A commutative higher cycle map into Deligne-Beilinson cohomology

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Introduction

Motivation As one of the steps towards an universal "motivic" cohomology theory, in 1985 Beilinson [2] defined rational motivic cohomology $H^i_{\mathcal{M}}(U, \mathbb{Q}(p))$ of an algebraic manifold $U/\mathbb{C}$ and defined the regulator map into the weight p Deligne-Beilinson cohomology of U,

$$\mathrm{reg} : H^i_{\mathcal{M}}(U, \mathbb{Q}(p)) \longrightarrow H^i_{\mathcal{D}}(U, \mathbb{Q}(p)).$$

The terms occurring on either side of this map arise as the cohomology of certain complexes. It was suggested by Goncharov in [24] and [25] that the regulator map should be induced by an explicitly defined map between these complexes. A cycle complex that computes motivic cohomology and even allows integral coefficients was already proposed 1986 by Bloch in [6]. He also constructed a map from the cohomology of this complex, the higher Chow groups, to the integral Deligne-Beilinson cohomology. His construction however was not very explicit.
Kerr, Lewis and Müller-Stach [39] gave the definition of a regulator map on complexes, using a variation of Bloch's cycle complex to compute motivic, and Jannsen's 3-term-complex [37] to compute Deligne-Beilinson cohomology.
Both motivic cohomology and Deligne-Beilinson cohomology carry the structure of a graded-commutative associative $\mathbb{Q}$-algebra. Their products can be partially defined on the underlying complexes – partially in the sense of "not everywhere defined". Although the regulator map of Kerr/Lewis/Müller-Stach induces a morphism of graded-commutative algebras on total cohomology (i.e., on the direct sum over $i, p \in \mathbb{Z}$), this map is not a strict homomorphism of algebras. That means, it is not compatible with the products on the underlying complexes. This is due to the fact that the product on the cycle complexes can be chosen to be associative and graded-commutative, while the (rather any) product on the 3-term complex lacks at least one of these properties.
The goal of this thesis is to describe a map of partially defined graded-commutative differential graded algebras (dg algebras) that on cohomology induces the same regulator as the one of [39].

Short sketch of the content We start by reviewing some complexes that compute motivic cohomology and Deligne-Beilinson cohomology. On the motivic side, we work with Bloch's cycle complexes and in particular with their refinement $z^p_{\mathbb{R}}(U, \bullet)$ of higher Chow chains that have proper intersection with respect to some "real faces". On the analytic side, we use (a variant of) Jannsen's 3-term complex $C_{\mathcal{D}}$ which comes with a whole family of products – but none of them is both associative and graded-commutative. Following an advice of Levine, we replace the 3-term complex by another complex $P_{\mathcal{D}}$ that actually has a graded-commutative product and is related to the complex $C_{\mathcal{D}}$ by means of the evaluation map ev, which turns out to be a quasi-isomorphism after extending coefficients to $\mathbb{Q}$.
These complexes actually depend not only on U, but on a choice of a good compactification (X, D) of U. Therefore, we put ourselves in a general setting and define an abstract (partially defined) regulator map from the complex of higher Chow chains to any family of complexes indexed by triples (X, D, p) that satisfies a list of properties. We verify these properties for

the complexes $C_{\mathcal{D}}$ and $P_{\mathcal{D}}$, thus obtaining two specialization of the abstract regulator map, the former of whom coincides with the regulator map in [39].

The diagram formed by these two regulators and ev,

$$\begin{tikzcd} & P_D^{2p-\bullet}(X,D,\mathbb{Z}(p)) \arrow[d, "ev"] \\ z^p_{\mathbb{R}}(U,\bullet) \arrow[ur, "\mathrm{reg}_P"] \arrow[r, "\mathrm{reg}_C"'] & C_{\mathcal{D}}^{2p-\bullet}(X,D,\mathbb{Z}(p)), \end{tikzcd}$$

although not commutative in general, commutes after passing to rational cohomology.

We equip the above complexes with partially defined intersection products, and it will turn out that the regulator maps are compatible with these products.

The product on the complex of higher chains is in general not graded-commutative. For rational coefficients however there exist subcomplexes $z^p_{\mathbb{R}}(U,\bullet)^{\mathrm{Alt}}_{\mathbb{Q}}$ of alternating chains that also compute motivic cohomology and that are endowed with a partially defined – this time graded-commutative – intersection product. It is convenient to regrade this complex and write $\mathcal{N}^{2p-\bullet}(U,p)^{\mathrm{Alt}}_{\mathbb{Q}} := z^p_{\mathbb{R}}(U,\bullet)^{\mathrm{Alt}}_{\mathbb{Q}}$, with the effect that the ($\mathbb{Q}$-linear extension of the) regulator maps become morphisms of cochain complexes.

The total complex $\oplus\mathcal{N}^{\bullet}(U,p)^{\mathrm{Alt}}_{\mathbb{Q}}$ is an associative graded-commutative partially defined dg algebra and the same holds for the total path complex, so that the restriction of the regulator reg_P to rational alternating chains is a map of associative graded-commutative partially defined dg algebras

$$\bigoplus_p \mathcal{N}^{\bullet}(U,p)^{\mathrm{Alt}}_{\mathbb{Q}} \longrightarrow \bigoplus_p P^{\bullet}_{\mathcal{D}}(X,D,\mathbb{Q}(p)).$$

This is the searched-for map. On cohomology, ev induces an isomorphism of algebras, so that this map is indeed equivalent to the regulator reg_C (and thus reg) on cohomology.

The passage from higher Chow chains to alternating Chow chains is realized by the alternating projection Alt. The regulator reg_P is invariant under alternation and even can be seen as an alternating version of reg_C. All this is formalized in the commutativity of the diagram below.

$$\begin{tikzcd} z^p_{\mathbb{R}}(U,n) \arrow[r, "\mathrm{reg}_P"] \arrow[d, "\mathrm{Alt}"'] & P_{\mathcal{D}}^{2p-n}(X,D,\mathbb{Z}(p)) \arrow[d, "ev"] \\ z^p_{\mathbb{R}}(U,n)^{\mathrm{Alt}}_{\mathbb{Q}} \arrow[ur, "\mathrm{reg}_P"] \arrow[r, "\mathrm{reg}_C"] & C_{\mathcal{D}}^{2p-n}(X,D,\mathbb{Q}(p)) \end{tikzcd}$$

One advantage of the regulator map between complexes is that one has explicit formulas for it. To illustrate that point, we give formulas for the regulator into $P_{\mathcal{D}}$ for small cubical degree and for a special class of cycles – namely the graph cycles. To see a concrete example, we apply the regulator map to a generalization of Totaro's cycle, which leads to dilogarithms.

Building on the construction of the higher Abel-Jacobi map in [39], we associate to both regulator maps an Abel-Jacobi map from higher Chow chains that are homologous to zero to a generalized Jacobian. It turns out that the Abel-Jacobi map for reg_P is a symmetrization of the Abel-Jacobi map for reg_C.

We also study the behaviour of the Abel-Jacobi map under exterior products and pullbacks along higher correspondences and provide explicit formulas.

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Conventions While in section 1 the higher Chow groups are defined over any field, outside of this section we mostly stick to the field $\mathbb{C}$ or $\mathbb{R}$. In this case, we work in the (complex) analytic setting, i.e., all spaces/sheaves are with respect to the analytic topology.
A complex algebraic manifold denotes a smooth quasi-projective algebraic variety over $\mathbb{C}$ together with the complex topology. In particular, it is hausdorff and second countable.
Usually, the notion of a "regulator" is reserved for mappings that are defined on higher K-theory and take values in some cohomology theory (here Deligne-Beilinson cohomology). We rather use "regulator" as a synonym for "higher cycle map", which is justified by Bloch's work [5], where he showed that – after tensoring with $\mathbb{Q}$ – higher Chow groups are isomorphic to higher K-theory.
A bullet $\bullet$ indicates that the object is a complex. By convention, complexes with lower indices have decreasing differential, while complexes with upper indices have increasing differential (degree $+1$). We omit the $\bullet$, if it is not necessary.
The notion of a partially defined product will be used in an informal way, meaning a product which is defined only on a certain (not further specified) subset.

1 Higher Chow groups

Higher Chow groups have been introduced by Spencer Bloch [5] in 1986 as a cycle-theoretic description of rational higher K-theory. They can be thought of as an algebraic version of singular (Borel-Moore) homology theory. In fact, they are known to compute motivic cohomology with integral coefficients. The name "higher Chow groups" is due to the fact that they contain the usual Chow groups as a special case.

We introduce a complex of cubical higher Chow chains whose cohomology groups are the higher Chow groups and describe their functoriality and their exterior and interior products. The passage to alternating chains allows to define a graded-commutative inner (intersection) product, with the drawback of needing rational coefficients. With view to the construction in 4.1, we also define higher Chow chains with good intersection with respect to some real faces and show in an appendix that they also compute the higher Chow groups.

1.1 Bloch's cycle complex

Following Totaro [62, p. 180], we define the cubical higher Chow groups for quasi-projective varieties over a field k. They can be seen as an algebraic analogon of the cubical singular homology groups. Define the (algebraic) n-cube as

$$\square^n := (\mathbb{P}_1(k) \setminus \{1\})^n .$$

The codimension 1 faces of the n-cube are defined to be the hyperplanes obtained by setting one of the coordinates z_i to be equal $z_i = 0$ or $z_i = \infty$. So for each i there are two i-th faces which can be identified with the images of the maps

$$\partial_{i,\epsilon} : \square^{n-1} \to \square^n, \quad \epsilon = 0, \infty$$

that insert ϵ at place i. Codimension r faces are intersections of r different 1 faces, i.e., obtained by setting r of the n coordinates to be 0 or ∞. The union of all codimension r faces is thus

$$\partial^r \square^n = \{(z_1, \ldots, z_n) \mid r \text{ different } z_i \in \{0, \infty\}\} .$$

One also has degeneracy maps $\pi_i : \square^n \to \square^{n-1}$ which omit the i-th coordinate and act as the identity on the other coordinates. The collection of all the cubes together with the various maps $\partial_{i,\epsilon}, \pi_j$ form a cubical object (in the sense of [9]) in the category of smooth quasi-projective varieties.

Now let U be a (smooth) quasi-projective variety over k. To mimic the construction of (cubical) singular homology one needs a replacement for the continuous maps $\square^n \to U$. Since there are too few algebraic maps from the n-cube to U, one instead considers algebraic cycles in $U \times \square^n$ (the fibre product over $\operatorname{Spec} k$). In general, the group of codimension p algebraic cycles in a quasi-projective variety V (over a field) is the free abelian group generated by the set of irreducible closed subvarieties of V, that is,

$$z^p(V) = \left\{ \sum_{\text{finite}} n_i Z_i \mid n_i \in \mathbb{Z}, Z_i \text{ closed irreducible subvariety of } V \right\}.$$

The faces of $\square^n$ extend to $U \times \square^n$ as $U \times$ face. To have a good notion of restriction to those faces, one works with a smaller set of cycles. Namely, an algebraic cycle is called *admissible*, if it intersects all faces properly, i.e., in the correct codimension[1]. The group generated by those algebraic cycles is

$$c^p(U,n) := \langle Z \in z^p(U \times \square^n) \text{ admissible } \rangle$$

whose members are the admissible (higher Chow) chains. The pullbacks of admissible chains along the face maps $\partial_{i,\epsilon}$ exist and give rise to a differential

$$\partial := \sum_{i=1}^{n} (-1)^{i+1} (\partial_{i,0}^* - \partial_{i,\infty}^*) : c^p(U,n) \to c^p(U,n-1).$$

The group of *degenerate cycles* $d^p(U,n)$ is the subgroup of all those algebraic cycles that are obtained as the pullback of admissible chains along some projection map π_j. This is a subgroup of admissible chains and one defines the group of *higher Chow chains* as the quotient

$$z^p(U,n) := c^p(U,n)/d^p(U,n).$$

This becomes a chain complex with the differential ∂, i.e., $\partial \circ \partial = 0$.
The *higher Chow groups* can be defined as the homology groups of $z^p(U, \bullet)$, or, cohomologically written,

$$CH^p(U,n) := H_n(z^p(U,\bullet)).$$

If A is a coefficient ring, then the higher Chow groups with coefficients in A are by definition the homology groups of $z^p(U,\bullet) \otimes_{\mathbb{Z}} A$. For $A = \mathbb{Q}$, this is the same as $\mathrm{CH}^p(U,\bullet) \otimes_{\mathbb{Z}} \mathbb{Q}$ by the universal coefficient theorem.

Remark. The original definition of Bloch's higher Chow groups uses simplicial chains instead of cubical ones. The corresponding homology groups however are the same [45, 4.7].

Higher Chow groups and motivic cohomology The "modern" approach to motivic cohomology of smooth schemes over a field k is to define somehow a triangulated category $DM(k)$ with special "Tate objects" $\mathbb{Z}(n)$ in it, and then define the motivic cohomology of U as

$$H^i_{\mathcal{M}}(U, \mathbb{Z}(p)) = \mathrm{Hom}_{DM(k)} \left(\mathbb{Z}(0), \mathbb{Z}_U(p)[i] \right),$$

where $\mathbb{Z}_U(p)$ is the "motive associated to U". All these categories have in common that (actually, it is a requirement for a good category DM) there is a functorial isomorphism for smooth U,

$$H^i_{\mathcal{M}}(U, \mathbb{Z}(j)) \cong \mathrm{CH}^j(U, 2j-i).$$

[1]Two algebraic cycles A, B are said to *intersect properly* if their intersection $A \cap B$ is either empty or lives in the right codimension, that is,

$$\mathrm{codim}(A \cap B) = \mathrm{codim} A + \mathrm{codim} B.$$

In this case, the intersection of A and B is a formal linear combination of the irreducible components of $A \cap B$ with suitable multiplicities. We again denote this intersection with $A \cap B$.

The maybe most prominent definition of $DM(k)$ comes from Voevodsky [64] (there are others by Levine, Hanamura,...) as a derived category of some category of motivic sheaves. In [63] Voevodsky verified that for U smooth over a field k his definition of motivic cohomology is the same thing as simplicial higher Chow groups, or which is the same, cubical higher Chow groups.

1.2 Properties of higher Chow chains

Functoriality

Let U be a quasi-projective variety over k that is equi-dimensional of (complex) dimension d. Then the abbreviation $z_k(U, n) := z^{d-k}(U, n)$ is well defined. Any map $f : U \to V$ extends to a map $f \times \mathrm{id} : U \times \square^n \to V \times \square^n$, which will also be denoted by f (for any n).

Proposition 1. *The proper pushforward resp. flat pullback of algebraic cycles [20, 3.3] induce morphisms of higher Chow chains such that*

- $z_p(U, \bullet)$ *is covariant functorial wrt. proper morphisms,*
- $z^q(U, \bullet)$ *is contravariant functorial wrt. flat morphisms.*

Sketch. The proof is given in [5, Prop. 1.3] (for simplicial chains), where it is shown that the usual pullback/pushforward of algebraic cycles respects proper intersection with boundaries. It also preserves degenerate chains and thus induces a morphism between higher Chow chains. Finally, the induced maps on higher Chow chains are compatible with the differential. □

The contravariant functoriality can be extended to morphisms between quasi-projective varieties by means of the formula

$$\begin{aligned} f^*Z :=& (\mathrm{pr}_{U\times\square^n})_*\Big((U \times \square^n \times Z) \cap \Gamma_{f\times \mathrm{id}_{\square^n}}\Big) \\ =& (\mathrm{pr}_{U\times\square^n})_*\Big((U \times Z) \cap \Gamma_f\Big), \end{aligned}$$

where Γ_f and $\Gamma_{f\times \mathrm{id}_{\square^n}}$ are the graphs of f and $f \times \mathrm{id}_{\square^n}$. But then not every cycle can be pulled back – one has to restrict to those cycles whose intersection with Γ_f exists.
For any finite set $\mathcal{S}$ of closed algebraic subsets of V, denote by

$$z_{\mathcal{S}}(V, \bullet) \subset z(V, \bullet)$$

the subcomplex of higher Chow chains that is obtained by imposing the additional condition that all cycles $S \times F$ are intersected properly for all $S \in \mathcal{S}$ and all faces F.

Lemma 2. *Let $f : U \to V$ be a morphism between smooth algebraic varieties. Then there exists a finite set $\mathcal{S}$ of closed algebraic subsets of V such that the pullback of cycles induces a morphism of complexes*

$$z_{\mathcal{S}}(V, \bullet) \to z(U, \bullet).$$

Proof. Let $\mathcal{S}$ be the union of the sets $S_i, i \geq 0$, where S_i is the set of points $v \in V$ such that $\dim f^{-1}(v) \geq i$. At least for $i > \dim V$ the sets S_i are empty and so $\mathcal{S}$ is finite. The proper intersection with all the S_i is enough to ensure proper intersection with the graph of f. See also [5] or [45, Cor. 4.9]. Indeed: Let $Z \in z^p_{\mathcal{S}}(V, \bullet)$ and consider an irreducible component

$Z' \subset (U \times Z) \cap (\Gamma_f \times F)$. Because the S_i, $i = 0, 1, \ldots$ form a filtration of V, there exists an i such that $Z' \subset U \times S_i \times F$ and $Z' \not\subseteq U \times S_{i+1} \times F$. It follows that

$$\dim Z' \leq \dim \big((U \times Z) \cap (f^{-1}(S_i) \times S_i \times F)\big) \leq \dim \big(Z \cap (S_i \times F)\big) + i.$$

Using this and the proper intersection of Z with the $S_i \times F$,

$$\begin{aligned} \operatorname{codim}_{\Gamma_f \times F} Z' &\geq \dim(\Gamma_f \times F) - \dim(Z \cap (S_i \times F)) - i \\ &= \dim(\Gamma_f \times F) - \dim(S_i \times F) + \operatorname{codim}_{S_i \times F}(Z \cap (S_i \times F)) - i \\ &= \dim(U) - \dim S_i + p - i. \end{aligned}$$

The claim follows because $\dim S_i + i \leq \dim U$. ☐

The pullback on chain level is only partially defined. The following "moving lemma" says that it is everywhere defined at least after passing to rational cohomology groups.

Proposition 3 (Moving lemma)**.** *Let U be a smooth quasi-projective variety over k. For any collection of finitely many closed subsets $S_1, \ldots S_l$ of U, every cohomology class in $\mathrm{CH}^p(U, n)_{\mathbb{Q}}$ has a representative that intersects all S_i properly.*
If U is projective or affine, this holds even for integral coefficients.

Proof. This is corollary 3.2 in [45] and theorem 14 in [46]. ☐

The product of higher Chow chains

One advantage of the cubical version of higher Chow groups (and the reason for their invention, see Totaro [62]) is, that the product of two cubes again is a cube. This allows an easy definition of an external product for higher Chow chains.
For two varieties U, V consider the isomorphism $\tau : U \times \square^n \times V \times \square^m \to U \times V \times \square^n \times \square^m = U \times V \times \square^{n+m}$ that exchanges the two middle factors. Together with the usual exterior product of algebraic cycles, this induces a morphism

$$z^p(U \times \square^n) \otimes z^q(V \times \square^m) \to z^{p+q}(U \times V \times \square^{n+m}).$$

The external product of two admissible cycles is again admissible (because faces on the product are just exterior products of faces in the respective factors) and the product of an admissible and a degenerate cycles is always a degenerate cycle. Consequently, one obtains an external product for higher Chow chains,

$$z^p(U, n) \otimes z^q(V, m) \to z^{p+q}(U \times V, m + n).$$

Taking the sum over the cubical indices gives rise to an exterior product on the complexes of higher Chow chains, i.e., the exterior product is compatible with Bloch's differential:

$$z^p(U, \bullet) \otimes z^q(V, \bullet) \to z^{p+q}(U \times V, \bullet).$$

This product is associative but not graded-commutative (in the sense of exterior products). For example, the exterior product of two graph cycles Γ_f, Γ_g (which are assumed to intersect the boundaries properly) is equal to $\Gamma_{f \times g} \neq \Gamma_{g \times f}$.

Now specialize to $U = V$ and consider the diagonal embedding $\Delta : U \to U \times U$. The composition of the external product with the pullback along Δ is not everywhere defined. It however gives rise to a partially defined map

$$(\Delta_U^{n,m})^* \ : \ z^p(U,n) \otimes z^q(U,m) \dashrightarrow z^{p+q}(U, n+m).$$

The intersection of higher Chow chains induces a well-defined product on higher Chow groups by defining the intersection of two higher cycle classes $\underline{Z}, \underline{Z}'$ with representing higher Chow cycles Z, Z' to be the unique class such that

$$\underline{Z} \cap \underline{Z}' := (\Delta_U^{n,m})^* \big(Z \times Z' + \partial B \big) \quad + \text{ boundaries}$$

for a higher Chow chain B for whom the pullback on the right hand side exists. If it exists, the product is well-defined on cohomology classes and is independent of the choice of B: This is a consequence of the identity $(\Delta_U^{n,m})^* \partial = \partial (\Delta_U^{n,m})^*$. Working with rational coefficients, the moving lemma implies that such a B always exists. In particular, the intersection product is everywhere defined on $\oplus_{p,n} \mathrm{CH}^p(U,n)_{\mathbb{Q}}$.
Marc Levine proved the following properties of this product in [45, Theorem 5.2].

Proposition 4. *For U smooth and quasi-projective over k, these products give $\bigoplus_{p,n} \mathrm{CH}^p(U,n)_{\mathbb{Q}}$ the structure of a bi-graded ring, which is graded-commutative with respect to n and commutative with respect to p. The restriction to $\bigoplus_p \mathrm{CH}^p(U,0)_{\mathbb{Q}}$ is the usual product structure on the rational Chow ring of U.*

That is, one has an associative, graded-commutative (with respect to n and m) product on the rational higher Chow groups (but not on the underlying complex, since one needs the moving lemma).

1.3 Alternating chains

The exterior product (and thus also the intersection product) of higher Chow chains is not graded-commutative – but becomes graded-commutative on rational cohomology. The alternating cycles were introduced to remedy this defect. They furthermore have the advantage that a degenerate alternating cycle is zero and thus the higher Chow groups can be defined without using a normalization (see [45, p. 36], [8]). They are defined as a subgroup of admissible chains that behave good with respect to some action on the coordinates.
Denote by G_n the subgroup of automorphisms of $\square^n$ generated by

$$\sigma : (z_1, \dots, z_n) \mapsto (z_{\sigma 1}, \dots, z_{\sigma n}), \quad \sigma \in S_n$$
$$\tau_i : (z_1, \dots, z_n) \mapsto (z_1, \dots, \tfrac{1}{z_i}, \dots, z_n).$$

Then G_n is just the group of automorphisms of the pair $(\mathbb{P}_1, 1)^n$ that map faces onto faces[2]. Acting identically on U, this gives rise to a group action of G_n on $U \times \square^n$, and hence an action on the algebraic cycles in $U \times \square^n$ via pullback. This action respects proper intersection with

[2]To see this, let φ be an automorphism of $\mathbb{P}_1^{\times n}$ that leaves $\mathbb{1}$ and $\{0,\infty\}^n$ invariant. Restrict it to some $\mathbb{P}_1 \subset \mathbb{P}_1^{\times n}$ and compose it with the projection to the k-th component. The resulting map $\mathbb{P}_1 \to \mathbb{P}_1$ is either constant or a Möbius transformation. There must exist a k such that this map is non-trivial and thus gives an automorphism of $(\mathbb{P}_1, \{1\})$. The assumption that it preserves $\{0,\infty\}$ leaves as the only possibilities the identity and $z \mapsto 1/z$.

faces and thus restricts to an action on admissible chains.

An admissible chain is called *alternating*, if G_n acts on it via the Sign representation, where $\mathrm{Sign} : G_n \to \{\pm 1\}$ denotes the unique homomorphism that extends the signum map on the symmetric group and maps $\tau_i \mapsto -1$ for all i. In other words, the set of alternating higher Chow chains is the subspace spanned by Sign-equivariant admissible algebraic cycles,

$$z^p(U,n)^{\mathrm{Alt}} := c^p(U,n)^{\mathrm{Sign}}.$$

The differential ∂ restricts to these subspaces and makes them into a chain complex with respect to the variable n.

The alternating projection The action of G_n comes with an associated projector

$$\begin{aligned} \mathrm{Alt} : c^p(U,n) &\to \mathbb{Q} \otimes c^p(U,n) \\ Z &\mapsto \frac{1}{|G_n|} \sum_{g \in G_n} \mathrm{Sign}(g) \cdot g^* Z, \end{aligned}$$

whose image is the space of G_n-alternating chains with rational coefficients. Bloch [8, 1.1] showed that Alt is compatible[3] with the differential ∂. After extension of scalars to the rationals, this yields an endomorphism of the space of rational admissible chains $c^p(U,\bullet)_{\mathbb{Q}}$, which turns out to be projector (i.e., $\mathrm{Alt} \circ \mathrm{Alt} = \mathrm{Alt}$) onto the subspace of alternating chains $z^p(U,\bullet)_{\mathbb{Q}}^{\mathrm{Alt}}$. The alternation of a degenerate chain is zero and thus Alt descends to a well-defined map on higher Chow chains

$$\mathrm{Alt} : z^p(U,\bullet) \to z^p(U,\bullet)_{\mathbb{Q}}^{\mathrm{Alt}}.$$

With rational coefficients, this becomes a quasi-isomorphism that is quasi-inverse to the inclusion $z^p(U,\bullet)_{\mathbb{Q}}^{\mathrm{Alt}} \subset z^p(U,\bullet)_{\mathbb{Q}}$. The proof uses an extended cubical structure on $z^p(U,\bullet)$, see the references in the proof of proposition 7.

Product for alternating chains The exterior product of admissible chains induces a product of alternating chains by composing it with the alternating projection,

$$z^p(U,n)_{\mathbb{Q}}^{\mathrm{Alt}} \otimes z^q(V,m)_{\mathbb{Q}}^{\mathrm{Alt}} \to z^{p+q}(U \times V, n+m)_{\mathbb{Q}}^{\mathrm{Alt}}.$$

This gives a graded-commutative associative exterior product, compatible with the differential on the complex of alternating higher Chow chains. Pullback along the diagonal yields a partially defined intersection product on the alternating chains with rational coefficients, which is associative and graded-commutative (with respect to the cubical degree n):

$$Z \cap Z' := \mathrm{Alt} \circ \Delta^* \left(Z \times Z' \right).$$

[3] To be honest, Bloch merely showed the compatibility of ∂ with the projector associated to the action of $S_n \subset G_n$. To see that it extends to G_n, it is to show that ∂ commutes with the projector associated to the $(\mathbb{Z}/2)^n$ action generated by the τ_i, i.e., $\frac{1}{2^n} \sum_{\tau \in (\mathbb{Z}/2)^n} (-1)^{|\tau|} \tau^* \partial^* = \frac{1}{2^{n+1}} \sum_{\tau \in (\mathbb{Z}/2)^{n+1}} (-1)^{|\tau|} \partial^* \tau^*$. Because τ_j interchanges 0 and ∞ in the j-th cube, one has $\partial_i \tau_j = \begin{cases} \tau_j \partial_i & j < i, \\ \tau_{j-1} \partial_i & j > i, \end{cases}$ and $\partial_j \tau_j = -\partial_j$. Using this, a computation gives that $\sum_{\substack{i=1\dots n+1 \\ J \subset (\mathbb{Z}/2)^{n+1}}} (-1)^{|J|+i+1} \partial_i^* \tau_J^* = 2 \sum_{\substack{i=1\dots n+1 \\ J \subset (\mathbb{Z}/2)^{n}}} (-1)^{|J|+i+1} \tau_J^* \partial_i^*$, from which the result follows.

With this product, the direct sum

$$\bigoplus_{p,n} z^p(U,n)_{\mathbb{Q}}^{\mathrm{Alt}}$$

becomes a partially defined, associative and graded-commutative dg algebra. By definition, the quasi-isomorphism Alt induces a morphism of partially defined algebras

$$\bigoplus_{p,n} z^p(U,n)_{\mathbb{Q}} \to \bigoplus_{p,n} z^p(U,n)_{\mathbb{Q}}^{\mathrm{Alt}}.$$

Remark. Alternating higher Chow chains can (and in fact this has been done in the preprint [67]) also be defined as higher Chow chains that alternate with respect to the action of the symmetric group (that is, without $\mathbb{Z}/2$-action, but modulo degenerates). In my opinion, this definition as a sub-quotient of a complex is undesirable. Thus here the alternating higher Chow chains are defined as admissible chains that are alternating with respect to the full automorphism group of the cube. To my knowledge, this approach is due to Hanamura.

1.4 Examples

- $\mathrm{CH}^p(U,0)$ is by definition the group of algebraic cycles on U modulo cycles of the form $\partial_0^* Z - \partial_\infty^* Z$, for Z running through admissible cycles in $U \times \square$. This is just the rational equivalence relation of algebraic cycles, and thus $\mathrm{CH}^p(U,0) = \mathrm{CH}^p(U)$ are the ordinary Chow groups. See also [21, Prop. 1.6 and the remark following].

- $\mathrm{CH}^p(U,n), n = 0,1,2$, is generated by graph cycles of functions that meet the faces in the correct codimension (the inverse image has the correct codimension), see 1.5. This means that while a general higher Chow chain corresponds to a multi-valued function, for $n \leq 2$ they can actually be represented by single-valued functions only.

  More precisely, for $n = 1$, any higher Chow cycle can be written as a finite sum $\sum_i (V_i, f_i)$ where $V_i \subset U$ is an irreducible algebraic cycle of codimension $p-1$, f_i a non-zero rational function on V_i and $\sum_i \mathrm{div}(f_i) = 0$ as a cycle of codimension p on U. The last condition just means that the multiplicities at 0 and ∞ cancel out [65].

- $\mathrm{CH}^0(U,n)$. The admissible chains here are generated by $U_i \times \square^n$ where U_i runs through the irreducible components of U. Since they are degenerate for $n > 0$, one gets

$$\mathrm{CH}^0(U,n) = \begin{cases} \mathbb{Z}^{\#\{\text{irred. comp.}\}}, & n = 0 \\ 0, & n > 0. \end{cases}$$

- $\mathrm{CH}^1(U,n)$. Spencer Bloch [5] showed that in codimension 1,

$$\mathrm{CH}^1(U,n) = \begin{cases} \mathrm{Pic}(U), & n = 0 \\ \Gamma(U, \mathcal{O}_U^*), & n = 1 \\ 0, & n \geq 2. \end{cases}$$

- Vanishing: One has the trivial vanishing

$$\mathrm{CH}^p(U,n) = 0 \qquad \text{for } p > n + \dim U.$$

As an extension of the vanishing in codimension $p = 0$ and $p = 1$, the conjectural Beilinson-Soulé vanishing says that

$$\mathrm{CH}^p(U,n) = 0 \quad \text{if either } (\tfrac{n}{p} \geq 2, p > 0) \text{ or } (n > 0, p = 0).$$

This is known to be true for rational coefficients and U the spectrum of a number field by Borel's computation of K-groups [11] and the comparison between rational higher Chow groups and rational K-theory [5].

- $U = \operatorname{Spec} k = \mathrm{pt}$. $\mathrm{CH}^p(\mathrm{pt}, n)$ is generated by codimension p algebraic cycles in $\square^n$ meeting all faces in the right dimension. By the work of Nesterenko/Suslin [54] and Totaro [62], $\mathrm{CH}^n(\mathrm{pt}, n) = K_n^M(k)$ are the Milnor K-groups. Together with the trivial vanishing, $\mathrm{CH}^p(\mathrm{pt}, n) = 0$ for $p > n$, the higher Chow groups are

$$\begin{array}{c|ccccccc}
p & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \\
3 & 0 & 0 & 0 & K_3^M & ? & ? & \cdots \\
2 & 0 & 0 & K_2^M & ? & ? & ? & \cdots \\
1 & 0 & k^* & 0 & 0 & 0 & 0 & \cdots \\
0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \quad n
\end{array}$$

For k a number field, Borel's computation [11] explicitly determines the rank of the higher Chow groups in terms of the number of real and complex embeddings of k. Thus the higher Chow groups are determined up to torsion.

1.5 Higher Chow chains as graphs

For small n (in fact $n = 0, 1, 2$) the higher Chow groups admit an alternative description via graph cycles.

Indeed, we sketch how the Gersten resolution for higher Chow groups [5, §10], the degeneration of the local to global spectral sequence for higher Chow groups [52, §5], and the Milnor-Chow homomorphism [62] imply that every class in $\mathrm{CH}^p(U,n)$, $n = 0, 1, 2$, can be represented by a sum of graph cycles.

Gersten resolution for higher Chow groups Let U be a smooth quasi-projective variety over k and denote by $\mathcal{CH}^p_U(n)$ the Zariski-sheafification of the functor $V \mapsto \mathrm{CH}^p(V,n)$. There is a local-to-global spectral sequence relating the cohomology of $\mathcal{CH}$ with the higher Chow groups. One can show that this spectral sequence partially degenerates and that this leads to

$$\mathrm{CH}^p(U,n) \cong H^{p-n}(U, \mathcal{CH}^p_U(p)), \qquad n = 0, 1, 2.$$

Furthermore, one has the Gersten resolution for higher Chow groups, which is a flasque resolution of sheaves (i_x denoting the skyscraper functor)

$$0 \to \mathcal{CH}^p_U(n) \to \bigoplus_{x\in U^{(0)}} i_x\,\mathrm{CH}^p(\operatorname{Spec} k(x), n) \to \bigoplus_{x\in U^{(1)}} i_x\,\mathrm{CH}^{p-1}(\operatorname{Spec} k(x), n-1) \to \dots$$
$$\to \bigoplus_{x\in U^{(p-1)}} i_x\,\mathrm{CH}^1(\operatorname{Spec} k(x), n-p+1) \to \bigoplus_{x\in U^{(p)}} i_x\,\mathrm{CH}^0(\operatorname{Spec} k(x), n-p) \to 0.$$

Consequently, an element in Bloch's higher Chow group $\mathrm{CH}^p(U,n)$, for $n = 0,1,2$, can be represented by an element in

$$\bigoplus_{x\in U^{(p-n)}} i_x\,\mathrm{CH}^n(\operatorname{Spec} k(x), n).$$

Milnor K-Theory For k a field, the Milnor k-group $K^M(k)$ is the quotient of the tensor algebra over k^*,

$$\mathbb{Z} \oplus k^* \oplus (k^* \otimes k^*) \oplus (k^* \otimes k^* \otimes k^*) \oplus \dots$$

by the so-called Steinberg relations, that is, the two-sided ideal generated by $a \otimes (1-a)$, where $a \neq 0, 1$. Note that the tensor algebra is $\mathbb{N}_0$-graded with $\mathbb{Z}$ in degree 0 and $(k^*)^{\otimes i}$ in degree i. The Steinberg relations are compatible with the grading and one obtains a $\mathbb{N}_0$-graded ring $K^M_\bullet(k)$. Totaro [62] showed that there is a ring isomorphism

$$K^M_\bullet(k) \cong \mathrm{CH}^\bullet(k, \bullet).$$

To describe the homomorphism $K^M_n(k) \to \mathrm{CH}^n(k,n)$, observe that an element in $\mathrm{CH}^n(k,n)$ is just a finite linear combination of points in $(\mathbb{P}_1(k) \setminus \{0,1,\infty\})^n$. The map sends a tuple $a = (a_1, \dots a_n)$ to the intersection $a \cap \square^n$. For example if $n = 1$, then the map sends $a \in k^* \setminus \{1\}$ to a and 1 to 0 (the empty sum).

Putting all these things together, an element in Bloch's higher Chow group $\mathrm{CH}^p(U,n)$, for $n = 0,1,2$, can be represented by elements in

$$\bigoplus_{x\in U^{(p-n)}} K^M_n(k(x))$$

that is, by a formal sum of symbols $(V, f_1 \otimes \dots \otimes f_n)$ with V an algebraic subvariety of codimension $p-n$ and $f_i \in k^*(V)$ a non-zero rational function.

1.6 Higher Chow chains with good real intersection

In the definition of the regulator map in 4.2, 4.3, we will have to restrict to higher Chow chains that are in good position with respect to the branch locus of the complex logarithm, which we choose to be $\mathbb{R}_- := [-\infty, 0] \subset \square$. We will see that every class in $\mathrm{CH}^p(U,n)_\mathbb{Q}$ is represented by a higher Chow cycle that intersects $\mathbb{R}^I_- \subset \square^n$ properly for all $I \subset \{0,1\}^n$.

A *real face* of $\square$ is one of the subsets $\{0\}, \{\infty\}, \mathbb{R}_-, \square$ and a real face of $\square^n$ is an n-fold exterior product of real faces of $\square$. For U a smooth quasi-projective variety over $\mathbb{C}$, define

$$c^p_\mathbb{R}(U,n) := \left\langle \begin{array}{l}\text{irreducible } Z \in z^p(U \times \square^n) \text{ such that } Z \text{ and } \partial Z \\ \text{intersect all real faces } U \times F \text{ properly}\end{array} \right\rangle.$$

Here proper intersection with a real face means that Z intersects all real strata of the face properly as real analytic varieties. For example, proper intersection with $\mathbb{R}_-$ means proper intersection with $(-\infty, 0)$, ∞ and 0.
Denote by $d^p_{\mathbb{R}}(U,n) \subset c^p_{\mathbb{R}}(U,n)$ the subgroup generated by those algebraic cycles with one factor being $= \square$ and define

$$z^p_{\mathbb{R}}(U,\bullet) := \frac{c^p_{\mathbb{R}}(U,\bullet)}{d^p_{\mathbb{R}}(U,\bullet)}$$

with boundary map ∂ as before.
In the same way, for $\mathcal{S} = \{S_1, \ldots, S_l\}$ any finite collection of closed subvarieties of U, define

$$c^p_{\mathcal{S},\mathbb{R}}(U,n) = \left\langle \begin{matrix} Z \in z^p(U \times \square^n) \text{ such that } Z \text{ and } \partial Z \text{ intersect } S_i \times F \text{ properly} \\ \text{as real analytic varieties, for all real faces } F \text{ and all } i \end{matrix} \right\rangle$$

and the quotient complex $z^p_{\mathcal{S},\mathbb{R}}(U,\bullet)$.

Theorem 5 (Moving Lemmas)**.**

- *The inclusions give rise to quasi-isomorphisms*

$$z^p_{\mathcal{S},\mathbb{R}}(U,\bullet)_{\mathbb{Q}} \hookrightarrow z^p_{\mathbb{R}}(U,\bullet)_{\mathbb{Q}} \hookrightarrow z^p(U,\bullet)_{\mathbb{Q}}.$$

- *For $D \subset X$ any closed subvariety of pure codimension q in a smooth projective variety X, the restriction to $X \setminus D$ induces a quasi-isomorphism*

$$\frac{z^p_{\mathbb{R}}(X,\bullet)_{\mathbb{Q}}}{z^{p-q}_{\mathbb{R}}(D,\bullet)_{\mathbb{Q}}} \longrightarrow z^p_{\mathbb{R}}(X \setminus D,\bullet)_{\mathbb{Q}}.$$

Proof. Originally, the moving lemma for higher Chow groups without subscript $\mathbb{R}$ has been proven by Bloch and Levine. For X projective, this is the collection of moving lemmas from [38, 8.14-8.16]. The proof of the first statement also works for quasi-projective X. □

The "alternating" version is

$$z^p_{\mathbb{R}}(U,n)^{\mathrm{Alt}} := \left\langle Z \in z^p(U,n)^{\mathrm{Alt}} \;\middle|\; \begin{matrix} Z \text{ and } \partial Z \text{ intersect all real faces} \\ \text{properly as real analytic varieties} \end{matrix} \right\rangle.$$

Proposition 6. *The diagram below formed by the alternating projection (horizontal) and inclusion (vertical) is a commutative diagram of partially defined dg algebras and each arrow is a quasi-isomorphism.*

$$\begin{tikzcd} \bigoplus_{p,n} z^p(U,n)_{\mathbb{Q}} \arrow[r] & \bigoplus_{p,n} z^p(U,n)^{\mathrm{Alt}}_{\mathbb{Q}} \\ \bigoplus_{p,n} z^p_{\mathbb{R}}(U,n)_{\mathbb{Q}} \arrow[r] \arrow[u] & \bigoplus_{p,n} z^p_{\mathbb{R}}(U,n)^{\mathrm{Alt}}_{\mathbb{Q}} \arrow[u] \end{tikzcd}$$

Proof. Commutativity of the diagram is obvious. That the maps in the diagram are compatible with the products is clear for the inclusions. For the alternating projections, this is equivalent to $\mathrm{Alt}(\mathrm{Alt}(Z) \cap \mathrm{Alt}(Z')) = \mathrm{Alt}(Z \cap Z')$, which can be verified by hand.
That the horizontal arrows are quasi-isomorphisms follows from general theory of (extended) cubical objects as in [47, Proposition 1.6], see also the appendix to this section. For the upper arrow this statement is also in [46, Lemma 29]. That the left vertical arrow is a quasi-isomorphism is content of the moving lemma 5. The quasi-isomorphicity of the right vertical

arrow finally follows from the commutativity of the diagram and the quasi-isomorphicity of the other three arrows. □

1.7 Appendix: Around cubical objects

In this appendix, the formalism of extended cubical objects [47] is applied to prove that the vertical maps in proposition 6 are quasi-isomorphisms.
Define the category of "abstract cubes" $\mathcal{C}ube$ to be the category with objects $\underline{n} := \{0,1\}^n$, $n = 0, 1, \ldots$ and morphisms given by

- $\mathrm{Aut}(\{0,1\}^n) = G_n$, generated by the symmetric group in n variables and the maps $\{0,1\} \to \{0,1\}, i \mapsto 1-i$.

- The other morphisms are generated by the inclusion of faces (the face maps) and linear projections (the degenerations).

Denote by $\mathcal{C}ube^{\mathrm{op}}$ the category with the same objects as $\mathcal{C}ube$, but all arrows reversed. A *cubical object* in a category $\mathcal{A}$ is a functor $\mathcal{C}ube^{\mathrm{op}} \to \mathcal{A}$. In the following, $\mathcal{A}$ will always stand for an abelian category.
$c^p(U,*)$ and $c^p_{\mathbb{R}}(U,*)$ are examples of cubical abelian groups. In the previous subsections, there have been two fundamental constructions of complexes of abelian groups out of a cubical object. The first one (normalized chain complex) divides the chains by degenerate elements, while the second one passes to the subgroup of alternating elements.
These two procedures can be defined and compared in an abstract setting.

Construction 1 If c is a cubical object and if c_{deg} denotes its degenerate elements, then the associated (normalized) complex is by definition the complex which in degree n is $\frac{c(\underline{n})}{c_{\mathrm{deg}}(\underline{n})}$ and whose differential is given by the alternating sum over the face maps.

Construction 2 By functoriality, any cubical object c comes with an action of G_n. Denote by Sign the sign representation of G_n and by $c(\underline{n})^{\mathrm{Sign}}$ the largest subobject of $c(\underline{n})$ on which G_n acts via the sign representation. This object always exists in an abelian category, and applying the above degree-wise, gives rise to a complex c^{Alt} whose differential is the same alternating sum as above.

In order to compare the two fundamental constructions above, define the category of extended cubes $\mathcal{E}\mathcal{C}ube$ to be the category $\mathcal{C}ube$ with additional morphisms generated by the map $\mu : \{0,1\}^2 \to \{0,1\}, (i,j) \mapsto i \cdot j$. A cubical object in $\mathcal{A}$ is called an *extended cubical object*, if it extends to a functor $\mathcal{E}\mathcal{C}ube^{\mathrm{op}} \to \mathcal{A}$. The mere existence of an extended cubical structure suffices to ensure the (quasi-)equivalence of the two constructions above:

Proposition 7. *For any extended cubical object $c : \mathcal{E}\mathcal{C}ube \to \mathcal{A}$ with values in a $\mathbb{Q}$-linear abelian category $\mathcal{A}$, the composition of the inclusion and the canonical map to the quotient*

$$c^{\mathrm{Alt}} \to c \to c/c_{\mathrm{deg}}$$

induces a quasi-isomorphism of complexes. A quasi-inverse is given by the alternating projection $\mathrm{Alt} = \frac{1}{|G_n|} \sum_{g \in G_n} (-1)^{\mathrm{Sign}(g)} g^*$.

Proof. The first statement is a combination of [47, Lemma 1.5] and [47, Prop 1.7]. For the second, note that Alt descends to a map $c/c_{\text{deg}} \to c^{\text{Alt}}$ that is split by the map in the statement. Thus Alt is necessarily a quasi-inverse. $\square$

Lemma 8. *The functors $\underline{n} \mapsto c^p(U, n)$ and $\underline{n} \mapsto c^p_{\mathbb{R}}(U, n)$ are extended cubical objects for any smooth quasi-projective U.*

Proof. We only prove the statement for $c^p_{\mathbb{R}}(U, *)$; the other follows analogously. We will show that it has an extended cubical structure induced by $\mu : \square^2 \to \square, (z, z') \mapsto z + z' - zz'$, which is agreed to be $\mu = \infty$ if either $z = \infty$ or $z' = \infty$. For this, it is to show that the proper intersection with real boundaries is preserved by the extended structure. It even suffices to check this for μ alone. Utilizing the isomorphism $(\square, \infty, 0) \cong (\mathbb{A}_1, 0, 1)$ given by $z \mapsto 1/(1-z)$, allows to work with cycles $Z \subset U \times \mathbb{A}_1^{\times 2}$. In these coordinates, the map μ is just the multiplication of the two components of $\mathbb{A}_1^{\times 2}$. It is to show that whenever Z intersects all real faces properly, then the same holds for

$$\mu^{-1}(Z) = \{(u, x, y) : (u, x \cdot y) \in Z\}.$$

This is done by showing that the various intersections of $\mu^{-1}(Z)$ with each of the real subvarieties in $U \times \{0, 1, (0, 1), \mathbb{A}_1\}^{\times 2}$ is either empty or has the correct real codimension.

- First consider the intersection with faces that contain the divisor $\{0\}$. For example, $\mu^{-1}(Z) \cap (x = 0) = \{(u, 0, y) \mid (u, 0) \in Z\} = (Z \cap (x = 0)) \times \mathbb{A}_1$. Because Z intersects $(x = 0)$ properly, it follows that the real codimension is $\operatorname{codim}_{\mathbb{R}}^{U \times \square^2} \mu^{-1}(Z) \cap (x = 0) = \operatorname{codim}_{\mathbb{R}}^{U \times \square}(Z \cap (x = 0)) \geq \operatorname{codim}_{\mathbb{R}}^{U \times \square}(Z) + 2$, as expected. Similarly, the intersection with $(y = 0)$ and all combinations of these two with other faces (e.g $0 \times (0, 1)$) is treated.

- Next consider faces containing the divisor 1. For example, $\mu^{-1}(Z) \cap (x = 1) = \{(u, 1, y) \mid (u, y) \in Z\} \cong Z$. In particular, its codimension in $X \times \mathbb{A}_1^2$ is equal to the codimension of Z plus one. Because Z intersects the real boundaries properly, an analogous argument shows that $\mu^{-1}(Z)$ intersects $1 \times F$ and $F \times 1$ properly for all real faces F.

- Now consider the face $U \times \mathbb{A}_1 \times (0, 1)$. The intersection with this face is

$$\begin{aligned} \mu^{-1}(Z) \cap (U \times \mathbb{A}_1 \times (0, 1)) &= \{(u, x, y) \mid (u, xy) \in Z, y \in (0, 1)\} \\ &\approx Z \times (0, 1), \end{aligned}$$

  where $\approx$ denotes the analytic isomorphism $(u, x, y) \mapsto (u, xy, y)$. Since the latter has real codimension at least $p + 1$, so does the intersection above. In exactly the same way, proceed with $U \times (0, 1) \times \mathbb{A}_1$.

- A similar reasoning can be done for the face $U \times (0, 1) \times (0, 1)$:

$$\begin{aligned} \mu^{-1}(Z) \cap (U \times (0, 1) \times (0, 1)) &= \{(u, x, y) \mid (u, xy) \in Z, x, y \in (0, 1)\} \\ &= \{(u, z, y) \mid (u, z) \in Z, y \in (0, 1), z/y \in (0, 1)\} \\ &\subset \{(u, z, y) \mid (u, z) \in Z, z, y \in (0, 1)\} \\ &= (Z \cap (U \times (0, 1))) \times (0, 1). \end{aligned}$$

  From the proper intersection of Z with the real faces, it follows that the right-hand side has real codimension at least $p + 2$, hence so does the left-hand side. $\square$

2 Currents

This section shall introduce to the basic theory of currents on manifolds. Currents are important for us, because they unify both the concepts of singular chains and differential forms. They have good covariant functorial properties, an associative external product (in contrast to the exterior product of singular chains) and contain appropriate subspaces which compute the integral and real (complex) cohomology of a manifold. Furthermore, there exists a partially defined intersection product that generalizes both the wedge product of differential forms and the intersection product of singular chains and that induces a well defined, everywhere defined product on cohomology.
Currents can be seen as either differential forms with distribution coefficients or as duals of differential forms. We adopt the second point of view, which emphasizes the homological character of currents, and we may think of them as "generalized manifolds with densities".
While currents are considered in huge generality in geometric measure theory, we restrict ourselves to currents on (real or complex) analytic manifolds. Excellent introductions into this area are the texts by Gillet/Soulé [23, §1], Demailly [12], Harvey [30], Jannsen [37] and, for the general notions of geometric measure theory, the fundamental works of Herbert Federer [17], [18] and the more recent reference work of Giaquinta/Modica/Soucek [22].
We start investigating currents on manifolds and afterwards turn to currents on pairs of spaces. Intersection theory for currents is the one introduced by Federer (Federer/Fleming [18]) and axiomatized by Hardt [28].

2.1 Currents on (complex) manifolds

Let M be a real analytic manifold and denote by $\mathcal{A}^k(M)$ the vector space of complex valued smooth differential k-forms on M. The subspace of compactly supported forms $\mathcal{A}^k_c(M)$ comes with the direct limit topology induced by the topology of uniform convergence (of all partial derivatives) on $\mathcal{A}^k(K)$ for all compact $K \subset M$. This is the uniquely defined topology in which a sequence $\omega_n \to \omega$ converges if and only if there is a compact subset K such that $\operatorname{spt}\omega_n, \operatorname{spt}\omega \subset K$ for all n and the sequence of partial derivatives $D^\alpha\omega_n(x) \to D^\alpha\omega(x)$ converges uniformly in x for each multi index α. The *currents of dimension* k on M by definition are the distributions on $\mathcal{A}^k_c(M)$, that is,

$$\mathcal{D}_k(M) = \mathbb{C}\text{-linear continuous functionals on } \mathcal{A}^k_c(M).$$

In other words, a current is a linear form on $\mathcal{A}_c(M)$ whose restriction to any $\mathcal{A}(K)$, $K \subset M$ compact, is a continuous functional. The space of currents on M,

$$\mathcal{D}(M) = \oplus_k \mathcal{D}_k(M),$$

is a graded $\mathbb{C}$-vector space concentrated in dimensions $[0, \dim M]$.
Let X be a complex analytic manifold of (complex) dimension m. Then there is an additional

decomposition of differential forms $\mathcal{A}_c^k(X) = \oplus_{p+q=k}\mathcal{A}_c^{p,q}(X)$, where $\mathcal{A}_c^{p,q}(X)$ is the vector space of complex valued smooth differential (p,q)-forms on X with compact support. The distributions on these subspaces with the inherited topology are the *currents of bidimension* (p,q) on X,

$$\mathcal{D}_{p,q}(X) = \text{ distributions on } \mathcal{A}_c^{p,q}(X).$$

Under the inclusion $\mathcal{D}_{p,q}(X) \hookrightarrow \mathcal{D}_{p+q}(X)$ (extension by zero), the currents of bidimension (p,q) are identified with those currents that pair to zero with all test form of bidegree $\neq (p,q)$. One obtains a decomposition

$$\mathcal{D}_k(X) = \bigoplus_{p+q=k} \mathcal{D}_{p,q}(X).$$

It is common to raise the indices by writing $\mathcal{D}^r = \mathcal{D}_{2d-r}$ and similarly $\mathcal{D}^{p,q} := \mathcal{D}_{m-p,m-q}$. The so obtained grading/bigrading is called the degree/bidegree. The degree of a current T will be denoted by $|T| = \deg(T)$ and the dimension by $\dim(T)$.

Examples of currents

The main examples of currents, considered more detailed in 2.7, are the following.

- If ω is a locally integrable differential r-form on an oriented manifold M (that is, a form whose integral over each compact set exists), then the integration gives a current $[\omega] \in \mathcal{D}^r(M)$ by setting

  $$[\omega](\eta) = \int_M \omega \wedge \eta.$$

  If $M = X$ is a complex manifold and ω is of bidegree (p,q), then $[\omega] \in \mathcal{D}^{p,q}(X)$.

- A singular k-chain $\gamma = \sum a_i\gamma_i$ of M is called (piecewise) smooth if each γ_i can be extended to a smooth map from a neighborhood of the k-simplex $\Delta_k \subset \mathbb{R}^{k+1}$ into M. The chain is oriented, if a orientation of Δ_k has been fixed. Differential forms on M can be integrated over piecewise smooth oriented chains by pullback

  $$\int_\gamma \eta = \sum_i a_i \int_{\Delta_k} \gamma_i^*\eta.$$

  This leads to a map from the group $C_k^\infty(M)$ of smooth oriented k-chains on M into k-dimensional currents on M which assigns to a smooth singular chain γ the integration current $[\gamma]$, defined by

  $$[\gamma](\eta) = \int_\gamma \eta.$$

  This even makes sense for locally finite smooth oriented chains (smooth Borel-Moore chains) and, in particular, for any smooth oriented real (sub)manifold (and in this case it recovers the integration current over this submanifold, see 2.7).

- [1] For a k-dimensional complex subvariety Z of a complex manifold X, the set Z_{reg} of manifold points is a smooth oriented real $2k$-dimensional manifold and thus induces a current of bidimension (k,k) by

  $$[Z](\eta) = \int_{Z_{\mathrm{reg}}} \eta.$$

[1] See [31, Prop. 1.6] or [40, Thm 3.1.1]

On the space of currents there are two differentials of degree $+1$,

$$\partial T(\eta) := T(d\eta) \quad \text{and} \quad dT(\eta) := (-1)^{deg(T)+1} T(d\eta)$$

that make $\mathcal{D}^\bullet$ into a complex, i.e. $\partial^2 = d^2 = 0$, concentrated in degrees $[0, 2m]$. With the differential d, the inclusion of smooth forms into currents $\mathcal{A}^\bullet(M) \to \mathcal{D}^\bullet(M)$ becomes a morphism of complexes. The differential ∂ on the other hand makes the map $C^\infty_\bullet(M) \to \mathcal{D}_\bullet(M)$ a morphism of complexes. Unless otherwise stated, we endow $\mathcal{D}^\bullet$ with the differential d.
Note that the formula $d[\omega] = [d\omega]$ usually fails for non-smooth forms ω, since in this case one additionally has to restrict the domain of integration. The log and dlog function for example are locally integrable on $\mathbb{P}_1(\mathbb{C})$ and give rise to currents

$$[\log z](\eta) = \int_{\mathbb{P}_1(\mathbb{C}) \setminus \mathbb{R}_-} \log z \cdot \eta$$
$$[\operatorname{dlog} z](\eta) = \int_{\mathbb{P}_1(\mathbb{C}) \setminus \{0,\infty\}} \operatorname{dlog} z \wedge \eta$$

where one naturally has to remove the poles and, in order to make log single valued, the negative real line $\mathbb{R}_- = [-\infty, 0]$. Give $\mathbb{R}_-$ the orientation such that $\partial\mathbb{R}_- = [0] - [\infty]$. Then an application of Stokes theorem [39, Lemma 4.1], shows that one has equations of currents,

$$\begin{aligned} d[\mathbb{R}_-] &= (z) = [0] - [\infty], \\ d[\log z] &= [\operatorname{dlog} z] - 2\pi i[\mathbb{R}_-], \\ d[\operatorname{dlog}] &= 2\pi i[(z)], \end{aligned}$$

where $(z) = \{0\} - \{\infty\}$ is the divisor of the coordinate function. More general for $f : \mathbb{P}_1 \to \mathbb{P}_1$ meromorphic, one has that $d[\log f] = [\operatorname{dlog} f] - 2\pi i[f^{-1}[-\infty, 0]]$ and $d[d\log f]$ is the divisor of f, i.e., the sum of zeroes minus the sum of poles of f (counted with multiplicities).

$\mathcal{A}$-module structure The *wedge product* of a current T with a smooth form ω is the current $T \wedge \omega$ given by

$$(T \wedge \omega)\eta = T(\omega \wedge \eta).$$

This is a current of degree $\deg T + \deg \omega$. The so defined map $(T, \omega) \mapsto T \wedge \omega$ satisfies the Leibniz rule with respect to d and thus gives a map of complexes,

$$\mathcal{D}(M)^r \otimes \mathcal{A}(M)^s \to \mathcal{D}(M)^{r+s}$$

i.e., a right $\mathcal{A}(M)$-module structure. The associated left $\mathcal{A}(M)$-module structure is obtained by forcing graded commutativity, that is, by the formula $\omega \wedge T := (-1)^{\deg \omega \cdot \deg T} T \wedge \omega$. The sign conventions again are adopted to the case of currents represented by differential forms, so that $[\omega_1] \wedge \omega_2 = [\omega_1 \wedge \omega_2]$. All this is compatible with the bidegree if $M = X$ is a complex manifold.

Exterior products of currents Given two currents $S \in \mathcal{D}^r(M), T \in \mathcal{D}^s(N)$ of degree r resp. s, there exist two slightly different notions of an outer product corresponding to the two different ways of looking at currents as homological or cohomological objects. Both products yield currents of degree $r + s$ on $M \times N$ (differing only by a sign). Such a current is uniquely determined by its values on test forms which are exterior products $\eta_1 \boxtimes \eta_2 = \operatorname{pr}_M^* \eta_1 \wedge \operatorname{pr}_N^* \eta_2$

of forms on M and N respectively. Indeed, the set of all forms $\eta_1 \boxtimes \eta_2$ is a dense sub vector space in $\mathcal{A}(M \times N)$, see Federer [18, 4.1.8].
The (homological) *cartesian product* is the current defined by

$$(S \times T)(\eta_1 \boxtimes \eta_2) = S(\eta_1) \cdot T(\eta_2)$$

and the (cohomological) *exterior product* is[2]

$$(S \boxtimes T)(\eta_1 \boxtimes \eta_2) = (-1)^{\deg(T)\deg(\eta_1)} S(\eta_1) \cdot T(\eta_2).$$

Proposition 9. *Both products are associative and bilinear. The exterior product satisfies the Leibniz rule with respect to the differential d,*

$$d(S \boxtimes T) = dS \boxtimes T + (-1)^{\deg S} S \boxtimes dT$$

and similarly, the cartesian product $S \times T$ satisfies the Leibniz rule with respect to the boundary map ∂.
Furthermore, both products are graded-commutative in the sense of exterior products. That is, if τ denotes the map that exchanges the two factors in $M \times N$, then one has

$$\tau_*(S \boxtimes T) = (-1)^{\deg S \deg T + mn} T \boxtimes S$$
$$\tau_*(S \times T) = (-1)^{\dim S \dim T} T \times S.$$

Especially for M and N complex manifolds, both products are graded commutative with respect to degree in the sense of exterior products. In this case, they are also additive with respect to bidegree.

Proof. This follows straight-forward from the definitions and the properties of the wedge product of forms. In case of the cartesian product this is done by Federer in [18, 4.1.8]. The statements for the tensor product can easily be derived from this. □

As a consequence, the exterior product of two closed currents is again closed and, if either S or T are exact, their exterior product is exact, too.

Examples.

- If S and T are represented by differential forms ω_i, then

$$[\omega_1] \boxtimes [\omega_2] = [\omega_1 \boxtimes \omega_2].$$

  Here $\omega_1 \boxtimes \omega_2 = \mathrm{pr}_1^* \omega \wedge \mathrm{pr}_2^* \omega$ is the exterior product of forms.

- For 1-forms ω_i, one has $[\omega_1] \times \ldots \times [\omega_n] = (-1)^{\binom{n}{2}} [\omega_1 \boxtimes \ldots \boxtimes \omega_n]$ by induction.

- If S and T are given by integration over submanifolds Y_1 and Y_2 respectively, then

$$S \times T = \int_{Y_1 \times Y_2},$$

[2] References are Demailly [12, §2.D.1] (for $\boxtimes$) and Federer [18, 4.1.8] (for $\times$). Demailly denotes the exterior product by the symbol $\otimes$.

that is, $[Y_1] \times [Y_2] = [Y_1 \times Y_2]$.

Thus $\boxtimes$ is an extension of the exterior product of forms to currents, while $\times$ extends the exterior product of manifolds/chains.
The compatibility of $\boxtimes$ and the wedge product is expressed by the

Lemma 10. *For currents T_1, T_2 and smooth forms ω_1, ω_2 on M,*

$$(T_1 \wedge \omega_1) \boxtimes (T_2 \wedge \omega_2) = (-1)^{|\omega_1||T_2|}(T_1 \boxtimes T_2) \wedge (\omega_1 \boxtimes \omega_2).$$

An analogue formula holds for the exterior product:

$$(T_1 \wedge \omega_1) \times (T_2 \wedge \omega_2) = (-1)^{|T_1 \wedge \omega_1||\omega_2|}(T_1 \times T_2) \wedge (\omega_1 \boxtimes \omega_2).$$

Note the various special cases obtained through $[M] \wedge \omega = [\omega]$ and $[Y] \wedge 1 = [Y]$.
For example,

$$[\omega] \times [Y] = [M \times Y] \wedge \mathrm{pr}^* \omega.$$

Proof. We give only the proof of the first formula. The second follows formally from the equality $T_1 \boxtimes T_2 = (-1)^{\deg T_2 \dim T_1} T_1 \times T_2$. Let $\eta = \eta_1 \boxtimes \eta_2$ be a test form. Then

$$\begin{aligned}
(S \boxtimes T) \wedge (\omega_1 \boxtimes \omega_2)(\eta_1 \boxtimes \eta_2) &= (-1)^{|\omega_2||\eta_1|}(S \boxtimes T)(\omega_1 \wedge \eta_1 \boxtimes \omega_2 \wedge \eta_2) \\
&= (-1)^{|\omega_2||\eta_1| + |T||\omega_1 \wedge \eta_1|} S(\omega_1 \wedge \eta_1) \cdot T(\omega_2 \wedge \eta_2) \\
&= (-1)^{|\omega_2||\eta_1| + |T||\omega_1 \wedge \eta_1|} S \wedge \omega_1(\eta_1) \cdot T \wedge \omega_2(\eta_2) \\
&= (-1)^{|\omega_2||\eta_1| + |T||\omega_1 \wedge \eta_1| + |T \wedge \omega_2||\eta_1|}(S \wedge \omega_1) \boxtimes (T \wedge \omega_2)(\eta_1 \boxtimes \eta_2) \\
&= (-1)^{|T||\omega_1|}(S \wedge \omega_1) \boxtimes (T \wedge \omega_2)(\eta_1 \boxtimes \eta_2).
\end{aligned}$$

□

2.2 Functorialities

Support of a current The *support* $\operatorname{spt} T$ *of a current* is the smallest closed subset C such that $T(w) = 0$ for all w with $\operatorname{spt}(w) \subset M \setminus C$. That is, it is the intersection of all closed subsets C such that $T|_{M \setminus C} = 0$.

Pushforward of currents Let $f : M \to N$ be a smooth map between two manifolds. The *pushforward along* f of a current $T \in \mathcal{D}(M)$ is the current given by the formula

$$(f_* T)(\eta) = T(f^* \eta).$$

The pushforward need not exist, because the pullback of a differential form with compact support in general is no longer compactly supported. However, it is well defined if f is proper. It even suffices that the restriction of f to the support of T is proper (i.e., $\operatorname{spt} T \cap f^{-1}(K)$ is compact for all $K \subset N$ compact). Then $f^* \eta$ is compactly supported in $\operatorname{spt} T$ and the formula above defines a current on N with the same dimension as T, i.e., $f_* T \in \mathcal{D}_k(N)$ whenever $T \in \mathcal{D}_k(M)$.
The following basic properties of the pushforward of currents are immediately obtained from the definition:

- $\operatorname{spt} f_* T \subset f(\operatorname{spt} T)$.

- The pushforward is compatible with composition of functions, that is, $(g \circ f)_* T = g_* f_* T$ if all occuring pushforwards exist.

- If f is a smooth map such that $f|_{\operatorname{spt} T}$ is proper, then $(f_* T) \wedge \omega = f_*(T \wedge f^* \omega)$.

- The pushforward is compatible with the boundary map, so that $\partial f_* T = f_* \partial T$. Thus, with the same notations as above, $f_* dT = (-1)^{\dim_{\mathbb{R}}(M)+\dim_{\mathbb{R}}(N)} d(f_* T)$. Especially for M and N even dimensional, the pushforward is compatible with the differential, provided that all pushforwards exist.

- If f is a holomorphic map between complex manifolds, then f_* preserves the bidimension, i.e., $f_* \mathcal{D}_{p,q}(M) \subset \mathcal{D}_{p,q}(N)$.

Moreover, one has

- If γ is a smooth oriented chain, then $f_*[\gamma] = [f \circ \gamma]$.

- If $i : M \subset N$ is a closed embedding of a submanifold (i.e., the image of the embedding is a closed submanifold) then the map $i_* : \mathcal{D}(M) \to \mathcal{D}(N)$ is injective. Indeed: A closed embedding is proper, hence the pushforward is defined. The injectivity of the push forward follows because the induced map on tangent spaces is injective.

Pushforward of forms The pushforward of currents can be used to define a pushforward for differential forms with compact support. Assume that $T = [\omega]$ is represented by a compactly supported differential form of degree q and that $f : M \to N$ is a smooth submersion (i.e., surjective and df is surjective at each point) between oriented manifolds. Then $f|_{\operatorname{spt} \omega}$ is always proper and the pushforward $f_*[\omega]$ exists as a current. By the proper submersion theorem, locally over N one has a product situation, the fibers being compact manifolds. Splitting up the integration and using Fubini's theorem, the current $f_*[\omega]$ can be written as

$$f_*[\omega](\eta) = \int_N \Big(\int_{f^{-1}(y)} \omega\Big) \wedge \eta(y), \qquad \eta \in \mathcal{A}_c^{\dim M - q}(N).$$

This expression has to be interpreted as zero whenever the degree of the vertical part of ω does not match the fiber dimension. As a consequence, $f_*[\omega]$ can be thought of as being represented by the differential form of degree $q - (\dim_{\mathbb{R}} M - \dim_{\mathbb{R}} N)$ that is given by integrating ω along fibers of f. This form is also denoted by $f_* \omega$:

$$f_* \omega(y) = \int_{f^{-1}(y)} \omega.$$

A very clear account on the so defined form can be found in the article of Stoll [60] and, more detailed, in the appendix of [59] of the same author. We merely state some of its properties.

- If ω is of class $\mathcal{C}^k$, then so is $f_* \omega$.

- If ω has L^1_{loc} coefficients, then so does $f_* \omega$.

- $\omega \mapsto f_* \omega$ is continuous in the $\mathcal{C}^s$ topology [12].

Pullback of currents For a submersion $f : M \to N$ and a current $T \in \mathcal{D}^p(N)$ the *pullback of T by f* is defined via

$$(f^*T)(\eta) = T(f_*\eta).$$

Here $f_*\eta$ is the pushforward of a differential form as above. $f_*\eta$ is a smooth form with compact support because the same holds for η and so the expression is well defined. One has the following properties.

1. $d(f^*T) = f^*(dT)$
2. If Z is a submanifold, then $f^*[Z] = [f^{-1}(Z)]$.
3. $f^*(T \wedge \omega) = f^*T \wedge f^*\omega$
4. If ω is a r-form with locally integrable coefficients, then $f^*[\omega] = [f^*\omega]$.
5. If f is holomorphic, then the pushforward along f preserves the bidimension, and so f^*T has the same bidegree as T.

Proof. The first three statements can be found in Demailly, [12, 2.14 ff]. The fourth follows from the third by choosing $T = [N]$. Note that the fourth property also follows directly from Fubini's theorem for manifolds (as stated in [22, 3.2 Prop. 3]):

$$f^*[\omega](\eta) = \int_N \omega \wedge f_*\eta = \int_M f^*\omega \wedge \eta = [f^*\omega](\eta). \qquad \square$$

Push 'n' Pull For later use, we state a result that follows from [60] and relates pushforward and pullback of forms: If there is a commutative diagram of smooth maps

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & M' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ N & \xrightarrow{\psi} & N' \end{array}$$

such that f and f' are submersions and φ is fiberwise an orientation preserving diffeomorphism (i.e., $\varphi|_{f^{-1}(n)}$ is a differomorphism onto $g^{-1}(\psi(n))$, then for each form η such that $f'_*\eta$ exists, $f_*\varphi^*\eta$ also exists and $f_*\varphi^*\eta = \psi^* f'_*\eta$. Dually, this shows that

$$\varphi_* f^* = f'^*\psi_* : \mathcal{D}^\bullet(N) \to \mathcal{D}^{\bullet-\delta}(M),$$

where δ is the real fiber dimension of f.

Exterior products of currents and maps

Lemma 11. *Let $f : M \to N$ and $g : M' \to N'$ be two smooth maps between manifolds. If f, g are proper then one has the equality of currents on $N \times N'$,*

$$(f \times g)_*(S \times T) = f_*S \times g_*T.$$

If f, g are submersions then

$$(f \times g)^*(S \times T) = f^*S \times g^*T$$

as currents on $M \times M'$.

Proof. Note that the product of two submersions/proper maps is again a submersion/proper (the latter is by no means obvious, see [61, Satz 19.8]). Let $\eta = \eta_1 \boxtimes \eta_2$ be an appropriate test form.

$$\begin{aligned}(f \times g)_*(S \times T)(\eta) &= (S \times T)(f^*\eta_1 \boxtimes g^*\eta_2) \\ &= S(f^*\eta_1)T(g^*\eta_2) \\ &= f_*S(\eta_1)g_*T(\eta_2) \\ &= (f_*S \times g_*T)(\eta_1 \boxtimes \eta_2).\end{aligned}$$

For the second statement let $\eta = \eta_1 \boxtimes \eta_2$ be a test form on $M \times M'$.

$$\begin{aligned}(f \times g)^*(S \times T)(\eta) &= (S \times T)((f \times g)_*\eta) \\ &= (S \times T)(f_*\eta_1 \boxtimes g_*\eta_2) \\ &= S(f_*\eta_1)T(g_*\eta_2) \\ &= (f^*S \times g^*T)(\eta_1 \boxtimes \eta_2).\end{aligned}$$ ☐

Similar formulas hold for the exterior product of currents: If f, g are smooth proper, then

$$(f \times g)_*(S \boxtimes T) = (-1)^{\dim S(\dim_{\mathbb{R}} N' - \dim_{\mathbb{R}} M')} f_*S \boxtimes g_*T,$$

and, if f, g are smooth submersions,

$$(f \times g)^*(S \boxtimes T) = (-1)^{(\dim_{\mathbb{R}} N - \dim_{\mathbb{R}} M)\deg T} f^*S \boxtimes g^*T.$$

Again the signs vanish in the complex setting.

2.3 Cohomology of currents

Fix a real analytic manifold M. Every open $U \subset M$ has an induced manifold structure and the inclusion gives rise to restriction maps ("pullback")

$$\mathrm{res}_{U,M} : \mathcal{D}(M) \to \mathcal{D}(U), \qquad \mathrm{res}_{U,M}(T)(\eta) = T(i_*\eta)$$

where $i_*\eta$ is the compactly supported form on M obtained as the extension by zero. The collection of all $\mathcal{D}(U)$ together with these restriction maps form a presheaf of $\mathcal{A}_M$-modules. This is actually a sheaf, denoted by $\mathcal{D}_M$, so that the sections in $\Gamma(U, \mathcal{D}_M) = \mathcal{D}_M(U)$ are again the continuous linear functional on $\Gamma_c(U, \mathcal{A}_M)$.

Remark. Note that $\Gamma_c(U, \mathcal{A}_M)$ consists of all those sections of $\mathcal{A}$ with compact support in U. This should not be mixed up with $\Gamma(U, \mathcal{A}_{M,c})$, which are the sections over M with locally compact support in U.

Theorem 12. *The inclusion of locally constant functions gives rise to a quasi-isomorphism*

$$\mathbb{C}_M \to \mathcal{D}_M^\bullet.$$

In particular ($\mathcal{D}$ is fine), there is an isomorphism for any integer n

$$\mathbb{H}^n(M, \mathbb{C}) \to H^n(\mathcal{D}^\bullet(M), d).$$

Sketch of proof. A proof is given in [27, p 382ff], and proceeds in two steps.

1. Establish an analogon of the Poincaré lemma for currents.

2. Use the $\mathcal{A}_M$-module structure to see that the sheaves $\mathcal{D}^k$ are fine.

The first step shows that the complex of sheaves of currents $\mathcal{D}_M^\bullet$ forms a resolution of the sheaf of locally constant functions $\mathbb{C}$ on M. By the second step, this resolution is acyclic and hence its global sections compute the (hyper-)cohomology of $\mathbb{C}$.

While the second part is obvious, the first part involves some work. One way to prove it is by approximating the currents by smooth forms and using the Poincaré lemma for forms, see Demailly [12]. □

On a complex manifold X, there exists an additional structure on forms and currents coming from the bigraduation on differential forms. It is convenient to state this in terms of a descending filtration on currents, the *(putative) Hodge filtration*

$$F^p\mathcal{D}^r(X) = \oplus_{s \geq p}\mathcal{D}^{s,r-s}(X).$$

Analogously, one defines a filtration for differential forms.

A more refined version of the previous theorem that includes the Hodge filtration is the following

Theorem 13. *The inclusions of forms into currents*

$$\Omega_X^\bullet \to \mathcal{A}_X^\bullet \to \mathcal{D}_X^\bullet$$

are filtered quasi-isomorphisms. If X is projective, then the induced filtration on cohomology

$$F^pH^n(X, \mathbb{C}) = im\big(H^nF^p\mathcal{D}^\bullet(X) \hookrightarrow H^n\mathcal{D}^\bullet(X)\big)$$

agrees with the classical Hodge filtration on cohomology[3]. Similarly for holomorphic resp. smooth differential forms (possibly with cohomology replaced by hypercohomology).

Sketch of proof. The two morphisms are both filtered morphisms and it is to show that the associated graded pieces with respect to the Hodge filtration are all quasi-isomorphisms. The p-th graded piece is $\Omega^p \to \mathcal{A}^{p,\bullet} \to \mathcal{D}^{p,\bullet}$. That these maps are quasi-isomorphisms is the content of the $\overline{\partial}$-lemma for forms and currents (also called Grothendieck-Dolbeault lemma since it has been proven independently by each of them, see [13], [57, pp. 3-4]).

If X is projective (or compact Kähler), then the spectral sequence associated to the filtration F (called the Fröhlicher spectral sequence) degenerates at E_1, and the putative filtration coincides with the filtration on the de Rham cohomology of X, which is the classical Hodge filtration (see also [55, Proposition 2.22]). □

[3] The Hodge filtration on the cohomology of X is the descending filtration on cohomology $F^pH(X, \mathbb{C})$ that is spanned by those classes that can be represented by harmonic differential forms of bidegree (r, s), with $r \geq p$.

2.4 Normal currents

The (real or complex valued) cohomology of a manifold can be computed by more geometrical currents – the locally normal currents. These are the currents that together with their differentials are representable by integration or, have finite mass in every compact subset. See also King [40] and Federer [18, 4.1.7].

Currents representable by integration and normal currents Denote by $\mathcal{CA}_M$ the sheaf of complex valued forms on M whose coefficients are continuous functions. Its space of sections carries a topology by declaring a sequence of forms $\omega_n \to \omega$ to converge, if for each coordinate neighbourhood and each compact subset K of this neighbourhood the coefficient functions $f_{I,n} \to f_I$ converge uniformly on K.
Let $\mathcal{CA}_{M,c}$ be the subsheaf of complex valued continuous forms with locally compact support. Give it the topology where a sequence $\omega_1, \omega_2, \ldots$ of forms is convergent whenever the sequence converges in the topology on $\mathcal{CA}_M$ and there exists a compact subset $K \subset M$ with $\operatorname{spt} \omega_j \subset K$ for all j.
A current is called *representable by integration*, if it can be extended to a continuous linear functional on $\mathcal{CA}_{M,c}$.
A *locally normal current* is a current T such that T and dT are both representable by integration. They form the subcomplex of sheaves $\mathcal{N}_M \subset \mathcal{D}_M$. A locally normal current is a *normal current* if its support is compact.

Examples

- $[\omega]$ for a locally integrable form ω is representable by integration.
- An arbitrary form with L^1_{loc}-coefficients need not to be normal. For example, the current $[\frac{1}{z}]$ on $\mathbb{C}$ is not locally normal since $d[\frac{1}{z}] = \delta_0 + [\frac{1}{z}dz]$ is not representable by integration. Currents of the form $[\omega]$ for ω a log form however are locally normal currents [41, p. 43]. They are normal if in addition $\operatorname{spt} \omega$ is compact.
- $[Z]$ for Z an oriented real subvariety is a locally normal current.
- $[\gamma]$ for γ a oriented singular chain is a locally normal current.

The mass norm Let X be a complex manifold[4] and denote by $< -, - >$ the associated Riemannian metric, that is, the metric that is locally given as the real part of the fundamental 2-form $\sum_j dz_j \wedge d\overline{z_j}$. This metric on the tangent space extends to a metric on the space of multi tangent vectors by defining the norm of a simple r-vector as $|v_1 \wedge \ldots \wedge v_r|^2 = \det(< v_i, v_j >)$. Recall that a multi vector is called *simple*, if it can be written as an exterior product of 1-vectors. Using this norm, one defines the *co-mass of a differential r-form ω* to be

$$\|\omega\| := \sup\{|\omega_x(v)| : x \in X,\ v \text{ is a simple r-vector at } x \text{ and } |v|_x \leq 1\}.$$

If the v_i form an orthonormal basis and ω is expanded in the dual basis $dv_{i_1} \wedge \ldots \wedge dv_{i_r}$, then $\|\omega\|$ is just the maximum of the absolut values of the coefficient functions.

[4] The following can be done more general for any Riemannian manifold.

Dually, the *mass of a current* T of dimension r is defined as

$$M(T) = \sup\{T(\eta) : \eta \in A_c^r(X) \text{ and } \|\eta\| \le 1\}.$$

If T is representable by integration, then the multiplication of T with the characteristic function of a Borel set B makes sense (e.g. as a limit $\lim_{U \supset B} \mathbb{1}_U \cdot T$) and the *mass of* T *in* B is defined as

$$M_B(T) = \sup\{|T(\mathbb{1}_B \eta)| : \|\eta\| \le 1\}$$

where $\|\eta\|$ is the co-mass from above. For example, if $T = [Z]$ is given by integration over a locally closed oriented submanifold, then $M_K[Z] = \mathrm{vol}(Z \cap K)$ [30].
The locally normal currents are exactly those currents T such that the mass of T and dT in every compact subset is finite.

Cohomology of normal currents On a complex manifold X, the Hodge filtration on $\mathcal{D}_X^\bullet$ induces a Hodge filtration on the subcomplex $\mathcal{N}_X^\bullet \subset \mathcal{D}_X^\bullet$ by

$$F^p\mathcal{N}_X := \mathcal{N}_X \cap F^p\mathcal{D}_X.$$

The $\mathbb{C}$-valued cohomology of X can also be computed by locally normal currents.

Theorem 14. *The following inclusions*

$$\mathbb{C} \to \Omega_X^\bullet \to \mathcal{A}_X^\bullet \to \mathcal{N}_X^\bullet \to \mathcal{D}_X^\bullet$$

are all filtered quasi-isomorphisms of sheaves on X.

Proof. This is [41, 4.1.1]. All maps are morphisms of filtered complexes of sheaves on X. For currents of order ≥ 1, the Dolbeault-Grothendieck lemma shows that the first two arrows and the map $\mathcal{A}_X \to \mathcal{D}_X$ are filtered quasi-isomorphisms. That the last inclusion is a filtered quasi-isomorphism is due to King, [41, 4.1.1]. More precisely, he showed in [41, 4.3 (for $W = \emptyset$)] that $\Omega \to \mathcal{N}$ is a filtered quasi-isomorphism, by proving a Poincaré lemma for d that preserves the Hodge filtration. □

Remark.

- $\mathcal{N}_X$ is <u>not</u> a bigraded subcomplex of $\mathcal{D}$, because the differential of the components can not be controlled. For example [41, p. 43], even if $T = T^{1,0} + T^{0,1}$ has differential $dT = 0$, one only knows that $dT^{1,0} = -dT^{0,1}$ which is not necessarily representable by integration.

- Locally normal currents compute the $\mathbb{C}$-valued cohomology of a much larger class of manifolds (but then there is no longer a Hodge filtration), see also [19, 5.11].

2.5 Integral currents

We interpret singular chains as currents to get a complex that calculates integral cohomology and has a graded commutative (external) product. The latter property is not true for singular chains (but for singular chains up to subdivision).

C^1 chains A *continuous differentiable (or C^1) k-chain* is a current $\gamma = \sum n_i \cdot (\gamma_i)_*[\Delta_k]$ where all γ_i are continuous differentiable maps defined on some open neighborhood of Δ_k. Note that the pushforward $(\gamma_i)_*[\Delta_k]$ is well defined because $[\Delta_k]$ is representable by integration. If all coefficients n_i occur to be integers, γ is called an integral chain. The chain is finite if $n_i = 0$ for almost all i.

Rectifiable currents For a compact subset $K \subset M$, the class $\mathcal{R}_{k,K}(M)$ of k-dimensional rectifiable currents in K consists of continuous linear functionals T on the space $\mathcal{A}(M)$ (of not necessarily compactly supported smooth forms) such that for each $\epsilon > 0$ there is a finite integral C^1 chain γ with support in K that satisfies $M(T - \gamma) < \epsilon$.
A k-dimensional *rectifiable current* in M is an element in the union $\bigcup_K \mathcal{R}_{k,K}(M)$, the index running over all compact $K \subset M$. A current T is called *locally rectifiable* if for each point x there exists a rectifiable current S such that $x \notin \operatorname{spt}(S - T)$.

Remark. Originally, Federer used Lipschitz chains instead of C^1 chains to define rectifiable currents. The here used definition is equivalent to Federer's, see [31, p.557].

For any Borel set B, the product $T \cdot \mathbb{1}_B$ of a rectifiable current with the characteristic function of B is again a rectifiable current (see [31, p. 558]). If B is an open subset, this is proven in [19, Remark 3.8 (3)].

Integral currents Rectifiable currents are not closed under the differential – a problem that is solved by introducing integral currents.
A *locally integral current* is a current T such that both T and dT are locally rectifiable. If moreover its support is compact, T is called an *integral current.*
For example, the current of integration over a subset E that has a locally finite C^1 triangulation (i.e., there is a locally finite representation $E = \sum n_i\gamma(\Delta_{k_i})$, $n_i \in \mathbb{Z}$) is a locally integral current. It is an integral current if the representation can chosen to be finite.
Any rectifiable current is representable by integration and thus any integral current is normal.

Cohomology of locally integral currents Denote by $\mathcal{I}^p_M$ the sheaf of locally integral currents of degree p on M. By definition, they form a subcomplex $\mathcal{I}^\bullet_M \subset \mathcal{D}^\bullet_M$, and even of the locally normal currents. $\mathcal{I}^0_M$ contains the sheaf of locally constant, integer valued functions $\mathbb{Z}_M$ and the inclusion $\mathbb{Z}_M \to \mathcal{I}^\bullet_M$ is a quasi-isomorphism of sheaves. Furthermore, the $\mathcal{I}^p_M$ are soft sheaves[5] and thus each $\mathcal{I}^p_M$ is acyclic ([41, 2.2]). $\mathcal{I}^\bullet_M$ then is an acyclic resolution of the constant sheaf and thus

$$H^r(M, \mathbb{Z}) \cong H^r(\Gamma(M, \mathcal{I}^\bullet_M)).$$

If more general $\mathcal{I}(M, \mathbb{Z}(p))$ denotes the global sections of $\mathcal{I}_M \otimes \mathbb{Z}(p)$, then this complex calculates the singular cohomology of M with values in $\mathbb{Z}(p)$.

Functoriality for integral currents By [19, Rmk 3.8], the pushforward along a smooth function $f : M \to N$ preserves integral currents, that is, $f_*\mathcal{I}_k(M) \subset \mathcal{I}_k(N)$. Moreover, the product $S \times T$

[5] Proof: Let $C \subset M$ be a closed subset and $T \in \mathcal{I}^p(U)$ for some open $U \supset C$. Choose V open such that $C \subset V \subset \overline{V} \subset U$. The current $\mathbb{1}_{\overline{V}} \cdot T$ is again locally rectifiable and has support in $\overline{V}$ (hence does not meet the boundary of U). In particular, its pushforward along the inclusion of the closed subset $\overline{V} \to M$ gives a locally rectifiable current on M. Its boundary $d(\mathbb{1}_{\overline{V}} \cdot T) = \mathbb{1}_{\partial\overline{V}} \cdot T + \mathbb{1}_{\overline{V}} \cdot dT$ is again locally rectifiable, hence the current is actually locally integral. Rmk: Maybe it is not true that $d\mathbb{1}_{\overline{V}} = \mathbb{1}_{\overline{V}\setminus V}$ but then one can move V a little bit so that $d\mathbb{1}_{\overline{V}} \cdot T$ is defined and locally rectifiable.

of two rectifiable currents is again rectifiable. In particular, pushforward and exterior products preserve integral currents.
This is enough to show that integral currents are preserved under pullbacks:

Lemma 15. *If $f : M \to N$ is a smooth submersion and f is locally integral, then f^*T is locally integral.*

Proof. Since f^* commutes with the differential (up to a sign), it suffices to show that f^*T is locally rectifiable whenever T is locally rectifiable. This can be checked locally. As f is a smooth submersion, it can locally (in M) be written as a projection. That is, for a sufficiently small open $U \subset M$ there is an open $V \subset N$ and coordinates around U and V such that f is a projection with respect to these coordinates. So we may assume that f is the restriction of a projection $\mathrm{pr} : W \times F \to W$ to some open subset $U \subset W \times F$, where the fibre F is an euclidean space. But in this case, $f^*T = \mathrm{pr}\,|_U^* T = (T \times [F])|_U$ is rectifiable. □

2.6 Flat currents

A current T of degree p is called[6] *locally flat* if and only if for each smooth function φ the current φT (defined by the module structure) can be expressed as $S + dR$ where S and R are currents represented by differential forms with locally integrable coefficients.
The locally flat currents form a sheaf on M, denoted by $\mathcal{F}_M$.

Examples.

- $\mathcal{I}^p \subset \mathcal{N}^p \subset \mathcal{F}^p$.
- $d\mathcal{F} \subset \mathcal{F}$.

Locally flat currents can also characterized as limits of locally normal currents (see [31, p. 316]): A locally flat current is a current T such that for all cut off functions φ there exists a compact set $K \supset \operatorname{spt}\varphi$ such that the current φT is a limit of locally normal currents supported on K with respect to the flat norm on K.
Here the *flat norm on* K of a current T is defined as

$$F_K(T) := \sup\big\{|T(\eta)| : \|\eta\|_K \le 1, \|d\eta\|_K \le 1\big\}.$$

The importance of flat currents stems from the following proposition due to Federer (see also [40, 2.1.8, 2.4.2]). Denote by $\mathcal{H}_k$ the k-dimensional Hausdorff measure.

Proposition 16 (Support theorems)**.** *Let $T \in \mathcal{F}_k(M)$ be a locally flat current.*

- *If $\mathcal{H}_k(\operatorname{spt} T) = 0$ then $T = 0$.* *(measure support theorem)*
- *Assume that $f, g : M \to N$ are smooth maps.*
  If $f|_{\operatorname{spt} T} = g|_{\operatorname{spt} T}$ then $f_*T = g_*T$. *(support theorem)*
- *In particular, if $M \overset{i}{\hookrightarrow} N$ is an embedded submanifold,*
  then $i_*\mathcal{F}(M) = \{T \in \mathcal{F}(N) \mid \operatorname{spt} T \subset M\}$. *(flatness theorem)*

[6] This definition is from [31].

Remark. The exterior product of (locally) normal currents is again (locally) normal, hence (locally) flat, see [19, 2.6]. This is no longer true for locally flat currents. That is, $S \times T$ need not be a locally flat current if S, T are merely locally flat. However, if S or T is normal and the other is flat, then $S \times T$ is flat [22, 5.1.3 Rmk 2.v)].

2.7 Integration currents

We now recall the constructions of the current of integration over a semianalytic set, mainly following [34]. This uses the simple extension of currents which has been introduced by Lelong [43] in order to show that the current of integration over a complex analytic set of codimension p is a closed locally integral (p, p) current.
We are interested in two applications of this theory. First, it follows that any algebraic cycle in a smooth complex algebraic manifold U gives rise to a current on a compactification X of U. Second, the integration over semianalytic sets allows to define the current represented by a semimeromorphic differential form.

Integration over manifolds

Let M be a connected oriented real analytic manifold. Choose a family of orientation preserving coordinate maps $\psi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ such that the sets U_α form a locally finite covering of M. Let τ_α be a differentiable partition of unity subordinate to the U_α.
For any compactly supported smooth differential form η on M, the *integral of η over M* is

$$\int_M \eta := \sum_\alpha \int_{\mathbb{R}^n} (\psi_\alpha)_*(\tau_\alpha \eta).$$

Note that $(\psi_\alpha)_* = (\psi_\alpha^{-1})^*$. This definition is independent of the choice of the chart and the chosen partition of unity.
For a not necessarily connected manifold $M = \sqcup M_\alpha$, the integral is extended linearly by

$$\int_M \eta = \sum_\alpha \int_{M_\alpha} \eta|_{M_\alpha}.$$

This gives rise to a dim(M)-dimensional current on M, the *current of integration over M*,

$$[M] : \eta \mapsto \int_M \eta.$$

If M is not orientable, by convention $[M] = 0$ is the zero current.

Integration over semianalytic subsets

Let M be a real analytic manifold. Let $\mathcal{O}_M(U)$ be the ring of real analytic functions on $U \subset M$, that is, the ring of functions on U that locally near each point can be represented by a convergent power series with real coefficients.

Definition ([4])**.** Define $S(U)$ to be the smallest family of subsets of U that contains all sets $\{f(x) > 0\}, f \in \mathcal{O}_M(U)$, and is stable under finite intersection, finite[7] union and complement.

[7] To define semianalytic sets, one can even allow locally finite union.

Every element in $S(U)$ can be written in the form $\bigcup_{i=1}^{p} \bigcap_{j=1}^{q} S_{ij}$ with basic semianalytic sets S_{ij} that are either equal to $\{f_{ij}(x) > 0\}$ or $\{f_{ij}(x) = 0\}$ for $f_{ij} \in \mathcal{O}_M(U)$.

Definition. A subset S of M is *semianalytic* if each $x \in M$ has a neighborhood U such that $S \cap U \in S(U)$.

For example, every real analytic subvariety of a manifold is semianalytic.
The closure, interior and the boundary of a semianalytic set are semianalytic [4, Corollary 2.8].

Definition. Let S be a semianalytic set in M. $x \in M$ is a *regular point* of S if there exists a neighborhood U of x such that $U \cap S$ is a connected real analytic submanifold of M. At each regular point there exists a tangent space and the *dimension*[8] *of the semianalytic set* S is defined as

$$\dim S := \sup\{\dim T_x S : x \text{ is a regular point}\}.$$

The regular points of a semianalytic set form a dense subset $S_{\text{reg}} \subset S$. The set $S_{\text{reg},d}$ of regular points of dimension d (points that locally look like a d-manifold) form a d-dimensional real analytic submanifold of M.
Integration over the top-dimensional piece of the regular part $S_{\text{reg},d}$, $d = \dim(S)$ defines a current $[S] \in \mathcal{D}(S_{\text{reg},d})$ of dimension d, the *current of integration over* S.

Remark. Herrera constructed more general currents $I(S,c)$ for each cohomology class $c \in H_p(S, \mathbb{R})$. Here we always set c to be the fundamental class of S (which always exists by Łojasiewicz' work on triangulations of semianalytic sets [49]).

Simple extension of currents

Let M be an oriented real analytic manifold of dimension m. If S is any semianalytic set in M, then from now one we write just $S_{\text{reg}} := S_{\text{reg},d}$ for the set of regular points of highest dimension $d = \dim S$. The inclusion

$$S_{\text{reg}} \hookrightarrow M \setminus (\overline{S} \setminus S_{\text{reg}}).$$

realizes S_{reg} as a closed submanifold of $M \setminus (\overline{S} \setminus S_{\text{reg}})$ (It is closed, because we removed the closure of S from M). Thus the inclusion is a proper map and the proper pushforward of the integration current of S_{reg} gives a current on $M \setminus (\overline{S} \setminus S_{\text{reg}})$ by the usual formula

$$[S](\eta) = \int_{S_{\text{reg}}} \eta|_{S_{\text{reg}}}.$$

The question whether this expression is well-defined for test forms on M instead of forms on $M \setminus (\overline{S} \setminus S_{\text{reg}})$ was studied by Lelong [43] (complex case) and Herrera [34] (semianalytic case). They start with a current T defined on an open set $U \subset M$. A current $\tilde{T} \in \mathcal{D}(M)$ is called an *extension of* T, if its restriction to U is T, i.e., if $\tilde{T}(\eta) = T(\eta)$ for all test forms η with compact support in U. In general, extensions need not exist nor need they be unique.
The obvious – or: most simple – extension would be to define

$$\tilde{T}(\eta) = T(\eta|_U).$$

To make sense of this expression ($\eta|_U$ is not necessary compactly supported), Lelong [43, §1] approximated U by a limit of compact subsets. He introduced

[8] For equivalent characterizations of the dimension of a semianalytic set, see also Hardt [28, p. 79].

Definition. The *simple extension* of a current T of order zero (i.e., representable by integation) is the current $\tilde{T}$ with

$$\tilde{T}(\eta) = \lim_{\epsilon \to 0} T(\alpha_\epsilon \eta)$$

where α_ϵ is a familiy of functions of a real parameter $\epsilon > 0$ such that

- each α_ϵ is a smooth function on M and $0 \leq \alpha_\epsilon(x) \leq 1$.
- there exist open sets $\omega'_\epsilon \subset \omega_\epsilon$ of M containing $D := M \setminus U$ such that $\alpha_\epsilon \equiv 0$ on ω'_ϵ and $\alpha_\epsilon \equiv 1$ on the complement of ω_ϵ.
- α_ϵ is monotonically increasing with $\epsilon \to 0$ and converges to $\mathbb{1}_U$ for $\epsilon \to 0$.

The simple extension need not exist, but if it exists, it is again representable by integration and extends T. In this case, it does not depend on the actual choice of the functions α_ϵ [43, 1.(6)].

Proposition 17 ([34, II.A.2.1])**.** *The simple extension of locally closed semianalytic sets exists.*

The cycle map into currents

Let U be a complex algebraic manifold and $X \supset U$ be a compactification, that is, a compact complex algebraic manifold such that $D = X \setminus U$ is a divisor.
The simple extension of the integration current over the regular part of a complex analytic (here algebraic) set in U has some remarkable properties, as shown by Lelong.

Proposition 18. *Let $Z \subset U$ be a codimension p irreducible algebraic subvariety. Then the simple extension $[Z]$ to X exists and is a d-closed integral (p,p)-current on X. It is equal to the current of integration over the closure $\overline{Z} \subset X$.*

Proof. The first statement is due to Lelong [43, Théorème 7]. The main point is the existence of the simple extension. Then it is clear that it is locally integral and of bidegree (p,p), because the integration is over a complex manifold. The current $[Z]$ has compact support in X, hence is integral. That the current is d-closed relies on the fact that the real codimension at the boundary $\overline{Z} \setminus Z$ drops by 2. Since the boundary of $[Z]$ is supported on $\overline{Z} \setminus Z$, it must by zero by the support theorem. Note that this last argument can be replaced by Stoke's formula for complex analytic sets, stating that $\int_Z d\eta = \int_{\overline{Z}\setminus Z} \eta$ for any test form η.
Finally, because $\overline{Z} \setminus Z$ consists only of components of codimension $> p$, the set of regular points of highest dimension in $\overline{Z}$ and in Z are the same. Thus $[\overline{Z}] = [Z]$. ☐

This extends linearly to algebraic cycles, that is, to finite linear combinations $\sum n_i Z_i$ where $Z_i \subset X \setminus D$ are closed algebraic subvarieties. The resulting integration current, $\sum n_i [Z_i]$, is a d-closed integral current on X (also called holomorphic chain). If the algebraic cycle is homogeneous of codimension p, then the associated holomorphic chain is of bidegree (p,p).
The *cycle map into currents* is the map

$$\mathrm{cl} : z^p(U) \to \mathcal{I}^{2p}(X, \mathbb{Z}(p))$$

which is defined as $\mathrm{cl}(Z) := (2\pi i)^p [Z]$.

Remark.
- Instead of using the regular part Z_{reg} to define the integration current, $[Z]$ could equivalently be defined as the proper pushforward $[Z] = \pi_*[\tilde{Z}]$ along a proper desingularization $\pi : \tilde{Z} \to Z$. Up to a subset of measure 0 the map π is locally biholomorphic and thus the two definitions indeed agree.

- Proposition 18 implies that the diagram below commutes, so that one may think of cl as "taking the closure in X".

$$\begin{tikzcd} z^p(U) \arrow[r, "\mathrm{cl}"] \arrow[d, "\text{closure}"'] & \mathcal{I}^{2p}(X,\mathbb{Z}(p)) \\ z^p(X) \arrow[ur, "\mathrm{cl}"'] & \end{tikzcd}$$

Consider now another pair $U' \subset X'$ of algebraic manifolds.

Lemma 19 (cl and $\boxtimes$)**.** *For any two algebraic cycles Z, Z' on $U \subset X$ and $U' \subset X'$,*

$$\mathrm{cl}(Z \times Z') = \mathrm{cl}(Z) \boxtimes \mathrm{cl}(Z').$$

Proof. It suffices to show that $[(Z \times Z')_{\mathrm{reg}}] = [Z_{\mathrm{reg}}] \boxtimes [Z'_{\mathrm{reg}}]$ as currents on $X \times X'$. By Fubini's theorem, the right hand side is equal to $[Z_{\mathrm{reg}} \times Z'_{\mathrm{reg}}]$, and it is enough to show the equality of sets $(Z \times Z')_{\mathrm{reg}} = Z_{\mathrm{reg}} \times Z'_{\mathrm{reg}}$.
Since the cartesian product of manifolds is a manifold, $\supset$ holds. Conversely, any regular point (z, z') in $Z \times Z'$ has a neighborhood in $X \times X'$ whose intersection with $Z \times Z'$ is isomorphic to a real analytic manifold. Shrinking this neighborhood if necessary, one can assume that it has the form $U \times U'$. The restriction of the isomorphism to $U \cong U \times \{z'\}$ and $U' \cong \{z\} \times U'$ shows that $(U \cap Z) \cong (U \cap Z) \times \{z'\}$ and $U' \cap Z' \cong \{z\} \times (U' \cap Z')$ are isomorphic real analytic manifolds, and so z, z' are regular points. ☐

Lemma 20 (cl and pushforward)**.** *Let $f : X \to X'$ be a smooth algebraic map that restricts to a proper map $f : U \to U'$. Then*

$$\mathrm{cl}(f_* Z) = f_* \mathrm{cl}(Z)$$

for all algebraic cycles $Z \subset U$. More precisely, there is a commutative diagram

$$\begin{tikzcd} z^p(U) \arrow[r, "\mathrm{cl}"] \arrow[d, "(f|_U)_*"] & \mathcal{I}^{2p}(X,\mathbb{Z}(p)) \arrow[d, "f_*"] \\ z^{p+\delta}(U') \arrow[r, "\mathrm{cl}"] & \mathcal{I}^{2p+2\delta}(X',\mathbb{Z}(p+\delta)) \end{tikzcd}$$

where $\delta = \dim X' - \dim X$.

Proof. Let Z be a closed irreducible subvariety of codimension p in U. Because $f|_U$ is proper, the same holds for the restriction $f : Z \to f(Z)$. The $2\pi i$-twists are the same so that it is to show that

$$[f_* Z] = f_*[Z].$$

Note that $f_* Z$ is defined as $d \cdot f(Z)$, where d is the degree of f. The dimension of $f(Z)$ is always less or equal than the dimension of Z. If the dimension of $f(Z)$ is strictly less than the dimension of Z, then $d = 0$ by definition, so that the left hand side vanishes. The right hand side is supported on $f(Z)$ and thus also vanishes by the support theorem.
So assume that f preserves the dimension. Then there exists an open subset $W' \subset f(Z)$ such that the restriction of f to $f^{-1}W' \cap Z \to W'$ is a locally biholomorphic cover and d is just the number of sheets of this cover.

Then for any test form η on W' one finds that

$$d\int_{W'}\eta = \int_{f^{-1}W'\cap Z} f^*\eta.$$

Note that $f^{-1}W'\cap Z \subset Z_{\mathrm{reg}}$ and $W' \subset f(Z)_{\mathrm{reg}}$ consist of regular points, and that the two sets are actually dense, that is, their complement is negligible for the integration. It follows that

$$\int_{f_*Z}\eta = d\int_{f(Z)_{\mathrm{reg}}}\eta = d\int_{W'}\eta = \int_{f^{-1}W'\cap Z} f^*\eta = \int_Z f^*\eta.$$

The open W' can be chosen such that the complement $f(Z)\setminus W'$ has measure zero by Sard's theorem (in our case, it consists of finitely many points). In particular, the statement is even true for test forms on $f(Z)_{\mathrm{reg}}$ and, since the simple extension of both sides exist, also for test forms on X'. □

Lemma 21 (cl and pullback)**.** *Let $f : X \to X'$ be a smooth algebraic map such that the restriction $f|_U : U \to U'$ is flat. Then the pullback along f (which exists, because smooth algebraic maps are submersions) satisfies*

$$\mathrm{cl}(f^*Z) = f^*\mathrm{cl}(Z).$$

In other words, the diagram below commutes

$$\begin{tikzcd}
z^p(U') \arrow[r, "\mathrm{cl}"] \arrow[d, "(f|_U)^*"] & \mathcal{I}^{2p}(X', \mathbb{Z}(p)) \arrow[d, "f^*"] \\
z^p(U) \arrow[r, "\mathrm{cl}"] & \mathcal{I}^{2p}(X, \mathbb{Z}(p))
\end{tikzcd}$$

Proof. By linearity of f^* and cl, it is enough to show the statement for Z closed irreducible. Since f^* preserves codimension, it is to show that

$$f^*[Z] = [f|_U^* Z].$$

Now, use two things: First, $f^*[Z_{\mathrm{reg}}] = [f^{-1}(Z_{\mathrm{reg}})]$. Second, $f|_U^*Z = f^{-1}(Z)$ by definition. This reduces the statement to

$$\int_{f^{-1}(Z_{\mathrm{reg}})}\eta = \int_{f^{-1}(Z)_{\mathrm{reg}}}\eta$$

for any test form η. But this is true because both integration domains differ only by a set of codimension $+1$. More precisely,

$$f^{-1}(Z_{\mathrm{reg}}) \subset f^{-1}(Z)_{\mathrm{reg}} \subset f^{-1}(Z)$$

and $f^{-1}(Z)\setminus f^{-1}(Z_{\mathrm{reg}}) = f^{-1}(Z_{\mathrm{sing}})$ has codimension $+1$. □

Remark. The lemmata 19, 20 and 21 can also be formulated in the complex analytic setting by replacing everywhere the word "algebraic" by "holomorphic". If X is projective, there is no difference since in this case every holomorphic function is algebraic and every analytic cycle is an algebraic cycle by Chow's lemma.

Semimeromorphic forms

Let X be a complex manifold and $D \subset X$ a divisor that is assumed to be defined locally by a single equation ($h = 0$). A *semimeromorphic form with polar set* D is a differential form ω on X which can locally at each point be expressed as $\omega = \frac{\tilde{\omega}}{h^q}$, for $\tilde{\omega}$ smooth and $q \geq 0$. To be more precise, ω is a smooth differential form on $X \setminus D$ in such a way that each point of X has a neighbourhood U where $U \cap D = (h = 0)$ for some holomorphic function h on U and $h^q\omega$ extends as a smooth differential form to U for some integer q.

A particular class of semimeromorphic forms are the differential *forms with logarithmic pole along* D, $\mathcal{A}(X, \log D)$. These are the forms ω such that locally for a reduced h (that is, h has no multiple factors) the forms $h\omega$ and $hd\omega$ extend even holomorphically to U.

To a semimeromorphic form ω one associates a current, the *principal value* $[\omega]$. This current is locally (in the above situation) defined by

$$[\omega](\eta) = \lim_{\epsilon \searrow 0} \int_{|h|>\epsilon} \omega \wedge \eta$$

where the integration is over the semianalytic set $|h| > \epsilon$. It is shown by Herrera/Lieberman [33] that the integral converges and defines a current on X independent of the chosen local representation of D and ω. A similar result has been obtained by Dolbeault [14]. The principal value of ω is just the simple extension of the current $[\omega] \in \mathcal{D}(X \setminus D)$ to X. From this point of view, Herrera/Lieberman showed that the simple extension of semimeromorphic differential forms exist.

We define the *wedge product of a locally normal current with a semimeromorphic form* as the simple extension

$$T \wedge \omega := \lim_{\epsilon \to 0} T \wedge (\alpha_\epsilon \omega)$$

whenever it exists.

The weak limit exists for example if $\operatorname{spt} T \cap D = \emptyset$. If ω is a smooth form, then the wedge product also exists and is equal to the usual wedge product. This needs that T is locally normal to ensure that $T(\mathbb{1}_{X\setminus D}\omega) = T(\omega)$.

If ω, ω' are semimeromorphic differential forms with polar set D resp. D', then $\omega \boxtimes \omega'$ is semimeromorphic with polar set $D \boxtimes D'$. Similarly, the wedge product of two semimeromorphic forms and the pullback of a semimeromorphic form along holomorphic functions are again semimeromorphic.

The restriction of a semimeromorphic form to a closed subset in general need not be a current. For log forms in good position however, this is true.

Lemma 22. *Let $D \subset X$ be a normal crossing divisor and ω a log-D form on X. For any algebraic cycle $Z \subset X$ that intersects D properly, the wedge product $[Z] \wedge \omega$ exists and is normal.*

Proof. The differential of the above functional will be of the same form, so that it suffices to show that the current exists and is representable by integration. Roughly, this means that the restriction of $\omega \wedge \eta$ to Z_{reg} is locally integrable for any continuous test form η. More precisely, it is to show that for any compact K (which can arranged to be contained in a coordinate chart of X) there exists a constant C such that for any continuous test form η with support in K one has

$$\left| \int_{Z\setminus D} \omega \wedge \eta \right| < C \cdot \|\eta\|_\infty.$$

Because Z and D intersect properly, one can choose coordinates $z_1, z_2, \ldots z_n$ in such a way that $Z = (z_1 = \ldots = z_p = 0)$ and such that the normal crossing divisor D has the form $D = (z_{p+1} \cdots z_{p+r} = 0)$. With respect to these coordinates, the integral becomes

$$|\int\limits_{0^p \times (\mathbb{C}^*)^r \times \mathbb{C}^{n-p-r}} \omega \wedge \eta|$$

for some continuous form η compactly supported in the above mentioned coordinate chart. The integral vanishes if $\omega \wedge \eta|_{X \setminus D}$ is not a multiple of the volume form of $X \setminus D$. Otherwise, it is a rational multiple of the volume form with denominator equal to the denominator of ω. Bounding the numerator of ω by a constant $C > 0$ from above, we get the upper bound

$$\int\limits_{(\mathbb{C}^*)^r \times \mathbb{C}^{n-p-r}} \Big| \frac{\mathbb{1}_K}{z_{p+1} \cdots z_{p+r}} \cdot dz_{p+1} \wedge d\overline{z}_{p+1} \wedge \ldots \wedge dz_n \wedge d\overline{z}_n \Big| \cdot C \cdot \|\eta\|_\infty.$$

A calculation in polar coordinates shows that the inner integrand is locally integrable: The main point in using polar coordinates $z = \rho e^{i\varphi}$ is that one obtains $dz \wedge d\overline{z} = 2\rho i d\varphi \wedge d\rho$ and therefore each factor of the form $\operatorname{dlog} z \wedge d\overline{z}$ has locally bounded coefficients. Because K is compact, the integral is also bounded by a constant independent of η and thus the lemma is proved. □

2.8 Intersection of currents

Slicing flat currents

Here we want to intersect a current T with the level set of a smooth function $f : M \to \mathbb{R}^n$. We follow [31] closely. Let $a \in \mathbb{R}^n$ and fix an "approximation of δ_a" as follows: Denote by Ω the volume form of $\mathbb{R}^n$ and choose a smooth function $\psi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with compact support such that $\int_{\mathbb{R}^n} \psi \cdot \Omega = 1$. For any $\rho > 0$ define a smooth form on $\mathbb{R}^n$ by

$$\Omega_{a,\rho}(x) := \frac{1}{\rho^n} \psi\left(\tfrac{x-a}{\rho}\right) \Omega(x).$$

These forms approximate the delta distribution δ_a in the sense that for any locally integrable function g one has $\lim_{\rho \to 0} \int g \Omega_{a,\rho} = g(a)$ for Lebesgue-almost all a (this essentially is Lebesgue's differentiation theorem).

For a current $T \in \mathcal{F}^p(M)$ the current $T \wedge f^*\Omega_{a,\rho}$ is locally flat of degree $p + n$. If the (weak) limit for $\rho \to 0$ exists, one writes

$$< T, f, a >= \lim_{\rho \to 0} T \wedge f^* \Omega_{a,\rho}$$

and calls this the *slice of T along f at a*. In fact, up to a set of Lebesgue measure zero, the slice at a exists and is independent of the choice of the approximation ψ, see [31, p. 565f].

Proposition 23 (Slicing theorem). *Let $f : M \to \mathbb{R}^n$ be a smooth map. If T is locally flat (locally normal, locally integral) then for Lebesgue almost all a the slice $<T, f, a>$ exists and is again locally flat (locally normal, locally integral).*

Proof. See [40, 2.3.4], or Federer [18, 4.3.2] for the local case. □

The next proposition gathers some properties of the slice of a current.

Proposition 24. *Assume that the slice* $<T, f, a>$ *exist. Then*

- $\operatorname{spt}<T, f, a> \subset \operatorname{spt} T \cap f^{-1}(a)$.
- $d<T, f, a> = <dT, f, a>$ *if* $|T| < \dim_{\mathbb{R}} M$.
- $g_*<T, f \circ g, a> = <g_*T, f, a>$ *whenever the left-hand side exists.*
- $<T, -f, 0> = (-1)^n<T, f, 0>$.

Proof. The first statement is trivial. Two and three are in Harvey-Shiffman [31, 1.3.9] and [31, Lemma 1.19]. For the fourth, choose ψ such that $\psi(-x) = \psi(x)$ and remark that for the volume form Ω on $\mathbb{R}^n$ one has $(-f)^*\Omega = (-1)^n f^*\Omega$. □

Intersection in $\mathbb{R}^n$

Consider currents S, T of degree r resp. s on some open $U \subset \mathbb{R}^n$ with $r + s \leq n$. Assume that the slice current $<S \times T, \xi, 0>$ along the difference map $\xi(x, y) = x - y$ exists and defines a locally flat current of degree $r + s$ on $U \times U$. Then the slice is supported on the diagonal $\Delta_U \subset \mathbb{R}^{2n}$ and hence, by the flatness theorem, there exists an unique current, the *intersection current* $S \cap T$, on U of degree $r + s$ such that

$$\Delta_*(S \cap T) = (-1)^{r(n-s)}<S \times T, f, 0>,$$

where $\Delta : U \to \mathbb{R}^{2n}$ is the diagonal embedding with image Δ_U.
The following properties are from [22, 5.3.4, prop 1].

Proposition 25.

i) If $S \cap T$ *exists, then* $\operatorname{spt}(S \cap T) \subset \operatorname{spt} S \cap \operatorname{spt} T$.

ii) $S \cap T$ *exists* $\iff$ $T \cap S$ *exists, and in this case* $S \cap T = (-1)^{\deg S \deg T} T \cap S$.

iii) Compatibility with differentials: If $S \cap T$ *and either* $dS \cap T$ *or* $S \cap dT$ *exist, then*

$$d(S \cap T) = dS \cap T + (-1)^{\deg S} S \cap dT.$$

iv) For $\varphi : U \to U'$ *an orientation preserving diffeomorphism,* $\varphi_* S \cap \varphi_* T = \varphi_*(S \cap T)$.

Intersection on manifolds

Let M be an oriented manifold and S, T be locally flat currents on M of degree r and s respectively. The intersection current of S, T on M is defined by localization, that is, by locally transferring the two currents into some $\mathbb{R}^n$ and intersecting them there.

Definition. The *intersection current* $S \cap T$ (if it exists) is a current on M such that for each orientation preserving coordinate chart $h : U \to \Omega \subset \mathbb{R}^n$ one has $h_*(S \cap T) = h_* S \cap h_* T$.

The intersection current in this definition is well defined. Indeed, if h' is another orientation preserving chart, then the coordinate change $\varphi = h' \circ h^{-1}$ is an orientation preserving diffeomorphism, hence $h'_*S \cap h'_*T = \varphi_*(h_*S \cap h_*T)$ by 25 iv). In particular, it suffices to test the condition for one specific atlas only.
Note that, due to the existence of a partition of unity for test forms, it suffices to construct $S \cap T$ locally. In particular, the intersection current is necessarily unique if it exists.
The relative notion of slicing allows to rewrite the definition as follows. This characterization has been taken as the definition of intersection in [28].

Lemma 26. *The intersection current $S \cap T$ (if it exists) is the locally flat current on M such that for each orientation preserving coordinate chart $h : U \to \Omega \subset \mathbb{R}^n$ one has*

$$\Delta_*(S \cap T|_U) = (-1)^{\deg S \dim T} < (S \times T)|_{U \times U}, \xi \circ (h \times h), 0 > .$$

Here the map ξ sends (u, v) to $u - v$.

Note that the right-hand side, if it exists, is supported on the diagonal of U and by the flatness theorem arises as the pushforward of a flat current on U.

Proof. We may assume that S, T are supported in some coordinate domain U. The intersection $h_*S \cap h_*T$ exists if and only if $<h_*S \times h_*T, \xi, 0>$ exists. Now apply proposition 24 to see that the existence of the latter is equivalent to the existence of $<S \times T, \xi \circ (h \times h), 0>$ (note that h is invertible). □

Particular cases of intersections

The intersection product of currents extends both the wedge product of differential forms and the intersection of subvarieties. This is content of the following proposition.

Proposition 27.

- *For a locally normal current T and a smooth form ω the intersection current $T \cap [\omega]$ exists and $T \cap [\omega] = T \wedge \omega$.*
- *For ω_1, ω_2 smooth differential forms of degree r and s with $r + s \leq n$, one has $[\omega_1] \cap [\omega_2] = [\omega_1 \wedge \omega_2]$.*
- *For M, N submanifolds of an oriented manifold that meet transversal, one has $[M] \cap [N] = [M \cap N]$.*
- *If A, B are two complex analytic sets in a complex manifold X such that $\operatorname{codim}(A \cap B) = \operatorname{codim} A + \operatorname{codim} B$, then $[A] \cap [B]$ exists and equals $[A \cap B]$.*

Proof. The first two claims can be found in Federer, p. 461 (and also in [22, 5.3.4 Prop. 3]). Note that the second is a consequence from the first one. The last two statements are from [58]. □

In particular, the intersection of currents extends the intersection of algebraic varieties (i.e. computes the correct multiplicities).

Intersection of algebraic varieties with log forms

The intersection of a normal current with a non-smooth form in general does not exist. For algebraic chains in sufficiently good position and logarithmic forms however, it exists.

Recall that a form with logarithmic poles along a normal crossing divisor $D \subset X$ is a smooth differential form ω on $X \setminus D$ such that for a reduced local defining equation $D = (h = 0)$ the forms $h\omega$ and $hd\omega$ extend to smooth forms on X.

Lemma 28. *For any complex subvariety $Z \subset X$ that intersects D properly and any log-D-form ω, the intersection $[Z] \cap [\omega]$ exists and $[Z] \cap [\omega] = [Z] \wedge \omega$.*

Proof. The statement can be checked locally, so that we may assume that $X = \mathbb{C}^n$. We show that on $\mathbb{C}^n \times \mathbb{C}^n$ (with coordinates (x, y))

$$<[Z] \boxtimes [\omega], \xi, 0> = \Delta_*([Z] \wedge \omega)$$

for the difference function $\xi(x, y) = x - y$. This is exactly the condition that $[Z] \wedge \omega$ is the intersection current of $[Z]$ and $[\omega]$.

For almost every $x \in Z_{\text{reg}}$, the current

$$\int_{y \in X \setminus D} \omega(y) \wedge \xi^* \Omega_{0,\rho}(x, y)$$

converges weakly to $\mathbb{1}_{Z \setminus D}\omega(x)$. Indeed, this is a smoothing of ω, and since ω is smooth outside D, every point in $Z \setminus D$ is a Lebesgue-point and hence the smoothing at this point converges to $\omega(x)$. Since $Z \cap D$ has positive codimension in Z (by the proper intersection condition), the convergence takes place almost everywhere on Z.

Because log forms are locally integrable, Lebesgue's majorized convergence theorem implies that

$$\lim_{\rho \to 0} \int_{Z_{\text{reg}}} \int_{X \setminus D} \omega(y) \wedge \xi^* \Omega_{0,\rho}(x, y) = \int_{Z_{\text{reg}}} \mathbb{1}_{Z \setminus D}(x)\omega(x) = \int_{Z \setminus D} \omega(x)$$

weakly on continuous test forms on $X \times X$. □

Intersection of real analytic chains

The last two points in proposition 27 have been generalized by Robert Hardt [28], who considered the intersection of real analytic chains. A *real analytic chain of dimension* t on M is a locally flat current T of dimension t such that $\dim(\operatorname{spt} T) \leq t$ and $\dim(\operatorname{spt} \partial T) \leq t - 1$.

Here the *real analytic dimension* of a non-empty subset $\emptyset \neq E \subset M$ of an oriented real analytic manifold M is defined as

$$\sup_{x \in M} \inf\{\dim \alpha : \alpha \text{ is the germ of an analytic variety at } x \text{ and } \alpha \text{ contains the germ of } E \text{ at } x\}.$$

The empty set has dimension $\dim \emptyset = -1$. If E is a (real) analytic variety, or even a semianalytic set, then this definition coincides with the analytic or measure theoretic definitions. That is, in this case $\dim E = \sup_{x \in E_{\text{reg}}} \dim T_x E$ and $\dim E = \sup\{r : \mathcal{H}_r(E) > 0\}$, see [28, 2.2].

In particular, every real analytic submanifold (even with boundary) and every semianalytic set is a real analytic chain.

Two analytic chains $\{S, T\}$ of degree r and s are said to *intersect suitably*, if

$$r + s \leq \dim M,$$
$$\operatorname{codim}(\operatorname{spt} S \cap \operatorname{spt} T) \geq r + s,$$
$$\operatorname{codim}\left[(\operatorname{spt} \partial S \cap \operatorname{spt} T) \cup (\operatorname{spt} S \cap \operatorname{spt} \partial T)\right] \geq r + s + 1.$$

Hardt proves that if $\{S, T\}$ intersect suitably, then the intersection $S \cap T$ of S and T exists and again is an analytic chain of degree $r + s$. Furthermore the following axioms are satisfied and they uniquely determine the intersection theory.

- $(iS) \cap T = i(S \cap T)$ for any integer i.
- $S \cap T = (-1)^{rs} T \cap S$.
- $(S \cap T)|_U = S|_U \cap T|_U$ for every open subset U.
- $\varphi_*(S \cap T) = \varphi_* S \cap \varphi_* T$ for every orientation-preserving analytic isomorphism between analytic manifolds.
- $\Delta_*(S \cap T) = (-1)^{\deg S \dim T}(S \times T) \cap \Delta_*[X]$.
- If R, S, T are analytic chains such that $\{R, S\}, \{S, T\}$ and $\{R, S, T\}$ intersect suitably, then $(R \cap S) \cap T = R \cap (S \cap T)$.
- If N is another oriented real analytic manifold, L an analytic chain in $M \times N$ and R an analytic chain in M such that $\operatorname{pr}_M |_{\operatorname{spt} L}$ is proper and $\{L, R \times [N]\}$ intersect suitably. Then $\operatorname{pr}_* L$ is an analytic chain on M, the intersection with R exists and $\operatorname{pr}_* L \cap R = \operatorname{pr}_*(L \cap R \times [N])$.
- $[0] \cap [0] = [0]$ in $\mathbb{R}^0 = \{0\}$.

To explain the statement about the associativity, more general a finite set of analytic chains $\{T_i\}_{i \in I}$ is said to intersect suitably, if

$$\sum \deg T_i \leq \dim M,$$
$$\operatorname{codim}(\cap_I \operatorname{spt} T_i) \geq \sum \deg(T_i),$$
$$\operatorname{codim}(\cup_i \Big(\operatorname{spt} \partial T_i \cap \bigcap_{j \neq i} \operatorname{spt} T_j \Big)) \geq \sum \deg(T_i) + 1.$$

General pullback of currents

The intersection product allows to (partially) define a general notion of a pullback along a smooth map $f : M \to N$ of oriented real analytic manifolds.
In fact, denote by $[\Gamma_f] = (\mathrm{id}, f)_*[M]$ the graph current of f. If $T \in \mathcal{D}(N)$ is a current such that the intersection $([M] \times T) \cap \operatorname{cl}(\Gamma_f)$ exists, then the *pullback of T along f* is the current

$$f^* T := (\operatorname{pr}_M)_* \Big(([M] \times T) \cap [\Gamma_f] \Big)$$

Note that the projection pr_M restricts to a proper map on Γ_f, and so the pushforward is defined. If f is a submersion, then this pullback is the same as the one defined in the previous subsection 2.2. If f is a morphism of algebraic manifolds, this extends the general pullback of algebraic cycles.

Inverse mapping formula

Hardt proved further properties of his intersection theory, for example the *inverse mapping formula*, which states that if $f : M \to N$ is an analytic mapping of oriented real analytic manifolds and T is an analytic chain on N such that $f^{-1}(\operatorname{spt} T)$ and $f^{-1}(\operatorname{spt} \partial T)$ have the expected codimensions, then the general pullback f^*T exists and is an analytic chain on M. If moreover S is an analytic chain on M such that $f|_{\operatorname{spt} S}$ is proper and $f^{-1}(\operatorname{spt} T) \cap \operatorname{spt} S$ and $(f^{-1}(\operatorname{spt} T) \cap \operatorname{spt} \partial S) \cup (f^{-1} \operatorname{spt}(\partial T) \cap \operatorname{spt} S)$ have the correct codimensions, then $T \cap f_* S$ exists and

$$T \cap f_* S = f_*((f^*T) \cap S).$$

If f is a projection, this formula is also called the *projection formula*.

Properties of intersection currents

Lemma 29 ($\boxtimes$ and $\cap$)**.** *Let M, N be oriented real analytic manifolds, S, P locally normal currents on M and T, Q locally normal currents on N. If the intersections $S \cap P$ and $T \cap Q$ exist, then $(S \boxtimes T) \cap (P \boxtimes Q)$ also exists and*

$$(S \boxtimes T) \cap (P \boxtimes Q) = (-1)^{|T||P|}(S \cap P) \boxtimes (T \cap Q).$$

Proof. For real analytic chains, such a formula was proven by Hardt [28]. Let m,n be the dimension of M,N and denote by s, p, t, q the degrees of S, P, T, Q respectively. Note that $\Delta_{M \times N} = \tau \circ (\Delta_M \times \Delta_N)$, where τ is the isomorphism $M \times M \times N \times N \to M \times N \times M \times N$ that exchanges the middle factors. It is to show that the local slices defining the left hand side exist and are equal to the pushforward of the right hand side along the diagonal. Since $M \times N$ is covered by product charts, it is enough to consider a coordinate chart (h, k) and the difference of the coordinates, $\xi_{(h,k)} = \xi \circ (h, k)$. It is related to the difference functions of the factors by $\xi_{(h,k)} \circ \tau = (\xi_h \times \xi_k)$. The function ψ that defines the slices can be chosen in such a way that $\Omega_{(0,0),\rho} = \Omega_{0,\rho} \boxtimes \Omega_{0,\rho}$. One then finds that $\xi^*_{(h,k)}\Omega_{(0,0),\rho} = \xi^*_h\Omega_{0,\rho} \boxtimes \xi^*_k\Omega_{0,\rho}$. Using this, the graded-commutativity of the exterior product (Prop. 9), the compatibility of slices under pushforward (Prop. 24), the definition of the slice, and the compatibility of exterior products under pullbacks (Lemma 11), one finds that

$$\begin{aligned}
<S \boxtimes T \boxtimes P \boxtimes Q, \xi_{(h,k)}, (0,0)> &= <\tau_*(S \boxtimes P \boxtimes T \boxtimes Q), \xi_{(h,k)}, (0,0)> \cdot (-1)^{|P||T|} \\
&= \tau_* <S \boxtimes P \boxtimes T \boxtimes Q, \xi_h \times \xi_k, (0,0)> \cdot (-1)^{|P||T|} \\
&= \tau_* \lim_{\rho \to 0} (S \boxtimes P \boxtimes T \boxtimes Q) \wedge (\xi^*_h \times \xi^*_k)\Omega_{(0,0),\rho} \cdot (-1)^{|P||T|} \\
&= \tau_* \lim_{\rho \to 0} (S \boxtimes P \boxtimes T \boxtimes Q) \wedge (\xi^*_h\Omega_{0,\rho} \boxtimes \xi^*_k\Omega_{0,\rho}) \cdot (-1)^{|P||T|+mn} \\
&= \tau_* \lim_{\rho \to 0} \Big((S \boxtimes T) \wedge \xi^*_h\Omega_{0,\rho} \boxtimes (P \boxtimes Q) \wedge \xi^*_k\Omega_{0,\rho}\Big) \cdot (-1)^c,
\end{aligned}$$

where the last equality is compatibility of exterior product and wedge product (Lemma 10), and $c = |P||T| + mn + |T \boxtimes Q|m)$.

Under the given assumptions, the two slices in the last row exist. By the continuity of $\boxtimes$, the slice in the first row (and thus the intersection current) also exists. It is left to show that the intersections is indeed the claimed one.

By definition of the intersection,

$$<S \boxtimes P, \xi_h, 0> = (-1)^{|S| \dim P + \dim S|P|} \Delta_*(S \cap P),$$
$$<T \boxtimes Q, \xi_k, 0> = (-1)^{|T| \dim Q + \dim T|Q|} \Delta_*(T \cap Q)$$

and we find with $c' = |S| \dim P + \dim S|P| + |T| \dim Q + \dim T|Q|$, and using lemma 11, that

$$\begin{aligned}
<S \boxtimes T \boxtimes P \boxtimes Q, \xi_{(h,k)}, (0,0)> &= \tau_*\Big(<S \boxtimes P, \xi_h, 0> \boxtimes <T \boxtimes Q, \xi_k, 0>\Big) \cdot (-1)^c \\
&= \tau_*\Big(\Delta_*(S \cap P) \boxtimes \Delta_*(T \cap Q)\Big) \cdot (-1)^{c+c'} \\
&= \tau_*(\Delta \times \Delta)_*\Big((S \cap P) \boxtimes (T \cap Q)\Big) \cdot (-1)^{c+c'+\dim(S\cap P)n} \\
&= \Delta_{M,N*}\Big((S \cap P) \boxtimes (T \cap Q)\Big) \cdot (-1)^{c+c'+\dim(S\cap P)n}.
\end{aligned}$$

Finally, one checks that $c + c' + \dim S \cap P + n \equiv |S \boxtimes T| \dim(P \boxtimes Q) + \dim(S \boxtimes T)|P \boxtimes Q|$ modulo 2, as required. □

For practical computations, the following extension of proposition 27 is very helpful.

Lemma 30 ($\cap$ and $\wedge$)**.** *Let $T, \tilde{T}$ be locally flat currents on a manifold M such that $T \cap \tilde{T}$ exists. Let $\omega, \tilde{\omega}$ be two smooth differential forms on M. Then $(T \wedge \omega) \cap (\tilde{T} \wedge \tilde{\omega})$ also exists and*

$$(T \wedge \omega) \cap (\tilde{T} \wedge \tilde{\omega}) = (-1)^{|\tilde{T}||\omega|} \cdot (T \cap \tilde{T}) \wedge (\omega \wedge \tilde{\omega}).$$

Proof. We may assume that $T, \tilde{T}$ are defined in some n-dimensional euclidean space. Denote by ξ the usual difference map. By lemma 10,

$$\begin{aligned}
<(T \wedge \omega) \boxtimes (\tilde{T} \wedge \tilde{\omega}), \xi, 0> &= \lim_{\rho \to 0} \Big[((T \wedge \omega) \boxtimes (\tilde{T} \wedge \tilde{\omega})) \wedge \xi^* \Omega_{0,\rho}\Big] \\
&= \lim_{\rho \to 0} \Big[(T \boxtimes \tilde{T}) \wedge (\omega \boxtimes \tilde{\omega}) \wedge \xi^* \Omega_{0,\rho}\Big] \cdot (-1)^{|\tilde{T}||\omega|} \\
&= \lim_{\rho \to 0} \Big[(T \boxtimes \tilde{T}) \wedge \xi^* \Omega_{0,\rho}\Big] \wedge (\omega \wedge \tilde{\omega}) \cdot (-1)^{|\tilde{T}||\omega| + n|\omega \wedge \tilde{\omega}|} \\
&= <T \boxtimes \tilde{T}, \xi, 0> \wedge (\omega \wedge \tilde{\omega}) \cdot (-1)^{|\tilde{T}||\omega| + n|\omega \wedge \tilde{\omega}|}.
\end{aligned}$$

Since the last term exists, so does the first term. Therefore, $(T \wedge \omega) \cap (\tilde{T} \wedge \tilde{\omega})$ also exists and one verifies that it indeed is equal to $(T \cap \tilde{T}) \wedge (\omega \wedge \tilde{\omega}) \cdot (-1)^{|\tilde{T}||\omega|}$, i.e., the signs that are obtained by replacing $\boxtimes$ by $\times$ and plugging in the definition of $\cap$ are the same on both sides. □

If $T = [Z]$ and $\tilde{T} = [\tilde{Z}]$ are integration over algebraic cycles that intersect properly and $\omega, \tilde{\omega}$ are semimeromorphic forms on Z_{reg} resp. $\tilde{Z}_{\text{reg}}$ such that $\omega \wedge \tilde{\omega}$ is semimeromorphic of $Z \cap \tilde{Z}$, then the above statement still holds true.

Lemma 31. *On a complex manifold X, the intersection product is compatible with the Hodge filtration, i.e., if the intersection of $S \in F^p\mathcal{D}(X)$ and $T \in F^q\mathcal{D}(X)$ exists, then it lies in $F^{p+q}\mathcal{D}(X)$.*

Proof. A current lies in $F^p\mathcal{D}(X)$ if and only if it vanishes on all test forms of type (r, s), where $r + p \geq \dim_{\mathbb{C}} X$. This can be verified locally. Since the exterior product is bi-additive in the degree, $S \times T \in F^{p+q}\mathcal{D}(X \times X)$. Whenever the slice along the difference map

$$<S \times T, \xi, 0> = \lim_{\rho \to 0} (S \times T) \wedge \xi^* \Omega_{0,\rho}$$

exists, then it lies in $F^{p+q+\dim_{\mathbb{C}} X}\mathcal{D}(X \times X)$, because $\xi^*\Omega_{0,\rho}$ is of pure type $(\dim_{\mathbb{C}} X, \dim_{\mathbb{C}} X)$ in the coordinates of the two factors. For the intersection then

$$(S \cap T)(\eta) = <S \times T, \xi, 0>(\mathrm{pr}_1^* \eta) = 0$$

whenever η is of type (r, s) and r satisfies $r + p + q + \dim_{\mathbb{C}}(X) \geq \dim_{\mathbb{C}}(X \times X) = 2\dim_{\mathbb{C}}(X)$. Thus $S \cap T \in F^{p+q}\mathcal{D}(X)$. □

Lemma 32 (cl and $\cap$)**.** *If Z, Z' are two algebraic cycles on U such that $Z \cap Z'$ exists, then* $\mathrm{cl}(Z) \cap \mathrm{cl}(Z')$ *also exists and equals* $\mathrm{cl}(Z \cap Z')$.

Proof. Consider the closures $\overline{Z}, \overline{Z}' \subset X$. Their intersection exists for example by using Serre's Tor formula to associate multiplicities to all components of the intersection. Because the fundamental class is given by integration over the closure, it suffices to show that $[\overline{Z} \cap \overline{Z'}]$ is an intersection current for the pair $[\overline{Z}], [\overline{Z'}]$. But this follows because the intersection of currents extends the intersection of algebraic cycles. □

Intersection on cohomology

We show (roughly following [22]) that the partially defined intersection product of currents induces an intersection product on cohomology that is everywhere well defined. The same holds for integral currents: While in general the intersection of integral currents is not neccessarily defined, and even if it is, the intersection need not be integral again (in fact it is not even rectifiable, see [22, p. 601]), cohomologically the intersection is well defined for integral currents as well.
We are going to show that any two currents on a compact complex manifold X can be brought into good position to each other by adding boundaries. We begin with the "local case" (compare [22, 5.4.2]).

Lemma 33. *Let S, T be flat currents on X such that T is d-closed and has support in the domain of some chart $h : U \to \mathbb{R}^n$. Then there exists a current $\tilde{T}$ such that $\tilde{T} = T + dR$ and $S \cap \tilde{T}$ exist. If T is integral/normal, then $\tilde{T}$ can be chosen such that $\tilde{T}$, R and $S \cap \tilde{T}$ are integral/normal as well.*

Proof. Without loss of generality, we may replace S by $S|_U$. Denote by $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ the translation by $a \in \mathbb{R}^n$ and define $T_a := (h^{-1} \circ \tau_a \circ h)_* T$. This is well defined for all $a \approx 0$ sufficiently small and is again normal (resp. integral) if T is. The intersection $h_* S \cap h_* T_a = h_* S \cap (\tau_a \circ h)_* T$ exists for almost all $a \approx 0$, because

$$\begin{aligned} <h_*S \times (\tau_a \circ h)_*T, \xi, 0> &= (\mathrm{id} \times \tau_a)_* <h_*S \times h_*T, \xi \circ (\mathrm{id} \times \tau_{-a}), 0> \\ &= (\mathrm{id} \times \tau_a)_* <h_*S \times h_*T, \tau_a \circ \xi, 0> \\ &= (\mathrm{id} \times \tau_a)_* <h_*S \times h_*T, \xi, -a> \end{aligned}$$

exists for almost all $a \approx 0$ by Federer's slicing theorem 23, and is integral/normal for almost all $a \approx 0$ if S and T are integral/normal. In particular, the intersection $h_* S \cap h_* T_a$ exists for almost all $a \approx 0$ and can chosen to be integral/normal, if S, T are integral/normal. Finally, since $h_* T$ and $h_* T_a$ are homotopic, one has $T_a = T + dR_a$ in U. Indeed, if $H : [0,1] \times X \to X$ is a smooth homotopy from $H(0,x) = h(x)$ to $H(1,x) = \tau_a \circ h(x)$, then

$$dR_a := dH_*\big([0,1] \boxtimes T\big) = (\tau_a \circ h)_* T - h_* T - H_*\big([0,1] \boxtimes dT\big) = h_* T_a - h_* T.$$

The current R_a is also integral/normal, if T is. The result follows with $\tilde{T} = T_a$ and $R = R_a$ for some good a (both extended to currents on X by zero). $\square$

Theorem 34. *Let S, T be two normal currents on X where T is d-closed. Then there exist normal currents $\tilde{T}$, R such that*

- *$T = \tilde{T} + dR$.*
- *The intersection $S \cap \tilde{T}$ exists and is normal.*
- *If S, T are integral, then $\tilde{T}$ can be chosen such that $\tilde{T}$, R and $S \cap \tilde{T}$ are also integral.*

Proof. Choose a (finite) covering $\mathcal{U} = \{U_i\}$ of $\operatorname{spt} T$ by coordinate charts. Choose a subordinate partition of unity χ_i by characteristic functions of Borel sets[9]. The χ_i can be chosen so that the $T_i := \chi_i \cdot T$ are normal (integral). This gives a decomposition $T = \sum T_i$ into currents supported in the coordinate charts U_i.
Now use lemma 33 to get normal (resp. integral) currents $\tilde{T_i}$ such that $\tilde{T_i} = T_i + dR_i$ and such that the intersection $S \cap \tilde{T_i}$ exists and is normal (integral) as well. For $\tilde{T} = \sum \tilde{T_i}$ the intersection $S \cap \tilde{T} = \sum S \cap \tilde{T_i}$ exists and furthermore $\tilde{T} = \sum \tilde{T_i} = \sum T_i + dR_i = T + d(\sum R_i)$. The statement follows with $R = \sum R_i$, which is normal (integral). $\square$

Given two cohomology classes $\underline{S}$, $\underline{T}$, there always exist normal (integral, if $\underline{S}, \underline{T}$ are integral) currents S, T representing these classes such that $S \cap T$ exists. The intersection $\underline{S} \cap \underline{T}$ then is defined as the class represented by $S \cap T$, that is,

$$\underline{S} \cap \underline{T} = S \cap T \text{ mod boundaries.}$$

This definition is independent of the choice of S and T. Indeed, assume that $S = S' + dR$ and $T = T' + dQ$ for some normal (integral) currents S', R, T', Q such that $S' \cap T'$ exists. Using theorem 34, one finds $Q' = Q +$ boundary and $R' = R +$ boundary such that the intersections $S' \cap Q'$, $R' \cap T'$, $R' \cap dQ$ all exist. Then, by proposition 25,

$$\begin{aligned} d\big((-1)^{|S'|} S' \cap Q' + R' \cap T' + R' \cap dQ\big) &= S' \cap dQ' + dR \cap T' + dR \cap dQ \\ &= S' \cap (T - T') + (S - S') \cap T' + (S - S') \cap (T - T') \\ &= S \cap T - S' \cap T'. \end{aligned}$$

Remarks.

- It follows from lemma 30 (or 27) that the quasi-isomorphism $\mathcal{A}(X) \to \mathcal{D}(X)$ is a homomorphism of algebras. If one were interested only in the intersection on complex valued cohomology, one could use the induced isomorphism to define a product on $H^\bullet\mathcal{D}$. The here used definition also applies to integral currents.
- The local moving lemma 33 and thus also theorem 34 extend to intersections with a countable family of flat currents $S_1, S_2, \ldots$.
- It is $\Delta_*(\underline{S} \cap \underline{T}) = \underline{S \boxtimes T} \cap \underline{[\Delta]} = \big((S \boxtimes T) + dR\big) \cap [\Delta]$ modulo boundaries, for representatives S, T and some current P. Conversely, the intersection can be defined by this last expression (there exist S, T, P such that the intersection exists). For currents, this

[9]See the proof of theorem A.9 in [32].

definition is equivalent to the definition given above. Because of a much weaker moving lemma, the cohomological intersection of higher Chow cycles has been defined by the second condition.

2.9 Log currents

Following King [41], we extend the notion of currents on manifolds to pairs (X, D) where X is a compact complex manifold and D is a normal crossing divisor on X. This is done in such a way that there is a quasi-isomorphism $\mathcal{A}_X(\log D) \to \mathcal{D}_X(\log D)$.

Null-D forms and on-D currents

For $D \subset X$ a complex subvariety, denote by $j : D_{\text{reg}} \to X$ the inclusion of its manifold points. The sheaf of *smooth null-D-forms* is the subsheaf $\mathcal{A}_X(\text{null}D) \subset \mathcal{A}_X$ consisting of those smooth forms on X that vanish on D_{reg}. More precisely, its sections are

$$\mathcal{A}_X(\text{null}D)(U) = \{\omega \in \mathcal{A}_X(U) : j^*\omega = 0 \text{ on } D_{\text{reg}} \cap U\}.$$

This indeed is a subsheaf and even a bigraded subcomplex of sheaves in $\mathcal{A}_X$. The usual wedge product of forms gives $\mathcal{A}_X(\text{null}D)$ the structure of an $\mathcal{A}_X$-module. In particular, it is a fine sheaf.

The *sheaf of on-D currents* is defined as the subsheaf of $\mathcal{D}_X(onD) \subset \mathcal{D}_X$ that is obtained by imposing the condition that $T \in \mathcal{D}(onD)_x \iff T \wedge \omega = 0$ for all $\omega \in \mathcal{A}(\text{null}D)_x$. Its sections over some open U will be denoted by $\mathcal{D}(U, onD)$. Then a current T lies in $\mathcal{D}(U, onD)$ if and only if $T \wedge \omega = 0$ for all $\omega \in \Gamma_c(U, \mathcal{A}(\text{null}D))$. From this description it follows easily that the on-D currents actually form a bigraded subcomplex of $\mathcal{D}$.

Remarks.

- The inclusion induces $\mathcal{D}(D_{\text{reg}}) \xrightarrow{i_*} \mathcal{D}(X, onD)$, since $(i_*T)(\eta) = T(i^*\eta) = T(0) = 0$.
- Every on-D current has support in D, but the converse in general is not true (see [41, remark after 1.3.9]). It is true however when restricted to rectifiable currents (e.g. integral currents). This is an instance of a flatness theorem, see [40, Thm 2.1.8].

Log currents

Let X be a compact complex manifold and $D \subset X$ be a (reduced) normal crossing boundary divisor. The sheaf of log-D currents of bidegree (p, q) is the quotient sheaf

$$\mathcal{D}_X^{p,q}(\log D) = \mathcal{D}_X^{p,q}/\mathcal{D}_X^{p,q}(onD).$$

The total grading is obtained by summation over all possible bidegrees. Since the on-D currents form a subcomplex, the differential descends and one ends up with the complex of log currents

$$\mathcal{D}_X^\bullet(\log D) = \bigoplus_{p+q=\bullet} \mathcal{D}_X^{p,q}(\log D).$$

Thus two currents agree as log currents if and only if they take the same value on all null-D test forms. The principal value associates to every differential form with logarithmic poles along

D a current on X. This assignment in general is not compatible with the differential. After projection to log currents however, it yields a map of complexes of sheaves

$$\mathcal{A}_X(\log D) \to \mathcal{D}_X(\log D).$$

After localization (partition of unity), the divisor D can assumed to be of the form $(h = 0)$ for a holomorphic function h. Then the current represented by a log form ω is

$$[\omega](\eta) = \lim_{\epsilon \searrow 0} \int_{|h|>\epsilon} \omega \wedge \eta.$$

Wedge product

For an on-D current T, the wedge product with a smooth differential form ω is again an on-D current. Indeed, $(T \wedge \omega)\eta = T(\omega \wedge \eta) = 0$, since the wedge product of a smooth form with a null-D test form η is again a null-D form. In particular, the $\mathcal{A}_X$-module structure descends to a module structure on $\log D$ currents. More general, King showed that the logarithmic currents carry a module structure for the $\log D$ forms, that is, there is a morphism of complexes of sheaves compatible with the bigrading,

$$\mathcal{D}_X(\log D) \otimes_{\mathcal{A}_X^0} \mathcal{A}_X(\log D) \to \mathcal{D}_X(\log D).$$

The construction can be given locally, so that one can assume $D = (h = 0)$ and $\omega = h^{-1}\omega'$ with a smooth form ω'. Let the log current be represented by the current T. By the work of Schwartz[10] [56], there exists a current S such that $T = S \wedge h$. Then one sets

$$T \wedge \omega := S \wedge \omega'.$$

The current S above is not unique, but the ambiguity vanishes modulo on-D currents. If T is an on-D current, then the result is again an on-D current and therefore the wedge product well defined as a log current (for details, see King, Thm 1.3.11).
If ω is smooth, this wedge product is equal to the usual module structure.
For an arbitrary log form, the wedge product can be represented by the simple extension of the wedge product with the smooth part of ω:

Claim. *As a log current,*

$$T \wedge \omega = \lim_{\epsilon \searrow 0} T \wedge \omega\alpha_\epsilon$$

where α_ϵ is any sequence of smooth functions on X taking values in $[0, 1]$, such that

- *each α_ϵ vanishes on D_{reg},*
- *α_ϵ is an increasing sequence for $\epsilon \to 0$, converging to $\mathbb{1}_{X \setminus D_{reg}}$ for $\epsilon \to 0$.*

Proof. Indeed, the statement can be proven in the local situation. So let h be a local equation for D, choose S such that $T = Sh$ and let η be a null-D test form. Then one has $T \wedge \omega\alpha_\epsilon(\eta) = S(h\alpha_\epsilon\omega \wedge \eta)$. For $\epsilon \to 0$, the argument converges to the (since η is null-D) smooth form $\mathbb{1}_{X \setminus D_{\text{reg}}} h\omega\eta = h\omega\eta$. This convergence takes place in the topology of the space of smooth test forms and, because S is continuous, the limit is $S(h\omega\eta) = T \wedge \omega(\eta)$. $\square$

[10] While Schwartz proved this statement for h holomorphic, this has been extended by Hörmander [36] (to h polynomial) and Lojasiewicz [48] (to h real analytic)

An important special case is that where T is the current of integration over a real analytic chain Z. In this case the functions α_ϵ does not even need to be smooth and can chosen to be characteristic functions. Thus the wedge product with a log form is represented by the weak limit

$$[Z] \wedge \omega = \lim_{\epsilon \searrow 0} \int_{Z_\epsilon} \omega,$$

where the semianalytic set Z_ϵ is the intersection of Z with the set $|h| > \epsilon$. This can also be shortly expressed as

$$[Z] \wedge \omega = \int_{Z \setminus D} \omega,$$

where the integral has to be computed as the limit over the regular part of an exhaustion.

Cohomology of log currents

The cohomology of log currents has been considered by King [41]. The key to its computation is the following proposition.

Proposition 35 ([41, 1.3.11])**.** *The wedge product induces a canonical isomorphism of sheaves $\Omega^p_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{D}^{0,q}_X \to \mathcal{D}^{p,q}_X(\log D)$.*

Proposition 36 (Cohomology of log currents)**.** *If D is a normal crossing divisor, then all vertical maps in the following commutative diagram of filtered complexes of sheaves are filtered quasi-isomorphisms.*

$$\begin{tikzcd}
\Omega_X \arrow[r, hook] \arrow[d] & \mathcal{A}_X \arrow[r, hook] \arrow[d] & \mathcal{D}_X \arrow[d] \\
\Omega_X(\log D) \arrow[r, hook] & \mathcal{A}_X(\log D) \arrow[r, hook] & \mathcal{D}_X(\log D)
\end{tikzcd}$$

Proving the proposition boils down to showing that the morphisms between the associated graded, $\Omega^p(\log D) \to \mathcal{A}^{p,\bullet}(\log D) \to \mathcal{D}^{p,\bullet}(\log D)$ are quasi-isomorphisms. Using proposition 35, this sequence can be obtained by tensoring the sequence $\mathbb{C} \to \mathcal{A}^{0,\bullet} \to \mathcal{D}^{0,\bullet}$ with the sheaf of holomorphic log forms $\Omega^p(\log D)$ (which is locally free). This latter sequence consists of quasi-isomorphisms by the $\overline{\partial}$-lemma for forms and currents. A detailed proof can be found in [41, Thm 2.1.2].

Cohomology of normal log currents

The complex of *normal log D currents* $\mathcal{N}_X(\log D)$ is the filtered complex defined as the image of $\mathcal{N}_X$ under the projection to $\mathcal{D}_X(\log D)$:

$$F^p\mathcal{N}_X(\log D) := image\Big(F^p\mathcal{N}_X \to \mathcal{D}_X(\log D)\Big).$$

In analogy to relative singular chains, we denote its space of sections over some open U by $F^p\mathcal{N}(U, D)$ and call its members *relative normal currents* (for the pair (X, D)).
Already Federer/Fleming [19] showed that the complex $\mathcal{N}(X, D)$ can be used to compute the $\mathbb{C}$-valued relative singular homology groups of the pair (X, D). The above defined filtration corresponds to the Holdge filtration. Indeed, log forms are locally normal by [41, p. 43], so that the map $\mathcal{A}(\log D) \to \mathcal{D}(\log D)$ factors over $\mathcal{N}(\log D)$. We have

Claim. *For D a simple normal crossing divisor, the inclusions*

$$\mathcal{A}^\bullet_X(\log D) \to \mathcal{N}^\bullet_X(\log D) \to \mathcal{D}^\bullet_X(\log D)$$

are all filtered quasi-isomorphisms.

Proof. For D smooth, this is proven by King in [41, 4.4.8] by comparing two long exact sequences. We reduce to this case. Both arrows are morphisms of filtered complexes. It suffices to show that the first arrow is a quasi-isomorphism. For $D = \sum D_i$ with D_i smooth, set $D_I := \cap_{i \in I} D_i$ and consider

$$\begin{tikzcd}
\mathcal{A}(\log D) \arrow[r] & \mathcal{N}(\log D) \\
\prod \mathcal{A}(\log D_i) \arrow[r] \arrow[u] & \prod \mathcal{N}(\log D_i) \arrow[u] \\
\prod \mathcal{A}(\log D_{ij}) \arrow[r] \arrow[u] & \prod \mathcal{N}(\log D_{ij}) \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

All D_I are smooth (D is simple). The vertical arrows are the alternating inclusions. They form a resolution of $\mathcal{A}(\log D)$: The top one is obviously surjective, and for the others note that, for a sequence of differential log forms $(\omega_I)_I$ that lies in the kernel, the dlog terms must cancel. But the I are all pairwise distinct, hence not all possible dlog occur in ω_I. Thus they have poles along some D_J with $J \supset I$. Consequently,

$$H \operatorname{Tot} \Big(\mathcal{A}(\log D) \leftarrow \prod \mathcal{A}(\log D_i) \leftarrow \dots \Big) = H\mathcal{A}(\log D).$$

The same holds with currents instead of differential forms, because normal currents are just differential forms with measure coefficients. Formally, use proposition 35 to get the result for log currents and note that the exactness descends to the subcomplex of normal currents. $\square$

Cohomology of integral log currents

Define the complex of *relative integral currents* $\mathcal{I}^\bullet(X, D)$ to be the quotient of all integral currents on X modulo those currents which have support on D. By the flatness theorem, the integral currents with support in D are just the integral currents on D pushed forward along the inclusion. Hence the inclusion of integral currents into currents induces an inclusion of complexes

$$\mathcal{I}(X, D) \hookrightarrow \mathcal{D}(X, \log D).$$

If A denotes a coefficient ring, the relative integral currents with coefficients in A are obtained by linear extension, that is,

$$\begin{aligned}
\mathcal{I}(X, D, A) :=& \mathcal{I}(X, D) \otimes_{\mathbb{Z}} A \\
=& \mathcal{I}(X)/\mathcal{I}(D) \otimes_{\mathbb{Z}} A.
\end{aligned}$$

The most important examples of relative integral currents are currents of integration over relative singular chains. In fact, every relative integral current is homologous to a relativ singular cycle. This is a consequence of the following proposition.

Proposition 37 ([19, Thm 5.11], [18, 4.4.5])**.** *If A and B are compact local Lipschitz neighborhood retracts in $\mathbb{R}^n$ with $A \supset B$, then the inclusion of singular chains into integral currents induces an isomorphism from the singular homology groups of (A, B) with integer coefficients to the homology groups of the chain complex $(\mathcal{I}_\bullet(A, B), \partial)$.*

Corollary 38. *For X a compact complex manifold and D a real analytic subvariety, the singular homology of (X, D) with values in A can be computed as the cohomology of the complex $\mathcal{I}(X, D, A)$. That is, $H_n^{\mathrm{sing}}(X, D, A) = H_n\mathcal{I}_\bullet(X, D, A)$ and, by duality, $H^n(X \setminus D, A) = H^n\mathcal{I}^\bullet(X, D, A)$.*

Proof. By Whitney's embedding theorem (see [42]), X can be embedded as a smooth submanifold in some $\mathbb{R}^n$. By assumption, X and hence D are compact. Because every submanifold and every real analytic subvariety of a manifold are local Lipschitz neighborhood retracts [18, 3.1.20], the proposition 37 is applicable and says that the integral singular homology of (X, D) is computed by $\mathcal{I}(X, D)$. Passing to arbitrary coefficients A in both theories (singular chains/currents) is declared by tensoring the $\mathbb{Z}$-complexes with A, and so the quasi-isomorphism holds for any coefficient ring. □

Functoriality of log currents

The functoriality of currents descends to the relative situation if it can be ensured that *on*-D currents are preserved.

Lemma 39 (Pushforward)**.** *Let $f : X \to X'$ be a holomorphic map between compact complex manifolds and $D \subset X$, $D' \subset X'$ divisors. If $f(D) \subset D'$, then the push-forward, defined by the usual formula, induces a morphism of complexes*

$$f_* : \mathcal{D}(X, \log D) \to \mathcal{D}(X', \log D').$$

It preserves relative normal (resp. relative integral) log currents and is functorial in the sense that $g_* \circ f_* = (g \circ f)_*$.

Proof. Because X is compact, the proper pushforward of currents on X exists and the only thing to show is that the pushforward descends to log currents, i.e., maps on-D currents to on-D' currents. This is essentially proven in [41, 1.1.10 c)] as follows. It is to show that the pullback of a null-D' form ω along f is a null-D form. Since ω is a null-D' form, the pullback $f^*\omega$ restricts to zero on $f^{-1}(D'_{\mathrm{reg}}) \cap D_{\mathrm{reg}}$. That it vanishes also on $f^{-1}(D'_{\mathrm{sing}}) \cap D_{\mathrm{reg}}$ follows from the fact that tangent vectors on D'_{sing} are limits of tangent vectors of D'_{reg}. □

Lemma 40 (Pullback)**.** *Let $f : X \to X'$ be a surjective submersion between compact complex manifolds and $D' \subset X'$ a divisor. Define $D := f^{-1}D'$. Then the pullback of currents induces a morphism of complexes*

$$f^* : \mathcal{D}(X', \log D') \to \mathcal{D}(X, \log D).$$

The pullback preserves relative integral currents and is functorial in the sense that $g^* \circ f^* = (f \circ g)^*$.

Proof. It is to show that on-D' currents are mapped to on-D currents. Since f is locally a projection, one has $D_{\mathrm{reg}} = f^{-1}(D'_{\mathrm{reg}})$. If η is any null-D test form on X, then η vanishes on the fibers over all regular points of D' and therefore the pushforward $f_*\eta$ is a null-D' test form on X' (with compact support). In particular, for a given on-D' current T one has $f^*T(\eta) = T(f_*\eta) = 0$. Hence, f^*T is an on-D current.
If T is integral, then f^*T is known to be locally integral. Since X is compact, f^*T has compact support, hence is integral. □

Lemma 41 (Exterior product of log currents)**.** *Let (X, D) and (X', D') be two pairs consisting of a compact complex manifold and a normal crossing divisor on it. The exterior product for currents induces an exterior product of log currents*

$$\boxtimes : \mathcal{D}(X, \log D) \otimes_{\mathbb{C}} \mathcal{D}(X', \log D') \to \mathcal{D}(X \times X', \log(D \boxtimes D')).$$

Here $D \boxtimes D' = X \times D' + D \times X'$ denotes the exterior product of the two divisors. This association is compatible with bidegree and the differential d.

Proof. It suffices to show that the exterior product is well defined. Then the assertions about the bidegree and the differential are clear. Thus it is to show that if S is an on-D current or if T is an on-D' current, then the exterior product $S \boxtimes T$ is an on-$D \boxtimes D'$ current. To check this, assume there is given a null-$(D \boxtimes D')$ form on $X \times X'$, which can be chosen to be of the form $\eta \boxtimes \eta'$. This form vanishes when restricted to $(D \boxtimes D')_{\mathrm{reg}} = (D_{\mathrm{reg}} \times X') \cup (X \times D'_{\mathrm{reg}})$. Thus η must be a null-D form and η' must be a null-D' form. Consequently, $(S \boxtimes T) \wedge (\eta \boxtimes \eta') = \pm (S \wedge \eta) \boxtimes (T \wedge \eta') = 0$ whenever S or T is a null form. □

Since the exterior product of two locally integral currents is again a locally integral current, we get immediately

Corollary 42. *The exterior product of log currents restricts to relative integral currents and gives*

$$\mathcal{I}(X, D, A(p)) \otimes_A \mathcal{I}(X', D', A(q)) \to \mathcal{I}(X \times X', D \boxtimes D', A(p+q)).$$

The lemma 10 also carries over to log currents without problems.

Lemma 43 ($\boxtimes$ and $\wedge$)**.** *For T, T' log currents and ω, ω' log forms, one has*

$$(T \wedge \omega) \boxtimes (T' \wedge \omega') = (-1)^{|T'||\omega|} (T \boxtimes T') \wedge (\omega \boxtimes \omega').$$

Intersection of log currents

It follows from lemma 30 that if the intersection $S \cap T$ exists, then so does $(\omega \wedge S) \cap T$ for any smooth form ω, and one has

$$\omega \wedge (S \cap T) = (\omega \wedge S) \cap T.$$

In particular, this implies that whenever the intersection of a current with an on-D-current is defined, it is again an on-D-current. Thus, the intersection descends to a partially defined $\mathbb{C}$-linear map

$$\mathcal{D}(X, \log D) \otimes_{\mathbb{C}} \mathcal{D}(X, \log D) \dashrightarrow \mathcal{D}(X, \log D).$$

That means that the intersection of two log currents exists if and only if they can be represented by currents whose intersection exists.

The intersection also restricts to a partially defined $\mathbb{Z}$-linear map of relative integral currents. Again, the intersection is everywhere well-defined on cohomology, as follows from the compatibility of the intersection product with the differential. To see that it is everywhere defined, choose two representatives S, T which are normal/integral. By the very theorem for usual currents, there exists $\tilde{T} = T + dR$ such that the intersection of $S, \tilde{T}$ exist. Considered as a log current, $\tilde{T}$ lies in the same cohomology class as T and so the intersection exists.
From this, all the properties of the intersection of currents carry over to the intersection of log currents. For example, the inverse mapping formula, the reduction to the diagonal, or the compatibility with $\boxtimes$.

3 Deligne-Beilinson cohomology

Deligne cohomology and Deligne-Beilinson cohomology are two bi-graded cohomology theories for algebraic varieties over $\mathbb{C}$ (or $\mathbb{R}$). Deligne cohomology was first introduced by Deligne as a cohomology theory that contains information about both the Hodge filtration and the integral structure of the cohomology of an algebraic manifold. An account on that theory can be found in [15] (Deligne did not publish anything himself).

For non-compact manifolds, Deligne cohomology does not give the "correct" cohomology groups. Beilinson [2] solved this issue by imposing growth conditions at the boundaries. This version is usually called Deligne-Beilinson cohomology. Beilinson also defined the dual theory – Deligne-Beilinson homology – and showed that the two are isomorphic to each other (Poincaré duality for smooth varieties). Details can be found in Jannsen's article [37].

Here we consider only smooth algebraic varieties, and indeed use the homological version to define Deligne(-Beilinson) cohomology. This has two advantages: First, the homology theory can be computed by very concrete complexes (of currents), while the cohomology theory is defined by complexes of sheaves. Second, the homological complexes have good covariant functorial properties (the same as higher Chow groups).

We start with the sheaf theoretic definition of Deligne cohomology and motivate the homological definition in term of currents. After that, we define Deligne-Beilinson cohomology using currents in the spirit of Jannsen [37], but use relative integral currents instead of relative singular chains (and with a cohomological Tate twist). Then we introduce the path complex and show that it can be used to compute rational Deligne-Beilinson cohomology. We explain the functoriality, the cycle maps, and the products on these two kinds of complexes.

In view of the to-be-defined regulator map, we summarize the properties of these complexes that are needed for the construction later in 4.1.

We close with a short appendix on total complexes.

3.1 Deligne cohomology

Let X be an m-dimensional compact complex manifold, and $A \subset \mathbb{R}$ be a coefficient ring. For any $p \in \mathbb{Z}$, denote by $A(p) := (2\pi i)^p A$ the p-th twist. Then the weight p Deligne cohomology of X with coefficients in $A(p)$ was originally defined (see [16]) as the hypercohomology of the complex of sheaves

$$A_X(p) \xrightarrow{\iota} \mathcal{O}_X \to \Omega^1_X \to \ldots \to \Omega^{p-1}_X$$

with Ω_X the sheaf of holomorphic differential forms on X and the unlabelled arrows being the usual differential. Equivalently, Deligne cohomology can be defined as the hypercohomology of the total complex of sheaves on X

$$\operatorname{Tot}\left(A_X(p) \oplus F^p\Omega^\bullet_X \to \Omega^\bullet_X\right)$$

where the unlabelled morphism is the difference of the two canonical inclusions. A quasi-isomorphism from this complex to the original complex is given by dividing out the inclusion $F^p\Omega^\bullet \to \Omega^\bullet$. This total complex (sometimes written as Cone(...)[−1]) is the starting point for other definitions of Deligne cohomology. Namely, one knows that the inclusions of holomorphic forms into smooth forms and of smooth forms into currents are filtered quasi-isomorphisms (see theorem 13)

$$(\Omega_X^\bullet, F^p) \to (\mathcal{A}_X^\bullet, F^p) \to (\mathcal{D}_X^\bullet, F^p).$$

Consequently, the (complex of) sheaves Ω in the total complex can be replaced by $\mathcal{A}$ or $\mathcal{D}$. Because the latter complexes are big enough to admit partitions of unity, they are acyclic with respect to the global sections functor and hence taking hypercohomology is reduced to taking cohomology of their global sections. When using currents, one moreover has a replacement for the constant sheaf $A_X(p)$: The sheaf of (locally) integral currents with coefficients in $A(p)$ is a soft sheaf and thus for X compact Γ-acyclic.[1] The inclusion of locally constant functions into integral currents is a quasi-isomorphism and hence gives rise to a Γ-acyclic resolution

$$A_X(p) \to \mathcal{I}_X \otimes_{\mathbb{Z}} A(p).$$

Replacing forms by currents is compatible with the defining map $A_X(p) \oplus F^p\Omega \to \Omega$ of Deligne cohomology in the sense that it gives a compatible resolution of the total complex of sheaves above. Taking global sections, one obtains the complex of A-modules

$$\begin{aligned} C_{\mathcal{D}}^\bullet(X, A(p)) :=& \operatorname{Tot}\left\{\mathcal{I}^\bullet(X, A(p)) \oplus F^p\mathcal{D}^\bullet(X) \xhookrightarrow{-\delta+\iota} \mathcal{D}^\bullet(X)\right\} \\ =& \mathcal{I}^\bullet(X, A(p)) \oplus F^p\mathcal{D}^\bullet(X) \oplus \mathcal{D}^{\bullet-1}(X) \end{aligned} \tag{3.1}$$

with differential

$$(a, b, c) \mapsto (da, db, b - a - dc).$$

So the Deligne cohomology groups can equivalently be defined as

$$H_{\mathcal{D}}^l(X, A(p)) := H^l(C_{\mathcal{D}}^\bullet(X, A(p))).$$

Remark. Some people use $A(p - m)$ as coefficient ring for the integral currents instead $A(p)$, thereby underlining the fact that the Deligne 3-term complex actually computes Deligne homology, and so is dual to the cohomological complex. We however stick to this notation and indeed think of p as something that keeps track of the complex codimension.

3.2 Deligne-Beilinson cohomology

Consider now a complex algebraic manifold U. By the works of Nagata and Hironaka, one can always find a *good compactification* $X \supset U$. That is, X is a compact algebraic manifold containing U such that $D = X \setminus U$ is a normal crossing divisor, i.e., at each point exist local coordinates $h_1, \ldots, h_m$ such that D is given by an equation $h_1 \cdots h_k = 0$ (for some $1 < k < m$). Thus one can equivalently study pairs (X, D) where X is a compact algebraic manifold and $D \subset X$ is a normal crossing divisor.
Beilinson [2] extended Deligne cohomology to such cases by replacing the complexes in definition

[1] See the section about integral currents 2.5.

(3.1) by their log versions. More precisely, the *Deligne-Beilinson complex of* (X, D) is defined to be the complex of A-modules

$$C^\bullet_{\mathcal{D}}(X, D, A(p)) := \operatorname{Tot}\Big(\mathcal{I}^\bullet(X, D, A(p)) \oplus F^p\mathcal{D}^\bullet(X, \log D) \to \mathcal{D}^\bullet(X, \log D)\Big).$$

This definition does not depend on the choice of the compactification, as will follow from the long exact sequence for Deligne-Beilinson cohomology (see paragraph below, or [37, 1.13]). For this reason, the compactification is sometimes omitted from the notation of the Deligne-Beilinson cohomology:

$$\begin{aligned} H^l_{\mathcal{D}}(U, A(p)) &:= H^l_{\mathcal{D}}(X, D, A(p)) \\ &:= H^l(C^\bullet_{\mathcal{D}}(X, D, A(p))). \end{aligned}$$

If $U = X$ is compact, one can choose $D = \emptyset$ and Deligne-Beilinson cohomology and Deligne cohomology agree. Even more, the complexes $C_{\mathcal{D}}(X, A(p)) = C_{\mathcal{D}}(X, \emptyset, A(p))$ are the same for X compact.

Note that, as a consequence of the fact that the inclusion of relative normal currents into log currents is a quasi-isomorphism, the Deligne-Beilinson cohomology can also be computed by the complex $\operatorname{Tot}\big(\mathcal{I}(X, D, A(p)) \oplus F^p\mathcal{N}(X, D) \to \mathcal{N}(X, D)\big)$. In other words, any cohomology class can be represented by triples (a, b, c) with a relative integral and b, c relative normal.

Remark 1. The definition of Deligne cohomology also makes sense for non-compact complex manifolds and then the restriction gives a natural map $C_{\mathcal{D}}(X, D, A(p)) \to C_{\mathcal{D}}(U, A(p))$. This map is a quasi-isomorphism whenever $U = X$ is compact, but in general not for $D \neq \emptyset$. In order to avoid notational confusion, here we consider Deligne cohomology for compact manifolds only.

The long exact sequence for Deligne-Beilinson cohomology

The most important property of Deligne-Beilinson cohomology is the existence of a long exact sequence that relates Deligne-Beilinson cohomology to singular resp. de Rham cohomology. The desire for such a long exact sequence also explains why the Deligne-Beilinson complex is defined as a total complex. In fact, total complexes are tools to generate long exact sequences on cohomology. So the complexes $C_{\mathcal{D}}$ give rise to long (not exact) sequences of complexes

$$\ldots \to C_{\mathcal{D}} \to \mathcal{I}(X, D, A(p)) \oplus F^p\mathcal{D}(X, \log D) \to \mathcal{D}(X, \log D) \to C_{\mathcal{D}}[1] \to \ldots$$

which become exact when passing to cohomology groups. Note that by proposition 36 the cohomology groups of the relative (integral) currents compute the Hodge structure on the cohomology of the pair (X, D) and thus the Hodge structure on U. That is, the induced long exact sequence on cohomology groups can be written as

$$\ldots \to H^\bullet_{\mathcal{D}}(X, D, A(p)) \to H^\bullet(U, A(p)) \oplus F^pH^\bullet(U, \mathbb{C}) \to H^\bullet(U, \mathbb{C}) \to H^\bullet_{\mathcal{D}}(X, D, A(p))[1] \to \ldots \tag{3.2}$$

Thus the long exact sequence relates Deligne-Beilinson cohomology with singular cohomology (Borel-Moore homology) and de Rham cohomology – and this even on the level of complexes. Compare also Jannsen [37, Cor. 1.13 b)].

Deligne-Beilinson cohomology and Jacobians

The long exact sequence (3.2) splits into short exact sequences, thereby revealing the Deligne-Beilinson cohomology groups as extensions of the integral classes of F^p-type,

$$0 \to J^{p,n}(U) \to H^{2p-n}_{\mathcal{D}}(X, D, \mathbb{Z}(p)) \to H^{2p-n}(U, \mathbb{Z}(p)) \cap F^p \to 0.$$

Here the leftmost term in the sequence is the p-th intermediate Jacobian of the mixed Hodge structure on $H^{2p-n-1}(U, \mathbb{C})$,

$$J^{p,n}(U) := \frac{H^{2p-n-1}(U, \mathbb{C})}{F^p H^{2p-n-1}(U, \mathbb{C}) + H^{2p-n-1}(U, \mathbb{Z}(p))}.$$

If $U = X$ is projective, then this is a generalized complex torus, that is, a complex vector space modulo an $\mathbb{R}$-linearly independent discrete subgroup.
Indeed, consider the composition of the canonical inclusion with the projection,

$$H^{2p-n-1}(X, \mathbb{R}(p)) \to \frac{H^{2p-n-1}(X, \mathbb{C})}{F^p H^{2p-n-1}(X, \mathbb{C})}.$$

The image of any class v under this map (also denoted by v) satisfies either $\overline{v} = v$ or $\overline{v} = -v$, depending on p. Thus any element in the kernel lies also $F^p \cap \overline{F^p} = 0$, so that the map is an injective map of real vector spaces. Because the $\mathbb{Z}(p)$-valued classes span a lattice in $H^{2p-n-1}(X, \mathbb{R}(p))$, they also form an $\mathbb{R}$-linear independent discrete subgroup in the right hand side and thus the quotient $J^{p,n}(X)$ is a generalized complex torus.
For $n = 0$, the Hodge decomposition implies that the subspace spanned by the integral classes has maximal rank in $H^{2p-1}(X, \mathbb{C})/F^p H^{2p-1}(X, \mathbb{C})$ and hence the Jacobian $J^{p,0}$ is a complex torus.

3.3 The path complex for Deligne-Beilinson cohomology

We now define complexes $P_{\mathcal{D}}(X, D, A(p))$ that are built up from paths in $\mathcal{D}(X, \log D)$ and that for $\mathbb{Q} \subset A$ compute the Deligne-Beilinson cohomology of the pair (X, D).
Denote by $\Lambda_A(x) := A[x, dx]$ the differential graded algebra of A-valued polynomial forms on the 1-simplex, that is, the free differential graded algebra over A generated by the variable x (in degree 0). The *path complex for Deligne-Beilinson cohomology* is the subcomplex of $\Lambda_A(x) \otimes_A \mathcal{D}(X, \log D)$ defined by

$$P_{\mathcal{D}}(X, D, A(p)) := \left\{ P \in \Lambda_A(x) \otimes_A \mathcal{D}(X, \log D) \text{ such that } \begin{array}{l} P_0 \in \mathcal{I}(X, D, A(p)), \\ P_1 \in F^p \mathcal{D}(X, \log D) \end{array} \right\}.$$

Here P_0, P_1 denote the images of P under the evaluation maps at 0 and 1 respectively. The evaluation at ϵ is the unique map that sends dx to zero and x to ϵ. For example, if $\omega(x) = a(x) + b(x)dx$ is an element in $\Lambda_A(x)$, then the evaluation at ϵ of an element $P = \omega(x) \otimes T$ is just $P_\epsilon = \omega(\epsilon)T = a(\epsilon)T$.
It follows from $(dP)_\epsilon = d(P_\epsilon)$ that $P_{\mathcal{D}}$ is indeed a subcomplex, that is, closed under the differential d of the tensor product. It moreover has an obvious A-module structure and we will see that it inherits a partially defined intersection product from $\Lambda_A(x) \otimes \mathcal{D}(X, \log D)$.

Remark 2. This complex can be obtained by applying the general procedure of Hinich/Schechtman [35, Theorem 4.1] of Thom-Sullivan cochains to the diagram $\mathcal{I}(X, D, A(p)) \oplus F^p\mathcal{D}(X, \log D) \to \mathcal{D}(X, \log D)$ (considered as a simplicial object). The quasi-isomorphism stated therein specializes to the comparison-isomorphism ev below.

Comparison to the 3-term-complex

The relation between the complexes $P_{\mathcal{D}}$ and the total complex $C_{\mathcal{D}}$ is given by means of the morphisms of A-modules

$$\begin{aligned} ev : P^n_{\mathcal{D}}(X, D, A(p)) &\to C^n_{\mathcal{D}}(X, D, A(p)) \\ \omega \otimes T &\mapsto \left(\omega(0)T\,,\, \omega(1)T\,,\, \int_0^1 \omega T\right). \end{aligned}$$

These maps are compatible with the differential and hence give rise to a morphism ev of complexes.

Lemma 44. *If $\mathbb{Q} \subset A$, then the morphism ev is a quasi-isomorphism. A quasi-inverse is induced by the maps*

$$\begin{aligned} s : C^n_{\mathcal{D}}(X, D, A(p)) &\to P^n_{\mathcal{D}}(X, D, A(p)) \\ (a, b, c) &\mapsto (1 - x) \otimes a + x \otimes b + dx \otimes c. \end{aligned}$$

Proof. s is indeed compatible with differentials, i.e., gives a map of complexes. It is obvious that s splits the map ev, that is, $ev \circ s = \mathrm{id}$. It suffices to show that the map $s \circ ev : P_{\mathcal{D}} \to P_{\mathcal{D}}$ is homotopic to the identity. Such a homotopy was given by Burgos Gil during a summer school in Freiburg 2013. Define the homotopy $h : P_{\mathcal{D}} \to P_{\mathcal{D}}[-1]$ by

$$h(\omega \otimes T) = x \int_{[0,1]} \omega \otimes T - \int_{[0,x]} \omega \otimes T.$$

This map is well defined because of $\mathbb{Q} \subset A$. One checks that $dh + hd = s \circ ev - \mathrm{id}$. In particular, $s \circ ev = \mathrm{id}$ on cohomology and so ev is a quasi-isomorphism. $\square$

3.4 Examples

- If X is a point, then the complexes for the Deligne (=Deligne-Beilinson) cohomology become relatively simple. Since the space of currents over a point is isomorphic to $\mathbb{C}$ (and a number $\lambda \in \mathbb{C}$ is identified with the integration current $\lambda[\mathrm{pt}]$), one obtains that the path complex is

$$P_{\mathcal{D}}(\mathrm{pt}, A(p)) = \left\{\omega \in \Lambda_{\mathbb{C}}(x) \,:\, \omega(0) \in A(p) \text{ and, if } p > 0,\, \omega(1) = 0\right\}.$$

This complex is concentrated in degree 0 and 1, as is the 3-term complex, which is

$$\begin{aligned} C_{\mathcal{D}}(\mathrm{pt}, A(p)) &= \mathrm{Tot}\left(A(p) \oplus F^p\mathbb{C} \xrightarrow{-\delta+\iota} \mathbb{C}\right) \\ &= \left(A(p) \oplus F^p\mathbb{C} \xrightarrow{(a,z)\mapsto z-a} \mathbb{C}\right). \end{aligned}$$

Note that $F^0\mathbb{C} = \mathbb{C}$ and $F^p\mathbb{C} = 0$ for $p > 0$.

The evaluation map $P_{\mathcal{D}} \to C_{\mathcal{D}}$ sends an element $\omega = f(x) + g(x)dx$ to the triple $\big(f(0), f(1), \int_0^1 g(x)dx\big)$.

The Deligne cohomology can now easily seen to be

$$H^l_{\mathcal{D}}(\mathrm{pt}, A(p)) = \begin{cases} A, & p = 0, l = 0 \\ \mathbb{C}/A(p), & p > 0, l = 1 \\ 0, & \text{else,} \end{cases}$$

as follows also from the sheaf-theoretic description of Deligne cohomology.

To conclude, there exists a sequence of quasi-isomorphisms

$$P_{\mathcal{D}}(\mathrm{pt}, A(p)) \xrightarrow{ev} C_{\mathcal{D}}(\mathrm{pt}, A(p)) \to \begin{cases} (A \to 0), & p = 0 \\ (0 \to \mathbb{C}/A(p)), & p > 0 \end{cases}$$

where the second arrow is induced by the projection to the first component resp. the third component (in the second case composed with the quotient map).

- For $p = 0$, Deligne- and Deligne-Beilinson cohomology are both equal to the Borel-Moore homology of U, that is, the singular cohomology of U.

- For $p = 1$, the Deligne complex of X is quasi-isomorphic to $\mathcal{O}^*_X[-1]$ by means of the short exact sequence $0 \to \mathbb{Z}_X(1) \to \mathcal{O}_X \to \mathcal{O}^*_X \to 0$. The Deligne-Beilinson complex of U (or, (X, D)) however is quasi-isomorphic to $\mathcal{O}^*_{U,\mathrm{alg}}[-1]$, where $\mathcal{O}^*_{U,\mathrm{alg}}$ denotes the sheaf of invertible regular functions on U [16, 2.12 iii)].

- For $p = 2$, the Deligne complex of X is quasi-isomorphic to $(\mathcal{O}^*_X \xrightarrow{\mathrm{dlog}} \Omega^1_X)[-1]$ by means of the commuting diagram

$$\begin{tikzcd} \mathbb{Z}(2) \arrow[r, hook] & \mathcal{O}_X \arrow[r, "d"] \arrow[d, "\exp\left((2\pi i)^{-1} z\right)"'] & \Omega^1_X \arrow[d, "(2\pi i)^{-1} z"] \\ & \mathcal{O}^*_X \arrow[r, "\mathrm{dlog}"] & \Omega^1_X \end{tikzcd}$$

  that gives rise to a quasi-isomorphism between its rows.

- Because $\mathbb{A}_n$ has the same cohomology groups as a point, the long exact sequence of Deligne-Beilinson cohomology shows that $H_{\mathcal{D}}(\mathbb{P}_n, \mathbb{P}_n \setminus \mathbb{A}_n, \mathbb{Z}(p)) = H_{\mathcal{D}}(\mathrm{pt}, \mathbb{Z}(p))$ for all p. In fact, this is an instance of the homotopy invariance of Deligne-Beilinson cohomology.

The second example states that Deligne-Beilinson cohomology is an enhancement of singular cohomology, whereas the first example shows that it carries strictly more information than singular cohomology. The third example indicates that the Deligne and the Deligne-Beilinson cohomology for non-compact manifolds in general are not the same.

3.5 Functoriality

One reason for using currents to define Deligne-Beilinson cohomology lies in their better functoriality under pushforwards. While the pushforward of forms is only defined along submersions (which have to be proper when restricted to the support of the considered form), the pushforward of currents is defined for any proper morphism. The functoriality of Deligne-Beilinson cohomology is summarized in

Proposition 45. *Let $f : U \to U'$ be a holomorphic map between complex manifolds.*

- *If f is proper, then the pushforward of currents induces a map*

$$f_* : H^l_{\mathcal{D}}(U, A(p)) \to H^{l+2\delta}_{\mathcal{D}}(U', A(p+\delta))$$

  where $\delta = \dim_{\mathbb{C}} U' - \dim_{\mathbb{C}} U$.

- *If f is a submersion, then the pullback of currents induces a map*

$$f^* : H^l_{\mathcal{D}}(U', A(p)) \to H^l_{\mathcal{D}}(U, A(p)).$$

We show more general that these statements hold for the two complexes $P_{\mathcal{D}}$ and $C_{\mathcal{D}}$. Since they are defined for pairs of spaces, we consider only *maps of pairs*, that is, mappings that restrict to a map between the distinguished subsets.

Theorem 46 (Pushforward)**.** *Let $f : (X, D) \to (X', D')$ be a holomorphic map of pairs. Then there is a commutative diagram of complexes*

$$\begin{tikzcd}
f_* := (2\pi i)^\delta \cdot \mathrm{id} \otimes f_* : \ P^\bullet_{\mathcal{D}}(X, D, A(p)) \arrow[r] \arrow[d, "ev"] & P^{\bullet+2\delta}_{\mathcal{D}}(X', D', A(p+\delta)) \arrow[d, "ev"] \\
f_* := (2\pi i)^\delta \cdot (f_*, f_*, f_*) : \ C^\bullet_{\mathcal{D}}(X, D, A(p)) \arrow[r] & C^{\bullet+2\delta}_{\mathcal{D}}(X', D', A(p+\delta))
\end{tikzcd}$$

where $\delta = \dim_{\mathbb{C}} X' - \dim_{\mathbb{C}} X$. The pushforward is compatible with composition of functions in the sense that $g_* \circ f_* = (g \circ f)_*$.

Proof. The pushforward of currents (defined, because X is compact) maps $\mathcal{I}^\bullet$ to $\mathcal{I}^{\bullet+2\delta}$ and F^p to $F^{p+\delta}$. The multiplication with $(2\pi i)^\delta$ ensures that the image has the correct twist. Thus the two vertical maps are well defined. Commutativity of the diagram follows because the pushforward acts on the underlying currents, while the evaluation map acts on the coefficient forms:

$$\begin{tikzcd}
\omega \otimes T \arrow[r, mapsto, "\mathrm{id} \otimes f_*"] \arrow[d, mapsto, "ev"] & \omega \otimes f_* T \arrow[d, mapsto, "ev"] \\
(\omega_0 T, \omega_1 T, \int_0^1 \omega T) \arrow[r, mapsto, "(f_*, f_*, f_*)"] & (\omega_0 f_* T, \omega_1 f_* T, \int_0^1 \omega f_* T)
\end{tikzcd}$$

Compatibility with composition of functions follows from the respective property of the pushforward between currents. To see that the pushforward is compatible with the differential d on $P_{\mathcal{D}}$, let ω be homogeneous of dx-degree $|\omega|$ and $\pm = (-1)^{|\omega|}$. Then

$$\begin{aligned}
f_* d(\omega \otimes T) &= f_*(d\omega \otimes T \pm \omega \otimes dT) \\
&= (2\pi i)^\delta \cdot \Big(d\omega \otimes f_* T \pm \omega \otimes f_* dT\Big) \\
&= (2\pi i)^\delta \cdot \Big(d\omega \otimes f_* T \pm \omega \otimes d f_* T\Big) \\
&= d(\omega \otimes f_* T),
\end{aligned}$$

where in the third equality it is used that f_* and d commute on complex manifolds (see 2.2). The proof that pushforward is compatible with the differential in $C_{\mathcal{D}}$ is similar, but easier and omitted. $\square$

Theorem 47 (Pullback)**.** *Let $f : (X', f^*D) \to (X, D)$ be a holomorphic submersion. Then the pullback of currents induces a commutative diagram of complexes*

$$\begin{tikzcd} f^* = \mathrm{id} \otimes f^* : \, P^\bullet_{\mathcal{D}}(X, D, A(p)) \arrow[r] \arrow[d, "ev"] & P^\bullet_{\mathcal{D}}(X', f^*D, A(p)) \arrow[d, "ev"] \\ f^* = (f^*, f^*, f^*) : \, C^\bullet_{\mathcal{D}}(X, D, A(p)) \arrow[r] & C^\bullet_{\mathcal{D}}(X', f^*D, A(p)). \end{tikzcd}$$

The pullback along submersions is compatible with composition in the sense that $f^* \circ g^* = (g \circ f)^*$.

Proof. The proof is similar to the proof of theorem 46 and follows from the properties of the pullback for log currents in lemma 40. $\square$

Remark 3. Rephrasing the two theorems, one could say that ev is a natural transformation $P_{\mathcal{D}} \to C_{\mathcal{D}}$ of functors with values in A-complexes when considered on both the category with morphisms the smooth submersions and the proper maps. In more fancy words one might say that ev is a natural transformation of twisted Poincaré duality complexes.

3.6 Cycle maps

One central aspect of Deligne-Beilinson cohomology is the existence of a cycle class map for all smooth algebraic manifolds. The cycle class in Deligne-Beilinson cohomology can be constructed by means of the long exact sequence of Deligne-Beilinson cohomology and the cycle class map in singular and de Rham cohomology, see [15, 2.2], [16, §7]. This construction can be made on the level of complexes and produces concrete fundamental cycles (and not only cycle classes).

Deligne-Beilinson fundamental cycle

Let (X, D) be a pair of a compact complex algebraic manifold X and a normal crossing divisor D. Write $U := X \setminus D$.

If Z is an algebraic cycle in U of codimension p, then denote by $[Z]$ (the simple extension of) the integration current associated to Z. That is, $[Z]$ is given by integration over the manifold points Z_{reg}. The support of $[Z]$ is the closure of Z in X, hence compact. Thus the current $[Z]$ is a d-closed integral current of codimension (p, p) on X.

The *Deligne-Belinson fundamental cycle in $P_{\mathcal{D}}$* associated to Z is the element

$$(2\pi i)^p [Z] \in P^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$$

which is just the fundamental current $\mathrm{cl}(Z)$ associated to Z with constant coefficient 1.

The cycle map in $P_{\mathcal{D}}$ induces a cycle map in the complex $C_{\mathcal{D}}$ by composing it with the evaluation map. The resulting *Deligne-Beilinson fundamental cycle in $C_{\mathcal{D}}$* associated to Z is then

$$(2\pi i)^p \left([Z], [Z], 0\right) \in C^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

Both fundamental cycles will be denoted by $\mathrm{cl}(Z)$. Note that it follows from $d[Z] = 0$ that the two fundamental cycles are indeed cycles, that is, have vanishing (total) differential.
If $D' \subset X$ is a normal crossing divisor, then we define the *Deligne-Beilinson fundamental cycle relative to* D' to be the image of the above defined fundamental cycle under the projection $P_{\mathcal{D}}(X, \mathbb{Z}(p)) \to P_{\mathcal{D}}(X, D', \mathbb{Z}(p))$, and similarly for $C_{\mathcal{D}}$. In other words, we consider $\mathrm{cl}(Z)$ up to on-D' currents.

Lemma 48. *The cycle maps in $P_{\mathcal{D}}$ and $C_{\mathcal{D}}$ are related by the commutative diagrams*

$$\begin{tikzcd}
 & P^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \arrow[d, "ev"] \\
z^p(U) \arrow[ur, "\mathrm{cl}"] \arrow[r, "\mathrm{cl}"'] & C^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)).
\end{tikzcd}$$

They are functorial in X with respect to proper pushforward and pullback along submersions in the following sence: If $f : X \to X'$ is a smooth algebraic map such that $f|_U$ is a (flat resp. proper) map $U \to U'$, then

$$\mathrm{cl} \circ f|_U^* = f^* \circ \mathrm{cl} \quad \textit{resp} \quad \mathrm{cl} \circ (f|_U)_* = f_* \circ \mathrm{cl}.$$

Proof. Commutativity of the diagram is obvious (by definition). It suffices to prove the functorialities for the complexes $P_{\mathcal{D}}$ only – the result for $C_{\mathcal{D}}$ then follows from this and the functoriality of ev. So it is to show that under the respective conditions on f, the corresponding diagram below commutes.

$$\begin{tikzcd}
z^p(U') \arrow[r, "\mathrm{cl}"] \arrow[d, "f^*"] & P^{2p}_{\mathcal{D}}(X', \mathbb{Z}(p)) \arrow[d, "f^*"] \\
z^p(U) \arrow[r, "\mathrm{cl}"] & P^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))
\end{tikzcd}
\quad \text{resp} \quad
\begin{tikzcd}
z^p(U) \arrow[r, "\mathrm{cl}"] \arrow[d, "f_*"] & P^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \arrow[d, "f_*"] \\
z^{p+\delta}(U') \arrow[r, "\mathrm{cl}"] & P^{2p+2\delta}_{\mathcal{D}}(X', \mathbb{Z}(p+\delta))
\end{tikzcd}$$

This follows from the compatility of the cycle map into currents with pushforward (lemma 20) and pullback (lemma 21). ☐

Remark 4.

- The composition of the cycle map on X with the projection to the Deligne-Beilinson complex of (X, D) induces a map

  $$z^p(X)/i_* z^{p-1}(D) \to P^{2p}_{\mathcal{D}}(X, D, \mathbb{Z}(p)).$$

  This is well defined because every algebraic cycle on D (that is, any cycle in $i_* z^{p-1}(D)$, i the inclusion of D) under the cycle map is send to an on-D current.

  This map extends the cycle map on U in the sense that the diagram below commutes.

  $$\begin{tikzcd}
  z^p(U) \arrow[r, "\mathrm{cl}"] \arrow[d, "\mathrm{closure}"'] & P^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \arrow[d] \\
  z^p(X)/i_* z^{p-1}(D) \arrow[r, "\mathrm{cl}"'] & P^{2p}_{\mathcal{D}}(X, D, \mathbb{Z}(p))
  \end{tikzcd}$$

- Note that if the compactness assumption of X is dropped, the definition of the Deligne-Beilinson cycle still makes sense as a *locally* integral cycle. This yields the *Deligne fundamental cycle.*

3.7 Exterior products

The cohomological exterior product $\boxtimes$ of currents induces exterior products on the two Deligne complexes, and in particular on Deligne-Beilinson cohomology.

The exterior products on $C_{\mathcal{D}}$

Alexander Beilinson introduced in his notes on absolute Hodge cohomology [3, 1.11] a whole family of products, depending on a real parameter $\alpha \in \mathbb{R}$. They are denoted by

$$\boxtimes_\alpha : C^r_{\mathcal{D}}(X, D, A(p)) \otimes_A C^s_{\mathcal{D}}(X', D', A(q)) \to C^{r+s}_{\mathcal{D}}(X \times X', D \boxtimes D', A(p+q)),$$

and defined by the formula

$$(a, b, c) \otimes (\tilde{a}, \tilde{b}, \tilde{c}) \mapsto \Big(a \boxtimes \tilde{a}, b \boxtimes \tilde{b}, \alpha c \boxtimes \tilde{a} + (-1)^r (1-\alpha) a \boxtimes \tilde{c} + (1-\alpha) c \boxtimes \tilde{b} + (-1)^r \alpha b \boxtimes \tilde{c}\Big).$$

A common way to write the product is in the form of a table. If r denotes the total degree of the triple (a, b, c), then the product with $(\tilde{a}, \tilde{b}, \tilde{c})$ is expressed by

	$\tilde{a}$	$\tilde{b}$	$\tilde{c}$
a	$a \boxtimes \tilde{a}$	0	$(-1)^r(1-\alpha) \cdot a \boxtimes \tilde{c}$
b	0	$b \boxtimes \tilde{b}$	$(-1)^r \alpha \cdot b \boxtimes \tilde{c}$
c	$\alpha \cdot c \boxtimes \tilde{a}$	$(1-\alpha) \cdot c \boxtimes \tilde{b}$	0

where one has to take care of the spaces the elements live in.

Proposition 49. *The products $\boxtimes_\alpha$ are compatible with the differential. For $\alpha = 0$ and $\alpha = 1$ it is associative, for $\alpha = \frac{1}{2}$ graded commutative. All products are homotopic to each other and, in particular, induce the same product on cohomology.*

Proof. See Beilinson, [3, Lemma 1.11]. □

The exterior product on $P_{\mathcal{D}}$

For the complex $P_{\mathcal{D}}$ there is a canonical choice of an exterior product coming from the product structures on the underlying tensor factors. It is given by

$$\begin{aligned}\boxtimes : P^r_{\mathcal{D}}(X, D, A(p)) \otimes_A P^s_{\mathcal{D}}(X', D', A(q)) &\to P^{r+s}_{\mathcal{D}}(X \times X', D \boxtimes D', A(p+q)) \\ (\omega \otimes S) \otimes (\eta \otimes T) &\mapsto (-1)^{|\eta||S|} \omega \wedge \eta \otimes (S \boxtimes T).\end{aligned}$$

Proposition 50. *The exterior product on $P_{\mathcal{D}}$ is associative, graded-commutative and compatible with the differential d.*

Proof. This is a straight-forward computation using the definition and the respective properties of the products on $\Lambda_A(x)$ and $\mathcal{D}(X, \log D)$. We omit the proof of the A-linearity and associativity, but show graded-commutativity.

If τ denotes the map that exchanges the factors in $X \times Y$, and if $\omega \otimes S$, $\eta \otimes T$ are elements in $P_{\mathcal{D}}(X, D, A(p))$ and $P_{\mathcal{D}}(X', D', A(q))$ respectively, then

$$\begin{aligned}(\omega \otimes S) \boxtimes (\eta \otimes T) &= (-1)^{|\eta||S|}(\omega \wedge \eta) \otimes (S \boxtimes T) \\ &= (-1)^{|\eta||S|+|\eta||\omega|+|S||T|}(\eta \wedge \omega) \otimes \tau_*(T \boxtimes S) \\ &= (-1)^{|\eta||S|+|\eta||\omega|+|S||T|}\tau_*\Big((\eta \wedge \omega) \otimes (T \boxtimes S)\Big) \\ &= (-1)^{(|\eta|+|T|)(|\omega|+|S|)}\tau_*\Big((\eta \otimes T) \boxtimes (\omega \otimes S)\Big).\end{aligned}$$

The compatibility with the differential follows from the Leibniz rule for $\wedge$, $\otimes$ and $\boxtimes$:

$$\begin{aligned}d((\omega \otimes S) \boxtimes (\eta \otimes T)) &= (-1)^{|\eta||S|}d(\omega \wedge \eta \otimes (S \boxtimes T)) \\ &= (-1)^{|\eta||S|}d(\omega \wedge \eta) \otimes (S \boxtimes T) + (-1)^{|\eta||S|+|\omega||\eta|}\omega \wedge \eta \otimes d(S \boxtimes T) \\ &= d(\omega \otimes S) \boxtimes (\eta \otimes T) + (-1)^{|\omega|+|S|}(\omega \otimes S) \boxtimes d(\eta \otimes T).\end{aligned}$$ □

Note that the exterior products on both complexes are compatible with pushforward and pullback (whenever defined).
The products of $C_{\mathcal{D}}$ and $P_{\mathcal{D}}$ however are not compatible with each other in the sense that the evaluation map ev induces a map of algebras on the direct sum over p. In fact, the product on $P_{\mathcal{D}}$ is graded-commutative and associative (at the same time), while the products on $C_{\mathcal{D}}$ are not. See also lemma 58

Remark 5. In contrast to the formulas in the standard reference [16], we have an additional sign before $a \boxtimes \tilde{c}$. This is due to the fact that [16] considers a constant sheaf in degree 0, while we work with arbitrary complexes. Our formulas are consistent with the original ones in [3].

Exterior products and fundamental cycles

The exterior product with fundamental cycles is commutative for both $C_{\mathcal{D}}$ and $P_{\mathcal{D}}$.

Lemma 51. *If Z is an algebraic cycle on U and $(a, b, c) \in C_{\mathcal{D}}(X', D', A)$, then for any $\alpha \in \mathbb{R}$,*

$$\mathrm{cl}(Z) \boxtimes_\alpha (a, b, c) = \tau_*\Big((a, b, c) \boxtimes_\alpha \mathrm{cl}(Z)\Big)$$

where $\tau : X' \times X \to X \times X'$ exchanges the coordinates. The same holds for the complex $P_{\mathcal{D}}$.

Proof. The integration current over Z is of even degree and thus by the graded-commutativity of the exterior product of currents,

$$\begin{aligned}\mathrm{cl}(Z) \boxtimes_\alpha (a, b, c) &= \big(\mathrm{cl}(Z), \mathrm{cl}(Z), 0\big) \boxtimes_\alpha (a, b, c) \\ &= \big(\mathrm{cl}(Z) \boxtimes a, \mathrm{cl}(Z) \boxtimes b, \mathrm{cl}(Z) \boxtimes c\big) \\ &= \tau_*\big(a \boxtimes \mathrm{cl}(Z), b \boxtimes \mathrm{cl}(Z), c \boxtimes \mathrm{cl}(Z)\big) \\ &= \tau_*\big((a, b, c) \boxtimes_\alpha \mathrm{cl}(Z)\big).\end{aligned}$$

Similarly for $\omega \otimes S \in P_{\mathcal{D}}(X', D', A)$:

$$\mathrm{cl}(Z) \boxtimes (\omega \otimes S) = \omega \otimes \big(\mathrm{cl}(Z) \boxtimes S\big) = \omega \otimes \tau_*\big(S \boxtimes \mathrm{cl}(Z)\big) = \tau_*\big((\omega \otimes S) \boxtimes \mathrm{cl}(Z)\big).$$ □

The cycle maps into $C_{\mathcal{D}}$ and $P_{\mathcal{D}}$ are compatible with the exterior products:

Lemma 52 (cl and $\times$)**.** *The $\boxtimes_\alpha$-product of fundamental cycles is independent of α and compatible with the exterior product of algebraic cycles, that is,* $\mathrm{cl} \circ \times = \boxtimes_\alpha \circ (\mathrm{cl} \times \mathrm{cl})$ *for all α. Moreover, the fundamental cycle* $\mathrm{cl}(pt) \in C^0_{\mathcal{D}}(pt, A(0))$ *is an identity element for all $\boxtimes_\alpha$.*
The same holds for the exterior product and the fundamental cycle in $P_{\mathcal{D}}$.

Proof. Let $Z \in z^p(U)$ and $Z' \in z^q(U')$ be two algebraic cycles. Note that for the exterior product of currents of even degree, $\times$ and $\boxtimes$, agree. First $C_{\mathcal{D}}$:

$$\begin{aligned}\mathrm{cl}(Z \times Z') &= (2\pi i)^{p+q}\big([Z \times Z'], [Z \times Z'], 0\big) = (2\pi i)^{p+q}\big([Z] \times [Z'], [Z] \times [Z'], 0\big)\\ &= (2\pi i)^{p+q}\big([Z] \boxtimes [Z'], [Z] \boxtimes [Z'], 0\big) = (2\pi i)^{p+q}\big([Z], [Z], 0\big) \boxtimes_\alpha \big([Z'], [Z'], 0\big)\\ &= \mathrm{cl}(Z) \boxtimes_\alpha \mathrm{cl}(Z')\end{aligned}$$

for all $\alpha \in \mathbb{R}$.
For $P_{\mathcal{D}}$:

$$\mathrm{cl}(Z \times Z') = (2\pi i)^{p+q}[Z \times Z'] = (2\pi i)^{p+q}([Z] \boxtimes [Z']) = \mathrm{cl}(Z) \boxtimes \mathrm{cl}(Z').$$

For the last statement, we tacitly identify currents on $X \times \mathrm{pt}$ with currents on X. Under this identification, $T \boxtimes [\mathrm{pt}] = T \in \mathcal{D}(X \times \mathrm{pt}) \cong \mathcal{D}(X)$, that is, $[\mathrm{pt}] \in \mathcal{D}(\mathrm{pt})$ is an identity element for the exterior product of currents.
Now note that if e is an identity element for $\boxtimes$, then $(e, e, 0)$ is an identity for $\boxtimes_\alpha$ for all α. Indeed, the product of an arbitrary triple with $e := \mathrm{cl}(\mathrm{pt})$ in $C_{\mathcal{D}}$ is

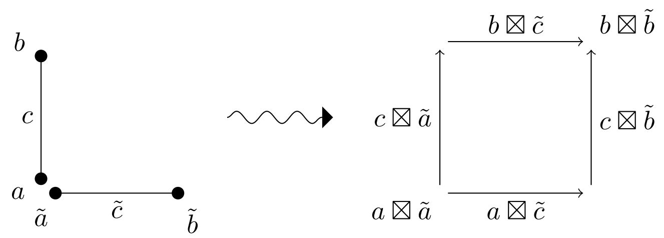
$$(a, b, c) \boxtimes_\alpha e = (a, b, c) \boxtimes_\alpha (e, e, 0) = (a \boxtimes e, b \boxtimes e, c \boxtimes e) \cong (a, b, c).$$

The same holds for the product on $P_{\mathcal{D}}$, since under the above mentioned identification,

$$(\omega \otimes S) \boxtimes \mathrm{cl}(\mathrm{pt}) = \omega \otimes (S \boxtimes [\mathrm{pt}]) = \omega \otimes S. \qquad \square$$

Aside: Interpretation of Beilinson's product

The quasi-isomorphism ev between $P_{\mathcal{D}}$ and $C_{\mathcal{D}}$ leads to a geometric interpretation of Beilinson's products on the latter.
Think of an element (a, b, c) in $C_{\mathcal{D}}$ as the startpoint a and endpoint b of a path with c the line segment connecting a and b (oriented from $a \to b$). Given two such triples, one can form the exterior product of the two paths, getting a square as drawn below



Now there are two possible ways to extract a path from the new startpoint $a \boxtimes \tilde{a}$ to the new endpoint $b \boxtimes \tilde{b}$ out of this diagram. Each is as good as the other and one has the freedom to combine them as one wishes using a parameter α.

For example, give the left upper path the weight α and the right lower path weight $1-\alpha$. Then the combined path is

$$\alpha \cdot [c \boxtimes \tilde{a} \pm b \boxtimes \tilde{c}] + (1-\alpha) \cdot \left[\pm a \boxtimes \tilde{c} + c \boxtimes \tilde{b}\right]$$

where we decorate each horizontal line segment with a sign $\pm = (-1)^r$. This in turn corresponds to a triple which is exactly the $\boxtimes_\alpha$ product of (a,b,c) with $(\tilde{a}, \tilde{b}, \tilde{c})$.

Remark 6.

- With this geometric construction in mind, the associativity and graded-commutativity properties of Beilinson's products are verified easily (e.g. by looking at a cube).
- There is no satisfying geometric explanation for this sign $(-1)^r$. This sign occurs when the endpoints a, b of the vertical path are passing $\tilde{c}$. Note that $r = \deg a = \deg b$ is the degree of these points.

3.8 Intersection products

Analogous to the exterior product, the intersection product of currents gives rise to intersection products $\cap_\alpha$ on $C_{\mathcal{D}}$ and $\cap$ on $P_{\mathcal{D}}$.
These products are defined in exactly the same way as the exterior products, with $\boxtimes$ replaced by $\cap$. That is, the product $\cap_\alpha$ on the 3-term complex, $\alpha \in \mathbb{R}$, sends $(a,b,c) \otimes (\tilde{a}, \tilde{b}, \tilde{c})$ (where the first tuple has total degree r) to

$$\left(a \cap \tilde{a}, b \cap \tilde{b}, \alpha c \cap \tilde{a} + (-1)^r (1-\alpha) a \cap \tilde{c} + (1-\alpha) c \cap \tilde{b} + (-1)^r \alpha b \cap \tilde{c}\right)$$

whenever all intersections are defined. Similarly, the intersection product on $P_{\mathcal{D}}$ is the one with

$$(\omega \otimes S) \cap (\eta \otimes T) = (-1)^{|\eta||S|} (\omega \wedge \eta) \otimes (S \cap T),$$

whenever the right hand side defines a valid element in $P_{\mathcal{D}}$.

Theorem 53. *The intersection induces partially defined maps of complexes*

$$\cap : P_{\mathcal{D}}(X, D, A(p)) \otimes_A P_{\mathcal{D}}(X, D, A(q)) \dashrightarrow P_{\mathcal{D}}(X, D, A(p+q))$$

and

$$\cap_\alpha : C_{\mathcal{D}}(X, D, A(p)) \otimes_A C_{\mathcal{D}}(X, D, A(q)) \dashrightarrow C_{\mathcal{D}}(X, D, A(p+q)).$$

The intersection with fundamental cycles is commutative and $\mathrm{cl}(U)$ *is an identity element. The intersections are preserved by pushforward along biholomorphic maps of pairs.*

Proof. If the underlying intersections exist, then both $\cap$ and $\cap_\alpha$ have the correct form (i.e., are additive in the (total) degree and the weight) and are A-bilinear. As for the exterior product, one sees that both types of intersection are compatible with the differential whenever all occurring intersections exist.
The commutativity of the intersection with fundamental cycles follows as in lemma 51 from the special form of this intersection together with the fact that the intersection with algebraic cycles is commutative.

The proof that $\mathrm{cl}(U)$ is the identity element for $\cap$ and $\cap_\alpha$ is again a straight forward calculation, based on the fact that the fundamental current $\mathrm{cl}(U) = [X]$ is an identity element for $\cap$. For $P_{\mathcal{D}}$ for example, $\mathrm{cl}(U) \cap (\omega \otimes T) = \omega \otimes ([X] \cap T) = \omega \otimes T$.

Let $\varphi : (X, D) \to (X', D')$ be a biholomorphic map of pairs. The pushforward on Deligne-Beilinson complexes is defined by applying the pushforward to all the currents involved (and taking care of the twists). Since the intersection of log currents is compatible with biholomorphic maps of pairs and because the intersection on Deligne-Beilinson complexes is built from the intersection of log currents, the last statement follows. $\square$

The evaluation map and $\cap$

Lemma 54. *The evaluation $ev : P_{\mathcal{D}}(X, D, A(p)) \to C_{\mathcal{D}}(X, D, A(p))$ is not an algebra homomorphism. For rational coefficients however, it induces an isomorphism of algebras on total cohomology.*

Proof. To see that the evaluation map is never a homomorphism of algebras, choose an integral current T and an arbitrary current S such that the intersection $S \cap T$ exists. For any integer $n > 0$ then

$$ev\big(dxS \cap (1-x)^n T\big) = ev\big((1-x)^n dxS \cap T\big) = \big(0, 0, \tfrac{-1}{n+1} S \cap T\big).$$

If the evaluation were a homomorphism of algebras, this would be equal to

$$\begin{aligned} ev\big(dxS\big) \cap_\alpha ev\big((1-x)^n T\big) &= \big(0, 0, S\big) \cap_\alpha \big(T, 0, 0\big) \\ &= \big(0, 0, \alpha S \cap T\big), \end{aligned}$$

which can not be true for arbitrary $n > 0$ (and a fixed α).

Now we show that ev is an isomorphism of algebras on rational cohomology that is, if $\mathbb{Q} \subset A$. In this case, ev induces an isomorphism on cohomology so that it suffices to show that this isomorphism is compatible with the intersection products on both complexes. Since all products $\cap_\alpha$ give rise to the same products on $HC_{\mathcal{D}}$, we may assume that $\alpha = \frac{1}{2}$.

It is enough to show that the pushforward of the product on $P_{\mathcal{D}}$ to $C_{\mathcal{D}}$ is equal to $\cap_{\frac{1}{2}}$, that is, $\cap_{\frac{1}{2}} = ev \circ \cap \circ (s \times s)$. Indeed, because of $s \circ ev \sim \mathrm{id}_P$, this implies that

$$s \circ \cap_{\frac{1}{2}} = s \circ ev \circ \cap \circ (s \times s) \sim \cap \circ (s \times s).$$

In other words, s induces an algebra isomorphism on the cohomology. ev is the inverse of s and hence the same holds for ev.

To finish the proof of the lemma, let $A = (S_0, S_1, S)$ and $B = (T_0, T_1, T)$ be two elements in $C_{\mathcal{D}}$ whose intersection exists. Then

$$
\begin{aligned}
ev \circ \cap \circ (s \times s)(A, B) &= ev\Big(\big((1-x)S_0 + xS_1 + dxS\big) \cap \big((1-x)T_0 + xT_1 + dxT\big)\Big) \\
&= \Big(S_0 \cap T_0, S_1 \cap T_1, \int_0^1 (1-x)dx\big(S \cap T_0 + (-1)^{|A|} S_0 \cap T\big) \\
&\qquad\qquad + \int_0^1 x dx\big(S \cap T_1 + (-1)^{|A|} S_1 \cap T\big)\Big) \\
&= \Big(S_0 \cap T_0, S_1 \cap T_1, \frac{1}{2}\big(S \cap T_0 + (-1)^{|A|} S_0 \cap T + S \cap T_1 + (-1)^{|A|} S_1 \cap T\big)\Big) \\
&= A \cap_{\frac{1}{2}} B.
\end{aligned}
$$
□

Compatibility of $\cap$ and cl

The intersection products on the Deligne complexes are compatible with the cycle maps, as follows from the respective property for log currents.

Lemma 55. *For two algebraic cycles $Z \in z^p(U)$, $Z' \in z^q(U)$ that intersect properly, one has*

$$\mathrm{cl}(Z \cap Z') = \mathrm{cl}(Z) \cap \mathrm{cl}(Z')$$

in $P_{\mathcal{D}}(X, D, \mathbb{Z}(p+q))$ and, for any $\alpha \in \mathbb{R}$,

$$\mathrm{cl}(Z \cap Z') = \mathrm{cl}(Z) \cap_\alpha \mathrm{cl}(Z')$$

in $C_{\mathcal{D}}(X, D, \mathbb{Z}(p+q))$.

Proof. For $P_{\mathcal{D}}$:

$$\mathrm{cl}(Z \cap Z') = (2\pi i)^{p+q}[Z \cap Z'] = (2\pi i)^{p+q}[Z] \cap [Z'] = \mathrm{cl}(Z) \cap \mathrm{cl}(Z').$$

For $C_{\mathcal{D}}$:

$$\mathrm{cl}(Z \cap Z') = (2\pi i)^{p+q}(Z \cap Z', Z \cap Z', 0) = (2\pi i)^{p+q}(Z, Z, 0) \cap_\alpha (Z', Z', 0) = \mathrm{cl}(Z) \cap_\alpha \mathrm{cl}(Z').$$

□

Any holomorphic map of pairs $f : X \to X'$ induces a general pullback on the Deligne complexes by means of the formula $f^*T := (\mathrm{pr}_X)_*\Big(\big(\mathrm{cl}(X \setminus D) \boxtimes T\big) \cap \mathrm{cl}\big(\Gamma_f\big)\Big)$, where Γ_f denotes the graph of f. The general pullback is partially defined and extends the pullback along a submersion.

Lemma 56 (cl and general pullback)**.** *Let $f : X \to X'$ be a holomorphic map of compact complex algebraic manifolds that restricts to a map $f|_U : U \to U'$. Then the cycle map* cl *is compatible with general pullback. That is, if $f|_U^* Z$ exists, then $f^*\mathrm{cl}(Z)$ also exists and equals $\mathrm{cl}(f|_U^* Z)$.*

Note that this includes the case where f is a submersion.

Proof. The general pullback of algebraic cycles is defined by essentially the same formula. Use that cl is compatible with pushforward, intersection of algebraic cycles and the exterior product, to conclude that

$$\begin{aligned}
\mathrm{cl}(f|_U^* Z) &= \mathrm{cl}\Big((\mathrm{pr}_U)_*\big((U \times Z) \cap \Gamma_{f|_U}\big)\Big)\\
&= (\mathrm{pr}_X)_*\mathrm{cl}\Big((U \times Z) \cap \Gamma_{f|_U}\Big)\\
&= (\mathrm{pr}_X)_*\Big(\mathrm{cl}(U \times Z) \cap \mathrm{cl}(\Gamma_{f|_U})\Big)\\
&= (\mathrm{pr}_X)_*\Big(\big(\mathrm{cl}(U) \boxtimes \mathrm{cl}(Z)\big) \cap \mathrm{cl}(\Gamma_{f|_U})\Big).
\end{aligned}$$

Since $\mathrm{cl}(\Gamma_{f|_U}) = \mathrm{cl}(\Gamma_f)$, the lemma follows. □

Compatibility of $\boxtimes$ and $\cap$

On the complex $P_{\mathcal{D}}$, the two products $\boxtimes$ and $\cap$ are compatible in the sense that for any $P, Q, P', Q' \in P_{\mathcal{D}}$ such that the intersections $P \cap Q$ and $P' \cap Q'$ exist, one has

$$(P \cap Q) \boxtimes (P' \cap Q') = (-1)^{|P'||Q|}(P \boxtimes P') \cap (Q \boxtimes Q').$$

The analogous claim for the complex $C_{\mathcal{D}}$ and the products $\boxtimes_\alpha$, $\cap_\alpha$ is false in general – even for $\alpha = 0$. It is true however, if one restricts to intersections with geometric currents, that is, to intersections with fundamental cycles.

Lemma 57. *Let $P \in C_{\mathcal{D}}(X, D, A(p)), P' \in C_{\mathcal{D}}(X', D', A(p'))$ and $Z \in z^q(U), Z' \in z^{q'}(U')$. For any $\alpha \in \mathbb{R}$ such that $P \cap_\alpha \mathrm{cl}(Z)$ and $P' \cap_\alpha \mathrm{cl}(Z')$ exist (the cycle map relative to D resp. D') one has*

$$\big(P \boxtimes_\alpha P'\big) \cap_\alpha \big(\mathrm{cl}(Z) \boxtimes_\alpha \mathrm{cl}(Z')\big) = \big(P \cap_\alpha \mathrm{cl}(Z)\big) \boxtimes_\alpha \big(P' \cap_\alpha \mathrm{cl}(Z')\big).$$

Proof. Write $P = (T_0, T_1, T), P' = (T_0', T_1', T')$.
The $\cap_\alpha$-product with $\mathrm{cl}(Z) \boxtimes_\alpha \mathrm{cl}(Z') = \mathrm{cl}(Z \times Z')$ has a very simple form:

$$(P \boxtimes_\alpha P') \cap_\alpha (\mathrm{cl}(Z) \boxtimes \mathrm{cl}(Z')) = \begin{pmatrix} (T_0 \boxtimes T_0') \cap \mathrm{cl}(Z \times Z') \\ (T_1 \boxtimes T_1') \cap \mathrm{cl}(Z \times Z') \\ (P \boxtimes_\alpha P')_3 \cap \mathrm{cl}(Z \times Z') \end{pmatrix}$$

where $(P \boxtimes P')_3 = T \boxtimes \big(\alpha T_0' + (1-\alpha)T_1'\big) + (-1)^{|T_0|}\big((1-\alpha)T_0 + \alpha T_1\big) \boxtimes T'$.
On the other hand,

$$(P \cap_\alpha \mathrm{cl}(Z)) \boxtimes_\alpha (P' \cap_\alpha \mathrm{cl}(Z')) = \begin{pmatrix} (T_0 \cap \mathrm{cl}(Z)) \boxtimes (T_0' \cap \mathrm{cl}(Z')) \\ (T_1 \cap \mathrm{cl}(Z)) \boxtimes (T_1' \cap \mathrm{cl}(Z')) \\ (*) \end{pmatrix}$$

where

$$\begin{aligned}
(*) = &\big(T \cap \mathrm{cl}(Z)\big) \boxtimes \big(\alpha T_0' \cap \mathrm{cl}(Z') + (1-\alpha)T_1' \cap \mathrm{cl}(Z')\big)\\
&+ (-1)^{|T_0 \cap Z|}\big((1-\alpha)T_0 \cap \mathrm{cl}(Z) + \alpha T_1 \cap \mathrm{cl}(Z)\big) \boxtimes \big(T' \cap \mathrm{cl}(Z')\big).
\end{aligned}$$

We now use the compatibility of $\boxtimes$ and $\cap$ for currents from lemma 29. Then the upper two components of the triples are obviously equal. One verifies that the lowest component in both cases are also equal. In fact, they are equal to

$$\begin{aligned}(*) &= \alpha(T \boxtimes T_0') \cap \mathrm{cl}(Z \times Z') \\ &+ (1-\alpha)(T \boxtimes T_1') \cap \mathrm{cl}(Z \times Z') \\ &+ (-1)^{|T_0|}(1-\alpha)(T_0 \boxtimes T') \cap \mathrm{cl}(Z \times Z') \\ &+ (-1)^{|T_0|}\alpha(T_1 \boxtimes T') \cap \mathrm{cl}(Z \times Z'). \end{aligned}$$ □

ev, s and products with algebraic cycles

The proof of lemma 57 is by a brute force computation. The result for $\alpha = \frac{1}{2}$ can more elegantly be deduced from the result for $P_{\mathcal{D}}$ using the evaluation and the splitting map. This follows from the important fact that, although the evaluation map in general is not compatible with the intersection or the exterior product, for the special case of cycles coming from geometry, it is:

Lemma 58. *The evaluation map is compatible with the (partially defined) intersection and the exterior product with algebraic cycles. For example, for any $P \in P_D$, any codimension p algebraic cycle Z and $\alpha \in \mathbb{R}$ one has*

$$ev(P \cap \mathrm{cl}(Z)) = ev(P) \cap_\alpha \mathrm{cl}(Z)$$

whenever all intersections exist. Similarly, the splitting map is compatible with intersection and exterior product with cycles:

$$s\big((a,b,c) \boxtimes_\alpha \mathrm{cl}(Z)\big) = s(a,b,c) \boxtimes \mathrm{cl}(Z)$$

and the same with $\cap_\alpha$, if all intersections exist.

Note the different meanings of the relative cycle class $\mathrm{cl}(Z)$.

Proof. We only show the statements that concern the intersection products, starting with the one for the evaluation map. Let $P = \omega \otimes T$ such that $T \cap [Z] =: S$ exists. Then

$$ev\big(P \cap \mathrm{cl}(Z)\big) = ev\big((2\pi i)^p \omega \otimes (T \cap [Z])\big) = (2\pi i)^p(\omega_0 S, \omega_1 S, \int_0^1 \omega S).$$

On the other hand, the intersection with $\mathrm{cl}(Z)$ in $C_{\mathcal{D}}$ is, independent of α, given by

$$\begin{aligned} ev(P) \cap_\alpha \mathrm{cl}(Z) &= (2\pi i)^p \big(\omega_0 T, \omega_1 T, \int_0^1 \omega T\big) \cap_\alpha \big([Z],[Z],0\big) \\ &= (2\pi i)^p \big(\omega_0 T \cap [Z], \omega_1 T \cap [Z], \int_0^1 \omega T \cap [Z]\big). \end{aligned}$$

For the statement about the splitting map, let $T = (T_1, T_2, T_3)$ be in $C_{\mathcal{D}}$ such that $T \cap_\alpha \mathrm{cl}(Z)$ exists. This intersection is $(T_1 \cap \mathrm{cl}(Z), T_2 \cap \mathrm{cl}(Z), T_3 \cap \mathrm{cl}(Z))$, independent of α. Thus we can conclude that

$$\begin{aligned} s(T \cap_\alpha \mathrm{cl}(Z)) &= (1-x)T_1 \cap \mathrm{cl}(Z) + xT_2 \cap \mathrm{cl}(Z) + dxT_3 \cap \mathrm{cl}(Z) \\ &= s(T) \cap \mathrm{cl}(Z). \end{aligned}$$ □

Reduction to the diagonal

Denote by $\Delta : X \to X \times X$ the diagonal mapping and $\mathrm{cl}(\Delta) := \mathrm{cl}(\Delta(X))$.

Lemma 59. *Reduction to the diagonal holds, that is, $S \cap T$ exists if and only if $(S \boxtimes T) \cap \mathrm{cl}(\Delta)$ exists. In this case, $\Delta_*(S \cap T) = (S \boxtimes T) \cap \mathrm{cl}(\Delta)$.*

Note that although stated merely in the absolute situation $(X, \emptyset)$, by the very definition of the intersection of relative currents, reduction to the diagonal also holds in the relative case.

Proof. For $C_{\mathcal{D}}$: Let $A = (a, b, c)$ and $\tilde{A} = (\tilde{a}, \tilde{b}, \tilde{c})$ consist of locally flat currents. Let $r = |a|$ be the total degree of A. If the $\cap_\alpha$-intersection of the two triples exist, then, with $\beta = 1 - \alpha$,

$$\begin{aligned}
\Delta_*(A \cap_\alpha \tilde{A}) &= \Big(\Delta_*(a \cap \tilde{a}), \Delta_*(b \cap \tilde{b}), \Delta_*\big(c \cap (\alpha\tilde{a} + \beta\tilde{b}) + (-1)^r(\beta a + \alpha b) \cap \tilde{c}\big)\Big) \cdot (2\pi i)^{\dim X} \\
&= \Big((a \boxtimes \tilde{a}) \cap \mathrm{cl}(\Delta), (b \boxtimes \tilde{b}) \cap \mathrm{cl}(\Delta), \\
&\qquad \big(c \boxtimes (\alpha\tilde{a} + \beta\tilde{b})\big) \cap \mathrm{cl}(\Delta) + (-1)^r\big((\beta a + \alpha b) \boxtimes \tilde{c}\big) \cap \mathrm{cl}(\Delta)\Big) \\
&= \Big(a \boxtimes \tilde{a}, b \boxtimes \tilde{b}, c \boxtimes (\alpha\tilde{a} + \beta\tilde{b}) + (-1)^r(\beta a + \alpha b) \boxtimes \tilde{c}\Big) \cap \mathrm{cl}(\Delta) \\
&= (A \boxtimes \tilde{A}) \cap_\alpha \mathrm{cl}(\Delta).
\end{aligned}$$

Conversely, if the last intersection exists, then it is supported on the diagonal and hence by the flatness theorem lies in the image of Δ_*.
For $P_{\mathcal{D}}$ the proof is again simpler:

$$\begin{aligned}
\Delta_*\Big((\omega \otimes T) \cap (\omega' \otimes T')\Big) &= (-1)^{|\omega'||T|} \cdot \Delta_*\Big(\omega \wedge \omega' \otimes (T \cap T')\Big) \\
&= (-1)^{|\omega'||T|}(2\pi i)^{\dim X} \cdot (\omega \wedge \omega') \otimes \Delta_*(T \cap T') \\
&= (-1)^{|\omega'||T|} \cdot (\omega \wedge \omega') \otimes \Big((T \boxtimes T') \cap \mathrm{cl}(\Delta)\Big) \\
&= (-1)^{|\omega'||T|} \cdot \Big((\omega \wedge \omega') \otimes (T \boxtimes T')\Big) \cap \mathrm{cl}(\Delta) \\
&= \Big((\omega \otimes T) \boxtimes (\omega' \otimes T')\Big) \cap \mathrm{cl}(\Delta).
\end{aligned}$$

By the reduction to the diagonal of usual currents, one side of the equation exists if and only if the other also exists. ☐

Intersection on cohomology

By the moving lemma for currents 34, any two normal currents can be moved in their cohomology class in such a way that their intersection exists. This is not true for cohomology classes of the Deligne complexes $C_{\mathcal{D}}$ and $P_{\mathcal{D}}$. They however can be brought in good position with respect to fundamental cycles.

Lemma 60. *Assume $D = \emptyset$. Given a fundamental cycle* $\mathrm{cl}(Z)$*, then any cohomology class in $HC_{\mathcal{D}}$ has a representative whose intersection with* $\mathrm{cl}(Z)$ *exists. This representative can chosen to be normal, that is, consist of normal currents only. For rational coefficients, the same is true for cohomology classes in $HP_{\mathcal{D}}$.*

Proof. Given a class in $HC_{\mathcal{D}}$, we can choose a representative (T_0, T_1, T) whose components are all normal currents (and T_0 is integral). The moving lemma gives a representation $T_0 =$

$\tilde{T}_0 + dR_0$ in such a way that $\tilde{T}_0 \cap [Z]$ exists. Moreover, there exists a smooth p-form ω such that $T_1 = [\omega] + dR_1$ for some $R_1 \in F^p\mathcal{D}(X)$. Then

$$(T_0, T_1, T) = (\tilde{T}_0, [\omega], T - R_1 + R_0) + d(R_0, R_1, 0).$$

Now choose a current $\tilde{T}$ such that $\tilde{T} \cap [Z]$ exists and $T - R_1 + R_0 = \tilde{T} - dR$. Consequently, the class of

$$(T_0, T_1, T) = (\tilde{T}_0, [\omega], \tilde{T}) + d(R_0, R_1, R)$$

can as well be represented by $(\tilde{T}_0, [\omega], \tilde{T})$ whose intersection with $\mathrm{cl}(Z)$ exists. Indeed, the intersection with a smooth differential form always exists and so

$$\big(\tilde{T}_0, [\omega], \tilde{T}\big) \cap \mathrm{cl}(Z) = \big(\tilde{T}_0 \cap \mathrm{cl}(Z), [\omega] \cap \mathrm{cl}(Z), \tilde{T} \cap \mathrm{cl}(Z)\big)$$

exists.
With rational coefficients, the evaluation induces a ring isomorphism on cohomology by lemma 54. This implies the statement for $P_\mathcal{D}$. □

The intersection of two cohomology classes $\underline{A}, \underline{B}$ in $HC_\mathcal{D}$ or $HP_\mathcal{D}$ is said to exist, if there exist two normal representatives A, B together with some normal current R such that $(A \boxtimes B + dR) \cap \mathrm{cl}(\Delta)$ exists. Then the intersection class is the unique cohomology class such that

$$\Delta_*(\underline{A} \cap \underline{B}) = (A \boxtimes B + dR) \cap \mathrm{cl}(\Delta) \quad + \text{boundaries}.$$

This definition is independent of the choice of the A, B, R: Whenever $\tilde{A}, \tilde{B}$ are other representatives of $\underline{A}, \underline{B}$ and $\tilde{R}$ is such that $(\tilde{A} \boxtimes \tilde{B} + d\tilde{R}) \cap \mathrm{cl}(\Delta)$ exists, then the difference of these expressions is equal to $d\tilde{\tilde{R}} \cap \mathrm{cl}(\Delta)$ for some $\tilde{\tilde{R}}$. It is to show that this current arises as the pushforward of a boundary along Δ. By the moving lemma 60, $\tilde{\tilde{R}} = Q + \text{boundary}$ such that $Q \cap \mathrm{cl}(\Delta)$ exists and − by the flatness theorem − is equal to $\Delta_*\tilde{Q}$. Hence $\Delta_* d\tilde{Q} = d\Delta_*\tilde{Q} = d(Q \cap \mathrm{cl}(\Delta)) = dQ \cap \mathrm{cl}(\Delta) = d\tilde{\tilde{R}} \cap \mathrm{cl}(\Delta)$ as required.
Note that lemma 60 assures that the intersection of any two cohomology classes always exists.

3.9 Requirements for the Deligne complexes

Denote by (X, D) a pair consisting of a smooth projective variety and a normal crossing divisor $D \subset X$. Assume that to every such pair, every coefficient ring $A \subset \mathbb{R}$ and every integer p (the weight) there is associated a complex of A-modules $C^\bullet_\mathcal{D}(X, D, A(p))$. We summarize all the properties of these complexes that are needed to apply the general construction of the regulator in 4.1.

(Coefficients) $C^\bullet_\mathcal{D}(X, D, A(p)) = C^\bullet_\mathcal{D}(X, D, \mathbb{Z}(p)) \otimes_\mathbb{Z} A$.

(Cycle map) There exist maps $\mathrm{cl} : z^p(X \setminus D) \to C^{2p}_\mathcal{D}(X, \emptyset, \mathbb{Z}(p))$ such that $d \circ \mathrm{cl} = 0$.

(Flat pullback) For any holomorphic submersion $Y \xrightarrow{f} X$ one has a morphism of complexes

$$f^* : C^\bullet_\mathcal{D}(X, D, A(p)) \to C^\bullet_\mathcal{D}(Y, f^{-1}D, A(p)).$$

(Proper pushforward) For any holomorphic map of pairs $(X, D) \xrightarrow{f} (X', D')$ one has a morphism of complexes

$$f_* : C^\bullet_{\mathcal{D}}(X, D, A(p)) \to C^{\bullet-2\delta}_{\mathcal{D}}(X', D', A(p-\delta))$$

where $\delta = \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} X'$ is the relative dimension. In particular, $df_* = f_* d$.

(Exterior products) There exist external products

$$C^\bullet_{\mathcal{D}}(X, D, A(p)) \otimes C^\bullet_{\mathcal{D}}(X', D', A(q)) \xrightarrow{\boxtimes} C^\bullet_{\mathcal{D}}(X \times X', D \boxtimes D', A(p+q))$$

in the category of complexes. Here $D \boxtimes D' = X \times D' + D \times X'$ denotes the exterior product of the divisors.

The above structures should satisfy the compatibility conditions listed below:

- cl is compatible with flat pullback and proper pushforward, that is,

$$\mathrm{cl} \circ f|^*_{X \setminus D} = f^* \circ \mathrm{cl} \quad \text{resp.} \quad \mathrm{cl} \circ (f|_{X\setminus D})_* = f_* \circ \mathrm{cl}$$

  whenever $f|_{X\setminus D}$ is flat resp. proper.

- For Z, Z' two algebraic cycles, $\mathrm{cl}(Z \times Z') = \mathrm{cl}(Z) \boxtimes \mathrm{cl}(Z')$.
- $\boxtimes$ is commutative if one factor is an algebraic cycle, i.e. $\mathrm{cl}(Z) \boxtimes T = \tau_*(T \boxtimes \mathrm{cl}(Z))$ where τ is the map that interchanges the two factors.
- Pushforward preserves $\boxtimes$.
- If φ is a biholomorphic map of pairs, then $\varphi_* = (\varphi^{-1})^*$.
- $\boxtimes$ is associative and cl(pt) is an identity element for $\boxtimes$.

(Intersection products) There exists a partially defined (internal) product

$$C^\bullet_{\mathcal{D}}(X, D, A(p)) \otimes C^\bullet_{\mathcal{D}}(X, D, A(q)) \overset{\cap}{\dashrightarrow} C^\bullet_{\mathcal{D}}(X, D, A(p+q))$$

such that

- If $S \cap T$ and either $dS \cap T$ or $S \cap dT$ exist, then $d(S \cap T) = dS \cap T + (-1)^{|S|} S \cap dT$.
- Intersection with algebraic cycles is commutative, that is, $T \cap \mathrm{cl}(Z) = \mathrm{cl}(Z) \cap T$ whenever one of the intersections exist.
- The intersection product is compatible with the intersection product of algebraic cycles. That is, whenever $Z \cap Z'$ exists, then so does $\mathrm{cl}(Z) \cap \mathrm{cl}(Z')$ and is equal to $\mathrm{cl}(Z \cap Z')$. In particular, the cycle map is compatible with general pullback, partially defined for $f : X \to X'$ by $f^*T := (\mathrm{pr}_X)_*\Big((\mathrm{cl}(X \setminus D) \boxtimes T) \cap \mathrm{cl}(\Gamma_f)\Big)$.
- Reduction to the diagonal: If $\Delta : X \to X \times X$ denotes the diagonal embedding and Δ_U the image of U, then

$$\Delta_*(S \cap T) = (S \boxtimes T) \cap \mathrm{cl}(\Delta_U).$$

- $\mathrm{cl}(X \setminus D)$ is an identity element for $\cap$.

- Compatibility of $\boxtimes$ and $\cap$ for cycles coming from geometry (i.e., of type $\mathrm{cl}(Z)$ for an algebraic cycle Z): If the intersections on the right-hand side exist, there is an equality

$$(S_1 \boxtimes S_2) \cap (\mathrm{cl}(Z_1) \boxtimes \mathrm{cl}(Z_2)) = (S_1 \cap \mathrm{cl}(Z_1)) \boxtimes (S_2 \cap \mathrm{cl}(Z_2)).$$

- $\cap$ is preserved by pushforward along biholomorphic maps of pairs and the trivial maps $(X, D) \to (X, D')$ for $D \subset D'$, that is, $\varphi_*(S \cap T) = \varphi_* S \cap \varphi_* T$.

Theorem 61. *The 3-term complex $C_{\mathcal{D}}$ (with $\boxtimes_0, \cap_0$ or $\boxtimes_1$, $\cap_1$) and the path complex $P_{\mathcal{D}}$ (with $\boxtimes$, $\cap$) as defined in this section satisfy all of the axioms above.*

Proof. This was done before: The cycle map is 3.6. Functoriality is subsection 3.5. The exterior and the intersection products are considered in 3.7 and 3.8. □

3.10 Regulator-specific properties

For the study of the regulator maps, we will need more precise information about the intersection theory of the class of Deligne-Beilinson chains formed by higher Chow cycles and some special elements. Endow the Deligne complexes $C_{\mathcal{D}}$ and $P_{\mathcal{D}}$ with their associative products and consider the special elements

$$\begin{aligned} R &= \big(2\pi i[\mathbb{R}_{-}], [\operatorname{dlog} z], [\log z]\big) && \in C^1_{\mathcal{D}}(\overline{\square}, \emptyset, \mathbb{Z}(1)) \\ R &= (1-x) \otimes 2\pi i[\mathbb{R}_{-}] + x \otimes [\operatorname{dlog} z] + dx \otimes [\log z] && \in P^1_{\mathcal{D}}(\overline{\square}, \emptyset, \mathbb{Z}(1)) \end{aligned}$$

and their exterior products R^n in the respective complexes. We are interested in the validity of associativity and inverse mapping formulas.

An inverse mapping formula for $C_{\mathcal{D}}$ and $P_{\mathcal{D}}$

The following lemma shows that the Deligne complexes (with any product) have no structural obstruction to an inverse mapping formula.

Lemma 62 (Formal inverse mapping formula)**.** *Let $f : X \to X'$ be a holomorphic map of pairs. Let $A, \tilde{A}$ be two chains in one of the Deligne complexes such that $f^*\tilde{A}$ and $A \cap f^*\tilde{A}$ exist. If the inverse mapping formula holds for all involved currents, then it also holds for A and $\tilde{A}$, that is, $f_*A \cap \tilde{A}$ exists and $f_*(A \cap f^*\tilde{A}) = f_*A \cap \tilde{A}$.*

Proof. Assume that $f^*(\tilde{a}, \tilde{b}, \tilde{c})$ and $(a, b, c) \cap f^*(\tilde{a}, \tilde{b}, \tilde{c})$ both exist. Denote by $\delta := \dim_{\mathbb{C}} X' - \dim_{\mathbb{C}} X$ the relative codimension of f. If the inverse mapping formula holds for all occuring currents, then, with $\beta := 1 - \alpha$,

$$\begin{aligned} f_*\big((a,b,c) \cap_\alpha f^*(\tilde{a}, \tilde{b}, \tilde{c})\big) &= f_*\big(a \cap f^*\tilde{a}, b \cap f^*\tilde{b}, c \cap (\alpha f^*\tilde{a} + \beta f^*\tilde{b}) + (-1)^{|a|}(\beta a + \alpha b) \cap f^*\tilde{c}\big) \\ &= \big(f_*a \cap \tilde{a}, f_*b \cap \tilde{b}, f_*c \cap (\alpha\tilde{a} + \beta\tilde{b}) + (-1)^{|a|}(\beta f_*a + \alpha f_*b) \cap \tilde{c}\big) \cdot (2\pi i)^\delta \\ &= (f_*a, f_*b, f_*c) \cap_\alpha (\tilde{a}, \tilde{b}, \tilde{c}) \cdot (2\pi i)^\delta \\ &= f_*(a,b,c) \cap_\alpha (\tilde{a}, \tilde{b}, \tilde{c}) \end{aligned}$$

also exists.

Similarly for $P_{\mathcal{D}}$: Assume that for two current S on X and T on Y both f^*T and $S \cap f^*T$ exist. Then

$$\begin{aligned}
f_*\big((\eta \otimes S) \cap f^*(\omega \otimes T)\big) &= f_*\big((\eta \otimes S) \cap (\omega \otimes f^*T)\big) \\
&= f_*\big((\eta \wedge \omega) \otimes (S \cap f^*T)\big) \cdot (-1)^{|\omega||S|} \\
&= (\eta \wedge \omega) \otimes f_*(S \cap f^*T)\big) \cdot (-1)^{|\omega||S|}(2\pi i)^{\delta} \\
&= (\eta \wedge \omega) \otimes (f_*S \cap T)\big) \cdot (-1)^{|\omega||S|}(2\pi i)^{\delta} \\
&= (\eta \otimes f_*S) \cap (\omega \otimes T) \cdot (2\pi i)^{\delta} \\
&= f_*(\eta \otimes S) \cap (\omega \otimes T). \qquad \square
\end{aligned}$$

We are interested in the following application:

Lemma 63. *Let $f : U \to V$ be an algebraic map that extends to a holomorphic map $X \to Y$ and consider $Z \in z_{\mathbb{R}}(U, n)$ and $Z' \in z_{\mathbb{R}}(V, n)$. For two disjoint tuples $I, J \subset \{0, 1\}^n$ (i.e., their componentwise addition satisfies $I + J \leq 1^n$) write $R^I = R^{i_1} \boxtimes R^{i_2} \boxtimes \ldots \boxtimes R^{i_n}$ and $Z_I := \mathrm{cl}(Z) \cap (\mathrm{cl}(U) \boxtimes R^I)$, and similarly for Z'. If both f^*Z' and $f^*Z' \cap Z$ exist and lie in $z_{\mathbb{R}}(U, n)$, then*

$$(f \times \mathrm{id})_*\Big((f \times \mathrm{id})^* Z'_J \cap Z_I\Big) = Z'_J \cap (f \times \mathrm{id})_* Z_I.$$

Proof. For clarity, we omit the notation of cl. By the formal inverse mapping formula above, it suffices to show the statement for currents. After a permutation of the coordinates, it suffices to show the equality for currents on $U \times \square^n$, $n = k + l + r + s$ with Z_I and Z'_J replaced by $Z \cap (U \boxtimes \mathbb{R}^k_- \boxtimes [\omega] \boxtimes \square^{r+s})$ and $Z' \cap (U \boxtimes \square^{k+l} \boxtimes \mathbb{R}^r_- \boxtimes [\omega'])$ respectively, where ω and ω' are holomorphic log forms on $\square^l$ resp $\square^s$ with poles along the real faces of the cube. Similar to the proof of lemma 28 one shows that these two intersection currents exist and are equal to the wedge products $Z \cap (U \boxtimes \mathbb{R}^k_- \boxtimes \square^{l+r+s}) \wedge \mathrm{pr}^* \omega$ and $Z' \cap (U \boxtimes \square^{k+l} \boxtimes \mathbb{R}^r_- \boxtimes \square^s) \wedge \mathrm{pr}^* \omega'$ respectively. Note that the wedge product with these forms commutes with pushforward and pullback along $f \times \mathrm{id}$. In particular, this reduces the statement to the case of real analytic chains. Because $f^*Z' \cap Z$ intersects the real faces properly, $(f \times \mathrm{id})^*(Z' \cap (U \boxtimes \square^{k+l} \boxtimes \mathbb{R}^r_- \boxtimes \square^s))$ and $Z \cap (U \boxtimes \mathbb{R}^k_- \boxtimes \square^{l+r+s})$ also intersect properly. Thus the projection formula for real analytic chains applies and finishes the proof. $\square$

Associativity statements

That a partially defined intersection product is associative does not imply that associativity holds for any triple whose intersection exists. We exhibit associativity in some particular cases.

Lemma 64. *Let Z, Z' be two properly intersecting algebraic cycles in $z_{\mathbb{R}}(U, n)$ such that $Z \cap Z'$ is again in $z_{\mathbb{R}}(U, n)$. Then*

$$\Big((\mathrm{cl}(U) \boxtimes R^n) \cap \mathrm{cl}(Z)\Big) \cap \mathrm{cl}(Z') = (\mathrm{cl}(U) \boxtimes R^n) \cap \mathrm{cl}(Z \cap Z').$$

Moreover, if $n = k + l$, then

$$\mathrm{cl}(Z \cap Z') \cap (\mathrm{cl}(U) \boxtimes R^{k+l}) = \Big(\mathrm{cl}(Z) \cap (\mathrm{cl}(U) \boxtimes R^k \boxtimes \mathrm{cl}(\square^l))\Big) \cap \Big(\mathrm{cl}(Z') \cap (\mathrm{cl}(U) \boxtimes \mathrm{cl}(\square^k) \boxtimes R^l)\Big).$$

Proof. Show the first equality for the 3-term complex: The intersection with fundamental cycles is nothing but componentwise intersection with the fundamental cycle in currents. This has the

effect that the equality can be checked for currents, with R^n replaced by any possible exterior product of $\mathbb{R}_-$, $\log(z)$ and $\mathrm{dlog}(z)$. Thus, after a permutation, the left hand side is an expression of the form $((\mathbb{R}_-^k \boxtimes [\omega]) \cap \mathrm{cl}(Z)) \cap \mathrm{cl}(Z')$ for $k \leq n$ and a log form ω with poles along the faces of the cube. Assume that ω is smooth and thus is irrelevant for the associativity question. That Z, Z' are properly intersecting higher Chow chains which (together with their intersection) are in good position with respect to the real faces is exactly the condition that $\{\mathrm{pr}^* \mathbb{R}_-^k, Z\}, \{Z, Z'\}$ and $\{\mathrm{pr}^* \mathbb{R}_-^k, Z, Z'\}$ intersect suitably and thus their intersection is associative. That this, the expression is equal to $(\mathbb{R}_-^k \boxtimes [\omega]) \cap (\mathrm{cl}(Z) \cap \mathrm{cl}(Z'))$, as was to be shown. If ω is a log form that is locally integrable on the given cycles (which again is ensured by the condition of good real intersection), then the statement extends by approximating ω by smooth forms.
Similarly the second equality can be handled. Considering the components of the triples translates the problem into currents, with each $R^?$ replaced by the exterior product of a real face and a log form. After approximating all log forms by smooth ones, the statement is again reduced to an associativity property for algebraic cycles and real faces. This is implied by the fact that $(Z \cap (U \times F_1 \times \square^l)) \cap (Z' \cap (U \times \square^k \times F_2))$ for any two real faces F_1, F_2 is equal to $(Z \cap Z') \cap (U \times F_1 \times F_2)$.
To prove the statement for $P_{\mathcal{D}}$, observe that the equalities there reduce to the same equalities of currents as in the case of $C_{\mathcal{D}}$. $\square$

3.11 Appendix: Total complexes

For a given morphism $u : A^\bullet \to B^\bullet$ of complexes, the *total complex* of u is defined to be the complex

$$\mathrm{Tot}(A^\bullet \xrightarrow{u} B^\bullet) := A^\bullet \oplus B[-1]^\bullet,$$

with the convention that the differential of the shifted complex is multiplied by -1. That is, the total complex has differential

$$\begin{aligned} A^q \oplus B^{q-1} &\to A^{q+1} \oplus B^q \\ (a, b) &\mapsto (da, u(a) - db). \end{aligned}$$

The total complex has a homological pendant, the mapping cone, to whom it is related by the formula

$$\mathrm{Cone}(A \xrightarrow{u} B)[-1] = \mathrm{Tot}(A \xrightarrow{-u} B).$$

The following lemma summarizes some properties of the total complex.

Lemma 65.

i) Any commutative diagram

$$\begin{tikzcd} A \arrow[r, "u"] & B \\ A' \arrow[u] \arrow[r, "u'"] & B' \arrow[u] \end{tikzcd}$$

induces a morphism of complexes $\mathrm{Tot}(u') \to \mathrm{Tot}(u)$ *(in the obvious way).*

ii) The inclusion resp. projection $B[-1] \to \mathrm{Tot}(u) \to A$ *are morphisms of complexes and give rise to a long exact sequence of cohomology groups*

$$\ldots \to H^{n-1}B \to H^n \mathrm{Tot}(u) \to H^n A \xrightarrow{u} H^n B \to \ldots$$

iii) The total complex $\mathrm{Tot}(u)$ *is acyclic if and only if* u *is a quasi-isomorphism.*

As a consequence of this lemma, replacing the map u by something "quasi-isomorphic" yields isomorphic total complexes.
Indeed, any commutative diagram as described in point i) of the lemma gives rise to a morphism between the long exact sequences from ii) for u' and u. If the vertical maps $A' \to A$ and $B' \to B$ are quasi-isomophisms, then the five-lemma implies that the induced map $\mathrm{Tot}(u') \to \mathrm{Tot}(u)$ is also a quasi-isomorphism.
For another example, assume that A' and B' are subcomplexes. Then $\mathrm{Tot}(u')$ is a subcomplex of $\mathrm{Tot}(u)$ and the corresponding quotient complex is just the total complex of the quotient map $\mathrm{Tot}(A/A' \xrightarrow{\overline{u}} B/B')$. If moreover u' is a quasi-isomorphism (that is, $\mathrm{Tot}(u')$ is acyclic), then the long exact sequence associated to the short exact sequence of the quotient shows that $\mathrm{Tot}(u) \to \mathrm{Tot}(\overline{u})$ is a quasi-isomorphism.

4 Regulator maps for higher Chow chains

We give a definition of a regulator map from higher Chow chains to any family of complexes $C_{\mathcal{D}}$ that satisfies the conditions in 3.9. We first investigate the properties of such an abstract regulator in some detail, before we become concrete and define regulators for the complexes $C_{\mathcal{D}}$ and $P_{\mathcal{D}}$ introduced in 3. We compare the two regulator maps and close with a discussion on the definition of the regulator map.

4.1 An abstract regulator

Assume that for each weight p there is given a functorial collection of complexes of abelian groups $C^{\bullet}_{\mathcal{D}}(X, D, \mathbb{Z}(p))$ for each pair of a projective algebraic manifold X and a normal crossing divisor $D \subset X$. Think of these complexes as "Deligne complexes", whose cohomology calculate the Deligne-Beilinson cohomology of $U := X \setminus D$. This collection should be covariant functorial for proper morphisms. Assume that there exist cycle maps cl, which to a codimension p algebraic cycle in U associate an element in $C^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) := C^{2p}_{\mathcal{D}}(X, \emptyset, \mathbb{Z}(p))$. We want to define higher cycle maps, or regulator maps, from higher Chow chains to these Deligne complexes,

$$\mathrm{reg}_{\mathcal{D}} : z^p(U, n) \dashrightarrow C^{2p-n}_{\mathcal{D}}(X, D, \mathbb{Z}(p)).$$

These maps should have some good properties. First of all, for $n = 0$ they should recover the relative cycle maps, that is, be equal to the composition of cl with the pushforward along $X \to (X, D)$. The regulators for varying n should be compatible, i.e., they should give rise to a morphism of complexes (suitably shifted). We are (in fact this was the starting point of this construction) interested in the multiplicative behavior of this map. Therefore we assume that on $\oplus_p C_{\mathcal{D}}(X, D, \mathbb{Z}(p))$ there exists an exterior product $\boxtimes$ which is additive in both the weight p and the degree, and which furthermore is unitary and associative in the sense of exterior products. Furthermore we assume that there exists a reasonable intersection product on these complexes. All these structures should be compatible in a sense which is made precise in subsection 3.9.
To define the regulator map $\mathrm{reg}_{\mathcal{D}}$, one may proceed as follows. First, compactify the cube to $\overline{\square} = \mathbb{P}_1$ with marked point $\{1\}$ as boundary divisor. The cycle map gives for each higher chain Z of codimension p an element

$$\mathrm{cl}(Z) \in C^{2p}_{\mathcal{D}}(X \times \overline{\square}^n, \mathbb{Z}(p)).$$

Then choose an element $R \in C^1_{\mathcal{D}}(\overline{\square}, \mathbb{Z}(1))$ and form the exterior products $R^n \in C^n_{\mathcal{D}}(\overline{\square}^n, \mathbb{Z}(n))$. Note that $R^0 \in C^0_{\mathcal{D}}(\mathrm{pt}, \mathbb{Z})$ is just the identity element for the external product. Finally define the value of a higher Chow chain Z under the map $\mathrm{reg}_{\mathcal{D}}$ to be

$$\mathrm{reg}_{\mathcal{D}}(Z) := \mathrm{pr}_{(X,D)*}\Big((\mathrm{cl}(U) \boxtimes R^n) \cap \mathrm{cl}(Z)\Big)$$

whenever the intersection exists.

Remark that the pushforward is always defined since the projection $\mathrm{pr}_{(X,D)} : (X \times \overline{\square}^n, \emptyset) \to (X, D)$ has compact fibres, hence is proper. Thus the only obstruction for the regulator map being defined is the non-existence of the intersection of $\mathrm{cl}(U) \boxtimes R^n$ with $\mathrm{cl}(Z)$.
In order to check that the construction is well-defined on higher Chow chains, let $Z = \mathrm{omit}_j^* \tilde{Z}$ be a degenerate cycle, obtained as the pullback of an admissible chain along the map that forgets the j-th cube. It is to show that the regulator $\mathrm{reg}_{\mathcal{D}}(Z)$ is zero. By the compatibility of $\boxtimes$ and intersection, $(\mathrm{cl}(U) \boxtimes R^n) \cap \mathrm{cl}(Z)$ is an exterior product with R in the j-th component. It is therefore enough to show that $(\mathrm{pr}_{\mathrm{pt}})_* R$ vanishes. This pushforward however has real dimension -1, hence must be zero. Consequently, the regulator map descends to a partially defined map on higher Chow chains.
The "abstract regulator map" serves as the prototype of a regulator. It becomes concrete after fixing a choice of the required data (the complexes $C_{\mathcal{D}}(X, D, \mathbb{Z}(p))$ together with products and the element R). This is done in later sections. First, some compatibility conditions are exhibited.

Remarks.

- By the moving lemma for higher Chow groups, the restriction induces a quasi-isomorphism of complexes $z^p(X, D, \bullet) := \frac{z^p(X,\bullet)}{i_* z^{p-1}(D,\bullet)} \to z^p(U, \bullet)$, and similar with subscript $\mathbb{R}$ (see [39, 5.9]). Thus our construction can be given completely in the "framework of pairs".

- Notice that the definition of the regulator map can be rewritten as

$$\mathrm{reg}_{\mathcal{D}}(Z) = \mathrm{pr}_{(X,D)*} \left(\left(\mathrm{pr}^{X \times \overline{\square}^n}_{\overline{\square}^n} \right)^* (R^n) \cap \mathrm{cl}(Z) \right).$$

  In this form, the regulator value of Z can be seen as the inverse image of the current R^n under the (analytic) correspondence given by $\mathrm{cl}(Z)$.

- The regulator is determined by R. Conversely, the image $\underline{R} \in C^1_{\mathcal{D}}(\overline{\square}, 1, \mathbb{Z}(1))$ of R can be recovered from the regulator on $U = \square$ as the value of the diagonal $\Delta \in z^1(\square, 1)$,

$$\underline{R} = \mathrm{reg}_{\mathcal{D}}(\Delta).$$

Domains of definition

In order to prove properties of the regulator map, we will need more specific properties of the intersection product of algebraic cycles subject to the defining element R of the regulator.
A *domain of definition* for $\mathrm{reg}_{\mathcal{D}}$ is a collection of subcomplexes $z^p_R(U, \bullet) \subset z^p(U, \bullet)$ for all p such that

- If $Z \in z^p_R(U, n)$, then $(\mathrm{cl}(U) \boxtimes R^n) \cap \mathrm{cl}(Z)$ exists (that is, $\mathrm{reg}_{\mathcal{D}}(Z)$ is defined).

- It is closed under the exterior product of higher Chow chains.

- Let $f : X \to Y$ be a holomorphic map of pairs, $Z \in z_R(U, n)$ and $Z' \in z_R(V, n)$. For two disjoint tuples $I, J \subset \{0, 1\}^n$ (i.e., their componentwise addition satisfies $I + J \leq 1^n$), write $R^I = R^{i_1} \boxtimes R^{i_2} \boxtimes \ldots \boxtimes R^{i_n}$ and $Z_I := \mathrm{cl}(Z) \cap (\mathrm{cl}(U) \boxtimes R^I)$, and similarly for Z'. If both $f^* Z'$ and $f^* Z' \cap Z$ exist and lie in $z_R(U, n)$, then

$$(f \times \mathrm{id})_* \Big((f \times \mathrm{id})^* Z'_J \cap Z_I \Big) = Z'_J \cap (f \times \mathrm{id})_* Z_I.$$

- For $Z, Z' \in z_R(U,n)$ that intersect properly, the expression $(\mathrm{cl}(U) \boxtimes R^n) \cap \mathrm{cl}(Z) \cap \mathrm{cl}(Z')$ is well defined (associative).

- For $Z, Z' \in z_R(U, n+m)$ whose intersection exists and lies in the domain of definition, $\mathrm{cl}(Z \cap Z') \cap (\mathrm{cl}(U) \boxtimes R^{n+m}) = (\mathrm{cl}(Z) \cap \mathrm{pr}^* R^n) \cap (\mathrm{cl}(Z') \cap \mathrm{pr}^* R^m)$.

For a coefficient ring A, the A-modules $z^p_R(U,n)_A$ are defined by linearity. They give rise to subcomplexes $z^p_R(U,\bullet)_A \subset z^p(U,\bullet)_A$ and to a regulator map with values in $C_{\mathcal{D}}(X, D, A(p))$.

For the following, we assume that a domain of definition has been fixed.

In the case of the particular examples in 4.2 and 4.3, $z_R = z_{\mathbb{R}}$ is the group of higher Chow chains which are in good position to the real faces. That these complexes form a valid domain of definition (for the respective defining elements) is verified in 3.10.

Remark. The third condition of a domain of the regulator implies in particular that

$$(f \times \mathrm{id})_* \Big((\mathrm{cl}(Z) \boxtimes R^n) \cap (f \times \mathrm{id})^* \mathrm{cl}(Z') \Big) = (f \times \mathrm{id})_* (\mathrm{cl}(Z) \boxtimes R^n) \cap \mathrm{cl}(Z'),$$
$$(f \times \mathrm{id})_* \Big((f \times \mathrm{id})^* (\mathrm{cl}(Z') \boxtimes R^n) \cap \mathrm{cl}(Z) \Big) = (\mathrm{cl}(Z') \boxtimes R^n) \cap (f \times \mathrm{id})_* \mathrm{cl}(Z).$$

Compatibility with differential

Whether the regulator is compatible with the differential or not depends crucially on the defining element R.

Lemma 66. *If the boundary of the defining element $dR = \mathrm{cl}((z))$ is the cycle associated to the divisor $(z) = 0 - \infty$, then the abstract regulator transforms the Bloch differential on higher Chow groups into the differential in $C_{\mathcal{D}}$. That is, $d \circ reg(Z) = reg \circ \partial(Z)$ holds for any higher Chow chain $Z \subset U \times \square^n$ in the domain of the regulator.*

Proof. After plugging in the definition of the regulator and abbreviating $\mathrm{pr} = \mathrm{pr}_{(X,D)}$ for any projection $X \times \overline{\square}^? \to (X, D)$, we have to show that

$$d\, \mathrm{pr}_* \Big((\mathrm{cl}(U) \boxtimes R^n) \cap \mathrm{cl}(Z) \Big) = \mathrm{pr}_* \Big((\mathrm{cl}(U) \boxtimes R^{n-1}) \cap \mathrm{cl}(\partial Z) \Big).$$

Since the differential commutes with pushforward, and using the product rule for $\boxtimes$ and $\cap$ (together with $d\mathrm{cl}(Z) = 0$), one gets

$$\begin{aligned} d\, \mathrm{pr}_* \big((\mathrm{cl}(U) \boxtimes R^n) \cap \mathrm{cl}(Z) \big) &= \mathrm{pr}_* \big((\mathrm{cl}(U) \boxtimes dR^n) \cap \mathrm{cl}(Z) \big) \\ &= \mathrm{pr}_* \sum_{i=1}^{n} (-1)^{i+1} \big(\mathrm{cl}(U) \boxtimes R^{i-1} \boxtimes dR \boxtimes R^{n-i} \big) \cap \mathrm{cl}(Z). \end{aligned}$$

Note that by assumption $dR = \mathrm{cl}((z))$ is the difference of the cycles associated to the two points 0 and ∞. Thus it can be interpreted as the pushforward of these points under the inclusion into $\overline{\square}$. Then using the projection formula one gets

$$
\begin{aligned}
&= \mathrm{pr}_* \sum_{i=1}^{n} (-1)^{i+1} (\partial_{i,0} - \partial_{i,\infty})_* \left(\mathrm{cl}(U) \boxtimes R^{n-1}\right) \cap \mathrm{cl}\,(Z) \\
&= \mathrm{pr}_* \sum_{i=1}^{n} (-1)^{i+1} (\mathrm{cl}(U) \boxtimes R^{n-1}) \cap (\partial_{i,0}^* - \partial_{i,\infty}^*) \mathrm{cl}\,(Z) \\
&= \mathrm{pr}_* \sum_{i=1}^{n} (-1)^{i+1} (\mathrm{cl}(U) \boxtimes R^{n-1}) \cap \mathrm{cl}\left((\partial_{i,0}^* - \partial_{i,\infty}^*) Z\right) \\
&= \mathrm{pr}_* \left((\mathrm{cl}(U) \boxtimes R^{n-1}) \cap \mathrm{cl}(\partial Z)\right)
\end{aligned}
$$

by definition of the differential on Bloch's complex. □

We remark that by additivity of the intersection product, the first equality in the preceding calculation shows that changing R^n by a boundary changes the resulting regulator only by a boundary.

Compatibility with intersection products

Next we show compatibility of the abstract regulator map with the product structures on higher Chow chains and on the complexes $C_{\mathcal{D}}$. It suffices to consider integral coefficients; The result for arbitrary coefficients and alternating chains then follows from the linearity of the regulator.

Lemma 67. *Let $Z \in z_R^p(U, n)$ and $Z' \in z_R^q(U, m)$ be two properly intersecting higher Chow chains such that $Z \cap Z'$ also lies in the domain of the regulator. Then the intersection of their regulator values exists and*

$$
\mathit{reg}_{\mathcal{D}}(Z \cap Z') = \mathit{reg}_{\mathcal{D}}(Z) \cap \mathit{reg}_{\mathcal{D}}(Z').
$$

Proof. Using reduction to the diagonal, the statement in question is transferred to the product space $(X \times X, D \boxtimes D)$, where $D \boxtimes D := D \times X + X \times D$ is the exterior self-product of the divisor D on X. In fact, it is to show that the intersection of $\mathrm{reg}_{\mathcal{D}}(Z) \boxtimes \mathrm{reg}_{\mathcal{D}}(Z')$ with the relative fundamental cycle $\mathrm{cl}\Delta_{(X,D)} := \mathrm{pr}_{(X\times X, D\boxtimes D)*}\,\mathrm{cl}(\Delta_U)$, Δ_U the diagonal in $U \times U$, exists and that

$$
\Delta_{(X,D)*} \mathrm{reg}_{\mathcal{D}}(Z \cap Z') = \left(\mathrm{reg}_{\mathcal{D}}(Z) \boxtimes \mathrm{reg}_{\mathcal{D}}(Z')\right) \cap \mathrm{cl}\Delta_{(X,D)}.
$$

Write $\Delta_U^{n,m}$ for the map $U \times \square^{n+m} \to U \times \square^n \times U \times \square^m$ induced by the diagonal, and likewise $\Delta_X^{n,m}$ for the map on the compactifications. Then the intersection product of the two higher Chow chains is given by the pullback $Z \cap Z' = (\Delta_U^{n,m})^*(Z \times Z')$.
Plug this into the definition of the regulator map and abbreviate $\mathrm{pr} := \mathrm{pr}_{(X\times X, D\boxtimes D)}$. Applying the compatibility of the cycle map with pullback and the projection formula then yields that

$$
\begin{aligned}
\Delta_{(X,D)*} \mathrm{reg}_{\mathcal{D}}(Z \cap Z') &= \Delta_{(X,D)*}\, \mathrm{pr}_{(X,D)*} \Big((\mathrm{cl}(U) \boxtimes R^{n+m}) \cap \mathrm{cl}(Z \cap Z') \Big) \\
&= \mathrm{pr}_*\, \Delta_{X*}^{n,m} \Big((\mathrm{cl}(U) \boxtimes R^{n+m}) \cap \mathrm{cl}(Z \cap Z') \Big) \\
&= \mathrm{pr}_*\, \Delta_{X*}^{n,m} \Big((\mathrm{cl}(U) \boxtimes R^{n+m}) \cap (\Delta_X^{n,m})^* \left[\mathrm{cl}(Z) \times \mathrm{cl}(Z')\right] \Big) \\
&= \mathrm{pr}_* \Big(\Delta_{X*}^{n,m} (\mathrm{cl}(U) \boxtimes R^{n+m}) \cap \mathrm{cl}(Z \times Z') \Big).
\end{aligned}
$$

In order to rewrite this expression, denote by τ the map that exchanges the two middle factors and by Δ_U the diagonal in $U \times U$. Use that the fundamental cycle $\mathrm{cl}(U \times U)$ is the identity element for the intersection product, the compatibility of $\cap, \boxtimes$, and that intersection is compatible with biholomorphisms to obtain

$$\begin{aligned}
(\Delta_X^{n,m})_*\big(\mathrm{cl}(U) \boxtimes R^{n+m}\big) &= \tau_*\Big[\mathrm{cl}(\Delta_U) \boxtimes R^n \boxtimes R^m\Big] \\
&= \tau_*\Big[\Big(\mathrm{cl}(U) \boxtimes \mathrm{cl}(U) \boxtimes R^n \boxtimes R^m\Big) \cap \Big(\mathrm{cl}(\Delta_U) \boxtimes \mathrm{cl}(\square^{n+m})\Big)\Big] \\
&= \Big(\mathrm{cl}(U) \boxtimes R^n \boxtimes \mathrm{cl}(U) \boxtimes R^m\Big) \cap \tau_*\Big(\mathrm{cl}(\Delta_U) \boxtimes \mathrm{cl}(\square^{n+m})\Big) \\
&= \Big(\mathrm{cl}(U) \boxtimes R^n \boxtimes \mathrm{cl}(U) \boxtimes R^m\Big) \cap (\mathrm{pr}_{X \times X})^* \mathrm{cl}(\Delta_U).
\end{aligned}$$

Apply this formula to the previous expression and use the commutativity and the associativity of the intersection with fundamental cycles. Then use the projection formula, the compatibility of $\times, \mathrm{cl}$, the distributivity of $\cap, \boxtimes$ and the compatibility of $\boxtimes$ with pushforward, to get

$$\begin{aligned}
&\mathrm{pr}_*\Big(\Delta_{X*}^{n,m}(\mathrm{cl}(U) \boxtimes R^{n+m}) \cap \mathrm{cl}(Z \times Z')\Big) \\
&= \mathrm{pr}_*\Big[(\mathrm{cl}(U) \boxtimes R^n \boxtimes \mathrm{cl}(U) \boxtimes R^m) \cap \mathrm{cl}(Z \times Z') \cap (\mathrm{pr}_{X \times X})^* \mathrm{cl}(\Delta_U)\Big] \\
&= \mathrm{pr}_*\Big[(\mathrm{cl}(U) \boxtimes R^n \boxtimes \mathrm{cl}(U) \boxtimes R^m) \cap (\mathrm{cl}(Z) \boxtimes \mathrm{cl}(Z'))\Big] \cap \mathrm{cl}(\Delta_U) \\
&= \mathrm{pr}_*\Big[\big((\mathrm{cl}(U) \boxtimes R^n) \cap \mathrm{cl}(Z)\big) \boxtimes \big((\mathrm{cl}(U) \boxtimes R^m) \cap \mathrm{cl}(Z')\big)\Big] \cap \mathrm{cl}(\Delta_U) \\
&= \Big[\mathrm{reg}_{\mathcal{D}}(Z) \boxtimes \mathrm{reg}_{\mathcal{D}}(Z')\Big] \cap \mathrm{pr}_{(X \times X, D \boxtimes D)*} \mathrm{cl}(\Delta_U).
\end{aligned}$$ $\square$

Another way to phrase the above compatibilities of the regulator map is by saying that the regulator induces a morphism between (suitable re-graded), partially defined differential graded algebras. Indeed, setting $\mathcal{N}^r(U,p) := z^p(U, 2p-r)$, the regulator becomes a (partially defined) degree preserving map of complexes

$$\mathrm{reg}_{\mathcal{D}} : \bigoplus_p \mathcal{N}^\bullet(U,p) \dashrightarrow \bigoplus_p C^\bullet_{\mathcal{D}}(X, D, \mathbb{Z}(p)).$$

Compatibility with products implies that this is a map of partially defined dg algebras.

It restricts to an everywhere defined map on $\mathcal{N}^r_R(U,p) := z^p_R(U, 2p-r)$, with the partial intersection product defined for those pairs of chains whose intersection exists and lies again in the domain of the regulator.

Functoriality

Assume there is given a smooth morphism $f : U \to U'$ of algebraic manifolds that is induced by a smooth morphism of pairs $f : (X,D) \to (X',D')$ of relative dimension δ. The map $f \times \mathrm{id} : X \times \overline{\square}^n \to X' \times \overline{\square}^n$ is proper, hence induces a pushforward on the Deligne complexes (by functoriality). If moreover the restriction $f|_U$ is proper, then there is also a pushforward on higher Chow chains and one may ask whether these two operations commute.

Lemma 68. *For $f|_U$ proper, the regulator map is compatible with these pushforwards, i.e.,*

$$\begin{tikzcd}
z^p(U,n) \arrow[r, "(f|_U)_*"] \arrow[d, dashed, "reg_{\mathcal{D}}"] & z^{p-\delta}(U',n) \arrow[d, dashed, "reg_{\mathcal{D}}"] \\
C_{\mathcal{D}}^{2p-n}(X,D,\mathbb{Z}(p)) \arrow[r, "f_*"] & C_{\mathcal{D}}^{2p-2\delta-n}(X',D',\mathbb{Z}(p-\delta))
\end{tikzcd}$$

commutes, where $\delta = \dim X - \dim X'$.

Proof. Let $Z \in c^p(U,n)$ such that $\mathrm{reg}_{\mathcal{D}}(Z)$ is defined. Then

$$\begin{aligned}
f_*\mathrm{reg}_{\mathcal{D}}(Z) &= f_* \circ \mathrm{pr}_{(X,D)*}\Big((\mathrm{cl}(U) \boxtimes R^n) \cap \mathrm{cl}(Z)\Big) \\
&= \mathrm{pr}_{(X',D')*} \circ (f \times \mathrm{id}_{\overline{\square}^n})_*\Big((\mathrm{cl}(U) \boxtimes R^n) \cap \mathrm{cl}(Z)\Big).
\end{aligned}$$

Noting that $\mathrm{cl}(U) \boxtimes R^n = (f \times \mathrm{id}_{\overline{\square}^n})^*(\mathrm{cl}(U') \boxtimes R^n)$, the projection formula yields

$$\begin{aligned}
&= \mathrm{pr}_{(X',D')*}\Big((\mathrm{cl}(U') \boxtimes R^n) \cap (f \times \mathrm{id}_{\overline{\square}^n})_*\mathrm{cl}(Z)\Big) \\
&= \mathrm{pr}_{(X',D')*}\Big((\mathrm{cl}(U') \boxtimes R^n) \cap \mathrm{cl}((f|_U)_*Z)\Big) \\
&= \mathrm{reg}_{\mathcal{D}}((f|_U)_*Z).
\end{aligned}$$

The last equality holds because cl is compatible with pushforward. $\square$

The functoriality of the regulator can be used to get rid of the dependence of the good compactification by taking the limit over all good compactifications of U. In fact, if $\mathrm{reg}_{\mathcal{D},(X,D)}$ denotes the regulator map with respect to the compactification (X,D), then the regulator of U can be defined as the inverse limit

$$\mathrm{reg}_{\mathcal{D},U}(Z) := \varprojlim_{(X,D)} \mathrm{reg}_{\mathcal{D},(X,D)}(Z),$$

where the limit is taken over the inverse system formed by proper morphisms $(X,D) \to (X',D')$ between good compactifications of U such that $D \to D'$ and $X \setminus D \to X' \setminus D'$. However, for the remaining part of this work, the regulator will always be considered relative to a fixed compactification.

Compatibility with G_n-action

The construction so far gives a map defined on higher Chow chains with $\mathbb{Z}$-coefficients and, by linearity, for coefficients in arbitrary rings.

Since one usually is more interested in alternating chains, one may want to compute the regulator of a chain that is given as the alternation of some chain Z. The following lemma gives an answer to this problem, stating that the regulator $\mathrm{reg}_{\mathcal{D}}(\mathrm{Alt}(Z)) = \mathrm{reg}_{\mathcal{D}}(Z)$ does not change at all (under some minor conditions that are satisfied e.g. for the regulator in $P_{\mathcal{D}}$ – but not $C_{\mathcal{D}}$). In particular, it says that the regulator value of such a chain has integral coefficients.

Lemma 69. *If the exterior product on $C_{\mathcal{D}}$ is graded-commutative and if R is mapped to $-R$ under $z \mapsto 1/z$, then the diagram below commutes.*

$$\begin{tikzcd}
\mathcal{N}^{\bullet}(U,p)_A \arrow[r, dashed, "reg_{\mathcal{D}}"] \arrow[d, "\mathrm{Alt}"'] & C^{\bullet}_{\mathcal{D}}(X,D,A(p)) \arrow[d, hook] \\
\mathcal{N}^{\bullet}(U,p)^{\mathrm{Alt}}_{A_{\mathbb{Q}}} \arrow[r, dashed, "reg_{\mathcal{D}}"] & C^{\bullet}_{\mathcal{D}}(X,D,A_{\mathbb{Q}}(p))
\end{tikzcd}$$

Proof. The automorphism group G_n of the n-cube acts on $X \times \overline{\square}^n$ and, by functorial pullback, also on the complexes $C_{\mathcal{D}}(X \times \overline{\square}^n, A(q))$. The cycle map is compatible with the action of G_n on higher Chow chains and the Deligne-Beilinson complex. Because G_n is generated by permutations and inversions of the coordinates, the assumptions guarantee that R^n is $(-1)^{\mathrm{Sign}}$-equivariant under G_n, and so

$$g^*(\mathrm{cl}(U) \boxtimes R^n) = (-1)^{\mathrm{Sign}(g)} \cdot \mathrm{cl}(U) \boxtimes R^n$$

for all $g \in G_n$. As a consequence, for any $g \in G_n$ and any $Z \in z^p(U,n)$ that lies in the domain of the regulator,

$$\begin{aligned}
\mathrm{reg}_{\mathcal{D}}(g^*Z) &= \mathrm{pr}_* \Big((\mathrm{cl}(U) \boxtimes R^n) \cap \mathrm{cl}(g^*Z) \Big) \\
&= \mathrm{pr}_* \Big((\mathrm{cl}(U) \boxtimes R^n) \cap g^*\mathrm{cl}(Z) \Big) \\
&= \mathrm{pr}_* \Big(g^*(\mathrm{cl}(U) \boxtimes R^n \cap \mathrm{cl}(Z)) \Big) \cdot (-1)^{\mathrm{Sign}(g)} \\
&= \mathrm{pr}_* \Big(\mathrm{cl}(U) \boxtimes R^n \cap \mathrm{cl}(Z) \Big) \cdot (-1)^{\mathrm{Sign}(g)} \\
&= \mathrm{reg}_{\mathcal{D}}(Z) \cdot (-1)^{\mathrm{Sign}(g)},
\end{aligned}$$

where the second-last equality is due to the fact that $(\mathrm{pr}_X)_* \circ g^* = (\mathrm{pr}_X \circ g^{-1})_* = (\mathrm{pr}_X)_*$. This implies $\mathrm{reg}_{\mathcal{D}} \circ \mathrm{Alt} = \mathrm{reg}_{\mathcal{D}}$ and thus proves the lemma. □

As another consequence of the above lemma, the regulator is (under the conditions stated in the lemma) also compatible with the product on the alternating complexes. That is, the restriction of the regulator map to alternating chains gives a partially defined map of (partially defined) graded-commutative algebras

$$\bigoplus_p \mathcal{N}^{\bullet}(U,p)^{\mathrm{Alt}}_{\mathbb{Q}} \dashrightarrow \bigoplus_p C^{\bullet}_{\mathcal{D}}(X,D,\mathbb{Q}(p)).$$

The same of course is also true for alternating chains in the domain of the regulator: That is, for alternating chains in $\mathcal{N}^r_R(U,p)_{\mathbb{Q}}$ where the product is the usual one on alternating higher Chow chains (restricted to those pairs whose intersection lies again in the domain of the regulator).

Compatibility with exterior products

Lemma 70. *For any two quasi-projective manifolds U, U' with good compactifications (X, D) and (X', D'), the following diagram commutes*

$$\begin{tikzcd} \bigoplus_{\substack{r+s=p\\k+l=n}} z^r(U,k)\otimes_{\mathbb{Z}} z^s(U',l) \arrow[r, dashed, "reg_{\mathcal{D}}"] \arrow[d, "\times"] & \bigoplus_{\substack{r+s=p\\k+l=n}} C^{2r-k}(X,D,\mathbb{Z}(r))\otimes_{\mathbb{Z}} C^{2s-l}_{\mathcal{D}}(X',D',\mathbb{Z}(s)) \arrow[d, "\boxtimes"] \\ z^p(U\times U',n) \arrow[r, dashed, "reg_{\mathcal{D}}"] & C^{2p-n}_{\mathcal{D}}(X\times X', D\boxtimes D',\mathbb{Z}(p)). \end{tikzcd}$$

In other words, $reg_C \circ \times = \boxtimes \circ reg_C$, that is, the regulator is compatible with external products.

Proof. Let $Z \in z^r(U,k)$ and $Z' \in z^s(U',l)$ higher Chow chains that lie in the domain of the regulator map. In order to simplify the notation, write $X = \mathrm{cl}(U)$, $X' = \mathrm{cl}(U')$, $\overline{\square} = \mathrm{cl}(\square)$ etc. The cycle class of the exterior product of the higher Chow chains is $\tau_*(Z \boxtimes Z')$, where τ denotes the obvious map $X \times \overline{\square}^k \times X' \times \overline{\square}^l \to X \times X' \times \overline{\square}^{k+l}$. The definition of the regulator map, the compatibility of the intersection with bihomolorphic maps and the commutativity of the exterior product with algebraic cycles shows that

$$\begin{aligned} \mathrm{reg}_{\mathcal{D}}(Z\times Z') &= \mathrm{pr}_{(X\times X',D\boxtimes D')*}\Big((X\boxtimes X'\boxtimes R^{k+l})\cap \tau_*(Z\boxtimes Z')\Big) \\ &= \mathrm{pr}_{(X\times X',D\boxtimes D')*}\Big((X\boxtimes R^{k}\boxtimes X'\boxtimes R^{l})\cap (Z\boxtimes Z')\Big) \\ &= \mathrm{pr}_{(X\times X',D\boxtimes D')*}\Big(\big[(X\boxtimes R^{k})\cap Z\big]\boxtimes \big[(X'\boxtimes R^{l})\cap Z'\big]\Big) \\ &= \mathrm{reg}_{\mathcal{D}}(Z)\boxtimes \mathrm{reg}_{\mathcal{D}}(Z'), \end{aligned}$$

where in the last equality it is used that $\mathrm{pr}_{(X\times X',D\boxtimes D')*} = \mathrm{pr}_{(X,D)*}\times \mathrm{pr}_{(X',D')*}$. $\square$

Compatibility with higher correspondences

Let X, Y be two smooth projective varieties. A higher Chow chain $C \in z^p(X\times Y, l)$ is called a *higher correspondence*, if one is not as much interested in the chain itself, but on its action on other higher Chow chains. For example, the *pullback along the higher correspondence C* is defined by

$$\begin{gathered} C^* : z^q(Y,n) \dashrightarrow z^{p+q-\dim(X)}(X, l+n) \\ Z \mapsto \Big(\mathrm{pr}^{X\times Y\times\square^l\times\square^n}_{X\times\square^{l+n}}\Big)_* \Big((\mathrm{pr}^{X\times Y\times\square^l\times\square^n}_{X\times Y\times\square^l})^* C \cap (\mathrm{pr}^{X\times Y\times\square^l\times\square^n}_{Y\times\square^n})^* Z\Big). \end{gathered}$$

In the case of Chow chains ($n = 0$), this is the usual pullback along a correspondence (see Fulton [21, Chapter 16]). The pullback along C can be extended to a pullback of alternating chains by composing C^* with the alternating projection.

Lemma 71. *The abstract regulator map is compatible with pullback along higher correspondences. That is, if $C \in z^p(X\times Y, l)$ and $Z \in z^q(Y,n)$ lie in the domain of the regulator and if the pullback C^*Z exists, then*

$$reg_{\mathcal{D}}(C^*Z) = reg_{\mathcal{D}}(C)^* reg_{\mathcal{D}}(Z).$$

The expression on the right-hand side is pullback along the analytic correspondence $reg_{\mathcal{D}}(C)$ *(defined in the obvious sense).*

Proof. In order to simplify the notation, we abbreviate $L = \overline{\square}^l$ and $N = \overline{\square}^n$ and use the "Fulton-way" of writing a pullback/pushforward. For the same reason, we write C instead of $\mathrm{cl}(C)$ and Z instead of $\mathrm{cl}(Z)$.

First, write down the definition of C^* and the regulator and use that the cycle class is compatible with pullback, pushforward and intersection. Then apply the projection formula to obtain elements living on $X \times Y \times L \times N$ and regroup the intersections (which is possible for cycles coming from geometry), to get that

$$\begin{aligned}
\mathrm{reg}_{\mathcal{D}}(C^*Z) &= \mathrm{pr}_{X*}\Big(\mathrm{cl}(C^*Z) \cap \mathrm{pr}_{LN}^{XLN*} R^{l+n}\Big) \\
&= \mathrm{pr}_{X*}\Big(\mathrm{pr}^{XYLN}_{XLN*}\Big(\mathrm{pr}_{XYL}^{XYLN*} C \cap \mathrm{pr}_{YN}^{XYLN*} Z\Big) \cap \mathrm{pr}_{LN}^{XLN*} R^{l+n}\Big) \\
&= \mathrm{pr}_{X*}\Big(\Big(\mathrm{pr}_{XYL}^{XYLN*} C \cap \mathrm{pr}_{YN}^{XYLN*} Z\Big) \cap \mathrm{pr}_{LN}^{XYLN*} R^{l+n}\Big) \\
&= \mathrm{pr}_{X*}\Big(\Big(\mathrm{pr}_{XYL}^{XYLN*} C \cap \mathrm{pr}_{L}^{XYLN*} R^{l}\Big) \cap \Big(\mathrm{pr}_{YN}^{XYLN*} Z \cap \mathrm{pr}_{N}^{XYLN*} R^{n}\Big)\Big).
\end{aligned}$$

Now use that intersection is compatible with pullback along projections (that is, with exterior products) and apply the projection formula several times to conclude

$$\begin{aligned}
&= \mathrm{pr}_{X*}\Big(\mathrm{pr}_{XYL}^{XYLN*}\Big(C \cap \mathrm{pr}_{L}^{XYL*} R^{l}\Big) \cap \mathrm{pr}_{YN}^{XYLN*}\Big(Z \cap \mathrm{pr}_{N}^{YN*} R^{n}\Big)\Big) \\
&= \mathrm{pr}_{X*}\Big(\Big(C \cap \mathrm{pr}_{L}^{XYL*} R^{l}\Big) \cap \mathrm{pr}^{XYLN}_{XYL*}\, \mathrm{pr}_{YN}^{XYLN*}\Big(Z \cap \mathrm{pr}_{N}^{YN*} R^{n}\Big)\Big) \\
&= \mathrm{pr}_{X*}\Big(\mathrm{pr}^{XYL}_{XY*}\Big(C \cap \mathrm{pr}_{L}^{XYL*} R^{l}\Big) \cap \mathrm{pr}_{Y}^{XY*}\, \mathrm{pr}^{YN}_{Y*}\Big(Z \cap \mathrm{pr}_{N}^{YN*} R^{n}\Big)\Big) \\
&= \mathrm{pr}_{X*}\Big(\mathrm{reg}_{\mathcal{D}}(C) \cap \mathrm{pr}_{Y}^{XY*}\, \mathrm{reg}_{\mathcal{D}}(Z)\Big) \\
&= \mathrm{reg}_{\mathcal{D}}(C)^*\mathrm{reg}_{\mathcal{D}}(Z).
\end{aligned}$$

□

More conceptually, define the pullback along the correspondence induced from a Deligne-Beilinson chain $E \in C^k_{\mathcal{D}}(X \times Y, \mathbb{Z}(p))$ to be the partially defined map

$$E^* : C^j_{\mathcal{D}}(Y, \mathbb{Z}(q)) \dashrightarrow C^{j+k-2m}_{\mathcal{D}}(X, \mathbb{Z}(p+q-m)),$$

$m = \dim_{\mathbb{C}} X$, that is given by the usual formula

$$E^*T := (\mathrm{pr}_X)_*\Big(E \cap (\mathrm{cl}(X) \boxtimes T)\Big).$$

The order of the integration in chosen in such a way that the Leibniz rule holds. Indeed, a simple calculation shows that for C, Z, E, T as above,

$$\begin{aligned}
d(E^*T) &= (dE)^*T + (-1)^k E^* dT, \\
\partial(C^*Z) &= (\partial C)^*Z + (-1)^l C^* \partial Z.
\end{aligned}$$

In particular, if C is ∂-closed (and so is a higher Chow cycle), then the pullbacks along C and $E = \mathrm{reg}(C)$ are compatible with respect to regulator map. That is, the commutative diagram deduced from lemma 71 is actually a commuting diagram of complexes,

$$\begin{tikzcd}
z^q(Y,\bullet) \arrow[rr, dashed, "C^*"] \arrow[d, dashed, "\mathrm{reg}_{\mathcal{D}}"] & & z^{p+q-m}(X,l+\bullet) \arrow[d, dashed, "\mathrm{reg}_{\mathcal{D}}"] \\
C_{\mathcal{D}}^{2q-\bullet}(Y,\mathbb{Z}(q)) \arrow[rr, dashed, "\mathrm{reg}_{\mathcal{D}}(C)^*"] & & C_{\mathcal{D}}^{2p+2q-2m-l-\bullet}(X,\mathbb{Z}(p+q-m)).
\end{tikzcd}$$

The abstract regulator on cohomology

In order to extend $\mathrm{reg}_{\mathcal{D}}$ to the whole cohomology, the following condition on R is crucial.

M: The inclusion $z^p_R(U,\bullet)_A \subset z^p(U,\bullet)_A$ is a quasi-isomorphism.

For the two instances of the regulator map we are going to consider, $z^p_R(U,n) = z^p_{\mathbb{R}}(U,n)$ is the group of higher Chow chains which are in good position with respect to the real faces. In this case, the moving lemma of Levine (adapted to the real analytic setting by Kerr/Lewis) states that **M** holds for $A \supset \mathbb{Q}$.

Lemma 72. *If $dR = \mathrm{cl}((z))$ and **M** holds, then $\mathrm{reg}_{\mathcal{D}}$ extends to an everywhere defined map on cohomology classes.*

Proof. Because the inclusion considered in **M** is quasi-surjective, any class $\underline{Z} \in \mathrm{CH}^p(U,n)$ is represented by some $Z \in z^p_R(U,n)$ and we define

$$\mathrm{reg}_{\mathcal{D}}(\underline{Z}) := \mathrm{reg}_{\mathcal{D}}(Z).$$

This definition does not depend on the choice of Z. Indeed, if $\tilde{Z}$ is another representative, then $Z - \tilde{Z}$ is the boundary of an element which, by **M** (quasi-injectivity), can chosen to be in $z^p_R(U,n+1)$. By the assumption on the defining element R, the regulator $\mathrm{reg}_{\mathcal{D}}$ is compatible with differentials, so that $\mathrm{reg}_{\mathcal{D}}(Z) - \mathrm{reg}_{\mathcal{D}}(\tilde{Z}) = \mathrm{reg}_{\mathcal{D}}(Z - \tilde{Z})$ is a boundary. $\square$

Recall that the intersection product of two higher cycle classes $\underline{Z}, \underline{Z}'$ with representing higher Chow cycles Z, Z' is defined to be the unique cohomology class

$$\underline{Z} \cap \underline{Z}' := (\Delta_U^{n,m})^*(Z \times Z' + \partial B) \quad \mathrm{mod} \quad \text{boundaries},$$

whenever there is a higher Chow chain B such that the pullback on the right hand side exists. We need a condition to ensure that B can be chosen in such a way that the representative on the right hand side lies in the domain of definition of the regulator. Therefore consider, for any finite set $\mathcal{S}$ of closed subsets of U,

$$z^p_{\mathcal{S},R}(U,n) := \left\{ Z \in z^p_R(U,n) \,\middle|\, Z \text{ intersects } S \times F \text{ properly } \forall S \in \mathcal{S} \text{ and all faces } F \subset \overline{\square}^n \right\}.$$

The associated complex $z^p_{\mathcal{S},R}(U,\bullet)$ is a subcomplex of $z^p_R(U,\bullet)$ and gives rise to a partially defined algebra $\oplus_p z^p_{\mathcal{S},R}(U,\bullet)$, where the intersection of two higher Chow chains is said to exist if and only if the intersection exists as higher Chow chains and lies again in $\oplus_p z^p_{\mathcal{S},R}(U,\bullet)$. This definition extends linearly to arbitrary coefficient rings A.
Consider the following strengthened form of condition **M**.

M$^+$: The inclusion $z^p_{\mathcal{S},R}(U,\bullet)_A \subset z^p(U,\bullet)_A$ is a quasi-isomorphism for any finite set $\mathcal{S}$.

If $\mathbf{M}^+$ holds, every cycle in $z^p_R(U,n)_A$ can be changed by a boundary in $z^p(U,n)_A$ such that the general pullback along a fixed map exists.
In the case of the regulators considered later, this condition will be satisfied for $A \supset \mathbb{Q}$.

Lemma 73. *If $dR = \mathrm{cl}((z))$ and $\mathbf{M}^+$ holds, then $reg_{\mathcal{D}}$ is compatible with the intersection on cohomology classes, that is,*

$$reg(\underline{Z} \cap \underline{Z}') = reg(\underline{Z}) \cap reg(\underline{Z}').$$

Proof. Choose representatives $Z \in z_R(U,n)$, $Z' \in z_R(U,m)$ of the given higher cycle classes and an admissible chain B such that $(Z \times Z') + \partial B$ can be pulled back along $\Delta^{n,m}_U$. Choose B' such that $\Delta^{n,m*}_U(Z \times Z' + \partial B) + \partial B'$ lies in the domain of the regulator. After replacing B by $B + \Delta^{n,m}_{U*}B'$, we can assume that $B' = 0$ and that the pullback of $Z \times Z' + \partial B$ along $\Delta^{n,m}_U$ lies in the domain of the regulator.
We again identify Z, Z' with the fundamental cycles they represent in $C_{\mathcal{D}}$ and write X instead of $\mathrm{cl}(U)$. Then, similar to lemma 67,

$$\begin{aligned}
&\Delta_* \mathrm{reg}_{\mathcal{D}}\Big(\Delta^{n,m*}_U(Z \times Z' + \partial B)\Big) \\
&\qquad = \mathrm{pr}_{(X\times X, D\boxtimes D)*}\, \Delta^{n,m}_{X*}\Big((X \boxtimes R^{n+m}) \cap \Delta^{n,m*}\big((Z \boxtimes Z') + \partial B\big)\Big) \\
&\qquad = \mathrm{pr}_{(X\times X, D\boxtimes D)*}\, \Big(\Delta^{n,m}_{X*}(X \boxtimes R^{n+m}) \cap \big((Z \boxtimes Z') + \partial B\big)\Big).
\end{aligned}$$

After writing the generalized diagonal as an intersection and noting the associativity of the intersection with algebraic cycles, this becomes

$$= \mathrm{pr}_{(X\times X, D\boxtimes D)*}\, \Big(\tau_*\big(\Delta(X) \boxtimes \overline{\square}^{n+m}\big) \cap \big(X \boxtimes R^n \boxtimes X \boxtimes R^m\big) \cap \big((Z \boxtimes Z') + \partial B\big)\Big)$$

and, using the projection formula, the compatibility of $\cap$ with projection to (X, D) and the definition,

$$\begin{aligned}
&= \mathrm{pr}_{(X\times X, D\boxtimes D)*}\, \Big(\Delta(X) \cap \mathrm{pr}_{(X\boxtimes X)*}\, \Big((X \boxtimes R^n \boxtimes X \boxtimes R^m) \cap ((Z \boxtimes Z') + \partial B)\Big)\\
&= \Delta(X, D) \cap \mathrm{pr}_{(X\times X, D\boxtimes D)*}\, \Big((X \boxtimes R^n \boxtimes X \boxtimes R^m) \cap ((Z \boxtimes Z') + \partial B)\Big)\\
&= \Delta(X, D) \cap \mathrm{reg}_{\mathcal{D}}(Z \boxtimes Z' + \partial B).
\end{aligned}$$

Since Z, Z' (hence $Z \times Z'$) are in the domain of the regulator, it follows from the compatibility of $\mathrm{reg}_{\mathcal{D}}$ with products and differentials that

$$= \Delta(X, D) \cap \Big(\mathrm{reg}_{\mathcal{D}}(Z) \boxtimes \mathrm{reg}_{\mathcal{D}}(Z') + \text{ boundary }\Big).$$

This implies that indeed $\mathrm{reg}_{\mathcal{D}}(\underline{Z} \cap \underline{Z}')$ is the cohomological intersection of $\mathrm{reg}_{\mathcal{D}}(Z)$ and $\mathrm{reg}_{\mathcal{D}}(Z')$. ∎

4.2 Regulator into the 3-term complex

We now apply the previous construction to the 3-term complex of currents from section 3. For U a quasi-projective manifold with smooth compactification X and normal crossing boundary divisor D, this is the complex

$$C_{\mathcal{D}}(X, D, A(p)) = \mathrm{Tot}\left(\mathcal{I}(X, D, A(p)) \oplus F^p\mathcal{D}(X, \log D) \xrightarrow{-\delta+\iota} \mathcal{D}(X, \log D)\right).$$

These complexes inherit the required functoriality properties from the functoriality of the underlying currents (componentwise applied). The pushforward along a morphism $f : X \to Y$ of relative dimension $\rho = \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} X$ has an additional twist, so that $f_*(a, b, c) = (2\pi i)^\rho(f_*a, f_*b, f_*c)$.

For an algebraic cycle Z of codimension p its associated Deligne-Beilinson fundamental cycle in the 3-term complex is given by the tuple $\mathrm{cl}(Z) = (2\pi i)^p([Z], [Z], 0)$, with $[Z]$ being the integral (p, p)-current of integration over the non-singular part Z_{reg}.

The exterior and the intersection product on the 3-term complex, $\boxtimes_\alpha$ and $\cap_\alpha$, has to be one of the associative ones (to make the construction well-defined), that is, only $\alpha = 0$ and $\alpha = 1$ are possible. We choose the parameter $\alpha = 0$ and so the product is given by Beilinson's formula

$$(a, b, c) \boxtimes_0 (\tilde{a}, \tilde{b}, \tilde{c}) = \left(a \boxtimes \tilde{a}\,,\, b \boxtimes \tilde{b}\,,\, c \boxtimes \tilde{b} + (-1)^r a \boxtimes \tilde{c}\right).$$

The intersection product is (partially) defined by the same formula with $\boxtimes$ replaced by $\cap$ (and exists if and only if the right hand side exists).

In order to apply the construction from the preceding section, we need to define a "base" element in $C^1_{\mathcal{D}}(\overline{\square}, \mathbb{Z}(1))$. For that, we copy from [39, 5.3]. Choose the logarithm on $\mathbb{P}_1$ branched over $\mathbb{R}_- = [-\infty, 0]$, with $\mathbb{R}_-$ oriented in such a way that its boundary $\partial\mathbb{R}_- = 0 - \infty = (z)$ is the divisor of the coordinate function. Then define

$$R := (2\pi i[\mathbb{R}_-], [\mathrm{dlog}\, z], [\log z]).$$

From the formula of currents $d[\log z] = [\mathrm{dlog}\, z] - 2\pi i[\mathbb{R}_-]$, as proved for example in [39], one obtains that the above element has differential $dR = 2\pi i\cdot((z), (z), 0)$, that is, the cycle associated to the divisor of the coordinate z.

The n-th exterior power of R is an element in the Deligne-Beilinson complex $C^n(\overline{\square}^n, \mathbb{Z}(n))$. Its components $R^n = (T^n, \Omega^n, L^n)$ can be computed to be (with signs coming from the comparison of the "homological" $(\times)$ and the "cohomological" $(\boxtimes)$ exterior product)

$$\begin{aligned}
T^n &= (-1)^{\binom{n}{2}}(2\pi i)^n[(\mathbb{R}_-)^{\times n}] \\
\Omega^n &= \Omega(z_1, \ldots, z_n) = [\mathrm{dlog}\, z_1 \wedge \ldots \wedge \mathrm{dlog}\, z_n] \\
L^n &= L(z_1, \ldots, z_n) = [\log z_1] \boxtimes \Omega(z_2, \ldots z_n) - 2\pi i[\mathbb{R}_-] \boxtimes L(z_2, \ldots, z_n) \\
&= \sum_{k=0}^{n-1} (-2\pi i)^k [\mathbb{R}_-]^{\boxtimes k} \boxtimes [\log z_{k+1}] \boxtimes [\mathrm{dlog}\, z_{k+2} \wedge \ldots \wedge \mathrm{dlog}\, z_n] \\
&= \sum_{k=0}^{n-1} (-1)^{\binom{k}{2}}(-2\pi i)^k [\underbrace{\mathbb{R}_- \times \ldots \times \mathbb{R}_-}_{k \text{ times}}] \boxtimes [\log z_{k+1}\, \mathrm{dlog}\, z_{k+2} \wedge \ldots \wedge \mathrm{dlog}\, z_n].
\end{aligned}$$

For example, the first exterior powers of R are

$$\begin{aligned}
R^2 &= \Big(-(2\pi i)^2[\mathbb{R}_- \times \mathbb{R}_-], [\mathrm{dlog}\, z_1 \wedge \mathrm{dlog}\, z_2], [\log z_1 \,\mathrm{dlog}\, z_2] - 2\pi i[\mathbb{R}_-] \boxtimes [\log z_2]\Big) \\
R^3 &= \Big(-(2\pi i)^3[\mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_-], [\mathrm{dlog}\, z_1 \wedge \mathrm{dlog}\, z_2 \wedge \mathrm{dlog}\, z_3], \\
&\qquad [\log z_1 \,\mathrm{dlog}\, z_2 \wedge \mathrm{dlog}\, z_3] - 2\pi i[\mathbb{R}_-] \boxtimes [\log z_2 \,\mathrm{dlog}\, z_3] - (2\pi i)^2[\mathbb{R}_- \times \mathbb{R}_-] \boxtimes [\log z_3]\Big).
\end{aligned}$$

Let $Z \in z^p(U, n)$ be a higher Chow chain that lies in the domain of the regulator map. Because $\cap_0$-multiplication with $([Z], [Z], 0)$ is just componentwise intersection with the current represented by Z_{reg}, the regulator of Z is

$$\mathrm{reg}_C(Z) = (2\pi i)^{p-n} \cdot (T_Z, \Omega_Z, L_Z)$$

where

$$\begin{aligned}
T_Z &= (2\pi i)^n \,\mathrm{pr}_{(X,D)*} \Big(([X] \boxtimes T^n) \cap [Z]\Big), \\
\Omega_Z &= \mathrm{pr}_{(X,D)*} \Big(([X] \boxtimes \Omega^n) \cap [Z]\Big), \\
L_Z &= \mathrm{pr}_{(X,D)*} \Big(([X] \boxtimes L^n) \cap [Z]\Big).
\end{aligned}$$

The resulting map agrees (up to $2\pi i$-factors and signs) with the map given in [39].
Note the the above expressions are well-defined for all Z that intersect the real faces (and thus the $\mathbb{R}_-$ components) properly. Thus the restriction to chains in $z^p_{\mathbb{R}}(U, n)$ is everywhere defined and, with $\mathcal{N}^{2p-n}_{\mathbb{R}}(U, p) := z^p(U, n)$, gives rise to a map of partially defined dg algebras

$$\mathrm{reg}_C : \bigoplus_p \mathcal{N}_{\mathbb{R}}(U, p) \to \bigoplus_p C_{\mathcal{D}}(X, D, \mathbb{Z}(p)).$$

Because the product $\bigcap_0$ is graded-commutative only up to homotopy, this regulator map however does not restrict to alternating chains (and thus is not a map of graded-commutative dg algebras).

Remark 7.
- More correctly one should replace the Z in the above formula by Z_{reg}. We make the convention that, before actually performing any integration, 1) the domain of undefinedness of the integrand should be removed from the integration domain, 2) the integration domain should be replaced by its manifold points.
- One could similarly proceed with an arbitrary parameter α, but since in this case the exterior product need not be associative, one has to choose explicitly how to evaluate the iterated products. Different choices give rise to different (though homological equivalent) regulators.

  For $\alpha = 1$, one obtains "reversed formulas", as follows from $t \cap_\alpha t' = (-1)^{|t||t'|} t' \cap_{1-\alpha} t$.

Regulator into the 2-term-complex

The weight p Deligne-Beilinson cohomology can alternatively be computed by a slightly smaller (quotient) complex of currents. Indeed, dividing the 3-term complex by the total complex over $\mathrm{id}_{F^p\mathcal{D}^\bullet(X,D)}$ one obtains the 2-term complex

$$\underline{C}_{\mathcal{D}}(X, D, A(p)) := \mathrm{Tot}\left(\mathcal{I}^\bullet(X, D, A(p)) \xrightarrow{-\delta} \sigma_p \mathcal{D}^\bullet(X, D)\right).$$

with quotient complex $\sigma_p\mathcal{D}^\bullet = \mathcal{D}^\bullet/F^p\mathcal{D}^\bullet$. The regulator map can now be defined by the same formula as in the previous section, with defining element

$$R := (2\pi i[\mathbb{R}_-], [\log z]).$$

But there exist no new interesting products on the 2-term complex (it is a total complex of the same form as the 3-term complex), so that one won't get anything interesting/new. All formulas are obtained from the ones in the 3-term complex by passing to the quotient.

4.3 The regulator into $P_\mathcal{D}$

In order to get a regulator map between (partially defined) graded-commutative differential graded algebras, we will now apply the construction of 4.1 to the complexes $P_\mathcal{D}(X, D, A(p))$. Recall that for (X, D) a good compactification of U, they have been defined as

$$P_\mathcal{D}(X, D, A(p)) := \left\{ \omega \otimes T \in \Lambda_A(x) \otimes_A \mathcal{D}(X, \log D) \text{ such that } \begin{matrix} \omega(0)T \in \mathcal{I}(X, D, A(p)), \\ \omega(1)T \in F^p\mathcal{D}(X, \log D) \end{matrix} \right\}.$$

These complexes inherit all the functorial properties from the complex of currents by letting a morphism act trivially on $\Lambda_A(x)$. To get the correct coefficients, the pushforward is twisted in exactly the same way as the pushforward for $C_\mathcal{D}$. A codimension p cycle Z in X is represented by the constant path $\mathrm{cl}(Z) = (2\pi i)^p[Z]$.

They are equipped with an A-linear exterior product coming from the wedge product on $\Lambda_A(x)$ and the exterior product of currents. Explicitly, $(\omega \otimes T) \boxtimes (\eta \otimes S) := (-1)^{|S||\eta|}\omega \wedge \eta \otimes (S \boxtimes T)$. Similarly, the intersection product is defined by replacing the exterior product in the above formula with the $\cap$ product. It is defined whenever the intersection of the underlying currents is defined and has the correct type (lies in $P_\mathcal{D}$). The two products are both graded-commutative in their sense.

For the underlying element of the regulator we choose the element in $P^1_\mathcal{D}(\overline{\square}, \mathbb{Z}(1))$ defined as

$$R := (1 - x)(2\pi i)[\mathbb{R}_-] + x[\mathrm{dlog}\, z] + dx[\log z].$$

Its relation to the element that defines the regulator for the 3-term complex is described in 4.4.

Exterior multiplication yields $R^n = R \boxtimes \ldots \boxtimes R$, which for small values of n is (with $2\pi i$-factors before $\mathbb{R}_-$ omitted) given by the currents on $\overline{\square}^n$:

$$\begin{aligned} R^2 =& (1-x)^2\mathbb{R}_- \boxtimes \mathbb{R}_- + x^2\,\mathrm{dlog} \boxtimes \mathrm{dlog} + x(1-x)[\mathbb{R}_- \boxtimes \mathrm{dlog} + \mathrm{dlog} \boxtimes \mathbb{R}_-] \\ &+ x dx[\log \boxtimes \mathrm{dlog} - \mathrm{dlog} \boxtimes \log] + (1-x)dx[\log \boxtimes \mathbb{R}_- - \mathbb{R}_- \boxtimes \log] \end{aligned}$$

$$\begin{aligned} R^3 =& (1-x)^3\mathbb{R}_- \boxtimes \mathbb{R}_- \boxtimes \mathbb{R}_- + (1-x)^2x[\mathbb{R}_- \boxtimes \mathbb{R}_- \boxtimes \mathrm{dlog} + \mathbb{R}_- \boxtimes \mathrm{dlog} \boxtimes \mathbb{R}_- + \mathrm{dlog} \boxtimes \mathbb{R}_- \boxtimes \mathbb{R}_-] \\ &+ (1-x)x^2[\mathbb{R}_- \boxtimes \mathrm{dlog} \boxtimes \mathrm{dlog} + \mathrm{dlog} \boxtimes \mathbb{R}_- \boxtimes \mathrm{dlog} + \mathrm{dlog} \boxtimes \mathrm{dlog} \boxtimes \mathbb{R}_-] + x^3\,\mathrm{dlog} \boxtimes \mathrm{dlog} \boxtimes \mathrm{dlog} \\ &+ x^2dx[\log \boxtimes \mathrm{dlog} \boxtimes \mathrm{dlog} - \mathrm{dlog} \boxtimes \log \boxtimes \mathrm{dlog} + \mathrm{dlog} \boxtimes \mathrm{dlog} \boxtimes \log] \\ &+ (1-x)x dx[\log \boxtimes \mathbb{R}_- \boxtimes \mathrm{dlog} - \mathbb{R}_- \boxtimes \log \boxtimes \mathrm{dlog} + \mathbb{R}_- \boxtimes \mathrm{dlog} \boxtimes \log] \\ &+ (1-x)x dx[\log \boxtimes \mathrm{dlog} \boxtimes \mathbb{R}_- - \mathrm{dlog} \boxtimes \log \boxtimes \mathbb{R}_- + \mathrm{dlog} \boxtimes \mathbb{R}_- \boxtimes \log] \\ &+ (1-x)^2dx[\log \boxtimes \mathbb{R}_- \boxtimes \mathbb{R}_- - \mathbb{R}_- \boxtimes \log \boxtimes \mathbb{R}_- + \mathbb{R}_- \boxtimes \mathbb{R}_- \boxtimes \log]. \end{aligned}$$

In general, $R^n = R_0^n + R_1^n$ where R_i^n consists of those summands of R^n whose dx-degree is i. They satisfy

$$\begin{aligned} R_0^{n+1} &= (1-x)2\pi i R_0^n \boxtimes \mathbb{R}_- + xR_0^n \boxtimes \mathrm{dlog} \\ R_1^{n+1} &= (-1)^n dx R_0^n \boxtimes \log + (1-x)2\pi i R_1^n \boxtimes \mathbb{R}_- + xR_1^n \boxtimes \mathrm{dlog}\,. \end{aligned}$$

Thus R_0^n consists of 2^n summands: all possible combinations built from dlog and $\mathbb{R}_-$. The degree one part R_1^n grows faster: It consists of $n2^{n-1}$ summands, 2^{n-1} for each position where the log term can be placed.

The intersection with R^n is well-defined for all higher Chow chains that properly intersect the real faces and so the regulator is the map $z^p_{\mathbb{R}}(U,n) \to P^{2p-n}_{\mathcal{D}}(X,D,\mathbb{Z}(p))$, defined by

$$\mathrm{reg}_P(Z) = (2\pi i)^p \,\mathrm{pr}_{(X,D)*}\big(([X] \boxtimes R^n) \cap [Z]\big).$$

Again using $d[\log z] = [\mathrm{dlog}\, z] - 2\pi i[\mathbb{R}_-]$, one verifies that $dR = 2\pi i \cdot [0-\infty] = \mathrm{cl}((z))$. This guaranties that reg_P induces a map of complexes. The inversion map $\tau : z \mapsto \frac{1}{z}$ is an automorphism of $\square$ that preserves the chosen branch $\square \setminus \mathbb{R}_-$ of the logarithm and reverses the orientation of $\mathbb{R}_-$. One finds that $\tau^*[\log z] = -[\mathbb{R}_-]$ and $\tau^*[\mathrm{dlog}\, z] = -[\mathrm{dlog}\, z]$, so that $\tau^* R = -R$. As a consequence, the sequence of lemmata in 4.1 implies that this indeed gives rise to a morphism of partially defined graded-commutative differential graded algebras

$$\mathrm{reg}_P : \bigoplus_p \mathcal{N}^\bullet_{\mathbb{R}}(U,p)^{\mathrm{Alt}}_{\mathbb{Q}} \to \bigoplus_p P^\bullet_{\mathcal{D}}(X,D,\mathbb{Q}(p)).$$

4.4 Comparison

We now compare the two regulator maps with values in $P_{\mathcal{D}}$ resp. $C_{\mathcal{D}}$. Recall from 3.3 that these two complexes are related by the evaluation homomorphism

$$\begin{aligned} ev : P^\bullet_{\mathcal{D}}(X,D,A(p)) &\to C^\bullet_{\mathcal{D}}(X,D,A(p)) \\ \omega \otimes T &\mapsto \left(\omega(0)T, \omega(1)T, \int_{[0,1]} \omega \cdot T\right). \end{aligned} \tag{4.1}$$

which turned out to be a quasi-isomorphism whenever $\mathbb{Q} \subset A$, a quasi-inverse being

$$\begin{aligned} s : C^\bullet_{\mathcal{D}}(X,D,A(p)) &\to P^\bullet_{\mathcal{D}}(X,D,A(p)) \\ (a,b,c) &\mapsto (1-x)\otimes a + x \otimes b + dx \otimes c. \end{aligned}$$

Note that the underlying elements of the two regulators (R^1_P and R^1_C, say) can be obtained from each other by applying the map ev resp. its splitting.

Recall moreover that although ev is just a map of vector spaces (not of algebras), transporting the intersection product from $P_{\mathcal{D}}$ to $C_{\mathcal{D}}$ using the splitting gives the product $\bigcap_{1/2}$. Thus on homology classes the product on $P_{\mathcal{D}}$ is equal to $\bigcap_{1/2}$ and hence to $\bigcap_0$, since the two products are homotopic. This showed that ev induces an isomorphism of algebras on homology.

For the construction of the two regulators, this means that the elements R^n_C, R^n_P in the definition of the two regulators differ only by a boundary. That is, $ev(R^n_P)$ equals R^n_C up to boundaries, and $s(R^n_P)$ equals R^n_P up to boundaries. Using this, we can show that the two regulator maps are isomorphic on cohomology:

Lemma 74. *For $\mathbb{Q} \subset A$, the non-commutative diagram of complexes*

$$\begin{tikzcd} z^p_{\mathbb{R}}(U,\bullet)_A \arrow[r, "reg_C"] \arrow[d, "reg_P"'] & C^{2p-\bullet}_{\mathcal{D}}(X,D,A(p)) \\ P^{2p-\bullet}_{\mathcal{D}}(X,D,A(p)) \arrow[ur, "qIso"'] & \end{tikzcd}$$

commutes after passage to (co-)homology.

Proof. ev is compatible with pushforward and, by lemma 58, ev is also compatible with the intersection/exterior product with cycle classes. For example, $ev(W \cap \mathrm{cl}(Z)) = ev(W) \cap_\alpha \mathrm{cl}(Z)$ for all $W \in P_{\mathcal{D}}$, $\alpha \in \mathbb{R}$ and all higher Chow chains Z. Note that the first $\mathrm{cl}(Z)$ denotes the fundamental cycle in $P_{\mathcal{D}}$ whereas the second denotes the fundamental cycle in $C_{\mathcal{D}}$. Using this,

$$\begin{aligned} ev \circ \mathrm{pr}_*\Big((\mathrm{cl}(U) \boxtimes R^n_P) \cap \mathrm{cl}(Z)\Big) &= \mathrm{pr}_*\Big(ev(\mathrm{cl}(U) \boxtimes R^n_P) \cap_0 \mathrm{cl}(Z)\Big) \\ &= \mathrm{pr}_*\Big((\mathrm{cl}(U) \boxtimes_0 ev(R^n_P)) \cap_0 \mathrm{cl}(Z)\Big) \\ &= \mathrm{pr}_*\Big((\mathrm{cl}(U) \boxtimes_0 (R^n_C + \text{boundary})) \cap_0 \mathrm{cl}(Z)\Big) \\ &= \mathrm{pr}_*\Big((\mathrm{cl}(U) \boxtimes_0 R^n_C) \cap_0 \mathrm{cl}(Z)\Big) + \text{boundary}. \end{aligned}$$

Thus one obtains $ev(\mathrm{reg}_P(Z)) \equiv \mathrm{reg}_C(Z)$ on cohomology. □

The above proof can be refined to give a result for chains instead of mere cycle classes. The key observation is the following.

Lemma 75.

$$ev(R^n_P) = \mathrm{Alt}_*(R^n_C).$$

Proof. Note that each generator of $(\mathbb{Z}/2)^n \subset G_n$ acts on R^n_C by a minus sign and thus $\mathrm{Alt}_* R^n_C = \mathrm{alt}_* R^n_C$, where alt denotes the alternation with respect to the action of the symmetric group S_n. Hence it suffices to show that $ev(R^n_P) = \mathrm{alt}_*(R^n_C)$. In order to do this, compare the first two and the last component of the two triples separately:

- Evaluating $x = 0$ in R^n_P makes all summands to zero except $(2\pi i)^n [\mathbb{R}_-]^{\boxtimes n}$, which is just the first component of R^n_C. Since this is symmetric, it is also equal to the first component of $\mathrm{alt}_* R^n_C$. Similar for the second component.

- Both $\mathrm{alt}_*(R^n_C)_3$ and the dx-degree-1-part of R^n_P are spanned by monomials of the form $g_* M$ where $g \in S_n$ and (we omit the $(2\pi i)^k$ factor)

  $$M = (\mathbb{R}_-)^{\boxtimes k} \boxtimes \log \boxtimes \mathrm{dlog}^{\boxtimes (n-k-1)}.$$

  In the former case the coefficients are rational numbers and in the latter case the coefficients are 1-forms ω_M. We have to compare the coefficients before $g_* M$ that occur in (the 3rd components of) $ev(R^n_P)$ and $\mathrm{alt}_*(R^n_C)$.

  On the one hand, the coefficient in $ev(R^n_P)_3$ is $\int_0^1 \omega_{g_* M}$, that is,

  $$\int_0^1 (1-x)^k x^{n-k-1} dx = \frac{k!(n-k-1)!}{n!}$$

  as follows from induction – or using properties of the beta function.

On the other hand, the coefficient before g_*M in $\mathrm{alt}_*(R_C^n)_3$ is just the coefficient before g_*M in alt_*g_*M for type reasons. This is equal to the coefficient of M in alt_*M, which is

$$\frac{|\text{stabilizer of } M|}{|S_n|} = \frac{k!(n-k-1)!}{n!}. \qquad \square$$

Lemma 76. *If $\mathbb{Q} \subset A$, one has*

$$ev \circ reg_P = reg_C \circ \mathrm{Alt}.$$

Moreover, for any A, the first two components of $ev \circ r_P$ and r_C are equal, that is,

$$ev \circ reg_P(Z) = \big(reg_C(Z)_0, reg_C(Z)_1,\ rest\ \big)$$

for any $Z \in z^p_{\mathbb{R}}(U, n)_A$.

Proof. The second statement follows immediately, as indicated in the proof of lemma 75, from the definitions and is omitted. Using lemma 75, the first statement can be proven as follows. Starting similar to the proof of lemma 74, compute

$$\begin{aligned} ev \circ r_P(Z) &= \mathrm{pr}_* \Big(ev(\mathrm{cl}(U) \boxtimes R_P^n) \cap_0 \mathrm{cl}(Z)\Big) \\ &= \mathrm{pr}_* \Big((\mathrm{cl}(U) \boxtimes_0 ev(R_P^n)) \cap_0 \mathrm{cl}(Z)\Big) \\ &= \mathrm{pr}_* \Big((\mathrm{cl}(U) \boxtimes_0 \mathrm{Alt}_*(R_C^n)) \cap_0 \mathrm{cl}(Z)\Big) \\ &= \mathrm{pr}_* \mathrm{Alt}_* \Big((\mathrm{cl}(U) \boxtimes_0 R_C^n) \cap_0 \mathrm{Alt}_*(\mathrm{cl}(Z))\Big). \end{aligned}$$

Observing that $\mathrm{pr} \circ \mathrm{Alt} = \mathrm{pr}$ and $\mathrm{Alt}_*\mathrm{cl}(Z) = \mathrm{cl}(\mathrm{Alt}Z)$ finishes the proof. $\square$

Remark 8. Lemma 75 can be sharpened such that $ev(R_P^n) = (\mathrm{id}, \mathrm{id}, \mathrm{alt})_*(R_C^n)$ even with integral coefficients. Similarly, lemma 76 has a refined version for integral coefficients saying that $ev \circ \mathrm{reg}_P(Z) = (2\pi i)^p \, \mathrm{pr}_* \big((\mathrm{id}, \mathrm{id}, \mathrm{alt})_*(\mathrm{cl}(U) \boxtimes R_C^n) \cap_0 \mathrm{cl}(Z)\big)$.

4.5 On the defining element of the regulator

A fundamental ingredient in the definition of the regulator map is – when the complexes $C_{\mathcal{D}}$ and products on them are already chosen – the choice of the defining element $R \in C^1_{\mathcal{D}}(\overline{\square}, \mathbb{Z}(1))$. This choice of course is not unique. If $\tilde{R}$ is another element such that $d\tilde{R} = \mathrm{cl}((z))$, then $R^1 - \tilde{R}^1$ is d-closed. Because $H^1_{\mathcal{D}}(\overline{\square}, \mathbb{Z}(1)) = H^0(\mathbb{P}_1, \mathcal{O}^*) = \mathbb{C}^*$, there is a whole $\mathbb{C}^*$-family of such elements. Since it can be seen from the construction of the regulator that any cohomological equivalent choice leads to a cohomological equivalent regulator map, this is essentially all the ambiguity that can happen.

We want to motivate the choice of R made in 4.2 and for that consider R as a triple in the relative complex $C^1_{\mathcal{D}}(\overline{\square}, 1, \mathbb{Z}(1))$. Having found a description of the latter, any two lifts to $C^1_{\mathcal{D}}(\overline{\square}, \mathbb{Z}(1))$ will differ only by an element "on-1" and which – under the assumption that we are working with more geometric (that is, normal) currents – will arise as the pushforward of an element in $C_{\mathcal{D}}(1, \mathbb{Z})$. Because R consists of currents of dimension ≥ 1, this pushforward has to be zero and thus the lift is unique.

Recall that $R \in C^1_{\mathcal{D}}(\overline{\square}, 1, \mathbb{Z}(1))$ can be recovered from the regulator map on complexes, and even from it's restriction to $z^1_{\mathbb{R}}(\square, 1) \to C^1_{\mathcal{D}}(\overline{\square}, 1, \mathbb{Z}(1))$ as the image of the diagonal $\Delta \subset \square^2$. As the diagonal is just the graph over the identity map on $\square$, giving the defining element, and thus the regulator, is the same as giving a map

$$\mathcal{O}_{\mathbb{R}}(\square) \to C^1_{\mathcal{D}}(\overline{\square}, 1, \mathbb{Z}(1))$$

from the set of those holomorphic functions on $\square$ whose graphs in $\square^2$ intersect the real faces properly.

In order to define such a map, start with the exponential sequence on $\square$,

$$0 \to \mathbb{Z}(1) \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 1.$$

This sequence can be seen as a quasi-isomorphism $(\mathbb{Z}(1) \to \mathcal{O}) \to \mathcal{O}^*[-1]$ from the sheaf-theoretic Deligne complex on $\square$ to the sheaf of non-vanishing holomorphic functions (shifted by -1). The left-hand side can be resolved by currents, by means of the quasi-isomorphisms

$$\mathbb{Z}(1) \xrightarrow{\sim} \mathcal{I}_{\square, \mathbb{Z}(1)} \quad \text{and} \quad \mathcal{O} \xrightarrow{\sim} \operatorname{Tot}\Big(F^1\mathcal{D}_{\square} \to \mathcal{D}_{\square}\Big)[1]$$

where the first quasi-isomorphism is induced by the inclusion of locally constant functions into locally integral currents, and the second one by the map $f \mapsto (df, f)$. Together, they induce a quasi-isomorphism of sheaves on $\square$,

$$(\mathbb{Z}(1) \to \mathcal{O}) \xrightarrow{\sim} C_{\mathcal{D}, \square, \mathbb{Z}(1)},$$

between the sheaf-theoretic Deligne complex and the sheafified 3-term complex.

The passage to sheaves on $\overline{\square}$ is accomplished by pushing these complexes forward along the inclusion $j : \square \to \overline{\square} = \square \cup \{1\}$. This results in the diagram

$$\begin{tikzcd}
 & & & j_*\big(\mathbb{Z}(1) \to \mathcal{O}\big) \arrow[dddlll] \arrow[dddrrr] & & & \\
 & & & \rotatebox[origin=c]{90}{$\in$} & & & \\
 & & & & & & \\
j_*\mathcal{O}^*_{\square}[-1] & \ni & \exp(f) & (n,f) \arrow[l, mapsto] \arrow[r, mapsto] & (n,[df],[f]) & \in & j_*C_{\mathcal{D},\square,\mathbb{Z}(1)} \\
 & & & & & & C_{\mathcal{D},(\overline{\square},1),\mathbb{Z}(1)} \arrow[u]
\end{tikzcd}$$

where the right lower map is the restriction map from the sheafified Deligne-Beilinson complex on $(\overline{\square}, 1)$ to the pushforward of the sheafified Deligne complex on $\square$.

After passing to degree 1 and global sections, this gives a relation between elements in $\mathcal{O}^*(\square)$ and elements in $C^1_{\mathcal{D}}(\overline{\square}, 1, \mathbb{Z}(1))$. In order to make this into a map from invertible holomorphic functions on $\square$ to $C^1_{\mathcal{D}}(\overline{\square}, 1, \mathbb{Z}(1))$, one has to make a choice of a logarithm, that is, one has to specify a branch locus (here $\mathbb{R}_{\_}$) and a branch (here $\log(1) = 0$). Then the logarithm function is well-defined for any function f whose image set does not meet the branch locus (i.e., $f^{-1}\mathbb{R}_{\_} = \emptyset$), and in this case f corresponds to the triple $(0, [\operatorname{dlog} f], [\log f])$.

Since we are working with currents, the expression $[\log f]$ even makes sense for holomorphic functions that meet the branch locus properly, that is, $f^{-1}\mathbb{R}_{\_}$ has the correct (real) codimension (or, no top-dimensional piece is mapped into $\mathbb{R}_{\_}$). These are just the functions in $\mathcal{O}_{\mathbb{R}}(\square)$. Since for such an f the resulting triple need no longer be d-closed, one has – in order to get a homomorphism of complexes – to correct it by the residue current in the first component. This yields the map

$$\mathcal{O}_{\mathbb{R}}(\square) \to C^1_{\mathcal{D}}(\overline{\square}, 1, \mathbb{Z}(1)), \quad f \mapsto \Big([f^{-1}(\mathbb{R}_{\text{-}}), [d\log f], [\log f]\Big).$$

The image of the identity id $\in \mathcal{O}_{\mathbb{R}}(\square)$ under this map gives exactly the formula for the element $R \in C^1(\overline{\square}, \mathbb{Z}(1))$ used in 4.2.

As a final remark, we come back to the ambiguities mentioned in the first two paragraphs of this section. The $\mathbb{C}^*$-action on cohomology occurring there comes from the $\mathbb{C}^*$-action on $\mathcal{O}^*$ so that, tracing the previous diagram, $\lambda \in \mathbb{C}^*$ acts on the 3-term complex by $(\gamma, \omega, f) \mapsto (\gamma, \omega, f + \log(\lambda))$. In particular, essentially all the other choices for R (for a fixed choice of logarithm) are

$$\Big([\mathbb{R}_{\text{-}}], [\mathrm{dlog}\, z], [\log(z) + \log(\lambda)]\Big)$$

where $\lambda \in \mathbb{C}^*$.

5 Examples

In this section, we compute some explicit formulas for the regulator map reg_P. First, we consider the (important) special case where U is a point. Then we give formulas for the regulator values of higher Chow chains in small cubical degree $n \leq 3$. One such chain is Totaro's cycle in $\square^3$, which we consider in more detail. After that, we consider the regulator values of graph cycles and give simple computable formulas for them.

5.1 Regulator of a point

Consider the special case of $U = X = \operatorname{Spec}\mathbb{C}$ being a point and coefficients $A = \mathbb{Z}$. Recall from section 3.4 that the space of currents over $\operatorname{Spec}\mathbb{C}$ identifies with the field of complex numbers $\mathbb{C}$, and that under this identification the complex $P_{\mathcal{D}}(\mathrm{pt}, \mathbb{Z}(p))$ becomes the subset of complex valued polynomial forms $\Lambda_{\mathbb{C}}(x) = \mathbb{C}[x, dx]$,

$$P_{\mathcal{D}}(\mathrm{pt}, \mathbb{Z}(p)) = \left\{ \omega \in \mathbb{C}[x, dx] \;\middle|\; \begin{array}{l} \omega(0) \in \mathbb{Z}(p) \text{ and} \\ \omega(1) = 0 \text{ if } p > 0 \end{array} \right\}.$$

The regulator into $P_{\mathcal{D}}$ is given by maps $z^p_{\mathbb{R}}(\mathrm{pt}, n) \to \mathbb{C}[x, dx]^{2p-n}$ for $n, p \geq 0$. In particular, there are only two non-trivial cases to consider: $n = 2p$ and $n = 2p - 1$. In each other case, the regulator value of a higher Chow chain Z is zero, because the intersection of $[Z]$ with R^n will have a degree which is too high or low to survive the integration step underlying the pushforward to the point. The two cases $n = 2p$ and $n = 2p - 1$ correspond to the two summands in the decomposition $R^n = R^n_0 + R^n_1$: the regulator

$$\mathrm{reg}_P(Z) = (2\pi i)^p \, \mathrm{pr}_* \left(R^n \cap [Z]\right)$$

in the first case is computed by the summand R^n_0 of R^n and in the second case by R^n_1. In both cases the intersection is seen to be zero dimensional, i.e., a sum of points (in the second case tensored with dx). Thus the pushforward is just the sum of the coefficients of these points, multiplied with the relative dimension $(2\pi i)^{-n}$.

In the first case only two kinds of currents occur. One can use shuffles to sort them and write

$$R^n_0 \cap [Z] = \sum_i \sum_{\sigma \in \mathrm{Sh}(i, n-i)} (-1)^{|\sigma|} (2\pi i)^i (1 - x)^i x^{n-i} \sigma_* \left(\mathbb{R}_-^{\boxtimes i} \boxtimes \mathrm{dlog}^{\boxtimes n-i}\right) \cap [Z].$$

Note that the summation index starts at $i = p$ since otherwise the resulting current lies in the $n - i + p > n$-th part of the Hodge filtration and thus is zero.

In the second case, an additional log term shows up. Thinking of this log term as belonging to the set of dlogs, we can again use shuffles to sort the terms in R^n_1. In fact, R^n_1 can be written as the sum over all currents of the form $\sigma_* \left(\mathbb{R}_- \boxtimes \ldots \boxtimes \mathbb{R}_- \boxtimes \mathrm{dlog} \boxtimes \ldots \underset{j}{\log} \ldots \boxtimes \mathrm{dlog}\right)$ (with suitable coefficients), with σ shuffling the forms and the $\mathbb{R}_-$ together. Becoming explicit,

$R_1^n \cap [Z]$ is the current

$$\sum_{\substack{i=0\dots n-1,\\ j=i\dots n-1}} \sum_{\sigma\in \mathrm{Sh}(i,n-i)} (-1)^{|\sigma|+j}(2\pi i)^i(1-x)^i x^{n-i-1}dx\otimes\sigma_*\left(\mathbb{R}_-^{\boxtimes i}\boxtimes \mathrm{dlog}^{\boxtimes j-i}\boxtimes\log\boxtimes\mathrm{dlog}^{\boxtimes n-j-1}\right)\cap[Z].$$

Recall – again from section 3.4 – that the Deligne cohomology of $\operatorname{Spec}\mathbb{C}$ can be computed by a very simple complex concentrated in a single degree. Indeed, the projection to resp. quotient by $\mathbb{Z}(p)$ give quasi-isomorphisms

$$\begin{aligned} C_{\mathcal{D}}(\mathrm{pt},\mathbb{Z}(p)) &= \big(\mathbb{Z}(p)\oplus F^p\mathbb{C}\to\mathbb{C}\big)\\ &\xrightarrow{qIso}\begin{cases}\mathbb{Z} & p=0,\\ \mathbb{C}/\mathbb{Z}(p)[-1] & p>0.\end{cases}\end{aligned}$$

Denote by $\widetilde{ev}$ the composition of the evaluation map with the above quasi-isomorphism. Then the regulator map into this complex is the composition

$$\mathcal{N}_{\mathbb{R}}^{\bullet}(\mathrm{pt},p)=z_{\mathbb{R}}^p(\mathrm{pt},2p-\bullet)\xrightarrow{\mathrm{reg}_P}P_{\mathcal{D}}^{\bullet}(\mathrm{pt},\mathbb{Z}(p))\xrightarrow{\widetilde{ev}}\begin{cases}\mathbb{Z} & p=0,\\ \mathbb{C}/\mathbb{Z}(p)[-1] & p>0.\end{cases}$$

This is the unique map that for $p=\bullet=0$ sends an element $k\cdot[\mathrm{pt}]$ to $k\in\mathbb{Z}$, and for $p>0$ sends a cycle $Z\in z^p(\mathrm{pt},2p-1)$ to $\int_0^1\mathrm{reg}_P(Z)$. In each other case, the map is zero. After passing to cohomology groups and taking into account that $\mathcal{N}_{\mathbb{R}}^{\bullet}$ computes motivic cohomology, one obtains that the induced regulator

$$\mathrm{reg}_P: H^l_{\mathcal{M}}(\mathrm{pt},\mathbb{Z}(p))\longrightarrow\begin{cases}\mathbb{Z} & p=0, l=0,\\ \mathbb{C}/\mathbb{Z}(p) & p>0, l=1,\\ 0 & \text{else.}\end{cases}$$

is given by

$$\begin{array}{ccccc} k\cdot\mathrm{pt} & \mapsto & k & & p=0, l=0,\\ Z & \mapsto & \int_0^1\mathrm{reg}_P(Z) & & p>0, l=1.\end{array}$$

For $n=p=1$ one obtains (see the formula in the next subsection) the logarithm map

$$\mathrm{reg}_P=\log: H^1_{\mathcal{M}}(\mathrm{pt},\mathbb{Z}(1))=\mathbb{C}^*\to\mathbb{C}/2\pi i\mathbb{Z}.$$

Formulas for $p>0$ can be obtained by evaluating (integrating over the formal variable x) the formula for $R_1^n\cap[Z]$ and pushing the result down to a point.

5.2 Formulas for $n\le 3$

We now calculate some low-dimensional examples of the regulator map reg_P and their images in the 3-term-complex under the evaluation map ev. They are easily read off from the computation of R^n in section 4.3.

- First of all, consider the case where $n=0$. Here reg_P is just the (relative) cycle map from the usual Chow chains on U to the Deligne-Beilinson complex $P_{\mathcal{D}}$ on (X,D). Composition with ev gives the cycle map into $C_{\mathcal{D}}$.

- For $n=1$ the regulator $\mathrm{reg}_P(Z)$ is the push-forward to (X,D) of

$$(1-x)(2\pi i)^{p+1}[U\times\mathbb{R}_{-}]\cap[Z]+x(2\pi i)^p[\mathrm{dlog}\,z]\cap[Z]+dx(2\pi i)^p[\log z]\cap[Z].$$

After writing the pushforward as a fiber integral over X twisted by $(2\pi i)^{-1}$, this becomes (with z the coordinate in $\square$)

$$(1-x)(2\pi i)^p\int_{(X\times\mathbb{R}_{-})\cap Z}+x(2\pi i)^{p-1}\int_Z \mathrm{dlog}\,z+dx(2\pi i)^{p-1}\int_Z \log z.$$

In all these formulas one has to exclude the poles of the integrand from the integration domain, that is, the two last integrals are over $Z_{\mathrm{reg}}\setminus(U\times\{0,\infty\})$ and $Z_{\mathrm{reg}}\setminus(U\times\mathbb{R}_{-})$ respectively.

In particular, the regulator into $P_{\mathcal{D}}$ contains for $n=1$ no new information compared to the regulator in the 3-term complex, which is given by the triple of log currents on X

$$(2\pi i)^{p-1}\left(2\pi i\int_{(X\times\mathbb{R}_{-})\cap Z},\int_Z \mathrm{dlog}\,z,\int_Z \log z\right).$$

- For $n=2$ the regulator value of $Z\in z^p_{\mathbb{R}}(U,2)$ in the complex $P_{\mathcal{D}}$ is:

$$\begin{aligned}
&-(2\pi i)^p(1-x)^2\int_{Z\cap(X\times\mathbb{R}_{-}\times\mathbb{R}_{-})}+(2\pi i)^{p-2}x^2\int_Z \mathrm{dlog}\,z_1\wedge\mathrm{dlog}\,z_2\\
&+(2\pi i)^{p-1}x(1-x)\left(\int_{Z\cap(X\times\mathbb{R}_{-}\times\overline{\square})}\mathrm{dlog}\,z_2-\int_{Z\cap(X\times\overline{\square}\times\mathbb{R}_{-})}\mathrm{dlog}\,z_1\right)\\
&+(2\pi i)^{p-2}x\,dx\left(\int_Z \log z_1\,\mathrm{dlog}\,z_2-\int_Z \log z_2\,\mathrm{dlog}\,z_1\right)\\
&+(2\pi i)^{p-1}(1-x)dx\left(\int_{Z\cap(X\times\overline{\square}\times\mathbb{R}_{-})}\log z_1-\int_{Z\cap(X\times\mathbb{R}_{-}\times\overline{\square})}\log z_2\right).
\end{aligned}$$

In the 3-term complex, its first two components are the currents

$$-(2\pi i)^p\int_{(X\times\mathbb{R}_{-}\times\mathbb{R}_{-})\cap Z},\qquad (2\pi i)^{p-2}\int_Z \mathrm{dlog}\,z_1\wedge\mathrm{dlog}\,z_2$$

and its third component is

$$\frac{(2\pi i)^{p-2}}{2}\left(\int_Z \log z_1\,\mathrm{dlog}\,z_2-\log z_2\,\mathrm{dlog}\,z_1+2\pi i\int_{Z\cap\{z_2\in\mathbb{R}_{-}\}}\log z_1-2\pi i\int_{Z\cap\{z_1\in\mathbb{R}_{-}\}}\log z_2\right).$$

This is a symmetrization of (a $\mathbb{C}$ version of) a formula found by Beilinson in [1].

On the other hand, the regulator value $\mathrm{reg}_{C,\cap_0}(Z)$ of Z in the 3-term complex is

$$(2\pi i)^{p-2}\Big(-(2\pi i)^2\int_{Z\cap\{z_1,z_2\in\mathbb{R}_{-}\}},\int_Z \mathrm{dlog}\,z_1\wedge\mathrm{dlog}\,z_2,\int_Z \log z_1\,\mathrm{dlog}\,z_2-2\pi i\int_{Z\cap\{z_1\in\mathbb{R}_{-}\}}\log z_2\Big).$$

- Examining the case $n=3$, we consider only the evaluation of the regulator value in the

3-term complex. The resulting triple has as first components

$$-(2\pi i)^p \int_{Z\cap(X\times(\mathbb{R}_-)^3)}, \qquad (2\pi i)^{p-3}\int_Z \operatorname{dlog} z_1 \wedge \operatorname{dlog} z_2 \wedge \operatorname{dlog} z_3,$$

and the third component $\int \operatorname{reg}_P(C(1))$ is given by

$$\begin{aligned}\tfrac{1}{3}\int_Z &\log z_1 \operatorname{dlog} z_2 \operatorname{dlog} z_3 - \log z_2 \operatorname{dlog} z_1 \operatorname{dlog} z_3 + \log z_3 \operatorname{dlog} z_1 \operatorname{dlog} z_2 \\ &+\tfrac{2\pi i}{6}\int_{Z\cap\{z_1\in\mathbb{R}_-\}} \log z_3 \operatorname{dlog} z_2 - \log z_2 \operatorname{dlog} z_3 \\ &+\tfrac{2\pi i}{6}\int_{Z\cap\{z_2\in\mathbb{R}_-\}} \log z_1 \operatorname{dlog} z_3 - \log z_3 \operatorname{dlog} z_1 \\ &+\tfrac{2\pi i}{6}\int_{Z\cap\{z_3\in\mathbb{R}_-\}} \log z_2 \operatorname{dlog} z_1 - \log z_1 \operatorname{dlog} z_2 \\ -\tfrac{(2\pi i)^2}{3}\Big[&\int_{Z\cap\{z_1,z_2\in\mathbb{R}_-\}} \log(z_3) - \int_{Z\cap\{z_1,z_3\in\mathbb{R}_-\}} \log(z_2) + \int_{Z\cap\{z_2,z_3\in\mathbb{R}_-\}} \log(z_1)\Big].\end{aligned}$$

To make this more concrete, we apply this formula to the cycle $C(1)$ considered by Burt Totaro in [62, §2], which by definition is the algebraic cycle in $\square^3$ parametrized by

$$\varphi(t) = (t, 1-\tfrac{1}{t}, 1-t), \qquad t \in \mathbb{P}_1 \setminus \{0,1,\infty\}.$$

The first and the last row vanishes, and each other term becomes $\pi^2/6$, so that

$$\int \operatorname{reg}_P(C(1)) = \tfrac{\pi^2}{6} = \operatorname{Li}_2(1)$$

is a special value of the dilogarithm function.

Remark. The importance of $C(1)$ lies in the fact that its regulator value $\frac{\pi^2}{6}$ is an element of order 24 in $\mathbb{C}/\mathbb{Z}(2) = \mathbb{C}/4\pi^2\mathbb{Z}$ and hence $C(1)$ is an element of order at least 24 in $\mathrm{CH}^2(\mathbb{Q},3)$. On the other hand, it has been shown by Bloch-Lichtenbaum [10, Thm 7.2] that $\mathrm{CH}^2(\mathbb{Q},3)$ is isomorphic to the indecomposable part of the K_3-group, that is, to the quotient $K_3^{\mathrm{ind}}(\mathbb{Q}) := K_3(\mathbb{Q})/K_3^M(\mathbb{Q})$ of the Quillen-K_3 by the (image of) Milnor-K_3. This group has been computed by [51] to be $\mathbb{Z}/24$, so that $\mathrm{CH}^2(\mathbb{Q},3) \cong K_3^{\mathrm{ind}}(\mathbb{Q}) \cong \mathbb{Z}/24\mathbb{Z}$. In particular, the order of $C(1)$ is at most 24 and consequently must be a generator of $\mathrm{CH}^2(\mathbb{Q},3)$.

5.3 The general Totaro cycle in $\square^3$

We continue the above example of the Totaro cycle and consider, following [39], more general for a parameter $a \in \mathbb{P}_1(\mathbb{C}) \setminus (\mathbb{R}_{\leq 0} \cup \mathbb{R}_{\geq 1})$ the parametrized subvarietiy

$$C(a) = \big\{(z, 1-\tfrac{a}{z}, 1-z) : z \in \mathbb{P}_1(\mathbb{C})\big\} \cap \square^3 .$$

It has Bloch boundary $\partial C(a) = -(a, 1-a)$ and thus is a higher Chow cycle if and only if $a \in \{0,1\}$. Nevertheless, for a as above, $C(a)$ intersects the real boundaries properly and we can calculate the regulator for such a. Using the formula from above (for $n = 3$) and noting that the first (by dimensionality) and the last (by the choice of a) row vanish, the computation of the regulator boils down to the computation of the integrals

$$\begin{aligned}
A &= \int_{-\infty}^{0} \log(1-z)\,\mathrm{dlog}(1-a/z) - \int_{-\infty}^{0} \log(1-a/z)\,\mathrm{dlog}(1-z)\\
B &= \int_{0}^{a} \log z\,\mathrm{dlog}(1-z) - \int_{0}^{a} \log(1-z)\,\mathrm{dlog}\, z\\
C &= \int_{\infty}^{1} \log(1-a/z)\,\mathrm{dlog}\, z - \int_{\infty}^{1} \log z\,\mathrm{dlog}(1-a/z).
\end{aligned}$$

Integration by parts reduces each of the above pairs of integrals to a single integral and a limit. Evaluation with mathematica gives

$$\begin{aligned}
A &= 2\int_{-\infty}^{0} \log(1-z)\,\mathrm{dlog}(1-\tfrac{a}{z}) - \lim_{h\nearrow 0}\Big[\log(1-\tfrac{a}{h})\log(1-h) - \log(1-ah)\log(1-\tfrac{1}{h})\Big]\\
&= 2\,\mathrm{Li}_2(a) + 2\log(a)\log(1-a).
\end{aligned}$$

Similarly for B, C:

$$\begin{aligned}
B &= -2\int_{0}^{a} \log(1-z)\,\mathrm{dlog}\, z + \lim_{h\nearrow 1}\Big[\log(ah)\log(1-ah))\Big] - \lim_{h\searrow 0}\Big[\log(h)\log(1-h)\Big]\\
&= 2\,\mathrm{Li}_2(a) + \log(a)\log(1-a)
\end{aligned}$$

$$\begin{aligned}
C &= 2\int_{\infty}^{1} \log(1-a/z)\,\mathrm{dlog}\, z - \lim_{h\nearrow\infty}\Big[\log(1-\tfrac{a}{1-\frac{1}{h}})\log(1-\tfrac{1}{h}) - \log(1-\tfrac{a}{h})\log(h)\Big]\\
&= 2\,\mathrm{Li}_2(a).
\end{aligned}$$

Thus the regulator in this case is

$$\begin{aligned}
\mathrm{reg}_P(C(a)) &= (2\pi i)^{2-3}\cdot(2\pi i)(1-x)xdx(A+B+C)\\
&= (1-x)xdx\big(6\,\mathrm{Li}_2(a) + 3\log(a)\log(1-a)\big).
\end{aligned}$$

In the 3-term complex, this becomes $\big(0, 0, \mathrm{Li}_2(a) + \frac{1}{2}\log(a)\log(1-a)\big)$. For $a \to 1$ this reduces to the value already computed. The KLM regulator on the other hand can easily computed to be

$$\mathrm{reg}_{C,\cap_0}(C(a)) = \big(0, 0, \mathrm{Li}_2(a) + \log(a)\log(1-a)\big).$$

In particular, the two regulators are not equal.

Remark. Following KLM [39], one can also consider the curve $D(b) := \{(1-z, 1-b/z, z)\} \cap \square^3$. Then $C(a) - D(1-a)$ is a higher Chow cycle. Since $D(b)$ is obtained from $C(b)$ by exchanging the first two components, the resulting regulator is $\mathrm{reg}_P(D(b)) = -\mathrm{reg}_P(C(b))$. Using this together with the transformation rules for the dilogarithm [50, (3.3)], shows that

$$\begin{aligned}
\mathrm{reg}_P(C(a) - D(1-a)) &= \mathrm{reg}_P(C(a)) + \mathrm{reg}_P(C(1-a))\\
&= 6x(1-x)dx\,\mathrm{Li}_2(1),
\end{aligned}$$

independent of a. The same happens with the regulator into the 3-term complex and in fact, one finds that $ev \circ \mathrm{reg}_P(C(a) - D(1-a)) = \mathrm{reg}_{C,\cap_0}(C(a) - D(1-a))$.

5.4 Regulator formulas for graph cycles

In this subsection, we consider higher Chow chains that arise as the graphs of rational functions and give formulas for their regulator values. Denote by reg either the regulator in 4.2, or 4.3. In the following, we fix a good compactification (X, D) of U.

Graph cycles over U Consider a non-zero rational function $f = (f_1, \dots f_n) : U \to \square^n$ on U that intersects the real faces properly (i.e., the inverse image of any real face under f exists and has the correct codimension). Then the graph Γ_f is a real higher Chow chain on U.

Lemma 77. *The regulator of the graph $\Gamma_f \in z^p_{\mathbb{R}}(U, n)$ is*

$$\mathit{reg}(\Gamma_f) = \mathrm{pr}_{(X,D)*}(f^* R^n).$$

Proof. Recall that a rational function $f : U \to V$ induces a general pullback of currents on good compactifications by $f^*T = \mathrm{pr}_{X*}(\mathrm{pr}_Y^* T \cap \mathrm{cl}(\Gamma_f))$. The lemma now follows immediately from the definitions:

$$\begin{aligned}\mathrm{reg}(\Gamma_f) &= \mathrm{pr}_{(X,D)*}\Big((\mathrm{cl}(U) \boxtimes R^n) \cap \mathrm{cl}(\Gamma_f)\Big)\\ &= \mathrm{pr}_{(X,D)*}\Big(\mathrm{pr}^*_{\overline{\square}^n} R^n \cap \mathrm{cl}(\Gamma_f)\Big)\\ &= \mathrm{pr}_{(X,D)*}(f^* R^n).\end{aligned}$$

□

If f is defined on a closed subvariety $V \subset U$ of codimension p, then the graph of f gives a higher Chow chain on U by pushforward along the inclusion $i : V \to U$. The regulator of such a graph is $\mathrm{reg}(i_*\Gamma_f) = i_*\mathrm{reg}(\Gamma_f)$, with i considered as a map $(V, D \cap V) \to (X, D)$. In particular, one gets a similar fomula as above. Indeed, if $\mathrm{pr}^V_{(X,D)}$ denotes the composition of the inclusion i with the projection, then

$$\mathrm{reg}(i_*\Gamma_f) = \mathrm{pr}^V_{(X,D)*}(f^* R^n).$$

Note that from a computational point of view the only difference to the previous formula is an additional restriction to V_{reg} and the multiplication with the factor $(2\pi i)^p$ coming from the pushforward along i.

More general: Let $V \subset U$ be a codimension p algebraic subvariety and $f : V \times \square^n \to \square^m$ a non-zero rational function such that its graph $\Gamma_f \subset U \times \square^{n+m}$ intersects all the real faces of $\square^{n+m}$ properly. Then the graph can be considered as an element in $z^{p+m}_{\mathbb{R}}(U, n+m)$, and the notation Γ_f/U will be used in order to emphasize that point.

Lemma 78. *For such functions f and $\mathrm{pr}^{V \times \overline{\square}^n}_{(X,D)*}$ the composition of the inclusion to $X \times \overline{\square}^n$ with the projection, one has*

$$\mathit{reg}(\Gamma_f/U) = \mathrm{pr}^{V \times \overline{\square}^n}_{(X,D)*}\Big(([X] \boxtimes R^n) \cap f^* R^m\Big).$$

Proof. The compatibility with pushforward allows to consider the case $V = U$ only. Write $[X]$ instead of $\mathrm{cl}(X)$ and $[\overline{\square}]$ instead of $\mathrm{cl}(\square)$ (think of $\mathrm{reg} = \mathrm{reg}_P$). Now use the definition of reg, compatibility of $\boxtimes$ and $\cap$ for geometric cycles, the associativity of $\cap$, the projection formula and the definition of the general pullback, to obtain

$$
\begin{aligned}
\operatorname{reg}(\Gamma_f/U) &= \operatorname{pr}_{(X,D)*}\Big(([X]\boxtimes R^{n+m})\cap \operatorname{cl}(\Gamma_f)\Big)\\
&= \operatorname{pr}_{(X,D)*}\Big(\big(([X]\boxtimes R^n\boxtimes[\overline{\square}]^m)\cap([X]\boxtimes[\overline{\square}]^n\boxtimes R^m)\big)\cap\operatorname{cl}(\Gamma_f)\Big)\\
&= \operatorname{pr}_{(X,D)*}\Big(([X]\boxtimes R^n\boxtimes[\overline{\square}]^m)\cap\big(([X]\boxtimes[\overline{\square}]^n\boxtimes R^m)\cap\operatorname{cl}(\Gamma_f)\big)\Big)\\
&= \operatorname{pr}_{(X,D)*}\Big(([X]\boxtimes R^n)\cap(\operatorname{pr}_{X\times[\overline{\square}]^n})_*\big(([X]\boxtimes[\overline{\square}]^n\boxtimes R^m)\cap\operatorname{cl}(\Gamma_f)\big)\Big)\\
&= \operatorname{pr}_{(X,D)*}\Big(([X]\boxtimes R^n)\cap f^*R^m\Big).
\end{aligned}
$$

$\square$

Examples:

a) If $f:\square^n\to\square^m$ is a rational function with components $f=(f_1,\dots,f_m)$, we obtain a very simple formula for the regulator:

$$\operatorname{reg}(\Gamma_f/\square^n)=\operatorname{pr}_{(\overline{\square}^n,\mathbb{1}^n)*}\Big(R^n\cap f_1^*R\cap\ldots\cap f_m^*R\Big).$$

Consider now $\operatorname{reg}=\operatorname{reg}_P$ and recall that for a rational function f with values in $\square$

$$f^*R=(1-x)T_f+x[\operatorname{dlog} f]+dx[\log f],$$

where $T_f=2\pi i[f^{-1}\mathbb{R}_-]$.

b) Important cycles in $z^p(U,1)$ are the cycles represented by graphs Γ_f, where f is a non-zero rational function defined on a codimension $p-1$ algebraic subvariety $i:V\subset U$. The regulator of such an element is

$$
\begin{aligned}
\operatorname{reg}(\Gamma_f/U) &= i_*\operatorname{reg}(\Gamma_f/V)\\
&= (2\pi i)^{p-1}\left(2\pi i(1-x)\int_{f^{-1}\mathbb{R}_-}+x\int_V\operatorname{dlog} f+dx\int_V\log f\right).
\end{aligned}
$$

c) For a pair of rational functions $(f,g):U\to\square^2$, the regulator is

$$
\begin{aligned}
\operatorname{reg}(\Gamma_{(f,g)}) =&\operatorname{pr}_{(X,D)*}\left(f^*R\cap g^*R\right)\\
=&(1-x)^2T_f\cap T_g+x^2[\operatorname{dlog} f\wedge\operatorname{dlog} g]\\
&+x(1-x)\Big(T_f\wedge\operatorname{dlog} g-T_g\wedge\operatorname{dlog} f\Big)\\
&+(1-x)dx\Big(T_g\wedge\log f-T_f\wedge\log g\Big)\\
&+xdx\Big([\log f\ \operatorname{dlog} g]-[\log g\ \operatorname{dlog} f]\Big).
\end{aligned}
$$

d) Using lemma 78, one easily recovers the formula for the regulator of the Totaro cycle $C(1)$. It can also be used to determine the regulator values of the higher dimensional "polylog cycles" $C_n(1), n\geq 1$, that will be introduced in 5.6 and that generalize $C(1)$ to higher dimensions. A tedious calculation (carried out in the appendix) shows that

$$\operatorname{reg}_P(C_2(1))=(1-x)^3x^2\operatorname{Li}_2(1)+30(1-x)^2x^2dx\operatorname{Li}_3(1)-(1-x)^2x^2dx2\pi i\operatorname{Li}_2(1).$$

In particular, its evaluation in $C_{\mathcal{D}}$ is $=\big(0,0,\operatorname{Li}_3(1)-\frac{2\pi i}{30}\operatorname{Li}_2(1)\big)$.

5.5 The relative Totaro cycle C_1

We now consider the general Totaro cycle $C_1(a)$ not for a fixed value of a, but let a vary in the real analytic variety $U := \mathbb{P}_1(\mathbb{C}) \setminus (\mathbb{R}_{\leq 0} \cup \mathbb{R}_{\geq 1})$, the space of parameters where $C_1(a)$ intersects the real faces properly. Formally, we consider the parametrized cycle

$$C_1 = [a, z, 1-z, 1-\frac{a}{1-z}] \subset U \times \square^3$$

as a higher Chow chain over U. Note that C_1 is the nothing but the relative graph cycle $\Gamma_{(f,g)}/U$ with respect to the pair of functions

$$(f,g) : U \times \square \to \square^2, \quad (a,z) \mapsto \big(1-z\,,\, 1-\frac{a}{1-z}\big).$$

Although U is not a valid (complex) algebraic variety, we nevertheless are going to compute the regulator map. Choose as a compactification of U the pair $(X, D) = (\mathbb{P}_1, \mathbb{R}_{\leq 0} \cup \mathbb{R}_{\geq 0})$. That is, even though we only considered complex subvarieties D in section 3, we here work with real analytic divisors. In other words, we consider currents on $\mathbb{P}_1$ up to currents on D.

To compute the regulator of C_1, observe that

$$\begin{aligned}
f^*R &= 2\pi i(1-x)\mathbb{P}_1 \boxtimes [\infty, 1] + x\,\mathrm{dlog}(1-z) + dx\log(1-z)\\
g^*R &= 2\pi i(1-x)\mathbb{P}_1 \boxtimes (1-a\cdot[0,1]) + x\,\mathrm{dlog}(1-\frac{a}{1-z}) + dx\log(1-\frac{a}{1-z}),
\end{aligned}$$

thereby using that $(\frac{a}{1-z} \in [\infty, 1]) = \mathbb{P}_1 \boxtimes (1-a\cdot[0,1])$.

Now substitute these expressions in the formula of lemma 78. Since currents supported on $(\mathbb{R}_{\leq 0} \cup \mathbb{R}_{\geq 1}) \times \overline{\square}$ are pushed down to on-D currents, they are treated as zero. Then the regulator becomes

$$\begin{aligned}
\mathrm{reg}_P(C_1) &= \mathrm{pr}_{(\mathbb{P}_1,D)*}\Big[(\mathbb{P}_1 \boxtimes R) \cap (f,g)^*R^2\Big]\\
&= \mathrm{pr}_{(\mathbb{P}_1,D)*}\Big[(\mathbb{P}_1 \boxtimes R) \cap \Big(x^2\,\mathrm{dlog}(1-z) \wedge \mathrm{dlog}(1-\frac{a}{1-z})\\
&\qquad + 2\pi i(1-x)x[\mathbb{P}_1 \boxtimes [\infty,1]\,\mathrm{dlog}(1-\frac{a}{1-z}) - \mathbb{P}_1 \boxtimes (1-a\cdot[0,1])\,\mathrm{dlog}(1-z)]\\
&\qquad + 2\pi i(1-x)dx[\mathbb{P}_1 \boxtimes (1-a\cdot[0,1])\log(1-z) - \mathbb{P}_1 \boxtimes [\infty,1]\log(1-\frac{a}{1-z})]\\
&\qquad + xdx[\log(1-z)\,\mathrm{dlog}(1-\frac{a}{1-z}) - \log(1-\frac{a}{1-z})\,\mathrm{dlog}(1-z)]\Big)\Big].
\end{aligned}$$

Now substitute R in this formula and expand. Again some terms vanish for degree reasons or because "$dx^2 = 0$". The intersection $(\mathbb{P}_1 \boxtimes [-\infty, 0]) \cap (\frac{a}{1-z} \in [\infty, 1]) = -[\infty, 1] \boxtimes [1-a, 0]$ vanishes as a log current after projection to the first component, and thus

$$\begin{aligned}
\mathrm{reg}_P(C_1) = \mathrm{pr}_{(\mathbb{P}_1,D)*}\Big(&2\pi i(1-x)x^2\mathbb{P}_1 \boxtimes [-\infty,0]\,\mathrm{dlog}(1-z) \wedge \mathrm{dlog}(1-\frac{a}{1-z})\\
&- 2\pi i(1-x)xdx\mathbb{P}_1 \boxtimes [-\infty,0]\log(1-z)\,\mathrm{dlog}(1-\frac{a}{1-z})\\
&+ 2\pi i(1-x)xdx\mathbb{P}_1 \boxtimes [-\infty,0]\log(1-\frac{a}{1-z})\,\mathrm{dlog}(1-z)
\end{aligned}$$

$$
\begin{aligned}
&+ 2\pi i(1-x)x^2\mathbb{P}_1 \boxtimes [\infty,1]\,\mathrm{dlog}(1-\frac{a}{1-z}) \wedge \mathrm{dlog}(z)\\
&+ 2\pi i(1-x)xdx\mathbb{P}_1 \boxtimes (1-a\cdot[0,1])\log(1-z)\,\mathrm{dlog}(z)\\
&- 2\pi i(1-x)xdx\mathbb{P}_1 \boxtimes [\infty,1]\log(1-\frac{a}{1-z})\,\mathrm{dlog}(z)\\
&+ x^2dx\log(1-z)\,\mathrm{dlog}(1-\frac{a}{1-z}) \wedge \mathrm{dlog}(z)\\
&+ x^2dx\log(z)\,\mathrm{dlog}(1-z) \wedge \mathrm{dlog}(1-\frac{a}{1-z})\\
&+ 2\pi i(1-x)xdx\mathbb{P}_1 \boxtimes [\infty,1]\log(z)\,\mathrm{dlog}(1-\frac{a}{1-z})\\
&- 2\pi i(1-x)xdx\mathbb{P}_1 \boxtimes (1-a[0,1])\log(z)\,\mathrm{dlog}(1-z)\Big).
\end{aligned}
$$

Finally, note that $\mathrm{dlog}(1-\frac{a}{1-z}) = \frac{da}{z-1+a} + \frac{adz}{(z-1+a)(1-z)}$ and that there is a Tate twist included in the pushforward to obtain

$$
\begin{aligned}
\mathrm{reg}_P(C_1) =&(1-x)x^2[\int_{-\infty}^{0} \tfrac{1}{z-1+a}\,\mathrm{dlog}(1-z)da - \int_{\infty}^{1} \tfrac{1}{z-1+a}\,\mathrm{dlog}(z)da]\\
&+ (1-x)xdx \int_{-\infty}^{0} \big[\log(1-\frac{a}{1-z})\,\mathrm{dlog}(1-z) - \log(1-z)\,\mathrm{dlog}(1-\frac{a}{1-z})\big]\\
&+ (1-x)xdx \int_{1}^{1-a} \big[\log(1-z)\,\mathrm{dlog}(z) - \log(z)\,\mathrm{dlog}(1-z)\big]\\
&- (1-x)xdx \int_{\infty}^{1} \big[\log(1-\frac{a}{1-z})\,\mathrm{dlog}(z) - \log(z)\,\mathrm{dlog}(1-\frac{a}{1-z})\big].
\end{aligned}
$$

The integrals in the last three rows have already been computed (do a substitution $z \mapsto 1-z$ first) during the investigation of the Totaro cycle $C_1(a)$. The integrals in the first row evaluate to logarithm functions so that

$$
\begin{aligned}
\mathrm{reg}_P(C_1) &= (1-x)x^2\big[\log(a)\,\mathrm{dlog}(1-a) - \log(1-a)\,\mathrm{dlog}(a)\big]\\
&\quad + (1-x)xdx\big[6\,\mathrm{Li}_2(a) + 3\log(1-a)\log(a)\big].
\end{aligned}
$$

As a test, one may compute the differential of $\mathrm{reg}_P(C_1)$ (as it is an element in the Deligne-Beilinson complex on U) and compare it with the regulator value of $\partial C_1 = [a, 1-a, a]$. We anticipate the result, which will be

$$
\begin{aligned}
d\mathrm{reg}_P(C_1) &= xdx\Big[\log(1-a)\,\mathrm{dlog}(a) - \log(a)\,\mathrm{dlog}(1-a)\Big]\\
&= \mathrm{reg}_P(\partial C_1).
\end{aligned}
$$

5.6 Higher Totaro cycles

The general Totaro cycle in $\square^3$ (5.3) has been extended by Bloch [7], [8] to higher dimensions. We will state them in a relative form, and for that make the convention that – in a parametrized notation – the variables in the base are separated from the variables in the fibres by a semicolon. For example, the general Totaro cycle over U will be written as $C_1 = [b; a, 1-b/a, 1-a]$.
The following definition of the higher Totaro cycle in $\square^{2n+1}$ is a slightly permutated (but equivalent as an alternating chain) representation of the cycle given by Bloch:

$$
C_n := \Big[a; z_n, 1-\tfrac{a}{z_n}, z_{n-1}, 1-\tfrac{z_n}{z_{n-1}}, z_{n-2}, \ldots, z_1, 1-\tfrac{z_2}{z_1}, 1-z_1\Big].
$$

This indeed is an admissible higher Chow chain over U.
Using lemma 78 and the graded-commutativity of $\cap$, the regulator of the general higher Totaro cycle can simply be written as

$$\mathrm{reg}_P(C_n) = \mathrm{pr}_{(\mathbb{P}_1,D)*}\Big(R_{z_1} \cap \ldots \cap R_{z_n} \cap R_{1-\frac{a}{z_n}} \cap R_{1-\frac{z_n}{z_{n-1}}} \cdots R_{1-\frac{z_2}{z_1}} \cap R_{1-z_1}\Big),$$

with the notation $R_f := f^*R = d[x\log(f)] + T_f$ for a rational function $f : \square^n \to \mathbb{P}_1$. This is a Deligne-Beilinson chain over $(\mathbb{P}_1, D)$, for D as in 5.5.

A recursion for reg(C_n)

We make the observation that the higher Totaro cycles C_n can also be obtained as iterated pullbacks along the higher correspondence $C := [\mu, \lambda; \lambda, 1 - \frac{\mu}{\lambda}]$, that is,

$$C_{n+1} = C^* C_n.$$

Indeed, starting with $C_0 = [a; 1-a]$ and writing $*$ to denote a free parameter, we obtain recursively

$$\begin{aligned}
C_1 &= \mathrm{pr}_1\Big([b,a;a,1-b/a,*] \cap [b,a;*,*,1-a]\Big)\\
&= [b;a,1-b/a,1-a]\\
C_2 &= \mathrm{pr}_1\Big([c,b;b,1-c/b,*,*,*] \cap [c,b;*,*,a,1-b/a,1-a]\Big)\\
&= [c;b,1-c/b,a,1-b/a,1-a]\\
C_3 &= \mathrm{pr}_1\Big([d,c;c,1-d/c,*,*,*,*,*] \cap [d,c;*,*,b,1-c/b,a,a-b/a,1-a]\Big)\\
&= [d;c,1-d/c,b,1-c/b,a,1-b/a,1-a]
\end{aligned}$$

etc.

It is easy to verify that the recursion indeed gives the cycles C_n stated above.
This observation, together with the compatibility of the regulator map into $P_{\mathcal{D}}$ with pullback along higher correspondences, gives an alternative way to compute the regulator value of the $C_n, n \geq 0$.

Negligible summands

In order to compute the regulator values of the higher Totaro chains, we have to compute

$$R^n \cap f^*_{a,z_n} R \cap f^*_{z_n,z_{n-1}} R \cap \ldots \cap f^*_{z_2,z_1} R \cap f^*_{z_1,1} R \tag{5.1}$$

where $f_{g,h}(\underline{z}) = 1 - \frac{g(\underline{z})}{h(\underline{z})}$ as a map $\square^n \dashrightarrow \square$ (defined outside the loci $h = \infty$ and $g = 0$). For such functions,

$$\begin{aligned}
T_{f_{g,h}} &= 2\pi i[g(z) \in \mathbb{R}_{\geq 1} \cdot h(z)],\\
\mathrm{dlog}(f_{g,h}) &= \frac{gdh - hdg}{h(h-g)}.
\end{aligned}$$

The expression (5.1) expands into various summands, some of whom may be zero for some reasons and thus are *negligible*. For example, consider for a given summand the number of occurrences (i.e., the exponents) of $(1-x)$, x and dx (or, equivalently, the number of occurences of $\mathbb{R}_-$, dlog and log in the summand). These numbers are denoted by k, l, m. Obviously, $m \in \{0, 1\}$. Since the Totaro cycle has complex dimension n, the wedge product of more than n dlogs is zero, so that we may assume $l \leq n$. Together with $k + l + m = 2n + 1$, these conditions are summarized in the table below

factor	$\mathbb{R}_-$	log	dlog	Σ
#	$\geq n$	≤ 1	$\leq n$	$2n+1$

Claim 1. *We may also assume that $k < 2n$, because the intersection of the Totaro cycle with at least $2n$ copies of $[\mathbb{R}_-]$ is empty.*

Proof. Fix a summand and note that each factor $\mathbb{R}_-$ in it corresponds to one of the following conditions on the coordinates.

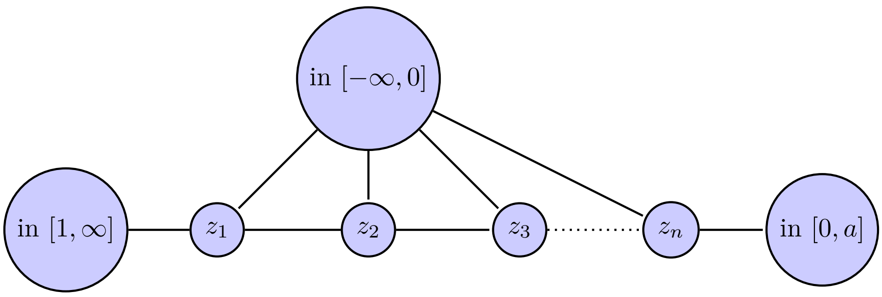
(I) $z_i \in [-\infty, 0]$, $i = 1, \ldots, n$

(II) $z_{i+1} \in z_i \cdot [1, \infty]$, $i = 1, \ldots, n-1$

(III) $z_n \in [0, a]$

(IV) $z_1 \in [1, \infty]$

We can visualize the situation as a graph with $n+3$ nodes: For each of the n variables $z_1, \ldots, z_n$ of the n-cube introduce a node with label "z_i". Also, add 3 nodes labelled with "in $[-\infty, 0]$", "in $[1, \infty]$" and "in $[0, a]$". Connect two nodes, if there is a factor that gives the respective condition. That is, (I) corresponds to the edge between "z_i" and "in $[-\infty, 0]$", (II) corresponds to the edge between "z_{i+1}" and "z_i" etc. The following graphic shows all the possible edges.



It is clear that the summand is negligible, if the resulting graph contains a path connecting two of the outer nodes. Obviously, one needs to remove at least two edges to get a non-negligible summand. □

Remark 9. The proof above indicates how to identify the summands in (5.1) with (sub-)graphs: For a given summand, connect two nodes by a black edge, if there is a "$\mathbb{R}_-$", with a red edge, if there is a "dlog", and with a yellow edge, if there is a "log" on the respective position in the summand. Conversely, any such colored graph uniquely determines a summand. The summand is negligible, if in the corresponding graph two of the outer nodes can be connected by a sequence of only black edges or a sequence of only red edges. The second condition relies on the fact that $\mathrm{dlog}(f) \wedge \mathrm{dlog}(g) = 0$, if f and g depend on the same (single) variable.

This gives a way to list all the non-negligible summands (the list may contain some negligible summands also).
In pseudo-code:

```
L = { }.
For all subgraphs G' of G do

    Color all edges of G' red.
    Color all edges of G \ G' black.

    If two different outer nodes are connected by a sequence of black edges,
        break.

    If two outer nodes are connected by a sequence of red edges,
        break.

    Add a new summand to L.

    For any edge e in G':
        Color e yellow.
        If no two outer nodes are connected by a sequence of red edges,
            add a new summand to L.
        Color e red.
```

5.7 Appendix: Blochs generalized Totaro cycle of dimension two

This appendix is devoted to the calculation of the regulator value of the two-dimensional Totaro cycle $C_2(1)$. This boils down to the computation of the expression (5.1) in the case where $n = 2$ and $a = 1$. Thus it is to compute $R^2 \cap f^*_{1,z_2}R \cap f^*_{z_1,z_2}R \cap f^*_{z_1,1}R$.
For this, note that

$$\begin{aligned}
f^*_{1,z_2}R &= (1-x)T_{1-\frac{1}{z_2}} + x\,\mathrm{dlog}(1-\frac{1}{z_2}) + dx\log(1-\frac{1}{z_2})\\
f^*_{z_2,z_1}R &= (1-x)T_{1-\frac{z_1}{z_2}} + x\,\mathrm{dlog}(1-\frac{z_1}{z_2}) + dx\log(1-\frac{z_1}{z_2})\\
f^*_{z_1,1}R &= (1-x)T_{1-z_1} + x\,\mathrm{dlog}(1-z_1) + dx\log(1-z_1)
\end{aligned}$$

and

$$\begin{aligned}
R^2 &= (1-x)^2 T_{z_1} \boxtimes T_{z_2} + (1-x)x[T_{z_1} \wedge \mathrm{dlog}\, z_2 - T_{z_2} \wedge \mathrm{dlog}\, z_1] + x^2[\mathrm{dlog}\, z_1 \wedge \mathrm{dlog}\, z_2]\\
&\quad + (1-x)dx[T_{z_2} \wedge \log z_1 - T_{z_1} \wedge \log z_2] + x dx[\log z_1\, \mathrm{dlog}\, z_2 - \log z_2\, \mathrm{dlog}\, z_1].
\end{aligned}$$

Using the preceding discussion, many of the terms in (5.1) vanish, and what remains after pushforward is

$$\mathrm{reg}_P(C_2(1)) = (1-x)^3x^2\,\mathrm{pr}_*\,A + (1-x)^3xdx\,\mathrm{pr}_*\,B + (1-x)^2x^2dx\,\mathrm{pr}_*\,C,$$

where A, B, C are the currents on $\square^2$ given by

$$A = T_{z_1}\cap T_{z_2}\cap \mathrm{dlog}(1-\frac{1}{z_2})\cap T_{1-\frac{z_2}{z_1}}\cap \mathrm{dlog}(1-z_1),$$

$$\begin{aligned} B = {} & T_{z_1}\cap T_{z_2}\cap \mathrm{dlog}(1-\frac{1}{z_2})\cap T_{1-\frac{z_2}{z_1}}\cap \log(1-z_1)\\ & + T_{z_1}\cap T_{z_2}\cap \log(1-\frac{1}{z_2})\cap T_{1-\frac{z_2}{z_1}}\cap \mathrm{dlog}(1-z_1), \end{aligned}$$

and

$$\begin{aligned} C = {} & \log(z_1)\cap T_{z_2}\cap \mathrm{dlog}(1-\frac{1}{z_2})\cap \mathrm{dlog}(1-\frac{z_2}{z_1})\cap T_{1-z_1}\\ & + \log(z_1)\cap T_{z_2}\cap \mathrm{dlog}(1-\frac{1}{z_2})\cap T_{1-\frac{z_2}{z_1}}\cap \mathrm{dlog}(1-z_1)\\ & + \log(z_1)\cap \mathrm{dlog}(z_2)\cap T_{1-\frac{1}{z_2}}\cap \mathrm{dlog}(1-\frac{z_2}{z_1})\cap T_{1-z_1}\\ & + \log(z_1)\cap \mathrm{dlog}(z_2)\cap T_{1-\frac{1}{z_2}}\cap T_{1-\frac{z_2}{z_1}}\cap \mathrm{dlog}(1-z_1)\\ & - T_{z_1}\cap \log(z_2)\cap \mathrm{dlog}(1-\frac{1}{z_2})\cap T_{1-\frac{z_2}{z_1}}\cap \mathrm{dlog}(1-z_1)\\ & - T_{z_1}\cap \log(z_2)\cap T_{1-\frac{1}{z_2}}\cap \mathrm{dlog}(1-\frac{z_2}{z_1})\cap \mathrm{dlog}(1-z_1)\\ & - \mathrm{dlog}(z_1)\cap \log(z_2)\cap \mathrm{dlog}(1-\frac{1}{z_2})\cap T_{1-\frac{z_2}{z_1}}\cap T_{1-z_1}\\ & - \mathrm{dlog}(z_1)\cap \log(z_2)\cap T_{1-\frac{1}{z_2}}\cap \mathrm{dlog}(1-\frac{z_2}{z_1})\cap T_{1-z_1}\\ & + T_{z_1}\cap T_{z_2}\cap \mathrm{dlog}(1-\frac{1}{z_2})\cap \mathrm{dlog}(1-\frac{z_2}{z_1})\cap \log(1-z_1)\\ & + T_{z_1}\cap \mathrm{dlog}(z_2)\cap T_{1-\frac{1}{z_2}}\cap \mathrm{dlog}(1-\frac{z_2}{z_1})\cap \log(1-z_1)\\ & + \mathrm{dlog}(z_1)\cap T_{z_2}\cap \mathrm{dlog}(1-\frac{1}{z_2})\cap T_{1-\frac{z_2}{z_1}}\cap \log(1-z_1)\\ & + \mathrm{dlog}(z_1)\cap \mathrm{dlog}(z_2)\cap T_{1-\frac{1}{z_2}}\cap T_{1-\frac{z_2}{z_1}}\cap \log(1-z_1)\\ & - \mathrm{dlog}(z_1)\cap \mathrm{dlog}(z_2)\cap T_{1-\frac{1}{z_2}}\cap \log(1-\frac{z_2}{z_1})\cap T_{1-z_1}\\ & - T_{z_1}\cap \mathrm{dlog}(z_2)\cap T_{1-\frac{1}{z_2}}\cap \log(1-\frac{z_2}{z_1})\cap \mathrm{dlog}(1-z_1)\\ & - T_{z_1}\cap T_{z_2}\cap \mathrm{dlog}(1-\frac{1}{z_2})\cap \log(1-\frac{z_2}{z_1})\cap \mathrm{dlog}(1-z_1)\\ & - \mathrm{dlog}(z_1)\cap T_{z_2}\cap \mathrm{dlog}(1-\frac{1}{z_2})\cap \log(1-\frac{z_2}{z_1})\cap T_{1-z_1}\\ & + \mathrm{dlog}(z_1)\cap \mathrm{dlog}(z_2)\cap \log(1-\frac{1}{z_2})\cap T_{1-\frac{z_2}{z_1}}\cap T_{1-z_1}\\ & + T_{z_1}\cap \mathrm{dlog}(z_2)\cap \log(1-\frac{1}{z_2})\cap T_{1-\frac{z_2}{z_1}}\cap \mathrm{dlog}(1-z_1)\\ & + \mathrm{dlog}(z_1)\cap T_{z_2}\cap \log(1-\frac{1}{z_2})\cap \mathrm{dlog}(1-\frac{z_2}{z_1})\cap T_{1-z_1}\\ & + T_{z_1}\cap T_{z_2}\cap \log(1-\frac{1}{z_2})\cap \mathrm{dlog}(1-\frac{z_2}{z_1})\cap \mathrm{dlog}(1-z_1). \end{aligned}$$

One finds that $(\mathrm{pr}_{\mathrm{pt}})_* A = -\frac{\pi^2}{6}$, and that $(\mathrm{pr}_{\mathrm{pt}})_* B = 0$ for dimension reasons.

In order to use the computer, we write the integrand explicitly, using that

$$\mathrm{dlog}(1-\frac{z_2}{z_1}) = \frac{z_2 dz_1 - z_1 dz_2}{z_1(z_1-z_2)}, \quad \mathrm{dlog}(1-\frac{1}{z_2}) = \frac{dz_2}{z_2(z_2-1)}, \quad \mathrm{dlog}(1-z_1) = \frac{dz_1}{z_1-1}$$

and obtain for C the representation

$$\begin{aligned}
-T_y \cap T_{1-x} \log(x) &\frac{dx \wedge dy}{x(x-y)(y-1)} \\
+T_y \cap T_{1-\frac{y}{x}} \log(x) &\frac{dx \wedge dy}{y(x-1)(y-1)} \\
+T_{1-x} \cap T_{1-\frac{1}{y}} \log(x) &\frac{dy \wedge dx}{x(x-y)} \\
-T_{1-\frac{y}{x}} \cap T_{1-\frac{1}{y}} \log(x) &\frac{dy \wedge dx}{y(x-1)} \\
-T_x \cap T_{1-\frac{y}{x}} \log(y) &\frac{dx \wedge dy}{y(x-1)(y-1)} \\
+T_x \cap T_{1-\frac{1}{y}} \log(y) &\frac{dx \wedge dy}{(x-1)(y-x)} \\
+T_{1-x} \cap T_{1-\frac{y}{x}} \log(y) &\frac{dx \wedge dy}{xy(y-1)} \\
-T_{1-x} \cap T_{1-\frac{1}{y}} \log(y) &\frac{dx \wedge dy}{x(y-x)} \\
-T_x \cap T_y \log(1-x) &\frac{dx \wedge dy}{x(x-y)(y-1)} \\
-T_x \cap T_{1-\frac{1}{y}} \log(1-x) &\frac{dy \wedge dx}{x(x-y)} \\
-T_y \cap T_{1-\frac{y}{x}} \log(1-x) &\frac{dx \wedge dy}{xy(y-1)} \\
-T_{1-\frac{y}{x}} \cap T_{1-\frac{1}{y}} \log(1-x) &\frac{dx \wedge dy}{yx} \\
+T_{1-x} \cap T_{1-\frac{1}{y}} \log(1-\frac{y}{x}) &\frac{dx \wedge dy}{xy} \\
+T_x \cap T_{1-\frac{1}{y}} \log(1-\frac{y}{x}) &\frac{dy \wedge dx}{y(x-1)} \\
+T_x \cap T_y \log(1-\frac{y}{x}) &\frac{dx \wedge dy}{y(x-1)(y-1)} \\
+T_y \cap T_{1-x} \log(1-\frac{y}{x}) &\frac{dx \wedge dy}{xy(y-1)} \\
-T_{1-x} \cap T_{1-\frac{y}{x}} \log(1-\frac{1}{y}) &\frac{dx \wedge dy}{xy} \\
-T_x \cap T_{1-\frac{y}{x}} \log(1-\frac{1}{y}) &\frac{dy \wedge dx}{y(x-1)} \\
-T_y \cap T_{1-x} \log(1-\frac{1}{y}) &\frac{dx \wedge dy}{x(y-x)} \\
-T_x \cap T_y \log(1-\frac{1}{y}) &\frac{dx \wedge dy}{(x-1)(y-x)}.
\end{aligned}$$

Now we plug in that

$$
\begin{aligned}
T_x \cap T_y &= [-\infty, 0] \boxtimes [-\infty, 0] = -[-\infty, 0] \times [-\infty, 0] \\
T_y \cap T_{1-x} &= -[\infty, 1] \boxtimes [-\infty, 0] = [\infty, 1] \times [-\infty, 0] \\
T_y \cap T_{1-\frac{y}{x}} &= -[0, y] \boxtimes [-\infty, 0] = [0, y] \times [-\infty, 0] \\
T_{1-x} \cap T_{1-\frac{1}{y}} &= -[\infty, 1] \times [0, 1] \\
T_{1-\frac{y}{x}} \cap T_{1-\frac{1}{y}} &= [0, y] \boxtimes [0, 1] = -[0, y] \times [0, 1] \\
T_x \cap T_{1-\frac{y}{x}} &= -[-\infty, 0] \times [-\infty, x] \\
T_{1-x} \cap T_{1-\frac{y}{x}} &= -[\infty, 1] \times [\infty, x],
\end{aligned}
$$

and use this to write C as a sum of iterated integrals.

Pushing the resulting currents down to the point yield the following integrals (in the left column) whose sum is the regulator value of $C_2(1)$. These integrals can be evaluated using Mathematica and the result is shown in the right column.

$$
\begin{array}{rr}
\displaystyle \int_\infty^1 \int_{-\infty}^0 \log(x) \frac{dy\, dx}{x(x-y)(y-1)} & 2\zeta(3) \\
\displaystyle -\int_{-\infty}^0 \int_0^y \log(x) \frac{dx\, dy}{y(x-1)(y-1)} & \zeta(3) - \dfrac{i\pi^3}{6} \\
\displaystyle -\int_\infty^1 \int_0^1 \log(x) \frac{dy\, dx}{x(x-y)} & \zeta(3) \\
\displaystyle \int_0^1 \int_0^y \log(x) \frac{dx\, dy}{y(x-1)} & 2\zeta(3) \\
\displaystyle -\int_{-\infty}^0 \int_{-\infty}^x \log(y) \frac{dy\, dx}{y(x-1)(y-1)} & \zeta(3) + \dfrac{i\pi^3}{6} \\
\displaystyle \int_{-\infty}^0 \int_0^1 \log(y) \frac{dy\, dx}{(x-1)(y-x)} & 2\zeta(3) \\
\displaystyle \int_\infty^1 \int_\infty^x \log(y) \frac{dy\, dx}{xy(y-1)} & 2\zeta(3) \\
\displaystyle -\int_\infty^1 \int_0^1 \log(y) \frac{dy\, dx}{x(y-x)} & \zeta(3) \\
\displaystyle -\int_{-\infty}^0 \int_{-\infty}^0 \log(1-x) \frac{dy\, dx}{x(x-y)(y-1)} & \zeta(3) \\
\displaystyle \int_{-\infty}^0 \int_0^1 \log(1-x) \frac{dy\, dx}{x(x-y)} & 2\zeta(3) \\
\displaystyle \int_{-\infty}^0 \int_0^y \log(1-x) \frac{dx\, dy}{xy(y-1)} & 2\zeta(3) \\
\displaystyle -\int_0^1 \int_0^y \log(1-x) \frac{dx\, dy}{yx} & \zeta(3)
\end{array}
$$

$$\int_{\infty}^{1}\int_{0}^{1}\log(1-\frac{y}{x})\frac{dy\,dx}{xy} \qquad \zeta(3)$$

$$-\int_{-\infty}^{0}\int_{0}^{1}\log(1-\frac{y}{x})\frac{dy\,dx}{y(x-1)} \qquad 2\zeta(3)$$

$$\int_{-\infty}^{0}\int_{-\infty}^{0}\log(1-\frac{y}{x})\frac{dy\,dx}{y(x-1)(y-1)} \qquad \zeta(3)-\frac{i\pi^3}{6}$$

$$-\int_{\infty}^{1}\int_{-\infty}^{0}\log(1-\frac{y}{x})\frac{dy\,dx}{xy(y-1)} \qquad 2\zeta(3)$$

$$-\int_{\infty}^{1}\int_{\infty}^{x}\log(1-\frac{1}{y})\frac{dy\,dx}{xy} \qquad \zeta(3)$$

$$\int_{-\infty}^{0}\int_{-\infty}^{x}\log(1-\frac{1}{y})\frac{dy\,dx}{y(x-1)} \qquad 2\zeta(3)$$

$$\int_{\infty}^{1}\int_{-\infty}^{0}\log(1-\frac{1}{y})\frac{dy\,dx}{x(y-x)} \qquad 2\zeta(3)$$

$$-\int_{-\infty}^{0}\int_{-\infty}^{0}\log(1-\frac{1}{y})\frac{dy\,dx}{(x-1)(y-x)} \qquad \zeta(3)-\frac{i\pi^3}{6}$$

We sum up all these numbers and obtain in the end that the regulator is

$$\mathrm{reg}_P(C_2(1))=(1-x)^3x^2\zeta(2)+(1-x)^2x^2dx\Big(30\zeta(3)-\tfrac{i\pi^3}{3}\Big).$$

6 The Abel-Jacobi map

This section shows how the regulators reg_C and reg_P give rise to Abel-Jacobi maps from higher Chow cycles homologous to zero to an intermediate Jacobian. This is done by reducing to the construction of the Abel-Jacobi map in [39] for the regulator reg_C.
In this section we assume that $U = X$ is a smooth projective complex manifold, that is, $D = 0$. For simplicity, we omit D from the notation and write $P_{\mathcal{D}}(X, \mathbb{Z}(p))$ for $P_{\mathcal{D}}(X, D, \mathbb{Z}(p))$ etc.

6.1 The Abel-Jacobi map for (higher) Chow groups

The classical Abel-Jacobi map As a tool to distinguish algebraic cycles on X, one has the cycle class map that assigns to each p-codimensional algebraic cycle its integral fundamental cycle in $H^{2p}(X, \mathbb{Z}(p))$. Because cycles algebraic equivalent to zero are mapped to zero in cohomology [66, Lemma 9.18], this descends to a cycle class map

$$\mathrm{CH}^p(X) \to H^{2p}(X, \mathbb{Z}(p)).$$

This map is in general[1] not injective, with examples given by each Griffith and Harris [29]. Its kernel, the set of equivalence classes of algebraic cycles which are homological equivalent to zero, is denoted by $\mathrm{CH}^p_{\mathrm{hom}}(X)$. The Abel-Jacobi map can be thought of as a refined invariant that tries to distinguish such cycles. If $m = \dim X$, it is the map

$$AJ : \mathrm{CH}^p_{\mathrm{hom}}(X) \to \frac{F^{m-p+1} H^{2m-2p+1}_{dR}(X)^\vee}{H^{2p-1}(X, \mathbb{Z}(p))}$$

that sends a cycle Z to the functional given by integration over γ, where $\gamma \subset X$ is any singular chain with boundary $\partial\gamma = Z$. The restriction to test forms in F^{m-p+1} is necessary to make the integration well defined on cohomology classes (and not only forms), and the quotient by $H^{2p-1}(X, \mathbb{Z}(p))$ is needed to be independent of the choice of γ.
The right-hand side is isomorphic to the p-th intermediate Jacobian $J^p(X)$ via Poincaré duality and the Abel-Jacobi map becomes a map

$$AJ : \mathrm{CH}^p_{\mathrm{hom}}(X) \to J^p(X) = \frac{H^{2p-1}(X, \mathbb{C})}{F^p H^{2p-1}(X, \mathbb{C}) + H^{2p-1}(X, \mathbb{Z}(p))}.$$

It is in this form that the map allows a generalization to higher Chow groups

The Abel-Jacobi map for higher Chow groups Kerr/Lewis/Müller-Stach [39] extended the definition of the Abel-Jacobi map to higher Chow groups. Starting with the higher cycle class map (the regulator) to Deligne cohomology,

$$\mathrm{CH}^p(X, n) \to H^{2p-n}_{\mathcal{D}}(X, \mathbb{Z}(p))$$

[1] For a survey on "algebraic vs. homological equivalence", see Murre's lectures on algebraic cycles and Chow groups [53, 9.4].

they use the long exact sequence for Deligne cohomology to define the notion of being homologous to zero for higher Chow cycles and also a map from those cycles to an intermediate Jacobian $J^{p,n}(X)$.
In short, their Abel-Jacobi map fits into a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}^p_{\mathrm{hom}}(X,n) \arrow[r] \arrow[d, "AJ"] & \mathrm{CH}^p(X,n) \arrow[r] \arrow[d, "\mathrm{reg}"] & \mathrm{coker} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J^{p,n}(X) \arrow[r, "\iota"] & H^{2p-n}_{\mathcal{D}}(X,\mathbb{Z}(p)) \arrow[r, "\mathrm{proj}"] & H^{2p-n}(X,\mathbb{Z}(p)) \cap F^p \arrow[r] & 0
\end{tikzcd}$$

such that $\mathrm{reg}(Z) = \iota \circ AJ(Z)$ for $Z \in \mathrm{CH}^p_{\mathrm{hom}}(X,n)$.
In other words, the Abel-Jacobi map contains at least as much information as the regulator map. Actually, both maps contain the same information, but $J^{p,n}$ is easier to understand than $H_{\mathcal{D}}$ in the sense that is has less relations: is a generalized complex torus.
For $n = 0$, this is just the classical Abel-Jacobi map on $\mathrm{CH}^p(X,0)_{\mathrm{hom}} = \mathrm{CH}^p_{\mathrm{hom}}(X)$ and the above diagram can already be found in Fouad el Zein/Steven Zucker's text [15, Proposition 1].

6.2 Higher Chow chains homologous to zero

Chains homologous to zero Following [39], a higher Chow chain $Z \in z^p_{\mathbb{R}}(X,n)$ is called *homologous to zero*, if $\mathrm{proj} \circ \mathrm{reg}_C(Z)$ is a boundary in $\mathcal{I}^{2p-n}(X,\mathbb{Z}(p)) \oplus F^p\mathcal{D}^{2p-n}(X,\mathbb{C})$, where proj denotes the projection from the 3-term complex $C_{\mathcal{D}}$ onto it's first two components. Equivalently, (by lemma 76) the composition $\mathrm{proj} \circ ev \circ \mathrm{reg}_P(Z)$ is a boundary.
Written as a diagram, the chains homologous to zero are exactly those chains that become boundaries in the rightmost term of the diagram below

$$\begin{tikzcd}
 & P^{2p-n}_{\mathcal{D}}(X,\mathbb{Z}(p)) \arrow[d, "ev"] & \\
z^p_{\mathbb{R}}(X,n) \arrow[ur, "\mathrm{reg}_P"] \arrow[r, "\mathrm{reg}_C"] & C^{2p-n}_{\mathcal{D}}(X,\mathbb{Z}(p)) \arrow[r, "\mathrm{proj}"] & \mathcal{I}^{2p-n}(X,\mathbb{Z}(p)) \oplus F^p\mathcal{D}^{2p-n}(X,\mathbb{C}).
\end{tikzcd}$$

The set of all higher Chow chains homologous to zero form a subgroup of the higher Chow chains, denoted by

$$z^p_{\mathbb{R},\mathrm{hom}}(X,n).$$

If Z is a boundary for Bloch's differential, then Z is automatically homologous to zero. Hence this notion makes sense on cohomology classes and we define $\mathrm{CH}^p_{\mathrm{hom}}(X,n)$ to be the cohomology classes represented by cycles homologous to zero. This is the kernel of the composition

$$\mathrm{CH}^p(X,n) \to H^{2p-n}_{\mathcal{D}}(X,\mathbb{Z}(p)) \to H^{2p-n}(X,\mathbb{Z}(p)) \oplus F^pH^{2p-n}(X,\mathbb{C})$$

that is induced by the regulator into the 3-term complex followed by the projection onto the first two components.

Alternative definitions of "homologous to zero" The notion of being "homologous to zero" depends on the regulator map. Even more, it depends only on the regulator into $C_{\mathcal{D}}$. One may wonder if the definition can be refined by working with the regulator to the complex $P_{\mathcal{D}}$ instead of passing to $C_{\mathcal{D}}$. For this, notice that the regulator $\mathrm{reg}_P(Z)$ of a higher Chow chain

$Z \in z^p_{\mathbb{R}}(X,n)$ can be written in the form

$$\mathrm{reg}_P(Z) = \sum_{i=0\ldots n} x^i(1-x)^{n-i} \otimes T_i + \sum_{i=0\ldots n-1} x^i(1-x)^{n-i-1} dx \otimes R_i$$

where by construction $T_0 \in \mathcal{I}^{2p-n}(X,\mathbb{Z}(p))$, $T_i \in F^{p+i-n}\mathcal{D}(X)$ and $R_i \in F^{p+i-n}\mathcal{D}(X)$ (same filtration, but different degree). Then Z is homologous to zero if and only if T_0 and T_n are boundaries. (Note the special case $n = 0$). One may also think of the seemingly stronger condition that all T_i have to be boundaries (in the respective spaces they live in).
The following lemma shows that (for ∂-closed cycles) this is equivalent to the definition of homologous to zero defined above. Even more, it says that being homologous to zero is already determined by T_0 alone (and so can be defined without mentioning the regulator map at all).

Lemma 79. *Let $Z \in z^p_{\mathbb{R}}(X,n)$ be a higher Chow cycle. Then the following statements are all equivalent:*

1. *Z is homologous to zero.*
2. *T_0 is a boundary in $\mathcal{I}^{2p-n}(X,\mathbb{Z}(p))$.*
3. *(T_0,T_n) is a boundary in $\mathcal{I}^{2p-n}(X,\mathbb{Z}(p)) \oplus F^p\mathcal{D}^{2p-n}(X)$.*
4. *$(T_0,T_1,\ldots,T_n)$ is a boundary in $\mathcal{I}^{2p-n}(X,\mathbb{Z}(p)) \oplus \bigoplus_{k=1}^{n} F^{p+k-n}\mathcal{D}^{2p-n}(X)$.*

Proof. We first consider the case $n = 0$. In this case, $T_0 = \mathrm{cl}(Z)$ and the theorem is equivalent to the statement that, whenever $T_0 = 0$ in integral cohomology, then T_0 vanishes also in $F^pH^{2p}(X,\mathbb{C})$. Assuming the former, then T_0 represents the zero class in $\mathbb{C}$-valued cohomology. Since T_0 is of (p,p)-type and by the injectivity of the map $F^pH^{2p}(X,\mathbb{C}) \to H^{2p}(X,\mathbb{C})$ (this is the $d'd''$-lemma [23, 1.2.1] for currents!) it is also zero in $F^pH^{2p}(X,\mathbb{C})$.
For $n > 0$:
1. $\iff$ 3. holds by definition, since $\mathrm{proj} \circ \mathrm{reg}_C = \mathrm{proj} \circ ev \circ \mathrm{reg}_P$.
Furthermore, the implications 4. $\Rightarrow$ 3. $\Rightarrow$ 2. are obvious. We use the assumption that Z is ∂-closed to show the converse implications. By this condition, the differential $d\,\mathrm{reg}_P(Z)$ and hence all of its components vanish. This implies that all the T_i are d-closed and, looking at the components "with dx":

$$\begin{aligned} 0 &= \sum d\left(x^i(1-x)^{n-i}\right) \otimes T_i - \sum x^i(1-x)^{n-i-1} dx \otimes dR_i \\ &= \sum i x^{i-1}(1-x)^{n-i} dx \otimes T_i - \sum (n-i) x^i(1-x)^{n-i-1} dx \otimes T_i - \sum x^i(1-x)^{n-i-1} dx \otimes dR_i. \end{aligned}$$

Comparing the coefficients of $x^i(1-x)^{n-i-1}dx$ gives that for all $i = 0 \ldots n-1$,

$$0 = (i+1)T_{i+1} - (n-i)T_i - dR_i. \tag{6.1}$$

In $\mathbb{C}$-valued cohomology, the above equations become

$$(i+1)T_{i+1} = (n-i)T_i \qquad \text{in } H^{2p-n}(X,\mathbb{C}).$$

If T_0 is a boundary in $\mathbb{Z}(p)$-valued cohomology, then also with $\mathbb{C}$ coefficients. It follows that all T_i are boundaries and vanish in cohomology with $\mathbb{C}$ coefficients. But the T_i, $i > 0$, lie in the image of $F^{p+i-n}H(X,\mathbb{C}) \to H(X,\mathbb{C})$ and by injectivity (the $d'd''$-lemma again) they also vanish in $F^{p+i-n}H^{2p-n}(X,\mathbb{C})$. $\square$

Denote by $ev_{(k)}$ the unique mapping from $P_{\mathcal{D}}$ to currents that sends $x^j(1-x)^{n-j}$ to 1, if $j = k$, and to zero, if $j \neq k$. Then, by the above lemma, the higher Chow cycles cohomologous to zero can be defined as those higher Chow cycles that become zero in one (then all) of the three groups of the right-hand side in the diagram below

$$\begin{tikzcd}
 & H^{2p-n}P_{\mathcal{D}}(X,\mathbb{Z}(p)) \arrow[r, "\oplus ev_{(k)}"'] \arrow[dd, "ev"] & H^{2p-n}(X,\mathbb{Z}(p)) \oplus \bigoplus_{k=1}^{n-1} H^{2p-n}(X) \oplus F^pH^{2p-n}(X) \arrow[dd, "\text{projection}"] \\
\mathrm{CH}^p(X,n) \arrow[ur, "\mathrm{reg}_P"] \arrow[dr, "\mathrm{reg}_C"'] & & \\
 & H^{2p-n}C_{\mathcal{D}}(X,\mathbb{Z}(p)) \arrow[r, "\text{projection}"] \arrow[dr, "\text{projection}"'] & H^{2p-n}(X,\mathbb{Z}(p)) \oplus F^pH^{2p-n}(X) \arrow[d, "\text{projection}"] \\
 & & H^{2p-n}(X,\mathbb{Z}(p)).
\end{tikzcd}$$

Remark 10.

- Although the triangle on the left does not commute, the maps from the first to the third column do commute.

- It is possible to get explicit formulas for the higher evaluation maps $ev_{(k)}$ on $P^n_{\mathcal{D}}(X,p)$. In fact, they can be constructed recursively as $ev_{(0)}(\omega(x) \otimes T) = \omega(0)T$ and

$$ev_{(k)}(f) = \frac{1}{k!}\left(\frac{d}{dx}\right)^k \Big|_{x=0} \left(f(x) - \sum_{i=0}^{k-1} x^i(1-x)^{n-i} ev_{(i)}(f) \right).$$

  These intermediate evaluations are chosen in such a way that the k-th evaluation map of an element $\sum_{i=0}^n x^i(1-x)^{n-i} \otimes T_i$ is just T_k. In particular, they extend the usual evaluation maps in the sense that $ev_{(0)} = ev_0$ and $ev_{(n)} = ev_1$. They are compatible with the differential, but not with the multiplicative structure.

6.3 Construction of the Abel-Jacobi map

The Abel-Jacobi map on cohomology Every total complex gives rise to a long exact sequence on cohomology. In particular, there is a long exact sequence associated to the total complex $C_{\mathcal{D}}(X,\mathbb{Z}(p))$. From this long exact sequence one can extract the exact sequence

$$0 \to J^{p,n}(X) \to H^{2p-n}C^{\bullet}_{\mathcal{D}}(X,\mathbb{Z}(p)) \xrightarrow{\text{proj}} H^{2p-n}(X,\mathbb{Z}(p)) \oplus F^pH^{2p-n}(X,\mathbb{C}),$$

where

$$J^{p,n}(X) := \frac{H^{2p-n-1}(X,\mathbb{C})}{H^{2p-n-1}(X,\mathbb{Z}(p)) + F^pH^{2p-n-1}(X,\mathbb{C})}$$

and the map from this Jacobian into the 3-term complex is induced by $T \mapsto (0,0,T)$. For any cycle homologous to zero, the regulator values $\mathrm{reg}_C(Z)$ and $ev \circ \mathrm{reg}_P(Z)$ both lie in the kernel of the projection, hence come from an element in $J^{p,n}(X)$. This uniquely defines the Abel-Jacobi maps with respect to the regulators reg_P and reg_C. The construction is summarized by the diagram

$$
\begin{tikzcd}
\mathrm{CH}^p_{\mathrm{hom}}(X,n) \arrow[r, hook] \arrow[d, shift right, "AJ_C"'] \arrow[d, shift left, "AJ_P"] & \mathrm{CH}^p(X,n) \arrow[d, "\mathrm{reg}_C"] \arrow[dr, "\mathrm{reg}_P"] & \\
 & & H^{2p-n}P_{\mathcal{D}}(X,\mathbb{Z}(p)) \arrow[dl, "ev"] \\
J^{p,n}(X) \arrow[r, hook] & H^{2p-n}C_{\mathcal{D}}(X,\mathbb{Z}(p)) &
\end{tikzcd}
$$

Note that although their projections onto the first two components $\mathrm{proj} \circ \mathrm{reg}_C(Z) = \mathrm{proj} \circ \mathrm{reg}_P(Z)$ are equal, the values of $\mathrm{reg}_C(Z)$ and $ev \circ \mathrm{reg}_P(Z)$ in general are different, hence give rise to different Abel-Jacobi maps.

Explicit formulas The Abel-Jacobi maps can be made explicit by observing that the image of $J^{p,n}(X)$ in $H_{\mathcal{D}}$ is spanned by those triples where at most the third component is $\neq 0$.

Thus the general procedure to construct formulas for the Abel-Jacobi map from a regulator value in the 3-term complex is to: First use that Z is homologous to zero to move all information into the third component by adding a boundary. Then project to the third component and finally pass to a quotient in order to be independent of the choices made for the boundary.

In case of the regulator reg_C, write the regulator value of a higher Chow chain Z as

$$\mathrm{reg}_C(Z) = (T_0, T_1, \mathrm{reg}_C(Z)_3).$$

If Z is homologous to zero, $T_0 = dS_0$ and $T_1 = dS_1$ for some $S_0 \in \mathcal{I}^{2p-n-1}(X, \mathbb{Z}(p))$ and $S_1 \in F^p\mathcal{D}^{2p-n-1}(X)$. Now the regulator value is equivalent to

$$\begin{aligned}
\mathrm{reg}_C(Z) &\equiv (T_0, T_1, r_C(Z)_3) - d(S_0, S_1, 0) \\
&= (0, 0, \mathrm{reg}_C(Z)_3 - S_1 + S_0)
\end{aligned}$$

and the resulting Abel-Jacobi map is given by

$$AJ_C(Z) = \mathrm{reg}_C(Z)_3 + S_0 - S_1.$$

Similarly, write

$$ev \circ \mathrm{reg}_P(Z) = (T_0, T_1, \int_0^1 \mathrm{reg}_P(Z)).$$

Note that by lemma 76, T_0, T_1 are exactly the equally named currents as above. Thus we can choose the same boundaries S_0, S_1 as before and obtain that the Abel-Jacobi map with respect to the regulator reg_P can be described by the formula

$$AJ_P(Z) = \int_0^1 \mathrm{reg}_P(Z) + S_0 - S_1. \tag{6.2}$$

Note that by lemma 76, one has

$$AJ_P(Z) = \mathrm{reg}_C(\mathrm{Alt} Z)_3 + S_0 - S_1,$$

which could somewhat sloppy be reformulated by writing

$$AJ_P = AJ_C \circ \mathrm{Alt}$$

with the observation that the equality holds integrally as maps

$$CH^p_{\mathrm{hom}}(X,n) \longrightarrow \frac{H^{2p-n-1}(X,\mathbb{C})}{H^{2p-n-1}(X,\mathbb{Z}(p)) + F^pH^{2p-n-1}(X,\mathbb{C})}$$

and not modulo $\mathbb{Q}(p)$ valued cohomology, as the alternation suggests.

Remark 11. For $n \geq p$ one has $F^p\mathcal{D}^{2p-n-1}(X) = 0$ for (bi-)degree reasons. Hence the Abel-Jacobi map lands in $\frac{H^{2p-n-1}(X,\mathbb{C})}{H^{2p-n-1}(X,\mathbb{Z}(p))}$, computed by the formula (6.2) with S_n set to zero.

Remark 12. The Abel-Jacobi maps can also be defined on quasi-projective varieties U, with the (absolute) cohomology of X replaced by the relative cohomology of (X, D). Note that lemma 79 does not hold in this case due to the failure of the $d'd''$-lemma, and that the statement in section 6.6 also breaks down.

6.4 Examples

- For $n = 0$ the regulator is just the cycle map and the cycles homologous to zero are

$$\mathrm{CH}^p_{\mathrm{hom}}(X,0) = \left\{Z \in \mathrm{CH}^p(X) : \mathrm{cl}(Z) = 0 \text{ in } H^{2p}(X,\mathbb{Z}(p))\right\}.$$

  The Abel-Jacobi map is $AJ(Z) = S_0 - S_1$, where S_0, S_1 are currents bounding $\mathrm{cl}(Z)$ such that S_0 is integral with coefficients in $\mathbb{Z}(p)$ and S_1 lies in $F^p\mathcal{D}^{2p-1}(X)$.

  Note that Poincaré duality induces an isomorphism $J^{p,0}(X) \cong \frac{F^{m-p+1}H^{2m-2p+1}(X,\mathbb{C})^\vee}{H^{2m-2p+1}(X,\mathbb{Z}(p))^\vee}$ and that the image of $AJ(Z)$ under this isomorphism is given by S_0 only (On the left-hand side of the isomorphism, S_1 was necessary to make $AJ(Z)$ d-closed. On the test forms on the right-hand side S_1 acts trivial). Finally, writing $S_0 = [\gamma]$ as the current of integration over a singular chain γ, one reobtains the classical Abel-Jacobi map of Griffiths.

- For $n = 1$ and any $Z \in z^p_{\mathbb{R}}(X,1)$ homologous to zero, its Abel-Jacobi value is the current on X

$$AJ(Z) = (2\pi i)^{p-1}\Big(\int_Z \log(z) + 2\pi i S_0 - S_1\Big).$$

  The currents S_i satisfy $dS_0 = [Z \cap X \times \mathbb{R}_-]$ and $dS_1 = [Z] \wedge \mathrm{dlog}(z)$.

- In particular, if $Z = \sum \Gamma_\alpha$ is the sum of graphs of meromorphic functions $f_\alpha : V_\alpha \to \mathbb{P}_1$, this recovers a version of Levine's formula ([44, p. 458] and [39, 4.5]): By remark 11, S_1 acts trivially in this case and thus, for γ any cycle with boundary $Z \cap (X \times \mathbb{R}_-)$,

$$AJ(Z) = (2\pi i)^{p-1}\Big(\sum_\alpha \int_{V_\alpha \setminus f_\alpha^{-1}\mathbb{R}_-} \log f_\alpha + 2\pi i \int_\gamma\Big).$$

- Totaro's cycle $C(1)$ is homologous to zero: T_0, T_1 are actually zero (not only boundaries) so that one can choose $S_0 = S_1 = 0$. It's Abel-Jacobi image is thus

$$AJ_P(C(1)) = \int \mathrm{reg}_P(Z) = \mathrm{Li}_2(1).$$

6.5 The Abel-Jacobi map and the exterior product

Let Z, Z' be two higher Chow cycles (on different spaces). Assume that Z is homologous to zero. Then lemma 70 assures that the exterior product $Z \times Z'$ is again homologous to zero. The lemma below gives a formula for the Abel-Jacobi value associated to this product. The result holds for both AJ_P and AJ_C so that we omit the subscript and write just AJ.

Lemma 80. *Let Z and Z' be higher Chow cycles on X and X' respectively such that Z is homologous to zero. Then*

$$AJ(Z \times Z') = AJ(Z) \boxtimes T_1',$$

where $\boxtimes$ denotes the exterior product of currents and T_1' denotes the second component of $reg_C(Z')$.

Proof. After eventually replacing reg by $ev \circ \mathrm{reg}$, we may assume that the regulator takes values in the 3-term complex. Write $\mathrm{reg}_C(Z') = (T_0', T_1', T')$, $\mathrm{reg}_C(Z) = (T_0, T_1, T)$ and $t = |T_0|$. Since Z is homologous to zero, $T_i = dS_i$, $i = 0, 1$.
From the compatibility of the regulator with exterior products, one finds that the first two components of $\mathrm{reg}_C(Z \times Z')$ are $T_i \boxtimes T_i' = d(S_i \boxtimes T_i')$, $i = 0, 1$, and that the third component is

$$\mathrm{reg}_C(Z \times Z')_3 = T \boxtimes T_1' + (-1)^t T_0 \boxtimes T'.$$

By definition of the Abel-Jacobi map,

$$AJ(Z) = T + S_0 - S_1$$

and for the product,

$$\begin{aligned} AJ(Z \times Z') &= \mathrm{reg}(Z \times Z')_3 + S_0 \boxtimes T_0' - S_1 \boxtimes T_1' \\ &= T \boxtimes T_1' + (-1)^t T_0 \boxtimes T' + S_0 \boxtimes T_0' - S_1 \boxtimes T_1' \\ &= AJ(Z) \boxtimes T_1' - S_0 \boxtimes T_1' + (-1)^t T_0 \boxtimes T' + S_0 \boxtimes T_0' \\ &= AJ(Z) \boxtimes T_1' - S_0 \boxtimes dT' + (-1)^t T_0 \boxtimes T', \end{aligned}$$

using that $dT' = T_1' - T_0'$ in the last equality. By the Leibniz rule, this is

$$= AJ(Z) \boxtimes T_1' + (-1)^t d(S_0 \boxtimes T'),$$

and so, modulo boundaries, the result follows. □

Remark 13.

- As a consequence of the lemma, the Abel-Jacobi image of $Z \times Z'$ vanishes if both Z and Z' are homologous to zero.
- Using $dT' = T_1' - T_0'$ and $dAJ(Z) = 0$, the lemma can equivalently stated as $AJ(Z \times Z') = AJ(Z) \boxtimes T_0'$, that is, with the second component replaced by the first component. The same is achieved by working with the product $\boxtimes_1$ on the 3-term complex instead of $\boxtimes_0$.
- If Z' is homologous to zero and T_0 denotes the first component of the regulator $\mathrm{reg}_C(Z)$, then an analogous reasoning shows that $AJ(Z \times Z') = (-1)^{|T_0|} T_0 \boxtimes AJ(Z')$.

6.6 The Abel-Jacobi map and higher correspondences

In this last section, we examine in which sense the Abel-Jacobi mappings AJ_C and AJ_P are compatible with higher correspondences.[2] Since $AJ_P = AJ_C \circ \mathrm{Alt}$, it suffices to consider $\mathrm{reg} = \mathrm{reg}_C$ and the corresponding Abel-Jacobi map $AJ = AJ_C$.

Recall from subsection 4.1 that the pullback along a ∂-closed higher correspondence $C \in z^p_{\mathbb{R}}(X \times Y, l)$ gives rise to a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^q(Y,n) \arrow[r, "C^*"] \arrow[d, "\mathrm{reg}"] & \mathrm{CH}^{p+q-m}(X, l+n) \arrow[d, "\mathrm{reg}"] \\
H^{2q-n}_{\mathcal{D}}(Y, \mathbb{Z}(q)) \arrow[r, "\mathrm{reg}(C)^*"] & H^{2p+2q-2m-l-n}_{\mathcal{D}}(X, \mathbb{Z}(p+q-m)).
\end{tikzcd}$$

We will see that the pullback along $\mathrm{reg}(C)$ restricts to a map between intermediate Jacobians. Indeed, note that the Jacobian can be identified with the set of those cohomology classes that are represented by triples of the form $(0,0,A)$. The pullback of such a triple along (E_0, E_1, E) is (we use the $\cap_0$ product on the Deligne complexes) just $(E_0, E_1, E)^*(0,0,A) = (0,0,(-1)^{|E_0|}E_0^*A)$. This shows that $\mathrm{reg}(C)^*$ restricts to a map

$$J^{q,n}(Y) \xrightarrow{\mathrm{reg}(C)^*} J^{p+q-m, l+n}(X)$$

and suggests that, for $\mathrm{reg}(C) = (E_0, E_1, E)$, we should have $AJ(C^*Z) = \pm E_0^* AJ(Z)$. We confirm this by calculating $AJ(C^*Z)$ using the explicit definition of the Abel-Jacobi map.

Theorem 81. *Let $C \in z^p_{\mathbb{R}}(X \times Y, l)$ be a ∂-closed higher correspondence and denote by E_0 the first component of $\mathrm{reg}_C(C)$ and by $e = |E_0|$ its degree. If then Z is any higher Chow chain homologous to zero such that C^*Z exists and intersects the real boundaries properly, then C^*Z is homologous to zero and*

$$AJ(C^*Z) = (-1)^e E_0^* AJ(Z).$$

Proof. Write $\mathrm{reg}(Z) = (T_0, T_1, T)$ and $\mathrm{reg}(C) = (E_0, E_1, E)$. Then, using the compatibility of the regulator map with pullback along higher correspondences,

$$\begin{aligned}
\mathrm{reg}(C^*Z) &= \mathrm{reg}(C)^*\mathrm{reg}(Z) \\
&= \mathrm{pr}_{X*}\Big((E_0, E_1, E) \cap_0 \mathrm{pr}_Y^{XY*}(T_0, T_1, T)\Big) \\
&= \Big(E_0^*T_0, E_1^*T_1, E^*T_1 + (-1)^e E_0^*T\Big).
\end{aligned}$$

For Z homologous to zero, we find $T_0 = dS_0$ and $T_1 = dS_1$. Since $d\mathrm{reg}(C) = 0$, this shows $(-1)^e d(E_i^*S_i) = E_i^*T_i$ for $i = 0,1$, so that C^*Z is homologous to zero. Moreover,

$$AJ(C^*Z) = E^*T_1 + (-1)^e E_0^*T + (-1)^e E_0^*S_0 - (-1)^e E_1^*S_1.$$

Now, using that $AJ(Z) = T + S_0 - S_1$, and that $dE = E_1 - E_0$ (because $\mathrm{reg}(C)$ is d-closed),

[2] As usual, $m = \dim_{\mathbb{C}} X$.

we obtain

$$\begin{aligned} AJ(C^*Z) &= E^*T_1 + (-1)^e E_0^* AJ(Z) + (-1)^e E_0^* S_1 - (-1)^e E_1^* S_1 \\ &= E^*T_1 + (-1)^e E_0^* AJ(Z) - (-1)^e (dE)^* S_1 \\ &= (-1)^e E_0^* AJ(Z) \end{aligned}$$

by the Leibniz rule. □

Remarks.

- The formula in the theorem could equivalently be stated as $AJ(C^*Z) = (-1)^e E_1^* AJ(Z)$. Indeed, this follows either from $dE = E_1 - E_0$ (use that $\mathrm{reg}_C(C)$ is d-closed) together with the Leibniz rule, or from the above proof with $\cap_1$ instead of $\cap_0$.
- The statement of the theorem is also true with $AJ = AJ_P$. When stated in terms of the regulator reg_P, it reads $AJ_P(C^*Z) = (-1)^e \mathrm{reg}_P(C)_0^* AJ_P(Z)$, where $\mathrm{reg}_P(C)_0$ denotes the evaluation at 0 and e the degree thereof.
- If C is a usual correspondence of (higher) Chow groups, then $e = 2p$ is even and the formula reads $AJ(C^*Z) = \mathrm{cl}(C)^* AJ(Z)$.
- Theorem 81 may be useful for the construction of non-trivial elements in the kernel of the higher Abel-Jacobi map. Indeed, if C is also homologous to zero, then the Abel-Jacobi image $AJ(C^*Z)$ is a boundary and so represents the zero class. Thus one only needs a criteria for the pullback C^*Z to be non-trivial. In the case of usual Chow groups such a procedure was carried out in [26].

Lebenslauf

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