Pseudodifferential Operators on Hilbert Space Riggings with Associated Ψ^*-Algebras and Generalized Hörmander Classes

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Summary

The present thesis is concerned with certain aspects of differential and pseudodifferential operators on infinite dimensional spaces. We aim to generalize classical operator theoretical concepts of pseudodifferential operators on finite dimensional spaces to the infinite dimensional case.

At first we summarize some facts about the canonical Gaussian measures on infinite dimensional Hilbert space riggings. Considering the naturally unitary group actions in $L^2(H_-, \gamma)$ given by weighted shifts and multiplication with $e^{i\langle t,\cdot\rangle_0}$ we obtain an unitary equivalence $\mathcal{F}$ between them. In this sense $\mathcal{F}$ can be considered as abstract Fourier transform. We show that $\mathcal{F}$ coincides with the Fourier-Wiener transform. Using the Fourier-Wiener Transform we define pseudodifferential operators in Weyl and Kohn-Nirenberg form on our Hilbert space rigging.

In the case of this Gaussian measure γ we discuss several possible Laplacians at first the Ornstein-Uhlenbeck operator and then pseudodifferential operators with negative definite symbol. In the second case, these operators are generators of L^2_γ-sub Markovian semi groups and L^2_γ-Dirichlet forms.

In **[67]** Gramsch, Ueberberg and Wagner described the construction of generalized Hörmander classes by commutator methods. Following this concept and the classical finite dimensional description of $\Psi^0_{\varrho,\delta}$ $(0 \leq \delta \leq \varrho \leq 1)$ in the C^*-algebra $\mathscr{L}(L^2)$ by Beals and Cordes we construct in both cases generalized Hörmander classes, which are Ψ^*-algebras. These classes act on a scale of Sobolev spaces, generated by our Laplacians.

In the case of the Ornstein-Uhlenbeck operator, we prove that a large class of continuous pseudodifferential operators considered by Albeverio and Dalecky **[2]** is contained in our generalized Hörmander class. Furthermore, in the case of a Laplacian with negative definite symbol, we develop a symbolic calculus for our operators. We show some Fredholm criteria for them and prove that these Fredholm operators are hypoelliptic. Moreover, in the finite dimensional case, using the Gaussian measure instead of the Lebesgue measure the index of these Fredholm operators is still given by Fedosov's formula.

Considering an infinite dimensional Heisenberg group rigging we discuss the connection of some representations of the Heisenberg group to pseudodifferential operators on infinite dimensional spaces. We use this connections to calculate the spectrum of pseudodifferential operators and to construct generalized Hörmander classes given by smooth elements which a spectrally invariant in $L^2(H_-, \gamma)$.

Finally, given a topological space X with Borel measure μ, a locally compact group G and a representation B of G in the group of all homeomorphisms of X, we construct a Borel measure μ_s on X which is invariant under $B(G)$.

Contents

Introduction

In this thesis we discuss certain aspects of differential and pseudodifferential operators on infinite dimensional Hilbert space riggings. We generalize operator-theoretical concepts of pseudodifferential operators on finite dimensional spaces to the infinite dimensional case. Infinite dimensional operators naturally arise in mathematical physics and in the theory of stochastic processes. For example infinite dimensional differential operators are used to describe the flow of energy in systems with infinitely many degrees of freedom and in stochastic calculus, they are used to construct diffusion operators (see [**21**], [**104**], [**105**] and [**111**]). However, there is also a strong mathematical interest in studying infinite dimensional spaces and analysis on them. They appear as spaces of functions, distributions and sequences.

In the classical finite dimensional theory pseudodifferential operators on $\mathbb{R}^n$ are defined by oscillatory integrals starting from symbols on $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$. These symbols $a(x,\xi)$ are $\mathscr{C}^\infty$-functions, which fulfill certain estimates. The class of operators attached to a certain class of symbols $S^m_{\varrho,\delta}$ $(0 \leq \delta \leq \varrho,\ \delta < 1)$ is the so called Hörmander-class $\Psi^m_{\varrho,\delta}$. In [**13**] Beals shows that one can describe the classes $\Psi^0_{\varrho,\delta}$ without using symbols, only by using commutators. More precisely, he shows that

$$\Psi^0_{\varrho,\delta} := \{a \in \mathscr{L}(H_0)\,|\,\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(\partial)a \in \bigcap_{s\in\mathbb{R}} \mathscr{L}(H^s, H^{s+\varrho|\alpha|-\delta|\beta|})\ \forall\,\alpha,\,\beta \in \mathbb{N}_0^n\},$$

where H^s are the Sobolev spaces.

Spectral invariance. Dealing with pseudodifferential operators in perturbation theory Gramsch introduced Ψ_0- and Ψ^*-algebras (see [**56**]). A Fréchet algebra $\mathcal{A}$, which is continuously embedded in a Banach algebra $\mathcal{B}$, is called Ψ_0-algebra, if $\mathcal{A}$ is locally spectrally invariant, i.e. if there exists an $\varepsilon > 0$ with

$$\{a \in \mathcal{A}\mid\ \|e-a\|_{\mathcal{B}} < \varepsilon\} \subseteq \mathcal{A}^{-1},$$

where $\mathcal{A}^{-1}$ denotes the group of invertible elements in Ψ. In addition we call $\mathcal{A}$ a Ψ-Algebra if $\mathcal{A}$ is spectrally invariant, i.e

$$\mathcal{A} \cap \mathcal{B}^{-1} = \mathcal{A}^{-1}.$$

Moreover, if $\mathcal{A}$ is a symmetric Ψ_0-sub algebra of a C^*-algebra B, we call $\mathcal{A}$ a Ψ^*-algebra. In this case $\mathcal{A}$ is spectrally invariant.

Once established a first and immediate consequence of the spectral invariance is the fact that a Ψ-Algebra $\mathcal{A}$ has an open group $\mathcal{A}^{-1}$, which is not true for general Fréchet algebras. In addition the inversion in $\mathcal{A}$ is continuous and Ψ- resp. Ψ^*-Algebras are stable under countable intersection. But the Ψ-property of an algebra $\mathcal{A}$ has many more consequences. For example Ψ-Algebras are stable with respect to the holomorphic functional calculus of Waelbroeck ([**131**]). In addition Gramsch showed that the Ψ-property is important for Oka's principle and in the perturbation and homotopy theory of Fredholm functions. Concerning the importance of these algebras in operator theory and the relevance of spectral invariance we refer also to [**25**], [**37**] [**58**], [**64**], [**96**], [**99**], [**106**], [**116**, chapter 4 and chapter 5] and [**123**]. Ψ_0- and Ψ^*-algebras and their applications have been considered in many publications during a long period of time. We will give a short overview over some of these topics at the beginning of chapter 3.

Until now the spectral invariance and the Ψ-property have been proved for many algebras cf. e.g [**5**], [**11**], [**13**], [**29**], [**30**], [**44**] [**56**], [**58**], [**96**], [**98**], [**100**], [**125**], [**123**], [**122**] and [**130**]. Moreover, spectral invariance plays an essential role in recent developments in infinite dimensional analysis, stochastic analysis and time-frequency analysis (cf. [**69**], [**70**], [**68**, §13, §14].

In [**67**] Gramsch, Ueberberg and Wagner described a construction of Ψ_0- resp. Ψ^*-algebras starting from closed derivations or closed operators. In addition, they developed a method to construct generalized Hörmander classes $\widetilde{\Psi}^0_{\varrho,\delta}$, which are sub multiplicative Ψ^*-algebras. We will describe these concepts in Chapter 3 more detailed .

Using Beal's description of $\Psi^0_{\varrho,\delta}$ by commutators Beals [**13**] and finally, Ueberberg [**130**] and Schrohe [**123**] showed that for $0 \leq \delta \leq \varrho \leq 1,\ \delta < 1$ the classes $\Psi^0_{\varrho,\delta}$ are sub multiplicative Ψ^*-algebras in $\mathscr{L}(L^2(\mathbb{R}^n,\lambda))$. Here $\mathscr{L}(L^2(\mathbb{R}^n,\lambda))$ stands for the space of all bounded linear operators on the L^2 space on $\mathbb{R}^n$ with Lebesgue measure.

Sub-multiplicativity. We call a Fréchet-algebra $\mathcal{A}$ sub multiplicative if there exists a system of semi-norms $\{\|\cdot\|_k\}$ on $\mathcal{A}$ which defines the topology of $\mathcal{A}$ such that

$$\|ab\|_k \leq \|a\|_k \|b\|_k \ \forall\, a,\, b \in \mathcal{A}.$$

Until now it is an open question whether every Ψ^*-Algebra is sub multiplicative. Zelasko showed in [**134**, Theorem 3] that there exist non commutative Fréchet algebras with open group which are not sub multiplicative. But for many operator algebras sub multiplicativity has been proved, for example Gramsch and Schrohe proved sub multiplicativity for Boutet de Movele's algebra (cf [**66**]) and Baldus showed in [**4**] sub multiplicativity of $\Psi(1, g)$ for all Hörmander metrices g.

Moreover, Gramsch [**59**] and Gramsch and Kaballo [**64**] used sub multiplicativity in connection with non abelian complex analytic cohomology and Oka's principle, Phillips [**113**] and Cuntz [**33**] used sub multiplicativity in connection with K- and KK-theory. Considering the case of a commutative Fréchet-algebra

$\mathcal{A}$ Mitiagin, Rolewicz and Zelazko [**108**] showed that sub multiplicativity is equivalent to the property that for every entire function $\varphi(z) = \sum_{n=1}^{\infty} a_n z^n$ and every $x \in \mathcal{A}$ the series $\sum_{n=1}^{\infty} a_n x^n$ is convergent.

Ornstein-Uhlenbeck operator. We aim to generalize Beals' description of $\Psi^m_{\varrho,\delta}$ to the infinite dimensional case. Looking at this characterization, infinite dimensional measure theory and analysis the following two questions immediately arise

- Which measure should we choose on an infinite dimensional Hilbert space?
- Having a measure, can we find a good candidate for a Laplace operator in these spaces?

Let us consider the first question. As a well known fact there is no Lebesgue measure on an infinite dimensional Hilbert space. Even worse, there exists no measure on an infinite dimensional Hilbert space for which all shifts are admissible, i.e. there always exists a shift such that the shifted measure is not absolutely continuous with regard to the original one. Furthermore, we do not find a measure in the infinite dimensional case, which can be called canonical. To deal with this first problem we consider quasi-nuclear Hilbert space riggings instead of single Hilbert spaces.

Definition 0.0.1. We call $H_+ \subseteq H_0 \subseteq H_-$ a quasi-nuclear Hilbert space rigging, if

(i) $H_+ \subseteq H_0 \subseteq H_-$ are dense real Hilbert spaces,
(ii) the embeddings $H_+ \hookrightarrow H_0$ and $H_0 \hookrightarrow H_-$ are quasi-nuclear,
(iii) H_+ is the dual space of H_- with regard to the inner product in H_0,
(iv) H_+ is separable, in particular H_0 and H_- are separable.

Considering only Gaussian measures we are able to find a measure which we can call canonical with respect to this rigging.

Answering the second question is even more complicated. Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis in an infinite dimensional Hilbert space. Then $\sum_{k=1}^{\infty} \frac{\partial^2}{\partial x^2} f$ does not necessarily converge, even if f is bounded, twice continuous differentiable and (e_k) is an orthonormal basis in H_-. Thus we have to find a Laplace operator on infinite dimensional Hilbert spaces to construct the Sobolev spaces.

In this thesis we discuss two possible ways of defining a good Laplace operator on infinite dimensional spaces. The first Laplace operator is considered in stochastic analysis for example by Berezanskii [**17**] and Malliavin [**104**]. We can define this Laplace operator by

$$\mathrm{L}_\gamma f(x) = -\frac{1}{2}(\mathrm{tr}_0 d^2 f(x) - 2\langle \nabla f(x), x\rangle_0) \quad \forall f \in \mathscr{C}_b^2(H_-),$$

(cf. [**2**],[**3**] and [**18**]). We show that this operator coincides with the well known Ornstein-Uhlenbeck operator, considered by Malliavin (cf. [**21**], [**104**], [**105**] and

[**111**]). Moreover, all real powers of the Laplacian are essential selfadjoint on the space $\mathscr{C}^{\infty}_{pol,cly}(H_-)$, the space of all cylindrical $\mathscr{C}^{\infty}$-functions such that all derivatives are bounded by polynomials. Starting with this Laplacian we define a scale of Sobolev spaces H^s.

Negative definite functions. A second possibility of constructing generalized Laplace operators is adapting the concept of negative definite functions to infinite dimensional Hilbert spaces. A function $\psi:\ \mathbb{R}^n \longrightarrow \mathbb{C}$ is called negative definite if $\psi(0) \geq 0$ and $e^{-t\psi}$ is a positive definite function for all $t > 0$. Let λ be the Lebesgue measure in $\mathbb{R}$. Then it is well known (cf.[**80**]) that we can consider every negative definite function as symbol of a pseudodifferential operator

$$\psi(D)u := \tilde{\mathcal{F}}^{-1}\psi(\xi)\tilde{\mathcal{F}}u,$$

for $u \in S(\mathbb{R}^n)$, where $\tilde{\mathcal{F}}$ denotes the Fourier-transform. Let A be the closure of this operator. Then $-A$ is a Dirichlet operator and generates a strongly continuous contraction sub Markovian semi group. Furthermore, if ψ is real-valued a symmetric Dirichlet form is defined by the closure of $\langle Au\,,\,u\rangle$ for $u \in D(A)$. More about the relevance of Dirichlet-Forms can be found in [**21**], [**45**] and [**103**].

Conversely, pseudodifferential operators with negative definite functions as symbols arise naturally as generators of Feller Groups and Dirichlet-forms. In both case these operators are also generators of a stochastic process. More precisely, every Levi process possesses as characteristic function a negative definite function and vice versa every negative definite function is a characteristic function of a Levi process. In addition, if μ_t is a convolution semi group then there exists a negative definite function ψ such that $\chi_{\mu_t} = e^{t\psi}$, where χ_{μ_t} denotes the characteristic function of μ_t (cf. [**6**] [**78**] [**80**], [**81**] and [**82**]).

At first we prove that some well know facts about negative definite functions remain valid if we replace $\mathbb{R}^n$ by a general Hilbert space H_-. We show that as in the finite dimensional case we still have a Petree's inequality for negative definite functions on H_-. Moreover, we are able to show that the inequality

$$|\psi(\xi)| \leq c_\psi(1 + \psi(\xi)^2)$$

remains valid, even in the infinite dimensional case, where the unit ball is not compact which is needed in the well known finite dimensional proof (cf. [**80**, 3.6.22]). Having this result we are able to define a pseudodifferential operator attached to a negative definite symbol ψ as in $\mathbb{R}^n$ with Lebesgue measure, but now using the Fourier-Wiener-transform $\mathcal{F}$ instead of the Fourier-transform. This Fourier-Wiener-transform is an unitary equivalence between the natural group action on $L^2(H_-,\gamma)$ by weighted unitary shifts and the multiplication with $e^{i\langle t\,,\,\rangle_0}$. Furthermore, if ψ has a Levi-Khinchin-representation with respect to our Hilbert space rigging we determine this pseudodifferential operator exactly on a subspace of all $\mathscr{C}^{\infty}$-functions on H_-. It turns out that the closure of the operator $-\psi(D)$

generates a semi group $(T_t)_{t>0}$. Here T_t is given by

$$T_t u = \mathcal{F}^{-1}\psi(\mathcal{F}u) \text{ for all } u \in L^2(H_-, \gamma).$$

Since we have to consider a Gaussian measure instead of the Lebesgue measure and the Fourier-Wiener instead of the Fourier-Transform it seems, in view of the connection between both, in the finite dimensional case (cf. Proposition 1.4.10) quite natural to adapt the concept of sub Markovian semi groups and Dirichlet-forms in the following way: We call a semi group $(S_t)_{t\in\mathbb{R}}$ an L^2_γ sub Markovian semi group if we have

$$0 \leq u \leq e^{\frac{\|\cdot\|^2}{2}} \text{ a.e. implies } 0 \leq S_t u \leq e^{\frac{\|\cdot\|^2}{2}} \text{ a.e.}$$

Using this notation we show that for a cylindrical function ψ T_t is an L^2_γ sub Markovian semi group (cf. 2.3.24). Furthermore $-\psi(D)$ extends to a L^2_γ-Dirichlet operator A. Concerning these adapted concept of Dirichlet operators we show, that the most important propositions remain valid in case of the Gaussian measure (see 2.3.15). Defining for $s > 0$ the Sobolev-space $H^s_\psi(H_-)$ as the space of all $u \in L^2(H_-, \gamma)$ such that

$$\|u\|_{\psi,s} := \left\|(1 + |\psi|)^{s/2}\mathcal{F}u\right\|_{L^2(H_-,\gamma)} < \infty$$

we are able to show that the domain of definition of the generator of T_t is $H^2_\psi(H_-)$. In addition this generator is our L^2_γ-Dirichlet operator A. If ψ is real-valued we associate a symmetric L^2_γ-Dirichlet-form to the L^2_γ-Dirichlet operator A. The domain of definition of this Dirichlet-form is given by $H^1_\psi(H_-)$.

The Weyl-correspondence. Having these Laplace operators and thus a scale of Sobolev spaces enables to us discuss pseudodifferential operators acting in this scale. Let us consider the case of the Ornstein-Uhlenbeck operator as Laplace operator first. Starting with symbols (functions) $a(x,p)$ on H^2_- and an the Fourier-Wiener-transform $\mathcal{F}$ Albeverio and Dalecky defined in [**2**] pseudodifferential operators $a(X,D)$ in Weyl form on infinite dimensional Hilbert space riggings $H_+ \subseteq H_0 \subseteq H_-$ by

$$a(X,D)u(x) := \mathcal{F}^{-1}_{p\to x}\mathcal{F}^{-1}_{y\to p}\left[a\left(\frac{x+y}{2}, p\right)u(y)\right].$$

In chapter 3 of this thesis we define generalized Hörmander classes $\widetilde{\Psi}^0_{\varrho,\delta}$ similarly to the characterization given by Beals, which contain the elements of a specific class of continuous pseudodifferential operators defined in [**2**]. These generalized Hörmander classes are sub multiplicative Ψ^*-Algebras.

Let $H_+ = H_0 = H_- = \mathbb{R}^n$. Consider the canonical Gaussian measure in $\mathbb{R}^n$ and let a be a symbol in $S^0_{0,0}$. Then the corresponding pseudodifferential operator defined in [**2**] is in our generalized Hörmander class $\widetilde{\Psi}^0_{0,0}$. Furthermore, in the case of the canonical Gaussian measure on $\mathbb{R}^n$, for any $\hat{a} \in \Psi^0 \subseteq \widetilde{\Psi}^0_{0,0}$ there exists an

$a \in S^0_{0,0}$ such that a is the associated symbol to $\hat{a}$. Here Ψ^0 is a sub multiplicative Ψ^*-algebra.

The Kohn-Nirenberg-correspondence. Now let us consider the case of a negative definite function as symbol for the Laplace operator. As in the finite dimensional theory we define classes of symbols by

$$S^{m,\psi}_{\varrho_k}(H_-) := \{q \in \mathbb{C}^\infty(H_- \times H_-) \mid \left|\partial^\alpha_\xi \partial^\beta_x q(x,\xi)\right| \leq c_{|\alpha|,|\beta|}(1+\psi(\xi))^{\frac{m-\varrho_k(|\alpha|)}{2}}\}$$

and

$$S^{m,\psi}_{\varrho,\delta}(H_-) := \{q \in \mathbb{C}^\infty(H_- \times H_-) \mid \left|\partial^\alpha_\xi \partial^\beta_x q(x,\xi)\right| \leq c'_{|\alpha|,|\beta|}(1+\psi(\xi))^{\frac{m-\varrho|\alpha|+\delta|\beta|}{2}}\},$$

where ψ is a negative definite real-valued function. For a function q in these classes we define the corresponding pseudodifferential operator $q(x,D)$ in Kohn-Nirenberg form by

$$q(x,D) := \mathcal{F}^{-1}_{\xi\to x}[q(x,\xi)(\mathcal{F}u)(\xi)],$$

where $\mathcal{F}$ denotes the Fourier Wiener-Transform. We write $\Psi^{m,\psi}_{\varrho_k}(H_-)$ resp. $\Psi^{m,\psi}_{\varrho,\delta}(H_-)$ for the corresponding classes of pseudodifferential operators.

For $H_+ = H_0 = H_- = \mathbb{R}^n$ and using the Lebesgue measure and the Fourier-transform instead of the Gaussian measure and the Fourier-Wiener transform Jacob showed in [**81**] that the operators defined by symbols in $S^{m,\psi}_{\varrho_k}(\mathbb{R}^n)$ are still continuous operators in a scale of Sobolev-Spaces. Furthermore, for this operators there still exists a symbolic calculus and a Gårding inequality.

We will show that this fact still holds in the case of the canonical Gaussian measure on $\mathbb{R}^n$. In addition we prove, that the description of the Hörmander classes by commutators is still true, if we replace the Lebesgue measure by the canonical Gaussian measure and the Fourier transform by the Fourier-Wiener transform. Thus we obtain that for $m = 0$ these generalized Hörmander classes are sub multiplicative Ψ^*-algebras. Even in the more general case of a Hörmander-metric, considered for example by Feffermann and Beals [**14**], [**15**] or Baldus [**6**] they use the Lebesgue measure and the Fourier-Transform.

Some of the facts mentioned above remain valid in the case of an infinite dimensional Hilbert space rigging. More precisely, we prove that in the case of cylindrical symbols or symbols depending only on ξ for the corresponding pseudo-differential operators there still exists some kind of symbolic calculus. Moreover, all these operators map $H^{s+m}_\psi(H_-)$ continuously to $H^s_\psi(H_-)$, where $H^s_\psi(H_-)$ is the scale of Sobolev-spaces mentioned above. In addition, for $q \in S^{m,\psi}_{\varrho_k,cyl}(H_-)$ the Gårding inequality remains valid.

Concerning some special negative-definite functions we show that each operator $q(x,D) \in \Psi^{m,\psi}_{\varrho,\delta}(H_-)$ being cylindrical or depending only on ξ fulfills that

$$ad^\alpha(M)\mathrm{ad}^\beta(D)(A) \in \bigcap_{s\in\mathbb{R}} \mathscr{L}(H^s_\psi(H_-), H^{s-m+\varrho|\alpha|-\delta|\beta|}_\psi(H_-)).$$

Thus these operators are contained in a generalized Hörmander class, constructed in [**67**].

In the finite dimensional case with Lebesgue measure every uniformly elliptic symbol leads to a Fredholm operator in $L^2(\mathbb{R}^n, \lambda)$. Schrohe showed in [**122**] that in this case the index of the Fredholm operator $q(x, D)$ is given by Fedosov's formula. Assuming some minimal growth condition on our negative definite function we prove the same result in the case of the Gaussian measure on $\mathbb{R}^n$. In addition we show in the finite and the infinite dimensional case that every Fredholm operator is hypoelliptic.

The Heisenberg Group. Some representations of the finite dimensional Heisenberg Group are used by Taylor [**129**] and Folland [**43**] to study pseudodifferential operators in Weyl-form. The connection between these representations $\pi_{\pm\lambda}$ and the pseudodifferential operators are given by

$$\pi_{\pm\lambda}(k) = \tilde{k}(\pm\lambda, \pm\sqrt{\lambda}X, \sqrt{\lambda}\tilde{D}),$$

where

$$\tilde{k}(\tau, y, \eta) = (2\pi)^{-\frac{2n+1}{2}} \int k(r, s, t) e^{i(t\tau + \langle s\,, y\rangle) + \langle r\,, \eta\rangle} \lambda(dt)\lambda^n(ds)\lambda^n(dr).$$

Here $\tilde{k}(\pm\lambda, \pm\sqrt{\lambda}X, \sqrt{\lambda}\tilde{D})$ denotes the pseudodifferential operator in Weyl form (cf. Definition 3.2.2) and $k \in L^1(\mathcal{H}_n, \lambda^{2n+1})$. In the finite dimensional case it is well known that λ^{2n+1} is the Haar measure on the Heisenberg group. Taylor [**129**] and Folland [**43**] use this connection to determine the spectrum of certain pseudodifferential operators. Furthermore, in 1979 Cordes [**29**] used a representation similary to $\pi_{\pm\lambda}$ of the finite dimensional Heisenberg Group in $L^2(\mathbb{R}^n, \lambda)$ to describe the Hörmander class $\Psi^0_{0,0}$ by smooth elements with respect to the mapping $(r, s) \longmapsto \pi(r, s, 0)A\pi(r, s, 0)^{-1}$ $(A \in \mathscr{L}(L^2(\mathbb{R}^n, \lambda)))$.

We aim to prove a similar connection between the Heisenberg group and pseudodifferential operators in the case of a Gaussian measure on an Hilbert space rigging. Let H be a Hilbert Space with inner product $\langle \cdot\,, \cdot\rangle$. Then as in the finite dimensional case the Heisenberg group $\mathcal{H}$ is defined by $\mathcal{H} := H \times H \times \mathbb{R}$ with group law $\odot$ given by

$$(r, s, t) \odot (r', s', t') = (r + r', s + s', t + t' + \frac{1}{2}\langle r\,, s'\rangle - \frac{1}{2}\langle r'\,, s\rangle).$$

We denote by $\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_-$ the corresponding rigging of Heisenberg groups to a rigging of Hilbert-spaces. In this case we can extend the group law to a continuous map $\mathcal{H}_+ \times \mathcal{H}_- \longrightarrow \mathcal{H}_-$.

Let us define a strongly continuous unitary representation of H_+ in $L^2(H_-, \gamma)$ by

$$\pi(r, s, t)f(x) := \sqrt{\varrho_r(x)} e^{i(t + \langle s\,, x\rangle_0 + \frac{1}{2}\langle r\,, s\rangle_0)} f(x + r),\ (r, s, t) \in \mathcal{H}_+.$$

Then we show, that these representation is irreducible. Set $\pi_{\pm\lambda}(r,s,t) := \pi(\sqrt{\lambda}r, \pm\sqrt{\lambda}s, \pm\lambda t)$. Then $\S\pi_{\pm\lambda}$ is a strongly continuous unitary irreducible representation again and no two different representations $\pi_{\pm\lambda}$ are unitary equivalent.

Once having established these representation we can prove in the finite dimensional case the same formula for the connection between pseudodifferential operators and the Heisenberg group as mentioned above. Having the equations above we are able to define $\pi_{\pm\lambda}(P)$ for some functions P even in the infinite dimensional case. Considering the well known Ornstein-Uhlenbeck operator we find that in the finite dimensional case the symbol of this operator is given by $\sigma(x,\xi) = \sum_{j=1}^{n} \frac{x_j+\xi_j^2-1}{2}$. In addition, we use the representation π to calculate the spectrum of some pseudodifferential operators in the infinite dimensional case.

Finally, we will construct generalized Hörmander classes given by smooth elements with respect to the the mapping $(r,s) \longmapsto \pi(r,s,0)A\pi(r,s,0)^{-1}$ ($A \in \mathscr{L}(L^2(H_-,\gamma))$), where r,s are elements of the infinite dimensional Heisenberg group $\mathcal{H}_+$ and show that these Hörmander classes a spectrally invariant in $L^2(H_-,\gamma)$ in the case of operators of order 0.

Organization on the text.

Chapter1. After giving a short introduction in the theory of cylindrical measures on quasi-nuclear Hilbert-Space riggings, we consider two important kinds of unbounded operators: the multiplication operators in coordinate directions and the operators of partial differentiation $\frac{\partial}{\partial_t}$. We determine the infinitesimal generator of a strongly continuous unitary translation group U_t and show that the family U_t ($t \in H_+$) is unitary equivalent to a family of multiplication operators $V_t = e^{i\langle t,\cdot\rangle_0}$ in the space $L^2(H_{-2},\gamma)$. Hence there exists an operator $\mathcal{F}$ such that $\mathcal{F}U_t = V_t\mathcal{F}$. Thus we can consider $\mathcal{F}$ as an abstract Fourier-transform. Finally, we prove that in the case of the canonical Gaussian measure $\mathcal{F}$ coincides with the Fourier-Wiener-Transform.

Chapter2. In this chapter we consider two possible ways of defining a Laplace operator on an quasi-nuclear Hilbert space rigging. In the first part we define an infinite dimensional Laplacian L_γ by

$$\mathrm{L}_\gamma f(x) = -\frac{1}{2}(\mathrm{tr}_0 d^2 f(x) - 2\langle \nabla f(x), x\rangle_0).$$

Then L_γ is positive, symmetric and densely defined. Moreover, we show that L_γ is essentially selfadjoint on $\mathscr{C}_b^2(H_-)$ and $\mathscr{C}_{b,cyl}^\infty(H_-)$ and coincides with the well known Ornstein-Uhlenbeck operator, considered by Malliavin. For all $s \in \mathbb{R}$ the space $\mathscr{C}_{pol,cyl}^\infty(H_-)$ is a space of essential selfadjointness of $(\mathrm{L}_\gamma+\mathrm{id})^s$. Furthermore, L_γ leaves the space $\mathscr{C}_{pol}^\infty(H_-)$ invariant.

In the second part of this chapter given a negative definite function $\psi : H_- \longrightarrow \mathbb{C}$ we examine the pseudodifferential operator $\psi(D)$ with symbol ψ. Then this pseudodifferential operator is defined by $\psi(D)u := \mathcal{F}^{-1}\psi(\cdot)\mathcal{F}u$. We show that this operator is closable and that the domain of definition of the closure A is the Sobolev-Space $H^2_\psi(H_-)$ attached to the negative definite function ψ. Moreover, after adapting the concept of Dirichlet operators and Dirichlet-forms to the case of Gaussian measures we obtain that $(-A)$ is a L^2_γ-Dirichlet operator in the case of a cylindrical function ψ. In addition, $-A$ generates a strongly continuous contraction L^2_γ-sub Markovian semi group $(T_t)_{t\geq 0}$ on $L^2(H_-,\gamma)$, where T_t is given by $T_t u := \mathcal{F}^{-1}e^{-t\psi}\mathcal{F}u$. Finally, we show that if ψ is real-valued there exists a symmetric L^2_γ-Dirichlet-form $(\mathcal{E}, D(\mathcal{E}))$, such that $D(\mathcal{E}) = H^1_\psi(H_-)$ and for $u \in D(A), v \in D(\mathcal{E})$ we have $\mathcal{E}(u,v) = \langle Au\,,\, v\rangle_{L^2(H_-,\gamma)}$.

Chapter3. In [**67**] Gramsch, Ueberberg and Wagner describe a general theory to construct Ψ_0- resp. Ψ^*-algebras. Starting from closed resp. symmetric operators they use iterated commutators. At first we summarize this theory and then we compute some commutators needed later on. Let $(e_j)_{j=1}^\infty \subset H_+$ be an orthonormal basis in H_0. Using the operators M_{e_j} and D_{e_j}, we define sub multiplicative Ψ^*-algebras $\Psi_n^{MD} \subseteq \mathscr{L}(L^2(H_-,\,\gamma))$ for all $n \in \mathbb{N}\cup\{\infty\}$, as in [**67**] and [**96**, chapter 2]. Let $\mathcal{H}^n_{MD}$ be the n-th Sobolev space attached to these operators. Then

$$\Psi_n^{MD} \times \mathcal{H}^n_{MD} \longrightarrow \mathcal{H}^n_{MD} \;:\; (a,\varphi) \longmapsto a(\varphi)$$

is continuous and bilinear. Furthermore,we define pseudodifferential operators in Weyl-form and show the in case of a Gaussian measure some of these operators are elements of Ψ_n^{MD}.

After that we consider the Ornstein-Uhlenbeck as Laplace operator and define the corresponding scale of Sobolev spaces H^s. Using some kind of the Malliavin calculus we obtain in the case of a Gaussian measure that the in H^0 closed annihilation and creation operators are continuous mappings from H^s to H^{s-1}. Moreover, we apply commutator methods to define generalized Hörmander classes $\widetilde{\Psi}^0_{\varrho,\delta}$. We show that this Hörmander classes are sub multiplicative Ψ^*-algebras. Finally, we reach a sub multiplicative Ψ^*-sub algebra of the Hörmander-class $\widetilde{\Psi}^0_{0,0}$ which contains certain multiplication operators, operators of the form $\mathcal{F}^{-1}M_g\mathcal{F}$, where $\mathcal{F}$ is the Fourier-Wiener-transform and M_g a certain multiplication operator. Moreover, this class contains a class of continuous pseudodifferential operators defined by Albeverio and Dalecky. In addition, in the finite dimensional case we completely characterize this Ψ^*-sub algebra of $\widetilde{\Psi}^0_{0,0}$ by symbols of operators from $\Psi^0_{0,0}$.

Chapter4. Let $\psi : H_- \longrightarrow \mathbb{R}$ be a negative definite function on a quasi-nuclear Hilbert-Space-Rigging $H_+ \subset H_0 \subset H_-$. We define classes of symbols as functions $q(x,\xi)$ on $H_-\times H_-$ which satisfy certain estimates with respect to the given negative definite real-valued function. For such a symbol, we define the corresponding pseudodifferential operator by $q(x,D) := \mathcal{F}^{-1}_{\xi\to x}[q(x,\xi)(\mathcal{F}u)(\xi)]$, where $\mathcal{F}$ denotes the Fourier Wiener-Transform. For these classes of pseudodifferential operators we show a symbolic calculus. Furthermore, we find that some sub classes of these operators extend to continuous operators in a scale of Sobolev-Spaces. Finally, we show that for cylindrical symbols and symbol depending only on ξ, $q(x,D)$ is contained in some generalized Hörmander-classe, which in case of operators of order 0 is a Ψ^*-Algebra. In the finite dimensional case under some additional assumptions $-q(x,D) - \lambda\mathrm{id}$ extends to a generator of a L^2_γ-sub Markovian-semi group. For $\psi = \|\xi\|^2$ we give a complete description of our classes of pseudodifferential operators by commutator estimates. Finally, we obtain on $\mathbb{R}^n$ sufficient criteria on the symbol of our pseudodifferential operator to be compact or a Fredholm operator.

Chapter5. Let γ denote the canonical Gaussian measure on H_- with respect to the given rigging and let $\mu := \gamma \otimes \gamma \otimes \lambda$. Then μ is a measure on $\mathcal{H}_- := \mathcal{H}_- \times H_- \times \mathbb{R}$. Using this two measures we define strongly unitary representations π of $\mathcal{H}_+$ in $L^2(H_-,\gamma)$ and κ of $\mathcal{H}_+$ in $L^2(\mathcal{H}_-,\mu)$. Moreover, we show that π is irreducible. We calculate the generators of the corresponding semi groups in coordinate directions and show that this generators fulfill the classical commutation relations for the Heisenberg Group. Using this representation π we examine pseudodifferential operators in Weyl-form on $\mathcal{H}_-$. In addition, we calculate the spectrum of some of pseudodifferential operators. Considering the classical Heisenberg-Laplacian in the finite dimensional case we can easily calculate the symbol and the spectrum of the Ornstein-Uhlenbeck operator. Furthermore, using results of Caps **[25]** we discuss the question for which symbols the pseudodifferential operator $q(X,D)$ is essential selfadjoint and for which perturbations $q(X,D)$ the operator $\mathrm{L}_\gamma + q(x,D)$ is essential selfadjoint on $S_\gamma(\mathbb{R}^n)$. Caps proved his results in the case of $\mathbb{R}^n$ with the Lebesgue measure using the Feffermann-Phong inequality. Finally, we construct generalized Hörmander classes and Ψ^*-algebras given by smooth elements with respect to the mapping $(r,s) \longmapsto \pi(r,s,0)A\pi(r,s,0)^{-1}$ $(A \in \mathscr{L}(L^2(H_-,\gamma)))$.

Chapter6. Given a topological space X with σ-finite Borel measure μ, a locally compact group G and a representation B of G in the group of all homeomorphisms of X, we examine how to construct a Borel measure μ_s on X which is invariant under $B(G)$ (Lemma 6.1.9). In many cases this construction leads to a non-trivial representation of G on $L^p(X,\mu_s)$. We define the notion of a $\mathcal{NF}_p$ measure. Under some additional conditions on G, X and the representation B we show that in the case where μ has the $\mathcal{NF}_p$-property, the symmetrized measure μ_s is a $\mathcal{NF}_p$ measure. Finally we give some examples and an application of our work leads to the construction of spectrally invariant algebras (Ψ^*- or Ψ_0-algebras, cf. **[56]**, **[65]**) of $\mathcal{C}^\infty$-elements in operator-algebras on L^p and L^2-spaces.

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CHAPTER 1

Unitary translation groups and an abstract Fourier-transform on infinite dimensional Hilbert space riggings

In this chapter we give an introduction to the theory of infinite dimensional cylindrical (quasi)measure and discuss some basic properties of these measures. In particular, we are interested in Gaussian measures in quasi-nuclear Hilbert space riggings $H_+ \subseteq H_0 \subseteq H_-$. In this case $L^2(H_-,\ \mu)$ possesses an orthonormal basis consisting of generalized Hermite-polynomials. Moreover, we consider two important kinds of unbounded operators - the multiplication operators in coordinate directions and the operators of partial differentiation. In addition, we define a commuting strongly continuous unitary translation group U_t. We show that the family U_t $(t \in H_+)$ is unitary equivalent to a family of multiplication operators $V_t = e^{i\langle t,\cdot\rangle_0}$ in the space $L^2(H_-,\ \gamma)$. Hence there exists an operator $\mathcal{F}$ such that $\mathcal{F}U_t = V_t\mathcal{F}$. Thus one can consider $\mathcal{F}$ as an abstract Fourier-transform.

1.1. Cylindrical measures in infinite dimensional spaces

At first we describe some basic facts about σ-algebras and (quasi)measures in infinite dimensional spaces. Moreover, we consider the Fourier-transform of these quasi measures and present some basic properties of the Fourier-transform. Let us start with a result, which is true for all σ-finite measures on a measure space $(\Omega,\ F)$.

LEMMA 1.1.1. *Let μ, ν, ϱ be measures on $(\Omega,\ F)$ such that the Radon-Nikodym derivatives $\frac{d\varrho}{d\mu}$, $\frac{d\nu}{d\mu}$ and $\frac{d\nu}{d\varrho}$ exist. Furthermore, let μ and ϱ be σ-finite. Then the following equality holds.*

$$\frac{d\varrho}{d\mu}\,\frac{d\nu}{d\varrho} = \frac{d\nu}{d\mu}.$$

PROOF. The Radon-Nikodym theorem (cf. [**8**, p. 116-118]) implies that there exist $f,\ g$ and h with $f,\ g,\ h \geq 0$, such that $\nu = f\mu,\ \nu = g\varrho$ *and* $\varrho = h\mu$. Let $A \in F$. Then we have

$$\int\limits_A f\ d\mu = \nu(A) = \int\limits_A g d\varrho = \int\limits_A g\ d(h\mu) = \int\limits_A gh\ d\mu.$$

Since $A \in F$ was arbitrary, we obtain

$$\frac{\partial \nu}{\partial \mu} = f = gh = \frac{d\nu}{d\varrho}\,\frac{d\varrho}{d\mu} \quad \mu - a.e. \qquad \square$$

NOTATIONS 1.1.2. Let X be a topological space. Then we write $\mathscr{B}(X)$ for the σ-algebra of all Borel sets, i.e. $\mathscr{B}(X)$ is the σ-algebra, which contains all open sets.

In the following we describe special σ-algebras and measures in infinite dimensional quasi-nuclear Hilbert space riggings, called cylindrical. Therefore we follow closely [**17**, chapter 2 section 1.4 and 1.9.]. Let $H_+ \subset H_0 \subset H_-$ be a quasi-nuclear Hilbert space rigging and $K \subset H_+$ finite dimensional. Moreover, let $\delta \in \mathscr{B}(K)$ be a Borel set. Then we define

$$\mathfrak{C}(K;\delta) := \{x \in H \mid P_K x \in \delta\},$$

where P_K is the orthogonal projection onto K in H_0. The set $\mathfrak{C}(K;\delta)$ is called cylindrical, K its coordinate and δ its base. Let $\mathcal{K}$ be a set of finite dimensional subspaces of H_+. Denote

$$\mathfrak{C}(\mathcal{K}, H_-) := \{\mathfrak{C}(K,\delta) \mid \delta \in \mathscr{B}(K),\ K \in \mathcal{K}\}. \tag{1}$$

LEMMA 1.1.3.

(i) *$\mathfrak{C}(\mathcal{K}, H)$ is an algebra of sets.*

(ii) *We have $\mathfrak{C}_\sigma(\mathcal{K}, H_-) = \mathscr{B}(H_-)$, where $\mathfrak{C}_\sigma(\mathcal{K}, H_-)$ is the σ-span of $\mathfrak{C}(\mathcal{K}, H_-)$.*

PROOF. See [**17**, page 97]. $\square$

Let $K \in \mathcal{K}$ and $\delta \in \mathscr{B}(H_-)$ fixed. Choosing an orthonormal basis $(e_k)_{k=1}^n$ in K, we can rewrite (1) by

$$\mathfrak{C}(K;\ \delta) = \{x \in H_- \mid (\langle x\,,\, e_1\rangle_{H_0}, \ldots, \langle x\,,\, e_n\rangle_{H_0}) \in \delta\}.$$

Moreover, if we choose arbitrary vectors h_k in K, the set

$$\{x \in H_- \mid (\langle x\,,\, e_1\rangle_{H_0}, \ldots, \langle x\,,\, e_n\rangle_{H_0}) \in \delta\}$$

is cylindrical, too (cf. [**17**, page 98 Remark1]). We aim to construct cylindrical measures on $\mathscr{B}(H_-)$. Therefore we fix $\mathcal{K}$. The function of sets

$$\mathfrak{C}(\mathcal{K}, H_-) \ni \mathfrak{C} \longrightarrow \mu(\mathfrak{C}) \in [0,1]$$

is called a cylindrical quasi measure, if $\mu(H) = 1$ and μ possesses the property of σ-additivity on the sets with fixed coordinate, i.e.

$$\mu\left(\bigcup_{j=1}^{\infty} \mathfrak{C}(K;\ \delta_j)\right) = \sum_{j=1}^{\infty} \mu(\mathfrak{C}(K;\ \delta_j)) \quad \delta_j \in \mathscr{B}(K)\ \forall j \in \mathbb{N}$$

for any $K \in \mathcal{K}$ and mutually disjoint sets $\mathfrak{C}(K;\ \delta_j)$. We call μ a cylindrical measure, if μ is $\sigma-$additive on $\mathfrak{C}(\mathcal{K}, H_-)$ and thus can be extended to a measure on $\mathscr{B}(H_-)$.

REMARK 1.1.4. Let μ be a cylindrical quasi measure on H_-.

(i) Then μ is always additive.
(ii) For $K \in \mathcal{K}$ (K finite dimensional) fixed the function of sets

$$\mathscr{B}(K) \ni \delta \longrightarrow \mu(\mathfrak{C}(K,\ \delta))$$

is a σ-additive measure. Thus the function of sets

$$\{\mathfrak{C}(K,\ \delta) \mid \delta \in \mathscr{B}(K)\} \ni \mathfrak{C} \longrightarrow \mu(\mathfrak{C}) \tag{2}$$

is a $\sigma-$ additive measure.

DEFINITION 1.1.5. Let μ be a cylindrical quasi measure on H_- and let $y \in \bigcup_{K \in \mathcal{K}} K$. Then we define the Fourier-transform or the characteristic function of the cylindrical quasi measure μ by

$$\chi_\mu(y) := \int e^{i\langle x,y\rangle}\ d\mu(x) := \int e^{i\langle x,y\rangle}\ d\mu^{(y)}(x),$$

where $\mu^{(y)}$ is the measure from (2) with $K = \mathit{span}\{y\}$. The last integral is well defined, since $\mu^{(y)}$ is $\sigma-$additive.

DEFINITION 1.1.6. A functional L on a topological vector space Φ is called positive semi-definite, if the following inequality holds for all $m \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ and $\varphi_1, \ldots \varphi_m, \in \Phi$.

$$\sum_{j,k=1}^{m} L(\varphi_j - \varphi_k)\alpha_j\overline{\alpha_k} \geq 0.$$

THEOREM 1.1.7. *Let $H_+ \subset H_0 \subset H_-$ be a quasi-nuclear Hilbert space rigging and let L be a functional on H_+. In order to be the Fourier-transform of a cylindrical quasi measure on H_-, it is necessary and sufficient, that L is positive semi-definite, continuous in H_+ and that we have $L(0) = 1$.*

PROOF. See [**46**, pp. 318-322] . □

We know many results about measures and measurable functions in finite dimensional spaces. Thus it is sometimes convenient to approximate measurable functions in infinite dimensional spaces by measurable functions in finite dimensional spaces. Therefore we now describe the concept of cylindrical functions.

Henceforth let $H_+ \subset H_0 \subset H_-$ be a quasi nuclear Hilbert space rigging. Let μ be a measure on H_-.

DEFINITION 1.1.8. A function $\mathcal{H}_- \ni \xi \longrightarrow f(\xi) \in \mathbb{C}$ measurable with regard to the $\sigma-$algebra $\mathscr{B}(H_-)$ is called cylindrical, if and only if, there exists a finite dimensional subspace $K \subset H_+$ such that f is measurable with regard to the σ-algebra $\mathfrak{C}(K,\ H_-)$, which is the $\sigma-$subalgebra of $\mathscr{B}(H_-)$ consisting of all sets with fixed coordinate K.

LEMMA 1.1.9. *Each cylindrical function f admits a representation $f(\xi) = F(\langle \xi, e_1\rangle_0, \ldots, \langle \xi, e_n\rangle_0)$, where $e_j \in H_+$ and F is a Borel function on $\mathbb{R}^n$. Furthermore, the e_j can be chosen orthonormal with regard to the inner product in H_0.*

PROOF. See [**17**, p. 126 Lemma 1.4]. □

LEMMA 1.1.10. *(c.f. Berezansky, Kondratiev, [**17**]) Let μ be a cylindrical measure on $\mathscr{B}(H_-)$. Then the set of bounded cylindrical functions is dense in each space $L^p := L^p(H_-, \mu)$.*

PROOF. We only have to show that we can approximate the characteristic function χ_A of an arbitrary set $A \in \mathscr{B}(H_-)$ by a bounded cylindrical function in L^p. Let $\varepsilon > 0$. Since $\mathscr{B}(H_-))$ is generated by the algebra $\mathfrak{C}(H_-)$, we can find a cylindrical set $A_\varepsilon \in \mathfrak{C}(H_-)$ such that $\mu(A\Delta A_\varepsilon) \leq \varepsilon$, where $A\Delta B := (A \setminus B) \cup (B \setminus A)$. Since χ_{A_ε} is a cylindrical function, we obtain

$$\int_{H_-} |\chi_A - \chi_{A_\varepsilon}|^p \, d\mu \leq \mu(A\Delta A_\varepsilon) \leq \varepsilon. \qquad \square$$

Now we will give an application of the theory of cylindrical functions.

PROPOSITION 1.1.11. *Let μ be a probability measure in $\mathbb{R}^n$. Then the set $\left\{e^{i\langle\cdot,t\rangle} \,|\, t \in \mathbb{R}^n\right\}$ is total in $L^2(\mathbb{R}^n, \mu)$.*

PROOF. See [**77**, p. 212/213, Lemma 3.14]. □

PROPOSITION 1.1.12. *Let $H_+ \subseteq H_0 \subseteq H_-$ be a quasi-nuclear Hilbert space rigging. Then the set $\left\{e^{i\langle\cdot,t\rangle_0} \,|\, \in H_+\right\}$ is total in $L^2(H_-, \mu)$.*

PROOF. Let $f \in L^2(H_-, \mu)$. Applying Lemma 1.1.10, for $\varepsilon > 0$ arbitrary, there exists a cylindrical function $g(x) := G(\langle x, \varphi_1\rangle_0, \cdots \langle x, \varphi_n\rangle_0)$, $\varphi_j \in H_+$, $(j = 1 \ldots n)$ orthogonal with respect to $\langle\cdot,\cdot\rangle_0$ with $\|f - g\|_{L^2(H_-,\,\mu)} \leq \frac{\varepsilon}{2}$, where $G(t) \in L^2(\mathbb{R}^n, \ \mu^{(y_1,\,\cdots,\,y_n)})$ and $\mu^{y_1,\,\cdots,\,y_n}$ is the measure in $\mathbb{R}^n$, obtained from the map $x \longrightarrow (y_1, \cdots y_n)$ with $y_k = \langle x, \varphi_k\rangle_0$. According to 1.1.11 there is a $P \in \operatorname{span}\{e^{i\langle\cdot,t\rangle_0}\}$ with $\|G - P\|_{L^2(\mathbb{R}^n,\,\mu^{(y_1,\cdots y_n)})} \leq \frac{\varepsilon}{2}$. Define $p(x) := P(\langle x, \varphi_1\rangle_0, \cdots \langle x, \varphi_n\rangle_0)$ $\forall x \in H_-$. Then we have

$$\|f - p\|_{L^2(H_-,\,\mu)} \leq \|f - g\|_{L^2(H_-,\,\mu)} + \|G - P\|_{L^2(\mathbb{R}^n,\,\mu^{(y_1,\cdots y_n)})} \leq \varepsilon. \qquad \square$$

Now our aim is to construct Gaussian measures in infinite dimensional space. These measures are extensions of cylindrical measures given by Gaussian measures in finite dimensional spaces. We present some basic facts about Gaussian measures and compute some integrals. To do this we follow closely [**17**, chapter 1 section 1.6, 1.7, 1.9] and use the notations introduced above.

We start by constructing Gaussian measures in quasi-nuclear Hilbert spaces-riggings. Therefore let S be a positive operator in $\mathscr{L}(H_0)$. Let $a \in H_0$ be fixed and S_K the restriction of $P_K S$ to K for $K \in \mathcal{K}$. We define a cylindrical measure γ on $\mathfrak{C}(\mathcal{K}, H)$ by setting

(3)

$$\gamma((K;\ \delta)) = \pi^{-\frac{1}{2}\dim\ K}(\det\ S_K)^{-\frac{1}{2}} \int\limits_{\delta} \exp\left(-\langle S_K^{-1}(x - P_K a), x - P_K a\rangle_0\right) d\lambda_K(x),$$

where λ_K is the Lebesgue measure in K induced by the metric of H_0. Then $\gamma(\mathfrak{C})$ is well defined for $\mathfrak{C} \in \mathfrak{C}(\mathcal{K}, H)$, i.e. the integral is independent of the choice of K and δ as long as $\mathfrak{C} = \mathfrak{C}(K, \delta)$ (cf. [**17**, page 106]).

THEOREM 1.1.13. *Formula (3) defines a cylindrical measure in H_-, which can be extended to a measure on $\mathscr{B}(H_-)$. The measure $\gamma_{S,a}$ obtained as result is called Gaussian measure with correlation operator S and mean value a. Moreover, the measure $\gamma_{S,a}$ is completely determined by the space H_0, the positive operator S and the mean value a. For $\varphi \in H_+$ the Fourier transform*

$$\chi_{\gamma_{S,0}}(\varphi) = \int\limits_{H_-} e^{i\langle x,\varphi\rangle_0} d\gamma_{S,0}(x)$$

is continuous and we have

$$\chi_{\gamma_{S,0}}(\varphi) = e^{\frac{1}{4}\langle S\varphi,\varphi\rangle_0}.$$

PROOF. See [**17**, page 111-113]. □

The Gaussian measure $\gamma := \gamma_1 := \gamma_{\mathrm{id},0}$ is called canonical Gaussian measure. We always write $\gamma_S := \gamma_{S,0}$.

THEOREM 1.1.14. *Let γ_S be a Gaussian measure in the Hilbert space H_- with positive nuclear operator S and mean value 0. Then γ_S can be represented as canonical Gaussian measure by a properly chosen quasi-nuclear Hilbert space rigging. Conversely every canonical Gaussian measure coincides with a Gaussian measure γ_S in H_-, where S is a positive nuclear operator.*

PROOF. See [**17**, p. 114 Theorem 1.9]. □

Once having this theorem we restrict ourself to the case of the canonical Gaussian measure. Throughout the rest of this thesis let γ denote the canonical Gaussian measure with respect to this rigging.

In the case of Gaussian measures in a finite dimensional space it is well known that the polynomials are dense in L^2. We show the same result in the case of Gaussian measures in infinite dimensional spaces. Throughout this section we follow closely [**17**, Chapter 2 Section 2.1]. We consider the quasi-nuclear Hilbert space riggings $H_+ \subset H_0 \subset H_-$.

DEFINITION 1.1.15. A measurable function on H_- is called measurable linear functional, if it is the limit of a γ-almost everywhere convergent sequence of continuous linear functionals

$$f(x) = \lim_{n\to\infty} \langle x, \varphi_n \rangle_0, \quad \gamma\text{-a.e.} \quad (\varphi_n \in H_+).$$

PROPOSITION 1.1.16. *Let $h \in H_0$ and let $(\varphi_j)_{j=1}^{\infty} \subset H_+$ be a sequence with $\varphi_j \xrightarrow[j\longrightarrow\infty]{H_0} h$. Then the measurable linear functional*

$$l_h(x) = \langle h, x \rangle_0 = \lim_{j\to\infty} \langle x, \varphi_j \rangle_0 \quad (x \in H_-)$$

is well defined. Moreover, we have $l_h \in \mathscr{L}^p(H_-, \gamma)$ for all $p \geq 1$.

PROOF. See [**35**]. □

DEFINITION 1.1.17. Let $\mathcal{P}_{cyl}(H_-)$ be the space of all continuous polynomials, which are cylindrical functions. These polynomials are called cylindrical polynomials.

PROPOSITION 1.1.18. *For all $p \geq 1$, the set of cylindrical polynomials $\mathcal{P}_{cyl}(H_-)$ is dense in $L^p(H_-, \gamma_S)$.*

PROOF. See [**17**, p.133]. □

A shift of a measure is called admissible, if the shifted measure is absolutely continuous with regard to the original one. In infinite dimensional spaces the following problem occurs: There are no measures for which all shifts are admissible. In the following section we describe the set of admissible shifts in the case of a Gaussian measure. Throughout this section we follow closely [**17**].

We define the shifted measure on $\mathfrak{C}_\sigma(H_-)$ for an arbitrary cylindrical measure μ. Therefore we introduce for $y \in H_-$ the mapping $T_y : H_- \longrightarrow H_-$ by $T_y x := x + y$. Then T_y is bijective and for the cylindrical set

$$\mathfrak{C} = \{x \in H_- \,|\, (\langle \varphi_1, x \rangle_0, \ldots, \langle \varphi_n, x \rangle_0) \in \delta\} \quad (\varphi_k \in \Phi, \delta \in \mathscr{B}(\mathbb{R}^n))$$

we have

$$\begin{aligned} T_y \mathfrak{C} &= \{z \in H_- \,|\, (\langle \varphi_1, z - y \rangle_0, \ldots, \langle \varphi_n, z - y \rangle_0) \in \delta\} \\ &= \{z \in H_- \,|\, (\langle \varphi_1, z \rangle_0, \ldots, \langle \varphi_n, z \rangle_0) \in \delta_y\}, \end{aligned}$$

where $\delta_y = \delta + (\langle \varphi_1, y \rangle_0, \ldots, \langle \varphi_n, y \rangle_0) \in \mathscr{B}(\mathbb{R}^n)$. This shows $T_y \mathfrak{C} \in \mathfrak{C}(H_-)$. Moreover, the σ-span of the sets $\mathfrak{C}$ is the σ-algebra $\mathfrak{C}_\sigma(H_-)$. This and the bijectivity of T_y show that

$$T_y \alpha = \{x + y \,|\, x \in \alpha\} \in \mathfrak{C}_\sigma(H_-)$$

for $\alpha \in \mathfrak{C}_\sigma(H_-)$. Now we define the measure μ_y by

$$\mu_y(\alpha) = \mu(T_y \alpha) \quad (\alpha \in \mathfrak{C}_\sigma(H_-);\ y \in H_-).$$

DEFINITION 1.1.19. Consider the Gaussian measure γ. For $y \in H_0$ we define

$$\varrho_y(\cdot) = \exp(-\langle y, y \rangle_0 - 2\langle y, \cdot \rangle_0).$$

LEMMA 1.1.20. *Let $y \in H_0$. Then $\varrho_y(\cdot) \in L^p(H_-,\ \gamma)$ for all $p \geq 1$.*

PROOF. See [**17**, p.154 Lemma 2.4]. □

THEOREM 1.1.21. *For $y \in H_0$ the measures γ and γ_y are mutually absolutely continuous and we have*

$$\frac{d\gamma_y}{d\gamma}(\cdot) = \varrho_y(\cdot).$$

Otherwise, the measures γ and γ_y are orthogonal.

PROOF. See [**17**, p.154-156 Theorem 2.45]. □

DEFINITION 1.1.22. We define the logarithmic derivative β_γ of the measure γ by

$$\beta_\gamma(t,x) = \lim_{h\to 0} \frac{\varrho_{ht}(x) - 1}{h} = \lim_{h\to 0} \frac{1}{h}\left(e^{-\langle ht,ht\rangle_0 - 2\langle ht,x\rangle_0} - 1\right) = -2\langle t\,,\, x\rangle_0, \tag{4}$$

with convergence in $L^p(H_-,\ \gamma)$ for all $1 \leq p < \infty$.

PROOF. See [**18**, page 251-252]. □

Finally we will give a proof of a result concerning arbitrary quasi-invariant measures.

LEMMA 1.1.23. *Let μ be a probability measure on $\mathscr{B}(H_-)$, quasi-invariant with respect to shifts by elements of H_+, i.e. for every $t \in H_+$ the Radon-Nikodym-derivative $\frac{d\mu(\cdot+t)}{d\mu(\cdot)} \in L^1(H_-,\ \mu)$ exists. Then for every open ball $B_R(x_0) = \{x \in H_- \mid \|x - x_0\|_{H_-} < R\}$ of radius $R > 0$ with center x_0 we have $\mu(B_R(x_0)) > 0$.*

PROOF. Suppose the assertion is wrong. Then there exists $x_0 \in H_-$ and a $R > 0$ with $\mu(B_R(x_0)) = 0$. Since H_+ is dense in H_-, we find a $\varphi \in H_+$, with $B_{R/2}(x) \subset B_R(x_0 + \varphi)$ for any $x \in H_-$. By assumption the measures $\mu(\cdot)$ and $\mu(\cdot + \varphi)$ are equivalent and hence we have $\mu(B_R(x_0 + \varphi)) = 0$. Since H_- is separable, we can cover H_- with countable many balls of radius $R/2$. But this implies $\mu(H_-) = 0$, in contradiction to our assumption $\mu(H_-) = 1$. □

In the following section we describe an orthonormal basis in the space $L^2(H_-,\gamma)$ consisting of generalized Hermite polynomials. At first, we note some basic facts about Hermite polynomials, following closely [**104**].

DEFINITION 1.1.24. (cf. [**104**, p. 230]) For $x \in \mathbb{R}$ we define the n-th Hermit polynomial by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}).$$

LEMMA 1.1.25. *Let $d\gamma_1(x) = \pi^{-1/2} e^{-x^2} dx$ be the canonical Gaussian measure in $\mathbb{R}$. Then we have*

(i) $\int\limits_{\mathbb{R}} H_n(x)H_m(x)\ d\gamma_1(x) = 2^n n!\delta_{nm}$.

(ii) $x^{2m} = \frac{(2m)!}{2^{2m}} \sum_{n=0}^{m} \frac{1}{(2n)!(m-n)!} H_{2n}(x)$.

(iii) $x^{2m+1} = \frac{(2m+1)!}{2^{2m+1}} \sum_{n=0}^{m} \frac{1}{(2n+1)!(m-n)} ! H_{2n+1}(x)$.

(iv) *The normalized Hermite polynomials*

$$h_n(x) = (2^n n!)^{-1/2} H_n(x)$$

form an orthonormal basis in $L^2(\mathbb{R},\ \gamma_1)$.

PROOF. See [**17**, page 138-139]. □

LEMMA 1.1.26. *(c.f.* [**104**, p. 230]*) Set* $\delta_x := -\frac{\partial}{\partial x} + 2x$ *and let* f *be in* $\mathscr{C}^2(\mathbb{R})$. *Then we have*

(i) $\delta_x H_n(x) = H_{n+1}(x)$,
(ii) $\frac{\partial}{\partial x}\delta_x\ f(x) - \delta_x \frac{\partial}{\partial x} f(x) = 2f(x)$,
(iii) $\frac{\partial}{\partial x} H_n(x) = 2nH_{n-1}(x)$ *for all* $n \in \mathbb{N}$,
(iv) $(\frac{\partial}{\partial x} + \delta_x)h_n(x) = 2x\ h_n(x)$,
(v) $xH_n(x) = \frac{1}{2}H_{n+1}(x) + nH_{n-1}(x)$ *for all* $n \in \mathbb{N}$,
(vi) $\frac{\partial}{\partial x} h_n(x) = \sqrt{2n}\ h_{n-1}(x)$,
(vii) $\delta_x h_n(x) = \sqrt{2(n+1)} h_{n+1}(x)$.

PROOF. (i) Suppose $n \in \mathbb{N}_0$ be arbitrary. Then we have

$$\begin{aligned} \delta_x H_n(x) &= (-\frac{\partial}{\partial x} + 2x)(-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \\ &= (-1)^{n+1} \left(e^{x^2} \frac{d^n}{dx^{n+1}}(e^{-x^2}) + 2xe^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) - 2xe^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \right) \\ &= (-1)^{n+1} e^{x^2} \frac{d^n}{dx^{n+1}}(e^{-x^2}) = H_{n+1}. \end{aligned}$$

(ii) For $f \in \mathscr{C}^2(\mathbb{R})$ we obtain

$$\frac{\partial}{\partial x}\delta_x f(x) - \delta_x \frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial x}(2x\ f(x)) - 2x\frac{\partial}{\partial x} f(x) = 2f(x).$$

(iii) An easy computation shows that $\frac{\partial}{\partial x} H_1(x) = \frac{\partial}{\partial x}(2x) = 2 = 2H_0(x)$. Let the assumption be right for $n-1 \in \mathbb{N}$. Then it follows

$$\begin{aligned} \frac{\partial}{\partial x} H_n(x) = \frac{\partial}{\partial x}\delta_x H_{n-1}(x) &= \delta_x \frac{\partial}{\partial x} H_{n-1}(x) + 2H_{n-1}(x) \\ &= 2(n-1)\delta_x H_{n-2}(x) + 2H_{n-1}(x) = 2nH_{n-1}(x). \end{aligned}$$

(iv) Clear by definition.
(v) For arbitrary $n \in \mathbb{N}_0$ we have

$$xH_n(x) = \frac{1}{2}(\delta_x + \frac{\partial}{\partial x})H_n(x) = \frac{1}{2}H_{n+1}(x) + nH_{n-1}(x).$$

(vi) For $n \in \mathbb{N}_0$ we get

$$\frac{\partial}{\partial x} h_n(x) = \frac{\partial}{\partial x} \frac{1}{\sqrt{2^n n!}} H_n(x) = \frac{2n}{\sqrt{2^n n!}} \, H_{n-1}(x) = \sqrt{2n} \, h_{n-1}(x).$$

(vii) Let $n \in \mathbb{N}_0$. The we obtain

$$\delta_x h_n(x) = \delta_x \frac{1}{\sqrt{2^n n!}} H_n(x) = \frac{1}{\sqrt{2^n n!}} H_{n+1}(x) = \sqrt{2(n+1)} h_{n+1}(x). \qquad \square$$

THEOREM 1.1.27. *Let $(e_j)_{j=1}^{\infty} \subset H_+$ be an orthonormal basis in H_0. For $\alpha \in \mathbb{N}_0^{\mathbb{N}}$ we set*

$$h_\alpha(x) = h_{\alpha_1}(\langle e_1, x \rangle_0) \cdots h_{\alpha_\nu}(\langle e_\nu, x \rangle_0).$$

Then the set $(h_\alpha)_{\alpha \in \mathbb{N}_0^N}$ is an orthornormal basis for $L^2(H_-, \gamma)$. In addition, we set $h_\alpha := h_\alpha^{\mathrm{id}}$.

PROOF. See [**17**, page 145-146 Theorem 2.2]. $\square$

1.2. Some closed operators

In the finite dimensional theory of pseudodifferential operators we have two important kinds of unbounded operators - the multiplication operators in coordinate directions and the operators of partial differentiations. In this section we define these operators for functions on an infinite dimensional Hilbert space and show that these operators are closed resp. closable.

Therefore let $H_+ \subseteq H_0 \subseteq H_-$ be a quasi-nuclear Hilbert spaces rigging. Moreover, let γ be the canonical Gaussian measure with respect to this rigging and $\varrho_t(\cdot)$ be defined as in 1.1.19.

DEFINITION 1.2.1. Suppose H and P are Hilbert spaces.

(i) Let $\mathscr{C}^k_{pol}(H, P)$ be the space of k times continuous differentiable maps $f : \ H \longrightarrow P$ with $\|d^n f(x)\|_{\mathscr{L}_n(H,P)} \leq C_n \ (1 + \|x\|_H)^{m_n}$ for all $n \in \mathbb{N}_0$, $n \leq k$ and suitable constants $C_n \in \mathbb{R}$ and $m_n \in \mathbb{N}_0$ depending on n.

(ii) Furthermore, we write $\mathscr{C}^k_b(H, P)$ for the space of k times continuous differentiable maps, with bounded derivatives. For $f \in \mathscr{C}^k_b(H, P)$ we define $\|f\|_{\mathscr{C}^k_b(H,P)} := \sum_{j=0}^n \|d^j f(x)\|_{\sup}$.

(iii) Let μ be a measure in H. Then $\mathscr{C}^k_{int}(H)$ denotes the space of k times continuous differentiable functions $f : \ H \longrightarrow \mathbb{C}$ such that

$$(x \longmapsto \|x\|_H^m \, \|d^n f(x)\|_{\mathscr{L}_n(H,\mathbb{C})}) \in L^2(H, \ \mu)$$

is bounded on bounded sets for all $n, m \in \mathbb{N}_0$, $n \leq k$.

(iv) We denote by $\mathscr{C}^k_{pol,cyl}(H)$, $(\mathscr{C}^k_{b,cyl}(H)$, $\mathscr{C}^k_{int,cyl}(H))$ the space of all differentiable cylindrical functions in $\mathscr{C}^k_{pol}(H, \mathbb{C})$, $(\mathscr{C}^k_b(H, \mathbb{C})$, $\mathscr{C}^k_{int}(H, \mathbb{C}))$.

(v) Let $S_\gamma(\mathbb{R}^n)$ be the space of all functions $f \in \mathscr{C}^\infty(\mathbb{R}^n)$ such that there exists a $g \in S(\mathbb{R}^n)$ with $f(x) = e^{\|x\|/2} g(x)$ for all $x \in \mathbb{R}^n$.

(vi) Let $S_{\gamma,cyl}(H_-)$ be the set of all cylindrical functions f such that there exist a function $F \in S_\gamma(\mathbb{R}^n)$ with $f(x) = F(\langle e_1\,,\, x\rangle_0 \cdot \langle e_n\,,\, x\rangle)$, where $(e_n)_{n\in\mathbb{N}} \subset H_+$ denotes an ONB of H_0.

DEFINITION 1.2.2. Let $t \in H_+$. Define $M_t : D(M_t) \longrightarrow L^2(H_-,\, \gamma)$ by

$$M_t f = \langle t, \cdot\rangle_0 f$$

for all

$$f \in D(M_t) = \{f \in L^2(H_-,\, \gamma) \,|\, \langle t, \cdot\rangle_0 f \in L^2(H_-,\, \gamma)\}.$$

Then $M_t :\ D(M_t) \longrightarrow L^2(H_-,\, \gamma)$ is selfadjoint.

LEMMA 1.2.3. *Let $f, g \in \mathscr{C}_b^1(H_-)$ and $t \in H_+$. Define* $\delta_t g(x) := -\frac{\partial g(x)}{\partial t} + 2\langle t\,,\, x\rangle_0 g(x)$. *Then we have*

$$\langle \frac{\partial}{\partial t} f, g\rangle_{L^2(H_-,\,\gamma)} = \langle f, \delta_t g\rangle_{L^2(H_-,\,\gamma)}.$$

PROOF. Using Lebesgue's theorem of dominated convergence we obtain

$$\begin{aligned}
&\int_{H_-} \frac{\partial f(x)}{\partial t}\overline{g(x)}d\gamma(x)\\
=\ & \lim_{h\to 0} \int_{H_-} \frac{f(x+ht)-f(x)}{h}\overline{g(x)}d\gamma(x)\\
=\ & \lim_{h\to 0} \frac{1}{h}\left(\int_{H_-} f(x+ht)\overline{g(x)}d\gamma(x) - \int_{H_-} f(x)\overline{g(x)}d\gamma(x)\right)\\
=\ & \lim_{h\to 0} \frac{1}{h}\left(\int_{H_-} f(x)\overline{g(x-ht)}\varrho_{-ht}(x)d\gamma(x) - \int_{H_-} f(x)\overline{g(x)}d\gamma(x)\right)\\
=\ & \lim_{h\to 0} \frac{1}{h}\int_{H_-} f(x)\left(\overline{g(x-ht)\varrho_{-ht} - g(x)}\right)d\gamma(x)\\
=\ & \lim_{h\to 0} \int_{H_-} f(x)\left(\overline{\frac{g(x-ht)-g(x)}{h}\varrho_{-ht}(x) + \frac{\varrho_{-ht}(x)-1}{h}g(x)}\right)d\gamma(x)\\
=\ & \int_{H_-} -f(x)\overline{\frac{\partial g(x)}{\partial t}} + 2f(x)\langle t\,,\, x\rangle_0\overline{g(x)}d\gamma(x).
\end{aligned}$$

Here we used $f, g \in \mathscr{C}_b^1(H_-)$. Thus the difference quotients are bounded. Moreover, for β_γ we have convergence in $L^2(H_-,\, \gamma)$ by assumption. □

PROPOSITION 1.2.4. *Let $t \in H_+$ be fixed. For $f \in \mathscr{C}_b^\infty(H_-)$ we define $\partial_t : \mathscr{C}_b^\infty(H_-) \longrightarrow L^2(H_-,\ \gamma)$ by $\partial_t f(x) = \frac{\partial}{\partial t} f(x)$. Then ∂_t is densely defined and closable in $L^2(H_-,\ \gamma)$. We will denote its closure by ∂_t again.*

PROOF. Set $\delta_t g(x) := -\frac{\partial g(x)}{\partial t} + 2\langle t\,,\, x\rangle_0 g(x)$ for $g \in \mathscr{C}_b^\infty(H_-)$. Let $(f_n)_{n=1}^\infty \subset \mathscr{C}_b^\infty$ sequence with $f_n \xrightarrow[n\to\infty]{L^2(H_-,\,\gamma)} 0$ and $\partial_t f_n \xrightarrow[n\to\infty]{L^2(H_-,\,\gamma)} f$. Thus for $g \in \mathscr{C}_b^\infty(H_-)$ we have

$$\langle f, g\rangle_{L^2(H_-,\,\gamma)} = \lim_{n\to\infty} \langle \partial_t f_n, g\rangle_{L^2(H_-,\,\gamma)} \overset{1.2.3}{=} \lim_{n\to\infty} \langle f_n, \delta_t g\rangle_{L^2(H_-,\,\gamma)} = 0.$$

Since $\mathscr{C}_b^\infty(H_-) \subset L^2(H_-,\ \gamma)$ is dense, it follows that $f = 0$. But this is our assertion. □

LEMMA 1.2.5. *For $f, g \in \mathscr{C}_{int}^1(H_-)$ and δ_t ($t \in H_+$) defined as in in 1.2.3 we have*

$$\langle \frac{\partial}{\partial t} f, g\rangle_{L^2(H_-,\,\gamma)} = \langle f, \delta_t g\rangle_{L^2(H_-,\,\gamma)}.$$

Moreover,we have $\mathscr{C}_{int}^\infty(H_-) \subset D(\partial_t)$ and for $f \in \mathscr{C}_{int}^\infty(H_-)$ we obtain

$$\partial_t f(x) = \frac{\partial}{\partial_t} f(x).$$

PROOF. Let $f, g \in \mathscr{C}_{int}^1(H_-)$. Assume $\zeta_n \in \mathscr{C}^\infty(\mathbb{R})$ having the following properties

(i) $\zeta_n(t) = 1 \quad \forall\, |t| \le n, \qquad \zeta_n(t) = 0 \quad \forall\, |t| \ge n+1,$
(ii) $|\zeta_n(t)| \le 1, \qquad |\zeta_n'(t)| \le c\ \forall n$, where $c > 0$.

For $x \in H_-$ define $h_n(x) := \zeta_n(\|x\|_-^2)$ and $f_n(x) := f(x)h_n(x)$ and $g_n(x) := g(x)h_n(x)$. Then we have $f_n(x) \xrightarrow[n\longrightarrow\infty]{} f(x)$ and

$$\frac{\partial f_n}{\partial t}(x) = \frac{\partial f}{\partial t}(x) h_n(x) + f(x) 2\langle x, t\rangle_- \zeta_n'(\|x\|_-^2) \xrightarrow[n\longrightarrow\infty]{} \frac{\partial f}{\partial t}(x)$$

pointwisely. The same equations hold for g and g_n. Moreover, we have

(5) $$\left| \frac{\partial f_n}{\partial t}(x) \overline{g_n(x)} \right| \le \|df(x)\|_{Op} \|t\|_- |g(x)| + 2\|x\|_- \|t\|_- |f(x)g(x)|$$

and

(6) $$|f_n(x)\beta_\gamma(t,x)g_n(x)| \le |f(x)2\langle x, t\rangle_0 g(x)| \le |f(x)g(x)|\, 2\, \|x\|_- \|t\|_+ .$$

Since $f,\ g,\ df,\ dg$ are bounded on bounded sets by assumption, $f_n,\ g_n,\ df_n,\ dg_n$ are bounded on H_- by definition. Now Lebesgue's theorem of dominated convergence implies

$$\begin{aligned}\int\limits_{H_-} \frac{\partial f}{\partial t}(x)\overline{g(x)}\, d\gamma(x) &= \lim_{n\to\infty}\int\limits_{H_-} \frac{\partial f_n}{\partial t}(x)\overline{g_n(x)}\, d\gamma(x) \\ &= \lim_{n\to\infty}\int\limits_{H_-} -f_n(x)\overline{\frac{\partial g_n(x)}{\partial t}} + f_n(x)\langle 2x, t\rangle_0\overline{g_n(x)}d\gamma(x) \\ &= \int\limits_{H_-} -f(x)\overline{\frac{\partial g(x)}{\partial t}} + f(x)\langle 2x, t\rangle_0\overline{g(x)}d\gamma(x).\end{aligned}$$

This shows the first assertion. Now we will prove the second assertion. Therefore we only have to show that $\partial_t f_n \xrightarrow[n\to\infty]{L^2(H_-,\,\gamma)} \frac{\partial}{\partial t}f$. According to (5) we have

$$|\partial f_n(x)| \le \|df(x)\|_{Op}\, \|t\|_- + 2\,\|x\|_-\,\|t\|_-\,|f(x)| \in L^2(H_-,\ \gamma).$$

Now our assertion follows by using Lebesgue's theorem of dominated convergence, since $\partial_t f_n(x) \xrightarrow[n\to\infty]{} \frac{\partial}{\partial t}f(x)$ pointwisely. □

REMARK 1.2.6. Let $f \in \mathscr{C}_b^\infty(H_-)$ and $t \in H_+$. We define

$$\delta_t f(x) := -\frac{\partial f(x)}{\partial t} + 2\langle t\,,\ x\rangle_0 f(x).$$

Then 1.2.4 and 1.2.5 remain valid for δ_t instead of $\frac{\partial}{\partial t}$ resp. ∂_t. We will write δ_t again for the closure of δ_t.

1.3. Unitary translation groups and their infinitesimal generator

In this section we introduce a unitary translation group, which is important to construct an abstract Fourier transform. In contrast to the theory of pseudodifferential operators in $\mathbb{R}^n$ we do not have any translation invariant measure in infinite dimensional spaces. Moreover, there exists only a dense subset of an infinite dimensional space such that the translated measure is absolute continuous with regard to a given measure. Therefore we only use shifts by elements of this dense subset. We also need the Radon-Nikodym derivative of the translated measures to define this unitary translation group. Let ϱ be defined as in 1.1.19.

REMARK 1.3.1. For $t,\ \tau \in H_+$ and $x \in H_-$ Lemma 1.1.1 implies

$$\varrho_{t+\tau}(x) = \frac{d\varrho(x+t+\tau)}{d\varrho(x)} = \frac{d\varrho(x+t)}{d\varrho(x)}\ \frac{d\varrho(x+t+\tau)}{d\varrho(x+t)} = \varrho_t(x)\varrho_\tau(x+t).$$

DEFINITION 1.3.2. For $t \in H_+$ and $\varphi \in L^2(H_-,\ \gamma)$ define U_t by

$$U_t\varphi(x) = \sqrt{\varrho_t(x)}\varphi(x+t),$$

where $\varrho_t(\cdot) = \frac{d\gamma(\cdot+t)}{d\gamma(\cdot)}$.

Lemma 1.3.3. *Let $t \in H_+$. Then U_t is unitary operator in $L^2(H_-,\ \gamma)$ and we have*

$$U_t{}^*\psi(x) = \sqrt{\varrho_{-t}(x)}\psi(x-t).$$

Proof. First we show that U_t is a bounded operator in $L^2(H_-,\ \gamma)$. For $\varphi \in L^2(H_-,\ \gamma)$ we have

$$\begin{aligned} \|U_t\varphi(x)\|^2_{L^2(H_-,\ \gamma)} &= \int \left|\varrho_t(x)\varphi(x+t)^2\right| d\gamma(x) \\ &= \int |\varrho_t(x-t)|\, |\varphi(x)|^2\, \varrho_{-t}(x)d\gamma(x) \\ &= \int |\varphi(x)|^2\, d\gamma(x) = \|\varphi\|^2_{L^2(H_-,\ \gamma)}. \end{aligned}$$

This shows that U_t $(t \in H_+)$ is a bounded operator in $L^2(H_-,\ \gamma)$. Now let us compute $U_t{}^*$. Therefore let $\varphi,\ \psi \in L^2(H_-,\ \gamma)$. Then it follows that

$$\begin{aligned} \langle U_t\varphi, \psi\rangle_{L^2(H_-,\ \gamma)} &= \int \sqrt{\varrho_t(x)}\varphi(x+t)\overline{\psi(x)}d\gamma(x) \\ &= \int \varphi(x)\overline{\sqrt{\varrho_t(x-t)}\psi(x-t)\varrho_{-t}(x)}d\gamma(x) \\ & \int \varphi(x)\overline{\sqrt{\varrho_{-t}(x)}\psi(x-t)}d\gamma(x) \\ &= \langle \varphi, U_t{}^*\psi\rangle_{L^2(H_-,\ \gamma)} \end{aligned}$$

with $U_t{}^*\psi(x) = \sqrt{\varrho_t(x-t)}\varrho_{-t}(x)\psi(x-t)$. Finally we show that U_t is a unitary operator.

$$\begin{aligned} U_t{}^*U_t\varphi(x) &= U_t{}^*(\sqrt{\varrho_t(x)}\varphi(x+t)) = \sqrt{\varrho_{-t}(x)}\sqrt{\varrho_t(x-t)}\varphi(x) = \varphi(x), \\ U_tU_t{}^*\varphi(x) &= U_t(\sqrt{\varrho_{-t}(x)}\varphi(x-t)) = \sqrt{\varrho_t(x)}\ \sqrt{\varrho_{-t}(x+t)}\varphi(x) = \varphi(x). \end{aligned}$$

But this is our assertion. □

Theorem 1.3.4. *Let $t \in H_+$ and U_t defined as in 1.3.2. Then U_t is a commuting strongly continuous unitary family in $L^2(H_-,\ \gamma)$ with $U_{t+s} = U_tU_s$ for all $s, t\ \in H_+$.*

Proof. For $t,\ s \in H_+$ and $\varphi \in L^2(H_-,\ \gamma)$ we have

$$\begin{aligned} U_{t+s}\varphi(x) = \sqrt{\varrho_{t+s}(x)}\varphi(x+t+s) &= \sqrt{\varrho_t(x)\varrho_s(x+t)}\varphi(x+t+s) \\ &= \sqrt{\varrho_t(x)}U_s\varphi(x+t) = U_tU_s\varphi(x). \end{aligned}$$

This shows that U_t $(t \in H_+)$ is a unitary group (note 1.3.3). For $\varphi \in \mathscr{C}_b(H_-)$, it follows that

$$\begin{aligned}\langle U_t\varphi, \varphi\rangle_{L^2(H_-,\,\gamma)} &= \int \sqrt{\varrho_t(x)}\varphi(x+t)\overline{\varphi(x)}\; d\gamma(x) \\ &\xrightarrow[t\to 0]{} \int |\varphi(x)|^2\; d\gamma(x) = \|\varphi\|^2_{L^2(H_-,\,\gamma)}.\end{aligned}$$

Here we used that φ is bounded and that $\sqrt{\varrho_t} \xrightarrow[t\to 0]{L^2(H_-,\,\gamma)} 1$. Hence it follows

$$\begin{aligned}\|(U_t - \mathrm{id})\varphi\|^2_{L^2(H_-,\,\gamma)} &= \langle (U_t - \mathrm{id})^*(U_t - \mathrm{id})\varphi,\ \varphi\rangle_{L^2(H_-,\,\gamma)} \\ &= \langle (2\,\mathrm{id} - U_t - U_t{}^*)\,\varphi,\ \varphi\rangle_{L^2(H_-,\,\gamma)} \\ &= 2\,\|\varphi\|^2_{L^2(H_-,\,\gamma)} - \langle U_t\varphi,\ \varphi\rangle_{L^2(H_-,\,\gamma)} - \langle \varphi,\ U_t\varphi\rangle_{L^2(H_-,\,\gamma)} \\ &= 2\,\|\varphi\|^2_{L^2(H_-,\,\gamma)} - 2Re\langle U_t\varphi,\ \varphi\rangle_{L^2(H_-,\,\gamma)} \xrightarrow[t\to 0]{} 0.\end{aligned}$$

Now we show the assertion. Therefore let $f \in L^2(H_-,\ \gamma)$ and $\varepsilon > 0$ arbitrary, but fixed. Then there exists a $\varphi \in \mathscr{C}_b(H_-)$, with $\|f - \varphi\| \leq \frac{\varepsilon}{3}$, since $\mathscr{C}_b(H_-) \subset L^2(H_-,\ \gamma)$ dense. The computation above shows that for $\varphi \in \mathscr{C}_b(H_-)$, there is a $\delta > 0$ such that $\|(U_t - \mathrm{id})\varphi\|_{L^2(H_-,\,\gamma)} \leq \frac{\varepsilon}{3}$ for all $t \in H_+$ with $\|t\|_+ \leq \delta$. Hence for all t with $\|t\|_+ \leq \delta$ we have

$$\begin{aligned}\|(U_t - \mathrm{id})f\|_{L^2(H_-,\,\gamma)} &\leq \|(U_t - \mathrm{id})(f-\varphi)\|_{L^2(H_-,\,\gamma)} + \|(U_t - \mathrm{id})\varphi\|_{L^2(H_-,\,\gamma)} \\ &\leq \|U_t - \mathrm{id}\|\,\|f - \varphi\|_{L^2(H_-,\,\gamma)} + \|(U_t - \mathrm{id})\varphi\|_{L^2(H_-,\,\gamma)} \\ &\leq 2\ \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Thus $\lim\limits_{t\longrightarrow 0} \|(U_t - \mathrm{id})f\|_{L^2(H_-,\,\gamma)} = 0$ and U_t $(t \in H_+)$ is strongly continuous. □

Remark 1.3.5. Let $t \in H_+$. Then $\mathbb{R} \ni h \longmapsto U_{ht}$ $(h \in \mathbb{R})$ is strongly continuous unitary one parameter group.

Now we compute the infinitesimal generator D_t $(t \in H_+)$ of the unitary one parameter groups defined in the previous section. Furthermore, we show that these infinitesimal generators define a family of commuting differential operators of order one. Finally, we determine a domain of essential selfadjointness of these infinitesimal generators.

Definition 1.3.6. Let D_t $(t \in H_+)$ denote the infinitesimal generator of the unitary C_0 group U_{ht} $(h \in \mathbb{R})$. For its domain of definition we write $D(D_t)$. According to the theorem of Stone (cf. [**117**, Theorem VIII.8]) we obtain that $-iD_t$ is selfadjoint.

Proposition 1.3.7. *Let* $t \in H_+$. *Then* $\mathscr{C}^1_b(H_-) \subseteq D(D_t)$ *and for* $\varphi \in \mathscr{C}^1_b(H_-)$ *we have*

$$D_t\varphi(x) = \frac{\partial}{\partial t}\varphi(x) - \langle t\,,\, x\rangle_0\varphi(x). \tag{7}$$

PROOF. For $t \in H_+$, $h \in \mathbb{R}$ and $\varphi \in \mathscr{C}_b^1(H_-)$ we get

$$\begin{aligned} \frac{U_{ht}\varphi(x) - \varphi(x)}{h} &= \frac{\sqrt{\varrho_{ht}(x)}\varphi(x+ht) - \varphi(x)}{h} \\ &= \sqrt{\varrho_{ht}(x)}\frac{\varphi(x+ht) - \varphi(x)}{h} + \frac{\sqrt{\varrho_{ht}(x)} - 1}{h}\varphi(x). \end{aligned}$$

Now we consider the two addends separately.

(i) For the first addend we have

$$\begin{aligned} &\sqrt{\varrho_{ht}(x)}\frac{\varphi(x+ht) - \varphi(x)}{h} - \frac{\partial}{\partial t}\varphi(x) \\ = \;& \sqrt{\varrho_{ht}(x)}\frac{\varphi(x+ht) - \varphi(x)}{h} - \sqrt{\varrho_{ht}(x)}\frac{\partial}{\partial t}\varphi(x) + \sqrt{\varrho_{ht}(x)}\frac{\partial}{\partial t}\varphi(x) - \frac{\partial}{\partial t}\varphi(x) \\ = \;& \sqrt{\varrho_{ht}(x)}\left(\frac{\varphi(x+ht) - \varphi(x)}{h} - \frac{\partial}{\partial t}\varphi(x)\right) + \left(\sqrt{\varrho_{ht}(x)} - 1\right)\frac{\partial}{\partial t}\varphi(x). \end{aligned}$$

Thus we obtain

$$\begin{aligned} &\left\|\sqrt{\varrho_{ht}(x)}\left(\frac{\varphi(x+ht) - \varphi(x)}{h} - \frac{\partial}{\partial t}\varphi(x)\right)\right\|^2_{L^2(H_-\gamma)} \\ \leq \;& \left(\int \left|\sqrt{\varrho_{ht}(x)}\right|^4 d\gamma(x)\right)^{1/2}\left(\int \left|\frac{\varphi(x+ht) - \varphi(x)}{h} - \frac{\partial}{\partial t}\varphi(x)\right|^4 d\gamma(x)\right)^{1/2} \\ \leq \;& \left(\int \varrho_{ht}(x)^2 d\gamma(x)\right)^{1/2}\left(\int \left|\frac{\varphi(x+ht) - \varphi(x)}{h} - \frac{\partial}{\partial t}\varphi(x)\right|^4 d\gamma(x)\right)^{1/2} \xrightarrow{h\to 0} 0 \end{aligned}$$

and

$$\left\|\left(\sqrt{\varrho_{ht}(x)} - 1\right)\frac{\partial}{\partial t}\varphi(x)\right\|_{L^2(H_-\gamma)} \leq c\,\|\sqrt{\varrho_{ht}}(x) - 1\|_{L^2(H_-\gamma)} \xrightarrow{h\to 0} 0$$

by assumption and Lebesgue's theorem of dominated convergence, since $\varphi \in \mathscr{C}_b^1(H_-)$.

(ii) Moreover, our assumptions imply directly the following equation

$$\begin{aligned} &\left\|\left(\frac{\sqrt{\varrho_{ht}(x)} - 1}{h} + \langle t\,,\,x\rangle_0\right)\varphi(x)\right\|_{L^2(H_-,\,\gamma)} \\ \leq \;& c\left\|\frac{\sqrt{\varrho_{ht}(x)} - 1}{h} + \langle t\,,\,x\rangle_0)\right\|_{L^2(H_-,\,\gamma)} \xrightarrow{h\longrightarrow 0} 0. \end{aligned}$$

This yields

$$D_t\varphi(x) = \frac{\partial}{\partial t}\varphi(x) - \langle t\,,\,x\rangle_0. \qquad \square$$

At next let describe more detailed spaces of essential selfadjointness for the operator D_t.

PROPOSITION 1.3.8. *For every* $\varphi \in \mathscr{C}^1_{int}(H_-)$.

$$D_t\varphi(x) = \frac{\partial}{\partial t}\varphi(x) - \langle t\,,\, x\rangle_0\varphi(x). \tag{8}$$

Moreover, we have $D_t(\mathscr{C}^\infty_{int}(H_-)) \subset \mathscr{C}^\infty_{int}(H_-)$, *and* $\mathscr{C}^\infty_{int}(H_-)$ *is a domain of essential selfadjointness of the operator* $-iD_t$.

PROOF. For $f \in \mathscr{C}^1_{int}(H_-)$ arbitrary define h_n and ζ_n as in Lemma 1.2.5. Set $f_n(x) = f(x)h_n(x)$. Then we have $f_n \xrightarrow[n\longrightarrow\infty]{} f \in L^2(H_-\gamma)$ and $f_n \in \mathscr{C}^1_b(H_-)$ and the following equality holds pointwisely.

$$\begin{aligned} D_t f_n(x) &= \frac{\partial}{\partial t} f_n(x) - \langle t, x\rangle_0 f_n(x) \\ &= \frac{\partial}{\partial t} f(x)h_n(x) + f(x)\frac{\partial}{\partial t}h_n(x) - \langle t, x\rangle_0 f_n(x) \\ &\xrightarrow[n\longrightarrow\infty]{ptw.} \frac{\partial}{\partial t} f(x) - \langle t, x\rangle_0 f(x). \end{aligned}$$

Moreover,

$$\left|\frac{\partial}{\partial t} f h_n + f\frac{\partial}{\partial t}h_n - \langle t, \cdot\rangle_0 f_n(x)\right| \le \left|\frac{\partial}{\partial t} f\right| + c\,|f| + |\langle t, \cdot\rangle_0 f(x)| \in L^2(H_-,\ \gamma).$$

Hence Lebesgue's theorem of dominated convergence implies that $D_t f_n$ convergences in $L^2(H_-,\ \gamma)$. Since D_t is closed, the first assertion is now a consequence of step one. Finally we have $\mathscr{C}^\infty_{int}(H_-) \subset L^2(H_-,\ \gamma)$ dense, U_{ht} unitary C_0 group and $U_{ht}(\mathscr{C}^\infty_{int}(H_-)) \subset \mathscr{C}^\infty_{int}(H_-)$. Hence the second assertion follows directly by the theorem of Nelson (cf. [**117**, Theorem VIII.10]). □

Moreover, the proof of Proposition 1.3.8 shows that $\mathscr{C}^\infty_b(H_-)$ is a domain of essential selfadjointness of iD_t.

REMARK 1.3.9. Furthermore, it is quite obvious that U_t leaves the space $S_\gamma(H_-)$ and $S_{\gamma,cly}(H_-)$. Thus both are domains of essential selfadjointnes for $-iD_t$ invariant.

1.4. An abstract Fourier transform

The Fourier transform in $\mathbb{R}^n$ is a unitary transform of function of $L^2(\mathbb{R}^n,\ \lambda)$, which is a unitary equivalence between the translation group and the group of multiplication with $e^{i\langle t,\cdot\rangle_0}$. Our aim is to find an unitary operator in infinite dimensional Hilbert spaces with similar properties as the Fourier transform in $\mathbb{R}^n$.

But at first let us note the following

LEMMA 1.4.1. *Let $\varphi \in H_+$. Then the following equation holds.*

$$\int\limits_{H_-} e^{\langle\varphi,x\rangle_0} d\gamma_S(x) = e^{S\langle\varphi,\varphi\rangle_0/4}. \tag{9}$$

PROOF. See [**35**]. □

LEMMA 1.4.2. *Let γ be the canonical Gaussian measure. Then U_φ ($\varphi \in H_+$) is cyclic with cycle vector 1.*

PROOF. Suppose $\varphi \in H_+$. Then we have

$$U_\varphi 1(x) = \sqrt{\varrho_\varphi(x)} = e^{-\frac{1}{2}\langle\varphi,\varphi\rangle_0 - \langle\varphi,x\rangle_0}$$

Set $M := \overline{span\{U_\varphi 1(x) \mid \varphi \in H_+\}}$. M contains all partial derivatives of $U_\varphi 1$ in all directions $\varphi \in H_+$ and thus all polynomials. Since the polynomials are dense in $L^2(H_-,\,\gamma)$, it follows that $M = L^2(H_-,\,\gamma)$. □

LEMMA 1.4.3. *For $\varphi \in H_+$ set $L(\varphi) = \langle U_\varphi 1, 1\rangle_{L^2(H_-,\,\gamma)}$. Then $L:\ H_+ \longrightarrow \mathbb{C}$ is continuous, positive semi definite and we have $L(0) = 1$. Moreover, we have*

$$L(\varphi) = e^{-\frac{1}{4}\langle\varphi,\varphi\rangle_0}.$$

PROOF. Let $\varphi_1 \ldots \varphi_n \in H_+$ and $\alpha_1 \ldots \alpha_n \in \mathbb{C}$. Then the following computation holds.

$$\begin{aligned}
\sum_{j,k=1}^{n} L(\varphi_j - \varphi_k)\alpha_j\overline{\alpha_k} &= \sum_{j,k=1}^{n} \langle U_{\varphi_j-\varphi_k} 1, 1\rangle_{L^2(H_-,\,\gamma)}\alpha_j\overline{\alpha_k} \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} \langle U_{\varphi_j} 1, U_{\varphi_k} 1\rangle_{L^2(H_-,\,\gamma)}\alpha_j\overline{\alpha_k} \\
&= \left\| \sum_{j=1}^{n} \alpha_j U_{\varphi_j} 1 \right\|^2_{L^2(H_-,\,\gamma)} \geq 0.
\end{aligned}$$

Therefore L is positive semi definite. The strong continuity of the family $U_t \quad (t \in H_+)$ implies directly the continuity of L. Furthermore, we have $L(0) = \langle 1, 1\rangle_{L^2(H_-,\,\gamma)} = 1$. In addition we find

$$\begin{aligned}
L(\varphi) &= \int U_\varphi 1(x) d\gamma(x) = \int \sqrt{\varrho_\varphi(x)} d\gamma(x) \\
&= e^{-\frac{1}{2}\langle\varphi,\varphi\rangle_0} \int e^{-\langle\varphi,x\rangle_0} d\gamma(x) \overset{1.4.1}{=} e^{-\frac{1}{2}\langle\varphi,\varphi\rangle_0} e^{\frac{1}{4}\langle\varphi,\varphi\rangle_0} = e^{-\frac{1}{4}\langle\varphi,\varphi\rangle_0}.
\end{aligned}$$

□

The following result is well known and used in many publications. But, since we have not found any references, we will give a complete proof.

PROPOSITION 1.4.4. *The family U_t ($t \in H_+$) is unitary equivalent to a family of multiplication operators $V_t = e^{i\langle t,\cdot\rangle_0}$ in the space $L^2(H_{-2},\ \gamma)$.*

PROOF. For $\varphi \in H_+$ let $L(\varphi) = \langle U_\varphi 1, 1\rangle_{L^2(H_-,\ \gamma)}$. Then 1.4.3 implies that L is continuous and positive semi-definite and we have $L(0) = 1$. Thus applying Proposition 1.1.7 and Lemma 1.4.3 we obtain that L is the Fourier-transform of the canonical Gaussian measure γ:

$$L(\varphi) = \int \exp(i\langle\varphi, x\rangle_0) d\gamma(x).$$

Now for $\varphi \in H_+$ we define $\mathcal{F}(U_\varphi 1) = e^{i\langle\varphi,\cdot\rangle_0}$ and extend $\mathcal{F}$ linearly to $span\{U_\varphi 1 \mid \varphi \in H_+\}$. For $f_j = \sum_{k_j=1}^{n_j} \lambda_{k_j}^{(j)} U_{\varphi_j} 1$ ($j = 1, 2$) we obtain

$$\begin{aligned}
\langle\mathcal{F}(f_1), \mathcal{F}(f_2)\rangle_{L^2(H_{-2},\ \gamma)} &= \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} \lambda_{k_1}^{(1)}\overline{\lambda_{k_2}^{(2)}}\langle e^{i\langle\varphi_{k_1},x\rangle_0}, e^{i\langle\varphi_{k_2},x\rangle_0}\rangle_{L^2(H_{-2},\ \gamma)} \\
&= \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} \lambda_{k_1}^{(1)}\overline{\lambda_{k_2}^{(2)}} \int e^{i\langle\varphi_{k_1}-\varphi_{k_2},x\rangle_0}\, d\gamma(x) \\
&= \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} \lambda_{k_1}^{(1)}\overline{\lambda_{k_2}^{(2)}} L(\varphi_{k_1} - \varphi_{k_2}) \\
&= \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} \lambda_{k_1}^{(1)}\overline{\lambda_{k_2}^{(2)}}\langle U_{\varphi_{k_1}-\varphi_{k_2}} 1, 1\rangle_{L^2(H_-,\gamma)} \\
&= \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} \lambda_{k_1}^{(1)}\overline{\lambda_{k_2}^{(2)}}\langle U_{\varphi_{k_1}} 1, U_{\varphi_{k_2}} 1\rangle_{L^2(H_-,\gamma)} \\
&= \langle f_1, f_2\rangle_{L^2(H_-,\gamma)}.
\end{aligned}$$

Thus $\mathcal{F}$ is well defined on $span\{U_\varphi 1 \mid \varphi \in H_+\}$ and an isometry. Since $span\{U_\varphi 1 \mid \varphi \in H_+\}$ is dense in $L^2(H_-,\ \gamma)$ $\mathcal{F}$ can be extended to a linear isometry from $L^2(H_-,\ \gamma)$ in $L^2(H_{-2},\ \gamma)$.
Now for $\varphi \in H_+$ we define $G(e^{i\langle\varphi,\cdot\rangle_0}) = U_\varphi 1$ and extend G linearly to $span\{e^{i\langle\varphi,\cdot\rangle_0} \mid \varphi \in H_+\} \subset L^2(H_-,\ \gamma)$. Similarly as above we see that G is an isometry. Therefore we can extend G to an isometric operator from $L^2(H_-,\ \gamma)$ in $L^2(H_-,\ \gamma)$, since Proposition 1.1.12 implies that $span\{e^{i\langle\varphi,\cdot\rangle_0} \mid \varphi \in H_+\}$ is dense in $L^2(H_{-2},\ \gamma)$.

For $h \in L^2(H_-,\ \gamma)$ there exists a sequence $h_k \in span\{e^{i\langle\varphi,\cdot\rangle_0} \mid \varphi \in H_+\}$ such that $h = \lim_{k\to\infty} h_k$. Thus we have

$$\mathcal{F}G\, h = \mathcal{F}G \lim_{k\to\infty} h_k = \lim_{k\to\infty} \mathcal{F}G\, h_k = \lim_{k\to\infty} h_k = h.$$

Furthermore, for $f \in L^2(H_-,\ \gamma)$ there exists a sequence $f_k \in span\{U_\varphi 1 \mid \varphi \in H_+\}$ such that $f = \lim_{k\to\infty} f_k$. Hence we have

$$G\mathcal{F}\ f = G\mathcal{F} \lim_{k\to\infty} f_k = \lim_{k\to\infty} G\mathcal{F}\ f_k = \lim_{k\to\infty} f_k = f.$$

This yields that $\mathcal{F}$ is bijective and $\mathcal{F}^{-1} = G$. Since

$$\mathcal{F}(U_\varphi f_1) = \sum_{k=1}^{n_1} \lambda_k^{(1)} \mathcal{F}(U_{\varphi+\varphi_k} g) = \sum_{k=1}^{n_1} \lambda_k^{(1)} e^{i\langle \varphi+\varphi_k, \cdot\rangle_0} = e^{i\langle \varphi, \cdot\rangle_0} \mathcal{F}(f_1)$$

and $span\{U_\varphi 1 \mid \varphi \in H_{+2}\}$ is dense in $L^2(H_-,\ \gamma)$, we have

$$\mathcal{F} U_\varphi = e^{i\langle \varphi, \cdot\rangle_0} \mathcal{F},$$

where $\mathcal{F}$ is isometry from $L^2(H_-,\ \gamma)$ onto $L^2(H_{-2},\ \gamma)$. □

At next we define the well known Fourier-Wiener-transform and show that in the case of canonical Gaussian measure our abstract Fourier-transform coincides with the Fourier-Wiener-transform.

Definition 1.4.5. For $w \in H_0$ and $f \in L^2(H,\ \gamma_2)$, where γ_2 is the Gaussian measure with correlations operator $2\,\mathrm{id}$. Then the Fourier-Wiener-transform is defined by

$$Wf(w) = e^{\frac{\|w\|^2}{2}} \int_{H_-} e^{-i\langle w, x\rangle_0} f(x)\ d\gamma_2(x).$$

Remark 1.4.6. In stochastic often the Fourier-Wiener transform is defined without the minus i.e. by $e^{\frac{\|w\|^2}{2}} \int_{H_-} e^{i\langle w, x\rangle_0} f(x)\ d\gamma_2(x)$.

Proposition 1.4.7. *Let $f(x) = F(\langle x, e_1\rangle_0, \dots \langle x, e_n\rangle_0)$ be a cylindrical function, where $e_1, \dots e_n \in H_+$ are mutually orthogonal in H_0 and $F \in L^2(\mathbb{R}^n,\ \gamma_2)$. Moreover, let P by orthogonal projection in H_0 onto $span\{e_1, \dots, e_n\}$ extended by continuity to H_-. Then Wf is also a cylindrical function*

$$Wf(w) = Wf(Pw) = e^{\frac{\|Pw\|^2}{2}} \int_{H_-} e^{-i\langle Pw, x\rangle_0} f(x)\ d\gamma_2(x).$$

Proof. See [**35**, page 72, Proposition 5.1]. □

Theorem 1.4.8. *The Fourier-Wiener-transform can be extended as a unitary operator Wf to $L^2(H_-,\ \gamma_1)$, where γ_1 is the canonical Gaussian measure in our Hilbert space rigging.*

Proof. See [**35**, page 73, Theorem 5.1]. □

Proposition 1.4.9. *The Fourier-Wiener-transform coincides with the transformation $\mathcal{F}$ defined in Proposition 1.4.4.*

Proof. For $\varphi \in H_+$ let U_φ be defined as in 1.3.2. Then $U_\varphi 1(x) = e^{-\frac{1}{2}\langle\varphi,\varphi\rangle_0 - \langle\varphi,x\rangle_0}$ is a cylindrical function. Let P_φ be the orthogonal projector onto $span\{\varphi\}$ in H_0 extended by continuity to H_-. Then we get

$$\begin{aligned} WU_\varphi 1(y) &= e^{\frac{\|P_\varphi y\|_0^2}{2}} \int_{H_-} e^{-i\langle x, P_\varphi y\rangle_0} e^{-\frac{\|\varphi\|_0^2}{2} - \langle\varphi,x\rangle_0}\, d\gamma_2(x) \\ &= e^{\frac{\|P_\varphi y\|_0^2}{2}} e^{-\frac{\|\varphi\|_0^2}{2}} \int_{H_-} e^{\langle -iP_\varphi y - \varphi, x\rangle_0}\, d\gamma_2(x) \\ &\overset{(9)}{=} e^{i\langle\varphi,y\rangle_0} = \mathcal{F}U_\varphi 1(y). \end{aligned}$$

But this is our assertion, since $span\{U_\varphi 1 \mid \varphi \in H_+\}$ is dense in $L^2(H_-,\ \gamma_1)$. □

Proposition and Definition 1.4.10. *For $u \in L^2(\mathbb{R}^n, \gamma)$ we define*

$$V_{G,n}u(x) := \pi^{-n/4} e^{-\frac{\|x\|^2}{2}} u(x).$$

Then $V_{G,n}$ is an isomorphism between $L^2(\mathbb{R}^n, \gamma)$ and $L^2(\mathbb{R}^n, \lambda)$ with inverse

$$V_{G,n}^{-1}u(x) := \pi^{n/4} e^{\frac{\|x\|^2}{2}} u(x).$$

Let $\tilde{\mathcal{F}}$ denote the Fourier-Transform on $L^2(\mathbb{R}^n, \lambda)$ given by

$$\tilde{\mathcal{F}}f(x) := (2\pi)^{-n/2} \int_{R^n} e^{-i\langle x\,,\,y\rangle} f(y) dy.$$

Then we have for all $u \in L^2(\mathbb{R}^n, \gamma)$ such that $V_{G,n}u \in L^1(\mathbb{R}^n, \lambda) \cap L^2(\mathbb{R}^n, \lambda)$

$$\mathcal{F}u(x) = [V_{G,n}^{-1}\tilde{\mathcal{F}}(V_{G,n}u)](x) = e^{\frac{\|x\|^2}{2}} \tilde{\mathcal{F}}(e^{-\frac{\|x\|^2}{2}} u)(x)$$

and thus

$$\mathcal{F}^{-1}u(x) = (V_{G,n}^{-1}\tilde{\mathcal{F}}^{-1}V_{G,n}u)(x) = e^{\frac{\|x\|^2}{2}} \tilde{\mathcal{F}^{-1}}(e^{-\frac{\|x\|^2}{2}} u)(x).$$

Proof. See [**84**, Example 13.5]. □

CHAPTER 2

Laplace operators in infinite dimensional spaces

In the classical finite dimensional theory the Laplace operator is given by $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x^2}$ and can be extended to a selfadjoint operator on $L^2(\mathbb{R}^n, \lambda)$. When trying to generalize this to the infinite dimensional theory several problems occur for example, there is no Lebesgue measure on an infinite dimensional Hilbert space. Even worse, there exists no measure on an infinite dimensional Hilbert space for which all shifts are admissible, i.e. there always exists a shift such that the shifted measure is not absolutely continuous with regard to the original one. Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis in an infinite dimensional Hilbert space. Then the operator $f \longmapsto \sum_{k=1}^{\infty} \frac{\partial^2}{\partial x^2} f$ does not necessarily convergence, even if f is bounded, twice continuous differentiable and (e_k) is an orthonormal basis in H_-. In this chapter we will consider two possible ways to solve this problems. In the first part we consider the Ornstein-Uhlenbeck operator which occurs naturally in stochastic as a kind of Laplacian (cf [**105**]). But as we show in Chapter 6 we are not able to find a symbol for this operator. In the second part we consider a slightly different way, i.e. we consider negative definite functions a symbols for a generalized Laplacian. As these operators are generators of L^2_γ-sub Markovian semi groups resp. L^2_γ-Dirichlet-forms it is also quite natural to use them as a replacement for the finite dimensional Laplace operator.

2.1. The Ornstein-Uhlenbeck operator as Laplacian

In this section it is our aim to define a first Laplace operator in $L^2(H_-,\ \gamma)$. In the finite dimensional case the Laplace operator is defined as sum of the second partial derivatives. Unfortunately, there is no Lebesgue measure in infinite dimensional space. Thus we have to consider a slightly modified operator to achieve selfadjointness of the Laplace operator. The same problem occurs in the case of a Gaussian measure in the finite dimensional case. Further on we discuss the problem of essential selfadjointness of this operator. In the last part of this section we show that the Laplace operator coincides with the well known Ornstein-Uhlenbeck operator. Moreover, we describe a domain of essential selfadjointness for all positive powers of the Laplace operator. We construct a Laplace operator in the case of infinite dimensional spaces. To guaranty the closability of the Dirichlet-form we have to realize something like 'integration by parts'. Some of the most important properties of this Laplace operator are discussed in [**18**]. Thus we follow [**18**, Chapter 6] to introduce the Laplace operator.

Let $H_+ \subseteq H_0 \subseteq H_-$ be a quasi-nuclear Hilbert space rigging and γ denote the canonical measure with respect to this rigging.

NOTATIONS 2.1.1.

(i) For $f \in \mathscr{C}^1(H_-, \mathbb{C})$ let $df(\cdot)$ be the Fréchet-derivative of f. Then ∇f denotes the realization of the Fréchet-derivative with regard to the inner product in H_0, i.e. for $h \in H_{-,\mathbb{C}}$ we have

$$df(\cdot)(h) = \langle \nabla f(\cdot), h \rangle_0, \quad \nabla f(\cdot) \in H_{+,\mathbb{C}}.$$

Sometimes we will write f' instead of ∇f.

(ii) Furthermore, for $f \in \mathscr{C}^2(H_-, \mathbb{C})$ let $d^2 f(\cdot)$ be the second derivative of f. Then f'' denotes the realization of $d^2 f$ with regard to the inner product in H_0, i.e. for $h, k \in H_{-,\mathbb{C}}$ we have

$$df(\cdot)(h,k) = \langle f''(\cdot)h, k \rangle_0, \quad f''(\cdot) \in \mathscr{L}(H_{-,\mathbb{C}}, H_{+,\mathbb{C}}).$$

DEFINITION 2.1.2. For $f,\ g \in \mathscr{C}^2_{b,cyl}(H_-)$ we set

$$d_\gamma(f,g) = \frac{1}{2} \int\limits_{H_-} \langle \nabla f, \nabla g \rangle_0 d\gamma$$

and for $f,\ g \in \mathscr{C}^2_b(H_-)$[1]

$$L_\gamma f = -\frac{1}{2}(\mathrm{tr}_0 d^2 f + 2\langle \nabla f, \cdot \rangle_0).$$

PROPOSITION 2.1.3. *Let $f,\ g \in \mathscr{C}^2_{b,cyl}(H_-)$. Then we have*

$$d_\gamma(f,g) = \frac{1}{2} \int\limits_{H_-} \langle \nabla f, \nabla g \rangle_0 d\gamma = \langle L_\gamma f, g \rangle = \langle f, L_\gamma g \rangle.^2$$

PROOF. See [**18**, p. 253 Theorem 3.1] □

LEMMA 2.1.4. *d_γ is non-negative and closable.*

PROOF. We only have to prove that d_γ is closable. Let $f_n \in \mathscr{C}^2_{b,cyl}(H_-)$ $n \in \mathbb{N}$ with $\|f_n\|_{L^2(H_-,\,\gamma)} \xrightarrow{n\to\infty} 0$. Then we have for $g \in \mathscr{C}^2_{b,cyl}(H_-)$

$$|d_\gamma(f_n, g)| = |\langle \mathrm{L}_\gamma f_n, g \rangle| = |\langle f_n,\ \mathrm{L}_\gamma g \rangle| \leq \|f_n\|_{L^2(H_-,\,\gamma)} \|\mathrm{L}_\gamma g\|_{L^2(H_-,\,\gamma)} \xrightarrow{n\to\infty} 0.$$

Hence d_γ is closable. □

[1] cf. also [**21**, page 72, Corollary 1.2.4].

[2] see also [**111**, page 62 Proposition 1.5.1]

PROPOSITION AND DEFINITION 2.1.5. *According to 2.1.4* d_γ *is closable and non-negative on* $\mathscr{C}^2_{b,cyl}(H_-)$. *Thus there exists a minimal closed extension of* d_γ. *We denote this extension by* d_γ *again and its domain of definition by* $D(d_\gamma)$. *According to 2.1.4* d_γ *is a non-negative closable form. Thus we have*

$$d_\gamma(f,g) = \langle \sqrt{\mathrm{L}^{Fr}_\gamma f}, \sqrt{\mathrm{L}^{Fr}_\gamma f}\ g \rangle \quad \forall f,\ g \in D(d_\gamma),$$

where L^{Fr}_γ *is the Friedrich's-extension*[3] *of* $\mathrm{L}_\gamma|_{\mathscr{C}^2_{b,cyl}}$ *and thus selfadjoint in* $L^2(H_-,\ \gamma)$.

LEMMA 2.1.6. *For* $f \in \mathscr{C}^2_b(H_-)$ *there exists a sequence* $(f_n)_{n=1}^\infty \subset \mathscr{C}^2_{b,cyl}(H_-)$ *such that* $f_n \xrightarrow[n\to\infty]{L^2(H_-,\,\gamma)} f$ *and*

$$\lim_{n\to\infty} \mathrm{L}_\gamma f_n = \mathrm{L}_\gamma f \in L^2(H_-,\ \gamma)$$

Thus for $f \in \mathscr{C}^2_b(H_-)$ *we have* $\mathrm{L}^{Fr}_\gamma f = \mathrm{L}_\gamma f$ *and we write* L_γ *instead of* L^{Fr}_γ.

PROOF. See [**3**, Lemma 6]. □

LEMMA 2.1.7. *The closure of* $(\mathrm{L}_\gamma, \mathscr{C}^\infty_{b,cyl}(H_-))$ *coincides with the closure of* $(\mathrm{L}_\gamma, \mathscr{C}^2_b(H_-))$ *and the closure of* $(\mathrm{L}_\gamma, \mathscr{C}^\infty_{0,cyl}(H_-))$.

PROOF. This Corollary follows by Lemma 2.1.6 and well known finite dimensional approximations. □

THEOREM 2.1.8. *The space* $\mathscr{C}^2_b(H_-)$ *is a domain of essential selfadjointness for* L_γ. *Thus* $\mathscr{C}^\infty_{b,cyl}(H_-)$ *and* $\mathscr{C}^\infty_{0,cyl}(H_-)$ *are domains of essential selfadjointness for the operator* L_γ.

PROOF. The first part is proved in [**18**, p. 275 Theorem 3.4]. Thus the second part follows by Lemma 2.1.7. □

REMARK 2.1.9. The Definition 2.1.2 can be formulated for $f, g \in \mathscr{C}^2_{pol,cyl}(H_-)$ resp. $f \in \mathscr{C}^2_{pol}(H_-)$ instead of $\mathscr{C}^2_{b,cyl}(H_-)$ resp. $\mathscr{C}^2_b(H_-)$ and 2.1.3 - 2.1.4 remain valid for $f, g \in \mathscr{C}^2_{pol,cyl}(H_-)$. Since d_γ is positive and closable, we obtain a Friedrichs-extension of $\mathrm{L}_{\gamma_1}|_{\mathscr{C}^2_{pol,cyl}(H_-)}$. For this selfadjoint extension we write L^{pol}_γ. Since L^{pol}_γ coincides with L_{γ_1} on $\mathscr{C}^2_{b,cyl}(H_-)$ and since $\mathscr{C}^2_{b,cyl}(H_-)$ is a domain of essential selfadjointness of L_{γ_1}, we obtain $\mathrm{L}^{pol}_\gamma = \mathrm{L}_{\gamma_1}$. Furthermore, Lemma 2.1.6 remains valid for $f \in \mathscr{C}^2_{pol}(H_-)$ and $f_n \in \mathscr{C}^2_{pol,cyl}(H_-)$. Thus as in Theorem 2.1.8 $\mathscr{C}^\infty_{pol,cyl}(H_-)$ and $\mathscr{C}^\infty_{pol}(H_-)$ are domains of essential selfadjointness of L_{γ_1}.

Next let us show that the Laplace operator defined above coincides with the so called Ornstein-Uhlenbeck operator. Moreover, we prove that the generalized Hermite polynomials are eigenvectors of the Laplace operator and that the span of the generalized Hermit polynomials is a domain of all essential selfadjointness of all positive powers of $\mathrm{L}_{\gamma_1} + \mathrm{id}$.

[3] cf. for example [**103**, page 22-23]

Lemma 2.1.10. *Let* $(e_j)_{j=1}^n \subset H_+$ *be an orthonormal basis in* $H_0 = H_{\mathrm{id}}$. *Furthermore, let* h_α *be defined as in 1.1.27. Then* h_α *is an eigenvector of* L_{γ_1} *with eigenvalue* $|\alpha|$, *i.e.*

$$\mathrm{L}_{\gamma_1} h_\alpha = |\alpha|\, h_\alpha.$$

Proof. Let $\alpha = (0, \cdots, 0, n, 0, \cdots)$, where the is in the k-th place, i.e. $h_\alpha = h_n(\langle e_k, x\rangle_0) = c_n H_n(\langle e_k, x\rangle_0)$. Then for n=0 we have $\mathrm{L}_{\gamma_1} 1 = 0$ and for n=1

$$\mathrm{L}_{\gamma_1} H_1(\langle e_k, x\rangle_0) = \langle e_k, x\rangle_0 \frac{\partial}{\partial x_k} H_1(x) = 2\langle e_k, x\rangle_0 = H_1(x).$$

Moreover, according to 1.1.26 we obtain for n>2

$$\begin{aligned}
&\mathrm{L}_{\gamma_1} H_n(\langle e_k, x\rangle_0)\\
&= -\frac{1}{2}\left(\frac{\partial^2}{\partial x_k^2} H_n(\langle e_k, x\rangle_0) - 2\langle e_k, x\rangle_0)\frac{\partial}{\partial x_k} H_n(\langle e_k, x\rangle_0)\right)\\
&= -\frac{1}{2}\,(4n(n-1)H_{n-2}(\langle e_k, x\rangle_0) - 4n\langle e_k, x\rangle_0 H_{n-1}(\langle e_k, x\rangle_0))\\
&= -\frac{1}{2}(4n(n-1)H_{n-2}(\langle e_k, x\rangle_0) - 4n(\frac{1}{2}H_n(\langle e_k, x\rangle_0) + (n-1)H_{n-2}(\langle e_k, x\rangle_0)))\\
&= nH_n(\langle e_k, x\rangle_0).
\end{aligned}$$

Let $\alpha \in \mathbb{N}_0^N$ arbitrary with $|\alpha| = n$. Then we have

$$h_\alpha = h_{\alpha_1}(\langle e_1, x\rangle_0)\cdots h_{\alpha_\nu}(\langle e_\nu, x\rangle_0)$$

and thus we obtain

$$\begin{aligned}
&\mathrm{L}_{\gamma_1} h_{\alpha_1}(\langle e_1, x\rangle_0)\cdots h_{\alpha_\nu}(\langle e_\nu, x\rangle_0)\\
=\;& -\frac{1}{2}\sum_{k=1}^{\nu}\left(\frac{\partial^2}{\partial x_k^2} - 2\langle e_k, x\rangle_0)\frac{\partial}{\partial x_k}\right) h_{\alpha_1}(\langle e_1, x\rangle_0)\cdots h_{\alpha_\nu}(\langle e_\nu, x\rangle_0)\\
=\;& \sum_{k=1}^{\nu} \alpha_k h_{\alpha_1}(\langle e_1, x\rangle_0)\cdots h_{\alpha_\nu}(\langle e_\nu, x\rangle_0)\\
=\;& |\alpha|\, h_{\alpha_1}(\langle e_1, x\rangle_0)\cdots h_{\alpha_\nu}(\langle e_\nu, x\rangle_0).
\end{aligned}$$

□

Definition 2.1.11 (Ornstein-Uhlenbeck operator). Let

$$D = \left\{ f \in L^2(H_-,\ \gamma_1) \mid \sum_{n=0}^{\infty} n^2 \left\|P_{\Gamma_n}(f)\right\|^2 < \infty \right\},$$

where P_{Γ_n} denotes the orthogonal projection on the closed linear span of h_α with $|\alpha| = n$. Then we define $\mathrm{L} : D \longrightarrow L^2(H_-,\ \gamma_1)$ by

$$\mathrm{L} = \sum_{n=0}^{\infty} n\ P_{\Gamma_n}$$

L is called Ornstein-Uhlenbeck operator[4].

PROPOSITION 2.1.12. *$span\{h_\alpha \,|\, \alpha \in \mathbb{N}_0^{\mathbb{N}}\}$ is a domain of essential selfadjointness for L^s and $(\mathrm{L} + \mathrm{id})^s$ for all $s > 0$.*

PROOF. The P_{Γ_n} are orthogonal projections with $\sum_{k=0}^{\infty} P_{\Gamma_n}(f) = f$. Thus the spectral theorem for unbounded operators implies that

$$D = \left\{ f \in L^2(H_-,\ \gamma_1) \,|\, \sum_{n=0}^{\infty} n^2 \left\| P_{\Gamma_n}(f) \right\|^2_{L^2(H_-,\ \gamma_1)} < \infty \right\}$$

is a domain of essential selfadjointness for

$$\mathrm{L} = \sum_{n=0}^{\infty} n \; P_{\Gamma_n}$$

resp. $L + \mathrm{id}$ and

$$D(\mathrm{L}^s) = \left\{ f \in L^2(H_-,\ \gamma_1) \,|\, \sum_{n=0}^{\infty} n^{2s} \left\| P_{\Gamma_n}(f) \right\|^2_{L^2(H_-,\ \gamma_1)} < \infty \right\}.$$

is domain of essential selfadjointness for

$$\mathrm{L}^s = \sum_{n=0}^{\infty} n^s \; P_{\Gamma_n}$$

resp. $(\mathrm{L} + \mathrm{id})^s$. Let $f \in D(\mathrm{L}^s)$ arbitrary and $j \in \mathbb{N}$. Since $\sum_{n=0}^{\infty} P_{\Gamma_n}(f) = f$, there exists a $n_1 > 0$, such that for all $k > n_1$ we have

$$\left\| \sum_{n=0}^{k} P_{\Gamma_n}(f) - f \right\|_{L^2(H_-,\ \gamma_1)} \le \frac{1}{2j}.$$

Furthermore, $f \in D(L^s)$ implies, that there exists $n_0 > n_1$ with

$$\sum_{n=n_0+1}^{\infty} n^{2s} \left\| P_{\Gamma_n}(f) \right\|^2_{L^2(H_-,\ \gamma_1)} < \frac{1}{2j}.$$

Due to the fact that $span\{h_\alpha \,|\, \alpha \in \mathbb{N}_0^{\mathbb{N}},\ |\alpha| = n\} = P_{\Gamma_n}(span\{h_\alpha \,|\, \alpha \in \mathbb{N}_0^{\mathbb{N}}\}) \subset P_{\Gamma_n}(L^2(H_-,\ \gamma_1))$ dense, there exists a $f_j \in span\{h_\alpha \,|\, \alpha \in \mathbb{N}_0^{\mathbb{N}},\ |\alpha| < n_0\}$ with

$$\left\| f_j - \sum_{n=0}^{n_0} P_{\Gamma_n}(f) \right\|_{L^2(H_-,\ \gamma_1)} \le \frac{1}{2j}$$

and

$$\left\| \sum_{n=0}^{n_0} n^{2s} P_{\Gamma_n}(f - f_j) \right\|_{L^2(H_-,\ \gamma_1)} \le \frac{1}{2j}.$$

[4] cf. also Bouleau, Hirsch [**21**, Proposition 1.2.7], Nualart [**111**, page 53] and Malliavin [**105**, page 10]

Overall, we have $\|f-f_j\| \leq \frac{1}{j}$ and $\|\mathrm{L}f-\mathrm{L}f_j\| \leq \frac{1}{j}$. This shows our assertion. □

PROPOSITION 2.1.13.

$$\mathrm{L} = \mathrm{L}_{\gamma_1}$$

PROOF. According to 2.1.5, L_{γ_1} is selfadjoint and due to 2.1.9 and 2.1.10 we have $\mathrm{L}f = \mathrm{L}_{\gamma_1}f$ for all $f \in span\{h_\alpha \,|\, \alpha \in \mathbb{N}_0^{\mathbb{N}}\}$. Since $span\{h_\alpha \,|\, \alpha \in \mathbb{N}_0^{\mathbb{N}}\}$ is a domain of essential selfadjointness of L and L_{γ_1} is a selfadjoint extension of $L|_{span\{h_\alpha \,|\, \alpha\in\mathbb{N}_0^{\mathbb{N}}\}}$, we obtain that $\mathrm{L} = \mathrm{L}_{\gamma_1}$. □

COROLLARY 2.1.14. *$\mathscr{C}_{pol}^\infty(H_-)$ is a domain of essential selfadjointness of $L_{\gamma_1}^s$ for all $s \in \mathbb{R}$ and L_{γ_1} leaves the space $\mathscr{C}_{pol}^\infty(H_-)$ invariant .*

PROOF. Let $f \in \mathscr{C}_{pol}^\infty(H_-)$. Our first step is to show by induction that for $k \in \mathbb{N}_0$ the following equation holds.

$$d^k(tr_{H_0}d^2f(x))(y_1,\dots,y_k) = tr_{H_0}d^2(d^kf(x)(y_1,\dots,y_k)), \tag{10}$$

where $y_1 \dots y_k \in H_-$ arbitrary. For $k=0$ this is clear. Therefore let the assumption be true for fixed $k \in \mathbb{N}_0$ and let $y_1 \dots y_{k+1} \in H_-$. Then the induction hypothesis implies

$$d^k(tr_{H_0}d^2f(x)(y_1,\dots,y_k) = tr_{H_0}d^2(d^kf(x)(y_1,\dots,y_k)).$$

Thus we obtain

$$\begin{aligned} d^{k+1}(tr_{H_0}d^2f(x)(y_1,\dots,y_{k+1}) &= d(d^k(tr_{H_0}d^2f(x)(y_1,\dots,y_k))(y_{k+1}) \\ &= d(tr_{H_0}d^2(d^kf(x)(y_1,\dots,y_k)))(y_{k+1}) \\ &= \frac{\partial}{\partial t}\sum_{n=1}^\infty g_n(t)\bigg|_{t=0}, \end{aligned}$$

where $g_n(t) = d^{k+2}f(x+ty_{k+1})(e_n,e_n,y_1,\dots,y_k)$. However, we have

$$\begin{aligned} g_n'(t) &= \frac{\partial}{\partial y_{k+1}}d^{k+2}f(x+ty_{k+1})(e_n,e_n,y_1,\dots,y_k) \\ &= d^{k+3}f(x+ty_{k+1})(e_n,e_n,y_1,\dots,y_{k+1}). \end{aligned}$$

Since $t \longrightarrow d^{k+3}f(x+ty_{k+1})$ continuous by assumption and $[-1,1]$ is compact, there exists a $c>0$ such that for all $t \in [-1,1]$

$$\sum_{n=1}^\infty |g_n'(t)| \leq \sum_{n=1}^\infty \left\|d^{k+3}f(x+ty_{k+1})\right\|_{Op} \|e_k\|_-^2 \|y_1\|_- \cdots \|y_{k+1}\|_- \leq c.$$

Hence it follows that

$$\begin{aligned} \frac{\partial}{\partial t}\sum_{n=1}^\infty g_n(t)\bigg|_{t=0} = \sum_{n=1}^\infty g_n'(0) = \sum_{n=1}^\infty g_n'(0) &= \sum_{n=1}^\infty d^{k+3}f(x)(e_n,e_n,y_1,\dots,y_{k+1}) \\ &= tr_{H_0}d^2(d^{k+1}f(x)(y_1,\dots,y_{k+1})). \end{aligned}$$

Altogether, we obtain

$$\left|d^k(tr_{H_0}d^2 f(x)(y_1,\dots,y_k)\right| \le c' \left\|d^{k+2}f(x)\right\|_{Op} \|y_1\|_- \cdots \|y_k\|_- ,$$

where $c' > 0$. Thus $tr_{H_0}d^2$ leaves the space $\mathscr{C}^\infty_{pol}(H_-)$ invariant. Moreover, we have $\langle f'(x), \beta_\gamma(x)\rangle = df(-2x)$ and hence L_{γ_1} leaves the space $\mathscr{C}^\infty_{pol}(H_-)$ invariant. Further on this shows that $\mathscr{C}^\infty_{pol}(H_-) \subset D((\mathrm{L}_{\gamma_1}+\mathrm{id})^k)$ for all $k \in \mathbb{N}$. The combination of the above and Proposition 2.1.12 yields $\mathscr{C}^\infty_{pol}(H_-)$ is a domain of essential selfadjointness of $(\mathrm{L}_{\gamma_1}+\mathrm{id})^s$ for all $s \in \mathbb{R}$. □

2.2. Infinite dimensional Laplace operators with negative definite functions as symbols

A function $\psi:\ \mathbb{R}^n \longrightarrow \mathbb{C}$ is called negative definite if $\psi(0) \ge 0$ and $e^{-t\psi}$ is a positive definite function for all $t > 0$. In the classical finite dimensional case according to **[80]** every negative definite functions gives raise to a pseudodifferential operator $\psi(D)$.

The closure $-A$ of $-\psi(D)$ is a Dirichlet operator and generates a strongly continuous contraction sub Markovian semi group. Furthermore, if ψ is real-valued, a symmetric Dirichlet form is defined by the closure of $\langle Au\,,\,u\rangle$ for $u \in D(A)$. Conversely, pseudodifferential operators with negative definite functions as symbols arise naturally as generators of Feller Groups and Dirichlet-forms (cf. **[6]** **[78]** **[80]**, **[81]**, **[82]**).

In this section we will replace $\mathbb{R}^n$ by an infinite dimensional Hilbert space. At first we prove that some well know facts about negative definite functions remain valid if we replace $\mathbb{R}^n$ by a general Hilbert Space H_- e.g. Petree's inequality and the fact that $|\psi(\xi)| \le c_\psi(1+\psi(\xi)^2)$. Now we are able to define a pseudodifferential operator attached to a negative definite symbol ψ by

$$\psi(D)u := \mathcal{F}^{-1}\psi(\xi)\mathcal{F}u,$$

where $\mathcal{F}$ denotes the Fourier-Wiener-transform. It turns out the some extension of the operator $-\psi(D)$ generates a semi group $(T_t)_{t>0}$.

Again let $H_+ \subset H_0 \subset H_-$ denote a quasi-nuclear Hilbert space rigging.

Definition 2.2.1. A function $\psi : H_- \longrightarrow \mathbb{C}$ belongs to the class $N(H_-)$ if for any choice of $k \in \mathbb{N}$ and vectors $\xi^1,\dots,\xi^k \in H_-$ the matrix

$$(\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l))_{j,l=1,\dots,k}$$

is positive Hermitian. Further we set $CN(H_-) := N(H_-) \cap C(H_-)$.

At first let us note some basic facts about negative definite functions.

Proposition 2.2.2. (i) *For $\psi \in N(H_-)$ we have $\psi(0) \ge 0$, $\psi(\xi) = \overline{\psi(-\xi)}$, $\mathfrak{Re}\,\psi(\xi) \ge \psi(0)$.*

(ii) *The set $N(H_-)$ is a convex cone which is closed under point wise convergence.*

(iii) *For* $\psi \in N(H_-)$, $\overline{\psi}$ *and* $\mathfrak{Re}\, \psi$ *belong to* $N(H_-)$.
(iv) *Any non-negative constant is an element of* $N(H_-)$.
(v) *For* $\psi \in N(H_-)$ *and* $\lambda > 0$ *the function* $\xi \longmapsto \psi(\lambda \xi)$ *belongs to* $N(H_-)$.
(vi) *We have* $\psi \in N(H_-)$ *if and only if*
  (a) $\psi(0) \geq 0$
  (b) $\psi(\xi) = \overline{\psi(-\xi)}$
  (c) *for any* $k \in \mathbb{N}$ *and any choice of vectors* $\xi^1, \dots, \xi^k \in H_-$ *and complex numbers* $c_1, \dots c_k$ *with* $\sum_{j=1}^k c_j = 0$ *we have* $\sum_{j,l=1}^k \psi(\xi^j - \xi^l) c_j \overline{c_l} \leq 0$.
(vii) *For* $\psi \in N(H_-)$ *the function* $\xi \longmapsto \psi(\xi) - \psi(0)$ *belongs also to* $N(H_-)$.
(viii) *Let* $u : H_- \longrightarrow \mathbb{C}$ *be a positive definite function. Then the function* $\xi \longmapsto u(0) - u(\xi)$ *is an element of* $N(H_-)$.
(ix) *A function* ψ *is an element of* $N(H_-)$ *if and only if* ψ *is negative definite in the sense that*
  (a) $\psi(0) \geq 0$
  (b) $\xi \longmapsto e^{-t\psi(\xi)}$ *is positive definite for* $t \geq 0$.
(x) *Let* $\psi \in N(H_-)$. *Then* $\frac{\psi}{\alpha + \beta\psi} \in N(H_-)$ *for all* $\alpha > 0$ *and* $\beta \geq 0$.
(xi) *For* $\psi \in N(H_-)$ *and* $\xi, \eta \in H_-$ *we have*
  (a) $\sqrt{|\psi(\xi + \eta)|} \leq \sqrt{|\psi(\xi)|} + \sqrt{|\psi(\eta)|}$
  (b) $\left| \sqrt{|\psi(\xi))|} - \sqrt{|\psi(\eta)|} \right| \leq \sqrt{|\psi(\xi - \eta)|}$
  (c) $|\psi(\xi) + \psi(\eta) - \psi(\xi - \eta)| \leq 2(\mathfrak{Re}\, \psi(\xi))^{1/2} (\mathfrak{Re}\, \psi(\eta))^{1/2}$
  (d) $\frac{1 + |\psi(\xi)|}{1 + |\psi(\eta)|} \leq 2(1 + |\psi(\xi - \eta))|)$
  (e) $1 + |\psi(\xi \pm \eta)| \leq (1 + |\psi(\xi)|)(1 + \sqrt{|\psi(\eta)|})^2$
(xii) *Let* $\psi \in N(H_-)$ *be continuous at* 0. *Then* $\psi \in CN(H_-)$.

Proof. The proof of this proposition can be found in [**80**, page 122-136] by writing H_- instead of $\mathbb{R}^n$ in the corresponding propositions. A complete proof of this proposition is also given in appendix A1. □

Proposition 2.2.3. *Let* $\psi \in N(H_-)$. *Moreover, we assume that there exists* $\varepsilon > 0$ *and a constant* $C > 0$ *such that* $|\psi(\xi)| \leq C$ *for all* $\xi \in B_\varepsilon(0)$. *Then there exist a constant* c_ψ *such that*

$$|\psi(\xi)| \leq c_\psi (1 + \|\xi\|_-^2).$$

Proof. Since ψ is bounded in $B_\varepsilon(0)$, it is sufficient to show that $|\psi(\xi)| \leq c' \|\xi\|_-^2$ for all $\xi \in H_- \setminus B_{\frac{1}{k}(0)}$, where $k \in \mathbb{N}$ is chosen such that $\frac{1}{k} \leq \frac{\varepsilon}{2}$. By Proposition 2.2.2(xi) we have $\psi(m\eta) \leq m^2 \psi(\eta)$ for all $\eta \in H_-$. Now let $\|\xi\|_- \geq \frac{1}{k}$. Then there exists $m_0 \in \mathbb{N}$ such that $\|\xi\|_- \in [\frac{m_0}{k}, \frac{m_0+1}{k})$. We obtain

$$\frac{1}{k} \leq \frac{\|\xi\|_-}{m_0} \leq \frac{m_0 + 1}{m_0 k} = (1 + \frac{1}{m_0}) \frac{1}{k} \leq \frac{2}{k} \leq \varepsilon$$

and thus

$$|\psi(\xi)| = \left|\psi(\frac{m_0}{m_0}\xi)\right| \le m_0^2 \left|\psi(\frac{\xi}{m_0})\right| \le Cm_0^2 \le Ck^2 \|\xi\|_-^2 . \qquad \square$$

Corollary 2.2.4. *For $\psi \in CN(H_-)$ there exists a constant c_ψ such that*

$$|\psi(\xi)| \le c_\psi(1 + \|\xi\|_-^2).$$

Proof. The continuity of ψ implies that there exists $\varepsilon > 0$ and $C > 0$ such that $|\psi(\xi)| \le C$ for all $\|\xi\|_- \le C$ and thus the assertion follows by Proposition 2.2.3. □

Definition 2.2.5. (i) Let us denote by $BN(H_-)$ the set of all functions $\psi \in N(H_-)$ for which there exists an $\varepsilon > 0$ and a $C > 0$ such that $|\psi(\xi)| \le C$ for all $\xi \in B_\varepsilon(0)$.
(ii) We say that a function ψ is a (continuous) negative definite function on H_- if $\psi \in N(H_-)(CN(H_-))$.

Example 2.2.6. Let us give some examples of functions in $N(H_-)$.
(i) Let $d \in H_+$. Then $(\xi \longmapsto i\langle d\,,\,\xi\rangle_0) \in CN(H_-)$.
(ii) Let $A \in \mathscr{L}(H_-, H_+)$ Then the mapping $\xi \longmapsto \langle A\xi\,,\,\xi\rangle$ belongs to $CN(H_-)$.
(iii) Let $x \in H_+$. Then we find that $\xi \longmapsto (1 - e^{i\langle x\,,\,\xi\rangle_0})$ is also an element of $CN(H_-)$.

Proof. The proof is similarly to [**80**, Example 3.6.18] and [**80**, Example 3.6.19]. □

Example 2.2.7. Let $(\mu_t)_{t\ge0}$ be a convolution semi group on H_+, e.g. for all $t \ge 0$ μ_t is a bounded Borel measure on H_+ with $\mu_t(H_+) \le 1$, $\mu_s * \mu_t = \mu_{s+t}$ and $\mu_t \longrightarrow \varepsilon_0$ vaguely as $t \longrightarrow 0$. Then there exists a negative definite function $\psi : H_- \longrightarrow \mathbb{C}$ such that $\hat{\mu_t}(\xi) = e^{-t\psi(\xi)}$ for all $\xi \in H_-$, where $\hat{\mu_t}$ denotes the Fourier-Transform of μ_t.

Proof. First let us note that the Fourier-Transform of a measure on H_+ is defined on $(H_+)' = H_-$. Now the rest of the proof is similar to [**80**, Theorem 3.6.4] □

Definition 2.2.8. We call $\psi : \; H_- \longrightarrow \mathbb{C}$ a negative definite function in Levi-Khinchin-Form with respect to the Hilbert space rigging $H_+ \subset H_0 \subset H_-$ if

$$\begin{aligned}\psi(\xi) = c \;&+\; i\langle d\,,\,\xi\rangle_0 + \langle A\xi\,,\,\xi\rangle_0 \\ &+ \int\limits_{H_+\setminus\{0\}} \left(1 - e^{-i\langle x\,,\,\xi\rangle_0} - \frac{i\langle x\,,\,\xi\rangle_0}{1+\|x\|^2}\right) \frac{1+\|x\|_+}{\|x\|_+^2}\mu(dx),\end{aligned}$$

where $c \ge 0$ is a positive constant, $d \in H_+$, $A \in \mathscr{L}(H_-, H_+)$, such that $\langle A\xi\,,\,\xi\rangle_0 \ge 0$ for all $\xi \in H_-$ and μ is a bounded Borel measure in H_+. We will denote by ν the measure given by

$$\nu(dx) = \frac{1+\|x\|_+}{\|x\|_+^2}\mu(dx). \tag{11}$$

LEMMA 2.2.9. *Let ψ be a negative definite function with respect to the Hilbert space rigging $H_+ \subset H_0 \subset H_-$ in Levi-Khinchin-Form. Then $\psi \in N(H_-)$.*

PROOF. Considering Example 2.2.6 this is obvious. □

LEMMA 2.2.10. *Let*

$$\psi(\xi) = \int\limits_{H_+\setminus\{0\}} \left(1 - e^{-i\langle x\,,\,\xi\rangle_0} - \frac{i\langle x\,,\,\xi\rangle_0}{1+\|x\|^2}\right)\nu(dx),$$

where ν is given by (11). Then we have

$$|\psi(\xi)| \le c(1+\|\xi\|_-^2).$$

PROOF. The idea of this proof can be found in [**80**, Theorem 3.7.7]. We have

$$\begin{aligned}
&\left|e^{-i\langle x\,,\,\xi\rangle_0} - 1 + \frac{i\langle x\,,\,\xi\rangle_0}{1+\|x\|_+^2}\right| \\
\le\ & \left|e^{-i\langle x\,,\,\xi\rangle_0} - 1 + i\langle x\,,\,\xi\rangle_0\right| + \left|i\langle x\,,\,\xi\rangle_0 - \frac{i\langle x\,,\,\xi\rangle_0}{1+\|x\|_+^2}\right| \\
\le\ & \frac{1}{2}\|x\|_+^2\|\xi\|_-^2 + \frac{\|x\|_+^2}{1+\|x\|_+^2}\|x\|_+\|\xi\|_-
\end{aligned}$$

and thus for $\|x\|_+ \le 1$

$$\begin{aligned}
\left|\left(e^{-i\langle x\,,\,\xi\rangle_0} - 1 + \frac{i\langle x\,,\,\xi\rangle_0}{1+\|x\|_+^2}\right)\frac{1+\|x\|_+^2}{\|x\|_+^2}\right| &\le \frac{1}{2}(1+\|x\|_+^2)\|\xi\|_-^2 + \|x\|_+\|\xi\|_- \\
&\le 2(1+\|\xi\|_-^2).
\end{aligned}$$

Moreover for $\|x\|_+ \ge 1$ we obtain

$$\left|\left(e^{-i\langle x\,,\,\xi\rangle_0} - 1 + \frac{i\langle x\,,\,\xi\rangle_0}{1+\|x\|_+^2}\right)\frac{1+\|x\|_+^2}{\|x\|_+^2}\right| \le 4 + \|\xi\|_- \le 4(1+\|\xi\|_-^2).$$

Since μ is a bounded Borel-measure it follows that

$$|\psi|(\xi) \le c(1+\|\xi\|_-^2)$$

where $c > 0$ is chosen suitable. □

COROLLARY 2.2.11. *For any negative definite Function ψ in Levi-Khinchin-Form with respect to the Hilbert space rigging $H_+ \subset H_0 \subset H_-$ there exists a constant $c > 0$ such that for all $\xi \in H_-$*

$$|\psi(\xi)| \leq c(1 + \|\xi\|_-^2).$$

COROLLARY 2.2.12. *Every negative definite Function ψ in Levi-Khinchin-Form with respect to the Hilbert space rigging $H_+ \subset H_0 \subset H_-$ is continuous.*

PROOF. According to 2.2.2(xii) we only have to show that ψ is continuous at 0. Moreover, we only have to check that the integral part of ψ is continuous. But this is clear by virtue of the proof of Lemma 2.2.10 and Lebesgue's theorem of dominated convergence. □

PROPOSITION 2.2.13. *Let $A \in \mathscr{L}(H_-, H_+)$ such that $\langle Ax\,,\,y\rangle_0 = \langle Ay\,,\,x\rangle_0$ for all $x,\ y \in H_-$ and $\langle Ax\,,\,x\rangle_0 \geq 0$. Then $A:\ H_0 \longrightarrow H_0$ is symmetric, non-negative and trace-class in H_0. Thus there exits an orthonormal basis $(f_j)_{j\in\mathbb{N}}$ in H_0 consisting of eigenvectors of A. For this eigenvectors we have $f_j \in H_+$ for all $j \in \mathbb{N}$. Moreover, we obtain*

(i) $\langle Ax\,,\,x\rangle = \sum_{j=1}^{\infty} \lambda_j \langle f_j\,,\,x\rangle_0^2$ *where λ_j denotes the eigenvalue of the eigenvector f_j.*

(ii) $\left|\frac{\partial}{\partial e_j}\langle Ax\,,\,x\rangle\right| \leq 2\sqrt{\lambda}\langle Ax\,,\,x\rangle_0$ *where λ denotes the largest eigenvalue of A.*

(iii) $\left|\frac{\partial}{\partial e_k}\frac{\partial}{\partial e_j}\langle Ax\,,\,x\rangle_0\right| \leq 2\,\|A\|_{\mathscr{L}(H_-,H_+)}$ *and all higher partial derivatives are 0.*

PROOF. Considering

$$A:\ H_0 \overset{H.-S.}{\hookrightarrow} H_- \xrightarrow{bounded} H_+ \overset{H.-S.}{\hookrightarrow} H_0$$

it follows that A is trace-class. Moreover, for all $x,\ y \in H_0$ we have $\langle Ax\,,\,y\rangle_0 = \langle Ay\,,\,x\rangle_0 = \langle x\,,\,Ay\rangle_0$. Thus A is symmetric. It is obvious that A is non-negative. Thus there exists an orthonormal basis in H_0 consisting of eigenvectors $(f_j)_{j\in\mathbb{N}}$ of A such that for the corresponding sequence of eigenvalues λ_j we have $\lambda_1 \geq \lambda_2 \geq \ldots$. Since $\lambda_j f_j = Af_j \in H_+$ it follows that $f_j \in H_+$. Now we obtain for $x \in H_-$

$$\begin{aligned}
\langle Ax\,,\,x\rangle &= \sum_{j,k=1}^{\infty} \langle Ae_j\,,\,e_k\rangle_0 \langle e_k\,,\,x\rangle_0 \langle e_j\,,\,x\rangle_0 \\
&= \sum_{j,k=1}^{\infty}\sum_{l=1}^{\infty} \lambda_l \langle f_l\,,\,e_k\rangle_0 \langle e_k\,,\,x\rangle_0 \langle f_l\,,\,e_j\rangle_0 \langle e_j\,,\,x\rangle_0 = \sum_{j=1}^{\infty} \lambda_j \langle f_j\,,\,x\rangle_0^2.
\end{aligned}$$

Furthermore we have

$$\left|\frac{\partial}{\partial e_j}\langle Ax\,,\,x\rangle_0\right|^2 = |\langle Ae_j\,,\,x\rangle_0 + \langle Ax\,,\,e_j\rangle_0|^2 = 4\,|\langle Ax\,,\,e_j\rangle_0|^2$$

$$\begin{aligned}
&= 4\left|\sum_{k=1}^{\infty}\lambda_k\langle f_k\,,\,x\rangle_0\langle f_k\,,\,e_j\rangle_0\right|^2\\
&\le 4\left(\sum_{k=1}^{\infty}|\lambda_k\langle f_k\,,\,x\rangle_0\langle f_k\,,\,e_j\rangle_0|\right)^2\\
&\le 4\left(\sum_{k=1}^{\infty}|\lambda_k\langle f_k\,,\,x\rangle_0|\right)^2 = 4\left(\sum_{k=1}^{\infty}|\langle Ax\,,\,f_k\rangle_0|\right)^2\\
&= 4\sum_{k=1}^{\infty}|\langle Ax\,,\,f_k\rangle_0|^2 = 4\sum_{k=1}^{\infty}\lambda_k^2\langle f_k\,,\,x\rangle_0^2\\
&\le 4\lambda_1\sum_{k=1}^{\infty}\lambda_k\langle f_k\,,\,x\rangle_0^2 = 4\lambda_1\langle Ax\,,\,x\rangle_0.
\end{aligned}$$

But this is our assertion number (ii). Now let us prove number (iii). We have

$$\left|\frac{\partial}{\partial e_j}\frac{\partial}{\partial e_k}\langle Ax\,,\,x\rangle_0\right| = 2\langle Ae_k\,,\,e_j\rangle_0 \le 2\,\|A\|_{\mathscr{L}(H_-,H_+)}\,\|e_j\|_-\,\|e_k\|_+ \le 2\,\|A\|_{\mathscr{L}(H_-,H_+)}\,.$$

But this is our proposition. □

THEOREM 2.2.14. *Let $\psi : H_- \longrightarrow \mathbb{R}$ be a real-valued negative definite function in Levi-Khinchin-Form with respect to the Hilbert space rigging $H_+ \subset H_0 \subset H_-$. Moreover, let us assume that for $2 \le l \le m$ all absolute H_0-moments of the Levy measure ν exist, i.e.*

$$M_l := \int_{H_+\setminus\{0\}} \|x\|_0^l\,\nu(dx) < \infty, \quad 2 \le l \le m. \tag{12}$$

Then for $\alpha \in \mathbb{N}_0^{\mathbb{N}}$ such that $|\alpha| \le m$ we have

$$\left|\partial_\xi^\alpha\psi(\xi)\right| \le c_{|\alpha|}\cdot\begin{cases}\psi(\xi), & \alpha = 0\\ \psi^{1/2}(\xi), & |\alpha| = 1\\ 1, & |\alpha| \ge 2.\end{cases}$$

In addition, if ψ is cylindric then it is m-times differentiable and for $m = \infty$ also of the class $S_{\gamma,cly}(H_-)$.

PROOF. Let us consider the function $\Phi(\xi) := \int_{H_+\setminus\{0\}}(1-\cos(\langle x\,,\,\xi\rangle_0))\nu(dx)$. Since all moments are finite we obtain by interchange of differentiation and integration for $|\alpha| \le m$

$$\partial_\xi^\alpha\Phi(\xi) = -\int_{H_+\setminus\{0\}} x^\alpha(\partial^\alpha\cos)(\langle x\,,\,\xi\rangle_0)\nu(dx).$$

For $|\alpha| = 1$ it follows by the Cauchy-Schwarz inequality

$$\begin{aligned} \left|\partial_{\xi_j}\Phi(\xi)\right| &\le \left(\int\limits_{H_+\setminus\{0\}} |x_j|^2\,\nu(dx)\right)^{1/2} \left(\int\limits_{H_+\setminus\{0\}} \sin^2(\langle x\,,\,\xi\rangle)\nu(dx)\right)^{1/2} \\ &\le \left(\int\limits_{H_+\setminus\{0\}} \|x\|_0^2\,\nu(dx)\right)^{1/2} \left(2\int\limits_{H_+\setminus\{0\}} (1-\cos(\langle x\,,\,\xi\rangle)\nu(dx)\right)^{1/2} \\ &= (2M_2)^{1/2}\Phi^{1/2}(\xi), \end{aligned}$$

and for $2 \le |\alpha| \le m$ we have

$$\left|\partial_\xi^\alpha\Phi(\xi)\right| \le \int\limits_{H_+\setminus\{0\}} |x^\alpha|\,|(\partial^\alpha\cos)(\langle x\,,\,\xi\rangle_0)|\,\nu(dx) \le \int\limits_{H_+\setminus\{0\}} \|x\|_0^{|\alpha|}\,\nu(dx) = M_{|\alpha|}.$$

For a constant c the result above is obvious, for the quadratic form we proved this in Proposition 2.2.13. □

Now let us consider negative definite functions as symbols for pseudodifferential operators. In infinite dimensional spaces pseudodifferential operators are defined in **[2]** as Weyl-quantization of the symbol. In the classic finite dimensional theory of pseudodifferential operators with negative definite symbol one always considers the Kohn-Nirenberg-quantization. However, since all symbols considered in this section are independent of x both quantizations coincide.

Definition 2.2.15. Let ψ be in $BN(H_-)$ and $f \in S_{\gamma,cyl}(H_-)$. Then we have

$$\psi(D)f := \mathcal{F}^{-1}\psi(\cdot)\mathcal{F}f \in L^2(H_-,\gamma).$$

Note that $\mathcal{F}$ leaves invariant the space $S_{\gamma,cyl}(H_-)$. Thus $\psi(\cdot)\mathcal{F}f \in L^2(H_-,\gamma)$ by Lemma 2.2.10.

Proposition 2.2.16. *Let*

$$\psi(\xi) = \int\limits_{H_+\setminus\{0\}} \left(1 - e^{-i\langle x\,,\,\xi\rangle_0} - \frac{i\langle x\,,\,\xi\rangle_0}{1+\|x\|^2}\right)\nu(dx),$$

where ν is given by (11) and $f \in S_{\gamma,cyl}(H_-)$. Then we get

$$\psi(D)f(\xi) = \int\limits_{H_+\setminus\{0\}} \mathcal{F}^{-1}\left(1 - e^{-i\langle x\,,\,\xi\rangle_0} - \frac{i\langle x\,,\,\xi\rangle_0}{1+\|x\|^2}\right)\mathcal{F}f(\xi)\nu(dx).$$

Proof. For ψ, f as above and $g \in L^2(H_-,\gamma)$ we obtain by Lemma 2.2.10, Definition 2.2.15 and Fubini's theorem

$$\begin{aligned} &\langle\mathcal{F}^{-1}\psi(\cdot)(\mathcal{F}f)\,,\,g\rangle_{L^2(H_-,\gamma)} \\ =\;& \langle\psi(\cdot)(\mathcal{F}f)\,,\,\mathcal{F}g\rangle_{L^2(H_-,\gamma)} \end{aligned}$$

$$
\begin{aligned}
&= \int_{H_-} \int_{H_+\setminus\{0\}} \left(1 - e^{-i\langle x\,,\,\xi\rangle_0} - \frac{i\langle x\,,\,\xi\rangle_0}{1+\|x\|^2}\right) \nu(dx)(\mathcal{F}f)(\xi)\overline{(\mathcal{F}g)(\xi)}\gamma(d\xi) \\
&= \int_{H_+\setminus\{0\}} \int_{H_-} \left(1 - e^{-i\langle x\,,\,\xi\rangle_0} - \frac{i\langle x\,,\,\xi\rangle_0}{1+\|x\|^2}\right) (\mathcal{F}f)(\xi)\overline{(\mathcal{F}g)(\xi)}\gamma(d\xi)\nu(dx) \\
&= \int_{H_+\setminus\{0\}} \langle \left(1 - e^{-i\langle x\,,\,\cdot\rangle_0} - \frac{i\langle x\,,\,\cdot\rangle_0}{1+\|x\|^2}\right) (\mathcal{F}f)\,,\,\mathcal{F}g\rangle_{L^2(H_-,\gamma)}\nu(dx) \\
&= \int_{H_+\setminus\{0\}} \langle \mathcal{F}^{-1} \left(1 - e^{-i\langle x\,,\,\cdot\rangle_0} - \frac{i\langle x\,,\,\cdot\rangle_0}{1+\|x\|^2}\right) (\mathcal{F}f)\,,\,g\rangle_{L^2(H_-,\gamma)}\nu(dx) \\
&= \int_{H_+\setminus\{0\}} \int_{H_-} (\mathcal{F}^{-1} \left(1 - e^{-i\langle x\,,\,\cdot\rangle_0} - \frac{i\langle x\,,\,\cdot\rangle_0}{1+\|x\|^2}\right) (\mathcal{F}f))(\xi)\overline{g(\xi)}\gamma(d\xi)\nu(dx) \\
&= \int_{H_-} \int_{H_+\setminus\{0\}} (\mathcal{F}^{-1} \left(1 - e^{-i\langle x\,,\,\cdot\rangle_0} - \frac{i\langle x\,,\,\cdot\rangle_0}{1+\|x\|^2}\right) (\mathcal{F}f))(\xi)\nu(dx)\overline{g(\xi)}\gamma(d\xi) \\
&= \langle \int_{H_+\setminus\{0\}} (\mathcal{F}^{-1} \left(1 - e^{-i\langle x\,,\,\cdot\rangle_0} - \frac{i\langle x\,,\,\cdot\rangle_0}{1+\|x\|^2}\right) (\mathcal{F}f))\nu(dx)\,,\,g\rangle_{L^2(H_-,\gamma)}.
\end{aligned}
$$

But this is our assertion since $g \in L^2(H_-, \gamma)$ is arbitrary. □

THEOREM 2.2.17. *Let $\psi : H_- \longrightarrow \mathbb{C}$ be an negative definite function in Levi-Khinchin-Form with respect to the Hilbert space rigging $H_+ \subset H_0 \subset H_-$. Then one can consider ψ as symbol on H_-^2 and the corresponding pseudodifferential operator $\widehat{\psi} = \psi(D)$ is given by*

$$
\begin{aligned}
\psi(D)u(x) &= \mathcal{F}^{-1}(\psi(\cdot)(\mathcal{F}u)(\cdot))(x) \\
&= cu(x) + D_d u(x) - Tr_0 A u''(x) + Tr_0 A u(x) \\
&\quad + \langle Ax\,,\,\nabla u(x)\rangle_0 + \langle A\nabla u(x)\,,\,x\rangle_0 - \langle Ax\,,\,x\rangle_0 u(x) \\
&\quad - \int_{H_+\setminus\{0\}} \sqrt{\varrho_y(x)}u(x-y) - u(x) + \frac{\langle \nabla u(x)\,,\,y\rangle_0 - \langle x\,,\,y\rangle_0}{1+\|y\|_+^2} d\nu(y))
\end{aligned}
$$

for all $u \in S_{\gamma,cyl}(H_-)$, where $u''(x) \in L^2(H_-, H_+)$ is defined by $\langle u''(x)h\,,\,k\rangle_0 = d^2u(x)(h,k)$ for all $h, k \in H_-$.

PROOF. Let us consider the four summands separately:

(i) It is clear, that $\mathcal{F}^{-1}c\mathcal{F} = c\,\mathrm{id}$ for any constant c.

(ii) Let $d \in H_+$. Then we obtain

$$\mathcal{F}^{-1}i\langle d\,,\,\cdot\rangle_0(\mathcal{F}u) = D_d u.$$

(iii) Let A as in definition 2.2.8. For $\xi \in H_-$ we obtain by continuity $\langle A\xi\,,\,\xi\rangle = \sum_k^\infty \lambda_k \xi_k^2$, where $\xi_k := \langle f_k\,,\,\xi\rangle$ and λ_k and f_k are defined as in Proposition 2.2.13. Thus as in (ii) it follows

$$
\begin{aligned}
&\mathcal{F}^{-1}\langle A\cdot\,,\,\cdot\rangle_0(\mathcal{F}u)(x)\\
=\ & -\sum_{k=1}^\infty \lambda_k D_k^2 u(x)\\
=\ & -\sum_{k=1}^\infty \lambda_k D_k\big(\frac{\partial}{\partial_{x_k}}u(x) - x_k u(x)\big)\\
=\ & -\sum_{k=1}^\infty \lambda_k \left(\frac{\partial^2}{(\partial_{x_k})^2}u(x) - u(x) - x_k\frac{\partial}{\partial_{x_k}}u(x) - x_k\frac{\partial}{\partial_{x_k}}u(x) + x_k^2 u(x)\right)\\
=\ & -Tr_0 A u''(x) + (Tr_0 A)u(x) + \langle Ax\,,\,\nabla u(x)\rangle_0\\
& \qquad + \langle A\nabla u(x)\,,\,x\rangle_0 - \langle Ax\,,\,x\rangle_0 u(x).
\end{aligned}
$$

(iv) In view of Proposition 2.2.16 we obtain

$$
\begin{aligned}
&\mathcal{F}^{-1}\left(\int_{H_+\backslash\{0\}}\left(1 - e^{-i\langle y\,,\,\xi\rangle_0} - \frac{i\langle y\,,\,\xi\rangle_0}{1+\|y\|^2}\right)\nu(dy)(\mathcal{F}u)(\xi)\right)(x)\\
=\ & \int_{H_+\backslash\{0\}} \mathcal{F}^{-1}\left(1 - e^{-i\langle y\,,\,\xi\rangle_0} - \frac{i\langle y\,,\,\xi\rangle_0}{1+\|y\|^2}(\mathcal{F}u)(\xi)\right)(x)\nu(dy)\\
=\ & \int_{H_+\backslash\{0\}} u(x) - \sqrt{\varrho_y(x)}u(x-y) - \frac{\sum_{j=1}^\infty y_j D_j u(x)}{1+\|y\|_+^2}\nu(dy)\\
=\ & -\int_{H_+\backslash\{0\}} \sqrt{\varrho_y(x)}u(x-y) - u(x) + \frac{\langle\nabla u(x)\,,\,y\rangle_0 - \langle x\,,\,y\rangle_0}{1+\|y\|_+^2}\nu(dy).
\end{aligned}
$$

Now (i)-(iv) yield our assertion. □

Definition 2.2.18. Let ψ be a continuous negative definite function on H_-. For $t \geq 0$ we define

$$T_t:\ L^2(H_-,\gamma) \longrightarrow L^2(H_-,\gamma)$$

by

$$T_t u := \mathcal{F}^{-1}e^{-t\psi(\cdot)}\mathcal{F}u.$$

It is obvious that T_t maps $L^2(H_-,\gamma)$ to $L^2(H_-,\gamma)$ continuously since $\left|e^{-t\psi(\cdot)}\right| \leq 1$.

PROPOSITION 2.2.19. *Let ψ be a negative definite function on H_- and T_t defined as in 2.2.18. Then T_t is a strongly continuous contraction semi group on $L^2(H_-, \gamma)$.*

PROOF. At first let us show that T_t is a semi group. Thus let $t, s \geq 0$ and $u \in L^2(H_-, \gamma)$. We obtain

$$\begin{aligned}(T_t \circ T_s)u &= \mathcal{F}^{-1} e^{-t\psi(\cdot)} \mathcal{F} \mathcal{F}^{-1} e^{-s\psi(\cdot)} \mathcal{F} u = \mathcal{F}^{-1} e^{-t\psi(\cdot)} e^{-s\psi(\cdot)} \mathcal{F} u \\ &= \mathcal{F}^{-1} e^{-(t+s)\psi(\cdot)} \mathcal{F} u = T_{t+s} u.\end{aligned}$$

Moreover T_t is a contraction since

$$\begin{aligned}\|T_t u\|_{L^2(H_-,\gamma)} &= \left\|\mathcal{F}^{-1} e^{-t\psi(\cdot)} \mathcal{F} u\right\|_{L^2(H_-,\gamma)} \\ &= \left\|e^{-t\psi(\cdot)} \mathcal{F} u\right\|_{L^2(H_-,\gamma)} \leq \|\mathcal{F} u\|_{L^2(H_-,\gamma)} = \|u\|_{L^2(H_-,\gamma)}.\end{aligned}$$

At last let us show that $T_t u$ is strongly continuous:

$$\begin{aligned}\|(T_t - \mathrm{id})u\|^2_{L^2(H_-,\gamma)} &= \left\|(\mathcal{F}^{-1} e^{-t\psi(\cdot)} \mathcal{F} - \mathcal{F}^{-1} 1 \mathcal{F})u\right\|^2_{L^2(H_-,\gamma)} \\ &= \left\|\mathcal{F}^{-1}(e^{-t\psi(\cdot)} - 1)\mathcal{F} u\right\|^2_{L^2(H_-,\gamma)} \\ &= \left\|(e^{-t\psi(\cdot)} - 1)\mathcal{F} u\right\|^2_{L^2(H_-,\gamma)} \\ &= \int_{H_-} \left|e^{-t\psi(\xi)} - 1\right|^2 |\mathcal{F} u(\xi)|^2 \gamma(d\xi) \longrightarrow 0\end{aligned}$$

by Lebesgue's Theorem of dominated convergence, since $\left|e^{-t\psi(\xi)} - 1\right|^2 \xrightarrow{t \longrightarrow 0} 0$ and $\left|e^{-t\psi(\xi)} - 1\right|^2 \leq 4$. $\square$

THEOREM 2.2.20. *Let ψ be a negative definite function on H_- such that $\psi \in BN(H_-)$. Moreover, let T_t defined as in 2.2.18 and denote by A the infinitesimal generator of T_t. Then for $u \in S_{\gamma,cyl}(H_-)$ we have*

$$Au := \lim_{t \to 0} \frac{T_t u - u}{t} = -\psi(D)u,$$

where $\psi(D)$ is defined as in 2.2.15.

PROOF. For $u \in S_{\gamma,cyl}(H_-)$ we obtain

$$\begin{aligned}\left\|\frac{(T_t u - u)}{t} + \psi(D)u\right\|^2_{L^2(H_-,\gamma)} &= \left\|\mathcal{F}^{-1} \frac{e^{-t\psi(\cdot)} - 1}{t} (\mathcal{F} u) + \mathcal{F}^{-1} \psi(\cdot)(\mathcal{F} u)\right\|^2_{L^2(H_-,\gamma)} \\ &= \left\|\mathcal{F}^{-1}(\frac{e^{-t\psi(\cdot)} - 1}{t} + \psi(\cdot))\mathcal{F} u\right\|^2_{L^2(H_-,\gamma)} \\ &= \left\|(\frac{e^{-t\psi(\cdot)} - 1}{t} + \psi(\cdot))\mathcal{F} u\right\|^2_{L^2(H_-,\gamma)}\end{aligned}$$

$$= \int_{H_-} \left| \frac{e^{-t\psi(\xi)} - 1 + t\psi\xi}{t} \right|^2 |\mathcal{F}u(\xi)|^2 \gamma(d\xi).$$

However, for $|t| \leq 1$ we find by Lemma 2.2.10 a constant $c > 0$ such that

$$\left| \frac{e^{-t\psi(\xi)} - 1 + t\psi\xi}{t} \right| \leq \frac{1}{2} |t| |\psi(\xi)|^2 \leq \frac{1}{2} |\psi(\xi)|^2 \leq c(1 + \|\xi\|_-^2)^2.$$

Of course, we have $\left| \frac{e^{-t\psi(\xi)}-1+t\psi\xi}{t} \right| \xrightarrow{t \to 0} 0$. Now, note that $\mathcal{F}$ leaves invariant the space $S_{\gamma,cyl}(H_-)$ and thus we obtain by Lebesgue's Theorem of dominate convergence

$$\left\| \frac{(T_t u - u)}{t} + \psi(D)u \right\|_{L^2(H_-,\gamma)}^2 = \int_{H_-} \left| \frac{e^{-t\psi(\xi)} - 1 + t\psi\xi}{t} \right|^2 |\mathcal{F}u(\xi)|^2 \gamma(d\xi) \xrightarrow{t \longrightarrow 0} 0.$$

But this is our assertion. □

DEFINITION 2.2.21. Let $f \in \mathscr{C}^\infty((0,\infty))$ be a real valued-function. We call f a Bernstein function if $f \geq 0$ and $(-1)^k \frac{d^k f(x)}{dx^k} \leq 0$ for all $n \in \mathbb{N} \setminus \{0\}$.

As shown in [**80**, Theorem 3.9.7] for every Bernstein function f there exits a unique convolution semi group $(\eta_t)_{t\geq 0}$ supported by $[0,\infty)$ such that

$$\mathcal{L}(\eta_t)(x) = e^{-tf(x)}, \quad x > 0 \text{ and } t > 0, \tag{13}$$

where $\mathcal{L}$ denotes the Laplace-Transform.

REMARK 2.2.22. Let ψ be a negative definite function and f a Bernstein function. Then as in [**80**, Lemma 3.9.9] $f \circ \psi$ is a negative definite function. Moreover if $\psi \in BN(H_-)$ then the same is true for $f \circ \psi$. If ψ is continuous then $f \circ \psi$ is continuous. In addition, according to [**80**, Example 3.9.16] for $\alpha \in [0,1]$ the function $f_\alpha(x) = x^\alpha$ is a Bernstein function. Thus we obtain that for any negative definite function ψ the function ψ^α is also a negative definite. This yields that functions of the form

$$\psi(\xi) = |\xi_1|^{\alpha_1} + |\xi_2|^{\alpha_2} + \ldots + |\xi_n|^{\alpha_n},$$

where $\alpha_j \in [0,2]$ are negative definite functions.

The following Theorem can be found in [**80**, Theorem 4.3.1]:

THEOREM 2.2.23. *Let f be a Bernstein function with corresponding convolution semi group η_t given by equation (13). Moreover, let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semi group on a Banach space $(X, \|\cdot\|_X)$. For $u \in X$ we define $T_t^f u$ by the Bochner-integral*

$$T_t^f u = \int_0^\infty T_s u \, \eta_t(ds).$$

Then the integral is well defined and $(T_t^f)_{t\geq 0}$ *is a strongly continuous contraction semi group on X. The semi group* T_t^f *is called subordinate to* T_t *with respect to* f.

THEOREM 2.2.24. *Let* ψ *be a negative definite function. Moreover, let* $(T_t)_{t\geq 0}$ *be the strongly continuous contraction semi group given by* ψ *as in Definition 2.2.18. Furthermore, let* f *be a Bernstein function with associated convolution semi group* $(\eta_t)_{t\geq 0}$ *given by equation (13). Then we obtain for the subordinated semi group* T_t^f *to* T_t *with respect to* f

$$T_t^f u = \mathcal{F}^{-1} e^{-tf\circ\psi(\cdot)} \mathcal{F} u$$

for all $u \in L^2(H_-, \gamma)$.

PROOF. For $u, v \in L^2(H_-, \gamma)$ we have

$$\begin{aligned}\int_0^\infty \int_{H_-} \left| \mathcal{F}^{-1} e^{-s\psi(\xi)} \mathcal{F}u(\xi)\overline{v(s)} \right| \gamma(d\xi)\eta_t(ds) &\leq \int_0^\infty \left\| \mathcal{F}^{-1} e^{-s\psi(\cdot)} \mathcal{F}u \right\| \|v\| \, \eta_t(ds) \\ &\leq \|u\| \, \|v\| \int_0^\infty \eta_t(ds) < \infty\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \int_{H_-} \left| e^{-s\psi(\xi)} \mathcal{F}u(\xi)\overline{\mathcal{F}v(s)} \right| \gamma(d\xi)\eta_t(ds) &\leq \int_0^\infty \|\mathcal{F}u\| \, \|\mathcal{F}v\| \, \eta_t(ds) \\ &\leq \|u\| \, \|v\| \int_0^\infty \eta_t(ds) < \infty.\end{aligned}$$

Hence we obtain by Fubini's theorem

$$\begin{aligned}\langle \int_0^\infty \mathcal{F}^{-1} e^{-s\psi(\cdot)} \mathcal{F}u \, \eta_t(ds) \, , \, v \rangle_{L^2(H_-,\gamma)} &= \int_{H_-} \int_0^\infty \mathcal{F}^{-1} e^{-s\psi(\xi)} \mathcal{F}u(\xi) \, \eta_t(ds)\overline{v(\xi)}\gamma(d\xi) \\ &= \int_0^\infty \int_{H_-} \mathcal{F}^{-1} e^{-s\psi(\xi)} \mathcal{F}u(\xi)\overline{v(\xi)}\gamma(d\xi)\eta_t(ds) \\ &= \int_0^\infty \int_{H_-} e^{-s\psi(\xi)} \mathcal{F}u(\xi)\overline{\mathcal{F}v(\xi)}\gamma(d\xi)\eta_t(ds) \\ &= \int_{H_-} \int_0^\infty e^{-s\psi(\xi)} \eta_t(ds) \mathcal{F}u(\xi)\overline{\mathcal{F}v(\xi)}\gamma(d\xi)\end{aligned}$$

$$= \langle \mathcal{L}(\eta_t)(\psi(\cdot))\mathcal{F}u \,,\, \mathcal{F}v\rangle_{L^2(H_-,\gamma)}$$
$$= \langle \mathcal{F}^{-1}e^{-tf\circ\psi(\cdot)}\mathcal{F}u \,,\, v\rangle_{L^2(H_-,\gamma)}.$$

But this is our assertion since $v \in L^2(H_-,\gamma)$ is arbitrary. □

REMARK 2.2.25. As shown in [**80**, Theorem 4.3.20] for a strongly continuous contraction semi group on a Banach space with generator $(A, D(A))$ and two Bernstein functions f_1, f_2 we have

(i) $A^{\alpha f_1} = \alpha A^{f_1}$ for all $\alpha > 0$
(ii) $A^{f_1+f_2} = \overline{A^{f_1} + A^{f_2}}$
(iii) $A^{f_1\circ f_2} = (A^{f_2})^{f_1}$
(iv) $A^{f_1\cdot f_2} = -A^{f_1} \circ A^{f_2} = -A^{f_2} \circ A^{f_1}$ if $f_1 \cdot f_2$ is also a Bernstein function.

2.3. L^2_γ-Sub-Markovian semi groups and Dirichlet-forms

Since we have to consider a Gaussian measure instead of the Lebesgue measure and the Fourier-Wiener instead of the Fourier-Transform it seems in view of Proposition 1.4.10 quite natural to adapt the concept of sub Markovian semi groups and Dirichlet-forms in the following way: We call a semi group $(S_t)_{t\in\mathbb{R}}$ an L^2_γ sub Markovian semi group if we have

$$0 \le u \le e^{\frac{\|P_n\cdot\|_0}{2}} \text{ a.e. implies } 0 \le S_t u \le e^{\frac{\|P_n\cdot\|_0}{2}} \text{ a.e.}$$

Using this notation we show that for a cylindrical function ψ, T_t is an L^2_γ sub Markovian semi group (cf. 2.3.24). Furthermore $-\psi(D)$ extends to a L^2_γ-Dirichlet operator A. Concerning these adapted concept of Dirichlet operators we show, that the most important propositions remain valid in case of the Gaussian-measure (see 2.3.15). Defining for $s > 0$ the Sobolev-space $H^s_\psi(H_-)$ as the space of all $u \in L^2(H_-,\gamma)$ such that

$$\|u\|_{\psi,s} := \left\|(1+|\psi|)^{s/2}\mathcal{F}u\right\|_{L^2(H_-,\gamma)} < \infty$$

we are able to show that the domain of definition of the generator of T_t is $H^2_\psi(H_-)$. In addition this generator is our L^2_γ-Dirichlet operator A. Finally, if ψ is real-valued we associate a symmetric L^2_γ-Dirichlet-form to the L^2_γ-Dirichlet operator A. The domain of definition of this Dirichlet-form is given by $H^1_\psi(H_-)$.

However, throughout the first part of this section we follow closely [**80**, 4.6 and 4,7] and transfer the necessary results, but refer to [**80**] concerning all general results. From now on let $(e_j)_{j\in\mathbb{N}} \subset H_+$ be an orthonormal basis in H_0 such that $(e_j)_{j\in\mathbb{N}}$ is orthogonal in H_+ and H_-. Moreover, we assume that we have for all $x \in H_+$ and $y \in H_-$

$$\langle x \,,\, y\rangle_0 = \sum_{j=1}^{\infty}\langle x \,,\, e_j\rangle_0\langle e_j \,,\, y\rangle_0$$

Definition 2.3.1. (i) Let P_n denote the orthogonal projection on the closed linear span of $\{e_j : j \leq n\}$ in H_0 extended by continuity to H_-.

(ii) Let S be in $\mathscr{L}(L^2(H_-,\gamma))$. Then we call S L^2_γ-sub Markovian if there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$0 \leq u \leq e^{\frac{\|P_n \cdot\|_0}{2}} \text{ a.e. implies } 0 \leq Su \leq e^{\frac{\|P_n \cdot\|_0}{2}} \text{ a.e.}$$

(iii) We call a semi group T_t an L^2_γ sub Markovian semi group if T_t is a contraction semi group and every operator T_t is sub Markovian.

During the rest of this chapter we always assume $n \geq n_0$.

Lemma 2.3.2. *Every L^2_γ-sub Markovian Operator S is positivity preserving.*

Proof. Let $n \geq n_0$ and $u \in L^2(H_-,\gamma)$ such that $u \geq 0$. We set $u_k := \min\{u, ke^{\frac{\|P_n \cdot\|_0}{2}}\}$. Then it is obvious that $u_k \xrightarrow{k\longrightarrow\infty} u$ in $L^2(H_-,\gamma)$ by Lebesgue's theorem of dominate convergence. For $v_k := \frac{u_k}{k}$ we have $0 \leq v_k \leq e^{\frac{\|P_n \cdot\|_0}{2}}$ and thus $0 \leq Sv_k \leq e^{\frac{\|P_n \cdot\|_0}{2}}$. Hence it follows $0 \leq Sv_k = \frac{1}{k}S(u_k)$. Since S is bounded we obtain $Su_k \xrightarrow{k\longrightarrow\infty} Su$ in $L^2(H_-,\gamma)$. Thus there exists a subsequence u_{k_l} such that $Su_{k_l} \xrightarrow{k\longrightarrow\infty} Su$ almost everywhere. This yields $0 \leq Su$ a.e. □

For $u \in L^2(H_-,\gamma;\mathbb{R})$ we denote $u^+ := \max\{u,0\}$, $u^- := \max\{-u,0\}$ and $u \wedge e^{\frac{\|P_n \cdot\|_0}{2}} := \min\{u, e^{\frac{\|P_n \cdot\|_0}{2}}\}$. Let S denote a L^2_γ-sub Markovian operator. Then since $u = (u - e^{\frac{\|P_n \cdot\|_0}{2}})^+ + u \wedge e^{\frac{\|P_n \cdot\|_0}{2}}$, $0 \leq |u| \wedge e^{\frac{\|P_n \cdot\|_0}{2}} - u \wedge e^{\frac{\|P_n \cdot\|_0}{2}}$ and $0 \leq |u| \wedge e^{\frac{\|P_n \cdot\|_0}{2}} \leq e^{\frac{\|P_n \cdot\|_0}{2}}$ we obtain

$$S(u \wedge e^{\frac{\|P_n \cdot\|_0}{2}}) \leq S(|u| \wedge e^{\frac{\|P_n \cdot\|_0}{2}}) \leq e^{\frac{\|P_n \cdot\|_0}{2}} \text{a.e.}$$

Now we can prove

Lemma 2.3.3. *Let $(T_t)_{t\geq 0}$ be an L^2_γ-sub Markovian contraction semi group with generator $(A, D(A))$ Then for all $u \in D(A)$ and $n \geq n_0$ we have*

$$\int_{H_-} (Au)(u - e^{\frac{\|P_n \cdot\|_0}{2}})^+ d\gamma(x) \leq 0. \tag{14}$$

Proof. Let $u \in L^2(H_-,\gamma)$ Then we have

$$\begin{aligned}
&\int_{H_-} (T_t u)(u - e^{\frac{\|P_n \cdot\|_0}{2}})^+ \, d\gamma \\
= &\int_{H_-} (T_t(u - e^{\frac{\|P_n \cdot\|_0}{2}})^+)(u - e^{\frac{\|P_n \cdot\|_0}{2}})^+ \, d\gamma + \int_{H_-} (T_t(u \wedge e^{\frac{\|P_n \cdot\|_0}{2}}))(u - e^{\frac{\|P_n \cdot\|_0}{2}})^+ \, d\gamma \\
\leq &\left\|(u - e^{\frac{\|P_n \cdot\|_0}{2}})^+\right\|^2_{H_-} + \int_{H_-} e^{\frac{\|P_n \cdot\|_0}{2}}(u - e^{\frac{\|P_n \cdot\|_0}{2}})^+ \, d\gamma
\end{aligned}$$

$$= \int_{H_-} (u - e^{\frac{\|P_n \cdot\|_0}{2}})(u - e^{\frac{\|P_n \cdot\|_0}{2}})^+ \, d\gamma + \int_{H_-} e^{\frac{\|P_n \cdot\|_0}{2}} (u - e^{\frac{\|P_n \cdot\|_0}{2}})^+ \, d\gamma$$

$$= \int_{H_-} u(u - e^{\frac{\|P_n \cdot\|_0}{2}})^+ \, d\gamma.$$

Thus we find $\int_{H_-} (T_t u - u)(u - e^{\frac{\|P_n \cdot\|_0}{2}})^+ \, d\gamma \leq 0$ which yields

$$\int_{H_-} (Au)(u - e^{\frac{\|P_n \cdot\|_0}{2}})^+ \, d\gamma = \lim_{t \to 0} \frac{1}{t} \int_{H_-} (T_t u - u)(u - e^{\frac{\|P_n \cdot\|_0}{2}})^+ \, d\gamma \leq 0. \qquad \square$$

Definition 2.3.4. (i) We call a closed densely defined Operator

$$A : L^2(H_-, \gamma; \mathbb{R}) \supseteq D(A) \longrightarrow L^2(H_-, \gamma; \mathbb{R})$$

an L^2_γ-Dirichlet operator if equation (14) is fulfilled for all $u \in D(A)$.

(ii) A linear Operator $A : L^2(H_-, \gamma) \supseteq D(A) \longrightarrow L^2(H_-, \gamma; \mathbb{R})$ is called negative definite in $L^2(H_-, \gamma; \mathbb{R})$ if

$$\int_{H_-} (Au)u \, d\gamma \leq 0. \tag{15}$$

Proposition 2.3.5. *Let $(A, D(A))$ be an linear densely defined Operator in $L^2(H_-, \gamma; \mathbb{R})$ which satisfies equation (14). Then A is negative definite in $L^2(H_-, \gamma; \mathbb{R})$.*

Proof. Let $u \in D(A)$, $n \geq n_0$ and $k > 0$. Then we have $(ku - e^{\frac{\|P_n \cdot\|_0}{2}})^+ = k(u - \frac{e^{\frac{\|P_n \cdot\|_0}{2}}}{k})^+$ and thus $\int_{H_-} (Au)(u - \frac{e^{\frac{\|P_n \cdot\|_0}{2}}}{k})^+ \, d\gamma \leq 0$. For $k \longrightarrow \infty$ we obtain be Lebesgue's Theorem of dominated convergence $\int_{H_-} (Au)u^+ \, d\gamma \leq 0$. Moreover, if we take $-u$ instead of u we have $\int_{H_-} (Au)u^- \, d\gamma \geq 0$. Now it follows that

$$\int_{H_-} (Au)u \, d\gamma = \int_{H_-} (Au)u^+ \, d\gamma - \int_{H_-} (Au)u^- \, d\gamma \leq 0. \qquad \square$$

Proposition 2.3.6. *Let $(A, D(A))$ be a negative operator on $L^2(H_-, \gamma)$. Then A is dissipative.*

Proof. See [**80**, Proposition 4.6.12] $\square$

Lemma 2.3.7. *A strongly continuous contraction semi group $(T_t)_{t>0}$ is L^2_γ-sub Markovian if and only if its resolvent $(R_\lambda)_{\lambda>0}$ fulfills the following condition:*

$$u \in L^2(H_-, \gamma) \text{ and } 0 \leq u \leq e^{\frac{\|P_n \cdot\|_0}{2}} \Longrightarrow 0 \leq \lambda R_\lambda u \leq e^{\frac{\|P_n \cdot\|_0}{2}}, \tag{16}$$

for all $n \geq n_0$. Moreover, in this case λR_λ is a contraction and we call $(R_\lambda)_{\lambda>0}$ a L^2_γ-sub Markovian resolvent.

PROOF. First let T_t be a L^2_γ sub Markovian semi group and $u \in L^2(H_-,\gamma)$ such that $0 \le u \le e^{\frac{\|P_n\cdot\|_0}{2}}$. Then the equation $R_\lambda u = \int_0^\infty e^{-\lambda t} T_t u dt$ yields

$$0 \le R_\lambda u \le \int\limits_0^\infty e^{-\lambda t} T_t dt \le \frac{1}{\lambda} e^{\frac{\|P_n\cdot\|_0}{2}} u$$

and $\|R_\lambda u\| \le \frac{1}{\lambda}$. □

Now let R_λ fulfill equation (16). Denote by $(A, D(A))$ the generator of T_t and let again $0 \le u \le e^{\frac{\|P_n\cdot\|_0}{2}}$. Let $T_t^{(\lambda)}$ be the semi group generated by the Yosida-Approximation A_λ of A. We have $T_t^{(\lambda)} u = e^{-\lambda t} \sum_{\nu=0}^\infty \frac{t\lambda}{\nu!}(\lambda R_\lambda)^\nu u$ which yields $0 \le T_t^{(\lambda)} u \le e^{-\lambda t} \sum_{\nu=0}^\infty \frac{t\lambda}{\nu!} e^{\frac{\|P_n\cdot\|_0}{2}} = e^{\frac{\|P_n\cdot\|_0}{2}}$. But since $T_t^{(\lambda)} u$ converges to $T_t u$ in $L^2(H_-,\gamma)$ we find a subsequence which converges almost everywhere. But this shows that T_t is a sub Markovian semi group.

PROPOSITION 2.3.8. *Let $(A, D(A))$ be a L^2_γ-Dirichlet operator which generates a strongly continuous contraction semi group. Then $(T_t)_{t\ge 0}$ is L^2_γ-sub Markovian.*

PROOF. Due to Lemma 2.3.7 it is sufficient to show that $(R_\lambda)_{\lambda>0}$ is a L^2_γ-sub Markovian resolvent. For $n \ge n_0$ and $u \in L^2(H_-,\gamma)$ such that $u \le e^{\frac{\|P_n\cdot\|_0}{2}}$ a.e. set $v := \lambda R_\lambda u \in D(A)$. Then we obtain

$$\begin{aligned}
&\lambda \int\limits_{H_-} v(v - e^{\frac{\|P_n\cdot\|_0}{2}})^+ d\gamma \\
= &\int\limits_{H_-} (\lambda v - Av)(v - e^{\frac{\|P_n\cdot\|_0}{2}})^+ d\gamma + \int\limits_{H_-} (Av)(v - e^{\frac{\|P_n\cdot\|_0}{2}})^+ d\gamma \\
= &\lambda \int\limits_{H_-} u(v - e^{\frac{\|P_n\cdot\|_0}{2}})^+ d\gamma + \int\limits_{H_-} (Av)(v - e^{\frac{\|P_n\cdot\|_0}{2}})^+ d\gamma \\
\le &\lambda \int\limits_{H_-} e^{\frac{\|P_n\cdot\|_0}{2}} (v - e^{\frac{\|P_n\cdot\|_0}{2}})^+ d\gamma,
\end{aligned}$$

which yields $\int_{H_-} ((v - e^{\frac{\|P_n\cdot\|_0}{2}})^+)^2 d\gamma = 0$ and thus $v \le e^{\frac{\|P_n\cdot\|_0}{2}}$ a.e. For $u \ge 0$ we have $-ku \le e^{\frac{\|P_n\cdot\|_0}{2}}$ for all $k \in \mathbb{N}$ and thus $v \ge -\frac{e^{\frac{\|P_n\cdot\|_0}{2}}}{k}$ for all $k \in \mathbb{N}$ which yields $v \ge 0$. □

Now we can state in view of the Hille-Yoshida-Theorem the following

THEOREM 2.3.9. (i) *Let A be a L^2_γ-Dirichlet operator with $R(\lambda id - A) = L^2(H_-,\gamma;\mathbb{R})$ for some $\lambda > 0$. Then A generates a L^2_γ-sub Markovian semi group on $L^2(H_-,\gamma)$.*

(ii) *Let A be a densely defined Operator which fulfills (14) for all* $u \in D(A)$. *Moreover, assume* $\overline{R(\lambda id - A)} = L^2(H_-, \gamma; \mathbb{R})$ *for some* $\lambda > 0$. *Then A is closable and its closure generates a* L^2_γ*-sub Markovian semi group on* $L^2(H_-, \gamma)$.

Using the same arguments as in [**80**, p. 385-388], only replacing $L^2(\mathbb{R}^n; \mathbb{R})$ by $L^2(H_-, \gamma; \mathbb{R})$ we obtain

THEOREM 2.3.10. (i) *Let* $(A, D(A))$ *be a densely defined operator on* $L^2(H_-, \gamma; \mathbb{R})$, *satisfying (15). Moreover we assume that*

$$|\langle -Au\,,\, v\rangle_{L^2(H_-,\gamma)}| \leq c(\langle -Au\,,\, u\rangle_{L^2(H_-,\gamma)})^{1/2}(\langle -Av\,,\, v\rangle_{L^2(H_-,\gamma)})^{1/2}. \tag{17}$$

Then there exists a closed densely defined linear form $(\mathcal{E}, D(\mathcal{E}))$ *on* $L^2(H_-, \gamma; \mathbb{R})$ *such that* $D(A) \subset D(\mathcal{E}) \subset L^2(H_-, \gamma; \mathbb{R})$. *For this form and* $u \in D(A)$ *and* $v \in D(\mathcal{E}))$ *we have* $\mathcal{E}(u, v) = \langle -Au\,,\, v\rangle_{L^2(H_-,\gamma)}$. *In addition we obtain*

$$|\mathcal{E}(u, v)| \leq c(\mathcal{E}_1(u, u))^{1/2}(\mathcal{E}_1(v, v))^{1/2}. \tag{18}$$

As usual for $\lambda > 0$ *we use the notation* $\mathcal{E}_\lambda(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot) + \lambda\langle \cdot\,,\, \cdot\rangle_{L^2(H_-,\gamma)}$

(ii) *The assertion of part (i) holds for every* L^2_γ*-Dirichlet operator which satisfies (18)*

(iii) *Moreover, in part (i) the operator* $(A, D(A))$ *is closable and its closure is a subspace of* $D(\mathcal{E})$.

THEOREM 2.3.11. *Let* $\mathcal{E}, (D(E))$ *be a densely defined bilinear form on* $L^2(H_-, \gamma; \mathbb{R})$ *which fulfills (18). Moreover, let us assume that* $\mathcal{E}$ *is positive definite. We denote by* $(R_\lambda)_{\lambda>0}$ *the corresponding resolvent (cf.* [**80**, 4.7.4]*). Suppose that* $(R_\lambda)_{\lambda>0}$ *is a* L^2_γ*-sub Markovian resolvent. Then for* $n \geq n_0$ *each of the following equivalent statements hold*

(i) *For all* $u \in D(\mathcal{E})$ *and all* $\lambda > 0$, $u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}} \in D(\mathcal{E})$ *and*

$$\mathcal{E}(u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}}, u - u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}}) \geq 0.$$

(ii) *For all* $u \in D(\mathcal{E})$, $u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}} \in D(\mathcal{E})$ *and*

$$\mathcal{E}(u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}}, u - u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}}) \geq 0.$$

(iii) *For all* $u \in D(\mathcal{E})$, $u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}} \in D(\mathcal{E})$ *and*

$$\mathcal{E}(u + u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}}, u - u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}}) \geq 0.$$

Conversely, if $(\mathcal{E}, D(\mathcal{E}))$ *satisfies one of the three conditions above for* $n \geq n_0$ *then* $(R_\lambda)_{\lambda>0}$ *is a* L^2_γ*-sub Markovian resolvent and the generator* $(A, D(A))$ *of* $(R_\lambda)_{\lambda>0}$ *is a Dirichlet operator.*

PROOF. (i) At first we show (i). Let $u \in D(\mathcal{E})$, $\lambda > 0$ and $\mu > 0$. For all $v, w \in H_-$ we define

$$\mathcal{E}^{(\mu)} := \mu\langle v - \mu R_\mu v\,,\, w\rangle_{L^2(H_-,\gamma)}.$$

Since $u = (u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+ + u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}}$ we get

$$\mathcal{E}^{(\mu)}(u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}}, u - (u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}})) = \mathcal{E}^{(\mu)}(u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}}, (u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+). \tag{19}$$

Remember that R_μ satisfies (16) for all $\mu > 0$. Thus using the same arguments as in the proof of 2.3.2 we obtain

$$\begin{aligned} &(u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+(u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}} - \mu R_\mu(u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}}))\\ &= (u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+(\lambda e^{\frac{\|P_n\cdot\|_0}{2}} - \mu R_\mu(u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}}))\\ &= (u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+(\lambda e^{\frac{\|P_n\cdot\|_0}{2}} - \mu R_\mu(|u| \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}})) \geq 0. \end{aligned} \tag{20}$$

Hence it follows that

$$\begin{aligned} &\mathcal{E}_1^\mu((u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+, (u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+)\\ =\;& \mathcal{E}_1^\mu(u, (u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+)\\ &-\mathcal{E}^\mu((u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}}), (u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+) - \langle u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}}\,,\, (u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+\rangle\\ \leq\;& \mathcal{E}_1^\mu(u, (u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+)\\ \leq\;& c(\mathcal{E}_1(u,u))^{1/2}(\mathcal{E}_1^\mu((u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+, (u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+))^{1/2} \end{aligned}$$

by Lemma [**80**, 4.7.17]. Now it follows that $\sup_{\mu>0} \mathcal{E}^\mu((u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+, (u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+) < \infty$ and we obtain by [**80**, Lemma 4.7.18] $u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}} = u - (u - \lambda e^{\frac{\|P_n\cdot\|_0}{2}})^+ \in D(\mathcal{E})$. Moreover, (19), (20) and [**80**, 4.7.18] imply that $\mathcal{E}(u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}}, u - (u \wedge \lambda e^{\frac{\|P_n\cdot\|_0}{2}})) \geq 0$.

(ii) Now let us show that (i) implies (ii). From (i) it follows that $u^+ = -((-u) \wedge 0) \in D(\mathcal{E})$ and thus $u^- \in D(\mathcal{E})$ and $u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}} \in D(\mathcal{E})$. Note that $u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}} = (u \wedge e^{\frac{\|P_n\cdot\|_0}{2}})^+$ and $u^- = (u \wedge e^{\frac{\|P_n\cdot\|_0}{2}})^-$. Then we have

$$\begin{aligned} &\mathcal{E}(u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}}, u - (u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}}))\\ =\;& \mathcal{E}(u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}}, u^+ - (u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}})) - \mathcal{E}(u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}}, u^-)\\ \geq\;& -\mathcal{E}(u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}}, (u \wedge e^{\frac{\|P_n\cdot\|_0}{2}})^-)\\ =\;& -\mathcal{E}((u \wedge e^{\frac{\|P_n\cdot\|_0}{2}})^+, (u \wedge e^{\frac{\|P_n\cdot\|_0}{2}})^+ - (u \wedge e^{\frac{\|P_n\cdot\|_0}{2}}))\\ =\;& \mathcal{E}((-(u \wedge e^{\frac{\|P_n\cdot\|_0}{2}})) \wedge 0, (-(u \wedge e^{\frac{\|P_n\cdot\|_0}{2}})) - (-(u \wedge e^{\frac{\|P_n\cdot\|_0}{2}})) \wedge 0) \geq 0, \end{aligned}$$

which yields (ii).

(iii) We obtain (iii) from (ii) by

$$\begin{aligned}&\mathcal{E}(u+u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}},u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}}))\\ =\ &\mathcal{E}(u^-\wedge e^{\frac{\|P_n\cdot\|_0}{2}},u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}}))\\ &+2\mathcal{E}(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}},u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}}))\geq 0.\end{aligned}$$

(iv) To finish the proof let us show that (iii) implies (16) for the resolvent. Thus let $v\in L^2(H_-,\gamma)$ such that $0\leq v\leq e^{\frac{\|P_n\cdot\|_0}{2}}$ a.e and set $u:=\lambda R_\lambda v$. At first it is easy to check that $u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}})=(u-e^{\frac{\|P_n\cdot\|_0}{2}})^++u\wedge 0$, which implies $\langle(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}})-v\,,\,u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}})\rangle_{L^2(H_-,\gamma)}\geq 0$. Then we obtain

$$\begin{aligned}0\ \geq\ &-2\mathcal{E}(u+u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}},u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}}))\\ &+\mathcal{E}(u-u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}},u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}}))\\ =\ &-\mathcal{E}(u,u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}}))\\ =\ &-\lambda\mathcal{E}_\lambda(\mathbb{R}_\lambda v,u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}}))+\lambda\langle u\,,\,u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}})\rangle_{L^2(H_-,\gamma)}\\ =\ &\lambda\langle u-v\,,\,u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}})\rangle_{L^2(H_-,\gamma)}\\ =\ &\lambda\left\|u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}})\right\|^2_{L^2(H_-,\gamma)}\\ &+\lambda\langle(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}})-v\,,\,u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}})\rangle_{L^2(H_-,\gamma)}.\end{aligned}$$

Thus we have $\lambda\left\|u-(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}})\right\|^2_{L^2(H_-,\gamma)}\leq 0$ which implies the assertion.

□

Definition 2.3.12. Let $(\mathcal{E},D(\mathcal{E}))$ be a closed linear form on $L^2(H_-,\gamma;\mathbb{R})$ such that $\mathcal{E}$ is continuous with respect to $\mathcal{E}_1^{sym}$ where $\mathcal{E}_1^{sym}(u,v):=\frac{1}{2}(\mathcal{E}_1(u,v)+\mathcal{E}_1(v,u))$ for all $u,v\in D(\mathcal{E})$.

(i) The form $(\mathcal{E},D(\mathcal{E}))$ is called a semi-L^2_γ-Dirichlet-form if for all $n\geq n_0$ and $u\in D(\mathcal{E})$ we have $(u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}})\in D(\mathcal{E})$ and

$$\mathcal{E}(u+u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}},u-u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}})\geq 0. \tag{21}$$

(ii) The form $(\mathcal{E},D(\mathcal{E}))$ is said to be a L^2_γ-Dirichlet-form if $(\mathcal{E},D(\mathcal{E}))$ is a semi-L^2_γ-Dirichlet and we have in addition

$$\mathcal{E}(u-u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}},u+u^+\wedge e^{\frac{\|P_n\cdot\|_0}{2}})\geq 0 \tag{22}$$

for all $n\geq n_0$.

(iii) If $(\mathcal{E}, D(\mathcal{E}))$ is symmetric and a L^2_γ-Dirichlet form then we call $(\mathcal{E}, D(\mathcal{E}))$ a symmetric L^2_γ-Dirichlet form. Note that for a symmetric form to be a L^2_γ-Dirichlet-form is equivalent to the condition that $u \in D(\mathcal{E})$ implies $(u^+ \wedge e^{\frac{\|P_n \cdot\|_0}{2}}) \in D(\mathcal{E})$ and
$$\mathcal{E}(u^+ \wedge e^{\frac{\|P_n \cdot\|_0}{2}}, u^+ \wedge e^{\frac{\|P_n \cdot\|_0}{2}}) \leq \mathcal{E}(u,u),$$
for all $n \geq n_0$.

PROPOSITION 2.3.13. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a closed form on $L^2(H_-, \gamma; \mathbb{R})$ such that $\mathcal{E}$ is continuous with respect to $\mathcal{E}_1^{sym}$. Assume that for every $\varepsilon > 0$ there exists a function $\varphi_\varepsilon : \mathbb{R} \longrightarrow [-\varepsilon, 1+\varepsilon]$ such that $\varphi_\varepsilon(t) = t$ for all $t \in [0,1]$ and that $t_1 \leq t_2$ implies $0 \leq \varphi_\varepsilon(t_2) - \varphi_\varepsilon(t_1) \leq t_2 - t_1$. In addition we suppose that for some $u \in D(\mathcal{E})$ we have $(\Phi_\varepsilon(u))(\cdot) := \varphi_\varepsilon(u(\cdot) e^{-\frac{\|P_n \cdot\|_0}{2}}) e^{\frac{\|P_n \cdot\|_0}{2}} \in D(\mathcal{E})$ such that*
$$\liminf_{\varepsilon \to 0} \mathcal{E}(u + \Phi_\varepsilon(u), u - \Phi_\varepsilon(u)) \geq 0 \tag{23}$$
and
$$\liminf_{\varepsilon \to 0} \mathcal{E}(u - \Phi_\varepsilon(u), u + \Phi_\varepsilon(u)) \geq 0. \tag{24}$$
Then we have $u^+ \wedge e^{\frac{\|P_n \cdot\|_0}{2}} \in D(\mathcal{E})$ and the equations (21) and (22) hold. Moreover, $(\mathcal{E}, D(\mathcal{E}))$ is a L^2_γ-Dirichlet form if and only if this assertion above holds for all $u \in D(\mathcal{E})$ and $n \geq n_0$.

PROOF. Adding the two inequalities above we obtain
$$\limsup_{\varepsilon \to 0} \mathcal{E}(\Phi_\varepsilon(u), \Phi_\varepsilon(u)) \leq \mathcal{E}(u,u).$$
Note that $\Phi_\varepsilon(u) \longrightarrow u^+ \wedge e^{\frac{\|P_n \cdot\|_0}{2}}$ in $L^2(H_-, \gamma; \mathbb{R})$. Now according to [**80**, Lemma 4.7.18] there exists a subsequence $(\varepsilon_k)_{k \in \mathbb{N}}$ such that $\varepsilon_k \longrightarrow 0$ and $\Phi_\varepsilon(u) \longrightarrow u^+ \wedge e^{\frac{\|P_n \cdot\|_0}{2}}$ weakly in $(D(\mathcal{E}), \mathcal{E}_1^{sym})$ and we have
$$\mathcal{E}(u^+ \wedge e^{\frac{\|P_n \cdot\|_0}{2}}, u^+ \wedge e^{\frac{\|P_n \cdot\|_0}{2}}) \leq \liminf_{k \to \infty} \mathcal{E}(\Phi_\varepsilon(u), \Phi_\varepsilon(u)).$$
Hence we obtain
$$\begin{aligned}
& \mathcal{E}(u + u^+ \wedge e^{\frac{\|P_n \cdot\|_0}{2}}, u - u^+ \wedge e^{\frac{\|P_n \cdot\|_0}{2}}) \\
\geq\ & \mathcal{E}(u,u) - \lim_{k \to \infty} \mathcal{E}(u, \Phi_\varepsilon(u)) + \lim_{k \to \infty} \mathcal{E}(\Phi_\varepsilon(u), u) - \liminf_{k \to \infty} \mathcal{E}(\Phi_\varepsilon(u), \Phi_\varepsilon(u)) \\
=\ & \limsup_{\varepsilon \to 0} \mathcal{E}(u + \Phi_\varepsilon(u), u - \Phi_\varepsilon(u)) \geq 0
\end{aligned}$$
and
$$\begin{aligned}
& \mathcal{E}(u - u^+ \wedge e^{\frac{\|P_n \cdot\|_0}{2}}, u + u^+ \wedge e^{\frac{\|P_n \cdot\|_0}{2}}) \\
\geq\ & \mathcal{E}(u,u) + \lim_{k \to \infty} \mathcal{E}(u, \Phi_\varepsilon(u)) - \lim_{k \to \infty} \mathcal{E}(\Phi_\varepsilon(u), u) - \liminf_{k \to \infty} \mathcal{E}(\Phi_\varepsilon(u), \Phi_\varepsilon(u)) \\
=\ & \limsup_{\varepsilon \to 0} \mathcal{E}(u - \Phi_\varepsilon(u), u + \Phi_\varepsilon(u)) \geq 0.
\end{aligned}$$

But this is the first part of our assertion. Finally we note that if $(\mathcal{E}, D(\mathcal{E}))$ is a L^2_γ-Dirichlet form, then the function $\varphi_\varepsilon(t) = t^+ \wedge 1$ fulfills the criterion and for this function we have $\Phi_\varepsilon(u)(x) = u^+ \wedge e^{\frac{\|P_n \cdot\|_0}{2}}$ for all $u \in D(\mathcal{E})$. Thus the assertion is proved. □

As in [**80**, p. 406] we obtain the following

LEMMA 2.3.14. *Suppose (21) and (22) or (23) and (24) hold only for a dense subset of $(D(\mathcal{E}), \mathcal{E}_1^{sym})$. Then they hold for all $u \in D(\mathcal{E})$.*

Summarizing our results above and using [**80**, 4.1, 4.6 and 4.7] we obtain

THEOREM 2.3.15. *Let $(A, D(A))$ be a L^2_γ-Dirichlet operator on $L^2(H_-, \gamma; \mathbb{R})$. Assume that A generates a L^2_γ-Sub-Markovian semi group T_t and satisfies equation (17). Then the bilinear form $(\mathcal{E}, D(\mathcal{E}))$ defined in Theorem 2.3.10 is a semi-L^2_γ-Dirichlet-form. If in addition $(A^*, D(A^*))$ is a L^2_γ-Dirichlet operator, then $(\mathcal{E}, D(\mathcal{E}))$ is a L^2_γ-Dirichlet-form. Moreover, if $(A, D(A))$ is selfadjoint then $(\mathcal{E}, D(E))$ is a symmetric L^2_γ-Dirichlet-form on $L^2(H_-, \gamma; \mathbb{R})$. Conversely, suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a semi-L^2_γ-Dirichlet-form on $L^2(H_-, \gamma; \mathbb{R})$. Then the operator $(A, D(A))$ defined in Theorem 2.3.11 is a L^2_γ-Dirichlet operator which generates a L^2_γ-sub Markovian semi group. If $(\mathcal{E}, D(\mathcal{E}))$ is a L^2_γ-Dirichlet-form then $(A^*, D(A^*))$ is a L^2_γ-Dirichlet operator, too. Furthermore if $(\mathcal{E}, D(E))$ is a symmetric L^2_γ-Dirichlet-form then $(A, D(A))$ is selfadjont and we have $D(\mathcal{E}) = D((-A)^{1/2})$ and $\mathcal{E}(u, v) = \langle (-A)^{-1/2} u\,,\, (-A)^{1/2} v\rangle_{L^2(H_-,\gamma)}$ for all $u, v \in D(\mathcal{E})$.*

Now we consider our pseudodifferential operator with negative definite symbol as some kind of generalized Laplace operator. We define scales of Sobolev-spaces attached to these operators. We show that they share some important properties with the classical Laplace operator. For example we determine $H^2_\psi(H_-)$ as domain of definition for $\psi(D)$. In addition these operators generate L^2_γ-Dirichlet-forms with domain $H^1_\psi(H_-)$

DEFINITION 2.3.16. Let ψ be a continuous negative definite function. Then we define for all $s \geq 0$ the generalized Sobolev-Space $H^s_\psi(H_-)$ as the space of all $u \in L^2(H_-, \gamma)$ such that

$$\|u\|_{\psi,s} := \left\|(1 + |\psi|)^{s/2} \mathcal{F} u\right\|_{L^2(H_-,\gamma)} < \infty.$$

Furthermore, we set $H^{-s}_\psi(H_-) := (H^s_\psi(H_-))'$, where the duality is given with respect to the inner product in $H^0_\psi(H_-) = L^2(H_-, \gamma)$. As usual we set $H^\infty_\psi(H_-) := \bigcap_{s\in\mathbb{R}} H^s_\psi(H_-)$ and $H^{-\infty}_(H_-) := \bigcup_{s\in\mathbb{R}} H^s_\psi(H_-)$.

PROPOSITION 2.3.17. *Let ψ be a negative definite function on H_-. Then the space $S_{\gamma,cyl}(H_-)$ is a dense subset of $H^s_\psi(H_-)$ for all $s \geq 0$.*

Proof. At first note that the Fourier-Wiener-transform leaves invariant the space $S_{\gamma,cyl}(H_-)$. Thus it is clear that $S_{\gamma,cyl}(H_-) \subset H^s_\psi(H_-)$. Now let $u \in H^s_\psi(H_-)$ arbitrary and $\varepsilon > 0$. Then there exists a function $w \in C_b(H_-)$ such that

$$\left\| w - (1+|\psi|^{s/2})\mathcal{F}u \right\|_{L^2(H_-,\gamma)} \leq \frac{\varepsilon}{2}.$$

Set $v := \frac{w}{(1+|\psi|)^{s/2}} \in C_b(H_-) \subset L^2(H_-,\gamma)$. Then there exists a sequence $(v_n)_{n\in\mathbb{N}} \subset \mathscr{C}^\infty_{b,cyl}(H_-)$ such that $v_n \xrightarrow{n\longrightarrow\infty} v$ in $L^2(H_-,\gamma)$ and almost everywhere. Moreover we can choose v_n such that $\|v_n\|_{\sup} \leq \|v\|_{\sup}$. Now it is obvious that $(1+|\psi|)^{s/2}v_n \xrightarrow{n\longrightarrow\infty} (1+|\psi|)^{s/2}v = w$ and $(1+|\psi(\xi)|)^{s/2}\,|v(\xi)| \leq |w(\xi)|$ for all $\xi \in H_-$. Hence we obtain by Lebesgue's Theorem of dominated convergence $(1+|\psi|)^{s/2}v_n \xrightarrow{n\longrightarrow\infty} w$ in $L^2(H_-,\gamma)$. Thus there exist a $n_0 \in \mathbb{N}$ such that

$$\left\|(1+|\psi|)^{s/2}v_{n_0} - w\right\|_{L^2(H_-,\gamma)} \leq \frac{\varepsilon}{2}.$$

Now we set $\tilde{u} := \mathcal{F}^{-1}v_{n_0}$. Since $\mathscr{C}^\infty_{b,cyl}(H_-) \subset S_{\gamma,cyl}(H_-)$ we obtain $\tilde{u} \in S_{\gamma,cyl}(H_-)$. Moreover, using the triangular inequality we find

$$\|\tilde{u} - u\|^2_{\psi,s} = \left\|(1+|\psi|)^{s/2}v_{n_0} - (1+|\psi|)^{s/2}\mathcal{F}(u)\right\|_{L^2(H_-,\gamma)} \leq \varepsilon.$$

But this is our assertion. □

Theorem 2.3.18. *Let $\psi_1,\ \psi_2 \in CN(H_-)$ be two continuous negative definite functions, such that $(1+|\psi_2(\xi)|) \leq c(1+|\psi_1(\xi)|)$ for all $\xi \in H_-$. Then for any $s \geq 0$ the embedding $H^s_{\psi_1}(H_-) \hookrightarrow H^s_{\psi_2}(H_-)$ is continuous. Conversely suppose that the embedding $H^s_{\psi_1}(H_-) \hookrightarrow H^s_{\psi_2}(H_-)$ is continuous for some $s > 0$. Then we have $(1+|\psi_2(\xi)|) \leq c(1+|\psi_1(\xi)|)$ for all $\xi \in H_-$.*

Proof. The first part is obvious by the definition of the norms. Now let $u \in H^s_{\psi_1}(H_-) \cap S_{\gamma,cyl}(H_-)$. Then there exists a constant $c_0 > 0$ such that $\|u\|_{\psi_2,s} \leq c_0\,\|u\|_{\psi_1,s}$. For $\eta \in H_+$ we set $u_\eta := \mathcal{F}^{-1}U_{-\eta}\mathcal{F}u$. Then we obtain by Peetre's inequality 2.2.2(xi)e)

$$\begin{aligned}\left|(1+|\psi_1(\xi)|^{s/2}\,\mathcal{F}u_\eta(\xi)\right| &\leq 2^{s/2}(1+|\psi_1(\eta)|)^{s/2}(1+|\psi_1(\xi-\eta)|)^{s/2}\,|\mathcal{F}u_\eta(\xi)| \\ \left|(1+|\psi_2(\xi)|^{s/2}\,\mathcal{F}u_\eta(\xi)\right| &\geq 2^{-s/2}(1+|\psi_2(\eta)|)^{s/2}(1+|\psi_2(\xi-\eta)|)^{-s/2}\,|\mathcal{F}u_\eta(\xi)|\,.\end{aligned}$$

Thus we find

$$\begin{aligned}\|u_\eta\|_{\psi_1,s} &= \left\|(1+|\psi_1(\cdot)|)^{s/2}\mathcal{F}u_\eta(\xi)\right\|_{L^2(H_-,\gamma)} \\ &\leq c_1(1+|\psi_1(\eta)|)^{s/2}\left\|(1+|\psi_1(\cdot-\eta)|)^{s/2}U_{-\eta}(\mathcal{F}u)(\cdot)\right\|_{L^2(H_-,\gamma)} \\ &= c_1(1+|\psi_1(\eta)|)^{s/2}\left\|U_{-\eta}(1+|\psi_1|)^{s/2}\mathcal{F}u\right\|_{L^2(H_-,\gamma)} \\ &= c_1(1+|\psi_1(\eta)|)^{s/2}\,\|u\|_{\psi_1,s}\end{aligned}$$

and similarly

$$\|u_\eta\|_{\psi_2,s} \geq c_2(1+|\psi_2(\eta)|)^{s/2} \|u\|_{\psi_2,s}.$$

Combining these two inequalities we obtain

$$(1+|\psi_2(\eta)|)^{s/2} \leq \frac{\|u_\eta\|_{\psi_2,s}}{c_2 \|u\|_{\psi_2,s}} \leq c_0 \frac{\|u_\eta\|_{\psi_1,s}}{c_2 \|u\|_{\psi_2,s}} \leq \left(c_0 \frac{c_1 \|u\|_{\psi_1,s}}{c_2 \|u\|_{\psi_2,s}}\right)(1+|\psi_1(\eta)|)^{s/2}.$$

Thus we have proved our inequality for all $\eta \in H_+$. But since $H_+ \subset H_-$ and ψ_1, ψ_2 are continuous it follows that for all $\xi \in H_-$ we have

$$(1+|\psi_2(\xi)|) \leq \left(c_0 \frac{c_1 \|u\|_{\psi_1,s}}{c_2 \|u\|_{\psi_2,s}}\right)^{2/s} (1+|\psi_1(\xi)|). \qquad \square$$

Proposition 2.3.19. *Let ψ be a continuous negative definite function. Then the operator $-\psi(D)$ with domain of definition $S_{\gamma,cyl}(H_-)$ defined in 2.2.15 is closable. Moreover, let A denote the closure of $-\psi(D)$. Then we obtain $D(A) = H^2_\psi(H_-)$*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset S_{\gamma,cyl}(H_-)$ be a sequence such that $u_n \xrightarrow{n\to\infty} 0$ in $L^2(H_-,\gamma)$ and $-\psi(D)u_n \xrightarrow{n\to\infty} u$ in $L^2(H_-,\gamma)$ for some $u \in L^2(H_-,\gamma)$. Then we obtain for all $v \in S_{\gamma,cyl}(H_-)$:

$$\langle u\,,\, v\rangle_{\psi,0} = \lim_{n\to\infty} \langle -\psi(D)u_n\,,\, v\rangle_{\psi,0} = \lim_{n\to\infty} \langle u_n\,,\, -\psi(D)v\rangle_{\psi,0} = 0,$$

which yields $u = 0$. Thus $\psi(D)$ is closable. Moreover, since $1+|\psi|^2 \leq (1+|\psi|)^2 \leq 2(1+|\psi|^2)$ we obtain for $u \in S_{\gamma,cyl}(H_-)$

$$\begin{aligned}
\|u\|^2_{\psi,0} + \|\psi(D)u\|^2_{\psi,0} &= \|\mathcal{F}u\|^2_{\psi,0} + \|\psi(\cdot)\mathcal{F}u\|^2_{\psi,0} \\
&= \int_{H_-} (1+|\psi(\cdot)|^2)\, |\mathcal{F}u|^2 \, d\gamma \\
&\leq \int_{H_-} (1+|\psi(\cdot)|)^2\, |\mathcal{F}u|^2 \, d\gamma \\
&\leq 2\int_{H_-} (1+|\psi(\cdot)|^2)\, |\mathcal{F}u|^2 \, d\gamma \\
&= 2(\|u\|^2_{\psi,0} + \|\psi D u\|^2_{\psi,0})
\end{aligned}$$

But this implies that the norm $\|\cdot\|_{\psi,2}$ and the graph norm of $\psi(D)$ are equivalent. Thus we obtain by 2.3.17

$$D(A) = \overline{S_{\gamma,cyl}(H_-)}^{\|\cdot\|_{graph}} = \overline{S_{\gamma,cyl}(H_-)}^{\|\cdot\|_{\psi,2}} = H^2_\psi(H_-) \qquad \square$$

THEOREM 2.3.20. *Let ψ be a continuous negative definite function on H_-. Moreover, let the strongly continuous semi group T_t be defined as in Definition 2.2.18. We denote by $(A, D(A))$ the generator of this semi group. Then we have*

$$A = -\psi(D) \text{ on } S_{\gamma,cyl}(H_-) \text{ and } D(A) = H^2_\psi(H_-).$$

PROOF. In view of Theorem 2.2.20, Proposition 2.3.17 and 2.3.19 and Proposition [**80**, 4.3.6] we only have to show that T_t leaves $H^2_\psi(H_-)$ invariant. Then $H^2_\psi(H_-)$ is a core for A, but since A is closed on $H^2_\psi(H_-)$ we are finished. However, we have for $u \in H^2_\psi(H_-)$

$$\|T_t u\|_{\psi,2} = \left\|(1+|\psi|)\mathcal{F}\mathcal{F}^{-1}e^{-t\psi}\mathcal{F}(u)\right\|_{\psi,0} \leq \|(1+|\psi|)\mathcal{F}(u)\|_{\psi,0} = \|u\|_{\psi,2} \leq \infty.$$

But this shows the assertion. □

PROPOSITION 2.3.21. *Let $\psi \in CN(\mathbb{R}^n)$. Then $T_t = \mathcal{F}^{-1}e^{-t\psi\cdot}\mathcal{F}$ is an L^2_γ-sub Markovian semi group.*

PROOF. Let $u \in \mathscr{C}_{pol}(\mathbb{R}^n)$, where $\mathscr{C}_{pol}(\mathbb{R}^n)$ denotes the space of all continuous polynomial bounded functions on $\mathbb{R}^n$. We obtain $e^{-\frac{\|\cdot\|^2}{2}}u \in L^2(\mathbb{R}^n,\lambda)\cap L^1(\mathbb{R}^n,\lambda)$ and thus $V_{G,n}u \in L^2(\mathbb{R}^n,\lambda)\cap L^1(\mathbb{R}^n,\lambda)$. Hence by 1.4.10 we obtain

$$\begin{aligned}
T_t u(x) &= \mathcal{F}^{-1}e^{t\psi(x)}\mathcal{F}u(x)\\
&= \mathcal{F}^{-1}e^{t\psi(x)}[V^{-1}_{G,n}\tilde{\mathcal{F}}V_{G,n}u](x)\\
&= \mathcal{F}^{-1}[e^{t\psi(\cdot)}V^{-1}_{G,n}\tilde{\mathcal{F}}V_{G,n}u](x)\\
&= V^{-1}_{G,n}\tilde{\mathcal{F}}^{-1}[V_{G,n}e^{t\psi(\cdot)}V^{-1}_{G,n}\tilde{\mathcal{F}}V_{G,n}u](x)\\
&= V^{-1}_{G,n}\tilde{\mathcal{F}}^{-1}e^{t\psi(\cdot)}\tilde{\mathcal{F}}V_{G,n}u(x)\\
&= e^{\frac{\|x\|^2}{2}}\left(\tilde{\mathcal{F}}^{-1}e^{-t\psi(\cdot)}\tilde{\mathcal{F}}(u(\cdot)e^{\frac{-\|\cdot\|^2}{2}})\right)(x) = e^{\frac{\|x\|^2}{2}}\tilde{T}_t(u(\cdot)e^{\frac{-\|\cdot\|^2}{2}})(x),
\end{aligned}$$

where $\tilde{\mathcal{F}}$ denotes the usual Fourier-Transform and $\tilde{T}_t$ the semi group associated to the negative definite function $\psi(\cdot)$ in $L^2(\mathbb{R}^n,\lambda)$. For $u \in \mathscr{C}_{pol}(\mathbb{R}^n)$and $0 \leq u \leq e^{\frac{\|P_n\cdot\|_0}{2}}$ a.e. we have $0 \leq u(\cdot)e^{\frac{-\|\cdot\|^2}{2}} \leq 1$. Thus since $\tilde{T}_t$ is sub Markovian (cf.[**80**, Example 4.6.29]) we get $0 \leq \tilde{T}_t u(\cdot)e^{\frac{-\|\cdot\|^2}{2}} \leq 1$ a.e. But this implies $0 \leq T_t u \leq e^{\frac{\|P_n\cdot\|_0}{2}}$ a.e. Now let $u \in L^2(\mathbb{R}^n,\gamma)$ arbitrary such that $0 \leq u(x) \leq e^{\frac{\|P_n x\|_0}{2}}$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathscr{C}_{pol}(\mathbb{R}^n)$ with $0 \leq u_n(x) \leq e^{\frac{\|x\|^2}{2}}$ such that $u_n \xrightarrow{n\longrightarrow\infty} u$ in $L^2(\mathbb{R}^n,\gamma)$. But since T_t is bounded we have $T_t u_n \xrightarrow{n\longrightarrow\infty} T_t u$ in $L^2(\mathbb{R}^n,\gamma)$. Hence there exists a subsequence u_{n_k} such that $T_t u_{n_k} \xrightarrow{k\longrightarrow\infty} T_t u$ pointwisely. But since $0 \leq T_t u_{n_k} \leq e^{\frac{\|x\|^2}{2}}$ a.e. we obtain $0 \leq T_t u \leq e^{\frac{\|x\|^2}{2}}$ a.e. □

According to [**35**, Rem 2.2, p. 45] we have $\gamma = \gamma_n \otimes \gamma_R$, where γ_n is the canonical Gaussian measure with respect to the Hilbert space rigging

$\mathbb{R}^n \cong P_n H_+ \subset P_n H_0 \subset P_n H_- \cong \mathbb{R}^n$. Furthermore, γ_R is the canonical Gaussian measure with respect to the rigging $H_+ \ominus P_n H_+ \cong H_+ \cap (H_0 \ominus P_n H_0) \subset H_0 \ominus P_n H_0 \subset \{x \in H_- \mid P_n x = 0\} \cong H_- \ominus P_n(H_-)$. Here P_n denotes the orthogonal projection on $span\{e_1, \cdots e_n\}$ in H_0. Now by [**19**, p.24] it follows that

$$L^2(H_-, \gamma) = L^2(\mathbb{R}^n, \gamma_n) \widehat{\otimes} L^2(H_- \ominus P_n H_-, \gamma_R),$$

where $\widehat{\otimes}$ denotes the topological tensor-product of Hilbert Spaces. Now let us note the following lemma, which we will prove in Lemma 4.3.3 in a more general case.

LEMMA 2.3.22. *Let $\psi(x) = \Psi(\langle e_1 , x\rangle_0, \cdots , \langle e_{n_0} , x\rangle_0)$ be a cylindrical negative definite function and $u = f \otimes g$ where $f \in S_\gamma(\mathbb{R}^n)$ and $g \in L^2(P_n(H_-), \gamma_R)$. Then we have*

$$T_t u(x) = \mathcal{F}_n^{-1} e^{-t\Psi(x_1,\cdots,x_{n_0})} \mathcal{F}_n f(x_1, \cdots , x_n) \otimes g(x_{n+1}, \ldots).$$

PROPOSITION 2.3.23. *Let $\psi \in CN(H_-)$ be cylindric. Then $T_t = \mathcal{F}^{-1} e^{-t\psi\cdot} \mathcal{F}$ is an L^2_γ-sub Markovian semi group.*

PROOF. For the ONB $(e_k)_{k=1}^\infty \subset H_+$ in H_0 there exits an n_0 such that $\psi(x) = \Psi(\langle e_1 , x\rangle_0, \cdots , \langle e_{n_0} , x\rangle_0)$ for all $x \in H_-$. Let $u \in L^2(H_-, \gamma)$ be with with $0 \leq u(x) \leq e^{\frac{\|P_n x\|_0}{2}}$ for $n > n_0$. Now we will prove this lemma in four steps. At first let us assume that $u(x) = f(\langle e_1 , x\rangle_0, \cdot, \langle e_n , x\rangle_0) \otimes \chi_U(x_{n+1}, \ldots)$, where χ_U is the characteristic function of a set U with $\gamma_R(U) > 0$. Then we obtain $0 \leq f(\langle e_1 , x\rangle_0, \ldots, \langle e_n , x\rangle_0) \leq e^{\frac{\|P_n x\|_0}{2}}$ and

$$T_t u(x) = \mathcal{F}_n^{-1} e^{-t\Psi(x_1,\cdots,x_{n_0})} \mathcal{F}_n f(x_1, \cdots , x_n) \chi_U(x_{n+1}, \ldots)$$

where F_n denotes the Fourier-Wiener-transform in $\mathbb{R}^n$. But since now Ψ is negative definite to we obtain by proposition 2.3.21 and the fact that $|\chi_U| \leq 1$

$$0 \leq T_t u \leq e^{\frac{\|P_n \cdot\|_0}{2}} \text{ a.e.}$$

In a second step let us assume that $u(x) = f(\langle e_1 , x\rangle_0, \cdot, \langle e_n , x\rangle_0) \otimes g(x_{n+1}, \ldots)$, where g is an elementary function in $L^2(P_n(H_-), \gamma_R)$, i.e. $v = \sum_{j=1}^m a_j \chi_{U_j}$, such that $U_j \cap U_k = \emptyset$ for $k \neq j$ and $\gamma_R(U_j) > 0$. Then we have $u(x) = \sum_{j=1}^m (a_j f(x_1, \ldots, x_n)) \chi_{U_j(x_{n+1},\ldots)}$, where $0 \leq a_j f(x_1, \ldots, x_n) \leq e^{\frac{\|P_n x\|_0}{2}}$. Thus step 1 implies that $0 \leq (a_j f(x_1, \ldots, x_n)) \chi_{U_j(x_{n+1},\ldots)} \leq e^{\frac{\|P_n x\|_0}{2}}$. Thus we find $0 \leq u(x) \leq e^{\frac{\|P_n x\|_0}{2}}$ since all U_j are disjoint.

In a third step we will assume that $u(x) = \sum_{j=1}^m f_j(\langle e_1 , x\rangle_0, \cdot, \langle e_n , x\rangle_0) \otimes g_j(x_{n+1}, \ldots)$, where the g_j are elementary functions as in step 2. Thus we have $u(x) = \sum_{j=1}^m \sum_{i=1}^{k_j} a_{j,k} f_j(\langle e_1 , x\rangle_0, \ldots, \langle e_n , x\rangle_0) \chi_{U_j}(x_{n+1}, \ldots)$. But this shows that we find disjoint sets W_j and functions $\tilde{f}_j$ such that $u(x) = \sum_{j=1}^l \tilde{f}_j(\langle e_1 , x\rangle_0, \ldots, \langle e_n , x\rangle_0) \chi_{W_j}(x_{n+1}, \ldots)$. Thus step 2 implies that $0 \leq u(x) \leq e^{\frac{\|P_n x\|_0}{2}}$.

Finally, let $u \in L^2(H_-, \gamma)$ arbitrary such that $0 \le u(x) \le e^{\frac{\|P_n x\|_0}{2}}$. Then there exists a sequence $(u_m)_{m\in\mathbb{N}}$ of functions described in (iii) with $0 \le u_m(x) \le e^{\frac{\|P_n x\|_0}{2}}$ such that $u_m \xrightarrow{m\longrightarrow\infty} u$ in $L^2(H_-, \gamma)$ and pointwisely a.e. But since T_t is bounded there we have $T_t u_n \xrightarrow{n\longrightarrow\infty} T_t u$ in $L^2(H_-\gamma)$. Hence there exists a subsequence u_{m_k} such that $T_t u_{m_k} \xrightarrow{k\longrightarrow\infty} T_t u$ pointwisely. But since $0 \le T_t u_{n_k} \le e^{\frac{\|P_n \cdot\|_0}{2}}$ a.e. we obtain $0 \le T_t u \le e^{\frac{\|P_n \cdot\|_0}{2}}$ a.e. □

THEOREM 2.3.24. *Let ψ be a cylindrical continuous negative definite function on H_-.*

(i) *Then the operator $-\psi(D)$ extends to selfadjoint L^2_γ-Dirichlet operator $(A, H^2_\psi(H_-, \mathbb{R}))$.*

(ii) *The form $\mathcal{E}(u,v) = \langle -Au\, ,\, v\rangle_{\psi,0}$ extends to a symmetric L^2_γ-Dirichlet-form on $D(\mathcal{E}) = H^1_\psi(H_-)$ and we have*

$$\mathcal{E}(u,v) = \langle \psi^{1/2}\mathcal{F}(u)\, ,\, \psi^{1/2}\mathcal{F}(u)\rangle_{\psi,0} = \langle [\psi(D)]^{1/2}(u)\, ,\, [\psi(D)]^{1/2}(u)\rangle_{\psi,0}$$

on $S_{\gamma,cyl}(H_-)$.

PROOF. Let us show that $A := -\overline{\psi(D)}$ is selfadjoint on $H^2_\psi(H_-)$. Thus assume that there exists a v, v^* such that for all $u \in H^2_\psi(H_-)$ we have $\langle v\, ,\, Au\rangle_{\psi,0} = \langle v^*\, ,\, u\rangle_{\psi,0}$. Then we obtain

$$\langle v^*\, ,\, u\rangle_{\psi,0} = \langle v\, ,\, Au\rangle_{\psi,0} = \langle \mathcal{F}v\, ,\, \psi\mathcal{F}u\rangle_{\psi,0} = \langle \mathcal{F}^{-1}\psi\mathcal{F}v\, ,\, u\rangle_{\psi,0}.$$

Since $H^2_\psi(H_-)$ is dense in $L^2(H_-, \gamma)$ we obtain $v \in H^2_\psi(H_-)$ and $v^* = Av$. Thus A is selfadjont. The rest of the first part is now clear by Proposition 2.3.23, Lemma 2.3.3 and Theorem 2.3.20. To prove the second part let us first show the equation (17) is fulfilled. For $u, v \in H^2_\psi(H_-)$ we obtain

$$\begin{aligned}
\langle Au\, ,\, v\rangle_{\psi,0} &= \langle \psi^{1/2}\mathcal{F}u\, ,\, \psi^{1/2}\mathcal{F}v\rangle_{\psi,0} \\
&\le \langle \psi^{1/2}\mathcal{F}u\, ,\, \psi^{1/2}\mathcal{F}u\rangle_{\psi,0}\langle \psi^{1/2}\mathcal{F}v\, ,\, \psi^{1/2}\mathcal{F}v\rangle_{\psi,0} \\
&= \langle \psi\mathcal{F}u\, ,\, \mathcal{F}u\rangle_{\psi,0}\langle \mathcal{F}v\, ,\, \mathcal{F}v\rangle_{\psi,0} = \langle Au\, ,\, u\rangle_{\psi,0}\langle Av\, ,\, v\rangle_{\psi,0}.
\end{aligned}$$

Now let us note that $f(s) = s^{1/2}$ is a Bernstein function and thus $\psi^{1/2}$ is negative definite too. Thus the assertion follows by the equation above, Remark 2.2.25, Proposition 2.3.23 and Theorem 2.3.15. □

CHAPTER 3

Ψ^*-Algebras and generalized Hörmander classes of pseudodifferential operators in Weyl form

This chapter is concerned with certain aspect of pseudodifferential operators on infinite dimensional Hilbert space riggings in Weyl form. In [**56**] B. Gramsch introduced Ψ_0- and Ψ^*-algebras. A Fréchet algebra Ψ, which is continuously embedded in a C^*-algebra B, is called Ψ^*-algebra, if Ψ is spectrally invariant and symmetric.

In this chapter we will construct generalized Hörmander classes and other Ψ^*-algebras of pseudodifferential operators on infinite dimensional Hilbert spaces. We define a scale of Sobolev Spaces using the Ornstein-Uhlenbeck operator. Then $\frac{\partial}{\partial t}$ $(t \in H_+)$ and the operator of multiplication with $\langle \cdot, t\rangle_0$ $(t \in H_+)$ are continuous from H^s to H^{s+1}.

Starting with symbols (functions) $a(x,p)$ on H_-^2 Albeverio and Dalecky defined in [**2**] pseudodifferential operators $a(X,D)$ in Weyl form on infinite dimensional Hilbert space riggings $H_+ \subseteq H_0 \subseteq H_-$. We define generalized Hörmander classes $\widetilde{\Psi}^0_{\varrho,\delta}$ and other Ψ^*-algebras of operators acting in the scale of Sobolev spaces. These generalized Hörmander classes contain certain multiplication and convolution operators. Moreover, we show that pseudodifferential operators $a(X,D)$ with symbol $a \in \mathcal{G}$ are elements of one of our generalized Hörmander classes, namely $\widetilde{\Psi}^0_{0,0}$ Here $\mathcal{G}$ denotes the space of functions which a Fourier transforms of certain complex valued measures on H_+^2. For $t \in H_+$ the unitary weighted translations in direction t are elements of $\mathcal{G}$. Thus we cannot expect that these operators considered by Albeverio and Dalecky are elements of $\widetilde{\Psi}^0_{\varrho,\delta}$ for $\varrho \neq 0$. In section 3.3 and 3.4 as well as in chapter 5 we will also discuss the case where $\varrho \neq 0$ and $\delta \neq 0$.

Finally, we consider the case $H_+ = H_0 = H_- = \mathbb{R}^n$. Let a be a symbol in $S^0_{0,0}$. Then the corresponding pseudodifferential operator defined in [**2**] is in our generalized Hörmander class $\widetilde{\Psi}^0_{0,0}$. Furthermore, for any $a(X,D) \in \Psi^0 \subseteq \widetilde{\Psi}^0_{0,0}$ there exists an $a \in S^0_{0,0}$ such that a is the associated symbol to $a(X,D)$. Here Ψ^0 is a sub multiplicative Ψ^*-algebra.

3.1. Ψ^*-algebras generated by closed operators

In [**67**] Gramsch, Ueberberg and Wagner describe a construction of Ψ_0- resp. Ψ^*-algebras, starting from closed derivations or closed resp. symmetric operators (cf. [**67**]). These concepts are generalized by Lauter in [**96**]. Before we will define some Ψ^*-algebras, we will describe these concepts of constructing Ψ^*-algebras. Throughout the first part of this section we will follow closely [**96**]. We omit all proofs, but refer to [**67**] and [**96**]. In the following let $\mathcal{A}^{-1}$ denote the group of all invertible elements of an algebra $\mathcal{A}$.

DEFINITION 3.1.1 (Gramsch, 1984). Let B be a Banach algebra with unit e, and $\mathcal{A}$ be a sub algebra of B with $e \in \mathcal{A}$. Then

(i) $\mathcal{A}$ is called locally spectrally invariant in B, if there exists an $\varepsilon > 0$ such that
$$\{a \in \mathcal{A} \mid \|e - a\|_B < \varepsilon\} \subseteq \mathcal{A}^{-1},$$
where $\mathcal{A}^{-1}$ denotes the group of invertible elements in $\mathcal{A}$.

(ii) $\mathcal{A}$ is called spectrally invariant in B, if $\mathcal{A} \cap B^{-1} = \mathcal{A}^{-1}$ holds for the groups $\mathcal{A}^{-1}$ resp. B^{-1} of invertible elements in $\mathcal{A}$ resp. B.

(iii) $\mathcal{A}$ is called a Ψ_0-algebra in B, if $\mathcal{A}$ is locally spectrally invariant in B and there is a topology $\mathcal{T}_{\mathcal{A}}$ on $\mathcal{A}$, which makes $(\mathcal{A}, \mathcal{T}_{\mathcal{A}}) \hookrightarrow B$ into a continuously embedded Fréchet algebra.

(iv) $\mathcal{A}$ is called a Ψ^*-algebra in B, if in addition, B is a C^*-algebra and $\mathcal{A}$ is a symmetric Ψ_0-algebra in B.

(v) $\mathcal{A}$ is called a sub multiplicative Ψ_0- resp. Ψ^*-algebra, if the topology $\mathcal{T}_{\mathcal{A}}$ on $\mathcal{A}$ can be generated by a sub multiplicative family of semi norms $(q_j)_{j\in\mathbb{N}_0}$, i.e. $q_j(xy) \le q_j(x)q_j(y)$ and $q_j(e) = 1$.

According to [**23**], [**110**], [**132**], [**131**] the algebra $\mathcal{A}$ is called spectral invariant, full or algèbre pleine if $\mathcal{A} \cap \mathcal{B}^{-1} = \mathcal{A}^{-1}$. The pair $(\mathcal{A}, B)$ is known as Wiener pair (cf. [**110**, chapt. III, pp.203, 214, 310], [**128**]).

REMARK 3.1.2.

(i) Let $\mathcal{A}$ be a dense locally spectrally invariant sub algebra of B. Then $\mathcal{A}$ is spectrally invariant. In particular, every Ψ^*-algebra $\mathcal{A}$ in a C^*-algebra B is spectrally invariant in B. In the definition of Ψ_0-Algebra one can actually always achieve $\varepsilon = 1$ (cf. [**56**, Lemma 5.3] and [**96**, p. 14]).

(ii) The class of (sub multiplicative) Ψ_0- resp. Ψ^*- algebras is stable with respect to countable intersection (cf. [**96**, p. 14]).

(iii) Let $\mathcal{A}$ be a Fréchet Algebra with open group $\mathcal{A}^{-1}$ of invertible elements. Then the inversion $\mathcal{A}^{-1} \ni b \mapsto b^{-1} \in \mathcal{A}$ is continuous (cf. [**132**]).

Before describing the construction of Ψ^*-Algebras by closed derivations let us make some remarks about the importance of Ψ^*-ALgebras.

REMARK 3.1.3. As mentioned in the introduction nowadays it is well known that the Hörmander classes $\Psi_{\varrho,\delta}(\mathbb{R}^n)$ $(0 \le \delta \le \varrho,\ \varrho < 1)$ are submultiplicative Fréchet operator algebras with spectral invariance in $\mathscr{L}(L^2(\mathbb{R}^n, \lambda))$. But it was a rather long process until these theorem was completely proved. There are contributions of a series of mathematicians including Hörmander, Seeley, Caldéron and Vailloncourt, Beals, Cordes, Fefferman, Boney and Chemin, Gramsch, Ueberberg, Schrohe and Wagner.

During the last twenty years many results for algebras of Ψ^*-type have been proved. With these notions it is possible to develop an operator theory for some Fréchet algebras in microlocal analysis. Special non linear methods have been developed which sharpen some results in the Banach and C^*-setting (cf. [**56**], [**88**], [**87**]).

An important point in the theory of Ψ^*-algebras $\mathcal{A}$ is that the Hilbert space Fredholm inverses are automatically in $\mathcal{A}$. Thus one can develop perturbation theory in these Fréchet algebras for holomorphic Fredholm functions, e.g. one has

- Oka principle for holomorphic maps with values in complex Fréchet Lie groups or in Fréchet manifolds of Fredholm and Semi Fredholm operators in Ψ^*-algebras of pseudodifferential operators.
- Division of operator valued distributions.
- Existence of global holomorphic projection valued function splitting of the kernel of holomorphic Fredholm functions with fixed dimension of the kernel.
- Meromorphic inversion and decomposition of holomorphic Semi Fredholm functions also on infinite dimensional regions

A similar development is under way concerning the L^p-theory based on the notion of Ψ_0- as well as algebras of $\mathscr{C}^\infty$-elements with respect to group representations (cf. [**48**]).

In the case of a Fréchet space the implicit function Theorem is not available. Thus in [**56**] there are developed rational methods which can be applied instead. In this connection it was shown in [**56**] that the set of relatively regular and idempotent elements in Ψ^*-algebras form analytic locally rational Fréchet manifolds. Furthermore, there are results on abstract hypo ellipticity [**65**], wave front sets and propagation of singularities in Ψ^*-algebras which are due to Gramsch.

In connection with [**56**] and [**61**] it was observed in K-theory using Karoubi's density theorem [**28**], [**89**] that a Ψ_0-algebras (resp. Ψ^*-algebra) has the same K-theory as its norm closure (resp. C^*-closure). Gramsch and Kaballo [**63**] pointed out as a contribution to additive complex analytic cohomology that an additive decomposition of meromorphic resolvents of semi Fredholm functions into a holomorphic part and meromorphic part which is a small ideal can be generalized to the setting of Ψ^*-algebras. In addition, they gave further results on the division problem for real analytic Fredholm functions and operator distributions in Ψ^*-algebras. In the setting of submultiplicative Ψ^*-algebras $\mathcal{A}$ there also is a corresponding multiplicative decomposition for holomorphic Fredholm functions with values in $\mathcal{A}^{-1}$ on a Stein manifold [**64**]. In addition Gramsch derives an extension of the Oka-principle to submultiplicative Ψ^*-Algebras [**57**].

Let us mention some results following [**67**], where the research is still in progress and far from being completed. For any Hilbert space H it was shown in [**95**] that every Ψ^*-algebra in $\mathscr{L}(H)$ contains its holomorphic functional calculus in the sense of J.L. Taylor [**96**], [**124**]. Moreover, this calculus applies to algebras of $n \times n$-matrices with elements in Ψ^*-algebras. Lorentz showed in [**102**] that any Jordan operator A in a Ψ^*-algebras $\mathcal{A} \subset \mathscr{L}(X)$ admits a Jordan decomposition within $\mathcal{A}$ and as a consequence one has a local similarity cross section for A in $\mathcal{A}$.

Furthermore, the Oka-principle leads also to isomorphisms between non-abelian groups of holomorphic objects on the one side and continuous objects on the other side. The strategy of proofs involves essentially non-linear functional analytic and complex analytic methods.

In 1954 Waelbrock ([**132**], [**131**]) introduced a holomorphic functional calculus for complete locally convex algebras with continuous inversion even for several variables. The holomorphic functional calculus for Ψ_0 and Ψ^*-algebras is an direct consequence of his results and play an import role in the theory of these algebras (cf. [**28**], [**55**] and [**91**]). In addition to the standard Hörmander classes there are lots of other examples of Ψ^*-algebras such as $\mathscr{C}^\infty$-elements in C^*-dynamical systems [**24**], [**32**], [**29**] and certain families of cross products [**86**], [**85**], [**126**]. Since the important concept of spectral invariance was stressed by B. Gramsch, the theory of Ψ^*-algebras has developed into a useful tool in the analysis of pseudodifferential operators and Fréchet operator algebras on singular spaces.

The construction methods of Ψ_0 and Ψ^*-algebras given in [**67**] which we will describe later on are a quite flexible tool and they even apply to operator algebras on fractal sets [**92**].

Frank Baldus developed for an appropriate in general non compact manifold $\mathcal{M}$ with metric g and a weight function M on $T^*\mathcal{M}$ an $S(M,g)$-pseudodifferential calculus. In [**7**] it was shown that the algebra of order zero operators is a submultiplicative Ψ^*-algebra in the sense of B. Gramsch in $\mathscr{L}(L^2(\mathcal{M}))$. Using the spectral invariance within the $S(M,g)$-calculus the author of [**6**] gives sufficient conditions for an operator to extend to a generator of a Feller semi group.

Spectral invariance generates strong connections between Ψ^*-algebras and their C^*-closure. While representation theory for C^*-algebras has been treated in [**36**] Lauter developed a representation theory for Ψ^*-algebras [**99**]. More precisely, using a result due to Gramsch on positive functional calculus it can be shown that there is a continuous, bijective map $\phi : \hat{\mathcal{A}} \longrightarrow \hat{\mathcal{B}}$, where $\mathcal{B}$ is the enveloping C^*-algebra of a Ψ^*-algebra $\mathcal{A}$ and $\hat{\mathcal{A}}$ resp. $\hat{\mathcal{B}}$ denotes the spectrum of $\mathcal{A}$ resp. $\mathcal{B}$.

In a paper of Chen and Wei [**27**], which follows a series of results of Schweitzer, Jolissaint and de la Harpe it was mentioned that the notion of spectral invariance plays an important role in the work of Connes-Moscovici on the Novikov conjecture as well as in Laffourges research on the Baum-Connes conjecture. In this connection it is of interest that for certain discrete groups G with length function l the Schwarz space $S_2^l(G)$ with respect to l is a spectral invariant dense subalgebra of the reduced group C^*-algebra $C_r^*(G)$. For more details we refer to [**27**].

It is a well known fact that the dense embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ of a Ψ^*-algebra $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$ induces an isomorphism in K-theory of $\mathcal{B}$. Hence on the one hand $\mathcal{A}$ is large enough to preserve the K-theory of $\mathcal{B}$ on the other hand it is better related to the differential structure than a C^*-algebra. This fact is used in [**91**] to prove a vanishing theorem for higher traces in cyclic cohomology for the spectral projections. Further there are given applications to the Quantum hall effect and related spectral gaps of operators.

There are approaches by Ditsche on localization results for special classes of solvable C^*-algebras on manifolds with corners Z. Let $\Psi^0_{b,cl}(Z)$ be the algebra of classical pseudodifferential operators of order zero and $B(Z)$ its C^*-closure in $\mathscr{L}(L^2(Z))$. Then it is known by results of Lauter, Melrose and Nistor that $B(Z)$ is a solvable C^*-algebra in the sense of [**39**]. Moreover, one can choose a solving series of minimal length for $B(Z)$, such that the geometry of Z is readily seen in this ideal chain. Since this is a global approach it should also be possible to localize this procedure, i.e. to show, that if we restrict our algebra to small open neighborhoods of arbitrary point on Z, only the underlying geometry of those neighborhoods give a contribution to the ideal chain. To do this, J. Ditsche analyzes algebras $\psi B(Z)\psi$, where ψ is a cut off function with $\operatorname{supp}\psi \subseteq U$ and U a neighborhood of $p \in Z$. Moreover, it is shown how to calculate the length of algebras of parameter dependent pseudodifferential operators on Z.

Furthermore, in the most recent research on Ψ^*-Fréchet algebras, there are approaches to Toeplitz operators. In the case of the Segal-Bargmann space $H^2(\mathbb{C}^n, \gamma)$ of Gaussian square integrable entire functions on $\mathbb{C}^n$ Bauer determined in [**11**] a class of vector-fields $\mathcal{Y}(\mathbb{C}^n)$ supported in cones $\mathcal{C} \subset C^n$. Showing that for any finite subset $\mathcal{V} \subset \mathcal{Y}(\mathbb{C}^n)$ the Toeplitz projection is a smooth element in a Ψ_0-algebra constructed by commutator methods with respect to $\mathcal{V}$ he obtains localized Ψ_0- and Ψ^*-algebras $\mathcal{F}$ in the cones $\mathcal{C}$. As an immediate consequence he obtains, that $\mathcal{F}$ contains all Toeplitz operators T_f with f bounded on $\mathbb{C}^n$ and smooth with bounded derivatives of all orders in a neighborhood of $\mathcal{C}$. In addition there is a natural unitary group action on $H^2(\mathbb{C}^n, \gamma)$ which is induced by weighted shifts and unitary groups on $\mathbb{C}^n$. W. Bauer examined the corresponding Ψ^*-algebras $\mathcal{A}$ of smooth elements in Toeplitz-C^*-algebras and gave sufficient conditions on the symbol f for T_f to belong to $\mathcal{A}$ in terms of estimates on its Berezin-transform $\tilde{f}$.

In a paper [**100**] which appeared 2005 Lauter, Monthubert and Nistor constructed algebras of pseudodifferential operators on a continuous family groupiod $\mathcal{G}$ that are closed under holomorphic functional calculus, contain the algebra of pseudodifferential operators of order 0 on $\mathcal{G}$ as a dense subalgebra and reflect the structure of the groupoid $\mathcal{G}$, when $\mathcal{G}$ is smooth. As an application they got a better understanding of the structure of inverse of elliptic pseudodifferential operators on classes of non-compact manifolds. For the construction of these algebras closed under holomorphic functional calculus they used commutator methods. Furthermore, they reduced the construction of spectrally invariant algebras of order 0 pseudodifferential operators to the analogous problem for regularizing operators. They introduced a generalized 'cusp'-calculi c_n, $n \geq 2$ on manifolds with boundary and with corners and embedded these calculi in Ψ^*-algebras consisting of smooth kernels.

Now let us describe the construction of Ψ^*-algebras by commutator methods.

Definition 3.1.4. For algebras $D(\delta)$ and $\mathcal{A}$ a linear mapping $\delta : D(\delta) \longrightarrow \mathcal{A}$ is called derivation, if δ fulfills

$$\delta(xy) = \delta(x)y + x\delta(y) \quad \forall x, y \in D(\delta).$$

Furthermore, if $D(\delta)$ and $\mathcal{A}$ are endowed with a $^*-$operation and if $\delta(x^*) = \delta(x)^*$ for all $x \in D(\delta)$, then δ is called a *-derivation. It is called an anti-*-derivation if $\delta(x^*) = -\delta(x)^*$ for all $x \in D(\delta)$. In addition, if $D(\delta)$ is a sub algebra of a Fréchet algebra $\mathcal{A}$ such that δ is a closed linear operator, then δ is said to be a closed derivation.

Definition 3.1.5. (cf. [**96**] p.27) Let B be a C^*-algebra with unit e, $(\mathcal{A},\ (q_j)_{j\in\mathbb{N}_0})$ be a sub multiplicative Ψ^*-algebra in B, and Δ be a finite set of closed derivations $\delta : \mathcal{A} \supseteq D(\delta) \longrightarrow \mathcal{A}$ with $e \in D(\delta)$. Put

(i) $\Psi_0^\Delta := \mathcal{A}$ with semi-norms $q_{0,j} := q_j$ for $j \in \mathbb{N}_0$.
(ii) $\Psi_1^\Delta := \bigcap\limits_{\delta\in\Delta} D(\delta)$.
(iii) $\Psi_n^\Delta := \{a \in \Psi_{n-1}^\Delta \,|\, \delta a \in \Psi_{n-1}^\Delta \text{ for all } \delta \in \Delta\}$, $n \geq 2$.
(iv) $\Psi_\infty^\Delta := \bigcap\limits_{n\in\mathbb{N}_0} \Psi_n^\Delta$.
(v) Endow Ψ_n^Δ for $n \geq 1$ with the system of seminorms

$$q_{n,j}(a) := q_{n-1,j}(a) + \sum_{\delta\in\Delta} q_{n-1,j}(\delta a) \text{ for } a \in \Psi_n^\Delta \subseteq \Psi_1^\Delta \text{ and } j \in \mathbb{N}_0$$

and Ψ_∞^Δ with the system $(q_{n,j})_{n\in\mathbb{N},j\in\mathbb{N}_0}$.

Proposition 3.1.6.

(i) $(\Psi_n^\Delta, (q_{n,j})_{j\in\mathbb{N}_0}) \hookrightarrow \mathcal{A}$ *is a continuously embedded Fréchet sub algebra of* $\mathcal{A}$ *and* $q_{n,j}$ *is a sub multiplicative seminorm on* Ψ_n^Δ.
(ii) $(\Psi_\infty^\Delta, (q_{n,j})_{n\in\mathbb{N},j\in\mathbb{N}_0}) \hookrightarrow \mathcal{A}$ *is a continuously embedded, sub multiplicative Fréchet algebra.*

(iii) Ψ_∞^Δ *is a sub multiplicative* Ψ_0*-algebra in* B.

PROOF. See [**96**, Proposition 2.4.3]. □

COROLLARY 3.1.7. *In addition, let each* $\delta \in \Delta$ *be a closed* **- or anti* **-derivation with respect to the* **operation induced by the* C^**-algebra* B*. Then*

(i) Ψ_n^Δ *is a symmetric sub algebra of* $\mathcal{A}$ *with respect to the* **operation induced by* B.
(ii) Ψ_∞^Δ *is a sub multiplicative* $\Psi^* - algebra$ *in* B.

PROOF. See [**96**, Corollary 2.4.4] and [**96**, Remark 2.4.5]. □

DEFINITION 3.1.8. Let H be a Hilbert space and $(\mathcal{A},\ (q_j)_{j\in\mathbb{N}_0}) \hookrightarrow \mathscr{L}(H)$ be a sub multiplicative Ψ^*-algebra. Without loss of generality we assume $q_0 = \|\cdot\|_{\mathscr{L}(H)}$. For a closed, densely defined operator $V\ :\ H \supseteq D(V) \longrightarrow H$ we define

(i) $\mathcal{J}(V) := \{a \in \mathcal{A}\,|\,a(D(V)) \subseteq D(V)\}$.
(ii) $\mathrm{ad}(V)\ :\ \mathcal{J}(V) \longrightarrow \mathscr{L}(D(V), H)$ by $\mathrm{ad}(V)(a)x = [V,\ a]x := (Va - aV)x$ for $a \in \mathcal{J}(V)$ and $x \in D(V)$ and recursively $\mathrm{ad}^j(V)(a) := \mathrm{ad}(V)(\mathrm{ad}^{j-1}(V)(a))$.
(iii) $\mathfrak{B}(V) := \{a \in \mathcal{J}(V)\,|\,ad(V)$ extends to a bounded linear operator $\delta_V a \in \mathcal{A}\}$. For $a \in \mathfrak{B}(V)$ $\delta_V a$ is uniquely determined by $\mathrm{ad}(V)a$, since $D(V) \subset H$ dense.
(iv) $\mathfrak{B}^*(V) := \{a \in \mathfrak{B}(V)\,|\,a^* \in \mathfrak{B}(V)\}$.

LEMMA 3.1.9.

(i) $\delta_V\ :\ \mathcal{A} \subset \mathfrak{B}(V) \longrightarrow \mathcal{A}\ :\ a \longmapsto \delta_V(a)$ *is a closed derivation.*
(ii) *If, in addition,* $V\ :\ H \supseteq D(V) \longrightarrow H$ *is symmetric, then* $\delta_V\ :\ \mathcal{A} \supseteq \mathfrak{B}^*(V) \longrightarrow \mathcal{A}$ *is a closed anti* **-derivation.*

PROOF. See [**96**, Lemma 2.4.7]. □

DEFINITION 3.1.10. Let E be a Banach space and $\mathcal{V}$ be a finite set of closed, densely defined operators $V\ :\ E \supseteq D(V) \longrightarrow E$. Then we define

(i) $\mathcal{H}_\mathcal{V}^0 := E$ with norm $p_0 := \|\cdot\|_E$.
(ii) $\mathcal{H}_\mathcal{V}^1 := \bigcap\limits_{V\in\mathcal{V}} D(V)$.
(iii) $\mathcal{H}_\mathcal{V}^n := \{\xi \in \mathcal{H}_\mathcal{V}^{n-1}\,|\,V\xi \in \mathcal{H}_\mathcal{V}^{n-1}$ for all $V \in \mathcal{V}\}$, $n \geq 2$.
(iv) $\mathcal{H}_\mathcal{V}^\infty := \bigcap\limits_{n\in\mathbb{N}} \mathcal{H}_\mathcal{V}^n$.
(v) Endow $\mathcal{H}_\mathcal{V}^n$ with the norm $p_n(\xi) := p_{n-1}(\xi) + \sum\limits_{V\in\mathcal{V}} p_{n-1}(V\xi)$, $\xi \in \mathcal{H}_\mathcal{V}^n$ and $\mathcal{H}_\mathcal{V}^\infty$ with the system of norms $(p_n)_{n\in\mathbb{N}_0}$.

LEMMA 3.1.11.

(i) $(\mathcal{H}_\mathcal{V}^n,\ p_n)$ *is a Banach space.*
(ii) $(\mathcal{H}_\mathcal{V}^\infty,\ (p_n)_{n\in\mathbb{N}_0})$ *is a Fréchet space.*

(iii) *The closed operator* $V \in \mathcal{V}$ *induces for each* $n \in \mathbb{N}$ *a operator* $\mathfrak{I}_n(V) \in \mathscr{L}(\mathcal{H}_{\mathcal{V}}^n, \mathcal{H}_{\mathcal{V}}^{n-1})$.

(iv) *If* E *is a Hilbert space, then there exists an equivalent norm* $\tilde{p_n}$ *on* $\mathcal{H}_{\mathcal{V}}^n$, *which makes* $(\mathcal{H}_{\mathcal{V}}^n,\ \tilde{p_n})$ *into a Hilbert space and* $(\mathcal{H}_{\mathcal{V}}^\infty,\ (\tilde{p_n})_{n\in\mathbb{N}_0})$ *into a Fréchet-Hilbert space.*

PROOF. See [**96**, Lemma 2.4.11] □

THEOREM 3.1.12. *Let* H *be a Hilbert space,* $(\mathcal{A},\ (q_j)_{j\in\mathbb{N}_0})$ *be a sub multiplicative* Ψ^**-algebra in* $\mathscr{L}(H)$, B *be a* C^**-algebra in* $\mathscr{L}(H)$ *with* $\mathcal{A} \subseteq B$ *and* $\mathcal{V}$ *a finite set of closed, densely defined operators* $V\ :\ H \supseteq D(V) \longrightarrow H$ *such that* V *or* iV *is symmetric. Furthermore, let*

(i) $\mathcal{H}_{\mathcal{V}}^n$ *resp.* $\mathcal{H}_{\mathcal{V}}^\infty$ *be as in Lemma 3.1.11.*

(ii) $\Delta := \Delta_{\mathcal{V}} := \{\delta_V \,|\, V \in \mathcal{V}\}$ *be the set of closed anti* **- or* **-derivations* $\delta_V\ :\ \mathcal{A} \supseteq D(\delta_V) \longrightarrow \mathcal{A}$ *with values in* $\mathcal{A}$, *constructed as in Lemma 3.1.9.*

(iii) $\Psi_n^{\mathcal{V}} := \Psi_n^{\Delta_{\mathcal{V}}}$ *resp.* $\Psi_\infty^{\mathcal{V}} := \Psi_\infty^{\Delta_{\mathcal{V}}}$ *be the scale of symmetric sub multiplicative Fréchet algebras constructed above corresponding to the set* $\Delta_{\mathcal{V}}$ *of closed (anti)***-derivations.*

Then we have

(i) $\Psi_\infty^{\mathcal{V}} \subseteq \Psi_n^{\mathcal{V}} \subseteq \mathcal{A} \subseteq B$ *for all* $n \in \mathbb{N}$.

(ii) $(\Psi_\infty^{\mathcal{V}},\ (q_{n,j})_{n\in\mathbb{N},j\in\mathbb{N}_0}) \hookrightarrow B$ *is a sub multiplicative* Ψ^**-algebra.*

(iii) $\Psi_n^{\mathcal{V}} \times \mathcal{H}_{\mathcal{V}}^n \longrightarrow \mathcal{H}_{\mathcal{V}}^n\ :\ (a,\varphi) \longmapsto a(\varphi)$ *is continuous and bilinear.*

(iv) $\Psi_\infty^{\mathcal{V}} \times \mathcal{H}_{\mathcal{V}}^\infty \longrightarrow \mathcal{H}_{\mathcal{V}}^\infty\ :\ (a,\varphi) \longmapsto a(\varphi)$ *is continuous and bilinear.*

(v) $\delta_V\ :\ \Psi_\infty^{\mathcal{V}} \longrightarrow \Psi_\infty^{\mathcal{V}}$ *is continuous.*

PROOF. See [**96**, Theorem 2.4.13]. □

PROPOSITION AND DEFINITION 3.1.13. *Let* $H_+ \subseteq H_0 \subseteq H_-$ *be a quasi-nuclear Hilbert space rigging and let* $(e_j)_{j\in\mathbb{N}} \subset H_+$ *be an orthonormal basis in* H_0. *Moreover, let* γ *be the canonical Gaussian measure with respect to this rigging. Let* $M_j := M_{e_j}$ *be defined as in Definition 1.2.2 and let* $D_j := D_{e_j}$ *be defined as in 1.3.6. Then we set*

$$\mathcal{V}_k := \{M_1,\ \ldots,\ M_k,\ D_1,\ \ldots,\ D_k\}.$$

Furthermore, we define $\mathcal{H}_{\mathcal{V}_k}^n$ *resp.* $\mathcal{H}_{\mathcal{V}_k}^\infty$ *and* $\Psi_n^{\mathcal{V}_k}$ *resp.* $\Psi_\infty^{\mathcal{V}_k}$ *as in Theorem 3.1.12, with* $\mathcal{A} = \mathscr{L}(L^2(H_-,\ \gamma))$. *Now we set for all* $n \in \mathbb{N}$

$$\mathcal{H}_{MD}^n := \bigcap_{k\in\mathbb{N}} \mathcal{H}_{\mathcal{V}_k}^n, \qquad \Psi_n^{MD} := \bigcap_{k\in\mathbb{N}} \Psi_n^{\mathcal{V}_k}$$

and

$$\mathcal{H}_{MD}^\infty := \bigcap_{k\in\mathbb{N}} \mathcal{H}_{\mathcal{V}_k}^\infty, \qquad \Psi^{MD} := \bigcap_{k\in\mathbb{N}} \Psi_\infty^{\mathcal{V}_k}.$$

Then Ψ_n^{MD} *and* Ψ^{MD} *are sub multiplicative* Ψ^* *algebras. Moreover, we have*

(i) $\Psi_n^{MD} \times \mathcal{H}_{MD}^n \longrightarrow \mathcal{H}_{MD}^n\ :\ (a,\varphi) \longmapsto a(\varphi)$ *is continuous and bilinear.*

(ii) $\Psi^{MD} \times \mathcal{H}^\infty_{MD} \longrightarrow \mathcal{H}^\infty_{MD} \; : \; (a, \varphi) \longmapsto a(\varphi)$ *is continuous and bilinear.*

PROOF. Since M_j is selfadjoint and iD_j is selfadjoint, δ_{M_j} is an anti-*-derivation and δ_{D_j} is a *-derivation. Thus Theorem 3.1.12 implies that $\Psi^{\mathcal{V}_k}_n$ and $\Psi^{\mathcal{V}_k}_\infty$ are sub multiplicative Ψ^*-algebras and hence, the first assertion follows with Remark 3.1.2. The rest is a direct consequence of Theorem 3.1.12. □

3.2. Commutators of pseudodifferential operators in Weyl-form with multiplication operators and partial derivations

In the first part of this section we give a definition of pseudodifferential operators starting from a symbol on the infinite dimensional Hilbert space H^2_-. Moreover, we show some basic properties of these operators and describe a class of continuous pseudodifferential operators. Throughout the first part of this section we follow closely [**2**].

Let $\mathcal{F} \; : \; L^2(H_-, \; \gamma) \longrightarrow L^2(H_-, \; \nu)$ denote the isometric isomorphism from 1.4.4 given by $\mathcal{F}U_t = V_t\mathcal{F}$. Moreover, for $\tau \in H_+$ define the family $W_\tau : \; L^2(H_-, \; \gamma) \longrightarrow L^2(H_-, \; \gamma)$ by

$$W_\tau f = e^{i\langle \tau, \cdot \rangle_0} f. \tag{25}$$

REMARK 3.2.1. The operators U_t and W_t satisfy the commutator relation in Weyl form

$$U_t W_\tau = e^{i\langle t, \tau \rangle_0} W_\tau U_t.$$

DEFINITION 3.2.2 (pseudodifferential Operator, Albeverio, Dalecky [**2**]). Let $a(x, p)$ be a symbol (a function) on H^2_-. Define the pseudodifferential operator $a(X, D)$ in $L^2(H_-, \; \gamma)$ by

$$a(X, D)\varphi(x) = \mathcal{F}^{-1}_{p \to x} \mathcal{F}_{y \to p} \left[a\left(\frac{x+y}{2}, p \right) \varphi(y) \right]. \tag{26}$$

The sign "$p \to x$" means that the corresponding operator is applied to a function of p and the result is considered as a function of x.

EXAMPLE 3.2.3. Let us compute some pseudodifferential operators. Thus let $t \in H_+$ be fixed.

(i) For $a(x, p) = \langle t, p \rangle_0$ and $\varphi \in \mathscr{C}^1_{int}(H_-)$ we obtain

$$a(X, D)\varphi(x) = \mathcal{F}^{-1}_{p \to x} \mathcal{F}_{y \to p}[\langle t, p \rangle_0 \varphi(y)] = F^{-1}_{p \to x}[\langle t, p \rangle_0 F(\varphi)(p)] = \frac{1}{i} D_t \varphi(x).$$

(ii) Let $a(x, p) = \langle t, x \rangle_0$ Then for $\varphi \in \mathscr{C}_{int}(H_-)$ we obtain

$$a(X, D)\varphi(x) = \frac{1}{2}\big(\langle t, x \rangle_0 \mathcal{F}^{-1}_{p \to x} \mathcal{F}_{y \to p} \varphi(y) + \mathcal{F}^{-1}_{p \to x} \mathcal{F}_{y \to p}[\langle t, y \rangle_0 \varphi(y)] \big) = \langle t, x \rangle_0 \varphi(x).$$

At next we describe what these operators look like in the finite dimensional case.

REMARK 3.2.4. Let $H_+ = H_0 = H_- = \mathbb{R}^n$. We assume that $\gamma = \gamma_1$ is the canonical Gaussian measure in $\mathbb{R}^n$. Moreover, let a be a symbol on $\mathbb{R}^{2n}$. Then

$$a(X,D)f(x) = e^{\frac{\|x\|^2}{2}} a(X,\tilde{D})(e^{-\frac{\|\cdot\|^2}{2}} f)(x),$$

where $a(X,\tilde{D})$ is the pseudodifferential operator in $\mathbb{R}^n$ given in Weyl-form[1], i.e.

$$a(X,\tilde{D})f(x) = \tilde{\mathcal{F}}^{-1}_{p\to x}\tilde{\mathcal{F}}_{y\to p}\left[a\left(\frac{x+y}{2},p\right)f(y)\right],$$

where $\tilde{\mathcal{F}}$ is the Fourier-transform in $\mathbb{R}^n$ with the Lebesgue measure.

PROOF. Applying 1.4.9 we obtain

$$\begin{aligned} a(X,D)f(x) &= \mathcal{F}^{-1}_{p\to x}\mathcal{F}_{y\to p}\left[a\left(\frac{x+y}{2},p\right)f(y)\right] \\ &= V^{-1}_{G,n}\tilde{\mathcal{F}}^{-1}_{p\to x}V_{G,n}V^{-1}_{G,n}\tilde{\mathcal{F}}_{y\to p}V_{G,n}\left[a\left(\frac{x+y}{2},p\right)f(y)\right] \\ &= e^{\frac{\|x\|^2}{2}}\tilde{\mathcal{F}}^{-1}_{p\to x}\tilde{\mathcal{F}}_{y\to p}\left[a\left(\frac{x+y}{2},p\right)e^{-\frac{\|y\|^2}{2}}f(y)\right]. \end{aligned}$$

Furthermore, for $a \in S^0_{\varrho,\delta}(\mathbb{R}^n)$ we obtain

$$\begin{aligned} \|a(X,D)f\|_{L^2(\mathbb{R}^n,\,d\gamma_1)} &= \left\|V^{-1}_{G,n}a(X,D)V_{G,n}f\right\|_{L^2(\mathbb{R}^n,\,d\gamma_1)} \\ &= \left\|a(X,D)V_{G,n}f\right\|_{L^2(\mathbb{R}^n,\,d\lambda)} \\ &\le c\left\|V_{G,n}f\right\|_{L^2(\mathbb{R}^n,\,d\lambda)} = c\left\|f\right\|_{L^2(\mathbb{R}^n,\,d\gamma_1)}, \end{aligned}$$

where λ denotes the Lebesgue measure in $\mathbb{R}^n$ and $c \le 0$ suitable. □

Now we consider a certain class of symbols. Our aim is to describe the pseudodifferential operators attached to such symbols more detailed. Thus we define the symbol we want to consider at next.

DEFINITION 3.2.5.

(i) Let $M_\infty(H^2_+,\mathbb{C})$ be the space of complex valued measures θ on $\mathcal{B}(H^2_+)$ such that

$$\int_{H^2_+} e^{a\|x\|_{H^2_+}} d\,|\theta|\,(x) < \infty \quad \forall a \in \mathbb{R}.$$

(ii) Furthermore, let $\mathcal{G}$ be the space of Fourier transforms of measures $\theta \in M_\infty(H^2_+)$, i.e.

$$\mathcal{G} = \left\{ a \,|\, a(x,p) = \int e^{i\langle x,x'\rangle_0 + i\langle p,p'\rangle_0} d\theta(x',p'), \quad \theta \in M_\infty(H^2_+,\mathbb{C}) \right\}.$$

[1]For more information about pseudodifferential operators in Weyl-form see for example Folland [**43**, chapter 2].

PROPOSITION 3.2.6. *For $a \in \mathcal{G}$ being the Fourier transform of a measure ξ the operator $a(X,D)$ is defined on $\mathscr{C}^{\infty}_{int}(H_-)$ and the following formula holds on $\mathscr{C}^{\infty}_{int}(H_-)$*

$$a(X,D)f = \int W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f \; d\xi(x',p') \tag{27}$$

or equivalently

$$a(X,D)f = \int W_{x'} U_{p'} e^{\frac{i}{2}\langle x',p\rangle_0} f \; d\xi(x',p'). \tag{28}$$

Furthermore, for $a \in \mathcal{G}$ the formula above holds for any $f \in L^2(H_-,\ \gamma)$.

PROOF. See [**2**, Proposition 3.7]. □

PROPOSITION 3.2.7. *Let $a \in \mathcal{G}$. Then $a(X,D)$ is a continuous linear operator in $L^2(H_-,\ \gamma)$.*

PROOF. Let $a(x,p) = \int e^{i\langle x,x'\rangle_0 + i\langle p,p'\rangle_0} d\xi(x',p')$. Then there exists a ξ-measurable function $g(x',p')$ with $|g| = 1$ and $d\xi = gd\,|\xi|$ (cf. [**22**]). Applying 3.2.6 we obtain

$$\begin{aligned}
&\|a(X,D)f\|^2_{L^2(H_-,\ \gamma)} \\
= &\int \left| \int e^{i\langle x,x'\rangle_0} U_{p'} e^{\frac{i}{2}\langle x',p'\rangle_0} f(x) g(x',p') d\,|\xi|\,(x',p') \right|^2 d\gamma(x) \\
\le &\int \int \left| e^{i\langle x,x'\rangle_0} e^{\frac{i}{2}\langle x,x'\rangle_0} g(x',p') \right|^2 d\,|\xi|\,(x',p') \int |U_{p'} f(x)|^2 \, d\,|\xi|\,(x',p') d\gamma(x) \\
= &\int 1 \; d\,|\xi|\,(x',p') \int \int |U_{p'} f(x)|^2 \, d\gamma(x) d\,|\xi|\,(x',p') \\
= &\; c \int \|U_{p'} f\|^2_{L^2(H_-,\ \gamma)} \; d\,|\xi|\,(x',p') \\
\le &\; c \int \|U_{p'}\|^2_{Op} \; \|f\|^2_{L^2(H_-,\ \gamma)} \; d\,|\xi|\,(x',p') \le \; c^2 \,\|f\|^2_{L^2(H_-,\ \gamma)}\,,
\end{aligned}$$

where $c > 0$ is chosen suitably. □

REMARK 3.2.8. Let $a \in \mathcal{G}$. Then we have $a(X,D)^* = b(X,D)$, where $b \in \mathcal{G}$. Moreover, if a is the Fourier transform of a positive measure ξ, we obtain $a(X,D)^* = \overline{a}(X,D)$.

PROOF. For $a \in \mathcal{G}$ there exists a measure $\xi \in M_\infty(H^2_+, \mathbb{C})$ such that $a(x,p) = \int e^{i\langle x,x'\rangle_0 + i\langle p,p'\rangle_0} \; d\; \xi(x',p')$. In addition, there exists a ξ-measurable function $h(x',p')$ such that $|h| = 1$ and $d\xi = hd\,|\xi|$. Hence for $f,\ g \in L^2(H_-,\ \gamma)$ we obtain

$$\langle a(X,D)f, g\rangle_{L^2(H_-,\ \gamma)} \; = \; \iint W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x) \overline{g(x)} \; d\xi(x',p') d\gamma(x)$$

$$
\begin{aligned}
&= \iint W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x)\overline{g(x)}\, d\gamma(x) d\xi(x',p') \\
&= \iint f(x) W_{\frac{x'}{2}} U_{p'}{}^* W_{\frac{x'}{2}} \overline{g(x)}\, d\gamma(x) d\xi(x',p') \\
&= \iint f(x) W_{\frac{x'}{2}} U_{p'}{}^* W_{\frac{x'}{2}} \overline{g(x)} h(x',p')\, d\,|\xi|\,(x',p') d\gamma(x) \\
&= \iint f(x) \overline{W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x) \overline{h(-x',-p')} d\widetilde{|\xi|}(x',p')} d\gamma(x) \\
&= \iint f(x) \overline{W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x) d\theta(x',p')} d\gamma(x) \\
&= \langle f, b(X,D) g\rangle_{L^2(H_-,\,\gamma)},
\end{aligned}
$$

where $\widetilde{|\xi|}(x',p')$ is the image of the measure $|\xi|$ under the mapping $(x',p') \longmapsto (-x',-p')$ and $d\theta(x',p') = \overline{h(-x',-p')} d\widetilde{|\xi|}(x',p')$. In the case of a positive measure we have $h \equiv 1$. Thus the second assertion is clear. □

Our aim is to show that for $a \in \mathcal{G}$ the operator $a(X,D)$ is an element of the Ψ^*-algebra defined in section 3.1. Therefore we have to study the commutators of $a(X,D)$ and the multiplication and partial differential operators in direction of elements of H_+. Let us start with the multiplication operators.

THEOREM 3.2.9. *Let $a \in \mathcal{F}$ and $t \in H_+$. Then $a(X,D)(D(M_t)) \subseteq D(M_t)$ and $[M_t, a(X,D)]$ can be extended to $L^2(H_-,\,\gamma)$ continuously, where M_t is defined as in 1.2.2. Moreover, for all $j \in \mathbb{N}$ $(\mathrm{ad}M_t)^j(a(X,D))$ can be extended to a continuous operator on $L^2(H_-,\,\gamma)$.*

PROOF. Let $f \in D(M_t)$. Then we have

$$
\begin{aligned}
\langle t, x\rangle_0 U_{p'} f(x) &= \langle t, x\rangle_0 \sqrt{\varrho_{p'}(x)} f(x+p') \\
&= \sqrt{\varrho_{p'}(x)} (\langle t, -p'\rangle_0 + \langle t, x+p'\rangle_0) f(x+p') \\
&= -\langle t, p'\rangle_0 U_{p'} f(x) + U_{p'}(\langle t, x\rangle_0 f(x)).
\end{aligned}
$$

It follows $[M_t,\ W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}}] f = W_{\frac{x'}{2}} [M_t,\ U_{p'}] W_{\frac{x'}{2}} f = -\langle t, p'\rangle_0 W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f$ and thus

$$
\begin{aligned}
&\|M_t a(X,D) f - a(X,D) M_t f\|^2_{L^2(H_-,\,\gamma)} \\
&= \int \left| \int \langle t, -p'\rangle_0 W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x) d\xi(x',p') \right|^2 d\gamma(x) \\
&\leq \int \int \|p'\|^2_{H_+} \|t\|^2_{H_-}\, d\,|\xi|\,(x',p') \int |U_{p'} f(x)|^2\, d\,|\xi|\,(x',p') d\gamma(x) \\
&\leq c' \int \|U_{p'} f\|^2_{L^2(H_-,\,\gamma)}\ d\,|\xi|\,(x',p') \leq c\ \|f\|^2_{L^2(H_-,\,\gamma)}.
\end{aligned}
$$

Now it follows that $\mathrm{ad}^j(M_t)(a(X,D))f(x) = \int \langle t, -p'\rangle_0^j W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x) d\xi(x',p')$.
Thus as above we obtain $\left\|\mathrm{ad}^j(M_t)(a(X,D))f(x)\right\|^2_{L^2(H_-,\,\gamma)} \leq c\ \|f\|^2_{L^2(H_-,\,\gamma)}$. □

For $t \in H_+$ we define D_t and ∂_t as in Proposition 1.3.8 and Proposition 1.2.4 and for $t \in H_-$ we denote by $\frac{\partial}{\partial t}$ the partial derivative.

LEMMA 3.2.10. *Let $x' \in H_+$ and $f \in \mathscr{C}^1_{int}(H_-)$. Then we find for $t \in H_-$ that $[\frac{\partial}{\partial t},\ W_{\frac{x'}{2}}]f(x) = i\langle \frac{x'}{2}, t\rangle_0 W_{\frac{x'}{2}} f(x)$ and for $t \in H_+$ that $[D_t,\ W_{\frac{x'}{2}}]f(x) = i\langle \frac{x'}{2}, t\rangle_0 W_{\frac{x'}{2}} f(x)$.*

PROOF. Let $t \in H_-$ and $f \in \mathscr{C}^1_{int}(H_-)$. Then the following equality holds.

$$\begin{aligned}
&[\frac{\partial}{\partial t},\ W_{\frac{x'}{2}}]f(x)\\
=\ & \frac{\partial}{\partial t}\left(e^{i\langle \frac{x'}{2},x\rangle_0} f(x)\right) - e^{i\langle \frac{x'}{2},x\rangle_0}\frac{\partial}{\partial t} f(x)\\
=\ & e^{i\langle \frac{x'}{2},x\rangle_0}\frac{\partial}{\partial t} f(x) + i\langle \frac{x'}{2}, t\rangle_0 e^{i\langle \frac{x'}{2},x\rangle_0} f(x) - e^{i\langle \frac{x'}{2},x\rangle_0}\frac{\partial}{\partial t} f(x) = i\langle \frac{x'}{2}, t\rangle_0 W_{\frac{x'}{2}} f(x).
\end{aligned}$$

Thus for $t \in H_+$ we get $[D_t,\ W_{\frac{x'}{2}}]f(x) = [\frac{\partial}{\partial t} - \langle t, \cdot\rangle_0,\ W_{\frac{x'}{2}}]f(x) = [\frac{\partial}{\partial t},\ W_{\frac{x'}{2}}]f(x)$.
□

LEMMA 3.2.11. *Let $p' \in H_+$ and $f \in \mathscr{C}^1_{int}(H_-)$. Then we have $[\frac{\partial}{\partial t},\ U_{p'}]f(x) = -\langle p', t\rangle_0 U_{p'} f(x)$ for $t \in H_-$. For $t \in H_+$ this yields $[D_t,\ U_{p'}]f(x) = 0$.*

PROOF. Let $p' \in H_+$ and $t \in H_-$ fixed. Then we have for $f \in \mathscr{C}^1_{int}(H_-)$

$$\begin{aligned}
[\frac{\partial}{\partial t},\ U_{p'}]f(x) &= \frac{\partial}{\partial t}\left(\sqrt{\varrho_{p'}(x)} f(x+p')\right) - U_{p'}\frac{\partial}{\partial t} f(x)\\
&= \frac{1}{2\sqrt{\varrho_{p'}(x)}}\varrho_{p'}(x) 2\langle p', t\rangle_0 f(x+p') = -\langle p', t\rangle_0 U_{p'} f(x).
\end{aligned}$$

Hence we get for $t \in H_+$ and $f \in \mathscr{C}^1_{int}(H_-)$

$$\begin{aligned}
&[D_t,\ U_{p'}]f(x)\\
=\ & [\frac{\partial}{\partial t} - \langle t, \cdot\rangle_0,\ U_{p'}]f(x)\\
=\ & [\frac{\partial}{\partial t},\ U_{p'}]f(x) - \langle t, x\rangle_0\sqrt{\varrho_{p'}(x)} f(x+p') + \sqrt{\varrho_{p'}(x)}\langle t, x+p'\rangle_0 f(x+p')\\
=\ & (-\langle p', t\rangle_0 + \langle t, p'\rangle_0) U_{p'} f(x) = 0,
\end{aligned}$$

since $\langle \cdot, \cdot\rangle_0$ is a real inner product. □

COROLLARY 3.2.12. *Let $x', p' \in H_+$ and $f \in \mathscr{C}^1_{int}(H_-)$. For $t \in H_-$ we have*

$$[\frac{\partial}{\partial t},\ W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}}]f(x) = (i\langle x', t\rangle_0 - \langle p', t\rangle_0)\, W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x).$$

Moreover, for $t \in H_+$ we get $[D_t,\ W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}}]f(x) = i\langle x', t\rangle_0 W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x)$.

Proof. Let $x', p' \in H_+$ fixed and $f \in \mathscr{C}^1_{int}(H_-)$. Then applying Lemma 3.2.10 and Lemma 3.2.11, for $t \in H_-$ the following equality holds.

$$\begin{aligned}
&[\frac{\partial}{\partial t},\ W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}}] f(x) \\
= \;& [\frac{\partial}{\partial t},\ W_{\frac{x'}{2}}] U_{p'} W_{\frac{x'}{2}} f(x) + W_{\frac{x'}{2}} [\frac{\partial}{\partial t},\ U_{p'}] W_{\frac{x'}{2}} f(x) + W_{\frac{x'}{2}} U_{p'} [\frac{\partial}{\partial t},\ W_{\frac{x'}{2}}] f(x) \\
= \;& (i\langle x', t\rangle_0 - \langle p', t\rangle_0)\, W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x).
\end{aligned}$$

Similarly, according to Lemma 3.2.10 and Lemma 3.2.11, we have for $t \in H_+$ and $f \in \mathscr{C}^1_{int}(H_-)$

$$[D_t,\ W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}}] f = i\langle x', t\rangle_0 W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f. \qquad \square$$

Considering Proposition 3.2.6, we have to show that the integral and the partial derivatives commute. Therefore we start with a technical estimation.

Lemma 3.2.13. *Let $f \in \mathscr{C}^1_{pol}(H_-)$, $t \in H_-$ and $x \in H_-$ arbitrary. Then there exist $K \geq 0$ and $a \geq 0$, such that*

$$\left| \frac{\partial}{\partial t} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(y) \right| \leq K\, e^{\,a(\|x'\| + \|p'\|)} \quad \forall y \in U_1(x).$$

Proof. Let $x \in H_-$ be fixed. Since $f \in \mathscr{C}^1_{pol}(H_-)$, there exist $K_1, m \geq 0$ such that $|f(y)| \leq K_1(1 + \|y\|_-)^m$ and $\left|\frac{\partial}{\partial t} f(y)\right| \leq K_1(1 + \|y\|_-)^m$ for all $y \in H_-$. Thus there exist $k,\ K,\ a \geq 0$ such that

$$\begin{aligned}
&\left| \frac{\partial}{\partial t} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(y) \right| \\
= \;& \left| (i\langle x', t\rangle_0 - \langle p', t\rangle_0)\, W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(y) + W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} \frac{\partial}{\partial t} f(y) \right| \\
\leq \;& \left(\|x'\|_+ \|t\|_- + \|p'\|_+ + \|t\|_-\right) \left| \sqrt{\varrho_{p'}(y)}\ \ f(y + p') \right| + \left| \sqrt{\varrho_{p'}(y)}\ \ \frac{\partial}{\partial t} f(y + p') \right| \\
\leq \;& (k(\|x'\|_+ + \|p'\|_+) + 1) \sqrt{e^{-\|p'\|_0^2}} \sqrt{e^{-2\langle p', y\rangle_0}} K_1 (1 + \|y + p'\|_-)^m \\
\leq \;& K\, e^{\,a(\|x'\| + \|p'\|)}.
\end{aligned} \qquad \square$$

Corollary 3.2.14. *For $\xi \in M_\infty(H_+^2)$, $f \in \mathscr{C}^1_{pol}(H_-)$ and $t \in H_-$ the following equality holds:*

$$\frac{\partial}{\partial t} \int W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x)\, d\xi(x', p') = \int \frac{\partial}{\partial t} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x)\, d\xi(x', p').$$

Moreover, the mapping $x \mapsto \int \frac{\partial}{\partial t} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x)\, d\xi(x', p')$ is continuous.

Proof. The assertion follows by the differentiation lemma and 3.2.13. $\square$

Corollary 3.2.15. *For $f \in \mathscr{C}^1_{pol}(H_-)$ and $a \in \mathcal{F}$ we have $a(X, D) f \in \mathscr{C}^1_{int}(H_-)$.*

PROOF. Let $M \subset H_-$ be bounded. Thus there exists $C > 0$ such that $\|x\|_- \leq C$ for all $x \in M$. Hence we obtain for all $x \in M$

$$a(X,D)f(x) \leq \int_{H_+} |U_{p'}f(x)| \; d|\xi|(x',p')$$

$$\leq C' \int_{H_+} \left| e^{\|x\|_- \|p'\|_+} (1 + \|x + p'\|_-)^m \right| \; d|\xi|(x',p')$$

$$\leq C' \int_{H_+} \left| e^{C\|p'\|_+} (1 + C + \|p'\|_+)^m \right| \; d|\xi|(x',p') \leq \tilde{C}.$$

Thus $a(X,D)f$ is bounded on bounded sets. Now let us consider the derivative. According to 3.2.14 we have

$$\begin{aligned} & \frac{\partial}{\partial t} a(X,D)f(x) \\ (29) \qquad & = \int \frac{\partial}{\partial t} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x) \; d\xi(x',p') \\ & = \int W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} \frac{\partial}{\partial t} f(x) + (\langle ix' - p', t \rangle_0) \, W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x) \; d\xi(x',p'). \end{aligned}$$

This yields $\frac{\partial}{\partial t} a(X,D)f(x)$ is continuous for all $t \in H_-$ (cf. Lemma 3.2.14) and the Fréchet derivative of $a(X,D)f$ exists. As above, we obtain that $d\, a(X,D)f$ is bounded on bounded sets and hence we have $a(X,D)f \in \mathscr{C}^1_{int}(H_-)$. □

REMARK 3.2.16. Applying the proof of Corollary 3.2.15 several times for each $f \in \mathscr{C}^\infty_{pol}(H_-)$ it follows $a(X,D)f \in \mathscr{C}^\infty_{int}(H_-)$.

PROPOSITION 3.2.17. *Let $a \in \mathcal{F}$, $t \in H_+$ and $f \in D(\partial_t)$. Then we have $a(X,D)(D(\partial_t)) \subseteq D(\partial_t)$ and*

$$[\partial_t, \; \hat{a}]f(x) = \int (i\langle x', t \rangle_0 - \langle p', t \rangle_0) \, W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x) \; d\xi(x',p').$$

As well, we get $a(X,D)(D(D_t)) \subseteq D(D_t)$ and for $f \in D(D_t)$ we obtain

$$[D_t, \; \hat{a}]f(x) = \int i\langle x', t \rangle_0 W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x) \; d\xi(x',p').$$

Thus $[\partial t, \; \hat{a}]$ and $[D_t, \; \hat{a}]$ can be extended continuously to $L^2(H_-, \; \gamma)$.

PROOF. Let $a \in \mathcal{F}$, $t \in H_+$ and $f \in \mathscr{C}^\infty_{pol}(H_-)$. According to Remark 3.2.16 and equation (29) we have $a(X,D)f \in \mathscr{C}^\infty_{int}(H_-)$ and $[\partial_t, \; \hat{a}]f(x) = \int (i\langle x', t \rangle_0 - \langle p', t \rangle_0) \, W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x) \; d\xi(x',p')$. Now let $f \in D(\partial_t)$ be arbitrary. Then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathscr{C}^\infty_b(H_-)$ such that $f_n \xrightarrow[n\to\infty]{} f$ and

$\partial_t f_n \xrightarrow[n\to\infty]{} \partial_t f$ in $L^2(H_-,\ \gamma)$. Remark 3.2.16 implies $a(X,D)f_n \in \mathscr{C}_{int}^\infty(H_-)$ and $\frac{\partial}{\partial t}a(X,D)f_n \in \mathscr{C}_{int}^\infty(H_-)$ and thus we have

$$\begin{aligned} & \partial_t a(X,D)f_n \\ = \quad & a(X,D)\partial_t f_n + \int (i\langle x',t\rangle_0 - \langle p',t\rangle_0)\, W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f_n(x)\, d\xi(x',p') \\ \xrightarrow[n\to\infty]{} \quad & a(X,D)\partial_t f + \int (i\langle x',t\rangle_0 - \langle p',t\rangle_0)\, W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x)\, d\xi(x',p'), \end{aligned}$$

since $a(X,D)$ is continuous. As $\frac{\partial}{\partial t}$ is closed this is our assertion. For $t \in H_+$ fixed we consider the operator D_t. Let $a \in \mathcal{F}$ and $f \in \mathscr{C}_b^\infty(H_-)$. Corollary 3.2.14 and 3.2.12 yield $[D_t,\ \hat{a}]f(x) = \int i\langle x',t\rangle_0 W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x)\, d\xi(x',p')$. Thus this assertion follows similarly to the first assertion. Moreover, the rest is similarly to 3.2.9 ☐

Notations 3.2.18. Let $\{e_j\}_{j=1}^\infty \subset H_+$ be an orthonormal basis of H_- and let $\beta \in \mathbb{N}_0^{\mathbb{N}}$. Furthermore, let ν denote the length of β. Then using the Notations of 3.2.18 we set

(i) $M^\beta := M_1^{\beta_1} M_2^{\beta_2} \ldots M_\nu^{\beta_\nu},\ \ \partial^\beta := \partial_1^{\beta_1}\partial_2^{\beta_2}\ldots\partial_\nu^{\beta_\nu},\ \ D^\beta := D_1^{\beta_1} D_2^{\beta_2}\ldots D_\nu^{\beta_\nu}$,
(ii) $A^\beta(p') := \langle e_1, -p'\rangle_0^{\beta_1} \langle e_2, -p'\rangle_0^{\beta_2} \ldots \langle e_\nu, -p'\rangle_0^{\beta_\nu}$,
(iii) $B^\beta(x') := (i\langle x', e_1\rangle_0)^{\beta_1} (i\langle x', e_2\rangle_0)^{\beta_2} \ldots (i\langle x', e_\nu\rangle_0)^{\beta_\nu}$,
(iv) $\mathcal{B}^\beta(x',p') := \left(i\langle x', e_1\rangle_0 - \langle S^{-1}p', e_1\rangle_0\right)^{\beta_1} \ldots \left(i\langle x', e_\nu\rangle_0 - \langle S^{-1}p', e_\nu\rangle_0\right)^{\beta_\nu}$.

Proposition 3.2.19. *Let $\alpha,\ \beta \in \mathbb{N}_0^{\mathbb{N}}$, $a \in \mathcal{F}$ and $f \in D(M^\alpha \partial^\beta)$ resp. $f \in D(M^\alpha D^\beta)$. Then we have*

$$\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(\partial)(a(X,D))f(x) = \int A^\alpha(p')\mathcal{B}^\beta(x',p') W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x) d\xi(x',p'),$$

$$\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(a(X,D))f(x) = \int A^\alpha(p') B^\beta(x') W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x) d\xi(x',p').$$

Proof. The assertion follows by induction similarly to 3.2.13, 3.2.14, 3.2.17 and can be found in [**71**, Prop. 3.4.4]. ☐

Theorem 3.2.20. *For $\alpha, \beta \in \mathbb{N}_0^N$ and $a \in \mathcal{F}$ $\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(\partial)(a(X,D))$ and $\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(a(X,D))$ can be extended to continuous linear operators on $L^2(H_-,\ \gamma)$.*

Proof. Let $g \in L^1(H_+^2, |\xi|)$ with $|g| = 1$ such that $gd\,|\xi| = d\xi$. For $a \in \mathcal{F}$ and $f \in D(M^\alpha D^\beta)$ we have

$$\begin{aligned} & \int \left|\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(a(X,D))f(x)\right|^2\ d\gamma(x) \\ \overset{3.2.19}{\leq} & \int\int \left|A^\alpha(p')B^\beta(x')\right|^2 d\,|\xi|\,(x',p') \int \left|W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x)\right|^2\ d\,|\xi|\,(x',p')\ d\gamma(x) \\ \overset{3.2.9,3.2.17}{\leq} & K \iint |U_{p'}f(x)|^2\ d\gamma(x) d\,|\xi|\,(x',p') \leq \tilde{K}\, \|f\|^2_{L^2(H_-,\ \gamma)}\,. \end{aligned}$$

Similarly we can show that $\mathrm{ad}^{\alpha}(M)\mathrm{ad}^{\beta}(\partial)(a(X,D))$ can be extended to an element of $\mathscr{L}(L^2(H_-,\ \gamma))$. □

COROLLARY 3.2.21. *Let $a \in \mathcal{G}$. Then we have $a(X,D) \in \Psi^{MD}$.*

PROOF. Follows obviously by Lemma 3.2.19 and Theorem 3.2.20. □

3.3. A scale of Sobolev spaces generated by the Ornstein-Uhlenbeck operator and generalized Hörmander classes

In the finite dimensional case we can describe the Ψ^*-algebra $\Psi^0_{\varrho,\delta}$ $(0 \leq \delta \leq \varrho \leq 1,\ \delta < 1)$ of pseudodifferential operators by $\Psi^0_{\varrho,\delta} := \{a \in \mathscr{L}(H^0) \,|\, ad^{\alpha}(M)ad^{\beta}(\frac{\partial}{\partial x})(a) \in \mathscr{L}(H^s, H^{s+\varrho|\alpha|-\delta|\beta|})\ \forall s \in \mathbb{R}\ \forall \alpha,\ \beta \in \mathbb{N}_0^n\}$, where H^s are the usual finite dimensional Sobolev spaces. In this chapter we construct a similar Ψ^*-algebra in the infinite dimensional case. Moreover, we define generalized Hörmander classes and show that these classes are Ψ^*-algebras in $\mathscr{L}(L^2(H_-,\ \mu))$. To define the Sobolev spaces we use the Laplace operator from the previous section. Furthermore, we show that the operator of partial differentiation maps H^s continuously to H^{s-1} for all $s \in \mathbb{R}$.

Henceforth let Λ be a strictly positive operator in a separable Hilbert space H. For $s \geq 0$ we set $H^s_\Lambda := D(\Lambda^s)$ and $H^{-s}_\Lambda := (H^s_\Lambda)'$. Moreover, for $f \in H^{-s}_\Lambda$, $g \in H^s_\Lambda$ we consider the pairing $\langle f, g\rangle_{H^0_\Lambda} := \langle \Lambda^{-s} f, \Lambda^s g\rangle_{H^0_\Lambda}$.

PROPOSITION 3.3.1. *Let $A \in \bigcap_{k\in\mathbb{Z}} \mathscr{L}(H^k_\Lambda)$. Then $A \in \bigcap_{s\in\mathbb{R}} \mathscr{L}(H^s_\Lambda)$ and there exists a unique operator $A^* \in \bigcap_{s\in\mathbb{R}} \mathscr{L}(H^s_\Lambda)$ such that $\langle Af, g\rangle_{H^0_\Lambda} = \langle f, A^*g\rangle_{H^0_\Lambda}$ for $f \in H^{-s}_\Lambda$ and $g \in H^s_\Lambda$.*

PROOF. According to [**31**, Prop. 6.4, p.32], there exists a unique operator $A^* \in \bigcap_{k\in\mathbb{Z}} \mathscr{L}(H^k_\Lambda)$ such that $\langle Af, g\rangle_{H_\Lambda} = \langle f, A^*g\rangle_{H_\Lambda}$ for $f \in H^{-k}_\Lambda$ and $g \in H^k_\Lambda$. Now for fixed $s > 0$, there exist $k \in \mathbb{N}$ and $0 < \theta < 1$ such that $s = \theta k$. Applying Theorem [**25**, Theorem 1.5.5] we get $A, A^* \in \mathscr{L}([H, H^k_\Lambda]_\theta)$, since $A, A^* \in \mathscr{L}(H) \cap \mathscr{L}(H^k_\Lambda)$. Thus Theorem [**25**, Theorem 1.5.10] implies that $A, A^* \in \mathscr{L}(H^s_\Lambda)$. Since $s > 0$ arbitrary, it follows that $A, A^* \in \bigcap_{s\geq 0} \mathscr{L}(H^s_\Lambda)$. For any fixed $s > 0$ the adjoint $(A^*)^*_s \in \mathscr{L}(H^{-s}_\Lambda)$ of A^* with respect to the inner product in H exists. For $f \in H$ and $g \in H^\infty_\Lambda := \bigcap_{s\in\mathbb{R}} H^s_\Lambda$ we get

$$\langle (A^*)^*_s f, g\rangle_H - \langle (A^*)^* f, g\rangle_H = \langle f, A^*g\rangle - \langle f, A^*g\rangle = 0.$$

Since H^∞_Λ is dense in H (cf. [**31**, p.30 Prop. 6.1]), it follows that $(A^*)^*_s f = (A^*)^* f = Af$ for $f \in H$. Hence A admits a continuous extension to H^{-s}_Λ. But this shows $A \in \bigcap_{s\in\mathbb{R}} \mathscr{L}(H^s_\Lambda)$. Now using [**31**, p.32 Prop. 6.4] again, we get $A^* \in \bigcap_{s\in\mathbb{R}} \mathscr{L}(H^s_\Lambda)$. □

COROLLARY 3.3.2. *Let $A, A^* \in \bigcap_{k\in\mathbb{N}} \mathscr{L}(H^k_\Lambda)$, where A^* is the adjoint of A in H^0_Λ. Then $A, A^* \in \bigcap_{s\in\mathbb{R}} \mathscr{L}(H^s_\Lambda)$.*

PROOF. Modifying the proof of Proposition 3.3.1 we obtain the result. □

Now we define a scale of Sobolev spaces H^s according to the Laplace operator defined in Chapter 4. Moreover, using commutator-methods we define Ψ^*-algebras and generalized Hörmander classes in $\mathscr{L}(H^0)$. We show that these operator algebras are subsets of $\mathscr{L}(H^s)$ for all $s \in \mathbb{R}$. For this purpose we use some results of the previous section. At the end we note a proposition about commutator estimates.

DEFINITION 3.3.3. Let $H_+ \subseteq H_0 \subseteq H_-$ be a quasi-nuclear Hilbert space rigging and γ the canonical Gaussian measure with respect to this measure. Moreover, let L_γ be defined as in 2.1.6. Then we set

$$\Lambda := (\mathrm{L}_\gamma + id)^{1/2}$$

and define

$$H^s := D(\Lambda^s) \text{ for } s \geq 0$$

with inner product

$$\langle f, g \rangle_{H^s} := \langle \Lambda^s f, \Lambda^s f \rangle_{L^2(H_-,\, \mu)} \quad \forall f, g \in H^s.$$

Since Λ is a strictly positive operator, H^s is a Hilbert space and the norm in H^s is equivalent to the norm defined in 3.1.10 for $k \in \mathbb{N}_0$. Furthermore, we set $H^{-s} := (H^s)'$, where the duality is given with respect to the inner product in H^0. In addition, we define

$$H^\infty = \bigcap_{s \in \mathbb{R}} H^s \qquad \text{and} \qquad H^{-\infty} = \bigcup_{s \in \mathbb{R}} H^s.$$

H^s is called Sobolev space of order s.[2]

DEFINITION 3.3.4. Let $0 < \varepsilon \leq 1$. Then we define

$$\begin{aligned} \mathcal{A}^\varepsilon &:= \Psi_\infty^{\{\Lambda^\varepsilon\}} \\ &= \{a \in \mathscr{L}(H^0) \,|\, a(H^\infty) \subseteq H^\infty \text{ and } \left\| \mathrm{ad}^j(\Lambda^\varepsilon)(a) f \right\|_{H^0} \leq c_j \left\| f \right\|_{H^0} \\ &\qquad\qquad \forall f \in H^\infty \ \forall j \in \mathbb{N}_0, \ \text{ and suitable } c_j \geq 0\}, \end{aligned}$$

as in Theorem 3.1.12. Since Λ^ε is selfadjoint, $\mathcal{A}^\varepsilon$ is a Ψ^*-algebra. Moreover, according to [**25**, Theorem 2.3.11], we have $\mathcal{A}^{\varepsilon'} \subseteq \mathcal{A}^\varepsilon$ for $0 < \varepsilon \leq \varepsilon' \leq 1$.

Our next aim is to show that $\mathcal{A}^\varepsilon \subseteq \bigcap_{s \in \mathbb{R}} \mathscr{L}(H^s)$. Therefore we prove the following result.

LEMMA 3.3.5. *Let H be a Hilbert space and $Z : D(Z) \longrightarrow H$ and $A : D(A) \longrightarrow H$ linear. Furthermore, we assume that there exists $D \subset D(Z) \cap D(A)$*

[2] The Sobolev spaces H^s coincide with the Sobolev spaces $\mathbb{D}_2^s$ introduced by Malliavin cf. [**21**, page 116]. Thus we have again that H^s is the completion of the polynomials with respect to the norm $\|\cdot\|_{H^s}$, at least in the case oft he canonical Gaussian measure.

such that $Z(D) \subseteq D$. Let $f \in D$ such that $AZ^j f \in D$ for all $j \in \mathbb{N}_0$. Then we have

$$Z^n Af = \sum_{k=0}^{n} \binom{n}{k} \mathrm{ad}^k(Z)(A) Z^{n-k} f.$$

PROOF. (by induction). For n=0 our hypothesis is true. Thus we assume that the induction hypothesis is true for $n \in \mathbb{N}$ fixed. For $f \in D$ we get

$$\begin{aligned} Z^{n+1}Af &= Z(Z^n Af) \\ &= \sum_{k=0}^{n} \binom{n}{k} Z\mathrm{ad}^k(Z)(A)Z^{n-k} f \\ &= \sum_{k=0}^{n} \binom{n}{k} (\mathrm{ad}^k(Z)(A)Z^{n-k+1} f + \mathrm{ad}^{k+1}(Z)(A)Z^{n-k} f) \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} \mathrm{ad}^k(Z)(A) Z^{n+1-k} f. \end{aligned}$$ □

COROLLARY 3.3.6. *Let $0 < \varepsilon \leq 1$ and $A \in \mathcal{A}^\varepsilon$. Then we have*

$$A \in \bigcap_{s \in \mathbb{R}} \mathscr{L}(H^s).$$

PROOF. Let $A \in \mathcal{A}^\varepsilon$. Since $\mathcal{A}^\varepsilon$ is a Ψ^*-algebra, $A^* \in \mathcal{A}^\varepsilon$. According to Lemma 3.3.5 we obtain $A,\ A^* \in \bigcap_{k \in \mathbb{N}_0} H^{\varepsilon k}$. Thus 3.3.2 implies

$$A \in \bigcap_{s \in \mathbb{R}} \mathscr{L}(H^s).$$ □

DEFINITION 3.3.7.
Let $H_+ \subseteq H_0 \subseteq H_-$ be a quasi-nuclear Hilbert space rigging and let $(e_j)_{j \in \mathbb{N}} \subset H_+$ be an orthonormal basis in H_0. Moreover, let μ be the measure on $\mathscr{B}(H_-)$, which fulfills the conditions of Proposition 1.3.7. Let $M_j := M_{e_j}$ be defined as in 1.2.2 and $D_j := D_{e_j}$ as in 1.3.6. Then we set

$$\mathcal{V}_k := \{M_1,\ \ldots,\ M_k,\ D_1,\ \ldots,\ D_k\}.$$

Furthermore, let $\mathcal{A} := \mathcal{A}^1$ be constructed as in 3.3.4. We define $\Psi_\infty^{\mathcal{V}_k}$ as in Theorem 3.1.12, i.e.

$$\Psi_\infty^{\mathcal{V}_k} = \bigcap_{n \in \mathbb{N}_0} \Psi_n^{\mathcal{V}_k},$$

where

- $\Psi_0^{\mathcal{V}_k} := \mathcal{A}$,
- $\Psi_1^{\mathcal{V}_k} := \bigcap_{V \in \mathcal{V}_k} D(\delta_V)$ (δ_V defined as in 3.1.8),
- $\Psi_n^{\mathcal{V}_k} := \{a \in \Psi_{n-1}^{\mathcal{V}_k} \mid \delta_V a \in \Psi_{n-1}^{\mathcal{V}_k} \text{ for all } V \in \mathcal{V}_k\}$, $n \geq 2$.

Now we set
$$\Psi^0 := \bigcap_{k\in\mathbb{N}_0} \Psi_\infty^{\mathcal{V}_k}.$$

THEOREM 3.3.8. *Let Ψ^0 be defined as in 3.3.7. Then Ψ^0 is a sub multiplicative Ψ^*-algebra in H^0, and*
$$\Psi^0 \subseteq \bigcap_{s\in\mathbb{R}} \mathscr{L}(H^s).$$
Moreover, $\Psi^0 \times H^\infty \longrightarrow H^\infty \;:\; (a,\varphi) \longmapsto a(\varphi)$ is continuous and bilinear.

PROOF. According to 3.1.12 $\Psi_\infty^{\mathcal{V}_k}$ is a sub multiplicative Ψ^*-algebra and thus Remark 3.1.2 implies that Ψ^0 is a Ψ^*-algebra. Moreover, $\Psi_\infty^0 \subseteq \mathcal{A}^1$ and thus $\Psi^0 \subset \bigcap_{s\in\mathbb{R}} \mathscr{L}(H^s)$. □

Now, according to [**67**, section 3] we define generalized Hörmander classes.

DEFINITION 3.3.9. Let $\alpha,\ \beta \in \mathbb{N}_0^{\mathbb{N}}$. Moreover, let $\mathrm{ad}^\alpha(M)$ and $\mathrm{ad}^\beta(D)$ be defined as in 3.2.18. For $0 \le \delta \le \varrho \le 1$ and $\delta < 1$ we define the generalized Hörmander-class $\widetilde{\Psi}^0_{\varrho,\delta}$ by
$$\widetilde{\Psi}^0_{\varrho,\delta} := \{A \in \mathcal{A}^{1-\delta} \,|\, \mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(A) \in \bigcap_{s\in\mathbb{R}} \mathscr{L}(H^s, H^{s+\varrho|\alpha|-\delta|\beta|}),\ \forall\, \alpha,\beta \in \mathbb{N}_0^{\mathbb{N}}\}.$$
Furthermore, let $\|\cdot\|_{\mathcal{A}^{1-\delta},l}$ be a fundamental system of sub multiplicative semi-norms on $\mathcal{A}^{1-\delta}$. Then for $A \in \widetilde{\Psi}^0_{\varrho,\delta}$ we define a system of semi-norm by
$$\|A\|_{k,0,0,0} := \|\cdot\|_{\mathcal{A}^{1-\delta},k}$$
and
$$\|A\|_{s,l,l',\nu} := \sup_{\substack{|\alpha|\le l,\ l(\alpha)\le\nu \\ |\beta|\le l',\ l(\beta)\le\nu}} \left\|\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(A)\right\|_{\mathscr{L}(H^s,H^{s+\varrho|\alpha|-\delta|\beta|})},$$
where $k,l,l',\nu \in \mathbb{N}$, $s\in\mathbb{R}$, $\alpha,\ \beta \in \mathbb{N}_0^{\mathbb{N}}$ and $l(\alpha)$ resp. $l(\beta)$ denotes the length of α resp. β.

THEOREM 3.3.10. *For $0 \le \delta \le \varrho \le 1$ and $\delta < 1$ $\widetilde{\Psi}^0_{\varrho,\delta}$ is a sub multiplicative Ψ^*-algebra in $\mathscr{L}(H^0)$. Furthermore, $\widetilde{\Psi}^0_{\varrho,\delta} \times H^\infty \longrightarrow H^\infty \;:\; (a,\varphi) \longmapsto a(\varphi)$ is continuous and bilinear.*

PROOF. See [**130**]. □

REMARK 3.3.11. It is clear that $\Psi^0 \subseteq \widetilde{\Psi}^0_{0,0}$.

REMARK 3.3.12. As mentioned in 3.1.3 it was a long way until it was proved that the classical Hörmander classes $\Psi^0_{\varrho,\delta}(\mathbb{R}^n)$ $(0 \le \delta \le \varepsilon,\ \delta < \varepsilon)$ are sub multiplicative Ψ^*-algebras. One important fact in the proof of the spectral invariance of these Hörmander classes is a result which is due to Ueberberg and Schrohe. Let $\mathcal{A}^\varepsilon$ be defined as in 3.3.4, but using the classical Laplace operator instead of the Orstein-Uhlenbeck operator, or equivalently let $\mathcal{A}^\varepsilon$ be defined as the set of all $\mathscr{C}^\infty$-elements with respect to the mapping $\alpha_t(a) : \mathbb{R} \longrightarrow \mathscr{L}(L^2(\mathbb{R}^n, \lambda))$ given by $\alpha_t(a) := e^{it\Lambda^\varepsilon} a e^{-it\Lambda^\varepsilon}$, $a \in \mathscr{L}(L^2(\mathbb{R}^n, \lambda))$. Then they showed that for $0 < \varepsilon \leq 1 - \delta$ $\Psi^0_{\varrho,\delta}(\mathbb{R}^n) \subset \mathcal{A}^\varepsilon$. Thus it was possible to prove that $\Psi^0_{\varrho,\delta}(\mathbb{R}^n)$ is a Ψ^*-algebra on the scale of Sobolev-spaces. Since starting with $\mathcal{A}^\varepsilon$ allows us to prove the spectral invariance of $\Psi_{\varrho,\delta}$ even if ε fixed and ϱ, δ arbitrary we always start with $\mathcal{A}^\varepsilon$.

At last we present a result about commutator estimates, which turns out to be very useful later on.

Proposition 3.3.13 (Caps, [**25**]). *Let $Z : D(Z) \to H$ be a strictly positive operator in a complex Hilbert space H and $m \in \mathbb{Z}$ fixed. Furthermore, let $H_Z^\infty = \bigcap_{k \in N} D(Z^k)$ and $A : H_Z^\infty \to H_Z^\infty$ be, such that for all $k, j \in \mathbb{N}_0$ there exists constants $a_{2k,j} \geq 0$, with*

$$\left\| Z^{2k} \mathrm{ad}^j(Z^2)(A)(x) \right\| \leq a_{2k,j} \left\| Z^{2k+m+j} x \right\|$$

for all $x \in H_Z^\infty$. Then for all $k \in \mathbb{Z}$, $j \in \mathbb{N}_0$, there exists $c_{k,j} \geq 0$ such that

$$\left\| Z^k \mathrm{ad}^j(Z)(A)(x) \right\| \leq c_{k,j} \left\| Z^{k+m} x \right\| \qquad \text{for all } x \in H_Z^\infty.$$

Proof. See [**25**, Proposition 2.3.8]. □

We show that the operators of partial differentiation, multiplying with coordinate functions and the generator of the translation-semigroup defined in Chapter 2 map continuously from H^{s+1} to H^s. Moreover, the operator norms of these operators are bounded by a constant independent of direction t as long as $t \in H_+$ and $\|t\|_0 = 1$.

Throughout this section let $(e_k)_{k=1}^\infty \subset H_+$ be an orthonormal basis in H_0. Furthermore, for all $k \in \mathbb{N}$ and $x \in H_-$ we define $x_k := \langle e_k \,, x \rangle_0$.

Proposition 3.3.14. *Let γ be the canonical Gaussian measure in a Hilbert space rigging $H_+ \subseteq H_0 \subseteq H_-$. Then we obtain for all $s \in \mathbb{R}$, $j \in \mathbb{N}_0$ and $f \in H^\infty$*

$$\left\| \Lambda^s \mathrm{ad}^j(\Lambda)(\partial_{e_k}) f \right\| \leq c_s \left\| \Lambda^{s+1} f \right\|,$$

where $c_s = \sqrt{2}$ for $s \geq 0$ and $c_s = 2^{\frac{-s+1}{2}}$ for $s < 0$. Moreover, the mapping $\partial_{e_k} : H^{s+1} \longrightarrow H^s$ is continuous for all $s \geq 0$ and can be extended for $s < 0$ to a continuous linear map.

Proof. The proof of this proposition will be given in several steps.

(i) At first we compute $\Lambda^s \mathrm{ad}^j(\Lambda)(\partial_{e_k}) h_\alpha$, where h_α is defined as in 1.1.27. Moreover, let h_n be the n-th normalized Hermite-polynomial. Using

1.1.26 we get

$$\begin{aligned}
&\Lambda\partial_{e_k}h_\alpha(x) - \partial_{e_k}\Lambda h_\alpha(x)\\
=&(\Lambda\partial_{e_k} - \partial_{e_k}\Lambda)(h_{\alpha_1}(x_1)\cdots h_{\alpha_\nu}(x_\nu))\\
=&(\sqrt{|\alpha|} - \sqrt{|\alpha|+1})(\sqrt{2\alpha_k}h_{\alpha_1}(x_1)\cdots h_{\alpha_k-1}(x_k)\cdots h_{\alpha_\nu}(x_\nu))\\
=&(\sqrt{|\alpha|} - \sqrt{|\alpha|+1})(\partial_{e_k}h_{\alpha_1}(x_1)\cdots h_{\alpha_k}(x_k)\cdots h_{\alpha_\nu}(x_\nu))\\
=&\frac{-1}{\sqrt{|\alpha|}+\sqrt{|\alpha|+1}}\partial_{e_k}h_\alpha(x)
\end{aligned}$$

and thus we obtain

$$\mathrm{ad}^j(\Lambda)(\partial_{e_k})h_\alpha(x) = \left(\frac{-1}{\sqrt{|\alpha|}+\sqrt{|\alpha|+1}}\right)^j (\partial_{e_k})h_\alpha(x)$$

and

$$\Lambda^s\mathrm{ad}^j(\Lambda)(\partial_{e_k})h_\alpha(x) = |\alpha|^{\frac{s}{2}}\left(\frac{-1}{\sqrt{|\alpha|}+\sqrt{|\alpha|+1}}\right)^j (\partial_{e_k})h_\alpha(x).$$

(ii) Now we prove the assertion on the linear span of the h_α. Hence let $f \in \mathscr{P} := span\{h_\alpha \,|\, \alpha \in \mathbb{N}_0^{\mathbb{N}}\}$, i.e. there exist $\alpha^{(1)}, \cdots, \alpha^{(n)} \in \mathbb{N}_0^{\mathbb{N}}$ and $a_l \in \mathbb{C},\ l = 1\dots n$ such that $f = \sum\limits_{l=1}^{n} a_l h_{\alpha^{(l)}}$. It follows that

$$\begin{aligned}
&\left\|\Lambda^s\mathrm{ad}^j(\Lambda)(\partial_{e_k})f\right\|^2_{L^2(H_-,\,\gamma)}\\
=&\sum_{l=1}^{n}\sum_{m=1}^{n} a_l\overline{a_m}\left|\alpha^{(l)}\right|^{\frac{s}{2}}\left(\frac{-1}{\sqrt{|\alpha^{(l)}|}+\sqrt{|\alpha^{(l)}|+1}}\right)^j\left(\frac{-1}{\sqrt{|\alpha^{(m)}|}+\sqrt{|\alpha^{(m)}|+1}}\right)^j\\
&\qquad\left|\alpha^{(m)}\right|^{\frac{s}{2}}\langle\partial_{e_k}h_\alpha^{(l)}\ ,\ \partial_{e_k}h_\alpha^{(m)}\rangle_{L^2(H_-,\,\gamma)}\\
=&\sum_{l=1}^{n}|a_l|^2\left|\alpha^{(l)}\right|^s\left(\frac{-1}{\sqrt{|\alpha^{(l)}|}+\sqrt{|\alpha^{(l)}|+1}}\right)^{2j} 2\alpha_k^{(l)}\\
\leq& c_s^2\sum_{l=1}^{n}|a_l|^2\left(\left|\alpha^{(l)}\right|+1\right)^{s+1}\langle he_\alpha^{(l)}\ ,\ h_\alpha^{(l)}\rangle_{L^2(H_-,\,\gamma)} = c_s^2\left\|\Lambda^{s+1}f\right\|^2_{L^2(H_-,\,\gamma)}.
\end{aligned}$$

(iii) Let us prove that $\partial_j\ :\ H^{s+1} \longrightarrow H^s$ is continuous for all $s \geq 0$. Thus let $s \geq 0$ fixed and $f \in H^{s+1}$ arbitrary. Then there exists a sequence $f_n \in \mathscr{P}$ such that $f_n \xrightarrow[n\longrightarrow\infty]{H^{s+1}} f$. Step 2 implies that

$$\|\partial_{e_k}f_n - \partial_{e_k}f_m\|_{H^s} \leq \sqrt{2}\,\|f_n - f_m\|_{H^{s+1}} \xrightarrow[n,m\to\infty]{} 0.$$

Thus f_n is a Cauchy-sequence in H^s and since H^s is complete, there exists $g \in H^s$ such that $\partial_{e_k}f_n \xrightarrow[n\longrightarrow\infty]{H^s} g$. Hence $\partial_{e_k}f_n \xrightarrow[n\longrightarrow\infty]{H^0} g$ and

$f_n \xrightarrow[n\longrightarrow\infty]{H^0} f$. Since ∂_{e_k} is closed we obtain $g = \partial_{e_k} f$. Thus $\partial_{e_k} f_n \xrightarrow[n\longrightarrow\infty]{H^s}$ $\partial_{e_k} f$ and

$$\|\partial_{e_k} f\|_{H^s} = \lim_{n\to\infty} \|\partial_{e_k} f_n\|_{H^s} \leq \sqrt{2} \lim_{n\to\infty} \|f_n\|_{H^{s+1}} = \sqrt{2}\,\|f\|_{H^{s+1}}\,.$$

(iv) For $s < 0$ $\partial_{e_k}|_{\mathscr{P}}$ has a continuous extension $\partial^s_{e_k}$ as an operator from H^{s+1} to H^s. We show that for $s \leq 0$ and $f \in D(\partial_{e_k})$ this extension coincides with ∂_{e_k}. At first let $s \leq -1$. Obviously, for any $f \in D(\partial_{e_k})$ there exists a sequence $(f_n)_{n=1}^\infty \subset \mathscr{P}$ such that $f_n \xrightarrow[n\to\infty]{H^0} f$ and $\partial_{e_k} f_n \xrightarrow[n\to\infty]{H^0}$ $\partial_{e_k} f$. Hence we obtain $\partial^s_{e_k} f = \lim_{n\to\infty} \partial^s_{e_k} f_n = \lim_{n\to\infty} \partial_{e_k} f_n = \partial_{e_k} f$ with convergence in H^s. Now let $-1 < s < 0$ and $f \in D(\partial_{e_k}) \cap H^{s+1}$. Then there exists a sequence $f_n \in \mathscr{P}$ such that $f_n \xrightarrow[k\to\infty]{H^{s+1}} f$ and $\partial^s_{e_k} f_n \xrightarrow[k\to\infty]{H^s}$ $\partial^s_{e_k} f$. Now we obtain $\partial^s_{e_k} f = \lim_{n\to\infty} \partial^s_{e_k} f_n = \lim_{n\to\infty} \partial_{e_k} f_n = \partial_{e_k} f$ with convergence in H^{-1}.

(v) Finally, let $f \in H^\infty$, $s \in \mathbb{R}$ and $j \in \mathbb{N}_0$ arbitrary. Then there exists a sequence $f_n \in \mathscr{P}$ such that $f_n \xrightarrow[n\longrightarrow\infty]{H^{s+2+j}} f$. According to (iii) and (iv) we get $\mathrm{ad}^j(\Lambda)(\partial_{e_k}) f_n \xrightarrow[n\longrightarrow\infty]{H^s} \mathrm{ad}^j(\Lambda)(\partial_{e_k}) f$ and thus

$$\left\|\mathrm{ad}^j(\Lambda)(\partial_{e_k}) f\right\|_{H^s} = \lim_{n\to\infty} \left\|\mathrm{ad}^j(\Lambda)(\partial_{e_k}) f_n\right\|_{H^s} \leq \lim_{n\to\infty} c_s \|f_n\|_{H^{s+1}} = c_s \|f\|_{H^{s+1}}\,.$$

□

PROPOSITION 3.3.15. *Let δ_{e_k} be defined as in 1.2.6. Then for all $s \in \mathbb{R}$ and all $j \in \mathbb{N}_0$ we get*

$$\left\|\Lambda^s \mathrm{ad}^j(\Lambda)(\delta_{e_k}) f\right\| \leq \tilde{c}_s \left\|\Lambda^{s+1} f\right\| \text{ for all } f \in H^\infty,$$

where $\tilde{c}_s = \sqrt{2}$ for $s \leq 0$ and $\tilde{c}_s = 2^{\frac{s+1}{2}}$ for $s > 0$. Furthermore, the mapping $\delta_{e_k} : H^{s+1} \longrightarrow H^s$ is continuous for $s \geq 0$ and can be extended for $s < 0$ to a continuous linear map. Moreover, we have

$$2\, M_{x_k} f = \delta_{e_k} f + \partial_{e_k} f \quad \forall f \in H^1.$$

PROOF. Since the proof of this assertion is similar to the proof of Proposition 3.3.14, we will only prove the first two steps. Thus let h_α be defined as in 1.1.27. Moreover, let h_n be the n-th normalized Hermite-polynomial. Using 1.1.26 we obtain

$$\begin{aligned}
&\Lambda\delta_{e_k}h_\alpha(x) - \delta_{e_k}\Lambda h_\alpha(x)\\
=&(\Lambda\delta_{e_k} - \delta_{e_k}\Lambda)(h_{\alpha_1}(x_1)\cdots h_{\alpha_\nu}(x_\nu))\\
=&(\sqrt{|\alpha|+2} - \sqrt{|\alpha|+1})(\sqrt{2(n+1)}h_{\alpha_1}(x_1)\cdots h_{\alpha_k+1}(x_k)\cdots h_{\alpha_\nu}(x_\nu))\\
=&(\sqrt{|\alpha|+2} - \sqrt{|\alpha|+1})(\delta_{e_k}h_{\alpha_1}(x_1)\cdots h_\alpha(x_k)\cdots h_{\alpha_\nu}(x_\nu))\\
=&\frac{1}{\sqrt{|\alpha|+2}+\sqrt{|\alpha|+1}}\delta_{e_k}h_\alpha(x).
\end{aligned}$$

Thus

$$\mathrm{ad}^j(\Lambda)(\delta_{e_k})h_\alpha(x) = \left(\frac{1}{\sqrt{|\alpha|+2}+\sqrt{|\alpha|+1}}\right)^j(\delta_{e_k})h_\alpha(x)$$

and

$$\Lambda^s\mathrm{ad}^j(\Lambda)(\delta_{e_k})h_\alpha(x) = (|\alpha|+2)^{\frac{s}{2}}\left(\frac{1}{\sqrt{|\alpha|+2}+\sqrt{|\alpha|+1}}\right)^j(\delta_{e_k})h_\alpha(x).$$

Now we will prove the assertion for the linear span of h_α. Hence let $f \in \mathscr{P}$, i.e. there exist $\alpha^{(1)},\cdots,\alpha^{(n)} \in \mathbb{N}_0^{\mathbb{N}}$ and $a_l \in \mathbb{C}$, $l = 1\ldots n$ such that $f = \sum\limits_{l=1}^{n} a_l h_{\alpha^{(l)}}$. Then we get

$$\begin{aligned}
&\left\|\Lambda^s\mathrm{ad}^j(\Lambda)(\delta_{e_k})h_\alpha\right\|^2_{L^2(H_-,\,\gamma)}\\
=&\sum_{l=1}^{n}\sum_{m=1}^{n} a_l\overline{a_m}\left|\alpha^{(l)}\right|^{\frac{s}{2}}\left(\frac{1}{\sqrt{|\alpha^{(l)}|+2}+\sqrt{|\alpha^{(l)}|+1}}\right)^j\\
&\qquad\left|\alpha^{(m)}\right|^{\frac{s}{2}}\left(\frac{1}{\sqrt{|\alpha^{(m)}|}+\sqrt{|\alpha^{(m)}|+1}}\right)^j\langle\delta_{e_k}h_\alpha^{(l)}\,,\ \delta_{e_k}h_\alpha^{(m)}\rangle_{L^2(H_-,\,\gamma)}\\
=&\sum_{l=1}^{n}|a_l|^2\left(\left|\alpha^{(l)}\right|+2\right)^s\left(\frac{1}{\sqrt{|\alpha^{(l)}|+2}+\sqrt{|\alpha^{(l)}|+1}}\right)^{2j}2(\alpha_k^{(l)}+1)\\
\leq&\tilde{c}_s^2\sum_{l=1}^{n}|a_l|^2\left(\left|\alpha^{(l)}\right|+1\right)^{s+1}\langle h_\alpha^{(l)}\,,\ h_\alpha^{(l)}\rangle_{L^2(H_-,\,\gamma)} = \tilde{c}_s^2\left\|\Lambda^{s+1}f\right\|^2_{L^2(H_-,\,\gamma)}.
\end{aligned}$$

The rest of this part is similar to the proof of 3.3.14. Finally, we show our last assertion. Thus let $f \in H_1$. Then there exists a sequence $(f_n)_{n=1}^\infty \subset \mathscr{P}$ such that $f_n \xrightarrow[n\to\infty]{H^1} f$. Thus we get $2M_{x_k}f_n = \delta_{e_k}f_n + \partial_{e_k}f_n \xrightarrow[n\to\infty]{H^0} \delta_{e_k}f + \partial_{e_k}f$. Since M_{x_k} is closed, we have $f \in D(M_{x_k})$ and $2M_{x_k}f = \delta_{e_k}f + \partial_{e_k}f$. $\square$

Corollary 3.3.16. *For all $s \in \mathbb{R}$ the mappings M_{x_k} and D_{e_k} are continuous from H^{s+1} to H^s with*

$$\left\|\mathrm{ad}^j(\Lambda)(M_{x_k})f\right\|_{H^s} \leq c'_s \|f\|_{H^{s+1}} \text{ for all } f \in H^\infty$$

and

$$\left\|\mathrm{ad}^j(\Lambda)(D_{e_k})f\right\|_{H^s} \leq c'_s \|f\|_{H^{s+1}} \text{ for all } f \in H^\infty,$$

where $c'_s \in \mathbb{R}$ is a constant depending only on s.

Proof. We have $2M_{x_k}f = \partial_{e_k}f + \delta_{e_k}f$ and $2D_{e_k}f = \partial_{e_k}f - \delta_{e_k}f$ for all $f \in H^\infty$ and thus we obtain

$$\|M_{x_k}f\|_{H^s} \leq \frac{1}{2}(\|\partial_{e_k}f\|_{H^s} + \|\delta_{e_k}f\|_{H^s}) \leq c'_s \|f\|_{H^{s+1}}$$

and

$$\|D_{e_k}f\|_{H^s} \leq \frac{1}{2}(\|\partial_{e_k}f\|_{H^s} + \|\delta_{e_k}f\|_{H^s}) \leq c'_s \|f\|_{H^{s+1}}.$$

□

Proposition 3.3.17. *For $k \in \mathbb{N}$ the following operators are elements of $\mathcal{A}^\varepsilon$ for all $0 < \varepsilon \leq 1$:*
(i) $\Lambda^{-1}\partial_{e_k}$, *(ii)* $\Lambda^{-1}\delta_{e_k}$, *(iii)* $\Lambda^{-1}M_{x_k}$, *(iv)* $\Lambda^{-1}D_{e_k}$
(v) $\partial_{e_k}\Lambda^{-1}$, *(vi)* $\partial_{e_k}\Lambda^{-1}$, *(vii)* $M_{x_k}\Lambda^{-1}$, *(viii)* $D_{e_k}\Lambda^{-1}$

Proof. We will prove this lemma only for $\Lambda^{-1}\partial_{e_k}$. For $f \in H^\infty$ we obtain

$$\begin{aligned} \mathrm{ad}(\Lambda)(\Lambda^{-1}\partial_{e_k})f &= [\Lambda,\ \Lambda^{-1}\partial_{e_k}]f \\ &= \Lambda\Lambda^{-1}\partial_{e_k}f - \Lambda^{-1}\partial_{e_k}\Lambda \\ &= \Lambda^{-1}\Lambda\partial_{e_k}f - \Lambda^{-1}\partial_{e_k}\Lambda \\ &= \Lambda^{-1}\mathrm{ad}(\Lambda)(\partial_{e_k})f \end{aligned}$$

and thus it follows by induction that

$$\mathrm{ad}^j(\Lambda)(\Lambda^{-1}\partial_{e_k})f = \Lambda^{-1}\mathrm{ad}^j(\Lambda)(\partial_{e_k})f.$$

Hence $\mathrm{ad}^j(\Lambda)(\Lambda^{-1}\partial_{e_k}) \in \mathscr{L}(H^0)$ for all $j \in \mathbb{N}_0$. Thus $\Lambda^{-1}\partial_{e_k} \in \mathcal{A}^1$ and by [**25**, Theorem 2.3.11] $\Lambda^{-1}\partial_{e_k} \in \mathcal{A}^\varepsilon$ for all $0 < \varepsilon \leq 1$. □

Now we will show that the iterated commutators of Λ^2 and ∂_{e_k} have order one.

Lemma 3.3.18. *Let γ be the canonical Gaussian measure and $f \in H^3$. Then we have*

$$[\Lambda^2,\ \partial_{e_k}]f(x) = -\partial_{e_k}f(x).$$

PROOF. Let h_α be defined as in 1.1.27. Then we get

$$\begin{aligned}
&\mathrm{L}_{\gamma_1}\partial_{e_k}h_\alpha - \partial_{e_k}\mathrm{L}_{\gamma_1}h_\alpha \\
&= \mathrm{L}_{\gamma_1}\partial_{e_k}(h_{\alpha_1}(x_1)\cdots h_{\alpha_\nu}(x_\nu)) - \partial_{e_k}\mathrm{L}_{\gamma_1}(h_{\alpha_1}(x_1)\cdots h_{\alpha_\nu}(x_\nu)) \\
&= \mathrm{L}_{\gamma_1}\sqrt{2\alpha_k}(h_{\alpha_1}(x_1)\cdots h_{\alpha_{k-1}}(x_k)\cdots h_{\alpha_\nu}(x_\nu)) - \partial_{e_k}\left|\alpha\right|(h_{\alpha_1}(x_1)\cdots h_{\alpha_\nu}(x_\nu) \\
&= \sqrt{2\alpha_k}(|\alpha|-1-|\alpha|)(h_{\alpha_1}(x_1)\cdots h_{\alpha_{k-1}}(x_k)\cdots h_{\alpha_\nu}(x_\nu)) \\
&= -\sqrt{2\alpha_k}h_{\alpha_1}(x_1)\cdots h_{\alpha_{k-1}}(x_k)\cdots h_{\alpha_\nu}(x_\nu) = -\partial_{e_k}h_\alpha.
\end{aligned}$$

Thus for all $f \in \mathscr{P}$ we obtain

$$[\Lambda^2,\ \partial_{e_k}]f(x) = -\partial_{e_k}f(x).$$

Let $f \in H^3$ arbitrary. Then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathscr{P}$ such that $f_n \xrightarrow[n\to\infty]{H^3} f$. Since $f_n \xrightarrow[n\to\infty]{H^1} f$ and $\Lambda^2 f_n \xrightarrow[n\to\infty]{H^1} \Lambda^2 f$, we obtain $\partial_{e_k}f_n \xrightarrow[n\to\infty]{H^0} \partial_{e_k}f$ and $\partial_{e_k}\Lambda^2 f_n \xrightarrow[n\to\infty]{H^0} \partial_{e_k}\Lambda^2 f$. Hence it follows that

$$\Lambda^2(\partial_{e_k}f_n) = \partial_{e_k}\Lambda^2 f_n - \partial_{e_k}f_n \xrightarrow[n\to\infty]{H^0} \partial_{e_k}\Lambda^2 f - \partial_{e_k}f.$$

Since Λ^2 is closed, this yields $\Lambda^2\partial_{e_k}f = -\partial_{e_k}f$. □

LEMMA 3.3.19. *Let γ be the canonical Gaussian measure and $f \in H^3$. Then we have*

$$[\Lambda^2,\ \delta_{e_k}]f(x) = \delta_{e_k}f(x).$$

PROOF. Let h_α be defined as in 1.1.27. Then we get

$$\begin{aligned}
&\mathrm{L}_{\gamma_1}\delta_{e_k}h_\alpha - \delta_{e_k}\mathrm{L}_{\gamma_1}h_\alpha \\
&= \mathrm{L}_{\gamma_1}\delta_{e_k}(h_{\alpha_1}(x_1)\cdots h_{\alpha_\nu}(x_\nu)) - \delta_{e_k}\mathrm{L}_{\gamma_1}(h_{\alpha_1}(x_1)\cdots h_{\alpha_\nu}(x_\nu)) \\
&= \mathrm{L}_{\gamma_1}\sqrt{2(\alpha_k+1)}(h_{\alpha_1}(x_1)\cdots h_{\alpha_{k+1}}(x_k)\cdots h_{\alpha_\nu}(x_\nu)) - \delta_{e_k}\left|\alpha\right|(h_{\alpha_1}(x_1)\cdots h_{\alpha_\nu}(x_\nu)) \\
&= (\sqrt{2(\alpha_k+1)}(|\alpha|+1-|\alpha|)(h_{\alpha_1}(x_1)\cdots h_{\alpha_{k+1}}(x_k)\cdots h_{\alpha_\nu}(x_\nu)) \\
&= \sqrt{2(\alpha_k+1)}h_{\alpha_1}(x_1)\cdots h_{\alpha_{k+1}}(x_k)\cdots h_{\alpha_\nu}(x_\nu) = \delta_{e_k}h_\alpha.
\end{aligned}$$

Thus for all $f \in \mathscr{P}$ we obtain

$$[\Lambda^2,\ \delta_{e_k}]f(x) = \delta_{e_k}f(x).$$

For $f \in H^3$ arbitrary the assertion follows similarly to Lemma 3.3.18. □

COROLLARY 3.3.20. *Let γ be the canonical Gaussian measure and $f \in H^\infty$. Then we have*

$$\mathrm{ad}^j(\Lambda^2)\partial_{e_k}f(x) = (-1)^j\partial_{e_k}f(x)$$

and

$$\mathrm{ad}^j(\Lambda^2)\delta_{e_k}f(x) = \delta_{e_k}f(x)$$

PROOF. (by induction). For $j = 1$ the hypothesis has been shown in Lemma 3.3.18 and 3.3.19. Thus let the hypothesis be true for fixed $j \in \mathbb{N}$. Then we get

$$\mathrm{ad}^{j+1}(\Lambda^2)(\partial_{e_k})f(x) = \mathrm{ad}(\Lambda^2)(\mathrm{ad}^j(\Lambda^2)(\partial_{e_k}))f(x) = (-1)^{j+1}\partial_{e_k}f(x).$$

Similarly we obtain

$$\mathrm{ad}^{j+1}(\Lambda^2)(\delta_{e_k})f(x) = \mathrm{ad}(\Lambda^2)(\delta_{e_k})f(x) = \delta_{e_k}f(x).$$ □

COROLLARY 3.3.21. *Let $f \in H^\infty$ and $j \in \mathbb{N}$. Then we have*

$$[\Lambda^2,\ (\partial_{e_k})^j]f = -j(\partial_{e_k})^j f$$

and

$$[\Lambda^2,\ (\delta_{e_k})^j]f = j(\delta_{e_k})^j f$$

PROOF. Let $f \in H^\infty$ and $j \in \mathbb{N}$. According to the Leibniz-rule, we obtain

$$[\Lambda^2,\ (\partial_{e_k})^j]f = \sum_{l=1}^{j}(\partial_{e_k})^{l-1}[\Lambda^2,\ \partial_{e_k}](\partial_{e_k})^{j-l}f = -j(\partial_{e_k})^j f.$$

The second assertion follows similarly. □

3.4. Multiplication and convolution operators as elements of the generalized Hörmander classes

Above we have defined the Ψ^*-algebra Ψ^0 and the generalized Hörmander classes $\widetilde{\Psi}^0_{\varrho,\delta}$. When constructing Ψ^*-algebras by commutator methods, it is a problem to show that these algebras are non-trivial. Thus we consider some multiplication operators and prove that these operators are elements of $\widetilde{\Psi}^0_{\varrho,\delta}$ for all $0 \le \delta \le \varrho \le 1$, $\delta < 1$. Throughout this section let $H_+ \subseteq H_0 \subseteq H_-$ be a quasi-nuclear Hilbert space rigging and let γ be the canonical Gaussian measure in this rigging. Let $(e_j)_{j=1}^n \subset H_+$ be an orthonormal basis in H_0. Furthermore, define $\mathscr{P} := span\{h_\alpha \mid \alpha \in \mathbb{N}_0^{\mathbb{N}}\}$.

DEFINITION 3.4.1. For $a \in \mathscr{C}_b^\infty(\mathbb{R})$ define the operator $M_{(a,j)} :\ H^0 \longrightarrow H^0$ by

$$M_{(a,j)}f(x) = a(\langle e_j, x\rangle_0)f(x).$$

Since a is bounded, $M_{(a,j)}$ is bounded. Moreover, for $x \in H_-$ we define $x_j := \langle x, e_j\rangle_0$ and $\partial_j := \partial_{e_j}$.

LEMMA 3.4.2. *For $M_{(a,j)}$ defined as in 3.4.1 and $f \in \mathscr{P}$ we have*

$$[\Lambda^2, M_{(a,j)}]f = -\frac{1}{2}M_{(a'',j)}f + M_{(a',j)}x_j f - M_{(a',j)}\partial_j f.$$

PROOF. Let $f \in \mathscr{P}$. Then we obtain

$$\begin{aligned}
& 2[\Lambda^2, M_{(a,j)}]f(x) \\
= \ & (-\partial_j^2 + 2x_j\partial_j)a(x_j)f(x) - a(x_j)(-\partial_j^2 + 2x_j\partial_j)f(x) \\
= \ & -\partial_j^2 a(x_j)f(x) - 2\partial_j a(x_j)\partial_j f(x) + 2x_j\partial_j a(x_j)f(x) \\
= \ & -M_{(a'',j)}f(x) + 2M_{(a',j)}x_j f(x) - 2M_{(a',j)}\partial_j f(x).
\end{aligned}$$

□

LEMMA 3.4.3. *Let $m \in \mathbb{N}$ be fixed and $a \in \mathscr{C}_b^\infty(\mathbb{R})$. Then there exist $a_{(l,k)} \in \mathscr{C}_b^\infty(\mathbb{R})$ $(l+k \leq m)$ such that*

$$\mathrm{ad}^m(\Lambda^2)(M_{(a,j)})f = \sum_{l+k\leq m} M_{(a_{(l,k)},j)} x_j^l \partial_j^k f \quad \forall f \in \mathscr{P},$$

where $M_{(a_{(l,k)},j)}$ is defined as in 3.4.1.

PROOF. (by induction) For $m = 1$ our hypothesis is true by Lemma 3.4.2. Let the hypothesis by true for fixed $m \in \mathbb{N}$. Then there exist $b_{(k,l)} \in \mathscr{C}_b^\infty(\mathbb{R})$ $(k+l \leq m)$ such that

$$\mathrm{ad}^m(\Lambda^2)(M_{(a,j)})f = \sum_{k+l\leq m} M_{(b_{(k,l)},j)} x_j^l (\partial_j)^k f \quad \forall f \in \mathscr{P}.$$

Thus we obtain

$$\begin{aligned}
& \mathrm{ad}^{m+1}(\Lambda^2)(M_{(a,j)})f(x) \\
= \ & [\Lambda^2, \sum_{k+l\leq m} M_{(b_{(k,l)},j)} x_j^l \partial_j^k] f(x) \\
= \ & \sum_{k+l\leq m} \Big(M_{(b_{(k,l)},j)} x_j^l [\Lambda^2,\ \partial_j^k] + M_{(b_{(k,l)},j)} [\Lambda^2,\ x_j^l]\partial_j^k + [\Lambda^2,\ M_{(b_{(k,l)},j)}] x_j^l \partial_j^k \Big) f(x) \\
= \ & \sum_{k+l\leq m} \Bigg(-kM_{(b_{(k,l)},j)} x_j^l \partial_j^k + M_{(b_{(k,l)},j)} \sum_{n=1}^{l} x_j^{n-1} [\Lambda^2,\ x_j] x_j^{l-n} \partial_j^k \\
& \qquad + (-\frac{1}{2} M_{(b''_{(k,l)},j)} + M_{(b'_{(k,l)},j)} x_j - M_{(b_{(k,l)},j)}\partial_j) x_j^l \partial_j^k \Bigg) f(x) \\
= \ & \sum_{k+l\leq m} \Bigg(-kM_{(b_{(k,l)},j)} x_j^l \partial_j^k + M_{(b_{(k,l)},j)} \sum_{n=1}^{l} x_j^{n-1} \frac{1}{2} [\Lambda^2,\ \partial_j + \delta_j] x_j^{l-n} \partial_j^k \\
& \qquad - \frac{1}{2} M_{(b''_{(k,l)},j)} x_j^l \partial_j^k + M_{(b'_{(k,l)},j)} x_j x_j^l \partial_j^k - M_{(b_{(k,l)},j)} \partial_j x_j^l \partial_j^k \Bigg) f(x) \\
= \ & \sum_{k+l\leq m} \Big(-kM_{(b_{(k,l)},j)} x_j^l \partial_j^k + M_{(b_{(k,l)},j)} \sum_{n=1}^{l} x_j^{n-1} (x_j - \partial_j) x_j^{l-n} \partial_j^k
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2}M_{(b''_{(k,l)},j)}x_j^l\partial_j^k + M_{(b'_{(k,l)},j)}x_j^{l+1}\partial_j^k \\
& -M_{(b_{(k,l)},j)}(lx_j^{l-1}\partial_j^k + x_j^l\partial_j^{k+1})\Big)f(x) \\
= \sum_{k+l\leq m} & \Big(-kM_{(b_{(k,l)},j)}x_j^l\partial_j^k + M_{(b_{(k,l)},j)}\sum_{n=1}^{l}(x_j^l - lx_j^{l-1}\partial_j^k + x_j^{l-1}\partial_j^{k+1}) \\
& -\frac{1}{2}M_{(b''_{(k,l)},j)}x_j^l\partial_j^k + M_{(b'_{(k,l)},j)}x_j^{l+1}\partial_j^k \\
& -M_{(b_{(k,l)},j)}(lx_j^{l-1}\partial_j^k + x_j^l\partial_j^{k+1})\Big)f(x).
\end{aligned}
$$

But this is our hypothesis. □

LEMMA 3.4.4. *Let $a \in \mathscr{C}_b^\infty(\mathbb{R})$ and $M_{(a,j)}$ defined as in 3.4.1. For all $k,\ m \in \mathbb{N}_0$, there exist $c_{k,2m} > 0$ such that for all $f \in H^\infty$*

$$
\left\|\Lambda^{2k}\mathrm{ad}^m(\Lambda^2)(M_{(a,j)})f\right\|_0 \leq c_{2k,m}\left\|\Lambda^{2k+m}f\right\|_0 .
$$

PROOF. In a first step let $f \in \mathscr{P}$. Then by Lemma 3.3.5 we have

$$
\begin{aligned}
\Lambda^{2k}\mathrm{ad}^m(\Lambda^2)(M_{(a,j)})f &= \sum_{n=0}^{k}\binom{k}{n}\mathrm{ad}^n(\Lambda^2)(\mathrm{ad}^m(\Lambda^2)(M_{(a,j)}))(\Lambda^2)^{k-n}f \\
&= \sum_{n=0}^{k}\binom{k}{n}\mathrm{ad}^{m+n}(\Lambda^2)(M_{(a,j)})(\Lambda^2)^{k-n}f \\
&= \sum_{n=0}^{k}\binom{k}{n}\sum_{i+l\leq m+n}M_{(b_{(i,l)},j)}x_j^l(\partial_j)^i(\Lambda^2)^{k-n}f,
\end{aligned}
$$

where $b_{(i,l)} \in \mathscr{C}_b^\infty(\mathbb{R})$ for all $l,\ i$. Thus there exists $c_{i,l} > 0$ and $c_{k,m} > 0$ such that

$$
\begin{aligned}
\left\|\Lambda^{2k}\mathrm{ad}^m(\Lambda^2)(M_{(a,j)})f\right\|_{H^0} &\leq \sum_{n=0}^{k}\binom{k}{n}\sum_{i+l\leq m+n}\left\|M_{(b_{(i,l)},j)}x_j^l(\partial_j)^i(\Lambda^2)^{k-n}f\right\|_{H^0} \\
&\leq \sum_{n=0}^{k}\binom{k}{n}\sum_{i+l\leq m+n}c\left\|x_j^l(\partial_j)^i(\Lambda^2)^{k-n}f\right\|_{H^0} \\
&\leq \sum_{n=0}^{k}\binom{k}{n}\sum_{i+l\leq m+n}c_{i,l}\left\|f\right\|_{H^{l+i+2k-2n}} \\
&\leq c_{2k,m}\left\|\Lambda^{2k+m}f\right\|_{H^0} .
\end{aligned}
$$

For $f \in H^\infty$ there exists a sequence $f_n \in \mathscr{P}$ such that $f_n \xrightarrow[n\longrightarrow\infty]{H^{2k+2m}} f$. Hence it follows that $\Lambda^{2k}\mathrm{ad}^m(\Lambda^2)(M_{(a,j)})f_n \xrightarrow[n\longrightarrow\infty]{H^0} \Lambda^{2k}\mathrm{ad}^m(\Lambda^2)(M_{(a,j)})f$ and thus

$$\left\|\Lambda^{2k}\mathrm{ad}^m(\Lambda^2)(M_{(a,j)})f\right\|_{H^0} \leq c_{2k,m}\left\|\Lambda^{2k,m}f\right\|_{H^0}. \qquad \square$$

PROPOSITION 3.4.5. *Let $j \in \mathbb{N}$, $a \in \mathscr{C}_b^\infty(\mathbb{R})$ and $M_{(a,j)}$ be defined as in 3.4.1. Then for each $k,\ m \in \mathbb{N}_0$, there exist $c_{k,m} \geq 0$ such that for all $f \in H^\infty$*

$$\left\|\Lambda^k\mathrm{ad}^m(\Lambda)(M_{(a,j)})f\right\|_{H^0} \leq c_{k,m}\left\|\Lambda^k f\right\|_{H^0}.$$

PROOF. This assertion follows directly by 3.3.13 and 3.4.4. $\square$

COROLLARY 3.4.6. *Let $j \in \mathbb{N}$, $a \in \mathscr{C}_b^\infty(\mathbb{R})$ and $M_{(a,j)}$ be defined as in 3.4.1. Then*

$$M_{(a,j)} \in \mathcal{A}^\varepsilon \quad \forall\, 0 < \varepsilon \leq 1.$$

PROOF. Proposition 3.4.5 implies that $M_{(a,j)} \in \mathcal{A}^1$ and thus our assertion follows by [**25**, Theorem 2.3.11]. $\square$

LEMMA 3.4.7. *Let $j \in \mathbb{N}$, $a \in \mathscr{C}_b^\infty(\mathbb{R})$ and $M_{(a,j)}$ be defined as in 3.4.1. Moreover, let $\alpha, \beta \in \mathbb{N}_0^{\mathbb{N}}$ and let $\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)$ be defined as in 3.2.18. Then we have for all $f \in H^\infty$*

$$\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(M_{(a,j)})f = \begin{cases} M_{(a^{(\beta_j)},j)}f & \text{if } \alpha = 0 \text{ and } \beta_k = 0 \text{ for all } k \neq j \\ 0 & \text{else.} \end{cases}$$

PROOF. For $f \in \mathscr{C}_{pol}^\infty(H_-)$ we obtain

$$[D_j,\ M_{(a,j)}]f(x) = [\partial_j,\ M_{(a,j)}]f(x) = \partial_j[(a(x_j)f(x)] - a(x_j)\partial_j f(x) = M_{(a',j)}f(x).$$

By induction it follows that $\mathrm{ad}(D_j)^k(M_{(a,j)})f = M_{(a^{(k)},j)}f$ The rest of our assertion is clear. $\square$

COROLLARY 3.4.8. *Let $j \in \mathbb{N}$, $a \in \mathscr{C}_b^\infty(\mathbb{R})$ and $M_{(a,j)}$ be defined as in 3.4.1. Furthermore, let Ψ^0 be defined as in 3.3.7. Then we have*

$$M_{(a,j)} \in \Psi^0.$$

PROOF. The assertion is clear by 3.3.7, Lemma 3.4.7 and Corollary 3.4.6. $\square$

PROPOSITION 3.4.9. *Let $j \in \mathbb{N}$, $a \in \mathscr{C}_b^\infty(\mathbb{R})$ and $M_{(a,j)}$ be defined as in 3.4.1. Moreover, for $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$ let $\widetilde{\Psi}^0_{\varrho,\delta}$ be defined as in 3.3.9. Then we have*

$$M_{(a,j)} \in \widetilde{\Psi}^0_{\varrho,\delta}.$$

PROOF. Lemma 3.4.6 implies that $M_{(a,j)} \in \mathcal{A}^{1-\delta}$. Thus the assertion follows by Lemma 3.4.7. $\square$

Let γ be the canonical Gaussian measure in the quasi-nuclear Hilbert space rigging $H_+ \subset H_0 \subset H_-$. Let $(e_j)_{j=1}^n \subset H_+$ be an orthonormal basis in H_0. Furthermore, define $\mathscr{P} := span\{h_\alpha \,|\, \alpha \in \mathbb{N}_0^{\mathbb{N}}\}$. Let $\mathcal{F}$ be the abstract Fourier-transform defined in 1.4.4. Then according to [**35**, p. 73, Theorem 5.1] we have $\mathcal{F}^{-1}f(x) = (f)(-x)$ and $\mathcal{F}^2 f(x) = f(-x)$ for all $f \in L^2(H_-, \gamma)$. Moreover, in [**17**, p.160] it is shown that $\mathcal{F}h_\alpha = (-i)^{|\alpha|}h_\alpha$. For $t \in H_+$ let U_t, V_t, D_t and M_t be defined as in 3.2.18. According to Proposition 1.4.4 we have $\mathcal{F}U_t = V_t\mathcal{F}$ and thus $\mathcal{F}D_t = M_t\mathcal{F}$ and $D_t\mathcal{F}^{-1} = M_f\mathcal{F}^{-1}$.

LEMMA 3.4.10. *Let L_γ be the Ornstein-Uhlenbeck operator (cf.* [**104**]*). For $\Lambda := (L_\gamma + id)^{\frac{1}{2}}$ we obtain*

$$[\mathcal{F},\ \Lambda^s]f = 0 \quad \textit{and} \quad [\mathcal{F}^{-1},\ \Lambda^s]f = 0$$

for all $f \in H^s$ and $s \in \mathbb{R}$. Moreover, for all $f \in H^s$ and $s \in \mathbb{R}$ this implies that $\|\mathcal{F}f\|_{H^s} = \|f\|_{H^s}$.

PROOF. For $\alpha \in \mathbb{N}_0^{\mathbb{N}}$ and $s \in \mathbb{R}$ we obtain $\mathcal{F}\Lambda^s h_\alpha = \mathcal{F}(|\alpha|+1)^{\frac{s}{2}}h_\alpha = (|\alpha|+1)^{\frac{s}{2}}(-i)^{|\alpha|}h_\alpha = \Lambda^s\mathcal{F}h_\alpha$. Thus we get $\mathcal{F}\Lambda^s = \Lambda^s\mathcal{F}$ for all $f \in \mathscr{P}$. Since $\mathscr{P}$ is dense in all H^s and Λ^s is closed we have $[\mathcal{F},\ \Lambda^s]f = 0$ for all f in H^s. But this implies $\|\mathcal{F}f\|_{H^s} = \|\Lambda^s\mathcal{F}f\|_{H^0} = \|\mathcal{F}\Lambda^s f\|_{H^0} = \|\Lambda^s f\|_{H^0} = \|f\|_{H^s}$. □

LEMMA 3.4.11. *Let $t \in H_+$ and $f \in H^1$. Then we have*

$$D_t\mathcal{F}f = \mathcal{F}M_t f \quad \textit{and} \quad M_t\mathcal{F}^{-1}f = \mathcal{F}^{-1}D_t f \tag{30}$$

PROOF. For $t \in H_+$ and $f \in H^1$ we obtain

$$D_t\mathcal{F}f(x) = D_t(\mathcal{F}^{-1}f)(-x) = (\mathcal{F}^{-1}M_t f)(-x) = \mathcal{F}M_t f(x)$$ □

Let $g \in \mathscr{C}_{b,cyl}^\infty(H_-)$ and $a(x,p) = g(p)$ be a symbol. According to Definition 3.2.2 we have $a(X,D)f(x) = [\mathcal{F}^{-1}M_g\mathcal{F}]f(x)$. Moreover, since there exists $c > 0$ such that $\|a(X,D)\|_{H^0} = \|\mathcal{F}^{-1}M_g\mathcal{F}f\|_{H^0} = \|M_g\mathcal{F}f\|_{H^0} \le c\,\|\mathcal{F}f\|_{H^0} = c\,\|f\|_{H^0}$, we obtain that $a(X,D)$ is a continuous linear operator in $L^2(H_-,\gamma)$. Moreover, according to Corollary 3.4.6 and Lemma 3.4.10 $a(X,D)$ leaves H^∞ invariant.

LEMMA 3.4.12. *Let $g \in \mathscr{C}_{b,cyl}^\infty(H_-)$, $a(x,p) = g(p)$ and $t \in H_+$. Then for $f \in H^\infty$ we have*

$$[D_t,\ a(X,D)]f = 0.$$

PROOF. Let $f \in H^\infty$. Then we obtain

$$\begin{aligned}[D_t,\ a(X,D)]f) &= \mathcal{F}^{-1}M_tM_g\mathcal{F}f - \mathcal{F}^{-1}M_g\mathcal{F}D_t f\\ &= \mathcal{F}^{-1}M_g\mathcal{F}D_t f - \mathcal{F}^{-1}M_g\mathcal{F}D_t f = 0.\end{aligned}$$

But this is our assertion. □

LEMMA 3.4.13. *Let $g \in \mathscr{C}_{b,cyl}^\infty(H_-)$, $a(x,p) = g(p)$ and $t \in H_+$. Then for $f \in H^\infty$ we have*

$$[M_t,\ a(X,D)]f = a_t(X,D)f,$$

where $a_t(x,p) := \partial_t g(p)$.

PROOF. Let $f \in H^\infty$. Then we obtain

$$\begin{aligned} & ([M_t,\ a(X,D)]f)(x) \\ = \ & M_t\mathcal{F}^{-1}M_g\mathcal{F}f(x) - \mathcal{F}^{-1}M_g\mathcal{F}M_tf(x) \\ = \ & \mathcal{F}^{-1}D_t(g(x)\mathcal{F}f(x)) - \mathcal{F}^{-1}M_g\mathcal{F}M_tf(x) \\ = \ & \mathcal{F}^{-1}\partial_t g(x)\mathcal{F}f(x) + \mathcal{F}^{-1}g(x)D_t\mathcal{F}f(x) - \mathcal{F}^{-1}M_g\mathcal{F}M_tf(x) \\ = \ & -a_t(X,D)f(x) + \mathcal{F}^{-1}M_g\mathcal{F}M_tf(x) - F^{-1}M_g\mathcal{F}M_tf(x) \\ = \ & -a_t(X,D)f(x). \end{aligned}$$

□

PROPOSITION 3.4.14. *Let* $g \in \mathscr{C}^\infty_{b,cyl}(H_-)$, $a(x,p) = g(p)$ *and* $t \in H_+$. *Moreover, let* $\alpha, \beta \in \mathbb{N}_0^{\mathbb{N}}$ *and let* $\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)$ *be defined as in 3.2.18. Then we have for all* $f \in H^\infty$

$$\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(a(X,D))f = \begin{cases} b(X,D)f & \beta = 0 \\ 0 & \textit{else}, \end{cases}$$

where $b(x,p) = \partial^\alpha g(p)$.

PROOF. The assertion follows directly by Lemma 3.4.12 and 3.4.13. □

LEMMA 3.4.15. *Let* $g \in \mathscr{C}^\infty_{b,cyl}(H_-)$, $a(x,p) = g(p)$. *Then we obtain*

$$[\Lambda^k,\ a(X,D)] = \mathcal{F}^{-1}[\Lambda^k,\ M_g]\mathcal{F}.$$

PROOF. For $g \in \mathscr{C}^\infty_{b,cyl}(H_-)$ and $a(x,p) = g(p)$ by Lemma 3.4.10 we have

$$\begin{aligned} [\Lambda^k,\ a(X,D)] &= [\Lambda^k,\ \mathcal{F}^{-1}M_g\mathcal{F}] \\ &= [\Lambda^k,\ \mathcal{F}^{-1}]M_g\mathcal{F} + \mathcal{F}^{-1}[\Lambda^k,\ M_g]\mathcal{F} + \mathcal{F}^{-1}M_g[\Lambda^k,\ \mathcal{F}] \\ &= \mathcal{F}^{-1}[\Lambda^k,\ M_g]\mathcal{F}. \end{aligned}$$

□

PROPOSITION 3.4.16. *Let* $g \in \mathscr{C}^\infty_{b,cyl}(H_-)$ *and* $a(x,p) = g(p)$. *Then for each* $k,\ m \in \mathbb{N}_0$, *there exist* $c_{k,m} \geq 0$ *such that for all* $f \in H^\infty$

$$\left\|\Lambda^k \mathrm{ad}^m(\Lambda)(a(X,D))f\right\|_{H^0} \leq c_{k,m}\left\|\Lambda^k f\right\|_{H^0}.$$

PROOF. For $f \in H^\infty$, $g \in \mathscr{C}^\infty_{b,cyl}(H_-)$ and $a(x,p) = g(p)$ and $k,m \in \mathbb{N}_0$ it follows

$$\Lambda^k\mathrm{ad}^m(\Lambda)(a(X,D))f = \Lambda^k\mathcal{F}^{-1}\mathrm{ad}^m(\Lambda)(M_g)\mathcal{F}f = \mathcal{F}^{-1}\Lambda^k\mathrm{ad}^m(\Lambda)(M_g)\mathcal{F}f,$$

by Lemma 3.4.10 and 3.4.15. Thus according to Proposition 3.4.5 there exist $c_{k,m}$ such that

$$\begin{aligned}\left\|\Lambda^k \mathrm{ad}^m(\Lambda)(a(X,D))f\right\|_{H^0} &= \left\|\Lambda^k \mathrm{ad}^m(\Lambda)(M_g)\mathcal{F}f\right\|_{H^0} \\ &\le c_{k,m}\left\|\Lambda^k \mathcal{F}f\right\|_{H^0} = c_{k,m}\left\|\Lambda^k f\right\|_{H^0},\end{aligned}$$

which shows our assertion. □

COROLLARY 3.4.17. *Let $g \in \mathscr{C}^\infty_{b,cyl}(H_-)$ and $a(x,p) = g(p)$. Then*

$$a(X,D) \in \mathcal{A}^\varepsilon \quad \forall\, 0 < \varepsilon \le 1.$$

PROOF. Proposition 3.4.16 implies that $a(X,D) \in \mathcal{A}^1$ and thus our assertion follows by [**25**, Theorem 2.3.11]. □

THEOREM 3.4.18. *Let $g \in \mathscr{C}^\infty_{b,cyl}(H_-)$ and $a(x,p) = g(p)$. Furthermore, let Ψ^0 be defined as in 3.3.7. Then we have*

$$a(X,D) \in \Psi^0.$$

PROOF. The assertion is clear by 3.3.7, Lemma 3.4.14 and Corollary 3.4.17. □

COROLLARY 3.4.19. *Let $g \in \mathscr{C}^\infty_{b,cyl}(H_-)$ and $a(x,p) = g(p)$. Let $\widetilde{\Psi}^0_{0,0}$ be defined as in 3.3.9. Then we have*

$$a(X,D) \in \widetilde{\Psi}^0_{0,0}.$$

THEOREM 3.4.20. *Let Ψ^0_{cyl} be the closed algebraic span of the operators $M_f\mathcal{F}^{-1}$ and $M_g\mathcal{F}$ in Ψ_0 where $f,\ g \in \mathscr{C}^\infty_{b,cyl}$ and $M_f,\ \mathcal{F}^{-1}M_g\mathcal{F}$ are defined as in 3.4.1 and 3.4.12. Then Ψ^0_{cyl} is a sub-multiplicative Ψ^*-algebra.*

3.5. Fourier operators of order 0 as elements of the generalized Hörmander classes

Let $a \in \mathcal{G}$. In Lemma 3.2.7 we have proved that $a(X,D)$ is a continuous linear operator in $L^2(H_-,\ \gamma)$. Moreover, in 3.2.21 we have shown that $a(X,D) \in \Psi^{MD}$ in the case of a Gaussian measure. Now it is our aim to show that under certain restrictions this operators are elements of $\widetilde{\Psi}^0_{0,0}$.

Throughout this section let $H_+ \subseteq H_0 \subseteq H_-$ be a Hilbert space rigging such that there exists an orthonormal basis $(e_j)_{j=1}^\infty \subset H_+$ in H_0 with

$$\langle x,y\rangle_0 = \sum_{j=1}^\infty \langle x,e_j\rangle_0\langle e_j,y\rangle_0 \quad \forall x \in H_-;\ y \in H_+. \tag{31}$$

Moreover, let γ be the canonical Gaussian measure on $\mathscr{B}(H_-)$.

LEMMA 3.5.1. *In this case Proposition 2.1.3 is true for all $f \in \mathscr{C}^\infty_{int}(H_-)$ and thus $\mathscr{C}^\infty_{int}(H_-)$ is a domain of essential selfadjointness of L_{γ_1}. Furthermore, L_γ leaves the space $\mathscr{C}^\infty_{int}(H_-)$ invariant.*

PROOF. Let $(e_j)_{j=1}^{\infty} \subset H_+$ be an orthonormal basis in H_0 such that relation (31) holds. Then the first assertion follows similarly to Proposition 2.1.3 (writing ∞ instead of n in the sum), together with Lebesgue's Theorem of dominated convergence, since

$$\sum_{k=1}^{\infty} \left|\frac{\partial f(x)}{\partial x_k}\right| \left|\frac{\partial g(x)}{\partial x_k}\right| = \sum_{k=1}^{\infty} |\langle f'(x), e_k\rangle_0| \, |\langle g'(x), e_k\rangle_0| \leq \|f'(x)\|_+ \|g'(x)\|_+ \sum_{k=1}^{\infty} \|e_k\|_-^2$$

and

$$\sum_{k=1}^{\infty} \left|\frac{\partial^2 f(x)}{\partial x_k^2}\right| \leq \sum_{k=1}^{\infty} |\langle f''(x)e_k, e_k\rangle_0| \leq \|f''(x)\|_{\mathscr{L}(H_-,H_+)} \sum_{k=1}^{\infty} \|e_k\|_-^2$$

and

$$\begin{aligned} \left|\sum_{k=1}^{n} \beta_\gamma(e_k, x)\frac{\partial f(x)}{\partial x_k}\right| &\leq \left|\sum_{k=1}^{n} \langle \beta_\gamma(x), e_k\rangle_0 \langle f'(x), e_k\rangle_0\right| \\ &= \left|\sum_{k=1}^{\infty} \langle \beta_\gamma(x), P_n e_k\rangle_0 \langle f'(x), e_k\rangle_0\right| \\ &= \left|\sum_{k=1}^{\infty} \langle P_n^* \beta_\gamma(x), e_k\rangle_0 \langle f'(x), e_k\rangle_0\right| \\ &= |\langle P_n^* \beta_\gamma(x), f'(x)\rangle_0| \\ &\leq \|\beta_\gamma(x)\|_- \|f'(x)\|_+ \in L^2(H_-,\ \gamma), \end{aligned}$$

where P_n is the orthogonal projection onto $span\{e_1, \dots e_n\}$ in H_+. Hence L_γ is symmetric and positive on $\mathscr{C}_{int}^{\infty}(H_-)$ and thus L_γ possesses a selfadjoint extension. Since $\mathscr{C}_b^{\infty}(H_-) \subset \mathscr{C}_{int}^{\infty}(H_-)$ is a domain of essential selfadjointness of L_γ, our second assertion follows directly. The third part is similar to Corollary 2.1.14 □

EXAMPLE 3.5.2. Let us give some examples of Hilbert spaces riggings $H_+ \subseteq H_0 \subseteq H_-$, for which there exists an orthonormal basis $(e_j)_{j=1}^{\infty} \subset H_+$ such that (31) holds.

- (i) As first example, let $w := (w_n)_{n\in\mathbb{N}}$ be a sequence such that $w_n > 0\ \forall n$ and $\sum_{k=1}^{\infty}(\frac{1}{w_k})^2 < \infty$. Then define $H_+ := l_w^2(\mathbb{N})$, $H_0 := l^2(\mathbb{N})$ and $H_- := l_{\frac{1}{w}}^2(\mathbb{N})$, i.e. H_- and H_+ are weighted sequence spaces.
- (ii) Moreover, we have such a situation, if we set $H_0 := L^2(S^1)$ and $H_+ := H^s$ $(s > 0)$, i.e. H^s is the Sobolev space of order s and $H_- := H^{-s}$ such that the rigging is quasi-nuclear.

LEMMA 3.5.3. *Let H be a separable Hilbert space and $t \in H$ with $\|t\|_H = 1$. Then there exist vectors $\{e_j\}_{j=1}^{\infty}$ such that $t,\ e_1,\ e_2,\ \dots$ form an orthonormal basis in H.*

PROOF. Let $t \in H$ with $\|t\|_H = 1$. Then we have $H = span\{t\} \oplus span\{t\}^\perp$. Let $(e_j)_{j=1}^\infty$ be an orthonormal basis in $span\{t\}^\perp$. Then $t,\ e_1,\ e_2,\ \ldots$ form an orthonormal basis in H. □

COROLLARY 3.5.4. *For $t \in H_+$ let M_t by defined as in 1.2.2.and let ∂_t by defined as in 1.2.5. Then the operators $M_t : H^{s+1} \longrightarrow H^s$ and $\partial_t : H^{s+1} \longrightarrow H^s$ are continuous, with $\|M_t\|_{\mathscr{L}(H^{s+1},H^s)} \le c'_s$, $\|D_t\|_{\mathscr{L}(H^{s+1},H^s)} \le c'_s$ and $\|\partial_t\|_{\mathscr{L}(H^{s+1},H^s)} \le c_s$, where c_s and c'_s are constants depending on s.*

PROOF. The assertion follows immediately by Lemma 3.5.3, Proposition 3.3.14 and Corollary 3.3.16. □

We will use Theorem 3.3.13 to show that for $a \in \mathcal{G}$ the pseudodifferential operator $a(X,D)$ is an element of the Ψ^*-algebra Ψ^0. Therefore we have to prove the assumption of this theorem. Let us start we a rather technical lemma.

LEMMA 3.5.5. *Let $H_+ \subseteq H_0 \subseteq H_-$ such that there exists an orthonormal basis $(e_\nu)_{\nu=1}^\infty \subset H_+$ with (31). Moreover, for $p',\ x' \in H_+$ let $W_{\frac{x'}{2}}$ and $U_{p'}$ be defined as in 1.3.2 and (25). Furthermore, let $j \in \mathbb{N}$ arbitrary and $f \in \mathscr{C}^\infty_{pol}(H_-)$. For $\alpha, \beta \in \mathbb{N}_0^n$ we set*

$$\mathcal{A}^\alpha(p') = \langle p', f_1\rangle_0^{\alpha_1} \cdots \langle p', f_\nu\rangle_0^{\alpha_\nu}$$

and

$$B^\alpha(x') = (i\langle x', f_1\rangle)_0^{\alpha_1} \cdots (i\langle x', f_\nu\rangle_0)^{\alpha_\nu},$$

where $(f_j)_{j=1}^n \subset H_+$ is an arbitrary orthonormal basis in H_-. For $\xi \in M_\infty(H_+^2)$ define $A : \mathscr{C}^\infty_{pol}(H_-) \longrightarrow \mathscr{C}^\infty_{int}(H_-)$ by

$$Af := \sum_{k=0}^{j} \int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} \sum_{l=0}^{n_k} \langle x', x'\rangle_0^{m_1(n_k)} \langle p', p'\rangle_0^{m_2(n_k)} V(x', p', n_k, l) f(x)\, d\xi(p', x'),$$

where

$$\begin{aligned} &V(x', p', n_k, l) f(x) \\ :=\ & a_{l,k} \mathcal{A}^\alpha(p') B^\beta(x') \langle x', x\rangle_0^{m_3(n_k)} \langle p', x\rangle_0^{m_4(n_k)} \langle x', p'\rangle_0^{m_5(n_k)} \partial^{n_k,l}_{x',p'} f(x). \end{aligned}$$

Moreover, we assume that $m_3(n_k) + m_4(n_k) + k \le j$, $a_{l,k} \in \mathbb{C}$, $m_1(n_k)$, $\ldots$, $m_5(n_k) \in \mathbb{N}_0$ and $\partial^{n_k,l}_{x',p'} f := \partial^{k_1(l)}_{x'} \partial^{k_2(l)}_{p'}$ with $k_1(l) + k_2(l) = k$. Then we obtain $Af \in \mathscr{C}^\infty_{int}(H_-)$ and

$$\begin{aligned} &[\Lambda^2,\ A] f(x) \\ =\ & \sum_{k=0}^{j+1} \int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} \sum_{l=0}^{\tilde n_k} \langle x', x'\rangle_0^{\tilde m_1(\tilde n_k)} \langle p', p'\rangle_0^{\tilde m_2(\tilde n_k)} \tilde V(x', p', \tilde n_k, l) f(x)\, d\xi(p', x'), \end{aligned}$$

where

$$\begin{aligned} & \tilde{V}(x',p',\tilde{n}_k,l)f(x) \\ := \; & b_{l,k}\mathcal{A}^{\alpha'}(p')B^{\beta'}(x')\langle x',x\rangle_0^{\tilde{m}_3(\tilde{n}_k)}\langle p',x\rangle_0^{\tilde{m}_4(\tilde{n}_k)}\langle x,p'\rangle_0^{\tilde{m}_5(\tilde{n}_k)}\partial_{(x',p')}^{(\tilde{n}_k,l)}f(x), \end{aligned}$$

such that $\tilde{m}_3(\tilde{n}_k)+\tilde{m}_4(\tilde{n}_k)+k \le j+1$, $b_{l,k}\in\mathbb{C}$, $\tilde{m}_1(\tilde{n}_k)$, $\ldots$, $\tilde{m}_5(\tilde{n}_k)\in\mathbb{N}_0$ *and* $\partial_{x',p'}^{\tilde{n}_k,l}f := \partial_{x'}^{k_1(l)}\partial_{p'}^{k_2(l)}$ *with* $k_1(l)+k_2(l)=k$ *and* α', $\beta'\in\mathbb{N}_0^{\mathbb{N}}$.

Proof. Let $f\in\mathscr{C}_{pol}^{\infty}(H_-)$ and

$$g(x',p') := \mathcal{A}^{\alpha}(p')B^{\beta}(x')\langle x',x'\rangle_0^{m_1(n_k)}\langle p',p'\rangle_0^{m_2(n_k)}\langle x',p'\rangle_0^{m_5(n_k)}.$$

Moreover, we write $\partial_\nu := \partial_{e_\nu}$ and $M_\nu := M_{e_\nu}$. Using Lemma 3.2.12 and 3.2.9 we obtain

$$\begin{aligned} & \mathrm{L}_\gamma \int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x') \\ = \; & \frac{-1}{2}\sum_{\nu=1}^{\infty}\partial_\nu^2\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x') \\ & +\sum_{\nu=1}^{\infty}M_\nu\partial_\nu\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x') \\ = \; & \frac{-1}{2}\sum_{\nu=1}^{\infty}\partial_\nu\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\partial_\nu[\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)]d\xi(p',x') \\ & +\frac{-1}{2}\sum_{\nu=1}^{\infty}\partial_\nu\int_{H_+^2}(i\langle x',e_\nu\rangle_0+\langle p',e_\nu\rangle_0) \\ & \qquad W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x') \\ & +\sum_{\nu=1}^{\infty}M_\nu\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\partial_\nu[\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)]\;d\xi(p',x') \\ & +\sum_{\nu=1}^{\infty}M_\nu\int_{H_+^2}(i\langle x',e_\nu\rangle_0+\langle p',e_\nu\rangle_0) \\ & \qquad W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x') \end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{-1}{2}\sum_{\nu=1}^{\infty}\int\limits_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\partial_\nu^2[\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)]\;d\xi(p',x') \\
& -\sum_{\nu=1}^{\infty}\int\limits_{H_+^2}(i\langle x',e_\nu\rangle_0+\langle p',e_\nu\rangle_0) \\
& \qquad W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\partial_\nu[\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)]\;d\xi(p',x') \\
& +\frac{-1}{2}\sum_{\nu=1}^{\infty}\int\limits_{H_+^2}(i\langle x',e_\nu\rangle_0+\langle p',e_\nu\rangle_0)^2 \\
& \qquad W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x') \\
& +\sum_{\nu=1}^{\infty}\int\limits_{H_+^2}W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')M_\nu\partial_\nu[\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)]\;d\xi(p',x') \\
& -\sum_{\nu=1}^{\infty}\int\limits_{H_+^2}\langle p',e_\nu\rangle_0 \\
& \qquad W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\partial_\nu[\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)]\;d\xi(p',x') \\
& +\sum_{\nu=1}^{\infty}M_\nu\int\limits_{H_+^2}(i\langle x',e_\nu\rangle_0+\langle p',e_\nu\rangle_0) \\
& \qquad W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x') \\
= \quad & -\frac{1}{2}\sum_{\nu=1}^{\infty}\int\limits_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',e_\nu\rangle_0^2 m_3(n_k)(m_3(n_k)-1)\langle x',x\rangle_0^{m_3(n_k)-2} \\
& \qquad\qquad \langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x') \\
& -\frac{1}{2}\sum_{\nu=1}^{\infty}\int\limits_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle p',e_\nu\rangle_0^2 m_4(n_k)(m_4(n_k)-1)\langle p',x\rangle_0^{m_4(n_k)-2} \\
& \qquad\qquad \langle x',x\rangle_0^{m_3(n_k)}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x') \\
& -\sum_{\nu=1}^{\infty}\int\limits_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',e_\nu\rangle_0\langle p',e_\nu\rangle_0 m_3(n_k)\langle x',x\rangle_0^{m_3(n_k)-1} \\
& \qquad\qquad m_4(n_k)\langle p',x\rangle_0^{m_4(n_k)-1}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x')
\end{aligned}
$$

$$-\sum_{\nu=1}^{\infty}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',e_\nu\rangle_0 m_3(n_k)\langle x',x\rangle_0^{m_3(n_k)-1}\langle p',x\rangle_0^{m_4(n_k)}$$

$$\partial_\nu\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x')$$

$$-\sum_{\nu=1}^{\infty}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle p',e_\nu\rangle_0\langle x',x\rangle_0^{m_3(n_k)}m_4(n_k)\langle p',x\rangle_0^{m_4(n_k)-1}$$

$$\partial_\nu\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x')$$

$$-\frac{1}{2}\sum_{\nu=1}^{\infty}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_\nu^2\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x')$$

$$-\sum_{\nu=1}^{\infty}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')(i\langle x',e_\nu\rangle_0+\langle p',e_\nu\rangle_0)\langle p',x\rangle_0^{m_4(n_k)}$$

$$m_3(n_k)\langle x',e_\nu\rangle_0\langle x',x\rangle_0^{m_3(n_k)-1}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x')$$

$$-\sum_{\nu=1}^{\infty}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')(i\langle x',e_\nu\rangle_0+\langle p',e_\nu\rangle_0)\langle x',x\rangle_0^{m_3(n_k)}$$

$$m_4(n_k)\langle p',e_\nu\rangle_0\langle p',x\rangle_0^{m_4(n_k)-1}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x')$$

$$-\sum_{\nu=1}^{\infty}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')(i\langle x',e_\nu\rangle_0+\langle p',e_\nu\rangle_0)$$

$$\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_\nu\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x')$$

$$+\frac{-1}{2}\sum_{\nu=1}^{\infty}\int_{H_+^2}(i\langle x',e_\nu\rangle_0+\langle p',e_\nu\rangle_0)^2$$

$$W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x')$$

$$+\sum_{\nu=1}^{\infty}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x,e_\nu\rangle_0\langle p',x\rangle_0^{m_4(n_k)}$$

$$m_3(n_k)\langle x',e_\nu\rangle_0\langle x',x\rangle_0^{m_3(n_k)-1}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x')$$

$$+\sum_{\nu=1}^{\infty}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x,e_\nu\rangle_0\langle x',x\rangle_0^{m_3(n_k)}$$

$$m_4(n_k)\langle p',e_\nu\rangle_0\langle p',x\rangle_0^{m_4(n_k)-1}\partial_{x',p'}^{n_k,l}f(x)\;d\xi(p',x')$$

$$
\begin{aligned}
&+\sum_{\nu=1}^{\infty}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)} \\
&\qquad (c_{n_k,l}\langle x',e_\nu\rangle_0\partial_{x',p'}^{\tilde{n}_k,\tilde{l}}+d_{n_k,l}\langle p',e_\nu\rangle_0\partial_{x',p'}^{\hat{n}_k,\hat{l}})\partial_\nu f(x)\ d\xi(p',x') \\
&+\sum_{\nu=1}^{\infty}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}M_\nu\partial_\nu f(x)\ d\xi(p',x') \\
&-\sum_{\nu=1}^{\infty}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle p',e_\nu\rangle_0\langle p',x\rangle_0^{m_4(n_k)} \\
&\qquad m_3(n_k)\langle x',e_\nu\rangle_0\langle x',x\rangle_0^{m_3(n_k)-1}\partial_{x',p'}^{n_k,l}f(x)\ d\xi(p',x') \\
&-\sum_{\nu=1}^{\infty}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle p',e_\nu\rangle_0\langle x',x\rangle_0^{m_3(n_k)} \\
&\qquad m_4(n_k)\langle p',e_\nu\rangle_0\langle p',x\rangle_0^{m_4(n_k)-1}\partial_{x',p'}^{n_k,l}f(x)\ d\xi(p',x') \\
&-\sum_{\nu=1}^{\infty}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle p',e_\nu\rangle \\
&\qquad \langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_\nu\partial_{x',p'}^{n_k,l}f(x)\ d\xi(p',x') \\
&+\sum_{\nu=1}^{\infty}\int_{H_+^2} (i\langle x',e_\nu\rangle_0+\langle p',e_\nu\rangle_0) \\
&\qquad W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')M_\nu\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)\ d\xi(p',x') \\
&+\sum_{\nu=1}^{\infty}\int_{H_+^2} (i\langle x',e_\nu\rangle_0+\langle p',e_\nu\rangle_0)\langle p',e_\nu\rangle_0 \\
&\qquad W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)\ d\xi(p',x') \\
= &-\frac{1}{2}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\,\|x'\|_0^2\,m_3(n_k)(m_3(n_k)-1)\langle x',x\rangle_0^{m_3(n_k)-2} \\
&\qquad \langle p',x\rangle_0^{m_4(n_k)}\partial_{x',p'}^{n_k,l}f(x)\ d\xi(p',x') \\
&-\frac{1}{2}\int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}g(x',p')\,\|p'\|_0^2\,m_4(n_k)(m_4(n_k)-1)\langle p',x\rangle_0^{m_4(n_k)-2} \\
&\qquad \langle x',x\rangle_0^{m_3(n_k)}\partial_{x',p'}^{n_k,l}f(x)\ d\xi(p',x')
\end{aligned}
$$

$$-\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') \langle x',p'\rangle_0 m_3(n_k) \langle x',x\rangle_0^{m_3(n_k)-1}$$

$$m_4(n_k)\langle p',x\rangle_0^{m_4(n_k)-1} \partial_{x',p'}^{n_k,l} f(x)\; d\xi(p',x')$$

$$-\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') m_3(n_k) \langle x',x\rangle_0^{m_3(n_k)-1} \langle p',x\rangle_0^{m_4(n_k)}$$

$$\langle x', (\partial_{x',p'}^{n_k,l} f)'(x)\rangle_0 \; d\xi(p',x')$$

$$-\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') \langle x',x\rangle_0^{m_3(n_k)} m_4(n_k) \langle p',x\rangle_0^{m_4(n_k)-1}$$

$$\langle p', (\partial_{x',p'}^{n_k,l} f)(x)'\rangle_0 \; d\xi(p',x')$$

$$+\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') \langle x',x\rangle_0^{m_3(n_k)} \langle p',x\rangle_0^{m_4(n_k)} \partial_{x',p'}^{n_k,l} \mathrm{L}_\gamma f(x)\; d\xi(p',x')$$

$$-\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') (i\langle x',x'\rangle_0 + \langle p',x'\rangle_0) \langle p',x\rangle_0^{m_4(n_k)}$$

$$m_3(n_k)\langle x',x\rangle_0^{m_3(n_k)-1} \partial_{x',p'}^{n_k,l} f(x)\; d\xi(p',x')$$

$$-\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') (i\langle x',p'\rangle_0 + \langle p',p'\rangle_0) \langle x',x\rangle_0^{m_3(n_k)}$$

$$m_4(n_k)\langle p',x\rangle_0^{m_4(n_k)-1} \partial_{x',p'}^{n_k,l} f(x)\; d\xi(p',x')$$

$$-\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') \langle x',x\rangle_0^{m_3(n_k)} \langle p',x\rangle_0^{m_4(n_k)}$$

$$(i\langle x', (\partial_{x',p'}^{n_k,l} f)'\rangle_0 + \langle p', (\partial_{x',p'}^{n_k,l} f)'(x)\rangle_0)\; d\xi(p',x')$$

$$-\frac{1}{2}\int_{H_+^2} (i\langle x',x'\rangle_0 + 2i\langle x',p'\rangle_0 + \langle p',p'\rangle_0)$$

$$W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') \langle x',x\rangle_0^{m_3(n_k)} \langle p',x\rangle_0^{m_4(n_k)} \partial_{x',p'}^{n_k,l} f(x)\; d\xi(p',x')$$

$$+\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') \langle x,x'\rangle_0 \langle p',x\rangle_0^{m_4(n_k)}$$

$$m_3(n_k)\langle x',x\rangle_0^{m_3(n_k)-1} \partial_{x',p'}^{n_k,l} f(x)\; d\xi(p',x')$$

$$
\begin{aligned}
&+\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') \langle x,p'\rangle_0 \langle x',x\rangle_0^{m_3(n_k)} \\
&\qquad\qquad m_4(n_k) \langle p',x\rangle_0^{m_4(n_k)-1} \partial_{x',p'}^{n_k,l} f(x)\; d\xi(p',x') \\
&+\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') \langle x',x\rangle_0^{m_3(n_k)} \langle p',x\rangle_0^{m_4(n_k)} \\
&\qquad\qquad (c_{n_k,l} \langle x', (\partial_{x',p'}^{\tilde{n}_k,\tilde{l}} f)'(x)\rangle_0 + d_{n_k,l} \langle p', (\partial_{x',p'}^{\hat{n}_k,\hat{l}} f)'(x)\rangle_0)\; d\xi(p',x') \\
&-\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') \langle p',x'\rangle_0 \langle p',x\rangle_0^{m_4(n_k)} \\
&\qquad\qquad m_3(n_k) \langle x',x\rangle_0^{m_3(n_k)-1} \partial_{x',p'}^{n_k,l} f(x)\; d\xi(p',x') \\
&-\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') \langle x',x\rangle_0^{m_3(n_k)} \\
&\qquad\qquad m_4(n_k) \langle p',p'\rangle_0 \langle p',x\rangle_0^{m_4(n_k)-1} \partial_{x',p'}^{n_k,l} f(x)\; d\xi(p',x') \\
&-\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') \\
&\qquad\qquad \langle x',x\rangle_0^{m_3(n_k)} \langle p',x\rangle_0^{m_4(n_k)} \langle p', (\partial_\nu \partial_{x',p'}^{n_k,l} f)'(x)\rangle_0\; d\xi(p',x') \\
&+\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') (i\langle x',x\rangle_0 + \langle p',x\rangle_0) \\
&\qquad\qquad \langle x',x\rangle_0^{m_3(n_k)} \langle p',x\rangle_0^{m_4(n_k)} \partial_{x',p'}^{n_k,l} f(x)\; d\xi(p',x') \\
&+\int_{H_+^2} (i\langle x',p'\rangle_0 + \langle p',p'\rangle_0) \\
&\qquad\qquad W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p') \langle x',x\rangle_0^{m_3(n_k)} \langle p',x\rangle_0^{m_4(n_k)} \partial_{x',p'}^{n_k,l} f(x)\; d\xi(p',x'),
\end{aligned}
$$

where $c_{n_k,l},\ d_{n_k,l} \in Z$ and $\partial_\nu := \partial_{e_\nu}$. Moreover, $\partial_{x',p'}^{\tilde{n}_k,\tilde{l}}$ (resp. $\partial_{x',p'}^{\hat{n}_k,\hat{l}}$) denotes differentiation one time less in direction x' (resp. p') as in $\partial_{x',p'}^{n_k,l}$. Taking note of the fact that $\langle t, f'(x)\rangle_0 = df(x)(t) = \partial_t f(x)$, the assertion follows directly, since L_γ is linear and the integral commutes with finite sums. To commute differentiation and integration is allowed as in 3.2.14. Now it remains to show that we are allowed to commute integral and series. This follows again by Lebesgue's theorem of dominated convergence. We will give an examples right now. Since we can assume x is fixed there exist $a,\ c > 0$ such that

$$\begin{aligned}
&\Bigg|\sum_{v=1}^{N} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p')\langle p',x\rangle_0^{m_4(n_k)} m_3(n_k)\langle x',x\rangle_0^{m_3(n_k)-1} \\
&\qquad\qquad \langle x',e_\nu\rangle_0 \langle x,e_\nu\rangle_0 \partial_{x',p'}^{n_k,l} f(x)\Bigg| \\
=\ &\Bigg|\sum_{v=1}^{\infty} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p')\langle p',x\rangle_0^{m_4(n_k)} m_3(n_k)\langle x',x\rangle_0^{m_3(n_k)-1} \\
&\qquad\qquad \langle x',e_\nu\rangle_0 \langle P_N^* x,e_\nu\rangle_0 \partial_{x',p'}^{n_k,l} f(x)\Bigg| \\
=\ &\Big|W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} g(x',p')\langle p',x\rangle_0^{m_4(n_k)} m_3(n_k)\langle x',x\rangle_0^{m_3(n_k)-1}\langle x', P_N^* x\rangle_0 \partial_{x',p'}^{n_k,l} f(x)\Big| \\
=\ &\Big|g(x',p')\sqrt{\varrho_{p'}}\langle p',x+p'\rangle_0^{m_4(n_k)} m_3(n_k)\langle x',x+p'\rangle_0^{m_3(n_k)-1} \\
&\qquad\qquad \langle x', P_N^* x+p'\rangle_0 \partial_{x',p'}^{n_k,l} f(x+p)\Big| \\
=\ &\Big|g(x',p')\sqrt{\varrho_{p'}}\langle p',x+p'\rangle_0^{m_4(n_k)} m_3(n_k)\langle x',x+p'\rangle_0^{m_3(n_k)-1} \\
&\qquad\qquad \langle x', P_N x\rangle_0 d^k f(x+p')(x',\ldots,x',p'\ldots p')\Big| \\
\leq\ &\Big|g(x',p')\sqrt{\varrho_p}\langle p',x+p'\rangle_0^{m_4(n_k)} m_3(n_k)\langle x',x+p'\rangle_0^{m_3(n_k)-1}\Big| \\
&\qquad\qquad \|x'\|_+ \|P_N x\|_- \left\|d^k(x+p')\right\|_{\mathscr{L}^k(H_-;\mathbb{C})} \|x'\|_-^{k_1} \|p'\|_-^{k_2} \\
\leq\ & ce^{a(\|x'\|_- \|p'\|_+)} \in L^1(H_+^2,\ \xi),
\end{aligned}$$

where P_N is the orthogonal projection onto $span\{e_1,\ldots,e_N\}$ in H_+. This estimate is independent of N. Thus by Lebesgue's Theorem we can commute series and integral. □

Corollary 3.5.6. *Let $H_+ \subseteq H_0 \subseteq H_-$ be a Hilbert space rigging such that (31) holds. Furthermore, let a be the Fourier-transform of $\xi \in M_\infty(H_+^2)$, i.e. $a \in \mathcal{G}$. For $f \in \mathscr{C}_{pol}^\infty(H_-)$ and $j \in \mathbb{N}_0$ we obtain*

$$\begin{aligned}
&\mathrm{ad}^j(\Lambda^2)(a(X,D))f(x) \\
=\ &\sum_{k=0}^{j}\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} \sum_{l=0}^{n_k}\langle x',x'\rangle_0^{m_1(n_k)}\langle p',p'\rangle_0^{m_2(n_k)} V(x',p',n_k,l)f(x)\ d\xi(p',x'),
\end{aligned}$$

where

$$V(x',p',n_k,l)f(x) := a_{l,k}\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\langle x',p'\rangle_0^{m_5(n_k)}\partial_{x',p'}^{n_k,l} f(x)$$

and $m_3(n_k)+m_4(n_k)+k \leq j$, $a_{l,k} \in \mathbb{C}$, $m_1(n_k),\ \ldots,\ m_5(n_k) \in \mathbb{N}_0$ and $\partial_{x',p'}^{n_k,l} f := \partial_{x'}^{k_1(l)}\partial_{p'}^{k_2(l)}$ such that $k_1(l)+k_2(l) = k$.

PROOF. We have $a(X,D)f(x) = \int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x)\ d\xi(p',x')$ by 3.2.6 . Thus our assertion follows by induction and 3.5.5, 3.2.16 and 3.5.1. □

COROLLARY 3.5.7. *Let $H_+ \subseteq H_0 \subseteq H_-$ be a Hilbert space rigging such that (31) holds. Moreover, let a be the Fourier-transform of $\xi \in M_\infty(H_+^2)$, i.e. $a \in \mathcal{G}$. In addition, let $f \in \mathscr{C}^\infty_{pol}(H_-)$ and $j \in \mathbb{N}_0$. For an arbitrary orthonormal basis $(f_\nu)_{\nu=1}^\infty \subset H_+$ in H_0 and $\alpha,\ \beta \in \mathbb{N}_0^{\mathbb{N}}$ define M^α, ∂^β as in 3.2.18. Let $\mathcal{A}^{\alpha'}(p'), B^{\beta'}(x')$ be defined as in 3.5.5 for $\alpha,\ \beta \in \mathbb{N}_0^n$. Then we obtain*

$$\begin{aligned}&\mathrm{ad}^j(\Lambda^2)(\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(a(X,D)))f(x)\\ =\ &\sum_{k=0}^{j}\int_{H_+^2} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} \sum_{l=0}^{n_k} \langle x',x'\rangle_0^{m_1(n_k)} \langle p',p'\rangle_0^{m_2(n_k)} V(x',p',n_k,l) f(x)\ d\xi(p',x'),\end{aligned}$$

where

$$\begin{aligned}&V(x',p',n_k,l)f(x)\\ :=\ &a_{l,k}(p')A^{\alpha'}B^{\beta'}(x')\langle x',x\rangle_0^{m_3(n_k)}\langle p',x\rangle_0^{m_4(n_k)}\langle x',p'\rangle_0^{m_5(n_k)}\partial^{n_k,l}_{x',p'}f(x)\end{aligned}$$

and $m_3(n_k)+m_4(n_k)+k \le j$, $a_{l,k} \in \mathbb{C}$, $m_1(n_k),\ \dots,\ m_5(n_k) \in \mathbb{N}_0$, $\partial^{n_k,l}_{x',p'} f := \partial_{x'}^{k_1(l)}\partial_{p'}^{k_2(l)}$ such that $k_1(l)+k_2(l)=k$.

PROOF. Let $A(p')$ be defined as in 3.2.18. Proposition 3.2.19 yields

$$\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(a(X,D))f(x) = \int_{H_+^2} A^\alpha(p')B^\beta(x') W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} f(x)\ d\xi(p',x').$$

Thus the assertion follows by induction, 3.5.5 and 3.5.1. □

At next we prove a technical result, which we need in what follows.

LEMMA 3.5.8. *Let E be a Banach space and $f \in \mathscr{C}^1(E)$ For $0 \neq t \in E$ we have*

$$\frac{\partial f}{\partial t}(x) = \|t\|\,\frac{\partial f}{\partial \frac{t}{\|t\|}}(x).$$

PROOF. For $f \in \mathscr{C}^1(E)$ and $0 \neq t \in E$ the following equality holds.

$$\frac{\partial f}{\partial t}(x) = df(x)(t) = \|t\|\, df(x)(\frac{t}{\|t\|}) = \|t\|\,\frac{\partial f}{\partial \frac{t}{\|t\|}}(x).$$ □

LEMMA 3.5.9. *Let $H_+ \subseteq H_0 \subseteq H_-$ be a Hilbert space rigging such (31) holds. Moreover, let $a \in \mathcal{G}$. Then for all $f \in H^\infty$, $k, j \in \mathbb{N}_0$ we have*

$$\left\|\Lambda^{2k}\mathrm{ad}^j(\Lambda^2)(a(X,D))f\right\|_0 \le a_{2k,j}\left\|\Lambda^{2k+j}f\right\|_0,$$

where $a_{2k,j} \ge 0$. Furthermore, $a(X,D) : H^{2k} \longrightarrow H^{2k}$ is continuous.

PROOF. At first let $f \in \mathscr{C}^\infty_{pol}(H_-)$ and $n, j \in \mathbb{N}_0$. Let $\partial^{n_k,l}_{x',p'} = \partial^{k_1}_{x'}\partial^{k_2}_{p'}$ with $k_1(l) + k_2(l) = k$. Then there exit $m_1(n_k), \ldots, m_5(n_k) \in \mathbb{N}_0$, $a_{l,k} \in \mathbb{C}$ such that $m_3(n_k) + m_4(n_k) + k \leq j$ and

$$\begin{aligned}
&(\Lambda^2)^n \mathrm{ad}^j(\Lambda^2)(a(X,D))f \\
= &\sum_{\nu=0}^{n} \binom{n}{\nu} (\mathrm{ad}^{\nu+j}(\Lambda^2)(a(X,D)))(\Lambda^2)^{n-\nu} f \\
= &\sum_{\nu=0}^{n} \binom{n}{\nu} \sum_{k=0}^{j+\nu} \int\limits_{H^2_+} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} \sum_{l=0}^{n_k} a_{l,k} \langle x', x'\rangle_0^{m_1(n_k)} \langle p', p'\rangle_0^{m_2(n_k)} \langle x', x\rangle_0^{m_3(n_k)} \\
&\qquad \langle p', x\rangle_0^{m_4(n_k)} \langle x', p'\rangle_0^{m_5(n_k)} \partial^{n_k,l}_{x',p'} (\Lambda^2)^{n-\nu} f(x)\, d\xi(p', x') \\
= &\sum_{\nu=0}^{n} \binom{n}{\nu} \sum_{k=0}^{j+\nu} \int\limits_{H^2_+} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} \sum_{l=0}^{n_k} a_{l,k} \langle x', x'\rangle_0^{m_1(n_k)} \langle p', p'\rangle_0^{m_2(n_k)}\, \|x'\|_0^{m_3(n_k)+k_1} \\
&\qquad \langle \frac{x'}{\|x'\|_0}, x\rangle_0^{m_3(n_k)}\, \|p\|_0^{m_4(n_k)+k_2}\, \langle \frac{p'}{\|p'\|_0}, x\rangle_0^{m_4(n_k)} \\
&\qquad \langle x', p'\rangle_0^{m_5(n_k)} \partial^{n_k,l}_{x'/\|x'\|, p'/\|p\|} (\Lambda^2)^{n-\nu} f(x)\, d\xi(p', x').
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\| \int\limits_{H^2_+} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} a_{l,k}\, \|x'\|_0^{2m_1(n_k)+m_3(n_k)+k_1} \|p'\|_0^{2m_2(n_k)+m_4(n_k)+k_2} \langle \frac{x'}{\|x'\|_0}, \cdot\rangle_0^{m_3(n_k)} \\
&\qquad \langle \frac{p'}{\|p'\|_0}, \cdot\rangle_0^{m_4(n_k)} \langle x', p'\rangle_0^{m_5(n_k)} \partial^{n_k,l}_{\frac{x'}{\|x'\|}, \frac{p'}{\|p\|}} (\Lambda^2)^{n-\nu} f\, d\xi(p', x')\|^2_{H^0} \\
= &\int\limits_{H_-} | \int\limits_{H^2_+} W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} a_{l,k}\, \|x'\|_0^{2m_1(n_k)+m_3(n_k)+k_1} \|p'\|_0^{2m_2(n_k)+m_4(n_k)+k_2} \langle \frac{x'}{\|x'\|_0}, x\rangle_0^{m_3(n_k)} \\
&\qquad \langle \frac{p'}{\|p'\|_0}, x\rangle_0^{m_4(n_k)} \langle x', p'\rangle_0^{m_5(n_k)} \partial^{n_k,l}_{\frac{x'}{\|x'\|}, \frac{p'}{\|p\|}} (\Lambda^2)^{n-\nu} f(x)\, d\xi(p', x')|^2\, d\gamma(x) \\
\leq &\int\limits_{H_-} \int\limits_{H^2_+} |a_{l,k}\, \|x'\|_0^{2m_1(n_k)+m_3(n_k)+k_1}\, \|p'\|_0^{2m_2(n_k)+m_4(n_k)+k_2}\, \langle x', p'\rangle_0^{m_5(n_k)}|^2\, d|\xi|(p', x') \\
&\qquad \int\limits_{H^2_+} |W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} [\langle \frac{x'}{\|x'\|_0}, \cdot\rangle_0^{m_3(n_k)} \langle \frac{p'}{\|p'\|_0}, \cdot\rangle_0^{m_4(n_k)} \partial^{n_k,l}_{\frac{x'}{\|x'\|}, \frac{p'}{\|p\|}} (\Lambda^2)^{n-\nu} f](x)|^2 \\
&\qquad\qquad d\,|\xi|\,(p', x')\; d\gamma(x)
\end{aligned}$$

$$
\begin{aligned}
&= c\int\limits_{H_+^2} \left\| W_{\frac{x'}{2}} U_{p'} W_{\frac{x'}{2}} \langle \frac{x'}{\|x'\|_0}, \cdot \rangle_0^{m_3(n_k)} \langle \frac{p'}{\|p'\|_0}, \cdot \rangle_0^{m_4(n_k)} \partial^{n_k,l}_{\frac{x'}{\|x'\|}, \frac{p'}{\|p\|}} (\Lambda^2)^{n-\nu} f \right\|^2_{L^2(H_-,\gamma)} d\,|\xi|\,(p',x') \\
&\leq \tilde{\tilde{c}} \int\limits_{H_+^2} \left\| \langle \frac{x'}{\|x'\|_0}, \cdot \rangle_0^{m_3(n_k)} \langle \frac{p'}{\|p'\|_0}, \cdot \rangle_0^{m_4(n_k)} \partial^{n_k,l}_{\frac{x'}{\|x'\|}, \frac{p'}{\|p\|}} (\Lambda^2)^{n-\nu} f \right\|^2_{L^2(H_-,\gamma)} d\,|\xi|\,(p',x') \\
&\leq c \int\limits_{H_+^2} \left\| (\Lambda^2)^{n-\nu} f \right\|^2_{H^{m_3(n_k)+m_4(n_k)+k}} d\,|\xi|\,(p',x') \leq \tilde{c}\, \|f\|^2_{H^{2n+j}}\,.
\end{aligned}
$$

Since $\mathscr{C}^{\infty}_{pol}(H_-) \subset H^s$ dense for all s and all operators are closed, we obtain the first assertion. The second is clear, since Λ^k is closed for all $k \in \mathbb{N}$. □

THEOREM 3.5.10. *Let $H_+ \subseteq H_0 \subseteq H_-$ be a quasi-nuclear Hilbert space rigging such that (31) holds. Moreover, let $a \in \mathcal{G}$. Then for all $f \in H^\infty$, $k, j \in \mathbb{N}_0$ we have*

$$\left\| \Lambda^k \mathrm{ad}^j(\Lambda)(a(X,D)) f \right\|_{H^0} \leq a_{2k,j} \left\| \Lambda^k f \right\|_{H^0},$$

where $a_{k,j} \geq 0$. Moreover, $a(X,D):\ H^k \longrightarrow H^k$ is continuous.

PROOF. The first part follows by 3.5.9 and Proposition 3.3.13. The second part follows by part one, since Λ is closed. □

THEOREM 3.5.11. *Let $H_+ \subseteq H_0 \subseteq H_-$ be a quasi-nuclear Hilbert space rigging such that (31) holds. Furthermore, let $a \in \mathcal{G}$. Then for all $f \in H^\infty$, $k, j \in \mathbb{N}_0$, $\alpha,\ \beta \in \mathbb{N}_0^{\mathbb{N}}$ we have*

$$\left\| \Lambda^k \mathrm{ad}(\Lambda)^j \mathrm{ad}(M)^\alpha \mathrm{ad}(D)^\beta (a(X,D)) f \right\|_{H^0} \leq a_{k,j} \left\| \Lambda^k f \right\|_{H^0},$$

where $a_{k,j} \geq 0$. Moreover, $\mathrm{ad}(M)^\alpha \mathrm{ad}(D)^\beta (a(X,D)):\ H^k \longrightarrow H^k$ is continuous.

PROOF. Similarly to 3.5.9 we obtain for all $f \in H^\infty$

$$\left\| \Lambda^{2k} \mathrm{ad}^j(\Lambda^2)(\mathrm{ad}(M)^\alpha \mathrm{ad}(D)^\beta (a(X,D))) f \right\|_{H^0} \leq a_{2k,j} \left\| \Lambda^{2k+j} f \right\|_{H^0},$$

and thus the assertion follows similarly to 3.5.10. □

COROLLARY 3.5.12. *Let $H_+ \subseteq H_0 \subseteq H_-$ be a quasi-nuclear Hilbert space rigging such that (31) holds. For $a \in \mathcal{G}$ we have*

$$a(X,D) \in \Psi^0,$$

and thus

$$a(X,D) \in \widetilde{\Psi}^0_{0,0}.$$

PROOF. The assertion follows by Theorem 3.5.10 and Theorem 3.5.11. □

3.6. The Ψ^*-Algebras in the finite dimensional case

Throughout this section let $H_+ = H_0 = H_- = \mathbb{R}^n$. Furthermore, we assume that $\gamma = \gamma_1$ is the canonical Gaussian measure and that λ is the Lebesgue measure in $\mathbb{R}^n$. Moreover, let $\langle \cdot, \cdot \rangle$ denote the euclidean inner product in $\mathbb{R}^n$ and let $(e_j)_{j=1}^n$ be the standard orthonormal basis in $\mathbb{R}^n$. For $0 \le \delta \le \varrho \le 1$, $\delta < 1$ and $m \in \mathbb{Z}$ we denote by $S^m_{\varrho,\delta}$ the class of all symbols $a \in \mathscr{C}^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_p)$ such that for all multi-index α, β there exists a constant $C_{\alpha,\beta}$ with

$$\left| \Big(\frac{\partial}{\partial p}\Big)^\alpha \Big(\frac{\partial}{\partial x}\Big)^\beta a(x,p) \right| \le C_{\alpha,\beta} \langle p \rangle^{m+\delta|\beta|-\varrho|\alpha|}, \tag{32}$$

where $\langle p \rangle = \sqrt{1+|p|^2}$. Moreover, we set $S^\infty_{\varrho,\delta} := \bigcup_{m\in\mathbb{Z}} S^m_{\varrho,\delta}$. For $a \in S^m_{\varrho,\delta}$ let $a(x, i\frac{\partial}{\partial x})$ be the pseudodifferential operator with symbol $a(x,p)$ given in Weyl-form.[3] Furthermore, we write $\mathscr{S}(\mathbb{R}^n)$ for the space of all Schwartz-functions on $\mathbb{R}^n$. Conferring to [**43**, page 86, Theorem 2.21] for $a \in \mathscr{S}^\infty_{\varrho,\delta}$ we have $a(x, i\frac{\partial}{\partial x})(\mathscr{S}(\mathbb{R}^n)) \subseteq \mathscr{S}(\mathbb{R}^n)$. Throughout this section let $\frac{\partial}{\partial x_k}$ denote the usual partial derivative, ∂_k the closure of $\frac{\partial}{\partial x_k}$ defined on $\mathscr{C}^\infty_b$in $L^2(\mathbb{R}^n,\ \gamma_1)$ and d_k the of closure $\frac{\partial}{\partial x_k}$ defined on $\mathscr{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n,\ \lambda)$.

Remark 3.6.1. Let $a \in S^0_{\varrho,\delta}$ $(\varrho > 0)$. For $\varphi,\ \psi \in \mathscr{C}^\infty_c(\mathbb{R})$ such that $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$, define $B := \varphi\, a(X,D)\, \psi$. According to 3.2.4 we have

$$B = e^{\frac{\|\cdot\|^2}{2}} \varphi(\cdot) a(x, i\frac{\partial}{\partial x}) \psi(\cdot) e^{-\frac{\|\cdot\|^2}{2}}.$$

Applying [**93**, Chapter 2, Theorem 2.7] we obtain that $\varphi(\cdot)a(x, i\frac{\partial}{\partial x})\psi(\cdot)$ maps $L^2(\mathbb{R}^n,\ \lambda)$ to $\mathscr{C}^\infty(\mathbb{R}^n)$ and thus B maps $L^2(\mathbb{R}^n,\ \gamma_1)$ to $\mathscr{C}^\infty(\mathbb{R}^n)$.

Proposition 3.6.2. *Let $D_j := D_{e_j}$ be defined as in 1.3.8, and $x_j := M_{e_j}$ be defined as in 1.2.2. Then for $a \in S^0_{0,0}$ and $f \in D(M_{e_j})$ resp. $f \in D(D_j)$ we have*

$$[x_j,\ a(X,D)]f = e^{\frac{\|\cdot\|^2}{2}} b(x, i\frac{\partial}{\partial x}) e^{-\frac{\|\cdot\|^2}{2}} f = b(X,D)f,$$

$$[D_j,\ a(X,D)]f = e^{\frac{\|\cdot\|^2}{2}} c(x, i\frac{\partial}{\partial x}) e^{-\frac{\|\cdot\|^2}{2}} f = c(X,D)f,$$

where $b, c \in S^0_{0,0}$. Thus according to 3.2.4 $[x_j,\ a(X,D)]$ and $[D_j,\ a(X,D)]$ can be extended to a continuous linear operators on $L^2(H_-,\ \gamma_1)$. Moreover, we obtain $[\delta_j,\ a(X,D)]f = c_1(X,D)f$ and $[\partial_j,\ a(X,D)]f = c_2(X,D)f$, where $c_1,\ c_2 \in S^0_{0,0}$. Let Ψ^{MD} be defined as in 3.1.13. Then for $a \in S^0_{0,0}$ we have $a(X,D) \in \Psi^{MD}$.

[3]For $a \in S^m_{\varrho,\delta}$ there exist $b, c \in S^m_{\varrho,\delta}$ such that $a(x, i\frac{\partial}{\partial x}) = b_{KN}(x, i\frac{\partial}{\partial x})$ and $a_{KN}(x, i\frac{\partial}{\partial x}) = c(x, i\frac{\partial}{\partial x})$, where $a_{KN}(x, i\frac{\partial}{\partial x}), b_{KN}(x, i\frac{\partial}{\partial x})$ are the pseudodifferential operator corresponding to to the symbols a, b in Kohn-Nirenberg-form, c.f. Appendix A.1 and [**43**].

PROOF. At first let $f \in \mathscr{C}^\infty_{pol}(\mathbb{R}^n)$. Then $e^{-\frac{\|\cdot\|^2}{2}} f(x) \in \mathscr{S}(\mathbb{R}^n)$ and thus we have $a(x, i\frac{\partial}{\partial x})e^{-\frac{\|x\|^2}{2}} f(x) \in \mathscr{S}(\mathbb{R}^n)$. But this implies $e^{\frac{\|x\|^2}{2}} a(x, i\frac{\partial}{\partial x})e^{-\frac{\|x\|^2}{2}} f(x) \in \mathscr{C}^\infty_{int}(\mathbb{R}^n)$. Hence according to 1.2.5 and 3.2.4 we obtain

$$\begin{aligned}
[x_j,\ a(X,D)]f(x) &= x_j a(X,D)f(x) - a(X,D)x_j f(x) \\
&= x_j e^{\frac{\|x\|^2}{2}} a(x, i\frac{\partial}{\partial x})e^{-\frac{\|x\|^2}{2}} f(x) - e^{\frac{\|x\|^2}{2}} a(x, i\frac{\partial}{\partial x})e^{-\frac{\|x\|^2}{2}} x_j f(x) \\
&= e^{\frac{\|x\|^2}{2}} [x_j,\ a(x, i\frac{\partial}{\partial x})] e^{-\frac{\|x\|^2}{2}} f(x) \\
&= e^{\frac{\|x\|^2}{2}} b(x, i\frac{\partial}{\partial x}) e^{-\frac{\|x\|^2}{2}} f(x),
\end{aligned}$$

where $b(x, i\frac{\partial}{\partial x}) = [x_j,\ a(x, i\frac{\partial}{\partial x})]$ and $b \in S^0_{0,0}$. Moreover, there exists a $c \in S^0_{0,0}$ such that $c(x, i\frac{\partial}{\partial x}) = [\frac{\partial}{\partial x_j},\ a(x, i\frac{\partial}{\partial x})]$ and

$$\begin{aligned}
& [D_j,\ a(X,D)]f(x) \\
= \ & [\frac{\partial}{\partial x_j} - x_j,\ a(X,D)]f(x) \\
= \ & \frac{\partial}{\partial x_j} e^{\frac{\|x\|^2}{2}} a(x, i\frac{\partial}{\partial x})e^{-\frac{\|x\|^2}{2}} f(x) - e^{\frac{\|x\|^2}{2}} a(x, i\frac{\partial}{\partial x})e^{-\frac{\|x\|^2}{2}} \frac{\partial}{\partial x_j} f(x) \\
& \qquad - [x_j,\ a(X,D)]f(x) \\
= \ & x_j e^{\frac{\|x\|^2}{2}} a(x, i\frac{\partial}{\partial x})e^{-\frac{\|x\|^2}{2}} f(x) + e^{\frac{\|x\|^2}{2}} \frac{\partial}{\partial x_j} a(x, i\frac{\partial}{\partial x})e^{-\frac{\|x\|^2}{2}} f(x) \\
& \qquad - e^{\frac{\|x\|^2}{2}} a(x, i\frac{\partial}{\partial x})e^{-\frac{\|x\|^2}{2}} \frac{\partial}{\partial x_j} f(x) - [x_j,\ a(X,D)]f(x) \\
= \ & x_j e^{\frac{\|x\|^2}{2}} a(x, i\frac{\partial}{\partial x})e^{-\frac{\|x\|^2}{2}} f(x) + e^{\frac{\|x\|^2}{2}} [\frac{\partial}{\partial x_j},\ a(x, i\frac{\partial}{\partial x})] e^{-\frac{\|x\|^2}{2}} f(x) \\
& + e^{\frac{\|x\|^2}{2}} a(x, i\frac{\partial}{\partial x}) \frac{\partial}{\partial x_j} e^{-\frac{\|x\|^2}{2}} f(x) - e^{\frac{\|x\|^2}{2}} a(x, i\frac{\partial}{\partial x})e^{-\frac{\|x\|^2}{2}} \frac{\partial}{\partial x_j} f(x) \\
& \qquad - [x_j,\ a(X,D)]f(x) \\
= \ & x_j e^{\frac{\|x\|^2}{2}} a(x, i\frac{\partial}{\partial x})e^{-\frac{\|x\|^2}{2}} f(x) + e^{\frac{\|x\|^2}{2}} [\frac{\partial}{\partial x_j},\ a(x, i\frac{\partial}{\partial x})] e^{-\frac{\|x\|^2}{2}} f(x) \\
& \qquad - e^{\frac{\|x\|^2}{2}} a(x, i\frac{\partial}{\partial x}) x_j e^{-\frac{\|x\|^2}{2}} f(x) - [x_j,\ a(X,D)]f(x) \\
= \ & e^{\frac{\|x\|^2}{2}} [\frac{\partial}{\partial x_j},\ a(x, i\frac{\partial}{\partial x})] e^{-\frac{\|x\|^2}{2}} f(x) \\
= \ & e^{\frac{\|x\|^2}{2}} c(x, i\frac{\partial}{\partial x}) e^{-\frac{\|x\|^2}{2}} f(x).
\end{aligned}$$

Thus $[D_j,\ a(X,D)]$ extends to a continuous linear operator in $L^2(\mathbb{R}^n,\ \gamma_1)$, since D_j is closed and $a(X,D)$ is continuous. The two last assertions follow from $\partial_j f = D_j f + x_j f$ and $\delta_j = x_k f - D_k f$ for $f \in \mathscr{C}^\infty_{int}(\mathbb{R}^n)$. □

LEMMA 3.6.3. *For $f \in \mathscr{C}^\infty_{pol}(\mathbb{R}^n)$ and $a \in S^0_{0,0}$ and all $j \in \mathbb{N}_0$ we have*

$$\mathrm{ad}^j(\mathrm{L}_{\gamma_1})(a(X,D))f = \sum_{l=1}^m c_l b(X,D)_l \partial^{\alpha_l}\delta_k^{\beta_l} f,$$

where c_l depends on j, α_l, β_l are multi-indices depending on j with $|\alpha_l| + |\beta_l| \leq j$ and $b_l \in S^0_{0,0}$.

PROOF. We will prove the assertion by induction. Let $f \in \mathscr{C}^\infty_{pol}(\mathbb{R}^n)$ and $a \in S^0_{0,0}$. Then we obtain

$$\begin{aligned}[2\mathrm{L}_{\gamma_1},\ a(X,D)]f &= \sum_{k=1}^n [\delta_k\partial_k,\ a(X,D)]f \\ &= \sum_{k=1}^n \delta_k[\partial_k,\ a(X,D)]f + [\delta_k,\ a(X,D)]\partial_k f \\ &= \sum_{k=1}^n b(X,D)_{1,k} f + b(X,D)_{2,k}\delta_k f + b(X,D)_{3,k}\partial_k f,\end{aligned}$$

where $b_{1,k},\ b_{2,k},\ b_{3,k} \in S^0_{0,0}$. Now let our hypothesis be true for fixed $j \in \mathbb{N}$. Then there exists $\alpha_l,\ \beta_l \in \mathbb{N}^n_0$ with $|\alpha_l| + |\beta_l| \leq j$ and $b_l \in S^0_{0,0}$ such that

$$\mathrm{ad}^{j+1}(\mathrm{L}_{\gamma_1})(a(X,D))f = [\mathrm{L}_{\gamma_1},\ \sum_{l=1}^m c_l b(X,D)_l \partial^{\alpha_l}\delta^{\beta_l}]f.$$

Since the commutator is additive, we have only to consider the summands.

$$\begin{aligned}&[\mathrm{L}_{\gamma_1},\ b(X,D)_l\partial^{\alpha_l}\delta^{\beta_l}]f \\ =\ &[\mathrm{L}_{\gamma_1},\ b(X,D)_l]\partial^{\alpha_l}\delta^{\beta_l} f + b(X,D)_l[\mathrm{L}_{\gamma_1},\ \partial^{\alpha_l}]\delta^{\beta_l} f + b(X,D)_l\partial^{\alpha_l}[L_\gamma,\ \delta^{\beta_l}]f \\ =\ &[\mathrm{L}_{\gamma_1},\ b(X,D)_l]\partial^{\alpha_l}\delta^{\beta_l} f + b(X,D)_l(-|\alpha|)\partial^{\alpha_l}\delta^{\beta_l} f + b(X,D)_l\partial^{\alpha_l}\,|\beta_l|\,\delta^{\beta_l} f.\end{aligned}$$

Now using the start of our induction the assertion follows from $\partial_k\delta_k - \delta_k\partial_k = 2\mathrm{id}$. □

LEMMA 3.6.4. *For $f \in H^\infty$ and $a \in S^0_{0,0}$ and all $j \in \mathbb{N}_0$ we have*

$$\left\|\Lambda^{2k}\mathrm{ad}^j(\Lambda^2)(a(X,D))f\right\|_{L^2(\mathbb{R}^n,\ \gamma_1)} \leq a_{2k,j}\left\|\Lambda^{2k+j} f\right\|_{L^2(\mathbb{R}^n,\ \gamma_1)},$$

where $a_{2k,j} \geq 0$. Furthermore, $a(X,D) : H^{2k} \longrightarrow H^{2k}$ is continuous.

PROOF. At first let $f \in \mathscr{C}^\infty_{pol}(H_-)$. Then we obtain

$$\begin{aligned}
&\left\|(\Lambda^2)^k \mathrm{ad}^j(\Lambda^2)(a(X,D))f\right\|_{L^2(\mathbb{R}^n,\,\gamma_1)}\\
=\ &\left\|\sum_{\nu=0}^{k}\binom{k}{\nu}\mathrm{ad}^\nu(\Lambda^2)(\mathrm{ad}^j(\Lambda^2)(a(X,D)))(\Lambda^2)^{k-\nu}f\right\|_{L^2(\mathbb{R}^n,\,\gamma_1)}\\
=\ &\left\|\sum_{\nu=0}^{k}\binom{k}{\nu}(\mathrm{ad}^{\nu+j}(\Lambda^2)(a(X,D)))(\Lambda^2)^{k-\nu}f\right\|_{L^2(\mathbb{R}^n,\,\gamma_1)}\\
=\ &\left\|\sum_{\nu=0}^{k}\binom{k}{\nu}\sum_{l=1}^{m_\nu}c_l b(X,D)_{l,\nu}\partial^{\alpha_l}\delta^{\beta_l}(\Lambda^2)^{k-\nu}f\right\|_{L^2(\mathbb{R}^n,\,\gamma_1)}\\
\le\ &\sum_{\nu=0}^{k}\binom{k}{\nu}\sum_{l=1}^{m_\nu}c_{l,\nu}\left\|\partial^{\alpha_l}\delta^{\beta_l}(\Lambda^2)^{k-\nu}f\right\|_{L^2(\mathbb{R}^n,\,\gamma_1)}\\
\le\ &\sum_{\nu=0}^{k}\binom{k}{\nu}\sum_{l=1}^{m_\nu}\tilde{c}_{l,\nu}\left\|(\Lambda^2)^{k-\nu}f\right\|_{H^{|\alpha_l|+|\beta_l|}}\le c\left\|\Lambda^{2k+j}f\right\|_{L^2(\mathbb{R}^n,\,\gamma_1)},
\end{aligned}$$

where $c_{l,\nu},\ \tilde{c}_{l,\nu}$ and $c \ge 0$ and α_l and β_l are multi-indices with $|\alpha_l| + |\beta_l| \le j$. The rest follows as in Lemma 3.5.9. □

THEOREM 3.6.5. *For $a \in S^0_{0,0}$ we have $a(X,D) \in \Psi^0$ and thus $a(X,D) \in \widetilde{\Psi}^0_{0,0}$.*

PROOF. As in Theorem 3.5.11 we now obtain

$$\left\|\Lambda^k \mathrm{ad}^j(\Lambda)(a(X,D))f\right\|_{H^0} \le a_{k,j}\left\|\Lambda^k f\right\|_{H^0}.$$

Thus using Lemma 3.6.2 the assertion follows as in 3.5.12. □

Our next aim is to show that for any operator $A \in \Psi^0$ there exists a symbol $a \in S^0_{0,0}$ such that $A = a(X,D)$. According to [**105**, p. 52 Prop. 5.5] and [**105**, p. 47 Theorem 4.3] we have $H^k \subseteq W^{2,k}_{loc}$, where $W^{2,k}$ denotes the usual Sobolev space in $\mathbb{R}^n$ with Lebesgue measure. Thus according to [**127**, p. 60 Corollary 7.4] we get $H^\infty \subseteq \mathscr{C}^\infty(\mathbb{R}^n)$. Let $a(X,D) \in \Psi^0$. Then by definition $a(X,D)$ leaves the space H^∞ invariant. Moreover, we define $\tilde{a}$ by

$$\tilde{a}f := e^{-\frac{\|\cdot\|^2}{2}}a(X,D)\,e^{\frac{\|\cdot\|^2}{2}}f \quad \forall f \in L^2(\mathbb{R}^n,\ \lambda).$$

LEMMA 3.6.6. *Let $a(X,D) \in \Psi^0$. Then $\tilde{a}$ is a continuous linear operator in $L^2(\mathbb{R}^n,\ \lambda)$.*

Proof. Let $f \in L^2(\mathbb{R}^n,\ \lambda)$. Then $e^{\frac{\|\cdot\|^2}{2}} f \in L^2(\mathbb{R}^n,\ \gamma_1)$ and thus we we obtain

$$\begin{aligned}\|\tilde{a} f\|_{L^2(\mathbb{R}^n,\ \lambda)} &= \pi^{\frac{n}{2}} \left\| a(X,D)\ e^{\frac{\|\cdot\|^2}{2}} f \right\|_{L^2(\mathbb{R}^n,\ \gamma_1)} \\ &\leq c\pi^{\frac{n}{2}} \left\| e^{\frac{\|\cdot\|^2}{2}} f \right\|_{L^2(\mathbb{R}^n,\ \gamma_1)} = c\, \|f\|_{L^2(\mathbb{R}^n,\ \lambda)},\end{aligned}$$

where $c > 0$ suitable. □

Lemma 3.6.7. *For $f \in \mathscr{C}^\infty_{int}(\mathbb{R}^n)$ and $a(X,D) \in \Psi^0$ we have*

$$\frac{\partial}{\partial x_k} a(X,D) f = \partial_k a(X,D) f = [\partial_k,\ a(X,D)] f + a(X,D) \frac{\partial}{\partial x_k} f.$$

Furthermore, $a(X,D)$ leaves the space $\mathscr{C}^\infty_{int}(\mathbb{R}^n)$ invariant and we have $H^\infty = \mathscr{C}^\infty_{int}(\mathbb{R}^n)$.

Proof. For $f \in H^\infty$ and $g \in \mathscr{C}^\infty_b(\mathbb{R}^n)$ it is obvious that

$$\partial_k (fg) = (\partial_k f) g + f \partial_k g.$$

Let $f \in \mathscr{C}^\infty_{int}(\mathbb{R}^n)$ and ζ_n be defined as in Lemma 1.2.5. Then we obtain $(a(X,D)f(x))\zeta_n(\|x\|^2) \xrightarrow[n\to\infty]{} a(X,D)f(x)$ pointwisely. Moreover, we have $\frac{\partial}{\partial x_k}((a(X,D)f(x))\zeta_n(\|x\|^2)) \xrightarrow[n\to\infty]{} \frac{\partial}{\partial x_k} a(X,D)f(x)$ pointwisely. By Lebesgue's theorem of dominated convergence we get $(a(X,D)f(x))\zeta_n(\|x\|^2) \xrightarrow[n\to\infty]{L^2(\mathbb{R}^n,\ \gamma_1)} a(X,D)f(x)$. However, we have

$$\begin{aligned}&\frac{\partial}{\partial x_k}((a(X,D)f(x))\zeta_n(\|x\|^2)) \\ =\ & \partial_k((a(X,D)f(x))\zeta_n(\|x\|^2)) \\ =\ & (\partial_k a(X,D)f(x))\zeta_n(\|x\|^2) + 2x_k(a(X,D)f(x))\zeta_n'(\|x\|^2).\end{aligned}$$

Since $a(X,D)f \in H^\infty$, we have $\partial_k a(X,D)f,\ x_k a(X,D)f \in L^2(\mathbb{R}^n,\ \gamma_1)$ and thus we get by Lebesgue's theorem of dominate convergence $\frac{\partial}{\partial x_k} a(X,D)f = \partial_k a(X,D)f$. The rest of this lemma follows by an easy induction. □

Lemma 3.6.8. *Let $f \in H^\infty$. Then we have $\frac{\partial}{\partial x_k}(e^{-\frac{\|\cdot\|^2}{2}} f) = d_k(e^{-\frac{\|\cdot\|^2}{2}} f)$.*

Proof. Let $f \in H^\infty$ and ζ_n be defined as in 1.2.5. Then we obtain

$$e^{-\frac{\|x\|^2}{2}} f(x)\zeta_n(\|x\|^2) \xrightarrow[n\to\infty]{} e^{-\frac{\|x\|^2}{2}} f(x)$$

and

$$\frac{\partial}{\partial x_k}(e^{-\frac{\|x\|^2}{2}} f(x))\zeta_n(\|x\|^2) \xrightarrow[n\to\infty]{} \frac{\partial}{\partial x_k} e^{-\frac{\|x\|^2}{2}} f(x)$$

pointwisely. Moreover, using Lebesgue's theorem of dominated convergence the convergence above is in $L^2(\mathbb{R}^n,\ \lambda)$. Since $e^{-\frac{\|x\|^2}{2}} f(x)\zeta_n(\|x\|^2)$ has compact support, it is an element of $\mathscr{S}(\mathbb{R}^n)$. But this is our assertion, since d_k is closed and coincides with $\frac{\partial}{\partial x_k}$ on $\mathscr{S}(\mathbb{R}^n)$. $\square$

LEMMA 3.6.9. *Then for $a(X,D) \in \Psi^0$ and $f \in D(x_j)$ resp. $f \in D(d_j)$ we have*

$$[x_j,\ \tilde{a}]f = e^{-\frac{\|\cdot\|^2}{2}} b(X,D)\ e^{\frac{\|\cdot\|^2}{2}} f = \tilde{b}f,$$

$$[d_j,\ \tilde{a}]f = e^{-\frac{\|\cdot\|^2}{2}} c(X,D)\ e^{\frac{\|\cdot\|^2}{2}} f = \tilde{c}f,$$

where $b(X,D), c(X,D) \in \Psi^0$. Thus according to Lemma 3.6.6 $[d_j,\ \tilde{a}]$ and $[x_j,\ \tilde{a}]$ can be extended to a continuous linear operator on $L^2(\mathbb{R}^n,\ \lambda)$.

PROOF. Let $f \in \mathscr{S}(\mathbb{R}^n)$. Then we have $e^{\frac{\|\cdot\|^2}{2}} f \in H^\infty$. Hence we obtain

$$[x_j,\ \tilde{a}]f = e^{-\frac{\|\cdot\|^2}{2}} [x_j,\ a(X,D)] e^{\frac{\|\cdot\|^2}{2}} f = \tilde{c}f$$

and

$$\begin{aligned}
&[\frac{\partial}{\partial x_j},\ \tilde{a}]f(x)\\
= \quad & -\frac{\|x\|^2}{2}(-x_j a(X,D)\ + [\partial_j,\ a(X,D)] + a(X,D)x_j)e^{\frac{\|x\|^2}{2}} f(x)\\
= \quad & e^{-\frac{\|x\|^2}{2}}([\partial_j,\ a(X,D)] - [x_j,\ a(X,D)])e^{\frac{\|x\|^2}{2}} f(x).
\end{aligned}$$

By definition of Ψ^0 we have $[x_j,\ a(X,D)]$, $[\partial_j,\ a(X,D)] \in \Psi^0$ and thus according to Lemma 3.6.6 $[d_j,\ \tilde{a}]$ and $[x_j,\ \tilde{a}]$ can be extended to a continuous linear operator. The assertion for $f \in D(x_j)$ resp. $D(d_j)$ is now obvious, since x_j and d_j are closed. $\square$

PROPOSITION 3.6.10. *Let Δ be the Laplace operator in $\mathbb{R}^n$, $a(X,D) \in \Psi^0$ and $j \in \mathbb{N}$. Then for all $f \in \mathscr{S}(\mathbb{R}^n)$ and $j \in \mathbb{N}$ we have*

$$\mathrm{ad}^j(\Delta)(\tilde{a})f = \sum_{l=1}^{m} c_l \tilde{b}_l d^{\alpha_l} f,$$

where α_l are multi-indices with $|\alpha_l| \leq j$ and $c_l \in \mathbb{Z}$ and $b(X,D) \in \Psi^0$. Furthermore, for $\Lambda_\Delta := (\mathrm{id} - \Delta)^{\frac{1}{2}}$ and all $j,k \in \mathbb{N}_0$ and $f \in H^\infty_{\Lambda_\Delta}$ we have

$$\left\|\Lambda_\Delta^{2k} \mathrm{ad}^j(\Lambda_\Delta^2)(\tilde{a})f\right\|_{L^2(\mathbb{R}^n,\ \lambda)} \leq a_{2k_j} \left\|\Lambda_\Delta^{2k+j} f\right\|_{L^2(\mathbb{R}^n,\ \lambda)}.$$

PROOF. (i) We will prove the first assertion by induction. For $f \in \mathscr{S}(\mathbb{R}^n)$ we get

$$[\Delta,\ \tilde{a}]f = \sum_{k=1}^{n} (d_k[d_k,\ \tilde{a}]f + [d_k,\ \tilde{a}]d_k f) = \sum_{k=1}^{n} \left(\tilde{b_k} f + \tilde{c_k} d_k f\right), \tag{33}$$

where $b(X,D)_k,\ c_k(X,D) \in \Psi^0$. Let our hypothesis be true for fixed $j \in \mathbb{N}$. Then we have $\mathrm{ad}^j(\Delta)(\tilde{a})f = \sum_{l=1}^m c_l \tilde{b}_l d^{\alpha_l} f$, where α_l are multi-indices with $|\alpha_l| \le j$, $b(X,D)_l \in \Psi^0$ and $c_l \in \mathbb{Z}$. Thus we obtain

$$\mathrm{ad}^{j+1}(\Delta)\tilde{a}f = [\Delta,\ \sum_{l=1}^m c_l \tilde{b}_l d^{\alpha_l}]f = \sum_{l=1}^m c_l[\Delta,\ \tilde{b}_l]d^{\alpha_l}f.$$

Using (33) this is our first assertion.

(ii) For $f \in \mathscr{S}(\mathbb{R}^n)$ there exist $c_{l,\nu}$, $c \ge 0$, $\alpha_l \in \mathbb{N}_0^n$ with $|\alpha_l| \le j$ and $b(X,D)_{l,\nu} \in \Psi^0$ such that

$$\begin{aligned}
&\left\|(\Lambda_\Delta^2)^k \mathrm{ad}^j(\Lambda_\Delta^2)(\tilde{a})f\right\|_{L^2(\mathbb{R}^n,\,\lambda)} \\
= &\left\|\sum_{\nu=0}^k \binom{k}{\nu} \mathrm{ad}^\nu(\Lambda_\Delta^2)(\mathrm{ad}^j(\Lambda_\Delta^2)(\tilde{a}))(\Lambda_\Delta^2)^{k-\nu}f\right\|_{L^2(\mathbb{R}^n,\,\lambda)} \\
= &\left\|\sum_{\nu=0}^k \binom{k}{\nu} (\mathrm{ad}^{\nu+j}(\Lambda_\Delta^2)(\tilde{a}))({\Lambda_\Delta}^2)^{k-\nu}f\right\|_{L^2(\mathbb{R}^n,\,\lambda)} \\
= &\left\|\sum_{\nu=0}^k \binom{k}{\nu} \sum_{l=1}^{m_\nu} c_l \tilde{b}_{l,\nu} d^{\alpha_l}({\Lambda_\Delta}^2)^{k-\nu}f\right\|_{L^2(\mathbb{R}^n,\,\lambda)} \\
\le &\sum_{\nu=0}^k \binom{k}{\nu} \sum_{l=1}^{m_\nu} c_{l,\nu} \left\|d^{\alpha_l}({\Lambda_\Delta}^2)^{k-\nu}f\right\|_{L^2(\mathbb{R}^n,\,\lambda)} \\
\le &\sum_{\nu=0}^k \binom{k}{\nu} \sum_{l=1}^{m_\nu} \tilde{c}_{l,\nu} \left\|({\Lambda_\Delta}^2)^{k-\nu}f\right\|_{H^{|\alpha_l|}_{\Lambda_\Delta}} \le c\left\|\Lambda_\Delta^{2k+j}f\right\|_{L^2(\mathbb{R}^n,\,\lambda)}.
\end{aligned}$$

The rest follows as in Lemma 3.5.9. □

THEOREM 3.6.11. *For $a(X,D) \in \Psi^0$ we have $\tilde{a} \in \Psi^0_{0,0}$, where $\Psi^0_{0,0} = \{a(x, i\frac{\partial}{\partial x}) \mid a \in S^0_{0,0}\}$.*

PROOF. As in Theorem 3.5.11 we now obtain

$$\left\|\Lambda_\Delta^k \mathrm{ad}^j(\Lambda_\Delta^2)(a(X,D))f\right\|_{L^2(\mathbb{R}^n,\,\lambda)} \le a_{k,j}\left\|\Lambda_\Delta^{k+j}f\right\|_{L^2(\mathbb{R}^n,\,\lambda)}.$$

Thus using Lemma 3.6.6 and 3.6.7 the theorem follows as in 3.5.12 by Beals' Theorem. □

COROLLARY 3.6.12. *In the finite dimensional case we have*

$$\Psi^0 = \{a(X,D) \mid a \in S^0_{0,0}\}.$$

Moreover, any $a(X,D) \in \Psi^0$ is given by $a(X,D) = e^{\frac{\|\cdot\|^2}{2}} a(x, i\frac{\partial}{\partial x_k})e^{-\frac{\|\cdot\|^2}{2}}$, where $a(x, i\frac{\partial}{\partial x_k})$ is the usual pseudodifferential operator in Weyl-form.

CHAPTER 4

A symbolic calculus for pseudodifferential operators in Kohn-Nirenberg form and applications to Ψ^*- Algebras

In this chapter we will deal with pseudodifferential operators on a quasi-nuclear Hilbert space rigging $H_+ \subset H_0 \subset H_-$ and on $\mathbb{R}^n$ given in Kohn-Nirenberg-form. Using these more general pseudodifferential operators in the classical finite dimensional theory and the case of the Lebesgue measure, it is shown in [**81**] that these operators are still continuous operators in a scale of Sobolev-Spaces. Furthermore, for there operators there still exists some kind of symbolic calculus and some kind of Gårding inequality. In addition we will show, that the description of the Hörmander classes by commutators is still true in the finite dimensional case if we replace the Lebesgue measure by the canonical Gaussian measure and the Fourier transform by the Fourier-Wiener transform.

We define classes of symbols similar to [**79**, Definition 2.4.4] and the classical case. For these symbols the corresponding pseudodifferential operator $q(x, D)$ is defined by

$$q(x, D) := \mathcal{F}^{-1}_{\xi\to x}[q(x, \xi)(\mathcal{F}u)(\xi)],$$

where $\mathcal{F}$ denotes the Fourier Wiener-Transform. We write $\Psi^{m,\psi}_{\varrho_k}(H_-)$ resp. $\Psi^{m,\psi}_{\varrho,\delta}(H_-)$ for the corresponding classes of pseudodifferential operators. We show that some well known results remain valid when dealing with the canonical Gaussian measure on an infinite dimensional Hilbert space rigging, e.g. we prove that in the case of cylindrical symbols or symbols depending only on ξ for the corresponding pseudodifferential operators there still exists some kind of symbolic calculus. Moreover, all these operators map $H^{s+m}_\psi(H_-)$ continuously to $H^s_\psi(H_-)$, where $H^s_\psi(H_-)$ is a scale of Sobolev-spaces. In addition, for $q \in S^{m,\psi}_{\varrho_k,cyl}(H_-)$ we have some kind of Gårding inequality.

Concerning some special negative-definite functions we show that each operator $q(x, D) \in \Psi^{m,\psi}_{\varrho,\delta}(H_-)$ being cylindrical or depending only on ξ is contained in a generalized Hörmander-class, constructed as in [**67**].

In the finite dimensional case, using a work of Schrohe (cf. [**122**]) we show under some minimal growth assumption on our negative definite function that every uniformly elliptic symbol $q \in \tilde{S}^{0,\psi}_{\varrho,\delta}$ defines a Fredholm operator $q(x, D)$ in $\mathscr{L}(H^s_\psi(\mathbb{R}^n))$, where $H^s_\psi(\mathbb{R}^n)$ stands for the Sobolev-space of order s, with respect to the negative definite function ψ and $\tilde{S}^{0,\psi}_{\varrho,\delta} \subset S^{0,\psi}_{\varrho,\delta}$. In addition we obtain that if $q \in \tilde{S}^{-\varepsilon,\psi}_{\varrho,\delta}$ $q(x,D)$ is compact in $L^2(\mathbb{R}^n,\gamma)$ and give a description of the finite dimensional operators.

4.1. Definition of symbols of pseudodifferential operators and generalized Hörmander classes

In this section we define classes of symbols with respect to a fixed negative definite function. In addition, using the Fourier-Wiener transform we define the corresponding classes of pseudodifferential operators. These pseudodifferential operators with negative definite symbols arise naturally as generators of translation invariant Feller semi groups and Dirichlet-forms. In both cases we can associate a stochastic process to these operators.

Definition 4.1.1. Let $k \in \mathbb{N} \cup \{\infty\}$ such that $k \geq 2$. We define the sub additive function $\varrho_k : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ by

$$l \longmapsto l \wedge k$$

Lemma 4.1.2. *Let ψ by a continuous negative definite function in Levi-Khinchin-form on $\mathbb{R}^n$ which satisfies* (12) *for all $k \in \mathbb{N}$. Then for all $m \in \mathbb{R}$ and all $\alpha \in \mathbb{N}_0^{\mathbb{N}}$ we have*

$$\left|\partial_\xi^\alpha (1+\psi(\xi))^{m/2}\right| \leq c_{|\alpha|}(1+\psi(\xi))^{\frac{m-\varrho_2(|\alpha|)}{2}} \tag{34}$$

Proof. See [**81**, Lemma 2.4.4]. □

Definition 4.1.3. (i) A real-valued negative definite $\mathscr{C}^\infty$-function $\psi : H_- \longrightarrow \mathbb{R}$ belongs to the class $\Lambda_k(H_-)$ if it satisfies

$$\left|\partial_\xi^\alpha (1+\psi(\xi))^{m/2}\right| \leq c_{|\alpha|}(1+\psi(\xi))^{\frac{m-\varrho_k(|\alpha|)}{2}}$$

for all $\alpha \in \mathbb{N}_0^{\mathbb{N}}$.

(ii) Let $\psi \in \Lambda_k$ and $m \in \mathbb{R}$. We call a $\mathscr{C}^\infty$-function $q : H_- \times H_- \longrightarrow \mathbb{C}$ a symbol in the class $S^{m,\psi}_{\varrho_k}(H_-)$ if for all $\alpha,\ \beta \in \mathbb{N}_0^{\mathbb{N}}$ there exists constants $c_{|\alpha|,|\beta|} \geq 0$ such that

$$\left|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)\right| \leq c_{|\alpha|,|\beta|}(1+\psi(\xi))^{\frac{m-\varrho_k(|\alpha|)}{2}}$$

for all $x \in H_-$ and all $\xi \in H_-$. We call m the order of the symbol $q(x,\xi)$.

(iii) Let $\psi \in \Lambda_k(H_-)$ and $m \in \mathbb{R}$. We call a $\mathscr{C}^\infty$-function $q : H_- \times H_- \longrightarrow \mathbb{C}$ a symbol in the class $S^{m,\psi}_0(H_-)$ if for all $\alpha,\ \beta \in \mathbb{N}_0^{\mathbb{N}}$ there exists constants $\tilde{c}_{|\alpha|,|\beta|} \geq 0$ such that

$$\left|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)\right| \leq \tilde{c}_{|\alpha|,|\beta|}(1+\psi(\xi))^{\frac{m}{2}}$$

for all $x \in H_-$ and all $\xi \in H_-$.

(iv) Let $0 \le \delta \le \varrho \le 1$, $\delta < 1$. For $\psi \in \Lambda_\infty(H_-)$ and $m \in \mathbb{R}$ we call a $\mathscr{C}^\infty$-function $q : H_- \times H_- \longrightarrow \mathbb{C}$ a symbol in the class $S^{m,\psi}_{\varrho,\delta}(H_-)$ if for all α, $\beta \in \mathbb{N}_0^{\mathbb{N}}$ there exists constants $c'_{|\alpha|,|\beta|} \ge 0$ such that

$$\left|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)\right| \le c'_{|\alpha|,|\beta|}(1+\psi(\xi))^{\frac{m-\varrho|\alpha|+\delta|\beta|}{2}}.$$

Moreover, we set $S^{-\infty,\psi}_{\varrho,\delta}(H_-) := \bigcap_{m\in\mathbb{R}} S^{m,\psi}_{\varrho,\delta}(H_-)$.

(v) We denote by $S^{m,\psi}_{\varrho_k,cyl}(H_-)$, $S^{m,\psi}_{0,cyl}(H_-)$ and $S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$ the set of all cylindrical symbols q in $S^{m,\psi}_{\varrho_k}(H_-)$, $S^{m,\psi}_{0}(H_-)$ resp. $S^{m,\psi}_{\varrho,\delta}(H_-)$.

Lemma 4.1.4. *Let $\psi \in \Lambda_k(H_-)$.*

(i) *The sets $S^{m,\psi}_{\varrho_k}(H_-)$ and $S^{m,\psi}_{0}(H_-)$ are vector spaces.*
(ii) *For $q_1 \in S^{m_1,\psi}_{\varrho_k}(H_-)$ and $q_2 \in S^{m,\psi}_{\varrho_k}(H_-)$ we have $q_1q_2 \in S^{m_1+m_2,\psi}_{\varrho_k}(H_-)$.*
(iii) *For $q_1 \in S^{m_1,\psi}_{\varrho,\delta}(H_-)$ and $q_2 \in S^{m,\psi}_{\varrho,\delta}(H_-)$ we have $q_1q_2 \in S^{m_1+m_2,\psi}_{\varrho,\delta}(H_-)$.*

Proof. The proof of (i) and (ii) are similar to [**81**, Lemma 2.4.9]. Thus let us prove (iii). Applying the Leibniz rule we obtain

$$\begin{aligned}
&\left|\partial_\xi^\alpha \partial_x^\beta (q_1q_2)(x,\xi)\right| \\
\le &\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} \left|\partial_\xi^{\alpha'} \partial_x^{\beta'} q_1(x,\xi)\right| \left|\partial_\xi^{\alpha''} \partial_x^{\beta''} q_2(x,\xi)\right| \\
\le &\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} c_{|\alpha'|,|\beta'|}(1+\psi(\xi))^{\frac{m_1-\varrho|\alpha'|+\delta|\beta'|}{2}} c_{|\alpha''|,|\beta''|}(1+\psi(\xi))^{\frac{m_2-\varrho|\alpha''|+\delta|\beta''|}{2}} \\
\le &\ \tilde{c}(1+\psi(\xi))^{\frac{m_1+m_2-\varrho|\alpha|+\delta|\beta|}{2}},
\end{aligned}$$

where $\tilde{c}$ depends only on $|\alpha|$ and $|\beta|$. □

Considering the general situation of Beals and Fefferman [**15**], Baldus showed in [**5**, Example 7.7.9] that for every continuous negative definite function ψ $\lambda := \sqrt{1+\psi}$ is an admissible weight with respect to the euclidean metric g_{eucl} as Hörmander metric. Thus we have $S^{m,\psi}_0(\mathbb{R}^n) = S(\lambda^m, g_{eucl})$. Moreover Baldus showed in [**5**, Example 7.7.9] that we have $S^{m,\psi}_{\varrho_k}(\mathbb{R}^n) \subset S_a(\lambda^m, \underline{g}, g_{eucl})$, where $\underline{g} := \sum_{j=1}^n (\|dx_j\|^2 + \frac{\|d\xi_j\|^2}{\lambda(\xi)^2}$ and $S_a(\lambda^m, \underline{g}, g_{eucl})$ is defined as in [**5**, Definition 1.3.15].

At next we define pseudodifferential operators.

Definition 4.1.5. Let ψ be in $\Lambda_k(H_-)$. For $q \in S^{m,\psi}_{\varrho_k}(H_-)$ or $q \in S^{m,\psi}_0(H_-)$ we define the pseudodifferential operator $q(x,D)$ on $S_{\gamma,cyl}(H_-)$ by

$$q(x,D)u(x) := \mathcal{F}^{-1}_{\xi\to x}[q(x,\xi)(\mathcal{F}u)(\xi)].$$

The sign '$\xi \to x$' means that the corresponding operator is applied to a function of ξ and the result is considered as a function of x. The classes of these operators are denote by $\Psi^{m,\psi}_{\varrho_k}(H_-)$ resp. $\Psi^{m,\psi}_0(H_-)$. For $\psi \in \Lambda_\infty(H_-)$ and $q \in S^{m,\psi}_{\varrho,\delta}(H_-)$

we denote the corresponding class of pseudodifferential operators by $\Psi^{m,\psi}_{\varrho,\delta}(H_-)$. In addition, let us denote by $\Psi^{m,\psi}_{\varrho_k,cyl}(H_-)$, $\Psi^{m,\psi}_{0,cyl}(H_-)$ and $\Psi^{m,\psi}_{\varrho,\delta,cyl}(H_-)$ the set of all operators corresponding to symbols in $S^{m,\psi}_{\varrho_k,cyl}(H_-)$, $S^{m,\psi}_{0,cyl}(H_-)$ resp. $S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$.

Remark 4.1.6. Here we have defined pseudodifferential operators in Kohn-Nirenberg form. In the classical case both are considered in many publications. For example in the classical Weyl calculus one has $a(X,\tilde{D})^* = \overline{a}(X,D)$. On the other hand in the case of the Kohn-Nirenberg form the symbol of the product and the commutator of two operators is much easy to calculate then in Weyl form. Moreover, having a symbol of the form $a(x,\xi) = \sum_{|\alpha|\leq n} a_\alpha(x)\xi^\alpha$ the Kohn-Nirenberg quantization leads to a differential operator given by $a(x,\tilde{D} = \sum_{|\alpha|\leq n} a_\alpha(x)(i\partial)^\alpha$. More about the connection between pseudodifferential operators in Weyl and in Kohn-Nierenberrg form can be found in Appendix A2.

Lemma 4.1.7. *For $q \in S^{m,\psi}_{0,cyl}(H_-)$ resp. $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$ or $q \in S^{m,\psi}_0(H_-)$ resp. $q \in S^{m,\psi}_{\varrho,\delta}(H_-)$ and $q(x,\xi) = p(\xi)$ we obtain that the operator $q(x,D)$ is well defined on $S_{\gamma,cyl}(H_-)$.*

Proof. Let us first note that the Fourier-Wiener-transform leaves the space $S_{\gamma,cyl}(H_-)$ invariant. Thus we find for fixed x $\mathcal{F}^{-1}[q(x,\xi)(\mathcal{F}u)(\xi)] \in S_{\gamma,cyl}(H_-)$. Hence, in the first case, $q(x,D)u(x)$ is well defined. In the second case the pseudodifferential operator is well defined since q is independent of x. □

Remark 4.1.8. For $\psi \in \Lambda_k(H_-)$ resp. $\psi \in \Lambda_\infty(H_-)$ we have

(i) $S^{m,\psi}_{\varrho_k}(H_-) \subset S^{m,\psi}_0(H_-)$ and thus $\Psi^{m,\psi}_{\varrho_k}(H_-) \subset \Psi^{m,\psi}_0(H_-)$,

(ii) $S^{m,\psi}_{\varrho',\delta}(H_-) \subset S^{m,\psi}_{\varrho,\delta}(H_-)$ and thus $\Psi^{m,\psi}_{\varrho',\delta}(H_-) \subset \Psi^{m,\psi}_{\varrho,\delta}(H_-)$ if $\varrho \leq \varrho'$,

(iii) $S^{m,\psi}_{\varrho,\delta'}(H_-) \subset S^{m,\psi}_{\varrho,\delta}(H_-)$ and thus $\Psi^{m,\psi}_{\varrho,\delta'}(H_-) \subset \Psi^{m,\psi}_{\varrho,\delta}(H_-)$ if $\delta' \leq \delta$,

(iv) $S^{m,\psi}_{1,0}(H_-) = S^{m,\psi}_{\varrho_\infty}(H_-)$.

Definition 4.1.9. Let ψ be in $\Lambda_k(\mathbb{R}^n)$. For $q \in S^{m,\psi}_{\varrho_k}(\mathbb{R}^n)$ or $q \in S^{m,\psi}_0(\mathbb{R}^n)$ we denote by $q(x,\widetilde{D})$ the pseudodifferential operator defined on $S(\mathbb{R}^n)$ by

$$q(x,\tilde{D})u(x) := \widetilde{\mathcal{F}}^{-1}_{\xi\to x}[q(x,\xi)(\widetilde{\mathcal{F}}u)(\xi)],$$

where $\widetilde{\mathcal{F}}$ denotes the Fourier-transform.

Definition 4.1.10. Let ψ be in $\Lambda_k(\mathbb{R}^n)$ and $m, m' \in \mathbb{R}$. We call a $\mathscr{C}^\infty$-function $q : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ a double-symbol in the class $S^{m,m',\psi}_0(\mathbb{R}^n)$ if for all $\alpha,\ \beta,\ \alpha',\ \beta' \in \mathbb{N}_0^{\mathbb{N}}$ there exist constants $c_{\alpha\beta\alpha'\beta'} \geq 0$ such that

$$\left|\partial^\alpha_\xi \partial^\beta_x \partial^{\alpha'}_{\xi'} \partial^{\beta'}_{x'} q(x,\xi;x',\xi')\right| \leq c_{\alpha\beta\alpha'\beta'}(1+\psi(\xi))^{\frac{m}{2}}(1+\psi(\xi'))^{\frac{m'}{2}}.$$

For $q \in S^{m,m',\psi}_0$ we define on S_γ the operator

$$q(x,D_x;x',D_{x'})u(x) := \mathcal{F}^{-1}_{\xi\to x}\mathcal{F}_{x'\to\xi}\mathcal{F}^{-1}_{\xi'\to x'}[q(x,\xi;x',\xi')(\mathcal{F}u)(\xi)]. \tag{35}$$

Moreover we denote by $q(x, \tilde{D}_x; x', \tilde{D}_{x'})$ the usual pseudodifferential operator on $S(\mathbb{R}^n)$ with double symbol q.

Now let $\psi \in \Lambda_\infty(\mathbb{R}^n)$ be a fixed negative definite function. In addition let $0 \leq \delta \leq \varrho \leq 1$. Let $\delta < \varrho$ and set $\varepsilon := 1 - \delta$. Moreover we set

$$\Lambda := (1 + \psi(D))^{1/2}. \tag{36}$$

Definition 4.1.11. We define

$$\begin{aligned} \mathcal{A}^{\psi,\varepsilon} \;=\; \{A \in \mathscr{L}(H^0_\psi(H_-)) \,|\, A(H^\infty_\psi(H_-)) \subseteq H^\infty_\psi(H_-) \text{ and } \\ \left\| \mathrm{ad}^j(\Lambda^\varepsilon)(a) f \right\|_{H^0_\psi} \leq c_j \left\| f \right\|_{H^0_\psi} \\ \forall f \in H^\infty_\psi(H_-)\ \forall j \in \mathbb{N}_0, \text{ and suitable } c_j \geq 0\}. \end{aligned}$$

Since Λ^ε is selfadjoint, $\mathcal{A}^{\psi,\varepsilon}$ is a Ψ^*-algebra. Moreover, according to [**25**, Theorem 2.3.11], we have $\mathcal{A}^{\psi,\varepsilon'} \subseteq \mathcal{A}^{\psi,\varepsilon}$ for $0 < \varepsilon \leq \varepsilon' \leq 1$.

Definition 4.1.12. Let $\alpha,\ \beta \in \mathbb{N}_0^{\mathbb{N}}$. Moreover, let $\mathrm{ad}^\alpha(M)$ and $\mathrm{ad}^\beta(D)$ be defined as in 3.2.18. We set $\varepsilon := 1-\delta$ and define the generalized Hörmander-class $\mathcal{A}^{\psi,m}_{\varrho,\delta}(H_-)$ by

$$\begin{aligned} \widetilde{\mathcal{A}}^{\psi,m}_{\varrho,\delta}(H_-) := \{A \in \Lambda^m \mathcal{A}^\varepsilon \,| A, A * (S_{\gamma,cyl}) \subseteq S_{\gamma,cyl}, \mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(A) \\ \in \bigcap_{s \in \mathbb{R}} \mathscr{L}(H^s_\psi(H_-), H_\psi^{s-m+\varrho|\alpha|-\delta|\beta|}(H_-)), \\ \forall\ \alpha, \beta \in \mathbb{N}_0^n\}. \end{aligned}$$

Furthermore, let $\|\cdot\|_{\mathcal{A}^{1-\delta},l}$ be a fundamental system of sub multiplicative semi norms on $\mathcal{A}^{1-\delta}$. Then for $A \in \widetilde{\mathcal{A}}^{\psi,0}_{\varrho,\delta}(H_-)$ we define a system of semi norm by

$$\|A\|_{k,0,0} := \|\cdot\|_{\mathcal{A}^{1-\delta},k}$$

and

$$\|A\|_{s,l,l'} := \sup_{\substack{|\alpha| \leq l \\ |\beta| \leq l'}} \left\| \mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(A) \right\|_{\mathscr{L}(H^s_\psi(H_-), H_\psi^{s+\varrho|\alpha|-\delta|\beta|}(H_-))},$$

where $k, l, l' \in \mathbb{N}$, $s \in \mathbb{R}$, $\alpha,\ \beta \in \mathbb{N}_0^n$. Finally, let $\mathcal{A}^{\psi,m}_{\varrho,\delta}(H_-)$ be the closure of $\widetilde{\mathcal{A}}^{\psi,m}_{\varrho,\delta}(H_-)$ with respect to the system of semi norms defined above.

Theorem 4.1.13. *$\mathcal{A}^{\psi,0}_{\varrho,\delta}(H_-)$ is a sub multiplicative Ψ^*-algebra in $\mathscr{L}(H^0)$. Furthermore, $\mathcal{A}^{\psi,0}_{\varrho,\delta}(H_-) \times H^\infty_\psi(H_-) \longrightarrow H^\infty_\psi(H_-)\ :\ (A, \varphi) \longmapsto A(\varphi)$ is continuous and bilinear.*

Proof. First let us note the following facts:

(i) $\mathcal{A}^{\psi,0}_{\varrho,\delta}(H_-) \subset \mathscr{L}(H^s_\psi(H_-), H^s_\psi(H_-))$ and we have $\mathrm{id} \in \mathcal{A}^{\psi,0}_{\varrho,\delta}(H_-)$.
(ii) $\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)\ :\ \mathcal{A}^{\psi,0}_{\varrho,\delta}(H_-) \longrightarrow \mathscr{L}(H^s, H^{s+\varrho|\alpha|-\delta|\beta|}) \forall s \in \mathbb{R}$ and
(iii) the Leibniz-rule is true for $\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)$.

Thus we obtain from [**67**, Lemma 3.9] that $\mathcal{A}^{\psi,0}_{\varrho,\delta}(H_-)$ is spectraly invariant in $\mathscr{L}(H^s_\psi(H_-), H^s_\psi(H_))$ for all $s \in \mathbb{R}$. □

4.2. An asymptotic expansion and estimates for pseudodifferential operators on $\mathbb{R}^n$ in Kohn-Nirenberg form

In this section we show some symbolic calculus for our pseudodifferential operator in the finite dimensional case. Furthermore, we use this calculus to show that some of our classes of pseudodifferential operators are algebras.

PROPOSITION 4.2.1. *Let ψ be in $\Lambda_k(\mathbb{R}^n)$. For $q \in S^{m,\psi}_{\varrho_k}(\mathbb{R}^n)$ or $q \in S^{m,\psi}_0(\mathbb{R}^n)$ we have*

$$q(x, D)u = V^{-1}_{G,n} q(x, \widetilde{D})(V_{G,n}u) \tag{37}$$

for all $u \in S_\gamma(\mathbb{R}^n)$.

PROOF. Let $u \in S_\gamma(\mathbb{R}^n)$. Then we have $V_{G,n}u \in S(\mathbb{R}^n)$ and obtain

$$\begin{aligned} q(x, D)u(x) &= \mathcal{F}^{-1}_{\xi\to x}[q(x, \xi)(\mathcal{F}u)(\xi)] \\ &= (V^{-1}_{G,n}\tilde{F}^{-1}V_{G,n})_{\xi\to x}[q(x, \xi)(V^{-1}_{G,n}\widetilde{\mathcal{F}}V_{G,n}u)(\xi)] \\ &= V^{-1}_{G,n}\widetilde{\mathcal{F}}^{-1}_{\xi\to x}[\tilde{q}(x, \xi)\widetilde{\mathcal{F}}V_{G,n}u(\xi)] \\ &= V^{-1}_{G,n}\tilde{q}(x, \widetilde{D})(V_{G,n}u)(x). \end{aligned}$$

But this is our proposition. □

THEOREM 4.2.2. *Let ψ be in $\Lambda_k(\mathbb{R}^n)$ and $m, m' \in \mathbb{R}$. For $q \in S^{m,m',\psi}_0(\mathbb{R}^n)$ and $u \in S_\gamma(\mathbb{R}^n)$ the operator in (35) is well defined and we have*

$$q(x, D_x; x', D_{x'})u(x) = V^{-1}_{G,n} q(x, \widetilde{D}_x; x', \widetilde{D}_{x'})(V_{G,n}u)(x). \tag{38}$$

PROOF. For $u \in S_\gamma(R^n)$ we get $V_{G,n}u \in S(\mathbb{R}^n)$. Thus we only have to show the equation above. However, this equation follows by

$$\begin{aligned} & V^{-1}_{G,n} q(x, \widetilde{D}_x; x', \widetilde{D}_{x'})(V_{G,n}u)(x) \\ = \ & V^{-1}_{G,n}[\widetilde{\mathcal{F}}^{-1}_{\xi\to x}\widetilde{\mathcal{F}}_{x'\to\xi'}\widetilde{\mathcal{F}}^{-1}_{\xi'\to x'}[\tilde{q}(x, \xi; x', \xi')\widetilde{\mathcal{F}}V_{G,n}u]](x) \\ = \ & \mathcal{F}^{-1}_{\xi\to x}V^{-1}_{G,n}\widetilde{\mathcal{F}}_{x'\to\xi'}[(\widetilde{\mathcal{F}}^{-1}_{\xi'\to x'}[q(x, \xi; \cdot, \xi')\widetilde{\mathcal{F}}V_{G,n}u])(\xi)] \\ = \ & \mathcal{F}^{-1}_{\xi\to x}\mathcal{F}_{x'\to\xi'}V^{-1}_{G,n}\widetilde{\mathcal{F}}^{-1}_{\xi'\to x'}[q(x, \xi; \cdot, \xi')\widetilde{\mathcal{F}}V_{G,n}u(\xi')] \\ = \ & \mathcal{F}^{-1}_{\xi\to x}\mathcal{F}_{x'\to\xi'}\mathcal{F}^{-1}_{\xi'\to x'}[(q(x, \xi; x', \cdot)V^{-1}_{G,n}\widetilde{\mathcal{F}}V_{G,n}u)(\xi')] \\ = \ & \mathcal{F}^{-1}_{\xi\to x}\mathcal{F}_{x'\to\xi'}\mathcal{F}^{-1}_{\xi'\to x'}[q(x, \xi; x', \xi')(V^{-1}_{G,n}\widetilde{\mathcal{F}}V_{G,n}u)(\xi')] \\ = \ & \mathcal{F}^{-1}_{\xi\to x}\mathcal{F}_{x'\to\xi}\mathcal{F}^{-1}_{\xi'\to x'}[q(x, \xi; x', \xi')(\mathcal{F}u)(\xi')] \\ = \ & q(x, D_x; x', D_{x'})u(x). \end{aligned}$$

Thus we have proved (38). □

THEOREM 4.2.3. *Let ψ be in $\Lambda_k(\mathbb{R}^n)$, $m, m' \in \mathbb{R}$ and $q \in S_0^{m,m',\psi}(\mathbb{R}^n)$. For $u \in S_\gamma(\mathbb{R}^n)$ the operator $q(x, D_x, x', D_{x'})$ defines a pseudodifferential operator in the class $\Psi_0^{m+m',\psi}(\mathbb{R}^n)$. This operator is given by $q(x, D_x, x', D_{x'}) = q_L(x, D)$, where q_L is the reduced symbol in $S_0^{m+m',\psi}(\mathbb{R}^n)$ given by*

$$q_L(x,\xi) = (2\pi)^{-n} Os - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y\,,\eta\rangle} q(x, \xi+\eta; x+y, \xi) dy d\eta. \tag{39}$$

PROOF. For $u \in S_\gamma(\mathbb{R}^n)$ we have by [**81**, Theorem 2.4.17]

$$\begin{aligned} q(x, D_x; x', D_{x'})u(x) &= V_{G,n}^{-1}\tilde{q}(x, \widetilde{D}_x; x', \widetilde{D}_{x'})(V_{G,n}u)(x) \\ &= V_{G,n}^{-1} q_L(x, \widetilde{D}_x)(V_{G,n}u)(x) \\ &= q_L(x, D). \end{aligned}$$

Thus our theorem is proved. □

PROPOSITION 4.2.4. *Let ψ be in $\Lambda_k(\mathbb{R}^n)$.*

(i) *If $q_j \in S_0^{m'_j,\psi}(\mathbb{R}^n)$ (for j=1,2) then we have $q_1(x, D) \circ q_2(x, D) \in \Psi_0^{m_1+m_2,\psi}(\mathbb{R}^n)$. Moreover, the symbol of $q_1(x, D) \circ q_2(x, D)$ is given by the reduced symbol $q_L(x, \xi)$ of the double-symbol $q(x, \xi; x', \xi') = q_1(x, \xi)q_2(x', \xi')$.*

(ii) *For any $q \in S_0^{m,\psi}(\mathbb{R}^n)$ there exists a $q^* \in S_0^{m,\psi}(\mathbb{R}^n)$ such that*

$$\langle q(x, D)u\,,\, v\rangle_0 = \langle u\,,\, q^*(x, D)v\rangle_0$$

for all $u, v \in S_\gamma(\mathbb{R}^n)$. Furthermore we obtain the symbol of q^* as reduced symbol of the double-symbol $q(x, \xi; x', \xi') = \overline{q(x', \xi)}$.

PROOF. Let $u \in S_\gamma(\mathbb{R}^n)$. Then we obtain $q_1(x, D) \circ q_2(x, D)u = V_{G,n}^{-1} q_1(x, \tilde{D}) \circ q_2(x, \tilde{D})(V_{G,n}u)$. But now (i) follows by [**81**, Corollary 2.4.19] and 4.2.1. Let us prove (ii). Again using [**81**, Corollary 2.4.19] and 4.2.1 we obtain

$$\begin{aligned} \langle q(x, D)u\,,\, v\rangle_0 &= \langle V_{G,n}^{-1} q(x, \tilde{D})(V_{G,n}u)\,,\, v\rangle_0 = \langle q(x, \tilde{D})(V_{G,n}u)\,,\, V_{G,n}v\rangle_\lambda \\ &= \langle V_{G,n}u\,,\, (q^*(x, \tilde{D})(V_{G,n}v)\rangle_\lambda = \langle u\,,\, V_{G,n}^{-1}(q^*(x, \tilde{D})(V_{G,n}v)\rangle_0. \end{aligned}$$

Note that in [**81**, Corollary 2.4.19] it is shown that the symbol of $q_1(x, \tilde{D}) \circ q_2(x, \tilde{D})$ is given as reduced symbol to the double symbol $q_1(x, \xi)q_2(x', \xi')$ and the symbol of $q^*(x, \tilde{D})$ by the reduced symbol of the double symbol $\overline{q(x', \xi)}$. Thus the two assertions follow directly by Theorem 4.2.3 . □

Let us note the following Lemma which can be found in [**81**, Lemma 2.4.21].

LEMMA 4.2.5. *Let ψ be in $\Lambda_k(\mathbb{R}^n)$, $m, m' \in \mathbb{R}$ and $q \in S_0^{m,m',\psi}(\mathbb{R}^n)$ such that $\partial_\xi^\alpha q(x, \xi; x', \xi') \in S_0^{m+\varrho_k(\alpha),m',\psi}(\mathbb{R}^n)$ holds for all $\alpha \in \mathbb{N}_0^n$. For all $N \in \mathbb{N}$ the* *simplified symbol q_L satisfies*

$$q_L(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} q_\alpha(x,\xi) \in S_0^{m+m'-\varrho_k(N),\psi}(\mathbb{R}^n), \tag{40}$$

where

$$q_\alpha(x,\xi) = (-i\partial_{x'})^\alpha \partial_\xi^\alpha q(x,\xi;x',\xi')\big|_{\substack{x'=x\\ \xi'=\xi}} \in S_0^{m+m'-\varrho_k(|\alpha|),\psi}(\mathbb{R}^n). \tag{41}$$

LEMMA 4.2.6. *Let ψ be in $\Lambda_k(\mathbb{R}^n)$, $m,m' \in \mathbb{R}$ and $q \in S_0^{m,m',\psi}(\mathbb{R}^n)$, such that $\partial_\xi^\alpha q(x,\xi;x',\xi') \in S_0^{m+\varrho_k(\alpha),m',\psi}(\mathbb{R}^n)$ holds for all $\alpha \in \mathbb{N}_0^n$ and $q_\alpha \in$ be defined as in (41). Assume that we have $k=\infty$ and $q_\alpha \in S_{\varrho_\infty}^{m+m'-|\alpha|,\psi}(\mathbb{R}^n)$. Then we obtain*

$$q_L \in S_{\varrho_\infty}^{m+m',\psi}(\mathbb{R}^n)$$

and

$$q_L(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} q_\alpha(x,\xi) \in S_{\varrho_\infty}^{m+m'-N,\psi}(\mathbb{R}^n). \tag{42}$$

PROOF. According to 4.2.5 there exist a $q_N \in S_0^{m+m'-N}(\mathbb{R}^n)$ such that

$$q_L(x,\xi) - \sum_{|\gamma|<N} \frac{1}{\gamma!} q_\gamma(x,\xi) = q_N(x,\xi).$$

Now let $\alpha, \beta \in N_0^n$ and choose $N = |\alpha|$ in the equation above. Then we obtain

$$\begin{aligned}
\left|\partial_\xi^\alpha \partial_x^\beta q_L(x,\xi)\right| &\leq \sum_{|\gamma|<|\alpha|} \frac{1}{\gamma!} \left|\partial_\xi^\alpha \partial_x^\beta q_\gamma(x,\xi)\right| + \left|\partial_\xi^\alpha \partial_x^\beta q_{|\alpha|}(x,\xi)\right| \\
&\leq \sum_{|\gamma|<|\alpha|} \frac{c_\gamma}{\gamma!} (1+\psi(\xi))^{\frac{m+m'-|\gamma|-|\alpha|}{2}} + c_{|\alpha|}(1+\psi(\xi))^{\frac{m+m'-|\alpha|}{2}} \\
&\leq c(1+\psi(\xi))^{\frac{m+m'-|\alpha|}{2}}.
\end{aligned}$$

Note that for $M \in \mathbb{N}$ we have $q_L(x,\xi) - \sum_{|\gamma|<N+M} \frac{1}{\gamma!} q_\gamma(x,\xi) = q_{N+M}(x,\xi)$, which yields

$$q_N(x,\xi) = \sum_{N \leq |\gamma| < N+M} \frac{1}{\gamma!} q_\gamma(x,\xi) + q_{N+M}(x,\xi).$$

Thus our second assertion follows by the same arguments as the first. □

LEMMA 4.2.7. *Let ψ be in $\Lambda_\infty(\mathbb{R}^n)$*

(i) *For $q \in S_{\varrho_\infty}^{m,\psi}(\mathbb{R}^n)$ and $p(x,\xi;x',\xi') = \overline{q(x',\xi)}$ we have $p \in S_0^{m,0,\psi}(\mathbb{R}^n)$ and all conditions of 4.2.5 are fulfilled with $k=\infty$. Moreover, we have $p_\alpha \in S_{\varrho_\infty}^{m-|\alpha|,\psi}(\mathbb{R}^n)$.*

(ii) *For* $q_1 \in S^{m,\psi}_{\varrho\infty}(\mathbb{R}^n)$, $q_2 \in S^{m'}_{\varrho\infty}(\mathbb{R}^n)$ *and* $p(x,\xi;x',\xi') = q_1(x,\xi)q_2(x,\xi)$ *we have* $p \in S^{m,m',\psi}_0(\mathbb{R}^n)$ *and all conditions of 4.2.5 are fulfilled with* $k = \infty$. *In addition, we have* $p_\alpha \in S^{m+m'-|\alpha|,\psi}_{\varrho\infty}(\mathbb{R}^n)$.

Proof. The first part of this lemma is obvious by equation (41). Now let us proof the second part. For $\alpha, \beta \in N_0^n$ we obtain

$$\begin{aligned}
&\left|\partial_\xi^\alpha \partial_x^\beta p_\gamma(x,\xi)\right| \\
= &\left|\partial_\xi^\alpha \partial_x^\beta \partial_\xi^\gamma q_1(x,\xi)(-i\partial_x)^\gamma q_2(x,\xi)\right| \\
= &\left|\sum_{\mu\le\beta}\sum_{\nu\le\alpha}\binom{\mu}{\beta}\binom{\nu}{\alpha}\partial_\xi^\nu \partial_x^\mu \partial_\xi^\gamma q_1(x,\xi)\partial_\xi^{\alpha-\nu}\partial_x^{\beta-\mu}(-i\partial_x)^\gamma q_2(x,\xi)\right| \\
\le &\sum_{\mu\le\beta}\sum_{\nu\le\alpha}\binom{\mu}{\beta}\binom{\nu}{\alpha} c_{\mu,\nu}(1+\psi(\xi))^{\frac{m-|\nu|-|\gamma|}{2}}(1+\psi(\xi))^{\frac{m'-|\alpha-\nu|}{2}} \\
= &\; c(1+\psi(\xi))^{\frac{m+m'-|\alpha|-|\gamma|}{2}}.
\end{aligned}$$

But this is our assertion. □

Corollary 4.2.8. *Let* ψ *be in* $\Lambda_\infty(\mathbb{R}^n)$ *and* $q_1(x,D) \in \Psi^m_{k_\infty}(\mathbb{R}^n)$, $q_2(x,D) \in \Psi^{m'}_{k_\infty}(\mathbb{R}^n)$. *Then we obtain* $q_1(x,D)^* \in \Psi^m_{k_\infty}(\mathbb{R}^n)$ *and* $q_1(x,D) \circ q_2(x,D) \in \Psi^{m+m'}_{k_\infty}(\mathbb{R}^n)$.

Theorem 4.2.9. *Let* $\psi \in \Lambda_\infty(\mathbb{R}^n)$.

(i) *For* $q_1 \in S^{m_1,\psi}_{\varrho,\delta}(R^n), q_2 \in S^{m_2,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ *we define*

$$p_\alpha(x,\xi) = (-i)^{|\alpha|}\partial_\xi^\alpha q_1(x,\xi)\partial_x^\alpha q_2(x,\xi) \in S^{m_1+m_2-(\varrho-\delta)|\alpha|,\psi}_{\varrho,\delta}(\mathbb{R}^n). \tag{43}$$

Then $q_1(x,D) \circ q_2(x,D)$ *belongs to the class* $\Psi^{m_1+m_2}_{\varrho,\delta}(\mathbb{R}^n)$ *and for all* $N \in \mathbb{N}$ *there exists an* $r_N \in S^{m_1+m_2-(\varrho-\delta)N,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ *such that*

$$q_1(x,D) \circ q_2(x,D) - \sum_{|\alpha|<N}\frac{1}{\alpha!}p_\alpha(x,D) = r_N(x,D).$$

(ii) *For* $q \in S^{m,\psi}_{\varrho,\delta}(R^n)$ *we define*

$$p^*_\alpha(x,\xi) = (-i)^{|\alpha|}\partial_\xi^\alpha \partial_x^\alpha \overline{q(x,\xi)} \in S^{m-(\varrho-\delta)|\alpha|,\psi}_{\varrho,\delta}(\mathbb{R}^n). \tag{44}$$

Then $q(x,D)^*$ *belongs to the class* $\Psi^{m_1}_{\varrho,\delta}(\mathbb{R}^n)$ *and for all* $N \in \mathbb{N}$ *there exists an* $r_N \in S^{m-(\varrho-\delta)N,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ *such that*

$$q(x,D)^* - \sum_{|\alpha|<N}\frac{1}{\alpha!}p^*_\alpha(x,D) = r_N(x,D).$$

Proof. We have

$$
\begin{aligned}
& q_1(x,D)\circ q_2(x,D) - \sum_{|\alpha|<N} \frac{1}{\alpha!} p_\alpha(x,D) \\
= \;& V_{G,n}^{-1} q_1(x,\tilde{D}) q_2(x,\tilde{D}) - \sum_{|\alpha|<N} \frac{1}{\alpha!} p_\alpha(x,\tilde{D}) V_{G,n} \\
= \;& V_{G,n}^{-1} q_1(x,\tilde{D}) q_2(x,\tilde{D}) - \sum_{|\alpha|<N} \frac{1}{\alpha!} (-i)^{|\alpha|} \partial_\xi^\alpha q_1(x,\tilde{D}) \partial_x^\alpha q_2(x,\tilde{D}) V_{G,n}.
\end{aligned}
$$

and

$$
\begin{aligned}
q(x,D)^* - \sum_{|\alpha|<N} p_\alpha^*(x,D) &= V_{G,n}^{-1} q(x,\tilde{D})^* - \sum_{|\alpha|<N} \frac{1}{\alpha!} p_\alpha^*(x,\tilde{D}) V_{G,n} \\
&= V_{G,n}^{-1} q(x,\tilde{D})^* - \sum_{|\alpha|<N} \frac{1}{\alpha!} (i)^{|\alpha|} \partial_\xi^\alpha \partial_x^\alpha \overline{q(x,\tilde{D})} V_{G,n}
\end{aligned}
$$

Thus our theorem follows from [**93**, Chapter 7, Theorem 1.4]. □

So far defining

$$
\Psi_0^{\infty,\psi}(\mathbb{R}^n) := \bigcup_{m\in\mathbb{R}} \Psi_0^{m,\psi}(\mathbb{R}^n), \tag{45}
$$

$$
\Psi_\infty^{\infty,\psi}(\mathbb{R}^n) := \bigcup_{m\in\mathbb{R}} \Psi_{\varrho\infty}^{m,\psi}(\mathbb{R}^n) \tag{46}
$$

and

$$
\Psi_{\varrho,\delta}^{\infty,\psi}(\mathbb{R}^n) := \bigcup_{m\in\mathbb{R}} \Psi_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n) \tag{47}
$$

we have proved

Theorem 4.2.10. *The sets* $\Psi_0^{0,\psi}(\mathbb{R}^n)$, $\Psi_\infty^{0,\psi}(\mathbb{R}^n)$, $\Psi_{\varrho,\delta}^{0,\psi}(\mathbb{R}^n)$ *and* $\Psi_0^{\infty,\psi}(\mathbb{R}^n)$, $\Psi_\infty^{\infty,\psi}(\mathbb{R}^n)$, $\Psi_{\varrho,\delta}^{\infty,\psi}(\mathbb{R}^n)$ *are algebras of pseudodifferential operators with composition as multiplication and involution* $*$*. In addition for* $\Psi_0^{\infty,\psi}(\mathbb{R}^n)$, $\Psi_\infty^{\infty,\psi}(\mathbb{R}^n)$ *and* $\Psi_{\varrho,\delta}^{\infty,\psi}(\mathbb{R}^n)$ *we have*

(i) $\lambda\Psi_0^{m,\psi}(\mathbb{R}^n) + \mu\Psi_0^{m,\psi}(\mathbb{R}^n) \subset \Psi_0^{m,\psi}(\mathbb{R}^n), \quad \lambda,\mu\in\mathbb{C}$
$\lambda\Psi_\infty^{m,\psi}(\mathbb{R}^n) + \mu\Psi_\infty^{m,\psi}(\mathbb{R}^n) \subset \Psi_\infty^{m,\psi}(\mathbb{R}^n), \quad \lambda,\mu\in\mathbb{C}$
$\lambda\Psi_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n) + \mu\Psi_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n) \subset \Psi_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n), \quad \lambda,\mu\in\mathbb{C}$

(ii) $\left(\Psi_0^{m,\psi}(\mathbb{R}^n)\right)^* \subset \Psi_0^{m,\psi}(\mathbb{R}^n)$
$\left(\Psi_\infty^{m,\psi}(\mathbb{R}^n)\right)^* \subset \Psi_\infty^{m,\psi}(\mathbb{R}^n)$
$\left(\Psi_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n)\right)^* \subset \Psi_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n)$

(iii) $\Psi_0^{m,\psi}(\mathbb{R}^n) \circ \Psi_0^{m',\psi}(\mathbb{R}^n) \subset \Psi_0^{m+m',\psi}(\mathbb{R}^n)$
$\Psi_\infty^{m,\psi}(\mathbb{R}^n) \circ \Psi_\infty^{m',\psi}(\mathbb{R}^n) \subset \Psi_\infty^{m+m',\psi}(\mathbb{R}^n)$
$\Psi_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n) \circ \Psi_{\varrho,\delta}^{m',\psi}(\mathbb{R}^n) \subset \Psi_{\varrho,\delta}^{m+m',\psi}(\mathbb{R}^n)$.

PROPOSITION 4.2.11. *Let $q \in S_0^{m,\psi}(\mathbb{R}^n)$ or $q \in S_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n)$ Moreover, let p be a polynomial. Then we obtain for $u \in S_\gamma(\mathbb{R}^n)$*

$$q(x,D)p(x)u(x) = \sum_\alpha \frac{1}{\alpha!}(-i\partial_x)^\alpha p(x)(\partial_\xi^\alpha q)(x,D)u(x). \tag{48}$$

PROOF. Since $p(x)u(x) \in S_\gamma(\mathbb{R}^n)$ we obtain by [**119**, Example 3.5 (ii)]

$$\begin{aligned} q(x,D)p(x)u(x) &= V_{G,n}^{-1} q(x,\tilde{D}) V_{G,n} p(x)u(x) \\ &= V_{G,n}^{-1} q(x,\tilde{D}) p(x) V_{G,n} u(x) \\ &= V_{G,n}^{-1} \sum_\alpha \frac{1}{\alpha!}(-i\partial_x)^\alpha p(x)(\partial_\xi^\alpha q)(x,\tilde{D}) V_{G,n} u(x) \\ &= \sum_\alpha \frac{1}{\alpha!}(-i\partial_x)^\alpha p(x)(\partial_\xi^\alpha q)(x,D)u(x), \end{aligned}$$

which proves our proposition. □

Now we prove that our pseudodifferential operators extend to continuous operators in a scale of Sobolev-spaces. Moreover, we show some kind of Gårding inequality and prove that under some additional conditions our operators extend to generators of L^2_γ-sub Markovian-semi groups and L^2_γ-sub Markovian-Dirichlet-forms.

DEFINITION 4.2.12. Let λ denote the Lebesgue-Measure in $\mathbb{R}^n$ and ψ be a continuous negative definite function. Then we define for all $s \geq 0$ the generalized Sobolev-space $H^s_{\psi,\lambda}(\mathbb{R}^n)$ as the space of all $u \in L^2(\mathbb{R}^n, \lambda)$ such that

$$\|u\|_{\psi,s,\lambda} := \left\| (1+|\psi|)^{s/2} \widetilde{\mathcal{F}} u \right\|_{L^2(\mathbb{R}^n,\lambda)} < \infty.$$

LEMMA 4.2.13. *For $u \in H^s_\psi(\mathbb{R}^n)$ we have $V_{G,n}u \in H^s_{\psi,\lambda}(\mathbb{R}^n)$ with*

$$\|u\|_{\psi,s} = \|V_{G,n}u\|_{\psi,s,\lambda}. \tag{49}$$

PROOF. For $u \in H^s_\psi(\mathbb{R}^n)$ we obtain by 1.4.10

$$\begin{aligned} \|u\|_{\psi,s} = \left\|(1+\psi(\cdot))^{s/2}\mathcal{F}u\right\|_{\psi,0} &= \left\|(1+\psi(\cdot))^{s/2} V_{G,n}(\mathcal{F}u)\right\|_{\psi,0,\lambda} \\ &= \left\|(1+\psi(\cdot))^{s/2}(\widetilde{\mathcal{F}} V_{G,n} u)\right\|_{\psi,0,\lambda} = \|V_{G,n}u\|_{\psi,s,\lambda}. \end{aligned}$$

This shows our assertion. □

THEOREM 4.2.14. *Let $q \in S_0^{m,\psi}(\mathbb{R}^n)$ or $q \in S_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n)$. We denote by $q(x,D)$ the corresponding pseudodifferential operator defined in 4.1.5. Then $q(x,D)$ maps $H_\psi^{s+m}(\mathbb{R}^n)$ continuously to $H_\psi^s(\mathbb{R}^n)$, i.e. there exists a $c > 0$ such that for all $u \in H_\psi^{s+m}(\mathbb{R}^n)$ we have*

$$\|q(x,D)u\|_{\psi,s} \leq c\,\|u\|_{\psi,s+m} \tag{50}$$

PROOF. Since by 2.3.17 $S_\gamma(\mathbb{R}^n)$ is dense in $H_\psi^s(\mathbb{R}^n)$ for all s we only have to prove (50) for all $u \in S_\gamma(\mathbb{R}^n)$. However, for $u \in S_\gamma(\mathbb{R}^n)$ we obtain by 4.2.13, 4.2.1 and [**81**, Theorem 2.5.4] resp. [**93**, Chapter 7 Theorem 1.6]

$$\begin{aligned}\|q(x,D)u\|_{\psi,s} = \|V_{G,n}q(x,D)u\|_{\psi,s,\lambda} &= \left\|V_{G,n}V_{G,n}^{-1}q(x,\tilde{D})V_{G,n}u\right\|_{\psi,s,\lambda}\\ &\leq \|V_{G,n}u\|_{\psi,s+m,\lambda} = c\,\|u\|_{\psi,s+m}\,,\end{aligned}$$

where $c > 0$. □

LEMMA 4.2.15. *Let $q \in S_0^{m,\psi}(\mathbb{R}^n)$ or $q \in S_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n)$ and $u, v \in S_\gamma(\mathbb{R}^n)$ Then the following equality holds:*

$$\langle q(x,D)u\,,\,v\rangle_{L^2(\mathbb{R}^n,\gamma)} = \langle q(x,\tilde{D})V_{G,n}u\,,\,V_{G,n}v\rangle_{L^2(\mathbb{R}^n,\gamma)}.$$

PROOF. For $u, v \in S_\gamma(\mathbb{R}^n)$ we obtain

$$\begin{aligned}\langle q(x,D)u\,,\,v\rangle_{L^2(\mathbb{R}^n,\gamma)} &= \langle V_{G,n}^{-1}q(x,\tilde{D})V_{G,n}u\,,\,v\rangle_{L^2(\mathbb{R}^n,\gamma)}\\ &= \langle V_{G,n}V_{G,n}^{-1}q(x,\tilde{D})V_{G,n}u\,,\,V_{G,n}u\rangle_{L^2(\mathbb{R}^n,\lambda)}\\ &= \langle q(x,\tilde{D})Vu\,,\,V_{G,n}u\rangle_{L^2(\mathbb{R}^n,\lambda)}.\end{aligned}$$

This shows our Lemma. □

PROPOSITION 4.2.16 (Gårding inequality). *Let $q \in S_{\varrho k}^{m,\psi}(\mathbb{R}^n)$ be non-negative. Then there exists a $K > 0$ such that for all $u \in S_\gamma(R^n)$*

$$\mathfrak{Re}\,\langle q(x,D)u\,,\,u\rangle_{L^2(\mathbb{R}^n,\gamma)} \geq -K\,\|u\|^2_{\psi,\frac{m-1}{2}}\,.$$

PROOF. Using 4.2.13 and [**81**, Theorem 2.5.5] we obtain for $u \in S_\gamma(\mathbb{R}^n)$

$$\begin{aligned}\mathfrak{Re}\,\langle q(x,D)u\,,\,u\rangle_{L^2(\mathbb{R}^n,\gamma)} &= \mathfrak{Re}\,\langle q(x,\tilde{D})V_{G,n}u\,,\,V_{G,n}u\rangle_{L^2(\mathbb{R}^n,\lambda)}\\ &\geq -K\,\|V_{G,n}u\|^2_{\psi,\frac{m-1}{2},\lambda} = -K\,\|u\|^2_{\psi,\frac{m-1}{2}}\,.\end{aligned}$$ □

DEFINITION 4.2.17. For $q \in S_0^{m,\psi}(\mathbb{R}^n)$ and $u, v \in S_\gamma(\mathbb{R}^n)$ we define the sesquilinear form B_q by

$$B_q(u,v) = \langle q(x,D)u\,,\,v\rangle_{L^2(\mathbb{R}^n,\gamma)}. \tag{51}$$

THEOREM 4.2.18. *Let $q \in S_{\varrho k}^{m,\psi}(\mathbb{R}^n)$ be real-valued and $m > 0$.*

(i) *Then we have*

$$|B_q(u,v)| \leq c\,\|u\|_{\psi,\frac{m}{2}}\,\|v\|_{\psi,\frac{m}{2}}$$

for all $u, v \in S_\gamma(\mathbb{R}^n)$ Thus we can extend B_q continuously to $H_\psi^{m/2}(\mathbb{R}^n)$.

(ii) *Moreover let us assume that there exists* $\mu_0 > 0$ *and* $R > 0$ *such that*

(52) $$q(x,\xi) \geq \mu_0(1+\psi(\xi))^{m/2} \text{ for } |\xi| \geq R,\ x \in \mathbb{R}^n$$

and

(53) $$\lim_{|\xi|\to\infty} \psi(\xi) = \infty.$$

Then we obtain for all $u \in H_\psi^{m/2}(\mathbb{R}^n)$ *the Gårding inequality*

$$\begin{aligned} \mathfrak{Re}\ B_q(u,u) &\geq \frac{\mu_0}{2} \|u\|^2_{\psi,\frac{m}{2}} - \lambda_0 \|u\|_0^2 , \\ \mathfrak{Re}\ B_q(u,u) &\geq \frac{\mu_0}{2} \|u\|^2_{\psi,\frac{m}{2}} - \lambda_1 \|u\|^2_{\psi,\frac{m-1}{2}} . \end{aligned}$$

(iii) *Under the assumptions of (ii) we obtain for* $s > -m$ *and for all* $u \in H_\psi^{s+m}(\mathbb{R}^n)$

$$\frac{\mu_0}{2} \|u\|^2_{\psi,m+s} \leq \|q(x,D)u\|^2_{\psi,s} + d\,\|u\|^2_{\psi,m+s-\frac{1}{2}}$$

Proof. Let $u, v \in S_\gamma(\mathbb{R}^n)$. Then by 4.2.13, 4.2.15 and [**81**, Theorem 2.5.6, Remark 2.5.7] we have

(i) $$\begin{aligned} B_q(u,v) &= \langle q(x,\tilde{D})V_{G,n}u\, ,\, V_{G,n}v\rangle_{L^2(\mathbb{R}^n,\lambda)} \\ &\leq c\,\|V_{G,n}u\|_{\psi,\frac{m}{2},\lambda}\,\|V_{G,n}v\|_{\psi,\frac{m}{2},\lambda} = c\,\|u\|_{\psi,\frac{m}{2}}\,\|v\|_{\psi,\frac{m}{2}} \end{aligned}$$

(ii) $$\begin{aligned} \mathfrak{Re}\ B_q(u,u) &= \mathfrak{Re}\ \langle q(x,\tilde{D})V_{G,n}u\, ,\, V_{G,n}u\rangle_{L^2(\mathbb{R}^n,\lambda)} \\ &\leq \tfrac{\mu_0}{2}\,\|V_{G,n}u\|^2_{\psi,\frac{m}{2},\lambda} - \lambda_0\,\|V_{G,n}u\|^2_{0,\lambda} \\ &= \tfrac{\mu_0}{2}\,\|u\|^2_{\psi,\frac{m}{2}} - \lambda_0\,\|u\|^2_0 \end{aligned}$$

(iii) $$\begin{aligned} \mathfrak{Re}\ B_q(u,u) &= \mathfrak{Re}\ \langle q(x,\tilde{D})V_{G,n}u\, ,\, V_{G,n}u\rangle_{L^2(\mathbb{R}^n,\lambda)} \\ &\leq \tfrac{\mu_0}{2}\,\|V_{G,n}u\|^2_{\psi,\frac{m}{2},\lambda} - \lambda_1\,\|V_{G,n}u\|^2_{\psi,\frac{m-1}{2},\lambda} \\ &= \tfrac{\mu_0}{2}\,\|u\|^2_{\psi,\frac{m}{2}} - \lambda_1\,\|u\|^2_{\psi,\frac{m-1}{2}} \end{aligned}$$

(iv) $$\begin{aligned} \tfrac{\mu_0}{2}\,\|u\|^2_{\psi,m+s} &= \tfrac{\mu_0}{2}\,\|V_{G,n}u\|^2_{\psi,m+s,\lambda} \\ &\leq \left\|q(x,\tilde{D})V_{G,n}u\right\|^2_{\psi,s,\lambda} + d\,\|V_{G,n}u\|^2_{\psi,m+s-\frac{1}{2},\lambda} \\ &= \|q(x,D)u\|^2_{\psi,s} + d\,\|u\|^2_{\psi,m+s-\frac{1}{2}} . \end{aligned}$$

This shows our theorem. □

Definition 4.2.19. Let $q \in S^{m,\psi}_{\varrho_k}(\mathbb{R}^n)$, $\mu \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^n,\gamma)$ Then we call $u \in H_\psi^{m/2}(\mathbb{R}^n)$ a variational solution of the equation

$$q_\mu(x,D)u := q(x,D)u + \mu u = f$$

if we have $B_{q_\mu}(u,\varphi) = \langle \varphi\, ,\, f\rangle_{L^2(\mathbb{R}^n,\gamma)}$ for all $\varphi \in H_\psi^{m/2}(\mathbb{R}^n)$

Lemma 4.2.20. *Let* $q \in S^{m,\psi}_{\varrho_k}(\mathbb{R}^n)$ *and* $\mu \in \mathbb{R}$. *For* $f \in L^2(\mathbb{R}^n,\gamma)$ *let* $u \in H_\psi^{m/2}(\mathbb{R}^n)$ *be a variational solution of* $q_\mu(x,D)u = f$. *Then* $V_{G,n}u$ *is a variational* *solution of* $q_\mu(x,\tilde{D})v = V_{G,n}f$. *Conversely, let* $f \in L^2(\mathbb{R}^n,\lambda)$ *and* $v \in H^{m/2}_{\psi,\lambda}(\mathbb{R}^n)$ *be a variational solution of* $q_\mu(x,\tilde{D})v = f$*, then* $V^{-1}_{G,n}v$ *is a variational solution of* $q_\mu(x,D)u = V^{-1}_{G,n}f$.

PROOF. Let $f \in L^2(\mathbb{R}^n,\gamma)$ and $u \in H^{m/2}_{\psi}(\mathbb{R}^n)$ be a variational solution of $q_\mu(x,D)u = f$. Then we obtain

$$\langle q_\mu(x,\tilde{D})V_{G,n}u\,,\,\varphi\rangle_{0,\lambda} = \langle q_\mu(x,D)V_{G,n}u\,,\,V^{-1}_{G,n}\varphi\rangle_0 = \langle V^{-1}\varphi\,,\,f\rangle_0 = \langle\varphi\,,\,V_{G,n}f\rangle_{0,\lambda}.$$

Conversely, let $f \in L^2(\mathbb{R}^n,\lambda)$ and $v \in H^{m/2}_{\psi,\lambda}(\mathbb{R}^n)$ be a variational solution of $q_\mu(x,\tilde{D})v = f$, then we have

$$\langle q(x,D)V^{-1}_{G,n}v\,,\,\varphi\rangle_0 = \langle q_\mu(x,\tilde{D}v\,,\,V\varphi\rangle_{0,\lambda} = \langle V\varphi\,,\,f\rangle_{0,\lambda} = \langle\varphi\,,\,V^{-1}f\rangle_0. \qquad \square$$

THEOREM 4.2.21. *Under the assumptions and with the notations of Theorem 4.2.18(ii) we obtain*

(i) *For all* $\mu \geq \mu_0$ *and* $f \in L^2(\mathbb{R}^n,\gamma)$ *there exists a unique variational solution of* $q_\mu(x,D)u = f$.
(ii) *Moreover, for* $m \geq 1$ *and* $f \in H^s_\psi(\mathbb{R}^n)$ $(s \geq 0)$ *any variational solution* $u \in H^{m/2}_\psi(\mathbb{R}^n)$ *of* $q_\mu(x,D)u = f$ *belongs to* H^{m+s}_ψ.

PROOF. Let $\mu \geq \mu_0$ and $f \in L^2(\mathbb{R}^n,\gamma)$. Then according to [**81**, Theorem 2.5.12] there exists a unique variational solution v of $q_\mu(x,\tilde{D})v = V_{G,n}f$. But in view of Lemma 4.2.20 $u := V^{-1}_{G,n}v$ is then the unique variational solution of $q_\mu(x,D)u = f$. To prove (ii) let u be a variational solution of $q_\mu(x,D)u = f$. Then $V_{G,n}f \in H^s_{\psi,\lambda}(\mathbb{R}^n)$ and we have $V_{G,n}u$ is a variational solution of $q_\mu(x,\tilde{D})v = V_{G,n}f$. Thus by [**81**, Theorem 2.5.13] $V_{G,n}u \in H^{m+s}_{\psi,\lambda}(\mathbb{R}^n)$. But this implies $u \in H^{m+s}_\psi(\mathbb{R}^n)$. $\square$

PROPOSITION 4.2.22. *Let* $\psi \in \Lambda_2$*, such that (53) holds. Moreover, let us assume that* $\psi(\xi) \geq c_0\,|\xi|^r$ *for some* $c_0 > 0, r > 0$ *and all* $|\xi| > R_1$. *Let* $q \in S^{2,\psi}_{\varrho_2}(\mathbb{R}^n)$ *such that* $\xi \longmapsto q(x,\xi)$ *is negative definite for all* $x \in \mathbb{R}^n$. *In addition, suppose that* q *fulfills (52). Finally let* $\mu > \mu_0$. *Then the operator* $(-q_\mu(x,D), H^{\psi,2}(\mathbb{R}^n,\mathbb{R}))$ *is generator of a contraction semi group in* $L^2(\mathbb{R}^n,\gamma)$.

PROOF. In view of the Hille-Yoshida theorem and Theorem 4.2.21 we only have to show that $-q(x,D)$ is dissipative. But by [**81**, Theorem 2.6.10] we obtain for $\nu > 0$

$$\|\nu u + q_\mu(x,D)\|_0 = \|\nu V_{G,n}u + q_\mu(x,D)V_{G,n}u\|_{0,\lambda} \geq \nu\,\|V_{G,n}u\|_{0,\lambda} = \nu\,\|u\|_0\,. \qquad \square$$

For $q \in S^{2,\psi}_{\varrho_2}(\mathbb{R}^n)$ let us denote by $\mathcal{E}$ the extension of B_{q_μ} to the space $H^1_\psi(\mathbb{R}^n)$ and by $\widetilde{\mathcal{E}}$ the extension of $\langle q_\lambda(x,\tilde{D})\cdot\,,\,\cdot\rangle_{0,\lambda}$ to the space $H^1_{\psi,\lambda}(\mathbb{R}^n)$. Then we obtain the following lemma

LEMMA 4.2.23. *Let* $(\widetilde{\mathcal{E}}, H^1_{\psi,\lambda}(\mathbb{R}^n))$ *be a semi-Dirichlet-form. Then the form* $(\mathcal{E}, H^1_\psi(\mathbb{R}^n))$ *is a* L^2_γ*-semi-Dirichlet-form.*

PROOF. The equation $\mathcal{E}(u,v) = \widetilde{\mathcal{E}}(V_{G,n}u, V_{G,n}v)$ extends by continuity from $H^2_\psi(\mathbb{R}^n)$ to $H^1_\psi(\mathbb{R}^n)$. By definition $\mathcal{E}$ is closed. Moreover since $\mathcal{E}^{sym}_1(u,v) = \widetilde{E}^{sym}_1(V_{G,n}u, V_{G,n}v)$ we obtain that $\mathcal{E}$ is continuous with respect to $\mathcal{E}^{sym}_1$. Finally we have

$$\begin{aligned} & \mathcal{E}(u + u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}}, u + u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}}) \\ = \ & \widetilde{\mathcal{E}}(V_{G,n}(u + u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}}), V_{G,n}(u + u^+ \wedge e^{\frac{\|P_n\cdot\|_0}{2}})) \\ = \ & \widetilde{\mathcal{E}}(V_{G,n}u + (V_{G,n}u^+) \wedge 1, V_{G,n}u + (V_{G,n}u^+) \wedge 1) \geq 0. \end{aligned}$$

This is our assertion. □

Now in view of [**81**, Theorem 2.6.10] we have finally proved and can state

THEOREM 4.2.24. *Let the assumptions of Proposition 4.2.22 hold. Then* $(-q_\mu(x,D), H^2_\psi(\mathbb{R}^n,\mathbb{R}))$ *is generator of a* L^2_γ *sub Markovian semi group. Moreover,* $(-q_\mu(x,D), H^2_\psi(\mathbb{R}^n,\mathbb{R}))$ *is a* L^2_γ*-Dirichlet operator and* $(B_{q_\mu},\ H^1_\psi(\mathbb{R}^n,\mathbb{R}))$ *is a* L^2_γ*-Dirichlet-form.*

4.3. An asymptotic expansion and estimates for pseudodifferential operators on quasi-nuclear Hilbert space riggings

In this section we develop a symbolic calculus for pseudodifferential operators on a quasi-nuclear Hilbert space rigging. Let us start with some relations between finite dimensional pseudodifferential operators and the infinite dimensional case.

NOTATIONS 4.3.1. For $x \in H_-$ and $m > n$ let us denote

(i) $P_n x := \sum_{k=1}^n \langle x\,,\, e_j\rangle_0 e_j$,

(ii) $\tilde{P}_n x := (\langle x\,,\, e_1\rangle_0, \dots, \langle x\,,\, e_n\rangle_0) \in \mathbb{R}^n$,

(iii) $\tilde{P}_{m,n} x := (\langle x\,,\, e_{n+1}\rangle_0, \dots, \langle x\,,\, e_m\rangle_0) \in \mathbb{R}^{n-m}$,

REMARK 4.3.2.
(i) Let $q \in S^{m,\psi}_{\varrho,\delta}(H_-)$ or $q \in S^{m,\psi}_0(H_-)$ be cylindrical such that $q(x,\xi) = q(P_n x, P_n \xi)$ for a fixed $n \in \mathbb{N}$. Define $\tilde{\psi} : \mathbb{R}^n \longrightarrow \mathbb{C}$ by $\tilde{\psi}(\xi) = \psi(\sum_{j=1}^n x_j e_j)$ Then there exits a function $\tilde{q} \in S^{m,\tilde{\psi}}_{\varrho,\delta}(\mathbb{R}^n)$ (resp. $S^{m,\tilde{\psi}}_0(\mathbb{R}^n)$) such that $q(x,\xi) = \tilde{q}(\tilde{P}_n x, \tilde{P}_n \xi)$.
(ii) Let $u \in S_{\gamma,cyl}(H_-)$. Then there exists functions $\tilde{u} \in S_\gamma(\mathbb{R}^n)$ such that $u(x) = \tilde{u}(\tilde{P}_n x)$.

LEMMA 4.3.3. *Let* $q \in S^{m,\psi}_{\varrho,\delta}(H_-)$ *or* $q \in S^{m,\psi}_0(H_-)$ *be cylindrical such that* $q(x,\xi) = q(P_n x, P_n \xi)$ *for a fixed* $n \in \mathbb{N}$. *Define* $\tilde{\psi} : \mathbb{R}^n \longrightarrow \mathbb{C}$ *by* $\tilde{\psi}(\xi) = \psi(\sum_{j=1}^n x_j e_j)$. *According to 4.3.2 there exits a cylindrical function* $\tilde{q} \in S^{m,\tilde{\psi}}_{\varrho,\delta}(\mathbb{R}^n)$ *(resp.* $S^{m,\tilde{\psi}}_0$*) such that* $q(x,\xi) = \tilde{q}(\tilde{P}_n x, \tilde{P}_n \xi)$ *Moreover, let* $u \in S_{\gamma,cyl}(H_-)$. *We* *assume that* $u(x) = f(x)g(x)$ *where* $f(x) = f(P_n x)$ *and* $g(x) = g((Id - P_n)x)$. *As above according to 4.3.2 there exists functions* $\tilde{f} \in S_\gamma(\mathbb{R}^n)$ *and* $\tilde{g} \in S_\gamma(\mathbb{R}^{m-n})$ *such that* $f(x) = \tilde{f}(\tilde{P}_n x)$ *and* $g(x) = (\tilde{P}_{n,m}x)$. *Then it follows that*

$$q(x,D)u(x) = [q(x,D)f(x)]g(x) = \tilde{q}(\tilde{P}_n x, \tilde{P}_n D)\tilde{f}(\tilde{P}_n x)\tilde{g}(\tilde{P}_{n,m}x), \tag{54}$$

where $\tilde{q}(\widetilde{P}_n x, \widetilde{P}_n D)$ *is the pseudodifferential-defined on* $\mathbb{R}^n$.

Proof. Let $x = (x_1, \cdots x_m) \in \mathbb{R}^m$. Then we define $Q_n(x) := (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $Q_{n,m}(x) := (x_{n+1}, \cdots, x_m) \in \mathbb{R}^{m-n}$ Let $u \in S_{\gamma,cyl}(H_-)$ such that $u(x) = f(x)g(x)$, where f and g are given as above. Let us denote by $\hat{\mathcal{F}}$ the Fourier-Wiener-Transform in $\mathbb{R}^m$. Then we have

$$\begin{aligned}
& q(x,D)u(x) \\
= \; & \mathcal{F}^{-1}_{\xi\to x} q(x,\xi)\mathcal{F}u(\xi) \\
= \; & \mathcal{F}^{-1}_{\xi\to x} q(P_n x, P_n\xi)\mathcal{F}_{y\to\xi}[f(P_n y)g((Id - P_n)y)] \\
= \; & \hat{\mathcal{F}}^{-1}_{\xi\to\tilde{P}_m x} \tilde{q}(\tilde{P}_n x, Q_n\xi)\hat{\mathcal{F}}_{y\to\xi}[\tilde{f}(Q_n y)\tilde{g}(Q_{n,m}y)] \\
= \; & e^{\frac{\|\tilde{P}_m x\|^2}{2}} \int\limits_{\mathbb{R}^m} e^{i\langle \tilde{P}_m x\,,\,\xi\rangle}\tilde{q}(\tilde{P}_n x, -Q_n\xi) \int\limits_{\mathbb{R}^m} e^{-i\langle\xi\,,\,y\rangle}\tilde{f}(Q_n y)\tilde{g}(Q_{n,m}y)e^{\frac{-\|y\|^2}{2}} \\
& \qquad d\lambda^m(y)d\lambda^m(\xi) \\
= \; & e^{\frac{\|\tilde{P}_n x\|^2}{2}} e^{\frac{\|\tilde{P}_{n,m}x\|^2}{2}} \int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} e^{i\langle\tilde{P}_n x\,,\,Q_n\xi\rangle} e^{i\langle\tilde{P}_{n,m}x\,,\,Q_{n,m}\xi\rangle}\tilde{q}(\tilde{P}_n x, -Q_n\xi) \\
& \qquad e^{-i\langle Q_n\xi\,,\,Q_n y\rangle}e^{-i\langle Q_{n,m}\xi\,,\,Q_{n,m}y\rangle}\tilde{f}(Q_n y)\tilde{g}(Q_{n,m}y) \\
& \qquad e^{-\frac{\|Q_n y\|^2}{2}} e^{-\frac{\|Q_{n,m}y\|^2}{2}} d\lambda^m(y)d\lambda^m(\xi) \\
= \; & e^{\frac{\|\tilde{P}_n x\|^2}{2}} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i\langle\tilde{P}_n x\,,\,\xi\rangle}\tilde{q}(\tilde{P}_n x, \xi)e^{-i\langle\xi\,,\,y\rangle}\tilde{f}(y)e^{-\frac{\|y\|^2}{2}} d\lambda^n(y)d\lambda^n(\xi) \\
& e^{\frac{\|\tilde{P}_{n,m}x\|^2}{2}} \int\limits_{\mathbb{R}^{m-n}}\int\limits_{\mathbb{R}^{m-n}} e^{i\langle\tilde{P}_{n,m}x\,,\,\xi\rangle}e^{-i\langle\xi\,,\,y\rangle}\tilde{g}(y)e^{-\frac{\|y\|^2}{2}} d\lambda^{m-n}(y)d\lambda^{m-n}(\xi) \\
= \; & \tilde{q}(\tilde{P}_n x, \tilde{P}_n D)\tilde{f}(\tilde{P}_n x)\tilde{g}(P_{n,m}x).
\end{aligned}$$

But this is our assertion. □

According to [**35**, Rem 2.2, p. 45] we obtain $\gamma = \gamma_n \otimes \gamma_R$, where γ_n is the canonical Gaussian measure with respect to the Hilbert space rigging $\mathbb{R}^n \cong P_n H_+ \subset P_n H_0 \subset P_n H_- \cong \mathbb{R}^n$. Furthermore, γ_R is the canonical Gaussian measure with respect to the rigging $H_+ \ominus P_n H_+ \cong H_+ \cap (H_0 \ominus P_n H_0) \subset H_0 \ominus P_n H_0 \subset \{x \in H_- \,|P_n x = 0\} \cong H_- \ominus P_n(H_-)$. Now by [**19**, p.24] it follows that

$$L^2(H_-,\gamma) = L^2(\mathbb{R}^n,\gamma_n)\widehat{\otimes}L^2(H_- \ominus P_n H_-,\gamma_R), \tag{55}$$

where $\widehat{\otimes}$ denotes the topological tensor-product of Hilbert Spaces. Then using Lemma 4.3.3 we obtain the following corollary.

COROLLARY 4.3.4. *Let* $q \in S^{m,\psi}_{\varrho,\delta}(H_-)$ *or* $q \in S^{m,\psi}_0(H_-)$ *be cylindrical such that* $q(x,\xi) = q(P_n x, P_n \xi)$ *for a fixed* $n \in \mathbb{N}$. *Moreover, let* $f \in S_\gamma(P_n H_-)$ *and* $g \in S_{\gamma,cyl}(H_- \ominus P_n H_-)$. *Then we can consider* $f \otimes g$ *as an element of* $L^2(H_-,\gamma)$ *and obtain*

$$\begin{aligned} q(x,D)(f\otimes g) &= [\tilde{q}(\tilde{P}_n x, \tilde{P}_n D) f] g \\ &= \tilde{q}(\tilde{P}_n x, \tilde{P}_n D) f \otimes g \\ &= (\tilde{q}(\tilde{P}_n x, \tilde{P}_n D) \otimes \mathrm{id})(f \otimes g). \end{aligned}$$

Now note that $S_\gamma(P_n H_-)$ is a dense subset of $L^2(\mathbb{R}^n,\gamma)$ and that $S_{\gamma,cyl}(H_- \ominus P_n H_-)$ is dense in $L^2(H_- \ominus P_n H_-, \gamma_R)$. Thus $S_\gamma(P_n H_-) \otimes S_{\gamma,cyl}(H_- \ominus P_n H_-)$ is dense in $L^2(\mathbb{R}^n,\gamma_n) \widehat{\otimes} L^2(H_- \ominus P_n H_-, \gamma_R)$. According to Theorem 4.2.14 $\tilde{q}(\tilde{P}_n x, \tilde{P}_n D)$ extends to a continuous linear operator on $L^2(\mathbb{R}^n,\gamma)$ and of course the identity is continuous in $L^2(H_- \ominus P_n H_-, \gamma_R)$. Now following [**19**, Theorem 2.1] $\tilde{q}(\tilde{P}_n x, \tilde{P}_n D) \otimes \mathrm{id}$ extends to a continuous linear operator in $L^2(\mathbb{R}^n,\gamma_n) \widehat{\otimes} L^2(H_- \ominus P_n H_-, \gamma_R)$ such that

$$\left\| \tilde{q}(\tilde{P}_n x, \tilde{P}_n D) \otimes \mathrm{id} \right\| \leq \left\| \tilde{q}(\tilde{P}_n x, \tilde{P}_n D) \right\| \|id\| .$$

Hence we can prove the following

THEOREM 4.3.5. *Let* $q \in S^{0,\psi}_{\varrho,\delta}(H_-)$ *or* $q \in S^{0,\psi}_0(H_-)$ *be cylindrical such that* $q(x,\xi) = q(P_n x, P_n \xi)$ *for a fixed* $n \in \mathbb{N}$. *Then* $q(x,D)$ *defined on* $S_{\gamma,cyl}(H_-)$ *extends to a continuous linear operator on* $L^2(H_-,\gamma)$.

PROOF. By the remarks above we only have to show that $q(x,D)$ and $\tilde{q}(\tilde{P}_n x, \tilde{P}_n D) \otimes \mathrm{id}$ coincide on $S_{\gamma,cyl}(H_-)$. To do this let $u \in S_{\gamma,cyl}(H_-)$. Then there exists a $m \geq n$ such that $u(x) = u(P_m x) = \tilde{u}(\tilde{P}_m x)$. According to Lemma 4.3.3 we have $q(x,D)u(x) = \tilde{q}(P_m x, P_m D)\tilde{u}(\tilde{P}_m x)$. Now choose a sequence $(\tilde{f}_k)_{k\in\mathbb{N}} \in S_\gamma(R^n) \otimes S_\gamma(\mathbb{R}^{m-n})$ such that $\tilde{f}_k \xrightarrow{n\longrightarrow\infty} \tilde{u}$ in $L^2(\mathbb{R}^m,\gamma_m)$. This is possible since $L^2(\mathbb{R}^m,\gamma_m) = L^2(\mathbb{R}^n,\gamma_n)\widehat{\otimes}L^2(\mathbb{R}^{m-n},\gamma_{m-n})$. We define $f_k(x)=\tilde{f}_k(\tilde{P}_m x)$ for all $x \in H_-$. Then we have $f_k \in S_\gamma(\mathbb{R}^n) \otimes S_{\gamma,cyl}(H_- \ominus P_n H_-)$ and $f_k \xrightarrow{k\longrightarrow\infty} u$ in $L^2(H_-,\gamma)$. Hence we obtain by Theorem 4.2.14 for $\mathbb{R}^m$

$$\begin{aligned} &\left\| (\tilde{q}(\tilde{P}_n x, \tilde{P}_n D) \otimes \mathrm{id})u - q(x,D)u \right\|_{L^2(H_-,\gamma)} \\ = &\lim_{k\to\infty} \left\| \tilde{q}((\tilde{P}_n x, \tilde{P}_n D) \otimes \mathrm{id}) f_k - q(x,D)u \right\|_{L^2(H_-,\gamma)} \\ = &\lim_{k\to\infty} \left\| \tilde{q}((\tilde{P}_n x, \tilde{P}_n D) \otimes \mathrm{id}_{\mathbb{R}^{m-n}}) \tilde{f}_k - \tilde{q}(P_m x, P_m D)\tilde{u} \right\|_{L^2(\mathbb{R}^m,\gamma_m)} \\ = &\lim_{k\to\infty} \left\| \tilde{q}(P_m x, P_m D)(\tilde{f}_k - \tilde{u}) \right\|_{L^2(\mathbb{R}^m,\gamma_m)} \leq c_m \lim_{k\to\infty} \left\| \tilde{f}_k - \tilde{u} \right\|_{L^2(\mathbb{R}^m,\gamma_m)} = 0, \end{aligned}$$

where c_m is a constant depending on q and m. But this is our assertion. □

PROPOSITION 4.3.6. *Let $q \in S^{0,\psi}_{\varrho,\delta}(H_-)$ or $q \in S^{0,\psi}_{0}(H_-)$ be cylindrical such that $q(x,\xi) = q(P_n x, P_n \xi) = \widetilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi)$ for a fixed $n \in \mathbb{N}$. Then for $u = f \otimes g$ where $f(x) = f(P_n x)$ and $g(x) = g((\mathrm{id} - P_n)x)$ and $f, g \in S_{\gamma,cyl}(H_-)$ we have*

$$[q(x,D)]^* u = [\widetilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi)]^* f \otimes g,$$

where $[\widetilde{q}(\widetilde{P}_n x, \widetilde{P}_n D)]^* \in \Psi^{0,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ resp. $[\widetilde{q}(\widetilde{P}_n x, \widetilde{P}_n D)]^* \in \Psi^{0,\psi}_{0}(\mathbb{R}^n)$.

PROOF. Let $q \in S^{0,\psi}_{\varrho,\delta}(H_-)$ or $q \in S^{0,\psi}_{0}(H_-)$ be cylindrical such that $q(x,\xi) = q(P_n x, P_n \xi) = \widetilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi)$ for a fixed $n \in \mathbb{N}$. Moreover, let $u = f_1 \otimes g_1$ and $v = f_2 \otimes g_2$ where $f_j(x) = f_j(\widetilde{P}_n x)$ and $g_j(x) = g_j((\mathrm{id} - P_n)x)$ and $f_j, g_j \in S_{\gamma,cyl}(H_-)$ $(j = 1,2)$. Then we obtain by Theorem 4.2.9 Proposition 4.2.4

$$\begin{aligned}
& \langle q(x,D)u\,,\, v\rangle_{L^2(H_-,\gamma)} \\
= \;& \langle q(x,D)(f_1 \otimes g_1)\,,\, f_2 \otimes g_2\rangle_{L^2(H_-,\gamma)} \\
= \;& \langle \widetilde{q}(\widetilde{P}_n x, \widetilde{P}_n D) f_1\,,\, f_2\rangle_{L^2(\mathbb{R}^n,\gamma_n)} \langle g_1\,,\, g_2\rangle_{L^2(H_- \ominus P_n H_-,\gamma_R)} \\
= \;& \langle f_1\,,\, [\widetilde{q}(\widetilde{P}_n x, \widetilde{P}_n D)]^* f_2\rangle_{L^2(\mathbb{R}^n,\gamma_n)} \langle g_1\,,\, g_2\rangle_{L^2(H_- \ominus P_n H_-,\gamma_R)} \\
= \;& \langle f_1 \otimes g_1\,,\, [q(\widetilde{P}_n x, \widetilde{P}_n D)]^* f_2 \otimes g_2\rangle,
\end{aligned}$$

where the symbol of $[q(\widetilde{P}_n x, \widetilde{P}_n D)]^*$ is an element of $S^{0,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ resp. $S^{0,\psi}_{0}(\mathbb{R}^n)$. But this is our assertion since $L^2(P_n H_-, \gamma_n) \otimes L^2(H_- \ominus P_n H_-, \gamma_R)$ is dense in $L^2(H_-, \gamma)$. □

Now let us start doing symbolic calculus. At first we will compute the following two concrete, but important examples:

PROPOSITION 4.3.7. *Let $q \in S^{m,\psi}_{0,cyl}(H_-)$ or $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$. Moreover let p be a cylindrical polynomial. Then we obtain for $u \in S_{\gamma,cyl}(H_-)$*

$$q(x,D)p(x)u(x) = \sum_{\alpha} \frac{1}{\alpha!} (-i\partial_x)^{\alpha} p(x) (\partial^{\alpha}_{\xi} q)(x,D) u(x). \tag{56}$$

PROOF. Let $q \in S^{m,\psi}_{0,cyl}(H_-)$ or $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$ and p be a cylindrical polynomial. Then there exist a $n \in \mathbb{N}$, a $\tilde{q} \in S^{m,\tilde{\psi}}_{0}(\mathbb{R}^n)$ resp. $\tilde{q} \in S^{m,\tilde{\psi}}_{\varrho,\delta}(\mathbb{R}^n)$ and a polynomial $\tilde{p}$ on $\mathbb{R}^n$ such that $q(x,\xi) = \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi)$ and $p(\xi) = \tilde{p}(\widetilde{P}_n \xi)$. According to Corollary 4.3.4 and Proposition 4.2.11 we obtain

$$\begin{aligned} q(x,D)p(x) &= \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n D)\tilde{p}(\widetilde{P}_n x) \otimes \mathrm{id}_{H_- \ominus P_n H_-} \\ &= \left(\sum_\alpha \frac{1}{\alpha!}(-i\partial_{\tilde{x}})^\alpha \tilde{p}(\widetilde{P}_n x)(\partial_{\tilde{\xi}}^\alpha q)(\widetilde{P}_n x, \widetilde{P}_n D) \right) \otimes \mathrm{id}_{H_- \ominus P_n H_-} \\ &= \sum_\alpha \frac{1}{\alpha!} \left((-i\partial_{\tilde{x}})^\alpha \tilde{p}(\widetilde{P}_n x)(\partial_{\tilde{\xi}}^\alpha q)(\widetilde{P}_n x, \widetilde{P}_n D) \otimes \mathrm{id}_{H_- \ominus P_n H_-} \right) \\ &= \sum_\alpha \frac{1}{\alpha!}(-i\partial_x)^\alpha p(x)(\partial_\xi^\alpha q)(x,D), \end{aligned}$$

which shows our proposition. □

PROPOSITION 4.3.8. *Let $q \in S^{m,\psi}_{0,cyl}(H_-)$ or $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$. Then we obtain for $u \in S_{\gamma,cyl}(H_-)$*

$$D_{x_j} q(x,D)u(x) = q(x,D)D_{x_j}u(x) + (\partial x_j q)(x,D)u(x). \tag{57}$$

PROOF. Let $q \in S^{m,\psi}_{0,cyl}(H_-)$ or $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$ and $u \in S_{\gamma,cyl}(H_-)$. Then there exist a $n \geq j$ such that $q(x,\xi) = q(P_n x, P_n \xi)$ and $u(x) = u(P_n \xi)$. Now using Lebesgue's Theorem of dominate convergence and [**35**, Proposition 5.1] we obtain

$$\begin{aligned} & D_{e_j} q(x,D)u(x) \\ = \ & D_{e_j} \mathcal{F}^{-1}_{\xi \to x} q(x,\xi)(\mathcal{F}u)(\xi) \\ = \ & \left(\frac{\partial}{\partial e_j} - \langle e_j \, , \, x \rangle \right) e^{\frac{\|P_n x\|^2}{2}} \int e^{i\langle P_n x \, , \, P_n \xi \rangle} q(P_n x, P_n \xi)(\mathcal{F}u)(P_n \xi) \\ = \ & e^{\frac{\|P_n x\|^2}{2}} \frac{\partial}{\partial e_j} \int e^{i\langle P_n x \, , \, P_n \xi \rangle} q(P_n x, P_n \xi)(\mathcal{F}u)(P_n \xi)\gamma(d\xi) \\ = \ & e^{\frac{\|P_n x\|^2}{2}} \int e^{i\langle P_n x \, , \, P_n \xi \rangle} \left(i\xi_j q(P_n x, P_n \xi) + (\partial_{x_j} q)(P_n x, P_n \xi) \right) \\ & \qquad (\mathcal{F}u)(P_n \xi)\gamma(d\xi) \\ = \ & q(x,D)D_{x_j}u(x) + (\partial x_j q)(x,D)u(x), \end{aligned}$$

which shows our proposition. □

Throughout the rest of this paper let $\psi \in \Lambda_{\varrho_k}(H_-)$ be a negative definite function such that there exist a $n \in \mathbb{N}_0$ with

$$\psi(P_n \xi) \leq c\psi(\xi) \text{ for all } n \in \mathbb{N}(n \geq n_0) \tag{58}$$

and

$$\psi((\mathrm{id} - P_n)\xi) \leq c\psi(\xi) \text{ for all } n \in \mathbb{N}(n \geq n_0). \tag{59}$$

In addition let us formulate the following condition, which we have to assume in some case later on. We call ψ a negative definite function of cylindrical growth if there exists a $n_0 \in \mathbb{N}$ such that

$$1 + \psi(\xi) \leq \tilde{c}_n(1 + \psi(P_n\xi))\ \forall n \geq n_0,\ \forall \xi \in H_-. \tag{60}$$

PROPOSITION 4.3.9. *Let $m \in \mathbb{R}$ and $q \in S^{m,\psi}_{\varrho,\delta}(H_-)$ resp. $q \in S^{m,\psi}_0(H_-)$ be cylindrical. Then there exists a $n \geq n_0$ such that $q(x,\xi) = \tilde{q}(\widetilde{P}_n, \widetilde{P}_n)$ and we have $\tilde{q} \in S^{m,\tilde{\psi}}_{\varrho,\delta}(\mathbb{R}^n)$ resp. $\tilde{q} \in S^{m,\tilde{\psi}}_0(\mathbb{R}^n)$.*
Conversely, let $\tilde{q}$ be in $S^{m,\tilde{\psi}}_{\varrho,\delta}(\mathbb{R}^n)$ resp. $S^{m,\tilde{\psi}}_0(\mathbb{R}^n)$ such that one of the following conditions hold

(i) *ψ fulfills equation (60), or*
(ii) *$\tilde{q}$ in $S^{m,\tilde{\psi}}_{0,0}(\mathbb{R}^n)$ resp. $S^{m,\tilde{\psi}}_0(\mathbb{R}^n)$ and $m \geq 0$.*

Then we obtain $q(x,\xi) := \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi) \in S^{m,\psi}_{\varrho,\delta}(H_-)$ resp. $q(x,\xi) \in S^{m,\psi}_0(H_-)$. In both cases it follows that $q(x,D) = \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n D) \otimes \mathrm{id}_{H_- \ominus P_n H_-}$.

PROOF. We only have to prove the middle part. At first note that for $\alpha, \beta \in \mathbb{N}_0^{\mathbb{N}}$ we have

$$\partial_\xi^\alpha \partial_x^\beta q(x,\xi) = \begin{cases} \partial_\xi^\alpha \partial_x^\beta \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi) & \text{if } \max(l(\alpha),\ l(\beta)) \leq n \\ 0 & \text{else,} \end{cases}$$

where $l(\alpha)$ denotes the length of α. Thus we obtain

$$\begin{aligned} \left|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)\right| &\leq \left|\partial_\xi^\alpha \partial_x^\beta \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi)\right| \\ &\leq c'_{|\alpha|,|\beta|}(1 + \psi(\widetilde{P}_n \xi))^{\frac{m-\varrho|\alpha|+\delta|\beta|}{2}} \\ &\leq \tilde{c}_{|\alpha|,|\beta|}(1 + \psi(\xi))^{\frac{m-\varrho|\alpha|+\delta|\beta|}{2}}. \end{aligned}$$

Similarly, we obtain the case $S^{m,\psi}_0(H_-)$. The rest of this proposition is now obvious. □

PROPOSITION 4.3.10. *Let ψ be in $\Lambda_k(H_-)$ such that (58), (59) hold.*

(i) *Let $q_j \in S^{m'_j,\psi}_{0,cyl}(H_-)$ (j=1,2) and assume that (60) holds or $m_1 + m_2 \geq 0$. Then we have $q_1(x,D) \circ q_2(x,D) \in \Psi^{m_1+m_2,\psi}_{0,cyl}(H_-)$. Moreover, let $n \geq n_0$ such that $q_j(x,\xi) = \tilde{q}_j(\widetilde{P}_n x, \widetilde{P}_n \xi)$. Then the symbol $q(x,\xi)$ of $q_1(x,D) \circ q_2(x,D)$ is given by $q(x,\xi) = \tilde{q}(\widetilde{P}x, \widetilde{P}\xi)$, where $\tilde{q}(\widetilde{P}x, \widetilde{P}\xi)$ is the reduced symbol $\tilde{q}_L(\tilde{x},\tilde{\xi})$ of the double-symbol $\tilde{q}(\tilde{x},\tilde{\xi};\tilde{x}',\tilde{\xi}') = \tilde{q}_1(\tilde{x},\tilde{\xi})\tilde{q}_2(\tilde{x}',\tilde{\xi}')$ in $\mathbb{R}^n$.*
(ii) *Let $m \geq 0$ or assume that equation (60) holds. Then for any $q \in S^{m,\psi}_{0,cyl}(H_-)$ there exists a $q^* \in S^{m,\psi}_{0,cyl}(H_-)$ such that*

$$\langle q(x,D)u\,,\, v\rangle_0 = \langle u\,,\, q^*(x,D)v\rangle_0$$

for all $u, v \in S_{\gamma,cyl}(H_-)$. *Furthermore let* $n \geq n_0$ *such that* $q(x,\xi) = \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi)$. *Then we obtain the symbol of* q^* *as* $q^*(x,\xi) = \tilde{q}^*(\widetilde{P}_n x, \widetilde{P}_n \xi)$ *where* $\tilde{q}^*$ *is the reduced symbol of the double-symbol* $\tilde{p}(\tilde{x},\tilde{\xi};\tilde{x}',\tilde{\xi}') = \overline{\tilde{q}(\tilde{x}',\tilde{\xi})}$ *in* $\mathbb{R}^n$.

Proof. To prove (i) let $q_j \in S^{m_j',\psi}_{0,cyl}(H_-)$ (j=1,2). According to Proposition 4.3.9 there exist a $n \in \mathbb{N}$ and $\tilde{q}_j \in S^{m_j',\tilde{\psi}}_0(\mathbb{R}^n)$ such that $q_j(x,\xi) = \tilde{q}_j(\widetilde{P}_n x \widetilde{P}_n \xi)$. For $f \otimes g \in S_\gamma(\mathbb{R}^n) \otimes S_{\gamma,cyl}(H_- \ominus P_n H_-)$ we obtain by Corollary 4.3.4

$$\begin{aligned} q_1(x,D) \circ q_2(x,D)(f \otimes g) &= q_1(x,D)((\tilde{q}_2(\widetilde{P}_n x, \widetilde{P}_n D)f) \otimes g) \\ &= (\tilde{q}_1(\widetilde{P}_n x, \widetilde{P}_n D) \circ \tilde{q}_2(\widetilde{P}_n x, \widetilde{P}_n D)f) \otimes g. \end{aligned}$$

According to Proposition 4.2.4(i) the symbol of $\tilde{q}_1(\widetilde{P}_n x, \widetilde{P}_n D) \circ \tilde{q}_2(\widetilde{P}_n x, \widetilde{P}_n D)$ is given by $\tilde{q}(\tilde{x},\tilde{\xi})$, where $\tilde{q}$ is the reduced symbol of the double symbol $\tilde{q}_D(\tilde{x},\tilde{\xi};\tilde{x}',\tilde{\xi}') := \tilde{q}_1(\tilde{x},\tilde{\xi})\tilde{q}_2(\tilde{x}',\tilde{\xi}')$. In addition we have $\tilde{q} \in S^{m_1+m_2,\tilde{\psi}}_0(\mathbb{R}^n)$. Now by Proposition 4.3.9 we obtain $q(x,\xi) := \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi) \in S^{m_1+m_2}_{0,cyl}(H_-)$. Thus Corollary 4.3.4 implies that q is the symbol of $q_1(x,D) \circ q_2(x,D)$.
Now let us prove (ii). According to Proposition 4.3.9 for $q \in S^{m,\psi}_{0,cyl}(H_-)$ there exits a $n \geq n_0$ and a $\tilde{q} \in S^{m,\tilde{\psi}}_0(\mathbb{R}^n)$ such that $q(x,\xi) = \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi)$. As in Proposition 4.3.6 we obtain

$$\langle q(x,D)u\,,\, v\rangle = \langle u\,,\, ([\tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi)]^* \otimes \mathrm{id}_{H_- \ominus P_n H_-})v\rangle.$$

Using Proposition 4.2.4(ii) we find that the symbol $\tilde{q}^* \in S^{m,\tilde{\psi}}_0(H_-)$ of the operator $[\tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi)]^*$ is given as the reduced symbol of the double symbol $\tilde{q}_D(\tilde{x},\tilde{\xi};\tilde{x}',\tilde{\xi}') := \overline{\tilde{q}(\tilde{x}',\tilde{\xi})}$. Again by Proposition 4.3.9 we obtain $q(x,\xi) := \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi) \in S^m_{0,cyl}(H_-)$ and the rest is clear by Corollary 4.3.4. □

Proposition 4.3.11. *Let us assume that (60) holds and let* ψ *be in* $\Lambda_\infty(H_-)$ *and* $q_1(x,D) \in \Psi^m_{k_\infty,cyl}(H_-)$, $q_2(x,D) \in \Psi^{m'}_{k_\infty,cyl}(H_-)$. *Then we obtain* $q_1(x,D)^* \in \Psi^m_{k_\infty,cyl}(H_-)$ *and* $q_1(x,D) \circ q_2(x,D) \in \Psi^{m+m'}_{k_\infty,cyl}(H_-)$.

Proof. Using the same arguments as in Proposition 4.3.10 we obtain by Corollary 4.2.8 that the symbols q of $q_1(x,D) \circ q_2(x,D)$ and q^* of $q_1(x,D)^*$ are given by

$$\begin{aligned} q(x,\xi) &= \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi) \\ q^*(x,\xi) &= \tilde{q}^*(\widetilde{P}_n x, \widetilde{P}_n \xi), \end{aligned}$$

where $\tilde{q} \in S^{m+m'}_{k_\infty}(\mathbb{R}^n)$ and $\tilde{q}^* \in S^m_{k_\infty}(\mathbb{R}^n)$. Now (59), (60), (61) and Proposition 4.3.9 imply that $q \in S^{m+m'}_{k_\infty,cyl}(H_-)$ and $q^* \in S^m_{k_\infty,cyl}(H_-)$. □

Theorem 4.3.12. *Let* $\psi \in \Lambda_\infty(H_-)$ *and assume that (60) holds.*

(i) *For* $q_1 \in S^{m_1,\psi}_{\varrho,\delta,cyl}(H_-), q_2 \in S^{m_2,\psi}_{\varrho,\delta,cyl}(H_-)$ *we define*

(61) $$p_\alpha(x,\xi) = (-i)^{|\alpha|}\partial^\alpha_\xi q_1(x,\xi)\partial^\alpha_x q_2(x,\xi) \in S^{m_1+m_2-(\varrho-\delta)|\alpha|,\psi}_{\varrho,\delta,cyl}(H_-).$$

Then $q_1(x,D)\circ q_2(x,D)$ *belongs to the class* $\Psi^{m_1+m_2}_{\varrho,\delta,cyl}(H_-)$ *and for all* $N \in \mathbb{N}$ *there exists a* $r_N \in S^{m_1+m_2-(\varrho-\delta)N,\psi}_{\varrho,\delta,cyl}(H_-)$ *such that*

$$q_1(x,D)\circ q_2(x,D) - \sum_{|\alpha|<N}\frac{1}{\alpha!}p_\alpha(x,D) = r_N(x,D).$$

(ii) *For* $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$ *we define*

(62) $$p^*_\alpha(x,\xi) = (-i)^{|\alpha|}\partial^\alpha_\xi\partial^\alpha_x\overline{q(x,\xi)} \in S^{m-(\varrho-\delta)|\alpha|,\psi}_{\varrho,\delta,cyl}(H_-).$$

Then $q(x,D)^*$ *belongs to the class* $\Psi^{m_1}_{\varrho,\delta,cyl}(H_-)$ *and for all* $N \in \mathbb{N}$ *there exists a* $r_N \in S^{m-(\varrho-\delta)N,\psi}_{\varrho,\delta,cyl}(H_-)$ *such that*

$$q(x,D)^* - \sum_{|\alpha|<N}\frac{1}{\alpha!}p^*_\alpha(x,D) = r_N(x,D).$$

Proof. First let us prove (i). As in Proposition 4.3.10 there exist a $n \geq n_0$, $\tilde{q}_1 \in S^{m_1,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ and $\tilde{q}_2 \in S^{m_2,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ such that $q_j(x,\xi) = \tilde{q}_j(\widetilde{P}_n x, \widetilde{P}_n\xi)$ $(j=1,2)$. Moreover, we have

$$q_1(x,D)\circ q_2(x,D) = (q_1(\widetilde{P}_n x,\widetilde{P}_n D)\circ q_2(\widetilde{P}_n x,\widetilde{P}_n D))\otimes \mathrm{id}_{H_-\ominus P_n H_-}.$$

Setting

$$\tilde{p}_\alpha(\tilde{x},\tilde{\xi}) = (-i)^{|\alpha|}\partial^\alpha_{\tilde{\xi}}\tilde{q}_1(\tilde{x},\tilde{\xi})\partial^\alpha_{\tilde{x}}\tilde{q}_2(\tilde{x},\tilde{\xi}) \in S^{m_1+m_2-(\varrho-\delta)|\alpha|,\psi}_{\varrho,\delta}(\mathbb{R}^n)$$

we obtain by Theorem 4.2.9

$$\tilde{q}_1(\widetilde{P}_n x,\widetilde{P}_n D)\circ q_2(\widetilde{P}_n x,\widetilde{P}_n D) - \sum_{|\alpha|<N}\frac{1}{\alpha!}\tilde{p}_\alpha(\widetilde{P}_n x,\widetilde{P}_n D) = \tilde{r}_N(\widetilde{P}_n x,\widetilde{P}_n D),$$

where $\tilde{r}_N \in S^{m_1+m_2-(\varrho-\delta)N,\tilde{\psi}}_{\varrho,\delta}(\mathbb{R}^n)$. Now we define $r_N(x,\xi) := \tilde{r}_N(\widetilde{P}_n x,\widetilde{P}_n\xi)$ and obtain by Proposition 4.3.6 $r_N \in S^{m_1+m_2-(\varrho-\delta)N,\psi}_{\varrho,\delta,cyl}(H_-)$. Obviously, we have $p_\alpha(x,\xi) := \tilde{p}_\alpha(\widetilde{P}_n x,\widetilde{P}_n\xi)$ and

$$q_1(x,D)\circ q_2(x,D) - \sum_{|\alpha|<N}\frac{1}{\alpha!}p_\alpha(x,D) = r_N(x,D).$$

But this proves the first part.
Now let us prove (ii). Again using Proposition 4.3.9 there exist a $n \geq n_0$, $\tilde{q} \in S^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ such that $q(x,\xi) = \tilde{q}(\widetilde{P}_n x,\widetilde{P}_n\xi)$. As before we have

$$\langle q(x,D)u\,,\,v\rangle = \langle u\,,\,([\tilde{q}(\widetilde{P}_n x,\widetilde{P}_n\xi)]^*\otimes \mathrm{id}_{H_-\ominus P_n H_-})v\rangle.$$

Setting

$$\tilde{p}^*_\alpha(\tilde{x},\tilde{\xi}) = (-i)^{|\alpha|}\partial^\alpha_{\tilde{\xi}}\partial^\alpha_{\tilde{x}}\overline{\tilde{q}(\tilde{x},\tilde{\xi})} \in S^{m-(\varrho-\delta)|\alpha|,\psi}_{\varrho,\delta}(\mathbb{R}^n)$$

we obtain by Theorem 4.2.9

$$\tilde{q}(\widetilde{P}_n x, \widetilde{P}_n D)^* - \sum_{|\alpha|<N} \frac{1}{\alpha!}\tilde{p}^*_\alpha(\widetilde{P}_n x, \widetilde{P}_n D) = \tilde{r}_N(\widetilde{P}_n x, \widetilde{P}_n D),$$

where $\tilde{r}_N \in S^{m-(\varrho-\delta)N,\psi}_{\varrho,\delta}(\mathbb{R}^n)$. Now again we set $r(x,\xi) := \tilde{r}(\widetilde{P}_n x, \widetilde{P}_n \xi)$ and obtain by Proposition 2.2.2 $r_N \in S^{m-(\varrho-\delta)N,\psi}_{\varrho,\delta,cyl}(H_-)$. Thus we find

$$q(x,D)^* - \sum_{|\alpha|<N} \frac{1}{\alpha!}p^*_\alpha(x,D) = r_N(x,D).$$

Finally, this shows our Theorem. □

Let us summarize the facts we proved about our pseudodifferential operators with cylindrical symbols in terms of graduated algebras.
Thus so far defining

$$\Psi^{\infty,\psi}_{0,cyl}(H_-) := \bigcup_{m\in\mathbb{R}} \Psi^{m,\psi}_{0,cyl}(H_-), \tag{63}$$

$$\Psi^{\infty,\psi}_{\infty,cyl}(H_-) := \bigcup_{m\in\mathbb{R}} \Psi^{m,\psi}_{\varrho_\infty,cyl}(H_-) \tag{64}$$

and

$$\Psi^{\infty,\psi}_{\varrho,\delta,cyl}(H_-) := \bigcup_{m\in\mathbb{R}} \Psi^{m,\psi}_{\varrho,\delta,cyl}(H_-) \tag{65}$$

we have proved

Theorem 4.3.13. *The sets $\Psi^{0,\psi}_{0,cyl}(H_-)$ and $\Psi^{\infty,\psi}_{0,cyl}(H_-)$ are algebras of pseudodifferential operators with composition as multiplication and involution $*$. Moreover if (60) holds $\Psi^{0,\psi}_{\infty,cyl}(H_-)$, $\Psi^{0,\psi}_{\varrho,\delta,cyl}(H_-)$, $\Psi^{\infty,\psi}_{\infty,cyl}(H_-)$ and $\Psi^{\infty,\psi}_{\varrho,\delta,cyl}(H_-)$ are also algebras of pseudodifferential operators with composition as multiplication and involution $*$. In addition we have*

(i) $\lambda\Psi^{m,\psi}_{0,cyl}(H_-) + \mu\Psi^{m,\psi}_{0,cyl}(H_-) \subset \Psi^{m,\psi}_{0,cyl}(H_-), \quad \lambda,\mu\in\mathbb{C}$;
$\lambda\Psi^{m,\psi}_{\infty,cyl}(H_-) + \mu\Psi^{m,\psi}_{\infty,cyl}(H_-) \subset \Psi^{m,\psi}_{\infty,cyl}(H_-), \quad \lambda,\mu\in\mathbb{C}$;
$\lambda\Psi^{m,\psi}_{\varrho,\delta,cyl}(H_-) + \mu\Psi^{m,\psi}_{\varrho,\delta,cyl}(H_-) \subset \Psi^{m,\psi}_{\varrho,\delta,cyl}(H_-), \quad \lambda,\mu\in\mathbb{C}$.

(ii) *Let $m\geq 0$ resp. $m+m'\geq 0$ or equation (60) hold. Then*
$\left(\Psi^{m,\psi}_{0,cyl}(H_-)\right)^* \subset \Psi^{m,\psi}_{0,cyl}(H_-)$;
$\Psi^{m,\psi}_{0,cyl}(H_-) \circ \Psi^{m',\psi}_{0,cyl}(H_-) \subset \Psi^{m+m',\psi}_{0,cyl}(H_-)$.

(iii) *Let equation (60) hold. Then*
$\left(\Psi^{m,\psi}_{\infty,cyl}(H_-)\right)^* \subset \Psi^{m,\psi}_{\infty,cyl}(H_-)$;
$\left(\Psi^{m,\psi}_{\varrho,\delta,cyl}(H_-)\right)^* \subset \Psi^{m,\psi}_{\varrho,\delta,cyl}(H_-)$;
$\Psi^{m,\psi}_{\infty,cyl}(H_-) \circ \Psi^{m',\psi}_{\infty,cyl}(H_-) \subset \Psi^{m+m',\psi}_{\infty,cyl}(H_-)$;
$\Psi^{m,\psi}_{\varrho,\delta,cyl}(H_-) \circ \Psi^{m',\psi}_{\varrho,\delta,cyl}(H_-) \subset \Psi^{m+m',\psi}_{\varrho,\delta,cyl}(H_-)$.

Now we aim to show, that some of the pseudodifferential operators on a quasi-nuclear Hilbert space rigging extend to continuous linear operators in a scale on Sobolev-Spaces. We will do this in three different cases.

The x independent case.
Before going on with the discussion of cylindrical symbols let us consider symbols depending only a ξ. Let us start with the following

THEOREM 4.3.14. *Let $q \in S^{m,\psi}_{\varrho,\delta}(H_-)$ or $q \in S^{m,\psi}_0(H_-)$ such that $q(x,\xi) = p(\xi)$. Then for all $s \in \mathbb{R}$ the corresponding pseudodifferential operator $p(D)$ is a continuous linear mapping from $H^{s+m}_\psi(H_-)$ to $H^s_\psi(H_-)$.*

PROOF. For $u \in H^{s+m}_\psi$ we obtain

$$\begin{aligned}
\|p(D)u\|_{H^s_\psi} &= \left\|(1+\psi(\cdot))^{s/2}\mathcal{F}p(D)u\right\|_{H^0_\psi} \\
&= \left\|(1+\psi(\cdot))^{s/2}\mathcal{F}\mathcal{F}^{-1}p(\cdot)\mathcal{F}u\right\|_{H^0_\psi} \\
&= \left\|(1+\psi(\cdot))^{s/2}p(\cdot)\mathcal{F}u\right\|_{H^0_\psi} \\
&\leq c\left\|(1+\psi(\cdot))^{s+m/2}\mathcal{F}u\right\|_{H^0_\psi} = c\,\|u\|_{H^{s+m}_\psi}.
\end{aligned}$$

This shows our assertion. □

LEMMA 4.3.15. *Let $q \in S^{m,\psi}_{\varrho,\delta}(H_-)$ or $q \in S^{m,\psi}_0(H_-)$ such that $q(x,\xi) = p(\xi)$. Then we obtain for $u \in S_{\gamma_c yl}(H_-)$*

$$p(D)u = \lim_{n\to\infty} p(P_nD)u,$$

where the convergence takes place in $L^2(H_-,\gamma)$.

PROOF. For $u \in S_{\gamma,cyl}(H_-)$ we have $\mathcal{F}u \in S_{\gamma,cyl}(H_-)$. Moreover, since $P_n\xi \xrightarrow{n\to\infty} \xi$ and p is continuous we obtain $p(P_n\xi)\mathcal{F}u(\xi) \xrightarrow{n\to\infty} p(\xi)\mathcal{F}u(\xi)$ for all $\xi \in H_-$. For $m > 0$ we have

$$p(P_n\xi)\mathcal{F}u(\xi) \leq c(1+\psi(P_n\xi))^{m/2}\mathcal{F}u(\xi) \leq \tilde{c}(1+\psi(\xi))^{m/2}\mathcal{F}u(\xi)$$

and for $m \leq 0$ we find $p(P_n\xi)\mathcal{F}u(\xi) \leq c\,\mathcal{F}u(\xi)$. Thus Lebesgue's Theorem of dominated convergence implies that

$$p \circ P_n\mathcal{F}u \xrightarrow[L^2(H_-,\gamma)]{n\to\infty} p\mathcal{F}u.$$

Now our assertion follows by the continuity of $\mathcal{F}^{-1}$. □

Lemma 4.3.16. *Let* $q \in S^{m,\psi}_{\varrho,\delta}(H_-)$ *or* $q \in S^{m,\psi}_{0}(H_-)$ *such that* $q(x,\xi) = p((\mathrm{id} - P_n)\xi) = \tilde{p}(P_{\infty,n}\xi)$. *Moreover let* $u = f \otimes g$ *where* $f(x) = f(P_n x)$ *and* $g(x) = g((\mathrm{id} - P_n)x)$ *and* $f, g \in S_{\gamma,cyl}(H_-)$. *Then we obtain by Lemma 4.3.15 and Lemma 4.3.3*

$$\begin{aligned} p(D)(f\otimes g) &= \lim_{m\to\infty} p(P_m(D))(f\otimes g) \\ &= \lim_{m\to\infty} f\otimes \tilde{p}(P_m P_{\infty,n} D) g = f \otimes \tilde{p}(P_{\infty,n}D)g. \end{aligned}$$

The case of a cylindrical function with cylindrical growth.
Now let us assume that (60) holds. Then we obtain the following

Proposition 4.3.17. *Let* $q \in S^{m,\psi}_{0,cyl}(H_-)$ *or* $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$. *Then* $q(x,D)$ *maps* $H^{s+m}(H_-)$ *continuously to* $H^s(H_-)$.

Proof. Let $q \in S^{m,\psi}_{0,cyl}(H_-)$ or $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$ and $u \in S_{\gamma,cyl}(H_-)$. Then there exists a $n \geq n_0$ such that $q(x,\xi) = q(P_n x, P_n \xi)$. Now by condition (60) and Proposition 4.3.9 we obtain $(1+\psi\circ P_n)^{-\frac{m}{2}} \in q \in S^{-m,\psi}_{0,cyl}(H_-)$. Thus we have by Theorem 4.3.12 resp. Proposition 4.3.10 $q(x,D)\circ(1+\psi)^{-\frac{m}{2}}(P_nD) \in S^{0,\psi}_{0,cyl}(H_-)$ resp. $q(x,D)\circ(1+\psi)^{-\frac{m}{2}}(P_nD) \in S^{0,\psi}_{\varrho,\delta,cyl}(H_-)$. Now we obtain by Theorem 4.3.5

$$\begin{aligned} \|q(x,D)u\|_0 &= \left\|q(x,D)(1+\psi)^{-\frac{m}{2}}(P_nD)(1+\psi)^{\frac{m}{2}}(P_nD)u\right\|_0 \\ &\leq c\left\|(1+\psi)^{\frac{m}{2}}(P_nD)u\right\|_0 \\ &= c\left\|(1+\psi)^{\frac{m}{2}}(P_n\cdot)\mathcal{F}u\right\|_0 \\ &\leq c'\left\|(1+\psi)^{\frac{m}{2}}(\cdot)\mathcal{F}u\right\|_0 = \|u\|_{\psi,m}, \end{aligned}$$

which proves our proposition. □

This proposition leads directly to the following

Theorem 4.3.18. *Let* $q \in S^{m,\psi}_{0,cyl}(H_-)$ *or* $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$. *Then* $q(x,D)$ *extends to a continuous linear mapping from* $H^{s+m}(H_-)$ *to* $H^s(H_-)$.

Proof. Let $q \in S^{m,\psi}_{0,cyl}(H_-)$ or $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$. Thus we have by Theorem 4.3.12 Proposition resp. 4.3.10 $(1+\psi)^{\frac{s}{2}}(P_nD)\circ q(x,D) \in S^{s+m,\psi}_{0,cyl}(H_-)$ resp. $(1+\psi)^{\frac{s}{2}}(P_nD)\circ q(x,D) \in S^{s+m,\psi}_{\varrho,\delta,cyl}(H_-)$. Now using Proposition 4.3.17 we obtain for $u \in S_{\gamma,cyl}(H_-)$

$$\|q(x,D)u\|_{\psi,s} = \left\|(1+\psi)^{\frac{s}{2}}(P_nD)u\right\|_0 \leq c\,\|u\|_{\psi,s+m},$$

which shows our theorem. □

For the next proposition we have to assume a stronger version of equation (60). Namely we have to assume that the constants $\tilde{c}_n$ are bounded i.e we assume that there exists constants $\tilde{c}$ such that

$$1+\psi(\xi) \leq \tilde{c}(1+\psi(P_n\xi))\ \forall n \geq n_0,\ \forall \xi \in H_-. \tag{66}$$

Proposition 4.3.19 (Gårding inequality). *Let $q \in S^{m,\psi}_{\varrho_k,cyl}(H_-)$ be non-negative. Then there exists a $K > 0$ such that for all $u \in S_{\gamma,cyl}(H_-)$*

$$\mathfrak{Re}\,\langle q(x,D)u\,,\,u\rangle_{L^2(H_-,\gamma)} \geq -K\,\|u\|^2_{\psi,\frac{m-1}{2}}\,.$$

Proof. Let $q \in S^{m,\psi}_{\varrho_k,cyl}(H_-)$ be non-negative and $u \in S_{\gamma,cyl}(H_-)$. Then der exists a $n \geq n_0$ such that $q(x,\xi) = \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi)$ and $u(x) = \tilde{u}(\widetilde{P}_n x)$. Thus we obtain

$$\begin{aligned}
\mathfrak{Re}\,\langle q(x,D)u\,,\,u\rangle_{L^2(H_-,\gamma)} &= \mathfrak{Re}\,\langle \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi)\tilde{u}\circ\widetilde{P}_n\,,\,\tilde{u}\circ\widetilde{P}_n\rangle_{L^2(H_-,\gamma)} \\
&= \mathfrak{Re}\,\langle \tilde{q}(\widetilde{P}_n x, \widetilde{P}_n \xi)\tilde{u}\,,\,\tilde{u}\rangle_{L^2(\mathbb{R}^n,\gamma_n)} \\
&\geq -K\,\|\tilde{u}\|^2_{\tilde{\psi},\frac{m-1}{2}} \\
&\geq -K'\,\|u\|^2_{\psi,\frac{m-1}{2}}
\end{aligned}$$

and so the Gårding inequality is proved. □

The case of a second order polynomial as negative definite function. Now let

$$\psi(\xi) := \langle A\xi\,,\,\xi\rangle, \text{ where } A \in \mathscr{L}(H_-,H_+) \text{ and } \psi(\xi) = \sum_{j=0}^{\infty} a_j \xi_j^2 \tag{67}$$

($\xi_j = \langle e_j\,,\,\xi\rangle_0$). We assume that $a_j \neq 0$ for all $j \in \mathbb{N}$. Moreover, in this second case we consider only the case $\delta = 0$. Note that in this part now $\varrho = 0$ is allowed. Thus we obtain

Lemma 4.3.20. *Under the assumptions above the symbol $q(x,\xi) := i\langle e_j\,,\,\xi\rangle_0$ is an element of $S^{1,\psi}_{\varrho,\delta,cyl}(H_-)$ and thus $D_{e_j} \in \Psi^{1,\psi}_{\varrho,\delta,cyl}(H_-)$. Using Theorem 4.3.14 we obtain that D_{e_j} is a continuous operator from $H^{s+1}_\psi(H_-)$ to $H^s_\psi(H_-)$ for all $s \in \mathbb{R}$.*

Proof. This follows directly from the fact that $a_j \neq 0$ for all $j \in \mathbb{N}$. □

Lemma 4.3.21. *Let $q(x,\xi) \in S^{0,\psi}_{\varrho,\delta,cyl}(H_-)$ such that $q(x,\xi) = q(P_n x, P_n \xi)$ Then we obtain for $u \in S_{\gamma,cyl}(H_-)$*

$$\begin{aligned}
&(1+\psi(D))q(x,D)u \\
=\;& q(x,D)(1+\psi(D))u + \sum_{j=1}^{n} a_j \left(2(\partial_{x_j}q)(x,D)D_{e_j} + ((\partial_{x_j})^2 q)(x,D)\right)u.
\end{aligned}$$

PROOF. Let $q(x,\xi) \in S^{0,\psi}_{\varrho,\delta,cyl}(H_-)$ such that $q(x,\xi) = q(P_n x, P_n \xi)$ and $u \in S_{\gamma,cyl}(H_-)$ Using Proposition 4.3.8 we obtain

$$\begin{aligned} & (D_{e_j})^2 q(x,D)u \\ = \; & (D_{e_j})[q(x,D)(D_{e_j})u + (\partial_{x_j} q)(x,D)u] \\ = \; & q(x,D)(D_{e_j})^2 u + (2\partial_{x_j} q)(x,D) D_{e_j} u + ((\partial_{x_j})^2 q)(x,D)u, \end{aligned}$$

where $\partial_{x_j} q = 0$ and $(\partial_{x_j})^2 q = 0$ for $j > n$. Thus for $k > n$ we obtain

$$\begin{aligned} & (1+\psi(P_k D))q(x,D)u \\ = \; & q(x,D)(1+\psi(P_k D))u + \sum_{j=1}^n a_j \left(2(\partial_{x_j} q)(x,D) D_{e_j} + ((\partial_{x_j})^2 q)(x,D)\right) u. \end{aligned}$$

Hence the assertion follows by Lemma 4.3.15 for $k \longrightarrow \infty$. □

PROPOSITION 4.3.22. *Let $q \in S^{0,\psi}_{\varrho,0,cyl}(H_-)$ such that $q(x,\xi) = q(P_n x, P_n \xi)$. Then for all $\alpha \in \mathbb{N}_0^{\mathbb{N}}$ there exists constants c_α and symbols $q_\alpha \in S^{0,\psi}_{\varrho,0,cyl}(H_-)$ with $q_\alpha(x,\xi) = q_\alpha(P_n x, P_n \xi)$ such that*

$$\mathrm{ad}^m(\Lambda^2)(q(x,D))u = \sum_{\substack{|\alpha| \le m \\ l(\alpha) \le n}} c_\alpha q_\alpha(x,D) D^\alpha u$$

for all $u \in S_{\gamma,cyl}(H_-)$.

PROOF. Let us prove this proposition by induction. For $m = 1$ this follows by Lemma 4.3.21, since $\partial_{x_j} q$ and $(\partial_{x_j})^2 q$ in $S^{0,\psi}_{\varrho,0,cyl}(H_-)$. Let our assertion now be true for a fixed $m \in \mathbb{N}$. Then we obtain

$$\begin{aligned} & \mathrm{ad}^{m+1}(\Lambda^2)(q(x,D))u \\ = \; & [\Lambda^2, \sum_{\substack{|\alpha| \le m \\ l(\alpha) \le n}} c_\alpha q_\alpha(x,D) D^\alpha] u \\ = \; & \sum_{\substack{|\alpha| \le m \\ l(\alpha) \le n}} c_\alpha ([\Lambda^2, \; q(x,D)] D^\alpha u + q(x,D)[\Lambda^2, \; D^\alpha] u) \\ = \; & \sum_{\substack{|\alpha| \le m \\ l(\alpha) \le n}} c_\alpha \sum_{j=1}^n a_j \left(2(\partial_{x_j} q)(x,D) D_{e_j} + ((\partial_{x_j})^2 q)(x,D)\right) D^\alpha u \\ = \; & \sum_{\substack{|\alpha| \le m+1 \\ l(\alpha) \le n}} \tilde{c}_\alpha \hat{q}_\alpha(x,D) D^\alpha u, \end{aligned}$$

which shows our proposition. □

PROPOSITION 4.3.23. *Let $q \in S^{0,\psi}_{\varrho,0,cyl}(H_-)$ such that $q(x,\xi) = q(P_n x, P_n \xi)$. Then for all $u \in H^\infty_\psi(H_-)$ and $k, m \in \mathbb{N}$ there exist $a_{k,m} \geq 0$ such that*

$$\left\|\Lambda^k \mathrm{ad}^m(\Lambda)(q(x,D))u\right\|_{\psi,0} \leq a_{m,j} \left\|\Lambda^k u\right\|_{\psi,0}.$$

PROOF. Let $q \in S^{0,\psi}_{\varrho,0,cyl}(H_-)$ such that $q(x,\xi) = q(P_n x, P_n \xi)$. Then for all $u \in S_{\gamma,cyl}(H_-)$ Proposition 4.3.23 implies that

$$\begin{aligned}
&\left\|\Lambda^{2k} \mathrm{ad}^m(\Lambda^2)(q(x,D))u\right\|_{\psi,0} \\
\leq\ & \sum_{\nu=0}^{m} \binom{k}{\nu} \left\|ad^{m+\nu}(\Lambda^2)(q(x,D))(\Lambda^2)^{k-\nu}u\right\|_{\psi,0} \\
\leq\ & \sum_{\nu=0}^{m} \binom{k}{\nu} \sum_{\substack{|\alpha|\leq m+\nu \\ l(\alpha)\leq n}} c_\alpha \left\|q_\alpha(x,D)D^\alpha(\Lambda^2)^{k-\nu}u\right\|_{\psi,0} \\
\leq\ & \sum_{\nu=0}^{m} \binom{k}{\nu} \sum_{\substack{|\alpha|\leq m+\nu \\ l(\alpha)\leq n}} \tilde{c}_\alpha \left\|(\Lambda^2)^{k-\nu}u\right\|_{\psi,|\alpha|} \leq \left\|\Lambda^{2k+m}u\right\|_{\psi,0}.
\end{aligned}$$

Now by Proposition2.3.17 Theorem 4.3.5 and 4.3.14 it is clear that we can extend this inequality to all $u \in H^\infty_\psi(H_-)$. Thus our proposition follows by [**25**, Proposition 2.3.8]. □

Thus we obtain the following Theorem:

THEOREM 4.3.24. *Let $q \in S^{0,\psi}_{\varrho,0,cyl}(H_-)$ such that $q(x,\xi) = q(P_n x, P_n \xi)$. Then $q(x,D)$ extends to a continuous linear mapping from $H^s_\psi(H_-)$ to $H^s_\psi(H_-)$.*

PROOF. For $s \in \mathbb{N}$ this follows by Proposition 4.3.23. Now note that $(1+\psi(D))^{1/2}$ is selfadjoint and strictly positive. Thus we obtain for $s > 0$ by interpolation $q(x,D) \in \mathscr{L}(H^s_\psi(H_-))$(cf. [**25**, p 61-62 Theorem 1.5.5]). Now let us consider the case $s < 0$. According to Proposition 4.3.10 there exists a symbol $q' \in S^{0,\psi}_{\varrho,0,cyl}(H_-)$ such that $q'(x,\xi) = q(P_n x, P_n \xi)$ and $q'(x,D) = [q(x,D]^*$ Now using the case above we obtain $q'(x,D) \in \mathscr{L}(H^{-s}(H_-))$ and thus

$$q(x,D) = [q'(x,D)]^* \in \mathscr{L}((H^{-s}_\psi)'(H_-)) =\in \mathscr{L}(H^s_\psi(H_-)).$$ □

4.4. Ψ^*-Algebras of pseudodifferential operators in the case of $\mathbb{R}^n$ and the Fredholm property

During this section let $\delta < \varrho$.

LEMMA 4.4.1. *Let $p(x,D) \in \Psi^{0,\psi}_{\varrho,\delta}(\mathbb{R}^n)$. Then we obtain*

$$[\Lambda^\varepsilon,\ p(x,D)] \in \Psi^{0,\psi}_{\varrho,\delta}(\mathbb{R}^n).$$

PROOF. Define $\lambda(\xi) := (1+\psi(\xi))^{1/2}$. Since $\varrho > \delta$ there exists a $N \in \mathbb{N}$ such that $N(\varrho-\delta) > 1$. According to Theorem 4.2.9 the symbol of the commutator $[\Lambda^\varepsilon,\ p(x,D)]$ is given by

$$\sum_{j=1}^{N} i^j \frac{1}{j!} \sum_{|\alpha|\le j} (\partial_\xi^\alpha \lambda^\varepsilon)(\xi)(\partial_x^\alpha q)(x,\xi) + r_{N+1}(x,\xi),$$

where $r_{N+1} \in S_{\varrho,\delta}^{1-(N+1)(\varrho-\delta),\psi}$. Now considering the summands separately we obtain

$$\begin{aligned} & \left|\partial_\xi^\gamma \partial_x^\beta (\partial_\xi^\alpha \lambda^\varepsilon)(\xi)(\partial_x^\alpha q)(x,\xi)\right| \\ = & \left|\sum_{\nu\le\gamma} \binom{\nu}{\gamma} \partial_\xi^\nu \partial_\xi^\alpha \lambda^\varepsilon(\xi) \partial_\xi^{\gamma-\nu} \partial_x^\beta (\partial_x)^\alpha q(x,\xi)\right| \\ \le & \sum_{\nu\le\gamma} \binom{\nu}{\gamma} c_\nu (1+\psi(\xi))^{\frac{\varepsilon-|\nu|-|\alpha|}{2}} (1+\psi(\xi))^{\frac{-(\varrho)|(\gamma-\nu)|+\delta|\alpha+\beta|}{2}} \\ \le & c(1+\psi(\xi))^{\frac{(1-\delta)(1-|\alpha|)-\varrho|\gamma|+\delta|\beta|}{2}} \le c(1+\psi(\xi))^{\frac{-\varrho|\gamma|+\delta|\beta|}{2}}. \end{aligned}$$

Thus our commutator is an element of $\Psi_{\varrho,\delta}^{0,\psi}(\mathbb{R}^n)$. □

Using Theorem 4.2.14 and Lemma 4.4.1 we immediately obtain

COROLLARY 4.4.2. *Let $p(x,D) \in \Psi_{\varrho,\delta}^{0,\psi}(\mathbb{R}^n)$. Then $p(x,D) \in \mathcal{A}^{\psi,\varepsilon}$.*

LEMMA 4.4.3. *For $q \in S_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n)$ we have*

$$[M_j,\ q(x,D)] \in \Psi_{\varrho,\delta}^{m-\varrho,\psi}(\mathbb{R}^n).$$

PROOF. In view of Proposition 4.2.11 we obtain for $q \in S_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n)$

$$\begin{aligned} [M_j,\ q(x,D)] &= M_j q(x,D) - x_j q(x,D) i(\partial_{\xi_j} q)(x,D) \\ &= i(\partial_{\xi_j} q)(x,D) \in \Psi_{\varrho,\delta}^{m-\varrho,\psi}. \end{aligned}$$

But this is our assertion. □

COROLLARY 4.4.4. *Let $q \in S_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n)$. Then it follows that for $\alpha \in \mathbb{N}_0^n$*

$$\mathrm{ad}^\alpha(M)(p(x,D)) \in \Psi_{\varrho,\delta}^{m-|\alpha|\varrho,\psi}(\mathbb{R}^n).$$

LEMMA 4.4.5. *Let $q \in S_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n)$. Then we obtain*

$$[D_j,\ q(x,D)] \in \Psi_{\varrho,\delta}^{m+\delta,\psi}(\mathbb{R}^n).$$

Proof. Let $q \in S^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ and $u \in S_\gamma(\mathbb{R}^n)$. Now using Lebesgue's Theorem of dominate convergence we obtain

$$\begin{aligned} & D_{e_j} q(x,D)u(x) \\ = \ & D_{e_j} \mathcal{F}^{-1}_{\xi\to x} q(x,\xi)(\mathcal{F}u)(\xi) \\ = \ & \left(\frac{\partial}{\partial e_j} - \langle e_j\,,\,x\rangle\right) e^{\frac{\|x\|^2}{2}} \int e^{i\langle x\,,\,\xi\rangle} q(x,\xi)(\mathcal{F}u)(\xi) \\ = \ & e^{\frac{\|x\|^2}{2}} \frac{\partial}{\partial e_j} \int e^{i\langle x\,,\,\xi\rangle} q(x,\xi)(\mathcal{F}u)(\xi)\gamma(d\xi) \\ = \ & e^{\frac{\|x\|^2}{2}} \int e^{i\langle x\,,\,\xi\rangle} \left(i\xi_j q(x,\xi) + (\partial_{x_j} q)(x,\xi)\right)(\mathcal{F}u)(\xi)\gamma(d\xi) \\ = \ & q(x,D)D_{x_j}u(x) + (\partial x_j q)(x,D)u(x). \end{aligned}$$

Thus we obtain $[D_j,\ q(x,D)] \in \Psi^{m+\delta,\psi}_{\varrho,\delta}(\mathbb{R}^n)$. □

Corollary 4.4.6. *Let $q \in S^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$. Then we have*

$$\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta D(p(x,D)) \in \Psi^{m-|\alpha|\varrho+|\beta|\delta,\psi}_{\varrho,\delta}(\mathbb{R}^n).$$

Thus according to Theorem 4.2.14 it follows

$$\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(A) \in \mathscr{L}(H^s_\psi(\mathbb{R}^n), H^{s-m+\varrho|\alpha|-\delta|\beta|}_\psi(\mathbb{R}^n))$$

for all $s \in \mathbb{R}$.

Now we can state the following

Theorem 4.4.7. *Let $\psi \in \Lambda_\infty(\mathbb{R}^n)$ be a negative definite function. For $0 \le \delta < \varrho \le 1$ let $\Psi^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ be defined as in Definition 4.1.5 and $\mathcal{A}^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ as in Definition 4.1.11. Then we have*

$$\Psi^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n) \subseteq \mathcal{A}^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n).$$

Proof. Let $q(x,D) \in \Psi^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$. Since $\Lambda^{-m} \in \Psi^{-m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ we obtain by Theorem 4.2.9 $\Lambda^{-m}q(x,D) \in \Psi^{0,\psi}_{\varrho,\delta}$. Thus according to Corollary 4.4.2 we have $q(x,D) \in \Lambda^m\mathcal{A}^{\varepsilon,\psi}$. Hence the Theorem follows directly by Corollary 4.4.6. □

Lemma 4.4.8. *Let $A \in \mathcal{A}^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$. Then for $u \in S(\mathbb{R}^n)$ we have*

$$V_{G,n}[M_j,\ A]V^{-1}_{G,n}u = [M_j,\ V_{G,n}AV^{-1}_{G,n}]u \tag{68}$$

and

$$V_{G,n}[D_j,\ A]V^{-1}_{G,n}u = [\partial_j,\ V_{G,n}AV^{-1}_{G,n}]u. \tag{69}$$

Thus for all $\alpha,\beta \in \mathbb{N}^n_0$ we find

$$\mathrm{ad}(M)^\alpha\mathrm{ad}(\partial)^\beta(V_{G,n}AV^{-1}_{G,n})u = V_{G,n}\mathrm{ad}(M)^\alpha\mathrm{ad}(D)^\beta(A)V^{-1}_{G,n}. \tag{70}$$

PROOF. For $A \in \mathcal{A}^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ and $u \in S(\mathbb{R}^n)$ we obtain

$$\begin{aligned} V_{G,n}[M_j,\ A]V_{G,n}^{-1}u &= (M_j V_{G,n}AV_{G,n}^{-1} - V_{G,n}AV_{G,n}^{-1}M_j)u \\ &= V_{G,n}(M_jA - AM_j)V_{G,n}^{-1}u = [M_j,\ V_{G,n}AV_{G,n}^{-1}]u \end{aligned}$$

and using the product rule for differentiation

$$\begin{aligned} V_{G,n}[D_j,\ A]V_{G,n}^{-1}u &= (V_{G,n}D_jAV_{G,n}^{-1} - V_{G,n}AD_jV_{G,n}^{-1})u \\ &= (V_{G,n}\partial_jAV_{G,n}^{-1} - x_jV_{G,n}AV_{G,n}^{-1} \\ &\quad -(V_{G,n}A\partial_jV_{G,n}^{-1} - x_jV_{G,n}AV_{G,n}^{-1}))u \\ &= (\partial_jV_{G,n}AV_{G,n}^{-1} - VAV_{G,n}^{-1}\partial_j)u = [\partial_j,\ V_{G,n}AV_{G,n}^{-1}]u. \end{aligned}$$

But this is our assertion. □

THEOREM 4.4.9. *Let $\psi(\xi) = \|\xi\|^2$. For $0 \leq \delta < \varrho \leq 1$ and $\mathcal{A}^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ as in Definition 4.1.11 we have*

$$\Psi^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n) = \mathcal{A}^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n).$$

PROOF. Because of Theorem 4.4.7 we only have to show that $\mathcal{A}^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n) \subset \Psi^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$. Thus for $A \in \mathcal{A}^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ let us consider $V_{G,n}AV_{G,n}^{-1}$. According to Lemma 4.4.8 we have $\mathrm{ad}(M)^\alpha\mathrm{ad}(\partial)^\beta(V_{G,n}AV_{G,n}^{-1})u = V_{G,n}\mathrm{ad}(M)^\alpha\mathrm{ad}(D)^\beta(A)V_{G,n}^{-1}$. Hence in view of Lemma 4.2.13 and Definition 4.1.11 we obtain

$$\begin{aligned} \left\|\mathrm{ad}(M)^\alpha\mathrm{ad}(\partial)^\beta(V_{G,n}AV_{G,n}^{-1})u\right\|_{\psi,s,\lambda} &= \left\|\mathrm{ad}(M)^\alpha\mathrm{ad}(D)^\beta(V_{G,n}^{-1})u\right\|_{\psi,s} \\ &\leq \left\|V_{G,n}^{-1}u\right\|_{\psi,s+m-\varrho|\alpha|+\delta|\beta|} \\ &= \|u\|_{\psi,s+m-\varrho|\alpha|+\delta|\beta|,\lambda}\,. \end{aligned}$$

Now using [**130**, Satz 1.8.c] we find a symbol $q \in S^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ such that $V_{G,n}AV_{G,n}^{-1} = q(x,\tilde{D})$. Thus we have $A = q(x,D)$. □

REMARK 4.4.10. Using the same proof as in Theorem 4.4.9 we obtain for $\psi(\xi) = \|\xi\|^2$ and $0 \leq \delta \leq \varrho \leq 1$ (even in the case $\varrho = \delta$) $\mathcal{A}^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n) \subset \Psi^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$. Furthermore, since we find by Proposition 4.2.1

$$\mathrm{ad}(M)^\alpha\mathrm{ad}(D)^\beta(q(x,D))u = V_{G,n}^{-1}\mathrm{ad}(M)^\alpha\mathrm{ad}(\partial)^\beta(\tilde{q}(x,\tilde{D}))V_{G,n}u$$

for all $u \in S_\gamma(\mathbb{R}^n)$ it follows again by [**130**, Satz 1.8.c] that $q(x,D) \in \Psi^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ satisfies

$$\mathrm{ad}(M)^\alpha\mathrm{ad}(D)^\beta(q(x,D)) \in \bigcap_{s\in\mathbb{R}} \mathscr{L}(H^s_\psi(\mathbb{R}^n), H^{s-m+\varrho|\alpha|-\delta|\beta|}_\psi(\mathbb{R}^n))$$

for all $\alpha,\beta \in \mathbb{N}_0^n$. Thus we find that a continuous operator A from $S_\gamma(\mathbb{R}^n)$ to $S'_\gamma(\mathbb{R}^n)$ is an element of $\Psi^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ if and only if

$$\mathrm{ad}(M)^\alpha\mathrm{ad}(D)^\beta(q(x,D)) \in \bigcap_{s\in\mathbb{R}} \mathscr{L}(H^s_\psi(\mathbb{R}^n), H^{s-m+\varrho|\alpha|-\delta|\beta|}_\psi(\mathbb{R}^n))$$

for all $\alpha, \beta \in \mathbb{N}_0^n$.

In this second part of the section we want to study compact and Fredholm pseudodifferential operators. Let us start with describing all finite dimensional operators in $\Psi_0^{m,\psi}(\mathbb{R}^n)$ and $\Psi_{\varrho,\delta}^{0,\psi}(\mathbb{R}^n)$ following an idea of Gramsch and Kalb [**65**].

PROPOSITION 4.4.11. *Let $q \in S_0^{m,\psi}(\mathbb{R}^n)$ resp. $q \in S_{\varrho,\delta}^{0,\psi}(\mathbb{R}^n)$ such that $q(x,D)$ has finite dimensional range. Then there exist $f_j,\ g_j \in S_\gamma(\mathbb{R}^n)$ $(j = 1,\dots,m)$ such that*

$$q(x,D)u = \sum_{j=1}^{m} \langle u\,,\ g_j\rangle_{L^2(\mathbb{R}^n,\gamma)} f_j.$$

PROOF. Let $(f_j)_{j=1..m}$ be a orthonormal basis of the range of $q(x,D)$. Then we obtain $q(x,D)u = \sum_{j=1}^m c_j(u) f_j$, where the c_j are continuous linear forms on $L^2(\mathbb{R}^n,\gamma)$. By the Riez' representation Theorem we find $0 \neq g_j \in L^2(\mathbb{R}^n,\gamma)$ such that $c_j(u) = \langle u\,,\ g_j\rangle_{L^2(\mathbb{R}^n,\gamma)}$. Now note that in view of equation (37) $q(x,D)$ maps $S_\gamma(\mathbb{R}^n)$ to $S_\gamma(\mathbb{R}^n)$. Since $S_\gamma(\mathbb{R}^n) \subset L^2(\mathbb{R}^n,\gamma)$ dense and $q(x,D)$ is continuous we obtain $q(x,D)(S_\gamma(\mathbb{R}^n)) \subset \mathrm{R}(\mathrm{q(x,D)})$ dense. However, $\mathrm{R}(q(x,D))$ finite dimensional implies $\mathrm{R}(q(x,D)) = q(x,D)(S_\gamma(\mathbb{R}^n)) \subset S_\gamma(\mathbb{R}^n)$. Thus $f_j \in S_\gamma(\mathbb{R}^n)$. Now consider the adjoint operator of $q(x,D)$. By Proposition 4.2.4 resp. Theorem 4.2.9 we obtain $[q(x,D)]^* \in \Psi_0^{m,\psi}(\mathbb{R}^n)$ resp. $[q(x,D)]^* \in \Psi_{\varrho,\delta}^{0,\psi}(\mathbb{R}^n)$. On the other hand for $u, v \in L^2(\mathbb{R}^n,\gamma)$ arbitrary we have

$$\begin{aligned}\langle q(x,D)u\,,\ v\rangle_{L^2(\mathbb{R}^n,\gamma)} &= \sum_{j=1}^{m} \langle u\,,\ g_j\rangle_{L^2(\mathbb{R}^n,\gamma)} \langle f_j\,,\ v\rangle_{L^2(\mathbb{R}^n,\gamma)} \\ &= \langle u\,, \sum_{j=1}^{m} \langle v\,,\ f_j\rangle_{L^2(\mathbb{R}^n,\gamma)} g_j\rangle_{L^2(\mathbb{R}^n,\gamma)},\end{aligned}$$

which implies $[q(x,D)]^* v = \sum_{j=1}^m \langle v\,,\ f_j\rangle_{L^2(\mathbb{R}^n,\gamma)} g_j$. However, as above we obtain $g_j \in S_\gamma(\mathbb{R}^n)$. □

Now we want to consider compact operators and Fredholm operators. Thus let us introduce as in the classical case (cf. [**93**] and [**122**]) the following symbol-classes

DEFINITION 4.4.12. Let $0 \le \delta \le \varrho \le 1,\ \delta < 1$. For $\psi \in \Lambda_\infty(\mathbb{R}^n)$ and $m \in \mathbb{R}$ we call a $\mathscr{C}^\infty$-function $q : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ a symbol in the class

(i) $\dot{S}_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n)$ if for all $\alpha,\ \beta \in \mathbb{N}_0^{\mathbb{N}}$ there exists a bounded function $c_{|\alpha|,|\beta|}(x)$ such that $c_{|\alpha|,|\beta|}(x) \longrightarrow 0$ as $x \longrightarrow \infty$ and

$$\left|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)\right| \le c_{|\alpha|,|\beta|}(x)(1+\psi(\xi))^{\frac{m-\varrho|\alpha|+\delta|\beta|}{2}}; \tag{71}$$

(ii) $\tilde{S}_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n)$ if for all $0 \neq \beta \in \mathbb{N}_0^{\mathbb{N}}$ $\partial_\xi^\beta q(x,\xi) \in \tilde{S}_{\varrho,\delta}^{m,\psi}(\mathbb{R}^n)$.

As in the classical case (cf. [**93**] and [**122**, Lemma 1.2]) we have

$$\dot{S}^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n) \subset \tilde{S}^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n) \subset S^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n). \tag{72}$$

To consider compact operators we need a minimal growth of our negative definite function. Thus we assume that there exists a $0 < r \le 1$ and a constant $c > 0$ such that

$$(1 + \|\xi\|^2)^r \le c(1 + \psi(\xi)) \quad \forall \xi \in \mathbb{R}^n \tag{73}$$

On the other hand in [**80**] it is shown that $1 + \psi(\xi) \le c_\psi(1 + \|\xi\|^2)$. Thus we find

LEMMA 4.4.13. *Let $q(x,\xi) \in S^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ $(\dot{S}^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n), \tilde{S}^{m,\psi}_{\varrho,\delta}(\mathbb{R}^n))$ Then we obtain*

(i) *$q(x,\xi) \in S^{m,\|\cdot\|^2}_{r\varrho,\delta}(\mathbb{R}^n)$ $(\dot{S}^{m,\|\cdot\|^2}_{r\varrho,\delta}(\mathbb{R}^n), \tilde{S}^{m,\|\cdot\|^2}_{r\varrho,\delta}(\mathbb{R}^n))$ if $m \ge 0$ and*

(ii) *$q(x,\xi) \in S^{rm,\|\cdot\|^2}_{r\varrho,\delta}(\mathbb{R}^n)$ $(\dot{S}^{rm,\|\cdot\|^2}_{r\varrho,\delta}(\mathbb{R}^n), \tilde{S}^{rm,\|\cdot\|^2}_{r\varrho,\delta}(\mathbb{R}^n))$ if $m < 0$.*

PROOF. This lemma follows directly by the following two estimates

(i) $(1+\psi(\xi))^{\frac{m-\varrho|\alpha|+\delta|\beta|}{2}} = (1+\|\xi\|^2)^{\frac{m+\delta|\beta|}{2}}(1+\|\xi\|^2)^{\frac{-r\varrho|\alpha|}{2}}$ for $m \ge 0$ and

(ii) $(1+\psi(\xi))^{\frac{m-\varrho|\alpha|+\delta|\beta|}{2}} = (1+\|\xi\|^2)^{\frac{\delta|\beta|}{2}}(1+\|\xi\|^2)^{\frac{rm-r\varrho|\alpha|}{2}}$ for $m < 0$.

□

Thus we obtain by [**93**, Chapter 3, Proposition 5.14]

LEMMA 4.4.14. *Let ψ and r as in equation (73). Moreover, assume $\delta \le r\varrho$. Then for each $q \in \dot{S}^{-\varepsilon,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ $(\varepsilon > 0)$ $q(x,\tilde{D})$ is a compact operator from $L^2(\mathbb{R}^n,\lambda)$ to $L^2(\mathbb{R}^n,\lambda)$.*

PROOF. Let $q \in \dot{S}^{-\varepsilon,\psi}_{\varrho,\delta}(\mathbb{R}^n)$. Then we obtain by Lemma 4.4.13 $q \in \dot{S}^{-r\varepsilon,\|\cdot\|^2}_{r\varrho,\delta}(\mathbb{R}^n)$, where $r\varepsilon > 0$. Thus our lemma follows by [**93**, Chapter 3, Proposition 5.14]. □

DEFINITION 4.4.15. We call a function $q \in S^{0,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ uniformly elliptic, if there are constants $R, C > 0$ such that for all $\|x\| + \|\xi\| > R$, $q(x,\xi)$ is invertible and $|p(x,\xi)^{-1}| \le C$.

Then we obtain using the same argument as in Lemma 4.4.14 by [**122**, Theorem 1.8]

PROPOSITION 4.4.16. *Assume ψ and r as in equation (73) and $\delta \le r\varrho$. Let $q \in \tilde{S}^{0,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ by uniformly elliptic. Then $q(x,\tilde{D})$ is a Fredholm operator in $L^2(\mathbb{R}^n,\lambda)$ and the index is given by Fedosov's- formula* [**42**]

$$\operatorname{ind} q(x,\tilde{D}) = -(-2\pi i)^{-n}\frac{(n-1)!}{(2n-1)!}\int_{\partial B} \operatorname{Tr}(q^{-1}dq)^{2n-1}, \tag{74}$$

where B is an open ball in $\mathbb{R}^{2n}$ such That $q(x,\xi)^{-1}$ exists and is bounded outside B. In addition $\mathbb{R}^{2n}$ is oriented by $dx_1 \wedge d\xi_1 \wedge \cdots \wedge dx_n \wedge d\xi_n > 0$.

REMARK 4.4.17. The trace in equation 74 is not necessary, since we consider only scalar valued symbols and no systems. But for historical reasons we will leave the trace in Fedosov's formula, since this form of the formula is very well known.

Since we want to deal with the case of a Gaussian measure let us state the following

LEMMA 4.4.18. *Let $A \in Hom(L^2(\mathbb{R}^n,\gamma))$ and $\tilde{A} \in Hom(L^2(\mathbb{R}^n,\gamma))$ such that*

$$A = V_{G,n}^{-1}\tilde{A}V_{G,n}.$$

Then we obtain

(i) $A \in \mathscr{L}(H^s_\psi(\mathbb{R}^n)) \Longleftrightarrow \tilde{A} \in \mathscr{L}(H^s_{\psi,\lambda}(\mathbb{R}^n))$
(ii) $\mathrm{N}(A) = V_{G,n}^{-1}\mathrm{N}(\tilde{A})$
(iii) $\mathrm{R}(A) = V_{G,n}^{-1}\mathrm{R}(\tilde{A})$
(iv) *A is compact in $L^2(\mathbb{R}^n,\gamma)$ if and only if $\tilde{A}$ is compact in $L^2(\mathbb{R}^n,\lambda)$*

PROOF. (i) Let $u \in H^s_{\psi,\lambda}(\mathbb{R}^n)$ and $A \in \mathscr{L}(H^s_\psi(\mathbb{R}^n))$. Then we have

$$\left\|\tilde{A}u\right\|_{\psi,\lambda,s} = \left\|V_{G,n}^{-1}\tilde{A}V_{G,n}V_{G,n}^{-1}u\right\|_{\psi,s} = \left\|AV_{G,n}^{-1}u\right\|_{\psi,s} \le c\left\|V_{G,n}^{-1}u\right\|_{\psi,s} = c\left\|u\right\|_{\psi,s,\lambda},$$

and conversely for $u \in H^s_{\psi,\gamma}(\mathbb{R}^n)$ and $A \in \mathscr{L}(H^s_{\psi,\lambda}(\mathbb{R}^n))$ we obtain

$$\left\|Au\right\|_{\psi,s} = \left\|V_{G,n}^{-1}\tilde{A}V_{G,n}u\right\|_{\psi,s} = \left\|\tilde{A}V_{G,n}u\right\|_{\psi,s,\lambda} \le c'\left\|V_{G,n}u\right\|_{\psi,s,\lambda} = c'\left\|u\right\|_{\psi,s}.$$

(ii) Let $u \in \mathrm{N(A)}$. Then we obtain $0 = Au = V_{G,n}^{-1}\tilde{A}V_{G,n}u$. Thus since $V_{G,n}^{-1}$ is invertible we find $\tilde{A}V_{G,n}u = 0$, which implies $u \in V_{G,n}^{-1}\mathrm{N}(\tilde{A})$. Conversely, let $u \in V_{G,n}^{-1}\mathrm{N}(\tilde{A})$. Then we obtain $\tilde{A}V_{G,n}u = 0$ and thus it follows that $0 = V_{G,n}^{-1}\tilde{A}V_{G,n}u = Au$.

(iii) Let $v \in \mathrm{R(A)}$. Then there exists a $u \in L^2(H_-,\gamma)$ such that $v = Au$. Now we obtain $V_{G,n}v = V_{G,n}V_{G,n}^{-1}\tilde{A}V_{G,n}u = \tilde{A}V_{G,n}u$, which shows $v \in V_{G,n}^{-1}\mathrm{R}(\tilde{A})$. Conversely, let $v \in V_{G,n}^{-1}\mathrm{R}(\tilde{A})$. Then there exists a $u \in L^2(H_-,\lambda)$ such that $V_{G,n}v = \tilde{A}u = V_{G,n}AV_{G,n}^{-1}u$. Thus we have $v = AV_{G,n}^{-1}u$.

(iv) Let A be compact and u_n a bounded sequence in $L^2(\mathbb{R}^n,\lambda)$. Then $V_{G,n}u_n$ is a bounded sequence in $L^2(\mathbb{R}^n,\gamma)$. Since A is compact there exists a subsequence $V_{G,n}u_{n_k}$ such that $AV_{G,n}u_{n_k}$ converges in $L^2(\mathbb{R}^n,\gamma)$. Thus $\tilde{A}u_{n_k} = V_{G,n}^{-1}AV_{G,n}u_{n_k}$ converges in $L^2(\mathbb{R}^n,\lambda)$. Conversely, let $\tilde{A}$ be compact and u_n a bounded sequence in $L^2(\mathbb{R}^n,\gamma)$. Then $V_{G,n}^{-1}u_n$ is a bounded sequence in $L^2(\mathbb{R}^n,\lambda)$. Since $\tilde{A}$ is compact there exists a subsequence $V_{G,n}^{-1}u_{n_k}$ such that $\tilde{A}V_{G,n}^{-1}u_{n_k}$ converges in $L^2(\mathbb{R}^n,\lambda)$. Thus $Au_{n_k} = V_{G,n}AV_{G,n}^{-1}u_{n_k}$ converges in $L^2(\mathbb{R}^n,\gamma)$.

□

This Lemma (4.4.18) together with Lemma 4.4.14 and Proposition 4.4.16 yield now for the Gaussian measure on $\mathbb{R}^n$

THEOREM 4.4.19. *Let ψ and r as in equation (73). Moreover, assume $\delta \leq r\varrho$. Then for each $q \in \dot{S}^{-\varepsilon,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ ($\varepsilon > 0$) $q(x, \tilde{D})$ is a compact operator from $L^2(\mathbb{R}^n, \gamma)$ to $L^2(\mathbb{R}^n, \gamma)$.*

THEOREM 4.4.20. *Assume ψ and r as in equation (73) and $\delta \leq r\varrho$. Let $q \in \tilde{S}^{0,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ by uniformly elliptic. Then $q(x, D)$ is a Fredholm operator in $L^2(\mathbb{R}^n, \gamma)$ and the index is given by Fedosov's- formula* [**42**]

$$\text{ind } q(x, D) = -(-2\pi i)^{-n} \frac{(n-1)!}{(2n-1)!} \int\limits_{\partial B} \text{Tr}(q^{-1}dq)^{2n-1},$$

where B is an open ball in $\mathbb{R}^{2n}$ such hat $q(x, \xi)^{-1}$ exists and is bounded outside B. In addition $\mathbb{R}^{2n}$ is oriented by $dx_1 \wedge d\xi_1 \wedge \cdots \wedge dx_n \wedge d\xi_n > 0$.

In view of Lemma 4.4.18 and Theorem [**122**, Theorem 1.8] we obtain

THEOREM 4.4.21. *Assume $\psi(\xi) = \|\xi\|^2$ and let $q \in \tilde{S}^{0,\psi}_{\varrho,\delta}(\mathbb{R}^n)$. Then $q(x, D)$ is a Fredholm operator in $L^2(\mathbb{R}^n, \gamma)$ if and only if $q(x, \xi)$ is uniformly elliptic.*

Finally, we show that every operator $q(x, D)$ with uniformly elliptic symbol $q \in \tilde{S}^{0,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ is a Fredholm operator in all $H^s_\psi(\mathbb{R}^n)$. Thus let us state the following proposition which can be found in [**56**, remark 5.7] and [**96**, 2.1.7 Proposition].

PROPOSITION 4.4.22. *Let H be a Hilbert space, $\mathcal{A}$ be a Ψ^*-algebra in $\mathscr{L}(H)$ and $a \in \mathcal{A}$ with closes range $\text{R}(a) \subset H$.*

(i) *If $p = p^2 = p^* \in \mathscr{L}(H)$ is the orthogonal projection onto $\text{N}(a) = \text{N}(a^*a)$, then one has $p \in \mathcal{A}$.*
(ii) *There exists a $b \in \mathcal{A}$ namely $b := (p + a^*a)^{-1}a^* \in \mathcal{A}$ such that*
   - *$p_1 := \text{id}_H - ba$ is the orthogonal projection onto $\text{N}(a)$*
   - *$p_2 := \text{id}_H - ab$ is the orthogonal projection onto $\text{R}(a)^\perp$*
   - *$aba = a$ and $bab = b$, i.e. b is a relative inverse of a*

   Note that in particular $\text{R}(a)$ is closed if $a : H \longrightarrow H$ is a Fredholm operator. In that case b is a Fredholm inverse of a.

In view of this lemma we can prove

THEOREM 4.4.23. *Assume ψ and r as in equation (73) and $\delta \leq r\varrho$. Let $q \in \tilde{S}^{0,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ by uniformly elliptic. Then $q(x, D)$ is a Fredholm operator in $\mathscr{L}(H^s_\psi(\mathbb{R}^n))$ for all $s \in \mathbb{R}$.*

PROOF. Since $q \in \tilde{S}^{0,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ we obtain $q(x, D) \in \mathcal{A}^{0,\psi}_{\varrho,\delta}(\mathbb{R}^n)$, which is a Ψ^*-algebra. Thus there exist by 4.4.22 $b, p_1, p_2 \in \mathcal{A}^{0,\psi}_{\varrho,\delta}(\mathbb{R}^n)$ such that p has finite dimensional range and $bq(x, D) = \text{id} + p_1$ and $q(x, D)b = \text{id} + p_2$. Thus b is inverse of $q(x,D)$ modulo finite dimensional operators in $H^s_\psi(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ and $q(x,D)$ is Fredholm in $\mathscr{L}(H^s_\psi(\mathbb{R}^n))$. □

4.5. Operators in Ψ^*-algebras of pseudodifferential operators in the case of the canonical Gaussian measure on quasi-nuclear Hilbert space riggings

Now we will show that our Ψ^*-Algebras defined above contain lots of our pseudodifferential operators we considered in this paper until now. To do this let us start with the following two lemmas.

Lemma 4.5.1. *For $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$ we have*

$$[M_j,\ q(x,D)] \in \Psi^{m-\varrho,\psi}_{\varrho,\delta,cyl}(H_-).$$

Proof. In view of Proposition 4.3.7 we obtain for $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$

$$\begin{aligned}[M_j,\ q(x,D)] &= M_j q(x,D) - x_j q(x,D) + i(\partial_{\xi_j} q)(x,D) \\ &= i(\partial_{\xi_j} q)(x,D) \in \Psi^{m-\varrho,\psi}_{\varrho,\delta,cyl}(H_-).\end{aligned}$$

But this is our assertion. □

Lemma 4.5.2. *Let $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$. Then we obtain*

$$[D_j,\ q(x,D)] \in \Psi^{m+\delta,\psi}_{\varrho,\delta,cyl}(H_-).$$

Proof. According to Proposition 4.3.8 we have for $u \in S_{\gamma,cyl}(H_-)$

$$[D_j,\ q(x,D)]u(x) = (\partial_{x_j} q)(x,D)u(x)$$

and thus obtain $[D_j,\ q(x,D)] \in \Psi^{m+\delta,\psi}_{\varrho,\delta,cyl}(H_-)$. □

At first let us show that these generalized Hörmander classes contain some Fourier-multipliers. Thus let us note first that we have for $q \in S^{m,\psi}_{\varrho,\delta}(H_-)$ such that $q(x,\xi) = p(\xi)$

$$\mathrm{ad}^m(\Lambda^\varepsilon)(p(D)) = 0 \quad \forall m \in \mathbb{N} \tag{75}$$

and

$$\mathrm{ad}^\beta(D)(p(D)) = 0 \quad \forall\, |\beta| \geq 1 \tag{76}$$

Lemma 4.5.3. *Let $q \in S^{m,\psi}_{\varrho,\delta}(H_-)$ such that $q(x,\xi) = p(\xi)$. Then we have for all $u \in S_{\gamma,cyl}(H_-)$*

$$ad^\alpha(M)(p(D))u = i^{|\alpha|}(\partial^\alpha p)(D)u,$$

where $(\partial^\alpha p)(D) \in \Psi^{m-|\alpha|\varrho,\psi}_{\varrho,\delta}(H_-)$.

Proof. For $n \in \mathbb{N}$ and $n > j$ we obtain by Lemma 4.5.1 $[M_j,\ p(P_nD)] = i(\partial_j p)(P_nD)$. But now Lemma 4.3.15 implies that $[M_j,\ p(D)] = i(\partial_j p)(D)$. Thus our assertion follows by induction. □

Combining the results above and Theorem 4.3.14 we obtain

THEOREM 4.5.4. *Let $q \in S^{m,\psi}_{\varrho,\delta}(H_-)$ such that $q(x,\xi) = p(\xi)$. Then*

$$p(D) \in \mathcal{A}^{\psi,m}_{\varrho,\delta}(H_-).$$

PROPOSITION 4.5.5. *Let $q \in S^{0,\psi}_{\varrho,\delta}(H_-)$ such that $q(x,\xi) = p(\xi)$. Moreover, we assume that there exists a constant $c > 0$ such that $p(\xi) > c$. Then $p(D)$ is invertible in $L^2(H_-,\gamma)$ and thus in $\mathcal{A}^{\psi,0}_{\varrho,\delta}(H_-)$ since $\mathcal{A}^{\psi,0}_{\varrho,\delta}(H_-)$ is a Ψ^*-algebra.*

PROOF. Our condition implies the $p(\xi)^{-1}$ is bounded. Thus we obtain that $p^{-1}(D) = \mathcal{F}^{-1}\frac{1}{p}\mathcal{F}$ is a bounded operator in $L^2(H_-,\gamma)$. In addition it is clear that $[p(D)]^{-1} = p^{-1}(D)$. □

At next let us consider the finite dimensional operators contained in this classes. Following an idea of Gramsch and Kalb [**65**] we obtain

PROPOSITION 4.5.6. *Let $a \in \mathcal{A}^{0,\psi}_{\varrho,\delta}(H_-)$ such that a has finite dimensional range. Then there exist $f_j,\, g_j \in H^\infty_\psi(H_-)$ $(j = 1,\dots,m)$ such that*

$$au = \sum_{j=1}^{m} \langle u\,,\, g_j\rangle_{L^2(H_-,\gamma)} f_j. \tag{77}$$

PROOF. Let $(f_j)_{j=1..m}$ be a orthonormal basis of the range of a. Then we obtain $a = \sum_{j=1}^m c_j(u) f_j$, where the c_j are continuous linear forms on $L^2(H_-,\gamma)$. By the Riez' representation Theorem we find $0 \neq g_j \in L^2(H_-,\gamma)$ such that $c_j(u) = \langle u\,,\, g_j\rangle_{L^2(H_-,\gamma)}$. Now note that a maps $H^\infty_\psi(H_-)$ to $H^\infty_\psi(H_-)$. Since $H^\infty_\psi(H_-) \subset L^2(H_-,\gamma)$ dense and a is continuous we obtain $a(H^\infty_\psi(H_-)) \subset \mathrm{R(a)}$ dense. However, $\mathrm{R}(a)$ finite dimensional implies $\mathrm{R}(a) = a(H^\infty_\psi(H_-)) \subset H^\infty_\psi(H_-)$. Thus $f_j \in H^\infty_\psi(H_-)$. Now consider the adjoint operator of a. By Theorem 4.1.13 we obtain $a^* \in \mathcal{A}^{m,\psi}_{\varrho,\delta}(H_-)$. On the other hand for $u, v \in L^2(H_-,\gamma)$ arbitrary we have

$$\begin{aligned}
\langle au\,,\, v\rangle_{L^2(H_-,\gamma)} &= \sum_{j=1}^{m} \langle u\,,\, g_j\rangle_{L^2(H_-,\gamma)} \langle f_j\,,\, v\rangle_{L^2(\mathbb{R}^n,\gamma)} \\
&= \langle u\,,\, \sum_{j=1}^{m} \langle v\,,\, f_j\rangle_{L^2(H_-,\gamma)} g_j\rangle_{L^2(H_-,\gamma)},
\end{aligned}$$

which implies $a^*v = \sum_{j=1}^m \langle v\,,\, f_j\rangle_{L^2(\mathbb{R}^n,\gamma)} g_j$. However, as above we obtain $g_j \in H^\infty_\psi(H_-)$. □

Since D_{e_j} and M_j are not necessarily continuous operators from $H^s_\psi(H_-)$ to $H^{s+m}_\psi(H_-)$ for some m and all s we are not able to prove that every operator of the form (77) is contained in $\mathcal{A}^{0,\psi}_{\varrho,\delta}(H_-)$. But if we choose f_j and g_j in $S_{\gamma,cyl}(H_-)$ we obtain the even stronger result

PROPOSITION 4.5.7. *Let $f_j,\ g_j \in S_{\gamma,cyl}(H_-)$ $(j = 1,\ldots,m)$ then the operator a defined by*

$$au = \sum_{j=1}^{k} \langle u\,,\, g_j\rangle_{L^2(H_-,\gamma)} f_j \quad (k \in \mathbb{N}) \tag{78}$$

is an element of $\mathcal{A}^{m,\psi}_{\varrho,\delta}(H_-)$.

PROOF. Let $f,\ g \in S_{\gamma,cyl}(H_-)$ and $au := \langle u\,,\, g\rangle_{L^2(H_-,\gamma)} f$.

(i) Then we find

$$\begin{aligned}
&\left\|[D_{e_j},\ a]u\right\|_{H^s_\psi}\\
\le\ & \left|\langle u\,,\, g\rangle_{L^2(H_-\gamma)}\right| \left\|D_{e_j} f\right\|_{s,\psi} + \left|\langle D_{e_j} u\,,\, g\rangle_{L^2(H_-\gamma)}\right| \|f\|_{H,\psi}\\
\le\ & \left|\langle u\,,\, g\rangle_{L^2(H_-\gamma)}\right| \left\|D_{e_j} f\right\|_{s,\psi} + \left|\langle u\,,\, D_{e_j} g\rangle_{L^2(H_-\gamma)}\right| \|f\|_{s,\psi}\\
\le\ & \|u\|_{s+m,\psi} \|g\|_{-s-m,\psi} \left\|D_{e_j} f\right\|_{s,\psi} + \|u\|_{s+m,\psi} \left\|D_{e_j} g\right\|_{-s-m,\psi} \|f\|_{s,\psi}\\
\le\ & c\, \|u\|_{s+m,\psi}
\end{aligned}$$

(ii) and

$$\begin{aligned}
&\left\|[M_j,\ a]u\right\|_{H^s_\psi}\\
\le\ & \left|\langle u\,,\, g\rangle_{L^2(H_-\gamma)}\right| \|M_j f\|_{s,\psi} + \left|\langle M_j u\,,\, g\rangle_{L^2(H_-\gamma)}\right| \|f\|_{H,\psi}\\
\le\ & \left|\langle u\,,\, g\rangle_{L^2(H_-\gamma)}\right| \|M_j f\|_{s,\psi} + \left|\langle u\,,\, M_j g\rangle_{L^2(H_-\gamma)}\right| \|f\|_{s,\psi}\\
\le\ & \|u\|_{s+m,\psi} \|g\|_{-s-m,\psi} \|M_j f\|_{s,\psi} + \|u\|_{s+m,\psi} \|M_j g\|_{-s-m,\psi} \|f\|_{s,\psi}\\
\le\ & c'\, \|u\|_{s+m,\psi}
\end{aligned}$$

(iii) and finally

$$\begin{aligned}
&\left\|[\psi(D),\ a]u\right\|_{H^s_\psi}\\
\le\ & \left|\langle u\,,\, g\rangle_{L^2(H_-\gamma)}\right| \|\psi(D) f\|_{s,\psi} + \left|\langle \psi(D) u\,,\, g\rangle_{L^2(H_-\gamma)}\right| \|f\|_{H,\psi}\\
\le\ & \left|\langle u\,,\, g\rangle_{L^2(H_-\gamma)}\right| \|\psi(D) f\|_{s,\psi} + \left|\langle u\,,\, \psi(D) g\rangle_{L^2(H_-\gamma)}\right| \|f\|_{s,\psi}\\
\le\ & \|u\|_{s+m,\psi} \|g\|_{-s-m,\psi} \|\psi(D) f\|_{s,\psi} + \|u\|_{s+m,\psi} \|\psi(D) g\|_{-s-m,\psi} \|f\|_{s,\psi}\\
\le\ & c''\, \|u\|_{s+m,\psi}\,.
\end{aligned}$$

Now note that D_{e_j} and M_j leave $S_{\gamma,cyl}$ invariant and $\psi(D)$ maps H^∞_ψ to H^∞_ψ. Thus our proposition follows by an easy induction and the linearity of the sum. □

Now we show that some of the pseudodifferential operators with cylindrical symbol on our quasi-nuclear Hilbert space riggings are contained in the generalizes Hörmander classes and Ψ^*-Algebras defined above . During this section let $\psi \in \Lambda_\infty(H_-)$ be a fixed negative definite function which fulfills the equations (58) (59). In addition let $0 \le \delta \le \varrho \le 1$ and set $\varepsilon := 1 - \delta$. Moreover we set $\Lambda := (1 + \psi(D))^{1/2}$.

The case of a cylindrical negative definite function.
At first let us now assume that ψ is a cylindrical negative definite function. Now let $\delta < \varrho$.

LEMMA 4.5.8. *Let $p(x,D) \in \Psi^{0,\psi}_{\varrho,\delta,cyl}(H_-)$. Then we obtain*

$$[\Lambda^\varepsilon,\ p(x,D)] \in \Psi^{0,\psi}_{\varrho,\delta,cyl}(H_-).$$

PROOF. Define $\lambda(\xi) := (1+\psi(\xi))^{1/2}$. Since $\varrho > \delta$ there exists a $N \in \mathbb{N}$ such that $N(\varrho-\delta) > 1$. According to Theorem 4.3.12 the symbol of the commutator $[\Lambda^\varepsilon,\ p(x,D)]$ is given by

$$\sum_{j=1}^{N} i^j \frac{1}{j!} \sum_{|\alpha|\le j} (\partial_\xi^\alpha \lambda^\varepsilon)(\xi)(\partial_x^\alpha q)(x,\xi) + r_{N+1}(x,\xi),$$

where $r_{N+1} \in S^{1-(N+1)(\varrho-\delta),\psi}_{\varrho,\delta,cyl}(H_-)$. Now considering the summands separately we obtain

$$\begin{aligned}
&\left|\partial_\xi^\gamma \partial_x^\beta (\partial_\xi^\alpha \lambda^\varepsilon)(\xi)(\partial_x^\alpha q)(x,\xi)\right| \\
= &\left|\sum_{\nu\le\gamma} \binom{\nu}{\gamma} \partial_\xi^\nu \partial_\xi^\alpha \lambda^\varepsilon(\xi) \partial_\xi^{\gamma-\nu} \partial_x^\beta (\partial_x)^\alpha q(x,\xi)\right| \\
\le &\sum_{\nu\le\gamma} \binom{\nu}{\gamma} c_\nu (1+\psi(\xi))^{\frac{\varepsilon-|\nu|-|\alpha|}{2}} (1+\psi(\xi))^{\frac{-(\varrho)|(\gamma-\nu)|+\delta|\alpha+\beta|}{2}} \\
\le &\ c(1+\psi(\xi))^{\frac{(1-\delta)(1-|\alpha|)-\varrho|\gamma|+\delta|\beta|}{2}} \le c(1+\psi(\xi))^{\frac{-\varrho|\gamma|+\delta|\beta|}{2}}.
\end{aligned}$$

Thus our commutator is an element of $\Psi^{0,\psi}_{\varrho,\delta,cyl}(H_-)$. □

Using Theorem 4.3.18 and Lemma 4.5.8 we immediately obtain

COROLLARY 4.5.9. *Let $p(x,D) \in \Psi^{0,\psi}_{\varrho,\delta,cyl}(H_-)$. Then $p(x,D) \in \mathcal{A}^{\psi,\varepsilon}$.*

Using Lemma 4.5.1 we obtain

COROLLARY 4.5.10. *Let $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$. Then it follows that for $\alpha \in \mathbb{N}_0^n$*

$$\mathrm{ad}^\alpha(M)(p(x,D)) \in \Psi^{m-|\alpha|\varrho,\psi}_{\varrho,\delta,cyl}(H_-).$$

In view of Lemma 4.5.2 we have

COROLLARY 4.5.11. *For $q \in S^{m,\psi}_{\varrho,\delta,cyl}(H_-)$ we have*

$$\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(p(x,D)) \in \Psi^{m-|\alpha|\varrho+|\beta|\delta,\psi}_{\varrho,\delta,cyl}(H_-).$$

Thus according to Theorem 4.3.18 it follows

$$\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(q(x,D)) \in \mathscr{L}(H_\psi^s(H_-), H_\psi^{s-m+\varrho|\alpha|-\delta|\beta|}(H_-))$$

for all $s \in \mathbb{R}$.

Now we can state the following

THEOREM 4.5.12. *Let $\psi \in \Lambda_\infty(H_-)$ be a cylindrical negative definite function. For $0 \leq \delta < \varrho \leq 1$ let $\Psi^{m,\psi}_{\varrho,\delta,cyl}(H_-)$ be defined as in Definition 4.1.5 and $\mathcal{A}^{m,\psi}_{\varrho,\delta}(H_-)$ as in Definition 4.1.12. Then we have*

$$\Psi^{m,\psi}_{\varrho,\delta,cyl}(H_-) \subseteq \mathcal{A}^{m,\psi}_{\varrho,\delta}(H_-).$$

PROOF. Let $q(x,D) \in \Psi^{m,\psi}_{\varrho,\delta,cyl}(H_-)$. Since $\Lambda^{-m} \in \Psi^{-m,\psi}_{\varrho,\delta,cyl}(H_-)$ we obtain by Theorem 4.3.12 $\Lambda^{-m}q(x,D) \in \Psi^{0,\psi}_{\varrho,\delta,cyl}$. Thus according to Corollary 4.5.9 we have $q(x,D) \in \Lambda^m \mathcal{A}^{\varepsilon,\psi}$. Hence the Theorem follows directly by Corollary 4.5.11. □

THEOREM 4.5.13. *Let us denote by $\widehat{\Psi}^{\psi,0}_{\varrho,\delta}(H_-)$ the closed algebraic span in $\mathcal{A}^{\psi,0}_{\varrho,\delta}(H_-)$ of $\Psi^{\psi,0,cyl}_{\varrho,\delta}(H_-)$, the set of all operators $q(x,D) \in \Psi^{\psi,0}_{\varrho,\delta}(H_-)$ such that $q(x,\xi) = p(\xi)$ and the set of all finite dimensional operators given by (78). Then $\widehat{\Psi}^{\psi,0}_{\varrho,\delta}(H_-)$ is a sub multiplicative Ψ^*-algebra in $\mathscr{L}(H^0)$. Furthermore,*

$$\widehat{\Psi}^{\psi,0}_{\varrho,\delta}(H_-) \times H^\infty_\psi(H_-) \longrightarrow H^\infty_\psi(H_-) \;:\; (A,\varphi) \longmapsto A(\varphi)$$

is continuous and bilinear.

The case of a second order polynomial as negative definite function. As a direct consequence of proposition 4.3.24 we obtain

THEOREM 4.5.14. *Let $q \in S^{0,\psi}_{\varrho,0,cyl}(H_-)$. Then $q(x,D) \in \mathcal{A}^{\psi,\varepsilon}$ for all $0 < \varepsilon \leq 1$, i.e*

$$\Psi^{0,\psi}_{\varrho,0,cyl}(H_-) \subset \mathcal{A}^{\psi,\varepsilon}.$$

Now taking into account Lemma 4.5.1 and Lemma 4.5.2 we obtain for $q \in S^{0,\psi}_{\varrho,0,cyl}(H_-)$ and $u \in S_{\gamma,cyl}(H_-)$ the following equation

$$\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)(q(x,D))u = (-i)^{|\alpha|}(\partial^\alpha_\xi \partial^\beta_x q)(x,D)u, \tag{79}$$

where $(\partial^\alpha_\xi \partial^\beta_x q) \in S^{0-\varrho|\alpha|,\psi}_{\varrho,0,cyl}(H_-) \subset S^{0|\alpha|,\psi}_{\varrho,0,cyl}(H_-)$. Thus combining Theorem 4.5.14 and Theorem 4.3.24 we obtain

THEOREM 4.5.15. *Let $\psi(\xi) = \langle A\xi\,,\,\xi\rangle$ such that ψ fulfills the assumptions above. Then we have*

$$\Psi^{0,\psi}_{0,0,cyl}(H_-) \subset \mathcal{A}^{\psi,m}_{0,0}(H_-).$$

THEOREM 4.5.16. *Let us denote by $\widehat{\Psi}^{\psi,0}_{0,0}(H_-)$ the closed algebraic span in $\mathcal{A}^{\psi,0}_{0,0}(H_-)$ of $\Psi^{\psi,0}_{0,0,cyl}(H_-)$, the set of all operators $q(x,D) \in \Psi^{\psi,0}_{0,0}(H_-)$ such that $q(x,\xi) = p(\xi)$ and the set of all finite dimensional operators given by (78). Then $\widehat{\Psi}^{\psi,0}_{0,0}(H_-)$ is a sub multiplicative Ψ^*-algebra in $\mathscr{L}(H^0)$. Furthermore,*

$$\widehat{\Psi}^{\psi,0}_{0,0}(H_-) \times H^\infty_\psi(H_-) \longrightarrow H^\infty_\psi(H_-) \;:\; (A,\varphi) \longmapsto A(\varphi)$$

is continuous and bilinear.

The case of negative definite functions which fulfill (58) and (59). Let ψ be a negative definite function.

PROPOSITION 4.5.17. *Let ψ be a negative definite function which fulfills (58) and (59). In addition let $q \in S^{0,\psi}_{0,cyl}(H_-)$. Then we obtain*

$$\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)q(x,D) \in \mathscr{L}(H^0(H_-)).$$

PROOF. According to Proposition 4.3.7 and 4.3.8 we have

$$\mathrm{ad}^\alpha(M)\mathrm{ad}^\beta(D)q(x,D)u = i^{|\alpha|}(\partial^\alpha_\xi\partial^\beta_x q)(x,D)u,$$

where $\partial^\alpha_\xi\partial^\beta_x q \in S^{0,\psi}_0(H_-)$. Thus our proposition follows by Theorem 4.3.5. □

COROLLARY 4.5.18. *Let ψ be a negative definite function which fulfills (58) and (59). Then we have*

$$\Psi^{0,\psi}_{0,cyl}(H_-) \subset \Psi^{MD}(H_-),$$

where, $\Psi^{MD}(H_-)$ is defined as in Theorem 3.1.12

As in Theorem 4.5.4 we have for $q \in S^{0,\psi}_0(H_-)$ such that $q(x,\xi) = p(\xi)$ $q(D) \in \Psi^{MD}_\psi(H_-)$. Thus we obtain

THEOREM 4.5.19. *Let us denote by $\widehat{\Psi}^{MD}_\psi(H_-)$ the closed algebraic span in $\mathscr{L}(H^0)$ of $\Psi^{\psi,0}_{0,cyl}(H_-)$, the set of all operators $q(x,D) \in \Psi^{\psi,0}_0(H_-)$ such that $q(x,\xi) = p(\xi)$ and the set of all finite dimensional operators given by (78). Then $\widehat{\Psi}^{MD}_\psi(H_-)$ is a sub multiplicative Ψ^*-algebra in $\mathscr{L}(H^0)$.*

Fredholm operators in $\widehat{\Psi}^{\psi,0}_{\varrho,\delta}(H_-)$ and $\widehat{\Psi}^{MD}_\psi(H_-)$.

PROPOSITION 4.5.20. *Let $q(x,\xi) \in S^{0,\psi}_{\varrho,\delta,cyl}(H_-)$ resp. $q(x,\xi) \in S^{0,\psi}_{0,cyl}(H_-)$ such that $q(x,\xi) = q(P_n x, P_n\xi)$. Let us denote by $\tilde{q}$ the function defined on $\mathbb{R}^{2n}$ by $\tilde{q}(\widetilde{P}_n x \widetilde{P}_n\xi) = q(x,\xi)$. Then according to 4.3.5 we have $q(x,D) \in \mathscr{L}(L^2(H_-,\gamma))$. Assume $\tilde{q}(x,\tilde{D})$ is invertible in $L^2(\mathbb{R}^n,\gamma_n)$ Then $q(x,D)$ is invertible in $\mathscr{L}(L^2(H_-,\gamma))$ and thus in $\widehat{\Psi}^{\psi,0}_{\varrho,\delta}(H_-)$ resp. $\widehat{\Psi}^{MD}_\psi(H_-)$.*

PROOF. Let $q(x,\xi) \in S^{0,\psi}_{\varrho,\delta,cyl}(H_-)$ resp. $q(x,\xi) \in S^{0,\psi}_{0,cyl}(H_-)$ such that $q(x,\xi) = q(P_n x, P_n\xi)$ and assume that $\tilde{q}(x,\tilde{D})$ is invertible in $L^2(\mathbb{R}^n,\gamma_n)$. Then we obtain $a := [\tilde{q}(x,D)]^{-1} \otimes \mathrm{id}_{H_-\ominus P_n H_-}$ is in $\mathscr{L}(L^2(H_-,\gamma))$, In addition for $u = f \otimes g \in L^2(\mathbb{R}^n,\gamma_n) \otimes L^2(H_- \ominus P_n H_-, \gamma_R)$ we have

$$\begin{aligned} q(x,D)au &= (\tilde{q}(x,D) \otimes \mathrm{id}_{H_-\ominus P_n H_-})([\tilde{q}(x,D)]^{-1} \otimes \mathrm{id}_{H_-\ominus P_n H_-})(f \otimes g) \\ &= \tilde{q}(x,D)[\tilde{q}(x,D)]^{-1} f \otimes g = u \end{aligned}$$

and similarly $aq(x,D)u = u$. □

In view of Proposition 4.4.22 we can prove

THEOREM 4.5.21. *Let $q, p \in S^{0,\psi}_{\varrho,\delta}(H_-)$ resp. $q, p \in S^{0,\psi}_0(H_-)$ such that q and p are cylindrical or depend only on ξ. Moreover, let a be in the closure in our Ψ^*-Algebra of the set of all operators c of the form $cu := \sum_{j=1}^n \langle u\,,\, g_j\rangle f_j$, where $f_j, g_j \in S_{\gamma,cyl}(H_-)$ and thus $a : L^2(H_-,\gamma) \longrightarrow L^2(H_-,\gamma)$ is compact. Let us assume that $q(x,D)$ is invertible in $L^2(H_-,\gamma)$ and that $\|p(x,D)\|_{\mathscr{L}(H^0)} < 1/\|q(x,D)\|^{-1}_{\mathscr{L}(H^0)}$. Now we define*

$$A := q(x,D) + p(x,D) + a.$$

Then A is Fredholm in $\mathscr{L}(H^s(H_-))$ for all $s \in \mathbb{R}$.

PROOF. Let A, q, p, a be defined as in our assertion. By 4.5.7 we obtain that a is contained in our Ψ^*-Algebra. Moreover, it is clear that a is compact in $L^2(H_-,\gamma)$ Since $\|p(x,D)\|_{\mathscr{L}(H^0)} < \|q(x,D)\|_{\mathscr{L}(H^0)}$ we obtain that $q(x,D) + p(x,D)$ is invertible in $L^2(H_-,\gamma)$ Let b denote this inverse. Then we have $Ab = \mathrm{id} + ab$ and $bA = \mathrm{id} + ba$. Thus A is invertible modulo compact operators in $L^2(H_-,\gamma)$ and thus Fredholm. Now since we have $A \in \widehat{\psi}^{0,\psi}_{\varrho,\delta}(H_-)$ (resp $\widehat{\Psi}^{MD}_\psi(H_-)$), which is a Ψ^*-Algebra there exist by 4.4.22 $d, p_1, p_2 \in \widehat{\psi}^{0,\psi}_{\varrho,\delta}(H_-)$ (resp $\widehat{\Psi}^{MD}_\psi(H_-)$) such that p_1 and p_2 have finite dimensional range and $dA = \mathrm{id} + p_1$ and $Ad = \mathrm{id} + p_2$. Thus d is inverse of A modulo finite dimensional operators in $H^s_\psi(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ and A is Fredholm in $\mathscr{L}(H^s_\psi(H_-))$. □

Now let us state a result on hypoellipticity which is due to Gramsch, Kalb [**65**].

THEOREM 4.5.22. *Let H be a Hilbert space, $D \subseteq H$ a dense subspace, and $\mathcal{A} \subseteq \mathscr{L}(H)$ a Ψ^*-Algebra in $\mathcal{L}(H)$. Assume that $a(D) \subseteq D$ holds for all $a \in \mathcal{A}$ and let $a \in \mathcal{A}$ be with closed range and finite dimensional kernel. Furthermore, let $\xi \in H$ be arbitrary. Then we have the following form of abstract hypoellipticity*

$$a\xi \in D \Longrightarrow \xi \in D.$$

PROOF. See [**97**, Theorem 2.11]. □

Since $H^\infty_\psi(H_-) \subset L^2(H_-,\gamma)$ dense it follows

COROLLARY 4.5.23. *Let $q \in S^{0,\psi}_{\varrho,\delta}(H_-)$ such that $q(x,D)$ is Fredholm. Then we have he following form of abstract hypoellipticity*

$$q(x,D)\xi \in H^\infty_\psi(H_-) \Longrightarrow \xi \in H^\infty_\psi(H_-).$$

Finally in this chapter, we state

THEOREM 4.5.24. *Let $\psi \in \Lambda_\infty(H_-)$ such that*

$$D_j \in \mathcal{A}^{m,\psi}_{\varrho,\delta}(H_-) \text{ for some } m > 0. \tag{80}$$

In addition, let $A \in \mathcal{A}^{0,\psi}_{\varrho,\delta}(H_-)$ such that A is Fredholm and $f \in L^2(H_-,\gamma)$ such that $D^m_j f \in L^2(H_-,\gamma)$ and

$$Au = f.$$

Then we have $D_j^m u \in L^2(H_-,\gamma)$.

PROOF. Since A is Fredholm there exists a $\tilde{A} \in \mathcal{A}^{0,\psi}_{\varrho,\delta}(H_-)$ such that $\tilde{A}A = \mathrm{id} - P$, where P has finite dimensional range. According to 4.5.6 we have $Pu \in H^\infty_\psi(H_-)$ and thus by (80) $D_j^m Pu \in L^2(H_-,\gamma)$. Now in view of Lemma 3.3.5 and the definition of $A \in \mathcal{A}^{0,\psi}_{\varrho,\delta}(H_-)$ we find that $D_j^m \tilde{A} f \in L^2(H_-,\gamma)$. Thus we obtain

$$D_j^m u = D_j^m(\tilde{A}Au + Pu) = D_j^m \tilde{A} f + D_j^m Pu \in L^2(H_-,\gamma). \qquad \square$$

At the end of this chapter let us describe how we can attach a symbol to an operator $A \in \mathscr{L}(L^2(H_-,\gamma))$ using a total family. Moreover, we will show how to get back the operator as 'pseudodifferential' operator using a special total family.

REMARK 4.5.25. Let $A \in \mathscr{L}(L^2(\mathbb{R}^n,\gamma))$. Using the total family $\{e_\xi = e^{i\langle \cdot\,,\xi\rangle} : \xi \in \mathbb{R}^n\}$ we define the formal e_ξ-symbol by

$$a(x,\xi) := e^{-i\langle x\,,\xi\rangle} A_{y\to x} e^{i\langle y\,,\xi\rangle}. \tag{81}$$

Then we obtain by Lebesgue's Theorem of dominated convergence that $a(x,\cdot) \in \mathscr{C}^\infty(\mathbb{R}^n)$ for fixed x, since all polynomials are integrable with respect to the canonical Gaussian measure. Moreover, if A maps $\mathscr{C}^\infty(\mathbb{R}^n)$ to $\mathscr{C}^k(\mathbb{R}^n)$ we find that $a(x,\xi) \in \mathscr{C}^k(\mathbb{R}^n \times \mathbb{R}^n)$. However, it is clear that $a \in (L^2(\mathbb{R}^n \times \mathbb{R}^n, \gamma\otimes\gamma))$ for fixed ξ. For such a e_ξ-symbol A we obtain

$$\begin{aligned}
& a(x,D)f(x) \\
= \;& \mathcal{F}^{-1}_{\xi\to x} a(x,\xi)(\mathcal{F}f)(\xi) \\
= \;& V^{-1}_{G,n}(\widetilde{\mathcal{F}}^{-1}_{\xi\to x} a(x,\xi)(\widetilde{\mathcal{F}}V_{G,n}f)(\xi))(x) \\
= \;& (\frac{1}{2\pi})^n V^{-1}_{G,n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x\,,\xi\rangle} a(x,\xi) e^{-i\langle z\,,\xi\rangle}(V_{G,n}f)(z)\lambda^n(dz)\lambda^n(d\xi) \\
= \;& (\frac{1}{2\pi})^n V^{-1}_{G,n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A_{y\to x} e^{i\langle y\,,\xi\rangle} e^{-i\langle z\,,\xi\rangle}(V_{G,n}f)(z)\lambda^n(dz)\lambda^n(d\xi) \\
= \;& V^{-1}_{G,n} A_{y\to x}\left[(\frac{1}{2\pi})^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle y\,,\xi\rangle} e^{-i\langle z\,,\xi\rangle}(V_{G,n}f)(z)\lambda^n(dz)\lambda^n(d\xi)\right] \\
= \;& V^{-1}_{G,n} A(V_{G,n}f)(x)
\end{aligned}$$

In this way it is possible to get the symbol back from our pseudodifferential operator in the finite dimensional case by the formula

$$V_{G,n} a(x,D)f = A(V_{G,n}f), \tag{82}$$

where $a(x,\xi)$ is the e_ξ-Symbol of A. Now let us come back to the infinite dimensional case. As shown in 1.1.12 the family $\{e_\xi = e^{i\langle \cdot\,,\xi\rangle} : \xi \in H_+\}$ is total in

$L^2(H_-, \gamma)$. Thus we define the e_ξ-symbol of an operator $A \in \mathscr{L}^2(H_-, \gamma)$ by

$$a(x,\xi) := e^{-i\langle x\,,\,\xi\rangle} A_{y\to x} e^{i\langle y\,,\,\xi\rangle} \qquad \forall x \in H_-,\ \xi \in H_+.$$

Let us now consider some special A, namely let us assume that $A = B \otimes \mathrm{id}$, where $B \in \mathscr{L}(L^2(P_n(H_-), \gamma_n))$ for some n. However, in this case we find

$$\begin{aligned} a(x,\xi) &= e^{-i\langle x\,,\,\xi\rangle} A_{y\to x} e^{i\langle y\,,\,\xi\rangle} \\ &= e^{-i\langle P_n x\,,\,\xi\rangle} e^{-i\langle (\mathrm{id}-P_n)x\,,\,\xi\rangle} A_{y\to x} e^{i\langle P_n y\,,\,\xi\rangle} e^{i\langle (\mathrm{id}-P_n)y\,,\,\xi\rangle} \\ &= e^{-i\langle P_n x\,,\,\xi\rangle} B_{y\to x} e^{i\langle P_n y\,,\,\xi\rangle} \otimes 1 \\ &= e^{-i\langle P_n x\,,\,P_n\xi\rangle} B_{y\to x} e^{i\langle P_n y\,,\,P_n\xi\rangle} \otimes 1. \end{aligned}$$

Thus for such A we can set

$$\tilde{a}(x,\xi) := \lim_{m\longrightarrow\infty} a(P_m x, P_m \xi) = a(P_n x, P_n \xi) \quad \text{ for all } x, \xi \in H_-.$$

Now using the same calculation as in the finite dimensional case we obtain by 4.3.3 for $\tilde{a}$

$$V_{G,n}\tilde{a}(x,D)f = BV_{G,n} \otimes id = AV_{G,n}f,$$

where $u = f \otimes g$ cylindrical and $f(x) = f(P_n)$ and $g(P_n x) = g(0)$. Hence in this way we are able to get our operator back from the e_ξ-symbol.

CHAPTER 5

Representations of infinite dimensional Heisenberg Groups with applications to pseudodifferential operators

Let H be a Hilbert Space with inner product $\langle \cdot\,,\,\cdot\rangle$. Then the Heisenberg group $\mathcal{H}$ is defined by $\mathcal{H} := H \times H \times \mathbb{R}$ and the group law is given as in the finite dimensional case. If $H = \mathbb{R}^n$ it is well known that the Haar measure on $\mathcal{H}$ is given by the Lebesgue measure on $\mathbb{R}^{2n+1}$. Moreover, in this finite dimensional case the irreducible representations of the Heisenberg group are well known and studied for example by Taylor [**129**] and Folland [**43**]. They use some representations of the Heisenberg Group to examine pseudodifferential operators in Weyl form. In this chapter we will do the same, but in the infinite dimensional case. But in the infinite dimensional case the classical construction of the Haar measure on $\mathcal{H}$ does not work.

In this chapter we consider a quasi-nuclear Hilbert space rigging $H_+ \subset H_0 \subset H_-$ and denote by $\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_-$ the corresponding rigging of Heisenberg groups. In this case we obtain a continuous bilinear map $\mathcal{H}_+ \times \mathcal{H}_- \longrightarrow \mathcal{H}_-$ given by

$$(r,s,t) \odot (r',s',t') = (r+r', s+s', t+t' + \frac{1}{2}\langle r\,,\, s'\rangle_0 - \frac{1}{2}\langle s\,,\, r'\rangle_0).$$

We will denote this map by $\odot$ again. For $(r,s,t) \in \mathcal{H}_+$ let us define

$$\pi(r,s,t):\; L^2(H_-,\ \gamma) \longrightarrow L^2(H_-,\ \gamma)$$

by

$$\pi(r,s,t)f(x) := \sqrt{\varrho_r(x)}e^{i(t+\langle s\,,\,x\rangle_0 + \frac{1}{2}\langle r\,,\,s\rangle_0)}f(x+r).$$

Then π is a strongly continuous unitary representation of H_+ in $L^2(H_-,\gamma)$. In addition we show, that these representation is irreducible. Furthermore, defining $\pi_{\pm\lambda}(r,s,t) := \pi(\sqrt{\lambda}r, \pm\sqrt{\lambda}s, \pm\lambda t)$ we show that no two different representations $\pi_{\pm\lambda}$ are unitary equivalent.

Let us denote by L_j, D_j and T the generators of the the semigroups $\pi(0,\tau e_j,0)$ $\pi(\tau e_j,0,0)$ and $\pi(0,0,\tau)$ where $e_j \subset H_+$ is an orthonormal basis of H_0. Then we obtain the Heisenberg commutation relations

$$[L_j,\ M_j] = -[M_j,\ L_j] = T$$

and

$$[L_j,\ M_i] = [L_j,\ T] = [M_j,\ T] = 0.$$

for $i \neq j$.

In the last two sections of this chapter we examine the connection between the representations $\pi_{\pm\lambda}$ of the Heisenberg group $\mathcal{H}_+$ and pseudodifferential operators in Weyl-Form on $L^2(H_-,\gamma)$. At first we consider the finite dimensional case. We determine the space $\mathscr{C}^\infty(\pi)$. In addition we show for $k \in L^1(\mathcal{H}_n, \lambda^{2n+1})$

$$\pi_{\pm\lambda}(k) = \tilde{k}(\pm\lambda, \pm\sqrt{\lambda}X, \sqrt{\lambda}D),$$

where

$$\tilde{k}(\tau, y, \eta) = (2\pi)^{-\frac{2n+1}{2}} \int k(r,s,t) e^{i(t\tau + \langle s\,,y\rangle) + \langle r\,,\eta\rangle} \lambda(dt)\lambda^n(ds)\lambda^n(dr)$$

and $\tilde{k}(\pm\lambda, \pm\sqrt{\lambda}X, \sqrt{\lambda}D)$ is the pseudodifferential operator in Weyl form, cf. Definition 3.2.2. Having the equations above we are able to define $\pi_{\pm\lambda}(P)$ for some functions P even in the infinite dimensional case. Considering the well known Ornstein-Uhlenbeck operator we find that in the finite dimensional case the symbol of this operator is given by $\sigma(x,\xi) = \sum_{j=1}^n \frac{x_j + \xi_j^2 - 1}{2}$ and describe perturbations for which $L_\gamma + q(X,D)$ is still essential selfadjoint. We use the representation π to calculate the spectrum of some pseudodifferential operators in the infinite dimensional case. Finally, we reach Ψ^*-algebras given by smooth elements with respect to the map

$$(r,s,t) \longmapsto \pi(r,s,t)A\pi(r,s,t)^{-1},$$

where A is an operator in a subalgebra of $\mathscr{L}(L^2(H_-,\gamma))$. Moreover, we are able to construct spectrally invariant generalized Hörmander classes given by smooth elements.

5.1. The infinite dimensional Heisenberg Group

Definition 5.1.1. Let H separable Hilbert space. Denote $\mathcal{H} := H \times H \times \mathbb{R}$. On $\mathcal{H}$ we define the multiplication $\odot$ by

$$(r,s,t) \odot (r',s',t') = (r + r', s + s', t + t' + \frac{1}{2}\langle r\,,\, s'\rangle - \frac{1}{2}\langle r'\,,\, s\rangle).$$

We call $\mathcal{H}$ the Heisenberg Group with respect to the Hilbert space H.

Lemma 5.1.2. *$(\mathcal{H},\odot)$ is a topological group with neutral element $(0,0,0)$ and we have*

$$(r,s,t)^{-1} = (-r,-s,-t).$$

Proof. Obviously $\mathcal{H}$ is closed under the multiplication $\odot$. Now let us proof that $\odot$ is associative:

$$\begin{aligned} &\big((r,s,t) \odot (r',s',t')\big) \odot (\tilde{r},\tilde{s},\tilde{t}) \\ = \;& (r + r', s + s', t + t' + \frac{1}{2}\langle r\,,\, s'\rangle_0 - \frac{1}{2}\langle r'\,,\, s\rangle_0)) \odot (\tilde{r},\tilde{s},\tilde{t}) \\ = \;& (r + r' + \tilde{r}, s + s' + \tilde{s}, t + t' + \tilde{t} + \frac{1}{2}\langle r\,,\, s'\rangle_0 - \frac{1}{2}\langle r'\,,\, s\rangle_0) + \frac{1}{2}\langle r + r'\,,\, \tilde{s}\rangle_0 \end{aligned}$$

$$-\frac{1}{2}\langle\tilde{r}\,,\,s+s'\rangle_0))$$

$$= (r+r'+\tilde{r}, s+s'+\tilde{s}, t+t'+\tilde{t}+\frac{1}{2}(\langle r\,,\,s'\rangle_0 - \langle r'\,,\,s\rangle_0 + \langle r\,,\,\tilde{s}\rangle_0 + \langle r'\,,\,\tilde{s}\rangle_0$$
$$-\langle\tilde{r}\,,\,s\rangle_0 - \langle\tilde{r}\,,\,s\rangle_0))$$
$$= (r+r'+\tilde{r}, s+s'+\tilde{s}, t+t'+\tilde{t}+\frac{1}{2}\langle r'\,,\,\tilde{s}\rangle_0 - \frac{1}{2}\langle\tilde{r}\,,\,s'\rangle_0 + \frac{1}{2}\langle r\,,\,s'+\tilde{s}\rangle_0$$
$$-\frac{1}{2}\langle r'+\tilde{r}\,,\,s\rangle_0)$$
$$= (r,s,t)\odot(r'+\tilde{r}, s'+\tilde{s}, t'+\tilde{t}+\frac{1}{2}\langle r'\,,\,\tilde{s}\rangle_0 - \frac{1}{2}\langle\tilde{r}\,,\,s'\rangle_0)$$
$$= (r,s,t)\odot\big((r',s',t')\odot(\tilde{r},\tilde{s},\tilde{t})\big).$$

An easy calculation shows that $(r,s,t)^{-1} = (-r,-s,-t)$ and that $(0,0,0)$ is the neutral element. Moreover, it is clear that the operation $\odot$ and the inversion is continuous with respect to the topology on $H\times H\times\mathbb{R}$. □

REMARK 5.1.3. (i) Occasionally one uses for the Heisenberg Group the group-law

$$(r,s,t)\widetilde{\odot}(r',s',t') = (r+r', s+s', t+t'+\langle r\,,\,s'\rangle)$$

for $(r,s,t),\ (r',s',t')\in H\times H\times\mathbb{R}$. In this case we will call $\mathcal{H}^{pol} = (H\times\mathcal{H}\times\mathbb{R},\widetilde{\odot})$ the polarized Heisenberg group. The polarized Heisenberg group is a topological group with neutral element $(0,0,0)$ and inverse $(r,s,t)^{\widetilde{\odot}-1} = (-r,-s,-t\langle r\,,\,s\rangle)$.

(ii) The map

$$(r,s,t)\longmapsto(r,s,t+\frac{1}{2}\langle r\,,\,s\rangle)$$

is an isomorphism from $\mathcal{H}$ to $\mathcal{H}^{pol}$.

PROOF. This remark can be proved as in the final dimensional case by an easy calculation. □

DEFINITION 5.1.4. Let $H_+\subset H_0\subset H_-$ be a quasi-nuclear Hilbert space rigging. We endow $\mathcal{H}_+$ with the topology induced by $\|(r,s,t)\|_\pm := \sqrt{\|r\|_\pm^2+\|s\|_\pm^2+|t|^2}$. In addition we call

$$\mathcal{H}_+\subset\mathcal{H}_0\subset\mathcal{H}_-$$

the quasi-nuclear Heisenberg group-rigging and $\mathcal{H}_+^{pol}\subset\mathcal{H}_0^{pol}\subset\mathcal{H}_-^{pol}$ the quasi-nuclear polarized Heisenberg group-rigging the with respect to the quasi-nuclear Hilbert space rigging $H_+\subset H_0\subset H_-$. It is clear that the embeddings $\mathcal{H}_+\hookrightarrow\mathcal{H}_0\hookrightarrow\mathcal{H}_-$ are again quasi-nuclear and dense.

LEMMA 5.1.5. *Let $H_+\subset H_0\subset H_-$ be a quasi-nuclear Hilbert space rigging.*

(i) *$(\mathcal{H}_+,\odot)$ is a subgroup of $(\mathcal{H}_0,\odot)$ and of $(\mathcal{H}_-,\odot)$.*

(ii) *$(\mathcal{H}_+^{pol}, \widetilde{\odot})$ is a subgroup of $(\mathcal{H}_0^{pol}, \widetilde{\odot})$ and of $(\mathcal{H}_-^{pol}, \widetilde{\odot})$.*

(iii) *Moreover, we obtain a continuous map*

$$\begin{aligned} \mathcal{H}_+ \times \mathcal{H}_- &\longrightarrow \mathcal{H}_- : \\ ((r,s,t),(r',s',t')) &\longmapsto (r+r', s+s', t+t' + \frac{1}{2}\langle r\,,\, s'\rangle_0 - \frac{1}{2}\langle s\,,\, r'\rangle_0). \end{aligned}$$

We will denote this map again by $\odot$.

(iv) *In addition, we obtain a continuous map*

$$\begin{aligned} \mathcal{H}_+^{pol} \times \mathcal{H}_-^{pol} &\longrightarrow \mathcal{H}_-^{pol} : \\ ((r,s,t),(r',s',t')) &\longmapsto (r+r', s+s', t+t' + \langle r\,,\, s'\rangle). \end{aligned}$$

We will denote this map again by $\widetilde{\odot}$.

Proof. The continuity of the map defined above is clear, since the topology in H_+ is stronger then the topology in H_-. □

From now on let $H_+ \subset H_0 \subset H_-$ be a quasi-nuclear Hilbert space rigging.

Proposition 5.1.6. *Let $(x', y', t') \in \mathcal{H}_+$ and $f(x,y,t)\varrho_{-x'}(x)\varrho_{-y}(y) \in L^1(\mathcal{H}_-, \mu)$. Then $f((x',y',t') \odot (x,y,t)) \in L^1(\mathcal{H}_-, \mu)$ and we obtain*

$$\int_{H_-} f((x',y',t') \odot (x,y,t)) d\mu(x,y,t) = \int_{H_-} f(x,y,t)\varrho_{-x'}(x)\varrho_{-y'}(y) d\mu(x,y,t).$$

Proof. For $f \geq 0$ we have

$$\begin{aligned} &\int_{\mathcal{H}_-} f((x',y',t') \odot (x,y,t)) d\mu(x,y,t) \\ &= \int_{\mathcal{H}_-} f(x+x', y+y' + t + t' + \frac{1}{2}\langle r\,,\, s'\rangle_0 - \frac{1}{2}\langle s\,,\, r'\rangle_0) d\mu(x,y,t) \\ &= \int_{H_-}\int_{H_-}\int_{\mathbb{R}} f(x+x', y+y', t+t' + \frac{1}{2}\langle r\,,\, s'\rangle_0 - \frac{1}{2}\langle s\,,\, r'\rangle_0) d\lambda(t) d\gamma(y) d\gamma(x) \\ &= \int_{H_-}\int_{H_-}\int_{\mathbb{R}} f(x+x', y+y', t) d\lambda(t) d\gamma(y) d\gamma(x) \\ &= \int_{\mathbb{R}}\int_{H_-}\int_{H_-} f(x+x', y+y', t) d\gamma(y) d\gamma(x) d\lambda(t) \\ &= \int_{\mathbb{R}}\int_{H_-}\int_{H_-} f(x,y,t)\varrho_{-x'}(x)\varrho_{-y'}(y) d\gamma(y) d\gamma(x) d\lambda(t) \end{aligned}$$

$$= \int_{H_-} f(x,y,t)\varrho_{-x'}(x)\varrho_{-y'}(y)d\mu(x,y,t)$$

using Tonelli's theorem. Now the proposition follows by Fubini's theorem. □

REMARK 5.1.7. Using the same arguments as above we obtain that for $(x',y',t') \in \mathcal{H}_+^{pol}$ and $f(x,y,t)\varrho_{-x'}(x)\varrho_{-y}(y) \in L^1(\mathcal{H}_-^{pol},\mu) f((x',y',t')\widetilde{\odot}(x,y,t)) \in L^1(\mathcal{H}_-^{pol},\mu)$. In addition we find again

$$\int_{H_-} f((x',y',t')\widetilde{\odot}(x,y,t))d\mu(x,y,t) = \int_{H_-} f(x,y,t)\varrho_{-x'}(x)\varrho_{-y'}(y)d\mu(x,y,t).$$

5.2. Unitary representations

In the section we will construct some unitary representations of $\mathcal{H}_+$ in $L^2(H_-,\gamma)$ and $L^2(\mathcal{H}_-,\mu)$.

DEFINITION 5.2.1. For $(r,s,\tau) \in \mathcal{H}_+$ we define

$$\kappa(r,s,\tau): \ L^2(\mathcal{H}_-,\ \mu) \longrightarrow L^2(\mathcal{H}_-,\ \mu)$$

by

$$\kappa(r,s,\tau)f(x,y,t) := \sqrt{\varrho_r(x)}\sqrt{\varrho_s(y)}f((r,s,\tau)\odot(x,y,t))$$

LEMMA 5.2.2. *$\kappa(r,s,\tau)$ $((r,s,\tau)\in\mathcal{H}_+)$ is a unitary representation of $\mathcal{H}_+$ in $L^2(\mathcal{H}_-,\mu)$.*

PROOF. At first let us show that $\kappa(r,s,\tau)$ is norm-preserving. Thus let $(r,s,\tau) \in \mathcal{H}_+$ and $f \in L^2(\mathcal{H}_-,\mu)$. Then we obtain

$$\begin{aligned} & \|\kappa(r,s,\tau)f\|^2_{L^2(H_-,\mu)} \\ = & \int_{\mathcal{H}_-} \varrho_r(x)\varrho_s(y)\left|f((r,s,\tau)\odot(x,y,t))\right|^2 d\mu(x,y,t) \\ \overset{5.1.6}{=} & \int_{\mathcal{H}_-} \varrho_r(x-r)\varrho_s(y-s)\left|f(x,y,t)\right|^2 \varrho_{-r}(x)\varrho_{-s}(y)d\mu(x,y,t) \\ = & \int_{\mathcal{H}_-} \left|f(x,y,t)\right|^2 d\mu(x,y,t) = \|f\|^2_{L^2(H_-,\mu)}. \end{aligned}$$

In addition for $f, g \in L^2(\mathcal{H}_-, \mu)$ we obtain

$$\begin{aligned}
&\langle \kappa(r,s,\tau) f \,,\, g\rangle \\
= & \int\limits_{H_-} \sqrt{\varrho_r(x)}\sqrt{\varrho_s(y)} f((r,s,\tau)\odot(x,y,t))\overline{g(x,y,t)} d\mu(x,y,t) \\
= & \int\limits_{H_-} \sqrt{\varrho_r(x-r)}\sqrt{\varrho_s(y-s)} f(x,y,t)\overline{g((r,s,\tau)^{-1}\odot(x,y,t))} \\
& \qquad\qquad \varrho_{-r}(x)\varrho_{-s}(y) d\mu(x,y,t) \\
= & \int\limits_{H_-} f(x,y,t)\overline{\sqrt{\varrho_{-r}(x)}\sqrt{\varrho_{-s}(y)} g((r,s,\tau)^{-1}\odot(x,y,t))} d\mu(x,y,t) \\
= & \langle f \,,\, \kappa((r,s,\tau)^{-1}) g\rangle.
\end{aligned}$$

Now it is clear that $\kappa^*(r,s,\tau)\kappa(r,s,\tau) = \mathrm{id}$ and $\kappa(r,s,\tau)\kappa^*(r,s,\tau) = \mathrm{id}$. For (r,s,τ) and (r',s',τ') in H_+ and $f \in L^2(\mathcal{H}_-, \mu)$ we obtain

$$\begin{aligned}
& \kappa(r,s,\tau)\kappa(r',s',\tau') f(x,y,z) \\
= & \sqrt{\varrho_r(x)}\sqrt{\varrho_s(y)}\sqrt{\varrho_r(x+r)}\sqrt{\varrho_s(y+s)} f((r,s,\tau)\odot(r',s',\tau')\odot(x,y,t)) \\
= & \sqrt{\varrho_{r+r'}(x)}\sqrt{\varrho_{s+s'}(y)} f((r,s,\tau)\odot(r',s',\tau')\odot(x,y,t)) \\
= & \kappa((r,s,\tau)\odot(r',s',\tau')) f(x,y,t),
\end{aligned}$$

which shows our assertion. □

PROPOSITION 5.2.3. $\kappa(r,s,\tau)$ $((r,s,\tau) \in \mathcal{H}_+)$ *is a strongly continuously family of unitary operators.*

THEOREM 5.2.4. $\kappa(r,s,\tau)$ $((r,s,\tau) \in \mathcal{H}_+)$ *is a strongly continuous unitary representation of* $\mathcal{H}_+$ *in* $L^2(\mathcal{H}_-, \mu)$.

PROOF. Let $f = f_1 \otimes f_2 \otimes f_3$ where $f_1,\ f_2 \in C_b(H_-)$ and $f_3 \in C_c(\mathbb{R})$. Moreover, we assume that f_1 and f_2 have bounded support, i.e. $\operatorname{supp} f_1 \cup \operatorname{supp} f_2 \subseteq B_R(0)$. Since $f_3 \in C_c(\mathbb{R})$ there exist a K such that $\operatorname{supp} f_3 \subseteq [-K, K]$. Let $(r,s,\tau) \in \mathcal{H}_+$ such that $\|r\|_+ \le 1$, $\|s\|_+ \le 1$ and $|\tau| \le 1$. Since $f(x+r, y+s, t+\tau+\frac{1}{2}\langle r\,,\,y\rangle - \frac{1}{2}\langle s\,,\,x\rangle) = 0$ for $|t| > 2(K+R+1)$ and $\|x\| > R+1$ and $\|y\| > R+1$ we obtain by Lebesgue's theorem of dominated convergence

$$\begin{aligned}
& \langle \kappa(r,s,\tau) f \,,\, f\rangle \\
= & \int\limits_{\mathcal{H}_-} \sqrt{\varrho_r(x)}\sqrt{\varrho_s(y)} f(x+r, y+s, t+\tau+\frac{1}{2}\langle r\,,\,y\rangle - \frac{1}{2}\langle s\,,\,x\rangle) \\
& \qquad\qquad \overline{f(x,y,t)} d\mu(x,y,t) \\
\xrightarrow{(r,s,\tau)\to 0} & \int\limits_{\mathcal{H}_-} |f(x,y,t)|^2\, d\mu(x,y,t) = \|f\|^2_{L^2(\mathcal{H}_-,\mu)}\,.
\end{aligned}$$

Thus for $f = \sum_{k=1}^{n} f_{(k,1)} \otimes f_{(k,2)} \otimes f_{(k,3)}$, where $f_{(k,1)},\ f_{(k,2)} \in C_b(H_-)$ and $f_{(k,3)} \in C_c(\mathbb{R})$ and $f_{(k,1)},\ f_{(k,2)}$ has bounded support we obtain

$$\langle \kappa(r,s,\tau) f\ ,\ f \rangle \xrightarrow{(r,s,\tau)\to 0} \|f\|^2_{L^2(\mathcal{H}_-,\mu)} .$$

Hence it follows

$$\begin{aligned} & \|\kappa(r,s,\tau) - \mathrm{id}) f\|^2_{L^2(\mathcal{H}_-,\,\mu)} \\ = \ & \langle (\kappa(r,s,\tau) - \mathrm{id})^* (\kappa(r,s,\tau) - \mathrm{id}) f,\ f \rangle_{L^2(\mathcal{H}_-,\,\mu)} \\ = \ & \langle (2\,\mathrm{id} - \kappa(r,s,\tau) - \kappa(r,s,\tau)^*)\, f,\ f \rangle_{L^2(\mathcal{H}_-,\,\mu)} \\ = \ & 2\,\|f\|^2_{L^2(\mathcal{H}_-,\,\mu)} - 2Re\langle \kappa(r,s,\tau) f,\ f \rangle_{L^2(\mathcal{H}_-,\,\mu)} \xrightarrow[(r,s,\tau)\to 0]{} 0. \end{aligned}$$

Now we show the assertion. Therefore let $g \in L^2(\mathcal{H}_-,\ \mu)$ and $\varepsilon > 0$ arbitrary, but fixed. Then there exists a f as above, with $\|g - f\| \le \frac{\varepsilon}{3}$. The computation above shows that for f, there is a $\delta > 0$ such that $\|(\kappa(r,s,\tau) - \mathrm{id}) f\|_{L^2(\mathcal{H}_-,\,\mu)} \le \frac{\varepsilon}{3}$ for all $t \in H_+$ with $\|t\|_{\mathcal{H}_+} \le \delta$. Hence for all t with $\|t\|_{\mathcal{H}_+} \le \delta$ we have

$$\begin{aligned} & \|(\kappa(r,s,\tau) - \mathrm{id}) g\|_{L^2(\mathcal{H}_-,\,\mu)} \\ \le \ & \|\kappa(r,s,\tau) - \mathrm{id}\|\, \|g - f\|_{L^2(\mathcal{H}_-,\,\mu)} + \|(\kappa(r,s,\tau) - \mathrm{id}) f\|_{L^2(\mathcal{H}_-,\,\mu)} \\ \le \ & 2\frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus our theorem follows by 5.2.2. □

Remark 5.2.5. For a moment let us consider the similar representation of $\mathcal{H}_+^{pol}$ in $L^2(\mathcal{H}_-^{pol},\mu)$. Thus we define for $(r,s,\tau) \in \mathcal{H}_+^{pol}$

$$\kappa^{pol}(r,s,\tau) :\ L^2(\mathcal{H}_-^{pol},\ \mu) \longrightarrow L^2(\mathcal{H}_-^{pol},\ \mu)$$

by

$$\kappa^{pol}(r,s,\tau) f(x,y,t) := \sqrt{\varrho_r(x)}\sqrt{\varrho_s(y)}\, f((r,s,\tau)\widetilde{\odot}(x,y,t)).$$

With the same arguments as above we obtain that $\kappa^{pol}(r,s,\tau)$ $((r,s,\tau) \in \mathcal{H}_+)$ is a strongly continuous unitary representation of $\mathcal{H}_+^{pol}$ in $L^2(\mathcal{H}_-^{pol},\mu)$.

Definition 5.2.6. For $(r,s,t) \in \mathcal{H}_+$ we define

$$\pi(r,s,t) :\ L^2(H_-,\ \gamma) \longrightarrow L^2(H_-,\ \gamma)$$

by

$$\pi(r,s,t) f(x) := \sqrt{\varrho_r(x)}\, e^{i(t + \langle s\,,\,x \rangle_0 + \frac{1}{2}\langle r\,,\,s \rangle_0)} f(x + r).$$

Now let us define a representation of $\mathcal{H}_+$ in $L^2(H_-,\gamma)$.

Lemma 5.2.7. *Let $(r,s,t) \in \mathcal{H}_+$ and (r',s',t') in $\mathcal{H}_+$ and $f \in L^2(H_-,\gamma)$ Then $\pi(r,s,t)$ is a unitary operator in $\mathscr{L}(L^2(H_-,\gamma))$ and we have*

$$\pi(r,s,t)\pi(r',s',t') f = \pi((r,s,t) \odot (r',s',t')) f.$$

PROOF. At first let us show that $\pi(r,s,t)$ is continuous. For $(r,s,t)\in\mathcal{H}_+$ and $f\in L^2(H_-,\gamma)$ we obtain

$$\begin{aligned}
\|\pi(r,s,t)f\|^2_{L^2(H_-,\gamma)} &= \int_{H_-} |\pi(r,s,t)f|^2\, d\gamma(x)\\
&= \int_{H_-} \varrho_r(x)\left|e^{i(t+\langle s\,,\,x\rangle_0+\frac{1}{2}\langle r\,,\,s\rangle_0)}\right|^2 |f(x+r)|^2\, d\gamma(x)\\
&= \int_{H_-} \varrho_r(x)\,|f(x+r)|^2\, d\gamma(x) = \|f\|^2_{L^2(H_-,\gamma)}\,.
\end{aligned}$$

Moreover, for $f,g\in L^2(H_-,\gamma)$ we obtain

$$\begin{aligned}
\langle\pi(r,s,t)f\,,\,g\rangle &= \int_{H_-} \sqrt{\varrho_r(x)}e^{i(t+\langle x\,,\,s\rangle_0)+\frac{1}{2}\langle r\,,\,s\rangle_0} f(x+r)\overline{g(x)}d\gamma(x)\\
&= \int_{H_-} \varrho_{-r}(x)\sqrt{\varrho_r(x-r)}e^{i(t+\langle x-r\,,\,s\rangle_0+\frac{1}{2}\langle r\,,\,s\rangle_0)} f(x)\overline{g(x-r)}d\gamma(x)\\
&= \int_{H_-} f(x)\overline{\sqrt{\varrho_{-r}(x)}e^{i(-t+\langle x\,,\,-s\rangle_0+\frac{1}{2}\langle -r\,,\,-s\rangle_0)}g(x-r)}d\gamma(x)\\
&= \langle f\,,\,\pi((r,s,t)^{-1})g\rangle.
\end{aligned}$$

To prove the second part of this lemma let (r,s,t) and (r',s',t') in $\mathcal{H}_+$ and $f\in L^2(H_-,\gamma)$. Then we obtain:

$$\begin{aligned}
&\pi(r,s,t)\pi(r',s',t')f(x)\\
=\;& \pi(r,s,t)\big(\sqrt{\varrho_{r'}(x)}e^{i(t'+\langle s'\,,\,x\rangle_0)+\frac{1}{2}\langle r'\,,\,s'\rangle_0} f(x+r')\big)\\
=\;& \sqrt{\varrho_r(x)}e^{i(t+\langle x\,,\,s\rangle_0+\frac{1}{2}\langle r\,,\,s\rangle_0)}\sqrt{\varrho_{r'}(x+r)}e^{i(t'+\langle s'\,,\,x+r\rangle_0+\frac{1}{2}\langle r'\,,\,s'\rangle_0)} f(x+r'+r)\\
=\;& \sqrt{\varrho_{r+r'}(x)}e^{i(t+t'+\frac{1}{2}\langle r\,,\,s'\rangle_0-\frac{1}{2}\langle r'\,,\,s\rangle_0+\frac{1}{2}\langle r+r'\,,\,s+s'\rangle_0+\langle s+s'\,,\,x\rangle_0)} f(x+r'+r)\\
=\;& \pi((r,s,t)\odot(r',s',t'))f(x).
\end{aligned}$$

□

PROPOSITION 5.2.8. $\pi(r,s,t)$ $((r,s,\tau)\in\mathcal{H}_+)$ *is a strongly continuously family of unitary operators.*

PROOF. Let $(r, s, \tau) \in \mathcal{H}_+$ and $f \in \mathscr{C}_b(H_-)$. By Lebesgue's theorem of dominated convergence we obtain

$$\begin{aligned} & \langle \pi(r,s,t)f\,,\ f\rangle \\ = & \int_{H_-} \sqrt{\varrho_r(x)} e^{i(t+\langle x\,,\,s\rangle + \frac{1}{2}\langle r\,,\,s\rangle_0)} f(x+r)\overline{f(x)} d\gamma(x) \\ \xrightarrow{(r,s,t)\to 0} & \int_{H_-} |f(x)|^2 \, d\gamma(x) = \|f\|^2_{L^2(H_-,\gamma)} \,. \end{aligned}$$

Hence as in Proposition 5.2.8 it follows

$$\begin{aligned} & \|\pi(r,s,t) - \mathrm{id})f\|^2_{L^2(H_-,\,\gamma)} \\ = \ & 2\,\|f\|^2_{L^2(H_-,\,\gamma)} - 2Re\langle \pi(r,s,t)f,\ f\rangle_{L^2(\mathcal{H}_-,\,\gamma)} \xrightarrow[(r,s,t)\to 0]{} 0. \end{aligned}$$

Now we show the assertion. Therefore let $g \in L^2(H_-,\ \gamma)$ and $\varepsilon > 0$ arbitrary, but fixed. Then there exists a f as above, with $\|g - f\| \le \frac{\varepsilon}{3}$. The computation above shows that for f, there is a $\delta > 0$ such that $\|(\pi(r,s,t) - \mathrm{id})f\|_{L^2(\mathcal{H}_-,\,\mu)} \le \frac{\varepsilon}{3}$ for all $t \in H_+$ with $\|t\|_{\mathcal{H}_+} \le \delta$. Hence for all t with $\|t\|_{\mathcal{H}_+} \le \delta$ we have as above $\|(\pi(r,s,t) - \mathrm{id})g\|_{L^2(H_-,\,\gamma)} \le \varepsilon$. □

THEOREM 5.2.9. $\pi(r,s,t)$ $((r,s,t) \in \mathcal{H}_+)$ *is a strongly continuous unitary representation of* $\mathcal{H}_+$ *in* $L^2(H_-,\gamma)$.

PROOF. This theorem follows directly by Lemma 5.2.7 and Proposition5.2.8 □

By using a version of Schur's lemma (cf. [**129**, Chapter 0 Proposition 4.1]) we will prove the following theorem

THEOREM 5.2.10. $\pi(r,s,t)$ *is an irreducible unitary representation of* $\mathcal{H}_+$ *in* $L^2(H_-,\gamma)$.

PROOF. By a version of Schur's lemma we have to show that for a bounded linear operator on $L^2(H_-,\gamma)$ such that $A\pi(r,s,t) = \pi(r,s,t)A$ it follows that $A = \lambda\mathrm{id}$. We want to reduce this proof to the well known finite dimensional case. Thus we define P_n as the orthogonal projection on $c.l.s\{e_1,\ldots,e_n\}$ in H_0 and P_{l_n} the orthogonal projection in $L^2(H_-,\gamma)$ on the $c.l.s\{h_\alpha\ :\ \mathrm{length}(\alpha) \le n\}$. Let $g \in \mathscr{C}_{int,cyl}(H_-)$ such that $g(x) = g(P_n x)$ and $u \in \mathscr{C}_b(H_-)$ with $u = \sum_\alpha a_\alpha h_\alpha$. Then we obtain $P_{l_n} g u = \sum_\alpha a_\alpha P_{l_n} g h_\alpha = \sum_\alpha a_\alpha g P_{l_n} h_\alpha = g P_{l_n} u$. Thus we have

$$[P_{l_n},\ M_g] = 0. \tag{83}$$

Now let A be a bounded linear operator on $L^2(H_-,\gamma)$ such that $A\pi(r,s,t) = \pi(r,s,t)A$. For $(r,s,t) = (0,s,0)$ where $s \in P_{l_n} H_+$ we obtain

$$Ae^{i\langle s\,,\,x\rangle_0} = e^{i\langle s\,,\,x\rangle_0} A,$$

which implies $P_{l_n}Ae^{i\langle s\,,x\rangle_0}P_{l_n} = P_{l_n}e^{i\langle s\,,x\rangle_0}AP_{l_n}$ and thus we find by (83)

$$P_{l_n}AP_{l_n}e^{i\langle s\,,x\rangle_0} = e^{i\langle s\,,x\rangle_0}P_{l_n}AP_{l_n}. \tag{84}$$

Let $\tilde{A}$ denote the continuous linear operator on $L^2(\mathbb{R}^n,\lambda)$ defined by

$$\tilde{A}u(\tilde{P}_n x) = P_{l_n}AP_{l_n}(V_{G,n}u)((\tilde{P}_n x)).$$

Now (84) leads to $e^{i\langle \widetilde{P}_{l_n}s\,,\cdot\rangle}\tilde{A}u = \tilde{A}e^{i\langle \widetilde{P}_{l_n}s\,,\cdot\rangle}u$ for all $u \in L^2(\mathbb{R}^n,\lambda)$. Using the first part of [**129**, Theorem 2.1 p. 46] we find that $\tilde{A}u(x) = \tilde{a}_n(x)u(x),\ x \in \mathbb{R}^n$. Thus it follows that there exists a function a on H_- such that $a(x) = a(P_n x)$ and $P_{l_n}AP_{l_n}u(x) = a_n(x)u(x)$ for all $u \in L^2(H_-,\gamma)$ such that $u(x) = u(P_n x)$. In a second step let us choose $(r,s,t) = (r,0,0)$ where $r \in P_n(H_+)$. Then as above $\pi(r,0,0)A = A\pi(r,0,0)$ leads to

$$\varrho_r(x)(P_{l_n}AP_{l_n}u)(x+r) = P_{l_n}AP_{l_n}\varrho_r(x)u(x+r)$$

for all $u \in L^2(H_-,\gamma)$ such that $u(x) = u(P_n x)$. But this implies

$$\varrho_r(x)a_n(x+r)u(x+r) = \varrho_r(x)a_n(x)u(x+r)$$

and thus we find $a_n(x) = \lambda_n$. Finally we note that P_{l_n} converges strongly and monotone to id. Thus we obtain that there exists a $\lambda \in \mathbb{C}$ such that $\lambda_n = \lambda$ for all $n \in \mathbb{N}$ and $A = \lambda \mathrm{id}$. □

At next we will construct some other representations of $\mathcal{H}_+$ in $L^2(H_-,\gamma)$. Thus we define for $\lambda > 0$

$$\delta_{\pm\lambda} :\ \mathcal{H}_+ \longrightarrow \mathcal{H}_+\ :\ (r,s,t) \longmapsto (\sqrt{\lambda}r, \pm\sqrt{\lambda}s, \pm\lambda t). \tag{85}$$

We find that $\delta_{\pm\lambda}$ is a continuous automorphism of $\mathcal{H}_+$.

Proposition and Definition 5.2.11. *For $\lambda > 0$ we define $\pi_{\pm\lambda}(r,s,t)$ by*

$$\pi_{\pm\lambda}(r,s,t) := \pi(\delta_{\pm\lambda}(r,s,t)) \tag{86}$$

for all $(r,s,t) \in \mathcal{H}_+$ Then $\pi_{\pm\lambda}(r,s,t)$ is a strongly continuous unitary irreducible representation of $\mathcal{H}_+$ in $L^2(H_-,\gamma)$. We call $\pi_{\pm\lambda}$ the Schrödinger representations of $\mathcal{H}_+$.

Proof. Since $\delta_{\pm\lambda}$ is continuous the strong continuity of $\pi_{\pm\lambda}$ follows directly by the strong continuity of π. Now note that $\{\pi_{\pm\lambda}(r,s,t)\ :\ (r,s,t) \in \mathcal{H}_+\} = \{\pi(r,s,t)\ :\ (r,s,t) \in \mathcal{H}_+\}$. Thus $\pi_{\pm\lambda}$ is irreducible since π is irreducible. □

Let us note that $\pi_{\pm\lambda}(r,s,t)$ is given explicitly on $L^2(H_-,\gamma)$ by

$$\pi_{\pm\lambda}(r,s,t)u(x) = \sqrt{\varrho_{\sqrt{\lambda}r}(x)}\,e^{i(\pm\lambda t + \langle \pm\sqrt{\lambda}s\,,x\rangle_0 + \frac{\pm\lambda}{2}\langle r\,,s\rangle_0)}u(x+r). \tag{87}$$

Obviously for $y,\eta \in H_-$ we obtain the one dimensional representations of $\mathcal{H}_+$ given by

$$\pi_{y,\eta}(r,s,t) = e^{i\langle r\,,y\rangle_0 + \langle s\,,\eta\rangle_0}. \tag{88}$$

We obtain the following

Proposition 5.2.12. *No two different representations of $\mathcal{H}_+$ given by (86) and (87) are unitary equivalent.*

Proof. We only have to show that if there exits an unitary F such that $F\pi_\lambda F^{-1} = \pi_{\lambda'}$ we have $\lambda = \lambda'$. Thus let us consider $(0,0,t) \in \mathcal{H}_+$. Then we have

$$Fe^{i\lambda t}F^{-1} = F\pi_\lambda(0,0,t)F^{-1} = \pi_{\lambda'} = e^{i\lambda' t}$$

for all $t \in \mathbb{R}$ and thus $\lambda = \lambda'$. □

Remark 5.2.13. Again let us consider the case of a quasi-nuclear polarized Heisenberg group rigging. Thus for $(r,s,t) \in \mathcal{H}_+^{pol}$ we define

$$\pi^{pol}(r,s,t): \ L^2(H_-, \ \gamma) \longrightarrow L^2(H_-, \ \gamma)$$

by

$$\pi^{pol}(r,s,t)f(x) := \sqrt{\varrho_r(x)}e^{i(t+\langle s \, , \, x\rangle_0}f(x+r).$$

Then again $\pi^{pol}(r,s,t)$ ($(r,s,t) \in \mathcal{H}_+^{pol}$) is a strongly continuous irreducible unitary representation of $\mathcal{H}_+^{pol}$ in $L^2(H_-,\gamma)$. For $\lambda > 0$ we find that $\delta_{\pm\lambda}$ is an automorphism of $\mathcal{H}_+^{pol}$. Hence as before we see that $\pi_{\pm\lambda}^{pol}(r,s,t) := \pi^{pol}(\delta_{\pm\lambda}(r,s,t))$ is a strongly continuous irreducible unitary representation of $\mathcal{H}_+^{pol}$ in $L^2(H_-^{pol},\gamma)$. Moreover, in this case no two different representations π_λ^{pol} are unitary equivalent.

Definition 5.2.14. For $(a,b,\tau) \in \mathcal{H}_+$ we define

$$V_t^{(a,b,\tau)} := \kappa(ta,tb,t\tau)$$

and

$$U_t^{(a,b,\tau)} := \pi(ta,tb,t\tau).$$

Lemma 5.2.15. *Let $(a,b,\tau) \in \mathcal{H}_+$ be fixed. Then $(V_t^{(a,b,\tau)})_{t\in\mathbb{R}}$ and $(U_t^{(a,b,\tau)})_{t\in\mathbb{R}}$ are unitary strongly continuous one parameter groups.*

Proof. Let $(a,b,\tau) \in \mathcal{H}_+$ and $t,s \in \mathbb{R}$. Then we obtain

$$\begin{aligned}
& (ta,tb,t\tau) \odot (sa,sb,s\tau) \\
= \ & ((t+s)a,(t+s)b,(t+s)\tau + \frac{1}{2}\langle ta \, , \, sb\rangle_0 - \frac{1}{2}\langle sa \, , \, tb\rangle_0) \\
= \ & ((t+s)a,(t+s)b,(t+s)\tau + \frac{1}{2}ts\langle a \, , \, b\rangle_0 - \frac{1}{2}ts\langle a \, , \, b\rangle_0) \\
= \ & ((t+s)a,(t+s)b,(t+s)\tau).
\end{aligned}$$

Thus we have $V_t^{(a,b,\tau)}V_s^{(a,b,\tau)} = V_{t+s}^{(a,b,\tau)}$ and $U_t^{(a,b,\tau)}U_s^{(a,b,\tau)} = U_{t+s}^{(a,b,\tau)}$. Now our assertion follows by Theorem 5.2.4 and Theorem 5.2.9. □

Definition 5.2.16. For $f \in \mathscr{C}^1(\mathcal{H}_-)$ and $t \in H_+$ we define D_t^1 and D_t^2 by

$$D_t^1 f(x,y,\tau) := \lim_{h\to 0}\frac{f(x+ht,y,\tau) - f(x,y,\tau)}{h} - \langle t \, , \, x\rangle_0 f(x,y,\tau)$$

and

$$D_t^2 f(x,y,\tau) := \lim_{h\to 0} \frac{f(x,y+ht,\tau)-f(x,y,\tau)}{h} - \langle t\,,\,y\rangle_0 f(x,y,\tau).$$

Moreover, we denote by $\mathscr{C}^k_{int,bs}(\mathcal{H}_-)$ the space of all $\mathscr{C}^k$-functions an $\mathcal{H}_-$, which have bounded support and satisfy $f(\cdot,y,\tau)\in\mathscr{C}^k_{int}(H_-)$ for every fixed y,τ and $f(x,\cdot,\tau)\in\mathscr{C}^k_{int}(H_-)$ for every fixed x,τ.

PROPOSITION 5.2.17. *Let $D_{(a,b,\tau)}$ $((a,b,\tau)\in\mathcal{H}_+)$ denote the infinitesimal generator of the unitary C_0 group $V_t^{(a,b,\tau)}$ $(t\in\mathbb{R})$. For its domain of definition we write $D(D_{(a,b,\tau)})$. According to the theorem of Stone (cf.* [**117**, Theorem VIII.8]*) we obtain that $-iD_{(a,b,\tau)}$ is self adjoint. For $f\in\mathscr{C}^1_{int,bs}(\mathcal{H}_-)$ we have*

$$\begin{aligned} & D_{(a,b,\tau)} f(x,y,s) \\ = \;& D_a^1 f(x,y,s) + D_b^2 f(x,y,s) + (\tau + \frac{1}{2}\langle a\,,\,y\rangle_0 - \frac{1}{2}\langle b\,,\,x\rangle_0)\frac{\partial}{\partial s} f(x,y,s). \end{aligned}$$

In addition , we have $D_{(a,b,\tau)}(\mathscr{C}^\infty_{int,bs}(\mathcal{H}_-))\subset\mathscr{C}^\infty_{int,bs}(\mathcal{H}_-)$, and $\mathscr{C}^\infty_{int,bs}(\mathcal{H}_-)$ is a domain of essential selfadjointness of the operator $-iD_{(a,b,\tau)}$.

PROOF. For $f\in\mathscr{C}^1_{b,bs}(\mathcal{H}_-)$ we obtain pointwisely

$$\begin{aligned}
& \frac{1}{t}(V_t^{(a,b,\tau)} f(x,y,s) - f(x,y,s)) \\
= \;& \frac{1}{t}\Bigg(\sqrt{\varrho_{ta}(x)}\sqrt{\varrho_{tb}(y)} f(x+ta, y+tb, s+t\tau+\frac{1}{2}t\langle a\,,\,y\rangle) \\
& \qquad\qquad -\frac{1}{2}t\langle b\,,\,x\rangle) - f(x,y,s)\Bigg) \\
= \;& \frac{1}{t}\Bigg(\sqrt{\varrho_{ta}(x)}\sqrt{\varrho_{tb}(y)} f(x+ta, y+tb, s+t\tau+\frac{1}{2}t\langle a\,,\,y\rangle) \\
& \qquad -\sqrt{\varrho_{tb}(y)} f(x, y+tb, s+t\tau+\frac{1}{2}t\langle a\,,\,y\rangle) - \frac{1}{2}t\langle b\,,\,x\rangle)\Bigg) \\
+ \;& \frac{1}{t}\Bigg(\sqrt{\varrho_{tb}(y)} f(x, y+tb, s+t\tau+\frac{1}{2}t\langle a\,,\,y\rangle - \frac{1}{2}t\langle b\,,\,x\rangle) \\
& \qquad -f(x,y,s+t\tau+\frac{1}{2}t\langle a\,,\,y\rangle - \frac{1}{2}t\langle b\,,\,x\rangle)\Bigg) \\
+ \;& \frac{1}{t}\Bigg(f(x,y,s+t\tau+\frac{1}{2}t\langle a\,,\,y\rangle - \frac{1}{2}t\langle b\,,\,x\rangle) - f(x,y,s)\Bigg) \\
\xrightarrow{t\to 0} \;& D_a^1 f(x,y,s) + D_b^2 f(x,y,s) + (\tau+\frac{1}{2}\langle a\,,\,y\rangle_0 - \frac{1}{2}\langle b\,,\,x\rangle_0)\frac{\partial}{\partial s} f(x,y,s)
\end{aligned}$$

According to Proposition 1.3.8 the first and the second addend also converge in $L^2(\mathcal{H}_-,\mu)$, the third converges in $L^2(\mathcal{H}_-,\mu)$ by Lebesgue's theorem of dominate convergence since f has bounded support. Now the rest of the first part of the proof follows similarly to Proposition 1.3.8. The second part is direct consequence of the theorem of Nelson (cf. [**117**, Theorem VIII.10]), since $V_t^{(a,b,\tau)}(\mathscr{C}^\infty_{int,bs}(\mathcal{H}_-)) \subset \mathscr{C}^\infty_{int,bs}(\mathcal{H}_-)$. □

Definition 5.2.18. Let $(e_j)_{j\in\mathbb{N}} \subset H_+$ be an orthonormal in H_0. Then we set:

$$\begin{aligned} L_j := \quad D_{(0,e_j,0)} = \quad & D^2_{e_j} - \frac{1}{2}\langle e_j\,,\,x\rangle_0 \frac{\partial}{\partial s}, \\ M_j := \quad D_{(e_j,0,0)} = \quad & D^1_{e_j} + \frac{1}{2}\langle e_j\,,\,y\rangle_0 \frac{\partial}{\partial s}, \\ T := \quad D_{(0,0,1)} = \quad & \frac{\partial}{\partial s}. \end{aligned}$$

Lemma 5.2.19. *We have*

$$[L_j,\ M_j] = -[M_j,\ L_j] = T$$

and

$$[L_j,\ M_i] = [L_j,\ T] = [M_j,\ T] = 0$$

for $i \neq j$.

Proof. We obtain:

$$\begin{aligned} [L_j,\ M_j] &= [D^2_{e_j} - \frac{1}{2}\langle e_j\,,\,x\rangle_0)\frac{\partial}{\partial s},\ D^1_{e_j} + \frac{1}{2}\langle e_j\,,\,y\rangle_0)\frac{\partial}{\partial s}] \\ &= [D^2_{e_j},\ \frac{1}{2}\langle e_j\,,\,y\rangle_0\frac{\partial}{\partial s}] + [-\frac{1}{2}\langle e_j\,,\,x\rangle_0)\frac{\partial}{\partial s},\ D^1_{e_j}] = \frac{1}{2}\frac{\partial}{\partial s} + \frac{1}{2}\frac{\partial}{\partial s} = T. \end{aligned}$$

The rest of the lemma is clear. □

Remark 5.2.20. For a moment let us have a short look at a polarized Heisenberg group rigging. Let $(e_j)_{j\in\mathbb{N}} \subset H_+$ be an orthonormal basis of H_0. Then we obtain:

(i) $\kappa^{pol}(te_j,0,0)$ is a strongly continuous one parameter group with infinitesimal generator $M_j^{pol}f(x,y,s) := D^1_{e_j}f(x,y,s) + \langle e_j\,,\,y\rangle_0\frac{\partial}{\partial s}$.
(ii) $\kappa^{pol}(0,te_j,0)$ is a strongly continuous one parameter group with infinitesimal generator $L_j^{pol}f(x,y,s) := D^1_{e_j}f(x,y,s)$.
(iii) $\kappa^{pol}(0,0,t)$ is a strongly continuous one parameter group with infinitesimal generator $T^{pol}f(x,y,s) := \frac{\partial}{\partial s}f(x,y,s)$.
(iv) Again we obtain the following commutation relations: $[L_j^{pol},\ M_j^{pol}] = -[M_j^{pol},\ L_j^{pol}] = T^{pol}$ and $[L_j^{pol},\ M_i^{pol}] = [L_j^{pol},\ T^{pol}] = [M_j^{pol},\ T^{pol}] = 0$.

Now let us consider the unitary representation π.

Theorem 5.2.21. *For $u \in \mathscr{C}^\infty_{int}(H_-)$ the infinitesimal generator $D^U_{a,b,\tau}$ of $(U_t^{(a,b,\tau)})_{t\in\mathbb{R}}$ is given by*

$$D^U_{a,b,\tau}u = D_a u + i\langle b\,,\,\cdot\rangle_0 + i\tau u,$$

where D_a is defined as in 1.3.2 Furthermore, $\mathscr{C}^{\infty}_{int}(H_-)$ is a domain of essential selfadjiontness for $D^U_{a,b,\tau}$.

PROOF. Let $u \in \mathscr{C}^{\infty}_{int}(H_-)$. Then we obtain by Lebesgue's Theorem of dominated convergence

$$\begin{aligned}
& \frac{U_t^{(a,b,\tau)} - \mathrm{id}}{t} u(x) \\
= \quad & \frac{1}{t}\left(\sqrt{\varrho_{ta}(x)} e^{i(t\tau + t\langle b\,,\, x\rangle_0 + \frac{t^2}{2}\langle a\,,\, b\rangle_0)} f(x+ta)u(x) - u(x)\right) \\
= \quad & e^{i(t\tau + t\langle b\,,\, x\rangle_0 + \frac{t^2}{2}\langle a\,,\, b\rangle_0)} \frac{\sqrt{\varrho_{ta}(x)} f(x+ta)u(x) - u(x)}{t} \\
& + e^{i(t\tau + \frac{t^2}{2}\langle a\,,\, b\rangle_0)} \frac{e^{it\langle b\,,\, x\rangle_0} u(x) - u(x)}{t} \\
& + e^{\frac{t^2}{2}\langle a\,,\, b\rangle_0)} \frac{e^{it\tau} u(x) - u(x)}{t} + \frac{e^{\frac{t^2}{2}\langle a\,,\, b\rangle_0} u(x) - u(x)}{t} \\
\xrightarrow{t \longrightarrow 0} \quad & D_a u(x) + i\langle b\,,\, x\rangle_0 u(x) + i\tau u(x).
\end{aligned}$$

Moreover, since $U_t^{(a,b,\tau)}$ leaves the space $\mathscr{C}^{\infty}_{int}(H_-)$ invariant we obtain by the Theorem of Stone and Nelson that $\mathscr{C}^{\infty}_{int}(H_-)$ is a domain of essential selfadjointness of $iD^U_{a,b,\tau}$. □

5.3. The Heisenberg Group and the Weyl calculus

The finite dimensional case. Since we do not have any Haar measure on an infinite dimensional Heisenberg Group let us first consider the finite dimensional case where $H_+ = H_0 = H_- = \mathbb{R}^n$ and denote by $\mathcal{H}_n$ the corresponding Heisenberg Group.

DEFINITION 5.3.1. Let us denote by $\tilde{\pi}_1$ (cf. [**129**, Chapter 1]) the irreducible unitary strongly continuous representation of $\mathcal{H}_n$ in $L^2(\mathbb{R}^n, \lambda^n)$ given by

$$\tilde{\pi}_1(r, s, t)u(x) = e^{i(t + \langle s\,,\, x\rangle + \frac{1}{2}\langle r\,,\, s\rangle)} u(x + r).$$

In [**129**, Chapter 1 Proposition 2.2] it is shown that

$$\mathscr{C}^{\infty}(\tilde{\pi}_1) = S(\mathbb{R}^n). \tag{89}$$

We will use this result to determine $\mathscr{C}^{\infty}(\pi)$. Thus let us prove the following

LEMMA 5.3.2. *Let $(r, s, t) \in \mathcal{H}_n$ and $u \in L^2(\mathbb{R}^n, \gamma)$. Then we obtain*

$$\pi(r, s, t)u(x) = e^{\frac{\|x\|^2}{2}} \tilde{\pi}_1(r, s, t)(e^{-\frac{\|x\|}{2}} u(x)).$$

PROOF. Let $(r,s,t) \in \mathcal{H}_n$ and $u \in L^2(\mathbb{R}^n,\gamma)$. Then we find since $\varrho_r(x) = e^{-\|r\|^2 - 2\langle r\,,x\rangle}$

$$\begin{aligned} & e^{\frac{\|x\|^2}{2}} \tilde{\pi}_1(r,s,t)(e^{-\frac{\|x\|}{2}} u(x)) \\ = \; & e^{\frac{\|x\|^2}{2}} e^{i(t+\langle s\,,x\rangle + \frac{1}{2}\langle r\,,s\rangle)} e^{-\frac{\|x+r\|}{2}} u(x+r) \\ = \; & e^{\frac{\|x\|^2}{2}} e^{i(t+\langle s\,,x\rangle + \frac{1}{2}\langle r\,,s\rangle)} \sqrt{\varrho_r(x)} e^{\frac{1}{2}\|r\|^2 + \langle r\,,x\rangle} e^{-(\frac{\|x\|}{2} + \langle r\,,x\rangle + \frac{\|r\|}{2})} u(x+r) \\ = \; & e^{i(t+\langle s\,,x\rangle + \frac{1}{2}\langle r\,,s\rangle)} \sqrt{\varrho_r(x)} u(x+r) = \pi(r,s,t)u(x). \end{aligned}$$

Thus our proposition is proved. □

Now in view of equation (89) we obtain the following proposition as corollary.

PROPOSITION 5.3.3. *We have*

$$\mathscr{C}^\infty(\pi) = S_\gamma(\mathbb{R}^n).$$

Let us consider the connection between pseudodifferential operators in Weyl form defined in 3.2.2 and our representation π.

PROPOSITION 5.3.4. *For a well-behaved symbol $a(x,\xi)$ (a symbol $a(x,\xi)$ such that all oscillatory integral in* [**129**, Proposition 3.1, Chapter 1] *exist) we obtain*

$$a(X,D) = (2\pi)^{-n} \int \hat{a}(s,r)\pi(r,s,0)\lambda^n(ds)\lambda^n(dr),$$

where $\hat{a}(s,r) = (2\pi)^{-n} \int a(x,\xi) e^{i(\langle x\,,s\rangle + \langle \xi\,,r\rangle)} \lambda^n(dx)\lambda^n(d\xi)$ is given by the Fourier-transform of $a(x,\xi)$.

PROOF. For $u \in S_\gamma(\mathbb{R}^n)$ we have by 3.2.4 and 5.3.2

$$\begin{aligned} a(X,D)u(x) &= V_{G,n}^{-1} a(X,\tilde{D}) V_{G,n} u(x) \\ &\overset{(*)}{=} V_{G,n}^{-1} (2\pi)^{-n} \int \hat{a}(s,r)\tilde{\pi}(r,s,0) V_{G,n} u(x) \lambda^n(ds)\lambda^n(dr) \\ &= (2\pi)^{-n} \int \hat{a}(s,r) V_{G,n}^{-1} \tilde{\pi}(r,s,0) V_{G,n} u(x) \lambda^n(ds)\lambda^n(dr) \\ &= (2\pi)^{-n} \int \hat{a}(s,r)\pi(r,s,0)u(x)\lambda^n(ds)\lambda^n(dr), \end{aligned}$$

where the equality $(*)$ follows from [**129**, Proposition 3.1, Chapter 1]. □

Let $\mathcal{E}'(\mathcal{H}_n)$ be the space of all compactly supported distributions on $\mathcal{H}_n$. Then by the general theory of Lie Groups $\pi(f) = \int_{\mathcal{H}_n} f(z)\pi(z)dz$ is defined on $\mathscr{C}^\infty(\pi)$. In view of Proposition 5.3.4 we obtain the following

THEOREM 5.3.5. *For $k \in L^1(\mathcal{H}_n, \lambda^{2n+1})$ we have*

$$\pi_{\pm\lambda}(k) = \tilde{k}(\pm\lambda, \pm\sqrt{\lambda} ix, \sqrt{\lambda} D) = \sigma_k(\pm\lambda)(X,D), \tag{90}$$

where

$$\tilde{k}(\tau, y, \eta) = (2\pi)^{-\frac{2n+1}{2}} \int k(r, s, t) e^{i(t\tau + \langle s\,, y\rangle) + \langle r\,, \eta\rangle} \lambda(dt)\lambda^n(ds)\lambda^n(dr) \tag{91}$$

and

$$\sigma_k(\pm\lambda)(x, \xi) = \tilde{k}(\pm\lambda, \pm\sqrt{\lambda}x, \pm\sqrt{\lambda}\xi). \tag{92}$$

Proof. Let $k \in L^1(\mathcal{H}_n, \lambda^{2n+1})$ and $u \in S_\gamma(\mathbb{R}^n)$. Then we obtain

$$\begin{aligned} \pi_{\pm\lambda}(k) &= \int k(t, q, p)\pi_{\pm\lambda}(r, s, t)\lambda(dt)\lambda^n(ds)\lambda^n(dr) \\ &= V_{G,n}^{-1} \int k(t, q, p)\tilde{\pi}_{\pm\lambda}(r, s, t)\lambda(dt)\lambda^n(ds)\lambda^n(dr)V_{G,n} \\ &\overset{(**)}{=} V_{G,n}^{-1}\tilde{k}(\pm\lambda, \pm\sqrt{\lambda}X, \sqrt{\lambda}\tilde{D})V_{G,n} \\ &= \tilde{k}(\pm\lambda, \pm\sqrt{\lambda}X, \sqrt{\lambda}D), \end{aligned}$$

where the equality $(**)$ follows from [**129**, (3.9), Chapter 1]. But this proves our Theorem. □

Remark 5.3.6. Again let us consider the case of the polarized Heisenberg Group. Let us remind, that in this case $\pi^{pol}_{\pm\lambda}$ is given by

$$\pi^{pol}_{\pm\lambda}(r, s, t)u(x) := \sqrt{\varrho_{\sqrt{\lambda}r}(x)}e^{i(\pm\lambda t + \langle\pm\sqrt{\lambda}s\,, x\rangle_0}u(x + \sqrt{\lambda}r).$$

Now we obtain

$$\int \hat{a}(s, r)\pi^{pol}(r, s, 0)\lambda^n(ds)\lambda^n(dr) = \mathcal{F}^{-1}_{\xi\to x}a(x, \xi)(\mathcal{F}u)(\xi) = a(x, D),$$

where $a(x, D)$ stands for the pseudodifferential operator given in terms of the Kohn-Nirenberg quantization in the case of a Gaussian measure. Consequently, we get

$$\pi^{pol}_{\pm\lambda}(k) = \tilde{k}(\pm\lambda, \pm\sqrt{\lambda}x, \sqrt{\lambda}D),$$

i.e. the operator $\pi^{pol}_{\pm\lambda}(k)$ is given in Kohn-Nirenberg form.

Let us denote by $\tilde{M}_j := \frac{\partial}{\partial r_j} + \frac{1}{2}s_j\frac{\partial}{\partial t}$, $\tilde{L}_j := \frac{\partial}{\partial s_j} - \frac{1}{2}r_j\frac{\partial}{\partial t}$ and $\tilde{T} = \frac{\partial}{\partial t}$ the well known basis of left invariant vector fields of the Lie-Algebra of $\mathcal{H}_n$ (cf. [**129**, Chapter 1]). Then we obtain the commutator relation

$$[\tilde{L}_j,\ \tilde{M}_j] = -[\tilde{M}_j,\ \tilde{L}_j] = -\tilde{T}.$$

Furthermore, we denote by

$$\tilde{\mathcal{L}}_0 := \sum_{j=1}^n (\tilde{L}_j^2 + \tilde{M}_j^2) \tag{93}$$

the classical Heisenberg-Laplacian. Then we have

$$\pi_{\pm\lambda}(\tilde{L}_j) = \pm i\sqrt{\lambda}\langle\cdot\,, e_j\rangle, \qquad \pi_{\pm\lambda}(\tilde{M}_j) = \sqrt{\lambda}D_{e_j} \qquad \pi_{\pm\lambda}(\tilde{T}) = \pm\lambda\mathrm{id} \tag{94}$$

and

$$\kappa(\tilde{L_j}) = L_j, \qquad \kappa(\tilde{M_j}) = M_j, \quad \kappa(\tilde{T}) = T. \tag{95}$$

Thus we find

$$\mathcal{L}_0 := \kappa(\mathcal{L}_0) = \sum_{j=1}^{n} (L_j^2 + M_j^2).$$

We call this operator the Gaussian-Heisenberg-Laplacian.

PROPOSITION 5.3.7. *For $\lambda > 0$ we find that*

$$\pi_{\pm_\lambda}(\tilde{\mathcal{L}}_0) = -\lambda(2L_\gamma + n\,\mathrm{id}),$$

where L_γ denotes the Ornstein-Uhlenbeck operator defined in (2.1.2).

PROOF. By the general theory of Lie Groups (cf. [**129**, Chapter 0]) it is clear that

$$\pi_{\pm_\lambda}(\tilde{\mathcal{L}}_0) = \sum_{j=1}^{n} \left((\sqrt{\lambda} D_{e_j})^2 + (\sqrt{\lambda} i \langle \cdot\,, e_j \rangle)^2 \right) = -\lambda \sum_{j=1}^{n} \left(-D_{e_j}^2 + \langle \cdot\,, e_j \rangle^2 \right).$$

Now we find

$$-D_{e_j}^2 + \langle \cdot\,, e_j\rangle^2 = -\left(\frac{\partial}{\partial e_j} - \langle \cdot\,, e_j\rangle\right)^2 + \langle \cdot\,, e_j\rangle^2 = -\frac{\partial^2}{(\partial e_j)^2} + 2\langle \cdot\,, e_j\rangle \frac{\partial}{\partial e_j} + id$$

and thus we obtain

$$-\frac{1}{\lambda}\pi_{\pm_\lambda}(\tilde{\mathcal{L}}_0) = \sum_{j=1}^{n} (-\frac{\partial^2}{(\partial e_j)^2} + 2\langle \cdot\,, e_j\rangle \frac{\partial}{\partial e_j} + id) = 2L_\gamma + n\,\mathrm{id}. \qquad \square$$

According to [**129**] the spectrum of $-\frac{1}{\lambda}\tilde{\pi}_{\pm_\lambda}(\tilde{\mathcal{L}}_0)$ is given by the set $\sigma = \{n + 2j \ : \ j \in \mathbb{N}\}$ and each $k \in \sigma$ is an eigenvalue. Moreover, in [**129**] it is shown that the eigenvectors of $-\frac{1}{\lambda}\tilde{\pi}_{\pm_\lambda}(\tilde{\mathcal{L}}_0)$ are the Hermite-functions, defined by $h_\alpha e^{-\frac{\|\cdot\|}{2}}$. Now, note again that $\pi_{\pm_\lambda}(\tilde{\mathcal{L}}_0) = V_{G,n}^{-1}\tilde{\pi}_{\pm_\lambda}(\tilde{\mathcal{L}}_0)V_{G,n}$. Thus we obtain again the well known fact

$$\sigma_{L^2(\mathbb{R}^n,\gamma)}(L_\gamma) = \mathbb{N}$$

and the eigenvectors are given by the Hermite-polynomials. Since we are later on also interested in the infinite dimensional case let us define more general operators. Thus let $b_1 \dots b_n$ be positive real numbers, $(b_{jk})_{j,k=1\dots 2n}$ be a positive definite matrix and $c \in \mathbb{R}$ a constant. Then we define the operators $\tilde{\mathcal{L}}_{b,c}$ and $\tilde{\mathcal{P}}_{b,c}$ by

$$\tilde{\mathcal{L}}_{b,c} = \sum_{j=1}^{n} b_j(\tilde{L}_j^2 + \tilde{M}_j^2) + c\,iT \tag{96}$$

and

$$\tilde{\mathcal{P}}_{b,c} \sum_{j,k=1}^{2n} b_{j,k} \tilde{Y}_j \tilde{Y}_k + c\, iT, \tag{97}$$

where $\tilde{Y}_j = \tilde{L}_j$ and $\tilde{Y}_{n+j} = \tilde{M}_j$ $(j = 1 \dots n)$. Let us start with a general result with shows again the connection between the Heisenberg Group and the Weyl calculus.

PROPOSITION 5.3.8. *For $\tilde{\mathcal{P}}_{b,c}$ being defined as in equation (97) we obtain*

$$\pi_{\pm\lambda}(\tilde{\mathcal{P}}_{b,c}) = -\lambda(Q(X, D) \pm c\,\mathrm{id}),$$

where $Q(x, \xi) := \sum_{j,k=1}^{2n} b_{j,k} \chi_j \chi_k$ and $\chi_j = x_j$ and $\chi_{n+j} = \xi$ $(j = 1 \dots n)$.

PROOF. Let $\tilde{\mathcal{P}}_{b,c}$ be defined as in equation (97). Considering [**129**, Chapter 0] and 5.3.2 we obtain $\pi_{\pm\lambda}(\tilde{\mathcal{P}}_{b,c}) = V_{G,n}^{-1} \tilde{\pi}_{\pm\lambda}(\tilde{\mathcal{P}}_{b,c}) V_{G,n}$. Thus we find [**129**, Chapter 1, (6.42)] and 3.2.4

$$\pi_{\pm\lambda}(\tilde{\mathcal{P}}_{b,c}) V_{G,n}^{-1} - \lambda(\lambda(Q(X, \tilde{D}) \pm c\,\mathrm{id})) V_{G,n} = -\lambda(\lambda(Q(X, D) \pm c\,\mathrm{id})). \qquad \square$$

Let us examine the operator $-\frac{1}{\lambda}\pi_{\pm\lambda}(\tilde{\mathcal{L}}_{b,c})$ more detailed. Thus we set

$$L_{\gamma,j} := -\frac{1}{2}\left(\frac{\partial}{\partial e_j} - 2\langle \cdot\,,\, e_j\rangle \frac{\partial}{\partial e_j}\right). \tag{98}$$

Then as noted above and proved in Lemma 2.1.10 we find

$$L_{\gamma,j} h_\alpha = \alpha_j h_\alpha. \tag{99}$$

Considering again Proposition 5.3.3 we obtain the following

COROLLARY 5.3.9. *For $\lambda > 0$ we have*

$$\begin{aligned} -\frac{1}{\lambda}\pi_{\pm\lambda}(\tilde{\mathcal{L}}_{b,c}) &= \sum_{j=1}^{n} b_j (2L_{\gamma,j} + \mathrm{id}) \pm c\,\mathrm{id} \\ &= 2\sum_{j=1}^{n} b_j L_{\gamma,j} + \left(\sum_{j=1}^{n} b_j \pm c\right)\mathrm{id} = Q(X, D) \pm c\,\mathrm{id}, \end{aligned}$$

where $Q(x, \xi) := \sum_{j=1}^{n} b_j (x_j^2 + \xi_j^2)$.

REMARK 5.3.10. The equation above shows that

$$-\lambda(Q(X, D)) = \pi_{\pm\lambda}(\tilde{\mathcal{L}}_0) = -\lambda(2L_\gamma + n\,\mathrm{id}),$$

where $Q(x, \xi) = \|x\|^2 + \|\xi\|^2$. Thus we obtain that the symbol of the Orntein-Uhlenbeck operator is given by

$$\sigma_{L_\gamma}(x, \xi) = \frac{1}{2}(\|x\|^2 + \|\xi\|^2) - n = \sum_{j=1}^{n} \frac{x_j^2 + \xi_j^2 - 1}{2}.$$

THEOREM 5.3.11. *Let $(h_\alpha)_{\alpha\in\mathbb{N}_0^n}$ be the basis consisting of the generalized Hermite polynomials of $L^2(\mathbb{R}^n,\gamma)$. Moreover let $Q(x,\xi) := \sum_{j=1}^n b_j(x_j^2+\xi_j^2)$, where $b_j > 0$ for $j = 1\dots n$. Then we have*

$$(Q(X,D)\pm c\,\mathrm{id})h_\alpha = \left(2\sum_{j=1}^n b_j\alpha_j + \sum_{j=1}^n b_j \pm c\right) h_\alpha.$$

In addition $Q(X,D)\pm c\,\mathrm{id}$ extends to a selfadjoint operator with domain of definition $D(Q)$ given by

$$D(Q) := \{f \in L^2(\mathbb{R}^n,\gamma) \;:\; \sum_{n=1}^\infty n^2 \left\|P_{\Gamma_n} f\right\|^2 \leq \infty\}, \tag{100}$$

where P_{Γ_n} is the orthogonal projection on the closed linear span of the set $\{h_\alpha : |\alpha| = n\}$. Moreover, we obtain that span$\{h_\alpha : \alpha\in\mathbb{N}_0^n\}$ is a domain of essential selfadjointnes In addition we have

$$\sigma_{L^2(H_-,\gamma)}(-\frac{1}{\lambda}\pi_{\pm\lambda}(\tilde{\mathcal{L}}_{b,c})) = \{2\sum_{j=1}^n b_j\alpha_j + \sum_{j=1}^n b_j \pm c \;:\; \alpha\in\mathbb{N}_0^n\}.$$

PROOF. In view of the spectral theorem for unbounded operators and (99) this Theorem is clear except for equation (100), the domain of definition. First let us note that we have $0 < b_j < \infty$ for all $(j = 1\dots n)$. Clearly, $D(Q)$ is given by

$$D(Q) := \{f \in L^2(\mathbb{R}^n,\gamma) \;:\; \sum_{\alpha\in\mathbb{N}^n}\left(\sum_{j=1}^n b_j\alpha_j\right)^2 \left\|P_\alpha f\right\|^2 \leq \infty\}, \tag{101}$$

where P_α is the orthogonal projection on $\{\lambda h_\alpha \;:\; \lambda\in\mathbb{C}\}$. Now let us define $\beta := \min b_j$ and $\gamma := \max b_j$. Then we find $\beta\,|\alpha| \leq \sum_{j=1}^n b_j\alpha_j \leq \gamma\,|\alpha|$. Hence we have

$$D(Q) := \{f \in L^2(\mathbb{R}^n,\gamma) \;:\; \sum_{\alpha\in\mathbb{N}^n} |\alpha|^2 \left\|P_\alpha f\right\|^2 \leq \infty\}.$$

Now since h_α is an orthonormal basis we obtain

$$\sum_{\alpha\in\mathbb{N}^n} |\alpha|^2 \left\|P_\alpha f\right\| = \sum_{n=1}^\infty \sum_{|\alpha|=n} n^2 \left\|P_\alpha f\right\| = \sum_{n=1}^\infty n^2 \left\|P_{\Gamma_n} f\right\|,$$

which proves our assertion. □

Essentisal selfadjointness in the finite dimensional case. In chapter 2 we have shown the the Ornstein-Uhlenbeck Operator is essential selfadjoint on $\mathscr{C}_{int}^\infty(\mathbb{R}^n)$. Above we have discussed the symbol of this operator. Now we try to answer the question which perturbations of this operator are still selfadjoint. Of course there is the

THEOREM 5.3.12 (Kato-Rellich). *Let $(A, D(A))$ be a selfadjoint operator on a Hilbert space H and let $(B, D(B))$ be symmetric with $D(A) \subset D(B)$. Moreover, let us assume that B is A-bounded with A-bound less then one, e.g. there exists an $a < 1$ such that $\|Bx\| \leq a\|Ax\| + b\|x\|$ for all $x \in D(A)$. Then $(A + B, D(A))$ is selfadjoint.*

But now the question arises for which pseudodifferential operators $q(X, D)$ $\mathrm{L}_\gamma + q(X, D)$ is selfadjoint or essential selfadjoint. Let us start with the following Lemma:

LEMMA 5.3.13. *Let $A : S(R^n) \longrightarrow L^2(\mathbb{R}^n, \lambda^n)$ be a linear operator and define $\tilde{A} := V_{G,n}^{-1} A V_{G,n}$. Then we have*

(i) $\tilde{A} : S_\gamma(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n, \gamma)$.
(ii) *$\tilde{A}$ is closable if and only if A is closable. In this case we have $D(\overline{\tilde{A}}) = V_{G,n}^{-1} D(\overline{A})$.*
(iii) *$\tilde{A}$ is symmetric if and only if A is symmetric.*
(iv) *$\tilde{A}$ is essential selfadjoint on $S_\gamma(\mathbb{R}^n)$ if and only if A is essential selfadjoint on $S(\mathbb{R}^n)$.*

PROOF. The first part follows since $V_{G,n}$ maps $S_\gamma(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$ and $V_{G,n}^{-1}$ maps $L^2(\mathbb{R}^n, \lambda^n)$ to $L^2(\mathbb{R}^n, \gamma)$. In addition for $f \in S_\gamma(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n, \gamma)$ we find

$$\langle \tilde{A} f\,,\, g\rangle_{L^2(\mathbb{R}^n,\gamma)} = \langle V_{G,n}\tilde{A} f\,,\, V_{G,n} g\rangle_{L^2(\mathbb{R}^n,\lambda^n)} = \langle A V_{G,n} f\,,\, V_{G,n} g\rangle_{L^2(\mathbb{R}^n,\lambda^n)}. \tag{102}$$

To prove (ii) let A be closable. Then for $(f_n)_{n\in\mathbb{N}} \subset S_\gamma(\mathbb{R})$ and $f \in L^2(\mathbb{R}^n, \gamma)$ such that $f_n \xrightarrow[L^2(\mathbb{R}^n,\gamma)]{n\to\infty} 0$ and $\tilde{A} f_n \xrightarrow[L^2(\mathbb{R}^n,\gamma)]{n\to\infty} f$ we find

$$\langle f\,,\, g\rangle_{L^2(\mathbb{R}^n,\gamma)} = \lim_{n\to\infty} \langle \tilde{A} f_n\,,\, g\rangle_{L^2(\mathbb{R}^n,\gamma)} \lim_{n\to\infty} \langle A V_{G,n} f_n\,,\, V_{G,n} g\rangle_{L^2(\mathbb{R}^n,\lambda)} = 0,$$

since $V_{G,n} f_n \xrightarrow[L^2(\mathbb{R}^n,\lambda)]{n\to\infty} 0$ and A is closable. This shows that $\tilde{A}$ is closable. Using the same arguments we obtain the only if part similarly. Now let us prove the statements about the domains of definition of the closure. Thus let $f \in D(\overline{\tilde{A}})$. Then there exists a sequence $f_n \subset S_\gamma(\mathbb{R}^n)$ such that $f_n \xrightarrow[L^2(\mathbb{R}^n,\gamma)]{n\to\infty} f$ and $\tilde{A} f_n \xrightarrow[L^2(\mathbb{R}^n,\gamma)]{n\to\infty} \tilde{A} f$. Then we obtain for $g \in L^2(\mathbb{R}^n, \lambda)$

$$\begin{aligned} \langle A V_{G,n} f_n\,,\, g\rangle_{2(\mathbb{R}^n,\lambda)} &= \langle \tilde{A} f_n\,,\, V^{-1} g\rangle_{L^2(\mathbb{R}^n,\gamma)} \\ &\xrightarrow{n\to\infty} \langle \tilde{A} f\,,\, V^{-1} g\rangle_{L^2(\mathbb{R}^n,\gamma)} = \langle V_{G,n}\tilde{A} f\,,\, g\rangle_{L^2(\mathbb{R}^n,\gamma)}. \end{aligned}$$

Thus we find $V_{G,n} f \in D(\overline{A})$ which implies $f \in V_{G,n}^{-1} D(\overline{A})$. The other inclusion follows by the same arguments. Equation (102) implies (iii) and thus (iv) follows from (ii) and (iii). □

DEFINITION 5.3.14. For $m, m' \in \mathbb{R}$ we define

(i) $\psi c^{(m,m')} := \{q \in \mathscr{C}^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) : \forall \alpha, \beta \in \mathbb{N}_0^n \exists c_{\alpha,\beta} \geq 0$
$$\left|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)\right| \leq c_{\alpha,\beta}(1 + \|\xi\|^2)^{\frac{m-|\alpha|}{2}}(1 + \|x\|^2)^{\frac{m'-|\beta|}{2}}\}.$$
(ii) $G^m := \{q \in \mathscr{C}^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) : \forall \alpha, \beta \in \mathbb{N}_0^n \exists c_{\alpha,\beta} \geq 0$
$$\left|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)\right| \leq c_{\alpha,\beta}(1 + \|\xi\|^2 + \|x\|^2)^{\frac{m-|\alpha|-|\beta|}{2}}\}.$$

Now we can state

THEOREM 5.3.15. (i) *Let $0 \leq \delta < \varrho \leq 1$ and $p \in S^{2(\varrho-\delta),\|\cdot\|^2}_{\varrho,\delta}(\mathbb{R}^n)$ be real valued and set $A := q(X, D)$.*
(ii) *Let $p \in G^4$, $q \in \psi c^{(2,2)}$ be real valued and set $A := p(X, D) + q(X, D)$.*
(iii) *Let $p \in \psi c^{2,0}$ be real valued and depending only on ξ, $q \in \psi c^{(1,1)}$ be real valued and $r \in \psi c^{(0,2)}$ be real valued and depending only on x. Then set $A := p(X) + q(X, D) + r(X)$.*
In all three cases A is essential selfadjoint on $S_\gamma(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n, \gamma)$.

PROOF. Using Lemma 5.3.13 and 3.2.4 this theorem follows by [**25**, Theorem 4.3.2], [**25**, Theorem 4.3.4] and [**25**, Theorem 4.3.20]. □

REMARK 5.3.16. O. Caps proved theses results in the case of $\mathbb{R}^n$ with Lebesgue measure instead of Gaussian measure and $S(\mathbb{R}^n)$ instead of $S_\gamma(\mathbb{R}^n)$ in [**25**] using the Feffermann-Phong inequality.

REMARK 5.3.17. In 5.3.10 we have seen that the symbol of the Ornstein-Uhlenbeck operator is given by $q(x,\xi) = \frac{1}{2}(\|x\|^2 + \|\xi\|^2) - n$. It is clear that $\frac{1}{2}\|x\|^2 - n \in \psi c^{(0,2)}$ and $\|\xi\|^2 \in \psi c^{(2,0)}$. Thus for every $p \in \psi c^{(1,1)}$ being real valued we obtain that $\mathrm{L}_\gamma + q(X, D)$ is essential selfadjoint on $S_\gamma(\mathbb{R}^n)$.

The infinite dimensional case. Now let us return to the infinite dimensional case. Considering an infinite dimensional Heisenberg Group-Rigging there exist no Haar measure on these Heisenberg Groups. Moreover, we don't know how to define an "infinite-dimensional Heisenberg-Laplacian" or what is meant by

$$\pi_{\pm\lambda}(\tilde{\mathcal{L}}_{b,c}), \text{ where } \tilde{\mathcal{L}}_{b,c} = \sum_{j=1}^{\infty} b_j(\tilde{L}_j^2 + \tilde{M}_j^2) + c\, iT.$$

On the other hand, if we $(b_n)_{n\in\mathbb{N}} \in l^1(\mathbb{N})$, $b_n > 0$ so that the symbol $Q(x,\xi) := \sum_{j=1}^n b_j(\langle x\,,\, e_j\rangle_0^2 + \langle \xi\,,\, e_j\rangle_0^2)$ exists for all $x, \xi \in H_-$. Then we obtain a pseudodifferential operator $Q(X, D)$ given by 3.2.2. Using this definition and the results above we try to define $\pi_{\pm\lambda}(\tilde{\mathcal{L}}_{b,c})$. Thus let us proof at first the following

PROPOSITION 5.3.18. *Let $(b_n)_{n\in\mathbb{N}} \in l^1(\mathbb{N})$, $b_n > 0$ be a sequence such that the symbol $Q(x,\xi) := \sum_{j=1}^\infty b_j(\langle x\,,\, e_j\rangle_0^2 + \langle \xi\,,\, e_j\rangle_0^2)$ exists for all $x, \xi \in H_-$ and we have $|Q(x,\xi)| \leq\ c\|x\|^\alpha + \|\xi\|^\alpha$ ($\alpha \in \mathbb{N}$). Then we obtain that $Q(X, D)$ maps*

$S_{\gamma,cyl}(H_-)$ to $S_{\gamma,cyl}(H_-)$. In addition we have

$$Q(X,D)u = 2\sum_{j=1}^{n} b_j L_{\gamma,j}u + \sum_{j=1}^{\infty} b_j u,$$

where n is chosen such that $u(x) = u(P_n x)$.

PROOF. Let $u \in S_{\gamma,cyl}(H_-)$ such that $u(x) = u(P_n x)$. Then we obtain by the continuity of the Fourier-Wiener-Transform and Lebesgue's Theorem of dominated convergence

$$\begin{aligned} Q(X,D)u(x) &= \mathcal{F}^{-1}_{\xi\to x}\mathcal{F}_{y\to\xi} Q(\frac{x+y}{2},\xi)u(y) \\ &= \mathcal{F}^{-1}_{\xi\to x}\mathcal{F}_{y\to\xi} \sum_{j=1}^{\infty} b_j(\langle\frac{x+y}{2}\,,\, e_j\rangle_0^2 + \langle\xi\,,\, e_j\rangle_0^2)u(y) \\ &= \mathcal{F}^{-1}_{\xi\to x}\mathcal{F}_{y\to\xi} \sum_{j=1}^{\infty} b_j\langle\frac{x+y}{2}\,,\, e_j\rangle_0^2 + \mathcal{F}^{-1}_{\xi\to x}\mathcal{F}_{y\to\xi} \sum_{j=1}^{\infty} b_j\langle\xi\,,\, e_j\rangle_0^2 u(y) \\ &= \sum_{j=1}^{\infty} b_j\left(\mathcal{F}^{-1}_{\xi\to x}\mathcal{F}_{y\to\xi}\right)\langle\frac{x+y}{2}\,,\, e_j\rangle_0^2 + \langle\xi\,,\, e_j\rangle_0^2 u(y) \\ &= \sum_{j=1}^{\infty} b_j\left(2L_{\gamma,j} + id\right)u(x) \\ &= 2\sum_{j=1}^{n} b_j L_{\gamma,j}u(x) + \sum_{j=1}^{\infty} b_j u(x), \end{aligned}$$

which shows our proposition. □

Thus in view of Proposition 5.3.18 and 5.3.8 we give the following

DEFINITION 5.3.19. Let $A \in \mathscr{L}(H_- \times H_-, H_+ \times H_+)$ be a linear operator such that $\langle A(x,\xi)\,,\,(x,\xi)\rangle_{H_0\times H_0} > 0$ for all $(0,0) \neq (x,\xi) \in H_- \times H_-$ Now we set $P_A(x,\xi) = \langle A(x,\xi)\,,\,(x,\xi)\rangle_{H_0\times H_0}$. Then for $c \in \mathbb{R}$ we define

$$\pi_{\pm\lambda}(\tilde{\mathcal{P}}_{A,c}) := -\lambda(P_A(X,D) \pm c\ \mathrm{id}).$$

REMARK 5.3.20. Let us note that using this definition we obtain

$$\pi_{\pm\lambda}(\tilde{\mathcal{L}}_{b,c})u(x) = \lim_{n\to\infty} \pi^{(n)}_{\pm\lambda}(\tilde{\mathcal{L}}^{(n)}_{b,c})\tilde{u}(\tilde{P}_n x),$$

where $\pi^{(n)}$ denotes the representation of the n-dimensional Heisenberg Group, $\tilde{\mathcal{L}}^{(n)}_{b,c} := \sum_{j=1}^{n} b_j(\tilde{L}_j^2 + \tilde{M}_j^2)$ and $\tilde{u} \in S_\gamma(\mathbb{R}^n)$ defined by $\tilde{u}(\tilde{P}_n x) := u(P_n x)$. In this sense we can even say setting "$\tilde{L}_\gamma := \sum_{j=1}^{\infty} \frac{1}{2}(\tilde{L}_j^2 + \tilde{M}_j^2 - 1)$" that "$\pi_{\pm\lambda}(\tilde{\mathcal{L}}_\gamma) = L_\gamma$"

and thus we will obtain for L_γ the formal Symbol

$$\sigma_{L_\gamma}(x,\xi) = \sum_{j=1}^{\infty} \frac{x_j^2 + \xi_j^2 - 1}{2}.$$

But even if L_γ is a well known operator, this symbol remains a formal series, which does not converge on $H_- \times H_-$.

Thus let us consider the operators, where the symbol converges, i.e. the case of

$$\pi_{\pm\lambda}(\tilde{\mathcal{L}}_{b,c}), \text{ where } \tilde{\mathcal{L}}_{b,c} = \sum_{j=1}^{\infty} b_j(\tilde{L}_j^2 + \tilde{M}_j^2) + c\, iT.$$

Let $Q(x,\xi)$ be the symbol of the corresponding pseudodifferential operator i.e.

$$Q(x,\xi) = \sum_{j=1}^{\infty} b_j(\langle x\,,\, e_j\rangle_0^2 + \langle \xi\,,\, e_j\rangle_0^2).$$

THEOREM 5.3.21. *Let $(h_\alpha)_{\alpha\in\mathbb{N}_0^{\mathbb{N}}}$ be the basis consisting of the generalized Hermite polynomials of $L^2(H_-,\gamma)$. Moreover let $Q(x,\xi) := \sum_{j=1}^{\infty} b_j(x_j^2 + \xi_j^2)$. Then we obtain*

$$(Q(X,D) \pm c\,\mathrm{id})h_\alpha = \left(2\sum_{j=1}^{\infty} b_j\alpha_j + \sum_{j=1}^{\infty} b_j \pm c\right) h_\alpha. \tag{103}$$

In addition $Q(X,D)\pm c\,\mathrm{id}$ defined on span$\{h_\alpha \,:\, \alpha\in\mathbb{N}_0^n\}$ extends to a selfadjoint operator with domain of definition $D(Q)$ given by

$$D(Q) := \{f\in L^2(\mathbb{R}^n,\gamma) \,:\, \sum_{\alpha\in\mathbb{N}^n}\left(\sum_{j=1}^{\infty} b_j\alpha_j\right)^2 \|P_\alpha f\|^2 \le \infty\}, \tag{104}$$

where P_α is the orthogonal projection on $\{\lambda h_\alpha \,:\, \lambda\in\mathbb{C}\}$. Moreover, we obtain that span$\{h_\alpha \,:\, \alpha\in\mathbb{N}_0^n\}$ is a domain of essential selfadjointnes for $Q(X,D)\pm c\,\mathrm{id}$ and thus $S_{\gamma,cyl}(H_-)$ is a domain of essential selfadjointnes for $-\frac{1}{\lambda}\pi_{\pm\lambda}(\tilde{\mathcal{L}}_{b,c})$.

PROOF. Equation (103) follows directly by Proposition 5.3.18 and (104) is a direct consequence of the spectral theorem for unbounded operators since the $(h_\alpha)_{\alpha\in\mathbb{N}_0^{\mathbb{N}}}$ form an orthonormal basis of $L^2(H_-,\gamma)$. Considering the integration by parts formula proved in [**71**, Proposition 4.1.5] we obtain that $Q(X,D)$ is a positive symmetric operator on $S_{\gamma,cyl}(H_-)$. Thus it has a selfadjoint extension. Now since $Q(x,D)$ is essential selfadjoint on *span*$\{h_\alpha \,:\, \alpha\in\mathbb{N}_0^{\mathbb{N}}\}\subset S_{\gamma,cyl}(H_-)$ this extension must coincide with $Q(X,D)$ and our theorem is proved. □

Let us now calculate the spectrum of our operator $Q(X,D)\pm c$id. Thus let us prove the following

LEMMA 5.3.22. *Let $(b_j)_{j\in\mathbb{N}}$ be a sequence, such that $b_j > 0$ for all $j\in\mathbb{N}$ and $\lim_{j\to\infty} b_j = 0$. Then the set $\{\sum_{j=1}^{\infty} b_j\alpha_j \,:\, \alpha\in\mathbb{N}_0^N\}$ is dense in $\mathbb{R}_+$.*

Proof. Since $\lim_{j\to\infty} b_j = 0$ there exists a strictly monotone decreasing subsequence $(b_{j_k})_{k\in\mathbb{N}}$ of $(b_j)_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} b_{j_k} = 0$. Now let $a \in \mathbb{R}_+$ be fixed. Then there exists a sequence $(a_k)_{k\in\mathbb{N}} \subset N_0$ such that

$$\sum_{k=1}^{n} a_k b_{j_k} \leq a \leq \sum_{k=1}^{n} a_k b_{j_k} + b_{j_n}.$$

This implies that

$$a - \sum_{k=1}^{n} a_k b_{j_k} \leq b_{j_n} \xrightarrow{n\longrightarrow\infty} 0.$$

Now we define a sequence $\alpha^{(n)} \in \mathbb{N}_0^{\mathbb{N}}$ by

$$\alpha_j^{(n)} := \begin{cases} a_k & \text{if } j = j_k \text{ and } k \leq n \\ 0 & \text{else.} \end{cases}$$

However, we obtain

$$\sum_{j=1}^{\infty} b_j \alpha_j^{(n)} \xrightarrow{n\longrightarrow\infty} a.$$ □

This lemma leads us directly to the following Theorem:

Theorem 5.3.23. *Let $(b_n)_{n\in\mathbb{N}} \in l^1(\mathbb{N})$, $b_n > 0$ be a sequence such the symbol $Q(x,\xi) := \sum_{j=1}^{\infty} b_j(\langle x\,,\, e_j\rangle_0^2 + \langle \xi\,,\, e_j\rangle_0^2)$ exists for all $x, \xi \in H_-$ and we have $|Q(x,\xi)| \leq c\,\|x\|^a + \|\xi\|^a$ $(a \in \mathbb{N})$. Then for $c \in \mathbb{R}$ we have*

$$\sigma(Q(X,D) \pm c\,\mathrm{id}) = \{\lambda \in \mathbb{R}\ \ \lambda \geq \sum_{j=1}^{\infty} b_j \pm c\}.$$

Proof. By Theorem 5.3.21 we obtain

$$\{2\sum_{j=1}^{\infty} b_j\alpha_j + \sum_{j=1}^{\infty} b_j \pm c\ :\ \alpha \in \mathbb{N}_0^{\mathbb{N}}\} \subset \sigma(Q(X,D) \pm c\,\mathrm{id}),$$

since $2\sum_{j=1}^{\infty} b_j\alpha_j + \sum_{j=1}^{\infty} b_j \pm c$ are the eigenvalues of $\sigma(Q(X,D) \pm c\ \mathrm{id})$. But now Lemma 5.3.22 implies that

$$\{\lambda \in \mathbb{R}\ \ \lambda \geq \sum_{j=1}^{\infty} b_j \pm c\} \subset \sigma(Q(X,D) \pm c\,\mathrm{id}),$$

since the spectrum of a closed operator is closed. On the other hand in view of Theorem 5.3.21 it is clear that for $\lambda < \sum_{j=1}^{\infty} b_j \pm c$ the operator $\lambda\mathrm{id} - (Q(X,D) \pm c\,\mathrm{id})$ is invertible. But this proves our theorem. □

5.4. Ψ^*-Algebras generated by a representation of the Heisenberg Group

Let us start this section with a theorem which is due to Cordes

THEOREM 5.4.1 (Cordes 1979, manuscripta mathematica (see [**29**])). *Let $X = L^2(\mathbb{R}, \lambda)$. For $A \in \mathscr{L}(X)$ we set*

$$\alpha_{r,s}(A) := \tilde{\pi}(r,s,0)A\tilde{\pi}(r,s,0)^*,$$

where $(r,s,0) \in \mathcal{H}_1$ and denote by

$$\begin{aligned} &\mathscr{C}^\infty(\alpha, \mathscr{L}(X)) \\ = \;& \{A \in \mathscr{L}(X) \;:\; \alpha_{r,s} \text{ is } \mathscr{C}^\infty \text{ with respect to} (r,s) \in \mathbb{R}^2 \text{ and values in } \mathscr{L}(X)\}. \end{aligned}$$

Then

$$\mathscr{C}^\infty(\alpha, \mathscr{L}(X)) = \Psi^0_{0,0}(\mathbb{R}),$$

where $\Psi^0_{0,0}(\mathbb{R})$ denotes the classical Hörmander class in one dimension.

Now let us note some general facts about smooth elements. We will follow [**96**, Apendix 3] resp. [**11**, section 1.3] and use the notations of section 3.1.

Let H be a Hilbert-Space and $\alpha_t (t \in \mathbb{R})$ a strongly continuous one parameter group on H and denote by

$$V : H \supseteq \mathcal{D}(V) := \{x \in H \;:\; \exists Vx := \lim_{t\to 0} \frac{\alpha_t x - x}{t} \in X\} \longrightarrow X \;:\; x \longmapsto Vx$$

its infinitesimal generator. Then V is a closed, densely defined linear operator on H satisfying $\alpha_t(\mathcal{D}(V)) \subseteq \mathcal{D}(V)$ and $\alpha_t V = V\alpha_t$. Using the notations of section 3.1 we set $\Delta := \{\delta_V\}$, where δ_V is the closed derivation given by $\delta_V : \mathscr{L}(H) \supseteq \mathcal{B}(V) \longrightarrow \mathscr{L}(H) : a \longmapsto \delta_V(a)$. If the group α is unitary then δ_V is a $*$-derivation. Let $(\mathcal{A},\ (q_j)_{j\in\mathbb{N}_0})$ be a sub multiplicative Ψ^*-algebra which is continuously embedded in $\mathscr{L}(H)$. Then we set $\Psi^\alpha_n[\mathcal{A}] := \Psi^\Delta_n$.

Let us consider the map φ defined by

$$\varphi : \mathbb{R} \longrightarrow \mathscr{L}(\mathscr{L}(H)) : t \longmapsto [\varphi(t) : \mathscr{L}(H) \ni a \longmapsto \alpha_t a \alpha_t^{-1}].$$

For $a \in \mathscr{L}(H)$ we denote by $\varphi_a : \mathbb{R} \longrightarrow (H)$ the map

$$\varphi_\alpha(t) := \varphi(t)(a) = \alpha_t a \alpha_t^{-1}.$$

We assume that $\mathcal{A} \subset \mathscr{L}(H)$ is a C^*-algebra in $\mathscr{L}(H)$ with the induced topology and let the maps φ_a only have values in $\mathcal{A}$ for all $a \in \mathcal{A}$. For $n \in \mathbb{N}_0$ we set

$$\Psi^n_\alpha[\mathcal{A}] := \{a \in \mathcal{A} : \varphi_a \in \mathscr{C}^n(\mathbb{R}, \mathcal{A})\} \quad \text{and} \quad \Psi^\infty_\alpha[\mathcal{A}] := \cap_{j\in\mathbb{N}} \Psi^j_\alpha[\mathcal{A}].$$

Then we obtain the following

THEOREM 5.4.2. *Let $(\alpha_t)_{t\in\mathbb{R}}$ be a C_0 group and $\mathcal{A} \subset \mathscr{L}(H)$ be a C^*-subalgebra in $\mathscr{L}(H)$. Then*

(i) $\Psi^n_\alpha[\mathcal{A}] \subset \Psi^\alpha_n[\mathcal{A}]$;
(ii) $\Psi^\alpha_{n+1}[\mathcal{A}] \subset \Psi^n_\alpha[\mathcal{A}]$;

(iii) $\Psi_\alpha^\infty[\mathcal{A}] = \Psi_\infty^\alpha[\mathcal{A}]$.

Proof. See [**11**, Theorem 1.3.1]. □

Before we return to our infinite dimensional Heisbenberg group let us state the following theorem which is due to Goodmann (cf. [**47**, Theorem 1.1]).

Theorem 5.4.3. *Let G be a Lie Group, g the corresponding Lie algebra with basis $X_1, \dots X_n$ and π be a strongly continuous unitary representation of G on a Hilbert Space H. Let us denote by $d\pi(X_i)$ the infinitesimal generator of the semi group $\pi(\exp(tX_i)$. Assume for that $a \in H$ $a \in d\pi(X_i)^m$ for all $i = 1, \dots, n$ and $m \in \mathbb{N}$. Then $a \in \mathscr{C}^\infty(\pi, H)$.*

Now let us return to the infinite dimensional Heisenberg group.

Lemma 5.4.4. *Let $(r, s, t) \in \mathcal{H}_+$ and $\pi(r, s, t)$ be defined as in* 5.2.6*. Then we have*

$$\pi(r, s, t)A\pi(r, s, t)^* = \pi(r, s, 0)A\pi(r, s, 0)^*$$

for all $A \in L^2(H_-, \gamma)$.

Proof. According to Definition 5.2.6 we have $\pi(r, s, t) = e^{it}\pi(r, s, 0)$. In addition Theorem 5.2.10 implies that

$$\pi(r, s, t)^* = \pi(r, s, t)^{-1} = \pi(-r, -s, -t),$$

which yields

$$\pi(r, s, t)A\pi(r, s, t) = e^{it}\pi(r, s, 0)Ae^{-it}\pi(r, s, 0)^* = \pi(r, s, 0)A\pi(r, s, 0)^*.$$ □

As a direct consequence of Lemma 5.4.4 we obtain

Corollary 5.4.5. *For $(r, s, t) \in \mathcal{H}_+$ and $(r', s', t') \in \mathcal{H}_+$ we have*

$$\begin{aligned} & \pi(r, s, t)\pi(r', s', t')A\pi(r', s', t')^*\pi(r, s, t)^* \\ = \; & \pi(r', s', t')\pi(r, s, t)A\pi(r, s, t)^*\pi(r', s', t')^* \end{aligned}$$

Definition 5.4.6. For $(t, s, 0) \in \mathcal{H}_+$ and $A \in \mathscr{L}(L^2(H_-, \gamma))$ we define according to the general theory

$$\varphi_{r,s}(t)(A) := \pi(tr, ts, 0)A\pi(tr, ts, 0)^* = U_t^{(r,s,0)}A(U_t^{(r,s,0)})^*$$

and

$$\Psi_{r,s} := \Psi^\infty_{U_t^{(r,s,0)}}[\mathscr{L}(L^2(H_-, \gamma))].$$

Then we obtain that $\Psi_{r,s}$ is a Ψ^*-algebra. In addition in view of Goodman's theorem we set

$$\Psi^U := \bigcap_{j\in\mathbb{N}} (\Psi_{e_j,0} \cap \Psi_{0,e_j}).$$

Since the intersection of Ψ^*-algebras is a Ψ^*-algebra we find that Ψ^U is a Ψ^*-algebra.

PROPOSITION 5.4.7. *For* $(r,s,0)\in\mathcal{H}_+$ *we have*

$$\pi(r,s,0)=a(X,D),$$

where $a(x,\xi)=e^{i\langle s\,,x\rangle_0+i\langle r\,,\xi\rangle_0}=\int_{H_+^2}e^{i\langle x'\,,x\rangle_0+i\langle p'\,,\xi\rangle_0}d(\delta_{(s,r)}(x',p'))$.

PROOF. Let $a(x,\xi)=e^{i\langle s\,,x\rangle_0+i\langle r\,,\xi\rangle_0}$ and $f\in\mathscr{C}^\infty_{int}(H_-)$. Then Proposition 3.2.6 yields

$$\begin{aligned}a(X,D)f(x) &= \int_{H_+^2} W_{\frac{x'}{2}}U_{p'}W_{\frac{x'}{2}}f(x)d(\delta_{(s,r)}(x',p'))\\ &= \int_{H_+^2} e^{i\frac{\langle x'\,,x\rangle_0}{2}}\sqrt{\varrho_{p'}(x)}e^{i\frac{\langle x'\,,x+p'\rangle_0}{2}}d(\delta_{(s,r)}(x',p'))\\ &= \sqrt{\varrho_r(x)}e^{i\langle s\,,x\rangle_0+\frac{\langle r\,,s\rangle_0}{2}}f(x+r)=\pi(r,s,0)f(x),\end{aligned}$$

which shows our proposition since $\mathscr{C}^\infty_{int}(H_-)\subset L^2(H_-,\gamma)$ dense. □

Combining now Proposition 5.4.7 and Theorem 3.5.11 we obtain

THEOREM 5.4.8. *For* $m\in\mathbb{R}$ *let* H^m *be defined as in 3.3.3. Then* π *leaves* H^m *invariant, i.e.*

$$\pi(\mathcal{H}_+)H^m\subseteq H^m.$$

In addition for $(r,s,0)\in\mathcal{H}_+$ *we find that* $\pi(r,s,0)\in\mathscr{L}(H^m)$.

This theorem implies that for $(r,s)\in H_+^2$ and $A\in\mathscr{L}(H^m)$ we have $\varphi_{r,s}(t)(A)\in\mathscr{L}(H^m(H_-))$. Let us finally define generalized Hörmander classes given by smooth elements.

DEFINITION 5.4.9. Let $(r,s,0)\in\mathcal{H}_+$ fixed, $0<\varepsilon\le 1$ and $\varrho,\delta\in\mathbb{R}$. Then we set

$$\begin{aligned}\Psi_{\varepsilon,\varrho,\delta}:=\{A\in\mathcal{A}^\varepsilon\ \ :\ \ &[(r,s,t)\mapsto\partial_s^\alpha\partial_r^\beta\varphi_{r,s}(A)\in\mathscr{C}(\mathcal{H}_+,\mathscr{L}(H^m,H^{m+\varrho|\alpha|-\delta|\beta|}))\\ &\forall m\in\mathbb{R}\,\forall\alpha,\beta\in\mathbb{N}_0^{\mathbb{N}}\}.\end{aligned}$$

THEOREM 5.4.10. $\Psi_{\varepsilon,\varrho,\delta}$ *is a symmetric ans spectrally invariant subalgebra of* $\mathscr{L}(L^2(H_-,\gamma))$.

PROOF. Let us first prove that $\Psi_{\varepsilon,\varrho,\delta}$ is spectrally invariant. We will do this in three steps. Thus let $A\in\Psi_{\varepsilon,\varrho,\delta}$, such that $A^{-1}\in L^2(H_-,\gamma)$ Then we obtain $A\in\mathcal{A}^\varepsilon$.

(i) For $(r,s,0)\in\mathcal{H}_+$ we have

$$\begin{aligned}\varphi_{r,s}(A^{-1}) &= \pi(r,s,0)A^{-1}\pi(r,s,0)^{-1}\\ &= (\pi(-r,-s,0)A\pi(-r,-s,0)^{-1})^{-1}=(\varphi_{-r,-s}(A))^{-1}\end{aligned}$$

But since the inversion is continuous in $\mathscr{L}(H^m)$ we obtain

$$[(r,s,t)\mapsto\varphi_{r,s}(A^{-1})]\in\mathscr{C}(\mathcal{H}_+,\mathscr{L}(H^m)).$$

(ii) Now let $t \in \mathbb{R}$ and e_j be fixed. Then we find

$$\begin{aligned} & \frac{1}{t}(\varphi_{r+te_j,s}(A^{-1}) - \varphi_{r,s}(A^{-1})) \\ = \quad & -\varphi_{r+te_j,s}(A^{-1})\frac{\varphi_{r+te_j,s}(A) - \varphi_{r,s}(A)}{t}\varphi_{r,s}(A^{-1}) \\ \xrightarrow{t\longrightarrow 0} \quad & -\varphi_{r,s}(A^{-1})\partial_{r,e_j}\varphi_{r,s}(A)\varphi_{r,s}(A^{-1}). \end{aligned}$$

(iii) Now we obtain by induction for all $\alpha,\ \beta \in \mathbb{N}_0^{\mathbb{N}}$

$$\begin{aligned} & \partial_s^\alpha \partial_r^\beta \varphi_{r,s}(A^{-1}) \\ = & \sum_{\substack{\alpha^{(1)}+\ldots+\alpha^{(l)}=\alpha \\ \beta^{(1)}+\ldots+\beta^{(l)}=\beta}} c_{\alpha^{(1)},\ldots,\alpha^{(l)},\beta^{(1)},\ldots,\beta^{(l)}}\varphi_{r,s}(A^{-1})(\partial_s^{\alpha^{(1)}}\partial_r^{\beta^{(1)}}\varphi_{r,s}(A))\varphi_{r,s}(A^{-1}) \\ & \qquad (\partial_s^{\alpha^{(2)}}\partial_r^{\beta^{(2)}}\varphi_{r,s}(A))\varphi_{r,s}(A^{-1})\cdots \\ & \qquad (\partial_s^{\alpha^{(l)}}\partial_r^{\beta^{(l)}}\varphi_{r,s}(A))\varphi_{r,s}(A^{-1}), \end{aligned}$$

where $c_{\alpha^{(1)},\ldots,\alpha^{(l)},\beta^{(1)},\ldots,\beta^{(l)}} \in \mathbb{Z}$. This shows that $\Psi_{\varepsilon,\varrho,\delta}$ is spectrally invariant.

To prove that $\Psi_{\varepsilon,\varrho,\delta}$ is symmetric let us note that

$$\begin{aligned} & \frac{1}{t}(\varphi(r+te_j,s)(A^*)\varphi(r+te_j,s)^{-1} - \varphi(r,s)(A^*)\varphi(r,s)^{-1}) \\ = \quad & -(\frac{1}{-t}(\varphi(-r-te_j,-s)(A^*)\varphi(-r-te_j,s)^{-1} - \varphi(r,s)(A^*)\varphi(r,s)^{-1}))^*. \end{aligned}$$

Thus our assertion follows again by induction. □

Proposition and Definition 5.4.11. *For $m \in \mathbb{R}$ we set*

$$\mathcal{A}^{\varepsilon,m} := \Lambda^{m/2}\mathcal{A}^\varepsilon\Lambda^{m/2}$$

and

$$\widetilde{\mathcal{A}}^\varepsilon := \bigcup_{m\in\mathbb{R}}\mathcal{A}^{\varepsilon,m} \subseteq \bigcup_{m\in\mathbb{R}}\bigcap_{s\in\mathbb{R}}\mathscr{L}(H^s, H^{s-m}).$$

Moreover, let us assume that $A : H^\infty \to H^\infty$ is invertible and $A \in \widetilde{\mathcal{A}}^\varepsilon$ then $A^{-1} \in \widetilde{\mathcal{A}}^\varepsilon$, more precisely, if A^{-1} has order $-m$ then $A^{-1} \in \mathcal{A}^{\varepsilon,-m}$.

Proof. Let $A : H^\infty \to H^\infty$ be invertible and $A \in \widetilde{A}^{\varepsilon,m}$. Then we find that $\Lambda^{-m/2}A\Lambda^{-m/2} \in \mathcal{A}^\varepsilon$. But since $\mathcal{A}^\varepsilon$ is a Ψ^*-algebra we find that $\Lambda^{m/2}A^{-1}\Lambda^{m/2} = (\Lambda^{-m/2}A\Lambda^{-m/2})^{-1} \in \mathcal{A}^\varepsilon$ and thus $A^{-1} \in \mathcal{A}^{\varepsilon,-m}$. But this shows our assertion. □

Definition 5.4.12. Let $(r,s,0)\in\mathcal{H}_+$ fixed, $0<\varepsilon\leq 1$ and $m,\varrho,\delta\in\mathbb{R}$. Then we set

$$\Psi^m_{\varepsilon,\varrho,\delta}:=\{A\in\mathcal{A}^{\varepsilon,m}\ :\ [(r,s,t)\mapsto\partial_s^\alpha\partial_r^\beta\varphi_{r,s}(A)\in\mathscr{C}(\mathcal{H}_+,\mathscr{L}(H^s,H^{s-m+\varrho|\alpha|-\delta|\beta|}))\ \forall s\in\mathbb{R}\,\forall\alpha,\beta\in\mathbb{N}_0^{\mathbb{N}}\}$$

and

$$\Psi^\infty_{\varepsilon,\varrho,\delta}:=\bigcup_{m\in\mathbb{R}}\Psi^m_{\varepsilon,\varrho,\delta}.$$

Theorem 5.4.13. *For every $A\in\Psi^\infty_{\varepsilon,\varrho,\delta}$ being invertible on H^∞ with order $-k$ we have $A^{-1}\in\Psi^\infty_{\varepsilon,\varrho,\delta}$.*

Proof. The proof of this theorem is similar to 5.4.10. We will do it again in three steps. Thus let $A\in\Psi^m_{\varepsilon,\varrho,\delta}$, such that A^{-1} exists on H^∞. Then be 5.4.11 we obtain $A\in\mathcal{A}^{\varepsilon,m}$.

(i) For $(r,s,0)\in\mathcal{H}_+$ we have

$$\begin{aligned}\varphi_{r,s}(A^{-1}) &= \pi(r,s,0)A^{-1}\pi(r,s,0)^{-1}\\ &= (\pi(-r,-s,0)A\pi(-r,-s,0)^{-1})^{-1}=(\varphi_{-r,-s}(A))^{-1}.\end{aligned}$$

Thus we obtain

$$[(r,s,t)\mapsto\varphi_{r,s}(A^{-1})]\in\mathscr{C}(\mathcal{H}_+,\mathscr{L}(H^{s-k},H^s)).$$

(ii) Now let $t\in\mathbb{R}$ and e_j be fixed. Then we find

$$\begin{aligned}&\frac{1}{t}(\varphi_{r+te_j,s}(A^{-1})-\varphi_{r,s}(A^{-1}))\\ =\quad& -\varphi_{r+te_j,s}(A^{-1})\frac{\varphi_{r+te_j,s}(A)-\varphi_{r,s}(A)}{t}\varphi_{r,s}(A^{-1})\\ \xrightarrow{t\longrightarrow 0}\quad& -\varphi_{r,s}(A^{-1})\partial_{r,e_j}\varphi_{r,s}(A)\varphi_{r,s}(A^{-1}).\end{aligned}$$

Since $\varphi_{r,s}(A^{-1})$ maps H^{s-k} to H^s and $\partial_{r,e_j}\varphi_{r,s}(A)$ from H^s to $H^{s-k-\delta}$ we obtain $[(r,s,t)\mapsto\partial_{r,e_j}\varphi_{r,s}(A^{-1})\in\in\mathscr{C}(\mathcal{H}_+,\mathscr{L}(H^{s-k},H^{s-\delta})$.

(iii) By induction we obtain the same formula as in the proof of 5.4.10(iii). In addition, using the same arguments as above we find that

$$[(r,s,t)\mapsto\partial_s^\alpha\partial_r^\beta\varphi_{r,s}(A^{-1})\in\mathscr{C}(\mathcal{H}_+,\mathscr{L}(H^{s-k},H^{s+\varrho|\alpha|-\delta|\beta|})).$$

But this is our assertion.

□

Remark 5.4.14. According to Remark 4.5.25 we can attach to every operator $A\in\Psi_{\varepsilon,\varrho,\delta}$ an e_ξ-symbol. Moreover, if $H_-=\mathbb{R}^n$ or $A=B\otimes\mathrm{id}$, where $B\in\mathscr{L}(L^2(\mathbb{R}^n,\gamma_n))$ we get our operator in $\Psi_{\varepsilon,\varrho,\delta}$ back as pseudodifferential operator defined as in 4.1.5.

CHAPTER 6

Invariant measures for special groups of homeomorphisms on infinite dimensional spaces

Given a topological space X with σ-finite Borel measure μ, a locally compact group G and a representation B of G in the group of all homeomorphisms of X, we examine how to construct a Borel measure μ_s on X which is invariant under $B(G)$ (Lemma 6.1.9). In many cases this construction leads to a non-trivial representation of G on $L^p(X, \mu_s)$. We define the notion of a $\mathcal{NF}_p$ measure. Under some additional conditions on G, X and the representation B we show that in the case where μ has the $\mathcal{NF}_p$-property, the symmetrized measure μ_s is a $\mathcal{NF}_p$ measure, as well (Theorem 6.1.18). Finally we give some examples and an application of our work leads to the construction of spectrally invariant algebras (Ψ^*- or Ψ_0-algebras, cf. **[56]**, **[65]**)) of $\mathcal{C}^\infty$-elements in operator-algebras on L^p and L^2-spaces.

This chapter is a joint work with Wolfram Bauer; the main idea arose when we considered the following two problems:

a) Let (W, μ) be an open subset of a Hilbert space H with Gaussian measure μ_g, where μ is the restriction of μ_g to W. Furthermore, let $(B_t)_{t \in G}$ be a (semi) group of homeomorphisms of W where G is a compact or locally compact group. Is it possible to find a measure $\tilde{\mu}$ on W invariant with respect to (B_t), namely $\tilde{\mu}(B_t(A)) = \tilde{\mu}(A)$ for all $\mu-$measurable sets $A \subset W$ and $t \in G$ such that $\tilde{\mu}(A) > 0$ for all open nonempty sets $A \subset W$?

b) Let H^m be a product of an infinite dimensional Hilbert-space H with a Gaussian measure μ (e.g. product of suitable Sobolev spaces). We assume that $H \subseteq \mathcal{C}(\overline{\Omega}, \mathbb{C})$, where $\overline{\Omega}$ is the closure of an open and bounded subset of $\mathbb{R}^n$ with nice boundary. Let U be a region in $\mathbb{C}^m$ and G a closed subgroup of the group $\mathrm{Aut}(U)$ of all biholomorphic maps of U. Let $W := \{f \in H^m \;:\; f(\overline{\Omega}) \subset U\}$. Is it possible to find an invariant measure $\tilde{\mu}$ on W such that $\tilde{\mu}(\alpha(A)) = \tilde{\mu}(A)$ for all μ-measurable sets $A \subset W$ and all $\alpha \in G$ such that $\tilde{\mu}(A) > 0$ for all open nonempty sets $A \subset W$?

Let (M, g) be a Riemannian manifold with metric g. Then it is well-known that each isometry Φ on M leaves the Riemannian measure m_R invariant (see **[75]**, p. 85) and so Φ leads to an isometry of the spaces $L^p(M, m_R)$ where $1 \leq p < \infty$. In particular, each semi group $(\alpha_t)_{t \geq 0}$ of isometries on M can be represented as a semi group of isometric *composition operators* $(C_t)_{t\geq 0}$ on $L^p(M, m_R)$ by setting $C_t(f) := f \circ \alpha_t$ for $f \in L^p(M, m_R)$. In the case where $(C_t)_{t\geq 0}$ is strongly continuous it follows from the general theory of semi groups on Banach spaces that it defines a closed generator A which is connected to the geometry of M.

If the underlying measure space X is not locally compact one has to be more careful about the existence of invariant measures even if we deal with quite natural groups of isomorphisms acting on X. It is well-known that on an infinite dimensional separable Hilbert space H there is no translation invariant Borel measure μ such that bounded sets have finite measure and it holds $\mu(U) > 0$ for all open nonempty sets $U \subset H$ (see [**94**]). Hence the group action of H on itself by translation does not lead to an unitary representation of H in $L^p(H, \mu)$ for any Borel measure μ on H with the described properties. Moreover, Oxtoby (cf.[**112**]) showed, that on a complete separable metric group $\mathcal{G}$, which is not locally compact, there exists no non-trivial left-invariant Borel measure μ such that μ is locally finite or $\mu(K) < \infty$ for all $K \subset \mathcal{G}$ compact.

In this paper we consider the case in between. A locally compact space G acts on a topological space X which not necessarily has to be locally compact. More precisely, starting with a measure μ on X and a representation $B : G \to Homeo(X)$ of a locally compact group G into the group of all homeomorphisms on X, we adapt μ such that it becomes invariant under all homeomorphisms $B_t \in B(G)$ (Lemma 6.1.9). This construction is quite general and, in particular, it applies to the case where X is an open subspace of a separable infinite dimensional Hilbert space or of a $\mathcal{DFN}$-space (the dual space of a nuclear Frèchet space) (Theorem 6.1.16). As a result we obtain an answer to problem a). The definitions will be as follows:

Denote by m a left invariant *Haar measure* m on G, which is finite if and only if G is compact (in this case we choose m such that $m(G) = 1$). Let μ be any positive and σ-finite Borel measure on X and assume that the map $G \ni t \mapsto \mu(B_t^{-1}C) \in [0, \infty]$ is Borel-measurable on G for all sets C in the Borel-σ-algebra $\mathcal{B}(X)$, then define $\mu_s(C) := \int_G \mu(B_t^{-1}C)dm(t)$. We obtain a measure μ_s which is invariant under the action of G on X (e.g. $\mu_s(B_t^{-1}C) = \mu_s(C)$ for all $t \in G$) and finite in the case where μ is finite and G is compact (in general μ_s not even has to be σ-finite). We show that the definition of μ_s is meaningful if X is a polish space (i.e. complete metric space with countable base of topology) or an open set in a $\mathcal{DFN}$-space. Let $\tilde{B}_t$ denote the induced group action on $L^p(X, \mu_s)$ defined by the composition operators $\tilde{B}_t f := f \circ B_t$ for $f \in L^p(X, \mu_s)$. Then in many cases $(\tilde{B}_t)_{t\in G}$ is a strongly continuous group representation if $(B_t)_{t\in G}$ is so (Proposition 6.1.19, 6.1.20, 6.1.24). Here we use some measure theoretic methods and theorems, e.g. *Kuratowski's Theorem* and the fact that every open subset U of a $\mathcal{DFN}$-space can be written as a countable union of compact metric spaces.

Our construction produces closed operators attached to infinite dimensional spaces (or manifolds). This leads to Fréchet operator algebras with spectral invariance ([**67**], [**56**], [**98**], [**99**]) respectively non-commutative geometries with prescribed properties using systems of closed operators also in the singularities of the underlying space.

Let $\mathcal{F} \subset \mathcal{C}(X)$ be a subspace of all continuous complex-valued functions on X. We define the notion of a $\mathcal{NF}_p$ measure μ. Roughly speaking μ is characterized by the property that the embedding $\tilde{\mathcal{F}} := \mathcal{F} \cap L^p(X,\mu) \hookrightarrow \mathcal{C}(X)$ is continuous if $\tilde{\mathcal{F}}$ carries the $L^p(X,\mu)$-topology and $\mathcal{C}(X)$ is equipped with the *compact-open topology* (topology of uniform convergence on all compact subsets of X). Hence in the case where $\mathcal{C}(X)$ is complete we can consider the closure $\tilde{\mathcal{F}}_c$ of $\tilde{\mathcal{F}}$ in $L^p(X,\mu)$ as a space of continuous functions on X.

We give conditions on X, the group G and the representation B under which the described process of symmetrization of a given $\mathcal{NF}_p$ measure μ again defines a $\mathcal{NF}_p$ measure μ_s (Theorem 6.1.18). Starting with a $B(G)$-invariant subspace $\mathcal{F} \subset \mathcal{C}(X)$ (i.e. $\tilde{B}_t(\mathcal{F}) \subset \mathcal{F}$ for all $t \in G$) this enables us to consider groups of composition operators acting on closed subspaces of $L^p(X,\mu_s)$.

In the case where $p = 2$ we can define the orthogonal projection from $L^2(X,\mu_s)$ onto $\tilde{\mathcal{F}}_c$. We show that P and all $\tilde{B}_t$ commute as operators on $L^2(X,\mu_s)$ (Corollary 6.3.3). We denote by $\mathcal{T}(S) \subset \mathcal{L}(\tilde{\mathcal{F}}_c)$ the C^*-Toeplitz algebra generated by operators $T_f := PM_f$ on $\tilde{\mathcal{F}}_c$ with symbols f in a space S of bounded measurable and $\mathcal{B}$-invariant symbols. It turns out that $\mathcal{T}(S)$ is invariant under the isomorphisms $\mathbf{B}_t \in \mathcal{L}(\mathcal{L}(L^2(X,\mu)))$ defined by $\mathbf{B}_t(A) := \tilde{B}_t A \tilde{B}_{t^{-1}}$ where $t \in G$. This fact in connection with the general theory of [**67**], [**56**], [**98**] and [**99**] gives the possibility to construct Ψ^*-algebras in $\mathcal{T}(S)$ defined by iterated commutators with the infinitesimal generator of $(\mathbf{B}_t)_{t\in G}$.

We give several examples how to obtain homeomorphisms $(B_t)_{t\in G}$ which can be used in the constructions described above. In particular, we discuss the case of measures on finite products of Hilbert spaces which are embedded in a space of continuous function, e.g. let us take Sobolev-spaces of continuous functions. In case of our constructions we give an answer to problem *b*) mentioned above.

By quite similar methods we show that we can lift strongly continuous semi groups $(B_t)_{t\geq 0}$ of invertible operators on Hilbert spaces to semi groups $(\tilde{B}_t)_{t\geq 0}$ of composition operators on $L^2(H,\mu_{s,\alpha})$ (Theorem 6.1.27). Here $\mu_{s,\alpha}$ ($\alpha > 0$) is a finite Borel measure on H arising from an infinite dimensional *Gaussian measure*. The semi group $(\tilde{B}_t)_{t\geq 0}$ fails to be unitary but we obtain $\|\tilde{B}_t\| \leq e^{\frac{\alpha}{2}t}$ for all $t \geq 0$. More general, instead of H we can take open or closed subsets U of H and assume that $(B_t)_{t\geq 0}$ is a semi group of homeomorphism of U.

Finally, by a different method using the eigen-functions of the Beltrami-Laplace operator we show how to construct Gaussian measures on L^2-spaces over a compact and connected Riemannian manifolds M which are invariant under all composition operators with isometries Φ on M (Proposition 6.2.12, Theorem 6.2.13). This construction is closely related to the theory of dynamical systems.

6.1. Symmetric Borel measures on topological spaces

Let (X, Σ_1, μ) and (Y, Σ_2, m) be measure spaces. We denote by $M(X,Y)$ the space of all measurable functions from X to Y. Let $M^{-1}(X,Y)$ be the subspace of $M(X,Y)$ consisting of all invertible functions $h : X \to Y$ such that h as well as its inverse are measurable. We often write $M(X)$ (resp. $M^{-1}(X)$) instead of $M(X,X)$ (resp. $M^{-1}(X,X)$). Let $Q \in M(X)$, then the measure μ is called *Q-invariant* (or *Q-preserving*) iff $\mu^Q = \mu$ where $\mu^Q(M) := \mu(Q^{-1}M)$ for all $M \in \Sigma_1$. Generalizing the notation of *Q-invariance* to families of measurable maps, we define:

Definition 6.1.1. Let $\mathcal{Q} \subset M(X)$, then we call μ a $\mathcal{Q}$-invariant (or $\mathcal{Q}$-preserving) measure, if μ is Q-invariant for all $Q \in \mathcal{Q}$.

In the following we write $\mathcal{M}_\sigma(X)$ for the space of all σ-finite measures on X. In the case where X also is considered as a topological space the σ-algebra Σ_1 always will be the Borel σ-algebra $\mathcal{B}(X)$ on X. We denote by $\Sigma_1 \otimes \Sigma_2$ the smallest σ-algebra in $X \times Y$ such that both projections $P_X : X \times Y \to X$ and $P_Y : X \times Y \to Y$ are measurable.

Assume in addition that X is a topological space and $\mathcal{F} \subset \mathcal{C}(X)$ is a linear subspace of the algebra of continuous complex-valued functions on X. The following definition can also be found in [**54**].

Definition 6.1.2. Let $p \geq 1$, then we call $\mu \in \mathcal{M}_\sigma(X)$ a $\mathcal{NF}_p$ measure iff for each compact set $K \subset X$ there is a compact set $H \subset X$ with $K \subset H$ and $C > 0$ such that for all $f \in \mathcal{F}$

$$\sup\Big\{|f(x)| : x \in K\Big\} \leq C \left[\int_H |f(z)|^p d\mu(z)\right]^{\frac{1}{p}} \tag{105}$$

holds. The space of all $\mathcal{NF}_p$ measures on X is denoted by $\mathcal{MF}_p(X)$. We call X a $\mathcal{NF}_p$-space if $\mathcal{MF}_p(X) \neq \emptyset$.

Example 6.1.3. We give examples for $\mathcal{NF}_p$-spaces X, where $\mathcal{F} := \mathcal{H}(X)$ is the spaces of holomorphic functions on X. (For the notion of holomorphic functions on topological spaces see e.g. [**36**].)

(a) Let $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ be open and denote by V the usual Lebesgue measure on U. Then for $1 \leq p \leq 2$ and $\mathcal{F} := \mathcal{H}(U)$ it is well-known that V is a $\mathcal{NF}_p$ measure and so U is a $\mathcal{NF}_p$-space.

(b) Let $P(x, D)$ be a hypo-elliptic differential operator. Then the solution space of $P(x, D)$ is a $\mathcal{NF}_2$-space (cf. [**54**]).

(c) Let E be a $\mathcal{DFN}$-space (i.e. the dual space of a nuclear Fréchet space with the strong topology) and $\Omega \subset E$ be open in E. For the space $\mathcal{F} := \mathcal{H}(\Omega)$ and $1 \leq p \leq 2$ it can be shown that $\mathcal{MF}_p(\Omega) \neq \emptyset$. Hence Ω is a $\mathcal{NF}_p$-space. (see [**9**], [**133**]).

Finally we remind of the notion of *group representations*. Let G be a locally compact group, then by $Homeo(X)$ we denote the space of all homeomorphisms of X. A group homomorphism $B : G \ni t \mapsto B_t \in Homeo(X)$ is called a *representation of* G *in* $Homeo(X)$. The representation B is said to be continuous (resp. measurable) iff the map $(t, x) \mapsto B_t x$ of $G \times X$ into X is continuous (resp. $\mathcal{B}(G \times X) - \mathcal{B}(X)$-measurable).

Now we explicitly compute how a weighted Lebesgue measure on an open subset of $\mathbb{R}^n$ can be adapted to a given group representation. We are making use of the transformation formula for the Lebesgue integral which in general is not available for arbitrary measure spaces.

Fix $n \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ be open and G a compact group with unit $e \in G$. By $\mathrm{Diff}(\Omega)$ we denote the group of all diffeomorphisms of Ω. Assume that $B : G \to \mathrm{Diff}(\Omega)$ is a continuous representation of G in $\mathrm{Diff}(\Omega)$. Starting with a weighted Lebesgue measure $\mu \in \mathcal{M}_\sigma(\Omega)$ we want to construct a measure $\mu_s \in \mathcal{M}_\sigma(\Omega)$ which is $B(G)$-invariant. This construction arises from a procedure of integration of μ along $B(G)$. For $i = 1, \cdots, n$ we denote by $\pi_i : \mathbb{R}^n \to \mathbb{R}$ the projection on the i-th component. Then we assume that all the maps given in (i) and (ii):

(i) $\Omega \ni z \mapsto \Big[G \ni t \mapsto \pi_i \circ B_{t^{-1}} z\Big] \in \mathcal{C}(G, \mathbb{R})$, for $i = 1, \cdots, n$;

(ii) $\Omega \ni z \mapsto \Big[G \ni t \mapsto \frac{\partial}{\partial z_j}\{\pi_i \circ B_{t^{-1}} z\}\Big] \in \mathcal{C}(G, \mathbb{R})$ for $i, j = 1, \cdots, n$

are well-defined and continuous on Ω if $\mathcal{C}(G, \mathbb{R})$ carries the topology of uniform convergence on G. Let m be the unique translation-invariant *Haar measure* on G with $m(G) = 1$ and assume that $g : \Omega \to \mathbb{R}^+$ is a positive and continuous weight-function. Let us consider $\mu \in \mathcal{M}_\sigma(\Omega)$ defined by $d\mu = g dV$, where V is the usual Lebesgue measure on Ω. We show that a $B(G)$-invariant measure μ_s on Ω is given by $d\mu_s := f dV$ where

$$f(z) := \int_G g \circ B_{t^{-1}}(z) \Big| \det[D_z B_{t^{-1}}](z) \Big| dm(t), \qquad z \in \Omega. \tag{106}$$

LEMMA 6.1.4. *Let $\Omega \subset \mathbb{R}^n$ be open and assume that $\mu_s \in \mathcal{M}_\sigma(\Omega)$ is defined by $d\mu_s = f dV$. Then μ_s is $B(G)$-invariant.*

PROOF. Let $t_0 \in G$ and $A \in \mathcal{B}(\Omega)$ be a Borel set in Ω. Then, using the *transformation formula* for the Lebesgue integral, we find with the characteristic function χ_A of A:

$$\begin{aligned}
& \mu_s(B_{t_0}^{-1}A) \\
= & \int\limits_\Omega \chi_{B_{t_0}^{-1}A}(z) f(z) dV(z) \\
= & \int\limits_G \int\limits_\Omega \chi_A \circ B_{t_0}(z) g \circ B_{t^{-1}}(z) \Big| \det[D_z B_{t^{-1}}](z) \Big| dV(z) dm(t) \\
= & \int\limits_G \int\limits_\Omega \chi_A(z) g \circ B_{(t_0 t)^{-1}}(z) \Big| \det[D_z B_{t^{-1}}](B_{t_0^{-1}}(z)) \det[D_z B_{t_0^{-1}}](z) \Big| dV(z) dm(t) \\
= & \int\limits_\Omega \chi_A(z) \int\limits_G g \circ B_{(t_0 t)^{-1}}(z) \Big| \det[D_z B_{(t_0 t)^{-1}}](z) \Big| dm(t) dV(z) \\
= & \int\limits_\Omega \chi_A(z) f(z) dV(z) = \mu_s(A).
\end{aligned}$$

Here we have used the translation invariance of m on G in the last equality. □

The question arises whether or not the measure μ_s is a $\mathcal{NF}_p$ measure for a subspace $\mathcal{F} \subset \mathcal{C}(\Omega)$, whenever μ has this property. We can prove:

LEMMA 6.1.5. *Let X be a topological space, $\mathcal{F} \subset \mathcal{C}(X)$ a subspace and $\mu \in \mathcal{MF}_p(X)$ where $p \geq 1$. If $g : X \to \mathbb{R}^+$ is a continuous positive function and $\tilde{\mu}$ is defined by $d\tilde{\mu} = g d\mu$, then $\tilde{\mu} \in \mathcal{MF}_p(X)$ as well.*

PROOF. Fix a compact set $K \subset X$. Then, by assumption, there is a compact set $H \subset X$ such that $K \subset H$ and $C > 0$ with

$$\sup \Big\{ |f(x)| : x \in K \Big\} \leq C \left[\int\limits_H |f(z)|^p d\mu(z) \right]^{\frac{1}{p}}.$$

for all $f \in \mathcal{F}$. Define $\varepsilon := \inf\{|g(z)| : z \in H\} > 0$, then inequality (105) holds with $\tilde{\mu}$ instead of μ and $C\varepsilon^{-1} > 0$ instead of C. □

REMARK 6.1.6. From Lemma 6.1.5 it is easy to see that for each continuous function $h : X \to \mathbb{C}$ and each finite measure $\mu \in \mathcal{MF}_p(X)$ it can be constructed $\tilde{\mu} \in \mathcal{MF}_p(X)$ such that h is $\tilde{\mu}$-integrable (use the weight $g(z) := (1 + |h(z)|)^{-1}$ for all $z \in X$).

For the next lemma let us assume that $\Omega \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$. Then we obtain with our notations above:

LEMMA 6.1.7. *Assume that $g : \Omega \to \mathbb{R}^+$ is uniformly continuous. Then μ as well as μ_s belong to $\mathcal{MF}_p(\Omega)$ where $\mathcal{F} := \mathcal{H}(\Omega)$ is the space of all holomorphic functions on Ω and $1 \leq p \leq 2$.*

PROOF. According to example (a) we have $V \in \mathcal{MF}_p(\Omega)$ for $1 \leq p \leq 2$. In order to show that μ_s is a $\mathcal{NF}_p$ measure it is enough to prove that $f : \Omega \to \mathbb{R}^+$ in (106) is continuous and positive (see Lemma 6.1.5). This easily follows from assumptions (i) and (ii) on B. □

If we deal with a topological space X (e.g. X is an infinite dimensional Hilbert space or a $\mathcal{DFN}$-space) in general we can not directly make use of the transformation formula. Let us find an equivalent definition for μ_s where μ is a finite Borel measure on X. For a Borel set $A \in \mathcal{B}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is open we have from our definitions above ($d\mu = gdV$):

$$\begin{aligned}
\mu_s(A) &= \int_\Omega \int_G \chi_A(z) g \circ B_{t^{-1}}(z) \Big| \det[D_z B_{t^{-1}}](z) \Big| dm(t) dV(z) \\
&= \int_G \int_\Omega \chi_A \circ B_t(z) g(z) \Big| \det[D_z B_{t^{-1}}](B_t z) \cdot \det[D_z B_t](z) \Big| dV(z) dm(t) \\
&= \int_G \int_\Omega \chi_{B_{t^{-1}} A}(z) \Big| \det[D_z B_{t^{-1}t}](z) \Big| g(z) dV(z) dm(t) \\
&= \int_G \mu(B_t^{-1} A) dm(t).
\end{aligned}$$

We have used that $B_{t^{-1}t} = B_e = id$. The expression on the right hand side also makes sense for a wider class of Borel measures $\tilde{\mu}$ on a topological space X, provided that the mapping $G \ni t \mapsto \tilde{\mu}(B_t^{-1} A) \in [0, \infty]$ is $\mathcal{B}(G)$-measurable.

DEFINITION 6.1.8. Let (X, Σ_1, μ) and (Y, Σ_2, m) be σ-finite measure spaces. Assume that there is a map $B : Y \to M^{-1}(X)$ such that

$$Y \ni t \mapsto \mu(B_t^{-1} A) \in [0, \infty] \tag{107}$$

is Σ_2- measurable for all $A \in \Sigma_1$. Then we define the *symmetrization* μ_s of μ w.r.t. to B to be the integral $\mu_s(A) := \int_Y \mu(B_t^{-1} A) dm(t)$.

In our applications we often assume that X is a topological space with Borel σ-algebra $\mathcal{B}(X)$ and μ is a finite or σ-finite Borel measure on X. For the measure space (Y, Σ_2, m) we choose a compact or locally compact group $G = Y$ with the translation invariant *Haar measure* m. The mapping $B : G \to M^{-1}(X)$ is a group homomorphism from G into $Homeo(X)$.

LEMMA 6.1.9. *The symmetrization μ_s defines a Borel measure on Σ_1. If in addition $Y = G$ is a locally compact group with left-invariant Haar measure m and $\Sigma_2 := \mathcal{B}(G)$ then μ_s is $B(G)$-invariant for a group homomorphism $B : G \to M^{-1}(X)$.*

PROOF. By assumption the map $Y \ni t \mapsto \mu(B_t^{-1} A) \in [0, \infty]$ is Σ_2-measurable for any set $A \in \Sigma_1$ and we conclude that μ_s is well-defined on Σ_1. We prove the

σ-additivity of μ_s. Let $(A_i)_{i\in\mathbb{N}} \subset \Sigma_1$ be a sequence such that $A_i \cap A_j = \emptyset$ for $i \neq j$. Because for each $t \in G$ the map B_t is one-to-one it follows $B_t^{-1}A_i \cap B_t^{-1}A_j = \emptyset$ for $i \neq j$ and $B_t^{-1}[\bigcup_i A_i] = \bigcup_i B_t^{-1}A_i$. Hence by the σ-additivity of μ we have

$$Y \ni t \mapsto \sum_i \mu(B_t^{-1}A_i) = \mu\Big(B_t^{-1}\Big[\bigcup_i A_i\Big]\Big) \in [0,\infty] \tag{108}$$

and the map (108) is Σ_2-measurable. Now, the theorem of dominated convergence applied to μ implies:

$$\mu_s\Big(\bigcup_i A_i\Big) = \int\limits_G \sum_i \mu(B_t^{-1}A_i)dm(t) = \sum_i \int\limits_G \mu(B_t^{-1}A_i)dm(t) = \sum_i \mu_s(A_i).$$

In the case where $Y = G$ is a locally compact group with left-invariant Haar measure m and $B : G \to M^{-1}(X)$ is a group homomorphism we can prove the $B(G)$-invariance of μ_s. Fix $t_0 \in G$ and $A \in \Sigma_1$, then it follows that

$$\mu_s^{B_{t_0}}(A) = \mu_s(B_{t_0^{-1}}A) = \int\limits_G \mu(B_{t^{-1}}B_{t_0^{-1}}A)dm(t) = \int\limits_G \mu(B_{(t_0 t)^{-1}}A)dm(t) = \mu_s(A)$$

by the left-translation invariance of the Haar measure m on G. □

With the notations of Definition 6.1.8 we want to find conditions under which the map (107) is Σ_2- measurable on Y for all $A \in \Sigma_1$.

Lemma 6.1.10. *Let $F : Y \times X \to X$ with $F(t,x) := B_t x$ be $\Sigma_2 \otimes \Sigma_1$-$\Sigma_1$-measurable. Then $Y \ni t \mapsto \mu(B_t^{-1}A) \in [0,\infty]$ is Σ_2-measurable for each $A \in \Sigma_1$.*

Proof. Let $A \in \Sigma_1$. By our assumption $\chi_A \circ F : Y \times X \to \mathbb{R}$ is $\Sigma_2 \otimes \Sigma_1$-measurable. Using *Tonelli's theorem* it follows that:

$$Y \ni t \mapsto \int\limits_X \chi_A \circ F(t,x)d\mu(x) = \int\limits_X \chi_{B_t^{-1}A}(x)d\mu(x) = \mu(B_t^{-1}A) \in [0,\infty]$$

is a Σ_2-measurable function (see [**8**]). □

We conclude that under the assumptions of Lemma 6.1.10 the symmetrization μ_s of μ is a well-defined measure on (X,Σ_1) (which does not have to be σ-finite again).

Let $\Omega \subset \mathbb{R}^n$ be open, $g : \Omega \to \mathbb{R}^+$ a continuous and strictly positive weight function and $\mu \in \mathcal{M}_\sigma(\Omega)$ defined by $d\mu = gdV$. Given a continuous representation B of a compact group G in Diff(Ω) with (i) and (ii) we have shown (see Lemma 6.1.4) that the $B(G)$-invariant measure μ_s is absolutely continuous w.r.t. the Lebesgue measure. The following example points out that this property does not hold in the more general setting of Definition 6.1.8. We give a finite measure μ on a Hilbert space H with the property $\mu(U) > 0$ for all open subsets $U \subset H$ and a group representation $B : \mathbb{R} \to Homeo(H)$ such that μ and μ_s are orthogonal (i.e. there is $X \subset H$ with $\mu(X) = 1$ and $\mu_s(X) = 0$, see [**35**, p. 60]).

EXAMPLE 6.1.11. Let H_1, H_2 be separable infinite dimensional Hilbert spaces. In addition we assume that there is a dense and continuous embedding $I : H_1 \hookrightarrow H_2$. Fix a Gaussian measure μ_1 on H_1 with the property $\mu_1(U) > 0$ for all open subsets $U \subset H_1$ and define the measure μ_2 on H_2 by $\mu_2(A) := \mu_1(A \cap H_1)$ for all $A \in \mathcal{B}(H_2)$. Then $\mu_2 = \mu_1^I$ and it is well known (see **[35]**, p. 44) that μ_2 is a Gaussian measure on H_2. Moreover, $\mu_2(H_1) = \mu_1(H_1) = 1$ and $\mu_2(V) > 0$ for all open sets $V \subset H_2$ because H_1 is dense in H_2. Choose $0 \neq a \in H_2 \setminus H_1$ and consider the representation $(B_t)_{t\in\mathbb{R}}$ of $\mathbb{R}$ in H_2 defined by $B_t y := y + ta$ for all $y \in H_2$. Because of $H_1 + ta \cap H_1 = \emptyset$ for $t \neq 0$ and $\mu_2(H_1) = \mu_2(H_2) = 1$ it follows $\mu_2(H_1 + ta) = 0$ for all $t \neq 0$. Let us choose $(X, \Sigma_1, \mu) = (H_2, \mathcal{B}(H_2), \mu_2)$ and $(Y, \Sigma_2, m) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), e^{-t^2}dt)$ in Definition 6.1.8. We obtain:

$$\mu_2(H_1) = 1, \qquad (\mu_2)_s(H_1) = \int\limits_{\mathbb{R}} \mu_2(H_1 + ta)\, e^{-t^2} dt = 0$$

and so the measures μ_2 and $(\mu_2)_s$ are orthogonal on H_2 with the desired properties.

Now let us describe how to integrate w.r.t μ_s. With the notations of Definition 6.1.8 we assume that the function $F : Y \times X \to X$ with $F(t,x) := B_t x$ is $\Sigma_2 \otimes \Sigma_1$-$\Sigma_1$-measurable.

LEMMA 6.1.12. *Let $f : X \to [0,\infty]$ be a non-negative Σ_1-measurable numerical function. Then with the product measure $m \otimes \mu$ on $\Sigma_2 \otimes \Sigma_1$ we have $\int_X f d\mu_s = \int_{Y\times X} f \circ F d(m \otimes \mu)$.*

PROOF. First let us assume that $g : X \to \mathbb{R}_0^+$ is a Σ_1-step-function on X. Then we can write $g = \sum_{i=1}^n \alpha_i \chi_{A_i}$ where $A_i \in \Sigma_1$ and $\alpha_i > 0$ for $i = 1, \cdots, n$. It follows:

$$\begin{aligned} \int\limits_X g d\mu_s &= \sum_{i=1}^n \alpha_i \mu_s(A_i) = \sum_{i=1}^n \alpha_i \int\limits_Y \int\limits_X \chi_{B_t^{-1}A_i}(x) d\mu(x) dm(t) \\ &= \sum_{i=1}^n \alpha_i \int\limits_Y \int\limits_X \chi_{A_i} \circ F(t,x) d\mu(x) dm(t) = \int\limits_{Y\times X} g \circ F d(m \otimes \mu). \end{aligned} \tag{109}$$

For an arbitrary Σ_1-measurable numerical function $f \geq 0$ let $(g_n)_{n\in\mathbb{N}}$ be a sequence of non-negative Σ_1-step-functions with $g_n \uparrow f$. Then $(g_n \circ F)_{n\in\mathbb{N}}$ is a sequence of $\Sigma_2 \otimes \Sigma_1$-step-functions with $g_n \circ F \uparrow f \circ F$. From equation (109) the assertion follows. □

In particular, under the conditions of Lemma 6.1.12 it follows that a Σ_1-measurable numerical function $f : X \to \mathbb{C}$ is μ_s-integrable iff $f \circ F : G \times X \to \mathbb{C}$ is $m \otimes \mu$-integrable and the integrals coincide. Let (X, Σ_1, μ) be a σ-finite measure space and let $Y := G$ be a locally compact group with left-invariant Haar measure m. If $B : G \to M^{-1}(X)$ is a representation of G such that the function $F : G \times X \to X$ in Lemma 6.1.10 is $\mathcal{B}(G) \otimes \Sigma_1$- Σ_1-measurable, then we can prove:

COROLLARY 6.1.13. *Let $t_1, t_2 \in G$ and $f : X \to \mathbb{C}$ be Σ_1-measurable. Then $f \circ B_{t_1}$ is μ_s-integrable iff $f \circ B_{t_2}$ is μ_s-integrable and in this case both integrals coincide.*

PROOF. By Lemma 6.1.12, *Fubini's Theorem* and using the translation invariance of m we find:

$$\begin{aligned}\int\limits_X \Big|f \circ B_{t_1}(z)\Big| d\mu_s(z) &= \int\limits_{G\times X} \Big|f \circ B_{t_1} \circ B_t(z)\Big| d(m \otimes \mu)(t, z) \\ &= \int\limits_X \int\limits_G \Big|f \circ B_{t_1 t}(z)\Big| dm(t) d\mu(z) \\ &= \int\limits_X \int\limits_G \Big|f \circ B_{t_2 t}(z)\Big| dm(t) d\mu(z) = \int\limits_X \Big|f \circ B_{t_2}(z)\Big| d\mu_s(z).\end{aligned}$$

Now the assertion follows from *Tonelli's theorem.* □

For a topological space Y denote by $O(Y)$ the family of all open sets in Y. A complete metric space Y with countable base $\mathcal{X} \subset O(Y)$ (i.e. each $A \in O(Y)$ is union of sets in the countable system $\mathcal{X}$) is called *polish space*. In general the inclusion $\mathcal{B}(Y) \otimes \mathcal{B}(X) \subset \mathcal{B}(Y \times X)$ holds, but if we restrict ourselves to *polish spaces* or *$\mathcal{DFN}$-spaces* we can prove:

PROPOSITION 6.1.14. *Let Y and X be polish spaces and consider $Y \times X$ with the product metric. Then we have $\mathcal{B}(Y \times X) = \mathcal{B}(Y) \otimes \mathcal{B}(X)$.*

PROOF. Fix countable bases $\mathcal{Y}$ (resp. $\mathcal{X}$) of open sets in Y (resp. in X). Consider the system

$$\mathcal{Y} \otimes \mathcal{X} := \Big\{U \times V : U \in \mathcal{Y} \text{ and } V \in \mathcal{X}\Big\} \subset O(Y \times X).$$

Then $\mathcal{Y} \otimes \mathcal{X}$ is a countable base for $Y \times X$ and so it generates $\mathcal{B}(Y \times X)$. On the other hand $\mathcal{Y}$ (resp. $\mathcal{X}$) generates $\mathcal{B}(Y)$ (resp. $\mathcal{B}(X)$) and so by Satz 22.1 in [**8**] we conclude that $\mathcal{Y} \otimes \mathcal{X}$ also generates $\mathcal{B}(Y) \otimes \mathcal{B}(X)$. Hence $\mathcal{B}(Y \times X) = \mathcal{B}(Y) \otimes \mathcal{B}(X)$. □

Now let us consider a $\mathcal{DFN}$-space E (i.e. E is the strong dual of a nuclear Fréchet space). In general there is no metric on E which induces the topology. But it is known (see [**109**]) that each open subset $U \subset E$ can be written as a countable union of compact metric spaces each with countable base (we call U *hemi-compact*).

PROPOSITION 6.1.15. *Let E be a $\mathcal{DFN}$-space and $U \subset E$ be open. If Y is a polish space and $Y \times U$ carries the product topology, then $\mathcal{B}(Y \times U) = \mathcal{B}(Y) \otimes \mathcal{B}(U)$.*

PROOF. Fix a fundamental system $(K_i)_{i\in\mathbb{N}} \subset U$ of compact sets (i.e. $K_i \subset K_{i+1}$ for $i \in \mathbb{N}$ and $U = \bigcup_i K_i$, see [**109**]). Then for each $i \in \mathbb{N}$ the complete metric space K_i has a countable base $\mathcal{K}_i \subset O(K_i) \subset \mathcal{B}(U)$. Fix a countable base $\mathcal{Y} \subset O(Y)$ of Y and consider the system

$$\mathcal{Y} \otimes \mathcal{K} := \bigcup_{i\in\mathbb{N}} \Big\{ Z \times V_i : Z \in \mathcal{Y} \text{ and } V_i \in \mathcal{K}_i \Big\}.$$

Then $\mathcal{Y}\otimes\mathcal{K}$ is a countable system of sets in $\mathcal{B}(Y \times U)$. Indeed, if $P_Y : Y \times U \to Y$ and $P_U : G \times U \to U$ denote the continuous projections, it follows:

$$Z \times V_i = P_Y^{-1}(Z) \cap P_U^{-1}(V_i) \subset \mathcal{B}(Y \times U), \qquad \forall\ Z \times V_i \in \mathcal{Y} \otimes \mathcal{K}.$$

Let $W \subset Y \times U$ be open and $(x, w) \in W$. Then fix $i \in \mathbb{N}$ with $(x, w) \in Y \times K_i$. Because $W \cap [Y \times K_i]$ is open in $Y \times K_i$ and Y and K_i are metric spaces we find $Z \times V_i \in \mathcal{Y} \otimes \mathcal{K}$ with

$$(x, w) \in Z \times V_i \subset W \cap [Y \times K_i] \subset W.$$

Hence $W = \bigcup\{Z \times V_i \in \mathcal{Y} \otimes \mathcal{K} : Z \times V_i \subset W\}$ is a countable union and so $\mathcal{B}(Y \times U)$ is generated by $\mathcal{Y} \otimes \mathcal{K}$. Because $\mathcal{Y}$ generates the Borel-σ-algebra $\mathcal{B}(Y)$ and $\bigcup_i\{V_i : V_i \in \mathcal{K}_i\}$ generates $\mathcal{B}(U)$ it follows from Satz 22.1 in [**8**] that $\mathcal{Y} \otimes \mathcal{K}$ also generates $\mathcal{B}(Y) \otimes \mathcal{B}(U)$. □

The well-known fact, that each compact space with countable base is metrizable together with Lemma 6.1.10, Proposition 6.1.14 and 6.1.15 now leads to:

THEOREM 6.1.16. *Let G be a compact group with countable base and assume that X is a polish space or an open set in a $\mathcal{DFN}$-space. Let $\mu \in \mathcal{M}_\sigma(X)$ be finite and $B : G \to M^{-1}(X)$ a measurable representation. Then for each $A \in \mathcal{B}(X)$ the map $G \ni t \mapsto \mu(B_t^{-1}A) \in \mathbb{R}^+$ is integrable over G.*

Application to group representations.
We show, that under some continuity conditions on $F : G \times X \to X$ with $F(t, x) := B_t x$ the space $\mathcal{MF}_p(X)$ is invariant under the symmetrization process. In this section, if nothing else is said, we assume that X is a polish space or an open subset of a $\mathcal{DFN}$-space with the Borel σ-algebra. Moreover, let G be a compact group with countable base and $B : G \to Homeo(X)$ a continuous group representation of G in the space of all homeomorphisms of X.

DEFINITION 6.1.17. A subspace $\mathcal{H} \subset M(X, \mathbb{C})$ is called $B(G)$-*invariant* iff for all $f \in \mathcal{H}$ we have the inclusion $\{f \circ B_t : t \in G\} \subset \mathcal{H}$.

For any $\mathcal{H} \subset M(X, \mathbb{C})$ consider $\mathcal{H}_G := \{f \circ B_t : f \in \mathcal{H},\ t \in G\}$. Then $\mathcal{H}_G$ is a $B(G)$-invariant space and $\mathcal{H}$ is $B(G)$-invariant itself iff $\mathcal{H} = \mathcal{H}_G$.

THEOREM 6.1.18. *Let $\mathcal{F} \subset M(X, \mathbb{C})$ be $B(G)$-invariant and $\mu \in \mathcal{MF}_p(X)$ where $p \geq 1$, then it follows that $\mu_s \in \mathcal{MF}_p(X)$ as well.*

Proof. According to Theorem 6.1.16 μ_s is well-defined. Let $K_1 \subset X$ be compact, then we conclude from the continuity of the representation B that the spaces $G \times K_1 \subset G \times X$ and $K_2 := F(G \times K_1) \subset X$ are compact, as well. Because $\mu \in \mathcal{MF}_p(X)$ and $\mathcal{F}$ is a $B(G)$-invariant space, there is $C > 0$ and a compact set K_3 with $K_2 \subset K_3 \subset X$ such that for all $f \in \mathcal{F}$ and $t \in G$:

$$\sup\Big\{|f \circ B_t(z)| : z \in K_2\Big\}^p \leq C \int\limits_{K_3} |f \circ B_t(u)|^p d\mu(u).$$

In particular, we have with $z \in K_1$ and $u := B_{t^{-1}}z \in K_2$ for all $t \in G$ the estimate:

$$\sup\Big\{|f(z)| : z \in K_1\Big\}^p \leq \sup\Big\{\Big|f \circ B_t(u)\Big| : u \in K_2\Big\}^p \leq C \int\limits_{K_3} |f \circ B_t(x)|^p d\mu(x).$$

Finally, integration over G together with $m(G) = 1$ and an application of Lemma 6.1.12 shows:

$$\sup\Big\{|f(z)| : z \in K_1\Big\}^p \leq C \int\limits_{G \times K_3} |f|^p \circ F d(m \otimes \mu) = C \int\limits_{K_3} |f(x)|^p d\mu_s(x)$$

and by definition it follows $\mu_s \in \mathcal{MF}_p(X)$. □

Let $p \geq 1$ and $\mathcal{H} \subset M(X, \mathbb{C})$ be a $B(G)$-invariant space. Assume that $B : G \to M^{-1}(X)$ is a measurable representation such that μ_s is well-defined the for any $\mu \in \mathcal{M}_\sigma(X)$. According to Corollary 6.1.13 the space $\mathcal{H}_p := \mathcal{H} \cap L^p(X, \mu_s)$ is $B(G)$-invariant. Denote by $\overline{\mathcal{H}_p}$ the L^p-closure of $\mathcal{H}_p$. Then we have shown that

$$\tilde{B} : G \ni t \mapsto \Big[\overline{\mathcal{H}_p} \in f \mapsto f \circ B_t \in \overline{\mathcal{H}_p}\Big] \in \mathcal{L}(\overline{\mathcal{H}_p}) \tag{110}$$

is well-defined. For all $t \in G$ the operators $\tilde{B}_t \in \mathcal{L}(\overline{\mathcal{H}_p})$ are bijective and isometric. In the case where $p = 2$ we obtain a group of unitary operators. Next we give some conditions under which $(\tilde{B}_t)_{t \in G}$ is strongly continuous.

Proposition 6.1.19. *Let $p \geq 1$ and assume that $\mathcal{H} \subset \mathcal{C}(X)$ is $B(G)$-invariant and $\mu \in \mathcal{M}_\sigma(X)$ is finite. For all $h \in \mathcal{H}_p$ let the convergence $h \circ B_t \to h$ hold uniformly on X as $t \to e$. Then $\tilde{B}$ is strongly continuous.*

Proof. Denote by $\|\cdot\|_p$ the $L^p(X, \mu_s)$-norm on X. Let $f \in \overline{\mathcal{H}_p}$ and $\varepsilon > 0$. Then choose $h \in \mathcal{H}_p$ with $\|f - h\|_p < \varepsilon$. It follows:

$$\begin{aligned} \|f \circ B_t - f\|_p &\leq \|(f - h) \circ B_t\|_p + \|h \circ B_t - h\|_p + \|h - f\|_p \\ &= 2\|f - h\|_p + \|h \circ B_t - h\|_p \leq 2\varepsilon + \|h \circ B_t - h\|_p. \end{aligned} \tag{111}$$

From Lebesgue's convergence theorem together with the uniform convergence $h \circ B_t \to h$ as t tends to $e \in G$ and $|h| + 1 \in L^p(X, \mu)$ it follows $\|h \circ B_t - h\|_p < \varepsilon$ for t in a suitable neighborhood of e. Using (111) this implies the strong continuity of (110). □

Let $\mathcal{C}_b(X)$ be the space of bounded complex-valued continuous functions. If we assume that $\mathcal{H} \subset \mathcal{C}_b(X)$, then by similar arguments we can prove for all finite measures $\mu \in \mathcal{M}_\sigma(X)$:

PROPOSITION 6.1.20. *Let $p \geq 1$ and let $\mathcal{H} \subset \mathcal{C}_b(X)$ be $B(G)$-invariant. Assume that $B_t x \to x$ as $t \to e$ for all $x \in X$. Then the group representation in (110) is strongly continuous.*

Let us choose $\mathcal{H} = \mathcal{C}_b(X)$. Under certain conditions we can show that $\overline{\mathcal{H}_p} = L^p(X, \mu_s)$ holds. One of these condition is that the topological space X is normal, e.g. that *Tietze's extension theorem* is true in X.

LEMMA 6.1.21. *Let Z be a metric space or a normal locally compact Hausdorff space. Moreover, let μ be a regular finite Borel measure on Z and $1 \leq p < \infty$. Then $\mathcal{C}_b(Z)$ is dense in $L^p(Z, \mu)$.*

PROOF. Choose $f \in L^p(Z, \mu)$ and $\varepsilon > 0$. Then there exists a step-function s, such that $\|f - s\|_p \leq \frac{\varepsilon}{2}$. Clearly s is bounded and according to [**41**, 2.3.6] there is $\tilde{u} \in \mathcal{C}(Z)$ with:

$$\mu\Big(\{x \mid s(x) \neq \tilde{u}(x)\}\Big) \leq \left(\frac{\varepsilon}{4\,\|s\|_\infty}\right)^p .$$

Now we define $u(x) :=\text{sgn}(\tilde{u}(x)) \min\{|\tilde{u}(x)|\,,\ \|s\|_\infty\}$. Then $u \in \mathcal{C}_b(Z)$ with $\|u\|_\infty \leq \|s\|_\infty$ and $\mu(B) \leq \left(\frac{\varepsilon}{4\|s\|_\infty}\right)^p$ where $B := \{x \mid s(x) \neq u(x)\}$. Now we obtain:

$$\|s - u\|_p^p = \int_B |u(x) - s(x)|^p \, d\mu(x) \leq 2^p \, \|s\|_\infty^p \, \mu(B) \leq \left(\frac{\varepsilon}{2}\right)^p .$$

This implies $\|f - u\|_p \leq \varepsilon$. □

If we assume that $p = 2$ and μ is a finite $\mathcal{NF}_2$ measure we can give another condition for the strong continuity of a group of composition operators. First we give some definitions:

DEFINITION 6.1.22. A topological locally convex space Z is called a k-space if $M \subset Z$ is open iff $M \cap K$ is open in K with the induced topology for each compact set $K \subset M$.

In terms of continuous maps we can characterize k-spaces as follows. The assertions (a) and (b) below are equivalent:

(a) Z is a k-space.
(b) A function $f : Z \to Y$, where Y is a topological space, is continuous iff its restriction to K is continuous for each compact set $K \subset Z$.

Examples of k-spaces are Hausdorff-spaces which are locally compact or satisfy the first axiom of countability. Moreover, all open or closed subsets of $\mathcal{DFN}$-spaces are k-spaces.

LEMMA 6.1.23. *Let Z be a k-space and $\mathcal{F} \subset \mathcal{C}(Z)$. Assume that μ is a $\mathcal{NF}_2$ measure on Z. Then for each $[g] \in \overline{\mathcal{F}_2}$ there is $f \in \mathcal{C}(Z)$ with $[g] = [f]$.*

PROOF. Let $([f_n])_n \subset \mathcal{F}_2$ be a fundamental sequence w.r.t. the L^2-topology. We conclude from (105) and $\mu \in \mathcal{MF}_2(Z)$ that $(f_n)_n$ is compact uniformly convergent to $f : Z \to \mathbb{C}$ which is continuous restricted to each compact subset $K \subset Z$. Because Z is a k-space by assumption, it follows $f \in \mathcal{C}(Z)$. Let $[g] \in L^2(Z, \mathbb{C})$ be the L^2-limit of $([f_n])_n$.Finally $(f_n)_n$ admits a subsequence which tends to g a.e. on Z we have $[f] = [g]$. □

From Lemma 6.1.23 it is clear that $\overline{\mathcal{F}_2}$ can be identified with a space of continuous complex-valued functions on Z.

PROPOSITION 6.1.24. *Let X be a k-space, $\mathcal{F} \subset \mathcal{C}(X)$ be $B(G)$-invariant and $\mu \in \mathcal{MF}_2(X)$. Then the unitary operator group (110) on $\overline{\mathcal{H}_2} := \overline{\mathcal{F}_2}$ is strongly continuous.*

PROOF. The space $\overline{\mathcal{H}_2} \subset L^2(X, \mu_s)$ is a Hilbert space and because μ_s is a $\mathcal{NF}_2$ measure by Theorem 6.1.18, the map $\overline{\mathcal{H}_2} \ni f \mapsto f(x) \in \mathbb{C}$ is continuous. By the Riesz-Fischer lemma there is $K : X \times X \to \mathbb{C}$ with $K(\cdot, x) \in \overline{\mathcal{H}_2}$ and for $x \in X$

$$f(x) = \Big\langle f, K(\cdot, x)\Big\rangle_2, \qquad \forall \; f \in \overline{\mathcal{H}_2}. \tag{112}$$

Because each $f \in \overline{\mathcal{H}_2}$ is continuous it follows that $\mathcal{D} := lh\{K(\cdot, x) : x \in X\}$ is a dense subspace of $\overline{\mathcal{H}_2}$. Now let $h = \sum_{i=1}^{n} \alpha_i K(\cdot, x_i) \in \mathcal{D}$ with $\alpha_i \in \mathbb{C}$ and $x_i \in X$ for $i = 1, \cdots, n$. Then we have:

$$\|h \circ B_t - h\|_2^2 = 2\Big[\|h\|_2^2 - \mathfrak{Re}\, \langle h \circ B_t, h\rangle_2\Big]$$

and so in order to prove $\|h \circ B_t - h\| \to 0$ as $t \to e$ it is sufficient to show that $\langle h \circ B_t, h\rangle_2 \to \|h\|_2^2$. Using (112) this follows from:

$$\begin{aligned} \langle h \circ B_t, h\rangle_2 &= \sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \Big\langle K(B_t \cdot, x_i), K(\cdot, x_j)\Big\rangle_2 \\ &= \sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} K(B_t x_j, x_i) \xrightarrow{t \to e} \|h\|_2^2. \end{aligned}$$

We have used that $K(\cdot, x_i) \in \mathcal{C}(X)$ and the continuity of $B : G \to Homeo(X)$. □

Representations of C_0-semi groups on L^2-spaces.
In this section let H be a separable Hilbert space and let $(B_t)_{t \geq 0} \subset \mathcal{L}^{-1}(H)$ be a C_0-semi group of invertible bounded operators on H. Assume that μ is a finite Borel measure on $\mathcal{G}$, where $\mathcal{G}$ is a G_δ-set in H (i.e. $\mathcal{G}$ is a countable intersection of open sets in H) and $B_t(\mathcal{G}) \subset \mathcal{G}$ for all $t \geq 0$. We construct a C_0-semigroup

$(\tilde{B}_t)_{t\geq 0} \subset \mathcal{L}^{-1}(\tilde{H})$ on $\tilde{H} := L^2(\mathcal{G}, \mu_s)$ of composition operators $\tilde{B}_t(f) := f \circ B_t$ where $f \in \tilde{H}$.

LEMMA 6.1.25. *The mapping* $\mathbb{R}^+ \times H \longrightarrow H : (t, z) \longmapsto B_t z$ *is continuous w.r.t. the product topology.*

PROOF. Since $(B_t)_{t\geq 0}$ is strongly continuous it is well-known that there exist $M > 1$ and $\beta > 0$ such that $\|B_t\| \leq Me^{\beta t}$. Let $(t, z) \in \mathbb{R}^+ \times H$ and let $(t_n, z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+ \times H$ be a sequence with $(t_n, z_n) \to (t, z)$ as $n \to \infty$. Then we obtain:

$$\|B_{t_n} z_n - B_t z\| \leq Me^{\beta t_n} \|z_n - z\| + \|B_{t_n} z - B_t z\| \xrightarrow{n\longrightarrow\infty} 0,$$

since $(B_t)_{t\geq 0}$ is strongly continuous. □

With the notations of Definition 6.1.8 let $(X,\ \Sigma_1,\ \mu) := (\mathcal{G},\ \mathcal{B}(\mathcal{G}),\ \mu)$, where $\mu \in \mathcal{M}_\sigma(\mathcal{G})$ is finite and define $(Y,\ \Sigma_2,\ m_\alpha) := (\mathbb{R}^+,\ \mathcal{B}(\mathbb{R}^+),\ e^{-t\alpha}dt)$ with $\alpha > 0$. Let $\mu_{s,\alpha}$ denote the symmetrization of μ (which is well-defined according to the lemma above and the fact that $\mathcal{G}$ (cf. [**115**, p. 150]) is a polish spaces) and define $\tilde{B}_t$ by $\tilde{B}_t(f) = f \circ B_t$ for all $t \in \mathbb{R}^+$.

As an example for the choice of $\mathcal{G}$ we can set $\mathcal{G} = H$ or $\mathcal{G}$ be an open ball in H centered in 0 and $(B_t)_{t\geq 0}$ be a semi group of unitary operators on H.

LEMMA 6.1.26. *For all* $t \geq 0$ *and* $f \in L^2(\mathcal{G}, \mu_{s,\alpha})$ *it holds* $\|\tilde{B}_t f\|_{s,\alpha} \leq e^{\frac{\alpha}{2}t} \|f\|_{s,\alpha}$, *where* $\|\cdot\|_{s,\alpha}$ *denotes the* $L^2(\mathcal{G}, \mu_{s,\alpha})$*-norm.*

PROOF. Let $t_0 \geq 0$ and $f \in L^2(\mathcal{G}, \mu_{s,\alpha})$. According to Lemma 6.1.12 we obtain:

$$\begin{aligned}
\int_{\mathcal{G}} |f \circ B_{t_0}|^2 \, d\mu_{s,\alpha} &= \int_{\mathcal{G}\times\mathbb{R}^+} |f(B_{t_0} B_t x)|^2 \, d(\mu \otimes m_\alpha)(x, t) \\
&= \int_{\mathcal{G}} \int_{[t_0,\infty)} |f(B_s x)|^2 \, e^{-\alpha(s-t_0)} ds d\mu(x) \\
&\leq e^{\alpha t_0} \int_{\mathcal{G}} \int_{\mathbb{R}^+} |f(B_s x)|^2 \, e^{-\alpha s} ds d\mu(x) = e^{\alpha t_0} \|f\|^2_{s,\alpha}.
\end{aligned}$$

This proves $\tilde{B}_{t_0} f \in L^2(\mathcal{G}, \mu_{s,\alpha})$ and the desired inequality. □

THEOREM 6.1.27. *Let* $\mathcal{G}$ *be a* G_δ*-set,* $\mu \in \mathcal{M}_\sigma(\mathcal{G})$ *and* $\alpha > 0$. *Moreover, we assume that* $(B_t)_{t>0} \subset \mathcal{L}^{-1}(H)$ *is a* C_0*-semi group of invertible bounded operators on* H *such that the inclusion* $B_t(\mathcal{G}) \subset \mathcal{G}$ *holds. For any* $t \geq 0$ *let* $\tilde{B}_t$ *be the isomorphism defined above, e.g.* $\tilde{B}_t f = f \circ B_t$. *Then* $(\tilde{B}_t)_{t\geq 0}$ *defines a* C_0*-semi group on* $L^2(\mathcal{G}, \mu_{s,\alpha})$.

Proof. It is obvious that $(\tilde{B}_t)_{t\geq 0}$ is a semi group of isomorphisms on $L^2(\mathcal{G}, \mu_{s,\alpha})$. Let $g \in \mathcal{C}_b(\mathcal{G})$, then we obtain for all $x \in H$:

$$[\tilde{B}_t g](x) - g(x) = g(B_t x) - g(x) \xrightarrow{t \longrightarrow 0} 0,$$

since $(B_t)_{t\geq 0}$ is strongly continuous and g is a continuous function. Moreover, g is bounded and thus by Lebesgue's Theorem of dominated convergence it follows:

$$\|\tilde{B}_t g - g\|_{s,\alpha} \xrightarrow{t \longrightarrow 0} 0. \tag{113}$$

Now let $f \in L^2(\mathcal{G}, \mu_{s,\alpha})$ be arbitrary and fix $\varepsilon > 0$. According to Lemma 6.1.21 there exists $g \in \mathcal{C}_b(\mathcal{G})$ with $\|f - g\|_{s,\alpha} \leq \varepsilon$. Furthermore (113) implies that there is $t_0 \leq 1$ such that for all $0 < t \leq t_0$ we have $\|\tilde{B}_t g - g\|_{L^2(\mathcal{G},\mu_{s,\alpha})} < \varepsilon$. Thus for $t \in [0, t_0]$ we get:

$$\begin{aligned} \|\tilde{B}_t f - f\|_{s,\alpha} &\leq \|\tilde{B}_t f - \tilde{B}_t g\|_{s,\alpha} + \|\tilde{B}_t g - g\|_{s,\alpha} + \|g - f\|_{s,\alpha} \\ &\leq \|\tilde{B}_t\|\varepsilon + 2\varepsilon \leq (e^{\alpha} + 2)\varepsilon \end{aligned}$$

which shows our assertion. □

6.2. Construction of group-actions induced by symmetries

We give examples how to construct representations $G \ni t \mapsto Homeo(X)$, where G is a compact group with countable base, X denotes a topological space and $Homeo(X)$ is the group of all homeomorphisms of X.

Examples of measurable representations on topological spaces. Let $\Omega \subset \mathbb{R}^n$ be open or closed and let $\omega : \Omega \to \mathbb{R}^+$ be a strictly positive and continuous weight function. With $f \in \mathcal{C}(\Omega)$ consider $\|f\|_\omega := \sup\{|f(x)|\omega(x) : x \in \Omega\}$. Define the Banach space $\mathcal{C}_\omega(\Omega)$ of continuous functions by

$$\mathcal{C}_\omega(\Omega) := \Big\{ f \in \mathcal{C}(\Omega), \ \ \|f\|_\omega < \infty \Big\}.$$

Assume that E is a topological space which is continuously embedded in $\mathcal{C}_\omega(\Omega)$. Fix $m \in \mathbb{N}$, then with the product topology on $\times_{i=1}^n \mathcal{C}_\omega(\Omega)$ and the topology on $\mathcal{C}(\Omega, \mathbb{C}^m)$ of *uniformly compact convergence* we have the continuous inclusions

$$E^m := \times_{i=1}^m E \hookrightarrow \mathcal{C}_\omega(\Omega)^m := \times_{i=1}^n \mathcal{C}_\omega(\Omega) \hookrightarrow \mathcal{C}(\Omega, \mathbb{C}^m).$$

Let $U \subset \mathbb{C}^m$ be open and bounded. For each set $A \subset U$ we denote by $\overline{A}$ the closure of A in $\mathbb{C}^m$. Now consider:

$$X_U := \Big\{ f = (f_1, \cdots, f_m) \in E^m : \overline{[f \cdot \omega](\Omega)} \subset U \Big\} \subset E^m. \tag{114}$$

Lemma 6.2.1. *The set $X_U \subset E^m$ defined in (114) is open in the product topology of E^m.*

PROOF. Because the embedding $E^m \hookrightarrow \mathcal{C}_\omega(\Omega)^m$ is continuous, it is enough to show that the set

$$\tilde{X}_U := \left\{ f \in \mathcal{C}_\omega(\Omega)^m : \overline{[f \cdot \omega](\Omega)} \subset U \right\} \subset \mathcal{C}_\omega(\Omega)^m \tag{115}$$

is open in $\mathcal{C}_\omega(\Omega)^m$. Fix $f \in \tilde{X}_U$ and let $\varepsilon := dist_1(\overline{[f \cdot \omega](\Omega)}, \partial U) > 0$ denote the distance of the compact set $\overline{[f \cdot \omega](\Omega)}$ to the topological boundary ∂U of U w.r.t. the 1-norm. Fix a function $g \in \mathcal{C}_\omega(\Omega)^m$ such that

$$\|g_1 - f_1\|_\omega + \cdots + \|g_m - f_m\|_\omega < \frac{\varepsilon}{2}.$$

It follows $[g \cdot \omega](\Omega) \subset \overline{[f \cdot \omega](\Omega)} + K_{\frac{\varepsilon}{2}}$ where $K_r \subset \mathbb{C}^m$ denotes the open r-ball ($r > 0$) w.r.t. the 1-norm centered in $0 \in \mathbb{C}^m$. Then $\overline{[g \cdot \omega](\Omega)} \subset U$ and by definition $g \in \tilde{X}_U$. □

Assume that $b : G \to Homeo(U)$ is a representation of G in the group of all homeomorphisms on U. With the notation of (115) let us define the *induced representation* $\tilde{B}_t : G \to Homeo(\tilde{X}_U)$ by $\tilde{B}_t f := (b_t \circ [f \cdot \omega]) \cdot \omega^{-1}$ for $f \in \tilde{X}_U$. Because of $\overline{[f \cdot \omega](\Omega)} \subset U$ and

$$\overline{[(\tilde{B}_t f) \cdot \omega](\Omega)} = \overline{b_t \circ [f \cdot \omega](\Omega)} = b_t \circ \overline{[f \cdot \omega](\Omega)} \subset U$$

for all $f \in \tilde{X}_U$ the map $\tilde{B}_t$ is well-defined. It is easy to check that it is a group homomorphism and for fixed $t \in G$ the map $\tilde{B}_t : \tilde{X}_U \to \tilde{X}_U$ is continuous.

REMARK 6.2.2. **Remark** If in addition for $t \in G$ the homeomorphism $b_t : U \to U$ extends to a linear map on $\mathbb{C}^m$ then we have $\tilde{B}_t f = b_t \circ f$.

With a bounded open set $U \subset \mathbb{C}^n$ we equip the space $Homeo(U)$ with the topology of uniform convergence on all compact subset $K \subset U$.

PROPOSITION 6.2.3. *Let $b : G \to Homeo(U)$ be a continuous representation, then the induced representation $\tilde{B} : G \to Homeo(\tilde{X}_U)$ is continuous as well.*

PROOF. Let $s, t \in G$ and $f, g \in \tilde{X}_U$. Then with the supremums-norm $\|\cdot\|_{\sup}$ on Ω and the product norm $\|\cdot\|_{\tilde{X}_U}$ on $\tilde{X}_U \subset \mathcal{C}_\omega(\Omega)^m$ we have:

$$\|\tilde{B}_t f - \tilde{B}_s g\|_{\tilde{X}_U} = \sum_{j=1}^m \Big\| b_t \circ [f \cdot \omega]_j - b_s \circ [g \cdot \omega]_j \Big\|_{\sup}. \tag{116}$$

Fix a sequence $(t_n, f_n)_{n \in \mathbb{N}} \subset G \times \tilde{X}_U$ with $(t_n, f_n) \to (t, f) \in G \times \tilde{X}_U$ as $(n \to \infty)$. By definition of the topology on $\tilde{X}_U$ we conclude that $(f_n \cdot \omega)_n$ converges to $f \cdot \omega$ uniformly on Ω. Hence we can choose a compact set $K \subset U$ and $n_0 \in \mathbb{N}$ such that $\overline{[f_n \cdot \omega](\Omega)} \subset K$ for all $k \geq n_0$ and $\overline{[f \cdot \omega](\Omega)} \subset K$. The continuity of the map $G \ni t \mapsto b_t \in Homeo(U)$ now implies:

$$\|b_{t_n} \circ [f_n \cdot \omega]_j - b_t \circ [f \cdot \omega]_j\|_{\sup} \xrightarrow{n \to \infty} 0$$

for all $j = 1, \cdots, m$. Together with (116) this finally implies $\tilde{B}_{t_n} f_n \to \tilde{B}_t f$ in $\tilde{X}_U$. $\square$

In order to define μ_s for $\mu \in \mathcal{M}_\sigma(X)$ and a polish space X we only need a measurable representation $B : G \to M^{-1}(X)$. With our notations above let $\tilde{V} \subset \mathcal{C}_\omega(\Omega)^m$ be open. In addition, assume that E is a polish space and define $V := \tilde{V} \cap E^m \subset E^m$. It is well-known that the spaces $\tilde{V}$ and V with the induced topologies are polish spaces as well (see [**8**]).

PROPOSITION 6.2.4. *Assume that $\tilde{B} : G \to M^{-1}(\tilde{V})$ is a measurable representation with $\tilde{B}_t(V) \subset V$ for all $t \in G$. Then $B : G \to M^{-1}(V)$ defined by $B_t := \tilde{B}_{t|_V}$ for $t \in G$ is measurable, as well.*

PROOF. For each $t \in G$ the map $B_t : V \to V$ is bijective. We show that it is measurable as well. Fix $A \in \mathcal{B}(V)$, then it follows from the continuous embedding $V \hookrightarrow \tilde{V}$, the fact that V and $\tilde{V}$ are polish spaces and *Kuratowski's Theorem* (see [**83**], p.420) that $A \in \mathcal{B}(\tilde{V})$. Because $B_t : V \to \tilde{V}$ is Borel-measurable we obtain $B_t^{-1}(A) \subset \mathcal{B}(V)$. Hence $B_t : V \to V$ is Borel-measurable for all $t \in G$ and so B is well-defined.

Now we prove that $G \times V \ni (t, z) \mapsto B_t z \in V$ is $\mathcal{B}(G \times V) - \mathcal{B}(V)$-measurable. As we have shown above $\mathcal{B}(V) \subset \mathcal{B}(\tilde{V})$ and by assumption the map

$$G \times \tilde{V} \to \tilde{V} : (t, z) \mapsto B_t z =: F(t, z)$$

is $\mathcal{B}(G \times \tilde{V})$-$\mathcal{B}(\tilde{V})$- measurable. Hence $F^{-1}(A) \in \mathcal{B}(G \times \tilde{V})$ and by the continuity of the embedding $G \times V \hookrightarrow G \times \tilde{V}$ and $F^{-1}(A) \subset G \times V$ we conclude that $F^{-1}(A) \in \mathcal{B}(G \times V)$. $\square$

Under some more conditions on $b : G \to Homeo(U)$ the restriction of $\tilde{B}_t$ to X_U leads to a continuous representation $B : G \to Homeo(X_U)$. Let us consider some special cases:

EXAMPLE 6.2.5. Let $\Omega \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ be open and bounded. We can consider the *Bergman space* $H := H^2(\Omega, V)$ defined as the $L^2(\Omega, V)$-closure of

$$\Big\{ f \in \mathcal{C}(\overline{\Omega}) : f_{|\Omega} : \Omega \to \mathbb{C} \text{ is holomorphic } \Big\}.$$

Denote by $K : \Omega \times \Omega \to \mathbb{C}$ the *Bergman kernel* of Ω and define the weight $\omega : \Omega \to \mathbb{R}^+$ by $\omega(x) := K(x, x)^{-\frac{1}{2}}$. It is well-known that ω is strictly positive and continuous on Ω. Moreover, for each $f \in H$ and $x \in \Omega$ we have:

$$|f(x)| \le \|f\|_2 K(x, x)^{\frac{1}{2}} = \|f\|_2 \omega(x)^{-1} \tag{117}$$

where $\| \cdot \|_2$ denotes the $L^2(\Omega, V)$-norm. Hence from (117) it follows that the inclusion $H^2(\Omega, V) \hookrightarrow \mathcal{C}_\omega(\Omega)$ is continuous. Let $U \subset \mathbb{C}^m$ be open and consider the space

$$GL(U) := \Big\{ A \in GL(\mathbb{C}^m) : A(U) = U \Big\}.$$

(More about this definition can be found in [**101**] and [**114**].) Let G be a compact group with countable base and $b : G \to GL(U)$ a measurable representation (e.g. U can be chosen to be the Euclidean ball in $\mathbb{C}^m$ and $G := \mathcal{U}(\mathbb{C}^m)$, the unitary group. Then a representation $b : G \to GL(U)$ is given by $b_U(z) := Uz$ with $U \in \mathcal{U}(\mathbb{C}^m)$ and $z \in U$.) Due to the remark above the induced representation

$$\tilde{B} : G \to Homeo(\tilde{X}_U)$$

(see (115)) is given by $\tilde{B}_t f = b_t \circ f$. If U is bounded, then $X_U = \tilde{X}_U \cap H^m$ is $B(G)$-invariant and by restriction of $\tilde{B}_t$ to X_U we obtain a representation $B : G \to Homeo(X_U)$ which is measurable according to Proposition 6.2.4. In the case where $b : G \to GL(U)$ is continuous it follows from standard arguments that B is even a continuous representation.

Example 6.2.6. Let $\Omega \subset \mathbb{R}^n$ be open or closed and bounded, such that the boundary fulfills e.g. the conditions of *Calderons's extension theorem*. Choose $s > \frac{n}{2}$ then, by well-known results, the Sobolev-space $H^s(\Omega)$ is a Banach-algebra and $H^s(\Omega) \hookrightarrow \mathcal{C}(\Omega)$. Let $U \subset \mathbb{C}^m$ be open and bounded and consider $\mathrm{Aut}(U)$, the group of biholomorphic mappings in U. Let G be a compact group with countable base and $b : G \to Aut(U)$ a representation. The induced representation

$$\tilde{B} : G \to Homeo(\tilde{X}_U)$$

is given by $B_t f = b_t \circ f$. Since $H := H^s(\Omega)$ is a Banach-algebra $X_U = \tilde{X}_U \cap H^m$ is $B(G)$- invariant by holomorphic functional calculus. Thus by restriction of $\tilde{B}_t$ to X_U we obtain a representation $B : G \to Homeo(X_U)$. Moreover, in the case where $b : G \to Aut(U)$ is continuous it follows again by holomorphic functional calculus that B is a continuous representation, as well.

Remark 6.2.7. Considering the group $\mathrm{Diff^k(U)}$ of $\mathcal{C}^k$-diffeomorphisms $(k > s)$ instead of $\mathrm{Aut}(U)$ we obtain again a representation $B : G \to Homeo(X_U)$ by well-known theorems about Sobolev-spaces.

Example 6.2.8. Let $U \subset \mathbb{C}^n$ be open or closed and G a compact group with countable base. Assume that $b : G \to Homeo(U)$ is a measurable representation of G. We might think e.g of U as a symmetric space and $b : G \to GL(U)$ where $GL(U)$ denotes the group of invertible homomorphisms leaving U invariant. With the usual Lebesgue measure V on U consider V_s defined by the representation b (see Definition 6.1.8 and Theorem 6.1.16). As we have remarked in (110) we obtain an unitary representation

$$\tilde{B} : G \ni t \mapsto \Big[L^2(U, V_s) \ni f \mapsto f \circ b_t \in L^2(U, V_s)\Big] \in \mathcal{L}(L^2(U, V_s)). \tag{118}$$

We have given several conditions under which the representation (118) is strongly continuous. If this is the case it is a continuous representation in our sense. Indeed, fix a sequences $(t_n, f_n)_n \subset G \times L^2(U, V_s)$ and (t, f) such that $t_n \to t$ in

G and $f_n \to f$ in $L^2(U, V_s)$ as $n \to \infty$, then:

$$\|B_{t_n} f_n - B_t f\|_{L^2} \leq \|f_n - f\|_{L^2} + \|B_{t_n} f - B_t f\|_{L^2} \xrightarrow{n \to \infty} 0$$

by the strong continuity of the unitary group $(B_t)_{t \in G}$. Fix any infinite dimensional finite Borel measure μ on $H := L^2(U, V_s)$ (e.g. let μ be a Gaussian measure), then we can consider the symmetrization μ_s of μ given by the representation (118). By the same construction we obtain an unitary representation $\tilde{\tilde{B}} : G \to \mathcal{L}(L^2(H, \mu_s))$. By continuing this process we build a sequence of unitary groups on Hilbert spaces induced by symmetries of the base space U.

As we have seen in Example 2 in general the measures μ and μ_s in Definition 6.1.8 are not equivalent. The following example is devoted to this question in our construction above. Here we choose μ to be a finite product of infinite dimensional Gaussian measures and B_t to be linear for all t. In this specific situation we obtain conditions under which μ_s is absolutely continuous w.r.t. μ. It turns out that these conditions are quite restrictive and in general absolute continuity of the measures fails or seems to be hard to prove.

Example 6.2.9. Let H be an infinite dimensional Hilbert space over $\mathbb{R}$ with Gaussian measure μ_B where B is the nuclear positive correlation operator (for definition see [**35**, pp. 40]). Fix $n \in \mathbb{N}$ and let us consider H^n with the product measure $\mu_n := \mu_B \times \cdots \times \mu_B$. For each invertible matrix $C \in \mathbb{C}^n$ we define $C : H^n \to H^n$ by matrix multiplication. The space H^n is a Hilbert space with norm

$$\|(z_1, \cdots, z_n)\|^2_{H^n} := \sum_{j=1}^{n} \|z_j\|^2.$$

For any finite Borel measure ν on H the characteristic function χ_ν is defined by the integral $\chi_\nu(z) = \int_H \exp(i\langle z, u\rangle) d\nu(u)$. In case of the Gaussian measure μ_B it is well-know that we have $\chi_{\mu_B}(z) = \exp(-\|B^{\frac{1}{2}} z\|^2)$ for $z \in H$ (see [**35**]) and so we obtain for the characteristic function of μ_n:

$$\chi_{\mu_n}((z_1, \cdots, z_n)) = \prod_{j=1}^{n} \chi_{\mu_B}(z_j) = \exp\left(-\left\|\left[\operatorname{diag}(B^{\frac{1}{2}})\right](z_1, \cdots, z_n)\right\|^2_{H^n}\right).$$

Here we denote by $\operatorname{diag}(B^{\frac{1}{2}})$ the map $(z_1, \cdots, z_n) \mapsto (B^{\frac{1}{2}} z_1, \cdots, B^{\frac{1}{2}} z_n)$ on H^n. Because μ_n is uniquely determined by χ_{μ_n} we conclude that it is a Gaussian measure with correlation operator $\operatorname{diag}(B)$. Now let us consider the measure μ_n^C on H^n defined by $\mu_n^C(X) = \mu_n(C^{-1}X)$ for all $X \in \mathcal{B}(H^n)$. It is shown (see [**35**], p. 42) that μ_n^C again is a Gaussian measure with correlation $C\operatorname{diag}(B)C^*$. We use the following general result about equivalence of infinite dimensional Gaussian measures μ_{B_1}, μ_{B_2} with nuclear positive correlations B_1, B_2 (see [**35**] remark 4.4, p. 66):

Let the operator $B_1^{-\frac{1}{2}}B_2B_1^{-\frac{1}{2}}$ be bounded and invertible. If $B_1^{-\frac{1}{2}}B_2B_1^{-\frac{1}{2}} - I$ is a Hilbert-Schmidt operator, then the measures μ_{B_1} and μ_{B_2} are equivalent. Otherwise they are orthogonal. (There is $X \subset H$ such that $\mu_{B_1}(X) = \mu_{B_1}(H) = 1$ and $\mu_{B_2}(X) = 0$.)

Let us apply this criterion to μ_n and μ_n^C. We set B_1 :=diag(B) and $B_2 := CB_1C^*$. It is easy to see that C and diag$(B^{\frac{1}{2}})$ commute and so it follows:

$$B_1^{-\frac{1}{2}}B_2B_1^{-\frac{1}{2}} = \text{diag}(B^{-\frac{1}{2}})C\text{diag}(B)C^*\text{diag}(B^{-\frac{1}{2}}) = CC^*.$$

Because C was invertible by assumption it follows that $B_1^{-\frac{1}{2}}B_2B_1^{-\frac{1}{2}}$ is invertible as well and so by the criterion above the operator $CC^* - I$ has to be Hilbert Schmidt for μ_n and μ_n^C to be equivalent. In the case where C is an unitary matrix it follows now that μ_n and μ_n^C are equivalent. If the matrix $CC^* - I$ is invertible on H^n itself (we can choose $C = tI$ with $t \in \mathbb{R} \setminus \{0,1\}$) both measures are orthogonal.

Now let us assume that $\Omega \subset \mathbb{R}^n$ is open and $H \subset \mathcal{C}_\omega(\Omega)$ where $\omega : \Omega \to \mathbb{R}^+$ is a strictly positive and continuous weight function. Denote by $U_r \subset \mathbb{C}^n$ the complex ball in $\mathbb{C}^n$ with radius r centered in 0 and consider the set $X_{U_r} \subset H^n$ defined as in (114) where $E = H$. Then according to Lemma 6.2.1 the set X_{U_r} is open and so $\mu_n(U_r) > 0$. In the following the restriction of μ_n to X_{U_r} is denoted by $\mu_{n,r}$. Let $\mathcal{N} \subset \mathcal{U}(\mathbb{C}^n)$ be a compact subgroup of the group $\mathcal{U}(\mathbb{C}^n)$ of all unitary matrices on $\mathbb{C}^n$ with Haar measure $m_{\mathcal{N}}$. There is a natural group action of $\mathcal{N}$ on X_{U_r} by $B_C(z) = C(z)$ for $C \in \mathcal{N}$. If we choose $(X, \Sigma_1, \mu) = (X_{U_r}, \mathcal{B}(X_{U_r}), \mu_{n,r})$ and $(Y, \Sigma_2, m) = (\mathcal{N}, \mathcal{B}(\mathcal{N}), m_{\mathcal{N}})$ in Definition 6.1.8, then we can prove:

THEOREM 6.2.10. *The measure $(\mu_{n,r})_s$ in Definition 6.1.8 w.r.t. $(B_C)_{C\in\mathcal{N}}$ is absolutely continuous w.r.t. $\mu_{n,r}$.*

PROOF. Let $C \in \mathcal{N}$ and choose a Borel set $N \subset X_{U_r}$ such that $\mu_{n,r}(N) = \mu_n(N) = 0$. It follows from our computations above that $\mu_{n,r}(C[N]) = \mu_n(C[N]) = 0$. Hence we obtain

$$[\mu_{n,r}]_s(N) = \int_{\mathcal{N}} \mu_{n,r}\Big(C[N]\Big)dm_{\mathcal{N}}(C) = 0. \tag{119}$$ □

Dynamical systems on L^2-spaces over Riemannian manifolds. In this section we show, how to construct a dynamical system $(H, \mathcal{B}(H), \mu, T)$ (for definition see [**83**]). Here H is a L^2-space over a Riemannian manifold, μ is an infinite dimensional *Gaussian measure* on H and $T : H \to H$ a *μ-preserving* (i.e. $\mu^T = \mu$) isomorphism. Unlike to our previous examples we are not symmetrizing a given measure by an integration process, but the μ-preserving property will follow more directly from our choice of parameters. Let us first remind of some general results in connection with infinite dimensional Gaussian measures.

Let H be an infinitely dimensional separable Hilbert space over $\mathbb{R}$ or $\mathbb{C}$ and $B \in \mathcal{L}(H)$ a non-negative nuclear operator on H. Let us denote by ν_B the *Gaussian measure* on H with characteristic function

$$\chi_{\nu_B}(z) = \int_H \exp\Big(2i\mathfrak{Re}\,\langle x, z\rangle\Big) d\nu_B(x) = \exp\Big(-\langle Bz, z\rangle\Big).$$

For each bounded operator $A \in \mathcal{L}(H)$ we consider the induced Borel measure ν_B^A defined by $\nu_B^A(M) := \nu_B(A^{-1}(M))$ for all $M \in \mathcal{B}(H)$. By a standard calculation using the transformation formula (see [**20**]) for integrals one finds for the characteristic function of $\mu := \nu_B^A$:

$$\chi_\mu(z) = \exp\Big(-\langle ABA^*z, z\rangle\Big), \qquad \forall\, z \in H.$$

Let us assume that $A \in \mathcal{L}(H)$ is unitary and $[A, B] = 0$. It follows $\chi_\mu = \chi_{\nu_B}$ and because the Gaussian measures are uniquely determined by its characteristic functions we conclude that $\nu_B^A = \nu_B$. Hence A is μ-preserving and in particular the composition operator

$$C_A : L^2(H, \nu_B) \to L^2(H, \nu_B) : f \mapsto f \circ A$$

is unitary. In order to find H, a Gaussian measure μ on H and isomorphisms $T \in \mathcal{L}(H)$ such that $(H, \mathcal{B}(H), \mu, T)$ becomes a dynamical system we restrict ourselves to L^2-Hilbert spaces H over a *Riemannian manifold*. Due to our remarks above we construct a nuclear operator B (which is naturally related to the geometry of H) as well as a family of unitary operators on H commuting with B.

Let (M, g) be a Riemannian manifold with metric g (for details see [**75**]) and denote by L the *Laplace-Beltrami operator* on M. A map $\Phi : M \to M$ is called an *isometry* of M if Φ is a diffeomorphism preserving the metric g. By this we mean that for each $p \in M$

$$g_p(u, v) = g_{\Phi(p)}\Big(d\Phi_p u, d\Phi_p v\Big), \qquad u, v \in M_p$$

where M_p denotes the tangent space to M at $p \in M$. In other words $d\Phi_p$ is an isometry of Euclidean vector spaces between (M_p, g_p) and $(M_{\Phi(p)}, g_{\Phi(p)})$. According to Proposition 1.3 in [**75**], p. 85 and the remark following it, the Riemannian measure m_R on M is invariant under isometries. Hence each isometry $\Phi : M \to M$ leads to an unitary *composition operator*

$$C_\Phi : L^2(M, m_R) \ni f \mapsto f \circ \Phi \in L^2(M, m_R).$$

There is the following characterization of diffeomorphisms of M which are isometries in terms of the Laplace-Beltrami operator L. A proof can be found in [**75**] Proposition 2.4:

THEOREM 6.2.11. *Let $\Phi : M \to M$ be a diffeomorphism of the Riemannian manifold M. Then Φ leaves the Laplace-Beltrami operator L invariant (i.e the commutator $[C_\Phi, L]$ vanishes) if and only if it is an isometry.*

From now on assume that (M,g) is a compact connected oriented Riemannian manifold. By the well-known *Hodge Theorem* (see [**118**]) it follows that there exists an orthonormal basis $[\varphi_n : n \in \mathbb{N}]$ of $L^2(M, m_R)$ consisting of eigen-functions of the Laplacian L. Moreover, all the eigen-values $(\lambda_n)_{n\in\mathbb{N}}$ are positive, except that zero is an eigen-value with multiplicity one. Each eigenvalue has finite multiplicity and they accumulate only at infinity. The asymptotic behavior of $(\lambda_n)_n$ is given by the formula

$$\lambda_n \sim n^{\frac{2}{\dim M}} \quad \text{as} \quad n \to \infty \tag{120}$$

which was discovered by H. Weyl and can be found in [**26**]. It also is a standard fact that the *heat operator* e^{-tL} on $L^2(M, m_R)$ with $t \in \mathbb{R}^+$ has a decomposition of the form:

$$e^{-tL}\varphi_n = e^{-\lambda_n t}\varphi_n.$$

for all $n \in \mathbb{N}$. Hence it follows from the asymptotic (120) that $Tr(e^{-tL}) < \infty$. Fix an isometry Φ on M. Because the composition operators C_Φ commutes with L, it also commutes with the compact operator e^{-tL} for all $t \in \mathbb{R}^+$. Moreover, e^{-tL} is positive for each $t > 0$ and so we can consider the Gaussian measure $\nu_{L,t}$ on H with characteristic function $\chi_{\nu_{L,t}}$ defined for $z \in H$ by

$$\chi_{\nu_{L,t}}(z) = \exp\Big(- \Big\langle e^{-tL}z, z \Big\rangle \Big).$$

From our remark above each composition operator C_Φ with an isometry $\Phi : M \to M$ fulfills $[C_\Phi, e^{-tL}] = 0$. Hence we obtain the following Proposition:

Proposition 6.2.12. *Let (M,g) be a Riemannian manifold and Φ be an isometry on (M,g). Moreover, let $\nu_{L,t}$ be the Gaussian measure defined above. Then C_Φ defined by*

$$C_\Phi f := f \circ \Phi$$

is $\nu_{L,t}$-preserving and we obtain the following unitary operators:

$$\mathbf{C}_{\Phi,t} : L^2(H, \nu_{L,t}) \to L^2(H, \nu_{L,t}) : f \mapsto f \circ \Phi.$$

In other words $(E := L^2(H, \nu_{L,t}), \mathcal{B}(E), \nu_{L,t}, \mathbf{C}_{\Phi,t})$ defines a dynamical system on E for each $t \in \mathbb{R}^+$. Let $Iso(M,g)$ be the isometry group of (M,g). Then $Iso(M,g)$ is a Lie group and compact if M is compact (cf. [**90**][ch. II Theorem 1.2]).

Theorem 6.2.13. *Let (M,g) be a Riemannian manifold and $Iso(M,g)$ be the isometry group of (M,g). Moreover, let $\nu_{L,t}$ be the Gaussian measure defined above, e.g. $\nu_{L,t}$ has the characteristic function $\chi_{\nu_{L,t}}(z) = \exp\Big(- \Big\langle e^{-tL}z, z \Big\rangle \Big)$, where L is the Laplace-Beltrami operator and $t > 0$. Then*

$$\mathbf{C}_t : Iso(M,g) \ni \Phi \mapsto \Big[L^2(H, \nu_{L,t}) \ni f \mapsto f \circ \Phi \in L^2(H, \nu_{L,t}) \Big] \in \mathcal{L}(L^2(H, \nu_{L,t}))$$

is an unitary group representation of the Lie group $Iso(M,g)$ on $\mathcal{L}(L^2(H, \nu_{L,t}))$.

6.3. Group action on generalized Toeplitz-algebras

Let X be a polish space or an open subset of a $\mathcal{DFN}$-space. In addition we assume that X is a k-space with $\mathcal{MF}_2(X) \neq \emptyset$ (see example 1). Assume that G is a compact group with countable base, $B : G \to Homeo(X)$ is a continuous representation and $\mathcal{H} \subset \mathcal{C}(X)$ is $B(G)$-invariant. Fix $\mu \in \mathcal{MF}_2(X)$, then according to Theorem 6.1.18 it follows that $\mu_s \in \mathcal{MF}_2(X)$ as well. With the notations in (110) and Proposition 6.1.24 we conclude that the unitary group:

$$\tilde{B} : G \ni t \mapsto \left[\overline{\mathcal{H}_2} \ni f \mapsto f \circ B_t \in \overline{\mathcal{H}_2}\right] \in \mathcal{L}(\overline{\mathcal{H}_2}) \tag{121}$$

is strongly continuous. By definition $\overline{\mathcal{H}_2}$ is a closed subspace of $L^2(X, \mu_s)$ consisting of continuous functions on X and we refer to it as $\mathcal{H}$-Bergman space over X. In the following we denote by $P : L^2(X, \mu_s) \to \overline{\mathcal{H}_2}$ the orthogonal projection (*Toeplitz projection*) onto $\overline{\mathcal{H}_2}$. Let us write $M_b(X, \mathbb{C})$ for the space of all bounded complex-valued measurable functions on X. Using our previous measure constructions we show how a representation of G in a generalized class of Toeplitz C^*-algebras can be defined.

Definition 6.3.1. Let $f \in M_b(X, \mathbb{C})$, then we denote by $T_f \in \mathcal{L}(\overline{\mathcal{H}_2})$ the Bergman-Toeplitz operator defined by $T_f(g) := P(fg)$ for all $g \in \overline{\mathcal{H}_2}$.

As we already have mentioned in the proof of Proposition 6.1.24, the *point evaluation* on X gives a continuous functional on $\overline{\mathcal{H}_2}$ and so there is a Bergman kernel $K : X \times X \to \mathbb{C}$ with (112).

Lemma 6.3.2. *For $x, y \in X$ and $t \in G$ we have the invariance $K(B_t x, y) = K(x, B_{t^{-1}} y)$ of the Bergman kernel.*

Proof. Let $[e_j : j \in \mathbb{N}]$ be an orthonormal base (ONB) of $\overline{\mathcal{H}_2}$. The group (121) acts unitarily on $\overline{\mathcal{H}_2}$ and so $[e_j \circ B_t : j \in \mathbb{N}]$ also defines an ONB of $\overline{\mathcal{H}_2}$. Let $x, y \in X$ and $t \in G$, then

$$K(x, y) = \sum_i e_i(x)\overline{e_i(y)} = \sum_i e_i \circ B_t(x)\overline{e_i \circ B_t(y)} = K(B_t x, B_t y). \tag{122}$$ □

Corollary 6.3.3. *For all $t \in G$ the commutator $[P, \tilde{B}_t] := P\tilde{B}_t - \tilde{B}_t P$ on $L^2(X, \mu_s)$ vanishes.*

Proof. Fix $f \in L^2(X, \mu_s)$, $t \in G$ and $z \in X$. Then by the reproducing kernel property of K and Lemma 6.3.2 we have:

$$\left[P\tilde{B}_t f\right](z) = \left\langle P\tilde{B}_t f, K(\cdot, z)\right\rangle_2 = \left\langle f, K(B_{t^{-1}}\cdot, z)\right\rangle_2 = \left[Pf\right](B_t z) = \left[\tilde{B}_t P f\right](z).$$

We conclude that $P\tilde{B}_t f = \tilde{B}_t P f$ for all $f \in L^2(X, \mu_s)$ and so $[P, \tilde{B}_t] = 0$. □

For each space $Y \subset X$ consider $\mathcal{H}_Y := \{f \in \mathcal{H} : f_{|_Y} = 0\}$. In the case where Y is $B(G)$-invariant it directly follows that $\mathcal{H}_Y$ is $\tilde{B}(G)$-invariant.

Lemma 6.3.4. *Let $x_0 \in X$ and $Y := \overline{\{B_t x_0 : t \in G\}}$. Assume that $\mathcal{H}_Y = \{0\}$, then there is $f_0 \in \overline{\mathcal{H}_2}$ such that $\overline{\mathcal{H}_2} = \overline{\{\tilde{B}_t f_0 : t \in G\}}$.*

Proof. Define $f_0 := K(\cdot, x_0) \in \overline{\mathcal{H}_2}$ and assume that $\overline{\{\tilde{B}_t f_0 : t \in G\}} \subsetneq \overline{\mathcal{H}_2}$. Then there is $g \in \overline{\mathcal{H}_2} \setminus \{0\}$ with $0 = \langle g, h\rangle_2$ for all $h \in \overline{\{\tilde{B}_t f_0 : t \in G\}}$. We conclude that

$$0 = \left\langle g, \tilde{B}_t f_0 \right\rangle_2 = \left\langle g, K(B_t \cdot, x_0)\right\rangle_2 = \left\langle g, K(\cdot, B_{t^{-1}} x_0)\right\rangle = g \circ B_{t^{-1}}(x_0)$$

for all $t \in G$. Hence $g \in \mathcal{H}_Y = \{0\}$ and we have received a contradiction. □

With a symbol $f \in M_b(X, \mathbb{C})$ we write $M_f \in \mathcal{L}(L^2(X, \mu_s))$ for the multiplication operator given by $M_f h := f \cdot h$ where $h \in L^2(X, \mu_s)$.

Lemma 6.3.5. *Let $f \in M_b(X, \mathbb{C})$, then for all $t \in G$ we have the identities $\tilde{B}_t M_f \tilde{B}_{t^{-1}} = M_{f \circ B_t}$ and $\tilde{B}_t T_f \tilde{B}_{t^{-1}} = T_{f \circ B_t}$.*

Proof. Let $h \in L^2(X, \mu_s)$ and $z \in X$. Then it follows for all $t \in G$:

$$[\tilde{B}_t M_f \tilde{B}_{t^{-1}} h](z) = [\tilde{B}_t (f \cdot h \circ B_{t^{-1}})](z) = f \circ B_t(z) \cdot h(z) = [M_{f \circ B_t} h](z).$$

This implies the first equation, the second follows from the first and Corollary 6.3.3 which shows $\tilde{B}_t T_f \tilde{B}_{t^{-1}} = \tilde{B}_t P M_f \tilde{B}_{t^{-1}} = P \tilde{B}_t M_f \tilde{B}_{t^{-1}} = P M_{f \circ B_t} = T_{f \circ B_t}$. □

Definition 6.3.6. Let $S \subset M_b(X, \mathbb{C})$, then we define by $\mathcal{T}(S) := \mathcal{C}^*\{T_f : f \in S\} \subset \mathcal{L}(\overline{\mathcal{H}_2})$ the Toeplitz C^*-algebra generated by all operators T_f with symbols $f \in S$.

Consider the representation of G in $\mathcal{L}(L^2(X, \mu_s))$ defined by

$$\mathbf{B} : G \ni t \mapsto \Big[\mathcal{L}(L^2(X, \mu_s)) \ni A \mapsto \tilde{B}_t A \tilde{B}_{t^{-1}} \in \mathcal{L}(L^2(X, \mu_s))\Big] \in \mathcal{L}(\mathcal{L}(L^2(X, \mu_s))).$$

Theorem 6.3.7. *Let $S \subset M_b(X, \mathbb{C})$ be $B(G)$-invariant. Then $\mathcal{T}(S)$ is $\mathbf{B}(G)$-invariant.*

Proof. Define $\overline{S} := \{\overline{f} : f \in S\}$ where $\bar{f}$ denotes the complex conjugate of f. Moreover, for all $n \in \mathbb{N}$ consider the space $W_n := \{T_{f_1} \cdots T_{f_n} : f_j \in S \cup \overline{S}\}$. It is easy to show that $T_f^* = T_{\bar{f}}$ and so it follows that the linear hull of $W := \bigcup_n W_n$ is invariant under the $*$-operation. Furthermore, we have with $t \in G$ and symbols $f_1, \cdots, f_n \in S \cup \overline{S}$:

$$\mathbf{B}_t(T_{f_1} \cdots T_{f_n}) = \mathbf{B}_t(T_{f_1}) \cdots \mathbf{B}_t(T_{f_n}) = T_{f_1 \circ B_t} \cdots T_{f_n \circ B_t} \in \mathcal{T}(S)$$

because $S \cup \overline{S}$ is $\mathcal{B}$-invariant. The linear hull of W is dense in $\mathcal{T}(S)$ and each $\mathbf{B}_t$ is continuous on $L^2(X, \mu_s)$. From this the assertion follows. □

Remark 6.3.8. With the result of Theorem 6.3.7 we can define a representation of G in the Toeplitz C^*-algebra $\mathcal{T}(S)$. This fact in connection with the general theory developed in [**67**], [**56**], [**98**] and [**99**] leads to the construction of Ψ^*-algebras in $\mathcal{T}(S)$ induced by the group action of $\mathbf{B}$ and iterated commutators.

APPENDIX A

A.1. A complete proof of Proposition 2.2.2

In this section we will give a complete proof of Proposition 2.2.2. During this chapter we will follow closely [**80**, Section 3.6]. Let us first recall the definition of a negative definite function. Moreover, we prove the most results for general vector-spaces over $\mathbb{R}$ or $\mathbb{C}$. Thus let V be such a vector space.

DEFINITION A.1.1. A function $\psi : V \longrightarrow \mathbb{C}$ belongs to the class $N(V)$ if for any choice of $k \in \mathbb{N}$ and vectors $\xi^1, \dots, \xi^k \in V$ the matrix

$$(\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l))_{j,l=1,\dots,k}$$

is positive Hermitian. Further for a topolgical vector space V we set $CN(V) := N(V) \cap C(V)$.

LEMMA A.1.2. *For $\psi \in N(V)$ we have $\psi(0) \geq 0$.*

PROOF. For $\xi = 0$ we find $0 \geq \psi(0) + \overline{\psi(0)} - \psi(0-0) = \overline{\psi(0)}$. Thus we have $\psi(0) \geq 0$. □

LEMMA A.1.3. *Let For $\psi \in N(V)$. Then we obtain $\psi(\xi) = \overline{\psi(-\xi)}$ and $\mathfrak{Re}\, \psi(\xi) \geq \psi(0)$.*

PROOF. Since for $\xi \in V$ the matrix

$$\begin{pmatrix} \psi(\xi) + \overline{\psi(\xi)} - \psi(0) & \psi(\xi) + \overline{\psi(0)} - \psi(\psi) \\ \psi(0) + \overline{\psi(\xi)} - \psi(-\xi) & \psi(0) + \overline{\psi(0)} - \psi(0) \end{pmatrix}$$

is positive we find $\psi(\xi) + \overline{\psi(0)} - \psi(\psi) = \overline{\psi(0)} + \psi(0) - \overline{\psi(0)}$ and thus $\psi(\xi) = \overline{\psi(-\xi)}$. Moreover, we have $\psi(\xi) + \overline{\psi(\xi)} - \psi(\xi - \xi) \geq 0$ and hence $\mathfrak{Re}\, \psi(\xi) \geq \psi(0)$. □

LEMMA A.1.4. *The set $N(V)$ is a convex cone which is closed under point wise convergence.*

PROOF. The convexity of $N(V)$ follows directly by the fact, that the sum of two positive Hermitian matrices is positive Hermitian again. Moreover, $N(V)$ is closed since the determinant on $\mathbb{R}^n$ is continuous. □

LEMMA A.1.5. *For $\psi \in N(V)$, $\overline{\psi}$ and $\mathfrak{Re}\, \psi$ belong to $N(V)$.*

PROOF. This follows directly by the fact that $\det(\overline{A}) = \overline{\det(A)}$ and $\det(\mathfrak{Re}\, A) = \mathfrak{Re}\, (\det A)$ for all matrices A. □

LEMMA A.1.6. *Any non-negative constant is an element of* $N(V)$ *and for* $\psi \in N(V)$ *and* $\lambda > 0$ *the function* $\xi \longmapsto \psi(\lambda\xi)$ *belongs to* $N(V)$.

PROOF. This is obvious. □

LEMMA A.1.7.
We have $\psi \in N(V)$ *if and only if*

(i) $\psi(0) \geq 0$,
(ii) $\psi(\xi) = \overline{\psi(-\xi)}$,
(iii) *for any* $k \in' \mathbb{N}$ *and any choice of vectors* $\xi^1, \dots, \xi^k \in V$ *and complex numbers* $c_1, \dots c_k$ *with* $\sum_{j=1}^k c_j = 0$ *we have* $\sum_{j,l=1}^k \psi(\xi^j - \xi^l) c_j \overline{c_l} \leq 0$

PROOF. Let ψ be a negative definite function. Then we have proved (i) and (ii) in A.1.2 and A.1.3. Let $(c_j)_{j=1..k} \in \mathbb{C}$ such that $\sum_{j=1}^k c_j = 0$. Then we have

$$\begin{aligned} 0 &\leq \sum_{j,l=1}^k (\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l)) c_j \overline{c_l} \\ &= \sum_{l=1}^k \overline{c_l} \left(\sum_{j=1}^k \psi(\xi^j) c_j \right) + \sum_{j=1}^k c_j \left(\sum_{l=1}^k \overline{\psi(\xi^j)} \overline{c_j} \right) - \sum_{j,l=1}^k (\psi(\xi^j - \xi_l) c_j \overline{c_l}) \\ &= -\sum_{j,l=1}^k (\psi(\xi^j - \xi_l) c_j \overline{c_l}). \end{aligned}$$

Conversely, let $\psi : V \longrightarrow \mathbb{C}$ be a function, which fulfills the assumptions (i) - (iii). Moreover, let $(\xi_j)_{j=1..k} \in V$ and $(c_j)_{j=1..k} \in \mathbb{C}$. Let us consider the vectors $0, (\xi_j)_{j=1..k} \in V$ and $c, (c_j)_{j=1..k} \in \mathbb{C}$, where $c = -\sum_{j=1}^k c_j$. Then (3) implies

$$\psi(0) \left|c\right|^2 + \sum j = 1^k \psi(\xi^j) c_j \overline{c} + \sum_{l=1}^k \psi(-\xi^l) c \overline{c_l} + \sum_{j,l=1}^k \psi(\xi^j - \xi^l) c_j \overline{c_l} \leq 0.$$

Using (1) and (2) we find

$$\sum_{j,l=1}^k (\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l)) c_j \overline{c_l} \geq \psi(0),$$

which proves our assertion. □

COROLLARY A.1.8. *For* $\psi \in N(V)$ *the function* $\xi \longmapsto \psi(\xi) - \psi(0)$ *belongs also to* $N(V)$.

PROOF. For $(\xi_j)_{j=1..k} \in V$ and $(c_j)_{j=1..k} \in \mathbb{C}$ such that $\sum_{j=1}^k c_j = 0$ we have

$$\sum_{j,l=1}^k \psi(\xi^j - \xi^l - \psi(0)) c_j \overline{c_l} = \sum_{j,l=1}^k \psi(\xi^j - \xi^l) c_j \overline{c_l} \leq 0.$$

In addition the conditions (1) and (2) of A.1.7 are obviously true for $\psi(\xi) - \psi(0)$. Thus we obtain our assertion by Lemma A.1.7. $\square$

COROLLARY A.1.9. *Let $u : V \longrightarrow \mathbb{C}$ be a positive definite function. Then the function $\xi \longmapsto u(0) - u(\xi)$ is an element of $N(V)$.*

PROOF. For $(\xi_j)_{j=1..k} \in V$ and $(c_j)_{j=1..k} \in \mathbb{C}$ such that $\sum_{j=1}^k c_j = 0$ we have

$$\sum_{j,l=1}^{k} (u(0) - u(\xi^j - \xi^l)c_j\overline{c_l}) = -\sum_{j,l=1}^{k} u(\xi^j - \xi^l)c_j\overline{c_l} \leq 0.$$

Furthermore, (1) and (2) of A.1.7 are satisfied, too. $\square$

THEOREM A.1.10. *A function ψ is an element of $N(V)$ if and only if ψ is negative definite in the sense that*

(i) $\psi(0) \geq 0$
(ii) *$\xi \longmapsto e^{-t\psi(\xi)}$ is positive definite for $t \geq 0$*

PROOF. Let $\psi \in N(V)$. Then (i) follows by Lemma A.1.2. To prove (ii) let $\xi^1, \ldots, \xi^k \in V$. Then for $t > 0$ the matrix

$$(t(\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l)))_{j,l=1,\ldots,k}$$

is positive Hermitian. Now [**80**, Lemma 3.5.9] implies that

$$(\exp(t(\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l))))_{j,l=1,\ldots,k}$$

is positive Hermitian. Let $c_1, \ldots c_l \in \mathbb{C}$ and set $c'_j := \exp(-t\psi(\xi^j))c_j$. Then we find

$$\begin{aligned}
&\sum_{j,l=1}^{k} \exp(-t\psi(\xi_j - \xi^l))c_j\overline{c_l} \\
= &\sum_{j,l=1}^{k} \exp(t(\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi_j - \xi^l)))\exp(-t\psi(\xi^j))\exp(-t\overline{\psi(\xi^l)})c_j\overline{c_l} \\
= &\sum_{j,l=1}^{k} \exp(t(\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l))c'_j\overline{c'_l} \geq 0.
\end{aligned}$$

This proves (ii). Conversely, (i) implies $\exp(-t\psi(0)) \leq 1$. Thus we obtain by A.1.6 and A.1.9 that the function

$$\xi \longmapsto \frac{1}{t}(1 - \exp(-t\psi(\xi))) = \frac{1}{t}(1 - \exp(-t\psi(0))) + \frac{1}{t}(\exp(-t\psi(0)) - \exp(-t\psi(\xi)))$$

is negative definite. Thus Lemma A.1.4 implies that

$$\psi(\xi) = \lim_{t \to 0} \frac{1}{t}(1 - \exp(-t\psi(\xi))) \in N(V).$$

$\square$

COROLLARY A.1.11. *Let $\psi \in N(V)$. Then $\frac{1}{\psi+\varepsilon}$ is a positive definite function for all $\varepsilon > 0$.*

PROOF. Lemma A.1.2 and A.1.3 imply $\mathfrak{Re}\,\psi(0) \geq \psi(0) \geq 0$ for all $\xi \in V$. Thus it is sufficient to prove the corollary for all ψ such that $\psi(0) > 0$. For $t > 0$ the function $\xi \longmapsto e^{-t\psi(\xi)}$ is positive definite and we have $\left|e^{-t\psi(\xi)}\right| \leq e^{-t\psi(0)}$. Thus, it follows that

$$\frac{1}{\psi} = \int_0^\infty e^{-t\psi(\xi)} dt,$$

□

which implies the corollary.

COROLLARY A.1.12. *Let $\psi \in N(V)$. Then $\frac{\psi}{\alpha+\beta\psi} \in N(V)$ for all $\alpha > 0$ and $\beta \geq 0$.*

PROOF. According to A.1.4 and A.1.6 $\alpha + \beta\psi \in N(V)$. Moreover, we have $\alpha + \beta\psi(0) \geq 0$. Thus A.1.11 implies that $\xi \mapsto \frac{1}{\alpha+\beta\psi}$ is positive definite, and hence by A.1.9 we obtain

$$\left(1 + \beta\frac{\psi(0)}{\alpha}\right)\frac{\psi}{\alpha+\beta\psi} = \frac{\psi - \psi(0)}{\alpha+\beta\psi} + \frac{\psi(0)}{\alpha} = \frac{1}{\alpha+\beta\psi(0)} - \frac{1}{\alpha+\beta\psi} + \frac{\psi(0)}{\alpha}$$

is negative definite and thus $\frac{\psi}{\alpha+\beta\psi} \in N(V)$. □

LEMMA A.1.13. *For $\psi \in N(V)$ and $\xi, \eta \in V$ we have*

(i) $\sqrt{|\psi(\xi+\eta)|} \leq \sqrt{|\psi(\xi)|} + \sqrt{|\psi(\eta)|}$,

(ii) $\left|\sqrt{|\psi(\xi))|} - \sqrt{|\psi(\eta)|}\right| \leq \sqrt{|\psi(\xi-\eta)|}$,

(iii) $|\psi(\xi) + \psi(\eta) - \psi(\xi-\eta)| \leq 2(\mathfrak{Re}\,\psi(\xi))^{1/2}(\mathfrak{Re}\,\psi(\eta))^{1/2}$.

PROOF. For $\xi, \eta \in V$ we have $\psi(0) \geq 0$, $\psi(\xi) = \overline{\psi(-\xi)}$ and

$$\det\begin{pmatrix} \psi(\xi) + \overline{\psi(\xi)} - \psi(0) & \psi(\xi) + \overline{\psi(\eta)} - \psi(\xi-\eta) \\ \psi(\eta) + \overline{\psi(\xi)} - \psi(\eta-\xi) & \psi(\eta) + \overline{\psi(\eta)} - \psi(0) \end{pmatrix} \geq 0,$$

which implies

$$\left|\psi(\xi) + \overline{\psi(\eta)} - \psi(\xi-\eta)\right| \leq 4\mathfrak{Re}\,\psi(\xi)\mathfrak{Re}\,\psi(\eta) \leq 4\,|\psi(\xi)|\,|\psi(\eta)|\,.$$

Using $-\eta$ instead of η and the fact that $|\psi(\eta)| = |\psi(-\eta)|$ we obtain

$$|\psi(\xi) + \psi(\pm\eta) - \psi(\xi\pm\eta)| \leq 4\mathfrak{Re}\,\psi(\xi)\mathfrak{Re}\,\psi(\eta) \leq 4\,|\psi(\xi)|\,|\psi(\eta)|\,,$$

which shows (iii) and yields

$$\begin{aligned} |\psi(\xi\pm\eta)| - |\psi(\xi)| - |\psi(\pm\eta)| &\leq |\psi(\xi\pm\eta)| - |\psi(\xi) + \psi(\pm\eta)| \\ \leq |\psi(\xi) + \psi(\pm\eta)\psi(\xi\pm\eta)| &\leq 2\,|\psi(\xi)|^{1/2}\,|\psi(\eta)|^{1/2}\,. \end{aligned}$$

This shows (i) and

$$\left|\sqrt{|\psi(\xi)|}-\sqrt{|\psi(\eta)|}\right|^2=|\psi(\xi)|+|\psi(\eta)|-2\sqrt{|\psi(\xi)|}\sqrt{|\psi(\eta)|}\le|\psi(\xi-\eta)|. \qquad \square$$

LEMMA A.1.14. *For $\psi\in N(V)$ and $\xi,\eta\in V$ we have*

$$\frac{1+|\psi(\xi)|}{1+|\psi(\eta)|}\le 2(1+|\psi(\xi-\eta)|).$$

PROOF. For $\eta,\zeta\in V$ we find

$$\begin{aligned}
& 2(1+|\psi(\eta)|)(1+|\psi(\zeta)|)\\
= & 2+2|\psi(\eta)|+2|\psi(\zeta)|+2|\psi(\eta)||\psi(\zeta)|\\
= & (1+|\psi(\eta)|+|\psi(\zeta)|+(|\psi(\eta)|+|\psi(\zeta)|))+(1+2|\psi(\eta)||\psi(\zeta)|)\\
\ge & 1+|\psi(\eta)||\psi(\zeta)|+2\sqrt{|\psi(\eta)||\psi(\zeta)|}\\
= & 1+\left(\sqrt{|\psi(\eta)|}+\sqrt{|\psi(\zeta)|}\right)^2.
\end{aligned}$$

Using A.1.13 we obtain

$$2(1+|\psi(\eta)|)(1+|\psi(\zeta)|)\ge 1+\sqrt{|\psi(\eta+\zeta)|}^2=1+|\psi(\eta+\zeta)|.$$

Taking $\zeta=\xi-\eta$ we finally find

$$2(1+|\psi(\xi-\eta))|)\ge\frac{1+|\psi(\xi)|}{1+|\psi(\eta)|}. \qquad \square$$

LEMMA A.1.15. *For $\psi\in N(V)$ and $\xi,\eta\in V$ we have*

$$1+|\psi(\xi\pm\eta)|\le(1+|\psi(\xi)|)(1+\sqrt{|\psi(\eta)|})^2.$$

PROOF. Using A.1.13 for $\xi,\eta\in V$ we find

$$\begin{aligned}
& 1+|\psi(\xi\pm\eta)|\\
= & 1+\sqrt{|\psi(\xi+\eta)|}^2\le 1+\left(\sqrt{|\psi(\xi)|}+\sqrt{|\psi(\eta)|}\right)^2\\
= & 1+|\psi(\xi)|+|\psi(\eta)|+2\sqrt{|\psi(\xi)|}\sqrt{|\psi(\eta)|}\\
\le & 1+|\psi(\xi)|+|\psi(\eta)|+2\sqrt{|\psi(\eta)|}(1+|\psi(\xi)|)\\
\le & 1+|\psi(\xi)|+|\psi(\eta)|+|\psi(\xi)||\psi(\eta)|+2\sqrt{|\psi(\eta)|}(1+|\psi(\xi)|)\\
= & (1+|\psi(\xi)|)\left(1+|\psi(\eta)|+2\sqrt{|\psi(\eta)|}\right)\\
= & (1+|\psi(\xi)|)\left(1+\sqrt{|\psi(\eta)|}\right)^2.
\end{aligned}$$

But this is our assertion. $\square$

COROLLARY A.1.16. *Let V by a topological vector spaces, such that continuity and sequential continuity are equivalent. For $\psi\in N(V)$ being continuous at 0 we obtain $\psi\in CN(V)$.*

Proof. By A.1.8 $\psi-\psi(0)$ is negatice definite, too. Moreover, ψ is continuous if and only if $\psi-\psi(0)$ is continuous. Thus we may assume $\psi(0)=0$. Taking in A.1.15 $-\xi$ instead of ξ we obtain

$$1+|\psi(\eta-\xi)|\leq(1+|\psi(\xi)|)(1+\sqrt{|\psi(\eta)|})^2$$

and substituting $\xi\mapsto\xi\eta$ we find

$$1+|\psi(\xi)|\leq(1+|\psi(\xi+\eta)|)(1+\sqrt{|\psi(\eta)|})^2.$$

This and A.1.15 imply that

$$\frac{1}{(1+\sqrt{|\psi(\eta)|})^2}\leq\frac{1+|\psi(\xi+\eta)|}{1+|\psi(\xi)|}\leq(1+\sqrt{|\psi(\eta)|})^2,$$

which yields our assertion for $\eta\longrightarrow 0$. □

A.2. Some remarks about the Kohn-Nirenberg and the Weyl correspondence

Let us make some remarks about the Kohn-Nirenberg and the Weyl Correspondence in the classical finite dimensional case.

Definition A.2.1. For $0\leq\delta\leq\varrho\leq 1$, $\delta<1$ and $m\in\mathbb{Z}$ we denote by $S^m_{\varrho,\delta}$ the class of all symbols $a\in\mathscr{C}^\infty(\mathbb{R}^n_x\times\mathbb{R}^n_p)$ such that for any multi-index α, β there exists a constant $C_{\alpha,\beta}$ with

$$\left|\partial_p^\alpha\partial_x^\beta a(x,p)\right|\leq C_{\alpha,\beta}\langle p\rangle^{m+\delta|\beta|-\varrho|\alpha|},$$

where $\langle p\rangle=\sqrt{1+|p|^2}$. Moreover, $S^m_{\varrho,\delta}$ is a Fréchet space with semi-norms

$$|a|_l^m=\max_{|\alpha|+|\beta|\leq l}\sup_{x,\xi}|\partial_p^\alpha\partial_x^\beta a(x,p)|\langle p\rangle^{-(m-\varrho|\alpha|+\delta|\beta|)}.$$

Definition A.2.2. For $m\in\mathbb{Z}$ and $0\leq\delta\leq\rho\leq 1$ the class $\Psi^m_{\rho,\delta}$ denotes the algebra of all pseudodifferential operators $a(x,D)$ given by

$$a(x,D)f(x)=\left(\frac{1}{2\pi}\right)^{n/2}\int_{\mathbb{R}^n}e^{i\langle x\,,p\rangle}a(x,p)\tilde{F}f(p)\,dp,$$

where $a\in S^m_{\varrho,\delta}$ and $f\in S(\mathbb{R}^n)$, $\tilde{F}f$ is the Fourier-transform of f, i.e.

$$\tilde{F}f(p)=\left(\frac{1}{2\pi}\right)^{n/2}\int e^{-i<y,p>}f(y)dy.$$

Here $S(\mathbb{R}^n)$ denotes the space of all Schwartz-functions. These pseudodifferential operators are called pseudodifferential operators in Kohn-Nirenberg form.

Definition A.2.3. Let $m \in \mathbb{Z}$ and $0 \leq \delta \leq \rho \leq 1$. For $a \in S^m_{\varrho,\delta}$ and $f \in S(\mathbb{R}^n)$ we define the pseudodifferential operator $a(X, D)$ by

$$a(X,D)f(x) = \left(\frac{1}{2\pi}\right)^n \iint a(\frac{1}{2}(x+y),p)e^{i\langle x-y\,,p\rangle}dy\,dp.$$

$a(X, D)$ is called pseudodifferential operator in Weyl form.

Proposition A.2.4. *For all $m \in \mathbb{Z}$, $0 \leq \delta \leq \rho \leq 1$ and all $a \in S^m_{\varrho,\delta}$ there exists a linear operator T such that*

$$a(x,D) = (Ta)(X,D).$$

Proof. See [**43**, p. 94]. □

Proposition A.2.5. *If $a \in S(\mathbb{R}^{2n})$ we have*

$$Ta(x,\xi) = 2^n \iint a(y,\eta)e^{4\pi i\langle x-y\,,\xi-\eta\rangle}d\eta dy.$$

Moreover, for $a \in S'(\mathbb{R}^{2n})$ we find

$$[\widetilde{\mathcal{F}}(Ta)](x,\xi) = e^{-\pi i\langle x\,,\xi\rangle}[\widetilde{\mathcal{F}}(a)](x,\xi).$$

Proof. See [**43**, p. 94]. □

Theorem A.2.6. *The operator T defined in Proposition A.2.4 maps all classes $S^m_{\varrho,\delta}$ ($0 \leq \delta \leq \rho \leq 1$, $\delta < 1$) into themselves, and is a Fréchet space isomorphism.*

Proof. See [**43**, p. 95]. □

Theorem A.2.7. *If $a \in S^m_{\varrho,\delta}$ with $\varrho > \delta$ then*

$$a - Ta \in S^{m-(\varrho-\delta)}_{\varrho,\delta} \text{ and } a(x,D) - a(X,D)\Psi^{m-(\varrho-\delta)}_{\varrho,\delta}.$$

Proof. See [**43**, p. 102]. □

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List of Symbols

Lebenslauf

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Mainz, 18.06.2007