Renormalization of Fermion Mixing

Dissertation zur Erlangung des Grades
„Doktor der Naturwissenschaften"
am Fachbereich Physik
der Johannes Gutenberg-Universität in Mainz

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Mainz, den 11. Mai 2007

Water and wait long enough, the avocado will grow.

Datum der mündlichen Prüfung: 11. Mai 2007

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Chapter 1

Introduction and Motivation

The renormalization of the electroweak Standard Model has already a successful history. In some of its sectors experimental measurements are sensitive to loop corrections and there, calculations with high precision are already mandatory. Surprisingly, there are still open problems related to the renormalization of the quark sector starting with the one-loop corrections. The presence of the quark mixing matrix that needs to be renormalized and the presence of unstable particles raise difficulties in defining correct and complete renormalized parameters that preserve all symmetries of the Lagrangian.

Going beyond the Standard Model, one has other models that allow for fermion mixing and there, a complete renormalization scheme is also required. For example, there are experimental evidences for neutrino oscillations and a model that describes the neutrino mixing has to be taken into account. One can also think of supersymmetric or other exotic particles. Mixing is present in the Standard Model for Dirac fermions (quarks), while beyond, it can involve also Majorana particles (possible candidates are neutrinos).

The necessity of the renormalization for fermion mixing was already recognised in the late '70s [Sirlin]. In 1982, Aoki et al. [Aok82] were describing an on-shell renormalization scheme that led to a diagonal fermion propagator. This prescription was used in 1990 by Denner and Sack [Den90a] to calculate the one-loop counter terms of external legs and of the quark mixing matrix. It was the first attempt to provide an analytical result for a mixing matrix counter term. In the next decade, few articles followed, trying to use a similar prescription to calculate the counter term for quark or neutrino mixing matrices, for models with stable particles. In 1999, in [Gam99], a gauge parameter dependence problem was recognized in the renormalization prescription of [Den90a]. Starting from 2000 on, many authors tried to find a prescription that will have as result a renormalized fermion mixing matrix that is unitary, gauge parameter independent and that will lead to an UV-finite amplitude. These requirements were successfully accomplished for the corrections to the quark mixing matrix in [Den04], but treating all quarks as stable particles.

The difficult point remains the renormalization of the theory involving unstable particles. In [Esp02], it was shown that a correct on-shell renormalization scheme that accounts for unstable particles and that includes the absorptive contributions from the self-energies leads to wave function renormalization constants not related by hermiticity (i.e. the constant renormalizing the incoming fermion and the outgoing antifermion is different from the one that renormalizes the outgoing fermion and the incoming antifermion). The lack of hermiticity can destroy in this case the unitarity of the renormalized mixing matrix.

The aim of this thesis is to provide a renormalization prescription for fermion mixing, in a general framework, a prescription that can be applied then to specific models. We base our calculations on a Lagrange density that describes generic interaction terms for Dirac and Majorana fermions, in models that allow for mixing. We also take into account the presence of unstable particles. The analysis of the renormalized Lagrangian is general, but the analytic formulas for the field renormalization constants and for the mass are restricted to the one-loop approximation. To fix all the counter terms of the coupling constants in the interaction Lagrangian, and in particular of the fermion mixing matrix, one needs to discuss specific models. We choose the mixing of the quarks in the electroweak Standard Model and the neutrino mixing in the seesaw mechanism. These examples provide enough information for an application to other theories.

In the general approach, as a prescription to separate the divergences resulting from fermion self-energies, we will use the on-shell scheme. With this scheme, one can easily identify the physical mass and also the decay width for unstable particles. However, we will not follow the 'classical' prescription and fix the field renormalization constants directly from the analysis of the full propagator, as done in [Esp02]. The wave function renormalization constants resulting in the on-shell scheme do not have to be identical with the field renormalization constants. We will explore different possibilities in defining the latter such that one obtains a hermitian Lagrangian or even diagonal renormalized mass terms.

We start in chapter 2 by describing the general fermion Lagrangian underlying our analysis. After briefly presenting the different procedures to diagonalize Dirac and Majorana mass terms, we enumerate all possible interaction terms involving fermions. The chapter ends with a list of Feynman rules and the prescription to evaluate different diagrams. The one-loop self-energies are calculated on account of the general couplings from chapter 2, in chapter 3. Chapter 4 is dedicated to the analysis of the fermion propagator and the extraction of the ultraviolet divergences from the one-loop self-energies. The divergences are absorbed by the counter term of the mass and by the wave function renormalization constants, determined such that the renormalized fermion propagator is diagonal on-shell. We will show that for a model that has particles with decay channels that lead to absorptive contributions at one-loop, the wave function renormalization constants will result in two independent sets, as stated in [Esp02]. One will contribute to the renormalization of the fermion field and the other one to the renormalization of the corresponding Dirac conjugated field. In general, a hermiticity relation between the two is not fulfilled.

As already emphasised above, in chapter 4 we just isolate the divergences that have to be absorbed by the renormalization constants. The fermion field renormalization constants are defined in chapter 5. Here, we investigate the possibility of defining the constants such that the hermiticity of the renormalized free Lagrangian is not destroyed by the presence of unstable particles. Since the presence of mixing matrices leads to non-diagonal field renormalization constants, we also explore the possibility of re-diagonalizing the renormalized mass term. In all these cases, we point out the consequences on self-energy corrections of external legs.

The renormalized fermion interaction Lagrangian is the subject of chapter 6. We analyse each possible coupling of fermions to vector or scalar bosons and at the end we derive the Feynman rules for the counter terms. Corrections to a generic process are presented in the last section of the chapter.

In all the enumerated chapters we will firstly present the current topic for general models with Dirac fermions and then, for models with Majorana fermions.

Chapters 7 and 8 are dedicated to the analysis of specific models. The renormalization of the quark fields and of the quark mixing matrix is described in chapter 7, as a direct application of chapter 6. We emphasise the consequences of the presence of absorptive imaginary contributions in amplitudes and we suggest a prescription to fix the quark mixing matrix counter terms from experimental measurements. The same method can be used for the neutrino mixing matrix. While for the quarks the one-loop corrections of the mixing matrix are negligible, this is no longer valid for the neutrinos. Therefore, in chapter 8, we investigate the renormalization in the neutrino seesaw mechanism. We start by defining the theoretical model and the differences introduced with respect to the Standard Model, especially in the Higgs sector. The renormalization of the massive Majorana neutrino fields and of their mixing matrix are treated in the second part of the chapter.

The last chapter is reserved for conclusions and for future perspectives of fermion field renormalization.

Chapter 2

Lagrange Density and Feynman Rules for Dirac and Majorana Fermions

This chapter is introducing the general notions and notations that we will use to describe particles with spin $\frac{1}{2}$. We start presenting the fermion Lagrange density with an emphasis on the mass term. Then, we give a general form for the interaction of fermions with vector and scalar bosons. At the end, we list the corresponding Feynman rules.

2.1 Remarks on Conventions and Notations

To describe Dirac and Majorana fermions, we always use 4-component spinors. If not additionally specified, we denote the Dirac fields with ψ and the particle type with indices i, j, k and so on. The Majorana ones will be given by χ with particle indices from the beginning of the Latin alphabet: a, b, c. When writing x, y or z as a subscript, we refer to both, Dirac and Majorana flavour indices.

An $n_r \times n_c$ matrix, where n_r is the number of rows and n_c the number of columns, will have elements generally indexed xy, with x=$1, ..., n_r$ and y=$1, ..., n_c$. Even if sometimes it will seem simpler to use block matrices instead of writing explicit indices, we will not do it. When considering particular cases, it is easier (and less confusing) to pick up one matrix element, one specific transformation, etc. if the indices are present.

Along this work, whenever a sum over particle indices will occur, we are going to write it explicitly. For the sum over Lorentz indices, we keep the Einstein summation convention. The Dirac indices will not be written explicitly.

2.2 Lagrange Density Mass Term

The mass is one of the important physical parameters that enter the Lagrangian. When one takes into consideration the mixing of the particles, the mass matrix in the Lagrangian has non-diagonal elements and the fields do not correspond to mass eigenstates. In order to identify the particles in the model, the mass matrix needs to be diagonalized.

In this section, we present the diagonalization procedure used for mass terms with either Dirac or Majorana particles. One can also combine the Dirac and Majorana structure of mass terms that will be presented in the first two subsections, in a so-called Dirac-Majorana mass term. The last situation can occur when none of the lepton numbers is conserved. It is especially important when describing the neutrinos with the seesaw mechanism. After shortly presenting here the Dirac-Majorana mass term and its diagonalization, we will later (in chapter 8) apply it for the seesaw mechanism.

2.2.1 Dirac Mass Term

We consider a Dirac field ψ_{0i}, with the left and right components introduced as

$$\psi_{0i}^L = \gamma_L \psi_{0i}, \tag{2.1}$$
$$\psi_{0i}^R = \gamma_R \psi_{0i}, \tag{2.2}$$

where

$$\gamma_L = \frac{1}{2}(1-\gamma_5), \tag{2.3}$$
$$\gamma_R = \frac{1}{2}(1+\gamma_5), \tag{2.4}$$

are the left and right projectors. As mentioned, the index i coming with the fields runs over the flavours. If we take the Standard Model, it can refer to the up or down-type quarks (i=u,c,t or i=d,s,b) or to the charged leptons (i=e,μ,τ), etc. Its range is not limited to 3 (3 up- or down-type quarks or 3 charged leptons or neutrinos as in the Standard Model). We will allow values ranging from 1 to n.

A general Dirac mass term can be written as

$$\mathcal{L}_{mass}^D = -\sum_{i,j} \overline{\psi_{0i}^R} M_{ij}^D \psi_{0j}^L + h.c.. \tag{2.5}$$

'$h.c.$' stands for the hermitian conjugated term and M^D is a complex square matrix. To have an interpretation for the elements of M^D as particle masses, the matrix needs to be diagonalized. Using the singular value decomposition, one can write

$$M^D_{ij} = \sum_k V_{ik} m_k (U_{jk})^*, \tag{2.6}$$

with U and V unitary matrices. The choice can be made such that m is a nonnegative, diagonal and real matrix.

Replacing the left and right-handed fields by

$$\psi^L_{0i} = \sum_j U_{ij} \psi^L_j, \tag{2.7}$$

$$\psi^R_{0i} = \sum_j V_{ij} \psi^R_j, \tag{2.8}$$

the mass term becomes diagonal:

$$\mathcal{L}^D_{mass} = -\sum_i \overline{\psi_i} m_i \psi_i. \tag{2.9}$$

ψ_i is given by $\psi_i = \psi^L_i + \psi^R_i$.

2.2.2 Majorana Mass Term

To define Majorana particles, we need to introduce the charge-conjugated field. For a fermion field χ it is given by

$$\chi^C = C\overline{\chi}^T, \tag{2.10}$$

where C is the charge-conjugation matrix. C is defined by

$$C(\gamma^\mu)^T C^{-1} = -\gamma^\mu, \tag{2.11}$$

such that $(\chi^C)^C = \chi$. This implies

$$\begin{aligned} C = -C^{-1} = -C^\dagger = -C^T, \\ C^2 = -1. \end{aligned} \tag{2.12}$$

In the following, we will need some of its properties, i.e.

$$\begin{aligned} C(\gamma_5)^T C^{-1} = \gamma_5, \\ C(\gamma^\mu \gamma_5)^T C^{-1} = \gamma^\mu \gamma_5. \end{aligned} \tag{2.13}$$

From (2.10), we find that

$$\overline{\chi^C} = -\chi^T C^{-1} \tag{2.14}$$

Following from the definition of C, for the Dirac spinors we can write

$$\begin{cases} u(p,s) = C\overline{v}^T(p,s) \\ v(p,s) = C\overline{u}^T(p,s) \end{cases} \Leftrightarrow \begin{cases} \overline{v}(p,s) = -u^T(p,s)C^{-1} \\ \overline{u}(p,s) = -v^T(p,s)C^{-1}. \end{cases} \tag{2.15}$$

This relation becomes useful when one wants to prove the equivalence of choosing any orientation for a fermion flow when evaluating a Feynman diagram, as it will be mentioned in the corresponding section.

Majorana fermions are described by

$$\chi = \chi^C, \tag{2.16}$$

i.e. particle and antiparticle are identical and therefore they are strictly neutral. Neutrinos can be Majorana particles if the lepton generation number is not a conserved quantity. Because of the definition (2.10), the left- and right-handed charge-conjugated field expressions are related to the field by

$$\begin{aligned} (\chi^L)^C &= \gamma_R \chi^C, \\ (\chi^R)^C &= \gamma_L \chi^C. \end{aligned} \tag{2.17}$$

This reversed behaviour between left and right can be seen easier if one defines directly the Majorana field for its projections:

$$\begin{aligned} (\chi^L)^C &= C\overline{\chi^L}^T, \\ (\chi^R)^C &= C\overline{\chi^R}^T, \end{aligned} \tag{2.18}$$

The equivalent expressions for the conjugated fields are:

$$\begin{aligned} \overline{(\chi^L)^C} &= -(\chi^L)^T C^{-1}, \\ \overline{(\chi^R)^C} &= -(\chi^R)^T C^{-1}. \end{aligned} \tag{2.19}$$

From (2.17) one can see that the condition (2.16) for Majorana particles implies in fact

$$\begin{aligned} (\chi^L)^C &= \chi^R, \\ (\chi^R)^C &= \chi^L. \end{aligned} \tag{2.20}$$

It means that the right-handed component of a Majorana particle can be identified with the antiparticle of the left-handed component. The two degrees of freedom of a Majorana fermion can be viewed as either particle or antiparticle or as left- and right-handed states of a particle. As a consequence, there are different ways to write a Majorana Lagrangian mass term. In terms of left-handed fields:

$$\mathcal{L}^M_{massL} = -\frac{1}{2}\sum_{a,b} \overline{(\chi^L_{0a})^C} M^L_{ab} \chi^L_{0b} + h.c.. \tag{2.21}$$

In terms of the right-handed ones, the equivalent expression is

$$\mathcal{L}^M_{massR} = -\frac{1}{2}\sum_{a,b} \overline{\chi^R_{0a}} M^R_{ab} (\chi^R_{0b})^C + h.c.. \tag{2.22}$$

Note that the whole Lagrangian mass term (taking also the hermitian conjugated part) has the same structure in both cases. The written part of the Lagrangian for the right-handed fields looks like the hermitian conjugated term of the previous expression (2.21), if we would exchange the upper script L with R and replace the mass. We write expressly one structure for the left-handed fields and the other one for the right components because we need exactly these forms in the Dirac-Majorana case. In contrast with Dirac particles, for Majorana fermions, the left and the right-handed fields can appear alone in the mass term. The factor 1/2 appears in fact in the entire free Lagrangian (i.e. also in the kinetic part) and it provides the correct normalisation of the kinetic energy. M_L and M_R are, as in the Dirac case, complex square matrices. As a consequence of the properties of Majorana fields, they can be also chosen symmetric as we will show now.

With the anticommutation properties of the fermion fields and the antisymmetry of the charge-conjugation matrix C, one can prove that any product of the type $\overline{(\chi_a^{L/R})^C}\chi_b^{L/R}$ is a symmetric tensor. As a consequence, using (2.19),

$$\sum_{a,b} \overline{(\chi^L_{0a})^C} M^L_{ab} \chi^L_{0b} = \sum_{a,b} \overline{(\chi^L_{0b})^C} M^L_{ab}, \chi^L_{0a}, \tag{2.23}$$

or equivalently,

$$\sum_{a,b} \overline{\chi^R_{0a}} M^R_{ab} (\chi^R_{0b})^C = \sum_{a,b} \overline{\chi^R_{0b}} M^R_{ab} (\chi^R_{0a})^C. \tag{2.24}$$

The relations (2.23) and (2.24) imply that it is possible to take both matrices symmetric.

The diagonalizing procedure for the mass term is similar to the Dirac case. However, we now have the advantage of a symmetric complex matrix. We can obtain a diagonal matrix using just one unitary matrix [Zum62]:

$$m^L = (U^L)^T M^L U^L. \tag{2.25}$$

Adding the particle indices, the expression can be written as

$$M^L_{ab} = \sum_c (U^L_{ac})^* m^L_c (U^L_{bc})^*, \tag{2.26}$$

where U^L is unitary and m^L a nonnegative, diagonal, real matrix. With

$$\chi^L_{0a} = \sum_b U^L_{ab} \chi^L_b, \tag{2.27}$$

the mass term expressed in terms of the left-handed fields becomes

$$\mathcal{L}^M_{massL} = -\frac{1}{2}\sum_a \overline{(\chi^L_a)^C} m^L_a \chi^L_a - \frac{1}{2}\sum_a \overline{\chi^L_a} m^L_a (\chi^L_a)^C. \tag{2.28}$$

We have written the hermitian conjugated part explicitly, to show that one can combine the two terms in (2.28), by defining

$$\chi = \chi^L + (\chi^L)^C. \tag{2.29}$$

This way, the mass term reaches the form

$$\mathcal{L}^M_{mass} = -\frac{1}{2}\sum_a \overline{\chi_a} m_a \chi_a, \tag{2.30}$$

where

$$m \equiv m^L. \tag{2.31}$$

The difference to the Dirac mass terms is in the factor 1/2.

Similarly, using right-handed fields, one can find a unitary matrix U^R and with

$$\chi^R_{0a} = \sum_b U^R_{ab} \chi^R_b, \tag{2.32}$$

$$\mathcal{L}^M_{massR} = -\frac{1}{2}\sum_a \overline{\chi^R_a} m^R_a (\chi^R_a)^C + h.c.. \tag{2.33}$$

Along this work, we will allow for specific models that might require to introduce both, left- and right-handed Majorana fields. For example, one may have to distinguish neutrinos which are members of isospin-doublets from neutrinos which are isospin singlets. It is then reasonable to refer in the first case to neutrinos as left-handed fields and use as symbol χ^L (together with the equations (2.21) or (2.30)), whereas for the isosinglets to use the right-handed fields χ^R (with mass terms as in (2.22) or (2.33)). This assignment has been used sometimes in the literature and will become relevant when considering the interaction terms; e.g. only isospin-doublets can interact with the W boson.

2.2.3 Dirac-Majorana Mass Term

Using the previous results, one can now easily discuss the case of a mixed Dirac-Majorana mass term:

$$\mathcal{L}^{D+M}_{mass} = \mathcal{L}^{M}_{massL} + \mathcal{L}^{D}_{mass} + \mathcal{L}^{M}_{massR}. \tag{2.34}$$

The terms are given by the equations (2.21), (2.5) and (2.22). We use χ_0^L to denote the left-handed fields and χ_0^R for the right-handed ones. The flavour of the fields will be indicated by i or j. For the start, the left and the right-handed fields are treated as independent. We assume n_L types of left-handed fields and n_R of right-handed ones. After collecting all the mass terms and diagonalizing them, we will end up with one Majorana field connected to the mass eigenstates. Written explicitly, (2.34) is

$$\mathcal{L}^{D+M}_{mass} = -\frac{1}{2} \sum_{\substack{i=\overline{1,n_L} \\ j=\overline{1,n_L}}} \overline{(\chi^L_{0i})^C} M^L_{ij} \chi^L_{0j} - \sum_{\substack{i=\overline{1,n_R} \\ j=\overline{1,n_L}}} \overline{\chi^R_{0i}} M^D_{ij} \chi^L_{0j} - \frac{1}{2} \sum_{\substack{i=\overline{1,n_R} \\ j=\overline{1,n_R}}} \overline{\chi^R_{0i}} M^R_{ij} (\chi^R_{0j})^C + h.c.. \tag{2.35}$$

The matrices M^L, M^R and M^D are all complex matrices, but, as shown in the previous part, the first two are in addition symmetric. Their dimensions are going to fit with the number of fields, i.e. M^L is $n_L \times n_L$, M^R is $n_R \times n_R$, M^D is $n_R \times n_L$. Using as summation indices i and j in all terms is confusing, but choosing different notations can be even more. Therefore, we indicate the range of values under each sum symbol. The overline implies that all flavours, from 1 to n_L or n_R are taken. If it is easier, one can ignore the sums and the indices and just remember the size of the matrices.

Since using (2.18) and (2.19) one can prove that

$$\sum_{\substack{i=\overline{1,n_R} \\ j=\overline{1,n_L}}} \overline{\chi^R_{0i}} M^D_{ij} \chi^L_{0j} = \sum_{\substack{i=\overline{1,n_R} \\ j=\overline{1,n_L}}} \overline{(\chi^L_{0j})^C} M^D_{ij} (\chi^R_{0i})^C, \tag{2.36}$$

the terms in the Lagrangian can be collected to

$$\mathcal{L}^{D+M}_{mass} = -\frac{1}{2} \begin{pmatrix} \overline{(\chi^L_0)^C} & \overline{\chi^R_0} \end{pmatrix} \begin{pmatrix} M^L & (M^D)^T \\ M^D & M_R \end{pmatrix} \begin{pmatrix} \chi^L_0 \\ (\chi^R_0)^C \end{pmatrix} + h.c.. \tag{2.37}$$

Due to its form and components, the $(n_L+n_R)\times(n_L+n_R)$ mass matrix is a complex symmetric one. Similarly to (2.26), one can find a decomposition such that

$$M^{D+M} = (U^\dagger)^T m U^\dagger. \tag{2.38}$$

M^{D+M} stands for the $(n_L+n_R)\times(n_L+n_R)$ mass matrix, U is a $(n_L+n_R)\times(n_L+n_R)$ unitary matrix and m a real, nonnegative diagonal one. A suitable separation of the unitary matrix U is

$$U = \begin{pmatrix} U^L \\ (U^R)^* \end{pmatrix}. \tag{2.39}$$

U^L has the dimension $n_L \times (n_L + n_R)$ and U^R $n_R \times (n_L + n_R)$. From the unitarity relation for U, we have

$$\begin{array}{ll} U^L(U^L)^\dagger = \mathbf{1}_{n_L}, & U^L(U^R)^T = \mathbf{0}_{n_L\times n_R}, \\ U^R(U^R)^\dagger = \mathbf{1}_{n_R}, & (U^R)^*(U^L)^\dagger = \mathbf{0}_{n_R\times n_L}, \end{array} \quad (U^L)^\dagger U^L + (U^R)^T(U^R)^* = \mathbf{1}_{n_L+n_R}. \tag{2.40}$$

If we write (2.38) for each block of mass matrices contained in M^{D+M}, then

$$\begin{aligned} M^L &= (U^L)^* m (U^L)^\dagger, \\ M^R &= U^R m (U^R)^T, \\ M^D &= U^R m (U^L)^\dagger. \end{aligned} \tag{2.41}$$

With U as in (2.39), one can write the independent left- and right-handed fields as linear superpositions of one Majorana field (χ).

$$\begin{aligned} \chi^L_{0i} &= \sum_{a=1}^{n_L+n_R} U^L_{ia}\chi^L_c \\ \chi^R_{0j} &= \sum_{a=1}^{n_L+n_R} U^R_{ja}\chi^R_c \end{aligned} \tag{2.42}$$

Combining χ^L and χ^R into $\chi = \chi^L + \chi^R$, the Lagrangian mass term becomes

$$\mathcal{L}^{D+M}_{mass} = -\frac{1}{2}\sum_a \overline{\chi_a} m_a \chi_a. \tag{2.43}$$

a takes values from 1 to $n_L + n_R$, but from this point on, there is no necessity in writing it explicitly.

Alternatively, due to the property of Majorana fields (2.20) , (2.43) can be written as

$$\begin{aligned} \mathcal{L}^{D+M}_{mass} &= -\frac{1}{2}\sum_a \overline{\chi^R_a} m_a \chi^L_a + h.c. \\ &= -\frac{1}{2}\sum_a \overline{(\chi^L_a)^C} m_a \chi^L_a - \frac{1}{2}\sum_a \overline{(\chi^R_a)^C} m_a \chi^R_a \\ &= -\frac{1}{2}\sum_a \overline{(\chi^L_a)^C} m_a (\chi^R_a)^C - \frac{1}{2}\sum_a \overline{(\chi^R_a)^C} m_a (\chi^L_a)^C, \text{ etc.} \end{aligned} \tag{2.44}$$

2.3 General Form for the Lagrange Density with Dirac and Majorana Fermions

As long as possible, we want to perform the calculations in a general manner. Therefore, we will write the interaction Lagrangian using generic Majorana and Dirac fields. Since in this thesis we are interested just in fermions, we restrict to their interactions. As a model for the terms with Majorana fermions, we use the appendix of [Hab85].

The free Lagrangian is given by

$$\mathcal{L}_0 = \frac{1}{2}\sum_a \overline{\chi_a}(i\partial\!\!\!/ - m_a)\chi_a + \sum_i \overline{\psi}_i(i\partial\!\!\!/ - m_i)\psi_i, \tag{2.45}$$

where, as in the previous section, χ_a and ψ_i are Majorana and Dirac fields, respectively. We assume that the mass matrices have been diagonalized as described before.

For the interaction of fermions with vector bosons we consider general V-A couplings, where $v_{xy,v}$ and $a_{xy,v}$ are the coupling constants. While the indices x or y stand for Majorana or Dirac fermions, v is indicating the vector bosons. Even if the electric charge e does not appear in every model as part of all the coupling constants, we factorise it to have a parameter to refer when considering the perturbation series. The general form of the charged current Lagrangian is

$$\begin{aligned}\mathcal{L}_{cc} =& e\sum_{a,i,v} \overline{\chi}_a\gamma^\mu(v_{ai,v} - a_{ai,v}\gamma_5)\psi_i\phi_{v,\mu} + e\sum_{a,i,v}\overline{\psi}_i\gamma^\mu(v^*_{ai,v} - a^*_{ai,v}\gamma_5)\chi_a\phi^*_{v,\mu} \\ &+ e\sum_{i,j,v}\overline{\psi}_i\gamma^\mu(v_{ij,v} - a_{ij,v}\gamma_5)\psi_j\phi_{v,\mu},\end{aligned} \tag{2.46}$$

where $\phi_{v,\mu}$ is describing the vector fields. Since here $\phi_{v,\mu}$ carries also charge, it has no coupling to two Majorana fields. For the interaction of the same type of fermions with vector bosons (for the present situation Dirac fermions with charged vector bosons), we do not write the hermitian conjugated term separately. By taking every possible combination of indices (in this case i, j and v), one includes it automatically. The hermiticity of the Lagrangian can be guaranteed if we restrict to couplings obeying

$$\begin{aligned} v_{yx,v} &= v^*_{xy,v}, \\ a_{yx,v} &= a^*_{xy,v}. \end{aligned} \tag{2.47}$$

The coupling constants for the vertices with fermions and a vector boson can be alternatively grouped to describe the couplings to the left and the right components of the field. The relation between the two sets of constants is:

$$\begin{aligned} g^L_{xy,v} &= v_{xy,v} + a_{xy,v}, \\ g^R_{xy,v} &= v_{xy,v} - a_{xy,v}. \end{aligned} \tag{2.48}$$

The upper script L or R indicates the couplings coming with γ_L or γ_R. From (2.47), we have

$$\begin{aligned} g^L_{yx,v} &= (g^L_{xy,v})^*, \\ g^R_{yx,v} &= (g^R_{xy,v})^*. \end{aligned} \tag{2.49}$$

Then, the expression (2.46) for the charged current term of the Lagrangian is equivalent to

$$\begin{aligned} \mathcal{L}_{cc} =& e\sum_{a,i,v} \overline{\chi}_a \gamma^\mu (g^L_{ai,v}\gamma_L + g^R_{ai,v}\gamma_R)\psi_i \phi_{v,\mu} + e\sum_{a,i,v} \overline{\psi}_i \gamma^\mu ((g^L_{ai,v})^*\gamma_L + (g^R_{ai,v})^*\gamma_R)\chi_a \phi^*_{v,\mu} \\ &+ e\sum_{i,j,v} \overline{\psi}_i \gamma^\mu (g^L_{ij,v}\gamma_L + g^R_{ij,v}\gamma_R)\psi_j \phi_{v,\mu}. \end{aligned} \tag{2.50}$$

Likewise, in the neutral current part we will allow for couplings of any pairs of Dirac or Majorana fields. Including flavour mixing in the couplings,

$$\begin{aligned} \mathcal{L}_{nc} &= \frac{1}{2} e\sum_{a,b,v} \overline{\chi}_a \gamma^\mu (v_{ab,v} - a_{ab,v}\gamma_5)\chi_b \phi_{v,\mu} + e\sum_{i,j,v} \overline{\psi}_i \gamma^\mu (v_{ij,v} - a_{ij,v}\gamma_5)\psi_j \phi_{v,\mu} \qquad (2.51) \\ &= \frac{1}{2} e\sum_{a,b,v} \overline{\chi}_a \gamma^\mu (g^L_{ab,v}\gamma_L + g^R_{ab,v}\gamma_R)\chi_b \phi_{v,\mu} + e\sum_{i,j,v} \overline{\psi}_i \gamma^\mu (g^L_{ij,v}\gamma_L + g^R_{ij,v}\gamma_R)\psi_j \phi_{v,\mu}. \end{aligned}$$

From the notation point of view, we do not distinguish between charged and neutral vector boson fields, or, as in the next relations, between charged and neutral scalar boson fields. However, one should keep in mind that in the Standard Model the coupling constants to neutral bosons are diagonal and a δ_{ij} will appear.

For the Yukawa couplings of the charged scalars to fermions, we can write

$$\begin{aligned} \mathcal{L}_{Yc} =& e\sum_{a,i,s} \overline{\chi}_a (c^L_{ai,s}\gamma_L + c^R_{ai,s}\gamma_R)\psi_i \phi_s + e\sum_{a,i,s} \overline{\psi}_i ((c^R_{ai,s})^*\gamma_L + (c^L_{ai,s})^*\gamma_R)\chi_a \phi^*_s \\ &+ e\sum_{i,j,s} \overline{\psi}_i (c^L_{ij,s}\gamma_L + c^R_{ij,s}\gamma_R)\psi_j \phi_s \qquad (2.52) \\ &+ e\sum_{i,j,s} \overline{\psi^C_i} (\tilde{c}^L_{ij,s}\gamma_L + \tilde{c}^R_{ij,s}\gamma_R)\psi_j \phi_s + e\sum_{i,j,s} \overline{\psi}_i ((\tilde{c}^R_{ji,s})^*\gamma_L + (\tilde{c}^L_{ji,s})^*\gamma_R)\psi^C_j \phi^*_s. \end{aligned}$$

ϕ_s is describing a generic scalar field and $c^L_{xy,s}$ and $c^R_{xy,s}$ its couplings to the fermions, such that

$$\begin{aligned} c^L_{yx,s} &= (c^R_{xy,s})^*, \\ c^R_{yx,s} &= (c^L_{xy,s})^*. \end{aligned} \tag{2.53}$$

This property preserves the hermiticity of the Lagrangian. The last two terms of the Lagrangian account for Yukawa interactions of a Dirac field with another explicitly charge conjugated one. To distingush the coupling constants in this case, we added a tilde above c. Such interactions do not appear in the Standard Model, but they are part, e.g., of the supersymmetric one. In the Feynman diagrams, they lead to vertices with clashing or diverging arrows. In this paper, we will come across such terms when considering the seesaw mechanism for neutrinos.

For the fermion coupling to neutral scalars, the possible terms entering the Lagrangian are

$$\mathcal{L}_{Yn} = \frac{1}{2} e \sum_{a,b,s} \overline{\chi}_a (c^L_{ab,s}\gamma_L + c^R_{ab,s}\gamma_R)\chi_b \phi_s + e \sum_{i,j,s} \overline{\psi}_i (c^L_{ij,s}\gamma_L + c^R_{ij,s}\gamma_R)\psi_j \phi_s. \tag{2.54}$$

With similar arguments like for (2.23) or (2.24), one can prove that for any fermion-fermion-boson interaction term

$$\sum_{a,b,v} \overline{\chi}_a \Gamma_{abv} \chi_b \phi_v = \sum_{a,b,v} \overline{(\chi_b)^C} C (\Gamma^T)_{abv} C^{-1} (\chi_a)^C \phi_v. \tag{2.55}$$

Here, we use χ as a generic notation for Dirac or Majorana fermionic fields and ϕ_v for any bosonic field. Γ_{abv} summarises all the interaction factors, like coupling constants and Dirac matrices. In this relation, one should be careful in understanding the transpose of Γ. This transpose refers just to Dirac matrices since it is a consequence of the interchange of the two fermionic fields. The particle indices in the coupling constants remain unmodified.

If both fermions are Majorana particles (2.16), the previous relation implies that

$$\Gamma_{bav} = C(\Gamma^T)_{abv} C^{-1}. \tag{2.56}$$

This constraint will be essential when writing the Feynman rules for vertices. In addition, since we were not using any property of the boson fields in proving it, Γ_{abv} can be replaced also by other expressions that describe the transition from a Majorana fermion with flavour a to one with flavour b. We will see that (2.56) holds also for the self-energy of Majorana fermions.

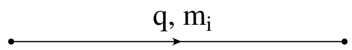
2.4 Feynman Rules

In the next chapter we evaluate fermion one-loop diagrams. For this we need a consistent set of Feynman rules for both, Dirac and Majorana fermions. To describe them, we use as framework the general Lagrangian given in the previous section. First, we start with the diagrams for Dirac fermions and then we add the interaction with the Majorana ones.

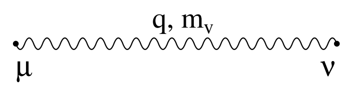
2.4.1 Feynman Rules with Dirac Particles

We start by enumerating the rules for the propagators for fermions, vectors and scalar bosons, and for the vertices including fermions. Then, we add the Feynman rules of the external bosons and at the end we particularise the fermion coupling constants for the Standard Model. In the next subsection we will complete the set of Feynman rules related to fermions, by adding the ones for external particles.

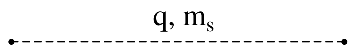
As starting point for the fermion part, we use the Dirac terms of the interaction Lagrangian given in section 2.3. The bosons' propagators are given in a R_ξ gauge.



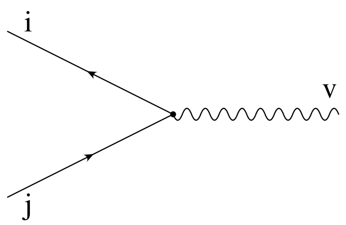
$$iS(q) = \frac{i}{\not{q} - m_i + i\rho}$$



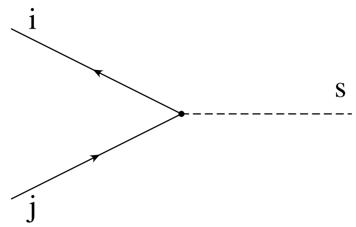
$$iD_v^{\mu\nu} = \frac{i}{q^2 - m_v^2 + i\rho}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2 - \xi m_v^2 + i\rho}(1-\xi)\right)$$



$$\frac{i}{q^2 - m_s^2 + i\rho}$$



$$\begin{aligned} i\Gamma^\mu_{ijv} &= ie\gamma^\mu(v_{ij,v} - a_{ij,v}\gamma_5) \\ &= ie\gamma^\mu(g^L_{ij,v}\gamma_L + g^R_{ij,v}\gamma_R) \end{aligned}$$



$$i\Gamma_{ijs} = ie(c^L_{ij,s}\gamma_L + c^R_{ij,s}\gamma_R) \tag{2.57}$$

The mass m_s in the scalar propagator stands as a common notation for both, physical and unphysical particles. In the final result of a calculation (e.g. self-energy), one can readjust the parameter m_s according to the scalar type. For example, if we have an unphysical scalar, a Goldstone boson corresponding to the longitudinal mode of a vector boson with mass m_v, than m_s has to be replaced with $\sqrt{\xi} m_v$.

For the external vector and scalar bosons, the Feynman rules read:

 $\varepsilon_\mu(p)$  $\varepsilon^*_\mu(p)$

 1 1

(2.58)

ε_μ is the polarisation vector of the boson and the momentum p flows from left to right.

In Table 2.1 we list the fermion coupling constants of the electroweak standard model. By $s_W = \sin\vartheta_W$ we denote the sine of the weak mixing angle, by c_W the

vertex	$g^L_{ij,v}$	$g^R_{ij,v}$	vertex	$c^L_{ij,s}$	$c^R_{ij,s}$
$\overline{\psi_i}\psi_j A$	$-Q\delta_{ij}$	$-Q\delta_{ij}$	$\overline{\psi_i}\psi_j\eta$	$-\frac{1}{2s_W}\frac{m_i}{m_W}\delta_{ij}$	$-\frac{1}{2s_W}\frac{m_i}{m_W}\delta_{ij}$
$\overline{\psi_i}\psi_j Z$	$\frac{I_3^W - s_W^2 Q}{s_W c_W}\delta_{ij}$	$-\frac{s_W}{c_W}Q\delta_{ij}$	$\overline{\psi_i}\psi_j\chi$	$-\frac{i}{2s_W}2I_3^W\frac{m_i}{m_W}\delta_{ij}$	$\frac{i}{2s_W}2I_3^W\frac{m_i}{m_W}\delta_{ij}$
$\overline{\nu_i}l_j W^+$	$\frac{1}{\sqrt{2}s_W}\delta_{ij}$	0	$\overline{\nu_i}l_j\Phi^+$	0	$-\frac{1}{\sqrt{2}s_W}\frac{m_j}{m_W}\delta_{ij}$
$\overline{l_j}\nu_i W^-$	$\frac{1}{\sqrt{2}s_W}\delta_{ij}$	0	$\overline{l_j}\nu_i\Phi^-$	$-\frac{1}{\sqrt{2}s_W}\frac{m_j}{m_W}\delta_{ij}$	0
$\overline{u_i}d_\alpha W^+$	$\frac{1}{\sqrt{2}s_W}V_{i\alpha}$	0	$\overline{u_i}d_\alpha\Phi^+$	$\frac{1}{\sqrt{2}s_W}\frac{m_i}{m_W}V_{i\alpha}$	$-\frac{1}{\sqrt{2}s_W}\frac{m_\alpha}{m_W}V_{i\alpha}$
$\overline{d_\alpha}u_i W^-$	$\frac{1}{\sqrt{2}s_W}(V_{i\alpha})^*$	0	$\overline{d_\alpha}u_i\Phi^-$	$-\frac{1}{\sqrt{2}s_W}\frac{m_\alpha}{m_W}(V_{i\alpha})^*$	$\frac{1}{\sqrt{2}s_W}\frac{m_i}{m_W}(V_{i\alpha})^*$

Table 2.1: Fermion coupling constants in the electroweak standard model.

cosine, Q is the electrical charge and I_3^W the third component of the weak isospin. ν is the neutrino field and l the charged lepton one. V stands for the quark-mixing matrix. To distinguish between up- and down-type quark flavours in the same vertex, we use different indices: i, j and so on for up, charm and top, and α, β, etc. for down, strange and bottom.

2.4.2 Feynman Rules with Dirac and Majorana Particles

To complete 2.4.1, we need to define generalised Feynman rules when one has a theory including Dirac and Majorana particles. As mentioned, we take the whole Lagrangian presented before and we add the new diagrams including Majorana fermions. Additionally, we introduce the notion of general fermion flow. As support, we use [Den92].

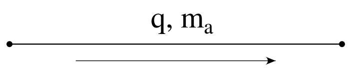
The notations for indices and couplings are like in section 2.3. To distinguish between the two types of fermions in the diagrams, the Dirac particles will carry an arrow for the fermion number flow, while Majorana ones will be represented without any.

For every fermion chain one can choose an arbitrary sense for the fermion flow and evaluate the diagrams according to it. This arbitrary fermion flow (the general fermion flow) can differ from the fermion *number* flow, i.e. one can choose to consider either the propagation of particles or of antiparticles. As long as one inserts the proper expressions, the resulting matrix element does not depend on the choice. For Majorana particles there is no change since particle and antiparticle are identical, while for Dirac fermions one has to find the connection between the two cases. For example

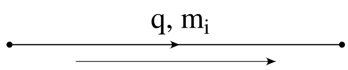
$$\Gamma \text{ becomes } (\Gamma)' = C\Gamma^T C^{-1}, \tag{2.59}$$

when taking the antiparticles in a vertex (see (2.55)).

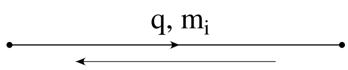
The general fermion flow is indicated in every diagram by an additional thinner arrow line. The momentum q runs as indicated by the arrow on the fermion line. As above, we insert the Feynman rules proceeding opposite to the fermion flow.



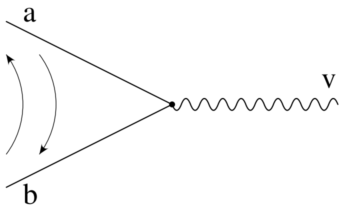
$$iS(q) = \frac{i}{\not{q} - m_a + i\epsilon}$$



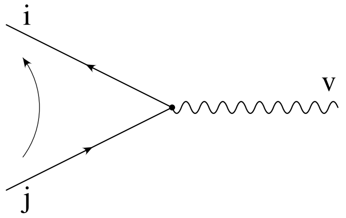
$$iS(q) = \frac{i}{\not{q} - m_i + i\epsilon}$$



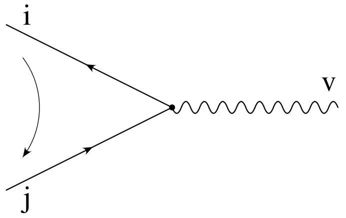
$$iS(-q) = \frac{i}{-\not{q} - m_i + i\epsilon}$$



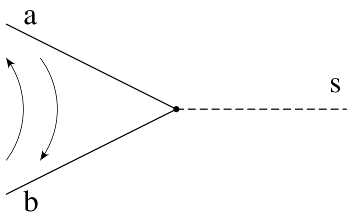
$$i\Gamma^{\mu}_{abv} = ie\gamma^{\mu}(v_{ab,v} - a_{ab,v}\gamma_5)$$
$$= ie\gamma^{\mu}(g^{L}_{ab,v}\gamma_L + g^{R}_{ab,v}\gamma_R)$$



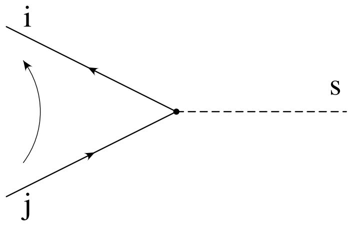
$$i\Gamma^{\mu}_{ijv} = ie\gamma^{\mu}(v_{ij,v} - a_{ij,v}\gamma_5)$$
$$= ie\gamma^{\mu}(g^{L}_{ij,v}\gamma_L + g^{R}_{ij,v}\gamma_R)$$



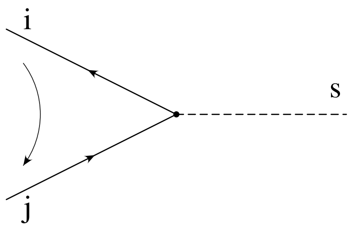
$$i(\Gamma^{\mu}_{ijv})' = iC(\Gamma^{\mu}_{ijv})^{T}C^{-1}$$
$$= -ie\gamma^{\mu}(v_{ij,v} + a_{ij,v}\gamma_5)$$
$$= -ie\gamma^{\mu}(g^{R}_{ij,v}\gamma_L + g^{L}_{ij,v}\gamma_R)$$



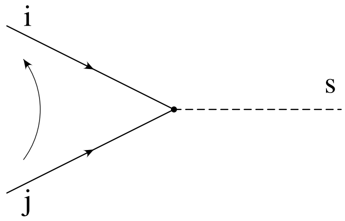
$$i\Gamma_{abs} = ie(c^{L}_{ab,s}\gamma_L + c^{R}_{ab,s}\gamma_R)$$



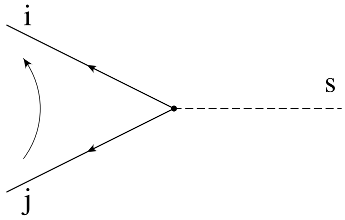
$$i\Gamma_{ijs} = ie(c^{L}_{ij,s}\gamma_L + c^{R}_{ij,s}\gamma_R)$$



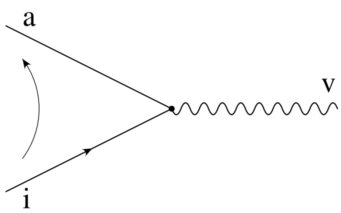
$$i(\Gamma_{ijs})' = iC(\Gamma_{ijs})^{T}C^{-1}$$
$$= ie(c^{L}_{ij,s}\gamma_L + c^{R}_{ij,s}\gamma_R)$$
$$= i\Gamma_{ijs}$$



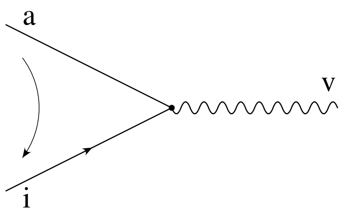
$$i\tilde{\Gamma}_{ijs} = ie(\tilde{c}^L_{ij,s}\gamma_L + \tilde{c}^R_{ij,s}\gamma_R)$$



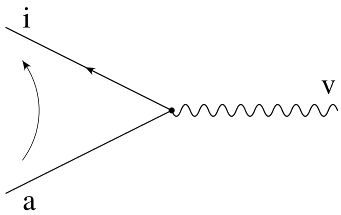
$$i\overline{\tilde{\Gamma}_{jis}} = ie((\tilde{c}^R_{ji,s})^*\gamma_L + (\tilde{c}^L_{ji,s})^*\gamma_R)$$



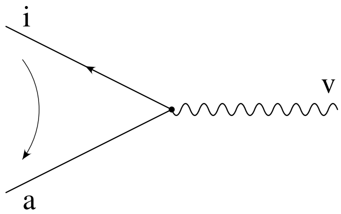
$$\begin{aligned} i\Gamma^\mu_{aiv} &= ie\gamma^\mu(v_{ai,v} - a_{ai,v}\gamma_5) \\ &= ie\gamma^\mu(g^L_{ai,v}\gamma_L + g^R_{ai,v}\gamma_R) \end{aligned}$$



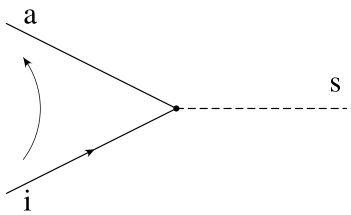
$$\begin{aligned} i(\Gamma^\mu_{aiv})' &= iC(\Gamma^\mu_{aiv})^T C^{-1} \\ &= -ie\gamma^\mu(v_{ai,v} + a_{ai,v}\gamma_5) \\ &= -ie\gamma^\mu(g^R_{ai,v}\gamma_L + g^L_{ai,v}\gamma_R) \end{aligned}$$



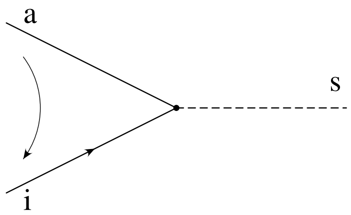
$$\begin{aligned} i\overline{\Gamma^\mu_{aiv}} &= ie\gamma^\mu(v^*_{ai,v} - a^*_{ai,v}\gamma_5) \\ &= ie\gamma^\mu((g^L_{ai,v})^*\gamma_L + (g^R_{ai,v})^*\gamma_R) \end{aligned}$$



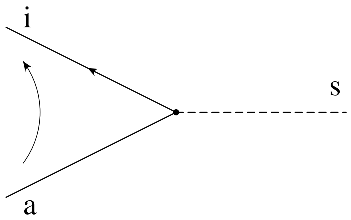
$$\begin{aligned} i(\overline{\Gamma^\mu_{aiv}})' &= iC(\overline{\Gamma^\mu_{aiv}})^T C^{-1} \\ &= -ie\gamma^\mu(v^*_{ai,v} + a^*_{ai,v}\gamma_5) \\ &= -ie\gamma^\mu((g^R_{ai,v})^*\gamma_L + (g^L_{ai,v})^*\gamma_R) \end{aligned}$$



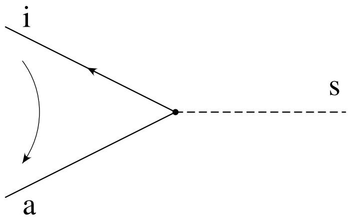
$$i\Gamma_{ais} = ie(c^L_{ai,s}\gamma_L + c^R_{ai,s}\gamma_R)$$



$$\begin{aligned} i(\Gamma_{ais})' &= iC(\Gamma_{ais})^T C^{-1} \\ &= ie(c^L_{ai,s}\gamma_L + c^R_{ai,s}\gamma_R) \\ &= i\Gamma_{ais} \end{aligned}$$

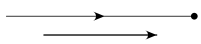
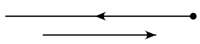
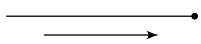


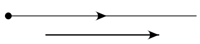
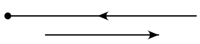
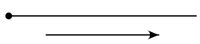
$$i\overline{\Gamma_{ais}} = ie((c^R_{ai,s})^*\gamma_L + (c^L_{ai,s})^*\gamma_R)$$

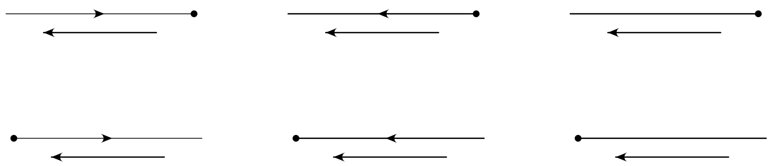


$$\begin{aligned} i(\overline{\Gamma_{ais}})' &= iC(\overline{\Gamma_{ais}})^T C^{-1} \\ &= ie((c^R_{ai,s})^*\gamma_L + (c^L_{ai,s})^*\gamma_R) \\ &= i\overline{\Gamma_{ais}} \end{aligned} \tag{2.60}$$

The external fermion lines are not needed in the calculation of self-energies, but they become important for the renormalization of the fields and further on for the amplitude of different processes. Therefore, we list also their Feynman rules. The momentum p runs from left to right.

   $u(p,s)$

   $\overline{u}(p,s)$

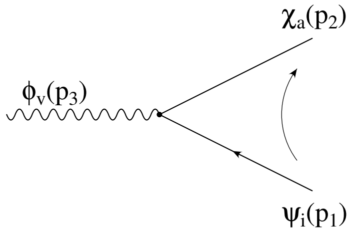


$\overline{v}(p, s)$

$v(p, s)$

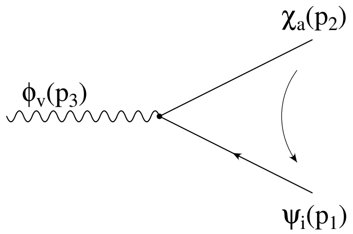
(2.61)

When inserting the spinors in the matrix element, one has to keep track of a reference order of the particles (given for example by the indices of the coupling constants in a vertex) and multiply the expression with the permutation parity of the spinors with respect to this order. This is an important condition for keeping the equivalence of the result when different choices for the fermion flow are made. For exemplification, let's take the amplitude of a decay of a vector boson into a Dirac and a Majorana particle. Along the fermion number flow, the amplitude is



$$\mathcal{T}_1 = i\overline{u}_a(p_2, s)\varepsilon_\mu(p_3)\Gamma^\mu_{aiv} v_i(p_1, s).$$

For the reversed general flow, we write



$$\mathcal{T}_2 = (-1)i\overline{u}_i(p_1, s)\varepsilon_\mu(p_3)(\Gamma^\mu_{aiv})' v_a(p_2, s).$$

In $\mathcal{T}_2$, the spinors changed positions compared to the reference order in $\mathcal{T}_1$ (permutation parity -1) and therefore the sign differs. Replacing $(\Gamma^\mu_{aiv})'$ by its equivalent $C(\Gamma^\mu_{aiv})^T C^{-1}$, recognising that $(\mathcal{T}_2)^T = \mathcal{T}_2$ since the amplitude is a number, using (2.12) and the relations (2.15) for the transposed Dirac spinors, one can prove that

$$\begin{aligned} \mathcal{T}_2 &= (-1)iv_a^T(p_2, s)C^{-1}\varepsilon_\mu(p_3)\Gamma^\mu_{aiv} C\overline{u}_i^T(p_1, s) \\ &\equiv \mathcal{T}_1. \end{aligned} \tag{2.62}$$

For more complicated diagrams, the proof is longer, but based on the same tricks. Other examples can be found in [Den92].

Chapter 3

One-loop Fermion Self-Energies

In the first section of this chapter, we will calculate the general expression for the fermion self-energy, considering a theory with fermions as Dirac particles. But, for our purposes, we need to have fermions as both, Dirac and Majorana particles. We will use the first result as a model for the next section when the Feynman rules including Majorana fermions will be inserted and all possible diagrams for self-energies will be considered. The results will be summarised in the last section.

3.1 General Expression for the Dirac Fermion Self-Energy

Repeating the same calculation for different combinations of particles and models can be avoided. Using generalised Feynman rules, one can express the self-energy as a function of several parameters, easy to identify when specific cases are considered. With the Feynman rules given in section 2.4.1, we calculate the self-energy separately for the case of a vector boson in the loop and a scalar one, respectively.

3.1.1 Diagrams with Vector Bosons

In order to obtain a general expression for the self-energy of a fermion, when a virtual vector boson is implied, one has to consider an one-loop diagram like in Figure 3.1. We take into account the possibility of mixing from one fermion of type i to one of type j.

In this case the self-energy is given by

$$-i\Sigma_{ij}^{V}(p) = \int \frac{d^4q}{(2\pi)^4} i\Gamma_{ikv,\mu} iS(p+q) i\Gamma_{kjv,\nu} iD_{v}^{\mu\nu}, \tag{3.1}$$

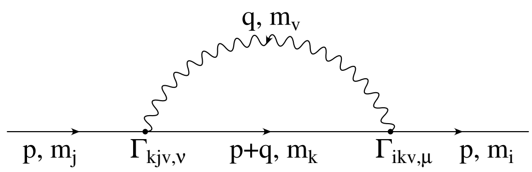


Figure 3.1: One-loop diagram with a fermion of mass m_k and a vector boson of mass m_v.

where p is the total momentum and q gives the momentum of the virtual boson. The upper index V is added to the self-energy symbol to emphasise that the internal boson is a vector one. For the scalar part we will use $\Sigma^S_{ij}(p)$. Notice that now we have just one general Dirac particle of type k and one general vector boson of type v, respectively. The sum over all possibilities will be taken into account later. To simplify the notation, we keep explicit just the indices for external fermions. The index V summaries in fact the indices for both internal particles, the vector boson and the fermion.

Inserting the expressions for Feynman rules (2.57) and eliminating the matrices in the denominator, the expression becomes:

$$-i\Sigma^V_{ij}(p) = e^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{((p+q)^2 - m_k^2)\,(q^2 - m_v^2)} \gamma_\mu (v_{ik,v} - a_{ik,v}\gamma_5)(\not{p} + \not{q} + m_k) \gamma_\nu (v_{kj,v} - a_{kj,v}\gamma_5) \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2 - \xi m_v^2}(1-\xi) \right). \quad (3.2)$$

As $i\Sigma^V_{ij}(p) \sim \int \frac{d^4q}{q^4} \sim \int \frac{dq}{q}$, the previous expression is logarithmically divergent. For its regularization we use the dimensional method (see Appendix A). Changing the notation as in (B.2) and rearranging the terms with the help of (A.7), (A.10), (A.11) and (A.14), our regularised self-energy is:

$$\Sigma_{ij}^{V}(p) = \frac{\alpha}{4\pi}\int \mathcal{D}q \frac{1}{((p+q)^2 - m_k^2)\,(q^2 - m_v^2)}$$

$$\Big(2(\varepsilon - 1)(\not{p} + \not{q})(v_{ik,v} - a_{ik,v}\gamma_5)(v_{kj,v} - a_{kj,v}\gamma_5) \qquad (1)$$

$$+2(2-\varepsilon)m_k(v_{ik,v} + a_{ik,v}\gamma_5)(v_{kj,v} - a_{kj,v}\gamma_5) \qquad (2)$$

$$-\frac{1-\xi}{q^2 - \xi m_v^2}(q^2 + 2pq)\not{q}(v_{ik,v} - a_{ik,v}\gamma_5)(v_{kj,v} - a_{kj,v}\gamma_5) \qquad (3a)$$

$$+\frac{1-\xi}{q^2 - \xi m_v^2}\not{p}q^2(v_{ik,v} - a_{ik,v}\gamma_5)(v_{kj,v} - a_{kj,v}\gamma_5) \qquad (3b)$$

$$-\frac{1-\xi}{q^2 - \xi m_v^2}m_k q^2(v_{ik,v} + a_{ik,v}\gamma_5)(v_{kj,v} - a_{kj,v}\gamma_5)\Big). \qquad (4)$$

$\alpha = \frac{e^2}{4\pi}$, the fine structure constant, was identified. We can make use of (A.14) without problems since anomalies do not occur here.

Further on, we introduce the notations

$$V_{ij}^{+} = v_{ik,v}v_{kj,v} + a_{ik,v}a_{kj,v}, \quad A_{ij}^{+} = v_{ik,v}a_{kj,v} + a_{ik,v}v_{kj,v}, \qquad (3.3)$$

$$V_{ij}^{-} = v_{ik,v}v_{kj,v} - a_{ik,v}a_{kj,v}, \quad A_{ij}^{-} = v_{ik,v}a_{kj,v} - a_{ik,v}v_{kj,v}, \qquad (3.4)$$

to evaluate the regularised self-energy term by term. As before, we omit the indices for the internal boson and fermion. With the help of the two-point integrals defined in Appendix B by (B.4) and (B.5), the first part of $\Sigma_{ij}^{V}(p)$ becomes

$$\begin{aligned}\Sigma_1 &= \frac{\alpha}{4\pi}2(\varepsilon - 1)\int \mathcal{D}q \frac{\not{p} + q_\mu\gamma^\mu}{((p+q)^2 - m_k^2)\,(q^2 - m_v^2)}(v_{ik,v} - a_{ik,v}\gamma_5)(v_{kj,v} - a_{kj,v}\gamma_5)\\ &= \frac{\alpha}{4\pi}2(\varepsilon - 1)\left(\not{p}B_0(p^2; m_v, m_k) + B_\mu(p^2; m_v, m_k)\gamma^\mu\right)(V_{ij}^{+} - A_{ij}^{+}\gamma_5). \qquad (3.5)\end{aligned}$$

With (B.6) and (B.10), we have

$$\Sigma_1 = -\frac{\alpha}{4\pi}2(\varepsilon - 1)\not{p}B_1(p^2; m_k, m_v)(V_{ij}^{+} - A_{ij}^{+}\gamma_5). \qquad (3.6)$$

Using the expansion in terms of ε (B.20):

$$\Sigma_1 = \frac{\alpha}{4\pi}\not{p}\left(1 + 2B_1(p^2; m_k, m_v)\right)(V_{ij}^{+} - A_{ij}^{+}\gamma_5) + \mathcal{O}(\varepsilon). \qquad (3.7)$$

Similarly one obtains

$$\begin{aligned}\Sigma_2 &= \frac{\alpha}{4\pi}2(2-\varepsilon)m_k\int\mathcal{D}q\frac{1}{\left((p+q)^2-m_k^2\right)(q^2-m_v^2)}(v_{ik,v}+a_{ik,v}\gamma_5)(v_{kj,v}-a_{kj,v}\gamma_5)\\ &= \frac{\alpha}{4\pi}2(2-\varepsilon)m_kB_0(p^2;m_v,m_k)(V_{ij}^- - A_{ij}^-\gamma_5) \qquad (3.8)\\ &= \frac{\alpha}{4\pi}m_k\left(4B_0(p^2;m_v,m_k)-2\right)(V_{ij}^- - A_{ij}^-\gamma_5)+\mathcal{O}(\varepsilon). \qquad (3.9)\end{aligned}$$

To evaluate the third part of $\Sigma_{ij}^V(p)$, one can consider the decomposition:

$$\frac{1}{(q^2-m_v^2)(q^2-\xi m_v^2)} = \frac{1}{m_v^2(1-\xi)}\left(\frac{1}{q^2-m_v^2}-\frac{1}{q^2-\xi m_v^2}\right) \qquad (3.10)$$

and the perfect square of the factor q^2+2pq, i.e. $(q+p)^2-p^2$. When adding and subtracting an m_k^2 one can simplify the factor $(p+q)^2-m_k^2$ in one term and obtain 0 for the corresponding integral, so

$$\begin{aligned}\Sigma_{3a} &= -\frac{\alpha}{4\pi}(1-\xi)\int\mathcal{D}q\frac{\not{q}(q^2+2pq)(v_{ik,v}-a_{ik,v}\gamma_5)(v_{kj,v}-a_{kj,v}\gamma_5)}{\left((p+q)^2-m_k^2\right)(q^2-m_v^2)(q^2-\xi m_v^2)} \qquad (3.11)\\ &= \frac{\alpha}{4\pi}\frac{p^2-m_k^2}{m_v^2}\int\mathcal{D}q\frac{\not{q}}{(p+q)^2-m_k^2}\left(\frac{1}{q^2-m_v^2}-\frac{1}{q^2-\xi m_v^2}\right)(V_{ij}^+ - A_{ij}^+\gamma_5).\end{aligned}$$

Now the two-point functions can be identified and with their covariant decomposition

$$\Sigma_{3a} = \frac{\alpha}{4\pi}\frac{p^2-m_k^2}{m_v^2}\not{p}\left(B_1(p^2;m_v,m_k)-B_1(p^2;\sqrt{\xi}m_v,m_k)\right)(V_{ij}^+ - A_{ij}^+\gamma_5). \qquad (3.12)$$

For the similar term with $\not{p}$, the same decomposition (3.10) is used and a $q^2+m_v^2-m_v^2$ or $q^2+\xi m_v^2-\xi m_v^2$ is going to be formed. Finally, one gets:

$$\begin{aligned}\Sigma_{3b} &= \frac{\alpha}{4\pi}(1-\xi)\not{p}\int\mathcal{D}q\frac{q^2(v_{ik,v}-a_{ik,v}\gamma_5)(v_{kj,v}-a_{kj,v}\gamma_5)}{\left((p+q)^2-m_k^2\right)(q^2-m_v^2)(q^2-\xi m_v^2)}\\ &= \frac{\alpha}{4\pi}\not{p}\left(B_0(p^2;m_v,m_k)-\xi B_0(p^2;\sqrt{\xi}m_v,m_k)\right)(V_{ij}^+ - A_{ij}^+\gamma_5). \qquad (3.13)\end{aligned}$$

The last part of $\Sigma_{ij}^V(p)$ looks almost identical to (3.13):

$$\begin{aligned}\Sigma_4 &= -\frac{\alpha}{4\pi}(1-\xi)m_k\int\mathcal{D}q\frac{q^2(v_{ik,v}+a_{ik,v}\gamma_5)(v_{kj,v}-a_{kj,v}\gamma_5)}{\left((p+q)^2-m_k^2\right)(q^2-m_v^2)(q^2-\xi m_v^2)}\\ &= -\frac{\alpha}{4\pi}m_k\left(B_0(p^2;m_v,m_k)-\xi B_0(p^2;\sqrt{\xi}m_v,m_k)\right)(V_{ij}^- - A_{ij}^-\gamma_5). \qquad (3.14)\end{aligned}$$

The final result for the self-energy, given by

$$\Sigma_{ij}^V(p) = \Sigma_1+\Sigma_2+\Sigma_{3a}+\Sigma_{3b}+\Sigma_4, \qquad (3.15)$$

can be decomposed to emphasise the vectorial and scalar parts:

$$\Sigma^V_{ij}(p) = \not{p}\Sigma^v_{ij}(p^2) + \not{p}\gamma_5\Sigma^a_{ij}(p^2) + \Sigma^s_{ij}(p^2) + \gamma_5\Sigma^p_{ij}(p^2), \tag{3.16}$$

$$\text{with}\quad \Sigma^v_{ij}(p^2) = \frac{\alpha}{4\pi}F^v(p^2; m_v, \sqrt{\xi}m_v, m_k)V^+_{ij}, \tag{3.17}$$

$$\Sigma^a_{ij}(p^2) = -\frac{\alpha}{4\pi}F^v(p^2; m_v, \sqrt{\xi}m_v, m_k)A^+_{ij}, \tag{3.18}$$

$$\Sigma^s_{ij}(p^2) = \frac{\alpha}{4\pi}m_k F^s(p^2; m_v, \sqrt{\xi}m_v, m_k)V^-_{ij}, \tag{3.19}$$

$$\Sigma^p_{ij}(p^2) = -\frac{\alpha}{4\pi}m_k F^s(p^2; m_v, \sqrt{\xi}m_v, m_k)A^-_{ij}. \tag{3.20}$$

If we collect the terms, from (3.6), (3.8), (3.12), (3.13) and (3.14) we can identify:

$$\begin{aligned}
F^v &= -2(\varepsilon-1)B_1(p^2; m_k, m_v) + \frac{p^2-m_k^2}{m_v^2}\Big(B_1(p^2; m_v, m_k) - B_1(p^2; \sqrt{\xi}m_v, m_k)\Big)\\
&\quad + B_0(p^2; m_v, m_k) - \xi B_0(p^2; \sqrt{\xi}m_v, m_k)\\
&= -\frac{\varepsilon}{p^2}\left(A(m_k^2) - A(m_v^2) - (p^2+m_k^2-m_v^2)B_0(p^2; m_k, m_v)\right)\\
&\quad + \frac{1}{p^2}A(m_k^2) + \frac{p^2-m_k^2-2m_v^2}{2m_v^2p^2}A(m_v^2) - \frac{p^2-m_k^2}{2m_v^2p^2}A(\xi m_v^2)\\
&\quad - \frac{1}{2m_v^2p^2}\left(\lambda(p^2, m_k^2, m_v^2) + 3m_v^2(p^2+m_k^2-m_v^2)\right)B_0(p^2; m_k, m_v)\\
&\quad + \frac{1}{2m_v^2p^2}\left(\lambda(p^2, m_k^2, \xi m_v^2) + \xi m_v^2(p^2+m_k^2-\xi m_v^2)\right)B_0(p^2; m_k, \sqrt{\xi}m_v),\\
F^s &= (-2\varepsilon+3)B_0(p^2; m_v, m_k) + \xi B_0(p^2; \sqrt{\xi}m_v, m_k).
\end{aligned} \tag{3.21}$$

Using (B.9), F^v was written as a function of $A(m^2)$ and B_0, the one- and two-point integrals explicitly calculated in B.2. λ is Kallen's function, as defined in (B.23).

If further on we consider the expansions in (3.7) and (3.9), F^v and F^s become

$$\begin{aligned}
F^v &= 1 + 2B_1(p^2; m_k, m_v) + \frac{p^2-m_k^2}{m_v^2}\Big(B_1(p^2; m_v, m_k) - B_1(p^2; \sqrt{\xi}m_v, m_k)\Big)\\
&\quad + B_0(p^2; m_v, m_k) - \xi B_0(p^2; \sqrt{\xi}m_v, m_k) + \mathcal{O}(\varepsilon)\\
&= 1 + \frac{1}{p^2}A(m_k^2) + \frac{p^2-m_k^2-2m_v^2}{2m_v^2p^2}A(m_v^2) - \frac{p^2-m_k^2}{2m_v^2p^2}A(\xi m_v^2)\\
&\quad - \frac{1}{2m_v^2p^2}\left(\lambda(p^2, m_k^2, m_v^2) + 3m_v^2(p^2+m_k^2-m_v^2)\right)B_0(p^2; m_k, m_v)\\
&\quad + \frac{1}{2m_v^2p^2}\left(\lambda(p^2, m_k^2, \xi m_v^2) + \xi m_v^2(p^2+m_k^2-\xi m_v^2)\right)B_0(p^2; m_k, \sqrt{\xi}m_v) + \mathcal{O}(\varepsilon),\\
F^s &= -2 + 3B_0(p^2; m_v, m_k) + \xi B_0(p^2; \sqrt{\xi}m_v, m_k) + \mathcal{O}(\varepsilon).
\end{aligned} \tag{3.22}$$

In terms of left and right projectors, the regularised self-energy expression is:

$$\Sigma^V_{ij}(p) = \not{p}\gamma_L\Sigma^L_{ij}(p^2) + \not{p}\gamma_R\Sigma^R_{ij}(p^2) + \gamma_L\Sigma^{DL}_{ij}(p^2) + \gamma_R\Sigma^{DR}_{ij}(p^2), \tag{3.23}$$

$$\text{with} \quad \Sigma^L_{ij}(p^2) = \frac{\alpha}{4\pi}F^v(p^2; m_v, \sqrt{\xi}m_v, m_k)(V^+_{ij} + A^+_{ij}), \tag{3.24}$$

$$\Sigma^R_{ij}(p^2) = \frac{\alpha}{4\pi}F^v(p^2; m_v, \sqrt{\xi}m_v, m_k)(V^+_{ij} - A^+_{ij}), \tag{3.25}$$

$$\Sigma^{DL}_{ij}(p^2) = \frac{\alpha}{4\pi}m_k F^s(p^2; m_v, \sqrt{\xi}m_v, m_k)(V^-_{ij} + A^-_{ij}), \tag{3.26}$$

$$\Sigma^{DR}_{ij}(p^2) = \frac{\alpha}{4\pi}m_k F^s(p^2; m_v, \sqrt{\xi}m_v, m_k)(V^-_{ij} - A^-_{ij}). \tag{3.27}$$

If we use the alternative way to express the couplings for the vertex, i.e. in terms of left- and right-handed components, with (2.48) the previous relations change to:

$$\Sigma^L_{ij}(p^2) = \frac{\alpha}{4\pi}F^v(p^2; m_v, \sqrt{\xi}m_v, m_k)g^L_{ik,v}g^L_{kj,v}, \tag{3.28}$$

$$\Sigma^R_{ij}(p^2) = \frac{\alpha}{4\pi}F^v(p^2; m_v, \sqrt{\xi}m_v, m_k)g^R_{ik,v}g^R_{kj,v}, \tag{3.29}$$

$$\Sigma^{DL}_{ij}(p^2) = \frac{\alpha}{4\pi}m_k F^s(p^2; m_v, \sqrt{\xi}m_v, m_k)g^R_{ik,v}g^L_{kj,v}, \tag{3.30}$$

$$\Sigma^{DR}_{ij}(p^2) = \frac{\alpha}{4\pi}m_k F^s(p^2; m_v, \sqrt{\xi}m_v, m_k)g^L_{ik,v}g^R_{kj,v}. \tag{3.31}$$

3.1.2 Diagrams with Scalar Bosons

The general one-loop Feynman diagram with a scalar boson is represented in Figure 3.2.

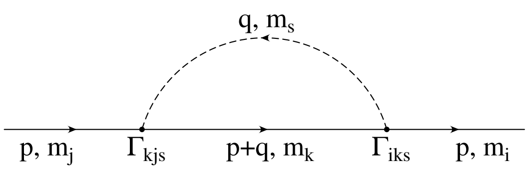


Figure 3.2: One-loop diagram with a fermion of mass m_k and a scalar boson of mass m_s.

The corresponding self-energy is written using the set of Feynman rules described in 2.57.

$$-i\Sigma^S_{ij}(p) = \int \frac{d^4q}{(2\pi)^4} i\Gamma_{iks} iS(p+q) i\Gamma_{kjs}\frac{i}{q^2 - m_s^2} \tag{3.32}$$

$$= e^2 \int \frac{d^4q}{(2\pi)^4} \frac{(c^L_{ik,s}\gamma_L + c^R_{ik,s}\gamma_R)(\not{p} + \not{q} + m_k)(c^L_{kj,s}\gamma_L + c^R_{kj,s}\gamma_R)}{((p+q)^2 - m_k^2)\,(q^2 - m_s^2)} \tag{3.33}$$

As mentioned in the previous subsection, the index S stands for the virtual boson of type s and also for the internal fermion k.

For its evaluation we follow similar steps as in the case of the fermion self-energy with vector bosons. After dimensional regularization and some rearranging of terms with the help of formulas presented in Appendix A, the self-energy becomes:

$$\Sigma_{ij}^{S}(p) = -\frac{\alpha}{4\pi}\int \mathcal{D}q \frac{(\not{p}+\not{q})(c_{ik,s}^{L}\gamma_R + c_{ik,s}^{R}\gamma_L)(c_{kj,s}^{L}\gamma_L + c_{kj,s}^{R}\gamma_R)}{((p+q)^2 - m_k^2)\,(q^2 - m_s^2)} \\ - \frac{\alpha}{4\pi} m_k \int \mathcal{D}q \frac{(c_{ik,s}^{L}\gamma_L + c_{ik,s}^{R}\gamma_R)(c_{kj,s}^{L}\gamma_L + c_{kj,s}^{R}\gamma_R)}{((p+q)^2 - m_k^2)\,(q^2 - m_s^2)}.$$

Expression (B.4), (B.5) and the covariant decomposition of $B_\mu(p; m_s, m_k)$, defined by (B.6) lead us to

$$\Sigma_{ij}^{S}(p) = -\frac{\alpha}{4\pi}\{\not{p}(B_0(p^2; m_s, m_k) + B_1(p^2; m_s, m_k))(c_{ik,s}^{R}c_{kj,s}^{L}\gamma_L + c_{ik,s}^{L}c_{kj,s}^{R}\gamma_R) \\ + m_k B_0(p^2; m_s, m_k)(c_{ik,s}^{L}c_{kj,s}^{L}\gamma_L + c_{ik,s}^{R}c_{kj,s}^{R}\gamma_R)\}.$$

With (B.10), the self-energy can be simplified to:

$$\Sigma_{ij}^{S}(p) = \frac{\alpha}{4\pi}\{\not{p}B_1(p^2; m_k, m_s)(c_{ik,s}^{R}c_{kj,s}^{L}\gamma_L + c_{ik,s}^{L}c_{kj,s}^{R}\gamma_R) \\ - m_k B_0(p^2; m_k, m_s)(c_{ik,s}^{L}c_{kj,s}^{L}\gamma_L + c_{ik,s}^{R}c_{kj,s}^{R}\gamma_R)\}.$$

Arguments of B_0 where changed according to their symmetry. Identifying

$$\Sigma_{ij}^{L}(p^2) = \frac{\alpha}{4\pi}B_1(p^2; m_k, m_s)c_{ik,s}^{R}c_{kj,s}^{L}, \tag{3.34}$$

$$\Sigma_{ij}^{R}(p^2) = \frac{\alpha}{4\pi}B_1(p^2; m_k, m_s)c_{ik,s}^{L}c_{kj,s}^{R}, \tag{3.35}$$

$$\Sigma_{ij}^{DL}(p^2) = -\frac{\alpha}{4\pi}m_k B_0(p^2; m_k, m_s)c_{ik,s}^{L}c_{kj,s}^{L}, \tag{3.36}$$

$$\Sigma_{ij}^{DR}(p^2) = -\frac{\alpha}{4\pi}m_k B_0(p^2; m_k, m_s)c_{ik,s}^{R}c_{kj,s}^{R}, \tag{3.37}$$

one can bring $\Sigma_{ij}^{S}(p)$ in a similar form to (3.23), i.e.

$$\Sigma_{ij}^{S}(p) = \not{p}\gamma_L\Sigma_{ij}^{L}(p^2) + \not{p}\gamma_R\Sigma_{ij}^{R}(p^2) + \gamma_L\Sigma_{ij}^{DL}(p^2) + \gamma_R\Sigma_{ij}^{DR}(p^2). \tag{3.38}$$

Using (B.9), we can write $B_1(p^2; m_k, m_s)$ in (3.34) and (3.35) as

$$B_1(p^2; m_k, m_s) = \frac{1}{2p^2}\left(A(m_k^2) - A(m_s^2) - (p^2 + m_k^2 - m_s^2)B_0(p^2; m_k, m_s)\right). \tag{3.39}$$

3.1.3 Analysis of Divergent and Imaginary Parts

After regularization, the divergent parts in the self-energy are comprised in the two-point integrals $B_0(p^2; m_1, m_2)$ and $B_1(p^2; m_1, m_2)$. In the appendix, we calculate and isolate these parts (see formula (B.19)). One can see that there is no mass or momentum dependence in the divergent parts of B_0 and B_1. To identify the divergences of the self-energy, we analyse each of its components. For loops with virtual bosons, we need first to replace the divergences of the two-point integrals in F^v and F^s, equation (3.22). We get

$$\text{div}[F^v(p^2; m_v, \sqrt{\xi}m_v, m_k)] = -\xi\Delta \tag{3.40}$$

$$\text{div}[F^s(p^2; m_v, \sqrt{\xi}m_v, m_k)] = (3+\xi)\Delta, \tag{3.41}$$

where Δ is given in (B.13). The divergent part for the self-energy components is given by

$$\begin{aligned}
\text{div}[\Sigma^L_{ij}(p^2)] &= -\frac{\alpha}{4\pi}\xi\Delta g^L_{ik,v}g^L_{kj,v},\\
\text{div}[\Sigma^R_{ij}(p^2)] &= -\frac{\alpha}{4\pi}\xi\Delta g^R_{ik,v}g^R_{kj,v},\\
\text{div}[\Sigma^{DL}_{ij}(p^2)] &= \frac{\alpha}{4\pi}m_k(3+\xi)\Delta g^R_{ik,v}g^L_{kj,v},\\
\text{div}[\Sigma^{DR}_{ij}(p^2)] &= \frac{\alpha}{4\pi}m_k(3+\xi)\Delta g^L_{ik,v}g^R_{kj,v},
\end{aligned} \tag{3.42}$$

when the internal boson is a vector, and by

$$\begin{aligned}
\text{div}[\Sigma^L_{ij}(p^2)] &= -\frac{\alpha}{4\pi}\frac{1}{2}\Delta c^R_{ik,s}c^L_{kj,s},\\
\text{div}[\Sigma^R_{ij}(p^2)] &= -\frac{\alpha}{4\pi}\frac{1}{2}\Delta c^L_{ik,s}c^R_{kj,s},\\
\text{div}[\Sigma^{DL}_{ij}(p^2)] &= \frac{\alpha}{4\pi}m_k\Delta c^L_{ik,s}c^L_{kj,s},\\
\text{div}[\Sigma^{DR}_{ij}(p^2)] &= \frac{\alpha}{4\pi}m_k\Delta c^R_{ik,s}c^R_{kj,s},
\end{aligned} \tag{3.43}$$

when it is a scalar. The terms of the self-energy carrying divergences have no momentum dependence. The parameters entering these expressions are just the mass of the internal fermion and products of coupling constants.

For the analysis of imaginary parts in the self-energy components, we do not display the contributions coming from complex-valued coupling constants and we focus only on the ones originating from the two-point functions.

According to (B.27)

$$\text{Im}B_0(p^2; m_1, m_2) = \frac{1}{p^2}\pi\sqrt{\lambda(p^2, m_1^2, m_2^2)}, \text{ if } (m_1+m_2)^2 < p^2. \tag{3.44}$$

Kallen's function λ is defined in (B.23). The restriction of the momentum (see also figure B.1) points to the kinematic requirement for a decay. From (B.9), we get

$$\mathrm{Im}B_1(p^2; m_1, m_2) = -\frac{p^2 + m_1^2 - m_2^2}{2p^4}\pi\sqrt{\lambda(p^2, m_1^2, m_2^2)}, \text{ if } (m_1 + m_2)^2 < p^2. \quad (3.45)$$

Replacing B_0 and B_1 with (3.44) and (3.45) in the formulas for the self-energy components, one can determine the imaginary contributions that come from the possible cuts through the one-loop diagrams. Such contributions to an amplitude give the absorptive parts. For the self-energy with internal vector bosons, the imaginary contributions from the n-point integrals are comprised in F^v and F^s. The imaginary part of their expressions as given in (3.22) is

$$\begin{aligned}
\mathrm{Im}F^v(p^2; m_v, \sqrt{\xi}m_v, m_k) =& -\frac{\pi}{2m_v^2p^4}\left(\lambda(p^2, m_k^2, m_v^2) + 3m_v^2(p^2 + m_k^2 - m_v^2)\right) \\
&\sqrt{\lambda(p^2, m_k^2, m_v^2)}\theta(p^2 - (m_k + m_v)^2) \\
&+ \frac{\pi}{2m_v^2p^4}\left(\lambda(p^2, m_k^2, \xi m_v^2) + \xi m_v^2(p^2 + m_k^2 - \xi m_v^2)\right) \\
&\sqrt{\lambda(p^2, m_k^2, \xi m_v^2)}\theta(p^2 - (m_k + \sqrt{\xi}m_v)^2) + \mathcal{O}(\varepsilon), \\
\mathrm{Im}F^s(p^2; m_v, \sqrt{\xi}m_v, m_k) =& \frac{\pi}{p^2}3\sqrt{\lambda(p^2, m_k^2, m_v^2)}\theta(p^2 - (m_k + m_v)^2) \qquad (3.46) \\
&+ \frac{\pi}{p^2}\xi\sqrt{\lambda(p^2, m_k^2, \xi m_v^2)}\theta(p^2 - (m_k + \sqrt{\xi}m_v)^2) + \mathcal{O}(\varepsilon),
\end{aligned}$$

where θ is the Heaviside step function. The absorptive contributions of the self-energy with internal scalar bosons result again from B_0 and B_1 as given in (3.44) and (3.45).

Note that the presence of imaginary contributions in the self-energy components is not only a consequence of the fact that particles are unstable. There are also imaginary parts arising from unphysical intermediate states. These ones are gauge dependent and they are present if, for example, we choose ξ such that $p^2 > m_k + \sqrt{\xi}m_v$. However, in the calculation of complete matrix elements, the ξ dependent parts will cancel eventually.

3.2 General Expressions for the Dirac and Majorana Fermion Self-Energies

Since in theories beyond the Standard Model also Majorana fermions appear, we need to evaluate the corresponding changes introduced in the self-energies. First, we consider the diagrams that have to be added to the Dirac fermion self-energy and then we evaluate the ones for the Majorana fermions. We take into account all allowed vertices, as described in (2.60).

3.2.1 Dirac Fermion Self-Energy

The cases considered in sections 3.1.1 and 3.1.2 do not cover anymore all the possible contributions to a Dirac fermion self-energy. In addition one has to include one-loop diagrams with a Majorana and a vector or scalar boson, like in Figure 3.3 and 3.4, respectively. Their evaluation is almost identical to the previous cases. As it will be seen in the next steps, just several parameters should be replaced.

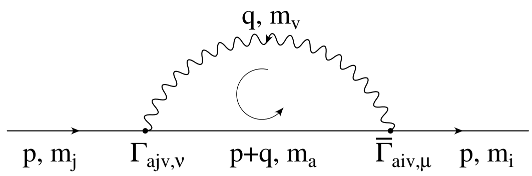


Figure 3.3: One-loop diagram with a Majorana fermion of mass m_a and a vector boson of mass m_v.

For each figure we insert the Feynman rules in a similar manner like before, this time paying attention to the changes imposed by the choice of the general fermion flow indicated by the additional arrow line. Using (2.57) and (2.60), the self-energy for Figure 3.3 is:

$$
\begin{aligned}
-i\Sigma_{ij}^{V}(p) &= \int \frac{d^4q}{(2\pi)^4} i\overline{\Gamma_{aiv,\mu}} iS(p+q) i\Gamma_{ajv,\nu} iD_v^{\mu\nu} \\
&= e^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{((p+q)^2 - m_a^2)\,(q^2 - m_v^2)} \gamma_\mu (v_{ai,v}^* - a_{ai,v}^* \gamma_5)(\not{p} + \not{q} + m_a) \\
&\quad \gamma_\nu (v_{aj,v} - a_{aj,v}\gamma_5) \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2 - \xi m_v^2}(1-\xi) \right).
\end{aligned}
\tag{3.47}
$$

If one compares this expression with (3.2), one can remark that replacing the Dirac fermion in the loop with a Majorana one does not bring significant changes. The result becomes identical when the mass m_k is changed to m_a and the coupling constants $v_{ik,v}$, $a_{ik,v}$ and $v_{kj,v}$, $a_{kj,v}$ are replaced with $v_{ai,v}^*$, $a_{ai,v}^*$ and $v_{aj,v}$, $a_{aj,v}$, respectively.

Here, we are not going to express the Feynman rules for the vertices also as couplings to the left and right projectors. It will be done for all diagrams in the last section of this chapter, when summarising the results.

For the similar diagram with a scalar boson (Figure 3.4), one also has to adjust the mass m_a for the internal fermion and accordingly the new coupling constants to get the result corresponding to (3.33).

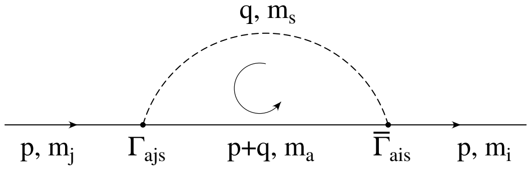


Figure 3.4: One-loop diagram with a Majorana fermion of mass m_a and a scalar boson of mass m_s.

$$
\begin{aligned}
-i\Sigma_{ij}^{S}(p) &= \int \frac{d^4q}{(2\pi)^4} i\overline{\Gamma_{ais}} iS(p+q) i\Gamma_{ajs} \frac{i}{q^2 - m_s^2} \\
&= e^2 \int \frac{d^4q}{(2\pi)^4} \frac{((c_{ai,s}^R)^*\gamma_L + (c_{ai,s}^L)^*\gamma_R)(\not{p} + \not{q} + m_a)(c_{aj,s}^L\gamma_L + c_{aj,s}^R\gamma_R)}{((p+q)^2 - m_a^2)\,(q^2 - m_s^2)}
\end{aligned} \tag{3.48}
$$

Because of the Feynman rules for Majorana and Dirac fermions, the coupling constants $c_{ik,s}^L$, $c_{ik,s}^R$ change to $(c_{ai,s}^R)^*$, $(c_{ai,s}^L)^*$ and $c_{kj,s}^L$, $c_{kj,s}^R$ to $c_{aj,s}^L$, $c_{aj,s}^R$. Note that in the first set of couplings, the left and the right components appear interchanged.

We have not excluded models that allow for boson interactions with a Dirac field and a charge conjugated one. In fact, as mentioned when defining the general form of the Lagrangian, we will encounter this type of interactions in the neutrino seesaw mechanism. There, a charged component of a Higgs triplet will couple to a charged lepton and a charge conjugated one. Then, diagrams as in Figure 3.5 will add to the charged lepton self-energy.

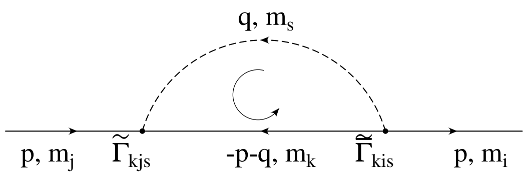


Figure 3.5: One-loop diagram with a charge conjugated Dirac fermion of mass m_k and a scalar boson of mass m_s.

The self-energy contribution of such a diagram is

$$
\begin{aligned}
-i\Sigma^S_{ij}(p) &= \int \frac{d^4q}{(2\pi)^4} i\overline{\tilde{\Gamma}_{kis}} iS(p+q) i\tilde{\Gamma}_{kjs} \frac{i}{q^2 - m_s^2} \\
&= e^2 \int \frac{d^4q}{(2\pi)^4} \frac{((\tilde{c}^R_{ki,s})^* \gamma_L + (\tilde{c}^L_{ki,s})^* \gamma_R)(\not{p} + \not{q} + m_k)(\tilde{c}^L_{kj,s}\gamma_L + \tilde{c}^R_{kj,s}\gamma_R)}{((p+q)^2 - m_a^2)\,(q^2 - m_s^2)}.
\end{aligned}
\tag{3.49}
$$

As previously, the coupling constants $c^L_{ik,s}$, $c^R_{ik,s}$ in (3.33) change to $(\tilde{c}^R_{ki,s})^*$, $(\tilde{c}^L_{ki,s})^*$ and $c^L_{kj,s}$, $c^R_{kj,s}$ to $\tilde{c}^L_{kj,s}$, $\tilde{c}^R_{kj,s}$.

In all cases we have seen that no additional calculation is required. A simple replacement of the mass and coupling parameters gives the final result for the self-energies.

3.2.2 Majorana Fermion Self-Energy

Calculating the self-energy for the Majorana fermions is not more complicated. We will end up with a rather tedious enumeration of masses and coupling constants. However, some special attention is required for two new specific cases that appear. The reader can also skip the next two subsections. The result will be summarised in section 3.3.

Like before, we start with the diagrams with virtual vector bosons and then we give the corresponding ones for scalar bosons. For the cases obviously similar to the ones from section 3.1, we are just writing the self-energy expression and emphasising the new parameters. When charge-conjugation is required, we will make extra remarks. In all the diagrams, the initial momentum p is flowing from left to right.

Diagrams with Vector Bosons

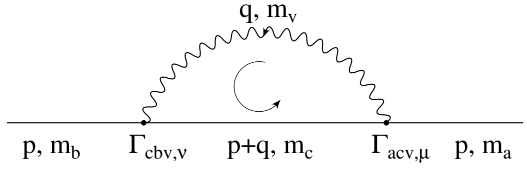


Figure 3.6: One-loop diagram with a Majorana fermion of mass m_c and a vector boson of mass m_v.

$$
\begin{aligned}
-i\Sigma^V_{ab}(p) &= \int \frac{d^4q}{(2\pi)^4} i\Gamma_{acv,\mu} iS(p+q) i\Gamma_{cbv,\nu} iD^{\mu\nu}_v \\
&= e^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{((p+q)^2 - m_c^2)\,(q^2 - m_v^2)} \gamma_\mu (v_{ac,v} - a_{ac,v}\gamma_5)(\not{p} + \not{q} + m_c) \\
&\quad \gamma_\nu (v_{cb,v} - a_{cb,v}\gamma_5) \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2 - \xi m_v^2}(1-\xi) \right) .
\end{aligned}
\tag{3.50}
$$

Comparing the Majorana fermion self-energy with a virtual vector boson and a Majorana fermion (formula (3.50)) with formula (3.2), one can identify m_k with m_c and $v_{ik,v}$, $a_{ik,v}$ and $v_{kj,v}$, $a_{kj,v}$ with $v_{ac,v}$, $a_{ac,v}$ and $v_{cb,v}$, $a_{cb,v}$, respectively.

The case when the internal Majorana fermion is replaced by a Dirac one having the fermion number flow from left to right (the 'normal' orientation) is considered in Figure 3.7.

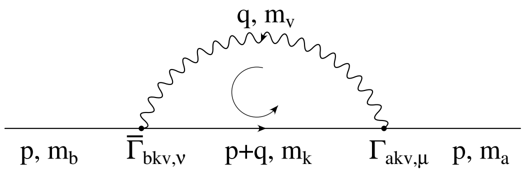


Figure 3.7: One-loop diagram with a Dirac fermion of mass m_k and a vector boson of mass m_v.

From our initial general expression for a self-energy with a vector boson given in (3.2) one has to change $v_{ik,v}$, $a_{ik,v}$ and $v_{kj,v}$, $a_{kj,v}$ to $v_{ak,v}$, $a_{ak,v}$ and $v^*_{bk,v}$, $a^*_{bk,v}$ to obtain (3.51).

$$
\begin{aligned}
-i\Sigma^V_{ab}(p) &= \int \frac{d^4q}{(2\pi)^4} i\Gamma_{akv,\mu} iS(p+q) i\overline{\Gamma_{bkv,\nu}} iD^{\mu\nu}_v \\
&= e^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{((p+q)^2 - m_k^2)\,(q^2 - m_v^2)} \gamma_\mu (v_{ak,v} - a_{ak,v}\gamma_5)(\not{p} + \not{q} + m_k) \\
&\quad \gamma_\nu (v^*_{bk,v} - a^*_{bk,v}\gamma_5) \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2 - \xi m_v^2}(1-\xi) \right) .
\end{aligned}
\tag{3.51}
$$

Majorana fermions allow a new type of diagram with reversed fermion number flow. Therefore, we have to consider also contributions as described in Figure 3.8.

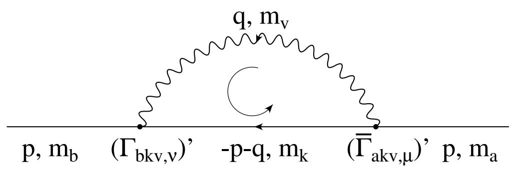


Figure 3.8: One-loop diagram with a Dirac fermion of mass m_k and an opposite fermion number flow and a vector boson of mass m_v.

In contrast with the Dirac case where in Figure 3.5 we have a new type of process, this one is physically equivalent to the one in Figure 3.7.

In Figure 3.8, the inner momentum flow was chosen to lead to an easy to identify expression when comparing with (3.2). Inserting the Feynman rules, one has to remember how we have set the initial momentum orientation. The vertices appearing now have a general fermion flow opposite to the internal Dirac propagator one. The self-energy is given by

$$\begin{aligned} -i\Sigma^V_{ab}(p) &= \int \frac{d^4q}{(2\pi)^4} i(\overline{\Gamma_{akv,\mu}})' iS(p+q) i(\Gamma_{bkv,\nu})' iD^{\mu\nu}_v \\ &= e^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{((p+q)^2 - m_k^2)\,(q^2 - m_v^2)} \gamma_\mu (v^*_{ak,v} + a^*_{ak,v}\gamma_5)(\not p + \not q + m_k) \\ &\quad \gamma_\nu (v_{bk,v} + a_{bk,v}\gamma_5) \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2 - \xi m_v^2}(1-\xi) \right) . \end{aligned} \tag{3.52}$$

It is nice to see that no real complications appear for this new case. Going back to (3.2), the couplings $v_{ik,v}$, $a_{ik,v}$; $v_{kj,v}$, $a_{kj,v}$ become $v^*_{ak,v}$, $-a^*_{ak,v}$; $v_{bk,v}$, $-a_{bk,v}$.

Diagrams with Scalar Bosons

Switching to the loop diagrams with scalar bosons, no new problems appear. The result obtained from (3.33) is sufficient to express a Majorana fermion self-energy with an internal scalar. In the first case (Figure 3.9), when one has an internal Majorana fermion, the old mass m_k becomes m_c and the couplings $c^L_{ik,s}$, $c^R_{ik,s}$; $c^L_{kj,s}$, $c^R_{kj,s}$ have just to change indices to $c^L_{ac,s}$, $c^R_{ac,s}$; $c^L_{cb,s}$, $c^R_{cb,s}$.

$$\begin{aligned} -i\Sigma^S_{ab}(p) &= \int \frac{d^4q}{(2\pi)^4} i\Gamma_{acs} iS(p+q) i\Gamma_{cbs} \frac{i}{q^2 - m_s^2} \\ &= e^2 \int \frac{d^4q}{(2\pi)^4} \frac{(c^L_{ac,s}\gamma_L + c^R_{ac,s}\gamma_R)(\not p + \not q + m_c)(c^L_{cb,s}\gamma_L + c^R_{cb,s}\gamma_R)}{((p+q)^2 - m_c^2)\,(q^2 - m_s^2)} \end{aligned} \tag{3.53}$$

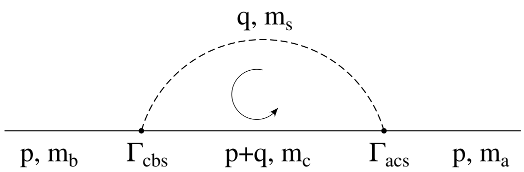


Figure 3.9: One-loop diagram with a Majorana fermion of mass m_c and a scalar boson of mass m_s.

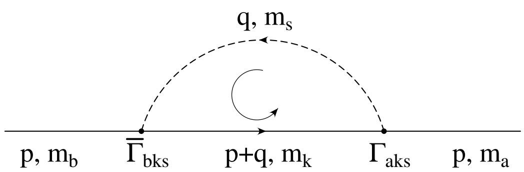


Figure 3.10: One-loop diagram with a Dirac fermion of mass m_k and a scalar boson of mass m_s.

For a Dirac fermion in the loop, with the regular fermion number flow, the self-energy is:

$$\begin{aligned} -i\Sigma^S_{ab}(p) &= \int \frac{d^4q}{(2\pi)^4} i\Gamma_{aks} iS(p+q) i\overline{\Gamma_{bks}} \frac{i}{q^2 - m_s^2} \\ &= e^2 \int \frac{d^4q}{(2\pi)^4} \frac{(c^L_{ak,s}\gamma_L + c^R_{ak,s}\gamma_R)(\not{p} + \not{q} + m_k)((c^R_{bk,s})^*\gamma_L + (c^L_{bk,s})^*\gamma_R)}{((p+q)^2 - m_k^2)\,(q^2 - m_s^2)}. \end{aligned} \tag{3.54}$$

The new internal fermion mass is still m_k and the vertex couplings $c^L_{ak,s}$, $c^R_{ak,s}$; $(c^R_{bk,s})^*$, $(c^L_{bk,s})^*$ take the role of $c^L_{ik,s}$, $c^R_{ik,s}$; $c^L_{kj,s}$, $c^R_{kj,s}$.

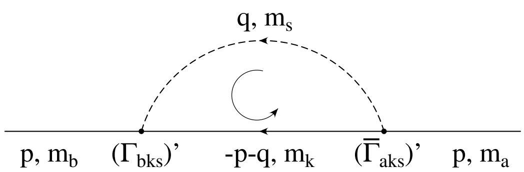


Figure 3.11: One-loop diagram with a Dirac fermion of mass m_k and an opposite fermion number flow and a scalar boson of mass m_s.

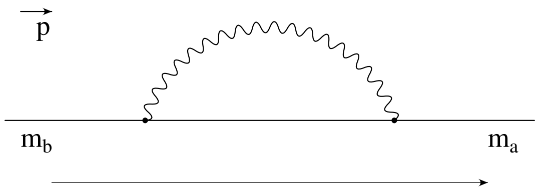
The new case given by the reversed fermion number flow for the Dirac particle (Figure 3.11) is simpler than before. Looking in (2.60) at the Feynman rules for these vertices we can see that since $C(\gamma_{L/R})^T C^{-1} = \gamma_{L/R}$, we have no change from the normal rules. The self-energy will be:

$$\begin{aligned}
-i\Sigma^S_{ab}(p) &= \int \frac{d^4q}{(2\pi)^4} i(\overline{\Gamma_{aks}})' iS(p+q) i(\Gamma_{bks})' \frac{i}{q^2 - m_s^2} \\
&= \int \frac{d^4q}{(2\pi)^4} i\overline{\Gamma_{aks}} iS(p+q) i\Gamma_{bks} \frac{i}{q^2 - m_s^2} \\
&= e^2 \int \frac{d^4q}{(2\pi)^4} \frac{((c^R_{ak,s})^* \gamma_L + (c^L_{ak,s})^* \gamma_R)(\not{p} + \not{q} + m_k)(c^L_{bk,s}\gamma_L + c^R_{bk,s}\gamma_R)}{((p+q)^2 - m_k^2)\,(q^2 - m_s^2)}.
\end{aligned} \tag{3.55}$$

With (3.33) one fits the result by taking $(c^R_{ak,s})^*$, $(c^L_{ak,s})^*$; $c^L_{bk,s}$, $c^R_{bk,s}$ instead of the couplings $c^L_{ik,s}$, $c^R_{ik,s}$; $c^L_{kj,s}$, $c^R_{kj,s}$.

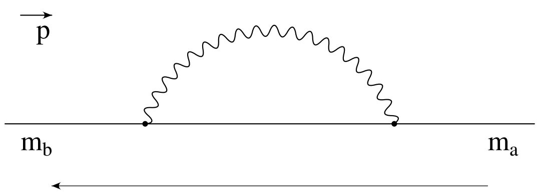
Properties

Because of the nature of particles, the Majorana fermion self-energy has an additional property. It is a consequence of the invariance of the result when changing the orientation of the general fermion flow. We consider a general one-loop diagram described by



$-i\Sigma_{ab}(p)$

The loop can contain any possible combination of particles. If we now evaluate the diagram changing the general fermion flow, we have



$-i\Sigma_{ba}(-p)$

Using the property (2.56) of Majorana couplings and the fact that

$$S(-q) = C(S(q))^T C^{-1}, \tag{3.56}$$

one can prove that the two self-energies are related by

$$\Sigma_{ba}(-p) = C\Sigma_{ab}^T(p)C^{-1}. \tag{3.57}$$

If we write the self-energies decomposed like in (3.23), i.e.

$$\Sigma_{ab}(p) = \not{p}\gamma_L\Sigma_{ab}^L(p^2) + \not{p}\gamma_R\Sigma_{ab}^R(p^2) + \gamma_L\Sigma_{ab}^{DL}(p^2) + \gamma_R\Sigma_{ab}^{DR}(p^2),$$

and we use the properties given in (2.13), (3.57) is equivalent to

$$\begin{aligned} \Sigma_{ba}^L(p^2) &= \Sigma_{ab}^R(p^2), \\ \Sigma_{ba}^{DL}(p^2) &= \Sigma_{ab}^{DL}(p^2), \\ \Sigma_{ba}^{DR}(p^2) &= \Sigma_{ab}^{DR}(p^2). \end{aligned} \tag{3.58}$$

3.3 Summary of Results for Fermion Self-Energies

In section 3.2, we have proved that calculating any fermion self-energy, with Dirac and/or Majorana fermions, the result is going to reduce to formulas of section 3.1. Here, we are summarising the possible Feynman diagrams and the parameters that enter the results.

For fermion self-energies with a virtual vector boson, the result is given by (3.16):

$$\Sigma_{xy}^V(p) = \not{p}\Sigma_{xy}^v(p^2) + \not{p}\gamma_5\Sigma_{xy}^a(p^2) + \Sigma_{xy}^s(p^2) + \gamma_5\Sigma_{xy}^p(p^2)$$

$$\begin{aligned} \text{with} \quad \Sigma_{xy}^v(p^2) &= \frac{\alpha}{4\pi}F^v(p^2; m_v, \sqrt{\xi}m_v, m_z)V_{xy}^+, \\ \Sigma_{xy}^a(p^2) &= -\frac{\alpha}{4\pi}F^v(p^2; m_v, \sqrt{\xi}m_v, m_z)A_{xy}^+, \\ \Sigma_{xy}^s(p^2) &= \frac{\alpha}{4\pi}m_zF^s(p^2; m_v, \sqrt{\xi}m_v, m_z)V_{xy}^-, \\ \Sigma_{xy}^p(p^2) &= -\frac{\alpha}{4\pi}m_zF^s(p^2; m_v, \sqrt{\xi}m_v, m_z)A_{xy}^-. \end{aligned}$$

V_{xy}^+, V_{xy}^-, A_{xy}^+ and A_{xy}^- are functions of coupling constants and they are the only parameters that change when considering different contributions to fermion self-energies. They are listed in Table 3.1 . The trivial replacement of the internal fermion mass m_z due to notation is not written. We do also not make additional remarks for other particle indices. It is obvious that xy refers to ij for a Dirac fermion self-energy and to ab for a Majorana one.

Alternatively, one can express the result using the decomposition of the self-energy in terms of left and right components (3.23):

$$\Sigma_{xy}^V(p) = \not{p}\gamma_L\Sigma_{xy}^L(p^2) + \not{p}\gamma_R\Sigma_{xy}^R(p^2) + \gamma_L\Sigma_{xy}^{DL}(p^2) + \gamma_R\Sigma_{xy}^{DR}(p^2),$$

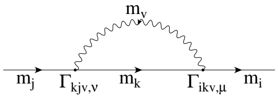
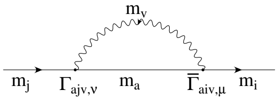
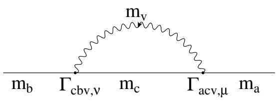
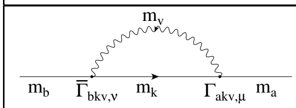
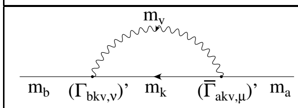
self-energy	$V_{xy}^{\pm}$	$A_{xy}^{\pm}$
	$v_{ik,v}v_{kj,v} \pm a_{ik,v}a_{kj,v}$	$v_{ik,v}a_{kj,v} \pm a_{ik,v}v_{kj,v}$
	$v^*_{ai,v}v_{aj,v} \pm a^*_{ai,v}a_{aj,v}$	$v^*_{ai,v}a_{aj,v} \pm a^*_{ai,v}v_{aj,v}$
	$v_{ac,v}v_{cb,v} \pm a_{ac,v}a_{cb,v}$	$v_{ac,v}a_{cb,v} \pm a_{ac,v}v_{cb,v}$
	$v_{ak,v}v^*_{bk,v} \pm a_{ak,v}a^*_{bk,v}$	$v_{ak,v}a^*_{bk,v} \pm a_{ak,v}v^*_{bk,v}$
	$v^*_{ak,v}v_{bk,v} \pm a^*_{ak,v}a_{bk,v}$	$-v^*_{ak,v}a_{bk,v} \mp a^*_{ak,v}v_{bk,v}$

Table 3.1: Fermion self-energies with a virtual vector boson.

$$\text{with} \quad \Sigma^L_{xy}(p^2) = \frac{\alpha}{4\pi}F^v(p^2; m_v, \sqrt{\xi}m_v, m_z)G^L_{xy}, \tag{3.59}$$

$$\Sigma^R_{xy}(p^2) = \frac{\alpha}{4\pi}F^v(p^2; m_v, \sqrt{\xi}m_v, m_z)G^R_{xy}, \tag{3.60}$$

$$\Sigma^{DL}_{xy}(p^2) = \frac{\alpha}{4\pi}m_z F^s(p^2; m_v, \sqrt{\xi}m_v, m_z)G^{DL}_{xy}, \tag{3.61}$$

$$\Sigma^{DR}_{xy}(p^2) = \frac{\alpha}{4\pi}m_z F^s(p^2; m_v, \sqrt{\xi}m_v, m_z)G^{DR}_{xy}. \tag{3.62}$$

We have defined:

$$\begin{aligned} G^L_{xy} &= V^+_{xy} + A^+_{xy} = g^L_{xz,v}g^L_{zy,v}, \\ G^R_{xy} &= V^+_{xy} - A^+_{xy} = g^R_{xz,v}g^R_{zy,v}, \\ G^{DL}_{xy} &= V^-_{xy} + A^-_{xy} = g^R_{xz,v}g^L_{zy,v}, \\ G^{DR}_{xy} &= V^-_{xy} - A^-_{xy} = g^L_{xz,v}g^R_{zy,v}. \end{aligned} \tag{3.63}$$

With the Feynman rules for vertices written as couplings to γ_L and γ_R (2.60) and with the help of formula (2.48) one can transform the parameters from Table 3.1 into G^L_{xy}, G^R_{xy}, G^{DL}_{xy} and G^{DR}_{xy}. The latter are listed in Table 3.2.

For the fermion self-energy with a virtual scalar boson, one has a similar table. The self-energy is given by (3.38), i.e.:

$$\Sigma^S_{xy}(p) = \not{p}\gamma_L\Sigma^L_{xy}(p^2) + \not{p}\gamma_R\Sigma^R_{xy}(p^2) + \gamma_L\Sigma^{DL}_{xy}(p^2) + \gamma_R\Sigma^{DR}_{xy}(p^2)$$

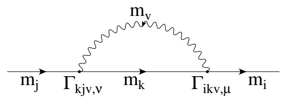
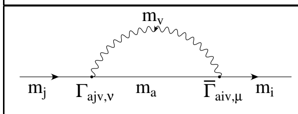
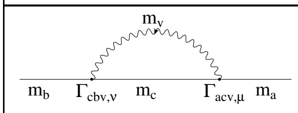
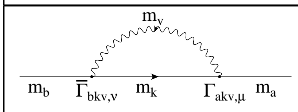
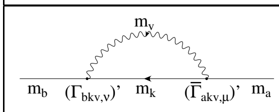
self-energy	G^L_{xy}	G^R_{xy}	G^{DL}_{xy}	G^{DR}_{xy}
	$g^L_{ik,v}g^L_{kj,v}$	$g^R_{ik,v}g^R_{kj,v}$	$g^R_{ik,v}g^L_{kj,v}$	$g^L_{ik,v}g^R_{kj,v}$
	$(g^L_{ai,v})^* g^L_{aj,v}$	$(g^R_{ai,v})^* g^R_{aj,v}$	$(g^R_{ai,v})^* g^L_{aj,v}$	$(g^L_{ai,v})^* g^R_{aj,v}$
	$g^L_{ac,v}g^L_{cb,v}$	$g^R_{ac,v}g^R_{cb,v}$	$g^R_{ac,v}g^L_{cb,v}$	$g^L_{ac,v}g^R_{cb,v}$
	$g^L_{ak,v}(g^L_{bk,v})^*$	$g^R_{ak,v}(g^R_{bk,v})^*$	$g^R_{ak,v}(g^L_{bk,v})^*$	$g^L_{ak,v}(g^R_{bk,v})^*$
	$(g^R_{ak,v})^* g^R_{bk,v}$	$(g^L_{ak,v})^* g^L_{bk,v}$	$(g^L_{ak,v})^* g^R_{bk,v}$	$(g^R_{ak,v})^* g^L_{bk,v}$

Table 3.2: Fermion self-energies with a virtual scalar boson.

$$\text{with} \quad \Sigma^L_{xy}(p^2) = \frac{\alpha}{4\pi} B_1(p^2; m_z, m_s) C^L_{xy}, \tag{3.64}$$

$$\Sigma^R_{xy}(p^2) = \frac{\alpha}{4\pi} B_1(p^2; m_z, m_s) C^R_{xy}, \tag{3.65}$$

$$\Sigma^{DL}_{xy}(p^2) = -\frac{\alpha}{4\pi} m_z B_0(p^2; m_z, m_s) C^{DL}_{xy}, \tag{3.66}$$

$$\Sigma^{DR}_{xy}(p^2) = -\frac{\alpha}{4\pi} m_z B_0(p^2; m_z, m_s) C^{DR}_{xy}. \tag{3.67}$$

We have replaced the different products of coupling constants by C^L_{xy}, C^R_{xy}, C^{DL}_{xy} and C^{DR}_{xy}.

$$\begin{aligned} C^L_{xy} &= c^R_{xz,s} c^L_{zy,s}, \\ C^R_{xy} &= c^L_{xz,s} c^R_{zy,s}, \\ C^{DL}_{xy} &= c^L_{xz,s} c^L_{zy,s}, \\ C^{DR}_{xy} &= c^R_{xz,s} c^R_{zy,s}. \end{aligned} \tag{3.68}$$

These parameters are used in Table 3.3 where we list the contributions from self-energies with internal scalars.

It is worth making a few additional remarks on some special features of the self-energy. These properties are not restricted to the case of Dirac fermions, therefore we will keep in the following, the generic notation.

Consider the sum of all possible one-loop self-energies (with virtual vector and

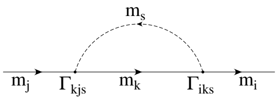
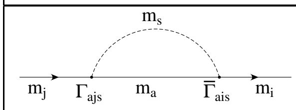
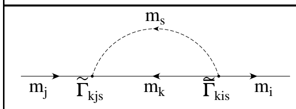
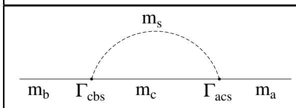
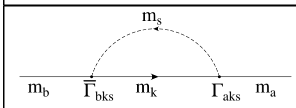
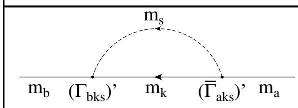
self-energy	C^L_{xy}	C^R_{xy}	C^{DL}_{xy}	C^{DR}_{xy}
	$c^R_{ik,s}c^L_{kj,s}$	$c^L_{ik,s}c^R_{kj,s}$	$c^L_{ik,s}c^L_{kj,s}$	$c^R_{ik,s}c^R_{kj,s}$
	$(c^L_{ai,s})^* c^L_{aj,s}$	$(c^R_{ai,s})^* c^R_{aj,s}$	$(c^R_{ai,s})^* c^L_{aj,s}$	$(c^L_{ai,s})^* c^R_{aj,s}$
	$(\tilde{c}^L_{ki,s})^* \tilde{c}^L_{kj,s}$	$(\tilde{c}^R_{ki,s})^* \tilde{c}^R_{kj,s}$	$(\tilde{c}^R_{ki,s})^* \tilde{c}^L_{kj,s}$	$(\tilde{c}^L_{ki,s})^* \tilde{c}^R_{kj,s}$
	$c^R_{ac,s}c^L_{cb,s}$	$c^L_{ac,s}c^R_{cb,s}$	$c^L_{ac,s}c^L_{cb,s}$	$c^R_{ac,s}c^R_{cb,s}$
	$c^R_{ak,s}(c^R_{bk,s})^*$	$c^L_{ak,s}(c^L_{bk,s})^*$	$c^L_{ak,s}(c^R_{bk,s})^*$	$c^R_{ak,s}(c^L_{bk,s})^*$
	$(c^L_{ak,s})^* c^L_{bk,s}$	$(c^R_{ak,s})^* c^R_{bk,s}$	$(c^R_{ak,s})^* c^L_{bk,s}$	$(c^L_{ak,s})^* c^R_{bk,s}$

Table 3.3: Fermion self-energies with a virtual scalar boson.

scalar bosons), for a given xy combination:

$$\Sigma_{xy}(p) = \not{p}\gamma_L \Sigma^L_{xy}(p^2) + \not{p}\gamma_R \Sigma^R_{xy}(p^2) + \gamma_L \Sigma^{DL}_{xy}(p^2) + \gamma_R \Sigma^{DR}_{xy}(p^2). \tag{3.69}$$

In the standard model, one can show that

$$\begin{aligned} \Sigma^{DL}_{xy}(p^2) &= m_x \Sigma^D_{xy}(p^2), \\ \Sigma^{DR}_{xy}(p^2) &= \Sigma^D_{xy}(p^2) m_y. \end{aligned} \tag{3.70}$$

$\Sigma^D_{xy}(p^2)$ stands for the sum of the contributions to the scalar part of the self-energy except the given mass factor. This property can be proven with the help of the Feynman rules for the Standard Model (Table 2.1) by explicit calculation. Taking each possible diagram and inserting the corresponding coupling constants, one can verify that (3.61) and (3.62) or (3.66) and (3.67) differ just by one mass factor. For the diagonal case, (3.70) implies

$$\Sigma^{DL}_{xx}(p^2) = \Sigma^{DR}_{xx}(p^2). \tag{3.71}$$

If we have models, for which there are no imaginary contributions from the

n-point integrals in the self-energy (see section 3.1.3), then

$$\begin{aligned}\Sigma^{L}_{xy}(p^2) &= (\Sigma^{L}_{yx}(p^2))^*,\\ \Sigma^{R}_{xy}(p^2) &= (\Sigma^{R}_{yx}(p^2))^*,\\ \Sigma^{DR}_{xy}(p^2) &= (\Sigma^{DL}_{yx}(p^2))^*.\end{aligned} \tag{3.72}$$

The relation can be easily proven for each possible combination of internal particles, using the hermiticity of the interaction Lagrangian reflected in the coupling constants, see equations (2.49) and (2.53).

If we refere to Majorana fermions, to all particular cases described above, the properties (3.58) additionally apply.

Chapter 4

Divergences of Fermion Self-Energies

In a Lagrangian, we have some parameters that we call 'masses' and 'couplings', but they are not identical with the corresponding observable quantities. Therefore, we have to carefully define the relations between these theoretical parameters and the corresponding experimental quantities. This procedure is called renormalization. An additional request for renormalization comes from the need to subtract the divergences occurring in perturbative calculations.

One convenient prescription to solve these problems is the on-shell renormalization scheme. In the following, we give a short description of the method and its results within one-loop accuracy. The purpose of this chapter is to study the properties of the full one-loop corrected propagator. We will write the propagator matrix in a form which allows us to directly read off the position of its poles and residues. This form will be convenient for the next chapter where we discuss the renormalization of the free Lagrangian and redefine fields and parameters to absorb divergences. Only then we can start to discuss the complete order α corrections.

The exact propagator of a particle is written as the sum of all one-particle-irreducible diagrams added up in a series like in Figure 4.1.

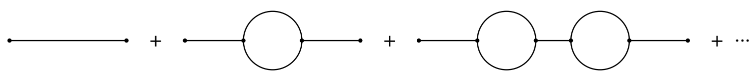


Figure 4.1: Dyson summation

We are interested in perturbative corrections to the fermion propagator and therefore, inserting the corresponding Feynman rules in Figure 4.1, one gets

$$iS(p) = \frac{i}{\not{p} - m}\left(1 - i\Sigma(p)\frac{i}{\not{p} - m} + (-i\Sigma(p))\,\frac{i}{\not{p} - m}\,(-i\Sigma(p))\,\frac{i}{\not{p} - m} + \ldots\right), \quad (4.1)$$

where $-i\Sigma(p)$ describes the loop. With the observation that the sum of all diagrams forms a geometric series, $iS(p)$ becomes

$$iS(p) = \frac{i}{\not p - m - \Sigma(p)}. \tag{4.2}$$

This way, the propagator has again a simple pole, but shifted due to $-\Sigma(p)$. The shift induces a change in the fermion mass. The physical mass (M) is given by the real part of the pole in the full propagator. By introducing two additional constants, $Z^{\frac{1}{2}}$ and $\overline{Z}^{\frac{1}{2}}$, one can rearrange $iS(p)$ and write it as

$$iS(p) = Z^{\frac{1}{2}} \frac{i}{\not p - M - \hat{\Sigma}(p)} \overline{Z}^{\frac{1}{2}}. \tag{4.3}$$

We will call $\hat{\Sigma}(p)$ the *subtracted self-energy.*

We denote the complex pole of the full propagator by $\mathfrak{M}$. Then,

$$S(p)\Big|_{p^2 \longrightarrow \mathfrak{M}^2} = Z^{\frac{1}{2}} \frac{\not p + \mathfrak{M}}{p^2 - \mathfrak{M}^2} \overline{Z}^{\frac{1}{2}}. \tag{4.4}$$

The real and the imaginary part of $\mathfrak{M}$ give

$$\begin{aligned} M &= \mathrm{Re}\mathfrak{M}, \\ \Gamma &= -2\mathrm{Im}\mathfrak{M}, \end{aligned} \tag{4.5}$$

where M will be identified with the physical mass and Γ with the width of the particle. $Z^{\frac{1}{2}}$ and $\overline{Z}^{\frac{1}{2}}$ will contribute to the residue of the propagator at the physical mass. All these constants, together with $\not p$ and $\hat{\Sigma}(p)$ are Dirac-matrices, but as before, we do not write it explicitly.

In the literature, $Z^{\frac{1}{2}}$ and $\overline{Z}^{\frac{1}{2}}$ are called wave function renormalization constants. In our approach, they do not necessarily contribute in a direct way to the renormalization of the fermion field. However, not to create confusion, we will refer to them in the same manner. From the start, we consider $Z^{\frac{1}{2}}$ and $\overline{Z}^{\frac{1}{2}}$ independent of each other, i.e. the hermiticity condition

$$\overline{Z}^{\frac{1}{2}} = \gamma^0 Z^{\frac{1}{2}\dagger} \gamma^0, \tag{4.6}$$

is not necessarily fulfilled. The precise definition of the constants will become more clear during this chapter.

We emphasise that the expression in (4.3) is achieved just by splitting $\Sigma(p)$ from (4.2) into $Z^{\frac{1}{2}}$, $\overline{Z}^{\frac{1}{2}}$, $\hat{\Sigma}(p)$ and a piece that relates the two mass parameters m and M. This way we separate the divergent parts from the finite, physical remainder:

only $Z^{\frac{1}{2}}$, $\overline{Z}^{\frac{1}{2}}$ and the difference $M - m$ will turn out to be divergent. These are the constants which have to be absorbed by the renormalization of parameters and fields in the Lagrangian.

In the discussion above we have assumed only one fermion. However, in reality we have more fermions which mix with each other. In the following section we will consider the separation of the wave function renormalization constants $Z^{\frac{1}{2}}$, $\overline{Z}^{\frac{1}{2}}$ and the difference $M - m$ for this more realistic situation.

4.1 Analysis of the One-loop Fermion Propagator

At one-loop, the fermion self-energy $\Sigma(p)$ is the sum of the one-particle-irreducible two-point functions given at the end of Chapter 3. Since we take into account mixing, we talk about matrices in flavour space and we have to add particle indices. If we take the inverse propagator in (4.2) and (4.3) and then we set equal the two expressions, we can write, including particle indices

$$(\not{p} - m_i)\delta_{ij} - \Sigma_{ij}(p) = \sum_{k,l} \left(\overline{Z}^{\frac{1}{2}}\right)^{-1}_{ik} \left((\not{p} - M_k)\delta_{kl} - \hat{\Sigma}_{kl}(p)\right) \left(Z^{\frac{1}{2}}\right)^{-1}_{lj}. \tag{4.7}$$

From here, the subtracted self-energy is written as

$$\hat{\Sigma}_{ij}(p) = (\not{p} - M_i)\delta_{ij} - \sum_{k,l} \overline{Z}^{\frac{1}{2}}_{ik} \left((\not{p} - m_k)\delta_{kl} - \Sigma_{kl}(p)\right) Z^{\frac{1}{2}}_{lj}. \tag{4.8}$$

$\hat{\Sigma}_{ij}(p)$ will emerge as a finite quantity provided that $Z^{\frac{1}{2}}$, $\overline{Z}^{\frac{1}{2}}$ and M are properly chosen.

As one can see from the calculations in Chapter 3, the self-energy is proportional to the fine-structure constant α. In the first order approximation, one neglects terms of order α^2 and higher. Therefore, it is reasonable to expand the constants in powers of α.

$$M_i = m_i + \delta m_i + \mathcal{O}(\alpha^2), \tag{4.9}$$

$$Z^{\frac{1}{2}}_{ij} = \delta_{ij} + \frac{1}{2}\delta Z_{ij} + \mathcal{O}(\alpha^2),$$

$$\overline{Z}^{\frac{1}{2}}_{ij} = \delta_{ij} + \frac{1}{2}\delta \overline{Z}_{ij} + \mathcal{O}(\alpha^2), \tag{4.10}$$

with δm_i real. Furthermore, one can consider the decomposition in left and right projectors:

$$\begin{aligned} \delta Z_{ij} &= \delta Z^L_{ij}\gamma_L + \delta Z^R_{ij}\gamma_R, \\ \delta \overline{Z}_{ij} &= \delta \overline{Z}^R_{ij}\gamma_L + \delta \overline{Z}^L_{ij}\gamma_R. \end{aligned} \tag{4.11}$$

Note that even if we don't assume that the hermiticity relation links the constants, for $\delta\overline{Z}_{ij}$ it is still convenient to choose the notation such that the index L comes with the constant related to the right projector and the other way around, like it would have been derived with (4.6), from δZ_{ij}. The notation will be helpful later, when we treat the renormalizion of the fields.

From equation (4.8), using the relations (4.9)-(4.11), in first order, the subtracted self-energy is

$$\hat{\Sigma}_{ij}(p) = \Sigma_{ij}(p) - \delta m_i \delta_{ij} - \frac{1}{2}\delta\overline{Z}_{ij}(\not{p} - m_j) - \frac{1}{2}(\not{p} - m_i)\delta Z_{ij} + \mathcal{O}(\alpha^2). \tag{4.12}$$

No summation over the indices is implied. For $\hat{\Sigma}_{ij}(p)$, we use a decomposition as in (3.23) or (3.38). Then:

$$\begin{aligned}
\hat{\Sigma}^L_{ij}(p^2) &= \Sigma^L_{ij}(p^2) - \frac{1}{2}(\delta Z^L_{ij} + \delta\overline{Z}^L_{ij}) + \mathcal{O}(\alpha^2),\\
\hat{\Sigma}^R_{ij}(p^2) &= \Sigma^R_{ij}(p^2) - \frac{1}{2}(\delta Z^R_{ij} + \delta\overline{Z}^R_{ij}) + \mathcal{O}(\alpha^2),\\
\hat{\Sigma}^{DL}_{ij}(p^2) &= \Sigma^{DL}_{ij}(p^2) - \delta m_i \delta_{ij} + \frac{1}{2}\delta\overline{Z}^R_{ij} m_j + \frac{1}{2} m_i \delta Z^L_{ij} + \mathcal{O}(\alpha^2),\\
\hat{\Sigma}^{DR}_{ij}(p^2) &= \Sigma^{DR}_{ij}(p^2) - \delta m_i \delta_{ij} + \frac{1}{2}\delta\overline{Z}^L_{ij} m_j + \frac{1}{2} m_i \delta Z^R_{ij} + \mathcal{O}(\alpha^2).
\end{aligned} \tag{4.13}$$

To determine the constants introduced above, one has to evaluate on-shell conditions. First, we will consider the diagonal elements of the propagator and then the off-diagonal ones. The approach is similar to [Esp02]. However, there, renormalized fields and masses are considered from the beginning, while we are going to derive the rules for the renormalization later.

One helpful remark is that the off-diagonal matrix elements of the inverse propagator

$$S^{-1}_{ij}(p) = (\not{p} - m_i)\delta_{ij} - \Sigma_{ij}(p) \quad\longrightarrow\quad S^{-1}_{ij}(p) = -\Sigma_{ij}(p),\ \text{for } i \neq j, \tag{4.14}$$

are proportional to α. In fact, we have to invert a matrix of type $a_i\delta_{ij} + \delta A_{ij}$, with $\delta A_{ij} \ll a_i$. Up to terms of order α^2, for each diagonal element, we will obtain the inverse of the initial term (see Appendix C.1, formulas (C.7) and (C.8)). Therefore, it is sufficient to invert

$$S^{-1}_{ii}(p) = \not{p} - m_i - \Sigma_{ii}(p). \tag{4.15}$$

Using a decomposition of the self-energy like in (3.23) and identifying

$$\begin{aligned}
A &= 1 - \Sigma^L_{ii}(p^2)\\
B &= 1 - \Sigma^R_{ii}(p^2)\\
C &= -m_i - \Sigma^{DL}_{ii}(p^2)\\
D &= -m_i - \Sigma^{DR}_{ii}(p^2),
\end{aligned} \tag{4.16}$$

one can rearrange

$$S_{ii}^{-1}(p) = \not{p}\gamma_L A + \not{p}\gamma_R B + \gamma_L C + \gamma_R D. \tag{4.17}$$

The inverse of this expression can be easily written as

$$S_{ii}(p) = \frac{1}{\not{p}(\gamma_L A + \gamma_R B) + \gamma_L C + \gamma_R D} = \frac{\not{p}(\gamma_L A + \gamma_R B) - \gamma_L D - \gamma_R C}{p^2 AB - CD}, \tag{4.18}$$

where the first fraction was simply expanded with $\not{p}(\gamma_L A + \gamma_R B) - \gamma_L D - \gamma_R C$.

As mentioned before, the propagator has again a simple pole, but shifted:

$$p^2 AB - CD = 0, \text{ for } p^2 \longrightarrow \mathfrak{M}_i^2. \tag{4.19}$$

Like in (4.5), we write the complex pole $\mathfrak{M}$ as

$$\begin{aligned} \mathfrak{M}_i &= M_i - i\frac{1}{2}\Gamma_i \\ &= m_i + \delta m_i - i\frac{1}{2}\Gamma_i + \mathcal{O}(\alpha^2). \end{aligned} \tag{4.20}$$

The physical mass M_i was replaced with (4.9). Note that in the last expression, Γ_i is of order α.

It is worth now making the observation that considering any element of the self-energy decomposition (3.23), one can expand it around $p^2 = m^2$ as

$$\Sigma(p^2) = \Sigma(m^2) + (p^2 - m^2)\Sigma'(m^2) + \mathcal{O}((p^2 - m^2)^2), \tag{4.21}$$

where $\Sigma'(m_i^2) = \left.\frac{\partial \Sigma(p^2)}{\partial p^2}\right|_{p^2 \to m_i^2} = \mathcal{O}(\alpha)$. For $p^2 = M^2$, where M is given by (4.9),

$$\Sigma(M^2) = \Sigma(m^2) + 2m\delta m \Sigma'(m^2) + \mathcal{O}(\alpha^2) = \Sigma(m^2) + \mathcal{O}(\alpha^2), \tag{4.22}$$

or for $p^2 = \mathfrak{M}^2$, written as in (4.20),

$$\begin{aligned} \Sigma(\mathfrak{M}^2) &= \Sigma(m^2) + 2m\left(\delta m - i\frac{1}{2}\Gamma_i\right)\Sigma'(m^2) + \mathcal{O}(\alpha^2) \\ &= \Sigma(m^2) + \mathcal{O}(\alpha^2) \\ &= \Sigma(M^2) + \mathcal{O}(\alpha^2). \end{aligned} \tag{4.23}$$

With this remark and inserting back (4.16), the vanishing condition for the denominator (4.19) will give the mass correction:

$$\delta m_i = \frac{1}{2}\mathrm{Re}\left[m_i \Sigma_{ii}^L(m_i^2) + m_i \Sigma_{ii}^R(m_i^2) + \Sigma_{ii}^{DL}(m_i^2) + \Sigma_{ii}^{DR}(m_i^2)\right], \tag{4.24}$$

and the width:

$$\Gamma_i = -\mathrm{Im}\left[m_i\Sigma_{ii}^L(m_i^2) + m_i\Sigma_{ii}^R(m_i^2) + \Sigma_{ii}^{DL}(m_i^2) + \Sigma_{ii}^{DR}(m_i^2)\right]. \tag{4.25}$$

On-shell, the diagonal matrix elements of the propagator will become:

$$S_{ii}(p)\Big|_{p^2\longrightarrow\mathfrak{M}_i^2} = Z_{ii}^{\frac{1}{2}}\frac{\not p+\mathfrak{M}_i}{p^2-\mathfrak{M}_i^2}\overline{Z}_{ii}^{\frac{1}{2}}, \tag{4.26}$$

$$\text{i.e.}\quad \left(\frac{\not p(\gamma_L A+\gamma_R B)-\gamma_L D-\gamma_R C}{p^2AB-CD}\right)\Bigg|_{p^2\longrightarrow\mathfrak{M}_i^2} = Z_{ii}^{\frac{1}{2}}\frac{\not p+\mathfrak{M}_i}{p^2-\mathfrak{M}_i^2}\overline{Z}_{ii}^{\frac{1}{2}} \tag{4.27}$$

If we replace $Z_{ii}^{\frac{1}{2}}$ and $\overline{Z}_{ii}^{\frac{1}{2}}$ with their expansions and decompositions (4.10) and (4.11) and rearrange the terms, in order α, (4.26) is:

$$\begin{aligned}S_{ii}(p)\Big|_{p^2\longrightarrow\mathfrak{M}_i^2} =& \frac{1}{p^2-\mathfrak{M}_i^2}\left(\not p\gamma_L\left(1+\frac{1}{2}\delta Z_{ii}^R+\frac{1}{2}\delta\overline{Z}_{ii}^R\right)+\not p\gamma_R\left(1+\frac{1}{2}\delta Z_{ii}^L+\frac{1}{2}\delta\overline{Z}_{ii}^L\right)\right.\\ &\left.+\gamma_L\mathfrak{M}_i\left(1+\frac{1}{2}\delta Z_{ii}^L+\frac{1}{2}\delta\overline{Z}_{ii}^R\right)+\gamma_R\mathfrak{M}_i\left(1+\frac{1}{2}\delta Z_{ii}^R+\frac{1}{2}\delta\overline{Z}_{ii}^L\right)\right).\end{aligned} \tag{4.28}$$

To identify the constants, one has to bring (4.18) into a similar form. Since we talk about the on-shell limit, we will expand the numerator and the denominator around $p^2 = M_i^2$, i.e. around the physical mass. The expansion looks similar to (4.21). A, B, C, and D will become:

$$\begin{aligned}A &= 1-\Sigma_{ii}^L(M_i^2)-(p^2-M_i^2)\Sigma_{ii}^{L'}(M_i^2)+\mathcal{O}((p^2-M_i^2)^2),\\ B &= 1-\Sigma_{ii}^R(M_i^2)-(p^2-M_i^2)\Sigma_{ii}^{R'}(M_i^2)+\mathcal{O}((p^2-M_i^2)^2),\\ C &= -m_i-\Sigma_{ii}^{DL}(M_i^2)-(p^2-M_i^2)\Sigma_{ii}^{DL'}(M_i^2)+\mathcal{O}((p^2-M_i^2)^2),\\ D &= -m_i-\Sigma_{ii}^{DR}(M_i^2)-(p^2-M_i^2)\Sigma_{ii}^{DR'}(M_i^2)+\mathcal{O}((p^2-M_i^2)^2),\end{aligned} \tag{4.29}$$

where $\Sigma'(M_i^2)$ is defined in analogy to (4.21). (4.27) requires in fact an expansion around $p^2 = \mathfrak{M}_i^2$, the complex pole. However, at $\mathcal{O}(\alpha)$, for the products of A, B, C, and D in the denominator of (4.27), M_i can be replaced with $\mathfrak{M}_i$. This works also for the numerator, where we need just the first terms of the expansions (4.29). Making use of (4.23) and in general, of the fact that terms of order α^2 and higher are neglected, one can prove that the propagator (4.18) can be written as

$$\begin{aligned}S_{ii}(p) = &\frac{1}{(p^2-\mathfrak{M}_i^2)T_{ii}+\mathcal{O}((p^2-\mathfrak{M}_i^2)^2)}\left(\not p\gamma_L(1-\Sigma_{ii}^L(M_i^2))+\not p\gamma_R(1-\Sigma_{ii}^R(M_i^2))\right.\\ &\left.+\gamma_L(m_i+\Sigma_{ii}^{DR}(M_i^2))+\gamma_R(m_i+\Sigma_{ii}^{DL}(M_i^2))+\mathcal{O}(p^2-\mathfrak{M}_i^2)\right),\end{aligned} \tag{4.30}$$

with

$$T_{ii} = 1 - \Sigma^L_{ii}(M_i^2) - \Sigma^R_{ii}(M_i^2) - M_i^2(\Sigma^{L'}_{ii}(M_i^2) + \Sigma^{R'}_{ii}(M_i^2)) \\ -M_i(\Sigma^{DL'}_{ii}(M_i^2) + \Sigma^{DR'}_{ii}(M_i^2)) + \mathcal{O}(\alpha^2).$$

In the limit $p^2 \longrightarrow \mathfrak{M}_i^2$, (4.30) has to be equal to (4.28) (condition (4.27)). In both expressions the denominators are scalar functions of p^2, so one has to separately compare the factors coming with $\not p\gamma_L$, $\not p\gamma_R$, γ_L and γ_R, respectively. We obtain for the diagonal elements of δZ and $\delta\overline{Z}$, the following system of equations:

$$\begin{cases} \delta Z^R_{ii} + \delta\overline{Z}^R_{ii} = 2\Sigma^R_{ii}(M_i^2) + 2\mathcal{D}_{ii}(M_i), \\ \delta Z^L_{ii} + \delta\overline{Z}^L_{ii} = 2\Sigma^L_{ii}(M_i^2) + 2\mathcal{D}_{ii}(M_i), \\ \delta Z^L_{ii} + \delta\overline{Z}^R_{ii} = \Sigma^L_{ii}(M_i^2) + \Sigma^R_{ii}(M_i^2) - \frac{1}{M_i}\left(\Sigma^{DL}_{ii}(M_i^2) - \Sigma^{DR}_{ii}(M_i^2)\right) + 2\mathcal{D}_{ii}(M_i), \\ \delta Z^R_{ii} + \delta\overline{Z}^L_{ii} = \Sigma^L_{ii}(M_i^2) + \Sigma^R_{ii}(M_i^2) + \frac{1}{M_i}\left(\Sigma^{DL}_{ii}(M_i^2) - \Sigma^{DR}_{ii}(M_i^2)\right) + 2\mathcal{D}_{ii}(M_i), \end{cases}$$

$$\mathcal{D}_{ii}(M_i) = M_i^2(\Sigma^{L'}_{ii}(M_i^2) + \Sigma^{R'}_{ii}(M_i^2)) + M_i(\Sigma^{DL'}_{ii}(M_i^2) + \Sigma^{DR'}_{ii}(M_i^2)). \tag{4.31}$$

The solution of this system of equations is not unique and we have to choose one free parameter. A convenient choice that leads to symmetric formulas is

$$\delta Z^L_{ii} + \delta Z^R_{ii} = \delta\overline{Z}^L_{ii} + \delta\overline{Z}^R_{ii} + 2\beta_i, \tag{4.32}$$

where β_i can be chosen finite. In this case, the solution is

$$\begin{cases} \delta Z^L_{ii} = \Sigma^L_{ii}(M_i^2) - \frac{1}{2M_i}\left(\Sigma^{DL}_{ii}(M_i^2) - \Sigma^{DR}_{ii}(M_i^2)\right) + \mathcal{D}_{ii}(M_i) + \frac{\beta_i}{2}, \\ \delta Z^R_{ii} = \Sigma^R_{ii}(M_i^2) + \frac{1}{2M_i}\left(\Sigma^{DL}_{ii}(M_i^2) - \Sigma^{DR}_{ii}(M_i^2)\right) + \mathcal{D}_{ii}(M_i) + \frac{\beta_i}{2}, \\ \delta\overline{Z}^L_{ii} = \Sigma^L_{ii}(M_i^2) + \frac{1}{2M_i}\left(\Sigma^{DL}_{ii}(M_i^2) - \Sigma^{DR}_{ii}(M_i^2)\right) + \mathcal{D}_{ii}(M_i) - \frac{\beta_i}{2}, \\ \delta\overline{Z}^R_{ii} = \Sigma^R_{ii}(M_i^2) - \frac{1}{2M_i}\left(\Sigma^{DL}_{ii}(M_i^2) - \Sigma^{DR}_{ii}(M_i^2)\right) + \mathcal{D}_{ii}(M_i) - \frac{\beta_i}{2}. \end{cases} \tag{4.33}$$

The free parameter β_i can not be fixed by on-shell conditions.

To complete the result, we have to determine also the off-diagonal elements of δZ and $\delta\overline{Z}$. Evaluating the real part of the pole in the propagator, we have identified M_i with the physical mass. Accordingly, the on-shell relations for the Dirac spinors are written with the help of M_i:

$$\left.\begin{aligned}(\not{p}-M_i)u_i(p,s)&=0,\\(\not{p}+M_i)v_i(p,s)&=0,\\\overline{u}_i(p,s)(\not{p}-M_i)&=0,\\\overline{v}_i(p,s)(\not{p}+M_i)&=0,\end{aligned}\right.\quad \text{for } p^2\longrightarrow M_i^2. \tag{4.34}$$

To avoid mixing in the corrections related to external particles, we impose additional on-shell conditions for the off-diagonal elements of $\hat{\Sigma}_{ij}(p)$ $(i\neq j)$. For incoming and outgoing particles, they are:

$$\hat{\Sigma}_{ij}(p)u_j(p,s)=0,\ \text{ for } p^2\longrightarrow M_j^2 \text{ and } i\neq j, \tag{4.35}$$

$$\overline{u}_i(p,s)\hat{\Sigma}_{ij}(p)=0,\ \text{ for } p^2\longrightarrow M_i^2 \text{ and } i\neq j. \tag{4.36}$$

These relations guarantee that mixing does not occur on external lines. In (4.35) and (4.36), the subtracted self-energy $\hat{\Sigma}_{ij}$ is given by the non-diagonal elements of (4.12):

$$\hat{\Sigma}_{ij}(p)=\Sigma_{ij}(p)-\frac{1}{2}\delta\overline{Z}_{ij}(\not{p}-M_j)-\frac{1}{2}(\not{p}-M_i)\delta Z_{ij},\ \text{ for } i\neq j. \tag{4.37}$$

Since in (4.12) the factors $\not{p}-m_i$ (m_j) are always multiplied with δZ_{ij} or $\delta\overline{Z}_{ij}$, we replaced the mass m_i by M_i (or accordingly m_j by M_j). This substitution can be also applied for the components of $\hat{\Sigma}_{ij}$, (4.13).

To understand the condition for an incoming fermion (4.35), consider the diagram in Figure 4.2.

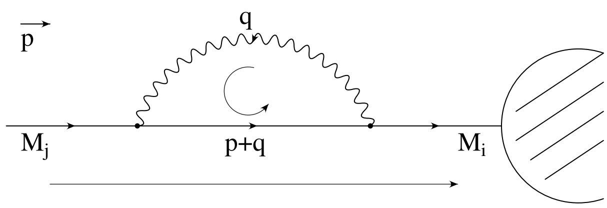


Figure 4.2: Self-energy correction to an incoming fermion.

The total momentum p flows from left to right and the general fermion flow is expressed by the additional arrow lines. Leaving apart the terms in (4.37) that contain the explicit wave function renormalization constants (we discuss them in the next chapter), the diagram above will lead to a correction term, at $\mathcal{O}(\alpha)$, given by

$$\Omega_i(p)\left(-i\hat{\Sigma}_{ij}(p)\right)u_j(p,s),\ \text{ for } p^2\longrightarrow M_j^2 \text{ and } i\neq j. \tag{4.38}$$

Since we are interested just in the incoming fermion, we write explicitly just part of the Feynman rules. $\Omega_i(p)$ represents the internal fermion propagator for the particle i and the shaded area of the diagram. If the non-diagonal components of δZ and $\delta\overline{Z}$ are chosen such that (4.35) is fulfilled, then this correction vanishes. A similar diagram can be used to explain the corresponding condition for the outgoing particle (4.36).

For antiparticles, one has to be careful in writing the on-shell requirements in a convenient way. We want to keep the same expression as before for $\hat{\Sigma}_{ij}(p)$ (no change in momentum sign or in particle indices) and to adjust just the external particles. To make it more clear, we choose as example the conditions for an incoming antiparticle related to the ones for incoming particle. For this case, the desired picture is obtained if we mirror the previous loop, reversing the sign for the external momentum p, see Figure 4.3.

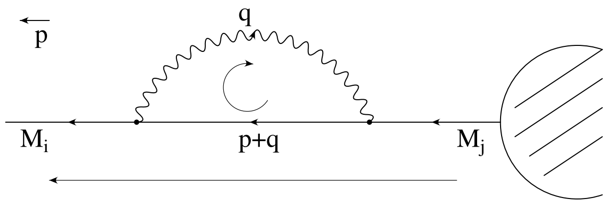


Figure 4.3: Self-energy correction to an incoming anti-fermion.

Comprising the Feynman rules that describe the part right to the self-energy in $\Omega_j(p)$ and leaving out the terms with δZ and $\delta\overline{Z}$ in (4.37), the diagram amounts to

$$\overline{v}_i(-p,s)\left(-i\hat{\Sigma}_{ij}(p)\right)\Omega_j(p), \text{ for } p^2 \longrightarrow M_i^2 \text{ and } i \neq j. \tag{4.39}$$

The on-shell condition is in this case:

$$\overline{v}_i(-p,s)\hat{\Sigma}_{ij}(p) = 0, \text{ for } p^2 \longrightarrow M_i^2 \text{ and } i \neq j. \tag{4.40}$$

For an outgoing antiparticle, the argument is similar and

$$\hat{\Sigma}_{ij}(p)v_j(-p,s) = 0, \text{ for } p^2 \longrightarrow M_j^2 \text{ and } i \neq j. \tag{4.41}$$

In order to evaluate (4.35) and (4.36) with the help of (4.34), we use the subtracted self-energy in the form (4.37). For $p^2 \longrightarrow M_j^2$, with the self-energy written on components, one gets from (4.35):

$$\begin{aligned}\left(\not{p}\gamma_L\Sigma_{ij}^L(M_j^2) + \not{p}\gamma_R\Sigma_{ij}^R(M_j^2) + \gamma_L\Sigma_{ij}^{DL}(M_j^2) + \gamma_R\Sigma_{ij}^{DR}(M_j^2)\right)u_j(p,s) \\ -\frac{1}{2}\delta\overline{Z}_{ij}(\not{p} - M_j)u_j(p,s) - \frac{1}{2}(\not{p} - M_i)\delta Z_{ij}u_j(p,s) = 0.\end{aligned} \tag{4.42}$$

On-shell, the last two terms vanish. The first two terms coming with a $\not{p}$ have to be rearranged in order to emphasise $(\not{p} - M_j)$ acting directly on $u_j(p,s)$. This is done by reversing the order between the left and right projectors and $\not{p}$ and in most cases just by adding and subtracting M_j from $\not{p}$. After dropping all the terms where (4.34) can be applied, we have to separately set to zero the factors of both, γ_L and γ_R. The following system of equations is obtained:

$$\begin{cases} -\frac{1}{2}M_i\delta Z^L_{ij} + \frac{1}{2}M_j\delta Z^R_{ij} = M_j\Sigma^R_{ij}(M_j^2) + \Sigma^{DL}_{ij}(M_j^2), \\ \frac{1}{2}M_j\delta Z^L_{ij} - \frac{1}{2}M_i\delta Z^R_{ij} = M_j\Sigma^L_{ij}(M_j^2) + \Sigma^{DR}_{ij}(M_j^2), \end{cases} \tag{4.43}$$

with the solution:

$$\begin{aligned} \delta Z^L_{ij} &= \frac{2}{M_j^2 - M_i^2}\left(M_j^2\Sigma^L_{ij}(M_j^2) + M_iM_j\Sigma^R_{ij}(M_j^2) + M_i\Sigma^{DL}_{ij}(M_j^2) + M_j\Sigma^{DR}_{ij}(M_j^2)\right), \\ \delta Z^R_{ij} &= \frac{2}{M_j^2 - M_i^2}\left(M_iM_j\Sigma^L_{ij}(M_j^2) + M_j^2\Sigma^R_{ij}(M_j^2) + M_j\Sigma^{DL}_{ij}(M_j^2) + M_i\Sigma^{DR}_{ij}(M_j^2)\right), \\ &\qquad \text{for} \quad i \neq j. \end{aligned} \tag{4.44}$$

Analogously, by using the condition for an outgoing particle one gets

$$\begin{cases} -\frac{1}{2}M_j\delta \overline{Z}^R_{ij} + \frac{1}{2}M_i\delta \overline{Z}^L_{ij} = M_i\Sigma^L_{ij}(M_i^2) + \Sigma^{DL}_{ij}(M_i^2), \\ \frac{1}{2}M_i\delta \overline{Z}^R_{ij} - \frac{1}{2}M_j\delta \overline{Z}^L_{ij} = M_i\Sigma^R_{ij}(M_i^2) + \Sigma^{DR}_{ij}(M_i^2), \end{cases} \tag{4.45}$$

with the solution:

$$\begin{aligned} \delta \overline{Z}^L_{ij} &= \frac{2}{M_i^2 - M_j^2}\left(M_i^2\Sigma^L_{ij}(M_i^2) + M_iM_j\Sigma^R_{ij}(M_i^2) + M_i\Sigma^{DL}_{ij}(M_i^2) + M_j\Sigma^{DR}_{ij}(M_i^2)\right), \\ \delta \overline{Z}^R_{ij} &= \frac{2}{M_i^2 - M_j^2}\left(M_iM_j\Sigma^L_{ij}(M_i^2) + M_i^2\Sigma^R_{ij}(M_i^2) + M_j\Sigma^{DL}_{ij}(M_i^2) + M_i\Sigma^{DR}_{ij}(M_i^2)\right), \\ &\qquad \text{for} \quad i \neq j. \end{aligned} \tag{4.46}$$

For antiparticles, the calculations should be performed similarly, but they are not necessary. Looking at (4.40) and (4.41), one sees that we need to adapt the sign of the momentum in (4.34):

$$\begin{aligned} &(\not{p} - M_i)v_i(-p,s) = 0, \\ &\overline{v}_i(-p,s)(\not{p} - M_i) = 0. \end{aligned} \tag{4.47}$$

This way the conditions (4.36) and (4.40) will lead to identical equations. The same is going to happen with (4.35) and (4.41).

In conclusion, the off-diagonal elements of δZ and $\delta\overline{Z}$ are given by (4.44) and (4.46) and the diagonal ones by (4.33). While off-diagonal the constants are fully determined, on-diagonal we have a free parameter (β_i). These constants guarantee that for external fermions, on-shell, the contributions coming from the non-diagonal self-energy $\hat{\Sigma}_{ij}(p)$ are vanishing.

Note that during the calculations of this section, even if we considered Dirac particles, the nature of the fermion was not really taken into consideration. It might seem so when explaining the conditions for the off-diagonal elements, but a similar picture can be presented for Majorana fermions and we end up with the same result. However, we have an additional property of the resulting constants. Looking at the solutions for the wave function renormalization constants (4.44) and (4.46) and considering also the constraint for the Majorana self-energy (3.58), one can see that we have

$$\begin{aligned} \delta Z^L_{ab} &= \delta\overline{Z}^R_{ba}, \\ \delta Z^R_{ab} &= \delta\overline{Z}^L_{ba}, \end{aligned} \qquad \text{for} \quad a \neq b. \tag{4.48}$$

Instead of four constants, we are left just with two. For $a = b$, from (4.33) we obtain a similar property if we restrict the free parameter, i.e.

$$\begin{aligned} \delta Z^L_{aa} &= \delta\overline{Z}^R_{aa}, \\ \delta Z^R_{aa} &= \delta\overline{Z}^L_{aa}, \end{aligned} \qquad \text{if} \quad \beta_a = 0. \tag{4.49}$$

In the next chapter, we show that as a consequence of these properties, the Majorana nature of the particle is preserved after renormalization.

4.2 Imaginary Parts, Divergences and Gauge Dependence

At the beginning of the chapter, we stated that the hermiticity condition for the wave function renormalization constants (4.6) does not automatically hold. We have now the explicit first order expressions of $Z^{\frac{1}{2}}$ and $\overline{Z}^{\frac{1}{2}}$ and we can check in which cases (4.6) is fulfilled and in which not. Let's start by writing the condition for the left and right components of what we called wave function renormalization constants. It reads

$$\begin{aligned} \delta\overline{Z}^L_{xy} &= (\delta Z^L_{yx})^*, \\ \delta\overline{Z}^R_{xy} &= (\delta Z^R_{yx})^*. \end{aligned} \tag{4.50}$$

For the fermion flavour indices we switch to our convention for the general notation of Dirac and Majorana fermions (see section 2.1). We make just the remark that due to (4.48) and (4.49), for Majorana fermions, one equation is enough. The two relations in (4.50) are related by complex conjugation if $x = a$ and $y = b$.

We start to check (4.50) for the non-diagonal constants. For the left component of the wave function renormalization constants, with (4.44) and (4.46) written for $i = x$ and $j = y$, we have

$$\begin{aligned}\delta\overline{Z}^L_{xy} - (\delta Z^L_{yx})^* = \frac{2}{M_x^2 - M_y^2} \Big(& M_x^2 \left(\Sigma^L_{xy}(M_x^2) - (\Sigma^L_{yx}(M_x^2))^*\right) \\ & + M_x M_y \left(\Sigma^R_{xy}(M_x^2) - (\Sigma^R_{yx}(M_x^2))^*\right) \quad \text{for} \quad x \neq y, \\ & + M_x \left(\Sigma^{DL}_{xy}(M_x^2) - (\Sigma^{DR}_{yx}(M_x^2))^*\right) \\ & + M_y \left(\Sigma^{DR}_{xy}(M_x^2) - (\Sigma^{DL}_{yx}(M_x^2))^*\right)\Big),\end{aligned} \tag{4.51}$$

and a similar expression for the difference of the right components:

$$\begin{aligned}\delta\overline{Z}^R_{xy} - (\delta Z^R_{yx})^* = \frac{2}{M_x^2 - M_y^2} \Big(& M_x M_y \left(\Sigma^L_{xy}(M_x^2) - (\Sigma^L_{yx}(M_x^2))^*\right) \\ & + M_x^2 \left(\Sigma^R_{xy}(M_x^2) - (\Sigma^R_{yx}(M_x^2))^*\right) \quad \text{for} \quad x \neq y. \\ & + M_y \left(\Sigma^{DL}_{xy}(M_x^2) - (\Sigma^{DR}_{yx}(M_x^2))^*\right) \\ & + M_x \left(\Sigma^{DR}_{xy}(M_x^2) - (\Sigma^{DL}_{yx}(M_x^2))^*\right)\Big),\end{aligned} \tag{4.52}$$

Looking back at the property of the total self-energy when the model consists of stable particles in the sense used in (3.72), one can see that in such a case the right parts of the equations (4.51) and (4.52) vanish and $Z^{\frac{1}{2}}$ and $\overline{Z}^{\frac{1}{2}}$ are related by (4.50). (3.72) is enough to check that also for x=y, (4.50) is preserved. From (4.33), we obtain

$$\begin{aligned}\delta\overline{Z}^L_{xx} - (\delta Z^L_{xx})^* = & \Sigma^L_{xx}(M_x^2) - (\Sigma^L_{xx}(M_x^2))^* \\ & + \frac{1}{2M_x}\left(\Sigma^{DL}_{xx}(M_x^2) - (\Sigma^{DR}_{xx}(M_x^2))^*\right) \\ & - \frac{1}{2M_x}\left(\Sigma^{DR}_{xx}(M_x^2) - (\Sigma^{DL}_{xx}(M_x^2))^*\right) \\ & + \mathcal{D}_{xx}(M_x) - (\mathcal{D}_{xx}(M_x))^* \\ & - \frac{1}{2}(\beta_x + \beta_x^*),\end{aligned} \tag{4.53}$$

$$\begin{aligned}\delta\overline{Z}^R_{xx} - (\delta Z^R_{xx})^* =& \Sigma^R_{xx}(M_x^2) - (\Sigma^R_{xx}(M_x^2))^* \\ &- \frac{1}{2M_x}\left(\Sigma^{DL}_{xx}(M_x^2) - (\Sigma^{DR}_{xx}(M_x^2))^*\right) \\ &+ \frac{1}{2M_x}\left(\Sigma^{DR}_{xx}(M_x^2) - (\Sigma^{DL}_{xx}(M_x^2))^*\right) \\ &+ \mathcal{D}_{xx}(M_x) - (\mathcal{D}_{xx}(M_x))^* \\ &- \frac{1}{2}(\beta_x + \beta_x^*).\end{aligned} \tag{4.54}$$

The contributions from $\mathcal{D}_{xx}(M_x)$ are (see (4.31)):

$$\begin{aligned}\mathcal{D}_{xx}(M_x) - (\mathcal{D}_{xx}(M_x))^* =& M_x^2\left(\Sigma^{L'}_{xx}(M_x^2) - (\Sigma^{L'}_{xx}(M_x^2))^*\right) \\ &+ M_x^2\left(\Sigma^{R'}_{xx}(M_x^2) - (\Sigma^{R'}_{xx}(M_x^2))^*\right) \\ &+ M_x\left(\Sigma^{DL'}(M_x^2) - (\Sigma^{DR'}_{xx}(M_x^2))^*\right) \\ &+ M_x\left(\Sigma^{DR'}_{xx}(M_x^2) - (\Sigma^{DL'}_{xx}(M_x^2))^*\right).\end{aligned} \tag{4.55}$$

Besides the relation for the self-energy (3.72), to cancel the terms in (4.53) and (4.54), we need to restrict the free parameter β_i from (4.32). Remember that for Majorana fermions, it is anyway equal to zero (equation (4.49)). We require:

$$\beta_i^* = -\beta_i, \tag{4.56}$$

i.e. β_i is purely imaginary, but still not determined. In fact, β_i imaginary is related to the freedom that one has in redefining phases of Dirac fermion mixing matrices.

In conclusion, for stable particles, $Z^{\frac{1}{2}}$ and $\overline{Z}^{\frac{1}{2}}$ are connected by hermiticity and just one of them is required to extract the divergences from the propagator. For unstable particles, (4.51)–(4.54) are different from zero and here, the presence of the two wave function renormalization constants is required.

Even if (4.50) does not hold for every model, we can still prove that the structure of the divergent parts of the self-energy allows us to verify that

$$\begin{aligned}\mathrm{div}[\delta\overline{Z}^L_{xy}] = (\mathrm{div}[\delta Z^L_{yx}])^*, \\ \mathrm{div}[\delta\overline{Z}^R_{xy}] = (\mathrm{div}[\delta Z^R_{yx}])^*.\end{aligned} \tag{4.57}$$

If we analyse the way the divergences enter the self-energy (formulas (3.42) and (3.43)), we see that they only depend on products of coupling constants and some constant factors. Any complex conjugated coupling constant is equal to the coupling constant describing the hermitian conjugated process (see (2.49) and (2.53)). Hence,

$$
\begin{aligned}
\mathrm{div}[\Sigma^L_{xy}(p^2)] &= (\mathrm{div}[\Sigma^L_{yx}(p^2)])^*,\\
\mathrm{div}[\Sigma^R_{xy}(p^2)] &= (\mathrm{div}[\Sigma^R_{yx}(p^2)])^*,\\
\mathrm{div}[\Sigma^{DR}_{xy}(p^2)] &= (\mathrm{div}[\Sigma^{DL}_{yx}(p^2)])^*,
\end{aligned}
\tag{4.58}
$$

for a hermitian Lagrangian and the divergent parts of the wave function renormalization constants are related by hermiticity.

(4.57) can be easily recognised if we detail the expressions in (4.51)−(4.54). For the calculation, we need the components of the self-energy expressed for every combination of internal particles in (3.59)−(3.62), if we talk about vector bosons and (3.64)−(3.67), if we have internal scalars. When writing the complex conjugated self-energy, we relate the coupling constants by (2.49) and (2.53). Remember that one has to sum over all possible internal states. The differences of wave function renormalization constants, (4.51)−(4.54) result in

- for $x \neq y$

$$
\begin{aligned}
\delta\overline{Z}^L_{xy} - (\delta Z^L_{yx})^* = i\frac{\alpha}{\pi}\frac{1}{M_x^2 - M_y^2}\Bigg(&\\
&\sum_{z,v}\mathrm{Im}F^v(M_x^2; m_v, \sqrt{\xi}m_v, m_z)\left(M_x^2 g^L_{xz,v}g^L_{zy,v} + M_xM_y g^R_{xz,v}g^R_{zy,v}\right)\\
&+\sum_{z,v} m_z\mathrm{Im}F^s(M_x^2; m_v, \sqrt{\xi}m_v, m_z)\left(M_x g^R_{xz,v}g^L_{zy,v} + M_y g^L_{xz,v}g^R_{zy,v}\right)\\
&+\sum_{z,s}\mathrm{Im}B_1(M_x^2; m_z, m_s)\left(M_x^2 c^R_{xz,s}c^L_{zy,s} + M_xM_y c^L_{xz,s}c^R_{zy,s}\right)\\
&-\sum_{z,s} m_z\mathrm{Im}B_0(M_x^2; m_z, m_s)\left(M_x c^L_{xz,s}c^L_{zy,s} + M_y c^R_{xz,s}c^R_{zy,s}\right)\Bigg),
\end{aligned}
\tag{4.59}
$$

$$\delta\overline{Z}^R_{xy} - (\delta Z^R_{yx})^* = i\frac{\alpha}{\pi}\frac{1}{M_x^2 - M_y^2}\Bigg(\sum_{z,v}\mathrm{Im}F^v(M_x^2; m_v, \sqrt{\xi}m_v, m_z)\left(M_x M_y g^L_{xz,v}g^L_{zy,v} + M_x^2 g^R_{xz,v}g^R_{zy,v}\right)$$
$$+ \sum_{z,v} m_z \mathrm{Im}F^s(M_x^2; m_v, \sqrt{\xi}m_v, m_z)\left(M_y g^R_{xz,v}g^L_{zy,v} + M_x g^L_{xz,v}g^R_{zy,v}\right)$$
$$+ \sum_{z,s}\mathrm{Im}B_1(M_x^2; m_z, m_s)\left(M_x M_y c^R_{xz,s}c^L_{zy,s} + M_x^2 c^L_{xz,s}c^R_{zy,s}\right)$$
$$- \sum_{z,s} m_z \mathrm{Im}B_0(M_x^2; m_z, m_s)\left(M_y c^L_{xz,s}c^L_{zy,s} + M_x c^R_{xz,s}c^R_{zy,s}\right)\Bigg), \quad (4.60)$$

- for $x = y$

$$\delta\overline{Z}^L_{xx} - (\delta Z^L_{xx})^* = i\frac{\alpha}{2\pi}\Bigg(\sum_{z,v}\mathrm{Im}F^v(M_x^2; m_v, \sqrt{\xi}m_v, m_z) g^L_{xz,v}g^L_{zx,v}$$
$$+ \frac{1}{2M_x}\sum_{z,v} m_z \mathrm{Im}F^s(M_x^2; m_v, \sqrt{\xi}m_v, m_z)\left(g^R_{xz,v}g^L_{zx,v} - g^L_{xz,v}g^R_{zx,v}\right)$$
$$+ \sum_{z,s}\mathrm{Im}B_1(M_x^2; m_z, m_s) c^R_{xz,s}c^L_{zx,s}$$
$$- \frac{1}{2M_x}\sum_{z,s} m_z \mathrm{Im}B_0(M_x^2; m_z, m_s)\left(c^L_{xz,s}c^L_{zx,s} - c^R_{xz,s}c^R_{zx,s}\right)\Bigg)$$
$$+ \mathcal{D}_{xx}(M_x) - (\mathcal{D}_{xx}(M_x))^*, \quad (4.61)$$

$$\delta\overline{Z}^R_{xx} - (\delta Z^R_{xx})^* = i\frac{\alpha}{2\pi}\Bigg(\sum_{z,v}\mathrm{Im}F^v(M_x^2; m_v, \sqrt{\xi}m_v, m_z) g^R_{xz,v}g^R_{zx,v}$$
$$- \frac{1}{2M_x}\sum_{z,v} m_z \mathrm{Im}F^s(M_x^2; m_v, \sqrt{\xi}m_v, m_z)\left(g^R_{xz,v}g^L_{zx,v} - g^L_{xz,v}g^R_{zx,v}\right)$$
$$+ \sum_{z,s}\mathrm{Im}B_1(M_x^2; m_z, m_s) c^L_{xz,s}c^R_{zx,s}$$
$$+ \frac{1}{2M_x}\sum_{z,s} m_z \mathrm{Im}B_0(M_x^2; m_z, m_s)\left(c^L_{xz,s}c^L_{zx,s} - c^R_{xz,s}c^R_{zx,s}\right)\Bigg)$$
$$+ \mathcal{D}_{xx}(M_x) - (\mathcal{D}_{xx}(M_x))^*. \quad (4.62)$$

The imaginary parts of F^v, F^s, B_0 and B_1 are given in section 3.1.3, by formulas (3.44)−(3.46). From the imaginary parts of $\mathcal{D}_{xx}(M_x)$, we have to add to the diagonal parts:

$$\begin{aligned}\mathcal{D}_{xx}(M_x) - (\mathcal{D}_{xx}(M_x))^* =& i\frac{\alpha}{2\pi}\left(M_x^2 \sum_{z,v} \left(g^L_{xz,v}g^L_{zx,v} + g^R_{xz,v}g^R_{zx,v}\right) \mathrm{Im}\,\frac{\partial}{\partial p^2}F^v\Big|_{p^2\to M_x^2}\right.\\ &+ M_x \sum_{z,v} m_z \left(g^R_{xz,v}g^L_{zx,v} + g^L_{xz,v}g^R_{zx,v}\right) \mathrm{Im}\,\frac{\partial}{\partial p^2}F^s\Big|_{p^2\to M_x^2}\\ &+ M_x^2 \sum_{z,s} \left(c^R_{xz,s}c^L_{zx,s} + c^L_{xz,s}c^R_{zx,s}\right) \mathrm{Im}\,\frac{\partial}{\partial p^2}B_1\Big|_{p^2\to M_x^2} \qquad (4.63)\\ &\left. - M_x \sum_{z,s} m_z \left(c^L_{xz,s}c^L_{zx,s} + c^R_{xz,s}c^R_{zx,s}\right) \mathrm{Im}\,\frac{\partial}{\partial p^2}B_1\Big|_{p^2\to M_x^2}\right).\end{aligned}$$

After calculating the derivative of F^v and F^s from (3.22), one can extract the imaginary parts of B_0' and B_1' from Appendix B.2.3, equations (B.40) and (B.42).

Note that all the imaginary parts given here are different from zero if $M_x^2 > (m_k + m_v)^2$ or $M_x^2 > (m_k + \sqrt{\xi} m_v)^2$ in the contributions from vector bosons, and if $M_x^2 > (m_k + m_s)^2$ in the ones for scalars. As expected, for models with unstable particles, the hermiticity relation for the wave function renormalization constants is broken by the imaginary parts of the self-energies that keep track of the possible decays. Moreover, these imaginary parts are not gauge independent and omitting them might lead to a gauge parameter dependence of the amplitude of a process.

δZ and $\delta\overline{Z}$ are gauge dependent parameters and the correct separation of the divergences of the full propagator will yield a gauge dependence that cancels in the total one-loop amplitude. The validity of this statement was checked for δZ and $\delta\overline{Z}$ equal to the ones in section 4.1 by [Esp02].

Additionally, the mass correction δm should be gauge parameter independent. For this, to the self-energies described in chapter 3, one needs to add tadpole diagrams. They cancel the gauge dependence in the real part of the diagonal self-energies.

The sum of imaginary contributions in the full propagator pole collected in the decay width Γ_i, (4.25), is gauge independent. The gauge dependent imaginary terms arising from self-energies with internal gauge bosons, formula (3.46), cancel with the ones in (3.44) and (3.45) when they account for the scalar Goldstone bosons corresponding to the longitudinal modes of vector bosons. The mass of these scalars will be given by $m_s = \sqrt{\xi} m_v$. One can show that adding up all the contributions in (4.25), Γ_i results gauge parameter independent.

To summarise the results of this chapter, we remind that in section 4.1 we determined δZ and $\delta\overline{Z}$ imposing conditions that lead to a fermion propagator that is diagonal on-shell and its residue equals to 1. The so-called wave function renormalization constants are not usually related by hermiticity, but their divergences are. Therefore, to assure a correct renormalization scheme based on on-shell conditions, one should keep δZ and $\delta\overline{Z}$ independent.

Chapter 5

Renormalization of the Free Fermionic Lagrangian

In the previous chapter, using the on-shell scheme, we calculated mass corrections and what we called wave function renormalization constants. Now we will start from the free Lagrange density and the components that need to be renormalized. We will try to identify the possible relations between the calculated corrections and the renormalization constants, which conceptually are different from what we previously determined.

Our calculations have as motivation the observation made in several articles (see for example [Kni96] or [Pil02]) that the wave function renormalization constants can be shifted by adding anti-hermitian, gauge-independent and UV-finite constant matrices. Such a shift does not damage the properties of the renormalized mixing matrix. However, other parts of the renormalized Lagrange density can be affected and in particular, the free field one.

We will consider several possibilities for defining the connection between the wave function renormalization constants and the constants that renormalize the fields and we will analyse the consequences on the propagator and its counter terms.

5.1 Renormalized Free Dirac Lagrangian

This section is structured in three parts. In the first one, we consider a general transformation of the fields and we derive the Feynman rules for counter terms. We calculate the first order in α contributions of self-energies to external lines and we analyse the changes implied by the shift of the wave function renormalization constants.

In the second part, we investigate the possibility of defining field renormalization constants related by hermiticity. We will find that this is impossible unless the fermion self-energies have some special properties.

As a step further, we test the consequences of a transformation on the renormalized fields that leads to a diagonal mass in the renormalized Lagrangian. In this case, we will obtain divergent contributions for the corrections to external fermion lines.

5.1.1 General Renormalized Free Lagrangian

As given in section 2.3, for Dirac fermions, the free Lagrangian is

$$\mathcal{L}_0^D = \sum_i \overline{\psi}_i (i\partial\!\!\!/ - m_i)\psi_i. \tag{5.1}$$

The first obvious change induced by renormalization is in the mass term. We express it as

$$m_i = Z_{Mi} M_i^r. \tag{5.2}$$

M_i^r is the renormalized mass, which later is identified with the physical one and

$$Z_{Mi} = 1 - \frac{\delta m_i^r}{M_i^r} + \mathcal{O}(\alpha^2). \tag{5.3}$$

We have denoted the mass correction proportional to α by δm_i^r. For the moment, δm_i^r is not necessarily equal to the shift δm_i that was defined in section 4.1.

The constants $Z^{\frac{1}{2}}$ and $\overline{Z}^{\frac{1}{2}}$ defined in the previous chapter as wave function renormalization constants can be absorbed into the normalisation of the fields. Let's consider the renormalized field described by ψ_i^r and the Dirac conjugated one by $\overline{\psi_i^r}$ and denote the field renormalization constants by $Z_{ij}^{r\frac{1}{2}}$ and $\overline{Z}_{ij}^{r\frac{1}{2}}$. We emphasise again that these constants are not identical to $Z^{\frac{1}{2}}$ and $\overline{Z}^{\frac{1}{2}}$. The role of $Z^{r\frac{1}{2}}$ and $\overline{Z}^{r\frac{1}{2}}$ is to renormalize the fields (this is why they get the upper index r), while $Z^{\frac{1}{2}}$ and $\overline{Z}^{\frac{1}{2}}$ had just to absorb the divergences coming in the full propagator. The first ones have to include these wave function renormalization constants, but they are not necessarily equal. The expansion and decomposition of the field renormalization constants is similar to (4.10) and (4.11),

$$Z_{ij}^{r\frac{1}{2}} = Z_{ij}^{r\frac{1}{2}L}\gamma_L + Z_{ij}^{r\frac{1}{2}R}\gamma_R = \delta_{ij} + \frac{1}{2}\delta Z_{ij}^{rL}\gamma_L + \frac{1}{2}\delta Z_{ij}^{rR}\gamma_R + \mathcal{O}(\alpha^2), \tag{5.4}$$

$$\overline{Z}_{ij}^{r\frac{1}{2}} = \overline{Z}_{ij}^{r\frac{1}{2}R}\gamma_L + \overline{Z}_{ij}^{r\frac{1}{2}L}\gamma_R = \delta_{ij} + \frac{1}{2}\delta\overline{Z}_{ij}^{rR}\gamma_L + \frac{1}{2}\delta\overline{Z}_{ij}^{rL}\gamma_R + \mathcal{O}(\alpha^2). \tag{5.5}$$

The unrenormalized field changes to

$$\psi_i = \sum_j Z_{ij}^{r\frac{1}{2}} \psi_j^r, \tag{5.6}$$

$$\overline{\psi}_i = \sum_j \overline{\psi_j^r} \overline{Z}_{ji}^{r\frac{1}{2}}. \tag{5.7}$$

Since the fermion self-energy $\Sigma_{ij}(p)$ can have off-diagonal elements $(i \neq j)$, and as a consequence, we have off-diagonal renormalization constants, we must allow mixing when transforming the fields. To start with, we do not relate $Z^{r\frac{1}{2}}$ and $\overline{Z}^{r\frac{1}{2}}$ by hermiticity, i.e.

$$\overline{Z}^{r\frac{1}{2}} \neq \gamma^0 (Z^{r\frac{1}{2}})^\dagger \gamma^0. \tag{5.8}$$

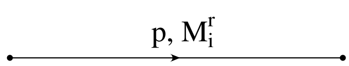
If we replace the bare parameters in (5.1) by the renormalized ones, we get

$$\mathcal{L}_0^D = \sum_{i,j,k} \overline{\psi_j^r} \overline{Z}_{ji}^{r\frac{1}{2}} (i\partial\!\!\!/ - m_i) Z_{ik}^{r\frac{1}{2}} \psi_k^r. \tag{5.9}$$

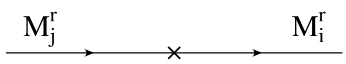
With (5.2)−(5.5), in first order:

$$\begin{aligned}\mathcal{L}_0^D =& \sum_i \overline{\psi_i^r}(i\partial\!\!\!/ - M_i^r)\psi_i^r \\ &+ \sum_{i,k} \overline{\psi_i^r}(i\partial\!\!\!/ - M_i^r)\frac{1}{2}\delta Z_{ik}^r \psi_k^r + \sum_{i,j} \overline{\psi_j^r}\frac{1}{2}\delta \overline{Z}_{ji}^r (i\partial\!\!\!/ - M_i^r)\psi_i^r \\ &+ \sum_{i,k} \overline{\psi_i^r}\delta m_i^r \delta_{ik}\psi_k^r + \mathcal{O}(\alpha^2).\end{aligned} \tag{5.10}$$

From here, one can read the new Feynman rules: the lowest order term will give the fermion propagator, while the terms of order α will be treated as two-point vertex counter terms,



$$iS(p) = \frac{i}{p\!\!\!/ - M_i^r + i\rho}$$

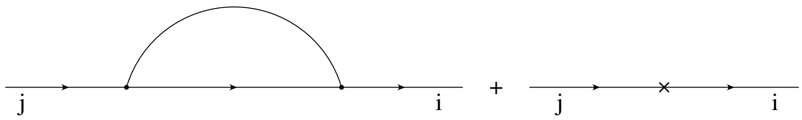


$$i(p\!\!\!/ - M_i^r)\ \frac{1}{2}\delta Z_{ij}^r + i\ \frac{1}{2}\delta \overline{Z}_{ij}^r (p\!\!\!/ - M_j^r) + i\delta m_i^r \delta_{ij}\ .$$

Now, the inverse full propagator is going to be given by

$$S_{ij}^{-1}(p) = (p\!\!\!/ - M_i^r)\delta_{ij} - \Sigma_{ij}^r(p), \tag{5.11}$$

where $\Sigma^r_{ij}(p)$ denotes the self-energy plus the counter terms and it is calculated as:

$-i\Sigma^r_{ij}(p) =$ 

The first diagram represents the unrenormalized self-energy, as given in chapter 3. Since it is already of order α, it does not get modifications at the one-loop level. Inserting the Feynman rules for the counter term, $\Sigma^r_{ij}(p)$ is equal to

$$-i\Sigma^r_{ij}(p) = -i\Sigma_{ij}(p) + i(\not{p} - M^r_i)\frac{1}{2}\delta Z^r_{ij} + i\frac{1}{2}\delta\overline{Z}^r_{ij}(\not{p} - M^r_j) + i\delta m^r_i\delta_{ij}. \tag{5.12}$$

At this point we can discuss the connection between the field renormalization, and the wave function renormalization constants calculated in section 4.1. At $\mathcal{O}(\alpha)$, the unrenormalized self-energy $\Sigma_{ij}(p)$ is identified from (4.12) with:

$$\Sigma_{ij}(p) = \hat{\Sigma}_{ij}(p) + \frac{1}{2}(\not{p} - M_i)\delta Z_{ij} + \frac{1}{2}\delta\overline{Z}_{ij}(\not{p} - M_j) + \delta m_i\delta_{ij}. \tag{5.13}$$

Choosing the renormalized mass M^r_i to be the same with the physical one from the previous chapter (M_i), then

$$\delta m_i = \delta m^r_i, \tag{5.14}$$

and (5.12) is equivalent to

$$\Sigma^r_{ij}(p) = \hat{\Sigma}_{ij}(p) + \frac{1}{2}(\not{p} - M_i)(\delta Z_{ij} - \delta Z^r_{ij}) + \frac{1}{2}(\delta\overline{Z}_{ij} - \delta\overline{Z}^r_{ij})(\not{p} - M_j). \tag{5.15}$$

We define:

$$Z^{\frac{1}{2}}_{ij} - Z^{r\frac{1}{2}}_{ij} = \frac{1}{2}\left(\delta Z_{ij} - \delta Z^r_{ij}\right) = \frac{1}{2}\varkappa_{ij}, \tag{5.16}$$

$$\overline{Z}^{\frac{1}{2}}_{ij} - \overline{Z}^{r\frac{1}{2}}_{ij} = \frac{1}{2}\left(\delta\overline{Z}_{ij} - \delta\overline{Z}^r_{ij}\right) = \frac{1}{2}\overline{\varkappa}_{ij}, \tag{5.17}$$

where $\varkappa_{ij}$ and $\overline{\varkappa}_{ij}$ are finite constants. In the definition, we take into account that at one-loop, the difference between the wave function, and the field renormalization constants is equal to the difference between their first order terms divided by 2. $\varkappa_{ij}$ and $\overline{\varkappa}_{ij}$ are in this case of order α. The introduced constants also have a decomposition in terms of the left and the right projectors:

$$\varkappa_{ij} = \varkappa^L_{ij}\gamma_L + \varkappa^R_{ij}\gamma_R, \tag{5.18}$$

$$\overline{\varkappa}_{ij} = \overline{\varkappa}^R_{ij}\gamma_L + \overline{\varkappa}^L_{ij}\gamma_R. \tag{5.19}$$

The components are then given by:

$$\begin{aligned} \varkappa_{ij}^{L} &= \delta Z_{ij}^{L} - \delta Z_{ij}^{rL}, \\ \varkappa_{ij}^{R} &= \delta Z_{ij}^{R} - \delta Z_{ij}^{rR}, \end{aligned} \tag{5.20}$$

$$\begin{aligned} \overline{\varkappa}_{ij}^{L} &= \delta \overline{Z}_{ij}^{L} - \delta \overline{Z}_{ij}^{rL}, \\ \overline{\varkappa}_{ij}^{R} &= \delta \overline{Z}_{ij}^{R} - \delta \overline{Z}_{ij}^{rR}. \end{aligned} \tag{5.21}$$

(5.15) is equivalent to

$$\begin{aligned} \Sigma_{ij}^{r}(p) &= \hat{\Sigma}_{ij}(p) + \frac{1}{2}(\not{p} - M_i)\varkappa_{ij} + \frac{1}{2}\overline{\varkappa}_{ij}(\not{p} - M_j) \\ &= \hat{\Sigma}_{ij}(p) + \mathcal{R}_{ij}(p). \end{aligned} \tag{5.22}$$

By $\mathcal{R}_{ij}(p)$ we denote the difference $\Sigma_{ij}^{r}(p) - \hat{\Sigma}_{ij}(p)$:

$$\mathcal{R}_{ij}(p) = \frac{1}{2}(\not{p} - M_i)\varkappa_{ij} + \frac{1}{2}\overline{\varkappa}_{ij}(\not{p} - M_j). \tag{5.23}$$

Taking into consideration a non-zero difference between the field renormalization constants and the wave function ones, we introduce back non-diagonal contributions to the fermion propagator. The relations describing the corrections from self-energy to external particles, i.e. (4.35), (4.36) and the corresponding ones for antiparticles are allowed not being zero any longer. For $\Sigma_{ij}^{r}(p)$, using (4.34), we have

- for $i \neq j$

$$\begin{aligned} \Sigma_{ij}^{r}(p)u_j(p,s)\Big|_{p^2\to M_j^2} &= \mathcal{R}_{ij}(p)u_j(p,s)\Big|_{p^2\to M_j^2} \\ &= \frac{1}{2}\big((-M_i\varkappa_{ij}^{L} + M_j\varkappa_{ij}^{R})\gamma_L \\ &\qquad +(-M_i\varkappa_{ij}^{R} + M_j\varkappa_{ij}^{L})\gamma_R\big)\, u_j(p,s)\Big|_{p^2\to M_j^2}, \end{aligned} \tag{5.24}$$

$$\begin{aligned} \overline{u}_i(p,s)\Sigma_{ij}^{r}(p)\Big|_{p^2\to M_i^2} &= \overline{u}_i(p,s)\mathcal{R}_{ij}(p)\Big|_{p^2\to M_i^2} \\ &= \overline{u}_i(p,s)\frac{1}{2}\big((-M_j\overline{\varkappa}_{ij}^{R} + M_i\overline{\varkappa}_{ij}^{L})\gamma_L \\ &\qquad +(-M_j\overline{\varkappa}_{ij}^{L} + M_i\overline{\varkappa}_{ij}^{R})\gamma_R\big)\Big|_{p^2\to M_i^2}. \end{aligned} \tag{5.25}$$

This difference to (4.35) and (4.36) enables us to impose (if possible), additional conditions on $\varkappa$ and $\overline{\varkappa}$ or, equivalently, on $Z^{r\frac{1}{2}}$ and $\overline{Z}^{r\frac{1}{2}}$. Since, as one can notice from above, these constants will contribute to predictions for measurable quantities, they have to be chosen finite as we mentioned in their definition. Thus, for the non-diagonal case, we have a finite expression that has to be taken into consideration when calculating the one-loop amplitude of a process.

We will now complete the discussion of section 4.1 also for the diagonal self-energy corrections of external legs. We will calculate the entire order α corrections, including the counter terms. These corrections are also described by diagrams as given in Figure 4.2 or 4.3. With the explicit propagator for the internal fermion, the on-shell relations to be evaluated are,

- for $i = j$

$$\begin{aligned}
\frac{1}{\not p - M_i}\Sigma^r_{ii}(p)u_i(p,s)\Big|_{p^2\to M_i^2} &= \frac{1}{\not p - M_i}\left(\hat{\Sigma}_{ii}(p) + \mathcal{R}_{ii}(p)\right)u_i(p,s)\Big|_{p^2\to M_i^2} \\
&= \frac{1}{\not p - M_i}\left(-i\frac{1}{2}\Gamma_i\right)u_i(p,s)\Big|_{p^2\to M_i^2} \\
&+ \frac{1}{2}\left((\varkappa^L_{ii} + \overline{\varkappa}^L_{ii})\gamma_L + (\varkappa^R_{ii} + \overline{\varkappa}^R_{ii})\gamma_R\right)u_i(p,s)\Big|_{p^2\to M_i^2} \\
+\frac{1}{2}M_i\frac{1}{\not p - M_i}&\left((\overline{\varkappa}^L_{ii} - \overline{\varkappa}^R_{ii})\gamma_L - (\overline{\varkappa}^L_{ii} - \overline{\varkappa}^R_{ii})\gamma_R\right)u_i(p,s)\Big|_{p^2\to M_i^2} \qquad (5.26) \\
\overline{u}_i(p,s)\Sigma^r_{ii}(p)\,\frac{1}{\not p - M_i}\Big|_{p^2\to M_i^2} &= \overline{u}_i(p,s)\left(\hat{\Sigma}_{ii}(p) + \mathcal{R}_{ii}(p)\right)\frac{1}{\not p - M_i}\Big|_{p^2\to M_i^2} \\
&= \overline{u}_i(p,s)\left(-i\frac{1}{2}\Gamma_i\right)\frac{1}{\not p - M_i}\Big|_{p^2\to M_i^2} \\
&+ \overline{u}_i(p,s)\Big|_{p^2\to M_i^2}\frac{1}{2}\left((\varkappa^R_{ii} + \overline{\varkappa}^R_{ii})\gamma_L + (\varkappa^L_{ii} + \overline{\varkappa}^L_{ii})\gamma_R\right) \\
+\overline{u}_i(p,s)\frac{1}{2}M_i&\left((\varkappa^R_{ii} - \varkappa^L_{ii})\gamma_L - (\varkappa^R_{ii} - \varkappa^L_{ii})\gamma_R\right)\frac{1}{\not p - M_i}\Big|_{p^2\to M_i^2}. \qquad (5.27)
\end{aligned}$$

For the diagonal case, part of the contribution is finite, but we are also left with terms that contain the factor $\frac{1}{\not p - M_i}$. There are two such terms in each equation. One is proportional to the particle width Γ_i, calculated in (4.25) and the other one is due to the difference between the the two sets of constants $Z^{\frac{1}{2}}$ and $Z^{r\frac{1}{2}}$.

The presence of the pole term proportional to Γ_i is a consequence of the way the perturbative expansion is performed. As one can see in equation (5.10), the expansion is based on the separation of the free Lagrangian describing a stable particle with a real mass M_i. The calculation indicates that the assumption of a stable particle was incorrect: the interaction allows the considered particle to decay. The discussion at the beginning of chapter 4 has shown that the Dyson summation will move the contributions from Γ_i into the denominator. According to (4.1)-(4.2) and (4.20):

$$\frac{i}{\not{p} - M_i}\left(1 - \frac{1}{2}\Gamma_i \frac{i}{\not{p} - M_i}\right) \longrightarrow \frac{i}{\not{p} - \left(M_i - i\frac{1}{2}\Gamma_i\right)}. \tag{5.28}$$

In fact, the re-summation of the imaginary parts is necessary to obtain well-defined matrix elements without additional divergent contributions, see also for example [Pes95]. The imaginary contributions from the self-energy are finite, see section 3.1.3. Therefore, setting the momentum on-shell, the denominator will be proportional just to finite contributions of Γ_i.

In the further calculations, we will not discuss this term any longer. Later, we will calculate decay rates for unstable particles and one will be able to check that due to complex conjugation, the $i\Gamma_i$ terms will drop out anyway in the squared amplitudes. Even in case it turns out that such terms contribute to measurable quantities, one will always be able to separately identify their contributions. Remember from the discussion at the end of section 4.2 that Γ_i is gauge parameter independent and therefore, the total amplitude remains gauge independent with or without this term.

Nevertheless, the pole related to $\varkappa$ and $\overline{\varkappa}$ has to cancel. Therefore, we need to require

$$\begin{aligned} \varkappa_{ii}^L &= \varkappa_{ii}^R, \\ \overline{\varkappa}_{ii}^L &= \overline{\varkappa}_{ii}^R. \end{aligned} \tag{5.29}$$

This restriction is telling us that we are allowed to shift the left and the right components of the wave function renormalization constants by the same amount. (5.20) and (5.21) imply that

$$\begin{aligned} \delta Z_{ii}^{rL} &= \delta Z_{ii}^{rR} + \left(\delta Z_{ii}^L - \delta Z_{ii}^R\right), \\ \delta \overline{Z}_{ii}^{rL} &= \delta \overline{Z}_{ii}^{rR} + \left(\delta \overline{Z}_{ii}^L - \delta \overline{Z}_{ii}^R\right). \end{aligned} \tag{5.30}$$

With (5.29) and without the term with Γ_i, the diagonal self-energy corrections to external legs come as

$$\left.\frac{1}{\not p - M_i}\Sigma^r_{ii}(p)u_i(p,s)\right|_{p^2\to M_i^2} = \frac{1}{2}\left(\varkappa^L_{ii} + \overline{\varkappa}^L_{ii}\right) u_i(p,s)\Big|_{p^2\to M_i^2}, \tag{5.31}$$

$$\left.\overline{u}_i(p,s)\Sigma^r_{ii}(p)\,\frac{1}{\not p - M_i}\right|_{p^2\to M_i^2} = \overline{u}_i(p,s)\frac{1}{2}\left(\varkappa^L_{ii} + \overline{\varkappa}^L_{ii}\right)\Big|_{p^2\to M_i^2}. \tag{5.32}$$

For completeness, we should consider the self-energy corrections of antiparticles. The expressions coming with the self-energies will be similar and (5.29) will be required, too. We mention that

• for $i \neq j$

$$\Sigma^r_{ij}(-p)v_j(p,s)\Big|_{p^2\to M_j^2} = \Sigma^r_{ij}(p)u_j(p,s)\Big|_{p^2\to M_j^2}, \tag{5.33}$$

$$\overline{v}_i(p,s)\Sigma^r_{ij}(-p)\Big|_{p^2\to M_i^2} = \overline{u}_i(p,s)\Sigma^r_{ij}(p)\Big|_{p^2\to M_i^2}, \tag{5.34}$$

• for $i = j$

$$\left.\frac{1}{-\not p - M_i}\Sigma^r_{ii}(-p)v_i(p,s)\right|_{p^2\to M_i^2} = \left.\frac{1}{\not p - M_i}\Sigma^r_{ii}(p)u_i(p,s)\right|_{p^2\to M_i^2}, \tag{5.35}$$

$$\left.\overline{v}_i(p,s)\Sigma^r_{ii}(-p)\,\frac{1}{-\not p - M_i}\right|_{p^2\to M_i^2} = \left.\overline{u}_i(p,s)\Sigma^r_{ii}(p)\,\frac{1}{\not p - M_i}\right|_{p^2\to M_i^2}. \tag{5.36}$$

Of course, if $\varkappa_{ij} = \overline{\varkappa}_{ij} = 0$, we have $\Sigma^r_{ij}(p)$ identical to $\hat{\Sigma}_{ij}(p)$ and no supplementary contributions to external legs arise. Still, we keep them non-zero since we might need to impose additional conditions on our Lagrangian.

5.1.2 Hermitian Renormalized Free Lagrangian

One of the possible additional requirements on the field renormalization constants concerns hermiticity. To keep the Lagrangian hermitian, we would prefer to require that $Z^{r\frac{1}{2}}_{ij}$ and $\overline{Z}^{r\frac{1}{2}}_{ij}$ have to fulfil the hermiticity condition (4.6). For the left and right components of the renormalization constants, the condition is equivalent to

$$\begin{aligned} \overline{Z}^{r\frac{1}{2}L}_{ji} &= (Z^{r\frac{1}{2}L}_{ij})^*, \\ \overline{Z}^{r\frac{1}{2}R}_{ji} &= (Z^{r\frac{1}{2}R}_{ij})^*. \end{aligned} \tag{5.37}$$

This requirement is adding a restriction to $\varkappa_{ij}$ and $\overline{\varkappa}_{ij}$. If one replaces the field renormalization constants from (5.20) and (5.21), i.e.

$$\begin{aligned} \delta Z_{ij}^{rL} &= \delta Z_{ij}^{L} - \varkappa_{ij}^{L}, \\ \delta Z_{ij}^{rR} &= \delta Z_{ij}^{R} - \varkappa_{ij}^{R}, \\ \delta \overline{Z}_{ij}^{rL} &= \delta \overline{Z}_{ij}^{L} - \overline{\varkappa}_{ij}^{L}, \\ \delta \overline{Z}_{ij}^{rR} &= \delta \overline{Z}_{ij}^{R} - \overline{\varkappa}_{ij}^{R}, \end{aligned} \tag{5.38}$$

in (5.37), the hermiticity condition implies:

$$\begin{aligned} \overline{\varkappa}_{ji}^{L} - (\varkappa_{ij}^{L})^{*} &= \delta \overline{Z}_{ji}^{L} - (\delta Z_{ij}^{L})^{*}, \\ \overline{\varkappa}_{ji}^{R} - (\varkappa_{ij}^{R})^{*} &= \delta \overline{Z}_{ji}^{R} - (\delta Z_{ij}^{R})^{*}. \end{aligned} \tag{5.39}$$

If we combine now this with the restriction (5.29), we obtain an overdetermined system that requires that the wave function renormalization constants fulfil

$$\delta \overline{Z}_{ii}^{L} - \delta \overline{Z}_{ii}^{R} = (\delta Z_{ii}^{L})^{*} - (\delta Z_{ii}^{R})^{*}. \tag{5.40}$$

The equation for the diagonal constants (4.33) leads us to

$$\begin{aligned} &\Sigma_{ii}^{L}(M_i^2) - \Sigma_{ii}^{R}(M_i^2) + \frac{1}{M_i}\left(\Sigma_{ii}^{DL}(M_i^2) - \Sigma_{ii}^{DR}(M_i^2)\right) = \\ &\qquad (\Sigma_{ii}^{L}(M_i^2))^{*} - (\Sigma_{ii}^{R}(M_i^2))^{*} - \frac{1}{M_i}\left((\Sigma_{ii}^{DL}(M_i^2))^{*} - (\Sigma_{ii}^{DR}(M_i^2))^{*}\right). \end{aligned} \tag{5.41}$$

If the absorptive parts of the self-energies are zero, then the properties (3.72) apply and the relation holds. In case we consider unstable external particles or we do not restrict the calculation to a special gauge, we have a non-zero imaginary part in the self-energy (see section 3.1.3). Imposing the hermiticity relation (5.37), we are left with poles in the amplitudes. (5.39) and (5.29) can not be simultaneously fulfilled, i.e. finiteness and hermiticity can not be concomitantly obtained. For this reason, in a lot of renormalization approaches, all external particles are treated as stable and the self-energy components are fixed as in (3.72). We will try not to do so.

5.1.3 Re-diagonalized Mass Term Approach

In chapter 2 we discussed possible mass terms in a Lagrangian. We start from non-diagonal ones and using unitary transformations of the fields we bring them in a diagonal form. In a second step, at the beginning of the previous chapter, we considered corrections of the propagator that lead to the renormalization of the fields. As a consequence of the fermion mixing, the diagonal form of the mass term is destroyed. In section 5.1.1 we were analysing the modifications that appear in the free Lagrangian and we were defining counter terms. Now we want to start again from the renormalized mass term, and as in section 2.2, to re-diagonalize it. It is instructive to investigate the possibility of defing a rotation of the renormalized fields such that the physical mass comes only in one diagonal mass term.

When inserting the renormalized fields (5.6) and (5.7) in the Lagrangian, the Dirac mass term

$$\mathcal{L}^D_{mass} = -\sum_i \overline{\psi^R_i} m_i \psi^L_i + h.c. = -\sum_{i,j,k} \overline{\psi^{rR}_j} \overline{Z}^{r\frac{1}{2}R}_{ji} m_i Z^{r\frac{1}{2}L}_{ik} \psi^{rL}_k + h.c. \tag{5.42}$$

is not diagonal anymore.

We define the new, real and nonnegative diagonal mass matrix by

$$\mathcal{M}_l = \sum_{i,j,k} (O^R_{jl})^* \overline{Z}^{r\frac{1}{2}R}_{ji} m_i Z^{r\frac{1}{2}L}_{ik} O^L_{kl}. \tag{5.43}$$

O^L and O^R are the two unitary matrices required by the singular value decomposition. One has to keep in mind that $\mathcal{M}_i$ is not the physical mass. We can write

$$\mathcal{M}_i = M_i + \delta M_i, \tag{5.44}$$

where we have M_i as the physical mass and δM_i as the first order in α correction.

To simplify the calculation details, we will assume that the field renormalization constants are related by hermiticity, i.e. (5.37). We know what are the drawbacks of this assumption, but still, the difficulties of the re-diagonalization approach for a general Lagrangian are due to something else, as we will see at the end of this subsection. To connect (5.43) to the Lagrange density, we introduce ψ'_i, the field related to mass eigenstates. Its left and right components are

$$\psi^{rL}_i = \sum_j O^L_{ij} \psi'^L_j, \tag{5.45}$$

$$\psi^{rR}_i = \sum_j O^R_{ij} \psi'^R_j. \tag{5.46}$$

Collecting all the transformations of the unrenormalized field (i.e. also (5.6)), we have

$$\psi^L_i = \sum_{j,k} Z^{r\frac{1}{2}L}_{ij} O^L_{jk} \psi'^L_k, \tag{5.47}$$

$$\psi^R_i = \sum_{j,k} Z^{r\frac{1}{2}R}_{ij} O^R_{jk} \psi'^R_k. \tag{5.48}$$

Replacing the relations in (5.42), the mass term becomes diagonal:

$$\begin{aligned}\mathcal{L}^D_{mass} &= -\sum_l \overline{\psi_l^{\prime R}} \mathcal{M}_l \psi_l^{\prime L} + h.c. \\ &= -\sum_l \overline{\psi_l^{\prime R}} M_l \psi_l^{\prime L} - \sum_l \overline{\psi_l^{\prime R}} \delta M_l \psi_l^{\prime L} + \mathcal{O}(\alpha^2) + h.c..\end{aligned} \tag{5.49}$$

Now, we look closer to the peculiarities of the re-diagonalization for one-loop corrections. If we use the expansions in powers of α of the renormalization constants (formulas (5.4) and (5.5)), the elements of the matrix to be diagonalized are given by

$$\sum_k \overline{Z}^{r\frac{1}{2}R}_{ik} m_k Z^{r\frac{1}{2}L}_{kj} = m_i \delta_{ij} + \frac{1}{2} \delta \overline{Z}^{rR}_{ij} m_j + m_i \frac{1}{2} \delta Z^{rL}_{ij} + \mathcal{O}(\alpha^2). \tag{5.50}$$

We have diagonal and real elements for the lowest order contributions and non-diagonal complex ones for the terms of order α and higher. The detailed diagonalization of such a matrix is described in Appendix C.3. Comparing the matrix with (C.15), we can identify

$$\begin{aligned} c_i &= m_i, \\ \delta C_{ij} &= \frac{1}{2} \left(m_i \delta Z^{rL}_{ij} + m_j \delta \overline{Z}^{rR}_{ij} \right), \end{aligned} \tag{5.51}$$

and make the corresponding replacements in the final result. For the hermitian conjugated δC, we use the hermiticity property of the renormalization constants (5.37):

$$\begin{aligned} (\delta C_{ji})^* &= \frac{1}{2} \left(m_j (\delta Z^{rL}_{ji})^* + m_i (\delta \overline{Z}^{rR}_{ji})^* \right) \\ &= \frac{1}{2} \left(m_j \delta \overline{Z}^{rL}_{ij} + m_i \delta Z^{rR}_{ij} \right). \end{aligned} \tag{5.52}$$

The role of $\mathcal{O}$ and $\mathcal{P}$ from the appendix is played by O^R and O^L, respectively.

Owing to (5.50), the required unitary matrices can be expanded as

$$\begin{aligned} O^L_{ij} &= \delta_{ij} + \delta O^L_{ij} + \mathcal{O}(\alpha^2), \\ O^R_{ij} &= \delta_{ij} + \delta O^R_{ij} + \mathcal{O}(\alpha^2). \end{aligned} \tag{5.53}$$

Making use of (C.14), we can write in first order approximation:

$$\begin{aligned} (O^L_{ji})^* &= \delta_{ij} - \delta O^L_{ij} + \mathcal{O}(\alpha^2), \\ (O^R_{ji})^* &= \delta_{ij} - \delta O^R_{ij} + \mathcal{O}(\alpha^2). \end{aligned} \tag{5.54}$$

The elements of O^L and O^R are related by (see C.38)

$$\delta O^L_{ij} = \delta O^R_{ij} + \frac{1}{2(m_i+m_j)}\left(-m_i\delta Z^{rL}_{ij} + m_i\delta Z^{rR}_{ij} + m_j\delta\overline{Z}^{rL}_{ij} - m_j\delta\overline{Z}^{rR}_{ij}\right). \quad (5.55)$$

For the off-diagonal elements, we have the explicit solution given in (C.39) and (C.37):

$$\left.\begin{aligned}\delta O^L_{ij} &= \frac{1}{2(m_j^2-m_i^2)}\left(m_i^2\delta Z^{rL}_{ij} + m_im_j\delta Z^{rR}_{ij} + m_j^2\delta\overline{Z}^{rL}_{ij} + m_im_j\delta\overline{Z}^{rR}_{ij}\right),\\ \delta O^R_{ij} &= \frac{1}{2(m_j^2-m_i^2)}\left(m_im_j\delta Z^{rL}_{ij} + m_i^2\delta Z^{rR}_{ij} + m_im_j\delta\overline{Z}^{rL}_{ij} + m_j^2\delta\overline{Z}^{rR}_{ij}\right),\end{aligned}\right\}\ \text{for } i\neq j, \quad (5.56)$$

while for the diagonal ones we have just the constraint (5.55).

From (C.40), we can identify the diagonal mass matrix $\mathcal{M}$.

$$\mathcal{M}_i = m_i + \frac{1}{4}m_i\left(\delta Z^{rL}_{ii} + \delta Z^{rR}_{ii} + \delta\overline{Z}^{rL}_{ii} + \delta\overline{Z}^{rR}_{ii}\right) + \mathcal{O}(\alpha^2) \quad (5.57)$$

If we identify the physical mass with the one given in (4.9) and replace the unrenormalized mass by $m_i{=}M_i{-}\delta m_i$, then

$$\delta M_i = -\delta m_i + \frac{1}{4}M_i\left(\delta Z^{rL}_{ii} + \delta Z^{rR}_{ii} + \delta\overline{Z}^{rL}_{ii} + \delta\overline{Z}^{rR}_{ii}\right). \quad (5.58)$$

At the end, one will be able to check that the real part of the pole in the full fermion propagator lies at $p^2 = M_i^2$.

Since $Z^{r\frac{1}{2}}_{ij}$ and $\overline{Z}^{r\frac{1}{2}}_{ij}$ are divergent, the same is true for O^L and O^R. By rediagonalization we remove divergent terms from the mass term. We know that unitary transformations are always possible without affecting the physical results and therefore these terms have to re-appear somewhere else.

From the free Lagrangian, we are left with the evaluation of the kinetic part. Since the algorithm for the term with left-handed fields is identical to the one with right-handed fields, we detail the analysis just for the first one:

$$\mathcal{L}^{D,L}_{kin} = i\sum_i \overline{\psi^L_i}\partial\!\!\!/\psi^L_i = i\sum_{i,j,k,l,m}\overline{\psi'^L_k}(O^L_{jk})^*\overline{Z}^{r\frac{1}{2}L}_{ji}\partial\!\!\!/ Z^{r\frac{1}{2}L}_{il}O^L_{lm}\psi'^L_m. \quad (5.59)$$

Because we want to identify the Feynman rules for the Dirac fermion propagator, we will expand the expression and separate the counter terms. With (5.53), (5.54) and the expansions of the field renormalization constants, the left part of the kinetic term is

$$\begin{aligned}\mathcal{L}_{kin}^{D,L} &= i\sum_i \overline{\psi_i'^L}\not{\partial}\psi_i'^L + i\sum_{i,j}\overline{\psi_i'^L}\left(-\delta O_{ij}^L + \frac{1}{2}\delta\overline{Z}_{ij}^{rL}\right)\not{\partial}\psi_j'^L \\ &\quad + i\sum_{i,j}\overline{\psi_i'^L}\not{\partial}\left(\delta O_{ij}^L + \frac{1}{2}\delta Z_{ij}^{rL}\right)\psi_j'^L + \mathcal{O}(\alpha^2) \\ &= i\sum_i \overline{\psi_i'^L}\not{\partial}\psi_i'^L + i\sum_{i,j}\overline{\psi_i'^L}\frac{1}{2}\delta\overline{Z}_{ij}^{rL}\not{\partial}\psi_j'^L + i\sum_{i,j}\overline{\psi_i'^L}\not{\partial}\frac{1}{2}\delta Z_{ij}^{rL}\psi_j'^L + \mathcal{O}(\alpha^2).\end{aligned} \tag{5.60}$$

For the right component of the kinetic term, we arrive to a similar expression:

$$\mathcal{L}_{kin}^{D,R} = i\sum_i \overline{\psi_i'^R}\not{\partial}\psi_i'^R + i\sum_{i,j}\overline{\psi_i'^R}\frac{1}{2}\delta\overline{Z}_{ij}^{rR}\not{\partial}\psi_j'^R + i\sum_{i,j}\overline{\psi_i'^R}\not{\partial}\frac{1}{2}\delta Z_{ij}^{rR}\psi_j'^R + \mathcal{O}(\alpha^2). \tag{5.61}$$

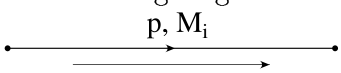
The free field Lagrange density is given by summing the mass and the kinetic terms. To write it more compact, we recombine the left and right components of the field:

$$\psi' = \psi'^L + \psi'^R. \tag{5.62}$$

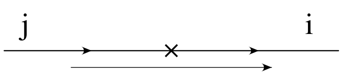
In first order approximation, the Dirac free Lagrangian is

$$\begin{aligned}\mathcal{L}_0^D &= \mathcal{L}_{kin}^D + \mathcal{L}_{mass}^D \\ &= \sum_i \overline{\psi_i'}(i\not{\partial} - M_i)\psi_i' + i\sum_{i,j}\overline{\psi_i'}\frac{1}{2}\delta\overline{Z}_{ij}^r\not{\partial}\psi_j' + i\sum_{i,j}\overline{\psi_i'}\not{\partial}\frac{1}{2}\delta Z_{ij}^r\psi_j' \\ &\quad - \sum_i \overline{\psi_i'}\delta M_i\psi_i' + \mathcal{O}(\alpha^2).\end{aligned} \tag{5.63}$$

The Feynman rules for propagator and counter terms are deduced in a similar way to subsection 5.1.1. At first order, we obtain them from the expanded expressions of the free field Lagrangian.



$$iS'(p) = \frac{i}{\not{p} - M_i}$$



$$i\not{p}\,\frac{1}{2}\delta Z_{ij}^r + i\,\frac{1}{2}\delta\overline{Z}_{ij}^r\not{p} - i\delta M_i\delta_{ij}.$$

The total self-energy, at one-loop, is the sum of the unrenormalized self-energy and the inverse propagator counter term:

$$-i\Sigma_{ij}'(p) = -i\Sigma_{ij}(p) + i\not{p}\frac{1}{2}\delta Z_{ij}^r + i\frac{1}{2}\delta\overline{Z}_{ij}^r\not{p} - i\delta M_i\delta_{ij}. \tag{5.64}$$

As a function of the wave function renormalization constants, using (4.12), (5.16) and (5.17), we can write

$$-\,i\Sigma'_{ij}(p) = -i\hat{\Sigma}_{ij}(p) - i\mathcal{C}_{ij}(p), \tag{5.65}$$

$$\text{with } \mathcal{C}_{ij}(p) = (\delta m_i + \delta M_i)\delta_{ij} - \frac{1}{2}m_i\delta Z_{ij} - \frac{1}{2}\delta\overline{Z}_{ij}m_j + \frac{1}{2}\not{p}\varkappa_{ij} + \frac{1}{2}\overline{\varkappa}_{ij}\not{p} \tag{5.66}$$

$$= (\delta m_i + \delta M_i)\delta_{ij} - \frac{1}{2}M_i\delta Z_{ij} - \frac{1}{2}\delta\overline{Z}_{ij}M_j + \frac{1}{2}\not{p}\varkappa_{ij} + \frac{1}{2}\overline{\varkappa}_{ij}\not{p}. \tag{5.67}$$

At this point we can evaluate the one-loop self-energy contributions from external legs, as we did in (5.24)−(5.27). Here, we start with the diagonal ones. We leave out the terms proportional to the decay width Γ_i that arise from imaginary parts of $\hat{\Sigma}_{ii}$ and we obtain:

- for $i = j$

$$\begin{aligned}
&\frac{1}{\not{p} - M_i}\Sigma'_{ii}(p)u_i(p,s)\Big|_{p^2\to M_i^2} = \frac{1}{\not{p} - M_i}\mathcal{C}_{ii}(p)u_i(p,s)\Big|_{p^2\to M_i^2} \\
&\quad = \frac{1}{\not{p} - M_i}\Big\{\Big(\delta m_i + \delta M_i + \frac{1}{2}M_i\Big(-\delta Z^L_{ii} - \delta\overline{Z}^R_{ii} + \varkappa^L_{ii} + \overline{\varkappa}^L_{ii}\Big)\Big)\gamma_L \\
&\qquad + \Big(\delta m_i + \delta M_i + \frac{1}{2}M_i\Big(-\delta Z^R_{ii} - \delta\overline{Z}^L_{ii} + \varkappa^R_{ii} + \overline{\varkappa}^R_{ii}\Big)\Big)\gamma_R\Big\}u_i(p,s)\Big|_{p^2\to M_i^2} \\
&\qquad + \frac{1}{2}\left((\varkappa^L_{ii} + \overline{\varkappa}^L_{ii})\gamma_L + (\varkappa^R_{ii} + \overline{\varkappa}^R_{ii})\gamma_R\right)u_i(p,s)\Big|_{p^2\to M_i^2},
\end{aligned} \tag{5.68}$$

$$\begin{aligned}
&\overline{u}_i(p,s)\Sigma'_{ii}(p)\,\frac{1}{\not{p} - M_i}\Big|_{p^2\to M_i^2} = \overline{u}_i(p,s)\mathcal{C}_{ii}(p)\frac{1}{\not{p} - M_i}\Big|_{p^2\to M_i^2} \\
&\quad = \overline{u}_i(p,s)\Big\{\Big(\delta m_i + \delta M_i + \frac{1}{2}M_i\Big(-\delta Z^L_{ii} - \delta\overline{Z}^R_{ii} + \varkappa^R_{ii} + \overline{\varkappa}^R_{ii}\Big)\Big)\gamma_L \\
&\qquad + \Big(\delta m_i + \delta M_i + \frac{1}{2}M_i\Big(-\delta Z^R_{ii} - \delta\overline{Z}^L_{ii} + \varkappa^L_{ii} + \overline{\varkappa}^L_{ii}\Big)\Big)\gamma_R\Big\}\frac{1}{\not{p} - M_i}\Big|_{p^2\to M_i^2} \\
&\qquad + \overline{u}_i(p,s)\frac{1}{2}\left((\varkappa^R_{ii} + \overline{\varkappa}^R_{ii})\gamma_L + (\varkappa^L_{ii} + \overline{\varkappa}^L_{ii})\gamma_R\right)\Big|_{p^2\to M_i^2}.
\end{aligned} \tag{5.69}$$

On-shell, the terms coming with the factor $\frac{1}{\not{p}-M_i}$ should vanish. This is equivalent with

$$\begin{cases} \delta M_i + \delta m_i = \frac{1}{2} M_i \left(\delta Z^L_{ii} + \delta \overline{Z}^R_{ii} - \varkappa^L_{ii} - \overline{\varkappa}^L_{ii} \right) \\ \delta M_i + \delta m_i = \frac{1}{2} M_i \left(\delta Z^R_{ii} + \delta \overline{Z}^L_{ii} - \varkappa^R_{ii} - \overline{\varkappa}^R_{ii} \right) \\ \delta M_i + \delta m_i = \frac{1}{2} M_i \left(\delta Z^L_{ii} + \delta \overline{Z}^R_{ii} - \varkappa^R_{ii} - \overline{\varkappa}^R_{ii} \right) \\ \delta M_i + \delta m_i = \frac{1}{2} M_i \left(\delta Z^R_{ii} + \delta \overline{Z}^L_{ii} - \varkappa^L_{ii} - \overline{\varkappa}^L_{ii} \right) \end{cases} \tag{5.70}$$

The system of equations has a solution if

$$\begin{cases} \delta Z^L_{ii} + \delta \overline{Z}^R_{ii} = \delta Z^R_{ii} + \delta \overline{Z}^L_{ii} \\ \varkappa^L_{ii} + \overline{\varkappa}^L_{ii} = \varkappa^R_{ii} + \overline{\varkappa}^R_{ii}. \end{cases} \tag{5.71}$$

While $\varkappa$ and $\overline{\varkappa}$ can be chosen such that (5.71) is fulfilled, the wave function renormalization constants are determined from the one-loop self-energy diagrams. Therefore, from (4.31), we see that the upper equation of (5.71) is true only if

$$\Sigma^{DL}_{ii}(M_i^2) = \Sigma^{DR}_{ii}(M_i^2). \tag{5.72}$$

As mentioned in section 3.3, in the Standard Model and some of its extensions, the self-energy has this property. If we had not required (5.37), we would have to separately diagonalize the mass term explicitly written in (5.42) and the hermitian conjugated one. Then, a δM_i^L and a δM_i^R would be needed and (5.72) would not be necessary.

With (5.71), one can see that taking δM_i from (5.58) with the field renormalization constants replaced by (5.38), all the terms coming with $\frac{1}{\not p - M_i}$ vanish. The contribution of the diagonal self-energy corrections to the external legs is reduced to a finite quantity determined by $\varkappa$ and $\overline{\varkappa}$, identical to (5.31) and (5.32).

Concerning the non-diagonal contributions, we find

• for $i \neq j$

$$\begin{aligned} \Sigma'_{ij}(p)u_j(p,s)\Big|_{p^2\to M_j^2} &= \mathcal{C}_{ij}(p)u_j(p,s)\Big|_{p^2\to M_j^2} \\ &= \frac{1}{2}\Big((-M_i\delta Z^L_{ij} - \delta\overline{Z}^R_{ij}M_j + \varkappa^R_{ij}M_j + \overline{\varkappa}^R_{ij}M_j)\gamma_L \\ &\quad + (-M_i\delta Z^R_{ij} - \delta\overline{Z}^L_{ij}M_j + \varkappa^L_{ij}M_j + \overline{\varkappa}^L_{ij}M_j)\gamma_R\Big)\, u_j(p,s)\Big|_{p^2\to M_j^2} \end{aligned} \tag{5.73}$$

$$\begin{aligned} \overline{u}_i(p,s)\Sigma'_{ij}(p)\Big|_{p^2\to M_i^2} &= \overline{u}_i(p,s)\mathcal{C}_{ij}(p)\Big|_{p^2\to M_i^2} \\ &= \overline{u}_i(p,s)\frac{1}{2}\Big((-M_i\delta Z^L_{ij} - \delta\overline{Z}^R_{ij}M_j + M_i\varkappa^L_{ij} + M_i\overline{\varkappa}^L_{ij})\gamma_L \\ &\quad + (-M_i\delta Z^R_{ij} - \delta\overline{Z}^L_{ij}M_j + M_i\varkappa^R_{ij} + M_i\overline{\varkappa}^R_{ij})\gamma_R\Big)\Big|_{p^2\to M_i^2}. \end{aligned} \tag{5.74}$$

Here, we could adjust $\varkappa$ and $\overline{\varkappa}$ to obtain zero corrections only if the wave function renormalization constant and implicitly the self-energy have some special properties. If we want to set the terms containing $\overline{\varkappa}^L_{ij} + \varkappa^L_{ij}$ to zero, then

$$\frac{M_i}{M_j}\delta Z^R_{ij} + \delta\overline{Z}^L_{ij} = \delta Z^L_{ij} + \delta\overline{Z}^R_{ij}\frac{M_j}{M_i}, \tag{5.75}$$

Inserting the sums for the wave function renormalization constants as functions of one-loop self-energy contributions (equations (4.43) and (4.45)), we get

$$\Sigma^L_{ij}(M_i^2) - \Sigma^L_{ij}(M_j^2) + \frac{1}{M_i}\Sigma^{DL}_{ij}(M_i^2) - \frac{1}{M_j}\Sigma^{DR}_{ij}(M_j^2) = 0. \tag{5.76}$$

An additional condition comes from the terms with $\varkappa^R_{ij} + \overline{\varkappa}^R_{ij}$:

$$\Sigma^R_{ij}(M_i^2) - \Sigma^R_{ij}(M_j^2) - \frac{1}{M_j}\Sigma^{DL}_{ij}(M_j^2) + \frac{1}{M_i}\Sigma^{DR}_{ij}(M_i^2) = 0. \tag{5.77}$$

Inspecting the results for one-loop self-energies given in chapter 3, we see that these conditions are not fulfilled. This time a trivial condition for the coupling constants can not help, since the problems come from the two-point integrals B_0, B_1 and their combination in the functions F^v and F^s. They all depend on momentum and setting first $p^2 = M_i^2$ and then $p^2 = M_j^2$ we can not obtain identical expressions for the different components of the self-energy.

One can hope that the contributions coming from the off-diagonal self-energies are finite, as in the first subsection, but it is not the case. We can easily analyse the divergences occurring in these terms. First one has to notice that there is no momentum dependence in the divergent parts of each component of the self-energy, as explicitly calculated in (3.42) and (3.43). Therefore, even summing over all one-loop diagrams, we will have

$$\begin{aligned}\mathrm{div}[\Sigma^L_{ij}(M_i^2)] &= \mathrm{div}[\Sigma^L_{ij}(M_j^2)],\\ \mathrm{div}[\Sigma^R_{ij}(M_i^2)] &= \mathrm{div}[\Sigma^R_{ij}(M_j^2)],\\ \mathrm{div}[\Sigma^{DL}_{ij}(M_i^2)] &= \mathrm{div}[\Sigma^{DL}_{ij}(M_j^2)],\\ \mathrm{div}[\Sigma^{DR}_{ij}(M_i^2)] &= \mathrm{div}[\Sigma^{DR}_{ij}(M_j^2)].\end{aligned} \tag{5.78}$$

Using the formulas for the non-diagonal wave function renormalization constants (4.44) and (4.46), the divergences in (5.73) and (5.74) will come from

$$\mathrm{div}[M_i \delta Z^L_{ij} + \delta \overline{Z}^R_{ij} M_j] = -2\mathrm{div}[\Sigma^{DL}_{ij}(M_i^2)] \neq 0, \tag{5.79}$$

$$\mathrm{div}[M_i \delta Z^R_{ij} + \delta \overline{Z}^L_{ij} M_j] = -2\mathrm{div}[\Sigma^{DR}_{ij}(M_i^2)] \neq 0. \tag{5.80}$$

To understand why such an approach causes these drawbacks related to divergent terms, one can discuss the operations we make. First start by looking at (5.45) or (5.46). Here we perform a rotation of the renormalized fields. The matrices we use are not UV-finite (5.56). If initially, the field renormalization constants were removing all the UV-divergences, with our additional transformation we introduce part of them back in the free field Lagrange density. The consequence is obvious in (5.64). Here we see that we have non-diagonal counter terms coming with $\not{p}$, but no non-diagonal terms coming with $\mathbf{1}$. This type of divergences resulting from the unrenormalized self-energy can not be absorbed.

Of course the transformation of the fields should be performed in the entire Lagrangian, but for the corrections to the external spinors, just the free part counts. If we add the interaction terms, we encounter also here the divergences reintroduced by the rotation of the fields. Overall, these divergences have to cancel. One should analyse a complete process to be able to check it. Since this is not too handy, we will not choose the fields defined here.

In this section we have seen that one can define Dirac field renormalization constants differing from the wave function renormalization ones by some finite quantities that have to fulfil (5.29). Such a transformation will imply that one has to take into account finite corrections for the external spinors. If we consider the general case (including the decay widths of the unstable particles), a hermiticity constraint on the field renormalization constants will lead to poles in the external line corrections. On the other hand, a supplementary rotation of the fields to re-diagonalize the mass term will bring divergent contributions from the non-diagonal self-energy corrections to external particles.

A summary of all the different choices we analysed is contained in Table 5.1. In the first column of the table we state the choice for the field renormalization constants, in the second one we mention if the hermiticity relation is fulfilled or not, and the last two columns describe the result from contributions of the self-energy on external legs. Here, we present the results for a general theory, with no special properties.

field ren. $(Z^{r\frac{1}{2}})$	hermiticity $\overline{Z}^{r\frac{1}{2}} = \gamma^0 (Z^{r\frac{1}{2}})^\dagger \gamma^0$	$\frac{1}{\not{p}-M_i}\Sigma^r_{ii}(p)u_i(p,s)\Big\vert_{p^2\to M_i^2}$ $\overline{u}_i(p,s)\Sigma^r_{ii}(p)\frac{1}{\not{p}-M_i}\Big\vert_{p^2\to M_i^2}$	$\Sigma^r_{ij}(p)u_j(p,s)\Big\vert_{p^2\to M_j^2}$ $\overline{u}_i(p,s)\Sigma^r_{ij}(p)\Big\vert_{p^2\to M_i^2}$
$Z^{\frac{1}{2}}$	no	0	0
$Z^{\frac{1}{2}} - \varkappa$	no	finite	finite
$Z^{\frac{1}{2}} - \varkappa$	yes	finite + pole term	finite
$Z^{\frac{1}{2}}$-$\varkappa$+re-diag	yes	finite + no/pole term	divergent

Table 5.1: Behaviour of the self-energy contribution to external legs when considering different field renormalization constants.

5.2 Renormalized Free Majorana Lagrangian

For Majorana fields the calculation does not differ too much from the one in the previous section and here we follow similar steps. In the second part of this section where we will discuss the hermiticity condition, we will see that for Majorana particles, the constraint on the field renormalization constants is not as restrictive as for Dirac particles. However, with the re-diagonalization of the mass term method we will still obtain divergences in the corrections to external legs. They will show up for the same reasons as in the Dirac case.

5.2.1 General Renormalized Free Lagrangian

For the unrenormalized Majorana fields we use a transformation similar to (5.6).

We begin by writing it separately for the left- and the right-handed fields:

$$\chi_a^L = \sum_b Z_{ab}^{r\frac{1}{2}L} \chi_b^{rL}, \tag{5.81}$$

$$\chi_a^R = \sum_b Z_{ab}^{r\frac{1}{2}R} \chi_b^{rR}, \tag{5.82}$$

although they are not independent since for Majorana fields $(\chi^L)^C = \chi^R$. For generality, we assume again that the hermiticity condition (4.6) is not necessarily fulfilled and the Dirac conjugated fields are transformed like

$$\overline{\chi_a^L} = \sum_b \overline{\chi_b^{rL}} \overline{Z}_{ba}^{r\frac{1}{2}L}, \tag{5.83}$$

$$\overline{\chi_a^R} = \sum_b \overline{\chi_b^{rR}} \overline{Z}_{ba}^{r\frac{1}{2}R}. \tag{5.84}$$

As for the Dirac case, $Z_{ab}^{r\frac{1}{2}L}$ and $Z_{ab}^{r\frac{1}{2}R}$ are the left and the right part of the field renormalization constant $Z_{ab}^{r\frac{1}{2}}$, and $\overline{Z}_{ba}^{r\frac{1}{2}R}$ and $\overline{Z}_{ba}^{r\frac{1}{2}L}$ are the left and the right components of $\overline{Z}_{ba}^{r\frac{1}{2}}$.

The unrenormalized left and right-handed Majorana fields are related by the condition (2.20). It is unavoidable to require that the renormalized fields are also Majorana, and that implicitly they obey the same condition. Using the properties (2.18) and (2.19) and imposing (2.20) to χ and χ^r, one finds the requirement

$$\begin{aligned} Z_{ab}^{r\frac{1}{2}R} &= \overline{Z}_{ba}^{r\frac{1}{2}L}, \\ \overline{Z}_{ba}^{r\frac{1}{2}R} &= Z_{ab}^{r\frac{1}{2}L}. \end{aligned} \tag{5.85}$$

For the wave function renormalization constants, as a consequence of the additional properties of the Majorana fermion self-energy, this condition was fulfilled ((4.48) and (4.49)). The renormalizing procedure does not (and it should not) change the nature of the particles. Inserting (5.85), the renormalized fields described by (5.81) and (5.82) are in fact defined by

$$\chi_a^L = \sum_b Z_{ab}^{r\frac{1}{2}L} \chi_b^{rL}, \tag{5.86}$$

$$\chi_a^R = \sum_b \overline{Z}_{ba}^{r\frac{1}{2}L} \chi_b^{rR}. \tag{5.87}$$

The conjugated fields become

$$\overline{\chi^L_a} = \sum_b \overline{\chi^{rL}_b}\,\overline{Z}^{r\frac{1}{2}L}_{ba}, \tag{5.88}$$

$$\overline{\chi^R_a} = \sum_b \overline{\chi^{rR}_b}\,Z^{r\frac{1}{2}L}_{ab}. \tag{5.89}$$

From here on, we will not use anymore $Z^{r\frac{1}{2}R}_{ab}$ and $\overline{Z}^{r\frac{1}{2}R}_{ab}$.

The modified Feynman rules presented for the Dirac case are valid for the Majorana one, too. We just have to remember the special feature of the field renormalization constant, i.e.

$$Z^{r\frac{1}{2}}_{ab} = Z^{r\frac{1}{2}L}_{ab}\gamma_L + \overline{Z}^{r\frac{1}{2}L}_{ba}\gamma_R = \delta_{ab} + \frac{1}{2}\delta Z^{rL}_{ab}\gamma_L + \frac{1}{2}\delta\overline{Z}^{rL}_{ba}\gamma_R + \mathcal{O}(\alpha^2), \tag{5.90}$$

$$\overline{Z}^{r\frac{1}{2}}_{ab} = Z^{r\frac{1}{2}L}_{ba}\gamma_L + \overline{Z}^{r\frac{1}{2}L}_{ab}\gamma_R = \delta_{ab} + \frac{1}{2}\delta Z^{rL}_{ba}\gamma_L + \frac{1}{2}\delta\overline{Z}^{rL}_{ab}\gamma_R + \mathcal{O}(\alpha^2). \tag{5.91}$$

If in the Dirac case, we had four different constants (considering the left and right decomposition), now we have just two since

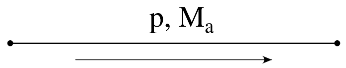
$$Z^{r\frac{1}{2}}_{ab} = \overline{Z}^{r\frac{1}{2}}_{ba}. \tag{5.92}$$

In terms of the renormalized fields and parameters, the Majorana part of the free Lagrangian (2.45) is

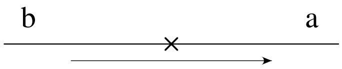
$$\begin{aligned}\mathcal{L}^M_0 = \frac{1}{2}\Bigg(&\sum_a \overline{\chi}^r_a(i\partial\!\!\!/ - M_a)\chi^r_a \\ &+ \sum_{a,b}\overline{\chi}^r_a(i\partial\!\!\!/ - M_a)\frac{1}{2}\delta Z^r_{ab}\chi^r_b + \sum_{a,b}\overline{\chi}^r_b\frac{1}{2}\delta Z^r_{ab}(i\partial\!\!\!/ - M_a)\chi^r_a \\ &+ \sum_{a,b}\overline{\chi}^r_a\delta m_a\delta_{ab}\chi^r_b\Bigg) + \mathcal{O}(\alpha^2).\end{aligned} \tag{5.93}$$

We have replaced the unrenormalized mass by the physical one, defined in chapter 4.

Remembering that the factor $\frac{1}{2}$ is responsable for avoiding the double counting in case of Majorana fermions, we see that the Feynman rules look as for Dirac fermions. The only difference comes from the definition of the Majorana fields, and more exactly from its consequence on the renormalization constant (5.92).



$$iS^r(p) = \frac{i}{p\!\!\!/ - M_a + i\rho}$$



$$i(\not{p} - M_a)\ \frac{1}{2}\delta Z^r_{ab} + i\ \frac{1}{2}\delta Z^r_{ba}(\not{p} - M_b) + i\delta m_a \delta_{ab}$$

As in (5.12), the total self-energy is given by

$$\Sigma^r_{ab}(p) = \Sigma_{ab}(p) - (\not{p} - M_a)\frac{1}{2}\delta Z^r_{ab} - \frac{1}{2}\delta Z^r_{ba}(\not{p} - M_b) - \delta m_a \delta_{ab}. \tag{5.94}$$

Similarly to the previous section, one can also define a connection to the wave function renormalization constants (calculated in 4.1) as in (5.16) and (5.17). The Majorana properties of δZ and $\delta\overline{Z}$ (4.48)−(4.49) and the condition (5.85) imply that $\varkappa_{ab}$ and $\overline{\varkappa}_{ab}$ obey:

$$\begin{cases} \varkappa^R_{ab} = \overline{\varkappa}^L_{ba} \\ \overline{\varkappa}^R_{ab} = \varkappa^L_{ba} \end{cases} \Leftrightarrow \quad \varkappa_{ab} = \overline{\varkappa}_{ba}. \tag{5.95}$$

We can identify

$$\Sigma^r_{ab}(p) = \hat{\Sigma}_{ab}(p) + \mathcal{R}_{ab}(p), \tag{5.96}$$

where

$$\mathcal{R}_{ab}(p) = \frac{1}{2}(\not{p} - M_a)\varkappa_{ab} + \frac{1}{2}\varkappa_{ba}(\not{p} - M_b). \tag{5.97}$$

To evaluate the contribution of $\Sigma^r_{ab}(p)$ on external spinors, we can take the relations for Dirac fermions (5.24)−(5.27) and replace $\varkappa^R_{ab}$ and $\overline{\varkappa}^R_{ab}$ as in (5.95). We obtain

- for $a \neq b$

$$\begin{aligned} \Sigma^r_{ab}(p)u_b(p,s)\Big|_{p^2\to M_b^2} &= \mathcal{R}_{ab}(p)u_b(p,s)\Big|_{p^2\to M_b^2} \\ &= \frac{1}{2}\Big((-M_a\varkappa^L_{ab} + M_b\overline{\varkappa}^L_{ba})\gamma_L \\ &\qquad +(-M_a\overline{\varkappa}^L_{ba} + M_b\varkappa^L_{ab})\gamma_R\Big)\, u_b(p,s)\Big|_{p^2\to M_b^2} \end{aligned} \tag{5.98}$$

$$\begin{aligned} \overline{u}_a(p,s)\Sigma^r_{ab}(p)\Big|_{p^2\to M_a^2} &= \overline{u}_a(p,s)\mathcal{R}_{ab}\Big|_{p^2\to M_a^2} \\ &= \overline{u}_a(p,s)\frac{1}{2}\Big((-M_b\varkappa^L_{ba} + M_a\overline{\varkappa}^L_{ab})\gamma_L \\ &\qquad +(-M_b\overline{\varkappa}^L_{ab} + M_a\varkappa^L_{ba}\gamma_R\Big)\Big|_{p^2\to M_a^2} \end{aligned} \tag{5.99}$$

and

- for $a = b$

$$
\begin{aligned}
\frac{1}{\not{p}-M_a}\Sigma^r_{aa}(p)u_a(p,s)\Big|_{p^2\to M_a^2} &= \frac{1}{\not{p}-M_a}\left(\hat{\Sigma}_{aa}(p)+\mathcal{R}_{aa}(p)\right)u_a(p,s)\Big|_{p^2\to M_a^2} \\
&= \frac{1}{\not{p}-M_a}\left(-i\frac{1}{2}\Gamma_a\right)u_a(p,s)\Big|_{p^2\to M_a^2} \\
&\quad + \frac{1}{2}\left(\varkappa^L_{aa}+\overline{\varkappa}^L_{aa}\right)u_a(p,s)\Big|_{p^2\to M_a^2} \qquad (5.100) \\
&\quad +\frac{1}{2}M_a\frac{1}{\not{p}-M_a}\left((\overline{\varkappa}^L_{aa}-\varkappa^L_{aa})\gamma_L-(\overline{\varkappa}^L_{aa}-\varkappa^L_{aa})\gamma_R\right)u_a(p,s)\Big|_{p^2\to M_a^2}
\end{aligned}
$$

$$
\begin{aligned}
\overline{u}_a(p,s)\Sigma^r_{aa}(p)\frac{1}{\not{p}-M_a}\Big|_{p^2\to M_a^2} &= \overline{u}_a(p,s)\left(\hat{\Sigma}_{aa}(p)+\mathcal{R}_{aa}(p)\right)\frac{1}{\not{p}-M_a}\Big|_{p^2\to M_a^2} \\
&= \overline{u}_a(p,s)\left(-i\frac{1}{2}\Gamma_a\right)\frac{1}{\not{p}-M_a}\Big|_{p^2\to M_a^2} \\
&\quad + \overline{u}_a(p,s)\Big|_{p^2\to M_a^2}\frac{1}{2}\left(\overline{\varkappa}^L_{aa}+\varkappa^L_{aa}\right) \qquad (5.101) \\
&\quad +\overline{u}_a(p,s)\frac{1}{2}M_a\left((\overline{\varkappa}^L_{aa}-\varkappa^L_{aa})\gamma_L-(\overline{\varkappa}^L_{aa}-\varkappa^L_{aa})\gamma_R\right)\frac{1}{\not{p}-M_a}\Big|_{p^2\to M_a^2}.
\end{aligned}
$$

The discussion of the term proportional to the decay width Γ_a would be identical to the one in the previous section, therefore we do not repeat it here.

The condition to cancel the other pole term in (5.101) is

$$\varkappa^L_{aa} = \overline{\varkappa}^L_{aa}. \qquad (5.102)$$

The restriction on field renormalization constants for Dirac fermions (5.30) is reduced to one equation:

$$\delta Z^{rL}_{aa} = \delta\overline{Z}^{rL}_{aa} + \left(\delta Z^L_{aa} - \delta\overline{Z}^L_{aa}\right), \qquad (5.103)$$

and the diagonal self-energy corrections to external legs are

$$\frac{1}{\not{p}-M_a}\Sigma^r_{aa}(p)u_a(p,s)\Big|_{p^2\to M_a^2} = \varkappa^L_{aa}u_a(p,s)\Big|_{p^2\to M_a^2}, \qquad (5.104)$$

$$\overline{u}_a(p,s)\Sigma^r_{aa}(p)\frac{1}{\not{p}-M_a}\Big|_{p^2\to M_a^2} = \overline{u}_a(p,s)\Big|_{p^2\to M_a^2}\varkappa^L_{aa}. \qquad (5.105)$$

5.2.2 Hermitian Renormalized Free Lagrangian

If in addition, we require that $Z^{r\frac{1}{2}}_{ab}$ and $\overline{Z}^{r\frac{1}{2}}_{ab}$ are related by hermiticity (equation (4.6)), from (5.85) we obtain

$$Z^{r\frac{1}{2}L}_{ab} = (\overline{Z}^{r\frac{1}{2}L}_{ba})^*. \tag{5.106}$$

The renormalized left- and right-handed Majorana fields are now given by

$$\chi^L_a = \sum_b Z^{r\frac{1}{2}L}_{ab} \chi^{rL}_b, \tag{5.107}$$

$$\chi^R_a = \sum_b (Z^{r\frac{1}{2}L}_{ab})^* \chi^{rR}_b. \tag{5.108}$$

The condition that $\varkappa$ and $\overline{\varkappa}$ have to fulfil in order to assure the hermiticity of the field renormalization constants is

$$\overline{\varkappa}^L_{ba} - (\varkappa^L_{ab})^* = \delta\overline{Z}^L_{ba} - (\delta Z^L_{ab})^*. \tag{5.109}$$

For Dirac fermions, the hermiticity condition implied two independent relations for the components of $\varkappa$ and $\overline{\varkappa}$ (see (5.39)). Since the Majorana condition relates left- and right-handed components, the two relations are not independent anymore. Here they are just the complex conjugated of each other and therefore we are left with one, namely (5.109).

Together with (5.102), we have for the diagonal part

$$\varkappa^L_{aa} - (\varkappa^L_{aa})^* = \delta\overline{Z}^L_{aa} - (\delta Z^L_{aa})^*. \tag{5.110}$$

This equation is telling us that the wave function renormalization constants δZ^L_{aa} and $\delta\overline{Z}^L_{aa}$ should have equal real parts and that the imaginary part of the field renormalization constants is

$$\mathrm{Im}[\delta Z^{rL}_{aa}] = \frac{1}{2}\left(\mathrm{Im}[\delta Z^L_{aa}] - \mathrm{Im}[\delta\overline{Z}^L_{aa}]\right). \tag{5.111}$$

With the help of the complex conjugated expression of (5.110),

$$(\varkappa^L_{aa})^* - \varkappa^L_{aa} = (\delta\overline{Z}^L_{aa})^* - \delta Z^L_{aa}, \tag{5.112}$$

we can write the conditions for the real part as

$$\delta\overline{Z}^L_{aa} - \delta Z^L_{aa} = (\delta Z^L_{aa} - \delta\overline{Z}^L_{aa})^*. \tag{5.113}$$

For the imaginary part:

$$\delta Z^{rL}_{aa} - (\delta Z^{rL}_{aa})^* = \frac{1}{2}\left(\delta Z^{L}_{aa} - (\delta Z^{L}_{aa})^* - \delta\overline{Z}^{L}_{aa} + (\delta\overline{Z}^{L}_{aa})^*\right). \tag{5.114}$$

At first order, from (4.33), for $\beta_a = 0$, the restriction (5.113) can be fulfilled if

$$\Sigma^{DL}_{aa}(M_a^2) - \Sigma^{DR}_{aa}(M_a^2) - (\Sigma^{DL}_{aa}(M_a^2) - \Sigma^{DR}_{aa}(M_a^2))^* = 0. \tag{5.115}$$

This means that in a model (like an extension of the Standard Model) for which (3.71) holds, we are able to shift the wave function renormalization constants by the same amount and define hermitian Majorana field renormalization constants. Self-energy contributions from external legs are non-zero, but finite.

5.2.3 Re-diagonalized Mass Term Approach

To analyse the modifications introduced by the renormalized fields for the Majorana sector, we start from the following form of the mass term:

$$\mathcal{L}^{M}_{mass} = -\frac{1}{2}\sum_a \overline{(\chi^L_a)^C} m_a \chi^L_a + h.c. = -\frac{1}{2}\sum_a \overline{\chi^R_a} m_a \chi^L_a + h.c.. \tag{5.116}$$

To switch between left- or right-handed charged conjugated fields, we use (2.20).

Replacing the fields with the renormalized ones given in (5.86) and (5.89), the Lagrangian mass term changes to

$$\mathcal{L}^{M}_{mass} = -\frac{1}{2}\sum_{a,b,c} \overline{\chi^{rR}_b} Z^{r\frac{1}{2}L}_{ab} m_a Z^{r\frac{1}{2}L}_{ac} \chi^{rL}_c + h.c.. \tag{5.117}$$

The factor to be transformed is now of type $Z^T m Z$ and, since $m^T = m$, symmetric. To rediagonalize it we need just one unitary matrix [Zum62], denoted here by A.

$$\mathcal{M}_d = \sum_{a,b,c} A_{bd} Z^{r\frac{1}{2}L}_{ab} m_a Z^{r\frac{1}{2}L}_{ac} A_{cd} \tag{5.118}$$

As a consequence, the transformation relation for the fields is

$$\chi^L_a = \sum_{b,c} Z^{r\frac{1}{2}L}_{ab} A_{bc} \chi'^L_c, \tag{5.119}$$

$$\chi^R_a = \sum_{b,c} \overline{Z}^{r\frac{1}{2}L}_{ba} (A_{bc})^* \chi'^R_c. \tag{5.120}$$

χ', with the left and right components given by the previous relations and related by (2.20), remains a Majorana field. The mass term gains the diagonal form:

$$\begin{aligned}\mathcal{L}^M_{mass} &= -\frac{1}{2}\sum_a \overline{(\chi'^L_a)^C}\mathcal{M}_a\chi'^L_a + h.c.\\ &= -\frac{1}{2}\sum_a \overline{(\chi'^L_a)^C}M_a\chi'^L_a - \frac{1}{2}\sum_a \overline{(\chi'^L_a)^C}\delta M_a\chi'^L_a + \mathcal{O}(\alpha^2) + h.c.,\end{aligned} \tag{5.121}$$

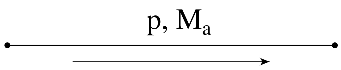
where M_a and not $\mathcal{M}_a$, is the physical mass. We assume that (5.106) is fulfilled, such that the 'h.c.' part is really the hermitian conjugated one.

We will not write the explicit expressions for A or δM. We will restrict the following discussion just to the steps necessary to show that this approach leads to divergent contributions from self-energy corrections on external legs, for the same reasons as in the Dirac case.

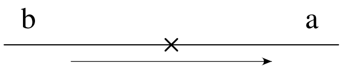
Looking at the kinetic term, we can see that the structure is similar to the one for Dirac fields. We can directly write the Majorana free Lagrangian as:

$$\begin{aligned}\mathcal{L}^M_0 = \frac{1}{2}\Bigg(&\sum_a \overline{\chi'_a}(i\overleftrightarrow{\partial\!\!\!/} - M_a)\chi'_a + i\sum_{a,b}\overline{\chi'_a}\frac{1}{2}\delta Z^r_{ba}\overleftrightarrow{\partial\!\!\!/}\chi'_b + i\sum_{a,b}\overline{\chi'_a}\overleftrightarrow{\partial\!\!\!/}\frac{1}{2}\delta Z^r_{ab}\chi'_b\\ &-\sum_a \overline{\chi'_a}\delta M_a\chi'_a\Bigg) + \mathcal{O}(\alpha^2).\end{aligned} \tag{5.122}$$

The Feynman rules for the Majorana propagator are:



$$iS'(p) = \frac{i}{p\!\!\!/ - M_a}$$



$$ip\!\!\!/\ \frac{1}{2}\delta Z^r_{ab} + i\ \frac{1}{2}\delta Z^r_{ba}p\!\!\!/ - i\delta M_a\delta_{ab}$$

As expected, the Majorana counter terms have the same structure as the Dirac ones. If we consider the action of the self-energy on external spinors, we will end with the same divergent contributions as in 5.1.3. The difference will come in the number of restrictions. They are reduced by a factor of two due to (5.85). But even so, we will not be able to absorb the divergences coming from the non-diagonal self-energy scalar structure and they will have to be cancelled by other interaction terms of the Lagrangian.

In section 5.1 and 5.2, we introduced the renormalized mass and fields for Dirac and Majorana fermions. We related the first ones to the wave function renormalization constants calculated in 4.1 and we assimilated the difference in $\varkappa$ and $\overline{\varkappa}$.

By analysing the renormalized free Lagrangian and the total self-energy, we have observed how much freedom one has in choosing $\varkappa$ or performing additional transformations on the renormalized fields.

In conclusion, from the technical point of view, one can shift the renormalization constants by some finite terms as long as one prevents reintroducing poles in the diagonal self-energy contributions to external spinors. When shifting them by terms that contain divergences, the effect is more destructive. These divergences will show back in the terms from where they have initially been removed by the renormalization procedure.

From now on, we will use the field renormalization constants, as defined in 5.1.1 and 5.2.1. When possible, we will try to impose the hermiticity condition. Nevertheless, one has to remember that the divergences of the wave function renormalization constants were related by hermiticity (4.57). Since $\varkappa$ and $\overline{\varkappa}$ are finite, we also have

$$\begin{aligned} \text{div}[\delta\overline{Z}^{rL}_{xy}] &= (\text{div}[\delta Z^{rL}_{yx}])^*, \\ \text{div}[\delta\overline{Z}^{rR}_{xy}] &= (\text{div}[\delta Z^{rR}_{yx}])^*, \end{aligned} \tag{5.123}$$

independent of the presence of absorptive imaginary parts in the self-energy.

5.3 Remarks on CPT Invariance

Along this chapter, we have proposed different definitions for the field renormalization constants. Part of the constants were not related by hermiticity and it might seem that the lack of this symmetry can damage other fundamental properties of the theory, like CPT invariance. In particular, since particles receive corrections from $Z^{r\frac{1}{2}}$ and antiparticles from $\overline{Z}^{r\frac{1}{2}}$, one might suspect that decay rates or cross sections of particles will be different from the ones for antiparticles once higher order corrections are included. Here we show that it is not the case. We follow a similar argumentation to [Esp02], based on the optical theorem.

The non-hermitian renormalization constants appear due to the presence of unstable particles in the theory and the imaginary parts that come in their self-energies. The optical theorem relates the imaginary part of the forward scattering amplitude to the total cross section of the scatterer. For a transition amplitude (not necessarily at one-loop), we have

$$2\text{Im}[\mathcal{T}(\psi_x \to \psi_x)] = \sum_f \int d\Pi_f \left| \mathcal{T}(\psi_x \to f) \right|^2. \tag{5.124}$$

By ψ_x we mean any fermion, Dirac or Majorana and by f any possible combination of particles in the final state, i.e. f represents any possible decay channel of ψ_x. $d\Pi$

is taken as

$$d\Pi = (2\pi)^4 \delta\left(p_x - \sum_k p_k\right) \prod_k \frac{d^3 p_k}{(2\pi)^3 2p_k^0}, \tag{5.125}$$

where p_x is the momentum of the initial particle and k counts the particles in the final state. If Γ_x is the total decay rate of the considered fermion, then

$$\Gamma_x = \frac{1}{M_x} \text{Im}[\mathcal{T}(\psi_x \to \psi_x)]. \tag{5.126}$$

For the CPT conjugated process, the theorem asserts

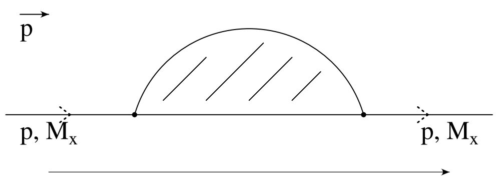
$$2\text{Im}[\mathcal{T}(\overline{\psi_x} \to \overline{\psi_x})] = \sum_f \int d\Pi_f \left| \mathcal{T}(\overline{\psi_x} \to f) \right|^2, \tag{5.127}$$

and

$$\Gamma_{\overline{x}} = \frac{1}{M_x} \text{Im}[\mathcal{T}(\overline{\psi_x} \to \overline{\psi_x})]. \tag{5.128}$$

To prove that the two decay rates are equal, it is sufficient to show that the imaginary parts of $\mathcal{T}(\psi_x \to \psi_x)$ and $\mathcal{T}(\overline{\psi_x} \to \overline{\psi_x})$ are equal. In the following, we will do a bit more. We will prove that individual contributions to the two amplitudes are equal, i.e. the relation is true for every Feynman diagram, independent of the process in the loop.

We start with the transition amplitude $\mathcal{T}(\psi_x \to \psi_x)$. A generic Feynman diagram is

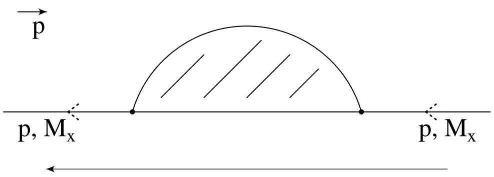


$$\mathcal{T}(\psi_x \to \psi_x) = \overline{u}_x(p,s)(-i\Gamma^r_{xx}(p))u_x(p,s).$$

$-i\Gamma^r_{xx}(p)$ describes whatever the shaded part may include. We will assume a general decomposition like we had for the self-energy:

$$\Gamma^r_{xx}(p) = \not{p}\gamma_L \Gamma^L_{xx}(p^2) + \not{p}\gamma_R \Gamma^R_{xx}(p^2) + \gamma_L \Gamma^{DL}_{xx}(p^2) + \gamma_R \Gamma^{DR}_{xx}(p^2). \tag{5.129}$$

The diagram for antiparticles is



$$\mathcal{T}\left(\overline{\psi_x} \to \overline{\psi_x}\right) = -\overline{v}_x(p,s)(-i\Gamma^r_{xx}(-p))v_x(p,s).$$

The extra minus-sign comes from the interchange of the fermion operators. To probe the equality of the two amplitudes, we need to remember some relations for the Dirac spinors and for the traces of Dirac matrices. For the spinors,

$$u_x(p,s)\overline{u}_x(p,s) = \frac{1}{2}(\not{p} + M_x)(1+\gamma_5\not{n}), \tag{5.130}$$
$$v_x(p,s)\overline{v}_x(p,s) = \frac{1}{2}(\not{p} - M_x)(1+\gamma_5\not{n}), \tag{5.131}$$

where $n^\mu = \left(\frac{\vec{p}\cdot\vec{n}}{M_x},\, \vec{n} + \frac{\vec{p}\cdot\vec{n}}{M_x(E_p+M_x)}\vec{p}\right)$ is the polarisation vector along the direction $\vec{n}$. For Dirac matrices, we need to know that

$$\mathrm{Tr}(\gamma_5) = \mathrm{Tr}(\gamma_\mu\gamma_5) = \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_5) = 0, \tag{5.132}$$

and that the trace of any odd product of Dirac matrices is zero. With these relations in mind, one can follow the proof:

$$\begin{aligned}
\mathcal{T}\left(\overline{\psi_x} \to \overline{\psi_x}\right) &= -\overline{v}_x(p,s)(-i\Gamma^r_{xx}(-p))v_x(p,s) \\
&= -\mathrm{Tr}\left(\frac{1}{2}(\not{p} - M_x)(1+\gamma_5\not{n})\right. \\
&\qquad \left.(-i)\left(-\not{p}\gamma_L\Gamma^L_{xx}(p^2) - \not{p}\gamma_R\Gamma^R_{xx}(p^2) + \gamma_L\Gamma^{DL}_{xx}(p^2) + \gamma_R\Gamma^{DR}_{xx}(p^2)\right)\right) \\
&= -\mathrm{Tr}\left(\frac{1}{2}(\not{p} - M_x)(-i)\frac{1}{2}\left(-\not{p}\Gamma^L_{xx}(p^2) - \not{p}\Gamma^R_{xx}(p^2) + \Gamma^{DL}_{xx}(p^2) + \Gamma^{DR}_{xx}(p^2)\right)\right) \\
&= -\mathrm{Tr}\left(\frac{1}{2}(-\not{p} - M_x)(-i)\frac{1}{2}\left(\not{p}\Gamma^L_{xx}(p^2) + \not{p}\Gamma^R_{xx}(p^2) + \Gamma^{DL}_{xx}(p^2) + \Gamma^{DR}_{xx}(p^2)\right)\right) \\
&= \mathrm{Tr}\left(\frac{1}{2}(\not{p} + M_x)(1+\gamma_5\not{n})(-i\Gamma^r_{xx}(p))\right) \\
&= \overline{u}_x(p,s)(-i\Gamma^r_{xx}(p))u_x(p,s) \\
&= \mathcal{T}\left(\psi_x \to \psi_x\right).
\end{aligned} \tag{5.133}$$

In the second line, we have used (5.131) and (5.129), in the third one we eliminated the terms with γ_5, then we have replaced $\not{p}$ with $-\not{p}$ (since the minus-sign comes just for odd products of Dirac matrices) and finally we inserted back γ_5 factors, such that with (5.130) we can reconstruct the amplitude for the particle.

Connecting now (5.126) and (5.128), we conclude that

$$\Gamma_{\overline{x}} = \Gamma_x. \tag{5.134}$$

In this proof we have not used any special property of $\Gamma^r_{xx}(p)$ and therefore, there is no special requirement for the renormalization constants. The CPT symmetry is preserved independent of their hermiticity.

Chapter 6

Renormalization of the Interaction Lagrangian

In chapter 5 we investigated the consequences of renormalization of fermionic fields in the free Lagrangian. It is obvious that the renormalization of this part is entirely based on the field and mass renormalization constants. When we move to the interaction terms, additional renormalization constants play a role, namely the ones coming with the electric charge, with coupling constants and with the bosonic field.

Since it is not our main theme, we will consider the renormalization constants for the couplings and the bosonic fields known. The only exception will be the part in the coupling that involves the fermion mixing matrices, since its determination is strongly related to the field renormalization. However, we do not determine it in this chapter. Here, we simply analyse the modification introduced by the renormalization of parameters in the interaction Lagrangian and at the end we give the Feynman rules for fermion propagators and vertices that involve fermion interactions. With the formal, complete renormalized Lagrangian, in the last section, we point out the contributions to the one-loop amplitude of a generic process with fermion mixing.

6.1 Interaction Terms with Dirac Fermions

In this section, we consider the interaction of Dirac fermions with bosons as described in the general Lagrangian of section 2.3. We analyse first the modifications in the terms with vector bosons and then we follow the same procedure for the scalar ones.

6.1.1 Interaction Terms with Vector Bosons

We start by writing the terms of the Lagrangian using the left and right decomposition of the fields. As introduced in section 2.3, such a term can be written in the general form as

$$\begin{aligned} \mathcal{L}^D_{vb} &= e\sum_{i,j,v}\overline{\psi}_i\gamma^\mu(g^L_{ij,v}\gamma_L + g^R_{ij,v}\gamma_R)\psi_j\phi_{v,\mu} \\ &= e\sum_{i,j,v}\overline{\psi^L_i}\gamma^\mu g^L_{ij,v}\gamma_L\psi^L_j\phi_{v,\mu} + e\sum_{i,j,v}\overline{\psi^R_i}\gamma^\mu g^R_{ij,v}\gamma_R\psi^R_j\phi_{v,\mu}. \end{aligned} \tag{6.1}$$

The index vb coming with the Lagrangian symbol stands for vector boson. Since the charged current term and the neutral one look similar, for the start, we analyse them together. However, keep in mind that for the charged current part, the two fermionic fields in (6.1) are referring to different fermion types (e.g. charged leptons and neutrinos or up and down-type quarks).

To have a complete picture we assume that all quantities appearing in the Lagrangian were renormalized. For the bosonic field we take

$$\phi_{v,\mu} = Z_v^{\frac{1}{2}}\phi^r_{v,\mu}. \tag{6.2}$$

All renormalization constant can be expanded in terms of the fine-structure constant α and for $Z_v^{\frac{1}{2}}$:

$$Z_v^{\frac{1}{2}} = 1 + \frac{1}{2}\delta Z_v + \mathcal{O}(\alpha^2). \tag{6.3}$$

Another parameter that gets corrected is the electric charge:

$$e = Z_e e^r, \tag{6.4}$$

with

$$Z_e = 1 + \frac{\delta e}{e^r} + \mathcal{O}(\alpha^2). \tag{6.5}$$

Depending on the interacting particles, the coupling constants $g^L_{ij,v}$ and $g^R_{ij,v}$ can suffer similar modifications due to the renormalization of their parameters. We assume a general transformation of type

$$\begin{aligned} g^L_{ij,v} &= Z^L_{ij,v}g^{rL}_{ij,v}, \\ g^R_{ij,v} &= Z^R_{ij,v}g^{rR}_{ij,v}. \end{aligned} \tag{6.6}$$

Remark that there is no summation over indices. $Z^L_{ij,v}$ and $Z^R_{ij,v}$ totals all constants coming with the renormalization of $g^L_{ij,v}$ or $g^R_{ij,v}$. For the Standard Model, $Z_{ij,v}$ can include the renormalization constant for the weak mixing angle and for the fermion (quark) mixing matrix.

Now we will split the renormalization constant for the coupling into two pieces. We want to distinguish the constant that comes with the renormalization of parameters that carry explicit fermion indices (e.g. the mixing matrix), from the one that renormalizes the other factors in the coupling. By 'other factors', we mean the common factors (like the weak mixing angle) that do not change when i or j indicate different flavours of the same type of fermions. The notation aesthetically worsens, but this splitting will be helpful when moving to particular cases.

We choose $Z_{(ij,v)}$ to denote the constant that renormalizes the common factors and $Z^r_{ij,v}$ to denote the one for the matrix. The subscript (ij,v) should be understood as an indication of the vertex type (e.g. a vertex with incoming down-type quarks, outgoing up ones and W). $Z_{(ij,v)}$ is independent of the values i or j as long we do not change the type of fermions, while $Z^r_{ij,v}$ is. Using the left and right decomposition, we write

$$\begin{aligned} Z^L_{ij,v} &= Z^L_{(ij,v)} Z^{rL}_{ij,v}, \\ Z^R_{ij,v} &= Z^R_{(ij,v)} Z^{rR}_{ij,v}, \end{aligned} \tag{6.7}$$

with

$$\begin{aligned} Z^{L/R}_{(ij,v)} &= 1 + \delta Z^{L/R}_{(ij,v)} + \mathcal{O}(\alpha^2), \\ Z^{rL/R}_{ij,v} &= 1 + \delta Z^{rL/R}_{ij,v} + \mathcal{O}(\alpha^2). \end{aligned} \tag{6.8}$$

If we refer to the Standard Model (Table 2.1), than $Z_{(ij,v)}$ can be equal to the weak mixing angle renormalization constant (for a vertex with Z or W bosons) and $Z_{ij,v}$ with the one for the quark mixing matrix (if we take W in the vertex). Explicitly, for the vertex with incoming down-type quarks denoted by d_α, outgoing up-type ones u_i and a W,

$$g^L_{i\alpha,W} = \frac{1}{\sqrt{2} s_W} V_{i\alpha} = \frac{1}{\sqrt{2} Z_{s_W} s^r_W} Z^{\text{CKM}}_{i\alpha} V^r_{i\alpha},$$

where we can identify

$$\begin{aligned} Z^L_{(i\alpha,W)} &= \frac{1}{Z_{s_W}}, \\ Z^{rL}_{i\alpha,W} &= Z^{\text{CKM}}_{i\alpha}. \end{aligned}$$

A detailed analysis of the case will be found in the next chapter. Here we state it just as an example for the identification of the two different constants $Z_{(ij,v)}$ and $Z^r_{ij,v}$.

As for the kinetic part, it is enough to detail just one of the terms of the Lagrangian. We choose again the part with left-handed fields and at the end, we fit the result to the right-handed ones, too. The renormalized fields are given as in (5.7) and (5.6) by

$$\begin{aligned}\overline{\psi_i^L} &= \sum_k \overline{\psi_k^{rL}} \overline{Z}_{ki}^{r\frac{1}{2}L}, \\ \psi_j^L &= \sum_l Z_{jl}^{r\frac{1}{2}L} \psi_l^{rL}.\end{aligned} \tag{6.9}$$

Inserting in (6.1) the expressions for all the transformed fields and also for the coupling constant, we get

$$\mathcal{L}_{vb}^{D,L} = Z_e e^r \sum_{\substack{i,j,v\\k,l}} \overline{\psi_k^{rL}} \overline{Z}_{ki}^{r\frac{1}{2}L} \left(\gamma^\mu Z_{(ij,v)}^L Z_{ij,v}^{rL} g_{ij,v}^{rL} \gamma_L\right) Z_{jl}^{r\frac{1}{2}L} \psi_l^{rL} \left(Z_v^{\frac{1}{2}} \phi_{v,\mu}^r\right). \tag{6.10}$$

As a consequence of the renormalization of the fields, we can define now a transformed coupling constant:

$$G_{kl,v}^{rL} = Z_e \sum_{i,j} \overline{Z}_{ki}^{r\frac{1}{2}L} Z_{(ij,v)}^L Z_{ij,v}^{rL} g_{ij,v}^{rL} Z_{jl}^{r\frac{1}{2}L} Z_v^{\frac{1}{2}}, \tag{6.11}$$

such that

$$\mathcal{L}_{vb}^{D,L} = e^r \sum_{i,j,v} \overline{\psi_i^{rL}} \gamma^\mu G_{ij,v}^{rL} \gamma_L \psi_j^{rL} \phi_{v,\mu}^r. \tag{6.12}$$

For an evaluation of the transformed coupling in first order in α, we have to insert the expansion of the renormalization constants. (6.11) can be rewritten as

$$\begin{aligned}G_{kl,v}^{rL} &= g_{kl,v}^{rL} + \frac{\delta e}{e^r} g_{kl,v}^{rL} + g_{kl,v}^{rL} \frac{1}{2}\delta Z_v + \delta Z_{(kl,v)}^L g_{kl,v}^{rL} + \delta Z_{kl,v}^{rL} g_{kl,v}^{rL} \\ &\quad + \sum_i \frac{1}{2}\delta \overline{Z}_{ki}^{rL} g_{il,v}^{rL} + \sum_j g_{kj,v}^{rL} \frac{1}{2}\delta Z_{jl}^{rL} + \mathcal{O}(\alpha^2) \\ &= g_{kl,v}^{rL} + \delta g_{kl,v}^L + \delta g_{kl,v}^{rL} + \mathcal{O}(\alpha^2).\end{aligned} \tag{6.13}$$

We denote by $\delta g_{kl,v}^L$ the corrections resulting from the renormalization of all the other parameters except the fermion fields and the explicit fermion flavour depending parameters in the coupling. $\delta g_{kl,v}^{rL}$ is the part that absorbs the fermion mixing. Therefore,

$$\delta g_{kl,v}^L = \left(\delta Z_{(kl,v)}^L + \frac{\delta e}{e^r} + \frac{1}{2}\delta Z_v\right) g_{kl,v}^{rL}, \tag{6.14}$$

$$\delta g_{kl,v}^{rL} = \delta Z_{kl,v}^{rL} g_{kl,v}^{rL} + \sum_i \frac{1}{2}\delta \overline{Z}_{ki}^{rL} g_{il,v}^{rL} + \sum_j g_{kj,v}^{rL} \frac{1}{2}\delta Z_{jl}^{rL}. \tag{6.15}$$

For the coupling coming with the right-handed fields we just have to exchange left with right in the renormalization constants, then

$$\begin{aligned} G^{rR}_{kl,v} &= g^{rR}_{kl,v} + \frac{\delta e}{e^r} g^{rR}_{kl,v} + g^{rR}_{kl,v}\frac{1}{2}\delta Z_v + \delta Z^{R}_{(kl,v)} g^{rR}_{kl,v} + \delta Z^{rR}_{kl,v} g^{rR}_{kl,v} \\ &\quad + \sum_i \frac{1}{2}\delta \overline{Z}^{rR}_{ki}\, g^{rR}_{il,v} + \sum_j g^{rR}_{kj,v}\frac{1}{2}\delta Z^{rR}_{jl} + \mathcal{O}(\alpha^2) \\ &= g^{rR}_{kl,v} + \delta g^{R}_{kl,v} + \delta g^{rR}_{kl,v} + \mathcal{O}(\alpha^2), \end{aligned} \tag{6.16}$$

with

$$\delta g^{R}_{kl,v} = \left(\delta Z^{R}_{(kl,v)} + \frac{\delta e}{e^r} + \frac{1}{2}\delta Z_v\right) g^{rR}_{kl,v}, \tag{6.17}$$

$$\delta g^{rR}_{kl,v} = \delta Z^{rR}_{kl,v} g^{rR}_{kl,v} + \sum_i \frac{1}{2}\delta \overline{Z}^{rR}_{ki}\, g^{rR}_{il,v} + \sum_j g^{rR}_{kj,v}\frac{1}{2}\delta Z^{rR}_{jl}. \tag{6.18}$$

At this point, one can make an additional remark. In chapter 5 we decided not to choose the formalism that includes the re-diagonalization of the mass term (section 5.1.3). In case we do it, instead of relation (6.9), we have

$$\begin{aligned} \overline{\psi^L_i} &= \sum_{k,l} \overline{\psi'^L_k} (P^L_{lk})^* \overline{Z}^{r\frac{1}{2}L}_{li}, \\ \psi^L_j &= \sum_{m,n} Z^{r\frac{1}{2}L}_{jm} O^L_{mn} \psi'^L_n, \end{aligned} \tag{6.19}$$

where O^L and P^L are the unitary matrices required in the diagonalization of the two fermionic fields in the vertex. Here, we wrote the ones coming with the left-handed fields. The right ones will look like in (5.48). The presence of the fields ψ' in the interaction Lagrangian, forces us to include the matrices O and P in the generic coupling G^r. These changes affect $\delta g^{rL}_{kl,v}$ and $\delta g^{rR}_{kl,v}$. They become:

$$\delta g^{rL}_{kl,v} = \delta Z^{rL}_{kl,v} g^{rL}_{kl,v} + \sum_i \left((\delta P^L_{ik})^* + \frac{1}{2}\delta \overline{Z}^{rL}_{ki}\right) g^{rL}_{il,v} + \sum_j g^{rL}_{kj,v}\left(\frac{1}{2}\delta Z^{rL}_{jl} + \delta O^L_{jl}\right), \tag{6.20}$$

$$\delta g^{rR}_{kl,v} = \delta Z^{rR}_{kl,v} g^{rR}_{kl,v} + \sum_i \left((\delta P^R_{ik})^* + \frac{1}{2}\delta \overline{Z}^{rR}_{ki}\right) g^{rR}_{il,v} + \sum_j g^{rR}_{kj,v}\left(\frac{1}{2}\delta Z^{rR}_{jl} + \delta O^R_{jl}\right). \tag{6.21}$$

The divergences reintroduced in the free part are found, as expected, in the interaction one, too. Contributions like the ones here appear in all terms describing the interaction of fermions. Because we do not intend to use this formalism, we will not write it for the other couplings. The transformation rule is simple anyway: wherever there is a fermion field renormalization constant, the contribution from the corresponding unitary matrix will be added.

The expressions given in this subsection are valid in general, for both, charged and neutral current interactions. We will give a more detailed description when considering particular cases. Our main aim is the renormalization of fermion mixing matrices and we will study such interaction terms in the separate chapters dedicated to the quark and neutrino mixing matrices.

6.1.2 Interaction Terms with Scalar Bosons

The Lagrangian term describing the interaction of Dirac fermions with scalar bosons will have similar modifications:

$$\begin{aligned}\mathcal{L}^D_{sb} &= e\sum_{i,j,s}\overline{\psi}_i(c^L_{ij,s}\gamma_L + c^R_{ij,s}\gamma_R)\psi_j\phi_s \\ &= e\sum_{i,j,s}\overline{\psi^R_i}c^L_{ij,s}\gamma_L\psi^L_j\phi_s + e\sum_{i,j,s}\overline{\psi^L_i}c^R_{ij,s}\gamma_R\psi^R_j\phi_s \\ &= e^r\sum_{i,j,s}\overline{\psi^{rR}_i}C^{rL}_{ij,s}\gamma_L\psi^{rL}_j\phi^r_s + e^r\sum_{i,j,s}\overline{\psi^{rL}_i}C^{rR}_{ij,s}\gamma_R\psi^{rR}_j\phi^r_s,\end{aligned} \tag{6.22}$$

where ϕ^r_s is the renormalized scalar field:

$$\phi_s = Z_s^{\frac{1}{2}}\phi^r_s. \tag{6.23}$$

The coupling constants were redefined to include, besides their own renormalization constant $Z_{ij,s}$, the factors coming from the renormalization of the fields:

$$C^{rL}_{kl,s} = Z_e\sum_{i,j}\overline{Z}^{r\frac{1}{2}R}_{ki}Z^L_{ij,s}c^{rL}_{ij,s}Z^{r\frac{1}{2}L}_{jl}Z_s^{\frac{1}{2}}, \tag{6.24}$$

$$C^{rR}_{kl,s} = Z_e\sum_{i,j}\overline{Z}^{r\frac{1}{2}L}_{ki}Z^R_{ij,s}c^{rR}_{ij,s}Z^{r\frac{1}{2}R}_{jl}Z_s^{\frac{1}{2}}. \tag{6.25}$$

$c^{rL}_{ij,s}$ and $c^{rR}_{ij,s}$ are given as in (6.6) by

$$\begin{aligned}c^L_{ij,s} &= Z^L_{ij,s}c^{rL}_{ij,s},\\ c^R_{ij,s} &= Z^R_{ij,s}c^{rR}_{ij,s}.\end{aligned} \tag{6.26}$$

In analogy to (6.7), we split the scalar coupling renormalization constants in one part related to common factors and one related to flavour indices:

$$\begin{aligned}Z^L_{ij,s} &= Z^L_{(ij,s)}Z^{rL}_{ij,s},\\ Z^R_{ij,s} &= Z^R_{(ij,s)}Z^{rR}_{ij,s}.\end{aligned} \tag{6.27}$$

The expansion in α is:

$$\begin{aligned} Z^{L/R}_{(ij,s)} &= 1 + \delta Z^{L/R}_{(ij,s)} + \mathcal{O}(\alpha^2), \\ Z^{rL/R}_{ij,s} &= 1 + \delta Z^{rL/R}_{ij,s} + \mathcal{O}(\alpha^2). \end{aligned} \tag{6.28}$$

For the scalar field renormalization constant we have

$$Z_s^{\frac{1}{2}} = 1 + \frac{1}{2}\delta Z_s + \mathcal{O}(\alpha^2), \tag{6.29}$$

and we can write (6.24) and (6.25) in first order in α as

$$\begin{aligned} C^{rL}_{kl,s} &= c^{rL}_{kl,s} + \frac{\delta e}{e^r} c^{rL}_{kl,s} + c^{rL}_{kl,s}\frac{1}{2}\delta Z_s + \delta Z^{L}_{(kl,s)} c^{rL}_{kl,s} + \delta Z^{rL}_{kl,s} c^{rL}_{kl,s} \\ &\quad + \sum_i \frac{1}{2}\delta \overline{Z}^{rR}_{ki}\, c^{rL}_{il,s} + \sum_j c^{rL}_{kj,s}\frac{1}{2}\delta Z^{rL}_{jl} + \mathcal{O}(\alpha^2) \\ &= c^{rL}_{kl,s} + \delta c^{L}_{kl,s} + \delta c^{rL}_{kl,s} + \mathcal{O}(\alpha^2), \end{aligned} \tag{6.30}$$

$$\begin{aligned} C^{rR}_{kl,s} &= c^{rR}_{kl,s} + \frac{\delta e}{e^r} c^{rR}_{kl,s} + c^{rR}_{kl,s}\frac{1}{2}\delta Z_s + \delta Z^{R}_{(kl,s)} c^{rR}_{kl,s} + \delta Z^{rR}_{kl,s} c^{rR}_{kl,s} \\ &\quad + \sum_i \frac{1}{2}\delta \overline{Z}^{rL}_{ki}\, c^{rR}_{il,s} + \sum_j c^{rR}_{kj,s}\frac{1}{2}\delta Z^{rR}_{jl} + \mathcal{O}(\alpha^2) \\ &= c^{rR}_{kl,s} + \delta c^{R}_{kl,s} + \delta c^{rR}_{kl,s} + \mathcal{O}(\alpha^2), \end{aligned} \tag{6.31}$$

with

$$\delta c^{L}_{kl,s} = \left(\delta Z^{L}_{(kl,s)} + \frac{\delta e}{e^r} + \frac{1}{2}\delta Z_s\right) c^{rL}_{kl,s}, \tag{6.32}$$

$$\delta c^{rL}_{kl,s} = \delta Z^{rL}_{kl,s} c^{rL}_{kl,s} + \sum_i \frac{1}{2}\delta \overline{Z}^{rR}_{ki}\, c^{rL}_{il,s} + \sum_j c^{rL}_{kj,s}\frac{1}{2}\delta Z^{rL}_{jl}, \tag{6.33}$$

and

$$\delta c^{R}_{kl,s} = \left(\delta Z^{R}_{(kl,s)} + \frac{\delta e}{e^r} + \frac{1}{2}\delta Z_s\right) c^{rR}_{kl,s}, \tag{6.34}$$

$$\delta c^{rR}_{kl,s} = \delta Z^{rR}_{kl,s} c^{rR}_{kl,s} + \sum_i \frac{1}{2}\delta \overline{Z}^{rL}_{ki}\, c^{rR}_{il,s} + \sum_j c^{rR}_{kj,s}\frac{1}{2}\delta Z^{rR}_{jl}. \tag{6.35}$$

In the scalar sector, we allowed for couplings of Dirac fermions to charge conjugated ones (see the last terms of equation (2.52)). Then, in the renormalized

Lagrangian, we also have explicit charge conjugated fields. For example, $\overline{\psi_i^C}$ is replaced by

$$\overline{\psi_i^C} = \sum_j \left(\overline{(\psi_j^{rL})^C} Z_{ij}^{r\frac{1}{2}L} \gamma_L + \overline{(\psi_j^{rR})^C} Z_{ij}^{r\frac{1}{2}R} \gamma_R \right) . \qquad (6.36)$$

Here, we have used the equivalent definition of the charge conjugated fields as given in (2.19). Accordingly, the transformed coupling constants equivalent to (6.24) and (6.25) will be

$$\tilde{C}_{kl,s}^{rL} = Z_e \sum_{i,j} Z_{ik}^{r\frac{1}{2}L} \tilde{Z}_{(ij,s)}^{L} \tilde{Z}_{ij,s}^{rL} \tilde{c}_{ij,s}^{rL} Z_{jl}^{r\frac{1}{2}L} Z_s^{\frac{1}{2}} , \qquad (6.37)$$

$$\tilde{C}_{kl,s}^{rR} = Z_e \sum_{i,j} Z_{ik}^{r\frac{1}{2}R} \tilde{Z}_{(ij,s)}^{R} \tilde{Z}_{ij,s}^{rR} \tilde{c}_{ij,s}^{rR} Z_{jl}^{r\frac{1}{2}R} Z_s^{\frac{1}{2}} . \qquad (6.38)$$

The transformed coupling constants from the last term of (2.52) can be deduced in a similar manner:

$$(\tilde{C}_{lk,s}^{rL})' = Z_e \sum_{i,j} \overline{Z}_{lj}^{r\frac{1}{2}R} \left(\tilde{Z}_{(ij,s)}^{R} \tilde{Z}_{ij,s}^{rR} \tilde{c}_{ij,s}^{rR} \right)^* \overline{Z}_{ki}^{r\frac{1}{2}R} \left(Z_s^{\frac{1}{2}} \right)^* , \qquad (6.39)$$

$$(\tilde{C}_{lk,s}^{rR})' = Z_e \sum_{i,j} \overline{Z}_{lj}^{r\frac{1}{2}L} \left(\tilde{Z}_{(ij,s)}^{L} \tilde{Z}_{ij,s}^{rL} \tilde{c}_{ij,s}^{rL} \right)^* \overline{Z}_{ki}^{r\frac{1}{2}L} \left(Z_s^{\frac{1}{2}} \right)^* . \qquad (6.40)$$

The prime signals the transformed couplings for the hermitian conjugated term, i.e. for the diagrams with diverging arrows. Along this work, we do not need these expressions, but we gave them for completeness.

6.2 Interaction Terms with Majorana Fermions

To complete the renormalization of the interaction Lagrangian described in section 2.3, we consider now the renormalized Majorana fields and their couplings to bosons. We begin the analysis with the interaction terms involving charged bosons and then, we move to the neutral current and Yukawa terms. For the first type of interactions, a Dirac fermion is required in the vertex, while for the second one we have two Majorana fermions coupling to the neutral bosons.

The redefinition of the coupling constants evolves like before. One has just to pay attention to the correct coupling between left and right handed fields. In the following, we enumerate the couplings and their definitions without repeating the whole procedure.

The interaction between Majorana fermions, Dirac fermions and vector bosons is included in the first two terms of the charged current Lagrangian (2.50). We replace the fermion fields with the renormalized ones, as defined in sections 5.1.1 and 5.2.1. For Dirac particles we need (5.6)−(5.7) and for Majorana fermions (5.86)−(5.89). All other renormalization constants are inserted similar to section 6.1.1, with the appropriate indices. For the transformed coupling constants related to vector bosons we obtain

$$G^{rL}_{bj,v} = Z_e \sum_{a,i} \overline{Z}^{r\frac{1}{2}L}_{ba} Z^L_{(ai,v)} Z^{rL}_{ai,v} g^{rL}_{ai,v} Z^{r\frac{1}{2}L}_{ij} Z^{\frac{1}{2}}_v, \tag{6.41}$$

$$G^{rR}_{bj,v} = Z_e \sum_{a,i} Z^{r\frac{1}{2}L}_{ab} Z^R_{(ai,v)} Z^{rR}_{ai,v} g^{rR}_{ai,v} Z^{r\frac{1}{2}R}_{ij} Z^{\frac{1}{2}}_v, \tag{6.42}$$

from the first term of (2.50) and from the second:

$$G^{rL}_{jb,v} = Z_e \sum_{i,a} \overline{Z}^{r\frac{1}{2}L}_{ji} Z^L_{(ia,v)} Z^{rL}_{ia,v} g^{rL}_{ia,v} Z^{r\frac{1}{2}L}_{ab} (Z^{\frac{1}{2}}_v)^*, \tag{6.43}$$

$$G^{rR}_{jb,v} = Z_e \sum_{i,a} \overline{Z}^{r\frac{1}{2}R}_{ji} Z^R_{(ia,v)} Z^{rR}_{ia,v} g^{rR}_{ia,v} \overline{Z}^{r\frac{1}{2}L}_{ba} (Z^{\frac{1}{2}}_v)^*. \tag{6.44}$$

The factorisation in the coupling renormalization constants evolved as in (6.7), such that all renormalization constants carrying explicit fermion indices (like a mixing matrix) are contained in Z^r. The two sets of renormalized coupling constants are related by complex conjugation as the unrenormalized ones, i.e.

$$\begin{aligned} g^{rL}_{ia,v} &= (g^{rL}_{ai,v})^*, \\ g^{rR}_{ia,v} &= (g^{rR}_{ai,v})^*. \end{aligned} \tag{6.45}$$

The relation is not valid for the transformed couplings G^{rL} and G^{rR} since the fermion field renormalization constants are not related by hermiticity.

The interaction of Majorana and Dirac fermions with scalar bosons is comprised in

$$C^{rL}_{bj,s} = Z_e \sum_{a,i} Z^{r\frac{1}{2}L}_{ab} Z^L_{(ai,s)} Z^{rL}_{ai,s} c^{rL}_{ai,s} Z^{r\frac{1}{2}L}_{ij} Z^{\frac{1}{2}}_s, \tag{6.46}$$

$$C^{rR}_{bj,s} = Z_e \sum_{a,i} \overline{Z}^{r\frac{1}{2}L}_{ba} Z^R_{(ai,s)} Z^{rR}_{ai,s} c^{rR}_{ai,s} Z^{r\frac{1}{2}R}_{ij} Z^{\frac{1}{2}}_s, \tag{6.47}$$

for the renormalization of the first term of (2.52). From the hermitian conjugated one, we obtain

$$C^{rL}_{jb,s} = Z_e \sum_{i,a} \overline{Z}^{r\frac{1}{2}R}_{ji} Z^L_{(ia,s)} Z^{rL}_{ia,s} c^{rL}_{ia,s} Z^{r\frac{1}{2}L}_{ab} (Z^{\frac{1}{2}}_s)^*, \tag{6.48}$$

$$C^{rR}_{jb,s} = Z_e \sum_{i,a} \overline{Z}^{r\frac{1}{2}L}_{ji} Z^R_{(ia,s)} Z^{rR}_{ia,s} c^{rR}_{ia,s} \overline{Z}^{r\frac{1}{2}L}_{ba} (Z^{\frac{1}{2}}_s)^*, \tag{6.49}$$

where

$$\begin{aligned} c^{rL}_{ia,s} &= (c^{rR}_{ai,s})^*, \\ c^{rR}_{ia,s} &= (c^{rL}_{ai,s})^*. \end{aligned} \tag{6.50}$$

The general Lagrangian of section 5.1.1 describes the Majorana fermions interaction with neutral vector bosons in the first term of equation (2.51) and with scalars in the first term of (2.54). The transformed coupling constants for the renormalized Lagrangian are

$$G^{rL}_{cd,v} = Z_e \sum_{a,b} \overline{Z}^{r\frac{1}{2}L}_{ca} Z^L_{(ab,v)} Z^{rL}_{ab,v} g^{rL}_{ab,v} Z^{r\frac{1}{2}L}_{bd} Z^{\frac{1}{2}}_v, \tag{6.51}$$

$$G^{rR}_{cd,v} = Z_e \sum_{a,b} Z^{r\frac{1}{2}L}_{ac} Z^R_{(ab,v)} Z^{rR}_{ab,v} g^{rR}_{ab,v} \overline{Z}^{r\frac{1}{2}L}_{db} Z^{\frac{1}{2}}_v, \tag{6.52}$$

for the coupling to vector bosons and

$$C^{rL}_{cd,s} = Z_e \sum_{a,b} Z^{r\frac{1}{2}L}_{ac} Z^L_{(ab,s)} Z^{rL}_{ab,s} c^{rL}_{ab,s} Z^{r\frac{1}{2}L}_{bd} Z^{\frac{1}{2}}_s, \tag{6.53}$$

$$C^{rR}_{cd,s} = Z_e \sum_{a,b} \overline{Z}^{r\frac{1}{2}L}_{ca} Z^R_{(ab,s)} Z^{rR}_{ab,s} c^{rR}_{ab,s} \overline{Z}^{r\frac{1}{2}L}_{db} Z^{\frac{1}{2}}_s, \tag{6.54}$$

for the neutral scalars.

When needed, one can use expansions for the renormalization constants and write the coupling constants as a series in α. The first order corrections of the transformed coupling constant can be grouped as in section 6.1. We will not enumerate here the formulas for each type of coupling. In the next section, we present a general formula that can be fitted to any combination of fermions and bosons.

6.3 Generic Fermion Interaction Terms

To summarise the results of the two sections allocated to the renormalization of the interaction Lagrangian, we refer to the general notation for the fermion indices x, y. A generic Lagrangian term for the interaction of fermions with vector bosons can be written as

$$\begin{aligned} \mathcal{L}_{vb} &= e \sum_{x,y,v} \overline{\psi}_x \gamma^\mu (g^L_{xy,v}\gamma_L + g^R_{xy,v}\gamma_R)\psi_y \phi_{v,\mu} \\ &= e^r \sum_{x,y,v} \overline{\psi^r_x} \gamma^\mu (G^{rL}_{xy,v}\gamma_L + G^{rR}_{xy,v}\gamma_R)\psi^r_y \phi^r_{v,\mu}. \end{aligned} \tag{6.55}$$

The second line introduces the renormalized parameters and the transformed coupling constants. The latter are

$$\begin{aligned} G^{rL}_{xy,v} &= Z_e \sum_{t,u} \overline{Z}^{r\frac{1}{2}L}_{xt} Z^{L}_{(tu,v)} Z^{rL}_{tu,v} g^{rL}_{tu,v} Z^{r\frac{1}{2}L}_{uy} Z^{\frac{1}{2}}_{v}, \\ G^{rR}_{xy,v} &= Z_e \sum_{t,u} \overline{Z}^{r\frac{1}{2}R}_{xt} Z^{R}_{(tu,v)} Z^{rR}_{tu,v} g^{rR}_{tu,v} Z^{r\frac{1}{2}R}_{uy} Z^{\frac{1}{2}}_{v}, \end{aligned} \tag{6.56}$$

where all renormalization constants presented before can be recognised. The expansions of $G^{rL}_{xy,v}$ and $G^{rR}_{xy,v}$ to first order are generally given by

$$\begin{aligned} G^{rL}_{xy,v} &= g^{rL}_{xy,v} + \delta g^{L}_{xy,v} + \delta g^{rL}_{xy,v} + \mathcal{O}(\alpha^2), \\ G^{rR}_{xy,v} &= g^{rR}_{xy,v} + \delta g^{R}_{xy,v} + \delta g^{rR}_{xy,v} + \mathcal{O}(\alpha^2), \end{aligned} \tag{6.57}$$

with

$$\begin{aligned} \delta g^{L}_{xy,v} &= \left(\delta Z^{L}_{(xy,v)} + \frac{\delta e}{e^r} + \frac{1}{2}\delta Z_v \right) g^{rL}_{xy,v}, \\ \delta g^{rL}_{xy,v} &= \delta Z^{rL}_{xy,v} g^{rL}_{xy,v} + \frac{1}{2}\sum_t \delta\overline{Z}^{rL}_{xt} g^{rL}_{ty,v} + \frac{1}{2}\sum_u g^{rL}_{xu,v} \delta Z^{rL}_{uy}, \\ \delta g^{R}_{xy,v} &= \left(\delta Z^{R}_{(xy,v)} + \frac{\delta e}{e^r} + \frac{1}{2}\delta Z_v \right) g^{rR}_{xy,v}, \\ \delta g^{rR}_{xy,v} &= \delta Z^{rR}_{xy,v} g^{rR}_{xy,v} + \frac{1}{2}\sum_t \delta\overline{Z}^{rR}_{xt} g^{rR}_{ty,v} + \frac{1}{2}\sum_u g^{rR}_{xu,v} \delta Z^{rR}_{uy}. \end{aligned} \tag{6.58}$$

The general version of an interaction term with scalar bosons is shortly written as

$$\begin{aligned} \mathcal{L}_{sb} &= e \sum_{x,y,s} \overline{\psi}_x (c^{L}_{xy,s}\gamma_L + c^{R}_{xy,s}\gamma_R)\psi_y \phi_s \\ &= e^r \sum_{x,y,s} \overline{\psi^r_x} (C^{rL}_{xy,s}\gamma_L + C^{rR}_{xy,s}\gamma_R)\psi^r_y \phi^r_s. \end{aligned} \tag{6.59}$$

The transformed couplings:

$$\begin{aligned} C^{rL}_{xy,s} &= Z_e \sum_{t,u} \overline{Z}^{r\frac{1}{2}R}_{xt} Z^{L}_{(tu,s)} Z^{rL}_{tu,s} c^{rL}_{tu,s} Z^{r\frac{1}{2}L}_{uy} Z^{\frac{1}{2}}_{s}, \\ C^{rR}_{xy,s} &= Z_e \sum_{t,u} \overline{Z}^{r\frac{1}{2}L}_{xt} Z^{R}_{(tu,s)} Z^{rR}_{tu,s} c^{rR}_{tu,s} Z^{r\frac{1}{2}R}_{uy} Z^{\frac{1}{2}}_{s}, \end{aligned} \tag{6.60}$$

expand to

$$
\begin{aligned}
C^{rL}_{xy,s} &= c^{rL}_{xy,s} + \delta c^{L}_{xy,s} + \delta c^{rL}_{xy,s} + \mathcal{O}(\alpha^2),\\
C^{rR}_{xy,s} &= c^{rR}_{xy,s} + \delta c^{R}_{xy,s} + \delta c^{rR}_{xy,s} + \mathcal{O}(\alpha^2),
\end{aligned}
\tag{6.61}
$$

where

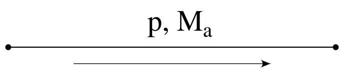
$$
\begin{aligned}
\delta c^{L}_{xy,s} &= \left(\delta Z^{L}_{(xy,s)} + \frac{\delta e}{e^r} + \frac{1}{2}\delta Z_s\right) c^{rL}_{xy,s},\\
\delta c^{rL}_{xy,s} &= \delta Z^{rL}_{xy,s} c^{rL}_{xy,s} + \frac{1}{2}\sum_t \delta \overline{Z}^{rR}_{xt}\, c^{rL}_{ty,s} + \frac{1}{2}\sum_u c^{rL}_{xu,s}\delta Z^{rL}_{uy},\\
\delta c^{R}_{xy,s} &= \left(\delta Z^{R}_{(xy,s)} + \frac{\delta e}{e^r} + \frac{1}{2}\delta Z_s\right) c^{rR}_{xy,s},\\
\delta c^{rR}_{xy,s} &= \delta Z^{rR}_{xy,s} c^{rR}_{xy,s} + \frac{1}{2}\sum_t \delta \overline{Z}^{rL}_{xt}\, c^{rR}_{ty,s} + \frac{1}{2}\sum_u c^{rR}_{xu,s}\delta Z^{rR}_{uy}.
\end{aligned}
\tag{6.62}
$$

According to the fermion in the vertex, one adapts the flavour indices and the renormalization constants. All coupling constants expressed until now can be recognised in (6.56) and (6.60). For Majorana fermions, one has to remember that $Z^{r\frac{1}{2}R}_{ab} = \overline{Z}^{r\frac{1}{2}L}_{ba}$ and $\overline{Z}^{r\frac{1}{2}R}_{ba} = Z^{r\frac{1}{2}L}_{ab}$ (see (5.85)), as already used in this chapter.

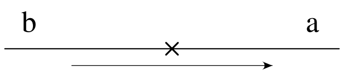
6.4 Feynman Rules Derived from the Renormalized Lagrange Density

If we look back at the first chapter, formally, there should not be many changes in the Feynman rules. The fields are now given by complicated expressions, the parameters in the Lagrangian changed, but if we use the new notation, the rules will be almost identical to section 2.4.2.

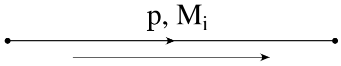
The term that requires additional care is the fermion propagator. Here, we still have to separately write counter terms coming from the free term of the Lagrange density. The Feynman rules for Dirac fermion propagator and counter terms are taken from section 5.1.1 and the ones for Majorana fermions from section 5.2.1.



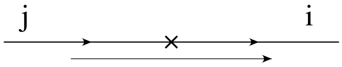
$$iS^r(p) = \frac{i}{\not p - M_a + i\rho}$$



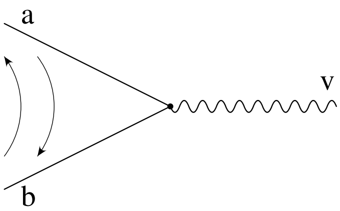
$$i(\not p - M_a)\ \frac{1}{2}\delta Z^r_{ab} + i\ \frac{1}{2}\delta Z^r_{ba}(\not p - M_b) + i\delta m_a \delta_{ab}$$



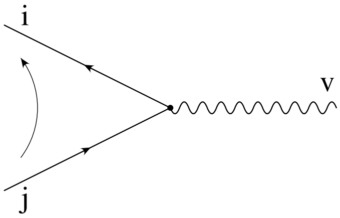
$$iS^r(p) = \frac{i}{\not{p} - M_i + i\rho}$$



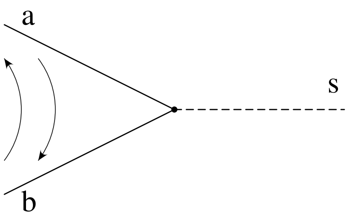
$$i(\not{p} - M_i)\ \frac{1}{2}\delta Z^r_{ij} + i\ \frac{1}{2}\delta \overline{Z}^r_{ij}(\not{p} - M_j) + i\delta m_i \delta_{ij}$$



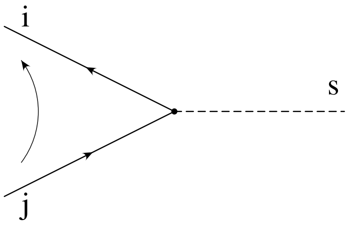
$$i\Gamma^{r\mu}_{abv} = ie^r \gamma^\mu (G^{rL}_{ab,v}\gamma_L + G^{rR}_{ab,v}\gamma_R)$$



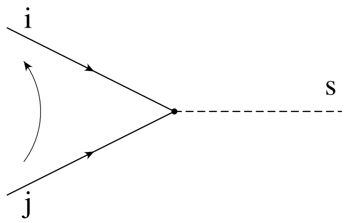
$$i\Gamma^{r\mu}_{ijv} = ie^r \gamma^\mu (G^{rL}_{ij,v}\gamma_L + G^{rR}_{ij,v}\gamma_R)$$



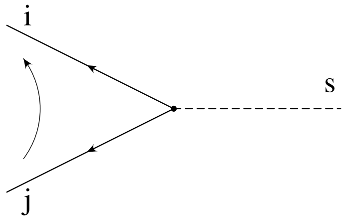
$$i\Gamma^r_{abs} = ie^r (C^{rL}_{ab,s}\gamma_L + C^{rR}_{ab,s}\gamma_R)$$



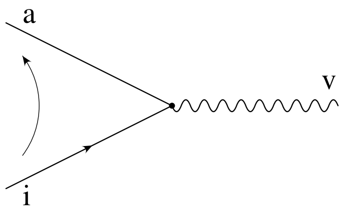
$$i\Gamma^r_{ijs} = ie^r (C^{rL}_{ij,s}\gamma_L + C^{rR}_{ij,s}\gamma_R)$$



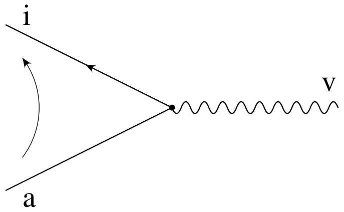
$$i\tilde{\Gamma}^r_{ijs} = ie^r (\tilde{C}^{rL}_{ij,s}\gamma_L + \tilde{C}^{rR}_{ij,s}\gamma_R)$$



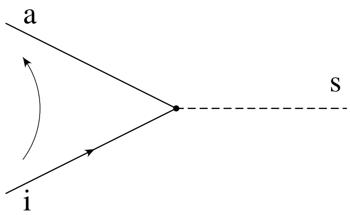
$$i(\tilde{\Gamma}^r_{ijs})' = ie^r((\tilde{C}^{rL}_{ij,s})'\gamma_L + (\tilde{C}^{rR}_{ij,s})'\gamma_R)$$



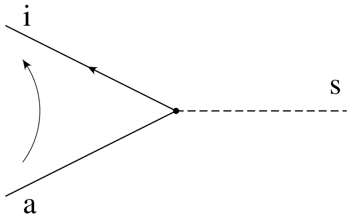
$$i\Gamma^{r\mu}_{aiv} = ie^r\gamma^\mu(G^{rL}_{ai,v}\gamma_L + G^{rR}_{ai,v}\gamma_R)$$



$$i\Gamma^{r\mu}_{iav} = ie^r\gamma^\mu(G^{rL}_{ia,v}\gamma_L + G^{rR}_{ia,v}\gamma_R)$$



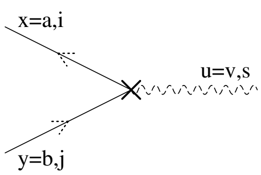
$$i\Gamma^r_{ais} = ie^r(C^{rL}_{ai,s}\gamma_L + C^{rR}_{ai,s}\gamma_R)$$



$$i\Gamma^r_{ias} = ie^r(C^{rL}_{ia,s}\gamma_L + C^{rR}_{ia,s}\gamma_R) \tag{6.63}$$

Each vertex diagram represents now the sum of the unrenormalized vertex plus the vertex counter term, i.e.

$$i\Gamma^r_{xyu} = i\Gamma_{xyu} +$$



(6.64)

With the generic notation for the indices of Dirac and Majorana fermion type, we distinguish

$$\begin{aligned}\Gamma^r_{xyv} &= e^r\gamma^\mu(G^{rL}_{xy,v}\gamma_L + G^{rR}_{xy,v}\gamma_R) \\ &= e^r\gamma^\mu\left(g^{rL}_{xy,v}\left(1+\frac{\delta g^L_{xy,v}}{g^{rL}_{xy,v}}+\frac{\delta g^{rL}_{xy,v}}{g^{rL}_{xy,v}}\right)\gamma_L + g^{rR}_{xy,v}\left(1+\frac{\delta g^R_{xy,v}}{g^{rR}_{xy,v}}+\frac{\delta g^{rR}_{xy,v}}{g^{rR}_{xy,v}}\right)\gamma_R\right),\end{aligned} \tag{6.65}$$

for any allowed interaction of Dirac and Majorana fermions with a vector boson, and

$$\begin{aligned}\Gamma^r_{xys} &= e^r(C^{rL}_{xy,s}\gamma_L + C^{rR}_{xy,s}\gamma_R) \\ &= e^r\left(c^{rL}_{xy,s}\left(1+\frac{\delta c^L_{xy,s}}{c^{rL}_{xy,s}}+\frac{\delta c^{rL}_{xy,s}}{c^{rL}_{xy,s}}\right)\gamma_L + c^{rR}_{xy,s}\left(1+\frac{\delta c^R_{xy,s}}{c^{rR}_{xy,s}}+\frac{\delta c^{rR}_{xy,s}}{c^{rR}_{xy,s}}\right)\gamma_R\right),\end{aligned} \tag{6.66}$$

for the fermion interaction with a scalar boson. The decomposition of the transformed coupling constants for the first order approximation follows the definitions given in each of the previous sections and summarised in (6.58) and (6.62).

The external fermions will be described by $u(p,s)$, $\overline{u}(p,s)$ and $v(p,s)$, $\overline{v}(p,s)$, the spinors related to the physical masses. The rules related to the fermion flow change remain as in section 2.4.2.

6.5 One-loop Corrections to Vertices

To investigate the contribution of the renormalized interaction terms of the Lagrangian in a physical process, we consider a generic scattering with two fermion fields, see Figure 6.1. As shown in the picture, in the process, there are contributions

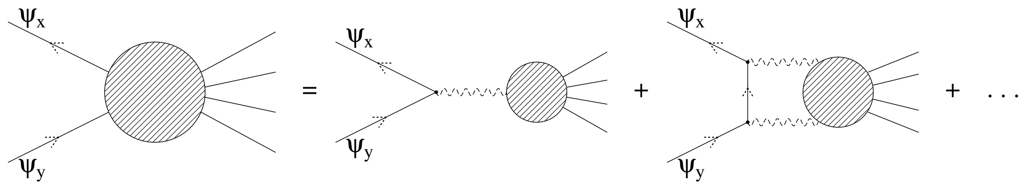
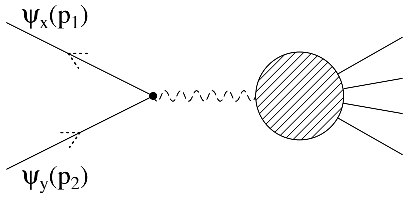


Figure 6.1: General scattering process that involves fermion-antifermion-boson vertices.

that involve vertices which connect the fermions by one single boson line to the rest of the diagram, but also contributions that involve other one-particle-irreducible diagrams. We will analyse the first type of contributions since at one-loop, the renormalized coupling constants play a role exactly in the fermion-antifermion-boson vertices .

The first diagram in the sum, i.e.



factorises into two pieces: one for the fermion-antifermion-boson vertex and one for everything belonging to the rest of the diagram. By everything we mean the boson propagator, the parts inside the shaded blob of the figure and the particles in the final state. We denote this second factor with V_μ when the interaction boson is vector like and with S for the scalar like. Like this, inserting the related Feynman rules (see section 2.4), the contribution of the diagram to the amplitude amounts to

$$\mathcal{T}_0^v = ie\overline{v}_x(p_2)\gamma^\mu(g^L_{xy,v}\gamma_L + g^R_{xy,v}\gamma_R)u_y(p_1)V_\mu(p_3), \tag{6.67}$$

for an interaction with a vector boson and to

$$\mathcal{T}_0^s = ie\overline{v}_x(p_2)(c^L_{xy,s}\gamma_L + c^R_{xy,s}\gamma_R)u_y(p_1)S(p_3), \tag{6.68}$$

for an interaction with a scalar boson. With p_1, p_2 we denote the momenta of the incoming particles and with p_3 the total momentum of the outgoing particles. If the vector boson is an external line, then $V_\mu(p_3)$ is replaced by the polarisation vector, while if we have an external scalar, $S(p_3)$ is equal to 1.

At one-loop, one adds the contributions from vertex counter terms, one-loop vertex diagrams and self-energy corrections of external legs. All the types of diagrams related to the corrections of the selected vertex and the incoming particles are sketched in Figure 6.2. For the diagrams that involve internal particles, one has to sum over all possibilities.

We comprise the one-loop contributions to the total amplitude from the other one-particle-irreducible diagrams of the process in $\mathcal{T}^{\mathrm{b}}_{xy}$. Then, the total amplitude, given as a sum of all diagrams is

$$\mathcal{T}^{tot}_{xy} = \mathcal{T}^r_{xy} + \mathcal{T}^{\mathrm{vertex}}_{xy} + \mathcal{T}^{y-se}_{xy} + \mathcal{T}^{x-se}_{xy} + \mathcal{T}^{\mathrm{b}}_{xy} + \mathcal{O}(\alpha^2). \tag{6.69}$$

$\mathcal{T}^r_{xy}$ is the contribution from the first two diagrams of Figure 6.2 added up, $\mathcal{T}^{\mathrm{vertex}}_{xy}$ is responsable for the sum of all vertex corrections (third type of diagram) and $\mathcal{T}^{y-se}_{xy}$ and $\mathcal{T}^{x-se}_{xy}$ stand for the self-energy corrections of the external fermion lines. In the following, we will separately discuss the detailed expressions of the one-loop corrections for the vector and the scalar boson vertices.

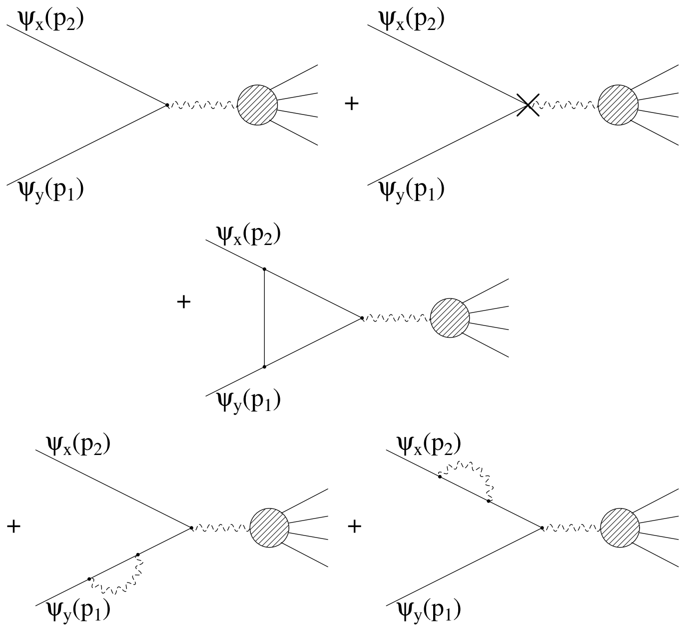


Figure 6.2: Diagrams contributing to the one-loop corrections of the vertex.

One-loop Amplitude for Vertices with Vector Bosons

For vector bosons, the first term of the total amplitude, $\mathcal{T}^r_{xy}$, reads:

$$\mathcal{T}^r_{xy} = ie^r \overline{v}_x(p_2)\gamma^\mu (G^{rL}_{xy,v}\gamma_L + G^{rR}_{xy,v}\gamma_R) u_y(p_1) V_\mu(p_3). \tag{6.70}$$

$G^{rL}_{xy,v}$ and $G^{rR}_{xy,v}$ are the transformed coupling constant that describe the sum of the lowest order vertex plus its counter term (see (6.64)). Their general expressions are given in (6.56). Using the expansions of these couplings as written in (6.65),

$$\begin{aligned}\mathcal{T}^r_{xy} = ie^r \overline{v}_x(p_2)\gamma^\mu \Bigg(& g^{rL}_{xy,v}\left(1 + \frac{\delta g^L_{xy,v}}{g^{rL}_{xy,v}} + \frac{\delta g^{rL}_{xy,v}}{g^{rL}_{xy,v}}\right)\gamma_L \\ & + g^{rR}_{xy,v}\left(1 + \frac{\delta g^R_{xy,v}}{g^{rR}_{xy,v}} + \frac{\delta g^{rR}_{xy,v}}{g^{rR}_{xy,v}}\right)\gamma_R \Bigg) u_y(p_1) V_\mu(p_3).\end{aligned} \tag{6.71}$$

The vertex corrections contribute to the amplitude of the given process by

$$\mathcal{T}^{\text{vertex}}_{xy} = ie^r \overline{v}_x(p_2)\delta^{\text{vert},\mu}_{xy} u_y(p_1) V_\mu(p_3). \tag{6.72}$$

Here, $\delta^{\text{vert},\mu}_{xy}$ totals the first order contributions from all diagrams.

The self-energy contributions from external legs have two sources. One is the decay width of the unstable particles and the other one the difference between the wave function renormalization constants and the field renormalization constants. As argued in section 5.1.1, we can leave out of the calculation the terms proportional to Γ_x. If the two sets of constants, the field and the wave function renormalization ones are equal, then $\mathcal{T}_{xy}^{y-se}$=$\mathcal{T}_{xy}^{x-se}$=0. If not, in the amplitudes, there is one remainder from the renormalized self-energy that does not vanish when acting on spinors. It is given by $\mathcal{R}_{xy}(p)$, as expressed in (5.23) for Dirac fermions and in (5.97) for Majorana. From the fermion of type y, the contribution to the total amplitude is

$$\mathcal{T}_{xy}^{y-se} = ie^r\overline{v}_x(p_2)\sum_u \gamma^\mu \left(g_{xu,v}^{rL}\gamma_L + g_{xu,v}^{rR}\gamma_R\right)\frac{1}{\not{p}_1 - M_u}\mathcal{R}_{uy}(p_1)u_y(p_1)V_\mu(p_3). \quad (6.73)$$

We sum over all possible internal fermions u that belong to the same family as y. Analogously, the amplitude of the self-energy corrections for the fermion x is

$$\mathcal{T}_{xy}^{x-se} = ie^r\overline{v}_x(p_2)\sum_t \mathcal{R}_{xt}(p_2)\frac{1}{\not{p}_2 - M_t}\gamma^\mu \left(g_{ty,v}^{rL}\gamma_L + g_{ty,v}^{rR}\gamma_R\right)u_y(p_1)V_\mu(p_3). \quad (6.74)$$

All the given amplitudes sum up in (6.82).

One-loop Amplitude for Vertices with Scalar Bosons

The list of the terms that are added up in the one-loop total amplitude (6.82) when the fermions interact with a scalar boson is similar to formulas (6.70)−(6.74). We have just to replace the part specific to vectors $\gamma^\mu V_\mu(p_3)$ with $S(p_3)$ and use the correct notation for the coupling constants. We obtain

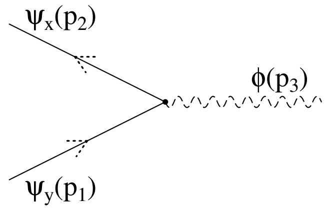
$$\begin{aligned}
\mathcal{T}_{xy}^{r} &= ie^r\overline{v}_x(p_2)(C_{xy,s}^{rL}\gamma_L + C_{xy,s}^{rR}\gamma_R)u_y(p_1)S(p_3) && (6.75)\\
&= ie^r\overline{v}_x(p_2)\left(c_{xy,s}^{rL}\left(1+\frac{\delta c_{xy,s}^{L}}{c_{xy,s}^{rL}}+\frac{\delta c_{xy,s}^{rL}}{c_{xy,s}^{rL}}\right)\gamma_L\right.\\
&\qquad \left.+c_{xy,s}^{rR}\left(1+\frac{\delta c_{xy,s}^{R}}{c_{xy,s}^{rR}}+\frac{\delta c_{xy,s}^{rR}}{c_{xy,s}^{rR}}\right)\gamma_R\right)u_y(p_1)S(p_3), && (6.76)\\
\mathcal{T}_{xy}^{\text{vertex}} &= ie^r\overline{v}_x(p_2)\delta_{xy}^{\text{vert}}u_y(p_1)S(p_3), && (6.77)\\
\mathcal{T}_{xy}^{y-se} &= ie^r\overline{v}_x(p_2)\sum_u \left(c_{xu,s}^{rL}\gamma_L + c_{xu,s}^{rR}\gamma_R\right)\frac{1}{\not{p}_1 - M_u}\mathcal{R}_{uy}(p_1)u_y(p_1)S(p_3), && (6.78)\\
\mathcal{T}_{xy}^{x-se} &= ie^r\overline{v}_x(p_2)\sum_t \mathcal{R}_{xt}(p_2)\frac{1}{\not{p}_2 - M_t}\left(c_{ty,s}^{rL}\gamma_L + c_{ty,s}^{rR}\gamma_R\right)u_y(p_1)S(p_3). && (6.79)
\end{aligned}$$

Divergent Contributions and the One-loop Amplitude

We cannot discuss in a more detailed way the total amplitude, but we can still make some remarks about the sources of ultraviolet divergences. At the end of chapter 5, we have decided to fix the field renormalization constants in such a way that they lead to finite contributions of self-energy corrections to external legs. It means that div$[\mathcal{T}_{xy}^{y-se}]$=div$[\mathcal{T}_{xy}^{x-se}]$=0. Then, the parts of the total amplitude that contain divergences are $\mathcal{T}_{xy}^{r}$, $\mathcal{T}_{xy}^{\text{vertex}}$ and $\mathcal{T}_{xy}^{\text{b}}$. Since the total amplitude is a finite quantity, we should have

$$\text{div}[\mathcal{T}_{xy}^{r} + \mathcal{T}_{xy}^{\text{vertex}} + \mathcal{T}_{xy}^{\text{b}}] = 0. \tag{6.80}$$

If a vertex with corrections like shown in Figure 6.2 is part of a complex diagram for which the boson is an internal particle, one can not continue the study of divergent parts without specifying the rest of the process. To carry the analysis a bit further, we reduce the process in Figure 6.1 to one for which the boson is an external line.



By ϕ we characterise the bosonic field, scalar or vector like. Such diagrams describe the heavy fermion decays that we will consider as examples when investigating the quark and neutrino mixing, but also gauge boson decays. It is just a matter of choosing the incoming and the outgoing particles.

Going to one-loop corrections, the diagrams shown in Figure 6.2 simplify to the ones in Figure 6.3. We can picture now also the one-loop contributions from the right part of the vertex, in this case the boson line with its self-energy. Therefore, Figure 6.3 contains all the contributions to the one-loop amplitude of our simplified process.

We will assume that the boson field renormalization was completed in such a way that the self-energy contributions of external legs are absorbed in its mass and field renormalization constants ($Z_v^{\frac{1}{2}}$ or $Z_s^{\frac{1}{2}}$). Therefore, we do not need to consider the last type of diagram. Its contributions are included in the counter terms (second diagram). Consequently, in this case, in formulas (6.70)−(6.74) related to the vector boson, $V_\mu(p_3)$ becomes $\varepsilon_\mu^*(p_3)$, and in those related to the scalar boson (6.75)−(6.79), $S(p_3)$=1.

There are no other one-particle-irreducible diagrams that contribute to the pro-

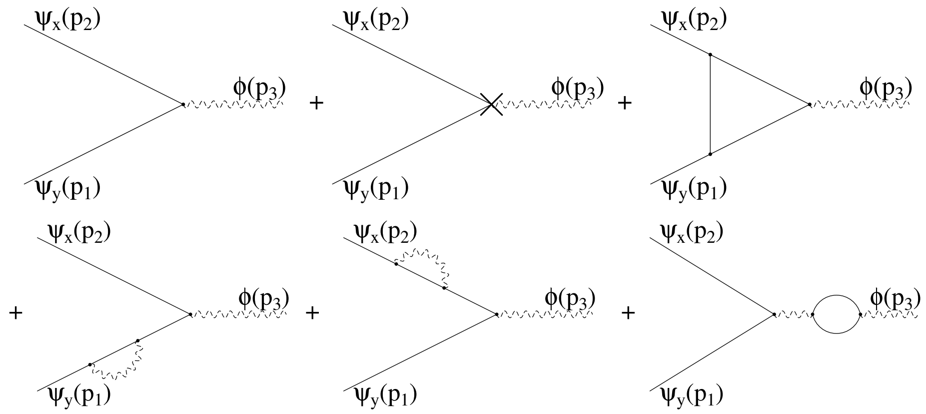


Figure 6.3: Diagrams contributing to the one-loop corrections of a fermion-antifermion-boson vertex

cess, like we had in the general case, hence,

$$\mathcal{T}^{\mathrm{b}}_{xy} = 0, \tag{6.81}$$

and the total amplitude is given by

$$\mathcal{T}^{tot}_{xy} = \mathcal{T}^{r}_{xy} + \mathcal{T}^{\mathrm{vertex}}_{xy} + \mathcal{T}^{y-se}_{xy} + \mathcal{T}^{x-se}_{xy} + \mathcal{O}(\alpha^2). \tag{6.82}$$

The relation characterising the divergences (6.80) is reduced to

$$\mathrm{div}[\mathcal{T}^{r}_{xy} + \mathcal{T}^{\mathrm{vertex}}_{xy}] = 0. \tag{6.83}$$

This condition will be essential for the definition of the mixing matrix corrections that can contribute to $\delta g^{rL}_{xy,v}$, $\delta g^{rR}_{xy,v}$ or $\delta c^{rL}_{xy,s}$ and $\delta c^{rR}_{xy,s}$, as it will follow in the next chapters.

Chapter 7

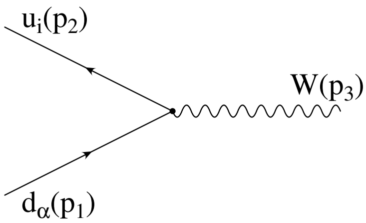
Renormalization of the Quark Mixing Matrix

The presence of a quark mixing matrix in the interaction Lagrangian leads to divergences that are not cancelled by the other renormalization constants that enter in a total higher-order amplitude. Therefore, a correction to the quark mixing matrix that absorbs the remaining divergent part is mandatory. This correction will appear in the renormalized coupling constant. The open question is how to define the renormalized quark mixing matrix in a complete and correct way, such that at the end we obtain a unitary, gauge independent renormalized mixing matrix. In the literature, there are two main ways to calculate the corrections: either by dropping out the absorptive parts and imposing unitarity, either by considering them and dropping the hermiticity condition for the field renormalization constants. The last situation has as a consequence a renormalized mixing matrix that is not unitary. Since the imaginary parts are an attribute of the theory, but also the unitarity of the mixing matrix, we try to keep both of them.

In this chapter we particularise the results of 6.1 and 6.5 for quark mixing and we discuss the alternatives for defining the renormalized quark mixing matrix. We start with the renormalization of the quark-antiquark-W vertex in a general approach and then, as a particular example, we study the corrections induced to the top decay.

7.1 Corrections to the Quark-Antiquark-W Vertex

In the Standard Model, the quark mixing matrix appears in the charged-current terms of the Lagrangian, as a part of the coupling between quarks and the W-boson. One can also find it in the coupling to charged scalars ($\Phi^{\pm}$), but we limit ourselves to the vertices with physical particles. Therefore, we consider processes involving vertices of the type



$$i\frac{e}{\sqrt{2}s_W}\, V_{i\alpha}\gamma^\mu\gamma_L.$$

The coupling constants related to such a vertex are, as listed in Table 2.1,

$$\begin{aligned} g^L_{i\alpha,W} &= \frac{1}{\sqrt{2}s_W}V_{i\alpha}, \\ g^R_{i\alpha,W} &= 0. \end{aligned} \tag{7.1}$$

7.1.1 Corrections from the Renormalized Parameters

Our purpose is to determine the corrections to the coupling constant and in particular to the mixing matrix, using the general result of section 6.1. For this, we need to identify the renormalization factors that enter in $g^L_{i\alpha,W}$.

In formula (7.1), one of the parameters that receives corrections is s_W. (The corrections can be related to the renormalized gauge boson masses W and Z.) We choose

$$s_W = Z_{s_W} s^r_W, \tag{7.2}$$

where Z_{s_W} can be expanded as

$$Z_{s_W} = 1 + \frac{\delta s_W}{s^r_W} + \mathcal{O}(\alpha^2). \tag{7.3}$$

As stated at the beginning of the chapter, the other parameter in $g^L_{i\alpha,W}$, i.e. the mixing matrix should also be readjusted. We denote the correction of order $\mathcal{O}(\alpha)$ by $\delta V_{i\alpha}$.

$$\begin{aligned} V_{i\alpha} &= V^r_{i\alpha} + \delta V_{i\alpha} + \mathcal{O}(\alpha^2) \\ &= \left(1 + \frac{\delta V_{i\alpha}}{V^r_{i\alpha}}\right) V^r_{i\alpha} + \mathcal{O}(\alpha^2) \\ &= Z^{\text{CKM}}_{i\alpha} V^r_{i\alpha}, \end{aligned} \tag{7.4}$$

where no summation over fermion indices occurs. The upper index CKM stays for Cabibbo-Kobayashi-Maskawa. Inserting all the renormalized parameters in (7.1), then

$$g^L_{i\alpha,W} = \frac{1}{\sqrt{2}Z_{s_W} s^r_W} Z^{\text{CKM}}_{i\alpha} V^r_{i\alpha}. \tag{7.5}$$

If we compare the renormalized quantities with the general parameters introduced in (6.6) and (6.7), we identify

$$g^{rL}_{i\alpha,W} = \frac{1}{\sqrt{2}s^r_W} V^r_{i\alpha}, \tag{7.6}$$

and

$$\begin{aligned} Z^L_{(i\alpha,W)} &= \frac{1}{Z_{s_W}}, \\ Z^{rL}_{i\alpha,W} &= Z^{\mathrm{CKM}}_{i\alpha}, \end{aligned} \tag{7.7}$$

as already mentioned in section 6.1.1. Substituting $Z^L_{(ij,v)}$, $Z^{rL}_{ij,v}$ and $g^{rL}_{ij,v}$ from (6.11) with the particular expressions from (7.6) and (7.7), we obtain the transformed coupling constant $G^{rL}_{j\beta,W}$, the constant that describes the vertex and its counter term:

$$G^{rL}_{j\beta,W} = \frac{1}{\sqrt{2}s^r_W} \frac{Z_e}{Z_{s_W}} Z_W^{\frac{1}{2}} \sum_{\substack{i=u,c,t \\ \alpha=d,s,b}} \overline{Z}^{r\frac{1}{2}L}_{ji} \left(Z^{\mathrm{CKM}}_{i\alpha} V^r_{i\alpha} \right) Z^{r\frac{1}{2}L}_{\alpha\beta}. \tag{7.8}$$

If we insert all the expansions of the renormalization constants, we can write the one-loop approximation for the coupling as

$$\begin{aligned} G^{rL}_{j\beta,W} =& \frac{1}{\sqrt{2}s^r_W} V^r_{j\beta} \left(1 - \frac{\delta s_W}{s^r_W} + \frac{\delta e}{e^r} + \frac{1}{2}\delta Z_W \right) + \frac{1}{\sqrt{2}s^r_W} \delta V_{j\beta} \\ &+ \frac{1}{\sqrt{2}s^r_W} \sum_{i=u,c,t} \frac{1}{2} \delta \overline{Z}^{rL}_{ji} V^r_{i\beta} + \frac{1}{\sqrt{2}s^r_W} \sum_{\alpha=d,s,b} V^r_{j\alpha} \frac{1}{2} \delta Z^{rL}_{\alpha\beta} + \mathcal{O}(\alpha^2). \end{aligned} \tag{7.9}$$

To simplify notations, we take

$$\delta_r = -\frac{\delta s_W}{s^r_W} + \frac{\delta e}{e^r} + \frac{1}{2}\delta Z_W, \tag{7.10}$$

for the coefficient of $g^{rL}_{j\beta,W}$ in $\delta g^L_{j\beta,W}$, (6.14). Then,

$$\delta g^L_{j\beta,W} = \delta_r g^{rL}_{j\beta,W}. \tag{7.11}$$

Note that δ_r is not necessarily real, since δZ_W might also include imaginary parts from absorptive contributions of the W-boson self-energy. As in (6.15), we identify the first order correction coming with the fermion mixing. It is comprised in

$$\delta g^{rL}_{j\beta,W} = \frac{1}{\sqrt{2}s^r_W} \left(\delta V_{j\beta} + \sum_{i=u,c,t} \frac{1}{2} \delta \overline{Z}^{rL}_{ji} V^r_{i\beta} + \sum_{\alpha=d,s,b} V^r_{j\alpha} \frac{1}{2} \delta Z^{rL}_{\alpha\beta} \right). \tag{7.12}$$

Finally, we can write the shorter version of the modified coupling constant that will describe the quark-antiquark-W vertex at one-loop:

$$\begin{aligned} G^{rL}_{i\alpha,W} &= \frac{1}{\sqrt{2}s^r_W} V^r_{i\alpha}(1+\delta_r) + \delta g^{rL}_{i\alpha,W} + \mathcal{O}(\alpha^2) \\ &= g^{rL}_{i\alpha,W}\left(1+\delta_r+\frac{\delta g^{rL}_{i\alpha,W}}{g^{rL}_{i\alpha,W}}\right) + \mathcal{O}(\alpha^2). \end{aligned} \tag{7.13}$$

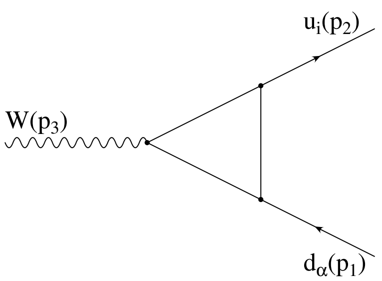
Inserting $G^{rL}_{i\alpha,W}$ in the general expression of the amplitude for the lowest order vertex plus the counter term (formula (6.71)), we obtain a first contribution to the total one-loop amplitude. For example, for the decay of the W-boson into a quark-antiquark pair we have

$$\mathcal{T}^r_{i\alpha} = i\frac{e^r}{\sqrt{2}s^r_W} V^r_{i\alpha}\overline{u}_i(p_2)\not{\varepsilon}(p_3)\left(1+\delta_r+\frac{\delta g^{rL}_{i\alpha,W}}{g^{rL}_{i\alpha,W}}\right)\gamma_L v_\alpha(p_1). \tag{7.14}$$

V_μ in (6.71) was replaced with ε_μ, the polarisation vector of the W-boson and the Dirac spinors were fitted according to the considered process. p_1, p_2, p_3 stand for the momenta of the particles.

7.1.2 Corrections from the One-loop Diagrams

Section 6.5 has offered a general description of the one-loop diagrams contributing to the amplitude. Here, we can be more specific. As example, we will continue with the evaluation of the amplitude for the W-boson decay into a quark-antiquark pair. However, one can easily switch to another process by changing the role of incoming and outgoing particles. The typical one-loop vertex diagram



gives the total contribution: $\mathcal{T}^{\text{vertex}}_{i\alpha} = \mathcal{T}^{\text{vert}}_{i\alpha} V^r_{i\alpha}$. (7.15)

By $\mathcal{T}^{\text{vert}}_{i\alpha}$ we denoted all factors in the amplitude for virtual vertex corrections, except the mixing matrix. To understand why we are able to factorise the quark mixing matrix, one can look at the diagrams contributing to the vertex corrections, given in Figure 7.1. It is easy to see that in the Standard Model, in every one-loop diagram, there is just one vertex that involves $V^r_{i\alpha}$.

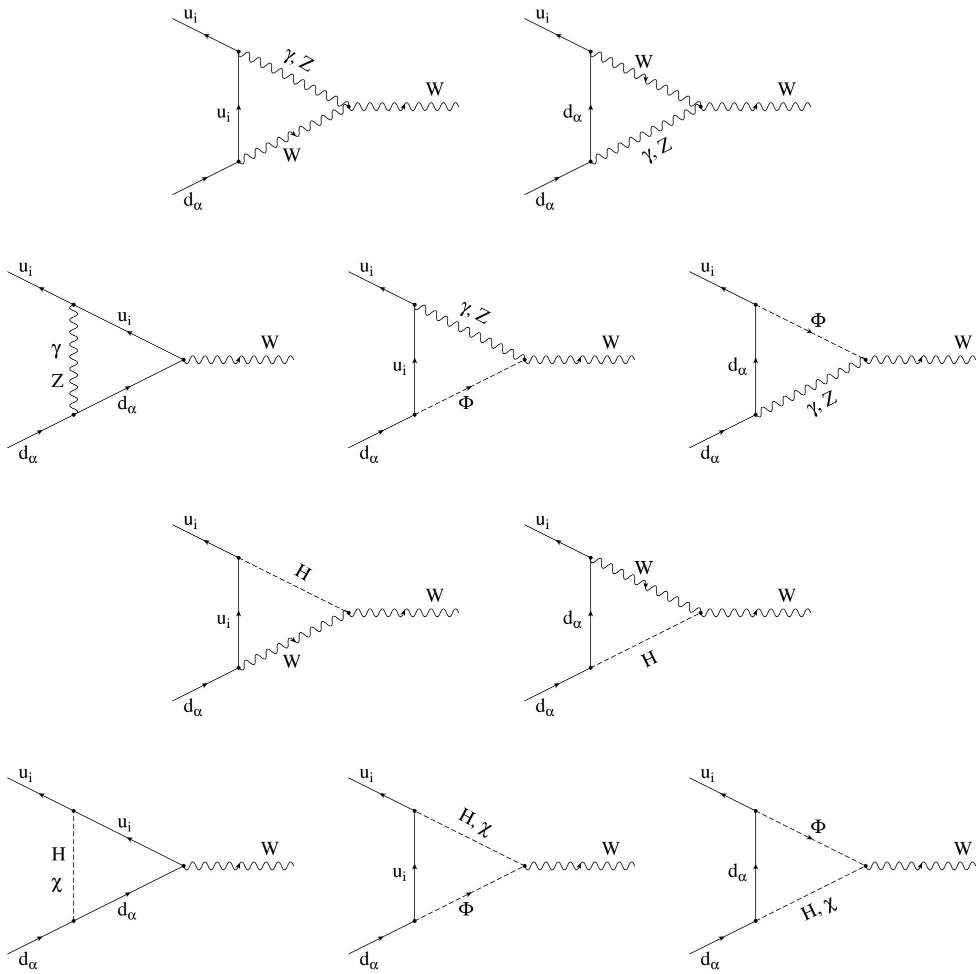


Figure 7.1: Vertex corrections for the quark-antiquark-W vertex

$\mathcal{T}_{i\alpha}^{\text{vert}}$ comes as a sum of four types of Dirac matrix elements:

$$\mathcal{T}_{i\alpha}^{\text{vert}} = \mathcal{T}_0\delta^0_{\text{vert}} + \mathcal{T}_1\delta^1_{\text{vert}} + \mathcal{T}_2\delta^2_{\text{vert}} + \mathcal{T}_3\delta^3_{\text{vert}}, \tag{7.16}$$

where $\mathcal{T}_0$ is the lowest order amplitude of the vertex, without quark mixing:

$$\mathcal{T}_0 = i\frac{e^r}{\sqrt{2}s_W^r}\overline{u}_i(p_2)\not{\epsilon}(p_3)\gamma_L v_\alpha(p_1), \tag{7.17}$$

and the other three matrix elements are:

$$\mathcal{T}_1 = i\frac{e^r}{\sqrt{2}s_W^r}\overline{u}_i(p_2)\not{\varepsilon}(p_3)\gamma_R v_\alpha(p_1), \tag{7.18}$$

$$\mathcal{T}_2 = i\frac{e^r}{\sqrt{2}s_W^r}\overline{u}_i(p_2)\gamma_L v_\alpha(p_1)(\varepsilon(p_3)p_2), \tag{7.19}$$

$$\mathcal{T}_3 = i\frac{e^r}{\sqrt{2}s_W^r}\overline{u}_i(p_2)\gamma_R v_\alpha(p_1)(\varepsilon(p_3)p_2). \tag{7.20}$$

We have directly replaced e and s_W with the renormalized parameters since all the matrix elements are multiplied with factors of order α. We will not calculate all these contributions. It is not our purpose and one can find complete results in the literature (see for example [Den90b]). We just spot that since the matrix elements $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ do not appear at the lowest order, δ^1_{vert}, δ^2_{vert} and δ^3_{vert} are finite and gauge independent.

Before writing the final total amplitude of the process, we still have to consider the terms coming from the self-energies of the external legs. The related diagrams are given in Figure 7.2. Rewriting (6.73) and (6.74) for the W decay into a quark-

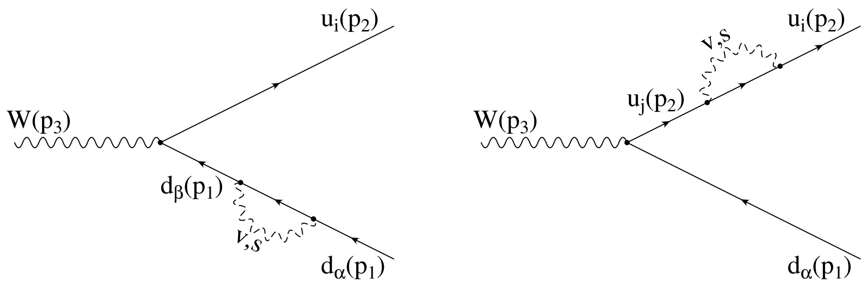


Figure 7.2: Self-energy corrections for the quark-antiquark-W vertex

antiquark pair, the contributions to the amplitude from the down- and up-type quark self-energy corrections are:

$$\mathcal{T}_{i\alpha}^{\alpha-se} = i\frac{e^r}{\sqrt{2}s_W^r}\overline{u}_i(p_2)\not{\varepsilon}(p_3)\gamma_L\sum_{\beta=d,s,b}V^r_{i\beta}\frac{1}{\not{p}_1 - M_\beta}\mathcal{R}_{\beta\alpha}(p_1)v_\alpha(p_1), \tag{7.21}$$

$$\mathcal{T}_{i\alpha}^{i-se} = i\frac{e^r}{\sqrt{2}s_W^r}\overline{u}_i(p_2)\sum_{i=u,c,t}\mathcal{R}_{ij}(p_2)\frac{1}{\not{p}_2 - M_j}V^r_{j\alpha}\not{\varepsilon}(p_3)\gamma_L v_\alpha(p_1). \tag{7.22}$$

The total amplitude is calculated as the sum of (7.14), (7.16) times the mixing matrix, (7.21) and (7.22) and it amounts to

$$\mathcal{T}_{i\alpha}^{tot} = \mathcal{T}_0 V_{i\alpha}^r \left(1 + \delta_r + \delta_{\text{vert}}^0 + \frac{\delta g_{i\alpha,W}^{rL}}{g_{i\alpha,W}^{rL}}\right) + V_{i\alpha}^r \sum_{k=1,2,3} \mathcal{T}_k \delta_{\text{vert}}^k + \mathcal{T}_{i\alpha}^{\alpha-se} + \mathcal{T}_{i\alpha}^{i-se} + \mathcal{O}(\alpha^2). \tag{7.23}$$

The corrections of order α are comprised in δ_r, δ_{vert}^k ($k = 0, 1, 2, 3$), $\delta g_{i\alpha,W}^{rL}$ and in the two terms that include the self-energies of external legs. To recapitulate: $\delta g_{i\alpha,W}^{rL}$ is defined in (7.12) as a sum over renormalization constants related to fermion fields and their mixing, δ_r, given in (7.10), adds up the remaining corrections from the fermion coupling to the W and δ_{vert}^k keeps track of the one-loop vertex diagram corrections.

7.1.3 Discussion of the One-loop Renormalized Quark Mixing Matrix

The unrenormalized quark mixing matrix is a unitary matrix, i.e. $V^\dagger V = V V^\dagger = \mathbf{1}$. The unitarity of the quark mixing matrix is a consequence of the invariance of the action under BRS (Becchi-Rouet-Stora) transformations, as shown in [Den04]. We will have to check whether the corrections performed on the matrix preserve its unitarity, i.e. whether

$$\begin{cases} \sum_{i=u,c,t} (V_{i\alpha}^r)^* V_{i\beta}^r = \delta_{\alpha\beta}, \\ \sum_{\alpha=d,s,b} V_{i\alpha}^r (V_{j\alpha}^r)^* = \delta_{ij}. \end{cases} \tag{7.24}$$

Starting from the unitarity of the unrenormalized mixing matrix, one can show that, for the first order terms, it implies:

$$\begin{cases} \sum_{i=u,c,t} (V_{i\alpha}^r)^* \delta V_{i\beta} = - \sum_{i=u,c,t} (\delta V_{i\alpha})^* V_{i\beta}^r, \\ \sum_{\alpha=d,s,b} V_{i\alpha}^r (\delta V_{j\alpha})^* = - \sum_{\alpha=d,s,b} \delta V_{i\alpha} (V_{j\alpha}^r)^*. \end{cases} \tag{7.25}$$

Now, let us look at the ultraviolet divergences that enter $\mathcal{T}_{i\alpha}^{tot}$. As discussed in the previous chapter (section 6.5), the sum of the divergences from all the correction constants in (7.23) has to be zero. For the quarks, the condition (6.83) is equivalent to

$$\text{div}\left[\delta_r + \delta_{\text{vert}}^0 + \frac{\delta g_{i\alpha,W}^{rL}}{g_{i\alpha,W}^{rL}}\right] = 0, \tag{7.26}$$

since all the other contributions in $\mathcal{T}_{i\alpha}^{tot}$ are finite.

To analyse the contributions in (7.26), we assume first a model for which the quark mixing matrix is equal to unity ($V_{i\alpha} = \delta_{i\alpha}$). Without mixing, the amplitude is finite (provided that δ_r is chosen properly) and the counter term δV is not required, hence $\delta V_{i\alpha}$=0. Note that in the vertex corrections we are able to factorise the mixing matrix and therefore, the divergences in δ^0_{vert} are independent of its presence. The same is valid for the divergences of δ_r, this time as a consequence of the unitarity of the mixing matrix. In both models, the Standard Model and the simplified one, the divergences from δ^0_{vert} and δ_r are equal. For the simplified case, (7.26) is equivalent to

$$\text{div}\left[\delta_r + \delta^0_{\text{vert}} + \left.\frac{\delta g^{rL}_{i\alpha,W}}{g^{rL}_{i\alpha,W}}\right|_{V_{i\alpha}=\delta_{i\alpha}}\right] = 0, \tag{7.27}$$

as calculated in [Den90b] and used in [Den90a]. In other words, if there is no mixing in the theory, one can prove that the total amplitude is finite. All the other divergences in (7.26) are on account of the presence of the quark mixing matrix. If we substract (7.27) from (7.26), we are left with the constraint

$$\text{div}\left[\frac{\delta g^{rL}_{i\alpha,W}}{g^{rL}_{i\alpha,W}} - \left.\frac{\delta g^{rL}_{i\alpha,W}}{g^{rL}_{i\alpha,W}}\right|_{V_{i\alpha}=\delta_{i\alpha}}\right] = 0. \tag{7.28}$$

To separate the correction of the quark mixing matrix, we insert $\delta g^{rL}_{i\alpha,W}$ and $g^{rL}_{i\alpha,W}$ from (7.12) and (7.6). Then,

$$\begin{aligned}\text{div}\Bigg[\delta V_{i\alpha} + \sum_{j=u,c,t}\frac{1}{2}\delta\overline{Z}^{rL}_{ij}V^r_{j\alpha} + \sum_{\beta=d,s,b}V^r_{i\beta}\frac{1}{2}\delta Z^{rL}_{\beta\alpha}& \\ -V_{i\alpha}\frac{1}{2}\left(\left.\delta\overline{Z}^{rL}_{ii}\right|_{V_{i\alpha}=\delta_{i\alpha}} + \left.\delta Z^{rL}_{\alpha\alpha}\right|_{V_{i\alpha}=\delta_{i\alpha}}\right)\Bigg] &= 0.\end{aligned} \tag{7.29}$$

For an unity mixing matrix, there are no non-diagonal fermion self-energies and therefore, the field renormalization constants in $\delta g^{rL}_{i\alpha,W}(V_{i\alpha} = \delta_{i\alpha})$ are diagonal. This is why for the renormalization constants calculated for the unity V we wrote just the diagonal parts.

We can conclude now that, as a minimum requirement, $\delta V_{i\alpha}$ should be defined such that (7.29) holds. So, we can fix its divergent part by

$$\text{div}[\delta V_{i\alpha}] = \frac{1}{2}\text{div}\left[V_{i\alpha}\left.\left(\delta\overline{Z}^{rL}_{ii} + \delta Z^{rL}_{\alpha\alpha}\right)\right|_{V_{i\alpha}=\delta_{i\alpha}} - \sum_{j=u,c,t}\delta\overline{Z}^{rL}_{ij}V^r_{j\alpha} - \sum_{\beta=d,s,b}V^r_{i\beta}\delta Z^{rL}_{\beta\alpha}\right]. \tag{7.30}$$

This property of δV was first shown by [Den90a].

At this point, we have to check that $\mathrm{div}[\delta V](V^r)^\dagger$ fulfils the anti-hermiticity condition (7.25) required for a unitary renormalized mixing matrix. Firstly, we select the anti-hermitian parts present in (7.30). For this, we construct anti-hermitian combinations of type $Z - Z^\dagger$. The terms in (7.30) can be rearranged to

$$\begin{aligned}\mathrm{div}[\delta V_{i\alpha}] =& \frac{1}{4}\mathrm{div}\left[\sum_{j=u,c,t}\left((\delta\overline{Z}^{rL}_{ji})^* - \delta\overline{Z}^{rL}_{ij}\right)V^r_{j\alpha} - \sum_{\beta=d,s,b}V^r_{i\beta}\left(\delta Z^{rL}_{\beta\alpha} - (\delta Z^{rL}_{\alpha\beta})^*\right)\right]\\ &-\frac{1}{4}\mathrm{div}\left[\sum_{j=u,c,t}\left((\delta\overline{Z}^{rL}_{ji})^* + \delta\overline{Z}^{rL}_{ij}\right)V^r_{j\alpha} + \sum_{\beta=d,s,b}V^r_{i\beta}\left(\delta Z^{rL}_{\beta\alpha} + (\delta Z^{rL}_{\alpha\beta})^*\right)\right]\\ &+\frac{1}{2}\mathrm{div}\left[V_{i\alpha}\left(\delta\overline{Z}^{rL}_{ii} + \delta Z^{rL}_{\alpha\alpha}\right)\Big|_{V_{i\alpha}=\delta_{i\alpha}}\right]. \end{aligned} \tag{7.31}$$

The first line of the equation contains the anti-hermitian pieces, the second line the remainder from their construction and the last line the unchanged part from (7.30) that was keeping track of the divergences already absorbed by δ_r and δ^0_{vert}. By direct computation[1] one can prove that the sum of the last two lines is zero, i.e. the hermitian piece (the second line) is exactly the divergence cancelled by δ_r and δ^0_{vert}. Hence,

$$\mathrm{div}[\delta V_{i\alpha}] = \frac{1}{4}\sum_{j=u,c,t}\mathrm{div}\left[(\delta\overline{Z}^{rL}_{ji})^* - \delta\overline{Z}^{rL}_{ij}\right]V^r_{j\alpha} - \frac{1}{4}\sum_{\beta=d,s,b}V^r_{i\beta}\mathrm{div}\left[\delta Z^{rL}_{\beta\alpha} - (\delta Z^{rL}_{\alpha\beta})^*\right]. \tag{7.32}$$

Let us repeat the calculation for the complex conjugated counter term $(\delta V_{i\alpha})^*$. We will consider a process for which the coupling is $\frac{1}{\sqrt{2}s_W}V^*_{i\alpha}$. Accordingly, the Dirac conjugated field will be the one for the down-type quark and the correction to the coupling constant from the fields (6.15) will be

$$\delta g^{rL}_{\alpha i,W} = \frac{1}{\sqrt{2}s^r_W}\left((\delta V_{i\alpha})^* + \sum_{\beta=d,s,b}\frac{1}{2}\delta\overline{Z}^{rL}_{\alpha\beta}(V^r_{i\beta})^* + \sum_{j=u,c,t}(V^r_{j\alpha})^*\frac{1}{2}\delta Z^{rL}_{ji}\right). \tag{7.33}$$

If we split the terms involving field renormalization constants into anti-hermitian

[1] We have calculated the renormalization constants using the GiNaC library for computer algebra [GiNaC].

and hermitian pieces as we did in (7.31), then

$$\delta g^{rL}_{\alpha i,W} = \frac{1}{\sqrt{2}s^r_W}\frac{1}{4}\left(\sum_{\beta=d,s,b}\left(\delta\overline{Z}^{rL}_{\alpha\beta} - (\delta\overline{Z}^{rL}_{\beta\alpha})^*\right)(V^r_{i\beta})^* - \sum_{j=u,c,t}(V^r_{j\alpha})^*\left((\delta Z^{rL}_{ij})^* - \delta Z^{rL}_{ji}\right)\right)$$
$$+ \frac{1}{\sqrt{2}s^r_W}\frac{1}{4}\left(\sum_{\beta=d,s,b}\left(\delta\overline{Z}^{rL}_{\alpha\beta} + (\delta\overline{Z}^{rL}_{\beta\alpha})^*\right)(V^r_{i\beta})^* + \sum_{j=u,c,t}(V^r_{j\alpha})^*\left((\delta Z^{rL}_{ij})^* + \delta Z^{rL}_{ji}\right)\right)$$
$$+ \frac{1}{\sqrt{2}s^r_W}(\delta V_{i\alpha})^*. \tag{7.34}$$

Based on the same arguments as above, we can identify the divergence of $(\delta V_{i\alpha})^*$ such that it cancels with the divergence of the anti-hermitian contribution of the field renormalization constants.

$$\mathrm{div}[(\delta V_{i\alpha})^*] = \frac{1}{4}\sum_{j=u,c,t}(V^r_{j\alpha})^*\mathrm{div}\left[(\delta Z^{rL}_{ij})^* - \delta Z^{rL}_{ji}\right]$$
$$- \frac{1}{4}\sum_{\beta=d,s,b}\mathrm{div}\left[\delta\overline{Z}^{rL}_{\alpha\beta} - (\delta\overline{Z}^{rL}_{\beta\alpha})^*\right](V^r_{i\beta})^* \tag{7.35}$$

To prove that (7.32) and (7.35) are related by complex conjugation and that the divergence of the mixing matrix counter term satisfies the unitarity requirement (7.25), we will take advantage of the properties of the divergences in the self-energy components. These properties were given in (4.58) and they were allowing us to check that the divergent parts of the field renormalization constants were fulfilling the hermiticity condition (see (5.123)). Here, we need the first relation of (5.123). For the up-type quarks we have

$$\mathrm{div}[\delta\overline{Z}^{rL}_{ij}] = (\mathrm{div}[\delta Z^{rL}_{ji}])^*, \tag{7.36}$$

and for the down ones

$$\mathrm{div}[\delta\overline{Z}^{rL}_{\beta\alpha}] = (\mathrm{div}[\delta Z^{rL}_{\alpha\beta}])^*. \tag{7.37}$$

With this feature of the divergences, one can prove that (7.35) is indeed the complex conjugated of (7.32) and that the restriction (7.25) from unitarity is true. From (7.32), one can finally write the divergence of the quark mixing matrix as

$$\mathrm{div}[\delta V_{i\alpha}] = \frac{1}{4}\sum_{j=u,c,t}\mathrm{div}\left[\delta Z^{rL}_{ij} - (\delta Z^{rL}_{ji})^*\right]V^r_{j\alpha} - \frac{1}{4}\sum_{\beta=d,s,b}V^r_{i\beta}\mathrm{div}\left[\delta Z^{rL}_{\beta\alpha} - (\delta Z^{rL}_{\alpha\beta})^*\right]. \tag{7.38}$$

In conclusion, we are able to prove that the divergences that are absorbed in $\delta V_{i\alpha}$ preserve the unitarity of the mixing matrix after renormalization. The question left is how to fix the finite parts of $\delta V_{i\alpha}$ such that we do not destroy the anti-hermiticity condition (7.25).

If not only the divergent parts of the field renormalization constants were related by hermiticity, but the constants themselves, it would be reasonable to extend (7.38) over the finite parts, too. This was the approach proposed firstly in [Den90a] (see also [Den93]). Like this, $\delta V_{i\alpha}$ absorbs all the anti-hermitian contributions from the field renormalization constants, including the finite ones. Later, it was discovered that even if this definition yields a unitary renormalized matrix, there is another problem: it leads to a gauge dependent δV [Gam99]. The consequence is a gauge parameter dependence of the physical matrix elements. The authors in [Gam99] claimed that this gauge dependence was a consequence of the on-shell scheme used in [Den90a] to fix field renormalization constants. They proposed then a renormalization scheme using field renormalization constants defined at zero-momentum.

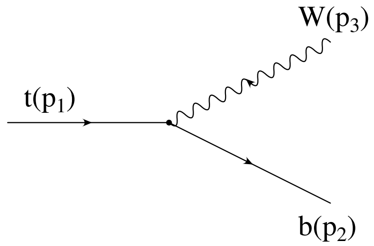
In the following years, other methods to deal with the gauge parameter dependence were introduced (e.g. [Bar00], [Yam01], [Die01], [Pil02], [Zho03]). Another on-shell approach can also be found in [Kni06]. As a solution to the scheme dependence problem, [Den04] proposes to fix δV directly on the physical matrix elements. Because of the problems following from a non-hermitian Lagrangian, the authors choose to treat just the dispersive parts of the self-energies.

It is of course not satisfactory to omit imaginary parts by a simple prescription. The imaginary parts are a prediction of the theory and describe important physical phenomena, e.g. decay processes of heavy quarks. At higher-orders, the situation becomes even more complicated. The imaginary contributions from the self-energy receive contributions from two sources: one is, as at the one-loop level, the loop integrals and the other one the complex coupling constants (like CP-violating phases in the quark mixing matrix). It is not obvious how these two sources can be separated. For now, the alternative is to determine how the quark mixing matrix correction is involved in a decay or a cross section and to identify observables that can be used to fix the counterterms, at least in principle. If all other parameters entering an amplitude are determined, one might be able to extract the contribution of the mixing matrix renormalization constant. In the next section, we consider as an example the one-loop corrections to the decay of a top quark.

7.2 Top Decay Rate

To study the changes induced by the one-loop corrections in a process, we choose the top decay into a W and a bottom quark. We select the bottom quark because this decay has the highest rate, but we do not really use its special properties. In principal, one can replace it with any down-type quark.

In lowest order, the Feynman diagram reads



$$i\frac{e}{\sqrt{2}s_W}\,V_{tb}^*\gamma^\mu\gamma_L.$$

The parameters in the coupling are renormalized according to (7.2) and (7.4), such that the renormalized coupling constant is

$$g_{bt,W}^{rL}=\frac{1}{\sqrt{2}s_W^r}(V_{tb}^r)^*. \tag{7.39}$$

Similar to (7.23), we write the total amplitude, in first order:

$$\begin{aligned}\mathcal{T}_{bt}^{tot}=&\mathcal{T}_0(V_{tb}^r)^*\left(1+\delta_r+\delta_{\text{vert}}^0+\frac{\delta g_{bt,W}^{rL}}{g_{bt,W}^{rL}}\right)+(V_{tb}^r)^*\sum_{k=1,2,3}\mathcal{T}_k\delta_{\text{vert}}^k\\&+\mathcal{T}_{bt}^{t-se}+\mathcal{T}_{bt}^{b-se}+\mathcal{O}(\alpha^2),\end{aligned} \tag{7.40}$$

where

$$\mathcal{T}_0=i\frac{e^r}{\sqrt{2}s_W^r}\overline{u}_b(p_2)\not{\varepsilon}^*(p_3)\gamma_L u_t(p_1), \tag{7.41}$$

$$\mathcal{T}_1=i\frac{e^r}{\sqrt{2}s_W^r}\overline{u}_b(p_2)\not{\varepsilon}^*(p_3)\gamma_R u_t(p_1), \tag{7.42}$$

$$\mathcal{T}_2=i\frac{e^r}{\sqrt{2}s_W^r}\overline{u}_b(p_2)\gamma_L u_t(p_1)(\varepsilon^*(p_3)p_1), \tag{7.43}$$

$$\mathcal{T}_3=i\frac{e^r}{\sqrt{2}s_W^r}\overline{u}_b(p_2)\gamma_R u_t(p_1)(\varepsilon^*(p_3)p_1), \tag{7.44}$$

$$\mathcal{T}_{bt}^{t-se}=i\frac{e^r}{\sqrt{2}s_W^r}\overline{u}_b(p_2)\not{\varepsilon}^*(p_3)\gamma_L\sum_{i=u,c,t}(V_{ib}^r)^*\frac{1}{\not{p}_1-M_i}\mathcal{R}_{it}(p_1)u_t(p_1), \tag{7.45}$$

$$\mathcal{T}_{bt}^{b-se}=i\frac{e^r}{\sqrt{2}s_W^r}\overline{u}_b(p_2)\sum_{\alpha=d,s,b}\mathcal{R}_{b\alpha}(p_2)\frac{1}{\not{p}_2-M_\alpha}(V_{t\alpha}^r)^*\not{\varepsilon}^*(p_3)\gamma_L u_t(p_1), \tag{7.46}$$

and

$$\delta g_{bt,W}^{rL}=\frac{1}{\sqrt{2}s_W^r}\left((\delta V_{tb})^*+\sum_{\alpha=d,s,b}\frac{1}{2}\delta\overline{Z}_{b\alpha}^{rL}(V_{t\alpha}^r)^*+\sum_{i=u,c,t}(V_{ib}^r)^*\frac{1}{2}\delta Z_{it}^{rL}\right). \tag{7.47}$$

In order to simplify the following relations, we note that in the differential decay rate, after summing all terms in the squared amplitude, one can factorise $\sum |\mathcal{T}_0|^2$. Therefore, we include all the vertex corrections in a factor denoted by $\delta_{\rm vert}$. Following the conventions in [Den90b][2],

$$\delta_{\rm vert} = \delta^0_{\rm vert} + \sum_{k=1,2,3} \delta^k_{\rm vert} \frac{G_k}{G_0}, \tag{7.48}$$

with

$$G_0 = \sum_{\rm spins} |\mathcal{T}_0|^2, \tag{7.49}$$

$$G_k = \sum_{\rm spins} \mathcal{T}_0 (\mathcal{T}_k)^\dagger. \tag{7.50}$$

This way, for the one-loop calculations, we write the squared amplitude for the vertices as

$$(\mathcal{T}_0)^\dagger V^r_{tb} \mathcal{T}^{\rm vertex}_{bt} + (\mathcal{T}^{\rm vertex}_{bt})^\dagger \mathcal{T}_0 (V^r_{tb})^* = |\mathcal{T}_0|^2 |V^r_{tb}|^2 \left(\delta_{\rm vert} + (\delta_{\rm vert})^*\right), \tag{7.51}$$

where

$$\mathcal{T}^{\rm vertex}_{bt} = (V^r_{tb})^* \left(\mathcal{T}_0 \delta^0_{\rm vert} + \sum_{k=1,2,3} \mathcal{T}_k \delta^k_{\rm vert} \right). \tag{7.52}$$

The differential decay rate, given as the rate of transition of the top quark into a W and a bottom quark is

$$d\Gamma^{tot}(t \to Wb) = \frac{1}{8\pi^2 m_t} \delta(p_1 - p_2 - p_3) \frac{d^3p_2}{2p^0_2} \frac{d^3p_3}{2p^0_3} \sum_{\rm spins} |\mathcal{T}^{tot}_{bt}|^2. \tag{7.53}$$

For this definition we used the conventions of [Nac90]. The sum over spins should be interpreted as an average over the spin directions of the incoming particle and as a summation over the ones in the final state. The total rate is given by the integral over the phase space.

We write the total squared amplitude as a sum of two terms to separate the contributions coming from the self-energy corrections of external legs:

$$|\mathcal{T}^{tot}_{bt}|^2 = \mathcal{P}^{\rm vert}_{bt} + \mathcal{P}^{se}_{bt}. \tag{7.54}$$

[2] We do not use exactly the same Dirac matrix elements since in our case, the role of some incoming and outgoing particles is interchanged (from $W \to t\bar{b}$ to $t \to Wb$, see also [Den91]). In addition, some mass factors explicitly written in [Den90b] are included for us in $\delta^i_{\rm vert}$.

From (7.40) we distinguish

$$\mathcal{P}_{bt}^{\text{vert}} = |\mathcal{T}_0|^2 |V_{tb}^r|^2 \left(1 + \delta_r + (\delta_r)^* + \delta_{\text{vert}} + (\delta_{\text{vert}})^* + \frac{\delta g_{bt,W}^{rL}}{g_{bt,W}^{rL}} + \left(\frac{\delta g_{bt,W}^{rL}}{g_{bt,W}^{rL}} \right)^* \right) \quad (7.55)$$

$$\mathcal{P}_{bt}^{se} = (\mathcal{T}_0)^\dagger V_{tb}^r (\mathcal{T}_{bt}^{t-se} + \mathcal{T}_{bt}^{b-se}) + (\mathcal{T}_{bt}^{t-se} + \mathcal{T}_{bt}^{b-se})^\dagger \mathcal{T}_0 (V_{tb}^r)^* + \mathcal{O}(\alpha^2). \quad (7.56)$$

In agreement with (7.54), the total one-loop decay rate can also be split into two terms: $\Gamma^{\text{vert}}(t \to Wb)$ coming with $\mathcal{P}_{bt}^{\text{vert}}$ and $\Gamma^{se}(t \to Wb)$ with $\mathcal{P}_{bt}^{se}$:

$$\Gamma^{tot}(t \to Wb) = \Gamma^{\text{vert}}(t \to Wb) + \Gamma^{se}(t \to Wb). \quad (7.57)$$

The lowest order total decay rate Γ_0^{tot} results from:

$$d\Gamma_0^{tot}(t \to Wb) = \frac{1}{8\pi^2 m_t} \delta(p_1 - p_2 - p_3) \frac{d^3 p_2}{2p_2^0} \frac{d^3 p_3}{2p_3^0} |V_{tb}|^2 \sum_{\text{spins}} |\mathcal{T}_0|^2. \quad (7.58)$$

Performing the averages and the sums over fermion helicities and W polarisations, calculating the traces and integrating

$$\begin{aligned} \Gamma_0^{tot}(t \to Wb) = & \frac{\alpha}{8 s_W^2} |V_{tb}|^2 \frac{1}{m_t} \sqrt{(m_t^2 - (m_b + m_W)^2)(m_t^2 - (m_b - m_W)^2)} \\ & \left(\frac{m_t^2 + m_b^2}{2m_t^2} + \frac{(m_t - m_b)^2}{2m_t^2 m_W^2} - \frac{m_W^2}{m_t^2} \right). \end{aligned} \quad (7.59)$$

Moving to one-loop, from the first line of (7.55), we identify the decay rate term that does not include self-energy corrections for external legs as

$$\Gamma^{\text{vert}}(t \to Wb) = \Gamma_0 |V_{tb}^r|^2 \left(1 + \delta_r + (\delta_r)^* + \delta_{\text{vert}} + (\delta_{\text{vert}})^* + \frac{\delta g_{bt,W}^{rL}}{g_{bt,W}^{rL}} + \left(\frac{\delta g_{bt,W}^{rL}}{g_{bt,W}^{rL}} \right)^* \right). \quad (7.60)$$

In Γ_0 we have collected all the terms independent of the mixing matrix:

$$\begin{aligned} \Gamma_0 = & \frac{\alpha^r}{8(s_W^r)^2} \frac{1}{m_t} \sqrt{(m_t^2 - (m_b + m_W)^2)(m_t^2 - (m_b - m_W)^2)} \\ & \left(\frac{m_t^2 + m_b^2}{2m_t^2} + \frac{(m_t - m_b)^2}{2m_t^2 m_W^2} - \frac{m_W^2}{m_t^2} \right). \end{aligned} \quad (7.61)$$

The quantities written here are the physical ones.

$\Gamma^{se}(t \to Wb)$ has to be calculated separately since in its expression one can not find a direct factorisation of Γ_0. First, we need to consider the explicit expression of the sums written in (7.45) and (7.46). For the non-diagonal terms, the action of $\mathcal{R}_{it}(p_1)$ and $\mathcal{R}_{b\alpha}(p_2)$ on external spinors will be expressed as in (5.24) and in (5.25), respectively. The diagonal terms of the sums (i.e. $\frac{1}{\not p_1 - M_t}\mathcal{R}_{tt}(p_1)$ and $\mathcal{R}_{bb}(p_2)\frac{1}{\not p_2 - M_b}$) act on external spinors as in (5.31) and (5.32). The relations are lengthy, but easy to evaluate. Therefore, we do not write other intermediate steps and we directly give the result. The term in the total decay rate, on account of the external fermion one-loop corrections is

$$\Gamma^{se}(t \to Wb) = \Gamma_0 |V^r_{tb}|^2 \frac{1}{2}\left(\frac{1}{(V^r_{tb})^*} X_{bt} + \frac{1}{V^r_{tb}}(X_{bt})^*\right), \tag{7.62}$$

where

$$\begin{aligned} X_{bt} = (V^r_{ub})^* \varkappa^L_{ut} + (V^r_{cb})^* \varkappa^L_{ct} + (V^r_{tb})^* (\varkappa^L_{tt} + \overline{\varkappa}^L_{tt}) \\ + \overline{\varkappa}^L_{bd}(V^r_{td})^* + \overline{\varkappa}^L_{bs}(V^r_{ts})^* + (\varkappa^L_{bb} + \overline{\varkappa}^L_{bb})(V^r_{tb})^*. \end{aligned} \tag{7.63}$$

Summing up (7.60) and (7.62), the final expression of the one-loop total rate of the decay of a top quark into a bottom and a W is

$$\begin{aligned} \Gamma^{tot}(t \to Wb) = \Gamma_0 |V^r_{tb}|^2 \Bigg(1 + \delta_r + (\delta_r)^* + \delta_{\text{vert}} + (\delta_{\text{vert}})^* \\ + \frac{(\delta V_{tb})^*}{(V^r_{tb})^*} + \frac{\delta V_{tb}}{V^r_{tb}} + \frac{1}{(V^r_{tb})^*} Y_{bt} + \frac{1}{V^r_{tb}}(Y_{bt})^*\Bigg). \end{aligned} \tag{7.64}$$

We have written explicitly the corrections of the mixing matrix hidden in $\delta g^{rL}_{bt,W}$ and we have collected all the other contributions from the fermion field renormalization constants in Y_{bt}.

$$Y_{bt} = \frac{1}{2}\left(\sum_{i=u,c,t} (V^r_{ib})^* \delta Z^{rL}_{it} + \sum_{\alpha=d,s,b} \delta \overline{Z}^{rL}_{b\alpha}(V^r_{t\alpha})^*\right) + \frac{1}{2} X_{bt}. \tag{7.65}$$

As a short check of our results, one can substitute the field renormalization constants in Y_{bt} with their expressions as a function of $\varkappa$, $\overline{\varkappa}$ and the wave function renormalization constants (5.38). The contributions from $\varkappa$ and $\overline{\varkappa}$ cancel with the ones from $\Gamma^{se}(t \to Wb)$, included in X_{bt}. Y_{bt} will then depend on the wave function renormalization constants and two remainders from X_{bt}, namely $\overline{\varkappa}^L_{tt}$ and $\varkappa^L_{bb}$. The presence of $\overline{\varkappa}^L_{tt}$ and $\varkappa^L_{bb}$ in the final result is a consequence of the freedom one has to set the diagonal field renormalization constants. This freedom was first recognized when calculating the wave function renormalization constants and there, (4.32), we introduced the parameter β_i.

With all the components of the total decay rate evaluated, we now discuss how one can fix the counter term of the quark mixing matrix. In (7.61), we have defined Γ_0 as the factor in the total decay rate that includes all the renormalized parameters except for V^r_{tb}. It means that with the experimental value of $\Gamma^{tot}(t \to Wb)$, we can define

$$|V^r_{tb}|^2 = \frac{1}{\Gamma_0}\Gamma^{tot}(t \to Wb), \tag{7.66}$$

having thus a value for the absolute value of the 'tb' component of the renormalized quark mixing matrix. As a consequence, the sum of all first order corrections left in (7.64) should be zero.

$$\delta_r + (\delta_r)^* + \delta_{\text{vert}} + (\delta_{\text{vert}})^* + \frac{(\delta V_{tb})^*}{(V^r_{tb})^*} + \frac{\delta V_{tb}}{V^r_{tb}} + \frac{1}{(V^r_{tb})^*}Y_{bt} + \frac{1}{V^r_{tb}}(Y_{bt})^* = 0 \tag{7.67}$$

From here, having all the other corrections evaluated, we determine the real part of δV_{tb}, including the finite contributions. This is

$$\text{Re}\frac{\delta V_{tb}}{V^r_{tb}} = -\frac{1}{2}\left(\delta_r + (\delta_r)^* + \delta_{\text{vert}} + (\delta_{\text{vert}})^* + \frac{1}{(V^r_{tb})^*}Y_{bt} + \frac{1}{V^r_{tb}}(Y_{bt})^*\right). \tag{7.68}$$

This is our final result for the complete definition of counter terms for the quark mixing matrix. Since the definition of δV_{tb} is based on the complete matrix element for a physical process, it is gauge independent by construction.

The dependence of Y_{bt} on δZ^{rL}_{it} and $\delta \overline{Z}^{rL}_{b\alpha}$ might raise some doubts regarding the result when the anti-top decay is considered. We remind the reader, that in section 5.3 we have proven that regardless of the particular expression of the self-energy, the total decay rate of the particle is equal to the decay rate of the antiparticle. For the present case, at one-loop, the optical theorem (5.126) can be simplified. Here, we are able to identify which processes in the self-energy give rise to imaginary parts and in addition, to select the combination of internal particles that are related to one specific decay channel. For the top decay into a W-boson and a bottom quark, the two top self-energy diagrams that should be considered are the ones with a virtual bottom, i.e. a bottom and a W and a bottom and a charged Higgs. The optical theorem reads

$$\Gamma^{tot}_0(t \to Wb) = \frac{1}{M_t}\text{Im}[\mathcal{T}(t \to (Wb + \Phi b) \to t)]. \tag{7.69}$$

From the equality of the transition amplitudes for particles and antiparticles (5.133), we have

$$\text{Im}[\mathcal{T}(t \to (Wb + \Phi b) \to t)] = \text{Im}[\mathcal{T}(\bar{t} \to (Wb + \Phi b) \to \bar{t})], \tag{7.70}$$

and with the optical theorem applied for the anti-top, results

$$\Gamma_0^{tot}(t \to Wb) = \Gamma_0^{tot}(\bar{t} \to W\bar{b}). \tag{7.71}$$

The same proof can be applied going to the one-loop corrections of the decay. We can select part by part which imaginary parts from the transition amplitude $\mathcal{T}(t \to t)$ contribute to the top decay and using the same argumentation, we have

$$\Gamma^{tot}(t \to Wb) = \Gamma^{tot}(\bar{t} \to W\bar{b}). \tag{7.72}$$

In conclusion, the absolute value of the '*tb*' component of the renormalized quark mixing matrix (7.66) and the real value of its counter term (7.68) do not change if they are calculated from the anti-top decay.

This was just one example for the determination of the renormalized quark mixing matrix from an experimental measurement. Similar algorithms can be applied to other decay rates or cross sections. It is true that the numerical effects from the quark mixing correction are expected to be small, but they are necessary for a complete renormalization scheme. Further on, when considering the neutrino mixing, the corresponding effects are not expected to be negligible any longer. Depending on the heavy neutrino mass scale, they can lead to corrections of $\mathcal{O}(10\%)$ [Kni96].

Chapter 8

Neutrino Seesaw Mechanism and the Renormalization of the Mixing Matrix

In this chapter, we apply the general renormalization approach beyond the Standard Model, to another up-to-date topic related to mixing, namely the neutrino one. We analyse the neutrinos in the framework of the seesaw mechanism.

The discovery of oscillations in atmospheric neutrinos (1998) [SKam98] opened the series of experimental results showing that the neutrinos change flavour when propagating a macroscopic distance. By now, we have experimental evidences of the neutrino oscillations coming from the study of atmospheric, solar, accelerator and reactor neutrinos [PDG06n]. The mass differences measured to be non-zero indicate that at least two of the neutrinos have masses. We still have to find out how many neutrinos are massive, which is the mass hierarchy, whether there are any sterile neutrinos and which model describes them since the Standard Model alone can not? An important issue related to the theoretical model concerns the nature of neutrinos: are they Dirac or Majorana particles? (The neutrinos' zero electric charge makes them candidates for Majorana fermions.)

A minimal extension of the Standard Model to generate massive neutrinos is the seesaw mechanism. This model can explain very well the smallness of the neutrino masses and it may include both, Dirac and Majorana masses. For seesaw neutrinos, the lepton sector of the Standard Model is enriched with massive right-handed neutrino fields to obtain the seesaw mechanism of type I. Additionally, one can generate a mass for the left-handed fields through a Higgs triplet and then we talk about seesaw mechanism of type II. In this chapter, we will start by describing the main features introduced by the model, i.e. the mass matrix, we write down the terms of the interaction Lagrangian and then we move to the renormalization of the neutrino mixing matrix.

8.1 Seesaw Mass Term

In the seesaw mechanism, neutrinos are described as Majorana particles, but since left- and right-handed fields enter the mass term of the Lagrangian, they can acquire also Dirac masses. We have a so-called Dirac-Majorana mass term. The general formalism was presented in section 2.2.3, but for the convenience of the reader we repeat the most important results in the following. As mentioned there, the situation is specific to models where the lepton flavour numbers L_e, L_μ and L_τ are not conserved.

The Lagrangian mass term is given as in (2.35). It is based on n_L left-handed neutrino fields and on n_R, independent, heavy right-handed neutrino fields. Denoting the neutrino Majorana fields by ν_0, the mass term is written as

$$\begin{aligned}\mathcal{L}^{ss\nu}_{mass} &= -\frac{1}{2}\sum_{\substack{i=\overline{e,n_L}\\ j=\overline{e,n_L}}} \overline{(\nu^L_{0i})^C} M^L_{ij}\nu^L_{0j} - \sum_{\substack{i=\overline{e,n_R}\\ j=\overline{e,n_L}}} \overline{\nu^R_{0i}} M^D_{ij}\nu^L_{0j} - \frac{1}{2}\sum_{\substack{i=\overline{e,n_R}\\ j=\overline{e,n_R}}} \overline{\nu^R_{0i}} M^R_{ij}(\nu^R_{0j})^C + h.c.\\ &= -\frac{1}{2}\begin{pmatrix}\overline{(\nu^L_0)^C} & \overline{\nu^R_0}\end{pmatrix}\begin{pmatrix}M^L & (M^D)^T\\ M^D & M^R\end{pmatrix}\begin{pmatrix}\nu^L_0\\ (\nu^R_0)^C\end{pmatrix} + h.c..\end{aligned} \tag{8.1}$$

The upper script $ss\nu$ indicates the seesaw neutrino model. The matrices M^L, M^D and M^R have the properties described in 2.2.3. They are all complex matrices and M^L and M^R are chosen symmetric. The sums over i and j are taken over the charged lepton flavours $e, \mu, \tau, ..., n_L/n_R$. The overline in the sum symbol indicates that all values in the specified range are taken. We emphasise that we do not restrict the number of the flavours to three or impose an equality between the number of left and right handed fields. This leaves an open door to the existence of additional sterile neutrinos, a possibility raised from experimental results (LSND [LSND96]) and yet under investigation (MiniBooNE [MBNE]).

The presence or the absence of M^L in (8.1) makes the difference between the two types of seesaw mechanisms. In the seesaw type I, all the terms with M^L are dropped. If one considers these terms (seesaw mechanism of type II), an additional Higgs triplet is required for the generation of masses. We will describe the difference between the two seesaw types, the Higgs fields required in both cases, as well as their consequences on the whole model, in the next sections.

The Majorana fields in (8.1) do not correspond to mass eigenstates. In order to work out the particle content of the model, one has to diagonalize the Dirac-Majorana neutrino mass matrix. This can be done with the help of a unitary matrix U, written as in (2.39):

$$U = \begin{pmatrix} U^L \\ (U^R)^* \end{pmatrix}. \tag{8.2}$$

The form of U is chosen such that the mass eigenstates result from the transformation

$$\begin{aligned} \nu_{0i}^L &= \sum_{a=1}^{n_L+n_R} U_{ia}^L \nu_a^L, \\ \nu_{0j}^R &= \sum_{a=1}^{n_L+n_R} U_{ja}^R \nu_a^R. \end{aligned} \tag{8.3}$$

Here, ν_a^L and ν_a^R are the left and the right components of the same Majorana field ν_a. The diagonal mass matrix elements will be m_a, with $a = 1, 2, 3, ..., n_L + n_R$. From here on, the neutrino mass eigenstates will be the ones carrying indices a, b, c, ... As mentioned, they will run until $n_L + n_R$ and we will not write this explicitly any longer. For the other indices, when confusion may arise, we will restate the range. Remember that U^L is $n_L \times (n_L + n_R)$, U^R is $n_R \times (n_L + n_R)$ and

$$\begin{aligned} M^L &= (U^L)^* m (U^L)^\dagger, \\ M^R &= U^R m (U^R)^T, \\ M^D &= U^R m (U^L)^\dagger. \end{aligned} \tag{8.4}$$

The unitarity relations for U (2.40) allow us to prove that

$$(U^R)^\dagger M^D = m(U^L)^\dagger - (U^L)^T M^L. \tag{8.5}$$

In the seesaw mechanism of type I, the conservation of the total lepton number is violated by the presence of the right-handed Majorana mass terms in the Lagrangian. The scale of these terms is assumed much bigger than the scale of the electroweak symmetry breaking, i.e. the elements of M^R are much bigger than the ones of M^D. This will lead to neutrino masses of order $-(M^D)^T(M^R)^{-1}M^D$ for $a = 1, ..., n_L$ and of the order of M^R for $a = n_L+1, ..., n_L+n_R$, if M^L=0. For the general case (seesaw mechanism of type II), we have additional contributions from the massive left-handed neutrinos. The first n_L elements of the mass matrix will then be characterised by $M^L-(M^D)^T(M^R)^{-1}M^D$. Here, the discussion is more complex, because depending on the parameters of the model, M^L or $(M^D)^T(M^R)^{-1}M^D$ can dominate the mass terms for $a = 1, ..., n_L$.

Note that U, (8.2), is not directly identified with the mixing matrix describing physical processes. As we will see in the next chapter, if we consider the charged lepton mass matrix diagonal, than U^L will directly take part in the interaction with the W-boson, giving thus the lepton mixing matrix. In U^L, the first n_L columns refer to the mixing of the light neutrinos. This $n_L \times n_L$ piece of U^L, for n_L=3 is called in the literature the neutrino mixing matrix[1]. This identification is valid up to phase conventions. Several parametrisations for the neutrino mixing matrix have been described in the literature, see for example [MNSP], [Gri01] or [PDG06n]. The light neutrino mixing matrix picked out from U^L is approximately unitary. Hence, in this chapter, we will not restrict the mixing to the light neutrinos and the unitarity relations will still be given by the complete set (2.40), i.e. considering also U^R.

We will not investigate further the implications of the mass scale and of the restrictions for the mixing matrix and we will procede in describing the Lagrangian in the two versions of the seesaw mechanism. We start with the seesaw mechanism of type I and then, we point out the terms to be added for the seesaw mechanism of type II.

8.2 Lagrange Density for Seesaw Type I

In the seesaw mechanism of type I, the neutrino mass term in the Lagrangian is described by (8.1) for M^L=0:

$$\mathcal{L}^{ss\nu I}_{mass} = -\sum_{\substack{i=\overline{e,n_R}\\ j=\overline{e,n_L}}} \overline{\nu^R_{0i}} M^D_{ij} \nu^L_{0j} - \frac{1}{2}\sum_{\substack{i=\overline{e,n_R}\\ j=\overline{e,n_R}}} \overline{\nu^R_{0i}} M^R_{ij} (\nu^R_{0j})^C + h.c. \tag{8.6}$$

The symbol I in the upper index of $\mathcal{L}$ refers to the seesaw mechanism type. Since we believe that the full theory should be gauge invariant (with the SU(2) of isospin being a subgroup of the gauge group), we need to have a mechanism to generate the neutrino masses. Because of its generality, we choose what was called by Grimus and Lavoura, a multi-Higgs-doublet Standard Model (an extension of the Standard Model by Higgs doublets). The details can be found in [GriLav]. Here, we state the various terms contributing to the Lagrangian and we give a short explanation for the Yukawa couplings[2].

As already stated in 8.1, we assume an extended standard model for the leptons, with n_L left-handed isospin doublets, the same number of right-handed charged isospin singlets, but n_R right-handed neutrino singlets. In addition, n_H Higgs doublets are responsable for the generation of Dirac masses. The Yukawa Lagrangian

[1] The matrix is often referred to as Maki-Nakagawa-Sakata matrix or Pontecorvo-Maki-Nakagawa-Sakata matrix after the scientists that firstly discussed the neutrino oscillations and the mixing of mass eigenstates.

[2] The coupling constants we use are equal to the ones in [GriLav] multiplied with $\frac{1}{e}$ because of the convention we have chosen in section 2.3.

of the lepton sector is

$$\mathcal{L}_Y^{ss\nu I} = -e\sum_{k=1}^{n_H}\left(\left(\varphi_k^-, \quad \varphi_k^{0*}\right)\overline{l^R}\Gamma_k + \left(\varphi_k^0, \quad -\varphi_k^+\right)\overline{\nu_0^R}\Delta_k\right)\begin{pmatrix}\nu_0^L \\ l^L\end{pmatrix} - \frac{1}{2}\overline{\nu_0^R}M^R(\nu_0^R)^C + h.c.. \quad (8.7)$$

By l we denote the charged lepton fields. To simplify expressions, we will sometimes omit the flavour indices, as it is the case here. Still one has to be careful with the dimension of matrices and vectors. In (8.7), the field multiplets l^L, l^R and ν_0^L are of size $n_L \times 1$, while ν_0^R is $n_R \times 1$. According to the number of left- and right-handed fields for each type of fermions, the size of the Yukawa coupling matrices will then be $n_L \times n_L$, for Γ_k and $n_R \times n_L$, for Δ_k.

The vacuum expectation values for each of the Higgs fields,

$$\langle 0|\varphi_k^0|0\rangle = \frac{v_k}{\sqrt{2}}, \quad (8.8)$$

will be considered as free parameters. Then, the charged lepton mass matrix will be given by

$$m^l = \frac{e}{\sqrt{2}}\sum_{k=1}^{n_H} v_k^*\Gamma_k, \quad (8.9)$$

and the Dirac neutrino one by

$$M^D = \frac{e}{\sqrt{2}}\sum_{k=1}^{n_H} v_k\Delta_k. \quad (8.10)$$

The interaction with all the neutral and charged scalars will be described later.

We choose a basis where the charged lepton mass matrix is diagonal, i.e.

$$m^l = \mathrm{diag}(m_e, \ m_\mu, \ m_\tau, \ ..., \ m_{n_L}). \quad (8.11)$$

The corresponding mass term in the Lagrangian is given by (2.9) and the kinetic and neutral current interaction terms stay as in the Standard Model.

For the neutrino part, we write the Lagrange density in terms of the mass eigenstate fields ν_a, where

$$\begin{aligned}\nu_a &= \nu_a^L + \nu_a^R \\ &= \nu_a^L + (\nu_a^L)^C.\end{aligned} \quad (8.12)$$

The relation of the left and right components ν^L and ν^R to the original field was given in (8.3). The terms of the Lagrangian will be presented in the following in the same order as in 2.3: first the free part, then the interaction with charged and neutral gauge bosons and at the end the Yukawa interaction terms. The couplings from all the interaction terms will be summarised in two tables given at the end of the next section. In Table 8.1 we will list the fermion couplings to the gauge bosons and in Table 8.2, the Yukawa couplings.

For free fields, the Lagrangian is equal to the Majorana part of equation (2.45):

$$\mathcal{L}_0^{ss\nu} = \frac{1}{2}\sum_a \overline{\nu_a}(i\partial\!\!\!/ - m_a)\nu_a. \tag{8.13}$$

Since (8.13) does not change its form when considering the type II seesaw model, we omitted the index I in $\mathcal{L}_0^{ss\nu}$.

The form of the charged and neutral current interaction terms remains also unmodified when switching from one model type to the other. The charged-current term looks similar to the one for quarks. It is given by

$$\mathcal{L}_{cc}^{ss\nu} = \frac{e}{\sqrt{2}s_W}\sum_{a,i}\bar{l}_i\gamma^\mu U_{ia}^L\gamma_L\nu_a W_\mu^- + \frac{e}{\sqrt{2}s_W}\sum_{a,i}\overline{\nu}_a\gamma^\mu(U_{ia}^L)^*\gamma_L l_i W_\mu^+, \tag{8.14}$$

where l describes the charged lepton fields. In the specific model considered here, the general coupling constants introduced in (2.50) become

$$\begin{aligned}(g_{ai,W}^L)^* &= g_{ia,W}^L = \frac{1}{\sqrt{2}s_W}U_{ia}^L,\\ (g_{ai,W}^R)^* &= g_{ia,W}^R = 0.\end{aligned} \tag{8.15}$$

$g_{ia,W}^L$ is determined by the matrix elements of U^L and therefore, in general, it is complex. As for the charged current coupling constants of quarks which are proportional to the Cabibbo-Kobayashi-Maskawa matrix elements, complex phases are related to CP-violating interactions.

For the interaction with the Z boson, the corresponding term in the Lagrangian is

$$\mathcal{L}_{nc}^{ss\nu} = \frac{e}{4s_W c_W}\sum_{a,b,i}\overline{\nu}_a\gamma^\mu\left((U_{ia}^L)^*U_{ib}^L\gamma_L - U_{ia}^L(U_{ib}^L)^*\gamma_R\right)\nu_b Z_\mu, \tag{8.16}$$

where $c_W = \cos\vartheta_W$ is the cosine of the weak mixing angle. Identifying the general coupling in (2.51):

$$\begin{aligned}g_{ab,Z}^L &= \frac{1}{2s_W c_W}\left((U^L)^\dagger U^L\right)_{ab} = \frac{1}{2s_W c_W}\sum_{i=e}^{n_L}(U_{ia}^L)^*U_{ib}^L,\\ g_{ab,Z}^R &= -\frac{1}{2s_W c_W}\left((U^L)^T(U^L)^*\right)_{ab} = -\frac{1}{2s_W c_W}\sum_{i=e}^{n_L}U_{ia}^L(U_{ib}^L)^* = -(g_{ab,Z}^L)^*.\end{aligned} \tag{8.17}$$

Observe that $g^L_{ab,Z}$ is hermitian and that the additional factor $\frac{1}{2}$ required in the Lagrangian by the Majorana nature of the ν is included (see 2.51). In contrast with the charged current case, these constants formally differ from the neutral current coupling constants for quarks. While in the Standard Model these couplings are real and diagonal, here, the presence of $(U^L)^\dagger U^L \neq 1$ allows for mixing also in the neutral part of the Lagrangian. The non-diagonal $g^L_{ab,Z}$ indicates that neutral-current interactions contribute to neutrino oscillation effects. We will discuss more about $g^L_{ab,Z}$ in section 8.5.

Now, we discuss all the Yukawa interaction terms resulting from (8.7). We start with the couplings to charged bosons and we end with the neutral Yukawa interactions. In general, there will be a non-diagonal mass matrix for the scalars which has first to be diagonalized. Since we are not interested in the details of the Higgs sector, we assume that the corresponding transformation is given. For the charged scalars, the mass eigenfields $S^\pm_s$ are defined by

$$\begin{aligned} \varphi^-_k &= \sum_s c^*_{k,s} S^-_s, \\ \varphi^+_k &= \sum_s c_{k,s} S^+_s, \end{aligned} \tag{8.18}$$

with known complex coefficients $c_{k,s} \in \mathbb{C}^{n_H}$. Consequently, the coupling matrices in (8.7) are re-defined by

$$\begin{aligned} \Gamma_s &= \sum_k c^*_{k,s} \Gamma_k, \\ \Delta_s &= \sum_k c_{k,s} \Delta_k. \end{aligned} \tag{8.19}$$

Then, the intercation of leptons with $S^\pm_s$ can be written as

$$\mathcal{L}^{ss\nu I}_{Yc} = e \sum_{a,i,s} \overline{l}_i \left(c^L_{ia,s}\gamma_L + c^R_{ia,s}\gamma_R \right) \nu_a S^-_s + e \sum_{a,i,s} \overline{\nu}_a \left((c^R_{ia,s})^*\gamma_L + (c^L_{ia,s})^*\gamma_R \right) l_i S^+_s, \tag{8.20}$$

with

$$\begin{aligned} c^L_{ia,s} &= -\left(\Gamma_s U^L\right)_{ia} = -\sum_{j=e}^{n_L} (\Gamma_s)_{ij} U^L_{ja}, \\ c^R_{ia,s} &= \ \ \left(\Delta^\dagger_s U^R\right)_{ia} = \ \ \sum_{j=e}^{n_R} (\Delta_s)^*_{ji} U^R_{ja}, \end{aligned} \tag{8.21}$$

or correspondingly

$$
\begin{aligned}
c^L_{ai,s} &= (c^R_{ia,s})^* = \sum_{j=e}^{n_R}(U^R_{ja})^*(\Delta_s)_{ji}, \\
c^R_{ai,s} &= (c^L_{ia,s})^* = -\sum_{j=e}^{n_L}(U^L_{ja})^*(\Gamma_s)^*_{ij}.
\end{aligned}
\tag{8.22}
$$

One pair of charged scalars that has to be part of the model is related to the W gauge boson. If, $S^\pm_s$=$S^\pm_{s_W}$, i.e. the Goldstone boson corresponding to the longitudinal mode of the W boson, Γ_s and Δ_s are

$$
\begin{aligned}
\Gamma_{s_W} &= \frac{1}{\sqrt{2}s_w}\frac{1}{m_W}m^l, \\
\Delta_{s_W} &= \frac{1}{\sqrt{2}s_w}\frac{1}{m_W}M^D.
\end{aligned}
\tag{8.23}
$$

Further, we consider the Yukawa interaction terms with neutral scalar bosons. In (8.7), the neutral scalar mass eigenfield is given by

$$
\varphi^0_k = \frac{1}{\sqrt{2}}\left(v_k + \sum_{s_0} c_{k,s_0}S^0_{s_0}\right). \tag{8.24}
$$

One can denote the coupling matrix of neutrinos to the neutral scalar eigenfields $S^0_{s_0}$ with

$$
\Delta_{s_0} = \sum_k c_{k,s_0}\Delta_k. \tag{8.25}
$$

Then, the related Lagrangian term is

$$
\mathcal{L}^{ss\nu I}_{Yn} = \frac{1}{2}e\sum_{a,b,s_0}\overline{\nu}_a\left(c^L_{ab,s_0}\gamma_L + (c^L_{ab,s_0})^*\gamma_R\right)\nu_b S^0_{s_0}, \tag{8.26}
$$

where

$$
\begin{aligned}
c^L_{ab,s_0} &= -\frac{1}{\sqrt{2}}\left((U^R)^\dagger\Delta_{s_0}U^L + (U^L)^T\Delta^T_{s_0}(U^R)^*\right)_{ab} \\
&= -\frac{1}{\sqrt{2}}\sum_{\substack{i=\overline{e,n_R}\\ j=\overline{e,n_L}}}\left((U^R_{ia})^*(\Delta_{s_0})_{ij}U^L_{jb} + U^L_{ja}(\Delta_{s_0})_{ij}(U^R_{ib})^*\right),
\end{aligned}
\tag{8.27}
$$

is symmetric. The factor $\frac{1}{2}$ in $\mathcal{L}^{ss\nu I}_{Yn}$ accounts for the Majorana nature of the neutrino mass eigenfields.

The interaction of charged leptons with neutral scalars can bring more terms in the Lagrangian then it did in the Standard Model. This depends on the number of Higgs bosons. In general, we have

$$\mathcal{L}^{ssl}_{Yn} = e \sum_{i,j=e}^{n_L} \bar{l}_i \left((\Gamma_{s_0})_{ij} \gamma_L + (\Gamma_{s_0})^*_{ji} \gamma_R \right) l_j S^0_{s_0}, \tag{8.28}$$

where

$$\Gamma_{s_0} = \sum_k c^*_{k,s_0} \Gamma_k. \tag{8.29}$$

If we consider the particular Goldstone boson coming with the longitudinal polarisation of the Z boson, the couplings (8.25) and (8.29) are:

$$\begin{aligned} \Gamma_{s_Z} &= i \frac{1}{\sqrt{2} s_w} \frac{1}{m_W} m^l, \\ \Delta_{s_Z} &= i \frac{1}{\sqrt{2} s_w} \frac{1}{m_W} M^D. \end{aligned} \tag{8.30}$$

For further details related to the general behaviour of the scalars, their Feynman rules and possible vertices, see [GriLav].

Summarising, in the interaction Lagrangian involving neutrinos, the coupling to W is described by (8.14), the coupling to Z by (8.16), the Yukawa interaction with charged scalars is described by (8.20), while the one with neutral scalars by (8.26). Regarding the couplings of charged leptons to neutral bosons, the Standard Model Lagrangian is enlarged to include all other terms from (8.28). (The interaction with $S_{s_Z}{=}\chi$ is already part of the Standard Model.) The complete list of coupling constants for neutrino and charged lepton interactions is included in Tables 8.1 and 8.2.

8.3 Lagrange Density for Seesaw Type II

The neutrino masses in the seesaw mechanism of type II are described by the general form of the Lagrangian (8.1). If we relate the mass term to the one in seesaw type I (8.6), we can write

$$\mathcal{L}^{ss\nu II}_{mass} = \mathcal{L}^{ss\nu}_{massL} + \mathcal{L}^{ss\nu I}_{mass}. \tag{8.31}$$

The difference is comprised in a Majorana mass term written for the left-handed fields:

$$\mathcal{L}^{ss\nu}_{massL} = -\frac{1}{2} \sum_{\substack{i=\overline{e,n_L} \\ j=\overline{e,n_L}}} \overline{(\nu^L_{0i})^C} M^L_{ij} \nu^L_{0j} + h.c.. \tag{8.32}$$

The masses in $\mathcal{L}^{ss\nu I}_{mass}$ can be generated by Higgs doublets as described in the previous section. But this type of Higgs fields can not be responsable for the masses in $\mathcal{L}^{ss\nu}_{massL}$. We have to enrich the Lagrangian in (8.7) with Yukawa interaction terms from Higgs triplets. The Yukawa Lagrangian is

$$\mathcal{L}^{ss\nu II}_{Y} = \mathcal{L}^{ss\nu}_{YL} + \mathcal{L}^{ss\nu I}_{Y}, \tag{8.33}$$

with $\mathcal{L}^{ss\nu}_{YL}$ described by (see e.g. [SeesawII])

$$\mathcal{L}^{ss\nu}_{YL} = -e\left(-\overline{(l^L)^C},\ \ \overline{(\nu^L_0)^C}\right) Y_\Delta \Delta_L \begin{pmatrix}\nu^L_0\\ l^L\end{pmatrix} + h.c.. \tag{8.34}$$

Y_Δ represents the Yukawa coupling constant matrix ($n_L \times n_L$) and Δ_L is the Higgs triplet[3]:

$$\Delta_L = \begin{pmatrix}\frac{1}{\sqrt{2}}\delta^+ & \delta^{++}\\ \delta^0 & -\frac{1}{\sqrt{2}}\delta^+\end{pmatrix}. \tag{8.35}$$

As usual, the upper script indicates the charge. Multiplying all matrices in (8.34), the contribution to the Yukawa intercation term is

$$\begin{aligned}\mathcal{L}^{ss\nu}_{YL} = -e\sum_{i,j}\overline{(\nu^L_{0i})^C}(Y_\Delta)_{ij}\nu^L_{0j}\delta^0 + \frac{e}{\sqrt{2}}\sum_{i,j}\overline{(\nu^L_{0i})^C}(Y_\Delta)_{ij}l^L_j\delta^+\\ + \frac{e}{\sqrt{2}}\sum_{i,j}\overline{(l^L_i)^C}(Y_\Delta)_{ij}\nu^L_{0j}\delta^+ + e\sum_{i,j}\overline{(l^L_i)^C}(Y_\Delta)_{ij}l^L_j\delta^{++} + h.c..\end{aligned} \tag{8.36}$$

Denoting by v_L the vacuum expectation value of the neutral component of the Higgs triplet δ^0,

$$\langle 0|\delta^0|0\rangle = \frac{v_L}{\sqrt{2}}, \tag{8.37}$$

we obtain the mass of the left-handed neutrino fields,

$$M^L = \sqrt{2}v_L e Y_\Delta. \tag{8.38}$$

One can see that a symmetric choice for the matrix M^L requires a symmetric Y_Δ.

Now we can proceed in analysing which new Yukawa interactions have to be taken into account and how their coupling constants look like. First, we replace the

[3]In general, one may consider models with more than one Higgs triplet. A possible mixing of the different triplets will only complicate the following formulas without introducing new aspects concerning the mixing of neutrinos.

flavour neutrino fields in (8.36) by the mass eigenfields defined in (8.3).

$$\mathcal{L}_{YL}^{ss\nu} = -e\sum_{a,b}\overline{\nu_a^C}((U^L)^T Y_\Delta U^L)_{ab}\gamma_L\nu_b\delta^0 + \frac{e}{\sqrt{2}}\sum_{a,i}\overline{\nu_a^C}((U^L)^T Y_\Delta)_{ai}\gamma_L l_i\delta^+ \\ + \frac{e}{\sqrt{2}}\sum_{i,a}\overline{l_i^C}(Y_\Delta U^L)_{ia}\gamma_L\nu_a\delta^+ + e\sum_{i,j}\overline{l_i^C}(Y_\Delta)_{ij}\gamma_L l_j\delta^{++} + h.c.. \quad (8.39)$$

The left projector γ_L was explicitly written and the indices indicating the left and the right field components were dropped. It is obvious that a new type of interaction occurs in the case of charged leptons. In the last term of (8.39), we have a δ^{++} that couples to l and $\overline{l^C}$. The charge conjugated lepton indicates an interaction of the type described in (2.52) by the coupling constants $\tilde{c}^L$ and $\tilde{c}^R$.

Before identifying the exact expression of the coupling constants, let us simplify the other terms in (8.39). For the third term, we can use the definition of the charge conjugated field as in (2.19) and 'move' the charge conjugation from one field to the other. We did the same sort of transformations at the end of section 2.4.2, when proving the equivalence of choosing any general fermion flow to evaluate a diagram. Here,

$$\begin{aligned}\overline{l_i^C}(Y_\Delta U^L)_{ia}\gamma_L\nu_a\delta^+ &= -(\nu_a)^T C^{-1}C(Y_\Delta U^L)_{ia}(\gamma_L)^T C^{-1} l_i\delta^+ \\ &= \overline{\nu_a^C}((U^L)^T Y_\Delta)_{ai}\gamma_L l_i\delta^+.\end{aligned} \quad (8.40)$$

The minus sign in the first line originates from the anticommutation of the fermionic fields. In the last line we replaced $Y_\Delta^T = Y_\Delta$. Taking into account the Majorana nature of the neutrino fields, we can identify $(\nu)^C = \nu$ (see (2.16)) and (8.39) is

$$\mathcal{L}_{YL}^{ss\nu} = -e\sum_{a,b}\overline{\nu_a}((U^L)^T Y_\Delta U^L)_{ab}\gamma_L\nu_b\delta^0 + e\sqrt{2}\sum_{a,i}\overline{\nu_a}((U^L)^T Y_\Delta)_{ai}\gamma_L l_i\delta^+ \\ + e\sum_{i,j}\overline{l_i^C}(Y_\Delta)_{ij}\gamma_L l_j\delta^{++} + h.c.. \quad (8.41)$$

The neutrinos and the antineutrinos are identical, therefore their charge conjugation does not introduce new types of interaction. In contrast, the presence of δ^{++} in the couplings to charged leptons requires an explicitly charge conjugated field.

Finally, we are able to write the total Yukawa interaction Lagrangian. For the interaction of neutrinos with charged scalars,

$$\mathcal{L}_{Yc}^{ss\nu II} = \mathcal{L}_{YLc}^{ss\nu} + \mathcal{L}_{Yc}^{ss\nu I}, \quad (8.42)$$

with $\mathcal{L}_{Yc}^{ss\nu I}$ given in (8.20) and the interaction with the singly charge scalars $\delta^\pm$

described by

$$\begin{aligned}\mathcal{L}_{YLc}^{ss\nu} &= e\sqrt{2}\sum_{a,i}\overline{\nu_a}((U^L)^T Y_\Delta)_{ai}\gamma_L l_i \delta^+ + h.c \\ &= e\sum_{a,i}\overline{\nu}_a(c_{ai,\delta}^L\gamma_L + c_{ai,\delta}^R\gamma_R)l_i\delta^+ + e\sum_{a,i}\overline{l_i}((c_{ai,\delta}^R)^*\gamma_L + (c_{ai,\delta}^L)^*\gamma_R)l_i\delta^-.\end{aligned} \tag{8.43}$$

The general coupling constants are

$$c_{ai,\delta}^L = \sqrt{2}((U^L)^T Y_\Delta)_{ai} = \sqrt{2}\sum_j U_{ja}^L (Y_\Delta)_{ji}, \tag{8.44}$$

$$c_{ai,\delta}^R = 0. \tag{8.45}$$

(8.42) does not describe all the interactions of the charged leptons with charged scalars. For them, we have to add the interaction with the doubly charged scalar fields:

$$\begin{aligned}\mathcal{L}_{YLc}^{ssl} &= e\sum_{i,j}\overline{l_i^C}(Y_\Delta)_{ij}\gamma_L l_j\delta^{++} + h.c.. \\ &= e\sum_{i,j}\overline{l_i^C}(\tilde{c}_{ij,\delta}^L\gamma_L + \tilde{c}_{ij,\delta}^R\gamma_R)l_j\delta^{++} + e\sum_{i,j}\overline{l}_i((\tilde{c}_{ij,\delta}^R)^*\gamma_L + (\tilde{c}_{ij,\delta}^L)^*\gamma_R)l_j^C\delta^{--}.\end{aligned} \tag{8.46}$$

When writing the hermitian conjugated term, we have used the fact that the coupling constants are symmetric since

$$\tilde{c}_{ij,\delta}^L = (Y_\Delta)_{ij} = (Y_\Delta)_{ji}, \tag{8.47}$$

$$\tilde{c}_{ij,\delta}^R = 0. \tag{8.48}$$

The neutral Yukawa interaction term for neutrinos is the sum of (8.26) from the seesaw type I and the corresponding term in (8.41), i.e.

$$\mathcal{L}_{Yn}^{ss\nu I} = \mathcal{L}_{YLn}^{ss\nu} + \mathcal{L}_{Yn}^{ss\nu I}. \tag{8.49}$$

In $\mathcal{L}_{YLn}^{ss\nu}$ we include

$$\begin{aligned}\mathcal{L}_{YLn}^{ss\nu} &= -e\sum_{a,b}\overline{\nu_a}((U^L)^T Y_\Delta U^L)_{ab}\gamma_L\nu_b\delta^0 + h.c. \\ &= \frac{1}{2}e\sum_{a,b}\overline{\nu}_a(c_{ab,\delta}^L\gamma_L + (c_{ab,\delta}^L)^*\gamma_R)\nu_b.\end{aligned} \tag{8.50}$$

The symmetric coupling constants $c_{ab,\delta}^L$ are

$$c_{ab,\delta}^L = -2((U^L)^T Y_\Delta U^L)_{ab} = -2\sum_{i,j}U_{ia}^L(Y_\Delta)_{ij}U_{ja}^L. \tag{8.51}$$

Concluding, for the seesaw type II, to the Lagrangian terms already described in section 8.2 one has to add the interaction of the neutrinos and charged leptons with the Higgs triplet Δ_L, contained in (8.43), (8.46) and (8.50). We have collected all coupling constants describing the interactions of charged leptons and neutrinos in the Standard Model extended with the seesaw mechanism in Tables 8.1 and 8.2. For the seesaw mechanism of type I, one has to leave out the interactions with δ from Table 8.2.

vertex	$g^L_{xy,v}$	$g^R_{xy,v}$
$\overline{l_i} l_j A$	δ_{ij}	δ_{ij}
$\overline{l_i} l_j Z$	$\frac{1}{s_W c_W}\left(-\frac{1}{2}+s_W^2\right)\delta_{ij}$	$\frac{s_W}{c_W}\delta_{ij}$
$\overline{\nu_a} \nu_b Z$	$\frac{1}{2 s_W c_W}\sum_{i=e}^{n_L}(U^L_{ia})^* U^L_{ib}$	$-\frac{1}{2 s_W c_W}\sum_{i=e}^{n_L} U^L_{ia}(U^L_{ib})^*$
$\overline{l_i} \nu_a W^-$	$\frac{1}{\sqrt{2} s_W} U^L_{ia}$	0
$\overline{\nu_a} l_i W^+$	$\frac{1}{\sqrt{2} s_W}(U^L_{ia})^*$	0

Table 8.1: Lepton coupling constants to vector bosons in the seesaw mechanism.

vertex	$c^L_{xy,s}$	$c^R_{xy,s}$
$\overline{l_i} l_j S^0_{s_0}$	$(\Gamma_{s_0})_{ij}$	$(\Gamma_{s_0})^*_{ji}$
$\overline{\nu_a} \nu_b S^0_{s_0}$	$-\frac{1}{\sqrt{2}} \sum_{\substack{i=\overline{e,n_R} \\ j=\overline{e,n_L}}} \{(U^R_{ia})^* (\Delta_{s_0})_{ij} U^L_{jb} + U^L_{ja} (\Delta_{s_0})_{ij} (U^R_{ib})^*\}$	$-\frac{1}{\sqrt{2}} \sum_{\substack{i=\overline{e,n_R} \\ j=\overline{e,n_L}}} \{U^R_{ia} (\Delta_{s_0})^*_{ij} (U^L_{jb})^* + (U^L_{ja})^* (\Delta_{s_0})^*_{ij} U^R_{ib}\}$
$\overline{\nu_a} l_i S^+_s$	$\sum_{j=e}^{n_R} (U^R_{ja})^* (\Delta_s)_{ji}$	$-\sum_{j=e}^{n_L} (U^L_{ja})^* (\Gamma_s)^*_{ij}$
$\overline{l_i} \nu_a S^-_s$	$-\sum_{j=e}^{n_L} (\Gamma_s)_{ij} U^L_{ja}$	$\sum_{j=e}^{n_R} (\Delta_s)^*_{ji} U^R_{ja}$
$\overline{\nu_a} \nu_b \delta_0$	$-2 \sum_{i,j=e}^{n_L} U^L_{ia} (Y_\Delta)_{ij} U^L_{jb}$	$-2 \sum_{i,j=e}^{n_L} (U^L_{ia})^* (Y_\Delta)^*_{ij} (U^L_{jb})^*$
$\overline{\nu_a} l_i \delta^+$	$\frac{2}{\sqrt{2}} \sum_{j=e}^{n_L} U^L_{ja} (Y_\Delta)_{ji}$	0
$\overline{l_i} \nu_a \delta^-$	0	$\frac{2}{\sqrt{2}} \sum_{j=e}^{n_L} (Y_\Delta)^*_{ij} (U^L_{ja})^*$
$\overline{l^C_i} l_j \delta^{++}$	$(Y_\Delta)_{ij}$	0
$\overline{l_i} l^C_j \delta^{--}$	0	$(Y_\Delta)^*_{ij}$

Table 8.2: Lepton coupling constants to scalar bosons in the seesaw mechanism.

8.4 Model Restrictions and Field Renormalization Constants

In section 5.2 we have shown that the field renormalization constants can be chosen hermitian, provided that

$$\Sigma^{DL}_{aa}(p^2) = \Sigma^{DR}_{aa}(p^2). \tag{8.52}$$

This property is not obviously fulfilled if we analyse the Lagrangian for the seesaw neutrinos.

The terms that enter in $\Sigma^{DL}_{aa}(p^2)$ are of type (3.61) for a virtual vector boson and of type (3.66) for a scalar one. For $\Sigma^{DR}_{aa}(p^2)$, the contributions are characterised by (3.62) and (3.67). These equations describe the structure of the self-energy for one given set of internal particles. For each combination of particles, there is one common factor in $\Sigma^{DL}_{aa}(p^2)$ and $\Sigma^{DR}_{aa}(p^2)$ that depends on the masses of the internal particles. Since these masses have to be considered independent parameters (e.g. the scalar masses depend on the details of the Higgs sector), we have to assume that the relation (8.52) has to be fulfilled by each diagram separately. The factor to be compared in each $\Sigma^{DL}_{aa}(p^2)$ and $\Sigma^{DR}_{aa}(p^2)$ is a product of coupling constants. Hence, (8.52) is satisfied if the combination of coupling constants obeys

$$\left.\begin{array}{l} G^{DL}_{aa} = G^{DR}_{aa}, \\ C^{DL}_{aa} = C^{DR}_{aa}, \end{array}\right. \quad \text{for } \forall \text{ internal fermions.} \tag{8.53}$$

G_{aa} refers to the couplings to vector bosons and C_{aa} to the ones to scalar bosons. Their expressions were listed in Tables 3.2 and 3.3, respectively. To facilitate the reading, we remind that

$$\begin{aligned} G^{DL}_{aa} &= g^R_{ax,v} g^L_{xa,v}, \\ G^{DR}_{aa} &= g^L_{ax,v} g^R_{xa,v}, \\ C^{DL}_{aa} &= c^L_{ax,s} c^L_{xa,s}, \\ C^{DR}_{aa} &= c^R_{ax,s} c^R_{xa,s}. \end{aligned} \tag{8.54}$$

In the following, we will investigate the restrictions on the coupling matrices following from the requirement (8.53). We will study each combination of internal particles in the self-energy using the coupling constants as given in Tables 8.1 and 8.2.

For a virtual W-boson, there is no coupling to the right-handed fields and therefore, both, G^{DL}_{aa} and G^{DR}_{aa} are zero. The situation changes when considering an internal Z. The relation to be fulfilled is

$$g^R_{ab,Z} g^L_{ba,Z} = g^L_{ab,Z} g^R_{ba,Z}, \quad \text{for } \forall\ a,\ b. \tag{8.55}$$

If we use the connection between the left and right coupling constants from (8.17), we see that, for the coupling to Z-bosons, G_{aa}^{DL} is the complex conjugated G_{aa}^{DR} for each possible internal neutrino.

$$G_{aa}^{DL} = (G_{aa}^{DR})^*, \quad \text{for } \forall\ a \text{ and internal } b \text{ and } Z. \tag{8.56}$$

The equality (8.53) implies then, that

$$G_{aa}^{DL/DR} = (G_{aa}^{DL/DR})^*, \quad \text{for } \forall\ a \text{ and internal } b \text{ and } Z. \tag{8.57}$$

In other words, G_{aa}^{DL} and G_{aa}^{DR} must be equal and also real. For the left coupling, (8.55) results as

$$\left((g_{ab,Z}^L)^*\right)^2 = \left(g_{ab,Z}^L\right)^2 \quad \text{for } \forall\ a,\ b. \tag{8.58}$$

Such an equality is true if $g_{ab,Z}^L$ is either real or purely imaginary. For the upper component of the unitary matrix U, (8.55) implies

$$\sum_{i=e}^{n_L} (U_{ia}^L)^* U_{ib}^L = \pm \sum_{i=e}^{n_L} U_{ia}^L (U_{ib}^L)^*, \quad \text{for } \forall\ a,\ b. \tag{8.59}$$

This restriction has to be compatible with the unitarity relations for U (2.40).

One simple choice for the matrix U^L is requiring equal phases for the elements belonging to the same row, i.e.

$$U_{ia}^L = u_{ia}^L e^{i\varphi_i}, \quad u_{ia}^L \in \Re. \tag{8.60}$$

In this case, (8.59) is satisfied for the plus sign. Because of the unitarity relation, we had $U^L(U^L)^\dagger$=1. This restriction implies for the phases in (8.60) that

$$\varphi_i = \varphi_j, \quad \text{for } \forall\ i,\ j. \tag{8.61}$$

and we can choose an overall phase factor $e^{i\varphi}$ for a U^L that otherwise has real elements.

Another choice is $(U^L)^\dagger U^L$ diagonal and either real or imaginary. Then (8.55) is fulfilled. For this solution there is no mixing owing to the interaction with Z. Anyway, without knowing all the details of the neutrino mass matrix, we can not decide whether (8.59) is fulfilled or not.

Going further to the scalar couplings of Table 8.2, one can see that C_{aa}^{DL} and C_{aa}^{DR} are also related by complex conjugation for each possible internal fermion and scalar boson. For the charged scalar bosons, if

$$c_{ai,s}^L (c_{ai,s}^R)^* = c_{ai,s}^R (c_{ai,s}^L)^*, \quad \text{for } \forall\ a,\ b, \tag{8.62}$$

then (8.53) holds. If we pick up the couplings from Table 8.2, we need

$$-\sum_{j=e}^{n_R}(U^R_{ja})^*(\Delta_s)_{ji}\sum_{j=e}^{n_L}(\Gamma_s)_{ij}U^L_{ja} = -\sum_{j=e}^{n_L}(U^L_{ja})^*(\Gamma_s)^*_{ij}\sum_{j=e}^{n_R}(\Delta_s)^*_{ji}U^R_{ja}, \tag{8.63}$$

for the coupling to each charged $S^\pm_s$. In the seesaw type I, we have specified the couplings to the Goldstone boson corresponding to the longitudinal mode of W (8.23). Using (8.5) for M^L=0, one can easily check (8.63). For the coupling to $\delta^\pm$, C^{DL}_{aa} and C^{DR}_{aa} are zero.

For the self-energies with internal neutral scalar bosons, the relation to be fulfilled is

$$(c^L_{ab,s_0})^2 = \left((c^L_{ab,s_0})^*\right)^2, \quad \text{for } \forall\ a,\ b. \tag{8.64}$$

As for the coupling to Z, this implies that c^L_{ab,s_0} should be either real or purely imaginary. Writing the coupling matrices for $S^0_{s_0}$, we need

$$\sum_{\substack{i=\overline{e,n_R}\\ j=\overline{e,n_L}}} \left((U^R_{ia})^*(\Delta_{s_0})_{ij}U^L_{jb} + U^L_{ja}(\Delta_{s_0})_{ij}(U^R_{ib})^*\right)$$
$$= \pm \sum_{\substack{i=\overline{e,n_R}\\ j=\overline{e,n_L}}} \left(U^R_{ia}(\Delta_{s_0})^*_{ij}(U^L_{jb})^* + (U^L_{ja})^*(\Delta_{s_0})^*_{ij}U^R_{ib}\right), \quad \text{for } \forall\ a,\ b. \tag{8.65}$$

Taking the neutrino coupling to the Goldstone boson corresponding to the longitudinal polarisation of Z for seesaw type I (8.30), we can prove that if the condition for U^L (8.59) holds, then (8.65), too. The equivalent condition for the coupling to δ^0 reads

$$\sum_{i,j=e}^{n_L} U^L_{ia}(Y_\Delta)_{ij}U^L_{jb} = \pm\sum_{i,j=e}^{n_L}(U^L_{ia})^*(Y_\Delta)^*_{ij}(U^L_{jb})^*. \tag{8.66}$$

Without a detailed description of the scalar bosons and their couplings to the leptons, we can not go further with the formal analysis of (8.63), (8.65) or (8.66). We also need a choice for the unitary matrix U, to discuss its elements and complex phases before we can prove that $\Sigma^{DL}_{aa}(p^2)$=$\Sigma^{DR}_{aa}(p^2)$ is valid for the Standard Model extended by the seesaw mechanism. As for the quarks, there are other symmetries that restrict the coupling constants such that (8.52) holds. However, the situation is more complicated than in the Standard Model since here there is a non trivial mixing present also in the interaction with neutral (vector or scalar) bosons. Without specifying the details of the full theory that generates mixing in the neutrino sector, it is impossible to prove (8.52).

For the following, we therefore simply assume that the coupling constants in Tables 3.2 and 3.3 result in the theory such that $\Sigma^{DL}_{aa}(p^2)=\Sigma^{DR}_{aa}(p^2)$. This relation allows us to define neutrino field renormalization constants related by hermiticity. As in 5.2.2, the renormalized fields are introduced by

$$\nu^L_a = \sum_b Z^{r\frac{1}{2}L}_{ab} \nu^{rL}_b, \tag{8.67}$$

with

$$\overline{Z}^{r\frac{1}{2}L}_{ba} = (Z^{r\frac{1}{2}L}_{ab})^*, \tag{8.68}$$

Consequently, the Dirac conjugated neutrino fields are connected to the renormalized one by

$$\overline{\nu^L_a} = \sum_b \overline{\nu^{rL}_b}(Z^{r\frac{1}{2}L}_{ab})^*. \tag{8.69}$$

The other set of field renormalization constants that we have to consider is the one for the charged lepton fields. As for the quarks, the presence of the neutrino mixing matrix will lead to non-diagonal self-energies and further on to non-diagonal field renormalization constants. To be able to decide if the latter are related by hermiticity (5.37), we have to analyse the imaginary parts in the self-energy, more exactly the absorptive ones (see section 3.1.3). Such imaginary contributions can appear in case there are Higgs particles with a mass lower than the mass of the charged leptons. The existence of these bosons is not entirely excluded [PDG06H], however we will assume that we do not have decay channels for charged leptons that lead to absorptive contributions at the one-loop level[4].

Another source of imaginary parts is (see 3.46) related to the choice of the gauge parameter ξ. If the gauge is chosen such that the squared mass of the charged lepton is bigger than $(m_x + \sqrt{\xi} m_v)^2$, where x is the internal fermion, then differences of type (4.59)−(4.62) do not cancel. Since the decay width are physical observables and, therefore, gauge invariant, imaginary parts of this type will eventually cancel in a complete calculation. The cancellation will involve contributions from internal gauge bosons and their corresponding Goldstone bosons. However, to simplify the relations, we will assume from the beginning that ξ is chosen such that $M_i^2 \leq (m_x + \sqrt{\xi} m_v)^2$, for $\forall\, i$, x and v. Then, the relation for the complex conjugated self-energy components (3.72) is valid and therefore, the hermiticity of Z, (5.37), too. The renormalized fields are introduced by

$$l_i = \sum_j Z^{r\frac{1}{2}}_{ij} l^r_j, \tag{8.70}$$

[4]The Standard Model decays $\mu \to e\overline{\nu}_e\nu_\mu$, etc. are responsable for imaginary parts at the two-loop level.

and

$$\overline{Z}^{r\frac{1}{2}}_{ji} = \gamma^0 (Z^{r\frac{1}{2}}_{ij})^* \gamma^0. \tag{8.71}$$

Note that a possible choice for $\varkappa_{ij}$ is $\varkappa_{ij}$=0, leading to field renormalization constants which are equal to the wave function ones as determined in section 4.1.

8.5 Corrections to the Neutrino Mixing Matrix and Unitarity

The necessity of a correction to the neutrino mixing matrix comes as in the quark case, from UV-divergent contributions that show up in interaction amplitudes, when mixing is present. We denote the renormalized neutrino mixing matrix by U^r and the general renormalization constant by Z^{MNS}. The upper script MNS stands for Maki-Nakagawa-Sakata. Remember that U is not the same as the Maki-Nakagawa-Sakata mixing matrix described in the literature. For components:

$$U_{ia} = Z^{\mathrm{MNS}}_{ia} U^r_{ia}, \tag{8.72}$$

for $\forall\ i, a = 1, ..., n_L + n_R$. At first order, we relate the renormalized matrix to the unrenormalized one by δU:

$$\begin{aligned} U = \begin{pmatrix} U^L \\ (U^R)^* \end{pmatrix} &= \begin{pmatrix} U^{Lr} \\ (U^{Rr})^* \end{pmatrix} + \begin{pmatrix} \delta U^L \\ (\delta U^R)^* \end{pmatrix} + \mathcal{O}(\alpha^2) \\ &= U^r + \delta U + \mathcal{O}(\alpha^2). \end{aligned} \tag{8.73}$$

As for quarks, the unitarity of the neutrino mixing matrix should be preserved after renormalization, too, i.e.

$$\begin{matrix} U^{Lr}(U^{Lr})^\dagger = \mathbf{1}_{n_L}, & U^{Lr}(U^{Rr})^T = \mathbf{0}_{n_L\times n_R}, \\ U^{Rr}(U^{Rr})^\dagger = \mathbf{1}_{n_R}, & (U^{Rr})^*(U^{Lr})^\dagger = \mathbf{0}_{n_R\times n_L}, \end{matrix} \quad (U^{Lr})^\dagger U^{Lr} + (U^{Rr})^T (U^{Rr})^* = \mathbf{1}_{n_L+n_R}. \tag{8.74}$$

For the first order correction of U, this constraint is translated into anti-hermiticity:

$$\begin{cases} (U^r)^\dagger \delta U = -(\delta U)^\dagger U^r \\ U^r(\delta U)^\dagger = -\delta U (U^r)^\dagger \end{cases} \Leftrightarrow \begin{cases} \displaystyle\sum_{i=1}^{n_L+n_R} (U^r_{ia})^* \delta U_{ib} = -\sum_{i=1}^{n_L+n_R} (\delta U_{ia})^* U^r_{ib} \\ \displaystyle\sum_{a=1}^{n_L+n_R} U^r_{ia} (\delta U_{ja})^* = -\sum_{a=1}^{n_L+n_R} \delta U_{ia} (U^r_{ja})^*. \end{cases} \tag{8.75}$$

If we write (8.75) for the first order corrections of the two parts of U, namely U^L and U^R, we have:

$$\begin{aligned} U^{Lr}(\delta U^L)^\dagger + \delta U^L (U^{Lr})^\dagger &= \mathbf{0}_{n_L\times n_L}, \\ U^{Rr}(\delta U^R)^\dagger + \delta U^R (U^{Rr})^\dagger &= \mathbf{0}_{n_R\times n_R}, \\ U^{Lr}(\delta U^R)^T + \delta U^L (U^{Rr})^T &= \mathbf{0}_{n_L\times n_R}, \\ (U^{Rr})^*(\delta U^L)^\dagger + (\delta U^R)^* (U^{Lr})^\dagger &= \mathbf{0}_{n_R\times n_L}, \\ (U^{Lr})^\dagger \delta U^L + (\delta U^L)^\dagger U^{Lr} + (U^{Rr})^T(\delta U^R)^* + (\delta U^R)^T (U^{Rr})^* &= \mathbf{0}_{n_L+n_R}. \end{aligned} \tag{8.76}$$

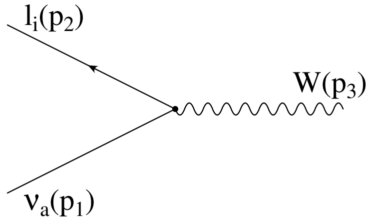
The upper relations can be deduced imposing the unitarity condition on U written as a function of U^{Lr}, U^{Rr} and their corrections, and taking into account that these two renormalized matrices also satisfy (8.74).

Another important discussion related to the matrix U concerns the number of parameters that describe the matrix. Later, we will need to know the number of measurements necessary to fix all the matrix elements. U has $(n_L + n_R)^2$ elements and implicitly, $2(n_L + n_R)^2$ real parameters (counting separately the real and the imaginary parts). Imposing all the unitarity constraints, we are left with $(n_L+n_R)^2$ real parameters. As one does for the Cabibbo-Kobayashi-Maskawa matrix, we have to count also the restrictions coming from field re-phasing. As for quark fields, one has the freedom to re-phase the charged lepton fields. This is no longer possible for the Majorana neutrinos, since this would introduce wrong phases in the mass terms. Therefore, from the interaction with the W-boson (8.14), we see that one has the freedom of absorbing n_L phases from U^L in the charged leptons. Overall, in U, there are $(n_L+n_R)^2 - n_L$ parameters: $\frac{(n_L+n_R)(n_L+n_R-1)}{2}$ mixing angles and $(n_L+n_R)^2 - n_L - \frac{(n_L+n_R)(n_L+n_R-1)}{2}$ CP-violating phases.

8.6 Corrections to the Lepton-W Vertex

As in the quark case, the analytical expression for the counter terms of the neutrino mixing matrix can be fixed only by calculating physical observables that involve corrections of vertices. In contrast with the Standard Model where the mixing matrix was part just in vertices with charged bosons, in the seesaw mechanism, the neutrino mixing is present also in the neutral current term or the neutral Yukawa interaction terms. There, we have products of mixing matrices and not only one element. This makes their analysis more complicated than for charged current interactions. In this section, we study the renormalized mixing matrix in the vertices with W. However, the algorithm can be applied in a similar way to neutrino interactions via the Z-boson or for the couplings to scalar bosons.

The vertex that describes the interaction between the neutrinos and the charged leptons, with exchange or emission of a W-boson is



$$i\frac{e}{\sqrt{2}s_W}\, U^L_{ia}\gamma^\mu\gamma_L.$$

Here, i indicates one of the n_L flavours of l. In general, this vertex is part of a Feynman diagram where the W-boson is an internal particle. The related terms in the interaction Lagrangian are given in (8.14). The vertex is similar to the one for quarks and formally, the corrections are alike.

Following section 6.5, we identify the contributions to the one-loop amplitude (6.82)

$$\mathcal{T}^{tot}_{ia} = \mathcal{T}^{r}_{ia} + \mathcal{T}^{\text{vertex}}_{ia} + \mathcal{T}^{a-se}_{ia} + \mathcal{T}^{i-se}_{ia} + \mathcal{O}(\alpha^2), \tag{8.77}$$

similar to equations (6.70)-(6.74), with

$$\mathcal{T}^{r}_{ia} = ie^r\overline{u}_i(p_2)\gamma^\mu G^{rL}_{ia,W}\gamma_L v_a(p_1)\varepsilon_\mu(p_3), \tag{8.78}$$

$$\mathcal{T}^{\text{vertex}}_{ia} = ie^r\overline{u}_i(p_2)\delta^{\text{vert},\mu}_{ia} v_a(p_1)\varepsilon_\mu(p_3), \tag{8.79}$$

$$\mathcal{T}^{a-se}_{ia} = ie^r\overline{u}_i(p_2)\sum_{b=1}^{n_L+n_R}\gamma^\mu g^{rL}_{ib,W}\gamma_L\frac{1}{\not{p}_1 - M_b}\mathcal{R}_{ba}(p_1)v_a(p_1)\varepsilon_\mu(p_3), \tag{8.80}$$

$$\mathcal{T}^{i-se}_{ia} = ie^r\overline{u}_i(p_2)\sum_{j=e}^{n_L}\mathcal{R}_{ij}(p_2)\frac{1}{\not{p}_2 - M_j}\gamma^\mu g^{rL}_{ja,W}\gamma_L v_a(p_1)\varepsilon_\mu(p_3). \tag{8.81}$$

The amplitude written here corresponds to the process $W\to\bar{\nu}l$, but an easy replacement of the Dirac spinors and/or of the polarisation vector ε_μ are enough to fit the result to other decays or cross sections.

The general coupling constant $G^{rL}_{ia,W}$, introduced as in (6.43), is:

$$\begin{aligned} G^{rL}_{ia,W} &= Z_e\sum_{j,b}\overline{Z}^{r\frac{1}{2}L}_{ij} Z^{L}_{(jb,W)} Z^{rL}_{jb,W} g^{rL}_{jb,W} Z^{r\frac{1}{2}L}_{ba} Z^{\frac{1}{2}}_{W} \\ &= Z_e\sum_{j,b}(Z^{r\frac{1}{2}L}_{ji})^* Z^{L}_{(jb,W)} Z^{rL}_{jb,W} g^{rL}_{jb,W} Z^{r\frac{1}{2}L}_{ba} Z^{\frac{1}{2}}_{W}. \end{aligned} \tag{8.82}$$

In the second line, the hermiticity of $Z^{r\frac{1}{2}}_{ij}$ (8.71) has been used to replace the second factor. From the unrenormalized coupling constant (8.15), i.e. $g^L_{ia,W}=\frac{1}{\sqrt{2}s_W}U^L_{ia}$, we identify

$$g^{rL}_{ia,W} = \frac{1}{\sqrt{2}s^r_W} U^{Lr}_{ia}, \tag{8.83}$$

$$Z^L_{(ia,W)} = \frac{1}{Z_{s_W}}, \tag{8.84}$$

$$Z^{rL}_{ia,W} = Z^{\text{MNS}L}_{ia}. \tag{8.85}$$

$Z^{\text{MNS}L}_{ia}$ refers to the upper part (the first n_L rows) of the mixing matrix renormalization constant Z^{MNS}_{ia} and it is equal to

$$Z^{\text{MNS}L}_{ia} = 1 + \frac{\delta U^L_{ia}}{U^{Lr}_{ia}} + \mathcal{O}(\alpha^2). \tag{8.86}$$

Expanded in α, the general coupling constant $G^{rL}_{ia,W}$ is

$$G^{rL}_{ia,W} = g^{rL}_{ia,W} + \delta g^L_{ia,W} + \delta g^{rL}_{ia,W} + \mathcal{O}(\alpha^2). \tag{8.87}$$

$\delta g^L_{ia,W}$ and $\delta g^{rL}_{ia,W}$ are defined for the general case in (6.58). The corrections not related to the fermion field renormalization are equal to

$$\delta g^L_{ia,W} = \frac{\delta_r}{\sqrt{2}s^r_W} U^{Lr}_{ia}, \tag{8.88}$$

with δ_r given as in the case of quarks by (7.10). We will focus on the last term in (8.87), the term that includes the corrections coming from the fermion field renormalization and from the mixing matrix, i.e.

$$\delta g^{rL}_{ia,W} = \frac{1}{\sqrt{2}s^r_W} \left(\delta U^L_{ia} + \sum_{j=e}^{n_L} \frac{1}{2} (\delta Z^{rL}_{ji})^* U^{Lr}_{ja} + \sum_{b=1}^{n_L+n_R} U^{Lr}_{ib} \frac{1}{2} \delta Z^{rL}_{ba} \right). \tag{8.89}$$

To identify the suitable parts to be absorbed in δU^L_{ia}, we need to rearrange the factors in the above formula. We have to keep in mind that $\delta U^L (U^{Lr})^\dagger$ is anti-hermitian (see (8.76)). A combination of type $Z - Z^\dagger$ multiplied with U^{Lr}, as we have already used for the divergences of the quark field renormalization constants, is anti-hermitian, too. (8.89) can be written as:

$$\begin{aligned} \delta g^{rL}_{ia,W} = {} & \frac{1}{\sqrt{2}s^r_W} \frac{1}{4} \left(- \sum_{j=e}^{n_L} \left(\delta Z^{rL}_{ij} - (\delta Z^{rL}_{ji})^* \right) U^{Lr}_{ja} + \sum_{b=1}^{n_L+n_R} U^{Lr}_{ib} \left(\delta Z^{rL}_{ba} - (\delta Z^{rL}_{ab})^* \right) \right) \\ & + \frac{1}{\sqrt{2}s^r_W} \frac{1}{4} \left(\sum_{j=e}^{n_L} \left(\delta Z^{rL}_{ij} + (\delta Z^{rL}_{ji})^* \right) U^{Lr}_{ja} + \sum_{b=1}^{n_L+n_R} U^{Lr}_{ib} \left(\delta Z^{rL}_{ba} + (\delta Z^{rL}_{ab})^* \right) \right) \\ & + \frac{1}{\sqrt{2}s^r_W} \delta U^L_{ia}. \end{aligned} \tag{8.90}$$

The similar separation in $\delta g^{rL}_{i\alpha,W}$ (hidden in equation (7.31)) and in $\delta g^{rL}_{\alpha i,W}$ (equation (7.34)) for quarks was allowing us to identify the anti-hermitian divergent contributions that are absorbed by the quark mixing counter term (see section 7.1.3). There, the expression for div$[\delta V]$, (7.38), could not be extended over the finite parts. One reason discussed there was the hermiticity problem related to the properties of the quark field renormalization constants. Here, thanks to the properties (8.68) and (8.71), we could extend the expression over the finite parts, too. Hence, one may try as a solution for the counter term of U^L:

$$\delta U^L_{ia} = \frac{1}{4}\sum_{j=e}^{n_L}\left(\delta Z^{rL}_{ij} - (\delta Z^{rL}_{ji})^*\right) U^{Lr}_{ja} - \frac{1}{4}\sum_{b=1}^{n_L+n_R} U^{Lr}_{ib}\left(\delta Z^{rL}_{ba} - (\delta Z^{rL}_{ab})^*\right). \tag{8.91}$$

Using the fact that $U^L(U^L)^\dagger = \mathbf{1}_{n_L}$, one can verify the first line of the requirement (8.76). Still, defining δU^L_{ia} by (8.91) can bring other disadvantages. As in [Den90a], δU^L_{ia} might turn out to be gauge parameter dependent. We introduced field renormalization constants differing from the wave function ones by a $\varkappa$. For neutrinos, we choose the $\varkappa$ and $\overline{\varkappa}$ such that the field renormalization constants are related by hermiticity. The imaginary parts that they have to absorb are also gauge parameter dependent (see equations (4.59) and (4.61)), hence one has to expect that one obtains a gauge parameter dependent counter term of the mixing matrix. Therefore, as for the quarks, we will determine these counter terms from physical observables.

Now one has to check that all divergences in (8.77) vanish. The condition written for the general case (6.83) reduces to

$$\begin{aligned} &\text{div}\left[\gamma^\mu G^{rL}_{ia,W}\gamma_L + \delta^{\text{vert},\mu}_{ia}\right] = 0 \\ \Leftrightarrow\ &\text{div}\left[\gamma^\mu\left(\frac{\delta_r}{\sqrt{2}s^r_W}U^{Lr}_{ia} + \delta g^{rL}_{ia,W}\right)\gamma_L + \delta^{\text{vert},\mu}_{ia}\right] = 0. \end{aligned} \tag{8.92}$$

$\delta^{\text{vert},\mu}_{ia}$, as for the quarks, will be a sum of different structures of Dirac matrix elements. However, divergences will appear just from terms that lead in (8.79) to a matrix structure identical with the one in the lowest order amplitude. We denote the vertex correction that contributes with a $\gamma^\mu\gamma_L$ structure with $\delta^{\text{vert}0}_{ia}$. Then,

$$\text{div}\left[\delta^{\text{vert},\mu}_{ia}\right] = \gamma^\mu \text{div}\left[\delta^{\text{vert}0}_{ia}\right]\gamma_L, \tag{8.93}$$

and

$$\text{div}\left[\frac{\delta_r}{\sqrt{2}s^r_W}U^{Lr}_{ia} + \delta g^{rL}_{ia,W} + \delta^{\text{vert}0}_{ia}\right] = 0. \tag{8.94}$$

If we could define the U^L counter term by (8.91), then we would know the exact structure of the divergences absorbed in δU^L. Since this is not allowed, we have to make a separation of the hermitian and anti-hermitian contributions in all the terms having the Dirac matrix elements of the lowest order amplitude and see which are the candidates for div$[\delta U^L]$. Remember from (8.76) that we need in fact, an anti-hermitian combination $\delta U^L (U^{Lr})^\dagger$ as we already obtained for $\delta g^{rL}_{ia,W}$, in (8.90). If we use (8.90) and we also separate the hermitian and anti-hermitian contribution from the vertex corrections and δ_r, the relation to analyse is

$$\frac{\delta_r}{\sqrt{2}s^r_W}U^{Lr}_{ia} + \delta g^{rL}_{ia,W} + \delta^{\text{vert0}}_{ia} = \frac{1}{\sqrt{2}s^r_W}\Bigg(\frac{1}{4}\sum_{j=e}^{n_L}\left(\delta Z^{rL}_{ij} + (\delta Z^{rL}_{ji})^*\right)U^{Lr}_{ja} + \frac{1}{4}\sum_{b=1}^{n_L+n_R} U^{Lr}_{ib}\left(\delta Z^{rL}_{ba} + (\delta Z^{rL}_{ab})^*\right) + \frac{1}{2}U^{Lr}_{ia}\left(\delta_r + (\delta_r)^*\right) + \sqrt{2}s^r_W\frac{1}{2}\left(\delta^{\text{vert0}}_{ia} + \sum_{j,b} U^{Lr}_{ib}(\delta^{\text{vert0}}_{jb})^* U^{Lr}_{ja}\right) - \frac{1}{4}\sum_{j=e}^{n_L}\left(\delta Z^{rL}_{ij} - (\delta Z^{rL}_{ji})^*\right)U^{Lr}_{ja} + \frac{1}{4}\sum_{b=1}^{n_L+n_R} U^{Lr}_{ib}\left(\delta Z^{rL}_{ba} - (\delta Z^{rL}_{ab})^*\right) + \frac{1}{2}U^{Lr}_{ia}\left(\delta_r - (\delta_r)^*\right) + \sqrt{2}s^r_W\frac{1}{2}\left(\delta^{\text{vert0}}_{ia} - \sum_{j,b} U^{Lr}_{ib}(\delta^{\text{vert0}}_{jb})^* U^{Lr}_{ja}\right) + \delta U^L_{ia}\Bigg). \quad (8.95)$$

As in (8.94), the divergence of the above piece has to cancel. If the first two lines within the parenthesis, i.e. if

$$\frac{1}{4}\sum_{j=e}^{n_L}\text{div}\left[\delta Z^{rL}_{ij} + (\delta Z^{rL}_{ji})^*\right]U^{Lr}_{ja} + \frac{1}{4}\sum_{b=1}^{n_L+n_R} U^{Lr}_{ib}\text{div}\left[\delta Z^{rL}_{ba} + (\delta Z^{rL}_{ab})^*\right] + \frac{1}{2}U^{Lr}_{ia}\text{div}\left[\delta_r + (\delta_r)^*\right] + \frac{1}{\sqrt{2}}s^r_W\text{div}\left[\delta^{\text{vert0}}_{ia} + \sum_{j,b} U^{Lr}_{ib}(\delta^{\text{vert0}}_{jb})^* U^{Lr}_{ja}\right] = 0 \quad (8.96)$$

then

$$\text{div}[\delta U^L_{ia}] = \frac{1}{4}\sum_{j=e}^{n_L}\text{div}\left[\delta Z^{rL}_{ij} - (\delta Z^{rL}_{ji})^*\right]U^{Lr}_{ja} - \frac{1}{4}\sum_{b=1}^{n_L+n_R} U^{Lr}_{ib}\text{div}\left[\delta Z^{rL}_{ba} - (\delta Z^{rL}_{ab})^*\right] - \frac{1}{2}U^{Lr}_{ia}\text{div}\left[\delta_r - (\delta_r)^*\right] - \frac{1}{\sqrt{2}}s^r_W\text{div}\left[\delta^{\text{vert0}}_{ia} - \sum_{j,b} U^{Lr}_{ib}(\delta^{\text{vert0}}_{jb})^* U^{Lr}_{ja}\right]. \quad (8.97)$$

div$[\delta U^L]$ identified like this, will fulfil (8.76). A similar definition, extended over the finite parts, is used in [Den04] to define the counter term of the quark mixing matrix.

$\delta_{ia}^{\text{vert},\mu}$ is not as easy to inspect as it was for quarks. The contributions from vertex diagrams contain sums over elements of the mixing matrix, which can not be factorised as in the quark vertex corrections (7.15). This is a consequence of the mixing in the neutral current and Yukawa interaction terms. In the one-loop vertex diagrams, we will have couplings of the neutrinos via the Z-boson or the neutral scalar $S^0_{s_0}$ that will introduce additional combinations of the neutrino mixing matrix. Therefore, we can not factorise U^{Lr}_{ia} without evaluating first all the vertex diagrams.

Moreover, a comparison with a model without mixing, as we did in section 7.1.3 is not advantageous in this case. While one can still prove that the divergences of δ_r are identical no matter the presence of the mixing matrix[5], the situation is different for the vertex corrections. The non-trivial sums over the mixing matrix components that appear in $\delta_{ia}^{\text{vert0}}$ yield different divergent contributions in the two models. One needs to consider each type of vertex diagrams in detail. This is possible only for specific models.

Further on, we need a better understanding of the divergences contained in the hermitian combinations of field renormalization constants from (8.96): $\delta Z^{rL}_{ab}+(\delta Z^{rL}_{ba})^*$ and $\delta Z^{rL}_{ij}+(\delta Z^{rL}_{ji})^*$. Their evaluation can be done using the formulas (4.31), (4.44) and (4.46) to express the sums of the wave function renormalization constants and the results of section 3.1.3 for the divergent parts in the self-energy (mainly as given in (5.78)). We obtain

$$\begin{aligned}\text{div}[\delta Z^{rL}_{ab}+(\delta Z^{rL}_{ba})^*] &= \text{div}[\delta Z^{L}_{ab}]+\text{div}[\delta \overline{Z}^{L}_{ab}] \\ &= 2\text{div}[\Sigma^L_{ab}(M_a^2)],\end{aligned} \quad \forall\; a,b, \tag{8.98}$$

for the neutrino field renormalization constants and

$$\begin{aligned}\text{div}[\delta Z^{rL}_{ij}+(\delta Z^{rL}_{ji})^*] &= \text{div}[\delta Z^{L}_{ij}]+\text{div}[\delta \overline{Z}^{L}_{ij}] \\ &= 2\text{div}[\Sigma^L_{ij}(M_i^2)],\end{aligned} \quad \forall\; i,j, \tag{8.99}$$

for the left components of the charged lepton constants. In our general model, it is formally impossible to prove that these expressions contain exactly the divergences that cancel the hermitian contributions from div$[\delta_r]$ and the vertex corrections, as required in (8.96). This is obvious when one considers the contributions involving scalar bosons since the parameters there are unrestricted. Again, we need a more specific model based on additional assumptions that allow one to investigate the cancellation of divergences in more detail.

[5] One can anyway prove that div$[\delta_r]$ is real.

As an example, one can use the multi-Higgs-doublet seesaw model described in [GriLav], with a 'softly' broken lepton number. This means that the lepton numbers (L_e, L_μ, L_τ) are separately conserved in the Yukawa couplings and that they are 'softly' broken by the mass matrix M^R of the right-handed neutrinos[6]. Then, the Yukawa coupling matrices Γ_k and Δ_k are simultaneously diagonal and therefore Γ_s, Δ_s, Γ_{s_0} and Δ_{s_0} are also diagonal. This implies that also the Dirac mass matrix M^D is diagonal, as well as the combinations

$$\begin{aligned} U^R m (U^L)^\dagger &= M^D, \\ U^L m^2 (U^L)^\dagger &= (M^D)^\dagger M^D. \end{aligned} \tag{8.100}$$

Another constraint on the model that will simplify the analysis is choosing U^L such that $(U^L)^\dagger U^L$ is diagonal. This is not an automatic consequence of the 'softly' broken lepton number version, but without it, the mixing in the neutral sector is hard to handle. With this choice, the model provides enough structure and additional relations between coupling constants to allow us to prove the validity of (8.96). If $(U^L)^\dagger U^L = 1$, one is able to follow the same argumentation as in section 7.1.3.

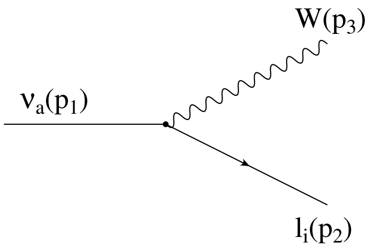
As we have already announced, even if we are able to identify the divergences of the neutrino mixing matrix counter term, we can not fix its finite parts similar to those of the divergent parts. Whether there is an obvious problem related to the hermiticity of the field renormalization constants, as it was for the quarks, or just a hidden one related to gauge parameter dependence, as we have here, we can not identify δU^L from the one-loop anti-hermitian corrections without referring to an observable and a corresponding complete calculation. In the next section, we will discuss as for the quarks, how one can fix the real part of the mixing matrix counter term from a decay rate.

8.7 Heavy Neutrino Decay Rate

The description of the neutrino decay follows the same steps as the top quark decay, section 7.2. Therefore, we will not repeat all the details here and we will emphasise only the differences introduced by the Majorana neutrinos.

For the tree diagram,

[6] In [GriLav] we have a seesaw mechanism of type I and $M^L = 0$.



$$i\frac{e}{\sqrt{2}s_W}U^L_{ia}\gamma^\mu\gamma_L$$

the amplitude is

$$\mathcal{T}(\nu_a \to W l_i) = i\frac{e}{\sqrt{2}s_W}\overline{u}_i(p_2)\not\epsilon^*(p_3)U^L_{ia}\gamma_L u_a(p_1). \tag{8.101}$$

The lowest order decay rate of a neutrino ν_a into a W and a charged lepton with flavour i can be written as in (7.59). One has to fit the masses to the present case: m_t becomes m_a and m_b, m_i. The result is

$$\begin{aligned}\Gamma^{tot}_0(\nu_a \to W l_i) =& \frac{\alpha}{8s^2_W}|U^L_{ia}|^2\frac{1}{m_a}\sqrt{(m^2_a-(m_i+m_W)^2)(m^2_a-(m_i-m_W)^2)} \\ &\left(\frac{m^2_a+m^2_i}{2m^2_a}+\frac{(m_a-m_i)^2}{2m^2_a m^2_W}-\frac{m^2_W}{m^2_a}\right).\end{aligned} \tag{8.102}$$

The one-loop amplitude is calculated as in (8.77). $v_a(p_1)$ in formulas (8.78)-(8.81) has to be replaced with $u_a(p_1)$. Then, the related decay rate, written as in (7.64), is

$$\begin{aligned}\Gamma^{tot}(\nu_a \to W l_i) = \Gamma_0|U^{Lr}_{ia}|^2\Bigg(&1+\delta_r+(\delta_r)^*+\delta^{\text{vert}}_{ia}+(\delta^{\text{vert}}_{ia})^* \\ &+\frac{\delta U^L_{ia}}{U^{Lr}_{ia}}+\frac{(\delta U^L_{ia})^*}{(U^{Lr}_{ia})^*}+\frac{1}{U^{Lr}_{ia}}Y_{ia}+\frac{1}{(U^{Lr}_{ia})^*}(Y_{ia})^*\Bigg).\end{aligned} \tag{8.103}$$

Γ_0 is obtained from (8.102), by leaving out $|U^L_{ia}|^2$ and replacing the electric charge and the sine of the weak mixing angle with their renormalized correspondents. $\delta^{\text{vert}}_{ia}$ sums up all the terms left out after factorising $\Gamma_0|U^{Lr}_{ia}|^2$ from the vertex correction contributions, as in (7.51). Finally, Y_{ia} is defined to be

$$Y_{ia} = \frac{1}{2}\sum_{j=e}^{n_L}(\delta Z^{rL}_{ji})^* U^{Lr}_{ja} + \frac{1}{2}\sum_{b=1}^{n_L+n_R}U^{Lr}_{ib}\delta Z^{rL}_{ba} + \frac{1}{2}X_{ia}. \tag{8.104}$$

X_{ia} collects the contributions from the self-energy corrections of external legs and similar to (7.63), reads

$$X_{ia} = \sum_{j\neq i}\overline{\varkappa}^L_{ij}U^{Lr}_{ja} + (\varkappa^L_{ii}+\overline{\varkappa}^L_{ii})U^{Lr}_{ia} + \sum_{b\neq a}U^{Lr}_{ib}\varkappa^L_{ba} + 2U^{Lr}_{ia}\varkappa^L_{aa}. \tag{8.105}$$

Remember that for Majorana particles, we considered $\varkappa^L_{aa}=\overline{\varkappa}^L_{aa}$ (5.102).

We define the absolute value of the ia element of U^L from the measured value of the neutrino decay rate, i.e.

$$|U^{Lr}_{ia}|^2 = \frac{1}{\Gamma_0}\Gamma^{tot}(\nu_a \to W l_i). \tag{8.106}$$

In analogy to (7.68), we set

$$\mathrm{Re}\frac{\delta U^L_{ia}}{U^{Lr}_{ia}} = -\frac{1}{2}\left(\delta_r + (\delta_r)^* + \delta^{\mathrm{vert}}_{ia} + (\delta^{\mathrm{vert}}_{ia})^* + \frac{1}{U^{Lr}_{ia}}Y_{ia} + \frac{1}{(U^{Lr}_{ia})^*}(Y_{ia})^*\right), \tag{8.107}$$

identifying thus the real part of the first order correction in the renormalization constant $Z^{\mathrm{MNS}L}_{ia}$, formula (8.86). The counter term is thefore fixed by relating it to measured values of U^{Lr}_{ia}. Again, by construction, δU^L_{ia} is gauge invariant since its definition is based on the calculation of a complete physical amplitude. The presence of δ_r and $\delta^{\mathrm{vert}}_{ia}$ in this formula is essential for the cancellation of gauge dependent terms contained in Y_{ia}.

Here, we have provided just an example of how we can use an experimental value of a decay rate to determine the real part of the correction to the mixing matrix. By heavy neutrino decays via the W-bosons we can fix the corrections to the elements from the (n_L+1)-th until (n_L+n_R)-th column of U^L. For the elements that describe the mixing of the light neutrinos (the neutrino mixing matrix as usually described in the literature), the study of the W decay is required. Measuring all the possible decays $W\to\bar{\nu}l$ and $\nu\to Wl$ we will be able to fix all the elements of δU^L and part from the ones of δU^R related to U^L by the unitarity constraints (8.76). For a complete set of processes that can allow us to fix all the $(n_L+n_R)^2 - n_L$ real parameters of U, we need observables involving scalar bosons. Because of the general approach we considered, we can not follow this here. The steps would be very similar, once we have a more detailed description of the model.

Some decay processes we mentioned are of course not measurable in the near future since we talk about particles too heavy to be produced. The corresponding parameters will remain unknown, but one expects that then, they will not play any role for the phenomenology of low-energy experiments. Still, from the theoretical point of view, we can and we have to discuss how these parameters should be fixed.

Chapter 9

Summary

Along this work, we have studied the renormalization of the fermionic Lagrangian for models that involve mixing. The first part of this thesis provided a general renormalization prescription for a Lagrangian with Dirac and Majorana fermions, while the second one was an application to two specific models: the quark mixing in the electroweak Standard Model and the neutrino mixing in the seesaw mechanism.

After shortly defining the general framework, we gave complete analytic results for the Dirac and Majorana fermion self-energies. We have isolated the ultraviolet divergences in each contribution to the self-energy and we have calculated the imaginary parts that arise from possible cuts through the one-loop diagrams. With the on-shell renormalization scheme we have separated the divergences in the full propagator and we have identified the physical mass and the decay width of the particle. So-called wave function renormalization constants were calculated such that the subtracted propagator is diagonal on-shell. As a consequence of the absorptive contributions from self-energies, these constants are grouped in two sets not related by hermiticity. As shown also in the literature, we proved that taking into account just the dispersive parts of the self-energies (which include also the UV-divergences) does not violate the hermiticity of the Lagrangian.

Instead of defining the field renormalization constants using directly the wave function renormalization ones, we proposed to differentiate the two by a set of finite constants. Using the additional freedom offered by this finite difference, we have tried to impose a hermiticity relation between the constants that directly renormalize the field and the constants that renormalize the Dirac conjugated field. We have shown that for Dirac fermions, unless the model has very special properties, this restriction leads to poles in the self-energy corrections to external legs. The requirement is less restrictive for the Majorana fermions and there, one has a better chance to fix the field renormalization constants such that the free Lagrangian is hermitian.

Another possibility we considered after renormalizing the fields was the re-diagonalization of the renormalized mass term of the Lagrangian. We have defined transformed fields related to mass eigenstates. In this case, part of the divergences present in the field renormalization constants were moved in other terms of the renormalized Lagrangian. The immediate consequence was a divergent contribution in the non-diagonal self-energy correction to external legs. Because of the complications brought in the calculation, we have decided not to follow this path.

The interaction terms of the renormalized fermionic Lagrangian were analysed for a general theory including vector and scalar bosons with arbitrary renormalizable interactions. We described the influence of the renormalization constants in each possible vertex involving Dirac and/or Majorana fermions and vector or scalar bosons. We have finished our analysis with the description of the total one-loop amplitude of a generic process that involves fermion mixing in a vertex.

This was the starting point for the study of the renormalization of fermion mixing matrices in the two models we have chosen. For the quark mixing in the electroweak Standard Model, we have taken into account quark field renormalization constants not related by hermiticity. This is necessary since heavy quarks may decay into lighter ones and imaginary parts are present in one-loop self-energies. We were able to determine the divergent contributions absorbed in the quark mixing matrix, such that the renormalized matrix is unitary. The lack of hermiticity was not permitting us to identify the full counter term of the mixing matrix (i.e. also the finite contributions) by only taking into account the anti-hermitian contributions to a quark-antiquark-W vertex amplitude. Therefore, we have proposed as an alternative method the determination of the counter term from experimental measurements of decays or cross sections. In principle, we have enough physical processes to determine all the parameters in the quark mixing matrix, even if present-day measurements are not sensitive enough to first order corrections.

The other interesting model we have studied was an extension of the Standard Model that describes neutrino mixing in the seesaw mechanism. The Majorana nature of the neutrinos and the special features of the model, made possible a renormalization scheme where hermiticity was achieved. Still, because of the gauge parameter dependence problem related to the counter term of fermion mixing matrices in general, we have shown that fixing the corrections directly from measurements is the better alternative.

Since the seesaw mechanism is just a theoretical hypothesis and there is still some way until we will know in detail the model that describes neutrinos, we were limited to a general analysis. While for quarks, using the analytical expressions we provide, one is able to calculate mass corrections and wave function renormalization constants, for neutrinos we can not go yet so far. However, we hope the examples we have presented offered a complete overview for possible situations that appear in the renormalization of models involving fermion mixing.

As we have shown, complete one-loop calculations for fermion mixing models, including unstable particles, are required before proceeding to higher orders. Starting from two-loop calculations, the two sources of imaginary parts in the amplitude, dispersive and absorptive, can not be easily separated. Since imaginary parts are related to the fact that particles are unstable, a solution is likely to be found only if one drops the assumption that amplitudes can be calculated treating fermions as external, free, on-shell particles.

Appendix A

Dimensional Regularization

The prevalent method to solve integrals when an UV divergence occurs is the dimensional regularization. This procedure preserves Lorentz and gauge invariance. Here, we make a brief presentation of the method to support our calculations. The convention keeps the notations of [Col98], where one can find detailed explanations. For another compact description one can also check [Hol00].

To be able to obtain a finite integral one changes the 4-dimensional integral to a D-dimensional one. The integral measure, defined as

$$\int \frac{d^4 q}{(2\pi)^4}, \text{ becomes} \int \frac{d^D q}{(2\pi)^D}. \tag{A.1}$$

For D small enough, the integrals with an UV divergence become convergent.

In order to keep the dimensions of the coupling constants independent of D, one introduces an arbitrary mass scale μ. This way

$$e^2 \to \mu^{4-D} e^2, \tag{A.2}$$

or in terms of the gauge coupling constant g

$$g \to \mu^{2-\frac{D}{2}} g. \tag{A.3}$$

With an integration on an arbitrary (non-integer) D-dimensional space, the known relations for Dirac matrices must be adapted to the case when the time-space index μ has an infinite range. The needed γ_μ matrices will also be infinite dimensional. The anti-commutation relations for the Dirac-matrices

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \mathbf{1} \tag{A.4}$$

and also the hermiticity properties

$$\gamma^{\mu\dagger} = \gamma_\mu = \begin{cases} \gamma^\mu \text{ if } \mu = 0, \\ -\gamma^\mu \text{ if } \mu \geq 1 \end{cases} \tag{A.5}$$

should be preserved.

Using the theory from the even dimensional case ($D = 2n$, n integer) [Aky73], one can extend it to any γ-algebra. The following relations come out:

$$g_\mu^\mu = g_{\mu\nu} g^{\mu\nu} = D \tag{A.6}$$

$$\gamma_\mu \gamma^\mu = D\mathbf{1} \tag{A.7}$$

$$\gamma_\mu \gamma_\nu \gamma^\mu = (2 - D)\gamma_\nu \tag{A.8}$$

$$\gamma_\mu \gamma_\nu \gamma_\rho \gamma^\mu = 4g_{\nu\rho} - (4 - D)\gamma_\nu \gamma_\rho \tag{A.9}$$

As a consequence:

$$\gamma_\mu \not{p} \gamma^\mu = (2 - D)\not{p}, \tag{A.10}$$

$$\{\not{p}, \not{q}\} = 2pq. \tag{A.11}$$

For the trace of γ-matrices we need a result for the unit matrix in the new space, i.e. to find $\mathrm{Tr}(\mathbf{1}) = f(D)$. For $D \to 4$, we have $\mathrm{Tr}(\mathbf{1}) = 4$. Together with matrix trace properties, one can determine

$$\begin{aligned} \mathrm{Tr}(\gamma_\mu \gamma_\nu) &= \mathrm{Tr}(\mathbf{1}) g_{\mu\nu} \\ &= f(D) g_{\mu\nu}. \end{aligned} \tag{A.12}$$

The trace of any odd number of γ-matrices is zero.

In the Standard Model one needs to consider chirality and therefore we need to redefine the γ_5 matrix. There are different prescriptions that try to avoid the problems implied by a general D-dimensional space. A brief description together with a new method can be found in [Jeg01] and [Kre94].

For our purpose, we can restrict to a definition that preserves Lorentz invariance (see [Kre94]) on the first four dimensions, but not for the full space.

$$\gamma^5 = \gamma_5 = \frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma, \tag{A.13}$$

with $\gamma_5^2 = \mathbf{1}$ and $\gamma_5^\dagger = \gamma_5$. The anti-commutation relations are preserved for $\mu = 0, 1, 2, 3$:

$$\{\gamma_5, \gamma^\mu\} = 0, \quad \text{for } \mu = 0, 1, 2, 3, \tag{A.14}$$

but otherwise $[\gamma_5, \gamma^\mu] = 0$ can be used.

Appendix B

One- and Two-point Integrals

The evaluation of one-loop diagrams using dimensional regularization leads to several types of integrals, as defined in [Hoo79] and [Pas79]. In the following, we will present the main ones, together with some relations to determine them.

B.1 Definitions

Since at the end, the dimension D (see (A.1)) should become again 4, one can parametrise it like:

$$D = 4 - 2\varepsilon, \tag{B.1}$$

where ε will go to 0. In the definition of the n-point integrals, we will use the following notation:

$$\mathcal{D}q = \frac{\mu^{4-D}}{i\pi^2} \frac{d^D q}{(2\pi)^{D-4}}, \tag{B.2}$$

where μ is the mass scale (A.2).

One-point integral:

$$\bullet\ A(m^2) = \int \mathcal{D}q \frac{1}{q^2 - m^2}. \tag{B.3}$$

Two-point integrals:

$$\bullet\ B_0(p^2; m_1, m_2) = \int \mathcal{D}q \frac{1}{(q^2 - m_1^2)((q+p)^2 - m_2^2)}, \tag{B.4}$$

$$\bullet\ B_\mu(p; m_1, m_2) = \int \mathcal{D}q \frac{q_\mu}{(q^2 - m_1^2)((q+p)^2 - m_2^2)}, \tag{B.5}$$

with the covariant decomposition:

$$B_\mu(p;m_1,m_2) = p_\mu B_1(p^2;m_1,m_2); \tag{B.6}$$

$$\bullet\ B_{\mu\nu}(p;m_1,m_2) = \int \mathcal{D}q \frac{q_\mu q_\nu}{(q^2-m_1^2)((q+p)^2-m_2^2)}, \tag{B.7}$$

and its decomposition:

$$B_{\mu\nu}(p;m_1,m_2) = g_{\mu\nu}B_{20}(p^2;m_1,m_2) + p_\mu p_\nu B_{21}(p^2;m_1,m_2). \tag{B.8}$$

It is worth to remark that, by definition, B_0 is symmetric in m_1 and m_2.

With some tricks, the vector integral (B.5) can be expressed in terms of $A(m)$ and $B_0(p^2;m_1,m_2)$:

$$\begin{aligned} B_\mu(p;m_1,m_2) &= \frac{p_\mu}{2p^2}\left(A(m_1^2) - A(m_2^2) - (p^2+m_1^2-m_2^2)B_0(p^2;m_1,m_2)\right) \\ \Rightarrow\ B_1(p^2;m_1,m_2) &= \frac{1}{2p^2}\left(A(m_1^2) - A(m_2^2) - (p^2+m_1^2-m_2^2)B_0(p^2;m_1,m_2)\right). \end{aligned} \tag{B.9}$$

Keeping in mind that $B_0(p^2;m_1,m_2) = B_0(p^2;m_2,m_1)$ one can deduce that

$$B_1(p^2;m_1,m_2) + B_0(p^2;m_1,m_2) = -B_1(p^2;m_2,m_1). \tag{B.10}$$

The elements of the decomposition (B.8) can also be written as function of $A(m)$ and $B_0(p^2;m_1,m_2)$. Note that we already have such an expression for $B_1(p^2;m_1,m_2)$ from (B.9).

$$B_{20}(p^2;m_1,m_2) = \frac{1}{D-1}\left(\frac{1}{2}A(m_2^2) + m_1^2B_0 + \frac{1}{2}(p^2+m_1^2-m_2^2)B_1\right) \tag{B.11}$$

$$B_{21}(p^2;m_1,m_2) = \frac{1}{2(D-1)p^2}\left((D-2)A(m_2^2) - 2m_1^2B_0 - D(p^2+m_1^2-m_2^2)B_1\right)$$

B.2 Evaluation of the One- and Two-point Integrals

For this part we select the important results for the one- and two-point integrals. Some hints for the calculations can be found in [Mut98] and [Hol00].

For the one-point integral, one gets:

$$A(m^2) = m^2\left(\Delta + 1 - \ln\frac{m^2}{\mu^2}\right) + \mathcal{O}(\varepsilon), \tag{B.12}$$

where

$$\Delta = \frac{1}{\varepsilon} - \gamma + \ln 4\pi, \tag{B.13}$$

and γ is Euler's constant ($\gamma = 0.577...$).

For the two-point integral, one obtains the following approximation:

$$B_0(p^2; m_1, m_2) = \Delta - \int_0^1 dx \ln \frac{x^2p^2 - x(p^2 + m_1^2 - m_2^2) + m_1^2}{\mu^2} + \mathcal{O}(\varepsilon). \tag{B.14}$$

In some particular cases, the integral is easy to evaluate:

$$B_0(0; 0, m) = \Delta + 1 - \ln \frac{m^2}{\mu^2} + \mathcal{O}(\varepsilon) = \frac{1}{m^2} A(m^2); \tag{B.15}$$

$$\begin{aligned} B_0(0; m_1, m_2) &= \Delta + 1 + \frac{m_1^2}{m_2^2 - m_1^2} \ln \frac{m_1^2}{\mu^2} - \frac{m_2^2}{m_2^2 - m_1^2} \ln \frac{m_2^2}{\mu^2} + \mathcal{O}(\varepsilon) \\ &= \frac{1}{2m_1^2} A(m_1^2) + \frac{1}{2m_2^2} A(m_2^2) + \frac{1}{2} \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \ln \frac{m_2^2}{m_1^2} + \mathcal{O}(\varepsilon); \end{aligned} \tag{B.16}$$

$$B_0(p^2; m, m) = \Delta - \ln \frac{m^2}{\mu^2} + \frac{p^2}{6m^2} + \mathcal{O}(\frac{p^2}{m^2}) + \mathcal{O}(\varepsilon), \text{ for } p^2 \text{ small}; \tag{B.17}$$

$$\Rightarrow \ B_0(0; m, m) = \Delta - \ln \frac{m^2}{\mu^2} + \mathcal{O}(\varepsilon). \tag{B.18}$$

Using the expansion in terms of ε, the expressions in (B.11) become

$$B_{20}(p^2; m_1, m_2) = \frac{1}{6} \left(A(m_2^2) + 2m_1^2 B_0 + (p^2 + m_1^2 - m_2^2) B_1 + (m_1^2 + m_2^2 - \frac{p^2}{3}) \right),$$

$$B_{21}(p^2; m_1, m_2) = \frac{1}{3p^2} \left(A(m_2^2) - m_1^2 B_0 - 2(p^2 + m_1^2 - m_2^2) B_1 - \frac{1}{2}(m_1^2 + m_2^2 - \frac{p^2}{3}) \right).$$

Terms of order $\mathcal{O}(\varepsilon)$ were omitted.

Looking at (B.12), (B.14) and (B.9) one can distinguish the divergent part of the n-point integrals.

$$\begin{aligned} \text{div}[A(m^2)] &= m^2 \Delta, \\ \text{div}[B_0(p^2; m_1, m_2)] &= \Delta, \\ \text{div}[B_1(p^2; m_1, m_2)] &= -\frac{1}{2} \Delta. \end{aligned} \tag{B.19}$$

From the same expressions, one obtains:

$$\begin{aligned} \varepsilon A &= m^2 + \mathcal{O}(\varepsilon), \\ \varepsilon B_0 &= \ \ 1 + \mathcal{O}(\varepsilon), \\ \varepsilon B_1 &= -\frac{1}{2} + \mathcal{O}(\varepsilon). \end{aligned} \tag{B.20}$$

B.2.1 Complete Evaluation of $B_0(p^2; m_1, m_2)$

For a complete result of the n-point integrals one needs to evaluate the integral in (B.14). The expression can be rewritten as

$$B_0(p^2; m_1, m_2) = \Delta - \int_0^1 dx \ln\left[\frac{p^2}{\mu^2}\left(x^2 - x(1 + \frac{m_1^2 - m_2^2}{p^2}) + \frac{m_1^2 - i\rho}{p^2}\right)\right]. \quad \text{(B.21)}$$

Here, ρ is a positive number, which tends to 0. From now on, $\mathcal{O}(\varepsilon)$ is not going to be written anymore, but it should be considered in every relation for B_0, since the following results are an expansion in ε.

In order to calculate the integral, we need to find out the sign for the argument of the logarithm:

$$x^2 - x(1 + \frac{m_1^2 - m_2^2}{p^2}) + \frac{m_1^2 - i\rho}{p^2} = (x - x_1)(x - x_2) \quad \text{(B.22)}$$

With the short notation for Kallen's function:

$$\lambda = \lambda(p^2, m_1^2, m_2^2) = p^4 + m_1^4 + m_2^4 - 2p^2m_1^2 - 2p^2m_2^2 - 2m_1^2m_2^2, \quad \text{(B.23)}$$

the solutions for the equation (B.22) are

$$x_{1,2} = \frac{1}{2p^2}\left(p^2 + m_1^2 - m_2^2 \pm \sqrt{\lambda + 4p^2 i\rho}\right). \quad \text{(B.24)}$$

The integral is evaluated easily, but a lot of care should be paid in case imaginary parts appear from the logarithm, i.e. when its argument is negative.

$$-\int_0^1 dx \ln\left(\frac{p^2}{\mu^2}(x - x_1)(x - x_2)\right) = -\ln\frac{p^2}{\mu^2} + 2 - (1 - x_1)\ln(1 - x_1) - x_1\ln(-x_1)$$
$$- (1 - x_2)\ln(1 - x_2) - x_2\ln(-x_2) + \eta\left(\frac{p^2}{\mu^2}(1 - x_1), (1 - x_2)\right) \quad \text{(B.25)}$$

The term with the function η comes from the decomposition rule for a logarithm of a product:

$$\ln(z_1 z_2) = \ln(z_1) + \ln(z_2) + \eta(z_1, z_2), \quad \text{(B.26)}$$

$$\eta(z_1, z_2) = 2\pi i\left[\theta(-\mathrm{Im}z_1)\theta(-\mathrm{Im}z_2)\theta(\mathrm{Im}(z_1z_2)) - \theta(\mathrm{Im}z_1)\theta(\mathrm{Im}z_2)\theta\left(-\mathrm{Im}(z_1z_2)\right)\right].$$

We have taken the logarithm branch cut along the negative real axis.

According to the values of x_1 and x_2 in different intervals for the square of the incoming momentum p, we have obtained compact expressions for (B.25). When necessary, formula (B.26) was used repeatedly to determine any possible imaginary terms. In the following, we just present the final results for the case $m_1 > m_2$. As we have mentioned, $B_0(p^2; m_1, m_2)$ is symmetric in m_1 and m_2, so our choice is arbitrary.

- $p^2 = 0$:

$$B_0(0; m_1, m_2) = \Delta + 1 - \frac{m_2^2}{m_2^2 - m_1^2} \ln \frac{m_2^2}{\mu^2} + \frac{m_1^2}{m_2^2 - m_1^2} \ln \frac{m_1^2}{\mu^2}$$

- $0 < p^2 < (m_1 - m_2)^2$:

$$B_0(p^2; m_1, m_2) = \Delta + 2 - \ln \frac{m_1 m_2}{\mu^2} - \frac{m_1^2 - m_2^2}{p^2} \ln \frac{m_1}{m_2} + \frac{1}{2p^2} \sqrt{\lambda} \ln \frac{m_1^2 + m_2^2 - p^2 + \sqrt{\lambda}}{m_1^2 + m_2^2 - p^2 - \sqrt{\lambda}}$$

- $p^2 = (m_1 - m_2)^2$:

$$B_0((m_1 - m_2)^2; m_1, m_2) = \Delta + 2 - \ln \frac{m_1 m_2}{\mu^2} - \frac{m_1 + m_2}{m_1 - m_2} \ln \frac{m_1}{m_2}$$

- $(m_1 - m_2)^2 < p^2 < (m_1 + m_2)^2$:

$$B_0(p^2; m_1, m_2) = \Delta + 2 - \ln \frac{m_1 m_2}{\mu^2} - \frac{m_1^2 - m_2^2}{p^2} \ln \frac{m_1}{m_2} + \frac{1}{p^2} \sqrt{|\lambda|} \left(\arctan \frac{m_1^2 + m_2^2 - p^2}{\sqrt{|\lambda|}} - \frac{\pi}{2} \right)$$

- $p^2 = (m_1 + m_2)^2$:

$$B_0((m_1 + m_2)^2; m_1, m_2) = \Delta + 2 - \ln \frac{m_1 m_2}{\mu^2} - \frac{m_1 - m_2}{m_1 + m_2} \ln \frac{m_1}{m_2}$$

- $(m_1 + m_2)^2 < p^2 < \infty$:

$$B_0(p^2; m_1, m_2) = \Delta + 2 - \ln \frac{m_1 m_2}{\mu^2} - \frac{m_1^2 - m_2^2}{p^2} \ln \frac{m_1}{m_2} + \frac{1}{2p^2} \sqrt{\lambda} \ln \frac{m_1^2 + m_2^2 - p^2 + \sqrt{\lambda}}{m_1^2 + m_2^2 - p^2 - \sqrt{\lambda}} + \frac{1}{p^2} \sqrt{\lambda} \pi i \quad \text{(B.27)}$$

The behaviour of B_0 is sketched in Figure B.1 for both, the real and the imaginary part. Here, we are interested only in the shape of B_0, therefore on the y-axis we

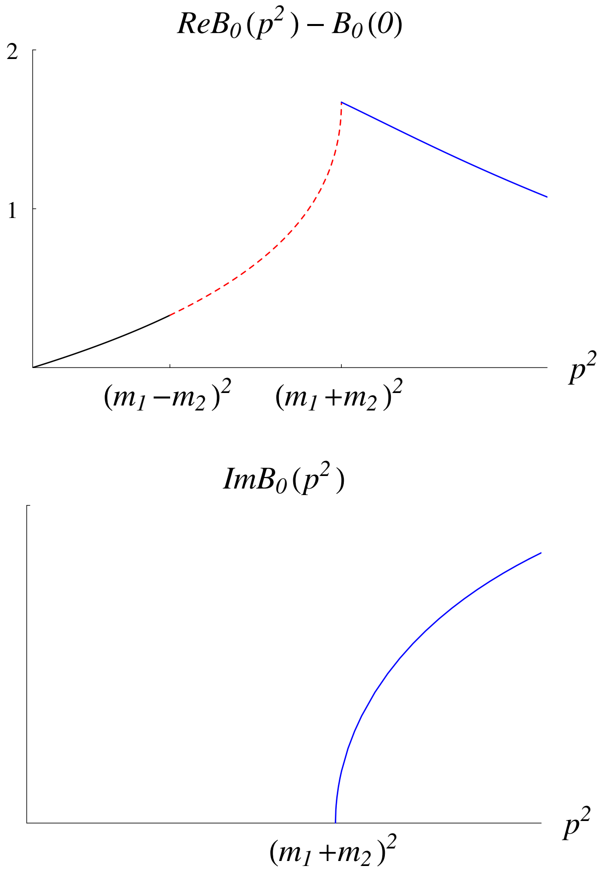


Figure B.1: $\mathrm{Re}B_0(p^2;m_1,m_2)-B_0(0;m_1,m_2)$ and $\mathrm{Im}B_0(p^2;m_1,m_2)$ as a function of p^2

specify a maximum for $\mathrm{Re}B_0(p^2;m_1,m_2)-B_0(0;m_1,m_2)$ and we do no write any ordinate for $\mathrm{Im}B_0(p^2;m_1,m_2)$. Note that at $p^2=(m_1+m_2)^2$, $B_0(p^2;m_1,m_2)$ has a critical point (maximum). The derivative of B_0 with respect to p^2 is not defined in this point.

For the particular case $m_1=m_2=m$, the previous results become simpler:

- $p^2=0$:

$$B_0(0;m,m)=\Delta-\ln\frac{m^2}{\mu^2}$$

- $0 < p^2 < 4m^2$:

$$B_0(p^2; m, m) = \Delta + 2 - \ln\frac{m^2}{\mu^2} + \sqrt{\left|1 - \frac{4m^2}{p^2}\right|}\left(\arctan\frac{2m^2 - p^2}{\sqrt{|4p^2m^2 - p^4|}} - \frac{\pi}{2}\right)$$

- $p^2 = 4m^2$:

$$B_0(4m^2; m, m) = \Delta + 2 - \ln\frac{m^2}{\mu^2}$$

- $4m^2 < p^2 < \infty$:

$$B_0(p^2; m, m) = \Delta + 2 - \ln\frac{m^2}{\mu^2} + \sqrt{1 - \frac{4m^2}{p^2}}\left(\ln\frac{1 - \sqrt{1 - \frac{4m^2}{p^2}}}{1 + \sqrt{1 + \frac{4m^2}{p^2}}} + \pi i\right) \tag{B.28}$$

B.2.2 Particular On-shell Cases for B_0, B_1 and Their Derivatives

In the expressions of the renormalization constants, we need to set the momentum in the self-energies on-shell. When taking the diagonal elements of the wave function renormalization constants, one will have first to differentiate the self-energies and then set the momentum on-shell. As all the fermion self-energies contain two-point integrals, their evaluation is mainly reduced to the evaluation of the integrals. Depending on the particles in the loop, one has several relations between the masses. In the following, we enumerate particular cases that are often encountered in calculations and we give the results, omitting the details.

- $p^2 \longrightarrow m^2,\ m_1 = m,\ m_2 = \delta,$ where $\delta \longrightarrow 0$:

$$B_0(m^2; m, \delta) = \Delta + 2 - \ln\frac{m^2}{\mu^2} + \mathcal{O}(\delta) + \mathcal{O}(\varepsilon), \tag{B.29}$$

$$B_0'(m^2; m, \delta) = -\frac{1}{m^2} - \frac{1}{2m^2}\ln\frac{\delta^2}{m^2} + \mathcal{O}(\delta) + \mathcal{O}(\varepsilon), \tag{B.30}$$

$$B_1(m^2; m, \delta) = -\frac{1}{2}\left(\Delta + 3 - \ln\frac{m^2}{\mu^2}\right) + \mathcal{O}(\delta) + \mathcal{O}(\varepsilon), \tag{B.31}$$

$$B_1'(m^2; m, \delta) = \frac{3}{2m^2} + \frac{1}{2m^2}\ln\frac{\delta^2}{m^2} + \mathcal{O}(\delta) + \mathcal{O}(\varepsilon), \tag{B.32}$$

- $p^2 \longrightarrow m^2,\ m_1 = m,\ m_2 = M,$ where $m^2 \ll M^2$:

$$B_0(m^2; m, M) = \Delta + 1 - \ln\frac{M^2}{\mu^2} + \mathcal{O}\left(\frac{m^2}{M^2}\right) + \mathcal{O}(\varepsilon), \tag{B.33}$$

$$B_0'(m^2; m, M) = \frac{1}{2M^2} + \mathcal{O}\left(\frac{m^2}{M^2}\right) + \mathcal{O}(\varepsilon), \tag{B.34}$$

$$B_1(m^2; m, M) = -\frac{1}{2}\left(\Delta - \ln\frac{M^2}{\mu^2}\right) - \frac{1}{4} + \mathcal{O}\left(\frac{m^2}{M^2}\right) + \mathcal{O}(\varepsilon), \tag{B.35}$$

$$B_1'(m^2; m, M) = -\frac{1}{6M^2} + \mathcal{O}\left(\frac{m^2}{M^2}\right) + \mathcal{O}(\varepsilon), \tag{B.36}$$

where $B_{0/1}'(m^2; m_1, m_2) = \left.\dfrac{\partial B_{0/1}(p^2; m_1, m_2)}{\partial p^2}\right|_{p^2 \to m^2}$.
For the calculation of B_0 and their derivatives, we started from the corresponding result in (B.27) and depending on the case, we either directly took limits or used Taylor expansions. The expressions of B_1 were first linked to the ones of B_0 by (B.9) and then calculated in a similar way.

B.2.3 Complete Evaluation of B_0' and B_1'

The complete result for the derivative of $B_0(p^2; m_1, m_2)$ can be obtain directly differentiating the expressions in (B.27) with respect to p^2. As an alternative method, one can start from the integral form of B_0 given in (B.21): first differentiate the expression with respect to p^2 and then integrate over x[1]. We obtain

$$\begin{aligned} B_0'(p^2; m_1, m_2) &= -\frac{1}{p^2} - \frac{1}{p^2}\int_0^1 dx \frac{x\dfrac{m_1^2 - m_2^2}{p^2} - \dfrac{m_1^2 - i\rho}{p^2}}{x^2 - x\left(1 + \dfrac{m_1^2 - m_2^2}{p^2}\right) + \dfrac{m_1^2 - i\rho}{p^2}} + \mathcal{O}(\varepsilon) \\ &= -\frac{1}{p^2}\left(1 + A\int_0^1 dx \frac{1}{x - x_1} + B\int_0^1 dx \frac{1}{x - x_2}\right), \end{aligned} \tag{B.37}$$

where A and B are defined as

$$\begin{aligned} A &= \frac{1}{x_1 - x_2}\left(x_1\frac{m_1^2 - m_2^2}{p^2} - \frac{m_1^2 - i\rho}{p^2}\right), \\ B &= \frac{1}{x_2 - x_1}\left(x_2\frac{m_1^2 - m_2^2}{p^2} - \frac{m_1^2 - i\rho}{p^2}\right). \end{aligned} \tag{B.38}$$

[1]Our function fullfills the conditions necessary to differentiate before integrating (the differentiating theorem for the dependence of the integral on a parameter).

x_1 and x_2 are the solutions of the quadratic equation (B.22), given in (B.24).

After integration, (B.37) can be written in the final form as

$$\begin{aligned} B_0'(p^2; m_1, m_2) = -\frac{1}{p^2} & (1 + A\ln(1-x_1) - A\ln(-x_1) \\ & + B\ln(1-x_2) - B\ln(-x_2)) \end{aligned} \tag{B.39}$$

and then evaluated considering the different intervals for p^2, as done in (B.27).

The general result for the derivative of $B_0(p^2; m_1, m_2)$ with respect to p^2 is

- $0 < p^2 < (m_1 - m_2)^2$:

$$\begin{aligned} B_0'(p^2; m_1, m_2) = & -\frac{1}{p^2} + \frac{m_1^2 - m_2^2}{p^4} \ln\frac{m_1}{m_2} \\ & + \frac{1}{2p^2\sqrt{\lambda}} \left(m_1^2 + m_2^2 - \frac{(m_1^2 - m_2^2)^2}{p^2} \right) \ln\frac{m_1^2 + m_2^2 - p^2 + \sqrt{\lambda}}{m_1^2 + m_2^2 - p^2 - \sqrt{\lambda}} \end{aligned}$$

- $p^2 = (m_1 - m_2)^2$:

$$B_0'(p^2; m_1, m_2) = -\frac{2}{(m_1 - m_2)^2} + \frac{m_1 + m_2}{(m_1 - m_2)^3} \ln\frac{m_1}{m_2}$$

- $(m_1 - m_2)^2 < p^2 < (m_1 + m_2)^2$:

$$\begin{aligned} B_0'(p^2; m_1, m_2) = & -\frac{1}{p^2} + \frac{m_1^2 - m_2^2}{p^4} \ln\frac{m_1}{m_2} \\ & - \frac{1}{p^2\sqrt{|\lambda|}} \left(m_1^2 + m_2^2 - \frac{(m_1^2 - m_2^2)^2}{p^2} \right) \left(\arctan\frac{m_1^2 + m_2^2 - p^2}{\sqrt{|\lambda|}} - \frac{\pi}{2} \right) \end{aligned}$$

- $(m_1 + m_2)^2 < p^2 < \infty$:

$$\begin{aligned} B_0'(p^2; m_1, m_2) = & -\frac{1}{p^2} + \frac{m_1^2 - m_2^2}{p^4} \ln\frac{m_1}{m_2} \\ & + \frac{1}{2p^2\sqrt{\lambda}} \left(m_1^2 + m_2^2 - \frac{(m_1^2 - m_2^2)^2}{p^2} \right) \ln\frac{m_1^2 + m_2^2 - p^2 + \sqrt{\lambda}}{m_1^2 + m_2^2 - p^2 - \sqrt{\lambda}} \\ & + \frac{1}{p^2\sqrt{\lambda}} \left(m_1^2 + m_2^2 - \frac{(m_1^2 - m_2^2)^2}{p^2} \right) \pi i \end{aligned} \tag{B.40}$$

B_0' as a function of p^2 is sketched in Figure B.2.

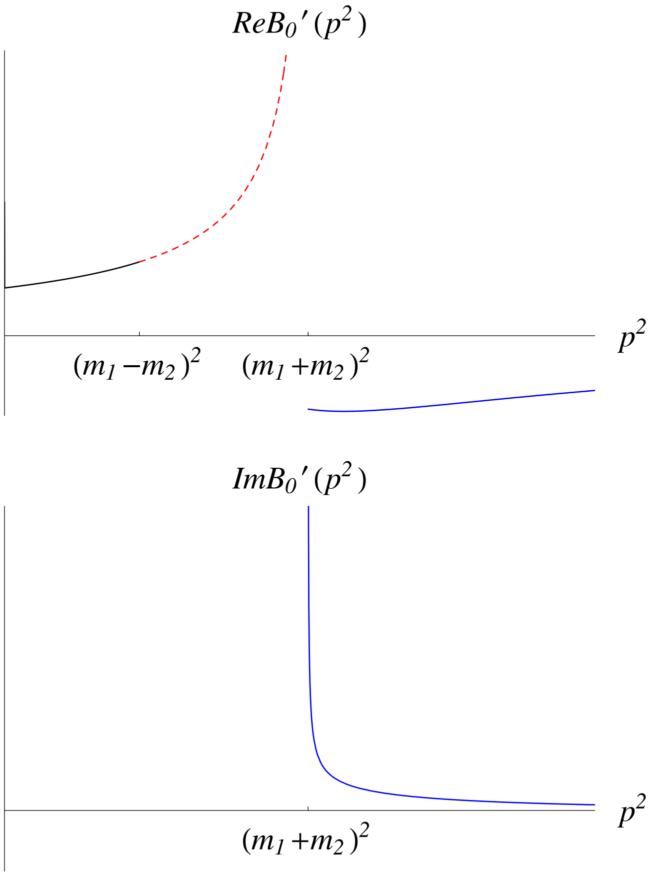


Figure B.2: $\mathrm{Re}B_0'(p^2; m_1, m_2)$ and $\mathrm{Im}B_0'(p^2; m_1, m_2)$ as a function of p^2

The derivative of $B_1(p^2; m_1, m_2)$ can be reduced to derivatives of $B_0(p^2; m_1, m_2)$. From (B.9), we get

$$B_1'(p^2; m_1, m_2) = -\frac{1}{2p^4}\left(A(m_1^2) - A(m_2^2) - (m_1^2 - m_2^2)B_0(p^2; m_1, m_2)\right) - \frac{p^2 + m_1^2 - m_2^2}{2p^2} B_0'(p^2; m_1, m_2) \tag{B.41}$$

$$= -\frac{1}{p^2} B_1(p^2; m_1, m_2) - \frac{1}{2p^2} B_0(p^2; m_1, m_2) - \frac{p^2 + m_1^2 - m_2^2}{2p^2} B_0'(p^2; m_1, m_2). \tag{B.42}$$

Appendix C

Matrix Manipulations for One-loop Calculations

When performing calculations in the first order approximation in α (at one-loop), we often deal with matrices whose elements are of type

$$\mathcal{A}_{ij} = a_i \delta_{ij} + \delta A_{ij} + \mathcal{O}(\alpha^2), \tag{C.1}$$

with $\delta A_{ij} \propto \alpha$ and $\delta A_{jk} \ll a_i$ for $\forall\ i,\ j,\ k$. Using this expansion, we present the results for the inverse matrix, some useful relations for unitary matrices and the matrix diagonalization procedure, together with some calculational details.

Besides this, sometimes we will need to take the square root of a matrix. (The matrix should be positive defined.) At first order, on components, we have

$$(\mathcal{A}^{\frac{1}{2}})_{ij} = \sqrt{a_i}\delta_{ij} + \frac{\delta A_{ij}}{\sqrt{a_i} + \sqrt{a_j}} + \mathcal{O}(\alpha^2). \tag{C.2}$$

If we consider the particular case $a_i = 1$ for $\forall\ i$, i.e.

$$(\mathcal{A}_0)_{ij} = \delta_{ij} + \delta A_{ij} + \mathcal{O}(\alpha^2), \tag{C.3}$$

then

$$(\mathcal{A}_0^{\frac{1}{2}})_{ij} = \delta_{ij} + \frac{1}{2}\delta A_{ij} + \mathcal{O}(\alpha^2). \tag{C.4}$$

C.1 Matrix Inversion

In most of the examples we will have 3×3 matrices and since the size makes the matrices very easy to operate by hand, we give the detailed result for this case.

$$\mathcal{A}_3 = \begin{pmatrix} a_1 + \delta A_{11} & \delta A_{12} & \delta A_{13} \\ \delta A_{21} & a_2 + \delta A_{22} & \delta A_{23} \\ \delta A_{31} & \delta A_{32} & a_3 + \delta A_{33} \end{pmatrix} + \mathcal{O}(\alpha^2) \tag{C.5}$$

In first order approximation, i.e. considering just terms up to δA_{ij} and neglecting every product between them, the inverse matrix is

$$\mathcal{A}_3^{-1} = \begin{pmatrix} \frac{1}{a_1} - \frac{\delta A_{11}}{a_1^2} & -\frac{\delta A_{12}}{a_1 a_2} & -\frac{\delta A_{13}}{a_1 a_3} \\ -\frac{\delta A_{21}}{a_1 a_2} & \frac{1}{a_2} - \frac{\delta A_{22}}{a_2^2} & -\frac{\delta A_{23}}{a_2 a_3} \\ -\frac{\delta A_{31}}{a_1 a_3} & -\frac{\delta A_{32}}{a_2 a_3} & \frac{1}{a_3} - \frac{\delta A_{33}}{a_3^2} \end{pmatrix} + \mathcal{O}(\alpha^2). \tag{C.6}$$

If we take the general form, with the index notation, we can write:

$$(\mathcal{A}^{-1})_{ij} = \frac{1}{a_i}\delta_{ij} - \frac{\delta A_{ij}}{a_i a_j} + \mathcal{O}(\alpha^2), \tag{C.7}$$

An important thing to notice is that each diagonal element of $\mathcal{A}^{-1}$ is the inverse of the corresponding term in $\mathcal{A}$, up to terms of order α, i.e.

$$\left(\frac{1}{a_i} - \frac{\delta A_{ii}}{a_i^2}\right)(a_i + \delta A_{ii}) = 1 + \mathcal{O}(\alpha^2). \tag{C.8}$$

For the particular matrix $\mathcal{A}_0$ (C.3), the inverse is given by

$$(\mathcal{A}_0)_{ij}^{-1} = \delta_{ij} - \delta A_{ij} + \mathcal{O}(\alpha^2). \tag{C.9}$$

Note that these results work for any $n \times n$ matrices and not only for the 3×3, as in our example.

C.2 Unitary Matrix

In the following, we enumerate several relations necessary for calculations where unitary matrices are involved. Let $\mathcal{U}$ be a unitary matrix with the expansion

$$\mathcal{U}_{ij} = u_{ij} + \delta U_{ij} + \mathcal{O}(\alpha^2). \tag{C.10}$$

The unitarity relation for $\mathcal{U}$ implies

$$\sum_j (\mathcal{U}_{ji})^* \mathcal{U}_{jk} = \sum_j \mathcal{U}_{ij} (\mathcal{U}_{kj})^* = \delta_{ik}. \tag{C.11}$$

Inserting (C.10), we obtain

$$\begin{aligned} \sum_j (u_{ji})^* u_{jk} = \sum_j u_{ij} (u_{kj})^* = \delta_{ik}, \\ \sum_j \left((\delta U_{ji})^* u_{jk} + (u_{ji})^* \delta U_{jk}\right) = \sum_j \left(u_{ij} (\delta U_{kj})^* + \delta U_{ij} (u_{kj})^*\right) = 0. \end{aligned} \tag{C.12}$$

If we assume $u_{ij} = u_i \delta_{ij}$, then the conditions become

$$\begin{aligned} (u_i)^* u_i &= 1, \\ (\delta U_{ji})^* u_j + (u_i)^* \delta U_{ij} &= u_i (\delta U_{ji})^* + \delta U_{ij} (u_j)^* = 0, \end{aligned} \quad \text{for } u_{ij} = u_i \delta_{ij}. \tag{C.13}$$

For $u_i = 1$, (C.13) leads to

$$\delta U_{ij} = -(\delta U_{ji})^*, \text{ for } u_{ij} = \delta_{ij}. \tag{C.14}$$

C.3 Matrix Diagonalization

For the diagonalizing procedure, we restrict to a complex matrix of type

$$\mathcal{C}_{ij} = c_i \delta_{ij} + \delta C_{ij} + \mathcal{O}(\alpha^2), \tag{C.15}$$

that has the lowest order elements c_i real and nonnegative. $\delta C_{ij} \in \mathbb{C}$ and $\delta C_{ij} \ll c_i$ for $\forall\ i,\ j$. The general case is not more complicated, but for our purpose it is not needed. The case of a symmetric one can be calculated similarly, using for example, the general prescription of [Zum62].

According to the singular value decomposition theorem, any complex matrix can be written in the form:

$$\mathcal{C} = \mathcal{O} \mathcal{D} \mathcal{P}^\dagger, \tag{C.16}$$

where $\mathcal{O}$ and $\mathcal{P}$ are two unitary matrices and $\mathcal{D}$ is a diagonal one. In addition, the two unitary matrices can be chosen such that the diagonal one has real and positive entries.

If $\mathcal{C}$ has an inverse, then the diagonal matrix is given by [Zum62]

$$\mathcal{D} = \mathcal{O}^\dagger \mathcal{C} \left(\mathcal{C}^\dagger \mathcal{C} \right)^{-\frac{1}{2}} \mathcal{C}^\dagger \mathcal{O}, \tag{C.17}$$

and $\mathcal{P}$ can be identified as

$$\mathcal{P} = \left(\mathcal{C}^\dagger \mathcal{C} \right)^{-\frac{1}{2}} \mathcal{C}^\dagger \mathcal{O}. \tag{C.18}$$

To perform the calculations up to terms of order α, we assume the following expansions for the unitary matrices:

$$\mathcal{O}_{ij} = o_{ij} + \delta O_{ij} + \mathcal{O}(\alpha^2), \tag{C.19}$$

$$\mathcal{P}_{ij} = p_{ij} + \delta P_{ij} + \mathcal{O}(\alpha^2). \tag{C.20}$$

Given the special form of $\mathcal{C}$ from (C.15), one can directly take

$$o_{ij} = p_{ij} = \delta_{ij}. \tag{C.21}$$

Still, for the intermediate steps, we keep the general assumption. At the end, we show why the ansatz (C.21) is reasonable.

From (C.17), we will try to determine $\mathcal{O}$, so that the diagonal matrix $\mathcal{D}$ has entries of type

$$\mathcal{D}_{ij} = (d_i + \delta D_i)\delta_{ij} + \mathcal{O}(\alpha^2). \tag{C.22}$$

d_i and δD_i are real and in addition, d_i is nonnegative for $\forall\ i$.

On components, (C.17) is

$$\mathcal{D}_{ij} = \sum_{k-n} (\mathcal{O}_{ki})^* \mathcal{C}_{kl} \left(\left(\mathcal{C}^\dagger \mathcal{C}\right)^{-\frac{1}{2}} \right)_{lm} (\mathcal{C}_{nm})^* \mathcal{O}_{nj}. \tag{C.23}$$

We start by evaluating the $\left(\mathcal{C}^\dagger\mathcal{C}\right)^{-\frac{1}{2}}$ factor. Up to first order terms,

$$\begin{aligned} \left(\mathcal{C}^\dagger\mathcal{C}\right)_{ij} &= \sum_k (\mathcal{C}_{ki})^* \mathcal{C}_{kj} \\ &= c_i^2 \delta_{ij} + c_i \delta C_{ij} + (\delta C_{ji})^* c_j + \mathcal{O}(\alpha^2). \end{aligned} \tag{C.24}$$

We used the fact that c_i is real. To simplify the notation, we denote $\mathcal{C}^\dagger\mathcal{C}$ by the general matrix $\mathcal{A}$ defined in (C.1). From (C.24), we identify the terms of the expansion as

$$\begin{aligned} a_i &= c_i^2, \\ \delta A_{ij} &= c_i \delta C_{ij} + (\delta C_{ji})^* c_j. \end{aligned} \tag{C.25}$$

Applying (C.2) and (C.7), we obtain

$$\begin{aligned} \left(\left(\mathcal{C}^\dagger\mathcal{C}\right)^{-\frac{1}{2}} \right)_{ij} &= \left(\mathcal{A}^{-\frac{1}{2}}\right)_{ij} \\ &= \frac{1}{\sqrt{a_i}} \left(\delta_{ij} - \frac{1}{\sqrt{a_j}} \frac{\delta A_{ij}}{\sqrt{a_i} + \sqrt{a_j}} \right) + \mathcal{O}(\alpha^2). \end{aligned} \tag{C.26}$$

Replacing all the matrices in (C.23) by their expansions, we have to solve

$$\begin{aligned} (d_i + \delta D_i)\delta_{ij} = \sum_{k-n} &\left((o_{ki})^* + (\delta O_{ki})^*\right) \left(c_k \delta_{kl} + \delta C_{kl}\right) \frac{1}{\sqrt{a_l}} \left(\delta_{lm} - \frac{1}{\sqrt{a_m}} \frac{\delta A_{lm}}{\sqrt{a_l} + \sqrt{a_m}} \right) \\ &\left(c_m \delta_{mn} + (\delta C_{nm})^*\right) \left(o_{nj} + \delta O_{nj}\right). \end{aligned} \tag{C.27}$$

After multiplication and inserting (C.25), we identify

$$d_i \delta_{ij} = \sum_k (o_{ki})^* c_k o_{kj}, \tag{C.28}$$

$$\begin{aligned} \delta D_i \delta_{ij} &= \sum_k \left((o_{ki})^* c_k \delta O_{kj} + (\delta O_{ki})^* c_k o_{kj} \right) \\ &\quad + \sum_{k,n} (o_{ki})^* \frac{1}{c_k + c_n} \left(c_n \delta C_{kn} + c_k (\delta C_{nk})^* \right) o_{nj}. \end{aligned} \tag{C.29}$$

We start by analysing (C.28). If we multiply from left with $\sum_i o_{li}$ and we use the unitarity relation (C.12), we obtain

$$o_{lj}d_j = c_l o_{lj} \;\Leftrightarrow\; (d_j - c_l)o_{lj} = 0. \tag{C.30}$$

For $j \neq l$, we have as solution $o_{lj} = 0$ and for the diagonal case, taking $o_{jj} \neq 0$,

$$d_j = c_j. \tag{C.31}$$

The dominant term of the unitary matrix $\mathcal{O}$ can be written now as $o_i\delta_{ij}$, i.e.

$$\mathcal{O}_{ij} = o_i\delta_{ij} + \delta O_{ij} + \mathcal{O}(\alpha^2). \tag{C.32}$$

The elements of the diagonal matrix $\mathcal{D}$ become

$$d_i\delta_{ij} = c_i\delta_{ij}, \tag{C.33}$$

$$\delta D_i\delta_{ij} = (o_i)^* c_i \delta O_{ij} + (\delta O_{ji})^* c_j o_j + (o_i)^* \frac{1}{c_i + c_j}\left(c_j\delta C_{ij} + c_i(\delta C_{ji})^*\right) o_j. \tag{C.34}$$

In particular, with the unitarity conditions (C.13),

$$\delta D_i = \frac{1}{2}\left(\delta C_{ii} + (\delta C_{ii})^*\right). \tag{C.35}$$

One can see that for the diagonalization of $\mathcal{C}$ (C.15), it is enough to choose a diagonal matrix for the dominant term of the unitary matrix $\mathcal{O}$. In addition, the phase factor o_i is not needed since we choose from the beginning c_i real. Therefore, we can set $o_i = 1$ and

$$\mathcal{O}_{ij} = \delta_{ij} + \delta O_{ij} + \mathcal{O}(\alpha^2). \tag{C.36}$$

δO_{ii} is also not constrained by diagonalization.

Inserting (C.36) in (C.34) we obtain the off-diagonal elements of $\mathcal{O}$.

$$\delta O_{ij} = \frac{1}{c_j^2 - c_i^2}\left(c_j\delta C_{ij} + c_i(\delta C_{ji})^*\right), \text{ for } i \neq j. \tag{C.37}$$

Returning to the expression of $\mathcal{P}$ from (C.18), we have

$$\begin{aligned}(\mathcal{P})_{ij} &= \sum_{k,l}\left(A^{-\frac{1}{2}}\right)_{ik}(\mathcal{C}_{lk})^*\mathcal{O}_{lj} \\ &= \delta_{ij} + \delta O_{ij} + \frac{1}{c_i + c_j}\left(-\delta C_{ij} + (\delta C_{ji})^*\right) + \mathcal{O}(\alpha^2).\end{aligned} \tag{C.38}$$

Together with (C.37), we get

$$\delta P_{ij} = \frac{1}{c_j^2 - c_i^2} \left(c_i \delta C_{ij} + c_j (\delta C_{ji})^* \right), \text{ for } i \neq j. \tag{C.39}$$

One can verify that δO_{ij} and δP_{ij} fulfil the unitarity condition (C.14).

To summarise, a complex matrix of type (C.15) can be diagonalized with the help of two related unitary matrices that have as dominant term the identity matrix and the off-diagonal elements given by (C.37) and (C.39). Their first order diagonal elements are not determined. The diagonal matrix that results has real, nonnegative elements given by

$$D_{ii} = c_i + \frac{1}{2} \left(\delta C_{ii} + (\delta C_{ii})^* \right) + \mathcal{O}(\alpha^2). \tag{C.40}$$

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