{kl}\left(\frac{\sigma_\infty^2}{2} + v\right) \\ b_j &= \partial_j v - (\partial_j G * v) \\ c &= -\Delta G * v. \end{aligned}$$

Therefore, we have

$$\|a_{kl}\|_{H^{2n}} \leq \frac{\sigma_\infty^2}{2} + R \tag{3.8}$$

$$\|b_j\|_{H^{2n}} \leq R(1 + \|\partial_j G\|_{L^1}) \tag{3.9}$$

$$\|c\|_{H^{2n}} \leq R\|\Delta G\|_{L^1}. \tag{3.10}$$

Exactly the same way, we can show that all derivatives of the coefficients a_{kl}, b_j and c are bounded by a constant independent of v (depending only on R). This implies the Lipschitz continuity of all this coefficients for some constant $\tilde{L} > 0$. Now, with theorem 3.1.14 we get

$$\|u\|^2_{H^2} \leq K\left(\|A(v)u\|^2_{L^2} + \|u\|^2_{L^2}\right) \leq K\|u\|^2_{1,v} \tag{3.11}$$

for all functions $u \in H^2(\mathbb{R}^d)$, where the constant K is independent of v since the constants $M, \tilde{L}, C_0$ do not depend on the choice of the function v as long as $\|v\|_{H^{2m+1}} \leq R$.

The second inequality in equation (3.7) follows from

$$\begin{aligned}
\|u\|_{1,v} &= \|u\|_{L^2} + \|A(v)u\|_{L^2} \\
&\leq \|u\|_{L^2} + \left\| \sum_{j=1}^{d} \left(\frac{\sigma_\infty}{2} + v \right) \partial_{jj} u \right\|_{L^2} + \left\| \sum_{j=1}^{d} \left(\partial_j v - (\partial_j G * v) \right) \partial_j u \right\|_{L^2} \\
&\quad + \|(\Delta G * v)u\|_{L^2} \\
&\leq \|u\|_{L^2} + \sum_{j=1}^{d} \|a_{jj}\|_\infty \|\partial_{jj} u\|_{L^2} + \sum_{j=1}^{d} \|b_j\|_\infty \|\partial_j u\|_{L^2} + \|c\|_\infty \|u\|_{L^2}
\end{aligned}$$

Using again the inequalities (3.8)-(3.10), we obtain

$$\|u\|_{1,v} \leq C\|u\|_{H^2}$$

with a constant $C > 0$ that depends only on σ_∞, R and the aggregation potential G. □

3.2.12 Remark. Corollary 3.2.11 shows that the spaces $H^n_{A(v)}$ are independent of the choice of the function $v \in H^{2m+1}_+(\mathbb{R}^d)$ as long as $\|v\|_{H^{2m+1}} \leq R$. And all the norms $\|\cdot\|_{n,v}$ are uniformly equivalent for all $\|v\|_{H^{2m+1}} \leq R$. Thus, we occasionally write $\|\cdot\|_n$ instead of $\|\cdot\|_{n,v}$.

3.2.13 Definition. Let $A_n(v)$ denote the part of $A(v)$ in $H^n_{A(v)}$, i.e. we have $A_n(v)f := A(v)f$ for all $f \in D(A_n(v)) := \{f \in H^n_{A(v)} | A(v)f \in H^n_{A(v)}\} = H^{n+1}_{A(v)}$.

3.2.14 Lemma. *Let $v \in H^{2m+1}(\mathbb{R}^d), L \leq 2n \leq 2m$. Then the linear operators $A_n(v), n \in \mathbb{N}$, are generators of C_0-semigroups $(T_n(t))_{t\geq 0}$ of contractions on $H^n_{A(v)}$ with $T_n(t) = T(t)|_{H^n_{A(v)}}$.*

Proof. Let $f \in H^{n+1}_{A(v)}$. We first show that $(T_n(t))_{t\geq 0}$ forms a C_0-semigroup of contractions on $H^n_{A(v)}$. Note that by general semigroup theory (see for example [25, p.5]) $f \in D(A(v))$ implies $T(t)f \in D(A(v))$, therefore $f \in H^{n+1}_{A(v)}$ implies $T(t)f \in H^{n+1}_{A(v)}$ and $T(t)A(v)f \in H^n_{A(v)}$. Theorem 3.2.8 gives the result in case $n = 1$. Thus, we get by induction

$$\begin{aligned}
\|T_{n+1}(t)f\|_{n+1} &= \|T_n(t)f\|_n + \|T_n(t)A(v)f\|_n \\
&\leq \|T_n(t)\|_{L(H^n_{A(v)})} \|f\|_n + \|T_n(t)\|_{L(H^n_{A(v)})} \|A(v)f\|_n \\
&\leq \|f\|_n + \|A(v)f\|_n = \|f\|_{n+1}.
\end{aligned}$$

And the strong continuity follows inductively from

$$\|T_{n+1}(t)f - f\|_{n+1} = \|T_n(t)f - f\|_n + \|T_n(t)A(v)f - A(v)f\|_n \underset{t\to 0}{\longrightarrow} 0$$

This shows $(T_n(t))_{t\geq 0}$ is a C_0-semigroup. Let C_n denote the generator of $(T_n(t))_{t\geq 0}, n \in \mathbb{N}$. Since the space $H^n_{A(v)}$ is continuously embedded in $L^2(\mathbb{R}^d)$ we get

$$\begin{aligned} D(C_n) &:= \left\{ u \in H^n_{A(v)} \;\middle|\; H^n_{A(v)}\text{-}\lim_{h\to 0} \frac{T_n(h)u - u}{h} \text{ exists} \right\} \\ &\subseteq \left\{ u \in H^n_{A(v)} \;\middle|\; L^2(\mathbb{R}^d)\text{-}\lim_{h\to 0} \frac{T_n(h)u - u}{h} \text{ exists} \right\} \\ &\subseteq D(A_n(v)) \end{aligned}$$

and

$$\begin{aligned} C_n u &= H^n_{A(v)}\text{-}\lim_{h\to 0} \frac{T_n(h)u - u}{h} \\ &= L^2(\mathbb{R}^d)\text{-}\lim_{h\to 0} \frac{T_n(h)u - u}{h} \\ &= A_n(v), \quad u \in D(C_n). \end{aligned}$$

Therefore C_n is a restriction of $A_n(v)$ and thus it remains only to show $D(A_n(v)) \subseteq D(C_n)$. Choose $\lambda > 0$ large enough such that $\lambda \in \rho(C_n) \cap \rho(A_n(v))$ and the resolvent $R^{C_n}_\lambda, R^{A_n(v)}_\lambda$ is given by (see [25, p.8])

$$R^{C_n}_\lambda u = \int_{s\geq 0} e^{-\lambda s} T(s)u \, ds = R^{A_n(v)}_\lambda u$$

for all $u \in H^n_{A(v)}$. Now we get for any $\tilde{u} \in D(A_n(v))$

$$\tilde{u} = R^{A_n(v)}_\lambda (\lambda - A(v))\tilde{u} = R^{C_n}_\lambda (\lambda - A(v))\tilde{u}$$

and since the resolvent $R^{C_n}_\lambda$ maps into $D(C_n)$ we have $\tilde{u} \in D(C_n)$. This completes the proof of $D(A_n(v)) \subseteq D(C_n)$. □

Combining all results of this section we arrive at:

3.2.15 Corollary. *For all $v \in H^{2m+1}_+(\mathbb{R}^d)$ the operator $A_n(v), L \leq 2n \leq 2m$, generates a positivity preserving C_0-semigroup of contractions on the Sobolev space $H^{2n}(\mathbb{R}^d)$.*

Proof. Lemma 3.2.14 shows that $A_n(v)$ generates C_0-semigroup of contractions on the Sobolev space $H^n_{A(v)}$. Changing to the equivalent norm on $H^{2n}(\mathbb{R}^d)$ preserves the contraction property. Due to theorem 3.2.8 the semigroups are positivity preserving. □

3.3 Model with diffusion

3.3.1 Lemma. *Let $L \leq 2n \leq 2m$. For all $v \in H_+^{2m+1}(\mathbb{R}^d)$ the operator*

$$A_n(v) : H^{2n+2}(\mathbb{R}^d) \to H^{2n}(\mathbb{R}^d)$$

is bounded. Furthermore, the operator norm $\|A_n(v)\|_{L(H^{2n+2},H^{2n})}$ is bounded by a constant that depends only on σ_∞, G and $\|v\|_{H^{2m+1}}$.

Proof. Let $\rho \in H^{2n+2}$ and $v \in H_+^{2m+1}(\mathbb{R}^d)$. Taking Proposition 1.4.4 (c) into account, we obtain

$$\begin{aligned}
\|A(v)\rho\|_{H^{2n}} &\leq \|\nabla \cdot (\frac{\sigma_\infty^2}{2} + v)\nabla\rho\|_{H^{2n}} + \|\nabla \cdot (\nabla G * v)\rho\|_{H^{2n}} \\
&\leq \|(\nabla v) \cdot (\nabla \rho)\|_{H^{2n}} + \|(\frac{\sigma_\infty^2}{2} + v)\Delta\rho\|_{H^{2n}} \\
&\quad + \|(\Delta G * v)\rho\|_{H^{2n}} + \|(\nabla G * v) \cdot (\nabla\rho)\|_{H^{2n}} \\
&\leq \|\nabla v\|_{H^{2n}} \|\nabla\rho\|_{H^{2n}} + \|\frac{\sigma_\infty^2}{2} + v\|_{H^{2n}} \|\Delta\rho\|_{H^{2n}} \\
&\quad + \|(\Delta G * v)\|_{H^{2n}} \|\rho\|_{H^{2n}} + \|\nabla G * v\|_{H^{2n}} \|\nabla\rho\|_{H^{2n}} \\
&\leq C\|\rho\|_{H^{2n+2}}.
\end{aligned}$$

Therefore, we see $A(v) \in L(H^{2n+2}(\mathbb{R}^d), H^{2n}(\mathbb{R}^d))$. Moreover, because of $\|\nabla v\|_{H^{2n}} \leq \|v\|_{H^{2m+1}}$ and $\|v\|_{H^{2n}} \leq \|v\|_{H^{2m+1}}$, the constant C depends only on σ_∞, G and $\|v\|_{H^{2m+1}}$. □

3.3.2 Lemma. *There exists a constant $C > 0$ such that for all $u, v \in H_+^{2m+1}(\mathbb{R}^d)$, we have*

$$\|A(v) - A(u)\|_{\mathcal{L}(H^{2n+2},H^{2n})} \leq C\|v - u\|_{H^{2n+1}}, \quad L \leq 2n \leq 2m.$$

Proof. Let $\rho \in H^{2n+2}(\mathbb{R}^d)$, then we get as in the proof of lemma 3.3.1

$$\begin{aligned}
\|A(v)\rho - A(u)\rho\|_{H^{2n}} &\leq \|\nabla \cdot (v-u)\nabla\rho\|_{H^{2n}} + \|\nabla \cdot (\nabla G * (v-u))\rho\|_{H^{2n}} \\
&\leq \|\nabla(v-u)\|_{H^{2n}} \|\nabla\rho\|_{H^{2n}} + \|v-u\|_\infty \|\Delta\rho\|_{H^{2n}} \\
&\quad + \|(\Delta G * (v-u))\|_{H^{2n}} \|\rho\|_{H^{2n}} \\
&\quad + \|\nabla G * (v-u)\|_{H^{2n}} \|\nabla\rho\|_{H^{2n}} \\
&\leq C\|v-u\|_{H^{2n+1}} \|\nabla\rho\|_{H^{2n}} + C\|v-u\|_{H^{2n+1}} \|\Delta\rho\|_{H^{2n}} \\
&\quad + C\|v-u\|_{H^{2n+1}} \|\rho\|_{H^{2n}} \\
&\quad + C\|v-u\|_{H^{2n+1}} \|\nabla\rho\|_{H^{2n}} \\
&\leq C\|v-u\|_{H^{2n+1}} \|\rho\|_{H^{2n+2}}.
\end{aligned}$$

Therefore, $\|A(v) - A(u)\|_{\mathcal{L}(H^{2n+2},H^{2n})} \leq C\|v-u\|_{H^{2n+1}}$. □

3.3.3 Lemma. *For all $v \in C([0,T], H^{2m}_+(\mathbb{R}^d)), L \leq 2m$, the family of linear operators $A(v(t)), t \in [0,T]$, is stable in $L^2(\mathbb{R}^d)$.*

Proof. By theorem 3.2.8 $A(v(t)), t \in [0,T]$, is a family of generators of contraction semigroups and therefore clearly stable. □

3.3.4 Lemma. *For all $v \in C([0,T], H^{2m+1}_+(\mathbb{R}^d)), L \leq 2m$, satisfying*

$$\|v(t) - v(s)\|_{H^{2m+1}} \leq e^{C|t-s|} - 1 \tag{3.12}$$

the family $A_n(v(t)), t \in [0,T]$, is a stable family of generators of C_0-semigroups on $H^{2n}(\mathbb{R}^d)$ for all $L \leq 2n \leq 2m$.

Proof. By corollary 3.2.15 every $A_n(v(t)), t \in [0,T]$, is a generator of a contraction semigroup on $H^{2n}(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{n,v(t)}$. To apply proposition 3.1.17 we have to show that

$$\|\rho\|_{n,v(t)} \leq \|\rho\|_{n,v(s)} e^{C_m|t-s|} \tag{3.13}$$

holds for all $t, s \in [0,T]$ and all $\rho \in H^{2n}(\mathbb{R}^d)$. Using corollary 3.2.11 and lemma 3.3.2, we obtain for all $L \leq 2n \leq 2m$

$$\begin{aligned}
\|A_{n-1}(v(t)) - A_{n-1}(v(s))\|_{L(H^n_{A(v(s))}, H^{n-1}_{A(v(s))})} &\leq C\|v(t) - v(s)\|_{H^{2n-1}} \\
&\leq C\|v(t) - v(s)\|_{H^{2m+1}} \\
&\leq C(e^{C|t-s|} - 1) \\
&\leq e^{C|t-s|} - 1
\end{aligned}$$

This gives

$$\|[A(v(t)) - A(v(s))]\rho\|_{n-1,v} \leq (e^{C|t-s|} - 1)\|\rho\|_{n,v} \tag{3.14}$$

Now, in case $n = 1$, equation (3.13) follows from

$$\begin{aligned}
\|\rho\|_{1,v(t)} &= \|\rho\|_{L^2} + \|A(v(t))\rho\|_{L^2} \\
&\leq \|\rho\|_{L^2} + \|A(v(s))\rho\|_{L^2} + \|[A(v(t)) - A(v(s))]\rho\|_{L^2} \\
&\leq \|\rho\|_{1,v(s)}(1 + e^{C|t-s|} - 1) \\
&\leq \|\rho\|_{1,v(s)} e^{C|t-s|}, \quad \rho \in H^2(\mathbb{R}^d).
\end{aligned}$$

In case $n > 1$, suppose that we have already shown that there exists a constant $C_{n-1} > 0$ such that

$$\|\tilde{\rho}\|_{n-1,v(t)} \leq \|\tilde{\rho}\|_{n-1,v(s)} e^{C_{n-1}|t-s|}.$$

holds for all $t, s \in [0,T]$ and all $\tilde{\rho} \in H^{2(n-1)}(\mathbb{R}^d)$. Since the operator $A(v(t))$ maps $H^{2n}(\mathbb{R}^d)$ into $H^{2(n-1)}(\mathbb{R}^d)$, we get for any $\rho \in H^{2n}(\mathbb{R}^d)$

$$\|A(v(t))\rho\|_{n-1,v(t)} \leq \|A(v(t))\rho\|_{n-1,v(s)} e^{C_{n-1}|t-s|}.$$

This together with equation (3.14) implies

$$\begin{aligned}
\|\rho\|_{n,v(t)} &= \|\rho\|_{n-1,v(t)} + \|A(v(t))\rho\|_{n-1,v(t)} \\
&\leq \left(\|\rho\|_{n-1,v(s)} + \|A(v(t))\rho\|_{n-1,v(s)}\right) e^{C_{n-1}|t-s|} \\
&\leq \Big(\|\rho\|_{n-1,v(s)} + \|A(v(s))\rho\|_{n-1,v(s)} \\
&\qquad\qquad + \|[A(v(t)) - A(v(s))]\rho\|_{n-1,v(s)}\Big) e^{C_{n-1}|t-s|} \\
&= \left(\|\rho\|_{n,v(s)} + \|[A(v(t)) - A(v(s))]\rho\|_{n-1,v(s)}\right) e^{C_{n-1}|t-s|} \\
&\leq \left(\|\rho\|_{n,v(s)} + (e^{C|t-s|} - 1)\|\rho\|_{n,v(s)}\right) e^{C_{n-1}|t-s|} \\
&\leq \|\rho\|_{n,v(s)}(1 + e^{C|t-s|} - 1)e^{C_{n-1}|t-s|} \leq \|\rho\|_{n,v(s)} e^{C_n|t-s|}
\end{aligned}$$

Note that the constants $C_n, n \in \mathbb{N}$, can be chosen such that they are independent of t and s since the continuity of v implies that $\|v(t)\|_{H^{2m+1}} \leq R$ for some $R \geq 0$. Thus, we can apply corollary 3.2.11. This gives us stability with respect to the norm $\|\cdot\|_{n,v(T)}$. Changing to the equivalent norm $\|\cdot\|_{H^{2n}}$ preserves the stability. □

3.3.5 Corollary. *Let $m, n \in \mathbb{N}_0$ such that $L \leq 2n \leq 2n+2 \leq 2m$. Then:*

(a) Let $\rho \in C([0,T], H_+^{2m+1}(\mathbb{R}^d))$ be a function with values in a bounded set B such that equation (3.12) holds. Then there exists a unique evolution system $\{U_\rho(t,s),\ 0 \leq s \leq t \leq T\}$ in $H^{2n}(\mathbb{R}^d)$ and constants ω, M such that

$$\|U_\rho(t,s)\|_{L(H^{2n}(\mathbb{R}^d))} \leq M e^{\omega(t-s)} \tag{3.15}$$

$$\partial_t^+ U_\rho(t,s)y\Big|_{t=s} = A(\rho(s))y \tag{3.16}$$

$$\partial_s U_\rho(t,s)y = -U_\rho(t,s)A(\rho(s))y \tag{3.17}$$

for all $0 \leq s \leq t \leq T$ and all $y \in H^{2n+2}(\mathbb{R}^d)$.

(b) $U_\rho(t,s)H^{2n+2}(\mathbb{R}^d) \subseteq H^{2n+2}(\mathbb{R}^d)$ and $\|U_\rho(t,s)\|_{H^{2n+2}} \leq \tilde{M}e^{\tilde{\beta}(t-s)}$ for some constants $\tilde{M} \geq 1, \tilde{\beta} \geq 0$. Furthermore, $U_\rho(t,s)$ is weakly continuous in t, s with respect to $H^{2n+2}(\mathbb{R}^d)$.

(c) $\partial_t^+ U_\rho(t,s)y = A(\rho(s))U_\rho(t,s)y$ for $y \in H^{2n+2}(\mathbb{R}^d)$ and $s \leq t$.

(d) $\partial_t U_\rho(t,s)y$ *exists for almost every* t *and*

$$\partial_t U_\rho(t,s)y = A(\rho(t))U_\rho(t,s)y, \qquad y \in H^{2n+2}(\mathbb{R}^d)$$

holds for almost every t.

(e) Furthermore, there exists a constant $K > 0$ *such that for all* $\rho_1, \rho_2 \in C([0,T], H_+^{2m+1}(\mathbb{R}^d))$ *with values in* B *and satisfying (3.12) and all* $y \in H^{2n+2}(\mathbb{R}^d)$ *we have*

$$\|U_{\rho_1}(t,s)y - U_{\rho_2}(t,s)y\|_{H^{2n}} \leq K\|y\|_{H^{2n+2}} \int_s^t \|\rho_1(r) - \rho_2(r)\|_{H^{2n}}\, dr \tag{3.18}$$

Proof. In lemma 3.3.4 it is shown that $(A_n(\rho(t)))_{t\in[0,T]}$ is a stable family of generators of C_0-semigroups on $H^{2n}(\mathbb{R}^d)$. Since $2n+2 \leq 2m$, corollary 3.2.15 gives us $H^{2n+2}(\mathbb{R}^d)$ is $A(\rho(t))$-admissible, $t \in [0,T]$, and the family of parts in $H^{2n+2}(\mathbb{R}^d)$ is given by linear operators $(A_{n+1}(\rho(t)))_{t\in[0,T]}$ which is again by lemma 3.3.4 a stable family of generators of C_0-semigroups on $H^{2n+2}(\mathbb{R}^d)$. Lemma 3.3.1 implies $A(\rho(t)) \in \mathcal{L}(H^{2n+2}(\mathbb{R}^d), H^{2n}(\mathbb{R}^d))$. Now let $t_1, t_2 \in [0,T]$. From lemma 3.3.2 we obtain

$$\|A(\rho(t_1)) - A(\rho(t_2))\|_{\mathcal{L}(H^{2n+2},H^{2n})} \leq C\|\rho(t_1) - \rho(t_2)\|_{H^{2n+1}} \underset{t_1\to t_2}{\longrightarrow} 0.$$

It follows that $t \mapsto A(\rho(t))$ is continuous with respect to $\|\cdot\|_{\mathcal{L}(H^{2n+2},H^{2n})}$. Hence conditions (i)-(iv) from theorem 3.1.19 are fulfilled.

It remains to show that equation (3.18) holds. This is proved in [25, lemma 4.4, p.202]. □

3.3.6 Definition. Let $m, n \in \mathbb{N}$ with $2n+2 \leq 2m$. A familiy of operators $(J_j)_{j\in\mathbb{N}}$ such that

(a) $J_j : H^{2n}(\mathbb{R}^d) \to H^{2m+1}(\mathbb{R}^d)$ is bounded for all $j \in \mathbb{N}$,

(b) $\|J_j\|_{L(H^{2n})} \leq C_1$ for all $j \in \mathbb{N}$,

(c) $\|J_{j+1} - J_j\|_{L(H^{2n+2},H^{2n})} \leq \frac{C}{j^2}$ for all $j \in \mathbb{N}$,

(d) $f \geq 0$ implies $J_j f \geq 0$ for all $j \in \mathbb{N}$,

(e) $J_j \underset{j\to\infty}{\longrightarrow} \mathrm{id}$ strongly in $L(H^{2n})$

is called a *positive mollifier*

In the following theorem we use $2m = 2n+2$ and a positive mollifier $(J_j)_j$.

3.3.7 Theorem. *Let $L \leq 2n$, $\rho_0 \in H_+^{2n+2}(\mathbb{R}^d)$ and let $B := B(\rho_0, r)$ be a ball in the hilbert space $H^{2n}(\mathbb{R}^d)$ around ρ_0 with radius $r > 0$. Then there exists a $\tilde{T} > 0$ such that the Cauchy problem*

$$\begin{aligned} \partial_t \rho(x,t) &= A(\rho(\cdot,t))\rho(x,t), \quad t \in [0,\tilde{T}] \\ \rho(x,0) &= \rho_0(x) \end{aligned} \tag{3.19}$$

has a unique solution $\rho \in C([0,\tilde{T}], H^{2n}(\mathbb{R}^d))$ with $\rho(t) \in B$ for all $t \in [0,\tilde{T}]$ in the sense that $\partial_t \rho(\cdot,t)$ exists for almost every t and (3.19) holds for almost every t.

Proof. (Step 0) For all $x \in C([0,T], H_+^{2n}(\mathbb{R}^d))$ with $x([0,T]) \subseteq B$ and such that equation (3.12) holds. let $U_{J_j x}$ denote the unique evolution system associated to the operator family $(A(J_j x(t))), t \in [0,T]$ that is given by theorem 3.3.5. Due to (a) of 3.3.6 $U_{J_j x}(t,s), 0 \leq s \leq t \leq T$ is well-defined.

(Step 1) Choose $\tilde{T} \leq T$ such that

(i) $\max_{0 \leq t \leq \tilde{T}} \|U_{\rho_0}(t,0)\rho_0 - \rho_0\|_{H^{2n}} \leq \frac{r}{3}$;

(ii) $\tilde{T} \leq \frac{1}{3}(K(C_1+1)\|\rho_0\|_{H^{2n+2}})^{-1}$ if $\|\rho_0\|_{H^{2n+2}} > 0$.

Here, the constant K in (ii) is the same as in equation (3.18) and C_1 is the constant from (b) of 3.3.6.

Choose j_0 such that

$$\|J_j \rho_0 - \rho_0\|_{H^{2n}} \leq C_1 r, \text{ for all } j \geq j_0. \tag{3.20}$$

Let $\mathcal{A}$ denote the set of all functions $\rho \in C([0,\tilde{T}], H^{2n}(\mathbb{R}^d))$ such that

(a) $\rho(0) = \rho_0$;

(b) $\rho([0,T]) \subseteq B$;

(c) $\rho(t)$ is non-negative, $t \in [0,\tilde{T}]$;

(d) there exists a $C > 0$ such that $\|\rho(t) - \rho(s)\|_{H^{2n}(\mathbb{R}^d)} \leq C|t-s|$, $t,s \in [0,\tilde{T}]$.

We define a family of mappings

$$\Phi_j : \mathcal{A} \to C([0,\tilde{T}], H^{2n}(\mathbb{R}^d)); \quad (\Phi_j \rho)(t) := U_{J_j \rho}(t,0)\rho_0, j \in \mathbb{N}. \tag{3.21}$$

Because of property (d) and 3.3.6 (a), we can apply corollary 3.3.5. Hence, Φ_j is well-defined.

(Step 2) Next, we want to show that Φ_j maps $\mathcal{A}$ into $\mathcal{A}$ for all $j \geq j_0$. Clearly, from property 3.1.18 (a) we get $\Phi_j\rho(0) = U_{J_j\rho}(0,0)\rho_0 = \rho_0$. The fact that $U_{J_j\rho}$ is positivity preserving, see theorem 3.1.19 and (d) of 3.3.6, directly implies $\Phi_j\rho(t)$ is non-negative for all t.

Furthermore, we have for all $j \geq j_0$

$$\begin{aligned}\|J_j\rho(\tau) - \rho_0(\tau)\|_{H^{2n}} &\leq \|J_j\rho(\tau) - J_j\rho_0(\tau)\|_{H^{2n}} + \|J_j\rho_0(\tau) - \rho_0(\tau)\|_{H^{2n}} \\ &\leq C_1 r + C_1 r\end{aligned}$$

This implies

$$\begin{aligned}\|(\Phi_j\rho)(t) - \rho_0\|_{H^{2n}} &= \|U_{J_j\rho}(t,0)\rho_0 - \rho_0\|_{H^{2n}} \\ &\leq \|U_{J_j\rho}(t,0)\rho_0 - U_{\rho_0}(t,0)\rho_0\|_{H^{2n}} + \|U_{\rho_0}(t,0)\rho_0 - \rho_0\|_{H^{2n}} \\ &\leq K\|\rho_0\|_{H^{2n+2}} \int_0^t \|J_j\rho(\tau) - \rho_0(\tau)\|_{H^{2n}}\, d\tau + \frac{r}{3} \\ &\leq \tilde{T}K\|\rho_0\|_{H^{2n+2}} 2C_1 r + \frac{r}{3} \\ &\leq \frac{2r}{3} + \frac{r}{3} = r,\end{aligned}$$

i.e., we have shown that $(\Phi_j\rho)([0,T]) \subseteq B$.

To see that $\Phi_j(\mathcal{A}) \subset \mathcal{A}$ holds it remains to show

$$\|(\Phi_j\rho)(t) - (\Phi_j\rho)(s)\|_{H^{2n}} \leq C|t-s|.$$

Using equation (3.17), we get $(s < t)$

$$\begin{aligned}(\Phi_j\rho)(t) - (\Phi_j\rho)(s) &= U_{J_j\rho}(t,0)\rho_0 - U_{J_j\rho}(s,0)\rho_0 \\ &= U_{J_j\rho}(t,s)U_{J_j\rho}(s,0)\rho_0 - U_{J_j\rho}(s,0)\rho_0 \\ &= -\int_s^t U_{J_j\rho}(t,\tau)A(J_j\rho(\tau))U_{J_j\rho}(s,0)\rho_0 d\tau.\end{aligned} \tag{3.22}$$

From equation (3.15) we obtain

$$\|U_{J_j\rho}(t,\tau)\|_{L(H^{2n}(\mathbb{R}^d))} \leq Me^{\omega|t-\tau|} \leq Me^{\omega\tilde{T}}.$$

And from proposition 3.3.5 (b) and $\|\rho_0\|_{H^{2n+2}} < \infty$, we get

$$\|U_{J_j\rho}(s,0)\rho_0\|_{H^{2n+2}(\mathbb{R}^d)} \leq C.$$

Using the last two inequalities and lemma 3.3.1, we can estimate equation (3.22) in $H^{2n}(\mathbb{R}^d)$:

$$\begin{aligned}\|(\Phi_j\rho)(t) - (\Phi_j\rho)(s)\|_{H^{2n}} &= \left\| \int_s^t U_{J_j\rho}(t,\tau)A(J_j\rho(\tau))U_{J_j\rho}(s,0)\rho_0 d\tau \right\|_{H^{2n}} \\ &\leq |t-s|CMe^{\omega\tilde{T}} \leq C|t-s|\end{aligned}$$

This completes the proof of $\Phi_j(\mathcal{A}) \subset \mathcal{A}$.

(Step 3) Let $\|\cdot\|_\infty$ denote the usual supremum norm on the Banach space $C([0,\tilde{T}], H^{2n}(\mathbb{R}^d))$, i.e.

$$\|\rho\|_\infty := \sup_{t\in[0,\tilde{T}]} \|\rho(t)\|_{H^{2n}(\mathbb{R}^d)}, \quad \rho \in C([0,\tilde{T}], H^{2n}(\mathbb{R}^d)).$$

Define $x_0 \in \mathcal{A}$ by $x_0(t) := \rho_0$ for all t and

$$x_{j+1} := \Phi_j(x_j) \in \mathcal{A} \text{ for all } j \in \mathbb{N}_0$$

Using equation (3.18), we obtain

$$\begin{aligned}
\|x_{j+1}(t) - x_j(t)\|_{H^{2n}} &= \|\Phi_j x_j(t) - \Phi_{j-1}x_{j-1}(t)\|_{H^{2n}} \\
&= \|U_{J_j x_j}(t,0)\rho_0 - U_{J_{j-1}x_{j-1}}(t,0)\rho_0\|_{H^{2n}} \\
&\le K\|\rho_0\|_{H^{2n+2}} \int_0^{\tilde{t}} \|J_j x_j(\tau) - J_{j-1}x_{j-1}(\tau)\|_{H^{2n}}\, d\tau \\
&\le K\tilde{T}\|\rho_0\|_{H^{2n+2}}(\|J_j - J_{j-1}\|_{L(H^{2n})}\|x_j\|_\infty \\
&\qquad + \|J_{j-1}\|_{L(H^{2n})}\|x_j - x_{j-1}\|_\infty)
\end{aligned}$$

Taking the supremum and using property (c) of definition 3.3.6, property (ii) from the definition of $\tilde{T}$ and $\|x_j\|_\infty \le r$, we get

$$\|x_{j+1} - x_j\|_\infty \le \frac{C}{j^2} + \frac{1}{3}\|x_j - x_{j-1}\|_\infty. \tag{3.23}$$

Applying lemma 3.1.20, we obtain

$$\sum_{j=0}^{\infty} \|x_{j+1} - x_j\|_\infty < \infty. \tag{3.24}$$

Thus, $(x_j)_j$ converges towards a function $\rho \in C([0,\tilde{T}], H^{2n}(\mathbb{R}^d))$. Moreover, ρ satisfies (a)-(c) from the definition of $\mathcal{A}$. Using corollary 3.3.5 (b) and (d), we get $\rho(t), x_j(t) \in H^{2n+2}(\mathbb{R}^d)$ and for almost every $t \in [0,\tilde{T}]$ uniformly

$$\begin{aligned}
&\|\partial_t x_j(t) - A(\rho(t))\rho(t)\|_{H^{2n}} \\
&\quad \le \|[A(J_{j-1}x_{j-1}(t)) - A(\rho(t))]x_j(t)\|_{H^{2n}} \\
&\qquad + \|A(\rho(t))[x_j(t) - \rho(t)]\|_{H^{2n}} \\
&\quad \le rC\|J_{j-1}x_{j-1}(t) - \rho(t)\|_{H^{2n+1}} + C\|x_j(t) - \rho(t)\|_{H^{2n}} \\
&\quad \underset{j\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Using corollary 3.3.5 (d) one can conclude that equation (3.19) holds for almost every t. □

3.3.8 Corollary. *Suppose assumption (AG) holds. Then* $\rho_0 \in H^{2n+2}(\mathbb{R}^d)$ *with* $2n > L + 1 + \frac{d}{2}$ *is a sufficient condition for (A1).*

Proof. This is a direct consequence of theorem 3.3.7. □

3.4 Predator system

In this section we examine the Eulerian limit partial differential equation for a system of particles with diffusion and a stochastic predator. Particle systems of this type were described in section 2.2. Most of the proofs in this section are very similar to the corresponding ones in case without a predator. The operator $A(v)$ from the previous sections gets replaced by $A(v, p)$ which is given by

$$\begin{aligned} A(v,p)\rho := \nabla \cdot [(\frac{\sigma_\infty^2}{2} + v)\nabla]\rho - \nabla \cdot ((\nabla G * v)\rho) \\ + \nabla \cdot ((\nabla H * \delta_p)\rho), \quad \rho \in D(A(v,p)), p \in \mathbb{R}^d, v \in H^1_+(\mathbb{R}^d). \end{aligned}$$

with $D(A(v,p)) := D(A(v)) = H^2(\mathbb{R}^d)$. We show the existence of a solution of the Eulerian limit system

$$\begin{aligned} \partial_t \rho(x,t) &= \nabla \cdot [(\frac{\sigma_\infty^2}{2} + \rho(x,t))\nabla]\rho(x,t) - \nabla \cdot ((\nabla G * \rho)\rho(x,t)) \\ &\quad + \nabla \cdot ((\nabla H * \delta_{P_\infty(t)})\rho(x,t)), \quad t \in [0,\infty) \\ \rho(x,0) &= \rho_0(x) \\ \partial_t P_\infty(t) &= \int_{\mathbb{R}^d} \rho(x,t) w(x - P_\infty(t))\, dx \\ P_\infty(0) &= p_0. \end{aligned} \tag{3.25}$$

3.4.1 Definition. Let $m \in \mathbb{N}$. We say a pair $(\rho(t), P_\infty(t)), t \in [0,T]$, of functions is a $H^{2m}(\mathbb{R}^d)-$valued solution of the system (3.25) if the following two conditions are satisfied.

(i) $\rho(t) \in C([0,T], H^{2m}(\mathbb{R}^d))$ and $P_\infty \in C^1([0,T], \mathbb{R}^d)$ and $\partial_t \rho(t)$ exists for almost every $t \in [0,T]$.

(ii) Equation (3.25) holds for the pair $(\rho(t), P_\infty(t))$ and almost every $t \in [0,T]$.

The following additional assumption on the repulsion potential H of the predator is required:

(AH) $\Delta H \le 0, \nabla H \in H^{2m+1}(\mathbb{R}^d)$ and

$$\|\nabla H(\cdot - x) - \nabla H(\cdot - y)\|_{H^{2m+1}} \le C|x-y| \text{ for all } x, y \in \mathbb{R}^d.$$

We decompose the operator $A(v,p)$ into

$$A(v,p) = A(v) + L(p).$$

with

$$L(p)\rho := \nabla \cdot ((\nabla H * \delta_p)\rho), \quad \rho \in D(L) \tag{3.26}$$

and $D(L) = D(A(v)) = D(A(v,p)) = H^2(\mathbb{R}^d)$. The operator $A(v)$ was already studied in detail in the previos sections.

3.4.2 Lemma. *For all $v \in H_+^L(\mathbb{R}^d)$, $u \in H^2(\mathbb{R}^d)$ and all $p \in \mathbb{R}^d$ we have*

$$\int_{\mathbb{R}^d} u\, A(v,p)u \; dx \le 0.$$

Proof. Taking lemma 3.2.3 into account, it only remains to show that

$$\int_{\mathbb{R}^d} u\, L(p)u \; dx \le 0.$$

Using partial integration and the identity $u(x)\nabla u(x) = \frac{1}{2}\nabla u^2(x)$, we obtain as in the proof of lemma 3.2.3

$$\begin{aligned}\int_{\mathbb{R}^d} u\,[\nabla \cdot (\nabla H * \delta_p)u] \; dx &= -\int_{\mathbb{R}^d} (\nabla u) \cdot [(\nabla H * \delta_p)u] \; dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^d} (\nabla H * \delta_p) \cdot \nabla u^2 \; dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} (\Delta H * \delta_p) u^2 \; dx.\end{aligned}$$

Assumption (AH) implies $(\Delta H * \delta_p) \le 0$. Hence, $\int_{\mathbb{R}^d} u\, L(p)u \; dx \le 0$. □

3.4.3 Lemma. *For all $v \in H_+^L(\mathbb{R}^d)$ and all $p \in \mathbb{R}^d$ the linear operator $A(v,p)$ is dissipative on $L^2(\mathbb{R}^d)$.*

Proof. Using lemma 3.4.2 and replacing $A(v)$ by $A(v,p)$ in the proof of lemma 3.2.4, we obtain that $A(v,p)$ is a dissipative operator. □

3.4.4 Lemma. *For all $v \in H^L_+(\mathbb{R}^d)$ and all $p \in \mathbb{R}^d$ the operator $A(v,p)$ satisfies the positive maximum principle.*

Proof. Choose $\rho \in C_c^\infty(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$ such that $\rho(x_0) = \sup_{x\in\mathbb{R}^d} \rho(x) \geq 0$. Because of lemma 3.2.7, it is again sufficient to give a proof for $L(p)$. Assumption (AH) implies $(\Delta H * \delta_p)(x_0) \leq 0$. Using this and $(\nabla\rho)(x_0)$, we obtain

$$(L(p)\rho)(x_0) = ((\Delta H * \delta_p)\rho)(x_0) + ((\nabla H * \delta_p) \cdot \nabla\rho)(x_0) \leq 0.$$

□

3.4.5 Lemma. *For all $v \in H^L_+(\mathbb{R}^d)$ and all $p \in \mathbb{R}^d$ there exists a $\lambda > 0$ such that the range, $R(\lambda - A(v,p))$, of $\lambda - A(v,p)$ is $L^2(\mathbb{R}^d)$.*

Proof. Writing the linear operator $A(v,p)$ in the form (3.2) we get $a_{kl} = \delta_{kl}\left(\frac{\sigma_\infty^2}{2} + v\right)$. Using the positivity of v, we can conclude exactly as in lemma 3.2.6

$$\sum_{k,l=1}^{d} a_{kl}(x)\xi_k\xi_l = \sum_{k=1}^{d}\left(\frac{\sigma_\infty^2}{2} + v\right)|\xi_k|^2 \geq \frac{\sigma_\infty^2}{2}|\xi|^2.$$

Since $\sigma_\infty > 0$, we obtain that $A(v,p)$ is a uniformly elliptic second order differential operator and the result follows from Theorem 3.1.5. □

3.4.6 Theorem. *For all $v \in H^L_+(\mathbb{R}^d)$ and all $p \in \mathbb{R}^d$ the operator $A(v,p)$ generates a positivity preserving C_0-semigroup of contractions on $L^2(\mathbb{R}^d)$.*

Proof. In lemma 3.4.3 we have shown that $A(v,p)$ is a dissipative linear operator on $L^2(\mathbb{R}^d)$. Furthermore lemma 3.4.5 says that $A(v,p)$ satisfies the range condition of the Lumer-Phillips-Theorem 3.1.9. Hence, $A(v,p)$ is the generator of a C_0-semigroup of contractions on $L^2(\mathbb{R}^d)$. From lemma 3.4.4 together with lemma 3.1.13 follows that this semigroup is positivity preserving. □

3.4.7 Definition. Let $m \in \mathbb{N}$ with $L \leq 2m$. For every $v \in H^{2m}_+(\mathbb{R}^d)$ and every point $p \in \mathbb{R}^d$ we inductively define a scale of abstract Sobolev spaces $(H^n_{A(v,p)}, \|\cdot\|_n), n \leq m$, associated to the linear operator $A(v,p)$ by $H^1_{A(v,p)} := H^2(\mathbb{R}^d)$ and

$$\begin{aligned} H^n_{A(v,p)} &:= D((A(v,p))^n) \\ &:= \{f \in D((A(v,p))^{n-1}) | A(v,p)f \in D((A(v,p))^{n-1})\}. \end{aligned}$$

The family of norms is given by

$$\|\cdot\|_{0,v,p} := \|\cdot\|_{L^2} \text{ and } \|\cdot\|_{n,v,p} := \|\cdot\|_{n-1,v,p} + \|A(v,p)\cdot\|_{n-1,v,p}$$

Since $A(v,p)$ is a uniformly elliptic second order differential operator, we can show the following analogous result to corollary 3.2.11.

3.4.8 Corollary. *Let $L \leq 2n \leq 2m$ and all $R > 0$ we have $H^{2n}(\mathbb{R}^d) = H^n_{A(v,p)}$ and there exist constants $C_{1;n}, C_{2;n} > 0$ such that for all $v \in H^{2m+1}_+(\mathbb{R}^d)$ with $\|v\|_{H^{2m+1}} \leq R$ and all $p \leq R$, we have*

$$C_{1;n}\|\cdot\|_{H^{2n}} \leq \|\cdot\|_{n,v,p} \leq C_{2;n}\|\cdot\|_{H^{2n}} \tag{3.27}$$

3.4.9 Definition. Let $A_n(v,p)$ denote the part of $A(v,p)$ in $H^n_{A(v,p)}$, i.e. we have $A_n(v,p)f := A(v,p)f$ for all

$$f \in D(A_n(v,p)) := \{f \in H^n_{A(v,p)} | A(v,p)f \in H^n_{A(v,p)}\} = H^{n+1}_{A(v,p)}.$$

3.4.10 Corollary. *For all $v \in H^{2m+1}_+(\mathbb{R}^d)$ and all $p \in \mathbb{R}^d$ the operator $A_n(v,p)$ generates a positivity preserving C_0-semigroup of contractions on the Sobolev space $H^{2n}(\mathbb{R}^d)$.*

Proof. Replacing $A(v)$ by $A(v,p)$ in the proof of 3.2.14 and of corollary 3.2.15, we obtain the result. □

3.4.11 Lemma. *Let $L \leq 2n \leq 2m$. For all $v \in H^{2m+1}_+(\mathbb{R}^d), p \in \mathbb{R}^d$ the operator*

$$A_n(v,p) : H^{2n+2}(\mathbb{R}^d) \to H^{2n}(\mathbb{R}^d)$$

is bounded. Moreover, the operator norm $\|A_n(v,p)\|_{L(H^{2n+2},H^{2n})}$ is bounded by a constant that depends only on σ_∞, G, H and $\|v\|_{H^{2m+1}}$.

Proof. Let $\rho \in H^{2n+2}(\mathbb{R}^d)$ and $v \in H^{2m+1}(\mathbb{R}^d)$. Taking lemma 3.3.1 into account it is sufficient to show $\|L(p)\rho\|_{H^{2n}} \leq C\|\rho\|_{H^{2n+2}}$. This follows from

$$\begin{aligned}\|L(p)\rho\|_{H^{2n}} &\leq \|\nabla\cdot(\nabla H * v)\rho\|_{H^{2n}}\\ &\leq \|(\Delta H * \delta_p)\rho\|_{H^{2n}} + \|(\nabla H * \delta_p)\cdot(\nabla\rho)\|_{H^{2n}}\\ &\leq \|(\Delta H * \delta_p)\|_{H^{2n}}\|\rho\|_{H^{2n}} + \|\nabla H * \delta_p\|_{H^{2n}}\|\nabla\rho\|_{H^{2n}}\\ &\leq C\|\rho\|_{H^{2n+2}}.\end{aligned}$$

with

$$C := \max\{\|(\Delta H * \delta_p)\|_{H^{2n}}, \|\nabla H * \delta_p\|_{H^{2n}}\} \leq \|\nabla H\|_{H^{2n+1}}.$$

Therefore, we see that $L(p) \in L(H^{2n+2}(\mathbb{R}^d), H^{2n}(\mathbb{R}^d))$. □

3.4.12 Lemma. *There exists a constant $C > 0$ such that for all $u, v \in H^{2m+1}_+(\mathbb{R}^d), L \leq 2m$ and all $p, q \in \mathbb{R}^d$ we have*

$$\|A(v,p) - A(u,q)\|_{\mathcal{L}(H^{2n+2},H^{2n})} \leq C(\|v-u\|_{H^{2n+1}} + |p-q|)), \; L \leq 2n \leq 2m.$$

Proof. Let $\rho \in H^{2n+2}(\mathbb{R}^d)$. Then using assumption (AH), we obtain

$$\begin{aligned} \|L(p)\rho - L(q)\rho\|_{H^{2n}} &\leq \|\nabla \cdot (\nabla H * \delta_p - \nabla H * \delta_q)\rho\|_{H^{2n}} \\ &\leq \|(\Delta H * \delta_p - \Delta H * \delta_q)\rho\|_{H^{2n}} \\ &\quad + \|(\nabla H * \delta_p - \nabla H * \delta_q) \cdot \nabla \rho\|_{H^{2n}} \\ &\leq \|\Delta H * \delta_p - \Delta H * \delta_q\|_{H^{2n}} \|\rho\|_{H^{2n}} \\ &\quad \|\nabla H * \delta_p - \nabla H * \delta_q\|_{H^{2n}} \|\nabla \rho\|_{H^{2n}} \\ &\leq C|p-q| \|\rho\|_{H^{2n+2}}. \end{aligned} \tag{3.28}$$

We get from equation (3.28) and lemma 3.3.2

$$\begin{aligned} \|A(v,p)\rho - A(v,p)\rho\|_{H^{2n}} &\leq \|A(v)\rho - A(v)\rho\|_{H^{2n}} + \|L(p)\rho - L(p)\rho\|_{H^{2n}} \\ &\leq C(\|v-u\|_{H^{2n+1}} + |p-q|). \end{aligned}$$

□

3.4.13 Lemma. *For all functions $x \in C([0,T], H_+^{2m+1}(\mathbb{R}^d)), L \leq 2m$ and all $p \in C([0,T], \mathbb{R}^d)$ satisfying*

$$\|x(t) - x(s)\|_{H^{2m+1}} + |p(t) - p(s)| \leq e^{C|t-s|} - 1 \tag{3.29}$$

the family $A_n(x(t), p(t)), t \in [0,T]$, is a stable family of generators of C_0-semigroups on $H^{2n}(\mathbb{R}^d)$ for all $L \leq 2n \leq 2m$.

Proof. One can conclude the result as in the proof of lemma 3.3.4. □

3.4.14 Corollary. *Let $m, n \in \mathbb{N}$ such that $L \leq 2n \leq 2n+2 \leq 2m$. Then:*

(a) Let $\rho \in C([0,T], H_+^{2m+1}(\mathbb{R}^d))$ be a function with values in a bounded set B and let $p \in C([0,T], \mathbb{R}^d)$ such that equation (3.29) holds. Define $\psi := (\rho, p)$. Then there exists a unique evolution system $\{U_\psi(t,s),\ 0 \leq s \leq t \leq T\}$ in $H^{2n}(\mathbb{R}^d)$ and constants ω, M such that

$$\|U_\psi(t,s)\|_{L(H^{2n}(\mathbb{R}^d))} \leq M e^{\omega(t-s)} \tag{3.30}$$

$$\partial_t^+ U_\psi(t,s) y\big|_{t=s} = A(\rho(s), p(s)) y \tag{3.31}$$

$$\partial_s U_\psi(t,s) y = -U_\psi(t,s) A(\rho(s), p(s)) y \tag{3.32}$$

for all $0 \leq s \leq t \leq T$ and all $y \in H^{2n+2}(\mathbb{R}^d)$.

(b) $U_\psi(t,s) H^{2n+2}(\mathbb{R}^d) \subseteq H^{2n+2}(\mathbb{R}^d)$ and $\|U_\psi(t,s)\|_{H^{2n+2}} \leq \tilde{M} e^{\tilde{\beta}(t-s)}$ for some constants $\tilde{M} \geq 1, \tilde{\beta} \geq 0$. Furthermore, $U_\psi(t,s)$ is weakly continuous in t, s with respect to $H^{2n+2}(\mathbb{R}^d)$.

(c) $\partial_t^+ U_\psi(t,s)y = A(\rho(s),p(t))U_\psi(t,s)y$ *for* $y \in H^{2n+2}(\mathbb{R}^d)$ *and* $s \leq t$.

(d) $\partial_t U_\psi(t,s)y$ *exists for almost every* t *and*

$$\partial_t U_\psi(t,s)y = A(\rho(t),p(t))U_\psi(t,s)y, \qquad y \in H^{2n+2}(\mathbb{R}^d)$$

holds for almost every t.

(e) *Furthermore, there exists a constant* $K > 0$ *such that for all* $\psi_1, \psi_2 \in C([0,T], H_+^{2n+1}(\mathbb{R}^d) \times \mathbb{R}^d)$ *such that* $\psi_i = (\rho_i, p_i)$ *and* $\rho_i([0,t]) \subseteq B, i = 1,2$ *and* ρ_i *satisfying (3.29) and all* $y \in H^{2n+2)}(\mathbb{R}^d)$ *we have*

$$\|U_{\psi_1}(t,s)y - U_{\psi_2}(t,s)y\|_{H^{2n}} \leq K\|y\|_{H^{2n+2}} \int_s^t \left(\|\rho_1(\tau) - \rho_2(\tau)\|_{H^{2n}} + |p_1(\tau) - p_2(\tau)|\right) \, dr \quad (3.33)$$

Proof. In lemma 3.4.13 it is shown that $(A_n(\rho(t),p(t)))_{t\in[0,T]}$ is a stable family of generators of C_0-semigroups on $H^{2n}(\mathbb{R}^d)$. Furthermore, in 3.4.10 we have seen that $H^{2n+2}(\mathbb{R}^d)$ is $A(\rho(t),p(t))$-admissible, $t \in [0,T]$, and the family of parts in $H^{2n+2}(\mathbb{R}^d)$ is given by linear operators $(A_{n+1}(\rho(t),p(t)))_{t\in[0,T]}$ which is again by lemma 3.4.13 a stable family of generators of C_0-semigroups on $H^{2n+2}(\mathbb{R}^d)$. Lemma 3.4.11 implies

$$A(\rho(t),p(t)) \in \mathcal{L}(H^{2n+2}(\mathbb{R}^d), H^{2n}(\mathbb{R}^d)).$$

Now let $t_1, t_2 \in [0,T]$. From lemma 3.4.12 we obtain

$$\|A(\rho(t_1),p(t_1))) - A(\rho(t_2),p(t_2))\|_{\mathcal{L}(H^{2n+2},H^{2n})} \leq C\|\rho(t_1) - \rho(t_2)\|_{H^{2n+1}} + |p(t_1) - p(t_2)| \underset{t_1\to t_2}{\longrightarrow} 0.$$

It follows that $t \mapsto A(\rho(t),p(t))$ is continuous with respect to $\|\cdot\|_{\mathcal{L}(H^{2n+2},H^{2n})}$. Hence conditions (i)-(iv) from theorem 3.1.19 are fulfilled.

Applying lemma 3.4.11 to [25, equation (4.14), p.202] gives (3.33). □

3.4.15 Theorem. *Let* $L \leq 2n$, $\rho_0 \in H^{2n+2}(\mathbb{R}^d), p_0 \in \mathbb{R}^d$ *and let* $B := B(\rho_0, r)$ *a ball in* $H^{2n}(\mathbb{R}^d)$ *around* ρ_0 *with radius* $r > 0$. *Then there exists* $\tilde{T} > 0$ *such that the system*

$$\begin{aligned} \partial_t\rho(x,t) &= A(\rho(\cdot,t), P_\infty(t))\rho(x,t), \quad t \in [0,\tilde{T}], \\ \rho(x,0) &= \rho_0(x), \\ \partial_t P_\infty(t) &= \int_{\mathbb{R}^d} \rho(x,t)w(x - P_\infty(t)) \, dx, \\ P_\infty(t) &= p_0, \end{aligned} \quad (3.34)$$

has a unique solution on $[0,\tilde{T}]$ *in the sense of definition 3.4.1 with* $\rho([0,\tilde{T}]) \subseteq B$.

Proof. This proof is closely related to the proof of the existence theorem in the case without a predator 3.3.7. The position of the predator P_∞ adds a dimension to the original Cauchy problem 3.1 that was considered in the previous section. Since the law of motion of the predator is different compared to the laws motion of the ordinary particles $X_N^k, k = 1, \dots, N$, we have to treat this additional dimension separately during this proof.

(Step 0) For all $\psi = (x, p) \in C([0,T], H_+^{2n}(\mathbb{R}^d) \times \mathbb{R}^d)$ with $x([0,T]) \subseteq B$ and such that equation (3.29) holds, let $U_{j,\psi}$ denote the unique evolution system associated to the operator family $(A(J_j x(t), p(t))), t \in [0,T]$ that is given by theorem 3.4.14. Moreover, let U_{ψ_0} denote the unique evolution system associated to the operator $A(\rho_0, p_0)$.

(Step 1) Choose $\tilde{T} \leq T$ such that

(i) $\max_{0 \leq t \leq \tilde{T}} \|U_{\psi_0}(t,0)\rho_0 - \rho_0\|_{H^{2n}} \leq \frac{r}{3}$;

(ii) $\tilde{T} \leq \frac{1}{4}(K(C_1 + 1)\|\rho_0\|_{H^{2n+2}})^{-1}$ if $\|\rho_0\|_{H^{2n+2}} > 0$;

(iii) $\tilde{T} \leq \frac{1}{8}(r\ell)^{-1}$;

(iv) $\tilde{T} \leq \frac{1}{8}\|w\|_{L^2}^{-1}$.

Here, the constant K in (ii) is the same as in equation (3.18), C_1 si from (b) of 3.3.6 and the constant ℓ in (iii) is given by the constant on the right-hand side of equation (2.51).

Choose j_0 such that

$$\|J_j\rho_0 - \rho_0\|_{H^{2n}} \leq C_1 r, \text{ for all } j \geq j_0. \tag{3.35}$$

Let $\mathcal{A}$ denote the set of all functions

$$\psi = (\rho, g) \in C([0,\tilde{T}], H_+^{2n}(\mathbb{R}^d) \times \mathbb{R}^d)$$

such that

(a) $\rho(0) = \rho_0$;

(b) $\rho([0,T]) \subseteq B$;

(c) $\rho(t)$ is non-negative for all $t \in [0,\tilde{T}]$;

(d) There exists a $C > 0$ such that $\|\rho(t) - \rho(s)\|_{H^{2n}} \leq C|t-s|$ for all $t, s \in [0,\tilde{T}]$;

(e) There exists a $C > 0$ such that $|g(t) - g(s)| \leq C|t-s|$ for all $t, s \in [0,\tilde{T}]$.

We define a family of mappings

$$\Phi_j : \mathcal{A} \to C([0,\tilde{T}], H^{2n}(\mathbb{R}^d) \times \mathbb{R}^d), j \in \mathbb{N};$$
$$(\Phi_j\psi)(t) := (U_{j,\psi}(t,0)\rho_0, \tilde{g}) \text{ for all } \psi = (\rho, g)$$

where $\tilde{g}$ is the solution of

$$\partial_t\tilde{g}(t) = \int_{\mathbb{R}^d} w(x - g(t))\rho(x,t)\, dx.$$
$$P_\infty(0) = p_0.$$

Due to property (d) and (e) and 3.3.6 (a), $(J_j\rho, g)$ satisfies equation (3.29). Hence, $U_{j,\psi}$ is well-defined.

(Step 2) One can conclude exactly as in Step 3 of the proof of theorem 3.3.7 that all points in the range of $\Phi_j, j \geq j_0$ satisty (a)-(d).

Moreover, we have $(t > s)$

$$\begin{aligned}
|\tilde{g}(t) - \tilde{g}(s)| &= \int_s^t \int_{RR^d} w(x - g(\tau))\rho(x,\tau)\, dx d\tau \\
&\leq (t-s) \sup_{\tau \leq \tilde{T}} \int_{RR^d} w(x - g(\tau))\rho(x,\tau)\, dx \\
&\leq (t-s) \sup_{\tau \leq \tilde{T}} \|w\|_{L^2}^{1/2} \|\rho(\cdot,\tau)\|_{L^2}^{1/2} \\
&\leq C(t-s)
\end{aligned}$$

Thus, (e) holds. This completes the proof of $\Phi_j(\mathcal{A}) \subseteq \mathcal{A}, j \geq j_0$.

(Step 3) Now, consider the space $H^{2n}(\mathbb{R}^d) \times \mathbb{R}^d$ equipped with the norm

$$\|\psi\|_{H^{2n} \times \mathbb{R}^d} = \|\rho\|_{H^{2n}(\mathbb{R}^d)} + |x|, \quad \text{for all } \psi = (\rho, x) \in H^{2n}(\mathbb{R}^d) \times \mathbb{R}^d. \quad (3.36)$$

Furthermore, let $\|\cdot\|_\infty$ denotes the supremum norm on the corresponding space of continuous functions. Obviously, $(C([0,\tilde{T}], H^{2n}(\mathbb{R}^d) \times \mathbb{R}^d), \|\cdot\|_\infty)$ is Banach space.

Define $\psi_0 \in \mathcal{A}$ by $\psi_0(t) := (\rho_0, p_0)$ for all t. and

$$\psi_{j+1} := (x_j, g) := \Phi_j(\psi_j) \in \mathcal{A} \text{ for all } t$$

Using equation (3.33), we obtain

$$\begin{aligned}\|x_{j+1}(t) - x_j(t)\|_{H^{2n}} &= \|U_{j,\psi_j}(t,0)\rho_0 - U_{j-1,\psi_{j-1}}(t,0)\rho_0\|_{H^{2n}} \\ &\leq K\|\rho_0\|_{H^{2n+2}} \int_0^{\tilde{t}} \Big(\|J_j x_j(\tau) - J_{j-1}x_{j-1}(\tau)\|_{H^{2n}} \\ &\qquad + |g_j(\tau) - g_{j-1}(\tau)| \Big)\, d\tau \\ &\leq K\tilde{T}\|\rho_0\|_{H^{2n+2}} \Big(\|J_j - J_{j-1}\|_{L(H^{2n})}\|x_j\|_\infty \\ &\qquad + \|J_{j-1}\|_{L(H^{2n})}\|x_j - x_{j-1}\|_\infty \\ &\qquad + \|g_j - g_{j-1}\|_\infty \Big)\end{aligned}$$

Taking the supremum and using property (c) of definition 3.3.6, property (ii) from the definition of $\tilde{T}$ and $\|x_j\|_\infty \leq r$, we get

$$\begin{aligned}\|x_{j+1} - x_j\|_\infty &\leq \frac{C}{j^2} + \frac{1}{4}\|x_j - x_{j-1}\|_\infty + \frac{1}{4}\|g_j - g_{j-1}\|_\infty \\ &\leq \frac{C}{j^2} + \frac{1}{2}\|\psi_j - \psi_{j-1}\|_\infty.\end{aligned} \tag{3.37}$$

We have

$$\begin{aligned}|\tilde{g}_{j+1}(t) - \tilde{g}_j(t)| &= \left| \int_0^t \partial_t \tilde{g}_{j+1}(s)\, ds - \int_0^t \partial_t \tilde{g}_j(s)\, ds \right| \\ &\leq \left| \int_0^t \Big[\int_{\mathbb{R}^d} x_j(x,s) w(x - g_j(s))\, dx \right. \\ &\qquad \left. - \int_{\mathbb{R}^d} x_{j-1}(x,s) w(x - g_{j-1}(s))\, dx \Big] ds \right| \\ &\leq I_1 + I_2\end{aligned} \tag{3.38}$$

with

$$\begin{aligned}I_1 := \left| \int_0^t \Big[\int_{\mathbb{R}^d} x_j(x,s) w(x - g_j(s))\, dx \right. \\ \left. - \int_{\mathbb{R}^d} x_{j-1}(x,s) w(x - g_j(s))\, dx \Big] ds \right|\end{aligned}$$

and

$$\begin{aligned}I_2 := \left| \int_0^t \Big[\int_{\mathbb{R}^d} x_{j-1}(x,s) w(x - g_j(s))\, dx \right. \\ \left. - \int_{\mathbb{R}^d} x_{j-1}(x,s) w(x - g_{j-1}(s))\, dx \Big] ds \right|.\end{aligned}$$

Using Schwarz inequality, we obtain for I_1

$$\begin{aligned} I_1 &\leq \tilde{T} \sup_{s\leq\tilde{T}} \|x_j(s) - x_{j-1}(s)\|_{L^2} \sup_{s\leq\tilde{T}} \|w(\cdot)\|_{L^2} \\ &\leq \frac{1}{8}\|x_j - x_{j-1}\|_\infty. \end{aligned}$$

And for the second part I_2 we obtain

$$\begin{aligned} I_2 &\leq \tilde{T} \sup_{s\leq\tilde{T}} \|x_{j-1}(s)\|_{L^2} \sup_{s\leq\tilde{T}} \|w(\cdot - g_j(s)) - w(\cdot - g_{j-1}(s))\|_{L^2} \\ &\leq \tilde{T} r\ell \|g_j - g_{j-1}\|_\infty \\ &\leq \frac{1}{8}\|\psi_1 - \psi_2\|_\infty. \end{aligned}$$

For the last inequality we used property (iii) of $\tilde{T}$. Therefore

$$|g_1(t) - g_2(t)| \leq \frac{1}{4}\|\psi_1 - \psi_2\|_\infty. \tag{3.39}$$

Combining the equations (3.37), (3.39), we get

$$\|\psi_{j+1} - \psi_j\|_\infty \leq \frac{C}{j^2} + \frac{3}{4}\|\psi_j - \psi_{j-1}\|_\infty.$$

Applying lemma 3.1.20, we obtain

$$\sum_{j=0}^{\infty} \|\psi_{j+1} - \psi_j\|_\infty < \infty. \tag{3.40}$$

Thus, $(\psi_j)_j$ converges towards a pair of functions ψ. Moreover, (a)-(c) from the definition of $\mathcal{A}$ hold for ψ. One can conclude that equation (3.34) holds for almost every t. □

3.4.16 Corollary. *Suppose assumption (AG) and (AH) with* $2m := 2n + 2$ *hold. Then* $\rho_0 \in H^{2n+2}(\mathbb{R}^d)$ *with* $2n > L + 1 + \frac{d}{2}$ *is a sufficient condition for (AS) from chapter 2.2.2.*

Proof. This is a direct consequence of theorem 3.4.15. □

Chapter 4

Simulation

In this chapter we discuss some simulation results of our individual-based models with a repulsive predator in dimension $d = 2$. The programming language `R` [26] was used for the computations. The simulated systems were of the type

$$dX_N^k(t) = F_N[X_N(t)](X_N^k(t))dt + \sigma_N d\mathbb{W}^k(t), \quad k = 1, \dots N. \tag{4.1}$$

with

$$F_N[X_N(t)](X_N^k(t)) = F^A[X_N(t)](X_N^k(t)) + F_N^R[X_N(t)](X_N^k(t)) \tag{4.2}$$
$$+ F^P[P_N(t)](X_N^k(t)) + F^0(X_N^k(t)) \tag{4.3}$$

where F^A and F_N^R model the aggregation and repulsion effects between different particles as described in chapter 1 and F^P is a repulsive force resulting from the presence of a predator in point $P_N(t)$. In addition to these terms we have $F^0(X_N^k(t)) := -\alpha X_N(t)$ for some constant $\alpha > 0$. This adds an attractive force of Ornstein–Uhlenbeck type towards 0. In our discussion of deterministic predators we made no assumptions on the sign of the resulting force, thus F^0 can be seen as a "predator" in 0 with an attraction potential. However, since $x \mapsto -\alpha x$ is an unbounded function, our convergence results can not be directly applied. The law of motion of the predator was stochastic as in chapter 2.2.2.

We simulated for a fixed number of particles $N = 50$. Due to very high computational demands we could not observe the behavior of system for large N. Instead we focused on stability effects for large time T. In fact, one can observe that after a short period of time the particle densities vary very little. Depending on the strength of all forces and the diffusion coefficients various interesting effects could be observed.

We used the following radial kernel functions:

$$\nabla G = \begin{cases} c_G \frac{x}{\|x\|}, & \|x\| \leq R, \\ 0, & \text{otherwise}, \end{cases}$$
$$\nabla V = c_V \frac{x}{\|x\|} \Gamma_{s,r}(\|x\|),$$
$$\nabla H = c_H \frac{x}{\|x\|} \Gamma_{s,r}(\|x\|).$$

Rescaling of the repulsion kernel was mimicked by the choice of parameters and constants. Here $\Gamma_{s,r}$ denotes a gamma density with shape parameter s and scale parameter r, i.e.

$$\Gamma_{s,r}(x) = \frac{1}{r^s \Gamma(s)} x^{s-1} \exp(-\frac{x}{r}).$$

The movement of the predator was given by

$$dP = c_P \sum_{k=1}^{N} (X_N^k(t) - P(t))dt + \sigma_P d\mathbb{W}_P$$

During our simulations we used the constants $\alpha = 0.1$,$s = \frac{1}{2}, \sigma_N = 30, R = 50$ and $r = 100$.

Please observe that these kernel functions do not satisfy the technical assumptions from chapter 1 and 2.

A very strong aggregation force ($c_G = 5, c_V = 5, c_H = 50, \sigma_P = 2, c_P = 0.01$) leads to strong concentration of all animals in one single group. This group and the predator move around the center on nearly circular trajectories, see figure 4.1 and 4.2.

If the aggregation force is very low ($c_G = 0.5, c_V = 5, c_H = 50, \sigma_P = 2, c_P = 0.01$), the animals remain on circle around the predator in a nearly uniform distribution, figure 4.3. No grouping can be seen.

Values in between these extreme settings lead to clustering in small groups of different size and different spatial distributions, see figures 4.4, 4.5 (both with $c_G = 3, c_V = 5, c_H = 20, \sigma_P = 5, c_P = 1$), 4.6 and 4.7 (both with $c_G = 2, c_V = 5, c_H = 20, \sigma_P = 5, c_P = 1$).

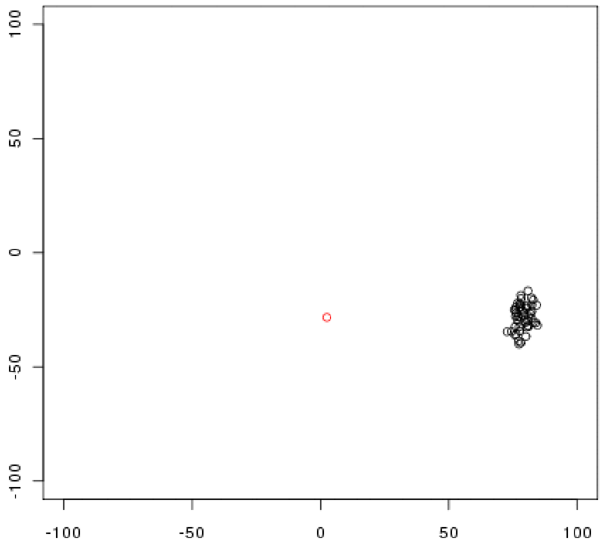

Figure 4.1

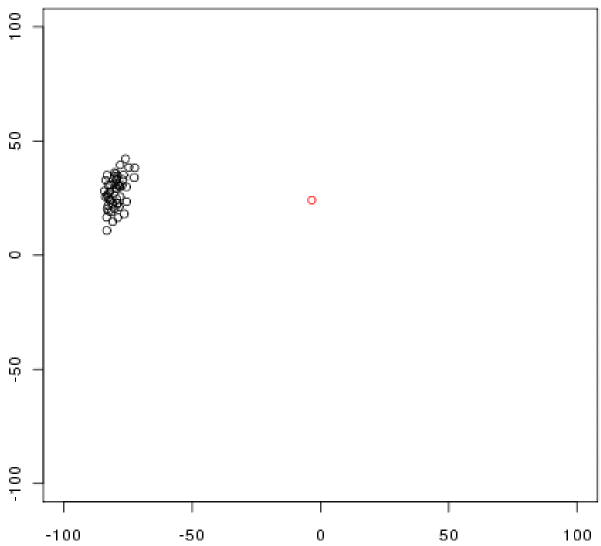

Figure 4.2

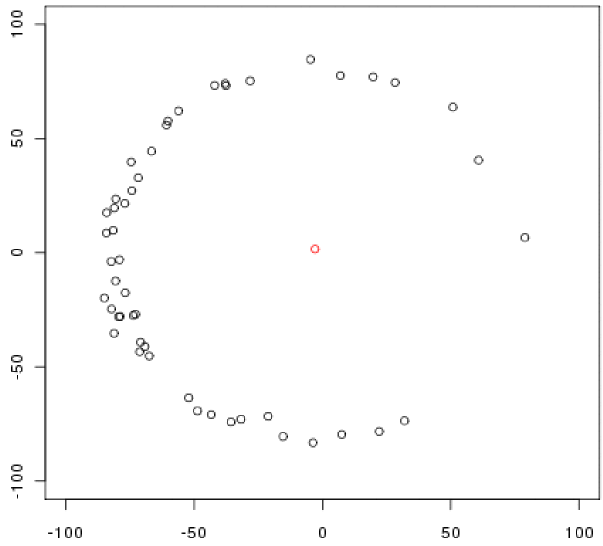

Figure 4.3

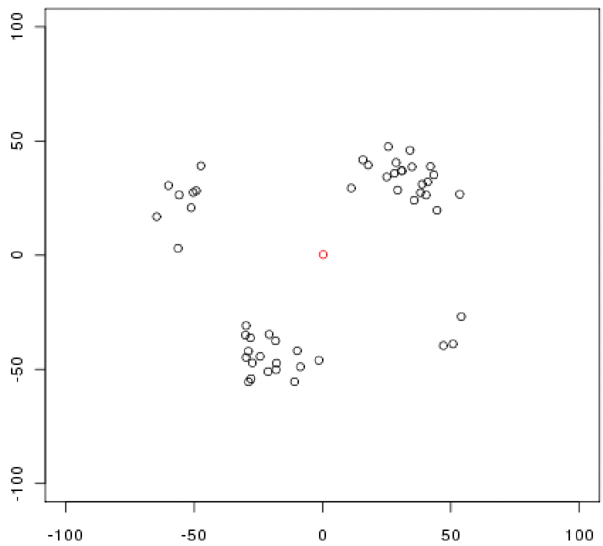

Figure 4.4

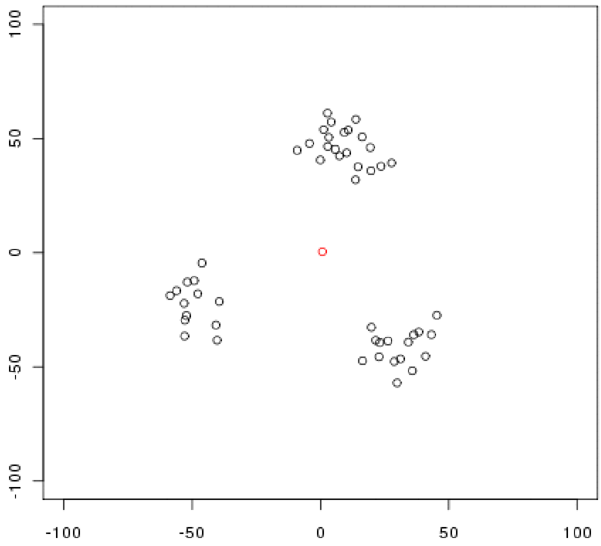

Figure 4.5

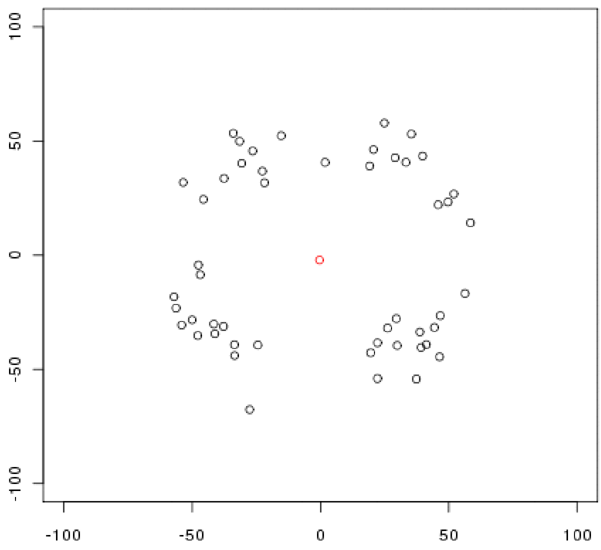

Figure 4.6

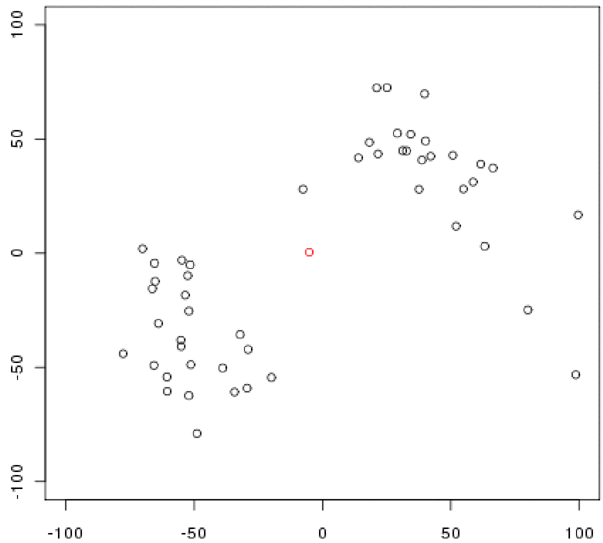

Figure 4.7

Appendix A

Completely monotone functions and positve definite functions

A.1 Definition. A function $\varphi \in C([0,\infty)) \cap C^\infty((0,\infty))$ such that for all $k \in \mathbb{N}_0$

$$(-1)^k \varphi^{(l)}(x) \geq 0, \quad x > 0, \tag{A.1}$$

is called *completely monotone* on $[0,\infty)$.

A.2 Definition. A function $f : \mathbb{R}^d \to \mathbb{R}$ sucht that there exists a function $\phi : [0,\infty) \to \mathbb{R}$ with $f(x) = \phi(|x|)$ is called a *radial function*.

A.3 Example. The function $\varphi(x) := (1+x)^{-r},\ r \geq 0$ is completely monotone on $[0,\infty)$.

Proof. For all $k \in \mathbb{N}_0$ we have

$$(-1)^k \varphi^{(k)}(x) = (-1)^{2k} r \cdot \dots \cdot (r+k-1)(1+x)^{-r-k} \geq 0. \tag{A.2}$$

□

A.4 Definition. A function $f : \mathbb{R}^d \to \mathbb{C}$ is called a *positive definite function* if for all $u_1, \dots, u_d \in \mathbb{R}^d$ the matrix $(f(u_i - u_j))_{(i,j)}$ is positive definite, i.e. we have

$$\sum_{i,j=1}^{d} c_i \overline{c_j} f(u_i - u_j) \geq 0 \tag{A.3}$$

for all $c_1, \dots c_d \in \mathbb{R}^d$.

A.5 Theorem. *A function φ is completely monotone on $[0,\infty)$ if and only if $\varphi(|\cdot|^2)$ is positive definite and radial on $\mathbb{R}^d$ for all $d \in \mathbb{N}$.*

Proof. Theorem 3 of [27] □

A.6 Theorem (Bochner's Theorem)**.** *A function $f : \mathbb{R}^d \to \mathbb{C}$ is the Fourier transform of a probability distribution if and only if the following three properties hold:*

(i) f is a positive definite function.

(ii) f is continuous at the origin.

(iii) $f(0) = (2\pi)^{-d/2}$.

A.7 Example. Let $r > \frac{d}{2}$ and $f(x) := (2\pi)^{-d/2}(1 + |x|^2)^{-r}$. Then f is the Fourier transform of a probability density function.

Proof. Example A.3 together with theorem A.5 shows that f is a positive definite function. Clearly, f is continuous in 0 and we have $f(0) = (2\pi)^{-d/2}$. Thus, by Bochner's Theorem A.6 f is the fourier transform of a probability distribution. Finally, $r > 0$ implies $f \in L^2(\mathbb{R}^d)$. Therefore, there exists a probability density function g with $\hat{g} = f$. □

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