Toeplitz Operators on finite and infinite dimensional spaces with associated Ψ^*-Fréchet Algebras

Dissertation
zur Erlangung des Grades
"Doktor
der Naturwissenschaften"

am Fachbereich "Physik, Mathematik und Informatik"
der Johannes Gutenberg-Universität
in Mainz

Wolfram Bauer

geb. in Wiesbaden

Mainz, den 13. Juni 2005

Tag der mündlichen Prüfung: Freitag, 25. November 2005.
Date of the oral examination:

(D77) Dissertation, Johannes Gutenberg-Universität Mainz

Summary

The present thesis is a contribution to the multi-variable theory of Bergman and Hardy Toeplitz operators on spaces of holomorphic functions over finite and infinite dimensional domains. In particular, we focus on certain spectral invariant Fréchet operator algebras $\mathcal{F}$ closely related to the local symbol behavior of Toeplitz operators in $\mathcal{F}$.

We summarize results due to the authors of [79] and [107] on the construction of Ψ_0- and Ψ^*-algebras in operator algebras and corresponding scales of generalized Sobolev spaces using commutator methods, generalized Laplacians and strongly continuous group actions.

In the case of the Segal-Bargmann space $H^2(\mathbb{C}^n, \mu)$ of Gaussian square integrable entire functions on $\mathbb{C}^n$ we determine a class of vector-fields $\mathcal{Y}(\mathbb{C}^n)$ supported in cones $\mathcal{C} \subset \mathbb{C}^n$. Further, we require that for any finite subset $\mathcal{V} \subset \mathcal{Y}(\mathbb{C}^n)$ the Toeplitz projection P is a smooth element in the Ψ_0-algebra constructed by commutator methods with respect to $\mathcal{V}$. As a result we obtain Ψ_0- and Ψ^*-operator algebras $\mathcal{F}$ localized in cones $\mathcal{C}$. It is an immediate consequence that $\mathcal{F}$ contains all Toeplitz operators T_f with f bounded on $\mathbb{C}^n$ and smooth with bounded derivatives of all orders in a neighborhood of $\mathcal{C}$.

There is a natural unitary group action on $H^2(\mathbb{C}^n, \mu)$ which is induced by weighted shifts and unitary groups on $\mathbb{C}^n$. We examine the corresponding Ψ^*-algebras $\mathcal{A}$ of smooth elements in Toeplitz-C^*-algebras. Among other results sufficient conditions on the symbol f for T_f to belong to $\mathcal{A}$ are given in terms of estimates on its Berezin-transform $\tilde{f}$.

Local aspects of the Szegö projection P_s on the Heisenbeg group and the corresponding Toeplitz operators T_f with symbol f are studied. In this connection we apply a result due to Nagel and Stein [117] which states that for any strictly pseudo-convex domain Ω the projection P_s is a pseudodifferential operator of exotic type $(\frac{1}{2}, \frac{1}{2})$.

The second part of this thesis is devoted to the infinite dimensional theory of Bergman and Hardy spaces and the corresponding Toeplitz operators. We give a new proof of a result observed by Boland [24], [25] and Waelbroeck [141]. Namely, that the space of all holomorphic functions $\mathcal{H}(U)$ on an open subset U of a $\mathcal{DFN}$-space (dual Fréchet nuclear space) is a $\mathcal{FN}$-space (Fréchet nuclear space) equipped with the compact open topology. Using the nuclearity of $\mathcal{H}(U)$ we obtain Cauchy-Weil-type integral formulas for closed subalgebras $\mathcal{A}$ in $\mathcal{H}^\infty(U)$, the space of all bounded holomorphic functions on U, where $\mathcal{A}$ separates points. Further, we prove the existence of Hardy spaces of holomorphic functions on U corresponding to the abstract Shilov boundary $\mathcal{S}_\mathcal{A}$ of $\mathcal{A}$ and with respect to a suitable boundary measure Θ on $\mathcal{S}_\mathcal{A}$.

Finally, for a domain U in a $\mathcal{DFN}$-space or a polish spaces we consider the symmetrizations μ_s of measures μ on U by suitable representations of a group G in the group of homeomorphisms on U. In particular, in the case where μ leads to Bergman spaces of holomorphic functions on U, the group G is compact and the representation is continuous we show that μ_s defines a Bergman space of holomorphic functions on U as well. This leads to unitary group representations of G on L^p- and Bergman spaces inducing operator algebras of smooth elements related to the symmetries of U.

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Introduction

The present thesis is a contribution to the construction of spectral invariant symmetric Fréchet subalgebras (Ψ^*-algebras) of Toeplitz-C^*-algebras over finite and infinite dimensional domains. Moreover, it is of general interest to give a notion of Bergman and Hardy spaces and the corresponding Toeplitz operators for domains Ω in certain infinite dimensional nuclear spaces E. This approach only uses the nuclearity of the Fréchet space $\mathcal{H}(\Omega)$ of all holomorphic functions on Ω equipped with the compact open topology.

The concept of Ψ^*-algebras is closely related to local aspects in operator theory. To give an idea of the kind of results we prove let us state the following theorems which can be found in chapter 2 and 6. Let μ be a Gaussian measure on $\mathbb{C}^n$ and fix open cones $\mathcal{C}_1 \subset \mathcal{C}_2$ in $\mathbb{C}^n$. We show how to obtain a rich variety of Ψ^*-algebras Ψ^{Δ}_{∞} in $\mathcal{L}(\, L^2(\mathbb{C}^n, \mu)\,)$ containing the Segal-Bargmann Toeplitz projection P and *localized in* $\mathcal{C}_1$ in the following sense (cf. Theorem 2.5.3 and Proposition 2.5.2):

Theorem 1 *Let $h \in L^\infty(\mathbb{C}^n)$ such that supp$(h) \subset \mathbb{C}^n \backslash \mathcal{C}_2$ or let us assume that $h \in \mathcal{C}^\infty_b(\mathcal{C}_2)$, then $T_h \in \Psi^{\Delta}_{\infty}$, where $T_h g := P(hg)$ for all $g \in L^2(\,\mathbb{C}^n, \mu\,)$.*

Hence the algebra Ψ^{Δ}_{∞} is invariant under perturbations by Toeplitz operators T_h with symbols supported outside of $\mathcal{C}_2$ and related to the regularity of h restricted to $\mathcal{C}_2$. In our constructions above we apply quite general ideas which were suggested by the authors of [79] and do not depend on the finite dimension of the underlying domains. Based on a combination of results in [25], [121] and [141] we prove that for any open subset U of a $\mathcal{DFN}$-space E (topological dual of a Fréchet nuclear space) there is a notion of Bergman space $\mathcal{H}^2(U, \mu)$. Moreover, let $\mathcal{A}$ be a closed subalgebra of the Banach algebra $\mathcal{H}^\infty(U)$ of bounded holomorphic functions on U which separates points and has the abstract Shilov boundary $\mathcal{S}_{\mathcal{A}}$. Then in addition we can prove the existence of a Hardy space of holomorphic functions on U (cf. Theorem 6.7.1) :

Theorem 2 *Let $\mu_1, \mu_2 \in \mathcal{MF}_2(U)$ be measures where $\mathcal{F} = \mathcal{H}(U)$ (cf. Definition 5.4.1). Assume that there is a diagram*

$$\mathcal{H}^\infty(U) \supset \mathcal{A} \xrightarrow{J_1} \mathcal{H}^2(U, \mu_1) \xrightarrow{J_2} \mathcal{H}^2(U, \mu_2)$$

where J_i are continuous embeddings and J_2 is nuclear. Then there is a Hardy space $\mathcal{H}^2(\mathcal{S}_{\mathcal{A}}, \Theta)$ containing $\mathcal{A}$ which admits a quasi-nuclear embedding into $\mathcal{H}^2(U, \mu_2)$.

We want to remark that our construction of Hardy spaces seems to be intrinsic and it requires no assumption on the boundary of U. For instance note that via biholomorphic equivalence for each simply connected domain U in the complex plane the Banach algebra of holomorphic functions on U with continuous extensions to $\bar{U}$ leads to a closed subalgebra of $\mathcal{H}^\infty(D)$ separating points where D denotes the open unit disc.

In the infinite dimensional setting the symmetries of the domains U resp. $\mathcal{S}_\mathcal{A}$ can be used to obtain Ψ^*-algebras of smooth elements in Toeplitz C^*-algebras (cf. chapter 7).

There is an extended theory on (locally) spectral invariant Fréchet algebras subsequent to [79]. Applications arise in the structural analysis of certain algebras of pseudodifferential operators, complex analysis, analytic perturbation theory of Fredholm operators, non-abelian cohomology or for analyzing isomorphisms of abelian groups in K-theory. As for a more detailed list and the consequences that follow for the algebras given in the present thesis we refer to Remark 1.0.2. We consider our work to be a first step to make the abstract theory of Ψ^*-algebra applicable to algebras of Bergman and Hardy Toeplitz operators.

Toeplitz operators and related algebras on classical spaces such as the Hardy space on the 1-torus or the Bergman space on the unit disc are well-understood and they play an important role in operator theory. In fact, there are many applications to function theory, integral equations and control theory. In the higher dimensional setting, the analysis of multi-variable Toeplitz operators is more complicated and is less well-known. Here the geometry of the underlying space is closely related to the associated Toeplitz C^*-algebras. For several classes of domains with not necessarily smooth boundary the spectral behavior, index theory and solvability of C^*-algebras is examined in [140],[83].

As a model space for quantum mechanical operators, I.E. Segal and V. Bargmann invented a space $\mathcal{F}_n := H^2(\mathbb{C}^n, \mu)$ of entire functions on $\mathbb{C}^n$ square integrable with respect to a Gaussian measure μ and canonically isomorphic to the Fock space [5], [133]. Fundamental concepts as the creation and annihilation operators can be represented as Toeplitz operators on $\mathcal{F}_n$ and there have been far reaching investigations of the corresponding operator theory by L.A. Coburn and C.A. Berger [21], [22], [23], [35], [36], [37] see also the results in [11], [12], [92], [145].

From a physical point of view, it is significant to extend the number of freedom corresponding to the complex dimension n to infinity. There also have been approaches on Toeplitz operators over infinite dimensional domains, motivated by quantum field theory [93], [90], [14]. J. Janas and K. Rudol have generalized the notion of Toeplitz operators on the Segal-Bargmann space $\mathcal{F}_n$ by replacing $\mathbb{C}^n$ by a separable complex Hilbert space H and μ by an infinite dimensional Gaussian measure on H. Some new phenomena arise, which have no counterpart in the case of $\mathcal{F}_n$ but also create difficulties in the analysis of [93], [90] and [14] which come from the measure theory on H. A second model, which is referred to in the literature as the *inductive limit approach*, only uses a so-called quasi Gaussian measure on H with non-nuclear correlation and it is based on Segal's model of the Fock space [132].

We focus on both, the multi-variable and the infinite dimensional theory of Bergman and Hardy Toeplitz operators. On the one hand it is of interest in operator theory to construct subalgebras of C^*algebras in $\mathcal{L}(H)$ where H is a Hilbert space which are related to local properties of its elements and acting on scales of Sobolev spaces. In this connection the notion of a Ψ-*algebra* and in particular of a Ψ^*-*algebra* has attached great importance (cf. the list of recent results at the beginning of chapter 1.) On the other hand, motivated by quantum mechanics, representation theory and infinite dimensional holomorphy we focus on the theory of Bergman and Hardy spaces over infinite dimensional domains. As for the area of Toeplitz operators our main interest is the following:

(I) The construction of spectral invariant Fréchet algebras and Ψ^*-algebras in particular in Toeplitz C^*-algebras for Bergman and Hardy Toeplitz operators T_f related to local properties of the symbol f.

(II) Bergman and Hardy spaces over certain infinite dimensional spaces related to the symmetry of the underlying domain and corresponding Toeplitz operators.

We describe what is meant by (I) and (II) above. In general, by passing from a ring of operators to its closure (a Banach or C^*-algebra) one looses local $\mathcal{C}^\infty$-properties such as pseudo- or micro-locality. Hence in order to keep control of the local behavior and, motivated by the ring of zero-order pseudodifferential operators, the notion of a Ψ-algebra $\mathcal{A}$ in a Banach algebra $\mathcal{B}$ was introduced (cf. [69]). In particular, if $\mathcal{A}$ is symmetric in a C^*-algebra $\mathcal{B}$ it is referred to as a Ψ^*-algebra. Essential in the definition is the notion of *spectral invariance* or *invariance under holomorphic functional calculus*:

$$\mathcal{A} \cap \mathcal{B}^{-1} = \mathcal{A}^{-1}$$

where $\mathcal{A}^{-1}$ (resp. $\mathcal{B}^{-1}$) denotes the group of invertible elements [17], [69], [79], [130], [139]. A spectrally invariant algebra $\mathcal{A}$ sometimes is called *full* or *algèbre plaine*, the pair of algebras $(\mathcal{A}, \mathcal{B})$ is said to be a *Wiener-pair* following ideas due to Bourbaki, Naimark and Waelbroeck [27], [118], [142], [143]. As an immediate consequence, a Ψ-algebra $\mathcal{A}$ has an open group $\mathcal{A}^{-1}$ which in general is not the case for an arbitrary Fréchet algebra. Moreover, we mention that the inversion in $\mathcal{A}$ is continuous and it is a useful property of the stability that countable intersections of Ψ- resp. Ψ^*- algebras are of this type again.

The concept of extracting the notion of spectral invariance from the operator theory and focusing on an abstract Ψ-algebras has been successful. An extensive study on spectral invariant Fréchet algebras starting with [69] led to many applications and among others we mention perturbation theory and homotopy theory of Fredholm functions or the holomorphic functional calculus of L. Waelbroeck.

Various methods of generating Ψ-algebras can be found in [79] and in part they are described in chapter 1 of this thesis. In our constructions later on we use *commutator methods* with vector fields supported in suitable sets, *generalized Laplacians* [44] and *unitary group actions* [28], [79], [107]. Let us describe the concept of commutator methods. It was shown by R. Beals [17] that the Hörmander classes $\Psi^0_{\rho,\delta}$ of pseudodifferential operators [1] can completely be characterized by conditions on iterated commutators with the multiplications M_{x_j} and the derivatives ∂_{x_j} of all orders. Using this result, it was shown by R. Beals [17] and finally by J. Ueberberg [139] and E. Schrohe [130] (cf. [26]) that the classes $\Psi^0_{\rho,\delta}$ are spectral invariant in $\mathcal{L}(H)$ where $H := L^2(\mathbb{R}^n, v)$ and so $\Psi^0_{\rho,\delta}$ is a Ψ^*-algebra in the sense

[1] $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$

of [79]. Moreover, as a result in [40] there are similar descriptions using commutators with smooth vector fields. The authors of [79] pointed out how to generalize this method to an abstract setting in order to construct Ψ-algebras $\mathcal{A}$ in certain subalgebras of the bounded operators on a Hilbert space H. These methods are using commutator characterizations fom begin on and so spectral invariance is an immediate consequence. Let $\mathcal{V}$ be a finite set of densely defined closed operators on H. Then, roughly speaking, for an operator $a \in \mathcal{L}(H)$ to be in $\mathcal{A} =: \Psi_\infty^{\mathcal{V}}$ we require that all the iterated commutators

$$\Big[\, [\,[\,a, V_1\,],\ V_2\,], \cdots \Big], \qquad V_j \in \mathcal{V} \tag{0.0.1}$$

are well-defined on a suitable dense subspace in H and admit extensions to bounded operators on H. Additionally, we remark that in the case where all operators $V \in \mathcal{V}$ are symmetric, $\mathcal{A}$ becomes a Ψ^*-algebra; associated to $\mathcal{V}$ there is a scale of abstract Sobolev spaces which is preserved by the operators in $\Psi_\infty^{\mathcal{V}}$.

As for question (I) we start our investigation with Toeplitz operators T_f on the Segal-Bargmann space $H^2(\mathbb{C}^n, \mu)$ of μ-square integrable entire functions on $\mathbb{C}^n$ introduced by I.E. Segal and V. Bargmann [5], [133] (for the theory of Toeplitz operators: [21], [22], [23], [37], [55], [56],[91], [92], [135]). Here f is an admissible measurable symbol and μ denotes a Gaussian measure. With the orthogonal projection P from $L^2(\mathbb{C}^n, \mu)$ onto $H^2(\mathbb{C}^n, \mu)$ the (in general unbounded) operator T_f is defined by $T_f(g) := P(fg)$ provided that the product fg is in $L^2(\mathbb{C}^n, \mu)$. We determine a subspace $\mathcal{Y}(\mathbb{C}^n)$ in the class $\mathcal{W}(\mathbb{C}^n)$ of all smooth vector fields on $\mathbb{C}^n$ and a suitable dense subspace $Z \subset L^2(\mathbb{C}^n, \mu)$ with the following properties:

(P1) For any finite system $\mathcal{V} \subset \mathcal{Y}(\mathbb{C}^n)$ of vector fields, with the Toeplitz projection $a := P$ and $V_j \in \mathcal{V}$ all the iterated commutators in (0.0.1) are well-defined on Z and they admit (unique) extensions to bounded operators on $L^2(\mathbb{C}^n, \mu)$.

(P2) Let A be an open set in the complex unit sphere ∂B_{2n} in $\mathbb{C}^n$. Then there exist vector-fields $0 \neq V \in \mathcal{Y}(\mathbb{C}^n)$ supported in $\mathcal{C}_A := \{\, z \in \mathbb{C}^n \setminus \{0\} : z \cdot \|z\|^{-1} \in A \,\}$.

As a first step, we will prove that the space $\mathcal{Y}_1$ of all vector fields with constant coefficients fullfils $(P1)$ with $Z := \mathcal{C}_c(\mathbb{C}^n)$. Now, in addition we want to fulfill $(P2)$ and so we have to enlarge $\mathcal{Y}_1$. As it is indicated by Example 2.3.1 one has to be careful. It turns out that the boundedness of commutators of P with a smooth vector field $V = \sum_{j=1}^n \{\, a_j \partial_j + b_j \bar{\partial}_j \,\}$ is closely related to the oscillation of the coefficients a_j and b_j at infinity. We prove that radial extensions to $\mathbb{C}^n$ of smooth functions on the complex sphere (which are cut off at 0) form a class of admissible coefficients a_j for V. Moreover, we can choose b_j to be smooth with bounded derivatives of all orders. This finally leads to a space $\mathcal{Y}(\mathbb{C}^n)$ of vector fields with (P1) and (P2) which, up-to a cut off at 0, contains $\mathcal{Y}_1$ as well as the derivatives $\partial_{\varphi_j} \in \mathcal{W}(\mathbb{C}^n)$ tangential to the complex sphere ∂B_{2n}.

As a consequence in our theory of operator algebras and with the notations above we conclude that $P \in \Psi_\infty^{\mathcal{V}} \subset \mathcal{L}(\, L^2(\mathbb{C}^n, \mu)\,)$ for all finite sets $\mathcal{V} \subset \mathcal{Y}(\mathbb{C}^n)$ and the Toeplitz projection P is said to be *smooth* with respect to $\mathcal{V}$. Let $\mathcal{C}_A$ be the cone over $A \subset \partial B_{2n}$

as it was defined in (P2) and assume that all vector fields $V_j \in \mathcal{V}$ are supported in $\mathcal{C}_A$. Then, for the multiplication $M_f \in \mathcal{L}(\, L^2(\mathbb{C}^n, \mu)\,)$ with a bounded symbol f to belong to $\Psi^{\mathcal{V}}_\infty$ only the behavior of f restricted to $\mathcal{C}_A$ is involved. Therefore, the same holds true for the Toeplitz operator $T_f = PM_f$. In particular, the Ψ-algebra $\Psi^{\mathcal{V}}_\infty$ is invariant under perturbations of Toeplitz operators with symbols supported outside of $\mathcal{C}_A$. The space $\mathcal{W}(\mathbb{C}^n)$ is not symmetric, but after some slight changes of notations we can give similar constructions of Ψ^*-algebras containing P. Due to the fact that P is a smooth element in $\mathcal{L}(\, L^2(\mathbb{C}^n, \mu)\,)$, the projection $P\Psi^{\mathcal{V}}_\infty P$ leads to classes of Ψ-algebras (resp Ψ^*-algebras) in the algebra $\mathcal{L}(\, H^2(\mathbb{C}^n, \mu)\,)$ localized by the symbol behaviour, as well.

Following the fairly general concepts in [79] we examine Ψ^*-algebras generated by unitary groups in Toeplitz C^*-algebras [28], [107]. The group of Weyl-operators on the Fock space have a well-known representation as unitary weighted shift operators W_x on both spaces $H^2(\mathbb{C}^n, \mu)$ and $L^2(\mathbb{C}^n, \mu)$. More precisely, we have

$$W_x f := k_x \cdot f\big(\,\cdot - x\,\big), \qquad \big(\, x \in \mathbb{C}^n\,\big)$$

where k_x denotes the normalized reproducing kernel on $H := H^2(\mathbb{C}^n, \mu)$. Moreover, it is a standard fact that the map

$$\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \ni (x,t) \mapsto W_{tx} \in \mathcal{L}\big(\, H^2(\mathbb{C}^n, \mu)\,\big) =: \mathcal{B}$$

leads to a an irreducible unitary representation of the Heisenberg group $\mathbb{H}^n$ in $\mathcal{B}$. As a link to quantum mechanics, it was shown that the C^*-algebra of canonical commutation relations over $\mathbb{C}^n$ (CCR($\mathbb{C}^n$)), generated by the Weyl-operators coincides with the C^*-algebra generated by Toeplitz operators with *almost periodic symbols* [23], [38].

For any $x \in \mathbb{C}^n$ we obtain a unitary strongly continuous group (C_0-group) $(W_{tx})_{t\in\mathbb{R}}$ of operators on $H^2(\mathbb{C}^n, \mu)$ with infinitesimal generator $V^{(x)}$ which coincides on a dense subspace in H with an (unbounded) Toeplitz operator T_p where p is a linear function. We can consider the corresponding map $\varphi_x : \mathbb{R} \to \mathcal{L}(\, \mathcal{L}(H)\,)$ defined for $t \in \mathbb{R}$ and $A \in \mathcal{L}(H)$ by conjugating with the Weyl-group:

$$\big[\, \varphi_x(t)\,\big](A) := W_{tx}\, A\, W_{-tx} \in \mathcal{L}(H).$$

In particular, if $A = T_f$ is a Toeplitz operator with (admissible) symbol f it can be shown that $W_x\, T_f\, W_{-x} = T_{f(\cdot - x)}$. Hence, all C^*-Toeplitz algebras $\mathcal{A} \subset \mathcal{L}(H)$ generated by Toeplitz operators with symbols in a class of functions invariant under shifts in direction x are invariant under the action of $\varphi_x(t)$. Let $a \in \mathcal{A}$ be fixed, then we define

$$\varphi_x^a : \mathbb{R} \to \mathcal{L}(H) : t \mapsto \big[\, \varphi_x(t)\,\big](a).$$

Under these notations and for all $k \in \mathbb{N}_0 \cup \{\infty\}$ let us consider the following algebras of $\mathcal{C}^k$-elements in $\mathcal{A}$, [28], [79], [107]:

$$\Psi^k_x[\,\mathcal{A}\,] := \Big\{\, a \in \mathcal{A} : \varphi^a_x \in \mathcal{C}^k\big(\,\mathbb{R}, \mathcal{L}(H)\,\big)\,\Big\} \qquad \text{and} \qquad \Psi^\infty_x[\,\mathcal{A}\,] := \bigcap_{j\in\mathbb{N}} \Psi^j_x[\,\mathcal{A}\,].$$

It is well-known that $\Psi_x^\infty[\mathcal{A}]$ can also be obtained by using iterated commutator methods with the one point system $\mathcal{V} := \{ V^{(x)} \}$ as described above [28], [107] and Theorem 1.3.1 in chapter 1. Due to the fact that the operator $iV^{(x)}$ is self-adjoint by *Stone's theorem* we conclude that $\Psi_x^\infty[\mathcal{A}]$ is a Ψ^*-algebras in the Toeplitz C^*-algebra $\mathcal{A}$. Now, let us assume that $\mathcal{A}$ is the Toeplitz C^*-algebra generated by all operators T_f with bounded symbol f on $\mathbb{C}^n$. Then we have $W_{tx} \in \mathcal{A}$ for all $t \in \mathbb{R}$ and $x \in \mathbb{C}^n$ and so $\mathcal{A}$ is invariant under $\varphi_x(t)$. We consider the problem

(P3) Can we describe a class of symbols $\mathcal{D}_k \subset L^\infty(\mathbb{C}^n, \mu)$ such that $T_f \in \Psi_x^k[\mathcal{A}]$ for all $f \in \mathcal{D}_k$ where $k \in \mathbb{N}_0 \cup \{\infty\}$?

Let us mention that from our definition above it follows that $a := T_f \in \Psi_x^k[\mathcal{A}]$ is described by a differentiability condition on the map

$$\mathbb{R} \ni t \mapsto \varphi_x^a(t) = T_{f(\cdot - tx)} \in \mathcal{A}$$

and so one might expect that some kind of smoothness of the symbol f in direction x is required. In fact this is not the case and we prove that under fairly loose assumptions based on the regularity of f we obtain operators in $\Psi_x^k[\mathcal{A}]$ and further that all the algebras $\Psi_x^k[\mathcal{A}]$ for $k \in \mathbb{N} \cup \{\infty\}$ coincide. Let us mention an effect that is closely related to this observation and which is described in detail at [19], [20], [82] and the end of chapter 3 of the present thesis:

There is a close connection between Toeplitz operators and pseudodifferential operators on compact manifolds M. By results in [82] the ring of pseudodifferential operators on M is isomorphic with the ring of Toeplitz operators on a appropriate Grauert tube about M. Via the Bargmann isometrie $\beta : L^2(\mathbb{R}^n, v) \to H^2(\mathbb{C}^n, \mu)$, the Segal-Bargmann space is isomorphic to $L^2(\mathbb{R}^n, v)$ where v denotes the usual Lebesgue measure. It was shown that each Toeplitz operator T_f on $H^2(\mathbb{C}^n, \mu)$ corresponds to a pseudodifferential operator W_σ with symbol σ in its Weyl form on $L^2(\mathbb{R}^n, v)$ under conjugation with the isometrie β [82], [35], [36]. Both symbols σ and f are related via the heat equation and Berezin's formula. Roughly speaking, the symbol σ of the Weyl operator corresponding to the Toeplitz operator T_f is the solution of the heat equation on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ at time $t = \frac{1}{8}$ with initial data f:

$$\sigma = \left(e^{-\frac{1}{8}\Delta} \right) f, \qquad \big(\text{Berezin's formula}\big).$$

Hence, for any $f \in L^\infty(\mathbb{C}^n)$ we conclude that σ is smooth with bounded derivatives of all orders. Heuristically, all information on the smoothness of f is lost by passing to the Weyl symbol σ.

We also study the analog of problem (P3) in the case of unitary C_0-groups induced by composition operators on $H^2(\mathbb{C}^n, \mu)$. In contrast to the Weyl group action, here on a dense subspace of the Segal-Bargmann space, the infinitesimal generator coincides with an (unbounded) Toeplitz operator T_q having polynomial symbol q of degree 2. The non-linearity of q causes some more trouble in the norm-estimates of the iterated commutators but we can give a characterization of classes $\mathcal{D}_k$ in (P3) by growth conditions on the Berezin transforms of $f \in \mathcal{D}_k$ (cf. Theorem 3.5.3).

Due to a result by A. Nagel and E.M. Stein [117], for a strictly pseudo-convex domain the Szegö-projection P_s is a pseudodifferential operator of exotic type $(\frac{1}{2}, \frac{1}{2})$. This already occurs in the case of the complex unit sphere in $\mathbb{C}^n$ of complex dimension n greater than 1, while on $S^1 \subset \mathbb{C}$ the operator P_s turns out to be of class $\Psi^0_{1,0}$ (cf. [103], pp. 178).

Via the biholomorphic equivalence of the unit ball $B_{2n+2} \subset \mathbb{C}^{n+1}$ with the upper half-space $\mathcal{H}_+$ in $\mathbb{C}^{n+1}$ and by an identification of its boundary $\partial\mathcal{H}_+$ with the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ the Szegö-projection P_s can be looked at as a convolution operator on $L^2(\mathbb{H}^n, v)$ with respect to the group structure of $\mathbb{H}^n$. Moreover, its localized version $\psi_1 \, P_s \psi_2$ where $\psi_j \in \mathcal{C}_c^\infty(\mathbb{H}^n)$ is of exotic type $(\frac{1}{2}, \frac{1}{2})$ and so it inherits all pseudo- and microlocal properties of pseudodifferential operators. On the other hand, in the case of exotic classes the full asymptotic calculus fails. The pseudodifferential techniques in Hörmander [88] break down and the L^2-boundedness result (which in fact is obvious for the Szegö-projection) is due to A.P Calderon and R. Vaillancourt [32]. In this setting we study the following problem:

(P4): Let $U \subset \mathbb{H}^n$ be an open subset. How can one define classes of spectral invariant Fréchet algebras or even Ψ^*-algebras $\mathcal{B}_U \subset \mathcal{L}(\, L^2(\mathbb{H}^n, v)\,)$ containing all pseudodifferential operators of type $(\frac{1}{2}, \frac{1}{2})$ such that $\mathcal{B}_U$ is localized on U in the following sense:

For any symbol $f \in L^\infty(\mathbb{H}^n)$ which is compactly supported and smooth in a neighborhood of $\bar{U}$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{H}^n)$ the Toeplitz operator φT_f is contained in $\mathcal{B}_U$?

In order to answer $(P4)$ we use commutator methods with finite systems of smooth and compactly supported vector fields. Moreover, we introduce a generalized Laplace operator [44] and a corresponding scale $(\mathcal{H}_j)_j$ of Sobolev spaces. As a result, we obtain an algebra $\mathcal{B}_U$ which solves $(P4)$ and operates on $(\mathcal{H}_j)_j$ without an order shift.

In the final part of chapter 4 we examine the Szegö-projection P on the unit sphere $S^{2n-1} = \partial B_{2n} \subset \mathbb{C}^n$. The Szegö kernel explicitly is known and we show that for any bounded function f smooth in an open set $U \subset S^{2n-1}$ the projection Pf restricted to U can be obtained by continuous extension of Pf from inside the ball B_{2n} (cf. Theorem 4.4.3). In our analysis the notion of *spherical harmonics* as well as the asymptotic of the eigenvalues of the *Beltrami-Laplace operator* on the complex sphere are involved. As a conjecture we consider many of the results proved in chapter 4 to be true in greater generality for strictly pseudo-convex domains in $\mathbb{C}^n$.

The chapters 5, 6 and 7 of this thesis are devoted to the infinite dimensional theory of Bergman- and Hardy Toeplitz operators (cf. (II) above). In order to define a closed subspace (*Bergman space*) of holomorphic functions in a L^2-space over an open manifold U one has to choose a Borel measure μ on U with the following property:

For each compact set $K \subset U$ there is a compact set $H \subset U$ with $K \subset H$ such that for all holomorphic $f \in \mathcal{H}(U)$ it holds

$$\sup\Big\{ \mid f(x) \mid : x \in K \Big\} \le C \Big[\int_H \mid f \mid^2 d\mu \Big]^{\frac{1}{2}}. \tag{0.0.2}$$

Here C is a suitable positive number which is related to K and independent of f.

Under loose topological assumptions on U (it has to be a *k-space* and *hemi-compact*) we generalize a result due to A. Pietsch in [121] to the infinite dimensional case. The existence of a measure μ with (0.0.2) is equivalent to the nuclearity of the Fréchet space $\mathcal{H}(U)$ of all holomorphic functions on U with respect to the *compact-open topology.* The measure μ with (0.0.2) is said to be a $\mathcal{NF}_2$-measure where $\mathcal{F} := \mathcal{H}(U)$ and in the following we call U a $\mathcal{NF}_2$-space if there exists a $\mathcal{NF}_2$-measure on U (cf. Definition 5.4.1).

On the one hand, in the case where U is an open set in $\mathbb{C}^n$, the nuclearity of $\mathcal{H}(U)$ is well-known, but it already fails if we choose U to be an infinite dimensional separable complex Hilbert space (cf. Corollary 6.2.1). On the other hand, there are results due to P.J. Boland and L. Waelbroeck [24], [25] and [141] on the nuclearity of $\mathcal{H}(U)$ in cases where U itself is a subset of a nuclear space. The following can be found as Theorem 5.5.2 in chapter 5:

Theorem 3 (P.J. Boland, L. Waelbroeck, [24], [25], [141]) *Let E be a $\mathcal{DFN}$-space (i.e. the strong dual of a Fréchet nuclear space) and $U \subset E$ be open. Then endowed with the compact-open topology the space $\mathcal{H}(U)$ of all holomorphic functions on U is a Fréchet nuclear space ($\mathcal{FN}$-space).*

Let E be a (complex) $\mathcal{DFN}$-space which without loss of generality is represented by a nuclear inductive limit of Hilbert spaces in the category of locally convex spaces and continuous linear mappings. Applying Gaussian measures on infinite dimensional spaces we explicitly construct a measure μ on E which, restricted to all open subsets $U \subset E$, has the property (0.0.2). Due to a generalization of Pietsch's theorem, this leads to a new proof of Theorem 3. Moreover, it follows that for all open subsets in $\mathcal{DFN}$-spaces we have the notion of *Bergman space* of holomorphic functions as well as the *Bergman Toeplitz projection.*

How can we obtain a generalization of *Hardy spaces* of square integrable holomorphic boundary values? Let us denote by $\mathcal{H}^\infty(U)$ the Banach algebra of all bounded holomorphic functions on U and fix a closed subalgebra $\mathcal{A}$ of $\mathcal{H}^\infty(U)$ which separates points with abstract Shilov boundary $\mathcal{S}_\mathcal{A}$. Given a finite measure μ on U with (0.0.2) we prove the existence of a kernel $\Phi_\mu : U \times \mathcal{S}_\mathcal{A} \to \mathbb{C}$ and a finite Radon measure ν on $\mathcal{S}_\mathcal{A}$ with the following properties:

(1) The map $U \ni z \mapsto \Phi_\mu(z, x) \in \mathbb{C}$ is holomorphic for all $x \in \mathcal{S}_\mathcal{A}$.

(2) For all $z \in U$ we have $\Psi_\mu(z) = [\, \Phi_\mu(z, \cdot) \,] \in L^1(\mathcal{S}_\mathcal{A}, \nu)$.

(3) There is $C > 0$ with $\| \Phi_\mu(z, \cdot) \|_{L^1(\mathcal{S}_\mathcal{A}, \nu)} \le C \cdot \| Eval(z) \|_{\mathcal{H}^2(U,\mu)'}$ for all $z \in U$.

(4) It holds $f(z) = \int_{\mathcal{S}_\mathcal{A}} x(f) \cdot \Phi_\mu(z, x)\, d\nu(x)$ for all $f \in \mathcal{A}$ and $z \in U$.

(cf. Theorem 6.6.1 and Remark 6.6.1). Partly, we follow some ideas in [61], where an analog result was proved in the finite dimensional case together with lifting-results due to R.G. Bartel and L.M. Graves [6]. We essentially use the nuclearity of $\mathcal{H}(U)$ and as an important ingredient in our proof (and suggested by B. Gramsch) Grothendieck's theorem 6.2.1 is applied. By an *abstract Hardy space* $\mathcal{H}^2(\mathcal{S}_\mathcal{A}, \Theta)$ we mean a closed subspace of $L^2(\mathcal{S}_\mathcal{A}, \Theta)$ where Θ is a suitable finite measure on the Shilov boundary $\mathcal{S}_\mathcal{A}$, which densely contains $\delta[\mathcal{A}]$. Here $\delta : \mathcal{A} \to \mathcal{C}(\mathcal{S}_\mathcal{A})$ is defined by $\delta f[x] = x(f)$ for all $x \in \mathcal{S}_\mathcal{A}$ and $f \in \mathcal{A}$. Moreover, we claim that there is a *Hardy kernel* $K : U \times \mathcal{S}_\mathcal{A} \to \mathbb{C}$ such that

(5) The map $K(z, \cdot) : \mathcal{S}_\mathcal{A} \to \mathbb{C}$ is bounded for fixed $z \in U$,

(6) For each compact set $H \subset U$ and $f \in \mathcal{A}$ it holds:

$$\sup\Big\{ \,|\, f(z) \,|\, : z \in H \Big\} \leq \|\, \delta(f) \,\|_{L^2(\mathcal{S}_\mathcal{A},\Theta)} \cdot \sup\Big\{ \,\|\, K(z, \cdot) \,\|_{L^2(\mathcal{S}_\mathcal{A},\Theta)} : z \in H \Big\}.$$

Note that from (6) we can identify $\mathcal{H}^2(\mathcal{S}_\mathcal{A}, \Theta)$ with a space of holomorphic functions on U. The Hardy kernel K corresponds to Φ_μ above which in general for fixed first component only leads to a function in $L^1(\mathcal{S}_\mathcal{A}, \nu)$. However, under a modification of μ (which always can be made) we can define $K = \Phi_\mu$ with (5) where $\Theta := \nu$. Property (6) is a direct consequence of (4). We mention that under these changes from (1)-(4) to (1), (3), (5) and (6) the inequality in (3) might get worse because of a new choice of the measure μ.

Even though we have formulated the construction of abstract Hardy spaces in a quite general setting, it is new and leads to interesting results for regions U in the complex plane with arbitrary boundary (let $U = D$ be the unit disc) and subalgebras $\mathcal{A} \subset \mathcal{H}^\infty(D)$.

What would be a canonical choice of a measure μ with (0.0.2) in our definition of Bergman- and Hardy spaces above if we deal with a domain X in an infinite dimensional spaces? In general, there is no analog to the Lebesgue measure on $\mathbb{C}^n$, but in the case where we have a group G and a (measurable) representation

$$\alpha : G \to \mathrm{Homeo}(X) \tag{0.0.3}$$

into the space of all homeomorphisms on X it seems to be natural to choose a $\mathcal{N}\mathcal{H}(U)_2$-measure μ_s, which is invariant under $\alpha(G)$. We study the existence of such a measure μ_s for compact groups G. Using the left-invariant Haar measure m on G and for X being a *polish space* or an open subset in a $\mathcal{DFN}$-spaces we give a quite general construction for μ_s. Given a function space $\mathcal{F} \subset \mathcal{C}(U)$ which is invariant under $\alpha(G)$, a continuous group representation α and a $\mathcal{N}\mathcal{F}_2$-measure μ we can prove that the symmetrization μ_s is a $\mathcal{N}\mathcal{F}_2$-measure again.

In many cases we obtain strongly continuous representations $\tilde{\alpha}_p$ of G in $L^p(X, \mu_s)$ and the construction produces closed operators (infinitesimal generators) attached to the symmetries of the underlying infinite dimensional spaces (or manifolds). Following [79] we can define Ψ-algebras with spectral invariance and prescribed properties [77], [107], [106]. We give several examples how to obtain representations α in (0.0.3) which could be used in our construction above. In the Hilbert space case $p = 2$ it can be shown that the group action $\tilde{\alpha}_2$ commutes with the Bergman Toeplitz projection P_{μ_s} with respect to the $\mathcal{NF}_2$-measure μ_s. As a consequence, G also can be represented in the corresponding Bergman spaces and we can define Ψ-algebras by group actions in the generalized Toeplitz C^*-algebras as well (cf. Remark 7.6.1).

Organization of the text:

CHAPTER 1: In the first chapter we provide some basic tools in the construction of topological algebras. Moreover, we set up notation and terminology. The notion of a Ψ-algebra in the sense of [69] and [79] is given as well as a brief exposition of the recent results in this area. We recall how to define scales of topological algebras and corresponding Sobolev spaces by commutator methods, parametrized families of automorphisms and smooth projections. For a detailed description we refer the reader to [77], [79] and [107].

CHAPTER 2: We consider the Hilbert space $H_1 := L^2(\mathbb{C}^n, \mu)$ where μ denotes a normed Gaussian measure. The *Segal-Bargmann space* $H_2 := H^2(\mathbb{C}^n, \mu)$ is the closed subspace of H_1 consisting of all entire functions square integrable with respect to μ. It is well-known that H_2 is a *reproducing kernel Hilbert space* and we prove some norm estimates for linear operators on H_2. The notion of the *Toeplitz operators* T_f for f in a class of (in general unbounded) symbols on $\mathbb{C}^n$ is introduced. In the case where $f_1, \cdots, f_m$ have only polynomial growth we can define products $T_{f_1} \cdots T_{f_m}$ of (unbounded) Toeplitz operators on a dense subspace of H_2 (resp. H_1). Hence all iterated commutators of such operators are meaningful. It was shown in chapter 1, [77], [79] how to construct a decreasing series of Fréchet operator algebras $(\Psi_k^{\Delta})_{k\in\mathbb{N}}$ with prescribed properties in $\mathcal{L}(H_1)$ using commutator methods with a finite set $\mathcal{V}$ of closed operators on H_1. We define a class $\mathcal{X}^{\Phi}(\mathbb{C}^n)$ of vector fields supported in cones such that the *Toeplitz projection* P from H_1 onto H_2 is contained in Ψ_k^{Δ} for all $k \in \mathbb{N}$ and all finite sets $\mathcal{V} \subset \mathcal{X}^{\Phi}(\mathbb{C}^n)$. This enables us to construct subalgebras of $\mathcal{L}(H_2)$ localized in cones $\mathcal{C} \subset \mathbb{C}^n$ and containing all Toeplitz operators T_f with a symbol $f \in L^{\infty}(\mathbb{C}^n)$ sufficiently smooth in $\mathcal{C}$ with bounded derivatives.

CHAPTER 3: We consider the C^*-algebra $\mathcal{A}$ generated by all Toeplitz operators with bounded symbols on the Segal-Bargmann space. Using the group of Weyl operators on $L^2(\mathbb{C}^n, \mu)$ where μ is a Gaussian measure we define a decreasing scale of Banach subalgebras $\Psi_n^{\alpha}[\mathcal{A}]$, $(n \in \mathbb{N})$ continuously embedded in $\mathcal{A}$ which is related to the smoothness of the operators as it was described in chapter 1. The intersection of all these algebras is denoted by $\Psi_{\infty}^{\alpha}[\,\mathcal{A}\,]$ and it coincides with the Ψ^*-algebra of smooth elements in $\mathcal{A}$ with respect to the Weyl group. We examine the class of bounded symbols g on $\mathbb{C}^n$ which leads to smooth Toeplitz operators $T_g \in \Psi_{\infty}^{\alpha}[\,\mathcal{A}\,]$. An example of an element in $\mathcal{A}$, which does not belong to $\Psi_1^{\alpha}[\mathcal{A}]$ is given. Via the Bargmann isometrie, the class of Toeplitz operators unitarily is equivalent to the so called Gabor-Daubechies windowed Fourier localization operators on

$L^2(\mathbb{R}^n, v)$ for certain windows resp. to pseudodifferential operators in their Weyl-form. We can reformulate some of our results in the setting of these operators. Finally, we examine the case where the Weyl group is replaced by a unitary C_0-group of composition operators on the Segal-Bargmann space. As a common result, we show that the Ψ^*-algebras of smooth operators are invariant under perturbations of the symbol by continuous functions with compact support.

Chapter 4: We examine some local aspects of the Szegö-projection P_s and the corresponding Toeplitz operators $T_f = P_s M_f$ with symbol f. Due to a result by A. Nagel and E.M. Stein for any strictly pseudo-convex domain Ω, the projection P_s is a pseudodifferential operator of exotic type $(\frac{1}{2}, \frac{1}{2})$. Using this fact and by the general theory in [79] a rich class of spectral invariant Fréchet sub-algebras $\mathcal{B}$ in $\mathcal{L}(\, L^2(\partial\Omega)\,)$ (or more generally in a Toeplitz C^*-algebra) containing P_s can be constructed by commutator methods. Hence conditions on the operator φT_f to belong to $\mathcal{B}$ (with a cut-off function φ) can be characterized by the (local) regularity of the symbol f. In the second part of this chapter we examine the question under which conditions the Szegö projection of a bounded function on the complex sphere locally admits a continuous extension to an analytic function on the unit ball. Mainly we are dealing with the upper half-space $\mathcal{H}_+$ in $\mathbb{C}^{n+1}$ and the complex unit sphere, but we consider many of the results to be true in greater generality for strictly pseudo-convex domains in $\mathbb{C}^n$.

Chapter 5: With a *Gaussian measure* μ_B on an infinite dimensional complex Hilbert space, we consider the space $H^2(V, \mu_B)$ of all square integrable holomorphic functions on an open subset $V \subset H$. We show that in many cases the L^2-closure of $H^2(V, \mu_B)$ can be identified with a space of holomorphic functions $(\mathcal{H}_{\mu_B}, \tau_\omega)$ defined on a dense submanifold in V. Here τ_ω denotes a topology which is finer than the *compact-open topology*. Given an open subset U in a $\mathcal{DFN}$*-space* (the topological dual of a nuclear Fréchet space) and using these results we construct a finite measure ν on U such that the *point evaluation map*

$$U \ni z \mapsto \delta_z \in \big[\, \mathcal{H}(\, U\,) \cap L^2(\, U, \nu\,)\,\big]'$$

is a holomorphic function on U. Finally, with this construction and by generalizing a method of A. Pietsch to the case of infinite dimensions (see [121]) we give a new proof of a result due to P. Boland and L. Waelbroeck ([25] and [141]). Namely, that the space $(\, \mathcal{H}(U), \tau_0\,)$ of holomorphic functions on U endowed with the *compact-open topology* is a $\mathcal{FN}$-space (Fréchet nuclear space).

Chapter 6: Let E be the dual of a Fréchet nuclear space ($\mathcal{DFN}$-space) and $U \subset E$ an open subset. We denote by $\mathcal{H}^\infty(U)$ the Banach algebra of all bounded holomorphic functions on U. For any closed subalgebra $\mathcal{A}$ of $\mathcal{H}^\infty(U)$ which separates points let $\mathcal{S}_\mathcal{A}$ be its abstract Shilov boundary. We prove the existence of an integral formula for $f \in \mathcal{A}$ similar to the Cauchy integral formula, which is a generalization of a result in [69] to U in an infinite dimensional nuclear space. Namely, given any $\mathcal{NF}_2$-measure μ on U where $\mathcal{F} = \mathcal{H}(U)$ is the Fréchet nuclear space of all holomorphic functions on U (see Definition 5.4.1), there is a finite Radon measure ν on $\mathcal{S}_\mathcal{A}$ and a complex-valued kernel Φ_μ on $U \times \mathcal{S}_\mathcal{A}$

such that $\Phi_\mu(z,\cdot)$ is ν-integrable for all $z \in U$ and $\Phi_\mu(\cdot,x)$ is holomorphic for all $x \in \mathcal{S}_\mathcal{A}$. Moreover, the point evaluation on $\mathcal{A}$ in $z \in U$ is given by an integral operator on $\mathcal{A}$, considered as a subset of $\mathcal{C}(\mathcal{S}_\mathcal{A})$ with kernel $\Phi_\mu(z,\cdot)$. We prove an estimate on the growth of Φ_μ of the form:

$$\left\| \Phi_\mu(\, z,\cdot)\, \right\|_{L^1(\mathcal{S}_\mathcal{A},\nu)} \leq C \cdot \left\| \operatorname{Eval}(\, z\,)\, \right\|_{\mathcal{H}^2(U,\mu)'}.$$

Here $C > 0$ is a number independent of z and *Eval* denotes the *point evaluation* in the generalized Bergman space $\mathcal{H}^2(U,\mu) := L^2(U,\mu) \cap \mathcal{H}(U)$. In the last section we define an abstract Hardy space $\mathcal{H}^2(\mathcal{S}_\mathcal{A},\Theta)$ of holomorphic functions on U by using the nuclearity of $\mathcal{H}(U)$ and following an idea in [72].

CHAPTER 7: Given a topological space X with σ-finite Borel measure μ, a locally compact group G and a measurable representation B of G in the group of all homeomorphisms of X, we examine how to construct a Borel measure μ_s on X which is invariant under $B(G)$ (Lemma 7.1.4). In many cases this construction leads to a non-trivial representation of G on $L^p(X,\mu_s)$. Under some additional conditions on G, X and the representation B we show that in the case where μ has the $\mathcal{NF}_p$-property, the symmetrized measure μ_s is a $\mathcal{NF}_p$-measure, as well (Theorem 7.2.1). Finally, we give some examples and an application of our work leading to the construction of spectrally invariant algebras (Ψ^*- or Ψ_0-algebras, cf. [69], [77])) of $\mathcal{C}^\infty$-elements in operator-algebras on L^p-spaces and Bergman spaces.

APPENDIX: We collect some basic results on dual Fréchet spaces E and in particular we are focusing on the case where E is of $\mathcal{DFN}$-type. Some standard facts on holomorphic functions on topological spaces are given. Following [134], the Heisenberg group and its action on the complex sphere in $\mathbb{C}^{n+1}$ is described and we define the Cauchy-Szegö projection P_s. We sketch some ideas and proofs in [117] concerning the symbol classes $\mathcal{S}^0_\rho$ which contains P_s for an appropriate choice of the pseudo-distance ρ. Finally, we collect results on the boundedness and compactness of Hankel operators on the Segal-Bargmann space and symmetric bounded domains in $\mathbb{C}^n$ [12], [16], [146], [148], [150].

Acknowledgements

The author is greatly indebted to his supervisor for valuable support during some years. He proposed both, the construction of Ψ-algebras in the setting of Toeplitz operators as well as the study of the infinite dimensional counterpart. In particular, I am grateful for his hints concerning infinite dimensional holomorphy.

Furthermore, I want to express my gratitude for the invitation to stay at the *State University of New York* (SUNY) at Buffalo in 02/03. I acknowledge the many helpful suggestions and explanations especially concerning the theory of Toeplitz operators I received from there. In addition, I thank the DAAD for financial support during my stay in Buffalo.

Chapter 1

Fréchet algebras with spectral invariance

Let us remind of the following general concept for the construction of spectral invariant algebras (Ψ^*-, and Ψ_0-algebras) with prescribed properties induced by a finite set of closed derivations or closed operators (for more details and some of the proofs see [77], [79], [107], [109]). We start with the definition of a Ψ_0- (resp. Ψ^*)-algebra.

Definition 1.0.1 (B. Gramsch, [69]) Let $\mathcal{B}$ be a Banach algebra with unit e and $\mathcal{F}$ a subalgebra of $\mathcal{B}$ with $e \in \mathcal{F}$. Then $\mathcal{F}$ is called *locally spectral invariant* in $\mathcal{B}$ if there exists an $\varepsilon > 0$ with

$$\{ a \in \mathcal{F} : \| e - a \|_{\mathcal{B}} < \varepsilon \} \subset \mathcal{F}^{-1}$$

where $\mathcal{F}^{-1}$ denotes the group of invertible elements in $\mathcal{F}$.

(a) $\mathcal{F}$ is called a Ψ_0-*algebra* in $\mathcal{B}$, if $\mathcal{F}$ is locally spectral invariant in $\mathcal{B}$ and there is a topology $\tau_{\mathcal{F}}$ on $\mathcal{F}$ which makes $(\mathcal{F}, \tau_{\mathcal{F}}) \hookrightarrow \mathcal{B}$ into a continuously embedded Fréchet algebra .

(b) $\mathcal{F}$ is called a Ψ^*-*algebra* in $\mathcal{B}$, if in addition $\mathcal{B}$ is a C^*-algebra and $\mathcal{F}$ is a symmetric Ψ_0-algebra in $\mathcal{B}$. (Due to an application of Lemma 1.0.1 this implies that $\mathcal{F} \cap \mathcal{B}^{-1} = \mathcal{F}^{-1}$).

(c) $\mathcal{F}$ is called a *submultiplicative* Ψ_0- resp. Ψ^*-algebra if the topology $\tau_{\mathcal{F}}$ can be generated by a submultiplicative family of semi-norms $(q_j)_{j \in \mathbb{N}_0}$, i.e.

$$q_j(\, xy \,) \leq q_j(\, x \,)\, q_j(\, y \,) \qquad \text{and} \qquad q_j(\, e \,) = 1.$$

Remark 1.0.1 We remark that according to [27], [118], [142] and [143] the algebra $\mathcal{F}$ is called *spectral invariant* , *full* or *algèbre pleine* if $\mathcal{F} \cap \mathcal{B}^{-1} = \mathcal{F}^{-1}$. The pair $(\mathcal{F}, \mathcal{B})$ is referred to as a *Wiener pair* (cf. [118], chapt. III, pp. 203, 214, 310).

It easily follows from the definition that the class of Ψ_0-algebras and Ψ^*-algebras is invariant under countable intersections. For a Fréchet algebra $\mathcal{F}$ with open group $\mathcal{F}^{-1}$ of invertible elements the inversion $\mathcal{F}^{-1} \ni b \mapsto b^{-1} \in \mathcal{F}$ is continuous (see [142]). Hence this applies to any Ψ_0-algebra. We give a useful result which was proved in [69].

Lemma 1.0.1 *Let $\mathcal{F}$ be a dense, locally spectral invariant topological subalgebra of the Banach algebra $\mathcal{B}$, then $\mathcal{F}$ is spectral invariant in $\mathcal{B}$.*

Because each symmetric closed subalgebra of a C^*-algebra $\mathcal{B}$ is spectral invariant it follows from Lemma 1.0.1 that every Ψ^*-algebra $\mathcal{F}$ in $\mathcal{B}$ is spectral invariant.

Remark 1.0.2 It was a rather long process until it had been completely proved that the Hörmander classes $\Psi^0_{\rho,\delta}(\mathbb{R}^n)^1$ are submultiplicative Fréchet operator algebras with spectral invariance in the algebra $\mathcal{L}(L^2(\mathbb{R}^n))$. There are contributions of a series of mathematicians including Hörmander, Seeley, Calderón and Vaillancourt, Beals, Cordes, Fefferman, Bony and Chemin, Ueberberg, Schrohe, Wagner.

During the last two decades it attracted attention that many consequences arise for algebras of Ψ^*-type. With these notions it is possible to develop an operator theory for some Fréchet algebras in the microlocal analysis. Special non-linear methods have been developed which sharpen some results in the Banach- and C^*-setting (cf. [69], [96], [97]). It is an essential point that in Ψ^*-algebras $\mathcal{A}$ the Hilbert space Fredholm inverses are automatically in $\mathcal{A}$. Therefore it is possible to develop *perturbation theory* in these Fréchet algebras namely for holomorphic Fredholm functions:

- Meromorphic inversion and decomposition of holomorphic Semi-Fredholm functions also on infinite dimensional regions.
- Oka-principle for holomorphic maps with values in complex Fréchet-Lie groups or in Fréchet manifolds of Fredholm and Semi-Fredholm operators in Ψ^*-algebras of pseudodifferential operators.
- Existence of global holomorphic projection-valued functions splitting off the kernel of holomorphic Fredholm functions with fixed dimension of the kernel.
- Division of operator-valued distributions.

A similar development is under way concerning the L^p-theory based on the notion of Ψ_0-algebras as well as algebras of $\mathcal{C}^\infty$-elements w.r.t. group representations [62].

The *Oka-principle* (resp. conjecture) leads also to isomorphisms between non-abelian groups of holomorphic objects on the one side and continuous objects on the other side. The strategy of proofs involves essentially non-linear functional analytic and complex analytic methods.

It is an important fact that Waelbroeck ([142] and [143], 1954) had introduced a *holomorphic functional calculus* for complete locally convex algebras with continuous inversion even for several complex variables. In particular, the holomorphic functional calculus for

Ψ_0- and Ψ^*-algebras is an immediate consequence of his result and it plays an essential role in a special case of algebras (cf. e.g. [42] and [71] in 1981 and recently [101], 2005) and in general Ψ^*-algebras. Due to this fact Ψ^*-type algebras are also known as *smooth algebras* or *algebras stable under holomorphic calculus* [101]. Besides the standard *Hörmander classes* $\Psi^0_{\rho,\delta}$ [1] of order 0 there are lots of other examples of Ψ^*-algebras such as C^∞-*elements* in C^*-dynamical systems [28], [43], [46] and certain families of *crossed products* [94], [95], [131]. Since the important concept of spectral invariance was stressed by B. Gramsch, the theory of Ψ^*-algebras has developed into a useful tool in the analysis of pseudodifferential operators and Fréchet operator algebras on singular spaces.

We give some of the results subsequent to [79] and it should be mentioned that the research is still in progress and far from being complete. For any Hilbert space X it was shown in [109] that every Ψ^*-algebra in $\mathcal{L}(X)$ contains its *holomorphic functional calculus* in the sense of J.L. Taylor [107], [128]. Moreover, this calculus applies to algebras of $n \times n$-matrices with elements in Ψ^*-algebras. It was shown in [112] that any *Jordan operator* A in a Ψ^*-algebra $\mathcal{A} \subset \mathcal{L}(X)$ admits a Jordan decomposition within $\mathcal{A}$ and as a consequence one has local similarity cross sections for A in $\mathcal{A}$.

As a contribution to *additive complex analytic cohomology* it was pointed out by B. Gramsch and W. Kaballo [76] that an additive decomposition of meromorphic resolvents M of semi-Fredholm functions into a holomorphic part and a meromorphic part which is in a small ideal can be generalized to the setting of Ψ^*-algebras. Furthermore, results on the *division problem* for real-analytic (semi-)Fredholm functions and operator distributions in Ψ-algebras are given. In the framework of a submultiplicative Ψ^*-algebra $\mathcal{A}$ there also is a corresponding multiplicative decomposition for holomorphic Fredholm functions with values in $\mathcal{A}^{-1}$ on a Stein manifold [75]. The first named author of [76] derives an extension of the *Oka-principle* to submultiplicative Ψ^*-algebras [68]. Since in the case of Fréchet spaces the *implicit function theorem* is not available in [69] there are developed *rational methods* which can be applied instead. In this connection it was shown in [69] that the set of idempotent or relatively regular elements in Ψ^*-algebras form *analytic locally rational Fréchet manifolds*. Let us mention that there are results on abstract *hypo-ellipticity* [77], *wave front sets* and *propagation of singularities* in Ψ^*-algebras which are due to B. Gramsch. In connection with [69] and [71] it was observed in K-theory around 1984 using *Karoubi's density theorem* [42], [98] that a Ψ_0-algebra (resp. Ψ^*-algebra) has the same K-theory as its norm-closure (resp. C^*-closure).

For appropriate triples $(\mathcal{M}, g, M)$ where $\mathcal{M}$ denotes an in general non-compact manifold, g is a metric and M a weight function on $T^*\mathcal{M}$ there is a $S(M,g)$-pseudo-differential calculus by F. Baldus. It was shown in [3] that the algebra of order zero operators is a submultiplicative Ψ^*-algebra in the sense of B. Gramsch in $\mathcal{L}(L^2(\mathcal{M}))$. Using the spectral invariance and within the $S(M,g)$-*calculus* the author of [4] gives sufficient conditions for an operator to extend to a generator of a *Feller semi-group*.

The construction methods of Ψ_0- and Ψ^*-algebras in [79] are quite flexible tools and they even apply to operator algebras on *fractal sets*, (K. Krohne, Mainz, 2003).

[1] $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$

In the most recent research on Fréchet-algebras there are approaches in *infinite dimensional analysis* by M. Höber. Let $H_+ \subset H_0 \subset H_-$ be an infinite dimensional Hilbert space rigging and by μ denote the corresponding canonical Gaussian measure. Using the Ornstein-Uhlenbeck operator as Laplace operator he defines a scale of Sobolev space and generalized Hörmander classes by commutator methods. Submultiplicative Ψ^*-subalgebras of the Hörmander classes are studied containing certain multiplications M_g and operators of the form $F^{-1}M_gF$, where F is the Fourier-Wiener-transform. Moreover, they contain continuous pseudodifferential operators defined by S. Albeverio and A. Daletskii [1].

Spectral invariance generates strong connections between Ψ^*-algebras and their C^*-closures and in some sense one can expect them to behave similar. While representation theory for C^*-algebras has been treated extensively [51] R. Lauter developed a *representation theory for* Ψ^**-algebras* [105]. More precisely, using a result due to B. Gramsch on positive functionals it can be shown that there is a continuous, bijective map $\Phi : \widehat{\mathcal{A}} \longrightarrow \widehat{\mathcal{B}}$, where $\mathcal{B}$ is the enveloping C^*-algebra of a Ψ^*-algebra $\mathcal{A}$ and $\widehat{\mathcal{A}}$ resp. $\widehat{\mathcal{B}}$ denotes the spectrum of $\mathcal{A}$ resp. $\mathcal{B}$.

There are approaches by J. Ditsche on localization results for special classes of *solvable* C^**-algebras* on manifolds with corners Z. Let $\Psi^0_{b,\,cl}(Z)$ be the algebra of classical pseudodifferential operators of order zero and $B(Z)$ its C^*-closure in $\mathcal{L}(L^2(Z))$. Then it is known by results of Lauter, Melrose and Nistor that $B(Z)$ is a solvable C^*-algebra in the sense of [53]. Moreover, one can choose a solving series of minimal length for $B(Z)$, such that the geometry of Z is readily seen in this ideal chain. Since this is a global approach it should also be possible, to localize this procedure, i.e. to show, that if we restrict our algebra to small open neighborhoods of arbitrary points on Z, only the underlying geometry of those neighborhoods give a contribution to the ideal chain. To do this, J. Ditsche analyzes algebras $\psi B(Z)\psi$, where ψ is a cut off function with $supp\psi \subseteq U$ and U an neighborhood of $p \in Z$. Moreover, it is shown how to calculate the length of algebras of parameter depended pseudodifferential operators on Z.

In a paper of X. Chen and S. Wei [34], (2003), which follows a series of results of (cf. [34]) L.B. Schweitzer, P. Jolissaint, and R. de la Harpe it was mentioned that the notion of spectral invariance plays an important role in the work of Connes-Moscovici on the *Novikov conjecture* as well as in Lafforgues research on the *Baum-Connes conjecture* (cf. *Noncommutative geometry*, Springer Lecture Notes 1831, (2004)). In this connection it is of interest that for certain discrete groups G with length function l the Schwartz space $S^l_2(G)$ with respect to l is a spectral invariant dense subalgebra of the reduced group C^*-algebra $C^*_r(G)$. For details we refer to [34] and the references given there.

It is known that the dense embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ of a Ψ^*-algebra $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$ induces an isomorphism in K-theory of $\mathcal{B}$. Hence on the one hand $\mathcal{A}$ is large enough to preserve the K-theory of $\mathcal{B}$ and on the other hand it is better related to differential structures than a C^*-algebra. This fact is exploited in [101] (2005) to prove a vanishing theorem for higher traces in *cyclic cohomology* for the spectral projections. Further the authors of [101] give applications to the *Quantum hall effect* and related *spectral gaps* of operators.

1.1 Fréchet algebras generated by closed derivations

In [79] it is shown how to construct Ψ_0- (resp. Ψ^*-) algebras with prescribed properties using closed derivations (resp. closed operators). We give the basic definitions which also can be found in [107]:

Definition 1.1.1 Let $\mathcal{B}$ be a C^*-algebra with unit $e \in \mathcal{B}$ and assume that $(\mathcal{F}, (q_j)_{j\in\mathbb{N}_0})$ is a submultiplicative Ψ^*-algebra in $\mathcal{B}$. Moreover, let Δ be a finite set of closed derivations $\delta : \mathcal{F} \supset \mathcal{D}(\delta) \to \mathcal{F}$ (i.e. $\delta(ab) = \delta(a)b + a\delta(b)$ such that $e \in \mathcal{D}(\delta)$). Define:

- $\Psi_0^\Delta := \mathcal{F}$ with the semi-norms $q_{0,j} := q_j$ for all $j \in \mathbb{N}_0$.
- $\Psi_1^\Delta := \bigcap_{\delta\in\Delta} \mathcal{D}(\delta)$.
- $\Psi_n^\Delta := \Big\{ a \in \Psi_{n-1}^\Delta : \delta a \in \Psi_{n-1}^\Delta \text{ for all } \delta \in \Delta \Big\}, \quad n \geq 2.$
- $\Psi_\infty^\Delta := \bigcap_{n\in\mathbb{N}_0} \Psi_n^\Delta$.
- We endow Ψ_n^Δ for $n \geq 1$ with the system of semi-norms defined by

$$q_{n,j}(\,a\,) := q_{n-1,j}(\,a\,) + \sum_{\delta\in\Delta} q_{n-1,j}(\,\delta a\,)$$

  for all $a \in \Psi_n^\Delta \subset \Psi_1^\Delta$ and $j \in \mathbb{N}_0$ and Ψ_∞^Δ with the system $(q_{n,j})_{n\in\mathbb{N},j\in\mathbb{N}_0}$.

Let $\mathcal{D}(\delta)$ and $\mathcal{A}$ be algebras with a $*$-operation, then a derivation $\delta : \mathcal{D}(\delta) \to \mathcal{A}$ is called a *$*$-derivation* if it holds $\delta(x^*) = \delta(x)^*$ for all $x \in \mathcal{D}(\delta)$. An *anti-$*$-derivation* is characterized by $\delta(x^*) = -\delta(x)^*$. Remark that δ is an anti-$*$-derivation iff $i\delta$ is a $*$-derivation. As a result of the construction above we obtain (see also [79] and [107], 2.4.3 Proposition and 2.4.4 Corollary):

Proposition 1.1.1 *For each $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$ the space Ψ_n^Δ is a subalgebra of $\mathcal{F}$ and $q_{n,j}$ is a submultiplicative semi-norm on Ψ_n^Δ. Moreover:*

(a) $(\,\Psi_n^\Delta, (q_{n,j}\,)_{j\in\mathbb{N}_0}) \hookrightarrow \mathcal{F}$ *and* $(\,\Psi_\infty^\Delta, (q_{n,j}\,)_{n\in\mathbb{N},j\in\mathbb{N}_0}) \hookrightarrow \mathcal{F}$ *are continuously embedded, submultiplicative Fréchet algebras.*

(b) Ψ_∞^Δ *is a submultiplicative Ψ_0-algebra in $\mathcal{B}$ and for each $\delta \in \Delta$ the map $\delta : \Psi_\infty^\Delta \to \Psi_\infty^\Delta$ is continuous.*

(c) If each $\delta \in \Delta$ is a closed $*$-derivation w.r.t. the $*$-operation induced by $\mathcal{B}$, then Ψ_n^Δ is a symmetric subalgebra of $\mathcal{F}$ and Ψ_∞^Δ is a submultiplicative Ψ^*-algebra in $\mathcal{B}$.

Proof (a) : We show that Ψ_n^Δ is an algebra for each $n \in \mathbb{N}_0 \cup \{\infty\}$. This is obvious for $n = 0, 1$ because $\mathcal{F}$ and $\mathcal{D}(\delta)$ with $\delta \in \Delta$ are algebras by definition and so we fix $a, b \in \Psi_n^\Delta$ where $n \geq 2$ and $\delta \in \Delta$. Then by induction

$$ab \in \Psi_{n-1}^\Delta \qquad \text{and} \qquad \delta(ab) = \delta(a)\, b + a\, \delta(b) \in \Psi_{n-1}^\Delta$$

which implies that $a, b \in \Psi_n^\Delta$. The algebra $\Psi_0^\Delta := \mathcal{F}$ is submultiplicative by assumption and for $a, b \in \Psi_n^\Delta$ and $n \geq 1$ we have:

$$\begin{aligned} q_{n,j}(ab) &= q_{n-1,j}(ab) + \sum_{\delta\in\Delta} q_{n-1,j}\big(\, \delta(ab)\, \big) \\ &\leq q_{n-1,j}(a) \cdot q_{n-1,j}(b) + \sum_{\delta\in\Delta} \Big\{\, q_{n-1,j}\big(\, \delta(a)\, \big) \cdot q_{n-1,j}(b) + q_{n-1,j}(a) \cdot q_{n-1,j}\big(\, \delta(b)\, \big) \,\Big\} \\ &\leq q_{n,j}(a) \cdot q_{n,j}(b). \end{aligned}$$

Using $\delta(e) = 0$ and $q_j(e) = 1$ for all $j \in \mathbb{N}_0$ we find $q_{n,j}(e) = 1$ for $n, j \in \mathbb{N}_0$. It remains to show that the algebras Ψ_n^Δ are complete for $n \in \mathbb{N}_0 \cup \{\infty\}$. We only give the proof in the case $n = 1$ and omit the induction step.

Let $(a_k)_{k\in\mathbb{N}}$ be a fundamental sequence in Ψ_1^Δ, then by construction of the corresponding topology and using the fact that $\mathcal{F}$ is complete for each $\delta \in \Delta$ there are $a, b_\delta \in \mathcal{F}$ such that

$$a_k \longrightarrow a \in \mathcal{F} \qquad \text{and} \qquad \delta(a_k) \longrightarrow b_\delta \in \mathcal{F}, \qquad \forall\ \delta \in \Delta.$$

Because the derivation $\delta : \mathcal{F} \supset \mathcal{D}(\delta) \to \mathcal{F}$ is closed by assumption it immediately follows that $a \in \mathcal{D}(\delta)$ and $b_\delta = \delta(a)$ for all $\delta \in \Delta$. Thus we obtain that $a \in \Psi_1^\Delta$ is the limit of $(a_k)_k$.

(b): According to Definition 1.0.1 it is sufficient to prove that for each $a \in \Psi_\infty^\Delta$ with the property $\|\, a \,\|_{\mathcal{B}} < 1$ it holds:

$$(\, e - a\,)^{-1} = \sum_{k=0}^{\infty} a^k \in \Psi_\infty^\Delta. \tag{1.1.1}$$

Let $\rho > 0$ with $\|\, a \,\|_{\mathcal{B}} < \rho < 1$. Then we show that there are numbers $c_{n,j}(a) > 0$ depending on a, n and j such that for all $k \in \mathbb{N}_0$:

$$q_{n,j}\left(a^k\right) \leq c_{n,j}(a) \cdot k^{2^n-1} \cdot \rho^{k-2^n+1}. \tag{1.1.2}$$

Using the continuous inversion in the Ψ^*-algebra $\mathcal{F}$ it can be shown that (1.1.2) holds in the case $n = 0$ (see [107], Lemma 2.1.8) and we only give the induction step. We apply the well-known formula $\delta(a^k) = \sum_{l=1}^{k} a^{l-1}\, \delta(a)\, a^{k-l}$ which is a generalization of the derivation property of δ to k-fold products:

$$\begin{aligned} q_{n,j}\left(a^k\right) &= q_{n-1,j}\left(a^k\right) + \sum_{\delta\in\Delta} q_{n-1,j}\left(\delta\,\left(a^k\right)\right) \\ &\leq q_{n-1,j}\left(a^k\right) + \sum_{\delta\in\Delta}\sum_{l=1}^{k} q_{n-1,j}\left(a^{l-1}\right)\cdot q_{n-1,j}(\,\delta(a)\,)\cdot q_{n-1,j}\left(a^{k-l}\right) \\ &\leq c_{n-1,j}(a)\,k^{2^{n-1}-1}\rho^{k-2^{n-1}+1} + \tilde{c}_{n,j}(a)\sum_{\delta\in\Delta}\sum_{l=1}^{k}(l-1)^{2^{n-1}-1}(k-l)^{2^{n-1}-1}\rho^{k-2^n+1} \\ &\leq c_{n,j}(a)\,k^{2^n-1}\rho^{k-2^n+1}. \end{aligned}$$

From (1.1.2) it now follows that the series in (1.1.1) converges in Ψ_n^Δ for all $n\in\mathbb{N}_0$ and so it converges in Ψ_∞^Δ. By construction $\delta:\Psi_\infty^\Delta\to\Psi_\infty^\Delta$ is closed and well-defined, hence it is continuous by the closed graph theorem.

(c) : By definition Ψ_j^Δ is symmetric under the $*$-operation for $j=0,1$. Let $n\geq 2$ and choose $a\in\Psi_n^\Delta$. Then by induction we have $a^*\in\Psi_{n-1}^\Delta$ and $\delta(a^*)=\delta(a)^*\in\Psi_{n-1}^\Delta$ and it follows that $a^*\in\Psi_n^\Delta$. Hence Ψ_∞^Δ is symmetric and the rest of the assertion is an application of (a) and (b) above. □

1.2 Operator algebras by commutator methods

Now we specialize our considerations to the case $\mathcal{B}=\mathcal{L}(H)$ where $(\,H,\langle\cdot,\cdot\rangle\,)$ is a Hilbert space. Let us describe how closed derivations (resp. $*$-derivations) can be obtained from closed (resp. symmetric closed) operators on H. Fix a closed and densely defined operator A on H with domain of definition $\mathcal{D}(A)\subset H$.

Definition 1.2.1 Let $(\,\mathcal{F},(q_j)_{j\in\mathbb{N}_0}\,)\hookrightarrow\mathcal{L}(H)$ be a submultiplicative Ψ^*-algebra in $\mathcal{L}(H)$. Without loss of generality we assume that $q_0=\|\cdot\|_{\mathcal{L}(H)}$. For any operator $a\in\mathcal{L}(H)$ such that $a(\,\mathcal{D}(A)\,)\subset\mathcal{D}(A)$ one defines:

- $\mathrm{ad}[\,A\,](a):=Aa-aA:\mathcal{D}(A)\to H$,
- $\mathcal{I}(A):=\Big\{\,a\in\mathcal{F}:a\,(\,\mathcal{D}(A)\,)\subset\mathcal{D}(A)\,\Big\}.$
- $\mathcal{B}(A):=\Big\{\,a\in\mathcal{I}(A):\mathrm{ad}[\,A\,](a)$ extends to a bounded linear operator $\delta_A a\in\mathcal{F}\,\Big\}.$
- $\mathcal{B}^*(A):=\Big\{\,a\in\mathcal{B}(A):a^*\in\mathcal{B}(A)\,\Big\}.$

Lemma 1.2.1 *The operator $\delta_A:\mathcal{F}\supset\mathcal{B}(A)\to\mathcal{F}$ with $\delta_A:a\mapsto\delta_A a$ is a closed derivation. If in addition A is symmetric closed, then $i\delta_A:\mathcal{F}\supset\mathcal{B}^*(A)\to\mathcal{F}$ is a closed $*$-derivation.*

Proof Let $a, b \in \mathcal{B}(A)$ and $x \in \mathcal{D}(A)$, then it follows for the product $ab \in \mathcal{I}(A)$:

$$\Big\{ \operatorname{ad}[\, A\,](\, ab\,) \Big\}(\, x\,) = Aab(\, x\,) - abA(\, x\,) = \Big\{ Aa - aA \Big\}\, b(\, x\,) + a \Big\{ Ab - bA \Big\}(\, x\,).$$

Hence the commutator $\operatorname{ad}[\, A\,](ab)$ extends to a bounded operator

$$\delta_A(\, ab\,) = \delta_A(a)\, b + a\, \delta_A(b)$$

and we obtain that $ab \in \mathcal{B}(A)$. Moreover, δ_A is a derivation on $\mathcal{B}(A)$. In order to show that δ_A is closed let $(a_k)_k \subset \mathcal{B}(A)$ be a sequence such that $a_k \to a \in \mathcal{F}$ and $\delta_A a_k \to b \in \mathcal{F}$ as $k \to \infty$ with respect to the topology of $\mathcal{F}$. It follows for any $x \in \mathcal{D}(A)$ and because of the continuous embedding $\mathcal{F} \hookrightarrow \mathcal{L}(H)$ that $\mathcal{D}(A) \ni a_k(x) \to a(x)$ as $k \to \infty$ and

$$Aa_k(\, x\,) = \Big\{ \delta_A a_k \Big\}\,(\, x\,) + a_k A(\, x\,) \longrightarrow b(\, x\,) + aA(\, x\,).$$

Because A is a closed operator by assumption this means that $a(x) \in \mathcal{D}(A)$ and $\{\operatorname{ad}[\, A\,](a)\}(x) = b(x)$. By our definition it holds $a \in \mathcal{B}(A)$ and $\delta_A a = b$ and δ_A is a closed derivation.

Now in addition we assume that $A : H \supset \mathcal{D}(A) \to H$ is a *symmetric operator*. By definition $\mathcal{B}^*(A)$ is a symmetric subalgebra of $\mathcal{F}$. Let $x, y \in \mathcal{D}(A)$ and $a \in \mathcal{B}^*(A)$, then it follows that:

$$\Big\langle\, \delta_A(a)x, y\, \Big\rangle = \Big\langle\, Aax - aAx, y\, \Big\rangle = \Big\langle\, x, a^*Ay - Aa^*y\, \Big\rangle = \Big\langle\, x, -\delta_A(a^*)y\, \Big\rangle.$$

From the fact that $\mathcal{D}(A)$ is dense in H it follows that $\delta_A(a)^* = -\delta_A(a^*)$ and so δ_A is an anti-$*$-derivation. We prove that $\delta_A : \mathcal{F} \supset \mathcal{B}^*(A) \to \mathcal{F}$ is closed. Let $(a_k)_k \subset \mathcal{B}^*(A)$ be a sequence with $a_k \to a \in \mathcal{F}$ and $\delta_A(a_k) \to b \in \mathcal{F}$ as k tends to infinity. Then we already have seen that $a \in \mathcal{B}(A)$ and $\delta_A(a) = b$. We have $a^* = \lim_{k\to\infty} a_k^* \in \mathcal{F}$ from the continuity of the $*$-operation and so

$$\delta_A\,(\, a_k^*\,) = -\delta_A\,(\, a_k\,)^* \longrightarrow -b^* \in \mathcal{F}.$$

Again, using the fact that $\delta_A : H \supset \mathcal{B}(A) \to H$ is closed it follows $a^* \in \mathcal{B}(A)$ and $\delta_A(a^*) = -\delta_A(a)^*$ which implies that $a \in \mathcal{B}^*(A)$. □

From the construction above we can associate to each finite system $\mathcal{V} \subset L(H)$ of closed (symmetric) operators on a Hilbert space H a system of closed (anti-$*$-) derivations. By Definition 1.1.1 this leads to a sequence of topological operator algebras

$$\Psi_k^\Delta[\, \mathcal{F}\,] := \Psi_k^\Delta \qquad \text{where} \qquad k \in \mathbb{N}_0 \cup \{\infty\}$$

with properties described in Proposition 1.1.1. More precisely:

Let $(\, \mathcal{F}, (q_j)_{j\in\mathbb{N}}\,)$ be a submultiplicative Ψ^*-algebra and $\mathcal{B} \subset \mathcal{L}(H)$ a C^*-algebra with continuous embedding $\mathcal{F} \hookrightarrow \mathcal{B}$. Assume that $\mathcal{V}$ is a finite set of closed, densely defined operators on H. In Definition 1.1.1 we set:

$$\Delta := \Big\{\, i\delta_A : \mathcal{F} \supset \mathcal{B}(A) \to \mathcal{F} : A \in \mathcal{V}\, \Big\}. \tag{1.2.1}$$

By Lemma 1.2.1 we obtain a system of $*$-derivations $i\delta_A$ in the case where all $A \in \mathcal{V}$ are symmetric. In general, for the construction of the topological algebras in Proposition 1.1.1 it is not important if we use $*$-derivations or anti-$*$-derivations. According to the Definitions 1.1.1 and 1.2.1 there is a decreasing sequence of topological algebras:

$$\mathcal{F} = \Psi_0^{\Delta}[\,\mathcal{F}\,] \supset \bigcap_{A \in \mathcal{V}} \mathcal{B}(A) = \Psi_1^{\Delta}[\,\mathcal{F}\,] \supset \cdots \supset \Psi_k^{\Delta}[\,\mathcal{F}\,] \supset \Psi_{k+1}^{\Delta}[\,\mathcal{F}\,] \supset \cdots \supset \Psi_{\infty}^{\Delta}[\,\mathcal{F}\,].$$

In the case of $*$-derivations we replace $\mathcal{B}(A)$ by $\mathcal{B}^*(A)$. In order to give a more concrete description of the operator algebras we define iterated commutators with systems of closed operators:

Definition 1.2.2 Let H be a Hilbert space and $M \subset H$ a linear subspace. Denote by $L(M)$ the space of all linear operators on M. For a finite system

$$\mathcal{A} := [\,A_1, \cdots, A_j\,] \qquad \text{where} \qquad A_1, \cdots A_j \in L(M)$$

and $B \in L(M)$ we call j the *length* of $\mathcal{A}$ and we inductively define:

$$\text{ad}\,[\,A_1\,]\,(B) = [\,A_1, B\,], \qquad \text{ad}\,[\,A_1, \cdots A_{r+1}\,]\,(B) := \text{ad}\,[\,A_{r+1}\,]\Big(\,\text{ad}\,[\,A_1, \cdots, A_r\,]\,(B)\,\Big)$$

where $r < j$. In addition we set $\text{ad}[\,\emptyset\,](B) = \text{ad}^0[\,A_1\,](B) := B$ if $\emptyset$ denotes the empty system and we call $\text{ad}[\,\mathcal{A}\,](B)$ the *commutator* of the operator B and the system $\mathcal{A}$.

From now on we assume that, given a finite set $\mathcal{V}$ of closed operators $A : H \supset \mathcal{D}(A) \to H$, there is a dense subspace $D \subset H$ such that each domain of definition $\mathcal{D}(A)$ is the closure of D with respect to the graph norm $\|\cdot\|_{\text{gr}} := \|\cdot\| + \|A\cdot\|$. We prove:

Proposition 1.2.1 *Let $k \in \mathbb{N} \cup \{\infty\}$ and $a \in \mathcal{F}$. With $D \subset H$ as above we assume that:*

(A): Let D be invariant under all the operators $A \in \mathcal{V}$ as well as under $a \in \mathcal{F}$ such that the commutators $ad\,[\,\mathcal{A}\,](a) : H \supset D \to H$ are well-defined for any system $\mathcal{A}$ in

$$\mathcal{S}_k(\,\mathcal{V}\,) := \Big\{\,[\,A_1, \cdots, A_j\,] : \ \textit{where } A_l \in \mathcal{V} \textit{ and } 1 \le j \le k\,\Big\}.$$

Assume that all iterated commutators $ad[\,\mathcal{A}\,](a)$ where $\mathcal{A} \in \mathcal{S}_k(\,\mathcal{V}\,)$ admit continuous extensions to operators $C(\mathcal{A}, a) \in \mathcal{F}$.

Then $a \in \Psi_k^{\Delta}[\,\mathcal{F}\,]$ and for any system $\mathcal{A} \in \mathcal{S}_k(\,\mathcal{V}\,)$ the operator $C(\mathcal{A}, a)$ is a continuous extension of $ad\,[\,\mathcal{A}\,](a) : H \subset \mathcal{D}(A) \to H$ for any $A \in \mathcal{V}$.

In order to prove Proposition 1.2.1 we give some results on extensions of operators. In the following assume that $(\,E_j, \|\cdot\|_j\,)$ for $j = 1, 2$ are normed spaces with a continuous embedding $E_1 \hookrightarrow E_2$. We start with the following easy observation:

Lemma 1.2.2 *Let $D \subset E_1$ be a dense subspace of E_1 and let $A \in \mathcal{L}(E_1, E_2)$. If the restriction $A_{|_D} : D \to E_2$ has a continuous extension $\tilde{A} \in \mathcal{L}(E_2)$, then it holds $\tilde{A}_{|_{E_1}} = A$. In particular, A is continuous in the topology of E_2.*

Proof Let $x \in E_1$ and choose a sequence $(y_n)_n \subset D$ such that $x = \lim_{n\to\infty} y_n$ in E_1. Then

$$\tilde{A}_{|_{E_1}} x = \lim_{n\to\infty} \tilde{A}_{|_{E_1}} y_n = \lim_{n\to\infty} A_{|_D} y_n = Ax. \qquad \Box$$

The following lemma is an immediate consequence of the *closed graph theorem*:

Lemma 1.2.3 *Assume that $(\, E_j, \|\cdot\|_j \,)$ are Banach spaces for $j = 1, 2$ and let $B \in \mathcal{L}(E_2)$ with $B(E_1) \subset E_1$, then it follows $B \in \mathcal{L}(E_1)$.*

Proof It is sufficient to prove that B is closed on E_1. We choose a sequence $(x_n)_n \subset E_1$ such that $\lim_{n\to\infty} x_n = y$ and $\lim_{n\to\infty} Bx_n = z \in E_1$ in the topology of E_1. From the continuous embedding $E_1 \hookrightarrow E_2$ and because of the continuity of B in the E_2-topology it follows that $By = \lim_{n\to\infty} Bx_n = z$. $\Box$

We will apply Lemma 1.2.3 in the case where $E_2 = H$ and $E_1 := (\, \mathcal{D}(A), \|\cdot\|_{\mathrm{gr}} \,)$ with a closed operator $A : H \supset \mathcal{D}(A) \to H$. As before let $\mathcal{D}(A)$ be the graph norm closure of a dense subspace $D \subset H$. We give a result on the invariance of $\mathcal{D}(A)$ under $a \in \mathcal{L}(H)$.

Lemma 1.2.4 *Let $a \in \mathcal{L}(H)$ be an operator such that $a(D) \subset \mathcal{D}(A)$ and assume that the commutator $ad[\,A\,](a) : H \supset D \to H$ admits an extension to $\delta_A a \in \mathcal{L}(H)$. Then it follows that $a(\,\mathcal{D}(A)\,) \subset \mathcal{D}(A)$.*

Proof Let $x \in \mathcal{D}(A)$ and choose a sequence $(y_k)_k \subset D$ such that $\|x - y_k\|_{\mathrm{gr}} \to 0$ as $k \to \infty$. It follows that $\lim_{k\to\infty} ay_k = ax$ and because $\mathrm{ad}[\,A\,](a)$ admits a bounded extension $\delta_A a$ to H the sequence

$$Aay_k = \mathrm{ad}[\,A\,](a)y_k + aAy_k$$

is convergent in H. By assumption A is closed and so it follows that $ax \in \mathcal{D}(A)$. $\Box$

Collecting our results above we can prove Corollary 1.2.1 which essentially will be used in the proof of Proposition 1.2.1:

Corollary 1.2.1 *Let $a \in \mathcal{L}(H)$ be an operator with $a(D) \subset \mathcal{D}(A)$ such that the commutator $ad[\,A\,](a) : H \supset D \to H$ has a bounded extension $\delta_A a \in \mathcal{L}(H)$. Then $\delta_A a$ is an extension of $ad[\,A\,](a) : H \supset \mathcal{D}(A) \to H$ as well.*

Proof Because of $a(D) \subset \mathcal{D}(A)$ we conclude from Lemma 1.2.4 that $\mathcal{D}(A)$ is invariant under a and so the commutator $\mathrm{ad}[\,A\,](a) : H \supset \mathcal{D}(A) \to H$ is well-defined. According to Lemma 1.2.3 with $E_2 := H$ and $E_1 := (\, \mathcal{D}(A), \|\cdot\|_{\mathrm{gr}} \,)$ there is $\alpha > 0$ such that for all $x \in \mathcal{D}(A)$ it holds $\|ax\|_{\mathrm{gr}} \leq \alpha \|x\|_{\mathrm{gr}}$. Hence for $x \in \mathcal{D}(A)$ it follows that:

$$\big\|\, \mathrm{ad}[\,A\,](a)x \,\big\| \leq \big\|\, Aax \,\big\| + \|\, a \,\| \,\big\|\, Ax \,\big\| \leq \big(\, \alpha + \|\, a \,\| \,\big) \,\|\, x \,\|_{\mathrm{gr}}.$$

Note that the inclusion $D \subset (\,\mathcal{D}(A), \|\cdot\|_{\mathrm{gr}}\,)$ is dense by our assumption on $\mathcal{D}(A)$. By Lemma 1.2.2 together with the fact that $\mathrm{ad}[\,A\,](a) : H \supset D \to H$ has a continuous extension to $\delta_A(a) \in \mathcal{L}(H)$ the assertion follows. □

Proof of Proposition 1.2.1 Let $k = 1$, then with our notations of the Definitions 1.1.1 and 1.2.1 we have:

$$\Psi_1^{\Delta}[\,\mathcal{F}\,] = \bigcap_{\delta \in \Delta} \mathcal{D}(\delta) = \bigcap_{A \in \mathcal{V}} \mathcal{B}(A) \subset \mathcal{L}(H).$$

We have to check the inclusion $a(\mathcal{D}(A)) \subset \mathcal{D}(A)$ for all $A \in \mathcal{V}$. This in fact follows from assumption (A) in Proposition 1.2.1 and the inclusion $D \subset \mathcal{D}(A)$ for all $A \in \mathcal{V}$ together with Lemma 1.2.4. Moreover, by Corollary 1.2.1 the operator $\delta_A a = C([A], a)$ is contained in $\mathcal{F}$ and a continuous extension of

$$\mathrm{ad}[\,A\,](a) : H \supset \mathcal{D}(A) \to H$$

for $A \in \mathcal{V}$ and so it follows that $a \in \Psi_1^{\Delta}[\,\mathcal{F}\,]$. Now we assume that $a \in \Psi_j^{\Delta}[\,\mathcal{F}\,]$ where $1 \leq j < k$. Let $A \in \mathcal{V}$ and consider $\mathrm{ad}[\,A\,](a) : H \supset \mathcal{D}(A) \to H$ which has an extension $\delta_A a \in \mathcal{F}$. Again the commutator:

$$\mathrm{ad}\,[\,\mathcal{A}_j\,]\,(\,\delta_A a\,) = \mathrm{ad}\,[\,A, \mathcal{A}_j\,]\,(a) : H \supset D \to D \subset H$$

has a continuous extension to $C(\,[\,A, \mathcal{A}_j\,], a\,) \in \mathcal{F}$ for all systems $\mathcal{A}_j \in \mathcal{S}_j(\,\mathcal{V}\,)$. Hence by induction we conclude that $\delta_A a \in \Psi_j^{\Delta}[\,\mathcal{F}\,]$ and it follows that $a \in \Psi_{j+1}^{\Delta}[\,\mathcal{F}\,]$. □

Next we define a sequence of generalized $\mathcal{V}$-*Sobolev spaces* in H corresponding to $\mathcal{V}$ and we describe in which sense it is related to the algebras Ψ_j^{Δ}.

Definition 1.2.3 Let H be a Hilbert space and assume that $\mathcal{V}$ is a finite set of closed, densely defined operators $A : H \supset \mathcal{D}(A) \to H$. Then we define:

- $\mathcal{H}_{\mathcal{V}}^0 := H$ with the norm $p_0 := \|\cdot\|_H$.
- $\mathcal{H}_{\mathcal{V}}^1 := \bigcap_{A \in \mathcal{V}} \mathcal{D}(A)$.
- $\mathcal{H}_{\mathcal{V}}^n := \Big\{ x \in \mathcal{H}_{\mathcal{V}}^{n-1} : Ax \in \mathcal{H}_{\mathcal{V}}^{n-1} \text{ for all } A \in \mathcal{V} \Big\}, \quad n \geq 2.$
- $\mathcal{H}_{\mathcal{V}}^{\infty} := \bigcap_{n \in \mathbb{N}} \mathcal{H}_{\mathcal{V}}^n$.

We endow $\mathcal{H}_{\mathcal{V}}^n$ with the norm $p_n(x) := p_{n-1}(x) + \sum_{A \in \mathcal{V}} p_{n-1}(Ax)$ for $x \in \mathcal{H}_{\mathcal{V}}^n$ and $\mathcal{H}_{\mathcal{V}}^{\infty}$ with the system of norms $(p_n)_{n \in \mathbb{N}_0}$.

Note that we also can replace H in Definition 1.2.3 by a Banach space. By using arguments similar to the proof of Proposition 1.1.1 it can be shown:

Lemma 1.2.5 *Let $n \in \mathbb{N}$, then*

(a) $(\,\mathcal{H}_{\mathcal{V}}^n, p_n\,)$ is a Banach spaces and $(\mathcal{H}_{\mathcal{V}}^{\infty}, (p_n)_{n \in \mathbb{N}})$ is a Fréchet space.

(b) Each operator $A \in \mathcal{V}$ induces a bounded operator $\tilde{A}_n : \mathcal{H}^n_{\mathcal{V}} \to \mathcal{H}^{n-1}_{\mathcal{V}}$.

Moreover, there are norms $\tilde{p}_n$ on $\mathcal{H}^n_{\mathcal{V}}$ equivalent to p_n such that $(\mathcal{H}^n_{\mathcal{V}}, \tilde{p}_n)$ turns into a Hilbert space and $(\mathcal{H}^\infty_{\mathcal{V}}, (\tilde{p}_n)_{n\in\mathbb{N}_0})$ into a Fréchet-Hilbert space.

Let us give a useful relation between the Fréchet operator algebras constructed with respect to a finite family $\mathcal{V}$ of closed (resp. closed and symmetric) densely defined operators on a Hilbert space H and the $\mathcal{V}$-Sobolev spaces introduced above. We are using our previous notations.

Theorem 1.2.1 *Each derivation $\delta_A \in \Delta$ where Δ is defined in (1.2.1) is closed and the maps in (a) and (b) below:*

(a) $\Psi^\Delta_n \times \mathcal{H}^n_{\mathcal{V}} \longrightarrow \mathcal{H}^n_{\mathcal{V}} : (\, a, x \,) \mapsto a(\, x \,)$,

(b) $\Psi^\Delta_\infty \times \mathcal{H}^\infty_{\mathcal{V}} \longrightarrow \mathcal{H}^\infty_{\mathcal{V}} : (\, a, x \,) \mapsto a(\, x \,)$

are continuous and bilinear. Moreover, $\delta_A : \Psi^\Delta_\infty \to \Psi^\Delta_\infty$ is continuous for all $A \in \mathcal{V}$. If each $A \in \mathcal{V}$ is symmetric, then Ψ^Δ_n is symmetric for all $n \in \mathbb{N}$ and Ψ^Δ_∞ is a Ψ^*-algebra.

Proof We only have to show the continuity in (a), then (b) follows as well. We prove for $n \in \mathbb{N}_0$ and each pair $(a, x) \in \Psi^\Delta_n \times \mathcal{H}^n_{\mathcal{V}}$ the inequality:

$$p_n\,[\, a(x)\,] \leq q_{n,0}(\, a\,) \cdot p_n[\, x\,].$$

Let $n = 0$, then $p_0[\, a(x)\,] = \|\, a(x)\,\|_H \leq \|\, a\,\| \cdot \|\, x\,\|_H = q_{0,0}(\, a\,) \cdot p_0(\, x\,)$. For $n \geq 1$ now the induction step follows from:

$$\begin{aligned}
p_n[\, a(x)\,] &= p_{n-1}[\, a(x)\,] + \sum_{A\in\mathcal{V}} p_{n-1}[\, Aa(x)\,] \\
&\leq p_{n-1}[\, a(x)\,] + \sum_{A\in\mathcal{V}} p_{n-1}\,[\, aA(x) + (\delta_A a)\,(x)\,] \\
&\leq q_{n-1,0}(a) \cdot p_{n-1}(x) + \sum_{A\in\mathcal{V}} [\, q_{n-1,0}(a) \cdot p_{n-1}[A(x)] + q_{n-1,0}\,(\delta_A a) \cdot p_{n-1}(x)\,] \\
&\leq q_{n,0}(\, a\,) \cdot p_n(\, x\,).
\end{aligned}$$

The continuity of $\delta_A : \Psi^\Delta_\infty \to \Psi^\Delta_\infty$ was proved in Proposition 1.1.1 and the last statement follows from Proposition 1.1.1 together with Lemma 1.2.1. □

We present a result on *abstract regularity* due to B. Gramsch and K.G. Kalb, cf. [77]. The proof makes use of the following lemma which can be found in [69], Remark 5.7:

Lemma 1.2.6 *Let H be a Hilbert space, Ψ be a Ψ^*-algebra in $\mathcal{L}(H)$ and $A \in \Psi$ with closed range $R(A) \subset H$. Then for the orthogonal projection $Q \in \mathcal{L}(H)$ onto $N(A)$ we have $Q \in \Psi$. In particular, there is $B = (Q + A^*A)^{-1}A^* \in \Psi$ such that $Q = I - BA$.*

We apply Lemma 1.2.6 to prove Theorem 1.2.2 below, which is a special case of 2.5.11 Proposition in [107] and originally can be found in [77]. In the following we assume that $\mathcal{V}$ is a finite set of symmetric closed operators on the Hilbert space H such that $\Psi^{\Delta}_{\infty}[\,\mathcal{F}\,]$ turns into a Ψ^*-algebra in $\mathcal{L}(H)$ by Proposition 1.1.1, (c). Moreover, let $\mathcal{H}^{\infty}_{\mathcal{V}}$ be dense in H which is fulfilled in most of the applications. Then it was shown in [77]:

Theorem 1.2.2 *Let $A \in \Psi^{\Delta}_{\infty}[\,\mathcal{F}\,]$ be a Fredholm operator. Let $u \in H$ be arbitrary such that $Au = f \in \mathcal{H}^k_{\mathcal{V}}$ for some $k \in \mathbb{N}_0 \cup \{\infty\}$. Then one has $u \in \mathcal{H}^k_{\mathcal{V}}$.*

Proof Because A is a Fredholm operator it follows that $R(A) \subset H$ is closed and by Lemma 1.2.6 one obtains $B \in \Psi^{\Delta}_{\infty}[\,\mathcal{F}\,]$ such that $Q = I - BA \in \Psi^{\Delta}_{\infty}[\,\mathcal{F}\,]$ is the orthogonal projection onto $N(A)$. From Theorem 1.2.1 we conclude that $Q(\,\mathcal{H}^{\infty}_{\mathcal{V}}\,) \subset \mathcal{H}^{\infty}_{\mathcal{V}}$ and because of $\dim R(Q) = \dim N(A) < \infty$ it follows that $Q(\mathcal{H}^{\infty}_{\mathcal{V}})$ is closed. Hence we have $R(Q) \subset \mathcal{H}^{\infty}_{\mathcal{V}}$ from the density of the inclusion $\mathcal{H}^{\infty}_{\mathcal{V}} \subset H$. Again by Theorem 1.2.1 we obtain $Bf \in \mathcal{H}^k_{\mathcal{V}}$. This implies that $u = BAu + Qu = Bf + Qu \in \mathcal{H}^k_{\mathcal{V}}$ which completes the proof. □

1.3 Smooth elements by C_0-group action

Let (H, α) be a pair consisting of a Hilbert space H and a strongly continuous one-parameter group α (C_0-group) acting on H. We want to define the algebra of smooth elements in $\mathcal{L}(H)$ with respect to α and we remind of the fact that the smoothness of operators can be characterized by means of the boundedness of iterated commutators with the infinitesimal generator of the group.

Let $\alpha : \mathbb{R} \to \mathcal{L}^{-1}(H)$ be a strongly continuous one-parameter group (C_0-group) of operators on H and denote by

$$V : H \supset \mathcal{D}(V) := \Big\{ x \in H : Vx := \lim_{t\to 0} t^{-1}\big(\,\alpha_t x - x\,\big) \in H \ \text{ exists} \Big\} \to H : x \mapsto Vx$$

its *infinitesimal generator*. Then by well-known results the operator V is closed and densely defined on H with $\alpha_t(\,\mathcal{D}(V)\,) \subset \mathcal{D}(V)$ and $[\,\alpha_t, V\,]\,x = 0$ for all $x \in \mathcal{D}(V)$. With our notation (1.2.1) in section 1.2 we set Δ simply to be $\{\delta_V\}$ where δ_V denotes the derivation given in Definition 1.2.1. If in addition the group α is unitary then by *Stone's theorem* we conclude that iV is essentially self adjoint and it is easy to check that δ_V is a $*$-derivation. Let

$$\big(\,\mathcal{A}, (q_j)_{j\in\mathbb{N}_0}\,\big) \hookrightarrow \mathcal{L}(H)$$

be a submultiplicative Ψ^*-algebra which continuously is embedded in $\mathcal{L}(H)$. Without loss of generality we assume that $q_0 = \|\cdot\|_{\mathcal{L}(H)}$. By the construction above we obtain a sequence of operator algebras $\Psi^{\alpha}_n[\,\mathcal{A}\,] := \Psi^{\Delta}_n$ with continuous embeddings:

$$\bigcap_{k\in\mathbb{N}} \Psi^{\alpha}_k[\,\mathcal{A}\,] =: \Psi^{\alpha}_{\infty}[\,\mathcal{A}\,] \subset \cdots \subset \Psi^{\alpha}_k[\,\mathcal{A}\,] \subset \Psi^{\alpha}_{k-1}[\,\mathcal{A}\,] \subset \cdots \subset \mathcal{B}(V) \subset \mathcal{A} \tag{1.3.1}$$

and corresponding $\mathcal{V}$-Sobolev space $\mathcal{H}^k_{\mathcal{V}}$ where $k \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{V} = \{ V \}$. In the case of a *unitary* C_0*-group* we replace $\mathcal{B}(V)$ in (1.3.1) by $\mathcal{B}^*(V)$. Then the algebras $\Psi^\alpha_n[\mathcal{A}]$ are symmetric for $n \in \mathbb{N}$ and $\Psi^\alpha_\infty[\mathcal{A}]$ is a Ψ^*-algebra.

Let us define a second scale of subspaces in $\mathcal{L}(H)$ which is generated by the action of the C_0-group $(\alpha_t)_{t\in\mathbb{R}}$. Consider the map $\varphi : \mathbb{R} \longrightarrow \mathcal{L}(\,\mathcal{L}(H)\,)$ defined for all $t \in \mathbb{R}$ and a bounded operator $a \in \mathcal{L}(H)$ by

$$\left[\,\varphi(t)\,\right](a) := \alpha_t\, a\, \alpha_t^{-1} \in \mathcal{L}(H).$$

For fixed $a \in \mathcal{L}(H)$ we denote by $\varphi_a : \mathbb{R} \to \mathcal{L}(H)$ the map

$$\varphi_a(\,t\,) := \left[\,\varphi(t)\,\right](\,a\,).$$

In addition we assume that $\mathcal{A} \subset \mathcal{L}(H)$ is a C^*-algebra in $\mathcal{L}(H)$ with the induced topology and let the maps φ_a only have values in $\mathcal{A}$ for all $a \in \mathcal{A}$. With $n \in \mathbb{N}_0 \cup \{\infty\}$ we define:

$$\Psi^n_\alpha[\,\mathcal{A}\,] := \Big\{\, a \in \mathcal{A} : \varphi_a \in \mathcal{C}^n\big(\,\mathbb{R}, \mathcal{A}\,\big)\,\Big\} \qquad \text{and} \qquad \Psi^\infty_\alpha[\,\mathcal{A}\,] := \bigcap_{j\in\mathbb{N}} \Psi^j_\alpha[\,\mathcal{A}\,].$$

Here $\mathcal{C}^n(\,\mathbb{R}, \mathcal{A}\,)$ denotes the space of n-times differentiable functions from $\mathbb{R}$ to $\mathcal{A}$. We directly can compute the derivatives of the functions $\varphi_a : \mathbb{R} \to \mathcal{A}$ where $a \in \Psi^n_\alpha[\,\mathcal{A}\,]$:

Lemma 1.3.1 *For $a \in \Psi^n_\alpha[\,\mathcal{A}\,]$ we set $b_n := \varphi_a^{(n)}(0) \in \mathcal{A}$, then it holds $\varphi_a^{(n)}(t) = \varphi_{b_n}(t)$ for all $t \in \mathbb{R}$.*

Proof In the case $n = 0$ there is nothing to show. For an arbitrary number $n \in \mathbb{N}$ and $t \in \mathbb{R}$ we obtain by induction:

$$\begin{aligned} h^{-1}\left[\,\varphi_a^{(n)}(\,t+h\,) - \varphi_a^{(n)}(\,t\,)\,\right] &= h^{-1}\left[\,\alpha_{t+h}\, b_n\, \alpha_{t+h}^{-1} - \alpha_t\, b_n\, \alpha_t^{-1}\,\right] \\ &= \alpha_t \left\{\, h^{-1}\left[\,\varphi_a^{(n)}(h) - \varphi_a^{(n)}(0)\,\right]\,\right\} \alpha_t^{-1}. \end{aligned}$$

The right hand side of this equation converges to $\varphi_{b_{n+1}}(t) = \alpha_t\, b_{n+1}\, \alpha_t^{-1}$ while the left hand side converges to $\varphi_a^{(n+1)}(t)$ as h tends to 0. $\square$

A version of the next theorem can be found in [107] and it describes the relation between the two scales of operators algebras $(\,\Psi^n_\alpha[\,\mathcal{A}\,]\,)_{n\in\mathbb{N}_0}$ and $(\,\Psi^\alpha_n[\,\mathcal{A}\,]\,)_{n\in\mathbb{N}_0}$.

Theorem 1.3.1 *Let $(\alpha_t)_{t\in\mathbb{R}}$ be a C_0-group and $\mathcal{A} \subset \mathcal{L}(H)$ a C^*-subalgebra. Then:*

(a) $\Psi^k_\alpha[\,\mathcal{A}\,] \subset \Psi^\alpha_k[\,\mathcal{A}\,]$ *and* $(\delta_V)^k(a) = \varphi_a^{(k)}(0)$ *for all* $a \in \Psi^k_\alpha[\,\mathcal{A}\,]$ *and* $k \in \mathbb{N}$.

(b) $\Psi^\alpha_{k+1}[\,\mathcal{A}\,] \subset \Psi^k_\alpha[\,\mathcal{A}\,]$ *for all* $k \in \mathbb{N}_0$.

(c) $\Psi^\infty_\alpha[\,\mathcal{A}] = \Psi^\alpha_\infty[\,\mathcal{A}\,]$.

Proof (a) : In the case where $\mathcal{A} = \mathcal{L}(H)$ Theorem 1.2.1 was proved in [107], pp. 222. Now, assume that $\mathcal{A}$ is a C^*-algebra in $\mathcal{B} := \mathcal{L}(H)$ and let $k = 1$. With

$$a \in \Psi^1_\alpha[\,\mathcal{A}\,] \subset \Psi^1_\alpha[\,\mathcal{B}\,]$$

and from $\varphi_a(\mathbb{R}) \subset \mathcal{A}$ we conclude that $a \in \Psi^\alpha_1[\,\mathcal{B}\,]$ and $\delta_V(a) = \varphi_a^{(1)}(0) \in \mathcal{A}$. It follows that a is contained in:

$$\Big\{\, b \in \mathcal{A} : b\,(\,\mathcal{D}(V)\,) \subset \mathcal{D}(V) \text{ and } \mathrm{ad}[\,V\,](b) \text{ extends to } \delta_V(b) \in \mathcal{A} \,\Big\} = \Psi^\alpha_1[\,\mathcal{A}\,].$$

Assume that (a) holds for $k \geq 1$ and let $a \in \Psi^{k+1}_\alpha[\,\mathcal{A}\,]$. It follows that $a \in \Psi^\alpha_k[\,\mathcal{A}\,]$ and

$$\delta_V(\,a\,) = \varphi_a^{(1)}(\,0\,) =: b \in \Psi^k_\alpha[\,\mathcal{A}\,] \subset \Psi^\alpha_k[\,\mathcal{A}\,].$$

Here we have used $\varphi_b(t) = \varphi_a^{(1)}(t)$ for all $t \in \mathbb{R}$ (see Lemma 1.3.1) and the fact that the function $\varphi_a^{(1)}$ is smooth up to the order k. By definition we conclude $a \in \Psi^\alpha_{k+1}[\,\mathcal{A}\,]$ and so we have proved (a).

(b) : Let $k \in \mathbb{N}_0$ and $a \in \Psi^\alpha_{k+1}[\,\mathcal{A}\,] \subset \Psi^\alpha_{k+1}[\,\mathcal{B}\,]$. Then $a \in \Psi^k_\alpha[\,\mathcal{B}\,] \cap \mathcal{A}$ follows from Theorem 1.2.1 in [107]. By definition we have $\varphi_a \in \mathcal{C}^k(\,\mathbb{R}, \mathcal{B}\,)$ with $\varphi_a(\mathbb{R}) \subset \mathcal{A}$. Hence we conclude that $\varphi_a \in \mathcal{C}^k(\,\mathbb{R}, \mathcal{A}\,)$ and again this implies $a \in \Psi^k_\alpha[\,\mathcal{A}\,]$. The statement (c) is a consequence of (a) and (b). □

Let H be a Hilbert space and $\mathcal{M} \subset \mathcal{L}(H)$ be a set of bounded operators. We denote by $C^*(\mathcal{M}) \subset \mathcal{L}(H)$ the C^*-algebra generated by $\mathcal{M}$. With

$$\mathcal{M}^* := \Big\{\, a^* \in \mathcal{L}(H) : a \in \mathcal{M} \,\Big\}$$

the algebra $C^*(\mathcal{M})$ is the closure of the linear hull $\mathrm{span}(W)$ where W is defined to be

$$W = \bigcup_{n \in \mathbb{N}} \Big\{\, a_1 a_2 \cdots a_n : a_j \in \mathcal{M} \cup \mathcal{M}^* \,\Big\}.$$

Definition 1.3.1 Let $b \in \mathcal{L}^{-1}(H)$ such that $b\,a\,b^{-1} \in \mathcal{M}$ for all $a \in \mathcal{M}$, then we call $\mathcal{M}$ invariant under b. With a group

$$\alpha = (\alpha_t)_{t \in \mathbb{R}} \subset \mathcal{L}^{-1}(H)$$

$\mathcal{M}$ is said to be *invariant under* α if $\mathcal{M}$ is invariant under α_t for all $t \in \mathbb{R}$.

Note that with our notations above $\mathcal{M}$ is invariant under the group α if and only if the map $\varphi_x : \mathbb{R} \to \mathcal{M}$ is well-defined for all $x \in \mathcal{M}$. Next we give an easy stability result for the described group action.

Lemma 1.3.2 *Let $\mathcal{M} \subset \mathcal{L}(H)$ be a subset and assume that $\mathcal{M} \cup \mathcal{M}^*$ is invariant under $b \in \mathcal{L}^{-1}(H)$. Then this also holds for the C^*-algebra $C^*(\mathcal{M})$.*

Proof Fix $j \in \mathbb{N}$ and finitely many operators $a_1, \cdots a_j \in \mathcal{M} \cup \mathcal{M}^*$. For all $t \in \mathbb{R}$ it follows that:

$$b \left\{ a_1 \cdots a_j \right\} b^{-1} = \left\{ b \, a_1 \, b^{-1} \right\} \cdots \left\{ b \, a_j \, b^{-1} \right\}$$

and so the space of all finite sums of finite products of operators in $\mathcal{M} \cup \mathcal{M}^*$ is invariant under the action of b. The full assertion follows from the continuity of the linear operator $\Phi_b \in \mathcal{L}(\mathcal{L}(H))$ defined by $\Phi_b(a) = b \, a \, b^{-1}$. □

1.4 Projections of algebras

Let $\mathcal{B}$ be a Banach algebra with unit e and $\mathcal{A} \subset \mathcal{B}$ be a locally spectral invariant subalgebra of $\mathcal{B}$. For any projection $p = p^2 \in \mathcal{A}$ we can consider $p \, \mathcal{A} \, p =: \mathcal{A}_p$. The following result can be found in [107]:

Lemma 1.4.1 *The algebra $\mathcal{A}_p$ is locally spectral invariant in $\mathcal{B}_p := p \, \mathcal{B} \, p$. If $\mathcal{A}$ is symmetric, $\mathcal{B}$ is a C^*-algebra and $p = p^2 = p^* \in \mathcal{A}$, then $\mathcal{A}_p$ is symmetric and so spectrally invariant.*

Proof Note that $\mathcal{B}_p$ is a Banach algebra with unit p. To prove the locally spectral invariance fix $\varepsilon > 0$ with

$$\Big\{ a \in \mathcal{A} : \| \, e - a \, \| < \varepsilon \Big\} \subset \mathcal{A}^{-1}$$

and consider $x \in \mathcal{A}_p$ such that $\| \, p - x \, \| \le \varepsilon$. Then with $b = x + e - p$ it follows $b - e = x - p$ and so $\| \, b - e \, \| < \varepsilon$. As a consequence b has an inverse in $\mathcal{A}$. Now

$$e = x \, b^{-1} + (\, e - p \,) \, b^{-1} = b^{-1} x + b^{-1} \, (\, e - p \,).$$

Finally, applying p from the left and the right hand side to this equation we obtain that x is invertible in $\mathcal{A}_p$ with $x^{-1} = p b^{-1} p \in \mathcal{A}_p^{-1}$. The last statement follows by an application of Lemma 1.0.1. □

Let $\mathcal{B} = \mathcal{L}(H)$ where H is a Hilbert space and let the algebra $\mathcal{A} \subset \mathcal{B}$ be constructed by commutator methods as it was described in section 1.2. We ask under which conditions the projected algebra $\mathcal{A}_p$ where $p = p^2 \in \mathcal{A}$ can be obtained by commutators with systems of closed operators as well. The general results we prove here will be applied later to the construction of Fréchet operator in C^*-Toeplitz algebras over *Bergman* and *Hardy spaces*. Let $Q \in \mathcal{L}(H)$ be an orthogonal projection onto a subspace $E := Q(H)$.

Lemma 1.4.2 *Let $A : H \supset \mathcal{D}(A) \to H$ be a closed operator and fix $Q \in \mathcal{L}(H)$ with $Q = Q^2 = Q^*$. We assume that:*

(a) $Q[\, \mathcal{D}(A) \,] \subset \mathcal{D}(A)$.

(b) The commutator $[\, A, Q \,] : H \supset \mathcal{D}(A) \to H$ admits a bounded extension to H.

Then $A_Q := Q \, A \, Q : E \supset \mathcal{D}(\, A_Q \,) := Q[\, \mathcal{D}(A) \,] \to E$ is closed on $E = Q(H)$.

Proof Let $(Qx_n)_n$ be a sequence in $Q[\,\mathcal{D}(A)\,]$ with $(x_n)_n \subset \mathcal{D}(A)$ such that it holds:

$$Qx_n \to y = Qy \qquad \text{and} \qquad Q\,A\,Qx_n \to z \qquad (\,n \to \infty\,).$$

Then from the boundedness of the commutator $[\,A, Q\,]$ we conclude that

$$A\,Qx_n = [\,A, Q\,]\,Qx_n + Q\,A\,Qx_n \in H$$

is convergent as n tends to infinity. Because A is closed by assumption it follows that $y \in \mathcal{D}(A)$ and so $y = Qy \in Q[\,\mathcal{D}(A)\,]$. Additionally it holds $\lim_{n\to\infty} AQx_n = AQy$. Now, using the continuity of Q we conclude that $\lim_{n\to\infty} QAQx_n = QAQy$. This proves the assertion. □

In the construction of operator algebras using of closed operators it is necessary to prove the continuity of certain commutators. With the orthogonal projection $Q \in \mathcal{L}(H)$ the linear space $E = Q(H)$ and a closed densely defined operator $A : H \supset \mathcal{D}(A) \to H$ as above we get:

Proposition 1.4.1 *Let $B \in \mathcal{L}(H)$ and assume that:*

(a) The inclusion $\mathcal{D}(A_Q) := Q[\,\mathcal{D}(A)\,] \subset \mathcal{D}(A)$ holds and $[\,A, Q\,] : H \subset \mathcal{D}(A) \to H$ has a bounded extension to an operator in $\mathcal{L}(H)$.

(b) The domain of definition $\mathcal{D}(A_Q)$ is invariant under $B_Q := Q\,B\,Q \in \mathcal{L}(E)$.

Then with $A_Q := Q\,A\,Q$ the commutator $[\,A_Q, B_Q\,]$ has a bounded extension C_1 from $\mathcal{D}(A_Q)$ to E if and only if $[A, B_Q]$ has a bounded extension C_2 from $\mathcal{D}(A)$ to H. Moreover, we have $C_1 = Q\,C_2\,Q$ and so if $\mathcal{D}(A)$ is invariant under C_2, then $\mathcal{D}(A_Q)$ is invariant under the extension C_1.

Proof From our assumption (a) and (b) it follows that both commutators $[\,A_Q, B_Q\,]$ (resp. $[\,A, B_Q\,]$) are well-defined on the dense sets $\mathcal{D}(A_Q)$ (resp. $\mathcal{D}(A)$) in E (resp. in H). Moreover, the following decomposition holds as operator equation on E or H:

$$\big[\,A, B_Q\,\big] = \big[\,A_Q, B_Q\,\big] + (I - Q)\,\big[\,A, Q\,\big]\,B_Q + B_Q\,\big[\,A, Q\,\big]\,(I - Q). \tag{1.4.1}$$

Because by (a) the commutator $[\,A, Q\,]$ has a bounded extension from $\mathcal{D}(A)$ to H the first assertion of Proposition 1.4.1 follows. From (1.4.1) we conclude that $C_1 = Q\,C_2\,Q$ for the bounded extension C_2 of the commutators $[\,A, B_Q\,]$ and C_1 of $[\,A_Q, B_Q\,]$. Because $\mathcal{D}(A)$ is invariant under Q and C_2 by assumption, it follows the invariance of $\mathcal{D}(A_Q)$ with respect to $C_1 \in \mathcal{L}(E)$. □

Under some additional assumptions we can replace (a) in Lemma 1.4.2 and Proposition 1.4.1 by a weaker condition. In many cases we may check the invariance of the domain of definition by proving it for some subspace. The proof is quite simple:

Lemma 1.4.3 *Let $B \in \mathcal{L}(H)$ and assume that there is a linear space $\mathcal{E} \subset \mathcal{D}(A)$ with:*

(a) $\mathcal{E}$ is dense in $\mathcal{D}(A)$ with respect to the graph norm $\| \cdot \|_{gr}$,

(b) It holds $B[\, \mathcal{E}\,] \subset \mathcal{D}(A)$,

(c) The commutator $[\, A, B\,]$ admits a continuous extension from $\mathcal{E}$ to H.

Then it follows the invariance $B[\, \mathcal{D}(A)\,] \subset \mathcal{D}(A)$.

Proof Let $g \in \mathcal{D}(A)$ and choose a sequence $(h_k)_k \subset \mathcal{E}$ such that $\| \, g - h_k \, \|_{\mathrm{gr}} \to 0$ as $k \to \infty$. It follows that $\lim_{k\to\infty} Bh_k = Bg$ and because $[\, A, B\,]$ admits a bounded extension to H the sequence

$$ABh_k = [\, A, B\,]h_k + BAh_k$$

is convergent in H. A is closed and so it follows that $Bg \in \mathcal{D}(A)$. □

Let $k \in \mathbb{N}$ and choose a finite set $\mathcal{V} := \{\, A_1, \cdots, A_k \,\}$ of closed and densely defined operators on the Hilbert space H with domains $\mathcal{D}(A_j)$ of definition. Further we assume that $(\, \mathcal{F}, (q_j)_{j\in\mathbb{N}}\,)$ is a submultiplicative Ψ^*-algebra in a C^*-algebra $\mathcal{B} \subset \mathcal{L}(H)$. In section 1.2 and with our notations in Definition 1.2.1 we have given a set of closed derivations Δ by

$$\Delta := \Big\{\, i\delta_A : \mathcal{F} \supset \mathcal{B}(A) \to \mathcal{F} : A \in \mathcal{V} \,\Big\}.$$

Given any orthogonal projection $Q = Q^2 = Q^* \in \mathcal{F}$ such that the conditions (a) and (b) in Lemma 1.4.2 are fulfilled we set $E := Q(H)$. According to Lemma 1.4.2 we can consider the following system $\mathcal{V}_Q$ of closed densely defined operators on E:

$$\mathcal{V}_Q := \Big\{\, A_Q := Q\, A\, Q : E \supset \mathcal{D}(\, A_Q\,) \to E : A \in \mathcal{V} \text{ and } \mathcal{D}(\, A_Q\,) := Q[\, \mathcal{D}(A)\,] \,\Big\}.$$

By Lemma 1.4.1 the algebra $\mathcal{F}_Q := Q\mathcal{F}Q$ is a Ψ^*-algebra in $\mathcal{B}_Q := Q\mathcal{B}Q$. Further, with the map

$$P : \mathcal{F} \to \mathcal{F}_Q : x \mapsto QxQ$$

let the topology on the algebra $\mathcal{F}_Q$ be generated by the systems of submultiplicative semi-norms $\tilde{q}_j := q_j \circ P$ for $j \in \mathbb{N}$. Because Q is an projection $\tilde{q}_j$ simply is the restriction of q_j to $\mathcal{F}_Q$. According to the submultiplicativity of $(q_j)_j$ it follows from

$$\tilde{q}_j(\, Px\,) = \tilde{q}_j(x) = q_j\big(\, QxQ\, \big) \leq q_j(\, Q\,)^2 \cdot q_j(\, x\,)$$

for all $x \in \mathcal{F}$ and all $j \in \mathbb{N}$ that P is continuous. It is easy to see that $(\mathcal{F}_Q, \tilde{q}_j)$ is closed in $(\mathcal{F}, q_j)$ and so it is a submultiplicative Ψ^*-algebra as well. Now let us set:

$$\Delta_Q := \Big\{\, i\delta_A : \mathcal{F}_Q \supset \mathcal{B}(\, A_Q\,) \to \mathcal{F}_Q : A \in \mathcal{V}_Q \,\Big\}.$$

Then by the constructions above we obtain a decreasing series of *Fréchet operator algebras* in the Ψ^*-algebra $\mathcal{F}_Q$ by defining:

$$\bigcap_{j\in\mathbb{N}} \Psi_j^{\Delta_Q}[\, \mathcal{F}_Q\,] =: \Psi_\infty^{\Delta_Q}[\, \mathcal{F}_Q\,] \subset \cdots \subset \Psi_k^{\Delta_Q}[\, \mathcal{F}_Q\,] \subset \cdots \subset \Psi_1^{\Delta_Q}[\, \mathcal{F}_Q\,] \subset \mathcal{F}_Q.$$

On the other hand, in the case where $Q \in \Psi_j^{\Delta}[\,\mathcal{F}\,]$ we can consider the locally spectral invariant and symmetric subalgebras

$$\Psi_j^{\Delta}[\,\mathcal{F}\,]_Q := Q\,\Psi_j^{\Delta}[\,\mathcal{F}\,]\,Q \subset \mathcal{F}_Q \qquad \big(\, j \in \mathbb{N}_0 \cup \{\infty\}\,\big)$$

(see Lemma 1.4.1). As before we set $E := Q(H)$, then we can prove the following relation between these two scales of algebras:

Proposition 1.4.2 *Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and assume that $Q \in \Psi_k^{\Delta}[\,\mathcal{F}\,]$. Then for each $j \leq k$ the equality $\Psi_j^{\Delta_Q}[\,\mathcal{F}_Q\,] = \Psi_j^{\Delta}[\,\mathcal{F}\,]_Q$ holds.*

Proof The case $k = 0$ is obvious and for $k = \infty$ the assertion immediately follows if it is proved for all $k \in \mathbb{N}$. Let $j = 1$ and fix

$$B \in \Psi_1^{\Delta_Q}\,[\,\mathcal{F}_Q\,] = \bigcap_{A \in \mathcal{V}} \mathcal{B}\,(\,A_Q\,) \subset \mathcal{F}_Q.$$

where $A_Q := Q\,A\,Q$ for any operator $A \in \mathcal{V}$. Then by definition $\mathcal{D}(\,A_Q\,)$ is invariant under the operator $B = Q\,B\,Q =: B_Q \in \mathcal{L}(E)$ and $[\,A_Q, B_Q\,]$ admits a bounded extension $C(\,A_Q, B_Q\,) \in \mathcal{F}_Q$ from $\mathcal{D}(\,A_Q\,)$ to E for all $A \in \mathcal{V}$. By assumption $Q \in \Psi_1^{\Delta}[\,\mathcal{F}\,]$ and so both conditions (a) and (b) of Proposition 1.4.1 are fulfilled. We conclude that the commutator $[A, B_Q]$ admits a bounded extension from $\mathcal{D}(A)$ to H and $\mathcal{D}(A)$ is invariant under B_Q for all $A \in \mathcal{V}$. It follows that

$$B_Q \in \bigcap_{A \in \mathcal{V}} \mathcal{B}(A) =: \Psi_1^{\Delta}[\,\mathcal{F}\,]$$

and so we have shown the inclusion $\Psi_1^{\Delta_Q}[\,\mathcal{F}_Q\,] \subset \Psi_1^{\Delta}[\mathcal{F}]_Q$.

Now, let $C_Q \in \Psi_1^{\Delta}[\mathcal{F}]_Q$. Because we assume that $Q \in \Psi_1^{\Delta}[\mathcal{F}]$ we also have $C_Q \in \Psi_1^{\Delta}[\mathcal{F}]$. In particular, for all $A \in \mathcal{V}$ the space $\mathcal{D}(A_Q)$ is invariant under C_Q and $[\,A, C_Q\,]$ admits a bounded extension $C(\,A, C_Q\,) \in \mathcal{F}$ from $\mathcal{D}(A)$ to H. Again by Proposition 1.4.1 the operators $[\,A_Q, C_Q\,]$ have bounded extensions $C(\,A_Q, C_Q\,) = C(\,A, C_Q\,)_Q \in \mathcal{F}_Q$ from $\mathcal{D}(A_Q)$ to E for all $A \in \mathcal{V}$ and so $C_Q \in \Psi_1^{\Delta_Q}[\,\mathcal{F}_Q\,]$. Hence in the case $j = 1$ it follows that:

$$\Psi_j^{\Delta_Q}\,[\,\mathcal{F}_Q\,] = \Psi_j^{\Delta}\,[\,\mathcal{F}\,]_Q\,. \tag{1.4.2}$$

Let us assume that (1.4.2) holds for $j < k$ and choose $B = B_Q \in \Psi_{j+1}^{\Delta_Q}[\,\mathcal{F}_Q\,]$. Then by definition the commutator $[A_Q, B_Q]$ has a continuous extension $C(A_Q, B_Q) \in \Psi_j^{\Delta_Q}[\mathcal{F}_Q]$ from $\mathcal{D}(A_Q)$ to H for all $A \in \mathcal{V}$. By induction there is $D \in \Psi_j^{\Delta}[\,\mathcal{F}\,]$ such that $C(A_Q, B_Q) = D_Q$. Moreover, the decomposition

$$\big[\,A, B_Q\,\big] = \big[\,A_Q, B_Q\,\big] + (I - Q)\,\big[\,A, Q\,\big]B_Q + B_Q\,\big[\,A, Q\,\big]\,(I - Q),$$

holds. From the fact that $[\,A, Q\,]$ has a continuous extension $C(\,A, Q\,) \in \Psi_j^{\Delta}[\,\mathcal{F}\,]$ and because of

$$B \in \Psi_{j+1}^{\Delta_Q}\,[\,\mathcal{F}_Q\,] \subset \Psi_j^{\Delta_Q}\,[\,\mathcal{F}_Q\,] \subset \Psi_j^{\Delta}\,[\,\mathcal{F}\,]$$

we conclude that for all $A \in \mathcal{V}$ the commutator $[\, A, B_Q\,]$ has a continuous extension $C(\, A, B_Q\,)$ from $\mathcal{D}(A)$ to H given by:

$$C\big(\, A_Q, B_Q\,\big)_Q + (I-Q)\, C(\, A, Q\,)\, B_Q + B_Q\, C(\, A, Q\,)\, (I-Q) \in \Psi_j^{\Delta}[\, \mathcal{F}\,].$$

and by definition it follows that $B_Q \in \Psi_{j+1}^{\Delta}[\, \mathcal{F}\,]$ and so

$$\Psi_{j+1}^{\Delta_Q}\big[\, \mathcal{F}_Q\,\big] \subset \Psi_{j+1}^{\Delta}\big[\, \mathcal{F}\,\big]_Q.$$

Finally, let us choose $C = C_Q \in \Psi_{j+1}^{\Delta}[\, \mathcal{F}\,]_Q \subset \Psi_{j+1}^{\Delta}[\, \mathcal{F}\,]$. Then for all $A \in \mathcal{V}$ the commutators $[\, A, C_Q\,]$ admit continuous extensions $C(\, A, C_Q\,) \in \Psi_j^{\Delta}[\, \mathcal{F}\,]$ from $\mathcal{D}(A)$ to H. Using Proposition 1.4.1 and by induction $[\, A_Q, C_Q\,]$ has a continuous extension to

$$C\,(\, A_Q, C_Q\,) = C\,(\, A, C_Q\,)_Q \in \Psi_j^{\Delta}[\, \mathcal{F}\,]_Q = \Psi_j^{\Delta_Q}\,[\, \mathcal{F}_Q\,]$$

and so it follows that $C_Q \in \Psi_{j+1}^{\Delta_Q}[\, \mathcal{F}_Q\,]$. $\square$

Remark 1.4.1 In order to guarantee that $\Psi_j^{\Delta}[\, \mathcal{F}\,]_Q$ is closed under the multiplication in $\mathcal{F}$ we have to claim $Q \in \Psi_j^{\Delta}[\, \mathcal{F}\,]$. From Proposition 1.4.2 it follows that in this case the algebras $\Psi_j^{\Delta_Q}[\, \mathcal{F}_Q\,]$ which are defined for $j \in \mathbb{N}_0 \cap \{\infty\}$ extend the finite series

$$\mathcal{F}_Q \supset \Psi_1^{\Delta}[\, \mathcal{F}\,]_Q \supset \cdots \supset \Psi_j^{\Delta}[\, \mathcal{F}\,]_Q$$

to infinity. Note that construction of $\Psi_j^{\Delta_Q}[\, \mathcal{F}_Q\,]$ only requires the weaker assumptions of Lemma 1.4.2. Hence the condition $Q \in \Psi_1^{\Delta}[\, \mathcal{F}\,]$ would be sufficient.

Let us examine the generalized $\mathcal{V}$-Sobolev spaces introduced in Definition 1.2.3. Corresponding to Proposition 1.4.2 and with our notations above we show:

Proposition 1.4.3 *Let $n \in \mathbb{N}$ and assume that the projection Q is contained in $\Psi_n^{\Delta}[\, \mathcal{F}\,]$, then for all $j \in \{\, 0, \cdots, n\,\}$ we have the equality $\mathcal{H}^j_{\mathcal{V}_Q} = Q\,[\, \mathcal{H}^j_{\mathcal{V}}\,]$.*

Proof Because of $\mathcal{D}(\, QA\,) = Q\,[\, \mathcal{D}(A)\,]$ for all closed operators $A \in \mathcal{V}$ the assertion is obvious in the case where $j = 0, 1$. Assume that

$$\mathcal{H}^j_{\mathcal{V}_Q} = Q\,[\, \mathcal{H}^j_{\mathcal{V}}\,] \qquad \text{for} \qquad j \in \{\, 1, \cdots, n-1\,\}$$

By Theorem 1.2.1 together with $Q \in \Psi_n^{\Delta}[\, \mathcal{F}\,]$ it follows that $Q\,[\, \mathcal{H}^{j+1}_{\mathcal{V}}\,] \subset \mathcal{H}^{j+1}_{\mathcal{V}}$. Hence for any $x \in Q\,[\, \mathcal{H}^{j+1}_{\mathcal{V}}\,]$ we have by definition:

$$\Big\{\, x, Ax : A \in \mathcal{V}\,\Big\} \subset \mathcal{H}^j_{\mathcal{V}} \quad \text{and so} \quad \Big\{\, x = Qx, QAQx : A \in \mathcal{V}\,\Big\} \subset Q\,\big[\, \mathcal{H}^j_{\mathcal{V}}\,\big] = \mathcal{H}^j_{\mathcal{V}_Q}.$$

This now implies that $x \in \mathcal{H}^{j+1}_{\mathcal{V}_Q}$ and the inclusion $Q\,[\, \mathcal{H}^{j+1}_{\mathcal{V}}\,] \subset \mathcal{H}^{j+1}_{\mathcal{V}_Q}$ is proved.

Conversely, let $x = Qx \in \mathcal{H}^{j+1}_{\mathcal{V}_Q}$, then we have by induction:

$$\Big\{ x, QAQx : A \in \mathcal{V} \Big\} \subset \mathcal{H}^{j}_{\mathcal{V}_Q} = Q \left[\mathcal{H}^{j}_{\mathcal{V}} \right] \subset \mathcal{H}^{j}_{\mathcal{V}}. \tag{1.4.3}$$

From $Q \in \Psi^{\Delta}_n[\, \mathcal{F} \,]$ it follows that $[\, A, Q \,]\, Qx \in \mathcal{H}^{j}_{\mathcal{V}}$ for all $A \in \mathcal{V}$ and so we conclude that:

$$AQx = Q\, A\, Qx + [\, A, Q \,]\, Qx \in \mathcal{H}^{j}_{\mathcal{V}} \qquad \text{for all} \qquad A \in \mathcal{V}.$$

By definition we obtain $x = Qx \in Q\,[\, \mathcal{H}^{j+1}_{\mathcal{V}} \,]$ and so we have $\mathcal{H}^{j+1}_{\mathcal{V}_Q} \subset Q\,[\, \mathcal{H}^{j+1}_{\mathcal{V}} \,]$. $\square$

Chapter 2

Cone localization of the Segal Bargmann projection

We consider the Hilbert space $H_1 := L^2(\mathbb{C}^n, \mu)$ where μ denotes a normed Gaussian measure. The *Segal-Bargmann space* $H_2 := H^2(\mathbb{C}^n, \mu)$ is the closed subspace of H_1 consisting of all entire functions square integrable with respect to μ. It is well-known that H_2 is a *reproducing kernel Hilbert space* and we prove some norm estimates for linear operators on H_2 which we need in our analysis later on.

The notion of a *Toeplitz operators* T_f for f in a class of (in general unbounded) symbols on $\mathbb{C}^n$ is introduced. In the case where $f_1, \cdots, f_m$ only have polynomial growth we can define products $T_{f_1} \cdots T_{f_m}$ of (unbounded) Toeplitz operators on a suitable dense subspace of H_2 (resp. H_1). Hence all iterated commutators of such operators are meaningful.

It was shown in chapter 1, [79], [77] and [107] how to construct a decreasing series of Fréchet operator algebras $(\Psi_k^\Delta)_{k\in\mathbb{N}}$ with prescribed properties in $\mathcal{L}(H_1)$ using commutator methods with a finite set $\mathcal{V}$ of closed operators. We define a class $\mathcal{X}^\Phi(\mathbb{C}^n)$ of smooth vector fields supported in cones such that the *Toeplitz projection* P from H_1 onto H_2 is contained in Ψ_k^Δ for all $k \in \mathbb{N}$ and all finite sets $\mathcal{V} \subset \mathcal{X}^\Phi(\mathbb{C}^n)$. This enables us to construct subalgebras of $\mathcal{L}(H_2)$ localized in cones $\mathcal{C} \subset \mathbb{C}^n$ and containing all Toeplitz operators T_f with a bounded symbol $f \in L^\infty(\mathbb{C}^n)$ sufficiently smooth in $\mathcal{C}$ with bounded derivatives.

Let $n \in \mathbb{N}$ be fixed and write $\|\cdot\|$ for the Euclidean norm on $\mathbb{C}^n$ (resp. on $\mathbb{R}^n$). Then we denote by μ the normed *Gaussian measure* on $\mathbb{C}^n$ with density $\pi^{-n}\exp(-\|\cdot\|^2)$ with respect to the usual Lebesgue measure v on $\mathbb{C}^n$. For the convenience of the reader we briefly recall the commutator methods introduced in chapter 1. Fix a finite set

$$\mathcal{V} \subset \Big\{ A : H_1 \supset \mathcal{D}(A) \longrightarrow H_1 : A \text{ is closed and densely defined} \Big\}$$

of closed operators on H_1. Let $A \in \mathcal{V}$ and $a \in \mathcal{L}(H_1)$ such that $a(\,\mathcal{D}(A)\,) \subset \mathcal{D}(A)$. Then the commutator $\mathrm{ad}[\,A\,](a) := A\,a - a\,A$ is well-defined on the dense subspace $\mathcal{D}(A)$ of H_1 and one sets:

- $\mathcal{I}(A) := \Big\{ a \in \mathcal{L}\,(\,H_1\,) : a\,(\,\mathcal{D}(A)\,) \subset \mathcal{D}(A) \Big\}.$

- $\mathcal{B}(A) := \Big\{ a \in \mathcal{I}(A) : \mathrm{ad}[\,A\,](a)$ extends to a bounded operator $\delta_A a \in \mathcal{L}\,(H_1)\Big\}$.

We have seen that $\Delta := \{\,\delta_A : \mathcal{L}(H_1) \supset \mathcal{B}(A) \to \mathcal{L}(H_1) : A \in \mathcal{V}\}$ is a finite set of closed derivations on $\mathcal{L}(H_1)$ (cf. Lemma 1.2.1). Now, we can introduce the following subalgebras of $\mathcal{L}(H_1)$:

- $\Psi_0^\Delta := \mathcal{L}\,(\,H_1\,)$.
- $\Psi_1^\Delta := \bigcap_{A\in\mathcal{V}} \mathcal{B}(A)$.
- $\Psi_k^\Delta := \Big\{ a \in \Psi_{n-1}^\Delta : \delta_A a \in \Psi_{n-1}^\Delta$ for all $A \in \mathcal{V}\Big\}, \quad k \geq 2$.
- $\Psi_\infty^\Delta := \bigcap_{k\in\mathbb{N}_0} \Psi_k^\Delta$.

It was shown in Definition 1.1.1 how to construct a norm q_k on Ψ_k^Δ such that all the spaces $(\,\Psi_k^\Delta,\, q_k\,)$ turn into Banach algebras and $(\,\Psi_\infty^\Delta,\, (q_k)_k\,)$ becomes a sub-multiplicative Fréchet- operator algebra (resp. a Ψ_0- or Ψ^*-algebra). In order to apply the general theory in the setting of Toeplitz operators on H_2 we always assume that the space $\mathcal{C}_c^\infty(\mathbb{C}^n)$ of smooth functions with compact support is dense in $\mathcal{D}(Z)$ with respect to the graph norm for all closed operators $Z \in \mathcal{V}$. Hence we choose $\mathcal{D}(Z)$ to be the graph norm closure of the test functions $\mathcal{C}_c^\infty(\mathbb{C}^n)$ in H_1. In our specific situation $\mathcal{V}$ consists of closed extensions of operators A, A^*, $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ where A is in a class of smooth vector fields $\mathcal{X}^\Phi(\mathbb{C}^n)$ on $\mathbb{C}^n$ which we will describe next.

We write B_{2n} for the Euclidean ball in $\mathbb{C}^n$ with radius one centered in 0 and by ∂B_{2n} we denote its boundary. With $\dot{\mathbb{C}}^n := \mathbb{C}^n \setminus \{0\}$ let us consider the space of all radial smooth functions $\mathcal{R}(\dot{\mathbb{C}}^n)$ defined by:

$$\mathcal{R}\left(\dot{\mathbb{C}}^n\right) := \Big\{ h : \dot{\mathbb{C}}^n \to \mathbb{C} : h(z) := f\left(z \cdot \|\,z\,\|^{-1}\right) \text{ where } f \in \mathcal{C}^\infty\,(\,\partial B_{2n}\,)\Big\}.$$

Choose $\Phi \in \mathcal{C}^\infty(\,\mathbb{C}^n, [\,0,1\,]\,)$ with $\Phi \equiv 0$ on $\frac{1}{2}B_{2n}$ and $\Phi \equiv 1$ on $\mathbb{C}^n \setminus B_{2n}$. Then we can define a space of smooth functions on $\mathbb{C}^n$ which are radial symmetric outside B_{2n} and vanishing in 0 by:

$$\mathcal{R}^\Phi\,(\,\mathbb{C}^n\,) := \Big\{ f : \mathbb{C}^n \to \mathbb{C} : f = h \cdot \Phi \text{ on } \dot{\mathbb{C}}^n \text{ where } h \in \mathcal{R}\big(\,\dot{\mathbb{C}}^n\,\big) \text{ and } f(\,0\,) = 0\Big\}.$$

For $k \in \mathbb{N}_0 \cup \{\infty\}$ let $\mathcal{C}_b^k(\mathbb{C}^n)$ denote the space of functions $f \in \mathcal{C}^k(\mathbb{C}^n)$ such that all derivatives of f up to the order k are bounded on $\mathbb{C}^n$. Let ∂_i (resp. $\bar{\partial}_i$) be the derivatives with respect to the complex coordinates z_i (resp. $\bar{z}_i$) and $(\,i = 1, \cdots, n\,)$, then we consider the subspace

$$\mathcal{X}^\Phi\big(\,\mathbb{C}^n\big) := \mathrm{span}\Big\{\, a_j\,\partial_j : a_j \in \mathcal{R}^\Phi\,(\,\mathbb{C}^n\,)\,\Big\} \oplus \mathrm{span}\Big\{\, b_j\,\bar{\partial}_j : b_j \in \mathcal{C}_b^\infty\,(\,\mathbb{C}^n\,)\Big\}.$$

in the space of all smooth vector fields on $\mathbb{C}^n$. Note that in particular $\mathcal{X}^\Phi(\mathbb{C}^n)$ contains all the vector fields $\Phi \cdot \partial_{\varphi_j}$ where

$$\partial_{\varphi_j} := \|\,z\,\|^{-1}\left[\,z_j\,\bar{\partial}_j - \bar{z}_j\,\partial_j\,\right], \qquad (\,j = 1, \cdots, n\,)$$

are the normed derivatives tangential to ∂B_{2n}. Our main results are connected to the smoothness of the *Segal-Bargmann* Toeplitz projection and the localization of operator algebras in cones. Let us give the basic notations.

We write P for the orthogonal projection (*Toeplitz projection*) from H_1 onto H_2. With any essentially bounded symbol $f \in L^\infty(\mathbb{C}^n)$ we denote by $M_f \in \mathcal{L}(H_1)$ the multiplication by f and we define the bounded Toeplitz operator $T_f := PM_f$ as an element of $\mathcal{L}(H_1)$ (resp. of $\mathcal{L}(H_2)$). Let $\mathcal{M}$ be the set

$$\mathcal{M} := \Big\{ Z, Z^*, \mathrm{Re}(Z), \mathrm{Im}(Z) : Z \in \mathcal{X}^\Phi(\mathbb{C}^n) \Big\}$$

where each element of $\mathcal{M}$ is considered as a closed operator on H_1 in the described way. With our notations above there are associated operator algebras Ψ_k^Δ for $k \in \mathbb{N} \cup \{\infty\}$ and we show (see Theorem 2.5.1):

Theorem 1 *Let $\mathcal{V}$ be any finite subset of $\mathcal{M}$ and Δ the associated space of closed derivations on $\mathcal{L}(H_1)$. Then $P \in \Psi_\infty^\Delta$.*

The Segal-Bargmann space H_2 is not invariant under the elements of $\mathcal{M}$, but by an application of Lemma 1.4.2 it follows that for $A \in \mathcal{M}$ the operators

$$PA : H_2 \supset \mathcal{D}(PA) := P\big[\mathcal{D}(A)\big] \longrightarrow H_2$$

are closed and so the same construction using $\mathcal{PM} := \{ PA : A \in \mathcal{M} \}$ instead of $\mathcal{M}$ and a finite subset $\mathcal{PV} \subset \mathcal{PM}$ leads to algebras in $\mathcal{L}(H_2)$. Let Δ_P be the associated space of closed derivations on $\mathcal{L}(H_2)$, then we show that each Toeplitz operator T_f with symbol $f \in \mathcal{C}_b^k(\mathbb{C}^n)$ where $k \in \mathbb{N} \cup \{\infty\}$ is contained in $\Psi_k^{\Delta_P}$ (see Theorem 2.5.2). Now, additionally we assume that all the operators

$$A = \sum_{j=1}^n \big[a_j \,\partial_j + b_j \,\bar{\partial}_j \big] + M_g \in \mathcal{V}$$

are supported in a cone $\mathcal{C} \subset \mathbb{C}^n$. By this we mean that the coefficients a_j, b_j and g are supported in $\mathcal{C}$ for $j = 1, \cdots, n$. Then the algebras Ψ_k^Δ and $\Psi_k^{\Delta_P}$, where $k \in \mathbb{N} \cup \{\infty\}$ are *localized in* $\mathcal{C}$ in the following sense (see Theorem 2.5.3):

Theorem 2 *Let $h \in L^\infty(\mathbb{C}^n)$ such that supp $(h) \subset \mathbb{C}^n \setminus \mathcal{C}$. Then $T_h \in \Psi_\infty^\Delta \subset \mathcal{L}(H_1)$ (resp. we have $T_h \in \Psi_\infty^{\Delta_P} \subset \mathcal{L}(H_2)$).*

In other words the algebras Ψ_∞^Δ and $\Psi_\infty^{\Delta_P}$ are invariant under perturbations by Toeplitz operators with symbols supported in $\mathbb{C}^n \setminus \mathcal{C}$. The proof of Theorem 2.5.3 essentially is based on the fact that the algebras $\Psi_j^{\Delta_P}$ and Ψ_j^Δ are related in the form $\Psi_j^{\Delta_P} = \{ PAP : A \in \Psi_j^\Delta \}$ which was proved in greater generality in Proposition 1.4.2 of chapter 1.

2.1 Toeplitz operators on the Segal-Bargmann space

For any $c > 0$ let us denote by μ_c the normed Gaussian measure on $\mathbb{C}^n$ with respect to the density

$$d\mu_c = c^n \pi^{-n} \exp\big(- c \,\|\cdot\|^2 \big)\, dv$$

where v denotes the usual Lebesgue measure on $\mathbb{C}^n$. In the present chapter we often will write $\mu := \mu_1$ and $\langle\cdot,\cdot\rangle_2$ (resp. $\|\cdot\|_2$) for the $L^2(\mathbb{C}^n,\mu)$-inner-product (resp. norm). An important tool in our analysis to show the L^p-boundedness of integral operators is the so called *Schur test* (see [84]) which we prove next for general measure spaces:

Theorem 2.1.1 *Let (X,ν) be a measure space and K a measurable function on $X\times X$. Suppose there are positive measurable functions p and q on X and positive numbers α and β such that*

$$\int_X |\,K(\,x,\cdot)\,|\,q\,d\nu \le \alpha\cdot p(\,x\,) \qquad \textit{for } [\nu]-a.e.\ x\in X, \tag{2.1.1}$$

$$\int_X |\,K(\cdot,y\,)\,|\,p\,d\nu \le \beta\cdot q(\,y\,) \qquad \textit{for } [\nu]-a.e.\ y\in X. \tag{2.1.2}$$

Then the integral operator A given by

$$[\,Af\,](\,x\,) = \int_X K(\,x,\cdot)\,f\,d\nu$$

with $f\in L^2(X,\nu)$and $x\in X$ defines a bounded linear operator from $L^2(X,\nu)$ into itself. Moreover, the norm of A can be estimated by $\|\,A\,\|^2\le\alpha\cdot\beta$.

Proof Fix a function $f\in L^2(X,\nu)$. If we apply the *Cauchy-Schwartz* inequality to the integral

$$|\,Af(\,x\,)\,| \le \int_X q^{-\frac{1}{2}}\,q^{\frac{1}{2}}\,|\,f\,|\,|\,K(\,x,\cdot)\,|\,d\nu$$

it follows that:

$$|\,Af(\,x\,)\,| \le \left[\int_X q\,|\,K(\,x,\cdot)\,|\,d\nu\right]^{\frac{1}{2}}\cdot\left[\int_X q^{-1}\,|\,f\,|^2\,|\,K(\,x,\cdot)\,|\,d\nu\right]^{\frac{1}{2}}.$$

From our assumption (2.1.1) we now obtain:

$$|\,Af(\,x\,)\,|^2 \le \alpha\cdot p(x)\cdot\int_X q^{-1}\,|\,f\,|^2\,|\,K(\,x,\cdot)\,|\,d\nu. \tag{2.1.3}$$

Finally, using *Fubini's Theorem* and (2.1.2) it follows by integrating (2.1.3) over X:

$$\begin{aligned}\int_X |\,Af\,|^2\,d\nu &\le \alpha\cdot\int_X q^{-1}(\,y\,)\cdot|\,f(\,y\,)\,|^2\int_X p(\,x\,)\,|\,K(\,x,y\,)\,|\,d\nu(\,x\,)\,d\nu(\,y\,)\\ &\le \beta\cdot\alpha\cdot\int_X |\,f\,|^2\,d\nu.\end{aligned}$$

Thus A is bounded on $L^2(X,\nu)$ with $\|\,A\,\|^2\le\alpha\cdot\beta$. □

Remark 2.1.1 Let h be a positive function and $\alpha > 0$. With $1 < r < \infty$ and $s > 0$ such that $s^{-1} + r^{-1} = 1$ holds we replace the conditions (2.1.1) and (2.1.2) by:

$$\int_X |\, K(\, x, \cdot)\,|\, h^r\, d\nu \leq \alpha \cdot h(\, x\,)^r \qquad \text{for } [\nu] - a.e.\ x \in X,$$
$$\int_X |\, K(\cdot, y\,)\,|\, h^s\, d\nu \leq \alpha \cdot h(\, y\,)^s \qquad \text{for } [\nu] - a.e.\ y \in X,$$

then by similar arguments it can be shown that A is bounded on $L^r(X, \nu)$ and for its operator norm we obtain $\|\, A\,\|_{\mathcal{L}(L^r(X,\nu))} \leq \alpha$.

The following result is an immediate consequence of the Schur test.

Proposition 2.1.1 *Let $L : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be a measurable function such that:*

$$|\, L(\, x, y\,)\,| \leq |\, F(\, x - y\,)\,| \exp\Big(Re \,\langle\, x, y\,\rangle \Big)$$

where $F \in L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})$. Then the integral operator A on $L^2(\mathbb{C}^n, \mu)$ defined by

$$\big[\, Af\,\big](\, z\,) := \int_{\mathbb{C}^n} L(\, z, \cdot)\, f\, d\mu, \qquad f \in L^2(\mathbb{C}^n, \mu)$$

with $z \in \mathbb{C}^n$ is bounded on $L^2(\mathbb{C}^n, \mu)$. Moreover, we have $\|\, A\,\| \leq 2^n \|\, F\,\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}$.

Proof In Theorem 2.1.1 we choose the positive function $p = q = \exp(\, \frac{1}{2} \|\cdot\|^2\,)$ on $\mathbb{C}^n$. Then it follows that:

$$\begin{aligned}
\int_{\mathbb{C}^n} |\, L(\,\cdot, y\,)\,|\, p\, d\mu &\leq \frac{1}{\pi^n} \int_{\mathbb{C}^n} |\, F(\,\cdot - y\,)\,|\, \exp\Big(\, \mathrm{Re}\,\langle\,\cdot, y\,\rangle - \frac{1}{2}\,\|\cdot\|^2\,\Big)\, dv \\
&= \frac{1}{\pi^n} \int_{\mathbb{C}^n} |\, F\,|\, \exp\Big(\, \mathrm{Re}\,\langle\,\cdot + y, y\,\rangle - \frac{1}{2} \|\cdot + y\,\|^2\,\Big)\, dv \\
&= 2^n\, p\,(y)\, \|\, F\,\|_{L^1(\,\mathbb{C}^n, \mu_{\frac{1}{2}}\,)}.
\end{aligned}$$

Similar we get $\int |\, L(\, x, \cdot\,)\,|\, p\, d\mu \leq 2^n\, p\,(x)\, \|\, F\,\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}$. Now, applying the Schur test (see Theorem 2.1.1), we obtain the desired result. □

Let us describe next the *Segal-Bargmann space* $H^2(\mathbb{C}^n, \mu)$, which is the closed subspace of $L^2(\mathbb{C}^n, \mu)$ consisting of all holomorphic and μ-square integrable functions. Moreover, we introduce the notion of *Weyl-operators* acting unitarily on both $L^2(\mathbb{C}^n, \mu)$ and $H^2(\mathbb{C}^n, \mu)$.

For $j := (\, j_1, \cdots, j_n\,) \in \mathbb{N}_0^n$ and $z \in \mathbb{C}^n$ define the monomials $m_j(z) = z_1^{j_1} \cdots z_n^{j_n}$. We sometimes also write $m_j = z^j$ and $j! := j_1! \cdots j_n!$. Then the system

$$\Big[\, e_j := (j!)^{-\frac{1}{2}} m_j : j \in \mathbb{N}_0^n\,\Big] \tag{2.1.4}$$

forms an orthonormal basis in $H^2(\mathbb{C}^n, \mu)$. Throughout this chapter $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product and $\| \cdot \|$ the Euclidean norm in $\mathbb{C}^n$. Because each *point evaluation* is a continuous functional on $H^2(\mathbb{C}^n, \mu)$ the *Segal-Bargmann space* is a Hilbert space with *reproducing kernel* K which is given by (cf. [82]):

$$K(\, z, w \,) := \exp\Big(\langle z, w \rangle \Big) \qquad\qquad z, w \in \mathbb{C}^n. \tag{2.1.5}$$

We also use the *normalized kernel function* defined by

$$k_w(\, z \,) := K(\, z, w \,) \cdot \| \, K(\cdot, w \,) \, \|_2^{-1} = \exp\Big(\langle z, w \rangle - \frac{1}{2} \, \| \, w \, \|^2 \Big) \tag{2.1.6}$$

for all $z, w \in \mathbb{C}^n$. It is well-known (and not hard to prove) that the reproducing kernel K is uniquely defined by the following properties:

(a) As a function of the first variable we have $K(\cdot, \zeta \,) \in H^2(\mathbb{C}^n, \mu)$ for all $\zeta \in \mathbb{C}^n$,

(b) K is conjugate-symmetric: $K(\, z, \zeta \,) = \overline{K(\, \zeta, z \,)}$ for all $(\, z, \zeta \,) \in \mathbb{C}^n \times \mathbb{C}^n$,

(c) *Reproducing property:* For all $f \in H^2(\mathbb{C}^n, \mu)$ and for all $z \in \Omega$:

$$f(z) = \int_\Omega f \; \cdot K(\, z, \cdot) \, d\mu. \tag{2.1.7}$$

An application of the Cauchy-Schwartz inequality to equality (2.1.6) leads to a growth condition for the functions in $H^2(\mathbb{C}^n, \mu)$. For $z \in \mathbb{C}^n$ a function f in the Segal-Bargmann space and with the point evaluation $\delta_z \in H^2(\mathbb{C}^n, \mu)'$ in $z \in \mathbb{C}^n$ defined by $\delta_z(f) = f(z)$ we have the equality $\| \, \delta_z \, \| = K(\, z, z \,)^{\frac{1}{2}}$ and so:

$$\delta_z(\, f \,) = | \, f(z) \, | \leq \| \, f \, \|_2 \cdot K(\, z, z \,)^{\frac{1}{2}} = \| \, f \, \|_2 \cdot \exp\Big(\frac{1}{2} \, \| \, z \, \|^2 \Big).$$

Let P denote the orthogonal projection from $L^2(\mathbb{C}^n, \mu)$ onto $H^2(\mathbb{C}^n, \mu)$. Then P can be written as an integral operator in the form (2.1.7). Moreover, for any bounded symbol in the space $f \in (\, L^\infty(\mathbb{C}^n), \| \cdot \|_\infty \,)$ we can define the *Toeplitz operator* T_f by:

$$T_f : H^2\big(\, \mathbb{C}^n, \mu \,\big) \to H^2\big(\, \mathbb{C}^n, \mu \,\big) : g \mapsto P\,(\, fg \,) = P \, M_f \, g. \tag{2.1.8}$$

where M_f is the multiplication by f. Then T_f is bounded with $\| \, T_f \, \| \leq \| \, f \, \|_\infty$. We also consider densely defined Toeplitz operators with unbounded symbols. Let us describe the details:

For $z, w \in \mathbb{C}^n$ let τ_z denote the *z-shift* on $\mathbb{C}^n$ given by $\tau_z(w) := z + w$. The Gaussian measure μ is not invariant under translations and we define the linear space:

$$\mathcal{T}\,(\, \mathbb{C}^n \,) := \Big\{ \, g \in L^2\,(\, \mathbb{C}^n, \mu \,) : g \circ \tau_x \in L^2\,(\, \mathbb{C}^n, \mu \,), \quad \forall \, x \in \mathbb{C}^n \, \Big\}. \tag{2.1.9}$$

It is easy to verify that a measurable function h on $\mathbb{C}^n$ belongs to $\mathcal{T}(\mathbb{C}^n)$ if and only if for every $x \in \mathbb{C}^n$:

$$\Big[\, \lambda \mapsto h(\,\lambda\,) \cdot K(\,\lambda, x\,) \,\Big] \in L^2\,(\,\mathbb{C}^n, \mu\,)\,.$$

Moreover, the linear span of the set of all *kernel functions* $\{\, K(\cdot, x\,) : x \in \mathbb{C}^n \,\}$ is dense in $H^2(\mathbb{C}^n, \mu)$. Hence the canonical domain of definition of the Toeplitz operator $T_h = PM_h$ given by

$$\mathcal{D}\big(\,T_h\,\big) := \Big\{\, g \in H^2\,(\,\mathbb{C}^n, \mu\,) : g \cdot h \in L^2\,(\,\mathbb{C}^n, \mu\,)\,\Big\} \tag{2.1.10}$$

is a dense, linear subspace of $H^2(\mathbb{C}^n, \mu)$ whenever the symbol h belongs to $\mathcal{T}(\mathbb{C}^n)$. In order to define finite products of Toeplitz operators T_h with symbols of polynomial growth we have to determine an invariant subspace of $H^2(\mathbb{C}^n, \mu)$. In the following we use the notations $H_1 := L^2(\mathbb{C}^n, \mu)$ and $H_2 := H^2(\mathbb{C}^n, \mu)$.

We inductively define a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_1 := \frac{1}{4}$ and $a_{n+1} := [\,4 \cdot (1 - a_n)\,]^{-1}$ for all numbers $n \geq 2$. It is an easy computation that $(a_n)_{n\in\mathbb{N}}$ has the following properties:

(a) $a_n < \frac{1}{2}, \quad \forall\, n \in \mathbb{N}$,

(b) $(a_n)_{n\in\mathbb{N}}$ is strictly increasing,

(c) $\lim_{n\to\infty} a_n = \frac{1}{2}$.

Now let us denote by $\mathbb{P}[\,\mathbb{C}^n\,]$ the linear space of all *polynomials* on $\mathbb{C}^n$ in the complex variables $z := (\,z_1, \cdots, z_n\,)$ and $\bar{z} := (\,\bar{z}_1, \cdots, \bar{z}_n\,)$. Furthermore, we write $\mathbb{P}_a[\,\mathbb{C}^n\,]$ for all *analytic polynomials* in z. We set

$$L_{\exp}\big(\,\mathbb{C}^n\,\big) := \Big\{\, f \in H_1 : \exists\, 0 < c < 2^{-1},\ \exists\, D > 0 \text{ with } |\,f(z)\,| \leq D\, \exp\big(\,c\,\|\,z\,\|^2\,\big) \text{ a.e.}\,\Big\}.$$

Then $L_{\exp}(\mathbb{C}^n)$ contains all polynomials $p \in \mathbb{P}[\,\mathbb{C}^n\,]$ which form a dense subspace of H_1. We define

$$H_{\exp}\big(\,\mathbb{C}^n\,\big) := \mathcal{H}\big(\,\mathbb{C}^n\,\big) \cap L_{\exp}\big(\,\mathbb{C}^n\,\big), \tag{2.1.11}$$

where $\mathcal{H}(\mathbb{C}^n)$ denotes the space of entire functions on $\mathbb{C}^n$. Similarly, the analytic polynomials $\mathbb{P}_a[\,\mathbb{C}^n\,]$ are contained in $H_{\exp}(\mathbb{C}^n)$ and both spaces are dense in H_2. With the Lebesgue measurable complex valued functions $M(\mathbb{C}^n)$ on $\mathbb{C}^n$ and $j \in \mathbb{N}$ we define:

(1) $\mathrm{Pol}_j(\,\mathbb{C}^n\,) := \Big\{\, f \in M(\,\mathbb{C}^n\,) : |\,f(z)\,|\,(\,1 + \|\,z\,\|^2\,)^{-\frac{j}{2}} \in L^\infty(\mathbb{C}^n)\,\Big\}$, the functions of order j at infinity,

(2) $\mathrm{Pol}(\,\mathbb{C}^n\,) := \bigcup_{j\in\mathbb{N}} \mathrm{Pol}_j\,(\,\mathbb{C}^n\,)$, all functions of finite order at infinity.

Proposition 2.1.2 *For any measurable symbol f in Pol$(\,\mathbb{C}^n\,)$ we have the inclusions:*

$$T_f\Big[\,H_{\exp}\,(\,\mathbb{C}^n\,)\,\Big] \subset H_{\exp}\,(\,\mathbb{C}^n\,) \subset \mathcal{D}\,(\,T_f\,)$$

where the domain of definition $\mathcal{D}(T_f)$ of the operator T_f was defined in (2.1.10).

Proof Because f has polynomial growth it is obvious that $H_{\exp}(\mathbb{C}^n) \subset \mathcal{D}(T_f)$. In order to prove the first inclusion fix $g \in H_{\exp}(\mathbb{C}^n)$. Then by definition we can choose numbers $0 < c < \frac{1}{2}$ and $D_1 > 0$ such that

$$|\, g(z)\,| \le D_1 \,\exp\big(c\,\|\,z\,\|^2\big) \qquad\qquad z \in \mathbb{C}^n.$$

By the properties (a), (b) and (c) and with the sequence $(a_n)_{n\in\mathbb{N}}$ defined above there is a number $n_0 \in \mathbb{N}$ such that $c < a_{n_0} < \frac{1}{2}$. Moreover, there is $D_2 > 0$ with

$$|\, f(z)\cdot g(z)\,| \le D_2 \,\exp\big(a_{n_0}\,\|\,z\,\|^2\big), \qquad\qquad \text{a.e. on } \mathbb{C}.$$

Using the transformation formula for the integral together with the reproducing property of K we obtain:

$$\begin{aligned}
\big|\, T_f g(z)\,\big| &\le \int_{\mathbb{C}^n} \big|\, f\, g\, \exp\big(\langle z,\cdot\rangle\big)\big|\, d\mu \\
&\le D_2\,\pi^{-n}\int_{\mathbb{C}^n} \exp\Big(\operatorname{Re}\langle\, z,\cdot\,\rangle - \big[\,1-a_{n_0}\,\big]\,\|\cdot\|^2\Big)\, dv \\
&= D_2\,\big(\,1-a_{n_0}\,\big)^{-n}\int_{\mathbb{C}^n} \exp\Big(\,2\operatorname{Re}\,\Big\langle\, 2^{-1} : \big(\,1-a_{n_0}\,\big)^{-\frac{1}{2}} z,\cdot\,\Big\rangle\,\Big)\, d\mu \\
&= D_3\,\exp\Big(\,\big[\,4\cdot(1-a_{n_0})\,\big]^{-1}\,\|\,z\,\|^2\Big) \\
&= D_3\,\exp\Big(\,a_{n_0+1}\,\|\,z\,\|^2\Big),
\end{aligned}$$

where $D_3 := D_2\cdot(\,1-a_{n_0}\,)^{-n}$. From (a) above we conclude that $T_f g \in H_{\exp}(\mathbb{C}^n)$. □

By Proposition 2.1.2 all finite products of Toeplitz operators with symbols in $\mathrm{Pol}(\mathbb{C}^n)$ are well-defined on the dense subspace $H_{\exp}(\mathbb{C}^n)$ of H_2. In general, these operators are unbounded, but in certain cases there are continuous extensions to H_2. We do not want to make any differences in the notations when we pass to these extensions.

Next we introduce the notion of *Weyl operators* on H_1 and H_2. They are weighted unitary shift operators which correspond to the translations in $L^2(\mathbb{R}^n, v)$.

Definition 2.1.1 For $x \in \mathbb{C}^n$ and $f \in L^2(\mathbb{C}^n,\mu)$ we define $W_x f := k_x \cdot f \circ \tau_{-x}$ where k_x is defined in (2.1.6). The operators $W_x \in \mathcal{L}(L^2(\mathbb{C}^n,\mu))$ are called *Weyl operators*.

All results on the Weyl operators in Lemma 2.1.1 below follow by straightforward computations and we omit the details.

Lemma 2.1.1 *With $x \in \mathbb{C}^n$ and a symbol $g \in \mathcal{T}(\mathbb{C}^n)$ the following relations between the operators W_x, P, M_g and T_g are valid:*

(1) W_x is an unitary operator with $W_x^* = W_{-x} = W_x^{-1}$.

(2) The commutator $[\,P, W_x\,] := P\,W_x - W_x\,P$ vanishes and $W_x(\,H_2\,) = H_2$.

(3) The composition rules $W_x^* \, M_g \, W_x = M_{g \circ \tau_x}$ *and* $W_x^* \, T_g \, W_x = T_{g\circ \tau_x}$.

(4) For $x, y \in \mathbb{C}^n$ *we have* $W_x \, W_y = \exp(-i Im \langle \, x, y \, \rangle \,) W_{x+y}$.

By another application of the Schur test we examine how to obtain norm estimates for operators on H_2. Let $S : H_2 \supset \mathcal{D}(S) \to H_2$ be densely defined such that with its adjoint operator S^* the inclusion holds:

$$\mathcal{M} := \operatorname{span}\Big\{ \, K(\cdot, \lambda) : \lambda \in \mathbb{C}^n \, \Big\} \subset \mathcal{D}(\, S \,) \cap \mathcal{D}\,(\, S^* \,)\,. \tag{2.1.12}$$

Then consider the measurable function $K_S : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by:

$$K_S(x, y) := \Big[\, S \, K(\cdot, y\,) \, \Big](\, x \,) = \overline{\Big[\, S^* K(\cdot, x\,) \, \Big](\, y \,)}, \qquad\qquad x, y \in \mathbb{C}^n.$$

By applying the reproducing property of K we obtain for $p \in \operatorname{span}\{ \, K(\cdot, \lambda \,) : \lambda \in \mathbb{C}^n \, \}$ and all $z \in \mathbb{C}^n$ the equation

$$\big[\, Sp \, \big](\, z \,) = \Big\langle \, Sp, K(\cdot, z\,) \, \Big\rangle_2 = \Big\langle \, p, S^* K(\cdot, z\,) \, \Big\rangle_2 = \int_{\mathbb{C}^n} p \, K_S(\, z, \cdot) \, d\mu.$$

Hence, S is an integral operator on a dense subspace of H_2 with kernel K_S. With these notations we prove:

Lemma 2.1.2 *With an operator* S *as above and the definition* $p := \exp(\frac{1}{2} \, \| \cdot \|^2 \,)$ *we have the equations*

$$\int_{\mathbb{C}^n} p \, | \, K_S(\, x, \cdot) \, | \, d\mu = 2^n \, p(\, x \,) \, \| \, S_x^* 1 \, \|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}, \tag{2.1.13}$$

$$\int_{\mathbb{C}^n} p \, | \, K_S(\cdot, y \,) \, | \, d\mu = 2^n \, p(\, y \,) \, \| \, S_y 1 \, \|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}. \tag{2.1.14}$$

Here $S_x^* := W_{-x} \, S^* \, W_x$ *and* $S_y := W_{-y} \, S \, W_y$. *If one of the integrals in (2.1.13) or (2.1.14) does not exist, then both sides of these equations will be* ∞.

Proof For all $x, y \in \mathbb{C}^n$ we have $K_S(\, x, y \,) = \overline{K_{S^*}(\, y, x\,)}$. Hence replacing S^* by S it is sufficient to prove (2.1.13).

$$\begin{aligned}
&\int_{\mathbb{C}^n} p\,(x) \, \Big| \, \big[\, S^* K(\cdot, x) \, \big](\, y \,) \, \Big| \, d\mu(\, x \,) \\
=& \exp\left(\frac{1}{2} \, \| \, x \, \|^2 \right) \int_{\mathbb{C}^n} | \, S^* W_x 1 \, | \, \exp\left(\frac{1}{2} \, \| \cdot \|^2 \right) \, d\mu \\
=& 2^n \, \exp\big(\, \| \, x \, \|^2 \, \big) \int_{\mathbb{C}^n} \Big| \, \big[\, W_{-x} S^* W_x 1 \, \big](\cdot - x \,) \, \Big| \, \Big| \, \exp(\, \langle \cdot - x, x \rangle \,) \, \Big| \, d\mu_{\frac{1}{2}} = (*).
\end{aligned}$$

After the shift $y \mapsto y + x$ we obtain:

$$\begin{aligned}(*) &= \frac{1}{\pi^n} \exp\Big(\frac{1}{2} \parallel x \parallel^2 \Big) \int_{\mathbb{C}^n} \big|\, W_{-x} S^* W_x 1 \,\big| \, \exp\Big(-\frac{1}{2} \parallel \cdot \parallel^2 \Big) \, dv \\ &= p\,(\,x\,)\, \frac{1}{\pi^n} \int_{\mathbb{C}^n} \big|\, S_x^* 1 \,\big| \, \exp\Big(-\frac{1}{2} \parallel \cdot \parallel^2 \Big) \, dv.\end{aligned}$$

From this equation (2.1.13) follows. Moreover, the integral on the left hand side of (2.1.13) exists if and only if the integral on the right hand side exists. □

Now, we want to apply the *Schur test* (see Theorem 2.1.1) to the integral operator with kernel K_S. We define $C(S) \in \mathbb{R} \cup \{\infty\}$ by

$$C(S) := 2^n \sup \Big\{ \parallel S_\lambda 1 \parallel_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})} : \lambda \in \mathbb{C}^n \Big\}. \tag{2.1.15}$$

The following result is a straightforward application of Theorem 2.1.1 and Lemma 2.1.2

Theorem 2.1.2 *Let $S : H_2 \supset \mathcal{D}(S) \to H_2$ be densely defined and linear with adjoint operator $S^* : H_2 \supset \mathcal{D}(S^*) \to H_2$ such that*

(i) it holds $\{\, K(\cdot, \lambda\,) : \lambda \in \mathbb{C}^n \,\} \subset \mathcal{D}(S) \cap \mathcal{D}(S^*)$ and

(ii) the numbers $C(S)$ and $C(S^*)$ defined in (2.1.15) are finite.

Then S has a continuous extension to an integral operator on H_1 which again is denoted by S with $\parallel S \parallel \leq \{\, C(S) \cdot C(S^*) \,\}^{\frac{1}{2}}$.

Let us describe the notion of *Berezin transform* for a (possibly unbounded) operator A on H_2. We want to mention that many of the concepts below have a meaning for Bergman spaces in general and they lead to many interesting questions concerning the operator theory.

Assume that $\mathcal{M} \subset \mathcal{D}(A)$ where $\mathcal{M}$ was defined in (2.1.12). Formally, we associate a *symbol* $\tilde{A}$ to A by:

$$\tilde{A} : \mathbb{C}^n \to \mathbb{C} : \tilde{A}(\lambda) := \Big\langle A k_\lambda, k_\lambda \Big\rangle_{L^2(\mathbb{C}^n, \mu)}. \tag{2.1.16}$$

Recall that for $\lambda \in \mathbb{C}^n$ the function k_λ is the *normalized Bergman kernel* of H_2 introduced in (2.1.6). The complex valued function $\tilde{A}$ is called *Berezin transform* of the operator A. It can be shown that for any $A \in \mathcal{L}(H_2)$ the Berezin transform $\tilde{A}$ is bounded and leads to the Toeplitz operator:

$$T_{\tilde{A}} = \int_{\mathbb{C}^n} W_{-t}\, A\, W_t \, d\mu(t) = \int_{\mathbb{C}^n} A_t \, d\mu(t)$$

under a process of integration in the sense of Bochner (cf. [22]). Hence $T_{\tilde{A}}$ is some kind of *average* of A. Without proofs we give two basic properties of the Berezin transform:

(I): If A is bounded, then $\| \tilde{A} \|_\infty \leq \| A \|$. In the case where A is compact it follows that:

$$\lim_{\lambda \to \infty} \tilde{A}(\lambda) = 0 \tag{2.1.17}$$

but in general (2.1.17) is not sufficient for A to be compact. Moreover, the operator A vanishes if and only if $\tilde{A} = 0$.

(II): For any symbol $f \in \mathcal{T}(\mathbb{C}^n)$ and $A = T_f$ it can be shown that $\mathcal{M} \subset \mathcal{D}(A)$ and the Berezin transform $\tilde{A}$ of a Toeplitz operator is given by:

$$\mathrm{Ber}(f) := \tilde{f}(\lambda) := \widetilde{T_f}(\lambda) = \int_{\mathbb{C}^n} f(\cdot + \lambda)\, d\mu.$$

In case of a bounded symbol $f \in L^\infty(\mathbb{C}^n)$ and for all $j \in \mathbb{N}$ and by [22] the iterated Berezin transform $Ber^j(f)$ is the solution of the *heat equation* at the time $t = \frac{1}{4}j$:

$$\left[\mathrm{Ber}^j(f) \right](x) = \frac{1}{(\pi j)^n} \int_{\mathbb{C}^n} f \,\exp\left(- \frac{\| x - \cdot \|^2}{j} \right) dv$$

Hence $\tilde{f}$ is a smooth bounded function on $\mathbb{C}^n$. For holomorphic f in $\mathcal{T}(\mathbb{C}^n)$ it yields $\widetilde{T_f} = f$. In general, the operator *Ber* is "smoothing".

Remark 2.1.2 There is an easy connection between the function $S_x 1$ which appears in Lemma 2.1.2 for $x \in \mathbb{C}^n$ and the Berezin transform of some operator S. Note that it holds $W_x 1 = k_x$ and so it follows that:

$$\Big[S_x 1 \Big](0) = \Big[W_{-x}\, S\, W_x 1 \Big](0) = \Big\langle W_{-x}\, S\, k_x, 1 \Big\rangle_2 = \Big\langle S k_x, W_x 1 \Big\rangle_2 = \tilde{S}(x).$$

Heuristically, this equation shows that $\tilde{S}$ only contains little information on the complex valued function $S_x 1$ which led to norm estimates on the operator S (see Theorem 2.1.2). This may count as a hint on the fact that boundedness of $\tilde{S}$ is not sufficient for the boundedness of the operator S. Nevertheless, the Berezin transform and the closely related notion of *mean oscillation* (cf. the appendix) of a symbol plays an important role for norm-estimates on operators (especially for Hankel operators). Some of these results can be found in the appendix and [55], [56].

In order to estimate the numbers $C(S)$ and $C(S^*)$ in Theorem 2.1.2 where S is a finite product of Toeplitz operators with symbols in $\mathrm{Pol}(\mathbb{C}^n)$ we need the following two lemma:

Lemma 2.1.3 *For $s \in [\, 4^{-1}, 2^{-1}\,]$ the Toeplitz projection P is bounded from $L^1(\mathbb{C}^n, \mu_{\varphi(s)})$ to $L^1(\mathbb{C}^n, \mu_s)$ where*

$$\varphi(s) := 1 - \frac{1}{4s} \qquad \text{and} \qquad \mu_0 := v.$$

In particular, it follows that P can be considered as an operator in $\mathcal{L}(L^1(\mathbb{C}^n, \mu_{\frac{1}{2}}))$ resp. from $L^1(\mathbb{C}^n, v)$ to $L^1(\mathbb{C}^n, \mu_{\frac{1}{4}})$.

Proof Let $s \in [\,4^{-1}, 2^{-1}\,]$ and fix a function $f \in L^1(\mathbb{C}^n, \mu_{\varphi(s)})$. Then we conclude that

$$\begin{aligned}\|\, Pf \,\|_{L^1(\mathbb{C}^n,\mu_s)} &\leq \frac{s^n}{\pi^n} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |\, f(\,u\,)\,| \exp\Big(\operatorname{Re}\langle z, u\rangle \Big)\, d\mu(u) \, \exp\big(-s \,\|\, z\,\|^2\,\big)\, dv(z)\\ &= \frac{s^n}{\pi^n} \int_{\mathbb{C}^n} |\, f(u)\,| \int_{\mathbb{C}^n} \exp\Big(\operatorname{Re}\langle z, u\rangle - s\,\|\, z\,\|^2 \Big)\, dv(z)\, d\mu(u)\\ &= \int_{\mathbb{C}^n} |\, f(u)\,| \int_{\mathbb{C}^n} \exp\Big(2 \operatorname{Re}\Big\langle z, \frac{1}{2}\, s^{-\frac{1}{2}} u \Big\rangle \Big)\, d\mu(z)\, d\mu(u).\end{aligned}$$

By the reproducing property of $\exp(\,\langle\cdot,\cdot\rangle\,)$ we obtain $\exp(\,[4s]^{-1}\,\|\,u\,\|^2\,)$ for the inner integral. Hence

$$\begin{aligned}\|\, Pf \,\|_{L^1(\mathbb{C}^n,\mu_s)} &\leq \pi^{-n} \int_{\mathbb{C}^n} |\, f\,| \exp\Big(-\Big[1 - \frac{1}{4s}\Big] \,\|\cdot\|^2 \Big)\, dv\\ &= \Big[1 - \frac{1}{4s}\Big]^{-n} \|\, f\,\|_{L^1(\mathbb{C}^n,\mu_{\varphi(s)})}.\end{aligned}$$

In particular, $P \in \mathcal{L}(\,L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})\,)$ follows from $\varphi(\frac{1}{2}) = \frac{1}{2}$ and the second assertion can be obtained for $s = \frac{1}{4}$. □

For $s \in (\,0, 2^{-1}\,]$ we conclude $s \geq \varphi(s)$ from

$$s\big(\,s - \varphi(\,s\,)\,\big) = \big(\,s - 2^{-1}\,\big)^2 \geq 0.$$

Consider the sequence $(\,x_n(s)\,)_{n\in\mathbb{N}} \subset \mathbb{R}$ recursively defined by

(i) $x_1(\,s\,) := s$,

(ii) $x_{n+1}(\,s\,) := \varphi\,(\,x_n(s)\,)$ for $x_n(\,s\,) > 0$.

In case $x_n(s) \leq 0$ let $x_{n+1}(\,s\,) := x_n(\,s\,)$. Then $(\,x_n(s)\,)_{n\in\mathbb{N}}$ is monotonely decreasing and bounded from top by $\frac{1}{2}$. Moreover, $x_m(\frac{1}{2}) = \frac{1}{2}$ for all $m \in \mathbb{N}$. Obviously all $x_m(s)$ continuously depend on s for s in a small neighborhood of $\frac{1}{2}$. Hence for each $c \in (\,0, \frac{1}{2}\,)$ and fixed $m \in \mathbb{N}$ we can choose $\varepsilon(\,m, c\,) > 0$ such that

$$c \leq x_j(\,s\,) \leq \frac{1}{2}, \qquad j \in \{\,1, \cdots, m\,\} \qquad \text{for} \qquad \frac{1}{2} - \varepsilon(\,m, c\,) \leq s \leq \frac{1}{2}.$$

Lemma 2.1.4 *Let $p \in \mathbb{P}[\,\mathbb{C}^n\,]$ a polynomial and assume that $0 \leq c_1 < c_2 \leq 2^{-1}$. Then the multiplication $M_p : L^1(\mathbb{C}^n, \mu_{c_1}) \to L^1(\mathbb{C}^n, \mu_{c_2})$ is continuous.*

Proof Choose a number $C_p > 0$ such that

$$|\, p\,(z)\,| \leq C_p \exp\big(\,[\,c_2 - c_1\,]\,\|\, z\,\|^2\,\big)$$

for all $z \in \mathbb{C}^n$. Then we have with $f \in L^1(\mathbb{C}^n, \mu_{c_1})$:

$$\begin{aligned}\|\, M_p f\,\|_{L^1(\mathbb{C}^n,\mu_{c_2})} &\leq \frac{c_2^n}{\pi^n} \int_{\mathbb{C}^n} |\, p\,| \exp\Big([c_1 - c_2]\,\|\cdot\|^2 \Big)\, |\, f\,| \exp\big(-c_1\,\|\cdot\|^2\,\big)\, dv\\ &\leq C \cdot \|\, f\,\|_{L^1(\mathbb{C}^n,\mu_{c_1})}.\end{aligned}$$

where $C > 0$ is the positive number defined by $C := C_p\,(\,c_1^{-1} \cdot c_2\,)^n$. □

Corollary 2.1.1 *Fix $m \in \mathbb{N}$ and let $p_1, \cdots, p_m \in \mathbb{P}[\,\mathbb{C}^n\,]$. Then for each $c \in (\,0, \frac{1}{2}\,)$ there exists $C > 0$ such that:*

$$\|\, T_{p_1} \cdots T_{p_m} f \,\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})} \leq C \cdot \|\, f \,\|_{L^1(\mathbb{C}^n, \mu_c)}$$

for all functions $f \in L^1(\mathbb{C}^n, \mu_c)$. Here the constant C is independent of f.

Proof With the notations above we can choose $\varepsilon(\,c, m\,) > 0$ and $s \in [\,2^{-1} - \varepsilon(\,c, m\,), 2^{-1}\,]$ such that $c \leq x_{j+1}(\,s\,) < x_j(\,s\,) < \frac{1}{2}$ for $j = 1, \cdots, 2m-1$. By Lemma 2.1.3 and Lemma 2.1.4 the operators

$$M_{p_l}, P : L^1\left(\,\mathbb{C}^n, \mu_{x_{j+1}(s)}\,\right) \longrightarrow L^1\left(\,\mathbb{C}^n, \mu_{x_j(s)}\,\right), \qquad (\,l = 1, \cdots, m\,)$$

are continuous for $j = 1, \cdots 2m-1$. Thus there are positive numbers C and $\tilde{C} > 0$ with

$$\|\, T_{p_1} \cdots T_{p_m} f \,\|_{L^1(\mathbb{C}^n,\, \mu_{\frac{1}{2}})} \leq \tilde{C} \cdot \|\, f \,\|_{L^1(\mathbb{C}^n,\, \mu_{x_{2m}(s)})} \leq C \cdot \|\, f \,\|_{L^1(\mathbb{C}^n,\, \mu_c)}.$$

This now proves Corollary 2.1.1. □

2.2 Commutators of P with linear vector fields

Denote by $C_c^\infty(\mathbb{C}^n)$ the space of smooth complex-valued functions on $\mathbb{C}^n$ with compact support. Let us consider P, $\partial_i := \frac{\partial}{\partial z_i}$ and $\overline{\partial}_i := \frac{\overline{\partial}}{\partial \overline{z}_i}$ as operators on $C_c^\infty(\mathbb{C}^n)$. With any multi-index $\alpha \in \mathbb{N}_0^n$ we write $\partial^\alpha := \partial_n^{\alpha_n} \cdots \partial_1^{\alpha_1}$ and $\bar{\partial}^\alpha := \bar{\partial}_n^{\alpha_n} \cdots \bar{\partial}_1^{\alpha_1}$. As we have seen before the Toeplitz projection P is an integral operator on $L^2(\mathbb{C}^n, \mu)$ given by:

$$P : H_1 \to H_2 : h \mapsto \int_{\mathbb{C}^n} h(u)\, \exp\left(\, \langle \cdot, u \rangle \,\right) d\mu(u). \tag{2.2.1}$$

Applying the *Schur test* we now can examine the boundedness of commutators of P with the derivatives ∂_i and $\bar{\partial}_i$:

Lemma 2.2.1 *With $0 \neq \alpha \in \mathbb{N}_0^n$ the following commutator relations between P, ∂^α and $\bar{\partial}^\alpha$ hold:*

(a) $[\,P, \partial^\alpha\,] := P\,\partial^\alpha - \partial^\alpha\, P = 0.$

(b) The commutator $[\,P, \bar{\partial}^\alpha\,] = P\bar{\partial}^\alpha$ has a bounded extensions to $L^2(\,\mathbb{C}^n, \mu\,)$.

Proof Fix $x \in \mathbb{C}^n$, a multi-index $0 \neq \alpha \in \mathbb{N}_0^n$ and $f \in C_c^\infty(\mathbb{C}^n)$. Then with the Segal-Bargmann kernel $K : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ given in (2.1.5) we have:

$$\begin{aligned} \partial^\alpha P f(\,x\,) &= \partial^\alpha \left\langle\, f, K(\cdot, x\,) \,\right\rangle_2 \qquad (2.2.2) \\ &= \int_{\mathbb{C}^n} f(\,u\,)\, \overline{u^\alpha K(\,u, x\,)}\, d\mu(\,u\,) = \left\langle\, \overline{u^\alpha}\, f, K(\cdot, x\,) \,\right\rangle_2. \end{aligned}$$

Now, integrating by parts together with (2.2.2) leads to:

$$\begin{aligned} P\partial^\alpha f(x) &= \Big\langle \partial^\alpha f, K(\cdot, x) \Big\rangle_2 \\ &= \frac{(-1)^{|\alpha|}}{\pi^n} \int_{\mathbb{C}^n} f(u) \frac{\partial^\alpha}{\partial u^\alpha} \exp\Big(\langle x, u\rangle - \| u \|^2 \Big)\, dv(u) \\ &= \Big\langle \overline{u^\alpha}\, f, K(\cdot, x) \Big\rangle_2 \\ &= \partial^\alpha P f(x). \end{aligned}$$

This proves (a). In order to show (b) we integrate by parts again:

$$\begin{aligned} P\bar{\partial}^\alpha f(x) &= \Big\langle \bar{\partial}^\alpha f, K(\cdot, x) \Big\rangle_2 \\ &= \frac{(-1)^{|\alpha|}}{\pi^n} \int_{\mathbb{C}^n} f(u) \frac{\bar{\partial}^\alpha}{\partial \bar{u}^\alpha} \exp\Big(\langle x, u\rangle - \| u \|^2 \Big)\, dv(u) \\ &= \int_{\mathbb{C}^n} f(u) \big[u - x \big]^\alpha K(x, u)\, d\mu(u). \end{aligned}$$

Because $\bar{\partial}^\alpha P = 0$ on $C_c^\infty(\mathbb{C}^n)$ for any $\alpha \neq 0$ it follows $[P, \overline{\partial}^\alpha] = P\,\overline{\partial}^\alpha$. Now we conclude from Proposition 2.1.1 with $F := \| \cdot \|^{|\alpha|} \in L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})$ that the commutators $[\, P, \overline{\partial}^\alpha\,]$ can be extended to continuous operators on H_1. □

Consider the space $\mathcal{X}_{\text{lin}}(\mathbb{C}^n) := \text{span}\{\, \partial_j, \bar{\partial}_j : j = 1, \cdots, n \,\}$ of all vector fields with constant coefficients on $\mathbb{C}^n$. Using the Lemma above we now can compute the iterated commutators of P with a finite system $\mathcal{Z}$ of vector fields in $\mathcal{X}_{\text{lin}}(\mathbb{C}^n)$. For the notations below see Definition 1.2.2.

Proposition 2.2.1 *Let $\mathcal{Z}$ be a finite system in $\mathcal{X}_{lin}(\mathbb{C}^n)$ of length r. Then we have*

$$ad\,[\, \mathcal{Z}\,]\,(P) \in span\Big\{\, P\bar{\partial}^\alpha :\ \alpha \in \mathbb{N}_0^n \ and\ |\,\alpha\,| = r \,\Big\} =: \mathcal{H}_r. \tag{2.2.3}$$

In particular, it follows that the commutator $ad[\, \mathcal{Z}\,](P)$ admits a bounded and linear extension to $L^2(\mathbb{C}^n, \mu)$.

Proof According to Lemma 2.2.1, (b) it is sufficient to prove (2.2.3). We use induction with respect to $r \in \mathbb{N}$. For a single vector field

$$Z_1 = \sum_{j=1}^n \big[\, a_j \partial_j + b_j \bar{\partial}_j \,\big] \in \mathcal{X}_{\text{lin}}(\mathbb{C}^n)$$

we obtain from Lemma 2.2.1, (a) and (b) :

$$[\, Z_1, P\,] = \sum_{j=1}^n a_j\, [\, \partial_j, P\,] + \sum_{j=1}^n b_j\, \big[\, \bar{\partial}_j, P\,\big] = -\sum_{j=1}^n b_j P\, \bar{\partial}_j.$$

and this proves (2.2.3) in the case $r=1$. Now, we assume that for $r \in \mathbb{N}$ the commutator $\mathrm{ad}[\,\mathcal{Z}\,](P)$ has the form $\sum_{|\alpha|=r} c_\alpha\, P\, \bar{\partial}^\alpha$ with coefficients $c_\alpha \in \mathbb{C}$ and we choose a linear vector field

$$Z_{r+1} := \sum_{j=1}^{n} \left[\, d_j \partial_j + e_j \bar{\partial}_j \,\right] \in \mathcal{X}_{\mathrm{lin}}\,(\,\mathbb{C}^n\,)\,.$$

Then it follows that:

$$\begin{aligned} \mathrm{ad}\,[\,\mathcal{Z}, Z_{r+1}\,]\,(P) &= \left[\, Z_{r+1}, \sum_{|\alpha|=r} c_\alpha\, P\, \bar{\partial}^\alpha \,\right] \\ &= \sum_{j=1}^{n} \sum_{|\alpha|=r} c_\alpha \Big\{\, d_j\, [\,\partial_j, P\bar{\partial}^\alpha\,] + e_j\, [\,\bar{\partial}_j, P\bar{\partial}^\alpha\,] \,\Big\}. \end{aligned}$$

For $j=1,\cdots,n$ we are using the following equalities which can be obtained from Lemma 2.2.1 and $\bar{\partial}^\alpha P = 0$ in the case $\alpha \neq 0$:

(i) $\left[\,\partial_j, P\bar{\partial}^\alpha\,\right] = P\left[\,\partial_j, \bar{\partial}^\alpha\,\right] + [\,\partial_j, P\,]\,\bar{\partial}^\alpha = 0,$

(ii) $\left[\,\bar{\partial}_j, P\bar{\partial}^\alpha\,\right] = P\left[\,\bar{\partial}_j, \bar{\partial}^\alpha\,\right] + \left[\,\bar{\partial}_j, P\,\right]\bar{\partial}^\alpha = -P\,\bar{\partial}_j\,\bar{\partial}^\alpha.$

Hence we have $\mathrm{ad}\,[\,\mathcal{Z}, Z_{r+1}\,]\,(P) = -\sum_{j=1}^{n}\sum_{|\alpha|=r} c_\alpha\, e_j\, P\, \bar{\partial}_j\, \bar{\partial}^\alpha \in \mathcal{H}_{r+1}$. □

2.3 Radial symmetric vector fields.

Our next aim is it to define a wider class $\mathcal{Y}(\,\mathbb{C}^n\,)$ of closable operators such that each finite system $\mathcal{A}$ in $\mathcal{Y}(\mathbb{C}^n)$ admits commutators $\mathrm{ad}[\,\mathcal{A}\,](P)$ which have bounded extensions from $\mathcal{C}_c^\infty(\mathbb{C}^n)$ to $H_1 := L^2(\mathbb{C}^n,\mu)$. In the previous section we have shown (see Proposition 2.2.1) that this property holds if we choose $\mathcal{Y}(\mathbb{C}^n)$ to be the space of all linear vector fields on $\mathbb{C}^n$. Following the construction in chapter 1 we now can associate a sequence of operator algebras containing P to each finite set in $\mathcal{X}_{\mathrm{lin}}(\mathbb{C}^n)$.

There are some disadvantages in using these vector fields. As we will see below (compare Lemma 2.4.1) for $j=1,\cdots,n$ the partial derivatives ∂_j and $\bar{\partial}_j$ are not symmetric on $\mathcal{C}_c^\infty(\mathbb{C}^n)$ with respect to $\langle\cdot,\cdot\rangle_2$. Instead it holds $\partial_j^* = z_j - \bar{\partial}_j$ and $\bar{\partial}_j^* = \bar{z}_j - \partial_j$ and because of the unbounded multiplication operators appearing in these formulas the space $\mathcal{X}_{\mathrm{lin}}(\mathbb{C}^n)$ is not invariant under the $*$-operation. One aim of our studies is it to find localizations of operator algebras to cones $\mathcal{C} \subset \mathbb{C}^n$. For this purpose we need vector fields which are supported in $\mathcal{C}$. Thus we have to enlarge $\mathcal{X}_{\mathrm{lin}}(\mathbb{C}^n)$ to a more appropriate space $\mathcal{Y}(\mathbb{C}^n)$ such that a boundedness result similar to Proposition 2.2.1 stays true if we replace $\mathcal{X}_{\mathrm{lin}}(\mathbb{C}^n)$ by the new class $\mathcal{Y}(\mathbb{C}^n)$.

Our first example shows that we can not simply replace $\mathcal{X}_{\mathrm{lin}}\,(\mathbb{C}^n)$ by the space of all smooth vector fields with bounded coefficients. It turns out that the boundedness of the commutators $[\,X,P\,]$ where $X = \sum\{\, a_j\,\partial_j + b_j\,\bar{\partial}_j \,\}$ is a smooth vector field is closely related to the oscillation of a_j and b_j at infinity. We give X with rapidly oscillating coefficients such that $[\,X,P\,]$ not even is bounded on $H_2 := H^2(\mathbb{C}^n,\mu)$.

Example 2.3.1 In case of dimension $n=1$ we define $f(z) := \exp(\,i|\,z\,|^2\,)$ for $z \in \mathbb{C}$. Let us consider the smooth vector field $X := f\,\partial$, then we show that the commutator

$$[\,X,P\,] = M_f\,[\,\partial,P\,] + [\,M_f,P\,]\,\partial = [\,M_f,P\,]\,\partial$$

does not admit a bounded extension from the dense subspace $P[\,\mathcal{C}_c^\infty(\,\mathbb{C}\,)\,]$ to H_2. With the Toeplitz operator $T_f = P\,M_f$ and the orthonormal basis of H_2 given in (2.1.4) it is a straightforward computation that T_f is diagonal of the form

$$T_f e_j = [\,1-i\,]^{-(\,j+1\,)}\,e_j, \qquad\qquad j \in \mathbb{N}_0.$$

Let us compute the norm of $[\,M_f,P\,]\,\partial e_j$. It follows with $\partial e_j = \sqrt{j}\,e_{j-1}$ for $j \in \mathbb{N}_0$ that:

$$\begin{aligned} [\,M_f,P\,]\,\partial e_j &= \sqrt{j}\ \{\,M_f - T_f\,\}\ e_{j-1} = \sqrt{j}\,\Big\{\,f - [\,1-i\,]^{-j}\,\Big\}\,e_{j-1}, \\ \|\,[\,M_f,P\,]\,\partial e_j\,\|_2 &\geq \sqrt{j}\,\Big\{\,\|\,fe_{j-1}\,\|_2 - 2^{-\frac{j}{2}}\,\|\,e_{j-1}\,\|_2\Big\} = \sqrt{j}\,\Big[\,1 - \,2^{-\frac{j}{2}}\,\Big]. \end{aligned}$$

Hence the numerical sequence $(\,\|\,[\,M_f,P\,]\,\partial e_j\,\|_2\,)_j$ is unbounded as j tends to infinity and so the operator $[\,X,P\,]$ does not admit a bounded extension to H_2.

We give the definition of a space $\mathcal{Y}(\mathbb{C}^n) = \mathcal{X}^\Phi(\mathbb{C}^n)$ which serves our purposes in a better way than $\mathcal{X}_{\text{lin}}(\mathbb{C}^n)$. It will consist of smooth vector fields with coefficients in $\mathcal{C}_b^\infty(\mathbb{C}^n)$. Let us start with some notations:

Let $B_n \subset \mathbb{R}^n$ be the Euclidean ball in $\mathbb{R}^n$ and let $U \subset \mathbb{R}^n$ be an open neighborhood of the boundary ∂B_n of B_n. With $\dot{\mathbb{R}}^n := \mathbb{R}^n \setminus \{0\}$ we consider

$$\Psi : \dot{\mathbb{R}}^n \to \partial B_n : x \mapsto x \cdot \|\,x\,\|^{-1}.$$

Let $f \in C^\infty(U)$ and set:

$$f_r : \dot{\mathbb{R}}^n \to \mathbb{C} : x \mapsto f \circ \Psi(\,x\,).$$

Then f_r is a smooth function on $\dot{\mathbb{R}}^n$ and in the following we refer to it as the *radial extension* of f. Choose $\Phi \in C^\infty(\,\mathbb{R}^n,[\,0,1\,]\,)$ with $\Phi \equiv 0$ on $\frac{1}{2}B_n$ and $\Phi \equiv 1$ on $\mathbb{R}^n \setminus B_n$ and define:

$$\begin{aligned} \mathcal{R}\Big(\,\dot{\mathbb{R}}^n\,\Big) &:= \Big\{\,h : \dot{\mathbb{R}}^n \to \mathbb{C} : h = f \circ \Psi \text{ where } U \supset \partial B_n \text{ and } f \in \mathcal{C}^\infty(U)\,\Big\}, \\ \mathcal{R}^\Phi\,(\,\mathbb{R}^n\,) &:= \Big\{\,h : \mathbb{R}^n \to \mathbb{C} : h = g \cdot \Phi \text{ on } \dot{\mathbb{R}}^n, \text{ where } g \in \mathcal{R}\big(\,\dot{\mathbb{R}}^n\,\big) \text{ and } h(0) = 0\,\Big\}. \end{aligned}$$

Proposition 2.3.1 *With $h \in \mathcal{R}(\,\dot{\mathbb{R}}^n\,)$ it holds $|\,h(\,x\,) - h(\,u\,)\,| \cdot \|\,u\,\| \leq C\|\,x-u\,\|$ for all $x,u \in \dot{\mathbb{R}}^n$ where $C > 0$ is independent of u and x.*

Proof Let $h \in \mathcal{R}(\dot{\mathbb{R}}^n)$ and without loss of generality we assume that h is real-valued. Then there exits an open set $U \subset \mathbb{R}^n$ with $\partial B_n \subset U$ and a function $f \in C^\infty(U)$ such that $h = f \circ \Psi$. First, we assume that $n > 2$. Choose $v, w \in \partial B_n$ and define:

$$a := \frac{1}{2}\,(\,v + w\,) \qquad \text{and} \qquad b := \frac{1}{2}\,(\,v - w\,).$$

It follows that $a \perp b$ and because of $n > 2$ we can choose a vector $c \in \operatorname{span}\{\,a, b\,\}^\perp$ such that $\|\,c\,\| = \|\,b\,\|$. Consider the smooth path $\gamma : [\,0, 1\,] \to \partial B_n$ given by

$$\gamma(\,t\,) := a + b\cos\pi t + c\sin\pi t.$$

It is easy to check, that γ is well-defined with $\gamma(0) = v$ and $\gamma(1) = w$. Moreover, by the mean value theorem we have:

$$\begin{aligned} |\,f(\,v\,) - f(\,w\,)\,| &\leq \int_0^1 \Big|\,\frac{d}{dt}[\,f\circ\gamma\,]\,(t)\Big|\,dt && (2.3.1)\\ &\leq \sup_{q\in\,\partial B_n} \|\,Df(\,q\,)\,\| \int_0^1 \Big\|\,\frac{d}{dt}\,\gamma(\,t\,)\,\Big\|\,dt. \end{aligned}$$

Because of $\int_0^1 \|\,\frac{d}{dt}\gamma(t)\,\| = \pi\,\|\,b\,\| = \frac{\pi}{2}\,\|\,v - w\,\|$ it follows from (2.3.1), that there is $C > 0$ with

$$|\,f(\,v\,) - f(\,w\,)\,| \leq C\,\|\,v - w\,\| \qquad \forall\ \ v, w \in \partial B_n. \tag{2.3.2}$$

In the case $n = 2$ choose $v = (\cos\varphi_1, \sin\varphi_1)$ and $w = (\cos\varphi_2, \sin\varphi_2)$ in ∂B_2. Without loss of generality we assume that $|\,\varphi_1 - \varphi_2\,| \leq \pi$. Then we define the path $\tilde{\gamma} : [\,0, 1\,] \to \partial B_2$ by

$$\tilde{\gamma}(t) := \Big(\ \cos\big(\,[\,1 - t\,]\,\varphi_1 + t\,\varphi_2\,\big),\ \sin\big(\,[\,1 - t\,]\,\varphi_1 + t\,\varphi_2\,\big)\Big).$$

Again it follows that $\tilde{\gamma}(0) = v$ and $\tilde{\gamma}(1) = w$ and we have $\|\,\frac{d}{dt}\tilde{\gamma}(\,t\,)\,\| = |\,\varphi_2 - \varphi_1\,|$ for all $t \in [\,0, 1\,]$. Moreover, an easy calculation shows that there is $c > 0$ independent of v and w such that

$$\begin{aligned} \|\,w - v\,\| &= \sqrt{2}\,\Big[\,1 - \cos\big(\,\,|\,\varphi_2 - \varphi_1\,|\,\,\big)\,\Big]^{\frac{1}{2}}\\ &= 2\,\sin\Big(\frac{1}{2}\,|\,\varphi_2 - \varphi_1\,|\,\Big) \geq c\ |\,\varphi_2 - \varphi_1\,|\,. \end{aligned}$$

Hence we conclude from (2.3.1) that (2.3.2) holds with a constant $C > 0$. Finally, we have for all dimensions $n \in \mathbb{N}$ and $x, u \in \dot{\mathbb{R}}^n$:

$$\begin{aligned} \Big|\,h(\,x\,) - h(\,u\,)\,\Big|\,\|\,u\,\| &= \Big|\,f\circ\Psi(\,x\,) - f\circ\Psi(\,u\,)\,\Big|\,\|\,u\,\|\\ &\leq C\,\Big\|\,\|\,u\,\|\cdot\|\,x\,\|^{-1}\cdot x - u\,\Big\|\\ &\leq C\,\Big|\,\|\,u\,\|\cdot\|\,x\,\|^{-1} - 1\,\Big|\,\|\,x\,\| + C\,\|\,x - u\,\|\\ &\leq C\,\Big|\,\|\,u\,\| - \|\,x\,\|\,\Big| + C\,\|\,x - u\,\|\\ &\leq 2\,C\,\|\,x - u\,\|. \end{aligned}$$

□

Using the result of Proposition 2.3.1 we now are able to prove a similar estimate on the functions in $\mathcal{R}^{\Phi}(\mathbb{R}^n)$.

Corollary 2.3.1 *For $h \in \mathcal{R}^{\Phi}(\mathbb{R}^n)$ it follows $|\, h(x) - h(u) \,| \cdot \|\, u \,\| \leq C\, [\, 1 + \|\, x - u \,\|\,]$ for all $x, u \in \mathbb{R}^n$. Here $C > 0$ is a positive constant independent of h.*

Proof Let $h \in \mathcal{R}^{\Phi}(\mathbb{R}^n)$, then there is $f \in \mathcal{R}(\dot{\mathbb{R}}^n)$ with $h(x) = [\, f \cdot \Phi\,](x)$ for all $x \neq 0$ and $h(0) = 0$. If $x = 0$ or $u = 0$ the inequality in Corollary 2.3.1 directly follows from the boundedness of h. Hence we assume that $x, u \in \dot{\mathbb{R}}^n$. From $0 \leq \Phi \leq 1$ we obtain:

$$|\, h(x) - h(u) \,| \leq |\, f(x) - f(u) \,| + \sup \Big\{ \, |f(z)| : z \in \dot{\mathbb{R}}^n \, \Big\} \cdot |\, \Phi(\, x\,) - \Phi(\, u\,) \,|. \tag{2.3.3}$$

By Proposition 2.3.1 we can find $C > 0$ with $|\, f(x) - f(u) \,| \cdot \|\, u \,\| \leq C\, \|\, x - u \,\|$. Moreover, we have:

$$\begin{aligned}\Big|\, \Phi(\, x\,) - \Phi(\, u\,) \,\Big| \, \|\, u \,\| &\leq \Big[\, |\, 1 - \Phi(\, x\,) \,| + |\, 1 - \Phi(\, u\,) \,| \,\Big] \, \Big[\, \|\, x \,\| + \|\, x - u \,\| \,\Big] \\ &\leq 1 + 2\|\, x - u \,\| + |\, 1 - \Phi(\, u\,) \,| \, \Big[\, \|\, u \,\| + \|\, x - u \,\| \,\Big] \\ &\leq 2 + 3\, \|\, u - x \,\|.\end{aligned}$$

Together with (2.3.3) now the desired result follows. $\square$

For $k \in \mathbb{Z}$ we write $f = \mathcal{O}(\, \|\, x \,\|^k\,)$ as $x \to \infty$ iff $f(x) \cdot \|\, x \,\|^{-k}$ is bounded for all x sufficiently large. The function f is said to be *of order k at infinity*.

Lemma 2.3.1 *With $\Psi := (\Psi_1, \cdots, \Psi_n)^T$ where $\Psi_j(x) := x_j \cdot \|\, x \,\|^{-1}$ any multi-index $\alpha \in \mathbb{N}_0^n$ and all $j = 1, \cdots, n$ we have $\frac{\partial^\alpha \Psi_j}{\partial x^\alpha} = \mathcal{O}(\, \|\, x \,\|^{-|\alpha|}\,)$ as $x \to \infty$.*

Proof For $\alpha = 0$ Lemma 2.3.1 is obvious. Now fix $0 \neq \alpha \in \mathbb{N}_0^n$ and $j \in \{\, 1, \cdots, n \,\}$, then we prove the following formula for the partial derivatives of Ψ_j:

$$\frac{\partial^\alpha \Psi_j}{\partial x^\alpha}(\, x\,) = \frac{p_{j,\alpha}\,(\, x\,)}{\|\, x \,\|^{2|\,\alpha\,|+1}} \tag{2.3.4}$$

where $p_{j,\alpha}$ is a polynomial on $\mathbb{R}^n$ of degree $|\alpha|+1$. First let $\alpha = e_i = (\delta_{i,j})_j$ for any number $i \in \{\, 1, \cdots, n \,\}$, then we obtain by a straightforward computation:

$$\frac{\partial \Psi_j}{\partial x_i}\,(\, x\,) = \frac{1}{\|\, x \,\|^3} \begin{cases} -x_j\, x_i & \text{for } i \neq j \\ \|\, x \,\|^2 - x_i^2 & \text{for } i = j. \end{cases}$$

This proves (2.3.4) in the case $|\, \alpha \,| = 1$. Now, assume that $|\, \alpha \,| > 1$ and choose l such that $\alpha_l \geq 1$. Then by induction we have with $\gamma := \alpha - e_l$ and a polynomial $p_{j,\gamma}$ of degree

$|\,\gamma\,| + 1 = |\,\alpha\,|$:

$$\begin{aligned}\frac{\partial^\alpha \Psi_j}{\partial x^\alpha}(x) &= \frac{\partial}{\partial x_l}\frac{\partial^\gamma \Psi_j}{\partial x^\gamma}(x)\\ &= \frac{\partial}{\partial x_l}\frac{p_{j,\gamma}(x)}{\|\,x\,\|^{2|\,\alpha\,|-1}}\\ &= \frac{\|\,x\,\|^2\,\frac{\partial p_{j,\gamma}}{\partial x_l}(x) - (2|\,\alpha\,|-1)\,p_{j,\gamma}(x)\,x_l}{\|\,x\,\|^{2|\,\alpha\,|+1}}.\end{aligned}$$

Moreover, the polynomial

$$p_{j,\alpha}(\,x\,) := \|\,x\,\|^2\,\frac{\partial p_{j,\gamma}}{\partial x_l}(\,x\,) - \big(\,2|\,\alpha\,| - 1\,\big)\,p_{j,\gamma}(\,x\,)\,x_l$$

is of degree $|\,\gamma\,| + 2 = |\,\alpha\,| + 1$. Using equation (2.3.4) for each $\alpha \in \mathbb{N}_0^n$ there is $C_\alpha > 0$ such that

$$\left|\,\frac{\partial^\alpha \Psi_j}{\partial x^\alpha}(\,x\,)\,\right| = \frac{|\,p_{j,\alpha}(\,x\,)\,|}{\|\,x\,\|^{|\,\alpha\,|+1}}\,\|\,x\,\|^{-|\,\alpha\,|} \leq C_\alpha\,\|\,x\,\|^{-|\,\alpha\,|}$$

for all $x \in \dot{\mathbb{R}}^n$ and this proves our assertion. □

Corollary 2.3.2 *For $h \in \mathcal{R}(\dot{\mathbb{R}}^n)$ and $\alpha \in \mathbb{N}_0^n$ we have $\frac{\partial^\alpha h}{\partial x^\alpha} = \mathcal{O}(\,\|\,x\,\|^{-|\,\alpha\,|}\,)$ as $x \to \infty$.*

Proof Let $h = f \circ \Psi$ where $f \in \mathcal{C}^\infty(U)$ and U is an open neighborhood of ∂B_n. In the case $\alpha = 0$ Corollary 2.3.2 follows from the boundedness of f on ∂B_n. Let us consider the case $|\,\alpha\,| = 1$. We obtain for $i \in \{\,1, \cdots, n\,\}$:

$$\frac{\partial h}{\partial x_i}(\,x\,) = \sum_{l=1}^n \frac{\partial f}{\partial x_l} \circ \Psi(\,x\,) \cdot \frac{\partial \Psi_l}{\partial x_i}(\,x\,). \tag{2.3.5}$$

Hence the assertion follows from the boundedness of all partial derivatives of f on ∂B_n and Lemma 2.3.1. Now assume that

$$\frac{\partial^\alpha h}{\partial x^\alpha} = \mathcal{O}\big(\,\|\,x\,\|^{-|\,\alpha\,|}\big) \qquad \text{as} \quad x \to \infty$$

for $\alpha \in \mathbb{N}_0^n$ with $|\,\alpha\,| \leq k$ and $k \in \mathbb{N}$. We choose $\gamma \in \mathbb{N}_0^n$ with $|\,\gamma\,| = k+1$, then there is an index $i \in \{\,1, \cdots, n\,\}$ such that $\gamma_i > 0$ and we set $\alpha := \gamma - e_i$. If we apply the *Leibniz rule* to (2.3.5), we obtain:

$$\begin{aligned}\frac{\partial^\gamma h}{\partial x^\gamma}(\,x\,) &= \frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial h}{\partial x_i}(\,x\,) \\ &= \sum_{l=1}^n \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{\partial^{\alpha-\beta}}{\partial x^{\alpha-\beta}}\left[\,\frac{\partial f}{\partial x_l} \circ \Psi\,\right](\,x\,) \cdot \frac{\partial^{\beta+e_i}\Psi_l}{\partial x^{\beta+e_i}}(\,x\,).\end{aligned} \tag{2.3.6}$$

Now, by induction and using Lemma 2.3.1 again it follows for $l = 1, \cdots, n$ and $\beta \leq \alpha$ as $x \to \infty$ that:

$$\frac{\partial^{\alpha-\beta}}{\partial x^{\alpha-\beta}} \left[\frac{\partial f}{\partial x_l} \circ \Psi \right] (\, x \,) = \mathcal{O}\left(\| \, x \, \|^{-|\,\alpha\,|+|\,\beta\,|} \right), \qquad \frac{\partial^{\beta+e_i} \Psi_l}{\partial x^{\beta+e_i}} (\, x \,) = \mathcal{O}\left(\| \, x \, \|^{-|\,\beta\,|-1} \right).$$

Hence we conclude from (2.3.6) that $\frac{\partial^\gamma h}{\partial x^\gamma}(\,x\,) = \mathcal{O}(\|\,x\,\|^{-|\,\gamma\,|})$. By induction the assertion follows. $\square$

Let M be one of the spaces $\mathbb{R}^n$ or $\dot{\mathbb{R}}^n$, then motivated by our results above and with numbers $j, k \in \mathbb{N}_0$ we define:

$$\begin{aligned} \mathcal{O}_k(M) &:= \left\{ f : M \to \mathbb{C} : f = \mathcal{O}\Big(\| \, x \, \|^{-k} \Big) \text{ as } x \to \infty \right\}, \qquad (2.3.7) \\ \mathcal{O}_k^b(M) &:= \left\{ f \in \mathcal{O}_k(M) : f \text{ is bounded } \right\}, \\ \mathcal{L}_j(M) &:= \left\{ f : M \to \mathbb{C} : \exists \; C > 0 \text{ with } |\, f(x) - f(u) \,| \cdot \| \, u \, \| \leq C \cdot \Big(1 + \| \, x - u \, \|^j \Big) \right\}. \end{aligned}$$

It is easy to see that for all $j \in \mathbb{N}$ the inclusion holds $\mathcal{L}_j(M) \subset \mathcal{L}_{j+1}(M)$. Let us denote by $\mathcal{L}(M)$ the union over j of all spaces $\mathcal{L}_j(M)$ and consider $\mathcal{O}_1^b(M)$ equipped with the sup-norm $\| \cdot \|_\infty$.

Proposition 2.3.2 *For all $j \in \mathbb{N}$ we have the inclusion $\mathcal{O}_1^b(M) \subset \mathcal{L}_j(M)$.*

Proof By our remarks above we only have to show that $\mathcal{O}_1^b(M) \subset \mathcal{L}_1(M)$. Fix $h \in \mathcal{O}_1^b(M)$ and choose $C > 0$ such that it holds

$$|\, h(\,x\,) \,| \cdot \big(1 + \| \, x \, \| \big) \leq C$$

for all $x \in M$. Then it follows with $x, u \in M$:

$$\begin{aligned} & \Big| \, h(\,x\,) - h(\,u\,) \, \Big| \cdot \| \, u \, \| \qquad (2.3.8) \\ \leq & |\, h(x) \,| \, (\, 1 + \| \, x \, \| \,) \left| \frac{\| \, u \, \|}{1 + \| \, x \, \|} - \frac{\| \, u \, \|}{1 + \| \, u \, \|} \right| + |\, h(x) \,| \, (1 + \| \, x \, \|) + |\, h(u) \,| \, \| \, u \, \| \\ \leq & C \Big| \, \| \, u \, \| - \| \, x \, \| \, \Big| + 2C \\ \leq & 2C \Big[1 + \| \, x - u \, \| \Big]. \end{aligned}$$

$\square$

In the following theorem we summarize our results on the spaces defined above:

Theorem 2.3.1 *The inclusions $\mathcal{R}(\dot{\mathbb{R}}^n) \subset \mathcal{L}_1(\dot{\mathbb{R}}^n)$ and $\mathcal{R}^\Phi(\mathbb{R}^n) \subset \mathcal{L}_1(\mathbb{R}^n)$ hold. Moreover, all partial derivatives of functions in $\mathcal{R}(\dot{\mathbb{R}}^n)$ (resp. of functions in $\mathcal{R}^\Phi(\mathbb{R}^n)$) are contained in the space $\mathcal{O}_1(\dot{\mathbb{R}}^n)$ (resp. in the space $\mathcal{O}_1^b(\mathbb{R}^n)$).*

We identify $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ in the canonical way. Let us denote by $C_b^k(\mathbb{C}^n)$ the space of all complex valued functions on $\mathbb{C}^n$ with continuous and bounded derivatives up to the order $k \in \mathbb{N}$ and write $\mathcal{C}_b^\infty(\mathbb{C}^n) := \bigcap_{j\in\mathbb{N}} \mathcal{C}_b^j(\mathbb{C}^n)$. Then we define:

$$\mathcal{X}^\Phi(\mathbb{C}^n) := \operatorname{span}\Big\{ a_j\,\partial_j : a_j \in \mathcal{R}^\Phi(\mathbb{C}^n) \Big\} \oplus \operatorname{span}\Big\{ b_j\,\bar{\partial}_j : b_j \in \mathcal{C}_b^\infty(\mathbb{C}^n) \Big\}.$$

Example 2.3.2 For $j = 1, \cdots, n$ we consider the vector fields ∂_{φ_j} defined by:

$$\partial_{\varphi_j} := \|\, z\,\|^{-1}\left[\, z_j\,\bar{\partial}_j - \bar{z}_j\,\partial_j \,\right]$$

which are the normed derivatives tangential to the sphere $\partial B_{2n} \subset \mathbb{C}^n$. An easy calculation shows that $a_j\partial_{\varphi_j} \in \mathcal{X}^\Phi(\mathbb{C}^n)$ for any function $a_j \in \mathcal{R}^\Phi(\mathbb{C}^n)$. Thus the space

$$\operatorname{span}\left\{\, a_j\partial_{\varphi_j} : \text{where } a_j \in \mathcal{R}^\Phi(\mathbb{C}^n) \text{ for } j = 1, \cdots, n \,\right\}$$

of *spherical vector fields* on $\mathbb{C}^n$ is naturally embedded into $\mathcal{X}^\Phi(\mathbb{C}^n)$.

2.4 Commutators of P with systems of vector fields

In the following section we want to prove a result analogous to Proposition 2.2.1 if the space $\mathcal{X}_{\text{lin}}(\mathbb{C}^n)$ of all linear vector fields is replaced by $\mathcal{X}^\Phi(\mathbb{C}^n)$. Namely, let us choose a finite system $\mathcal{A}$ in $\{\, Z, Z^*, \operatorname{Re}(Z), \operatorname{Im}(Z) : Z \in \mathcal{X}^\Phi(\mathbb{C}^n) \,\}$. Then we show that all the iterated commutators

$$\operatorname{ad}[\,\mathcal{A}\,](P) : \mathcal{C}_c^\infty(\mathbb{C}^n) \to \mathcal{C}^\infty(\mathbb{C}^n) \tag{2.4.1}$$

admit bounded extensions to H_1. Note that (2.4.1) is well-defined because each $A \in \mathcal{A}$ leaves $\mathcal{C}_c^\infty(\mathbb{C}^n)$ invariant. In order to prove this result we will show that the operators in (2.4.1) are integral operators. We give estimates on their kernels and apply the *Schur test* to prove boundedness.

In the following let us denote by $\partial_{j,u}$ (resp. $\partial_{j,z}$) and $\bar{\partial}_{j,u}$ (resp. $\bar{\partial}_{j,z}$) the partial derivatives with respect to the variables u (resp. z). Instead of the notation $M_{a(z)}$ for the multiplication operator with a function $z \mapsto a(z)$ we often simply write $a(z)$. We start with some general remarks:

Lemma 2.4.1 *Let $k \in \{\,1, \cdots, n\,\}$. Then we have for all $h \in C_c^\infty(\mathbb{C}^n)$ and $g \in C^\infty(\mathbb{C}^n)$:*

$$(i):\ \Big\langle \bar{\partial}_k h, g \Big\rangle_2 = \Big\langle h, [\,\bar{z}_k - \partial_k\,]\,g \Big\rangle_2, \qquad (ii):\ \Big\langle \partial_k h, g \Big\rangle_2 = \Big\langle h, [\,z_k - \bar{\partial}_k\,]\,g \Big\rangle_2.$$

Hence we write $\partial_k^* := z_k - \bar{\partial}_k$ and $\bar{\partial}_k^* := \bar{z}_k - \partial_k$ for the formal adjoints of ∂_k and $\bar{\partial}_k$.

Proof (i): Let $h \in C_c^\infty(\mathbb{C}^n)$ and $g \in C^\infty(\mathbb{C}^n)$, then integration by parts leads to:

$$\begin{aligned}
\Big\langle \bar{\partial}_k h, g \Big\rangle_2 &= \frac{1}{\pi^n}\int_{\mathbb{C}^n} \bar{\partial}_k h(\,z\,)\,\bar{g}(\,z\,)\,\exp\big(-\|\,z\,\|^2\big)\,dv(\,z\,)\\
&= -\frac{1}{\pi^n}\int_{\mathbb{C}^n} h(\,z\,)\left[\bar{\partial}_k\overline{g(\,z\,)} - z_k\overline{g(\,z\,)}\right]\exp\big(-\|\,z\,\|^2\big)\,dv(\,z\,)\\
&= \Big\langle\, h, [\,\bar{z}_k - \partial_k\,]\,g \,\Big\rangle_2.
\end{aligned}$$

The equality (ii) follows by a similar computation. $\square$

Let $Z = \sum_{j=1}^{n} \{ a_j \partial_j + b_j \bar{\partial}_j + \partial_j c_j + \bar{\partial}_j d_j \} + e$ where a_j, b_j, c_j, d_j, e are smooth complex valued functions on $\mathbb{C}^n$ for $j = 1, \cdots, n$. Then we write:

$$\bar{Z} := \sum_{j=1}^{n} \{ \bar{a}_j \bar{\partial}_j + \bar{b}_j \partial_j + \bar{\partial}_j \bar{c}_j + \partial_j \bar{d}_j \} + \bar{e},$$

$$Z^* := \sum_{j=1}^{n} \{ \partial_j^* \bar{a}_j + \bar{\partial}_j^* \bar{b}_j + \bar{c}_j \partial_j^* + \bar{d}_j \bar{\partial}_j^* \} + \bar{e}.$$

We call $\bar{Z}$ the *conjugate operator* and Z^* the *formally adjoint operator*. Note that both operations $*$ and $-$ are involutions on the space

$$\mathcal{L} := \text{span}\Big\{ a\partial_j, b\bar{\partial}_j, \partial_j c, \bar{\partial}_j d, e : \text{ where } a, b, c, d, e \in \mathcal{C}^\infty (\mathbb{C}^n) \text{ and } j = 1, \cdots, n \Big\}.$$

From $\overline{(\partial_j^*)} = (\bar{\partial}_j)^*$ it easily follows that $\overline{Z^*} = (\bar{Z})^*$ for all $Z \in \mathcal{L}$. We directly compute these operators in Lemma 2.4.2 for vector fields $Z := \sum_{j=1}^{n} \{ a_j \partial_j + b_j \bar{\partial}_j \} \in \mathcal{X}^\Phi(\mathbb{C}^n)$. Let us associate to Z the functions $f_{Z,1}$ and $f_{Z,2}$ on $\mathbb{C}^n$ defined by:

$$f_{Z,1}(z) := \sum_{j=1}^{n} \Big\{ z_j \, \bar{a}_j(z) + \bar{z}_j \, \bar{b}_j(z) \Big\} \tag{2.4.2}$$

$$f_{Z,2}(z) := \sum_{j=1}^{n} \Big\{ [\partial_j a_j] (z) + [\bar{\partial}_j b_j] (z) \Big\}.$$

Remark 2.4.1 In the case $Z \in \mathcal{X}^\Phi(\mathbb{C}^n)$ it follows from Theorem 2.3.1 that $f_{Z,2} \in \mathcal{C}_b^\infty(\mathbb{C}^n)$ while in general the function $f_{Z,1}$ is unbounded on $\mathbb{C}^n$.

Next we prove a decomposition formula for the operators $\bar{Z}^*$ and Z^* in terms of Z, $\bar{Z}$ and the functions defined above.

Lemma 2.4.2 *Let $Z = \sum_{j=1}^{n} \{ a_j \partial_i + b_j \bar{\partial}_j \} \in \mathcal{X}^\Phi(\mathbb{C}^n)$. Then we have $\bar{Z}^* = -Z - f_{Z,2} + \overline{f_{Z,1}}$ and $Z^* = -\bar{Z} - \overline{f_{Z,2}} + f_{Z,1}$.*

Proof From the equality $\partial_j \, a_j = a_j \, \partial_j + M_{\partial_j a_j}$ for all $j = 1, \cdots, n$ we obtain:

$$\begin{aligned} \bar{Z}^* &= \sum_{j=1}^{n} \Big\{ \bar{\partial}_j^* a_j + \partial_j^* b_j \Big\} \\ &= \sum_{j=1}^{n} \Big[\Big\{ \bar{z}_j - \partial_j \Big\} a_j + \Big\{ z_j - \bar{\partial}_j \Big\} b_j \Big] = -Z - f_{Z,2} + \overline{f_{Z,1}}. \end{aligned}$$

The second formula for Z^* follows from the first and $Z^* = \overline{(\bar{Z}^*)}$. $\square$

Let us denote the *real part* (resp. the *imaginary part*) of Z by

$$\mathrm{Re}(\, Z\,) := \frac{1}{2}\,[\, Z + Z^* \,]\,, \qquad\qquad \mathrm{Im}(\, Z\,) = \frac{1}{2i}\,[\, Z - Z^* \,]\,.$$

Now, we compute the iterated commutators of the operators $Z, Z^*, \mathrm{Re}(Z)$ and $\mathrm{Im}(Z)$ with the Toeplitz projection P. For $A \in \mathcal{L}$ we write A_z to indicate that we consider A as an operator with respect to the variable z.

Proposition 2.4.1 *Let $r \in \mathbb{N}$ and let $\mathcal{A} := [\, A_i : i = 1, \cdots, r\,]$ be a finite system of operators in $\mathcal{L}$. Then the commutator ad$[\,\mathcal{A}\,](P)$ is an integral operators with kernel $L_{\mathcal{A}}$ and*

$$\overline{L_{\mathcal{A}}(\, u, z\,)} = \big(\, \bar{A}_{r,z} - A^*_{r,u}\,\big) \cdots \big(\, \bar{A}_{1,z} - A^*_{1,u}\,\big)\, K(\, u, z\,) \tag{2.4.3}$$

for all $u, z \in \mathbb{C}^n$ with respect to the measure μ. As before we write $K(u,z) = \exp(\langle u, z\rangle)$.

Proof Let $r = 1$, then it follows for any function $g \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$:

$$\begin{aligned}\Big\{\, [\, A_1, P\,]g \,\Big\}(\, z\,) &= A_{1,z}\Big\langle\, g, K(\cdot, z)\Big\rangle_2 - \Big\langle A_{1,u}g, K(\cdot, z\,)\,\Big\rangle_2\\ &= \Big\langle\, g, \big[\, \bar{A}_{1,z} - A^*_{1,u}\,\big]\, K\,(\cdot, z\,)\,\Big\rangle_2.\end{aligned}$$

Thus $[\, A_1, P\,]$ has an integral kernel $L_{\mathcal{A}}$ with

$$\overline{L_{\mathcal{A}}(\, u, z\,)} = \big[\, \bar{A}_{1,z} - A^*_{1,u}\,\big]\, K(\, u, z\,).$$

Let $r > 1$ be an entire number and by induction let us assume that equation (2.4.3) holds for all $m \in \{\, 1, \cdots, r-1\,\}$ and the system $\mathcal{A}_m := [\, A_j : j = 1, \cdots, m\,]$. Then:

$$\begin{aligned}\Big\{\, \mathrm{ad}\,[\,\mathcal{A}_m, A_{m+1}\,]\;(P)\, g\,\Big\}(\, z\,) &= \Big\{\, \Big[\, A_{m+1}, \mathrm{ad}\,[\,\mathcal{A}_m\,]\;(P)\,\Big]\, g\,\Big\}(\, z\,)\\ &= A_{m+1,z}\Big\langle\, g, \overline{L_{\mathcal{A}_m}(\cdot, z\,)}\,\Big\rangle_2 - \Big\langle\, A_{m+1,u}g, \overline{L_{\mathcal{A}_m}(\cdot, z\,)}\,\Big\rangle_2\\ &= \Big\langle\, g, \big[\, \bar{A}_{m+1,z} - A^*_{m+1,u}\,\big]\, \overline{L_{\mathcal{A}_m}(\cdot, z\,)}\,\Big\rangle_2\\ &= \Big\langle\, g, \overline{L_{\mathcal{A}_{m+1}}(\cdot, z\,)}\,\Big\rangle_2.\end{aligned}$$

for all $z \in \mathbb{C}^n$ and by induction with $\mathcal{A}_r = \mathcal{A}$ the assertion follows. □

With a finite system $\mathcal{A}$ of operators in

$$\Big\{\, Z, Z^*, \mathrm{Re}(Z), \mathrm{Im}(Z) : Z \in \mathcal{X}^\Phi(\mathbb{C}^n)\,\Big\} \subset \mathcal{L}$$

we want to estimate the growth of the kernels $L_{\mathcal{A}}$. Using the *Schur test* (see Theorem 2.1.1) this will lead to the boundedness results for the iterated commutators. With $Z \in \mathcal{X}^\Phi(\mathbb{C}^n)$ consider the following derivations on $\mathcal{C}^\infty(\mathbb{C}^n \times \mathbb{C}^n)$:

$$(i):\quad \delta_Z := \bar{Z}_z + \bar{Z}_u, \qquad\qquad (ii):\quad \delta_{Z^*} := -Z_z - Z_u = -\overline{\delta_Z}. \tag{2.4.4}$$

Using these notations we prove:

Lemma 2.4.3 *Let $Z \in \mathcal{X}^\Phi(\mathbb{C}^n)$, then the following decompositions hold:*

(a) $\bar{Z}_z - Z_u^* = \delta_Z - \left[\, f_{Z,1} - \overline{f_{Z,2}} \,\right] (\, u \,)$,

(b) $\overline{(\, Z^* \,)}_z - (\, Z^* \,)_u^* = \delta_{Z^*} + \left[\, \overline{f_{Z,1}} - f_{Z,2} \,\right] (\, z \,)$,

(c) $\overline{Im(\, Z \,)}_z - Im(\, Z \,)_u^* = \frac{1}{2i} \left[\, \delta_{Z^*} - \delta_Z \,\right] + \frac{1}{2i} \left[\, f_{Z,1} - \overline{f_{Z,2}} \,\right] (\, u \,) + \frac{1}{2i} \left[\, \overline{f_{Z,1}} - f_{Z,2} \,\right] (\, z \,)$,

(d) $\overline{Re(\, Z \,)}_z - Re(\, Z \,)_u^* = \frac{1}{2} \left[\, \delta_Z + \delta_{Z^*} \,\right] - \frac{1}{2} \left[\, f_{Z,1} - \overline{f_{Z,2}} \,\right] (\, u \,) + \frac{1}{2} \left[\, \overline{f_{Z,1}} - f_{Z,2} \,\right] (\, z \,)$.

Proof The identities (a) and (b) directly follow from Lemma 2.4.2. We only prove (d) and remark that (c) can be computed in the same way.

$$\begin{aligned}\overline{\mathrm{Re}(\, Z \,)}_z - \mathrm{Re}(\, Z \,)_u^* &= \frac{1}{2} \overline{[\, Z_z + Z_z^* \,]} - \frac{1}{2} [\, Z_u + Z_u^* \,]^* \\ &= \frac{1}{2} \left[\, \bar{Z}_z - Z_u^* \,\right] + \frac{1}{2} \left[\, \overline{(Z^*)}_z - (Z_u^*)^* \,\right].\end{aligned}$$

Finally, we apply the formulas (a) and (b) to obtain (d). □

According to the decomposition in Lemma 2.4.3 let us write:

$$(iii) \ \ \delta_{\mathrm{Re}(Z)} := \frac{1}{2} \left[\, \delta_Z + \delta_{Z^*} \,\right] \qquad\qquad (vi) \ \ \delta_{\mathrm{Im}(Z)} := \frac{1}{2i} \left[\, \delta_{Z^*} - \delta_Z \,\right]. \tag{2.4.5}$$

With $K(\, u, z \,) = \exp(\, \langle u, z \rangle \,)$ let us consider the following functions on $\mathbb{C}^n \times \mathbb{C}^n$:

(a) $T_Z(\, u, z \,) := \Big\{\, K^{-1} \cdot [\, \delta_Z - f_{Z,1}(\, u \,) \,] \ K \Big\} (\, u, z \,)$,

(b) $T_{Z^*}(\, u, z \,) := \Big\{\, K^{-1} \cdot \left[\, \delta_{Z^*} + \overline{f_{Z,1}}(\, z \,) \,\right] K \,\Big\} (\, u, z \,)$,

(c) $T_{\mathrm{Im}(Z)}(\, u, z \,) := \Big\{ K^{-1} \cdot \left[\, \delta_{\mathrm{Im}(Z)} + \frac{1}{2i} f_{Z,1}(\, u \,) + \frac{1}{2i} \overline{f_{Z,1}}(\, z \,) \,\right] \ K \,\Big\} (\, u, z \,)$,

(d) $T_{\mathrm{Re}(Z)}(\, u, z \,) := \Big\{\, K^{-1} \cdot \left[\, \delta_{\mathrm{Re}(Z)} - \frac{1}{2} f_{Z,1}(\, u \,) + \frac{1}{2} \overline{f_{Z,1}}(\, z \,) \,\right] \ K \,\Big\} (\, u, z \,)$.

With these definitions we have the identities:

$$T_{\mathrm{Re}(Z)} = \frac{1}{2} \left[\, T_Z + T_{Z^*} \,\right], \qquad\qquad T_{\mathrm{Im}\,(Z)} = \frac{1}{2i} \left[\, T_{Z^*} - T_Z \,\right]. \tag{2.4.6}$$

We directly compute the functions $T_Z, T_{Z^*}, T_{\mathrm{Re}(Z)}$ and $T_{\mathrm{Im}(Z)}$. Because of equation (2.4.6) we only consider T_Z and T_{Z^*}. Let

$$Z = \sum_{j=1}^{n} \left\{\, a_j \partial_j + b_j \bar{\partial}_j \,\right\},$$

then using the identities $\partial_{j,z}K = \bar{\partial}_{j,u}K = 0$ for the Segal Bargmann kernel K we obtain:

$$\begin{aligned}
T_Z(u,z) &= \left\{ K^{-1} \cdot \sum_{j=1}^{n} \left[\bar{a}_j(z)\,\bar{\partial}_{j,z} + \bar{b}_j(u)\,\partial_{j,u} \right] K \right\}(u,z) - f_{Z,1}(u), \\
&= \sum_{j=1}^{n} u_j \left[\bar{a}_j(z) - \bar{a}_j(u) \right] + \sum_{j=1}^{n} \bar{b}_j(u) \left[\bar{z}_j - \bar{u}_j \right] \\
&= \left[F_Z + G_Z \right](u,z), \\
T_{Z^*}(u,z) &= \left\{ -K^{-1} \cdot \sum_{j=1}^{n} \left[b_j(z)\,\bar{\partial}_{j,z} + a_j(u)\,\partial_{j,u} \right] K \right\}(u,z) + \overline{f_{Z,1}}(z) \\
&= \sum_{j=1}^{n} \bar{z}_j \left[a_j(z) - a_j(u) \right] + \sum_{j=1}^{n} b_j(z) \left[z_j - u_j \right] \\
&= -\left[\bar{F}_Z + \bar{G}_Z \right](z,u)
\end{aligned}$$

where we define for all $(u,z) \in \mathbb{C}^n \times \mathbb{C}^n$:

$$F_Z(u,z) := \sum_{j=1}^{n} u_j \left[\bar{a}_j(z) - \bar{a}_j(u) \right], \qquad G_Z(u,z) := \sum_{j=1}^{n} \bar{b}_j(u) \left[\bar{z}_j - \bar{u}_j \right]. \tag{2.4.7}$$

In particular, if we choose $Z \in \mathcal{X}^\Phi(\mathbb{C}^n)$ and apply Theorem 2.3.1 it follows that

$$a_j \in \mathcal{R}^\Phi(\mathbb{C}^n) \subset \mathcal{L}_1(\mathbb{C}^n) \qquad \text{and} \qquad b_j \in \mathcal{C}_b^\infty(\mathbb{C}^n) \qquad \text{for} \quad (j = 1, \cdots, n).$$

Hence there are constants $c_1, c_2 \geq 0$ such that:

$$(i)\ \left| F_Z(u,z) \right| \leq c_1 \cdot \left(1 + \| u - z \| \right), \qquad (ii)\ \left| G_Z(u,z) \right| \leq c_2 \cdot \| u - z \|. \tag{2.4.8}$$

for all $(u,z) \in \mathbb{C}^n \times \mathbb{C}^n$. In the following proposition we summarize our results on the functions T_A where $A \in \{ Z, Z^*, \mathrm{Re}(Z), \mathrm{Im}(Z) \}$.

Proposition 2.4.2 *Let $Z \in \mathcal{X}^\Phi(\mathbb{C}^n)$, then with the definitions above we have:*

(a) $T_Z(u,z) = \left[F_Z + G_Z \right](u,z)$,

(b) $T_{Z^*}(u,z) = -\left[\bar{F}_Z + \bar{G}_Z \right](z,u)$,

(c) $T_{Re(Z)}(u,z) = \frac{1}{2}\left[F_Z(u,z) + G_Z(u,z) - \bar{F}_Z(z,u) - \bar{G}_Z(z,u) \right]$,

(d) $T_{Im(Z)}(u,z) = \frac{i}{2}\left[F_Z(u,z) + G_Z(u,z) + \bar{F}_Z(z,u) + \bar{G}_Z(z,u) \right]$.

Applying these formulas we can give a new description of the integral kernels in Proposition 2.4.1. Assume that $\mathcal{A} := [\, A_1, \cdots, A_r \,]$ is a finite system of operators in

$$\Big\{ \, Z, Z^*, \mathrm{Re}(Z), \mathrm{Im}(Z) : Z \in \mathcal{X}^{\Phi}(\, \mathbb{C}^n \,) \, \Big\}.$$

With the derivations in (2.4.4) and (2.4.5) and $k \in \{\, 1, \cdots, r \,\}$ we define $\delta_{\mathcal{A}}^{(1)} := id$ and in the case where $k \geq 2$ we set $\delta_{\mathcal{A}}^{(k)} := \delta_{A_k} \cdots \delta_{A_2}$. Let $\mathbf{A}$ be the set consisting of all permutation of the systems $\mathcal{A}$. Then we consider the algebra $\mathcal{M}^{\mathcal{A}}(\mathbb{C}^{2n})$ of smooth functions on $\mathbb{C}^n \times \mathbb{C}^n$ generated by the set:

$$\mathcal{V} := \Big\{ \, 1, \delta_{\mathcal{B}}^{(k)} T_{B_1} : \text{ with } k \leq r \text{ and } [\, \mathcal{B}, B_1 \,] \in \mathbf{A} \Big\}.$$

Proposition 2.4.3 *Let $\mathcal{A}$ be a finite system in $\{\, Z, Z^*, Re(Z), Im(Z) : Z \in \mathcal{X}^{\Phi}(\mathbb{C}^n) \,\}$. Then the integral kernel $L_{\mathcal{A}}$ of the iterated commutator $ad[\, \mathcal{A} \,](P)$ has the form*

$$\overline{L_{\mathcal{A}}(\, u, z \,)} = \sum_j \Big\{ B_j \cdot T_j \cdot K \Big\} (\, u, z \,) \tag{2.4.9}$$

where $T_j \in \mathcal{M}^{\mathcal{A}}(\, \mathbb{C}^{2n} \,)$ and $B_j \in \mathcal{C}_b^{\infty}(\, \mathbb{C}^{2n} \,)$ for all j and the sum in (2.4.9) is finite.

Proof We prove (2.4.9) by induction with respect to the length r of $\mathcal{A}$. By Proposition 2.4.1, Lemma 2.4.3 and Remark 2.4.1 for a single operator $\mathcal{A} = [\, A_1 \,]$ we have:

$$\overline{L_{\mathcal{A}}(\, u, z \,)} = \big(\, \bar{A}_{1,z} - A_{1,u}^* \,\big)\, K(\, u, z \,) = T_{A_1}(\, u, z \,)\, K(\, u, z \,) + h(\, u, z \,)\, K(\, u, z \,)$$

where $h \in \mathcal{C}_b^{\infty}(\mathbb{C}^{2n})$ and $T_{A_1} = \delta_{\mathcal{A}}^{(1)} T_{A_1} \in \mathcal{M}^{\mathcal{A}}(\mathbb{C}^{2n})$. We are done in the case $r = 1$. Now, assume that (2.4.9) holds for systems of length $r - 1$, then we define

$$\mathcal{A}_{r-1} := [\, A_1, \cdots, A_{r-1} \,]\,.$$

Again, it follows from Lemma 2.4.2 that there are two functions $h_1 \in \mathcal{C}^{\infty}(\mathbb{C}^{2n})$ and $h_2 \in \mathcal{C}_b^{\infty}(\mathbb{C}^{2n})$ such that $\bar{A}_{r,z} - A_{r,u}^* = \delta_{A_r} + h_1 + h_2$. Proposition 2.4.1 now implies with the system $\mathcal{A} = [\, \mathcal{A}_{r-1}, A_r \,]$ that:

$$\begin{aligned}
\overline{L_{\mathcal{A}}(\, u, z \,)} &= \big(\, \bar{A}_{r,z} - A_{r,u}^* \,\big)\, \overline{L_{\mathcal{A}_{r-1}}(\, u, z \,)} \\
&= \Big[\, \delta_{A_r} + h_1 \,\Big]\, \overline{L_{\mathcal{A}_{r-1}}(\, u, z \,)} + h_2(\, u, z \,) \cdot \overline{L_{\mathcal{A}_{r-1}}(\, u, z \,)}.
\end{aligned}$$

Because of $h_2 \in \mathcal{C}_b^{\infty}(\mathbb{C}^{2n})$ and by induction we conclude that the second summand on the right hand side already has the form of (2.4.9). Let us compute the first one:

$$\Big[\, \delta_{A_r} + h_1 \,\Big]\, \overline{L_{\mathcal{A}_{r-1}}(\, u, z \,)} = \sum_j \Big[\, \delta_{A_r} + h_1 \,\Big] \Big\{\, B_j \cdot T_j \cdot K \,\Big\}\, (\, u, z \,).$$

Let us consider the functions H_j in this expression defined by

$$H_j(\, u, z \,) := [\, \delta_{A_r} + h_1 \,]\, \{\, B_j \cdot T_j \cdot K \,\}\, (\, u, z \,).$$

Because the operator δ_{A_r} is a derivation we find by using $[\,\delta_{A_r}+h_1\,]\,K = T_{A_r}K$:

$$H_j(\,u,z\,) = \Big[\Big\{\,(\,\delta_{A_r}B_j\,)\,T_jK\,\Big\} + \Big\{\,B_j\,(\,\delta_{A_r}T_j\,)\,K\,\Big\} + \Big\{\,B_jT_jT_{A_r}K\,\Big\}\Big]\,(\,u,z\,).$$

for all j. We have $\delta_{A_r}B_j \in \mathcal{C}_b^\infty(\mathbb{C}^{2n})$ and because δ_{A_r} is a derivation it is easy to see that the functions $\delta_{A_r}T_j$ and T_{A_r} are contained in $\mathcal{M}^{\mathcal{A}}(\mathbb{C}^{2n})$ again. It follows that H_j has the form (2.4.9). $\square$

We want to prove some estimates on the growth of the functions in $\mathcal{M}^{\mathcal{A}}(\mathbb{C}^{2n})$ where $\mathcal{A}$ is a finite system in $\{\,Z, Z^*, \mathrm{Re}(Z), \mathrm{Im}(Z) : Z \in \mathcal{X}^\Phi(\mathbb{C}^n)\,\}$. With our definition in (2.3.7) we set:

$$\mathcal{SO}(\mathbb{C}^n) := \Big\{\,f \in \mathcal{O}_1^b(\mathbb{C}^n) : \text{ all partial derivatives of } f \text{ belong to } \mathcal{O}_1^b(\mathbb{C}^n)\,\Big\}.$$

By the Leibniz formula $\mathcal{SO}(\mathbb{C}^n)$ is a $\mathcal{C}_b^\infty(\mathbb{C}^n)$-module. According to Theorem 2.3.1 all partial derivatives of functions in $\mathcal{R}^\Phi(\mathbb{C}^n)$ are contained in $\mathcal{SO}(\mathbb{C}^n)$. Moreover, it is easy to see that for all vector fields $Z \in \mathcal{X}^\Phi(\mathbb{C}^n)$ and $Y \in \{\,Z, \bar{Z}\,\}$ the inclusion

$$Y\Big(\,\mathcal{SO}\,(\,\mathbb{C}^n\,)\,\Big) \subset \mathcal{SO}\,(\,\mathbb{C}^n\,)$$

holds. Let us define the involution κ on $\mathbb{C}^n \times \mathbb{C}^n$ given by $\kappa(\,z,u\,) = (\,u,z\,)$ and set:

$$\mathcal{S}_1\left(\mathbb{C}^{2n}\right) := \left\{\,f, f\circ\kappa, \bar{f}, \bar{f}\circ\kappa : f(u,z) = \sum_{j=1}^n u_j \cdot [\,c_j(z) - c_j(u)\,] \text{ and } c_j \in \mathcal{SO}(\mathbb{C}^n)\right\},$$

$$\mathcal{S}_2\left(\mathbb{C}^{2n}\right) := \left\{\,g, g\circ\kappa, \bar{g}, \bar{g}\circ\kappa : g(u,z) = \sum_{j=1}^n d_j(u) \cdot [\,\bar{z}_j - \bar{u}_j\,] \text{ and } d_j \in \mathcal{C}_b^\infty(\mathbb{C}^n)\right\}.$$

Then for $j = 1,2$ the spaces $\mathcal{S}_j(\mathbb{C}^{2n})$ are contained in $\mathcal{C}^\infty(\mathbb{C}^{2n})$. Let us prove an equality for functions in $\mathcal{S}_j(\mathbb{C}^{2n})$ where $j = 1,2$ which we will need later on in order to estimate the integral kernels $L_{\mathcal{A}}$ in Proposition 2.4.3.

Lemma 2.4.4 *Let $f_j \in \mathcal{S}_j(\mathbb{C}^{2n})$ for $j = 1,2$, then there are constants $c_j > 0$ such that it holds*

$$(i):\quad |\,f_1(u,z)\,| \le c_1\left(1 + \|\,u-z\,\|\right), \qquad (ii):\quad |\,f_2(u,z)\,| \le c_2\,\|\,u-z\,\|$$

where $(\,u,z\,) \in \mathbb{C}^n \times \mathbb{C}^n$.

Proof The assertion (ii) immediately follows from the definition of $\mathcal{S}_2(\mathbb{C}^{2n})$. The inequality (i) can be obtained from $\mathcal{SO}(\mathbb{C}^n) \subset \mathcal{O}_1^b(\mathbb{C}^n) \subset \mathcal{L}_1(\mathbb{C}^n)$ (cf. Proposition 2.3.2) and the fact that the right hand side of (i) is invariant under κ. $\square$

With a vector field

$$Z = \sum_{j=1}^n \{\,a_j\,\partial_j + b_j\,\bar{\partial}_j\,\} \in \mathcal{X}^\Phi\,(\mathbb{C}^n)$$

and the functions $F_Z, G_Z \in \mathcal{C}^\infty(\mathbb{C}^{2n})$ defined in (2.4.7) we obtain by applying the derivation $\delta_Z = \bar{Z}_z + \bar{Z}_u$:

$$\delta_Z F_Z = \sum_{j=1}^{n} \Big\{ \bar{b}_{j,u} \, [\, \bar{a}_{j,z} - \bar{a}_{j,u} \,] + u_j \, \big[\, (\bar{Z}\bar{a}_j)_z - (\bar{Z}\bar{a}_j)_u \,\big] \Big\}, \tag{2.4.10}$$

$$\delta_Z G_Z = \sum_{j=1}^{n} \Big\{ \bar{b}_{j,u} \, [\, \bar{a}_{j,z} - \bar{a}_{j,u} \,] + (\bar{Z}\bar{b}_j)_u \, [\, \bar{z}_j - \bar{u}_j \,] \Big\}, \tag{2.4.11}$$

$$\delta_Z \left[\, \bar{F}_Z \circ \kappa \,\right] = \sum_{j=1}^{n} \Big\{ \bar{a}_{j,z} \, [\, a_{j,u} - a_{j,z} \,] + \bar{z}_j \, \big[\, (\bar{Z}a_j)_u - (\bar{Z}a_j)_z \,\big] \Big\}, \tag{2.4.12}$$

$$\delta_Z \left[\, \bar{G}_Z \circ \kappa \,\right] = \sum_{j=1}^{n} \Big\{ b_{j,z} \, \big[\, \bar{b}_{j,u} - \bar{b}_{j,z} \,\big] + (\bar{Z}b_j)_z \, [\, u_j - z_j \,] \Big\}. \tag{2.4.13}$$

Here we have used the abbreviation $a_z(u,z) := a(z)$ and $a_u(u,z) := a(u)$ for any function $a \in \mathcal{C}^\infty(\mathbb{C}^n)$. We conclude that:

Lemma 2.4.5 *Let $Z \in \mathcal{X}^\Phi(\mathbb{C}^n)$ and $A \in \{\, Z, Z^*, Re(Z), Im(Z) \,\}$. Then both the functions $\delta_A F_Z$ and $\delta_A \big[\, \bar{F}_Z \circ \kappa \,\big]$ (resp. the functions $\delta_A G_Z$ and $\delta_A \big[\, \bar{G}_Z \circ \kappa \,\big]$) are contained in $\mathcal{C}_b^\infty(\mathbb{C}^{2n}) + \mathcal{S}_1(\mathbb{C}^{2n})$ (resp. in $\mathcal{C}_b^\infty(\mathbb{C}^{2n}) + \mathcal{S}_2(\mathbb{C}^{2n})$). Moreover, for $j = 1,2$ the spaces $\mathcal{C}_b^\infty(\mathbb{C}^{2n}) + \mathcal{S}_j(\mathbb{C}^{2n})$ are invariant under δ_A.*

Proof In the case $A = Z$ the first assertion directly follows from equations (2.4.10)-(2.4.13) and the fact that

$$\bar{Z}\,[\, \mathcal{C}_b^\infty(\mathbb{C}^n) \,] \subset \mathcal{C}_b^\infty(\mathbb{C}^n) \qquad \text{and} \qquad \bar{Z}\left[\, \mathcal{R}^\Phi(\mathbb{C}^n) \,\right] \subset \mathcal{SO}(\mathbb{C}^n)$$

which can easily be seen from Theorem 2.3.1. The case $A = Z^*$ can be obtained by a similar calculation and using $\delta_{Z^*} = -\overline{\delta_Z}$. For an operator $A \in \{\, \mathrm{Re}(Z), \mathrm{Im}(Z) \,\}$ our assertion is a consequence of the decompositions (2.4.5).

Let us prove the last statement. Again we only have to consider the case $A \in \{\, Z, Z^* \,\}$. Because $\mathcal{C}_b^\infty(\mathbb{C}^{2n})$ is invariant under δ_Z and δ_{Z^*} it is sufficient to prove that the inclusion

$$\delta_Y \left[\, \mathcal{S}_j\left(\mathbb{C}^{2n}\right) \,\right] \subset \mathcal{S}_j\left(\mathbb{C}^{2n}\right) + \mathcal{C}_b^\infty\left(\mathbb{C}^{2n}\right)$$

holds for $j = 1,2$ and $Y \in \{Z, Z^*\}$. This follows with an calculation analogous to the one in (2.4.10) and (2.4.11) from $X\,[\, \mathcal{SO}(\mathbb{C}^n) \,] \subset \mathcal{SO}(\mathbb{C}^n)$ together with $X\,[\, \mathcal{C}_b^\infty(\mathbb{C}^n) \,] \subset \mathcal{C}_b^\infty(\mathbb{C}^n)$ for any $X \in \{\, Z, \bar{Z} \,\}$. $\square$

Corollary 2.4.1 *Let $\mathcal{A} := [\, A_1, \cdots, A_r \,]$ be a system in*

$$\Big\{\, Z, Z^*, Re(Z), Im(Z) : Z \in \mathcal{X}^\Phi(\mathbb{C}^n) \,\Big\}$$

where $r \in \mathbb{N}$ and let $\delta_{\mathcal{A}}^{(k)}$ be defined as above Proposition 2.4.3. Then, there is a positive constant c_k such that for any $k \in \{\, 1, \cdots, r \,\}$ and for all $u, z \in \mathbb{C}^n \times \mathbb{C}^n$

$$\Big|\, \delta_{\mathcal{A}}^{(k)} T_{A_1}(u,z) \,\Big| \leq c_k\big(\, 1 + \|\, u - z \,\|\, \big). \tag{2.4.14}$$

Proof Let $k = 1$, then from Proposition 2.4.2 we conclude that $T_{A_1} = \delta^{(1)}_{\mathcal{A}} T_{A_1}$ is a linear combination of F_Z, G_Z, $\bar{F}_Z \circ \kappa$ and $\bar{G}_Z \circ \kappa$. Hence the existence of $c > 0$ with (2.4.14) follows from our inequalities in (2.4.8). Let $r \geq 2$ then for $k = 2$ we obtain from Lemma 2.4.5 that:

$$\delta^{(2)}_{\mathcal{A}} T_{A_1} = \delta_{A_2} T_{A_1} \in \mathcal{S}_1\left(\mathbb{C}^{2n}\right) + \mathcal{S}_2\left(\mathbb{C}^{2n}\right) + \mathcal{C}^\infty_b\left(\mathbb{C}^{2n}\right)$$

and so we conclude (2.4.14) from Lemma 2.4.4. Because we have shown in Lemma 2.4.5 that the space $\mathcal{S}_1(\mathbb{C}^{2n}) + \mathcal{S}_2(\mathbb{C}^{2n}) + \mathcal{C}^\infty_b(\mathbb{C}^{2n})$ is invariant under δ_A where $A \in \mathcal{A}$, the inequality (2.4.14) follows for all $k > 2$ by induction and Lemma 2.4.4. □

Proposition 2.4.4 *Let $\mathcal{A}$ be a finite system in $\{ Z, Z^*, Re(Z), Im(Z) : Z \in \mathcal{X}^\Phi(\mathbb{C}^n) \}$. Then there exists $m \in \mathbb{N}$ and $c > 0$ such that*

$$|\, L_{\mathcal{A}}(u,z) \,| \leq c \left(1 + \| \, u - z \, \|^m \right) \cdot \exp\left(Re \, \langle u, z\rangle \right),$$

where $(\, u, z \,) \in \mathbb{C}^n \times \mathbb{C}^n$ and $L_{\mathcal{A}}$ denotes the integral kernel of $ad[\, \mathcal{A} \,](P)$.

Proof This directly follows from Corollary 2.4.1, the form of $L_{\mathcal{A}}$ given in Proposition 2.4.3 and the fact that there is $\tilde{c}_l > 0$ with $(1+t)^l \leq \tilde{c}_l(1+t^l)$ for all $t > 0$ and fixed $l \in \mathbb{N}$. □

Remark 2.4.2 Analyzing the proof of Proposition 2.4.3 it is easy to see that the power m in Proposition 2.4.4 can be chosen to be the length of the system $\mathcal{A}$.

Now we can prove our main theorem in this section concerning the boundedness of the iterated commutators which is a generalization of Proposition 2.2.1.

Theorem 2.4.1 *Let $\mathcal{A}$ be a finite system in $\{ Z, Z^*, Re(Z), Im(Z) : Z \in \mathcal{X}^\Phi(\mathbb{C}^n) \}$. Then the commutator $ad\,[\, \mathcal{A} \,](P)$ admits a bounded extension from $\mathcal{C}^\infty_c(\mathbb{C}^n)$ to $L^2(\mathbb{C}^n, \mu)$.*

Proof As we have seen in the Propositions 2.4.1 and 2.4.4 the commutators $\mathrm{ad}[\, \mathcal{A} \,](P)$ are integral operators on $\mathcal{C}^\infty_c(\mathbb{C}^n)$ with kernels $L_{\mathcal{A}}$ and there is $m \in \mathbb{N}$ and $c > 0$ such that:

$$|\, L_{\mathcal{A}}(\, u, z \,) \,| \leq c \cdot \left(1 + \| \, u - z \, \|^m \right) \cdot \exp\left(\mathrm{Re} \, \langle u, z\rangle \right)$$

holds for all $u, z \in \mathbb{C}^n$. Because the fact that the function $p := (1 + \| \cdot \|^m)$ is contained in $L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})$ the assertion follows from Proposition 2.1.1. □

Note that in our definition of the vector fields in $\mathcal{X}^\Phi(\mathbb{C}^n)$ we have to claim more on the coefficients $a_l := Zz_l$ than on $b_l := Z\bar{z}_l$ in order to prove Theorem 2.4.1.

2.5 Fréchet algebras localized in cones

For $f \in L^\infty(\mathbb{C}^n)$ we consider the Toeplitz operators T_f with symbol f which was defined in (2.1.8). Depending on the context T_f can be viewed as a bounded operator on H_1 as well as on H_2. With our definitions above let us assume that $\mathcal{A}$ is a finite system in

$$\left\{ Z, Z^*, \mathrm{Re}(\, Z \,), \mathrm{Im}(\, Z \,) : Z \in \mathcal{X}^\Phi\,(\, \mathbb{C}^n \,) \right\}.$$

To begin with, we consider the commutators of $\mathcal{A}$ and M_f where $f \in \mathcal{C}_b^k(\mathbb{C}^n)$ and $k \in \mathbb{N}$ as operators on the space $\mathcal{C}_c^\infty(\mathbb{C}^n)$.

Lemma 2.5.1 *Assume that $\mathcal{A}$ has length r and let $f \in \mathcal{C}_b^r(\mathbb{C}^n)$. Then as an operator equation on $\mathcal{C}_c^\infty(\mathbb{C}^n)$ it holds*

$$ad[\,\mathcal{A}\,]\,(M_f) = M_h$$

where $h \in \mathcal{C}_b(\mathbb{C}^n)$. In particular, the commutator $ad[\,\mathcal{A}\,](M_f)$ admits a bounded extension from $\mathcal{C}_c^\infty(\mathbb{C}^n)$ to H_1.

Proof We have shown that for each $A \in \mathcal{A}$ there is $p \in \mathcal{C}^\infty(\mathbb{C}^n)$ and a vector fields Y with coefficients in $\mathcal{C}_b^\infty(\mathbb{C}^n)$ such that $A = Y + M_p$. In the case where $\mathcal{A} = [\,A\,]$ only consist of a single operator it follows:

$$\mathrm{ad}\,[\,\mathcal{A}\,]\,(M_f) = [\,A, M_f\,] = [\,Y + M_p, M_f\,] = [\,Y, M_f\,] = M_{Yf}.$$

and $Yf \in \mathcal{C}_b(\mathbb{C}^n)$. This proves Lemma 2.5.1 in the case $r = 1$. The full assertion follows by induction. $\square$

Let us consider T_f as an operators on the Segal-Bargmann space H_2. Because in general this space is not invariant under the vector fields $Z \in \mathcal{X}^\Phi(\mathbb{C}^n)$ we use operators on H_2 of the form

$$\Big\{\, PZ, PZ^*, P\mathrm{Re}(\,Z\,), P\mathrm{Im}(\,Z\,) : Z \in \mathcal{X}^\Phi\,(\,\mathbb{C}^n\,)\,\Big\}. \tag{2.5.1}$$

to build commutators. We reformulate PZ and PZ^* in terms of (unbounded) Toeplitz operators. For

$$Z = \sum_{j=1}^n \big\{\, a_j\, \partial_j + b_j\, \bar{\partial}_j \,\big\} \in \mathcal{X}^\Phi(\mathbb{C}^n)$$

we define the functions $g := f_{Z,2}$ and $h := f_{Z,1}$ as in (2.4.2). It is easy to verify that on H_2 it holds $\partial_j = T_{\bar{z}_j}$ and so it follows from Lemma 2.4.2 that:

$$PZ = \sum_{j=1}^n T_{a_j} T_{\bar{z}_j}, \qquad \text{and} \qquad PZ^* = -\sum_{j=1}^n T_{\bar{b}_j} T_{\bar{z}_j} + T_{h-\bar{g}}. \tag{2.5.2}$$

Let $f \in L^\infty(\mathbb{C}^n)$ and $\mathcal{A}_P$ a finite system in (2.5.1), then we obtain by equation (2.5.2) together with Proposition 2.1.2 that all commutators $\mathrm{ad}[\,\mathcal{A}_P\,](T_f)$ are well-defined as operators on $P\,[\,\mathcal{C}_c^\infty(\mathbb{C}^n)\,] \subset H_{\exp}(\mathbb{C}^n)$.

In the following definition we denote by B_{2n} the Euclidean ball in $\mathbb{C}^n$ of radius 1 centered in $0 \in \mathbb{C}^n$ and let ∂B_{2n} be its boundary. We want to define what we understand by $A_j \in \mathcal{A}$ (or $\mathcal{A}$) to be supported in a *cone* in $\mathbb{C}^n$.

Definition 2.5.1 Let $N \subset \partial B_{2n}$ be an arbitrary subset and define the *cone* $\mathcal{C}_N$ over N in $\mathbb{C}^n$ to be the set

$$\mathcal{C}_N := \Big\{\, z \in \dot{\mathbb{C}}^n : z \cdot \|z\|^{-1} \in N \,\Big\} \subset \mathbb{C}^n.$$

We say that an operator $A \in \{ Z, Z^*, \mathrm{Re}(Z), \mathrm{Im}(Z) : Z \in \mathcal{X}^{\Phi}(\mathbb{C}^n) \}$ has *support* in $\mathcal{C}_N$ if all the coefficients $a_j := Zz_j$ and $b_j := Z\bar{z}_j$ for have support in $\mathcal{C}_N$. The system $\mathcal{A}$ has *support* in $\mathcal{C}_N$ if this holds for each $A \in \mathcal{A}$.

All the unbounded operators we have considered above are closable on H_1. For the rest of chapter 2 we use the following convention:

General assumption: We identify $A \in \{ Z, Z^*, \mathrm{Re}(Z), \mathrm{Im}(Z) : Z \in \mathcal{X}^{\Phi}(\mathbb{C}^n) \}$ with the closed and densely defined extension $\bar{A} : H_1 \subset \mathcal{D}(\bar{A}) \to H_1$ of A where

$$\mathcal{D}(\bar{A}) := \text{closure of } \mathcal{C}_c^{\infty}(\mathbb{C}^n) \text{ in } H_1 \text{ w.r.t. the graph norm } \|\cdot\|_{\mathrm{gr}} := \|A\cdot\|_2 + \|\cdot\|_2.$$

To avoid confusing notations we write $\bar{A} = A$ and $\mathcal{D}(\bar{A}) = \mathcal{D}(A)$. By construction $\mathcal{C}_c^{\infty}(\mathbb{C}^n)$ is dense in $\mathcal{D}(A)$ with respect to the graph norm. Hence using this convention we can consider any finite system

$$\mathcal{A} \subset \Big\{ Z, Z^*, \mathrm{Re}(\,Z\,), \mathrm{Im}(\,Z\,) : Z \in \mathcal{X}^{\Phi}\,(\,\mathbb{C}^n\,) \Big\}$$

as a system of closed and densely defined operators on the space $H_1 = L^2(\mathbb{C}^n, \mu)$.

We say $[\,f\,] \in H_1$ has *support* in $C \subset \mathbb{C}^n$ if there is a function $g \in [\,f\,]$ such that $\mathrm{supp}(g) \subset C$. Now, we recall the following easy result on multiplication operators:

Lemma 2.5.2 *Assume that N_1, N_2 are open in ∂B_{2n} such that $N_1 \subset N_2$. If $f \in H$ has support in $\mathbb{C}^n \setminus \mathcal{C}_{N_2}$, then:*

(a) For $A \in \mathcal{A}$ and $f \in \mathcal{D}(A)$ the function Af has support in $\{ \mathbb{C}^n \setminus \mathcal{C}_{N_1} \} \cap supp(A)$.

(b) If $A \in \mathcal{A}$ has support in $\mathcal{C}_{N_1}$, then $f \in \mathcal{D}(A)$ and $Af = 0$. In particular, the commutator $[\,A, M_f\,] : H \subset \mathcal{D}(A) \to H$ is well-defined and identically zero.

Proof (a) Let $f \in \mathcal{D}(A)$ with support in $\mathbb{C}^n \setminus \mathcal{C}_{N_2}$. Then we choose $(g_n)_n \subset \mathcal{C}_c^{\infty}(\mathbb{C}^n)$ such that

$$\mathrm{supp}\,(\,g_n\,) \subset \mathbb{C}^n \setminus \mathcal{C}_{N_1} \qquad \text{and} \qquad \lim_{n\to\infty} \|\,g_n - f\,\|_{\,\mathrm{gr}} = 0.$$

It follows for $n \in \mathbb{N}$ that Ag_n has support in $\{ \mathbb{C}^n \setminus \mathcal{C}_{N_1} \} \cap \mathrm{supp}(A)$ and because $(Ag_n)_n$ admits a subsequence which converges to Af almost everywhere on $\mathbb{C}^n$ we obtain the inclusion $\mathrm{supp}(Af) \subset \{ \mathbb{C}^n \setminus \mathcal{C}_{N_1} \} \cap \mathrm{supp}(A)$.

(b) Choose a sequence $(f_n)_n \subset \mathcal{C}_c^{\infty}(\mathbb{C}^n)$ with $\lim_{n\to\infty} \|f_n - f\|_2 = 0$ such that each f_n has support in $\mathbb{C}^n \setminus \mathcal{C}_{N_1}$. Because A has support in $\mathcal{C}_{N_1}$ it follows that $Af_n = 0$. By assumption A is closed and we conclude that $f \in \mathcal{D}(A)$ and $Af = 0$. Because $g \in M_f[\,\mathcal{D}(A)\,]$ has support in $\mathbb{C}^n \setminus \mathcal{C}_{N_2}$ it follows that $\mathcal{D}(A)$ is invariant under M_f. Hence $[\,A, M_f\,]$ is well defined and the composition AM_f vanishes on $\mathcal{D}(A)$ by (a). Moreover, from (a) it is clear that $\mathrm{supp}(Ag) \subset \mathcal{C}_{N_1}$ for all $g \in \mathcal{D}(A)$ and so $M_f A = 0$ on $\mathcal{D}(A)$. This proves (b). □

Let us assume that the operator $A : H \supset \mathcal{D}(A) \to H$ is the closed and densely defined extension of $A \in \{ Z, Z^*, \mathrm{Re}(Z), \mathrm{Im}(Z) : Z \in \mathcal{X}^{\Phi}(\mathbb{C}^n) \}$ in the sense described above.

Lemma 2.5.3 *For $h \in \mathcal{C}_b^1(\mathbb{C}^n)$ the domain of definition $\mathcal{D}(A)$ is invariant under both operators M_h and P.*

Proof Let us write $A = Y + M_f$ where $f \in \mathcal{C}^\infty(\mathbb{C}^n)$ and Y is a smooth vector field in $\mathcal{X}^\Phi(\mathbb{C}^n)$. Because of the commutator equation

$$[\, A, M_h \,] = M_{Yh} \qquad \text{and} \qquad Yh \in \mathcal{C}_b\,(\, \mathbb{C}^n \,)$$

the commutator $[\, A, M_h \,]$ admits a bounded extension from $\mathcal{C}_c^\infty(\mathbb{C}^n)$ to H_1. By definition $\mathcal{C}_c^\infty(\mathbb{C}^n)$ is dense in $\mathcal{D}(A)$ with respect to the graph norm $\|\cdot\|_{\mathrm{gr}} = \|\cdot\| + \|A\cdot\|$. Hence by Lemma 1.4.3 in chapter 1 we only have to prove that

$$M_h\Big[\, \mathcal{C}_c^\infty\,(\, \mathbb{C}^n \,) \,\Big] \subset \mathcal{D}(A).$$

Let $g \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ and fix a sequence $(\psi_k)_k \subset \mathcal{C}_c^\infty(\mathbb{C}^n)$ with $\psi_k \to h$ and $Y\psi_k \to Yh$ uniformly on the compact set $\mathrm{supp}(g)$ as k tends to infinity. Then by Lebesgue's Theorem it follows that $\psi_k \cdot g \to M_h g$ in H_1 and as $k \to \infty$

$$A[\, \psi_k \cdot g \,] = \psi_k \cdot Ag + \big\{\, Y\psi_k \,\big\} \cdot g$$

is convergent in H_1. Because A is closed by assumption we obtain $M_h g \in \mathcal{D}(A)$.

We show that $\mathcal{D}(A)$ is invariant under P. By Lemma 1.4.3 and because the commutator $[\,P, A\,]$ has a bounded extension from $\mathcal{C}_c^\infty(\mathbb{C}^n)$ to H (see Theorem 2.4.1) again we only have to prove that $P[\,\mathcal{C}_c^\infty(\mathbb{C}^n)\,] \subset \mathcal{D}(A)$. Choose a *cut-off function* $\varphi \in \mathcal{C}^\infty(\mathbb{C}^n)$ such that $\varphi \equiv 1$ on B_{2n} and $\varphi \equiv 0$ on $\mathbb{C}^n \setminus 2B_{2n}$ and define $\varphi_k := \varphi(k^{-1}\cdot)$ for $k \in \mathbb{N}$. Then $\varphi_k(z) \to 1$ for all $z \in \mathbb{C}^n$ as k tends to infinity and because of

$$Y\varphi_k = k^{-1}\,[\, Y\varphi \,]\,\big(\, k^{-1}\cdot\big)$$

the sequence $(\, Y\varphi_k \,)_k$ is uniformly bounded and tends to 0 as $k \to \infty$. Let $g \in \mathcal{C}_c^\infty(\mathbb{C}^n)$, then $\lim_{k\to\infty} \varphi_k \cdot Pg = Pg$ in H_1 and

$$A\Big(\, \varphi_k \cdot \{\, Pg \,\} \,\Big) = \varphi_k \cdot \Big\{\, [\, A, P \,]\, g + PAg \,\Big\} + \{\, Y\varphi_k \,\} \cdot Pg. \tag{2.5.3}$$

By the continuity of $[\, A, P \,]$ it follows that $[\, A, P \,]g \in H_1$. Now Lebesgue's convergence theorem implies that the right hand side of (2.5.3) converges in H_1. Because A is closed by assumption we conclude that $Pg \in \mathcal{D}(A)$. $\square$

We prove a result on commutators analogous to Lemma 2.5.3 which we will need later on in our definition of Fréchet operator algebras. Let $\mathcal{A}$ be a finite systems of closed operators in $\{\, Z, Z^*, \mathrm{Re}(Z), \mathrm{Im}(Z) : Z \in \mathcal{X}^\Phi(\mathbb{C}^n) \,\}$.

Proposition 2.5.1 *For each $A \in \mathcal{A}$ the space $\mathcal{D}(A)$ is invariant under the closed extension of the commutator $ad[\, \mathcal{A} \,](P)$.*

Proof Let $A \in \mathcal{A}$, then by Theorem 2.4.1 the commutator $\mathrm{ad}[\mathcal{A}, A](P)$ admits a bounded extension from $\mathcal{C}_c^\infty(\mathbb{C}^n)$ to H_1. Lemma 1.4.3 now implies that it is sufficient to show:

$$\mathrm{ad}\,[\,\mathcal{A}\,]\,(P) : \mathcal{C}_c^\infty\,(\mathbb{C}^n) \to \mathcal{D}(A).$$

Fix $g \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ and let $(\varphi_k)_k$ be the sequence of functions in the proof of Lemma 2.5.3. Then we have

$$\lim_{k\to\infty} \Big\{\, \varphi_k \cdot \mathrm{ad}[\,\mathcal{A}\,](P)g \,\Big\} = \mathrm{ad}[\,\mathcal{A}\,](P)g$$

in H_1. Furthermore, there is a function $f \in \mathcal{C}^\infty(\mathbb{C}^n)$ and a smooth vector field Y such that $A = Y + M_f$. Hence

$$A\Big(\, \varphi_k \cdot \mathrm{ad}[\,\mathcal{A}\,](P)g \,\Big) = \varphi_k \cdot \Big\{\, \mathrm{ad}\,[\,\mathcal{A}, A\,]\,(P)g + \mathrm{ad}[\,\mathcal{A}\,](P)Ag \,\Big\} + [\,Y\varphi_k\,] \cdot \mathrm{ad}[\,\mathcal{A}\,](P)g.$$

The right hand side of this equation is convergent in H_1 as k tends to infinity and by the fact that A is closed we conclude that $\mathrm{ad}[\,\mathcal{A}\,](P)g$ is contained in $\mathcal{D}(A)$. □

As a first application of our calculations above we show that the *Segal-Bargmann* Toeplitz projection P is a smooth element with respect to all finite sets $\mathcal{V}$ of closed extensions of operators contained in $\{\,Z, Z^*, \mathrm{Re}(Z), \mathrm{Im}(Z) : Z \in \mathcal{X}^\Phi(\mathbb{C}^n)\,\}$ (our general assumption on the domains of definitions apply). More precisely, let us choose $\mathcal{B} = \mathcal{F} = \mathcal{L}(H)$ in Definition 1.2.1. Then we define a chain

$$\big(\, \Psi_k^\Delta := \Psi_k^\Delta[\,\mathcal{F}\,]\,\big)_k \qquad \text{with} \qquad k \in \mathbb{N}_0 \cup \{\infty\}$$

of Fréchet algebras as it was described in section 1.2. With our definition above Proposition 2.1.2 let us consider the following subspace $\mathcal{D}$ of $\mathcal{C}^\infty(\mathbb{C}^n)$:

$$\mathcal{D} := \Big\{\, f \in L_{\exp}(\mathbb{C}^n\,) \cap \mathcal{C}^\infty(\,\mathbb{C}^n\,) : \text{all partial derivatives of } f \text{ belong to } L_{\exp}(\,\mathbb{C}^n\,)\,\Big\}.$$

By Lemma 2.4.2 and the Leibniz formula it is easy to see that $\mathcal{D}$ is invariant under all operators $Z \in \mathcal{V}$. With $f \in \mathcal{D}$ let us examine the Toeplitz projection $Pf \in L_{\exp}(\mathbb{C}^n\,)$. For any multi-index $\alpha \in \mathbb{N}_0^n$ and with $m_\alpha(z) := z^\alpha$ we have shown in (2.2.2) that

$$\partial^\alpha Pf = P\big[\,\bar{m}_\alpha f\,\big].$$

Because $\bar{m}_\alpha f \in L_{\exp}(\mathbb{C}^n)$ and using the fact that $L_{\exp}(\mathbb{C}^n)$ is invariant under P (cf. Proposition 2.1.2) we conclude that $\mathcal{D}$ is invariant under P.

The closed extensions of operators $\mathrm{Re}(Z)$ and $\mathrm{Im}(Z)$ where $Z \in \mathcal{X}^\Phi(\mathbb{C}^n)$ are symmetric and by an application of Lemma 1.2.1 and Proposition 1.1.1 it follows

Lemma 2.5.4 *Let $\mathcal{V}$ be a symmetric subset of $\{\,Z, Z^*, Re(Z), Im(Z) : Z \in \mathcal{X}^\Phi(\mathbb{C}^n)\,\}$, i.e. $A \in \mathcal{V}$ iff $A^* \in \mathcal{V}$, then Ψ_∞^Δ and $P\Psi_\infty^\Delta P$ are Ψ^*-algebras in $\mathcal{L}(\,H_1\,)$ resp. in $\mathcal{L}(\,H_2\,)$.*

Proof It is easy to see that we can replace $\mathcal{V}$ by the system $\{\,\mathrm{Re}(Z), \mathrm{Im}(Z) : Z \in \mathcal{V}\,\}$ of symmetric closed operators without changing the algebra Ψ_∞^Δ. The second assertion directly follows from the first and $P = P^* \in \Psi_\infty^\Delta$. □

Theorem 2.5.1 *It holds $P \in \Psi_\infty^\Delta$. In particular, the space $\mathcal{H}_\mathcal{V}^\infty$ in Definition 1.2.3 is invariant under the Toeplitz projection P.*

Proof To prove Theorem 2.5.1 we have to check assumption (A) in Proposition 1.2.1 for each $k \in \mathbb{N}$. We choose $D := \mathcal{D}$ defined above. Then by our general assumption and $\mathcal{C}_c^\infty(\mathbb{C}^n) \subset \mathcal{D}$ for all $A \in \mathcal{V}$ this is a dense subspace of $\mathcal{D}(A)$ with respect so the graph norm. (A) directly follows from Theorem 2.4.1. □

Proposition 2.5.2 *Let $k \in \mathbb{N}$ and $h \in \mathcal{C}_b^k(\mathbb{C}^n)$. Then the multiplication operator M_h as well as the Toeplitz operators T_h are contained in Ψ_k^Δ.*

Proof Because Ψ_k^Δ is an algebra and $P \in \Psi_\infty^\Delta \subset \Psi_k^\Delta$ it is sufficient to prove that $M_h \in \Psi_k^\Delta$. Let $k = 1$ and fix $A \in \mathcal{V}$. Then by Lemma 2.5.3 it follows that $\mathcal{D}(A)$ is invariant under M_h. Because all the commutators $[A, M_h]$ are multiplications with bounded continuous functions they admit bounded extensions from $\mathcal{D}(A)$ to H_1. By definition we have $M_h \in \Psi_1^\Delta$.

Note that in the case $k > 1$ the commutators $[\,A, M_h\,]$ are multiplications with symbols in $\mathcal{C}_b^{k-1}(\mathbb{C}^n)$ and so the full assertion follows by induction. □

Let $f \in L^\infty(\mathbb{C}^n)$ and consider the Toeplitz operator T_f restricted to H_2. With a finite set $\mathcal{V}_P$ contained in (2.5.1) we immediately obtain from the Propositions 2.5.2 and 1.4.2 with $Q := P$ and the notations introduced there:

Theorem 2.5.2 *Let $h \in \mathcal{C}_b^k(\mathbb{C}^n)$ where $k \in \mathbb{N} \cup \{\infty\}$, then it holds:*

$$T_h \in \Psi_k^{\Delta_P} := \Psi_k^{\Delta_P}\,[\,\mathcal{L}\,(H_2)\,]\,.$$

In particular, the $\mathcal{PV}$-Sobolev space $\mathcal{H}_{\mathcal{V}_P}^k \subset H_2$ is invariant under T_h.

Finally, we show that the Fréchet operator algebras Ψ_k^Δ (resp. $\Psi_k^{\Delta_P}$) are localized in the following sense:

Let N_1, N_2 be open in ∂B_{2n} such that $N_1 \subset N_2$. Choose a finite set $\mathcal{V}$ of operators in

$$\Big\{\, Z, Z^*, \mathrm{Re}(\,Z\,), \mathrm{Im}(\,Z\,) : Z \in \mathcal{X}^\Phi\,(\,\mathbb{C}^n\,)\,,\ \text{and } Z \text{ has support in } \mathcal{C}_{N_1} \Big\}.$$

We prove that the constructed Fréchet-algebras are invariant under perturbations of Toeplitz operators with symbols supported in the cone $\mathbb{C}^n \setminus \mathcal{C}_{N_2}$.

Theorem 2.5.3 *Let $h \in L^\infty(\mathbb{C}^n)$ such that supp$(h) \subset \mathbb{C}^n \setminus \mathcal{C}_{N_2}$. Then $T_h \in \Psi_\infty^\Delta \subset \mathcal{L}(H_1)$ (resp. we have $T_h \in \Psi_\infty^{\Delta_P} \subset \mathcal{L}(H_2)$).*

Proof We only have to show $T_h \in \Psi_\infty^\Delta$, then $T_h \in \Psi_\infty^{\Delta_P}$ immediately follows from Proposition 1.4.2. According to Theorem 2.5.1 we have $P \in \Psi_\infty^\Delta$ and so it is sufficient to prove $M_h \in \Psi_\infty^\Delta$. Let $\mathcal{A}$ be a finite system of operators in $\mathcal{V}$ and $A \in \mathcal{A}$. Then by Lemma 2.5.2 (b) the space $\bigcap_{A\in\mathcal{A}} \mathcal{D}(A)$ is invariant under M_h and the commutator $[\,A, M_h\,]$ vanishes on $\mathcal{D}(A)$ because A and h have disjoint support by assumption. By definition we have $M_h \in \Psi_1^\Delta$ and because the commutator $[\,A, M_h\,]$ vanishes for all $A \in \mathcal{V}$ it follows that $M_h \in \Psi_k^\Delta$ for all $k \in \mathbb{N}$. □

Chapter 3

Smooth elements in an algebra of Toeplitz operators generated by unitary groups.

For $n \in \mathbb{N}$ we denote by μ the Gaussian measure on the complex space $\mathbb{C}^n$ given by the density $d\mu = \pi^{-n} \exp(-\| \cdot \|^2)\, dv$ where v is the usual Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. In the previous chapter we have introduced the Segal-Bargmann space $H_2 := H^2(\mathbb{C}^n, \mu)$ as the closed subspace of $H_1 := L^2(\mathbb{C}^n, \mu)$ consisting of all entire functions which are square integrable with respect to μ. It is well-known that H_2 is a reproducing kernel Hilbert space and some of the basic results on the operator theory on it can be found in [5] as well as in [23], [21] and [37].

Let P denote the orthogonal projection from H_1 onto H_2. With a symbol $g \in \mathcal{T}(\mathbb{C}^n)$ (for definition see below) the Toeplitz operator

$$T_g : H_2 \supset \mathcal{D}(\, T_g \,) \to H_2$$

is the densely defined map given by $T_g f := P(\, gf \,)$ for $f \in \mathcal{D}(\, T_g \,)$. For $g \in L^\infty(\mathbb{C}^n)$ it is obvious that T_g is bounded and we denote by $\mathcal{A} \subset \mathcal{L}(H_2)$ the C^*-algebra generated by all Toeplitz operators with bounded symbols. In the following we also consider Toeplitz operators with unbounded symbols and we refer to [89] and [91]. There are unitary groups acting on H_1 and preserving H_2 corresponding to the symmetries of $\mathbb{C}^n$, among those the Weyl group. For each $x \in \mathbb{C}^n$ the unitary Weyl operator W_x on H_2 is a weighted shift operators in the direction x and it admits an extension to an unitary operator on H_1. We consider the C_0-group $(W_{tx})_{t\in\mathbb{R}}$ which acts on $\mathcal{A}$.

By the construction due to the authors of [79] which we have described in detail in chapter 1, we obtain a scale of Banach-subalgebras $\Psi_n^\alpha[\, \mathcal{A}\,] \subset \mathcal{A}$ which is characterized by the continuity of iterated commutators with the infinitesimal generator $V^{(x)}$ of $(W_{tx})_{t\in\mathbb{R}}$. Due to our remarks in section 1.3 this scale is related to the smoothness of the operators in $\mathcal{A}$ with respect to the Weyl group action. The Ψ^*-algebra $\Psi_\infty^\alpha[\mathcal{A}]$ of smooth elements in $\mathcal{A}$ with respect to $(W_{tx})_{t\in\mathbb{R}}$ is obtained by the intersection of the scale $(\, \Psi_n^\alpha[\, \mathcal{A}\,]\,)_n$. It turns out that under quite weak conditions on the bounded symbols g the operators $T_g \in \mathcal{A}$ (and hence all finite sums of finite products) belong to $\Psi_\infty^\alpha[\,\mathcal{A}\,]$.

Denote by $\mathcal{B}$ the C^*-algebra of all multiplication operators M_g on H_1 with bounded symbols g. Then the Weyl group also acts on $\mathcal{B}$ and we obtain a similar scale of subalgebras in this space. Moreover, we denote by Φ the projection from $\mathcal{B}$ to $\mathcal{A}$ defined by

$$\Phi\,(\,M_g\,) := T_g = P\,M_g\,P.$$

From the fact that P commutes with W_x for all $x \in \mathbb{C}^n$ we conclude that $P \in \Psi_\infty^\alpha[\,\mathcal{B}\,]$ and so it clearly holds:

$$\Phi\big(\,\Psi_\infty^\alpha[\,\mathcal{B}\,]\,\big) \subset \Psi_\infty^\alpha[\,\mathcal{A}\,].$$

The question arises if Φ preserves the scales. Let us denote by $\mathcal{I}_x := \mathcal{I}(V^{(x)}) \subset \mathcal{B}$ the space of all operators in $\mathcal{B}$ which leave the domain of definition $\mathcal{D}(V^{(x)}) \subset H_1$ invariant. Then we have

$$\mathcal{L}(\,H_1\,) \supset \mathcal{I}_x \supset \Psi_1^\alpha[\,\mathcal{B}\,] \supset \Psi_2^\alpha[\,\mathcal{B}\,] \supset \cdots \supset \Psi_\infty^\alpha[\,\mathcal{B}\,] = \bigcap_{j\in\mathbb{N}} \Psi_j^\alpha[\,\mathcal{B}\,].$$

Let $\mathcal{I}_x \subset \mathcal{L}(H_1)$ carry the uniform topology of $\mathcal{L}(H_1)$. We prove that the map:

$$\Phi : \mathcal{I}_x \longrightarrow \Psi_\infty^\alpha[\,\mathcal{A}\,] = \bigcap_{j\in\mathbb{N}} \Psi_j^\alpha[\,\mathcal{A}\,]$$

is well-defined and continuous in the Fréchet topology of $\Psi_\infty^\alpha[\,\mathcal{A}\,]$. Moreover, we show that the algebra $\Psi_\infty^\alpha[\,\mathcal{A}\,]$ contains all Toeplitz operators T_f with continuous symbol f vanishing at infinity and it follows that the algebra of compact smooth elements is uniformly dense in $\mathcal{K}(H_2)$. Our results show that T_f belongs to $\Psi_\infty^\alpha[\mathcal{A}]$ under quite weak assumptions on the symbol f. We can give an example of an operator $A \in \mathcal{A}$ which does not even belong to the algebra $\Psi_1^\alpha[\,\mathcal{A}\,]$.

Via the *Bargmann transform* $\beta : L^2(\mathbb{R}^n, v) \to H_2$ (cf. [5], [58], [82]) and with a bounded symbol φ on $\mathbb{C}^n$ we consider the *Gabor-Daubechies* windowed Fourier localization operators $L_\varphi^{v_0}$ on $L^2(\mathbb{R}^n, v)$ defined by

$$L_\varphi^{v_0} = \beta^{-1}\,T_\varphi\,\beta,$$

where the window v_0 is a fixed normalized Gaussian function on $L^2(\mathbb{R}^n, v)$. For $a \in \mathbb{C}^n$ the Weyl group $(W_{ta})_{t\in\mathbb{R}}$ transforms to an unitary group of operators on $L^2(\mathbb{R}^n, v)$ which in the case of $a \in \mathbb{R}^n$ coincides with a usual shift in direction a. The corresponding algebras of C^k-elements in the C^*-algebra $\mathcal{D} \subset \mathcal{L}(L^2(\mathbb{R}^n, v))$ generated by

$$\Big\{\,L_\varphi^{v_0} : \varphi \in L^\infty\,(\,\mathbb{C}^n\,)\,\Big\}$$

is denoted by $\Psi_k^{p,q}[\,\mathcal{D}\,]$ where $a = p + iq$ and $k \in \mathbb{N}_0 \cup \{\infty\}$. They can be obtained from $\Psi_k^\alpha[\,\mathcal{A}\,]$ by conjugation with β. There is a natural scale of corresponding generalized Sobolev spaces $\mathcal{H}_{p,q}^k \subset L^2(\mathbb{R}^n, v)$ and as an application of our results on Toeplitz operators we prove that $L^{v_0}_{1+\varphi}$ is in $\Psi^{p,q}_{\infty}[\,\mathcal{D}\,]$ for all continuous symbols φ on $\mathbb{C}^n$ vanishing at infinity. Moreover, the general theory leads to a result on regularity. If there is $u \in L^2(\mathbb{R}^n, v)$ such that

$$L^{v_0}_{1+\varphi} u = f \in \mathcal{H}^k_{p,q}$$

for some $k \in \mathbb{N}_0 \cup \{\infty\}$, then we show that $u \in \mathcal{H}^k_{p,q}$. The operators $L^{v_0}_{\varphi} = \beta^{-1}\, T_{\varphi}\, \beta$ also can be viewed as pseudo-differential operators on $L^2(\mathbb{R}^n, v)$ in its Weyl form (cf. [58], [82]) with Weyl symbol σ. Under the canonical identification of $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ both symbols φ and σ are connected to each other via the heat equation on $\mathbb{R}^{2n}$. Roughly speaking the Weyl symbol σ of $L^{v_0}_{\varphi}$ is the solution of the heat equation with initial data φ at time $t = \frac{1}{8}$:

$$\varphi^{(\frac{1}{8})} := \sigma_0 = \left(e^{-\frac{1}{8}\,\Delta} \right) \varphi \qquad \text{where} \qquad \sigma_0(\, q, p\,) = \sigma(-p, 2q).$$

From the fact that the heat equation is smoothing we find that σ_0 has high regularity even if we start with an only bounded symbol φ. If we restrict ourselves to symbols $\varphi \in \mathcal{C}_c(\mathbb{C}^n)$ with compact support, then we can show that the map

$$\{\, \mathcal{C}_c\,(\, \mathbb{C}^n\,)\,, \|\cdot\|_{\infty}\,\} \ni \varphi \mapsto \varphi^{(\frac{1}{2})} \in \mathcal{S}^{-\infty}_{\rho,\delta} := \bigcap_{m \in \mathbb{R}} \mathcal{S}^m_{\rho,\delta}$$

is well-defined and continuous for $0 \leq \delta \leq \rho \leq 1$ if the right hand side carries the projective (Fréchet)-topology induced by the symbol spaces $\mathcal{S}^m_{\rho,\delta}$ for $m \in \mathbb{R}$. Note that these observations have close connections to our results on the smoothness of Toeplitz operators on H_2.

With an unitary C_0-group $(u_t)_{t\in\mathbb{R}} \subset \mathcal{L}(\mathbb{C}^n)$ we consider the corresponding group on H_2 of composition operators $U_t : H_2 \ni f \mapsto f \circ u_t \in H_2$. We prove that on a dense subspace of H_2 its infinitesimal generator A_2 coincides with an unbounded Toeplitz operator. Denote by μ_c the Gaussian measure on $\mathbb{C}^n$ with correlation $c \in (0, \frac{1}{2})$ and let a be the infinitesimal generator of $(u_t)_{t\in\mathbb{R}}$. Then for $f \in L^{\infty}(\mathbb{C}^n)$ and $j \in \mathbb{N}$ we define

$$\|\, f\,\|_{c,j} := \sup \Big\{\, (\, 1 + \|\, a\lambda\,\|\,)^j \cdot \|\, f \circ \tau_{\lambda}\,\|_{L^1(\mathbb{C}^n,\mu_c)} : \lambda \in \mathbb{C}^n \,\Big\}.$$

Note that the group $(U_t)_t$ acts isometric with respect to $\|\cdot\|_{c,j}$ and so we can define the corresponding U_t-invariant normed spaces:

$$\mathcal{F}_{c,j}(\mathbb{C}^n) := \Big\{\, f \in L^{\infty}(\mathbb{C}^n) : \|\, f\,\|_{c,j} < \infty \,\Big\}.$$

For $n \in \mathbb{N} \cup \{\infty\}$ denote by $\Psi^U_n[\,\mathcal{A}\,] \subset \mathcal{A}$ the scale of Banach-algebras in $\mathcal{A}$ defined by the action of the unitary C_0-group $(U_t)_{t\in\mathbb{R}}$, then for $j \in \mathbb{N}$ and $c \in (0, \frac{1}{2})$ we prove that the space

$$\Big\{\, T_f : f \in \mathcal{F}_{c,j}\ (\,\mathbb{C}^n\,)\,,\ \mathbb{R} \ni t \mapsto t^{-1}\,[\, f \circ u_t - f\,] \in \mathcal{F}_{c,j}\,(\mathbb{C}^n) \text{ is continuous in } t = 0 \,\Big\}$$

is contained in $\Psi^U_j[\,\mathcal{A}\,]$. As a corollary we conclude that the space of Toeplitz operators in $\Psi^U_{\infty}[\,\mathcal{A}\,]$ is invariant under perturbations of the symbols by continuous functions with compact support. Finally, we compare smoothness with respect to rotation and translation. We give an example of a function $f \in L^{\infty}(\mathbb{C})$ such that $T_f \in \Psi^{\alpha}_{\infty}[\,\mathcal{A}\,]$ but with a rotation $(u_t)_t$ in $\mathbb{C}$ the group action $\mathbb{R} \ni t \mapsto U_t^{-1} \circ T_f \circ U_t \in \mathcal{L}(H_2)$ not even is continuous.

3.1 Smooth Toeplitz operators generated by the Weyl group action

For any $x \in \mathbb{C}^n$ we have introduced the unitary group $U_t := (W_{tx})_{t\in\mathbb{R}}$ of Weyl operators in Definition 2.1.1. We want to describe conditions on the symbol f of a Toeplitz operator T_f which are sufficient for T_f to be smooth with respect to the corresponding group action (cf. section 1.3). According to Theorem 1.3.1 the smoothness of operators equivalently can be described by iterated commutators with the infinitesimal generator V of the group U_t. In our situation it turns out that V is an unbounded Toeplitz operator and so we start with some commutators formulas for Toeplitz operators. Similar to chapter 2 an important tool for the boundedness results is the Schur test (cf. Theorem 2.1.1 and Proposition 2.1.1).

With our notation in (2.1.11) we define the following subspace $S_{\text{Lip}}(\mathbb{C}^n)$ of $H_{\exp}(\mathbb{C}^n)$. By $M(\mathbb{C}^n)$ we denote the space of all measurable functions on $\mathbb{C}^n$:

$$S_{\text{Lip}}(\mathbb{C}^n) := \Big\{ f \in M(\mathbb{C}^n) : \exists\, c, D > 0 \text{ with } |\, f(z) - f(w) \,| \leq D \cdot \exp\big(c \,\|\, z - w \,\| \big) \Big\}.$$

and we write $SP_{\text{Lip}}(\mathbb{C}^n)$ for the space of all functions in $S_{\text{Lip}}(\mathbb{C}^n)$ of polynomial growth:

$$\text{SP}_{\text{Lip}}(\, \mathbb{C}^n \,) := S_{\text{Lip}}(\, \mathbb{C}^n \,) \cap \text{Pol}(\, \mathbb{C}^n \,)$$

Let $L_{\exp}(\mathbb{C}^n)$ be the subspace of H_1 defined above Proposition 2.1.2. For any linear space E we denote by $L(E)$ the class of all linear operators on E. We can prove:

Lemma 3.1.1 *Let $f \in SP_{Lip}(\mathbb{C}^n)$. Then we have the inclusions*

$$M_f \in L\,(\, L_{\exp}(\mathbb{C}^n)\,) \qquad \textit{and} \qquad P\,[\, L_{\exp}(\, \mathbb{C}^n \,)\,] \subset H_{\exp}(\, \mathbb{C}^n \,).$$

Moreover, the commutators $ad^j[\, M_f \,](P) \in L(\, L_{\exp}(\mathbb{C}^n)\,)$ have a continuous extension to H_1 for all $j \in \mathbb{N}$.

Proof The first inclusion is obvious from our assumption on the polynomial growth of the symbol f and $P[\, L_{\exp}(\, \mathbb{C}^n \,)\,] \subset H_{\exp}(\mathbb{C}^n)$ follows with the same computation as in Proposition 2.1.2. In order to show that for all $j \in \mathbb{N}$ the commutators $ad^j[\, M_f \,](P)$ have a continuous extension to H_1 we use the well-known formula

$$\text{ad}^j\,[\, M_f \,]\,(P) = \sum_{l=0}^{j} (-1)^l \binom{j}{l} M_{f^{j-l}} P M_{f^l}. \tag{3.1.1}$$

Consider one of the operators $A_{j,l} := M_{f^{j-l}} P M_{f^l}$ where $j \geq l$. Because the Toeplitz projection P is an integral operator on H_1 with integral kernel $K(z,u) = \exp(\langle z, u\rangle)$ we conclude that $A_{j,l}$ is an integral operator as well. For its kernel K_j^l we compute

$$K_j^l(\, z, u \,) = f(z)^{j-l} \cdot f(\, u \,)^l \cdot \exp\Big(\langle z, u \rangle \Big), \qquad \forall\, z, u \in \mathbb{C}^n. \tag{3.1.2}$$

Hence from (3.1.2) we conclude that the iterated commutators $\mathrm{ad}^j[M_f](P)$ are integral operators with kernel

$$\begin{aligned} K_j(z,u) &= \sum_{l=0}^{j}(-1)^l \binom{j}{l} f(z)^{j-l} f(u)^l \exp\Big(\langle z,u\rangle\Big) \\ &= [\,f(z)-f(u)\,]^j \exp\big(\langle z,u\rangle\big). \end{aligned}$$

Because of $f \in SP_{\mathrm{Lip}}(\mathbb{C}^n)$ we can choose constants $c, D > 0$ such that

$$\big|\, f(z) - f(u) \,\big| \leq D \exp\Big(c \,\|\, z - u \,\|\Big)$$

and we obtain the following estimate for the kernel of the iterated commutator:

$$|\, K_j(z,u)\,| \leq D^j \exp\Big(cj \,\|\, z-u \,\| + \mathrm{Re}\,\langle z,u\rangle \Big).$$

The function $F_j := D^j \exp(\, cj \,\|\cdot\|\,)$ is in $L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})$ for all $j \in \mathbb{N}$ and according to Proposition 2.1.1 we conclude that $\mathrm{ad}^j[\,M_f\,](P)$ has a continuous extension to H_1 for all $j \in \mathbb{N}$ with:

$$\Big\|\, \mathrm{ad}^j[\,M_f\,](P) \,\Big\| \leq 2^n \,\|\, F_j \,\|_{L^1(\mathbb{C}^n,\mu_{\frac{1}{2}})}\,. \qquad \square$$

Lemma 3.1.2 *With functions $f, g \in Pol(\mathbb{C}^n)$ and all $j \in \mathbb{N}$ the iterated commutators*

$$A_j(f,g) := ad^j\,[\,M_f\,]\Big(\,[\,P, M_g\,]\Big) \in L\,(\,L_{\exp}(\mathbb{C}^n)\,)$$

are well-defined. Moreover, we have

$$ad^j[\,M_f\,]\Big(\,[\,P, M_g\,]\Big) = \Big[\,ad^j[\,M_f\,](P), M_g\,\Big].$$

Proof We conclude from Proposition 2.1.2 that for all $j \in \mathbb{N}$ the operators $A_j(f,g)$ are well-defined on $L_{\exp}(\mathbb{C}^n)$. Let $j = 1$, then we have

$$\begin{aligned} \Big[\,[\,M_f, P\,], M_g\Big] &= [\,M_f P, M_g\,] - [\,P\,M_f, M_g\,] \\ &= M_f\,[\,P, M_g\,] - [\,P, M_g\,]\,M_f = \mathrm{ad}\,[\,M_f\,]\Big(\,[\,P, M_g\,]\Big). \end{aligned}$$

Now, assume that $\mathrm{ad}^j[\,M_f\,](\,[\,P, M_g\,]\,) = [\,\mathrm{ad}^j[\,M_f\,](P), M_g\,]$ holds for $j \in \mathbb{N}$, then we find by induction:

$$\begin{aligned} \mathrm{ad}^{j+1}\,[\,M_f\,]\Big(\,[\,P, M_g\,]\Big) &= \Big[\,M_f, \mathrm{ad}^j[\,M_f\,]\Big(\,[\,P, M_g\,]\Big)\Big] \\ &= \Big[\,M_f, \Big[\,\mathrm{ad}^j[\,M_f\,](P), M_g\,\Big]\Big] \\ &= \Big[\,M_f, \mathrm{ad}^j[\,M_f\,](P)\,M_g\,\Big] - \Big[\,M_f, M_g\,\mathrm{ad}^j[\,M_f\,](P)\,\Big] \\ &= \Big[\,M_f, \mathrm{ad}^j[\,M_f\,](P)\,\Big]\,M_g - M_g\,\Big[\,M_f, \mathrm{ad}^j[\,M_f\,](P)\,\Big] \\ &= \Big[\,\mathrm{ad}^{j+1}\,[\,M_f\,](P), M_g\,\Big]. \end{aligned} \qquad \square$$

Corollary 3.1.1 *Let $g \in L^\infty(\mathbb{C}^n)$ and $f \in SP_{Lip}(\mathbb{C}^n)$. Then for all $j \in \mathbb{N}$ the operators $A_j(f,g)$ admit continuous extensions to bounded operators on H_1. Moreover, all the maps*

$$L^\infty(\mathbb{C}^n) \ni g \mapsto A_j(f,g) \in \mathcal{L}(H_1) \tag{3.1.3}$$

are continuous and linear between Banach spaces.

Proof According to Lemma 3.1.1 all commutators $\mathrm{ad}^j[\, M_f \,](P)$ admit continuous extensions to H_1 for $j \in \mathbb{N}$. Further, from Lemma 3.1.2 we conclude that (3.1.3) is linear with

$$\|\, A_j(f,g) \,\| \leq 2 \,\|\, g \,\|_\infty \, \Big\| \mathrm{ad}^j[\, M_f \,](P) \Big\| . \qquad \square$$

Fix a symbol $g \in L^\infty(\mathbb{C}^n)$ and let $f \in SP_{\mathrm{Lip}}(\mathbb{C}^n)$. Then by Lemma 3.1.2 we have for all numbers $j \in \mathbb{N}$

$$\mathrm{ad}^{j+1}[\, M_f \,]\ (P) = \Big[\, M_f, \mathrm{ad}^j[\, M_f \,](P) \Big] = \mathrm{ad}^j\, [\, M_f \,] \Big(\ [\, M_f, P \,]\ \Big). \tag{3.1.4}$$

With a finite set

$$X := \Big\{\, X_1, \cdots, X_n \,\Big\} \subset \mathcal{L}(\, H_1 \,)$$

of bounded operators on H_1 we denote by $\mathcal{A}(\, X_1, \cdots, X_n\,)$ the sub-algebra of $\mathcal{L}(H_1)$ generated by the elements of X. Moreover, let

$$\mathcal{M}_P\,(\, X_1, \cdots, X_n \,) := \Big\{\, PCP \in \mathcal{L}(H_2) : C \in \mathcal{A}(\, X_1, \cdots, X_n \,) \,\Big\}$$

be the corresponding subspace in $\mathcal{L}(H_2)$. Using Lemma 3.1.1 and Corollary 3.1.1 and according to (3.1.4) we can consider $B_0(f) := [\, P, M_f \,]$, $A_0(f,g) := [\, P, M_g \,]$ and

$$B_j(f) := \mathrm{ad}^j[\, M_f \,]\Big(\, [\, P, M_f \,]\Big), \qquad A_j\,(f,g) := \mathrm{ad}^j\,[\, M_f \,]\Big(\ [\, P, M_g \,]\ \Big),$$

for all $j \in \mathbb{N}$ as bounded operators on H_1. Moreover, from Lemma 3.1.2 and using (3.1.4) we conclude that the equality holds:

$$A_j(f,g) = [\, M_g, B_{j-1}(f) \,]$$

for all $j \geq 1$. Hence for all $n \in \mathbb{N}$ we have the inclusions

$$\mathcal{A}\Big(\, B_j(f), A_j(f,g) : j = 1, \cdots, n \,\Big) \subset \mathcal{A}\Big(\, B_j(f), M_g : j = 0, \cdots, n \,\Big) \subset \mathcal{L}(H_1).$$

and the corresponding result is true if we replace $\mathcal{A}$ by $\mathcal{M}_P$ and H_1 by H_2.

Theorem 3.1.1 *Let $g \in L^\infty(\mathbb{C}^n)$ and fix a symbol $f \in SP_{Lip}(\mathbb{C}^n)$. Then for all $j \in \mathbb{N}$ the iterated commutators*

$$C_j(f,g) := ad^j\,[\, T_f \,]\ (T_g) \in L\,(\, H_{\exp}(\mathbb{C}^n)\,)$$

are well-defined and with a finite set $L \subset \mathbb{N}$ they have the form

$$C_j(\, f, g\,) := \sum_{l \in L} P\, a_l\, b_l\, c_l\, P \tag{$*$}$$

where

(1) $a_l, c_l \in \mathcal{A}_j(f) := \mathcal{A}\Big(\, B_r(f), I : r = 0, \cdots, j-1 \,\Big)$ *and*

(2) $b_l \in \Big\{\, A_r(f,g) : r = 0, \cdots, j-1 \,\Big\}$.

In particular, we have:

$$C_j(\, f, g\,) \in \mathcal{M}_P\Big(\, B_r(f), A_r(f,g) : r = 0, \cdots, j-1 \,\Big)$$

and each operator $C_j(f,g)$ can be considered as an element of $\mathcal{L}(H_2)$.

Proof By Proposition 2.1.2 the commutators $C_j(f,g)$ are well-defined for all $j \in \mathbb{N}$. It is a straightforward computation that

$$C_1(f,g) = [\, T_f, T_g \,] = P \Big[\, [\, P, M_f \,], [\, P, M_g \,] \,\Big]\, P = P \Big[\, B_0(f), A_0(f,g) \,\Big]\, P$$

and this proves $(*)$ in the case $j = 1$. Assume that $C_j(f,g)$ has the form $(*)$ for $j \in \mathbb{N}$. Then there are operators $a_l, c_l \in \mathcal{A}_j(f)$ such that

$$C_{j+1}(f,g) = [\, T_f, C_j(f,g) \,] = \sum_{l \in L} [\, T_f, P\, a_l\, b_l\, c_l\, P \,]\,.$$

To prove $(*)$ in the case $j+1$ it is sufficient to show that there is a finite set $\tilde{L} \subset \mathbb{N}$ and operators $\tilde{a}_{k,l}, \tilde{c}_{k,l} \in \mathcal{A}_{j+1}(f)$ and $\tilde{b}_{k,l} \in \{\, A_r(f,g) : r = 0, \cdots, j \,\}$ such that

$$[\, T_f, P\, a_l\, b_l\, c_l\, P \,] = \sum_{k \in \tilde{L}} P\, \tilde{a}_{k,l} \tilde{b}_{k,l}\, \tilde{c}_{k,l}\, P, \qquad\qquad (l \in L).$$

This follows from $T_f\, P\, a_l\, b_l\, c_l\, P = P\, M_f\, P\, a_l\, b_l\, c_l\, P$ together with

$$[\, M_f, A_r(f,g) \,] = A_{r+1}(f,g)$$

for all $r \in \{\, 0, \cdots, j-1 \,\}$ and $[\, M_f, P \,], [\, M_f, a_l \,], [\, M_f, c_l \,] \in \mathcal{A}_{j+1}(f)$ for all $l \in L$. □

Corollary 3.1.2 *Let $g \in L^\infty(\mathbb{C}^n)$ and $f \in SP_{Lip}(\mathbb{C}^n)$. Then with a finite set $L \subset \mathbb{N}$ and operators $a_l, c_l \in \mathcal{A}_j(f)$ the commutators $C_j(f,g)$ have the form*

$$C_j(f,g) := \sum_{l \in L} P\, a_l\, M_g\, c_l\, P.$$

for all $j \in \mathbb{N}$. In particular, the maps

$$L^\infty(\mathbb{C}^n) \ni g \mapsto C_j\,(f,g) \in \mathcal{L}\,(H_2)$$

are continuous and linear between Banach spaces.

Proof In the case $j = 1$ this follows from

$$C_1(g, f) = P \Big([\, M_f, P \,]\; M_g + M_g\; [\, M_f, P \,] \Big) P$$

and for all $j > 1$ the assertion is a direct consequence of Theorem 3.1.1 together with the formula $A_j(f, g) = [\, M_g, B_{j-1}(f) \,]$. □

For each $x \in \mathbb{C}^n$ we consider the unitary *Weyl operators* $W_x \in \mathcal{L}(H_1)$ we have introduced in Definition 2.1.1. As we have remarked in Lemma 2.1.1 the operators W_x commute with the Toeplitz projection P and so H_2 is an invariant subspace for W_x. Hence we can consider W_x as an unitary operator on H_2. It is an easy computation that both the spaces $L_{\exp}(\mathbb{C}^n)$ and $H_{\exp}(\mathbb{C}^n)$ are invariant for all the Weyl operators. In fact, in the case of $H_{\exp}(\mathbb{C}^n)$ this directly follows by Proposition 2.1.2 and the observation that W_x is a Toeplitz operator with bounded symbol:

$$W_x = \exp\left(\frac{1}{2} \parallel x \parallel^2 \right) T_{\exp(\, 2i\, \mathrm{Im} \langle \cdot, x \rangle \,)}.$$

We prove this interesting relation. With $f \in H_2$ and $y \in \mathbb{C}^n$ and by the reproducing property it follows that:

$$\begin{aligned}
\Big[T_{\exp(\, 2i\, \mathrm{Im} \langle\, \cdot, x \,\rangle\,)} f \Big](\, y \,) &= \Big\langle \exp\big(2i\, \mathrm{Im}\, \langle\, \cdot, x \,\rangle \big)\, f, \exp\big(\langle\, \cdot, y \,\rangle \big) \Big\rangle_2 \\
&= \Big\langle \exp\big(\langle\, \cdot, x \,\rangle \big)\, f, \exp\big(\langle\, \cdot, y - x \,\rangle \big) \Big\rangle_2 \\
&= \exp\big(\langle\, y - x, x \,\rangle \big) \cdot f(\, y - x \,) \\
&= \exp\big(-2^{-1} \parallel x \parallel^2 \big)\; \big[\, W_x f \,\big](\, y \,).
\end{aligned}$$

Thus, for each operator $A \in L(D)$ where D is one of the spaces $H_{\exp}(\mathbb{C}^n)$ or $L_{\exp}(\mathbb{C}^n)$ the conjugation

$$A_x := W_{-x}\, A\, W_x \in L(D)$$

is well-defined linear and in general leads to an unbounded operator on D. We give some easy commutator formulas involving A_x:

Lemma 3.1.3 *Let $f \in Pol(\mathbb{C}^n)$ and $A \in L(\, L_{\exp}(\mathbb{C}^n)\,)$. Then we have with $x \in \mathbb{C}^n$ and all numbers $j \in \mathbb{N}$:*

(a) $\Big(ad^j[\, M_f \,]\, (A) \Big)_x = ad^j[\, M_{f \circ \tau_x} \,]\; (A_x).$

(b) If in addition f is a linear function, then $ad^j[M_{f \circ \tau_x}](A) = ad^j[M_f](A)$. In particular, it follows that $\Big[\, ad^j[\, M_f \,]\, (P), W_x \Big] = 0.$

Proof By Lemma 2.1.1 we have $(M_g)_x = M_{g\circ\tau_x}$ and using formula (3.1.1) with A instead of P it follows that:

$$\Big(\mathrm{ad}^j [\, M_f \,]\, (A) \Big)_x = \sum_{l=0}^{j} (-1)^l \binom{j}{l} \, (\, M_{f^{j-l}} \,)_x \, A_x \, (\, M_{f^l} \,)_x = \mathrm{ad}^j [\, M_{f\circ\tau_x} \,]\, (A_x).$$

Now, in addition let $f : \mathbb{C}^n \to \mathbb{C}$ be linear. Then for $j = 1$ we conclude that:

$$\mathrm{ad}\, [\, M_{f\circ\tau_x} \,]\, (A) = \Big[\, M_{f+f(x)}, A \,\Big] = [\, M_f, A \,] = \mathrm{ad}\, [\, M_f \,]\, (A).$$

For an arbitrary $j \in \mathbb{N}$ the first statement in (b) follows by induction. The second assertion can be obtained from the first one, (a) and Lemma 2.1.1 which implies that $P_x = P$ holds for all $x \in \mathbb{C}^n$. $\square$

Fix $x \in \mathbb{C}^n$ with $\|\, x \,\| = 1$ and let $H \in \{\, H_1, H_2 \,\}$. With the space $\mathcal{U}(H) \subset \mathcal{L}(H)$ of all unitary operators on H we consider the group representation:

$$\mathbb{R} \ni t \mapsto U^{(x)}(t) := W_{tx} \in \mathcal{U}(\, H \,). \tag{3.1.5}$$

By Lemma 2.1.1 (4) equation (3.1.5) defines a group of unitary operators on H. It is well-known that this group is strongly continuous (C_0-group). Hence, its *infinitesimal generator*

$$V^{(x)} : H \supset \mathcal{D}(V^{(x)}) := \left\{ \, h \in H : V^{(x)}h := \lim_{t\to 0} \frac{U^{(x)}(t)h - h}{t} \in H \text{ exits in } H \, \right\} \longrightarrow H$$

is a closed, densely defined linear operator on H satisfying

$$U^{(x)}(t) \big[\, \mathcal{D}(V^{(x)}) \,\big] \subset \mathcal{D}(V^{(x)}) \qquad \text{and} \qquad \Big[\, U^{(x)}(t), V^{(x)} \,\Big] = 0$$

for all $t \in \mathbb{R}$. In addition by *Stone's Theorem* $iV^{(x)}$ is self-adjoint. In the following let $H := H_2$. If we fix an entire function $h \in \mathcal{D}(V^{(x)}) \subset H_2$, then by definition the limit $\lim_{t\to 0} F(\, t, x, h \,)$ exists in H_2 where

$$F(\, t, x, h \,) := \frac{1}{t} \Big[\, U^{(x)}(\, t \,)\, h - h \,\Big] \in H_2, \qquad t \in \mathbb{R} \setminus \{0\}. \tag{3.1.6}$$

Hence, if t tends to 0 the functions $(\, F(t, x, h) \,)_{t\in\mathbb{R}\setminus\{0\}} \subset H_2$ converge uniformly on each compact subset $K \subset \mathbb{C}^n$ to

$$\begin{aligned} F(\, 0, x, h \,)(\, z \,) :&= \frac{d}{dt} \Big[\, k_{tx}(\, z \,)\, h(\, z - tx \,) \,\Big]_{|t=0} \\ &= \langle z, x \rangle \cdot h(\, z \,) + \frac{d}{dt}\, h(\, z - tx \,)_{|t=0}. \end{aligned} \tag{3.1.7}$$

In particular, let $x = e_j = (\delta_{i,j})_{i=1,\cdots,n} \in \mathbb{C}^n$ for $j = 1, \cdots, n$ and define the infinitesimal generators corresponding to the coordinate directions $V_j := V^{(e_j)}$ corresponding to the coordinate directions. Then it follows from (3.1.7) and with $h \in \mathcal{D}(V_j)$ that

$$V_j\, h = F(\, 0, e_j, h \,) = \Big[\, M_{z_j} - \partial_j \,\Big]\, h \in H_2, \qquad (\, j = 1, \cdots, n \,).$$

Let us show that $V^{(x)}$ coincides with an unbounded Toeplitz operator restricted to certain subspaces of $\mathcal{D}(\, V^{(x)} \,)$. We need some preparations:

Lemma 3.1.4 *Let $\alpha \in \mathbb{N}_0^n$. Then we have with $x \in \mathbb{C}^n$, each multi-index $\zeta \in \mathbb{N}_0^n$ and the definition $P_{\alpha,\zeta}(t) := \langle K(\cdot, tx)\, m_\alpha \circ \tau_{-tx}, m_\zeta \rangle_2$ where $m_\alpha(z) = z^\alpha$ for all $t \in \mathbb{R}$:*

(i) $P_{\alpha,\alpha}(t) = \alpha! - t^2 Q_1(t)$ *where Q_1 is a polynomial with $Q_1(0) = \alpha! \sum_{j=1}^n \alpha_j |\, x_j\, |^2$.*

(ii) For any number $j \in \{ 1, \cdots, n \}$ we have $P_{\alpha,\alpha+e_j}(t) = t\, Q_{2,j}(t)$, where $Q_{2,j}$ is a polynomial with $Q_{2,j}(0) = \alpha!\, (\, \alpha_j + 1\,)\, \bar{x}_j$.

(iii) Let $j \in \{ 1, \cdots, n \}$ with $\alpha_j \geq 1$, then $P_{\alpha,\alpha-e_j}(\, t\,) = t\, Q_{3,j}(\, t\,)$, where $Q_{3,j}$ is a polynomial with $Q_{3,j}(0) = -\alpha!\, x_j$.

Proof Using the Taylor-expansion of K we obtain for all $\alpha, \zeta \in \mathbb{N}_0^n$ and $t \in \mathbb{R}$:

$$\begin{aligned} P_{\alpha,\zeta}(t) &= \sum_{\beta \in \mathbb{N}_0^n} \frac{1}{\beta!} m_\beta(\, t\bar{x}\,) \big\langle\, m_\beta \cdot m_\alpha \circ \tau_{-tx}, m_\zeta\, \big\rangle_2 \\ &= \sum_{\beta \leq \zeta} \sum_{\gamma \leq \alpha} \frac{1}{\beta!} \binom{\alpha}{\gamma} m_\beta(\, \bar{x}\,)\, m_{\alpha-\gamma}(\, -x\,)\, t^{|\,\beta\,| + |\,\alpha-\gamma\,|}\, \zeta!\, \delta_{\beta+\gamma,\zeta}. \end{aligned} \tag{3.1.8}$$

To prove (i) we set $\zeta = \alpha$. In the case where $|\, \beta\, | + |\, \alpha - \gamma\, | = 1$ we necessarily have $\beta \neq 0$ and $\alpha = \gamma$ or $\beta = 0$ and $\alpha \neq \gamma$. In both situations we find $\delta_{\beta+\gamma,\alpha} = 0$ and so (3.1.8) considered as a function of t is a polynomial without linear term. Moreover, $t^{|\beta|+|\alpha-\gamma|}\delta_{\beta+\gamma,\alpha} = t^2$ is only possible in the cases where $|\, \beta\, | = 1$ and $\gamma = \alpha - \beta$. From (3.1.8) we obtain $Q_1(0) = \alpha! \sum_{j=1}^n \alpha_j\, |\, x_j\, |^2$.

To prove (ii) let $\zeta := \alpha + e_j$ where $j \in \{ 1, \cdots, n \}$. From (3.1.8) we find $P_{\alpha,\alpha+e_j}(0) = 0$ and we can choose a polynomial $Q_{2,j}$ with $P_{\alpha,\alpha+e_j}(t) = t\, Q_{2,j}(t)$. The equality

$$t^{|\,\beta\,|+|\,\alpha-\gamma\,|}\delta_{\beta+\gamma,\alpha+e_j} = t$$

is only possible in the case where $\beta = e_j$ and $\gamma = \alpha$. We conclude that

$$Q_{2,j}(\, 0\,) = \alpha!\, \big(\, \alpha_j + 1\, \big)\, \bar{x}_j.$$

Finally, consider $j \in \{ 1, \cdots, n \}$ such that $\alpha_j \geq 1$ and let $\zeta = \alpha - e_j$. Then we obtain from (3.1.8) that $P_{\alpha,\alpha-e_j}(0) = 0$. Choose a polynomial $Q_{3,j}$ such that $P_{\alpha,\alpha-e_j}(t) = t\, Q_{3,j}(t)$. Moreover, the equality $t^{|\,\beta\,|+|\,\alpha-\gamma\,|}\delta_{\beta+\gamma,\alpha-e_j} = t$ implies $\beta = 0$ and $\gamma = \alpha - e_j$. Hence we obtain $Q_3(0) = -\alpha!\, x_j$. □

Now, we can prove that the restriction of the infinitesimal generator $V^{(x)}$ to the space of holomorphic polynomials coincides with an unbounded Toeplitz operator.

Lemma 3.1.5 *Let $x \in \mathbb{C}^n$ with $\|x\| = 1$. Then $\mathbb{P}_a[\mathbb{C}^n] \subset \mathcal{D}(V^{(x)})$ and for each $p \in \mathbb{P}_a[\mathbb{C}^n]$ we have $V^{(x)}p = T_{2iIm\langle\cdot,x\rangle}p$.*

Proof It is sufficient to show that $m_\alpha \in \mathcal{D}(V^{(x)})$ for all $\alpha \in \mathbb{N}_0^n$. Let us define the function $G_{x,\alpha} : \mathbb{R} \to H_2$ by

$$G_{x,\alpha}(t) := \begin{cases} t^{-1}\left[\, U^{(x)}(\,t\,)\, m_\alpha - m_\alpha \,\right], & t \in \mathbb{R} \setminus \{0\} \\ \sum_{j=1}^n [\, \bar{x}_j\, m_{\alpha+e_j} - \alpha_j\, x_j\, m_{\alpha-e_j}\,], & t = 0. \end{cases}$$

where we set $m_{\alpha-e_j} := 0 \in H_2$ for $j \in \{\,1, \cdots, n\,\}$ such that $\alpha_j = 0$. We show that $G_{x,\alpha}$ is continuous in $t = 0$. It is a straightforward computation using $\langle\, m_\alpha, m_\beta \rangle_2 = \delta_{\alpha,\beta}\, \alpha!$ that

$$\|\, G_{x,\alpha}(0)\, \|_2^2 = \sum_{j=1}^n |\, x_j\,|^2\, \alpha!\, \big(\, 2\alpha_j + 1\,\big) = 2\alpha! \sum_{j=1}^n \alpha_j\, |\, x_j\,|^2 + \alpha!.$$

With the notations of Lemma 3.1.4 and because the operators $U^{(x)}(t)$ are unitary for all $t \in \mathbb{R}$ we have with $t \neq 0$:

$$\begin{aligned} \|\, G_{x,\alpha}(\,t\,)\, \|_2^2 &= \frac{2}{t^2}\left[\, \|\, m_\alpha\, \|_2^2 - \mathrm{Re}\left\langle U^{(x)}(\,t\,)\, m_\alpha, m_\alpha \right\rangle_2 \right] \\ &= \frac{2}{t^2}\left(\, \|\, m_\alpha\,\|_2^2 - \exp\left(-\frac{t^2}{2}\right) \mathrm{Re}\, P_{\alpha,\alpha}(\,t\,)\right) \\ &= \frac{2}{t^2}\left(\alpha! - \exp\left(-\frac{t^2}{2}\right)\left[\, \alpha! - t^2\tilde{Q}(t)\,\right]\right) \end{aligned}$$

where $\tilde{Q}(\,t\,) := \mathrm{Re}\, Q_1(\,t\,)$ with Q_1 as in Lemma 3.1.4. By using standard analysis we find:

$$\lim_{t\to\infty} \|\, G_{x,\alpha}(\,t\,)\,\|_2^2 = \alpha! + 2\tilde{Q}(0) = 2\alpha! \sum_{j=1}^n \alpha_j\, |\, x_j\,|^2 + \alpha! = \|\, G_{x,\alpha}(0)\,\|_2^2. \tag{3.1.9}$$

Next we compute the inner-product $\langle\, G_{x,\alpha}(t), G_{x,\alpha}(0)\,\rangle_2$ for $t \in \mathbb{R} \setminus \{0\}$. Using Lemma 3.1.4 again we obtain

$$\begin{aligned} &\left\langle G_{x,\alpha}(\,t\,), G_{x,\alpha}(0) \right\rangle_2 \\ =&\frac{1}{t}\left\langle U^{(x)}(\,t\,)m_\alpha - m_\alpha, \sum_{j=1}^n \left[\, \bar{x}_j m_{\alpha+e_j} - \alpha_j x_j m_{\alpha-e_j}\,\right] \right\rangle_2 \\ =&\frac{1}{t}\sum_{j=1}^n \left\{ x_j \left\langle U^{(x)}(\,t\,)\, m_\alpha, m_{\alpha+e_j} \right\rangle_2 - \alpha_j \bar{x}_j \left\langle U^{(x)}(\,t\,)\, m_\alpha, m_{\alpha-e_j} \right\rangle_2 \right\} \\ =&\frac{1}{t}\exp\left(-\frac{t^2}{2}\right) \sum_{j=1}^n \left\{ x_j\, P_{\alpha,\alpha+e_j}(\,t\,) - \alpha_j\, \bar{x}_j\, P_{\alpha,\alpha-e_j}(\,t\,) \right\} \\ =&\exp\left(-\frac{t^2}{2}\right) \sum_{j=1}^n \left\{ x_j\, Q_{2,j}(\,t\,) - \alpha_j\, \bar{x}_j\, Q_{3,j}(\,t\,) \right\}. \end{aligned}$$

Here we define $P_{\alpha,\alpha-e_j} \equiv 0$ for $\alpha_j = 0$. Using the notations above it follows that:

$$\lim_{t\to 0} \Big\langle\, G_{x,\alpha}(\,t\,), G_{x,\alpha}(\,0\,)\,\Big\rangle_2 = \sum_{j=1}^n \Big[\, x_j\, Q_{2,j}(\,0\,) - \alpha_j\, \bar{x}_j\, Q_{3,j}(\,0\,)\,\Big] \qquad (3.1.10)$$

$$= 2\alpha! \sum_{j=1}^n \alpha_j \mid x_j \mid^2 + \alpha!.$$

From equations (3.1.9) and (3.1.10) we conclude that

$$\| \, G_{x,\alpha}(t) - G_{x,\alpha}(0) \, \|_2^2 = \Big(\, \| \, G_{x,\alpha}(t) \, \|_2^2 - 2\mathrm{Re} \Big\langle \, G_{x,\alpha}(t), G_{x,\alpha}(0) \, \Big\rangle_2 + \| \, G_{x,\alpha}(0) \, \|_2^2 \Big)$$

tends to 0 as $t \to 0$ and so we obtain $m_\alpha \in \mathcal{D}(V^{(x)})$ with $V^{(x)} m_\alpha = G_{x,\alpha}(0)$ for all $\alpha \in \mathbb{N}_0^n$. Finally, to prove the identity

$$G_{x,\alpha}(0) = T_{2i\,\mathrm{Im}\langle\cdot,x\rangle} m_\alpha$$

we use the well-known formula $T_{\bar{z}_j} m_\alpha = \frac{\partial}{\partial z_j} m_\alpha$, which holds for all $\alpha \in \mathbb{N}_0^n$ and $j = 1, \cdots, n$.

$$T_{2i\,\mathrm{Im}\langle\cdot,x\rangle} m_\alpha = T_{\langle\cdot,x\rangle} m_\alpha - T_{\langle x,\cdot\rangle} m_\alpha \qquad (3.1.11)$$

$$= \langle \cdot, x \,\rangle \cdot m_\alpha - \sum_{j=1}^n x_j\, T_{\bar{z}_j}\, m_\alpha$$

$$= \sum_{j=1}^n \left(\bar{x}_j m_{\alpha+e_j} - x_j \frac{\partial}{\partial z_j} m_\alpha \right) \qquad (3.1.12)$$

$$= \sum_{j=1}^n \big(\, \bar{x}_j m_{\alpha+e_j} - \alpha_j x_j m_{\alpha-e_j} \, \big) = G_{x,\alpha}(0). \qquad \square$$

Now, let us consider the Toeplitz operator $T_{2i\,\mathrm{Im}\langle\cdot,x\rangle}$, $(x \in \mathbb{C}^n)$ which appears in Lemma 3.1.5. We define the function:

$$\Phi_x := 2i\,\mathrm{Im}\langle\cdot, x\rangle : \mathbb{C}^n \to \mathbb{C}.$$

The natural domain of definition of T_{Φ_x} is given by

$$\mathcal{D}\,(T_{\Phi_x}) := \Big\{ \, f \in H_2 : f \cdot \Phi_x \in H_1 \, \Big\}.$$

Lemma 3.1.6 *The Toeplitz operator* $T_{\Phi_x} : H_2 \supset \mathcal{D}(T_{\Phi_x}) \to H_2$ *is unbounded, densely defined and closed.*

Proof Because Φ_x is a linear function we have $\Phi_x \in SP_{\mathrm{Lip}}(\mathbb{C}^n)$ and the following equation holds:

$$M_{\Phi_x} = T_{\Phi_x} + [\, M_{\Phi_x}, P \,] : \mathcal{D}(T_{\Phi_x}) \subset H_2 \longrightarrow H_1. \qquad (3.1.13)$$

By Lemma 3.1.1 with $j = 1$ the commutator $[\, M_{\Phi_x}, P \,]$ has a continuous extension to H_2. Choose a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\, T_{\Phi_x} \,)$ such that:

(i) $\lim_{n\to\infty} f_n = f \in H_2$,

(ii) $\lim_{n\to\infty} T_{\Phi_x} f_n = g \in H_2$,

then we conclude from the continuity of $[\, M_{\Phi_x}, P\,]$ and (3.1.13) that

$$\Phi_x f = \lim_{n\to\infty} \Phi_x f_n \in H_1$$

exists. Hence $f \in \mathcal{D}(T_{\Phi_x})$ by definition with

$$g = \lim_{n\to\infty} T_{\Phi_x} f_n = \Phi_x f - \lim_{n\to\infty} [\, M_{\Phi_x}, P\,]\, f_n = T_{\Phi_x} f.$$

It follows that the Toeplitz operator T_{Φ_x} is closed. $\square$

Consider the space $\mathcal{D}(T_{\Phi_x})$ with the graph norm $\|\, f\,\|_x := \|\, f\,\|_2 + \|\, T_{\Phi_x} f\,\|_2$. Then it is clear that $(\, \mathcal{D}(T_{\Phi_x}), \|\cdot\|_x\,)$ is a Banach space which contains the spaces $\mathbb{P}_a[\,\mathbb{C}^n\,]$ of holomorphic polynomials as well as the space $H_{\exp}(\mathbb{C}^n)$.

Lemma 3.1.7 *Equipped with the graph norm $\|\cdot\|_x$ the embedding $\mathbb{P}_a[\,\mathbb{C}^n\,] \hookrightarrow H_{\exp}(\mathbb{C}^n)$ is dense for all $x \in \mathbb{C}^n$.*

Proof Let $f \in H_{\exp}(\mathbb{C}^n)$, then by definition there are numbers $c_1 \in (\,0, 2^{-1}\,)$ and $D_1 > 0$ such that it holds

$$|\, f(z)\,| \leq D_1 \cdot \exp\big(\, c_1 \,\|\, z\,\|^2\,\big)$$

for all $z \in \mathbb{C}^n$. Hence we have $f \in L^2(\mathbb{C}^n, \mu_r)$ for all $r \in (2c_1, 1)$. Now, let us fix two positive numbers c_2, c_3 with $2c_1 < c_2 < c_3 < 1$ and choose a constant $D_2 > 0$ with

$$\|\, z\,\|^2 \leq D_2 \,\exp\Big(\, [\, c_3 - c_2\,]\, \|\, z\,\|^2\Big)$$

for all $z \in \mathbb{C}^n$. Then we conclude for all analytic polynomials $p \in \mathbb{P}_a[\,\mathbb{C}^n\,]$:

$$\begin{aligned}\|\, T_{\Phi_x}(f-p)\,\|_2^2 \leq \|\, \Phi_x(\, f-p\,)\,\|_2^2 \leq 2\,\|\, x\,\|^2 \int_{\mathbb{C}^n} \|\cdot\|^2\, |\, f-p\,|^2\, d\mu \\ \leq 2D_2\,\|\, x\,\|^2\, r^{-n}\,\|\, f-p\,\|^2_{L^2(\mathbb{C}^n,\mu_r)} < \infty\end{aligned}$$

where $r = 1 - c_3 + c_2 \in (\, 2c_1, 1\,)$. Because $\mathbb{P}_a[\,\mathbb{C}^n\,]$ is dense in $L^2(\mathbb{C}^n, \mu_r) \cap \mathcal{H}(\mathbb{C}^n)$ for all $r > 0$ the assertion follows. $\square$

Theorem 3.1.2 *Let $x \in \mathbb{C}^n$ with $\|x\| = 1$. Then we have the inclusion*

$$H_{\exp}(\mathbb{C}^n) \subset \mathcal{D}\left(V^{(x)}\right) \cap \mathcal{D}\left(T_{\Phi_x}\right)$$

and it holds $V^{(x)} f = T_{\Phi_x} f$ for all $f \in H_{\exp}(\mathbb{C}^n)$.

Proof This immediately follows from $T_{\Phi_x}p = V^{(x)}p$ for $p \in \mathbb{P}_a[\mathbb{C}^n]$ and Lemma 3.1.7 which implies that

$$H_{\exp}(\mathbb{C}^n) \subset \operatorname{clos}(\,\mathbb{P}_a[\,\mathbb{C}^n\,], \|\cdot\|_x\,) \subset \mathcal{D}\left(V^{(x)}\right) \cap \mathcal{D}\,(T_{\Phi_x})$$

and the continuity of $V^{(x)}, T_{\Phi_x} : (\,\mathbb{P}_a[\,\mathbb{C}^n\,], \|\cdot\|_x\,) \to H_2$ in the graph norm. □

Let $V_2^{(x)} := V^{(x)}$ and denote by $V_1^{(x)}$ the infinitesimal generator of $(U_x(t))_{t\in\mathbb{R}}$ considered as operators on H_1. Then $V_2^{(x)}$ is the restriction of $V_1^{(x)}$ to $\mathcal{D}(V_2^{(x)}) = \mathcal{D}(V_1^{(x)}) \cap H_2$.

Lemma 3.1.8 *Fix $x \in \mathbb{C}^n$ with $\|x\| = 1$. If $r \in \mathbb{N}$ and $\mathcal{A}_r(\Phi_x) \subset \mathcal{L}(H_1)$ denotes the algebra defined in Theorem 3.1.1, then we have*

$$A\left[\,\mathcal{D}\left(V_1^{(x)}\right)\,\right] \subset \mathcal{D}\left(V_1^{(x)}\right)$$

and the commutator $[\,A, V_1^{(x)}\,]$ vanishes for all operators $A \in \mathcal{A}_r(\Phi_x)$.

Proof It is sufficient to show that for all $j \in \mathbb{N}$ the space $\mathcal{D}(V_1^{(x)})$ is invariant under the operators

$$B_j(\,\Phi_x\,) := \operatorname{ad}^j[\,M_{\Phi_x}\,]\Big(\,[\,P, M_{\Phi_x}\,]\,\Big)$$

According to Lemma 3.1.3, (b) together with the linearity of $\Phi_x = 2i\operatorname{Im}\langle\cdot, x\rangle$ we conclude that the commutator $[\,U^{(x)}(t), B_j(\Phi_x)\,]$ vanishes for all $t \in \mathbb{R}$. Fix $h \in \mathcal{D}(V_1^{(x)})$ then:

$$\frac{1}{t}\Big\{\,U^{(x)}(t)\,B_j(\Phi_x)h - B_j(\Phi_x)h\,\Big\} = B_j(\Phi_x)\left\{\,\frac{1}{t}\left(\,U^{(x)}(t)h - h\,\right)\right\} \to B_j\,(\Phi_x)\;V_1^{(x)}h$$

as t tends to 0. Here we have used the fact that the operators $B_j(\Phi_x)$ have a continuous extension to H_1 (see Corollary 3.1.1). □

Remark 3.1.1 Let V be any subspace of H_2 with $H_{\exp}(\mathbb{C}^n) \subset V$. Assume that $A : V \to V$ is an integral operator with kernel K_A and let the restrictions of A

$$A : H_{\exp}(\,\mathbb{C}^n\,) \to H_{\exp}(\,\mathbb{C}^n\,)$$

be well-defined with adjoint operator $A^* : H_{\exp}(\mathbb{C}^n) \to H_{\exp}(\mathbb{C}^n)$. Then by the formula

$$\left[\,AK(\,\cdot, \lambda\,)\,\right](z) = \overline{\left[\,A^*K(\,\cdot, z\,)\,\right](\lambda)} = K_A(z, \lambda)$$

and from $K(\cdot, \lambda) \in H_{\exp}(\mathbb{C}^n)$ it follows that K_A is determined by the restriction of A to the space $H_{\exp}(\,\mathbb{C}^n\,)$. Hence A has a continuous extension from V to an (integral)-operator on H_2 iff this is true for A considered as operators on $H_{\exp}(\,\mathbb{C}^n\,)$ and both extensions coincide as operators on H_2.

Lemma 3.1.9 *Let $g \in L^\infty(\mathbb{C}^n)$ such that $M_g[\,\mathcal{D}(V_1^{(x)})\,] \subset \mathcal{D}(V_1^{(x)})$. Then for all numbers $j \in \mathbb{N}$ the domain of definition $\mathcal{D}(V_2^{(x)})$ is an invariant subspace for the continuous extension $\tilde{C}_j(\Phi_x, g) \in \mathcal{L}(H_2)$ of the commutators $ad^j[T_{\Phi_x}](T_g)$ which coincides with a continuous extension of $ad^j[\,V_2^{(x)}\,](\,T_g\,)$.*

Proof By Corollary 3.1.2 there are operators $a_l, c_l \in \mathcal{A}_j(\Phi_x) \subset \mathcal{L}(H_1)$ with $l \in L$ where $L \subset \mathbb{N}$ is a finite set such that

$$\tilde{C}_j(\Phi_x, g) = \sum_{l \in L} P\, a_l\, M_g\, c_l\, P.$$

Hence it is sufficient to prove that $\mathcal{D}(V_2^{(x)})$ is invariant for each operator $P\, a_l\, M_g\, c_l\, P$. This is obvious with Lemma 3.1.8 and $P[\,\mathcal{D}(V_1^{(x)})\,] = \mathcal{D}(V_2^{(x)})$ which follows from the fact that P commutes with the Weyl operators $U^{(x)}(\,t\,)$.

It is easy to see that $V_2^{(x)}$ is an integral operator on $\mathcal{D}(\,V_2^{(x)}\,)$ and so by remark 3.1.1 the last assertion follows. □

In the following theorem we collect all the results we have received in Lemma 3.1.9, Theorem 3.1.2 and Corollary 3.1.2.

Theorem 3.1.3 *Fix $x \in \mathbb{C}^n$ with $\|\, x \,\| = 1$ and let $g \in L^\infty(\mathbb{C}^n)$ such that*

$$M_g \left[\, \mathcal{D}\left(\, V_1^{(x)} \right) \right] \subset \mathcal{D}\left(\, V_1^{(x)} \right).$$

Then we have $T_{\Phi_x} f = V_2^{(x)} f$ for all $f \in H_{\exp}(\mathbb{C}^n)$ and

(i) The iterated commutators

$$ad^j\,[\,T_{\Phi_x}\,]\,(T_g) \in L\,(\,H_{\exp}(\,\mathbb{C}^n\,)\,) \quad \text{and} \quad ad^j\,[\,V_1^{(x)}\,]\,(T_g) \in L\left(\mathcal{D}(\,V_1^{(x)}\,)\right)$$

have continuous extensions to H_2 for all $j \in \mathbb{N}$. Moreover, these extensions coincide and they will be denoted by $ad^j[\,T_{\Phi_x}\,](T_g)$ as well.

(ii) The map $L^\infty(\mathbb{C}^n) \ni g \mapsto ad^j[\,T_{\Phi_x}\,](T_g) \in \mathcal{L}(H_2)$ is continuous.

(iii) The space $\mathcal{D}(V_2^{(x)})$ is invariant under the extension of $ad^j[\,T_{\Phi_x}\,](T_g)$ for all $j \in \mathbb{N}$.

3.2 Ψ^*-algebras generated by the Weyl group

In the following we denote by $\mathcal{B}$ the C^*-algebra $\{\, M_f : f \in L^\infty(\mathbb{C}^n)\,\} \subset \mathcal{L}(H_1)$. For the Toeplitz C^*-algebra in $\mathcal{L}(H_2)$ generated by all Toeplitz operators with bounded symbols we use the notation $\mathcal{A}$. For $x \in \mathbb{C}^n$ with $\|\, x \,\| = 1$ let

$$\alpha_t := U^{(x)}(t), \qquad (\, t \in \mathbb{R}\,) \tag{3.2.1}$$

be the unitary group of *Weyl operators* on H_1 resp. H_2.

Lemma 3.2.1 *The Toeplitz C^*-algebra $\mathcal{A}$ is invariant under the Weyl group action, i.e. $\alpha_t A \alpha_{-t} \in \mathcal{A}$ for any $A \in \mathcal{A}$.*

Proof This follows by an application of Lemma 1.3.2 and 2.1.1, (3). $\square$

We use our notations in section 1.3 of chapter 1 and fix a complex direction $x \in \mathbb{C}^n$ where $\| x \| = 1$. With the unitary C_0-group $(\alpha_t)_t$ in (3.2.1) we can define a decreasing sequence of spectral invariant Fréchet algebras

$$\Psi^j_\alpha[\mathcal{B}] \qquad \text{resp.} \qquad \Psi^j_\alpha[\mathcal{A}], \qquad j \in \mathbb{N}_0 \cap \{\infty\}$$

of C^j-elements in $\mathcal{A}$ resp. in $\mathcal{B}$. The operators in $\Psi^\infty_\alpha[\mathcal{B}]$ resp. $\Psi^\infty_\alpha[\mathcal{A}]$ are called *smooth* with respect to the group $(\alpha_t)_t$. We have seen in Theorem 1.3.1 that there is an analog description of smooth elements using commutators with the infinitesimal generator of the group $(\alpha_t)_t$ which we have determined in the section below. This enables us to derive conditions on the symbol f which are sufficient for T_f to belong to

$$\Psi^\infty_\alpha[\mathcal{A}] = \Psi^\alpha_\infty[\mathcal{A}]$$

It turns out that this already is the case for quite weak regularity of the function f (cf. Theorem 3.2.1).

Now, our main theorem states, that the scale of algebras in $\mathcal{B}$ generated by the Weyl group action for all directions $x \in \mathbb{C}^n$

$$\mathcal{L}(H_1) \supset \mathcal{B} = \Psi^\alpha_0[\mathcal{B}] \supset \mathcal{I}\left(V_1^{(x)}\right) \supset \Psi^\alpha_1[\mathcal{B}] \supset \cdots \supset \Psi^\alpha_\infty[\mathcal{B}]$$

breaks down under the canonical map $\mathcal{P} : \mathcal{B} \ni M_g \mapsto T_f \in \mathcal{A}$ in the sense that

$$\mathcal{P}\left[\mathcal{I}\left(V_1^{(x)}\right)\right] \subset \Psi^\alpha_\infty[\mathcal{A}].$$

Theorem 3.2.1 *Let $x \in \mathbb{C}^n$ with $\| x \| = 1$ and fix a function $g \in L^\infty(\mathbb{C}^n)$ such that the domain of definition $\mathcal{D}(V_1^{(x)})$ is invariant under $M_{\bar{g}}$ and M_g. Then we have*

$$T_g \in \Psi^\alpha_\infty[\mathcal{A}] = \Psi^\infty_\alpha[\mathcal{A}].$$

Moreover, the map

$$\mathcal{I} := \left\{ g \in L^\infty(\mathbb{C}^n) : M_g, M_{\bar{g}} \textit{ leave } \mathcal{D}\left(V_1^{(x)}\right) \textit{ invariant } \right\} \ni g \mapsto T_g \in \Psi^\alpha_\infty[\mathcal{A}]$$

is continuous in the Fréchet topology of $\Psi^\alpha_\infty[\mathcal{A}]$ if $\mathcal{I}$ carries the topology of $L^\infty(\mathbb{C}^n)$.

Proof By Theorem 3.1.3 (i) and (iii) with $V_2^{(x)} = T_{\Phi_x}$ on $H_{\exp}(\mathbb{C}^n)$ we conclude that for all $j \in \mathbb{N}$ the commutators

$$\text{ad}^j\left[V_2^{(x)}\right](T_g) \qquad \text{and} \qquad \text{ad}^j\left[V_2^{(x)}\right](T_g)^* = \text{ad}^j\left[V_2^{(x)}\right](T_{\bar{g}})$$

have continuous extensions to elements in $\mathcal{L}(H_2)$. Moreover, these extensions which we have denoted by $\delta^j_{V_2^{(x)}}(T_g)$ resp. $\delta^j_{V_2^{(x)}}(T_{\bar{g}})$ leave the domain of definition $\mathcal{D}(V_2^{(x)})$ invariant. By definition it follows that

$$\delta^j_{V_2^{(x)}}(T_h) \in \mathcal{I}\left(V_2^{(x)}\right)$$

where $h \in \{ g, \bar{g} \}$ and $j \in \mathbb{N}$. Now, from Theorem 1.3.1 we conclude that

$$T_g \in \Psi_\infty^\alpha [\mathcal{L}(H_2)] = \Psi_\alpha^\infty [\mathcal{L}(H_2)].$$

In addition, Theorem 1.3.1 (a) together with $\varphi_{T_h} : \mathbb{R} \to \mathcal{A}$ implies that

$$\delta^j_{V_2^{(x)}} (T_h) = \varphi^{(j)}_{T_h}(0) \in \mathcal{A} \qquad \text{for} \quad h \in \{ g, \bar{g} \}$$

and we obtain $T_g \in \Psi_\infty^\alpha[\mathcal{A}] = \Psi_\alpha^\infty[\mathcal{A}]$. Finally, the continuity of the symbol map

$$L^\infty (\mathbb{C}^n) \ni g \mapsto T_g \in \Psi_\infty^\alpha[\mathcal{A}]$$

directly follows from Theorem 3.1.3 (ii) and together with the definition of the Fréchet topology of $\Psi_\infty^\alpha[\mathcal{A}]$. □

Next we give an example of a class of bounded functions g such that for all $x \in \mathbb{C}^n$ with $\| x \| = 1$ the domain of definition $\mathcal{D}(V_1^{(x)})$ is an invariant subspace for M_g and $M_{\bar{g}}$. Denote by $\mathcal{C}_b(\mathbb{C}^n)$ the space of all complex valued continuous bounded functions. For all $j \in \{ 1, \cdots, n \}$ and $z \in \mathbb{C}^n$ we write $z_j := x_j + iy_j$ and with $\alpha, \beta \in \mathbb{N}_0^n$

$$z^{\alpha,\beta} := x_1^{\alpha_1} \cdots x_n^{\alpha_n} y_1^{\beta_1} \cdots y_n^{\beta_n}, \qquad \partial^{\alpha,\beta} := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \partial_{y_1}^{\beta_1} \cdots \partial_{y_n}^{\beta_n}.$$

For each $m \in \mathbb{N}$ consider the space $\mathcal{C}_b^m(\mathbb{C}^n)$ of bounded complex valued functions on $\mathbb{C}^n$ defined by:

$$\mathcal{C}_b^m (\mathbb{C}^n) := \Big\{ f \in \mathcal{C}^m (\mathbb{C}^n) : \partial^{\alpha,\beta} f \in \mathcal{C}_b (\mathbb{C}^n) \text{ for } \alpha, \beta \in \mathbb{N}_0^n \text{ with } | \alpha | + | \beta | \leq m \Big\}.$$

Lemma 3.2.2 *Let $x \in \mathbb{C}^n$ with $\| x \| = 1$ and $g \in \mathcal{C}_b^2(\mathbb{C}^n)$. Then M_g leaves $\mathcal{D}(V_1^{(x)})$ invariant.*

Proof Choose $h \in \mathcal{D}(V_1^{(x)})$ and without loss of generality assume that g is real valued. By Lemma 2.1.1 it follows that

$$\frac{1}{t}\Big[U^{(x)}(t) M_g h - M_g h \Big] = \frac{1}{t} \Big[M_{g \circ \tau_{-tx}} - M_g \Big] U^{(x)}(t) h + M_g \frac{1}{t}\Big[U^{(x)}(t) h - h \Big]. \tag{3.2.2}$$

Because of $h \in \mathcal{D}(V_1^{(x)})$ the second term converges in H_1 as $t \to 0$ by the continuity of M_g. In order to treat the first term we define $\tilde{g}_x(z) := -\langle \operatorname{grad} [g](z), x \rangle_{\mathbb{R}^{2n}}$ for $z \in \mathbb{C}^n$. Then it follows by our choice of g that $\tilde{g}_x \in L^\infty(\mathbb{C}^n)$ and applying the Taylor formula we get:

$$\begin{aligned} C_{t,x} : &= \Big\| \frac{1}{t} \Big[M_{g \circ \tau_{-tx}} - M_g \Big] - M_{\tilde{g}_x} \Big\| \\ &= \Big\| \frac{1}{t} \Big[g \circ \tau_{-tx} - g \Big] - \tilde{g}_x \Big\|_\infty \leq \sum_{| \alpha | + | \beta | = 2} \frac{| t |}{(\alpha + \beta)!} \, \big\| \partial^{\alpha,\beta} g \big\|_\infty \, | x^{\alpha,\beta} |. \end{aligned}$$

Hence $\lim_{t\to 0} C_{t,x} = 0$ and so it follows that

$$\Big\| \frac{1}{t} \Big[M_{g\circ\tau_{-tx}} - M_g \Big] U^{(x)}(t)h - M_{\tilde{g}_x} h \Big\|_2 \leq C_{t,x} \| h \|_2 + \| M_{\tilde{g}_x} \| \, \Big\| U^{(x)}(t)h - h \Big\|_2$$

tends to 0 as $t \to 0$. We obtain that also the limit of the first term in (3.2.2) exists if t tends to 0 and this now proves $M_g h \in \mathcal{D}(V_1^{(x)})$. $\square$

Remark 3.2.1 Each Toeplitz operator T_f with symbol $f \in \mathcal{C}_c^2(\mathbb{C}^n)$ leads to a smooth element in $\mathcal{A}$ with respect to the Weyl group action.

Because the test functions $\mathcal{C}_c^\infty(\mathbb{C}^n)$ are uniformly dense in $\mathcal{C}_c(\mathbb{C}^n)$, the space of all continuous functions with compact support we obtain:

Theorem 3.2.2 *Let $x \in \mathbb{C}^n$ with $\|x\| = 1$ and denote by $\mathcal{C}_0(\mathbb{C}^n)$ the space of all continuous functions vanishing at infinity. Then $\{ T_f : f \in \mathcal{C}_0(\mathbb{C}^n) \} \subset \Psi_\alpha^\infty[\mathcal{A}]$.*

Proof According to Theorem 3.2.1 and Lemma 3.2.2 the symbol map

$$\mathcal{C}_c^2 (\mathbb{C}^n) \ni g \mapsto T_g \in \Psi_\alpha^\infty[\mathcal{A}]$$

is continuous with respect to the uniform topology on $\mathcal{C}_c^2(\mathbb{C}^n)$. Hence the assertion follows from the fact that $\mathcal{C}_c^2(\mathbb{C}^n)$ is uniformly dense in $\mathcal{C}_0(\mathbb{C}^n)$. $\square$

Corollary 3.2.1 *Let $\mathcal{K}(H_2) \subset \mathcal{L}(H_2)$ denote the ideal of all compact operators on H_2. Then we have the inclusions*

$$\mathcal{K}(H_2) \subset clos\Big\{ A \in \Psi_\alpha^\infty[\mathcal{A}] \Big\} \subset \mathcal{A}.$$

Here the closure of $\Psi_\alpha^\infty[\mathcal{A}]$ is taken with respect to the uniform topology on $\mathcal{L}(H_2)$.

Proof This follows with Theorem 3.2.2 and $\mathcal{K}(H_2) = \mathrm{clos}\{ T_g : g \in \mathcal{C}_c(\mathbb{C}^n) \}$ which is proved in Theorem 9 in [21]. $\square$

Example 3.2.1 We construct an operator $A \in \mathcal{A}$ which is not contained in $\Psi_1^\alpha[\mathcal{A}]$. Let $n = 1$ and $x \in \mathbb{C}$ such that $|x| = 1$. Then for $j \in \mathbb{N}$ we denote by $P_j \in \mathcal{L}(H_2)$ the rank one projection onto the space $\mathrm{span}\{ m_j := z^j \}$. With a sequence $a := (a_n)_{n\in\mathbb{N}} \in c_0(\mathbb{N})$ we now define the diagonal operator

$$A := \sum_{j\in\mathbb{N}} a_j P_j \in \mathcal{L}(H_2).$$

Then A is compact and we conclude from Corollary 3.2.1 that $A \in \mathcal{A}$. With the function $\Phi_x := 2i \,\mathrm{Im}\langle \cdot, x\rangle$ let us compute the commutator $[T_{\Phi_x}, A]\, m_j = [V^{(x)}, A]\, m_j$ for all $j \in \mathbb{N}$. Using Lemma 3.1.5 we obtain

$$\begin{aligned} [T_{\Phi_x}, A] \; m_j &= a_j \, T_{\Phi_x} \, m_j - A\big[\bar{x} \, m_{j+1} - j \, x \, m_{j-1} \big] \\ &= a_j \, \big(\bar{x} \, m_{j+1} - j \, x \, m_{j-1} \big) - \big(a_{j+1} \, \bar{x} \, m_{j+1} - j \, a_{j-1} \, x \, m_{j-1} \big) \\ &= \big(a_j - a_{j+1} \big) \, \bar{x} \, m_{j+1} - j \, x \, \big(a_j - a_{j-1} \big) \, m_{j-1}. \end{aligned}$$

Now, define $e_j := (j!)^{-\frac{1}{2}} z^j$ as in (2.1.4). Then we have $\langle e_j, e_l \rangle_2 = \delta_{l,j}$ for all $j, l \in \mathbb{N}$. Hence it follows that

$$\| [T_{\Phi_x}, A](e_j) \|_2^2 = (j+1) \, | a_j - a_{j+1} |^2 + | a_j - a_{j-1} |^2 . \tag{3.2.3}$$

We choose the sequence $a \in c_0(\mathbb{N})$ such that the right hand side of (3.2.3) tends to infinity for $j \to \infty$. This can be done by the choice of an oscillating sequence

$$a_j := (-1)^j j^{-\frac{1}{4}}.$$

Then it follows

$$\left| a_j - a_{j+1} \right|^2 = \left| j^{-\frac{1}{4}} + (j+1)^{-\frac{1}{4}} \right|^2 \leq 4 j^{-\frac{1}{2}}$$

and so the right hand side of (3.2.3) is unbounded for $j \to \infty$. We conclude that $[T_{\Phi_x}, A]$ has no bounded extension to H_2 and so $A \notin \Psi_1^\alpha[\mathcal{A}]$. By Theorem 1.3.1 we also have $A \notin \Psi_\alpha^1[\mathcal{A}]$.

3.3 Berezin Toeplitz and Gabor-Daubechies Windowed Fourier localization operators

We describe the connection between the Berezin Toeplitz operators on H_2 and the class of *Gabor-Daubechies* localization operators L_φ^w with *window* $w \in L^2(\mathbb{R}^n, v)$ and symbol φ (see [35]). They are operators on $L^2(\mathbb{R}^n, v)$ defined by

$$\left\langle L_\varphi^w f, g \right\rangle_{L^2(\mathbb{R}^n, v)} = \frac{1}{\pi^n} \int_{\mathbb{C}^n} \varphi(a) \left\langle \beta f, W_a \beta w \right\rangle_2 \left\langle W_a \beta w, \beta g \right\rangle_2 dv(a)$$

where $\beta : L^2(\mathbb{R}^n, v) \to H_2 = H^2(\mathbb{C}^n, \mu)$ denotes the *Bargmann isometrie*. Here β is an integral operator and it is well-known [57], [35] that for $f \in L^2(\mathbb{R}^n, v)$

$$[\beta f](z) = (2\pi)^{-\frac{n}{4}} \int_{\mathbb{R}^n} f(x) \exp\left(xz - \frac{1}{4} x^2 - \frac{1}{2} z^2 \right) dv(x) \tag{3.3.1}$$

where we define $zy := z_1 y_1 + \cdots + z_n y_n$ for all $z, y \in \mathbb{C}^n$. Let $a := p + iq \in \mathbb{C}^n$ with $p, q \in \mathbb{R}^n$, then it is an easy computation that with $H := L^2(\mathbb{R}^n, v)$:

$$\beta^{-1} W_a \beta =: F_p^q : H \to H : f \mapsto \Big[x \mapsto f\big(x - 2p\big) \exp\big(iq [p - x] \big) \Big]. \tag{3.3.2}$$

In particular, for $a \in \mathbb{R}^n$ we conclude that F_a^0 is the usual unitary shift by $-2a$. The Segal-Bargmann space H_2 canonically can be identified with the *Boson-Fock space*

$$\mathcal{F}(\mathbb{C}^n) := \bigoplus_{j \geq 0} [\mathbb{C}^n]_s^j$$

where $\left[\mathbb{C}^n\right]_s^j$ denotes the symmetric Hilbert space tensor product $\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$ of length j. It can be shown that the operators on H_2 corresponding to the *creation* and *annihilation* operators on $\mathcal{F}(\mathbb{C}^n)$ are given by:

$$M_{z_i} = T_{z_i} \qquad \text{and} \qquad \partial_{z_i} = T_{\bar{z}_i}, \qquad\qquad (i = 1, \cdots, n).$$

By a straightforward computation using the integral formula (3.3.1) one verifies that the corresponding creation and annihilation operators on $L^2(\mathbb{R}^n, v)$ have the form:

$$\beta^{-1}\, M_{z_i}\, \beta = \frac{1}{2}\, x_i - \partial_{x_i} \qquad \text{and} \qquad \beta^{-1}\, \partial_{z_i}\, \beta = \frac{1}{2}\, x_i + \partial_{x_i}. \tag{3.3.3}$$

With the orthonormal basis $[\, e_j : j \in \mathbb{N}_0^n \,]$ in H_2 consisting of monomial on $\mathbb{C}^n$ which we have defined in (2.1.4) the *Hermite functions* are given by

$$w_j := \beta^{-1}\, (\, e_j \,) \in L^2\, (\mathbb{R}^n, v)\, .$$

In the following for $z \in \mathbb{C}^n$ and the normalized kernels $k_z \in H_2$ in (2.1.6) we write:

$$v_z := \beta^{-1}\, (\, k_z \,) \in L^2(\, \mathbb{R}^n, v\,). \tag{3.3.4}$$

We can calculate the function $v_z \in L^2(\mathbb{R}^n, v)$ for $z \in \mathbb{C}^n$ more explicitly. The following result was proved in [82].

Lemma 3.3.1 *For $z \in \mathbb{C}^n$ there is a constant $c_z \in \mathbb{C}^n$ not depending on $x \in \mathbb{R}^n$ such that it holds*

$$v_z(x) = c_z \exp\left(\, \bar{z}x - \frac{1}{4}\, x^2 \right). \tag{3.3.5}$$

Proof The normalized kernel $k_z \in H_2$ for the Gaussian measure μ was given by:

$$k_z(\, \lambda\,) := \exp\left(\, \langle \lambda, z \rangle - \frac{1}{2} \parallel z \parallel^2 \right)$$

and so it fulfills the differential equations $\partial_{\lambda_i} k_z = \bar{z}_i k_z$ for $i = 1, \cdots, n$. From the identity in (3.3.3) we conclude that

$$\left[\, \frac{1}{2}\, x_i + \partial_{x_i} \right] v_z = \beta^{-1}\, \partial_{\lambda_i}\, \beta v_z = \beta^{-1}\, \partial_{\lambda_i} k_z = \bar{z}_i v_z. \tag{3.3.6}$$

The most general solution of equation (3.3.6) is given by (3.3.5). $\square$

The connection between the *Gabor-Daubechies localization operators* L_φ^w with window $w = v_z$ and the *(Berezin) Toeplitz operators* is given by the following lemma.

Lemma 3.3.2 *Let $z \in \mathbb{C}^n$ and let $\varphi \in L^\infty(\mathbb{C}^n)$. Then $\beta\, L_\varphi^{v_z}\, \beta^{-1} = T_{\varphi \circ \tau_{-z}}$.*

Proof Let $f, g \in H_2$, then by an application of Lemma 2.1.1 it follows with $W_z 1 = k_z$ for all $z \in \mathbb{C}^n$:

$$\begin{aligned}
\Big\langle \beta L_\varphi^{v_z} \beta^{-1} f, g \Big\rangle_2 &= \frac{1}{\pi^n} \int_{\mathbb{C}^n} \varphi(\,a\,) \Big\langle f, W_a k_z \Big\rangle_2 \Big\langle W_a k_z, g \Big\rangle_2 dv(\,a\,) \\
&= \int_{\mathbb{C}^n} \varphi(\,a\,)\; [\,W_{-z} f\,](\,a\,) \overline{[\,W_{-z} g\,](\,a\,)}\, d\mu(\,a\,) \\
&= \Big\langle T_\varphi W_{-z} f, W_{-z} g \Big\rangle_2 \\
&= \Big\langle T_{\varphi \circ \tau_{-z}} f, g \Big\rangle_2.
\end{aligned}$$

Because $f, g \in H_2$ were arbitrary this implies the assertion. □

Let us denote by $\mathcal{D} \subset \mathcal{L}(\, L^2(\mathbb{R}^n, v)\,)$ the C^*-algebra in $\mathcal{L}(\, L^2(\mathbb{R}^n, v)\,)$ generated by the Gabor-Daubechies operators $\{\, L_\varphi^{v_0} : \varphi \in L^\infty(\mathbb{C}^n)\,\}$. Then by Lemma 3.3.2 we have with the Toeplitz algebra $\mathcal{A} \subset \mathcal{L}(H_2)$

$$\mathcal{D} = \beta^{-1}\, \mathcal{A}\, \beta := \Big\{\, \beta^{-1}\, A\, \beta : A \in \mathcal{A} \,\Big\} \subset \mathcal{L}\left(\, L^2\left(\mathbb{R}^n, v\right)\,\right).$$

Fix $p, q \in \mathbb{R}^n$ and consider the unitary C_0-group $(\, F_{tp}^{tq}\,)_{t \in \mathbb{R}} \subset \mathcal{D}$. By V_p^q we denote its infinitesimal generator. Let $a := p + iq \in \mathbb{C}^n$ and $V^{(a)}$ be the generator of the unitary Weyl group $(W_{ta})_{t \in \mathbb{R}} \subset \mathcal{A}$, then

$$\mathcal{D}\left(\, V_p^q\,\right) = \beta^{-1}\Big[\, \mathcal{D}\left(V^{(a)}\right)\,\Big] \qquad \text{and} \qquad V_p^q = \beta^{-1}\, V^{(a)}\, \beta.$$

According to section 3.2 we can define the Fréchet algebras $\Psi_k^\alpha[\,\mathcal{A}\,] \subset \mathcal{A}$ given by the group $(W_{ta})_t$ and the corresponding scale of algebras:

$$\Psi_k^{p,q}\,[\,\mathcal{D}\,] \subset \mathcal{D}$$

generated by $(\, F_{tp}^{tq}\,)_{t \in \mathbb{R}}$. It is easy to check that for all $k \in \mathbb{N}_0 \cup \{\infty\}$ it holds:

$$\Psi_k^{p,q}\,[\,\mathcal{D}\,] = \beta^{-1}\, \Psi_k^\alpha[\,\mathcal{A}\,]\, \beta := \Big\{\, \beta^{-1}\, A\, \beta : A \in \Psi_k^\alpha[\,\mathcal{A}\,]\,\Big\}.$$

Similarly, for all $k \in \mathbb{N}$ we can define the corresponding Sobolev spaces in the sense of Definition 1.2.3 by $\mathcal{H}_\alpha^k := \mathcal{D}(\,(V^{(a)})^k\,)$ and $\mathcal{H}_{p,q}^k := \mathcal{D}(\,(V_p^q)^k\,)$. In the case $k = \infty$ we set:

$$\mathcal{H}_\alpha^\infty := \bigcap_{k \in \mathbb{N}} \mathcal{H}_\alpha^k, \qquad \text{and} \qquad \mathcal{H}_{p,q}^\infty := \bigcap_{k \in \mathbb{N}} \mathcal{H}_{p,q}^k.$$

Then for all $k \in \mathbb{N}_0 \cup \{\infty\}$ we have: $\mathcal{H}_{p,q}^k = \beta^{-1} \mathcal{H}_\alpha^k$.

In particular, let $q = 0$ and $p \in \mathbb{R}^n$ with $\|\, p\,\| = \frac{1}{2}$. Then due to (3.3.2) the group $(F_{tp}^0)_{t \in \mathbb{R}}$ is the usual *shift operator* in direction p. Hence the infinitesimal generator V_p^0

is a closed extension of the partial derivative $\frac{\partial}{\partial p} : \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, v)$ in direction p. Moreover, the inclusions

$$\mathcal{C}_c^\infty (\mathbb{R}^n) \subset \mathcal{H}_{p,q}^\infty \subset L^2 (\mathbb{R}^n, v)$$

are dense in $L^2(\mathbb{R}^n, v)$. In the following we want to prove a result on the regularity of a solution $u \in L^2(\mathbb{R}^n, v)$ of the equation

$$L_\varphi^{v_0} u = f, \qquad \text{with symbol} \quad \varphi \in L^\infty(\mathbb{C}^n)$$

provided f is in one of the generalized Sobolev spaces $\mathcal{H}_{p,q}^k$ where $k \in \mathbb{N}_0 \cup \{\infty\}$. The next lemma can be found in [69], Remark 5.7 or in [107].

Lemma 3.3.3 *Let H be a Hilbert space, Ψ be a Ψ^*-algebra in $\mathcal{L}(H)$ and $A \in \Psi$ with closed range $R(A) \subset H$. Then for the orthogonal projection $Q \in \mathcal{L}(H)$ onto $N(A)$ we have $Q \in \Psi$. In particular, there exists an operator $B \in \Psi$, namely $B = (Q + A^*A)^{-1} A^*$ such that $Q = I - BA$.*

We apply Lemma 3.3.3 to prove the following theorem, which is a special case of 2.5.11 Proposition in [107]. Originally it can be found in [77].

Theorem 3.3.1 *Let $p, q \in \mathbb{R}^n$ and $A \in \Psi_\infty^{p,q}[\mathcal{D}]$ be a Fredholm operator. Let $u \in L^2(\mathbb{R}^n, v)$ be arbitrary such that*

$$Au = f \in \mathcal{H}_{p,q}^k$$

for some $k \in \mathbb{N}_0 \cup \{\infty\}$. Then one has $u \in \mathcal{H}_{p,q}^k$.

Proof Because A is a Fredholm operator it follows that $R(A) \subset L^2(\mathbb{R}^n, v)$ is closed and by Lemma 3.3.3 one obtains $B \in \Psi_\infty^{p,q} [\mathcal{D}]$ such that

$$Q = I - BA \in \Psi_\infty^{p,q} [\mathcal{D}]$$

is the orthogonal projection onto $N(A)$. From Theorem 1.2.1 we conclude that the inclusion holds $Q(\mathcal{H}_{p,q}^\infty) \subset \mathcal{H}_{p,q}^\infty$. Because of $\dim R(Q) = \dim N(A) < \infty$ it follows that $Q(\mathcal{H}_{p,q}^\infty)$ is closed. Hence we have $R(Q) \subset \mathcal{H}_{p,q}^\infty$ from the density of $\mathcal{H}_{p,q}^\infty \subset L^2(\mathbb{R}^n, v)$. Again by Theorem 1.2.1 we obtain $Bf \in \mathcal{H}_{p,q}^k$. This now implies that

$$u = BAu + Qu = Bf + Qu \in \mathcal{H}_{p,q}^k$$

which completes the proof. □

If we combine Theorem 3.2.2 and Theorem 3.3.1 we obtain the following result on the regularity of Gabor-Daubechies operators $L_\varphi^{v_0}$.

Corollary 3.3.1 *Let $a = p + iq \in \mathbb{C}^n$ with $\|a\| = 1$. Then for all $\varphi \in \mathcal{C}_0(\mathbb{C}^n)$ we have*

$$L_{1+\varphi}^{v_0} \in \Psi_\infty^{p,q} [\mathcal{D}].$$

Moreover, if $u \in L^2(\mathbb{R}^n, v)$ such that $L_{1+\varphi}^{v_0} u = f \in \mathcal{H}_{p,q}^k$ for some $k \in \mathbb{N}_0 \cup \{\infty\}$, then one has $u \in \mathcal{H}_{p,q}^k$.

Proof According to Theorem 3.2.2 and Lemma 3.3.2 we have for all $\varphi \in \mathcal{C}_0(\mathbb{C}^n)$

$$L^{v_0}_{1+\varphi} = \beta^{-1} \left(I + T_\varphi \right) \beta \in \beta^{-1} \, \Psi^{\alpha}_{\infty}[\, \mathcal{A} \,] \, \beta = \Psi^{p,q}_{\infty} \, [\, \mathcal{D} \,].$$

It is well-known, that T_φ is compact and so $L^{v_0}_{1+\varphi} = I + \beta^{-1} T_\varphi \beta$ is a Fredholm operator. Now the assertion follows with Theorem 3.3.1. □

3.4 Berezin Toeplitz operator and Weyl quantization

Next we give some applications to the *Weyl-quantization* and for further details we refer to [82]. Let us recall the notion of *Weyl pseudodifferential operators* on $L^2(\mathbb{R}^n, v)$.

Let $\Phi \in L^1(\mathbb{R}^{2n}, v)$ be an integrable function, then arising from a representation of the Heisenberg group (cf. [82]) we define for any $u \in (\mathbb{R}^{2n})^*$:

$$\sigma(\, u \,) := \int_{\mathbb{R}^{2n}} \Phi \, \exp(\, i \, \langle u, \cdot \rangle \,) \, dv.$$

With a suitable interpretation let us consider the (distribution) kernel W given by the Fourier transform:

$$W_\sigma(\, x, y \,) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sigma \left(p, \, \frac{x+y}{2} \right) \, \exp \Big(\, i \, p \, (\, x - y \,) \Big) \, dv(\, p \,)$$

which makes sense as a function for $\sigma \in L^1(\mathbb{R}^{2n}, v)$. The map $\sigma \mapsto W_\sigma$ is an isomorphism on the rapidly decreasing functions $\mathcal{S}(\, \mathbb{R}^{2n} \,)$, the tempered distributions $\mathcal{S}'(\, \mathbb{R}^{2n} \,)$ and on the space $L^2(\, \mathbb{R}^{2n}, v \,)$ (cf. [58], p. 80). We call σ the *Weyl-symbol* of the integral operator

$$A_\sigma f(\, x \,) := \int_{\mathbb{R}^n} f \cdot W_\sigma(\, x, \cdot \,) \, dv, \qquad\qquad f \in \mathcal{S}\,(\, \mathbb{R}^n \,) \tag{3.4.1}$$

and A_σ is said to be the *Weyl pseudodifferential operator* with symbol σ. Via the Bargmann isometrie (3.3.1) the operator A_σ correspond to an operator a_σ on H_2, which was determined in [82]. We only have to derive the corresponding Berezin transform $\widetilde{a}_\sigma$. In analogy to our notation in (3.3.4) we define for all $\lambda \in \mathbb{C}^n$:

$$V_\lambda := \beta^{-1} K(\cdot, \lambda \,) \qquad \text{where} \qquad K(\, a, b \,) = \exp(\, \langle a, b \rangle \,).$$

According to (2.1.16) and in the case where A_σ is well defined on $\{ \, V_\lambda : \lambda \in \mathbb{C}^n \, \}$ we obtain for all $\lambda \in \mathbb{C}^n$:

$$\widetilde{a}_\sigma(\lambda) = \Big\langle a_\sigma k_\lambda, k_\lambda \Big\rangle_2 = \| \, V_\lambda \, \|^{-2}_{L^2(\mathbb{R}^n, v)} \, \Big\langle A_\sigma V_\lambda, V_\lambda \Big\rangle_{L^2(\mathbb{R}^n, v)}. \tag{3.4.2}$$

For $\lambda \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$ let us define the function $f_\lambda \in L^2(\mathbb{R}^n, v)$ by:

$$f_\lambda(x) := \exp \left(\, \bar{\lambda} \, x - \frac{1}{4} \, x^2 \right). \tag{3.4.3}$$

Then by a computation similar to the proof of Lemma 3.3.1 there is a constant $c_\lambda \in \mathbb{C}$ independent of $x \in \mathbb{R}^n$ such that it holds $V_\lambda = c_\lambda \cdot f_\lambda$. By this (3.4.2) transforms to:

$$\widetilde{a}_\sigma(\,\lambda\,) = \|\, f_\lambda \,\|^{-2}_{L^2(\mathbb{R}^n,v)} \;\Big\langle\, A_\sigma f_\lambda, f_\lambda \,\Big\rangle_{L^2(\mathbb{R}^n,v)}. \tag{3.4.4}$$

Let us explicitly compute the norm $\|\, f_\lambda \,\|^2_{L^2(\mathbb{R}^n,v)}$ for $\lambda \in \mathbb{R}^n$. By definition of f_λ it follows:

$$\begin{aligned} \|\, f_\lambda \,\|^2 &= \int_{\mathbb{R}^n} \exp\Big(\, 2x\,\mathrm{Re}\lambda - \frac{1}{2}\,x^2 \Big)\; dv(x) \\ &= \exp\big(\, 2\,[\,\mathrm{Re}\lambda\,]^2\,\big) \int_{\mathbb{R}^n} \exp\Big(-\frac{1}{2}\;[\,x - 2\mathrm{Re}\lambda\,]^2 \Big)\; dv(x) \\ &= (\,2\pi\,)^{\frac{n}{2}} \cdot \exp\big(\,2\,[\,\mathrm{Re}\lambda\,]^2\,\big)\,. \end{aligned} \tag{3.4.5}$$

Moreover, for the inner-product $\langle\, A_\sigma f_\lambda, f_\lambda \,\rangle_{L^2(\mathbb{R}^n,v)}$ we obtain the integral formula:

$$\Big\langle A_\sigma f_\lambda, f_\lambda \Big\rangle_{L^2(\mathbb{R}^n,v)} = \int_{\mathbb{R}^{3n}} \sigma\Big(\,p, \frac{x+y}{2}\Big)\; F_1(\lambda,x,y)\;\exp\Big(\,i\,p\,(\,x-y\,)\Big)\,dv(x,y,p) = (*)$$

where the kernel F_1 is defined by the expression:

$$F_1(\,\lambda,x,y\,) := \frac{1}{(2\pi)^n} \cdot \exp\Big(\,\bar{\lambda}\,y + \lambda\,x - \frac{1}{4}\,\big(\,x^2+y^2\,\big)\,\Big).$$

For fixed x and p let us change the variables $q := \frac{1}{2}\,(\,x+y\,)$, then it follows for the integral $(*)$ above:

$$(*) = \int_{\mathbb{R}^{2n}} \sigma(\,p,\;q\,)\cdot F_2(\,\lambda,\;p,\;q\,)\;\exp\big(\,2q\,\bar{\lambda} - 2i\,p\,q\,\big)\;dv(\,p,\;q\,).$$

Here the function $F_2(\,\lambda\,,p,\;q\,)$ is defined by the integral:

$$\begin{aligned} F_2(\,\lambda,\;p,\;q\,) :&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \exp\Big(\,x\,(\,\lambda-\bar{\lambda}\,) - \frac{1}{4}\,x^2 - \frac{1}{4}\,(\,2q-x\,)^2\; + 2i\,p\,x\Big)\,dv(x) \\ &= \exp\big(-q^2\,\big)\;\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp\Big(\,\gamma\,x - \frac{1}{2}\,x^2\Big)\,dv(x) = (**) \end{aligned}$$

where $\gamma = q + 2i\;[\,\mathrm{Im}\lambda + p\,]$. Hence we obtain that:

$$\begin{aligned} (**) &= \exp\Big(\,-q^2 + \frac{1}{2}\,\gamma^2\Big)\;\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp\Big(-\frac{1}{2}\,(\,x-\gamma\,)^2\Big)\;dv(x) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\cdot\exp\Big(\,-q^2 + \frac{1}{2}\,\gamma^2\Big). \end{aligned}$$

Substituting this into the integral above we obtain with $a := 2\mathrm{Re}\lambda$ and $b := 2\mathrm{Im}\lambda$:

$$\begin{aligned}&\Big\langle A_\sigma f_\lambda, f_\lambda \Big\rangle_{L^2(\mathbb{R}^n,v)}\\ =&\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^{2n}} \sigma(\,p,q\,)\,\exp\left(\,-q^2+\frac{1}{2}\,\gamma^2+2\,q\bar{\lambda}-2i\,pq\right)\,dv(\,p,\;q\,)\\ =&\frac{1}{(2\pi)^{\frac{n}{2}}}\,\exp\left(\,2^{-1}\,a^2\,\right)\int_{\mathbb{R}^{2n}}\sigma(p,q)\exp\left(-2\,\left\{\frac{1}{2}\,q-\frac{1}{2}\,a\right\}^2-2\,\left\{p+\frac{1}{2}\,b\,\right\}^2\right)\,dv(p,q).\end{aligned}$$

After dividing this by (3.4.5) and the transformations $p \mapsto -p$ and $q \mapsto 2q$ it follows for the Berezin transform $\tilde{a}_\sigma$ of a_σ:

$$\tilde{a}_\sigma(\,\lambda\,)=\frac{2^n}{\pi^n}\int_{\mathbb{R}^{2n}}\sigma\,(\,-p,2q\,)\exp\Big(\,2\,\big\|\,(\,q,p\,)-(\,\mathrm{Re}\lambda,\mathrm{Im}\lambda\,)\,\big\|^2\,\Big)\;dv(\,p,q\,).$$

We can interpret the right hand side as the solution of the heat equation on $\mathbb{R}^{2n}$ with initial data

$$\mathbb{R}^{2n}\ni(\,q,p\,)\mapsto\sigma\,(\,-p,2q\,)=:\sigma_0(\,q,p\,)$$

at time $t=\frac{1}{8}$ in $\lambda := (\,\mathrm{Re}\lambda, \mathrm{Im}\lambda\,)\in\mathbb{R}^{2n}\cong\mathbb{C}^n$. It follows Berezin's formula (cf. [19]):

$$\tilde{a}_\sigma\,(\,\lambda_1+i\lambda_2\,)=\Big(\,e^{-\frac{1}{8}\,\Delta}\sigma_0\Big)\,(\,\lambda_1,\lambda_2\,)\,.$$

As a_σ was a Toeplitz operator T_ρ with symbol ρ, then we would have for its Berezin transform:

$$\tilde{a}_\sigma\,(\,\lambda_1+i\lambda_2\,)=\widetilde{T}_\rho\,(\,\lambda_1+i\lambda_2\,)=\tilde{\rho}\,(\,\lambda_1+i\lambda_2\,)=\Big(\,e^{-\frac{1}{4}\Delta}\rho\Big)\,(\,\lambda_1,\lambda_2\,)\,.$$

Then the (transformed) Weyl symbol σ_0 of the Weyl operator A_σ on $L^2(\mathbb{R}^n, v)$ corresponding to the Toeplitz operator on H_2 with symbols ρ is the solution of the heat equation on $\mathbb{R}^{2n}\cong\mathbb{C}^n$ at time $t=\frac{1}{8}$ with initial data ρ.

$$\sigma_0=\Big(\,e^{-\frac{1}{8}\,\Delta}\,\Big)\,\rho\qquad\text{and}\qquad\Big(\,e^{\frac{1}{8}\,\Delta}\,\Big)\,\sigma_0=\rho. \tag{3.4.6}$$

To determine the Weyl symbol of a given Toeplitz operator we only have to solve the heat equation. On the other hand in order to present a Weyl pseudo-differential operator as a Toeplitz operator on H_2 the *retrograde heat equation* has to be solved which in general is not possible. Nevertheless, if we restrict ourselves to a certain class of symbols with asymptotic expansion there is a one-to-one correspondence between Weyl-operators on $L^2(\mathbb{R}^n, v)$ and Toeplitz operators on H_2. For the details we refer to [82].

Let $f \in L^\infty(\mathbb{C}^n, \mu)$ be a bounded function and $t=\frac{1}{8}$, then we denote by $f^{(t)}$ the solution of the heat equation with initial data f at time $t>0$. For $t=\frac{1}{2}$ it is given by the integral formula:

$$f^{(\frac{1}{8})}(\,\xi,x\,)=\frac{2^n}{\pi^n}\int_{\mathbb{R}^{2n}}f\;\exp\left(\,-2\,\|\,(\,\xi,x\,)-\cdot\,\|^2\,\right)\;dv. \tag{3.4.7}$$

For $\alpha, \beta \in \mathbb{N}_0^n$ and applying standard facts on parameter integrals there is a polynomial Q_α^β on $\mathbb{R}^{2n}$ depending only on α and β such that for the partial derivatives of (3.4.7) it holds:

$$\begin{aligned}\partial_\xi^\alpha \partial_x^\beta f^{(\frac{1}{8})}(\, \xi, x\,) &= \frac{2^n}{\pi^n} \int_{\mathbb{R}^{2n}} f\, Q_\alpha^\beta\big(\, (\, \xi, x\,) - \cdot\,\big) \, \exp\big(-2 \,\|\, (\, \xi, x\,) - \cdot\, \|^2\,\big)\, dv \\ &= \frac{2^n}{\pi^n} \int_{\mathbb{R}^{2n}} f\big(\, \cdot + (\, \xi, x\,)\,\big)\, Q_\alpha^\beta \, \exp\big(\, -2 \,\|\cdot\|^2\,\big)\, dv\end{aligned}$$

From this equation it follows that for all $\alpha, \beta \in \mathbb{N}_0^n$ there is a constant $C_\alpha^\beta > 0$ which is independent of f such that:

$$\Big|\, \partial_\xi^\alpha \partial_x^\beta\, f^{(\frac{1}{8})}(\, \xi, x\,)\, \Big| \leq \|\, f\,\|_\infty \frac{2^n}{\pi^n} \int_{\mathbb{R}^{2n}} |\, Q_\alpha^\beta\,| \, \exp\big(\, -2 \,\|\cdot\|^2\,\big)\, dv = C_\alpha^\beta \,\|\, f\,\|_\infty.$$

Hence as a pseudo-differential symbol we find that $f^{(\frac{1}{8})} \in \mathcal{S}_{0,0}^0$. In particular, if we assume that f has compact support $K \subset \mathbb{R}^{2n}$ then there is $c > 0$ with:

$$\begin{aligned}\Big|\, \partial_\xi^\alpha \partial_x^\beta\, f^{(\frac{1}{8})}(\, \xi, x\,)\, \Big| &\leq \|\, f\,\|_\infty \frac{2^n}{\pi^n} \int_{K-(\xi,x)} |\, Q_\alpha^\beta\,| \, \exp\big(\, -2 \,\|\cdot\|^2\,\big)\, dv \\ &\leq c\, \|\, f\,\|_\infty \, \exp\big(\, -\|\, \xi\,\|^2 - \|\, x\,\|^2\,\big).\end{aligned} \tag{3.4.8}$$

Hence in this case we have

$$f^{(\frac{1}{8})} \in \cap_{m \in \mathbb{R}} \mathcal{S}_{\rho,\delta}^m =: \mathcal{S}_{\rho,\delta}^{-\infty} \qquad \text{where} \qquad (\, 0 \leq \delta \leq \rho \leq 1\,)$$

and so the Weyl operator corresponding to the Toeplitz operator T_f with symbol f under the canonical identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ belongs to the space

$$\mathrm{OP}\mathcal{S}_{\rho,\delta}^{-\infty} := \Big\{\, A_\sigma : \sigma \in \mathcal{S}_{\rho,\delta}^{-\infty}\,\Big\}$$

where A_σ is defined as in (3.4.1). Note that by the inequality (3.4.8) the map

$$\Big(\, \mathcal{C}_c\,(\mathbb{C}^n)\,, \|\cdot\|_\infty\,\Big) \ni f \mapsto f^{(\frac{1}{8})} \in \mathcal{S}_{\rho,\delta}^{-\infty} \tag{3.4.9}$$

is continuous if the space $\mathcal{S}_{\rho,\delta}^m$ carry the Fréchet topology induced by the family of norms:

$$\|\, \sigma\,\|_{[j]} = \sup_{|\,\alpha\,|+|\,\beta\,| \leq j} \sup_{\xi, x} |\, \partial_\xi^\alpha \partial_x^\beta\, \sigma(\, \xi, x\,)\,|\, \lambda(\xi)^{-m+\rho|\,\alpha\,|-\delta\,|\,\beta\,|}$$

for $j \in \mathbb{N}_0$ and $\sigma \in \mathcal{S}_{\rho,\delta}^m$ [1] where $\lambda(\xi) = (\, 1 + \xi^2\,)^{\frac{1}{2}}$. The space $\mathcal{S}_{\rho,\delta}^{-\infty}$ equipped with the projective topology of $\mathcal{S}_{\rho,\delta}^m$, $m \in \mathbb{R}$ is a Fréchet space as well. Hence the map (3.4.9) extents to $\mathcal{C}_0(\mathbb{C}^n)$ equipped with the sup-norm. Note that this result has close connection to Theorem 3.2.2 in the case of Toeplitz operators on H_2.

[1] $0 \leq \delta \leq \rho \leq 1$

3.5 Algebras by groups of composition operators

Denote by $\mathcal{U}(\mathbb{C}^n)$ the group of unitary matrices on $\mathbb{C}^n$ and and let us fix a unitary C_0-group

$$\mathbb{R} \ni t \mapsto u_t \in \mathcal{U}\left(\, \mathbb{C}^n \,\right)$$

with infinitesimal generator $a \in \mathcal{L}(\mathbb{C}^n)$. Then for each of the Hilbert spaces $H \in \{\, H_1, H_2 \,\}$ where $H_1 := L^2(\mathbb{C}^n, \mu)$ and $H_2 := \mathcal{H}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, \mu)$ we define a group $(U_t)_{t\in\mathbb{R}} \subset \mathcal{L}(H)$ of composition operators by:

$$U_t : \mathbb{R} \longrightarrow \mathcal{L}(H) : t \mapsto \Big[\, H \ni f \mapsto U_t(\, f\,) := f \circ u_t \in H \,\Big]. \tag{3.5.1}$$

Lemma 3.5.1 *The map $(U_t)_{t\in\mathbb{R}} \subset \mathcal{L}(H_2)$ given in (3.5.1) defines a C_0-group of unitary operators on H_2.*

Proof Because the Gaussian measure μ is invariant under the unitary transformations $(u_t)_{t\in\mathbb{R}}$ it follows that $(U_t)_{t\in\mathbb{R}}$ defines an unitary group of operators on H_2. In order to prove the strong continuity it is sufficient to show that

$$\lim_{t\to 0} \big\langle\, U_t f - f, f \,\big\rangle_2 = \lim_{t\to 0} \big\langle\, f \circ u_t - f, f \,\big\rangle_2 = 0, \qquad \forall\;\, f \in H_2.$$

Let $\varepsilon > 0$ and note that the linear hull $\operatorname{span}\{\, K(\cdot, \lambda) : \lambda \in \mathbb{C}^n \,\} \subset H_2$ is dense. Thus we can choose $h := \sum_{j=1}^m \alpha_j\, K(\cdot, \lambda_j\,)$ with $\lambda_1, \cdots, \lambda_m \in \mathbb{C}^n$ and $\alpha_1, \cdots, \alpha_m \in \mathbb{C}$ such that it holds $\|\, f - h \,\|_2 < \varepsilon$. We obtain:

$$\begin{aligned} \big|\, \big\langle\, f \circ u_t - f, f \,\big\rangle_2 \,\big| &= \big|\, \big\langle\, f \circ u_t - f, f - h \,\big\rangle_2 \,\big| + \big|\, \big\langle\, f \circ u_t - f, h \,\big\rangle_2 \,\big| \\ &\le 2\varepsilon \,\|\, f \,\|_2 + \sum_{j=1}^m \bar{\alpha}_j \Big[\, f \circ u_t(\lambda_j) - f(\lambda_j) \,\Big] < 3\varepsilon \,\|\, f \,\|_2 \end{aligned}$$

for suitable small $|t| > 0$ and by the strong continuity of $(u_t)_{t\in\mathbb{R}}$ together with the continuity of $f \in H_2$. □

Denote by A, the infinitesimal generator of the unitary group $(U_t)_{t\in\mathbb{R}}$ on H_2. Then

$$A : \mathcal{D}(A) := \left\{\, f \in H_2 : Af := \lim_{t\to 0} \frac{1}{t} \left(\, U_t f - f \,\right) \in H_2 \text{ exists } \right\} \subset H_2 \to H_2.$$

is an unbounded and densely defined operator on $\mathcal{L}(H_2)$. Moreover, by *Stone's Theorem* iA is self-adjoint. The next lemma follows by a straightforward computation.

Lemma 3.5.2 *Let $(U_t)_{t\in\mathbb{R}} \subset \mathcal{U}(H_1)$ be the unitary group defined above and let $g \in \mathcal{T}(\mathbb{C}^n)$ (for definition see (2.1.9)).*

(a) For all $t \in \mathbb{R}$ we have $(U_t)^* = (U^*)_t = U_{-t}$ where $(U^*)_{t\in\mathbb{R}}$ denotes the group defined by the action of

$$\mathbb{R} \ni t \mapsto u_t^* = u_{-t} \in \mathcal{U}\left(\mathbb{C}^n\right).$$

(b) The commutator $[\,P, U_t\,]$ *vanishes. In particular we have* $U_t[\,H_2\,] = H_2$.

(c) For all $t \in \mathbb{R}$ *we have* $U_t^*[\,\mathcal{D}(M_g)\,] = \mathcal{D}(\,M_{U_t^* g}\,)$ *and* $U_t^* M_g U_t = M_{U_t^* g}$.

(d) For all $t \in \mathbb{R}$ *we have* $U_t^*[\,\mathcal{D}(T_g)\,] = \mathcal{D}(\,T_{U_t^* g}\,)$ *and* $U_t^* T_g U_t = T_{U_t^* g}$.

Next we give a composition formula for certain classes of Toeplitz operators which can be found in [37]. For $\mathcal{C}^\infty$-functions f, g we consider the (formal) *twisted product.*

$$f \diamond g := \sum_k \frac{(-1)^{|\,k\,|}}{k!}(\,\partial^k f\,)(\,\overline{\partial}^k g\,).$$

A straightforward adaption of Theorem 2 in [37] to the measure μ we are using here leads to:

Theorem 3.5.1 *For polynomials* $\varphi, \psi \in \mathbb{P}[\mathbb{C}^n]$ *we have* $T_\varphi T_\psi$ *defined on a the dense linear hull span* $\{\,p\,\exp(\langle \cdot, a\rangle) : a \in \mathbb{C}^n$ *and* $p \in \mathbb{P}[\,\mathbb{C}^n\,]\,\}$. *On this domain* $T_\varphi T_\psi = T_{\varphi \diamond \psi}$ *and the new symbol* $\varphi \diamond \psi$ *is contained in* $\mathbb{P}[\,\mathbb{C}^n\,]$.

After this preparation we compute the infinitesimal generator A on a dense subspace of its domain of definition $\mathcal{D}(A) \subset H_2$.

Lemma 3.5.3 *The inclusion* $\mathbb{P}_a[\,\mathbb{C}^n\,] \subset \mathcal{D}(A) \subset H_2$ *holds and for any given polynomial* $p \in \mathbb{P}_a[\,\mathbb{C}^n\,]$ *we have*

$$Ap = \Big[\,T_{\langle az,z\rangle} - n \cdot tr(a) I\,\Big]\;p, \tag{3.5.2}$$

where $T_{\langle az,z\rangle}$ *is an unbounded Toeplitz operator.*

Proof It is sufficient to prove (3.5.2) in the case where $p = m_\alpha = z^\alpha$ with $\alpha \in \mathbb{N}_0^n$. For $z \in \mathbb{C}^n$ let us define the function

$$F_\alpha(z) := \frac{d}{dt}[\,m_\alpha \circ u_t(\,z\,)\,]_{|t=0} \in \mathbb{C}.$$

Then, by Theorem 3.5.1 we compute with $m_\alpha \circ u_t(z) = \langle\, m_\alpha, \exp(\,\langle \cdot, u_t z\,\rangle\,)\,\rangle_2$

$$\begin{aligned} F_\alpha(z) &= \Big\langle\, m_\alpha, \frac{d}{dt}\Big[\,\exp(\,\langle \cdot, u_t z\,\rangle\,)\,\Big]_{|t=0}\,\Big\rangle_2 \\ &= \Big\langle\, m_\alpha \langle az, \cdot\rangle, \exp(\,\langle \cdot, z\,\rangle\,)\,\Big\rangle_2 \\ &= \Big[\,\sum_{j=1}^n T_{[\,az\,]_j} T_{\overline{z}_j} m_\alpha\,\Big](\,z\,) = \Big[\,\sum_{i=1}^n T_{[\,az\,]_j \diamond \overline{z}_j} m_\alpha\,\Big](\,z\,). \end{aligned}$$

Moreover, for the $\diamond$-product we obtain $[\,az\,]_j \diamond \bar{z}_j = [\,az\,]_j\,\bar{z}_j - \mathrm{tr}(a)$ for $j \in \{\,1, \cdots, n\,\}$ and this leads to:

$$F_\alpha(\,z\,) = \Big[\,T_{\langle az,z\rangle} m_\alpha - ntr(a)\,m_\alpha\,\Big](\,z\,).$$

In the following we denote by $[\, d_j : j = 1, \cdots n\,]$ the standard orthonormal basis in $\mathbb{C}^n$. By using the Taylor formula we obtain for all $z \in \mathbb{C}^n$ and $t \in \mathbb{R} \setminus \{0\}$:

$$F_\alpha(t,z) := \frac{1}{t}\Big[\, m_\alpha \circ u_t(z) - m_\alpha(z)\,\Big] = \frac{1}{t}\sum_{\substack{\gamma\le\alpha\\ \gamma\neq 0}} \binom{\alpha}{\gamma} m_{\alpha-\gamma}(z)\, m_\gamma \circ [\, u_t - I\,]\,(z). \tag{3.5.3}$$

From $m_\gamma \circ [\, u_t - I\,](z) = \prod_{j=1}^n \langle\, z, [\, u_t^* - I\,]\, d_j\,\rangle^{\gamma_j}$ we conclude that

$$|\, m_\gamma \circ [\, u_t - I\,]\,(z)\,| \le \|\, z\,\|^{|\,\gamma\,|} \cdot \|\, u_t^* - I\,\|^{|\,\gamma\,|}$$

and similar $|\, m_{\alpha-\gamma}(z)\,| \le \|\, z\,\|^{|\,\alpha\,|-|\,\gamma\,|}$. If we insert this in equation (3.5.3) it follows that there is a positive number $c > 0$ such that

$$|\, F_\alpha(t,z)\,| \le c\,\|\, z\,\|^{|\,\alpha\,|}\, \frac{1}{|\,t\,|}\,\|\, u_t^* - I\,\|\,. \tag{3.5.4}$$

By the strong continuity of the unitary group $(u_t)_t$ the map

$$[\,-1,1\,] \setminus \{\,0\,\} \ni t \mapsto \frac{1}{|\,t\,|}\,\|\, u_t^* - I\,\| \in \mathbb{R}$$

is bounded and we conclude from (3.5.4) that there is $C > 0$ with $|\, F_\alpha(t,z)\,| \le C\|\, z\,\|^{|\alpha|}$ for all $t \in [\,-1,1\,] \setminus \{\,0\,\}$. Moreover, we have:

$$F_\alpha(\,z\,) = \lim_{t\to 0} F_\alpha(\,t,z\,) = \Big[\, T_{\langle az,z\rangle}\, m_\alpha - n\,\mathrm{tr}(a) m_\alpha\,\Big](\,z\,).$$

By Lebesgue's theorem it follows $\lim_{t\to 0} F_\alpha(t,\cdot) = F_\alpha$ in $L^2(\mathbb{C}^n,\mu)$ and this implies that $m_\alpha \in \mathcal{D}(A)$ with $Am_\alpha = [\, T_{\langle az,z\rangle} - n\,\mathrm{tr}(a) I\,]\, m_\alpha$. □

Let us define $\lambda(z) := (\,1 + \|\, z\,\|^2\,)^{\frac{1}{2}}$ and consider the norm $\|\cdot\|^\sim := \|M_{\lambda^2}\cdot\|_2$ on $\mathbb{P}_a[\mathbb{C}^n]$. By $\|\cdot\|_{\mathrm{gr}} := \|\cdot\|_2 + \|A\cdot\|_2$ we denote the graph norm on $\mathcal{D}(A)$.

Lemma 3.5.4 *There is $C > 0$ such that for all $p \in \mathbb{P}_a[\,\mathbb{C}^n\,]$ we have $\|\, p\,\|_{gr} \le C\,\|\, p\,\|^\sim$.*

Proof Because of $A = T_{\langle az,z\rangle} - n\cdot \mathrm{tr}(a) I$ on $\mathbb{P}_a[\,\mathbb{C}^n\,]$ it follows for $p \in \mathbb{P}_a[\,\mathbb{C}^n\,]$:

$$\begin{aligned}\|\, p\,\|_{\mathrm{gr}} &= \|\, p\,\|_2 + \|\, Ap\,\|_2\\ &\le \|\, p\,\|^\sim + \|\, \langle az,z\rangle p\,\|_2 + n\cdot\mathrm{tr}(a)\,\|\, p\,\|^\sim\\ &\le \Big(1 + n\cdot \mathrm{tr}(a) + \|\, a\,\|\,\Big)\|\, p\,\|^\sim.\end{aligned}$$ □

By a similar computation than we have done in the proof of Lemma 3.1.7 we find that the inclusion $\mathbb{P}_a[\,\mathbb{C}^n\,] \subset H_{\exp}(\mathbb{C}^n)$ is dense with respect to $\|\cdot\|^\sim$.

Corollary 3.5.1 *With the unbounded Toeplitz operator $B := T_{\langle az,z\rangle} - n\cdot tr(a) I$ we have the inclusion $H_{\exp}(\mathbb{C}^n) \subset \mathcal{D}(A)\cap\mathcal{D}(B) \subset H_2$ and $A = B$ on $H_{\exp}(\mathbb{C}^n)$.*

Proof From Lemma 3.5.4 and with $\mathbb{P}_a[\,\mathbb{C}^n\,] \subset \mathcal{D}(A)$ we obtain the following inclusions

$$\mathbb{P}_a\,[\,\mathbb{C}^n\,] \subset H_{\exp}\,(\mathbb{C}^n) \subset \operatorname{clos}\,(\,\mathbb{P}_a\,[\,\mathbb{C}^n\,]\,,\|\cdot\|^{\sim}\,) \subset \operatorname{clos}\,(\,\mathbb{P}_a[\,\mathbb{C}^n\,],\|\cdot\|_{\mathrm{gr}}\,) \subset \mathcal{D}(A) \cap \mathcal{D}(B).$$

Both operators A and B are continuous on $\mathbb{P}_a[\,\mathbb{C}^n\,]$ with respect to $\|\cdot\|^{\sim}$ and they coincide on $\mathbb{P}_a[\,\mathbb{C}^n\,]$. Hence they also coincide on $H_{\exp}(\mathbb{C}^n) \subset \mathcal{D}(A) \cap \mathcal{D}(B)$. $\square$

In the following let $A : H_2 \supset \mathcal{D}(A) \to H_2$ (resp. $a \in \mathcal{L}(\mathbb{C}^n)$) denote the infinitesimal generator of the unitary group defined in (3.5.1) (resp. of $(u_t)_{t\in\mathbb{R}} \subset \mathcal{U}(\mathbb{C}^n)$).

Lemma 3.5.5 *Let $\lambda \in \mathbb{C}^n$. Then for $j \in \mathbb{N}$ and each symbol $f \in L^\infty(\mathbb{C}^n)$ we have*

$$W_{-\lambda}\, ad^j[\,A\,]\,(T_f)\; W_\lambda 1 = \sum_{|\,\alpha\,|+|\,\beta\,|\leq j} [\,a\lambda\,]^\alpha\, \overline{[\,a\lambda\,]}^\beta A_{\alpha,\beta}(\,\lambda\,)1 \tag{3.5.5}$$

where $A_{\alpha,\beta}(\,\lambda\,) \in N_{j,f} := \{\,T_{p_1}\cdots T_{p_k}T_{f\circ\tau_\lambda}T_{p_{k+1}}\cdots T_{p_j} : p_l \in \mathbb{P}[\,\mathbb{C}^n\,]\,\} \subset L(\,H_{\exp}(\mathbb{C}^n)\,)$.

Proof By Proposition 2.1.2 and for all $\alpha,\beta \in \mathbb{N}_0^n$ the operators $A_{\alpha,\beta}(\,\lambda\,)$ are well-defined and using Corollary 3.5.1 we have

$$A = T_{\langle az,z\rangle} - n\cdot \operatorname{tr}(a)I : H_{\exp}\,(\,\mathbb{C}^n\,) \to H_{\exp}\,(\,\mathbb{C}^n\,)\,.$$

It follows from Lemma 3.1.3 and for all $\lambda \in \mathbb{C}^n$ that:

$$\begin{aligned} W_{-\lambda}\,\operatorname{ad}^j[\,A\,]\,(T_f)\; W_\lambda\,1 &= W_{-\lambda}\,\operatorname{ad}^j\Big[\,T_{\langle\,az,z\,\rangle}\,\Big]\,(T_f)\; W_\lambda\,1 \\ &= \operatorname{ad}^j\Big[\,T_{\langle\,a(\,z+\lambda\,),\,z+\lambda\,\rangle}\,\Big](\,T_{f\circ\tau_\lambda}\,)\,1 \\ &= \operatorname{ad}^j\Big[\,T_{\langle az,z\rangle} + \sum_{j=1}^n [\,a\lambda\,]_j\,T_{\bar{z}_j} + \sum_{j=1}^n \overline{[\,a\lambda\,]_j}\,T_{z_j}\,\Big]\,(T_{f\circ\tau_\lambda})\;1. \end{aligned}$$

Now, from the formula

$$\operatorname{ad}^j[\,C\,](D) = \sum_{l=0}^j (-1)^l \binom{j}{l}\; C^{j-l}\,D\,C^l$$

for operators D, C we obtain the assertion. $\square$.

Corollary 3.5.2 *Let $0 < c < \frac{1}{2}$ and $f \in L^\infty(\mathbb{C}^n)$. Then there is $C > 0$ independent of f such that for all $\lambda \in \mathbb{C}^n$:*

$$\Big\|\,W_{-\lambda} ad^j[\,A\,]\,(T_f)\; W_\lambda\,1\,\Big\|_{L^1(\,\mathbb{C}^n,\mu_{\frac{1}{2}}\,)} \leq C\Big[\,1 + \|\,a\lambda\,\|\,\Big]^j\; \|\,f\circ\tau_\lambda\,\|_{L^1(\mathbb{C}^n,\mu_c)}$$

Proof According to Lemma 3.5.5 there are operators $A_{\alpha,\beta}(\lambda) \in N_{j,f}$ such that:

$$\left\| W_{-\lambda}\, \mathrm{ad}^j[\,A\,]\,(T_f)\; W_\lambda\, 1\,\right\|_{L^1(\mathbb{C}^n,\mu_{\frac{1}{2}})} \leq \sum_{|\alpha|+|\beta|\leq j} \big|\,[\,a\lambda\,]^\alpha\,\big|\,\big|\,[\,a\lambda\,]^\beta\,\big| \cdot \|\, A_{\alpha,\beta}(\,\lambda\,)\,1\,\|_{L^1(\mathbb{C}^n,\mu_{\frac{1}{2}})}\,.$$

Let $0 \leq k \leq j$ and fix polynomials $p_{k+1}, \cdots, p_j \in \mathbb{P}[\mathbb{C}^n]$. Then by an easy computation we find $T_{p_{k+1}} \cdots T_{p_j} 1 \in \mathbb{P}[\,\mathbb{C}^n\,] \subset L^1(\mathbb{C}^n, \mu_c)$. Hence by Corollary 2.1.1 there are positive numbers $C_{\alpha,\beta}$ independent of f such that

$$\|\, A_{\alpha,\beta}(\,\lambda\,)\,1\,\|_{L^1(\mathbb{C}^n,\mu_{\frac{1}{2}})} \leq C_{\alpha,\beta}\,\|\, f\circ\tau_\lambda\,\|_{L^1(\mathbb{C}^n,\mu_c)}.$$

We conclude that it exists $C > 0$ with

$$\left\|\, W_{-\lambda}\, \mathrm{ad}^j[\,A\,]\,(T_f)\; W_\lambda\, 1\,\right\|_{L^1(\mathbb{C}^n,\mu_{\frac{1}{2}})} \leq C\,\Big[\,1 + \|\,a\lambda\,\|\,\Big]^j\; \|\, f\circ\tau_\lambda\|_{L^1(\mathbb{C}^n,\mu_c)}\,. \qquad \square$$

Because of Corollary 3.5.1 and Proposition 2.1.2 for $f \in L^\infty(\mathbb{C}^n)$ we have the inclusions

$$A\,[\,H_{\exp}\,(\mathbb{C}^n)\,] \subset H_{\exp}\,(\mathbb{C}^n) \qquad \text{and} \qquad T_f\,[\,H_{\exp}\,(\mathbb{C}^n)\,] \subset H_{\exp}\,(\mathbb{C}^n)\,.$$

Hence for each $j \in \mathbb{N}$ the operator $\mathrm{ad}^j[\,A\,]\,(T_f) : H_{\exp}\,(\mathbb{C}^n) \subset H_2 \longrightarrow H_2$ is well-defined on a dense subspace of H_2. For its adjoint operator we prove:

Lemma 3.5.6 *Let $j \in \mathbb{N}$ and $\lambda \in \mathbb{C}^n$. For $f \in L^\infty(\mathbb{C}^n)$ and the Segal Bargmann kernel K we have*

$$K(\cdot,\lambda\,) \in \mathcal{D}\left(\, ad^j[\,A\,]\,(T_f)^*\,\right)$$

and the equality $ad^j[\,A\,](\,T_f\,)^* K(\cdot,\lambda\,) = ad^j[\,A\,](\,T_{\bar{f}}\,)K(\cdot,\lambda\,)$ holds.

Proof For $\lambda \in \mathbb{C}^n$ and all functions $g \in H_{\exp}(\mathbb{C}^n) = \mathcal{D}(\,\mathrm{ad}^j[\,A\,](T_f)\,)$ we have to show that

$$\left\langle\, \mathrm{ad}^j[\,A\,]\,(T_f)\,g, K(\cdot,\lambda\,)\,\right\rangle_2 = \left\langle\, g, \mathrm{ad}^j[\,A\,]\,\big(T_{\bar{f}}\big)\;K(\,\cdot,\lambda\,)\,\right\rangle_2.$$

Let $g \in H_{\exp}(\mathbb{C}^n)$, then:

$$\begin{aligned}
\left\langle\, \mathrm{ad}^j[\,A\,]\,(T_f)\;g,\; K(\cdot,\lambda\,)\,\right\rangle_2 &= \sum_{l=0}^{j} \binom{j}{l} (-1)^l \left\langle\, A^{j-l}\,T_f\,A^l\,g, K(\cdot,\lambda\,)\,\right\rangle_2 \\
&= \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} \left\langle\, g, A^l\,T_{\bar{f}}\,A^{j-l}\,K(\cdot,\lambda\,)\,\right\rangle_2 \\
&= \left\langle\, g, \mathrm{ad}^j[\,A\,]\,\big(T_{\bar{f}}\big)\;K(\cdot,\lambda\,)\,\right\rangle_2.
\end{aligned}$$

Here we have used the invariance of $H_{\exp}(\mathbb{C}^n)$ under the operators A and T_f as well as the formulas $A^* = -A$ and $T_f^* = T_{\bar{f}}$. $\square$

Definition 3.5.1 Let $j \in \mathbb{C}^n$ and $c > 0$. For each $f \in L^\infty(\mathbb{C}^n)$ we define the norms:

$$\| f \|_{c,j} := \sup \Big\{ \Big[1 + \| a\lambda \| \Big]^j \ \| f \circ \tau_\lambda \|_{L^1(\,\mathbb{C}^n, \mu_c\,)} : \lambda \in \mathbb{C}^n \Big\} \in \mathbb{R} \cup \{\infty\}.$$

By $\mathcal{F}_{c,j}(\mathbb{C}^n)$ we denote the normed linear space

$$\mathcal{F}_{c,j}\,(\,\mathbb{C}^n\,) := \Big\{ f \in L^\infty\,(\mathbb{C}^n) : \| f \|_{c,j} < \infty \Big\}.$$

Note that the group $(U_t)_t$ acts isometric with respect to $\| \cdot \|_{c,j}$ and so all the spaces $\mathcal{F}_{c,j}(\mathbb{C}^n)$ are U_t-invariant.

Theorem 3.5.2 *Let $f \in \mathcal{F}_{c,j}(\mathbb{C}^n)$ with $j \in \mathbb{N}$ and assume that $c \in (\,0, \frac{1}{2}\,)$. Then the iterated commutator $ad^j[\,A\,](T_f)$ has a continuous extension to an operator on H_1 such that there is $C > 0$ with:*

$$\| \, ad^j[\,A\,]\,(T_f) \ \| \leq C \, \| f \|_{c,j}.$$

Proof Because $\bar{f} \in \mathcal{F}_{c,j}(\mathbb{C}^n)$ as well, we conclude from Corollary 3.5.2 that there is $C > 0$ with

$$\sup_{\lambda \in \mathbb{C}^n} \Big\| \, W_{-\lambda} \, \mathrm{ad}^j[\,A\,]\,(T_h)\, W_\lambda \, 1 \, \Big\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})} \leq C \, \| f \|_{c,j} < \infty$$

for $h \in \{\, f, \bar{f}\,\}$. Hence by Lemma 3.5.6 assumptions (i) and (ii) in Theorem 2.1.2 hold for the iterated commutator $\mathrm{ad}^j[\,A\,](T_f)$ and we conclude that it has a continuous extension to H_1 such that $\| \, \mathrm{ad}^j[\,A\,](T_f) \, \| \leq C \| \, f \, \|_{c,j}$. □

With the infinitesimal generator $a \in \mathcal{L}(\mathbb{C}^n)$ of the unitary group $(u_t)_{t \in \mathbb{R}}$ we denote by $N(a)$ the kernel of a. For each $r > 0$ and $x \in \mathbb{C}^n$ let $K_r(x)$ be the ball in $\mathbb{C}^n$ centered in x with radius r.

Corollary 3.5.3 *Let $f \in L^\infty(\mathbb{C}^n)$ such that there is a radius $r > 0$ with*

$$supp f \subset N(a) + K_r(0).$$

Then for each $j \in \mathbb{C}^n$ we have $f \in \mathcal{F}_{\frac{1}{4},j}(\mathbb{C}^n)$. In particular, the iterated commutator $ad^j[\,A\,](T_f)$ has a continuous extension to H_2.

Proof Let $c := \frac{1}{4}$ and denote by P_a the orthogonal projection from $\mathbb{C}^n$ onto $N(a)$. Then we have for all $\lambda \in \mathbb{C}^n$ with $\lambda_1 := (\,I - P_a\,)\lambda$ and $\lambda_2 := P_a\lambda$:

$$\begin{aligned}
&\|f \circ \tau_\lambda\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{4}})} \\
=&\frac{1}{(4\pi)^n} \int_{N(a)} \int_{N(a)^\perp} |\, f(z_1 + z_2 + \lambda)\,| \exp\Big(-\frac{1}{4} \ \| \, z_1 \, \|^2 - \frac{1}{4} \ \| \, z_2 \, \|^2 \ \Big) \, dv\,(z_1) \ dv\,(z_2) \\
=&\frac{1}{(4\pi)^n} \int_{N(a)} \int_{N(a)^\perp} |\, f(z_1 + z_2)\,| \exp\Big(-\frac{1}{4} \ \| \, z_1 - \lambda_1 \, \|^2 - \frac{1}{4} \, \| \, z_2 - \lambda_2 \, \|^2 \Big) \, dv\,(z_1) \ dv\,(z_2) \\
\leq& C \, \| \, f \, \|_\infty \int_{K_r(0)} \exp\Big(-\frac{1}{4} \ \| \, z_1 - \lambda_1 \, \|^2 \ \Big) \, dv\,(z_1)
\end{aligned} \tag{3.5.6}$$

where the constant C is given by the integral

$$C := \frac{1}{(4\pi)^n} \int_{N(a)} \exp\left(-\frac{1}{4} \parallel z \parallel^2\right) \, dv(z) > 0.$$

Now, we estimate the integral on the right hand side of (3.5.6) as a function of λ_1. There is $\tilde{C} > 0$ such that

$$\begin{aligned}
&\int_{K_r(0)} \exp\Big(-\frac{1}{4} \parallel z - \lambda_1 \parallel^2\Big) \, dv(z) \\
\leq &\exp\Big(-\frac{1}{4} \parallel \lambda_1 \parallel^2\Big) \int_{K_r(0)} \exp\left(\frac{1}{2} \operatorname{Re}\langle \lambda_1, z\rangle - \frac{1}{4} \parallel z \parallel^2\right) \, dv(z) \\
\leq &\tilde{C} \, \exp\Big(-\frac{1}{4} \parallel \lambda_1 \parallel^2 + r \parallel \lambda_1 \parallel\Big).
\end{aligned}$$

With a suitable positive number D now it follows that

$$\parallel f \circ \tau_\lambda \parallel_{L^1(\mathbb{C}^n, \mu_{\frac{1}{4}})} \leq D \parallel f \parallel_\infty \exp\left(-\frac{1}{4} \parallel \lambda_1 \parallel^2 + r \parallel \lambda_1 \parallel\right).$$

This implies that $f \in \bigcap_{j \in \mathbb{N}} \mathcal{F}_{c,j}(\mathbb{C}^n)$ and with Theorem 3.5.2 the assertion follows. □

As we already mentioned above for each $c \in (0, 2^{-1})$ the spaces $\mathcal{F}_{c,j}(\mathbb{C}^n)$ are invariant under the action of the unitary group $(u_t)_{t \in \mathbb{R}} \subset \mathcal{L}(\mathbb{C}^n)$. Moreover, the map

$$\mathcal{F}_{c,j}(\mathbb{C}^n) \ni f \mapsto f \circ u_t \in \mathcal{F}_{c,j}(\mathbb{C}^n)$$

defines isometries for all $j \in \mathbb{N}$ and $t \in \mathbb{R}$. This easily follows for $f \in \mathcal{F}_{c,j}(\mathbb{C}^n)$ from the equality

$$\parallel f \circ u_t \circ \tau_\lambda \parallel_{L^1(\mathbb{C}^n, \mu_c)} = \parallel f \circ \tau_{u_t \lambda} \parallel_{L^1(\mathbb{C}^n, \mu_c)}$$

and $[a, u_t] = 0$ for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}^n$ which implies that:

$$\begin{aligned}
\parallel f \circ u_t \parallel_{c,j} &= \sup_{\lambda \in \mathbb{C}^n} \Big[1 + \parallel a\lambda \parallel\Big]^j \parallel f \circ u_t \circ \tau_\lambda \parallel_{L^1(\mathbb{C}^n, \mu_c)} \\
&= \sup_{\lambda \in \mathbb{C}^n} \Big[1 + \parallel a u_t \lambda \parallel\Big]^j \parallel f \circ \tau_{u_t \lambda} \parallel_{L^1(\mathbb{C}^n, \mu_c)} = \parallel f \parallel_{c,j}.
\end{aligned}$$

Lemma 3.5.7 *Let $j \in \mathbb{N}$ and $f \in \mathcal{F}_{c,j}(\mathbb{C}^n)$. Assume that the map*

$$\mathbb{R} \ni t \mapsto t^{-1} [f \circ u_t - f] \in \mathcal{F}_{c,j}(\mathbb{C}^n) \tag{3.5.7}$$

is continuous in $t = 0$. Then the continuous extension $\tilde{C}_j(A, T_f)$ of $ad^j[A](T_f)$ leaves the space $\mathcal{D}(A)$ invariant.

Proof Let $h \in \mathcal{D}(A)$, then we have with $j \in \mathbb{N}$ and for all $t \in \mathbb{R}$:

$$\frac{1}{t}\ [\, U_t - I\,]\ \tilde{C}_j\,(\,A, T_f\,)\ h = \frac{1}{t}\Big[\,U_t, \tilde{C}_j\,(\,A, T_f\,)\,\Big]\,h + \tilde{C}_j\,(\,A, T_f\,)\,\frac{1}{t}\ [\,U_t - I\,]\ h. \quad (3.5.8)$$

Because of $h \in \mathcal{D}(A)$ the limit of the second term in (3.5.8) exists if t tends to zero. Hence we only have to show the existence of

$$\lim_{t \to 0} \frac{1}{t}\ \Big[\,U_t, \tilde{C}_j\,(\,A, T_f\,)\Big]\ h \in H_2. \quad (3.5.9)$$

From $[\,U_t, A\,] = 0$ on $\mathcal{D}(A)$ and $U_t\, T_h\, U_{-t} = T_{h \circ u_t}$ for all $t \in \mathbb{R}$ it follows that

$$\frac{1}{t}\Big[\,U_t, \tilde{C}_j\,(\,A, T_f\,)\,\Big]h = \tilde{C}_j\,\Big(\,A, T_{\frac{1}{t}[\,f \circ u_t - f\,]}\Big)\,U_t h.$$

From Theorem 3.5.2 the map $\mathcal{F}_{c,j}(\mathbb{C}^n) \ni g \mapsto \tilde{C}_j(A, T_g) \in \mathcal{L}(H_2)$ is continuous and by the continuity of (3.5.7) in $t = 0$ we now obtain that

$$G : \mathbb{R} \mapsto \mathcal{L}\,(H_2) : t \mapsto \tilde{C}_j\Big(A, T_{\frac{1}{t}[\,f \circ u_t - f\,]}\Big) \in \mathcal{L}\,(H_2)$$

is continuous in $t = 0$. Let us define $G(0) := \lim_{t \to 0} G(t) \in \mathcal{L}(H_2)$, then the existence of the limit (3.5.9) follows from

$$\|\,G(t)\,U_t\,h - G(0)\,h\,\|_2 \leq \|\,G(t) - G(0)\,\| \cdot \|\,h\,\|_2 + \|\,G(0)\,\| \cdot \|\,U_t\,h - h\,\|_2$$

and the convergence $\lim_{t \to 0} \|\,U_t\,h - h\,\|_2 = 0$. □

Denote by $\mathcal{A}$ the C^*-algebra in $\mathcal{L}(H_2)$ generated by all Toeplitz operators with bounded symbols. Then for functions $g_1, \cdots, g_m \in L^\infty(\mathbb{C}^n)$ and the unitary group $(U_t)_{t \in \mathbb{R}} \subset \mathcal{L}(H_2)$ it follows for all $t \in \mathbb{R}$ by Lemma 3.5.2

$$U_t\, T_{g_1} \cdots T_{g_m}\, U_{-t} = T_{g_1 \circ u_t} \cdots T_{g_m \circ u_t} \in \mathcal{A}.$$

We conclude that for all $t \in \mathbb{R}$ the space $\mathcal{A}$ is invariant under $\varphi : \mathbb{R} \mapsto \mathcal{L}(\,\mathcal{L}(H_2)\,)$ which is defined by

$$\varphi(t)X := \big[\,\mathcal{L}\,(H_2) \ni X \mapsto U_t\, X\, U_{-t}\,\big].$$

For $n \in \mathbb{N} \cup \{\infty\}$ consider the scales of algebras

$$\big(\,\Psi_U^n[\,\mathcal{A}\,]\,\big)_n \subset \mathcal{A} \qquad \text{and} \qquad \big(\,\Psi_n^U\,[\,\mathcal{A}\,]\,\big)_n \subset \mathcal{A}$$

defined by the action of the group $(U_t)_t$ as it was described in section 3.2. By the remarks above it follows that Theorem 1.3.1 holds for these scales and Lemma 3.5.7 together with Theorem 3.5.2 imply:

Theorem 3.5.3 *Let $c \in (\,0, 2^{-1}\,)$ and $j \in \mathbb{N}$. With $g(f,t) := t^{-1}\,[\, f \circ u_t - f\,]$ it holds the inclusion*

$$\tilde{\mathcal{F}}_{c,j}(\mathbb{C}^n) := \Big\{\, T_f : f \in \mathcal{F}_{c,j}(\mathbb{C}^n),\ \mathbb{R} \ni t \mapsto g(f,t) \in \mathcal{F}_{c,j}(\mathbb{C}^n) \text{ is continuous } \Big\} \subset \Psi_j^U[\,\mathcal{A}\,].$$

Moreover, the map

$$\tilde{\mathcal{F}}_{c,j}\,(\mathbb{C}^n) \ni f \mapsto T_f \in \Psi_j^U[\,\mathcal{A}\,]$$

is continuous if $\tilde{\mathcal{F}}_{c,j}(\mathbb{C}^n)$ carries the topology induced by $\mathcal{F}_{c,j}(\mathbb{C}^n)$.

We give a class of functions which belongs to the spaces $\mathcal{F}_{c,j}(\mathbb{C}^n)$ for all $c \in (\,0, \frac{1}{2}\,)$ and each number $j \in \mathbb{N}$.

Corollary 3.5.4 *It holds $\{\, T_f : f \in \mathcal{C}_c^2(\mathbb{C}^n)\,\} \subset \Psi_\infty^U[\,\mathcal{A}\,] = \Psi_U^\infty[\,\mathcal{A}\,]$.*

Proof Without loss of generality we can assume that f is real valued. Choose $r > 0$ such that *suppf* $\subset K_r(0)$ and define the function

$$g(z) := \big\langle\, \mathrm{grad} f(z), az\,\big\rangle.$$

Because $(u_t)_t \in \mathcal{L}(\mathbb{C}^n)$ is unitary the support of $t^{-1}\,[\, f \circ u_t - f\,] - g$ is contained in $K_r(0)$ for all $t > 0$. With a similar computation than the one in Corollary 3.5.3 there is $C_1 > 0$ with

$$\begin{aligned} &\Big\|\, \frac{1}{t}\ [\, f \circ u_t - f\,] \circ \tau_\lambda - g \circ \tau_\lambda \,\Big\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{4}})} \\ &\leq C_1 \Big\|\, \frac{1}{t}\ [\, f \circ u_t - f\,] - g\,\Big\|_\infty \exp\Big(-\frac{1}{4}\,\|\,\lambda\,\|^2 + r\,\|\,\lambda\,\|\Big). \end{aligned} \tag{3.5.10}$$

Consider the function $F : [\,-1,1\,] \times \mathbb{C}^n \to \mathbb{R}$ defined by $F(\,t,z\,) := f \circ u_t(z)$. Then for all $\|\,z\,\| = \|\,u_t z\,\| > r$ and $t \in [\,-1,1\,]$ it follows that $F(\,t,z\,) = 0$. This shows that

$$\frac{d^2}{dt^2}\, F \in \mathcal{C}_c^2\Big(\,[\,-1,1\,] \times \mathbb{C}^n, \mathbb{R}\,\Big).$$

Hence there is a number $C > 0$ such that

$$\sup\Big\{\, \frac{d^2}{dt^2} F(t,z) : (t,z) \in [-1,1] \times \mathbb{C}^n \,\Big\} \leq C < \infty.$$

By the Taylor formula we conclude from $g(z) = \frac{d}{dt}F(0,z) \in \mathbb{R}$ for $|\,t\,| < 1$ and all $z \in \mathbb{C}^n$:

$$\Big|\, \frac{1}{t}\ [\, f \circ u_t(z) - f(z)\,] - g(z) \Big| \leq \frac{t}{2}\, \frac{d^2}{dt^2}\ |\, F(\,\theta, z\,)\,| \leq \frac{t}{2}\, C, \qquad |\,\theta\,| < 1.$$

Hence $t^{-1}[\, f \circ u_t - f\,] - g$ converges to 0 uniformly on $\mathbb{C}^n$ if t tends to 0. By inequality (3.5.10) together with Corollary 3.5.3 we obtain that

$$f \in \bigcap_{j \in \mathbb{N}} \tilde{F}_{\frac{1}{4},j}\,(\,\mathbb{C}^n\,) \subset \Psi^U_\infty[\,\mathcal{A}\,] = \Psi^\infty_U[\,\mathcal{A}\,]. \qquad \square$$

As a corollary we conclude that the space of smooth Toeplitz operators with respect to $(u_t)_t$ is invariant under perturbations of the symbols by continuous functions with compact support.

Theorem 3.5.4 *It holds* $\{\, T_f : f \in \mathcal{C}_c(\mathbb{C}^n)\,\} \subset \Psi^U_\infty[\,\mathcal{A}\,] = \Psi^\infty_U[\,\mathcal{A}\,]$.

Proof Let $f \in \mathcal{C}_c(\mathbb{C}^n)$ and $r > 0$ with supp$f \subset K_r(0)$. Analyzing the proof of Corollary 3.5.3 and using Corollary 3.5.4 it follows that the map

$$\mathcal{M} := \Big\{\, g \in \mathcal{C}^2_c\,(\mathbb{C}^n) : \operatorname{supp} g \subset K_{r+1}(0) \,\Big\} \ni g \mapsto T_g \in \Psi^U_\infty[\,\mathcal{A}\,]$$

is continuous if $\mathcal{M}$ carries the sup-norm. The assertion now follows from the fact that f can be approximated uniformly by functions $g \in \mathcal{M}$. $\square$

Finally, we give an example of an Toeplitz operator in $\Psi^\infty_\alpha[\,\mathcal{A}]$ with a bounded symbol, which is not even continuous with respect to a rotation. To simplify the computations we restrict ourselves to the case $n = 1$.

Example 3.5.1 For the dimension $n = 1$ we consider the unitary C_0-group

$$u_t : \mathbb{R} \to \mathcal{L}(\mathbb{C}) : t \mapsto u_t(\,z\,) := \exp(\,it\,) \cdot z.$$

Then, the corresponding unitary C_0-group $(U_t)_{t\in\mathbb{R}} \subset \mathcal{L}(H_2)$ of composition operators has the infinitesimal generator A with

$$Ap = i\,\big[\,T_{|z|^2} - I\,\big]\,p \qquad \text{for all} \qquad p \in \mathbb{P}_a\,[\,\mathbb{C}\,].$$

Let $x \in \mathbb{C}$ and consider $g_x : \mathbb{C} \to \mathbb{C}$ defined by $g_x(z) := \exp(\,2i\,\mathrm{Im}\langle z, x\rangle\,)$. For $f \in H_2$ and $z \in \mathbb{C}$ it follows that:

$$\begin{aligned}
[\,T_{g_x} f\,]\,(\,z\,) &= \Big\langle\, f \exp\big(\,\langle \cdot, x\rangle - \langle x, \cdot\,\rangle\,\big),\ \exp\big(\,\langle \cdot, z\rangle\,\big)\,\Big\rangle_2 \\
&= \Big\langle\, f \exp\big(\,\langle \cdot, x\rangle\,\big), \exp\big(\,\langle \cdot, z - x\rangle\,\big)\,\Big\rangle_2 \\
&= \exp\big(\,\langle z - x, x\rangle\,\big)\, f \circ \tau_{-x}(z) = \exp\Big(-\frac{1}{2}\,|\,x\,|^2\Big)\,[\,W_x f\,]\,(\,z\,).
\end{aligned}$$

Hence we conclude that $T_{g_x} = \exp(-\frac{1}{2}\,|\,x\,|^2\,)\,W_x$ with the Weyl operator W_x. Next consider the function $\Phi_x : \mathbb{R} \to \mathcal{L}(H_2)$ defined by:

$$\begin{aligned}
\Phi_x(t) :&= U_t\,T_{g_x}\,U_{-t} = T_{g_x \circ u_t} = T_{g_{u_t^* x}} = \exp\Big(-\frac{1}{2}\,|\,x\,|^2\Big)\,W_{u_t^* x} \\
&= \exp\Big(-\frac{1}{2}\,|\,x\,|^2\Big)\,W_{\exp(\,-it\,)x}.
\end{aligned}$$

For $t \in \mathbb{R}$ now we show that $\| \Phi_1(t) - \Phi_1(0) \| = \exp(-\frac{1}{2}) \| W_{\exp(-it)} - W_1 \|$ stays away from 0. Let $w \in \mathbb{C}$, then we have:

$$\begin{aligned} & \Big\| W_{\exp(-it)x} - W_x \Big\| \qquad (3.5.11) \\ \geq & \Big| \Big\langle \Big[W_{\exp(-it)x} - W_x \Big] k_w,\ k_w \Big\rangle_2 \Big| \\ = & \exp\Big(-\frac{1}{2} | x |^2 - | w |^2 \Big) \Big| K(w, u_t^* x) K(w - u_t^* x, w) - K(w, x) K(w - x, w) \Big| \\ = & \exp\left(-\frac{1}{2} | x |^2 \right) \Big| \exp(2i \operatorname{Im}\langle w, u_t^* x \rangle) - \exp(2i \operatorname{Im}\langle w, x \rangle) \Big|. \end{aligned}$$

Now we assume that $x = 1$ and $t \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$. Then we can define

$$w_t := (\sin t)^{-1} \exp(-it) \in \mathbb{C}.$$

It follows that $\langle w_t, u_t^* 1 \rangle = (\sin t)^{-1}$ and $\langle w_t, 1 \rangle = (\sin t)^{-1} \exp(-it)$. Using equation (3.5.11) we obtain

$$\Big\| W_{\exp(-it)} - W_1 \Big\| \geq \exp\Big(-\frac{1}{2} \Big) | 1 - \exp(2i) | > 0, \qquad \forall\ t \in \Big[-\frac{\pi}{2}, \frac{\pi}{2} \Big] \setminus \{0\}.$$

Hence the map $\Phi_1 : \mathbb{R} \mapsto \mathcal{L}(H_2)$ is not continuous in $t = 0$. Moreover, because of

$$g_1 \circ \tau_t = g_1 \qquad \text{for all} \qquad t \in \mathbb{R}$$

it is easy to see that $[W_t, M_{g_1}] = 0$ and with the infinitesimal generator $v^{(1)}$ of the Weyl group $(W_t)_{t \in \mathbb{R}}$ it follows that

$$M_h \left[\mathcal{D}\left(V^{(1)}\right) \right] \subset \mathcal{D}\left(V^{(1)}\right)$$

where $h \in \{ g_1, \overline{g_1} \}$ and so $T_{g_1} \in \Psi^\alpha_\infty[\mathcal{A}] = \Psi^\infty_\alpha[\mathcal{A}]$, the smooth elements with respect to the Weyl group action.

Chapter 4

Fréchet algebras by localized commutator methods and Szegö Toeplitz operators

We examine some local aspects of the Szegö-projection P_s and the corresponding Toeplitz operators $T_f = P_s M_f$ with symbol f. Due to a result by A. Nagel and E. M. Stein for any strictly pseudo-convex domain Ω the projection P_s is a pseudo-differential operator of exotic type $(\frac{1}{2}, \frac{1}{2})$. Using this fact and by the general theory in [79] a rich class of spectral invariant Fréchet sub-algebras $\mathcal{B}$ in $L^2(\partial\Omega)$ (or more generally in a Toeplitz C^*-algebra) containing P_s can be constructed by commutator methods. Hence conditions on the operator φT_f to belong to $\mathcal{B}$ with a cut-off function φ can be characterized by the (local) regularity of the symbol f. In the second part of this chapter we examine the question under which conditions the Szegö projection of a bounded function on the complex sphere locally admits a continuous extension to an analytic function on the unit ball. Mainly we are dealing with the upper half-space $\mathcal{H}_+$ in $\mathbb{C}^{n+1}$ and the complex unit sphere, but we consider many of the results to be true in wider generality for strictly pseudo-convex domains in $\mathbb{C}^n$.

Let $\mathcal{H}_+$ be the upper half-space in $\mathbb{C}^{n+1}$ and by P_s denote the orthogonal projection (Szegö-projection) of $L^2(\partial\mathcal{H}_+)$ onto the closed subspace of square integrable boundary values of holomorphic functions in $\mathcal{H}_+$. By identifying the boundary $\partial\mathcal{H}_+$ of $\mathcal{H}_+$ with the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ we can look at the Szegö-projection as a (singular) convolution operator on $H := L^2(\mathbb{H}^n, \beta)$ with respect to the group action on $\mathbb{H}^n$ ([134], [117], the appendix). Here β denotes the left-invariant *Haar measure* on $\mathbb{H}^n$ which coincides with the $2n+1$-dimensional Lebesgue measure under the canonical identification $\mathbb{H}^n \cong \mathbb{R}^{2n+1}$. The theory of singular integrals and pseudo-differential operators are closely related. The localized version $\psi_1 P_s \psi_2 = \psi_1 T_{\psi_2}$ of P_s where $\psi_1, \psi_2 \in \mathcal{C}_c^\infty(\mathbb{H}^n)$ is an operator in the *exotic class* $\Psi^0_{\frac{1}{2},\frac{1}{2}}$ (see [134], [117]). More general, by results due to A. Nagel and E. M. Stein (see [117]), the Szegö-projection on any strictly pseudo-convex domain is a pseudo-differential operator of type $(\frac{1}{2}, \frac{1}{2})$ and so it inherits many nice properties like *pseudo locality.* It is a well-known fact that symbols in $\Psi^0_{1,1}$ in general do not lead to bounded operators on $L^2(\mathbb{R}^n, v)$. On the other hand for $\Psi^0_{\rho,\rho}$ with $0 \leq \rho < 1$ the almost-orthogonality principle applies (see [134]) and so operators in this class are bounded on $L^2(\mathbb{R}^n, v)$ (which in fact is obvious in the case of $\rho = \frac{1}{2}$ and the Szegö-projection). A difficulty one has to deal with in the case of exotic classes is the fact that the full asymptotic calculus fails. For operators of type $(\frac{1}{2}, \frac{1}{2})$ the pseudo-differential techniques in Hörmander [88] break down; here the L^2-boundedness result is due to A.P. Calderón and R. Vaillancourt [32]. Given $f \in L^\infty(\mathbb{H}^n)$ we examine in which sense local smoothness of the symbol f is reflected by the Toeplitz operator $T_f := P_s M_f$ We consider the following problem:

(P): Let $U \subset \mathbb{H}^n$ be an open subset. How can one define classes of spectral invariant Fréchet algebras or even Ψ^*-algebras $\mathcal{B}_U \subset \mathcal{L}(\ L^2(\mathbb{H}^n, \beta)\ )$ containing all pseudo-differential operators of type $(\frac{1}{2}, \frac{1}{2})$ such that $\mathcal{B}_U$ is localized on U in the following sense:

For any symbol $f \in L^\infty(\mathbb{H}^n)$ which is compactly supported and smooth in a neighborhood of $\bar{U}$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{H}^n)$ the Toeplitz operator φT_f is contained in $\mathcal{B}_U$?

It was shown by R. Beals [17] that the Hörmander classes $\Psi^0_{\rho,\delta}$ of pseudo-differential operators where $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$ completely can be characterized by conditions on iterated commutators with the multiplications M_{x_j} and the derivatives ∂_{x_j} of all orders. In fact this observation led to a proof of spectral invariance of $\Psi^0_{\rho,\delta}$ in $\mathcal{L}(\ L^2(\mathbb{R}^n, v)\ )$. Moreover, by a result in [40] the class $\Psi^0_{\frac{1}{2},\frac{1}{2}}$ has an analog description using commutators with smooth vector fields. We solve the problem (P) by following a general construction in [79], motivated by the above mentioned properties of pseudo-differential operators in $\Psi^0_{\rho,\delta}$. We are using commutator methods with finite systems of smooth and compactly supported vector fields and as a result the algebras $\mathcal{B}_U$ operate on a given scale of Sobolev spaces without order shift. The multiplication M_f as well as the operator $\psi_1 T_{\psi_2}$ are contained in $\mathcal{B}_U$ for any bounded symbol f smooth in a neighborhood of $\bar{U}$ and $\psi_1, \psi_2 \in \mathcal{C}_c^\infty(\mathbb{H}^n)$.

Finally we examine the Szegö-projection P on the unit sphere S^{2n-1} in $\mathbb{C}^n$. The Szegö-kernel is explicitly known and P is an integral operator on $V := L^2(S^{2n-1}, \sigma)$ with the usual surface measure σ. Here P maps V onto the Hardy-space $H^2(S^{2n-1})$ of holomorphic functions on the ball $B_{2n} \subset \mathbb{C}^n$ with square integrable boundary values. We show that the Szegö-projection of any smooth function f on S^{2n-1} is smooth again and it admits a continuous analytic extension to B_{2n}. The proof uses the fact that there is a canonical orthonormal basis of $H^2(S^{2n-1})$ consisting of eigenfunctions of the Beltrami Laplace operator B on $\mathcal{C}^\infty(S^{2n-1})$ and the asymptotics of the corresponding eigenvalues. This result has a local version if we only claim the smoothness of f on open subsets of S^{2n-1} (see Theorem 4.4.3).

Chapter 4 is organized as follows: we summarize some important facts on the Hörmander classes $\Psi^m_{\rho,\delta}$ of pseudo-differential operators (see [103], [134], [88]) and we study general scales of Sobolev spaces generated by generalized Laplacians (see [44]). The algebras $\mathcal{A}$ and its local version $\mathcal{A}_\Phi$ as well as sub-algebras $\Psi_\rho[\mathcal{A}]$ and $\Psi_\rho[\mathcal{A}_\Phi]$ defined by commutator methods with smooth vector-fields are introduced. We show that for appropriate choices of $\Phi \in \mathcal{C}_c^\infty(\mathbb{H}^n)$ the algebras $\mathcal{B}_U := \Psi_{\frac{1}{2}}[\, \mathcal{A}_\Phi \,]$ solve our problem (P) above. We prove spectral invariance (for further detail we refer to [79] or chapter 1) and under additional conditions on the defining vector-fields symmetry of the algebras is obtained. Hence in this case $\mathcal{B}_U$ is a Ψ^*-algebra in the sense of [79] (cf Definition 1.0.1). In the fourth part we recall the definition of the Szegö-projection on the Heisenberg group $\mathbb{H}^n$ as a singular convolution operator with respect to the group structure and we prove that it preserves local smoothness. The discussion of the Szegö-projection P on the unit sphere S^{2n-1} in $\mathbb{C}^n$ is contained in the last section. There we give the results on the continuous extensions of Pf to the interior of the complex ball provided f is smooth in some open subset $U \subset S^{2n-1}$.

4.1 Pseudodifferential operators and commutator methods

In the following let $\mathcal{S}^m_{\rho,\delta}$ for $m \in \mathbb{R}$ and $(0 \leq \delta \leq \rho \leq 1)$ denote the standard symbol classes of all smooth functions $a(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ such that for $\alpha, \beta \in \mathbb{N}_0^n$:

$$\left| \, \partial_x^\beta \, \partial_\xi^\alpha \, a(x,\xi) \, \right| \leq C_{\alpha,\beta} \cdot \lambda(\,\xi\,)^{m-\rho|\,\alpha\,|+\delta|\,\beta\,|}$$

where $\lambda(\xi) := (\,1 + \|\,\xi\,\|^2\,)^{\frac{1}{2}}$ and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$ resp. $\mathbb{C}^n$. For any symbol $a \in \mathcal{S}^m_{\rho,\delta}$ the corresponding *pseudo-differential operator* R_a on $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ (the rapidly decreasing functions) is given by:

$$R_a f(\,x\,) := (2\pi)^{-n} \int_{\mathbb{R}^n} a(\,x,\xi\,)\, e^{ix\cdot\xi}\, \hat{f}(\,\xi\,)\, d\xi. \tag{4.1.1}$$

Here $\hat{f}$ denotes the *Fourier transform* of $f \in \mathcal{S}$ defined by $\hat{f}(\xi) = \int f(x)\, e^{-ix\cdot\xi}\, dx$. The class of all operators R_a with symbol $a \in \mathcal{S}^m_{\rho,\delta}$ is denoted by $\Psi^m_{\rho,\delta}$ and R_a can be considered as a continuous linear map on $\mathcal{S}$. The operators in $\Psi^0_{\rho,\delta}$ are said to be of type $(\,\rho,\delta\,)$. Moreover, we write $\Psi^{-\infty} := \bigcap_{m\in\mathbb{N}} \Psi^m_{\rho,\delta}$ which is independent of ρ and δ. In general, the following inclusions hold (see [103]):

$$\Psi^{-\infty} \subset \Psi^m_{1,0} \subset \Psi^m_{\rho,\delta} \subset \Psi^{m'}_{\rho',\delta'}, \qquad (\,m \leq m',\;\; \rho \geq \rho',\;\; \delta \leq \delta'\,).$$

With the operator $\Lambda^s := R_{\lambda^s} \in \Psi^s_{1,0}$ and $s \in \mathbb{R}$ (see [103]) and the $L^2(\mathbb{R}^n, v)$-norm $\|\cdot\|_2$ we obtain the usual scale of *Sobolev spaces*:

$$H^s = H^s\,(\,\mathbb{R}^n\,) := \Big\{\, u \in \mathcal{S}' : \Lambda^s u \in L^2\,(\,\mathbb{R}^n\,)\,, \;\; \|\,u\,\|_{H^s} := \|\,\Lambda^s u\,\|_2 \,\Big\}$$

and we write $H^\infty := \bigcap_{t\in\mathbb{R}} H^t$ and $H^{-\infty} := \bigcup_{t\in\mathbb{R}} H^t$. It is a well-known fact, that the class $\Psi^m_{\rho,\delta}$ with $(\,0 \leq \delta \leq \rho \leq 1\,)$ and $\delta < 1$ acts on H^s for $s > 0$ as follows (see [32], [103], p. 224 remark 1, [130], [139]):

Theorem 4.1.1 *Let $R_a \in \Psi^m_{\rho,\delta}$ where $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$, then for all $s \in \mathbb{R}$ there is a continuous extension of R_a from H^{s+m} to H^s.*

In particular, in the case $m = 0$ it follows that R_a is a continuous operator on $L^2(\mathbb{R}^n, v)$. Let us mention two more structural results which give characterizations of the Hörmander classes by commutators and can be found in [17] resp. [40]. For multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ and with $D_{x_i} := -i\frac{\partial}{\partial_{x_i}} = -i\partial_{x_i}$ we introduce the iterated commutators

$$\mathrm{ad}[\,-ix\,]^\alpha \mathrm{ad}[\,D_x\,]^\beta := \mathrm{ad}[-ix_1]^{\alpha_1}\cdots \mathrm{ad}[\,-ix_n\,]^{\alpha_n}\mathrm{ad}[\,D_{x_1}\,]^{\beta_1}\cdots \mathrm{ad}[\,D_{x_n}\,]^{\beta_n}.$$

The following result is due to R. Beals [17].

Theorem 4.1.2 *Let $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. Assume that $B : \mathcal{S} \to \mathcal{S}'$ is a continuous operator. Then (a) and (b) below are equivalent:*

(a) B is of the class $\Psi^0_{\rho,\delta}$.

(b) The commutators $ad[\,-ix\,]^\alpha ad[\,D_x\,]^\beta(B) : H^{s-\rho|\,\alpha\,|+\delta|\,\beta\,|} \to H^s$ have extensions to continuous operators for all $\alpha, \beta \in \mathbb{N}_0^n$.

Using this result, it was shown by R. Beals [17] and finally by J Ueberberg [139] and E. Schrohe [130] that the classes $\Psi^0_{\rho,\delta}$ are spectral invariant in $\mathcal{L}(H)$ where $H := L^2(\mathbb{R}^n, v)$ and so $\Psi^0_{\rho,\delta}$ is a Ψ^*-algebra in the sense of [79]. There is a characterization of pseudo-differential operators of class $\Psi^0_{\frac{1}{2},\frac{1}{2}}$ by iterated commutators with smooth vector fields on $\mathbb{R}^n$ due to R. Coifman and Y. Meyer [40]. With a finite system $\mathcal{V}_n := [\,X_1, \cdots, X_n\,]$ of vector fields and an operator T we inductively define, provided the compositions make sense:

$$\mathrm{ad}[\,X_1\,](T) = [\,X_1, T\,] = X_1T - TX_1 \quad \text{and} \quad \mathrm{ad}[\,\mathcal{V}_j\,](T) := \Big[\,X_j, \mathrm{ad}[\,\mathcal{V}_{j-1}\,](T)\,\Big].$$

Theorem 4.1.3 *([40]) Let $K \subset \mathbb{R}^n$ be compact and $T : \mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathcal{C}_c^\infty(\mathbb{R}^n)$ a continuous operator with supp $(T) \subset K$. Then (a) and (b) below are equivalent:*

(a) T is of the class $\Psi^0_{\frac{1}{2},\frac{1}{2}}$.

(b) For all $s \in \mathbb{R}$ there is a continuous extension $T : H^s \to H^s$ of T. Moreover, for any vector fields $\mathcal{V}_m := [\,X_1, \cdots, X_m\,]$, $m \in \mathbb{N}$ with $\mathcal{C}^\infty$-coefficients all the iterated commutators $ad[\,\mathcal{V}_m\,](T)$ have continuous extensions to $T_m : H^s \to H^{s-\frac{m}{2}}$.

Remark 4.1.1 For any smooth and compactly supported vector-field Z there is a smooth and compactly supported vector-field $\bar{Z}$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $Z^* = \bar{Z} + M_\varphi$. Because for $T \in \Psi^0_{\frac{1}{2},\frac{1}{2}}$ the commutator $[\,M_\varphi, T\,]$ is of class $\Psi^0_{\frac{1}{2},\frac{1}{2}}$ again, we conclude in this case that the implication $(a) \Rightarrow (b)$ in Theorem 4.1.3 still holds if we replace $\mathcal{V}_m := [\,X_1, \cdots, X_m\,]$ by $\mathcal{V}_m^* := [\,X_1^*, \cdots, X_m^*\,]$.

4.2 Localization of operator algebras and Sobolev spaces

Following the theory developed in chapter 1 we construct a scale of Sobolev spaces starting with certain positive self-adjoint operators, the generalized Laplacians (see [44]). In the following we use the notation $H := L^2(\mathbb{R}^n, v)$ and we write $\|\cdot\|_2$ for the L^2-norm on H. As a result we obtain a localized version of the Sobolev spaces H^s and corresponding Ψ^*-algebras in $\mathcal{L}(H)$. Let $\Phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be a cut-off function with $\Phi \geq 0$, then we consider the positive (semi-bounded below) operators

$$\Lambda_\Phi := I + \Phi\, \Lambda^{\frac{1}{2}}\, \Phi. \tag{4.2.1}$$

Note that $\Lambda_\Phi \in \Psi^{\frac{1}{2}}_{1,0} \subset \Psi^{\frac{1}{2}}_{\rho,\delta}$ for $(\, 0 \leq \delta \leq \rho \leq 1\,)$ and $\delta < 1$. By general results on semi-bounded operators (see [44], Theorem 2.7) it follows that Λ_Φ admits a self-adjoint extension (*Friedrichs extension*) which is denoted by Λ_Φ again. In analogy to the classical Sobolev norms $\|\cdot\|_{H^s}$ for $s \in \mathbb{R}$ we define with $u \in \mathcal{D}\,(\Lambda_\Phi^{2s})$

$$\|\, u\,\|_{\Phi,s} := \left\|\, \Lambda_\Phi^{2s} u\, \right\|_2$$

(for details see [44]). Let $\mathcal{H}_\Phi^s$ be the completion of $\mathcal{D}(\,\Lambda_\Phi^{2s}\,)$ with respect to $\|\cdot\|_{\Phi,s}$. Then by [44], p. 30 we have $\mathcal{H}_\Phi^0 = H$ and $\mathcal{H}_\Phi^s \subset \mathcal{H}_\Phi^t$ if $s > t$. We introduce the locally convex spaces

$$\mathcal{H}_\Phi^\infty := \bigcap_{s\in\mathbb{R}} \mathcal{H}_\Phi^s \qquad \text{and} \qquad \mathcal{H}_\Phi^{-\infty} := \bigcup_{s\in\mathbb{R}} \mathcal{H}_\Phi^s.$$

where $\mathcal{H}_\Phi^{-\infty}$ carries the inductive limit topology, while $\mathcal{H}_\Phi^\infty$ is a Fréchet space with a topology induced by all the norms $\|\cdot\|_{\Phi,k}$ where $k \in \mathbb{N}$. A collection of spaces $\mathcal{H}_\Phi^s$ for $s \in \mathbb{R}$ constructed above will be referred to as an *$\mathcal{HS}$-chain*.

In order to continue with our construction we have to consider commutators of pseudo-differential operators of class $\Psi^0_{\rho,\delta}$. For $P_j \in \Psi^{m_j}_{\rho,\delta}$ where $j = 1,2$ the product P_1P_2 as well as the commutator $[\, P_1, P_2\,]$ are pseudo-differential operators again. More precisely, they are of class $P_1P_2 \in \Psi^{m_1+m_2}_{\rho,\delta}$ and $[\, P_1, P_2\,] \in \Psi^{m_1+m_2-(\rho-\delta)}_{\rho,\delta}$ (see [103], pp. 59-60). Hence in the exotic case $\rho = \delta$ and $m_1 = m_2 = 0$ the order of the commutator $[\, P_1, P_2\,]$ in general does not improve. For $P_1 = \Lambda^s$ or a multiplication by $\Phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ we can prove a stronger result.

Proposition 4.2.1 *Let $0 \leq \delta \leq \rho \leq 1$ where $\delta < 1$ and fix $R \in \Psi^0_{\rho,\delta}$. Then it holds:*

(a) For $\Phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ the commutator $[\,\Phi, R\,]$ is of class $\Psi^{-\rho}_{\rho,\delta}$.

(b) For all $s \in \mathbb{R}$ the commutator $[\,\Lambda^s, R\,]$ is of class $\Psi^{s-(1-\delta)}_{\rho,\delta}$.

For $\delta < \rho$ Proposition 4.2.1 easily can be proved using the asymptotic expansion for the symbol of the commutators. In fact, this is not possible for the exotic classes $\Psi^0_{\rho,\rho}$ where $\rho < 1$ and so we have to calculate the symbols more directly. In order to proceed we introduce the *double symbols* of Kumano-go [103].

Let $m, m' \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$, then we denote by $\mathcal{S}^{m,m'}_{\rho,\delta}$ (*double symbols*) the class of all $\mathcal{C}^\infty$-functions on $\mathbb{R}^{4n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi'}$ such that

$$\left| \partial^\alpha_\xi \partial^{\alpha'}_{\xi'} D^\beta_x D^{\beta'}_{x'} a(x, \xi, x', \xi') \right| \leq C_{\alpha,\alpha',\beta,\beta'} \cdot \lambda(\xi)^{m+\delta|\beta|-\rho|\alpha|} \lambda(\xi, \xi')^{\delta|\beta'|} \lambda(\xi')^{m'-\rho|\alpha'|}$$

where $\lambda(\xi, \xi') := (1+|\xi|^2+|\xi'|^2)^{\frac{1}{2}}$. Using this inequality the space $\mathcal{S}^{m,m'}_{\rho,\delta}$ can be equipped with a Fréchet topology in the standard way. To any symbol $a \in \mathcal{S}^{m,m'}_{\rho,\delta}$ we can associate an operator R on $\mathcal{S}$ in the sense of [103], p. 65. The class of all operators with symbol in $\mathcal{S}^{m,m'}_{\rho,\delta}$ is denoted by $\Psi^{m,m'}_{\rho,\delta}$. Let us assume that $a \in \mathcal{S}^{m,0}_{\rho,\delta}$ does not depend on ξ', then by Corollary 3, p. 66 in [103] for any $f \in \mathcal{S}$ the corresponding operator $R \in \Psi^{m,0}_{\rho,\delta}$ has the form of an double integral:

$$Rf(x) := (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a(x, \xi, y) f(y)\, dy\, d\xi. \tag{4.2.2}$$

In fact, the operators with double symbols are pseudo-differential operators in the usual sense. The following relation between both classes holds (see [103] Theorem 2.5, p.73).

Theorem 4.2.1 *Let $m, m' \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, then $\Psi^{m,m'}_{\rho,\delta} \hookrightarrow \Psi^{m+m'}_{\rho,\delta}$. Moreover, equipped with the Fréchet topologies the embedding is continuous.*

Now, using well-known results on the symbols of composed pseudo-differential we can prove Proposition 4.2.1 above:

Proof of Proposition 4.2.1 In order to prove (a) and (b) we compute the symbols of both commutators. Let $\Phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ and $R \in \Psi^0_{\rho,\delta}$ with symbol $b \in \mathcal{S}^0_{\rho,\delta}$, then $[\Phi, R]$ is of the form (4.2.2) where

$$a\big(x, \xi, y\big) := \Phi(x) \cdot b(x, \xi) - b(x, \xi) \cdot \Phi(y) \in \mathcal{S}^{0,0}_{\rho,\delta}.$$

Using the Taylor formula together with $a(x, \xi, x) = 0$ we rewrite a in the form

$$a(x, \xi, y) = \sum_{k=1}^n \big(y_k - x_k\big) \int_0^1 \partial_{y_k} a\big(x, \xi, x + \theta(y - x)\big)\, d\theta.$$

If we put this in (4.2.2) and replace $(y_k - x_k) \cdot e^{i(x-y)\cdot\xi}$ by $i\partial_{\xi_k} e^{i(x-y)\cdot\xi}$, then integration by parts implies that we can replace a in (4.2.2) by

$$\tilde{a}\big(x, \xi, y\big) = -i \sum_{k=1}^n \int_0^1 \partial_{\xi_k} \partial_{y_k} a\big(x, \xi, x + \theta(y - x)\big)\, d\theta. \tag{4.2.3}$$

Let us compute the integrand in (4.2.3):

$$\partial_{\xi_k}\partial_{y_k} a\big(x, \xi, x+\theta(y-x)\big) = -\partial_{\xi_k} b(x, \xi) \cdot \partial_{y_k}\Phi\big(x+\theta(y-x)\big).$$

The first factor of the term on the right hand side is in $\mathcal{S}^{-\rho}_{\rho,\delta}$ while the second factor is in $\mathcal{C}^\infty_c(\mathbb{R}^n)$ as a function of x and y. Hence if we consider $\tilde{a}$ as a double symbol it follows from Corollary 2 in [103], p. 66 that $\tilde{a} \in \mathcal{S}^{-\rho,0}_{\rho,\delta}$. By Theorem 4.2.1 we conclude that $[\Phi, R] \in \mathcal{S}^{-\rho}_{\rho,\delta}$ and we have proved (a).

A similar argument works in the proof of (b). For $s \in \mathbb{R}$ and using Theorem 2.6, p. 74 in [103] it is easy to see that the symbol of $R \circ \Lambda^s$ is given by $a(x, \xi) \cdot \lambda^s(\xi)$. Furthermore by applying Corollary 2, p. 66 in [103] the double symbol $p(x, \xi, x', \xi')$ of $\Lambda^s \circ R$ has the form:

$$p\big(x, \xi, x', \xi' \big) = \lambda^s(\xi) \cdot a\big(x', \xi' \big) \in \mathcal{S}^{s,0}_{\rho,\delta}.$$

It follows from Theorem 3.1, p. 75 in [103] and $p(x, \xi, x, \xi) = \lambda^s(\xi) \cdot a(x, \xi)$ that $R \circ \Lambda^s$ is a pseudo-differential operator with single symbol given by:

$$p_L\big(x, \xi \big) := \lambda^s(\xi) \cdot a(x, \xi) + \sum_{k=1}^n \int_0^1 r_{k,\theta}(x, \xi)\, d\theta$$

where

$$r_{k,\theta}\big(x, \xi \big) := (2\pi)^{-n} \cdot \mathrm{Os} - \int\!\!\int e^{-iy\cdot\eta}\partial_{\xi_k} D_{x'_k} p\big(x, \xi+\theta\eta, x+y, \xi \big)\, dy\, d\eta$$

is an oscillatory integral. We conclude that the commutator $[\Lambda^s, R]$ has the (single) symbol

$$\sigma\big([\Lambda^s, R] \big) = \sum_{k=1}^n \int_0^1 r_{k,\theta}(x, \xi)\, d\theta. \tag{4.2.4}$$

Let us compute the integrands appearing in $r_{k,\theta}$. We obtain from Corollary 2, p. 66 in [103] together with $\partial_{\xi_k}\lambda^s(\xi) \in \mathcal{S}^{s-1}_{1,0} \subset \mathcal{S}^{s-1}_{\rho,\delta}$ and $\partial_{x'_k} a(x', \xi') \in \mathcal{S}^{\delta}_{\rho,\delta}$ that

$$\partial_{\xi_k}\partial_{x'_k} p\big(x, \xi, x', \xi' \big) = \partial_{\xi_k}\lambda^s(\xi) \cdot \partial_{x'_k} a\big(x', \xi' \big) \in \mathcal{S}^{s-1,\delta}_{\rho,\delta}.$$

Finally, applying Lemma 2.4, p. 69 in [103] it follows that $\{ r_{k,\theta}(x, \xi)\}_{|\,\theta\,|\leq 1}$ is a bounded subset of $\mathcal{S}^{s-(1-\delta)}_{\rho,\delta}$ in the Fréchet topology. Hence by (4.2.4) the function $\sigma([\Lambda^s, R])$ is a symbol in $\mathcal{S}^{s-(1-\delta)}_{\rho,\delta}$. □

Corollary 4.2.1 *Let $R \in \Psi^0_{\frac{1}{2},\frac{1}{2}}$ and assume that X is one of the operators $\Lambda^{\frac{1}{2}}$ or Λ_Φ. Then all commutators $ad^l[X](R)$ where $l \in \mathbb{N}_0$ are of class $\Psi^0_{\frac{1}{2},\frac{1}{2}}$. In particular, they admit bounded extensions to operators in $\mathcal{L}(H)$.*

Proof By induction it is sufficient to show that $[X, R]$ is of class $\Psi^0_{\frac{1}{2},\frac{1}{2}}$. For $X := \Lambda^{\frac{1}{2}}$ this directly follows from Proposition 4.2.1, (b). Now we assume that $X = \Lambda_\Phi$, then

$$[\Lambda_\Phi, R] = \Phi\, \Lambda^{\frac{1}{2}}\, [\Phi, R] + \Phi \left[\Lambda^{\frac{1}{2}}, R \right] \Phi + [\Phi, R]\, \Lambda^{\frac{1}{2}}\, \Phi.$$

Applying Proposition 4.2.1 it follows that $[\Lambda^{\frac{1}{2}}, R] \in \Psi^0_{\frac{1}{2},\frac{1}{2}}$ and $[\Phi, R] \in \Psi^{-\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}$. From the composition rule for pseudo-differential operators and $\Lambda^{\frac{1}{2}} \in \Psi^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}$ the assertion follows. □

Now, following the general theory in [79] (see also [30] and Proposition 1.2.1 with $D := H^\infty$) for any $s > 0$ we can define the algebra $\mathcal{A}^s$ by:

$$\mathcal{A}^s := \Big\{ a \in \mathcal{L}(H) : a\big(H^\infty \big) \subset H^\infty \text{ and for all } j \in \mathbb{N}_0 \\ \big\| \operatorname{ad}^j \big[\Lambda^{\frac{s}{2}} \big] (a)\, f \big\|_2 \leq \alpha_j \cdot \| f \|_2, \quad \forall\, f \in H^\infty \text{ and } \alpha_j \geq 0 \text{ suitable} \Big\}.$$

It follows from $\Lambda^{\frac{s}{2}} \in \Psi^{\frac{s}{2}}_{1,0}$ and Theorem 4.1.1 that H^∞ is invariant under $\Lambda^{\frac{s}{2}}$. Hence all the commutators $\operatorname{ad}^j \left[\Lambda^{\frac{s}{2}} \right] (a)$ and powers $\Lambda^{\frac{js}{2}}$ with $j \in \mathbb{N}$ are well-defined on H^∞. Because $\Lambda^{\frac{s}{2}}$ is a symmetric closed operator it follows by standard arguments that $\mathcal{A}^s$ is a Ψ^*-algebra in the sense of [79].

Using a cut-off function $\Phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\Phi \geq 0$ and the corresponding self-adjoint operator Λ_Φ in (4.2.1) we can define a local version $\mathcal{A}^s_\Phi$ of $\mathcal{A}^s$ for $s > 0$.

$$\mathcal{A}^s_\Phi := \Big\{ a \in \mathcal{L}(H) : a\big(\mathcal{H}^\infty_\Phi \big) \subset \mathcal{H}^\infty_\Phi \text{ and for all } j \in \mathbb{N}_0 \\ \big\| \operatorname{ad}^j [\Lambda^s_\Phi] (a)\, f \big\|_2 \leq \alpha_j \cdot \| f \|_2, \quad \forall\, f \in \mathcal{H}^\infty_\Phi \text{ and } \alpha_j \geq 0 \text{ suitable} \Big\}.$$

We obtain with the definitions $\mathcal{A} := \mathcal{A}^1$ and $\mathcal{A}_\Phi := \mathcal{A}^1_\Phi$:

Lemma 4.2.1 *Let $\Phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ be a positive function. Then the algebra $\Psi^0_{\frac{1}{2},\frac{1}{2}}$ of pseudo-differential operators is contained in both $\mathcal{A}$ and $\mathcal{A}_\Phi$.*

Proof The inclusion $\Psi^0_{\frac{1}{2},\frac{1}{2}} \subset \mathcal{A}$ directly follows from Corollary 4.2.1 and Theorem 4.1.1. Let us show that $\Psi^0_{\frac{1}{2},\frac{1}{2}} \subset \mathcal{A}_\Phi$. Considered as operators on the scale H^s, $s \in \mathbb{R}$ of Sobolev spaces a direct computation with $a \in \Psi^0_{\frac{1}{2},\frac{1}{2}}$ shows:

$$\Lambda^{2m}_\Phi a = \sum_{l=0}^{2m} \binom{2m}{l} \operatorname{ad}^l [\Lambda_\Phi] (a)\, \Lambda^{2m-l}_\Phi. \tag{4.2.5}$$

Let $u \in \mathcal{H}^m_\Phi \subset H$ where $m \in \mathbb{N}$, then $\Lambda^j_\Phi u \in H$ for $j = 0, \cdots, 2m$. Hence by Corollary 4.2.1 together with (4.2.5) it follows that $\Lambda^{2m}_\Phi a u \in H$ and so $a(u) \in \mathcal{H}^m_\Phi$. Because $m \in \mathbb{N}$

was arbitrary we conclude that $a(\mathcal{H}_\Phi^\infty) \subset \mathcal{H}_\Phi^\infty$. The boundedness of all commutators $\mathrm{ad}^l[\,\Lambda_\Phi\,](a)$ follows from Corollary 4.2.1. □

Let us show that $\mathcal{A}_\Phi$ contains all the multiplication operators with symbols smooth in a neighborhood of *supp* (Φ).

Lemma 4.2.2 *Let $f \in L^\infty(\mathbb{R}^n)$ be smooth in a open neighborhood U of supp (Φ). Then the multiplication operator M_f is contained in $\mathcal{A}_\Phi$.*

Proof Let us compute a single commutator $[\,\Lambda_\Phi, f\,]$. Because the product $f \cdot \Phi$ belongs to $\mathcal{C}_c^\infty(\mathbb{R}^n)$ the following equality is well-defined:

$$[\,\Lambda_\Phi, f\,] = \left[\,\Lambda^{\frac{1}{2}}, |\,\Phi\,|^2 f\,\right] + \Phi f \left[\,\Phi, \Lambda^{\frac{1}{2}}\,\right] + \left[\,\Phi, \Lambda^{\frac{1}{2}}\,\right] \Phi f.$$

From Proposition 4.2.1 it follows that $[\,\Lambda^{\frac{1}{2}}, h\,] \in \Psi^{-\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} \subset \Psi^0_{\frac{1}{2},\frac{1}{2}}$ for any $h \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and so we conclude that $[\,\Lambda_\Phi, f\,] \in \Psi^0_{\frac{1}{2},\frac{1}{2}}$. By induction and using Corollary 4.2.1 all iterated commutator $\mathrm{ad}^l[\,\Lambda_\Phi\,](f)$ are contained in $\Psi^0_{\frac{1}{2},\frac{1}{2}}$. In particular, they admit a bounded extension to H. Now, the assertion follows with an argument similar to the proof of Lemma 4.2.1. □

Note that the algebras $\mathcal{A}^s$ and $\mathcal{A}^s_\Phi$ operate on the corresponding scales of Sobolev spaces without an order shift. For $\mathcal{A}^s_\Phi$ this follows from

$$\Lambda_\Phi^{ms} a = \sum_{l=0}^{m} \binom{m}{l} \mathrm{ad}^l\,[\,\Lambda_\Phi^s\,]\,(a)\,\Lambda_\Phi^{(\,m-l\,)s}. \tag{4.2.6}$$

considered as an operator equation on $\mathcal{H}_\Phi^\infty$. For $b \in \mathcal{A}^s_\Phi$ and $a \in \{\,b, b^*\,\} \subset \mathcal{A}^s_\Phi$ we conclude from (4.2.6) for all $m \in \mathbb{N}_0$ and with a suitable positive constant c and $f \in \mathcal{H}_\Phi^\infty$ that:

$$\|\,af\,\|_{\Phi,ms} = \left\|\,\Lambda_\Phi^{2ms} af\,\right\|_2 \leq \sum_{l=0}^{2m} \binom{2m}{l} \cdot \alpha_l \cdot \left\|\,\Lambda_\Phi^{(2m-l)s} f\,\right\|_2 \leq c \cdot \|\,f\,\|_{\Phi,ms}.$$

Let $\rho \in \mathbb{R}$ and Y^s, $(s \in \mathbb{R} \cup \{\infty, -\infty\})$ be one of the $\mathcal{HS}$-chains H^s or $\mathcal{H}^s_\Phi$. In the following we denote by

$$\mathcal{O}(\,\rho\,) := \bigcap_{t \in \mathbb{R}} \mathcal{L}\big(\,Y^t, Y^{t-\rho}\,\big)$$

the *order class* of all linear operators $a \in L(Y^\infty)$ such that, for every $t \in \mathbb{R}$ there is a bounded extension a_t of a from Y^t to $Y^{t-\rho}$. It is a well-known fact (see [44], Theorem 6.3) that for any $\rho \in \mathbb{R}$ the space $\mathcal{O}(\,\rho\,)$ equipped with the norms

$$\|\,a\,\|_{Y^t, Y^{t-\rho}} := \sup\Big\{\,\|\,a_t u\,\|_{Y^{t-\rho}} : u \in Y^t, \quad \|\,u\,\|_{Y^t} \leq 1\Big\}, \qquad t \in \mathbb{R}$$

is a Fréchet space. It follows that $\mathcal{A}^s \subset \bigcap_{m \in \mathbb{N}_0} \mathcal{L}(\,H^{\frac{ms}{2}}\,)$ and $\mathcal{A}^s_\Phi \subset \bigcap_{m \in \mathbb{N}} \mathcal{L}(\,\mathcal{H}_\Phi^{ms}\,)$. By standard arguments in *interpolation theory* (see [44], Theorem 6.3) we obtain for $s > 0$:

$$\mathcal{A}^s \subset \bigcap_{t \in \mathbb{R}} \mathcal{L}\left(\,H^t\,\right) = \mathcal{O}(\,0\,) \qquad \text{and} \qquad \mathcal{A}^s_\Phi \subset \bigcap_{t \in \mathbb{R}} \mathcal{L}\left(\,\mathcal{H}^t_\Phi\,\right) = \mathcal{O}(\,0\,). \tag{4.2.7}$$

Proposition 4.2.2 *Let $\Phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be a positive function, then for all $k \in \mathbb{N}_0$ the embedding $H^{\frac{k}{2}} \hookrightarrow \mathcal{H}_\Phi^{\frac{k}{2}}$ is well-defined and continuous.*

Proof For $k = 0$ the assertion is obvious. Let us assume that $\| v \|_{\Phi, \frac{k}{2}} \leq c \| v \|_{H^{\frac{k}{2}}}$ for $k \in \mathbb{N}_0$, with a suitable number $c > 0$ and $v \in H^{\frac{k}{2}}$. Then from Lemma 4.2.2 and (4.2.7) the multiplication M_Φ is an operator in $\mathcal{L}(\mathcal{H}_\Phi^s)$ for all $s \geq 0$ and so there is $\tilde{c} > 0$ such that with any $u \in \mathcal{H}_\Phi^{\frac{k+1}{2}}$

$$\| u \|_{\Phi, \frac{k+1}{2}} = \left\| \left(1 + \Phi \Lambda^{\frac{1}{2}} \Phi \right) u \right\|_{\Phi, \frac{k}{2}} \leq \tilde{c} \left(\| u \|_{\Phi, \frac{k}{2}} + \left\| [\Lambda^{\frac{1}{2}}, \Phi] u \right\|_{\Phi, \frac{k}{2}} + \left\| \Lambda^{\frac{1}{2}} u \right\|_{\Phi, \frac{k}{2}} \right).$$

From Proposition 4.2.1 it follows that $X := [\Lambda^{\frac{1}{2}}, \Phi] \in \Psi^0_{\frac{1}{2}, \frac{1}{2}}$ and by Theorem 4.1.1 X is continuous on $H^{\frac{k}{2}}$. By induction there are $c_1, c_2 > 0$ such that

$$\| u \|_{\Phi, \frac{k+1}{2}} = c_1 \left(\| u \|_{H^{\frac{k}{2}}} + \left\| \Lambda^{\frac{1}{2}} u \right\|_{H^{\frac{k}{2}}} \right) \leq c_2 \| u \|_{H^{\frac{k+1}{2}}}.$$

From this inequality we conclude that $H^{\frac{k+1}{2}} \hookrightarrow \mathcal{H}_\Phi^{k+1}$ is embedded continuously. $\square$

Remark 4.2.1 By identifying the spaces H^{-s} for $s \geq 0$ with the topological dual space $(H^s)'$ (resp. by identifying $\mathcal{H}_\Phi^{-s}$ and $(\mathcal{H}_\Phi^s)'$) with respect to the inner-product on H in the sense of [44] there is a continuous embedding of Sobolev spaces $\mathcal{H}_\Phi^{-\frac{k}{2}} \hookrightarrow H^{-\frac{k}{2}}$ for $k \in \mathbb{N}$.

Under some additional conditions we can show that - localized in the support of Φ - the Sobolov spaces H^s and $\mathcal{H}_\Phi^s$ coincide. To be more precise let us make the following assumptions on the cut-off functions $\Phi, \Theta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\Phi, \Theta \geq 0$:

(a) Let $U \subset \mathbb{R}^n$ be open with $\Phi \equiv 1$ on U.

(b) With $V \subset\subset W \subset\subset U$ we assume that $\Theta \equiv 1$ on V and *supp* $\Theta \subset W$.

Here we write $V \subset\subset W$ iff V is an open and bounded subset of W such that the closure $\bar{V}$ of V is contained in W.

Theorem 4.2.2 *Under the assumptions (a) and (b) above the multiplication M_Θ is a continuous operator from $\mathcal{H}_\Phi^{\frac{k}{2}}$ to $H^{\frac{k}{2}}$ (resp. from $H^{-\frac{k}{2}}$ to $\mathcal{H}_\Phi^{-\frac{k}{2}}$) for all $k \in \mathbb{N}$.*

Proof By duality it is sufficient to prove the first assertion. Let $u \in \mathcal{H}_\Phi^{\frac{k}{2}} \subset H$ where $k \in \mathbb{N}_0$, then we prove the following decomposition:

$$\left(\Phi \Lambda^{\frac{1}{2}} \Phi \right)^k M_\Theta u = \Lambda^{\frac{k}{2}} M_\Theta u + A_k u \tag{4.2.8}$$

where A_k is an operator in $\Psi^{-\infty}$. Let $k = 1$, then we obtain with $\Phi \cdot \Theta = \Theta$:

$$\Phi \Lambda^{\frac{1}{2}} \Phi \left(\Theta u \right) = \Phi \Lambda^{\frac{1}{2}} \left(\Theta u \right) = \Lambda^{\frac{1}{2}} \left(\Theta u \right) + \left(\Phi - 1 \right) \Lambda^{\frac{1}{2}} \left(\Theta u \right).$$

By our choice of Φ and Θ it follows that dist $\Big(\operatorname{supp}(\Phi-1), \operatorname{supp}\Theta\Big) > 0$ and so by Theorem 2.7, p. 75 in [103] we conclude that

$$A_1 := \big(\,\Phi - 1\,\big)\Lambda^{\frac{1}{2}}\Theta \in \Psi^{-\infty}.$$

Let us assume that (4.2.8) holds for $k \in \mathbb{N}$. Then by induction and using a similar argument:

$$\begin{aligned}\Big(\,\Phi\Lambda^{\frac{1}{2}}\Phi\,\Big)^{k+1}\big(\,\Theta u\,\big) &= \Big(\,\Phi\Lambda^{\frac{1}{2}}\Phi\,\Big)\Lambda^{\frac{k}{2}}\big(\,\Theta u\,\big) + \Big(\,\Phi\Lambda^{\frac{1}{2}}\Phi\,\Big)A_k(\,u\,)\\ &= \Phi\Lambda^{\frac{k+1}{2}}\big(\,\Theta u\,\big) + \Phi\Lambda^{\frac{1}{2}}\big(\,\Phi - 1\,\big)\Lambda^{\frac{k}{2}}\big(\,\Theta u\,\big) + \Big(\Phi\Lambda^{\frac{1}{2}}\Phi\Big)A_k(\,u\,)\\ &= \Lambda^{\frac{k+1}{2}}\big(\,\Theta u\,\big) + A_{k+1}(\,u\,)\end{aligned}$$

where $A_{k+1} \in \Psi^{-\infty}$ and this proves (4.2.8). It is easy to see that $(\,\Phi\Lambda^{\frac{1}{2}}\Phi\,)^k$ maps $\mathcal{H}_\Phi^{\frac{k}{2}}$ continuously to $\mathcal{H}_\Phi^0 = H$. Because of $M_\Theta \in \mathcal{O}(0)$ it follows that $\Theta u \in \mathcal{H}_\Phi^{\frac{k}{2}}$ and from

$$\Lambda^{\frac{k}{2}}\big(\,\Theta u\,\big) = \Big(\,\Phi\Lambda^{\frac{1}{2}}\Phi\,\Big)^k\big(\,\Theta u\,\big) - A_k(\,u\,) \in \mathcal{H}_\Phi^0 = H$$

we conclude that $\Theta u \in H^{\frac{k}{2}}$. Due to the fact that A_k is a continuous operator on H for all $k \in \mathbb{N}$ together with Lemma 4.2.2 and (4.2.7) we obtain:

$$\begin{aligned}\|\,M_\Theta u\,\|_{H^{\frac{k}{2}}} = \Big\|\,\Lambda^{\frac{k}{2}}\Theta u\,\Big\|_2 &\le c_1\,\|\,M_\Theta u\,\|_{\Phi,\frac{k}{2}} + \|\,A_k u\,\|_2\\ &\le c_2\Big\{\,\|\,M_\Theta u\,\|_{\Phi,\frac{k}{2}} + \|\,u\,\|_{\Phi,0}\Big\} \le c_3\|\,u\,\|_{\Phi,\frac{k}{2}}\end{aligned}$$

and this finally proves the continuity of M_Θ from $\mathcal{H}_\Phi^{\frac{k}{2}}$ to $H^{\frac{k}{2}}$ for $k \in \mathbb{N}_0$. $\square$

Let a be an operator of type $(\frac{1}{2}, \frac{1}{2})$ with compact support, then according to Theorem 4.1.3 for any finite system $\mathcal{V}_m := [\,Z_1, \cdots, Z_m\,]$ of smooth vector fields the commutator ad$[\,\mathcal{V}_m\,](a)$ on H^∞ can be extended to a continuous operator from H^s to $H^{s-\frac{m}{2}}$ where $s \in \mathbb{R}$. We show that a similar result holds true if we replace the Sobolev spaces H^s by the local version $\mathcal{H}_\Phi^k$ for $k \in \mathbb{N}$. Here the function Φ is as in (a) and we assume that all the coefficients of Z_j are supported in $V \subset\subset U$.

Proposition 4.2.3 *Let $\tilde{V} \subset\subset U$ and fix a finite system $\mathcal{V}_m := [\,Z_1, \cdots, Z_m\,]$ of smooth vector fields supported in $\tilde{V}$. Assume that a and a^* are are operators with compact support and of type $(\frac{1}{2}, \frac{1}{2})$. Then for all $k \in \mathbb{Z}$ the iterated commutator ad$[\,\mathcal{V}_m\,](a)$ on $\mathcal{H}_\Phi^\infty$ has a continuous extensions from $\mathcal{H}_\Phi^k$ to $\mathcal{H}_\Phi^{k-\frac{m}{2}}$.*

Proof First we assume that $k \in \mathbb{N}$ with $k \ge \frac{m}{2}$. Let $u \in \mathcal{H}_\Phi^\infty$ and choose open sets V and W with $\tilde{V} \subset\subset V \subset\subset W \subset\subset U$. Fix a cut-off function $\Theta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ as in (b) above i.e. $\Theta \equiv 1$ on V and $\mathrm{supp}(\Theta) \subset W$. Then for $m \in \mathbb{N}$ we have by our assumptions on the support of Z_m:

$$\mathrm{ad}\,[\,\mathcal{V}_m\,]\,(a)u = \mathrm{ad}\,[\,\mathcal{V}_m\,]\,(a)\; M_\Theta u + Z_m\,\mathrm{ad}\,[\,\mathcal{V}_{m-1}\,]\,(a)M_{(1-\Theta)}u.$$

The second term of the right hand side is an operator in $\Psi^{-\infty}$ and so it maps $\mathcal{H}^k_\Phi$ to $\mathcal{H}^{k-m}_\Phi$ continuously. From Theorem 4.2.2 we conclude that M_Θ is bounded from $\mathcal{H}^k_\Phi$ to H^k and by the classical Theorem 4.1.3 the commutators $\mathrm{ad}\,[\,\mathcal{V}_m\,]\,(a)$ cause an order shift of Sobolev spaces from H^k to $H^{k-\frac{m}{2}}$. Finally, the assertion follows from the continuous embedding $H^{k-\frac{m}{2}} \hookrightarrow \mathcal{H}^{k-\frac{m}{2}}_\Phi$.

Let us define the system $\mathcal{V}^*_m := [\, Z^*_1, \cdots, Z^*_m\,]$. By similar arguments and applying remark 4.1.1 it follows that the iterated commutator $\mathrm{ad}[\,\mathcal{V}^*_m\,](a^*)$ maps $\mathcal{H}^k_\Phi$ continuously into $\mathcal{H}^{k-\frac{m}{2}}_\Phi$ for all integers $k \geq \frac{m}{2}$. By duality and Proposition 6.4 in [44] we obtain that

$$\mathrm{ad}[\,\mathcal{V}_m\,](a) = (-1)^m \Big\{ \,\mathrm{ad}\,[\,\mathcal{V}^*_m\,]\,(a^*)\; \Big\}^*$$

is continuous from $\mathcal{H}^{-k}_\Phi$ to $\mathcal{H}^{-k-\frac{m}{2}}_\Phi$ for $k \geq 0$. The full assertion now follows by interpolation theory (see [44], Theorem 6.3). $\square$

Let $\mathcal{V} := \{\, Z_1, \cdots, Z_k \,\}$ be a finite set of compactly supported smooth vector-fields. Then we have $Z_j \in \Psi^1_{1,0}$ and according to Theorem 4.1.1 for all $s \in \mathbb{R}$ there are continuous extensions of Z_j from H^{s+1} to H^s. For all positive parameters $\rho > 0$ we define a sub-algebra $\Psi_\rho\,[\,\mathcal{A}\,]$ of $\mathcal{A}$ by:

$$\Psi_\rho\,[\,\mathcal{A}\,] := \Big\{\, a \in \mathcal{A} : \mathrm{ad}\,[\,Z_{i_1}, \cdots Z_{i_m}\,]\,(a) \in \bigcap_{s\in\mathbb{Z}} \mathcal{L}\Big(\, H^s, H^{s-m\rho}\,\Big) \;\text{ for } Z_{i_l} \in \mathcal{V}\,\Big\}.$$

In the same manner we construct a local version of $\Psi_\rho[\,\mathcal{A}\,]$ in $\mathcal{A}_\Phi$ by using commutator methods. Let us assume that Φ is a cut-off function defined as in (a) above Theorem 4.2.2 and $\mathcal{W} := \{\, Y_1, \cdots, Y_k \,\}$ denotes a finite set of vector-fields supported in U.

$$\Psi_\rho\,[\,\mathcal{A}_\Phi\,] := \Big\{\, a \in \mathcal{A}_\Phi : \mathrm{ad}\,[\,Y_{i_1}, \cdots Y_{i_m}\,]\,(a) \in \bigcap_{s\in\mathbb{Z}} \mathcal{L}\Big(\, \mathcal{H}^s_\Phi, \mathcal{H}^{s-m\rho}_\Phi\,\Big) \;\text{ for } Y_{i_l} \in \mathcal{W}\,\Big\}.$$

Theorem 4.2.3 *The class $\Psi^0_{\frac{1}{2},\frac{1}{2}}$ of pseudo-differential operators with compact support is contained in both algebras $\Psi_{\frac{1}{2}}\,[\,\mathcal{A}\,]$ and $\Psi_{\frac{1}{2}}\,[\,\mathcal{A}_\Phi\,]$. Moreover, let $f \in L^\infty(\mathbb{H}^n)$ be smooth in a neighborhood N of supp(Φ), then the multiplication M_h is an operator in $\Psi_{\frac{1}{2}}[\,\mathcal{A}_\Phi\,]$.*

Proof The first assertion directly can be derived from Lemma 4.2.1, Theorem 4.1.3 and Proposition 4.2.3. It was shown in Lemma 4.2.2 that $M_h \in \mathcal{A}_\Phi$. The commutator conditions follow from the fact that each vector-field $Y \in \mathcal{W}$ is supported in $U \subset\subset N$ which implies from $\Phi \equiv 1$ on U that $YM_h = YM_{h\,\Phi}$ with $h\,\Phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. $\square$

A sub-algebra $\mathcal{B}$ in $\mathcal{L}(H)$ is called *spectral invariant* if it holds

$$\mathcal{B} \cap \mathcal{L}(H)^{-1} = \mathcal{B}^{-1}$$

where we denote by $\mathcal{B}^{-1}$ the group of all invertible elements in $\mathcal{B}$. It can be shown that both $\Psi_\rho[\mathcal{A}]$ and $\Psi_\rho[\mathcal{A}_\Phi]$ are algebras but additionally the spectral invariance follows from the construction:

Proposition 4.2.4 *The algebras $\Psi_\rho[\mathcal{A}]$ and $\Psi_\rho[\mathcal{A}_\Phi]$ are spectral invariant in $\mathcal{L}(H)$.*

Proof Because the proof only uses general facts on commutators we only treat the case $\Psi_\rho[\mathcal{A}]$. Let $a \in \mathcal{A}$ such that a^{-1} exists in $\mathcal{L}(H)$. From the fact that $\mathcal{A}$ is a Ψ^*-algebra in $\mathcal{L}(H)$ it follows that the inverse a^{-1} is contained in $\mathcal{A}$. Let us verify the commutator conditions on a^{-1}. For any finite system $[Z_{i_1}, \cdots, Z_{i_m}]$ in $\mathcal{V}$ we define a function Γ by:

$$\Gamma\left(a^{-1}\right) = \Gamma(a) := 0 \qquad\qquad \Gamma\Big(\operatorname{ad}[Z_{i_1}, \cdots, Z_{i_m}](a)\Big) := m.$$

Let us extend Γ to finite products $b_1 \cdot b_2 \cdots b_k$ of operators a, a^{-1} and $\operatorname{ad}[Z_{i_1}, \cdots, Z_{i_j}](a)$ by the rule:

$$\Gamma(b_1 \cdot b_2 \cdots b_k) := \Gamma(b_1) + \Gamma(b_2) + \cdots + \Gamma(b_k).$$

Assume that $b := b_1 \cdot b_2, \cdots b_k$ with $\Gamma(b) = m$. With $Z_j \in \mathcal{V}$ we prove for the single commutator $[Z_j, b]$ that:

$$[Z_j, b] \in \operatorname{span}\Big\{ c := c_1 \cdot c_2 \cdots c_l : \Gamma(c) = m+1 \text{ and } c_i \in \left\{ a^{-1}, \operatorname{ad}[Z_{i_1}, \cdots, Z_{i_r}](a) \right\} \Big\}$$

and we denote the linear space appearing on the right hand side by V_{m+1}. In fact, this assertion directly follows from the functional equation on Γ and the well-known product rule:

$$[Z_j, b] = \sum_{l=1}^{k} b_1 \cdots b_{l-1} \cdot [Z_j, b_l] \cdot b_{l+1} \cdots b_k$$

together with $[Z_j, a^{-1}] = -a^{-1}[Z_j, a]\,a^{-1}$ which implies that $\Gamma([Z_j, a^{-1}]) = 1$. In particular, by induction we conclude that

$$\operatorname{ad}[Z_{i_1}, \cdots Z_{i_m}]\left(a^{-1}\right) \in V_m. \tag{4.2.9}$$

Considered as operators on the scale of Sobolev spaces defined above the elements in V_m cause an order shift by $-m\rho$. Hence from (4.2.9) it follows that $a^{-1} \in \Psi_\rho[\mathcal{A}]$ and this proves spectral invariance. □

Under some more restrictions on the vector-fields in $\mathcal{V}$ resp. $\mathcal{W}$ we can prove that the algebras $\Psi_\rho[\mathcal{A}]$ and $\Psi_\rho[\mathcal{A}_\Phi]$ are symmetric (Ψ^*-algebras).

Lemma 4.2.3 *Assume that $Z_j \in \mathcal{V}$ (resp. $Y_j \in \mathcal{W}$) have real valued coefficients and with $\rho > 0$ let $\mathcal{B}$ be one of the algebras $\Psi_\rho[\mathcal{A}]$ or $\Psi_\rho[\mathcal{A}_\Phi]$. Then $\mathcal{B}$ is symmetric, i.e if $a \in \mathcal{B}$, then $a^* \in \mathcal{B}$. In particular, $\mathcal{B}$ is a Ψ^*-algebra in $\mathcal{L}(H)$.*

Proof We only give the proof in the case where $\mathcal{B} = \Psi_\rho[\, \mathcal{A}_\Phi \,]$. Let $a \in \Psi_\rho[\, \mathcal{A}_\Phi \,]$, then in particular it follows that $a, a^* \in \mathcal{A}_\Phi$ are in the class $\mathcal{O}(0)$ with respect to the scale $\mathcal{H}^j_\Phi$. Hence the space $\mathcal{H}^\infty_\Phi$ is invariant under a^* and by Proposition 6.4 in [44] together with our notations above it holds $(a_t)^* = (a^*)_{-t}$ for all $t \in \mathbb{R}$. Considered as an operator on $\mathcal{H}^\infty_\Phi$ we prove that there is $b \in \mathcal{O}(\,(m-1)\rho\,)$ such that

$$\mathrm{ad}\,[\, \mathcal{V}_m \,]\,(a^*) = \Big\{\, \mathrm{ad}\,[\, \mathcal{V}_m \,]\,(a) \,\Big\}^* + b \in \mathcal{O}(\, m\rho \,). \tag{4.2.10}$$

Let $m = 1$, then with $Z_{i_1} = \sum_{j=1}^n c_j^{i_1}\, \partial_{x_j}$ and $f, g \in \mathcal{H}^\infty_\Phi$ it follows from Theorem 4.2.2 that it holds $c_j^{i_1} \cdot g \in H^\infty \subset \mathcal{C}^\infty(\mathbb{R}^n)$. Because by assumption all the coefficients $c_j^{i_1}$ are compactly supported and we conclude that $c_j^{i_1} \cdot g \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. By a straightforward computation:

$$\Big\langle\, [\, Z_{i_1}, a \,]\, f, g \,\Big\rangle = \Big\langle f, \Big\{\, [\, Z_{i_1}, a^* \,] + [\, M_{h_1}, a^* \,] \,\Big\}\, g \Big\rangle$$

where $h_1 := \sum_{j=1}^n \partial_{x_j} c_j^{i_1} \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. According to Lemma 4.2.2 it follows that the commutator $[\, M_{h_1}, a^* \,]$ is contained in $\mathcal{A}_\Phi$ and so $[\, Z_{i_1}, a \,]^* = [\, Z_{i_1}, a^* \,] + b$ where $b \in \mathcal{A}_\Phi = \mathcal{O}(0)$. Now by induction and with $h_m := \sum_{j=1}^n \partial_j\, c_j^{i_m}$ we obtain with a similar calculation:

$$\Big\{\, \mathrm{ad}\,[\, \mathcal{V}_m \,]\,(a) \,\Big\}^* = \mathrm{ad}\,[\, \mathcal{V}_m \,]\,(a^*) + \Big[\, M_{h_m}, \mathrm{ad}\,[\, \mathcal{V}_{m-1} \,]\,(a^*) \,\Big] + [\, Z_{i_m} + M_{h_m}, b \,].$$

where $b \in \mathcal{O}((m-2)\rho)$ by our inductional assumption. Hence, the second and the third term on the right hand side belong to the class $\mathcal{O}(\,(m-1)\rho\,)$ and so (4.2.10) is proved. By Proposition 6.4 in [44] for any $A \in \mathcal{O}(s)$ it follows that $A^* \in \mathcal{O}(s)$ and so we conclude from (4.2.10) and for all $m \in \mathbb{N}$ that $\mathrm{ad}[\, \mathcal{V}_m \,](a^*) \in \mathcal{O}(m\rho)$. Thus by definition we obtain that $a^* \in \Psi_\rho[\, \mathcal{A}_\Phi \,]$. □

Our next aim is it to define a topology on the algebras $\Psi_\rho[\, \mathcal{A} \,]$ and $\Psi_\rho[\, \mathcal{A}_\Phi \,]$. Recall that both spaces $\mathcal{A}$ and $\mathcal{A}_\Phi$ already carry the projective topology of a Fréchet algebra with sub-multiplicative semi-norms $(p_k)_{k \in \mathbb{N}}$ (see [79], [30]). For $t \in \mathbb{R}$ and $k \in \mathbb{N}$ we define

$$\|\, a \,\|_{t,k} := \sup \Big\{\, \|\, \mathrm{ad}\,[\, Z_{i_1}, \cdots, Z_{i_k} \,]\,(a)\, \|_{H^t, H^{t-k\rho}} : Z_{i_j} \in \mathcal{V} \,\Big\}$$

where $a \in \Psi_\rho[\, \mathcal{A} \,]$. We equip $\Psi_\rho[\, \mathcal{A} \,]$ with the projective topology given by the system of semi-norms

$$\Big\{\, p_i, \|\cdot\|_{t,k} : t \in \mathbb{R},\ i, k \in \mathbb{N} \,\Big\}. \tag{4.2.11}$$

Applying interpolation theory (see [44], Theorem 6.3) we can replace (4.2.11) by a countable system of semi-norms generating the same topology. According to [79] (Proposition 3.5 and 3.6 remark 1.) we can assume that these semi-norms are sub-multiplicative. The same construction leads to a Fréchet topology for $\Psi_\rho[\, \mathcal{A}_\Phi \,]$.

Theorem 4.2.4 *Let $\rho > 0$ and $\mathcal{B}$ one of the algebras $\Psi_\rho[\mathcal{A}]$ or $\Psi_\rho[\mathcal{A}_\Phi]$. Then $\mathcal{B}$ is spectral invariant in $\mathcal{L}(H)$ and sub-multiplicative. Moreover,*

$$b : \mathcal{B} \times W \to W : (\, a, x \,) \mapsto a(\, x \,) \tag{4.2.12}$$

is continuous with respect to the product topology where $W = H^\infty$ (resp. $W = \mathcal{H}_\Phi^\infty$). If all vector-fields in $\mathcal{V}$ (resp. $\mathcal{W}$) have real valued coefficients, then $\mathcal{B}$ is a Ψ^*-algebra.

Proof We only have to prove the continuity of b in (4.2.12). This follows from the general theory in [79] and the fact that by definition the algebra $\mathcal{B}$ is continuously embedded in $\mathcal{A}$ (resp. $\mathcal{A}_\Phi$.) □

4.3 The Szegö-projection in the theory of pseudo-differential operators

Let us give the definition of the Szegö-projection on the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ with invariant Haar measure β. For further details and the notations below the reader is referred to [117], [134]. We recall that the boundary of the upper half-space $\mathcal{H}_+$ in $\mathbb{C}^{n+1}$ can be identified canonically with $\mathbb{H}^n$ (see the appendix). Hence the Szegö-projection P_s from $L^2(\mathcal{H}_+, v)$ onto the space of square integrable boundary values of holomorphic functions on $\mathcal{H}_+$ can be considered as an operator on $L^2(\mathbb{H}^n) := L^2(\mathbb{H}^n, \beta)$ where β denotes the left invariant Haar measure on $\mathbb{H}^n$. It can be shown that β coincides with the usual Lebesgue measure on $\mathbb{H}^n$ under the identification $\mathbb{H}^n \cong \mathbb{R}^{2n+1}$. If we denote by $y \cdot x$ for $x = [\eta, s]$ and $y := [\zeta, t]$ in $\mathbb{H}^n$ the Heisenberg product given by

$$[\zeta, t] \cdot [\eta, s] := [\zeta + \eta, t + s + 2\,\mathrm{Im}(\zeta \cdot \bar{\eta})],$$

then according to [134], p. 540 the operator P_s can be written in the sense of an $L^2(\mathbb{H}^n)$-limit as:

$$P_s f(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{H}^n} K_\varepsilon(y^{-1} \cdot x)\, f(y)\, dy \tag{4.3.1}$$

where $f \in L^2(\mathbb{H}^n)$. For $\varepsilon > 0$ and with $c := 2^{n-1} i^{n+1}\, n!\, \pi^{-(n+1)}$ and the Euclidean inner product $\|\cdot\|$ on $\mathbb{C}^n$ the kernel K_ε is given by

$$K_\varepsilon(y) := \frac{c}{(t + i \|\zeta\|^2 + i\varepsilon)^{n+1}}, \qquad y := [\zeta, t] \in \mathbb{H}^n.$$

Note that the integral in (4.3.1) is defined by identifying $(\mathbb{H}^n, \beta)$ with $\mathbb{R}^{2n+1}$ equipped with the usual Lebesgue measure. It follows with $\Theta(x, y) := x \cdot y^{-1}$ and

$$|\det D\Phi(x, \cdot)| = 1 \qquad \text{for all} \qquad x \in \mathbb{H}^n$$

that:

$$P_s f(x) = K\Big[f \circ \Theta(x, \cdot) \Big] := \lim_{\varepsilon \to 0} \int_{\mathbb{H}^n} K_\varepsilon(y) \cdot f \circ \Theta(x, y)\, dy. \tag{4.3.2}$$

Hence, formally for $f \in \mathcal{S}(\mathbb{H}^n)$ and with respect to the Heisenberg product the Szegö-projection P_s is given by the convolution formula $P_s f(x) = f * K(x)$ where K is the distribution $\lim_{\varepsilon \to 0} K_\varepsilon$. We can write K_ε in the form:

$$K_\varepsilon(y) = -\frac{\partial}{\partial t} \Big(\frac{c}{n} \cdot \big[t + i \|\zeta\|^2 + i\,\varepsilon \big]^{-n} \Big). \tag{4.3.3}$$

Let us consider the norm function $\gamma : \mathbb{H}^n \to \mathbb{R}^+$ defined by

$$\gamma(x) := \max \left\{ \| \zeta \|, | t |^{\frac{1}{2}} \right\}.$$

It can be shown (see [134], p. 542), that there is $\alpha > 0$ such that for any non-negative measurable function $F : (0, \infty) \to \mathbb{R}^+$

$$\int_{\mathbb{H}^n} F \circ \gamma(x) \, dx = \alpha \int_0^\infty F(r) \, r^{2n+1} \, dr. \tag{4.3.4}$$

Because of

$$\gamma(y) \leq \left(t^2 + \| \zeta \|^4 \right)^{\frac{1}{4}} \leq 2^{\frac{1}{4}} \, \gamma(y)$$

the map $g_\rho[t, \zeta] := [\, t + i \cdot \| \zeta \|^2 \,]^{-\rho}$ is integrable at infinity iff $\rho > n+1$. Furthermore, g_ρ is locally integrable iff $\rho < n+1$. From the fact that g_n is locally integrable, we can consider the distribution K given by

$$K(x) = -\frac{\partial}{\partial t} \left(\frac{c}{n} \cdot \left[\, t + i \, \| \zeta \|^2 \, \right]^{-n} \right) = -\frac{c}{n} \cdot \partial_t \, g_n. \tag{4.3.5}$$

Note that K coincides with $c \cdot g_{n+1}$ away from the origin. For any $f \in L^2(\mathbb{H}^n)$ supported in a compact set and with $x \notin supp\,(f)$ we can write:

$$P_s f(x) = \int_{\mathbb{H}^n} K\left(y^{-1} \cdot x \right) \cdot f(y) \, dy. \tag{4.3.6}$$

Let us consider $G : (\mathbb{H}^n \times \mathbb{H}^n) \setminus \Delta \to \mathbb{C}$ where $\Delta := \{ \, (x, x) : x \in \mathbb{H}^n \, \}$ denotes the diagonal defined for $y := [\, \zeta, t \,]$ and $x := [\, \eta, s \,]$ by:

$$G(x, y) := K\left(y^{-1} \cdot x \right) = \frac{c}{\left(t - s - 2 \, \mathrm{Im}(\zeta \cdot \bar{\eta}) + i \| \eta - \zeta \|^2 \right)^{n+1}}.$$

The following important result which is due to A. Nagel and E.M Stein can be found in [117] and [134] and the appendix.

Theorem 4.3.1 *Let $\varphi, \psi \in \mathcal{C}_c^\infty(\mathbb{H}^n)$, then the localized form $\varphi P_s \psi$ of the Szegö-projection is an pseudo-differential operator in $\Psi^0_{\frac{1}{2},\frac{1}{2}}$.*

In particular, in the case where supp $(\varphi) \cap \mathrm{supp}(\psi) = \emptyset$ it follows that $\varphi P_s \psi \in \Psi^{-\infty}$. In the following lemma we prove that a global version of this result holds true for the Toeplitz projection P_s.

Lemma 4.3.1 *Let $U \subset \mathbb{H}^n$ be open and $[\, f \,] \in L^2(\mathbb{H}^n)$ with supp $(f) \cap U = \emptyset$, then the Toeplitz projection $P_s f$ is a smooth function on U.*

Proof For any compact set $K \subset U$ and $x \in K \subset \mathbb{H}^n \setminus \mathrm{supp}(f)$ it follows by equation (4.3.6) that $P_s f$ has the form:

$$P_s f(\, x\,) = \int_{\mathbb{H}^n} G(\, x, y\,) \cdot f(\, y\,)\, dy. \tag{4.3.7}$$

We show that there are numbers $\tilde{c}, r > 0$ such that for all $(x, y) \in K \times \{\, \|\, y\,\| \geq r\,\}$ it holds:

$$\big|\, G(\, x, y\,)\,\big| \leq \tilde{c} \cdot \gamma(\, y\,)^{-2n-2}. \tag{4.3.8}$$

Note that with the dilation δ on the Heisenberg group $\mathbb{H}^n$ defined by

$$\delta\big[\, \zeta, t\,\big] := \big[\, \delta\, \zeta, \delta^2\, t\,\big]$$

we have $\gamma \circ \delta(x) = \delta \cdot \gamma(x)$ for all $x \in \mathbb{H}^n$ and $\delta > 0$. Because of

$$\big(\, t^2 + \|\, \zeta\,\|^4\,\big)^{\frac{1}{4}} > \frac{1}{2} \cdot \gamma(\, y\,)$$

for all $y = [\, \zeta, t\,] \in \mathbb{H}^n$ we can choose $\delta > 0$ sufficiently large such that

$$\left|\, t - \frac{s}{\delta^2} - \frac{2}{\delta}\, \mathrm{Im}(\, \zeta \cdot \bar{\eta}\,)\,\right|^2 + \left\|\, \frac{1}{\delta}\eta - \zeta\,\right\|^4 \geq \frac{1}{16} \cdot \gamma(\, y\,)^4$$

for all $(\, x = [\, \eta, s\,], y = [\, \zeta, t\,]\,) \in K \times \{\, \|\, y\,\| = 1\,\}$. Hence multiplying this inequality by δ^4 it follows that:

$$\big|\, \delta^2 t - s - 2\, \mathrm{Im}(\, \delta\zeta \cdot \bar{\eta}\,)\,\big|^2 + \|\, \eta - \delta\zeta\,\|^4 \geq \frac{1}{16} \cdot \gamma \circ \delta(\, y\,)^4.$$

Thus there are $\delta_0, \tilde{c} > 0$ such that for $(\, x, y\,) \in M_{\delta_0} := K \times \{\, \delta(z) : \|\, z\,\| = 1,\ \delta \geq \delta_0\,\}$ we obtain (4.3.8):

$$\big|\, G(\, x, y\,)\,\big| = \frac{c}{\Big(\, |\, t - s - 2\, \mathrm{Im}(\, \zeta \cdot \bar{\eta}\,)\,|^2 + \|\, \eta - \zeta\,\|^4\,\Big)^{\frac{n+1}{2}}} \leq \tilde{c}\, \gamma(\, y\,)^{-(2n+2)}. \tag{4.3.9}$$

By writing G in the form

$$G(\, x, y\,) = c\,\Big(\, t - s - 2i\, \bar{\zeta} \cdot \eta + i\|\, \eta\,\|^2 + i\|\, \zeta\,\|^2\,\Big)^{-(n+1)}$$

it is easy to see that for $k \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^n$ it holds:

$$\partial_s^k\, \partial_{\bar{\eta}}^{\beta} G(\, x, y\,) = c_{n,\beta}\, \eta^{\beta}\, G(\, x, y\,)^{\frac{n+1+k+|\,\beta\,|}{n+1}}$$

where $c_{n,\beta} > 0$. By induction it follows that there are positive numbers c_γ for $\gamma \in \mathbb{N}_0^n$ such that

$$\partial_\eta^\alpha\, \partial_s^k\, \partial_{\bar{\eta}}^{\beta} G(\, x, y\,) = \sum_{\gamma \leq \alpha, \gamma \leq \beta} c_\gamma\, \eta^{\beta - \gamma}\, \Big(\, - 2i\bar{\zeta} + i\bar{\eta}\,\Big)^{\alpha - \gamma}\, G(\, x, y\,)^{\frac{n+1+k+|\,\beta\,|+|\,\alpha\,|-|\,\gamma\,|}{n+1}}.$$

Hence there is $r > 0$ such that for all $\alpha, \beta \in \mathbb{N}_0^n$ and $k \in \mathbb{N}_0$ it exits a positive number $c_{\alpha,\beta,k}$ independent of $y \in \mathbb{H}^n$ with:

$$\Big| \partial_\eta^\alpha\, \partial_s^k\, \partial_{\bar\eta}^\beta G(\, x, y\,) \Big| \leq c_{\alpha,\beta,k} \sum_{\gamma \leq \alpha, \gamma \leq \beta} \big(1 + \|\, \zeta \,\| \big)^{|\,\alpha\,|+|\,\beta\,|-2\,|\,\gamma\,|} |\, G(\, x, y\,)\,|^{\frac{n+1+k+|\,\beta\,|+|\,\alpha\,|-|\,\gamma\,|}{n+1}} = (*)$$

for all $x \in K$ and $\|\, y \,\| > r$. By our estimate in (4.3.9) there are $\delta_0 > 0$ and $\tilde{c}_{\alpha,\beta,k} > 0$ such that for all $(x, y) \in M_{\delta_0}$ it holds

$$(*) \leq \tilde{c}_{\alpha,\beta,k} \cdot \gamma(\, y\,)^{|\,\alpha\,|+|\,\beta\,|} \cdot \gamma(\, y\,)^{-2(n+1+k+|\,\beta\,|+|\,\alpha\,|)} \leq \tilde{c}_{\alpha,\beta,k} \cdot \gamma(\, y\,)^{-2(n+1+k)-|\,\alpha\,|-|\,\beta\,|}.$$

According to (4.3.4) the power $\gamma(\, y\,)^{-2m}$ is square integrable at infinity if and only if $m > \frac{1}{2}(\, n+1\,)$ and we conclude that for any $f \in L^2(\mathbb{H}^n)$ with $\mathrm{dist}(\, K, \mathrm{supp}(f)\,) > 0$:

$$\Big[\mathbb{H}^n \ni y \mapsto \sup_{x \in K} \Big| \partial_s^k\, \partial_\eta^\alpha\, \partial_{\bar\eta}^\beta G(\, x, y\,) \cdot f(\, y\,) \Big| \Big] \in L^1\,(\, \mathbb{H}^n\,).$$

where $\alpha, \beta \in \mathbb{N}_0^n$ and $k \in \mathbb{N}_0$. By well-known results on parameter integrals we conclude from equation (4.3.7) that the restriction $P_s f_{|K}$ is in $\mathcal{C}^\infty(K)$ and because $K \subset U$ was arbitrary the assertion follows. □

Now, we can prove that the Szegö-projection P_s does not increase the singular support of a function $f \in L^2(\mathbb{H}^n)$.

Theorem 4.3.2 *Let $[\, f\,] \in L^2(\mathbb{H}^n)$ and $U \subset \mathbb{H}^n$ an open set such that $f_{|U} \in \mathcal{C}^\infty(U)$. Then there is $[\, g\,] \in L^2(\mathbb{H}^n)$ such that g is smooth on U and $[\, g\,] = [\, P_s f\,] \in L^2(\mathbb{H}^n)$.*

Proof Choose $x \in U$ and fix open and bounded neighborhoods V_1 and V_2 of $x \in U$ such that $x \in V_1 \subset\subset V_2 \subset\subset U$. Let $\varphi_1, \varphi_2 \in \mathcal{C}_c^\infty(\mathbb{H}^n)$ be functions with $\varphi_j \equiv 1$ on V_j for $j = 1, 2$. Moreover, we assume that $\mathrm{supp}(\varphi_1) \subset V_2$ and $\mathrm{supp}(\varphi_2) \subset U$. Then we have the equality $\varphi_2 \cdot \varphi_1 = \varphi_1$ and for $x \in V_1$ it follows that:

$$\varphi_1\, P_s f = \varphi_1\, P_s\, \varphi_2 \big(\, \varphi_1\, f\,\big) + \varphi_1\, P_s \big(\, 1 - \varphi_1\,\big)\, f. \tag{4.3.10}$$

By Theorem 4.3.1 the operator $\varphi_1 P_s \varphi_2$ is a pseudo-differential operator of class $\Psi^0_{\frac{1}{2},\frac{1}{2}}$. Because of $\varphi_1\, f \in \mathcal{C}_c^\infty(\mathbb{H}^n)$ the first summand of (4.3.10) is smooth on $\mathbb{H}^n$ with compact support. By Lemma 4.3.1 the second term is smooth restricted to all open sets $V_0 \subset\subset V_1$ with $x \in V_0$. □

For a bounded symbol $h \in L^\infty(\mathbb{H}^n)$ and with $H := L^2(\mathbb{H}^n)$ we consider the Toeplitz operator $T_h := P_s M_h \in \mathcal{L}(H)$. As an immediate consequence of Theorem 4.3.2 we have:

Corollary 4.3.1 *Let $[\, f\,] \in L^2(\mathbb{H}^n)$ and $x \in \mathbb{H}^n$. Assume that f and the symbol h are smooth in a neighborhood of x, then the same holds true for $T_h f$.*

Let us consider the spectral invariant Fréchet algebra $\Psi_{\frac{1}{2}}[\,\mathcal{A}_\Phi\,]$ we have constructed above Theorem 4.2.3. Then it was shown that $\Psi_{\frac{1}{2}}[\,\mathcal{A}_\Phi\,]$ contains the class of compactly supported operators in $\Psi^0_{\frac{1}{2},\frac{1}{2}}$ as well as all multiplication operators with symbols smooth in a neighborhood of supp(Φ) (see Theorem 4.2.3). An application of Theorem 4.3.1 now proves that $\Psi_{\frac{1}{2}}[\,\mathcal{A}_\Phi\,]$ solves problem (P) of the introduction

Theorem 4.3.3 *(Localization of Toeplitz operators) Let $\varphi \in \mathcal{C}^\infty_c(\mathbb{H}^n)$ and $h \in L^\infty(\mathbb{H}^n)$ compactly supported such that h is smooth in a neighborhood of supp (Φ), then for the Toeplitz operator it follows that $\varphi\, T_h \in \Psi_{\frac{1}{2}}[\,\mathcal{A}_\Phi\,]$.*

Finally, we want to mention that a result analog to Theorem 4.3.1 holds in greater generality. The authors of [117] introduce new classes of pseudo-differential operators which allow the description of the parametricies of some non-elliptic pseudo-differential operators such as the Szegö-projection for strictly pseudo-convex domains. As a main result it has been shown:

Theorem 4.3.4 *([6], [117]) The Szegö-projection on a strictly pseudo-convex domain has a symbol of class $\mathcal{S}^0_{\frac{1}{2},\frac{1}{2}}$.*

4.4 Expansion of smooth functions on odd spheres

We consider the Hardy-Toeplitz projection P on the complex sphere S^{2n-1} in $\mathbb{C}^n$. As it was mentioned in Theorem 4.3.4 above P is a pseudo-differential operator of class $\Psi^0_{\frac{1}{2},\frac{1}{2}}(S^{2n-1})$ and so it has the pseudo-local property. Let $f \in L^2(S^{2n-1})$ and assume that f restricted to an open subset $U \subset S^{2n-1}$ is in $\mathcal{C}^\infty(U)$. The Hardy Toeplitz projection Pf of f can be considered as a holomorphic function f_h on the open unit ball $B_{2n} \subset \mathbb{C}^n$. Under our assumptions on f the question arises if f_h admits a continuous extension to U which coincides with Pf on S^{2n-1}. In particular, it is of interest if the Fourier expansion of f_h is convergent in U. As a fundamental ingredient in our proofs we use the asymptotic of the eigenvalues for the Beltrami-Laplace operator on the sphere. It is likely that some of the results corresponding to Theorem 4.4.2 and 4.4.3 below are valid for strictly pseudo-convex domains in $\mathbb{C}^n$. To begin with and for the convenience of the reader we give some facts on harmonic polynomials in $\mathbb{R}^n$.

Let $n \geq 2$ and denote by $S^{n-1} := \{\, z \in \mathbb{R}^n : \|\, z\,\| = 1\,\}$ the unit sphere in $\mathbb{R}^n$. With the usual Laplace operator $\Delta := \partial^2_{x_1} + \cdots + \partial^2_{x_n}$ on $\mathbb{R}^n$ where $\partial_{x_j} := \frac{\partial}{\partial x_j}$ we consider the space $\mathcal{SH}^{\mathbb{R}}_l(n)$ of homogeneous polynomials of degree $l \in \mathbb{N}_0$ on $\mathbb{R}^n$ contained in the kernel of Δ. A polynomial $p \in \mathcal{SH}^{\mathbb{R}}_l(n)$ is called (regular) *spherical harmonics* of order l.

Applying *Green's formula* where σ denotes the usual surface measure on S^{n-1} we obtain for $h_j \in \mathcal{SH}_j^{\mathbb{R}}(n)$ and $j \in \mathbb{N}_0$:

$$\begin{aligned} 0 &= \int_{\|\, x \,\|<1} \Big\{ h_l(\, x \,) \cdot \Delta \overline{h_m}(\, x \,) - \overline{h_m}(\, x \,) \cdot \Delta h_l(\, x \,) \Big\}\, dx \\ &= \int_{S^{2n-1}} \big(\, m - l \,\big) \cdot h_l(\, \xi \,) \cdot \overline{h_m}(\, \xi \,)\, d\sigma(\, \xi \,). \end{aligned}$$

Here we have used the fact that the normal derivative of a $\mathbb{R}$-homogeneous polynomial Q of degree l on S^{n-1} is given by $\frac{d}{dt}Q(t\,\xi)_{|t=1} = l \cdot Q(\xi)$. Thus the spherical harmonics of distinct degrees are orthogonal with respect to the $L^2(S^{n-1}, \sigma)$-inner product. In the following we identify $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ in the canonical way:

$$z = (\, z_1, \cdots, z_n \,) = (\, x_1, \cdots, x_n \,) + i(\, y_1, \cdots, y_n \,), \qquad \bar{z} := (\, \bar{z}_1, \cdots, \bar{z}_n \,).$$

With the complex derivatives $\partial_{z_k} := 2^{-1}(\, \partial_{x_k} + i\partial_{y_k} \,)$ and $\partial_{\bar{z}_k} := 2^{-1}(\, \partial_{x_k} - i\partial_{y_k} \,)$ the complex Laplacian $\Box$ has the form:

$$\Box := 2 \sum_{k=1}^{n} \partial_{z_k}\, \partial_{\bar{z}_k} = \frac{1}{2} \sum_{k=1}^{n} \big\{\, (\partial_{x_k})^2 + (\partial_{y_k})^2 \,\big\}\,.$$

and so for all dimensions $n \in \mathbb{N}$ we have the notion of spherical harmonics on the complex sphere $S^{2n-1} \cong \{\, z \in \mathbb{C}^n \,:\, \|\, z \,\| = 1 \,\}$. Let us write $\mathcal{SH}_j^{\mathbb{C}}(n)$ for the space of all $\mathbb{R}$-homogeneous polynomials in z and $\bar{z}$ on $\mathbb{C}^n$ which are in the kernel of $\Box$. Examples of spherical harmonics in $\mathcal{SH}_j^{\mathbb{C}}(n)$ are given by the monomials on $\mathbb{C}^n$ of degree j:

$$z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \qquad \alpha \in \mathbb{N}_0^n, \quad \text{and} \quad |\, \alpha \,| = j.$$

Here the defining equation $\Box z^\alpha = 0$ is obvious from $\partial_{\bar{z}_k} z^\alpha = 0$. It follows from our computation above that the monomials z^α and z^β of distinct orders $|\, \alpha \,| \neq |\, \beta \,|$ are orthogonal in $L^2(S^{2n-1}, \sigma)$. In fact this is even true for $\alpha \neq \beta$.

Let us return to the general case of $S^{n-1} \subset \mathbb{R}^n$. We denote by B the *Beltrami Laplace operator* on $\mathcal{C}^\infty(S^{n-1})$. The following result can be found in [138]:

Theorem 4.4.1 *The operator B is essentially self-adjoint with $\langle B\varphi, \varphi\rangle_{L^2(S^{n-1},\sigma)} > 0$ for any function $\varphi \in \mathcal{D}(B) = \mathcal{C}^\infty(S^{n-1})$. The closure $\bar{B}$ of B is an operator with a pure point spectrum, and its eigenvalues are $\gamma_l := l\,(l+n-2)$ for $l \in \mathbb{N}_0$. The spherical harmonics $\mathcal{SH}_l^{\mathbb{R}}(n)$ of degree l form the eigenfunctions of $\bar{B}$ with eigenvalue γ_l.*

In particular, we conclude that with our identification of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ above the restrictions of the monomials z^α for $\alpha \in \mathbb{N}_0$ to the complex sphere $S^{2n-1} \subset \mathbb{C}^n$ are eigenfunctions of B with corresponding eigenvalues

$$\lambda_\alpha := |\, \alpha \,| \cdot \big(\, |\, \alpha \,| + 2n - 2 \,\big). \tag{4.4.1}$$

From the fact that $\mathcal{C}^\infty(S^{2n-1}) = \mathcal{D}(B^k)$ for all powers $k \in \mathbb{N}$ we can prove an asymptotic behavior of the Fourier coefficients of $\mathcal{C}^\infty$-functions on the complex sphere. In the following denote by $\langle \cdot, \cdot \rangle_\sigma$ the L^2-inner-product on S^{2n-1}.

Lemma 4.4.1 *Let $f \in \mathcal{C}^\infty(S^{2n-1})$, then for all $k > 0$ there is $c_k > 0$ such that for all multi-indices $\alpha \in \mathbb{N}_0^n$ the inequality $|\langle f, z^\alpha\rangle_\sigma| \le c_k \cdot (1 + |\alpha|^2)^{-k} \cdot \|z^\alpha\|_\sigma$ holds.*

Proof For $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ it follows from Theorem 4.4.1 and the eigen-value λ_α of B corresponding to z^α that

$$\left\langle B^k f, z^\alpha \right\rangle_\sigma = \left\langle f, B^k z^\alpha \right\rangle_\sigma = \lambda_\alpha^k \cdot \langle f, z^\alpha \rangle_\sigma .$$

From (4.4.1) it is clear that there is $c > 0$ with $\lambda_\alpha \ge c \cdot (1 + |\alpha|^2)$ for all $0 \ne \alpha \in \mathbb{N}_0^n$. Hence it follows that:

$$|\langle f, z^\alpha \rangle_\sigma| \le \lambda_\alpha^{-k} \cdot \left\| B^k f \right\|_\sigma \cdot \| z^\alpha \|_\sigma \le c_k \big(1 + |\alpha|^2\big)^{-k} \cdot \| z^\alpha \|_\sigma , \quad (\alpha \ne 0).$$

where $c_k > 0$ does not depend on α. To include the case $\alpha = 0$ we may increase c_k. □

Let us consider the Hardy space $H^2(S^{2n-1})$ on the unit sphere which can be defined to be the closure of $\mathrm{span}\{ z^\alpha : \alpha \in \mathbb{N}_0^n \}$ in $L^2(S^{2n-1}, \sigma)$. Then $H^2(S^{2n-1})$ consists of square integrable boundary values of holomorphic functions in $B_{2n} := \{ z \in \mathbb{C}^n : \|z\| < 1 \}$. By P we denote the orthogonal projection (*Szegö-projection*) from $L^2(S^{2n-1}, \sigma)$ onto the Hardy space and we write $e_\alpha := z^\alpha \cdot \| z^\alpha \|_\sigma^{-1}$ for $\alpha \in \mathbb{N}_0^n$. The system $[e_\alpha : \alpha \in \mathbb{N}_0^\alpha]$ forms an orthonormal basis of $H^2(S^{2n-1})$ (see [39]) and we can prove:

Theorem 4.4.2 *Let $f \in \mathcal{C}^\infty(S^{2n-1})$, then the expansion $g(z) := \sum_{\alpha \in \mathbb{N}_0^n} \langle f, e_\alpha \rangle_\sigma \cdot e_\alpha$ defines a holomorphic function on B_{2n}. All (formal) derivatives of g and g itself are convergent on the closed ball $\overline{B}_{2n}$ and they represent continuous functions on $\overline{B}_{2n}$. Furthermore, the restriction of g to S^{2n-1} coincides with $Pf \in \mathcal{C}^\infty(S^{2n-1})$.*

Proof By a standard calculation for $\alpha \in \mathbb{N}_0^n$ (see [39]) the L^2-norm $\|z^\alpha\|_\sigma^2$ of the monomials z^α is given by:

$$\| z^\alpha \|_\sigma^2 = 2\,\pi^n \frac{\alpha!}{(n + |\alpha| - 1)!}. \tag{4.4.2}$$

Hence it follows that for all $z \in B_{2n}$ the Fourier expansion of g can be written in the following form:

$$g(z) = \frac{1}{2\,\pi^n} \cdot \sum_{\alpha \in \mathbb{N}_0^n} \langle f, z^\alpha \rangle_\sigma \frac{(n + |\alpha| - 1)!}{\alpha!} \cdot z^\alpha.$$

With any $\beta \in \mathbb{N}_0^n$ we write $\partial_z^\beta = \partial_{z_n}^{\beta_n} \cdots \partial_{z_1}^{\beta_1}$. Then with the usual multi-index notation we have

$$\partial_z^\beta z^\alpha = \frac{\alpha!}{(\alpha - \beta)!}\, z^{\alpha - \beta}$$

and so it follows for $z \in B_{2n}$ with $c_\alpha(f) := \langle f, z^\alpha \rangle_\sigma$ and by an index shift as a function on the open unit ball:

$$\partial_z^\beta g(z) = \frac{1}{2\,\pi^n} \cdot \sum_{\alpha \in \mathbb{N}_0^n} c_{\alpha+\beta}(f) \cdot \frac{(n + |\alpha| + |\beta| - 1)!}{\alpha!} \cdot z^\alpha.$$

Now, we use our estimate on the Fourier coefficients $c_\alpha(f)$ of f. By Lemma 4.4.1 and for any $k \in \mathbb{N}$ there is $c_k > 0$ such that:

$$\big|\, c_{\alpha+\beta}(f) \,\big| \leq c_k \cdot \big(1 + |\, \alpha + \beta \,|^2 \big)^{-k} \cdot \left\{ \frac{(\, \alpha + \beta \,)!}{(\, n + |\, \alpha \,| + |\, \beta \,| - 1 \,)!} \right\}^{\frac{1}{2}}.$$

Applying this estimate to the series above we obtain:

$$\sum_{\alpha \in \mathbb{N}_0^n} |\, c_{\alpha+\beta}(f) \,| \frac{(\, n + |\, \alpha \,| + |\, \beta \,| - 1 \,)!}{\alpha!} \cdot |\, z^\alpha \,|$$

$$\leq c_k \cdot \sum_{\alpha \in \mathbb{N}_0^n} \big(1 + |\, \alpha + \beta \,|^2 \big)^{-k} \cdot \left\{ \frac{(\, n + |\, \alpha \,| + |\, \beta \,| - 1 \,)!}{\alpha!} \right\}^{\frac{1}{2}} \cdot \left\{ \frac{(\, \alpha + \beta \,)!}{\alpha!} \right\}^{\frac{1}{2}} \cdot |\, z^\alpha \,|.$$

We show that the right hand side of this inequality is convergent on S^{2n-1} for an integer k sufficiently large. By the Cauchy-Schwartz inequality we only have to prove the convergence of

(i) $G_k(\, z \,) := \sum_{\alpha \in \mathbb{N}_0^n} (\, 1 + |\, \alpha + \beta \,|^2 \,)^{-k} \cdot \frac{(\, \alpha+\beta \,)!}{\alpha!}$ and

(ii) $F_k(\, z \,) := \sum_{\alpha \in \mathbb{N}_0^n} (\, 1 + |\, \alpha + \beta \,|^2 \,)^{-k} \cdot \frac{(\, n+|\, \alpha \,|+|\, \beta \,|-1 \,)!}{\alpha!} \cdot |\, z^\alpha \,|^2$.

Let us start with (i). For large $k \in \mathbb{N}$ convergence follows from standard calculations together with the estimate

$$\frac{(\, \alpha + \beta \,)!}{\alpha!} \leq |\, \alpha + \beta \,|^{|\, \beta \,|}.$$

In order to treat (ii) we use the multi-nominal formula $\|\, z \,\|^{2l} = \sum_{|\, \alpha \,|=l} \frac{l!}{\alpha!} |\, z^\alpha \,|^2$ for $l \in \mathbb{N}_0$ and $z \in \mathbb{C}^n$. We compute

$$\begin{aligned} F_k(\, z \,) &= \sum_{l=0}^{\infty} \big(1 + [\, l + |\, \beta \,| \,]^2 \big)^{-k} \cdot \frac{(\, n + l + |\, \beta \,| - 1 \,)!}{l!} \cdot \sum_{|\, \alpha \,|=l} \frac{l!}{\alpha!} \cdot |\, z^\alpha \,|^2 \\ &\leq \sum_{l=0}^{\infty} \big(1 + l^2 \big)^{-k} \cdot \big|\, n + l + |\, \beta \,| - 1 \,\big|^{n+|\, \beta \,|-1} \cdot \|\, z \,\|^{2l}. \end{aligned}$$

Let $c > 0$ be a constant with

$$\big|\, n + l + |\, \beta \,| - 1 \,\big|^{n+|\, \beta \,|-1} \leq c \cdot \big(1 + l^2 \big)^{n+|\, \beta \,|-1}$$

for our fixed β and all $l \in \mathbb{N}$. Then we obtain with $z \in S^{2n-1}$:

$$F_k(\, z \,) \leq c \sum_{l=0}^{\infty} \big(1 + l^2 \big)^{-k-1+n+|\, \beta \,|} < \infty$$

for $k > n + |\beta| - \frac{1}{2}$. Hence we have proved that all formal derivatives of g are convergent for all z in the closed unit ball $\overline{B}_{2n}$. In particular, $\partial_z^\beta g$ is bounded on B_{2n} for all $\beta \in \mathbb{N}_0^n$ and so g together with its derivatives are equi-continuous on B_{2n}. By the general theory it follows that $\partial_z^\beta g$ have continuous extensions h_β to $\overline{B_{2n}}$. From *Abel's theorem* we conclude that for all $z \in S^{2n-1}$:

$$h_\beta(z) = \lim_{r \uparrow 1} \partial_z^\beta g(rz) = \partial_z^\beta g(z)$$

where the right hand side denotes the formal derivative of g on S^{2n-1} which converges by our computation above. It is well-known that Pf can be obtained a.e. on S^{2n-1} by non-tangential limit of g from the inside of the ball. From the continuity of g up to the boundary it follows that g coincides with Pf a.e. on S^{2n-1}. □

Finally, we want to prove a localized version of Proposition 4.4.2. Let us remind of the *Szegö-kernel* $K_s : B_{2n} \times S^{2n-1} \to \mathbb{C}$ for the sphere which gives an integral representation of g in the open ball B_{2n}. According to [102] the kernel K_s can be calculated explicitly

$$K_s(z, \zeta) := \frac{(n-1)!}{2\pi^n} \cdot \frac{1}{\big(1 - z \cdot \bar{\zeta}\big)^n} \tag{4.4.3}$$

Given any $f \in L^2(S^{2n-1}, \sigma)$ the *Szegö-projection* Pf can be considered as holomorphic function on B_{2n} by the integral formula:

$$Pf(z) = \int_{S^{2n-1}} K_s(z, \zeta) \cdot f(\zeta)\, d\sigma(\zeta), \qquad (z \in B_{2n}). \tag{4.4.4}$$

Fix a point $x \in S^{2n-1}$ and assume that f vanishes in a ε-neighborhood $U_\varepsilon(x) \subset S^{2n-1}$ of x, then there is an open $\tilde{\varepsilon}$-neighborhood $U_{\tilde{\varepsilon}}(x) \subset \mathbb{C}^n$ such that for all $\alpha \in \mathbb{N}_0^n$:

$$\sup \Big\{ \, |\partial_z^\alpha K_s(z, \zeta)| : z \in U_{\tilde{\varepsilon}}(x),\ \zeta \in S^{2n-1} \setminus U_\varepsilon(x) \Big\} =: c_\alpha < \infty.$$

In fact, this follows from the explicit form of the kernel K_s in (4.4.3). We conclude that the integral expression

$$\Gamma : U_{\tilde{\varepsilon}}(x) \cap B_{2n} \ni z \mapsto \int_{S^{2n-1}} K_s(z, \zeta) \cdot f(\zeta)\, d\sigma(\zeta) \tag{4.4.5}$$

admits an analytic extensions $\bar{\Gamma}$ to $U_{\tilde{\varepsilon}}(x)$. By standard arguments the restriction of $\bar{\Gamma}$ to $S^{2n-1} \cap U_{\tilde{\varepsilon}}(x)$ coincides with Pf in this set. Now we can prove:

Theorem 4.4.3 *Let $f \in L^2(S^{2n-1}, \sigma)$ be smooth in an open subset $U \subset S^{2n-1}$. Then the Toeplitz projection Pf of f is smooth on U as well. Moreover, Pf admits a continuous extension to an analytic function on an open neighborhood of U in $\overline{B}_{2n} \subset \mathbb{C}^n$.*

Proof Let $x \in U$ and choose $V \subset\subset U$ with $x \in V$. Fix $\varphi \in \mathcal{C}^\infty(S^{2n-1})$ such that $\varphi \equiv 1$ on V and $\varphi \equiv 0$ on $S^{2n-1} \setminus U$. We consider the decomposition

$$Pf = P\big(1 - \varphi\big)f + P\varphi f. \tag{4.4.6}$$

Because φf is smooth on S^{2n-1} it follows from Proposition 4.4.2 that $P\varphi f$ is contained in $\mathcal{C}^\infty(S^{2n-1})$ and it admits a continuous extensions to a function analytic in B_{2n}. By our remarks above the first term on the right hand side of (4.4.6) is the restriction of an analytic function on a neighborhood $U_\varepsilon(x) \subset \mathbb{C}^n$ of x to $U_\varepsilon(x) \cap S^{2n-1}$. $\square$

Chapter 5

Gaussian measures and holomorphic functions on open subsets of $\mathcal{DFN}$-spaces.

With a *Gaussian measure* μ_B on an infinite dimensional complex Hilbert space, we consider the space $H^2(V, \mu_B)$ of all square integrable holomorphic functions on an open subset $V \subset H$. We show that in many cases the L^2-closure of $H^2(V, \mu_B)$ can be identified with a space of holomorphic functions $(\mathcal{H}_{\mu_B}, \tau_\omega)$ defined on a dense submanifold in V. Here τ_ω denotes a topology which is finer than the *compact-open topology*. Given an open subset U in a $\mathcal{DFN}$*-space* (the topological dual of a nuclear Fréchet space) and using these results we construct a finite measure ν on U such that the *point evaluation map*

$$U \ni z \mapsto \delta_z \in \left[\mathcal{H}(U) \cap L^2(U, \nu) \right]'$$

is a holomorphic function on U. Finally, with this measure construction and by generalizing a method of A. Pietsch to the case of infinite dimensions (see [121]) we give a new proof of a result due to P. Boland and L. Waelbroeck (cf. [25] and [141]). Namely, that the space $(\mathcal{H}(U), \tau_0)$ of holomorphic functions on U endowed with the *compact-open topology* is a $\mathcal{FN}$-space (nuclear Fréchet space).

Let Ω be a bounded domain in $\mathbb{C}^n$ and denote by v the usual Lebesgue measure under the canonical identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Then it is a well-known fact (cf. [140], p. 79 and Proposition 5.2.1 below), that for any compact subset $K \subset \Omega$, the restriction map:

$$L^2(\Omega, v) \cap \mathcal{H}(\Omega) \ni f \mapsto f_{|K} \in \mathcal{C}(K) \tag{5.0.1}$$

is continuous. Here $\mathcal{C}(K)$ denotes the *Banach-algebra* of continuous functions on K and $\mathcal{H}(\Omega)$ is the space of all holomorphic functions on Ω. By a result due to A. Pietsch (see [121]) the continuity of (5.0.1) implies the *nuclearity* of the Fréchet space $(\mathcal{H}(\Omega), \tau_0)$ with respect to the *compact-open topology* τ_0.

Let us replace $\mathbb{C}^n$ by an infinite dimensional separable complex Hilbert space H and instead of the Lebesgue measure v we use a fully σ-additive *Gaussian measure* μ_B on H.

Moreover, Ω is replaced by an open (not necessary bounded) subset $V \subset H$. In this setting it turns out that $L^2(V, \mu_B)$-convergence in $L^2(V, \mu_B) \cap \mathcal{H}(V)$ does not imply convergence in the *compact-open topology* of $\mathcal{H}(V)$ anymore. What can be proved is, that there exists an infinite dimensional subspace $H_1 \subset H$ nuclear embedded into H and a topology τ_ω on the space $\mathcal{H}(V \cap H_1)$ of holomorphic functions which is finer than τ_0 such that the restriction map

$$L^2\big(V, \mu_B \big) \cap \mathcal{H}\big(V \big) \ni f \mapsto f_{|V \cap H_1} \in \big(\mathcal{H}\,(\, V \cap H_1 \,)\,, \tau_\omega \,\big)$$

is continuous. Moreover, in the case where V is a balanced open set it is a well-known fact that the space $(\, \mathcal{H}(V \cap H_1), \tau_\omega \,)$ is locally convex and complete [51]. In this sense the closure of $L^2(V, \mu_B) \cap \mathcal{H}(V)$ in $L^2(V, \mu_B)$ can be identified with a space of holomorphic functions on a dense sub-manifold of V.

Finally, we consider the Fréchet space of holomorphic functions on an open subset U of a $\mathcal{DFN}$*-space* E (dual of a Fréchet nuclear space) equipped with the compact-open topology. We represent E as a nuclear inductive limit of dense Hilbert space embeddings in the category of locally convex spaces and continuous mappings. Using the above mentioned results on holomorphic functions on Hilbert spaces, we construct a measure ν on E such that (5.0.1) is valid, where Ω and v are replaced by U and ν. A generalization of a result in [121] to the infinite dimensional case now implies the nuclearity of the Fréchet space $(\mathcal{H}(U), \tau_0)$. By different methods this already was proved in [25] and [141].

5.1 Gaussian measures on Hilbert spaces

Let $(\, H, \langle \cdot, \cdot \rangle \,)$ be an infinite dimensional separable complex Hilbert space with Borel σ-algebra $\mathcal{B}(H)$. A complex valued function Ψ on H is called *positive definite* if for every finite collections $y_1, \cdots, y_n \in H$ and any complex numbers $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$ the following inequality holds:

$$\sum_{k,j=1}^{n} \Psi\big(y_j - y_k \big)\, \alpha_j\, \overline{\alpha_k} \geq 0.$$

Let us denote by $\mathcal{F}$ the collection of all finite dimensional subspaces of H. Moreover, for any $G \in \mathcal{F}$ we write P_G for the orthogonal projection from H onto G. Consider the subalgebra

$$\mathcal{U} := \bigcup \Big\{ \mathcal{U}_G : G \in \mathcal{F} \Big\} \subset \mathcal{B}(H) \qquad \text{where} \qquad \mathcal{U}_G := \Big\{ P_G^{-1}(A) \subset H : A \in \mathcal{B}(G) \Big\}.$$

An additive set function $\mu : \mathcal{U} \to \mathbb{R}_0^+$ is called *non-negative quasi-measure* on $\mathcal{U}$ if for each $G \in \mathcal{F}$ the restriction $\mu_{|\mathcal{U}_G}$ is a measure on the σ-algebra $\mathcal{U}_G$. A function $f : H \to \mathbb{C}$ is called *cylindrical* if there is $G \in \mathcal{F}$ such that $f = f \circ P_G$ and in addition f is $\mathcal{U}_G$-measurable. In this case we can define the integral of f by

$$\int_H f \, d\mu := \int_G f \circ P_G \, d\mu_G$$

where μ_G is the measure on $\mathcal{B}(G)$ defined by $\mu_G(A) := \mu(\, P_G^{-1}(A)\,)$ for $A \in \mathcal{B}(G)$. Let $y \in H$ then the map $\exp(\, i\mathrm{Re}\langle \cdot, y\rangle\,)$ is cylindrical on H and we can define the *characteristic function* $\chi_\mu : H \to \mathbb{C}$ of the quasi-measure μ by:

$$\chi_\mu(\, x\,) := \int_H \exp\big(\, i\mathrm{Re}\,\langle\, z, x\,\rangle\,\big)\, d\mu(\, z\,).$$

It is easy to check that χ_μ is positive definite. In general, the relation between positive definite functions on H and quasi-measures on $\mathcal{U}$ is given by the following result which is a generalization of *Bochner's Theorem* and can be found in [48].

Theorem 5.1.1 *A necessary and sufficient condition for a complex-valued function Ψ on H to be a characteristic function of a non-negative quasi-measure μ on $(H,\mathcal{U})$ is that it is positive definite and that all its restriction to finite dimensional subspaces of H are continuous at* 0.

Let $B \in \mathcal{L}(H)$ be a non-negative linear operator, then we consider the corresponding function $\Psi_B : H \to \mathbb{C}$ defined for all $z \in H$ by:

$$\Psi_B(\, z\,) := \exp\big(-\langle\, Bz, z\,\rangle\,\big) = \exp\big(-\|\, B^{\frac{1}{2}}z\,\|^2\,\big).$$

It can be shown that Ψ_B is positive definite on H and so using Theorem 5.1.1 it is the characteristic function of a positive quasi-measure μ_B on $\mathcal{U}$. We call μ_B the *Gaussian quasi-measure* with *correlation operator* B. Because of

$$\mu_B(\, H\,) = \chi_{\mu_B}(\, 0\,) = \Psi_B(\, 0\,) = 1$$

μ_B is a probability measure. The question arises whether or not μ_B is σ-additive on $\mathcal{U}$. The following necessary and sufficient condition can be found in [48]:

Theorem 5.1.2 *The quasi-measure μ_B is σ-additive if and only if the correlation B is a non-negative nuclear operator on H.*

In the following we only use σ-additive *Gaussian quasi-measures* μ_B. In this case μ_B can be extended to a measure on the σ-algebra $\mathcal{B}(H)$ and we call μ_B *Gaussian measure* on the Hilbert space H. From now on we assume that the correlation operator B be positive and nuclear. Let $[\, b_j : j \in \mathbb{N}\,]$ be an orthonormal basis of H consisting of eigenvalues of B with corresponding sequence $(\lambda_j)_j \subset \mathbb{R}^+$ of eigenvalues. Then for any $n \in \mathbb{N}$ we define a function $\Phi_n : H \to \mathbb{C}^n$ by:

$$\Phi_n(\, z\,) := \Big(\, \langle z, b_1\rangle, \cdots, \langle z, b_n\rangle\,\Big)^T \in \mathbb{C}^n, \quad \text{and} \quad \beta_n := \mathrm{diag}\big(\, \lambda_1, \cdots, \lambda_n\,\big) \in \mathbb{R}^{n\times n}.$$

Note that $\Phi_n : P_nH \to \mathbb{C}^n$ is an isometrie with $\beta_n^s \circ \Phi_n = \Phi_n \circ B^s$ for $s \in \mathbb{R}$. For any dimension $n \in \mathbb{N}$ let v be the usual Lebesgue measure on $\mathbb{C}^n$ under the canonical identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Then we denote by μ_{β_n} the *Gaussian measure* on $\mathbb{C}^n$ with the density:

$$d\mu_{\beta_n}(\,z\,) = \frac{1}{\pi^n \det\beta_n} \exp\big(-\big\langle\, \beta_n^{-1} z, z\,\big\rangle_n\,\big)\, dv(\,z\,).$$

Here we write $\langle\cdot,\cdot\rangle_n$ for the *Euclidean norm* on $\mathbb{C}^n$. With $F \in L^1(\mathbb{C}^n, \mu_{\beta_n})$ consider the function f on H defined by $f := F \circ \Phi_n : H \to \mathbb{C}$. Then we have $f \in L^1(H, \mu_B)$ with:

$$\int_H f\, d\mu_B = \int_{\mathbb{C}^n} F\, d\mu_{\beta_n}. \tag{5.1.1}$$

It can be shown that for a linear subspace V of H only the cases $\mu_B(V) \in \{0,1\}$ are possible (cf. [48]). For later applications we examine the case $V = B^s H$ with $s \in [\,0,1\,]$.

Proposition 5.1.1 *Let μ_B be the Gaussian measure with positive nuclear correlation operator $B \in \mathcal{L}(H)$. Then*

$$\mu_B(B^s H) = \begin{cases} 1 & \textit{if } \; B^{1-2s} \textit{ is nuclear and positive} \\ 0 & \textit{else.} \end{cases}$$

In particular, it holds $\mu_B(\,B^s H\,) = 0$ for $s \in [\,\frac{1}{2}, 1\,]$.

Proof Assume that $s \in [\,0,1\,]$ with $\mu_B(\,B^s H\,) = 1$. Then we denote by μ_s the restriction of μ_B to $(\,B^s H, \|B^{-s}\cdot\|\,)$. Let us compute the characteristic function of μ_s.

$$\chi_{\mu_s}(\,x\,) = \int_{B^s H} \exp\big(\,i\,\mathrm{Re}\big\langle\, B^{-s}\cdot, B^{-s}x\,\big\rangle\,\big)\, d\mu_s, \qquad (\,x \in B^s H\,).$$

Because the integrand is defined almost everywhere on H for $x \in B^s H$ and by our definition of μ_s we can write:

$$\chi_{\mu_s}(\,x\,) = \int_H \exp\big(\,i\,\mathrm{Re}\big\langle\, B^{-s}\cdot, B^{-s}x\,\big\rangle\,\big)\, d\mu_B = \chi_{\mu_B}\big(\,B^{-2s}x\,\big) = \exp\big(-\big\|\,B^{\frac{1}{2}-s}x\,\big\|_s^2\,\big)$$

where $\|\cdot\|_s := \|B^{-s}\cdot\|$ and $x \in B^s H$. Because μ_s is σ-additive it directly follows from Theorem 5.1.2 that B^{1-2s} has to be a nuclear operator. Note that the nuclearity of a positive power of B does not depend on the space $B^s H$ where it is defined.

Conversely, let $B^{1-2s} > 0$ be nuclear on $(\,B^s H, \|\cdot\|_s\,)$ and denote by μ_s the σ-additive Gaussian measure on $B^s H$ with correlation B^{1-2s}. Then from the continuous embedding $B^s H \hookrightarrow H$ we can define the transported measure $\tilde{\mu}_s$ on H by:

$$\tilde{\mu}_s(\,A\,) := \mu_s\big(\,A \cap B^s H\,\big) \qquad \text{for all} \qquad A \in \mathcal{B}(H).$$

By the transformation formula for integrals we find for the characteristic function $\chi_{\tilde{\mu}_s}(x)$ with $x \in H$:

$$\begin{aligned}\chi_{\tilde{\mu}_s}(\,x\,) &= \int_H \exp\big(\,i\,\mathrm{Re}\langle\,x,\cdot\,\rangle\,\big)\,d\tilde{\mu}_s\\ &= \int_{B^sH} \exp\big(\,i\,\mathrm{Re}\langle\,B^{2s}x,\cdot\,\rangle_s\big)\,d\mu_s\\ &= \exp\Big(\,-\|\,B^{\frac{1}{2}-s}B^{2s}x\,\|_s^2\,\Big) = \exp\Big(\,-\|\,B^{\frac{1}{2}}x\,\|^2\,\Big).\end{aligned}$$

Because the measure $\tilde{\mu}_s$ is uniquely determined by its characteristic function (cf. [48]) it follows that $\tilde{\mu}_s = \mu_B$ and so $\mu_B(\,B^sH\,) = \mu_s(\,B^sH\,) = 1$. □

5.2 Integral estimates for holomorphic functions

We prove an estimate on the *point evaluation* for a Bergman space over an open subset in $\mathbb{C}^n$, which can be generalized to the case of an infinite dimensional Hilbert space with σ-additive Gaussian measure. Let μ_n denote the Gaussian measure on $\mathbb{C}^n$ defined by the density:

$$d\mu_n(\,z\,) = \pi^{-n}\exp\big(\,-\,\|\,z\,\|_n^2\,\big)\,dv(\,z\,)$$

where $\|\cdot\|_n$ is the *Euclidean norm* and v denotes the usual Lebesgue measure on $\mathbb{C}^n$. With the translation $\tau_z(x) := z + x$ for $z, x \in \mathbb{C}^n$ and the *Euclidean inner product* $\langle\cdot,\cdot\rangle_n$ one easily verifies:

Lemma 5.2.1 *Let $M \subset \mathbb{C}^n$ be a measurable set and $f \in L^1(M,\mu_n)$, then for each $z \in \mathbb{C}^n$ we have*

$$\int_M f\,d\mu_n = \exp\big(\,-\,\|\,z\,\|_n^2\,\big)\int_{\tau_{-z}(M)} f\circ\tau_z\cdot\exp\big(\,-\,2\,Re\,\langle\cdot,z\,\rangle_n\,\big)\,d\mu_n.$$

Let $\mathcal{H}(\Omega)$ denote the space of all holomorphic functions on the open set $\Omega \subset \mathbb{C}^n$. It is well-known that the Bergman-space

$$H^2\big(\,\Omega,\mu_n\,\big) := L^2\big(\,\Omega,\mu_n\,\big)\cap\mathcal{H}(\,\Omega\,)$$

is a *reproducing kernel* Hilbert space and so for each $z \in \Omega$ the *point evaluation map* $\delta_z : H^2(\,\Omega,\mu_n\,) \to \mathbb{C}$ defined by $\delta_z(f) := f(z)$ is a continuous functional. Proposition 5.2.1 provides an estimate for the norm of δ_z.

Proposition 5.2.1 *Let $\Omega \subset \mathbb{C}^n$ be open and $x \in \Omega$. If $r_1,\cdots,r_n$ are nonnegative real numbers such that $Q_x := \{y \in \mathbb{C}^n : |y_j - x_j| < r_j\} \subset \Omega$. Then we obtain for $f \in H^2(\Omega,\mu_n)$:*

$$|\,f(\,x\,)\,| \leq \|\,f\,\|_{L^2(\Omega,\mu_n)}\cdot\exp\big(\,2^{-1}\|\,x\,\|_n^2\,\big)\cdot\prod_{j=1}^n\Big[\,1-\exp\big(\,-\,r_j^2\,\big)\,\Big]^{-\frac{1}{2}}. \tag{5.2.1}$$

Proof Using the transformation formula in Lemma 5.2.1 it follows that:

$$\exp\big(\,\|\,x\,\|_n^2\,\big)\int_\Omega |\,f\,|^2\,d\mu_n \geq \exp\big(\,\|\,x\,\|_n^2\,\big)\int_{Q_x} |\,f\,|^2\,d\mu_n$$
$$= \int_{Q_0} |\,f\circ\tau_x\,|^2\,\exp\big(-2\mathrm{Re}\langle\cdot,x\,\rangle_n\,\big)\,d\mu_n = (*).$$

Now, using the *Taylor expansion* of $\exp(-\langle z,x\rangle_n)\cdot f\circ\tau_x(z) = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha z^\alpha$ as a function of z we obtain:

$$(*) = \int_{Q_0}\Big|\,\exp\big(-\langle\cdot,x\,\rangle_n\,\big)f\circ\tau_x\,\Big|^2\,d\mu_n(\,z\,) \overset{(+)}{=} \sum_{\alpha\in\mathbb{N}_0^n}\Big|\,a_\alpha\,\Big|^2\int_{Q_0}\Big|\,z^\alpha\,\Big|^2\,d\mu_n. \tag{5.2.2}$$

In $(+)$ we have used the fact that the monomials z^α with $\alpha\in\mathbb{N}_0^n$ are mutual orthogonal with respect to the Gaussian measure on Q_0. Now, equation (5.2.2) leads to the inequality

$$\int_\Omega |\,f\,|^2\,d\mu_n \geq \exp\big(-\|\,x\,\|_n^2\,\big)\,\Big|\,a_0\,\Big|^2\prod_{j=1}^n \pi^{-1}\int_{|\,u\,|\leq r_j}\exp\big(-|\,u\,|^2\,\big)\,dv_1(\,u\,). \tag{5.2.3}$$

The integrals on the right hand side of the inequality (5.2.3) can be computed easily by using *polar coordinates*:

$$\int_{|\,u\,|\leq r_j}\exp\big(-|\,u\,|^2\,\big)\,dv_1(\,u\,) = \int_0^{2\pi}\int_0^{r_j} r\exp\big(-r^2\,\big)\,dr\,d\varphi$$
$$= \pi\Big[\,1-\exp\big(-r_j^2\,\big)\,\Big].$$

This together with $|\,a_0\,| = |\,f(x)\,|$ and (5.2.3) implies Proposition 5.2.1. $\square$

Corollary 5.2.1 *Let $\Omega\subset\mathbb{C}^n$ be open and $K:\Omega\times\Omega\to\mathbb{C}$ the reproducing kernel function for $H^2(\,\Omega,\mu_n\,)$. With the notation of Proposition 5.2.1 and $x\in\Omega$ such that $Q_x\subset\Omega$ we have:*

$$K\big(\,x,x\,\big) \leq \exp\big(\,\|\,x\,\|_n^2\,\big)\prod_{j=1}^n\big[\,1-\exp\big(-r_j^2\,\big)\,\big]^{-1} \tag{5.2.4}$$

Moreover, it holds $\|\,\delta_x\,\|_{H^2(\Omega,\mu_n)}^2 \leq K(\,x,x\,)$ for all $x\in\Omega$.

Proof Because of $K(\cdot,x\,)\in H^2(\,\Omega,\mu_n\,)$ for all $x\in\Omega$ and $K(\,x,x\,)>0$ Proposition 5.2.1 implies that:

$$\Big|\,K(\,x,x\,)\,\Big| \leq \|\,K(\cdot,x\,)\,\|_{L^2(\Omega,\mu_n)}\exp\big(\,2^{-1}\|\,x\,\|_n^2\,\big)\cdot\prod_{j=1}^n\Big[\,1-\exp\big(-r_j^2\,\big)\,\Big]^{-\frac{1}{2}}. \tag{5.2.5}$$

By applying the reproducing kernel property of K we have $\|K(\cdot,x)\|_{L^2(\Omega,\mu_n)} = K(x,x)^{\frac{1}{2}}$ and using (5.2.5) the inequality (5.2.4) follows. The second assertion can be obtained be the reproducing kernel property of K by standard arguments. $\square$

We want to prove an estimate analogous to Proposition 5.2.2 in the case of an infinite dimensional Gaussian measure on a complex Hilbert space H. We will see that an inequality corresponding to (5.2.1) only holds for x in a certain sub-manifolds of the open set V where the measure is supported. Let V be an open subset of the complex and separable infinite dimensional Hilbert space H. Fix a Gaussian measure μ_B on H with positive nuclear correlation $B \in \mathcal{L}(H)$. As before, by $H^2(V, \mu_B)$ we denotes the closure of all holomorphic polynomials in $L^2(V, \mu_B)$. In the following we use all the notations of section 5.1 and in addition we assume that $\mathrm{tr}(B^{\frac{1}{2}}) < \infty$ for the trace tr of $B^{\frac{1}{2}}$.

If $r > 0$ and $z \in \mathbb{C}^j$ (resp. $x \in H$), then let $K_j(r, z) \subset \mathbb{C}^j$ (resp. $K_\infty(r, x) \subset H$) be the open ball in the *Euclidean norm* $\| \cdot \|_j$ (resp. the *Hilbert space norm* $\| \cdot \|$) with radius r centered in z (resp. x). Finally, we denote by

$$P_j : H \to \mathrm{span}\{\, b_1, \cdots, b_j \,\}$$

the orthogonal projection onto the linear hull of the eigen-vectors $b_1, \cdots, b_j$ of B.

Lemma 5.2.2 *Let $r > 0$ and $x \in H$. If $\chi_{r,x}$ denotes the characteristic function of the closed ball $\overline{K_\infty(\, r, x\,)}$ in H. Then we have*

$$\chi_{r,x,j} \circ P_j(\, y\,) \xrightarrow{j \to \infty} \chi_{r,x}(\, y\,), \qquad \forall\, y \in H. \tag{5.2.6}$$

where $M_j := \overline{K_\infty(\, r, P_j x\,)} \cap P_j H$ for $j \in \mathbb{N}$ and $\chi_{r,x,j}$ is the characteristic function of M_j.

Proof To begin with assume that $y \notin \overline{K_\infty(\, r, x\,)}$, then there is a number $j_0 \in \mathbb{N}$ such that $\| \, P_{j_0}(\, y - x\,) \, \| > r$ and so it follows that $\chi_{r,x,j} \circ P_j(y) = 0$ for $j \geq j_0$.

For $y \in \overline{K_\infty(r, x)}$ we have $\| \, P_j(\, y - x\,) \, \| \leq \| \, y - x \, \| \leq r$ for all $j \in \mathbb{N}$ and so $P_j y \in M_j$. Hence $\chi_{r,x,j} \circ P_j(y) = 1$ for all $j \in \mathbb{N}$. □

Now, using Lemma 5.2.2 and the inequalities in Proposition 5.2.1 we can give the desired norm estimates for certain *point evaluations* δ_z on $\mathcal{H}(V) \cap L^2(V, \mu_B)$.

Theorem 5.2.1 *Let $R > 0$ and $x = B^{\frac{1}{2}} y \in K_\infty(R, 0)$ with $y \in H$. If we fix a number r with $0 < r < dist(x, \partial K_\infty(R, 0))$, then for any holomorphic function f on $K_\infty(R, 0)$ it holds*

$$|\, f(x) \,| \leq C(\, B, r\,) \cdot \exp\left(\, 2^{-1} \, \| \, y \, \|^2 \,\right) \cdot \| \, f \, \|_{L^2(K_\infty(R,0), \mu_B)}$$

where

$$C(\, B, r\,) = \prod_{j=1}^{\infty} \left[\, 1 - \exp\left(-r^2 \cdot \lambda_j^{-\frac{1}{2}} \cdot tr\left(B^{\frac{1}{2}} \right)^{-1} \right) \right]^{-\frac{1}{2}}$$

is a finite number independent of f.

Proof Each function $f \in \mathcal{H}(\, K_\infty(R, 0)\,)$ is weakly continuous on $\overline{K_\infty(r, x)}$ and so f is bounded there. Using Lebesgue's Theorem, the continuity of $|f|^2$ and Lemma 5.2.2 we obtain that:

$$\begin{aligned}
&\| f \|^2_{L^2(\overline{K_\infty(r,x)},\mu_B)} \\
&= \lim_{j\to\infty} \int_H \left| f \circ P_j \right|^2 \cdot \chi_{r,x,j} \circ P_j \, d\mu_B(z) \\
&= \lim_{j\to\infty} \frac{1}{\pi^j \det \beta_j} \int_{\overline{K_j(r,\Phi_j\circ P_j x)}} \left| f \circ \Phi_j^{-1}(z) \right|^2 \cdot \exp\left(- \left\langle \beta_j^{-1} z, z \right\rangle_j \right) dv_j(z).
\end{aligned}$$

Because of $\overline{K_\infty(r, x)} \subset K_\infty(R, 0)$ and by the transformation formula for the integral (see Lemma 5.2.1) we conclude with $y_j := \Phi_j \circ P_j(x) \in \mathbb{C}^j$ for $j \in \mathbb{N}$:

$$\begin{aligned}
&\| f \|^2_{L^2(K_\infty(R,0),\mu_B)} \qquad (5.2.7)\\
\geq &\| f \|^2_{L^2(\overline{K_\infty(r,x)},\mu_B)} \\
= &\lim_{j\to\infty} \int_{\beta_j^{-\frac{1}{2}}(\overline{K_j(r,y_j)})} \left| f \circ \Phi_j^{-1} \circ \beta_j^{\frac{1}{2}}(z) \right|^2 d\mu_j(z) \\
= &\lim_{j\to\infty} \exp\left(-\| \beta_j^{-\frac{1}{2}} y_j \|_j^2 \right) \int_{\beta_j^{-\frac{1}{2}}(\overline{K_j(r,0)})} \left| F_j(z) \exp\left(- \left\langle z, \beta_j^{-\frac{1}{2}} y_j \right\rangle_j \right) \right|^2 d\mu_j(z).
\end{aligned}$$

Here $F_j(z) := f \circ \Phi_j^{-1} \circ \beta_j^{\frac{1}{2}} \left(z + \beta_j^{-\frac{1}{2}} y_j \right)$ and μ_j denotes the *Gaussian measure* on $\mathbb{C}^j$ with the density $\pi^{-j} \exp(-\| \cdot \|_j^2)$ with respect to the Lebesgue measure. Because β_j is a diagonal matrix, it is easy to see that

$$\beta_j^{-\frac{1}{2}} \overline{K_j(r, 0)} = \left\{ z \in \mathbb{C}^j : \sum_{l=1}^{j} \lambda_l \cdot | z_l |^2 \leq r^2 \right\}.$$

Now, we consider the poly-disc $S_j \subset \mathbb{C}^j$ defined by:

$$S_j := \left\{ z \in \mathbb{C}^j : | z_l |^2 \leq r^2 \cdot \lambda_l^{-\frac{1}{2}} \mathrm{tr} \left(B^{\frac{1}{2}} \right)^{-1} \quad \text{for} \quad l = 1, \cdots, j \right\}.$$

Then for each $j \in \mathbb{N}$ we have the inclusion $(*)$ $S_j \subset \beta_j^{-\frac{1}{2}} \overline{K_j(r, 0)}$. With $z \in S_j$ this directly follows from:

$$\sum_{l=1}^{j} \lambda_l \cdot | z_l |^2 \leq \sum_{l=1}^{j} \lambda_l \cdot \frac{r^2}{\lambda_l^{\frac{1}{2}} \mathrm{tr} \left(B^{\frac{1}{2}} \right)} = \frac{r^2}{\mathrm{tr} \left(B^{\frac{1}{2}} \right)} \cdot \sum_{l=1}^{j} \lambda_l^{\frac{1}{2}} < r^2.$$

Using $(*)$ and inequality (5.2.7) we obtain now with $\| \beta_j^{-\frac{1}{2}} y_j \|_j = \| B^{-\frac{1}{2}} \circ P_j x \|$ and for all dimensions $j \in \mathbb{N}$:

$$\| f \|^2_{L^2(K_\infty(R,0),\mu_B)} \geq \limsup_{j\to\infty} \left\{ \exp\left(-\| B^{-\frac{1}{2}} \circ P_j x \|^2 \right) \cdot \int_{S_j} | G_j |^2 d\mu_j \right\}$$

where $G_j := F_j \cdot \exp(-\langle \cdot, \beta_j^{-\frac{1}{2}} y_j \rangle_j)$. Because G_j is holomorphic in a neighborhood of S_j an application of Proposition 5.2.1 with $G_j(0) = F_j(0)$ leads to:

$$\int_{S_j} | G_j |^2 \, d\mu_j \geq \big| F_j(0) \big|^2 \cdot \prod_{l=1}^{j} \left[1 - \exp\left(-r^2 \cdot \lambda_l^{-\frac{1}{2}} \cdot \operatorname{tr}\left(B^{\frac{1}{2}} \right)^{-1} \right) \right]. \tag{5.2.8}$$

Using $F_j(0) = f \circ P_j(x)$ and the continuity of f, inequality (5.2.8) together with the convergence $\lim_{j\to\infty} \exp(-\| B^{-\frac{1}{2}} \circ P_j x \|^2) = \exp(-\| y \|^2)$ implies that:

$$\| f \|^2_{L^2(K_\infty(R,0),\mu_B)} \geq \exp\big(- \| y \|^2 \big) \cdot | f(x) |^2 \cdot \prod_{l=1}^{\infty} \left[1 - \exp\left(-r^2 \cdot \lambda_l^{-\frac{1}{2}} \cdot \operatorname{tr}\left(B^{\frac{1}{2}} \right)^{-1} \right) \right].$$

In order to finish the proof we have to show that the right hand side does not vanish. Because of

$$0 < c_l := \exp\left(-r^2 \cdot \lambda_l^{-\frac{1}{2}} \cdot \operatorname{tr}\left(B^{\frac{1}{2}} \right)^{-1} \right) \leq \frac{\lambda_l^{\frac{1}{2}} \operatorname{tr}\left(B^{\frac{1}{2}} \right)}{\lambda_l^{\frac{1}{2}} \operatorname{tr}\left(B^{\frac{1}{2}} \right) + r^2} < 1$$

and $(\lambda_l)_l \in l^{\frac{1}{2}}(\mathbb{N})$ it follows that $(c_l)_l \in l^1(\mathbb{N})$ and so the infinite product converges to a positive number $C(B, r)^{-2}$. □

In the following we define $D := \bigcup_{j\in\mathbb{N}} P_j H \subset H$. Then D is a dense subspace of H and Theorem 5.2.1 can be generalized in the following sense.

Corollary 5.2.2 *Let $R > 0$ and $u \in D$. For $x = B^{\frac{1}{2}} y \in K_\infty(R, u)$ where $y \in H$ and any number $0 < r < dist(x, \partial K_\infty(R, u))$ we have with $f \in \mathcal{H}(K_\infty(R, u))$:*

$$\big| f(x) \big| \leq C(B, r) \cdot \exp\big(2^{-1} \| y \|^2 \big) \cdot \| f \|_{L^2(K_\infty(R,u),\mu_B)}, \tag{5.2.9}$$

where $C(B, r)$ is the positive constant defined in Theorem 5.2.1.

Proof Because of $D \subset B(H)$ it is known that the Gaussian measure μ_B is quasi-invariant under translations in direction u (cf. [48] and Lemma 5.2.1 for the finite dimensional case) and

$$\int_{K_\infty(R,u)} |f|^2 d\mu_B = \exp\left(-\left\| B^{-\frac{1}{2}} u \right\|^2 \right) \int_{K_\infty(R,0)} \left| f \circ \tau_u \cdot \exp\left(- \left\langle \cdot, B^{-1} u \right\rangle \right) \right|^2 d\mu_B. \tag{5.2.10}$$

By assumption we have $x - u \in K_\infty(R, 0)$ and

$$0 < r < \operatorname{dist}\big(x - u, \partial K_\infty(R, 0) \big) = \operatorname{dist}\big(x, \partial K_\infty(R, u) \big).$$

If we apply Theorem 5.2.1 to the holomorphic function $F_u := f \circ \tau_u \exp(-\langle \cdot, B^{-1} u \rangle)$ for the point $\tilde{x} := x - u \in K_\infty(R, 0)$ it follows that:

$$\big| f(x) \big| \leq C(B, r) \cdot \exp\left(2^{-1} \left\| y - B^{-\frac{1}{2}} u \right\|^2 + \operatorname{Re} \left\langle x - u, B^{-1} u \right\rangle \right) \cdot \| F_u \|_{L^2(K_\infty(R,0),\mu_B)}.$$

Finally, (5.2.10) together with the identity $\langle\, x-u, B^{-1}u \,\rangle = \langle\, y - B^{-\frac{1}{2}}u, B^{-\frac{1}{2}}u \,\rangle$ lead to inequality (5.2.9). $\square$

In the following we denote by H_1 the complex Hilbert space $B^{\frac{1}{2}}H$ endowed with the inner product $\langle \cdot,\cdot \rangle_1 := \langle\, B^{-\frac{1}{2}}\cdot, B^{-\frac{1}{2}}\cdot \,\rangle$. Then $B^{-\frac{1}{2}} : H_1 \to H$ is an isometrie. Let $V \subset H$ be open. The next lemma provides a locally uniform estimate on H_1 of holomorphic functions with respect to the $L^2(V,\mu_B)$- norm.

Proposition 5.2.2 *Let $V \subset H$ be an open set and $x \in V \cap H_1$. Then there is an open neighborhood $W_x \subset V \cap H_1$ of x with respect to the topology of H_1 and a constant $C_x > 0$ such that for each $f \in \mathcal{H}(V)$ the following inequality holds*

$$\sup\Big\{\, |\, f(z)\,| : z \in W_x \,\Big\} \leq C_x \cdot \|\, f \,\|_{L^2(V,\mu_B)}$$

where C_x is independent of the holomorphic function f.

Proof Because $D \subset H$ is dense we can choose $z \in D$ and $r_1 > 0$ with $x \in K_\infty(r_1, z) \subset V$. Further, choose $r_2 > 0$ with $K_\infty(\, r_2, x\,) \subset K_\infty(\, r_1, z\,) \subset V$ and consider

$$W_x := K_\infty\left(\frac{1}{2}\cdot r_2, x\right) \cap K_\infty^{(1)}(\, 1, x\,) \subset H_1 \cap V$$

where $K_\infty^{(1)}(\, 1, x\,)$ denotes the open ball in H_1 with radius 1 centered in x. Because the embedding $H_1 \hookrightarrow H$ is continuous, W_x is an open neighborhood of x in $H_1 \cap V$.

Let $0 < r_3 < \frac{1}{2}\cdot r_2$, then $r_3 < \operatorname{dist}(\, y, \partial K_\infty(r_1, z)\,)$ for all $y \in W_x$ and Corollary 5.2.2 implies that:

$$\big|\, f(y)\,\big| \leq C\big(\, B, r_3\,\big)\cdot \exp\big(\, 2^{-1}\|\, y\,\|_1^2 \,\big)\cdot \|\, f\,\|_{L^2(K_\infty(\, r_1, z\,),\mu_B)} \leq C_x \cdot \|\, f\,\|_{L^2(V,\mu_B)}$$

with $C_x := \exp\big(\, 2^{-1}(\, \|\, x\,\|_1 + 1\,)^2\,\big)\cdot C(\, B, r_3\,)$. Because $y \in W_x$ was arbitrary Proposition 5.2.2 is proved. $\square$

Lemma 5.2.3 *There is a countable collection $(W_n)_{n\in\mathbb{N}}$ of open sets in $V \cap H_1$ with respect to the topology of H_1 and a sequence $(C_n)_{n\in\mathbb{N}}$ of positive numbers such that*

(i) $V \cap H_1 = \bigcup_{n\in\mathbb{N}} W_n$.

(ii) The inequality $\sup\{\, |\, f(z)\,| : z \in W_n \,\} \leq C_n \cdot \|\, f\,\|_{L^2(V,\mu_B)}$ *holds for all $f \in \mathcal{H}(V)$ and all $n \in \mathbb{N}$.*

Proof We can choose a sequence $(z_j)_{j\in\mathbb{N}} \subset D \cap V$ which is dense V. Hence for $j \in \mathbb{N}$ there are numbers $r_j > 0$ such that

$$H_1 \cap V = \bigcup_{j\in\mathbb{N}} \big[\, K_\infty(\, r_j, z_j\,) \cap H_1 \,\big].$$

From this we conclude that it is sufficient to prove Lemma 5.2.3 in the case of open balls $V := K_\infty(r,z) \subset H$ where $r>0$ and $z \in D \cap V$. Define

$$W_n := K_\infty\left(\left[1 - \frac{1}{n} \right] r, z \right) \cap K_\infty^{(1)}(\, n, z\,), \qquad n \in \mathbb{N}.$$

Then we have $\bigcup_{n\in\mathbb{N}} W_n = K_\infty(r,z) \cap H_1$ and by the same argument we have used in the proof of Proposition 5.2.2 we find $C_n > 0$ such that

$$\sup\Big\{ \mid f(z) \mid : z \in W_n \Big\} \le C_n \cdot \| f \|_{L^2(V,\mu_B)}$$

holds for all $f \in \mathcal{H}(V)$ and all $n \in \mathbb{N}$. □

Let U be an open subset in an locally convex space E. Consider the space $\mathcal{H}(U)$ of all (continuous) holomorphic functions on U endowed with the *compact-open topology* which is generated by the semi-norms

$$p_K(\, f\,) := \sup\Big\{ \mid f(x) \mid : x \in K \Big\}.$$

Here K ranges over the compact subsets of U. We denote this topology by τ_0. If U is an open subset of an infinite dimensional Banach space, then $(\, \mathcal{H}(U), \tau_0\,)$ is not barreled and thus it is not a Fréchet space (see [51], p. 168) as in the case of finite dimensions. In the following we define a topology τ_ω on $\mathcal{H}(U)$ which is finer then τ_0. In particular, for any infinite dimensional Banach space E and each open subset $U \subset E$ we have $\tau_0 \subsetneqq \tau_\omega$. The following definition can be found in [51].

Definition 5.2.1 A semi-norm p on $\mathcal{H}(U)$ is *ported* by the compact subset K of U if for every open set V with $K \subset V \subset U$, there exists $c_V > 0$ such that

$$p(\, f\,) \le c_V \cdot \sup\Big\{ \mid f(x) \mid : x \in V \Big\}, \qquad \forall\, f \in \mathcal{H}(U).$$

The τ_ω-topology on $\mathcal{H}(U)$ is the topology generated by the semi-norms ported by the compact subsets of U.

Lemma 5.2.4 *Let H be a complex separable Hilbert space and $V \subset H$ be an open subset. Then the restriction map*

$$R : \mathcal{H}(\, V\,) \cap L^2\big(\, V, \mu_B\, \big) \longrightarrow \big(\, \mathcal{H}(\, V \cap H_1\,), \tau_\omega\, \big)$$

defined by $R(f) := f_{|_{V\cap H_1}}$ is continuous with respect to the $L^2(V,\mu_B)$-topology.

Proof Choose a countable family $(W_n)_n$ of open sets in $V \cap H_1$ and a sequence $(C_n)_n$ of positive numbers such that (i) and (ii) in Lemma 5.2.3 hold. If necessary after replacing W_n by

$$\tilde{W}_n := \bigcup_{j=1}^{n} W_j \subset H_1 \cap V, \qquad (\, n \in \mathbb{N}\,)$$

we can assume without loss of generality that the family $(W_n)_n$ is an increasing countable cover of $V \cap H_1$ with (ii). Let $K \subset V \cap H_1$ be compact, then there is $n_0 \in \mathbb{N}$ with $K \subset W_{n_0}$ and for each τ_ω-continuous semi-norm p on $\mathcal{H}(V \cap H_1)$ we have

$$p(\, f\,) \leq c \cdot \sup \Big\{ \mid f(x) \mid \, : z \in W_{n_0} \Big\} \leq c \cdot C_{n_0} \cdot \| \, f \, \|_{L^2(V,\mu_B)}$$

where $c > 0$ is a suitable constant. From this the result follows. □

The next lemma as well as some more topological properties of the space $(\, \mathcal{H}(V), \tau_\omega \,)$ can be found in [51], p. 216.

Lemma 5.2.5 *If U is a balanced open subset of a Fréchet space, then $(\, \mathcal{H}(U), \tau_\omega \,)$ is a complete locally convex space.*

Let V be a balanced open subset of H, and $\mathcal{M} \subset \mathcal{H}(V) \cap L^2(V, \mu_B)$ be an arbitrary set of holomorphic functions. Then by Lemma 5.2.4 and Lemma 5.2.5 the closure of $\mathcal{M}$ in $L^2(V, \mu_B)$ can be identified with a subspace of $(\, \mathcal{H}(V \cap H_1), \tau_\omega \,)$. This is analog to the well-known result, that in the case of finite dimensions the Bergman space on bounded subsets of $\mathbb{C}^n$ can be considered as a space of holomorphic functions. n space on bounded subsets of $\mathbb{C}^n$ can be considered as a space of holomorphic functions.

5.3 Some topological properties of $\mathcal{DFN}$-spaces

In this section we want to list some well-known topological properties of $\mathcal{DFN}$-spaces (dual of a Fréchet nuclear space). A more detailed description and most of the proofs can be found in [51], [60], [116] and our appendix. In the following let E be a $\mathcal{DFN}$-space then with the *compact-open topology* E can be represented as a nuclear inductive countable spectrum of Banach spaces in the category of locally convex spaces and continuous linear mappings. Some important examples of $\mathcal{DFN}$-spaces are given by:

(1) The space s' where s denotes the rapidly decreasing sequences.

(2) The tempered distributions $\mathcal{S}'(\mathbb{R}^n)$.

(3) The dual of $\mathcal{C}^\infty(\Omega)$ where Ω is an open subset of $\mathbb{R}^n$.

(4) The dual of the space $\mathcal{H}(U)$ of all holomorphic functions on U where U is an open subset in $\mathbb{C}^n$ or in a $\mathcal{DFN}$-space.

(5) Any countable nuclear inductive spectrum of Banach spaces.

Remark 5.3.1 Each $\mathcal{DFN}$-space can be densely embedded into a Hilbert-space (cf. the appendix). Moreover, in the appendix we give an example of a class of $\mathcal{DNF}$-spaces which canonically are embedded in a Bergman space of holomorphic functions on open subsets in $\mathbb{C}^n$. Let us mention that there are many examples of $\mathcal{DFN}$-spaces which in addition are algebras ($\mathcal{DFN}$-algebras).

We use the following notations (cf. Definition A.1.1):

Definition 5.3.1 Let X be a topological locally convex space and $U \subset X$ be open.

(1) A sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets $K_n \subset U$ is called *fundamental sequence*, if for each compact set $K \subset U$ there is $n_0 \in \mathbb{N}$ such that $K \subset K_{n_0}$. The set U is called *hemi-compact* if it contains a fundamental sequence of compact sets.

(2) The open set U is called *Lindelöf* if each open cover of U admits a countable subcover.

(3) X is called *k-space*, if $M \subset X$ is open if and only if $M \cap K$ is open in K, with the induced topology, for each compact subset $K \subset X$.

The proof of Theorem 5.3.1 can be found in [116] (p. 513, 7.6 *Theorem*) and Corollary A.1.3. Together with Lemma 5.3.1 it implies that the space $\mathcal{H}(U)$ (resp. $\mathcal{C}(U)$) of holomorphic (resp. continuous) functions on an open subset $U \subset E$ endowed with the *compact-open* topology are Fréchet spaces.

Theorem 5.3.1 *Let E be a $\mathcal{DFN}$-space. Then all the open and all the closed subsets of E are k-spaces.*

As another useful property of $\mathcal{DFN}$-spaces we mention the following lemma which is proved in the appendix, Lemma A.1.2. l property of $\mathcal{DFN}$-spaces we mention the following lemma which is proved in the appendix, Lemma A.1.2.

Lemma 5.3.1 *Let $U \subset E$ be an open set in a $\mathcal{DFN}$-space E, then U is hemi-compact.*

As an easy consequence of Lemma 5.3.1 we obtain:

Corollary 5.3.1 *Let E be a $\mathcal{DFN}$-space and $U \subset E$ be open, then U is Lindelöf.*

For the construction of a measure ν on E such that for all open subsets $U \subset E$ the point evaluations in $L^2(U,\nu) \cap \mathcal{H}(U)$ is continuous we use the fact that E can be represented as a particular inductive spectrum of Hilbert spaces.

Lemma 5.3.2 *Any $\mathcal{DFN}$-space E is isomorphic to a nuclear inductive spectrum $\overrightarrow{\lim_{m\to\infty}} H_m$ of Hilbert spaces H_m. Moreover, the defining maps $\pi_{n,m} : H_n \hookrightarrow H_m$ are (nuclear) dense embeddings for $n, m \in \mathbb{N}$ with $m > n$.*

Proof Let the Fréchet nuclear space F be represented strictly by a countable nuclear projective spectrum $\{ B_n, \rho \}$ of Banach spaces ([60], pp. 108). If $E := F_b'$ denotes the dual space of F with the *strong topology* it follows from Satz 2.4, p. 146 in [60] that E is the inductive limit $E = (\overleftarrow{\operatorname{proj}_{m\to\infty}} B_m)_b' \cong \overrightarrow{\lim_{m\to\infty}} B_m'$. For all $n, m \in \mathbb{N}$ with $n < m$ the spectral maps $\pi_{n,m} : B_n' \to B_m'$ are nuclear. Every nuclear map between complete local convex spaces can be factorized over $l^2(\mathbb{N})$ (see [60], p. 102). From this fact we obtain the following commuting diagram:

$$\begin{tikzcd}
B_1' \arrow[rr, "\pi_{1,2}"] \arrow[dr, "\alpha_1"'] & & B_2' \arrow[rr, "\pi_{2,3}"] \arrow[dr, "\alpha_2"] & & B_3' \cdots \arrow[dr, "\alpha_3"] & \\
& l^2(\mathbb{N}) \arrow[ur, "\beta_1"'] \arrow[rr, "T_1"'] & & l^2(\mathbb{N}) \arrow[ur, "\beta_2"'] \arrow[rr, "T_2"'] & & l^2(\mathbb{N}) \cdots
\end{tikzcd}$$

with $T_j := \alpha_{j+1} \circ \beta_j$ for $j \in \mathbb{N}$. Denote by $\{ \tilde{H}_n, \pi \}$ the *nuclear inductive spectrum* defined by the Hilbert spaces $l^2(\mathbb{N})$ and the nuclear maps T_j with $j \in \mathbb{N}$. The commuting diagram together with the reflexivity of F with respect to the strong topology now shows that:

$$E \cong \big(\overleftarrow{\operatorname{proj}}_{m\to\infty} B_m \big)_b' \cong \overrightarrow{\lim_{m\to\infty}} B_m' \cong \overrightarrow{\lim_{m\to\infty}} \tilde{H}_m \qquad F \cong \big(\overrightarrow{\lim_{m\to\infty}} \tilde{H}_m \big)_b' \cong \overleftarrow{\operatorname{proj}}_{m\to\infty} \tilde{H}_m'$$

where $\tilde{H}_m'$ are Hilbert spaces. Using ([60], p. 143) there is a reduced nuclear projective spectrum $\{H_m, \rho\}$ of Hilbert spaces H_m [i.e. $\rho_n(\overleftarrow{\operatorname{proj}}_{m\to\infty} H_m)$ is dense in H_n for all $n \in \mathbb{N}$] with $\overleftarrow{\operatorname{proj}}_{m\to\infty} \tilde{H}_m' \cong \overleftarrow{\operatorname{proj}}_{m\to\infty} H_m$. We obtain that

$$E \cong \big(\overleftarrow{\operatorname{proj}}_{m\to\infty} H_m \big)_b' \cong \overrightarrow{\lim_{m\to\infty}} H_m'$$

and with ([60], p. 145) for all $n, m \in \mathbb{N}$ with $n < m$ the defining maps $\pi_{n,m} : H_n' \to H_m'$ are one-to-one. Consider the following commutative diagram:

$$\begin{tikzcd}
H_1' \arrow[r, "\pi_{1,2}"] & H_2' \arrow[r, "\pi_{2,3}"] & H_3' \arrow[r, "\pi_{3,4}"] & H_4' \cdots & & \\
& H_1' \arrow[u, "\pi_{1,2}"] \arrow[r, "\pi_{1,2}"] & \overline{\pi_{1,2}H_1'} \arrow[u, "\pi_{2,3}"] \arrow[r, "\pi_{2,3}"] & \overline{\pi_{2,3}H_2'} \arrow[u, "\pi_{3,4}"] \arrow[r, "\pi_{3,4}"] & \overline{\pi_{3,4}H_3'} \cdots & \\
& & H_1' \arrow[u, "\pi_{1,2}"] \arrow[r, "\pi_{1,2}"] & H_2' \arrow[u, "\pi_{2,3}"] \arrow[r, "\pi_{2,3}"] & H_3' \arrow[u, "\pi_{3,4}"] \arrow[r, "\pi_{3,4}"] & H_4' \ \cdots
\end{tikzcd} \tag{5.3.1}$$

Here the spaces $\overline{\pi_{j,j+1}H_j'}$ with $j \in \mathbb{N}$ denote the H_{j+1}'-closure of $\pi_{j,j+1}H_j'$. Using ([60], p. 122) we conclude that the nuclear inductive spectrum $\{ H_n', \pi \}$ is isomorphic to the nuclear inductive spectrum

$$H_1' \xrightarrow{\pi_{1,2}} \overline{\pi_{1,2}H_1'} \xrightarrow{\pi_{2,3}} \overline{\pi_{2,3}H_2'} \cdots \ . \tag{5.3.2}$$

Thus it is sufficient to prove that the inductive limit generated by the spectrum (5.3.2) is isomorphic to the inductive limit of a nuclear spectrum of Hilbert space embeddings.

We consider:

$$\begin{tikzcd}
H_1' \arrow[r, "\pi_{1,2}"] \arrow[d, "\mathrm{id}"'] & \overline{\pi_{1,2}H_1'} \arrow[r, "\pi_{2,3}"] \arrow[d, "\tilde{\pi}_{1,2}^{-1}"'] & \overline{\pi_{2,3}H_2'} \cdots \arrow[d, "\tilde{\pi}_{2,3}^{-1}"'] \\
H_1' \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \widetilde{(H_1', \|\pi_{1,2}\cdot\|)} \arrow[r, "\tilde{\pi}_{1,2}"] \arrow[d, "\tilde{\pi}_{1,2}"'] & \widetilde{(H_2', \|\pi_{2,3}\cdot\|)} \cdots \arrow[d, "\tilde{\pi}_{2,3}"'] \\
H_1' \arrow[r, "\pi_{1,2}"] & \overline{\pi_{1,2}H_1'} \arrow[r, "\pi_{2,3}"] & \overline{\pi_{2,3}H_2'} \cdots
\end{tikzcd}$$

The spaces $\widetilde{(H_j', \|\pi_{j,j+1}\cdot\|)}$ denote the completions of the spaces H_j' with respect to the Prae-Hilbert space norm $\|\pi_{j,j+1}\cdot\|$ and the maps $\tilde{\pi}_{j,j+1}$ (resp. $\tilde{\pi}_{j,j+1}^{-1}$) are the continuous continuations of $\pi_{j,j+1}$ (resp. $\pi_{j,j+1}^{-1}$). It is easy to show that all the maps are continuous and one-to-one. Moreover H_1' is dense in $\widetilde{(H_1', \|\pi_{1,2}\cdot\|)}$.

After repeating this procedure we obtain the diagram where all the spaces $H_j^{(l)}$ with $l, j \in \mathbb{N}$ are Hilbert spaces and the vertical arrows denote mutual inverse continuous maps.

$$\begin{tikzcd}
H_1' \arrow[r, "\pi_{1,2}"] \arrow[d, leftrightarrow] & \overline{\pi_{1,2}H_1'} \arrow[r, "\pi_{2,3}"] \arrow[d, leftrightarrow] & \overline{\pi_{2,3}H_2'} \cdots \arrow[d, leftrightarrow] & & \\
H_1^{(1)} \arrow[r, "\mathrm{id}"] & H_2^{(1)} \arrow[r, "\tilde{\pi}_{1,2}"] \arrow[d, leftrightarrow] & H_3^{(1)} \arrow[r, "\tilde{\pi}_{3,4}"] \arrow[d, leftrightarrow] & H_4^{(1)} \cdots \arrow[d, leftrightarrow] & \\
 & H_1^{(2)} \arrow[r, "\mathrm{id}"] & H_2^{(2)} \arrow[r, "\tilde{\tilde{\pi}}_{1,2}"] & H_3^{(2)} \arrow[r, "\tilde{\tilde{\pi}}_{3,4}"] & H_4^{(2)} \cdots
\end{tikzcd}$$

Here $H_1^{(1)} = H_1'$ and $H_2^{(j)} = H_1^{(j+1)}$. The spaces $H_1^{(j)}$ are dense in $H_2^{(j)}$ for all $j \in \mathbb{N}$. From this we have the diagram

$$\begin{tikzcd}
H_1' \arrow[r, "\pi_{1,2}"] \arrow[d, leftrightarrow] & \overline{\pi_{1,2}H_1'} \arrow[r, "\pi_{2,3}"] \arrow[d, leftrightarrow] & \overline{\pi_{2,3}H_2'} \cdots \arrow[d, leftrightarrow] \\
H_1^{(1)} \arrow[r, "\mathrm{id}"] & H_1^{(2)} \arrow[r, "\mathrm{id}"] & H_1^{(3)} \cdots
\end{tikzcd}$$

Because all the maps $\pi_{j,j+1}$ are nuclear, it follows that the inductive spectrum (5.3.2) is isomorphic to the following nuclear spectrum of Hilbert spaces

$$H_1^{(1)} \xrightarrow{\mathrm{id}} H_1^{(2)} \xrightarrow{\mathrm{id}} H_1^{(3)} \cdots$$

where all the defining maps are embeddings with dense range. □

Remark 5.3.2 After replacing the norms $\|\cdot\|_n$ in the Hilbert spaces H_n in Lemma 5.3.2 by the norms $c_n\|\cdot\|_n$ where $c_n > 0$ are suitable positive constants we can assume without lost of generality that $\|\pi_{n,m}\| < 1$ for all $m, n \in \mathbb{N}$ with $n < m$.

5.4 $\mathcal{NF}_p$-measures on open sets of $\mathcal{DFN}$-spaces

Using the results of section 5.1 we construct a finite Borel measure ν on the $\mathcal{DFN}$-space E with a topological property which is closely related to the nuclearity of $\mathcal{H}(E)$. To be more precise we have to define the notion of a $\mathcal{NF}_p$-measure for a real number $p \geq 1$.

Let X be a locally convex space and assume that $\mathcal{F} \subset \mathcal{C}(X)$ is a linear subspace in the algebra $\mathcal{C}(X)$ of continuous complex-valued functions on X. By $\mathcal{M}_\sigma(X)$ we denote the space of all σ-finite Borel-measures on X. We define:

Definition 5.4.1 Let $p \geq 1$, then we call $\mu \in \mathcal{M}_\sigma(X)$ a $\mathcal{NF}_p$-measure iff for each compact set $K \subset X$ there is a compact set $H \subset X$ with $K \subset H$ and $C > 0$ such that for all $f \in \mathcal{F}$

$$\sup\Big\{ \mid f(x) \mid : x \in K \Big\} \leq C \left[\int_H \mid f \mid^p d\mu \right]^{\frac{1}{p}} . \tag{5.4.1}$$

The space of all $\mathcal{NF}_p$-measures on X is denoted by $\mathcal{MF}_p(X)$. We call X a $\mathcal{NF}_p$-space if $\mathcal{MF}_p(X) \neq \emptyset$.

Example 5.4.1 We give examples for $\mathcal{NF}_p$-spaces X, where $\mathcal{F} := \mathcal{H}(X)$ is the spaces of holomorphic functions on X.

(a) Let $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ be open and denote by v the usual Lebesgue-measure on U. Then for $1 \leq p \leq 2$ and $\mathcal{F} := \mathcal{H}(U)$ it is well-known that v is a $\mathcal{NF}_p$-measure and so U is a $\mathcal{NF}_p$-space (cf. Proposition 5.2.1).

(b) Let $P(x, D)$ be a hypo-elliptic differential operator. Then the solution space of $P(x, D)$ is a $\mathcal{NF}_2$-space (cf. [72]).

We will prove later on:

(c) Let E be a $\mathcal{DFN}$-space and $\Omega \subset E$ be open in E. For the space $\mathcal{F} := \mathcal{H}(\Omega)$ and $1 \leq p \leq 2$ it can be shown that $\mathcal{MF}_p(\Omega) \neq \emptyset$. Hence Ω is a $\mathcal{NF}_p$-space.

We prove a permanence property of $\mathcal{NF}_2$-measures. Let X_j for $j = 1, 2$ be locally convex spaces and assume that $\mathcal{F}^j \subset \mathcal{C}(X_j)$ are linear subspaces. With the usual definition

$$f_1 \otimes f_2(z, w) := f_1(z) \cdot f_2(w), \qquad (z, w) \in X_1 \times X_2$$

for $f_j \in \mathcal{F}^j$ for $j = 1, 2$ the *algebraic tensor product* $\mathcal{F}^1 \otimes \mathcal{F}^2$ can be considered as a subspace of $\mathcal{C}(X_1 \times X_2)$ where $X_1 \times X_2$ carries the topology of a product space. The following properties of $\mathcal{NF}_2$-measures can be shown:

Proposition 5.4.1 *Let ν_j be $\mathcal{NF}^j_2$-measures for $j = 1, 2$, then the product measure $\nu_1 \otimes \nu_2$ is a $\mathcal{NG}_2$-measure where $\mathcal{G} = \mathcal{F}^1 \otimes \mathcal{F}^2$.*

Proof Fix a compact set $K \subset X_1 \times X_2$ and for $j = 1, 2$ consider the canonical continuous projections $\pi_j : X_1 \times X_2 \to X_j$. Then both spaces $K_j := \pi_j(K) \subset X_j$ are compact and the inclusion holds:

$$K \subset K_1 \times K_2 \subset X_1 \times X_2.$$

Because by our assumption ν_j are $\mathcal{NF}_2^j$-measures for $j = 1, 2$ we can choose compact subsets H_j of X_j with $K_j \subset H_j$ and constants $M_j > 0$ such that for $j = 1, 2$:

$$\sup \Big\{ \mid f_j(x) \mid : x \in K_j \Big\} \leq M_j \cdot \left[\int_{H_j} \mid f_j \mid^2 d\nu_j \right]^{\frac{1}{2}}. \tag{5.4.2}$$

for all $f_j \in \mathcal{F}^j$. By restriction we can consider $\mathcal{F}^j$ as subspaces of $L^2(H_j, \nu_j)$. Let us denote its Hilbert space closure by $\mathcal{F}_c^j$. According to (5.4.2) we can regard elements of $\mathcal{F}_c^j$ as continuous functions on K_j and for any $x \in K_j$ the point evaluation

$$\delta_x^j : \mathcal{F}^j \to \mathbb{C} : f_j \mapsto f_j(\, x\,)$$

have a continuous extension to a functional in $(\, \mathcal{F}_c^j\,)'$ which again we denote by δ_x^j. From the Riesz Lemma there are kernel function $B_j : H_j \times K_j \to \mathbb{C}$ for $j = 1, 2$ with

$$f_j(\, x\,) = \delta_x^j(\, f_j\,) = \Big\langle f_j, B_j(\cdot, x\,) \Big\rangle_{L^2(H_j,\nu_j)} \qquad \text{and} \qquad B_j(\cdot, x\,) \in \mathcal{F}_c^j. \tag{5.4.3}$$

Let us denote by $\langle \cdot, \cdot \rangle_j$ the inner-product in $L^2(H_j, \nu_j)$, then from equation (5.4.3) it is easy to see that:

$$\big\| \, \delta_x^j \, \big\| = \| \, B_j(\cdot, x\,) \, \|_j = B_j(\, x, x\,)^{\frac{1}{2}}, \qquad \big(\, x \in K_j\, \big).$$

Moreover, by equation (5.4.2) the map $K_j \ni x \mapsto B_j(\, x, x\,)^{\frac{1}{2}} \in \mathbb{R}^+$ is bounded by M_j. Fix $(\, x, y\,) \in K_1 \times K_2$ and $h = \sum_{i=1}^n f_i \otimes g_j$ in $\mathcal{G}$, then:

$$\begin{aligned} \mid h(\, x, y\,) \mid &= \Big| \sum_{i=1}^{n} \Big\langle f_i, B_1(\cdot, x\,) \Big\rangle_1 \Big\langle g_i, B_2(\cdot, y\,) \Big\rangle_2 \Big| \\ &= \Big| \int_{H_1 \times H_2} h\big(\, z_1, z_2\, \big) \cdot \overline{B_1(\, z_1, x\,)} \cdot \overline{B_2(\, z_2, y\,)}\, d(\, \nu_1 \otimes \nu_2\,)(\, z_1, z_2\,) \Big| \\ &\leq \Big[\int_{H_1 \times H_2} \mid h \mid^2 d(\nu_1 \otimes \nu_2) \Big]^{\frac{1}{2}} \cdot B_1(\, x, x\,)^{\frac{1}{2}} \cdot B_2(\, y, y\,)^{\frac{1}{2}} \end{aligned}$$

Finally, from the boundedness of B_j on the diagonal in $K_1 \times K_2$ an the fact that $H_1 \times H_2$ is compact in $X_1 \times X_2$ we obtain that $\nu_1 \otimes \nu_2$ is a $\mathcal{NG}_2$-measure. □

Let $\mathcal{C}(X_j)$ be a Fréchet space with respect to the compact open topology and let us equip $\mathcal{F}^1 \otimes \mathcal{F}^2$ with the π-topology. By the universal property of the π-tensor-product the embedding

$$\mathcal{F}^1 \otimes_\pi \mathcal{F}^2 \hookrightarrow \mathcal{C}\,(\, X_1 \times X_2\,) \tag{5.4.4}$$

is continuous. Hence the completion $\hat{\mathcal{G}} := \mathcal{F}_1 \hat{\otimes}_\pi \mathcal{F}_2$ of $\mathcal{F}_1 \otimes_\pi \mathcal{F}_2$ can be considered as a subspace of $\mathcal{C}(\, X_1 \times X_2\,)$.

Corollary 5.4.1 *With our notations above, the product $\nu_1 \otimes \nu_2$ is a $\mathcal{N}\hat{\mathcal{G}}_2$-measure.*

Proof Let $h \in \hat{\mathcal{G}}$ and choose $(h_n)_n \subset \mathcal{G}$ converging to h with respect to the π-topology. According to the continuous embedding (5.4.4) the sequence $(h_n)_n$ tends to h uniformly on all compact subsets of $X_1 \times X_2$. Let $K \subset X_1 \times X_2$ be compact, then it follows from Proposition 5.4.1 that we can fix a constant $M > 0$ and $H \subset X_1 \times X_2$ compact such that $K \subset H$ and

$$\sup \Big\{ \, | \, h_n(\, x, y \,) \, | : (\, x, y \,) \in K \Big\} \leq M \cdot \left[\int_H | \, h_n \, |^2 \, d(\, \nu_1 \otimes \nu_2 \,) \right]^{\frac{1}{2}} \tag{5.4.5}$$

for all $n \in \mathbb{N}$. If n tends to infinity we obtain inequality (5.4.5) for the function h instead of h_n and this proves our assertion. $\square$

In our setting we choose X to be an open set U in a $\mathcal{DFN}$-space E and $\mathcal{F} := \mathcal{H}(U)$. By the explicit construction of a measure ν with (5.4.1) we show that U is a $\mathcal{NF}_2$-space. Finally, inequality (5.4.1) together with a generalization of a result due to A. Pietsch (see [121]) to infinite dimensions leads to the nuclearity of the Fréchet space $\mathcal{H}(U)$ with respect to the compact open topology.

Lemma 5.4.1 *Let $(H_j, \langle \cdot, \cdot \rangle_j)$ for $j = 1, 2$ be separable complex Hilbert spaces with nuclear and dense embedding $I : H_1 \hookrightarrow H_2$. Then there is a nuclear positive operator $B \in \mathcal{L}(H_2)$ such that $H_1 \subset B^{\frac{1}{2}} H_2$ and the embedding $J : H_1 \hookrightarrow (\, B^{\frac{1}{2}} H_2, \| \, B^{-\frac{1}{2}} \cdot \|_2 \,)$ is continuous with the norm estimate $\| \, J \, \| \leq tr(B)$.*

Proof Because $I(H_1) \subset H_2$ is a dense subspace the map $I^* : H_2 \to H_1$ is one-to-one. From the nuclearity of I we conclude that I^* is nuclear as well and so there are orthonormal bases $[\, e_j : j \in \mathbb{N} \,]$ and $[\, d_j : j \in \mathbb{N} \,]$ of H_2 resp. H_1 such that

$$I^* x = \sum_{j \in \mathbb{N}} \alpha_j \cdot \langle \, x, e_j \, \rangle_2 \, d_j$$

for all $x \in H_2$. Without loss of generality we can assume that $\alpha_i \geq \alpha_j > 0$ for $i \leq j$. Moreover $(\alpha_j)_j \in l^1(\mathbb{N})$. Now we consider the operator:

$$Bx := \sum_{j \in \mathbb{N}} \alpha_j \cdot \langle \, x, e_j \, \rangle_2 \, e_j.$$

Then $B \in \mathcal{L}(H_2)$ is nuclear and positive. Fix a point $z = \sum_{j \in \mathbb{N}} \langle z, e_j \rangle_2 \, e_j \in H_1$, then we have $z \in B^{\frac{1}{2}} H_2$ if there exists $y \in H_2$ such that $\alpha_j^{\frac{1}{2}} \, \langle y, e_j \rangle_2 = \langle z, e_j \rangle_2$ for all $j \in \mathbb{N}$. In order to prove the existence of such an $y \in H_2$ it is enough to show $(\alpha_j^{-\frac{1}{2}} \langle z, e_j \rangle_2)_j \in l^2(\mathbb{N})$ which follows from

$$\alpha_j^{-1} | \, \langle z, e_j \rangle_2 \, |^2 = \alpha_j^{-1} | \, \langle Iz, e_j \rangle_2 \, |^2 = \alpha_j^{-1} | \, \langle z, I^* e_j \rangle_1 \, |^2 \leq \alpha_j^{-1} \| \, z \, \|_1^2 \, \| \, I^* e_j \, \|_1^2 = \alpha_j \cdot \| \, z \, \|_1^2.$$

Moreover, by summarizing this inequality over $j \in \mathbb{N}$ we obtain:

$$\| B^{-\frac{1}{2}} z \|_2^2 = \Big\| \sum_{j\in\mathbb{N}} \alpha_j^{-\frac{1}{2}} \cdot \langle z, e_j \rangle_2 \, e_j \Big\|_2^2 = \sum_{j\in\mathbb{N}} \alpha_j^{-1} \cdot | \langle z, e_j \rangle_2 |^2 \leq \| z \|_1^2 \sum_{j\in\mathbb{N}} \alpha_j.$$

Because of $(\alpha_j)_j \in l^1(\mathbb{N})$ this proves the continuity of $H_1 \hookrightarrow (B^{\frac{1}{2}} H_2, \| B^{-\frac{1}{2}} \cdot \|_2)$ together with the norm estimate $\| J \| \leq \sum_j \alpha_j = \mathrm{tr}(B)$. $\square$

Remark 5.4.1 Consider the Hilbert space triple $(H_j, \langle \cdot, \cdot \rangle_j)$ for $j = 1, 2, 3$ where $H_j \subset H_l$ whenever $j < l$. Assume that the embeddings $I_j : H_j \to H_{j+1}$ are nuclear for $j = 1, 2$ and define $I_3 := I_2 \circ I_1 : H_1 \to H_3$. Because the adjoint operators $I_2^* : H_3 \to H_2$ and $I_1^* : H_2 \to H_1$ are nuclear as well it follows with *Theorem 8.3.3.* in [120] that:

$$I_2^* \in l^1\big(H_3, H_2 \big) \qquad \text{and} \qquad I_1^* \in l^1\big(H_2, H_1 \big).$$

Using Theorem 8.2.7. in [120] we conclude that $I_3^* = I_1^* \circ I_2^* \in l^{\frac{1}{2}}(H_3, H_1)$. Fix orthonormal bases $[\, \tilde{e}_j : j \in \mathbb{N} \,]$ and $[\, \tilde{d}_j : j \in \mathbb{N} \,]$ of H_3 resp. H_1 such that (as in Lemma 5.4.1) the operator I_3^* has the form:

$$I_3^* x = \sum_{j\in\mathbb{N}} \tilde{\alpha_j} \cdot \langle\, x, \tilde{e_j} \,\rangle_3 \, \tilde{d}_j, \qquad (x \in H_3)$$

where $\tilde{\alpha_j} > 0$ for all $j \in \mathbb{N}$ and $\tilde{\alpha_j} \leq \tilde{\alpha_l}$ whenever $j \geq l$. By Theorem 8.3.2. in [120] we find that $\tilde{\alpha_j}$ is the *jth approximation number* of I_3^* and so it follows from $I_3^* \in l^{\frac{1}{2}}(H_3, H_1)$ that it holds $(\tilde{\alpha_j})_j \in l^{\frac{1}{2}}(\mathbb{N})$. We conclude that the positive nuclear operator B which we have constructed in Lemma 5.4.1 can be chosen such that $\mathrm{tr}(B^{\frac{1}{2}}) < \infty$ in the case where the embedding I_3 factorizes into two nuclear embeddings I_1 and I_2.

Let E be a $\mathcal{DFN}$-space. According to Lemma 5.3.2 and the following remark E can be represented as a nuclear inductive spectrum $\{ H_n, \pi \}_{n\in\mathbb{N}_0}$ of separable complex Hilbert spaces H_n in the category of locally convex spaces and continuous mappings. In addition all the maps $\pi_{n,n+1} : H_n \hookrightarrow H_{n+1}$ are dense nuclear embeddings with $\| \pi_{n,n+1} \| < 1$ for $n \in \mathbb{N}_0$. Without loss of generality we can assume that each embedding $\pi_{n,n+1}$ factorizes in two nuclear embeddings.

According to Lemma 5.4.1 and Remark 5.4.1 there is a sequence $(B_n)_{n\in\mathbb{N}}$ of positive nuclear operators with $B_n \in \mathcal{L}(H_n)$ and $\mathrm{tr}(B_n^{\frac{1}{2}}) < \infty$ such that $H_{n-1} \subset B_n^{\frac{1}{2}} H_n$ for all $n \in \mathbb{N}$. Moreover, the embeddings

$$H_{n-1} \hookrightarrow \Big(B_n^{\frac{1}{2}} H_n, \| B_n^{-\frac{1}{2}} \cdot \|_n \Big)$$

are continuous. Let ν_n, $n \in \mathbb{N}$ be the normed *Gaussian measure* on H_n with the *correlation operator* B_n. Then all the assumptions on B_n in section 5.1 hold. With the continuous embedding

$$\pi_n : H_n \hookrightarrow \bigcup_{n\in\mathbb{N}} H_n = E = \overrightarrow{\lim_{n\to\infty}} H_n$$

and with a sequence $(\gamma_n)_{n\in\mathbb{N}} \in l^1(\mathbb{N})$ such that $\gamma_n > 0$ for all $n \in \mathbb{N}$ we can consider the finite Borel-measure ν on E defined by:

$$\nu(A) := \sum_{n\in\mathbb{N}} \gamma_n \cdot \nu_n\big(\, \pi_n^{-1}(A)\,\big), \qquad \big(\, A \in \mathcal{B}(E)\,\big). \tag{5.4.6}$$

Here $\mathcal{B}(E)$ denotes the Borel σ-algebra of E and we have used the continuity of π_n for all $n \in \mathbb{N}$. With $x \in H_n$ and $r > 0$ let $K^{(n)}(r,x)$ be the ball in H_n with radius r centered in x.

Lemma 5.4.2 *Consider $E := \bigcup_{n\in\mathbb{N}} H_n$ with the inductive limit topology. With the nuclear embeddings $\pi_{n,n+1} : H_n \hookrightarrow H_{n+1}$ we assume that $\|\, \pi_{n,n+1}\,\| \le 1$ for all $n \in \mathbb{N}$. If $U \subset E$ is an open subset, then we have for all $n \in \mathbb{N}$:*

(a) $\partial_n\,(\, U \cap H_n\,) \subset \partial_{n+1}\,(\, U \cap H_{n+1}\,)$.

(b) Let $T \subset U \cap H_n$, then $dist_{n+1}\,[\, T, \partial_{n+1}\,(\, U \cap H_{n+1}\,)\,] \le dist_n\,[\, T, \partial_n(\, U \cap H_n\,)\,]$.

Here ∂_n denotes the boundary and $dist_n$ the distance in the Hilbert space H_n.

Proof In order to prove (a) let $x \in \partial_n(\, U \cap H_n\,)$ and $r > 0$. Then we have by definition:

$$(i):\ \ K^{(n)}(r,x) \cap U \ne \emptyset \qquad \text{and} \qquad (ii):\ \ K^{(n)}(r,x) \cap (\, H_n \setminus U\,) \ne \emptyset.$$

Because of $\|\, \pi_{n,n+1}\,\| \le 1$ it follows that $K^{(n)}(r,x) \subset K^{(n+1)}(r,x)$ and from (i) we conclude that $K^{(n+1)}(r,x) \cap U \ne \emptyset$. Finally using $H_n \subset H_{n+1}$ we have for all $n \in \mathbb{N}$

$$\emptyset \ne K^{(n+1)}(r,x) \cap (\, H_{n+1} \setminus U\,)\,.$$

and it follows $x \in \partial_{n+1}\,(U \cap H_{n+1})$. For the proof of (b) we apply (a) and $\|\cdot\|_{n+1} \le \|\cdot\|_n$:

$$\begin{aligned}\operatorname{dist}_{n+1}[\,T, \partial_{n+1}(U\cap H_{n+1})\,] &= \inf\Big\{\ \|x-y\|_{n+1} : x \in T,\ y \in \partial_{n+1}(U\cap H_{n+1})\ \Big\}\\ &\le \inf\Big\{\ \|x-y\|_n : x \in T,\ y \in \partial_n(U\cap H_n)\ \Big\}\\ &= \operatorname{dist}_n[\,T, \partial_n(U\cap H_n)\,]\,.\end{aligned}$$

□

Theorem 5.4.1 *Let E be as in Lemma 5.4.2 and $U \subset E$ an open subset. Then the measure ν is a $\mathcal{NF}_2$-measure where $\mathcal{F} = \mathcal{H}(U)$.*

Proof Let $K_1 \subset U$ be compact. Then we can choose $n \in \mathbb{N}$ such that $K_1 \subset U \cap H_n$ is a compact set in the topology of H_n. Because K_1 also is compact in $U \cap H_{n+2}$, it follows that:

$$\tilde{\epsilon} := \operatorname{dist}_{n+2}\big[\, K_1, \partial_{n+2}\,(\, U \cap H_{n+2}\,)\ \big] > 0.$$

Fix $0 < \epsilon < \tilde{\epsilon}$ and for $j \ge n$ define

$$U(\,\epsilon, j\,) := \Big\{\ y \in U \cap H_j :\ \operatorname{dist}_j\big(\, y, K_1\,\big) < \epsilon\ \Big\}.$$

By Lemma 5.4.2 (b) we have the inclusions

$$U(\,\epsilon, n\,) \subset U(\,\epsilon, n+1\,) \subset U(\,\epsilon, n+2\,) \subset U \cap H_{n+2}$$

and because $U(\epsilon, n+1)$ is bounded in H_{n+1} by nuclearity of $\pi_{n+1,n+2}$ it is relative compact in H_{n+2}. Now, define K_2 to be the closure of $U(\epsilon, n+1)$ in H_{n+2}. Then we have

$$K_1 \subset K_2 \subset U \cap H_{n+2} \subset U$$

and K_2 is a compact subset of U. Let $f \in \mathcal{H}(U)$, then by restriction it defines a holomorphic function on $U(\epsilon, n+1)$ which is bounded due to the compactness of $K_2 \subset U$. Hence by Proposition 5.2.2 for each $x \in K_1$ there is an open neighborhood $W_x \subset U \cap B_{n+1}^{\frac{1}{2}} H_{n+1}$ with the topology induced by $\|\, B_{n+1}^{-\frac{1}{2}} \cdot \|_{n+1}$ and a constant $M_x > 0$ such that

$$\sup\Big\{\, |\, f(z)\,| : z \in W_x \,\Big\} \leq M_x \cdot \|\, f\,\|_{L^2(U(\epsilon,n+1),\, \nu_{n+1})}. \tag{5.4.7}$$

By the continuity of the embedding $H_n \hookrightarrow (\, B_{n+1}^{\frac{1}{2}} H_{n+1}, \|B_{n+1}^{-\frac{1}{2}} \cdot \|_{n+1}\,)$ (see Lemma 5.4.1) the sets $\tilde{W}_x := W_x \cap H_n$ are open in H_n and equation (5.4.7) holds with $\tilde{W}_x$ instead of W_x.

Finally, due to the compactness of K_1 in $U \cap H_n$ and because of $U(\,\epsilon, n+1\,) \subset K_2$ we can find $\tilde{M} > 0$ with

$$\sup\Big\{\, |\, f(z)\,| : z \in K_1 \,\Big\} \leq \tilde{M} \cdot \|\, f\,\|_{L^2(K_2,\nu_{n+1})} \leq \frac{\tilde{M}}{\gamma_{n+1}} \|\, f\,\|_{L^2(K_2,\nu)}.$$

This implies Theorem 5.4.1. □

5.5 The Nuclearity of $\mathcal{H}(U)$

Let E be a $\mathcal{DFN}$-space and $U \subset E$ an open subset. In [25], [141] it was proved that the space $\mathcal{H}(U)$ of all continuous holomorphic functions on U with the *compact-open topology* is a $\mathcal{FN}$-space (nuclear Fréchet space). Using the $\mathcal{NH}(U)_2$-measure ν on U which we have constructed above as well as a generalization of a result in [120], we give a new proof of this fact.

Theorem 5.5.1 *Let U be an open set in a $\mathcal{DFN}$-space E and let $\mathcal{F} := \mathcal{N}(U)$ be a locally convex subspace of the Fréchet space $\mathcal{C}(U)$. Then $\mathcal{F}$ is a nuclear space if and only if $\mathcal{F}$ is a $\mathcal{NF}_p$-space with $1 \leq p \leq 2$.*

Proof Let $\mathcal{N}(U)$ be a nuclear space. Then for each compact subset $K \subset U$ we consider the normed spaces

$$\mathcal{N}(K) := \Big\{\, f_{|K} : K \to \mathbb{C} : f \in \mathcal{N}(U),\ \|\, f\,\|_K := \sup\{\, |\, f(x)\,| : x \in K\,\}\,\Big\}.$$

By assumption using the nuclearity of $\mathcal{N}(U)$ we conclude that there is a compact set H with $K \subset H \subset U$ such that the restriction map

$$\pi_{H,K} : \mathcal{N}(H) \to \mathcal{N}(K) : f \mapsto f_{|K}$$

is nuclear. This implies that there are sequences $(\phi_j)_{j\in\mathbb{N}} \subset \mathcal{N}(H)'$ in the topological dual of $\mathcal{N}(H)$ and $(g_j)_{j\in\mathbb{N}} \subset \mathcal{N}(K)$ with $\| g_i \|_K \leq 1$ such that:

$$f(\, z\,) = \sum_{j=1}^{\infty} \phi_j(\, f\,) \cdot g_j(\, z\,) \qquad \text{and} \qquad \sum_{j=1}^{\infty} \| \, \phi_j \, \|_{\mathcal{N}(H)'} < \infty \tag{5.5.1}$$

for all $f \in \mathcal{N}(H)$ and all $z \in K$. By the Hahn-Banach Theorem we can choose a sequence of Radon measures $(\mu_j)_{j\in\mathbb{N}} \subset \mathcal{C}(H)'$ extending ϕ_j to $\mathcal{C}(H)$ and $\| \mu_j \|_{\mathcal{C}(H)'} = \| \phi_j \|_{\mathcal{N}(H)'}$ for all $j \in \mathbb{N}$. By (5.5.1) the definition $\mu_H := \sum_{j=1}^{\infty} | \mu_j |$ leads to a positive Radon measure μ_H on H. For each $z \in K$ and $f \in \mathcal{N}(H)$ we obtain:

$$|\, f(z) \,| \leq \sum_{j=1}^{\infty} \big|\, \phi_j(f) \,\big| \leq \sum_{j=1}^{\infty} \big|\, \mu_i \,\big|(\, |f| \,) = \int_H |\, f \,| \, d\mu_H.$$

According to Lemma 5.3.1 U is hemi-compact and so we can fix a fundamental sequence $(K_i)_{i\in\mathbb{N}} \subset U$ of compact sets. In the described way we can find a sequence $(H_i)_{i\in\mathbb{N}} \subset U$ of compact sets H_i with $K_i \subset H_i \subset U$ and positive Radon measures $(\tilde{\mu}_i)_{i\in\mathbb{N}}$ with:

$$\sup \Big\{ \, |\, F(z) \,| : z \in K_i \, \Big\} \leq \int_{H_i} |\, F \,| \, d\tilde{\mu}_i, \qquad \forall\, F \in \mathcal{N}(U), \quad \forall\, i \in \mathbb{N}.$$

Moreover, if we define $\mu := \sum_{j=1}^{\infty} 2^{-j} \cdot \tilde{\mu}_j(\, H_j \,)^{-1} \cdot \tilde{\mu}_i$ we obtain a finite positive Radon measure μ on U with:

$$\begin{aligned} \sup \Big\{ \, |\, F(z) \,| : z \in K_j \, \Big\} &\leq 2^j \cdot \tilde{\mu}_j\big(\, H_j \,\big) \int_{H_j} |\, F \,| \, d\mu \\ &\leq 2^j \cdot \tilde{\mu}_j\big(\, H_j \,\big)^{1+\frac{1}{q}} \left[\int_{H_j} |\, F \,|^p \, d\mu \right]^{\frac{1}{p}} \end{aligned}$$

where $j \in \mathbb{N}$ and $1 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. In other words μ is a $\mathcal{NF}_p$-measure.

Now, assume that $\mathcal{F} = \mathcal{N}(U)$ is a $\mathcal{NF}_p$-space with $1 \leq p \leq 2$. Then there is a measure μ on U such that the *compact-open* topology on $\mathcal{N}(U)$ is generated by the semi-norms

$$p_K(\, f\,) := \left[\int_K |\, f \,|^2 \, d\mu \right]^{\frac{1}{2}}, \qquad f \in \mathcal{N}(U)$$

where K ranges over the compact subsets of U. Denote by $\mathcal{N}^{\,2}_{\mu}(K)$ the closure of $\mathcal{N}(K)$ in the space $L^2(K, \mu)$. In order to prove the nuclearity of $\mathcal{N}(U)$ it is enough to show that for each compact set $K \subset U$ there is a compact set H with $K \subset H \subset U$ such that the restriction operator

$$\tilde{\pi}_{H,K} : \mathcal{N}^{\,2}_{\mu}(H) \rightarrow \mathcal{N}^{\,2}_{\mu}(K)$$

is *quasi-nuclear* (Hilbert Schmidt). For each compact set $K \subset U$ we can find a compact set H with $K \subset H \subset U$ such that for all $z \in K$ the point evaluation $\delta_z : \mathcal{N}(H) \rightarrow \mathbb{C}$ defined by $\delta_z(f) := f(z)$ is continuous with respect to the $L^2(H, \mu)$-topology. Hence for all $z \in K$ the mapping δ_z can be considered as a continuous functional on $\mathcal{N}^{\,2}_{\mu}(H)$. Moreover, there is $M > 0$ with:

$$\| \delta_z \| = \sup \Big\{ \, | f(z) | : f \in \mathcal{N}(H), \ \| f \|_{L^2(H,\mu)} \leq 1 \, \Big\} \leq M, \qquad (z \in K).$$

We denote by $\langle \cdot, \cdot \rangle_H$ the inner product of $L^2(H, \mu)$. Then by the *Hahn-Banach Theorem* for each $z \in K$ there is $f_z \in L^2(H, \mu)$ such that for all $g \in \mathcal{N}^{\,2}_{\mu}(H)$

$$\delta_z(\, g \,) = \langle \, g, f_z \, \rangle_H, \qquad \text{and} \qquad \| \, f_z \, \|_{L^2(H,\mu)} = \| \, \delta_z \, \| \leq M.$$

Choose an orthonormal basis $[\, e_j : j \in \mathbb{N} \,] \subset \mathcal{N}(H)$ of $\mathcal{N}^2_{\mu}(H)$. Then with the *Bessel inequality* we obtain for all $z \in K$

$$\sum_{j=1}^{\infty} \Big| \, e_j(z) \, \Big|^2 = \sum_{j=1}^{\infty} \Big| \, \delta_z(e_j) \, \Big|^2 = \sum_{j=1}^{\infty} \Big| \, \langle e_j, f_z \rangle_H \, \Big|^2 \leq \| \, f_z \, \|^2_{L^2(H,\mu)} = \| \, \delta_z \, \|^2 \leq M^2. \quad (5.5.2)$$

By integrating inequality (5.5.2) with respect to μ over the compact set K and using the monotone convergence theorem we conclude that

$$\sum_{j=1}^{\infty} \| \, \tilde{\pi}_{H,K} e_j \, \|^2_{L^2(K,\mu)} \leq M^2 \cdot \mu(K) < \infty.$$

Hence the restriction map $\tilde{\pi}_{H,K} : \mathcal{N}^2_{\mu}(H) \rightarrow \mathcal{N}^2_{\mu}(K)$ is *quasi-nuclear.* $\square$

In Theorem 5.4.1 we have proved that ν is a $\mathcal{NF}_2$-measure for $\mathcal{F} := \mathcal{H}(U)$ where U is an open subset of a $\mathcal{DFN}$-space. Theorem 5.5.1 now implies:

Theorem 5.5.2 *Let $U \subset E$ be an open subset of a $\mathcal{DFN}$-space E. Then, endowed with the compact-open topology, the space $\mathcal{H}(U)$ of all holomorphic functions on U is a nuclear Fréchet space ($\mathcal{FN}$-space).*

Chapter 6

The Cauchy-Weil theorem and abstract Hardy spaces for open subsets of $\mathcal{DFN}$-spaces

Let E be the dual of a Fréchet nuclear space ($\mathcal{DFN}$-space) and $U \subset E$ an open subset. We denote by $\mathcal{H}^\infty(U)$ the Banach algebra of all bounded holomorphic functions on U. For any closed subalgebra $\mathcal{A}$ of $\mathcal{H}^\infty(U)$ which separates points let $\mathcal{S}_\mathcal{A}$ be its abstract Shilov boundary. We prove the existence of an integral formula for $f \in \mathcal{A}$ which is a generalization of a result in [69] to U in an infinite dimensional nuclear space. Namely, given any $\mathcal{NF}_2$-measure μ on U where $\mathcal{F} = \mathcal{H}(U)$ is the nuclear Fréchet space of all holomorphic functions on U (cf. Definition 5.4.1), there is a finite Radon measure ν on $\mathcal{S}_\mathcal{A}$ and a complex-valued kernel Φ_μ on $U \times \mathcal{S}_\mathcal{A}$ such that $\Phi_\mu(z, \cdot)$ is ν-integrable for all $z \in U$ and $\Phi_\mu(\cdot, x)$ is holomorphic for all $x \in \mathcal{S}_\mathcal{A}$. Moreover, the point evaluation on $\mathcal{A}$ in $z \in U$ is given by an integral operator on $\mathcal{A}$ considered as a subset of $\mathcal{C}(\mathcal{S}_\mathcal{A})$ with kernel $\Phi_\mu(z, \cdot)$. We prove an estimate on the growth of Φ_μ of the form:

$$\| \, \Phi_\mu(\, z, \cdot) \, \|_{L^1(\mathcal{S}_\mathcal{A}, \nu)} \leq C \cdot \| \, \mathrm{Eval}(\, z \,) \, \|_{\mathcal{H}^2(U,\mu)'}.$$

Here $C > 0$ is a number independent of z and *Eval* denotes the point evaluation in the generalized Bergman space $\mathcal{H}^2(U, \mu) := L^2(U, \mu) \cap \mathcal{H}(U)$. In the last section we define an abstract Hardy space $\mathcal{H}^2(\mathcal{S}_\mathcal{A}, \Theta)$ of holomorphic functions on U by using the nuclearity of $\mathcal{H}(U)$ and following an idea in [72].

It was shown in [25], [142] and chapter 5 that the space $\mathcal{H}(U)$ of all holomorphic functions on U equipped with the *compact-open topology* is a nuclear Fréchet space (for the notion of holomorphic functions on U we refer to [51]). With any measure space $(X, \mathcal{S}, \nu)$ and $1 \leq p < \infty$ we define $B_p := L^p(X, \nu)$. Given a holomorphic B_p-valued function $G \in \mathcal{H}(U, B_p)$ we prove by applying a result due to A. Grothendieck [81]:

Theorem 1 There is a kernel function $\Phi : U \times X \to \mathbb{C}$ with the following properties:

(a) For all $z \in U$ we have $G(z) = [\, X \ni x \mapsto \Phi(z, x) \in \mathbb{C} \,] \in B_p$.

(*b*) For all $x \in X$ the map $U \ni z \mapsto \Phi(z, x) \in \mathbb{C}$ is holomorphic.

The existence of Φ is closely related to the nuclearity of $\mathcal{H}(U)$. We give an example which shows that in the case of an open subset V in an infinite dimensional separable complex Hilbert space H a kernel Φ with (*a*) and (*b*) above in general does not exist. As a corollary it follows that $\mathcal{H}(H)$ can not be a nuclear Fréchet space with respect to any topology which is finer than the topology of pointwise convergence on H.

In section 6.3 we show how to construct a holomorphic lifting of Banach space valued functions with certain growth condition by the use of $\mathcal{NF}_2$-measures. Before we state the result we give some notations.

In Definition 5.4.1 we have introduce the notion of $\mathcal{NF}_2$-spaces. We call a finite Borel measure μ on U a $\mathcal{NF}_2$-measure iff for each compact set $K \subset U$ there exists a compact set $H \subset U$ with $K \subset H$ and a constant $C > 0$ such that for all $f \in \mathcal{F} = \mathcal{H}(U)$ it holds:

$$\sup\Big\{ \mid f(z) \mid : z \in K \Big\} \leq C \cdot \left[\int_H \mid f \mid^2 d\mu \right]^{\frac{1}{2}}. \tag{6.0.1}$$

Let $\mathcal{MF}_2(U)$ be the space of all $\mathcal{NF}_2$ -measures on U. In the case where $\mathcal{MF}_2(U) \neq \emptyset$ we call U a $\mathcal{NF}_2$-space. In Theorem 5.5.1 we have proved a generalization of a result due to A. Pietsch to infinite dimensional domains (see [121]). Namely, that for the nuclearity of $\mathcal{F} = \mathcal{H}(U)$ it is sufficient and necessary that U is a $\mathcal{NF}_2$-space.

Let us fix a measure $\mu \in \mathcal{MF}_2(U)$, then by using the estimate (6.0.1) above it turns out that the space $\mathcal{H}^2(U, \mu)$ of all holomorphic functions which are square integrable with respect to μ is closed in $L^2(U, \mu)$. Let us denote by P_μ the orthogonal projection of $L^2(U, \mu)$ onto $\mathcal{H}^2(U, \mu)$. By general results P_μ is an integral operator with kernel $K_\mu : U \times U \to \mathbb{C}$.

With a pair of complex Banach spaces A and B over $\mathbb{C}$, a linear operator η of A onto B and $f \in \mathcal{H}(U, B)$ - the space of holomorphic B-valued functions on U - we prove the existence of $\lambda \in \mathcal{H}(U, A)$ which fulfills a certain growth condition and solves the lifting problem $f = \eta \circ \lambda$:

$$\begin{tikzcd}
& A \arrow[d, "\eta"'] \\
U \arrow[ur, "\lambda"] \arrow[r, "f"'] & B \arrow[u, bend right, "\xi"']
\end{tikzcd}$$

where $\xi : B \to A$ is a continuous (not necessary linear) left inverse of η which exists by a result due to R.G. Bartle and L.M. Graves (see [6]). Again the proof essentially uses the nuclearity of $\mathcal{H}(U)$. We obtain λ by integrating the continuous lifting $\tilde{\lambda} := \xi \circ f$ with respect to the integral kernel $K_\mu : U \times U \to \mathbb{C}$ of the projection P_μ.

Theorem 2 There is $\lambda \in \mathcal{H}(U, A)$ solving the lifting problem $f = \eta \circ \lambda$. Moreover, for any measure $\mu \in \mathcal{MF}_2(U)$ such that $\| f \|_B$ is μ-square integrable over U we can choose λ with

$$\| \lambda(z) \|_A \leq c \cdot \| \operatorname{Eval}(z) \|_{\mathcal{H}^2(U,\mu)'} = c \cdot K_\mu(\, z, z \,)^{\frac{1}{2}}$$

where $Eval(z) \in \mathcal{H}^2(U,\mu)'$ is the evaluation in $z \in U$ and c is a suitable positive number depending only on f and η.

Let $\mu \in \mathcal{MF}_2(U)$, then for any $f \in \mathcal{H}^2(U,\mu)$ and $z \in U$ we have an integral formula for the evaluation given by:

$$f(\, z\,) = \int_U f \cdot K_\mu(\, z, \cdot\,)\, d\mu. \tag{6.0.2}$$

The question arises whether μ with property (6.0.2) can be concentrated to the boundary ∂U (or more generally to an abstract boundary of U) if we restrict ourselves to functions $f \in \mathcal{H}^2(U,\mu)$ which admit certain extensions to $\overline{U}$. Let $\mathcal{H}^\infty(U)$ be the Banach algebra of all bounded holomorphic functions on U and denote by $\mathcal{A} \subset \mathcal{H}^\infty(U)$ a closed subalgebra which separates the points of U with abstract Shilov boundary $\mathcal{S}_A$. Then we prove a generalization of the results in [69] to the infinite dimensional case:

Theorem 3 For any $\mu \in \mathcal{MF}_2(U)$, there is a finite positive Radon measure ν on $\mathcal{S}_\mathcal{A}$ and a kernel function $\Phi_\mu : U \times \mathcal{S}_\mathcal{A} \to \mathbb{C}$ with the properties:

- The map $U \ni z \mapsto \Phi_\mu(z,x) \in \mathbb{C}$ is holomorphic for all $x \in \mathcal{S}_\mathcal{A}$.
- For all $z \in U$ we have $\Phi_\mu(z,\cdot) \in L^1(\mathcal{S}_\mathcal{A},\nu)$.
- There is $c > 0$ with $\|\, \Phi_\mu(z,\cdot)\, \|_{L^1(\mathcal{S}_\mathcal{A},\nu)} \leq c \cdot \|\, \mathrm{Eval}(z)\, \|_{\mathcal{H}^2(U,\mu)'}$ for all $z \in U$.
- It holds $f(\, z\,) = \int_{\mathcal{S}_\mathcal{A}} x(\, f\,) \cdot \Phi_\mu(\, z, x\,)\, d\nu(x)$ for all $f \in \mathcal{A}$ and $z \in U$.

The proof of Theorem 3 involves our results on kernels in Theorem 1 as well as on holomorphic liftings in Theorem 2 stated above.

In the last part of the present chapter we define an abstract Hardy space $\mathcal{H}^2(\mathcal{S}_\mathcal{A},\Theta)$. By this we mean a closed subspace of $L^2(\mathcal{S}_\mathcal{A},\Theta)$ where Θ is a suitable measure on the Shilov boundary $\mathcal{S}_\mathcal{A}$ which densely contains $\delta[\mathcal{A}]$. Here $\delta : \mathcal{A} \to \mathcal{C}(\mathcal{S}_\mathcal{A})$ is given by $\delta(f)[x] := x(f)$ for all $x \in \mathcal{S}_\mathcal{A}$ and $f \in \mathcal{A}$. Moreover, we claim that there is a kernel $K : U \times \mathcal{S}_\mathcal{A} \to \mathbb{C}$ such that

(i) The map $K(z,\cdot) : \mathcal{S}_\mathcal{A} \to \mathbb{C}$ is bounded for fixed $z \in U$,

(ii) For each compact set $H \subset U$ and $f \in \mathcal{A}$ it holds:

$$\sup\Big\{\, |\, f(z)\,| : z \in H\,\Big\} \leq \|\, \delta(f)\, \|_{L^2(\mathcal{S}_\mathcal{A},\Theta)} \cdot \sup\Big\{\, \|\, K(z,\cdot)\, \|_{L^2(\mathcal{S}_\mathcal{A},\Theta)} : z \in H\,\Big\}.$$

From (ii) it is clear that we can identify $\mathcal{H}^2(\mathcal{S}_\mathcal{A},\Theta)$ with a space of holomorphic functions on U. We want to mention that this construction is even new and leads to non-trivial results for regions U in the complex plane with arbitrary boundary (or the case where U is the unit disc) and subalgebras $\mathcal{A} \subset \mathcal{H}^\infty(U)$. Given finite measures $\mu_1, \mu_2 \in \mathcal{MF}_2(U)$ where $\mathcal{F} = \mathcal{H}(U)$ such that there exists a nuclear embedding $\mathcal{H}^2(U,\mu_1) \hookrightarrow \mathcal{H}^2(U,\mu_2)$ we can construct an abstract Hardy space $\mathcal{H}^2(\mathcal{S}_\mathcal{A},\Theta)$ in the sense described above which has a quasi-nuclear (Hilbert-Schmidt) embedding into $\mathcal{H}^2(U,\mu_2)$.

6.1 Nuclearity and generalized Bergman spaces

As a general assumption in the present chapter let E be a $\mathcal{DFN}$-space and $U \subset E$ an open subset. It is well-known that a function f from U to any topological space is continuous on U iff its restriction to each compact subset $K \subset U$ equipped with the induced topology is continuous (cf. [116] or Lemma A.1.1 and Corollary A.1.3 in our appendix). We call the class of topological spaces sharing this property *k-spaces*. Further, we have mentioned in Lemma 5.3.1 that U admits a fundamental sequence of compact subsets. We call U *hemi-compact* and using these properties it easily can be shown that, equipped with the *compact-open topology*, the linear space $\mathcal{F} := \mathcal{H}(U)$ of complex valued holomorphic functions on U is a Fréchet space. By Theorem 5.5.2 (cf. [25], [142]) a result similar to the case of open subsets in $\mathbb{C}^n$ holds for $\mathcal{DFN}$-spaces. Namely, the Fréchet topology on $\mathcal{F}$ is nuclear.

Generalizing a result due to A. Pietsch in [121] to the infinite dimensional setting it was shown in Theorem 5.5.1 that the nuclearity of $\mathcal{F}$ implies that $\mathcal{MF}_2(U) \neq \emptyset$ and so according to Definition 5.4.1 it follows that U is a $\mathcal{NF}_2$-space. In chapter 5 we explicitly have given the construction of a finite measure $\mu \in \mathcal{MF}_2(U)$ on U by using infinite dimensional Gaussian measures on Hilbert spaces. It is easy to see that μ is not unique; once we have proved its existence it can be chosen in the following way:

Lemma 6.1.1 *Let $\mathcal{MF}_2(U) \neq \emptyset$, then for any $g \in \mathcal{C}(U)$ there is a measure $\mu \in \mathcal{MF}_2(U)$ such that $g \in L^2(U, \mu)$.*

Proof Let $\gamma \in \mathcal{MF}_2(U)$ and define a measure μ by $d\mu := [\, 1 + |\, g\,|^2\,]^{-1}\, d\gamma$. Because γ is finite by definition it follows that $g \in L^2(U, \mu)$. For any compact set $K \subset U$ there is a compact set $H \subset U$ with $K \subset H$ and $C > 0$ such that (5.4.1) holds with $p = 2$, all $f \in \mathcal{F}$ and with γ instead of μ. We define $D := \sup\{\, 1 + |\, g(x)\,|^2 : x \in H\,\} < \infty$. Then it follows:

$$\sup\Big\{\, |\, f(z)\,| : z \in K\,\Big\} \leq C \cdot D^{\frac{1}{2}} \cdot \left[\, \int_H |\, f\,|^2\, d\mu\, \right]^{\frac{1}{2}}$$

for all $f \in \mathcal{F}$ and by definition we conclude that $\mu \in \mathcal{MF}_2(U)$. □

Let $\mu \in \mathcal{MF}_2(U)$, then by our remarks above it is easy to see that the linear space

$$\mathcal{H}^2(U, \mu) := L^2(U, \mu) \cap \mathcal{H}(U)$$

is closed in $L^2(U, \mu)$. Moreover, for $z \in U$ the evaluation $Eval(z) \in \mathcal{H}^2(U, \mu)'$ defined for all $f \in \mathcal{H}^2(U, \mu)$ by $[\, Eval(z)\,](f) := f(z)$ is a continuous functional. By standard arguments $\mathcal{H}^2(U, \mu)$ is a Hilbert space with reproducing kernel function $K_\mu : U \times U \to \mathbb{C}$ and $Eval(z) = \langle\, \cdot, K_\mu(\cdot, z\,)\,\rangle_{L^2(U,\mu)}$. Let us denote by

$$P_\mu : L^2(\, U, \mu\,) \to \mathcal{H}^2(\, U, \mu\,)$$

the orthogonal projection (Toeplitz projection) onto $\mathcal{H}^2(U,\mu)$. Then it is easy to see that P_μ is an integral operator on U with

$$[\, P_\mu f\,](\, z\,) = \int_U f\, K_\mu(\, z, \cdot)\, d\mu$$

for all $z \in U$ and $f \in L^2(U,\mu)$. The reproducing kernel K_μ is continuous on U in each variable separately, but using the nuclearity of $\mathcal{F} = \mathcal{H}(U)$ we also can prove:

Lemma 6.1.2 *For any $\mu \in \mathcal{MF}_2(U)$ the map $F_\mu : U \to \mathbb{R}^+$ defined by $F_\mu(z) := K_\mu(z,z)$ is continuous on U.*

Proof Because U is a k-space it is sufficient to prove the continuity of F_μ restricted to each compact set $K \subset U$. With the induced topology K is a metric space with metric d. Let us fix an orthonormal basis $[\,\varphi_j : j \in \mathbb{N}\,]$ of $\mathcal{H}^2(U,\mu)$, then the kernel K_μ has the form:

$$K_\mu(\, z, \zeta\,) = \sum_{j\in\mathbb{N}} \varphi_j(\, z\,) \cdot \overline{\varphi_j(\, \zeta\,)}.$$

By the Riesz-Fischer and the Riesz representation theorem we obtain for each $z \in U$:

$$\begin{aligned} K_\mu(z,z) = \sum_{j\in\mathbb{N}} |\, \varphi_j(z)\, |^2 &= \sup\left\{ \left| \left\langle a, (\, \varphi_j(z)\,) \right\rangle_{l^2(\mathbb{N})} \right|^2 : a \in l^2(\mathbb{N}),\ \|\, a\, \|_{l^2(\mathbb{N})} = 1 \right\} \\ &= \sup\left\{ |\, f(z)\, | \ : \ f \in \mathcal{H}^2(U,\mu),\ \|\, f\, \|_{L^2(U,\mu)} = 1 \right\}^2. \end{aligned} \tag{6.1.1}$$

Because μ is a $\mathcal{NF}_2$-measure the set $\mathcal{M} := \{\, f \in \mathcal{H}^2(U,\mu) : \|\, f\, \|_{L^2(U,\mu)} = 1\, \}$ is bounded in $\mathcal{F} := \mathcal{H}(U)$. Now, using the nuclearity of $\mathcal{H}(U)$ it follows that the space $\mathcal{M}_K := \{\, f_{|K} \ : \ f \in \mathcal{M}\, \}$ is relatively compact in

$$\mathcal{H}(K) := \left\{ g_{|K} : K \to \mathbb{C} : g \in \mathcal{H}(U),\ \|\, f\, \|_K := \sup\{\, |\, f(z)\, | : z \in K\, \} \right\}.$$

The compactness of $K \subset U$ and the theorem of Arzela Ascoli imply that $\mathcal{M}_K$ is an equicontinuous and bounded family. Hence for $\epsilon > 0$ and $z_1 \in K$ there is $\delta > 0$ such that for all $z_2 \in K$ with $d(\, z_1, z_2\,) < \delta$:

$$\sup\left\{ |\, f(z_1) - f(z_2)\, | : f \in \mathcal{M} \right\} < \epsilon.$$

Without loss of generality we now assume that $K_\mu(z_1,z_1) \geq K_\mu(z_2,z_2)$. Then the inequality (6.1.1) shows in the case of $d(\, z_1, z_2\,) < \delta$:

$$\begin{aligned} \left| K_\mu(z_1,z_1)^{\frac{1}{2}} - K_\mu(z_2,z_2)^{\frac{1}{2}} \right| &= \sup_{f\in\mathcal{M}} |\, f(z_1)\, | - \sup_{f\in\mathcal{M}} |\, f(z_2)\, | \\ &\leq \sup_{f\in\mathcal{M}} \left\{ \ |\, f(z_1) - f(z_2)\, | + |f(z_2)| \ \right\} - \sup_{f\in\mathcal{M}} |\, f(z_2)\, | \\ &\leq \sup_{f\in\mathcal{M}} |\, f(z_1) - f(z_2)\, | < \epsilon. \end{aligned}$$

This finally proves the continuity of $F_\mu : U \to \mathbb{R}^+$. □

6.2 Grothendiecks Theorem and Nuclearity

With a measure space $(X, \mathcal{S}, \nu)$ and $1 \leq p < \infty$ we define $B_p := L^p(X, \nu)$ and we consider the space $\mathcal{H}(U, B_p)$ of all holomorphic B_p-valued functions on U. Given $G \in \mathcal{H}(U, B_p)$ we construct a kernel $\Phi : U \times X \to \mathbb{C}$ with the properties:

(*a*) For all $z \in U$ we have $G(z) = [\, X \ni x \mapsto \Phi(z, x) \in \mathbb{C}\,] \in B_p$.

(*b*) For all $x \in X$ the map $U \ni z \mapsto \Phi(z, x) \in \mathbb{C}$ is holomorphic on U.

In order to prove the existence of such a map Φ we essentially use the fact that $\mathcal{H}(U)$ carries the topology of a nuclear Fréchet space together with an application of a theorem due to Grothendieck (see [72], [81]). Let Λ be a set and assume that $\mathcal{N}$ is a nuclear Fréchet space of complex valued functions on Λ with a topology finer than the topology of pointwise convergence.

Theorem 6.2.1 *(Grothendieck) Assume that B is a Banach space and let B' be its topological dual. Let $F : U \to B$ be a function with the weak extension property: For each continuous functional $\varphi \in B'$ there is $h \in \mathcal{N}$ such that*

$$h(\,\lambda\,) = \varphi \circ F(\,\lambda\,), \qquad\qquad (\,\lambda \in \Lambda\,).$$

Then F has an expansion $F(\lambda) = \sum_{j=1}^{\infty} \alpha_j \cdot h_j(\,\lambda\,) \cdot f_j$ where $(h_j)_j \subset \mathcal{N}$ and $(f_j)_j \subset B$ tends to zero. Moreover, the sequence $(\alpha_j)_j \subset \mathbb{R}^+$ can be chosen rapidly decreasing.

Remark 6.2.1 Let $\tilde{F} : U \to B'$ be a function with the weak extension property: For each $y \in B$ there is $\tilde{h} \in \mathcal{N}$ such that $\tilde{h}(\,\lambda\,) = F(\,\lambda\,)y$ for all $\lambda \in \Lambda$. Then $\tilde{F}$ has an expansion similar to the one in Theorem 6.2.1 with a sequence $(f_j)_j \in B'$ tending to zero.

In the following let us denote by $\|\cdot\|_p$ the B_p-norm, then we prove:

Proposition 6.2.1 *For $1 \leq p < \infty$ let $([\,f_j\,])_{j\in\mathbb{N}} \subset B_p$ and fix a sequence $\lambda = (\lambda_j) \in l^1(\mathbb{N})$ with $\lambda_j > 0$. Then there is a set $D_\lambda \subset X$ of measure zero such that for all $z \in X \setminus D_\lambda$ and for almost all $j \in \mathbb{N}$ the inequality $|f_j(z)| \leq \lambda_j^{-\frac{1}{p}} \cdot \|\, f_j\,\|_p$ holds.*

Proof Because for $[\,f_j\,] \in B_p$ the function $|\,f_j\,|^p$ is ν-integrable over X it is sufficient to prove Proposition 6.2.1 in the case $p = 1$. Let $t > 0$ and assume that $g \in L^1(X, \nu)$. With the measurable set $T_{t,g} := \{\, x \in X : |\,g(x)\,| > t \cdot \|\,g\,\|_1 \,\}$ we obtain:

$$\nu\,(\,T_{t,g}\,) \cdot t \cdot \|\,g\,\|_1 \leq \int_X |\,g\,|\,d\nu = \|\,g\,\|_1. \tag{6.2.1}$$

In case $g = 0$ we have $\nu(T_{t,g}) = 0$ and for $g \neq 0$ it follows from (6.2.1) that $\nu(T_{t,g}) \leq t^{-1}$. With the sequence $(f_j)_j$ we obtain:

$$\nu\Big(\bigcup_{j\in\mathbb{N}} T_{\lambda_j^{-1}, f_j}\Big) \leq \sum_{j\in\mathbb{N}} \lambda_j < \infty.$$

Hence the set $D_\lambda := \bigcap_{l\in\mathbb{N}} \bigcup_{j=l}^{\infty} T_{\lambda_j^{-1}, f_j}$ has measure zero and for each point $z \in X \setminus D_\lambda$ we have the inequality $|\, f_j(z)\,| > \lambda_j^{-1} \cdot \|\, f_j \,\|_1$ for only finitely many $j \in \mathbb{N}$. $\square$

Applying Theorem 6.2.1 we can prove the existence of $\Phi : U \times X \to \mathbb{C}$ representing the holomorphic function $G \in \mathcal{H}(U, B_p)$.

Theorem 6.2.2 *Let $1 \le p < \infty$ and fix a holomorphic function $G \in \mathcal{H}(U, B_p)$. Then there exists $\Phi : U \times X \to \mathbb{C}$ with (a) and (b) above.*

Proof Let us choose $\mathcal{N} := \mathcal{H}(U)$ in Theorem 6.2.1 (for nuclearity see chapter 5, [25] or [142]). Then for each $\varphi \in B_p'$ we have $\varphi \circ G \in \mathcal{H}(U)$. Hence by Theorem 6.2.1 there are sequences $(\,[\, f_j \,]\,)_j \subset B_p$ and $(\, h_j \,)_j \subset \mathcal{H}(U)$ tending to zero and $(\alpha_j)_j \subset \mathbb{R}^+$ rapidly decreasing such that the following expansion holds:

$$G(\, z\,) = \sum_{j=1}^{\infty} \alpha_j \cdot h_j(\, z\,) \cdot [\, f_j \,]\,.$$

Choose $\lambda = (\lambda_j) \in l^1(\mathbb{N})$ with $\lambda_j > 0$ for all $j \in \mathbb{N}$ such that $(\,\lambda_j^{-\frac{1}{p}} \cdot \alpha_j\,)_j \in l^1(\mathbb{N})$ (e.g. we can define (λ_j) with $\lambda_j := j^{-2p}$ for $j \in \mathbb{N}$). For each $j \in \mathbb{N}$ let f_j be a representation of $[\, f_j \,] \in B_p$. Then with Proposition 6.2.1 and the boundedness of the numerical sequence $(\, \|\, [f_j] \,\|_p \,)_j$ we can determine a set $D_\lambda \subset X$ of measure zero and $C > 0$ such that for all $x \in X \setminus D_\lambda$ and for almost all $j \in \mathbb{N}$ the inequality $|\, \lambda_j^{\frac{1}{p}} \cdot f_j(x) \,| \le \|\, f_j \,\|_p \le C$ is valid. Let us write

$$G(\, z\,) = \sum_{j=1}^{\infty} \left(\lambda_j^{-\frac{1}{p}} \cdot \alpha_j \right) \cdot h_j(\, z\,) \cdot \left[\, \lambda_j^{\frac{1}{p}} \cdot f_j \,\right].$$

For any $(z, x) \in U \times X$ we define:

$$\Phi(\, z, x\,) := \begin{cases} \sum_{j=1}^{\infty} \alpha_j \cdot h_j(\, z\,) \cdot f_j(\, x\,) & x \in X \setminus D_\lambda \\ 0 & x \in D_\lambda. \end{cases} \tag{6.2.2}$$

Because of $(\,\lambda_j^{-\frac{1}{p}} \cdot \alpha_j\,)_j \in l^1(\mathbb{N})$ and from the fact that $(h_j)_j$ tends to zero uniformly on all compact subsets of U we conclude that the series in (6.2.2) as a function of z is convergent in $\mathcal{H}(U)$ for all $x \in X \setminus D_\lambda$ and so (b) follows. Moreover, for fixed $z \in U$ the expansion 6.2.2 converges to $G(z)$ a.e. on X and this finally implies (a). $\square$

The following example shows that the nuclearity condition on $\mathcal{H}(U)$ in Theorem 6.3.1 is essential for existence of $\Phi : U \times X \to \mathbb{C}$ with (a) and (b). We choose $p = 2$ and replace U in Theorem 6.3.1 by an infinite dimensional separable complex Hilbert space. In our construction below infinite dimensional Gaussian measures are involved; for details and some proofs we refer to chapter 5, [14] or [48].

Example 6.2.1 Let $(H, \langle\cdot,\cdot\rangle)$ be an infinite dimensional separable complex Hilbert space. Fix a Gaussian measure μ on H with nuclear positive *correlation operator* $B > 0$, i.e. μ is the unique probability measure on H with characteristic function

$$\chi_\mu(z) := \int_H \exp\Big(i \cdot \mathrm{Re}\langle z, x\rangle \Big)\, d\mu(x) = \exp\Big(- \| B^{\frac{1}{2}} z \|^2 \Big).$$

We denote by $[\, e_j : j \in \mathbb{N}\,]$ an orthonormal basis of H consisting of eigenvectors of B and we write $(\lambda_j)_j \in l^1(\mathbb{N})$ for the corresponding sequence of eigenvalues. In addition let us assume that the operator $B^{\frac{1}{2}} > 0$ is not nuclear (iff $(\sqrt{\lambda_j})_j \notin l^1(\mathbb{N})$), then by Proposition 5.1.1 it follows that $\mu(B^{\frac{1}{4}} H) = 0$. For $l \in \mathbb{N}$ we consider $g_l : H \times H \to \mathbb{C}$ defined by:

$$g_l(z, x) := \sum_{j=1}^{l} \lambda_j^{-\frac{1}{2}} \cdot \langle\, z, e_j \,\rangle \cdot \langle\, e_j, x \,\rangle\, .$$

Using the formula $\int_H \langle\cdot, e_j\,\rangle \cdot \langle\, e_l, \cdot\rangle\, d\mu = \lambda_j \cdot \delta_{j,l}$ (cf. [48]) we conclude for $l, m \in \mathbb{N}$ with $l \geq m$ and all $z \in H$ that:

$$\Big\| g_l(z, \cdot) - g_m(z, \cdot) \Big\|^2_{L^2(H,\mu)} = \sum_{j=m+1}^{l} |\, \langle\, z, e_j \,\rangle\, |^2$$

and so $G(z) := \lim_{l\to\infty} g_l(z, \cdot) \in L^2(H, \mu)$ is well-defined for each fixed $z \in H$. Moreover, the map $H \ni z \mapsto G(z) \in L^2(H, \mu)$ is isometric:

$$\| G(z) \|_{L^2(H,\mu)} = \lim_{l\to\infty} \Big\| g_l(z, \cdot) \Big\|_{L^2(H,\mu)} = \| z \|.$$

With $B_2 := L^2(H, \mu)$ it follows that $G \in \mathcal{L}(H, B_2) \subset \mathcal{H}(H, B_2)$. Let us assume that there is a function $\Phi : H \times H \to \mathbb{C}$ such that (a) and (b) above Theorem 6.2.1 holds. With $z_1, z_2 \in H$ and $\lambda \in \mathbb{C}$ we consider the sets:

$$\begin{aligned} A_{z_1,z_2,} : &= \Big\{\, x \in H : \Phi(z_1 + z_2, x) - \Phi(z_1, x) - \Phi(z_2, x) = 0 \,\Big\}, \\ A_{z_1,\lambda} : &= \Big\{\, x \in H : \Phi(\lambda \cdot z_1, x) - \lambda \cdot \Phi(z_1, x) = 0 \,\Big\}. \end{aligned}$$

Because $G : H \to B_2$ is linear, we conclude that $\mu(A_{z_1,z_2}) = \mu(A_{z_1,\lambda}) = 1$. Choose countable dense subsets $D_1 \subset H$ and $D_2 \subset \mathbb{C}$ and define

$$A := \bigcap_{z_1,z_2\in D_1} A_{z_1,z_2} \cap \bigcap_{z_1\in D_1, \lambda\in D_2} A_{z_1,\lambda}.$$

Then $\mu(A) = 1$ and by the continuity of $H \ni z \mapsto \Phi(z, x)$ it follows that $\Phi(\cdot, x)$ is linear for all $x \in A$. Let us define $\tilde{\Phi}(z, x) := \Phi(z, x)$ for all $(z, x) \in H \times A$ and $\tilde{\Phi}(z, x) := 0$ in the case $x \in H \setminus A$. By this we have constructed a function with (a) such that $\tilde{\Phi}(\cdot, x)$ is a continuous functional on H for all $x \in H$. Hence there is a map $f : H \to H$ such that for fixed $x \in H$:

$$\tilde{\Phi}(\, z, x \,) = \langle\, z, f(x) \,\rangle = \sum_{j=1}^{\infty} \langle\, e_j, f(x) \,\rangle \cdot \langle\, z, e_j \,\rangle\,.$$

In particular, for $l \in \mathbb{N}$ there is a set $C_l \subset H$ of measure one such that for all $x \in C_l$ the equality:

$$\langle\, e_l, f(x) \,\rangle = \tilde{\Phi}(\, e_l, x \,) = G(\, e_l, x \,) = \lambda_l^{-\frac{1}{2}} \,\langle\, e_l, x \,\rangle$$

holds. Hence $\tilde{\Phi}(z,x) = \sum_{j=1}^{\infty} \lambda_j^{-\frac{1}{2}} \cdot \langle e_j, x\rangle \cdot \langle z, e_j\rangle$ for all $z \in H$ and $x \in C := \bigcap_{l \in \mathbb{N}} C_l$ where $\mu(C) = 1$. In our construction above we have chosen the correlation B with $\mu(B^{\frac{1}{4}}H) = 0$. Thus $\tilde{C} := (H \setminus B^{\frac{1}{4}}H) \cap C$ has measure 1. In particular, we conclude that $\tilde{C} \neq \emptyset$ and with $x \in \tilde{C}$ it follows that:

$$\infty > \tilde{\Phi}(\, x, x \,) = \sum_{j=1}^{\infty} \Big|\, \Big\langle\, x, \lambda_j^{-\frac{1}{4}} e_j \,\Big\rangle \,\Big|^2 = \Big\|\, B^{-\frac{1}{4}} x \,\Big\|^2.$$

This is a contradiction to our assumption $x \notin B^{\frac{1}{4}}H$ and we conclude that a map Φ with (a) and (b) above does not exist.

It follows from Example 6.2.1, Theorem 6.2.1 and the proof of Theorem 6.2.2:

Corollary 6.2.1 *Let H be an infinite dimensional separable complex Hilbert space. Then $\mathcal{H}(H)$ is not nuclear with respect to any Fréchet topology which is finer than the topology of pointwise convergence on H.*

6.3 Holomorphic liftings for Banach space valued functions

Let $(\, A, \|\cdot\|_A \,)$ and $(\, B, \|\cdot\|_B \,)$ be Banach spaces and $\eta \in \mathcal{L}(A, B)$ a continuous operator of A onto B. It was shown in [6] that there is a continuous right inverse $\xi \in \mathcal{C}(B, A)$ of η, i.e. $\eta \circ \xi = \mathrm{id}_{|B}$. Let E be a $\mathcal{DFN}$-space and $U \subset E$ an open subset, then for $f \in \mathcal{C}(U, B)$ the lifting problem $f = \eta \circ \lambda$

$$\begin{tikzcd} & A \arrow[d, "\eta"'] \\ U \arrow[ur, "\lambda"] \arrow[r, "f"'] & B \arrow[u, bend right, "\xi"'] \end{tikzcd} \tag{6.3.1}$$

has the continuous solution $\lambda := \xi \circ f : U \to A$. Using a suitable measure $\mu \in \mathcal{MF}_2(U)$ with corresponding reproducing kernel function $K_\mu : U \times U \to \mathbb{C}$ the following result shows that for holomorphic functions $f \in \mathcal{H}(U, B)$ the lifting λ with $f = \eta \circ \lambda$ can be chosen holomorphic with certain growth conditions depending on μ.

Theorem 6.3.1 *Let $f \in \mathcal{H}(U,B)$, then there is $\lambda \in \mathcal{H}(U,A)$ solving $\eta \circ \lambda = f$. Moreover, for any measure $\mu \in \mathcal{MF}_2(U)$ such that the function $\| f \|_B$ is μ-square integrable over U the lifting λ can be chosen with*

$$\| \lambda(z) \|_A \leq c \cdot \| Eval(z) \|_{\mathcal{H}^2(U,\mu)'} \tag{6.3.2}$$

where $Eval(z) \in \mathcal{H}^2(U,\mu)'$ is the evaluation in $z \in U$ and $c > 0$ only depends on f and ξ.

Proof Let $\xi \in \mathcal{C}(B,A)$ be the continuous (possibly non-linear) Bartle and Graves right inverse with $\eta \circ \xi = \mathrm{id}_{|_B}$ and define $\tilde{\lambda} := \xi \circ f$. Then $\tilde{\lambda}$ is a continuous solution of the lifting problem (6.3.1). Consider the continuous map:

$$G : U \ni z \mapsto \| f(z) \|_B \in \mathbb{R}_0^+.$$

Using Lemma 6.1.1 there is a measure $\mu \in \mathcal{MF}_2(U)$ with $G \in L^2(U,\mu)$. The authors of [6] have shown that the inequality $\| \xi(y) \|_A \leq \beta \cdot \| y \|_B$ holds for all $y \in B$ where

$$\beta = \rho \cdot \sup \Big\{ \inf \{ \| x \|_B : \xi(x) = y \} : \| y \|_A = 1 \Big\}, \qquad (\rho > 1).$$

In particular, from our assumption on G it follows that $\| \tilde{\lambda}(\cdot) \|_A \in L^2(U,\mu)$. As before we denote by $K_\mu : U \times U \to \mathbb{C}$ the reproducing kernel for $\mathcal{H}^2(U,\mu)$. For each $z \in U$ the function $K_\mu(z,\cdot)$ is square integrable over U with respect to μ and so

$$U \ni y \mapsto \Big\| K_\mu(z, y) \cdot \tilde{\lambda}(y) \Big\|_A \leq \beta \cdot \Big| K_\mu(z, y) \Big| \cdot G(y) \in \mathbb{R}_0^+ \tag{6.3.3}$$

is μ-integrable. Because $F_z : U \ni y \mapsto K_\mu(z,y) \cdot \tilde{\lambda}(y) \in A$ is continuous for all $z \in U$ we conclude from the general theory and (6.3.3) that F_z is Bochner μ-integrable. Now, define the map $\lambda : U \to A$ by

$$\lambda(z) := \int_U K_\mu(z, \cdot)\, \tilde{\lambda}\, d\mu = \int_U F_z \, d\mu.$$

We show that λ is holomorphic with $\eta \circ \lambda = f$. Using well-known properties of the Bochner integral we obtain for each $\varphi \in A'$

$$\varphi \circ \lambda(z) = \int_U \varphi \circ \tilde{\lambda} \cdot K_\mu(z, \cdot)\, d\mu = P_\mu\Big(\varphi \circ \tilde{\lambda} \Big)(z)$$

and thus $\varphi \circ \lambda \in \mathcal{H}(U)$. Moreover, for $z_1, z_2 \in U$ with the $L^2(U,\mu)$-norm $\| \cdot \|$ the Cauchy-Schwartz inequality shows that:

$$\begin{aligned} \| \lambda(z_1) - \lambda(z_2) \|_A &\leq \int_U \Big| K_\mu(z_1,\cdot) - K_\mu(z_2,\cdot) \Big| \cdot \Big\| \tilde{\lambda} \Big\|_A d\mu \\ &\leq \beta \cdot \| K_\mu(z_1,\cdot) - K_\mu(z_2,\cdot) \| \cdot \| G \| \\ &= \beta \cdot \Big[K_\mu(z_1,z_1) - 2\mathrm{Re}\, K_\mu(z_1,z_2) + K_\mu(z_2,z_2) \Big]^{\frac{1}{2}} \cdot \| G \|. \end{aligned}$$

Using Lemma 6.1.2 the continuity of $\lambda : U \to A$ follows and so $\lambda \in \mathcal{H}(U, A)$. Let $\psi \in B'$ and $z \in U$, then we have $|\, \psi \circ f(z) \,| \le \|\, \psi \,\|_{B'} \cdot G(\, z\,)$ and because G is square integrable over U with respect to μ and f is holomorphic we obtain $\psi \circ f \in \mathcal{H}^2(U, \mu)$. Thus

$$\psi \circ \eta \circ \lambda(\, z\,) = \int_U K_\mu(z, \cdot) \, \Big[\, \psi \circ \eta \circ \tilde{\lambda} \,\Big] \, d\mu = \int_U K_\mu(z, \cdot) \Big[\, \psi \circ f \,\Big] \, d\mu = \psi \circ f(\, z\,).$$

Because $\psi \in B'$ was arbitrary this proves $\eta \circ \lambda = f$. Finally, the inequality (6.3.2) follows from

$$\|\, \lambda(z) \,\|_A \le \int_U |\, K_\mu(z, \cdot) \,| \, \big\|\, \tilde{\lambda} \,\big\|_A \, d\mu \le \beta \cdot c \cdot \big\|\, K_\mu(z, \cdot) \,\big\| = \beta \cdot c \cdot \big\|\, \delta_z \,\big\|_{\mathcal{H}^2(U,\mu)'}$$

where β only depends on η while $c^2 := \int_U \|\, f \,\|_B^2 \, d\mu$ is depending on f. $\square$

6.4 Holomorphic liftings on an inductive nuclear spectrum of Hilbert spaces

In Theorem 6.3.1 the norm of the point evaluation map on $\mathcal{H}^2(U, \mu)$ plays an important role for the estimate of the holomorphic liftings. In the case of an open subset U in an nuclear inductive spectrum of Hilbert space embeddings we explicitly can estimate the growth of K_μ on the diagonal. We remark that by Lemma 5.3.2 each $\mathcal{DFN}$-space has representation as such an inductive spectrum and so this assumption is not quite a restriction. In the following we use some calculations we have done in chapter 5.

Let $(\, H_j, \|\cdot\|_j \,)$ for $j \in \mathbb{N}_0$ be a sequence of infinite dimensional separable complex Hilbert spaces with nuclear embeddings $\pi_{j,j+1} : H_j \to H_{j+1}$. Then with the topology of the inductive limit

$$E := \bigcup_{j \in \mathbb{N}_0} H_j$$

becomes a $\mathcal{DFN}$-space. Without loss of generality we can assume that each embedding $\pi_{j,j+1}$ factorizes in two nuclear embeddings. According to Lemma 5.4.1 in chapter 5 there is a sequence $(B_n)_{n \in \mathbb{N}}$ of positive nuclear operators $B_n \in \mathcal{L}(H_n)$ with $tr(B_n^{\frac{1}{2}}) < \infty$ and $H_{n-1} \subset B_n^{\frac{1}{2}} H_n$ for all $n \in \mathbb{N}$. Moreover, all the embeddings

$$H_{n-1} \hookrightarrow \Big(\, B_n^{\frac{1}{2}} H_n, \big\|\, B_n^{-\frac{1}{2}} \cdot \big\|_n \,\Big)$$

are continuous. Let ν_n be the normed *Gaussian measure* on H_n with *correlation operator* B_n. Then, with the embedding $\pi_n : H_n \hookrightarrow E$ and any sequence $(\gamma_n)_n \in l^1(\mathbb{N})$ such that $\gamma_n > 0$ for all $n \in \mathbb{N}$ we can consider the finite Borel measure ν on U defined by

$$\nu(A) := \sum_{n \in \mathbb{N}} \gamma_n \cdot \nu_n\big(\, \pi_n^{-1}(A) \,\big)$$

for A in the Borel-σ-algebra of E (see 5.4.6). We have shown in chapter 5 Theorem 5.4.1, that for any open subset $U \subset E$ the restriction of ν to U is contained in $\mathcal{MF}_2(U)$ where $\mathcal{F} := \mathcal{H}(U)$. As before let P_ν be the orthogonal projection from $L^2(U,\nu)$ onto $\mathcal{H}^2(U,\nu)$ with integral kernel function $K_\nu : U \times U \to \mathbb{C}$. Now we want to estimate the growth of

$$U \ni z \mapsto K_\nu(\, z, z\,) \in \mathbb{R}^+$$

in terms of the nuclear embeddings B_n and the boundary distance of z to ∂U_n where $U_n := U \cap H_n$ in a suitable Hilbert space H_n. By an application of Corollary 5.2.2 we have:

Lemma 6.4.1 *For $x \in U_{n-1} \subset B_n^{\frac{1}{2}} H_n$ and $0 < r < dist_n(\, x, \partial U_n\,)$ where the distance and the boundary are taken in H_n we have:*

$$|\, f(x)\,| \leq \exp\left(2^{-1} \cdot \left\| \, B_n^{-\frac{1}{2}} x \, \right\|_n^2 \right) \cdot C(B_n, r) \cdot \|\, f\,\|_{L^2(U_n,\nu_n)} \tag{6.4.1}$$

for $f \in \mathcal{H}^2(U_n,\nu_n) := \mathcal{H}(U_n) \cap L^2(U_n,\nu_n)$. Here $C(B_n,r) > 0$ only depends on B_n and r.

The constant $C(B_n,r)$ explicitly was given in Theorem 5.2.1. With the sequence $(\lambda_j)_j$ of eigenvalues of the operator B_n:

$$\begin{aligned} C\,(\,B_n, r\,) &= \prod_{j=1}^{\infty} \left[\, 1 - \exp\left(-r^2 \cdot \lambda_j^{-\frac{1}{2}} \cdot \left[\, \mathrm{tr}\, B_n^{\frac{1}{2}} \,\right]^{-1} \right) \,\right]^{-\frac{1}{2}} \\ &\leq \exp\left(\frac{1}{2} \cdot \sum_{j=1}^{\infty} \log \left[\, 1 + r^{-2} \cdot \lambda_j^{\frac{1}{2}} \cdot \mathrm{tr}\, B_n^{\frac{1}{2}} \,\right] \right) \leq \exp\left(\frac{1}{2} \cdot r^{-2} \cdot \left[\, \mathrm{tr}\, B_n^{\frac{1}{2}} \,\right]^2 \right). \end{aligned}$$

Here we have used the inequalities $[\,1 - \exp(-t)\,]^{-\frac{1}{2}} \leq [\,1 + t^{-1}\,]^{\frac{1}{2}}$ and $\log(\,1 + t\,) \leq t$ which hold for all positive numbers t.

According to inequality (6.4.1) the evaluation map $Eval(z)$ on $\mathcal{H}^2(U_n,\nu_n)$ has an extension to a continuous functional on the L^2-closure $\mathcal{H}^2_c(U_n,\nu_n)$ of $\mathcal{H}^2(U_n,\nu_n)$ in $L^2(U_n,\nu_n)$ for all $z \in U_{n-1}$. Hence for each $n \in \mathbb{N}$ there is a kernel

$$k_n : U_n \times U_{n-1} \to \mathbb{C},$$

such that $k_n(\cdot, z) \in \mathcal{H}^2_c(U_n,\nu_n)$ for all $z \in U_{n-1}$ and

$$\mathrm{Eval}(\, z\,) = \left\langle\, \cdot, k_n(\cdot, z\,)\,\right\rangle_{L^2(U_n,\nu_n)}.$$

We can identify $\mathcal{H}^2_c(U_n,\nu_n)$ with a subspace of $\mathcal{H}(U_{n-1})$ and for all $f \in \mathcal{H}^2_c(U_n,\nu_n)$ and $z \in U_{n-1}$ we have:

$$f(\, z\,) = \int_{U_n} f\, \overline{k_n(\cdot, z\,)}\, d\nu_n.$$

Example 6.4.1 Let $U_n = H_n$ for $n \in \mathbb{N}$. Then the kernel $k_n : H_n \times H_{n-1} \to \mathbb{C}$ explicitly is given by

$$k_n(\cdot, z) = \exp \circ G_{B^{-\frac{1}{2}} z} \qquad \text{with} \qquad G_y := \left\langle B_n^{-\frac{1}{2}} \cdot, y \right\rangle_n$$

for any $y \in H_n$. Here G_y can be considered as an element in $L^2(H_n, \nu_n)$ (cf. our computations in Example 6.2.1).

For each $n \in \mathbb{N}$ we define a function $\tilde{K}_n : U \times U_{n-1} \to \mathbb{C}$ with $\tilde{K}_n(\cdot, z) \in L^2(U, \nu)$ in the following way. Fix $z \in U_{n-1}$, then we set:

$$\tilde{K}_n(\, u, z\,) := \begin{cases} k_n(\, u, z\,) & \text{for } u \in U_n \setminus U_{n-1} \\ 0 & \text{else.} \end{cases}$$

Finally, we define the kernel $K_n : U \times U_{n-1} \to \mathbb{C}$. Let $(u, z) \in U \times U_{n-1}$, then for $n \in \mathbb{N}$ we write:

$$K_n(\, u, z\,) := \gamma_n^{-1} \cdot P_\nu \Big[\, \tilde{K}_n(\cdot, z\,) \,\Big](\, u\,).$$

Lemma 6.4.2 *For all $n \in \mathbb{N}$ and $(u, z) \in U \times U_{n-1}$ it holds $K_\nu(u, z) = K_n(u, z)$.*

Proof Let $z \in U_{n-1}$, then the restriction of $K_\nu(\cdot, z)$ to U_n belongs to $\mathcal{H}^2(U_n, \nu_n)$. From the reproducing property of $k_n(\cdot, z) \in \mathcal{H}_c^2(U_n, \nu_n)$ and for all $(u, z) \in U \times U_{n-1}$ it follows that:

$$\begin{aligned} K_\nu(\, z, u\,) &= \Big\langle\, K_\nu(\cdot, u), k_n(\cdot, z) \,\Big\rangle_{L^2(U_n, \nu_n)} = \gamma_n^{-1} \cdot \Big\langle\, K_\nu(\cdot, u), \tilde{K}_n(\cdot, z) \,\Big\rangle_{L^2(U, \nu)} \\ &= \Big\langle\, K_\nu(\cdot, u), K_n(\cdot, z) \,\Big\rangle_{L^2(U, \nu)}. \end{aligned}$$

Here we have used $\nu_j(H_{j-1}) = 0$ for all $j \in \mathbb{N}$ (cf. [48] or Proposition 5.1.1 together with $H_{n-1} \subset B_n^{\frac{1}{2}} H_n$). Hence from the reproducing property of K_ν on $\mathcal{H}^2(U, \nu)$ it follows for all $(u, z) \in U \times U_{n-1}$:

$$\Big\langle\, K_\nu(\cdot, u), K_\nu(\cdot, z) - K_n(\cdot, z) \,\Big\rangle_{L^2(U, \nu)} = 0.$$

Because the linear hull of $\{\, K_\nu(\cdot, u\,) \,:\, u \in U \,\}$ is a dense subspace of $\mathcal{H}^2(U, \nu)$ we conclude that $K_\nu(u, z) = K_n(u, z)$ for all $(u, z) \in U \times U_{n-1}$. □

Now, we can prove the desired estimate on the point evaluation in $\mathcal{H}^2(U, \nu)$. Let $n \in \mathbb{N}$ and $z \in U_{n-1}$, then it follows with our notations above:

$$\begin{aligned} \|\, \mathrm{Eval}(z) \,\|^2_{\mathcal{H}^2(U, \nu)'} &\leq K_\nu(\, z, z\,) \\ &= \gamma_n^{-2} \cdot \Big\|\, P_\nu \tilde{K}_n(\cdot, z\,) \,\Big\|^2_{L^2(U, \nu)} \\ &\leq \gamma_n^{-1} \cdot \|\, k_n(\cdot, z) \,\|^2_{L^2(U_n, \nu_n)} = \gamma_n^{-1} \cdot \|\, \mathrm{Eval}_n(z) \,\|^2_{\mathcal{H}_c^2(U_n, \nu_n)}. \end{aligned}$$

where $Eval_n(z)$ denotes the evaluation on $\mathcal{H}_c^2(U_n, \nu_n)$ in $z \in U_{n-1}$. The following proposition is an immediate consequence of this calculation, Lemma 6.4.1 and our estimate on $C(B_n, r)$ above:

Proposition 6.4.1 *Let $z \in U_{n-1}$ where $n \in \mathbb{N}$ and $0 < r < dist_n(z, \partial U_n)$. The distance and the boundary are taken with respect to the topology of H_n. Then for the reproducing kernel $K_\nu : U \times U \to \mathbb{C}$ it follows that:*

$$K_\nu(z, z)^{\frac{1}{2}} = \| \mathit{Eval}(z) \|_{\mathcal{H}^2(U,\nu)'} \le \gamma_n^{-\frac{1}{2}} \cdot \exp\Big(2^{-1} \cdot \| B_n^{-\frac{1}{2}} z \|_n^2 + \frac{1}{2r^2} \cdot \Big[\, tr\, B_n^{\frac{1}{2}} \Big]^2 \Big).$$

Remark 6.4.1 In the case where $U = E$ we can choose $r > 0$ arbitrarily and it follows from Proposition 6.4.1 that $K_\nu(z, z)^{\frac{1}{2}} \le \gamma_n^{-\frac{1}{2}} \exp(2^{-1} \cdot \| B_n^{-\frac{1}{2}} z \|_n^2)$ for all $z \in H_{n-1}$.

6.5 The Shilov boundary

Let $\mathcal{A}$ be a Banach algebra with unit $e \in \mathcal{A}$ and denote by $\mathcal{M}(\mathcal{A})$ the space of multiplicative functionals (maximal ideals) in $\mathcal{A}$. Then endowed with the weak*-topology

$$\mathcal{U}(\mathcal{A}) := \big\{ a^* \in \mathcal{M}(\mathcal{A}) : \| a^* \|_{\mathcal{M}(\mathcal{A})} = 1 \big\}$$

is a compact space. Moreover, for each $a \in \mathcal{A}$ the *evaluation* $\delta_a : \mathcal{U}(\mathcal{A}) \to \mathbb{C}$ in a defined by $\delta_a(a^*) = a^*(a)$ is continuous. With any $\mathcal{I} \subset \mathcal{U}(\mathcal{A})$ and $g \in \mathcal{C}(\mathcal{U}(\mathcal{A}))$ we define:

$$\| g \|^{\mathcal{I}} := \sup \Big\{ \, | \, g(a^*) \, | : a^* \in \mathcal{I} \, \Big\}.$$

Let us consider the space $\mathcal{C}(\mathcal{U}(\mathcal{A}))$ with the usual norm $\| \cdot \| := \| \cdot \|^{\mathcal{U}(\mathcal{A})}$. Then the evaluation map $\delta : \mathcal{A} \ni a \mapsto \delta_a \in \mathcal{C}(\mathcal{U}(\mathcal{A}))$ is linear and continuous:

$$\Big\| \, \delta_a \, \Big\| = \sup \Big\{ \, | \, \delta_a(a^*) \, | = | \, a^*(a) \, | : a^* \in \mathcal{U}(\mathcal{A}) \, \Big\} \le \| \, a \, \|_{\mathcal{A}}. \tag{6.5.1}$$

From now on we suppose that $\mathcal{A}$ is an unital subalgebra of the bounded continuous functions on some Hausdorff space X. Consider the map:

$$\mathrm{Eval} : X \longrightarrow \mathcal{U}(\mathcal{A}) : x \mapsto \big[\, \mathcal{A} \ni a \mapsto a(\, x \,) \in \mathbb{C} \, \big].$$

Then *Eval* is continuous and it is one-to-one if and only if $\mathcal{A}$ separates the points of X. With $a \in \mathcal{A}$ it yields $\delta_a \circ \mathit{Eval} = a$ and we obtain that:

$$\begin{aligned} \| \, a \, \|_{\mathcal{A}} &= \sup \Big\{ \, \Big| \, \delta_a \circ \mathrm{Eval}(x) \, \Big| : x \in X \, \Big\} \\ &\le \sup \Big\{ \, \Big| \, \delta_a(a^*) \, \Big| : a^* \in \mathcal{U}(\mathcal{A}) \, \Big\} = \Big\| \, \delta_a \, \Big\|. \end{aligned} \tag{6.5.2}$$

For function algebras $\mathcal{A}$ the inequalities (6.5.1) and (6.5.2) prove that δ is an isometry. It turns out that among all the compact subsets of $\mathcal{U}(\mathcal{A})$ there is at least one minimal compact set $\mathcal{S}$ for which $\| \cdot \|^{\mathcal{S}} = \| \cdot \|$ (see [118]). $\mathcal{S}$ is called the *Shilov boundary* of $\mathcal{A}$. Using our inequalities above it follows for $a \in \mathcal{A}$:

$$\| \delta_a \| \leq \| a \|_{\mathcal{A}} = \sup \Big\{ \, | \, \delta_a \circ \mathrm{Eval}(x) \, | : x \in X \Big\} = \| \delta_a \|^{\mathrm{Eval}(X)} \leq \| \delta_a \|.$$

This shows that $\| \delta_a \| = \| \delta_a \|^{\mathcal{I}}$ where $\mathcal{I} = Eval(X)$ and we obtain that

$$\mathcal{S} \subset \overline{\mathrm{Eval}(\, X \,)} \subset \mathcal{U}(\, \mathcal{A} \,).$$

If in addition X is compact and $\mathcal{A}$ separates the points the mapping $Eval$ is one-to-one and $Eval(X)$ is compact as well. In this case $Eval$ is a homeomorphism between X and $Eval(X)$ and $\mathcal{S}$ can be considered as a subset of X. If X is compact but $\mathcal{A}$ does not separate the points then $Eval$ is not one-to-one. Every function in $\mathcal{A}$ achieves its norm on the set $Eval^{-1}(\mathcal{S})$, but this set needs not to be minimal. In general, there will be many minimal compact subsets of X which carry the norms of the functions in $\mathcal{A}$.

For $j = 1, 2$ let $\mathcal{A}_j$ be unital subalgebras in the space of all bounded continuous functions on X with Shilov boundaries $\mathcal{S}_j$. Assume that there is an isometry $I : \mathcal{A}_1 \to \mathcal{A}_2$ of algebras which is onto, then the adjoint map $I^* : \mathcal{M}(\mathcal{A}_2) \to \mathcal{M}(\mathcal{A}_1)$ is isometric and onto as well with $(I^*)^{-1} = (I^{-1})^*$. In particular, $\mathcal{U}(\mathcal{A}_1)$ is the range of $\mathcal{U}(\mathcal{A}_2)$ under I^*. We define $\mathcal{I}_1 := I^*(\mathcal{S}_2) \subset \mathcal{U}(\mathcal{A}_1)$. Then by the continuity of I^* we conclude that $\mathcal{I}_1$ is compact and it is a straightforward computation that with $a_1 \in \mathcal{A}_1$ it holds:

$$\Big\| \delta_{a_1} \Big\|^{\mathcal{I}_1} = \Big\| \delta_{Ia_1} \Big\|^{\mathcal{S}_2} = \Big\| \delta_{Ia_1} \Big\| = \Big\| Ia_1 \Big\|_{\mathcal{A}_2} = \Big\| a_1 \Big\|_{\mathcal{A}_1} = \Big\| \delta_{a_1} \Big\|.$$

This now implies $\mathcal{S}_1 \subset \mathcal{I}_1$ and similar $\mathcal{S}_2 \subset (I^{-1})^*(\mathcal{S}_1)$. It follows that $\mathcal{I}_1 \subset \mathcal{S}_1$ and we obtain the identity $I^*(\mathcal{S}_2) = \mathcal{S}_1$.

Example 6.5.1 Let X_1 and X_2 be Hausdorff spaces and $\mathcal{A}_2 \subset \mathcal{C}_b(X_2)$ be an unital subalgebra of the space of all bounded continuous functions on X_2. Let $i : X_1 \to X_2$ be a continuous function which is one-to-one and with dense range. Then we consider the induced subalgebra $\mathcal{A}_1$ of $\mathcal{C}_b(X_1)$ defined by:

$$\mathcal{A}_1 := \Big\{ \, I(\, a_2 \,) := a_2 \circ i \in \mathcal{C}_b(\, X_1 \,) : a_2 \in \mathcal{A}_2 \Big\}.$$

Because the map i has dense range by assumption it follows that $I : \mathcal{A}_2 \to \mathcal{A}_1$ is an isometry and onto. With the Shilov boundaries $\mathcal{S}_j$ of $\mathcal{A}_j$ for $j = 1, 2$ we obtain with our remark above $\mathcal{S}_2 = I^*(\mathcal{S}_1)$.

Now, we specialize to the case of functions algebras on open subsets U of a $\mathcal{DFN}$-space E. There are no non-trivial open sets in E with compact closure and so we cannot identify the Shilov boundary of subalgebras in $\mathcal{C}_b(\bar{U})$ with a subset of $\bar{U}$ in the described way. Let us assume that there is a separable complex Hilbert-space $(\, H, \langle \cdot, \cdot \rangle \,)$ and a continuous and dense embedding $i : E \hookrightarrow H$. In fact, by Corollary A.2.1 such an embedding can always be constructed in a canonical way using the Bergman kernel on U with respect to a $\mathcal{NF}_2$-measure.

Example 6.5.2 Assume that the closure $\overline{U}_H \subset H$ of U in the topology of H is bounded in H. Then $\overline{U}_H$ is a weakly compact set and so we can consider the dense and continuous embedding $i : U \hookrightarrow \overline{U}_H$. Moreover, let us write $\mathcal{H}_b(\overline{U}_H)$ for the space of all weakly continuous functions $f : \overline{U}_H \to \mathbb{C}$ such that the restriction of f to U is in $\mathcal{H}(U)$. Then $\mathcal{H}_b(\overline{U}_H)$ is a Banach algebra of continuous functions on a compact space. Let us remark that each (continuous) holomorphic function on an open subset of a Hilbert space automatically is weakly continuous. Hence as a subspace $\mathcal{H}_b(\overline{U}_H)$ contains all functions on $\overline{U}_H$ which admit a holomorphic extension to an open neighborhood of $\overline{U}_H$. As in example 6.5.1 we define the following algebra of bounded holomorphic functions on U:

$$\mathcal{H}_{b,H}(\,U\,) := \Big\{\, I(f) := f \circ i : U \to \mathbb{C} : \text{ with } f \in \mathcal{H}_b(\overline{U}_H) \,\Big\}.$$

In other words $\mathcal{H}_{b,H}(U)$ is the space of holomorphic functions on U which admit a weakly continuous extension to $\overline{U}_H$. In particular, $\mathcal{H}_b(\overline{U}_H)$ contains all restrictions to $\overline{U}_H$ of linear functionals on H and it follows that both $\mathcal{H}_b(\overline{U}_H)$ and $\mathcal{H}_{b,H}(U)$ separate points. Let us write $\mathcal{A}_1 := \mathcal{H}_b(\overline{U}_H)$ and $\mathcal{A}_2 := \mathcal{H}_{b,H}(U)$. By our remarks above the maps $I : \mathcal{A}_1 \to \mathcal{A}_2$ and $I^* : \mathcal{M}(\mathcal{A}_2) \to \mathcal{M}(\mathcal{A}_1)$ are isometries and onto. With these notations we define:

Definition 6.5.1 We call a set $A \subset E \cap \overline{U}_H$ which is closed in the topology of E a boundary of $\mathcal{A}_2$ iff for all $f \in \mathcal{A}_1$:

$$\sup\Big\{\, |\, f(z)\,| : z \in A \,\Big\} = \|\, \delta_f \,\|^{\mathcal{U}(\mathcal{A}_1)} = \|\, \delta_{I(f)} \,\|^{\mathcal{U}(\mathcal{A}_2)}. \qquad (*)$$

Hence f archives his norm on the closed set A.

We denote by $\mathcal{S}_1 \subset \mathcal{U}(\mathcal{A}_1)$ the Shilov boundary of $\mathcal{A}_1$. By the fact that $\mathcal{A}_1$ is a subalgebra of the space of all weakly continuous functions on the weakly compact set $\overline{U}_H$ we can identify the compact space $\mathcal{S}_1$ with a subset of $\overline{U}_H$. More precisely, there is a minimal compact set V in $\overline{U}_H$ such that $\mathcal{S}_1 = \mathrm{Eval}(V)$.

Lemma 6.5.1 *Assume that $A := V \cap E$ is dense in $V \subset H$ with respect to the weak topology of H. Then A is a boundary of $\mathcal{A}_2$.*

Proof Because V is closed in H and the embedding $E \hookrightarrow H$ is continuous, it follows that $A = V \cap E$ is closed in the topology of E. In order to prove that A is a boundary of $\mathcal{A}_2$ we fix $f \in \mathcal{A}_1$. We obtain that:

$$\begin{aligned}\sup\Big\{\, |\, f(z)\,| : z \in A \,\Big\} &= \sup\Big\{\, |\, f(z)\,| : z \in V \,\Big\} \\ &= \|\, \delta_f \,\|^{\mathcal{S}_1} = \|\, \delta_f \,\|^{\mathcal{U}(\mathcal{A}_1)} = \|\, \delta_{I(f)} \,\|^{\mathcal{U}(\mathcal{A}_2)}.\end{aligned}$$

from the fact that A is weakly dense in V and f is weakly continuous on $\overline{U}_H$. □

6.6 The abstract Cauchy-Weil theorem

We remind of some well-known results on the topological dual $\mathcal{C}(X)'$ of all continuous and complex valued functions on a compact Hausdorff space X. Consider a finite positive Radon measure ν on X, then there is a canonical isometric embedding J of $L^1(X,\nu)$ into $\mathcal{C}(X)'$. The proof of the following lemma can be found in [69]:

Lemma 6.6.1 *For every subset M of $\mathcal{C}(X)'$ which is separable in the norm topology there is a finite positive Radon measure ν on X such that $M \subset J(\, L^1(X,\nu)\,)$.*

Let U be an open subset of a $\mathcal{DFN}$-space E. We denote by $\mathcal{H}^\infty(U)$ the Banach algebra of all bounded and holomorphic functions in U. With a closed subalgebra $\mathcal{A}$ of $\mathcal{H}^\infty(U)$ which separates points of U we write $\mathcal{S}_\mathcal{A}$ for the *Shilov boundary* of $\mathcal{A}$ and $\mathcal{F} := \mathcal{H}(U)$. Moreover, let $\mathcal{A}'$ be the topological dual of $\mathcal{A}$:

Theorem 6.6.1 *Let $\mu \in \mathcal{MF}_2(U)$, then there is a finite positive Radon measure ν on $\mathcal{S}_\mathcal{A}$ and a kernel $\Psi = \Psi_\mu \in \mathcal{H}(\,U, L^1(\mathcal{S}_\mathcal{A},\nu)\,)$ such that for all $f \in \mathcal{A}$ and $z \in U$ it holds:*

$$f(\,z\,) = \int_{\mathcal{S}_\mathcal{A}} \delta_f \cdot \Psi(\,z\,)\, d\nu \tag{6.6.1}$$

Moreover, it exists $c > 0$ with $\|\, \Psi(z)\,\|_{L^1(\mathcal{S}_\mathcal{A},\nu)} \leq c \cdot \|\, Eval(z)\,\|_{\mathcal{H}^2(U,\mu)'}$ for all $z \in U$.

Proof With our previous notations we consider the isometry $\delta : \mathcal{A} \to \mathcal{C}(\mathcal{S}_\mathcal{A})$ which is defined by $\delta(f) := \delta_f$ and we write $\delta^* : \mathcal{C}(\mathcal{S}_\mathcal{A})' \to \mathcal{A}'$ for its adjoint. Let $\varphi \in \mathcal{A}'$, then by the *Hahn Banach theorem* there is $\tau \in \mathcal{C}(\mathcal{S}_\mathcal{A})'$ extending $\varphi \circ \delta^{-1} \in \delta[\,\mathcal{A}\,]'$. Hence it follows that $\delta^*(\tau) = \varphi$ and δ^* is onto. Now, consider the evaluation:

$$\mathrm{Eval} : U \to \mathcal{M}(\mathcal{A}) \subset \mathcal{A}'.$$

Then $\|\, Eval(z)\,\|_{\mathcal{A}'} \leq 1$ for all $z \in U$ and $Eval$ is holomorphic on U by Theorem 6.2.1. We obtain a diagram:

$$\begin{tikzcd} & \mathcal{C}(\mathcal{S}_A)' \arrow[d, "\delta^*"'] \\ U \arrow[ur, "\tilde{\Psi}"] \arrow[r, "\mathrm{Eval}"'] & \mathcal{A}' \arrow[u, bend right, "\xi"'] \end{tikzcd} \tag{6.6.2}$$

where ξ is a continuous right inverse of δ^* (see [6]). According to Theorem 6.3.1 for any measure $\mu \in \mathcal{MF}_2(U)$ there is a holomorphic lifting $\tilde{\Psi} : U \to \mathcal{C}(\mathcal{S}_\mathcal{A})'$ with $Eval = \delta^* \circ \tilde{\Psi}$ and it exists $c > 0$ such that

$$\left\|\, \tilde{\Psi}(z)\,\right\|_{\mathcal{C}(\mathcal{S}_\mathcal{A})'} \leq c \cdot \left\|\, \mathrm{Eval}(z)\,\right\|_{\mathcal{H}^2(U,\mu)'}. \tag{6.6.3}$$

In the last inequality we write $Eval$ for the evaluation map in $\mathcal{H}^2(U,\mu)'$. Because U is separable and $\tilde{\Psi} : U \to \mathcal{C}(\mathcal{S}_\mathcal{A})'$ is continuous the range $\tilde{\Psi}(U)$ is separable in $\mathcal{C}(\mathcal{S}_\mathcal{A})'$ as well.

From Lemma 6.6.1 we conclude that there is a positive Radon measure ν on $\mathcal{S}_\mathcal{A}$ such that with the canonical isometric embedding J from $L^1(\mathcal{S}_\mathcal{A},\nu)$ into $\mathcal{C}(\mathcal{S}_\mathcal{A})'$ the inclusion holds:

$$\tilde{\Psi}(U) \subset J\big[\,L^1(\mathcal{S}_\mathcal{A},\nu)\,\big] \subset \mathcal{C}(\mathcal{S}_\mathcal{A})'.$$

We can define $\Psi \in \mathcal{H}(\,U, L^1(\mathcal{S}_\mathcal{A},\nu)\,)$ by $\Psi := J^{-1}\circ\tilde{\Psi}$. Because the embedding J is isometric inequality (6.6.3) holds for Ψ instead of $\tilde{\Psi}$. Finally, we have for $f\in\mathcal{A}$ and all $z\in U$ using (6.6.2):

$$\int_{\mathcal{S}_\mathcal{A}} \delta_f\cdot\Psi(\,z\,)\,d\nu = \big[\,J\circ\Psi(z)\,\big](\,\delta_f\,) = \big[\,\tilde{\Psi}(z)\,\big](\,\delta_f\,) = \big[\,\delta^*\circ\tilde{\Psi}(z)\,\big](\,f\,) = f(\,z\,). \qquad \square$$

Remark 6.6.1 Using Theorem 6.2.2 and with the notations of Theorem 6.6.1 for any finite measure $\mu\in\mathcal{MF}_2(U)$ there is a kernel $\Phi_\mu : U\times\mathcal{S}_\mathcal{A}\to\mathbb{C}$ such that:

- The map $U\ni z\mapsto\Phi_\mu(z,x)\in\mathbb{C}$ is holomorphic for all $x\in\mathcal{S}_\mathcal{A}$.
- For all $z\in U$ we have $\Psi_\mu(z) = [\,\Phi_\mu(z,\cdot)\,]\in L^1(\mathcal{S}_\mathcal{A},\nu)$.
- There is $C>0$ with $\|\,\Phi_\mu(z,\cdot)\,\|_{L^1(\mathcal{S}_\mathcal{A},\nu)} \le C\cdot\|\,Eval(z)\,\|_{\mathcal{H}^2(U,\mu)'}$ for all $z\in U$.
- It holds $f(\,z\,) = \int_{\mathcal{S}_\mathcal{A}}\delta_f\cdot\Phi_\mu(\,z,\cdot)\,d\nu$ for all $f\in\mathcal{A}$ and $z\in U$.

Example 6.6.1 Let us assume that there is a separable complex Hilbert space H and a dense embedding $i: E\hookrightarrow H$. With an open set $U\subset E$, which is bounded in H we use our notations in Example 6.5.2. Then there is a compact set $V\subset\partial\overline{U}_H\subset H$, the boundary in H, such that:

$$\kappa := \big(\,I^*\,\big)^{-1}\circ\mathrm{Eval} : H\supset V \xrightarrow{\mathrm{Eval}} \mathcal{S}_{\mathcal{A}_1} \xrightarrow{(\,I^*\,)^{-1}} \mathcal{S}_{\mathcal{A}_2}$$

is a homeomorphism. For $\mu\in\mathcal{MF}_2(U)$ let ν be the corresponding measure in Theorem 6.6.1 on $\mathcal{S}_{\mathcal{A}_2}$ and let $\Phi_\mu : U\times\mathcal{S}_{\mathcal{A}_2}\to\mathbb{C}$ be as in Remark 6.6.1. Then we can define the transported measure $\tilde{\nu} := \nu[\,\kappa(\,\cdot\,)\,]$ on V. We obtain the integral formula:

$$f(\,z\,) = \int_V f(\,x\,)\cdot\Phi_\mu\Big(\,z,\kappa(\,x\,)\,\Big)\,d\tilde{\nu}(\,x\,).$$

for all $f\in\mathcal{A}_2$ by the transformation formula for integrals.

6.7 Abstract Hardy spaces for domains with arbitrary boundary

In the following section we define an abstract Hardy space for open domains U in a $\mathcal{DFN}$-space E with arbitrary boundary by exploiting an idea in [72]. We want to emphasize that this construction even leads to new results for regions U in the complex plane with arbitrary boundary and algebras $\mathcal{A} \subset \mathcal{H}^\infty(U)$. For instance we can choose U to be a simply connected domain with arbitrary boundary. Then the Banachalgebra of holomorphic functions on U which are continuous up to the boundary of U leads via biholomorphic equivalence to a closed subalgebra of $\mathcal{H}^\infty(D)$ where $D \subset \mathbb{C}$ denotes the complex unit disc.

Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{H}^\infty(U)$ which separates points and denote by $\mathcal{S}_\mathcal{A}$ its abstract Shilov-boundary. Then, via the isometry $\delta : \mathcal{A} \to \mathcal{C}(\mathcal{S}_\mathcal{A})$ we can identify $\mathcal{A}$ with a closed subspace of the $\mathcal{C}(\mathcal{S}_\mathcal{A})$. Our aim is to construct a finite measure Θ on $\mathcal{S}_\mathcal{A}$ and a kernel $K : U \times \mathcal{S}_\mathcal{A} \to \mathbb{C}$ such that an integral formula holds for $f \in \mathcal{A}$ and all $z \in U$ similar to (6.6.1) in Theorem 6.6.1 and Remark (6.6.1). Namely, for all $z \in U$ we claim:

$$f(\,z\,) = \int_{\mathcal{S}_\mathcal{A}} \delta_f \cdot K(z,\cdot)\, d\Theta, \qquad\qquad (\,f \in \mathcal{A}\,)$$

Moreover, we want to choose K such that $K(z,\cdot)$ is bounded on $\mathcal{S}_\mathcal{A}$ for fixed $z \in U$ or more generally such that

$$\Phi : U \to L^2\,(\,\mathcal{S}_\mathcal{A}, \Theta\,) : z \mapsto K(\,z,\cdot\,)$$

is well-defined and holomorphic on U. Comparing this construction to Theorem 6.6.1 where we only have $\Psi(z,\cdot) \in L^1(\mathcal{S}_\mathcal{A}, \nu)$ for all $z \in U$ we loose some information on the growth of

$$U \ni z \mapsto \|\, K(z,\cdot)\,\|_2 \in \mathbb{R}^+$$

near the boundary of U. Here $\|\cdot\|_2$ is the $L^2(\mathcal{S}_\mathcal{A}, \Theta)$-norm for. Similar to the classical situation the abstract Hardy space $\mathcal{H}^2(\mathcal{S}_\mathcal{A}, \Theta)$ now is defined to be the Hilbert space closure of $\delta[\,\mathcal{A}\,] \subset \mathcal{C}(\mathcal{S}_\mathcal{A})$ in $L^2(\mathcal{S}_\mathcal{A}, \Theta)$. We show that for all compact sets $H \subset U$ there is an inequality of the form:

$$\sup\Big\{\,|\,f(z)\,| : z \in H\,\Big\} \leq \|\,\delta_f\,\|_{L^2(\mathcal{S}_\mathcal{A},\Theta)} \cdot \sup\Big\{\,\|\,K(z,\cdot)\,\|_{L^2(\mathcal{S}_\mathcal{A},\Theta)} : z \in H\,\Big\} < \infty.$$

Hence $\mathcal{H}^2(\mathcal{S}_\mathcal{A}, \Theta)$ can be identified with a Hilbert space of holomorphic functions on U. Finally, given Bergman spaces $\mathcal{H}^2(U, \mu_j)$ with $j = 1, 2$ and continuous embeddings

$$\mathcal{A} \xrightarrow{J_1} \mathcal{H}^2(\,U, \mu_1\,) \xrightarrow{J_2} \mathcal{H}^2(\,U, \mu_2\,)$$

where J_2 is nuclear we show that it exists an abstract Hardy space in between $\mathcal{A}$ and $\mathcal{H}^2(U, \mu_2)$ which has a quasi-nuclear (Hilbert-Schmidt) embedding into $\mathcal{H}^2(U, \mu_2)$.

Let us start with a lemma which involves some elementary results in measure theory (cf. [125]). Assume that X is a compact space and μ is a finite measure on X. We denote by $D \subset \mathbb{C}$ the closed unit disc and fix $f \in L^1(X, \mu)$. Then according to 1.40 Theorem in [125] it follows from the condition

$$\mu(E)^{-1} \cdot \int_E f\, d\mu \in D \tag{6.7.1}$$

for all Borel sets E with $\mu(E) > 0$ that $f(x) \in D$ for μ-almost all $x \in X$.

In the following we write $\mathcal{M}(X)$ for the Banach space of all complex measures on X equipped with the norm $\| \nu \| := | \nu | (X)$ (total variation of ν). Let $(\Phi_j)_j \subset \mathcal{M}(X)$ be a sequence with $\sum_{k=1}^{\infty} \| \Phi_k \| < \infty$ and define the positive Borel-measure

$$\mu := \sum_{k=1}^{\infty} | \Phi_k | \in \mathcal{M}(X).$$

Lemma 6.7.1 *([72], 2.5 Satz 3) There exists a sequence $(f_j)_j \in L^\infty(X, \mu)$ of complex valued functions such that $\| f_j \|_\infty \leq 1$ with $d\Phi_j = f_j \, d\mu$ for all $j \in \mathbb{N}$.*

Proof It is trivial that $\Phi_j < \mu$ for all $j \in \mathbb{N}$ and therefore the Radon-Nikodym theorem guarantees the existence of $f_j \in L^1(X, \mu)$ such that $d\Phi_j = f_j \, d\mu$ holds. In order to show (6.7.1) fix a Borel set E with $\mu(E) > 0$, then

$$\mu(E)^{-1} \cdot \Big| \int_E f_j \, d\mu \Big| = \Big[\sum_{j \in \mathbb{N}} | \Phi_j |(E) \Big]^{-1} \cdot | \Phi_j(E) | \leq 1.$$

By the criterion mentioned above it follows that $| f_j(x) | \leq 1$ for μ-almost all $x \in X$ and $j \in \mathbb{N}$. Hence $\| f \|_\infty \leq 1$. □

Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{H}^\infty(U)$ which separates points. Following an idea in [72] we consider the point evaluation $Eval : U \to \mathcal{A}'$. We apply Remark 6.2.1 with the nuclear Fréchet space $\mathcal{N} := \mathcal{H}(U)$. Then there are sequences $(\alpha_j)_j \subset \mathbb{R}^+$ rapidly decreasing, $(h_j)_j \in \mathcal{H}(U)$ and $(\varphi_j)_j \in \mathcal{A}'$ tending to zero and an expansion:

$$\mathrm{Eval}(\, z\,) = \sum_{j \in \mathbb{N}} \alpha_j \cdot h_j(\, z\,) \cdot \varphi_j. \tag{6.7.2}$$

As we have seen in the proof of Theorem 6.6.1, the adjoint map $\delta^* : \mathcal{C}(\mathcal{S}_\mathcal{A})' \to \mathcal{A}'$ is onto and so it admits a continuous right inverse $\xi : \mathcal{A}' \to \mathcal{C}(\mathcal{S}_\mathcal{A})'$ (cf. [6]) which in general is not linear but fulfills the norm inequality $\| \xi(y) \| \leq \lambda \cdot \| y \|$ with suitable $\lambda > 0$.

We can choose $\mu_j = \xi(\varphi_j) \in \mathcal{C}(\mathcal{S}_\mathcal{A})'$, a sequence $(\sigma_j)_j \in l^1(\mathbb{N})$ with $\sigma_j > 0$ and $(\beta_j)_j$ rapidly decreasing such that $\alpha_j = \sigma_j \cdot \beta_j$ for all $j \in \mathbb{N}$. Then $(\mu_j)_j$ tends to zero and for all $z \in U$:

$$\mathrm{Eval}(\, z\,) = \sum_{j \in \mathbb{N}} \beta_j \cdot h_j(\, z\,) \cdot \big[\, \sigma_j \, \delta^*(\, \mu_j\,)\, \big] \in \mathcal{A}'.$$

Consider the positive measure $\Theta := \sum_{j=1}^{\infty} \sigma_j \cdot | \mu_j |$. According to Lemma 6.7.1 there is a sequence $(f_j) \in L^\infty(\mathcal{S}_\mathcal{A}, \Theta)$ of complex valued functions with $|f_j(x)| \leq 1$ for all $x \in \mathcal{S}_\mathcal{A}$ such that $\sigma_j \, d\mu_j = f_j \, d\Theta$ for all $j \in \mathbb{N}$. Let us define the kernel:

$$K : U \times \mathcal{S}_\mathcal{A} \longrightarrow \mathbb{C} : (z, u) \mapsto \sum_{j \in \mathbb{N}} \beta_j \cdot h_j(z) \cdot f_j(u).$$

For any compact set $H \subset U$ it follows that

$$\sup\Big\{ \mid K(z,u) \mid : z \in H,\ u \in \mathcal{S}_{\mathcal{A}} \Big\} < \infty.$$

Moreover, we obtain for $f \in \mathcal{A}$ the integral formula:

$$\begin{aligned} \int_{\mathcal{S}_{\mathcal{A}}} \delta_f \cdot K(\,z, \cdot\,)\, d\Theta &= \sum_{j \in \mathbb{N}} \beta_j \cdot h_j(\,z\,) \cdot \int_{\mathcal{S}_{\mathcal{A}}} \delta_f \cdot f_j \, d\Theta \\ &= \sum_{j \in \mathbb{N}} \alpha_j \cdot h_j(\,z\,) \cdot \int_{\mathcal{S}_{\mathcal{A}}} \delta_f \, d\mu_j \\ &= \sum_{j \in \mathbb{N}} \alpha_j \cdot h_j(\,z\,) \cdot \delta^*(\mu_j)[\,f\,] = \mathrm{Eval}(z)[\,f\,] = f(\,z\,). \end{aligned} \tag{6.7.3}$$

Now, we can define the Hardy space $\mathcal{H}^2(\mathcal{S}_{\mathcal{A}}, \Theta)$ by:

$$\mathcal{H}^2(\,\mathcal{S}_{\mathcal{A}}, \Theta\,) := \text{ closure of } \delta[\,\mathcal{A}\,] \subset \mathcal{C}(\mathcal{S}_{\mathcal{A}}) \text{ in } L^2(\mathcal{S}_{\mathcal{A}}, \Theta).$$

From our computations above and using the Cauchy-Schwartz inequality we obtain for each compact set $H \subset U$ and $f \in \mathcal{A}$:

$$\sup\Big\{ \mid f(z) \mid : z \in H \Big\} \leq \parallel \delta_f \parallel_{L^2(\mathcal{S}_{\mathcal{A}},\Theta)} \cdot \sup\Big\{ \parallel K(z,\cdot) \parallel_{L^2(\mathcal{S}_{\mathcal{A}},\Theta)} : z \in H \Big\} < \infty. \tag{6.7.4}$$

Hence we can consider $\mathcal{H}^2(\mathcal{S}_{\mathcal{A}}, \Theta)$ as a space of holomorphic functions on U. Let us show how an expansion (6.7.2) can be obtained by using Bergman spaces $\mathcal{H}^2(U,\mu)$ with a measure $\mu \in \mathcal{MF}_2(U)$ where $\mathcal{F} = \mathcal{H}(U)$. Under some additional assumptions on μ we can prove that the abstract Hardy space $\mathcal{H}^2(\mathcal{S}_{\mathcal{A}}, \Theta)$ admits a Hilbert-Schmidt (quasi-nuclear) embedding into $\mathcal{H}^2(U,\mu)$. We want to generate the diagram

$$\mathcal{H}^\infty(U) \supset \mathcal{A} \xrightarrow{\ I_1\ } \mathcal{H}^2(\mathcal{S}_{\mathcal{A}}, \Theta) \xrightarrow{\ I_2\ } \mathcal{H}^2(U,\mu) \tag{6.7.5}$$

where I_j are embeddings for $j = 1, 2$ and I_2 is of Hilbert-Schmidt type. Hence it is natural to claim some kind of nuclearity condition on the embbeding $I_3 : \mathcal{A} \to \mathcal{H}^2(U,\mu)$. In order to proceed we remind of the following result on operators between Hilbert spaces which factorize over a Banach space of continuous functions. Let $\omega : U \to \mathbb{R}^+$ be continuous, then we define the Banach space

$$\mathcal{C}_\omega(U) := \Big\{ f \in \mathcal{C}(U) : \parallel f \parallel_\omega := \sup\{ \mid f(z) \mid \cdot\, \omega(z) : z \in U \} < \infty \Big\}.$$

Assume that there are separable complex Hilbert spaces $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$ and a continuous operator $A : H_1 \to \mathcal{C}_\omega(U)$. Let $I : \mathcal{C}_\omega(U) \hookrightarrow H_2$ be a continuous embedding, then we can prove:

Lemma 6.7.2 *Let $H_2 := L^2(U,\mu)$ where μ is a measure such that $\omega^{-1} \in L^2(U,\mu)$. Then the operator $T := I \circ A \in \mathcal{L}(H_1, H_2)$ is of Hilbert-Schmidt type.*

Proof Fix an orthonormal basis $[\, e_k : k \in \mathbb{N}\,]$ of H_1 and denote by $\varepsilon_x : \mathcal{C}_\omega(U) \to \mathbb{C}$ the evaluation map in $x \in U$. Then ε_x is continuous with $\|\, \varepsilon_x \,\| \le \omega(x)^{-1}$. Consider the continuous functional $\Phi_x := \varepsilon_x \circ A \in H_1'$ and choose $g_x \in H_1$ with $\Phi_x = \langle \cdot, g_x \rangle_1$. Then we obtain:

$$\|\, g_x \,\|_1^2 = \sum_{k=1}^{\infty} \big|\, \langle e_k, g_x \rangle_1 \,\big|^2 = \sum_{k=1}^{\infty} \big|\, [\, \varepsilon_x \circ A\,](\, e_k\,) \,\big|^2 = \sum_{k=1}^{\infty} \big|\, [\, Ae_k \,](x) \,\big|^2 \tag{6.7.6}$$

for all $x \in H_1$. On the other hand we compute for the norm of g_x:

$$\|\, g_x \,\|_1^2 = \big|\, \langle g_x, g_x \rangle \,\big| = \big|\, \Phi_x(\, g_x\,) \,\big| = \big|\, \big[\, \varepsilon_x \circ A \,\big](\, g_x\,) \,\big| \le \|\, \varepsilon_x \,\| \, \|\, A \,\| \, \|\, g_x \,\|_1 \tag{6.7.7}$$

and this implies that $\|\, g_x \,\|_1 \le \|\, \varepsilon_x \,\| \, \|\, A \,\| \le \omega(x)^{-1} \,\|\, A \,\|$. Using (6.7.6) and (6.7.7) together we find:

$$\sum_{k=1}^{\infty} \big|\, [\, Te_k \,](x) \,\big|^2 = \sum_{k=1}^{\infty} \big|\, [\, Ae_k \,](x) \,\big|^2 = \|\, g_x \,\|_1^2 \le \omega^{-2}(\, x\,) \,\|\, A \,\|^2.$$

By our assumption the right hand side of this inequality is μ-integrable over U and so:

$$\sum_{k=1}^{\infty} \|\, Te_k \,\|_2^2 = \int_U \sum_{k=1}^{\infty} \big|\, [\, Te_k \,](x) \,\big|^2 d\mu(x) < \infty.$$

Hence $T : H_1 \to H_2$ is a Hilbert-Schmidt operator. □

Now we can construct the embeddings (6.7.5). The idea is to give an expansion (6.7.2) where we can control the behavior of the holomorphic parts $h_j \in \mathcal{H}(U)$ near the boundary.

Theorem 6.7.1 *Let U be an open subset of a $\mathcal{DFN}$-space E and fix two finite measures $\mu_1, \mu_2 \in \mathcal{MF}_2(U)$ where $\mathcal{F} = \mathcal{H}(U)$. Assume that there is a diagram*

$$\mathcal{H}^\infty(U) \supset \mathcal{A} \xrightarrow{\ J_1\ } \mathcal{H}^2(U, \mu_1) \xrightarrow{\ J_2\ } \mathcal{H}^2(U, \mu_2)$$

where J_i are continuous embeddings and J_2 is nuclear. Then there is a Hardy space $\mathcal{H}^2(\mathcal{S}_\mathcal{A}, \Theta)$ which admits a quasi-nuclear embedding into $\mathcal{H}^2(U, \mu_2)$.

Proof Fix an orthonormal basis $[\, e_j : j \in \mathbb{N}\,]$ of the Bergman space $\mathcal{H}^2(U, \mu_2)$ and let us denote by $k_2 : U \times U \to \mathbb{C}$ its Bergman kernel. Then for $z \in U$ and $f \in \mathcal{H}^2(U, \mu_1)$ it follows that:

$$f(\, z\,) = \Big\langle \,[\, J_2 f\,], k_2(\cdot, z\,) \Big\rangle_{\mu_2} = \Big\langle\, f, J_2^* \, k_2(\cdot, z\,) \Big\rangle_{\mu_1}. \tag{6.7.8}$$

Using the expansion $J_2^* \, k_2(\cdot, z\,) = \sum_{k=1}^\infty [\, J_2^* e_k \,] \cdot \overline{e_k(z)}$ which is convergent in $\mathcal{H}^2(U, \mu_1)$ we obtain from (6.7.8):

$$f(\, z\,) = \sum_{k=1}^{\infty} e_k(\, z\,) \cdot \Big\langle\, f, J_2^* e_k \Big\rangle_{\mu_1}.$$

Consider the holomorphic map $Eval : U \to \mathcal{A}'$. Because of $\| Eval(z) \|_{\mathcal{A}'} \leq 1$ the following expression is well defined in the sense of Bochner-integrals:

$$\text{Eval}(z) = \sum_{k=1}^{\infty} e_k(z) \cdot \int_U \text{Eval} \cdot \overline{[\, J_2^* e_k \,]} \, d\mu_1 \in \mathcal{A}'. \tag{6.7.9}$$

Let us denote by φ_k for $k \in \mathbb{N}$ the integrals in the expansion (6.7.9). Then we have written the evaluation $Eval(z) = \sum_{k=1}^{\infty} e_k(z) \cdot \varphi_k$ in the form (6.7.2) with the additional information that $e_k \in \mathcal{H}^2(U, \mu_2)$ with norm 1 for all $k \in \mathbb{N}$. Moreover, from the nuclearity of $J_2^* : \mathcal{H}^2(U, \mu_2) \to \mathcal{H}^2(U, \mu_1)$ and

$$\| \varphi_k \|_{\mathcal{A}'} \leq \int_U \| \text{Eval} \|_{\mathcal{A}'} \cdot | \, J_2^* e_k \, | \, d\mu_1 \leq c \cdot \| \, J_2^* e_k \, \|_{L^2(U,\mu_1)}.$$

where $c > 0$ we conclude that $\sum_{k=1}^{\infty} \| \varphi_k \|_{\mathcal{A}'} < \infty$. Now, we proceed as it was described before in the general construction of $\mathcal{H}^2(\mathcal{S}_{\mathcal{A}}, \Theta)$. We can choose a sequence of finite complex measures $\nu_j := \xi(\varphi_j) \in \mathcal{C}'(\mathcal{S}_{\mathcal{A}})$ such that

$$\sum_{j=1}^{\infty} \| \nu_j \|_{\mathcal{C}'(\mathcal{S}_{\mathcal{A}})} \leq c \cdot \sum_{j=1}^{\infty} \| \varphi_j \|_{\mathcal{A}'} \leq \infty \qquad \text{and} \qquad \Theta := \sum_{j=1}^{\infty} | \, \nu_j \, | \in \mathcal{M}(\mathcal{S}_{\mathcal{A}}).$$

Moreover, there is a sequence of pointwise defined functions $(f_k)_k \in L^{\infty}(\mathcal{S}_{\mathcal{A}}, \Theta)$ such that $| \, f_k(x) \, | \leq 1$ for all $x \in \mathcal{S}_{\mathcal{A}}$ and $d\nu_k = f_k \, d\Theta$. Consider the map

$$\Phi : U \to L^2\big(\mathcal{S}_{\mathcal{A}}, \Theta \big) : z \mapsto \sum_{k=1}^{\infty} e_k(z) \cdot f_k.$$

We show that Φ is well-defined and holomorphic on U. Because of $| \, f_k(\, x \,) \, | \leq 1$ it follows that:

$$\int_{\mathcal{S}_{\mathcal{A}}} | \, f_k \, |^2 \, d\Theta \leq \int_{\mathcal{S}_{\mathcal{A}}} | \, f_k \, | \, d\Theta = \| \, \nu_k \, \|_{\mathcal{C}(\mathcal{S}_{\mathcal{A}})'} \leq c \cdot \| \, \varphi_j \, \|_{\mathcal{A}'}$$

where c is a suitable positive number and so we immediately have from our inequalities above:

$$\int_{\mathcal{S}_{\mathcal{A}}} | \, \Phi(z) \, |^2 \, d\Theta \leq \sum_{k=1}^{\infty} \big| \, e_k(\, z \,) \, \big|^2 \sum_{k=1}^{\infty} \int_{\mathcal{S}_{\mathcal{A}}} | \, f_k \, |^2 \, d\Theta \leq c \cdot k_2(\, z, z \,) \sum_{k=1}^{\infty} \| \, \varphi_j \, \|_{\mathcal{A}'} \leq \infty.$$

Applying Theorem 6.2.2 there is a kernel function $K : U \times \mathcal{S}_{\mathcal{A}} \to \mathbb{C}$ such that both (a) and (b) below are fulfilled:

(a) For all $z \in U$ we have $\Phi(z) = [\, \mathcal{S}_{\mathcal{A}} \ni x \mapsto K(\, z, x \,) \in \mathbb{C} \,] \in L^2(\, \mathcal{S}_{\mathcal{A}}, \Theta \,)$.

(b) For all $x \in \mathcal{S}_{\mathcal{A}}$ the map $U \ni z \mapsto K(\, z, x \,) \in \mathbb{C}$ is holomorphic.

Let us define $\omega : U \to \mathbb{R}^+$ by $\omega(z) = \| K(z,\cdot) \|^{-1}_{L^2(\mathcal{S}_\mathcal{A},\Theta)} = \| \Phi(z) \|^{-1}_{L^2(\mathcal{S}_\mathcal{A},\Theta)}$, then by (6.7.4) we obtain a continuous embedding

$$j : \mathcal{H}^2\big(\mathcal{S}_\mathcal{A}, \Theta \big) \hookrightarrow \mathcal{C}_\omega(U).$$

Next we prove that $\omega^{-1} \in L^2(U,\mu_2)$. This follows by the monotone convergence theorem and:

$$\begin{aligned}
\int_U \omega^{-2}\, d\mu_2 &= \int_U \int_{\mathcal{S}_\mathcal{A}} \big| K(z,x) \big|^2 \, d\Theta(x)\, d\mu_2(z) \\
&= \int_{\mathcal{S}_\mathcal{A}} \int_U \Big| \sum_{k=1}^\infty e_k(z) \cdot f_k(x) \Big|^2 \, d\mu_2(z)\, d\Theta(x) \\
&= \int_{\mathcal{S}_\mathcal{A}} \sum_{k=1}^\infty | f_k |^2 \, d\Theta \\
&\le \sum_{k=1}^\infty \int_{\mathcal{S}_\mathcal{A}} | f_k | \, d\Theta \\
&= \sum_{k=1}^\infty \| \nu_k \|_{\mathcal{C}(\mathcal{S}_\mathcal{A})'} \\
&\le c \cdot \sum_{k=1}^\infty \| \varphi_k \|_{\mathcal{A}'} < \infty.
\end{aligned}$$

Hence the embedding $\mathcal{H}^2(\mathcal{S}_\mathcal{A},\Theta) \hookrightarrow \mathcal{H}^2(U,\mu_2)$ is well-defined and it factorizes over the space $\mathcal{C}_\omega(U)$. From Lemma 6.7.2 the assertion follows. □

Remark 6.7.1 Let $\mu_1 \in \mathcal{MF}_2(U)$, then it is always possible to construct $\mu_2 \in \mathcal{MF}_2(U)$ such that the embedding $J_2 : \mathcal{H}^2(U,\mu_1) \hookrightarrow \mathcal{H}^2(U,\mu_2)$ is nuclear. Because the composition of quasi-nuclear operators is nuclear it is sufficient to prove the existence of a quasi-nuclear embedding J_2. Let $K_{\mu_1} : U \times U \to \mathbb{C}$ be the Bergman kernel of μ_1, then we have for all $z \in U$:

$$\omega(z) := K_{\mu_1}(z, z) = \| \operatorname{Eval}(z) \|^2_{\mathcal{H}^2(U,\mu_1)'} \ge \mu_1(U)^{-2} > 0$$

and by Lemma 6.1.2 it follows that ω is continuous on U. Hence we can define a finite measure $\mu_2 \in \mathcal{MF}_2(U)$ by $d\mu_2 := \omega^{-1}\, d\mu_1$. Let $[\, e_j : j \in \mathbb{N}\,]$ be an orthonormal basis of $\mathcal{H}^2(U,\mu_1)$, then

$$\| J_2 \|^2_{HS} = \sum_{k\in\mathbb{N}} \| J_2 e_k \|^2_{L^2(U,\mu_2)} = \int_U \omega \, d\mu_2 = \mu_1(U) < \infty.$$

In the following example we compute the l^p-type of embedding from the classical Hardy space over the n-dimensional unit sphere ∂B_n into the Bergman space over the unit ball B_n. We write $\| \cdot \|$ for the Euclidean norm on $\mathbb{C}^n$.

Example 6.7.1 Let $B_n := \{ z \in \mathbb{C}^n : \| z \| < 1 \}$ be the open unit ball in $\mathbb{C}^n$ and write ∂B_n for its boundary. Equipped with the standard area measure σ on ∂B_n and the Lebesgue measure v on B_n we consider the classical Hardy space $\mathcal{H}^2(\partial B_n, \sigma)$ and the corresponding Bergman space $\mathcal{H}^2(B_n, v)$. It is well-known that the monomials $[\, z^\alpha : \alpha \in \mathbb{N}_0^n \,]$ form a complete orthogonal system in both spaces and by a standard calculation (see [102]) there is a constant $c > 0$ such that for all $\alpha \in \mathbb{N}_0^n$ the following equality holds:

$$\| z^\alpha \|_{L^2(\partial B_n, \sigma)} = c \cdot \big(\,|\, \alpha \,|+ n \,\big)^{\frac{1}{2}} \cdot \| z^\alpha \|_{L^2(B_n, v)}.$$

Because of $\sum_{\alpha \in \mathbb{N}_0^n} (\,|\, \alpha \,| + n \,)^{-\frac{p}{2}} < \infty$ for all $p > 2n$ we conclude that in the case of dimension $n \in \mathbb{N}$ the embedding $J : \mathcal{H}^2(\partial B_n, \sigma) \hookrightarrow \mathcal{H}^2(B_n, v)$ has l^p-type $p > 2n$. In particular, it follows that J is not quasi-nuclear even in the case $n = 1$.

Chapter 7

Invariant measures for special groups of homeomorphisms on infinite dimensional spaces

Given a topological space X with σ-finite Borel measure μ, a locally compact group G and a representation B of G in the group of all homeomorphisms of X, we examine how to construct a Borel measure μ_s on X which is invariant under $B(G)$ (Lemma 7.1.4). In many cases this construction leads to a non-trivial representation of G on $L^p(X, \mu_s)$. Under some additional conditions on G, X and the representation B we show that in the case where μ has the $\mathcal{NF}_p$-property, the symmetrized measure μ_s is a $\mathcal{NF}_p$-measure, as well (Theorem 7.2.1). Finally we give some examples and an application of our work leads to the construction of spectrally invariant algebras (Ψ^*- or Ψ_0-algebras, cf. [69], [77])) of $\mathcal{C}^\infty$-elements in operator-algebras on L^p-spaces and Bergman spaces.

This chapter is a joint work with M. Hoeber; the main idea arose when we considered the following two problems:

(a) Let (W, μ) be an open subset of a Hilbert space H with Gaussian measure μ_g, where μ is the restriction of μ_g to W. Furthermore, let $(B_t)_{t\in G}$ be a (semi)-group of homeomorphisms of W where G is a compact or locally compact group. Is it possible to find a measure $\tilde{\mu}$ on W invariant with respect to (B_t), namely $\tilde{\mu}(B_t(A)) = \tilde{\mu}(A)$ for all $\mu-$measurable sets $A \subset W$ and $t \in G$ such that $\tilde{\mu}(A) > 0$ for all open nonempty sets $A \subset W$?

(b) Let H^m be a product of an infinite dimensional Hilbert-space H with a Gaussian measure μ (e.g. product of suitable Sobolev spaces). We assume that $H \subseteq \mathcal{C}(\overline{\Omega}, \mathbb{C})$, where $\overline{\Omega}$ is the closure of an open and bounded subset of $\mathbb{R}^n$ with nice boundary. Let U be a region in $\mathbb{C}^m$ and G a closed subgroup of the group $\mathrm{Aut}(U)$ of all biholomorphic maps of U. Let $W := \{ f \in H^m \ : \ f(\overline{\Omega}) \subset U \}$. Is it possible to find an invariant measure $\tilde{\mu}$ on W such that $\tilde{\mu}(\alpha(A)) = \tilde{\mu}(A)$ for all μ-measurable sets $A \subset W$ and all $\alpha \in G$ such that $\tilde{\mu}(A) > 0$ for all open nonempty sets $A \subset W$?

Let (M, g) be a Riemannian manifold with metric g. Then it is well-known that each isometry Φ on M leaves the Riemannian measure m_R invariant (see [85], p. 85) and so Φ leads to an isometry of the spaces $L^p(M, m_R)$ where $1 \leq p < \infty$. In particular, each semi-group $(\alpha_t)_{t\geq 0}$ of isometries on M can be represented as a semi-group of isometric *composition operators* $(C_t)_{t\geq 0}$ on $L^p(M, m_R)$ by setting $C_t(f) := f \circ \alpha_t$ for $f \in L^p(M, m_R)$. In the case where $(C_t)_{t\geq 0}$ is strongly continuous it follows from the general theory of semi-groups on Banach spaces that it defines a closed generator A which is connected to the geometry of M.

If the underlying measure space X is not locally compact one has to be more careful about the existence of invariant measures even if we deal with quite natural groups of isomorphisms acting on X. It is well-known that on an infinite dimensional separable Hilbert-space H there is no translation invariant Borel measure μ such that bounded sets have finite measure and it holds $\mu(U) > 0$ for all open nonempty sets $U \subset H$ (see [104]). Hence the group action of H on itself by translation does not lead to an unitary representation of H in $L^p(H, \mu)$ for any Borel measure μ on H with the described properties. Moreover, Oxtoby (cf.[119]) showed, that on a complete separable metric group $\mathcal{G}$, which is not locally compact, there exists no non-trivial left-invariant Borel measure μ such that μ is locally finite or $\mu(K) < \infty$ for all $K \subset \mathcal{G}$ compact.

Here we consider the case in between. A locally compact space G acts on a topological space X which not necessarily has to be locally compact. More precisely, starting with a measure μ on X and a representation $B : G \to Homeo(X)$ of a locally compact group G into the group of all homeomorphisms on X, we adapt μ such that it becomes invariant under all homeomorphisms $B_t \in B(G)$ (Lemma 7.1.4). This construction is quite general and, in particular, it applies to the case where X is an open subspace of a separable infinite dimensional Hilbert-space or of a $\mathcal{DFN}$-space (the dual space of a nuclear Frèchet space) (Theorem 7.1.1). As a result we obtain an answer to problem (a). The definitions will be as follows:

Denote by m a left invariant *Haar measure* m on G, which is finite if and only if G is compact (in this case we choose m such that $m(G) = 1$). Let μ be any positive and σ-finite Borel measure on X and assume that the map $G \ni t \mapsto \mu(B_t^{-1}C) \in [\, 0, \infty\,]$ is Borel-measurable on G for all sets C in the Borel-σ-algebra $\mathcal{B}(X)$, then define

$$\mu_s(C) := \int_G \mu\big(B_t^{-1}C \big)\, dm(t).$$

We obtain a measure μ_s which is invariant under the action of G on X (e.g. $\mu_s(B_t^{-1}C) = \mu_s(C)$ for all $t \in G$) and finite in the case where μ is finite and G is compact (in general μ_s not even has to be σ-finite). We show that the definition of μ_s is meaningful if X is a polish space (i.e. complete metric space with countable base of topology) or an open set in a $\mathcal{DFN}$-space. Let $\tilde{B}_t$ denote the induced group action on $L^p(X, \mu_s)$ defined by the composition operators $\tilde{B}_t f := f \circ B_t$ for $f \in L^p(X, \mu_s)$. Then in many cases $(\tilde{B}_t)_{t\in G}$ is a strongly continuous group representation if $(B_t)_{t\in G}$ is so (Proposition 7.2.1, 7.2.2, 7.2.3). Here we use some measure theoretic methods, e.g. *Kuratowski's Theorem* and the fact that every open subset U of a $\mathcal{DFN}$-space can be written as a countable union of compact metric spaces (we say U is *hemi-compact*).

Our construction produces closed operators attached to infinite dimensional spaces (or manifolds). This leads to Fréchet operator algebras with spectral invariance ([79], [69], [107], [106]) respectively non-commutative geometries with prescribed properties using systems of closed operators also in the singularities of the underlying space.

Let $\mathcal{F} \subset \mathcal{C}(X)$ be a subspace of all continuous complex-valued functions on X. In chapter 5, Definition 5.4.1 we have introduced the notion of a $\mathcal{NF}_p$-measure μ. Roughly speaking μ is characterized by the property that the embedding

$$\tilde{\mathcal{F}} := \mathcal{F} \cap L^p(X, \mu) \hookrightarrow \mathcal{C}(X)$$

is continuous if $\tilde{\mathcal{F}}$ carries the $L^p(X,\mu)$-topology and $\mathcal{C}(X)$ is equipped with the *compact-open topology* (topology of uniform convergence on all compact subsets of X). Hence in the case where $\mathcal{C}(X)$ is complete we can consider the closure $\tilde{\mathcal{F}}_c$ of $\tilde{\mathcal{F}}$ in $L^p(X,\mu)$ as a space of continuous functions on X.

We give conditions on X, the group G and the representation B under which the described process of symmetrization of a given $\mathcal{NF}_p$-measure μ again defines a $\mathcal{NF}_p$-measure μ_s (Theorem 7.2.1). Starting with a $B(G)$-invariant subspace $\mathcal{F} \subset \mathcal{C}(X)$ (i.e. $\tilde{B}_t(\mathcal{F}) \subset \mathcal{F}$ for all $t \in G$) this enables us to consider groups of composition operators acting on closed subspaces of $L^p(X, \mu_s)$.

In the case where $p = 2$ we can define the orthogonal projection from $L^2(X, \mu_s)$ onto $\tilde{\mathcal{F}}_c$. We show that P and all $\tilde{B}_t$ commute as operators on $L^2(X, \mu_s)$ (Corollary 7.6.1). We denote by $\mathcal{T}(S) \subset \mathcal{L}(\tilde{\mathcal{F}}_c)$ the C^*-Toeplitz algebra generated by operators $T_f := PM_f$ on $\tilde{\mathcal{F}}_c$ with symbols f in a space S of bounded measurable and $\mathcal{B}$-invariant symbols. It turns out that $\mathcal{T}(S)$ is invariant under the isomorphisms $\mathbf{B}_t \in \mathcal{L}(\ \mathcal{L}(L^2(X,\mu))\ )$ defined by

$$\mathbf{B}_t(A) := \tilde{B}_t\, A\, \tilde{B}_{t^{-1}}, \qquad (t \in G).$$

This fact in connection with the general theory of [79], [69], [107] and [106] gives the possibility to construct Ψ^*-algebras in $\mathcal{T}(S)$ defined by iterated commutators with the infinitesimal generator of $(\mathbf{B}_t)_{t\in G}$.

We give several examples how to obtain homeomorphisms $(B_t)_{t\in G}$ which can be used in the constructions described above. In particular, we discuss the case of measures on finite products of Hilbert-spaces which are embedded in a space of continuous function, e.g. let us take Sobolev-spaces of continuous functions. In case of our constructions we give an answer to problem (b) mentioned above.

By quite similar methods we show that we can lift strongly continuous semi-groups $(B_t)_{t\geq 0}$ of invertible operators on Hilbert-spaces to semi-groups $(\tilde{B}_t)_{t\geq 0}$ of composition operators on $L^2(H, \mu_{s,\alpha})$ (Theorem 7.3.1). Here $\mu_{s,\alpha}$ ($\alpha > 0$) is a finite Borel measure on H arising from an infinite dimensional *Gaussian measure*. The semi-group $(\tilde{B}_t)_{t\geq 0}$ fails to be unitary but we obtain $\|\,\tilde{B}_t\,\| \leq e^{\frac{\alpha}{2}t}$ for all $t \geq 0$. More general, instead of H we can take open or closed subsets U of H and assume that $(B_t)_{t\geq 0}$ is a semi-group of homeomorphism of U.

Finally, by a different method using the eigen-functions of the *Beltrami-Laplace operator* we show how to construct Gaussian measures on L^2-spaces over a compact and connected Riemannian manifolds M which are invariant under all composition operators with isometries Φ on M (Proposition 7.5.1, Theorem 7.5.2). This construction is closely related to the theory of dynamical systems.

7.1 Symmetric Borel measures on topological spaces

Let (X, Σ_1, μ) and (Y, Σ_2, m) be measure spaces. We denote by $M(X,Y)$ the space of all measurable functions from X to Y. Let $M^{-1}(X,Y)$ be the subspace of $M(X,Y)$ consisting of all invertible functions $h : X \to Y$ such that h as well as its inverse are measurable. We often write $M(X)$ (resp. $M^{-1}(X)$) instead of $M(X,X)$ (resp. $M^{-1}(X,X)$). Let $Q \in M(X)$, then the measure μ is called *Q-invariant* (or *Q-preserving*) iff $\mu^Q = \mu$ where $\mu^Q(M) := \mu(Q^{-1}M)$ for all $M \in \Sigma_1$. Generalizing the notation of *Q-invariance* to families of measurable maps, we define:

Definition 7.1.1 Let $\mathcal{Q} \subset M(X)$, then we call μ a $\mathcal{Q}$-invariant (or $\mathcal{Q}$-preserving) measure, if μ is Q-invariant for all $Q \in \mathcal{Q}$.

In the following we write $\mathcal{M}_\sigma(X)$ for the space of all σ-finite measures on X. In the case where X also is considered as a topological space the σ-algebra Σ_1 always will be the Borel σ-algebra $\mathcal{B}(X)$ on X. We denote by $\Sigma_1 \otimes \Sigma_2$ the smallest σ-algebra in $X \times Y$ such that both projections $P_X : X \times Y \to X$ and $P_Y : X \times Y \to Y$ are measurable.

We remind of the notion of *group representations*. Let G be a locally compact group, then by $Homeo(X)$ we denote the space of all homeomorphisms of X. A group homomorphism

$$B : G \ni t \mapsto B_t \in \mathrm{Homeo}(X)$$

is called a *representation of G in $Homeo(X)$*. The representation B is said to be continuous (resp. measurable) iff the map $(t,x) \mapsto B_t x$ of $G \times X$ into X is continuous (resp. iff this map is $\mathcal{B}(G \times X) - \mathcal{B}(X)$-measurable).

To begin with, we explicitly compute how a weighted Lebesgue measure on an open subset of $\mathbb{R}^n$ can be adapted to a given group representation. We are making use of the transformation formula for the Lebesgue integral which in general is not available for arbitrary measure spaces.

Fix $n \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ be open and G a compact group with unit $e \in G$. By Diff(Ω) we denote the group of all diffeomorphisms of Ω. Assume that $B : G \to \mathrm{Diff}(\Omega)$ is a continuous representation of G in Diff(Ω). Starting with a weighted Lebesgue measure $\mu \in \mathcal{M}_\sigma(\Omega)$ we want to construct a measure $\mu_s \in \mathcal{M}_\sigma(\Omega)$ which is $B(G)$-invariant. This construction arises from a procedure of integration of μ along $B(G)$. For $i = 1, \cdots, n$ we denote by

$\pi_i : \mathbb{R}^n \to \mathbb{R}$ the projection on the i-th component. Then we assume that all the maps given in (i) and (ii):

(i) $\Omega \ni z \mapsto \Big[G \ni t \mapsto \pi_i \circ B_{t^{-1}} z \Big] \in \mathcal{C}(G, \mathbb{R})$ for $i = 1, \cdots, n$;

(ii) $\Omega \ni z \mapsto \Big[G \ni t \mapsto \frac{\partial}{\partial z_j} \{ \pi_i \circ B_{t^{-1}} z \} \Big] \in \mathcal{C}(G, \mathbb{R})$ for $i, j = 1, \cdots, n$

are well-defined and continuous on Ω if $\mathcal{C}(G, \mathbb{R})$ carries the topology of uniform convergence on G. Let m be the unique translation-invariant *Haar-measure* on G with $m(G) = 1$ and assume that $g : \Omega \to \mathbb{R}^+$ is a positive and continuous weight-function. Let us consider $\mu \in \mathcal{M}_\sigma(\Omega)$ defined by $d\mu = g\, dv$, where v is the usual Lebesgue measure on Ω. We show that a $B(G)$-invariant measure μ_s on Ω is given by $d\mu_s := f\, dv$ where

$$f(z) := \int_G g \circ B_{t^{-1}}(z) \Big| \det \big[D_z B_{t^{-1}} \big](z) \Big| \, dm(t), \qquad (z \in \Omega). \tag{7.1.1}$$

Lemma 7.1.1 *Let $\Omega \subset \mathbb{R}^n$ be open and assume that $\mu_s \in \mathcal{M}_\sigma(\Omega)$ is defined by $d\mu_s = f\, dv$. Then μ_s is $B(G)$-invariant.*

Proof Let $t_0 \in G$ and $A \in \mathcal{B}(\Omega)$ be a Borel set in Ω. Then, using the *transformation formula* for the Lebesgue integral, we find with the characteristic function χ_A of A:

$$\begin{aligned}
&\mu_s(B_{t_0}^{-1} A) \\
&= \int_\Omega \chi_{B_{t_0}^{-1} A}(z) f(z)\, dv(z) \\
&= \int_G \int_\Omega \chi_A \circ B_{t_0}(z)\, g \circ B_{t^{-1}}(z) \Big| \det \big[D_z B_{t^{-1}} \big](z) \Big| \, dv(z)\, dm(t) \\
&= \int_G \int_\Omega \chi_A(z) g \circ B_{(t_0 t)^{-1}}(z) \Big| \det \big[D_z B_{t^{-1}} \big](B_{t_0^{-1}}(z)) \cdot \det \big[D_z B_{t_0^{-1}} \big](z) \Big| \, dv(z)\, dm(t) \\
&= \int_\Omega \chi_A(z) \int_G g \circ B_{(t_0 t)^{-1}}(z) \Big| \det \big[D_z B_{(t_0 t)^{-1}} \big](z) \Big| \, dm(t)\, dv(z) \\
&= \int_\Omega \chi_A(z) f(z)\, dv(z) = \mu_s(A).
\end{aligned}$$

Here we have used the translation invariance of m on G in the last equality. □

Let X be a topological space, $\mathcal{F} \subset \mathcal{C}(X)$ a subspace. In Definition 5.3.1 we have introduced the notion of $\mathcal{NF}_p$-measures and $\mathcal{NF}_p$-spaces for $1 \le p < \infty$. The question arises whether or not the invariant measure μ_s is a $\mathcal{NF}_p$-measure for $\mathcal{F} \subset \mathcal{C}(\Omega)$, whenever μ has this property. We can prove the following easy lemma:

Lemma 7.1.2 *Let $\mu \in \mathcal{MF}_p(X)$ where $p \ge 1$. If $g : X \to \mathbb{R}^+$ is a continuous positive function and $\tilde{\mu}$ is defined by $d\tilde{\mu} = g \cdot d\mu$, then $\tilde{\mu} \in \mathcal{MF}_p(X)$ as well.*

Proof Fix a compact set $K \subset X$. Then, by assumption, there is a compact set $H \subset X$ such that $K \subset H$ and $C > 0$ with

$$\sup\Big\{ \,|\, f(x) \,|\, : x \in K \Big\} \leq C \left[\int_H |\, f\,|^p \, d\mu \right]^{\frac{1}{p}}.$$

for all $f \in \mathcal{F}$. Define $\varepsilon := \inf\{ \,|\, g(z) \,|\, : z \in H \} > 0$, then inequality (5.4.1) holds with $\tilde{\mu}$ instead of μ and $C\varepsilon^{-\frac{1}{p}} > 0$ instead of C. $\square$

Remark 7.1.1 From Lemma 7.1.2 it is easy to see that for each continuous function $h : X \to \mathbb{C}$ and each finite measure $\mu \in \mathcal{MF}_p(X)$ it can be constructed $\tilde{\mu} \in \mathcal{MF}_p(X)$ such that h is $\tilde{\mu}$-integrable (use the continuous positive weight $g(z) := (\, 1 + |\, h(z) \,| \,)^{-1}$ for all $z \in X$ and cf. Lemma 6.1.1).

For the next lemma let us assume that $\Omega \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$. Then we obtain with our notations above:

Lemma 7.1.3 *Assume that $g : \Omega \to \mathbb{R}^+$ is uniformly continuous. Then μ as well as μ_s belong to $\mathcal{MF}_p(\Omega)$ where $\mathcal{F} := \mathcal{H}(\Omega)$ is the space of all holomorphic functions on Ω and $1 \leq p \leq 2$.*

Proof According to example 5.4.1 (a) we have $v \in \mathcal{MF}_p(\Omega)$ for $1 \leq p \leq 2$. In order to show that μ_s is a $\mathcal{NF}_p$-measure it is enough to prove that $f : \Omega \to \mathbb{R}^+$ in (7.1.1) is continuous and positive (see Lemma 7.1.2). This easily follows from assumptions (i) and (ii) on B together with uniform estimates on g. $\square$

If we deal with a topological space X (e.g. X is an infinite dimensional Hilbert-space or a $\mathcal{DFN}$-space) in general we can not directly make use of the transformation formula. Let us find an equivalent definition for μ_s where μ is a finite Borel measure on X. For a Borel set $A \in \mathcal{B}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is open we have from our definitions above ($d\mu = g\, dv$):

$$\begin{aligned}
\mu_s(A) &= \int_\Omega \int_G \chi_A(z)\, g \circ B_{t^{-1}}(z) \Big|\, \det \big[\, D_z B_{t^{-1}} \,\big](z) \,\Big|\, dm(t)\, dv(z) \\
&= \int_G \int_\Omega \chi_A \circ B_t(z)\, g(z) \Big|\, \det \big[\, D_z B_{t^{-1}} \,\big](\, B_t z \,) \cdot \det \big[\, D_z B_t \,\big](z) \,\Big|\, dv(z)\, dm(t) \\
&= \int_G \int_\Omega \chi_{B_{t^{-1}} A}(z) \Big|\, \det \big[\, D_z B_{t^{-1} t} \,\big](z) \,\Big|\, g(z)\, dv(z)\, dm(t) \\
&= \int_G \mu\big(\, B_t^{-1} A \,\big)\, dm(t).
\end{aligned}$$

We have used that $B_{t^{-1}t} = B_e = id$. The expression on the right hand side also makes sense for a wider class of Borel measures $\tilde{\mu}$ on a topological space X, provided that the mapping

$$G \ni t \mapsto \tilde{\mu}\big(\, B_t^{-1} A \,\big) \in [\, 0, \infty \,]$$

is $\mathcal{B}(G)$-measurable. We intend to examine this question in greater generality for measure spaces which not necessarily are carrying a group structure:

Definition 7.1.2 Let (X, Σ_1, μ) and (Y, Σ_2, m) be σ-finite measure spaces. Assume that there is a map $B : Y \to M^{-1}(X)$ such that

$$Y \ni t \mapsto \mu\big(B_t^{-1} A \big) \in [\,0, \infty\,] \tag{7.1.2}$$

is Σ_2- measurable for all $A \in \Sigma_1$. Then we define the *symmetrization* μ_s of μ w.r.t. to B to be the integral

$$\mu_s(A) := \int_Y \mu\big(B_t^{-1} A \big)\, dm(t).$$

In our applications we often assume that X is a topological space with Borel σ-algebra $\mathcal{B}(X)$ and μ is a finite or σ-finite Borel measure on X. For the measure space (Y, Σ_2, m) we choose a compact or locally compact group $G = Y$ with the translation invariant *Haar-measure* m. The mapping $B : G \to M^{-1}(X)$ is a group homomorphism from G into $Homeo(X)$.

Lemma 7.1.4 *The symmetrization μ_s defines a Borel measure on Σ_1. If in addition the space $Y = G$ is a locally compact group with left-invariant Haar-measure m and $\Sigma_2 := \mathcal{B}(G)$ then μ_s is $B(G)$-invariant for a group homomorphism $B : G \to M^{-1}(X)$.*

Proof By assumption the map $Y \ni t \mapsto \mu(B_t^{-1} A) \in [\,0, \infty\,]$ is Σ_2-measurable for any set $A \in \Sigma_1$ and we conclude that μ_s is well-defined on Σ_1. We prove the σ-additivity of μ_s. Let $(A_i)_{i \in \mathbb{N}} \subset \Sigma_1$ be a sequence such that $A_i \cap A_j = \emptyset$ for $i \neq j$. Because for each $t \in G$ the map B_t is one-to-one it follows that:

$$B_t^{-1} A_i \cap B_t^{-1} A_j = \emptyset \quad \text{for} \quad i \neq j \qquad \text{and} \qquad B_t^{-1}\Big[\bigcup_i A_i \Big] = \bigcup_i B_t^{-1} A_i.$$

Hence by the σ-additivity of μ we have:

$$Y \ni t \mapsto \sum_i \mu\big(B_t^{-1} A_i \big) = \mu\Big(B_t^{-1}\Big[\bigcup_i A_i \Big] \Big) \in [\,0, \infty\,] \tag{7.1.3}$$

and the map (7.1.3) is Σ_2-measurable. Now, the theorem of dominated convergence applied to μ implies that:

$$\begin{aligned} \mu_s\Big(\bigcup_i A_i \Big) &= \int_G \sum_i \mu\big(B_t^{-1} A_i \big)\, dm(t) \\ &= \sum_i \int_G \mu\big(B_t^{-1} A_i \big)\, dm(t) = \sum_i \mu_s\big(A_i \big). \end{aligned}$$

In the case where $Y = G$ is a locally compact group with left-invariant Haar-measure m and $B : G \to M^{-1}(X)$ is a group homomorphism we can prove the $B(G)$-invariance of the measure μ_s. Fix $t_0 \in G$ and $A \in \Sigma_1$, then it follows that:

$$\mu_s^{B_{t_0}}(A) = \mu_s\big(B_{t_0^{-1}} A \big) = \int_G \mu\big(B_{t^{-1}} B_{t_0^{-1}} A \big)\, dm(t) = \int_G \mu\big(B_{(t_0 t)^{-1}} A \big)\, dm(t) = \mu_s(A)$$

by the left-translation invariance of the Haar-measure m on G. □

With the notations of Definition 7.1.2 we want to find conditions under which the map (7.1.2) is Σ_2- measurable on Y for all $A \in \Sigma_1$.

Lemma 7.1.5 *Let $F : Y \times X \to X$ defined by $F(t,x) := B_t x$ be $\Sigma_2 \otimes \Sigma_1$-$\Sigma_1$-measurable. Then $Y \ni t \mapsto \mu\big(B_t^{-1} A \big) \in [\,0,\infty\,]$ is Σ_2-measurable for each $A \in \Sigma_1$.*

Proof Let $A \in \Sigma_1$. By our assumption $\chi_A \circ F : Y \times X \to \mathbb{R}$ is $\Sigma_2 \otimes \Sigma_1$-measurable. Using *Tonelli's theorem* it follows that:

$$Y \ni t \mapsto \int_X \chi_A \circ F(\,t,\cdot)\, d\mu = \int_X \chi_{B_t^{-1}A}\, d\mu = \mu\big(B_t^{-1} A \big) \in [\,0,\infty\,]$$

is a Σ_2-measurable function (cf. [7]). □

We conclude that under the assumptions of Lemma 7.1.5 the symmetrization μ_s of μ is a well-defined measure on $(\,X,\Sigma_1\,)$ (which does not have to be σ-finite again).

Let $\Omega \subset \mathbb{R}^n$ be open, $g : \Omega \to \mathbb{R}^+$ a continuous and strictly positive weight function and the measure $\mu \in \mathcal{M}_\sigma(\Omega)$ be defined by $d\mu = g\,dv$. Given a continuous representation B of a compact group G in Diff(Ω) with (i) and (ii) above we have shown (see Lemma 7.1.1) that the $B(G)$-invariant measure μ_s is absolutely continuous w.r.t. the Lebesgue measure. The following example points out that this property does not hold in the more general setting of Definition 7.1.2. We give a finite measure μ on a Hilbert space H with the property $\mu(U) > 0$ for all open subsets $U \subset H$ and a group representation $B : \mathbb{R} \to Homeo(H)$ such that μ and μ_s are *orthogonal* (i.e. there is $X \subset H$ with $\mu(X) = 1$ and $\mu_s(X) = 0$, see [48, p. 60]).

Example 7.1.1 Let H_1, H_2 be separable infinite dimensional Hilbert spaces. In addition we assume that there is a dense and continuous embedding $I : H_1 \hookrightarrow H_2$. Fix a Gaussian measure μ_1 on H_1 with the property $\mu_1(U) > 0$ for all open subsets $U \subset H_1$ and define the measure μ_2 on H_2 by $\mu_2(A) := \mu_1(A \cap H_1)$ for all $A \in \mathcal{B}(H_2)$. Then $\mu_2 = \mu_1^I$ and it is well-known (see [48], p. 44) that μ_2 is a Gaussian measure on H_2. Moreover,

$$\mu_2(H_1) = \mu_1(H_1) = 1 \qquad \text{and} \qquad \mu_2(V) > 0 \tag{7.1.4}$$

for all open sets $V \subset H_2$ from the fact that H_1 is dense in H_2. Choose $0 \neq a \in H_2 \setminus H_1$ and consider the representation $(B_t)_{t\in\mathbb{R}}$ of $\mathbb{R}$ in H_2 defined by $B_t y := y + ta$ for all $y \in H_2$. Because of $H_1 + ta \cap H_1 = \emptyset$ for $t \neq 0$ and $\mu_2(H_1) = \mu_2(H_2) = 1$ it follows that

$$\mu_2\,(\,H_1 + ta\,) = 0 \qquad \forall\;\; t \neq 0.$$

Let us choose $(\,X,\Sigma_1,\mu\,) = (\,H_2,\mathcal{B}(H_2),\mu_2\,)$ and $(\,Y,\Sigma_2,m\,) = (\,\mathbb{R},\mathcal{B}(\mathbb{R}),e^{-t^2}dt\,)$ in Definition 7.1.2. We obtain:

$$\mu_2\,(H_1) = 1, \qquad\qquad (\mu_2)_s\,(H_1) = \int_{\mathbb{R}} \mu_2\big(\,H_1 + ta\,\big)e^{-t^2}dt = 0$$

and so the measures μ_2 and $(\mu_2)_s$ are orthogonal on H_2 with the desired properties.

Now, let us describe how to integrate w.r.t μ_s. With the notations of Definition 7.1.2 we assume that the function $F : Y \times X \to X$ with $F(t,x) := B_t x$ is $\Sigma_2 \otimes \Sigma_1$- Σ_1-measurable.

Lemma 7.1.6 *Let $f : X \to [\,0,\infty\,]$ be a non-negative Σ_1-measurable numerical function. Then:*

$$\int_X f \, d\mu_s = \int_{Y\times X} f \circ F \, d(\, m \otimes \mu\,)$$

where we denote by $m \otimes \mu$ the product measure on $\Sigma_2 \otimes \Sigma_1$.

Proof First let us assume that $g : X \to \mathbb{R}_0^+$ is a Σ_1-step-function on X. Then we can write $g = \sum_{i=1}^n \alpha_i \cdot \chi_{A_i}$ where $A_i \in \Sigma_1$ and $\alpha_i > 0$ for $i = 1, \cdots, n$. It follows that:

$$\begin{aligned}\int_X g \, d\mu_s &= \sum_{i=1}^n \alpha_i \cdot \mu_s(A_i) \qquad (7.1.5)\\ &= \sum_{i=1}^n \alpha_i \int_Y \int_X \chi_{B_t^{-1}A_i}(x) \, d\mu(x) \, dm(t)\\ &= \sum_{i=1}^n \alpha_i \int_Y \int_X \chi_{A_i} \circ F(\,t,x\,) \, d\mu(x) \, dm(t) = \int_{Y\times X} g \circ F \, d(\, m\otimes\mu\,).\end{aligned}$$

For an arbitrary Σ_1-measurable numerical function $f \geq 0$ let $(g_n)_{n\in\mathbb{N}}$ be a sequence of non-negative Σ_1-step-functions with $g_n \uparrow f$. Then $(g_n \circ F)_{n\in\mathbb{N}}$ is a sequence of $\Sigma_2 \otimes \Sigma_1$-step-functions with $g_n \circ F \uparrow f \circ F$. From equation (7.1.5) the assertion follows. □

In particular, under the conditions of Lemma 7.1.6 it follows that a Σ_1-measurable numerical function $f : X \to \mathbb{C}$ is μ_s-integrable iff $f \circ F : G \times X \to \mathbb{C}$ is $m \otimes \mu$-integrable and the integrals coincide. Let $(\, X, \Sigma_1, \mu\,)$ be a σ-finite measure space and let $Y := G$ be a locally compact group with left-invariant Haar-measure m. If $B : G \to M^{-1}(X)$ is a representation of G such that $F : G \times X \to X$ in Lemma 7.1.5 is $\mathcal{B}(G) \otimes \Sigma_1$- Σ_1-measurable, then we can prove:

Corollary 7.1.1 *Let $t_1, t_2 \in G$ and $f : X \to \mathbb{C}$ be Σ_1-measurable. Then $f \circ B_{t_1}$ is μ_s-integrable iff $f \circ B_{t_2}$ is μ_s-integrable and in this case both integrals coincide.*

Proof By Lemma 7.1.6, *Fubini's Theorem* and the translation invariance of m we find:

$$\begin{aligned}\int_X \big|\, f \circ B_{t_1} \big| \, d\mu_s &= \int_{G\times X} \big|\, f \circ B_{t_1} \circ B_t(z) \big| \, d(m\otimes\mu)(t,z)\\ &= \int_X \int_G \big|\, f \circ B_{t_1 t}(z) \big| \, dm(t) \, d\mu(z)\\ &= \int_X \int_G \big|\, f \circ B_{t_2 t}(z) \big| \, dm(t) \, d\mu(z) = \int_X \big|\, f \circ B_{t_2} \big| \, d\mu_s.\end{aligned}$$

Now, the assertion follows from *Tonelli's theorem.* □

For a topological space Y denote by $O(Y)$ the family of all open sets in Y. A complete metric space Y with countable base $\mathcal{X} \subset O(Y)$ (i.e. each $A \in O(Y)$ is union of sets in the countable system $\mathcal{X}$) is called *polish space*. In general the inclusion:

$$\mathcal{B}(Y) \otimes \mathcal{B}(X) \subset \mathcal{B}(Y \times X)$$

holds, but if we restrict ourselves to *polish spaces* or $\mathcal{DFN}$*-spaces* we can prove:

Proposition 7.1.1 *Let Y and X be polish spaces and consider $Y \times X$ with the product metric. Then we have $\mathcal{B}(Y \times X) = \mathcal{B}(Y) \otimes \mathcal{B}(X)$.*

Proof Fix countable bases $\mathcal{Y}$ (resp. $\mathcal{X}$) of open sets in Y (resp. in X) and consider the system:

$$\mathcal{Y} \otimes \mathcal{X} := \Big\{ U \times V : U \in \mathcal{Y} \text{ and } V \in \mathcal{X} \Big\} \subset O(\, Y \times X \,).$$

Then $\mathcal{Y} \otimes \mathcal{X}$ is a countable base for $Y \times X$ and so it generates $\mathcal{B}(Y \times X)$. On the other hand $\mathcal{Y}$ (resp. $\mathcal{X}$) generates $\mathcal{B}(Y)$ (resp. $\mathcal{B}(X)$) and so by Satz 22.1 in [7] we conclude that $\mathcal{Y} \otimes \mathcal{X}$ also generates $\mathcal{B}(Y) \otimes \mathcal{B}(X)$. Hence $\mathcal{B}(Y \times X) = \mathcal{B}(Y) \otimes \mathcal{B}(X)$. ☐

Let us consider a $\mathcal{DFN}$-space E. In general there is no metric on E which induces the topology. But it is known (cf. Lemma 5.3.1 and Lemma A.1.2) that each open subset $U \subset E$ can be written as a countable union of compact metric spaces each with countable base (we have called U *hemi-compact*).

Proposition 7.1.2 *Let E be a $\mathcal{DFN}$-space and $U \subset E$ be open. If Y is a polish space and $Y \times U$ carries the product topology, then $\mathcal{B}(Y \times U) = \mathcal{B}(Y) \otimes \mathcal{B}(U)$.*

Proof Fix a fundamental system $(K_i)_{i \in \mathbb{N}} \subset U$ of compact sets (i.e. $K_i \subset K_{i+1}$ for $i \in \mathbb{N}$ and $U = \bigcup_i K_i$, see [116]). Then for each $i \in \mathbb{N}$ the complete metric space K_i has a countable base $\mathcal{K}_i \subset O(K_i) \subset \mathcal{B}(U)$. Fix a countable base $\mathcal{Y} \subset O(Y)$ of Y and consider the system:

$$\mathcal{Y} \otimes \mathcal{K} := \bigcup_{i \in \mathbb{N}} \Big\{ Z \times V_i : Z \in \mathcal{Y} \text{ and } V_i \in \mathcal{K}_i \Big\}.$$

Then $\mathcal{Y} \otimes \mathcal{K}$ is a countable system of sets in $\mathcal{B}(Y \times U)$. Indeed, if $P_Y : Y \times U \to Y$ and $P_U : G \times U \to U$ denote the continuous projections, it follows:

$$Z \times V_i = P_Y^{-1}(Z) \cap P_U^{-1}(V_i) \subset \mathcal{B}(Y \times U), \qquad \forall\ Z \times V_i \in \mathcal{Y} \otimes \mathcal{K}.$$

Let $W \subset Y \times U$ be open and $(x, w) \in W$. Then fix $i \in \mathbb{N}$ with $(x, w) \in Y \times K_i$. Because the intersection $W \cap [\, Y \times K_i \,]$ is open in $Y \times K_i$ and Y and K_i are metric spaces we find a set-product $Z \times V_i \in \mathcal{Y} \otimes \mathcal{K}$ with

$$(x, w) \in Z \times V_i \subset W \cap \big[\, Y \times K_i \,\big] \subset W.$$

Hence the open set

$$W = \bigcup \Big\{ Z \times V_i \in \mathcal{Y} \otimes \mathcal{K} : Z \times V_i \subset W \Big\}$$

is a countable union and so $\mathcal{B}(Y \times U)$ is generated by $\mathcal{Y} \otimes \mathcal{K}$. Because $\mathcal{Y}$ generates the Borel-σ-algebra $\mathcal{B}(Y)$ and $\bigcup_i \{ V_i : V_i \in \mathcal{K}_i \}$ generates $\mathcal{B}(U)$ it follows from Satz 22.1 in [7] that $\mathcal{Y} \otimes \mathcal{K}$ also generates $\mathcal{B}(Y) \otimes \mathcal{B}(U)$. □

The well-known fact, that each compact space with countable base is metrizable together with Lemma 7.1.5, Proposition 7.1.1 and 7.1.2 now leads to:

Theorem 7.1.1 *Let G be a compact group with countable base and assume that X is a polish space or an open set in a $\mathcal{DFN}$-space. Let $\mu \in \mathcal{M}_\sigma(X)$ be finite and $B : G \to M^{-1}(X)$ a measurable representation. Then for each $A \in \mathcal{B}(X)$ the map $G \ni t \mapsto \mu(\, B_t^{-1} A\,) \in \mathbb{R}^+$ is integrable over G.*

7.2 Group representations and symmetric measures

We show, that under some continuity conditions on $F : G \times X \to X$ with $F(t,x) := B_t x$ the space $\mathcal{MF}_p(X)$ is invariant under the symmetrization process. In this section, if nothing else is said, we assume that X is a polish space or an open subset of a $\mathcal{DFN}$-space with the Borel σ-algebra. Moreover, let G be a compact group with countable base and $B : G \to Homeo(X)$ a continuous group representation of G in the space of all homeomorphisms of X.

Definition 7.2.1 A subspace $\mathcal{H} \subset M(X, \mathbb{C})$ is called $B(G)$-*invariant* iff for all $f \in \mathcal{H}$ the inclusion $\{ f \circ B_t : t \in G \} \subset \mathcal{H}$ holds.

For any $\mathcal{H} \subset M(X, \mathbb{C})$ we can consider $\mathcal{H}_G := \{ f \circ B_t : f \in \mathcal{H},\ t \in G \}$. Then $\mathcal{H}_G$ is a $B(G)$-invariant space and $\mathcal{H}$ is $B(G)$-invariant itself iff $\mathcal{H} = \mathcal{H}_G$.

Theorem 7.2.1 *Let $\mathcal{F} \subset M(X, \mathbb{C})$ be $B(G)$-invariant and $\mu \in \mathcal{MF}_p(X)$ where $p \geq 1$, then it follows that $\mu_s \in \mathcal{MF}_p(X)$ as well.*

Proof According to Theorem 7.1.1 μ_s is well-defined. Let $K_1 \subset X$ be compact, then we conclude from the continuity of the representation B that the spaces $G \times K_1 \subset G \times X$ and $K_2 := F(G \times K_1) \subset X$ are compact, as well. Because $\mu \in \mathcal{MF}_p(X)$ and $\mathcal{F}$ is a $B(G)$-invariant space, there is $C > 0$ and a compact set K_3 with $K_2 \subset K_3 \subset X$ such that for all $f \in \mathcal{F}$ and $t \in G$:

$$\sup \Big\{ \,|\, f \circ B_t(z) \,|\, : z \in K_2 \Big\}^p \leq C \int_{K_3} |\, f \circ B_t \,|^p \, d\mu.$$

In particular, we have with $z \in K_1$ and $u := B_{t^{-1}} z \in K_2$ for all $t \in G$ the estimate:

$$\sup \Big\{ \,|\, f(z) \,|\, : z \in K_1 \Big\}^p \leq \sup \Big\{ \,|\, f \circ B_t(u) \,|\, : u \in K_2 \Big\}^p \leq C \int_{K_3} |\, f \circ B_t \,|^p \, d\mu.$$

Finally, integration over G together with $m(G) = 1$ and an application of Lemma 7.1.6 shows that:

$$\sup\Big\{ \mid f(z) \mid : z \in K_1 \Big\}^p \leq C \int_{G\times K_3} \mid f \mid^p \circ F \, d(m \otimes \mu) = C \int_{K_3} \mid f \mid^p d\mu_s$$

and by definition it follows that $\mu_s \in \mathcal{MF}_p(X)$. □

Let $p \geq 1$ and $\mathcal{H} \subset M(X, \mathbb{C})$ be a $B(G)$-invariant space. Assume that $B : G \to M^{-1}(X)$ is a measurable representation such that μ_s is well-defined for any $\mu \in \mathcal{M}_\sigma(X)$. According to Corollary 7.1.1 the space $\mathcal{H}_p := \mathcal{H} \cap L^p(X, \mu_s)$ is $B(G)$-invariant. Denote by $\overline{\mathcal{H}_p}$ the L^p-closure of $\mathcal{H}_p$. Then we have shown that

$$\tilde{B} : G \ni t \mapsto \Big[\overline{\mathcal{H}_p} \in f \mapsto f \circ B_t \in \overline{\mathcal{H}_p} \Big] \in \mathcal{L}(\overline{\mathcal{H}_p}) \tag{7.2.1}$$

is well-defined. For all $t \in G$ the operators $\tilde{B}_t \in \mathcal{L}(\overline{\mathcal{H}_p})$ are bijective and isometric. In the case where $p = 2$ we obtain a group of unitary operators. Next we give some conditions under which $(\tilde{B}_t)_{t\in G}$ is strongly continuous.

Proposition 7.2.1 *Let $p \geq 1$ and assume that $\mathcal{H} \subset \mathcal{C}(X)$ is a $B(G)$-invariant space and $\mu \in \mathcal{M}_\sigma(X)$ is finite. For all $h \in \mathcal{H}_p$ let the convergence $h \circ B_t \to h$ hold uniformly on X as $t \to e$. Then $\tilde{B}$ is strongly continuous.*

Proof Denote by $\| \cdot \|_p$ the $L^p(X, \mu_s)$-norm on X. Let $f \in \overline{\mathcal{H}_p}$ and $\varepsilon > 0$. Then choose $h \in \mathcal{H}_p$ with $\| f - h \|_p < \varepsilon$. It follows that:

$$\begin{aligned} \Big\| f \circ B_t - f \Big\|_p &\leq \Big\| (f - h) \circ B_t \Big\|_p + \Big\| h \circ B_t - h \Big\|_p + \Big\| h - f \Big\|_p \\ &= 2 \Big\| f - h \Big\|_p + \Big\| h \circ B_t - h \Big\|_p \\ &\leq 2\varepsilon + \Big\| h \circ B_t - h \Big\|_p . \end{aligned} \tag{7.2.2}$$

From Lebesgue‘s convergence theorem together with the uniform convergence $h \circ B_t \to h$ as t tends to $e \in G$ and $\mid h \mid + 1 \in L^p(X, \mu)$ it follows that $\| h \circ B_t - h \|_p < \varepsilon$ for t in a suitable neighborhood of e. Using (7.2.2) this implies the strong continuity of (7.2.1). □

Let $\mathcal{C}_b(X)$ be the space of bounded complex-valued continuous functions. If we assume that $\mathcal{H}$ is a subspace of $\mathcal{C}_b(X)$, then by similar arguments we can prove for all finite measures $\mu \in \mathcal{M}_\sigma(X)$:

Proposition 7.2.2 *Let $p \geq 1$ and let $\mathcal{H} \subset \mathcal{C}_b(X)$ be $B(G)$-invariant. Assume that the convergence $B_t x \to x$ as $t \to e$ holds for all $x \in X$. Then the group representation in (7.2.1) is strongly continuous.*

Let us choose $\mathcal{H} = \mathcal{C}_b(X)$. Under certain additional assumptions we can show that $\mathcal{H}_p$ is dense in $L^p(X, \mu_s)$. One of these condition is that the topological space X is normal, e.g. *Tietze's extension theorem* applies.

Lemma 7.2.1 *Let Z be a metric or normal locally compact Hausdorff space. With a regular finite Borel measure μ on Z and $1 \le p < \infty$ the space $\mathcal{C}_b(Z)$ is dense in $L^p(Z,\mu)$.*

Proof Choose $f \in L^p(Z,\mu)$ and $\varepsilon > 0$. Then there exists a step-function s, such that $\| f - s \|_p \le \frac{\varepsilon}{2}$. Clearly s is bounded and according to [57, 2.3.6] there is $\tilde{u} \in \mathcal{C}(Z)$ with:

$$\mu\Big(\{ x : s(x) \neq \tilde{u}(x) \} \Big) \le \left(\frac{\varepsilon}{4\| s \|_\infty} \right)^p .$$

Now we define $u(x) :=\mathrm{sgn}(\tilde{u}(x)) \min\{ | \tilde{u}(x) |, \| s \|_\infty \}$. Then by definition $u \in \mathcal{C}_b(Z)$ with $\| u \|_\infty \le \| s \|_\infty$ and $\mu(B) \le \left(\frac{\varepsilon}{4 \| s \|_\infty} \right)^p$ where $B := \{ x : s(x) \neq u(x) \}$. Now, we obtain:

$$\| s - u \|_p^p = \int_B | u - s |^p \, d\mu \le 2^p \| s \|_\infty^p \, \mu(B) \le \left(\frac{\varepsilon}{2} \right)^p .$$

This implies $\| f - u \|_p \le \varepsilon$. □

If we assume that $p = 2$ and μ is a finite $\mathcal{NF}_2$-measure we can give another condition for the strong continuity of a group of composition operators. For the notion of *k-spaces* see Definition 5.3.1 and Lemma A.1.1

Lemma 7.2.2 *Let Z be a k-space and $\mathcal{F} \subset \mathcal{C}(Z)$. Assume that μ is a $\mathcal{NF}_2$-measure on Z. Then for each $[\, g\,] \in \overline{\mathcal{F}_2}$ there is $f \in \mathcal{C}(Z)$ with $[\, g\,] = [\, f\,]$.*

Proof Let $([\, f_n\,])_n \subset \mathcal{F}_2$ be a fundamental sequence with respect to the L^2-topology. We conclude from (5.4.1) and $\mu \in \mathcal{MF}_2(Z)$ that $(f_n)_n$ is compact uniformly convergent to a function $f : Z \to \mathbb{C}$ which is continuous restricted to each compact subset $K \subset Z$. Because Z is a k-space by assumption, it follows that $f \in \mathcal{C}(Z)$. Let $[\,g\,] \in L^2(Z,\mu)$ be the L^2-limit of $([\, f_n\,])_n$. From the fact that $(f_n)_n$ admits a subsequence which tends to g a.e. on Z we conclude that $[\, f\,] = [\, g\,]$. □

From Lemma 7.2.2 it is clear that $\overline{\mathcal{F}_2}$ can be identified with a space of continuous complex-valued functions on Z.

Proposition 7.2.3 *Let X be a k-space, $\mathcal{F} \subset \mathcal{C}(X)$ be $B(G)$-invariant and $\mu \in \mathcal{MF}_2(X)$. Then the unitary operator-group (7.2.1) on $\overline{\mathcal{H}_2} := \overline{\mathcal{F}_2}$ is strongly continuous.*

Proof The space $\overline{\mathcal{H}_2} \subset L^2(X,\mu_s)$ is a Hilbert-space and because μ_s is a $\mathcal{NF}_2$-measure by Theorem 7.2.1, the map $\overline{\mathcal{H}_2} \ni f \mapsto f(x) \in \mathbb{C}$ is continuous for any $x \in X$. By the Riesz-Fischer lemma there is $K : X \times X \to \mathbb{C}$ with $K(\cdot,x) \in \overline{\mathcal{H}_2}$ and for $x \in X$

$$f(\, x\,) = \Big\langle f, K(\cdot,x) \Big\rangle_2, \qquad \forall \; f \in \overline{\mathcal{H}_2}. \tag{7.2.3}$$

Because each $f \in \overline{\mathcal{H}_2}$ is continuous it follows that $\mathcal{D} := \mathrm{span}\{ K(\cdot,x) : x \in X \}$ is a dense subspace of $\overline{\mathcal{H}_2}$. By an argument similar to the proof of Proposition 7.2.1 it is sufficient to prove the strong continuity of (7.2.1) on $\mathcal{D}$. Let

$$h = \sum_{i=1}^n \alpha_i \cdot K\big(\cdot, x_i\,\big) \in \mathcal{D}$$

with coefficients $\alpha_i \in \mathbb{C}$ and $x_i \in X$ for $i = 1, \cdots, n$. Then we have:

$$\left\| h \circ B_t - h \right\|_2^2 = 2\Big[\| h \|_2^2 - \mathrm{Re}\langle h \circ B_t, h \rangle_2 \Big]$$

and so we only have to show that $\langle h \circ B_t, h \rangle_2 \to \| h \|_2^2$ as $t \to e$. Using (7.2.3) this follows from:

$$\begin{aligned} \langle h \circ B_t, h \rangle_2 &= \sum_{i,j=1}^{n} \alpha_i \cdot \overline{\alpha_j} \cdot \Big\langle K\big(B_t\cdot, x_i \big), K\big(\cdot, x_j \big) \Big\rangle_2 \\ &= \sum_{i,j=1}^{n} \alpha_i \cdot \overline{\alpha_j} \cdot K\big(B_t x_j, x_i \big) \xrightarrow{t \to e} \| h \|_2^2. \end{aligned}$$

We have used that $K(\cdot, x_i) \in \mathcal{C}(X)$ and the continuity of $B : G \to Homeo(X)$. □

7.3 Representations of C_0-semi-groups on L^2-spaces

In this section let H be a separable Hilbert-space and let $(B_t)_{t\geq 0} \subset \mathcal{L}^{-1}(H)$ be a C_0-semi-group of invertible bounded operators on H. Assume that μ is a finite Borel measure on $\mathcal{G}$, where $\mathcal{G}$ is a G_δ-set in H (i.e. $\mathcal{G}$ is a countable intersection of open sets in H) such that $B_t(\mathcal{G}) \subset \mathcal{G}$ for all $t \geq 0$. We construct a C_0-semigroup $(\tilde{B}_t)_{t\geq 0} \subset \mathcal{L}^{-1}(\tilde{H})$ on $\tilde{H} := L^2(\mathcal{G}, \mu_s)$ of composition operators $\tilde{B}_t(f) := f \circ B_t$ where $f \in \tilde{H}$.

Lemma 7.3.1 *The mapping* $\mathbb{R}^+ \times H \longrightarrow H : \ (t, z) \longmapsto B_t z$ *is continuous with respect to the product topology.*

Proof Since $(B_t)_{t\geq 0}$ is strongly continuous it is well-known that there exist $M > 1$ and $\beta > 0$ such that $\| B_t \| \leq M e^{\beta t}$. Let $(t, z) \in \mathbb{R}^+ \times H$ and $(t_n, z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+ \times H$ be a sequence with $(t_n, z_n) \to (t, z)$ as $n \to \infty$. Then we obtain:

$$\left\| B_{t_n} z_n - B_t z \right\| \leq M e^{\beta t_n} \left\| z_n - z \right\| + \left\| B_{t_n} z - B_t z \right\| \xrightarrow{n \longrightarrow \infty} 0,$$

since $(B_t)_{t\geq 0}$ is strongly continuous. □

With the notations of Definition 7.1.2 let $(X, \Sigma_1, \mu) := (\mathcal{G}, \mathcal{B}(\mathcal{G}), \mu)$, where $\mu \in \mathcal{M}_\sigma(\mathcal{G})$ is finite and define $(Y, \Sigma_2, m_\alpha) := (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), e^{-t\alpha} dt)$ with $\alpha > 0$. Let $\mu_{s,\alpha}$ denote the symmetrization of μ (which is well-defined according to the lemma above and the fact that $\mathcal{G}$ (cf. [123, p. 150]) is a polish spaces) and define $\tilde{B}_t$ by $\tilde{B}_t(f) = f \circ B_t$ for all $t \in \mathbb{R}^+$.

As an example for the choice of $\mathcal{G}$ we can set $\mathcal{G} = H$ or $\mathcal{G}$ to be an open ball in H centered in 0 and $(B_t)_{t\geq 0}$ be a semi-group of unitary operators on H.

Lemma 7.3.2 *For all* $t \geq 0$ *and* $f \in L^2(\mathcal{G}, \mu_{s,\alpha})$ *it holds* $\| \tilde{B}_t f \|_{s,\alpha} \leq e^{\frac{\alpha}{2} t} \| f \|_{s,\alpha}$*, where* $\| \cdot \|_{s,\alpha}$ *denotes the* $L^2(\mathcal{G}, \mu_{s,\alpha})$*-norm.*

Proof Let $t_0 \geq 0$ and $f \in L^2(\mathcal{G}, \mu_{s,\alpha})$. According to Lemma 7.1.6 we obtain:

$$\begin{aligned}
\int_{\mathcal{G}} \big| f \circ B_{t_0} \big|^2 d\mu_{s,\alpha} &= \int_{\mathcal{G}\times\mathbb{R}^+} \big| f(B_{t_0} B_t x) \big|^2 d(\mu \otimes m_\alpha)(x,t) \\
&= \int_{\mathcal{G}} \int_{[\, t_0, \infty\,)} \big| f(B_s x) \big|^2 e^{-\alpha(s-t_0)}\, ds\, d\mu(x) \\
&\leq e^{\alpha t_0} \int_{\mathcal{G}} \int_{\mathbb{R}^+} \big| f(B_s x) \big|^2 e^{-\alpha s}\, ds\, d\mu(x) = e^{\alpha t_0} \| f \|_{s,\alpha}^2 .
\end{aligned}$$

This proves $\tilde{B}_{t_0} f \in L^2(\mathcal{G}, \mu_{s,\alpha})$ and the desired inequality. □

Theorem 7.3.1 *Let $\mathcal{G}$ be a G_δ-set, $\mu \in \mathcal{M}_\sigma(\mathcal{G})$ and $\alpha > 0$. Moreover, we assume that $(B_t)_{t\geq 0} \subset \mathcal{L}^{-1}(H)$ is a C_0-semi-group of invertible bounded operators on H such that the inclusion $B_t(\mathcal{G}) \subset \mathcal{G}$ holds. For any $t \geq 0$ let $\tilde{B}_t$ be the isomorphism defined above, e.g. $\tilde{B}_t f = f \circ B_t$. Then $(\tilde{B}_t)_{t\geq 0}$ defines a C_0-semi-group on $L^2(\mathcal{G}, \mu_{s,\alpha})$.*

Proof It is obvious that $(\tilde{B}_t)_{t\geq 0}$ is a semi-group of isomorphisms on $L^2(\mathcal{G}, \mu_{s,\alpha})$. Let us fix a function $g \in \mathcal{C}_b(\mathcal{G})$, then we obtain for all $x \in H$:

$$\big[\, \tilde{B}_t g \,\big](x) - g(x) = g\big(B_t x \big) - g(x) \xrightarrow{t \longrightarrow 0} 0,$$

since $(B_t)_{t\geq 0}$ is strongly continuous and g is a continuous function. Moreover, g is bounded and thus by Lebesgue's Theorem of dominated convergence it follows that:

$$\big\| \tilde{B}_t g - g \big\|_{s,\alpha} \xrightarrow{t \longrightarrow 0} 0. \tag{7.3.1}$$

Now let $f \in L^2(\mathcal{G}, \mu_{s,\alpha})$ be arbitrary and fix $\varepsilon > 0$. According to Lemma 7.2.1 there exists $g \in \mathcal{C}_b(\mathcal{G})$ with $\| f - g \|_{s,\alpha} \leq \varepsilon$. Furthermore (7.3.1) implies that there is $t_0 \leq 1$ such that for all $0 < t \leq t_0$ we have $\| \tilde{B}_t g - g \|_{s,\alpha} < \varepsilon$. Thus for $t \in [\, 0, t_0 \,]$ we get:

$$\begin{aligned}
\big\| \tilde{B}_t f - f \big\|_{s,\alpha} &\leq \big\| \tilde{B}_t f - \tilde{B}_t g \big\|_{s,\alpha} + \big\| \tilde{B}_t g - g \big\|_{s,\alpha} + \big\| g - f \big\|_{s,\alpha} \\
&\leq \big\| \tilde{B}_t \big\| \varepsilon + 2\varepsilon \leq \big(e^\alpha + 2 \big) \varepsilon.
\end{aligned}$$

□

7.4 Group actions induced by symmetries

We give examples how to construct representations $G \ni t \mapsto Homeo(X)$, where G is a compact group with countable base, X denotes a topological space and $Homeo(X)$ is the group of all homeomorphisms of X.

Let $\Omega \subset \mathbb{R}^n$ be open or closed and let $\omega : \Omega \to \mathbb{R}^+$ be a strictly positive and continuous weight function. With $f \in \mathcal{C}(\Omega)$ consider $\| f \|_\omega := \sup\{ \, | f(x) | \cdot \omega(x) : x \in \Omega \, \}$. Define the Banach space $\mathcal{C}_\omega(\Omega)$ of continuous functions by

$$\mathcal{C}_\omega(\Omega) := \Big\{ f \in \mathcal{C}(\Omega), \;\; \| f \|_\omega < \infty \Big\}.$$

Assume that E is a topological space which is continuously embedded in $\mathcal{C}_\omega(\Omega)$. Fix $m \in \mathbb{N}$, then with the product topology on $\times_{i=1}^n \mathcal{C}_\omega(\Omega)$ and the topology on $\mathcal{C}(\Omega, \mathbb{C}^m)$ of *uniformly compact convergence* we have the continuous inclusions

$$E^m := \times_{i=1}^m E \hookrightarrow \mathcal{C}_\omega(\,\Omega\,)^m := \times_{i=1}^n \mathcal{C}_\omega(\Omega) \hookrightarrow \mathcal{C}\,(\,\Omega, \mathbb{C}^m\,)\,.$$

Let $U \subset \mathbb{C}^m$ be open and bounded. For each set $A \subset U$ we denote by $\overline{A}$ the closure of A in $\mathbb{C}^m$. Now consider:

$$X_U := \Big\{\, f = (\,f_1, \cdots, f_m\,) \in E^m : \overline{[\,f \cdot \omega\,](\Omega)} \subset U \,\Big\} \subset E^m. \tag{7.4.1}$$

Lemma 7.4.1 *The set $X_U \subset E^m$ defined in (7.4.1) is open with respect to the product topology of E^m.*

Proof Because the embedding $E^m \hookrightarrow \mathcal{C}_\omega(\Omega)^m$ is continuous, it is sufficient to show that the set

$$\tilde{X}_U := \Big\{\, f \in \mathcal{C}_\omega(\Omega)^m : \overline{[\,f \cdot \omega\,](\Omega)} \subset U \,\Big\} \subset \mathcal{C}_\omega(\Omega)^m \tag{7.4.2}$$

is open in $\mathcal{C}_\omega(\Omega)^m$. Fix $f \in \tilde{X}_U$ and let $\varepsilon := \mathrm{dist}_1(\,\overline{[\,f \cdot \omega\,](\Omega)}, \partial U\,) > 0$ denote the distance of the compact set $\overline{[\,f \cdot \omega\,](\Omega)}$ to the topological boundary ∂U of U with respect to the 1-norm. Fix a function $g \in \mathcal{C}_\omega(\Omega)^m$ such that

$$\|\, g_1 - f_1 \,\|_\omega + \cdots + \|\, g_m - f_m \,\|_\omega < \frac{\varepsilon}{2}.$$

It follows that $[\,g \cdot \omega\,](\Omega) \subset \overline{[\,f \cdot \omega\,](\Omega)} + K_{\frac{\varepsilon}{2}} \subset U$ where $K_r \subset \mathbb{C}^m$ denotes the open r-ball where $r > 0$ with respect to the 1-norm centered in $0 \in \mathbb{C}^m$. Then $\overline{[\,g \cdot \omega\,](\Omega)} \subset U$ and so by definition $g \in \tilde{X}_U$. □

Assume that $b : G \to \mathit{Homeo}(U)$ is a representation of G in the group of all homeomorphisms on U. With the notation of (7.4.2) let us define the *induced representation*

$$\tilde{B}_t : G \to \mathrm{Homeo}\left(\tilde{X}_U\right) : f \mapsto \big(\, b_t \circ [\,f \cdot \omega\,]\,\big) \cdot \omega^{-1}$$

for $f \in \tilde{X}_U$. Because of $\overline{[\,f \cdot \omega\,](\Omega)} \subset U$ and

$$\overline{[\,(\tilde{B}_t f) \cdot \omega\,](\Omega)} = \overline{b_t \circ [\,f \cdot \omega\,](\Omega)} = b_t\left(\,\overline{[\,f \cdot \omega\,](\Omega)}\,\right) \subset U$$

for all $f \in \tilde{X}_U$ the map $\tilde{B}_t$ is well-defined. It is easy to check that it is a group homomorphism and for fixed $t \in G$ the map $\tilde{B}_t : \tilde{X}_U \to \tilde{X}_U$ is continuous.

Remark 7.4.1 If in addition for $t \in G$ the homeomorphism $b_t : U \to U$ extends to a linear map on $\mathbb{C}^m$, then we have $\tilde{B}_t f = b_t \circ f$.

With a bounded open set $U \subset \mathbb{C}^n$ we equip the space $\mathit{Homeo}(U)$ with the topology of uniform convergence on all compact subset $K \subset U$.

Proposition 7.4.1 *Let $b : G \rightarrow Homeo(U)$ be a continuous representation, then the induced representation $\tilde{B} : G \rightarrow Homeo(\tilde{X}_U)$ is continuous as well.*

Proof Let $s, t \in G$ and $f, g \in \tilde{X}_U$. Then with the supremums-norm $\| \cdot \|_{\sup}$ on Ω and the product norm $\| \cdot \|_{\tilde{X}_U}$ on $\tilde{X}_U \subset \mathcal{C}_\omega(\Omega)^m$ we have:

$$\left\| \tilde{B}_t f - \tilde{B}_s g \right\|_{\tilde{X}_U} = \sum_{j=1}^{m} \left\| b_t \circ [\, f \cdot \omega \,]_j - b_s \circ [\, g \cdot \omega \,]_j \right\|_{\sup}. \tag{7.4.3}$$

Fix a sequence $(t_n, f_n)_{n \in \mathbb{N}} \subset G \times \tilde{X}_U$ with $(t_n, f_n) \rightarrow (t, f) \in G \times \tilde{X}_U$ as $(n \rightarrow \infty)$. By definition of the topology on $\tilde{X}_U$ we conclude that $(\, f_n \cdot \omega \,)_n$ converges to $f \cdot \omega$ uniformly on Ω. Hence we can choose a compact set $K \subset U$ and $n_0 \in \mathbb{N}$ such that $\overline{[\, f_n \cdot \omega \,](\Omega)} \subset K$ for all $k \geq n_0$ and $\overline{[\, f \cdot \omega \,](\Omega)} \subset K$. The continuity of the map $G \ni t \mapsto b_t \in Homeo(U)$ now implies that:

$$\left\| b_{t_n} \circ [\, f_n \cdot \omega \,]_j - b_t \circ [\, f \cdot \omega \,]_j \right\|_{\sup} \xrightarrow{n \rightarrow \infty} 0$$

for all $j = 1, \cdots, m$. Together with (7.4.3) this finally implies $\tilde{B}_{t_n} f_n \rightarrow \tilde{B}_t f$ in $\tilde{X}_U$. $\square$

In order to define μ_s for $\mu \in \mathcal{M}_\sigma(X)$ and a polish space X we only need a measurable representation $B : G \rightarrow M^{-1}(X)$. With our notations above let $\tilde{V} \subset \mathcal{C}_\omega(\Omega)^m$ be open. In addition, assume that E is a polish space and define $V := \tilde{V} \cap E^m \subset E^m$. It is well-known that the spaces $\tilde{V}$ and V with the induced topologies are polish spaces as well (see [7]).

Proposition 7.4.2 *Let $\tilde{B} : G \rightarrow M^{-1}(\tilde{V})$ be measurable with $\tilde{B}_t(V) \subset V$ for all $t \in G$. Then $B : G \rightarrow M^{-1}(V)$ defined by $B_t := \tilde{B}_{t|_V}$ for $t \in G$ is measurable, as well.*

Proof For each $t \in G$ the map $B_t : V \rightarrow V$ is bijective. We show that it is measurable as well. Fix $A \in \mathcal{B}(V)$, then it follows from the continuous embedding $V \hookrightarrow \tilde{V}$, the fact that V and $\tilde{V}$ are polish spaces and *Kuratowski's Theorem* (see [89], p.420) that $A \in \mathcal{B}(\tilde{V})$. Because $B_t : V \rightarrow \tilde{V}$ is Borel-measurable we obtain $B_t^{-1}(A) \subset \mathcal{B}(V)$. Hence $B_t : V \rightarrow V$ is Borel-measurable for all $t \in G$ and so B is well-defined.

Now, we prove that $G \times V \ni (\, t, z \,) \mapsto B_t z \in V$ is $\mathcal{B}(\, G \times V \,) - \mathcal{B}(V)$-measurable. As we have shown above $\mathcal{B}(V) \subset \mathcal{B}(\tilde{V})$ and by assumption the map

$$G \times \tilde{V} \rightarrow \tilde{V} : (\, t, z \,) \mapsto B_t z =: F(\, t, z \,)$$

is $\mathcal{B}(\, G \times \tilde{V} \,)$-$\mathcal{B}(\tilde{V})$- measurable. Hence $F^{-1}(A) \in \mathcal{B}(G \times \tilde{V})$ and by the continuity of the embedding $G \times V \hookrightarrow G \times \tilde{V}$ together with the inclusion $F^{-1}(A) \subset G \times V$ we conclude that $F^{-1}(A) \in \mathcal{B}(G \times V)$. $\square$

Under some more conditions on $b : G \rightarrow Homeo(U)$ the restriction of $\tilde{B}_t$ to X_U leads to a continuous representation $B : G \rightarrow Homeo(X_U)$. Let us consider some special cases:

Example 7.4.1 Let $\Omega \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ be open and bounded. We can consider the *Bergman space* $H := H^2(\Omega, v)$ defined as the $L^2(\Omega, v)$-closure of

$$\Big\{\, f \in \mathcal{C}\left(\, \overline{\Omega} \,\right) : f_{|\Omega} : \Omega \rightarrow \mathbb{C} \text{ is holomorphic } \Big\}.$$

Denote by $K : \Omega \times \Omega \to \mathbb{C}$ the *Bergman kernel* of Ω and define the weight $\omega : \Omega \to \mathbb{R}^+$ by $\omega(\, x\,) := K(\, x, x\,)^{-\frac{1}{2}}$. It is well-known that ω is strictly positive and continuous on Ω. Moreover, for each $f \in H$ and $x \in \Omega$ we have:

$$|\, f(x)\,| \leq \|\, f\,\|_2\, K(x,x)^{\frac{1}{2}} = \|\, f\,\|_2\, \omega(x)^{-1} \tag{7.4.4}$$

where $\|\cdot\|_2$ denotes the $L^2(\Omega, v)$-norm. Hence from (7.4.4) it follows that the inclusion $H^2(\Omega, v) \hookrightarrow \mathcal{C}_\omega(\Omega)$ is continuous. Let $U \subset \mathbb{C}^m$ be open and consider the space

$$GL(U) := \Big\{\, A \in GL\,(\,\mathbb{C}^m\,) : A(U) = U\,\Big\}.$$

(More about this definition can be found in [111] and [122].) Let G be a compact group with countable base and $b : G \to GL(U)$ a measurable representation (e.g. U can be chosen to be the Euclidean ball in $\mathbb{C}^m$ and $G := \mathcal{U}(\mathbb{C}^m)$, the unitary group. Then a representation $b : G \to GL(U)$ is given by $b_U(z) := Uz$ with $U \in \mathcal{U}(\mathbb{C}^m)$ and $z \in U$.) Due to Remark 7.4.1 the induced representation

$$\tilde{B} : G \to \mathrm{Homeo}\big(\,\tilde{X}_U\,\big)$$

(see (7.4.2)) is given by $\tilde{B}_t f = b_t \circ f$. If U is bounded, then $X_U = \tilde{X}_U \cap H^m$ is $B(G)$-invariant and by restriction of $\tilde{B}_t$ to X_U we obtain a representation $B : G \to Homeo(X_U)$ which is measurable according to Proposition 7.4.2. In the case where $b : G \to GL(U)$ is continuous it follows from standard arguments that B is a continuous representation.

Example 7.4.2 Let $\Omega \subset \mathbb{R}^n$ be open or closed and bounded, such that the boundary fulfills e.g. the conditions of *Calderons's extension theorem*. Choose $s > \frac{n}{2}$ then, by well-known results, the Sobolev-space $H^s(\Omega)$ is a Banach-algebra and $H^s(\Omega) \hookrightarrow \mathcal{C}(\Omega)$. Let $U \subset \mathbb{C}^m$ be open and bounded and consider $\mathrm{Aut}(U)$, the group of biholomorphic mappings in U. Let G be a compact group with countable base and $b : G \to Aut(U)$ a representation. The induced representation

$$\tilde{B} : G \to \mathrm{Homeo}\big(\,\tilde{X}_U\,\big)$$

is given by $B_t f = b_t \circ f$. Since $H := H^s(\Omega)$ is a Banach-algebra $X_U = \tilde{X}_U \cap H^m$ is $B(G)$-invariant by holomorphic functional calculus. Thus by restriction of $\tilde{B}_t$ to X_U we obtain a representation $B : G \to Homeo(\, X_U\,)$.

Considering the group $\mathrm{Diff^k(U)}$ of $\mathcal{C}^k$-diffeomorphisms ($k > s$) instead of $\mathrm{Aut}(U)$ we obtain again a representation $B : G \to Homeo(\, X_U\,)$ by well-known theorems on Sobolev-spaces.

Example 7.4.3 Let $U \subset \mathbb{C}^n$ be open or closed and G a compact group with countable base. Assume that $b : G \to Homeo(U)$ is a measurable representation of G. We might think of U as a symmetric space and $b : G \to GL(U)$ where $GL(U)$ denotes the group of invertible homomorphisms leaving U invariant. With the usual Lebesgue measure v on U

consider v_s defined by the representation b (see Definition 7.1.2 and Theorem 7.1.1). As we have remarked in (7.2.1) we obtain an unitary representation

$$\tilde{B} : G \ni t \mapsto \Big[L^2(\, U, v_s \,) \ni f \mapsto f \circ b_t \in L^2(\, U, v_s \,) \Big] \in \mathcal{L}(\, L^2(U, v_s)\,). \tag{7.4.5}$$

We have given several conditions under which the representation (7.4.5) is strongly continuous. If this is the case it is a continuous representation in our sense (cf. Lemma 7.3.1). Indeed, fix a sequences $(\, t_n, f_n\,)_n \subset G \times L^2(U, v_s)$ and $(\, t, f\,)$ such that $t_n \to t$ in G and $f_n \to f$ in $L^2(U, v_s)$ as $n \to \infty$, then:

$$\| \, B_{t_n} f_n - B_t f \, \|_{L^2} \leq \| \, f_n - f \, \|_{L^2} + \| \, B_{t_n} f - B_t f \, \|_{L^2} \xrightarrow{n \to \infty} 0$$

by the strong continuity of the unitary group $(B_t)_{t \in G}$. Fix any infinite dimensional finite Borel measure μ on $H := L^2(U, v_s)$ (e.g. let μ be a Gaussian measure), then we can consider the symmetrization μ_s of μ given by the representation (7.4.5). By the same construction we obtain an unitary representation $\tilde{\tilde{B}} : G \to \mathcal{L}(\, L^2(H, \mu_s)\,)$. By continuing this process we build a sequence of unitary groups on Hilbert-spaces induced by symmetries of the base space U.

As we have seen in Example 7.1.1 in general the measures μ and μ_s in Definition 7.1.2 are not equivalent. The following example is devoted to this question in our construction above. Here we choose μ to be a finite product of infinite dimensional Gaussian measures and B_t to be linear for all t. In this specific situation we obtain conditions under which μ_s is absolutely continuous w.r.t. μ. It turns out that these conditions are quite restrictive and in general absolute continuity of the measures fails.

Example 7.4.4 Let H be an infinite dimensional Hilbert space over $\mathbb{R}$ with Gaussian measure μ_B where B is the nuclear positive correlation operator (for the definition we refer to chapter 5 or [48, pp.40]). Fix $n \in \mathbb{N}$ and let us consider H^n with the product measure $\mu_n := \mu_B \otimes \cdots \otimes \mu_B$. For each invertible matrix $C \in \mathbb{C}^n$ we define $C : H^n \to H^n$ by matrix multiplication. The space H^n is a Hilbert space with norm

$$\| \, (\, z_1, \cdots, z_n \,) \|_{H^n}^2 := \sum_{j=1}^n \| \, z_j \, \|^2.$$

For any finite Borel measure ν on H we recall that the characteristic function χ_ν is defined by the integral $\chi_\nu(z) = \int_H \exp(\, i\langle z, \cdot \rangle\,)\, d\nu$. In case of the Gaussian measure μ_B it is well-know that we have

$$\chi_{\mu_B}(z) = \exp\left(-\| \, B^{\frac{1}{2}} z \, \|^2 \right)$$

for $z \in H$ (see [48]) and so we obtain for the characteristic function of μ_n:

$$\chi_{\mu_n}(\, z_1, \cdots, z_n \,) = \prod_{j=1}^n \chi_{\mu_B}(\, z_j \,) = \exp\left(-\| \, [\, \mathrm{diag}(\, B^{\frac{1}{2}} \,)\,](\, z_1, \cdots, z_n \,) \|_{H^n}^2 \right).$$

Here we denote by diag($B^{\frac{1}{2}}$) the map ($z_1, \cdots, z_n$) $\mapsto$ ($B^{\frac{1}{2}}z_1, \cdots, B^{\frac{1}{2}}z_n$) on H^n. Because μ_n is uniquely determined by χ_{μ_n} we conclude that it is a Gaussian measure with correlation operator diag(B). Now let us consider the measure μ_n^C on H^n defined by $\mu_n^C(X) = \mu_n(C^{-1}X)$ for all $X \in \mathcal{B}(H^n)$. It is shown (see [48], p. 42) that μ_n^C again is a Gaussian measure with correlation $C\mathrm{diag}(B)C^*$. In what follows we use the following general result about equivalence of infinite dimensional Gaussian measures μ_{B_1}, μ_{B_2} with nuclear positive correlations B_1, B_2 (see [48] remark 4.4, p. 66):

Let the operator $B_1^{-\frac{1}{2}}B_2B_1^{-\frac{1}{2}}$ be bounded and invertible. If $B_1^{-\frac{1}{2}}B_2B_1^{-\frac{1}{2}} - I$ is a Hilbert-Schmidt operator, then the measures μ_{B_1} and μ_{B_2} are equivalent. Otherwise they are orthogonal i.e. there is $X \subset H$ such that $\mu_{B_1}(X) = \mu_{B_1}(H) = 1$ and $\mu_{B_2}(X) = 0$.

Let us apply this criterion to μ_n and μ_n^C. We set B_1 :=diag(B) and $B_2 := CB_1C^*$. It is easy to see that diag($B^{\frac{1}{2}}$) commutes with C and C^* and so it follows:

$$B_1^{-\frac{1}{2}}B_2B_1^{-\frac{1}{2}} = \mathrm{diag}\Big(B^{-\frac{1}{2}} \Big)\, C\, \mathrm{diag}(\, B\,)\, C^*\, \mathrm{diag}\Big(B^{-\frac{1}{2}} \Big) = C\, C^*.$$

Because C was invertible by assumption it follows that $B_1^{-\frac{1}{2}}B_2B_1^{-\frac{1}{2}}$ is invertible as well and so by the criterion above the operator $CC^* - I$ has to be Hilbert Schmidt for μ_n and μ_n^C to be equivalent. In the case where C is an unitary matrix it follows now that μ_n and μ_n^C are equivalent. If the matrix $CC^* - I$ is invertible on H^n itself (we can choose $C = tI$ with $t \in \mathbb{R} \setminus \{0, 1\}$) both measures are orthogonal.

Now, let us assume that $\Omega \subset \mathbb{R}^n$ is open and $H \subset \mathcal{C}_\omega(\Omega)$ where $\omega : \Omega \to \mathbb{R}^+$ is a strictly positive and continuous weight function. Denote by $U_r \subset \mathbb{C}^n$ the complex ball in $\mathbb{C}^n$ with radius r centered in 0 and consider the set $X_{U_r} \subset H^n$ defined as in (7.4.1) where $E = H$. Then according to Lemma 7.4.1 the set X_{U_r} is open and so $\mu_n(U_r) > 0$. In the following the restriction of μ_n to X_{U_r} is denoted by $\mu_{n,r}$. Let $\mathcal{N} \subset \mathcal{U}(\mathbb{C}^n)$ be a compact subgroup of the group $\mathcal{U}(\mathbb{C}^n)$ of all unitary matrices on $\mathbb{C}^n$ with Haar measure $m_{\mathcal{N}}$. There is a natural group action of $\mathcal{N}$ on X_{U_r} by $B_C(z) = C(z)$ for $C \in \mathcal{N}$. If we choose (X, Σ_1, μ) = ($X_{U_r}, \mathcal{B}(X_{U_r}), \mu_{n,r}$) and (Y, Σ_2, m) = ($\mathcal{N}, \mathcal{B}(\mathcal{N}), m_{\mathcal{N}}$) in Definition 7.1.2, then we can prove:

Theorem 7.4.1 *The measure ($\mu_{n,r}$)$_s$ in Definition 7.1.2 w.r.t. (B_C)$_{C\in\mathcal{N}}$ is absolutely continuous w.r.t. $\mu_{n,r}$.*

Proof Let $C \in \mathcal{N}$ and choose a Borel set $N \subset X_{U_r}$ such that $\mu_{n,r}(N) = \mu_n(N) = 0$. It follows from our computations above that $\mu_{n,r}(C[N]) = \mu_n(C[N]) = 0$. Hence we obtain

$$[\, \mu_{n,r}\,]_s\, (N) = \int_{\mathcal{N}} \mu_{n,r}\Big(C[\, N\,] \Big)\, dm_{\mathcal{N}}(C) = 0. \qquad \square$$

7.5 Dynamical systems on L^2-spaces over Riemannian manifolds

In this section we show, how to construct a dynamical system $(\, H, \mathcal{B}(H), \mu, T\,)$ (for definition see [89]). Here H is a L^2-space over a Riemannian manifold, μ is an infinite dimensional *Gaussian measure* on H and $T : H \to H$ a *μ-preserving* (i.e. $\mu^T = \mu$) isomorphism. Unlike to our previous examples we are not symmetrizing a given measure by an integration process, but the μ-preserving property will follow more directly from our choice of parameters. Let us first remind of some general results in connection with infinite dimensional Gaussian measures.

Let H be an infinite dimensional separable Hilbert-space over $\mathbb{R}$ or $\mathbb{C}$ and $B \in \mathcal{L}(H)$ a non-negative nuclear operator on H. Let us denote by ν_B the *Gaussian measure* on H with characteristic function

$$\chi_{\nu_B}(z) = \int_H \exp\Big(\, i\mathrm{Re}\,\langle \cdot, z\,\rangle \Big)\, d\nu_B = \exp\big(-\langle\, Bz, z\,\rangle\,\big).$$

For each bounded operator $A \in \mathcal{L}(H)$ we consider the induced Borel measure ν_B^A defined by $\nu_B^A(M) := \nu_B(\,A^{-1}(M)\,)$ for all $M \in \mathcal{B}(H)$. By a standard calculation using the transformation formula (see [14]) for integrals one finds for the characteristic function of $\mu := \nu_B^A$:

$$\chi_\mu(z) = \exp\big(-\langle\, ABA^*z, z\,\rangle\,\big), \qquad \forall\, z \in H.$$

Let us assume that $A \in \mathcal{L}(H)$ is unitary and $[\,A, B\,] = 0$. It follows $\chi_\mu = \chi_{\nu_B}$ and because the Gaussian measures are uniquely determined by its characteristic functions we conclude that $\nu_B^A = \nu_B$. Hence A is μ-preserving and in particular the composition operator

$$C_A : L^2(H, \nu_B) \to L^2(H, \nu_B) : f \mapsto f \circ A$$

is unitary. In order to find H, a Gaussian measure μ on H and isomorphisms $T \in \mathcal{L}(H)$ such that $(\,H, \mathcal{B}(H), \mu, T\,)$ becomes a dynamical system we restrict ourselves to L^2-Hilbert-spaces H over a *Riemannian manifold*. Due to our remarks above we construct a nuclear operator B (which is naturally related to the geometry of H) as well as a family of unitary operators on H commuting with B.

Let $(\,M, g\,)$ be a Riemannian manifold with metric g (for details see [85]) and denote by L the *Beltrami-Laplace operator* on M. A map $\Phi : M \to M$ is called an *isometry* of M if Φ is a diffeomorphism preserving the metric g. By this we mean that for each $p \in M$

$$g_p(\,u, v\,) = g_{\Phi(p)}\big(\,d\Phi_p u, d\Phi_p v\,\big), \qquad (\,u, v \in M_p\,)$$

where M_p denotes the tangent space to M at $p \in M$. In other words $d\Phi_p$ is an isometry of Euclidean vector spaces between $(\,M_p, g_p\,)$ and $(\,M_{\Phi(p)}, g_{\Phi(p)}\,)$. According to Proposition 1.3 in [85], p. 85 and the remark following it, the Riemannian measure m_R on M is invariant under isometries. Hence each isometry $\Phi : M \to M$ leads to an unitary *composition operator*

$$C_\Phi : L^2(M, m_R) \ni f \mapsto f \circ \Phi \in L^2(M, m_R).$$

There is the following characterization of diffeomorphisms of M which are isometries in terms of the Beltrami-Laplace operator L. A proof can be found in [85] Proposition 2.4:

Theorem 7.5.1 *Let $\Phi : M \to M$ be a diffeomorphism of the Riemannian manifold M. Then Φ leaves the Beltrami-Laplace operator L invariant (i.e the commutator $[\, C_\Phi, L\,]$ vanishes) if and only if it is an isometry.*

From now on assume that (M, g) is a compact connected oriented Riemannian manifold. By the well-known *Hodge Theorem* (see [124]) it follows that there exists an orthonormal basis $[\varphi_n : n \in \mathbb{N}]$ of $L^2(M, m_R)$ consisting of eigen-functions of the Laplacian L. Moreover, all the eigen-values $(\lambda_n)_{n\in\mathbb{N}}$ are positive, except that zero is an eigen-value with multiplicity one. Each eigenvalue has finite multiplicity and they accumulate only at infinity. The asymptotic behavior of $(\lambda_n)_n$ is given by the formula

$$\lambda_n \sim n^{\frac{2}{\dim M}} \quad \text{as} \quad n \to \infty \tag{7.5.1}$$

which was discovered by H. Weyl and can be found in [33]. It also is a standard fact that the *heat operator* e^{-tL} on $L^2(M, m_R)$ with $t \in \mathbb{R}^+$ has a decomposition of the form:

$$e^{-tL}\varphi_n = e^{-\lambda_n t}\varphi_n.$$

for all $n \in \mathbb{N}$. Hence it follows from the asymptotic (7.5.1) that $\mathrm{tr}(\, e^{-tL}\,) < \infty$. Fix an isometry Φ on M. Because the composition operators C_Φ commutes with L, it also commutes with the compact operator e^{-tL} for all $t \in \mathbb{R}^+$. Moreover, e^{-tL} is positive for each $t > 0$ and so we can consider the Gaussian measure $\nu_{L,t}$ on H with characteristic function $\chi_{\nu_{L,t}}$ defined for $z \in H$ by

$$\chi_{\nu_{L,t}}(\, z\,) = \exp\big(- \big\langle\, e^{-tL}z, z\,\big\rangle\,\big).$$

From our remark above each composition operator C_Φ with an isometry $\Phi : M \to M$ fulfills $[\, C_\Phi, e^{-tL}\,] = 0$. Hence we obtain the following Proposition:

Proposition 7.5.1 *Let (M, g) be a Riemannian manifold and Φ be an isometry on (M, g). Moreover, let $\nu_{L,t}$ be the Gaussian measure defined above. Then C_Φ defined by*

$$C_\Phi f := f \circ \Phi$$

is $\nu_{L,t}$-preserving and we obtain the following unitary operators:

$$\mathbf{C}_{\Phi,t} : L^2\big(\, H, \nu_{L,t}\,\big) \to L^2\big(\, H, \nu_{L,t}\,\big) : f \mapsto f \circ \Phi.$$

In other words $(\, E := L^2(H, \nu_{L,t}), \mathcal{B}(E), \nu_{L,t}, \mathbf{C}_{\Phi,t}\,)$ defines a dynamical system on E for each $t \in \mathbb{R}^+$. Let $Iso(M, g)$ be the isometry-group of (M, g). Then $Iso(M, g)$ is a Lie-group and compact if M is compact (cf. [100][ch. II Theorem 1.2]).

Theorem 7.5.2 *Let $(\, M, g\,)$ be a Riemannian manifold and $Iso(M,g)$ be the isometry-group of (M,g). Moreover, let $\nu_{L,t}$ be the Gaussian measure defined above, e.g. $\nu_{L,t}$ has the characteristic function $\chi_{\nu_{L,t}}(z) = \exp\big(-\big\langle e^{-tL}z, z\big\rangle\big)$, where L is the Beltrami-Laplace operator and $t > 0$. Then*

$$\mathbf{C}_t : Iso(\, M, g\,) \ni \Phi \mapsto \Big[\, L^2(\, H, \nu_{L,t}\,) \ni f \mapsto f \circ \Phi \in L^2(\, H, \nu_{L,t}\,)\,\Big] \in \mathcal{L}(\, L^2(H, \nu_{L,t})\,)$$

is an unitary group representation of the Lie-group $Iso(M,g)$ on $\mathcal{L}(\, L^2(H,\nu_{L,t})\,)$.

7.6 Group action on generalized Toeplitz-algebras

Let X be a polish space or an open subset of a $\mathcal{DFN}$-space. In addition we assume that X is a k-space with $\mathcal{MF}_2(X) \neq \emptyset$, (cf. Example 5.4.1). Assume that G is a compact group with countable base, $B : G \to Homeo(X)$ is a continuous representation and $\mathcal{H} \subset \mathcal{C}(X)$ is $B(G)$-invariant. Fix $\mu \in \mathcal{MF}_2(X)$, then according to Theorem 7.2.1 it follows that $\mu_s \in \mathcal{MF}_2(X)$ as well. With the notations in (7.2.1) and Proposition 7.2.3 we conclude that the unitary group:

$$\tilde{B} : G \ni t \mapsto \Big[\, \overline{\mathcal{H}_2} \ni f \mapsto f \circ B_t \in \overline{\mathcal{H}_2}\,\Big] \in \mathcal{L}\big(\, \overline{\mathcal{H}_2}\,\big) \tag{7.6.1}$$

is strongly continuous. By definition and Lemma 7.2.2 the space $\overline{\mathcal{H}_2}$ is closed in $L^2(X, \mu_s)$ and it consists of continuous functions on X. We refer to it as *$\mathcal{H}$-Bergman space* over X. In the following we denote by $P : L^2(X, \mu_s) \to \overline{\mathcal{H}_2}$ the orthogonal projection (*Toeplitz projection*) onto $\overline{\mathcal{H}_2}$. Using our previous measure constructions we show how a representation of G in a generalized class of Toeplitz C^*-algebras can be defined.

Definition 7.6.1 Let $f \in L^\infty(X)$, then we denote by $T_f \in \mathcal{L}(\, \overline{\mathcal{H}_2}\,)$ the Bergman-Toeplitz operator defined by $T_f g := P(\, fg\,)$ for all $g \in \overline{\mathcal{H}_2}$.

As we already have mentioned in the proof of Proposition 7.2.3, the *point evaluation* on X gives a continuous functional on $\overline{\mathcal{H}_2}$ and so there is a Bergman kernel $K : X \times X \to \mathbb{C}$ with (7.2.3).

Lemma 7.6.1 *For $x, y \in X$ and $t \in G$ we have the invariance $K(\, B_t x, y\,) = K(\, x, B_{t^{-1}} y\,)$ of the Bergman kernel.*

Proof Let $[\, e_j : j \in \mathbb{N}\,]$ be an orthonormal base of $\overline{\mathcal{H}_2}$. The group (7.6.1) acts unitarily on $\overline{\mathcal{H}_2}$ and so $[\, e_j \circ B_t : j \in \mathbb{N}\,]$ also defines an ONB of $\overline{\mathcal{H}_2}$. Let $x, y \in X$ and $t \in G$, then

$$K(\, x, y\,) = \sum_i e_i(\, x\,)\, \overline{e_i(\, y\,)} = \sum_i e_i \circ B_t(\, x\,)\, \overline{e_i \circ B_t(\, y\,)} = K\big(\, B_t x, B_t y\,\big). \qquad \square$$

Corollary 7.6.1 *For all $t \in G$ the commutator $[\, P, \tilde{B}_t\,] := P\tilde{B}_t - \tilde{B}_t P$ on $L^2(X, \mu_s)$ vanishes.*

Proof Fix $f \in L^2(X, \mu_s)$, $t \in G$ and $z \in X$. Then by the reproducing kernel property of K and Lemma 7.6.1 we have:

$$\begin{aligned} \big[\, P\tilde{B}_t f \,\big](\, z\,) &= \big\langle\, P\tilde{B}_t f, K(\cdot, z\,)\,\big\rangle_2 \\ &= \big\langle\, f, K\big(\, B_{t^{-1}}\cdot, z\,\big)\,\big\rangle_2 \\ &= \big[\, Pf\,\big]\big(\, B_t z\,\big) = \big[\, \tilde{B}_t Pf\,\big](\, z\,). \end{aligned}$$

We conclude that $P\tilde{B}_t f = \tilde{B}_t Pf$ for all $f \in L^2(X, \mu_s)$ and so $[\, P, \tilde{B}_t\,] = 0$. □

For each space $Y \subset X$ consider $\mathcal{H}_Y := \{\, f \in \mathcal{H} : f_{|_Y} = 0\,\}$. In the case where Y is $B(G)$-invariant it directly follows that $\mathcal{H}_Y$ is $\tilde{B}(G)$-invariant.

Lemma 7.6.2 *Let $x_0 \in X$ and $Y := \{\, B_t x_0 : t \in G\,\}$. Assume that $\mathcal{H}_Y = \{\,0\,\}$, then there is $f_0 \in \overline{\mathcal{H}_2}$ such that $\overline{\mathcal{H}_2}$ is the closure of span$\{\, \tilde{B}_t f_0 : t \in G\,\}$.*

Proof Define $f_0 := K(\cdot, x_0) \in \overline{\mathcal{H}_2}$ with the reproducing kernel K and assume that

$$V := \overline{\mathrm{span}\{\, \tilde{B}_t f_0 : t \in G\,\}} \subsetneqq \overline{\mathcal{H}_2}.$$

Then there is $0 \neq g \in \overline{\mathcal{H}_2}$ with $0 = \langle g, h\rangle_2$ for all $h \in V$. In particular, we conclude that:

$$0 = \big\langle\, g, \tilde{B}_t f_0\,\big\rangle_2 = \big\langle\, g, K\big(\, B_t\cdot, x_0\,\big)\,\big\rangle_2 = \big\langle\, g, K\big(\cdot, B_{t^{-1}} x_0\,\big)\,\big\rangle_2 = g \circ B_{t^{-1}}(\, x_0\,)$$

for all $t \in G$. Hence $g \in \mathcal{H}_Y = \{\,0\,\}$ and we have received a contradiction. □

With a symbol $f \in L^\infty(X)$ we write $M_f \in \mathcal{L}(\, L^2(X, \mu_s)\,)$ for the multiplication operator given by $M_f h := f \cdot h$ where $h \in L^2(X, \mu_s)$.

Lemma 7.6.3 *Let $f \in L^\infty(X)$, then for all $t \in G$ we have the identities:*

(a) $\tilde{B}_t\, M_f\, \tilde{B}_{t^{-1}} = M_{f \circ B_t}$,

(b) $\tilde{B}_t\, T_f\, \tilde{B}_{t^{-1}} = T_{f \circ B_t}$.

Proof Let $h \in L^2(X, \mu_s)$ and $z \in X$. Then it follows for all $t \in G$:

$$\Big[\, \tilde{B}_t\, M_f\, \tilde{B}_{t^{-1}} h\,\Big](\, z\,) = \Big[\, \tilde{B}_t\big(\, f \cdot h \circ B_{t^{-1}}\,\big)\,\Big](\, z\,) = f \circ B_t(z) \cdot h(\, z\,) = [\, M_{f \circ B_t} h\,](\, z\,).$$

This implies (a) and (b) follows from the (a) and Corollary 7.6.1 which shows that:

$$\tilde{B}_t\, T_f\, \tilde{B}_{t^{-1}} = \tilde{B}_t\, P\, M_f\, \tilde{B}_{t^{-1}} = P\, \tilde{B}_t\, M_f\, \tilde{B}_{t^{-1}} = P\, M_{f \circ B_t} = T_{f \circ B_t}.$$ □

Definition 7.6.2 Let $S \subset L^\infty(X)$ a space of symbols, then we define the Toeplitz C^*-algebra

$$\mathcal{T}(S) := \mathcal{C}^*\Big\{\, T_f : f \in S\,\Big\} \subset \mathcal{L}\left(\, \overline{\mathcal{H}_2}\,\right)$$

to be the C^*-algebra generated by all operators T_f with symbols $f \in S$.

Consider the representation of G in $\mathcal{L}(\overline{\mathcal{H}_2})$ defined by:

$$\mathbf{B} : G \ni t \mapsto \Big[\mathcal{L}\left(\overline{\mathcal{H}_2}\right) \ni A \mapsto \tilde{B}_t A \tilde{B}_{t^{-1}} \in \mathcal{L}\left(\overline{\mathcal{H}_2}\right) \Big] \in \mathcal{L}\left(\mathcal{L}\left(\overline{\mathcal{H}_2}\right)\right).$$

Theorem 7.6.1 *Let $S \subset L^\infty(X)$ be $B(G)$-invariant. Then $\mathcal{T}(S)$ is $\mathbf{B}(G)$-invariant.*

Proof Define $\overline{S} := \{ \overline{f} : f \in S \}$ where $\bar{f}$ denotes the complex conjugate of f. Moreover, for all $n \in \mathbb{N}$ consider the space

$$W_n := \Big\{ T_{f_1} \cdots T_{f_n} : f_j \in S \cup \overline{S} \Big\}.$$

It is easy to show that $T_f^* = T_{\bar{f}}$ and so it follows that the linear hull of $W := \bigcup_n W_n$ is invariant under the $*$-operation. Further, we have with $t \in G$ and $f_1, \cdots, f_n \in S \cup \overline{S}$:

$$\mathbf{B}_t\big(T_{f_1} \cdots T_{f_n} \big) = \mathbf{B}_t\big(T_{f_1} \big) \cdots \mathbf{B}_t\big(T_{f_n} \big) = T_{f_1 \circ B_t} \cdots T_{f_n \circ B_t} \in \mathcal{T}(S)$$

because $S \cup \overline{S}$ is $\mathcal{B}$-invariant. The linear hull of W is dense in $\mathcal{T}(S)$ and each $\mathbf{B}_t$ is continuous on $\mathcal{L}\left(\overline{\mathcal{H}_2}\right)$. From this the assertion follows. $\square$

Remark 7.6.1 By Theorem 7.6.1 we can define a representation of G in the Toeplitz C^*-algebra $\mathcal{T}(S)$. This fact in connection with the general theory developed in [79], [69], [107] and [106] leads to the construction of Ψ^*-algebras in $\mathcal{T}(S)$ induced by the group action of $\mathbf{B}$ and iterated commutators.

Chapter A

Appendix

In this appendix we collect the basic tools in our analysis before. Some of the proofs can be found here and we give references for more detailed informations.

A.1 On the topology of $\mathcal{DFN}$-spaces

We prove some topological results on open or closed submanifolds of $\mathcal{DFN}$-spaces. A more detailed description and some of the proofs also can be found in ([51], [60], [116]). In the following let E be a dual Fréchet nuclear space ($\mathcal{DFN}$-space). Then, due to Lemma 5.3.2 with respect to the *compact-open topology* E can be represented as a nuclear inductive countable spectrum of Banach spaces in the category of locally convex spaces and continuous linear mappings. Moreover, the nuclear maps can be chosen to be embeddings.

Definition A.1.1 Let X be a topological locally convex space and $U \subset X$ and open subset of X.

(1) A sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets $K_n \subset U$ is called *fundamental sequence*, if for each compact set $K \subset U$ there is $n_0 \in \mathbb{N}$ such that $K \subset K_{n_0}$. The set U is called *hemi-compact* if it contains a fundamental sequence of compact sets.

(2) The open set U is called *Lindelöf* if each open cover of U admits a countable subcover.

(3) X is called *k-space*, if $M \subset X$ is open if and only if $M \cap K$ is open in each compact subset $K \subset X$ with the induced topology.

An equivalent characterization of k-spaces using continuous mappings is given by the following lemma.

Lemma A.1.1 *Let (X,τ) and (Y,σ) be topological spaces then (a) and (b) are equivalent*

(a) X is a k-space.

(b) A mapping $f : X \to Y$ is continuous if the restriction of f to K is continuous for each compact subset K of X.

Proof Assume that (a) holds and $f : X \to Y$ has the property that its restriction to each compact set $K \subset X$ is continuous. Let $U \subset Y$ be an open subset, then

$$f^{-1}(U) \cap K = \left(f_{|_K} \right)^{-1}(U) \subset K$$

is an open set in K for all compact sets K in X. By (a) we conclude that $f^{-1}(U) \subset X$ is open and (b) follows . Now, assume that (b) holds. Consider the topology $\tilde{\tau}$ on X given by the open sets

$$\Big\{ U \subset X : U \cap K \subset K \ \text{ is open in } K \text{ for all compact sets } K \text{ of } X \Big\}.$$

Then obviously $\tilde{\tau}$ is finer than τ and the restriction of $(*)$ $\mathrm{id} : (X, \tau) \to (X, \tilde{\tau})$ to all compact sets $K \subset (X, \tau)$ is continuous. By assumption (b) the map $(*)$ is continuous and it follows that τ is finer than $\tilde{\tau}$. This now implies (a). □

Important examples of k-spaces are Hausdorff spaces which are locally compact or satisfy the first axiom of countability (cf. [99], p. 231, 13 Theorem). In general, subspaces of k-spaces do not have to be k-spaces again. An example of a space which is not a k-space is given by the product of uncountably many copies of the real line (cf. [99], p. 240).

Lemma A.1.2 *Let $U \subset E$ be an open set in a $\mathcal{DFN}$-space E, then U is hemi-compact.*

Proof Without loss of generality we consider an inductive spectrum $E = \{ E_n, \pi \}$ of Banach spaces $(E_n, \| \cdot \|_n)$ such that the map $\pi_{n,n+1} : E_n \hookrightarrow E_{n+1}$ are nuclear embeddings and we define with $U_n := U \cap E_n$ for $m, n \in \mathbb{N}$:

$$K_{n,m} := \left\{ x \in U_n : \| x \|_n \leq m, \ \text{ and } \ \mathrm{dist}_{n+1} (x, \partial U_{n+1}) \geq \frac{1}{m} \right\}.$$

Here ∂U_n denotes the boundary of U_n in E_n and we write dist_n for the distance function in E_n. For all $n, m \in \mathbb{N}$ the sets $K_{n,m}$ are bounded in E_n and so they are relatively compact in E_{n+1}. Let us write $K^a_{n,m}$ for the closure of $K_{n,m}$ in E_{n+1} and fix a compact set $C \subset U$. Then there is $n \in \mathbb{N}$ such that $C \subset E_n \cap U = U_n \subset U_{n+1}$ is compact (cf. [60]). Because U_{n+1} is an open subset of E_{n+1} it follows that

$$\mathrm{dist}_{n+1} (C, \partial U_{n+1}) > 0.$$

Hence there are $n, m \in \mathbb{N}$ such that $C \subset K_{n,m} \subset K^a_{n,m}$. Let $j : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection, then we define

$$K_l := \bigcup_{r=1}^{l} K^a_{j(r)}.$$

We obtain a fundamental sequence (K_l) of compact sets such that $K_{l_1} \subset K_{l_2}$ for $l_1 \leq l_2$ and $U = \bigcup_{l \in \mathbb{N}} K_l$. □

Corollary A.1.1 *Let E be a $\mathcal{DFN}$-space and $U \subset E$ be open, then U is Lindelöf.*

Proof Assume that $(*)$ $U \subset \bigcup_{\alpha\in I} U_\alpha$ is an open cover of U where I is any index set. By Lemma A.1.2 we can fix a fundamental sequence $(K_l)_{l\in\mathbb{N}}$ of compact sets in U such that U is the countable union of all K_l for $l \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$ it is clear that $(*)$ is an open cover of K_n and so there are finitely many indices $\alpha(n,1), \cdots, \alpha(n, m_n) \in I$ such that

$$K_n \subset \bigcup_{l=1}^{m_n} U_{\alpha(\,n,\,l\,)}.$$

Hence we conclude that the sets $\{\, U_{\alpha(\,n,\,j\,)} : n \in \mathbb{N},\ j = \alpha(\,n, 1\,), \cdots, \alpha(\,n, m_n\,)\,\}$ form a countable sub-cover of $(*)$. □

Remark A.1.1 In Lemma A.1.2 all we have used was the compactness of the embeddings

$$\pi_{n,n+1} : E_n \hookrightarrow E_{n+1}, \qquad\qquad (\, n \in \mathbb{N}\,)$$

and so we can conclude that all open sets in dual Fréchet Montel spaces ($\mathcal{DFM}$-spaces) are *hemi-compact* as well. In general, if E is a $\mathcal{DFC}$-space (i.e. $E = F_c'$ is the dual of a Frèchet space endowed with the topology of compact convergence), then the compact sets of F_c' are metric and hence separable. Moreover, each open subset of E is *hemi-compact* and *Lindelöf* and for the proof of this fact we refer to [116], p. 511, 7.2 Proposition. The following result can be found in [116], p. 513, 7.6 *Theorem.* Because it is essential for our construction of the reproducing kernel Hilbert spaces in the infinite dimensional setting we give the proof.

Theorem A.1.1 *Let E be a $\mathcal{DFC}$-space. Then each closed subset of E is a k-space.*

Proof Let X be a closed subset of E and $U \subset X$ such that $U \cap K$ is open in K for each compact subset K of X. We have to show, that U is an open neighborhood of all its elements $\xi \in U$. Without loss of generality we assume that $\xi = 0 \in U$. Let E be the dual of the Fréchet space F with the compact open topology. For the proof we construct a compact set L of F such that $L^\circ \cap X \subset U$ where

$$L^\circ := \Big\{\, x \in E : \sup\{\, |\, \langle x, \varphi\rangle_{E,F}\, | : \varphi \in L\,\} \leq 1 \,\Big\}$$

denotes the *polar* of L in E and $\langle \cdot, \cdot \rangle_{E,F}$ is given by the duality between E and F. Then the assertion follows from the fact that L° is an open neighborhood of 0 in E and so $L^\circ \cap X$ is a 0-neighborhood in X.

Let $(W_m)_{m\in\mathbb{N}}$ be a decreasing 0-neighborhood base in F with $F = W_0$. We define

$$K_m := {W_m}^\circ \cap X \subset X.$$

We show that there exists a sequence of finite sets $A_m \subset W_m$ such that with the union $B_m := A_0 \cup \cdots \cup A_m$ we have

$$ {B_m}^\circ \cap K_{m+1} \subset U. \tag{A.1.1}$$

To prove (A.1.1) notice first that the sets K_m are compact in X and hence E induces on each K_m the weak topology $\sigma(E, F)$ (cf. Proposition 32.7 and 32.8 in [137]). Since the set $U \cap K_1$ is a 0-neighborhood in K_1 by assumption we can find a finite set $A_0 \subset F = W_0$ such that

$$A_0{}^\circ \cap K_1 \subset U \cap K_1 \subset U.$$

Suppose that we have already found finite sets $A_j \subset W_j$ for $j = 0, \cdots, m$ such that (A.1.1) holds and assume that

$$\emptyset \neq (B_m \cup P)^\circ \cap K_{m+2} \not\subset U$$

for any finite set $P \subset W_{m+1}$. Let K be the complement of $U \cap K_{m+2}$ in K_{m+2}, then K is a compact subset of K_{m+2}. For each finite set $P \subset W_{m+1}$ the set

$$(B_m \cup P)^\circ \cap K = B_m{}^\circ \cap P^\circ \cap K \tag{A.1.2}$$

is a non-void closed subset of K and so compact itself. If $\mathcal{P}$ is a finite family of finite subsets $P \subset W_{m+1}$ then it follows from

$$\bigcap_{P \in \mathcal{P}} \{ B_m{}^\circ \cap P^\circ \cap K \} = B_m^\circ \cap \Big[\cup \{ P : P \in \mathcal{P} \} \Big]^\circ \cap K \neq \emptyset$$

that the sets in (A.1.2) have the finite intersection property, and therefore by compactness

$$B_m^\circ \cap W_{m+1}^\circ \cap K = \bigcap \Big\{ B_m{}^\circ \cap P^\circ \cap K : P \subset W_{m+1}, \ \ P \text{ finite} \Big\} \neq \emptyset.$$

Because of $K \subset X$ this leads to $B_m{}^\circ \cap K_{m+1} = B_m{}^\circ \cap W_{m+1}{}^\circ \cap X \not\subset U$ which is a contradiction to (A.1.1). Now, we define

$$B := \bigcup_m A_m = \bigcup_m B_m \subset F.$$

Since by our choice $A_m \subset W_m$ for every $m \in \mathbb{N}$ and because each A_m is finite, we conclude that B is the range of a null sequence in F. Let $L := B \cup \{0\}$, then L is a compact subset of F and according to (A.1.1)

$$L^\circ \cap K_{m+1} \subset B_m{}^\circ \cap K_{m+1} \subset U$$

for every $m \in \mathbb{N}$. Since $X = \bigcap_m K_m$ we conclude that $L^\circ \cap X \subset U$. □

Corollary A.1.2 *Let E be a $\mathcal{DFC}$-space. Then each open subset of E is a k-space.*

Proof Let $V \subset E$ be open. Consider $f : V \to Y$ into a topological space Y such that the restriction of f to each compact subset K of V is continuous. By Theorem A.1.1 and Lemma A.1.1 the restriction of f to each subset X of V, which is closed in E, is continuous. Since each point of V admits a neighborhood base formed by closed sets the mapping f is continuous. Again, using Lemma A.1.1 we conclude that V is a k-space. □

Let F be a nuclear Frèchet space, then the bounded subsets of F are relative compact and so F' equipped with the *compact-open* topology is a $\mathcal{DFC}$-space. From Theorem A.1.1 and Lemma A.1.1 we immediately obtain:

Corollary A.1.3 *Let E be a $\mathcal{DFN}$-space then both, the closed and the open subsets of E are k-spaces.*

Definition A.1.2 An open set U in a locally convex space X is said to be *uniformly open* if there exists a continuous semi-norm p on X such that U is open in the semi-normed space (X, p).

Using these topological results on $\mathcal{DFN}$-spaces we can now prove

Theorem A.1.2 *Let E be a $\mathcal{DFN}$-space, then every open set in E is uniformly open.*

Proof Let $U \subset E$ be an open subset. For each $x \in U$ we can choose a continuous semi-norm p_x on E such that $x + B_{p_x}(1) \subset U$ where

$$B_{p_x}(1) := \Big\{ y \in E : p_x(y) < 1 \Big\}$$

is the open unit ball with respect to p_x. According to Corollary A.1.1 the set U is Lindelöf and so there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset U$ such that

$$U = \bigcup_{n\in\mathbb{N}} \Big\{ x_n + B_{p_{x_n}}(1) \Big\}.$$

The space U is hemi-compact by Lemma A.1.2 and so there is an increasing fundamental system $(K_n)_n$ of compact sets in U. For $n \in \mathbb{N}$ choose $\lambda_n > 0$ with $K_n \subset \lambda_n B_{p_{x_n}}(1)$ and let us define $V := \bigcap_{n\in\mathbb{N}} \lambda_n B_{p_{x_n}}(1)$. If $B \subset E$ is a bounded set, then B is relatively compact in E and there is $n_0 \in \mathbb{N}$ with

$$B \subset K_n \subset \lambda_n B_{p_{x_n}}(1) \qquad \text{for all} \quad n \geq n_0.$$

Hence V absorbs all bounded subsets of E and because E is *bornological* it follows that V is a neighborhood of zero. If we denote by p the *Minkowsky functional* defined by V we have $p_{x_n} \leq \lambda_n p$ or equivalently $B_p(1) \subset \lambda_n B_{p_{x_n}}(1)$ for all $n \in \mathbb{N}$. Therefore U is an open set in (E, p). □

A.2 $\mathcal{DFN}$-spaces of holomorphic functions

In chapter 5 we have given several examples of $\mathcal{DFN}$-spaces. Here we want to describe a class of such spaces consisting of holomorphic functions on open subsets of $\mathbb{C}^n$. More precisely, we define an inductive nuclear spectrum of Hilbert spaces which is continuously embedded into a Bergman space. Moreover, we show that each $\mathcal{DFN}$-space can be embedded continuously and dense into a suitable Hilbert space.

Let $m \in \mathbb{N}$ and $\Omega_\infty \subset \mathbb{C}^m$ be an open set. Fix a fundamental sequence $(\Omega_n)_{n\in\mathbb{N}}$ of bounded and open subsets of Ω_∞ satisfying the conditions:

$$\overline{\Omega_n} \subset \Omega_{n+1} \qquad \text{and} \qquad \Omega_\infty = \bigcup_{n\in\mathbb{N}} \Omega_n \qquad (n \in \mathbb{N}).$$

Let us consider the *Bergman spaces* of holomorphic functions on Ω_n defined by:

$$H^2(\,\Omega_n, v\,) := \mathcal{H}(\,\Omega_n\,) \cap L^2(\,\Omega_n, v\,)$$

with respect to the usual Lebesgue measure v and the space $\mathcal{H}(\Omega_n)$ of all holomorphic functions on Ω_n. For $j < n$ we can extend the functions in $H^2(\Omega_j, v)$ to $\Omega_n \setminus \Omega_j$ by 0. Then we obtain canonical isometric embeddings:

$$I_{j,n} : H^2\,(\,\Omega_j, v\,) \hookrightarrow L^2\,(\,\Omega_n, v\,) \qquad \text{for all} \qquad j \leq n.$$

Let us write $I_n := I_{n,\infty}$ for $n \in \mathbb{N}$. The Bergman spaces $H^2(\,\Omega, v\,)$ are closed subspaces of $L^2(\,\Omega_n, v\,)$ and we denote by

$$P_n : L^2(\,\Omega_n, v\,) \rightarrow H^2(\,\Omega_n, v\,) \qquad \text{and} \qquad P := P_\infty$$

the orthogonal projections. By $\langle \cdot, \cdot \rangle_{\Omega_j}$ for $j \in \mathbb{N} \cup \{\infty\}$ we indicate the $L^2(\,\Omega_j, v\,)$-inner product. The range

$$H_n := P \circ I_n \left[\, H^2\,(\,\Omega_n, v\,)\,\right] \subset H^2\,(\,\Omega_\infty, v\,)$$

is a space of holomorphic functions on Ω_∞ for all $n \in \mathbb{N}$. It is a well-known fact that each of the Bergman spaces $H^2(\,\Omega_n, v\,)$ is a reproducing kernel Hilbert-space. Denote by

$$K_n : \Omega_n \times \Omega_n \rightarrow \mathbb{C}$$

for $n \in \mathbb{N} \cup \{\infty\}$ the corresponding *Bergman kernel*. As an easy observation we prove:

Lemma A.2.1 *Fix a number $n \in \mathbb{N}$ and $(u, v) \in \Omega_n \times \Omega_\infty$. Then for all $k > n$ we have the identity $P \circ I_n\,[\,K_n(\,\cdot, u\,)\,]\,(v) = P \circ I_k\,[\,K_k(\cdot, u\,)\,]\,(v)$.*

Proof With $z \in \Omega_\infty$ we obtain by the reproducing property of K_n:

$$\Big\langle\, K_\infty(\cdot, z), P \circ I_n\,[\,K_n(\cdot, u)\,]\,\Big\rangle_{\Omega_\infty} = \Big\langle\, K_\infty(\cdot, z), K_n(\cdot, u)\,\Big\rangle_{\Omega_n} = K_\infty(\,u, z\,). \qquad \text{(A.2.1)}$$

Equation (A.2.1) is independent of $n \in \mathbb{N}$ and so we conclude that

$$K_\infty(\cdot, z) \perp P\left[\, I_n K_n(\cdot, u\,) - I_k\, K_k(\cdot, u\,)\,\right]$$

for all $z \in \Omega_\infty$. From the fact that the linear hull span$\{\,K_\infty(\cdot, z) : z \in \Omega_\infty\,\}$ is dense in the Bergman space $H^2(\,\Omega_\infty, v\,)$ the assertion follows. $\square$

For all numbers $n, j \in \mathbb{N} \cup \{\infty\}$ with $j \leq n$ we consider the restriction map:

$$R_{n,j} : H^2\,(\,\Omega_n, v\,) \longrightarrow H^2\,(\,\Omega_j, v\,) : f \mapsto f_{|\Omega_j},$$

and we define $R_j := R_{\infty,j}$. Fix a point $z \in \Omega_n$ and $f \in H^2(\,\Omega_j, v\,)$, then we compute for the adjoint operator $R^*_{n,j}$:

$$\begin{aligned}\big[\, R^*_{n,j} f \,\big]\,(z) &= \Big\langle\, R^*_{n,j} f, K_n(\cdot, z\,) \,\Big\rangle_{\Omega_n} \\ &= \Big\langle\, f, R_{n,j}\, K_n(\cdot, z\,) \,\Big\rangle_{\Omega_j} \\ &= \Big\langle\, I_{j,n} f, K_n(\cdot, z\,) \,\Big\rangle_{\Omega_n} = [\, P_n \circ I_{j,n} f \,]\,(z).\end{aligned}$$

Hence we find the identities $R^*_{n,j} = P_n \circ I_{j,n}$ and $R^*_k = P \circ I_k$ for all $k \in \mathbb{N}$.

Lemma A.2.2 *Assume that the range $ran(R_j)$ is dense in $H^2(\,\Omega_j, v\,)$ for all $j \in \mathbb{N}$. Then the operator $P \circ I_n : H^2(\,\Omega_n, v\,) \to H_n := ran(\,P \circ I_n\,)$ is an isomorphism.*

Proof We only have to prove that $P \circ I_n$ is injective. This follows from the computation above with $P \circ I_n = R^*_n$ and the fact that R_n has dense range in $H^2(\Omega_n, v)$ which implies that it adjoint map is injective. □

In the following we assume that R_n have dense range in $H^2(\Omega_n, v)$ for all $n \in \mathbb{N}$ in order to apply Lemma A.2.2. This assumption is not too restrictive, in particular it holds if the holomorphic polynomials are dense in $H^2(\Omega_j, v)$ for all $j \in \mathbb{N}$.

Via the isomorphism $P \circ I_n$ in Lemma A.2.2 we can consider the topology on the space $H_n := \mathrm{ran}(\,P \circ I_n\,)$ induced by $H^2(\,\Omega_n, v\,)$. Then obviously H_n becomes a Hilbert space for each $n \in \mathbb{N}$ with the inner-product given by:

$$\langle \cdot, \cdot \rangle_n := \Big\langle\, [\, P \circ I_n\,]^{-1} \cdot, [\, P \circ I_n\,]^{-1} \cdot \,\Big\rangle_{\Omega_n}.$$

Lemma A.2.3 *For $j \in \mathbb{N} \cup \{\infty\}$ and $n \geq j$ the inclusions $J_{j,n} : H_j \hookrightarrow H_n$ are well-defined and continuous with $\|\, J_{j,n} \,\| \leq 1$.*

Proof Let $n \in \mathbb{N}$ and $j \leq n$, then by the computation above we have the equalities:

$$P \circ I_j = R^*_j = \Big[\, R_{n,j} \circ R_n \,\Big]^* = R^*_n \circ R^*_{n,j} = P \circ I_n \circ P_n \circ I_{j,n}. \tag{A.2.2}$$

Let $f = P \circ I_j\, g \in H_j$ with $g \in H^2(\,\Omega_j, v\,)$ and define

$$h := P_n \circ I_{j,n}\, g \in H^2\,(\,\Omega_n, v\,)\,.$$

By equation (A.2.2) it follows that $f = P \circ I_n\, h \in H_n$ and so the embedding $H_j \hookrightarrow H_n$ is well-defined. The continuity follows from:

$$\|\, J_{j,n} f \,\|_n = \|\, f \,\|_n = \|\, h \,\|_{\Omega_n} \leq \|\, I_{j,n} g \,\|_{\Omega_n} = \|\, g \,\|_{\Omega_j} = \|\, f \,\|_j$$

which in addition implies that $\|\, J_{j,n} \,\| \leq 1$. □

From the construction above we obtain a continuous spectrum of Hilbert spaces of holomorphic functions on Ω_∞.

$$H_0 \xrightarrow{J_{0,1}} H_1 \xrightarrow{J_{1,2}} \cdots \longrightarrow H_j \xrightarrow{I_j} H^2(\Omega_\infty, v). \tag{A.2.3}$$

We show that (A.2.3) is nuclear and so equipped with the inductive topology we obtain a $\mathcal{DFN}$-space

$$E := \bigcup_{n\in\mathbb{N}} H_n \subset H^2(\Omega_\infty, v)$$

of square-integrable holomorphic functions on Ω_∞.

Lemma A.2.4 *The spectrum (A.2.3) is nuclear, i.e. the embeddings $J_{j,n} : H_j \to H_n$ are of Hilbert-Schmidt type for all $n \in \mathbb{N}$ and $j < n$.*

Proof Fix $n \in \mathbb{N}$, then according to equation (A.2.2) we have the following commutative diagram:

$$\begin{array}{ccc}
H^2(\Omega_n, v) & \xrightarrow{P_{n+1}I_{n,n+1}} & H^2(\Omega_{n+1}, v) \\
{\scriptstyle PI_n}\Big\downarrow & & {\scriptstyle PI_{n+1}}\Big\downarrow \\
H_n & \xrightarrow{J_{n,n+1}} & H_{n+1}.
\end{array}$$

From the fact that all the maps $P \circ I_j$ from $H^2(\Omega_j, v)$ to H_j are isometric isomorphisms it is sufficient to prove that $P_{n+1} \circ I_{n,n+1} = R^*_{n+1,n}$ is of Hilbert-Schmidt type. This again holds iff $R_{n+1,n}$ is Hilbert-Schmidt. Let $[\, e_j : j \in \mathbb{N}\,]$ be an orthonormal basis in the space $H^2(\Omega_{n+1}, v)$, then it is well-known that

$$K_{n+1}(z, w) = \sum_{j\in\mathbb{N}} e_j(z) \cdot \overline{e_j(w)}, \qquad (z, w) \in \Omega_{n+1} \times \Omega_{n+1}.$$

Moreover, the map $\Omega_{n+1} \ni z \mapsto K_{n+1}(z, z) =: \omega(z) \in \mathbb{R}^+$ is continuous by Lemma 6.1.2. From the theorem of monotone convergence we find that:

$$\begin{aligned}
\| R_{n+1,n} \|_{HS} &= \sum_{j\in\mathbb{N}} \int_{\Omega_n} |\, R_{n+1,n} e_j \,|^2 \, dv \\
&= \int_{\Omega_n} \sum_{j\in\mathbb{N}} |\, R_{n+1,n} e_j \,|^2 \, dv = \int_{\Omega_n} \omega \, dv < \infty.
\end{aligned}$$

Here $\| \cdot \|_{HS}$ denotes the Hilbert-Schmidt norm. For the existence of the last integral we have used the compactness of $\overline{\Omega_n} \subset \mathbb{C}^n$ and the continuity of ω on $\overline{\Omega_n}$. □

We have constructed a countable nuclear inductive spectrum of Hilbert space embeddings which is contained in a Bergman space of holomorphic functions on an open set Ω_∞ in $\mathbb{C}^n$. Let us show that each spectrum of this type can densely and continuously be embedded into a suitable separable complex Hilbert space.

Let E be a $\mathcal{DFN}$-space. According to Lemma 5.3.2 and Remark 5.3.2 E can be represented as a nuclear inductive spectrum $\{H_n, \pi_n\}_{n\in\mathbb{N}_0}$ of separable complex Hilbert spaces H_n with norms $\|\cdot\|_n$ in the category of locally convex spaces and continuous mappings. In addition, all the maps $\pi_{n,n+1} : H_n \hookrightarrow H_{n+1}$ are dense and nuclear embeddings with $\| \pi_{n,n+1} \| < 1$ for all $n \in \mathbb{N}_0$. Without loss of generality we can assume that each $\pi_{n,n+1}$ factorizes into two nuclear embeddings. According to Lemma 5.4.1 and Remark 5.4.1 there is a sequence $(B_n)_{n\in\mathbb{N}}$ of positive nuclear operators with $B_n \in \mathcal{L}(H_n)$ and $tr(B_n^{\frac{1}{2}}) < \infty$ such that $H_{n-1} \subset B_n^{\frac{1}{2}} H_n$ for all $n \in \mathbb{N}$. Moreover, all the embeddings

$$i_{n-1} : H_{n-1} \hookrightarrow \left(B_n^{\frac{1}{2}} H_n, \left\| B_n^{-\frac{1}{2}} \cdot \right\|_n \right)$$

are continuous with $\| i_{n-1} \| \leq \mathrm{tr}(B_n)$. Let ν_n, $n \in \mathbb{N}$ be the normed *Gaussian measure* on H_n with *correlation operator* B_n. With the embedding

$$\pi_n : H_n \hookrightarrow \bigcup_{n\in\mathbb{N}} H_n = E = \overrightarrow{\lim_{n\to\infty}} H_n$$

and any sequence $(\gamma_n)_{n\in\mathbb{N}} \in l^1(\mathbb{N}, \mathbb{N}_+)$ we can consider the finite Borel-measure ν on E defined on the Borel σ-algebra $\mathcal{B}(E)$ of E by

$$\nu(A) := \sum_{n\in\mathbb{N}} \gamma_n \cdot \nu_n \left(\pi_n^{-1}(A) \right), \qquad \text{for all} \qquad A \in \mathcal{B}(E). \tag{A.2.4}$$

By our computations in Example 6.2.1 it follows that there is a canonical isometry of H_n into $L^2(H_n, \nu_n)$ given by:

$$H_n \ni z \mapsto \left\langle B_n^{-\frac{1}{2}} \cdot, z \right\rangle_n =: G_{z,n} \in L^2 (H_n, \nu_n) .$$

In general, by standard calculations (cf. [48]) all the powers $G_{z,n}^k$ where $k \in \mathbb{N}_0$ are ν_n-integrable and they are mutual orthogonal in $L^2(H_n, \nu_n)$ with:

$$\left\| G_{z,n}^k \right\|_{L^2(H_n,\nu_n)} = \sqrt{k!} \cdot \| z \|_n^k.$$

Hence for any fixed $z \in H_{n-1} \subset B^{\frac{1}{2}} H_n$ we can define a map $\Phi_{z,n} : \mathbb{R} \to L^2(H_n, \nu_n)$ by:

$$\Phi_{z,n}(t) := \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot G_{y,n}^k = \exp (t \cdot G_{y,n}), \qquad \text{with} \quad y := B_n^{-\frac{1}{2}} z \in H_n. \tag{A.2.5}$$

where due to the orthogonality of the powers $G_{y,n}^k$ and equation (A.2.5) the series is convergent in $L^2(H_n, \nu_n)$ with

$$\| \Phi_{z,n}(t) \|_{L^2(H_n,\nu_n)}^2 = \exp \left(t^2 \cdot \| y \|_n^2 \right)$$

for all $t \in \mathbb{R}$. Moreover, it can be shown that $\Phi_{z,n}(t)$ has the reproducing property:

$$f(t \cdot z) = \left\langle f, \Phi_{z,n}(t) \right\rangle_{L^2(H_n,\nu_n)} \qquad \text{for all} \qquad f \in \mathcal{H} (H_n) \cap L^2 (H_n, \nu_n) . \tag{A.2.6}$$

It directly follows from (A.2.5) that:

Lemma A.2.5 *Let $n \in \mathbb{N}$ and $y = B_n^{-\frac{1}{2}} z \in H_n$ where $z \in H_{n-1}$. Then the mapping $\Phi_{z,n}$ is differentiable in $t_0 = 0$ with $\Phi'_{z,n}(0) = G_{y,n} \in L^2(H_n, \nu_n)$.*

Now, using the Toeplitz projection we construct a complex Hilbert space H_E together with a continuous embedding $E \hookrightarrow H_E$. The topology on H_E is induced by the Bergman metric on E. Via restriction to E the space of holomorphic polynomials on H_E will be contained in $H^2(U, \nu)$ for all open subsets $U \subset E$. Let

$$P : L^2(E, \nu) \to H^2(E, \nu) := \mathcal{H}(E) \cap L^2(E, \nu)$$

be the orthogonal projection onto the Bergman space. For each $n \in \mathbb{N}$ we define a sesquilinear form $\beta_n : H_n \times H_n \to \mathbb{C}$ by:

$$\beta_n(w, z) := \gamma_{n+1}^{-2} \cdot \Big\langle \, P\, G_{y,n+1}, P\, G_{v,n+1} \Big\rangle_{L^2(E,\nu)}.$$

where $y := B_{n+1}^{-\frac{1}{2}} z$ and $v := B_{n+1}^{-\frac{1}{2}} w$. Here we extend both function $G_{y,n+1}$ and $G_{v,n+1}$ from H_{n+1} to E by zero.

Lemma A.2.6 *Let $n \in \mathbb{N}$ and fix $(z, w) \in H_n \times H_n$. Then β_n is continuous and $\beta_m(w, z)$ is independent of $m \geq n$. Moreover, if $E' \cap H^2(E, \nu)$ separates the points of E then β_n is positive definite.*

Proof The continuity of β_n in each component directly follows from the continuity of the operators P, the correlation $B_{n+1}^{-\frac{1}{2}} : H_n \to H_{n+1}$ and the map

$$H_{n+1} \ni y \mapsto G_{y,n+1} \in L^2 \left(H_{n+1}, \nu_{n+1} \right) \hookrightarrow L^2(E, \nu).$$

With $z, w \in H_n$ let us define $y := B_{n+1}^{-\frac{1}{2}} z$ and $v := B_{n+1}^{-\frac{1}{2}} w$. Then we have $v, y \in H_{n+1}$ and according to Lemma A.2.5 it follows for $t \in \mathbb{R}$ and the $L^2(E, \nu)$-inner product $\langle \cdot, \cdot \rangle$ that:

$$\begin{aligned} \beta_n(w, z) &= \gamma_{n+1}^{-2} \Big\langle \, P\, \Phi'_{z,n+1}(0), P\, \Phi'_{w,n+1}(0) \Big\rangle \\ &= \frac{\partial^2}{\partial t\, \partial s} \Big\langle \, \gamma_{n+1}^{-1} P\, \Phi_{z,n+1}(t), \gamma_{n+1}^{-1} P\, \Phi_{w,n+1}(s) \Big\rangle_{|s=t=0}. \end{aligned}$$

Note that for $t \in \mathbb{R}$ the functions $\Phi_{z,n+1}(t)$ coincides with $k_{n+1}(\cdot, tz)$ for $U = E$ where k_{n+1} was defined in Example 6.4.1. Hence with the Bergman kernel $K_E : E \times E \to \mathbb{C}$ of E we conclude from Lemma 6.4.2 that:

$$\beta_n(w, z) = \frac{\partial^2}{\partial t\, \partial s} K_E(sw, tz)_{|t=s=0}.$$

In particular, $\beta_n(w, z)$ is independent of $m \geq n$. Assume that $E' \cap H^2(E, \nu)$ separates the points of E and there is $0 \neq z_0 \in H_n$ such that $\beta_n(z_0, z_0) = 0$. Choose a continuous functional

$$\varphi \in E' \cap H^2(E, \nu) \qquad \text{with} \qquad \varphi(z_0) \neq 0.$$

Because the restriction of φ to H_{n+1} is contained in $H^2(\, H_{n+1}, \nu_{n+1}\,)$ we have by (A.2.6) and with $y_0 := B_{n+1}^{-\frac{1}{2}} z_0 \in H_{n+1}$:

$$\begin{aligned} 0 \neq |\, \varphi(z_0)\,| &= \Big|\, \langle\, \varphi, \Phi_{z_0,\, n+1}(1)\,\rangle_{L^2_{n+1}} \Big| \\ &= \Big|\, \langle\, \varphi, G_{y_0,n+1}\,\rangle_{L^2_{n+1}} \Big| \\ &= \gamma_{n+1}^{-1} \cdot \Big|\, \langle\, \varphi, P\, G_{y_0,n+1}\,\rangle \Big| \\ &\leq \gamma_{n+1}^{-1} \cdot \|\, \varphi\,\| \cdot \|\, P\, G_{y_0,n+1}\,\| = \|\varphi\| \cdot \beta_n^{\frac{1}{2}}(z_0, z_0) = 0. \end{aligned}$$

By this contradiction we necessarily have $\beta_n(z_0, z_0) > 0$ and so the form β_n is positive definite. $\square$

Example A.2.1 We show that the sequence $(\gamma_n)_n \in l^1(\mathbb{N}, \mathbb{N}_+)$ can be chosen such that with the corresponding measure ν in (A.2.4) the space $E' \cap H^2(E, \nu)$ separates the points of E.

Because $E \setminus \{0\}$ is hemi-compact by Lemma A.1.2 there is an increasing sequence $(K_n)_{n\in\mathbb{N}} \subset E \setminus \{0\}$ of compact subsets such that

$$E \setminus \{0\} = \bigcup_{n\in\mathbb{N}} K_n.$$

Due to the compactness of K_n there are finitely many points $x_1^{(n)}, \cdots, x_{m_n}^{(n)} \in K_n$ and corresponding functionals $\varphi_1^{(n)}, \cdots, \varphi_{m_n}^{(n)} \in E'$ such that $\varphi_j^{(n)}(x_j^{(n)}) = 1$ for all $j = 1, \cdots, m_n$ and

$$K_n \subset \bigcup_{j=1}^{m_n} \Big\{\, x \in E : |\, \varphi_j^{(n)}(x)\,| > \frac{1}{2}\,\Big\}.$$

Let $\varphi \in E'$ and ν be the finite measure defined in (A.2.4). Then for each $n \in \mathbb{N}$ there is a constant $c_n > 0$ such that $|\, \varphi(x)\,| \leq c_n \cdot \|\, x\,\|_n$ for all $x \in H_n$. We obtain for the L^2-norm of φ:

$$\int_E |\, \varphi\,|^2\, d\nu \leq \sum_{n\in\mathbb{N}} \gamma_n \cdot c_n^2 \int_{H_n} \|\cdot\|_n^2\, d\nu_n = \sum_{n\in\mathbb{N}} \gamma_n \cdot c_n^2 \cdot \mathrm{tr}\,(\, B_n\,)$$

where the right hand side of this inequality is finite for a suitable choice of the sequence $(\gamma_n) \in l^1(\mathbb{N}, \mathbb{N}^+)$. Hence let $(\gamma_j^{(n)})_{j\in\mathbb{N}} \subset l^1(\mathbb{N}, \mathbb{N}_+)$ be with

$$\Big\{\, \varphi_1^{(n)}, \cdots, \varphi_{m_n}^{(n)}\,\Big\} \subset H^2\,\big(\, E, \nu^{(n)}\,\big)$$

where we denote by $\nu^{(n)}$ the measure in $(A.2.4)$ defined by $(\gamma_j^{(n)})_j$. Without loss of generality we can assume that

$$\gamma_j^{(n)} \geq \gamma_{j+1}^{(n)} \qquad \text{and} \qquad \gamma_j^{(n)} \geq \gamma_j^{(n+1)}, \qquad \forall\, (\, n, j\,) \in \mathbb{N}^2.$$

Finally consider the diagonal sequence $\gamma_j := \gamma_j^{(j)}$. Then $(\gamma_j)_{j\in\mathbb{N}} \in l^1(\mathbb{N},\mathbb{N}_+)$ and the inclusion

$$\bigcup_{n\in\mathbb{N}} \left\{ \varphi_1^{(n)}, \cdots, \varphi_{m_n}^{(n)} \right\} \subset H^2(\, E, \nu\,)$$

holds where ν denotes the measure corresponding to $(\gamma_j)_j$. Moreover, this set of continuous functionals separates the points of E.

Definition A.2.1 We assume that $E' \cap H^2(\, E, \nu\,)$ separates the points of E. Further, let the map $\beta : E \times E \to \mathbb{C}$ be given by

$$\beta(\, w, z\,) := \beta_n(\, w, z\,) \qquad \text{for} \qquad (\, w, z\,) \in H_n \times H_n.$$

Then from Lemma A.2.6 we conclude that β is a well-defined positive sesquilinear form on E. Let us denote by $(\, H_E, \beta\,)$ the Hilbert space completion of $(\, E, \beta\,)$.

For all $n \in \mathbb{N}$ and each $w \in H_n$ we have with our notations above:

$$\begin{aligned} \beta(\, w, w\,)^{\frac{1}{2}} &= \gamma_{n+1}^{-1} \cdot \|\, P\, G_{y,n+1}\, \| \\ &\le \gamma_{n+1}^{-\frac{1}{2}} \cdot \|\, G_{y,n+1}\, \|_{L^2_{n+1}} \\ &= \gamma_{n+1}^{-\frac{1}{2}} \cdot \|\, y\, \|_{n+1} = \gamma_{n+1}^{-\frac{1}{2}} \cdot \operatorname{tr}(B_{n+1}) \cdot \|\, w\, \|_n \end{aligned} \tag{A.2.7}$$

where $y := B_{n+1}^{-\frac{1}{2}} w \in H_{n+1}$. Now, this proves that E can be considered as a subspace of the Hilbert space H_E:

Corollary A.2.1 *For all $n \in \mathbb{N}$ the embeddings $i_n : H_n \hookrightarrow H_E$ are continuous and hence nuclear. In particular, the inclusion $E \hookrightarrow H_E$ is continuous and dense.*

Denote by $\mathbb{P}\,[\, \beta(\cdot, w) : w \in E\,]$ the space of all holomorphic polynomials on E in with "variables" $\beta(\cdot, w)$ for $w \in E$. It is well-known (cf. [48]) that all powers $\|\cdot\|_n^k$ with $k \in \mathbb{N}$ are integrable over H_n with respect to the Gaussian measure ν_n. In the following Lemma let $H^2(\, H_n, \nu_n\,)$ be the closure of all holomorphic and ν_n-square integrable functions over H_n in $L^2(\, H_n, \nu_n\,)$. From the Cauchy-Schwartz inequality and (A.2.7) it follows that by restriction from E to H_n we can consider $\mathbb{P}\,[\, \beta(\cdot, w) : w \in E\,]$ as a subspace of $H^2(\, H_n, \nu_n\,)$.

Lemma A.2.7 *The restriction map $R : \mathbb{P}\,[\, \beta(\cdot, w) : w \in E\,] \to H^2(\, H_n, \nu_n\,)$ is dense for all numbers $n \in \mathbb{N}$.*

Proof Because for each $n \in \mathbb{N}$ the inclusion $j_n : H_n \hookrightarrow H_E$ is nuclear, there are orthonormal bases $[\, e_j : j \in \mathbb{N}\,]$ of H_n and $[\, d_l : l \in \mathbb{N}\,] \subset E$ of H_E and $(\lambda_j)_j \in l^1(\mathbb{N})$ with $\lambda_j \neq 0$ such that:

$$j_n(z) = \sum_{l=1}^{\infty} \lambda_l \cdot \langle z, e_l \rangle_n\, d_l \qquad \text{and so} \qquad \beta(z, d_i) = \lambda_i \langle z, e_i \rangle_n \tag{A.2.8}$$

for all $z \in H_n$ and $i \in \mathbb{N}$. It is known (cf. [48]) that the polynomials $\mathbb{P}\,[\, \langle \cdot, e_j \rangle_n : j \in \mathbb{N}\,]$ in the infinitely many variables $\langle \cdot, e_j \rangle_n$ form a dense subspace in $H^2(\, H_n, \nu_n\,)$. From (A.2.8) we conclude that $\mathbb{P}\,[\, \langle \cdot, e_j \rangle_n : j \in \mathbb{N}\,] \subset \mathbb{P}\,[\, \beta(\cdot, w) : w \in E\,]$ and the assertion follows. $\square$

A.3 Holomorphy on topological spaces

For the convenience of the reader we want to give some result on holomorphic functions on topological spaces which we use throughout this thesis. We are following closely [51].

We define *$\mathcal{G}$-holomorphic functions* on so-called *finitely open* subsets of a vector space X over $\mathbb{C}$ into a locally convex space. This approach only uses the locally convex topology of uniform convergence over finite dimensional compact subsets of U and it enjoys great generality. However, the definition of *holomorphic* or *$\mathcal{F}$-holomorphic* functions also involves the given locally convex topology on the domain space.

Definition A.3.1 A subset U of a complex linear space X is said to be *finitely open* if the section $U \cap F \subset L$ is open in the Euclidean topology of L for each finite dimensional subspace L of X.

Using the notion of *finitely open sets* we now can define the space of *$\mathcal{G}$-holomorphic* functions on U. (cf. [51], pp. 144)

Definition A.3.2 Let X be a complex linear space, U a finitely open subset of X and F a locally convex space. A function $f : U \subset X \to F$ is called *$\mathcal{G}$- holomorphic* if for each $\xi \in U$, $\eta \in X$ and $\Phi \in F'$ the $\mathbb{C}$-valued function of one complex variable

$$\mathbb{C} \supset V \ni \lambda \mapsto \Phi \circ f\big(\xi + \lambda\eta \big) \in \mathbb{C}$$

is holomorphic on some neighborhood V of 0 in $\mathbb{C}$. We let $\mathcal{H}_{\mathcal{G}}(U, F)$ denote the set of all $\mathcal{G}$-holomorphic functions from U into F. In the case $F = \mathbb{C}$ we simply write $\mathcal{H}_{\mathcal{G}}(U)$ in place of $\mathcal{H}_{\mathcal{G}}(U, F)$.

Some important results on $\mathcal{G}$-holomorphic functions can be found in [51]. In this thesis we mostly deal with holomorphic functions on vector spaces equipped with a locally convex topology. Then we use the notion of a *holomorphic function* or *$\mathcal{F}$-holomorphic function* in the following sense:

Definition A.3.3 If X and F are locally convex spaces over $\mathbb{C}$ and U is an open subset of X, then $f : U \to F$ is *holomorphic* or *$\mathcal{F}$-holomorphic* if $f \in \mathcal{H}_{\mathcal{G}}(U, F)$ and f is continuous. We let $\mathcal{H}(U, F)$ denote the set of all holomorphic mappings from U into F and write $\mathcal{H}(U)$ in place of $\mathcal{H}(U, \mathbb{C})$.

Denote by $\mathcal{P}_a(X, F)$ (resp. $\mathcal{P}_a^{(n)}(X, F)$) the vector space of all polynomials (resp. n-homogeneous polynomials) from X into F and let $\mathcal{P}(X, F)$ (resp. $\mathcal{P}^{(n)}(X, F)$) be the space of all continuous polynomials (resp. n-homogeneous continuous polynomials). Then it is shown in [51] that each $\mathcal{G}$-holomorphic function can be expanded into a power series.

Note that in our definitions above we do not make assumption on the completeness of the topological spaces. However, in the following theorem we denote by $\tilde{F}$ the completion of a locally convex space F.

Theorem A.3.1 *([51]) Let X be a complex linear space and F denotes a locally convex space over $\mathbb{C}$. With a finitely open subset $U \subset X$ and $\xi \in U$ define*

$$B_\xi := \Big\{ z \in X : \xi + \lambda z \in U : |\lambda| \le 1 \Big\}.$$

If $f : U \to F$, then $f \in \mathcal{H}_{\mathcal{G}}(U, F)$ if and only if for every $\xi \in U$ there exists a unique sequence of homogeneous polynomials, $(P_{n,\xi,f})_{n\in\mathbb{N}}$ such that $P_{n,\xi,f} \in \mathcal{P}_a^{(n)}(X, \tilde{F})$ and

$$f(\xi + z) = \sum_{n=0}^{\infty} P_{n,\xi,f}(z), \qquad where \qquad P_{n,\xi,f}(z) = \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{f(\xi + \lambda z)}{\lambda^{n+1}}\, d\lambda. \qquad \text{(A.3.1)}$$

for all $z \in B_\xi$ and all $n \in \mathbb{N}$. Moreover, if W is a balanced subset of X and $rW \subset B_\xi$ for $r > 0$, then

$$\| P_{n,\xi,f} \|_{\beta,W} \le \frac{1}{r^n} \| f \|_{\beta,\xi+rW}$$

for all continuous semi-norms β on F and with $\| f \|_{\beta,W} := \sup\{ \beta \circ f(z) : z \in W \}$.

For $f \in \mathcal{H}(U, F)$ the polynomials $P_{n,\xi,f}$ in Theorem A.3.1 can be chosen to be continuous and the power series in (A.3.1) converges uniformly on a suitable neighborhood of ξ. If in addition F is a normed space, then the continuity of a function $f \in \mathcal{H}_{\mathcal{G}}(U, F)$ follows from a condition which seems to be weaker:

Theorem A.3.2 *([51]) If U is an open subset of a locally convex space E and F is a normed linear space, then $f \in \mathcal{H}_{\mathcal{G}}(U, F)$ is holomorphic iff it is locally bounded.*

Under additional conditions on X and F the next theorem, which can be found as example 3.8, (a) and (b) in [51] shows that the continuity of a function $f \in \mathcal{H}_{\mathcal{G}}(U, F)$ where $U \subset X$ follows from apparently rather weak assumptions

Lemma A.3.1 *Let X and F be Banach spaces and U be a connected open subset of X. Assume that $f \in \mathcal{H}_{\mathcal{G}}(U, F)$, then $f \in \mathcal{H}(U, F)$ if one of the following conditions holds*

(i) There is $\xi \in U$ such that the n-homogeneous polynomials $P_{n,\xi,f}$ in the power series expansion of f in Theorem A.3.1 are continuous on X.

(ii) f is continuous at a single point in U.

Let E be the inductive limit of Banach spaces in the category of topological spaces and continuous mappings (such spaces are called *super inductive*). Let $U \subset E$ be a connected open subset of E, F a Banach space and $f \in H_{\mathcal{G}}(U, F)$. Then again the criterion in Lemma A.3.1 for the continuity of f holds. In particular, Lemma A.3.1 is true for connected open subsets of $\mathcal{DFN}$-spaces E and Banach spaces F but in this case we even have a stronger result ([51], example 3.8.(e)):

Lemma A.3.2 *([51]) Let E be a $\mathcal{DFN}$-space and F a Banach space. If $U \subset E$ is an open set and $f \in \mathcal{H}_{\mathcal{G}}(\, U, F\,)$, then f is $\mathcal{F}$-holomorphic if it is bounded on compact sets. Moreover, a collection of holomorphic functions which is uniformly bounded on compact sets is locally bounded or equi-continuous.*

If E is a $\mathcal{DFN}$-space and $U \subset E$ an open subset, then denote by τ_0 the topology on $\mathcal{H}(U, F)$ of uniform convergence on all compact subsets of U (*compact-open topology*). According to Theorem A.1.2 the space U is uniformly open and so there is a continuous semi-norm p on E such that U is open in the semi-normed space $(\, E, p\,)$. Denote by U_p the set U considered as an open subset of $(\, E, p\,)$.

Corollary A.3.1 *Let E be a $\mathcal{DFN}$-space, F a Banach space and assume that $U \subset E$ is an open subset. If $\mathcal{F}$ is a τ_0-bounded subset in $(\, \mathcal{H}(\, U, F\,), \tau_0\,)$, then we can choose the semi-norm p in such a way that $\mathcal{F} \subset \mathcal{H}(\, U_p, F\,)$ is locally bounded.*

Proof By Lemma A.3.2 we can choose the sequences $(x_n)_{n\in\mathbb{N}}$ and $(p_{x_n})_{n\in\mathbb{N}}$ in the proof of Theorem A.1.2 in such a way that with $V_n := B_{p_{x_n}}(1)$ it follows

$$U = \bigcup_{n\in\mathbb{N}} \Big\{\, x_n + V_n \,\Big\} \qquad \text{and} \qquad \sup \Big\{\, \|\, f \,\|_{x_n+V_n} : f \in \mathcal{F} \,\Big\} < \infty, \qquad (\, n \in \mathbb{N}\,)$$

where we define $\|\, f \,\|_{x_n+V_n} := \sup\{\, \|\, f(z) \,\|_F : z \in x_n + V_n \,\}$. Because for all $n \in \mathbb{N}$ we have

$$B_p(1) \subset \lambda_n V_n$$

it follows that $\mathcal{F}$ is a locally bounded subset of $\mathcal{H}(U_p, F)$. □

A.4 Heisenberg group and Hardy spaces over the ball

Let D be the open unit disc in $\mathbb{C}$, then by classical analysis D is biholomorphically equivalent to the *upper half-plane* $\mathcal{H}_+ := \{\, z \in \mathbb{C} : \operatorname{Im}(z) > 0\,\}$. The real line acts on $\mathcal{H}_+$ by translation and it can be identified with the boundary of $\mathcal{H}_+$. There is a generalization of this fact to the case $\mathbb{C}^{n+1}$ for $n \in \mathbb{N}$ which we want to describe here. We closely follow the book of Stein, [134]. Let $\mathcal{B}_{n+1}$ denote the open unit ball in $\mathbb{C}^{n+1}$ with respect to the Euclidean topology. Then the *upper half-space* $\mathcal{H}_+ \subset \mathbb{C}^{n+1}$ is given by:

$$\mathcal{H}_+ := \Big\{\, (\, z, z_{n+1}\,) \in \mathbb{C}^{n+1} : \; z \in \mathbb{C}^n \text{ and } \operatorname{Im} z_{n+1} > \|\, z \,\|^2 \,\Big\}$$

where we write $\|\cdot\|$ for the Euclidean norm in $\mathbb{C}^n$. The domains $\mathcal{B}_{n+1}$ and $\mathcal{H}_+$ are biholomorphically equivalent under the map $F = (\, F_1, \cdots, F_{n+1}\,) : \mathcal{H}_+ \to \mathcal{B}_{n+1}$ with:

$$F_{n+1}(\, z\,) = \frac{i - z_{n+1}}{i + z_{n+1}} \qquad F_k(\, z\,) = \frac{2iz_k}{i + z_{n+1}}, \qquad k = 1, \cdots, n$$

$$F^{-1}_{n+1}(\, w\,) = \frac{i - iw_{n+1}}{1 + w_{n+1}} \qquad F^{-1}_k(\, w\,) = \frac{w_k}{1 + w_{n+1}}, \qquad k = 1, \cdots, n.$$

Moreover, it is easy to check that via F the boundary $\partial\mathcal{H}_+$ corresponds to the boundary $\partial\mathcal{B}_{n+1}$ of the complex unit ball except for the *south pole* $(\, 0, \cdots, 0, -1\,)$.

A.5 Symmetries of the ball

By *symmetries* or *automorphisms* of a domain Ω in $\mathbb{C}^{n+1}$ we denote the Lie-group $\mathrm{Aut}(\Omega)$ of biholomorphic self-mappings. In the case of the upper half-space $\Omega = \mathcal{H}_+$ we describe the subgroups in $\mathrm{Aut}(\mathcal{H}_+)$ of *dilation*, *rotation* and *translation*. Let $z \in \mathbb{C}^{n+1}$, then in the following we use the notation:

$$z = \big(z', z_{n+1} \big), \qquad \text{with} \qquad z' := \big(z_1, \cdots, z_n \big) \in \mathbb{C}^n \qquad \text{and} \qquad z_{n+1} \in \mathbb{C}.$$

For any positive number $\delta > 0$ and a unitary transformation $u \in \mathcal{L}(\mathbb{C}^n)$ it is easy to see that both maps below are symmetries of $\mathcal{H}_+^{n+1}$:

$$\begin{aligned} \delta\,[\,z\,] &:= \delta \Big[\big(z', z_{n+1} \big) \Big] := \big(\delta \cdot z',\ \delta^2 \cdot z_{n+1} \big), && \big(\text{ dilation }\big), \\ U\,[\,z\,] &:= U \Big[\big(z', z_{n+1} \big) \Big] = \big(u(\,z'\,),\ z_{n+1} \big) && \big(\text{ rotation }\big). \end{aligned}$$

In order to describe the *translation* we have to introduce the *Heisenberg group* $\mathbb{H}^n$. As a set we define:

$$\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R} = \Big\{ \,[\,\zeta, t\,] : \zeta \in \mathbb{C}^n, t \in \mathbb{R}\, \Big\}.$$

We endow $\mathbb{H}^n$ with the multiplication given for $[\,\zeta, t\,], [\,\eta, s\,] \in \mathbb{H}^n$ by the rule:

$$\big[\,\zeta, t\,\big] \cdot \big[\,\eta, s\,\big] := \big[\,\zeta + \eta,\ t + s + 2 \cdot \mathrm{Im}\,(\zeta \cdot \bar{\eta})\,\big]$$

where as usual $\zeta \cdot \bar{\eta} := \zeta_1 \cdot \bar{\eta}_1 + \cdots + \zeta_n \cdot \bar{\eta}_n$. Then $\mathbb{H}^n$ becomes a group with identity $e := [\,0, 0\,]$ and inverse

$$\big[\,\zeta, t\,\big]^{-1} = \big[\,-\zeta, -t\,\big].$$

There is a map $g : \mathbb{H}^n \to \mathrm{Aut}(\,\mathcal{H}_+\,)$ into the symmetries of the upper half space given for each $[\,\zeta, t\,] \in \mathbb{H}^n$ by:

$$g\,\big[\,\zeta, t\,\big] : \mathcal{H}_+ \to \mathcal{H}_+ : \big(z',\ z_{n+1} \big) \mapsto \big(z' + \zeta,\ z_{n+1} + t + 2i\, z' \cdot \bar{\zeta} + i \|\,\zeta\,\|^2 \big). \tag{A.5.1}$$

It is easy to check that the automorphisms $g\,[\,\zeta, t\,]$ are well-defined and they preserve the boundary $\partial\mathcal{H}_+$. Moreover, it holds:

$$g \Big\{ \,[\,\zeta, t\,] \cdot [\,\eta, s\,]\, \Big\} = g\,\big[\,\zeta, t\,\big] \circ g\,\big[\,\eta, s\,\big] \tag{A.5.2}$$

and so g is a group homomorphism which is *faithful*. The action of g is *simply transitive* on $\partial\mathcal{H}_+$, i.e. for any two points $z, y \in \partial\mathcal{H}_+$ there exists a unique element $[\,\zeta, t\,] \in \mathbb{H}^n$ such that $g\,[\,\zeta, t\,](z) = y$. Hence via its action on the origin we can identify the Heisenberg group with the boundary $\partial\mathcal{H}_+$ of the upper half-space by:

$$G : \mathbb{H}^n \ni [\,\zeta, t\,] \mapsto \big(\zeta, t + i \|\,\zeta\,\|^2 \big) = g\,[\,\zeta, t\,]\,(0, 0) \in \partial\mathcal{H}_+. \tag{A.5.3}$$

By this identification and the fact that $\mathbb{H}^n$ acts on $\partial\mathcal{H}_+$ we obtain an action of $\mathbb{H}^n$ on itself which simply is given by the *left translation*

$$\mathbb{H}^n \ni b \mapsto a\,b \in \mathbb{H}^n.$$

It can be shown that the *left-invariant Haar measure* μ on $\mathbb{H}^n$ is given by the usual Lebesgue measure under the identification $\mathbb{H}^n \cong \mathbb{R}^{2n+1}$. Hence μ also is *right-invariant*.

A.6 Cauchy-Szegö projection and Hardy spaces

As before we denote by $\mathcal{H}_+$ the upper half-space in $\mathbb{C}^{n+1}$ with boundary $\partial\mathcal{H}_+$ which was identified with the Heisenberg group $\mathbb{H}^n$. By transporting the *Haar measure* μ on $\mathbb{H}^n$ (which simply is the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$) to $\partial\mathcal{H}_+$ we obtain a measure β on $\partial\mathcal{H}_+$. Hence for all $p \in [\,1, \infty\,)$ we can identify the spaces $L^p(\,\mathbb{H}^n\,) := L^p(\,\mathbb{H}^n, \mu\,)$ and $L^p(\,\partial\mathcal{H}_+, \beta\,)$.

We define the *Hardy space* of holomorphic functions with square integrable boundary values as a closed subspace of $L^2(\,\partial\mathcal{H}_+, \beta\,)$. Let F be a function on $\mathcal{H}_+$ and $\varepsilon > 0$, then with $\mathbf{i} := (\,0, \cdots, 0, i\,) \in \mathbb{C}^{n+1}$

$$F_\varepsilon(z) := F\big(\,z + \varepsilon \cdot \mathbf{i}\,\big)$$

is defined in a neighborhood of $\mathcal{H}_+$. Denote by $H^2(\,\mathcal{H}_+)$ the space of all holomorphic functions F on $\mathcal{H}_+$ such that:

$$\begin{aligned}\|\,F\,\|^2_{H^2} : &= \sup_{\varepsilon>0} \int_{\partial\mathcal{H}_+} |\,F_\varepsilon\,|^2\, d\beta \\ &= \sup_{\varepsilon>0} \int_{\mathbb{C}^n\times\mathbb{R}} \Big|\,F\,\big(\,z',\, t + i\,\big\{\,\varepsilon + \|\,z'\,\|^2\,\big\}\,\big)\,\Big|^2\, dz'\,dt < \infty.\end{aligned}$$

The functions in $H^2(\,\mathcal{H}_+)$ can be identified with their boundary values. A proof of Theorem A.6.1 can be found in [134]:

Theorem A.6.1 *Let $F \in H^2(\,\mathcal{H}_+)$, then:*

(i) There is $f \in L^2(\,\partial\mathcal{H}_+, \beta\,)$ such that $\lim_{\varepsilon\to 0} F(\,z + \varepsilon \cdot \mathbf{i}\,)_{|\partial\mathcal{H}_+} = f$ with respect to the L^2-topology.

(ii) The space of all boundary values f of functions F in (i) is a closed subspace of the Hilbert space $L^2(\,\partial\mathcal{H}_+, \beta\,)$. Moreover, $\|\,f\,\|_{L^2(\,\partial\mathcal{H}_+,\beta\,)} = \|\,F\,\|_{H^2}$.

For each $z \in \mathcal{H}_+$ the *evaluation* δ_z in the topological dual $H^2(\,\mathcal{H}_+)'$ of $H^2(\,\mathcal{H}_+)$ given by $\delta_z(F) = F(z)$ is continuous. There is a unique function

$$S : \mathcal{H}_+ \times \mathcal{H}_+ \to \mathbb{C},$$

the so called *Cauchy-Szegö kernel*, with the following properties (cf. [134]):

(a) For each $w \in \mathcal{H}_+$ it holds $S(\cdot, w) \in H^2(\,\mathcal{H}_+)$.

(b) For each pair $z, w \in \mathcal{H}_+$ the function S is *symmetric*, i.e. $S(z, w) = \overline{S(w, z)}$.

(c) The kernel S has the *reproducing property*: For all $z \in \mathcal{H}_+$ and $F \in H^2(\mathcal{H}_+)$ it holds:

$$F(z) = \int_{\partial\mathcal{H}_+} F \cdot S(\,z, \cdot\,)\, d\beta$$

where we have identified F with its boundary values in the sense of Theorem A.6.1.

For any function $F = f \in L^2(\,\partial\mathcal{H}_+,\beta\,)$ the integral formula in (c) gives the orthogonal projection onto the Hardy space $H^2(\,\mathcal{H}_+)$. In the case of the upper half-space we can calculate the *Cauchy-Szegö kernel* explicitly. A proof of the next result can be found in [134].

Proposition A.6.1 *With* $c_n := (4\,\pi^{n+1})^{-1}\,n!$ *it holds* $S(z,w) = c_n \cdot [\,r(z,w)\,]^{-n-1}$ *where*

$$r(z,w) := \frac{i}{2}\cdot\left(\,\bar{w}_{n+1} - z_{n+1}\,\right) - \sum_{k=1}^{n} z_k \cdot \bar{w}_k$$

and S *is the unique function that enjoys the properties* (a), (b) *and* (c) *above.*

It is a straightforward computation that the *Cauchy-Szegö kernel* transforms in the following way under the following symmetries of $\mathcal{H}_+$.

(a) dilation: For $\delta > 0$ we have $S(\,\delta\circ z, \delta\circ w\,) = \delta^{-2n-2}\cdot S(\,z,w\,)$.

(b) rotation: For any unitary rotation u on $\mathbb{C}^n$ we have $S(\,u(z),u(w)\,) = S(\,z,w\,)$.

(c) translation: For $h \in \mathbb{H}^n$ we have $S(\,h(z),h(w)\,) = S(\,z,w\,)$ where we identify $\mathbb{H}^n$ with a group of symmetries in the described way.

As it was mentioned before we can write the orthogonal projection P_s of $L^2(\,\partial\mathcal{H}_+,\beta\,)$ onto the Hardy space $H^2(\,\mathcal{H}_+)$ as an integral operator:

$$\begin{aligned}\left[\,P_s f\,\right](z) &= \Big\langle\, f, S(\cdot, z\,)\Big\rangle_{L^2(\partial\mathcal{H}_+,\beta\,)}\\ &= \lim_{\varepsilon\to 0}\int_{\partial\mathcal{H}_+} f\,S(\,z+\varepsilon\cdot\mathbf{i},\cdot\,)\,d\beta \end{aligned}\tag{A.6.1}$$

where the limit exists in the $L^2(\,\partial\mathcal{H}_+,\beta\,)$-norm. From the identification of $\mathbb{H}^n$ and $\partial\mathcal{H}_+$ and the fact that β is the transported version of the invariant Haar measure $\mu = v$ on $\mathbb{H}^n$ we can rewrite P_s as a (convolution) integral operator on $L^2(\,\mathbb{H}^n, v\,)$ as follows. We use the invariance of S under *translation* by the Heisenberg group and the bijective map

$$G : \mathbb{H}^n \to \partial\mathcal{H}_+ : h \mapsto h(\,0\,)$$

in (A.5.3). Here we identify $\mathbb{H}^n$ with its action on the boundary $\partial\mathcal{H}_+$. For any two points $z,w \in \partial\mathcal{H}_+$ we can choose $g,h \in \mathbb{H}^n$ such that $w = G(g) = g(0)$ and $z = G(h) = h(0)$. Moreover, we have:

$$g^{-1}\big(\,z+\varepsilon\cdot\mathbf{i}\,\big) = g^{-1}(\,z\,)+\varepsilon\cdot\mathbf{i}$$

and so we obtain:

$$S\big(\,z+\varepsilon\cdot\mathbf{i},\ w\,\big) = S\big(\,h(\,0\,)+\varepsilon\cdot\mathbf{i},\ g(\,0\,)\,\big) = S\big(\,g^{-1}h(\,0\,)+\varepsilon\cdot\mathbf{i},\ 0\,\big).$$

For $h \in \mathbb{H}^n$ we define $K_\varepsilon : \mathbb{H}^n \to \mathbb{C}$ by $K_\varepsilon(h) := S(h(0)+\varepsilon\cdot\mathbf{i}, 0)$. A direct computation using Proposition A.6.1 leads to

$$K_\varepsilon\big(\,[\,\zeta,t\,]\,\big) = \frac{c}{(\,t+i\|\,\zeta\,\|^2+i\varepsilon\,)^{n+1}}, \qquad \text{where} \qquad c = \big(\,2^{n-1}i^{n+1}n!\,\big)\,\pi^{-n-1}. \tag{A.6.2}$$

A.7 Cauchy-Szegö projection and exotic classes

Let us examine convolution operators on $\mathbb{H}^n$ with respect the multiplication on the Heisenberg group. For $y := [\,\zeta, t\,]$ and $x := [\,\eta, s\,] \in \mathbb{H}^n$ we define:

$$\begin{aligned}\Theta\big(\, y^{-1}, x^{-1}\,\big) = L_x\big(\, x - y\,\big) &:= y^{-1} \cdot x \\ &= \big[\, -\zeta, -t\,\big] \cdot \big[\,\eta, s\,\big] = \big[\,\eta - \zeta, s - t - 2 \cdot \mathrm{Im}(\zeta \cdot \bar{\eta})\,\big]\end{aligned} \tag{A.7.1}$$

where the mapping $L_x(z) = z + [\,0,\ 2 \cdot \mathrm{Im}(z' \cdot \bar{\eta})\,]$ with $z = [\,z', r\,]$ is linear with $\det L_x = 1$ and $x \mapsto L_x(y)$ varies smoothly. We want to show that a convolution operator T on the Heisenberg group $\mathbb{H}^n$ given by:

$$\begin{aligned}\big[\,Tf\,\big](x) &= \int_{\mathbb{H}^n} k\big(\, y^{-1} \cdot x\,\big) \cdot f(\,y\,)\, dy = \int_{\mathbb{H}^n} f\big(\, x \cdot y^{-1}\,\big) \cdot k(\,y\,)\, dy \\ &= \int_{\mathbb{H}^n} f \circ \Theta(\,x, \cdot\,)\, k\, dv =: \big(\, f * k\,\big)(\,x\,)\end{aligned} \tag{A.7.2}$$

has a realization as a pseudo-differential operator. Moreover, we can compute its symbol. In this section we use the following convention for the Fourier transform (which slightly differs from our formulas in chapter 4). For any $f \in \mathcal{S}(\,\mathbb{R}^n\,)$ and $\xi \in \mathbb{R}^n$ we write:

$$\big[\,\mathcal{F}f\,\big](\,\xi\,) = \hat{f}(\,\xi\,) := \int_{\mathbb{R}^n} f(\,x\,)\, e^{-2\pi i x \cdot \xi}\, dv(x).$$

Proposition A.7.1 *([117]) Let $k \in L^1(\,\mathbb{H}^n\,)$, then (A.7.2) defines a pseudodifferential operator with symbol $a(\,x, \xi\,) = \hat{k} \circ \tilde{L}_x(\,\xi\,)$ where $\tilde{L}_x = (\,L_x^t\,)^{-1}$.*

Proof For $f \in \mathcal{S}(\,\mathbb{H}^n\,)$ a change of variables to $\eta = L_x^t(\,\xi\,)$ and the inversion formula for the Fourier transform imply that:

$$\begin{aligned}\big(\, f * k\,\big)(\,x\,) &= \int_{\mathbb{H}^n} f(\,y\,) \cdot k \circ L_x(\,x - y\,)\, dy \\ &= \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} f(\,y\,) \cdot e^{2\pi i\, L_x(\,x-y\,)\cdot\xi} \cdot \hat{k}(\,\xi\,)\, dy\, d\xi \\ &= \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} f(\,y\,) \cdot e^{2\pi i\,(\,x-y\,)\cdot\eta} \cdot \hat{k} \circ \tilde{L}_x(\,\eta\,)\, dy\, d\eta \\ &= \int_{\mathbb{H}^n} \hat{k} \circ \tilde{L}_x(\,\eta\,) \cdot e^{2\pi i\, x\cdot\eta} \cdot \hat{f}(\,\eta\,)\, d\eta.\end{aligned}$$

Comparing this formula with (A.7.2) implies the assertion. □

For $\varepsilon > 0$ we want to apply this result to the kernels $k = K_\varepsilon$ defined in (A.6.2). Hence we have to compute the Fourier transform of K_ε. For $\varepsilon \geq 0$ and $(\,\xi', \xi_{2n+1}\,) \in \mathbb{C}^n \times \mathbb{R}$ let us consider the function

$$F_\varepsilon\big(\,\xi', \xi_{2n+1}\,\big) := \begin{cases} \exp\Big(\, -\frac{\pi}{2\,\xi_{2n+1}} \cdot \|\,\xi'\,\|^2 - 2\,\pi\,\varepsilon\,\xi_{2n+1}\Big), & \xi_{2n+1} > 0 \\ 0 & \text{else.}\end{cases}$$

Lemma A.7.1 *There is a constant $\tilde{c} \in \mathbb{C}$ such that for $\varepsilon > 0$ and with the kernel K_ε in (A.6.2) it holds $F_\varepsilon = \tilde{c} \cdot \widehat{K_\varepsilon}$.*

Proof For $\varepsilon > 0$ let us compute the inverse Fourier transform $\mathcal{F}^{-1}(F_\varepsilon)$ of F_ε. Fix an element $z = [\, x, t\,] \in \mathbb{H}^n$, then

$$\begin{aligned}
&\big[\, \mathcal{F}^{-1} F_\varepsilon \,\big](z) \\
&= \int_0^\infty \int_{\mathbb{C}^n} \exp\Big\{ \frac{-\pi \,\|\, \xi' \,\|^2}{2\, \xi_{2n+1}} - 2\pi \, \big(\, \varepsilon \, \xi_{2n+1} - i \operatorname{Re}\big(x \cdot \overline{\xi'}\big) - it \, \xi_{2n+1} \, \big) \Big\} \, d\xi' \, d\xi_{2n+1} \\
&= \int_0^\infty \exp\Big\{ - 2\pi \, \xi_{2n+1}(\, \varepsilon - it \,) \Big\} \int_{\mathbb{C}^n} \exp\Big\{ - \frac{\pi \,\|\, \xi' \,\|^2}{2\, \xi_{2n+1}} + 2\pi i \operatorname{Re}(\, x \cdot \overline{\xi'} \,) \Big\} \, d\xi' \, d\xi_{2n+1}
\end{aligned}$$

Now using the well-known formula where $a > 0$:

$$\mathcal{F}^{-1}\big(\, \exp\big\{ -a \,\| \cdot \|^2 \big\} \, \big)(\, x \,) = \frac{\pi^n}{a^n} \, \exp\Big\{ - \frac{\pi^2}{a} \,\|\, x \,\|^2 \Big\} .$$

with $a := \frac{\pi}{2\, \xi_{2n+1}}$, then it follows that:

$$\begin{aligned}
\big[\, \mathcal{F}^{-1} F_\varepsilon \,\big](z) &= 2^n \int_0^\infty \xi_{2n+1}^n \exp\Big\{ - \xi_{2n+1} 2\, \pi \, \big(\,\|\, x \,\|^2 + \varepsilon - i\, t \, \big) \Big\} \, d\xi_{2n+1} \\
&= \frac{n!}{2\pi^{n+1}} \cdot \frac{1}{(\,\|\, x \,\|^2 + \varepsilon - i\, t \,)^{n+1}} = \frac{\tilde{c}}{(\, i\|\, x \,\|^2 + i\varepsilon + t \,)^{n+1}} .
\end{aligned}$$

Comparing this with formula (A.6.2) our assertion follows. □

With our notations above and according to [134], p. 540, for $f \in L^2(\mathbb{H}^n)$ the Toeplitz projection P_s can be viewed in the sense of a $L^2(\mathbb{H}^n)$-limit as:

$$[\, P_s f \,](\, x \,) = \lim_{\varepsilon \to 0} \int_{\mathbb{H}^n} K_\varepsilon\big(\, y^{-1} \cdot x \, \big) \, f(\, y \,) \, dy$$

with K_ε as in (A.6.2). Moreover, in the case where $\varphi \in \mathcal{S}(\mathbb{H}^n)$ and $x \in \mathbb{H}^n$ we write:

$$[\, P_s \varphi \,](\, x \,) = K \Big[\, \varphi \circ \Theta\, (\, x, \cdot \,) \Big] = \lim_{\varepsilon \to 0} \int_{\mathbb{H}^n} K_\varepsilon(\, y \,) \cdot \varphi \circ \, \Theta(\, x, y \,) \, dy.$$

Hence formally and with respect to the Heisenberg product P_s is given by the convolution formula $P_s \varphi(x) = \varphi * K(x)$ where K is the distribution

$$K = \lim_{\varepsilon \to 0} K_\varepsilon. \tag{A.7.3}$$

Let $\varphi \in \mathcal{S}(\mathbb{H}^n)$, then we obtain from Lemma A.7.1 that:

$$\widehat{K}(\varphi) = K(\hat{\varphi}) = \lim_{\varepsilon \to 0} K_\varepsilon(\hat{\varphi}) = \lim_{\varepsilon \to 0} \widehat{K_\varepsilon}(\varphi) = \tilde{c} \cdot \lim_{\varepsilon \to 0} F_\varepsilon(\varphi) = \tilde{c} \cdot F_0(\varphi).$$

and so for the *Cauchy-Szegö kernel* K one obtains with a constant $\tilde{c} > 0$:

$$\widehat{K} = F_0 = \begin{cases} \tilde{c} \cdot \exp\Big\{ -\frac{\pi}{2\,\xi_{2n+1}} \cdot \|\,\xi'\,\|^2 \Big\}, & \xi_{2n+1} > 0 \\ 0 & \text{else.} \end{cases} \tag{A.7.4}$$

Note that for any $\delta > 0$ the kernel $\widehat{K}$ is homogeneous of degree 0 with respect to the dilations:

$$\delta : \mathbb{H}^n \to \mathbb{H}^n : [\,\xi, \xi_{2n+1}\,] \mapsto [\,\delta\xi, \delta^2 \xi_{2n+1}\,]. \tag{A.7.5}$$

Moreover, because F_0 is a bounded function if follows from (A.6.2) and Proposition A.7.1 for $\varphi \in \mathcal{S}(\mathbb{H}^n)$ that

$$\begin{aligned}[\,P_s\varphi\,](x) &= \lim_{\varepsilon \to 0} \varphi * K_\varepsilon(x) \\ &= \lim_{\varepsilon \to 0} \int_{\mathbb{H}^n} \widehat{K_\varepsilon} \circ \tilde{L}_x(\eta) \cdot e^{2\pi i\, x \cdot \eta}\, \hat{\varphi}(\eta)\, d\eta \\ &= \tilde{c}^{-1} \int_{\mathbb{H}^n} F_0 \circ \tilde{L}_x(\eta) \cdot e^{2\pi i\, x \cdot \eta}\, \hat{\varphi}(\eta)\, d\eta.\end{aligned} \tag{A.7.6}$$

Hence we conclude that the *Cauchy-Szegö projection* P_s is a pseudo-differential operator with symbol $a(x, \eta) = F_0 \circ \tilde{L}_x(\eta)$ where F_0 is homogeneous of degree 0. In the following section we will show that localized in x the symbol $a(x, \eta)$ belongs to the class $\mathcal{S}^0_{\frac{1}{2},\frac{1}{2}}$ (cf. section 4.1).

A.8 On the symbol class $\mathcal{S}^0_\rho$

Starting with a pseudo-distance ρ on $\mathbb{H}^n$ we describe a related class $\mathcal{S}^0_\rho$ of symbols on the space $\mathbb{H}^n_x \times \mathbb{H}^n_\xi \cong \mathbb{R}^{4n+2}$ which first was introduced by A. Nagel and E. M. Stein (cf. [117]). For an appropriate choice of ρ the inclusion

$$\mathcal{S}^0_\rho \subset \mathcal{S}^0_{\frac{1}{2},\frac{1}{2}}$$

holds (cf. Theorem A.8.1) where $\mathcal{S}^0_{\frac{1}{2},\frac{1}{2}}$ denotes the class of exotic symbols of Hörmander type defined in section 4.1. Let us give the definition of the *pseudo-distance* ρ on $\mathbb{H}^n$ which we will need for the estimates on the symbols $a \in \mathcal{S}_\rho$. We identify both spaces $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ and $\mathbb{R}^{2n+1}$ in the following manner. For $\xi = (\xi_1, \cdots, \xi_n, \xi_{2n+1}) \in \mathbb{H}^n$ we write

$$\zeta_j := \mathrm{Re}(\xi_j), \qquad \zeta_{j+n} := \mathrm{Im}(\xi_j) \quad \text{and} \quad \zeta_{2n+1} = \xi_{2n+1}, \qquad (j = 1, \cdots, n).$$

Moreover, with $\zeta := (\zeta', \zeta_{2n+1}) = (\zeta_1, \cdots, \zeta_{2n}, \zeta_{2n+1}) \in \mathbb{R}^{2n+1} \cong \mathbb{H}^n$ we define a *pseudo-distance*:

$$\rho_0 : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}^+ : \rho_0(x, \zeta) = \rho_0(\zeta) := \left\{ \| \zeta' \|^2 + \| \zeta \|^2 \right\}^{\frac{1}{4}}. \tag{A.8.1}$$

where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{H}^n$ resp in $\mathbb{C}^n$. Hence ρ_0 does only depend on the (cotangent) ζ-space. With our notations in (A.7.1) and Proposition (A.7.1) we define a x-depended distance function $\tilde{\rho}$ by writing:

$$\tilde{\rho} : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}^+ : \tilde{\rho}(x, \zeta) := \rho_0 \circ \tilde{L}_x(\zeta) \tag{A.8.2}$$

where again we use the described identification of $\mathbb{H}^n$ and $\mathbb{R}^{2n+1}$.

Definition A.8.1 ([117]) Let ρ be a *pseudo-distance* as in (A.8.2) or (A.8.1), then we denote by $\hat{\mathcal{S}}^m_\rho$ the collection of all $a(x, \zeta) \in \mathcal{C}^\infty(\mathbb{H}^n \times \mathbb{H}^n)$ with compact support in x such that for all $\lambda^{(1)}, \cdots, \lambda^{(l)} \in \mathbb{H}^n$ it holds:

$$\left| (\lambda^{(1)}, \partial\zeta) \cdots (\lambda^{(l)}, \partial\zeta)\, a(x, \zeta) \right| \leq C_l \cdot \rho(x, \zeta)^m \cdot \prod_{j=1}^{l} \theta \left\{ \frac{\rho(x, \lambda^{(j)})}{\rho(x, \zeta)} \right\} \tag{A.8.3}$$

where $\theta(t) = t + t^2$ and we denote by $(\lambda_j, \partial\zeta)$ the vector fields $\sum_{r=1}^{2n+1} \lambda_r^{(j)} \cdot \partial_{\zeta_r}$. The class $\mathcal{S}^m_\rho$ is the collection of symbols $a \in \hat{S}^m_\rho$ such that

$$\partial_{x_j} a(x, \zeta) = \sum_{k=1}^{2n+1} a_k(x, \zeta) \cdot \zeta_k + a_0(x, \zeta) \tag{A.8.4}$$

for $j = 1, \cdots, 2n+1$ where $a_k \in \hat{\mathcal{S}}^{m-1}_\rho$ and $a_0 \in \hat{\mathcal{S}}^m_\rho$. Moreover, we claim that $\partial_{x_j} a_k$ and $\partial_{x_j} a_0$ have a decomposition of the form (A.8.4) again. The same has to be true for the new symbols in this decomposition and so on ad infinitum.

The next theorem is a special case of Proposition 16 in [117]. We assume that $\rho = \tilde{\rho}$ is the pseudo-distance in (A.8.2).

Theorem A.8.1 *([117]) Let $a(x, \zeta) \in \mathcal{S}^0_\rho$, then locally in x the symbol $a(x, \zeta)$ is in $\mathcal{S}^0_{\frac{1}{2}, \frac{1}{2}}$.*

Proof By definition we have to prove that for any $\alpha, \beta \in \mathbb{N}_0^{2n}$ and a symbol $a(x, \zeta)$ in $\mathcal{S}^0_\rho$ having compact support in x it holds:

$$\left| \partial_\zeta^\alpha \partial_x^\beta a(x, \zeta) \right| \leq c_{\alpha,\beta} (1 + \| \zeta \|)^{-\frac{|\alpha|}{2} + \frac{|\beta|}{2}}$$

From Definition A.8.1 it follows that for $\beta \in \mathbb{N}_0^{2n}$ the derivatives $\partial_x^\beta a$ are of the form:

$$\partial_x^\beta a(x, \zeta) = \sum_{|\gamma| \leq |\beta|} a_\gamma(x, \zeta) \cdot \zeta^\gamma$$

where $a_\gamma \in \hat{\mathcal{S}}_\rho^{-|\gamma|}$. Our next aim is to compute the derivatives $\partial_\zeta^\alpha\{ a_\gamma(x,\zeta)\cdot\zeta^\gamma \}$. They are made up of terms like

$$\{ \partial_\zeta^{\alpha_1} a_\gamma(x,\zeta) \} \cdot \{ \partial_\zeta^{\alpha_2}\zeta^\gamma \}, \qquad \text{where} \qquad \alpha_1+\alpha_2=\alpha.$$

It is easy to verify that $\|\zeta\|^{\frac{1}{2}} \leq \rho(x,\zeta) \leq \|\zeta\|$ for large ζ. For functions p and q we write $p(x) \lesssim q(x)$ if there is a positive constant c with $p(x) \leq c\cdot q(x)$. With this notation we have:

$$\begin{aligned} \left| \partial_\zeta^{\alpha_1} a_\gamma(x,\zeta) \right| \cdot \left| \partial_\zeta^{\alpha_2}\zeta^\gamma \right| &\lesssim \rho(x,\zeta)^{-|\gamma|-|\alpha_1|}\|\zeta\|^{|\gamma|-|\alpha_2|} \\ &\lesssim \big(1+\|\zeta\|\big)^{-\frac{|\gamma|}{2}-\frac{|\alpha_1|}{2}+|\gamma|-|\alpha_2|} \lesssim \big(1+\|\zeta\|\big)^{\frac{|\beta|}{2}-\frac{|\alpha|}{2}} \end{aligned}$$

where we have used the fact that $|\gamma| \leq |\beta|$. □

We define the notion of *homogeneous distributions*. As before δ denotes the dilation on $\mathbb{H}^n$ of homogeneous dimension $2n+2$ defined in (A.7.5).

Definition A.8.2 A distribution $K \in \mathcal{S}'(\mathbb{H}^n)$ is said to be *homogeneous of degree* λ if it holds

$$\delta^{-(2n+2)} \cdot K\big(\varphi\circ\delta^{-1}\big) = \delta^\lambda \cdot K(\varphi)$$

for all test functions $\varphi \in \mathcal{S}(\mathbb{H}^n)$. Moreover, we call K a $\mathcal{C}^\infty$*-distribution* in an open set $\Omega \subset \mathbb{H}^n$ if there is a function $f \in \mathcal{C}^\infty(\Omega)$ such that $K(\varphi) = \int f\cdot\varphi$ for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$.

Remark A.8.1 By a straightforward computation it follows that the distribution K in (A.7.3) is homogeneous of degree $\lambda = -2n-2$. Hence we have:

$$K\left(\varphi\circ\delta^{-1}\right) = K(\varphi), \qquad \text{for} \qquad \varphi \in \mathcal{S}\big(\mathbb{H}^n\big).$$

Because of $\hat{\varphi}\circ\delta^{-1} = \delta^{2n+2}\cdot\widehat{\varphi\circ\delta}$ it follows that $\delta^{2n+2}\cdot\widehat{K}(\varphi\circ\delta) = \widehat{K}(\varphi)$ and so $\widehat{K}$ is a distribution of degree 0. It can be shown that K is the sum of a function of mean value zero and a multiple of the delta distribution at the origin (see [117]).

Locally we can write the Cauchy-Szegö projection as an *Fourier-integral operator*. Let K be the distribution in (A.7.3) which is $\mathcal{C}^\infty$ away from the origin. Choose a cut-off function $\psi \in \mathcal{C}_c^\infty(\mathbb{H}^n)$ with $\psi \equiv 1$ in a neighborhood of 0 and define $K_0 := K\cdot\psi$. We recall of the definition

$$\Theta\big(y^{-1},x^{-1}\big) := L_x\big(x-y\big) = y^{-1}\cdot x, \qquad (x,y\in\mathbb{H}^n)$$

in (A.7.1). With cut-off functions $a,b \in \mathcal{C}_c^\infty(\mathbb{H}^n)$ we assume that

$$a(x)\cdot\big\{ K\circ\Theta(x,y) \big\}\cdot b(y) = a(x)\cdot\big\{ K_0\circ\Theta(x,y) \big\}\cdot b(y)$$

and we define $G := \widehat{K_0} \in \mathcal{C}^\infty(\mathbb{H}^n)$ (note that K_0 has compact support). From Proposition A.7.1 and our notations there together with (A.7.6) it follows for the operator P_s:

$$\Big[\, aP_s bf \,\Big](\,x\,) = \tilde{c}^{-1} \int_{\mathbb{H}^n} a(x)\, G \circ \tilde{L}_x(\,\xi\,)\, e^{2\pi i\, x\cdot\xi}\, \widehat{f \cdot b}(\,\xi\,)\, d\xi \tag{A.8.5}$$
$$= \tilde{c}^{-1} \int_{\mathbb{H}^n} a(\,x\,)\, G(\,\xi\,)\, e^{2\pi i\,(x\cdot L_x^t \xi)}\, \widehat{f \cdot b}\big(\,L_x^t \xi\,\big)\, d\xi$$
$$= \tilde{c}^{-1} \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} a(\,x\,)\, G(\,\xi\,)\, e^{2\pi i\,(\,x-y\,)\cdot L_x^t \xi}\, f(\,y\,)\, b(\,y\,)\, dy\, d\xi$$
$$= \tilde{c}^{-1} \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} a(\,x\,)\, G(\,\xi\,)\, e^{2\pi i\, L_x(\,x-y\,)\cdot\xi}\, f(\,y\,)\, b(\,y\,)\, dy\, d\xi.$$

Hence the localized version aP_sb of the Cauchy-Szegö projection has the form of a Fourier-integral operator with symbol $\tilde{a}(\,x,\xi\,) = a(\,x\,)\,G(\,\xi\,)$. Using the fact that the function $a(x) \in \mathcal{C}_c^\infty(\mathbb{H}^n)$ has compact support the following result directly can be obtained from Theorem 18 in [117]. The distance function ρ_0 appearing in the next theorem was given in (A.8.1).

Theorem A.8.2 *([117]) If $G \in \mathcal{S}^0_{\rho_0}$, then the operator aP_sb in (A.8.5) is a pseudodifferential operator $b(\,x,D\,)$ with symbol $b \in \mathcal{S}^0_{\tilde{\rho}}$ where $\tilde{\rho}(\,x,\xi\,) := \rho_0 \circ \tilde{L}_x(\xi)$.*

Hence we have to show that the function G is in the symbol class $\mathcal{S}^0_{\rho_0}$. Because G and ρ_0 do not depend on the space variable x we only have to prove that for all $\alpha \in \mathbb{N}_0^{2n}$ and $k \in \mathbb{N}_0$ there is a constant $C_{\alpha,k} > 0$ with

$$\big|\, \partial_\zeta^\alpha\, \partial_{\xi_{2n+1}}^k\, G(\,\xi\,)\,\big| \leq C_{\alpha,k} \cdot \prod_{j=1}^{|\alpha|+k} \theta\big\{\, \rho_0(\,\xi\,)^{-1}\,\big\} \tag{A.8.6}$$

for $\xi = (\,\zeta, \xi_{2n+1}\,) \in \mathbb{R}^{2n+1} \cong \mathbb{H}^n$ and $\theta(\,t\,) = t + t^2$.

Lemma A.8.1 *The function $G := \widehat{K_0} \in \mathcal{C}^\infty(\mathbb{H}^n)$ fulfills (A.8.6) for $\alpha \in \mathbb{N}_0^{2n}$ and $k \in \mathbb{N}_0$.*

Proof To estimate G let us fix a test function $\varphi \in \mathcal{S}(\,\mathbb{H}^n\,)$. It follows from integration by parts that:

$$K_0\big(\,\varphi\,\big) = \frac{c}{n} \int_{\mathbb{R}^{2n+1}} \frac{1}{(\,\|\,\zeta\,\|^2 - it\,)^n}\, \frac{\partial}{\partial t}\big[\,\psi \cdot \varphi\,\big]\,(\,\zeta,t\,)\, d\zeta\, dt.$$
$$= \frac{c}{n} \int_{\mathbb{R}^{2n+1}} \tilde{F}(\,y\,) \cdot \varphi(\,y\,)\, dy + \frac{c}{n} \int_{\mathbb{R}^{2n+1}} \frac{\psi(\,\zeta,t\,)}{(\,\|\,\zeta\,\|^2 - it\,)^n}\, \frac{\partial}{\partial t}\varphi(\,\zeta,t\,)\, d\zeta\, dt$$

where $\tilde{F}(\,\zeta,t\,) := (\,\|\,\zeta\,\|^2 - it\,)^{-n} \cdot \frac{\partial}{\partial t}\psi(\,\zeta,t\,)$ is contained in $\mathcal{C}_c^\infty(\mathbb{H}^n)$. Because the Fourier transform of the first term is in $\mathcal{S}(\,\mathbb{H}^n\,)$ it is sufficient to prove the estimates (A.8.6) for the singular part:

$$\tilde{G}(\,\xi\,) := \frac{c}{n} \int_{\mathbb{R}^{2n+1}} \frac{\psi(\,\zeta,t\,)}{(\,\|\,\zeta\,\|^2 - it\,)^n}\, \frac{\partial}{\partial t} \exp\Big(\,2\pi i\,\big\{\,\xi' \cdot \zeta + \xi_{2n+1} \cdot t\,\big\}\,\Big)\, d\zeta\, dt$$

where $\xi = (\xi', \xi_{2n+1}) \in \mathbb{R}^{2n+1} \cong \mathbb{H}^n$. With the dilation $\delta(\xi) = (\delta\xi', \delta^2 t)$ we compute the partial derivatives

$$\left[\partial_{\xi'}^{\alpha} \partial_{\xi_{2n+1}}^{k} \tilde{G} \right] \circ \delta(\xi) \tag{A.8.7}$$

for all $\alpha \in \mathbb{N}_0^{2n}$ and $k \in \mathbb{N}_0$. Because the map $\mathbb{H}^n \ni (\zeta, t) \mapsto (\|\zeta\|^2 - it)^{-n}$ is homogeneous of degree $2n$ it is easy to verify that:

$$\begin{aligned} &\left[\partial_{\xi'}^{\alpha}\, \partial_{\xi_{2n+1}}^{k} \tilde{G} \right] \circ \delta(\xi) \\ &\quad = \tilde{c} \cdot \delta^{-|\alpha|-2k-2} \cdot \int_{\mathbb{R}^{2n+1}} \frac{\psi \circ \delta^{-1}(\zeta, t)}{(\|\zeta\|^2 - it)^n} \cdot \zeta^{\alpha}\, t^k \, \exp\Big(-2\pi i \{ \xi' \cdot \zeta + \xi_{2n+1} \cdot t \} \Big)\, d\xi'\, dt \end{aligned}$$

where $\tilde{c} > 0$ is independent of δ. From this computation it follows for the asymptotic behavior of (A.8.7)

$$\left[\partial_{\xi'}^{\alpha}\, \partial_{\xi_{2n+1}}^{k} \tilde{G} \right] \circ \delta(\xi) = \mathcal{O}\left(\delta^{-|\alpha|-2k-2} \right), \qquad \text{as} \quad (\delta \to \infty).$$

Using the fact that $\rho_0 \circ \delta(\xi)^{-1} = \mathcal{O}(\delta^{-1})$ as $\delta \to \infty$ the right hand side of (A.8.6) is of order $-|\alpha| - k$ as $\delta \to \infty$. Hence we conclude that $G \in \mathcal{S}_{\rho_0}^0$. $\square$

Therefore, it follows from Theorem A.8.1 and Theorem A.8.2:

Theorem A.8.3 *Let $a, b \in \mathcal{C}_c^{\infty}(\mathbb{H}^n)$, then the localized form $aP_s b$ of the Cauchy-Szegö projection is an pseudo-differential operator with symbol in $\mathcal{S}_{\frac{1}{2},\frac{1}{2}}^0$.*

This result can be generalized to domains in $\mathbb{C}^{n+1}$ with smooth strictly pseudo-convex boundary but the details are more technically. The proof can be found in [117], Theorem 20.

Theorem A.8.4 *The Cauchy-Szegö projection on a strictly pseudo-convex domain has a symbol of class $\mathcal{S}_{\frac{1}{2},\frac{1}{2}}^0$.*

Remark A.8.2 In our summary above we were concerned with the case of the Cauchy-Szegö projection P_s on the unit ball in $\mathbb{C}^{n+1}$ of complex dimension a least two. In case of the unit circle

$$S^1 := \Big\{ z \in \mathbb{C} : |z| = 1 \Big\}$$

it is well-known that P_s is a pseudo-differential operator of class $\mathcal{S}_{1,0}^0$ (see [103], pp.178).

A.9 Hankel operators and mean oscillation

Let $n \in \mathbb{N}$ and denote by v the usual Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Then we write μ for the normalized *Gaussian measure* on $\mathbb{C}^n$ with the density

$$d\mu := \pi^{-n} \cdot \exp\left(-\|\cdot\|^2 \right) dv.$$

The *Segal Bargmann* space $H_2 := H^2(\mathbb{C}^n, \mu)$ of all μ-square-integrable holomorphic functions on $\mathbb{C}^n$ is a closed subspace in $H_1 := L^2(\mathbb{C}^n, \mu)$. Moreover, all *point evaluations* are continuous and so H_2 is a reproducing kernel Hilbert space with reproducing kernel K given by:

$$K(x, \lambda) := \exp\big(\langle x, \lambda \rangle \big), \qquad (x, \lambda \in \mathbb{C}^n).$$

Let P be the orthogonal projection (*Toeplitz projection)* from H_1 onto H_2 and write $Q := (I - P)$. We define a *space of symbols*:

$$\mathcal{T}(\mathbb{C}^n) := \Big\{ g \in H_1 : g \circ \tau_x \in H_1, \quad \forall\, x \in \mathbb{C}^n \Big\}$$

where $\tau_x(z) := z + x$ denotes the *translation* by x. (Note that μ is not invariant under translations!). For any $f \in \mathcal{T}(\mathbb{C}^n)$ let M_f be the (in general unbounded) *multiplication* by f on H_1. Formally:

$$M_f = \begin{pmatrix} T_f & H^*_{\bar{f}} \\ H_f & Q\, M_f\, Q \end{pmatrix} : \begin{matrix} H_2 \\ \oplus \\ H_2^\perp \end{matrix} \to \begin{matrix} H_2 \\ \oplus \\ H_2^\perp \end{matrix}$$

with:

- $T_f := P\, M_f\, P : H_2 \to H_2$, the *Toeplitz operator* with symbol f.
- $H_f := Q\, M_f\, P : H_2 \to H_2^\perp$, the *Hankel operator* with symbol f.
- $H^*_{\bar{f}} := P\, M_f\, Q : H_2^\perp \to H_2$.

Remark A.9.1 The facts (a) and (b) below can be checked easily:

(a) The space $\mathcal{T}(\mathbb{C}^n)$ contains $\mathbb{P}[z, \bar{z}]$, the polynomials in the variables z and $\bar{z}$ on $\mathbb{C}^n$.

(b) For each symbol $f \in \mathcal{T}(\mathbb{C}^n)$ and with the subspace $\mathcal{D}$ of H_2 defined by:

$$\mathcal{D} := \Big\{ g \in H_2 : f \cdot g \in H_1 \Big\}$$

both operators T_f and H_f on $\mathcal{D}$ are densely defined (and in general unbounded).

Let us consider the linear and dense subspace $\mathcal{M} \subset H_2$ with:

$$\mathcal{M} := \text{span}\Big\{ K(\cdot, \lambda) : \lambda \in \mathbb{C}^n \Big\} \subset H_2.$$

Let A be a (possibly unbounded) operator on H_2 with domain of definition $\mathcal{D}(A)$ such that $\mathcal{M} \subset \mathcal{D}(A)$. Formally, we associate a *symbol* $\tilde{A}$ to A by:

$$\tilde{A} : \mathbb{C}^n \to \mathbb{C} : \tilde{A}(\lambda) := \Big\langle A k_\lambda, k_\lambda \Big\rangle_{H_2} \tag{A.9.1}$$

where for $z, \lambda \in \mathbb{C}^n$ (see [20] and [19]):

$$k_\lambda(z) := \frac{K(z, \lambda)}{\| K(\cdot, \lambda) \|_{H_2}} = \exp \big(\langle z, \lambda \rangle - 2^{-1} \| \lambda \|^2 \big)$$

is the *normalized Bergman kernel* of H_2. The complex valued function $\tilde{A}$ is called the *Berezin transform* of the operator A. For functions $f \in \mathcal{T}(\mathbb{C}^n)$ the Berezin transform of T_f is well-defined and we shortly write:

$$\tilde{f} := \widetilde{T_f} : \mathbb{C}^n \to \mathbb{C}. \tag{A.9.2}$$

By a straightforward calculation it can be shown that $\tilde{f}$ is the solution of the heat equation on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ at time $t = \frac{1}{4}$ with initial data f. Moreover, by the analysis in [22] for any operator $A \in \mathcal{L}(H_2)$ it holds in the sense of a Bochner integral:

$$T_{\tilde{A}} = \int_{\mathbb{C}^n} W_t A W_{-t} \, d\mu(t) \tag{A.9.3}$$

with the unitary Weyl operators $\{ W_t : t \in \mathbb{C}^n \}$ of weighted shifts on $\mathbb{C}^n$ in Definition 2.1.1. Hence the Toeplitz operator $T_{\tilde{A}}$ is some kind of *average* of A. The formula (A.9.1) is meaningful for all Bergman spaces and so the Berezin transform can be defined in greater generality. For some of its basic properties in the case of H_2 we refer to our remarks below Theorem 2.1.2 and [21], [22].

Next we introduce the notion of *oscillation* and *mean oscillation*. Consider the spaces:

$$\begin{aligned} \mathcal{BC}\,(\mathbb{C}^n) &:= \text{ bounded continuous functions on } \mathbb{C}^n, \\ \mathcal{C}_0\,(\mathbb{C}^n) &:= \text{ continuous functions on } \mathbb{C}^n \text{ vanishing at } \infty. \end{aligned}$$

For $f \in \mathcal{BC}(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$ the *oscillation* is defined to be:

$$\mathrm{Osc}_z(f) := \sup \Big\{ \, | f(z) - f(w) | : \| z - w \| < 1 \Big\}.$$

We call a function f of *bounded oscillation* and we write $f \in \mathcal{BO}(\mathbb{C}^n)$ if and only if:

$$\big[z \mapsto \mathrm{Osc}_z(f) \big] \in \mathcal{BC}\,(\mathbb{C}^n)\,.$$

Similar we call f of *vanishing oscillation* and we write $f \in \mathcal{VO}(\mathbb{C}^n)$ if and only if:

$$\big[z \mapsto \mathrm{Osc}_z(f) \big] \in \mathcal{C}_0\,(\mathbb{C}^n)\,.$$

Furthermore, for symbols f such that $| f |^2 \in \mathcal{T}(\mathbb{C}^n)$ and all $z \in \mathbb{C}^n$ we define:

$$\mathrm{MO}\big(f, z \big) := \widetilde{| f |^2}(z) - | \tilde{f}(z) |^2 \quad \text{ and } \quad \| f \|_{\mathrm{BMO}} := \sup \Big\{ \mathrm{MO}\big(f, z \big)^{\frac{1}{2}} : z \in \mathbb{C}^n \Big\}.$$

We call f a function of *bounded mean oscillation* and we write $f \in \mathcal{BMO}(\mathbb{C}^n)$ if and only if $\| f \|_{\mathrm{BMO}} < \infty$. Similar, f is said to be of *vanishing mean oscillation* and in the following we write $f \in \mathcal{VMO}(\mathbb{C}^n)$ if and only if

$$\mathrm{MO}(f, \cdot) \in \mathcal{C}_0(\mathbb{C}^n).$$

All spaces defined above are linear. The functions in $\mathcal{BMO}(\mathbb{C}^n)$ and $\mathcal{VMO}(\mathbb{C}^n)$ can be unbounded.

Example A.9.1 ([12]) Let $p \in \mathbb{P}[z, \bar{z}] \subset \mathcal{T}(\mathbb{C}^n)$ be a non-constant polynomial in z and $\bar{z}$ of degree ρ, then $\mathrm{MO}(p, \cdot)$ is a polynomial in $\mathbb{P}[z, \bar{z}]$ of maximal degree $2\rho - 2$. In particular, the *mean oscillation* of any linear polynomial is a constant function. Hence:

$$\Big\{ f \in \mathbb{P}[z, \bar{z}] : f \text{ linear} \Big\} \subset \mathcal{BMO}(\mathbb{C}^n).$$

We want to give sufficient and necessary conditions on symbols $f \in \mathcal{T}(\mathbb{C}^n)$ for H_f and $H_{\bar{f}}$ to be bounded (resp. compact).

Lemma A.9.1 *Let $g \in \mathcal{T}(\mathbb{C}^n)$ and $\lambda \in \mathbb{C}^n$, then it follows that:*

$$MO\big(g, \lambda\big)^{\frac{1}{2}} \leq \sqrt{2} \cdot max\{ \| H_g \|, \| H_{\bar{g}} \| \}.$$

Hence if the Hankel operators H_g and $H_{\bar{g}}$ are simultaneously bounded on H_2, then it holds that $g \in \mathcal{BMO}(\mathbb{C}^n)$.

The question arises if $g \in \mathcal{BMO}(\mathbb{C}^n)$ is sufficient for the boundedness of H_g and $H_{\bar{g}}$. One step toward an answer of this problem is given by the following result:

Proposition A.9.1 *Let $g \in \mathcal{BMO}(\mathbb{C}^n)$ and $a, b \in \mathbb{C}^n$, then it follows that:*

$$\big| \tilde{g}(a) - \tilde{g}(b) \big| \leq 2 \cdot \| g \|_{BMO} \| a - b \|.$$

In particular, we have $\tilde{g} \in \mathcal{BO}(\mathbb{C}^n)$ and $\| Osc_z(\tilde{g}) \|_\infty \leq 2 \cdot \| g \|_{BMO}$.

Using the observation above we can give a quite useful decomposition of both the spaces $\mathcal{BMO}(\mathbb{C}^n)$ and $\mathcal{VMO}(\mathbb{C}^n)$. Let us define:

$$\mathcal{F} := \Big\{ f \in \mathcal{T}(\mathbb{C}^n) : \widetilde{|f|^2} \in \mathcal{BC}(\mathbb{C}^n) \Big\}$$
$$\mathcal{I} := \Big\{ f \in \mathcal{T}(\mathbb{C}^n) : \widetilde{|f|^2} \in \mathcal{C}_0(\mathbb{C}^n) \Big\}.$$

Then for $f \in \mathcal{BMO}(\mathbb{C}^n)$, the trivial identity $f = \tilde{f} + (f - \tilde{f})$ leads to:

$$\mathcal{BMO}(\mathbb{C}^n) = \mathcal{BO}(\mathbb{C}^n) + \mathcal{F} \qquad \text{and} \qquad \mathcal{VMO}(\mathbb{C}^n) = \mathcal{VO}(\mathbb{C}^n) + \mathcal{I}.$$

It can be shown (cf. [12]) that for $g \in \mathcal{BO}(\mathbb{C}^n) \cup \mathcal{F}$ the *Hankel operator* H_g is bounded. Similar for $h \in \mathcal{VO}(\mathbb{C}^n) \cup \mathcal{I}$ the operator H_h is compact. Hence from the decomposition above we conclude that:

Theorem A.9.1 *([12]) For $f \in \mathcal{T}(\mathbb{C}^n)$ the following are equivalent:*

(a) H_f *and* $H_{\bar{f}}$ *are bounded operators,*

(b) The commutator $[\, M_f, P\,] = M_f P - P M_f$ *is bounded on* $L^2(\,\mathbb{C}^n, \mu\,)$,

(c) $f \in \mathcal{BMO}(\,\mathbb{C}^n\,) = \mathcal{BO}(\,\mathbb{C}^n\,) + \mathcal{F}$.

All the quantities $\|\,[\,M_f, P\,]\,\|$, *max*$\{\, \|\,H_f\,\|, \|\,H_{\bar{f}}\,\|\,\}$ *and* $\|\,f\,\|_{BMO}$ *are equivalent. The corresponding equivalence of* (a), (b) *and* (c) *is true if we replace "bounded" by "compact" and the space* $\mathcal{BMO}(\,\mathbb{C}^n\,)$ *by* $\mathcal{VMO}(\,\mathbb{C}^n\,)$.

What are the bounded (resp. compact) Hankel operators with *anti-holomorphic* resp. *polynomial* symbols?

Corollary A.9.1 *([12]) For* $p \in \mathbb{P}[\,z, \bar{z}\,]$ *and* $f \in \mathcal{T}(\,\mathbb{C}^n\,)$ *holomorphic the assertions* (a) *and* (b) *and the assertions* (c) *and* (d) *are equivalent:*

(a) f *is a linear polynomial in* z.

(b) The Hankel operator $H_{\bar{f}}$ *is bounded.*

and

(c) p *is a linear polynomial in* z *and* $\bar{z}$.

(d) The Hankel operators H_p *and* $H_{\bar{p}}$ *are bounded.*

Compact Hankel operator H_f with *polynomial* or *anti-holomorphic* symbols f are only possible in the trivial case $H_f = 0$.

There are similar results on Hankel operators in the case of *bounded symmetric domains* in [16]. In general the *oscillation* has to be measured with respect to the *Bergman metric*. Note the *Segal-Bargmann space* is *flat* in the sense that

Bergman metric = Euclidean metric.

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain (a bounded subset of $\mathbb{C}^n$ which contains 0 and is invariant under

$$z \mapsto \lambda z, \qquad \lambda \in \mathbb{C}, \qquad |\,\lambda\,| = 1.)$$

Then the class of anti-holomorphic functions leading to bounded resp. compact Hankel operators may be richer. A result analog to Corollary A.9.1 can be found in [16]. With a holomorphic symbol f the Hankel operator $H_{\bar{f}}$ is bounded if an only if f is in the *Bloch space* $B(\,\Omega\,)$ and $H_{\bar{f}}$ is compact if and only if f is in the *little Bloch space* $B_0(\,\Omega\,)$. We recall the definition of $B(\,\Omega\,)$ and $B_0(\,\Omega\,)$. With the Bergman kernel K corresponding to Ω let us write:

$$G_z := \frac{1}{2} \left(\frac{\partial^2}{\partial z_j \partial \bar{z}_j} \log K(\,z, z\,) \right)_{i,j}$$

for the Bergman metric. We define the expression:n metric. We define the expression:

$$Q_f(\, z\,) := \sup \Big\{ \,|\, \langle\, \nabla_z f, \bar{x} \,\rangle \,|\, \cdot \langle\, G_z x, x \,\rangle^{-\frac{1}{2}} : 0 \neq x \,\Big\}.$$

where $\Delta_z := (\, \partial_{z_1}, \cdots, \partial_{z_n} \,)$. Then the following characterizations hold:

$$f \in B(\,\Omega\,) \quad \Longleftrightarrow \quad \|\, Q_f \,\|_{\sup} < \infty.$$

$$f \in B_0(\,\Omega\,) \quad \Longleftrightarrow \lim_{z \to \partial\Omega} Q_f(\, z\,) = 0$$

There are some interesting results on Hankel operators and *Schatten-p-classes* which we give next:

For and $1 \leq p < \infty$ the *Schatten-p-class* $\mathcal{S}_p$ consists of operators T with $\|\, T\,\|_p < \infty$, where

$$\|\, T\,\|_p := \Big[\, \mathrm{tr} \Big\{ \, (T^*T)^{\frac{p}{2}} \, \Big\} \Big]^{\frac{1}{p}}.$$

Closely related to the results before it holds in case of the *Segal-Bargmann space*:

Theorem A.9.2 *(J.Xia, D. Zheng, (2001), [145]). Let* $1 \leq p < \infty$ *and* $f \in \mathcal{T}(\, \mathbb{C}^n\,)$, *then* (a) *and* (b) *below are equivalent:*

(a) H_f *and* $H_{\bar{f}}$ *are in* $\mathcal{S}_p$ *simultaneously.*

(b) The following integral is finite:

$$\int_{\mathbb{C}^n} MO(\, f, \cdot)^{\frac{p}{2}} \, d\mu < \infty.$$

Following [145] we are comparing Theorem A.9.2 to the case of *Bergman spaces over the unit ball*:

$$B_n := \Big\{ \, z \in \mathbb{C}^n : \|\, z \,\| < 1 \, \Big\} \subset \mathbb{C}^n.$$

Let $f \in L^2(\, B_n, v\,)$, then in the case of B_n the *mean oscillation* $\mathrm{MO}(\, f, \cdot\,)$ is given by the formula:

$$MO\big(\, f, z\,\big) = \widetilde{|\, f \,|^2}(z) - |\, \tilde{f}(z)\,|^2$$

$$= \int_{B_n} \big|\, f - \tilde{f}(z)\,\big|^2 \, \frac{(\, 1 - \|\, z\,\|^2\,)^{n+1}}{|\, 1 - \langle\, z, \cdot \rangle\,|^{2n+2}} \, dv.$$

There is a result due to K. Zhu which is analog to Theorem A.9.2 above and actually was proved earlier.

Theorem A.9.3 *(K. Zhu, [148]) For* $2 \leq p < \infty$ *and* $f \in L^2(\, B_n, v\,)$ *the statements* (a) *and* (b) *below are equivalent:*

(a) H_f and $H_{\bar{f}}$ are in $\mathcal{S}_p$ simultaneously.

(b) The following integral is finite:

$$\int_{B_n} MO\big(f, z \big)^{\frac{p}{2}} \cdot \frac{1}{\big(1 - \| z \|^2 \big)^{n+1}} \, dv(z) < \infty.$$

Remark A.9.2 By comparing both Theorems A.9.2 and A.9.3 we find that from the point of operator theory the Bergman spaces over the ball and the Segal Bargmann space differ in some points:

- Theorem A.9.2 involves the *Möbius invariant* measure $d\gamma = (1 - \| z \|^2)^{-(n+1)} \, dv$. The *Möbius group* corresponds to the group of *translations* $\{ \tau_x : x \in \mathbb{C}^n \}$ on $\mathbb{C}^n$.
- Theorem A.9.2 only holds for $2 \leq p < \infty$. It was extended by *J. Xia* to the case:

  $$c_n := \frac{2n}{n+1} < p < 2.$$

  It is easy to see that for $1 \leq p \leq c_n$ condition (b) of Theorem A.9.3 is sufficient but not necessary for (a) to be true. Because of

  $$\lim_{n \to \infty} \frac{2n}{n+1} = 2$$

  Theorem A.9.3 is the best one can say in the case of general dimensions $n \in \mathbb{N}$.

If we restrict ourselves to the case of bounded symbols

$$L^\infty(\mathbb{C}^n) \subset \mathcal{T}(\mathbb{C}^n),$$

then for Hankel operators on the *Segal-Bargmann space* a specific effect arises, which might be seen as an operator theoretical analogy to *Liouville's Theorem.* Namely, that there are no non-constant bounded entire functions.

Theorem A.9.4 *(A. Berger, L.A. Coburn, K. Stroethoff, [22], [135]) For any bounded symbol $f \in L^\infty(\mathbb{C}^n)$ the Hankel operator H_f is compact if and only if $H_{\bar{f}}$ is compact.*

There are many other ideals above the compact operators such as the *Schatten-p-classes.* The following questions arise:

(a) Does for bounded symbols f and $1 \leq p < \infty$ it hold that:

$$H_f \in \mathcal{S}_p \quad \Longleftrightarrow \quad H_{\bar{f}} \in \mathcal{S}_p\,?$$

(b) If (a) holds true, is there any connection between the norms $\| H_f \|_p$ and $\| H_{\bar{f}} \|_p$? (maybe " = " ?)

Problem (a) was proved by J. Xia and D. Zheng for the complex plane $n = 1$ and $p = 2$ in [145]. A generalization to all dimensions $n \in \mathbb{N}$ and an answer to (b) for the case $p = 2$ is given by the author:

Theorem A.9.5 *([11]) The assertion (a) holds for $\mathcal{S}_2$ (the Hilbert-Schmidt operators) and all dimensions $n \in \mathbb{N}$. Moreover, for a Hankel operator $H_f \in \mathcal{S}_2$ with $f \in L^\infty(\mathbb{C}^n)$:*

$$\| H_{\bar f} \|_2 \leq 2 \cdot \| H_f \|_2. \tag{A.9.4}$$

Theorem A.9.5 is easy to prove in the case of symbols $f \in L^2(\mathbb{C}^n, v)$ because then the operators H_f, T_f and $T_{\bar f}$ are in $\mathcal{S}_2$ and $T_{|f|^2}$ is in $\mathcal{S}_1$. From

$$H_f^* H_f = T_{|f|^2} - T_{\bar f} T_f$$

it follows that:

$$\| H_f \|_2 = \mathrm{tr}\big(H_f^* H_f \big) = \mathrm{tr}\left(T_{|f|^2} \right) - \mathrm{tr}\left(T_{\bar f} T_f \right) = \| H_{\bar f} \|_2.$$

It is easy to see that Theorem A.9.5 fails in the case of unbounded symbols $f \in \mathcal{T}(\mathbb{C}^n)$. We do not know if the constant "2" in the estimate $(A.9.4)$ is sharp. By our knowledge the case $p \neq 2$ in (a) above still is an open problem, even for the complex plane.

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