Polarization in heavy quark decays

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Abstract

In this thesis I concentrate on the angular correlations in top quark decays and their next–to–leading order (NLO) QCD corrections. I also discuss the leading–order (LO) angular correlations in unpolarized and polarized hyperon decays.

In the first part of the thesis I calculate the angular correlation between the top quark spin and the momentum of decay products in the rest frame decay of a polarized top quark into a charged Higgs boson and a bottom quark in Two-Higgs-Doublet-Models: $t(\uparrow) \to b + H^+$. The decay rate in this process is split into an angular independent part (unpolarized) and an angular dependent part (polar correlation). I provide closed form formulae for the $\mathcal{O}(\alpha_s)$ radiative corrections to the unpolarized and the polar correlation functions for $m_b \neq 0$ and $m_b = 0$. The results for the unpolarized rate agree with the existing results in the literature. The results for the polarized correlations are new. I found that, for certain values of $\tan\beta$, the $\mathcal{O}(\alpha_s)$ radiative corrections to the unpolarized, polarized rates, and the asymmetry parameter can become quite large.

In the second part I concentrate on the semileptonic rest frame decay of a polarized top quark into a bottom quark and a lepton pair: $t(\uparrow) \to X_b + \ell^+ + \nu_\ell$. I analyze the angular correlations between the top quark spin and the momenta of the decay products in two different helicity coordinate systems: system 1a with the z–axis along the charged lepton momentum, and system 3a with the z–axis along the neutrino momentum. The decay rate then splits into an angular independent part (unpolarized), a polar angle dependent part (polar correlation) and an azimuthal angle dependent part (azimuthal correlation). I present closed form expressions for the $\mathcal{O}(\alpha_s)$ radiative corrections to the unpolarized part and the polar and azimuthal correlations in system 1a and 3a for $m_b \neq 0$ and $m_b = 0$. For the unpolarized part and the polar correlation I agree with existing results. My results for the azimuthal correlations are new. In system 1a I found that the azimuthal correlation vanishes in the leading order as a consequence of the $(V - A)$ nature of the Standard Model current. The $\mathcal{O}(\alpha_s)$ radiative corrections to the azimuthal correlation in system 1a are very small (around 0.24% relative to the unpolarized LO rate). In system 3a the azimuthal correlation does not vanish at LO. The $\mathcal{O}(\alpha_s)$ radiative corrections decreases the LO azimuthal asymmetry by around 1%.

In the last part I turn to the angular distribution in semileptonic hyperon decays. Using the helicity method I derive complete formulas for the leading order joint angular decay distributions occurring in semileptonic hyperon decays including lepton mass and polarization effects. Compared to the traditional covariant calculation the helicity method allows one to organize the calculation of the angular decay distributions in a very compact and efficient way. This is demonstrated by the specific example of the polarized hyperon decay $\Xi^0(\uparrow) \to \Sigma^+ + l^- + \bar{\nu}_l$ $(l^- = e^-, \mu^-)$ followed by the nonleptonic decay $\Sigma^+ \to p + \pi^0$, which is described by a five–fold angular decay distribution.

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Chapter 1

Introduction

The top quark, the most massive fundamental particle in the Standard Model (SM), was discovered in 1995 by the CDF and DØ collaborations at the Tevatron collider at Fermilab [1, 2]. The top quark mass was measured at CDF and DØ to be $m_t = 174.2 \pm 3.3\,\text{GeV}$ (pole mass)[3]. The existence of the the top quark was predicted as the electroweak isospin partner of the b–quark (see Table 1.1), which was discovered in 1977 [4]. Although the top quark discovery was anticipated the large mass of the top quark was a big surprise. The top quark has a mass slightly less than the mass of the gold atom, approximately twice that of W and Z bosons, the carriers of the electroweak force, and thirty–five times that of the next most massive fermion, the b–quark. The SM neither predicts nor explains the observed mass hierarchy.

At the top mass scale the strong coupling constant is small, $\alpha_s(m_t) \sim 0.1$. Therefore QCD effects involving the top quark are well behaved in the perturbative sense, i.e. the top quark decays provide an ideal tool for studying perturbative QCD.

The interplay between the large top mass and its spin is of crucial importance in studying the SM. The decay width of the top quark is dominated by the two–body channel $t \to b + W^+$ because the top quark has a mass above the Wb threshold, and the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix element V_{tb} is close to unity with other channels heavily suppressed. The top quark decay width increases with the top quark mass. The top quark decay width $\Gamma_t = 1.39\,\text{GeV}$ for the pole mass of 174 GeV was calculated for this channel in the SM to second order in QCD [5] and to first order in EW [6] corrections. Consequently the SM result of the top quark lifetime is $\tau_t \approx 0.5 \times 10^{-24}\,\text{s}$. This is much shorter than the typical time for the formation of QCD bound states $\tau_{QCD} \approx 1/\Lambda_{QCD} \approx 3 \times 10^{-24}\,\text{s}$, i.e. the top quark decays long before it can hadronize [7]. Therefore top decays provide a very clean source of information about the structure of the SM. In particular the time scale is much shorter than the typical time required for the QCD interactions to randomize its spin. Therefore the polarization of the top quark is preserved in the decay and can be studied through the angular correlations between the direction of the top quark spin and the momenta of the decay products. Measuring the top quark polarization and comparing it to the theoretical predictions would present a clean test of the SM.

Until recently the precision of the measurements of the top quark have been limited because of the relatively small top production cross section leading to a small number of

fermions				bosons
	1^{st} generation	2^{nd} generation	3^{rd} generation	γ
leptons	$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L , \quad e_R$	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L , \quad \mu_R$	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L , \quad \tau_R$	$W^{\pm}, Z^0$
quarks	$\begin{pmatrix} u \\ d \end{pmatrix}_L , \quad u_R \, , d_R$	$\begin{pmatrix} c \\ s \end{pmatrix}_L , \quad c_R \, , s_R$	$\begin{pmatrix} t \\ b \end{pmatrix}_L , \quad t_R \, , b_R$	g

Table 1.1: The Standard Model of the elementary particles. The matter is made up of the fermions, the quarks and lepton which are divided into three families or generations. The left–handed fermions form isospin doublets while right–handed fermions are isospin singlets. The interaction between fermions are mediated by gauge bosons.

events. At present the world's only source of top quarks is the Fermilab Tevatron collider. At the Tevatron the top quarks are produced mainly in pairs by $p\bar{p}$ collision at a center–of–mass energy of $\sqrt{s} = 1.96\,\text{TeV}$. At the Tevatron energy top quark pairs are produced from quark annihilation ($q\bar{q} \to t\bar{t}$) 85% of the time and from gluon annihilation ($gg \to t\bar{t}$) 15% of the time (see Fig. 1.1). Single top quark production occur via electroweak production mechanism $q\bar{q}' \to W^* \to \bar{b}t$ or $qg \to q'W^* \to q'\bar{b}t$ with roughly a factor 3 suppression [9].

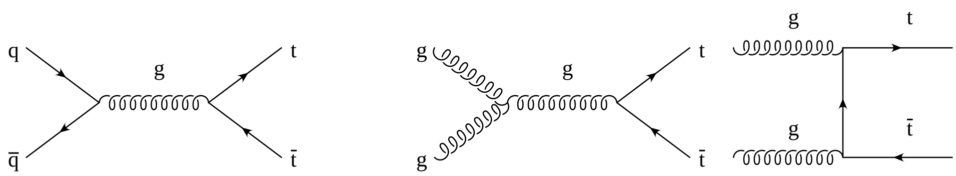

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Figure 1.1: Leading order Feynman diagrams for top quark pair production via $q\bar{q} \to t\bar{t}$ or $gg \to t\bar{t}$ at the Tevatron. The total production cross section is ca. 7 pb.

The top quark pair production cross section depends strongly on the top quark mass. The theoretical predictions for the top quark pair production cross section at QCD NLO are around 6.7 pb [13, 14] (at $m_t = 175\,\text{GeV}$, $\sqrt{s} = 1.96\,\text{TeV}$). These predictions are in good agreement with the measurements [1] at CDF of $7.32 \pm 0.85\,\text{pb}$ [15] and at DØ of $7.1^{+1.9}_{-1.7}\,\text{pb}$ [16]. With such a pair production cross section, in order to produce large number of top quarks for precision studies, the collider must have a very high luminosity.

In the Run I period (1992-1996) there were around 1200 $t\bar{t}$ pairs in the interaction region where the detectors DØ and CDF are situated. In the Run II period (2001 till now) this number increased dramatically (see Table 1.2). The future colliders are also powerful top quark factories. The Large Hadron Collider (LHC) at CERN is planned to start in 2007, initial measurements in 2008, and the first precision measurements perhaps in 2009 with integrated luminosity between 1 and 10 fb^{-1}. The planned International Linear Collider, the ILC [8] will hopefully run from 2015. Highly polarized top quarks

[1]Preliminary results, not yet submitted for publication as of September 2006.

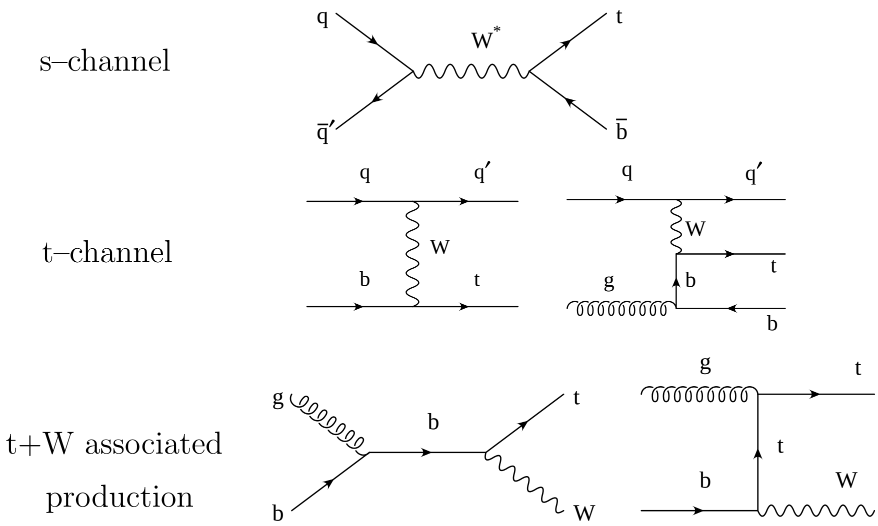


Figure 1.2: Leading order Feynman diagrams for the single top quark production (electroweak interaction) at the Tevatron. The production cross section for the s–channel is ca. 0.9 pb, for the t–channel is ca. 2.0 pb, and for the $t + W$ associated production is ca. 0.12 pb.

colliders	$t\bar{t}$-pair/y	center–of–mass energy	method	start up time
TEVATRON Run II	$5 - 6 \times 10^3$	$\sqrt{s} = 1.8$ TeV	$p\bar{p}$	from 2001
LHC*	$10^7 - 10^8$	$\sqrt{s} = 14.0$ TeV	pp	from 2007
ILC*	$1 - 4 \times 10^5$	$\sqrt{s} = 300 - 800$ GeV	e^+e^-	from 2015

* Planned colliders

Table 1.2: Present and planned $t\bar{t}$ factories.

will become available in singly produced top quarks at hadron colliders (see e.g. [9–12]) and in top quark pairs produced at future linear $e^+ - e^-$–colliders (see e.g. [17–23]), providing enough statistics for precision studies of polarized top quark decays.

The angular correlation in the semileptonic decay of a polarized top quark is the consequence of the pure left–chiral $(V - A)$ structure of the charged current in the SM. Precision measurements of these angular correlations will reveal important informations about the SM and its $(V - A)$ coupling. To match the ever improving data from the experiments it is necessary to calculate the NLO corrections to the angular correlations. The radiative corrections to the angular correlation between the top quark polarization vector $\vec{P}_t$ and the final state leptons in the semileptonic rest frame decay of a polarized top quark $t(\uparrow) \to b + \ell^+ + \nu_\ell$ have been studied extensively before [24–28]. In this thesis we present the *first* NLO QCD corrections to the azimuthal correlations in polarized top quark decays.

Another important aspect about the top quark is its Yukawa coupling. The top quark is the only fermion with a mass close to the electroweak symmetry breaking scale. The couplings involving the top quark can be important in determining if the SM mechanism for the electroweak symmetry breaking is the correct one. The Yukawa coupling of the top quark and the Higgs boson[29] is among the most likely places where the new physics may manifest itself. The experimental searches performed at LEP allowed one to put a lower bound on the SM Higgs boson mass $m_H > 114\,\mathrm{GeV}$ at 95% CL [35]. A global SM fit to electroweak precision measurements has been used to obtain an upper bound of $m_H < 219\,\mathrm{GeV}$ at 95% CL [36]. If charged Higgs particles exist as predicted in models with two Higgs doublets (2HDM), they will play an important role in the production and the decay of the top quark. If the charged Higgs mass is less than the difference between the top mass and the b–quark mass, the top quark can decay as: $t \rightarrow H^+ + b$. Depending on $\tan\beta$ (the ratio of the vacuum expectation values of the two Higgs doublets) the decay rate for $t \rightarrow H^+ + b$ can be comparable to that of the dominant decay mode $t \rightarrow W^+ + b$. The $O(\alpha_s)$ corrections to the decay rate $t \rightarrow H^+ + b$ have been calculated previously in [37–39] and in [40–43], and have been found to be important. This thesis provides the *first* closed form expression of the $O(\alpha_s)$ radiative corrections to the asymmetry parameter in polarized top decay $t(\uparrow) \rightarrow H^+ + b$.

The emphasis in this thesis is to obtain closed form expressions for the differential and integrated rates. The advantage of closed form expression is that one can easily discuss limiting cases such as setting masses to zero, or the limiting behavior at threshold or kinematic end points. Also it is simple to recalculate the numerical results for different sets of parameters in the process such as different mass and coupling values. To ensure the correctness of our closed form results we checked all the integrations numerically.

This thesis is structured as following.

In chapter 2 we present the $\mathcal{O}(\alpha_s)$ radiative corrections to the rest frame decay of a polarized top quark into a charged Higgs and a bottom quark in Two-Higgs-Doublet-Models: $t(\uparrow) \rightarrow b + H^+$. We calculate the LO unpolarized and polarized rates and their NLO QCD corrections which consist of the virtual one–loop contribution and the real gluon emission (tree graph) contribution. We use dimensional regularization ($D = 4 - 2\,\delta$ with $\delta \ll 1$) to regularize the ultraviolet divergences of the virtual one–loop corrections. We regularize the infrared divergences in the virtual one–loop corrections by introducing a finite (small) gluon mass $m_g \neq 0$ in the gluon propagator. In the real emission corrections the phase space boundary becomes deformed away from the IR singular point through the introduction of a (small) gluon mass. The logarithmic gluon mass dependence resulting from the regularization procedure cancels out when adding the virtual one–loop and real emission contributions. Our result for the $\mathcal{O}(\alpha_s)$ radiative corrections to the unpolarized rate agrees with calculations in the literature [37, 38]. The $\mathcal{O}(\alpha_s)$ radiative corrections to the polarized rate are new. We have checked consistency with the Goldstone boson equivalence theorem which, in the limit $m_{W^+}/m_t \rightarrow 0$ and $m_{H^+}/m_t \rightarrow 0$ relates unpolarized and polarized rates for $t \rightarrow H^+ + b$ to the unpolarized and polarized longitudinal rates in the decay $t \rightarrow W^+ + b$ calculated in [44]. The closed form expressions for the $\mathcal{O}(\alpha_s)$ radiative corrections for the unpolarized and polarized rates are compact especially in the $m_b = 0$ limit. They can be used to scan the predictions of the 2HDM for the $(m_H, \tan\beta)$ parameter space. We found that, for certain values of $\tan\beta$, the $\mathcal{O}(\alpha_s)$ radiative corrections to the unpolarized, polarized rates, and the asymmetry parameter can become quite large.

In chapter 3 and 4 we calculate polar and azimuthal angular correlations in the semileptonic rest frame decay of a polarized top quark $t(\uparrow) \rightarrow X_b + \ell^+ + \nu_\ell$. These angular correlations are usually analyzed in three different helicity coordinate systems with the event plane defined in the (x, z) plain and the z–axes along (i) the charged lepton ℓ^+, (ii) the W^+–boson and (iii) the neutrino ν_ℓ, which are denoted as system 1a, 2'a and 3a respectively. The top quark polarization vector is parameterized with polar and azimuthal angles in these frames. The decay rate then consists of a piece with no angular dependence (the unpolarized rate), a piece with polar angle dependence (polar correlation or polar rate) and a piece with azimuthal angle dependence (azimuthal correlation or azimuthal rate). In chapter 3 I present the angular correlations in system 1a and their $\mathcal{O}(\alpha_s)$ corrections. The "narrow–width" approximation is used for the W–propagator. We obtain closed form expressions for the unpolarized and polar differential rate (differential w.r.t. the charged lepton energy). In the next step we integrate over the charged lepton energy to obtain respective closed form expressions for the integrated rates. For the radiative corrections to the unpolarized and polar rates we have found agreement with the results in [24, 26–28]. The radiative corrections to the azimuthal correlation function have not been done before. The azimuthal correlation between the planes formed by the vectors $(\vec{p}_\ell, \vec{p}_{X_b})$ and $(\vec{p}_\ell, \vec{P}_t)$ belongs to a class of polarization obsevables which vanish at the Born term level in the SM. Vanishing of the LO azimuthal correlation is a consequence of the $(V-A)$ nature of the SM currents. The azimuthal correlations occur in at NLO. We present a *first* calculation of the $\mathcal{O}(\alpha_s)$ correction to this azimuthal correlation. A numerical result for the charged lepton spectrum of the azimuthal correlation is given. The integrated azimuthal correlation function is presented in closed form. We show that the $\mathcal{O}(\alpha_s)$ correction to the azimuthal correlation in system 1a is tiny compared to the LO unpolarized result.

In chapter 4 we calculate the $\mathcal{O}(\alpha_s)$ radiative corrections to the above mentioned angular correlations in system 3a. We first present closed form expressions for the unpolarized and polar differential rates (differential w.r.t. the neutrino energy). The closed form integrated rates are then obtained by performing the neutrino energy integration. In helicity system 3a the LO azimuthal correlation does not vanish. As in the case of system 1a, the neutrino spectrum for the azimuthal correlation is obtained numerically. The integrated azimuthal correlation is presented in closed form. For the unpolarized and polar rates we agree with existing results [28]. Again we provide a *first* calculation of the $\mathcal{O}(\alpha_s)$ corrections to the azimuthal correlation functions in system 3a. As in the case of system 1a the neutrino spectrum for the azimuthal correlation is obtained numerically. The integrated azimuthal correlation is presented in closed form.

It is important to realize that the angular correlations in the helicity systems 1a and 3a constitute independent measurements of the angular structure of polarized top quark decays.

In chapter 5 we turn to angular decay distributions in semileptonic hyperon decays. Using the helicity method we derive complete formulas for the LO joint angular decay distributions occurring in semileptonic hyperon decays including lepton mass and polarization effects. Compared to the traditional covariant calculation the helicity method allows one to organize the calculation of the angular decay distributions in a very compact and efficient way. In the helicity method the angular analysis is of cascade type, i.e. each decay in the decay chain is analyzed in the respective rest system of that particle. Such an approach is ideally suited as input for a Monte Carlo event generation program.

Although we used the helicity method the results are checked by doing full-fledged covariant calculations. As a specific example we take the decay $\Xi^0 \to \Sigma^+ + l^- + \bar{\nu}_l$ ($l^- = e^-, \mu^-$) followed by the nonleptonic decay $\Sigma^+ \to p + \pi^0$. These results are also applicable to the semileptonic and nonleptonic decays of the ground state charm and bottom baryons, and to the decays of the top quark.

Chapter 2

Polarization Effects in $t(\uparrow) \to b + H^+$

Charged Higgs particles appear in generic two–Higgs–doublet models (2HDM) in the Standard Model (SM) and also in the Minimal Supersymmetric Standard Model (MSSM) [29]. Provided $m_t > m_H + m_b$ the top quark can decay into the charged Higgs and the bottom quark: $t \to H^+ + b$.

First we define our notation. For the general case including the NLO decay process $t \to b + H^+ + g$ we define the four–momenta of top quark, bottom quark, charged Higgs and gluon by p_t, p_b, p_H and k. We also define the four–momentum of the quark–gluon combined system by $P = p_b + k$. This is of use in the calculation of the real emission of a gluon from a quark. The scaled masses are defined as

$$y = \frac{m_H}{m_t}, \qquad \epsilon = \frac{m_b}{m_t}, \qquad \Lambda = \frac{m_g}{m_t}, \qquad z = \frac{P^2}{m_t^2}. \tag{2.1}$$

The scaled kinematic variables for the real emission are defined as

$$\begin{aligned} p_0(z) &= \frac{1}{2}(1 - y^2 + z), & w_0(z) &= \frac{1}{2}(1 + y^2 - z), \\ p_3(z) &= \frac{1}{2}\sqrt{\lambda(1, y^2, z)}, & w_3(z) &= p_3, \\ p_\pm(z) &= p_0 \pm p_3, & w_\pm(z) &= w_0 \pm w_3, \\ Y_p(z) &= \frac{1}{2}\ln\frac{p_+}{p_-}, & Y_w(z) &= \frac{1}{2}\ln\frac{w_+}{w_-}. \end{aligned} \tag{2.2}$$

where the Källén function $\lambda(a, b, c)$ is defined by

$$\lambda(a, b, c) := a^2 + b^2 + c^2 - 2(a\,b + b\,c + c\,a)\,. \tag{2.3}$$

Note that p_0 and p_3 are the energy and the momentum modulus of the quark–gluon system $P = p_b + k$ scaled to the top mass. Similarly w_0 and w_3 are the scaled energy and the momentum modulus of the charged Higgs. We recover the LO kinematics if $k = 0$ i.e. $z = \epsilon^2$. The LO scaled kinetic variables are defined by

$$\begin{aligned} \bar{p}_0 &= p_0(\epsilon^2), & \bar{p}_3 &= p_3(\epsilon^2), & \bar{p}_\pm &= p_\pm(\epsilon^2), & \bar{Y}_p &= Y_p(\epsilon^2), \\ \bar{w}_0 &= w_0(\epsilon^2), & \bar{w}_3 &= w_3(\epsilon^2), & \bar{w}_\pm &= w_\pm(\epsilon^2), & \bar{Y}_w &= Y_w(\epsilon^2). \end{aligned} \tag{2.4}$$

The variables $\bar{p}_0$, $\bar{p}_3$, $\bar{w}_0$ and $\bar{w}_3$ are the scaled b–quark energy, b–quark momentum modulus, charged Higgs energy and charged Higgs momentum modulus in the LO case.

2.1 The Born approximation

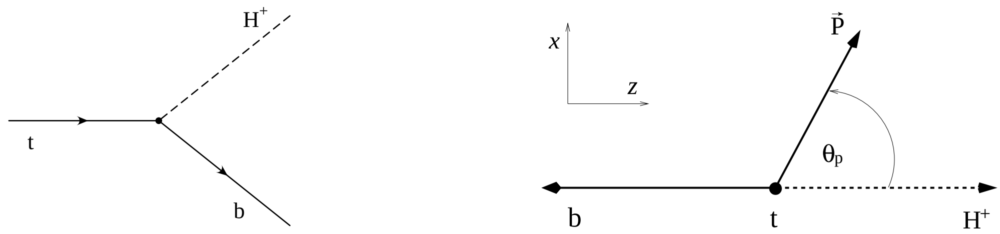


Figure 2.1: The Feynman diagram (left) and the reference frame (right) for $t(\uparrow) \to b + H^+$ in the Born approximation. $\vec{P}$ is the polarization vector of the top quark and θ_p is the polar angle between $\vec{P}$ and $\vec{p}_H$

The coupling of the charged Higgs boson to the top and bottom quark momentum in the MSSM can either be expressed as a superposition of scalar and pseudoscalar coupling factors or as a superposition of right– and left–chiral coupling factors. The Born term amplitude is thus given by

$$\mathcal{M}_0 = \bar{u}(p_b, s_b)(a\mathbb{1} + b\gamma_5)u(p_t, s_t) \tag{2.5}$$

$$= \bar{u}(p_b, s_b)\left\{g_t\frac{\mathbb{1}+\gamma_5}{2} + g_b\frac{\mathbb{1}-\gamma_5}{2}\right\}u(p_t s_t), \tag{2.6}$$

where $a = \frac{1}{2}(g_t + g_b)$ and $b = \frac{1}{2}(g_t - g_b)$. The inverse relation reads $g_t = a + b$ and $g_b = a - b$.

In order to avoid flavor changing neutral currents (FCNC) the generic Higgs coupling to all quarks has to be restricted. In the notation of [29] in model 1 the doublet H_1 couples to all bosons and the doublet H_2 couples to all quarks. This leads to the coupling factors

$$\boxed{\text{model 1:}} \qquad a = \frac{g_w}{2\sqrt{2}m_W}V_{tb}(m_t - m_b)\cot\beta,$$

$$b = \frac{g_w}{2\sqrt{2}m_W}V_{tb}(m_t + m_b)\cot\beta, \tag{2.7}$$

where V_{tb} is the CKM-matrix element and $\tan\beta = v_2/v_1$ is the ratio of the vacuum expectation values of the two electrically neutral components of the two Higgs doublets. The weak coupling factor g_w is related to the usual Fermi coupling constant G_F by

$$g_w^2 = 4\sqrt{2}m_W^2 G_F\,. \tag{2.8}$$

In model 2, the doublet H_1 couples to the right–chiral down–type quarks and the doublet H_2 couples to the right–chiral up–type quarks. Model 2 leads to the coupling factors

$$\boxed{\text{model 2:}} \qquad a = \frac{g_w}{2\sqrt{2}m_W}V_{tb}(m_t\cot\beta + m_b\tan\beta),$$

$$b = \frac{g_w}{2\sqrt{2}m_W}V_{tb}(m_t\cot\beta - m_b\tan\beta). \tag{2.9}$$

Writing the coupling constants g_t and g_b explicitly one has:

$$
\begin{aligned}
g_t = a + b &= \frac{g_w}{\sqrt{2} m_W} V_{tb} \tan\beta\, m_t \,, \\
g_b = a - b &= -\frac{g_w}{\sqrt{2} m_W} V_{tb} \tan\beta\, m_b \,,
\end{aligned} \tag{2.10}
$$

for model 1 and

$$
\begin{aligned}
g_t = a + b &= \frac{g_w}{\sqrt{2} m_W} V_{tb} \tan\beta\, m_t \,, \\
g_b = a - b &= \frac{g_w}{\sqrt{2} m_W} V_{tb} \cot\beta\, m_b \,,
\end{aligned} \tag{2.11}
$$

for model 2. The coupling constants g_t and g_b highlight an important fact about fermion–Higgs boson couplings: the strength of the coupling is proportional to the mass of the fermion.

The LO amplitude squared is then given by

$$
\begin{aligned}
\overline{|\mathcal{M}_0|^2} &= \sum_{s_b} |\mathcal{M}_0|^2 = \sum_{s_b} \mathcal{M}_0 \mathcal{M}_0^\dagger \\
&= \sum_{s_b} \bar{u}(p_b, s_b)\,(a + b\gamma_5)\,u(p_t, s_t)\Big[\bar{u}(p_b, s_b)\,(a + b\gamma_5)\,u(p_t, s_t)\Big]^\dagger \\
&= \sum_{s_b} \bar{u}(p_b, s_b)\,(a + b\gamma_5)\,u(p_t, s_t)\Big[u^\dagger(p_b, s_b)\gamma_0\,(a + b\gamma_5)\,u(p_t, s_t)\Big]^\dagger \\
&= \sum_{s_b} \bar{u}(p_b, s_b)\,(a + b\gamma_5)\,u(p_t, s_t) u(p_t, s_t)^\dagger \gamma_0 (a + b\gamma_0\gamma_5^\dagger\gamma_0) u(p_b, s_b) \\
&= \sum_{s_b} \bar{u}(p_b, s_b)\,(a + b\gamma_5)\,u(p_t, s_t) \bar{u}(p_t, s_t)(a - b\gamma_5) u(p_b, s_b)) \\
&= \mathrm{Tr}\Big[(\not{p}_b + m_b)(a + b\gamma_5)(\not{p}_t + m_t)(\frac{1 + \gamma_5 \not{s}_t}{2})(a - b\gamma_5)\Big] \\
&= 2(a^2 + b^2)(p_t \cdot p_b) + 2(a^2 - b^2) m_t m_b + 4ab\, m_t (p_b \cdot s_t).
\end{aligned} \tag{2.12}
$$

The scalar products that do not involve the polarization vector can be obtained using four–momentum conservation and choosing a convenient frame without having to specify the directions. For example $(p_t - p_b)^2 = p_H^2$ gives $(p_t \cdot p_b) = m_t E_b$ in the top rest frame. In the polarized case the direction of the frame should be defined. We parameterize the momenta and the polarization vector in the top–rest frame as (see Fig 2.1)

$$
\begin{aligned}
p_t &= (m_t; 0, 0, 0), \\
p_b &= (E_b; 0, 0, -|\vec{p}_b|), \\
p_H &= (E_H; 0, 0, |\vec{p}_b|), \\
s_t &= P(0; \sin\theta_P \cos\phi, \sin\theta_P \sin\phi, \cos\theta_P),
\end{aligned} \tag{2.13}
$$

where θ_P and ϕ are the polar and azimuthal angles of the polarization vector of the top quark. The parameter P is the degree of the polarization with $0 \leqslant P \leqslant 1$. $P = 0$ corresponds to an unpolarized top quark while $P = 1$ corresponds to 100% top quark polarization. The energy and the momentum modulus of the bottom quark and the Higgs particle are given by

$$\begin{aligned} E_b &= \frac{m_t^2 + m_b^2 - m_H^2}{2m_t} = \frac{1}{2} m_t (1 - y^2 + \epsilon^2) := m_t \, \bar{p}_0, & (2.14) \\ |\vec{p}_b| &= \sqrt{E_b^2 - m_b^2} = m_t \frac{\sqrt{\lambda(1, y^2, \epsilon^2)}}{2} := m_t \, \bar{p}_3 \,, & (2.15) \end{aligned}$$

where $\bar{p}_0$ and $\bar{p}_3$ are defined in (2.4). With these parameterizations we obtain

$$\begin{aligned} (p_t \cdot p_b) &= m_t E_b = \frac{1}{2} m_t^2 (1 - y^2 + \epsilon^2) = m_t^2 \, \bar{p}_0 & (2.16) \\ (p_b \cdot s_t) &= |\vec{p}_b| \cos\theta_P = m_t^2 \frac{\sqrt{\lambda(1, y^2, \epsilon^2)}}{2} \cos\theta_P = m_t^2 \, \bar{p}_3 \cos\theta_P \,. & (2.17) \end{aligned}$$

The decay rate of a particle with a mass m and momentum p into a given final state of particles $(p_1, p_2, \ldots, p_n)$ is

$$\begin{aligned} d\Gamma &= \frac{1}{2m} \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_i} (2\pi)^4 \delta^{(4)}(p - \sum_{i=1}^{n} p_i) |M(m \to \{p_i\})|^2 \\ &= \frac{1}{2m} \frac{1}{(2\pi)^{3n-4}} dR_n(p; p_1, p_2, \ldots, p_n) |M(m \to \{p_i\})|^2 \,, & (2.18) \end{aligned}$$

where the n–body phase space is denoted by

$$R_n \left(p; p_1, p_2, \ldots, p_n\right) = \int dR_n \left(p; p_1, p_2, \ldots, p_n\right) = \int \prod_{i=1}^{n} \frac{d^3 \vec{p}_i}{2E_i} \delta^4 \left(p - \sum_{i=1}^{n} p_i \right) . \quad (2.19)$$

In the present case the differential rate is given by

$$d\Gamma = \frac{1}{2m_t} \frac{1}{(2\pi)^2} dR_2(p_t; p_b, p_H) \overline{|\mathcal{M}_0|^2} \quad (2.20)$$

where the two–body phase space reads

$$dR_2(p_t; p_b, p_H) = \frac{d^3 \vec{p}_b}{2E_b} \frac{d^3 \vec{p}_H}{2E_H} \delta^4(p_t - p_b - p_H) \,. \quad (2.21)$$

We use the relations

$$\begin{aligned} \int \frac{d^3 \vec{p}}{2E} &= \int d^4 p \delta(p^2 - m^2) \Theta(E) \,, & (2.22) \\ d^3 \vec{p} &= d\Omega |\vec{p}|^2 d|\vec{p}| \,, & (2.23) \\ |\vec{p}| d|\vec{p}| &= E dE \,, & (2.24) \end{aligned}$$

to write the two–body phase space as

$$\begin{aligned}
dR_2(p_t;p_b,p_H) &= \frac{d^3\vec{p}_b}{2E_b} d^4p_H \delta(p_H^2-m_H^2)\Theta(E_H)\delta^4(p_t-p_b-p_H) \\
&= \frac{d^3\vec{p}_b}{2E_b}\delta\big((p_t-p_b)^2-m_H^2\big) \\
&= \frac{1}{2}|\vec{p}_b| d\Omega\, dE_b\, \delta\big(m_t^2+m_b^2-m_H^2-2(p_t\cdot p_b)\big) \\
&\downarrow \quad \text{in top quark rest frame with the polarization vector of the top quark } \vec{P} \text{ along the } z\text{–axis} \\
&= \frac{1}{2}|\vec{p}_b| d\Omega\, dE_b\, \delta\big(m_t^2+m_b^2-m_H^2-2m_tE_b\big)\ . \\
&= \frac{1}{4m_t}|\vec{p}_b| d\Omega\,, \qquad (2.25)
\end{aligned}$$

In the last step the δ–function is integrated out using

$$\delta\left(f(x)\right) = \sum_{i=1}^{n} \frac{1}{|f'(x_i)|}\delta(x-x_i)\,, \quad f(x_i)=0, \quad f'(x_i)\neq 0\,. \qquad (2.26)$$

If the process is symmetric in space as for the unpolarized decay case the angular integration will be a simple factor: $\int d\Omega = d\cos\theta d\phi = 4\pi$ where θ and ϕ are the polar and the azimuthal angles of the bottom quark in the top–rest frame. We then define a *two–body phase space factor* PS_2

$$\begin{aligned}
\mathrm{PS}_2 &:= \frac{1}{2m_t}\int \frac{d^3p_b}{(2\pi^3)}\frac{1}{2E_b}\frac{d^3p_H}{(2\pi^3)}\frac{1}{2E_H}(2\pi)^4\delta^{(4)}(p_t-p_b-p_H) \\
&= \frac{1}{2m_t}\frac{1}{(2\pi)^2}\, R_2(p_t;p_b,p_H) \\
&= \frac{\bar{p}_3}{8\pi m_t}\,. \qquad (2.27)
\end{aligned}$$

However, the polarized calculation is not spherically symmetric because of the polarization vector. The polar angle θ of the b–quark and the angle θ_P between the charged Higgs momentum and the polarization vector are related by $\theta = \pi + \theta_P$ (see Fig. 2.1), i.e. $d\cos\theta = d\cos\theta_P$ up to a minus sign which is of no relevance. Then the angular integral takes the form:

$$d\Omega = 2\pi\, d\cos\theta = 2\pi\, d\cos\theta_P\,, \qquad (2.28)$$

where the integration of the azimuthal angle gives 2π since $\overline{|\mathcal{M}_0|^2}$ does not have an azimuthal dependence. However $\overline{|\mathcal{M}_0|^2}$ does have a polar dependence, i.e. the polar angle cannot be simply integrated out via $\int d\cos\theta_P = 2$. Thus the two–body phase space is

$$dR_2(p_t;p_b,p_H) = \frac{\bar{p}_3}{4}\, 2\pi\, d\mathrm{cos}\,\theta_P\,. \qquad (2.29)$$

Substituting (2.29) equation into (2.20) and inserting the scalar products in $\overline{|\mathcal{M}_0|^2}$ from (2.16) and (2.17) we obtain

$$\frac{d\Gamma_{Born}}{d\cos\theta_P} = \frac{1}{2}\left(\Gamma^{unpol}_{Born} + P\,\Gamma^{pol}_{Born}\cos\theta_P\right), \tag{2.30}$$

where the unpolarized and polarized Born term rates are

$$\Gamma^{unpol}_{Born} = \frac{\bar{p}_3}{8\pi m_t} m_t^2 \Big\{ 2(a^2+b^2)\bar{p}_0 + 2(a^2-b^2)\epsilon \Big\}, \tag{2.31}$$

$$\Gamma^{pol}_{Born} = \frac{\bar{p}_3}{8\pi m_t} m_t^2 \Big\{ 4ab\bar{p}_3 \Big\}. \tag{2.32}$$

We define an asymmetry parameter α_H by

$$\alpha_H = \frac{\Gamma^{pol}}{\Gamma^{unpol}}\,. \tag{2.33}$$

Alternatively one can define a forward-backward asymmetry A_{FB} by writing

$$A_{FB} = \frac{\Gamma_F - \Gamma_B}{\Gamma_F + \Gamma_B}, \tag{2.34}$$

where Γ_F and Γ_B are the rates into the forward and backward hemispheres, respectively. The two measures are related by $A_{FB} = \frac{1}{2}\mathrm{P}\alpha_H$. In our numerical results we shall always set $\mathrm{P} = 1$ for simplicity.

Since $m_b \ll m_t$ it is interesting to take the $m_b = 0$ limit. As can be seen from (2.7) and (2.9) the b–quark mass can be safely neglected in model 1. But in model 2 the left–chiral coupling term is proportional to $m_b \tan\beta$, and become comparable to the right–chiral coupling term $m_t \cot\beta$ when $\tan\beta$ becomes large. One can therefore naively set $m_b = 0$ in all expressions. One has to distinguish between the cases when the scale of the b–quark mass is set by $m_t \cot\beta$ and when the scale of b–quark mass is set by m_t. In the former case one has to keep the b–quark mass finite. In the latter case the b–quark mass can safely be set to zero except for the logarithmic terms proportional to $\ln(m_b/m_t)$ that appear in NLO calculations. This way of taking $m_b = 0$ limit i.e. neglecting m_b when it is set by m_t but keeping it when it is set by $m_t \cot\beta$ is referred to as "kinematical" $m_b = 0$ approximation.

In the $m_b \to 0$ case the unpolarized and polarized Born term rates simplify to

$$\Gamma^{unpol}_{Born}(m_b \to 0) = (a^2+b^2)\left(1 + \frac{a^2-b^2}{a^2+b^2}\frac{2\epsilon}{1-y^2}\right)\hat{\Gamma}, \tag{2.35}$$

$$\Gamma^{pol}_{Born}(m_b \to 0) = 2ab\,\hat{\Gamma}, \tag{2.36}$$

where $\hat{\Gamma} = m_t(1-y^2)^2/(16\pi)$. The asymmetry parameter is then simply given by

$$\alpha_H(m_b \to 0) = \frac{\Gamma^{pol}_{Born}(m_b = 0)}{\Gamma^{unpol}_{Born}(m_b = 0)} = \frac{2ab}{a^2+b^2}\left(1 + \frac{a^2-b^2}{a^2+b^2}\frac{2\epsilon}{1-y^2}\right)^{-1}. \tag{2.37}$$

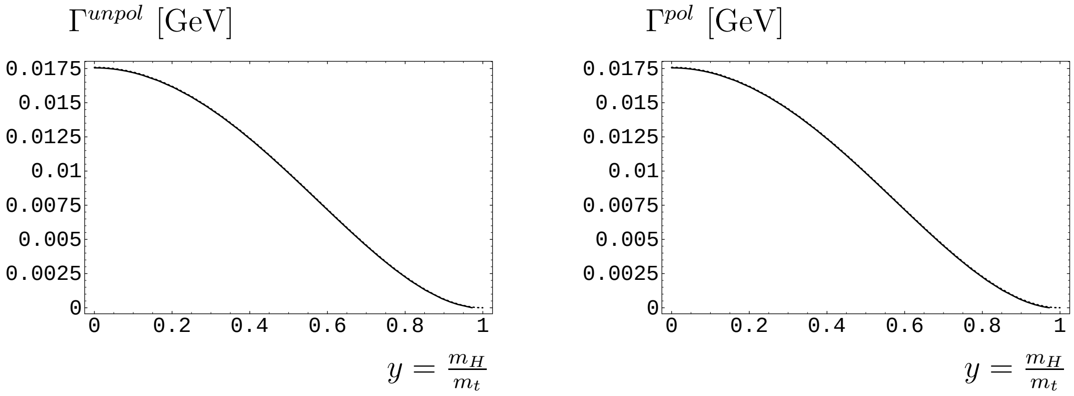


Figure 2.2: The LO unpolarized (left) and polarized (right) rates as functions of $y = m_H/m_t$ for model 1 with $m_b = 4.8$ GeV, $m_t = 175$ GeV and $\tan\beta = 10$. The $m_b \to 0$ curves can not be discerned at the scale of the figure.

As was mentioned before the ϵ–dependent terms in Eqs. (2.35) and (2.37) can be safely dropped in model 1 but not in model 2. To make this more explicit we rewrite the relevant second term in the round brackets of Eq. (2.35) and (2.37) for model 2. One has

$$\frac{a^2 - b^2}{a^2 + b^2}\frac{2\epsilon}{1 - y^2} = \frac{4}{1 - y^2}\left(\frac{m_b}{m_t \cot\beta}\right)^2 \frac{1}{1 + \epsilon^2 \tan^4\beta}\,. \tag{2.38}$$

It is quiet clear that the contribution of this term can not be neglected if m_b becomes comparable to $m_t \cot\beta$.

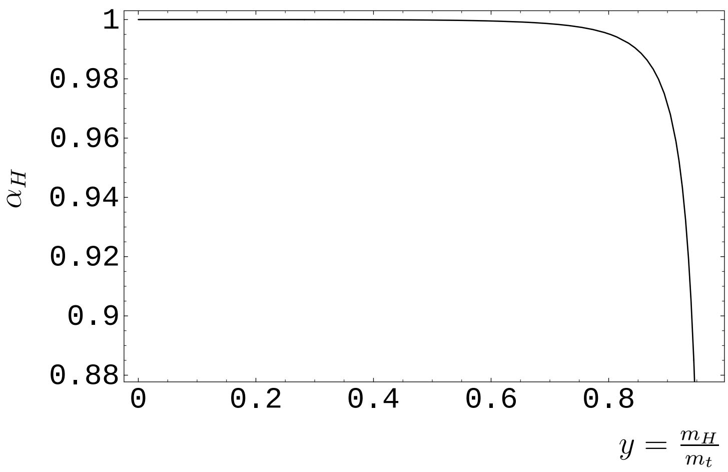


Figure 2.3: Asymmetry parameter α_H as function of $y = m_H/m_t$ for model 1 with $m_b = 4.8$ GeV, $m_t = 175$ GeV and $\tan\beta = 10$.

In model 1 the asymmetry parameter α_H is independent of $\tan\beta$ and the value of the Higgs mass for $m_b = 0$. It attains its maximal value $\alpha_H = 1$ for $m_b \to 0$ and, as Fig. 2.3 shows, stays very close to its maximal value for non–zero bottom masses except for the

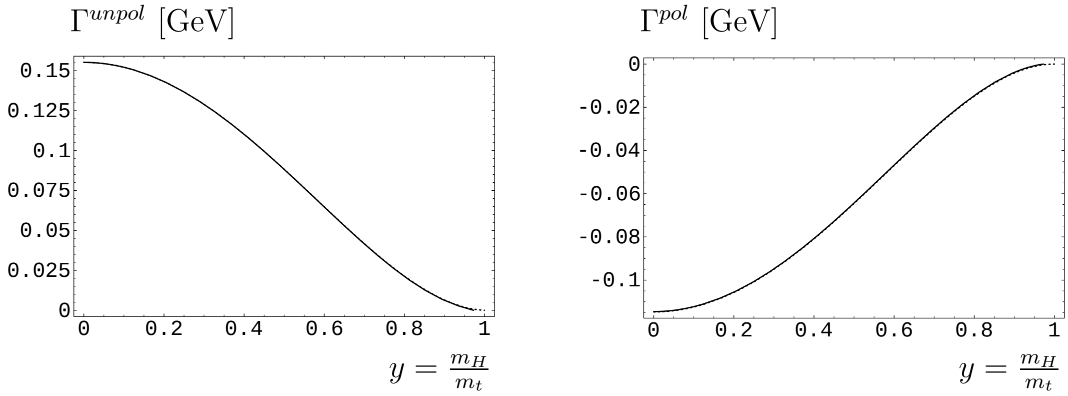


Figure 2.4: The LO unpolarized (left) and polarized (right) decay rates as functions of $y = m_H/m_t$ for model 2 with $m_b = 4.8$ GeV, $m_t = 175$ GeV and $\tan\beta = 10$. The $m_b \to 0$ curves can not be discerned at the scale of the figure.

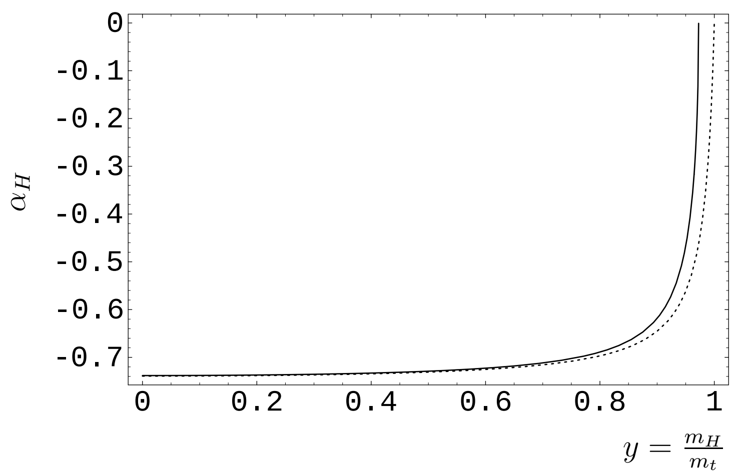


Figure 2.5: Asymmetry parameter α_H as function of $y = m_H/m_t$ for model 2 with $m_b = 4.8$ GeV, $m_t = 175$ GeV and $\tan\beta = 10$. The dotted line shows the corresponding $m_b \to 0$ curves.

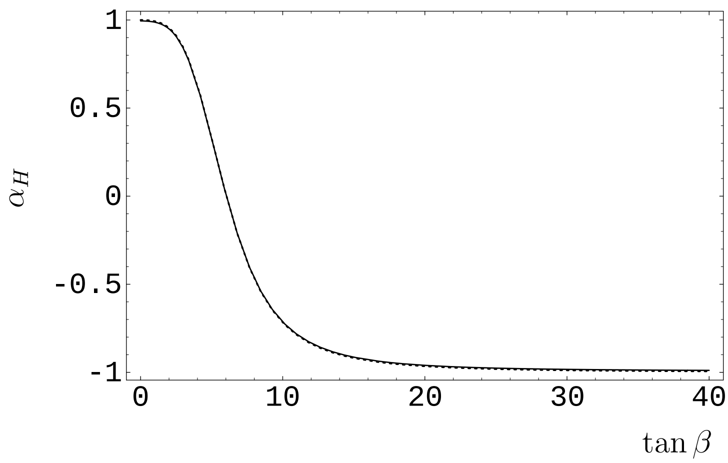


Figure 2.6: The asymmetry parameter as a function of $\tan\beta$ in model 2, with m_H=120 GeV. The barely visible dotted line shows the corresponding $m_b \to 0$ curves.

endpoint region when m_H approaches m_t. Contrary to this, in model 2 the asymmetry parameter α_H shows a strong dependence on the value of $\tan\beta$ as shown in Fig. 2.6. The main qualitative features of the behaviour of α_H in model 2 can again be extracted from the simple $m_b \to 0$ formula Eq. (2.37). The asymmetry parameter α_H is positive/negative for small/large values of $\tan\beta$ and goes through zero for $\tan\beta = \sqrt{m_t/m_b}$. More details of the numerical analysis will be given in section 2.4 where the Born term results are discussed together with their radiatively corrected counterparts.

2.2 Virtual one–loop corrections

The virtual one–loop corrections arises from a virtual gluon exchange between the top quark and bottom quark legs, and from emission and absorption of a virtual gluon from the same quark leg. The former case is called the vertex correction and the latter is referred to as the quark self–energy diagram. The Feynman graphs contributing to the virtual one–loop QCD corrections are shown in Fig. 2.7.

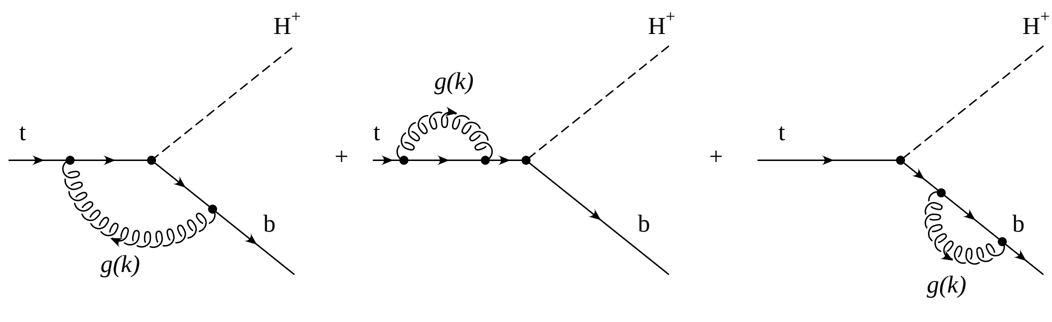


Figure 2.7: Feynman diagrams for the virtual one–loop corrections in $t \to H^+ + b$.

2.2.1 Vertex corrections

For the vertex correction shown in Fig. 2.7 (left) the Feynman rules give

$$\mathcal{M}_v = \bar{u}(p_b, s_b) \sum_{m,n=1}^{8} \int \frac{d^4k}{(2\pi)^4} \left(-ig_s\gamma_\mu \frac{\lambda^m}{2}\right) \frac{i}{(\not{p}_b - \not{k}) - m_b}(a + b\,\gamma_5)\frac{i}{(\not{p}_t - \not{k}) - m_t}$$
$$\times \left(-ig_s\gamma_\nu \frac{\lambda^n}{2}\right)\left(-i\delta_{mn}\frac{g^{\mu\nu}}{k^2}\right) u(p_t, s_t)$$
$$= -ig_s^2\,\bar{u}(p_b, s_b)\left(\frac{\lambda^n}{2}\,\frac{\lambda_n}{2}\right)\int \frac{d^4k}{(2\pi)^4}\frac{\gamma_\mu(\not{p}_b - \not{k} + m_b)(a + b\,\gamma_5)(\not{p}_t - \not{k} + m_t)\gamma^\mu}{\left[(p_t - k)^2 - m_t^2\right]\left[(p_b - k)^2 - m_b^2\right]k^2} u(p_t, s_t)\,, \tag{2.39}$$

where g_s is the strong coupling constant and λ^n $(n = 1, 2, \ldots, 8)$ denote the Gell–Mann matrices (see Appendix A.3). Writing the color indices explicitly for the relevant terms one has[1]

$$\frac{1}{3}\sum_{i,j,k=1}^{3}\sum_{n=1}^{8}\bar{u}_i(p_b, s_b)\frac{(\lambda^n)_{ij}}{2}\,\frac{(\lambda_n)_{jk}}{2}\,u_k(p_t, s_t) = C_F\,\frac{1}{3}\sum_{i=1}^{3}\bar{u}_i(p_b, s_b)u_i(p_t, s_t)\,, \tag{2.40}$$

where the color indices $i, j, k = 1, 2, 3$ denote the three color charges red, green and blue. The sum over the color indices are averaged by dividing it by three. The color factor $C_F = 4/3$ arises from summing the Gell–Mann matrices as given in (A.29):

$$\sum_{n=1}^{8}\sum_{j=1}^{3}\frac{(\lambda^n)_{ij}}{2}\,\frac{(\lambda_n)_{jk}}{2} = \frac{4}{3}\,\delta_i^k \equiv C_F\,I. \tag{2.41}$$

With (2.40) we write the matrix element for the virtual one–loop QCD correction in (2.39) as

$$\mathcal{M}_v = -ig_s^2\,C_F\,\bar{u}(p_b, s_b)\int \frac{d^4k}{(2\pi)^4}\frac{\gamma_\mu(\not{p}_b - \not{k} + m_b)(a + b\,\gamma_5)(\not{p}_t - \not{k} + m_t)\gamma^\mu}{\left[(p_t - k)^2 - m_t^2\right]\left[(p_b - k)^2 - m_b^2\right]k^2} u(p_t, s_t)\,. \tag{2.42}$$

The color indices as well as the 3×3 unit matrix I are implicit in this expression.

The integral (2.42) is both IR and UV divergent. We use *dimensional regularization* for the UV–divergences taking the space time dimension as $D = 4 - 2\delta$ where $0 < \delta \ll 1$ is the regularization parameter. A small gluon mass m_g is used to regulate the IR–divergences. Thus we have

$$\mathcal{M}_v = -ig_s^2 C_F\,\bar{u}(p_b, s_b)\int \frac{d^Dk}{(2\pi)^D}\frac{\gamma_\mu(\not{p}_b - \not{k} + m_b)(a + b\,\gamma_5)(\not{p}_t - \not{k} + m_t)\gamma^\mu}{\left[(p_t - k)^2 - m_t^2\right]\left[(p_b - k)^2 - m_b^2\right](k^2 - m_g^2)} u(p_t, s_t)\,. \tag{2.43}$$

[1] We write the summation symbol explicitly only for emphasis. Generally we adopt the rule that repeated indices are summed over.

The numerator of this integral can be simplified using the anti–commutation rules for the γ–matrices and the Dirac equation for the quark spinors, i.e.

$$\gamma_\mu \not{p} = 2p_\mu - \not{p}\gamma_\mu, \tag{2.44}$$

$$(\not{p} - m)u(p,s) = 0\,. \tag{2.45}$$

We take γ_5 to anticommute with γ^μ, i.e. $\gamma_5\gamma^\mu = -\gamma^\mu\gamma_5$, even though γ^μ is a D–dimensional object. This is known to be a viable γ_5–prescription, at least at one–loop order [45]. With (2.44) and (2.45) the numerator of (2.43) can be simplified to

$$\begin{aligned}
&\bar{u}(p_b,s_b)\gamma_\mu(\not{p}_b - \not{k} + m_b)(a + b\,\gamma_5)(\not{p}_t - \not{k} + m_t)\gamma^\mu u(p_t,s_t)\\
=&\bar{u}(p_b,s_b)(2p_{b\mu} - \not{p}_b - \gamma_\mu\not{k} + m_b\gamma_\mu)(a + b\,\gamma_5)(2p_t^\mu - \gamma^\mu\not{p}_t - \not{k}\gamma^\mu + \gamma^\mu m_t)u(p_t,s_t)\\
=&\bar{u}(p_b,s_b)(2p_{b\mu} - \not{k}\gamma_\mu)(a + b\,\gamma_5)(2p_t^\mu - \gamma^\mu\not{k})u(p_t,s_t)\\
=&\bar{u}(p_b,s_b)(a + b\,\gamma_5)(2p_{b\mu} - \gamma_\mu\not{k})(2p_t^\mu - \not{k}\gamma^\mu)u(p_t,s_t)\\
=&\bar{u}(p_b,s_b)(a + b\,\gamma_5)\big[4(p_t\cdot p_b) - 2\not{k}\not{p}_b - 2\not{p}_t\not{k} + D\,k^2\big]u(p_t,s_t)\,.
\end{aligned} \tag{2.46}$$

With these simplifications the vertex correction takes the form

$$\mathcal{M}_v = -ig_s^2 C_F\,\bar{u}(p_b,s_b)(a + b\,\gamma_5)\int\frac{d^Dk}{(2\pi)^D}\frac{4(p_t\cdot p_b) - 2\not{k}\not{p}_b - 2\not{p}_t\not{k} + D\,k^2}{\big[(p_t-k)^2-m_t^2\big]\big[(p_b-k)^2-m_b^2\big](k^2-m_g^2)}u(p_t,s_t)\,. \tag{2.47}$$

The denominator is the product of the denominators of three propagators: t–quark, b–quark and gluon propagator. The numerator contains tensors of the integration variable k, namely 1, k_μ and k^2. The term k^2 in the numerator can be further simplified:

$$\begin{aligned}
&\frac{k^2}{\big[(p_t-k)^2-m_t^2\big]\big[(p_b-k)^2-m_b^2\big](k^2-m_g^2)}\\
=&\frac{k^2 - m_g^2 + m_g^2}{\big[(p_t-k)^2-m_t^2\big]\big[(p_b-k)^2-m_b^2\big](k^2-m_g^2)}\\
=&\frac{1}{\big[(p_t-k)^2-m_t^2\big]\big[(p_b-k)^2-m_b^2\big]} + \frac{m_g^2}{\big[(p_t-k)^2-m_t^2\big]\big[(p_b-k)^2-m_b^2\big](k^2-m_g^2)}.
\end{aligned}$$

Thus the loop integrals that we need are the three–point one–loop scalar and vector integrals C_0 and C^μ,

$$C_0(q,m_t,m_t,m_b,m_g) := \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{1}{\big[(p_t-k)^2-m_t^2\big]\big[(p_b-k)^2-m_b^2\big](k^2-m_g^2)}, \tag{2.48}$$

$$= \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{1}{(k^2-m_t^2)\big[(q-k)^2-m_b^2\big]\big[(p_t-k)^2-m_g^2\big]}, \tag{2.49}$$

$$C^\mu(q,m_t,m_t,m_b,m_g) := \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{k^\mu}{\big[(p_t-k)^2-m_t^2\big]\big[(p_b-k)^2-m_b^2\big](k^2-m_g^2)} \tag{2.50}$$

$$= \frac{\mu^{4-D}}{i\pi^2} \int \frac{d^D k}{(2\pi)^{D-4}} \frac{k^\mu}{(k^2 - m_t^2)\left[(q-k)^2 - m_b^2\right]\left[(p_t - k)^2 - m_g^2\right]}, \quad (2.51)$$

and the two–point one–loop scalar and vector integrals B_0 and B_0^μ,

$$B_0(q, m_t, m_b) := \frac{\mu^{4-D}}{i\pi^2} \int \frac{d^D k}{(2\pi)^{D-4}} \frac{1}{\left[(p_t - k)^2 - m_t^2\right]\left[(p_b - k)^2 - m_b^2\right]}, \quad (2.52)$$

$$= \frac{\mu^{4-D}}{i\pi^2} \int \frac{d^D k}{(2\pi)^{D-4}} \frac{1}{[(q-k)^2 - m_t^2](k^2 - m_b^2)}, \quad (2.53)$$

$$B^\mu(q, m_t, m_b) := \frac{\mu^{4-D}}{i\pi^2} \int \frac{d^D k}{(2\pi)^{D-4}} \frac{1}{\left[(p_t - k)^2 - m_t^2\right]\left[(p_b - k)^2 - m_b^2\right]}, \quad (2.54)$$

$$= \frac{\mu^{4-D}}{i\pi^2} \int \frac{d^D k}{(2\pi)^{D-4}} \frac{k^\mu}{[(q-k)^2 - m_t^2](k^2 - m_b^2)}, \quad (2.55)$$

where[2] $q = p_t - p_b$. The loop integrals $B_0(q, m_t, m_b)$, $B^\mu(q, m_t, m_b)$, $C_0(q, m_t, m_t, m_b, m_g)$ and $C^\mu(q, m_t, m_t, m_b, m_g)$ occur in the calculations frequently. Therefore we drop the arguments and denote them simply by B_0, B^μ, C_0 and C^μ respectively.

In the later calculations we will also have one–point one–loop scalar integral A_0:

$$A_0(m) := \frac{\mu^{4-D}}{i\pi^2} \int \frac{d^D k}{(2\pi)^{D-4}} \frac{1}{k^2 - m^2}. \quad (2.56)$$

The specific scalar loop integrals that we need are calculated in Appendix B.2. General methods for the calculation of the loop integrals can be found in [46–50].

The tensor loop integrals can be reduced to scalar integrals with the help of the Passarino–Veltman method. The integral C^μ, a Lorentz vector (rank–one tensor), is a function of the unintegrated momenta in the integrand of (2.50), i.e. p_t and p_b. It can thus be expanded as

$$C^\mu = C_1\, p_t^\mu + C_2\, p_b^\mu\,, \quad (2.57)$$

where C_1 and C_2 are scalar coefficients. Define scalar functions R_i $(i = 1, 2)$:

$$R_1 := p_t^\mu C_\mu\,, \qquad R_2 := p_b^\mu C_\mu\,. \quad (2.58)$$

Contracting (2.57) with $p_{t\mu}$ and $p_{b\mu}$ one obtains

$$\begin{aligned} R_1 = C^\mu p_{t,\mu} &= C_1\, m_t^2 + C_2\, (p_t \cdot p_b)\,, \\ R_2 = C^\mu p_{b,\mu} &= C_1\, (p_t \cdot p_b) + C_2\, m_b^2\,, \end{aligned} \quad (2.59)$$

or in matrix form,

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} m_t^2 & (p_t \cdot p_b) \\ (p_t \cdot p_b) & m_b^2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \quad (2.60)$$

The inverse of this matrix gives the C_i in terms of the R_i. One has

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{4}{\lambda} \begin{pmatrix} -m_b^2 & (p_t \cdot p_b) \\ (p_t \cdot p_b) & -m_t^2 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}, \quad (2.61)$$

[2]Although $p_t - p_b = p_H$ in the present case we define a new variable $q = p_t - p_b$ to keep generality.

or writing explicitly

$$C_1 = \frac{4}{\lambda}\left[(p_t\cdot p_b)\,R_2 - m_b^2\,R_1\right]\,, \qquad C_2 = \frac{4}{\lambda}\left[(p_t\cdot p_b)\,R_1 - m_t^2\,R_2\right]\,, \tag{2.62}$$

where $\lambda := \lambda(m_t^2, m_b^2, q^2) = \frac{1}{4}[(p_t\cdot p_b)^2 - m_t^2\,m_b^2]$. As before $q = p_t - p_b$. As for the integrand of C^μ, the numerators of the contractions $R_1 = C^\mu p_{t,\mu}$ and $R_2 = C^\mu p_{b,\mu}$ contain the products $k\cdot p_t$ and $k\cdot p_b$. To cancel the terms in the denominator we write these products in the following form:

$$\begin{aligned} k\cdot p_t &= -\frac{1}{2}\left\{\left[(p_t-k)^2 - m_t^2\right] - (k^2 - m_g^2) - m_g^2\right\}, \\ k\cdot p_b &= -\frac{1}{2}\left\{\left[(p_b-k)^2 - m_b^2\right] - (k^2 - m_g^2) - m_g^2\right\}. \end{aligned} \tag{2.63}$$

Substituting $k\cdot p_t$ into R_1 gives

$$\begin{aligned} R_1 = C^\mu\,p_{t,\mu} &= -\frac{1}{2}\frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{[(p_t-k)^2 - m_t^2] - (k^2-m_g^2) - m_g^2}{\left[(p_t-k)^2 - m_t^2\right]\left[(p_b-k)^2 - m_b^2\right]\left(k^2 - m_g^2\right)} \\ &= -\frac{1}{2}\frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{1}{\left[(p_b-k)^2 - m_b^2\right]\left(k^2 - m_g^2\right)} \\ &\quad +\frac{1}{2}\frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{1}{\left[(p_t-k)^2 - m_t^2\right]\left[(p_b-k)^2 - m_b^2\right]} \\ &\quad +\frac{1}{2}m_g^2\frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{1}{\left[(p_t-k)^2 - m_t^2\right]\left[(p_b-k)^2 - m_b^2\right]\left(k^2 - m_g^2\right)} \\ &= -\frac{1}{2}\Big\{B_0(p_b, m_b, m_g) - B_0(q, m_t, m_b) - m_g^2 C_0\Big\}\,. \end{aligned} \tag{2.64}$$

Since $B_0(p_b, m_b, m_g)$ is IR–convergent we can set $m_g = 0$. One obtains

$$R_1 = C^\mu\,p_{t,\mu} = \frac{1}{2}\Big\{B_0(q, m_t, m_b) - B_0(p_b, m_b, 0)\Big\}\,. \tag{2.65}$$

Similarly

$$R_2 = C^\mu\,p_{b,\mu} = \frac{1}{2}\Big\{B_0(q, m_t, m_b) - B_0(p_t, m_t, 0)\Big\}\,. \tag{2.66}$$

The scalar loop integrals $B_0(q, m_t, m_b)$ and $B_0(p_t, m_t, 0)$ are given in (B.23) and (B.24). Finally substituting (2.65) and (2.66) into (2.62) we obtain

$$\begin{aligned} C_1 &= \frac{1}{q^2}\left[\ln\left(\frac{m_b}{m_t}\right) + \frac{m_t^2 - m_b^2 - q^2}{2\sqrt{\lambda}}\ln\left(\frac{m_t^2 + m_b^2 - q^2 + \sqrt{\lambda}}{m_t^2 + m_b^2 - q^2 - \sqrt{\lambda}}\right)\right], \\ C_2 &= -\frac{1}{q^2}\left[\ln\left(\frac{m_b}{m_t}\right) + \frac{m_t^2 - m_b^2 + q^2}{2\sqrt{\lambda}}\ln\left(\frac{m_t^2 + m_b^2 - q^2 + \sqrt{\lambda}}{m_t^2 + m_b^2 - q^2 - \sqrt{\lambda}}\right)\right]. \end{aligned} \tag{2.67}$$

In terms of the scaled variables the coefficients become

$$C_1 = \frac{1}{m_t^2}\frac{1}{y^2}\left[\ln\epsilon + \frac{1}{2\,\bar{p}_3}\left(1 - y^2 - \epsilon^2\right)\bar{Y}_p\right], \tag{2.68}$$

$$C_2 = -\frac{1}{m_t^2}\frac{1}{y^2}\left[\ln\epsilon + \frac{1}{2\,\bar{p}_3}\left(1 + y^2 - \epsilon^2\right)\bar{Y}_p\right]. \tag{2.69}$$

The integral in the amplitude for the vertex correction (2.47) is written in terms of the loop integrals as

$$\begin{aligned}
&\frac{16\pi^2}{i}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{4(p_t\cdot p_b) - 2\not{k}\not{p}_b - 2\not{p}_t\not{k} + D\,k^2}{[(\not{p}_b - \not{k}) - m_b][(\not{p}_t - \not{k}) - m_t][k^2 - m_g^2]} \\
&= 4(p_t\cdot p_b)C_0 - 2(\not{p}_tC_1 + \not{p}_bC_2)\not{p}_b - 2\not{p}_t(\not{p}_tC_1 + \not{p}_bC_2) + D\,B_0 + D\,m_g^2C_0 \\
&= \left[4(p_t\cdot p_b) + Dm_g^2\right]C_0 - 2m_t^2C_1 - 2m_b^2C_2 + DB_0 - 2(C_1 + C_2)\not{p}_t\not{p}_b\,.
\end{aligned}$$

Thus the amplitude (2.47) becomes

$$\begin{aligned}
\mathcal{M}_v =& -ig_s^2C_F\,\bar{u}(p_b,s_b)(a + b\,\gamma_5)\frac{i}{16\pi^2} \\
&\Big\{\left[4(p_t\cdot p_b) + Dm_g^2\right]C_0 - 2m_t^2C_1 - 2m_b^2C_2 + DB_0 - 2(C_1 + C_2)\not{p}_t\not{p}_b\Big\}u(p_t,s_t)\,.
\end{aligned} \tag{2.70}$$

The term $D\,m_g^2$ can be dropped because we anticipate that the logarithmic singularity in C_0 is in the form of $\ln\frac{m_g}{m_t}$ and thus $D\,m_g^2$ vanishes in the $m_g \to 0$ limit. Also g_s^2 is replaced using $g_s^2 = \alpha_s\,4\pi$. Then one has

$$\begin{aligned}
\mathcal{M}_v =& \frac{\alpha_s}{4\pi}C_F\,\bar{u}(p_b,s_b)(a + b\,\gamma_5)\Big\{4(p_t\cdot p_b)C_0 - 2m_t^2C_1 - 2m_b^2C_2 + DB_0\Big\}u(p_b,s_b) \\
&- \frac{\alpha_s}{4\pi}C_F\,\bar{u}(p_b,s_b)\Big\{(a + b\,\gamma_5)2(p_t\cdot p_b) + (a - b\,\gamma_5)m_tm_b\Big\}u(p_b,s_b) \\
=& \frac{\alpha_s}{2\pi}C_F\,\bar{u}(p_b,s_b)\Big\{(a + b\,\gamma_5)\big[2(p_t\cdot p_b)C_0 - 2C_1\left[m_t^2 + m_tm_b + 2(p_t\cdot p_b)\right] \\
&- 2C_2\left[m_b^2 + m_tm_b + 2(p_t\cdot p_b)\right] + \frac{D}{2}B_0\big] - 2\,a\,(C_1 + C_2)m_tm_b\Big\}u(p_t,s_t) \\
=& \;\bar{u}(p_b,s_b)\Big\{(a + b\,\gamma_5)\mathbf{\Lambda_1} + a\mathbf{\Lambda_2}\Big\}u(p_t,s_t)\,,
\end{aligned} \tag{2.71}$$

where Λ_1 and Λ_2 are given by

$$\begin{aligned}
\mathbf{\Lambda_1} = \frac{\alpha_s}{2\pi}C_F\Big\{&2(p_t\cdot p_b)C_0 - 2C_1\left[m_t^2 + m_tm_b + 2(p_t\cdot p_b)\right] \\
&-2C_2\left[m_b^2 + m_tm_b + 2(p_t\cdot p_b)\right] + 2B_0 - 1\Big\}\,,
\end{aligned} \tag{2.72}$$

$$\mathbf{\Lambda_2} = \frac{\alpha_s}{2\pi}C_F\Big\{-2\,(C_1 + C_2)m_tm_b\Big\}\,. \tag{2.73}$$

Note that $\frac{D}{2}\,B_0 = (2 - \delta)B_0 = 2B_0 - 1$ where we used the identity

$$\lim_{\delta\to 0} B_0\,\delta = 1\,. \tag{2.74}$$

This identity is the result of the UV–divergent property of B_0 which is given in (B.23).

2.2.2 Quark self–energy

The self–energy for the top quark comes from the Fig. 2.7 (center). The Feynman rules give

$$\begin{aligned}\mathcal{M}_{\rm t,self} =& \bar{u}(p_b, s_b)(a + b\,\gamma_5)\frac{i}{\not{p}_t - m_t} \\ & \sum_{m,n}\int\frac{d^4k}{(2\pi)^4}\left(-ig_s\gamma_\mu\frac{\lambda^m}{2}\right)\frac{i}{(\not{p}_t - \not{k}) - m_t}\left(-ig_s\gamma_\nu\frac{\lambda^n}{2}\right)\left(-i\delta_{mn}\frac{g^{\mu\nu}}{k^2}\right)u(p_t, s_t) \\ =& \bar{u}(p_b, s_b)(a + b\,\gamma_5)\frac{i}{\not{p}_t - m_t}\left(-ig_s^2 C_F\int\frac{d^4k}{(2\pi)^4}\frac{\gamma_\mu(\not{p}_t - \not{k} + m_t)\gamma^\mu}{[(p_t - k)^2 - m_t^2]k^2}\right)u(p_t, s_t) \\ =& \bar{u}(p_b, s_b)(a + b\,\gamma_5)\frac{i}{\not{p}_t - m_t}\mathbf{\Sigma_t}\,u(p_t, s_t)\,, \end{aligned} \tag{2.75}$$

with

$$\mathbf{\Sigma_t} := -ig_s^2 C_F\int\frac{d^4k}{(2\pi)^4}\frac{\gamma_\mu(\not{p}_t - \not{k} + m_t)\gamma^\mu}{[(p_t - k)^2 - m_t^2]k^2}\,. \tag{2.76}$$

The self–energy for the bottom quark comes from the Fig. 2.7 (right). For the b-quark self–energy we have

$$\mathcal{M}_{\rm b,self} = \bar{u}(p_b, s_b)\mathbf{\Sigma_b}\frac{i}{\not{p}_b - m_b}(a + b\,\gamma_5)\,u(p_t, s_t)\,, \tag{2.77}$$

with

$$\mathbf{\Sigma_b} := -ig_s^2 C_F\int\frac{d^4k}{(2\pi)^4}\frac{\gamma_\mu(\not{p}_b - \not{k} + m_b)\gamma^\mu}{[(p_b - k)^2 - m_b^2]k^2}\,. \tag{2.78}$$

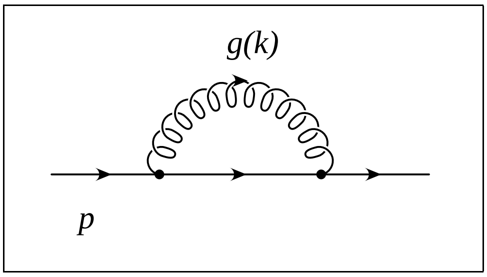


Figure 2.8: The quark self–energy.

Let us turn to the quark self-energy (see Fig. 2.8) with general mass m and momentum p, which we denote by Σ. The self–energy diagram has UV–divergences. For example, in the UV case, i.e. for the large integration momentum ($|\vec{k}| \gg m_t, |\vec{p}_t|$), the terms with no gluon momentum k in the numerator of the self–energy (2.78) behave as

$$\int d^4k\frac{1}{(k^2)^2}$$

We can read off from (B.10) that

$$\int d^4k \frac{1}{(k^2)^2} = i\pi^2 \int dk^2 \frac{1}{k^2}\,,$$

and therefore the integral is logarithmically divergent. Dimensional regularization is used for the UV–divergences. Although the self–energy itself is IR–convergent the differentiation w.r.t. p^2 is IR–divergent. This differentiation appears in the on–shell renormalization of the virtual loop corrections. We regularize the IR–divergences by a gluon mass m_g. We therefore generically write

$$\mathbf{\Sigma}(\not{p}, \mathbf{m}) = -ig_s^2 C_F \mu^{4-D} \int \frac{d^D k}{(2\pi)^{D-4}} \frac{\gamma_\mu(\not{p} - \not{k} + m)\gamma^\mu}{[(p-k)^2 - m^2](k^2 - m_g^2)}\,, \tag{2.79}$$

where in this application p could be p_t or p_b. Changing the dimension of the integral will also change its mass dimension. This effect is compensated by a factor μ^{4-D}, where μ has the dimension of a mass. Using the Dirac algebra in D–dimensions and the contractions

$$\gamma_\mu\gamma^\mu = D, \quad \gamma_\mu\not{p}\gamma^\mu = (2-D)\not{p}\,, \tag{2.80}$$

we have

$$\Sigma(\not{p}, m) = -ig_s^2 C_F \mu^{4-D} \int \frac{d^D k}{(2\pi)^{D-4}} \frac{(2-D)(\not{p} - \not{k}) + D\,m}{[(p-k)^2 - m^2](k^2 - m_g^2)}\,. \tag{2.81}$$

In terms of the integrals (2.52) and (2.54) the quark self–energy $\Sigma(\not{p}, m)$ takes the form

$$\Sigma(\not{p}, m) = \frac{g_s^2}{16\pi^2} C_F \Big\{ \big[(2-D)\not{p} + D\,m\big]\, B_0(p, m, m_g) - (2-D)\gamma_\mu B^\mu(p, m, m_g) \Big\}. \tag{2.82}$$

As before we observe that B^μ depends on the unintegrated momentum p^μ. We can therefore make the ansatz

$$B^\mu(p, m, m_g) = B_1 p^\mu\,, \tag{2.83}$$

where B_1 is a scalar coefficient. Contracting the above equation with p_μ gives

$$B_1 = \frac{1}{p^2} B^\mu p_\mu = \frac{\mu^{4-D}}{i\pi^2} \frac{1}{p^2} \int \frac{d^D k}{(2\pi)^{D-4}} \frac{(p\cdot k)}{[(p-k)^2 - m^2](k^2 - m_g^2)}\,, \tag{2.84}$$

where $p\cdot k = \frac{1}{2}(p^2 + k^2 - (p-k)^2)$. To cancel the terms in the denominator we add and subtract m_g^2 and m^2 and write

$$p\cdot k = \frac{1}{2}\big[(p^2 - m^2 + m_g^2) + (k^2 - m_g^2) - ((p-k)^2 - m^2)\big].$$

Substituting this back into B_μ we have

$$B^\mu(p, m, m_g) = \frac{p^\mu}{2p^2}\Big[(p^2 - m^2 + m_g^2) B_0(p, m, m_g) + A_0(m) - A_0(m_g)\Big]\,, \tag{2.85}$$

where the scalar one–point one–loop integral is defined as

Substituting the vector loop integral B_μ given in (2.85) into (2.82) we obtain

$$\Sigma(\not{p}, m) = \frac{g_s^2}{16\pi^2} C_F \Big\{ \not{p} \frac{(2-D)}{2p^2} \Big[(p^2 + m^2 - m_g^2) B_0(p, m, m_g) + A_0(m_g) - A_0(m) \Big] + m D B_0(p, m, m_g) \Big\}$$
$$:= \not{p}\, \Sigma^V(p^2, m^2) + m\, \Sigma^S(p^2, m^2), \quad (2.86)$$

where V and S denote the vector and scalar part of the self–energy. The scalar loop integrals A_0 and B_0 are calculated in Appendix B. In the limit of vanishing gluon mass one has $A_0(m_g = 0) = 0$ (see (B.14)). Thus we have

$$\Sigma^V(p^2, m^2) = \frac{g_s^2}{16\pi^2} C_F \frac{(2-D)}{2p^2} \Big[(p^2 + m^2) B_0(p, m, 0) - A_0(m) \Big], \quad (2.87)$$

$$\Sigma^S(p^2, m^2) = \frac{g_s^2}{16\pi^2} C_F\, D B_0(p, m, 0). \quad (2.88)$$

Results for $A_0(m)$ and $B_0(p, m, 0)$ are given in Appendix (B.14) and (B.24), respectively. Putting everything together the quark self–energy simplifies to

$$\Sigma^V(p^2, m^2) = -\frac{\alpha_s}{4\pi} C_F \left\{ \Delta + \ln\left(\frac{\mu^2}{m^2}\right) + \frac{m^2 + p^2}{p^2} \left[1 + \frac{m^2 - p^2}{p^2} \ln\left(\frac{m^2 - p^2}{p^2}\right) \right] \right\}, \quad (2.89)$$

$$\Sigma^S(p^2, m^2) = \frac{\alpha_s}{4\pi} 4C_F \left[\Delta + \frac{3}{2} + \ln\left(\frac{\mu^2}{m^2}\right) + \frac{m^2 - p^2}{p^2} \ln\left(\frac{m^2 - p^2}{p^2}\right) \right], \quad (2.90)$$

with

$$\Delta = \frac{1}{\delta} - \gamma_E + \ln(4\pi),$$
$$\alpha_s = \frac{g_s^2}{4\pi}.$$

Now that the regularization has been done, we turn to the renormalization.

2.2.3 Renormalization

As we have seen in the last section the loop integrals involve UV–divergences. These UV–divergences can be eliminated by redefining the fields and parameters. The redefinition of the fields and the basic parameters in the theory is called *renormalization*. The formal development of a general theory of renormalization can be found , for example, in [53, 54]. Here we calculate the renormalization constants for the quark masses and the quark fields. These are all the renormalization constants needed for the one–loop renormalization of the quark self–energy and the vertex graph.

Generally the renormalization process proceeds as follows:

- Renormalization constants are defined by introducing the renormalization transformations. These will give the relation between the unrenormalized and renormalized quantities.
- Renormalization conditions on the renormalized quantities are imposed. These conditions are given in different renormalization schemes.
- By renormalization transformations the renormalization conditions on the renormalized quantities become conditions on renormalization constants. Thus the renormalization constants are fixed and the divergences are absorbed into them.

We begin with the renormalization transformation for the masses and quark fields:

$$\psi = \sqrt{Z_2}\psi_R, \quad m = Z_m m_R. \tag{2.91}$$

where ψ and m are the unrenormalized or *bare* field and mass and ψ_R and m_R are the renormalized field and mass. In this subsection we determine the quark field and mass renormalization constants Z_2 and Z_m. In perturbation theory we write

$$Z_2 = 1 + \delta Z_2, \quad Z_m = 1 + \delta Z_m. \tag{2.92}$$

The full quark propagator is given by the geometric series shown in Fig. 2.9 where Σ_{1PI} denotes the 1–particle irreducible contributions.

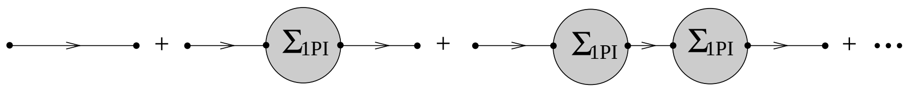


Figure 2.9: Contribution of 1-particle irreducible diagram to a fermion propagator.

$$\begin{aligned}
\Delta\left(\not p, m\right) &= \frac{i}{\not p - m} + \frac{i}{\not p - m}\left[-i\Sigma_{1PI}\left(\not p, m\right)\right]\frac{i}{\not p - m} \\
&\quad + \frac{i}{\not p - m}\left[-i\Sigma_{1PI}\left(\not p, m\right)\right]\frac{i}{\not p - m}\left[-i\Sigma_{1PI}\left(\not p, m\right)\right]\frac{i}{\not p - m} + \cdots \\
&= \frac{i}{\not p - m}\left\{1 + \frac{\Sigma_{1PI}\left(\not p, m\right)}{\not p - m} + \left[\frac{\Sigma_{1PI}\left(\not p, m\right)}{\not p - m}\right]^2 + \cdots\right\} \\
&= \frac{i}{\not p - m}\left[1 - \frac{\Sigma_{1PI}\left(\not p, m\right)}{\not p - m}\right]^{-1} = \frac{i}{\not p - m - \Sigma_{1PI}\left(\not p, m\right)}.
\end{aligned} \tag{2.93}$$

The renormalized propagator can now be written as

$$\Delta_R\left(\not p, m_R\right) = \frac{i}{\not p - m_R - \Sigma_{1PI,R}\left(\not p, m_R\right)}. \tag{2.94}$$

The renormalization transformation for the quark field will relate the unrenormalized propagator $\Delta(\not{p}, m)$ and renormalized propagator $\Delta_R(\not{p}, m_R)$ by

$$\Delta(\not{p}, m) = Z_2 \Delta_R(\not{p}, m_R). \tag{2.95}$$

Inserting $\Delta(\not{p}, m)$ and $\Delta_R(\not{p}, m_R)$ from (2.93) and (2.94) into (2.95) leads to

$$\Sigma_{1PI,R}(\not{p}, m_R) = Z_2 \Sigma_{1PI}(\not{p}, m) - (\not{p} - m_R)(Z_2 - 1) + Z_2(Z_m - 1)m_R. \tag{2.96}$$

The renormalization condition imposed on $\Delta_R(\not{p}, m_R)$ fixes the renormalization constants through (2.96). Now let us turn to the renormalization conditions. In the "on–shell scheme" the renormalization condition is imposed as

$$\Delta_R(\not{p}, m_R) \stackrel{!}{=} \frac{i}{\not{p} - m_R}. \tag{2.97}$$

The LO fermion propagator $\dfrac{i}{\not{p} - m}$ has a pole at the fermion mass m and its residue is 1. The renormalization condition renders the renormalized propagator to have the same properties: the renormalized propagator has its pole at the renormalized mass m_R and it has the residue 1. To demonstrate this we expand $\Sigma_{1PI,R}(\not{p}, m)$ around the renormalized mass m_R

$$\Sigma_{1PI,R}(\not{p}, m) = \Sigma_{1PI,R}(\not{p} = m_R) + (\not{p} - m_R)\frac{d\Sigma_{1PI,R}(\not{p}, m_R)}{d\not{p}}|_{\not{p}=m_R} + O((\not{p} - m_R)^2). \tag{2.98}$$

Substituting the above equation into (2.94) we have

$$\Delta_R\left(\not{p}, m_R\right) = \frac{i}{(\not{p} - m_R)\left[1 - \frac{d\Sigma_{1PI,R}(\not{p}, m_R)}{d\not{p}}|_{\not{p}=m_R}\right] - \Sigma_{1PI,R}(\not{p} = m_R)}. \tag{2.99}$$

It follows from (2.97) that

$$\Sigma_{1PI,R}(\not{p} = m_R) = 0, \tag{2.100}$$

$$\frac{d\Sigma_{1PI,R}(\not{p}, m_R)}{d\not{p}}|_{\not{p}=m_R} = 0. \tag{2.101}$$

Eq. (2.100) fixes the pole of the propagator at m_R while Eq. (2.101) fixes the residue of the quark propagator to be 1.

A point we would like to mention is that after imposing the renormalization conditions (2.100) and (2.101) the non-vanishing term is

$$\Sigma_R\left(\not{p}, m_R\right) = \mathcal{O}\left(\left(\not{p} - m_R\right)^2\right) \tag{2.102}$$

which means,

$$\frac{\Sigma_R\left(\not{p}, m_R\right)}{\not{p} - m_R} = \mathcal{O}\left(\not{p} - m_R\right).$$

The self–energy contributions of the quarks are shown by the last two diagrams in Fig. 2.7. After renormalization they have the form

$$\frac{\Sigma_R\left(\not{p}, m_R\right)}{\not{p} - m_R} u_R(p, s) \tag{2.103}$$

where $u_R(p, s)$ is the renormalized quark spinor. Since the mass is renormalized the renormalized quark spinor satisfies the Dirac equation $(\not{p} - m_R)u_R = 0$. This leads to the vanishing of the renormalized self–energy contribution:

$$\frac{\Sigma_R\left(\not{p}, m_R\right)}{\not{p} - m_R} u_R(p, s) = \mathcal{O}\left(\not{p} - m_R\right) u_R(p, s) = 0. \tag{2.104}$$

Now we proceed to fix the renormalization constants. Inserting (2.100) and (2.101) into (2.96) gives

$$\begin{cases} Z_m = 1 - \Sigma_{1PI}(\not{p} = m_R) \\ Z_2^{-1} = 1 - \dfrac{d\Sigma_{1PI}(\not{p}, m_R)}{d\not{p}}|_{\not{p}=m_R} \end{cases} \tag{2.105}$$

For the quark self–energy the 1–particle irreducible graph is shown in Fig. 2.8. The quark propagator at one–loop order is the geometric sum of all graphs in Fig. 2.10 where

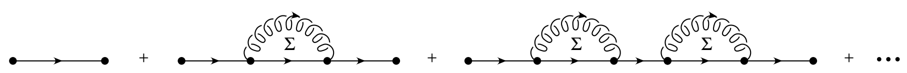


Figure 2.10: Contribution of 1-particle irreducible diagram to a quark propagator.

Σ_{1PI} equals the quark self–energy Σ given in (2.86). Using the perturbative form of the renormalization constant $Z = 1 + \delta Z$, we obtain the quark mass renormalization constant and the quark field renormalization constant as

$$\begin{aligned} \delta Z_m &= -\Sigma(\not{p} = m_R) \\ \delta Z_2 &= \frac{d\Sigma(\not{p}, m_R)}{d\not{p}}|_{\not{p}=m_R}. \end{aligned} \tag{2.106}$$

Inserting $\Sigma = \not{p}\Sigma^V + m\Sigma^S$ into (2.106) we have

$$\delta Z_m = -m_R\left[\Sigma^V(p^2 = m_R^2) + \Sigma^S(p^2 = m_R^2)\right] \tag{2.107}$$

$$\delta Z_2 = \Sigma^V(p^2 = m_R^2) + 2m_R^2\frac{\partial}{\partial p^2}\left[\Sigma^V(p^2) + \Sigma^S(p^2)\right]\Big|_{p^2=m_R^2}. \tag{2.108}$$

We make use of $\not{p}^2 = p^2$ and use the following change of variables in the differentiation in the second equation:

$$\begin{aligned}\frac{d}{d\not{p}}\Big|_{\not{p}=m_R} &= \frac{\partial}{\partial\not{p}}\Big|_{\not{p}=m_R} + \frac{\partial p^2}{\partial\not{p}}\frac{\partial}{\partial p^2}\Big|_{\not{p}=m_R} \\ &= \frac{\partial}{\partial\not{p}}\Big|_{\not{p}=m_R} + \frac{\partial\not{p}^2}{\partial\not{p}}\frac{\partial}{\partial p^2}\Big|_{\not{p}=m_R} \\ &= \frac{\partial}{\partial\not{p}}\Big|_{\not{p}=m_R} + 2m_R\frac{\partial}{\partial p^2}\Big|_{\not{p}=m_R}. \end{aligned} \tag{2.109}$$

The mass renormalization constant δZ_m is obtained by substituting Σ^V and Σ^S from (2.89) and (2.90). Writing this in a general way, we have

$$\delta Z_m^q = \frac{\alpha_s}{4\pi} C_F\, m_q \left\{ \frac{3}{\delta} - 3\gamma_E + 3\ln\frac{4\pi\mu^2}{m_q^2} + 4 \right\}, \tag{2.110}$$

where $q = t, b$ denote the t–quark and the b–quark, respectively.

The quark field renormalization constant δZ_2 is both UV- and IR-divergent in the on–shell scheme. It involves the differentiation $\frac{\partial}{\partial p^2}B_0$ which is given in Appendix (B.26). Performing the differentiation and then collecting all the terms we obtain

$$\delta Z_2^q = -\frac{\alpha_s}{4\pi} C_F \left\{ \frac{1}{\delta} - \gamma_E + \ln\frac{4\pi\mu^2}{m_q^2} + 2\ln\frac{m_g^2}{m_q^2} + 4 \right\}. \tag{2.111}$$

where $q = t, b$ denotes the t–quark and the b–quark respectively as before.

2.2.4 Renormalized virtual one–loop corrections

In the Standard model the fermions acquire mass through the Higgs mechanism. The same term in the Lagrangian will give rise to both the fermion mass and the fermion–Higgs coupling. Therefore the renormalization of the fermion coupling to the Higgs is not independent of the renormalization of its mass. The $\bar{b}tH$ vertex, as shown in (2.5), reads

$$\bar{u}(p_b, s_b) \left\{ g_t \frac{\mathbb{1} + \gamma_5}{2} + g_b \frac{\mathbb{1} - \gamma_5}{2} \right\} u(p_t, s_t). \tag{2.112}$$

As explained before the explicit form of the coupling constants g_t and g_b differ in Model I and Model II. However, the common property of both models is that g_t and g_b are proportional to m_t and m_b respectively. This allows one to represent the coupling factors by

$$g_t = c_t\, g_w\, m_t, \quad g_b = c_b\, g_w\, m_b. \tag{2.113}$$

The detailed forms of the constants c_t and c_b are not relevant for the renormalization procedure. They can be read off from (2.10) and (2.11). The term in the Lagrangian which gives rise to the coupling in (2.112) is then:

$$g_w\, \bar{\psi}_b \left\{ c_t\, m_t \frac{\mathbb{1} + \gamma_5}{2} + c_b\, m_b \frac{\mathbb{1} - \gamma_5}{2} \right\} \psi_t H^- , \tag{2.114}$$

where ψ_b, ψ_b and H^- denote the top quark field, the bottom quark field and the Higgs boson field., respectively. For the purpose of renormalization we introduce renormalized quark fields and masses. They are related to the unrenormalized ones by the following renormalization transformations[3]:

$$\begin{aligned} \psi_t &= \sqrt{Z_2^t}\psi_{t,R} = (1 + \frac{1}{2}\delta Z_2^t)\psi_{t,R}\,, \\ \psi_b &= \sqrt{Z_2^b}\psi_{b,R} = (1 + \frac{1}{2}\delta Z_2^b)\psi_{b,R}\,, \\ m_t &= Z_2^t m_{t,R} = (1 + \delta Z_m^t)m_{t,R}\,, \\ m_b &= Z_2^b m_{b,R} = (1 + \delta Z_m^b)m_{b,R}\,, \end{aligned} \tag{2.115}$$

where we have used $Z = 1 + \delta Z$. The Higgs field and the weak coupling constant g_w are not renormalized at $\mathcal{O}(\alpha_s)$.

Writing the coupling (2.114) in terms of the renormalized quark fields and masses using (2.115) we have

$$\begin{aligned} & g_w\bar{\psi}_{b,R}(1 + \frac{1}{2}\delta Z_2^b)\Big\{c_t\,(1 + \delta Z_m^t)m_{t,R}\frac{\mathbb{1} + \gamma_5}{2} + c_b\,(1 + \delta Z_m^b)m_{b,R}\frac{\mathbb{1} - \gamma_5}{2}\Big\}(1 + \frac{1}{2}\delta Z_2^t)\psi_{t,R}H^- \\ &= g_w\bar{\psi}_{b,R}\Big\{c_t\, m_{t,R}\frac{\mathbb{1} + \gamma_5}{2} + c_b\, m_{b,R}\frac{\mathbb{1} - \gamma_5}{2}\Big\}\psi_{t,R}H^- \\ &\quad + g_w\,\bar{\psi}_{b,R}\Big\{c_t\, m_{t,R}\frac{\mathbb{1} + \gamma_5}{2} + c_b\, m_{b,R}\frac{\mathbb{1} - \gamma_5}{2}\Big\}(\frac{1}{2}\delta Z_2^t + \frac{1}{2}\delta Z_2^t)\psi_{t,R}H^- \\ &\quad + g_w\bar{\psi}_{b,R}\Big\{c_t m_{t,R}\frac{1}{2}\delta Z_2^t\frac{\mathbb{1} + \gamma_5}{2} + c_b m_{b,R}\frac{1}{2}\delta Z_2^b\frac{\mathbb{1} + \gamma_5}{2}\Big\}\psi_{t,R}H^- \\ &= g_w\,\bar{\psi}_{b,R}\Big\{g_{t,R}\frac{\mathbb{1} + \gamma_5}{2} + g_{b,R}\frac{\mathbb{1} - \gamma_5}{2}\Big\}\psi_{t,R}H^- \\ &\quad + g_w\,\bar{\psi}_{b,R}\Big\{g_{t,R}(\frac{1}{2}\delta Z_2^t + \frac{1}{2}\delta Z_2^b + \delta Z_m^t)\frac{\mathbb{1} + \gamma_5}{2} + (\frac{1}{2}\delta Z_2^t + \frac{1}{2}\delta Z_2^b + \delta Z_m^b)\frac{\mathbb{1} - \gamma_5}{2}\Big\}\psi_{t,R}H^-\,. \end{aligned} \tag{2.116}$$

Therefore the counter term for the vertex is

$$g_w\,\bar{\psi}_{t,R}\Big\{g_{t,R}(\frac{1}{2}\delta Z_2^t + \frac{1}{2}\delta Z_2^b + \delta Z_m^t)\frac{\mathbb{1} + \gamma_5}{2} + g_{b,R}(\frac{1}{2}\delta Z_2^t + \frac{1}{2}\delta Z_2^b + \delta Z_m^b)\frac{\mathbb{1} - \gamma_5}{2}\Big\}\psi_{b,R}H^-\,. \tag{2.117}$$

Changing back to the notation in terms of a and b using

$$g_t = a + b, \quad g_b = a - b$$

[3]The subscript "R" denotes the renormalized quantities. It should not be confused with the subscript "R" which denotes the right–handed chiral projection.

we write the renormalized coupling term in the Lagrangian as

$$\begin{aligned}
&\bar{\psi}_{b,R}(a_R + b_R\gamma_5)\psi_{t,R}\,H^- \\
&\quad +\bar{\psi}_{b,R}\Big\{(a_R+b_R)\left(\frac{1}{2}\delta Z_2^t + \frac{1}{2}\delta Z_2^b + \delta Z_m^t\right)\frac{1\!\!1+\gamma_5}{2} \\
&\quad +(a_R-b_R)\left(\frac{1}{2}\delta Z_2^t + \frac{1}{2}\delta Z_2^b + \delta Z_m^b\right)\frac{1\!\!1-\gamma_5}{2}\Big\}\psi_{t,R}\,H^- \\
&= \bar{\psi}_{b,R}(a_R + b_R\gamma_5 + \delta\Lambda)\psi_{t,R}\,H^-
\end{aligned} \tag{2.118}$$

with

$$\delta\Lambda = (a+b)\left(\frac{1}{2}\delta Z_2^t + \frac{1}{2}\delta Z_2^b + \delta Z_m^t\right)\frac{1\!\!1+\gamma_5}{2} + (a-b)\left(\frac{1}{2}\delta Z_2^t + \frac{1}{2}\delta Z_2^b + \delta Z_m^b\right)\frac{1\!\!1-\gamma_5}{2}\,. \tag{2.119}$$

We have dropped the subscript “R” which denote the renormalized quantities, since, from now on, everything is renormalized.

Finally the renormalized $\mathcal{O}(\alpha_s)$ virtual correction is

$$\mathcal{M}_{v,R} = \bar{u}(p_b,s_b)\Big\{(a+b\,\gamma_5)\boldsymbol{\Lambda_1} + a\boldsymbol{\Lambda_2} + \delta\boldsymbol{\Lambda}\Big\}u(p_t,s_t) \tag{2.120}$$

with Λ_1, Λ_2 and $\delta\Lambda$ given in (2.72), (2.73) and (2.119), respectively.

Summarizing the renormalization procedure one finds that the quark self–energy serves to fix the renormalization constants. After renormalization the quark self–energy contributions vanish (see 2.104) and the one–loop vertex correction is modified to the form (2.120).

At $\mathcal{O}(\alpha_s)$ the full amplitude is the sum of the amplitudes of the Born term, virtual one–loop and the real emissions:

$$\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_{v,R} + \mathcal{M}_r\,. \tag{2.121}$$

Squaring the full amplitude one has

$$\begin{aligned}
|\mathcal{M}|^2 &= (\mathcal{M}_0 + \mathcal{M}_{v,R} + \mathcal{M}_r)^\dagger(\mathcal{M}_0 + \mathcal{M}_{v,R} + \mathcal{M}_r) \\
&= \mathcal{M}_0^\dagger\mathcal{M}_0 + \mathcal{M}_0^\dagger\mathcal{M}_{v,R} + \mathcal{M}_{v,R}^\dagger\mathcal{M}_0 + \mathcal{M}_r^\dagger\mathcal{M}_r + \mathcal{O}(\alpha_s^2) \\
&= |\mathcal{M}_0|^2 + (\mathcal{M}_{v,R})^2 + |\mathcal{M}_r|^2 + \mathcal{O}(\alpha_s^2)\,,
\end{aligned} \tag{2.122}$$

where

$$|\mathcal{M}_0|^2 = \mathcal{M}_0^\dagger\mathcal{M}_0\,, \tag{2.123}$$

$$(\mathcal{M}_{v,R})^2 = \mathcal{M}_0^\dagger\mathcal{M}_{v,R} + \mathcal{M}_{v,R}^\dagger\mathcal{M}_0\,, \tag{2.124}$$

$$|\mathcal{M}_r|^2 = \mathcal{M}_r^\dagger\mathcal{M}_r\,. \tag{2.125}$$

Now that the amplitude squared is at hand we write the rate for the virtual one–loop contribution

$$d\Gamma_v = \frac{1}{2m_t}\frac{1}{(2\pi)^2}dR_2(p_t;p_b,p_H)(\mathcal{M}_{v,R})^2 \tag{2.126}$$

in the usual form:

$$\frac{d\Gamma_v}{d\cos\theta_P} = \frac{1}{2}\left(\Gamma_v + P\,\Gamma_v^P \cos\theta_P\right)\,, \tag{2.127}$$

where the unpolarized and polarized virtual rates are

$$\begin{aligned}
\Gamma_v^{unpol} &= \Gamma_{Born}^{unpol}\left(\delta Z_2^t + \delta Z_2^b + \delta Z_m^t + \delta Z_m^b + 2\Lambda_1\right) + \\
&\quad + \frac{\sqrt{\lambda(m_t^2, m_b^2, m_H^2)}}{16\pi m_t^3}\Big\{2\,a^2\,\Lambda_2\left[(m_t + m_b)^2 - m_H^2\right] \\
&\quad - 2ab\left(m_t^2 + m_b^2 - m_H^2\right)\left(\delta Z_m^t - \delta Z_m^b\right)\Big\} \\
&= \Gamma_{Born}^{unpol}\left(\delta Z_2^t + \delta Z_2^b + \delta Z_m^t + \delta Z_m^b + 2\Lambda_1\right) \\
&\quad + \frac{\bar{p}_3}{8\pi} m_t \Big\{2\,a^2\,\Lambda_2\left[(1+\epsilon)^2 - y^2\right] + 4ab\,\bar{p}_0\left(\delta Z_m^t - \delta Z_m^b\right)\Big\}\,,
\end{aligned} \tag{2.128}$$

$$\begin{aligned}
\Gamma_v^{pol} &= \Gamma_{Born}^{pol}\left(\delta Z_2^t + \delta Z_2^b + \delta Z_m^t + \delta Z_m^b + 2\Lambda_1\right) \\
&\quad + \frac{\sqrt{\lambda(m_t^2, m_b^2, m_H^2)}}{16\pi m_t^3}\left[2ab\,\Lambda_2 + (a^2 + b^2)\left(\delta Z_m^t - \delta Z_m^b\right)\right] \\
&= \Gamma_{Born}^{pol}\left(\delta Z_2^t + \delta Z_2^b + \delta Z_m^t + \delta Z_m^b + 2\Lambda_1\right) \\
&\quad + \frac{\bar{p}_3^2}{4\pi} m_t\left[2ab\,\Lambda_2 - (a^2 + b^2)\left(\delta Z_m^t - \delta Z_m^b\right)\right].
\end{aligned} \tag{2.129}$$

The renormalized virtual one–loop correction to the unpolarized rate (2.128) is in agreement with Ref. [39]. The result for the virtual one–loop correction to the polarized rate (2.129) is new. The infrared divergent terms residing in the renormalization factor Z_2^q and in the integral term C_0 in Λ_1 are proportional to the Born term rates Γ_{Born} and Γ_{Born}^{pol}, respectively.

2.3 Real gluon emission

At NLO QCD the real emission (tree graph) contributions come from the real gluon emissions from the quark legs. At the $\mathcal{O}(\alpha_s)$ level one real gluon is emitted from each of the quark legs as shown in Fig. 2.11. The three–body phase space integration leads to IR–divergences which are regularized by a small gluon mass m_g. In this section the amplitude squared is split into IR–divergent and IR–convergent pieces. The phase space integrations of the two pieces are performed separately. In the end the results for the IR–divergent and IR–convergent parts are summed up to give the total results.

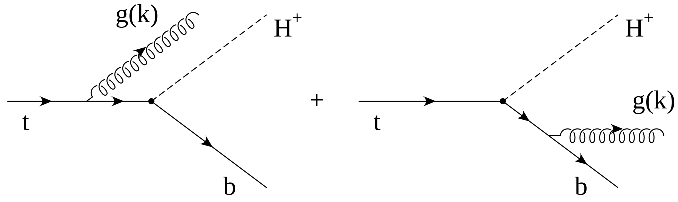


Figure 2.11: The Feynman graphs for the real gluon emission contributions at the $\mathcal{O}(\alpha_s)$ in the decay $t(\uparrow) \to H^+ + b$.

2.3.1 The amplitude squared for the real emissions

The Feynman graphs for the real gluon emission at $\mathcal{O}(\alpha_s)$ are shown in Fig. 2.11. The Feynman rules give the transition amplitude [4]

$$\mathcal{M}_r = \bar{u}(p_b, s_b)\,(a\mathbb{1} + b\gamma_5)\left[i\frac{\not{p}_t - \not{k} + m_t}{(p_t - k)^2 - m_t^2}\right]\left(-ig_s\frac{\lambda^n}{2}\gamma_\sigma\right)\varepsilon^{\sigma *}(k,\lambda)u(p_t, s_t)$$

$$+\,\bar{u}(p_b, s_b)\left(-ig_s\frac{\lambda^n}{2}\gamma_\sigma\right)\varepsilon^{\sigma *}(k,\lambda)\left[i\frac{\not{p}_b + \not{k} + m_b}{(p_b + k)^2 - m_b^2}\right](a\mathbb{1} + b\gamma_5)\,u(p_t, s_t), \quad (2.130)$$

where $\varepsilon_\sigma(k,\lambda)$ is the polarization vector of the gluon with momentum k and spin λ. In the Feynman–'tHooft gauge the polarization sum for the gluon gives

$$\sum_\lambda \epsilon_\mu(k,\lambda)\epsilon_\nu^*(k,\lambda) = -g_{\mu\nu}. \quad (2.131)$$

Using the mass-shell conditions $p_t^2 = m_t^2$, $p_b^2 = m_b^2$ and $k^2 = 0$ in the denominator of the quark propagators, and Dirac relations as well as the Dirac equations in the numerator the amplitude simplifies to

$$\mathcal{M}_r = \left(-ig_s\frac{\lambda^n}{2}\right)\bar{u}(p_b, s_b)\left\{\frac{2p_t^\sigma - \not{k}\gamma^\sigma}{-2k\cdot p_t} + \frac{2p_b^\sigma + \gamma^\sigma\not{k}}{2k\cdot p_b}\right\}(a\mathbb{1}+b\gamma_5)u(p_t, s_t)\,\varepsilon_\sigma^*(k,\lambda). \quad (2.132)$$

The IR-behavior of the transition amplitude has to be kept in mind when performing the phase space integration. We regularize the IR-divergence with a gluon mass and this will considerably complicate the 3–body phase space kinematics in the decay $p_t \to p_H + p_b + k$ considerably. Therefore it is advantageous to separate the IR-convergent and IR-divergent parts of the amplitude. For the IR–convergent part we can safely set the gluon mass to zero which leads to a considerable simplification of the phase space kinematics. Inspection of the amplitude (2.132) shows that the terms without the gluon momenta will lead to IR-divergences after squaring and doing the phase space integration. We thus write the

[4] We shall refer to the tree graph contributions as the *real contributions*. Accordingly we shall denote the tree graph contributions with a suffix "r".

amplitude as

$$\mathcal{M}_r = (-ig_s \frac{\lambda^n}{2})\bar{u}(p_b, s_b) \left\{ \frac{2p_t^\sigma - \not{k}\gamma^\sigma}{-2k\cdot p_t} + \frac{2p_b^\sigma + \gamma^\sigma \not{k}}{2k\cdot p_b} \right\} (a\mathbb{1} + b\gamma_5) u_t(p_t, s_t) \varepsilon^*_\sigma(k, \lambda)$$

$$= (-ig_s \frac{\lambda^n}{2})\bar{u}_b(p_{b,s_b}) \left\{ (\frac{2p_b^\sigma}{2k\cdot p_b} - \frac{2p_t^\sigma}{2k\cdot p_t}) + (\frac{\not{k}\gamma^\sigma}{2k\cdot p_t} + \frac{\gamma^\sigma\not{k}}{2k\cdot p_b}) \right\} (a\mathbb{1} + b\gamma_5) u(p_t, s_t) \varepsilon^*_\sigma(k, \lambda),$$

separating the terms with and without a gluon momenta k in the numerator. For the amplitude squared we obtain

$$|\mathcal{M}_r|^2 = \sum_{s_b}\sum_{\lambda} \mathcal{M}_r \mathcal{M}_r^\dagger = \sum_{s_b}\sum_{\lambda}$$

$$\left\{ (-ig_s \frac{\lambda^n}{2})\bar{u}(p_b, s_b) \Big[(\frac{2p_b^\sigma}{2k\cdot p_b} - \frac{2p_t^\sigma}{2k\cdot p_t}) + (\frac{\not{k}\gamma^\sigma}{2k\cdot p_t} + \frac{\gamma^\sigma\not{k}}{2k\cdot p_b}) \Big] (a\mathbb{1} + b\gamma_5)\, \epsilon^*_\sigma(k, \lambda) u(p_t, s_t) \right\}$$

$$\left\{ (-ig_s \frac{\lambda^n}{2})\bar{u}(p_b, s_b) \Big[(\frac{2p_{b\rho}}{2k\cdot p_b} - \frac{2p_{t\rho}}{2k\cdot p_t}) + (\frac{\not{k}\gamma_\rho}{2k\cdot p_t} + \frac{\gamma_\rho\not{k}}{2k\cdot p_b}) \Big] (a\mathbb{1} + b\gamma_5)\, \epsilon^\rho(k, \lambda) u(p_t, s_t) \right\}^\dagger$$

$$= -g_s^2 C_F \,\mathrm{Tr}\Big\{ (\not{p}_b + m_b) \Big[\underbrace{(\frac{2p_b^\sigma}{2p_b\cdot k} - \frac{2p_t^\sigma}{2p_t\cdot k})}_{term\,1} \gamma_\mu + \frac{\gamma^\sigma\not{k}\gamma_\mu}{2p_b\cdot k} + \frac{\gamma_\mu\not{k}\gamma^\sigma}{2p_t\cdot k} \Big] (a\mathbb{1} + b\gamma_5) \times$$

$$\times \Big(\not{p}_t + m_t\Big) \frac{1}{2}(1 - \gamma_5 \not{s}_t) \Big[\underbrace{(\frac{2p_{b\sigma}}{2p_b\cdot k} - \frac{2p_{t\sigma}}{2p_t\cdot k})}_{term\,2} \gamma_\nu + \frac{\gamma_\sigma\not{k}\gamma_\nu}{2p_b\cdot k} + \frac{\gamma_\nu\not{k}\gamma_\sigma}{2p_t\cdot k} \Big] (a\mathbb{1} - b\gamma_5) \Big\}. \quad (2.133)$$

In the above expression the multiplication of the "term 1" and the "term 2" gives

$$\begin{aligned} &= -\alpha_s\, 4\pi\, C_F (\frac{2p_t^\sigma}{2k\cdot p_t} - \frac{2p_b^\sigma}{2k\cdot p_b})(\frac{2p_{t\sigma}}{2k\cdot p_t} - \frac{2p_{b\sigma}}{2k\cdot p_b}) \times \\ &\quad \times \sum_{s_b} \mathrm{Tr} \left\{ (\not{p}_b + m_b)\,(a\mathbb{1} + b\gamma_5) \Big(\not{p}_t + m_t\Big) \frac{1}{2}(1 - \gamma_5\not{s}_t)\,(a\mathbb{1} - b\gamma_5) \right\} \\ &= -\alpha_s\, 4\pi\, C_F \left\{ \frac{m_t^2}{(k\cdot p_t)^2} + \frac{m_b^2}{(k\cdot p_b)^2} - 2\frac{p_b\cdot p_t}{(k\cdot p_b)(k\cdot p_t)} \right\} \times \\ &\quad \times \sum_{s_b} \mathrm{Tr} \left\{ (\not{p}_b + m_b)\,(a\mathbb{1} + b\gamma_5) \Big(\not{p}_t + m_t\Big) \frac{1}{2}(1 - \gamma_5\not{s}_t)\,(a\mathbb{1} - b\gamma_5) \right\} \\ &= |\widetilde{\mathcal{M}}_0|^2 |M|^2_{SGF} \end{aligned}$$

where

$$\begin{aligned} |\widetilde{\mathcal{M}}_0|^2 &= \sum_{s_b} \mathrm{Tr} \left\{ (\not{p}_b + m_b)\,(a\mathbb{1} + b\gamma_5) \Big(\not{p}_t + m_t\Big) \frac{1}{2}(1 - \gamma_5\not{s}_t)\,(a\mathbb{1} - b\gamma_5) \right\}, \\ |M|^2_{SGF} &= -4\pi\alpha_s\, C_F \left\{ \frac{m_t^2}{(k\cdot p_t)^2} + \frac{m_b^2}{(k\cdot p_b)^2} - 2\frac{p_b\cdot p_t}{(k\cdot p_b)(k\cdot p_t)} \right\}. \end{aligned} \quad (2.134)$$

The IR–divergent factor $|M|^2_{SGF}$ is known as the soft–gluon (or eikonal) factor. $|\widetilde{\mathcal{M}}_0|^2$ is the Born term amplitude squared but evaluated for $p_t = p_b + p_H + k$, i.e. $|\widetilde{\mathcal{M}}_0|^2(k=0) = |\mathcal{M}_0|^2$. The rest of the real emission amplitude squared is IR–convergent and is denoted by $|\mathcal{M}_r^{rest}|^2$. Thus the real emission amplitude squared is split into a IR–convergent and a IR–divergent part according to

$$|\mathcal{M}_r|^2 = |\mathcal{M}_r^{rest}|^2 + |\widetilde{\mathcal{M}}_0|^2 |M|^2_{SGF}. \tag{2.135}$$

The integration of the term $|\widetilde{\mathcal{M}}_0|^2|M|^2_{SGF}$ is complicated by the fact that $|\widetilde{\mathcal{M}}_0|^2$ depends on the gluon momentum. However $(|\widetilde{\mathcal{M}}_0|^2 - |\mathcal{M}_0|^2)|M|^2_{SGF}$ is convergent and can be integrated without the gluon mass regulator. We further isolate the IR–divergent part by writing

$$\begin{aligned} |\mathcal{M}_r|^2 &= \left\{|\mathcal{M}_r^{rest}|^2 + \Big(|\widetilde{\mathcal{M}}_0|^2 - |\mathcal{M}_0|^2\Big)|M|^2_{SGF}\right\} + |\mathcal{M}_0|^2|M|^2_{SGF} \\ &= |\mathcal{M}_{r,conv}|^2 + |\mathcal{M}_0|^2|M|^2_{SGF}\,, \end{aligned} \tag{2.136}$$

with

$$|\mathcal{M}_{r,conv}|^2 = |\mathcal{M}_r^{rest}|^2 + \Big(|\widetilde{\mathcal{M}}_0|^2 - |\mathcal{M}_0|^2\Big)|M|^2_{SGF}\,. \tag{2.137}$$

We will see in the phase space integration that adding and subtracting a term as done above is closely related to what is referred to as the *plus prescription*. Now the IR-divergent piece $|M|^2_{SGF}$ can be integrated separately. The same universal soft gluon factor appears in the calculation of the radiative corrections to $t \to W^+ + b$ in [44] and to $t \to b + \ell^+ + \nu_\ell$ in [25].

For the IR–convergent part of $|\mathcal{M}_{r,conv}|^2$ we obtain

$$\begin{aligned} |\mathcal{M}_{r,conv}|^2 = 8\pi\alpha_s C_F \frac{k\cdot p_H}{(k\cdot p_t)(k\cdot p_b)} \Bigg\{ &(a^2+b^2)(k\cdot p_H) + \\ &- 2abm_t\Bigg[\left(\frac{m_t^2 + m_{H^+}^2 - m_b^2}{2k\cdot p_t} - 1 - \frac{m_{H^+}^2}{k\cdot p_H}\right)(k\cdot s_t) + (p_H\cdot s_t) \\ &+ \left(\frac{m_t^2}{k\cdot p_t} - \frac{m_b^2}{k\cdot p_b} - 2 - \frac{m_{H^+}^2}{k\cdot p_H}\right)\Big((p_H\cdot s_t) - (p_H\cdot s_t)|_{k=0}\Big)\Bigg]\Bigg\}. \end{aligned} \tag{2.138}$$

The unpolarized and polarized parts of (2.138) are not difficult to identify. We denote them by $|\mathcal{M}_{r,conv}^{unpol}|^2$ and$|\mathcal{M}_{r,conv}^{pol}|^2$, respectively. Writing them in their explicit form one has

$$|\mathcal{M}_{r,conv}^{unpol}|^2 = 8\pi\,\alpha_s\,C_F(a^2+b^2)\frac{(k\cdot p_H)^2}{(k\cdot p_t)(k\cdot p_b)}\,, \tag{2.139}$$

$$\begin{aligned} |\mathcal{M}_{r,conv}^{pol}|^2 = 8\pi\,\alpha_s\,C_F(-2a\,b\,m_t)\frac{k\cdot p_H}{(k\cdot p_t)(k\cdot p_b)}\Bigg[&\left(\frac{m_t^2 + m_{H^+}^2 - m_b^2}{2k\cdot p_t} - 1 - \frac{m_{H^+}^2}{k\cdot p_H}\right)(k\cdot s_t) \\ + (p_H\cdot s_t) + &\left(\frac{m_t^2}{k\cdot p_t} - \frac{m_b^2}{k\cdot p_b} - 2 - \frac{m_{H^+}^2}{k\cdot p_H}\right)\Big((p_H\cdot s_t) - (p_H\cdot s_t)|_{k=0}\Big)\Bigg]. \end{aligned} \tag{2.140}$$

The unpolarized part has a very simple structure. It is quite remarkable that the unpolarized and polarized pieces of the convergent tree-level contribution are proportional to $(a^2 + b^2)$ and $2ab$, respectively.

How we proceed with the kinematics of the amplitude squared depends on how we calculate the phase space integration, which will be discussed in the next section.

2.3.2 The phase space integration

The differential rate for the real emission contribution is given by

$$\begin{aligned} d\Gamma &= \frac{1}{2m_t}\frac{1}{(2\pi)^3}\frac{d^3\vec{p}_H}{2E_H}\frac{1}{(2\pi)^3}\frac{d^3\vec{p}_b}{2E_b}\frac{1}{(2\pi)^3}\frac{d^3\vec{k}}{2E_g}(2\pi)^4\delta^4(p_t - p_b - p_H - k)\,|\mathcal{M}_{tree}|^2 \\ &= \frac{1}{2m_t}\frac{1}{(2\pi)^5}dR_3(p_t; p_H, p_b, k)\,|\mathcal{M}_r|^2, \end{aligned}$$

where $dR_3(p_t; p_H, p_b, k)$ is the three–body phase space:

$$dR_3(p_t; p_b, k, p_H) = \frac{d^3\vec{p}_H}{2E_H}\frac{d^3\vec{p}_b}{2E_b}\frac{d^3\vec{k}}{2E_g}\delta^4(p_t - p_b - p_H - k) \tag{2.141}$$

One way of doing the three body phase integration is to use $\frac{d^3\vec{p}}{2E} = \frac{1}{2}d\Omega|\vec{p}|\,dE$ and to integrate over the gluon energy k_0 and the Higgs boson energy E_H as was done in [55]. We choose to perform the phase space integration by factorizing the three body phase space into two two–body phase spaces (see Fig. 2.12) as described in [56]. The momentum

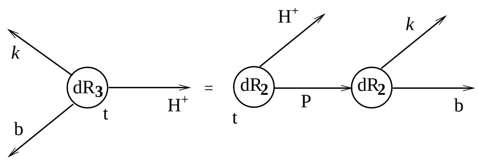


Figure 2.12: The decay of the top quark into three–particle final states as a sequence of two two–particle final state decays.

of the combined system of the b–quark and the gluon is defined by $P = p_b + k$. We also use $z = \frac{P^2}{m_t^2}$. With the help of the relations

$$\int \frac{d^3\vec{p}}{2E} = \int d^4p\delta\left(p^2 - m^2\right)\Theta\left(p^0\right) \tag{2.142}$$

$$1 = \int dP^2\delta\left(P^2 - m_P^2\right) \tag{2.143}$$

$$1 = \int d^4P\delta^4\left(P - p_b - k\right) \tag{2.144}$$

we write the three body phase space as

$$\begin{aligned} dR_3\left(p_t;p_H,p_b,k\right) &= \frac{d^3\vec{p}_H}{2E_H}\frac{d^3\vec{p}_b}{2E_b}\frac{d^3\vec{k}}{2E_g}\delta^4(p_t-p_b-p_H-k) \\ &= \frac{d^3\vec{p}_H}{2E_H}\frac{d^3\vec{p}_b}{2E_b}\frac{d^3\vec{k}}{2E_g}\delta^4(p_t-p_b-p_H-k)\times \\ &\qquad \times dP^2\delta\left(P^2-m_P^2\right)\,d^4P\delta^4\left(P-p_b-k\right) \\ &= dP^2\left\{\frac{d^3\vec{p}_H}{2E_H}\frac{d^3\vec{P}}{2E_P}\delta^4(p_t-p_H-P\right\}\left\{\frac{d^3\vec{p}_b}{2E_b}\frac{d^3\vec{k}}{2E_g}\delta^4\left(P-p_b-k\right)\right\} \\ &= dP^2 dR_2\left(p_t;p_H,P\right)dR_2\left(P;p_b,k\right) \\ &= m_t^2 dz\, dR_2\left(p_t;p_H,P\right)\, dR_2\left(P;p_b,k\right)\ , \end{aligned} \tag{2.145}$$

where the integration limits for z are

$$(\epsilon+\Lambda)^2 \leqslant z \leqslant (1-y)^2. \tag{2.146}$$

The two–body phase space integration $\int dR_2\left(p_t;p_H,P\right)$ is done in the top–rest frame with $\vec{p}_H$ defining the z–axis. Analogous to (2.29) we have

$$dR_2\left(p_t;p_H,P\right) = \frac{\sqrt{\lambda(1,y^2,z)}}{8}2\pi\, d\cos\theta_P\,. \tag{2.147}$$

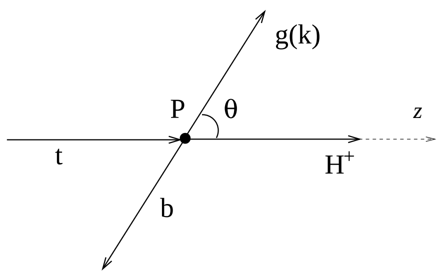


Figure 2.13: The P–rest frame.

The two–body phase space integration $dR_2\left(P;p_b,k\right)$ is done in the P–rest frame with $\vec{p}_H$ defining the z–axis (see Fig. 2.13). We denote the bottom quark energy and momentum modulus, and gluon energy and momentum modulus in the P–rest frame by E_b^*, p_b^*, E_g^*

and p_g^*. For $dR_2\,(P;p_b,k)$ one has

$$\begin{aligned} dR_2\,(P;p_b,k) &= \frac{d^3p_b^*}{2E_b^*}d^4k\delta(k^2-m_g^2)\delta^4(P-p_b^*-k) \\ &= \frac{1}{2E_b}|p_b^*|^2\,d\phi\,d\mathrm{cos}\theta\,d|p_b^*|\delta((P-p_b^*)^2-m_g^2) \\ &= \frac{1}{2}|p_b^*|(2\pi)\,d\cos\theta\,dE_b^*\,\delta(P^2+m_b^2-2P\cdot p_b^*-m_g^2)\,, \\ &\downarrow \qquad \text{in the P-rest frame} \\ &= \frac{1}{2}|p_b^*|(2\pi)\,d\cos\theta\,dE_b^*\,\delta(P^2+m_b^2-2m_t\sqrt{z}E_b^*-m_g^2)\,, \\ &= \frac{\pi p_g^*}{2m_t\sqrt{z}}\,d\cos\theta. \end{aligned} \tag{2.148}$$

The azimuthal angle ϕ is integrated out since the amplitude squared has no azimuthal dependence. After the Dirac delta integration the scaled energies and the momenta of the b–quark and gluon are fixed in terms of z, y, Λ:

$$\begin{aligned} \hat{p}_g^* &:= \frac{1}{m_t}p_g^* = \frac{\sqrt{\lambda(z,y^2,\Lambda^2)}}{2\sqrt{z}}\,, \\ \hat{p}_b^* &:= \frac{1}{m_t}p_b^* = \hat{p}_g^*\,, \\ \hat{E}_b^* &:= \frac{1}{m_t}E_b^* = \frac{1}{2}(z^2+\epsilon^2-\Lambda^2)\,, \\ \hat{E}_b^* &:= \frac{1}{m_t}E_g^* = \frac{1}{2}(z^2-\epsilon^2+\Lambda^2)\,. \end{aligned} \tag{2.149}$$

The result in (2.148) shows that phase space integration $dR_2\,(P;p_b,k)$ simplifies to the integration over the polar angle θ of the gluon in the P–rest frame. We therefore examine the $\cos\theta$–dependence of the scalar products that appear in $|\mathcal{M}_r|^2$.

The four momenta involved in the decay process are $p_t,\ p_H,\ p_b,\ k$ and the top quark polarization four vector s_t. We choose to eliminate p_H using momentum conservation $p_H = p_t - P$. We also substitute p_b by $p_b = P - k$. Then we remain with scalar products $(p_t \cdot P)$, $(k \cdot P)$ and $(p_t \cdot k)$. In the P–rest frame they are given by

$$\begin{aligned} (p_t\!\cdot\!P) &= \frac{1+z-y^2}{2}m_t^2, \\ (k\!\cdot\!P) &= \frac{z+\Lambda^2-\epsilon^2}{2}m_t^2, \\ (p_t\!\cdot\!k) &= \frac{\hat{E}_g^*\,p_0+p_3\,\hat{p}_g^*\cos\theta}{\sqrt{z}}m_t^2\,, \end{aligned} \tag{2.150}$$

where p_0, p_3 and $\hat{p}_g^*$, $\hat{E}_g^*$ are listed in (2.4) and (2.149). In the IR–convergent case we set

$\Lambda = 0$ and the invariant scalar products simplify to

$$\begin{aligned}(p_t\cdot P) &= \frac{1+z-y^2}{2}m_t^2,\\ (k\cdot P) &= \frac{z-\epsilon^2}{2}m_t^2,\\ (p_t\cdot k) &= \frac{(P\cdot k)(p_t\cdot P)+m_t^2(P\cdot k)p_3\cos\theta}{m_t^2 z}.\end{aligned} \tag{2.151}$$

From the scalar products in (2.151) it is clear that only $(p_t \cdot k)$ is relevant in the $d\cos\theta$ integration because only $(p_t \cdot k)$ depends on the angle θ. Thus the two body phase space integration $dR_2(P; p_b, k)$ boils down to following set of integrals

$$I_n := \int dR_2(P;p_b,k)(p_t\cdot k)^n = \frac{\pi\hat{p}_g^*}{2\sqrt{z}}\int d\cos\theta\ (p_t\cdot k)^n\,, \tag{2.152}$$

with, as we will see later on, $-2 \leqslant n \leqslant 1$. The calculation of these basic integrals is deferred to Appendix D.1. The integrations for the IR–divergent and IR–convergent cases are different. The results for the IR–divergent basic integrals are denoted by $\hat{I}_n$.

The polarization dependent scalar products are $(P \cdot s_t)$ and $(k \cdot s_t)$. In the top–rest frame $(P\cdot s_t)$ is given by

$$(P\cdot s_t) = \frac{1}{2}\sqrt{\lambda(1,y^2,z)}\,m_t\cos\theta_P. \tag{2.153}$$

Since the product $(P \cdot s_t)$ is not dependent on $\cos\theta$, it is a constant w.r.t. the two–body phase space integration $dR_2(P; p_b, k)$. We write $(k \cdot s_t) = k_\mu s_t^\mu$ which will give rise to integrals of the form

$$I_n^\mu := \int dR_2(P;p_b,k)(p_t\cdot k)^n k^\mu = \frac{\pi\hat{p}_g^*}{2\sqrt{z}}\int d\cos\theta\ (p_t\cdot k)^n k^\mu\,. \tag{2.154}$$

Note that these vector integrals are IR–convergent because of the gluon momentum k^μ in the numerator. Their calculation is also given in Appendix D.1.

Substituting the results for the two–body phase spaces $dR_2(p_t; p_H, P)$ and $dR_2(P; p_b, k)$ from (2.147) and (2.148) into the three–body phase space $dR_3(p_t; p_H, p_b, k)$ in (2.145) we obtain

$$\begin{aligned}dR_3(p_t;p_H,p_b,k) &= m_t^2\,dz\,dR_2(p_t;p_H,P)\cdot dR_2(P;p_b,k)\\ &= m_t^2\,dz\,\frac{\sqrt{\lambda(1,y^2,z)}}{8}2\pi\,d\cos\theta_P\cdot\frac{\pi\hat{p}_g^*}{2\sqrt{z}}\,d\cos\theta\,.\end{aligned} \tag{2.155}$$

The real emission matrix squared has been split into IR–divergent and IR–convergent pieces in (2.136). Let us begin with the unpolarized IR–convergent part given in (2.139):

$$\begin{aligned}d\Gamma^{unpol}_{r,conv} &= \frac{1}{2m_t}\frac{1}{(2\pi)^5}dR_3(p_t;p_H,p_b,k)\,|\mathcal{M}^{unpol}_{r,conv}|^2\\ &= \frac{1}{2m_t}\frac{1}{(2\pi)^5}m_t^2\,dz\,\frac{\sqrt{\lambda(1,y^2,z)}}{8}2\pi\,d\cos\theta_P\ \times\\ &\qquad \times dR_2(P;p_b,k)\,8\pi\,\alpha_s\,C_F(a^2+b^2)\frac{(k\cdot p_H)^2}{(k\cdot p_t)(k\cdot p_b)}\,.\end{aligned} \tag{2.156}$$

In the IR–convergent case the scalar products $(p_H \cdot k)$ and $(p_b \cdot k)$ simplify to $(k^2 = 0)$

$$\begin{aligned} (p_H \cdot k) &= (p_t \cdot k) - (k \cdot P) \\ (p_b \cdot k) &= (P \cdot k) - k^2 = (k \cdot P). \end{aligned} \tag{2.157}$$

Substituting (2.157) into (2.156) and performing the two–body phase space integration $dR_2(P; p_b, k)$ with the help of the basic integrals defined in (2.152) and (2.154) we obtain

$$\begin{aligned} \int dR_2(P; p_b, k) \frac{(k \cdot p_H)^2}{(k \cdot p_t)(k \cdot p_b)} &= \int dR_2(P; p_b, k) \Big[\frac{(p_t \cdot k)}{(k \cdot P)} + \frac{(k \cdot P)}{(p_t \cdot k)} - 2 \Big] \\ &= \frac{1}{(k \cdot P)} I_1 + (k \cdot P) I_{-1} - 2 I_0 \\ &= \frac{z - \epsilon^2}{4} \pi \left\{ \frac{1 - 3z - y^2}{z^2} + \frac{2}{\sqrt{\lambda(1, y^2, z)}} \ln \left(\frac{1 - y^2 + z + \sqrt{\lambda(1, y^2, z)}}{1 - y^2 + z - \sqrt{\lambda(1, y^2, z)}} \right) \right\}. \end{aligned} \tag{2.158}$$

We insert the above result into the (2.156) and obtain

$$\begin{aligned} d\Gamma^{unpol}_{r,conv} &= \frac{1}{2m_t} \frac{1}{(2\pi)^5} dR_3(p_t; p_b, k, p_H) \, |\mathcal{M}^{unpol}_{r,conv}|^2 \\ &= \frac{1}{2m_t} \frac{1}{(2\pi)^5} m_t^2 \, dz \, \frac{\sqrt{\lambda(1, y^2, z)}}{8} 2\pi \, d\cos\theta_P \cdot 8\pi \, \alpha_s \, C_F (a^2 + b^2) \times \\ &\quad \times \frac{1}{4} \pi \left\{ \frac{1 - 3z - y^2}{z^2} + \frac{2}{\sqrt{\lambda(1, y^2, z)}} \ln \left(\frac{1 - y^2 + z + \sqrt{\lambda(1, y^2, z)}}{1 - y^2 + z - \sqrt{\lambda(1, y^2, z)}} \right) \right\}. \end{aligned} \tag{2.159}$$

Next the z integration is done with the help of the following basic integrals defined by:

$$\begin{aligned} R(n) &:= \int_{\epsilon^2}^{(1-y)^2} dz \frac{1}{(z - \epsilon^2) \sqrt{\lambda(1, y^2, z)^n}}, \\ R(m, n) &:= \int_{\epsilon^2}^{(1-y)^2} dz \frac{z^m}{\sqrt{\lambda(1, y^2, z)^n}}, \\ S(n) &:= \int_{\epsilon^2}^{(1-y)^2} dz \frac{1}{(z - \epsilon^2) \sqrt{\lambda(1, y^2, z)^n}} \ln \left(\frac{1 - y^2 + z + \sqrt{\lambda(1, y^2, z)}}{1 - y^2 + z - \sqrt{\lambda(1, y^2, z)}} \right), \\ S(m, n) &:= \int_{\epsilon^2}^{(1-y)^2} dz \frac{z^m}{\sqrt{\lambda(1, y^2, z)^n}} \ln \left(\frac{1 - y^2 + z + \sqrt{\lambda(1, y^2, z)}}{1 - y^2 + z - \sqrt{\lambda(1, y^2, z)}} \right). \end{aligned} \tag{2.160}$$

The needed basic integrals are calculated in the Appendix D.2. Finally, the IR-convergent unpolarized contribution for the rate is given by

$$\frac{d\Gamma^{unpol}_{r,conv}}{d\cos\theta_P} = -\frac{1}{8\pi m_t}\frac{\alpha_s}{4\pi}C_F\, m_t^2(a^2+b^2)\Bigg\{\frac{3}{4}R(0,-1)+\frac{\epsilon^2(1-y^2)}{4}R(-2,-1) -\frac{1-y^2+3\epsilon^2}{4}R(-1,-1)+\frac{1}{2}\epsilon^2 S(0,0)-\frac{1}{2}S(1,0)\Bigg\}. \quad (2.161)$$

Note that the angle θ_P can be integrated out as $\int d\cos\theta_P = 2$ since there is no angular dependence in the unpolarized rate.

We now turn to the IR–divergent part given in (2.136).

$$\begin{aligned} d\Gamma_{r,div} &= \frac{1}{2m_t}\frac{1}{(2\pi)^5}R_3(p_t;p_b,k,p_H)\,|\mathcal{M}_0|^2\,|M|^2_{SGF} \\ &= \frac{1}{2m_t}\frac{1}{(2\pi)^5}|\mathcal{M}_0|^2\, m_t^2\, dz\,\frac{\sqrt{\lambda(1,y^2,z)}}{8}2\pi\, d\cos\theta_P R_2(P;p_b,k)\,|M|^2_{SGF} \\ &= \frac{-1}{4\pi m_t}|\mathcal{M}_0|^2\,\frac{\alpha_s}{8\pi}C_F\,\frac{m_t^2}{2\pi}\,d\cos\theta_P\int\limits_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz\,\sqrt{\lambda(1,y^2,z)}\int dR_2(P;p_b,k)\,\Delta_{SGF}\,, \\ &= \frac{1}{4\pi m_t}|\mathcal{M}_0|^2\,\frac{\alpha_s}{8\pi}C_F\,S(\Lambda)\,d\cos\theta_P \end{aligned} \quad (2.162)$$

where we have defined Δ_{SGF} and $S(\Lambda)$ as

$$\Delta_{SGF} := -(4\pi\alpha_s C_F)^{-1}|M|^2_{SGF}\,, \quad (2.163)$$

$$S(\Lambda) := -\frac{m_t^2}{2\pi}\int\limits_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz\,\sqrt{\lambda(1,y^2,z)}\int dR_2(P;p_b,k)\,\Delta_{SGF}\,. \quad (2.164)$$

In Δ_{SGF} we replace p_b by $p_b = P-k$. Because Δ_{SGF} is IR–divergent we can not neglect the gluon mass k^2 completely as we did in the IR–convergent case. However the gluon mass in $(p_b\cdot k)=(k\cdot P)-k^2\sim(P\cdot k)$ can be neglected since $k^2\ll(k\cdot P)$. As before k^2 in the numerator can be dropped since we have at most a logarithmic IR–divergence of $\ln(m_g/m_t)$. Thus Δ_{SGF} becomes

$$\begin{aligned} \Delta_{SGF} &= \frac{m_t^2}{(k\cdot p_t)^2}+\frac{(P-k)^2}{(k\cdot P-k^2)^2}-2\frac{P\cdot p_t-k\cdot p_t}{(k\cdot P-k^2)(k\cdot p_t)} \\ &\approx \frac{m_t^2}{(k\cdot p_t)^2}+\frac{P^2-2P\cdot k}{(k\cdot P)^2}-2\frac{P\cdot p_t-k\cdot p_t}{(k\cdot P)(k\cdot p_t)} \\ &\approx \frac{m_t^2}{(k\cdot p_t)^2}+\frac{P^2}{(k\cdot P)^2}-2\frac{P\cdot p_t}{(k\cdot P)(k\cdot p_t)}. \end{aligned} \quad (2.165)$$

In the P–rest frame $(k \cdot P) = m_t\sqrt{z}E_g^*$. The two–body phase space integration $dR_2(P;p_b,k)$ is done the with help of the basic integrals I_n defined in (2.152). Thus we obtain

$$m_t^2 \int dR_2(P;p_b,k)\Delta_{SGF} = \hat{I}_{-2}(z,\Lambda) + \frac{z}{z\,\hat{E}_g^{*2}}\hat{I}_0(z,\Lambda) - \frac{1-y^2+z}{\sqrt{z}\,\hat{E}_g^*}\hat{I}_{-1}(z,\Lambda)\,, \qquad (2.166)$$

where[5] the basic integrals $\hat{I}_n(z,\Lambda)$ are listed in Appendix D.1. The *hat* denotes that these basic integrals are IR–divergent. When we do the z integration we would like to eliminate the z–dependent terms multiplying the IR–divergent basic integrals $\hat{I}_n(z,\Lambda)$ using the "plus" prescription. There are two types of z–dependent terms. One is $\sqrt{\lambda(1,y^2,z)}$ coming from the phase space integration and the others are the coefficients of $\hat{I}_n(z,\Lambda)$. Let us begin with $\sqrt{\lambda(1,y^2,z)}$.

$$\begin{aligned} m_t^2 \int\limits_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz\,\sqrt{\lambda(1,y^2,z)} \int dR_2(P;p_b,k)\,\Delta_{SGF} \\ = \underbrace{\int\limits_{\epsilon^2}^{(1-y)^2} dz\,\sqrt{\lambda(1,y^2,z)}\Big[m_t^2 \int dR_2(P;p_b,k)\,\Delta_{SGF}\Big]_+}_{IR-convergent} \\ + m_t^2\sqrt{\lambda(1,y^2,\epsilon^2)} \int\limits_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz \int dR_2(P;p_b,k)\,\Delta_{SGF}\,, \end{aligned} \qquad (2.167)$$

where the "plus" prescription is defined by

$$\begin{aligned} \int dz\,\sqrt{\lambda(1,y^2,z)}\Big[m_t^2 \int dR_2(P;p_b,k)\,\Delta_{SGF}\Big]_+ \\ := \int dz\Big(\sqrt{\lambda(1,y^2,z)} - \sqrt{\lambda(1,y^2,\epsilon^2)}\Big)\, m_t^2 \int dR_2(P;p_b,k)\Delta_{SGF}\,. \end{aligned} \qquad (2.168)$$

The integral in (2.168) is IR–convergent. We perform the two–body phase space integration $R_2(P;p_b,k)$ with the help of the basic integrals $I_n(z)$ defined in 2.152 and obtain

$$m_t^2 \int dR_2(P;p_b,k)\Delta_{SGF}(\Lambda=0) = I_{-2} + \frac{z}{z\,\hat{E}_g^{*2}} I_0 - \frac{1-y^2+z}{\sqrt{z}\,\hat{E}_g^*} I_{-1}\,. \qquad (2.169)$$

Substituting the IR–convergent basic integrals I_{-2}, I_{-1} and I_0 from (D.13), (D.14) and (D.15) one obtains

$$m_t^2 \int dR_2(P;p_b,k)\Delta_{SGF} = \frac{2\pi}{z-\epsilon^2}\left\{2 - \frac{1+z-y^2}{\sqrt{(1,y^2,z)}} \ln\left(\frac{1-y^2+z+\sqrt{\lambda(1,y^2,z)}}{1-y^2+z-\sqrt{\lambda(1,y^2,z)}}\right)\right\}. \qquad (2.170)$$

[5]Note that the factor z in $\frac{z}{z\,\hat{E}_g^{*2}}\hat{I}_0(z,\Lambda)$ has not been canceled because we want to calculate a more general form of integral $\frac{1}{z\,\hat{E}_g^{*2}}\hat{I}_0(z,\Lambda)$

The z–integration of (2.168) is performed with the help of the basic integrals defined in (2.160):

$$\int_{\epsilon^2}^{(1-y)^2} dz\,\sqrt{\lambda(1,y^2,z)}\Big[m_t^2\int dR_2(P;p_b,k)\Delta_{SGF}\Big]_+ = 2\pi\Big\{2\,R(-1)-(1+\epsilon^2-y^2)\,S(0)$$
$$-S(0,0)+\sqrt{\lambda(1,y^2,\epsilon^2)}\,\left[(1+\epsilon^2-y^2)\,S(1)+S(0,1)-2\,R(0)\right]\Big\}. \qquad (2.171)$$

Now let us eliminate the z terms in the coefficients of the basic integrals $\hat{I}_n(z,\Lambda)$ in (2.167) using the subtraction technique familiar from the plus prescription:

$$m_t^2\int_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz\,dR_2(P;p_b,k)\,\Delta_{SGF}$$
$$=\int_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz\left[\hat{I}_{-2}(z,\Lambda)+\frac{z}{z\,\hat{E}_g^{*2}}\hat{I}_0(z,\Lambda)-\frac{1-y^2+z}{\sqrt{z}\,\hat{E}_g^*}\hat{I}_{-1}(z,\Lambda)\right]$$
$$=\int_{\epsilon^2}^{(1-y)^2} dz\left[\frac{z-\epsilon^2}{z\,\hat{E}_g^{*2}}I_0(z)-\frac{z-\epsilon^2}{\sqrt{z}\,\hat{E}_g^*}I_{-1}(z)\right]$$
$$+\int_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz\left[\hat{I}_{-2}(z,\Lambda)+\frac{\epsilon^2}{z\,\hat{E}_g^{*2}}\hat{I}_0(z,\Lambda)-\frac{1-y^2+\epsilon^2}{\sqrt{z}\,\hat{E}_g^*}\hat{I}_{-1}(z,\Lambda)\right]$$
$$=\pi\,\Sigma(\Lambda,z_m)\Big|_{z_m=(1-y)^2}+\underbrace{\int_{\epsilon^2}^{(1-y)^2} dz\left[\frac{z-\epsilon^2}{z\,\hat{E}_g^{*2}}I_0(z)-\frac{z-\epsilon^2}{\sqrt{z}\,\hat{E}_g^*}I_{-1}(z)\right]}_{IR-convergent}, \qquad (2.172)$$

where[6]

$$\Sigma(\Lambda,z_m):=\frac{1}{\pi}\int_{(\epsilon+\Lambda)^2}^{z_m} dz\left[\hat{I}_{-2}(z,\Lambda)+\frac{\epsilon^2}{z\,\hat{E}_g^{*2}}\hat{I}_0(z,\Lambda)-\frac{1-y^2+\epsilon^2}{\sqrt{z}\,\hat{E}_g^*}\hat{I}_{-1}(z,\Lambda)\right]. \qquad (2.173)$$

For the IR–convergent part we substitute the basic integrals I_n from Appendix D.1. The

[6]The hat on the basic integrals $\hat{I}_n$ for the two–body phase space integration $dR_2(P;p_b,k)$ denote that they are IR–divergent (see Appendix D.1).

z integration is done with the help (2.160). One obtains

$$\int\limits_{\epsilon^2}^{(1-y)^2} dz \left[\frac{z-\epsilon^2}{z\,\hat{E}_g^{*2}} I_0(z) - \frac{z-\epsilon^2}{\sqrt{z}\,\hat{E}_g^{*}} I_{-1}(z) \right] \tag{2.174}$$

$$= 2\pi \int dz \left\{ \frac{1}{z} - \frac{1}{\sqrt{(1,y^2,z)}} \ln\left(\frac{1-y^2+z+\sqrt{\lambda(1,y^2,z)}}{1-y^2+z-\sqrt{\lambda(1,y^2,z)}} \right) \right\}$$

$$= 2\pi \Big\{ R(-1,0) - S(0,1) \Big\}. \tag{2.175}$$

Substituting (2.175) into (2.172) we have

$$m_t^2 \int\limits_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz\, dR_2(P;p_b,k)\, \Delta_{SGF} = \pi\, \Sigma(\Lambda, z_m)\Big|_{z_m=(1-y)^2} + 2\pi \Big\{ R(-1,0) - S(0,1) \Big\}. \tag{2.176}$$

Note that, when defining the IR–divergent integral $\Sigma(\Lambda, z_m)$, we take a general upper limit z_m instead of the specific value $z_m = (1-y)^2$. As we will see in other decays which have a similar hadronic structure, as e.g. in $t \to b + \ell^+ + \nu_\ell$, the same IR–divergent integrals will appear but with different upper integration limits. Therefore we calculate these integrals with a general upper limit z_m and substitute specific values for z_m in specific applications.

Collecting everything we obtain

$$\begin{aligned} S(\Lambda) &= \frac{m_t^2}{2\pi} \int dz \sqrt{\lambda(1,y^2,z)}\, dR_2(P;p_b,k) \Delta_{SGF} \\ &= 2R(-1) - S(0,0) - (1-y^2+\epsilon^2) S(0) \\ &\quad + \sqrt{\lambda(1,y^2,\epsilon^2)} \left[R(-1,0) - 2R(0) + (1-y^2+\epsilon^2) S(1) + \frac{1}{2} \Sigma(\Lambda, z_m)\Big|_{z_m=(1-y)^2} \right], \end{aligned} \tag{2.177}$$

The calculations of the IR–divergent integral $\Sigma(\Lambda, z_m)$ will be done in the next subsection.

2.3.3 Integration of the soft gluon factor

The IR–divergent soft gluon factor $\Sigma(\Lambda, z_m)$ contains the following three basic integrals:

$$\int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz\, I_{-2}(z,\Lambda), \quad \int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz\, \frac{\epsilon^2}{z\,\hat{E}_g^{*2}} I_0(z,\Lambda), \quad \int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz\, \frac{1}{\sqrt{z}\,\hat{E}_g^{*}} I_{-1}(z,\Lambda)$$

We calculate them one by one.

(i) $\mathcal{I}_0 := \dfrac{1}{\pi} \displaystyle\int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz\, \frac{\epsilon^2}{z\,\hat{E}_g^{*2}} I_0(z,\Lambda)$

For this integral it turns out that integrand is simpler with z in the numerator instead of ϵ^2 because the denominator also has a z factor. These two cases differ by a simple term.

$$\begin{aligned}\frac{1}{\pi}\int_{(\epsilon+\Lambda)^2}^{z_m} dz\,\frac{\epsilon^2}{z\,\hat{E}_g^{*2}} I_0(z,\Lambda) &= \frac{1}{\pi}\int_{(\epsilon+\Lambda)^2}^{z_m} dz\,\frac{z}{z\,\hat{E}_g^{*2}} I_0(z,\Lambda) - \frac{1}{\pi}\int_{\epsilon^2}^{z_m} dz\,\frac{z-\epsilon^2}{z\,\hat{E}_g^{*2}} I_0(z) \\ &= \frac{1}{\pi}\int_{(\epsilon+\Lambda)^2}^{z_m} dz\,\frac{1}{\hat{E}_g^{*2}} I_0(z,\Lambda) - \frac{1}{\pi}2\pi\int_{\epsilon^2}^{z_m} dz\,\frac{1}{z} \\ &= -2\ln(\frac{z_m}{\epsilon^2}) + \frac{1}{\pi}\int_{(\epsilon+\Lambda)^2}^{z_m} dz\,\frac{1}{\hat{E}_g^{*2}} I_0(z,\Lambda)\,. \end{aligned} \tag{2.178}$$

Note that we obtain $\frac{z-\epsilon^2}{\hat{E}_g^{*2}} I_0(z) = 2\pi$ using $I_0(z)$ from (D.15). Substituting $I_0(z,\Lambda)$ from D.10 and $\hat{E}_g^*$ from (2.149) we have

$$\frac{1}{\pi}\int dz\,\frac{z}{z\,\hat{E}_g^{*2}} I_0(z,\Lambda) = \int dz\, 2\sqrt{z^2 + (\epsilon^2-\Lambda^2)^2 - 2\,z\,(\epsilon^2+\Lambda^2)}\big(z-\epsilon^2+\Lambda^2\big)^2\,. \tag{2.179}$$

After a change of variables $z \longrightarrow t + \epsilon^2 - \Lambda^2$, the integral becomes

$$\int dt\,\sqrt{t^2 - 4\,t\,\Lambda^2 - 4\,\epsilon^2\,\Lambda^2 + 4\,\Lambda^4}t^2\,.$$

The result for of this integration is given by [57]

$$\int dt\,\frac{\sqrt{a\,t^2+b\,t+c}}{t^2} = -\frac{\sqrt{X}}{t} - \frac{b}{2\sqrt{-c}}\arcsin(\frac{2\,c+b\,t}{t\sqrt{-\Delta}}) + \sqrt{a}\,\ln\Big(b+2\,a\,t+2\sqrt{a\,X}\Big)\,, \tag{2.180}$$

for $a > 0,\ \Delta < 0$ where, in the present case[7],

$$a = 1,\ b = -4\Lambda^2,\ c = -4\Lambda^2(\epsilon^2-\Lambda^2),\ X = a\,t^2 + b\,t + c,\ \Delta = 4a\,c - b^2.$$

The integral (2.180)is reexpressed in terms of $z = z(t)$. Then take the integration limit and Laurent series expand the expression in Λ around $\Lambda = 0$ up to $\mathcal{O}(\Lambda^0)$. This will give

$$\frac{1}{\pi}\int_{(\epsilon+\Lambda)^2}^{z_m} dz\,\frac{1}{\hat{E}_g^{*2}} I_0(z,\Lambda) = 2\ln(\frac{z_m-\epsilon^2}{\epsilon\,\Lambda}) - 2. \tag{2.181}$$

Inserting (2.181) into (2.178) we obtain

$$\mathcal{I}_0(\Lambda, z_m) = 2\ln(\frac{z_m-\epsilon^2}{\epsilon\,\Lambda}) - 2\ln(\frac{z_m}{\epsilon^2}) - 2. \tag{2.182}$$

[7] $\Delta < 0$ since $a > 0,\ c < 0$.

$$\boxed{\textbf{(ii)} \quad \mathcal{I}_{-2} := \frac{1}{\pi} \int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz\, I_{-2}(z,\Lambda)}$$

Inserting $I_{-2}(z,\Lambda)$ from Appendix (D.12) we have

$$\mathcal{I}_{-2} = \frac{1}{\pi} \int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz \, \frac{2\sqrt{\left(z-\epsilon^2\right)^2 - 2\,(z+\epsilon^2)\,\Lambda^2 + \Lambda^4}}{\left(z-\epsilon^2\right)^2 + (1+y^4+z^2-2\,y^2\,(1+z)-2\,\epsilon^2)\,\Lambda^2+\Lambda^4} \tag{2.183}$$

After a change of the integration variable $z \to v$ with

$$z = \epsilon^2 + \Lambda^2 + \frac{(1+v)\,\epsilon\,\Lambda}{\sqrt{v}}\,, \tag{2.184}$$

the integrand becomes

$$\frac{1}{v}\frac{(v-1)^2\epsilon^2}{\epsilon^2 + v[(1-y^2)^2+(v-2y^2)\epsilon^2+\epsilon^4] + \Lambda(\epsilon+v\epsilon+\sqrt{v}\Lambda)[2\sqrt{v}(1-y^2+\epsilon^2)+\Lambda(\epsilon+v\epsilon+\sqrt{v}\Lambda)]},$$

with the integration limits $1 < v < \frac{\left(z_m-\epsilon^2\right)^2}{\epsilon^2\Lambda^2}$ for the new variable v in the $\Lambda \to 0$ limit. Since $\Lambda \ll 1 < v$ we can neglect any term linear in Λ that is added to v. An inspection of the denominator shows that it is completely safe to set all Λ to zero. Thus the integral simplifies to

$$\mathcal{I}_{-2} = \int dv \left[\frac{1}{v} - \frac{\left(1-y^2+\epsilon^2\right)^2}{v^2\epsilon^2 + v(1-2y^2+y^4-2y^2\epsilon^2+\epsilon^4)+\epsilon^2}\right], \tag{2.185}$$

which can be integrated using the general form [57]

$$\int dv \, \frac{1}{a\,v^2+b\,v+c} = \frac{1}{\sqrt{-\Delta}} \ln(\frac{b+2\,a\,v-\sqrt{-\Delta}}{b+2\,a\,v+\sqrt{-\Delta}}) \tag{2.186}$$

for $\Delta = 4a\,c - b^2 < 0$. In our case[8]

$$a = \epsilon^2,\;\; b = 1-2y^2+y^4-2y^2\epsilon^2+\epsilon^4,\;\; c=\epsilon^2,\;\; \Delta = -\left(1-y^2+\epsilon^2\right)^2[(1-y^2+\epsilon^2)^2-4\epsilon^2].$$

The inverse of $z = z(v)$ in (2.184) is given by

$$v = \frac{\left(z-\epsilon^2\right)^2 - 2\,z\,\Lambda^2 + \Lambda^4 + \sqrt{-4\,\epsilon^4\,\Lambda^4 + \left(-z^2+2\,z\,\epsilon^2-\epsilon^4+2\,z\,\Lambda^2-\Lambda^4\right)^2}}{2\,\epsilon^2\,\Lambda^2}\,.$$

The terms a, b, c and Δ are also expressed in terms of z. We then take the integration limit and Taylor expand the expression in Λ around $\Lambda = 0$ up to $\mathcal{O}(\Lambda^0)$. This will give

$$\mathcal{I}_{-2}(\Lambda, z_m) = 2\ln(\frac{z_m-\epsilon^2}{\epsilon\,\Lambda}) - 2\frac{1-y^2+\epsilon^2}{\sqrt{\lambda(1,y^2,\epsilon^2)}} \ln\left(\frac{1-y^2+\epsilon^2+\sqrt{\lambda(1,y^2,\epsilon^2)}}{1-y^2+\epsilon^2-\sqrt{\lambda(1,y^2,\epsilon^2)}}\right). \tag{2.187}$$

[8] $\Delta < 0$ since
$m_t^4\Delta = -m_t^4\left(1-y^2+\epsilon^2\right)^2[(1-y^2+\epsilon^2)^2-4\epsilon^2] = -(2E_b)^2\left[(2E_b)^2-4m_b^2\right] = -16E_b^2|\vec{p}_b|^2 < 0$

$$\textbf{(iii)} \quad \mathcal{I}_{-1} = \frac{1}{\pi} \int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz \, \frac{1}{\sqrt{z}\,\hat{E}_g^*} I_{-1}(z,\Lambda)$$

Inserting the basic integral I_{-1} from (D.11) and $\hat{E}_g^*$ from (2.149) gives

$$\mathcal{I}_{-1} = \int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz \, \frac{2}{(z-\epsilon^2+\Lambda^2)\sqrt{\lambda(1,y^2,z)}} \times \times \ln\left[\frac{(1-y^2+z)(z-\epsilon^2+\Lambda^2)+\sqrt{\lambda(1,y^2,z)}\sqrt{\lambda(z,\epsilon^2,\Lambda^2)}}{(1-y^2+z)(z-\epsilon^2+\Lambda^2)-\sqrt{\lambda(1,y^2,z)}\sqrt{\lambda(z,\epsilon^2,\Lambda^2)}}\right]. \tag{2.188}$$

This integral has a complicated structure with the logarithmic function and different square roots. We simplify the integrand of this integral by further isolating the IR–divergent part of the integrand with the help of a change in the integration variable $z \to t$ where $z = t\,\epsilon\,\Lambda + (\epsilon+\Lambda)^2$. We then Laurent series expand the integrand in Λ around $\Lambda = 0$ and obtain the IR–divergent part:

$$\frac{2}{\Lambda\,\epsilon(2+t)\sqrt{\lambda(1,y^2,\epsilon^2)}} \ln\left[\frac{\epsilon(2+t)(1-y^2+\epsilon^2)+\sqrt{4t\epsilon^2+t^2\epsilon^2}\sqrt{\lambda(1,y^2,\epsilon^2)}}{\epsilon(2+t)(1-y^2+\epsilon^2)-\sqrt{4t\epsilon^2+t^2\epsilon^2}\sqrt{\lambda(1,y^2,\epsilon^2)}}\right]. \tag{2.189}$$

The IR–divergence is explicit in the Laurent series with negative powers in Λ. Changing back to the variable z, the IR–divergent part (2.189) is given by

$$\mathcal{I}_{-1}^{div} = \frac{1}{\sqrt{\lambda(1,y^2,\epsilon^2)}} \int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz \, \frac{2}{(z-\epsilon^2-\Lambda^2)} \times \times \ln\left[\frac{(1-y^2+\epsilon^2)(z-\epsilon^2-\Lambda^2)+\sqrt{\lambda(1,y^2,\epsilon^2)}\sqrt{\lambda(z,\epsilon^2,\Lambda^2)}}{(1-y^2+\epsilon^2)(z-\epsilon^2-\Lambda^2)-\sqrt{\lambda(1,y^2,\epsilon^2)}\sqrt{\lambda(z,\epsilon^2,\Lambda^2)}}\right]. \tag{2.190}$$

Subtracting the divergent part from the total integral gives the convergent part $\mathcal{I}_{-1}^{conv}$. Then the integral $\mathcal{I}_{-1}$ is written as the sum of the divergent and convergent pieces as

$$\mathcal{I}_{-1} = \mathcal{I}_{-1}^{div} + \mathcal{I}_{-1}^{conv} \tag{2.191}$$

For the convergent part we set Λ to zero and obtain

$$\mathcal{I}_{-1}^{conv} = \int\limits_{\epsilon^2}^{z_m} dz \, \frac{2}{(z-\epsilon^2)\sqrt{\lambda(1,y^2,z)}} \ln\left(\frac{1-y^2+z+\sqrt{\lambda(1,y^2,z)}}{1-y^2+z-\sqrt{\lambda(1,y^2,z)}}\right) + \frac{2}{\sqrt{\lambda(1,y^2,z)}} \ln\left(\frac{1-y^2+\epsilon^2+\sqrt{\lambda(1,y^2,\epsilon^2)}}{1-y^2+\epsilon^2-\sqrt{\lambda(1,y^2,\epsilon^2)}}\right) \int dz \, \frac{1}{z-\epsilon^2}$$

$$= \quad 2\tilde{S}(0) - \frac{2}{\sqrt{\lambda(1,y^2,z)}} \ln\left(\frac{1-y^2+\epsilon^2+\sqrt{\lambda(1,y^2,\epsilon^2)}}{1-y^2+\epsilon^2-\sqrt{\lambda(1,y^2,\epsilon^2)}}\right)\tilde{R}(0)\,, \quad (2.192)$$

where

$$\tilde{R}(0) \quad := \int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz \frac{1}{z-\epsilon^2}\,, \quad (2.193)$$

$$\tilde{S}(0) \quad := \int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz \frac{1}{(z-\epsilon^2)\sqrt{\lambda(1,y^2,z)}} \ln\left(\frac{1-y^2+z+\sqrt{\lambda(1,y^2,z)}}{1-y^2+z-\sqrt{\lambda(1,y^2,z)}}\right). \quad (2.194)$$

The tilde notation is used to emphasize that the integrals have a general upper limit z_m as compared to (2.160) where the upper integration limit $z_m = (1-y)^2$ has been used. The basic integrals $\tilde{R}(0)$ and $\tilde{S}(0)$ are calculated in (D.53) and (D.54) in Appendix D.2. Both $\tilde{R}(0)$ and $\tilde{S}(0)$ are divergent at $z = \epsilon^2$. These divergences are the result of partial fractioning of the convergent integral $\mathcal{I}_{-1}^{conv}$. As usual we regulate these divergences by a gluon mass Λ. The divergent terms cancel and we obtain

$$\mathcal{I}_{-1}^{conv} \quad = \quad \frac{1}{\bar{p}_3}\Big\{\text{Li}_2(1-\frac{p_-(z_m)}{\bar{p}_-}) + \text{Li}_2(1-\frac{p_+(z_m)}{\bar{p}_-}) - \text{Li}_2(1-\frac{p_-(z_m)}{\bar{p}_+})$$
$$+\text{Li}_2(1-\frac{\bar{p}_-}{\bar{p}_+}) - \text{Li}_2(1-\frac{p_+(z_m)}{\bar{p}_+}) - \text{Li}_2(1-\frac{\bar{p}_+}{\bar{p}_-})\Big\}. \quad (2.195)$$

Let us now turn to the divergent part. To eliminate the square root involving the integration variable z, we change the integration variable $z \to v$ where

$$z = \epsilon^2 + \Lambda^2 + \frac{(1+v)\,\epsilon\,\Lambda}{\sqrt{v}}. \quad (2.196)$$

The integral becomes

$$\mathcal{I}_{-1}^{div} \quad = \quad \int dv \frac{1}{\sqrt{\lambda(1,y^2,\epsilon^2)}}\left(-\frac{1}{v}+\frac{2}{1+v}\right)\ln(\frac{\bar{p}_- + v\,\bar{p}_+}{v\,\bar{p}_- + \bar{p}_+})$$
$$= \quad \frac{1}{\sqrt{\lambda(1,y^2,\epsilon^2)}}\int dv \left\{\frac{\ln(v\,\bar{p}_- + \bar{p}_+)}{v} - \frac{2\,\ln(v\,\bar{p}_- + \bar{p}_+)}{1+v}\right.$$
$$\left. -\frac{\ln(\bar{p}_- + v\,\bar{p}_+)}{v} + \frac{2\,\ln(\bar{p}_- + v\,\bar{p}_+)}{1+v}\right\}. \quad (2.197)$$

The integration limit for the new variable v is $1 < v < \frac{\left(z_m-\epsilon^2\right)^2}{\epsilon^2\Lambda^2}$ in the $\Lambda \to 0$ limit. These integrations can easily be done and the result is

$$\mathcal{I}_{-1}^{div} = \frac{1}{\sqrt{\lambda(1,y^2,\epsilon^2)}}\Bigg\{ \ln(v)\,\ln(\bar{p}_+) - \ln(v)\,\ln(\bar{p}_-) - 2\,\ln(1+v)\,\ln(\bar{p}_+ - \bar{p}_-)$$
$$+2\,\ln\Big[\frac{(1+v)\,\bar{p}_+}{\bar{p}_+ - \bar{p}_-}\Big]\,\ln(\bar{p}_- + v\,\bar{p}_+) + 2\,\mathrm{Li}_2\Big[\frac{(1+v)\,\bar{p}_-}{\bar{p}_- - \bar{p}_+}\Big]$$
$$-\mathrm{Li}_2\Big(-\frac{v\,\bar{p}_-}{\bar{p}_+}\Big) + \mathrm{Li}_2\Big(-\frac{v\,\bar{p}_+}{\bar{p}_-}\Big) + 2\,\mathrm{Li}_2\Big[\frac{\bar{p}_- + v\,\bar{p}_+}{\bar{p}_- - \bar{p}_+}\Big]\Bigg\}\Bigg|_1^{\frac{(z_m-\epsilon^2)^2}{\epsilon^2\Lambda^2}}. \qquad (2.198)$$

We insert the integration limits and Laurent series expand the result in Λ around $\Lambda = 0$ up to $\mathcal{O}(\Lambda^0)$. After expansion and simplification using dilog identities we write the final result in terms of the compact kinematic variables defined in (2.2) and (2.4):

$$\mathcal{I}_{-1}^{div} = \frac{1}{\bar{p}_3}\Bigg\{\mathrm{Li}_2(1 - \frac{p_+(z_m)}{\bar{p}_-}) + 2\,\ln(\frac{z_m - \epsilon^2}{\epsilon\,\Lambda})\,\bar{Y}_p + \bar{Y}_p^2\Bigg\}. \qquad (2.199)$$

Collecting the IR–convergent and IR–divergent result from (2.195) and (2.199) and inserting them into (2.191) we have

$$\mathcal{I}_{-1}(\Lambda, z_m) = \frac{1}{\bar{p}_3}\Bigg\{\mathrm{Li}_2(1 - \frac{p_-(z_m)}{\bar{p}_-}) + \mathrm{Li}_2(1 - \frac{p_+(z_m)}{\bar{p}_-}) - \mathrm{Li}_2(1 - \frac{p_-(z_m)}{\bar{p}_+})$$
$$+\mathrm{Li}_2(1 - \frac{\bar{p}_-}{\bar{p}_+}) - \mathrm{Li}_2(1 - \frac{p_+(z_m)}{\bar{p}_+}) + 2\,\ln(\frac{z_m - \epsilon^2}{\epsilon\,\Lambda})\,\bar{Y}_p + \bar{Y}_p^2\Bigg\}. \qquad (2.200)$$

Finally substituting the results of (**i**), (**ii**) and (**iii**) into (2.173) we obtain

$$\Sigma(\Lambda, z_m) = \frac{2\bar{p}_0}{\bar{p}_3}\Big[\mathrm{Li}_2(1 - \frac{p_-(z_m)}{\bar{p}_+}) - \mathrm{Li}_2(1 - \frac{p_-(z_m)}{\bar{p}_-}) - \mathrm{Li}_2(1 - \frac{p_+(z_m)}{\bar{p}_-})$$
$$+\mathrm{Li}_2(1 - \frac{p_+(z_m)}{\bar{p}_+}) - \mathrm{Li}_2(1 - \frac{\bar{p}_-}{\bar{p}_+}) - \bar{Y}_p\,(1 + \bar{Y}_p)\Big]$$
$$+4\ln(\frac{z_m - \epsilon^2}{\epsilon\Lambda})\Big[1 - \frac{\bar{p}_0\bar{Y}_p}{\bar{p}_3}\Big] - 2\ln(\frac{z_m}{\epsilon^2}) - 2. \qquad (2.201)$$

In the decay $t \to b + H^+$ the upper limit of the z–integration is $z_m = (1-y)^2$. For $z_m = (1-y)^2$ one obtains

$$\hat{\Sigma}(\Lambda) := \Sigma(\Lambda, z_m)\Big|_{z_m=(1-y)^2} = \frac{2\bar{p}_0}{\bar{p}_3}\Big[2\mathrm{Li}_2(1 - \frac{1-y}{\bar{p}_+}) - 2\mathrm{Li}_2(1 - \frac{1-y}{\bar{p}_-}) - \mathrm{Li}_2(1 - \frac{\bar{p}_-}{\bar{p}_+})$$
$$-\bar{Y}_p\,(1 + \bar{Y}_p)\Big] + 4\ln(\frac{(1-y)^2 - \epsilon^2}{\epsilon\Lambda})\Big[1 - \frac{\bar{p}_0\bar{Y}_p}{\bar{p}_3}\Big] - 4\ln(\frac{1-y}{\epsilon}) - 2. \qquad (2.202)$$

Thus all the necessary integrations in (2.177) have been done. Collecting everything together and further simplifying the results using standard dilog identities we obtain

$$S(\Lambda) = 4\bar{p}_3\Big[\ln(\frac{4\bar{p}_3^2}{\epsilon y}) - 2 - \ln(\Lambda)\Big] - 2\bar{Y}_w + 2\epsilon^2\left(\bar{Y}_p + \bar{Y}_w\right) \tag{2.203}$$

$$+2\bar{p}_0\Big\{2\mathrm{Li}_2(1-\frac{\bar{p}_-}{\bar{p}_+}) - \mathrm{Li}_2(1-\frac{\bar{w}_-}{\bar{w}_+}) + \mathrm{Li}_2(\frac{2\bar{p}_3}{\bar{p}_+\bar{w}_+}) - 2\bar{Y}_p\Big[\frac{1}{2} - \ln(\Lambda) + \ln(\frac{4\bar{p}_3^2}{\bar{p}_+\bar{w}_+})\Big] + \bar{Y}_p^2\Big\}.$$

Next we rewrite the IR–divergent rate given in (2.162). Using

$$|\mathcal{M}_0|^2 = \mathrm{PS}_2^{-1}\,\Gamma^{(0)}\,, \tag{2.204}$$

$$\frac{d\Gamma^{(0)}}{d\cos\theta_P} = \frac{1}{2}\left(\Gamma_{Born}^{unpol} + P\,\Gamma_{Born}^{pol}\cos\theta_P\right)\,, \tag{2.205}$$

we obtain

$$\begin{aligned} d\Gamma_{r,div} &= -\frac{1}{8\pi m_t}\frac{\alpha_s}{4\pi}C_F\,d\cos\theta_P\,|\mathcal{M}_0|^2\,S(\Lambda) \\ &= \frac{1}{2}\left(\Gamma_{r,div}^{unpol} + P\,\Gamma_{r,div}^{pol}\cos\theta_P\right)\,d\cos\theta_P\,, \end{aligned} \tag{2.206}$$

where

$$\Gamma_{r,div}^{unpol} = -\frac{1}{4\pi m_t}\frac{\alpha_s}{4\pi}C_F\,\mathrm{PS}_2^{-1}\,\Gamma_{Born}^{unpol}\,S(\Lambda)\,, \tag{2.207}$$

$$\Gamma_{r,div}^{pol} = -\frac{1}{4\pi m_t}\frac{\alpha_s}{4\pi}C_F\,\mathrm{PS}_2^{-1}\,\Gamma_{Born}^{pol}\,S(\Lambda)\,. \tag{2.208}$$

What remains to be done is to integrate the IR–convergent polarized contribution. Its matrix element squared $|\mathcal{M}_{r,conv}^{pol}|^2$ is given in (2.140). Similar to the unpolarized case, we write $|\mathcal{M}_{r,conv}^{pol}|^2$ in terms of scalar products involving p_t, P, k and s_t. Since $p_t\cdot s_t = 0$ one has $p_H\cdot s_t = -P\cdot s_t$. We also have $p_H\cdot s_t|_{k=0} = -P\cdot s_t|_{z=\epsilon^2}$. Substituting the scalar product $P\cdot s_t$ from (2.153) we obtain

$$\begin{aligned} |\mathcal{M}_{r,conv}^{pol}|^2 = {} & 8\pi\,\alpha_s\,C_F(2a\,b\,m_t)\times \\ & \Big\{\Big[m_t^2\frac{1+y^2-\epsilon^2}{2}\frac{1}{(p_t\cdot k)^2} - \Big(1+m_t^2\frac{1-y^2-\epsilon^2}{k\cdot P}\Big)\frac{1}{p_t\cdot k} + \frac{1}{k\cdot P}\Big](k\cdot s_t) \\ & +m_t\frac{\sqrt{\lambda(1,y^2,z)}}{2}\cos\theta_P\Big(\frac{1}{k\cdot P} - \frac{1}{p_t\cdot k}\Big) \\ & +\frac{1}{2m_t}\cos\theta_P\Big[\sqrt{\lambda(1,y^2,z)} - \sqrt{\lambda(1,y^2,\epsilon^2)}\Big]\Delta_{SGF}(\Lambda=0)\Big\}. \end{aligned} \tag{2.209}$$

Then the differential rate is

$$\begin{aligned} d\Gamma_{r,conv}^{pol} &= \frac{1}{2m_t}\frac{1}{(2\pi)^5}dR_3(p_t;p_b,k,p_H)\,|\mathcal{M}_{r,conv}^{pol}|^2 \\ &= \frac{1}{2m_t}\frac{1}{(2\pi)^5}m_t^2\,dz\,\frac{\sqrt{\lambda(1,y^2,z)}}{8}2\pi\,d\cos\theta_P dR_2(P;p_b,k)|\mathcal{M}_{r,conv}^{pol}|^2\,. \end{aligned} \tag{2.210}$$

The two body phase space integration $R_2(P;p_b,k)$ is performed with the help of the basic integrals (2.152) and (2.154):

$$d\Gamma^{pol}_{r,conv} = \frac{1}{2m_t}\frac{1}{(2\pi)^5}m_t^2\,dz\,\frac{\sqrt{\lambda(1,y^2,z)}}{8}2\pi\,d\cos\theta_P 8\pi\,\alpha_s\,C_F(2a\,b\,m_t)\times$$
$$\Bigg\{\Big[m_t^2\frac{1+y^2-\epsilon^2}{2}I^\mu_{-2} - \Big(1+m_t^2\frac{1-y^2-\epsilon^2}{k\cdot P}\Big)I^\mu_{-1} + \frac{1}{k\cdot P}I^\mu_0\Big]s_{t\,\mu}$$
$$+\ m_t\frac{\sqrt{\lambda(1,y^2,z)}}{2}\cos\theta_P\Big(\frac{1}{k\cdot P}I_0 - I_{-1}\Big)$$
$$+\ \frac{\cos\theta_P}{2m_t}\Big[\sqrt{\lambda(1,y^2,z)} - \sqrt{\lambda(1,y^2,\epsilon^2)}\Big]\int dR_2(P;p_b,k)\Delta_{SGF}(\Lambda=0)\Bigg\}\,, \qquad (2.211)$$

where $\int dR_2(P;p_b,k)\Delta_{SGF}(\Lambda=0)$ is given in (2.170). We substitute the basic integrals from Appendix D.1. Finally the z–integrations are obtained in terms of the basic integrals defined in (2.160). One obtains

$$\begin{aligned} d\Gamma^{pol}_{r,conv} \;=\; & -\frac{1}{8\pi m_t}\frac{\alpha_s}{4\pi}C_F 2ab\,m_t^2\Bigg\{-4\bar p_3\,R(-1) + 8\bar p_3^2\,R(0) + \frac{1}{4}(1-y^2)^2\epsilon^2 R(-2,0) \\ & -\frac{1}{4}\Big[(1-y^2)^2 + 2\,(3-y^2)\,\epsilon^2\Big]\,R(-1,0) - \frac{1}{4}(2+10\,y^2-\epsilon^2)\,R(0,0) + \frac{7}{4}R(1,0) \\ & +(1-y^2+\epsilon^2)\,2\bar p_3\,S(0) - (1-y^2+\epsilon^2)\,4\bar p_3^2\,S(1) + 2\bar p_3 S(0,0) \\ & +\frac{1}{2}\Big[4\,y^2\,\big(1-y^2\big) + \big(7+5\,y^2\big)\,\epsilon^2 - 2\,\epsilon^4\Big]\,S(0,1) \\ & -\frac{1}{2}\big(3-3\,y^2+\epsilon^2\big)\,S(1,1) - \frac{1}{2}S(2,1)\Bigg\}\cos\theta_P\,d\cos\theta_P\,. \end{aligned} \qquad (2.212)$$

2.3.4 Full real emission contributions

We write the full real emission contribution as

$$\frac{d\Gamma_r}{d\cos\theta_P} = \frac{1}{2}\left(\Gamma_r^{unpol} + P\,\Gamma_r^{pol}\cos\theta_P\right)\,. \qquad (2.213)$$

The unpolarized and the polarized parts are the sum of their respective IR–divergent and IR–convergent pieces.

$$\Gamma_r^{unpol} = \Gamma^{unpol}_{r,div} + \Gamma^{unpol}_{r,conv} \qquad (2.214)$$

$$\Gamma_r^{pol} = \Gamma^{pol}_{r,div} + \Gamma^{pol}_{r,conv} \qquad (2.215)$$

where $\Gamma^{unpol}_{r,div}$, $\Gamma^{unpol}_{r,conv}$, $\Gamma^{pol}_{r,div}$ and $\Gamma^{pol}_{r,conv}$ are given in (2.207), (2.161), (2.207) and (2.208), respectively. Collecting all of them we obtain

$$\begin{aligned}
\Gamma_r^{unpol} &= -\frac{1}{4\pi m_t}\frac{\alpha_s}{4\pi}C_F\Big\{m_t^2(a^2+b^2)\Big[\frac{3}{4}R(0,-1)+\frac{\epsilon^2(1-y^2)}{4}R(-2,-1) \\
&\quad -\frac{1-y^2+3\epsilon^2}{4}R(-1,-1)+\frac{1}{2}\epsilon^2 S(0,0)-\frac{1}{2}S(1,0)\Big]+\mathrm{PS}_2^{-1}\,\Gamma_{Born}^{unpol}\,S(\Lambda)\Big\}. \qquad (2.216)
\end{aligned}$$

$$\begin{aligned}
\Gamma_r^{pol} &= -\frac{1}{4\pi m_t}\frac{\alpha_s}{4\pi}C_F\Big\{2ab\,m_t^2\Big[-4\bar{p}_3\,R(-1)+8\bar{p}_3^2\,R(0)+\frac{1}{4}(1-y^2)^2\epsilon^2 R(-2,0) \\
&\quad -\frac{1}{4}\Big((1-y^2)^2+2\,(3-y^2)\,\epsilon^2\Big)\,R(-1,0)-\frac{1}{4}(2+10\,y^2-\epsilon^2)\,R(0,0)+\frac{7}{4}R(1,0) \\
&\quad -+(1-y^2+\epsilon^2)\,2\bar{p}_3\,S(0)-(1-y^2+\epsilon^2)\,4\bar{p}_3{}^2\,S(1)+2\bar{p}_3 S(0,0) \\
&\quad +\frac{1}{2}\Big(4\,y^2\,\left(1-y^2\right)+\left(7+5\,y^2\right)\,\epsilon^2-2\,\epsilon^4\Big)\,S(0,1)-\frac{1}{2}\left(3-3\,y^2+\epsilon^2\right)\,S(1,1) \\
&\quad -\frac{1}{2}S(2,1)\Big]+\mathrm{PS}_2^{-1}\,\Gamma_{Born}^{pol}\,S(\Lambda)\Big\}. \qquad (2.217)
\end{aligned}$$

2.4 Total $\mathcal{O}(\alpha_s)$ results

We write the total rate up to $\mathcal{O}(\alpha_s)$ as

$$\frac{d\Gamma}{d\cos\theta_P}=\frac{1}{2}\left(\Gamma^{unpol}+P\,\Gamma^{pol}\cos\theta_P\right) \qquad (2.218)$$

where the unpolarized and polarized parts are the sums of the respective LO rate and the NLO QCD corrections:

$$\Gamma^{unpol} = \Gamma_{Born}^{unpol}+\Gamma_{NLO}^{unpol} \qquad (2.219)$$

$$\Gamma_{NLO}^{pol} = \Gamma_{Born}^{pol}+\Gamma_{NLO}^{pol}\,. \qquad (2.220)$$

The LO rates Γ_{Born}^{unpol} and Γ_{Born}^{pol} are given in (2.31) and (2.32). The NLO QCD corrections Γ_{NLO}^{unpol} and Γ_{NLO}^{pol} are obtained by combining the respective virtual one–loop and the real contributions

$$\Gamma_{NLO}^{unpol} = \Gamma_v^{unpol}+\Gamma_r^{unpol} \qquad (2.221)$$

$$\Gamma_{NLO}^{pol} = \Gamma_v^{pol}+\Gamma_r^{pol}\,. \qquad (2.222)$$

The rates Γ_v^{unpol}, Γ_r^{unpol}, Γ_v^{pol} and Γ_r^{pol} are given in (2.128), (2.216), (2.129) and (2.217) respectively.

When simplifying the combination we first sum up the virtual one loop results with the IR–divergent results of the real emission. All the dilogs, double logs and the IR–divergences are contained in this sum. The dilogs are simplified using standard dilog identities [58]. The IR–divergences cancel between virtual and IR–divergent real contributions. The unpolarized $\mathcal{O}(\alpha_s)$ corrections are

$$\Gamma^{unpol}_{NLO} = \frac{\alpha_s}{8\pi^2} m_t C_F \Big[(a^2+b^2)\,\mathcal{U}_+ + \epsilon(a^2-b^2)\,\mathcal{U}_- - 6ab\,\bar{p}_0\bar{p}_3\,\ln\epsilon\Big]\,, \tag{2.223}$$

with the coefficients

$$\begin{aligned}
\mathcal{U}_+ &= 4\bar{p}_0^2\,\mathcal{D} + \bar{p}_0\left\{\frac{2\bar{p}_3}{y^2}\big(1-\epsilon^2+y^2\big)\ln\epsilon + \bar{p}_3\Big[\frac{9}{2} - 4\ln(\frac{4\bar{p}_3^2}{\epsilon y})\Big] + 2\,(1-\epsilon^2)\,\bar{Y}_w\right\} \\
&\quad + \frac{1}{4y^2}\big(2-2\epsilon^2-2\epsilon^4+2\epsilon^6-y^2-4\epsilon^2y^2-5\epsilon^4y^2-4y^4+3y^6\big)\,\bar{Y}_p\,, \\
\mathcal{U}_- &= 4\bar{p}_0\,\mathcal{D} + \frac{2\bar{p}_3}{y^2}\big(1-\epsilon^2+y^2\big)\ln\epsilon + \bar{p}_3\Big[6+4\ln(\frac{4\bar{p}^2}{\epsilon y})\Big] + 2(1-\epsilon^2)\,\bar{Y}_w \\
&\quad + \frac{\bar{Y}_p}{y^2}\,\big(1-2\epsilon^2+\epsilon^4-y^2-3\epsilon^2y^2\big)\,, \\
\mathcal{D} &= \mathrm{Li}_2(\bar{p}_+) - \mathrm{Li}_2(\bar{p}_-) - 2\,\mathrm{Li}_2(1-\frac{\bar{p}_-}{\bar{p}_+}) + \ln(\frac{4\,y\,\bar{p}_3^2}{\bar{p}_+^2})\,\bar{Y}_p - \ln\epsilon\,\bar{Y}_w\,.
\end{aligned} \tag{2.224}$$

Our unpolarized result (2.223) is in complete agreement with that of [38]. We do not, however, agree with the results of [39].

The polarized $\mathcal{O}(\alpha_s)$ corrections are

$$\begin{aligned}
\Gamma^{pol}_{NLO} &= \frac{\alpha_s}{8\pi^2} m_t C_F \Bigg\{ -3(a^2+b^2)\bar{p}_3^2\ln\epsilon \\
&\quad + ab\Bigg[\frac{1}{4}\Big(-11+28y-16y^2-8y^3+7y^4+\epsilon^2(4+8y-14y^2)+7\epsilon^4\Big) \\
&\quad + \Big(2-9y^2+y^4-\epsilon^2(4+3y^2)+2\epsilon^4\Big)\frac{\bar{p}_3}{y^2}\bar{Y}_p + \frac{8\bar{p}_3^2\bar{w}_0}{y^2}\ln\epsilon \\
&\quad + 8\bar{p}_3^2\ln\Big(\frac{1-y}{(1-y)^2-\epsilon^2}\Big) + \Big(3-3y^2+2\epsilon^2(4+y^2)-2\epsilon^4\Big)\ln(\frac{1-y}{\epsilon}) \\
&\quad + 4\bar{p}_0\bar{p}_3\Big(2\mathrm{Li}_2(1-\frac{1-y}{\bar{p}_-}) - 2\mathrm{Li}_2(1-\frac{1-y}{\bar{p}_+}) - \mathrm{Li}_2(\bar{w}_-) \\
&\quad + \mathrm{Li}_2(\bar{w}_-) + 2\ln(\frac{(1-y^2)-\epsilon^2}{\epsilon^2})\bar{Y}_p\Big) \\
&\quad - \Big(2+y^2-\epsilon^2(3+2y^2)+\epsilon^4\Big)\Big(2\mathrm{Li}_2(y) - \mathrm{Li}_2(\bar{w}_-) - \mathrm{Li}_2(\bar{w}_+)\Big)\Bigg]\Bigg\}\,.
\end{aligned} \tag{2.225}$$

Next we discuss various limiting cases for the unpolarized and polarized rates which, among others, serve to check on the correctness of our results.

The limit $m_H \to 0$ is of interest since, according to the Goldstone equivalence theorem, the unpolarized and polarized rates for $t \to H^+ + b$ become related to the unpolarized and polarized longitudinal rates for $t \to W^+ + b$ when $m_W \to 0$. For $m_H \to 0$ one has

$$\begin{aligned}
\lim_{m_H\to 0} \Gamma^{unpol} &= \frac{m_t^3}{8\pi}\frac{G_F}{\sqrt{2}}|V_{tb}|^2(1-\epsilon^2)^3\Bigg\{1+\frac{\alpha_s}{\pi}C_F\frac{1+\epsilon^2}{1-\epsilon^2}\Bigg[\frac{5-22\epsilon^2+5\epsilon^4}{4(1-\epsilon^4)} \\
&-2\ln\epsilon\ln(1-\epsilon^2)-2\frac{1-\epsilon^2}{1+\epsilon^2}\ln\Big(\frac{1-\epsilon^2}{\epsilon^2}\Big) \\
&-\frac{4-5\epsilon^2+7\epsilon^4}{(1-\epsilon^2)(1-\epsilon^4)}\ln\epsilon-2\mathrm{Li}_2(1-\epsilon^2)\Bigg]\Bigg\} \qquad (2.226)
\end{aligned}$$

$$\begin{aligned}
\lim_{m_H\to 0} \Gamma^{pol} &= \frac{m_t^3}{8\pi}\frac{G_F}{\sqrt{2}}|V_{tb}|^2(1-\epsilon^2)^3\Bigg\{1+\frac{\alpha_s}{\pi}C_F\frac{1}{1-\epsilon^2}\Bigg[-\frac{3}{4}(5+\epsilon^2) \\
&-2(1+\epsilon^2)\ln\epsilon\ln(1-\epsilon^2)-\frac{4+5\epsilon^2}{1-\epsilon^2}\ln\epsilon \\
&-2(1-\epsilon^2)\ln\Big(\frac{1-\epsilon^2}{\epsilon^2}\Big)+(1-2\epsilon^2)\mathrm{Li}_2(1-\epsilon^2)\Bigg]\Bigg\}. \qquad (2.227)
\end{aligned}$$

We have checked that for model 1 and $\tan\beta = 1$ the limiting expressions Eqs. (2.226) and (2.227) agree exactly with the $m_{W^+} \to 0$ limit of the corresponding longitudinal and polarized longitudinal rates in the process $t \to W^+ + b$ listed in [44]. This is nothing but the statement of the Goldstone equivalence theorem. Our unpolarized result (2.226) agrees with the corresponding result in [38].

Taking both $m_H \to 0$ and $m_b \to 0$ one obtains

$$\lim_{m_H\to 0} \Gamma^{unpol} = \frac{m_t^3}{8\pi}\frac{G_F}{\sqrt{2}}|V_{tb}|^2\cot^2\beta\Bigg\{1+\frac{\alpha_s}{2\pi}C_F\Bigg[\frac{5}{2}-\frac{2\pi^2}{3}\Bigg]\Bigg\}, \qquad (2.228)$$

$$\lim_{m_H\to 0} \Gamma^{pol} = \frac{m_t^3}{8\pi}\frac{G_F}{\sqrt{2}}|V_{tb}|^2\cot^2\beta\Bigg\{1-\frac{\alpha_s}{2\pi}C_F\Bigg[\frac{15}{2}-\frac{\pi^2}{3}\Bigg]\Bigg\} \qquad (2.229)$$

which, when setting $\cot\beta = 1$, agree exactly with Eqs. (48) and (49) of [44]. The unpolarized rate in this limit agrees with the corresponding results in [37, 38, 40].

When m_H approaches m_t for $m_b \to 0$ one has

$$\lim_{m_H\to m_t} \frac{\Gamma^{unpol}}{\Gamma^{unpol}_{Born}} = 1+\frac{\alpha_s}{2\pi}C_F\Bigg[\frac{13}{2}-\frac{4\pi^2}{3}-3\ln(1-y^2)\Bigg], \qquad (2.230)$$

$$\lim_{m_H\to m_t} \frac{\Gamma^{pol}}{\Gamma^{pol}_{Born}} = 1+\frac{\alpha_s}{2\pi}C_F\Bigg[1-\pi^2-3\ln(1-y^2)\Bigg]. \qquad (2.231)$$

For the unpolarized rate the limiting expression agrees with the corresponding limit given in [38].

Finally, we consider the limit $m_b \to 0$ keeping the charged Higgs mass finite with the proviso discussed after (2.34). This results in very compact expressions for the unpolarized and polarized rates. Due to the smallness of the bottom quark mass and the fact that the bottom quark mass corrections are of $O(m_b^2/m_t^2)$ the $m_b \to 0$ formulae give quite good approximations to the exact formulae for Higgs masses as long as the Higgs mass is not close to the top quark mass. One obtains

$$\begin{aligned}\Gamma^{unpol}(m_b \to 0) &= \frac{m_t}{16\pi}(1-y^2)^2(a^2+b^2)\Bigg\{1+\frac{a^2-b^2}{a^2+b^2}\frac{2\epsilon}{1-y^2}\\ &+\frac{\alpha_s}{2\pi}C_F\Bigg(\frac{9}{2}-\frac{2\pi^2}{3}-\frac{4y^2}{1-y^2}\ln y+\Big(\frac{2-5y^2}{y^2}-4\ln y\Big)\ln(1-y^2)\\ &-4\mathrm{Li}_2(y^2)+\frac{(a-b)^2}{a^2+b^2}3\ln\epsilon\Bigg)\Bigg\} \qquad (2.232)\end{aligned}$$

and

$$\begin{aligned}\Gamma^{pol}(m_b \to 0) &= \frac{m_t}{16\pi}(1-y^2)^2 2ab\Bigg\{1+\frac{\alpha_s}{2\pi}C_F\Bigg(\frac{11-6y-7y^2}{2(1+y)^2}+\frac{1+2y^2}{(1-y^2)^2}\frac{\pi^2}{3}\\ &+\frac{2-9y^2+y^4}{(1-y^2)y^2}\ln(1+y)+\frac{2-5y^2}{y^2}\ln(1-y)-4\mathrm{Li}_2(y)\\ &+\frac{8+4y^4}{(1-y^2)^2}\mathrm{Li}_2(-y)-\frac{(a-b)^2}{2ab}3\ln\epsilon\Bigg)\Bigg\}. \qquad (2.233)\end{aligned}$$

Note that the seemingly mass singular terms proportional to $\ln\epsilon$ in (2.232) and (2.233) are not in fact mass singular since they are multiplied by the factor $(a-b)^2$ which is proportional to m_b^2 in both models 1 and 2. Although the contributions proportional to $\epsilon^2\ln\epsilon$ formally vanish for $m_b \to 0$ as expected from the Lee–Nauenberg theorem [61], they can become numerically quite large for $m_b = 4.8$ GeV in model 2 depending, of course, on the value of $\tan\beta$. This can be seen by calculating the ratios of the coupling factor expressions in model 2, i.e.

$$\frac{(a-b)^2}{a^2+b^2} = \frac{2\epsilon^2\tan^4\beta}{1+\epsilon^2\tan^4\beta} \qquad (2.234)$$

$$\frac{(a-b)^2}{2ab} = \frac{2\epsilon^2\tan^4\beta}{1-\epsilon^2\tan^4\beta}. \qquad (2.235)$$

For example, for $\tan\beta = 10$ one finds $(a-b)^2/(a^2+b^2) = 1.77$ and $(a-b)^2/(2ab) = -2.31$. With a little bit of algebra one finds that the NLO corrections in model 2 are in fact dominated by the $\epsilon^2\ln\epsilon$ contributions for larger values of $\tan\beta$ as also noted in [38]. This is evident in Fig. 2.16 where the radiative corrections to the Born term result can be seen to be as large as -50% compared to the $\approx -10\%$ expected from the corresponding corrections in the decay $t \to W^+ + b$ [44].

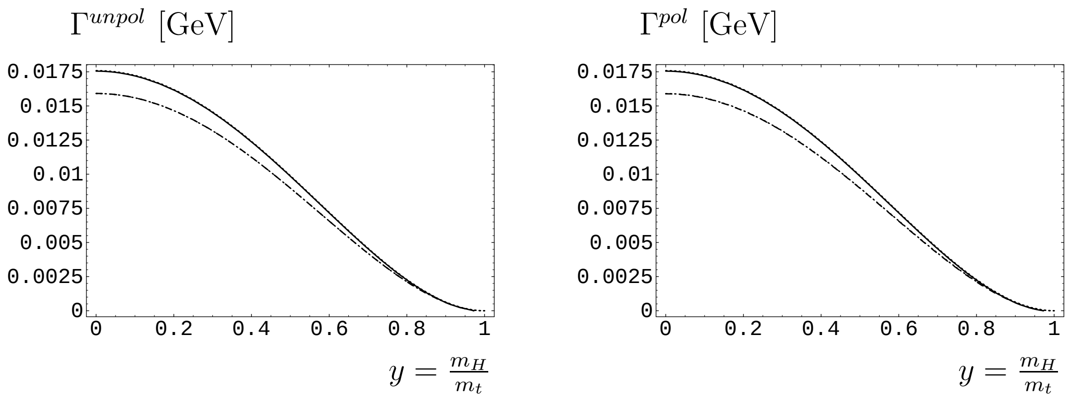


Figure 2.14: The LO (solid lines) unpolarized (left) and polarized (right) decay rates and their $\mathcal{O}(\alpha_s)$ correction (dashed lines) as functions of $y = m_H/m_t$ for model 1 with $m_b = 4.8$ GeV, $m_t = 175$ GeV and $\tan\beta = 10$. The dotted line for the $m_b \to 0$ approximation is not visible for the LO at the scale of this graph. For the $\mathcal{O}(\alpha_s)$ the dotted line overlaps with the dashed line making a dot–dashed curve.

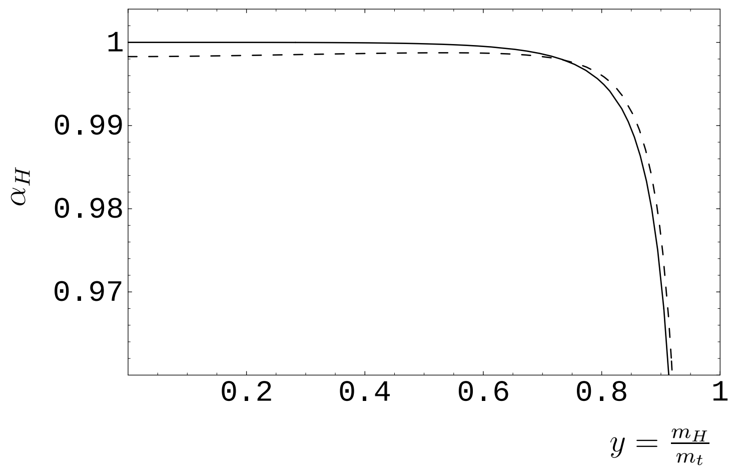


Figure 2.15: The LO (full line) asymmetry parameter α_H and its $\mathcal{O}(\alpha_s)$ correction (dashed line) as functions of $y = m_H/m_t$ for model 1 with $m_b = 4.8$ GeV and $m_t = 175$ GeV as function of $y = \frac{m_H}{m_t}$ for $\tan\beta = 10$.

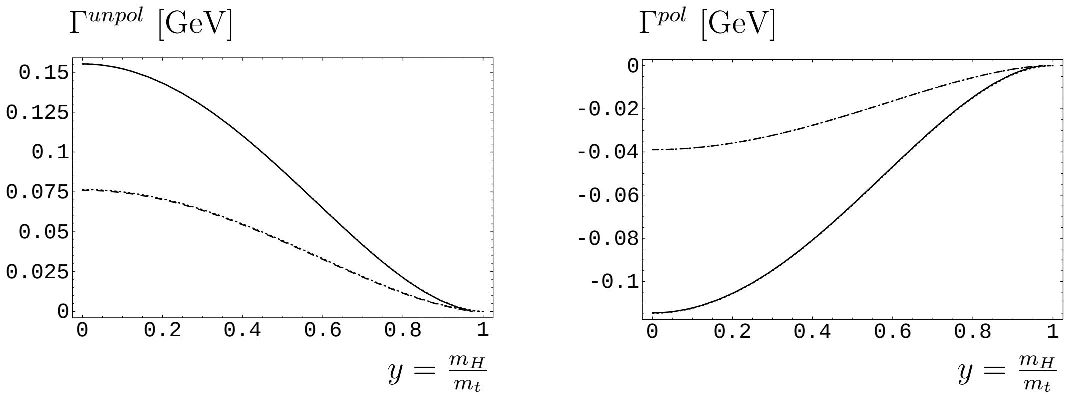


Figure 2.16: The LO (solid lines) unpolarized (left) and polarized (right) decay rates and their $\mathcal{O}(\alpha_s)$ correction (dashed lines) as functions of $y = m_H/m_t$ for model 2 with $m_b = 4.8$ GeV, $m_t = 175$ GeV and $\tan\beta = 10$. The dotted line for the "kinematical" $m_b \to 0$ approximation is not visible for the LO at the scale of this graph. For the $\mathcal{O}(\alpha_s)$ the dotted line overlaps with the dashed making a dot–dashed curve.

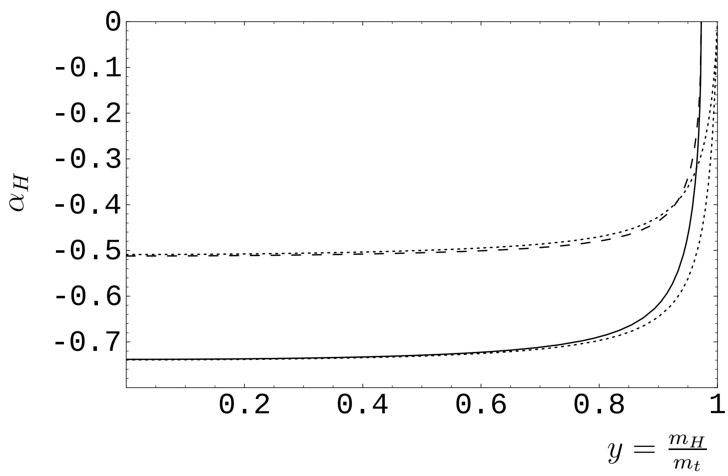


Figure 2.17: The LO (full line) asymmetry parameter α_H and its $\mathcal{O}(\alpha_s)$ corrections as functions of $y = m_H/m_t$ for model 2 with $m_b = 4.8$ GeV and $m_t = 175$ GeV as function of $y = \frac{m_H}{m_t}$ for $\tan\beta = 10$. The dotted lines show the corresponding "kinematical" $m_b \to 0$ curves.

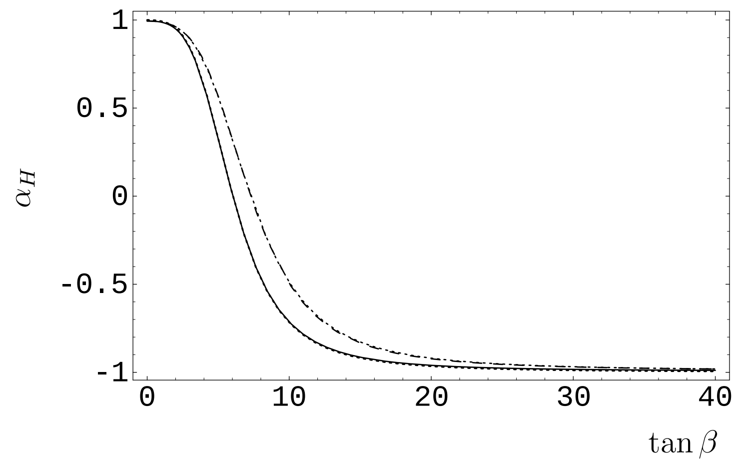


Figure 2.18: $\tan\beta$ dependence of symmetry parameter for model 2 with $m_t = 175$ GeV, $m_b = 4.8$ GeV and $m_H = 120$ GeV. The dotted line for the "kinematical" $m_b \to 0$ approximation is not visible for the LO at the scale of this graph. For the $\mathcal{O}(\alpha_s)$ the dotted line overlaps with the dashed line making a dot–dashed curve.

We now turn to our numerical results.

In Fig. 2.15 we show the radiative corrections to the polarization asymmetry α_H in model 1 as a function of $y = m_H/m_t$. The radiative corrections are quite small and lower α_H by only $\approx 2‰$ for the most range of $y = m_H/m_t$. We do not present a curve for the dependence of α_H on $\tan\beta$ because the radiative corrections are again quite small $\approx 2‰$ over a large range of $\tan\beta$–values and thus do not change the flat Born term behaviour. Figs. 2.16 show the LO and NLO model 2 results for the unpolarized and polarized rates as a function of the mass ratio $y = m_H/m_t$ for $\tan\beta = 10$. As input values for our numerical evaluation we use $m_b = 4.8$ GeV and $m_t = 175$ GeV. The strong coupling constant is evolved from $\alpha_s(M_Z) = 0.1175$ to $\alpha_s(m_t) = 0.1070$ using two-loop running. The unpolarized and polarized rates are largest for $m_H = 0$ and become smaller towards the phase-space boundary $y = 1 - \epsilon$. The radiative corrections are substantial due to the $\epsilon^2 \ln\epsilon$ contribution discussed above. The dotted curves in Figs. 2.16 are drawn using the "kinematical" $m_b \to 0$ approximations Eqs. (2.35) (LO) and Eqs. (2.232) and (2.233) (NLO). In these equations the bottom quark mass has been set to zero whenever the scale of m_b is set by m_t as in the kinematical factors and not by $m_t \cot\beta$ as in the coupling factors. As Fig. 2.16 shows the "kinematical" $m_b \to 0$ approximation is an excellent approximation for both the unpolarized and polarized rate.

In Fig. 2.17 we show the model 2 LO and NLO results for the asymmetry parameter α_H where we have again fixed $\tan\beta$ at $\tan\beta = 10$ and show the dependence of α_H on the mass ratio $y = m_H/m_t$. The asymmetry parameter is large and negative with only little dependence on the Higgs mass except for the region close to the phase-space boundary where α_H approaches zero. The radiative corrections to the LO Born term result are substantial and reduce the size of the asymmetry parameter by $\approx 25\%$ over much of the range of the Higgs mass. As anticipated from the results for the unpolarized and polarized rate the $m_b \to 0$ approximation is excellent. At the scale of Fig. 2.17 the

$m_b \neq 0$ corrections are barely visible. In Fig. 2.18 we fix the mass of the charged Higgs boson at $m_H = 120$ GeV and vary $\tan\beta$ between 0 and 40. For small values of $\tan\beta$ the asymmetry parameter α_H is negative and rapidly approaches zero around $\tan\beta = 7$. The LO zero position $\tan\beta = \sqrt{m_t/m_b}$ is shifted upward by approximately one unit by the radiative corrections. Beyond the zero position the asymmetry parameter rapidly approaches values close to $\alpha_H = -1$. The radiative corrections are largest around the zero position of $\tan\beta$ at $\tan\beta \approx 7$.

Before turning to the next chapter let us give a brief summary.

We have calculated the $O(\alpha_s)$ radiative corrections to polarized top quark decay into a charged Higgs and a bottom quark in two variants of the Two-Higgs-Doublet model. We have checked our unpolarized results against known ones and found agreement. With the same techniques we have calculated the polarized rate. Using a particular limit of our polarized rate for $t(\uparrow) \rightarrow H^+ + b$ we were able to compare our result with the corresponding limit for the decay $t \rightarrow W^+ + b$ appealing to the Goldstone equivalence theorem. Because of our numerous cross-checks we are quite confident that our new results on the polarized rates are correct. We have found very compact $m_b = 0$, $O(\alpha_s)$ expressions for the unpolarized and polarized rates which can be usefully employed to scan the predictions of the 2HDM $(m_H, \tan\beta)$ parameter space. We have found that the radiative corrections to the unpolarized and polarized rates, and the asymmetry parameter of the decay can become quite large.

Chapter 3

The angular correlations for the decay $t(\uparrow) \to b + \ell^+ + \nu_\ell$ in the helicity system 1a

3.1 Introduction

In the rest frame decay of a top quark decaying into a jet X_b, a charged lepton ℓ^+ and a neutrino ν_ℓ, the final state particles X_b, ℓ^+ and ν_ℓ define an event frame. Relative to this event plane one can then define the polarization direction of the polarized top quark. There are various choices of possible coordinate systems relative to the event plane where one differentiates between helicity systems with the z–axis in the event plane and transversity systems with the z–axis perpendicular to the event plane. We analyze angular correlations in different helicity coordinate systems according to the orientation of the z–axis. Also one has to specify the orientation of the x–axis for which one has two possible choices for each system. Thus one can define coordinate systems as (see Fig. 3.1):

$$\text{system } 1: \quad \vec{p}_\ell \parallel z\ ; \quad a: (\vec{p}_\nu)_x \geqslant 0 \qquad b: (\vec{p}_b)_x \geqslant 0 \tag{3.1}$$

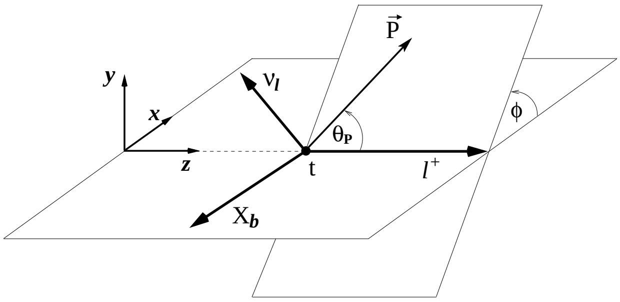
$$\text{system } 2': \quad \vec{W} \parallel z\ ; \quad a: (\vec{p}_\ell)_x \geqslant 0 \qquad b: (\vec{p}_\nu)_x \geqslant 0 \tag{3.2}$$

$$\text{system } 3: \quad \vec{p}_\nu \parallel z\ ; \quad a: (\vec{p}_b)_x \geqslant 0 \qquad b: (\vec{p}_\ell)_x \geqslant 0 \tag{3.3}$$

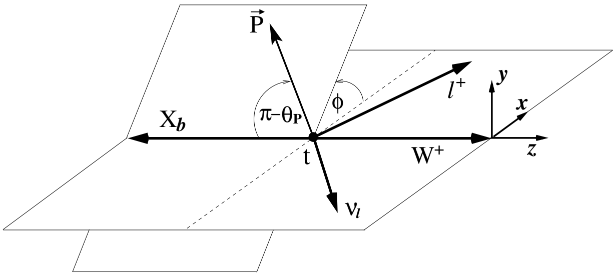
The systems a and b are related by a rotation of the azimuthal angles by π, i.e. $\phi \to \pi + \phi$. Therefore they are equivalent. In this thesis we shall only deal with system 1a and system 3a. The polarized top quark decay in system 1a is discussed in this chapter. In the next chapter we discuss the polarized top quark decay in system 3a.

The four–momenta of top quark, bottom quark, charged lepton, neutrino and gluon are denoted by $p_t,\ p_b,\ p_\ell,\ p_\nu$ and k. The hadronic sector of the decay $t \to b + \ell^+ + \nu_\ell$ is identical to that of the decay $t \to H^+ + b$ discussed in chapter 2 except for the different vertex structures in the two cases. Similar to the case of the decay $t \to H^+ + b$ we denote the quark–gluon combined system as $P = p_b + k$. We define scaled masses by

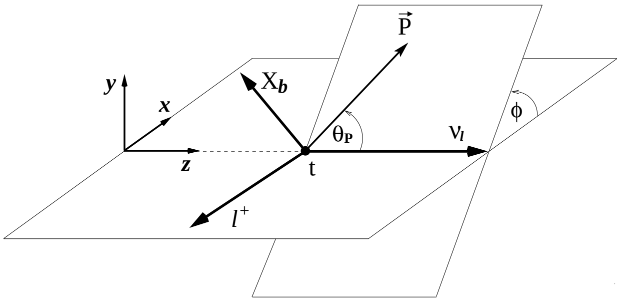
$$\epsilon = \frac{m_b}{m_t}, \qquad \Lambda = \frac{m_g}{m_t}, \qquad z = \frac{P^2}{m_t^2}\,. \tag{3.4}$$



(1a)



(2'a)



(3a)

Figure 3.1: The definition of the polar angle θ_P and the azimuthal angle ϕ in the rest frame decay of a polarized top quark in three different helicity systems. The event plane defines the (x, z)–plane with (1a) $\vec{p}_\ell \parallel z$ and $(\vec{p}_\nu)_x \geqslant 0$, $(2'a)$ $\vec{W} \parallel z$ and $(\vec{p}_\ell)_x \geqslant 0$, and (3a) $\vec{p}_\nu \parallel z$ and $(\vec{p}_b)_x \geqslant 0$.

The charged lepton is taken as massless. The W–boson four momentum is denoted by $W = p_\ell + p_\nu$. The scaled charged lepton energy, the scaled neutrino energy, the scaled invariant mass of the virtual W–boson and the scaled mass of the W–boson are defined by

$$x = \frac{2E_\ell}{m_t}, \qquad x_\nu = \frac{2E_\nu}{m_t}, \qquad y = \sqrt{\frac{W^2}{m_t^2}}, \qquad \hat{y} = \frac{m_W}{m_t}. \tag{3.5}$$

The scaled kinematic variables are the same[1] as in (2.2):

$$\begin{aligned}
p_0(z) &= \frac{1}{2}(1-y^2+z), & w_0(z) &= \frac{1}{2}(1+y^2-z),\\
p_3(z) &= \frac{1}{2}\sqrt{\lambda(1,y^2,z)}, & w_3(z) &= p_3,\\
p_\pm(z) &= p_0 \pm p_3, & w_\pm(z) &= w_0 \pm w_3,\\
Y_p(z) &= \frac{1}{2}\ln\frac{p_+}{p_-}, & Y_w(z) &= \frac{1}{2}\ln\frac{w_+}{w-}.
\end{aligned} \tag{3.6}$$

where the Källén function $\lambda(a,b,c)$ is defined in 2.3. If $z = \epsilon^2$ i.e. $k = 0$ we recover the leading order kinematics. We define

$$\begin{aligned}
\bar{p}_0 &= p_0(\epsilon^2), & \bar{p}_3 &= p_3(\epsilon^2), & \bar{p}_\pm &= p_\pm(\epsilon^2), & \bar{Y}_p &= Y_p(\epsilon^2),\\
\bar{w}_0 &= w_0(\epsilon^2), & \bar{w}_3 &= w_3(\epsilon^2), & \bar{w}_\pm &= w_\pm(\epsilon^2), & \bar{Y}_w &= Y_w(\epsilon^2).
\end{aligned} \tag{3.7}$$

Note that p_0 and p_3 are the energy and momentum modulus of the quark– gluon system $P = p_b + k$ scaled to the top mass whereas $\bar{p}_0$ and $\bar{p}_3$ are the scaled energy and the momentum modulus of the bottom quark in the leading order case. Similarly w_0 and w_3 are the scaled energy and momentum modulus of the W–boson for the real emission case and $\bar{w}_0$ and $\bar{w}_3$ for the leading order case.

3.2 The Born approximation

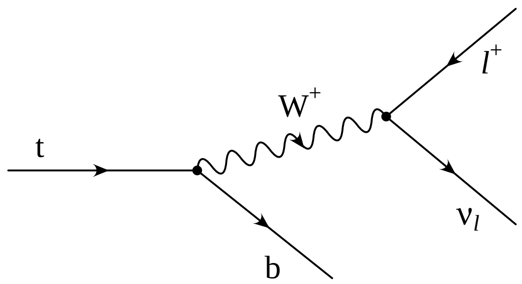


Figure 3.2: Feynman diagrams for the decay $t(\uparrow) \to b + \ell^+ + \nu_\ell$ at LO.

At LO the Feynman graph shown in Fig 3.2 gives the amplitude

$$M_0 = \bar{u}(p_b,s_b)\left(-i\frac{g_w}{\sqrt{2}}\gamma^\mu\frac{1-\gamma_5}{2}V_{tb}\right)u(p_t,s_t)\left(-i\frac{g_{\mu\nu}-W_\mu W_\nu/m_W^2}{W^2-m_W^2+i\varepsilon}\right) \tag{3.8}$$
$$\times\,\bar{u}(p_\nu,s_\nu)\left(-i\frac{g_w}{\sqrt{2}}\gamma^\nu\frac{1-\gamma_5}{2}\right)v(\ell,s_\ell)\,.$$

[1]Note that y is the scaled invariant mass of the W–boson, $y = \sqrt{\frac{W^2}{m_t^2}}$ in the decay $t \to b + \ell^+ + \nu_\ell$ whereas, in $t \to b + H^+$, y is the scaled mass of the Higgs boson, $y = m_H/m_t$.

The $W_\mu W_\nu/m_W^2$–term in the W–boson propagator gives rise to the terms proportional to $m_t m_\ell/m_W^2$ or $m_t m_\ell/m_W^2$, where m_ℓ denotes the lepton mass. Since the W–boson mass is much bigger than the lepton mass the $W_\mu W_\nu/m_W^2$ term can be neglected. The intermediate W–boson is a resonance with a width of $\Gamma_W = 2.134 \pm 0.079$ Gev [3]. In the present application we keep the full propagator dependence of the W^+ in the Breit-Wigner (BW) form [53] and write the W–propagator as

$$-i\frac{g_{\mu\nu}}{W^2 - m_W^2 + i\varepsilon} \to -i\frac{g_{\mu\nu}}{W^2 - m_W^2 + i\, m_W \Gamma_W} = -i\frac{g_{\mu\nu}}{m_W^2}\frac{1}{y^2/\hat{y}^2 - 1 + i\gamma}\,, \tag{3.9}$$

where $\gamma = \frac{\Gamma_W}{m_W}$. Thus the LO amplitude takes the form

$$\begin{aligned} M_0 &= i\frac{g_w^2 V_{tb}}{8m_W^2}\frac{1}{y^2/\hat{y}^2 - 1 + i\gamma}\bar{u}(b, s_b)\gamma_\mu\left(1-\gamma_5\right)u(t, s_t)\bar{u}(p_\nu, s_\nu)\gamma^\mu\left(1-\gamma_5\right)v(\ell, s_\ell) \\ &= i\frac{G_F V_{tb}}{\sqrt{2}\left(y^2/\hat{y}^2 - 1 + i\gamma\right)}H_\mu^{(0)}L^\mu\,, \end{aligned} \tag{3.10}$$

where the hadron current $H_\mu^{(0)}$ and the leptonic current L_μ are defined by

$$H_\mu^{(0)} = \bar{u}(p_b, s_b)\gamma_\mu\left(1-\gamma_5\right)u(p_t, s_t), \tag{3.11}$$

$$L_\mu = \bar{u}(p_\nu, s_\nu)\gamma_\mu\left(1-\gamma_5\right)v(p_\ell, s_\ell)\,. \tag{3.12}$$

Squaring the amplitude we have

$$\overline{|M_0|^2} = \sum_{s_b, s_\ell, s_\nu}|M_0|^2 = \sum_{s_b, s_\ell, s_\nu}M_0^\dagger M_0 = \frac{G_F^2|V_{tb}|^2}{2\left[(1-y^2/\hat{y}^2)^2+\gamma^2\right]}H_{\mu\nu}^{(0)}L^{\mu\nu} \tag{3.13}$$

where the hadron tensor $H_{\mu\nu}^{(0)}$ and the leptonic tensor $L_{\mu\nu}$ are defined by

$$H_{\mu\nu}^{(0)} = \sum_{s_b}H_\mu^{(0)}(H_\nu^{(0)})^\dagger, \qquad L_{\mu\nu} = \sum_{s_\ell, s_\nu}L_\mu L_\nu^\dagger\,. \tag{3.14}$$

Summing up the relevant spins in the hadron tensor and using the properties of the Dirac matrices and their traces we have

$$\begin{aligned} H_{\mu\nu}^{(0)} &= \sum_{s_b}\bar{u}(p_b, s_b)\gamma_\mu\left(1-\gamma_5\right)u(p_t, s_t)\bar{u}(t, s_t)\gamma_\nu\left(1-\gamma_5\right)u(p_b, s_b) \\ &= \mathrm{Tr}\left[\left(\not{p}_b + m_b\right)\gamma_\mu\left(1-\gamma_5\right)\left(\not{p}_t + m_t\right)\left(\frac{1+\gamma_5\not{s}_t}{2}\right)\gamma_\nu\left(1-\gamma_5\right)\right] \\ &= \mathrm{Tr}\left[\not{p}_b\gamma_\mu\left(\not{p}_t - m_t\not{s}_t\right)\gamma_\nu\left(1-\gamma_5\right)\right] \\ &= \mathrm{Tr}\left[\not{p}_b\,\gamma_\mu\tilde{\not{p}}_t\,\gamma_\nu\left(1-\gamma_5\right)\right] \\ &= p_b^\alpha\tilde{p}_t^\beta\,\mathrm{Tr}\left[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu - \gamma_5\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\right] \\ &= p_b^\alpha\tilde{p}_t^\beta\cdot 4\left[g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\nu}g_{\mu\beta} - i\epsilon_{\alpha\mu\beta\nu}\right] \\ &= 4p_b^\alpha\tilde{p}_t^\beta\,T_{\alpha\beta\mu\nu}, \end{aligned} \tag{3.15}$$

where $\tilde{p}_t$ and $T_{\mu\nu\alpha\beta}$ are defined by

$$\begin{aligned}\tilde{p}_t &= p_t - m_t s_t, &(3.16)\\ T_{\mu\nu\alpha\beta} &= g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\nu}g_{\mu\beta} - i\epsilon_{\alpha\mu\beta\nu}\,. &(3.17)\end{aligned}$$

Similarly the lepton tensor is given by

$$L_{\mu\nu} = 8p_\ell^\sigma p_\nu^\rho T_{\sigma\rho\mu\nu}\,. \tag{3.18}$$

The contraction of the lepton and hadron tensors involves the contraction

$$T_{\mu\nu\alpha\beta}T^{\sigma\rho\mu\nu} = 4\delta_\alpha^\sigma\delta_\beta^\rho\,. \tag{3.19}$$

With (3.15), (3.18) and (3.19) we obtain

$$H_{\mu\nu}^{(0)}L^{\mu\nu} = 128\,(\tilde{p}_t \cdot p_\ell)\,(p_b \cdot p_\nu) \tag{3.20}$$

The scalar products which appear in the LO and in the QCD NLO virtual one–loop contributions are

$$\begin{aligned}p_t \cdot p_\ell &= \frac{1}{2}\,x\,m_t^2\,,\\ p_t \cdot p_\nu &= \frac{1-x+y^2-\epsilon^2}{2}m_t^2\,,\\ p_t \cdot p_b &= \frac{1-y^2+\epsilon^2}{2}m_t^2\,,\\ p_b \cdot p_\nu &= \frac{1-x-\epsilon^2}{2}m_t^2\,,\\ p_b \cdot p_\ell &= \frac{x-y^2}{2}m_t^2\,,\\ p_\nu \cdot p_\ell &= \frac{1}{2}\,y^2\,m_t^2. &(3.21)\end{aligned}$$

In the polarized calculations we need to define the polarization vector in a specific frame. In system 1a (top quark rest frame with the charged lepton momentum defining the z–axis) the polarization vector and the momenta read

$$\begin{aligned}p_t &= m_t(1;0,0,0)\,, &(3.22)\\ p_\ell &= \frac{m_t}{2}x(1;0,0,1)\,,\\ p_\nu &= \frac{m_t}{2}(1-x+y^2-\epsilon^2)(1;\sin\theta_{\ell\nu},0,\cos\theta_{\ell\nu})\,,\\ p_b &= \frac{m_t}{2}(1-y^2+\epsilon^2)(1;\sin\theta_{\ell b},0,\cos\theta_{\ell b})\,,\\ s_t &= P(0;\sin\theta_P\cos\phi,\sin\theta_P\sin\phi,\cos\theta_P)\,,\end{aligned}$$

where

$$\begin{aligned} \cos\theta_{\ell\nu} &= \frac{x(1-x+y^2-\epsilon^2)-2y^2}{x(1-x+y^2-\epsilon^2)}\,, \qquad (3.23) \\ \cos\theta_{\ell b} &= \frac{2y^2-x(1+y^2-\epsilon^2)}{x\sqrt{\lambda(1,y^2,\epsilon^2)}}\,. \end{aligned}$$

Thus the scalar products involving the polarization vector are given by

$$\begin{aligned} p_b\cdot s_t &= \frac{-2y^2+x\,(1+y^2-\epsilon^2)}{2x}\,m_t\,P\cos\theta_P \\ &\quad +\frac{\sqrt{y^2\,(x-x^2-y^2+xy^2-x\epsilon^2)}}{x}m_t\,P\cos\phi\sin\theta_P\,, \\ p_\nu\cdot s_t &= \frac{x^2+2y^2-x\,(1+y^2-\epsilon^2)}{2x}\,m_t\,P\cos\theta_P \\ &\quad -\frac{\sqrt{y^2\,(x-x^2-y^2+xy^2-x\epsilon^2)}}{x}\,m_t\,P\cos\phi\sin\theta_P\,, \\ p_\ell\cdot s_t &= \frac{x}{2}\,m_t\,P\cos\theta_P\,. \qquad (3.24) \end{aligned}$$

Substituting the scalar products into the contraction (3.20) we have

$$H^{(0)}_{\mu\nu}L^{\mu\nu} = 128\frac{m^4}{4}(1+P\cos\theta_P)x(1-x-\epsilon^2)\,. \qquad (3.25)$$

The above derivation shows that the LO result $\Gamma \sim (1+\cos\theta_P)$ does not depend on the mass of the bottom quark. It does, however, depend on the mass of the lepton. The lepton mass effect can be easily calculated from (3.20). One obtains $|M|^2 \sim 1 + (1 - \frac{1}{2}m_\ell^2/E_\ell^2 + \ldots)\cos\theta_P$. The lepton mass correction is thus negligibly small since, in the narrow–width approximation for the W^+, which we will discuss later, the minimal lepton energy is given by $E_\ell^{\min} = (m_W^4 + m_t^2 m_\ell^2)/(2m_t m_W^2)$ and is thus very much larger than the lepton mass appearing in the lepton mass correction.

It is convenient to collect the terms in the lepton–hadron contraction according to their angular dependence. One has

$$H^{(0)}_{\mu\nu}L^{\mu\nu} = 128\frac{m^4}{4}\Big[M_0^A(x) + M_0^B(x)\,P\cos\theta_P + M_0^C(x)\,P\sin\theta_P\cos\phi\Big], \qquad (3.26)$$

where M_0^A is not dependent on the angular parameters of the polarization vector of the top quark. M_0^B and M_0^C denote the polar and the azimuthal angle dependent pieces of the lepton–hadron contraction, respectively. They are called the unpolarized, polar and azimuthal correlation terms or A, B and C terms respectively. Comparing (3.25) and (3.26) one has

$$M_0^A(x) = M_0^B(x) = x(1-x-\epsilon^2), \qquad M_0^C(x) = 0. \qquad (3.27)$$

The differential rate is

$$\begin{aligned} d\Gamma &= \frac{1}{2m_t}\frac{1}{(2\pi)^3}\frac{d^3\vec{p}_b}{2E_b}\frac{1}{(2\pi)^3}\frac{d^3\vec{p}_\ell}{2E_\ell}\frac{1}{(2\pi)^3}\frac{d^3\vec{p}_\nu}{2E_\nu}(2\pi)^4\delta^4(p_t-p_b-p_\ell-p_\nu)\,\overline{|M|}^2 \\ &= \frac{1}{2m_t}\frac{1}{(2\pi)^5}dR_3(p_t;p_b,p_\ell,p_\nu)\,\overline{|M|}^2\,, \qquad (3.28) \end{aligned}$$

where

$$dR_3(p_t; p_b, p_\ell, p_\nu) = \frac{d^3\vec{p}_b}{2E_b}\frac{d^3\vec{p}_\ell}{2E_\ell}\frac{d^3\vec{p}_\nu}{2E_\nu}\delta^4(p_t - p_b - p_\ell - p_\nu)\,. \tag{3.29}$$

We factorize the three–body phase space into two two–body phase spaces $dR_3(p_t; p_b, p_\ell, p_\nu)$ as we have done in the case of the three–body phase space for the real gluon emission in the decay $t \to H^+ + b$ (see 2.145):

$$R_3(p_t; p_b, p_\ell, p_\nu) = \int dW^2\, dR_2(p_t; p_b, W)\, dR_2(W; p_\ell, p_\nu)\,. \tag{3.30}$$

We can proceed in analogy to the three–body phase space integration of $t \to H^+ + b + g$. The two–body phase space integration $dR_2(p_t; p_b, W)$ is done in the top–rest frame with the polarization vector of the top quark $\vec{P}$ defining the z–axis. Analogous to (2.29) we have

$$\begin{aligned}\int dR_2\,(p_t; p_b, W) &= \int \frac{d^3\vec{p}_b}{2E_b}\frac{d^3\vec{W}}{2E_W}\delta^4(p_t - p_b - W)\\ &= \int \frac{1}{4m_t}|\vec{W}|\, 2\pi\, d\cos\theta_P,\end{aligned} \tag{3.31}$$

where θ_P denotes the angle between W–momentum and the polarization vector of the top quark. The azimuthal angle of the W–momentum is integrated out. The energy and momentum modulus of the W–boson and the bottom quark are fixed by the integrating over the δ–function:

$$\begin{aligned}E_W &= m_t\frac{1 + y^2 - \epsilon^2}{2} = m_t\,\bar{w}_0, \qquad E_b = m_t\frac{1 - y^2 + \epsilon^2}{2} = m_t\,\bar{p}_0,\\ |\vec{W}| &= |\vec{p}_b| = m_t\frac{\sqrt{\lambda(1, y^2, \epsilon^2)}}{2} = m_t\,\bar{p}_3.\end{aligned} \tag{3.32}$$

The two–body phase space integration $dR_2(W; p_\ell, p_\nu)$ is also done in the top quark rest frame with the W–momentum along the z–axis:

$$\begin{aligned}\int dR_2(W; p_\ell, p_\nu) &= \int \frac{d^3\vec{p}_\ell}{2E_\ell}\frac{d^3p_\nu}{2E_\nu}\delta^4(W - p_\ell - p_\nu)\\ &= \int \frac{d^3\vec{p}_\ell}{2E_\ell}d^4p_\nu\delta(p_\nu^2)\delta^4(W - p_\ell - p_\nu)\\ &= \int \frac{d^3\vec{p}_\ell}{2E_\ell}\delta((W - p_\ell)^2)\\ &= \int \frac{d^3\vec{p}_\ell}{2E_\ell}\delta(W^2 - 2\,W\cdot p_\ell)\end{aligned} \tag{3.33}$$

We change from the Cartesian coordinate system to polar coordinate system using

$$d^3\vec{p}_\ell = |\vec{p}_\ell|^2 d\cos\theta_{\ell W}d\phi = E_\ell^2 d\cos\theta_{\ell W}d\phi\,, \tag{3.34}$$

where $\theta_{\ell W}$ and ϕ are the polar and azimuthal angles of the charged lepton. Then one has

$$\begin{aligned}\int dR_2(W; p_\ell, p_\nu) &= \int \frac{E_\ell}{2} dE_\ell \int d\cos\theta_{\ell W}\, d\phi\, \delta(W^2 - 2E_w E_\ell + 2E_\ell |\vec{W}| \cos\theta_{\ell W}) \\ &= \int \frac{E_\ell}{2} dE_\ell\, d\phi \, \frac{1}{2E_\ell |\vec{W}|}\, d\cos\theta_{\ell W}\, \delta(\cos\theta_{\ell W}) \\ &= \frac{1}{4|\vec{W}|} \int dE_\ell\, d\phi\,. \end{aligned} \tag{3.35}$$

The δ–function fixes the angle $\theta_{\ell W}$ as

$$\cos\theta_{\ell W} = -\frac{2y^2 - x(1 + y^2 - \epsilon^2)}{x\sqrt{\lambda(1, y^2, \epsilon^2)}}. \tag{3.36}$$

Since, in the rest frame of the top quark, the W–boson and the bottom quark are produced back–to–back, $\theta_{\ell b} + \theta_{\ell W} = \pi$, i.e. $\cos\theta_{\ell b} = -\cos\theta_{\ell W}$, as can be seen from (3.23).

Inserting the results for $dR_2(p_t; p_b, W)$ and $dR_2(W; p_\ell, p_\nu)$ into (3.30) we have

$$dR_3(p_t; p_b, p_\ell, p_\nu) = \frac{1}{4\pi} d\cos\theta_P d\phi\, \frac{\pi^2}{4} m_t^2 dy^2 dx\,. \tag{3.37}$$

The integration limits depend on the order of integration. One has

$$\int_0^{(1-\epsilon)^2} dy^2 \int_{\bar{w}_-}^{\bar{w}_+} dx\,, \qquad \text{or} \qquad \int_0^{1-\epsilon^2} dx \int_0^{\frac{x}{1-x}(1-x-\epsilon^2)} dy^2\,. \tag{3.38}$$

Comparing the parameterization of the three–body phase space in (3.35) with the three different parameterizations corresponding to the helicity systems shown in Fig 3.1, we find that (3.37) corresponds to system 2'a. Note that system 1a is obtained by a anticlockwise rotation of system 2'a by an angle $\theta_{\ell W}$ around the y–axis followed by a reflection of the neutrino and p_b w.r.t. the (z, y)–plane. In system 1a, the three-body phase space $R_3(p_t; p_b, p_\ell, p_\nu)$ is parameterized by

$$dR_3(p_t; p_b, p_\ell, p_\nu) = \frac{1}{4\pi} d\cos\theta_P d\phi\, \frac{\pi^2}{4} m_t^2 dy^2 dx\,. \tag{3.39}$$

where θ_P is the angle between the charged lepton and the top quark polarization vector. Angle ϕ is the azimuthal angle between the $(\vec{P}_t, \vec{p}_\ell)$ and $(\vec{p}_\ell, \vec{p}_{X_b})$ planes.

Collecting the lepton and hadron tensor contraction from (3.26) and the phase space integration from (3.39), the differential rate is given by

$$\begin{aligned} d\Gamma &= \frac{1}{2m_t} \frac{1}{(2\pi)^5} dR_3(p_t; p_b, p_\ell, p_\nu)\, \overline{|M|}^2 \\ &= \frac{1}{2m_t} \frac{1}{(2\pi)^5} \frac{1}{4\pi} d\cos\theta_P d\phi\, \frac{\pi^2}{4} m_t^2 dy^2\, dx \frac{G_F^2 |V_{tb}|^2}{2\,[(1 - y^2/\hat{y}^2)^2 + \gamma^2]} H_{\mu\nu}^{(0)} L^{\mu\nu} \\ &= \Gamma_F\, \frac{12}{[(1 - y^2/\hat{y}^2)^2 + \gamma^2]}\, dy^2 dx \frac{1}{4\pi} d\cos\theta_P d\phi\, \times \\ &\qquad \Big[M_0^A(x) + M_0^B(x)\, P\cos\theta_P + M_0^C(x)\, P\sin\theta_P \cos\phi \Big]\,, \end{aligned} \tag{3.40}$$

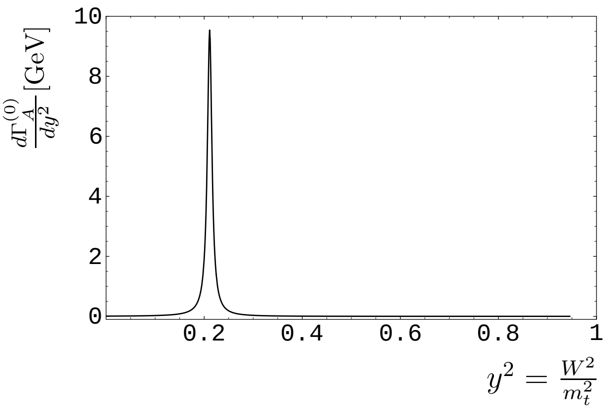


Figure 3.3: The LO unpolarized differential decay rate w.r.t. y^2, i.e. the y^2–spectrum for the decay $t(\uparrow) \to b + \ell^+ + \nu_\ell$.

where

$$\Gamma_F = \frac{G_F^2 m_t^5}{192\pi^3} |V_{tb}|^2 \tag{3.41}$$

is a reference rate corresponding to a (hypothetical) point-like four-Fermion interaction (with massless final state particles). It is obvious that the general form of the differential rate can be written as

$$\frac{d\Gamma}{dx d\cos\theta_P d\phi} = \frac{1}{4\pi} \left(\frac{d\Gamma_A}{dx} + \frac{d\Gamma_B}{dx} P \cos\theta_P + \frac{d\Gamma_C}{dx} P \sin\theta_P \cos\phi \right), \tag{3.42}$$

where Γ_A denote the unpolarized rate. Γ_B and Γ_C denote the polar and azimuthal correlations. For simplicity we refer to Γ_A, Γ_B and Γ_C as unpolarized, polar and azimuthal rate, or A, B, and C terms, respectively. Substituting (3.27) into (3.40) one has

$$\begin{aligned} \frac{d\Gamma_A}{dx} &= \frac{d\Gamma_B}{dx} = \Gamma_F \int dy^2 \frac{12\, x\, (1 - x - \epsilon^2)}{(1 - y^2/\hat{y}^2)^2 + \gamma^2} \\ \frac{d\Gamma_C}{dx} &= 0\,, \end{aligned} \tag{3.43}$$

The y^2–spectrum is shown in Fig 3.3. It has a peak at $y = \frac{m_W}{m_t}$. Because of the large mass (80.423 GeV)[3] and the narrow width (2.134 GeV)[3] of the W boson it is natural to use the "narrow–width" approximation for the W–propagator. In the amplitude squared the W–propagator becomes

$$\left| \frac{1}{W^2 - m_W^2 + i\Gamma_W m_W} \right|^2 = \frac{1}{M_W^4} \frac{\pi}{\gamma} \frac{1}{\pi} \frac{\gamma}{(1 - y^2/\hat{y}^2)^2 + \gamma^2}\,. \tag{3.44}$$

Using the Lorentz representation of the δ–function,

$$\delta(x) = \lim_{a \to 0} \frac{1}{\pi} \frac{a}{x^2 + a^2} \tag{3.45}$$

we write in the narrow–width approximation

$$\left| \frac{1}{W^2 - m_W^2 + i\Gamma_W m_W} \right|^2 \quad \overset{\frac{\Gamma_W}{m_W} \to 0}{\Longrightarrow} \quad \frac{1}{m_W^4} \pi \frac{m_W}{\Gamma_W} \delta\left(y^2 - \hat{y}^2\right). \tag{3.46}$$

In the narrow–width approximation the differential rate becomes

$$\begin{aligned} d\Gamma^{(0)} &= \frac{1}{2m_t}\frac{1}{(2\pi)^5} dR_3(p_t; p_b, p_\ell, p_\nu)\,\overline{|M|}^2 \\ &= \frac{1}{2m_t}\frac{1}{(2\pi)^5}\frac{\pi^2}{4} m_t^2 \int\limits_0^{(1-\epsilon)^2} dy^2 \int\limits_{\bar{w}_-}^{\bar{w}_+} dx \frac{G_F^2 |V_{tb}|^2}{2\left[(1-y^2/\hat{y}^2)^2+\gamma^2\right]} H^{(0)}_{\mu\nu} L^{\mu\nu} \\ &\overset{\frac{\Gamma_W}{m_W}\to 0}{\Longrightarrow} \quad \Gamma_F\, 2\pi \frac{m_W}{\Gamma_W}\frac{3}{16}\frac{\hat{y}^2}{m_t^4}\int\limits_{\bar{w}_-}^{\bar{w}_+} dx\, H^{(0)}_{\mu\nu} L^{\mu\nu}\,. \end{aligned} \tag{3.47}$$

From now on y^2 should be replaced by $\hat{y}^2$ everywhere because we use the narrow–width approximation. Therefore we drop the hat in $\hat{y}$ and use instead $y = \frac{m_W}{m_t}$, which should not be confused with the y defined in (3.5).

The differential unpolarized, polar and azimuthal rates in the narrow–width approximation are

$$\frac{d\Gamma_A^{(0)}}{dx} = \frac{d\Gamma_B^{(0)}}{dx} = \Gamma_F\, 2\pi \frac{m_W}{\Gamma_W} 6y^2\, x(1-x-\epsilon^2), \tag{3.48}$$
$$\frac{d\Gamma_C^{(0)}}{dx} = 0.$$

The integrated rates are ($\bar{w}_- \leqslant x \leqslant \bar{w}_+$):

$$\Gamma_A^{(0)} = \Gamma_B^{(0)} = \Gamma_F\, 2\pi \frac{m_W}{\Gamma_W} 2y^2 \bar{p}_3 \left[(1-\epsilon^2)^2 + y^2(1+\epsilon^2) - 2y^4\right], \tag{3.49}$$
$$\Gamma_C^{(0)} = 0.$$

In Fig 3.4 we show the unpolarized spectra as a function of the scaled lepton energy x with and without the narrow resonance approximation.

$\Gamma_A^{(0)}$(*no approx.*)/$\Gamma_A^{(0)}$(*no approx.*)	1
$\Gamma_A^{(0)}$(*narrow width approx.*)/$\Gamma_A^{(0)}$(*no approx.*)	1.01595
$\Gamma_A^{(0)}$(*narrow width approx. with* $m_b = 0$)/$\Gamma_A^{(0)}$(*no approx.*)	1.01618

Table 3.1: The difference resulting from the narrow–width approximation and the $m_b = 0$ limit for LO unpolarized rate in system 1a.

Numerical change in the rate is given in Table 3.1. The integrated unpolarized result in the narrow width approximation increases by 1.6% relative to the exact integrated rate.

It is quite safe to take the bottom mass to zero. The difference in the spectra for $m_b = 0$ and $m_b \neq 0$ with $m_b = 4.8$ GeV is invisible on the scale of Fig. 3.4. In the narrow width approximation the integrated rate for $m_b = 0$ increases by 2‰ relative to the integrated rate for $m_b = 4.8$ GeV. Clearly narrow width approximation and $m_b = 0$ limit are very good approximations.

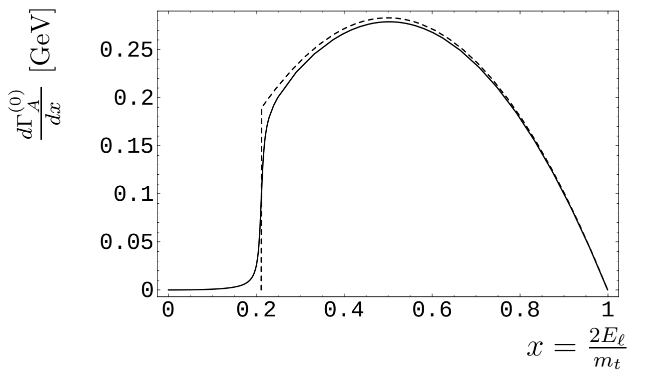


Figure 3.4: The differential rate w.r.t. the charged lepton energy (charged lepton spectrum). The dashed line corresponds to the spectrum in the narrow width approximation. The solid line corresponds to the spectrum without the narrow width (NW) approximation. The spectrum in the narrow–width approximation lies slightly above the Breit–Wigner spectrum except for the region very close to $x = y^2$.

With the narrow width approximation and a massless bottom quark we write the leading order differential rates as:

$$\begin{aligned} \frac{d\Gamma_A^{(0)}}{dx} &= \frac{d\Gamma_B^{(0)}}{dx} = \Gamma_F\, 2\pi \frac{m_W}{\Gamma_W} 6y^2\, x(1-x), \\ \frac{d\Gamma_C^{(0)}}{dx} &= 0. \end{aligned} \tag{3.50}$$

The integrated rates for $m_b = 0$ are ($y^2 \leqslant x \leqslant 1$):

$$\begin{aligned} \Gamma_A^{(0)} &= \Gamma_B^{(0)} = \Gamma_F\, 2\pi \frac{m_W}{\Gamma_W} y^2 (1-y^2)^2 (1+2y^2), \\ \Gamma_C^{(0)} &= 0\,. \end{aligned} \tag{3.51}$$

One can read off from (3.51) that the width of the top quark is enhanced by a factor of $2\pi m_W/\Gamma_W \cdot y(1-y)^2(1+2y) = 44.09$ compared to a point-like four-Fermion interaction due to the presence of the W–pole.

Let us return to (3.48). The fact that $\Gamma_A = \Gamma_B$ means that the proposed polar correlation measurement has 100% analyzing power to analyze the polarization of the top quark whereas the azimuthal correlation measurement has zero analyzing power. In the following we shall present some simple arguments to show that $\Gamma_A = \Gamma_B$ can be directly traced to the fact that we are dealing with a $(V-A)(V-A)$ current–current structure in this transition. Once this is established we then present a physics argument that $\Gamma_C = 0$ necessarily follows from $\Gamma_A = \Gamma_B$.

Let us rewrite the original $(V-A)(V-A)$ SM form into a more convenient form using the Fierz transformation of the second kind which transforms the $(V-A)(V-A)$ form into a $(S+P)(S-P)$ form (see e.g. [59]):

$$\begin{aligned} M_0 &= \bar{u}(p_b, s_b)\gamma^\mu(1-\gamma_5)u(p_t, s_t)\ \bar{u}(p_\nu, s_\nu)\gamma^\mu(1-\gamma_5)v(p_\ell, s_\ell) & (3.52)\\ &= 2\bar{u}(p_b, s_b)(1+\gamma_5)C\bar{u}^T(p_\nu, s_\nu)\ v^T(p_\ell, s_\ell)C^{-1}(1-\gamma_5)u(p_t, s_t) & (3.53)\\ &= 2\bar{u}(p_b, s_b)(1+\gamma_5)v(p_\nu, s_\nu)\ \bar{u}(p_\ell, s_\ell)(1-\gamma_5)u(p_t, s_t) & (3.54)\end{aligned}$$

where we have used $C\bar{u}^T(p_\nu, s_\nu) = v(p_\nu, s_\nu)$ and $v^T(p_\ell, s_\ell)C^{-1} = \bar{u}(p_\ell, s_\ell)$. The advantage of the form (3.54) is that the spinors of the top quark and the lepton are now connected by one Dirac string. In particular this means that there is no correlation between the top quark spin and the momenta of the b-quark jet or the neutrino, i.e. there will be no azimuthal correlation term. Returning to the spinor amplitude $\bar{u}(p_\ell, s_\ell)(1-\gamma_5)u(p_t, s_t)$ one notes that the combination $(1-\gamma_5)$ acts to project out the positive helicity spinor of the (massless) lepton. One can evaluate the amplitude $\bar{u}(p_\ell, s_\ell)(1-\gamma_5)u(p_t, s_t)$ for a top quark polarized in the (θ_P, ϕ)–direction (see Fig.1) using $u_+(p_t, s_t)^T = \sqrt{2m_t}(\cos\theta_P/2, e^{i\phi}\sin\theta_P/2, 0, 0)$ and $\bar{u}_+(p_\ell, s_\ell) = \sqrt{E_\ell}(1, 0, -1, 0)$ for a positive helicity lepton moving in the z–direction. One obtains

$$\bar{u}_+(p_\ell, s_\ell)(1-\gamma_5)u_+(p_t, s_t) = 2\sqrt{2E_\ell m_t}\cos\frac{\theta_P}{2}\,. \tag{3.55}$$

On squaring the amplitude (3.55) one finally obtains

$$|\bar{u}_+(p_\ell, s_\ell)(1-\gamma_5)u_+(p_t, s_t)|^2 = 4E_\ell m_t(1+\cos\theta_P)\,, \tag{3.56}$$

which leads to $d\Gamma_A^{(0)} = d\Gamma_B^{(0)}$ and $d\Gamma_C = 0$ as calculated in (3.48) using the trace method.

The above derivation shows that the LO result $\Gamma \sim (1+\cos\theta_P)$ does not depend on the mass of the bottom quark. It does, however, depend on the mass of the lepton. The lepton mass effect can be easily calculated from (3.20). One obtains $|M_0|^2 \sim (1 + (1-\frac{1}{2}m_\ell^2/E_\ell^2 + ...)\cos\theta_P)$. The lepton mass correction is thus negligibly small since, in the narrow resonance approximation for the W^+, the minimal lepton energy is given by $(m_W^4 + m_t^2 m_\ell^2)/(2m_t m_W^2)$ and is thus very much larger than the lepton mass appearing in the lepton mass correction.

Returning to the original current-current form (3.52) and its Fierz–transformed form (3.54) it is clear that there will be no azimuthal correlation, i.e. one has $\Gamma_C = 0$ at the Born term level. It is nevertheless instructive and interesting to go through the exercise to show that $\Gamma_C = 0$ directly follows from $\Gamma_A = \Gamma_B$ if the rate is to remain positive definite over all of phase space. We use a short–hand notation and write A for $d\Gamma_A/dx$ and B for $d\Gamma_B/dx$ etc.. With $A = B$ the angular decay distribution is given by (we set P=1)

$$\Gamma \sim A(1+\cos\theta_P + \frac{C}{A}\sin\theta_P\cos\phi)\,. \tag{3.57}$$

From the structure of (3.57) one can immediately conclude that the ratio C/A necessarily has to vanish if the rate is to remain positive definite over all of angular phase space. This can be seen in the following way. Assume first that C/A is positive. Set $\cos\phi = -1$ and expand the resulting decay distribution around $\theta_P = \pi$ $(\theta_P \leq \pi)$. One obtains

$$\Gamma \sim A(\pi-\theta_P)(\frac{\pi-\theta_P}{2} - \frac{C}{A}) \tag{3.58}$$

For any given value of C/A the piece $(\pi - \theta_P)/2$ can always be chosen small enough to render the rate to become negative. If C/A is assumed to be negative one chooses $\cos\phi = +1$ and goes through the same steps of arguments as before. The upshot is that C has to be zero if one has $A = B$ in order for the rate to be positive definite everywhere. As mentioned before the explicit calculation using the form (3.52) or more directly (3.54) of course confirms this conclusion. We have showed that the vanishing of the LO azimuthal correlation is the direct result of the $(V - A)$ structure of the SM $(V - A)$ current. Non-zero LO azimuthal correlation can be induced by a non–SM right–handed $(V + A)$ current [60]. NLO QCD corrections will also give rise to the azimuthal correlations as we will see in the next sections.

3.3 Virtual one–loop corrections

3.3.1 Vertex corrections

At one–loop level one has three Feynman diagrams (see Fig. 3.5) contributing to the virtual corrections.

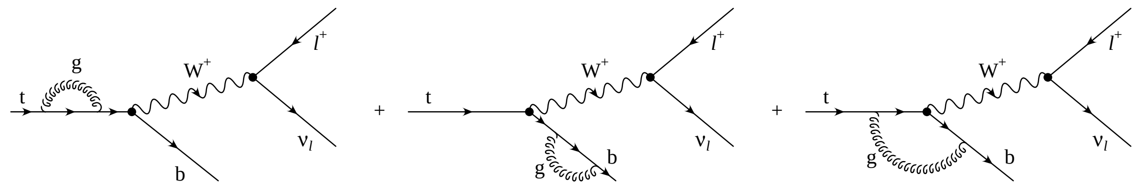


Figure 3.5: Feynman diagrams for the virtual one–loop corrections for the decay $t \to b + \ell + \nu_\ell$.

For the vertex correction (Fig 3.5 right–most) the Feynman rules give

$$
\begin{aligned}
H^{(1)}_{v,\mu} &= \bar{u}(p_b, s_b) \sum_{m,n} \int \frac{d^4k}{(2\pi)^4} \left(-ig_s\gamma_\alpha \frac{\lambda^m}{2}\right) \frac{i}{\left(\not{p}_b - \not{k}\right) - m_b} \gamma_\mu (1-\gamma_5) \frac{i}{\left(\not{p}_t - \not{k}\right) - m_t} \\
&\quad \times \left(-ig_s\gamma_\beta \frac{\lambda^n}{2}\right)\left(-i\delta_{mn}\frac{g^{\alpha\beta}}{k^2}\right) u(p_t, s_t) \\
&= \bar{u}(p_b, s_b)\left[-ig_s^2 C_F \int \frac{d^4k}{(2\pi)^4}\gamma_\alpha \frac{\not{p}_b - \not{k} + m_b}{(p_b - k)^2 - m_b^2}\gamma_\mu (1-\gamma_5) \frac{\not{p}_t - \not{k} + m_t}{(p_t-k)^2 - m_t^2}\gamma^\alpha \frac{1}{k^2}\right] u(p_t, s_t) \\
&= \bar{u}(p_b, s_b)\Lambda_\mu u(p_t, s_t)\,,
\end{aligned}
\tag{3.59}
$$

where

$$
\Lambda_\mu = -ig_s^2 C_F \int \frac{d^4k}{(2\pi)^4}\gamma_\alpha \frac{(\not{p}_b - \not{k} + m_b)\gamma_\mu (1-\gamma_5)(\not{p}_t - \not{k} + m_t)}{[(p_b-k)^2 - m_b^2][(p_t-k)^2 - m_t^2]k^2}\gamma^\alpha \tag{3.60}
$$

The integral Λ_μ has both IR and UV divergences. As in section 2.2 dimensional regularization is used for the UV divergences with space time dimension taken as $D = 4 - 2\delta$. A small gluon mass m_g is used for the IR–divergences. We rewrite Λ_μ as

$$\Lambda_\mu = -ig_s^2 C_F \int \frac{d^D k}{(2\pi)^D} \frac{4(p_t\cdot p_b)\gamma_\mu - 2\gamma_\mu \not{k}\not{p}_b - 2\not{p}_t\not{k}\gamma_\mu + (2-D)(2k_\mu\not{k} - k^2\gamma_\mu)}{[(p_b-k)^2 - m_b^2][(p_t-k)^2 - m_t^2](k^2 - m_g^2)} (1-\gamma_5)\,. \tag{3.61}$$

The numerator of Λ_μ can be simplified using the Dirac algebra, the Dirac equation and the contraction of the Dirac γ–matrices:

$$\begin{aligned}
&\gamma_\alpha \left(\not{p}_b - \not{k} + m_b\right) \gamma_\mu (1-\gamma_5) \left(\not{p}_t - \not{k} + m_t\right) \gamma^\alpha \\
&= (2p_{b\alpha} - \gamma_\alpha \not{k})\, \gamma_\mu (1-\gamma_5)\, (2p_t^\alpha - \not{k}\gamma^\alpha) \\
&= (2p_{b\alpha} - \gamma_\alpha \not{k})\, \gamma_\mu\, (2p_t^\alpha - \not{k}\gamma^\alpha)(1-\gamma_5) \\
&= \left[4\,(p_t\cdot p_b)\,\gamma_\mu - 2\gamma_\mu\not{k}\not{p}_b - 2\not{p}_t\not{k}\gamma_\mu + \gamma_\alpha\not{k}\gamma_\mu\not{k}\gamma^\alpha\right](1-\gamma_5) \\
&= \left[4(p_t\cdot p_b)\gamma_\mu - 2\gamma_\mu\not{k}\not{p}_b - 2\not{p}_t\not{k}\gamma_\mu + (2-D)(2k_\mu\not{k} - k^2\gamma_\mu)\right](1-\gamma_5)\,,
\end{aligned} \tag{3.62}$$

Inserting the above result back into Λ_μ gives

$$\Lambda_\mu = -ig_s^2 C_F \int \frac{d^4 k}{(2\pi)^4} \frac{4(p_t\cdot p_b)\gamma_\mu - 2\gamma_\mu \not{k}\not{p}_b - 2\not{p}_t\not{k}\gamma_\mu + (2-D)(2k_\mu\not{k} - k^2\gamma_\mu)}{[(p_b-k)^2 - m_b^2]\left[(p_t-k)^2 - m_t^2\right] k^2} (1-\gamma_5)\,. \tag{3.63}$$

Thus one has a two–point one–loop scalar integral B_0, a three–point one–loop scalar integral C_0 and a three–point one–loop vector integral C^μ defined by (2.48–2.52). In addition one also has a three–point one–loop tensor integral $C^{\mu\nu}$ defined by

$$C^{\mu\nu} := \frac{\mu^{4-D}}{i\pi^2} \int \frac{d^D k}{(2\pi)^{D-4}} \frac{k^\mu k^\nu}{\left[(p_b-k)^2 - m_b^2\right]\left[(p_t-k)^2 - m_t^2\right]\left(k^2 - m_g^2\right)}. \tag{3.64}$$

The tensor integrals are reduced to scalar integrals using the Passarino–Veltman method [46, 47, 50]:

$$\begin{aligned}
C^\mu &= C_1\, p_t^\mu + C_2\, p_b^\mu \\
C^{\mu\nu} &= C_{11}\, p_t^\mu p_t^\nu + C_{22}\, p_b^\mu p_b^\nu + C_{12}\, (p_t^\mu p_b^\nu + p_b^\mu p_t^\nu) + C_{00}\, g^{\mu\nu}.
\end{aligned} \tag{3.65}$$

The coefficients C_1 and C_2 for the vector integral C^μ are given in (2.68). Now we turn to the reduction of the tensor integral $C^{\mu\nu}$. The rank two symmetric tensor $C^{\mu\nu}$ can be expanded as

$$C^{\mu\nu} = C_1\, p_t^\mu\, p_t^\nu + C_{22}\, p_b^\mu\, p_b^\nu + C_{12}\, (p_t^\mu\, p_b^\nu + p_b^\mu\, p_t^\nu) + C_{00}\, g^{\mu\nu}, \tag{3.66}$$

where C_{00}, C_{11}, C_{22} and C_{12} are scalar coefficient functions. The contraction of $C^{\mu\nu}$ with $p_{t,\mu}$ or $p_{b,\mu}$ is a vector. Let us define

$$S_{1\nu} := p_1^\mu C_{\mu\nu}, \qquad S_{2\nu} := p_2^\mu C_{\mu\nu}. \tag{3.67}$$

The vectors $S_{1\nu}$ and $S_{2\nu}$ can be expanded in terms of the momenta p_t and p_b,

$$S_{1\nu} = p_1^\mu \, C_{\mu\nu} \quad := \quad R_{11}\, p_{1\nu} + R_{12}\, p_{2\nu} \tag{3.68}$$

$$S_{2\nu} = p_2^\mu \, C_{\mu\nu} \quad := \quad R_{21}\, p_{1\nu} + R_{22}\, p_{2\nu} \tag{3.69}$$

where R_{11}, R_{12}, R_{21} and R_{22} are scalar coefficients. Introduction of these intermediate scalars R_{ij} is advantageous. Since they involve only a single contraction they are simpler to write in terms of scalar loop integrals.

Contraction of $S_{1\nu}$ and $S_{2\nu}$ with the four–momenta p_t^ν and p_b^ν will determine the scalar coefficients R_{ij} in terms of C_{ij}.

$$\begin{aligned} p_t^\nu S_{1\nu} = p_t^\nu p_1^\mu \, C_{\mu\nu} &= m_t^4 \, C_{11} + (p_t \cdot p_b)^2 \, C_{22} + 2m_t^2 (p_t \cdot p_b) \, C_{12} + m_t^2 \, C_{00} \\ &= R_{11}\, m_t^2 + R_{12}\, (p_t \cdot p_b) \\ p_b^\nu S_{1\nu} = p_b^\nu p_1^\mu \, C_{\mu\nu} &= m_t^2 (p_t \cdot p_b) \, C_{11} + m_b^2 (p_t \cdot p_b) \, C_{22} \\ &\quad +2 \left[(p_t \cdot p_b) + m_t^2 m_b^2 \right] \, C_{12} + (p_t \cdot p_b) \, C_{00} \\ &= (p_t \cdot p_b) \, R_{11} + m_b^2 \, R_{12} \end{aligned} \tag{3.70}$$

Solving these equations with respect to R_{11} and R_{12} one obtains

$$R_{11} \;=\; C_{00} + m_1^2 \, C_{11} + (p_t \cdot p_b) \, C_{12}, \tag{3.71}$$

$$R_{12} \;=\; m_1^2 \, C_{12} + (p_t \cdot p_b) \, C_{22} \, . \tag{3.72}$$

With a similar calculation for $S_{2\nu}$ one obtains

$$R_{21} \;=\; (p_t \cdot p_b) \, C_{11} + m_2^2 \, C_{12}, \tag{3.73}$$

$$R_{22} \;=\; C_{00} + (p_t \cdot p_b) \; C_{12} + m_2^2 \, C_{22}. \tag{3.74}$$

We also define R_{00} as a contraction of $C_{\mu\nu}$ with the metric tensor $g_{\mu\nu}$. In D–dimensional space–time we have ($g_{\mu\nu} g^{\mu\nu} = D$):

$$R_{00} := C_{\mu\nu} g^{\mu\nu} = D \, C_{00} + m_1^2 \, C_{11} + m_2^2 \, C_{22} + (m_1^2 + m_2^2 - m_0^2) \, C_{12}. \tag{3.75}$$

Solving the group of five equations formed by (3.71 – 3.75) with respect to C_{ij} we obtain

$$C_{00} \;=\; \frac{R_{00} - R_{11} - R_{22}}{D-2}, \tag{3.76}$$

$$C_{11} \;=\; \frac{4}{\lambda} \Big[m_2^2 \, (C_{00} - R_{11}) + (p_t \cdot p_b) \, R_{21} \Big], \tag{3.77}$$

$$C_{12} \;=\; \frac{4}{\lambda} \Big[(p_t \cdot p_b) \, (R_{22} - C_{00}) - m_2^2 \, R_{12} \Big], \tag{3.78}$$

$$C_{22} \;=\; \frac{4}{\lambda} \Big[m_1^2 \, (C_{00} - R_{22} + (p_t \cdot p_b) \, R_{12} \Big] \, . \tag{3.79}$$

Next we write the scalar coefficients R_{ij} in terms of the scalar loop integrals.

$$\begin{aligned}
R_{00} = g_{\mu\nu}C^{\mu\nu} &= \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{k^2}{\left[(p_t-k)^2-m_t^2\right]\left[(p_b-k)^2-m_b^2\right]\left(k^2-m_g^2\right)} \\
&= \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{(k^2-m_g^2)+m_g^2}{\left[(p_t-k)^2-m_t^2\right]\left[(p_b-k)^2-m_b^2\right]\left(k^2-m_g^2\right)} \\
&= B_0(q,m_t,m_b)+m_g^2\,C_0(q,m_t,m_t,m_b,m_g) \\
&\downarrow \qquad \text{in the } m_g \to 0 \text{ limit} \\
&= B_0(q,m_t,m_b)\,, \qquad (3.80)
\end{aligned}$$

where[2] $q = p_t - p_b$. The coefficients R_{11} and R_{12} are determined by reducing the vector integral $S_{1,\nu}$ to loop integrals:

$$\begin{aligned}
S_{1\nu} = p_{t\nu}C^{\mu\nu} &= \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{(p_t\cdot k)k^\nu}{\left[(p_t-k)^2-m_t^2\right]\left[(p_b-k)^2-m_b^2\right]\left(k^2-m_g^2\right)} \\
&= -\frac{1}{2}\frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{\left\{[(p_t-k)^2-m_t^2]-(k^2-m_g^2)-m_g^2\right\}k^\mu}{\left[(p_t-k)^2-m_t^2\right]\left[(p_b-k)^2-m_b^2\right]\left(k^2-m_g^2\right)} \\
&= -\frac{1}{2}\frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{k^\nu}{\left[(p_b-k)^2-m_b^2\right]\left(k^2-m_g^2\right)} \\
&\quad +\frac{1}{2}\frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{k^\nu+p_b^\nu}{\left[(p_t-p_b-k)^2-m_t^2\right]\left[k^2-m_b^2\right]} \\
&\quad +\frac{1}{2}\frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{m_g^2\,k^\nu}{\left[(p_t-k)^2-m_t^2\right]\left[(p_b-k)^2-m_b^2\right]\left(k^2-m_g^2\right)} \\
&= -\frac{1}{2}B^\nu(m_b,m_b,m_g)+\frac{1}{2}B^\nu(q,m_t,m_b)+\frac{1}{2}B^\nu(q,m_t,m_b)p_b^\nu \\
&\quad +\frac{1}{2}m_g^2\,C^\nu(q,m_t,m_t,m_b,m_g)\,. \qquad (3.81)
\end{aligned}$$

In the $m_g \to 0$ limit $m_g^2\,C_0 = 0$. We also expand B^ν in term of p_t and p_b:

$$\begin{aligned}
B^\nu(m_b,m_b,0) &= B_1(m_b,m_b,0)p_b^\nu\,, \\
B^\nu(q,m_t,m_b) &= B_1(q,m_t,m_b)q^\nu \\
&= B_1(q,m_t,m_b)(p_t^\nu-p_b^\nu)\,. \qquad (3.82)
\end{aligned}$$

[2]Although $W = p_t - p_b$ in the present case we define, as before, $q = p_t - p_b$ for generality.

Substituting (3.82) into $S_{1\nu}$ and collecting terms in p_t and p_b we have

$$\begin{aligned} S_{1\nu} &= \frac{1}{2}B_1(q,m_t,m_b)p_{t\nu} \\ &+\frac{1}{2}\Big[B_0(q,m_t,m_b)+B_1(q,m_t,m_b)-B_1(m_b,m_b,0)\Big]p_b^\nu. \end{aligned} \tag{3.83}$$

Comparing this with the definition of $S_{1\nu}$ in (3.67) we obtain

$$R_{11} = \frac{1}{2}B_1(q,m_t,m_b), \tag{3.84}$$

$$R_{12} = \frac{1}{2}\Big[B_0(q,m_t,m_b)+B_1(q,m_t,m_b)-B_1(m_b,m_b,0)\Big]. \tag{3.85}$$

Repeating the same process for $S_{2\nu}$ we obtain

$$R_{22} = \frac{1}{2}\Big[B_0(q,m_t,m_b)-B_1(q,m_t,m_b)\Big]. \tag{3.86}$$

Thus all of the coefficient function R_{11}, R_{12}, R_{21} and R_{22} have been calculated. Substituting them into (3.76) we obtain R_{11}, R_{12}, R_{21} and R_{22}.

$$\begin{aligned} C_{00} &= \frac{R_{00}-R_{11}-R_{22}}{D-2} = \frac{1}{2(4-2\delta)}B_0(q,m_t,m_b) \\ &= \frac{1}{4}\left(\frac{1}{1-\delta}\right)B_0(q,m_t,m_b) = \frac{1}{4}(1+\delta)\,B_0(q,m_t,m_b) \\ &= \frac{1}{4}\Big[1+B_0(q,m_t,m_b)\Big], \end{aligned} \tag{3.87}$$

where we have used $\varepsilon B_0(q,m_t,m_b)=1$. We take B_0 from (B.23) and obtain

$$\begin{aligned} C_{00} &= \frac{1}{4}\Bigg[\Delta+3+\frac{m_t^2-m_b^2}{q^2}\ln\left(\frac{m_b}{m_t}\right)+\ln\left(\frac{\mu^2}{m_t\,m_b}\right) \\ &\qquad +\frac{\sqrt{\lambda}}{2\,q^2}\ln\left(\frac{m_t^2+m_b^2-q^2+\sqrt{\lambda}}{m_t^2+m_b^2-q^2-\sqrt{\lambda}}\right)\Bigg]. \end{aligned} \tag{3.88}$$

Similarly we obtain

$$\begin{aligned} C_{11} &= \frac{1}{4}\Bigg\{\frac{4\,m_b^2}{\lambda}+\frac{2}{q^2}\left(1-\frac{m_t^2-m_b^2}{q^2}\right)\ln\left(\frac{m_b}{m_t}\right) \\ &\quad -\frac{2\,q^2}{\lambda}\left[1-\frac{2\,m_t^2}{q^2}+\frac{(m_t^2-m_b^2)^2}{(q^2)^2}\right]\left[1+\frac{\sqrt{\lambda}}{2\,q^2}\ln\left(\frac{m_t^2+m_b^2-q^2+\sqrt{\lambda}}{m_t^2+m_b^2-q^2-\sqrt{\lambda}}\right)\right]\Bigg\}, \end{aligned}$$

$$C_{12} = \frac{1}{4}\Bigg\{-\frac{2}{\lambda}(m_t^2+m_b^2-q^2)+\frac{2}{q^2}\,\frac{m_t^2-m_b^2}{q^2}\,\ln\left(\frac{m_b}{m_t}\right)$$

$$+ \frac{2}{\lambda}\left[\frac{(m_t^2 - m_b^2)^2}{q^2} - (m_t^2 + m_b^2)\right]\left[1 + \frac{\sqrt{\lambda}}{2\,q^2}\ln\left(\frac{m_t^2 + m_b^2 - q^2 + \sqrt{\lambda}}{m_t^2 + m_b^2 - q^2 - \sqrt{\lambda}}\right)\right]\Bigg\}\,,$$

$$C_{22} = \frac{1}{4}\Bigg\{\frac{4\,m_t^2}{\lambda} - \frac{2}{q^2}\left(1 + \frac{m_t^2 - m_b^2}{q^2}\right)\ln\left(\frac{m_b}{m_t}\right)$$
$$- \frac{2\,q^2}{\lambda}\left[1 - \frac{2\,m_b^2}{q^2} + \frac{(m_t^2 - m_b^2)^2}{(q^2)^2}\right]\left[1 + \frac{\sqrt{\lambda}}{2\,q^2}\ln\left(\frac{m_t^2 + m_b^2 - q^2 + \sqrt{\lambda}}{m_t^2 + m_b^2 - q^2 - \sqrt{\lambda}}\right)\right]\Bigg\}\,. \tag{3.89}$$

Now that all the necessary loop integrals are at hand we write Λ_μ in (3.63) in terms of scalar loop integrals and their coefficients. The correspondence between the variables in the numerator of Λ_μ (left column in (3.90)) and the loop integrals (right column in (3.90)) can be summarized as

$$\begin{aligned}
1 &\longrightarrow C_0\,,\\
k^2 &\longrightarrow B_0\,,\\
k^\mu &\longrightarrow C_1\,p_t^\mu + C_2\,p_b^\mu\,,\\
k^\mu k^\nu &\longrightarrow C_{11}\,p_t^\mu p_t^\nu + C_{22}\,p_b^\mu p_b^\nu + C_{12}\,(p_t^\mu p_b^\nu + p_b^\mu p_t^\nu) + C_{00}\,g^{\mu\nu}\,,\\
k^\mu \not{k} &\longrightarrow C_{11}\,p_t^\mu \not{p}_t + C_{22}\,p_b^\mu \not{p}_b + C_{12}\,\left(p_t^\mu \not{p}_b + p_b^\mu \not{p}_t\right) + C_{00}\,\gamma^\mu\,.
\end{aligned} \tag{3.90}$$

Thus for the integral Λ_μ we obtain

$$\begin{aligned}
\Lambda_\mu &= -ig_s^2 C_F\frac{i}{16\pi^2}\Big\{4(p_t\cdot p_b)C_0\,\gamma_\mu - 2\gamma_\mu(C_1\,\not{p}_t + C_2\,\not{p}_b)\not{p}_b - 2\not{p}_t(C_1\,\not{p}_t + C_2\,\not{p}_b)\gamma_\mu \qquad (3.91)\\
&+(2-D)\Big[2(C_{11}\,p_t^\mu\not{p}_t + C_{22}\,p_b^\mu\not{p}_b + C_{12}\,\left(p_t^\mu\not{p}_b + p_b^\mu\not{p}_t\right) + C_{00}\,\gamma^\mu) - B_0\gamma_\mu\Big]\Big\}\,(1-\gamma_5)\\
&= -ig_s^2 C_F\frac{i}{16\pi^2}\Big\{4(p_t\cdot p_b)C_0\,\gamma_\mu - 2\gamma_\mu(C_1\,\not{p}_t\not{p}_b + C_2\,m_b^2) - 2(C_1\,m_t^2 + C_2\,\not{p}_t\not{p}_b)\gamma_\mu\\
&+(2-D)\Big[2C_{11}\,p_t^\mu\not{p}_t + 2C_{22}\,p_b^\mu\not{p}_b + 2C_{12}\,\left(p_t^\mu\not{p}_b + p_b^\mu\not{p}_t\right) + (2C_4 - B_0)\gamma_\mu\Big]\Big\}\,(1-\gamma_5)\,.
\end{aligned}$$

Since Λ_μ appears between quark spinors we can use the Dirac equations. We commute $\not{p}_t$ to the right and $\not{p}_b$ to the left by

$$\begin{aligned}
\gamma^\mu\not{p}_t\not{p}_b &= 2(p_t\cdot p_b)\gamma^\mu - 2p_b^\mu\not{p}_t + \not{p}_b\gamma^\mu\not{p}_t\,,\\
\not{p}_t\not{p}_b\gamma^\mu &= 2(p_t\cdot p_b)\gamma^\mu - 2p_t^\mu\not{p}_b + \not{p}_b\gamma^\mu\not{p}_t\,,
\end{aligned}$$

and use the Dirac equation $\not{p}_t u(p_t,s_t) = m_t u(p_t,s_t)$ and $\bar{u}(p_b,s_b)\not{p}_b = m_b\bar{u}(p_b,s_b)$. We also use $p_t\cdot p_b = \frac{1}{2}(m_t^2 + m_b^2 - q^2)$. We then collect coefficient functions that multiply γ^μ, p_t^μ, p_b^μ, $\gamma^\mu\gamma_5$, $p_t^\mu\gamma_5$ and $p_b^\mu\gamma_5$ according to

$$\Lambda^\mu = (V_g\gamma_\mu + V_t p_{t\mu} + V_b p_{b\mu}) - (A_g\gamma_\mu + A_t p_{t\mu} + A_b p_{b\mu})\,\gamma_5\,. \tag{3.92}$$

The coefficient functions (called form factors) are given by [3]

$$V_g = \frac{\alpha_s}{4\pi} C_F \Big\{ 2\,(C_0 - C_1 - C_2)\,\big(m_t^2 + m_b^2 - q^2\big) \\ - 2\Big[C_1\, m_t^2 + C_2\, m_b^2 + (C_1 + C_2)\, m_t m_b\Big] + (D-2)\,(B_0 - 2C_{00})\Big\}, \tag{3.93}$$

$$A_g = \frac{\alpha_s}{4\pi} C_F \Big\{ 2\,(C_0 - C_1 - C_2)\,\big(m_t^2 + m_b^2 - q^2\big) \\ - 2\Big[C_1\, m_t^2 + C_2\, m_b^2 - (C_1 + C_2)\, m_t m_b\Big] + (D-2)\,(B_0 - 2C_{00})\Big\}, \tag{3.94}$$

$$V_t = \frac{\alpha_s}{4\pi} C_F \Big[4C_2 m_b - 2\,(D-2)\,(C_{11} m_t + C_{12} m_b)\Big], \tag{3.95}$$

$$A_t = \frac{\alpha_s}{4\pi} C_F \Big[4C_2 m_b - 2\,(D-2)\,(-C_{11} m_t + C_{12} m_b)\Big], \tag{3.96}$$

$$V_b = \frac{\alpha_s}{4\pi} C_F \Big[4C_1 m_t - 2\,(D-2)\,(C_{12} m_t + C_{22} m_b)\Big], \tag{3.97}$$

$$A_b = \frac{\alpha_s}{4\pi} C_F \Big[-4C_1 m_t - 2\,(D-2)\,(-C_{12} m_t + C_{22} m_b)\Big]. \tag{3.98}$$

Note that the only difference between the respective vector and axial vector form factors are that the terms with m_t change sign. This can be traced back to the occurrence of γ_5 in the axial vector form factors.

3.3.2 Renormalization

The renormalization is similar to the decay $t \to H^+ + b$. At one–loop order in the on–shell renormalization scheme the quark self energy diagrams from the first two graphs of (3.5) determine the renormalization constants for the quark masses and the quark fields as discussed in subsection 2.2.3. The self energy contributions vanish as explained in (2.104) and we are left with the renormalization of the vertex corrections. At $\mathcal{O}(\alpha_s)$ we write the hadron current for the vertex corrections as

$$H^\mu = \bar{u}(p_b, s_b)\,\Gamma^\mu\, u(p_t, s_t)\,. \tag{3.99}$$

The vertex Γ^μ is given by

$$\Gamma^\mu = \gamma^\mu\,(1 - \gamma_5) + \Lambda^\mu\,, \tag{3.100}$$

where $\gamma^\mu\,(1-\gamma_5)$ is the LO vertex and Λ^μ is the NLO one–loop contribution given in (3.92)–(3.98). The renormalization prescription relates the unrenormalized vertex Λ^μ with the renormalized vertex Λ_R^μ by

$$\Gamma^\mu = Z_1 \Gamma_R^\mu\,. \tag{3.101}$$

[3] g_s^2 is substituted by $g_s^2 = 4\pi\,\alpha_s$.

Inserting the above into (3.100) and using $Z_1 = 1 + \delta Z_1$ one has

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu (1-\gamma_5) + \Lambda^\mu \\ &= (1+\delta Z_1)\Big[\gamma^\mu (1-\gamma_5) + \Lambda^\mu_R\Big] \\ &= \gamma^\mu (1-\gamma_5) + \Lambda^\mu_R + \delta Z_1 \, \gamma^\mu (1-\gamma_5) + \mathcal{O}(\alpha_s^2)\,. \end{aligned} \tag{3.102}$$

Neglecting the higher order terms in α_s gives

$$\Lambda^\mu_R = \Lambda^\mu - \delta Z_1 \, \gamma^\mu (1-\gamma_5)\;. \tag{3.103}$$

We substitute Λ^μ given in (3.92) into (3.103) and obtain the renormalized vertex as

$$\Lambda^\mu_R = \Big((V_g - \delta Z_1)\gamma^\mu + V_t\, p_t^\mu + V_b\, p_b^\mu\Big) - \Big((A_g - \delta Z_1)\gamma^\mu + A_t\, p_t^\mu + A_b\, p_b^\mu\Big)\gamma_5. \tag{3.104}$$

Note that only the form factors V_g and A_g are renormalized, i.e. renormalization is only relevant for V_g and A_g. We can now write the renormalized vertex in terms of the renormalized form factors:

$$\Lambda^\mu_R := (V_{g,R}\gamma^\mu + V_{t,R}\, p_t^\mu + V_{b,R}\, p_b^\mu) - (A_{g,R}\gamma^\mu + A_{t,R}\, p_t^\mu + A_{b,R}\, p_b^\mu)\gamma_5,$$

Comparing (3.104) and (3.105) we obtain the renormalized form factors:

$$\begin{aligned} V_{g,R} &= V_g - \delta Z_1, & A_{g,R} &= A_g - \delta Z_1, \\ V_{t,R} &= V_t, & A_{t,R} &= A_t, \\ V_{b,R} &= V_b, & A_{b,R} &= A_b. \end{aligned} \tag{3.105}$$

The Ward-Takahashi identity [53, 54] gives the relation between the remormalization constants for the quark field and the coupling constant as

$$Z_1 = \sqrt{Z_2^t}\sqrt{Z_2^b}. \tag{3.106}$$

That is, on the one–loop level,

$$\delta Z_1 = \frac{1}{2}\delta Z_2^t + \frac{1}{2}\delta Z_2^b \tag{3.107}$$

Inserting δZ_2 from (2.111) one has

$$\delta Z_1 = -\frac{\alpha_s}{4\pi} C_F \left[\Delta + 4 + \ln\left(\frac{\mu^2}{m_t m_b}\right) + 2\ln\left(\frac{m_g^2}{m_t m_b}\right)\right]. \tag{3.108}$$

Substituting δZ_1 from (3.108) and the unrenormalized form factors from (3.93) into the renormalized form factors (3.105) the NLO renormalized vertex reads

$$\Lambda^\mu = (V_g\gamma^\mu + V_t\, p_t^\mu + V_b\, p_b^\mu) - (A_g\gamma^\mu + A_t\, p_t^\mu + A_b\, p_b^\mu)\gamma_5. \tag{3.109}$$

where the renormalized form factors are given by

$$V_g = \frac{\alpha_s}{4\pi} C_F \left\{ \frac{2\bar{p}_0}{\bar{p}_3} \left[\mathrm{Li}_2 \left(1 - \frac{\bar{w}_-}{\bar{w}_+} \right) - \mathrm{Li}_2 \left(1 - \frac{\bar{p}_- \bar{w}_-}{\bar{p}_+ \bar{w}_+} \right) + (2 \ln \Lambda - \ln \epsilon) \bar{Y}_p \right. \right.$$
$$\left. - \left(\bar{Y}_p + 2\bar{Y}_w \right) \left(\bar{Y}_p + \ln \epsilon \right) \right] - 4 - 2 (2 \ln \Lambda - \ln \epsilon) - \frac{1-\epsilon^2}{y^2} \ln \epsilon$$
$$\left. + \left[\frac{(1+\epsilon)^2 - y^2}{\bar{p}_3} - \frac{2\bar{p}_3}{y^2} \right] \bar{Y}_p \right\}, \tag{3.110}$$

$$V_t = \frac{\alpha_s}{4\pi} C_F \frac{1}{m_t} \frac{(1-\epsilon)}{y^2} \left\{ 2 - 2 \left(\frac{1+2\epsilon}{1-\epsilon} - \frac{1-\epsilon^2}{y^2} \right) \ln \epsilon \right.$$
$$\left. + \left[\frac{4\bar{p}_3}{y^2} - \frac{\epsilon}{1-\epsilon} \frac{(1-\epsilon)(3+\epsilon) + y^2}{\bar{p}_3} \right] \bar{Y}_p \right\}, \tag{3.111}$$

$$V_b = -\frac{\alpha_s}{4\pi} C_F \frac{1}{m_t} \frac{(1-\epsilon)}{y^2} \left\{ 2 - 2 \left(\frac{2+\epsilon}{1-\epsilon} - \frac{1-\epsilon^2}{y^2} \right) \ln \epsilon \right.$$
$$\left. + \left[\frac{4\bar{p}_3}{y^2} - \frac{1}{1-\epsilon} \frac{(1-\epsilon)(1+3\epsilon) - y^2}{\bar{p}_3} \right] \bar{Y}_p \right\}, \tag{3.112}$$

$$A_g = \frac{\alpha_s}{4\pi} C_F \left\{ \frac{2\bar{p}_0}{p_3} \left[\mathrm{Li}_2 \left(1 - \frac{\bar{w}_-}{\bar{w}_+} \right) - \mathrm{Li}_2 \left(1 - \frac{\bar{p}_- \bar{w}_-}{\bar{p}_+ \bar{w}_+} \right) + (2 \ln \Lambda - \ln \epsilon) \bar{Y}_p \right. \right.$$
$$\left. - \left(\bar{Y}_p + 2\bar{Y}_w \right) \left(\bar{Y}_p + \ln \epsilon \right) \right] - 4 - 2 (2 \ln \Lambda - \ln \epsilon) - \frac{1-\epsilon^2}{y^2} \ln \epsilon$$
$$\left. + \left[\frac{(1-\epsilon)^2 - y^2}{\bar{p}_3} - \frac{2\bar{p}_3}{y^2} \right] \bar{Y}_p \right\}, \tag{3.113}$$

$$A_t = -\frac{\alpha_s}{4\pi} C_F \frac{1}{m_t} \frac{(1+\epsilon)}{y^2} \left\{ 2 - 2 \left(\frac{1-2\epsilon}{1+\epsilon} - \frac{1-\epsilon^2}{y^2} \right) \ln \epsilon \right.$$
$$\left. + \left[\frac{4\bar{p}_3}{y^2} + \frac{\epsilon}{1+\epsilon} \frac{(1+\epsilon)(3-\epsilon) + y^2}{\bar{p}_3} \right] \bar{Y}_p \right\}, \tag{3.114}$$

$$A_b = \frac{\alpha_s}{4\pi} C_F \frac{1}{m_t} \frac{(1+\epsilon)}{y^2} \left\{ 2 - 2 \left(\frac{2-\epsilon}{1+\epsilon} - \frac{1-\epsilon^2}{y^2} \right) \ln \epsilon \right.$$
$$\left. + \left[\frac{4\bar{p}_3}{y^2} - \frac{1}{1+\epsilon} \frac{(1+\epsilon)(1-3\epsilon) - y^2}{\bar{p}_3} \right] \bar{Y}_p \right\}. \tag{3.115}$$

Finally the renormalized hadron current for the one–loop virtual correction reads

$$H^{(1)}_{v,\mu} = \bar{u}(p_b, s_b) \Lambda_\mu u(p_t, s_t) \,, \tag{3.116}$$

where the renormalized vertex Λ_μ is given in (3.109). Note that in (3.109)–(3.116) we have dropped the subscript R denoting renormalized quantities, since from this point on we shall only work with renormalized quantities. As before we use scaled variables. The UV divergent terms in unrenormalized form factors V_g and A_g, or more specifically in B_0, are canceled by the renormalization constant δZ_1, as they should. The IR–divergent terms remain. Note that in the hadron current H^μ the IR–divergent terms come with V_g and A_g. Consequently squaring the current and contracting with the lepton tensor, the IR–divergent terms will be proportional to the leading order contribution. We shall see that the IR–divergent terms in the real emission contribution is proportional to the leading order rate. When we sum up virtual and real corrections the IR–divergences will also cancel, as required by the Lee–Nauenberg theorem [61].

3.3.3 Renormalized virtual one–loop correction

The full $\mathcal{O}(\alpha_s)$ hadron current is given by

$$H_\mu = H_\mu^{(0)} + H_{v,\mu}^{(1)} + H_{r,\mu}^{(1)}\,, \tag{3.117}$$

where $H_\mu^{(0)}$ is the LO hadron current. $H_{v,\mu}^{(1)}$ and $H_{r,\mu}^{(1)}$ are the hadron currents for the virtual one–loop and real emission corrections at $\mathcal{O}(\alpha_s)$. For the hadron tensor one has

$$\begin{aligned} H_{\mu\nu} &= \sum_{s_b} H_\mu H_\nu^\dagger = \sum_{s_b} \left(H_\mu^{(0)} + H_{v,\mu}^{(1)} + H_{r,\mu}^{(1)}\right)\left(H_\nu^{(0)} + H_{v,\mu}^{(1)} + H_{r,\mu}^{(1)}\right)^\dagger \\ &= \sum_{s_b} H_\mu^{(0)} H_\nu^{(0)\dagger} + \sum_{s_b}\left\{H_\mu^{(0)} H_{v,\nu}^{(1)\dagger} + H_{v,\mu}^{(1)} H_\nu^{(0)\dagger} + H_{r,\mu}^{(1)} H_{r,\nu}^{(1)\dagger}\right\} + \mathcal{O}(\alpha_s^2) \\ &= H_{\mu\nu}^{(0)} + H_{v,\mu\nu}^{(1)} + H_{r,\mu\nu}^{(1)} + \mathcal{O}(\alpha_s^2), \end{aligned} \tag{3.118}$$

where the hadron tensor for the NLO virtual one–loop and real emission corrections are given by

$$H_{v,\mu\nu}^{(1)} = \sum_{s_b} \left(\, H_\mu^{(0)} H_{v,\nu}^{(1)\dagger} + H_{v,\mu}^{(1)} H_\nu^{(0)\dagger}\,\right)\,, \tag{3.119}$$

$$H_{r,\mu\nu}^{(1)} = \sum_{s_b} H_{r,\mu}^{(1)} H_{r,\nu}^{(1)\dagger}\,. \tag{3.120}$$

The LO hadron current $H_\mu^{(0)}$ and the hadron current for the virtual one–loop correction $H_{v,\mu}^{(1)}$ are given in (3.11) and (3.116).

The vertex amplitude squared is given by

$$\left(M_v^{(1)}\right)^2 = \frac{G_F^2 |V_{tb}|^2}{2\left[(1-y^2/\hat{y}^2)^2 + \gamma^2\right]} H_{v,\mu\nu}^{(1)} L^{\mu\nu}. \tag{3.121}$$

Substituting the renormalized vertex into the hadron tensor we have

$$
\begin{aligned}
H^{(1)}_{v,\mu\nu} = \sum_{s_b} \Big[& \bar{u}(p_b, s_b)\gamma_\mu (1-\gamma_5)\, u(p_t, s_t)\bar{u}(p_t, s_t)\gamma_0 \Lambda^\dagger_\nu \gamma_0 u(p_b, s_b) \\
& + \bar{u}(p_b, s_b)\Lambda_\mu u(p_t, s_t)\bar{u}(p_t, s_t)\,(1+\gamma_5)\,\gamma_\nu u(p_b, s_b)\Big] \\
= \mathrm{Tr}\Big[& \left(\not{p}_b + m_b\right)\gamma_\mu (1-\gamma_5) \left(\not{p}_t + m_t\right) \frac{1}{2}\left(1+\gamma_5 \not{s}_t\right) \\
& (V_g\gamma_\nu + V_t p_{t\nu} + V_b p_{b\nu} - A_g\gamma_\nu\gamma_5 + A_t p_{t\nu}\gamma_5 + A_b p_{b\nu}\gamma_5)\Big] \\
+ \mathrm{Tr}\Big[& \left(\not{p}_b + m_b\right)(V_g\gamma_\mu + V_t p_{t\mu} + V_b p_{b\mu} - A_g\gamma_\mu\gamma_5 - A_t p_{t\mu}\gamma_5 - A_b p_{b\mu}\gamma_5) \\
& \left(\not{p}_t + m_t\right)\frac{1}{2}\left(1+\gamma_5\not{s}_t\right)\gamma_\nu (1-\gamma_5)\Big] \\
= 4\,(V_g + A_g) & \left[\tilde{p}_{t\mu} p_{b\nu} + \tilde{p}_{t\nu} p_{b\mu} - (\tilde{p}_t \cdot p_b)\, g_{\mu\nu} - i\epsilon_{\alpha\beta\mu\nu}\tilde{p}_t^\alpha p_b^\beta\right] \\
+ 4\,(V_g - A_g) & \frac{m_b}{m_t}\left(m_t^2 g_{\mu\nu} + i\epsilon_{\alpha\beta\mu\nu}\tilde{p}_t^\alpha p_t^\beta\right) \\
+ 2\,(V_t + A_t) & \, m_b\,(\tilde{p}_t^\mu p_t^\nu + \tilde{p}_t^\nu p_t^\mu) \\
+ 2\,(V_b + A_b) & \, m_b\,(\tilde{p}_t^\mu p_b^\nu + \tilde{p}_t^\nu p_b^\mu) \\
+ 2\,(V_t - A_t) & \frac{1}{m_t}\Big[(p_t\cdot p_b)\,(\tilde{p}_{t\mu} p_{t\nu} + \tilde{p}_{t\nu} p_{t\mu}) + m_t^2\,(p_{t\mu} p_{b\nu} + p_{t\nu} p_{b\mu}) \\
& - 2\,(\tilde{p}_t \cdot p_b)\, p_{t\mu} p_{t\nu} + \tilde{p}_t^\alpha p_t^\beta p_b^\gamma\,(i\epsilon_{\alpha\beta\gamma\nu} p_{t\mu} - i\epsilon_{\alpha\beta\gamma\mu} p_{t\nu})\Big] \\
+ 2\,(V_b - A_b) & \frac{1}{m_t}\Big[(p_t\cdot p_b)\,(\tilde{p}_{t\mu} p_{b\nu} + \tilde{p}_{t\nu} p_{b\mu}) - (\tilde{p}_t\cdot p_b)\,(p_{t\mu} p_{b\nu} + p_{t\nu} b\mu) \\
& + 2m_t^2 p_{b\mu} p_{b\nu} \;+ \tilde{p}_t^\alpha p_t^\beta p_b^\gamma\,(i\epsilon_{\alpha\beta\gamma\nu} p_{b\mu} - i\epsilon_{\alpha\beta\gamma\mu} p_{b\nu})\Big],
\end{aligned} \tag{3.122}
$$

with the usual short–hand notation $\tilde{p}_t = p_t - m_t\, s_t$. Contracting the hadron tensor $(H_v^{(1)})_{\mu\nu}$ with the lepton tensor (3.18) one has[4]

$$
\begin{aligned}
H^{(1)}_{v,\mu\nu} L^{\mu\nu} = -256\Big\{ & (-\frac{V_g + A_g}{2})\,(\tilde{p}_t\cdot p_\ell)\,(p_b\cdot p_\nu) \\
+ \epsilon^2 & \left(-m_t \frac{V_t + V_b + A_t + A_b}{4\epsilon}\right)(\tilde{p}_t\cdot p_\ell)\,(p_t\cdot p_\nu) \\
+ & \left(-m_t\frac{V_t + V_b - A_t - A_b}{4}\right)(p_b\cdot p_\nu)\left[(p_b\cdot p_\ell) + \frac{(\tilde{p}_t\cdot p_\ell)\,(p_t\cdot p_b) - (\tilde{p}_t\cdot p_b)\,(p_t\cdot p_\ell)}{m_t^2}\right]
\end{aligned}
$$

[4](3.123) agrees with Eq. (2.7) of [26].

$$+\frac{1}{2}\epsilon^2 m_t^2 \left(\frac{V_g - A_g}{4\epsilon}\right)\left[(p_\ell \cdot p_\nu) + \frac{(\tilde{p}_t \cdot p_\ell)(p_t \cdot p_\nu) - (p_t \cdot p_\ell)(\tilde{p}_t \cdot p_\nu)}{m_t^2}\right]\Bigg\}. \qquad (3.123)$$

The coefficients of the scalar products are the linear combinations of the form factors. The form factors appearing in (3.123) have been worked out in (3.110)-(3.115). Although the form factors $V_{g,t,b}$ and $A_{g,t,b}$ are somewhat lengthy the coefficients as the linear combinations of the form factors are relatively simpler.

$$\begin{aligned} -\frac{V_g + A_g}{2} = \left(\frac{\alpha_s}{4\pi}C_F\right)\Bigg\{ \frac{2\bar{p}_0}{\bar{p}_3}\Bigg[\mathrm{Li}_2\left(1 - \frac{\bar{p}_-\bar{w}_-}{\bar{p}_+\bar{w}_+}\right) - \mathrm{Li}_2\left(1 - \frac{\bar{w}_-}{\bar{w}_+}\right) - \bar{Y}_p\left(\bar{Y}_p + 1\right) \\ + 2\left(\bar{Y}_p + \ln\epsilon\right)\left(\bar{Y}_p + \bar{Y}_w\right)\Bigg] + \frac{1}{y}\Big[2\bar{p}_3\bar{Y}_p + \left(1 - 2y - \epsilon^2\right)\ln\epsilon\Big] \\ - 4\left(\frac{\bar{p}_0}{\bar{p}_3}\bar{Y}_p - 1\right)\ln\Lambda + 4\Bigg\}, \end{aligned} \qquad (3.124)$$

$$-m_t\frac{V_t + V_b + A_t + A_b}{4\epsilon} = \left(\frac{\alpha_s}{4\pi}C_F\right)\left\{\frac{\bar{Y}_p}{2\bar{p}_3}\left(1 + \frac{1-\epsilon^2}{y}\right) + \frac{1}{y}\ln\epsilon\right\}, \qquad (3.125)$$

$$-m_t\frac{V_t + V_b - A_t - A_b}{4} = \left(\frac{\alpha_s}{4\pi}C_F\right)\left\{\frac{\bar{Y}_p}{2\bar{p}_3}\left(1 - \frac{1-\epsilon^2}{y}\right) - \frac{1}{y}\ln\epsilon\right\}, \qquad (3.126)$$

$$\frac{V_g - A_g}{4\epsilon} = \left(\frac{\alpha_s}{4\pi}C_F\right)\frac{\bar{Y}_p}{\bar{p}_3}. \qquad (3.127)$$

The scalar products in (3.123) have been worked out in (3.21) and (3.24). Then, upon collecting terms according to their angular dependence, we write

$$H^{(1)}_{v,\mu\nu}L^{\mu\nu} = \left(-\frac{\alpha_s}{4\pi}C_F\right)256\frac{m^4}{4}\Big[M_v^A(x) + M_v^B(x)\,P\cos\theta_P + M_v^C(x)\,P\sin\theta_P\cos\phi\Big], \quad (3.128)$$

where M_v^A, M_v^B and M_v^C correspond to the unpolarized, polar and azimuthal contributions respectively. It is quite remarkable that the unpolarized and polar contributions are equal to each other. This is not manifest in (3.123) but can be worked out using the relations (3.21) and (3.24). Separating the IR–convergent and IR–divergent parts we have

$$M_v^A(x) = M_v^B(x) = M_0^A(x)M_{v,div} + M_{v,conv}(x) \qquad (3.129)$$

where $M_0^A(x)$ is given in (3.27), and

$$\begin{aligned} M_{v,div} &= \left(-\frac{\alpha_s}{4\pi}C_F\right)^{-1}\left(\frac{V_g + A_g}{2}\right) \\ &= \frac{2\bar{p}_0}{\bar{p}_3}\Bigg[\mathrm{Li}_2\left(1 - \frac{\bar{p}_-\bar{w}_-}{\bar{p}_+\bar{w}_+}\right) - \mathrm{Li}_2\left(1 - \frac{\bar{w}_-}{\bar{w}_+}\right) \\ &\quad -\bar{Y}_p\left(\bar{Y}_p + 1\right) + 2\left(\bar{Y}_p + \ln\epsilon\right)\left(\bar{Y}_p + \bar{Y}_w\right)\Bigg] \\ &\quad + \frac{1}{y^2}\Big[2\bar{p}_3\bar{Y}_p + \left(1 - 2y^2 - \epsilon^2\right)\ln\epsilon\Big] - 4\left(\frac{\bar{p}_0}{\bar{p}_3}\bar{Y}_p - 1\right)\ln\Lambda + 4\,, \end{aligned} \qquad (3.130)$$

$$\begin{aligned}
M_{v,conv}(x) &= \frac{1}{y^2}\Big[(1-x)\left(y^2-x\right)+\left(2x-x^2-y^2+xy^2\right)\epsilon^2-x\epsilon^4\Big]\ln(\epsilon) \\
&+\frac{\bar{Y}_p}{2y^2\bar{p}_3}\Big[\left(1-y^2\right)\left(-x+x^2+y^2-xy^2\right) \\
&-\left(2x^2-3x+2y^2-2xy^2+x^2y^2-3y^4-xy^4\right)\epsilon^2 \\
&+\left(x^2-3x+y^2-2xy^2\right)\epsilon^4+x\epsilon^6\Big]. \qquad (3.131)
\end{aligned}$$

The azimuthal contribution is not IR–divergent. This is because the IR–divergence in the loop is proportional to the leading order rate and the leading order azimuthal rate is zero. For the azimuthal loop contribution we have

$$\begin{aligned}
M_v^C(x) &= -\sqrt{y^2\left(x-x^2-y+xy^2-x\epsilon^2\right)}\Big\{\left(1-x-\epsilon^2\right)\frac{\ln(\epsilon)}{y^2} \\
&+\big[(1-x)\left(1-y^2\right)-\left(2-x-3y^2\right)\epsilon^2+\epsilon^4\big]\frac{\bar{Y}_p}{2y^2\,\bar{p}_3}\Big\}. \qquad (3.132)
\end{aligned}$$

We can now give the virtual one–loop contribution to the differential rate,

$$\begin{aligned}
d\Gamma_v^{(1)} &= \frac{1}{2m_t}\frac{1}{(2\pi)^5}dR_3(p_t;p_b,p_\ell,p_\nu)\left(M_v^{(1)}\right)^2 \\
&= \frac{1}{2m_t}\frac{1}{(2\pi)^5}\frac{\pi^2}{4}m_t^2\,\frac{1}{4\pi}d\cos\theta_P d\phi\, dy^2\, dx\frac{G_F^2|V_{tb}|^2}{2\left[(1-y^2/\hat{y}^2)^2+\gamma^2\right]}H_{v,\mu\nu}^{(1)}L^{\mu\nu} \\
&\downarrow \quad \text{in the narrow–width approximation} \\
&= \Gamma_F 2\pi\frac{m_W}{\Gamma_W}C_F(-\frac{\alpha_s}{2\pi})6y^2\Big[M_v^A+M_v^B\,P\cos\theta_P+M_v^C\,P\sin\theta_P\cos\phi\Big]dx\,d\cos\theta_P d\phi.
\end{aligned}$$

As usual we write the differential rate in the form

$$\begin{aligned}
\frac{d\Gamma_v^{(1)}}{dxd\cos\theta_P d\phi} &= \Gamma_F 2\pi\frac{m_W}{\Gamma_W}C_F(-\frac{\alpha_s}{2\pi})6y^2\times \\
&\times\frac{1}{4\pi}\Big[M_v^A(x)+M_v^B(x)\,P\cos\theta_P+M_v^C(x)\,P\sin\theta_P\cos\phi\Big], \qquad (3.133)
\end{aligned}$$

where M_v^i ($i=A,\,B,\,C$) are given in (3.129)–(3.132).

3.4 Real gluon emissions

3.4.1 The amplitude squared for the real emissions

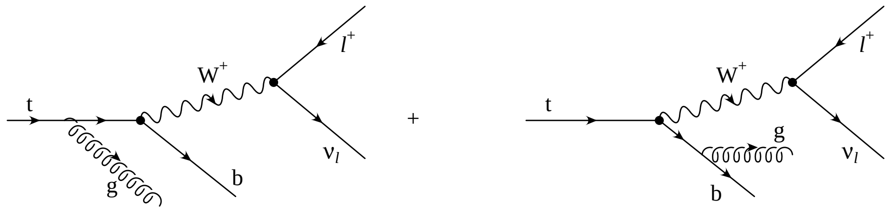


Figure 3.6: Feynman diagrams for the NLO real gluon emissions for the decay $t \to b + \ell^+ + \nu_\ell$.

The Feynman diagrams for the real gluon emissions at $\mathcal{O}(\alpha_s)$ are shown in Fig 3.6. The Feynman rules give the hadron current as

$$H^{(1)}_{r,\mu} = \bar{u}(p_b, s_b)\gamma_\mu (1-\gamma_5)\, i\frac{\not{p}_t - \not{k} + m_t}{(\not{p}_t - \not{k})^2 - m_t^2}\Big(-ig_s\frac{\lambda^n}{2}\gamma_\sigma\Big)\epsilon^{\sigma *}(k,\lambda)u(p_t, s_t)$$

$$+\,\bar{u}(p_b, s_b)\Big(-ig_s\frac{\lambda^n}{2}\gamma_\sigma\Big)\epsilon^{\sigma *}(k,\lambda) i\frac{\not{p}_b + \not{k} + m_b}{(\not{p}_b + \not{k})^2 - m_b^2}\gamma_\mu (1-\gamma_5)\, u(p_t, s_t). \quad (3.134)$$

In the numerator $\not{p}_t$ is commuted to the right and $\not{p}_b$ to the left using the commutation relations of the γ–matrices. Then one uses the Dirac equation of the quark spinors $\not{p}_t u(p_t, s_t) = m_t u(p_t, s_t)$ and $\bar{u}(p_b, s_b)\not{p}_b = m_b\bar{u}(p_b, s_b)$. In the denominator one uses the "mass–shell" condition $p_t^2 = m_t^2$ and $p_b^2 = m_b^2$. Thus the hadron current simplifies to

$$H^{(1)}_{r,\mu} = (-ig_s\frac{\lambda^n}{2})\bar{u}(p_b, s_b)\left[\frac{\gamma_\mu\not{k}\gamma_\sigma - 2p_{t\sigma}\gamma_\mu}{2p_t\cdot k} + \frac{\gamma_\sigma\not{k}\gamma_\mu + 2p_{b\sigma}\gamma_\mu}{2p_b\cdot k}\right](1-\gamma_5)\,\epsilon^{\sigma *}(k,\lambda)u(p_t, s_t). \quad (3.135)$$

We separate the terms with and without a gluon momentum in the numerator and write

$$H^{(1)}_{r,\mu} = (-ig_s\frac{\lambda^n}{2})\bar{u}(p_b, s_b)\left[(\frac{p_{b\sigma}}{p_b\cdot k} - \frac{p_{t\sigma}}{p_t\cdot k})\gamma_\mu + \frac{\gamma_\sigma\not{k}\gamma_\mu}{2p_b\cdot k} + \frac{\gamma_\mu\not{k}\gamma_\sigma}{2p_t\cdot k}\right](1-\gamma_5)\,\epsilon^{\sigma *}(k,\lambda)u(p_t, s_t). \quad (3.136)$$

For the hadron tensor one obtains

$$\begin{aligned} H^{(1)}_{r,\mu\nu} &= \sum_{s_b}\sum_{\lambda} H^{(1)}_{r,\mu}\,(H^{(1)}_{r,\nu})^\dagger \\ &= -g_s^2 C_F\,\mathrm{Tr}\Big\{(\not{p}_b + m_b)\Big[(\frac{p_{b\sigma}}{p_b\cdot k} - \frac{p_{t\sigma}}{p_t\cdot k})\gamma_\mu + \frac{\gamma_\sigma\not{k}\gamma_\mu}{2p_b\cdot k} + \frac{\gamma_\mu\not{k}\gamma_\sigma}{2p_t\cdot k}\Big](1-\gamma_5)\times \\ &\times\Big(\not{p}_t + m_t\Big)\frac{1}{2}(1-\gamma_5\not{s}_t)\Big[(\frac{p_b^\sigma}{p_b\cdot k} - \frac{p_t^\sigma}{p_t\cdot k})\gamma_\nu + \frac{\gamma^\sigma\not{k}\gamma_\nu}{2p_b\cdot k} + \frac{\gamma_\nu\not{k}\gamma^\sigma}{2p_t\cdot k}\Big](1-\gamma_5)\Big\}, \end{aligned} \quad (3.137)$$

where we have used the gluon polarization sum given in (2.131). Similar to the analysis in section 2.3.1, the terms without a gluon momentum in the numerator are IR–divergent:

$$\begin{aligned}
H^{div}_{\mu\nu} &= -g_s^2 C_F \,\mathrm{Tr}\Big\{ (\not{p}_b + m_b)\Big[(\frac{p_{b\sigma}}{p_b\cdot k} - \frac{p_{t\sigma}}{p_t\cdot k})\gamma_\mu\Big](1-\gamma_5)\,\times \\
&\quad \times\Big(\not{p}_t + m_t\Big)\frac{1}{2}(1-\gamma_5\not{s}_t)\Big[(\frac{p_b^\sigma}{p_b\cdot k} - \frac{p_t^\sigma}{p_t\cdot k})\gamma_\nu\Big](1-\gamma_5)\Big\} \\
&= -\alpha_s\, 4\pi\, C_F\Big\{\frac{m_t^2}{(k\cdot p_t)^2} + \frac{m_b^2}{(k\cdot p_b)^2} - 2\frac{p_b\cdot p_t}{(k\cdot p_b)(k\cdot p_t)}\Big\}\times \\
&\quad \times \mathrm{Tr}\Big[\Big(\not{p}_b + m_b\Big)\gamma_\mu\,(1-\gamma_5)\Big(\not{p}_t + m_t\Big)\Big(\frac{1+\gamma_5\not{s}_t}{2}\Big)\gamma_\nu\,(1-\gamma_5)\Big] \\
&= \widetilde{H}^{(0)}_{\mu\nu}\,|M|^2_{SGF}
\end{aligned}$$

where

$$\widetilde{H}^{(0)}_{\mu\nu} = 4p_b^\alpha \tilde{p}_t^\beta\, T_{\alpha\beta\mu\nu}\,, \tag{3.138}$$

with the usual abbreviation $\tilde{p}_t = p_t - m_t s_t$. $|M|^2_{SGF}$ is the IR–divergent soft–gluon factor defined in (2.134). The hadron tensor $\widetilde{H}^{(0)}_{\mu\nu}$ has the same form as the leading order hadron tensor (3.15) but must be evaluated for $p_t = p_b + p_\ell + p_\nu + k$. Thus $\widetilde{H}^{(0)}_{\mu\nu}$ has a residual gluon momentum dependence. In the limit $k \to 0$ one recovers the leading order hadron tensor $H^{(0)}_{\mu\nu}$. The remaining piece of the real emission hadron tensor is IR–convergent and we denote it by $H^{conv}_{\mu\nu}$. Accordingly we have

$$H^{(1)}_{r,\mu\nu} = H^{conv}_{\mu\nu} + \widetilde{H}^{(0)}_{\mu\nu}\,|M|^2_{SGF} \tag{3.139}$$

where $\widetilde{H}^{(0)}_{\mu\nu}\,|M|^2_{SGF}$ is divergent. For the convergent piece one obtains

$$\begin{aligned}
H^{conv,\mu\nu} &= 4\pi\alpha_s\, C_F\,\frac{4}{(k\cdot p_t)(k\cdot p_b)}\Big\{-\frac{k\cdot p_t}{k\cdot p_b}\Big[(p_b\cdot p_b)(k^\mu \bar{p}_t^\nu + k^\nu \bar{p}_t^\mu - k\cdot\bar{p}_t\, g^{\mu\nu}) \\
&+ i\Big(\epsilon^{\sigma\rho\mu\nu}\,(p_b-k)\cdot\tilde{p}_t - \epsilon^{\sigma\rho\gamma\nu}(p_b-k)^\mu\,\tilde{p}_{t,\gamma} + \epsilon^{\sigma\rho\gamma\mu}(p_b-k)^\nu\,\tilde{p}_{t,\gamma}\Big)k_\sigma\, p_{b,\rho}\Big] \\
&+ \frac{k\cdot p_b}{k\cdot p_t}\Big[(\tilde{p}_t\cdot p_t)\Big(k^\mu\, p_b^\nu + k^\nu\, p_b^\mu - k\cdot p_b\, g^{\mu\nu} - i\,\epsilon^{\sigma\rho\mu\nu}k_\sigma\, p_{b,\rho}\Big) \\
&- (\tilde{p}_t\cdot k)\Big((p_t-k)^\mu\, p_b^\nu + (p_t-k)^\nu\, p_b^\mu - (p_t-k)\cdot p_b\, g^{\mu\nu} - i\,\epsilon^{\sigma\rho\mu\nu}(p_t-k)_\sigma\, p_{b,\rho}\Big)\Big] \\
&- (\tilde{p}_t\cdot p_b)\Big(k^\mu\, p_b^\nu + k^\nu\, p_b^\mu - k\cdot p_b\, g^{\mu\nu} - i\,\epsilon^{\sigma\rho\mu\nu}k_\sigma\, p_{b,\rho}\Big) + (p_t\cdot p_b)\Big(k^\mu\,\tilde{p}_t^\nu + k^\nu\,\tilde{p}_t^\mu - k\cdot\tilde{p}_t\, g^{\mu\nu}\Big) \\
&- (k\cdot p_b)\Big(p_t^\mu\,\tilde{p}_t^\nu + p_t^\nu\,\tilde{p}_t^\mu - p_t\cdot\tilde{p}_t\, g^{\mu\nu}\Big) + (k\cdot p_t)\Big((p_b+k)^\mu\,\tilde{p}_t^\nu + (p_b+k)^\nu\,\tilde{p}_t^\mu - (p_b+k)\cdot\tilde{p}_t\, g^{\mu\nu}\Big) \\
&+ (k\cdot\tilde{p}_t)\Big(2p_b^\mu\, p_b^\nu - p_b\cdot p_b g^{\mu\nu}\Big) + i\Big(\epsilon^{\sigma\rho\mu\nu}\,(k\cdot\tilde{p}_t) + \epsilon^{\sigma\rho\gamma\mu}\, k^\nu\tilde{p}_{t,\gamma} - \epsilon^{\sigma\rho\gamma\nu}\, k^\mu\tilde{p}_{t,\gamma}\Big)p_{b,\sigma}\, p_{t,\rho} \\
&+ i\Big(\epsilon^{\sigma\rho\mu\nu}\,(p_t\cdot\tilde{p}_t) + \epsilon^{\sigma\rho\gamma\mu}\, p_t^\nu\tilde{p}_{t,\gamma} - \epsilon^{\sigma\rho\gamma\nu}\, p_t^\mu\tilde{p}_{t,\gamma}\Big)k_\sigma\, p_{b,\rho}\Big\}\,.
\end{aligned} \tag{3.140}$$

Finally, the rate for the real gluon emission is

$$\begin{aligned} d\Gamma_r^{(1)} &= \frac{1}{2m_t}\frac{1}{(2\pi)^3}\frac{d^3\vec{p}_b}{2E_b}\frac{1}{(2\pi)^3}\frac{d^3\vec{p}_\ell}{2E_\ell}\frac{1}{(2\pi)^3}\frac{d^3\vec{p}_\nu}{2E_\nu}\frac{1}{(2\pi)^3}\frac{d^3\vec{k}}{2E_g}(2\pi)^4\delta^4(p_t - p_b - p_\ell - p_\nu - k)\,\overline{|M_r^{(1)}|}^2 \\ &= \frac{1}{2m_t}\frac{1}{(2\pi)^8} dR_4(p_t; p_b, p_\ell, p_\nu, k)\,\overline{|M_r^{(1)}|}^2, \end{aligned} \tag{3.141}$$

where the tree amplitude squared is

$$\overline{|M_r^{(1)}|}^2 = \frac{G_F^2 |V_{tb}|^2}{2\left[(1 - y^2/\hat{y}^2)^2 + \gamma^2\right]} H^{(1)}_{r,\mu\nu} L^{\mu\nu}\,. \tag{3.142}$$

The hadron tensor for the real emission is given in (3.139). This contraction is a linear combination of the scalar products of the relevant four–momenta of the particles of the process. The parameterizations of the four–momenta depend on how we proceed with the phase space integration, which we will discuss in the next section.

3.4.2 Phase space integration

The four–body phase space $dR_4(p_t; p_b, p_\ell, p_\nu, k)$ is factorized according to

$$\begin{aligned} dR_4\,(p_t; p_b, p_\ell, p_\nu, k) &= dP^2\, dR_3(p_t; P, p_\ell, p_\nu)\, dR_2(P; p_b, k) \\ &= m_t^2 dz\, dR_3(p_t; P, p_\ell, p_\nu)\, dR_2(P; p_b, k) \end{aligned} \tag{3.143}$$

The three-body phase space integration $R_3(p_t; P, p_\ell, p_\nu)$ is calculated in a similar way as in (3.39). The only difference is that the bottom quark momentum p_b in (3.39) is replaced by $P = p_b + k$. This gives

$$dR_3(p_t; P, p_\ell, p_\nu) = \frac{1}{4\pi} d\cos\theta_P d\phi\, \frac{\pi^2}{4} m_t^2\, dy^2\, dx\,. \tag{3.144}$$

Adopting the narrow–width approximation, the differential rate simplifies to

$$\begin{aligned} d\Gamma_r^{(1)} &= \frac{1}{2m_t}\frac{1}{(2\pi)^8} dR_4(p_t; p_b, p_\ell, p_\nu, k)\,\overline{|M_r^{(1)}|}^2 \qquad (3.145) \\ &= \Gamma_F\, 2\pi \frac{m_W}{\Gamma_W}\, y^2 \frac{3}{128\pi^3 m_t^4}\, \frac{1}{4\pi} d\cos\theta_P d\phi\, m_t^2 \int dz\, dx\, dR_2(P; p_b, k)\, H^{(1)}_{r,\mu\nu} L^{\mu\nu} \end{aligned}$$

where the integration limits depend on the order of the x and z integrations according to

$$\int_{\bar{w}_-}^{\bar{w}_+} dx \int_{(\epsilon+\Lambda)^2}^{(1-x)(1-y^2/x)} dz \qquad \text{or} \qquad \int_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz \int_{w_-}^{w_+} dx\,. \tag{3.146}$$

Note that we use $y = \frac{m_W}{m_t}$ after applying the narrow–width approximation. Inserting the hadron tensor from (3.139) gives the lepton–hadron contraction:

$$H^{(1)}_{r,\mu\nu} L^{\mu\nu} = H^{conv}_{\mu\nu}\, L^{\mu\nu} + \widetilde{H}^{(0)}_{\mu\nu} L^{\mu\nu} |M|^2_{SGF}\,. \tag{3.147}$$

We rewrite the factor $\widetilde{H}^{(0)}_{\mu\nu} L^{\mu\nu}$ multiplying the IR–divergent $|M|^2_{SGF}$ as follows:

$$\begin{aligned}
\widetilde{H}^{(0)}_{\mu\nu} L^{\mu\nu} &= 128\,(\tilde{p}_t \cdot p_\ell)\,(p_b \cdot p_\nu) \\
&\downarrow \quad p_b = P - k \\
&= 128\,(\tilde{p}_t \cdot p_\ell)\,(P \cdot p_\nu) - 128\,(\tilde{p}_t \cdot p_\ell)\,(k \cdot p_\nu) \\
&= 128\,(\tilde{p}_t \cdot p_\ell)\,(P \cdot p_\nu) - 128 p_{\ell\alpha} p_{\nu\beta}\, \tilde{p}^\alpha_t k^\beta\,.
\end{aligned} \tag{3.148}$$

The second term in the last row of (3.148) contains the gluon momentum. Therefore $p_{\ell\alpha} p_{\nu\beta}\, \tilde{p}^\alpha_t k^\beta |M|^2_{SGF}$ is IR–convergent. Separating the IR–convergent and IR–divergent part we write the lepton–hadron contraction in the following form:

$$H^{(1)}_{r,\mu\nu} L^{\mu\nu} = \underbrace{H^{conv}_{\mu\nu}\, L^{\mu\nu} - 128 p_{\ell\alpha} p_{\nu\beta}\, \tilde{p}^\alpha_t k^\beta |M|^2_{SGF}}_{\text{IR–convergent}} + \underbrace{128\,(\tilde{p}_t \cdot p_\ell)\,(P \cdot p_\nu)\,|M|^2_{SGF}}_{\text{IR–divergent}}\,. \tag{3.149}$$

The two–body phase space integration $dR_2(P; p_b, k)$ is done separately for the IR–divergent and IR–convergent pieces of the hadron–lepton contraction.

Let us concentrate on the IR–convergent part first. The two–body phase space integration $dR_2(P; p_b, k)$ is independent of the lepton four–momenta. We use (3.18) to write

$$\begin{aligned}
dR_2&(P; p_b, k)(H^{conv}_{\mu\nu} L^{\mu\nu} - 128 p_{\ell\alpha} p_{\nu\beta}\, \tilde{p}^\alpha_t k^\beta\, |M|^2_{SGF}) \\
&= dR_2(P; p_b, k)(H^{conv}_{\mu\nu}\, 8 T^{\mu\nu\alpha\beta} p_{\ell\alpha} p_{\nu,\beta} - 128 p_{\ell\alpha} p_{\nu\beta}\, \tilde{p}^\alpha_t k^\beta\, |M|^2_{SGF}) \\
&= 8 p_{\ell\alpha} p_{\nu\beta}\, dR_2(P; p_b, k)(H^{conv}_{\mu\nu}\, T^{\mu\nu\alpha\beta} - 128\, \tilde{p}^\alpha_t k^\beta\, |M|^2_{SGF}) \\
&= 8 p_{\ell\alpha} p_{\nu\beta}\, t^{\alpha\beta},
\end{aligned} \tag{3.150}$$

where the tensor integral $t^{\alpha\beta}$ is defined by

$$t^{\alpha\beta} := dR_2(P; p_b, k)(H^{conv}_{\mu\nu}\, T^{\mu\nu\alpha\beta} - 128\, \tilde{p}^\alpha_t k^\beta\, |M|^2_{SGF})\,. \tag{3.151}$$

Similar to (2.148) the two–body phase space integration $dR_2(P; p_b, k)$ simplifies to the integration over the polar angle θ of the gluon in the P–rest frame. The only θ–dependent scalar product is $p_t \cdot k$. Therefore the integration can be written in terms of the following basic integrals:

$$\begin{aligned}
I_n &= \int dR_2(P, p_b, k)(p_t \cdot k)^n, \\
I^\mu_n &= \int dR_2(P, p_b, k)(p_t \cdot k)^n k^\mu, \\
I^{\mu\nu}_n &= \int dR_2(P, p_b, k)(p_t \cdot k)^n k^\mu k^\nu, \\
I^{\mu\nu\gamma}_n &= \int dR_2(P, p_b, k)(p_t \cdot k)^n k^\mu k^\nu k^\gamma.
\end{aligned} \tag{3.152}$$

These basic integrals are calculated in Appendix D.1. We perform the two–body phase space $dR_2(P, p_b, k)$ with the help of these basic integrals. The contraction $p_{\ell\alpha} p_{\nu\beta}\, t^{\alpha\beta}$ is a linear combination of the scalar products[5]

$$
\begin{aligned}
p_t \cdot p_\ell &= \frac{1}{2}\, x\, m_t^2 , \\
p_t \cdot p_\nu &= \frac{1 - x + y^2 - z}{2} m_t^2 , \\
p_t \cdot P &= \frac{1 - y^2 + z}{2} m_t^2 , \\
P \cdot p_\nu &= \frac{1 - x - z}{2} m_t^2 , \\
P \cdot p_\ell &= \frac{x - y^2}{2} m_t^2 , \\
p_\nu \cdot p_\ell &= \frac{1}{2}\, y^2\, m_t^2 \\
P \cdot s_t &= \frac{-2y^2 + x\,(1 + y^2 - z)}{2x}\, m_t\, \cos\theta_P \\
&\quad + \frac{\sqrt{y^2\,(x - x^2 - y^2 + xy^2 - xz)}}{x} m_t\, \cos\phi \sin\theta_P\, , \\
p_\nu \cdot s_t &= \frac{x^2 + 2y^2 - x\,(1 + y^2 - z)}{2x}\, m_t\, \cos\theta_P \\
&\quad - \frac{\sqrt{y^2\,(x - x^2 - y^2 + xy^2 - xz)}}{x}\, m_t\, \cos\phi \sin\theta_P\, , \\
p_\ell \cdot s_t &= \frac{x}{2}\, m_t \cos\theta_P\, . \qquad (3.153)
\end{aligned}
$$

Note that, for $z = \epsilon^2$ in the above relations, one recovers the leading order scalar products (3.21) and (3.24). After substituting the scalar products we perform the z–integration.

The z–integration is done with the help of the indefinite integrals (or primitives) defined by

$$
\begin{aligned}
\mathcal{R}(m, n) &:= \int dz \frac{z^m}{\lambda(1, y^2, z)^n} , \\
\mathcal{S}(m, n) &:= \int dz \frac{2\, z^m}{\lambda(1, y^2, z)^{n+\frac{1}{2}}} \ln\left(\frac{1 - y^2 + z + \sqrt{\lambda(1, y^2, z)}}{1 - y^2 + z - \sqrt{\lambda(1, y^2, z)}}\right) . \qquad (3.154)
\end{aligned}
$$

The same kind of basic integrals are defined for the decay $t(\uparrow) \to b + H^+$ in (2.160). Note that they are not identical. The important difference is that the integrals in (3.154) are indefinite integrals, i.e. the integration limits are not substituted. It is convenient to present the results of the z–integration in terms of indefinite integrals since, depending on where we use them, we will have to take different integration limits for these integrals. Note also the factor $\frac{1}{2}$ in the exponent of the denominator of the integrand of $\mathcal{S}(m, n)$ in

[5] The last three scalar product are polarization dependent with the top quark polarization vector parameterized in system 1a.

(3.154). It is because the logarithmic term always come together with $\sqrt{\lambda(1,y^2,z)}$ in odd powers. The reason is explained in the discussion following (3.195).

All the necessary basic integrals are calculated and listed in Appendix D.3. For the unpolarized and polar parts we perform the z–integration using the basic integrals (3.154) and write

$$m_t^2 \int dz\, dR_2(P;p_b,k)\, H^{conv}_{\mu\nu} L^{\mu\nu} = (-4\pi\alpha_s\, C_F)\, 128\frac{m^4}{4}\pi \times \\ \times \left\{ \Big[M^A_{r,conv}(x,z) + M^B_{r,conv}(x,z)\, P\cos\theta_P \Big]\Big|_{z=z_m}^{z=\epsilon^2} + \int dz\, M^C_r(x,z)\, P\sin\theta_P\cos\phi \right\}, \tag{3.155}$$

where[6] $z_m = (1-x)(1-\frac{y^2}{x})$. The unpolarized and polar contributions $M^A_{r,conv}(x,z)$ and $M^B_{r,conv}(x,z)$ are given in terms of the indefinite basic integrals.

$$\begin{aligned} M^A_{r,conv}(x,z) \;=\;& \mathcal{R}(-2,0)\,\rho_a(-2,0) + \mathcal{R}(-1,0)\,\rho_a(-1,0) + \mathcal{R}(0,0)\,\rho_a(0,0) \\ &+\mathcal{R}(0,1)\,\rho_a(0,1) + \mathcal{R}(0,2)\,\rho_a(0,2) \\ &+\mathcal{R}(1,1)\,\rho_a(1,1) + \mathcal{R}(1,2)\,\rho_a(1,2) \\ &+\mathcal{S}(0,0)\,\sigma_a(0,0) + \mathcal{S}(0,1)\,\sigma_a(0,1) + \mathcal{S}(0,2)\,\sigma_a(0,2) \\ &+\mathcal{S}(1,0)\,\sigma_a(1,0) + \mathcal{S}(1,1)\,\sigma_a(1,1) + \mathcal{S}(1,2)\,\sigma_a(1,2), \end{aligned} \tag{3.156}$$

$$\begin{aligned} M^B_{r,conv}(x,z) \;=\;& \mathcal{R}(-2,0)\,\rho_b(-2,0) + \mathcal{R}(-1,0)\,\rho_b(-1,0) + \mathcal{R}(0,0)\,\rho_b(0,0) \\ &+\mathcal{R}(0,1)\,\rho_b(0,1) + \mathcal{R}(0,2)\,\rho_b(0,2) + \mathcal{R}(0,3)\,\rho_b(0,3) \\ &+\mathcal{R}(1,1)\,\rho_b(1,1) + \mathcal{R}(1,2)\,\rho_b(1,2) + \mathcal{R}(1,3)\,\rho_b(1,3) \\ &+\mathcal{S}(0,0)\,\sigma_b(0,0) + \mathcal{S}(0,1)\,\sigma_b(0,1) + \mathcal{S}(0,2)\,\sigma_b(0,2) \\ &+\mathcal{S}(0,3)\,\sigma_b(0,3) + \mathcal{S}(1,0)\,\sigma_b(1,0) + \mathcal{S}(1,1)\,\sigma_b(1,1) \\ &+\mathcal{S}(1,2)\,\sigma_b(1,2) + \mathcal{S}(1,3)\,\sigma_b(1,3), \end{aligned} \tag{3.157}$$

where the indefinite basic integrals $\mathcal{R}(m,n)$ and $\mathcal{S}(m,n)$ are listed in Appendix D.3. The coefficient functions $\rho_a(m,n)$ and $\sigma_a(m,n)$ for the unpolarized contribution $M^A_{r,conv}$, and $\rho_b(m,n)$ and $\sigma_b(m,n)$ for the polar contribution $M^B_{r,conv}$ are listed in Appendix D.4. The integration limits for the indefinite integrals are $\epsilon^2 \leqslant z \leqslant (1-x)(1-y^2/x)$ according to the fact that we perform the z–integration first and the x–integration second.

The z–integration for the azimuthal part is not discussed in this chapter. Because the integrals that appear in the azimuthal calculations are more involved than the ones in the unpolarized and polar cases, the azimuthal contribution will be treated separately in section 3.6.

For the unpolarized part, substituting the basic integrals and their coefficients, we obtain

$$M^A_{r,conv}(x,z) = h^a_1 z + \frac{1}{z}\frac{\epsilon^2 h^a_2}{2} + \frac{h^a_3 + h^a_4\, z}{8p_3^2} + \frac{(h^a_5 + h^a_6\, z)\, Y_p}{8p_3^3} + \frac{(h^a_7 + h^a_8\, z)\, Y_p}{4p_3} \\ h^a_9\, p_3 Y_p + \frac{1}{4} h^a_{10} \ln z + \Big[\mathrm{Li}_2(w_+) + \mathrm{Li}_2(w_-) \Big] h^a_{11}\,, \tag{3.158}$$

[6]The integration limits in (3.155) are substituted as $M^{A,B}_{r,conv}(x,z)|^{z=\epsilon^2}_{z=z_m}$ instead of usual $|^{z=z_m}_{z=\epsilon^2}$. It is done so for the ease of comparison with [25] and [26].

with the coefficients

$$h_1^a = -6x + \frac{y^2}{2}\,, \tag{3.159}$$

$$h_2^a = (1-x)\left[x + \frac{(x-y^2)\,\epsilon^2}{1-y^2}\right]\,, \tag{3.160}$$

$$\begin{aligned} h_3^a = &\left(1-y^2\right)^2\left(5x^2 + (5-2x)\,y^2\right) \\ &- 4\left[x^2 + \left(1-5x+2x^2\right)y^2 + (2-x)\,y^4\right]\epsilon^2 \\ &- \frac{1}{1-y^2}\left[x^2 + \left(1-6x+3x^2\right)y^2 + (3-2x)\,y^4\right]\epsilon^4\,, \end{aligned} \tag{3.161}$$

$$\begin{aligned} h_4^a = &-5x^2 - \left(5+18x+3x^2\right)y^2 - (3-2x)\,y^4 \\ &+ 4\left[x^2 + (1-x)\,y^2\right]\epsilon^2 + \frac{1}{1-y^2}\left[x^2 + (1-2x)\,y^2\right]\epsilon^4\,, \end{aligned} \tag{3.162}$$

$$\begin{aligned} h_5^a = &-\left(1-y^2\right)^2\left[5x^2 + \left(5+8x-x^2\right)y^2 - y^4\right] \\ &+ 2\left(1-y^2\right)\left[2x^2 + \left(2-6x+x^2\right)y^2 + y^4\right]\epsilon^2 \\ &+ \left(x^2 + \left(1-4x+x^2\right)y^2 + y^4\right)\epsilon^4\,, \end{aligned} \tag{3.163}$$

$$\begin{aligned} h_6^a = &\,5x^2 + \left(5+28x+12x^2\right)y^2 + (6-x)\,(2+x)\,y^4 - y^6 \\ &- 2\left(2x^2 + \left(2+2x-x^2\right)y^2 - y^4\right)\epsilon^2 - \left(x^2+y^2\right)\epsilon^4\,, \end{aligned} \tag{3.164}$$

$$\begin{aligned} h_7^a = &-5 + 10x + 17y^2 + 8xy^2 + 8x^2y^2 - 3y^4 - 2xy^4 - y^6 \\ &+ 2\left(2 - 5x - 2x^2 - 2y^2 + 5xy^2\right)\epsilon^2 + \left(1 - 4x + y^2\right)\epsilon^4\,, \end{aligned} \tag{3.165}$$

$$h_8^a = 5 + 10x - 4x^2 + 2y^2 + 2xy^2 + y^4 - 2\,(2+3x)\,\epsilon^2 - \epsilon^4\,, \tag{3.166}$$

$$h_9^a = 8x\,, \tag{3.167}$$

$$h_{10}^a = -5 + 20x - 12x^2 + 2y^2 - 6xy^2 + y^4 + 4\,(1-x)\,\epsilon^2 + \epsilon^4\,, \tag{3.168}$$

$$h_{11}^a = -x\,(1+2x) - x\epsilon^2\,. \tag{3.169}$$

For the polar part we obtain

$$\begin{aligned} M^B_{r,conv}(x,z) = &\, h_1^b z + \frac{1}{z}\frac{\epsilon^2 h_2^b}{2x(1-y^2)^2} + \frac{3y^2}{8x\,p_2^4}(h_3^b + h_4^b\,z) + \frac{h_5^b + h_6^b\,z}{8x(1-y^2)^2 p_3^2} \\ &+ \frac{Y_p}{4x\,p_3}\left(3y^2\frac{h_7^b + h_8^b\,z}{2p_3^4} + \frac{h_9^b + h_{10}^b\,z}{2p_3^2} + h_{11}^b + h_{12}^b\,z\right) \\ &+ h_{13}^b\,p_3 Y_p + \frac{1}{4x}h_{14}^b \ln z + \Big[\mathrm{Li}_2(w_+) + \mathrm{Li}_2(w_-)\Big]h_{15}^b\,, \end{aligned} \tag{3.170}$$

with the coefficients

$$h_1^b = -6x + \frac{y^2}{2}\,, \tag{3.171}$$

$$h_2^b = (1-x)\left[x^2\left(1-y^2\right)^2 - \left(x-y^2\right)\left(x - 2y^2 + xy^2\right)\epsilon^2\right]\,, \tag{3.172}$$

$$h_3^b = -\left(1-y^2\right)^2\left(3x^2+2x^3+y^2+6xy^2+3x^2y^2+y^4\right)+2\left(1-y^2\right)^2\left(3x^2+y^2\right)\epsilon^2 + \left(-3x^2+2x^3-y^2+6xy^2-3x^2y^2-y^4\right)\epsilon^4\,, \tag{3.173}$$

$$h_4^b = 3x^2+4x^3+y^2+12xy^2+18x^2y^2+4x^3y^2+6y^4+12xy^4+3x^2y^4+y^6 - 2\left(3x^2+2x^3+y^2+6xy^2+3x^2y^2+y^4\right)\epsilon^2+\left(3x^2+y^2\right)\epsilon^4\,, \tag{3.174}$$

$$h_5^b = \left(1-y^2\right)^2(-6x^2+3x^3-2y^2-7xy^2+46x^2y^2+18x^3y^2 + 18y^4+86xy^4+34x^2y^4+3x^3y^4+6y^6-7xy^6-2x^2y^6+2y^8) + 4\left(1-y^2\right)^2\left(3x^2-2x^3+y^2-5xy^2-14x^2y^2-2x^3y^2-6y^4-xy^4+2x^2y^4\right)\epsilon^2 + (-6x^2+5x^3-2y^2+27xy^2-6x^2y^2-2x^3y^2-2y^4-30xy^4-6x^2y^4 + 5x^3y^4-2y^6+27xy^6-6x^2y^6-2y^8)\epsilon^4\,, \tag{3.175}$$

$$h_6^b = \left(1-y^2\right)^2\left(6x^2+x^3+2y^2+31xy^2+30x^2y^2+x^3y^2+10y^4+15xy^4+2x^2y^4-2y^6\right) - 4\left(1-y^2\right)^2\left(3x^2+y^2+7xy^2+2x^2y^2\right)\epsilon^2-(-6x^2+x^3-2y^2+3xy^2 + 6x^2y^2+x^3y^2+2y^4+3xy^4-6x^2y^4-2y^6)\epsilon^4\,,$$

$$h_7^b = \left(1-y^2\right)^2(1+x)\left(3x^2+y^2+8xy^2+x^2y^2+3y^4\right) - 2\left(1-y^2\right)^2\left(3x^2+x^3+y^2+3xy^2\right)\epsilon^2+\left(1-y^2\right)\left(3x^2-x^3+y^2-3xy^2\right)\epsilon^4\,, \tag{3.176}$$

$$h_8^b = -3x^2-5x^3-y^2-15xy^2-30x^2y^2-10x^3y^2-10y^4-30xy^4 - 15x^2y^4-x^3y^4-5y^6-3xy^6+2(1+x)\left(3x^2+y^2+8xy^2+x^2y^2+3y^4\right)\epsilon^2 - \left(3x^2+x^3+y^2+3xy^2\right)\epsilon^4\,, \tag{3.177}$$

$$h_9^b = 6x^2-x^3+2y^2+19xy^2-38x^2y^2-33x^3y^2-14y^4-145xy^4 - 122x^2y^4-15x^3y^4-38y^6-19xy^6+10x^2y^6+x^3y^6+2y^8+xy^8 - 2(6x^2-2x^3+2y^2+2xy^2-42x^2y^2-11x^3y^2-16y^4-43xy^4+x^3y^4 + 2y^6+5xy^6)\epsilon^2+\left(6x^2-3x^3+2y^2-15xy^2-18x^2y^2+x^3y^2-6y^4+9xy^4\right)\epsilon^4\,, \tag{3.178}$$

$$h_{10}^b = -6x^2-3x^3-2y^2-43xy^2-68x^2y^2-8x^3y^2-24y^4-72xy^4-26x^2y^4-x^3y^4 - 2y^6-xy^6+2\left(6x^2+2x^3+2y^2+26xy^2+20x^2y^2+x^3y^2+6y^4+5xy^4\right)\epsilon^2 - \left(6x^2+x^3+2y^2+9xy^2\right)\epsilon^4\,, \tag{3.179}$$

$$h_{11}^b = -2+3x-2x^2-8x^3-2y^2-47xy^2-28x^2y^2+8x^3y^2-2y^4+5xy^4 - 2x^2y^4-2y^6-xy^6-2\left(-2+2x-9x^2+2x^3-2y^2-10xy^2-3x^2y^2\right)\epsilon^2 - \left(2-x+4x^2+2y^2-xy^2\right)\epsilon^4\,, \tag{3.180}$$

$$h_{12}^b = 2-3x-10x^2-4x^3-6xy^2+2x^2y^2+2y^4+xy^4 - 2\left(2-2x+x^2\right)\epsilon^2+(2-x)\epsilon^4\,, \tag{3.181}$$

$$h_{13}^b = 8x\,, \tag{3.182}$$

$$h_{14}^b = -2+3x+20x^2-12x^3-6xy^2-6x^2y^2+2y^4+xy^4 \tag{3.183}$$

$$+\,4\,(1-x)\,\epsilon^2 - (2-x)\,\epsilon^4\,,$$

$$h^b_{15} = -x\,(3+2x) - x\epsilon^2\,. \tag{3.184}$$

Next we turn to the IR–divergent part (see 3.139):

$$128\,(\tilde{p}_t\cdot p_\ell)\,(P\cdot p_\nu)\,|M|^2_{SGF} = 128\frac{m_t^4}{4}(1+P\cos\theta_P)x(1-x-z)|M|^2_{SGF}\,, \tag{3.185}$$

where the scalar products are evaluated in (3.153). From the above expression it is clear that the azimuthal contribution is not IR–divergent. Collecting terms according to their angular dependence we write

$$128\,(\tilde{p}_t\cdot p_\ell)\,(P\cdot p_\nu)\,|M|^2_{SGF} = 128\frac{m_t^4}{4}\Big[\widetilde{M}^A_0(x,z) + \widetilde{M}^B_0(x,z)P\cos\theta_P\Big]|M|^2_{SGF}\,, \tag{3.186}$$

where

$$\widetilde{M}^A_0(x,z) = \widetilde{M}^B_0(x,z) = x(1-x-z). \tag{3.187}$$

As discussed in subsection 2.3.1 the z–dependent terms multiplying the IR–divergent terms will complicate the integration. Therefore we use the subtraction method to further isolate the IR–divergent piece $|M|^2_{SGF}$:

$$\widetilde{M}^A_0(x,z)|M|^2_{SGF} = \Big[\widetilde{M}^A_0(x,z) - M^A_0(x)\Big]|M|^2_{SGF} + M^A_0(x)|M|^2_{SGF} \tag{3.188}$$

where the LO term $M^A_0(x) = \widetilde{M}^A_0(z\!=\!\epsilon^2)$ is given in (3.27).

The phase space integration for the IR–divergent contribution reads

$$\begin{aligned} m_t^2\int dz\,dR_2(P;p_b,k)\,\widetilde{M}^A_0(x,z)|M|^2_{SGF} \\ = \underbrace{m_t^2\int dz\,\Big[\widetilde{M}^A_0(x,z) - M^A_0(x)\Big]dR_2(P;p_b,k)\,|M|^2_{SGF}}_{IR-conv.} \\ +\underbrace{m_t^2\int dz\,M^A_0(x)dR_2(P;p_b,k)|M|^2_{SGF}}_{IR-div.}\,. \end{aligned} \tag{3.189}$$

The IR–convergent piece is calculated without a gluon mass regulator, i.e. we take $\Lambda = 0$:

$$m_t^2\int\limits_{\epsilon^2}^{z_m} dz\,\Big[\widetilde{M}^A_0(x,z) - M^A_0(x)\Big]dR_2(P;p_b,k)\,|M|^2_{SGF} \tag{3.190}$$

$$= 128\frac{m_t^4}{4}(-4\pi\alpha_s C_F)\,2\pi\,x\,\Big[2\,\mathcal{R}(0,0) - 2\,\big(1-y^2\big)\,\mathcal{S}(0,0) - 2\,\mathcal{S}(1,0)\Big]\Big|^{\epsilon^2}_{z=z_m}$$

The IR–divergent piece is integrated with a non–zero gluon mass, i.e. $\Lambda \neq 0$:

$$m_t^2\int dz\,M^A_0(x)dR_2(P;p_b,k)|M|^2_{SGF} = (-4\pi\alpha_s C_F)M^A_0(x)\,m_t^2\int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz\,dR_2(P;p_b,k)\,\Delta_{SGF}. \tag{3.191}$$

Analogous to (2.176) we obtain

$$m_t^2 \int_{(\epsilon+\Lambda)^2}^{z_m} dz\, dR_2(P; p_b, k)\, \Delta_{SGF} = -2\pi \Big[\mathcal{R}(-1,0) - 2\mathcal{S}(0,0)\Big]\Big|_{z=z_m}^{z=\epsilon^2} + \pi\Sigma(\Lambda, z_m). \quad (3.192)$$

The IR–divergent piece $\Sigma(\Lambda, z_m)$ is identical to the one calculated for $t \to H^+ + b$ in 2.201. One has

$$\begin{aligned}\Sigma(\Lambda, z_m) = {} & 4\left(1 - \frac{\bar{p}_0}{\bar{p}_3}\bar{Y}_p\right) \ln\left(\frac{z_m - \epsilon^2}{\epsilon\Lambda}\right) - 2 \\ & + \frac{2\bar{p}_0}{\bar{p}_3}\left[\mathrm{Li}_2\left(1 - \frac{p_+(z_m)}{\bar{p}_+}\right) + \mathrm{Li}_2\left(1 - \frac{p_-(z_m)}{\bar{p}_+}\right) - \mathrm{Li}_2\left(1 - \frac{p_+(z_m)}{\bar{p}_-}\right)\right. \\ & \left. - \mathrm{Li}_2\left(1 - \frac{p_-(z_m)}{\bar{p}_-}\right) - \mathrm{Li}_2\left(1 - \frac{\bar{p}_+}{\bar{p}_-}\right) - \bar{Y}_p\left(\bar{Y}_p + 1\right)\right]. \end{aligned} \quad (3.193)$$

When substituting the maximal value of z in (3.193), i.e. $z_m = (1-x)(1-y^2/x)$, the values of kinematic variables depend on the different regions of x. For $x^2 \leqslant y^2$ we have

$$\begin{aligned} & p_+(z_m) = 1 - x, \quad p_-(z_m) = 1 - \frac{y^2}{x}, \quad w_+(z_m) = \frac{y^2}{x}, \quad w_-(z_m) = x, \\ & p_3(z_m) = -\frac{1}{2}x(1 - \frac{y^2}{x^2}), \quad Y_p = -\frac{1}{2}\ln\left(\frac{x - y^2}{x - x^2}\right). \end{aligned} \quad (3.194)$$

For $y^2 \leqslant x^2$ we have

$$\begin{aligned} & p_+(z_m) = 1 - \frac{y^2}{x}, \quad p_-(z_m) = 1 - x, \quad w_+(z_m) = x, \quad w_-(z_m) = \frac{y^2}{x}, \\ & p_3(z_m) = \frac{1}{2}x(1 - \frac{y^2}{x^2}), \quad Y_p = \frac{1}{2}\ln\left(\frac{x - y^2}{x - x^2}\right). \end{aligned} \quad (3.195)$$

It is easy to see that exchanging the regions of x ($x^2 \leqslant y^2 \leftrightarrow y^2 \leqslant x^2$) will exchange the values $p_+(z_m) \leftrightarrow p_-(z_m)$ and $w_+(z_m) \leftrightarrow w_-(z_m)$. However, the result does not depend on where x lies since (3.193) is symmetric under the exchange of $p_+(z_m) \leftrightarrow p_-(z_m)$ and $w_+(z_m) \leftrightarrow w_-(z_m)$. Also Y_p and p_3 always come in product form. Although they are separately different in the different regions of x, their product that appears in the expressions are the same for both $x^2 \leqslant y^2$ and $y^2 \leqslant x^2$. In order to maintain this symmetry one deduces that any terms proportional to Y_p and p_3 or $p_3^2 Y_p$ should not appear in the expressions. The substitution $z_m = (1-x)(1-y^2/x)$ and further simplifications give

$$\begin{aligned} \Sigma(\Lambda) := {} & \Sigma(\Lambda, z_m)\Big|_{z_m=(1-x)(1-y^2/x)} \\ = {} & -4\left(1 - \frac{\bar{p}_0}{\bar{p}_3}\bar{Y}_p\right)\ln\left[\frac{(1-y^2)(1-y^2/x) - \epsilon^2}{\epsilon\Lambda}\right] - 2 \\ & + \frac{2\bar{p}_0}{\bar{p}_3}\left[\mathrm{Li}_2\left(1 - \frac{1-x}{\bar{p}_+}\right) + \mathrm{Li}_2\left(1 - \frac{1-y^2/x}{\bar{p}_+}\right) - \mathrm{Li}_2\left(1 - \frac{1-x}{\bar{p}_-}\right)\right. \\ & \left. - \mathrm{Li}_2\left(1 - \frac{1-y^2/x}{\bar{p}_-}\right) - \mathrm{Li}_2\left(1 - \frac{\bar{p}_+}{\bar{p}_-}\right) - \bar{Y}_p\left(\bar{Y}_p + 1\right)\right]. \end{aligned} \quad (3.196)$$

Thus we obtain

$$
\begin{aligned}
m_t^2 \int & dz\, dR_2(P; p_b, k)\, 128\, (\tilde{p}_t \cdot p_\ell)\, (P \cdot p_\nu)\, |M|^2_{SGF} \\
&= 128 \frac{m_t^4}{4} \pi\, (1 + P \cos\theta_P) m_t^2 \int dz dR_2(P; p_b, k)\, \widetilde{M}_0^A(x,z) |M|^2_{SGF} \\
&= 128 \frac{m_t^4}{4} \pi (-4\pi\alpha_s C_F)\, (1 + P\cos\theta_P) \Big[\hat{M}^A_{conv}(x,z) \Big|_{z=z_m}^{z=\epsilon^2} + M_0^A(x)\Sigma(\Lambda) \Big] \,,
\end{aligned}
\tag{3.197}
$$

where

$$
\begin{aligned}
\hat{M}^A_{conv}(x,z) &= -2x \left(1 - x - \epsilon^2\right) \mathcal{R}(-1,0) + 4x\mathcal{R}(0,0) \\
&\quad -4x \left(x - y^2 + \epsilon^2\right) \mathcal{S}(0,0) - 4x\mathcal{S}(1,0) \\
&= 6\, x\, z - 2\, x\, \left(2 - x - y^2 - \epsilon^2\right) \ln(z) \\
&\quad +2\, x\, \left(1 + x + \epsilon^2\right) \Big(\mathrm{Li}_2(w_-) + \mathrm{Li}_2(w_+) \Big) - 8\, x\, p_3\, Y_p \,.
\end{aligned}
\tag{3.198}
$$

Since the leading order azimuthal rate vanishes there is no IR–divergent azimuthal contribution from real emission. This is also expected since there is no IR–divergent azimuthal contributions from the virtual loop corrections.

Finally we sum up the IR–divergent (3.197) and the IR–convergent (3.155) pieces and write the total real emission contributions as

$$
\begin{aligned}
\frac{d\Gamma_r^{(1)}}{dx d\cos\theta_P d\phi} &= \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F (-\frac{\alpha_s}{2\pi}) 6y^2\, \frac{1}{4\pi} \times \\
&\quad \times \Big[M_r^A(x) + M_r^B(x)\, P\cos\theta_P + M_r^C(x)\, P\sin\theta_P\cos\phi \Big] \,,
\end{aligned}
\tag{3.199}
$$

with

$$
M_r^A(x) = \Big[M^A_{r,conv}(x,z) + \hat{M}^A_{conv}(x,z) \Big] \Big|_{z=z_m}^{z=\epsilon^2} + M_0^A(x)\Sigma(\Lambda), \tag{3.200}
$$

$$
M_r^B(x) = \Big[M^B_{r,conv}(x,z) + \hat{M}^A_{conv}(x,z) \Big] \Big|_{z=z_m}^{z=\epsilon^2} + M_0^A(x)\Sigma(\Lambda), \tag{3.201}
$$

$$
M_r^C(x) = \int dz\, M_r^C(x,z) \,. \tag{3.202}
$$

where $M_0^A(x)$, $M^A_{r,conv}(x,z)$, $M^B_{r,conv}(x,z)$, $\hat{M}^A_{r,conv}(x,z)$ and $\Sigma(\Lambda)$ are given in (3.27), (3.158), (3.170), (3.198) and (3.196), respectively. The calculation of the azimuthal contribution is deferred to section 3.6.

3.5 Total $\mathcal{O}(\alpha_s)$ results for system 1a

The total $\mathcal{O}(\alpha_s)$ correction is the sum of the virtual one–loop and the real emissions contributions. The general form of the $\mathcal{O}(\alpha_s)$ correction is

$$
\frac{d\Gamma^{(1)}}{dx d\cos\theta_P d\phi} = \frac{1}{4\pi} \left(\frac{d\Gamma_A^{(1)}}{dx} + \frac{d\Gamma_B^{(1)}}{dx}\, P\cos\theta_P + \frac{d\Gamma_C^{(1)}}{dx}\, P\sin\theta_P\cos\phi \right) . \tag{3.203}
$$

Summing up the virtual one–loop and the real emission contributions from (3.133) and (3.199) we write the total $\mathcal{O}(\alpha_s)$ result as:

$$\begin{aligned}\frac{d\Gamma^{(1)}}{dx\,d\cos\theta_P d\phi} &= \frac{d\Gamma_v^{(1)}}{dx\,d\cos\theta_P d\phi} + \frac{d\Gamma_r^{(1)}}{dx\,d\cos\theta_P d\phi} \\ &= \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F(-\frac{\alpha_s}{2\pi}) 6y^2 \frac{1}{4\pi}\Big[M^A(x) + M^B(x)\,P\cos\theta_P + M^C(x)\,P\sin\theta_P\cos\phi\Big]\,,\end{aligned} \tag{3.204}$$

where

$$M^A(x) = M_v^A(x) + M_0^A(x)\Sigma(\Lambda) + \Big[M^A_{r,conv}(x,z) + \hat{M}^A_{conv}(x,z)\Big]\Big|_{z=z_m}^{z=\epsilon^2}\,, \tag{3.205}$$

$$M^B(x) = M_v^A(x) + M_0^A(x)\Sigma(\Lambda) + \Big[M^B_{r,conv}(x,z) + \hat{M}^A_{conv}(x,z)\Big]\Big|_{z=z_m}^{z=\epsilon^2}\,, \tag{3.206}$$

$$M^C(x) = M_v^C(x) + \int dz\, M_r^C(x,z)\,. \tag{3.207}$$

Comparing (3.203) and (3.204) we have

$$\frac{d\Gamma_i^{(1)}}{dx} = \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F(-\frac{\alpha_s}{2\pi}) 6y^2 \frac{1}{4\pi}\, M^i(x)\,, \tag{3.208}$$

where $i = A, B, C$ represent the unpolarized, polar and azimuthal contributions, respectively.

In this section we concentrate on the unpolarized and polar parts, leaving the azimuthal part to section 3.6. We first sum up the virtual one–loop contribution with the divergent part of the real emission. The IR–divergences cancel in the sum. Both in the virtual loop and the IR–divergent part of the real emission the unpolarized part equals the polar part. Therefore it is sufficient to calculate the unpolarized contributions.

$$\begin{aligned}M_v^A(x) + M_0^A(x)\Sigma(\Lambda) &= M^A_{v,conv}(x) + M_0^A(x) M^A_{v,div}(x) + M_0^A(x)\Sigma(\Lambda) \\ &= M^A_{v,conv}(x) + M_0^A(x)\Big[M^A_{v,div}(x) + \Sigma(\Lambda)\Big]\,.\end{aligned} \tag{3.209}$$

We then take $M^A_{v,div}$ and $\Sigma(\Lambda)$ from (3.130) and (3.196) and substitute them into (3.209). After some simplifications one obtains

$$\begin{aligned}&M^A_{v,div}(x) + \Sigma(\Lambda) = \\ &\quad \frac{2\bar{p}_0}{\bar{p}_3}\Big[\mathrm{Li}_2\left(1 - \frac{1-x}{\bar{p}_+}\right) + \mathrm{Li}_2\left(1 - \frac{1-y^2/x}{\bar{p}_+}\right) - \mathrm{Li}_2\left(1 - \frac{1-x}{\bar{p}_-}\right) \\ &\quad - \mathrm{Li}_2\left(1 - \frac{1-y^2/x}{\bar{p}_-}\right) + \mathrm{Li}_2(\bar{w}_-) - \mathrm{Li}_2(\bar{w}_+) + 4\bar{Y}_p \ln\epsilon\Big] \\ &\quad + 4\left(1 - \frac{\bar{p}_0}{\bar{p}_3}\bar{Y}_p\right)\ln\left(z_m - \epsilon^2\right) + 2 - 2\,\ln(z_m) \\ &\quad + \frac{1}{y^2}\left(1 - 2\,y^2 - \epsilon^2\right)\,\ln(\epsilon) - \left(4\,\bar{p}_0 - \frac{2\,\bar{p}_3^2}{y^2}\right)\,\frac{\bar{Y}_p}{\bar{p}_3}\,.\end{aligned} \tag{3.210}$$

The dilog identity

$$\mathrm{Li}_2\left(1-\frac{\bar{w}_-\bar{p}_-}{\bar{w}_+\bar{p}_+}\right)-\mathrm{Li}_2\left(1-\frac{\bar{w}_-}{\bar{w}_+}\right)-\mathrm{Li}_2\left(1-\frac{\bar{p}_-}{\bar{p}_+}\right)=\mathrm{Li}_2\left(\bar{w}_-\right)-\mathrm{Li}_2\left(\bar{w}_+\right)-2\bar{Y}_w\left(\bar{Y}_p+\ln\epsilon\right) \tag{3.211}$$

has been used to obtain the simple form of (3.210). We collect all the dilog and double log terms (multiplication of two logarithmic terms), and denote them by Φ_0:

$$\begin{aligned}\Phi_0 = \frac{2\bar{p}_0}{\bar{p}_3}\Bigg[&\mathrm{Li}_2\left(1-\frac{1-x}{\bar{p}_+}\right)+\mathrm{Li}_2\left(1-\frac{1-y^2/x}{\bar{p}_+}\right)-\mathrm{Li}_2\left(1-\frac{1-x}{\bar{p}_-}\right) \\ &-\mathrm{Li}_2\left(1-\frac{1-y^2/x}{\bar{p}_-}\right)+\mathrm{Li}_2(\bar{w}_-)-\mathrm{Li}_2(\bar{w}_+)+4\bar{Y}_p\ln\epsilon\Bigg]+4\left(1-\frac{\bar{p}_0}{\bar{p}_3}\bar{Y}_p\right)\ln\left(z_m-\epsilon^2\right)\,.\end{aligned} \tag{3.212}$$

The sum is then written as

$$\begin{aligned}&M_0^A(x)M_{v,div}^A(x)+M_0^A(x)\Sigma(\Lambda)=M_0^A\Phi_0 \\ &\quad +M_0^A\left[2-2\,\ln(z_m)+\frac{1}{y^2}\left(1-2\,y^2-\epsilon^2\right)\,\ln(\epsilon)-\left(4\,\bar{p}_0-\frac{2\,\bar{p}_3^2}{y^2}\right)\,\frac{\bar{Y}_p}{\bar{p}_3}\right].\end{aligned} \tag{3.213}$$

Substituting this back into (3.209) and inserting $M_{v,conv}^A$ of (3.131) we obtain

$$\begin{aligned}&M_v^A(x)+M_0^A(x)\Sigma(\Lambda) \\ &= M_0^A\Phi_0+2x\left(1-x-\epsilon^2\right)(1-\ln z_m) \\ &\quad +\frac{\bar{Y}_p}{\bar{p}_3}\left[\left(1-6x+5x^2\right)\left(1-y^2\right)-\left(2-2x-5x^2-3y^2+4xy^2\right)\epsilon^2+(1+4x)\,\epsilon^4\right] \\ &\quad +\left[(1-2x)\,(1-x)-(1-3x)\,\epsilon^2\right]\ln(\epsilon).\end{aligned} \tag{3.214}$$

The IR–convergent parts of the unpolarized and polar real emission contributions are obtained by a straightforward insertion of the integration limits, i.e.

$$\left[M_{r,conv}^A(x,z)+\hat{M}_{conv}^A(x,z)\right]\Big|_{z=z_m}^{z=\epsilon^2}, \tag{3.215}$$

and

$$\left[M_{r,conv}^B(x,z)+\hat{M}_{conv}^A(x,z)\right]\Big|_{z=z_m}^{z=\epsilon^2}. \tag{3.216}$$

Finally we collect the above results and insert them into (3.205) and (3.206), and write the total $\mathcal{O}(\alpha_s)$ corrections to the differential rate as

$$\begin{aligned}\frac{d\Gamma^{(1)}}{dxd\cos\theta_P d\phi} &= \Gamma_F 2\pi\frac{m_W}{\Gamma_W}C_F(-\frac{\alpha_s}{2\pi})6y^2\frac{1}{4\pi}\times \\ &\times\left[M^A(x)+M^B(x)\,P\cos\theta_P+M^C(x)\,P\sin\theta_P\cos\phi\right].\end{aligned} \tag{3.217}$$

The unpolarized part is given by[7]

$$
\begin{aligned}
M^A(x) = M_0^A\,\Phi_0 &+ x\left(1+\epsilon^2\right)\left[\mathrm{Li}_2(\bar{w}_+) + \mathrm{Li}_2(\bar{w}_-) - \mathrm{Li}_2(x) - \mathrm{Li}_2(\frac{y^2}{x})\right] \\
&- \frac{\bar{Y}_p}{\bar{p}_3}\left[\bar{p}_3^2\left(3+2x+y^2+\epsilon^2\right) + 2\epsilon^2 y^2\right] \\
&- \frac{1}{2}\left[\left(1-y^2\right)\left(3+2x+y^2\right) + 2\epsilon^2\left(1+5x\right) + \epsilon^4\right]\ln\epsilon \\
&+ \frac{1}{2}\left[5-13x+8x^2-\epsilon^2\left(4-3x\right)-\epsilon^4\right]\ln\left(1-x\right) \\
&+ \frac{1}{2}(x+4x^2-2xy^2-2y^2-y^4+\epsilon^2 x)\ln\left(1-\frac{y^2}{x}\right) \\
&+ \frac{1}{2}\left(-4xy^2+4y^2-y^4+\frac{y^4}{x}\right) + \frac{1}{2}\epsilon^2\left(x+y^2-\frac{x^2}{x-y^2}\right).
\end{aligned} \tag{3.218}
$$

The polar part reads

$$
\begin{aligned}
M^B(x) = M_0^A\,\Phi_0 &- x\left(1-\epsilon^2\right)\left[\mathrm{Li}_2(\bar{w}_+) + \mathrm{Li}_2(\bar{w}_-) - \mathrm{Li}_2(x) - \mathrm{Li}_2(\frac{y^2}{x})\right] \\
&- \frac{\bar{Y}_p}{\bar{p}_3}\Big[\bar{p}_3^2\left(5-2x-y^2-\frac{2y^2}{x}-\frac{2}{x}\right) - \bar{p}_3^2\epsilon^2\left(1-\frac{2}{x}\right) \\
&- \epsilon^2\left(x+xy^2-2y^2+\frac{y^2}{x}+\frac{y^4}{x}\right) + \epsilon^4\left(x+\frac{y^2}{x}\right)\Big] \\
&- \frac{1}{2}\Big[\left(1-y^2\right)\left(5-2x-y^2-\frac{2y^2}{x}-\frac{2}{x}\right) \\
&- 2\epsilon^2\left(3-7x-\frac{2}{x}\right) + \epsilon^4\left(1-\frac{2}{x}\right)\Big]\ln\epsilon \\
&+ \frac{1}{2}\Big[-3-7x+8x^2+\frac{2}{x}+\epsilon^2\left(4-x-\frac{4}{x}\right) - \epsilon^4\left(1-\frac{2}{x}\right)\Big]\ln\left(1-x\right) \\
&+ \frac{1}{2}(-5x+4x^2-2xy^2+6y^2-y^4-\frac{2y^4}{x}+\epsilon^2 x)\ln\left(1-\frac{y^2}{x}\right) \\
&+ \frac{1}{2}\left(2-2x^2+2y^2-3y^4-\frac{2y^2}{x}+\frac{3y^4}{x}\right) \\
&- \frac{1}{2}\epsilon^2\left(4-x-y^2-\frac{2y^2}{x}+\frac{x^2}{x-y^2}\right) + \epsilon^4.
\end{aligned} \tag{3.219}
$$

The endpoint behavior of $d\Gamma^{(1)}/dx$ in (3.217) is interesting. The term $\ln(z_m - \epsilon^2)$ in Φ_0 is divergent at the endpoints (integration limits for x) $x = \bar{w}_+$ and $x = \bar{w}_-$:

$$
\ln(z_m - \epsilon^2) = \ln\left[(1-x)(1-\frac{y^2}{x}) - \epsilon^2\right] = \ln\left[\frac{(\bar{w}_+ - x)(x - \bar{w}_-)}{x}\right]. \tag{3.220}
$$

[7] 3.218 and 3.219 agree with $F_1^+(x,y)$ of Eq.(9) and $J_1^+(x,y)$ of Eq.(10) in [28]. Note that our variable y^2 correspond to y of [28].

When we integrate this term w.r.t. x and insert the integration limits we use a cut–off:

$$\int_{\bar{w}_-}^{\bar{w}_+} \dots dx = \lim_{\delta_1 \to 0, \delta_2 \to 0} \int_{\bar{w}_- + \delta_2}^{\bar{w}_+ - \delta_1} \dots dx \,. \tag{3.221}$$

The endpoint divergences then manifest themselves in the form of $\ln \delta_1$ and $\ln \delta_2$. But all these logarithmic divergences will cancel out in the final result after x–integration.

In the limit of $m_b = 0$ the terms the integration limit of x is $y^2 \leqslant x \leqslant 1$. The terms $\ln(1-x)$, $\ln(1-y^2/x)$ are divergent at the endpoints. Again one can use cut–offs in the integration limits and all these divergences cancel out after integration. Also the mass divergent terms $\ln \epsilon$ will cancel out.

Since the differential rate functions $M^A(x)$ and $M^B(x)$ are mass convergent for $m_b \to 0$ we can therefore safely take $m_b \to 0$ limits of (3.218) and (3.219):

$$\begin{aligned} \frac{d\Gamma^{(1)}}{dx d\cos\theta_P d\phi} &= \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F (-\frac{\alpha_s}{2\pi}) 6y^2 \frac{1}{4\pi} \times \\ &\times \Big[M^A(x) + M^B(x)\, P\cos\theta_P + M^C(x)\, P \sin\theta_P \cos\phi \Big], \end{aligned} \tag{3.222}$$

where[8]

$$\begin{aligned} \mathcal{M}^A(x) &:= \lim_{m_b \to 0} M^A(x) \\ &= \frac{y^2}{2}(4 - 4x - y^2 + \frac{y^2}{x}) + x(1-x)\Big[\frac{\pi^2}{3} + \ln^2(\frac{x-y^2}{x-x^2}) + 2\mathrm{Li}_2(x) + 2\mathrm{Li}_2(\frac{y^2}{x})\Big] \\ &\quad + x\Big[\frac{\pi^2}{6} - \mathrm{Li}_2(x) + \mathrm{Li}_2(y^2) - \mathrm{Li}_2(\frac{y^2}{x})\Big] - \frac{1}{2}(3 + 2x + y^2)\left(1 - y^2\right)\ln(1-y^2) \\ &\quad + \frac{5}{2}(1-x)\ln(1-x) + \frac{1}{2}(9x - 4x^2 - 2y^2 - 2xy^2 - y^4)\ln(1 - \frac{y^2}{x}), \end{aligned} \tag{3.223}$$

$$\begin{aligned} \mathcal{M}^B(x) &:= \lim_{m_b \to 0} M^B(x) \\ &= \frac{1}{2}(2 - 2x^2 + 2y^2 - \frac{2y^2}{x} - 3y^4 + \frac{3y^4}{x}) \\ &\quad + x(1-x)\Big[\frac{\pi^2}{3} + \ln^2(\frac{x-y^2}{x-x^2}) + 2\mathrm{Li}_2(x) + 2\mathrm{Li}_2(\frac{y^2}{x})\Big] \\ &\quad - x\Big[\frac{\pi^2}{6} - \mathrm{Li}_2(x) + \mathrm{Li}_2(y^2) - \mathrm{Li}_2(\frac{y^2}{x})\Big] \\ &\quad + \frac{1}{2}(5 - \frac{2}{x} - 2x - y^2 - \frac{2y^2}{x})\left(1 - y^2\right)\ln(1-y^2) \\ &\quad + \frac{1}{2}(3x - 4x^2 + 6y^2 - 2xy^2 - y^4 - \frac{2y^2}{x})\ln(1 - \frac{y^2}{x}). \end{aligned} \tag{3.224}$$

[8] 3.223 and 3.224 agrees with $F_1^+(x,y)$ of Eq. (43) and $J_1^+(x,y)$ of Eq. (45) in [28]. Note that our variable y^2 correspond to y of [28].

Returning to the $m_b \neq 0$ case, we integrate the differential rate (3.217) over the energy spectrum ($\bar{w}_- \leqslant x \leqslant \bar{w}_+$) and obtain

$$\frac{d\Gamma^{(1)}}{d\cos\theta_P d\phi} = \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F(-\frac{\alpha_s}{2\pi})6y^2 \frac{1}{4\pi}\Big[M^A + M^B\, P\cos\theta_P + M^C\, P\sin\theta_P\cos\phi\Big], \quad (3.225)$$

where[9]

$$\begin{aligned} M^A &= \int_{\bar{w}_-}^{\bar{w}_+} dx\, M^A(x) \\ &= k_0^a\Omega_0 + k_1^a\Omega_1 + k_2^a\bar{Y}_w + k_3^a\bar{Y}_p + k_4^a\bar{p}_3\ln\epsilon + k_5^a\bar{p}_3, \end{aligned} \quad (3.226)$$

$$\begin{aligned} M^B &= \int_{\bar{w}_-}^{\bar{w}_+} dx\, M^B(x) \\ &= k_0^b\Omega_0 + k_1^b\Omega_1 + k_2^b\bar{Y}_w + k_3^b\bar{Y}_p + k_4^b\bar{p}_3\ln\epsilon + k_5^b\bar{p}_3 + k_6^b\bar{Y}_w\bar{Y}_p + k_7^b\bar{Y}_w\ln\epsilon \end{aligned} \quad (3.227)$$

with the variables

$$\begin{aligned} \Omega_0 &= \frac{2\bar{p}_0}{\bar{p}_3}\left[4\mathrm{Li}_2\left(1 - \frac{\bar{p}_-}{\bar{p}_+}\right) - 4\bar{Y}_p\ln\left(1 - \frac{\bar{p}_-}{\bar{p}_+}\right) + \bar{Y}_p\ln y^2 - 2\bar{Y}_w\ln\epsilon\right] + 8\ln(2\bar{p}_3) - 2\ln y^2, \\ \Omega_1 &= 4\mathrm{Li}_2(\bar{w}_-) - 4\mathrm{Li}_2(\bar{w}_+). \end{aligned} \quad (3.228)$$

The coefficients for M^A are given by

$$k_0^a = 4\bar{p}_3\left[\left(1-\epsilon^2\right)^2 + y^2\left(1+\epsilon^2\right) - 2y^4\right], \quad (3.229)$$

$$k_1^a = 4\bar{p}_0\left[\left(1-\epsilon^2\right)^2 + y^2\left(1+\epsilon^2\right) - 2y^4\right], \quad (3.230)$$

$$k_2^a = -8\left(1-\epsilon^2\right)\Big[1 + y^2 - 4y^4 - \epsilon^2\left(2-y^2\right) + \epsilon^4\Big], \quad (3.231)$$

$$\begin{aligned} k_3^a = -2\Big[&3 + 6y^2 - 21y^4 + 12y^6 \\ &- \epsilon^2\left(1 + 12y^2 + 5y^4y^2\right) + \epsilon^4\left(1 - 11 + 2y^2\right) - \epsilon^6\Big], \end{aligned} \quad (3.232)$$

$$k_4^a = -12\Big[1 + 3y^2 - 4y^4 - \epsilon^2\left(4-y^2\right) + 3\epsilon^4\Big], \quad (3.233)$$

$$k_5^a = -2\Big[5 + 9y^2 - 6y^4 - \epsilon^2\left(22 - 9y^2\right) + 5\epsilon^4\Big]. \quad (3.234)$$

[9] 3.226 and 3.229 agree with $\mathcal{F}_1(y)$ of Eq. (31) and $\mathcal{J}_1^+(y)$ of Eq. (32) in [28]. Note that our variable y^2 correspond to y of [28].

The coefficients for M^B are given by

$$k_0^b = 4\bar{p}_3\left[\left(1-\epsilon^2\right)^2 + y^2\left(1+\epsilon^2\right) - 2y^4\right], \tag{3.235}$$
$$k_1^b = 5 - 9y^4 + 4y^6 - \epsilon^2\left(8 - 8y^2 + 6y^4\right) + \epsilon^4 + 2\epsilon^6, \tag{3.236}$$
$$k_2^b = -8\left[1 + 4y^2 - y^4 - \epsilon^2\left(3 + 3y^2 - 4y^4\right) + \epsilon^4\left(3 - y^2\right) - \epsilon^6\right], \tag{3.237}$$
$$k_3^b = -2\left[3\left(1-y^2\right)^2\left(1+4y^2\right) + \epsilon^2\left(11 - 5y^4\right) - \epsilon^4\left(13 - 2y^2\right) - \epsilon^6\right], \tag{3.238}$$
$$k_4^b = -12\left[\left(1-y^2\right)\left(1+4y^2\right) - \epsilon^2\left(4 - y^2\right) + 3\epsilon^4\right], \tag{3.239}$$
$$k_5^b = 2\left[15 - y^2 + 2y^4 - \epsilon^2\left(12 + 7y^2\right) - 3\epsilon^4\right], \tag{3.240}$$
$$k_6^b = -24\left(1 + y^2 - \epsilon^2\right)\left(2\bar{p}_3 + \frac{\epsilon^2 y^2}{\bar{p}_3}\right), \tag{3.241}$$
$$k_7^b = -24\left(1 - y^4 - 2\epsilon^2 + \epsilon^4\right). \tag{3.242}$$

For $m_b = 0$ the total rate simplifies to

$$\frac{d\Gamma^{(1)}}{d\cos\theta_P d\phi} = \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F\left(-\frac{\alpha_s}{2\pi}\right) 6y^2 \frac{1}{4\pi}\left[\mathcal{M}^{\mathcal{A}} + \mathcal{M}^{\mathcal{B}}\, P\cos\theta_P + \mathcal{M}^{\mathcal{C}}\, P\sin\theta\cos\phi\right], \tag{3.243}$$

where[10]

$$\begin{aligned}
\mathcal{M}^{\mathcal{A}} &:= \lim_{m_b\to 0} M^A \\
&= \left(y^2-1\right)\left(5 + 9y^2 - 6y^4\right) \\
&\quad + 2(1-y^2)^2(1+2y^2)\left[\frac{2\pi^2}{3} + 4\ln(1-y^2)\ln(y) + 4\mathrm{Li}_2(y^2)\right] \\
&\quad + 8y^2\left(1 - y^2 - 2y^4\right)\ln(y) + 2\left(1-y^2\right)^2\left(5+4y^2\right)\ln(1-y^2)
\end{aligned} \tag{3.244}$$

$$\begin{aligned}
\mathcal{M}^{\mathcal{B}} &:= \lim_{m_b\to 0} M^B \\
&= \left(1-y^2\right)\left(15 - y^2 + 2y^4\right) - \frac{2\pi^2}{3}\left(1-y^2\right)\left(1+y^2+4y^4\right) \\
&\quad + 2\left(1-y^2\right)^2\left(5+4y^2\right)\ln(1-y^2) + 16y^2\left(2+y^2-y^4\right)\ln(y) \\
&\quad + 16\left(1-y^2\right)\left(2+2y^2-y^4\right)\ln(1-y^2)\ln(y) \\
&\quad + 4\left(1-y^2\right)\left(5+5y^2-4y^4\right)\mathrm{Li}_2(y^2).
\end{aligned} \tag{3.245}$$

As we will see in section 3.7, the lepton energy spectra for $m_b \neq 0$ and for $m_b = 0$ are not distinguishable on the scale of the plots. Both the total unpolarized rate and the polar rate for $m_b = 0$ limit increase by 1.4% relative to the $m_b \neq 0$ results. If we scale this difference w.r.t. the unpolarized LO rate it is around 1‰. This shows that setting $m_b = 0$ is a very good approximation.

The azimuthal contributions will be calculated in the next section.

[10] 3.244 and 3.245 agree with $\mathcal{F}_1(y)$ of Eq. (60) and $\mathcal{J}_1^+(y)$ of Eq. (61) in [28]. Note that our variable y^2 correspond to y of [28].

3.6 Azimuthal correlation at $\mathcal{O}(\alpha_s)$

Compared to the unpolarized and polar contributions the NLO azimuthal contribution from real gluon emission has an additional square root factor $\sqrt{y^2\,(x - x^2 - y^2 + xy^2 - xz)}$ which leads to integrals of the form

$$\int\!dx\, x^k \!\int\!dz\, \frac{z^m \sqrt{y^2\,(x - x^2 - y^2 + xy^2 - xz)}}{\sqrt{\lambda(1,y^2,z)^n}} \ln\left(\frac{1 - y^2 + z + \sqrt{\lambda(1,y^2,z)}}{1 - y^2 + z - \sqrt{\lambda(1,y^2,z)}}\right). \quad (3.246)$$

This square root factor $\sqrt{y^2\,(x - x^2 - y^2 + xy^2 - xz)}$ comes from the scalar products $P\cdot s_t$ and $p_\nu \cdot s_t$ (see 3.153) contributing to the azimuthal term. In order to obtain the energy spectrum one has to perform a z–integration of the functions such as those in (3.246). But these integrations can not be done in closed form.

However, the total integration of (3.246) can be done in closed form by changing the order of the integration, i.e. by doing the x–integration fist and then the z–integration:

$$\int\!dz\, \frac{z^m}{\sqrt{\lambda(1,y^2,z)^n}} \ln\left(\frac{1 - y^2 + z + \sqrt{\lambda(1,y^2,z)}}{1 - y^2 + z - \sqrt{\lambda(1,y^2,z)}}\right)\!\int\!dx\, x^k \sqrt{y^2\,(x - x^2 - y^2 + xy^2 - xz)},$$

with the integration limits[11]

$$\int\limits_{\epsilon^2}^{(1-y)^2} dz \int\limits_{w_-}^{w_+} dx\,. \quad (3.247)$$

The x–integration is done with the help of basic integrals defined by

$$\mathcal{X}_n := \int\limits_{w_-}^{w_+} dx\; x^n \sqrt{y^2\,(x - x^2 - y^2 + xy^2 - x\epsilon^2)}\,. \quad (3.248)$$

The necessary basic integrals $\mathcal{X}_n$ are calculated and listed in Appendix D.6.

The differential rate for the $\mathcal{O}(\alpha_s)$ azimuthal contribution can be read off from (3.208):

$$\frac{d\Gamma_C^{(1)}}{dxdz} = \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F(-\frac{\alpha_s}{2\pi}) 6y^2 \left(M_v^C(x)\delta(x) + M_r^C(x,z)\right), \quad (3.249)$$

where the virtual one–loop contribution is given in (3.132). The real emission contribution $M_r^C(x,z)$ comes from the contraction of the IR–convergent part of the hadron tensor with the lepton tensor as shown in (3.155).

Note that the $\mathcal{O}(\alpha_s)$ azimuthal part does not have an IR–divergence. The reason is that the IR–divergence in the NLO is proportional to the leading order rate. Since the azimuthal rate vanishes in the leading order the NLO azimuthal contribution is IR–convergent.

[11] $w_\pm = \frac{1}{2}\big(1 + y^2 - z \pm \sqrt{\lambda(1,y^2,z)}\,\big)$ as defined in (3.6).

To prepare for the x–integration $M_r^C(x,z)$ is written as a polynomial of x (including negative power of x):

$$M_r^C(x,z) = -\sqrt{y^2(1-x)(x-y^2) - xy^2 z}\left\{\frac{y^2(z-\epsilon^2)}{xz^2\lambda^3}j_1 + \frac{1}{z^2\lambda^3}j_2 + \frac{x}{z^2\lambda^3}j_3 + 2\left[\frac{6y^2(z-\epsilon^2)}{x\lambda^{7/2}}j_4 + \frac{1}{\lambda^{7/2}}j_5 + \frac{x}{\lambda^{7/2}}j_6\right]\ln\left(\frac{1-y^2+z+\sqrt{\lambda}}{1-y^2+z-\sqrt{\lambda}}\right)\right\}, \quad (3.250)$$

where $\lambda = \lambda(1, y^2, z)$. The coefficients j_i are given by

$$\begin{aligned} j_1 = {} & \left(1-y^2\right)^4 z + \left(1-y^2\right)^2\left(25+2y^2-y^4\right)z^2 \\ & - 4\left(1-y^2\right)\left(11+y^2+y^4\right)z^3 + 2\left(4-8y^2-3y^4\right)z^4 + \left(11+4y^2\right)z^5 - z^6 \\ & + \epsilon^2\Big[\left(1-y^2\right)^4 - \left(1-y^2\right)^2\left(11+6y^2+y^4\right)z - 4\left(2+10y^2+5y^4-y^6\right)z^2 \\ & + 2\left(22+24y^2-3y^4\right)z^3 - \left(25-4y^2\right)z^4 - z^5\Big], \end{aligned} \quad (3.251)$$

$$\begin{aligned} j_2 = {} & -\left(1-y^2\right)^4\left(11+2y^2\right)z^2 + 2\left(1-y^2\right)^3\left(13-2y^2\right)z^3 \\ & - 4\left(3+2y^2+3y^4\right)z^4 - 2\left(5-23y^2+2y^4\right)z^5 + \left(7+2y^2\right)z^6 \\ & + \epsilon^2\Big[\left(1-y^2\right)^4\left(2+y^2\right)z + 4\left(1-y^2\right)^2\left(6+16y^2+3y^4\right)z^2 \\ & - 2\left(40-41y^2-34y^4+19y^6\right)z^3 + 4\left(19-22y^2+8y^4\right)z^4 \\ & - \left(18+3y^2\right)z^5 - 4z^6\Big] + \epsilon^4\Big[\left(1-y^2\right)^4\left(1+y^2\right) \\ & - 2\left(1-y^2\right)^2\left(1+y^2\right)\left(5+4y^2\right)z + 2\left(6-37y^2-36y^4+3y^6\right)z^2 \\ & + 2\left(7+53y^2+10y^4\right)z^3 - \left(29+31y^2\right)z^4 + 12z^5\Big], \end{aligned} \quad (3.252)$$

$$\begin{aligned} j_3 = {} & 12\left(1-y^2\right)^4 z^2 - 2\left(1-y^2\right)^2\left(6+7y^2\right)z^3 - 4\left(3-13y^2+2y^4\right)z^4 \\ & + 2\left(6+5y^2\right)z^5 + \epsilon^2\Big[3\left(1-y^2\right)^4 z + 2\left(1-y^2\right)^2\left(15+13y^2\right)z^2 \\ & - 4\left(15-20y^2+13y^4\right)z^3 + 2\left(9+7y^2\right)z^4 + 9z^5\Big] + \epsilon^4\Big[-\left(1-y^2\right)^4 \\ & + 2\left(1-y^2\right)^2\left(5+4y^2\right)z + 4y^2\left(15+y^2\right)z^2 - 2\left(13+14y^2\right)z^3 + 17z^4\Big], \end{aligned} \quad (3.253)$$

$$\begin{aligned} j_4 = {} & -\left(1-y^2\right)^3 - y^2\left(1-y^2\right)z + 3z^2 - 2z^3 \\ & + \epsilon^2\Big[\left(1-y^2\right)\left(2+2y^2+y^4\right) - \left(3-y^2-3y^6\right)z - 3y^2z^2 + z^3\Big], \end{aligned} \quad (3.254)$$

$$\begin{aligned} j_5 = {} & 2\left(1-y^2\right)^5 + 5y^2\left(1-y^2\right)^3 z - \left(1-y^2\right)\left(11-23y^2+4y^4\right)z^2 \\ & + \left(13-19y^2+14y^4\right)z^3 - \left(3+10y^2\right)z^4 - z^5 \\ & + \epsilon^2\Big[-\left(1-y^2\right)^3\left(8+11y^2+2y^4\right) + \left(1-y^2\right)\left(2+y^2\right)\left(7-27y^2+8y^4\right)z \\ & + \left(6+37y^2-43y^4+12y^6\right)z^2 - \left(22-9y^2+8y^4\right)z^3 + 2\left(5+y^2\right)z^4\Big] \\ & + \epsilon^4\Big[\left(1-y^2\right)\left(7+30y^2+21y^4+2y^6\right) - \left(19+14y^2-37y^4-8y^6\right)z \end{aligned}$$

$$+\left(15-17y^2-12y^4\right)z^2-\left(1-8y^2\right)z^3-2z^4\Big], \tag{3.255}$$

$$\begin{aligned}
j_6 = & -2\left(1-y^2\right)^5-\left(1-y^2\right)^3\left(4-5y^2\right)z \\
& +\left(1-y^2\right)\left(12-11y^2-3y^4\right)z^2-\left(4+15y^2-y^4\right)z^3-\left(2+y^2\right)z^4 \\
& +\epsilon^2\Big[5\left(1-y^2\right)^3\left(2+y^2\right)-7\left(1-y^2\right)^2\left(1-2y^2\right)z-15\left(15-19y^2+12y^4\right)z^2 \\
& +\left(11+2y^2\right)z^3+z^4\Big]+3\epsilon^4\Big[\left(1-y^2\right)\left(3+6y^2+y^4\right) \\
& -\left(5-4y^2-3y^6\right)z+\left(1-3y^2\right)z^2+z^3\Big].
\end{aligned} \tag{3.256}$$

Performing the x–integration with the help of the basic integrals $\mathcal{X}_n$ we obtain

$$\begin{aligned}
\int_{w_-}^{w_+} dx\, M_r^C(x,z) = & -\frac{15\pi y^3\left[(1+y)^2-\epsilon^2\right]^2}{4\left[(1+y)^2-z\right]^3}-\frac{3\pi y\left[(1-y)^2-\epsilon^2\right]^2}{64\left[(1-y)^2-z\right]} \\
& -\frac{\pi y}{16}\left(2+2y^2+\epsilon^2\right)-\frac{\pi\epsilon^4 y\,(1-y)^2\left(1+4y+y^2\right)}{16\,(1+y)^2 z^2} \\
& +\frac{\pi y^2\left[(1+y)^2-\epsilon^2\right]^2 v_1}{16\,(1+y)^2\left[(1+y)^2-z\right]^2}+\frac{\pi y\, v_2}{64\,(1+y)^4\left[(1+y)^2-z\right]} \\
& +\frac{\pi\epsilon^2 y\, v_3}{16\,(1+y)^4 z} \\
& +\pi Y_p\sqrt{\lambda(1,y^2,z)}\left\{-\frac{15y^2(1+y)\left[(1+y)^2-\epsilon^2\right]^2}{8\left[(1+y)^2-z\right]^4}\right. \\
& +\frac{3(1-y)\left[(1-y)^2-\epsilon^2\right]^2}{128\left[(1-y)^2-z\right]^2}+\frac{3y\, v_4}{16\left[(1+y)^2-z\right]^3} \\
& +\frac{3v_5}{128\left[(1+y)^2-z\right]^2}+\frac{v_6}{256y\left[(1+y)^2-z\right]} \\
& \left.+\frac{v_7}{256y\left[(1-y)^2-z\right]}\right\},
\end{aligned} \tag{3.257}$$

where the coefficients v_i are given by

$$v_1 = (1+y)^2\left(27+54y-25y^2\right)-\epsilon^2\left(27+30y+23y^2\right), \tag{3.258}$$

$$\begin{aligned}
v_2 = & (1+y)^5\left(51-63y+73y^2-21y^3\right) \\
& -2\epsilon^2(1+y)^4\left(51-6y-5y^2\right)+\epsilon^4\left(51+204y+202y^2+108y^3+11y^4\right),
\end{aligned} \tag{3.259}$$

$$v_3 = -\left(1-y^2\right)(1+y)^4+\epsilon^2\left(5+20y+4y^2-4y^3-5y^4\right), \tag{3.260}$$

$$v_4 = (2-y)(1+y)^3(1+5y)-2\epsilon^2(1+y)\left(2+7y-y^2\right)+\epsilon^4(2+3y), \tag{3.261}$$

$$v_5 = (1+y)\left(21+4y-2y^2+20y^3-11y^4\right)$$

$$- 2\epsilon^2 \left(21 + 15y - y^2 - 3y^3\right) + \epsilon^4 \left(21 + 5y\right), \tag{3.262}$$

$$\begin{aligned} v_6 = {} & \left(33 - 34y + 18y^2 - 36y^3 + 53y^4 + 30y^5\right) \\ & - 2\epsilon^2 \left(3 + y\right) \left(11 - 11y + 10y^2\right) + \epsilon^4 \left(33 - 10y\right), \end{aligned} \tag{3.263}$$

$$\begin{aligned} v_7 = {} & - \left(1 - y\right)^2 \left(33 + 32y + 49y^2 + 30y^3\right) \\ & + 2\epsilon^2 \left(33 - 22y - 13y^2 + 10y^3\right) - \epsilon^4 \left(33 - 10y\right). \end{aligned} \tag{3.264}$$

Next we perform the z–integration. Inspecting (3.257) we define the following basic integrals for the z–integration:

$$U(n) = \int dz \frac{\sqrt{\lambda\left(1, y^2, z\right)}}{\left[\left(1 + y\right)^2 - z\right]^n} Y_p \tag{3.265}$$

$$V(n) = \int dz \frac{\sqrt{\lambda\left(1, y^2, z\right)}}{\left[\left(1 - y\right)^2 - z\right]^n} Y_p\,. \tag{3.266}$$

The necessary basic integrals are calculated and listed in Appendix D.6. After z–integration, we combine the result with the integrated virtual one–loop contributions. The differential virtual one–loop contribution $M_v^C(x)$ is given in (3.132). The x–integration is performed with the help of the basic integrals $\mathcal{X}_n$. The result is

$$\int\limits_{\bar{w}_-}^{\bar{w}_+} M_v^C(x) dx = - \frac{\pi \bar{p}_3^2 \ln \epsilon}{4y} \left(1 - y^2 - \epsilon^2\right) - \frac{\pi \bar{p}_3 \bar{Y}_p}{8y} \left[\left(1 - \epsilon^2\right)^2 - 2y^2 \left(1 - 3\epsilon^2\right) + y^4\right]. \tag{3.267}$$

Finally substituting the virtual one–loop and real emission contributions into (3.249) we have

$$\begin{aligned} \Gamma_C^{(1)} &= \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F \left(-\frac{\alpha_s}{2\pi}\right) 6y^2 \int dz \int dx \Big(M_v^C(x)\delta(x) + M_r^C(x, z)\Big) \\ &= \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F \left(-\frac{\alpha_s}{2\pi}\right) 6y^2 \frac{1}{12} \Big\{ d_1 \left[\mathrm{Li}_2\left(\bar{p}_+\right) + \mathrm{Li}_2\left(\bar{p}_-\right) + 2\mathrm{Li}_2\left(y\right) - 2\bar{Y}_w \bar{Y}_p + \ln y^2 \ln \epsilon\right] \\ &\qquad + d_2\, \bar{p}_3 \bar{Y}_p + d_3 \ln\left(1 - y\right) + d_4 \ln \epsilon + d_5 \Big\}, \end{aligned} \tag{3.268}$$

with the coefficients

$$d_1 = - \frac{3\pi y}{2} \left[4 + 3y^2 - 3y^4 - \epsilon^2 \left(5 - 2y^2\right) + \epsilon^4\right], \tag{3.269}$$

$$\begin{aligned} d_2 = {} & - \frac{3\pi}{2y} \Big[1 + 16y - 16y^2 - 16y^3 - 9y^4 \\ & - 2\epsilon^2 \left(1 + 8y - 4y^2\right) + \epsilon^4 + \frac{32y^3 \left(1 + y\right)^2}{\left(1 + y\right)^2 - \epsilon^2}\Big], \end{aligned} \tag{3.270}$$

$$d_3 = \frac{3\pi}{2} \Big[8 - 7y + 2y^3 - 8y^4 + 5y^5$$

$$- \epsilon^2 \left(16 - 19y + 3y^3\right) + 2\epsilon^4 (4 - 3y)\Big], \tag{3.271}$$

$$d_4 = -\frac{3\pi}{4y}\Big[1 + 16y - 17y^2 + 7y^4 - 16y^5 + 9y^6$$
$$- \epsilon^2 \left(3 + 32y - 40y^2 + 5y^4\right) + \epsilon^4 \left(3 + 16y - 11y^2\right) - \epsilon^6\Big], \tag{3.272}$$

$$d_5 = \frac{\pi y}{2}\Big\{3 + 4\pi^2 - 9y + 3\left(1 + \pi^2\right) y^2 + 9y^3 - 3\left(2 + \pi^2\right) y^4$$
$$- \epsilon^2\Big[15 + 5\pi^2 - 27y + 2\left(3 - \pi^2\right) y^2\Big] + \epsilon^4 \left(12 + \pi^2\right)\Big\}. \tag{3.273}$$

For $m_b = 0$ the azimuthal contribution simplifies considerably. One has

$$\begin{aligned}\Gamma_C^{(1)} &= \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F(-\frac{\alpha_s}{2\pi})\frac{3}{8}\pi y^2\Big\{2y(-4 - 3y^2 + 3y^4)\left(2\mathrm{Li}_2(y) - \mathrm{Li}_2(y^2)\right) \\ &\quad -2(1-y^2)(8 - 7y + 8y^2 - 5y^3)\ln(1+y) - \frac{(1-y^2)^3}{y}\ln(1-y^2) \\ &\quad +\frac{1}{3}y\left[6(1-y)^2(1-y-2y^2) + \pi^2(4 + 3y^2 - 3y^4)\right]\Big\}. \end{aligned} \tag{3.274}$$

Note that the totally integrated NLO virtual one–loop and real azimuthal contributions are separately mass singular. When summing the two contributions the mass singularities cancel as expected from the Lee–Nauenberg theorem [61]. Numerically the total azimuthal rate in the $m_b = 0$ limit increases by 1.7% relative to the $m_b \neq 0$ case. If we scale this difference w.r.t. the unpolarized LO rate it is around 0.04‰. Therefore $m_b = 0$ limit is a very good approximation.

3.7 Summary

The total differential decay rate at $\mathcal{O}(\alpha_s)$ is the sum of the LO and QCD NLO differential rates:

$$\frac{d\Gamma}{dx\, d\cos\theta_P\, d\phi} = \frac{1}{4\pi}\left[\left(\frac{d\Gamma_A^{(0)}}{dx} + \frac{d\Gamma_A^{(1)}}{dx}\right) + \left(\frac{d\Gamma_B^{(0)}}{dx} + \frac{d\Gamma_B^{(1)}}{dx}\right) P\cos\theta_P + \frac{d\Gamma_C^{(1)}}{dx} P\sin\theta_P\cos\phi\right]. \tag{3.275}$$

We have used the narrow–width approximation for the W propagator. As discussed at the end of section 3.5 and 3.6 $m_b = 0$ is a very good approximation. Therefore we set $m_b = 0$ in the numerical results in this section. The differential unpolarized, polar and azimuthal rates are shown in Fig. 3.7, Fig. 3.8 and Fig. 3.9 respectively (for $m_b = 0$). It is easy to see that the QCD NLO corrections decrease the differential rate. Note also that the $\mathcal{O}(\alpha_s)$ unpolarized and the polar spectra look almost identical. This is because the LO unpolarized rate equals the polarized rate and the QCD NLO corrections for the unpolarized and polar differential rates are close to each other.

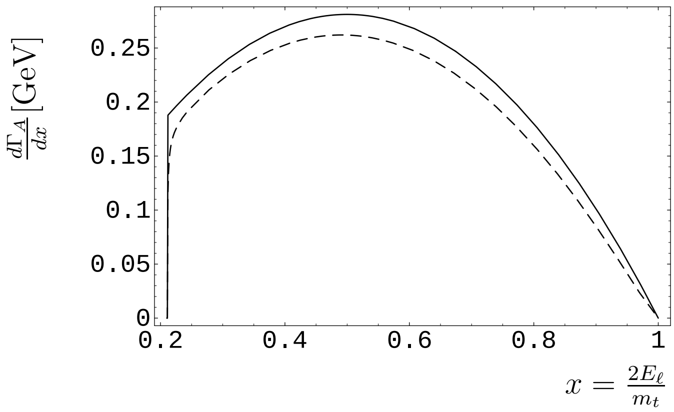


Figure 3.7: The unpolarized differential rates w.r.t. the charged lepton energy (charged lepton spectra) in system 1a. The solid line is the LO spectrum and the dashed line is the spectrum with QCD NLO corrections.

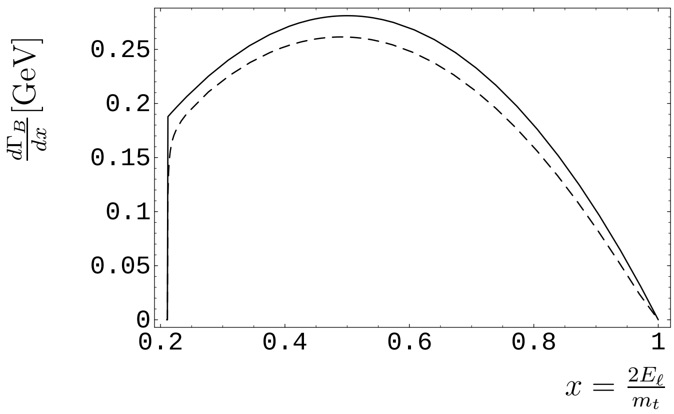


Figure 3.8: The differential polar rates w.r.t. the charged lepton energy (charged lepton spectrum). The solid line is the LO spectrum and the dashed line is the spectrum with QCD NLO corrections

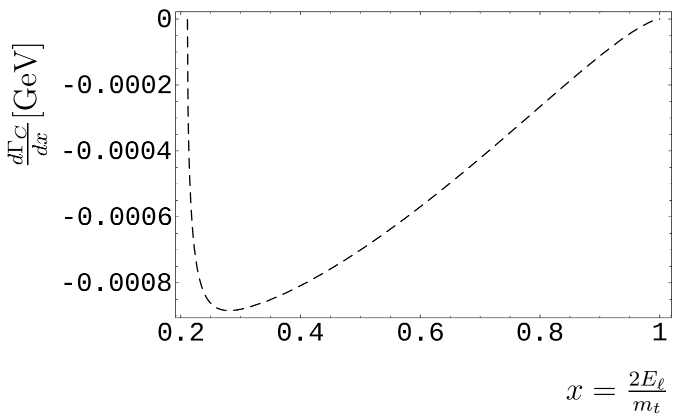


Figure 3.9: The azimuthal differential rate w.r.t. the charged lepton energy (charged lepton spectrum). The contribution comes only from QCD NLO corrections since the LO rate vanishes.

The azimuthal spectrum is obtained by numerical integration. The closed form integrated azimuthal rate is obtained by performing the x–integration first and then the

z–integration. In the numerical integration we perform the z–integration first to obtain the charged lepton energy spectrum. Then we perform the x–integration to obtain the integrated rate. These two different ways are non–trivial check for each other.

The total integrated rate can be written as

$$\frac{d\Gamma}{d\cos\theta_P\,d\phi} = \frac{1}{4\pi}\left[\left(\Gamma_A^{(0)} + \Gamma_A^{(1)}\right) + \left(\Gamma_B^{(0)} + \Gamma_B^{(1)}\right) P\cos\theta_P + \Gamma_C^{(1)} P\sin\theta_P\cos\phi\right]. \quad (3.276)$$

To highlight the change of the LO result by the QCD NLO correction we write the total rate in the form

$$\frac{d\Gamma}{d\cos\theta_P\,d\phi} = \frac{\Gamma_A^{(0)}}{4\pi}\left[\left(1 + \frac{\Gamma_A^{(1)}}{\Gamma_A^{(0)}}\right) + \left(1 + \frac{\Gamma_B^{(1)}}{\Gamma_A^{(0)}}\right) P\cos\theta_P + \frac{\Gamma_C^{(1)}}{\Gamma_A^{(0)}} P\sin\theta_P\cos\phi\right]. \quad (3.277)$$

Taking values $|V_{tb}| = 0.999$, $m_t = 175\,\mathrm{GeV}$, and neglecting the b–quark mass we obtain

$$\boxed{\frac{d\Gamma}{d\cos\theta_P\,d\phi} = \frac{\Gamma_A^{(0)}}{4\pi}\Big[(1 - 8.54\,\%) + (1 - 8.71\,\%)\,P\cos\theta_P - 0.24\,\%\,P\sin\theta_P\cos\phi\Big].} \quad (3.278)$$

We can also write the rate in the following form:

$$\frac{1}{\Gamma_A}\frac{d\Gamma}{d\cos\theta_P\,d\phi} = \frac{1}{4\pi}\left[1 + P\,\kappa_\ell^B\cos\theta_P + P\,\kappa_\ell^C\sin\theta_P\cos\phi\right], \quad (3.279)$$

where

$$\kappa_\ell^B := \frac{\Gamma_B}{\Gamma_A}, \qquad \kappa_\ell^C := \frac{\Gamma_C}{\Gamma_A}. \quad (3.280)$$

The coefficients κ_ℓ^B and κ_ℓ^C are called the *polar analyzing power* or *polar asymmetry* and *azimuthal analyzing power* or *azimuthal asymmetry* of the charged lepton, respectively.

At LO we have $\kappa_\ell^B = 1$ and $\kappa_\ell^C = 0$ in system 1a (see 3.51), corresponding to 100% analyzing power for the polar correlation and zero analyzing power for the azimuthal correlation.

At NLO the analyzing powers in system 1a are given by

$$\kappa_\ell^B = \frac{\Gamma_B^{(0)} + \Gamma_B^{(1)}}{\Gamma_A^{(0)} + \Gamma_A^{(1)}}, \quad (3.281)$$

$$\kappa_\ell^C = \frac{\Gamma_C^{(1)}}{\Gamma_A^{(0)} + \Gamma_A^{(1)}}. \quad (3.282)$$

For the numerical result at NLO we obtain

$$\boxed{\frac{1}{\Gamma_A^{(0)} + \Gamma_A^{(1)}}\frac{d\Gamma}{d\cos\theta_P\,d\phi} = \frac{1}{4\pi}\Big(1 + 99.8\%\,P\cos\theta_P - 0.26\%\,P\sin\theta_P\cos\phi\Big).} \quad (3.283)$$

Compared to the analyzing power at LO the polar analyzing power has changed from the LO value 1 to the NLO value 0.998 and the azimuthal analyzing power has changed from the LO value zero to 0.0026, i.e. the analyzing powers are changed very little by the radiative corrections.

The positivity of the total rate can be shown by writing

$$\frac{d\Gamma}{d\cos\theta_P\, d\phi} = \frac{\Gamma_A^{(0)}}{4\pi}\left[\left(1+\frac{\Gamma_A^{(1)}}{\Gamma_A^{(0)}}\right) + P\sqrt{\left(1+\frac{\Gamma_B^{(1)}}{\Gamma_A^{(0)}}\right)^2 + \left(\frac{\Gamma_C^{(1)}}{\Gamma_A^{(0)}}\cos\phi\right)^2}\,\sin\left(\theta_P+\delta\right)\right]. \tag{3.284}$$

with

$$\tan\delta = \frac{\Gamma_A^{(0)}+\Gamma_B^{(1)}}{\Gamma_C^{(1)}\cos\phi}, \qquad -\frac{\pi}{2} \leqslant \delta \leqslant \frac{\pi}{2}. \tag{3.285}$$

The total rate takes its minimum value for $\cos\phi = \pm 1$, $\sin(\delta+\theta_P) = -1$ and $P = 1$:

$$\begin{aligned}
&\left(1+\frac{\Gamma_A^{(1)}}{\Gamma_A^{(0)}}\right) - \sqrt{\left(1+\frac{\Gamma_B^{(1)}}{\Gamma_A^{(0)}}\right)^2 + \left(\frac{\Gamma_C^{(1)}}{\Gamma_A^{(0)}}\right)^2} \\
&= (1-8.54\,\%) - \sqrt{(1-8.71\,\%)^2 + (0.24\,\%)^2}\,, \\
&= 0.17\,\%
\end{aligned} \tag{3.286}$$

i.e., the differential rate is positive in the whole domain as it must be.

Finally we give a summary of this chapter.

We have calculated the $O(\alpha_s)$ corrections to the unpolarized, polar and azimuthal correlation observable in polarized top quark decay in the helicity system 1a, where the event plane lies in the (x,z)–plane and the charged lepton momentum is along the z–axis. The LO unpolarized and polar rates are equal to each other. The $\mathcal{O}(\alpha_s)$ corrections decrease both the unpolarized and polar rates by $\sim 9\%$. For the unpolarized and polar $\mathcal{O}(\alpha_s)$ corrections we found agreement with existing results in the literature. Our azimuthal results are new. We have seen that the SM (V-A) current predicts zero azimuthal correlations at LO. The $\mathcal{O}(\alpha_s)$ corrections to the zero LO result amount to $\sim -0.24\%$ relative to the LO unpolarized rate. If top quark decays reveal a violation of the LO SM $(V-A)$ current structure in the azimuthal correlation function which exceeds the 1% level, the violation must have a non–SM origin.

Chapter 4

The angular correlations for the decay $t(\uparrow) \to b + \ell^+ + \nu_\ell$ in the helicity system 3a

In this chapter we calculate the neutrino spectrum in system 3a (see Fig. 4.1), in which the neutrino momentum is along the z–axis with $(p_\ell)_x \leqslant 0$. From the experimental point of view the identification of the momentum direction of the invisible neutrino will be more difficult than the determination of the momentum direction of the charged lepton in system 1a but can be done using four–momentum conservation.

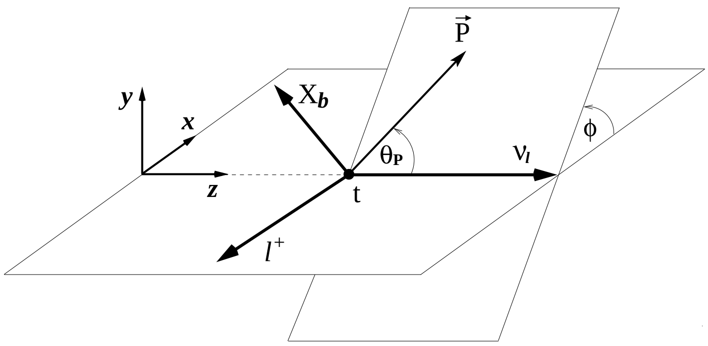


Figure 4.1: The helicity system 3a for the rest–frame decay of as polarized top quark: $t(\uparrow) \to b + \ell^+ + \nu_\ell$. $\vec{P}$ is the polarization vector of the top quark. θ_P and ϕ are the polar and azimuthal angles respectively.

The neutrino spectrum of the up-type quark in system 3a can be seen to be identical to the charged lepton spectrum in the down-type quark in system 1a shown in Fig. 3.1. This follows from the structure of the lepton tensors in these two cases. In the up–type quark decay the lepton tensor is

$$L_{\mu\nu} = 8 p_\ell^\sigma p_\nu^\rho \, T_{\sigma\rho\mu\nu}, \tag{4.1}$$

while, for the down–type quark decay, the lepton tensor is

$$L_{\mu\nu} = 8 p_\nu^\sigma p_\ell^\rho \, T_{\sigma\rho\mu\nu}. \tag{4.2}$$

It is easy to see that they are related by a permutation of the charged lepton and the neutrino momenta: $p_\ell \leftrightarrow p_\nu$.

4.1 The Born approximation

The LO differential rate is given by

$$d\Gamma^{(0)} = \frac{1}{2m_t}\frac{1}{(2\pi)^5} dR_3(p_t; p_b, p_\ell, p_\nu)\, \overline{|M_0|}^2 \tag{4.3}$$

Similar to (3.47), for the LO differential rate in the narrow–width approximation, one obtains

$$\frac{d\Gamma^{(0)}}{dx_\nu d\cos\theta_P d\phi} = \Gamma_F\, 2\pi \frac{m_W}{\Gamma_W}\,\frac{3}{16}\frac{\bar{y}^2}{m_t^4}\int dx_\nu\, H^{(0)}_{\mu\nu} L^{\mu\nu}\,, \tag{4.4}$$

where, as defined in (3.4), x_ν is the scaled energy of the neutrino: $x_\nu = \frac{2E_\nu}{m_t}$. We have decided to use the neutrino energy as energy variable in analyzing the decay in system 3a. We could have equally well done the analysis using again the charged lepton energy as energy variable as in system 1a. Experimentally the analysis in system 3a requires the reconstruction of the momentum direction of the neutrino which entails a knowledge of the energy of the neutrino. In addition one can make use of our results in the charged lepton energy spectrum in the down–type quark decay (e.g. in the decay of $b \to c+\ell+\nu_\ell$) as discussed at the beginning of this chapter.

In this chapter we will adhere to the narrow–width approximation from the beginning, i.e. we use $y = \frac{m_W}{m_t}$ instead of $y = \sqrt{\frac{W^2}{m_t^2}}$. The hadron–lepton contraction is given in (3.20):

$$H^{(0)}_{\mu\nu} L^{\mu\nu} = 128\,(\tilde{p}_t \cdot p_\ell)\,(p_b \cdot p_\nu)\ . \tag{4.5}$$

At LO the explicit representation of the momenta in system 3a can be obtained from system 1a by the exchange of the labels $l \leftrightarrow \nu$ (except for sign changes) such that e.g. $x \leftrightarrow x_\nu$, $p_\ell \leftrightarrow p_\nu$. One has

$$\begin{aligned}
p_t &= m_t(1;0,0,0)\,, \\
p_\nu &= \frac{m_t}{2}x_\nu(1;0,0,1)\,, \\
p_\ell &= \frac{m_t}{2}(1-x_\nu+y^2-\epsilon^2)(1;-\sin\theta_{\nu\ell},0,\cos\theta_{\nu\ell})\,, \\
p_b &= \frac{m_t}{2}(1-y^2+\epsilon^2)(1;-\sin\theta_{\nu b},0,\cos\theta_{\nu b})\,, \\
s_t &= P(0;\sin\theta_P\cos\phi,\sin\theta_P\sin\phi,\cos\theta_P)\,,
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
\cos\theta_{\nu\ell} &= \frac{x_\nu(1-x_\nu+y^2-\epsilon^2)-2y^2}{x_\nu(1-x+y^2-\epsilon^2)}\,, \\
\cos\theta_{\nu b} &= \frac{2y^2-x_\nu(1+y^2-\epsilon^2)}{x_\nu\sqrt{\lambda(1,y^2,\epsilon^2)}}\,.
\end{aligned} \tag{4.7}$$

The parameterizations in (4.6) gives the scalar products that appear in the LO and virtual one–loop QCD corrections:

$$\begin{aligned}
p_t \cdot p_\nu &= \frac{1}{2}\, x_\nu\, m_t^2\,, \\
p_t \cdot p_\ell &= \frac{1 - x_\nu + y^2 - \epsilon^2}{2} m_t^2\,, \\
p_t \cdot p_b &= \frac{1 - y^2 + \epsilon^2}{2} m_t^2\,, \\
p_b \cdot p_\ell &= \frac{1 - x_\nu - \epsilon^2}{2} m_t^2\,, \\
p_b \cdot p_\nu &= \frac{x_\nu - y^2}{2} m_t^2\,, \\
p_\ell \cdot p_\nu &= \frac{1}{2}\, y^2\, m_t^2\,, \\
p_b \cdot s_t &= \frac{-2y^2 + x_\nu\,(1 + y^2 - \epsilon^2)}{2x_\nu}\, m_t\, \cos\theta_P \\
&\quad + \frac{\sqrt{y^2\,(x_\nu - x_\nu^2 - y^2 + x_\nu y^2 - x_\nu \epsilon^2)}}{x_\nu} m_t\, \cos\phi \sin\theta_P\,, \\
p_\ell \cdot s_t &= \frac{x_\nu^2 + 2y^2 - x_\nu\,(1 + y^2 - \epsilon^2)}{2x_\nu}\, m_t\, \cos\theta_P \\
&\quad - \frac{\sqrt{y^2\,(x_\nu - x_\nu^2 - y^2 + x_\nu y^2 - x_\nu \epsilon^2)}}{x_\nu}\, m_t\, \cos\phi \sin\theta_P\,, \\
p_\nu \cdot s_t &= \frac{x_\nu}{2}\, m_t \cos\theta_P\,.
\end{aligned} \tag{4.8}$$

Note that these scalar products are also related to the corresponding scalar products in system 1a (see 3.21 and 3.24) by permutations $p_\ell \leftrightarrow p_\nu$ and $x \leftrightarrow x_\nu$. The scalar products in the lepton–hadron contraction (4.5) can be taken from (4.8). Then substituting the lepton–hadron contraction into the differential rate in (4.4) and collecting terms according to their angular dependence one has

$$\frac{d\Gamma^{(1)}}{dx_\nu d\cos\theta_P d\phi} = \frac{1}{4\pi}\left(\frac{d\Gamma_A^{(0)}}{dx_\nu} + \frac{d\Gamma_B^{(0)}}{dx_\nu}\, P\cos\theta_P + \frac{d\Gamma_C^{(0)}}{dx_\nu}\, P\sin\theta_P\cos\phi\right), \tag{4.9}$$

where, as usual,

$$\frac{d\Gamma_i^{(0)}}{dx_\nu} = 2\pi\Gamma_F \frac{m_W}{\Gamma_W} 6y^2\, M_0^i(x_\nu)\,. \tag{4.10}$$

The LO unpolarized, polar and the azimuthal contributions $M_0^A(x_\nu)$, $M_0^B(x_\nu)$ and $M_0^C(x_\nu)$

read

$$M_0^A(x_\nu) = (1 - x_\nu + y^2 - \epsilon^2)(x_\nu - y^2)\,, \tag{4.11}$$

$$M_0^B(x_\nu) = \frac{(x_\nu - y^2)}{x_\nu}(x_\nu(1 - x_\nu + y^2 - \epsilon^2) - 2y^2)\,, \tag{4.12}$$

$$M_0^C(x_\nu) = \frac{2y(x_\nu - y^2)}{x_\nu}\sqrt{x_\nu(1 - x_\nu + y^2 - \epsilon^2) - y^2}\,. \tag{4.13}$$

The unpolarized and polar rate functions differ from each other in contrast to the charged lepton spectrum where the unpolarized rate equals the polar rate. Also the LO azimuthal contribution does not vanish as in system 1a. This will lead to non–vanishing IR–divergent contributions both from the virtual one–loop and real gluon emission, thus complicating the azimuthal calculations considerably.

Upon x_ν integration of (4.9) one has

$$\Gamma_A^{(0)} = 2\pi\Gamma_F\frac{m_W}{\Gamma_W}2y^2\bar{p}_3\left[(1-\epsilon^2)^2 + y^2(1+\epsilon^2) - 2y^4\right]\,, \tag{4.14}$$

$$\Gamma_B^{(0)} = 2\pi\Gamma_F\frac{m_W}{\Gamma_W}y^2\Big\{2\bar{p}_3\big[(1-\epsilon^2)^2 - y^2(11-\epsilon^2) - 2y^4\big] + 24y^4\bar{Y}_w\Big\}\,, \tag{4.15}$$

$$\Gamma_C^{(0)} = 2\pi\Gamma_F\frac{m_W}{\Gamma_W}y^2\frac{3\pi}{2}y\Big\{(1-y)^3(1+3y) - 2(1-y^2)\epsilon^2 + \epsilon^4\Big\}\,. \tag{4.16}$$

Because of the square root factor in the azimuthal differential rate we have used the basic integrals $\mathcal{X}_n$ defined in Appendix D.6 [1]:

$$\begin{aligned} M_0^C &= \int_{\bar{w}_-}^{\bar{w}_+} dx_\nu\, M_0^C(x_\nu) = 2y(\mathcal{X}_0 - y^2\mathcal{X}_{-1}) \\ &= \frac{\pi}{4}y\Big[(1-y)^3(1+3y) - 2(1-y^2)\epsilon^2 + \epsilon^4\Big]\,. \end{aligned} \tag{4.17}$$

The integrated unpolarized rate $\Gamma_A^{(0)}$ equals the one in system 1a given in 3.49 as it should. In the limit $m_b = 0$ one has

$$\Gamma_A^{(0)} = 2\pi\Gamma_F\frac{m_W}{\Gamma_W}y^2(1-y^2)^2(1+2y^2)\,, \tag{4.18}$$

$$\Gamma_B^{(0)} = 2\pi\Gamma_F\frac{m_W}{\Gamma_W}y^2\Big((1-y^2)(1-11y^2-2y^4) - 12y^4\ln y^2\Big)\,, \tag{4.19}$$

$$\Gamma_C^{(0)} = 2\pi\Gamma_F\frac{m_W}{\Gamma_W}y^2\frac{3\pi}{2}y(1-y)^3(1+3y)\,. \tag{4.20}$$

The change caused by the $m_b = 0$ for the unpolarized, polar and azimuthal rates relative to their LO values are 0.27%, −0.16% and 0.36%, respectively. Therefore $m_b = 0$ is an excellent approximation. In this approximation, for the ratios of the rates we have ($y^2 = m_W^2/m_t^2 = 0.211$):

$$\Gamma_A^{(0)} : \Gamma_B^{(0)} : \Gamma_C^{(0)} = 1 : -0.318 : 0.919 \tag{4.21}$$

[1] Note that for the present LO case $z = \epsilon^2$ in the basic integrals $\mathcal{X}_n$.

4.2 Virtual one–loop correction

The renormalized hadron tensor for the virtual one–loop QCD correction is given in (3.122). For the differential rate for the virtual one–loop correction one has

$$d\Gamma_v^{(1)} = \Gamma_F\, 2\pi \frac{m_W}{\Gamma_W}\, \frac{3}{16}\frac{y^2}{m_t^4} \int dx_\nu\, H^{(1)}_{v,\mu\nu} L^{\mu\nu}\,, \tag{4.22}$$

where the lepton–hadron contraction is given in (3.123). Substituting the scalar products from (4.8) and separating terms according to their angular dependence we have

$$H^{(1)}_{v,\mu\nu} L^{\mu\nu} = \Big(-\frac{\alpha_s}{4\pi} C_F\Big)\, 256 \frac{m^4}{4} \Big[M_v^A(x_\nu) + M_v^B(x_\nu)\, P\cos\theta_P + M_v^C(x_\nu)\, P\sin\theta_P\cos\phi\Big], \tag{4.23}$$

where $M_v^A(x_\nu)$, $M_v^B(x_\nu)$ and $M_v^C(x_\nu)$ correspond to the unpolarized, polar and azimuthal contributions, respectively. Separating the IR–convergent and IR–divergent parts we write

$$M_v^A(x_\nu) = M_0^A(x_\nu) M_{v,div} + M^A_{v,conv}(x_\nu)\,, \tag{4.24}$$

$$M_v^B(x_\nu) = M_0^B(x_\nu) M_{v,div} + M^B_{v,conv}(x_\nu)\,, \tag{4.25}$$

$$M_v^C(x_\nu) = M_0^C(x_\nu) M_{v,div} + M^C_{v,conv}(x_\nu)\,. \tag{4.26}$$

The divergent contribution $M_{v,div}$ is universal and can be taken from (3.130). Unlike the charged lepton spectrum in system 1a, the azimuthal contribution in system 3a has IR–divergences since the LO azimuthal rate does not vanish in system 3a. The convergent unpolarized, polar and azimuthal parts can be extracted from (4.23). One has

$$\begin{aligned} M^A_{v,conv}(x_\nu) =& \frac{\ln\epsilon}{y^2}\Big[\,(1-x_\nu)\left(y^2-x_\nu\right) + \left(2x_\nu - x_\nu^2 - y^2 + x_\nu y^2\right)\epsilon^2 - x_\nu\epsilon^4\Big] \\ &+ \frac{\bar{Y}_p}{2y^2\bar{p}_3}\Big[\left(1-y^2\right)\left(-x_\nu + x_\nu^2 + y^2 - x_\nu y^2\right) \\ &- \left(2x_\nu^2 - 3x_\nu + 2y^2 - 2x_\nu y^2 + x_\nu^2 y^2 - 3y^4 - x_\nu y^4\right)\epsilon^2 \\ &+ \left(x_\nu^2 - 3x_\nu + y^2 - 2x_\nu y^2\right)\epsilon^4 + x_\nu\epsilon^6\Big] \end{aligned} \tag{4.27}$$

$$\begin{aligned} M^B_{v,conv}(x_\nu) =& \frac{\ln\epsilon}{y^2}\Big[(1-\frac{1}{x_\nu})(x_\nu^2 - 3x_\nu y^2 + 2y^4) + (2x_\nu - x_\nu^2 - 3y^2 + x_\nu y^2)\epsilon^2 - x_\nu\epsilon^4\Big] \\ &+ \frac{\bar{Y}_p}{2y^2\bar{p}_3}\Big[(y^2-1)(x_\nu - x_\nu^2 - 3y^2 + 3x_\nu y^2 + 2\frac{y^4}{x_\nu} - 2y^4) \\ &+ (3x_\nu - 2x_\nu^2 - 6y^2 + 4x_\nu y^2 - x_\nu^2 y^2 + 2\frac{y^4}{x_\nu} - 5y^4 + x_\nu y^4)\epsilon^2 \\ &- (3x_\nu - x_\nu^2 - 3y^2 + 2x_\nu y^2)\epsilon^4 + x_\nu\epsilon^6\Big] \end{aligned} \tag{4.28}$$

$$\begin{aligned} M^C_{v,conv}(x_\nu) =& -\sqrt{y^2(x_\nu - x_\nu^2 - y^2 + x_\nu y^2 - x_\nu\epsilon^2)}\Bigg\{\frac{\ln\epsilon}{y^2}\Big[(2-x_\nu)\,(1-\frac{y^2}{x_\nu}) - 2\,\epsilon^2\Big] \\ &+ \Big[(1-y^2)\,(2 - x_\nu + y^2 - \frac{2\,y^2}{x_\nu}) - (4 - x_\nu + 5\,y^2 - \frac{2\,y^2}{x_\nu})\,\epsilon^2 + 2\,\epsilon^4\Big]\frac{\bar{Y}_p}{2\,y^2\,\bar{p}_3}\Bigg\}. \end{aligned} \tag{4.29}$$

4.3 Real gluon emissions

Analogous to (3.145) the differential rate for the $\mathcal{O}(\alpha_s)$ real gluon emission in the narrow–with approximation is given by

$$\begin{aligned} d\Gamma_r^{(1)} &= \frac{1}{2m_t}\frac{1}{(2\pi)^8} dR_4(p_t; p_b, p_\ell, p_\nu, k)\, \overline{|M_r^{(1)}|}^2 \\ &= \Gamma_F\, 2\pi \frac{m_W}{\Gamma_W}\, y^2 \frac{3}{128\pi^3 m_t^4}\, m_t^2 \int dz\, dx_\nu\, dR_2(P; p_b, k)\, H^{(1)}_{r,\mu\nu} L^{\mu\nu}\,, \end{aligned} \tag{4.30}$$

where the integration limits are

$$\int\limits_{\bar{w}_-}^{\bar{w}_+} dx \int\limits_{(\epsilon+\Lambda)^2}^{(1-x)(1-y^2/x)} dz \qquad \text{or} \qquad \int\limits_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz \int\limits_{w_-}^{w_+} dx\,. \tag{4.31}$$

The hadron tensor will again be given as a sum of a IR–divergent and a IR–convergent piece:

$$H^{(1)}_{r,\mu\nu} = H^{conv}_{\mu\nu} + \widetilde{H}^{(0)}_{\mu\nu}\, |M|^2_{SGF}\,, \tag{4.32}$$

where the hadron tensor $\widetilde{H}^{(0)}_{\mu\nu}$ is given in (3.138). The IR–divergent soft gluon factor $|M|^2_{SGF}$ is defined in (2.134). The remaining IR–convergent part is listed in (3.139).

Contracting the hadron tensor of the real emissions with the lepton tensor from (4.1) one has

$$\begin{aligned} H^{(1)}_{r,\mu\nu} L^{\mu\nu} &= H^{conv}_{\mu\nu}\, L^{\mu\nu} + \widetilde{H}^{(0)}_{\mu\nu} L^{\mu\nu} |M|^2_{SGF} \\ &= H^{conv}_{\mu\nu}\, L^{\mu\nu} + 128\, (\tilde{p}_t \cdot p_\ell)\, (p_b \cdot p_\nu)\, |M|^2_{SGF} \end{aligned} \tag{4.33}$$

Substituting p_b by $p_b = P - k$ one has[2]:

$$H^{(1)}_{r,\mu\nu} L^{\mu\nu} = \underbrace{H^{conv}_{\mu\nu}\, L^{\mu\nu} - 128 p_{\ell\alpha} p_{\nu\beta}\, \tilde{p}_t^\alpha k^\beta |M|^2_{SGF}}_{\text{IR–convergent}} + \underbrace{128\, (\tilde{p}_t \cdot p_\ell)\, (P \cdot p_\nu)\, |M|^2_{SGF}}_{\text{IR–divergent}}\,. \tag{4.34}$$

Let us begin with the IR–convergent piece. The two–body phase space integration $dR_2(P; p_b, k)$ is done in a similar way as in the charged lepton spectrum case in section 3.4. The result of the two–body phase space integration is a linear combination of the scalar

[2]The term $p_{\ell\alpha} p_{\nu\beta}\, \tilde{p}_t^\alpha k^\beta |M|^2_{SGF}$ is IR–convergent because of the gluon momentum k

products for the real emissions. These scalar products are given by

$$
\begin{aligned}
p_t \cdot p_\nu &= \frac{1}{2}\, x_\nu\, m_t^2, \\
p_t \cdot p_\ell &= \frac{1 - x_\nu + y^2 - z}{2} m_t^2, \\
p_t \cdot P &= \frac{1 - y^2 + z}{2} m_t^2, \\
P \cdot p_\ell &= \frac{1 - x_\nu - z}{2} m_t^2, \\
P \cdot p_\nu &= \frac{x_\nu - y^2}{2} m_t^2, \\
p_\nu \cdot p_\ell &= \frac{1}{2}\, y^2\, m_t^2 \\
P \cdot s_t &= \frac{-2y^2 + x_\nu\,(1 + y^2 - z)}{2x_\nu}\, m_t\, \cos\theta_P \\
&\quad + \frac{\sqrt{y^2\,(x_\nu - x_\nu^2 - y^2 + x_\nu y^2 - x_\nu z)}}{x_\nu} m_t\, \cos\phi \sin\theta_P\,, \\
p_\ell \cdot s_t &= \frac{x_\nu^2 + 2y^2 - x_\nu\,(1 + y^2 - z)}{2x_\nu}\, m_t\, \cos\theta_P \\
&\quad - \frac{\sqrt{y^2\,(x_\nu - x_\nu^2 - y^2 + x_\nu y^2 - x_\nu z)}}{x_\nu}\, m_t\, \cos\phi \sin\theta_P\,, \\
p_\nu \cdot s_t &= \frac{x_\nu}{2}\, m_t \cos\theta_P\,.
\end{aligned}
\tag{4.35}
$$

Note that, for $z = \epsilon^2$ in the above relations, one recovers the leading order scalar products (4.8). Also note that (4.35) are related to the corresponding scalar products in system 1a (see 3.153) by permutations $p_\ell \leftrightarrow p_\nu$ and $x \leftrightarrow x_\nu$. These scalar products are substituted into the result of the two–body phase space integration $dR_2(P; p_b, k)$. We do not present any intermediate steps of the calculation because they are quite similar to those in section 3.4. We thus proceed directly to the results. Collecting terms according to their angular dependence we write

$$
\begin{aligned}
& m_t^2 \int dz\, dR_2(P; p_b, k)\, (H_{\mu\nu}^{conv} L^{\mu\nu} - 128 p_{\ell\alpha} p_{\nu\beta}\, \tilde{p}_t^{\alpha} k^{\beta} |M|^2_{SGF}) \\
&= (-4\pi\alpha_s C_F)\, 128 \frac{m^4}{4} \Big\{ \Big[M_{r,conv}^{A}(x_\nu, z) + M_{r,conv}^{B}(x_\nu, z)\, P\cos\theta_P \Big] \Big|_{z=z_m}^{z=\epsilon^2} \\
&\qquad + \int dz\, M_{r,conv}^{C}(x_\nu, z)\, P \sin\theta_P \cos\phi \Big\}\,,
\end{aligned}
\tag{4.36}
$$

where[3] $z_m = (1 - x_\nu)(1 - \frac{y^2}{x_\nu})$. For the unpolarized and polar parts we obtain

$$
M_{r,conv}^{A}(x_\nu, z) \;=\; \mathcal{R}(-2,0)\, \rho_a(-2,0) + \mathcal{R}(-1,0)\, \rho_a(-1,0) + \mathcal{R}(0,0)\, \rho_a(0,0)
$$

[3]The integration limits in (4.36) are substituted as $M_{r,conv}^{A,B}(x_\nu, z)|_{z=z_m}^{z=\epsilon^2}$ instead of usual $|_{z=\epsilon^2}^{z=z_m}$. It is done so for the ease of comparison with [25] and [26].

$$
\begin{aligned}
&+\mathcal{R}(0,1)\,\rho_a(0,1) + \mathcal{R}(0,2)\,\rho_a(0,2) + \mathcal{R}(1,1)\,\rho_a(1,1) + \mathcal{R}(1,2)\,\rho_a(1,2) \\
&+\mathcal{S}(0,0)\,\sigma_a(0,0) + \mathcal{S}(0,1)\,\sigma_a(0,1) + \mathcal{S}(0,2)\,\sigma_a(0,2) + \mathcal{S}(1,0)\,\sigma_a(1,0) \\
&+\mathcal{S}(1,1)\,\sigma_a(1,1) + \mathcal{S}(1,2)\,\sigma_a(1,2), \qquad (4.37)
\end{aligned}
$$

$$
\begin{aligned}
M^B_{r,conv}(x_\nu, z) \;=\; &\mathcal{R}(-2,0)\,\rho_b(-2,0) + \mathcal{R}(-1,0)\,\rho_b(-1,0) + \mathcal{R}(0,0)\,\rho_b(0,0) \\
&+\mathcal{R}(0,1)\,\rho_b(0,1) + \mathcal{R}(0,2)\,\rho_b(0,2) + \mathcal{R}(0,3)\,\rho_b(0,3) + \mathcal{R}(1,1)\,\rho_b(1,1) \\
&+\mathcal{R}(1,2)\,\rho_b(1,2) + \mathcal{R}(1,3)\,\rho_b(1,3) + \mathcal{S}(0,0)\,\sigma_b(0,0) + \mathcal{S}(0,1)\,\sigma_b(0,1) \\
&+\mathcal{S}(0,2)\,\sigma_b(0,2) + \mathcal{S}(0,3)\,\sigma_b(0,3) + \mathcal{S}(1,0)\,\sigma_b(1,0) + \mathcal{S}(1,1)\,\sigma_b(1,1) \\
&+\mathcal{S}(1,2)\,\sigma_b(1,2) + \mathcal{S}(1,3)\,\sigma_b(1,3), \qquad (4.38)
\end{aligned}
$$

where the coefficient functions $\rho_a(m,n)$ and $\sigma_a(m,n)$ for the unpolarized contribution $M^A_{r,conv}(x_\nu, z)$, and $\rho_b(m,n)$ and $\sigma_b(m,n)$ for the polar contribution $M^B_{r,conv}(x_\nu, z)$ are listed in Appendix D.5.

The azimuthal contribution has to be treated separately in section 4.5, because of the complicated structure of the involved integrals.

For the unpolarized part substituting the basic integrals we obtain

$$
\begin{aligned}
M^A_{r,conv}(x_\nu, z) = h^a_1 z + \frac{1}{z}\frac{\epsilon^2 h^a_2}{2} + \frac{h^a_3 + h^a_4 z}{8p_3^2} + \frac{(h^a_5 + h^a_6 z)\,Y_p}{8p_3^3} + \frac{(h^a_7 + h^a_8 z)\,Y_p}{4p_3} \\
h^a_9\, p_3 Y_p + \frac{1}{4} h^a_{10} \ln z + \Big[\mathrm{Li}_2(w_+) + \mathrm{Li}_2(w_-)\Big] h^a_{11}, \qquad (4.39)
\end{aligned}
$$

where the coefficients h^a_i are given by

$$
h^a_1 = -6x_\nu + 5y^2\,, \qquad (4.40)
$$

$$
h^a_2 = x_\nu - x_\nu^2 - y^2 + 2x_\nu y^2 - y^4 + \frac{1}{1-y^2}(1-x_\nu)\left(x_\nu - y^2\right)\epsilon^2\,, \qquad (4.41)
$$

$$
\begin{aligned}
h^a_3 = &\left(1-y^2\right)^2\left(5x_\nu^2 + 5y^2 - 2x_\nu y^2\right) \qquad (4.42)\\
&- 4\left(x_\nu^2 + y^2 - 5x_\nu y^2 + 2x_\nu^2 y^2 + 2y^4 - x_\nu y^4\right)\epsilon^2 \\
&+ \frac{2}{1-y^2}\left(-x_\nu^2 - y^2 + 6x_\nu y^2 - 3x_\nu^2 y^2 - 3y^4 + 2x_\nu y^4\right)\epsilon^4\,,
\end{aligned}
$$

$$
\begin{aligned}
h^a_4 = &-5x_\nu^2 - 5y^2 - 18x_\nu y^2 - 3x_\nu^2 y^2 - 3y^4 + 2x_\nu y^4 \qquad (4.43)\\
&+ 4\left(x_\nu^2 + y^2 - x_\nu y^2\right)\epsilon^2 + \frac{1}{1-y^2}\left(x_\nu^2 + y^2 - 2x_\nu y^2\right)\epsilon^4\,,
\end{aligned}
$$

$$
\begin{aligned}
h^a_5 = &\left(1-y^2\right)^2\left(-5x_\nu^2 - 5y^2 - 8x_\nu y^2 + x_\nu^2 y^2 + y^4\right) \qquad (4.44)\\
&+ 2\left(1-y^2\right)\left(2x_\nu^2 + 2y^2 - 6x_\nu y^2 + x_\nu^2 y^2 + y^4\right)\epsilon^2 \\
&+ \left(x_\nu^2 + y^2 - 4x_\nu y^2 + x_\nu^2 y^2 + y^4\right)\epsilon^4\,,
\end{aligned}
$$

$$
\begin{aligned}
h^a_6 = &\,5x_\nu^2 + 5y^2 + 28x_\nu y^2 + 12x_\nu^2 y^2 + 12y^4 + 4x_\nu y^4 - x_\nu^2 y^4 - y^6 \qquad (4.45)\\
&- 2\left(2x_\nu^2 + 2y^2 + 2x_\nu y^2 - x_\nu^2 y^2 - y^4\right)\epsilon^2 - \left(x_\nu^2 + y^2\right)\epsilon^4\,,
\end{aligned}
$$

$$
h^a_7 = -5 + 10x_\nu + 5y^2 + 24x_\nu y^2 + 8x_\nu^2 y^2 + 5y^4 - 18x_\nu y^4 + 3y^6 \qquad (4.46)
$$

$$+\left(2-5x_\nu-2x_\nu^2-4y^2+9x_\nu y^2-2y^4\right)\epsilon^2+\left(1-4x_\nu+y^2\right)\epsilon^4\,,$$

$$h_8^a=5+10x_\nu-4x_\nu^2+14y^2+10x_\nu y^2-3y^4-2\left(2+3x_\nu-2y^2\right)\epsilon^2-\epsilon^4\,, \tag{4.47}$$

$$h_9^a=4\left(2x_\nu-y^2\right)\,, \tag{4.48}$$

$$h_{10}^a=-5+20x_\nu-12x_\nu^2-16y^2+14x_\nu y^2-y^4+2\left(2-2x_\nu-y^2\right)\epsilon^2+\epsilon^4\,, \tag{4.49}$$

$$h_{11}^a=-x_\nu-2x_\nu^2+3y^2+2x_\nu y^2-y^4-\left(x_\nu-y^2\right)\epsilon^2\,. \tag{4.50}$$

For the polar part we obtain

$$\begin{aligned}M^B_{r,conv}(x_\nu,z)&=h_1^b z+\frac{1}{z}\frac{\epsilon^2}{2x_\nu}h_2^b+3y^2\frac{h_3^b+h_4^b z}{8p_3^2}+\frac{h_5^b+h_6^b z}{8x_\nu p_3^2}\\&+\frac{Y_p}{8x_\nu p_3^3}\Big[3y^2\frac{h_7^b+h_8^b z}{8x_\nu p_3^4}+\frac{h_9^b+h_{10}^b z}{4p_3^2}+2h_{11}^b+2z\,h_{12}^b\Big]\\&+h_{13}^b\,p_3Y_p+\frac{1}{4x_\nu}h_{14}^b\ln z+\Big[\mathrm{Li}_2(w_+)+\mathrm{Li}_2(w_-)\Big]\frac{h_{15}^b}{x_\nu}\,,\end{aligned} \tag{4.51}$$

where the coefficients h_i^b are given by

$$h_1^b=6(-x_\nu+\frac{y^2}{2})\,, \tag{4.52}$$

$$h_2^b=(x_\nu-y^2)(x_\nu-x_\nu^2-2y^2+x_\nu y^2)-\frac{(1-x_\nu)(x_\nu-y^2)(x_\nu-2y^2+x_\nu y^2)\epsilon^2}{(1-y^2)^2}\,, \tag{4.53}$$

$$\begin{aligned}h_3^b&=-(1-y^2)^2(3x_\nu^2+2x_\nu^3+y^2+6x_\nu y^2+3x_\nu^2y^2+y^4)\\&+2(1-y^2)^2(3x_\nu^2+y^2)\epsilon^2+(-3x_\nu^2+2x_\nu^3-y^2+6x_\nu y^2-3x_\nu^2y^2-y^4)\epsilon^4\,,\end{aligned} \tag{4.54}$$

$$\begin{aligned}h_4^b&=3x_\nu^2+4x_\nu^3+y^2+12x_\nu y^2+18x_\nu^2y^2+4x_\nu^3y^2+6y^4+12x_\nu y^4+3x_\nu^2y^4+y^6\\&-2(3x_\nu^2+2x_\nu^3+y^2+6x_\nu y^2+3x_\nu^2y^2+y^4)\epsilon^2+(3x_\nu^2+y^2)\epsilon^4\,,\end{aligned} \tag{4.55}$$

$$\begin{aligned}h_5^b&=-6x_\nu^2+3x_\nu^3-2y^2+x_\nu y^2+38x_\nu^2y^2+18x_\nu^3y^2+10y^4\\&+70x_\nu y^4+50x_\nu^2y^4+3x_\nu^3y^4+22y^6+x_\nu y^6-10x_\nu^2y^6-6y^8\\&-4(-3x_\nu^2+2x_\nu^3-y^2+7x_\nu y^2+10x_\nu^2y^2+2x_\nu^3y^2+2y^4+11x_\nu y^4-4x_\nu^2y^4-2y^6)\epsilon^2\\&+\frac{\epsilon^4}{(1-y^2)^2}(-6x_\nu^2+5x_\nu^3-2y^2+27x_\nu y^2-6x_\nu^2y^2-2x_\nu^3y^2-2y^4-30x_\nu y^4-6x_\nu^2y^4\\&+5x_\nu^3y^4-2y^6+27x_\nu y^6-6x_\nu^2y^6-2y^8)\,,\end{aligned} \tag{4.56}$$

$$\begin{aligned}h_6^b&=6x_\nu^2+x_\nu^3+2y^2+23x_\nu y^2+30x_\nu^2y^2+x_\nu^3y^2+10y^4+39x_\nu y^4+10x_\nu^2y^4+6y^6\\&-4(3x_\nu^2+y^2+5x_\nu y^2+4x_\nu^2y^2+2y^4)\epsilon^2+\\&\frac{\epsilon^4}{(1-y^2)^2}(6x_\nu^2-x_\nu^3+2y^2-3x_\nu y^2-6x_\nu^2y^2-x_\nu^3y^2-2y^4-3x_\nu y^4+6x_\nu^2y^4+2y^6)\,,\end{aligned} \tag{4.57}$$

$$h_7^b=(1-y^2)^2(3x_\nu^2+3x_\nu^3+y^2+9x_\nu y^2+9x_\nu^2y^2+x_\nu^3y^2+3y^4+3x_\nu y^4) \tag{4.58}$$

$$
\begin{aligned}
&- 2(1-y^2)^2(3x_\nu^2 + x_\nu^3 + y^2 + 3x_\nu y^2)\epsilon^2 \\
&+ (3x_\nu^2 - x_\nu^3 + y^2 - 3x_\nu y^2 - 3x_\nu^2 y^2 + x_\nu^3 y^2 - y^4 + 3x_\nu y^4)\epsilon^4 \,,
\end{aligned}
$$

$$
\begin{aligned}
h_8^b = &-3x_\nu^2 - 5x_\nu^3 - y^2 - 15x_\nu y^2 - 30x_\nu^2 y^2 - 10x_\nu^3 y^2 - 10y^4 - 30x_\nu y^4 - 15x_\nu^2 y^4 \\
&- x_\nu^3 y^4 - 5y^6 - 3x_\nu y^6 + 2(1+x_\nu)(3x_\nu^2 + y^2 + 8x_\nu y^2 + x_\nu^2 y^2 + 3y^4)\epsilon^2 \\
&- (3x_\nu^2 + x_\nu^3 + y^2 + 3x_\nu y^2)\epsilon^4 \,,
\end{aligned} \tag{4.59}
$$

$$
\begin{aligned}
h_9^b = &\,6x_\nu^2 - x_\nu^3 + 2y^2 + 11x_\nu y^2 - 34x_\nu^2 y^2 - 33x_\nu^3 y^2 - 10y^4 - 113x_\nu y^4 - 130x_\nu^2 y^4 \\
&- 15x_\nu^3 y^4 - 46y^6 - 59x_\nu y^6 + 14x_\nu^2 y^6 + x_\nu^3 y^6 + 6y^8 + 17x_\nu y^8 - 2(6x_\nu^2 - 2x_\nu^3 + 2y^2 \\
&- 2x_\nu y^2 - 36x_\nu^2 y^2 - 11x_\nu^3 y^2 - 10y^4 - 47x_\nu y^4 - 6x_\nu^2 y^4 + x_\nu^3 y^4 - 4y^6 + 13x_\nu y^6)\epsilon^2 \\
&+ (6x_\nu^2 - 3x_\nu^3 + 2y^2 - 15x_\nu y^2 - 18x_\nu^2 y^2 + x_\nu^3 y^2 - 6y^4 + 9x_\nu y^4)\epsilon^4 \,,
\end{aligned} \tag{4.60}
$$

$$
\begin{aligned}
h_{10}^b = &-6x_\nu^2 - 3x_\nu^3 - 2y^2 - 35x_\nu y^2 - 64x_\nu^2 y^2 - 8x_\nu^3 y^2 - 20y^4 - 96x_\nu y^4 - 46x_\nu^2 y^4 \\
&- x_\nu^3 y^4 - 22y^6 - 17x_\nu y^6 + 2(6x_\nu^2 + 2x_\nu^3 + 2y^2 + 22x_\nu y^2 + 22x_\nu^2 y^2 \\
&+ x_\nu^3 y^2 + 8y^4 + 13x_\nu y^4)\epsilon^2 - (6x_\nu^2 + x_\nu^3 + 2y^2 + 9x_\nu y^2)\epsilon^4 \,,
\end{aligned} \tag{4.61}
$$

$$
\begin{aligned}
h_{11}^b = &-2 + 3x_\nu + 10x_\nu^2 - 8x_\nu^3 + 10y^2 - 63x_\nu y^2 - 40x_\nu^2 y^2 + 8x_\nu^3 y^2 - 14y^4 - 3x_\nu y^4 \\
&- 10x_\nu^2 y^4 - 10y^6 + 7x_\nu y^6 - 2(-2 + 2x_\nu - 9x_\nu^2 + 2x_\nu^3 - 2y^2 - 10x_\nu y^2 \\
&- 7x_\nu^2 y^2 - 4y^4 + 4x_\nu y^4)\epsilon^2 - (2 - x_\nu + 4x_\nu^2 + 2y^2 - x_\nu y^2)\epsilon^4 \,,
\end{aligned} \tag{4.62}
$$

$$
\begin{aligned}
h_{12}^b = &\,2 - 3x_\nu - 22x_\nu^2 - 4x_\nu^3 - 12y^2 - 30x_\nu y^2 + 2x_\nu^2 y^2 + 2y^4 - 7x_\nu y^4 \\
&- 2(2 - 2x_\nu + x_\nu^2 - 4x_\nu y^2)\epsilon^2 + (2 - x_\nu)\epsilon^4 \,,
\end{aligned} \tag{4.63}
$$

$$
h_{13}^b = 4(2x_\nu - y^2) \,, \tag{4.64}
$$

$$
\begin{aligned}
h_{14}^b = &-2 + 3x_\nu + 32x_\nu^2 - 12x_\nu^3 + 12y^2 - 48x_\nu y^2 + 6x_\nu^2 y^2 + 14y^4 - 5x_\nu y^4 \\
&+ 2(2 - 2x_\nu + x_\nu y^2)\epsilon^2 - (2 - x_\nu)\epsilon^4 \,,
\end{aligned} \tag{4.65}
$$

$$
h_{15}^b = -3x_\nu^2 - 2x_\nu^3 - 7x_\nu y^2 + 2x_\nu^2 y^2 + 2y^4 - x_\nu y^4 - x_\nu(x_\nu - y^2)\epsilon^2 \,. \tag{4.66}
$$

Next we turn to the IR–divergent part of the real emission (see 4.34):

$$
128\,(\tilde{p}_t \cdot p_\ell)\,(P \cdot p_\nu)\;|M|^2_{SGF} \,. \tag{4.67}
$$

Substituting the scalar products from (4.35) and collecting terms according to their angular dependence we write

$$
\begin{aligned}
&128\,(\tilde{p}_t \cdot p_\ell)\,(P \cdot p_\nu)\;|M|^2_{SGF} \\
&\quad = 128\frac{m^4}{4}\Big[\widetilde{M}_0^A(x_\nu, z) + \widetilde{M}_0^B(x_\nu, z) P\cos\theta_P + \widetilde{M}_0^C(x_\nu, z) P\sin\theta_P\cos\phi\Big]\,|M|^2_{SGF},
\end{aligned} \tag{4.68}
$$

where

$$
\widetilde{M}_0^A(x_\nu, z) = (1 - x_\nu + y^2 - z)(x_\nu - y^2) \,, \tag{4.69}
$$

$$
\widetilde{M}_0^B(x_\nu, z) = \frac{(x_\nu - y^2)}{x_\nu}(x_\nu(1 - x_\nu + y^2 - z) - 2y^2) \,, \tag{4.70}
$$

$$
\widetilde{M}_0^C(x_\nu, z) = \frac{2y(x_\nu - y^2)}{x_\nu}\sqrt{x_\nu(1 - x_\nu + y^2 - z) - y^2} \,. \tag{4.71}
$$

We concentrate on the unpolarized and the polar part here. The calculation of the azimuthal part is done separately in the section 4.5. Using the “plus” prescription the IR–divergent phase space integration is reduced to $\Sigma(\Lambda, z_m)$ given in (2.201). The remaining IR–convergent terms are denoted by $\hat{M}_{r,conv}(x_\nu, z)$. The details of the calculation are similar to the ones in system 1a. One obtains

$$\begin{aligned}
&m_t^2 \int dz\, dR_2(P; p_b, k)\, \widetilde{H}^{(0)}_{\mu\nu} L^{\mu\nu} |M|^2_{SGF} \\
&= 128 \frac{m^4}{4} \pi(-4\pi\alpha_s C_F) \left\{ M_0^A(x_\nu)\, \Sigma(\Lambda) + \hat{M}^A_{r,conv}(x_\nu, z) \Big|^{z=\epsilon^2}_{z=z_m} \right. \\
&\qquad + \left(M_0^B(x_\nu)\, \Sigma(\Lambda) + \hat{M}^A_{r,conv}(x_\nu, z) \Big|^{z=\epsilon^2}_{z=z_m} \right) P \cos\theta_P \\
&\qquad \left. + \int dz\, M^C_{r,div}(x_\nu, z)\, P \sin\theta_P \cos\phi \right\}
\end{aligned} \tag{4.72}$$

where the IR–divergent integral $\Sigma(\Lambda)$ the is obtained from the IR–divergent integral $\Sigma(\Lambda, z_m)$ calculated in (2.201):

$$\begin{aligned}
\Sigma(\Lambda) &:= \Sigma(\Lambda, z_m)\Big|_{z_m=(1-x_\nu)(1-y^2/x_\nu)} \\
&= -4\left(1 - \frac{\bar{p}_0}{\bar{p}_3}\bar{Y}_p\right) \ln\left[\frac{(1-y)(1-y^2/x_\nu) - \epsilon^2}{\epsilon\Lambda}\right] - 2 \\
&\quad + \frac{2\bar{p}_0}{\bar{p}_3}\left[\mathrm{Li}_2\left(1 - \frac{1-x_\nu}{\bar{p}_+}\right) + \mathrm{Li}_2\left(1 - \frac{1-y^2/x_\nu}{\bar{p}_+}\right) - \mathrm{Li}_2\left(1 - \frac{1-x_\nu}{\bar{p}_-}\right)\right. \\
&\qquad \left. - \mathrm{Li}_2\left(1 - \frac{1-y^2/x_\nu}{\bar{p}_-}\right) - \mathrm{Li}_2\left(1 - \frac{\bar{p}_+}{\bar{p}_-}\right) - \bar{Y}_p\left(\bar{Y}_p + 1\right)\right].
\end{aligned} \tag{4.73}$$

For the azimuthal part which will be treated in section 4.5 we have

$$M^C_{r,div}(x_\nu, z) = m_t^2 \int dR_2(P; p_b, k) \widetilde{M}^C_0(x_\nu, z) \Delta_{SGF} \tag{4.74}$$

The remaining IR–convergent pieces are given by

$$\begin{aligned}
\hat{M}^A_{r,conv}(x_\nu, z) &= 6(x_\nu - y^2)z - 2(x_\nu - y^2)(2 - x_\nu - \epsilon^2)\ln(z) \\
&\quad +2(x_\nu - y^2)(1 + x_\nu - y^2 + \epsilon^2)\Big[\mathrm{Li}_2(w_-) + \mathrm{Li}_2(w_+)\Big] \\
&\quad -8(x_\nu - y^2)p_3 Y_p\,,
\end{aligned} \tag{4.75}$$

$$\begin{aligned}
\hat{M}^B_{r,conv}(x_\nu, z) &= 6(x_\nu - y^2)z - 2(x_\nu - y^2)(2 - x_\nu - \epsilon^2 - 2\frac{y^2}{x_\nu})\ln(z) \\
&\quad +2(x_\nu - y^2)(1 + x_\nu + 2\frac{y^2}{x_\nu} - y^2 + \epsilon^2)\Big[\mathrm{Li}_2(w_-) + \mathrm{Li}_2(w_+)\Big] \\
&\quad -8(x_\nu - y^2)p_3 Y_p\,.
\end{aligned} \tag{4.76}$$

Finally we sum up the IR–divergent and the IR–convergent pieces and write the total real emission contributions as

$$\begin{aligned}\frac{d\Gamma_r^{(1)}}{dx_\nu d\cos\theta_P d\phi} &= \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F(-\frac{\alpha_s}{2\pi})6y^2 \frac{1}{4\pi} \times \\ &\times\Big[M_r^A(x_\nu) + M_r^B(x_\nu)\, P\cos\theta_P + M_r^C(x_\nu)\, P\sin\theta_P\cos\phi\Big], \quad (4.77)\end{aligned}$$

where

$$M_r^A(x_\nu) = \Big[M_{r,conv}^A(x_\nu, z) + \hat{M}_{r,conv}^A(x_\nu, z)\Big]\Big|_{z=z_m}^{z=\epsilon^2} + M_0^A(x_\nu)\Sigma(\Lambda), \quad (4.78)$$

$$M_r^B(x_\nu) = \Big[M_{r,conv}^B(x_\nu, z) + \hat{M}_{r,conv}^B(x_\nu, z)\Big]\Big|_{z=z_m}^{z=\epsilon^2} + M_0^B(x_\nu)\Sigma(\Lambda), \quad (4.79)$$

$$M_r^C(x_\nu) = \int dz\,\big(M_{r,conv}^C(x_\nu, z) + M_{r,div}^C(x_\nu, z)\big). \quad (4.80)$$

The terms $M_0^A(x_\nu)$, $M_0^B(x_\nu)$, $M_{r,conv}^A(x_\nu, z)$, $M_{r,conv}^B(x_\nu, z)$, $\hat{M}_{r,conv}^A(x_\nu, z)$, $\hat{M}_{r,conv}^B(x_\nu, z)$ and $\Sigma(\Lambda)$ are given in (4.11), (4.12), (4.39), (4.51), (4.75), (4.76) and (4.73), respectively. The azimuthal contribution $M_r^C(x_\nu)$ will be discussed in section 4.5.

4.4 Total $\mathcal{O}(\alpha_s)$ results for system 3a

The general form of the $\mathcal{O}(\alpha_s)$ correction is

$$\frac{d\Gamma^{(1)}}{dx_\nu\, d\cos\theta_P\, d\phi} = \frac{1}{4\pi}\left(\frac{d\Gamma_A^{(1)}}{dx_\nu} + \frac{d\Gamma_B^{(1)}}{dx_\nu} P\cos\theta_P + \frac{d\Gamma_C^{(1)}}{dx_\nu} P\sin\theta_P\cos\phi\right). \quad (4.81)$$

Summing up the virtual one–loop and real emission contributions from (4.22) and (4.77) gives the total $\mathcal{O}(\alpha_s)$ result:

$$\begin{aligned}\frac{d\Gamma^{(1)}}{dx_\nu d\cos\theta_P d\phi} &= \frac{d\Gamma_v^{(1)}}{dx_\nu d\cos\theta_P d\phi} + \frac{d\Gamma_r^{(1)}}{dx_\nu d\cos\theta_P d\phi} \quad (4.82)\\ &= \Gamma_F 2\pi\frac{m_W}{\Gamma_W}C_F(-\frac{\alpha_s}{2\pi})6y^2\frac{1}{4\pi}\Big[M^A(x_\nu) + M^B(x_\nu)\, P\cos\theta_P + M^C(x_\nu)\, P\sin\theta_P\cos\phi\Big],\end{aligned}$$

where

$$\begin{aligned}M^A(x_\nu) &= M_v^A(x_\nu) + M_0^A(x_\nu)\Sigma(\Lambda)\\ &\quad + \Big[M_{r,conv}^A(x_\nu, z) + \hat{M}_{r,conv}^A(x_\nu, z)\Big]\Big|_{z=z_m}^{z=\epsilon^2}, \quad (4.83)\end{aligned}$$

$$\begin{aligned}M^B(x_\nu) &= M_v^A(x_\nu) + M_0^B(x_\nu)\Sigma(\Lambda)\\ &\quad + \Big[M_{r,conv}^B(x_\nu, z) + \hat{M}_{r,conv}^B(x_\nu, z)\Big]\Big|_{z=z_m}^{z=\epsilon^2}, \quad (4.84)\end{aligned}$$

$$\begin{aligned}M^C(x_\nu) &= M_0^C(x_\nu)M_{v,div} + M_{v,conv}^C(x_\nu)\\ &\quad + \int dz\,\Big(M_{r,conv}^C(x_\nu, z) + \widetilde{M}_0^C(x_\nu, z)\Delta_{SGF}\Big). \quad (4.85)\end{aligned}$$

Comparing (4.81) and (4.82) we have

$$\frac{d\Gamma_i^{(1)}}{dx_\nu} = \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F(-\frac{\alpha_s}{2\pi})6y^2 \frac{1}{4\pi} M^i(x_\nu)\,, \tag{4.86}$$

where $i = A, B, C$ represent the unpolarized, polar and azimuthal contributions, respectively.

Let us begin with the unpolarized part. We first sum up the virtual one–loop contribution and the divergent part of the real emission:

$$M_v^A(x_\nu) + M_0^A(x_\nu)\Sigma(\Lambda) = M_{v,conv}^A(x_\nu) + M_0^A(x_\nu)M_{v,div} + M_0^A(x_\nu)\Sigma(\Lambda)\,. \tag{4.87}$$

Substituting $M_{v,div}$ and $\Sigma(\Lambda)$ from (3.130) and (4.73) one has

$$\begin{aligned} M_{v,div} + \Sigma(\Lambda) \;&=\; \Phi_0 + 2 - 2\,\ln\left[(1-x_\nu)(1-y^2/x_\nu)\right] \\ &\quad + \frac{1}{y^2}\left(1 - 2\,y^2 - \epsilon^2\right)\,\ln(\epsilon) - \left(4\,\bar{p}_0 - \frac{2\,\bar{p}_3^2}{y^2}\right)\frac{\bar{Y}_p}{\bar{p}_3}\,, \end{aligned} \tag{4.88}$$

where Φ_0 is the sum of the double–log and dilogs:

$$\begin{aligned} \Phi_0 = \frac{2\bar{p}_0}{\bar{p}_3}\Bigg[&\mathrm{Li}_2\left(1 - \frac{1-x_\nu}{\bar{p}_+}\right) + \mathrm{Li}_2\left(1 - \frac{1-y^2/x_\nu}{\bar{p}_+}\right) - \mathrm{Li}_2\left(1 - \frac{1-x_\nu}{\bar{p}_-}\right) \\ &-\mathrm{Li}_2\left(1 - \frac{1-y^2/x_\nu}{\bar{p}_-}\right) + \mathrm{Li}_2(\bar{w}_-) - \mathrm{Li}_2(\bar{w}_+) + 4\bar{Y}_p \ln\epsilon\Bigg] \\ &+ 4\left(1 - \frac{\bar{p}_0}{\bar{p}_3}\bar{Y}_p\right)\ln\left[(1-x_\nu)(1-y^2/x_\nu) - \epsilon^2\right]. \end{aligned} \tag{4.89}$$

The IR–divergences cancel in the sum as expected. Substituting Φ_0 into (4.87) one has for the unpolarized part,

$$\begin{aligned} &M_v^A(x_\nu) + M_0^A(x_\nu)\Sigma(\Lambda) \\ &\quad= M_0^A\Phi_0 + 2\left(x_\nu - y^2\right)\left(1 - x_\nu + y^2 - \epsilon^2\right)\left(1 - \ln\left[(1-x_\nu)(1-y^2/x_\nu)\right]\right) \\ &\qquad - \frac{\bar{Y}_p}{\bar{p}_3}\Big[\left(1-y^2\right)\left(4x_\nu - 5x_\nu^2 - 4y^2 + 10x_\nu y^2 - 5y^4\right) \\ &\qquad - \left(1 - 2x_\nu + 5x_\nu^2 + 5y^2 - 16x_\nu y^2 + 11y^4\right)\epsilon^2 + \left(2 - 6x_\nu + 7y^2\right)\epsilon^4 - \epsilon^6\Big] \\ &\qquad - \Big[x_\nu - 2x_\nu^2 - y^2 + 4x_\nu y^2 - 2y^4 - \left(1 + x_\nu - y^2\right)\epsilon^2 + \epsilon^4\Big]\ln(\epsilon)\,. \end{aligned} \tag{4.90}$$

Similarly, for the polar part, one has

$$
\begin{aligned}
&M_v^B(x_\nu) + M_0^B(x_\nu)\Sigma(\Lambda) \\
&= M_0^B\Phi_0 + 2\left(x_\nu - y^2\right)\left(1 - x_\nu + y^2 - \frac{2y^2}{x_\nu} - \epsilon^2\right)\left(1 - \ln\left[(1-x_\nu)(1-y^2/x_\nu)\right]\right) \\
&\quad -\frac{\bar{Y}_p}{\bar{p}_3}\Big[\left(1-y^2\right)\left(6x_\nu - 5x_\nu^2 - 16y^2 + 10x_\nu y^2 - 5y^4 + \frac{10y^4}{x_\nu}\right) \\
&\quad -\left(1 + 5x_\nu^2 + 9y^2 - \frac{2y^2}{x_\nu} - 16x_\nu y^2 + 11y^4 - \frac{12y^4}{x_\nu}\right)\epsilon^2 \\
&\quad +\left(2 - 6x_\nu + 7y^2 - \frac{2y^2}{x_\nu}\right)\epsilon^4 - \epsilon^6\Big] \\
&\quad -\Big[3x_\nu - 2x_\nu^2 - 7y^2 + 4x_\nu y^2 - 2y^4 + \frac{4y^4}{x_\nu} \\
&\quad -\left(1 + x_\nu - y^2 - \frac{2y^2}{x_\nu}\right)\epsilon^2 + \epsilon^4\Big]\ln(\epsilon). \qquad (4.91)
\end{aligned}
$$

Finally we substitute the above results into (4.83) and (4.84) and write the total $\mathcal{O}(\alpha_s)$ corrections to the differential rate as

$$
\begin{aligned}
\frac{d\Gamma^{(1)}}{dx_\nu d\cos\theta_P d\phi} &= \Gamma_F 2\pi\frac{m_W}{\Gamma_W}C_F(-\frac{\alpha_s}{2\pi})6y^2\frac{1}{4\pi}\times \\
&\quad \times\Big[M^A(x_\nu) + M^B(x_\nu)\,P\cos\theta_P + M^C(x_\nu)\,P\sin\theta_P\cos\phi\Big], \qquad (4.92)
\end{aligned}
$$

where the unpolarized part is given by[4]

$$
\begin{aligned}
M^A(x_\nu) &= M_0^A\Phi_0 + \\
&\quad +\left(x_\nu + y^2 - 2x_\nu y^2 + y^4 + x_\nu\epsilon^2 - y^2\epsilon^2\right)\left[\mathrm{Li}_2(\bar{w}_+) + \mathrm{Li}_2(\bar{w}_-) - \mathrm{Li}_2(x_\nu) - \mathrm{Li}_2(\frac{y^2}{x_\nu})\right] \\
&\quad -\frac{\bar{Y}_p}{\bar{p}_3}\Big[\left(5 - 2x_\nu + 3y^2 - \epsilon^2\right)\bar{p}_3^2 + 2y^2\epsilon^2\Big] \\
&\quad -\frac{1}{2}\Big[\left(5 - 2x_\nu + 3y^2\right)\left(1-y^2\right) - 6\left(1 + x_\nu - 2y^2\right)\epsilon^2 + \epsilon^4\Big]\ln\epsilon \\
&\quad +\frac{1}{2}\Big[5 - 5x_\nu + 3y^2 + 4x_\nu y^2 - 5y^4 - \frac{2y^4}{x_\nu} - \left(4 + 5x_\nu - 11y^2\right)\epsilon^2 - \epsilon^4\Big]\ln\left(1 - x_\nu\right) \\
&\quad +\frac{1}{2}(9x_\nu - 4x_\nu^2 - 11y^2 + 6x_\nu y^2 - 2y^4 + \frac{2y^4}{x_\nu} + 7\left(-x_\nu + y^2\right)\epsilon^2)\ln\left(1 - \frac{y^2}{x_\nu}\right) \\
&\quad +\frac{1}{2}\left(2y^2 + 3x_\nu y^2 - 3y^4 - \frac{2y^4}{x_\nu}\right) + \frac{1}{2}\left(3y^2 - \frac{y^2}{1-x_\nu}\right)\epsilon^2. \qquad (4.93)
\end{aligned}
$$

[4] 4.93 and 4.94 agree with $F_1^-(x,y)$ of Eq.(9) and $J_1^-(x,y)$ of Eq.(10) in [28]. Note that our variable y^2 correspond to y of [28].

The polar part is given by

$$
\begin{aligned}
M^B(x_\nu) = {} & M_0^B \Phi_0 + \\
& + \left(x_\nu - y^2\right)\left(1 - x_\nu + y^2 - \frac{2y^2}{x_\nu} - \epsilon^2\right)\Big[\mathrm{Li}_2(\bar{w}_+) + \mathrm{Li}_2(\bar{w}_-) - \mathrm{Li}_2(x_\nu) - \mathrm{Li}_2(\frac{y^2}{x_\nu})\Big] \\
& + \frac{\bar{Y}_p}{\bar{p}_3}\Big[\left(3 - \frac{2}{x_\nu} + 10x_\nu + y^2 + \frac{10y^2}{x_\nu} + \epsilon^2 + \frac{2\epsilon^2}{x_\nu}\right)\bar{p}_3^2 \\
& + \left(x_\nu - 2y^2 + \frac{y^2}{x_\nu} + x_\nu y^2 + \frac{y^4}{x_\nu}\right)\epsilon^2 - \left(x_\nu + \frac{y^2}{x_\nu}\right)\epsilon^4\Big] \\
& + \frac{1}{2}\Big[\left(1 - y^2\right)\left(3 - \frac{2}{x_\nu} + 10x_\nu + y^2 + \frac{10y^2}{x_\nu}\right) \\
& - 2\left(1 - \frac{2}{x_\nu} - 5x_\nu + 4y^2 + \frac{2y^2}{x_\nu}\right)\epsilon^2 - \left(1 + \frac{2}{x_\nu}\right)\epsilon^4\Big]\ln\epsilon \\
& + \frac{1}{2}\Big[-3 + \frac{2}{x_\nu} + x_\nu - 7y^2 - \frac{12y^2}{x_\nu} + 12x_\nu y^2 - y^4 + \frac{8y^4}{x_\nu} \\
& + \left(4 - \frac{4}{x_\nu} - 9x_\nu + 7y^2\right)\epsilon^2 + \left(-1 + \frac{2}{x_\nu}\right)\epsilon^4\Big]\ln(1 - x_\nu) \\
& + \frac{1}{2}\left(-9x_\nu - 4x_\nu^2 - y^2 + 6x_\nu y^2 - 2y^4 + \frac{10y^4}{x_\nu} - 7\left(x_\nu - y^2\right)\epsilon^2\right)\ln\left(1 - \frac{y^2}{x_\nu}\right) \\
& + \frac{1}{2}\left(2 - 2x_\nu^2 + 2y^2 - \frac{2y^2}{x_\nu} - 5x_\nu y^2 + 7y^4 - \frac{2y^4}{x_\nu}\right) \\
& - \frac{1}{2}\left(4 + 9y^2 - \frac{y^2}{1 - x_\nu} - \frac{2y^2}{x_\nu}\right)\epsilon^2 + \epsilon^4 . \qquad (4.94)
\end{aligned}
$$

The spectrum functions have divergences at the endpoints (the integration limits for x_ν). But with similar steps discussed after 3.219 we learn that these divergences cancel after x_ν–integration. Therefore one can thus safely take the $m_b \to 0$ limit which results in considerable simplifications. One has

$$
\begin{aligned}
\frac{d\Gamma^{(1)}}{dx_\nu d\cos\theta_P d\phi} &= \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F(-\frac{\alpha_s}{2\pi}) 6y^2 \frac{1}{4\pi} \times \\
& \times \Big[\mathcal{M}^A(x_\nu) + \mathcal{M}^B(x_\nu)\, P\cos\theta_P + \mathcal{M}^C(x_\nu)\, P\sin\theta_P\cos\phi\Big], \qquad (4.95)
\end{aligned}
$$

where the unpolarized part is given by[5]

$$
\begin{aligned}
\mathcal{M}^A(x_\nu) &:= \lim_{m_b \to 0} M^A(x_\nu) \\
&= (x_\nu - y^2)(1 - x_\nu - y^2)\Big[\frac{\pi^2}{3} + \ln^2(\frac{x_\nu - y^2}{x_\nu - x_\nu^2}) + 2\mathrm{Li}_2(x_\nu) + 2\mathrm{Li}_2(\frac{y^2}{x_\nu})\Big] \\
& \quad + (x_\nu + y^2 - 2\, x_\nu\, y^2 + y^4)\Big[\frac{\pi^2}{6} - \mathrm{Li}_2(x_\nu) + \mathrm{Li}_2(y^2) - \mathrm{Li}_2(\frac{y^2}{x_\nu})\Big]
\end{aligned}
$$

[5]3.223 and 3.224 agrees with $F_1^-(x,y)$ of Eq. (44) and $J_1^-(x,y)$ of Eq. (46) in [28]. Note that our variable y^2 correspond to y of [28].

$$
\begin{aligned}
&-\frac{1}{2}(5-2x_\nu+3y^2)\left(1-y^2\right)\ln(1-y^2)\\
&+\frac{1}{2}\left(5-5\,x_\nu+3\,y^2+4\,x_\nu\,y^2-5\,y^4-\frac{2\,y^4}{x_\nu}\right)\ln(1-x_\nu)\\
&+\frac{1}{2}\left(9\,x_\nu-4\,x_\nu^2-11\,y^2+6\,x_\nu\,y^2-2\,y^4+\frac{2\,y^4}{x_\nu}\right)\ln(1-\frac{y^2}{x_\nu})\\
&+\frac{y^2}{2}(2+3x_\nu-3y^2-2\frac{y^2}{x_\nu})\,. \qquad (4.96)
\end{aligned}
$$

The polar part is given by

$$
\begin{aligned}
\mathcal{M}^B(x_\nu) &:= \lim_{m_b\to 0} M^B(x_\nu)\\
&= \left(x_\nu-y^2\right)\left(1-x_\nu+y^2-\frac{2\,y^2}{x_\nu}\right)\left[\frac{\pi^2}{3}+\ln^2(\frac{x_\nu-y^2}{x_\nu-x_\nu^2})+2\mathrm{Li}_2(x_\nu)+2\mathrm{Li}_2(\frac{y^2}{x_\nu})\right]\\
&+\left(-x_\nu-5\,y^2-2\,x_\nu\,y^2+y^4-\frac{2\,y^4}{x_\nu}\right)\left[\frac{\pi^2}{6}-\mathrm{Li}_2(x_\nu)+\mathrm{Li}_2(y^2)-\mathrm{Li}_2(\frac{y^2}{x_\nu})\right]\\
&+\frac{1}{2}\left(3-\frac{2}{x_\nu}+10\,x_\nu+y^2+\frac{10\,y^2}{x_\nu}\right)\left(1-y^2\right)\ln(1-y^2)\\
&\frac{1}{2}\left(-3+\frac{2}{x_\nu}+x_\nu-7\,y^2-\frac{12\,y^2}{x_\nu}+12\,x_\nu\,y^2-y^4+\frac{8\,y^4}{x_\nu}\right)\ln(1-x_\nu)\\
&+\frac{1}{2}\left(-9\,x_\nu-4\,x_\nu^2-y^2+6\,x_\nu\,y^2-2\,y^4+\frac{10\,y^4}{x_\nu}\right)\ln(1-\frac{y^2}{x_\nu})\\
&+\frac{1}{2}\left(2-2\,x_\nu^2+2\,y^2-\frac{2\,y^2}{x_\nu}-5\,x_\nu\,y^2+7\,y^4-\frac{2\,y^4}{x_\nu}\right). \qquad (4.97)
\end{aligned}
$$

Returning to the $m_b \neq 0$ case, we integrate the differential rate (4.92) over the energy spectrum ($\bar{w}_- \leqslant x_\nu \leqslant \bar{w}_+$) and obtain

$$
\frac{d\Gamma^{(1)}}{d\cos\theta_P d\phi} = \Gamma_F 2\pi\frac{m_W}{\Gamma_W}C_F(-\frac{\alpha_s}{2\pi})6y^2\frac{1}{4\pi}\Big[M^A+M^B\,P\cos\theta_P+M^C\,P\sin\theta_P\cos\phi\Big]. \quad (4.98)
$$

where

$$
M^A = \int_{\bar{w}_-}^{\bar{w}_+} dx_\nu\, M^A(x_\nu)\,. \qquad (4.99)
$$

The result for (4.99) is identical to that of system 1a given in (3.226) as it should, because the unpolarized integrated rates are frame independent. This provides for necessary mutual check on the unpolarized results derived in system 1a and system 3a.

The polar part is given by[6]

$$
\begin{aligned}
M^B = \int_{\bar{w}_-}^{\bar{w}_+} dx_\nu\, M^A(x_\nu) &= k_0^b\Omega_0+k_1^b\Omega_1+k_2^b\bar{Y}_w+k_3^b\bar{Y}_p+k_4^b\bar{p}_3\ln\epsilon+k_5^b\bar{p}_3+k_6^b\bar{Y}_w\bar{Y}_p\\
&+k_7^b\bar{Y}_w\ln\epsilon+48y^4(k_8^b+k_9^b\frac{\bar{p}_0}{\bar{p}_3})\,, \qquad (4.100)
\end{aligned}
$$

[6] 4.100 agrees with $\mathcal{J}_1^-(y)$ of Eq.(33) in [28]. Note that our variable y^2 correspond to y of [28].

with the variables

$$\Omega_0 = \frac{2\bar{p}_0}{\bar{p}_3}\left[4\mathrm{Li}_2\left(1-\frac{\bar{p}_-}{\bar{p}_+}\right) - 4\bar{Y}_p\ln\left(1-\frac{\bar{p}_-}{\bar{p}_+}\right) + \bar{Y}_p\ln y^2 - 2\bar{Y}_w\ln\epsilon\right] + 8\ln(2\bar{p}_3) - 2\ln y^2,$$
$$\Omega_1 = 4\mathrm{Li}_2(\bar{w}_-) - 4\mathrm{Li}_2(\bar{w}_+)\,. \tag{4.101}$$

The coefficients for M^B are given by

$$k_1^b = 5 - 42y^2 + 45y^4 + 4y^6 - 2\left(4+8y^2+3y^4\right)\epsilon^2 + \epsilon^4 + 2\epsilon^6\,, \tag{4.102}$$

$$k_2^b = 8\left[-1+8y^2+10y^4+\left(3-9y^2-4y^4\right)\epsilon^2 - \left(3-y^2\right)\epsilon^4+\epsilon^6\right]\,, \tag{4.103}$$

$$k_3^b = -6\left(1-y^2\right)\left(3-23y^2-4y^4\right) + 2\left(1+24y^2+5y^4\right)\epsilon^2$$
$$+ 2\left(7-2y^2\right)\epsilon^4 + 2\epsilon^6\,, \tag{4.104}$$

$$k_4^b = -12\left[3-23y^2-4y^4-\left(6-y^2\right)\epsilon^2+3\epsilon^4\right]\,, \tag{4.105}$$

$$k_5^b = 2\left[15-37y^2+2y^4-\left(12+7y^2\right)\epsilon^2-3\epsilon^4\right]\,, \tag{4.106}$$

$$k_6^b = 24\left(\frac{y^2\epsilon^2\left(1+y^2-\epsilon^2\right)}{\bar{p}_3} - 2\left(1-5y^2-\epsilon^2\right)\bar{p}_3\right)\,, \tag{4.107}$$

$$k_7^b = -24\left[\left(1-5y^2\right)\left(1-y^2\right) - 2\left(1-y^2\right)\epsilon^2+\epsilon^4\right]\,, \tag{4.108}$$

$$k_8^b = 4\mathrm{Li}_2(\frac{\bar{p}_-\bar{w}_-}{\bar{p}_+\bar{w}_+}) - 4\mathrm{Li}_2(\frac{\bar{p}_-}{\bar{p}_+}) - \mathrm{Li}_3(\bar{w}_-) + \mathrm{Li}_3(\bar{w}_+) + 8\ln(\bar{w}_+)\bar{Y}_p$$
$$- \left[\mathrm{Li}_2(\bar{w}_-) + \mathrm{Li}_2(\bar{w}_+) - 2\ln(y^2) - 4\ln(\bar{w}_+) + 8\ln(2\bar{p}_3)\right]\bar{Y}_w\,, \tag{4.109}$$

$$k_9^b = 2\bar{Y}_w\left(\mathrm{Li}_2(\bar{w}_+) - \mathrm{Li}_2(\bar{w}_-) - 4\mathrm{Li}_2(\frac{2\bar{p}_3}{\bar{p}_+}) - 4\ln(\bar{w}_+)\bar{Y}_p + 4\ln(\frac{2\bar{p}_3}{\bar{p}_+})\bar{Y}_p + 2\ln(\bar{p}_+)\bar{Y}_w\right)$$
$$+ 4\left(\mathrm{Li}_3(\frac{\bar{p}_-\bar{w}_-}{\bar{p}_+\bar{w}_+}) - \mathrm{Li}_3(\frac{\bar{p}_-}{\bar{p}_+}) - \mathrm{Li}_3(\frac{\bar{w}_-}{\bar{w}_+}) + \mathrm{Li}_3(1)\right)$$
$$+ 4\left(\mathrm{Li}_2(\frac{\bar{p}_-\bar{w}_-}{\bar{p}_+\bar{w}_+}) - \mathrm{Li}_2(\frac{\bar{p}_-}{\bar{p}_+})\right)\bar{Y}_p\,. \tag{4.110}$$

For $m_b = 0$ the total integrated rates simplify to

$$\frac{d\Gamma^{(1)}}{d\cos\theta_P d\phi} = \Gamma_F 2\pi\frac{m_W}{\Gamma_W}C_F(-\frac{\alpha_s}{2\pi})6y^2\frac{1}{4\pi}\left(\mathcal{M}^A + \mathcal{M}^B P\cos\theta_P + \mathcal{M}^C P\sin\theta\cos\phi\right)\,. \tag{4.111}$$

The total unpolarized rate $\mathcal{M}^A$ is identical to the one in system 1a given in (3.244). For the total polar rate one obtains[7]

$$\mathcal{M}^B := \lim_{m_b\to 0} M^B \;=\; \left(1-y^2\right)\left(15-37\,y^2+2\,y^4\right) - \frac{2\,\pi^2}{3}\left(1+6\,y^2+9\,y^4-4\,y^6\right)$$
$$+4\,y^2\left[4-\left(19+\pi^2\right)y^2-2\,y^4\right]\ln(y^2)$$

[7] 4.112 agrees with $\mathcal{J}_1^-(y)$ of Eq.(62) in [28]. Note that our variable y^2 correspond to y of [28].

$$
\begin{aligned}
&-2\,\left(1-y^2\right)\,\left(1+19\,y^2+4\,y^4\right)\,\ln(1-y^2)\\
&+8\,\left(1-y^2\right)\,\left(2-13\,y^2-y^4\right)\,\ln(y^2)\,\ln(1-y^2)\\
&+4\left[5-42\,y^2+45\,y^4+4\,y^6+18\,y^4\,\ln(y^2)\right]\mathrm{Li}_2(y^2)\\
&+240\,y^4\left[\mathrm{Li}_3(1)-\mathrm{Li}_3(y^2)\right]. \qquad (4.112)
\end{aligned}
$$

Numerically the total unpolarized rate in $m_b = 0$ limit increases by 1.4% relative to the $m_b \neq 0$ case while the polar total rate for $m_b = 0$ decreases by 2.7% relative to the corresponding $m_b \neq 0$ result. If we scale these changes w.r.t. the unpolarized LO rate they are around 1‰ and 0.2‰ respectively. Therefore $m_b = 0$ limit is a very good approximation.

The azimuthal contribution will be calculated in the next section.

4.5 Azimuthal correlation at $\mathcal{O}(\alpha_s)$

The differential rate for the $\mathcal{O}(\alpha_s)$ azimuthal contribution can be read off from (4.86):

$$
\frac{d\Gamma_i^{(1)}}{dx_\nu dz} = \Gamma_F 2\pi \frac{m_W}{\Gamma_W} C_F(-\frac{\alpha_s}{2\pi})6y^2\left(M_v^C(x_\nu)\delta(z) + \mathcal{M}_r^C(x_\nu,z)\right), \qquad (4.113)
$$

where the virtual one–loop contribution $M_v^C(x)$ is given in (4.26), and the real emission contribution will be given in (4.115). As explained in section 3.6 the phase space integration for the real emission contribution to the azimuthal correlation has a complicated structure that can not be integrated in closed form. Therefore we can not obtain a closed form expression for the neutrino spectrum in system 3a. Of course, a numerical integration is possible and we obtain the plot for the azimuthal neutrino spectrum as shown in Fig. 4.2.

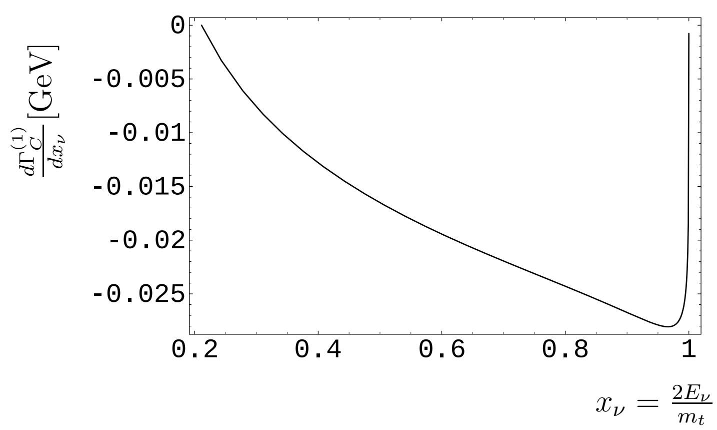


Figure 4.2: The QCD NLO corrections to the azimuthal differential rate w.r.t. the neutrino energy (neutrino spectrum).

However the closed form integrated azimuthal rate can be obtained by changing the order of integration in the phase space integration of the real emission, i.e. performing the x–integration first and then the z–integration according to[8]

$$\int\limits_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz \int\limits_{w_-}^{w_+} dx_\nu \,. \tag{4.114}$$

The real emission contribution for the azimuthal rate is IR–divergent. The real emission hadron tensor is split into a IR–convergent and a IR–divergent piece in (4.32). This leads to the total azimuthal contribution involving integrations of the form

$$\int dz \int dx_\nu \, M_r^C(x_\nu, z) = \int dz \int dx_\nu \left(M_{r,conv}^C(x_\nu, z) + M_{r,div}^C(x_\nu, z) \right), \tag{4.115}$$

as can be seen from (4.80).

Let us begin with the IR–convergent piece. We obtain $M_{r,conv}^C(x_\nu, z)$ from the two–body phase space integration $R_2(P; p_b, k)$ of the IR–convergent part of the lepton–hadron contraction in (4.36). To prepare for the x_ν–integration $M_{r,conv}^C(x_\nu, z)$ is written as a polynomial of x_ν:

$$\begin{aligned} M_{r,conv}^C(x_\nu, z) = -\sqrt{y^2 (1-x_\nu)(x_\nu - y^2) - x_\nu y^2 z} \Bigg\{ & \frac{y^2}{z^2 \lambda^3} \frac{1}{x_\nu} j_1 + \frac{1}{z^2\lambda^3} j_2 + \frac{x_\nu}{z^2 \lambda^3} j_3 \\ & + 2 \left[\frac{y^2}{\lambda^{7/2}} \frac{1}{x_\nu} j_4 + \frac{1}{\lambda^{7/2}} j_5 + \frac{x_\nu}{\lambda^{7/2}} j_6 \right] \ln \left(\frac{1 - y^2 + z + \sqrt{\lambda}}{1 - y^2 + z - \sqrt{\lambda}} \right) \Bigg\} \end{aligned} \tag{4.116}$$

where $\lambda = \lambda(1, y^2, z)$. The coefficients j_i are given by

$$\begin{aligned} j_1 = & -z(1 - 6y^2 + 15y^4 - 20y^6 + 15y^8 - 6y^{10} + y^{12} - 35z + 124y^2 z - 146y^4 z \\ & + 44y^6 z + 29y^8 z - 16y^{10} z + 92z^2 - 80y^2 z^2 - 58y^4 z^2 - 12y^6 z^2 + 58y^8 z^2 - 116z^3 \\ & - 96y^2 z^3 - 80y^4 z^3 - 92y^6 z^3 + 95z^4 + 110y^2 z^4 + 73y^4 z^4 - 41z^5 - 28y^2 z^5 + 4z^6) \\ & + (1 - 6y^2 + 15y^4 - 20y^6 + 15y^8 - 6y^{10} + y^{12} - 10z + 32y^2 z - 28y^4 z - 8y^6 z \\ & + 22y^8 z - 8y^{10} z - 23z^2 + 32y^2 z^2 + 22y^4 z^2 - 48y^6 z^2 + 17y^8 z^2 + 52z^3 - 24y^2 z^3 \\ & + 44y^4 z^3 - 8y^6 z^3 - 5z^4 - 2y^2 z^4 - 13y^4 z^4 - 10z^5 + 16y^2 z^5 - 5z^6)\epsilon^2 \\ & - (1 - 4y^2 + 6y^4 - 4y^6 + y^8 - 11z + 16y^2 z - 4y^6 z - y^8 z - 8z^2 + 58y^8 z^2 \\ & - 116z^3 - 40y^2 z^2 - 80y^2 z^2 - 58y^4 z^2 - 12y^6 z^2 - 20y^4 z^2 + 4y^6 z^2 + 44z^3 \\ & + 48y^2 z^3 - 6y^4 z^3 - 25z^4 + 4y^2 z^4 - z^5)\epsilon^4 \,, \end{aligned} \tag{4.117}$$

$$\begin{aligned} j_2 = & \, z(1 - 6y^2 + 15y^4 - 20y^6 + 15y^8 - 6y^{10} + y^{12} - 26z + 67y^2 z - 8y^4 z \\ & - 118y^6 z + 122y^8 z - 37y^{10} z + 84z^2 - 90y^2 z^2 + 46y^4 z^2 - 158y^6 z^2 + 118y^8 z^2 - 114z^3 \\ & - 82y^2 z^3 - 70y^4 z^3 - 150y^6 z^3 + 83z^4 + 148y^2 z^4 + 97y^4 z^4 - 36z^5 - 37y^2 z^5 + 8z^6) \\ & - (1 - 6y^2 + 15y^4 - 20y^6 + 15y^8 - 6y^{10} + y^{12} - 9z + 24y^2 z - 6y^4 z - 36y^6 z \\ & + 39y^8 z - 12y^{10} z - 6z^2 - 56y^2 z^2 + 128y^4 z^2 - 64y^6 z^2 - 2y^8 z^2 + 58z^3 \\ & - 12y^2 z^3 - 14y^4 z^3 + 64y^6 z^3 - 63z^4 + 30y^2 z^4 - 75y^4 z^4 + 15z^5 + 20y^2 z^5 + 4z^6)\epsilon^2 \end{aligned} \tag{4.118}$$

[8] $w_\pm = \frac{1}{2}(1 + y^2 - z \pm \sqrt{\lambda(1, y^2, z)})$ as defined in (3.6).

$$
\begin{aligned}
&+ (1 - 3y^2 + 2y^4 + 2y^6 - 3y^8 + y^{10} - 10z + 2y^2 z + 18y^4 z - 2y^6 z \\
&- 8y^8 z + 12z^2 - 74y^2 z^2 - 72y^4 z^2 + 6y^6 z^2 + 14z^3 + 106y^2 z^3 \\
&+ 20y^4 z^3 - 29z^4 - 31y^2 z^4 + 12z^5)\epsilon^4 ,
\end{aligned}
$$

$$
\begin{aligned}
j_3 = {}& 2z^2(6 - 24y^2 + 36y^4 - 24y^6 + 6y^8 - 6z + 5y^2 z + 8y^4 z - 7y^6 z \\
&- 6z^2 + 26y^2 z^2 - 4y^4 z^2 + 6z^3 + 5y^2 z^3) - z(3 - 12y^2 + 18y^4 - 12y^6 + 3y^8 + 30z \\
&- 34y^2 z - 22y^4 z + 26y^6 z - 60z^2 + 80y^2 z^2 - 52y^4 z^2 + 18z^3 + 14y^2 z^3 + 9z^4)\epsilon^2 \\
&- (1 - 4y^2 + 6y^4 - 4y^6 + y^8 - 10z + 12y^2 z + 6y^4 z - 8y^6 z \\
&- 60y^2 z^2 - 4y^4 z^2 + 26z^3 + 28y^2 z^3 - 17z^4)\epsilon^4 ,
\end{aligned} \tag{4.119}
$$

$$
\begin{aligned}
j_4 = {}& -6 + 28y^2 - 50y^4 + 40y^6 - 10y^8 - 4y^{10} + 2y^{12} + 7z + y^2 z - 34y^4 z \\
&+ 26y^6 z + 11y^8 z - 11y^{10} z + 3z^2 - 18y^2 z^2 - 10y^4 z^2 + 25y^8 z^2 - 10z^3 - 20y^2 z^3 \\
&- 26y^4 z^3 - 30y^6 z^3 + 14z^4 + 28y^2 z^4 + 20y^4 z^4 - 9z^5 - 7y^2 z^5 + z^6 \\
&(13 - 29y^2 + 10y^4 + 14y^6 - 7y^8 - y^{10} - 9z + 10y^2 z - 24y^4 z + 18y^6 z + 5y^8 z \\
&- 16z^2 + 2y^2 z^2 - 12y^4 z^2 - 10y^6 z^2 + 8z^3 - 2y^2 z^3 + 10y^4 z^3 + 3z^4 - 5y^2 z^4 + z^5)\epsilon^2 \\
&- 6\left(2 - y^4 - y^6 - 3z + y^2 z + 3y^4 z - 3y^2 z^2 + z^3\right)\epsilon^4 ,
\end{aligned} \tag{4.120}
$$

$$
\begin{aligned}
j_5 = {}& 4 - 14y^2 + 10y^4 + 20y^6 - 40y^8 + 26y^{10} - 6y^{12} - 7z + 15y^2 z - 31y^4 z \\
&+ 73y^6 z - 78y^8 z + 28y^{10} z - 5z^2 + 20y^2 z^2 - 15y^4 z^2 + 54y^6 z^2 - 54y^8 z^2 + 19z^3 \\
&+ 7y^2 z^3 + 30y^4 z^3 + 56y^6 z^3 - 17z^4 - 40y^2 z^4 - 34y^4 z^4 + 8z^5 + 12y^2 z^5 - 2z^6 \\
&- 3(3 - y^2 - 13y^4 + 15y^6 - 2y^8 - 2y^{10} - 6z + 13y^2 z - 12y^4 z - 3y^6 z + 8y^8 z \\
&- 9y^2 z^2 + 9y^4 z^2 - 12y^6 z^2 + 6z^3 - y^2 z^3 + 8y^4 z^3 - 3z^4 - 2y^2 z^4)\epsilon^2 \\
&+ (7 + 23y^2 - 9y^4 - 19y^6 - 2y^8 - 19z - 14y^2 z + 37y^4 z + 8y^6 z \\
&+ 15z^2 - 17y^2 z^2 - 12y^4 z^2 - z^3 + 8y^2 z^3 - 2z^4)\epsilon^4 ,
\end{aligned} \tag{4.121}
$$

$$
\begin{aligned}
j_6 = {}& -2 + 10y^2 - 20y^4 + 20y^6 - 10y^8 + 2y^{10} - 4z + 17y^2 z - 27y^4 z + 19y^6 z \\
&- 5y^8 z + 12z^2 - 23y^2 z^2 + 8y^4 z^2 + 3y^6 z^2 - 4z^3 - 15y^2 z^3 + y^4 z^3 - 2z^4 - y^2 z^4 \\
&+ (10 - 25y^2 + 15y^4 + 5y^6 - 5y^8 - 7z + 28y^2 z - 35y^4 z \\
&+ 14y^6 z - 15z^2 + 19y^2 z^2 - 12y^4 z^2 + 11z^3 + 2y^2 z^3 + z^4)\epsilon^2 \\
&- 3\left(3 + 3y^2 - 5y^4 - y^6 - 5z + 4y^2 z + 3y^4 z + z^2 - 3y^2 z^2 + z^3\right)\epsilon^4 .
\end{aligned} \tag{4.122}
$$

The x_ν–integration is done with the help of the basic integrals $\mathcal{X}_n$ given in Appendix D.6. Thus we obtain

$$
\begin{aligned}
\int\limits_{w_-(z)}^{w_+(z)} dx_\nu \, M^C_{r,conv}(x_\nu, z) = {}& -\frac{15\pi y^3 \left[(1+y)^2 - \epsilon^2\right]^2}{4\left[(1+y)^2 - z\right]^3} - \frac{3\pi y \left[(1-y)^2 - \epsilon^2\right]^2}{64\left[(1-y)^2 - z\right]} \\
&+ \frac{\pi y}{16}\left(20 - 12y^2 - \epsilon^2\right) - \pi y z \\
&+ \frac{\pi y^2 \left[(1+y)^2 - \epsilon^2\right] v_1}{16\,(1+y)^2 \left[(1+y)^2 - z\right]^2} + \frac{\pi y \, v_2}{64\,(1+y)^4 \left[(1+y)^2 - z\right]}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\pi y \epsilon^2 (1-y)^2}{16 (1+y)^2 z^2} v_3 + \frac{\pi y\, v_4}{16 (1+y)^4 z} \\
&+ \pi Y_p \sqrt{\lambda(1,y^2,z)} \Bigg\{ - \frac{15 y^2 (1+y) \left[(1+y)^2 - \epsilon^2\right]^2}{8 \left[(1+y)^2 - z\right]^4} \\
&+ \frac{3 (1-y) \left[(1-y)^2 - \epsilon^2\right]^2}{128 \left[(1-y)^2 - z\right]^2} + y + \frac{3y\, v_5}{16 \left[(1+y)^2 - z\right]^3} \\
&+ \frac{v_6}{128 \left[(1+y)^2 - z\right]^2} + \frac{v_7}{256 y \left[(1+y)^2 - z\right]} \\
&+ \frac{v_8}{256 y \left[(1-y)^2 - z\right]} \Bigg\} , \qquad (4.123)
\end{aligned}
$$

with the coefficients v_i given by

$$v_1 = 27 + 60y + 110y^2 + 148y^3 + 71y^4 - \left(27 + 30y + 23y^2\right) \epsilon^2 , \qquad (4.124)$$

$$
\begin{aligned}
v_2 = &(1+y)^4 \left(51 - 12y - 438y^2 + 180y^3 - 85y^4\right) \\
&- 2(1+y)^2 \left(51 + 96y - 30y^2 - 16y^3 - 37y^4\right) \epsilon^2 \\
&+ \left(51 + 204y + 202y^2 + 108y^3 + 11y^4\right) \epsilon^4 , \qquad (4.125)
\end{aligned}
$$

$$v_3 = 2 (1-y) (1+y)^2 (1+3y) - \left(1 + 4y + y^2\right) \epsilon^2 , \qquad (4.126)$$

$$
\begin{aligned}
v_4 = &-2(1-y)^3 (1+y)^4 (1+3y) - (1-y) (1+y)^2 \left(7 + 21y + y^2 + 11y^3\right) \epsilon^2 \\
&+ \left(5 + 20y + 4y^2 - 4y^3 - 5y^4\right) \epsilon^4 , \qquad (4.127)
\end{aligned}
$$

$$v_5 = (1+y)^3 \left(2 + y + 11y^2\right) - 2 (1+y) \left(2 + 3y + 7y^2\right) \epsilon^2 + (2+3y) \epsilon^4 , \qquad (4.128)$$

$$
\begin{aligned}
v_6 = &(1+y) \left(63 - 36y - 358y^2 + 172y^3 - 65y^4\right) \\
&- 2 \left(63 + 21y + 5y^2 - 25y^3\right) \epsilon^2 + 3 (21 + 5y) \epsilon^4 , \qquad (4.129)
\end{aligned}
$$

$$
\begin{aligned}
v_7 = &33 - 26y - 374y^2 - 60y^3 - 419y^4 + 14y^5 \\
&- 2 \left(33 - 18y + 23y^2 + 2y^3\right) \epsilon^2 + (33 - 10y) \epsilon^4 , \qquad (4.130)
\end{aligned}
$$

$$
\begin{aligned}
v_8 = &-(1-y)^2 \left(33 + 40y + 57y^2 + 14y^3\right) \\
&+ 2 \left(33 - 18y - 9y^2 + 2y^3\right) \epsilon^2 - (33 - 10y) \epsilon^4 . \qquad (4.131)
\end{aligned}
$$

Performing the z–integration similar to that of (3.257) we have

$$
\begin{aligned}
M^C_{r,conv} = -\frac{\pi}{8} \Big\{ &c_1 \Big(\mathrm{Li}_2 (\bar{p}_+) + \mathrm{Li}_2 (\bar{p}_-) + 2\mathrm{Li}_2 (y) - 2\bar{Y}_w \bar{Y}_p + \ln y^2 \ln \epsilon \Big) \\
&+ c_2\, \bar{p}_3 \bar{Y}_p + c_3 \ln \frac{1-y}{\epsilon} + c_4 \Big\} , \qquad (4.132)
\end{aligned}
$$

with the coefficients

$$c_1 = y \Big(2 - 11y^2 + 16y^3 - 3y^4 - \left(3 - 2y^2\right) \epsilon^2 + \epsilon^4 \Big) , \qquad (4.133)$$

$$c_2 = 2 \left(8 - 3y - 8y^2 - y^3 - (8 - 5y) \epsilon^2 - \frac{16 (2-3y) (1+y) y^2}{(1+y)^2 - \epsilon^2} \right) , \qquad (4.134)$$

$$c_3 = -(1-y)\left(8+3y-45y^2+49y^3-7y^4\right) + (1-y)\left(16-y-y^2\right)\epsilon^2 - 2(4-3y)\epsilon^4\,, \qquad (4.135)$$

$$c_4 = \frac{1}{3}y\Big(-18-2\pi^2+15y+69y^2+11\pi^2y^2-111y^3-16\pi^2y^3 + 45y^4+3\pi^2y^4+\left(45+3\pi^2-33y-18y^2-2\pi^2y^2\right)\epsilon^2-\left(27+\pi^2\right)\epsilon^4\Big)\,. \qquad (4.136)$$

Let us now turn to the IR–divergent part. We know from (4.74) that the IR–divergent azimuthal contribution is:

$$\int dz \int dx_\nu\, M^C_{r,div}(x_\nu,z) = m_t^2 \int dz \int dx_\nu\, dR_2(P;p_b,k)\, \widetilde{M_0^C}(x_\nu,z)\Delta_{SGF}\,, \qquad (4.137)$$

where

$$\widetilde{M_0^C}(x_\nu,z) = 2y\left(1-\frac{y^2}{x_\nu}\right)\sqrt{x_\nu(1-x_\nu+y^2-z)-y^2}\,. \qquad (4.138)$$

The x_ν–integration is done with the help of the basic integrals $\mathcal{X}_n$ given in D.6. Since $(z-\epsilon^2)|M|^2_{SGF}$ is IR–convergent, we write the result as a polynomial of $(z-\epsilon^2)$:

$$\begin{aligned}\int_{w_-(z)}^{w_+(z)} dx_\nu\, \widetilde{M_0^C}(x_\nu,z) &= \int_{w_-(z)}^{w_+(z)} dx_\nu\, 2y\left(1-\frac{y^2}{x_\nu}\right)\sqrt{x_\nu(1-x_\nu+y^2-z)-y^2}\\ &= 2y\left(\mathcal{X}_0 - y^2\mathcal{X}_{-1}\right)\\ &= M_0^C + a_1(z-\epsilon^2)+a_2(z-\epsilon^2)^2\,, \end{aligned} \qquad (4.139)$$

where M_0^C is the LO azimuthal part given in (4.17), and

$$a_1 = -\frac{1}{2}\pi y(1-y^2-\epsilon^2)\,, \qquad a_2 = \frac{1}{4}\pi y\,. \qquad (4.140)$$

Substituting back into (4.137)

$$\begin{aligned} m_t^2 \int_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz dR_2(P;p_b,k)\Delta_{SGF} \int_{w_-(z)}^{w_+(z)} dx_\nu\, \widetilde{M_0^C}(x_\nu,z) \\ = m_t^2 \int_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz\, dR_2(P;p_b,k)\Big[M_0^C + a_1(z-\epsilon^2)+a_2(z-\epsilon^2)^2\Big]\Delta_{SGF} \\ = M_0^C\, m_t^2 \int_{(\epsilon+\Lambda)^2}^{(1-y)^2} dz dR_2(P;p_b,k)\Delta_{SGF} \\ + m_t^2 \int_{\epsilon^2}^{(1-y)^2} dz dR_2(P;p_b,k)\Big[a_1(z-\epsilon^2)+a_2(z-\epsilon^2)^2\Big]\Delta_{SGF}\,. \end{aligned} \qquad (4.141)$$

The integration in the last line is IR–convergent. For the IR–divergent integral, similar to (2.176)[9], we obtain

$$m_t^2 \int dz\, dR_2(P;p_b,k)\,\Delta_{SGF} = 2\pi\left[\mathcal{R}(-1,0) - 2\mathcal{S}(0,0)\right]\Big|_{\epsilon^2}^{(1-y)^2} + \pi\,\hat{\Sigma}(\Lambda)\,, \qquad (4.142)$$

where $\hat{\Sigma}(\Lambda)$ is given in (2.202) with $y = m_W/m_t$. The IR–convergent integrals are

$$m_t^2 \int dz (z-\epsilon^2)\, dR_2(P;p_b,k)\Delta_{SGF} = 4\,\pi\,\left[\mathcal{R}(0,0) - \left(1-y^2\right)\,\mathcal{S}(0,0) - \mathcal{S}(1,0)\right]\Big|_{\epsilon^2}^{(1-y)^2},$$

$$m_t^2 \int dz\,(z-\epsilon^2)^2\, dR_2(P;p_b,k)\Delta_{SGF} = -4\,\pi\left[\epsilon^2\,\mathcal{R}(0,0) - \mathcal{R}(1,0) - \left(1-y^2\right)\,\epsilon^2\,\mathcal{S}(0,0) \right.$$
$$\left. + \left(1-y^2-\epsilon^2\right)\,\mathcal{S}(1,0) + \mathcal{S}(2,0)\right]\Big|_{\epsilon^2}^{(1-y)^2}. \qquad (4.143)$$

Substituting all these integrals into (4.137) we have

$$\int dz \int dx_\nu\, M^C_{r,div}(x_\nu,z) = M_0^C\,\hat{\Sigma}(\Lambda) + \hat{M}^C_{r,conv}\,, \qquad (4.144)$$

where

$$\hat{M}^C_{r,conv} = k_0\Big[\mathrm{Li}_2(\bar{w}_+) + \mathrm{Li}_2(\bar{w}_-) - 2\mathrm{Li}_2(y)\Big] + k_1 \ln(1-y) + k_2\,\bar{p}_3\bar{Y}_p + k_3\ln(\epsilon) + k_4\,, \qquad (4.145)$$

with the coefficients

$$k_0 = \frac{1}{2}\pi\,y\,\left(1 - 8\,y^3 + 3\,y^4 - \epsilon^4 - 2\,y^2\,\left(1+\epsilon^2\right)\right)\,, \qquad (4.146)$$
$$k_1 = \frac{1}{2}\pi\,y\,\left(1 + 16\,y^3 - y^4 - 6\,\epsilon^2 + 2\,\epsilon^4 - 2y^2\,\left(8 - 3\,\epsilon^2\right)\right)\,, \qquad (4.147)$$
$$k_2 = \pi\,y\,\left(1 + 5\,y^2 + 3\,\epsilon^2\right)\,, \qquad (4.148)$$
$$k_3 = -\frac{1}{2}\pi\,y\,\left(1 + 16\,y^3 - y^4 - 6\,\epsilon^2 + 2\,\epsilon^4 - 2\,y^2\,\left(8 - 3\,\epsilon^2\right)\right)\,, \qquad (4.149)$$
$$k_4 = -\frac{1}{8}\pi\,y\,\Big[11 + 76\,y^3 - 33\,y^4 - 28\,\epsilon^2 + 17\,\epsilon^4$$
$$- 2\,y^2\,\left(21 - 8\,\epsilon^2\right) - 12\,y\,\left(1 - 2\,\epsilon^2\right)\Big]. \qquad (4.150)$$

Inserting (4.132) and (4.144) into (4.115) gives the total NLO real emission contribution:

$$M_r^C = \int dz \int dx_\nu\, M_r^C(x_\nu,z) = M^C_{r,conv} + \hat{M}^C_{r,conv} + M_0^C\,\hat{\Sigma}(\Lambda)\,, \qquad (4.151)$$

where M_0^C, $M^C_{r,conv}$, $\hat{M}^C_{r,conv}$ and $\hat{\Sigma}(\Lambda)$ are given in (4.17), (4.132), (4.145) and (2.202).

[9]The basic integrals $\mathcal{R}(m,n)$ and $\mathcal{S}(m,n)$ here are defined in (3.154). They are different from those of (2.176), where the basic integrals defined in (2.160) are used.

The virtual one–loop azimuthal contribution to the differential rate is given in (4.26). Performing the x_ν–integration with the help of the basic integrals $\mathcal{X}_n$ we obtain

$$\begin{aligned} M_v^C &= \int dx_\nu M_v^C(x_\nu) \\ &= M_{v,div} \int dx_\nu M_0^C(x_\nu) + \int dx_\nu M_{v,conv}^C(x_\nu) \\ &= M_{v,div} M_0^C + M_{v,conv}^C\,, \end{aligned} \tag{4.152}$$

where

$$\begin{aligned} M_{v,conv}^C &= \int dx_\nu M_{v,conv}^C(x_\nu) = -\pi \frac{(1-y)^2 - \epsilon^2}{32y} \times \\ &\times \Big\{ 2\Big(3 + 6y - 12y^2 + 2y^3 + y^4 - 2\left(3 + 3y + 2y^2\right)\epsilon^2 + 3\epsilon^4\Big)\ln(\epsilon) \\ &+ \Big[(1-y)^2\left(3 + 12y + 6y^2 - 4y^3 - y^4\right) \\ &- \left(9 + 12y - 2y^2 + 20y^3 + 9y^4\right)\epsilon^2 + \left(9 + 6y + 13y^2\right)\epsilon^4 - 3\epsilon^6\Big]\frac{\bar{Y}_p}{\bar{p}_3}\Big\}. \end{aligned} \tag{4.153}$$

The terms $M_{v,div}$ and M_0^C are given in (3.130) and (4.17).

The total $\mathcal{O}(\alpha_s)$ corrections are the sum of the virtual and real emission corrections. Collecting the virtual one–loop and the real emission contributions and substituting them into (4.113) we obtain

$$\begin{aligned} \Gamma_C^{(1)} &= \Gamma_F\, 2\pi \frac{m_W}{\Gamma_W} C_F \left(-\frac{\alpha_s}{2\pi}\right) 6y^2 \int dz \int dx_\nu \left\{ M_v^C(x_\nu)\delta(z) + M_r^C(x_\nu, z) \right\} \\ &= \Gamma_F\, 2\pi \frac{m_W}{\Gamma_W} C_F \left(-\frac{\alpha_s}{2\pi}\right) 6y^2 \Big\{ M_0^C M_{v,div} + M_{v,conv}^C + M_{r,conv}^C + \hat{M}_{conv}^C + M_0^C \Sigma(\Lambda) \Big\}. \end{aligned} \tag{4.154}$$

As before we first sum up and simplify the IR–divergent part of the virtual and real emission terms $M_0^C M_{v,div}$ and $M_0^C \Sigma(\Lambda)$. Similar to (4.88) we write

$$M_0^C M_{v,div} + M_0^C \hat{\Sigma}(\Lambda) = M_0^C \hat{\Phi}_0 + M_0^C \Big[2 + \frac{1}{y^2}\left(1 - 2\,y^2 - \epsilon^2\right)\ln(\epsilon) - \Big(4\,\bar{p}_0 - \frac{2\,\bar{p}_3^2}{y^2}\Big)\frac{\bar{Y}_p}{\bar{p}_3}\Big], \tag{4.155}$$

where

$$\begin{aligned} \hat{\Phi}_0 &= \frac{2\bar{p}_0}{\bar{p}_3}\Big[2\mathrm{Li}_2(1 - \frac{1-y}{\bar{p}_+}) + 2\mathrm{Li}_2(1 - \frac{1-y}{\bar{p}_+}) + \mathrm{Li}_2(1 - \bar{w}_-) - \mathrm{Li}_2(1 - \bar{w}_+) - 4\ln\epsilon\bar{Y}_p\Big] \\ &\quad + 4\ln\left((1-y)^2 - \epsilon^2\right)\Big(1 - \frac{\bar{p}_0\bar{Y}_p}{\bar{p}_3}\Big) - 4\ln(1-y). \end{aligned} \tag{4.156}$$

Inserting (4.155) back into (4.154) and also substituting the IR–convergent terms we obtain

$$\begin{aligned}\Gamma^C = & \Gamma_F 2\pi \frac{M_W}{\Gamma_W} C_F \left(-\frac{\alpha_s}{2\pi}\right) 6y^2 \Big\{ M_0^C \hat{\Phi}_0 + \\ & + d_0 \Big[2\,\ln(y)\,\ln(\epsilon) + 2\,\mathrm{Li}_2(y) + \mathrm{Li}_2(\bar{p}_+) + \mathrm{Li}_2(\bar{p}_-) - 2\,\bar{Y}_p\,\bar{Y}_w \Big] \\ & + d_1 \Big[\mathrm{Li}_2(\bar{w}_+) + \mathrm{Li}_2(\bar{w}_-) - 2\,\mathrm{Li}_2(y) \Big] + d_2\,\ln(1-y) \\ & + d_3\,\ln(\epsilon) + d_4\,\frac{\bar{Y}_p}{\bar{p}_3} + d_5 \Big\}, \end{aligned} \tag{4.157}$$

where the coefficients d_i are given by

$$d_0 = -\frac{1}{8}\pi y(2 - 11y^2 - 3y^4 + 16y^3 - 3\epsilon^2 + 3y^2\epsilon^2 + \epsilon^4), \tag{4.158}$$

$$d_1 = \frac{1}{2}\pi y \left(1 - 8y^3 + 3y^4 - \epsilon^4 - 2y^2\left(1 + \epsilon^2\right)\right), \tag{4.159}$$

$$\begin{aligned} d_2 = & \frac{1}{8}\pi \Big[8 - 48y^2 + 72y^4 + 7y - 18y^3 - 21y^5 \\ & - \left(16 + 23y - 39y^3\right)\epsilon^2 + (8 + 10y)\,\epsilon^4 \Big], \end{aligned} \tag{4.160}$$

$$\begin{aligned} d_3 = & \frac{1}{16y}\pi \Big[1 - 9y^2 - 9y^4 + 17y^6 - 16y + 96y^3 - 80y^5 \\ & - \left(3 - 48y^2 + 45y^4 - 32y + 32y^3\right)\epsilon^2 + \left(3 - 27y^2 - 16y\right)\epsilon^4 - \epsilon^6 \Big], \end{aligned} \tag{4.161}$$

$$\begin{aligned} d_4 = & \frac{1}{32y}\pi \left((1-y)^2 - \epsilon^2\right) \Big[(1+y)^2 \left(1 - 8y^2 - 17y^4 - 16y + 80y^3\right) \\ & - \left(3 - 54y^2 - 33y^4 - 28y\right)\epsilon^2 + \left(3 - 15y^2 - 14y\right)\epsilon^4 - \epsilon^6 \Big], \end{aligned} \tag{4.162}$$

$$\begin{aligned} d_5 = & \frac{1}{24}\pi y \Big[-3\left(1 - 6y^2 - 5y\right)(1-y)^2 + \pi^2\left(2 - 11y^2 - 3y^4 + 16y^3\right) \\ & + \left(15 - 3\pi^2 - 6y^2 + 2\pi^2 y^2 - 39y\right)\epsilon^2 - \left(12 - \pi^2\right)\epsilon^4 \Big]. \end{aligned} \tag{4.163}$$

Finally for the integrated azimuthal rate in the $m_b = 0$ limit we have

$$\begin{aligned} \Gamma_C^{(1)} = & \Gamma_F 2\pi \frac{M_W}{\Gamma_W} C_F \left(-\frac{\alpha_s}{2\pi}\right) \frac{3}{8}\pi y \Big[\frac{\pi^2}{3} y^2 \left(10 - 43y^2 + 16y^3 - 3y^4\right) \\ & -2(1-y)^2 y^2 \left(1 - 5y - 6y^2\right) \\ & + \left(1 - 16y - 9y^2 + 96y^3 - 9y^4 - 80y^5 + 17y^6\right)\ln(1+y) \\ & + \left(1 + 5y^2 - 45y^4 + 64y^5 - 25y^6\right)\ln(1-y) \\ & -4y^2\left(2 + 5y^2 - 48y^3 + 21y^4\right)\mathrm{Li}_2(y) \\ & +2y^2\left(6 - 19y^2 - 16y^3 + 9y^4\right)\mathrm{Li}_2(y^2) \Big]. \end{aligned} \tag{4.164}$$

We checked numerically that the total azimuthal rate for m_b increases by 1.6% relative to the $m_b \neq 0$ case. If we scale this change w.r.t. the unpolarized LO rate it is around 1‰. Therefore taking $m_b = 0$ is a very good approximation.

4.6 Summary

The differential decay rate at $\mathcal{O}(\alpha_s)$ is the sum of the LO and QCD NLO contributions:

$$\frac{d\Gamma}{dx_\nu\, d\cos\theta_P\, d\phi} = \\ \frac{1}{4\pi}\left[\left(\frac{d\Gamma_A^{(0)}}{dx_\nu}+\frac{d\Gamma_A^{(1)}}{dx_\nu}\right)+\left(\frac{d\Gamma_B^{(0)}}{dx_\nu}+\frac{d\Gamma_B^{(1)}}{dx_\nu}\right)P\cos\theta_P+\left(\frac{d\Gamma_C^{(0)}}{dx_\nu}+\frac{d\Gamma_C^{(1)}}{dx_\nu}\right)P\sin\theta_P\cos\phi\right]. \tag{4.165}$$

Since setting $m_b = 0$ is a good approximation we give the numerical results in this limit. The differential unpolarized, polar and azimuthal rates (for $m_b = 0$) are shown in Fig. 4.3, Fig. 4.4 and Fig. 4.5 respectively.

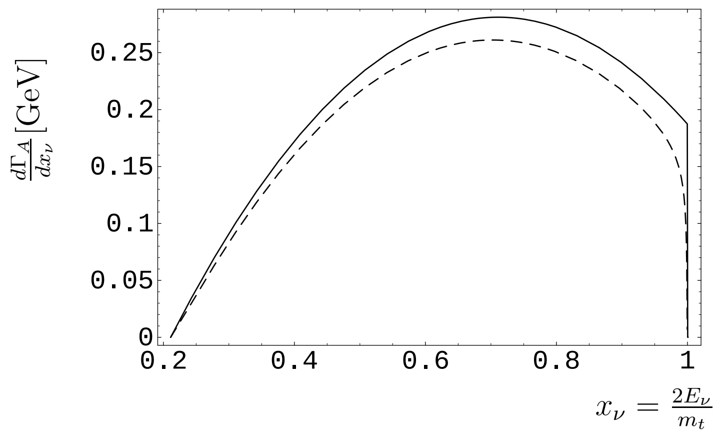


Figure 4.3: The unpolarized differential rate w.r.t. the neutrino energy (neutrino spectrum) in system 3a. The solid line is the LO spectrum and the dashed line is the spectrum with QCD NLO corrections.

The total integrated rate can be written as

$$\frac{d\Gamma}{d\cos\theta_P\, d\phi} = \frac{1}{4\pi}\left[\left(\Gamma_A^{(0)}+\Gamma_A^{(1)}\right)+\left(\Gamma_B^{(0)}+\Gamma_B^{(1)}\right)P\cos\theta_P+\left(\Gamma_C^{(0)}+\Gamma_C^{(1)}\right)P\sin\theta_P\cos\phi\right]. \tag{4.166}$$

As in the case of system 1a, to highlight the change of the LO result by the QCD NLO correction, we write the total rate in the following form

$$\frac{d\Gamma}{d\cos\theta_P\, d\phi} = \frac{\Gamma_A^{(0)}}{4\pi}\left[\left(1+\frac{\Gamma_A^{(1)}}{\Gamma_A^{(0)}}\right)+\left(\frac{\Gamma_B^{(0)}}{\Gamma_A^{(0)}}+\frac{\Gamma_B^{(1)}}{\Gamma_A^{(0)}}\right)P\cos\theta_P+\left(\frac{\Gamma_C^{(0)}}{\Gamma_A^{(0)}}+\frac{\Gamma_C^{(1)}}{\Gamma_A^{(0)}}\right)P\sin\theta_P\cos\phi\right]. \tag{4.167}$$

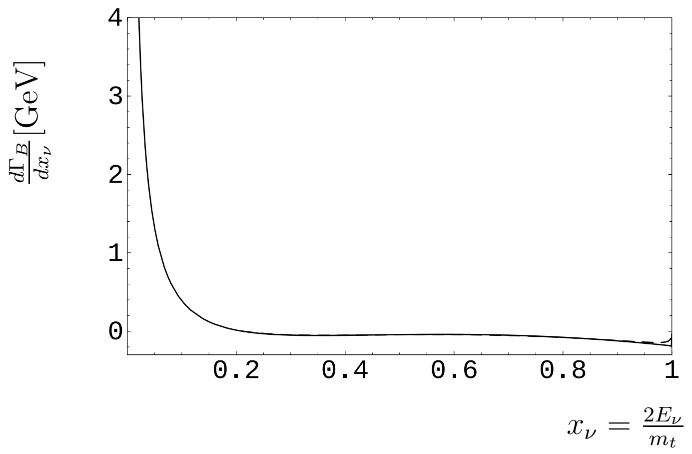


Figure 4.4: The polarized differential rate w.r.t. the neutrino energy (neutrino spectrum) in system 3a. QCD NLO corrections can not be discerned at the scale of this plot.

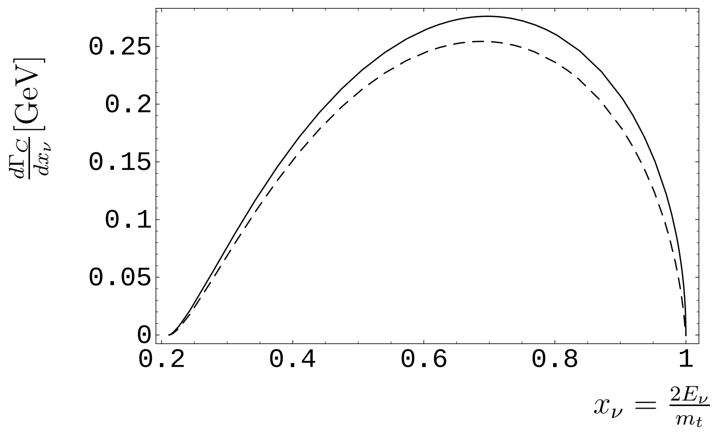


Figure 4.5: The azimuthal differential rate w.r.t. the neutrino energy (neutrino spectrum) in system 3a. The solid line is the LO spectrum and the dashed line is the spectrum with QCD NLO corrections.

Taking numerical values $|V_{tb}| = 0.999$, $m_t = 175\,\text{GeV}$, and neglecting the b–quark mass we obtain

$$\boxed{\begin{aligned}&\frac{d\Gamma}{d\cos\theta_P\,d\phi} = \Gamma_A^{(0)}\frac{1}{4\pi}\times\\ &\quad\times\Big[(1-8.54\,\%) + (-0.318+1.03\,\%)\,P\cos\theta_P + (0.919-8.65\,\%)\,P\sin\theta_P\cos\phi\Big]\,.\end{aligned}} \tag{4.168}$$

The NLO correction for the unpolarized rate equals to the one in system 1a (see 3.278) as it should since the unpolarized rate is the same for the different helicity frames. Calculation of the unpolarized rates in two different frames provides for a non–trivial check of each other. It is interesting to note that the NLO polar correction only increases by around 1%. This is much smaller than the change in the unpolarized and polar rates by the NLO corrections.

In terms of the analyzing powers (asymmetry parameters) defined in (3.279) we write the total rate as

$$\boxed{\frac{1}{\Gamma_A^{(0)} + \Gamma_A^{(1)}} \frac{d\Gamma}{d\cos\theta_P\, d\phi} = \frac{1}{4\pi}\Big[1 - 33.63\%\, P\cos\theta_P + 91.03\%\, P\sin\theta_P\cos\phi\Big]\,.} \tag{4.169}$$

The LO and $\mathcal{O}(\alpha_s)$ analyzing powers are shown in Table 4.169. The change in the analyzing powers by the NLO corrections are very small.

	polar analyzing power	azimuthal analyzing power
LO	-0.3180	0.9189
NLO	-0.3363	0.9103

Table 4.1: The LO and QCD NLO analyzing powers in system 3a.

The positivity of the total rate can be shown by writing the total rate as

$$\begin{aligned}\frac{d\Gamma}{d\cos\theta_P\, d\phi} = \frac{\Gamma_A^{(0)}}{4\pi}\Bigg[&\left(1 + \frac{\Gamma_A^{(1)}}{\Gamma_A^{(0)}}\right) \\ &+ \frac{1}{\Gamma_A^{(0)}} P \sqrt{\left(\Gamma_B^{(0)} + \Gamma_B^{(1)}\right)^2 + \cos^2\phi\left(\Gamma_C^{(0)} + \Gamma_C^{(1)}\right)^2}\,\sin\left(\theta_P + \delta\right)\Bigg]\,,\end{aligned} \tag{4.170}$$

with

$$\tan\delta = \frac{\Gamma_B^{(0)} + \Gamma_B^{(1)}}{(\Gamma_C^{(0)} + \Gamma_C^{(1)})\cos\phi}, \qquad -\frac{\pi}{2} \leqslant \delta \leqslant \frac{\pi}{2}. \tag{4.171}$$

The total rate takes minimum value for $\cos\phi = \pm 1$, $\sin(\delta + \theta_P) = -1$ and $P = 1$:

$$\begin{aligned}&\left(1 + \frac{\Gamma_A^{(1)}}{\Gamma_A^{(0)}}\right) - \frac{1}{\Gamma_A^{(0)}}\sqrt{\left(\Gamma_B^{(0)} + \Gamma_B^{(1)}\right)^2 + \left(\Gamma_C^{(0)} + \Gamma_C^{(1)}\right)^2} \\ &= (1 - 8.54\,\%) - \sqrt{(-0.318 + 1.03\,\%)^2 + (0.919 - 8.65\,\%)^2} \\ &= 2.7\,\%\,.\end{aligned} \tag{4.172}$$

This shows that the total rate is positive definite over the whole angular domain.

Finally, we give a summary of this chapter.

We have calculated the $O(\alpha_s)$ corrections to the angular correlations in polarized top quark decay in the helicity system 3a, where the neutrino momentum is along the z–axis and the event plane lies in the (x, z)–plane. At LO the azimuthal correlation does not vanish in system 3a, unlike the case in system 1a, where the LO unpolarized and polar rates are equal to each other and the azimuthal rate vanishes. We present closed form expressions for differential (w.r.t. neutrino energy) and integrated rates for the unpolarized and polar correlations. They agree with existing results. Our results for the azimuthal rates are new. The $O(\alpha_s)$ corrections to the differential azimuthal rate are obtained numerically. The $O(\alpha_s)$ corrections to the integrated azimuthal rate are obtained in closed form. The $O(\alpha_s)$ corrections reduce both the total unpolarized and azimuthal rates by around 9% relative to the LO rates while the polar rate increase by 1%. With QCD NLO corrections the magnitude of the polar analyzing power increases by around 6% relative to the LO value whereas the azimuthal analyzing power decreases by roughly 1%.

Chapter 5

Helicity Analysis of Semileptonic Hyperon Decays Including Lepton Mass Effects

5.1 Introduction

In this chapter we present the results for the full angular decay distribution in the semileptonic decay of a polarized hyperon: $\Xi^0(\uparrow) \to \Sigma^+(\to p\,\pi^0) + \ell^- + \bar{\nu}_\ell$. Contrary to the top decays discussed in the previous chapters this hyperon decay is analyzed in cascade fashion, i.e. we analyze the decay in three different rest frames in the cascade decay process:

(i) $\Xi^0 \to \Sigma^+ + W^-_{\rm off-shell}$ in the Ξ^0–rest frame,

(ii) $\Sigma^+ \to p + \pi^0$ in the Σ^+–rest frame,

(iii) $W^-_{\rm off-shell} \to \ell^- + \bar{\nu}_\ell$ in the $W^-_{\rm off-shell}$–rest frame.

Semileptonic hyperon decays have traditionally been analyzed in the rest frame of the decaying parent hyperon using fully covariant methods based on either four-component Dirac spinor methods [62–65] or on two-component Pauli spinor methods [66–68]. In this chapter we employ helicity methods to analyze semileptonic hyperon decays. In the muonic mode it is quite important to incorporate lepton mass effects in the analysis since e.g. in the decay $\Xi^0 \to \Sigma^+ + \ell^- + \bar{\nu}_l$ the mass difference between the parent and daughter hyperon $M_{\Xi^0} - M_{\Sigma^+} = (1314.83 - 1189.37)\,\mathrm{MeV} = 125.46\,\mathrm{MeV}$ is comparable to the muon mass $m_\mu = 105.658\,\mathrm{MeV}$.

Cascade–type analysis have been quite popular some time ago in the strong interaction sector when analyzing the decay chains of the strong interaction baryonic and mesonic resonances (see e.g. [69–72]). In the weak interaction sector cascade–type analysis were applied before to nonleptonic decays [73–80], to semileptonic decays of heavy mesons and baryons [74, 76, 77, 81–88], and to rare decays of heavy mesons [88, 89] and heavy baryons [90]. A new feature appears in semileptonic decays compared to nonleptonic decays when one includes lepton mass effects. In this case one has new interference contributions coming from the time-components of the vector and axial vector currents interfering with the usual three-vector components of the currents (see e.g. [76, 82]).

Although we frequently refer to the specific semileptonic cascade decay $\Xi^0 \to \Sigma^+(\to p + \pi^0) + W^-_{\rm off-shell}(\to \ell^- + \bar{\nu}_l)$ the spin–kinematical analysis presented in this thesis is quite general and can be equally well applied to the semileptonic decays of heavy charm and bottom baryons, and for that matter, also to the semileptonic decay of the top quark. In order to facilitate such further applications we have always included the necessary sign changes when going from the $(\ell^-, \bar{\nu}_l)$ case to the (ℓ^+, ν_l) case as occurs e.g. in the semileptonic hyperon decay $\Sigma^+ \to \Lambda + e^+ + \nu_e$, in semileptonic $c \to s$ charm baryon decays or in semileptonic top quark decays [92–95]. When sign changes are indicated the upper sign will always refer to the $(\ell^-, \bar{\nu}_l)$ case, which is the main concern of this thesis, whereas the lower sign will refer to the (ℓ^+, ν_l) case. We also mention that we have always assumed that the amplitudes are relatively real and have therefore dropped azimuthal correlation contributions coming from the imaginary parts. Put in a different language this means that we have not considered T–odd contributions in our angular analysis which could result from final state interaction effects or from truly CP–violating effects. By keeping the imaginary parts in the azimuthal correlation terms one can easily write down the relevant T–odd contributions if needed by using the formulas of this thesis. This is discussed for a specific example in Appendix E.1.

This chapter is structured as follows. In section 5.2 we introduce the helicity amplitudes and relate them to a standard set of invariant form factors. In order to estimate the size of the helicity amplitudes for the $\Xi^0 \to \Sigma^+$ current–induced trasition we provide some simple estimates for the invariant form factors and their q^2–dependence which we shall refer to as the minimal form factor model. We also recount how the two-body decay of a polarized particle is treated in the helicity formalism. This two-body decay enters as a basic building block in our quasi-factorized master formulae in the main text which describe the various cascade–type angular decay distributions presented in this paper. In section 5.3 we derive the unpolarized decay rate written in terms of bilinear forms of the helicity amplitudes. Section 5.4 is devoted to the discussion of the rate ratio $\Gamma(e)/\Gamma(\mu)$ in semileptonic hyperon decays. In section 5.5 we discuss single spin polarization effects including spin–momentum correlation effects between the polarization of the parent baryon and the momenta of the decay products. Section 5.6 treats momentum-momentum correlations between the momenta of the decay products in the cascade decay $\Xi^0 \to \Sigma^+(\to p + \pi^0) + W^-_{\mathrm{off-shell}}(\to \ell^- + \bar{\nu}_l)$ for an unpolarized Ξ^0. Section 5.7 contains our summary and our conclusions.

We collected some technical notes in Appendix E. In Appendix E.1 we go through a specific example and identify a specific T–odd term for the joint angular decay distribution written down in section 5.6. The example is easily generalized to other cases. In Appendix E.2 we list the full five–fold angular decay distribution for the cascade decay $\Xi^0 \to \Sigma^+(\to p + \pi^0) + W^-_{\mathrm{off-shell}}(\to l^- + \bar{\nu}_l)$ for a polarized parent hyperon Ξ^0. The full five–fold angular decay distribution reduces to the other decay distributions presented in this chapter after integration or after setting the relevant parameters to zero.

5.2 The helicity amplitudes

The momenta and masses in the semileptonic hyperon decays are denoted by $B_1(p_1, M_1) \to B_2(p_2, M_2) + \ell(p_\ell, m_\ell) + \nu_\ell(p_\nu, 0)$. For the hadronic transitions described by the helicity amplitudes it is not necessary to distinguish between the cases $(\ell^-, \bar{\nu}_\ell)$ and (ℓ^+, ν_ℓ). The matrix elements of the vector and axial vector currents $J_\mu^{V,A}$ between the spin 1/2 states are written as

$$M_\mu^V =< B_2|J_\mu^V|B_1 >= \bar{u}_2(p_2)\left[F_1^V(q^2)\gamma_\mu + \frac{F_2^V(q^2)}{M_1}\sigma_{\mu\nu}q^\nu + \frac{F_3^V(q^2)}{M_1}q_\mu\right]u_1(p_1)\ , \tag{5.1}$$

$$M_\mu^A =< B_2|J_\mu^A|B_1 >= \bar{u}_2(p_2)\left[F_1^A(q^2)\gamma_\mu + \frac{F_2^A(q^2)}{M_1}\sigma_{\mu\nu}q^\nu + \frac{F_3^A(q^2)}{M_1}q_\mu\right]\gamma_5 u_1(p_1)\ , \tag{5.2}$$

where $q = p_1 - p_2$ is the four–momentum transfer. As in [65] we take

$$\sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \qquad\qquad \gamma_5 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ . \tag{5.3}$$

The other γ matrices are defined as in Bjorken-Drell (see Appendix A.1).

Next we express the vector and axial vector helicity amplitudes $H^{V,A}_{\lambda_2\lambda_W}$ ($\lambda_{1,2} = \pm 1/2$, $\lambda_W = t, \pm 1, 0; \lambda_1 = \lambda_2 - \lambda_W$) in terms of the invariant form factors, where the $\lambda_W = t, \pm 1, 0$ are the helicity components of the $W_{\rm off-shell}$. Since we take into account lepton mass effects the time–component $''t''$ of the four-currents $J_\mu^{V,A}$ needs to be retained. Concerning the transformation properties of the four components of the currents one notes that, in the rest frame of the $W_{\rm off-shell}$ ($\vec{q} = 0$), the three space–components $\lambda_W = \pm 1, 0$ transform as $J = 1$ whereas the time-component transforms as $J = 0$. We employ a short–hand notation such that we always write $\lambda_W = t, \pm 1, 0$ for $\lambda_W = 0(J = 0), \pm 1(J = 1), 0(J = 1)$. Whenever we write $\lambda_W = t$ this has to be understood as $\lambda_W = 0(J = 0)$.

One then needs to calculate the expressions

$$H^{V,A}_{\lambda_2\lambda_W} = M_\mu^{V,A}(\lambda_2)\bar{\epsilon}^{*\mu}(\lambda_W)\ . \tag{5.4}$$

We do not explicitly annote the helicity of the parent hyperon λ_1 in the helicity amplitudes since λ_1 is fixed by the relation $\lambda_1 = \lambda_2 - \lambda_W$. It is very important to detail the phase conventions when evaluating the expression (5.4). This is because the angular decay distributions to be discussed later on contain interference contributions between different helicity amplitudes which depend on the relative signs of the helicity amplitudes. We shall work in the rest frame of the parent baryon B_1 with the daughter baryon B_2 moving in the positive z–direction. The baryon spinors are then given by [96]

$$\begin{aligned} \bar{u}_2(\pm\tfrac{1}{2}, p_2) &= \sqrt{E_2 + M_2}\left(\chi_\pm^\dagger, \frac{\mp|\vec{p}_2|}{E_2 + M_2}\chi_\pm^\dagger\right)\ , \\ u_1(\pm\tfrac{1}{2}, p_1) &= \sqrt{2M_1}\begin{pmatrix} \chi_\pm \\ 0 \end{pmatrix}\ , \end{aligned} \tag{5.5}$$

where $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the usual Pauli two-spinors. For the four polarization four-vectors of the currents we have [96]

$$\begin{aligned} \bar{\epsilon}^\mu(t) &= \frac{1}{\sqrt{q^2}}\,(q_0; 0, 0, -p)\ , \\ \bar{\epsilon}^\mu(\pm 1) &= \frac{1}{\sqrt{2}}\,(0; \pm 1, -i, 0)\ , \\ \bar{\epsilon}^\mu(0) &= \frac{1}{\sqrt{q^2}}\,(p; 0, 0, -q_0)\ , \end{aligned} \tag{5.6}$$

where the bar over the polarization four-vectors reminds one that the m quantum numbers of the currents are quantized along the negativ z–direction. They are obtained from the polarization four-vectors quantized along the positive z–axis by a 180° rotation around the y–axis. Using the spinors (5.5) and the polarization vectors (5.6) one obtains ($\lambda_1 = \lambda_2 - \lambda_W$)

$$
\begin{aligned}
H^V_{\frac{1}{2}t} &= \frac{\sqrt{Q_+}}{\sqrt{q^2}}\Big((M_1 - M_2)F_1^V + q^2/M_1 F_3^V\Big)\,, \\
H^V_{\frac{1}{2}1} &= \sqrt{2Q_-}\Big(-F_1^V - (M_1 + M_2)/M_1 F_2^V\Big)\,, \\
H^V_{\frac{1}{2}0} &= \frac{\sqrt{Q_-}}{\sqrt{q^2}}\Big((M_1 + M_2)F_1^V + q^2/M_1 F_2^V\Big)\,,
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
H^A_{\frac{1}{2}t} &= \frac{\sqrt{Q_-}}{\sqrt{q^2}}\Big(-(M_1 + M_2)F_1^A + q^2/M_1 F_3^A\Big)\,, \\
H^A_{\frac{1}{2}1} &= \sqrt{2Q_+}\Big(F_1^A - (M_1 - M_2)/M_1 F_2^A\Big)\,, \\
H^A_{\frac{1}{2}0} &= \frac{\sqrt{Q_+}}{\sqrt{q^2}}\Big(-(M_1 - M_2)F_1^A + q^2/M_1 F_2^A\Big)\,.
\end{aligned}
\tag{5.8}
$$

We use the abbreviation

$$Q_\pm = (M_1 \pm M_2)^2 - q^2 \tag{5.9}$$

From parity or from an explicit calculation one has

$$
\begin{aligned}
H^V_{-\lambda_2,-\lambda_W} &= H^V_{\lambda_2,\lambda_W}\,, \\
H^A_{-\lambda_2,-\lambda_W} &= -H^A_{\lambda_2,\lambda_W}\,.
\end{aligned}
\tag{5.10}
$$

In order to get a feeling about the size of the helicity amplitudes we make a simple minimal ansatz for the invariant amplitudes at zero momentum transfer using $SU(3)$ symmetry. The analysis is greatly simplified by the fact that the C.G. coefficients for the $(n \to p)$–transition are the same as those for the $(\Xi^0 \to \Sigma^+)$–transition. One thus has $F_1^V(0) = 1$ and $F_1^A(0) = 1.267$. For the magnetic form factor $F_2^V(0)$ we take $F_2^V(0) = M_{\Xi^0}(\mu_p + \mu_n)/(2M_p) = 2.6$ as in [65]. The second class current contributions are set to zero, i.e. we take $F_3^V(0) = F_2^A(0) = 0$. Note that a first class quark current can in principle populate the second class form factors F_3^V and F_2^A. For example, in the covariant spectator quark model calculation of [99, 100] one finds $F_3^V(0) = (M_1 - M_2)/(6M_2)$ and $F_2^A(0) = 0$. However, since these contributions would have to be proportional to the mass difference $M_1 - M_2$ we set them to zero for consistency reasons. For $F_3^A(0)$ we use the Goldberger-Treiman relation $F_3^A(0) = M_{\Xi^0}(M_{\Xi^0} + M_{\Sigma^+})F_1^A(0)/(m_{K^-})^2 = 17.1$ (see e.g. [97]).

For the q^2–dependence of the invariant form factors we take a Veneziano–type ansatz which has given a good description of the q^2–dependence of the electromagnetic form

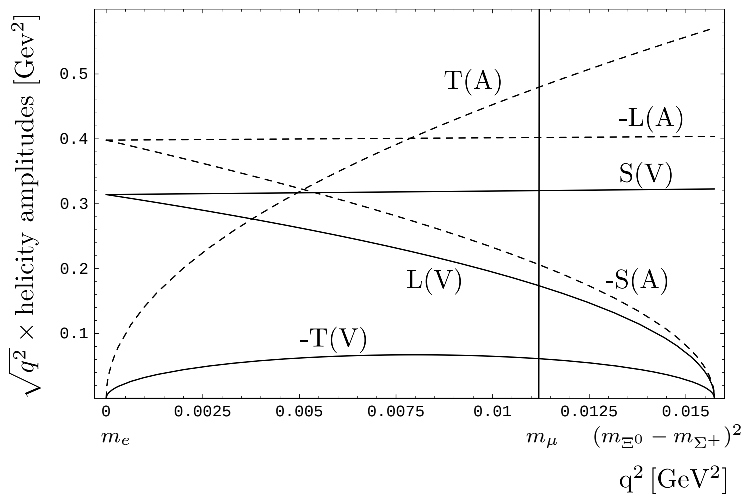


Figure 5.1: q^2–dependence of the six independent helicity amplitudes $S(V,A) := \sqrt{q^2}H^{V,A}_{\frac{1}{2}t}$, $T(V,A) := \sqrt{q^2}H^{V,A}_{\frac{1}{2}1}$ and $L(V,A) := \sqrt{q^2}H^{V,A}_{\frac{1}{2}0}$ multiplied by $\sqrt{q^2}$ for the e-mode (full range) and for the μ–mode (to the right of the vertical line $q^2 = m_\mu^2$).

factors of the neutron and proton [101]. We write

$$F_i^{V,A}(q^2) = F_i^{V,A}(0)\prod_{n=0}^{n_i}\frac{1}{1-\frac{q^2}{m^2_{V,A}+n\alpha'^{-1}}} \approx F_i^{V,A}(0)\left(1+q^2\sum_{n=0}^{n_i}\frac{1}{m^2_{V,A}+n\alpha'^{-1}}\right). \quad (5.11)$$

For $F_1^V(q^2)$ and $F_2^V(q^2)$ we use $m_V = m_{K^*(892)} = 0.892\,\mathrm{GeV}$ which is the lowest lying strange vector meson with $J^P = 1^-$ quantum numbers. Correspondingly we use $m_A = m_{K^*(1.270)} = 1.273 GeV$ $(J^P = 1^+)$ for $F_1^A(q^2)$ and $m_A = m_K = 0.494\,\mathrm{GeV}$ $(J^P = 0^-)$ for $F_3^A(q^2)$, respectively. The slope of the Regge trajectory is taken as $\alpha' = 0.9$ GeV^{-2}. The number of poles in (5.11) is determined by the large q^2 power counting laws. One thus has $n_{0,1} = 1$ and $n_3 = 2$. For the slopes of the form factors we thus have 1.781, 0.983 and 5.241 GeV^{-2} for $F^V_{1,2}, F_1^A$ and F_3^A, respectively. The q^2–dependence of the form factors introduces only small effects since the range of q^2 is so small for the $\Xi^0 \to \Sigma^+$ transitions. For the largest q^2–value at zero recoil, the form factors have only increased by 2.8% $(F^V_{1,2})$, 1.5% (F_1^A) and 8.2% (F_3^A) from their $q^2 = 0$ values.

Based on these estimates for the invariant form factors we show in Fig.5.1 a plot of the q^2–dependence of the six helicity amplitudes. For easier comparability we have plotted the quantities $\sqrt{q^2}H^{V,A}_{\lambda_2\lambda_W}$. Close to the lower boundary $q^2 = m_l^2$ the longitudinal and scalar helicity amplitudes dominate, with $H^V_{\frac{1}{2}0} \approx H^V_{\frac{1}{2}t}$ and $H^A_{\frac{1}{2}0} \approx H^A_{\frac{1}{2}t}$. Close to the upper boundary at the zero recoil point $q^2 = (M_1 - M_2)^2$, which is relevant for the μ–mode, the orbital S–wave contributions $H^V_{\frac{1}{2}t}$ and $H^A_{\frac{1}{2}1} = -\sqrt{2}H^A_{\frac{1}{2}0}$ are the dominant amplitudes.

We emphasize that our ansatz for the form factors is only meant to implement the gross features of the dynamics of the semileptonic hyperon decays $\Xi^0 \to \Sigma^+ + \ell^- + \bar{\nu}_l$ which will eventually be superseded by the results of a careful analysis of the decay data. We shall nevertheless use the above minimal model for the $\Xi^0 \to \Sigma^+$ form factors to calculate the rate ratio $\Gamma(e)/\Gamma(\mu)$ (Sec. 5.4), the longitudinal and transverse polarization of the lepton (Sec. 5.5) and a mean azimuthal correlation parameter (Sec. 5.6) for this decay.

The angular decay distributions in the subsequent sections will be written in terms of the sum of the vector and axial vector helicity amplitudes

$$H_{\lambda_2\lambda_W} = H^V_{\lambda_2\lambda_W} + H^A_{\lambda_2\lambda_W} \ . \tag{5.12}$$

From an inspection of Fig. 5.1 one can see that for the lower q^2–values the combinations $H_{-\frac{1}{2}0}$ and the helicity flip $H_{-\frac{1}{2}0}$ are the prominent helicity amplitudes. For the higher q^2–values the helicity amplitudes $H_{\frac{1}{2}1} \approx -H_{-\frac{1}{2}1}$ become competitive.

Using the helicity formalism we now derive the two–body decay of a polarized spin J particle. Consider the two particle decay $a \to b + c$ of a spin J_a particle where the polarization of particle a in the frame (x_0, y_0, z_0) is given by $\rho^0_{\lambda_a\lambda'_a}$. Consider a second frame (x, y, z) obtained from (x_0, y_0, z_0) by the rotation $R(\theta, \phi, 0)$ and whose z–axis is defined by particle b. The polarization density matrix ρ in the frame (x, y, z) is obtained by a "rotation" of the density matrix ρ^0 from the frame (x_0, y_0, z_0) to the frame (x, y, z). The rate for $a \to b + c$ is then given by the the sum of the decay probabilities $|H_{\lambda_b\lambda_c}|^2$ (with $\lambda_a = \lambda_b - \lambda_c$) weighted by the diagonal terms of the density matrix ρ of particle a in the frame (x, y, z). Thus we find

$$\Gamma_{a\to b+c}(\theta,\phi) \propto \sum_{\lambda_a,\lambda'_a,\lambda_b,\lambda_c} D^{J*}_{\lambda_a,\lambda_b-\lambda_c}(\theta,\phi)\rho^0_{\lambda_a,\lambda'_a} D^J_{\lambda'_a,\lambda_b-\lambda_c}(\theta,\phi)|H_{\lambda_b\lambda_c}|^2 \ , \tag{5.13}$$

where

$$D^J_{m,m'}(\theta,\phi) = e^{-im\phi} d^J_{m\,m'}(\theta) \ . \tag{5.14}$$

In the conventions of Rose [112], for the Wigner d^J–functions, one has

$$J = \frac{1}{2}: \qquad d^{1/2}_{mm'}(\theta) = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} , \tag{5.15}$$

$$J = 1: \qquad d^1_{mm'}(\theta) = \begin{pmatrix} \frac{1}{2}(1+\cos\theta) & -\frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1-\cos\theta) \\ \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{2}(1-\cos\theta) & \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1+\cos\theta) \end{pmatrix} . \tag{5.16}$$

The rows and columns are labeled in the order $(1/2, -1/2)$ and $(1, 0, -1)$, respectively. All the master formulas written down in the subsequent sections can be obtained by a repeated application of the basic two-body formula Eq.(5.13).

5.3 Unpolarized decay rate

The differential decay rate is given by (see e.g. [82])

$$\frac{d\Gamma}{dq^2 dE_\ell} = \frac{G^2}{(2\pi)^3}|V_{us}|^2 \frac{1}{8M_1^2} L_{\mu\nu}H^{\mu\nu} \ , \tag{5.17}$$

where $L_{\mu\nu}$ is the usual lepton tensor ($\varepsilon_{0123} = +1$)

$$L^{\mu\nu} = p_\ell^\mu p_\nu^\nu + p_\ell^\nu p_\nu^\mu - \frac{q^2 - m_\ell^2}{2} g^{\mu\nu} \pm i\varepsilon^{\mu\nu\alpha\beta} p_{l,\alpha} p_{\nu,\beta} \; . \tag{5.18}$$

The hadron tensor $H_{\mu\nu}$ is given by the tensor product of the vector and axial vector matrix elements defined in Eqs.(5.1) and (5.2), cf.

$$H_{\mu\nu} = (M^V + M^A)_\mu \, (M^V + M^A)_\nu^\dagger \; . \tag{5.19}$$

Eq.(5.17) shows that $L_{\mu\nu}H^{\mu\nu}$ determines the dynamical weight function in the (q^2, E_ℓ) Dalitz plot (see e.g. the discussion in [56]). In a Monte Carlo generator one would thus have to generate events according to the weight $L_{\mu\nu}H^{\mu\nu}$ in the (q^2, E_ℓ) Dalitz plot.

The differential q^2–distribution can be obtained from (5.17) by E_ℓ–integration, where the limits are given by (see e.g. [82]) $E_\ell^- \leqslant E_\ell \leqslant E_\ell^+$ with

$$E_\ell^\pm = \frac{1}{2q^2}\Big(q_0(q^2 + m_\ell^2) \pm p(q^2 - m_\ell^2)\Big). \tag{5.20}$$

q_0 is the energy of the off-shell W–boson in the rest system of the parent baryon B_1 given by

$$q_0 = (M_1^2 - M_2^2 + q^2)/(2M_1) \tag{5.21}$$

and p is the the magnitude of three–momentum of the daughter baryon B_2 (or the off-shell W–boson) in the same system given by

$$p = |\vec{p}_2| = \frac{\sqrt{Q_+ Q_-}}{2M_1} = \frac{\sqrt{\lambda(M_1^2, M_2^2, q^2)}}{2M_1} \, , \tag{5.22}$$

where the Källán function $\lambda(a, b, c)$ is defined in (2.3). Finally, in order to get the total rate one has to integrate over q^2 in the integration limits $m_\ell^2 \leqslant q^2 \leqslant (M_1 - M_2)^2$.

On reversing the order of integrations, the differential lepton energy distribution can be obtained from (5.17) by q^2–integration. The relevant integration limits can be obtained from the inverse of (5.20). One obtains (see e.g. [82]) $q_-^2 \leqslant q^2 \leqslant q_+^2$ with

$$q_\pm^2 = \frac{1}{a}(b \pm \sqrt{b^2 - ac}) \, , \tag{5.23}$$

where

$$\begin{aligned}
a &= M_1^2 + m_\ell^2 - 2M_1 E_\ell \, , \\
b &= M_1 E_\ell (M_1^2 - M_2^2 + m_\ell^2 - 2M_1 E_\ell) + m_\ell^2 M_2^2 \, , \\
c &= m_\ell^2 \Big((M_1^2 - M_2^2)^2 + m_\ell^2 M_1^2 - (M_1^2 - M_2^2) 2M_1 E_\ell \Big) \, .
\end{aligned}$$

Using

$$\begin{aligned}
E_\ell(\max) &:= E_{\max} = \frac{M_1^2 - M_2^2 + m_\ell^2}{2M_1} \\
E_\ell(\min) &:= E_{\min} = m_\ell
\end{aligned} \tag{5.24}$$

one can simplify (5.23) to write

$$q_{\pm}^2 = \frac{2M_1^2}{2M_1(E_{\max} - E_\ell) + M_2^2} \left((E_{\max} - E_\ell) \left(E_\ell \pm \sqrt{E_\ell^2 - E_{\min}^2} \right) + \frac{m_\ell^2 M_2^2}{2M_1^2} \right) . \quad (5.25)$$

Finally, in order to get the total rate, one has to integrate over the lepton energy in the limits $m_\ell \leq E_\ell \leq (M_1^2 - M_2^2 + m_\ell^2)/2M_1$.

As it turns out the two–dimensional integration becomes much simpler if one considers the two–fold differential rate w.r.t. the variables q^2 and $\cos\theta$ instead, where θ is the polar angle of the lepton in the (ℓ, ν_ℓ) c.m. system relative to the momentum direction of the $W_{\text{off}-\text{shell}}$. E_ℓ and $\cos\theta$ are related by (see e.g. [82])

$$\cos\theta = \frac{2q^2 E_\ell - q_0(q^2 + m_\ell^2)}{p(q^2 - m_\ell^2)} . \quad (5.26)$$

Differentiating Eq.(5.26) one has

$$\frac{d\cos\theta}{dE_\ell} = \frac{2q^2}{p(q^2 - m_\ell^2)} \quad (5.27)$$

which leads to the differential decay distribution

$$\frac{d\Gamma}{dq^2 d\cos\theta} = \frac{G^2}{(2\pi)^3} |V_{us}|^2 \frac{(q^2 - m_\ell^2)p}{16 M_1^2 q^2} L_{\mu\nu} H^{\mu\nu} . \quad (5.28)$$

It is clear from comparing Eqs.(5.17) and (5.28) that, when writing a Monte Carlo program, one should *not* generate events in the $(q^2, \cos\theta)$ Dalitz plot according to the weight $L_{\mu\nu} H^{\mu\nu}$.

The $\cos\theta$ dependence of $L_{\mu\nu} H^{\mu\nu}$ can be easily worked out by following the methods described in [82] which is based on the completeness relation for the polarization four–vectors

$$\sum_{m,m'=t,\pm,0} \bar{\epsilon}^\mu(m) \bar{\epsilon}^{*\nu}(m') g_{mm'} = g^{\mu\nu} . \quad (5.29)$$

The tensor $g_{mm'} = \text{diag}(+,-,-,-)$ is the spherical representation of the metric tensor where the components are ordered in the sequence $m, m' = t, \pm 1, 0$. One can then rewrite the contraction of the lepton and hadron tensors $L_{\mu\nu} H^{\mu\nu}$ as

$$\begin{aligned} L_{\mu\nu} H^{\mu\nu} &= L^{\mu'\nu'} g_{\mu'\mu} g_{\nu'\nu} H^{\mu\nu} = \sum_{m,m',n,n'} L^{\mu'\nu'} \bar{\epsilon}_{\mu'}(m) \bar{\epsilon}^*_\mu(m') g_{mm'} \bar{\epsilon}^*_{\nu'}(n) \bar{\epsilon}_\nu(n') g_{nn'} H^{\mu\nu} \\ &= \sum_{m,m',n,n'} \left(L^{\mu'\nu'} \bar{\epsilon}_{\mu'}(m) \bar{\epsilon}^*_{\nu'}(n) \right) \left(H^{\mu\nu} \bar{\epsilon}^*_\mu(m') \bar{\epsilon}_\nu(n') \right) g_{mm'} g_{nn'} . \end{aligned} \quad (5.30)$$

We shall refer to the second line of (5.30) as the semi-covariant representation of the angular decay distribution.

One has to remember that Eq.(5.30) refers to the differential rate of the decay of an unpolarized parent hyperon into a daughter baryon whose spin is not observed. This means that one has to take into account the additional conditions $\lambda_1 = \lambda_1'$ and $\lambda_2 = \lambda_2'$ in Eq.(5.30). Angular momentum conservation then implies that not all index pairs $m = m'$ and $n = n'$ in Eq.(5.30) can be realized. Taking angular momentum conservation into account one has diagonal contributions $m = m' = n = n' = t, \pm 1, 0$ as well as nondiagonal contributions with $m = m' = t$ and $n = n' = 0$ and vice versa.

The point of writing $L_{\mu\nu}H^{\mu\nu}$ in the factorized form of Eq.(5.30) is that each of the two factors in the second line of Eq.(5.30) is Lorentz invariant and can thus be evaluated in different Lorentz frames. The leptonic part will be evaluated in the (ℓ, ν_ℓ) CM frame (or $W_{\text{off-shell}}$–rest frame) bringing in the decay angle θ, whereas the hadronic part will be evaluated in the Ξ^- rest frame bringing in the helicity amplitudes defined in section 5.2.

Turning to the (ℓ, ν_ℓ) CM system the lepton momenta in the (x, y, z)–system read (see Fig. 5.4)

$$\begin{aligned} p_\ell^\mu &= (E_\ell\,; p_\ell \sin\theta, 0, -p_\ell \cos\theta)\,, \\ p_\nu^\mu &= p_\ell\,(1, -\sin\theta, 0, \cos\theta)\,. \end{aligned} \tag{5.31}$$

The angle θ is always measured w.r.t. the direction of the lepton ℓ, regardless of whether we are dealing with the $(\ell^-, \bar{\nu}_\ell)$ or the (ℓ^+, ν_ℓ) case. Since the orientation in the (x, y)–plane need not be specified in the present problem we have chosen the lepton momenta to lie in the (x, z)–plane. E_ℓ and p_ℓ are the energy and the magnitude of the three-momentum of the charged lepton in the (ℓ, ν_ℓ) CM system, resp., given by $E_\ell = (q^2 + m_\ell^2)/2\sqrt{q^2}$ and $p_\ell = (q^2 - m_\ell^2)/2\sqrt{q^2}$. The longitudinal and time-component polarization four–vectors take the form $\bar{\epsilon}^\mu(0) = (0; 0, 0, -1)$ and $\bar{\epsilon}^\mu(t) = (1; 0, 0, 0)$ whereas the transverse parts are unchanged from (5.6). Using the explicit form of the lepton tensor Eq.(5.18) it is then not difficult to evaluate (5.30) in terms of the helicity amplitudes $H_{\lambda_2\lambda_W}$ of section 5.2. One obtains

$$\begin{aligned} L_{\mu\nu}H^{\mu\nu} &= \frac{2}{3}(q^2 - m_\ell^2)\Bigg[\frac{3}{8}(1 \mp \cos\theta)^2|H_{\frac{1}{2}1}|^2 + \frac{3}{8}(1 \pm \cos\theta)^2|H_{-\frac{1}{2}-1}|^2 \\ &+ \frac{3}{4}\sin^2\theta(|H_{\frac{1}{2}0}|^2 + |H_{-\frac{1}{2}0}|^2) \\ &+ \frac{m_\ell^2}{2q^2}\Big\{\frac{3}{2}(|H_{\frac{1}{2}t}|^2 + |H_{-\frac{1}{2}t}|^2) + \frac{3}{4}(|H_{\frac{1}{2}1}|^2 + |H_{-\frac{1}{2}-1}|^2)\sin^2\theta \\ &+ \frac{3}{2}\cos^2\theta(|H_{\frac{1}{2}0}|^2 + |H_{-\frac{1}{2}0}|^2) - 3\cos\theta(H_{\frac{1}{2}t}H_{\frac{1}{2}0} + H_{-\frac{1}{2}t}H_{-\frac{1}{2}0})\Big\}\Bigg]\,, \end{aligned} \tag{5.32}$$

where the $H_{\lambda_2\lambda_W}$ are the sums of the corresponding vector and axial vector helicity amplitudes defined in Eq.(5.12). We mention that the helicity flip factor $m_\ell/2q^2$ does not give rise to a singularity since $q^2 \geq m_\ell^2$.

By explicit verification, or by hindsight, one can show that Eq.(5.30) can be written very compactly in terms of Wigner's d^J–functions. One has what we shall refer to as our first master formula

$$L_{\mu\nu}H^{\mu\nu} = \frac{1}{8}\sum_{\lambda_\ell,\lambda_W,\lambda'_W,J,J',\lambda_2}(-1)^{J+J'}|h^l_{\lambda_\ell\lambda_\nu=\pm\frac{1}{2}}|^2 \\ \delta_{\lambda_2-\lambda_W,\lambda_2-\lambda'_W}d^J_{\lambda_W,\lambda_\ell\mp\frac{1}{2}}(\theta)d^{J'}_{\lambda'_W,\lambda_\ell\mp\frac{1}{2}}(\theta)H_{\lambda_2\lambda_W}H^*_{\lambda_2\lambda'_W}\ . \tag{5.33}$$

Except for the phase factor $(-1)^{J+J'}$ the master formula can in fact be derived by repeated application of the basic two–body decay formula in (5.13). The Kronecker δ-function $\delta_{\lambda_2-\lambda_W,\lambda_2-\lambda'_W}$ in (5.33) expresses the fact that we are dealing with the decay of an unpolarized parent hyperon. One has to remember that $\lambda_W = 0$ and $\lambda_W = t$ both refer to the helicity projection 0 (see section 5.2). Therefore there are nondiagonal interference contributions between $J = 1, \lambda_W = 0$ and $J = 0, \lambda_W = t$ because they are allowed by the angular momentum conservation condition $\lambda_2 - \lambda_W = \lambda_2 - \lambda'_W$ implying $\lambda_W = \lambda'_W$. The interference contributions carry an extra minus sign as can be seen from the phase factor $(-1)^{J+J'}$ in (5.33). The phase factor $(-1)^{J+J'}$ comes in because of the pseudo–Euclidean nature of the spherical metric tensor $g_{mm'}$ defined after (5.29).

The sign change in the first line of Eq.(5.32) going from the $(\ell^-,\bar\nu_\ell)$ to the (ℓ^+,ν_ℓ) case can now be seen to result from the products of the relevant elements of the Wigner's d^1–functions. For example, for $\lambda_2 = 1/2$, $\lambda_W = 1$ the nonflip contributions ($\lambda_\ell = -\lambda_\nu = \mp1/2$) are proportional to $(d^1_{1,\mp1})^2 = (\frac{1}{2}(1\mp\cos\theta))^2$. There are no corresponding sign changes in the other lines of Eq.(5.32).

The $h_{\lambda_\ell\lambda_\nu}$ are the helicity amplitudes of the final lepton pair in the (ℓ,ν) c.m. system. For example, for the $(\ell^-,\bar\nu)$ case with $\vec p_{\ell^-}$ along the positive z-axis, they can be worked out by using Eq.(5.5), the negative energy spinor of the massless antineutrino with helicity $\lambda_{\bar\nu} = \frac{1}{2}$ given by

$$v_{\bar\nu}(\tfrac{1}{2}) = \sqrt{E_\nu}\begin{pmatrix}\chi_+\\ -\chi_+\end{pmatrix}, \tag{5.34}$$

and the SM form of the lepton current ($\lambda_W = \lambda_{\ell^-} - \lambda_{\bar\nu}$)

$$h_{\lambda_{\ell^-}=\mp\frac{1}{2}\lambda_{\bar\nu}=\frac{1}{2}} = \bar u_{\ell^-}(\mp\tfrac{1}{2})\gamma^\mu(1+\gamma_5)v_{\bar\nu}(\tfrac{1}{2})\left\{\begin{matrix}\epsilon_\mu(-1)\\ \epsilon_\mu(t),\epsilon_\mu(0)\end{matrix}\right\}. \tag{5.35}$$

We shall refer to the upper case $\lambda_{\ell^-} = -\frac{1}{2}$ as the nonflip transition and to the lower case $\lambda_{\ell^-} = \frac{1}{2}$ as the flip transition. Note the unconventional form of the SM lepton current which is due to the γ_5 definition in section 5.2. The polarization four–vectors are given by $\epsilon^\mu(t) = (1;0,0,0)$, $\epsilon^\mu(0) = (0;0,0,1)$ and $\epsilon^\mu(\pm1) = (0;\mp1,-i,0)/\sqrt{2}$. The flip contribution is identical for $\lambda_W = t$ and $\lambda_W = 0$. A similar expression can be written down for the case (ℓ^+,ν_ℓ) which we shall not work out in explicit form. For the moduli squared of the helicity amplitudes one finally obtains

$$\text{nonflip } (\lambda_{\rm W} = \mp1):\qquad |{\rm h}_{\lambda_\ell=\mp\frac{1}{2}\lambda_\nu=\pm\frac{1}{2}}|^2 = 8(q^2-m_\ell^2) \tag{5.36}$$

$$\text{flip } (\lambda_{\rm W} = {\rm t},0):\qquad |{\rm h}_{\lambda_\ell=\pm\frac{1}{2}\lambda_\nu=\pm\frac{1}{2}}|^2 = 8\frac{m_\ell^2}{2q^2}(q^2-m_\ell^2)\ . \tag{5.37}$$

In Eq.(5.33) the sum over J, J' runs over 0 and 1 and the index λ_W, λ'_W runs over the four components $t, \pm1, 0$. As remarked on before one has to remember to include the interference contribution from $(J=0;\lambda_W=t)$ and $(J=1;\lambda_W=0)$ giving an extra minus sign. The matrix $d^1_{mm'}$, finally, is Wigner's d^1–function ($d^0_{mm'}=1$ for $m,m'=t$) shown in (5.15).

The form Eq.(5.33) affords a ready physics interpretation. $H_{\lambda_2\lambda_W}H^*_{\lambda_2\lambda'_W}$ determines the density matrix of the $W_{\rm off-shell}$ (which happens to be block-diagonal in the present application). The density matrix is then "rotated" into the direction of the lepton in the (ℓ,ν_ℓ) c.m. system with the help of the d^1–functions whence the squared helicity amplitudes $|h_{\lambda_\ell\lambda_\nu}|^2$ determine the helicity dependent rates into the lepton pair.

Performing the sum in (5.33) ($\lambda_\ell=\pm1/2;\lambda_W=t,\pm1,0;J=0,1;J'=0,1;\lambda_2=\pm1/2$) one recalculates Eq.(5.32). Note that the flip contribution proportional to $m_\ell^2/2q^2$ and nonflip contributions are clearly separated in Eq.(5.32). This separation facilitates the determination of the longitudinal polarization of the lepton to be discussed in section 5.5.

The differential rate $d\Gamma/dq^2$ is obtained from Eqs.(5.28) and (5.32) by $\cos\theta$–integration which, in a sense, is trivial. One obtains

$$\begin{aligned}\frac{d\Gamma}{dq^2} &= \frac{1}{3}\frac{G^2}{(2\pi)^3}|V_{us}|^2\frac{(q^2-m_\ell^2)^2p}{8M_1^2q^2}\Bigg[|H_{\frac12 1}|^2+|H_{-\frac12-1}|^2+|H_{\frac12 0}|^2+|H_{-\frac12 0}|^2 \qquad (5.38)\\ &+\frac{m_\ell^2}{2q^2}\Big\{3\Big(|H_{\frac12 t}|^2+|H_{-\frac12 t}|^2\Big)+|H_{\frac12 1}|^2+|H_{-\frac12-1}|^2+|H_{\frac12 0}|^2+|H_{-\frac12 0}|^2\Big\}\Bigg].\end{aligned}$$

We conclude this section with a comment on the relative merits of the two equivalent decay formulas (5.30) and (5.33). In the semi–covariant representation Eq.(5.30) the origin of the phase factor $(-1)^{J+J'}$ is clearly identified. Also, (5.30) does not depend on the phase conventions chosen for the polarisation four-vectors since they always enter in squared form. This is different in the master formula (5.33) and the master formulas written down in the following sections. They depend on the correct choice of phases for the polarisation four-vectors and for the matrix elements of Wigner's d^J–functions. Judging from the fact that there exist different conventions for these phases in the literature the reader can appreciate what a hazardous enterprise it can be to get all the signs correct in the angular decay distributions if one has to rely solely on master formulas without explication of phase conventions. Whereas the signs of the polar correlations can usually be checked by angular momentum considerations there is no easy way to check on the signs of the azimuthal correlations to be discussed in the subsequent sections. In fact, we have repeatedly used the semi–covariant representation Eq.(5.30) to check on the correctness of the phase conventions for the polarisation four-vectors and Wigner's d^J–functions used in the different master formulas in this chapter.

5.4 The rate ratio $\Gamma(e)/\Gamma(\mu)$

We begin our discussion of the rate ratio $\Gamma(e)/\Gamma(\mu)$ in a very simplified setting which, however, gets very close to the correct result. Namely, we use SM–type couplings and set $F_1^V(q^2)=F_1^A(q^2)=1$ and $F_{2,3}^{V,A}(q^2)=0$ in the helicity amplitude relations Eqs.(5.7,5.8). This corresponds to free quark decays in the SM. We then use Eq.(5.38) to derive expressions for the SM differential rate $d\Gamma^{SM}/dq^2$ and the SM integrated rate Γ^{SM}. This affords us the opportunity to check that our rate results agree with known expressions for SM quark decays that are available in the literature.

In order to cast our expressions into compact forms we define following scaled variables:

$$\hat{p} = p/M_1\,, \qquad \hat{q}^2 = q^2/M_1^2\,, \qquad \rho^2 = M_2^2/M_1^2\,, \qquad \eta^2 = m_\ell^2/M_1^2 \tag{5.39}$$

where $\hat{p} = \frac{1}{2}\sqrt{\lambda(1,\hat{q}^2,\rho^2)}$ is the scaled magnitude of the daughter baryon's three momentum in the rest frame of the parent baryon. Also, for compactness we introduce the Born term rate

$$\Gamma_0 = \frac{G^2|V_{us}|^2 M_1^5}{192\pi^3} \tag{5.40}$$

which represents the Standard Model decay of a massive parent fermion into three massles fermions, i.e. $M_1 \neq 0$ and $M_2, m_\ell, m_\nu = 0$.

Using Eq.(5.38) and the SM-type couplings described above one obtains

$$\frac{d\Gamma^{SM}}{d\hat{q}^2} = \Gamma_0 \frac{(\hat{q}^2-\eta^2)^2}{\hat{q}^4} 4\hat{p}\left\{ -2\hat{q}^4 + \hat{q}^2(1+\rho^2) + (1-\rho^2)^2 + \frac{\eta^2}{2\hat{q}^2}\left[-2\hat{q}^4 - 2\hat{q}^2(1+\rho^2) + 4(1-\rho^2)^2 \right] \right\} \tag{5.41}$$

which agrees with the SM result given e.g. in [98].

Integrating over $\hat{q}^2$ $(\eta^2 \leq \hat{q}^2 \leq (1-\rho)^2)$ one obtains

$$\begin{aligned} \Gamma^{SM} &= \Gamma_0 \Bigg(R\left(\frac{1}{2} - 7\rho^2 - 7\rho^4 + \rho^6 + 6\eta^2\rho^2 - 7\eta^4\rho^2\right) \\ &\quad -24\rho^4(1-\eta^4)\ln\left(\frac{1+\rho^2-\eta^2-R}{2\rho}\right) + (\rho \leftrightarrow \eta) \Bigg)\,, \end{aligned} \tag{5.42}$$

where $R = (1+\rho^4+\eta^4-2\rho^2-2\eta^2-2\rho^2\eta^2)^{1/2}$. The symmetrization $\rho \leftrightarrow \eta$ in (5.42) must be done for both the logarithmic and nonlogarithmic terms. The symmetry of the rate expression (5.42) under the exchange $(\rho \leftrightarrow \eta)$ reflects the simple Fierz property of the SM $(V-A)$ coupling.

It is tempting to try and estimate lepton mass effects in semileptonic hyperon decays by using the SM-type rate expression Eq.(5.42) in order to obtain a first estimate of the rate ratio $\Gamma(e)/\Gamma(\mu)$. For the two cases $\Lambda \to p + \ell^- + \bar{\nu}_\ell$ and $\Xi^0 \to \Sigma^+ + \ell^- + \bar{\nu}_\ell$ one obtains

$$\frac{\Gamma^{SM}_{\Lambda\to p}(e)}{\Gamma^{SM}_{\Lambda\to p}(\mu)} = 6.19 \;\; (5.30 \pm 1.18\,[3]) \tag{5.43}$$

$$\frac{\Gamma^{SM}_{\Xi^0\to\Sigma^+}(e)}{\Gamma^{SM}_{\Xi^0\to\Sigma^+}(\mu)} = 118.9 \;\; (55.6^{+22.2}_{-16.7}\,[103]) \tag{5.44}$$

We have added the corresponding experimental ratios and their errors in brackets. For the $\Lambda \to p$ case the SM-type rate ratio is within the error bar of the experimental value whereas for the $\Xi^0 \to \Sigma^+$ case the SM-type rate ratio is off by more than two standard deviations [1].

[1] The NA48 Collaboration cites a preliminary value of 114.1 ± 19.4 [104] for the rate ratio (5.44).

The spin flip contribution to the rate proportional to the charged lepton mass is shown in (5.38). Irrespective to the lepton mass dependence of the matrix element squared (which is expected to be small) the overwhelming effect in the rate reduction when going from the electron to the muon is due to the reduction in the phase space as shown in Fig. 5.2. As a result the rate decreases with increasing charged lepton mass.

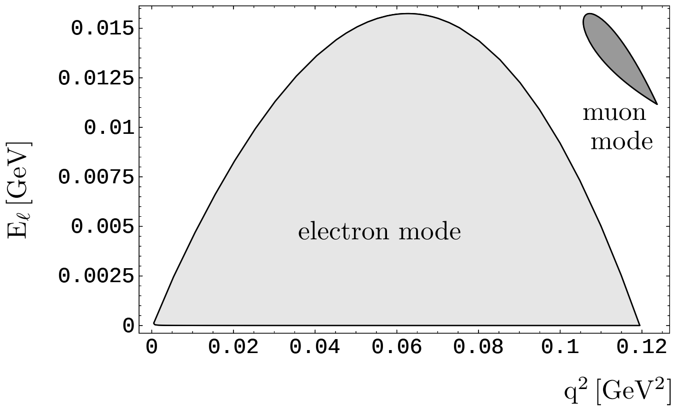


Figure 5.2: The phase spaces for the electron mode and the muon mode in the decay $\Xi \to \Sigma^+ + \ell^- + \bar{\nu}_\ell$.

Even though the SM rate expression Eq.(5.42) incorporates the correct treatment of three-particle phase space one might doubt its validity for a realistic estimate of lepton mass effects in semileptonic hyperon decays since the form factors $F_1^V(q^2)$ and $F_1^A(q^2)$ in semileptonic hyperon decays are not always close to the SM values on which the estimates (5.43, 5.44) are based on. Also, one has neglected a possible q^2 dependence of the form factors $F_1^{V,A}$ as well as the contributions of the form factors $F_{2,3}^{V,A}$. As concerns the omission of the form factors $F_{2,3}^{V,A}$ one notes from Eqs.(5.7) and (5.8) that these contribute only in higher orders of the mass difference $(M_1 - M_2)$. The same holds true for the omission of the q^2 dependence since the kinematical range of q^2 is of the order $(M_1 - M_2)^2$. Altogether, if one neglects the contributions of the form factors $F_{2,3}^{V,A}$ and assumes that the form factors $F_1^{V,A}$ are flat one can show that, up to a very good approximation, the rate ratio $\Gamma(e)/\Gamma(\mu)$ is independent of the actual values of F_1^V and F_1^A. Under these assumptions one obtains [59]

$$\Gamma/\Gamma_0 = \frac{16}{5}(1-\rho)^5(\frac{1+\rho}{2})^3|F_1^V|^2 \left(1 + 3\left(\frac{|F_1^A|}{|F_1^V|}\right)^2\right) r(x) + O(\delta^2)\,, \tag{5.45}$$

where $\delta = (1-\rho)/(1+\rho)$. The function $r(x)$ is given by

$$r(x) = \frac{1}{2}\sqrt{1-x^2}\left(2 - 9x^2 - 8x^4\right) + \frac{15}{4}x^4 \ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\,, \tag{5.46}$$

where $x = \eta/(1-\rho)$. Eq.(5.45) shows that the rate ratio of the $e-$ and μ–modes is independent of the actual values of $F_1^V(0)$ and $F_1^A(0)$ to an approximation of $O(\delta^2 \approx 0.0025)$ when the above assumptions are used. Using Eqs.(5.45) and Eqs.(5.46) one obtains

$\Gamma(e)/\Gamma(\mu) = 6.16$ for $\Lambda \to p+\ell^-+\bar{\nu}_\ell$ and $\Gamma(e)/\Gamma(\mu) = 118.7$ for $\Xi^0 \to \Sigma^+ + \ell^- + \bar{\nu}_\ell$ which is rather close to the SM–type values in (5.43). Finally, using the minimal model for the form factors written down in section 5.2, one finds $\Gamma(e)/\Gamma(\mu) = 118.07$ for $\Xi^0 \to \Sigma^+ + \ell^- + \bar{\nu}_\ell$ which again is quite close to the SM value. All in all one finds that the rate ratio $\Gamma(e)/\Gamma(\mu)$ is not very sensitive to the details of the underlying dynamics. It will be interesting to find out how much the theoretical values of the rate ratios are affected by radiative corrections.

5.5 Single spin polarization effects

5.5.1 Polarization of the daughter baryon

The lepton-hadron contraction $L_{\mu\nu}H^{\mu\nu}$ given in Eqs.(5.32) and (5.33) can be separated into contributions of positive and negative helicities of the daughter baryon denoted by $L_{\mu\nu}H^{\mu\nu}_{\pm\pm}$. They are given by

$$\begin{aligned} L_{\mu\nu}H^{\mu\nu}_{++}(\theta) &= \frac{2}{3}(q^2-m_\ell^2)\left[\frac{3}{8}(1\mp\cos\theta)^2|H_{\frac{1}{2}1}|^2 + \frac{3}{4}\sin^2\theta|H_{\frac{1}{2}0}|^2\right. \\ &+ \frac{m_\ell^2}{2q^2}\left\{\frac{3}{2}|H_{\frac{1}{2}t}|^2 + \frac{3}{4}|H_{\frac{1}{2}1}|^2\sin^2\theta\right. \\ &\left.\left. + \frac{3}{2}\cos^2\theta|H_{\frac{1}{2}0}|^2 - 3\cos\theta H_{\frac{1}{2}t}H_{\frac{1}{2}0}\right\}\right] \end{aligned} \tag{5.47}$$

$$\begin{aligned} L_{\mu\nu}H^{\mu\nu}_{--}(\theta) &= \frac{2}{3}(q^2-m_\ell^2)\left[\frac{3}{8}(1\pm\cos\theta)^2|H_{-\frac{1}{2}-1}|^2 + \frac{3}{4}\sin^2\theta|H_{-\frac{1}{2}0}|^2\right. \\ &+ \frac{m_\ell^2}{2q^2}\left\{\frac{3}{2}|H_{-\frac{1}{2}t}|^2 + \frac{3}{4}|H_{-\frac{1}{2}-1}|^2\sin^2\theta\right. \\ &\left.\left. + \frac{3}{2}\cos^2\theta|H_{-\frac{1}{2}0}|^2 - 3\cos\theta H_{-\frac{1}{2}t}H_{-\frac{1}{2}0}\right\}\right] \end{aligned} \tag{5.48}$$

This allows one to compute the component P_z of the polarization vector along the direction of $\vec{p}_2$ in the rest system of B_1. One obtains

$$P_z(\theta) = \frac{L_{\mu\nu}H^{\mu\nu}_{++} - L_{\mu\nu}H^{\mu\nu}_{--}}{L_{\mu\nu}H^{\mu\nu}_{++} + L_{\mu\nu}H^{\mu\nu}_{--}} \tag{5.49}$$

In a similar vein the polarization of the daughter baryon in the x–direction can be obtained from Eq.(5.33) by leaving the helicity label λ_2 unsummed. The product of helicity amplitudes now reads $H_{\lambda_2\lambda_W}H^*_{\lambda_2'\lambda_W'}$ and the δ–function turns into $\delta_{\lambda_2-\lambda_W,\lambda_2'-\lambda_W'}$ because, again, the parent baryon is taken to be unpolarized. As before, $\lambda_W = t$ and $\lambda_W = 0$ have zero helicity but transform as $J=1$ and $J=0$, respectively. One obtains

$$P_x(\theta) = \frac{2L_{\mu\nu}H^{\mu\nu}_{+-}}{L_{\mu\nu}H^{\mu\nu}_{++} + L_{\mu\nu}H^{\mu\nu}_{--}}, \tag{5.50}$$

where

$$2L_{\mu\nu}H^{\mu\nu}_{+-}(\theta) \;=\; -\frac{2}{3}(q^2 - m_\ell^2) \tag{5.51}$$
$$\Bigg[\frac{3}{2\sqrt{2}}\sin\theta(\pm 1 - \cos\theta)H_{\frac{1}{2}1}H_{-\frac{1}{2}0} + \frac{3}{2\sqrt{2}}\sin\theta(\pm 1 + \cos\theta)H_{\frac{1}{2}0}H_{-\frac{1}{2}-1}$$
$$+\frac{m_\ell^2}{2q^2}\Big\{\frac{3}{\sqrt{2}}\sin\theta\cos\theta(H_{\frac{1}{2}1}H_{-\frac{1}{2}0} - H_{\frac{1}{2}0}H_{-\frac{1}{2}-1}) - \frac{3}{\sqrt{2}}\sin\theta(H_{\frac{1}{2}1}H_{-\frac{1}{2}t} - H_{\frac{1}{2}t}H_{-\frac{1}{2}-1})\Big\}\Bigg] .$$

Of course, if one does not define a transverse reference direction the specification of P_x does not make physical sense per se. Such a transverse reference direction is e.g. provided by the transverse momemtum of the lepton in the semileptonic decay. In fact, we shall see in section 5.6 how the density matrix of the daughter baryon enters the joint angular decay distribution of the cascade decay $\Xi^0 \to \Sigma^+(\to p+\pi^+)+\ell^-+\bar{\nu}_\ell$ where the transverse reference direction is defined by the decay $\Sigma^+ \to p+\pi^+$. The polarization component P_y is zero because we assume that the invariant amplitudes and thereby the helicity amplitudes are relatively real.

5.5.2 Polarization of the lepton

The lepton–side flip– and nonflip–contributions to $L_{\mu\nu}H^{\mu\nu}$ are clearly identifiable as can be seen by an inspection of Eqs.(5.33) and (5.38). One can thus directly write down the longitudinal polarization of the lepton for the decay of an unpolarized parent hyperon at no extra cost. One has

$$P_z^{(\ell)} = \pm\frac{L_{\mu\nu}H^{\mu\nu}(\text{flip}) - \mathrm{L}_{\mu\nu}\mathrm{H}^{\mu\nu}(\text{nonflip})}{L_{\mu\nu}H^{\mu\nu}(\text{flip}) + \mathrm{L}_{\mu\nu}\mathrm{H}^{\mu\nu}(\text{nonflip})} \tag{5.52}$$

For the decay $\Xi^0 \to \Sigma^+ + \ell^- + \bar{\nu}_\ell$ the longitudinal polarization of the electron is $\approx -100\%$ over most of the range of q^2 because $m_e \approx 0$. This changes only for q^2–values very close to the threshold $q^2 = m_e^2$. For the μ^-–mode the longitudinal polarization is quite small and negative and remains below $\approx -30\%$ over the whole q^2–range as Fig.5.3 shows. On average one has $\langle P_z^{(\mu^-)}\rangle = -0.16$. Judging from the fact that $P_z^{(\mu^-)}$ is small the helicity flip and nonflip contributions are of almost equal importance for the μ^-–mode.

It is important to realize that the longitudinality of the polarization $P_z^{(\ell)}$ is defined w.r.t. the momentum direction of the lepton in the (ℓ, ν_ℓ) c.m. system and *not* w.r.t. the momentum direction of the lepton in the rest system of the parent baryon Ξ^0. If one needs to avail of the longitudinal polarization in the latter frame this can also be done using the helicity method as has been shown in [85].

As before, the transverse polarization of the lepton can also be obtained from Eq.(5.33) by leaving the helicity lable λ_ℓ in (5.33) unsummed. One then obtains the density matrix of the lepton which we write as $(L_{\mu\nu}H^{\mu\nu})_{\lambda_\ell\lambda_{l'}}$. This allows one to extract also the transverse polarization of the lepton $P_x^{(\ell)}$. One obtains [98]

$$P_x^{(\ell)}(\theta) = \frac{2(L_{\mu\nu}H^{\mu\nu})_{+-}}{(L_{\mu\nu}H^{\mu\nu})_{++} + (L_{\mu\nu}H^{\mu\nu})_{--}}\,. \tag{5.53}$$

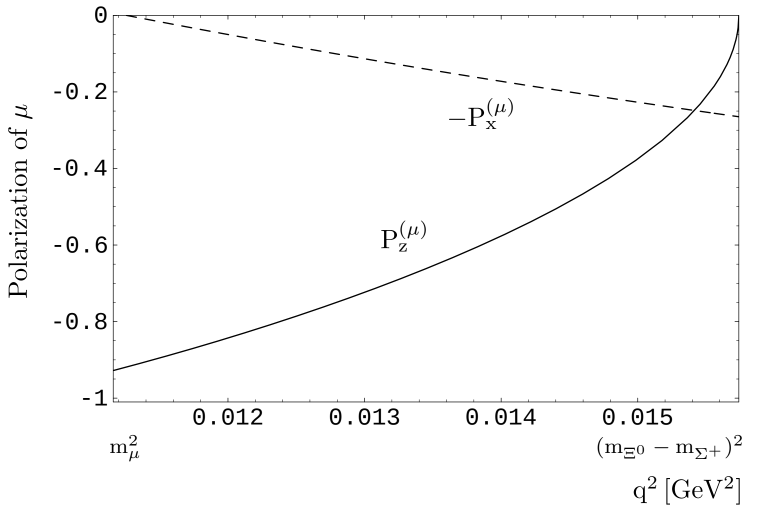


Figure 5.3: Longitudinal and transverse polarization of the μ^- in the $(\mu^-, \bar{\nu}_\mu)$ c.m. system.

In order to evaluate (5.53) for the $(\ell^-, \bar{\nu})$ case one needs the relation $h_{\frac{1}{2}\frac{1}{2}} = \sqrt{m_\ell^2/2q^2}\; h_{-\frac{1}{2}\frac{1}{2}}$. In Fig.(5.3) we show the q^2–dependence of the transverse polarization of the μ^- in the decay $\Xi^0 \to \Sigma^+ + \ell^- + \bar{\nu}_\ell$. The transverse polarization starts off at rather high positive values close to $q^2_{\rm min} = m_\mu^2$ and drops to zero at the zero recoil point $q^2_{\rm max} = (M_1 - M_2)^2$. For the e^- the transverse polarization is practically zero over the whole q^2–range. Because of the lack of structure in the e^-–case we do not show a plot of the polarization of the electron.

5.5.3 Decay of a polarized parent baryon

In this subsection we consider the decay of a polarized parent baryon and in turn determine the angular decay distributions of the leptonic side and the hadronic side relative to the polarization of the parent baryon. The polarization of the parent baryon is described by the density matrix

$$\rho_{\lambda_1\lambda_1'} = \frac{1}{2}\begin{pmatrix} 1 + \mathrm{P}\cos\theta_\mathrm{P} & \mathrm{P}\sin\theta_\mathrm{P} \\ \mathrm{P}\sin\theta_\mathrm{P} & 1 - \mathrm{P}\cos\theta_\mathrm{P} \end{pmatrix} \tag{5.54}$$

where we have assumed that the polarization vector of the parent baryon lies in the (x, z)–plane with positive x–component as shown in Figs. 5.4 and 5.5. The rows and columns in the matrix (5.54) are labeled in the order $(1/2, -1/2)$.

Lepton side as polarization analyzer

The angular decay distribution is a straightforward generalization of Eq.(5.32) where one now has to include the density matrix of the decaying parent baryon B_1. Also, the rotation of the density matrix of the $W_{\rm off-shell}$ into the direction of the lepton now involves also the azimuthal angle χ. This brings in the phase factor $e^{i(\lambda_W - \lambda_W')(\pi-\chi)}$. The appropiate angle entering the phase factor is $(\pi - \chi)$ since the azimuthal angle has to be specified in

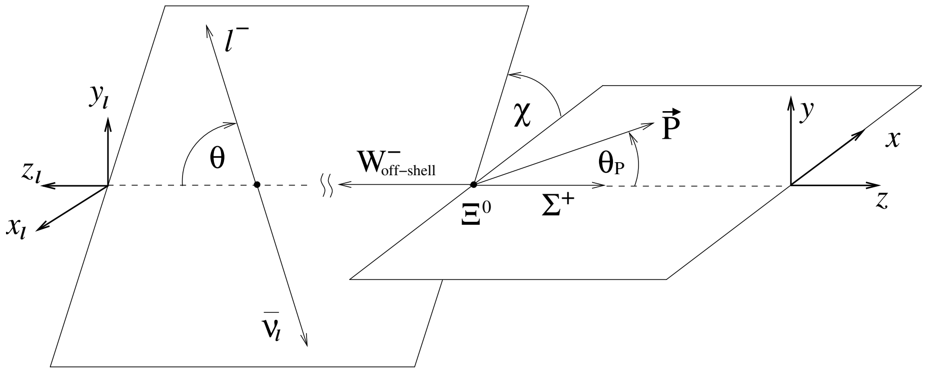


Figure 5.4: Definition of the polar angles θ and θ_P, and the azimuthal angle χ describing the decay of a polarized Ξ^0 using the lepton side as polarization analyzer. $\vec{P}$ denotes the polarization vector of the Ξ^0. The coordinate system (x_ℓ, y_ℓ, z_ℓ) is obtained from the coordinate system (x, y, z) by a $180°$ rotation around the y–axis.

the leptonic (x_ℓ, y_ℓ)–plane (see Fig. 5.4). Using (5.13) one obtains the master formula

$$W(\theta,\chi,\theta_B) \propto \sum_{\lambda_\ell,\lambda_W,\lambda'_W,J,J',\lambda_2} \rho_{\lambda_2-\lambda_W,\lambda_2-\lambda'_W}(\theta_B)(-1)^{J+J'}|h^l_{\lambda_\ell\lambda_\nu=\pm 1/2}|^2 e^{i(\lambda_W-\lambda'_W)(\pi-\chi)} d^J_{\lambda_W,\lambda_\ell-\lambda_\nu}(\theta) d^{J'}_{\lambda'_W,\lambda_\ell-\lambda_\nu}(\theta) H_{\lambda_2\lambda_W} H^*_{\lambda_2\lambda'_W} \tag{5.55}$$

where $\lambda_\nu = \pm 1/2$ ($\lambda_\nu = 1/2$ for $(\ell^-, \bar{\nu}_\ell)$ and $\lambda_\nu = -1/2$ for (ℓ^+, ν_ℓ)).

Doing the helicity sums and putting in the correct normalization one obtains

$$\begin{aligned}
\frac{d\Gamma}{dq^2 d\cos\theta d\chi d\cos\theta_P} &= \frac{1}{6}\frac{G^2}{(2\pi)^4}|V_{us}|^2 \frac{(q^2-m_\ell^2)^2 p}{8M_1^2 q^2} \qquad (5.56)\\
&\Big[\frac{3}{8}(1\mp\cos\theta)^2|H_{\frac12 1}|^2(1-P\cos\theta_P)\\
&+\frac{3}{8}(1\pm\cos\theta)^2|H_{-\frac12 -1}|^2(1+P\cos\theta_P)\\
&+\frac{3}{4}\sin^2\theta\Big(|H_{\frac12 0}|^2(1+P\cos\theta_P)+|H_{-\frac12 0}|^2(1-P\cos\theta_P)\Big)\\
&\pm\frac{3}{2\sqrt2}P\sin\theta\cos\chi\sin\theta_P\Big((1\mp\cos\theta)H_{\frac12 1}H_{\frac12 0}\\
&+(1\pm\cos\theta)H_{-\frac12 -1}H_{-\frac12 0}\Big)\\
&+\frac{m_\ell^2}{2q^2}\Big\{\frac32|H_{\frac12 t}|^2(1+P\cos\theta_P)+\frac32|H_{-\frac12 t}|^2(1-P\cos\theta_P)\\
&-3\cos\theta(H_{\frac12 t}H_{\frac12 0}(1+P\cos\theta_P)+H_{-\frac12 t}H_{-\frac12 0}(1-P\cos\theta_P))\\
&+\frac32\cos^2\theta(|H_{\frac12 0}|^2(1+P\cos\theta_P)+|H_{-\frac12 0}|^2(1-P\cos\theta_P))\\
&+\frac34\sin^2\theta(|H_{\frac12 1}|^2(1-P\cos\theta_P)+|H_{-\frac12 -1}|^2(1+P\cos\theta_P))
\end{aligned}$$

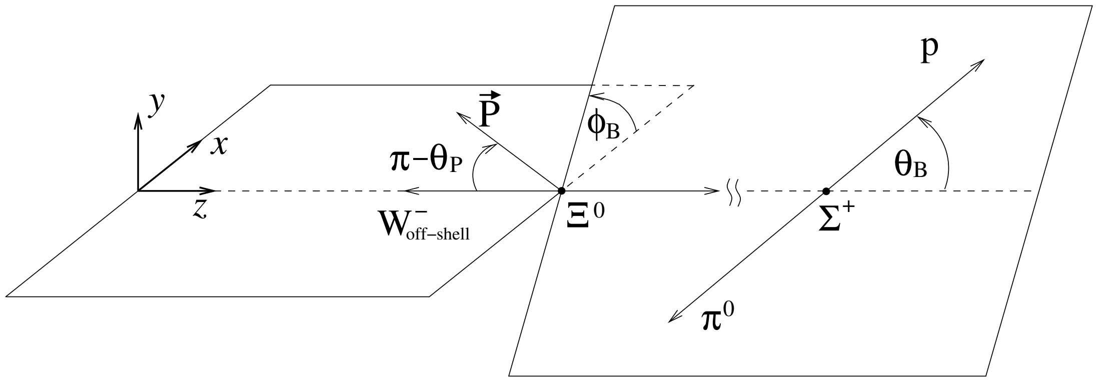


Figure 5.5: Definition of the polar angles θ_B and θ_B and the azimuthal angle ϕ_B in the cascade decay of a polarized Ξ^0 using the hadron side as polarization analyzer

$$
-\frac{3}{\sqrt{2}} P \sin\theta \cos\chi \sin\theta_P (H_{\frac{1}{2}1} H_{\frac{1}{2}t} - H_{-\frac{1}{2}-1} H_{-\frac{1}{2}t})
$$
$$
+\frac{3}{\sqrt{2}} P \sin\theta \cos\theta \cos\chi \sin\theta_P (H_{\frac{1}{2}1} H_{\frac{1}{2}0} - H_{-\frac{1}{2}-1} H_{-\frac{1}{2}0}) \Big\} \Big]
$$

A similar result was published in [91]. However, the signs of the azimuthal correlation terms in [91] do not agree with the corresponding signs in Eq.(5.56).

Hadron side as polarization analyzer

Following the familiar procedure of building up the cascade decay in a quasi-factorized form one obtains the master formula

$$
\begin{aligned}
W(\theta_B, \phi_B, \theta_P) \;\propto\; & \sum_{\lambda_\ell, \lambda_W, \lambda'_W, J, J', \lambda_2, \lambda'_2, \lambda_3} (-1)^{J+J'} \rho_{\lambda_2-\lambda_W, \lambda'_2-\lambda'_W}(\theta_P) H_{\lambda_2\lambda_W} H^*_{\lambda'_2\lambda'_W} \\
& \int_0^{2\pi} d\phi_\ell \int_{-1}^{1} d\cos\theta \; |h^l_{\lambda_\ell\lambda_\nu=\pm 1/2}|^2 e^{i(\lambda_W-\lambda'_W)\phi_\ell} d^J_{\lambda_W, \lambda_\ell-\lambda_\nu}(\theta) d^{J'}_{\lambda'_W, \lambda_\ell-\lambda_\nu}(\theta) \\
& e^{i(\lambda_2-\lambda'_2)\phi_B} d^{\frac{1}{2}}_{\lambda_2\lambda_3}(\theta_B) d^{\frac{1}{2}}_{\lambda'_2\lambda_3}(\theta_B) \; |h^B_{\lambda_3 0}|^2 \;,
\end{aligned}
\tag{5.57}
$$

where the $h^B_{\lambda_3 0}$ are the helicity amplitudes of the decay $B_2 \to B_3 + \pi$. Latter decay is as usual characterized by the asymmetry parameter

$$
\alpha_B = \frac{|h^B_{\frac{1}{2}0}|^2 - |h^B_{-\frac{1}{2}0}|^2}{|h^B_{\frac{1}{2}0}|^2 + |h^B_{-\frac{1}{2}0}|^2} \;. \tag{5.58}
$$

The asymmetry parameter for the nonleptonic decay $\Sigma^+ \to p + \pi^0$ relevant in this calculation is given by $\alpha_B = -0.98^{+0.017}_{-0.015}$ [3]. Note that the phase factor in Eq.(5.57) now is $\exp[\mathrm{i}(\lambda_2 - \lambda'_2)\phi_\mathrm{B}]$ which is appropiate for the azimuthal angle ϕ_B measured relative to the (x, z)–plane (see Fig. 5.5).

Doing the helicity sum and the integration in Eq.(5.57), and putting in the correct normalization one obtains

$$\begin{aligned}
\frac{d\Gamma}{dq^2 d\cos\theta_B d\phi_B d\cos\theta_P} &= B(B_2 \to B_3 + \pi)\frac{1}{12}\frac{G^2}{(2\pi)^4}|V_{us}|^2 \frac{(q^2 - m_\ell^2)^2 p}{8M_1^2 q^2} \\
&\Big[(1+\frac{m_\ell^2}{2q^2})(1+\alpha_B\cos\theta_B)(1-P\cos\theta_P)|H_{\frac{1}{2}1}|^2 \\
&+(1+\frac{m_\ell^2}{2q^2})(1-\alpha_B\cos\theta_B)(1+P\cos\theta_P)|H_{-\frac{1}{2}-1}|^2 \\
&+(1+\frac{m_\ell^2}{2q^2})(1+\alpha_B\cos\theta_B)(1+P\cos\theta_P)|H_{\frac{1}{2}0}|^2 \\
&+(1+\frac{m_\ell^2}{2q^2})(1-\alpha_B\cos\theta_B)(1-P\cos\theta_P)|H_{-\frac{1}{2}0}|^2 \\
&+2P\alpha_B\sin\theta_B\cos\phi_B\sin\theta_P H_{\frac{1}{2}0}H_{-\frac{1}{2}0} \\
+\frac{m_\ell^2}{2q^2}&\Big\{(1+\alpha_B\cos\theta_B)(1+P\cos\theta_P)\,3\,|H_{\frac{1}{2}t}|^2 \\
&+(1-\alpha_B\cos\theta_B)(1-P\cos\theta_P)\,3\,|H_{-\frac{1}{2}t}|^2 \\
&+2P\alpha_B\sin\theta_B\cos\phi_B\sin\theta_P(H_{\frac{1}{2}0}H_{-\frac{1}{2}0} + 3H_{\frac{1}{2}t}H_{-\frac{1}{2}t})\Big\}\Big]\ ,
\end{aligned} \tag{5.59}$$

where $B(B_2 \to B_3 + \pi)$ is the branching fraction of the nonleptonic decay $B_2 \to B_3 + \pi$.

5.6 Joint angular decay distribution

Following the familiar procedure the joint angular decay distribution for the semileptonic cascade decay $B_1 \to B_2(\to B_3 + \pi) + \ell + \nu_\ell$ of an unpolarized parent baryon B_1 can be derived from the master formula

$$\begin{aligned}
W(\theta,\chi,\theta_B) \;\propto\; & \sum_{\lambda_\ell,\lambda_W,\lambda'_W,J,J',\lambda_2,\lambda'_2,\lambda_3} (-1)^{J+J'}|h^l_{\lambda_\ell\lambda_\nu=\pm1/2}|^2 e^{i(\lambda_W-\lambda'_W)(\pi-\chi)} \\
& \delta_{\lambda_2-\lambda_W,\lambda'_2-\lambda'_W} d^J_{\lambda_W,\lambda_\ell-\lambda_\nu}(\theta) d^{J'}_{\lambda'_W,\lambda_\ell-\lambda_\nu}(\theta) H_{\lambda_2\lambda_W} H^*_{\lambda'_2\lambda'_W} \\
& d^{\frac{1}{2}}_{\lambda_2\lambda_3}(\theta_B) d^{\frac{1}{2}}_{\lambda'_2\lambda_3}(\theta_B)|h^B_{\lambda_3 0}|^2\ .
\end{aligned} \tag{5.60}$$

The δ–function in Eq.(5.60) expresses the fact that we are dealing with the decay of an unpolarized parent hyperon which implies $\lambda_2 - \lambda_W = \lambda'_2 - \lambda'_W$.

When writing down the corresponding normalized decay distribution we shall as before assume that the helicity amplitudes are relatively real. One obtains

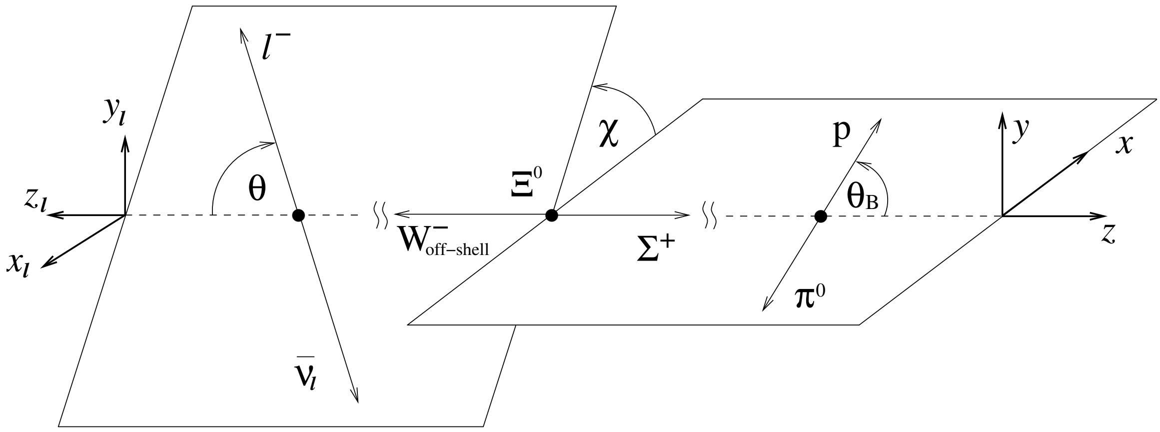


Figure 5.6: Definition of the polar angles θ and θ_B, and the azimuthal angle χ in the joint angular decay distribution of an unpolarized Ξ^0 in the cascade decay $\Xi^0 \to \Sigma^+(\to p+\pi^0)+\ell^-+\bar{\nu}_\ell$. The coordinate system (x_ℓ, y_ℓ, z_ℓ) is obtained from the coordinate system (x,y,z) by a $180°$ rotation around the y–axis.

$$\begin{aligned}
\frac{d\Gamma}{dq^2 d\cos\theta d\chi d\cos\theta_B} &= B(B_2\to B_3+\pi)\frac{1}{6}\frac{G^2}{(2\pi)^4}|V_{us}|^2\frac{(q^2-m_\ell^2)^2 p}{8M_1^2 q^2} \qquad (5.61)\\
&\Big[\frac{3}{8}(1\mp\cos\theta)^2|H_{\frac12 1}|^2(1+\alpha_B\cos\theta_B)\\
&+\frac{3}{8}(1\pm\cos\theta)^2|H_{-\frac12 -1}|^2(1-\alpha_B\cos\theta_B)\\
&+\frac{3}{4}\sin^2\theta(|H_{\frac12 0}|^2(1+\alpha_B\cos\theta_B)+|H_{-\frac12 0}|^2(1-\alpha_B\cos\theta_B))\\
&\pm\frac{3}{2\sqrt2}\alpha_B\sin\theta\cos\chi\sin\theta_B((1\mp\cos\theta)H_{-\frac12 0}H_{\frac12 1}\\
&+(1\pm\cos\theta)H_{\frac12 0}H_{-\frac12 -1})\\
&+\frac{m_\ell^2}{2q^2}\Big\{\frac{3}{2}|H_{\frac12 t}|^2(1+\alpha_B\cos\theta_B)+\frac{3}{2}|H_{-\frac12 t}|^2(1-\alpha_B\cos\theta_B)\\
&-3\cos\theta(H_{\frac12 t}H_{\frac12 0}(1+\alpha_B\cos\theta_B)+H_{-\frac12 t}H_{-\frac12 0}(1-\alpha_B\cos\theta_B))\\
&+\frac{3}{2}\cos^2\theta(|H_{\frac12 0}|^2(1+\alpha_B\cos\theta_B)+|H_{-\frac12 0}|^2(1-\alpha_B\cos\theta_B))\\
&+\frac{3}{4}\sin^2\theta(|H_{\frac12 1}|^2(1+\alpha_B\cos\theta_B)+|H_{-\frac12 -1}|^2(1-\alpha_B\cos\theta_B))\\
&-\frac{3}{\sqrt2}\alpha_B\sin\theta\cos\chi\sin\theta_B(H_{-\frac12 t}H_{\frac12 1}-H_{\frac12 t}H_{-\frac12 -1})\\
&+\frac{3}{\sqrt2}\alpha_B\sin\theta\cos\theta\cos\chi\sin\theta_B(H_{-\frac12 0}H_{\frac12 1}-H_{\frac12 0}H_{-\frac12 -1})\Big\}\Big]\ .
\end{aligned}$$

We have performed several checks on the correctness of the signs of the azimuthal correlation terms by using the semi–covariant representation Eq.(5.30) and by doing a full-fledged covariant calculation. The overall sign of the nonflip azimuthal correlation terms (fifth and sixth line in (5.61)) corrects the sign mistake in [86]. Note the reciprocity of the angular decay distributions Eq.(5.56) and Eq.(5.61). One obtains (5.61) from (5.56) by the substitutions $(1 + sign\{\lambda_2 - \lambda_W\}P\cos\theta_P \to (1 + sign\{\lambda_2\}\alpha_B\cos\theta_B)$ for the polar correlation terms and $P\sin\theta_P H_{\lambda_2\lambda_W}H_{\lambda_2\lambda'_W} \to \alpha_B\sin\theta_B H_{\lambda_2\lambda_W}H_{-\lambda_2\lambda'_W}$ in the azimuthal correlation terms.

Eq.(5.61) can be cast into a form where the dependence on the polarization vector of the daughter baryon becomes explicit. One has

$$\frac{d\Gamma}{dq^2 d\cos\theta d\chi d\cos\theta_B} = B(B_2 \to B_3 + \pi)\frac{1}{4}\frac{G^2}{(2\pi)^4}|V_{us}|^2\frac{(q^2 - m_\ell^2)p}{8M_1^2q^2}L_{\mu\nu}H^{\mu\nu} \qquad (5.62)$$
$$(1 + P_z\alpha_B\cos\theta_B + P_x\alpha_B\cos(\pi - \chi)\sin\theta_B) \ ,$$

where $L_{\mu\nu}H^{\mu\nu}$, P_x and P_z are given in Eqs.(5.32, 5.50) and (5.49), respectively. When integrating Eq.(5.62) over $\cos\theta$, $\cos\theta_B$ and q^2 one can define a mean azimuthal correlation parameter $\langle\gamma\rangle$ through the relation $\Gamma \sim 1 + \langle\gamma\rangle\cos\chi$. Using again the minimal model of section 5.2 one finds the numerical values $\langle\gamma\rangle = -0.14$ and $\langle\gamma\rangle = -0.12$ in the $e-$ and μ–modes, resp., for the mean azimuthal correlation parameter.

At zero recoil on finds a rather simple expression for the above azimuthal correlation parameter. It reads

$$\gamma = \frac{\alpha_B\,\pi^2}{8}\frac{1 - 2\sqrt{2}\,\epsilon_\ell\,H^V_{\frac{1}{2}t}/H^A_{\frac{1}{2}1}}{1 + \epsilon_\ell(1 + 2|H^V_{\frac{1}{2}t}|^2/|H^A_{\frac{1}{2}1}|^2)}\,, \qquad (5.63)$$

where $\epsilon_\ell = m_\ell^2/2q^2$. Eq.(5.63) shows that, in the e–mode and at zero recoil, the azimuthal correlation parameter is a constant independent of the form factors as stated before in [86]. In the μ–mode, however, the azimuthal correlation parameter at zero recoil does depend on the form factors through the ratio $H^V_{\frac{1}{2}t}/H^A_{\frac{1}{2}1}$. Since $H^V_{\frac{1}{2}t}/H^A_{\frac{1}{2}1} \approx F_1^V/(\sqrt{2}F_1^A)$ this would afford the opportunity to determine the ratio F_1^V/F_1^A through a zero recoil measurement of the azimuthal correlation parameter in the μ–mode.

5.7 Summary and conclusions

We have worked out the angular decay distributions that govern the semileptonic cascade decay $\Xi^0 \to \Sigma^+(\to p + \pi^0) + W^-_{\rm off-shell}(\to \ell^- + \bar{\nu}_\ell)$ using a cascade-type analysis. The cascade-type analysis has certain advantages, the main advantage being that one obtains the decay distributions in a compact quasi-factorized form. This leads to rather compact forms for the decay distributions. In our analysis we have included lepton mass effects as well as polarization effects of the decaying parent hyperon. We have always indicated the necessary sign changes when going from the $(\ell^-, \bar{\nu}_\ell)$ case to the (ℓ^+, ν_ℓ) case. Our angular decay formulae are thus applicable also to the semileptonic hyperon decay $\Sigma^+ \to \Lambda + e^+ + \nu_e$, or to semileptonic charm baryon decays induced by the transition $c \to s + \ell^+ + \nu_\ell$ and also to the decays $t \to b + \ell^+ + \nu_\ell$. It should be clear that our angular decay formula are also applicable to the corresponding nonleptonic baryon decays involving vector mesons $(\lambda_W = \pm, 0)$ or pseudoscalar mesons $(\lambda_W = t)$. In this case one has to omit the interference contributions between the time-component and the space-components of the currents.

Of interest are also the corresponding semileptonic antihyperon decays. The angular decay distributions of semileptonic antihyperon decays can be obtained from the corresponding angular decay distributions of the semileptonic hyperon decays by the replacements $H_{\lambda_2\lambda_W}(B) \rightarrow H_{\lambda_2\lambda_W}(\bar{B})$, $\alpha_B \rightarrow \alpha_{\bar{B}}$ and changing from the $(\ell^-, \bar{\nu}_\ell)$ to the (ℓ^+, ν_ℓ) case (or vice versa). Neglecting again CP–violating effects one has from CP-invariance $H_{\lambda_2,\lambda_W}(\bar{B}) = H_{-\lambda_2,-\lambda_W}(B)$ and $\alpha_{\bar{B}} = -\alpha_B$. One can verify that the decay distributions (5.56), (5.59) and (5.61) are form invariant under $H_{\lambda_2\lambda_W} \rightarrow H_{-\lambda_2-\lambda_W}$, $\alpha_B \rightarrow -\alpha_B$ and $P \rightarrow -P$ as follows from CP–invariance.

We have summed over the helicity states of the final particles assuming that their polarization go unobserved. This corresponds to taking the trace of the density matrix of the final particles. It is clear that one can equally well calculate the density matrix of the final state particles by leaving the relevant helicity index unsummed. This was illustrated for the density matrix of the final lepton in the semileptonic decay process.

Doing the helicity sums in the master formulas listed in this thesis by hand can become quite cumbersome. However, this task can be automated using *Mathematica* [116]. To ensure the correctness of our results we checked them by doing full–fledged covariant calculations. We mention also that the helicity frame analysis used in this paper can be easily transcribed to a transversity frame analysis (see e.g. [82]) where the z–axis is perpendicular to the hadron plane . In fact, any choice of z–axis in the analysis will provide the same total amount of information on the dynamics of the process enscribed in the invariant amplitudes. It is then a question of experimental exigiency of whether to analyze angular decay distributions in the helicity frame or the transversity frame, or, for that matter, in any other frame.

For the sake of conciseness we have written our results in terms of bilinear products of the helicity amplitudes $H_{\lambda_2\lambda_W} = H^V_{\lambda_2\lambda_W} + H^A_{\lambda_2\lambda_W}$ instead of bilinear products of the vector and axial vector helicity amplitudes $H^V_{\lambda_2\lambda_W}$ and $H^A_{\lambda_2\lambda_W}$. Writing the decay distributions in terms of $H^V_{\lambda_2\lambda_W}$ and $H^A_{\lambda_2\lambda_W}$ can be quite illuminating if one wishes to identify the overall parity nature of the observables that multiply the angular terms in the angular decay distributions.

The radiative QED corrections to this decay process are much more complicated than the radiative QCD corrections to the decays $t \rightarrow H^+ + b$ and $t \rightarrow b + \ell^+ + \nu_\ell$. calculated in the first part of this thesis. This is so since in semileptonic hyperon decays the virtual photon can couple to both the parent and the daughter hyperon as well as to the charged lepton. The couplings to the parent and the daughter hyperon bring in unknown form factor behaviors. For a discussions of the radiative QED corrections to semileptonic hyperon decays see [65].

Appendix A

Notations

A.1 Dirac matrices

Dirac matrices satisfy the following anticommutation relations[1]:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}. \tag{A.1}$$

It is possible to construct following 16 linearly independent 4×4 matrices

$$\begin{aligned}
\text{one scalar}: \ \Gamma^S &= \mathbb{1}, \\
\text{four vectors}: \ \Gamma^V &= \gamma_\mu, \\
\text{one pseudo scalar}: \ \Gamma^P &= \gamma_5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3, \\
\text{four axial vectors}: \ \Gamma^A &= \gamma_5\gamma_\mu, \\
\text{six tensors}: \ \Gamma^T &= \sigma_{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right].
\end{aligned} \tag{A.2}$$

Note that γ_5 is defined to commute with any of the γ matrices. These 16 linearly independent matrices form the basis for the Clifford space for 4×4 matrices. Any 4×4 matrix can be written in terms of the Γ^n. We can determine the Lorentz transformation properties of the bilinear forms $\bar{\psi}\Gamma^n\psi$ (ψ is a Dirac spinor) constructed from the 16 Γ^n.

The *adjoint spinor* is defined by

$$\bar{\psi} := \psi^\dagger\gamma^0\,. \tag{A.3}$$

The *Dirac conjugation* of a matrix is defined as follows:

$$\bar{\Gamma}^n = \gamma^0(\Gamma^n)^\dagger\gamma^0\,. \tag{A.4}$$

Some frequently used Dirac conjugate matrices are

$$\bar{\gamma}^\mu = \gamma^\mu, \qquad \bar{\gamma}_5 = -\gamma_5\,, \qquad \bar{\sigma}_{\mu\nu} = \sigma_{\mu\nu}\,. \tag{A.5}$$

[1] We follow the conventions of Bjorken, Drell [115].

The scalar product with γ matrices is

$$\not{A} = \gamma \cdot A = \gamma_\mu A^\mu = g_{\mu\nu}\gamma^\mu A^\nu = \gamma^\nu A_\nu = g^{\mu\nu}\gamma_\mu A_\nu = \gamma^0 A^0 - \boldsymbol{\gamma} \cdot \mathbf{A}\,. \tag{A.6}$$

The Dirac representation of the γ–matrices is written as ($\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$)

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{A.7}$$

where the Pauli matrices σ^i are three hermitian , unitary, traceless 2×2-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.8}$$

The Pauli matrices satisfy

$$\sigma_i \sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k, \tag{A.9}$$

where $\mathbb{1}$ is the 2×2 unit matrix, δ_{ij} is the Kronecker symbol, ϵ_{ijk} is the totally antisymmetric third order Levi-Cività tensor.

The trace of a product of odd number of γ–matrices is zero. Some useful trace results are (in four dimensions):

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = 4g^{\mu\nu}, \tag{A.10}$$

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right), \tag{A.11}$$

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right) = 0, \tag{A.12}$$

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4i\epsilon^{\mu\nu\rho\sigma}. \tag{A.13}$$

The totally antisymmetric fourth order Levi-Civitá tensor

$$\epsilon^{0123} = -\epsilon_{0123} = 1 \tag{A.14}$$

gives following contractions

$$\begin{aligned} \epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\mu\nu} &= -24, \\ \epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\mu\rho} &= -6\delta^\nu{}_\rho, \\ \epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} &= -2\left(\delta^\mu{}_\rho\delta^\nu{}_\sigma - \delta^\mu{}_\sigma\delta^\nu{}_\rho\right). \end{aligned} \tag{A.15}$$

For the contractions of the γ-matrices in D–dimensions ($\mu = \{0, 1, 2, \ldots, D-1\}$) one has

$$\begin{aligned} \gamma^\mu\gamma_\mu &= D\mathbb{1}_{D\times D}, \\ \gamma^\mu\gamma^\nu\gamma_\mu &= -\left(D-2\right)\gamma^\nu, \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4g^{\nu\rho} - \left(4-D\right)\gamma^\nu\gamma^\rho, \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + \left(4-D\right)\gamma^\nu\gamma^\rho\gamma^\sigma. \end{aligned} \tag{A.16}$$

A.2 Dirac equation

The Dirac equation

$$i\gamma^\mu \partial_\mu \psi(x) - m\psi(x) = 0 \tag{A.17}$$

is the equation of motion for the relativistic fermion with a mass m and spin 1/2. Dirac spinors $u(p,s)$ and $v(p,s)$ describe the solution of the Dirac equation with positive and negative energy respectively:

$$\begin{aligned} \left(\not{p} - m\right) u(p,s) &= 0, & \left(\not{p} + m\right) v(p,s) &= 0, \\ \bar{u}(p,s)\left(\not{p} - m\right) &= 0, & \bar{v}(p,s)\left(\not{p} + m\right) &= 0. \end{aligned} \tag{A.18}$$

The orthonormality of the spinors gives

$$\begin{aligned} \bar{u}(p,s)u(p,s') &= 2m\delta_{ss'}, & \bar{v}(p,s)v(p,s') &= -2m\delta_{ss'} \\ \bar{u}(p,s)v(p,s') &= 0, & \bar{v}(p,s)u(p,s') &= 0, \end{aligned} \tag{A.19}$$

and the completeness gives

$$\begin{aligned} \sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) &= \sum_s \left[\left(\not{p} + m\right)\left(\frac{\mathbb{1} + \gamma_5 \not{s}}{2}\right)\right]_{\alpha\beta} = \left(\not{p} + m\right)_{\alpha\beta} \\ \sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) &= \sum_s \left[\left(\not{p} - m\right)\left(\frac{\mathbb{1} + \gamma_5 \not{s}}{2}\right)\right]_{\alpha\beta} = \left(\not{p} - m\right)_{\alpha\beta}. \end{aligned} \tag{A.20}$$

In the rest frame of the Dirac particle, the four component spin vector is $s^\mu = (0, \vec{s})$ with

$$s_\mu s^\mu = -1, \qquad p_\mu s^\mu = 0, \qquad \vec{s}^{\,2} = 1. \tag{A.21}$$

The spin vector for a Dirac particle with momentum p can be obtained by a boost and reads

$$s^\mu = \left(\frac{\vec{s}\cdot\vec{p}}{m},\ \vec{s} + \frac{(\vec{s}\cdot\vec{p})\,\vec{p}}{m\,(E+m)}\right). \tag{A.22}$$

A.3 Gell–Mann matrices

It is believed that the strong interaction of quarks follow a local gauge principle with the gauge group $SU(3)$, i.e. the gauge group of QCD is $SU(3)$. The $SU(3)$ symmetry together with the requirement of renormalizability and Lorentz invariance fixes the structure of the QCD action uniquely. The symmetry group $SU(3)$ has $3^2 - 1 = 8$ independent generators. In the fundamental representation these will be traceless 3×3 Hermitian matrices, and since this is rank–2 Lie algebra two of the eight generators may be simultaneously diagonalized. The canonical representation of the group $SU(3)$ is

$$U = e^{i\frac{\lambda_a}{2}\alpha_a} \qquad (a = 1, 2, 3, \ldots, 8), \tag{A.23}$$

where the group generators are $\frac{\lambda_a}{2}$. The standard choice for these generators are the eight Gell–Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{A.24}$$

with λ_3 and λ_8 are chosen as the diagonal generators. The λ_a satisfy the Lie algebra

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = \frac{i}{2} f_{abc}\, \lambda_c \,. \tag{A.25}$$

The structure constants f_{abc} in this representation are completely symmetric under exchange of indices a, b and c. The non–vanishing structure constants in this representation are

$$\begin{aligned} f_{123} &= 1\,, \qquad f_{147} = f_{246} = f_{257} = f_{345} = \frac{1}{2}\,, \\ f_{156} &= f_{367} = -\frac{1}{2} \qquad f_{458} = f_{678} = \frac{\sqrt{3}}{2}. \end{aligned} \tag{A.26}$$

The generators $\frac{\lambda_a}{2}$ are traceless and normalized such that

$$\mathrm{Tr}\left(\frac{\lambda_a}{2}\frac{\lambda_b}{2}\right) = \frac{1}{2}\delta_{ab}\,. \tag{A.27}$$

In QCD calculations color is treated similar to the spin degree of freedom: one averages over initial colors and sums over final colors. Then, as we have seen in (2.39) and (2.133), the Feynman rules will always lead to paired color indices for the squared matrix element. One finds that, for every squared QCD matrix element, the color structure of the particles is reduced to a number which multiplies a colorless squared matrix element. This number is called the *color factor*. We have the following identity for the color sum over the paired color indices of the Gell–Mann matrices:

$$\frac{(\lambda^a)_i^j}{2}\,\frac{(\lambda_a)_k^l}{2} = \frac{1}{2}\left(\delta_i^l\,\delta_k^j - \frac{1}{N_C}\,\delta_i^j\,\delta_k^l\right), \tag{A.28}$$

where a is the color index ($a = 1, 2, 3, \ldots, 8$). N_C is the number of colors. The *color factor* is defined as

$$C_F = \frac{N_C^2 - 1}{2\,N_C}\,. \tag{A.29}$$

For the multiplication of two Gell–Mann matrices one has

$$\frac{(\lambda^a)_i^j}{2}\,\frac{(\lambda_a)_j^l}{2} = \frac{N_C^2-1}{2\,N_C}\,I = C_F\,I\,. \tag{A.30}$$

The color factor takes the value $C_F = \frac{4}{3}$ for $N_C = 3$.

A.4 The CKM matrix

The quark weak eigenstates and the their mass eigenstates are related by a 3×3 unitary matrix called Cabbibo–Kobayashi–Maskawa (CKM) matrix.

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{A.31}$$

This matrix characterizes the strength of flavour-changing weak decays. One possible parameterization of this matrix is realized by using three angles θ_1, θ_2, θ_3 and a phase δ:

$$V = \begin{pmatrix} c_1 & s_1\,c_3 & s_1\,s_3 \\ -\,s_1\,c_2 & c_1\,c_2\,c_3 - s_2\,s_3\,e^{i\delta} & c_1\,c_2\,s_3 + s_2\,c_3\,e^{i\delta} \\ -\,s_1\,s_2 & c_1\,s_2\,c_3 + c_2\,s_3\,e^{i\delta} & c_1\,s_2\,s_3 - c_2\,c_3\,e^{i\delta} \end{pmatrix}, \tag{A.32}$$

with $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$. A precise determination of these matrix elements is one of the most important fields in the theoretical and experimental physics [3].

A.5 Feynman rules

In this section the Feynman rules used in this thesis are listed for the sake of completeness.

A.5.1 Outer lines

incoming fermion : $u(p,k)$,

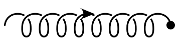
outgoing fermion : $\bar{u}(p,k)$;

incoming antifermion : $\bar{v}(p,k)$,

outgoing antifermion : $v(p,k)$;

incoming boson : $\varepsilon_\mu(k,\lambda)$  ,

outgoing boson : $\varepsilon^*_\mu(k,\lambda)$  ;

incoming gluon : $\varepsilon_\mu(k,\lambda)C^a$  ,

outgoing gluon : $\varepsilon^*_\mu(k,\lambda)(C^a)^*$  ;

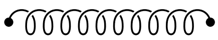
incoming scalar : 1 ; outgoing scalar : 1 ;

A.5.2 Propagators in the Feynman gauge

The $i\epsilon$ prescription for the propagators are implicit.

fermion propagator $\frac{i}{\not{p} - m}$,

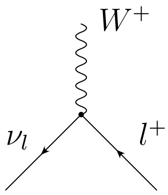
boson propagator  $\frac{i(-g^{\mu\nu} + q^\mu q^\nu / M^2)}{q^2 - M^2}$,

gluon propagator  $\frac{-i g^{\mu\nu}}{k^2}$.

A.5.3 Vertices

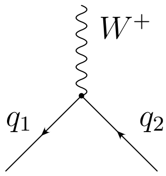
QED vertices

W–boson and lepton vertex:



$$-i \frac{g_w}{\sqrt{2}} \gamma^\mu \frac{1 - \gamma_5}{2} ,$$

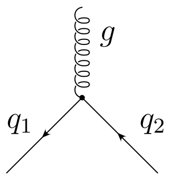
W–boson and quark vertex:



$$-i \frac{g_w}{\sqrt{2}} \gamma^\mu \frac{1 - \gamma_5}{2} V_{q_1 q_2} .$$

QCD vertices

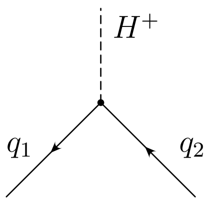
quark–gluon vertex:



$$-i g_s \gamma^\mu \frac{\lambda^a}{2} .$$

MSSM vertices

Charged Higgs and quark vertex:



$$a + b\,\gamma_5 ,$$

where the constants a and b take the following values in the two models discussed in the main text:

$$\boxed{\text{model 1:}} \qquad a = \frac{g_w}{2\sqrt{2}m_W} V_{tb}(m_t - m_b)\cot\beta, \tag{A.33}$$

$$b = \frac{g_w}{2\sqrt{2}m_W} V_{tb}(m_t + m_b)\cot\beta, \tag{A.34}$$

$$\boxed{\text{model 2:}} \qquad a = \frac{g_w}{2\sqrt{2}m_W} V_{tb}(m_t \cot\beta + m_b \tan\beta), \tag{A.35}$$

$$b = \frac{g_w}{2\sqrt{2}m_W} V_{tb}(m_t \cot\beta - m_b \tan\beta). \tag{A.36}$$

Appendix B

Calculation of the loop integrals

A general one–loop integral is defined as[50]

$$\frac{\mu^{4-D}}{i\pi^2}\int d^D k \frac{1}{D_0 D_1 \dots D_{n-1}}\,, \tag{B.1}$$

with the denominators

$$D_0 = k^2 - m_0^2 + i\varepsilon\,, \qquad D_i = (p_i + k)^2 - m_i^2 + i\varepsilon, \qquad i = 1, \dots, n-1. \tag{B.2}$$

The outline of our loop calculation is the following:

- $\frac{1}{D_0 D_2 \dots D_{n-1}} \to \frac{1}{(k^2-M)^n}$ with Feynman parameterization,
- $\int d^D k \frac{1}{(k^2-M)^n} \to \int d^D k \frac{1}{(k^2+M)^n}$ with Wick rotation,
- Finally $\int d^D k \frac{1}{(k^2+M)^n}$ is calculated using Euler's beta function.

B.1 Feynman parameterization

Generally the Feynman parameterization is given by

$$\frac{1}{A_1^{a_1} A_2^{a_2} \cdots A_n^{a_n}} = \frac{\Gamma(a_1 + a_2 + \cdots + a_n)}{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_n)} \int dx_1 dx_2 \cdots dx_n \delta\left(x_1 + x_2 + \cdots + x_n - 1\right) \times$$
$$\times \frac{x_1^{a_1-1}\, x_2^{a_2-1} \dots x_n^{a_n-1}}{(x_1 A_1 + x_2 A_2 \dots x_n A_n)^{a_1+a_2\dots a_n}} \tag{B.3}$$

with the integration limits

$$0 \leqslant x_n \leqslant 1 - \sum_{i=1}^{n-1} x_i, \quad \dots, \quad 0 \leqslant x_2 \leqslant 1 - x_1, \quad 0 \leqslant x_1 \leqslant 1. \tag{B.4}$$

For the two point and three point one–loop integrals we need Feynman parameterizations of the form

$$\frac{1}{AB} = \int_0^1 \frac{dx}{\left[Ax + B\,(1-x)\right]^2}\,, \tag{B.5}$$

$$\frac{1}{ABC} = \int_0^1 dx \int_0^{1-x} \frac{dy}{\left[Ax + By + C\,(1-x-y)\right]^3}\,. \tag{B.6}$$

B.2 Scalar loop integrals

As we discussed the Feynman parameterization changes the loop integrals to

$$\mathcal{I}_n = \int d^D k \frac{1}{(k^2 - M + i\epsilon)^n}\,. \tag{B.7}$$

Let us first calculate this integral to make notations and the techniques clear. We take the space–time dimension D, constant M and parameter ϵ as real numbers and k as D–dimensional vector: $k_\mu = (k_0; k_1, k_2 \ldots k_{D-1})$. In Minkowski space $k^2 = k_0^2 - \vec{k}^2$. The k_0 integration can be performed with the help of Cauchy's residue theorem. The analytic continuation of the k_0–integration can be written as

$$\int_C dk_0 \frac{1}{(k_0^2 - \vec{k}^2 - M + i\epsilon)^n}\,,$$

along a contour C in the complex k_0 plane. The integrand has two poles at $\pm\omega_0 = \pm\sqrt{\vec{k}^2 + M - i\epsilon}$. As shown in Fig B.1 we have chosen the contour C such that it does not enclose any of the poles. Cauchy's residue theorem gives

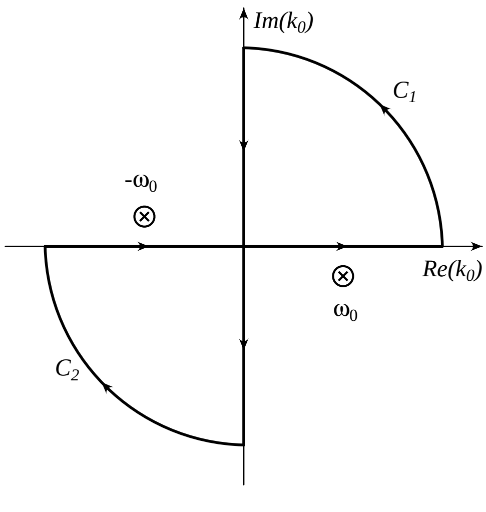


Figure B.1: The contour of the k_0–integration

$$\int_C dk_0 \frac{1}{(k_0^2 - \vec{k}^2 - M + i\epsilon)^n} = 0\,.$$

For the contour integral $\int_C = \int_{-\infty}^{+\infty} + \int_{C_1} + \int_{+i\infty}^{-i\infty} + \int_{C_2}$. We shall eventually let the radii of C_1 and C_2 go to infinity and the contour integrals along C_1 and C_2 vanish in the limit. Then we have

$$\int_{-\infty}^{\infty} dk_0 \frac{1}{(k_0^2 - \vec{k}^2 - M + i\epsilon)^n} = \int_{-i\infty}^{i\infty} dk_0 \frac{1}{(k_0^2 - \vec{k}^2 - M + i\epsilon)^n}\,.$$

Now we perform a change of variables for the integral along the imaginary k_0 axis

$$k^0 \to k^{0'} = i\mathcal{K}^0, \quad k^i \to k^{i'} = \mathcal{K}^i\,,$$

where $\mathcal{K}^0$ and $\mathcal{K}^i$ are components of a Euclidean four vector. This change of variables is called a Wick rotation. Then we have

$$\int_{-\infty}^{\infty} dk_0 \frac{1}{(k_0^2 - \vec{k}^2 - M + i\epsilon)^n} = i \int_{-\infty}^{\infty} d\mathcal{K}_0 \frac{1}{(-{\mathcal{K}_0}^2 - \vec{\mathcal{K}}^2 - M + i\epsilon)^n}\,.$$

The Wick rotation changes the D–dimensional Minkowski vector k into a D–dimensional Euclidean vector $\mathcal{K}$ with $\mathcal{K}^2 = {\mathcal{K}_0}^2 + \vec{\mathcal{K}}^2$. This gives

$$\int_{-\infty}^{\infty} dk_0 \frac{1}{(k_0^2 - \vec{k}^2 - M + i\epsilon)^n} = i(-1)^n \int_{-\infty}^{\infty} d\mathcal{K}_0 \frac{1}{(\mathcal{K}^2 + M)^n}.$$

The parameter ϵ can now be dropped since $\mathcal{K}^2$ and M are positive. Inserting this result into (B.7) we obtain

$$\begin{aligned}
\mathcal{I}_n &= \int d^D k \frac{1}{(k^2 - M + i\epsilon)^n} \\
&= i(-1)^n \int d^D \mathcal{K} \frac{1}{(\mathcal{K}^2 + M)^n} \\
&= i(-1)^n \int d\Omega_D\, \mathcal{K}^{D-1} d\mathcal{K} \frac{1}{(\mathcal{K}^2 + M)^n}\,.
\end{aligned}$$

Note that the integrand has rotational symmetry so the angular integration can be done. Making use of following identity,

$$\begin{aligned}
\left(\sqrt{\pi}\right)^D &= \left(\int_{-\infty}^{+\infty} dx\, e^{-x^2} \right)^D = \int d^D x \exp\left(-\sum_{i=1}^{D} x_i^2 \right) \\
&= \int d\Omega_D \int_0^{\infty} dx\, x^{D-1} e^{-x^2} = \left(\int d\Omega_D \right) \frac{1}{2} \int_0^{\infty} dx^2\, \left(x^2\right)^{\frac{D}{2}-1} e^{-x^2} \\
&= \left(\int d\Omega_D \right) \frac{1}{2} \Gamma\left(\frac{D}{2} \right)\,,
\end{aligned} \tag{B.8}$$

one has

$$\int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}\,. \tag{B.9}$$

Only the radial integral is left. Changing the integration variable $\mathcal{K}$ to $\mathcal{K}^2$ by

$$\mathcal{K}^{D-1}d\mathcal{K} = \frac{1}{2}(\mathcal{K}^2)^{\frac{D}{2}-1}d(\mathcal{K}^2)\,,$$

we have

$$\mathcal{I}_n = i(-1)^n\frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}\int\limits_0^\infty \frac{1}{2}(\mathcal{K}^2)^{\frac{D}{2}-1}d(\mathcal{K}^2)\frac{1}{(\mathcal{K}^2+M)^n}\,. \tag{B.10}$$

This integral is done with the help of Euler's beta function in the form

$$B(m,n) = \int\limits_0^\infty dx\frac{x^{n-1}}{(1+x)^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\,. \tag{B.11}$$

One obtains

$$\mathcal{I}_n = i(-1)^n\pi^{\frac{D}{2}}\frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)}M^{\frac{D}{2}-n}\,. \tag{B.12}$$

Next we turn to the scalar loop integrals.

Scalar one–point one–loop integral $A_0(m^2)$

The result of the scalar one–loop integral can be read off directly from $\mathcal{I}_n$ with $M = m^2$ and $n = 1$:

$$\begin{aligned} A_0(m) &= \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{1}{k^2-m^2} \\ &= \frac{\mu^{4-D}}{i\pi^2}\frac{1}{(2\pi)^{D-4}}(-i)\pi^{\frac{D}{2}}\Gamma\left(1-\frac{D}{2}\right)m^{D-2} \\ &= -m^2\,\Gamma\left(\delta-1\right)\left(\frac{4\pi\mu^2}{m^2}\right)^\delta\,. \end{aligned} \tag{B.13}$$

In the last step the space time dimension D is replaced by $D = 4-2\delta$. Making use of the expansions

$$\begin{aligned} \Gamma(\delta-1) &= -\frac{1}{\delta}+\gamma_E-1+\mathcal{O}(\delta)\,, \\ \left(\frac{4\pi\mu^2}{m^2}\right)^\delta &= 1+\delta\ln(4\pi)+\delta\ln\left(\frac{\mu^2}{m^2}\right)+\mathcal{O}(\delta^2)\,, \end{aligned}$$

where γ_E is the Euler–Mascheroni constant, we finally obtain

$$A_0(m) = m^2\left[\frac{1}{\delta}-\gamma_E+\ln(4\pi)+1+\ln\left(\frac{\mu^2}{m^2}\right)\right] = m^2\left[\Delta+1+\ln\left(\frac{\mu^2}{m^2}\right)\right]\,, \tag{B.14}$$

with

$$\Delta = \frac{1}{\delta} - \gamma_E + \ln(4\pi)\,. \tag{B.15}$$

Note that $A_0(m_g) = 0$ in the $m_g \to 0$ limit.

Scalar two–point one–loop integral $B_0(q, m_t, m_b)$

Scalar two–point one–loop integral is defined in (2.52).

$$
\begin{aligned}
B_0(q,m_t,m_b) &= \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\frac{1}{\left[(q-k)^2-m_t^2\right]\left[k^2-m_b^2\right]}\\
&\downarrow \qquad \text{Feynman parameterization}\\
&= \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\int_0^1 dx\frac{1}{\left[((q-k)^2-m_t^2)x+(k^2-m_b^2)(1-x)\right]^2}\\
&= \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\int_0^1 dx\frac{1}{\left[k^2-2x\,q\,k+x(q^2-m_t^2)-(1-x)m_b^2\right]^2}\\
&= \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\int_0^1 dx\frac{1}{\left[(k-q\,x)^2-q^2x^2+(q^2-m_t^2)x-(1-x)m_b^2\right]^2}\\
&\downarrow \qquad k-q\,x\to k\\
&= \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\int_0^1 dx\frac{1}{\left[k^2-\left(q^2x^2+x(m_t^2-m_b^2-q^2)+m_b^2\right)\right]^2}\\
&\downarrow \qquad \text{Euler's beta function}\\
&= \frac{(2\pi\mu)^{4-D}}{\pi^2}\int_0^1 dx\,\pi^{\frac{D}{2}}\Gamma\left(2-\frac{D}{2}\right)\left[q^2x^2+x(m_t^2-m_b^2-q^2)+m_b^2\right]^{\frac{D}{2}-2}\\
&\downarrow \quad D=4-2\delta\\
&= \frac{(2\pi\mu)^{2\delta}}{\pi^2}\int_0^1 dx\,\pi^{2-\delta}\Gamma(\delta)\left[q^2x^2+x(m_t^2-m_2^2-q^2)+m_b^2\right]^{-\delta}\\
&= \int_0^1 dx\,\Gamma(\delta)\left[\frac{q^2x^2+x(m_t^2-m_b^2-q^2)+m_b^2}{4\pi\mu^2}\right]^{-\delta}
\end{aligned}
$$

$$\downarrow \quad A^{-\delta} = e^{-\delta \ln A} = 1 - \delta \ln A + \mathcal{O}(\delta^2)$$

$$= \int_0^1 dx\, \Gamma(\delta) \Big[1 - \delta \ln \left(\frac{q^2 x^2 + x(m_t^2 - m_b^2 - q^2) + m_b^2}{4\pi\mu^2} \right) + \mathcal{O}(\delta^2) \Big]$$

$$= \frac{1}{\delta} - \gamma_E + \ln 4\pi - \int_0^1 dx\, \ln \left[\frac{q^2}{\mu^2} \left(x^2 + x \frac{m_t^2 - m_b^2 - q^2}{q^2} + \frac{m_2^2}{q^2} \right) \right] + \mathcal{O}(\delta)$$

$$\downarrow \quad \mathcal{O}(\delta) \text{ is dropped;} \quad \Delta = \frac{1}{\delta} - \gamma_E + \ln 4\pi; \quad m^2 \to m^2 - i\varepsilon$$

$$= \Delta - \int_0^1 dx\, \ln \left[\frac{q^2}{\mu^2} \left(x^2 + x \frac{m_t^2 - m_b^2 - q^2}{q^2} + \frac{m_b^2 - i\varepsilon}{q^2} \right) \right] . \tag{B.16}$$

We have written the $i\varepsilon$ prescription explicitly in the last step. This prescription is necessary when the argument of the logarithm is negative. But we have $0 < q^2 \leqslant (m_t - m_b)^2$ in the present calculation, which leads to a positive argument of the logarithm. Therefore we can drop the $i\varepsilon$ prescription.

To perform the x–integration we write the argument of the logarithm as a product.

$$\left(x^2 + x \frac{m_t^2 - m_b^2 - q^2}{q^2} + \frac{m_b^2}{q^2} \right) = (x + v_+)(x + v_-) \,, \tag{B.17}$$

with solutions

$$v_\pm = \frac{m_t^2 - m_b^2 - q^2 \pm \sqrt{\lambda}}{2q^2} \qquad \text{with} \quad \lambda := \lambda(m_t^2, m_b^2, q^2) \,. \tag{B.18}$$

One has

$$\int_0^1 dx\, \ln[\frac{q^2}{\mu^2}(x + v_+)(x + v_-)] = \ln(\frac{q^2}{\mu^2}) + \int_0^1 dx \ln(x + v_+) + \int_0^1 dx \ln(x + v_-)$$
$$+ \int_0^1 dx\, \eta\Big(\frac{q^2}{\mu^2}(x + v_+), (x + v_-) \Big) \,, \tag{B.19}$$

where the η–function comes from the decomposition rule for a logarithm of a product. If the logarithm branch cut is taken along the negative real axis, the decomposition rule is[1]

$$\ln(z_1\, z_2) = \ln z_1 + \ln z_2 + \eta(z_1, z_2) \tag{B.20}$$

with

$$\eta(z_1, z_2) = 2\pi i \, \big[\theta(-\mathrm{Im} z_1) \theta(-\mathrm{Im} z_2) \theta\big(\mathrm{Im}(z_1 z_2)\big) - \theta(\mathrm{Im} z_1) \theta(\mathrm{Im} z_2) \theta\big(- \mathrm{Im}(z_1 z_2)\big) \big] \,, \tag{B.21}$$

[1] See for example [46]

where $\theta(x)$ is the Heaviside Step Function. Note that $v_\pm$ are positive in the present case. Therefore all the η's from the decompositions vanish and we have no imaginary part in the present calculation. Thus we have

$$\int_0^1 dx\, \ln[\frac{q^2}{\mu^2}(x+v_+)(x+v_-)] = \ln(\frac{q^2}{\mu^2}) - 2 + \ln[(1+v_+)(1+v_-)]$$
$$- \frac{v_+ + v_-}{2} \ln\left(\frac{v_+ v_-}{(1+v_+)(1+v_-)}\right) + \frac{v_+ - v_-}{2} \ln\left(\frac{v_+(1+v_-)}{v_-(1+v_-)}\right)$$
$$= -2 + \ln(\frac{q^2}{4\pi\mu^2}) - \frac{m_t^2 - m_b^2 - q^2}{2q^2} \ln\left(\frac{m_t^2}{m_b^2}\right) - \frac{\sqrt{\lambda}}{2q^2} \ln\left(\frac{m_t^2 + m_b^2 - q^2 + \sqrt{\lambda}}{m_t^2 + m_b^2 - q^2 - \sqrt{\lambda}}\right). \tag{B.22}$$

Substituting the above back into (B.16) we obtain

$$B_0(q, m_t, m_b) = \Delta + 2 + \ln\left(\frac{\mu^2}{m_t m_b}\right) + \frac{m_t^2 - m_b^2}{q^2} \ln\left(\frac{m_b}{m_t}\right)$$
$$+ \frac{\sqrt{\lambda}}{2q^2} \ln\left(\frac{m_t^2 + m_b^2 - q^2 + \sqrt{\lambda}}{m_t^2 + m_b^2 - q^2 - \sqrt{\lambda}}\right)$$
$$= \Delta + 2 + \ln\left(\frac{\mu^2}{m_t m_b}\right) + \frac{m_t^2 - m_b^2}{q^2} \ln\left(\frac{m_b}{m_t}\right)$$
$$+ \frac{\sqrt{\lambda}}{2q^2} \ln\left(\frac{m_t^2 + m_b^2 - q^2 + \sqrt{\lambda}}{m_t^2 + m_b^2 - q^2 - \sqrt{\lambda}}\right). \tag{B.23}$$

In the loop calculations we also need $B_0(q, m^2, 0)$, $B_0(m, m, 0)$ and $\frac{\partial}{\partial p^2} B_0(p^2, m, 0)$.

$$B_0(q, m, 0) = \frac{1}{\delta} - \gamma_E - \int_0^1 dx\, \ln\left(\frac{p^2 x^2 + x(m^2 - p^2)}{4\pi\mu^2}\right)$$
$$= \Delta + 2 + \ln\left(\frac{\mu^2}{m^2}\right) + \frac{m^2 - p^2}{p^2} \ln\left(\frac{m^2 - p^2}{m^2}\right), \tag{B.24}$$
$$B_0(m, m, 0) = \Delta + 2 + \ln\left(\frac{\mu^2}{m^2}\right). \tag{B.25}$$

Now we turn to the derivative of B_0. The derivative of B_0 in p^2 affects the powers of the loop integration momentum k and thus the IR and UV behavior of the loop integral. In fact we will see that differentiating w.r.t. p^2 turns the IR–convergent but UV–divergent integral $B_0(p^2, m^2, 0)$ into a IR–divergent but UV–convergent one. Therefore it should be performed before truncating the series expansion in the regularization parameter, e.g δ in our case. We chose to perform the series expansion before the k–integration is done.

$$\frac{\partial}{\partial p^2} B_0(p,m,m_g) = \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\int_0^1 dx\,\frac{\partial}{\partial p^2}\frac{1}{\left[k^2-p^2x^2+(p^2-m^2)x-(1-x)m_g^2\right]^2}$$

$$= \frac{\mu^{4-D}}{i\pi^2}\int\frac{d^Dk}{(2\pi)^{D-4}}\int_0^1 dx\,\frac{x(1-x)}{2\left[k^2-p^2x^2+(p^2-m^2)x-(1-x)m_g^2\right]^3}$$

$$= \frac{(2\pi\mu)^{4-D}}{\pi^2}\int_0^1 dx\,\pi^{\frac{D}{2}}\,\frac{\Gamma\left(3-\frac{D}{2}\right)}{\Gamma(3)}\,\frac{x(1-x)}{\left[p^2x^2+x(m^2-m_g^2-p^2)+m_g^2\right]^{3-\frac{D}{2}}}$$

$\downarrow$ safe to set D=4

$$= \int_0^1 dx\,\frac{x(1-x)}{p^2x^2+x(m^2-m_g^2-p^2)+m_g^2}\,.$$

Using the on–shell condition $p^2 = m^2$, performing the x–integration and expanding the result in terms of m_g^2 around $m_g = 0$ we obtain

$$\frac{\partial}{\partial p^2} B_0(p,m,m_g)\Big|_{p^2=m^2} = \frac{1}{m^2}\Big[-1-\frac{1}{2}\ln\left(\frac{m_g^2}{m^2}\right)+\mathcal{O}(m_g^2)\Big]$$
$$= -\frac{1}{m^2}\Big[1+\ln\left(\frac{m_g}{m}\right)\Big]\,. \qquad \text{(B.26)}$$

Scalar three–point one–loop integral $C_0(q, m_t, m_t, m_b, m_g)$

The scalar three–point one–loop integral is defined in (2.48). Power counting shows that C_0 is UV–convergent, therefore we take $D = 4$. The calculation proceeds in analogy to the evaluation of B_0.

$$C_0 = \frac{1}{i\pi^2}\int d^4k\,\frac{1}{\left[(p_b-k)^2-m_b^2\right]\left[(p_t-k)^2-m_t^2\right]\left(k^2-m_g^2\right)}\,.$$

$$= \frac{1}{i\pi^2}\int d^4k\,\frac{1}{(k^2-2p_t\cdot k)\,(k^2-2p_b\cdot k)\left(k^2-m_g^2\right)}$$

$\downarrow$ Feynman parameterization

$$= \frac{1}{i\pi^2}\int d^4k\int_0^1 dx\int_0^{1-x} dy\,\frac{\Gamma(3)}{\left[x\,(k^2-2p_t\cdot k)+y\,(k^2-2p_b\cdot k)+(1-x-y)\left(k^2-m_g^2\right)\right]^3}$$

$$= \frac{1}{i\pi^2} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[(k - x\,p_t - y\,p_b)^2 - (x\,p_t + y\,p_b)^2 - m_g^2\,(1-x-y)\right]^3}$$

$$= \frac{2}{i\pi^2} \int_0^1 dx \int_0^{1-x} dy \int d^4k \frac{1}{\left[k^2 - (x\,p_t + y\,p_b)^2 - m_g^2\,(1-x-y)\right]^3}$$

$\downarrow$ Euler's beta function

$$= - \int_0^1 dx \int_0^{1-x} dy \frac{1}{(x\,p_t + y\,p_b)^2 + m_g^2\,(1-x-y)} \,. \tag{B.27}$$

We perform a change of integration variables

$$x = uv, \qquad y = u(1-v), \tag{B.28}$$

with its Jacobi determinant

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u \,. \tag{B.29}$$

We obtain

$$C_0 = - \int_0^1 dv \int_0^1 du \frac{u}{u^2[v\,p_t + (1-v)p_b]^2 + m_g^2\,(1-u)}$$

$\downarrow$ for simplicity we define $z = v\,p_t + (1-v)p_b$

$$= - \int_0^1 dv \int_0^1 du \frac{u}{u^2 z^2 + m_g^2\,(1-u)}$$

$$= - \int_0^1 dv \left(\frac{m_g}{z^2\sqrt{4z^2 - m_g^2}} \arctan\left(\frac{2z^2 u - m_g^2}{m_g\sqrt{4z^2 - m_g^2}}\right) + \frac{1}{2z^2} \ln\left(m_g^2(1-u) + z^2u^2\right)\right)\Bigg|_{u=0}^{u=1} . \tag{B.30}$$

We expand the first term up to $\mathcal{O}(m_g)$,

$$\frac{m_g}{z^2\sqrt{4z^2 - m_g^2}} \arctan\left(\frac{2z^2 u - m_g^2}{m_g\sqrt{4z^2 - m_g^2}}\right)\Bigg|_{u=0}^{u=1} = \frac{\pi m_g}{4z^2\sqrt{z^2}} + \mathcal{O}(m_g) \,. \tag{B.31}$$

Therefore the first term in (B.30) vanishes in $m_g = 0$ limit. The second term gives

$$C_0 \;=\; - \int_0^1 dv \, \frac{1}{2\,[v\,p_t + (1-v)p_b]^2} \ln\left(\frac{[v\,p_t + (1-v)p_b]^2}{m_g^2}\right) + \mathcal{O}(m_g) \,.$$

To perform the v–integration we write the argument of the integrand in product form

$$[v\,p_t + (1-v)p_b]^2 = (p_t - p_b)^2(v+v_+)(v+v_-) = q^2(v+v_+)(v+v_-)$$

with

$$v_\pm = \frac{m_t^2 - m_b^2 - q^2 \pm \sqrt{\lambda}}{2q^2}, \quad q = p_t - p_b, \quad \lambda = \lambda(m_t^2, m_b^2, q^2)\,. \tag{B.32}$$

Similar to the case of B_0, after the decomposition, the arguments of the logarithms are positive in the present calculation. Therefore no η–functions will appear and there is no imaginary part.

Then one has (terms of $\mathcal{O}(m_g)$ are dropped)

$$\begin{aligned}
C_0 &= -\frac{1}{2}\int_0^1 dv\,\frac{1}{q^2(v+v_+)(v+v_-)}\ln\left[\frac{q^2(v+v_+)(v+v_-)}{m_g^2}\right] \\
&= -\frac{1}{2q^2}\int_0^1 dv\left[\ln\left(\frac{q^2}{m_g^2}\right) + \ln(v+v_+) + \ln(v+v_-)\right] \\
&= -\frac{1}{2q^2}\frac{1}{v_+-v_-}\left\{\ln\left(\frac{v_+(1+v_-)}{v_-(1+v_+)}\right)\ln\left(\frac{m_t m_b}{m_g^2}\right)\right. \\
&+ \left.\ln\left(\frac{v_+}{1+v_+}\right)\ln\left(\frac{v_+(1+v_+)}{v_-(1+v_-)}\right) - 2\mathrm{Li}_2\left(\frac{v_+-v_-}{1+v_+}\right) + 2\mathrm{Li}_2\left(\frac{v_+-v_-}{v_+}\right)\right\} \\
&= -\frac{1}{\sqrt{\lambda(m_t^2, m_b^2, m_g^2)}}\left[\ln\left(\frac{m_t^2+m_b^2-q^2+\sqrt{\lambda}}{m_t^2+m_b^2-q^2-\sqrt{\lambda}}\right)\ln\left(\frac{m_t m_b}{m_g^2}\right)\right. \\
&+ \ln\left(\frac{m_t^2+m_b^2-q^2+\sqrt{\lambda}}{2m_t^2}\right)\ln\left(\frac{(m_t^2-m_b^2+\sqrt{\lambda})^2-(q^2)^2}{(m_t^2-m_b^2-\sqrt{\lambda})^2-(q^2)^2}\right) \\
&- \left.2\mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{m_t^2-m_b^2+q^2+\sqrt{\lambda}}\right) + 2\mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{m_t^2-m_b^2-q^2+\sqrt{\lambda}}\right)\right].
\end{aligned}$$

In terms of the scaled kinetic variables we write

$$\begin{aligned}
C_0(q, m_t, m_t, m_b, m_g) = -\frac{1}{4\,\bar{p}_3}\Bigg\{ &\left(\ln\epsilon + \bar{Y}_p\right)\left(2\,\bar{Y}_p + 4\,\bar{Y}_w\right) + 2\,\left(\ln\epsilon - 2\,\ln\Lambda\right)\,\bar{Y}_p \\
&- 2\,\mathrm{Li}_2\!\left(1 - \frac{\bar{w}_-}{\bar{w}_+}\right) + 2\,\mathrm{Li}_2\!\left(1 - \frac{\bar{p}_-\,\bar{w}_-}{\bar{p}_+\,\bar{w}_+}\right)\Bigg\}\,.
\end{aligned} \tag{B.33}$$

Appendix C

Polylogarithm

The polylogarithm is defined by

$$\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n} \qquad (|x| \leqslant 1)\,. \tag{C.1}$$

An alternative definition is

$$\mathrm{Li}_n(x) = \int_0^x \frac{\mathrm{Li}_{n-1}(t)}{t} dt\,, \tag{C.2}$$

with

$$\mathrm{Li}_1(x) = -\ln(1-x). \tag{C.3}$$

The calculations in this thesis involve dilogarithms (dilogs) $\mathrm{Li}_2(x)$ and trilogarithms (trilogs) $\mathrm{Li}_3(x)$:

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t}\, dt = \sum_{k=1}^{\infty} \frac{x^k}{k^2}\,, \qquad (|x| \leqslant 1)\,. \tag{C.4}$$

$$\mathrm{Li}_3(x) = \int_0^x \frac{\mathrm{Li}_2(t)}{t}\, dt \tag{C.5}$$

In the following we list some frequently used dilog identities. A more detailed discussion about polylogarithm can be found in [58].

Dilog identities with one variable:

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \ln(x)\,\ln(1-x) \tag{C.6}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(-\frac{x}{1-x}\right) = -\frac{1}{2}\ln^2(1-x)\,, \qquad x < 1 \tag{C.7}$$

$$\mathrm{Li}_2(1-x) - \mathrm{Li}_2\left(\frac{1}{x}\right) = \frac{1}{2}\ln(x)\,\ln\left[\frac{x}{(x-1)^2}\right] - \frac{\pi^2}{6}\,, \qquad x > 1 \tag{C.8}$$

$$\mathrm{Li}_2\left(\frac{1}{1+x}\right) - \mathrm{Li}_2(-x) = \frac{\pi^2}{6} - \frac{1}{2}\ln(1+x)\ln\left(\frac{1+x}{x^2}\right) \tag{C.9}$$

$$\mathrm{Li}_2(-x) + \mathrm{Li}_2\left(-\frac{1}{x}\right) = -\frac{1}{2}\ln^2(x) - \frac{\pi^2}{6}\,, \qquad x > 0 \tag{C.10}$$

$$\mathrm{Li}_2(-x) - \mathrm{Li}_2(1-x) = -\frac{1}{2}\mathrm{Li}_2\left(\frac{1}{x^2}\right) + \ln(x)\ln\left(\frac{x-1}{x}\right)\,, \qquad x > 1 \tag{C.11}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \frac{1}{2}\mathrm{Li}_2(x^2) \tag{C.12}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{1}{x}\right) = -\frac{1}{2}\ln^2(x) + \frac{\pi^2}{3} - i\pi\ln(x)\,, \qquad x > 1 \tag{C.13}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{x}{x-1}\right) = -\frac{1}{2}\ln^2(x-1) + \frac{\pi^2}{2} + i\pi\ln\left(\frac{x-1}{x^2}\right), \quad x > 1 \tag{C.14}$$

$$\begin{aligned}\mathrm{Li}_2(x) - \mathrm{Li}_2(-x) &= \mathrm{Li}_2\left(\frac{x-1}{x+1}\right) - \mathrm{Li}_2\left(\frac{1-x}{1+x}\right) \\ &\quad + \ln\left(\frac{1+x}{1-x}\right)\ln(x) + \frac{\pi^2}{4}\,.\end{aligned} \tag{C.15}$$

Dilog identities with two variables:

The Abel identity:

$$\begin{aligned}\mathrm{Li}_2\left(\frac{x}{1-x}\,\frac{y}{1-y}\right) &= \mathrm{Li}_2\left(\frac{x}{1-y}\right) + \mathrm{Li}_2\left(\frac{y}{1-x}\right) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) \\ &\quad - \ln(1-x)\ln(1-y)\end{aligned} \tag{C.16}$$

The Hill identity:

$$\begin{aligned}\mathrm{Li}_2(x\,y) &= \mathrm{Li}_2(x) + \mathrm{Li}_2(y) - \mathrm{Li}_2\left[\frac{x\,(1-y)}{1-x\,y}\right] - \mathrm{Li}_2\left[\frac{y\,(1-x)}{1-x\,y}\right] \\ &\quad - \ln\left(\frac{1-x}{1-x\,y}\right)\ln\left(\frac{1-y}{1-x\,y}\right)\end{aligned} \tag{C.17}$$

The Schaeffer identity:

$$\begin{aligned}\mathrm{Li}_2\left[\frac{y\,(1-x)}{x\,(1-y)}\right] &= \mathrm{Li}_2(x) - \mathrm{Li}_2(y) + \mathrm{Li}_2\left(\frac{y}{x}\right) + \mathrm{Li}_2\left(\frac{1-x}{1-y}\right) \\ &\quad + \ln(x)\ln\left(\frac{1-x}{1-y}\right) - \frac{\pi^2}{6}\end{aligned} \tag{C.18}$$

Some special values for the dilogs:

$$\mathrm{Li}_2(0) = 0\,, \tag{C.19}$$

$$\mathrm{Li}_2(1) = \frac{\pi^2}{6}\,, \tag{C.20}$$

$$\mathrm{Li}_2(-1) = -\frac{\pi^2}{12}\,. \tag{C.21}$$

Appendix D

Basic integrals for the phase space integrations

D.1 Two–body phase space integration $R_2(P;p_b,k)$ in the real gluon emissions

For the real gluon emissions in both the decay $t \to H^+ + b$ (see subsection 2.3.2) and $t \to b + \ell^+ + \nu_\ell$ (see subsection 3.4.2) we need to perform a two–body phase space integration $R_2(P;p_b,k)$ for the hadronic part. It is convenient to calculate $R_2(P;p_b,k)$ in the P–rest frame. The parameterization in the rest frame of the top quark with the $-\vec{P}$–momentum[1] defining the z–axis is

$$p_t = (m_t; 0,0,0), \quad P = (p_0; 0,0,-p_3). \tag{D.1}$$

Boosting to the P–rest frame one has

$$p_t = (m_t\frac{p_0}{\sqrt{z}}; 0,0,-\frac{p_3}{\sqrt{z}}), \quad P = (m_t\sqrt{z}; 0,0,0). \tag{D.2}$$

A convenient way of parameterization of the gluon momentum in P–rest frame is

$$k = (E_g^*; p_g^* \sin\theta\cos\phi,\, p_g^*\sin\theta\sin\phi,\, p_g^*\cos\theta)\,. \tag{D.3}$$

The angles θ and ϕ are the polar and azimuthal angles of the gluon momentum in the P–rest frame. The energy and the momentum modulus of the gluon in the P–rest frame are given by

$$E_g^* = \frac{z-\epsilon^2+\Lambda^2}{2\sqrt{z}}m_t, \qquad p_g^* = \frac{\sqrt{\lambda(z,\epsilon^2,\Lambda^2)}}{2\sqrt{z}}m_t\,, \tag{D.4}$$

where we have introduced a gluon mass m_g ($\Lambda = m_g/m_t$) for regularization purposes.

As discussed in subsection 2.3.2, the two–body phase space integration $dR_2(P;p_b,k)$ boils down to an angular integral $d\cos\theta$. The only θ dependent scalar product is $p_t\cdot k$. Using the parameterization in the P–rest frame (D.3) and (D.4) gives

$$p_t\cdot k = \frac{p_0E_g^* + p_3p_g^*\cos\vartheta}{\sqrt{z}}m_t. \tag{D.5}$$

[1]The vector $\vec{P}$ here should not be confused with the polarization vector of the top quark $\vec{P}$

The basic integrals for the two–body phase space integration $dR_2(P; p_b, k)$ are defined by

$$I_n := \int dR_2(P; p_b, k)\, (p_t \cdot k)^n \tag{D.6}$$

$$I_n^{\mu} := \int dR_2(P; p_b, k)\, (p_t \cdot k)^n k^{\mu} \tag{D.7}$$

$$I_n^{\mu\nu} := \int dR_2(P; p_b, k)\, (p_t \cdot k)^n k^{\mu} k^{\nu} \tag{D.8}$$

$$I_n^{\mu\nu\gamma} := \int dR_2(P; p_b, k)\, (p_t \cdot k)^n k^{\mu} k^{\nu} k^{\gamma}\,. \tag{D.9}$$

For our purpose we need basic integrals with $-2 \leqslant n \leqslant 1$. I_0 is given in (2.148). The remaining basic integrals are calculated in a similar way. These integrals are different for the IR–divergent and IR–convergent cases. For the IR–divergent integration [2] we have

$$\hat{I}_0 = \frac{1}{m_t} \frac{\pi p_g^*}{\sqrt{z}}\,, \tag{D.10}$$

$$\hat{I}_{-1} = \frac{1}{m_t^2} \frac{\pi}{2p_3} \ln\left(\frac{p_0\, E_g^* + p_3\, p_g^*}{p_0\, E_g^* - p_3\, p_g^*} \right)\,, \tag{D.11}$$

$$\hat{I}_{-2} = \frac{1}{m_t^3} \frac{\pi \sqrt{z}\, p_g^*}{p_0^2\, E_g^{*2} - p_3^2\, p_g^{*2}}\,. \tag{D.12}$$

For the IR–convergent integration we have

$$I_{-2} = \frac{1}{m_t^2} \frac{\pi}{(k \cdot P)}, \tag{D.13}$$

$$I_{-1} = \frac{1}{m_t^2} \frac{\pi}{p_3} Y_p, \tag{D.14}$$

$$I_0 = \frac{1}{m_t^2} \frac{\pi (k \cdot P)}{z}, \tag{D.15}$$

$$I_1 = \frac{1}{m_t^4} \frac{\pi (p_t \cdot P)(k \cdot P)^2}{z^2}. \tag{D.16}$$

The tensor integrals I_n^{μ}, $I_n^{\mu\nu}$ and $I_n^{\mu\nu\gamma}$ are IR convergent because of the gluon momentum in the numerator. They can be decomposed in terms of the scalar integrals I_n. For I_n^{μ} the Lorentz invariance allows one to write

$$I_n^{\mu} = a_1\,(n) p_t^{\mu} + a_2(n)\, P^{\mu}. \tag{D.17}$$

Contracting both sides with $p_{t\mu}$ and P_{μ} and solving the linear equations one has

$$a_1(n) = -\frac{1}{m_t^4 p_3^2} \Big[m_t^2 z\, I_{n+1} - (p_t \cdot P)(k \cdot P)\, I_n \Big]\,,$$

$$a_2(n) = \frac{1}{m_t^4 p_3^2} \Big[m_t^2 (k \cdot P)\, I_{n+1} - (p_t \cdot P)\, I_n \Big]\,. \tag{D.18}$$

[2] The hat is used to denote the IR–divergent basic integrals.

Similarly we write

$$I_n^{\mu\nu} = b_1(n)\, p_t^\mu p_t^\nu + b_2(n)\, P^\mu P^\nu + b_3(n)\, (p_t^\mu P^\nu + p_t^\nu P^\mu) + b_4(n) g^{\mu\nu}. \tag{D.19}$$

Contracting again both sides with P_μ and $p_{t\mu}$ and solving the linear equations gives

$$b_1(n) = \frac{1}{2(p_t\cdot P) m_t^4 p_3^2}\Big\{4m_t^2 z\,(p_t\cdot P)\, a_1(n+1) + m_t^4 z^2\, a_2(n+1) \\ - (k\cdot P)\Big[m_t^4 z + 2(p_t\cdot P)^2\Big]\, a_1(n)\Big\}, \tag{D.20}$$

$$b_2(n) = \frac{1}{2(p_t\cdot P) m_t^4 p_3^2}\Big\{4m_t^2 (p_t\cdot P)\, a_1(n+1) \\ + \Big[3m_t^4 z - 2(p_t\cdot P)^2\Big] a_2(n+1) - 3m_t^4 (k\cdot P) a_1(n)\Big\}, \tag{D.21}$$

$$b_3(n) = \frac{1}{2m_t^4 p_3^2}\Big[-4(p_t\cdot P)\, a_1(n+1) - m_t^2 z\, a_2(n+1) + 3m_t^2 (k\cdot P)\, a_1(n)\Big], \tag{D.22}$$

$$b_4(n) = -a_1(n+1) - \frac{m_t^2 z}{2(p_t\cdot P)}\, a_2(n+1) + \frac{m_t^2 (k\cdot P)}{2(p_t\cdot P)} a_1(n)\,. \tag{D.23}$$

In a similar way we write

$$I_n{}^{\mu\nu\gamma} = c_1(n) p_t^\mu p_t^\nu p_t^\gamma + c_2(n) P^\mu P^\nu P^\gamma + c_3(n)\, (p_t^\mu p_t^\nu P^\gamma + p_t^\mu P^\nu p_t^\gamma + P^\mu p_t^\nu p_t^\gamma) \\ + c_4(n)\, (p_t^\mu P^\nu P^\gamma + p_t^\mu P^\nu P^\gamma + P^\mu P^\nu p_t^\gamma) + c_5(n)\, (p_t^\mu g^{\nu\gamma} + p_t^\nu g^\mu \gamma + p_t^\gamma g^{\mu\nu}) \\ + c_6(n)\, (P^\mu g^{\nu\gamma} + P^\nu g^\mu \gamma + P^\gamma g^{\mu\nu})\,. \tag{D.24}$$

with coefficients given by

$$c_1(n) = -\frac{1}{m_t^4 p_3^2}\Big[m_t^2 z\, b_1(n+1) - (p_t\cdot P)(k\cdot P)\, b_1(n) - 2m_t^2 z\, c_5(n)\Big], \tag{D.25}$$

$$c_2(n) = -\frac{1}{m_t^4 p_3^2}\Big[m_t^2 (k\cdot P)\, b_2(n) - (p_t\cdot P)\, b_2(n+1) - 2m_t^2\, c_6(n)\Big], \tag{D.26}$$

$$c_3(n) = -\frac{1}{m_t^4 p_3^2}\Big[m_t^2 (k\cdot P)\, b_1(n) - (p_t\cdot P)\, b_1(n+1) + 2(p_t\cdot P)\, c_5(n)\Big], \tag{D.27}$$

$$c_4(n) = -\frac{1}{m_t^4 p_3^2}\Big[m_t^2 z\, b_2(n+1) - (p_t\cdot P)(k\cdot P)\, b_2(n) + 2(p_t\cdot P)\, c_6(n)\Big], \tag{D.28}$$

$$c_5(n) = -\frac{1}{m_t^4 p_3^2}\Big[m_t^2 z\, b_4(n+1) - (p_t\cdot P)(k\cdot P)\, b_4(n)\Big], \tag{D.29}$$

$$c_6(n) = -\frac{1}{m_t^4 p_3^2}\Big[m_t^2 (k\cdot P)\, b_4(n) - (p_t\cdot P)\, b_4(n+1)\Big]. \tag{D.30}$$

D.2 Basic z–integrals for $t(\uparrow) \to H^+ + b$

The basic z–integrations for the decay $t(\uparrow) \to H^+ + b$ are defined by (see 3.154):

$$R(n) := \int_{\epsilon^2}^{(1-y)^2} dz \frac{1}{(z-\epsilon^2)\sqrt{\lambda(1,y^2,z)^n}},$$

$$R(m,n) := \int_{\epsilon^2}^{(1-y)^2} dz \frac{z^m}{\sqrt{\lambda(1,y^2,z)^n}},$$

$$S(n) := \int_{\epsilon^2}^{(1-y)^2} dz \frac{1}{(z-\epsilon^2)\sqrt{\lambda(1,y^2,z)^n}} \ln\left(\frac{1-y^2+z+\sqrt{\lambda(1,y^2,z)}}{1-y^2+z-\sqrt{\lambda(1,y^2,z)}}\right),$$

$$S(m,n) := \int_{\epsilon^2}^{(1-y)^2} dz \frac{z^m}{\sqrt{\lambda(1,y^2,z)^n}} \ln\left(\frac{1-y^2+z+\sqrt{\lambda(1,y^2,z)}}{1-y^2+z-\sqrt{\lambda(1,y^2,z)}}\right). \tag{D.31}$$

The important point in the calculation of these integrals is the change of the integration variable ($z \to u$) with

$$z = 1+y^2 - \frac{y(1+u^2)}{u}, \quad \text{where} \quad 1 \leqslant u \leqslant \frac{1+y^2-\epsilon^2+\sqrt{\lambda(1,y^2,\epsilon)^2}}{2y} \tag{D.32}$$

Some of the basic integrals are divergent when substituting the integration limits. The reason is that the IR–convergent integrals are split into IR–divergent integrals by partial fractioning. This divergence is artificial. We use a small cut–off in the integration limits by writing

$$\int_{\epsilon^2}^{(1-y)^2} dz = \lim_{\delta_1,\delta_2\to 0} \int_{\epsilon^2+\delta_1}^{(1-y)^2-\delta_2} dz, \tag{D.33}$$

and the divergences in the form of $\ln\delta_1$ and $\ln\delta_2$ will cancel in the sum.

Now we list all the basic integrals needed for the decay $t(\uparrow) \to H^+ + b$.

$$R(-1) = -2\bar{p}_3 + 2\bar{p}_3 \ln(\frac{2\bar{w}_+\bar{p}_3}{y^2\epsilon_2}) - \left(1-\epsilon^2+y^2\right)\bar{Y}_w\,, \tag{D.34}$$

$$R(0) = \frac{1}{2}\ln\left[\frac{(1-y)^2-\epsilon^2}{(1+y)^2-\epsilon^2}\right] + \ln(\frac{\bar{w}_+}{y\epsilon_2})\,, \tag{D.35}$$

$$R(0,0) = (1-y)^2 - \epsilon^2\,, \tag{D.36}$$

$$R(0,-1) = \bar{p}_3\left(1-\epsilon^2+y^2\right) - 2y^2\bar{Y}_w\,, \tag{D.37}$$

$$R(-1,0) = 2\ln(\frac{1-y}{\epsilon})\,, \tag{D.38}$$

$$R(-1,-1) = -2\bar{p}_3 + 2\left(1-y^2\right)\bar{Y}_p - 2y^2\bar{Y}_w\,, \tag{D.39}$$

$$R(1,-1) = -\frac{8\bar{p}_3^3}{3} + \left(1+y^2\right)\Big[\left(1-\epsilon^2+y^2\right)\bar{p}_3 - 2y^2\bar{Y}_w\Big]\,, \tag{D.40}$$

$$R(-2,0) = \epsilon^{-2} - (1-y)^{-2}\,, \tag{D.41}$$

$$R(-2,-1) = \frac{2\bar{p}_3}{\epsilon^2} - \frac{1}{1-y^2}\Big[\left(1+y^2\right)\bar{Y}_p + y^2\bar{Y}_w\Big]\,, \tag{D.42}$$

$$R(1,0) = \frac{(1-y)^4}{2} - \frac{\epsilon^4}{2}\,, \tag{D.43}$$

$$\begin{aligned} S(0) &= \mathrm{Li}_2(\bar{w}_+) - \mathrm{Li}_2(\bar{w}_-) - 2\mathrm{Li}_2(1 - \frac{\bar{p}_-}{\bar{p}_+}) \\ &+ 2\ln(\frac{2\bar{w}_+\bar{p}_3}{y^2\epsilon_2})\bar{Y}_p - 2\bar{Y}_p^2 + 2\ln(\epsilon)\bar{Y}_w\,, \end{aligned} \tag{D.44}$$

$$S(1) = \frac{1}{\bar{p}_3}\Big\{\mathrm{Li}_2(-\frac{y}{\bar{w}_+}) - \mathrm{Li}_2(-\frac{y\bar{p}_-}{\bar{p}_+\bar{w}_+}) + \ln(\frac{2\bar{p}_3}{y\epsilon_2})\bar{Y}_p - \bar{Y}_p^2\Big\}\,, \tag{D.45}$$

$$\begin{aligned} S(0,-1) &= -\frac{1}{4}\big[(1-y)^2 - \epsilon^2\big]\big[\epsilon^2 - (3-y)\,(1+y)\,\big] - 4\left(1-y^4\right)\ln(\frac{1-y}{\epsilon}) \\ &+ 8y^2\Big[2\mathrm{Li}_2(y) - \mathrm{Li}_2(\bar{w}_-) - \mathrm{Li}_2(\bar{w}_+)\Big] + 8\left(1-\epsilon^2+y^2\right)\bar{p}_3\bar{Y}_p\,, \end{aligned} \tag{D.46}$$

$$S(0,0) = 2\left(\bar{p}_3 - \epsilon^2\bar{Y}_p - y^2\bar{Y}_w\right)\,, \tag{D.47}$$

$$S(0,1) = \mathrm{Li}_2(\bar{w}_-) + \mathrm{Li}_2(\bar{w}_+) - 2\mathrm{Li}_2(y)\,, \tag{D.48}$$

$$S(1,0) = \frac{\bar{p}_3}{2}\left(1+\epsilon^2+5y^2\right) - \epsilon^4\bar{Y}_p - y^2\left(2+y^2\right)\bar{Y}_w\,, \tag{D.49}$$

$$\begin{aligned} S(1,1) &= \epsilon^2 - (1-y)^2 + 2\left(1-y^2\right)\ln(\frac{1-y}{\epsilon}) \\ &+ \left(1+y^2\right)\Big[\mathrm{Li}_2(\bar{w}_-) + \mathrm{Li}_2(\bar{w}_+) - 2\mathrm{Li}_2(y)\Big] - 4\bar{p}_3\bar{Y}_p\,, \end{aligned} \tag{D.50}$$

$$\begin{aligned} S(2,0) &= \frac{1}{9}\sqrt{\lambda(1,y^2,\epsilon^2)}\left(1+\epsilon^2+\epsilon^4+19y^2+4\epsilon^2y^2+10y^4\right) \\ &+ \frac{1}{3}\left(1+9y^2+9y^4+y^6\right)\ln(\frac{y\bar{p}_-}{\bar{p}_+\bar{w}_+}) \\ &+ \frac{1}{3}\left(1+3y^2-3y^4+y^6\right)\ln(\frac{\bar{w}_+}{y}) \\ &- \frac{1}{3}\big[\epsilon^6 - \left(1+y^2\right)\left(1+8y^2+y^4\right)\big]\ln(\frac{y^2-\bar{w}_+}{\bar{w}_+^2-\bar{w}_+})\,, \end{aligned} \tag{D.51}$$

$$\begin{aligned} S(2,1) &= \frac{1}{4}\big[\epsilon^2 - (1-y)^2\big]\big[4+\epsilon^2+(1-y)^2+8y^2\big] + 3\left(1-y^4\right)\ln(\frac{1-y}{\epsilon}) \\ &- \left(1+4y^2+y^4\right)\Big[2\mathrm{Li}_2(y) - \mathrm{Li}_2(\bar{w}_-) - \mathrm{Li}_2(\bar{w}_+)\Big] \\ &- 2\bar{p}_3\left(3+\epsilon^2+3y^2\right)\bar{Y}_p\,. \end{aligned} \tag{D.52}$$

The two integrals used in the calculation of the soft gluon factor are the following:

$$\tilde{R}(0) := \int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz\frac{1}{z-\epsilon^2} = \ln\left(\frac{z_m - \epsilon^2}{2\epsilon\,\Lambda}\right) \tag{D.53}$$

$$\tilde{S}(0) := \int\limits_{(\epsilon+\Lambda)^2}^{z_m} dz \frac{1}{(z-\epsilon^2)\sqrt{\lambda(1,y^2,z)}} \ln\left(\frac{1-y^2+z+\sqrt{\lambda(1,y^2,z)}}{1-y^2+z-\sqrt{\lambda(1,y^2,z)}}\right)$$
$$= \frac{1}{2\,\bar{p}_3}\Big\{\mathrm{Li}_2(1-\frac{p_-(z_m)}{\bar{p}_-}) + \mathrm{Li}_2(1-\frac{p_+(z_m)}{\bar{p}_-}) - \mathrm{Li}_2(1-\frac{p_-(z_m)}{\bar{p}_+})$$
$$+ \mathrm{Li}_2(1-\frac{\bar{p}_-}{\bar{p}_+}) - \mathrm{Li}_2(1-\frac{p_+(z_m)}{\bar{p}_+}) - \mathrm{Li}_2(1-\frac{\bar{p}_+}{\bar{p}_-}) + 2\ln\left(\frac{z_m-\epsilon^2}{2\epsilon\,\Lambda}\right)\bar{Y}_p\Big\}. \quad (D.54)$$

D.3 Basic z–integrals for $t(\uparrow) \to b + \ell^+ + \nu_\ell$

The basic z–integrations for the decay $t(\uparrow) \to b + \ell^+ + \nu_\ell$ are defined by the indefinite integrals (see 3.154):

$$\mathcal{R}(m,n) := \int dz\, \frac{z^m}{\lambda(1,y^2,z)^n},$$
$$\mathcal{S}(m,n) := \int dz\, \frac{2\,z^m}{\lambda(1,y^2,z)^{n+\frac{1}{2}}} \ln\left(\frac{1-y^2+z+\sqrt{\lambda(1,y^2,z)}}{1-y^2+z-\sqrt{\lambda(1,y^2,z)}}\right). \quad (D.55)$$

The necessary indefinite integrals $\mathcal{R}(m,n)$ and $\mathcal{S}(m,n)$ are the following:

$$\mathcal{R}(-2,0) = -\frac{1}{z}\,, \quad (D.56)$$

$$\mathcal{R}(-1,0) = \ln(z)\,, \quad (D.57)$$

$$\mathcal{R}(0,0) = z\,, \quad (D.58)$$

$$\mathcal{R}(0,1) = \frac{1}{4y}\ln(\frac{w_0+y}{w_0-y})\,, \quad (D.59)$$

$$\mathcal{R}(0,2) = -\frac{1}{32y^3}\ln(\frac{w_0+y}{w_0-y}) + \frac{w_0}{16y^2p_3^2}\,, \quad (D.60)$$

$$\mathcal{R}(0,3) = \frac{3}{512y^5}\ln(\frac{w_0+y}{w_0-y}) - \frac{w_0}{1024y^4p_3^4}\Big[3y^4+3(1-z)^2-2y^2\,(7+3z)\Big]\,, \quad (D.61)$$

$$\mathcal{R}(1,1) = \frac{\ln(4p_3^2)}{2} + \frac{(1+y^2)}{4y}\ln(\frac{w_0+y}{w_0-y})\,, \quad (D.62)$$

$$\mathcal{R}(1,2) = -\frac{(1+y^2)}{32y^3}\ln(\frac{w_0+y}{w_0-y}) + \frac{zw_0}{16y^2p_3^2}\,, \quad (D.63)$$

$$\mathcal{R}(1,3) = \frac{3\,(1+y^2)}{512y^5}\ln(\frac{w_0+y}{w_0-y}) - \frac{1}{2048y^4p_3^4}\Big[\left(1-y^2\right)^2\left(3-2y^2+3y^4\right)$$
$$-\left(1+y^2\right)\left(9-2y^2+9y^4\right)z + 9\left(1+y^2\right)^2z^2 - 3\left(1+y^2\right)z^3\Big]\,, \quad (D.64)$$

$$\mathcal{S}(0,0) = -\frac{1}{2}\Big[\mathrm{Li}_2(w_-)+\mathrm{Li}_2(w_+)\Big]\,, \quad (D.65)$$

$$\mathcal{S}(0,1) = \frac{1}{8\,(1-y^2)\,y^2}\Big[\,\big(1+y^2\big)\ln(z) - y^2\ln(4p_3^2) + y\ln(\frac{w_0+y}{w_0-y})\Big] + \frac{w_0 Y_p^2}{4y^2 p_3}\,, \quad \text{(D.66)}$$

$$\begin{aligned}
\mathcal{S}(0,2) = {} & \frac{-\,(1+y^2)^2}{192\,(1-y^2)\,y^4 z} - \frac{(1+y^2)\,(1-4y^2+y^4)}{48(1-y^2)^3 y^4}\ln(z) \\
& - \frac{(3-y^2)}{48(1-y^2)^3}\ln(4p_3^2) - \frac{(3-14y^2+3y^4)}{192(1-y^2)^3 y^3}\ln(\frac{w_0+y}{w_0-y}) \\
& + \frac{(1-y^2-z)\,{w_0}^2}{192y^4 z p_3^2} + \frac{(2y^2-4p_3^2)\,w_0 Y_p^2}{96y^4 p_3^3}\,, \qquad \text{(D.67)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}(0,3) = {} & \frac{-(1+y^2)^3}{5120\,(1-y^2)\,y^6 z^2} + \frac{(1+y^2)^2\,(25-98y^2+25y^4)}{15360(1-y^2)^3 y^6 z} \qquad \text{(D.68)} \\
& + \frac{(1+y^2)\,(1-6y^2+16y^4-6y^6+y^8)}{240(1-y^2)^5 y^6}\ln(z) - \frac{(10-5y^2+y^4)}{240(1-y^2)^5}\ln(4p_3^2) \\
& + \frac{(45-260y^2+814y^4-260y^6+45y^8)}{15360(1-y^2)^5 y^5}\ln(\frac{w_0+y}{w_0-y}) \\
& + \frac{(1-y^2-z)\,{w_0}^2}{2560y^4 z p_3^4} + \frac{(3-3y^4-25z+25y^2 z+22z^2)\,{w_0}^2}{15360y^6 z^2 p_3^2} \\
& + \frac{1}{3840y^6 p_3^5}\Big[1-5y^2+10y^4+10y^6-5y^8+y^{10} \\
& \quad - 5\left(1-2y^2-2y^6+y^8\right)z + 10\left(1+y^6\right)z^2 \\
& \quad - 10\left(1+y^2+y^4\right)z^3 + 5\left(1+y^2\right)z^4 - z^5\Big]Y_p^2\,,
\end{aligned}$$

$$\mathcal{S}(1,0) = -\frac{z}{2} + \frac{(1-y^2)}{2}\ln(z) - \frac{(1+y^2)}{2}\Big[\mathrm{Li}_2(w_-) + \mathrm{Li}_2(w_+)\Big] + 2p_3 Y_p^2\,, \quad \text{(D.69)}$$

$$\mathcal{S}(1,1) = \frac{1-y^2}{8y^2}\ln(z) + \frac{1}{8}\ln(4p_3^2) + \frac{1}{8y}\ln(\frac{w_0+y}{w_0-y}) + \frac{(1-y^2)^2-(1+y^2)\,z}{8y^2 p_3}Y_p^2\,, \quad \text{(D.70)}$$

$$\begin{aligned}
\mathcal{S}(1,2) = {} & \frac{1-y^2}{192y^4} - \frac{(3+y^2)}{192\,(1-y^2)\,y^3}\ln(\frac{w_0+y}{w_0-y}) - \frac{(1+y^4)\ln(z)}{48\,(1-y^2)\,y^4} \qquad \text{(D.71)} \\
& + \frac{\ln(4p_3^2)}{48\,(1-y^2)} + \frac{1-y^2-z}{192y^2 p_3^2} - \frac{1}{192y^4 p_3^3}\Big[\left(1-y^2\right)^2\left(1+y^4\right) \\
& - 3\left(1+y^2\right)\left(1+y^4\right)z + 3\big(1+y^2\big)^2 z^2 - \left(1+y^2\right)z^3\Big]Y_p^2\,, \qquad \text{(D.72)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}(1,3) = {} & \frac{(1-2y^2-2y^6+y^8)}{240(1-y^2)^3 y^6}\ln(z) + \frac{(2-y^2)}{240(1-y^2)^3}\ln(4p_3^2) \qquad \text{(D.73)} \\
& - \frac{(45-95y^2-41y^4+27y^6)}{15360(-1+y^2)^3 y^5}\ln(\frac{w_0+y}{w_0-y}) + \frac{1-y^2-z}{2560y^2 p_3^4} \\
& - \frac{13-8y^2+27y^4-13z+5y^2 z}{15360\,(1-y^2)\,y^4 p_3^2}
\end{aligned}$$

$$+ \frac{1}{3840y^6p_3^5}\Big[\left(1-y^2\right)^2\left(1-2y^2-2y^6+y^8\right)$$
$$-5\left(1-y^2-2y^4-2y^6-y^8+y^{10}\right)z+10\big(1+y^2\big)^2\left(1-y^2+y^4\right)z^2$$
$$-10\left(1+2y^2+2y^4+y^6\right)z^3+5\big(1+y^2\big)^2z^4-\left(1+y^2\right)z^5\Big]Y_p^2\,.$$

D.4 Coefficient functions for the decay $t \to b + \ell + \nu_\ell$ in system 1a

The coefficient functions for the unpolarized part of the real emissions in the decay $t \to b + \ell + \nu_\ell$ in system 1a are the following:

$$\rho_a(-2,0) = -x(1-x)\epsilon^2 - \frac{1}{2(1-y^2)}(1-x)(x-y^2)\epsilon^4\,, \tag{D.74}$$

$$\begin{aligned}\rho_a(-1,0) = &\frac{1}{2}x(1-x)\\ &-\frac{1}{2(1-y^2)^2}(-3x+2x^2+4y^2-6xy^2+4x^2y^2+2y^4-3xy^4)\epsilon^2\\ &+\frac{1}{2(1-y^2)^3}(-2x+x^2+2y^2+x^2y^2-y^4-2xy^4+y^6)\epsilon^4\,,\end{aligned} \tag{D.75}$$

$$\rho_a(0,0) = -4x + \frac{y^2}{2}\,, \tag{D.76}$$

$$\begin{aligned}\rho_a(0,1) = &\frac{1}{2}(-10x+17x^2+18y^2+14xy^2+15x^2y^2-y^4-4xy^4-y^6)\\ &-\frac{1}{(1-y^2)^2}(-4x+5x^2+10y^2-12xy^2-6y^4+7x^2y^4+8y^6-8xy^6)\epsilon^2\\ &-\frac{1}{2(1-y^2)^3}(-2x+3x^2+6y^2-12xy^2+2x^2y^2-3y^4\\ &+6xy^4+3x^2y^4+4y^6-8xy^6+y^8)\epsilon^4\,,\end{aligned} \tag{D.77}$$

$$\begin{aligned}\rho_a(0,2) = &3y^2(1-y^2)^2(-10x+x^2+y^2)\\ &+6y^2(4x-5x^2-5y^2+8xy^2-x^2y^2-y^4)\epsilon^2\\ &+\frac{3y^2}{1-y^2}(2x-3x^2-3y^2+6xy^2-x^2y^2-y^4)\epsilon^4\,,\end{aligned} \tag{D.78}$$

$$\begin{aligned}\rho_a(1,1) = &\frac{1}{2}(10x-7x^2+2y^2+2xy^2+y^4)\\ &+\frac{1}{(1-y^2)^2}(-4x+x^2+2y^2+2xy^2+2x^2y^2+y^4-4xy^4)\epsilon^2\\ &-\frac{1}{2(1-y^2)^3}(2x-x^2-2y^2-x^2y^2+y^4+2xy^4-y^6)\epsilon^4]\,,\end{aligned} \tag{D.79}$$

$$\rho_a(1,2) = 3y^2(10x + 9x^2 + 9y^2 + 6xy^2 - x^2y^2 - y^4) - 6y^2(4x - x^2 - y^2)\epsilon^2 - \frac{3y^2}{1-y^2}(2x - x^2 - y^2)\epsilon^4\,, \quad \text{(D.80)}$$

$$\sigma_a(0,0) = 2x(-1 + 2x - 2y^2) + 2x\epsilon^2\,, \quad \text{(D.81)}$$

$$\sigma_a(0,1) = 2(5x - 6x^2 - 4y^2 - 28xy^2 - 8x^2y^2 - 8y^4 - xy^4 + 2x^2y^4) - 2(4x - 3x^2 - 6y^2 - 7xy^2 + 3x^2y^2 + 4y^4 - xy^4)\epsilon^2 + 2(-x + x^2 + 2y^2 - xy^2)\epsilon^4\,, \quad \text{(D.82)}$$

$$\sigma_a(0,2) = 12y^2(-1 + y^2)^2(5x + 2x^2 + 2y^2 - xy^2) - 12(-4x + 3x^2 + 3y^2 - 2xy^2)(-y^2 + y^4)\epsilon^2 + 12y^2(-x + x^2 + y^2 - xy^2)\epsilon^4\,, \quad \text{(D.83)}$$

$$\sigma_a(1,0) = 4x\,, \quad \text{(D.84)}$$

$$\sigma_a(1,1) = -2(5x - x^2 + 6y^2 + 5xy^2 + x^2y^2) + 2(4x + x^2 + 2y^2 - xy^2)\epsilon^2 + 2x\epsilon^4\,, \quad \text{(D.85)}$$

$$\sigma_a(1,2) = -12y^2(5x + 7x^2 + 7y^2 + 12xy^2 + x^2y^2 + y^4 - xy^4) + 12y^2(4x + x^2 + y^2 - 2xy^2)\epsilon^2 + 12xy^2\epsilon^4 \quad \text{(D.86)}$$

The coefficient functions for the polarized part of the real emissions in the decay $t \to b + \ell + \nu_\ell$ in system 1a are the following:

$$\rho_b(-2,0) = x(x-1)\epsilon^2 + \frac{1-x}{(1-y^2)^2}(x - 3y^2 + xy^2 - y^4 + 2\frac{y^4}{x})\epsilon^4\,, \quad \text{(D.87)}$$

$$\rho_b(-1,0) = x(x-1) + \frac{1}{(1-y^2)^2}(3x - 2x^2 - 4y^2 + 6xy^2 - 4x^2y^2 - 2y^4 + 3xy^4)\epsilon^2 + \frac{1}{(1-y^2)^4}(-4x + 3x^2 + 14y^2 - 18xy^2 + 8x^2y^2 + 15y^4 - 12xy^4 + x^2y^4 + 6y^6 - 2xy^6 + y^8 - 10\frac{y^4}{x} - 2\frac{y^8}{x})\epsilon^4\,, \quad \text{(D.88)}$$

$$\rho_b(0,0) = -8x + y^2\,, \quad \text{(D.89)}$$

$$\rho_b(0,1) = \frac{1}{x(1-y^2)^4}\big[(1-y^2)^4(-12x^2 + 7x^3 - 50xy^2 - 62x^2y^2 + 11x^3y^2 - 14y^4 - 11xy^4 - 4x^2y^4 - xy^6) + 2x(1-y^2)^2(14x - 3x^2 + 18y^2 - 8xy^2 - 4x^2y^2 - 50y^4 + 10xy^4 + 5x^2y^4 - 20y^6 - 8xy^6)\epsilon^2 + (-16x^2 + 7x^3 + 30xy^2 - 18x^2y^2 + 21x^3y^2 - 22y^4 + 59xy^4 - 96x^2y^4 + 17x^3y^4 - 16y^6 + 33xy^6 - 2x^2y^6 + 3x^3y^6 - 6y^8 + 21xy^8 - 12x^2y^8 - 4y^{10} + xy^{10})\epsilon^4\big]\,, \quad \text{(D.90)}$$

$$\rho_b(0,2) = \frac{1}{x(1-y^2)^2}\big[2(1-y^2)^2(6x^3 + 36xy^2 + 6x^2y^2 - 67x^3y^2 + 6y^4 - 243xy^4 - 372x^2y^4 - 64x^3y^4 - 132y^6 - 162xy^6 + 6x^2y^6 + 5x^3y^6 + 6y^8 + 9xy^8)$$

$$- 4(1-y^2)^2(6x^3 + 36xy^2 - 60x^2y^2 - 31x^3y^2 - 18y^4 - 153xy^4 - 42x^2y^4 + 5x^3y^4 - 4y^6 + 21xy^6)\epsilon^2 + 2(6x^3 + 36xy^2 - 126x^2y^2 + 17x^3y^2 - 42y^4 + 33xy^4 + 120x^2y^4 + 12x^3y^4 + 40y^6 + 18xy^6 - 114x^2y^6 + 5x^3y^6 - 38y^8 + 33xy^8)\epsilon^4\big] , \tag{D.91}$$

$$\rho_b(0,3) = \frac{1}{x}\big[120y^2(1-y^2)^2(x^3 + 3xy^2 + 6x^2y^2 + x^3y^2 + 2y^4 + 3xy^4) - 240x(1-y^2)^2y^2(x^2+3y^2)\epsilon^2 + 120y^2(x^3 + 3xy^2 - 6x^2y^2 + x^3y^2 - 2y^4 + 3xy^4)\epsilon^4\big] , \tag{D.92}$$

$$\rho_b(1,1) = \frac{1}{x(1-y^2)^4}\big[-(1-y^2)^4(7x^3 + 6xy^2 - 2x^2y^2 - 2y^4 - xy^4) + 2x(1-y^2)^2(-2x + x^2 + 2y^2 - 2xy^2 + 2x^2y^2 + y^4 - 2xy^4)\epsilon^2 - (-4x^2 + 3x^3 + 14xy^2 - 18x^2y^2 + 8x^3y^2 - 10y^4 + 15xy^4 - 12x^2y^4 + x^3y^4 + 6xy^6 - 2x^2y^6 - 2y^8 + xy^8)\epsilon^4\big] , \tag{D.93}$$

$$\rho_b(1,2) = \frac{1}{x(1-y^2)^2}\big[-2(1-y^2)^2(6x^3 + 36xy^2 + 114x^2y^2 + 23x^3y^2 + 42y^4 + 177xy^4 + 84x^2y^4 + 5x^3y^4 + 16y^6 + 9xy^6) + 4(1-y^2)^2(6x^3 + 36xy^2 + 48x^2y^2 + 5x^3y^2 + 18y^4 + 21xy^4)\epsilon^2 - 2(6x^3 + 36xy^2 - 18x^2y^2 - x^3y^2 - 6y^4 - 39xy^4 - 12x^2y^4 + 5x^3y^4 - 4y^6 + 33xy^6)\epsilon^4\big] , \tag{D.94}$$

$$\rho_b(1,3) = \frac{1}{x}\big[-120y^2(x^3 + 3xy^2 + 12x^2y^2 + 6x^3y^2 + 4y^4 + 18xy^4 + 12x^2y^4 + x^3y^4 + 4y^6 + 3xy^6) + 240y^2(x^3 + 3xy^2 + 6x^2y^2 + x^3y^2 + 2y^4 + 3xy^4)\epsilon^2 - 120xy^2(x^2 + 3y^2)\epsilon^4\big] , \tag{D.95}$$

$$\sigma_b(0,0) = 4x(1 + 2x - 2y^2) + 4x\epsilon^2 , \tag{D.96}$$

$$\sigma_b(0,1) = \frac{1}{x}\big[4(3x^2 + 32xy^2 + 66x^2y^2 + 10x^3y^2 + 24y^4 + 64xy^4 + 15x^2y^4 + 2x^3y^4) - 4(8x^2 + 5x^3 + 50xy^2 + 33x^2y^2 + 3x^3y^2 + 12y^4 + 4xy^4 - x^2y^4)\epsilon^2 + 4x(5x + x^2 + 10y^2 - xy^2)\epsilon^4\big] , \tag{D.97}$$

$$\sigma_b(0,2) = \frac{1}{x}\big[-24(x^3 + 6xy^2 + 10x^2y^2 - 11x^3y^2 + 4y^4 - 32xy^4 - 106x^2y^4 - 31x^3y^4 - 38y^6 - 104xy^6 - 28x^2y^6 + x^3y^6 - 6y^8 + 10xy^8 + 4x^2y^8) - 24(-2x^3 - 12xy^2 + 2x^2y^2 + 19x^3y^2 + 75xy^4 + 68x^2y^4 + 3x^3y^4 + 22y^6 - 3xy^6 - 10x^2y^6 - 2y^8)\epsilon^2 + 24(-x^3 - 6xy^2 + 12x^2y^2 + 4x^3y^2 + 4y^4 + 15xy^4 - 6x^2y^4 - 2y^6)\epsilon^4\big] , \tag{D.98}$$

$$\sigma_b(0,3) = \frac{1}{x}\big[-240(1-y^2)^2y^2(x^3 + 3xy^2 + 9x^2y^2 + 3x^3y^2 + 3y^4 + 9xy^4 + 3x^2y^4 + y^6) + 480(1-y^2)^2y^2(x^3 + 3xy^2 + 3x^2y^2 + y^4)\epsilon^2$$

$$+ 240(x^3 + 3xy^2 - 3x^2y^2 - y^4)(-y^2 + y^4)\epsilon^4\big] , \tag{D.99}$$

$$\sigma_b(1,0) = 8x , \tag{D.100}$$

$$\begin{aligned}\sigma_b(1,1) = {}& 4(3x + 3x^2 + 14y^2 + 9xy^2 - x^2y^2) \\ & + 4(-4x + x^2 - 6y^2 - xy^2)\epsilon^2 + 4x\epsilon^4 ,\end{aligned} \tag{D.101}$$

$$\begin{aligned}\sigma_b(1,2) = {}& \frac{1}{x}\big[24(x^3 + 6xy^2 + 28x^2y^2 + 11x^3y^2 + 10y^4 + 65xy^4 + 54x^2y^4 + 6x^3y^4 \\ & + 23xy^6 + 4x^2y^6) - 24(2x^3 + 12xy^2 + 34x^2y^2 + 7x^3y^2 + 12y^4 \\ & + 37xy^4 + 10x^2y^4 + 2y^6)\epsilon^2 + 24(x^3 + 6xy^2 + 6x^2y^2 + 2y^4)\epsilon^4\big] ,\end{aligned} \tag{D.102}$$

$$\begin{aligned}\sigma_b(1,3) = {}& \frac{1}{x}\big[240y^2(x^3 + 3xy^2 + 15x^2y^2 + 10x^3y^2 + 5y^4 \\ & + 30xy^4 + 30x^2y^4 + 5x^3y^4 + 10y^6 + 15xy^6 + 3x^2y^6 + y^8) \\ & - 480y^2(x^3 + 3xy^2 + 9x^2y^2 + 3x^3y^2 + 3y^4 + 9xy^4 + 3x^2y^4 + y^6)\epsilon^2 \qquad (D.103) \\ & + 240y^2(x^3 + 3xy^2 + 3x^2y^2 + y^4)\epsilon^4\big] .\end{aligned} \tag{D.104}$$

D.5 Coefficient functions for the real emissions in the decay $t \to b + \ell + \nu_\ell$ in system 3a

The coefficient functions for unpolarized part of the real emissions in the decay $t \to b+\ell+\nu_\ell$ in system 3a are the following:

$$\rho_a(-2,0) = \left(-x_\nu + x_\nu^2 + y^2 - 2x_\nu y^2 + y^4\right)\epsilon^2 - \frac{(1-x_\nu)(x_\nu - y^2)\,\epsilon^4}{1-y^2} , \tag{D.105}$$

$$\begin{aligned}\rho_a(-1,0) = {}& -\frac{1}{(1-y^2)^3}\big[\left(1-y^2\right)^3\left(-x_\nu + x_\nu^2 + y^2 - 2x_\nu y^2 + y^4\right) \\ & + \left(1-y^2\right)\left(-3x_\nu + 2x_\nu^2 + 3y^2 - 2x_\nu y^2 + 4x_\nu^2y^2 - 7x_\nu y^4 + 3y^6\right)\epsilon^2 \\ & + \left(-2x_\nu + x_\nu^2 + 2y^2 + x_\nu^2y^2 - y^4 - 2x_\nu y^4 + y^6\right)\epsilon^4\big] ,\end{aligned} \tag{D.106}$$

$$\rho_a(0,0) = -8\left(x_\nu - y^2\right) , \tag{D.107}$$

$$\begin{aligned}\rho_a(0,1) = {}& \frac{1}{(1-y^2)^3}\big[\left(1-y^2\right)^3(-10x_\nu + 17x_\nu^2 + 18y^2 + 2x_\nu y^2 + 15x_\nu^2y^2 \\ & + 27y^4 - 24x_\nu y^4 + 3y^6) + 2\left(-1+y^2\right)(-4x_\nu + 5x_\nu^2 + 10y^2 \\ & - 10x_\nu y^2 - 12y^4 + 4x_\nu y^4 + 7x_\nu^2y^4 + 12y^6 - 14x_\nu y^6 + 2y^8)\epsilon^2 \\ & - (-2x_\nu + 3x_\nu^2 + 6y^2 - 12x_\nu y^2 + 2x_\nu^2y^2 - 3y^4 + 6x_\nu y^4 \\ & + 3x_\nu^2y^4 + 4y^6 - 8x_\nu y^6 + y^8)\epsilon^4\big] ,\end{aligned} \tag{D.108}$$

$$\begin{aligned}\rho_a(0,2) = {}& \frac{1}{1-y^2}\big[6y^2\left(1-y^2\right)^3\left(-10x_\nu + x_\nu^2 + y^2\right) \\ & - 12y^2\left(1-y^2\right)\left(-4x_\nu + 5x_\nu^2 + 5y^2 - 8x_\nu y^2 + x_\nu^2y^2 + y^4\right)\epsilon^2\end{aligned}$$

$$- 6y^2 \left(-2x_\nu + 3x_\nu^2 + 3y^2 - 6x_\nu y^2 + x_\nu^2 y^2 + y^4\right) \epsilon^4\Big] , \tag{D.109}$$

$$\rho_a(1,1) = \frac{1}{(1-y^2)^3}\Big[\left(1-y^2\right)^3 \left(10x_\nu - 7x_\nu^2 + 2y^2 + 14x_\nu y^2 - 3y^4\right)$$
$$- 2\left(1-y^2\right)\left(4x_\nu - x_\nu^2 - 2y^2 - 4x_\nu y^2 - 2x_\nu^2 y^2 + y^4 + 6x_\nu y^4 - 2y^6\right)\epsilon^2$$
$$- \left(2x_\nu - x_\nu^2 - 2y^2 - x_\nu^2 y^2 + y^4 + 2x_\nu y^4 - y^6\right)\epsilon^4\Big] , \tag{D.110}$$

$$\rho_a(1,2) = -\frac{1}{1-y^2}\Big[6y^2\left(1-y^2\right)\left(-10x_\nu - 9x_\nu^2 - 9y^2 - 6x_\nu y^2 + x_\nu^2 y^2 + y^4\right)$$
$$+ 12y^2\left(1-y^2\right)\left(4x_\nu - x_\nu^2 - y^2\right)\epsilon^2 - 6y^2\left(-2x_\nu + x_\nu^2 + y^2\right)\epsilon^4\Big] , \tag{D.111}$$

$$\sigma_a(0,0) = 4\left(-x_\nu + 2x_\nu^2 - 2y^2 - 4x_\nu y^2 + 2y^4\right) + 4\left(x_\nu - y^2\right)\epsilon^2 , \tag{D.112}$$

$$\sigma_a(0,1) = 4(5x_\nu - 6x_\nu^2 - 4y^2 - 22x_\nu y^2 - 8x_\nu^2 y^2 - 16y^4 - 5x_\nu y^4 + 2x_\nu^2 y^4 + 8y^6$$
$$- 2x_\nu y^6) - 4\left(4x_\nu - 3x_\nu^2 - 6y^2 - 9x_\nu y^2 + 3x_\nu^2 y^2 + 8y^4 - 3x_\nu y^4\right)\epsilon^2$$
$$+ 4\left(-x_\nu + x_\nu^2 + 2y^2 - x_\nu y^2\right)\epsilon^4 , \tag{D.113}$$

$$\sigma_a(0,2) = 24y^2\left(-1+y^2\right)^2\left(5x_\nu + 2x_\nu^2 + 2y^2 - x_\nu y^2\right)$$
$$- 24\left(-4x_\nu + 3x_\nu^2 + 3y^2 - 2x_\nu y^2\right)\left(-y^2 + y^4\right)\epsilon^2$$
$$+ 24y^2\left(-x_\nu + x_\nu^2 + y^2 - x_\nu y^2\right)\epsilon^4 , \tag{D.114}$$

$$\sigma_a(1,0) = 4\left(2x_\nu - y^2\right) , \tag{D.115}$$

$$\sigma_a(1,1) = -4\left(5x_\nu - x_\nu^2 + 6y^2 + 11x_\nu y^2 + x_\nu^2 y^2 + 4y^4 - 2x_\nu y^4\right)$$
$$+ 4\left(4x_\nu + x_\nu^2 + 2y^2 - 3x_\nu y^2\right)\epsilon^2 + 4x_\nu\epsilon^4 , \tag{D.116}$$

$$\sigma_a(1,2) = -24y^2\left(5x_\nu + 7x_\nu^2 + 7y^2 + 12x_\nu y^2 + x_\nu^2 y^2 + y^4 - x_\nu y^4\right)$$
$$+ 24y^2\left(4x_\nu + x_\nu^2 + y^2 - 2x_\nu y^2\right)\epsilon^2 + 24x_\nu y^2\epsilon^4 . \tag{D.117}$$

The coefficient functions for polarized part of the real emissions in the decay $t \to b + \ell + \nu_\ell$ in system 3a are the following:

$$\rho_b(-2,0) = \left(-x_\nu^2 + x_\nu^3 + 3x_\nu y^2 - 2x_\nu^2 y^2 - 2y^4 + x_\nu y^4\right)\epsilon^2$$
$$+ \frac{1-x_\nu}{x_\nu\left(1-y^2\right)^2}\left(x_\nu^2 - 3x_\nu y^2 + x_\nu^2 y^2 + 2y^4 - x_\nu y^4\right)\epsilon^4 , \tag{D.118}$$

$$\rho_b(-1,0) = -\left(-x_\nu^2 + x_\nu^3 + 3x_\nu y^2 - 2x_\nu^2 y^2 - 2y^4 + x_\nu y^4\right) - \frac{1}{x_\nu(1-y^2)^2}(-3x_\nu^2 + 2x_\nu^3$$
$$+ 11x_\nu y^2 - 14x_\nu^2 y^2 + 4x_\nu^3 y^2 - 8y^4 + 16x_\nu y^4 - 7x_\nu^2 y^4 - 4y^6 + 3x_\nu y^6)\epsilon^2$$
$$+ \frac{1}{x_\nu(1-y^2)^4}(-4x_\nu^2 + 3x_\nu^3 + 14x_\nu y^2 - 18x_\nu^2 y^2 + 8x_\nu^3 y^2 - 10y^4$$
$$+ 15x_\nu y^4 - 12x_\nu^2 y^4 + x_\nu^3 y^4 + 6x_\nu y^6 - 2x_\nu^2 y^6 - 2y^8 + x_\nu y^8)\epsilon^4 , \tag{D.119}$$

$$\rho_b(0,0) = -4\left(2x_\nu - y^2\right) , \tag{D.120}$$

$$\rho_b(0,1) = \frac{1}{x_\nu(1-y^2)^4}\big[\big(1-y^2\big)^4(-12x_\nu^2 + 7x_\nu^3 - 6x_\nu y^2 - 94x_\nu^2 y^2 + 11x_\nu^3 y^2 - 46y^4 - 7x_\nu y^4 - 24x_\nu^2 y^4 - 20y^6 + 7x_\nu y^6) - 2\big(1-y^2\big)^2(-14x_\nu^2 + 3x_\nu^3 - 4x_\nu y^2 - 6x_\nu^2 y^2 + 4x_\nu^3 y^2 - 14y^4 + 58x_\nu y^4 - 10x_\nu^2 y^4 + 5x_\nu^3 y^4 + 2x_\nu y^6 - 18x_\nu^2 y^6 - 10y^8 + 4x_\nu y^8)\epsilon^2 + (-16x_\nu^2 + 7x_\nu^3 + 30x_\nu y^2 - 18x_\nu^2 y^2 + 21x_\nu^3 y^2 - 22y^4 + 59x_\nu y^4 - 96x_\nu^2 y^4 + 17x_\nu^3 y^4 - 16y^6 + 33x_\nu y^6 - 2x_\nu^2 y^6 + 3x_\nu^3 y^6 - 6y^8 + 21x_\nu y^8 - 12x_\nu^2 y^8 - 4y^{10} + x_\nu y^{10})\epsilon^4\big]\,, \tag{D.121}$$

$$\rho_b(0,2) = \frac{1}{x_\nu(1-y^2)^2}\big[2\big(1-y^2\big)^2(6x_\nu^3 + 36x_\nu y^2 - 6x_\nu^2 y^2 - 67x_\nu^3 y^2 - 6y^4 - 195x_\nu y^4 - 348x_\nu^2 y^4 - 64x_\nu^3 y^4 - 108y^6 - 258x_\nu y^6 - 6x_\nu^2 y^6 + 5x_\nu^3 y^6 - 6y^8 + 57x_\nu y^8) - 4\big(1-y^2\big)^2(6x_\nu^3 + 36x_\nu y^2 - 66x_\nu^2 y^2 - 31x_\nu^3 y^2 - 24y^4 - 105x_\nu y^4 - 72x_\nu^2 y^4 + 5x_\nu^3 y^4 - 34y^6 + 45x_\nu y^6)\epsilon^2 + 2(6x_\nu^3 + 36x_\nu y^2 - 126x_\nu^2 y^2 + 17x_\nu^3 y^2 - 42y^4 + 33x_\nu y^4 + 120x_\nu^2 y^4 + 12x_\nu^3 y^4 + 40y^6 + 18x_\nu y^6 - 114x_\nu^2 y^6 + 5x_\nu^3 y^6 - 38y^8 + 33x_\nu y^8)\epsilon^4\big]\,, \tag{D.122}$$

$$\rho_b(0,3) = \frac{1}{x_\nu}\big[120\big(1-y^2\big)^2 y^2\left(x_\nu^3 + 3x_\nu y^2 + 6x_\nu^2 y^2 + x_\nu^3 y^2 + 2y^4 + 3x_\nu y^4\right) - 240x_\nu\big(1-y^2\big)^2 y^2\left(x_\nu^2 + 3y^2\right)\epsilon^2 + 120y^2\left(x_\nu^3 + 3x_\nu y^2 - 6x_\nu^2 y^2 + x_\nu^3 y^2 - 2y^4 + 3x_\nu y^4\right)\epsilon^4\big]\,, \tag{D.123}$$

$$\rho_b(1,1) = \frac{1}{x_\nu(1-y^2)^4}\big[-\big(1-y^2\big)^4\left(7x_\nu^3 + 34x_\nu y^2 - 6x_\nu^2 y^2 - 6y^4 + 7x_\nu y^4\right) + 2\big(1-y^2\big)^2(-2x_\nu^2 + x_\nu^3 + 8x_\nu y^2 - 6x_\nu^2 y^2 + 2x_\nu^3 y^2 - 4y^4 + 3x_\nu y^4 - 4x_\nu^2 y^4 - 2y^6 + 4x_\nu y^6)\epsilon^2 - (-4x_\nu^2 + 3x_\nu^3 + 14x_\nu y^2 - 18x_\nu^2 y^2 + 8x_\nu^3 y^2 - 10y^4 + 15x_\nu y^4 - 12x_\nu^2 y^4 + x_\nu^3 y^4 + 6x_\nu y^6 - 2x_\nu^2 y^6 - 2y^8 + x_\nu y^8)\epsilon^4\big]\,, \tag{D.124}$$

$$\rho_b(1,2) = \frac{1}{x_\nu(1-y^2)^2}\big[-2\big(1-y^2\big)^2(6x_\nu^3 + 36x_\nu y^2 + 102x_\nu^2 y^2 + 23x_\nu^3 y^2 + 30y^4 + 177x_\nu y^4 + 120x_\nu^2 y^4 + 5x_\nu^3 y^4 + 52y^6 + 57x_\nu y^6) + 4\big(1-y^2\big)^2\left(6x_\nu^3 + 36x_\nu y^2 + 42x_\nu^2 y^2 + 5x_\nu^3 y^2 + 12y^4 + 45x_\nu y^4\right)\epsilon^2 - 2(6x_\nu^3 + 36x_\nu y^2 - 18x_\nu^2 y^2 - x_\nu^3 y^2 - 6y^4 - 39x_\nu y^4 - 12x_\nu^2 y^4 + 5x_\nu^3 y^4 - 4y^6 + 33x_\nu y^6)\epsilon^4\big]\,, \tag{D.125}$$

$$\rho_b(1,3) = \frac{1}{x_\nu}\big[-120y^2(x_\nu^3 + 3x_\nu y^2 + 12x_\nu^2 y^2 + 6x_\nu^3 y^2 + 4y^4 + 18x_\nu y^4 + 12x_\nu^2 y^4 + x_\nu^3 y^4 + 4y^6 + 3x_\nu y^6) + 240y^2\left(x_\nu^3 + 3x_\nu y^2 + 6x_\nu^2 y^2 + x_\nu^3 y^2 + 2y^4 + 3x_\nu y^4\right)\epsilon^2 - 120x_\nu y^2\left(x_\nu^2 + 3y^2\right)\epsilon^4\big]\,, \tag{D.126}$$

$$\sigma_b(0,0) = \frac{4}{x_\nu}\big[\left(x_\nu^2 + 2x_\nu^3 + 8x_\nu y^2 - 4x_\nu^2 y^2 - 2y^4 + 2x_\nu y^4\right) + 4x_\nu\left(x_\nu - y^2\right)\epsilon^2\big]\,, \tag{D.127}$$

$$\begin{aligned}\sigma_b(0,1) = \frac{4}{x_\nu}\Big[&(3x_\nu^2 + 14x_\nu y^2 + 68x_\nu^2 y^2 + 10x_\nu^3 y^2 + 26y^4 + 92x_\nu y^4 + 27x_\nu^2 y^4 \\ &+ 2x_\nu^3 y^4 + 12y^6 + 14x_\nu y^6 - 2x_\nu^2 y^6 - 2y^8) - 4(8x_\nu^2 + 5x_\nu^3 + 40x_\nu y^2 \\ &+ 35x_\nu^2 y^2 + 3x_\nu^3 y^2 + 14y^4 + 26x_\nu y^4 - 3x_\nu^2 y^4 - 2y^6)\epsilon^2 \\ &+ 4x_\nu\left(5x_\nu + x_\nu^2 + 10y^2 - x_\nu y^2\right)\epsilon^4\Big]\,, \end{aligned} \tag{D.128}$$

$$\begin{aligned}\sigma_b(0,2) = \frac{24}{x_\nu}\Big[&-(x_\nu^3 + 6x_\nu y^2 + 8x_\nu^2 y^2 - 11x_\nu^3 y^2 + 2y^4 - 28x_\nu y^4 - 98x_\nu^2 y^4 - 31x_\nu^3 y^4 \\ &- 30y^6 - 112x_\nu y^6 - 38x_\nu^2 y^6 + x_\nu^3 y^6 - 16y^8 + 14x_\nu y^8 + 8x_\nu^2 y^8 + 4y^{10}) \\ &- (-2x_\nu^3 - 12x_\nu y^2 + 4x_\nu^2 y^2 + 19x_\nu^3 y^2 + 2y^4 + 63x_\nu y^4 + 70x_\nu^2 y^4 \\ &+ 3x_\nu^3 y^4 + 24y^6 + 9x_\nu y^6 - 14x_\nu^2 y^6 - 6y^8)\epsilon^2 + (-x_\nu^3 - 6x_\nu y^2 \\ &+ 12x_\nu^2 y^2 + 4x_\nu^3 y^2 + 4y^4 + 15x_\nu y^4 - 6x_\nu^2 y^4 - 2y^6)\epsilon^4\Big]\,, \end{aligned} \tag{D.129}$$

$$\begin{aligned}\sigma_b(0,3) = \frac{1}{x_\nu}\Big[&-240\big(1-y^2\big)^2 y^2\left(x_\nu^3 + 3x_\nu y^2 + 9x_\nu^2 y^2 + 3x_\nu^3 y^2 + 3y^4 + 9x_\nu y^4 + 3x_\nu^2 y^4 + y^6\right) \\ &+ 480\big(1-y^2\big)^2 y^2\left(x_\nu^3 + 3x_\nu y^2 + 3x_\nu^2 y^2 + y^4\right)\epsilon^2 \\ &+ 240\left(x_\nu^3 + 3x_\nu y^2 - 3x_\nu^2 y^2 - y^4\right)\left(-y^2 + y^4\right)\epsilon^4\Big]\,, \end{aligned} \tag{D.130}$$

$$\sigma_b(1,0) = 4\left(2x_\nu - y^2\right)\,, \tag{D.131}$$

$$\begin{aligned}\sigma_b(1,1) = \frac{1}{x_\nu}\Big[&4\left(3x_\nu^2 + 3x_\nu^3 + 24x_\nu y^2 + 15x_\nu^2 y^2 - x_\nu^3 y^2 + 6y^4 + 10x_\nu y^4 + 2x_\nu^2 y^4 + 2y^6\right) \\ &+ 4\left(-4x_\nu^2 + x_\nu^3 - 8x_\nu y^2 - 3x_\nu^2 y^2 - 2y^4\right)\epsilon^2 + 4x_\nu^2\epsilon^4\Big]\,, \end{aligned} \tag{D.132}$$

$$\begin{aligned}\sigma_b(1,2) = \frac{1}{x_\nu}\Big[&24(x_\nu^3 + 6x_\nu y^2 + 26x_\nu^2 y^2 + 11x_\nu^3 y^2 + 8y^4 + 61x_\nu y^4 \\ &+ 60x_\nu^2 y^4 + 6x_\nu^3 y^4 + 22y^6 + 43x_\nu y^6 + 8x_\nu^2 y^6 + 4y^8) \\ &- 24\left(2x_\nu^3 + 12x_\nu y^2 + 32x_\nu^2 y^2 + 7x_\nu^3 y^2 + 10y^4 + 41x_\nu y^4 + 14x_\nu^2 y^4 + 6y^6\right)\epsilon^2 \\ &+ 24\left(x_\nu^3 + 6x_\nu y^2 + 6x_\nu^2 y^2 + 2y^4\right)\epsilon^4\Big]\,, \end{aligned} \tag{D.133}$$

$$\begin{aligned}\sigma_b(1,3) = \frac{1}{x_\nu}\Big[&240y^2(x_\nu^3 + 3x_\nu y^2 + 15x_\nu^2 y^2 + 10x_\nu^3 y^2 + 5y^4 \\ &+ 30x_\nu y^4 + 30x_\nu^2 y^4 + 5x_\nu^3 y^4 + 10y^6 + 15x_\nu y^6 + 3x_\nu^2 y^6 + y^8) \\ &- 480y^2\left(x_\nu^3 + 3x_\nu y^2 + 9x_\nu^2 y^2 + 3x_\nu^3 y^2 + 3y^4 + 9x_\nu y^4 + 3x_\nu^2 y^4 + y^6\right)\epsilon^2 \\ &+ 240y^2\left(x_\nu^3 + 3x_\nu y^2 + 3x_\nu^2 y^2 + y^4\right)\epsilon^4\Big]\,. \end{aligned} \tag{D.134}$$

D.6 Basic integrals for the azimuthal calculation

The basic integrals for the x–integration are defined by

$$\mathcal{X}_0 := \int\limits_{w_-}^{w_+} dx\sqrt{(1-x)\,(x-y^2) - xz} = \frac{1}{8}\pi\,\lambda\left(1, y^2, z\right)\,, \tag{D.135}$$

$$\mathcal{X}_1 := \int\limits_{w_-}^{w_+} dx\, x\sqrt{(1-x)(x-y^2)-xz} = \frac{1}{16}\pi\left(1+y^2-z\right)\lambda\left(1,y^2,z\right), \tag{D.136}$$

$$\mathcal{X}_{-1} := \int\limits_{w_-}^{w_+} dx \frac{\sqrt{(1-x)(x-y^2)-xz}}{x} = \frac{1}{2}\pi\left[(1-y)^2-z\right]. \tag{D.137}$$

The basic integrals for the z–integration in the azimuthal calculation are defined by

$$U(n) = \int dz \frac{\sqrt{\lambda(1,y^2,z)}}{\left[(1+y)^2-z\right]^n} Y_p\,, \qquad V(n) = \int dz \frac{\sqrt{\lambda(1,y^2,z)}}{\left[(1-y)^2-z\right]^n} Y_p\,. \tag{D.138}$$

Calculating these indefinite integrals one has

$$\begin{aligned} U(4) = &-\frac{y}{5(1+y)\left[(1+y)^2-z\right]^2} + \frac{7-4y+y^2}{30(1+y)^3\left[(1+y)^2-z\right]} \\ &- \frac{(15-5y+5y^2+y^3)\ln\left((1+y)^2-z\right)}{60(1+y)^5} - \frac{(1-y)^3(1+8y+y^2)\ln z}{120y^2(1+y)^5} \\ &- \frac{6y^3+y^4+6y(1-z)+(1-z)^2-2y^2(7+z)}{30y^2\left[(1+y)^2-z\right]^3} p_3 Y_p\,, \end{aligned} \tag{D.139}$$

$$\begin{aligned} U(3) = &-\frac{2y}{3(1+y)\left[(1+y)^2-z\right]} - \frac{(3+y^2)\ln\left((1+y)^2-z\right)}{6(1+y)^3} - \frac{(1-y)^3\ln z}{12y(1+y)^3} \\ &- \frac{(1-y)^2-z}{3y\left[(1+y)^2-z\right]^2} p_3 Y_p\,, \end{aligned} \tag{D.140}$$

$$\begin{aligned} U(2) = &\frac{8y\ln\left((1+y)^2-z\right) + \left[4-4y+(1+y)\ln y^2\right]\ln z}{4(1+y)} \\ &+ \frac{1}{2}\Big[\mathrm{Li}_2(p_+)+\mathrm{Li}_2(p_-)\Big] + \left(\frac{4p_3}{(1+y)^2-z} - Y_w\right)Y_p\,, \end{aligned} \tag{D.141}$$

$$U(1) = \frac{1}{2}\Big[z-\left(1+y-y^2\right)\ln z\Big] - y\Big[\mathrm{Li}_2(p_+)+\mathrm{Li}_2(p_-)\Big] - 2\left(p_3 - y\,Y_w\right)Y_p\,, \tag{D.142}$$

$$\begin{aligned} U(0) = &\frac{1}{8}\left(4z-z^2\right) - \frac{1}{4}\left(1-2y^4+y^2\ln y^2\right)\ln z - y^2\Big[\mathrm{Li}_2(p_+)+\mathrm{Li}_2(p_-)\Big] \\ &+ 2y^2 Y_w Y_p - \left(1+y^2-z\right)p_3 Y_p\,, \end{aligned} \tag{D.143}$$

$$\begin{aligned} V(2) = &-\frac{8y\ln\left((1-y)^2-z\right) - \left[4+4y+(1-y)\ln y^2\right]\ln z}{4(1-y)} \\ &+ \frac{1}{2}\Big[\mathrm{Li}_2(p_+)+\mathrm{Li}_2(p_-)\Big] + \left(\frac{4p_3}{(1+y)^2-z} - Y_w\right)Y_p\,, \end{aligned} \tag{D.144}$$

$$V(1) = \frac{1}{2}\Big[z-\left(1-y-y^2\right)\ln z\Big] + y\Big[\mathrm{Li}_2(p_+)+\mathrm{Li}_2(p_-)\Big] - 2\left(p_3 + y\,Y_w\right)Y_p\,. \tag{D.145}$$

Appendix E

Some technical notes on the semileptonic hyperon decays

E.1 T–odd contributions

In the main text we have assumed that the invariant form factors and thereby the helicity amplitudes are relatively real. If one allows for relative phases between the helicity amplitudes one will obtain so–called T–odd contributions in the angular decay distributions. They appear in the azimuthal correlation terms as can be seen by the following example taken from the joint angular decay distribution in Sec.6. One of the azimuthal correlation terms derives from the helicity configurations ($\lambda_2 = -1/2, \lambda_W = 0; \lambda'_2 = 1/2, \lambda'_W = 1$) and ($\lambda_2 = 1/2, \lambda_W = 1; \lambda'_2 = -1/2, \lambda'_W = 0$). Picking out the relevant terms in the master formula (5.60) one has

$$H_{\frac{1}{2}1}H^*_{-\frac{1}{2}0}\, e^{i(\pi-\chi)} + H_{-\frac{1}{2}0}H^*_{\frac{1}{2}1}\, e^{-i(\pi-\chi)} = -2\cos\chi \mathrm{ReH}_{\frac{1}{2}1}\mathrm{H}^*_{-\frac{1}{2}0} - 2\sin\chi \mathrm{ImH}_{\frac{1}{2}1}\mathrm{H}^*_{-\frac{1}{2}0}\ . \quad \text{(E.1)}$$

The $\cos\chi$ dependent term already appears in (5.61) whereas the $\sin\chi$ dependent term has been dropped in (5.61) because of the relative reality assumption for the helicity amplitudes. Adding the relevant θ and θ_B dependent trigonometric functions in the above azimuthal correlation term one has the two angle dependent T–odd terms ($\sin\theta \sin\chi \sin\theta_B$ $\mathrm{ImH}_{\frac{1}{2}1}\mathrm{H}^*_{-\frac{1}{2}0}$) and ($\cos\theta \sin\theta \sin\chi \sin\theta_B\, \mathrm{ImH}_{\frac{1}{2}1}\mathrm{H}^*_{-\frac{1}{2}0}$) proportional to $\sin\chi$.

Next we rewrite the product of angular factors in terms of scalar and pseodoscalar products using the momentum representations in the (x, y, z)–system (see Fig.5). For the normalized momenta one has ($\hat{p}^2 = 1$)

$$\begin{aligned} \hat{p}_{l^-} &= (\sin\theta\cos\chi, \sin\theta\sin\chi, -\cos\theta) \\ \hat{p}_W &= (0, 0, -1) \\ \hat{p}_{\Sigma^+} &= (0, 0, 1) \\ \hat{p}_p &= (\sin\theta_B, 0, \cos\theta_B)\ , \end{aligned} \quad \text{(E.2)}$$

where the momenta have unit length indicated by a hat notation. The above angular factors can then be rewritten as

$$\begin{aligned} \sin\theta\sin\chi\sin\theta_B &= \hat{p}_W \cdot (\hat{p}_{l^-} \times \hat{p}_p) \\ \cos\theta\sin\theta\sin\chi\sin\theta_B &= (\hat{p}_{l^-} \cdot \hat{p}_W)\,[\hat{p}_W \cdot (\hat{p}_{l^-} \times \hat{p}_p)] \end{aligned} \quad \text{(E.3)}$$

Under time reversal ($t \to -t$) one has ($p \to -p$). Since the T–odd momenta invariants in (E.3) involve an odd number of momenta they change sign under time reversal. This has led to the notion of the so–called T–odd obsevables. Observables that multiply T–odd momenta invariants are called T–odd obsevables. They can be contributed to by true CP–violating effects or by final state interaction effects unless either or both change all helicity amplitudes by a common phase. One may distinguish between the two sources of T–odd effects by comparing with the corresponding antihyperon decays since phases from CP–violating effects change sign whereas phases from final state interaction effects do not change sign when going from hyperon to antihyperon decays.

From the above example it should be clear how to obtain the T–odd contributions from the master formulas for the other cases. In practise what one has to do is to add terms where the real part of the bilinear forms of helicity amplitudes is replaced by the corresponding imaginary part and the cosine of the azimuthal angle is replaced by the sine with a possible sign change.

E.2 Full five-fold angular decay distribution

In this section we write down the full five-fold angular decay distribution for the semileptonic cascade decay of a polarized hyperon. There are now altogether three polar angles θ, θ_B and θ_P, where θ_P describes the polar orientation of the polarization vector of the parent hyperon as shown in Fig. E.1 (which is directly taken from [76]). Since there are now two planes in the cascade decay, there is one more azimuthal angle which we choose as ϕ_ℓ as shown in Fig. E.2. It is important to note that Fig. E.2 shows a special configuration where the momentum of the proton lies in the first quadrant and the momentum of the lepton lies in the second quadrant. It is clear that, for this special configuration, the three azimuthal angles ϕ_ℓ, ϕ_B and χ add up to π ($\phi_\ell + \phi_B + \chi = \pi$). For other configurations it may happen that the three angles add up to $\pi + \mathrm{mod}(2\pi)$ if the rotation sense of the angles in Fig. E.2 is kept. This will be of no consequence for the angular decay distribution which is invariant under azimuthal 2π shifts.

The full five-fold angular decay distribution can be directly taken from [76] after including the appropiate sign changes going from the (ℓ^+, ν_ℓ) to the $(\ell^-, \bar{\nu}_\ell)$ case[1]. We have simplified the corresponding expressions in [76] by assuming as before that the helicity amplitudes are real. For completeness we shall also write down the decay distribution in explicit form using Wigner's d^J–functions as before. One has the master formula

$$
\begin{aligned}
W(\theta, \theta_P, \theta_B, \phi_B, \phi_\ell) \;\; \propto \;\; & \sum_{\lambda_\ell, \lambda_W, \lambda'_W, J, J', \lambda_2, \lambda'_2, \lambda_3} (-1)^{J+J'} |h^l_{\lambda_\ell \lambda_{\nu=\pm 1/2}}|^2 e^{i(\lambda_W - \lambda'_W)\phi_\ell} \\
& \rho_{\lambda_2 - \lambda_W, \lambda'_2 - \lambda'_W} d^J_{\lambda_W, \lambda_\ell - \lambda_\nu}(\theta) d^{J'}_{\lambda'_W, \lambda_\ell - \lambda_\nu}(\theta) H_{\lambda_2 \lambda_W} H^*_{\lambda'_2 \lambda'_W} \\
& e^{i(\lambda_2 - \lambda'_2)\phi_B} d^{\frac{1}{2}}_{\lambda_2 \lambda_3}(\theta_B) d^{\frac{1}{2}}_{\lambda'_2 \lambda_3}(\theta_B) |h^B_{\lambda_3 0}|^2
\end{aligned}
\tag{E.4}
$$

[1] Apart from listing angular decay distributions [76] contains much additional useful material as e.g. a discussion of the statistical tensors of the processes and their bounds, HQET results for the form factors etc..

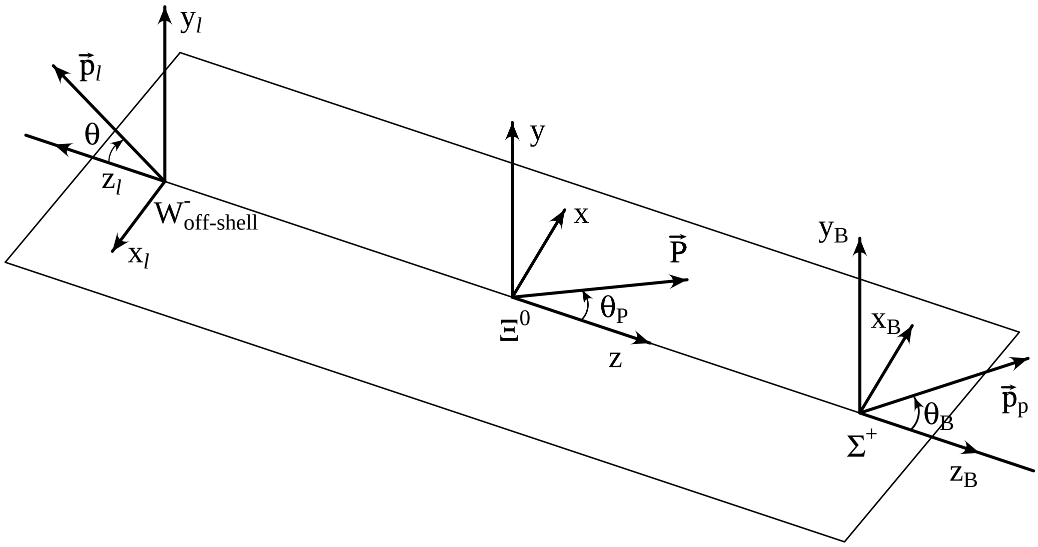


Figure E.1: Definition of the three polar angles θ, θ_B and θ_P in the semileptonic decay of a polarized Ξ^0 into $\Sigma^+ + \ell^- + \bar{\nu}_\ell$ followed by the nonleptonic decay $\Sigma^+ \to p + \pi^0$. The polarization vector of the parent baryon $\vec{P}$ lies in the (x,z)–plane with positive P_x component.

For the normalized five-fold angular decay distribution one finds

$$\frac{d\Gamma}{dq^2 d\cos\theta_B d\cos\theta d\cos\theta_P d\chi d\phi_\ell} = B(B_2 \to B_3 + M)\frac{1}{12}\frac{G^2}{(2\pi)^5}|V_{us}|^2\frac{(q^2-m_\ell^2)^2 p}{8M_1^2 q^2}$$
$$\Big[\; b_{00}^{00} + 3\cos\theta\; b_{00}^{01} + \cos\theta_B\; b_{00}^{10} \tag{E.5}$$
$$+(\frac{3}{2}\cos^2\theta - \frac{1}{2})\; b_{00}^{02}$$
$$-3\sqrt{2}\sin\theta\cos\phi_\ell\; b_{01}^{01}$$
$$-2\sin\theta_B\cos\phi_B\; b_{10}^{10}$$
$$-\frac{3}{\sqrt{2}}\sin 2\theta\cos\phi_\ell\; b_{01}^{02}$$
$$+3\cos\theta\cos\theta_B\; b_{00}^{11}$$
$$+(\frac{3}{2}\cos^2\theta - \frac{1}{2})\cos\theta_B\; b_{00}^{12}$$
$$-\frac{3}{2}\sqrt{2}\sin\theta\sin\theta_B\cos\chi\; b_{11}^{11}$$
$$-\frac{3}{4}\sqrt{2}\sin\theta_B\sin 2\theta\cos\chi\; b_{11}^{12}$$
$$+\frac{3}{2}\sin^2\theta\sin\theta_B\cos(\chi-\phi_\ell)\; b_{12}^{12}$$
$$-6\cos\theta\sin\theta_B\;\; \cos\phi_B\; b_{10}^{11}$$
$$-3\sqrt{2}\sin\theta\cos\theta_B\cos\phi_\ell\; b_{01}^{11}$$

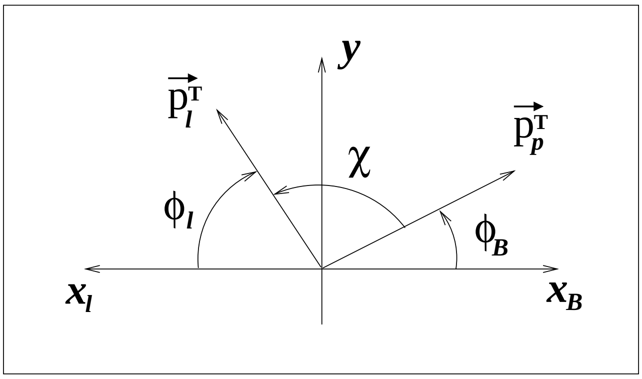


Figure E.2: Definition of the three azimuthal angles ϕ_ℓ, ϕ_B and χ ($\phi_\ell + \phi_B + \chi = \pi$) in the semileptonic decay of a polarized Ξ^0. Fig. E.2 is a view of Fig. E.1 from the right along the negative z–direction. $\vec{p}_\ell^T$ and $\vec{p}_p^T$ denote the transverse components of the momentum of the lepton and proton, respectively.

$$-\frac{3}{2}\sqrt{2}\sin 2\theta \cos\theta_B \cos\phi_\ell \; b_{01}^{12}$$
$$-\sin\theta_B(3\cos^2\theta - 1)\cos\phi_B b_{10}^{12} \Big]$$

It is important that the rotation sense of the azimuthal angles in Fig. E.2 is kept. We have used the relation $\phi_\ell + \phi_B + \chi = \pi + \mathrm{mod}(2\pi)$ to rewrite $\cos(\phi_B + \phi_\ell) = -\cos\chi$ and $\cos(\phi_B + 2\phi_\ell) = -\cos(\chi - \phi_\ell)$. Note that (E.5) contains the redundant angle ϕ_B. As before one can reexpress $\cos\phi_B$ as $\cos\phi_B = -\cos(\phi_\ell + \chi)$.

The coefficients b_{ij}^{kl} in (E.5) are given by [2]

$$\begin{aligned}
b_{00}^{00} &= ((1+\epsilon_\ell)(|H_{-\frac{1}{2}-1}|^2 + |H_{\frac{1}{2}0}|^2) + 3\epsilon_\ell|H_{\frac{1}{2}t}|^2)\,\rho_{\frac{1}{2}\frac{1}{2}} \\
&\quad +((1+\epsilon_\ell)(|H_{\frac{1}{2}1}|^2 + |H_{-\frac{1}{2}0}|^2) + 3\epsilon_\ell|H_{-\frac{1}{2}t}|^2)\,\rho_{-\frac{1}{2}-\frac{1}{2}} \\
b_{00}^{01} &= \frac{1}{2}(\mp|H_{\frac{1}{2}1}|^2 - 4\epsilon_\ell\; H_{-\frac{1}{2}0}H_{-\frac{1}{2}t})\,\rho_{-\frac{1}{2}-\frac{1}{2}} \\
&\quad -\frac{1}{2}(\mp|H_{-\frac{1}{2}-1}|^2 + 4\epsilon_\ell\; H_{\frac{1}{2}0}H_{\frac{1}{2}t})\,\rho_{\frac{1}{2}\frac{1}{2}} \\
b_{00}^{10} &= \alpha_B(-(1+\epsilon_\ell)(|H_{-\frac{1}{2}0}|^2 - |H_{\frac{1}{2}1}|^2) - 3\epsilon_\ell|H_{-\frac{1}{2}t}|^2)\,\rho_{-\frac{1}{2}-\frac{1}{2}} \\
&\quad +\alpha_B((1+\epsilon_\ell)(-|H_{-\frac{1}{2}-1}|^2 + |H_{\frac{1}{2}0}|^2) + 3\epsilon_\ell|H_{\frac{1}{2}t}|^2)\,\rho_{\frac{1}{2}\frac{1}{2}} \\
b_{00}^{02} &= \frac{1-2\epsilon_\ell}{2}(-2|H_{-\frac{1}{2}0}|^2 + |H_{\frac{1}{2}1}|^2)\,\rho_{-\frac{1}{2}-\frac{1}{2}} \\
&\quad +\frac{1-2\epsilon_\ell}{2}(|H_{-\frac{1}{2}-1}|^2 - 2|H_{\frac{1}{2}0}|^2)\,\rho_{\frac{1}{2}\frac{1}{2}}, \\
b_{01}^{01} &= \frac{1}{2}(2\epsilon_\ell H_{-\frac{1}{2}t}H_{-\frac{1}{2}-1} \pm H_{-\frac{1}{2}0}H_{-\frac{1}{2}-1}
\end{aligned} \tag{E.6}$$

[2]The coefficient b_{10}^{11} takes twice the value as compared to the corresponding coefficient in [76]. Also in (43) of [76] concerning the overall normalization one has to do the replacement $q^2 \to (q^2 - m_\ell^2)^2/(2q^2)$.

$$
\begin{aligned}
& -2\epsilon_\ell H_{\frac{1}{2}1}H_{\frac{1}{2}t} \pm H_{\frac{1}{2}1}H_{\frac{1}{2}0})\,\rho_{\frac{1}{2}-\frac{1}{2}},\\
b^{10}_{10} &= -\alpha_B(3\epsilon_\ell H_{\frac{1}{2}t}H_{-\frac{1}{2}t} + (1+\epsilon_\ell)H_{\frac{1}{2}0}H_{-\frac{1}{2}0})\,\rho_{-\frac{1}{2}\frac{1}{2}}\\
b^{02}_{01} &= \frac{1-2\epsilon_\ell}{2}(H_{-\frac{1}{2}0}H_{-\frac{1}{2}-1} - H_{\frac{1}{2}1}H_{\frac{1}{2}0})\,\rho_{\frac{1}{2}-\frac{1}{2}},\\
b^{11}_{00} &= \frac{\alpha_B}{2}(\mp|H_{\frac{1}{2}1}|^2 + 4\epsilon_\ell\; H_{-\frac{1}{2}0}H_{-\frac{1}{2}t})\,\rho_{-\frac{1}{2}-\frac{1}{2}}\\
& +\frac{\alpha_B}{2}(\mp|H_{-\frac{1}{2}-1}|^2 - 4\epsilon_\ell\; H_{\frac{1}{2}0}H_{\frac{1}{2}t})\,\rho_{\frac{1}{2}\frac{1}{2}}\\
b^{12}_{00} &= \frac{\alpha_B}{2}(1-2\epsilon_\ell)(2|H_{-\frac{1}{2}0}|^2 + |H_{\frac{1}{2}1}|^2)\,\rho_{-\frac{1}{2}-\frac{1}{2}}\\
& -\frac{\alpha_B}{2}(1-2\epsilon_\ell)(|H_{-\frac{1}{2}-1}|^2 + 2|H_{\frac{1}{2}0}|^2)\,\rho_{\frac{1}{2}\frac{1}{2}},\\
b^{11}_{11} &= \alpha_B(2\epsilon_\ell H_{\frac{1}{2}1}H_{-\frac{1}{2}t} \mp H_{\frac{1}{2}1}H_{-\frac{1}{2}0})\,\rho_{-\frac{1}{2}-\frac{1}{2}}\\
& +\alpha_B(-2\epsilon_\ell H_{\frac{1}{2}t}H_{-\frac{1}{2}-1} \mp H_{\frac{1}{2}0}H_{-\frac{1}{2}-1})\,\rho_{\frac{1}{2}\frac{1}{2}},\\
b^{12}_{11} &= \alpha_B(1-2\epsilon_\ell)(H_{\frac{1}{2}1}H_{-\frac{1}{2}0}\,\rho_{-\frac{1}{2}-\frac{1}{2}} - H_{\frac{1}{2}0}H_{-\frac{1}{2}-1}\,\rho_{\frac{1}{2}\frac{1}{2}}),\\
b^{12}_{12} &= -\alpha_B(1-2\epsilon_\ell)H_{\frac{1}{2}1}H_{-\frac{1}{2}-1}\,\rho_{\frac{1}{2}-\frac{1}{2}},\\
b^{11}_{10} &= \alpha_B\,\epsilon_\ell\,(H_{\frac{1}{2}0}H_{-\frac{1}{2}t} + H_{\frac{1}{2}t}H_{-\frac{1}{2}0})\,\rho_{-\frac{1}{2}\frac{1}{2}},\\
b^{11}_{01} &= \frac{\alpha_B}{2}(-2\epsilon_\ell H_{-\frac{1}{2}t}H_{-\frac{1}{2}-1} \mp H_{-\frac{1}{2}0}H_{-\frac{1}{2}-1}\\
& -2\epsilon_\ell H_{\frac{1}{2}1}H_{\frac{1}{2}t} \pm H_{\frac{1}{2}1}H_{\frac{1}{2}0})\,\rho_{\frac{1}{2}-\frac{1}{2}},\\
b^{12}_{01} &= -\frac{\alpha_B}{2}(1-2\epsilon_\ell)(H_{-\frac{1}{2}0}H_{-\frac{1}{2}-1} + H_{\frac{1}{2}1}H_{\frac{1}{2}0})\,\rho_{\frac{1}{2}-\frac{1}{2}},\\
b^{12}_{10} &= \alpha_B(1-2\epsilon_\ell)H_{\frac{1}{2}0}H_{-\frac{1}{2}0}\,\rho_{-\frac{1}{2}\frac{1}{2}},
\end{aligned}
$$

We have introduced the abbreviation $\epsilon_\ell = m_\ell^2/2q^2$ for the leptonic flip suppression factor. As in the main text the upper signs in the coefficients b^{kl}_{ij} hold for the case $(\ell^-, \bar\nu_\ell)$ relevant to the cascade decay $\Xi^0 \to \Sigma^+(\to p + \pi^0) + \ell^- + \bar\nu_\ell$ treated in this thesis. The lower signs hold for the case (ℓ^+, ν_ℓ) as was discussed in [76]. Finally, $\rho_{\lambda_1\lambda_1'}$ is the spin density matrix of the parent hyperon given in (5.54).

We have performed various checks on (E.5). First we found it to agree with the angular decay distribution derived from the master formula (E.4). We further checked that (E.5) reduces to the decay distributions listed in the main text after integration or after setting the relevant parameters to zero. We thus checked that (E.5) reduces to (5.61) when setting $P = 0$. There is a factor of 4π from the integration over $\cos\theta_P$ and ϕ_ℓ. Further (E.5) reduces to (5.56) when setting $\alpha_B = 0$, dropping the branching ratio factor $B(B_2 \to B_3 + M)$ and replacing ϕ_ℓ by $(\pi - \chi)$. Also there is a factor 4π from the integration over $\cos\theta_B$ and ϕ_B. Finally, (E.5) reduces to (5.59) when integrating over ϕ_ℓ and $\cos\theta$. As mentioned before we have assumed that the helicity amplitudes (or the invariant amplitudes) are relatively real. Nonzero relative phases between the helicity amplitudes could arise from final state interaction effects or from extensions of the SM that bring in CP–violating phases (see e.g. [113, 114]). In such a case one would have to keep the full phase structure contained in the master formula (E.4).

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