Mutually Catalytic Branching at Infinite Rate

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Summary

The purpose of this doctoral thesis is to prove existence for a mutually catalytic random walk with infinite branching rate on countably many sites. The process is defined as a weak limit of an approximating family of processes. An approximating process is constructed by adding jumps to a deterministic migration on an equidistant time grid. As law of jumps we need to choose the invariant probability measure of the mutually catalytic random walk with a finite branching rate in the recurrent regime. This model was introduced by Dawson and Perkins (1998) and this thesis relies heavily on their work. Due to the properties of this invariant distribution, which is in fact the exit distribution of planar Brownian motion from the first quadrant, it is possible to establish a martingale problem for the weak limit of any convergent sequence of approximating processes. We can prove a duality relation for the solution to the mentioned martingale problem, which goes back to Mytnik (1996) in the case of finite rate branching, and this duality gives rise to weak uniqueness for the solution to the martingale problem. Using standard arguments we can show that this solution is in fact a Feller process and it has the strong Markov property.

For the case of only one site we prove that the model we have constructed is the limit of finite rate mutually catalytic branching processes as the branching rate approaches infinity. Therefore, it seems natural to refer to the above model as an infinite rate branching process. However, a result for convergence on infinitely many sites remains open.

Key words and phrases. martingale problem, mutually catalytic branching, infinite branching rate, dual process, super-random walk, weak convergence

Contents

Introduction

Motivation and main results

Branching random walks are processes that describe populations of particles that are placed in some site space. There are two kinds of dynamics on this population: motion and branching. Particles move independently through the space and reproduce or die according to some branching law independent of the motion. Traditionally in branching theory, the basic assumption is that disjoint parts develop independently. This independence assumption allows the use of a lot of mathematical tools, which has made the development of a huge mathematical theory possible; see, for example, the lecture notes [Da93], [Eth00] and [Per02]. If we consider two types of populations (or substances), though, there is the possibility to introduce interaction between both substances, and this interaction is meant to violate the basic independence assumptions. In this thesis we assume interaction of both types via a linear influence on the opposite substance's branching rate. If the interaction is only one-sided, which means substance 1 is assumed to evolve autonomously whereas the branching of substance 2 is assumed to be controlled by substance 1, then the terms catalyst and reactant stand for both substances, respectively. In this case the catalyst makes it possible for the reactant to grow (or die) – see for instance [GKW99] and [DF91] – hence the names. Yet, for this one-sided interaction conditional independence is retained.

In 1998 Dawson and Perkins, see [DP98], introduced and studied a mutually catalytic branching model. In their model both substances catalyze each other; that is to say, the branching rate of each type at a site is proportional to the amount of the other type present at that site. This true interaction of types destroys the usual independence assumption in branching theory. In particular, this model is not a superprocess (if the set of sites is the real line) or a super-random walk (if the set of sites is countable, e.g. the lattice $\mathbb{Z}^d$) in its standard definition. See [DF99] for a survey and a more detailed introduction to catalytic and mutually catalytic models.

In this thesis we concentrate on (and use) the results of the semi-discrete model of Dawson and Perkins: the site space is countable and the population size of both substances on one site is continuous, i.e. a pair of non-negative real numbers. We will go into greater detail in Section 1.1, in which the model of Dawson and Perkins on the lattice is described. As in [DF00], with an abuse of language we call this model super-random walk. Nevertheless, note that the model of Dawson and Perkins has a finite branching rate. Our aim is to establish a version of this model with an infinite branching rate.

Now, we first specialize the model of Dawson and Perkins for one colony. We name this colony

0 and indicate it as a subscript. Let the pair $(Z^\gamma_{1,0,t}, Z^\gamma_{2,0,t}) \in [0,\infty)^2$ describe the size of both populations, namely types 1 and 2, on colony 0 at time $t \geq 0$. We consider a drift towards some point $\Theta = (\theta_1, \theta_2) \in [0,\infty)^2$. Then, the evolution in time is governed by the SDE

$$dZ^\gamma_{\alpha,0,t} \;=\; \big(\theta_\alpha \;-\; Z^\gamma_{\alpha,0,t}\big)dt \;+\; \sqrt{\gamma\, Z^\gamma_{1,0,t}\, Z^\gamma_{2,0,t}}\; dB_{\alpha,0,t}\,, \tag{1}$$

for all times $t \geq 0$ and types $\alpha \in \{1,2\}$, where $(B_{\alpha,0,t})_{t\geq 0}$ are two independent standard Brownian motions. We indicate the dependence on the constant $\gamma > 0$ as a superscript. In the context of ordinary Feller diffusions γ is called the branching rate and it indicates the variance of the offspring distribution of approximating Galton-Watson branching processes; see, for example, [EK86] Theorem 9.1.3 on p.388.

One idea to establish a version of (1) with $\gamma = \infty$, which means with an infinite branching rate (or with infinite variance), is to trade time for variance, i.e. we consider $t \to \infty$ instead of $\gamma \to \infty$. It is best pictured in Equation (2.24) in the proof of Lemma 2.6 that this argumentation makes good sense. At this point we note in addition that Dawson and Perkins investigated the long-term behaviour of their model and established a limit distribution, which has full expectation but infinite variance (under some recurrence assumption). It turns out that this limit distribution is given by the exit distribution of planar Brownian motion from the first quadrant. We denote this distribution by DP; that means, we set $DP_x(d\xi) := P_x\big[\, B_T \in d\xi \,\big]$ if planar Brownian motion $B_t = (B_{1,t}, B_{2,t})$ starts in $x \in [0,\infty)^2$ and where $T = \inf\{\, t > 0 : B_{1,t}\, B_{2,t} = 0\}$. In fact, we will construct a process $X = (X_{1,0,t}, X_{2,0,t})$ such that

$$\mathcal{L}[X_{\cdot,0,t}] \;=\; DP_{(\mu,\nu)}\,, \tag{2}$$

with parameters $\mu, \nu \in [0,\infty)$ depending on $t \geq 0$ and on the initial value of X. And, moreover, we can show that for any $t \geq 0$

$$\lim_{\gamma\to\infty} \mathcal{L}[Z^\gamma_{\cdot,0,t}] \;=\; \mathcal{L}[X_{\cdot,0,t}]\,, \tag{3}$$

provided both processes have the same initial condition. Recall that the DP-distribution only charges the boundary of the first quadrant. Hence, the appropriate state space for X on site 0 is $L := \partial[0,\infty)^2$. In particular, in the case of infinite rate branching, at a fixed time only one type can live at site 0.

The proof of Equations (2) and (3) above involves a duality relation for X and Z^γ, respectively, which goes back to Mytnik, see [My98b] or [My96]. We will describe this duality below, see Equation (7), for countably many sites. However, to establish this duality for X and Z^γ as above it is convenient to adjoin an auxiliary site, named 1, say. We choose the size of both types on colony 1 constant and equal to (θ_1, θ_2). Therefore, we consider $X_t = \big((X_{1,0,t}, X_{2,0,t})\,,\,(\theta_1,\theta_2)\big)$ and $Z^\gamma_t = \big((Z^\gamma_{1,0,t}, Z^\gamma_{2,0,t})\,,\,(\theta_1,\theta_2)\big)$ as processes on two colonies. In this context it becomes more obvious how to adopt the idea of Mytnik to establish a duality relation and how to find the proper dual process.

The infinite variance process $(X_{1,0,t}\,,\,X_{2,0,t})$ on site 0 with drift towards (θ_1,θ_2), which is a Markov process, can also be characterised by its generator $\mathcal{A}_\Theta$. It is a Lévy-type generator of the following form: Let $x = (x_1, x_2) \in L$ and $f : L \longrightarrow \mathbb{R}$ be a smooth function. If $x_\alpha > 0$ (the other type being zero) then

$$\mathcal{A}_\Theta f(x) \;=\; \theta_\alpha \int\limits_L \Big[f(y) \,-\, f(x) \,-\, (y_\alpha - x_\alpha)\,\partial_\alpha f(x) \Big]\, \eta(x,dy) \;+\; (\theta_\alpha - x_\alpha)\,\partial_\alpha f(x)\,, \tag{4}$$

for some σ-finite (jump) measure η on L with singularity at x. By $\partial_\alpha f(x)$ we denote the partial derivate $\frac{\partial}{\partial x_\alpha} f(x)$ of f. See Proposition 2.1 for an explicit representation of η. Here, we only remark that the map $x \mapsto \eta(x, A)$ for some open set A containing $0 \in L$ is not continuous (at zero).

Next, we turn our attention to a countably infinite site space S. In this context it will be necessary to impose conditions on the migration of particles, and to restrict the class of configurations which are permitted. To this end we introduce Liggett-Spitzer-type spaces $\mathbb{E}_\gamma \subseteq \big([0,\infty)\times[0,\infty)\big)^S$ and $\mathbb{L}_\gamma \subseteq L^S$ with respect to some weight function γ on S; see page 29 for a definition. – Please do not mistake the weight function γ on S for the branching parameter γ as in Equation (1). – Since the spaces $\mathbb{E}_\gamma$ and $\mathbb{L}_\gamma$ are not locally compact we cannot use the usual Hille-Yoshida machinery to establish existence of the process with state space $\mathbb{L}_\gamma$. Instead, we have to construct 'by hands' a family of approximating processes $\big\{ \tilde{X}^\varepsilon \,:\, 0 < \varepsilon \le 1 \big\}$ with configurations in $\mathbb{L}_\gamma$. Such a process $\tilde{X}^\varepsilon$ is piecewise constant and has jumps on an equidistant time grid with grid size $\varepsilon > 0$. The law of jumps is given by the DP-distribution. Using standard tightness arguments we can show that a subsequence $(\tilde{X}^{\varepsilon_n})_n$ converges to some process X, i.e.

$$\tilde{X}^{\varepsilon_n} \implies X, \quad \text{as } \varepsilon_n \searrow 0, \tag{5}$$

in the sense of weak convergence of processes with paths in $D_{\mathbb{L}_\gamma}[0,\infty)$, the space of càdlàg functions with values in $\mathbb{L}_\gamma$. In order to show this we define truncated processes $\tilde{X}^{K,\varepsilon}$ which are bounded by some arbitrary large $K > 0$, and hence, these processes possess a finite second moment. However, $\tilde{X}^{K,\varepsilon}$ and $\tilde{X}^\varepsilon$ coincide on a set with probability close to one. For the definition of the truncated processes we need a bounded variant of the DP-distribution – that is, the exit distribution of planar Brownian motion when it leaves the box $[0,K]^2$.

We will identify the limit X in Equation (5) as the solution to a martingale problem. To this end, we need to define processes X^ε with configurations in $\mathbb{E}_\gamma$ instead of $\mathbb{L}_\gamma$, but which are close to $\tilde{X}^\varepsilon$ in some sense; see Equation (3.21) on page 38. Then, the statement of Equation (5) remains valid with X^ε instead of $\tilde{X}^\varepsilon$. For Mytnik's duality functions $F(\cdot, y)$, indexed by $y \in \mathbb{L}_b$, where $\mathbb{L}_b$ is some subset of $\mathbb{L}_\gamma$ – see Equation (3.25) on page 39 for definitions – and some operator $\mathscr{A}$ given by Equation (3.27), the processes X^ε allow

$$t \;\mapsto\; F(X^\varepsilon_t, y) \;-\; F(X^\varepsilon_0, y) \;-\; \int_0^t \mathscr{A} F(X^\varepsilon_s, y)\, ds \tag{6}$$

to be a martingale. X inherits this martingale property since we can show that the family of processes $\big\{ X^\varepsilon \,:\, 0 < \varepsilon \le 1 \big\}$ has uniformly bounded moments of order p, where $1 \le p < 2$. All these facts are due to properties of the DP-distribution.

Furthermore, Mytnik's duality remains valid. There is a dual process Y with the same dynamics as X, but with configurations in the smaller set $\mathbb{L}_b$, such that whenever X starts in $x \in \mathbb{L}_\gamma$ and Y starts in $y \in \mathbb{L}_b$ then for all $t \ge 0$

$$E_x\big[\, F(\, X_t \,,\, y\,)\,\big] \;=\; E_y\big[\, F(\, x \,,\, Y_t\,)\,\big]. \tag{7}$$

Mytnik's duality functions $F(\cdot, y)$, which we sometimes refer to as mixed Laplace-Fourier transforms, separate distributions on $\mathbb{L}_\gamma$ if $y \in \mathbb{L}_b$, hence, relation (7) uniquely determines the one-dimensional distributions of X. Since X satisfies the martingale property of Equation (6) we infer that X is a solution to the martingale problem for $\mathscr{A}$ and functions $F(\cdot, y)$, $y \in \mathbb{L}_b$. Then (7) implies uniqueness of the finite-dimensional distributions of X. Exploiting further duality (7) we prove the Feller property and the strong Markov property for X. We can summarise the results of Chapter 3 with the following statement.

The processes $\{ X^\varepsilon : 0 < \varepsilon \leq 1 \}$ weakly converge, as $\varepsilon \searrow 0$, to the unique solution of the $D_{\mathbb{L}_\gamma}[0, \infty)$-martingale problem associated with Equation (6). This solution has the Feller property and is a strong Markov process.

This process is the "mutually catalytic super-random walk" we wished to construct.

Outline

This thesis is basically composed of four chapters. Firstly, in Section 1.1, we present very briefly and rather informally the model of Dawson and Perkins, which is the catalytic branching random walk with finite branching rate. Then, we define the DP-distribution and investigate its properties in Section 1.2. Chapter 2 is devoted to the model with infinite branching rate on one colony. Tedious calculations for the model's pregenerator of Equation (4) are deferred to the Appendix. The conception of two colonies allows us to find a natural dual process and due to this we are able to compute the model's transition probabilities in Section 2.2; compare with Equation (2). By means of duality we can show in Lemma 2.6 that the infinite rate branching model can be achieved as the limit of finite rate branching, as stated in Equation (3).

The main part of this thesis is Chapter 3. In Section 3.1 we define the family of approximating processes $\{ X^\varepsilon : 0 < \varepsilon \leq 1 \}$ on countably many sites and show that this family is tight. To establish tightness in Section 3.2 we need bounded versions of the approximating processes. Therefore, Section 1.3 introduces a truncated version of the DP-distribution. This idea is picked up at the end of sections 2.1 and 3.1. In Section 3.3 we establish a martingale problem, as indicated in Equation (6), and Mytnik's duality for the weak limit of the approximating processes, cf. Equation (7). Duality will imply uniqueness for the martingale problem and its solution has the Feller and the strong Markov property; see Lemma 3.23 below.

At the end Chapter 4 gives suggestions for further development in this area.

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I am very grateful to (...) who did not hesitate to proofread. And finally, I wish to thank the great (...) and his institute in Erlangen for some other reasons.

[1]Please note that the Johannes Gutenberg-University of Mainz does not allow to name the referees of this thesis or any other persons in the acknowledgement of this online publication. Thus all names have been withdrawn.

Chapter 1

Preliminaries

1.1 The model of Dawson and Perkins

In this first section, we would like to review some facts of D. Dawson's and E. Perkins' model for *mutually catalytic branching with finite variance*, cf. [DP98]. Thereby we will introduce our notation. The stochastic process Z describes a two-type "infinitesimal mass" interacting particle system on $\mathbb{Z}^d$, i.e. Z is a vector of pairs (or a pair of vectors), namely $Z_t = (\, Z_{1,k,t}\,,\, Z_{2,k,t}\,)_{k\in\mathbb{Z}^d}$, where $Z_{\alpha,k,t} \in [0,\infty)$ denotes the amount of mass of particle type $\alpha \in \{1,2\}$ at site $k \in \mathbb{Z}^d$ at time $t \geq 0$. In the sequel we will sloppily abbreviate for example, $Z_{1,t}$ or $Z_{\alpha,\cdot,t}$ for $(Z_{1,k,t})_{k\in\mathbb{Z}^d}$ and at times we simply write Z_t for the configuration $Z_{\cdot,\cdot,t}$ at time $t \geq 0$. We might even write $Z_{1,k}(t)$ or $Z_{1,t}(k)$ instead of $Z_{1,k,t}$, which is more close to the usual notation. However, we will use the letters t, s or r to label time and k, j or l to mark the site, so there should be no mistake in name.

Formally the process is given by the following system of integral equations. For each type $\alpha \in \{1,2\}$ we have

$$Z_{\alpha,k,t} \;=\; Z_{\alpha,k,0} \;+\; \int_0^t (Z_{\alpha,\cdot,s}Q)_k \, ds \;+\; \int_0^t \sqrt{\gamma\, Z_{1,k,s}\, Z_{2,k,s}}\; dB_{\alpha,k,s}\,, \tag{1.1}$$

for $t \geq 0$ and $k \in \mathbb{Z}^d$, where $\{\, (B_{\alpha,k,t})_{t\geq 0} : \alpha \in \{1,2\},\, k \in \mathbb{Z}^d \,\}$ is a family of independent one-dimensional Brownian motions, $\gamma > 0$, and, $Q = (q_{jk})_{j,k\in\mathbb{Z}^d}$ denotes the Q-matrix of a continuous time $\mathbb{Z}^d$-valued Markov chain, that is, q_{jk} denotes the jump rate from j to k. The matrix multiplication reads as $(Z_{\alpha,\cdot,s}Q)_k := \sum_{j\in\mathbb{Z}^d} Z_{\alpha,j,s}q_{jk}$ for each $k \in \mathbb{Z}^d$.

Equation (1.1) can be interpreted in the following way: The particles migrate according to Q, each type independent of the other. Since the process Z describes the behaviour of infinitely many particles with infinitesimal mass, we should rather say that the mass of each particle type flows independently of the other type. In addition, independently on each site $k \in \mathbb{Z}^d$, the mass of each type fluctuates randomly according to Feller's branching diffusion. The parameter $\gamma > 0$ represents the variance of the branching mechanism; see [EK86] Chap.9, p.386-389 for the simplest case. Here, however, the diffusion rate is in addition proportional to the amount of mass of the other type at that particular site. Hence, branching of type 1 is only possible in the presence of type 2, and vice versa.

Theorems 1.1 and 2.2 of [DP98] ensure existence of this process Z, provided the entries of Q

satisfy some exponential growth conditions; cf. the conditions (H0) to (H2) on page 1090 of that paper. Z then has continuous paths with values in a subspace of $\big([0,\infty)\times[0,\infty)\big)^{\mathbb{Z}^d}$ such that the configurations only grow 'tempered', i.e. the term $\langle Z_{\alpha,\cdot,t}, e^{-\lambda|\cdot|}\rangle := \sum_{k\in\mathbb{Z}^d} Z_{\alpha,k,t}\, e^{-\lambda|k|}$ is finite for all $\lambda > 0$, where $|k| = \sum_{i=1}^d |k_i|$ for $k = (k_1,\ldots,k_d) \in \mathbb{Z}^d$.

This model (especially the continuous site version; that means if the set of sites is the real line, $\mathbb{R}$, or the plane, $\mathbb{R}^2$ – instead of $\mathbb{Z}^d$) has attracted attention of various well-known authors, from C. Mueller, see [MP00], who, according to Dawson and Perkins, suggested this model, to A. Etheridge, K. Fleischmann and J. Xiong, cf. the trilogy [DE02a], [DE02b] and [DE03c], or Cox and Klenke, see [CKP99] and [CK00], to mention a few. There is also a version with more than two types of particles; see [FX01] and [DFX05].

Equivalently, it is possible to rewrite Equation (1.1) via the process's generator $\mathscr{A}$, which, applied to Mytnik's duality functions $F(\cdot,y)$ – see Equation (3.25) on page 39 for a definition – reads as

$$\begin{aligned}\mathscr{A}F(\cdot,y)(z) \;=\; F(z,y)\Big[&-\langle z_{1,\cdot}+z_{2,\cdot}\,,\, Q^*(y_{1,\cdot}+y_{2,\cdot})\rangle \\ &+ i\langle z_{1,\cdot}-z_{2,\cdot}\,,\, Q^*(y_{1,\cdot}-y_{2,\cdot})\rangle \\ &+ 4\gamma \sum_{k\in\mathbb{Z}^d} z_{1,k}\, z_{2,k}\, y_{1,k}\, y_{2,k}\Big]\end{aligned} \tag{1.2}$$

for appropriate $z,y \in \big([0,\infty)\times[0,\infty)\big)^{\mathbb{Z}^d}$, where Q^* denotes the transpose of Q. Compare with Equation (2.1) in [DP98] and Remark 2.5 of the same reference. The martingale problem for $\mathscr{A}$ and Mytnik's duality functions then implies the following duality. Let Z, starting in z, satisfy Equation (1.1) then Theorem 2.4 of [DP98] gives for $t \geq 0$

$$E_z\Big[F\big(Z_t\,,\,y\big)\Big] \;=\; E_y\Big[F\big(z\,,\,Y_t\big)\Big], \tag{1.3}$$

for a dual process Y (with initial condition y) of the same type as Z; that means Y also satisfies Equation (1.1) with Q replaced by Q^*. In [DP98] the dual process Y has to have configurations that are rapidly decreasing to make the expressions in (1.3) well defined; for example, let $(y_{1,k}, y_{2,k}) = (0,0)$ for all but finitely many $k \in \mathbb{Z}^d$. By Lemma 2.3 of [DP98] the class of functions $F(\cdot,y)$ considered in (1.3) separates measures and is convergence determining. Hence, because of this self-duality, which is due to Mytnik (see [My98b] or [My96]), it is possible to show weak uniqueness for Equation (1.1) and, amongst others, the strong Markov property of Z, see Theorem 2.4 and Corollary 2.7 of [DP98]. Moreover, the existence of an equilibrium distribution for Z will be an easy consequence of the duality relation (1.3); see Theorem 1.4 of [DP98]. And under some recurrence assumptions (on Q, satisfied, for example, by simple symmetric random walk in dimension $d = 1$ or $d = 2$) this limit law is explicitly known:

For a real number $a \geq 0$ denote by $\overline{a} : \mathbb{Z}^d \longrightarrow [0,\infty)$ the map which is constant and equal to a. Fix $x_1, x_2 \geq 0$ and let $B_t = \big(B_{1,t}\,,\,B_{2,t}\big)$ be a planar Brownian motion starting at $x = (x_1,x_2) \in [0,\infty)^2$ under $\mathbb{P}_{(x_1,x_2)}$. Define the stopping time $T := \inf\big\{\,t:\; B_{1,t}B_{2,t} \,=\, 0\,\big\}$. Then Theorem 1.5 of [DP98] says that for the process Z of Equation (1.1) with initial configuration $(\overline{x_1}, \overline{x_2})$, i.e. all components of type $\alpha \in \{1,2\}$ equal x_α,

$$P_{(\bar{x}_1,\bar{x}_2)}\big[\,Z_t \,\in\, \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \mathbb{P}_{(x_1,x_2)}\big[\big(\overline{B}_{1,T}\,,\,\overline{B}_{2,T}\big)\in\,\cdot\,\big] \tag{1.4}$$

in the sense of weak convergence of probabilities (on the space of tempered configurations). In particular, $Z_{1,\cdot,\infty} = 0$ or $Z_{2,\cdot,\infty} = 0$ $P_{(\bar{x}_1,\bar{x}_2)}$-a.s. while the other type, which did not die out, is constant but random. Note that Dawson and Perkins assume Q to be symmetric with row sums equal to zero.

Statements (1.2) to (1.4) will be the key ingrediences to construct a variant of Z which has infinite variance, very loosely speaking, with $\gamma = \infty$ in (1.1). The exit distribution of planar Brownian motion from the upper right quadrant in Equation (1.4) will be of particular importance. We therefore introduce the following notation. For $z = (z_1, z_2) \in (0,\infty)^2$ we set

$$DP_z(d\xi) \;:=\; \mathbb{P}_{(z_1,z_2)}\big[\big(B_{1,T}\,,\,B_{2,T}\big) \in d\xi\,\big]. \tag{1.5}$$

The distribution DP_z charges $L := [0,\infty)^2 \setminus (0,\infty)^2 = [0,\infty)\times\{0\} \cup \{0\}\times[0,\infty)$. And if $z \in L$ we can consistently set

$$DP_z \;:=\; \delta_z, \tag{1.6}$$

the Dirac measure at z.

1.2 The DP-distribution

We investigate properties of the DP-distribution defined above. First, observe that if the Brownian motion starts in $x \in (0,\infty)^2$ then DP_x is absolutely continuous w.r.t. Lebesgue measure. The next lemma gives the density; compare with [DP98] p.1094.

1.1 Lemma *Let $x = (x_1, x_2) \in (0,\infty)^2$. Then*

$$\begin{aligned} DP_x(d\xi_1, d\xi_2) \;:=\; &\frac{1}{\pi}\,\frac{4x_1\,x_2\,\xi_1}{4\,x_1^2\,x_2^2 + \big(\xi_1^2 + x_2^2 - x_1^2\big)^2}\;{}_{[0,\infty)\times\{0\}}(\xi_1,\xi_2)\,d\xi_1 \\ &+ \quad \frac{1}{\pi}\,\frac{4x_1\,x_2\,\xi_2}{4\,x_1^2\,x_2^2 + \big(\xi_2^2 + x_1^2 - x_2^2\big)^2}\;{}_{\{0\}\times[0,\infty)}(\xi_1,\xi_2)\,d\xi_2 \end{aligned} \tag{1.7}$$

Proof. Let $B_t = \big(B_{1,t}\,,\,B_{2,t}\big)$ be planar Brownian Motion starting at $(0,a) \in \mathbb{R}^2$, $a > 0$. Let S be the time when B first hits the x-axis, that means $S := \inf\big\{\,t > 0 :\; B_{2,t} \,=\, 0\,\big\}$. Then we have, see [RY91] p.108 Proposition III.3.11,

$$B_{1,S} \overset{d}{=} a \cdot C\,,$$

where C is a standard Cauchy random variable. Hence, starting at (b,a), for $b \in \mathbb{R}$, $B_{1,S}$ has a distribution with density f given by

$$f(\xi) \;:=\; \frac{1}{\pi}\,\frac{a}{a^2 + (\xi - b)^2}\,, \qquad \xi \in \mathbb{R}.$$

Now we apply the conformal mapping $z \mapsto \sqrt{z}$. Just note $(x_1 + ix_2)^2 \,=\, x_1^2 - x_2^2 \,+\, i\,2x_1x_2$ to obtain for $\xi > 0$

$$\frac{1}{\pi}\,\frac{2x_1x_2}{(2x_1x_2)^2 \,+\, \big(\xi^2 \,-\, (x_1^2 - x_2^2)\big)^2}\,\frac{1}{|\frac{1}{2\xi}|} \;=\; \frac{1}{\pi}\,\frac{4x_1x_2\xi}{4x_1^2x_2^2 \,+\, \big(\xi^2 + x_2^2 - x_1^2\big)^2}\,.$$

For $\xi < 0$ we have to consider $-\xi^2$. This gives (1.7). □

As square root of a Cauchy distribution, it is immediate that the DP_x-distribution has finite moments up to order stricly smaller than two, and the second moment is infinite if $x \in (0,\infty)^2$.

The DP-distribution and Mytnik's duality functions are perfectly suited for each other. Here, we give a definiton of these functions for pairs of nonnegative real numbers. To this end let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be elements of $[0,\infty)^2$. Set

$$F(x,y) := \exp\big\{-(x_1+x_2)(y_1+y_2) + i(x_1-x_2)(y_1-y_2)\big\}. \tag{1.8}$$

And recall $L = [0,\infty)\times\{0\} \cup \{0\}\times[0,\infty)$. Then F has the following invariance properties under the DP-distribution.

1.2 Lemma

(a) For all $y \in [0,\infty)^2$ and all $x \in L$,

$$F(x,y) = \int DP_y(dz) F(x,z). \tag{1.9}$$

(b) For all $x, y \in [0,\infty)^2$,

$$\int DP_y(d\xi) F(x,\xi) = \int DP_x(d\zeta) F(\zeta,y). \tag{1.10}$$

Proof. (a) Let $x_1, x_2 \geq 0$. According to the definition of the DP-distribution we have

$$\int DP_y(dz)F(x,z) = E_y\left[\exp\Big\{-(x_1+x_2)(B_{1,T}+B_{2,T}) + i(x_1-x_2)(B_{1,T}-B_{2,T})\Big\}\right],$$

where $B_t = (B_{1,t}, B_{2,t})$ is planar Brownian motion starting at $y = (y_1,y_2)$ and $T = \inf\{t : B_{1,t}B_{2,t} = 0\}$. Note that

$$M_t := e^{-4x_1x_2\,t}\exp\Big\{-(x_1+x_2)(B_{1,t}+B_{2,t}) + i(x_1-x_2)(B_{1,t}-B_{2,t})\Big\} \tag{1.11}$$

is a martingal since

$$\begin{aligned} E_y[M_t|\mathscr{F}_s] &= M_s\, e^{-4x_1x_2\,(t-s)}\, E_y\bigg[\exp\Big\{-(x_1+x_2)(B_{1,t}-B_{1,s}+B_{2,t}-B_{2,s})\Big\} \\ &\qquad\qquad \times \exp\Big\{i(x_1-x_2)(B_{1,t}-B_{1,s}-(B_{2,t}-B_{2,s}))\Big\}\;\Big|\;\mathscr{F}_s\bigg] \\ &= M_s\, e^{-4x_1x_2\,(t-s)}\, E_0\left[\exp\Big\{-(x_1+x_2)B_{1,t-s} + i(x_1-x_2)B_{1,t-s}\Big\}\right] \\ &\qquad\qquad \times E_0\left[\exp\Big\{-(x_1+x_2)B_{2,t-s} - i(x_1-x_2)B_{2,t-s}\Big\}\right] \\ &= M_s\,. \end{aligned}$$

For the last equality observe that $B_{1,t-s}$ is normally distributed with mean 0 and variance $t-s$, hence $E_0\big[e^{i\lambda B_{1,t-s}}\big] = e^{-\frac{1}{2}\lambda^2(t-s)}$ is analytic, and in particular for $\lambda_1 = (x_1-x_2)+i(x_1+x_2)$ and $\lambda_2 = -(x_1-x_2)+i(x_1+x_2)$, resp., we have

$$E_0\left[\exp\Big\{-(x_1+x_2)B^{(1)}_{t-s} + i(x_1-x_2)B^{(1)}_{t-s}\Big\}\right] = e^{-\frac{1}{2}\big[\,(x_1-x_2)+i(x_1+x_2)\big]^2(t-s)},$$

$$E_0\left[\exp\Big\{-(x_1+x_2)B^{(2)}_{t-s} - i(x_1-x_2)B^{(2)}_{t-s}\Big\}\right] = e^{-\frac{1}{2}\big[-(x_1-x_2)+i(x_1+x_2)\big]^2(t-s)},$$

and $e^{-\frac{1}{2}\left[(x_1-x_2)+i(x_1+x_2)\right]^2(t-s)}\,e^{-\frac{1}{2}\left[-(x_1-x_2)+i(x_1+x_2)\right]^2(t-s)} = e^{\left[(x_1+x_2)^2-(x_1-x_2)^2\right](t-s)}$.

Now observe that the family $(M_{t\wedge T})_{t\geq 0}$ is absolutely bounded by 1, hence, is uniformly integrable. Since $M_{t\wedge T}\to M_T$ a.s. as $t\to\infty$ we have according to the opptional stopping theorem

$$e^{-(x_1+x_2)(y_1+y_2)+i(x_1-x_2)(y_1-y_2)} = E_y[M_0] = \lim_{n\to\infty} E_y[M_{T\wedge n}] = E_y[M_T], \tag{1.12}$$

provided $x_1x_2=0$.

(b) Now we choose $x=(x_1,x_2)\in[0,\infty)^2$ and $y=(y_1,y_2)\in[0,\infty)^2$. Let B be planar Brownian motion under P^x starting in x with stopping time T, and W an independent copy starting in y under P^y and stopping time T' as above, but for W. Since $|F|\leq 1$ we infer

$$\begin{aligned}
E_x\big[F(B_T,y)\big] &= \int F\big(B_T(\omega),y\big)P^x(d\omega)\\
&= \int E_y\big[F\big(B_T(\omega),W_{T'}\big)\big]P^x(d\omega)\\
&= \int\int F\big(B_T(\omega),W_{T'}(\omega')\big)P^y(d\omega')P^x(d\omega) = E_{(x,y)}\big[F\big(B_T,W_{T'}\big)\big]\\
&= \int E_x\big[F\big(B_T,W_{T'}(\omega')\big)\big]P^y(d\omega')\\
&= \int F\big(x,W_{T'}(\omega')\big)P^y(d\omega')\\
&= E_y\big[F(x,W_{T'})\big]
\end{aligned}$$

which is equation (1.10). □

1.3 Lemma *Let $x\in[0,\infty)^2$ and $c>0$. Then, for any integrable function $f:[0,\infty)^2\longrightarrow\mathbb{C}$,*

$$\int DP_{c\,x}(dz)f(z) = \int DP_x(dz)f(c\,z).$$

Proof. Simply use the density in (1.7) and substitute. □

As a consequence of Lemma 1.2 the family of functions $\{F(\cdot,y)\,:\,y\in L\}$ separates measures which charge only L.

1.4 Lemma

(a) If μ_1 and μ_2 are probability measures on $[0,\infty)^2$ such that $\int F(x,y)\mu_1(dx) = \int F(x,y)\mu_2(dx)$ for all $y\in[0,\infty)^2$ then $\mu_1=\mu_2$.

(b) If μ_1 and μ_2 are probability measures on L such that $\int F(x,y)\mu_1(dx) = \int F(x,y)\mu_2(dx)$ for all $y\in L$ then $\mu_1=\mu_2$.

Proof. (a) Let $E=[0,\infty)^2$ and consider the compactification $\hat{E} = E\cup\{\infty\}$. Then the complex linear span of $\mathcal{D}=\{F(\cdot,y):y\in E\}$ is a sub-algebra of the space of continuous functions with limits at infinity which separates points, and hence, is dense in $\mathcal{C}_b(\hat{E},\mathbb{C})$ according to the Stone-Weierstraß Theorem, cf. [Bau90] § 23 Remark 3, on p.198. This implies $\mu_1=\mu_2$, see [Kl06] Corollary 15.3, p.283 or [Bau92] Corollary 29.1, p.214, if we take into account that $F(\cdot,y)$ characterizes the distribution of $(x_1+x_1\,,\,x_1-x_2)$ which uniquely corresponds with (x_1,x_2).

(b) For measures on E which charge only $L \subset E$ we can use Lemma 1.2(a) and Fubini. In fact, for any $y \in E$ we have

$$\begin{aligned} \int F(x,y)\mu_1(dx) &= \int_L \int_L F(x,\xi)\mu_1(dx)\, DP_y(d\xi) \\ &= \int_L \int_L F(x,\xi)\mu_2(dx)\, DP_y(d\xi) \;=\; \int F(x,y)\mu_2(dx). \end{aligned}$$

Hence, $\mu_1 = \mu_2$ by part (a). □

We present an estimate on the p-th moment of the DP-Distribution.

1.5 Lemma *Let $B_t = (B_{1,t}, B_{2,t})_{t\geq 0}$ be planar Brownian motion starting in $(u,v) \in (0,\infty)^2$ and $T = \inf\{ t > 0 : B_{1,t}\, B_{2,t} = 0 \}$. For $1 < p < 2$ we have*

$$E_{(u,v)}\Big[\Big(\sup_{t\geq 0} |B_{\alpha,t\wedge T}|\Big)^p\Big] \;<\; \infty$$

for each $\alpha \in \{1,2\}$. Yet, we have the following upper bounds:

$$E_{(u,v)}\Big[\Big(\sup_{t\geq 0} |B_{1,t\wedge T} - u|\Big)^p\Big] \;\leq\; C_p \min\{ (u+1)v\,,\, (v+1)u \}, \tag{1.13}$$

$$E_{(u,v)}\Big[\Big(\sup_{t\geq 0} |B_{2,t\wedge T} - v|\Big)^p\Big] \;\leq\; C_p \min\{ (u+1)v\,,\, (v+1)u \}. \tag{1.14}$$

for some constant C_p, which only depends on p.

Proof. Let planar Brownian motion $B_t = (B_{1,t}, B_{2,t})_{t\geq 0}$ start in $(u,v) \in (0,\infty)^2$, $T = \inf\{ t > 0 : B_{1,t}\, B_{2,t} = 0 \}$ and $T_u = \inf\{ t > 0 : B_{1,t} = 0 \}$ and $T_v = \inf\{ t > 0 : B_{2,t} = 0 \}$. Note that $T = \min(T_u\,, T_v)$. The stopping time T_u has Lévy distribution with parameters u^2 and 0, i.e. the density

$$P\big[T_u \in ds \big] \;=\; \frac{u}{\sqrt{2\pi}}\, e^{-\frac{u^2}{2s}}\, s^{-3/2}\, ds\,,$$

where $s > 0$ and $u > 0$, compare [KS91] Section 2.8 Equation(8.5) p.96. The same is true for T_v with u replaced by v. Since $e^{-\frac{u^2}{2s}} \leq 1$ we can estimate

$$P\big[T_u > t \big] \;\leq\; \int_t^\infty \frac{u}{\sqrt{2\pi}}\, s^{-3/2}\, ds \;=\; \sqrt{\frac{2}{\pi}}\, \frac{u}{\sqrt{t}}\,, \tag{1.15}$$

for $t > 0$. Hence, we have

$$P\big[T > t \big] \;=\; P\big[T_u > t \big]\, P\big[T_v > t \big] \;\leq\; \frac{2}{\pi}\, \frac{uv}{t}. \tag{1.16}$$

Then, for $\frac{1}{2} < r < 1$, there exists a constant C_r which only depends on r, such that

$$E[T^r] \;=\; \int_0^\infty P\big[T^r > t \big]\, dt \;=\; \int_0^\infty P\big[T > \sqrt[r]{t} \big]\, dt \;\leq\; \begin{cases} C_r\,(u+1)v \\ C_r\,(v+1)u \end{cases} \tag{1.17}$$

since

$$\int_1^\infty P\big[T > \sqrt[r]{t} \big]\, dt \;\leq\; \int_1^\infty \frac{2}{\pi}\, \frac{uv}{t^{1/r}}\, dt \;=\; \frac{2}{\pi}\, u\, v\, \frac{r}{1-r}$$

by (1.16) and

$$\int_0^1 P\big[\,T > \sqrt[r]{t}\,\big]\,dt \;\leq\; \int_0^1 P\big[\,T_u > \sqrt[r]{t}\,\big]\,dt \;\leq\; \int_0^1 \sqrt{\frac{2}{\pi}}\,\frac{u}{t^{1/2r}}\,dt \;=\; \sqrt{\frac{2}{\pi}}\,u\,\frac{2\,r}{2\,r\,-\,1}$$

by (1.15), and similar with v.

Finally, we can apply the Burkholder-Gandy inequality, see for instance [LSh89] p.75 (Chap. 1 §9 Theorem 7). Note that we can estimate the quadratic variation, $\langle B_{\alpha,\cdot\wedge T}\rangle_t \;\leq\; T$, and then, with $r = \frac{p}{2}$, we have

$$E\Big[\Big(\sup_{t\geq 0}|B_{1,t\wedge T}\,-\,u|\Big)^p\Big] \;\leq\; c_p\,E\Big[\,\langle B_{1,\cdot\wedge T}\rangle_\infty^{p/2}\,\Big] \;\leq\; c_p\,E\big[\,T^r\,\big].$$

The same is true for $B_{2,t\wedge T} - v$. Thus, we can apply (1.17) and we are done. □

The above result implies that the martingale $(B_{1,t\wedge T}\,,\,B_{2,t\wedge T})_{0\leq t\leq\infty}$ is uniformly integrable. Therefore we can state the following.

1.6 Corollary *With the notation from above.*

$$\lim_{t\to\infty} E_{(u,v)}\big[\,(\,B_{1,t\wedge T}\,,\,B_{2,t\wedge T}\,)\,\big] \;=\; E_{(u,v)}\big[\,(\,B_{1,T}\,,\,B_{2,T}\,)\,\big] \;=\; (u,v).$$

In particular, a random variable which is $DP_{(u,v)}$-distributed has mean u in the first coordinate and mean v in the second.

The next Lemma gives a finer estimate on the p-th moment of the DP-distribution. It exploits Equations (1.13) and (1.14) of Lemma 1.5.

1.7 Lemma *For $1 < p < 2$ there exists a constant C_p such that for $(u,v)\;\in\;[0,\infty)^2$ we have*

$$\int DP_{(u,v)}(d\xi)\,(\xi_1 - u)^p \;\leq\; 2\,C_p\,\min\big\{\,u^{p-1}\,v\,,\;u\,v^{p-1}\,\big\}, \tag{1.18}$$

$$\int DP_{(u,v)}(d\xi)\,(\xi_2 - v)^p \;\leq\; 2\,C_p\,\min\big\{\,u^{p-1}\,v\,,\;u\,v^{p-1}\,\big\}. \tag{1.19}$$

The constant C_p is the same as in Lemma 1.5.

Proof. Let $u > 0$ and $v > 0$. By Lemma 1.3 and the estimates in Lemma 1.5 we obtain

$$\int DP_{(u,v)}(d\xi)(\xi_1 - u)^p \;=\; \int DP_{(1,\frac{v}{u})}(d\xi)\,u^p(\xi_1 - 1)^p \;\leq\; 2\,C_p\,u^{p-1}v$$

$$\int DP_{(u,v)}(d\xi)(\xi_1 - u)^p\,, = \;\int DP_{(\frac{u}{v},1)}(d\xi)\,v^p\big(\xi_1 - \frac{u}{v}\big)^p \;\leq\; 2\,C_p\,v^{p-1}u\,.$$

The same arguments work for the integrand $(\xi_2 - v)^p$. Finally, note that if $uv = 0$ then $DP_{(u,v)} = \delta_{(u,v)}$ and the estimates are trivial. □

1.3 The $\mathrm{DP^K}$-distribution

Unfortunately, the distribution DP_y, which is the exit distribution of planar Brownian motion starting in $y \in [0,\infty)^2$ from the first quadrant, is not bounded. To overcome difficulties that arise with this fact we pen in Brownian motion in a finite box.

Let $K > 0$. Consider planar Brownian motion $B_t = (B_{1,t}, B_{2,t})$ starting in $y = (y_1, y_2) \in (0,K)^2$. Define the stopping times, for $\alpha \in \{1,2\}$,

$$\begin{aligned} S_\alpha^K &:= \inf\{\, t > 0 \,:\, B_{\alpha,t} \geq K \,\}\,, \\ T_\alpha^0 &:= \inf\{\, t > 0 \,:\, B_{\alpha,t} \leq 0 \,\}\,, \\ \tau_\alpha^K &:= \inf\{\, t > 0 \,:\, B_{\alpha,t} \leq 0 \text{ or } B_{\alpha,t} \geq K \,\} \;=\; T_\alpha^0 \wedge S_\alpha^K\,, \end{aligned}$$

set $\tau^K := \tau_1^K \wedge \tau_2^K$ and note that $P-a.s$ holds $\tau^K \longrightarrow T := \inf\{\, t > 0 \,:\, B_{1,t}B_{2,t} = 0 \,\}$ as $K \to \infty$. The distributions of, for example S_α^K and τ_α^K are well know and can be found in [KS91] Section 2.8: See for instance Equation (8.5) and (8.6), Equation (8.24) gives the distribution of $T_\alpha^0 \wedge S_\alpha^K$. In particular $P^y[T_\alpha^0 < S_\alpha^K] = \frac{K-y_\alpha}{K}$ and $P^y[T_\alpha^0 > S_\alpha^K] = \frac{y_\alpha}{K}$, and $E_y[T_\alpha^0 \wedge S_\alpha^K] = y_\alpha(K-y_\alpha)$; see 2.8.13 and 2.8.14. Proposition 2.8.10 of [KS91] gives the distribution of the random variable B_{1,τ_1^K}.

1.8 Remark *It is possible to compute the exit distribution, $\mathcal{L}[B_{\tau^K}]$, of Brownian motion from the square. Namely, let $\mathcal{Q} := (0,K)^2$ be the K-square and consider the Dirichlet problem*

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \mathcal{Q}\,, \\ u_{|\partial\mathcal{Q}} &= g \quad \text{on } \partial\mathcal{Q}\,, \end{aligned} \qquad (*)$$

where $\partial\mathcal{Q}$ is the boundary of $\mathcal{Q}$ and the function g is defined on the sides $\partial\mathcal{Q}_j$, $j = 1,2,3,4$, of the square by continuous elementary functions $g_1(x_1)$, $g_2(x_2)$, $g_3(x_1)$ and $g_4(x_2)$. The linear PDE $(*)$ has solution $u(x) = \int_{\partial\mathcal{Q}} \tilde\kappa(x,dy)g(y)$, and we can write

$$u(x) \;=\; \int_{\partial\mathcal{Q}} \kappa(x,y)g(y)dy\,,$$

if the kernel $\tilde\kappa(x,dy)$, which exists, has a density $\kappa(x,y)$ w.r.t. Lebesgue measure. On the other hand, u has representation

$$u(x) \;=\; E_x\big[g(B_{\tau^K})\big]$$

where B is planar Brownian motion and, as defined above, $\tau^K = \inf\{\, t > 0 \,:\, B_t \in \mathbb{R}^2 \setminus \mathcal{Q} \,\}$, see [KS91] Section 4.2. Note that all points in $\partial\mathcal{Q}$ are regular. That means $P_x[\, B_{\tau^K} \in dy \,] = \tilde\kappa(x,dy) = \kappa(x,y)dy$. In fact, if the boundary conditions are: $u(x_1,0) = g_1(x_1)$ for $0 < x_1 < K$, $u(K,x_2) = g_2(x_2)$ for $0 < x_2 < K$, $u(x_1,K) = g_3(x_1)$ for $0 < x_1 < K$ and $u(0,x_2) = g_4(x_2)$ for $0 < x_2 < K$, then the solution for $(*)$ is given by

$$u(x_1,x_2) \;=\; \sum_{j=1}^{4} u_j(x_1,x_2)$$

where

$$u_1(x_1,x_2) \;=\; \sum_{n=1}^{\infty} c_{1,n}\, \sin\Big(\frac{n\pi}{K}\,x_1\Big)\sinh\Big(n\pi\big(1-\frac{x_2}{K}\big)\Big)\,,$$
$$u_2(x_1,x_2) \;=\; \sum_{n=1}^{\infty} c_{2,n}\, \sin\Big(\frac{n\pi}{K}\,x_2\Big)\sinh\Big(\frac{n\pi}{K}\,x_1\Big)\,,$$
$$u_3(x_1,x_2) \;=\; \sum_{n=1}^{\infty} c_{3,n}\, \sin\Big(\frac{n\pi}{K}\,x_1\Big)\sinh\Big(\frac{n\pi}{K}\,x_2\Big)\,,$$
$$u_4(x_1,x_2) \;=\; \sum_{n=1}^{\infty} c_{4,n}\, \sin\Big(\frac{n\pi}{K}\,x_2\Big)\sinh\Big(n\pi\big(1-\frac{x_1}{K}\big)\Big)\,,$$

with coefficients $c_{j,n}$ *for* $j \in \{1,2,3,4\}$ *and* $n \in \mathbb{N}$*, given by*

$$c_{j,n} \;=\; \frac{2}{K\sinh(n\pi)}\int_0^K g_j(s)\sin\Big(\frac{n\pi}{K}\,s\Big)ds\,;$$

compare with [Gu80] Section 2.1 p. 118 and Section 2.2 pp. 131-134. ◊

1.9 Definition *Let* B *be planar Brownian motion starting in* $y=(y_1,y_2) \in (0,K)^2$ *and* τ^K *as above. For the random variable* B_{τ^K} *with values in* $\partial(0,K)^2$ *we set*

$$DP_y^K(d\xi) \;:=\; P_y\big[\,B_{\tau^K} \in d\xi\,\big]\,. \tag{1.20}$$

For $y \in [0,\infty)^2 \setminus (0,K)^2$ *we simply set* $DP_y^K = \delta_y$ *.*

Next, we show that this distribution has similar properties as the DP_y-distribution in Lemma 1.2.

1.10 Lemma

(a) *For all* $y \in [0,\infty)^2$ *and all* $x \in L$*,*

$$F(x,y) \;=\; \int DP_y^K(dz)F(x,z) \;=\; E_y\big[F(x,B_{\tau^K})\big]. \tag{1.21}$$

(b) *For all* $x,y \in [0,\infty)^2$*,*

$$\int\!\!\int DP_y(dv)DP_x^K(du)F(u,v) \;=\; \int\!\!\int DP_x(du)DP_y^K(dv)F(u,v). \tag{1.22}$$

(c) *Let* $x,y \in [0,\infty)^2$ *and let* B *and* W *be independent planar Brownian motions starting in* x *and* y*, resp. Set* $T = \inf\{\,t>0 : B_{1,t}B_{2,t}=0\,\}$ *and* $T' = \inf\{\,t>0 : W_{1,t}W_{2,t}=0\,\}$*. Define*

$$\tilde F(x,y) \;:=\; E_{(x,y)}\big[F(B_T,W_{T'})\big] \;=\; \int\!\!\int DP_x(du)DP_y(dv)F(u,v)\,.$$

Then

$$E_x\big[\,\tilde F(B_{\tau^K},y)\,\big] \;=\; E_y\big[\,\tilde F(x,W_{\tau'^K})\,\big]\,. \tag{1.23}$$

Proof. (a) By Equation (1.11) in the proof of Lemma 1.2(a)

$$M_t \;:=\; e^{-4x_1x_2\,t}\exp\Big\{-(x_1+x_2)(B_{1,t}+B_{2,t}) \;+\; i(x_1-x_2)(B_{1,t}-B_{2,t})\Big\}$$

is a martingale. Then the optional stopping theorem yields for $x \in L$,

$$F(x,y) \;=\; E_y[M_0] \;=\; \lim_{n\to\infty} E_y[M_{\tau^K\wedge n}] \;=\; E_y[M_{\tau^K}] \;=\; E_y\big[F(x,B_{\tau^K})\big].$$

(b) Let $x,y \in [0,\infty)^2$. Then

$$\begin{aligned}\int\!\!\int DP_y(dv)DP_x^K(du)F(u,v) \;&=\; \int DP_y(dv)F(x,v)\\ &=\; \int DP_x(du)F(u,y) \qquad\qquad \text{(by (1.10))}\\ &=\; \int DP_x(du)DP_y^K(dv)F(u,v).\end{aligned}$$

Note that the computation is even valid for $x,y \in [0,\infty)^2\setminus(0,K)^2$ since in this case $DP_x^K = \delta_x$ and $DP_y^K = \delta_y$, so the first and the last equalities are evident.
(c) By Lemma 1.2(b) the function $\tilde F$ satisfies for all $x,y \in [0,\infty)^2$ the following identities

$$\tilde F(x,y) \;=\; \int DP_x(du)F(u,y) \;=\; \int DP_y(dv)F(x,v).$$

Then we use (1.22)

$$\begin{aligned}E_x\big[\,\tilde F(B_{\tau^K},y)\,\big] \;&=\; \int DP_x^K(du)\tilde F(u,y)\\ &=\; \int DP_x^K(du)\int DP_y(dv)F(u,v)\\ &=\; \int DP_y^K(dv)\int DP_x(du)F(u,v)\\ &=\; \int DP_y^K(dv)\tilde F(x,v)\\ &=\; E_y\big[\,\tilde F(x,W_{\tau'^K})\,\big]\end{aligned}$$

which proves (1.23). □

For easy reference we state the following equation.

1.11 Lemma

$$\int DP_x(dy)DP_z^K(dx) \;=\; DP_z(dy)$$

Proof. Apply the Markov property to planar Brownian motion. □

Planar Brownian motion stopped when it leaves the square $[0,K]^2$ is a bounded process. So, in contrast to the former section, we have a finite second moment for the DP^K-distribution. The next aim will be to give an upper bound for the variance of the DP^K-distribution.

In the spirit of Remark 1.8 we consider the Poisson equation

$$\begin{aligned} \frac{1}{2}\Delta u &= -g \quad \text{in } \mathcal{Q}\,, \\ u_{|\partial\mathcal{Q}} &= f \quad \text{on } \partial\mathcal{Q}\,, \end{aligned} \tag{1.24}$$

for some bounded continuous functions $g : \mathcal{Q} \to \mathbb{R}$ and $f : \partial\mathcal{Q} \to \mathbb{R}$; recall $\partial\mathcal{Q}$ is the boundary of $\mathcal{Q} := (0, K)^2$. Then, cf. [KS91] Eq. (2.27) of Problem 2.25 on p. 253, the solution u of (1.24) has representation

$$u(x) \;=\; E_x\Big[f(B_{\tau^K}) \;+\; \int_0^{\tau^K} g(B_t)\, dt \Big]$$

for all $x \in \overline{\mathcal{Q}}$. In particular, for $g \equiv 1$ and $f \equiv 0$ in (1.24) we have

$$E_x\big[\,\tau^K\,\big] \;=\; u(x). \tag{1.25}$$

Since $(B_{t\wedge\tau^K})_{t\geq 0}$ is bounded and $(B^2_{\alpha,t} \;-\; t)_{t\geq 0}$, with $\alpha \in \{1, 2\}$, as well as $(B_{1,t}\, B_{2,t})_{t\geq 0}$ are martingales, we infer

$$E_x\big[\,\tau^K\,\big] \;=\; E_x\big[\,B^2_{\alpha,\tau^K} \;-\; x_\alpha\,\big] \tag{1.26}$$

and

$$E_x\big[\,(B^2_{1,\tau^K} - x_1)(B^2_{2,\tau^K} - x_2)\,\big] \;=\; E_x\big[\,(B^2_{1,\tau^K} - x_1)\,\big]\, E_x\big[\,(B^2_{2,\tau^K} - x_2)\,\big] \;=\; 0. \tag{1.27}$$

To obtain a Fourier series expansion for the function $u(x_1, x_2) \;=\; E_x\big[\,\tau^K\,\big]$, where $x = (x_1, x_2) \in \mathcal{Q}$, compare with [Gu80] p.158/159. We arrive at

$$u(x_1, x_2) \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{d_{mn}}{m^2 + n^2}\; \sin\Big(\frac{m\,\pi}{K}\, x_1\Big)\; \sin\Big(\frac{n\,\pi}{K}\, x_2\Big),$$

where the coefficients d_{mn} are given by

$$d_{mn} \;:=\; \frac{8K^2}{\pi^4} \int_0^{\pi}\!\int_0^{\pi} \sin(m\xi_1)\, \sin(n\xi_2)\, d\xi_1\, d\xi_2 \;=\; \begin{cases} 0 & \text{if } m \text{ or } n \text{ is even}, \\ \frac{32\,K^2}{\pi^4\, mn} & \text{if } m \text{ and } n \text{ are odd}. \end{cases}$$

Hence, we obtain as unique solution to the Dirichlet-Poisson equation (1.24), with $g \equiv 1$ and $f \equiv 0$,

$$u(x_1, x_2) \;=\; \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\; \sin\Big(\frac{(2m+1)\,\pi}{K}\, x_1\Big)\; \sin\Big(\frac{(2n+1)\,\pi}{K}\, x_2\Big), \tag{1.28}$$

where a_{mn}, for $m, n \in \mathbb{N}_0$, are given by

$$a_{mn} \;=\; \frac{32\,K^2}{\pi^4}\; \frac{\frac{1}{2m+1}\,\frac{1}{2n+1}}{(2m+1)^2 \;+\; (2n+1)^2}\,. \tag{1.29}$$

Next, we need to present a result of Klenke and Mytnik, see [KM07] Corollary 3.4. We therefore introduce the following notation. For $K = 1$ set

$$V(x_1, x_2) \;:=\; E_{(x_1,x_2)}\big[\,\tau^1\,\big]\,, \tag{1.30}$$

where $x = (x_1, x_2) \in [0,1]^2$. Then the formulars above yield, for arbitrary $K > 0$ and for $x = (x_1, x_2) \in (0,K)^2$ the relations

$$E_{(x_1,x_2)}\big[\,\tau^K\,\big] \;=\; K^2\, V\Big(\,\frac{x_1}{K}\,,\,\frac{x_2}{K}\,\Big), \tag{1.31}$$

$$\mathrm{Cov}_{(x_1,x_2)}\big(\,B_{\alpha,\tau^K}\,,\,B_{\beta,\tau^K}\,\big) \;=\; K^2\, V\Big(\,\frac{x_1}{K}\,,\,\frac{x_2}{K}\,\Big)\delta_{\alpha\beta}\,, \qquad \alpha,\beta \in \{1,2\}, \tag{1.32}$$

which also can be seen by Brownian scaling. Then, Klenke and Mytnik show the following upper bound for the second moment of the DP^K-distribution. We include a detailed proof.

1.12 Lemma *There exists a constant $C > 0$ such that for each $K > 0$ and for planar Brownian motion B in $[0,K]^2$, with initial value $x = (x_1, x_2) \in (0,K)^2$, $\alpha, \beta \in \{1,2\}$,*

$$\mathrm{Cov}_{(x_1,x_2)}\big(\,B_{\alpha,\tau^K}\,,\,B_{\beta,\tau^K}\,\big) \;\le\; C\, x_1\, x_2\,\Big[\,1 \,+\, \log(K) \,+\, \log(1/x_1) \wedge \log(1/x_2)\,\Big]\,\delta_{\alpha\beta}\,. \tag{1.33}$$

Proof. By Equations (1.31) and (1.32) we can restrict our attention to the case $K = 1$. And, then, by symmetry in x_1 and x_2 it is enough to show

$$V(x_1,x_2) \;\le\; C\, x_1\, x_2\,\big[\,1 \,+\, \log(1/x_1)\big], \tag{1.34}$$

for $x_1, x_2 \in (0,1)$.

Fix x_2 and let $x_1 \ge \frac{1}{3}$. Note that $\min\limits_{x_1\in[\frac{1}{3},1]} x_1\big[1 + \log(1/x_1)\big] \;=\; \frac{1+\log(3)}{3} \;=:\; \frac{1}{C}$. Then, recall statements at the beginning of this section and estimate

$$V(x_1,x_2) \;\le\; E_{x_2}\big[\,\tau_2^1\,\big] \;=\; x_2(1-x_2) \;\le\; x_2 \;\le\; C\, x_1\, x_2\big[\,1 \,+\, \log(1/x_1)\big]\,.$$

Now, let $x_1 \in (0, \frac{1}{3}]$. Choose $M \in \mathbb{N}$ maximal with $x_1 \le \frac{1}{2M+1}$. Split the series expansion for $V(x_1,x_2)$ in Equation (1.28) in two parts, namely, $V(x_1,x_2) \;=\; I_M(x_1,x_2) \,+\, I_\infty(x_1,x_2)$, where

$$I_M(x_1,x_2) \;:=\; \sum_{m=0}^{M-1}\sum_{n=0}^{\infty} \frac{32}{\pi^4}\, \frac{\frac{1}{2m+1}\,\frac{1}{2n+1}}{(2m+1)^2 \,+\, (2n+1)^2}\, \sin\big((2m+1)\pi\, x_1\big)\, \sin\big((2n+1)\pi\, x_2\big),$$

$$I_\infty(x_1,x_2) \;:=\; \sum_{m=M}^{\infty}\sum_{n=0}^{\infty} \frac{32}{\pi^4}\, \frac{\frac{1}{2m+1}\,\frac{1}{2n+1}}{(2m+1)^2 \,+\, (2n+1)^2}\, \sin\big((2m+1)\pi\, x_1\big)\, \sin\big((2n+1)\pi\, x_2\big).$$

The two terms will be estimated separately. For I_M note that $\sin\big((2m+1)\pi\, x_1\big) \;\le\; (2m+1)\pi\, x_1$ (and similarly with n and x_2). Then

$$\begin{aligned}
I_M(x_1,x_2) \;&\le\; \frac{32}{\pi^2}\, x_1\, x_2 \sum_{m=0}^{M-1}\sum_{n=0}^{\infty} \frac{1}{(2m+1)^2 \,+\, (2n+1)^2} \\
&\le\; \frac{32}{\pi^2}\, x_1\, x_2 \sum_{m=0}^{M-1}\sum_{n=1}^{\infty} \frac{1}{(2m+1)^2 \,+\, n^2} \\
&\le\; \frac{32}{\pi^2}\, x_1\, x_2 \sum_{m=0}^{M-1} \int_0^{\infty} \frac{1}{(2m+1)^2 \,+\, t^2}\, dt \\
&\le\; \frac{32}{\pi^2}\, x_1\, x_2 \sum_{m=0}^{M-1} \frac{1}{(2m+1)}\, \frac{\pi}{2} \\
&\le\; \frac{16}{\pi}\, x_1\, x_2 \,\Big[\,1 \,+\, \int_0^{M-1} \frac{1}{s+1}\, ds\,\Big] \;=\; \frac{16}{\pi}\, x_1\, x_2 \,\Big[\,1 \,+\, \log(M)\,\Big].
\end{aligned}$$

For $I_\infty(x_1\,,\,x_2)$ note that $\sin\big((2m+1)\pi\,x_1\big)\;\leq\;1$ and $\sin\big((2n+1)\pi\,x_2\big)\;\leq\;(2n+1)\pi\,x_2$. Then, similar as above,

$$\begin{aligned}
I_\infty(x_1\,,\,x_2) \;&\leq\; \frac{32}{\pi^3}\,x_2\,\sum_{m=M}^{\infty}\frac{1}{2m+1}\sum_{n=0}^{\infty}\frac{1}{(2m+1)^2\;+\;(2n+1)^2}\\
&\leq\; \frac{32}{\pi^3}\,x_2\,\sum_{m=M}^{\infty}\frac{1}{(2m+1)^2}\,\frac{\pi}{2}\\
&\leq\; \frac{16}{\pi^2}\,x_2\int_{M-1}^{\infty}\frac{1}{(2s+1)^2}\,ds \;=\; \frac{8}{\pi^2}\,x_2\,\frac{1}{2M-1}\;\leq\;\frac{8}{\pi^2}\,\frac{x_2}{M}\,.
\end{aligned}$$

The maximality of M implies $x_1\;\geq\;\frac{1}{2(M+1)+1}\;\geq\;\frac{1}{5M}$, hence,

$$I_\infty(x_1\,,\,x_2)\;\leq\;\frac{40}{\pi^2}\,x_1\,x_2\;\leq\;\frac{40}{\pi^2}\,x_1\,x_2\Big[1\;+\;\log(M)\Big].$$

Since log is monotone increasing and $M\;\leq\;\frac{1}{2\,x_1}\;\leq\;\frac{1}{x_1}$ both bounds (for I_M and I_∞, respectively) together imply (1.34) and the proof is complete. □

Chapter 2

Infinite branching rate on one colony

As a first step we will present a process with an infinite branching rate on one colony. We compute its generator and check existence and a duality relation. The duality implies weak uniqueness for this process. Then, we identify the one-dimensional distributions of the infinite rate model. In addition, we can show that the transition probabilities of the model with finite variance converge to the transition probabilities for the process with infinite variance as the variance approaches infinity. The DP-distribution will be ubiquitous.

2.1 The generator

We describe the local dynamics of the model. As already mentioned in Chapter 1, the DP-distribution of Section 1.2 will be the key ingredient. The idea is as follows:

We consider two sites, denoted by 0 and 1. Site 1 represents an infinitely big environment which is not affected by any kind of local dynamics or fluctuations. This means that the amount of mass of particle type 1 is constant for all times and it equals θ_1, say. The amount of mass of particle type 2 is constant and equals θ_2. That means in particular we chose variance zero for site 1. In contrast, on site 0 the dynamics is governed by the DP-distribution which is triggered by the amount of mass that migrates, namely, the mass that leaves site 0 or pours in from outside, i.e. from site 1. Since site 1 neither loses nor gains any mass we have to choose the following migration kernel

$$Q = \begin{pmatrix} -\kappa & 0 \\ \rho & 0 \end{pmatrix}, \tag{2.1}$$

where $\kappa, \rho > 0$.

To establish a pregenerator for the dynamics on site 0 we consider a two-step procedure:
Step 1: During a small time interval mass of both types migrates. That means mass immigrates from site 1 to site 0 or leaves site 0. For simplicity, we assume that this shifts the initial condition linearly in time. Assume, for example, $(x_1, 0) \in L$ as initial condition (on site 0). Denote by $\acute{u}, \acute{v}$

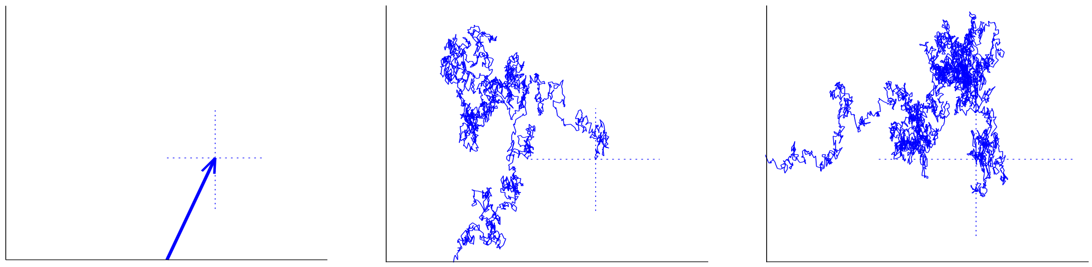

Figure 2.1: two-step procedure

the amount of mass of type 1 and 2, respectively, that migrates during a unit time interval, where $\acute{u} \in \mathbb{R}$ and $\acute{v} \geq 0$. Then, we obtain $(x_1 + \acute{u}t, \acute{v}t)$, for $t \geq 0$.
Step 2: From this linearly perturbed condition we randomly choose a new point on L. More precisely, we choose a point on L according to planar Brownian motion stopped when it leaves the first quadrant, i.e. we consider the random variable $D = (D_1, D_2)$ with distribution $DP_{x_1+\acute{u}t,\acute{v}t}$.

Both steps are illustrated in Figure 2.1. The leftmost picture shows the linear shift and the other two display how Brownian motion picks a new point on L. Note that there are two possible outcomes for the second step. Either Brownian motion hits the same axis where the initial condition was located or Brownian motion hits the other axis. In the later case we can say that the dynamics forces a change of types (on site 0). Of course, the smaller the pertubation in the initial condition the smaller the probability for a change of types. But the change of types can not be neglected as we consider $t \searrow 0$. A precise picture of this matter describes the pregenerator $\mathcal{A}_{\acute{u},\acute{v}}$ associated with the pertubation $(\acute{u}, \acute{v})$. To identify $\mathcal{A}_{\acute{u},\acute{v}}$ we have to compute

$$\mathcal{A}_{\acute{u},\acute{v}} f(x_1, 0) \;=\; \lim_{t \searrow 0} \frac{E_{x_1+\acute{u}t,\acute{v}t}\Big[f\big(D_1, D_2\big)\Big] \;-\; f(x_1, 0)}{t}\,, \tag{2.2}$$

for some function f on L. The calculations for (2.2) are done in Section A.1 of the Appendix. They are purely analytic and we strongly advise the reader to skip this part at any time of reading. But note that according to the 'infinite variance assumption' on site 0, coexistence of both types is not possible, i.e., the appropriate state space for the process (on site 0) with pregenerator $\mathcal{A}_{\acute{u},\acute{v}}$ has to be L.

Next, we combine the results for (2.2) with the migration kernel Q of (2.1). Recall that on site 1 the process is constant and equals $\Theta := (\theta_1, \theta_2) \in [0, \infty)^2$ for all times $t \geq 0$. Then, to correctly represent the migration in step 1 above we need to choose $\acute{u} = \rho\theta_1 - \kappa x_1$ for particle type 1 and $\acute{v} = \rho\theta_2 - \kappa x_2$ for particle type 2 if we evaluate $\mathcal{A}_{\acute{u},\acute{v}} f$ at $x = (x_1, x_2) \in L$. To indicate the dependency on Θ we write $\mathcal{A}_\Theta$ instead of $\mathcal{A}_{\acute{u},\acute{v}}$. By combining all we infer the following result.

2.1 Proposition *Let $x = (x_1, x_2) \in L$ and Q be given by (2.1). Let $f \in \mathcal{C}_b^2(L, \mathbb{R})$ and set*

$$h(y) := \frac{4}{\pi} \frac{y}{(y-1)^2(y+1)^2} \qquad \text{and} \qquad g(y) := \frac{4}{\pi} \frac{y}{(y^2+1)^2}\,,$$

for $y \geq 0$. Then we have the following.

If $x \in (0,\infty)\times\{0\}$ *then*

$$\mathcal{A}_\Theta f(x) \;=\; \frac{\rho\theta_2}{x_1^2}\int_0^\infty \Big[f(\xi,0) \;-\; f(x_1,0) \;-\; (\xi - x_1)\,\partial_1 f(x_1,0)\Big]\, h\Big(\frac{\xi}{x_1}\Big)\, d\xi$$

$$+\quad \frac{\rho\theta_2}{x_1^2}\int_0^\infty \Big[f(0,\zeta) \;-\; f(x_1,0) \;+\; x_1\,\partial_1 f(x_1,0)\Big]\, g\Big(\frac{\zeta}{x_1}\Big)\, d\zeta$$

$$+\quad (\rho\theta_1 - \kappa x_1)\cdot \partial_1 f(x_1,0)\,,$$

if $x \in \{0\}\times(0,\infty)$ *then*

$$\mathcal{A}_\Theta f(x) \;=\; \frac{\rho\theta_1}{x_2^2}\int_0^\infty \Big[f(\xi,0) \;-\; f(0,x_2) \;+\; x_2\,\partial_2 f(0,x_2)\Big]\, g\Big(\frac{\xi}{x_2}\Big)\, d\xi$$

$$+\quad \frac{\rho\theta_1}{x_2^2}\int_0^\infty \Big[f(0,\zeta) \;-\; f(0,x_2) \;-\; (\zeta - x_2)\,\partial_2 f(0,x_2)\Big]\, h\Big(\frac{\zeta}{x_2}\Big)\, d\zeta$$

$$+\quad (\rho\theta_2 - \kappa x_2)\cdot \partial_2 f(0,x_2)\,,$$

and, if $x = (0,0)$ *then*

$$\mathcal{A}_\Theta f(x) \;=\; \rho\theta_1\cdot\partial_1 f(0,0) \;+\; \rho\theta_2\cdot\partial_2 f(0,0)\,.$$

To see that this result makes good sense note the similarities with generators for Lévy processes with no Gaussian component. For example, take a closer look at the first case of Proposition 2.1. The first integral describes the jump from $x = (x_1, 0)$, with $x_1 > 0$, back on the x_1-axis. The Lévy measure $h(\xi/x_1)\,d\xi$ has a peak at x_1, and therefore the integrand $f(\xi,0) - f(x_1,0)$ has to be made integrable by subtracting the mean of the jumps. Recall that a random variable D, with $D \sim DP_{(u,v)}$, $u, v \geq 0$, has mean (u, v); cf. Corollary 1.6. A change of types, i.e. a jump to the x_2-axis, only occurs with finite intensity. These jumps are represented by the second integral. The drift term $(\rho\theta_1 - \kappa x_1)\cdot\partial_1 f(x_1,0)$ neutralises the subtraction of the mean. Note that in contrast to ordinary Lévy processes our jump measures, $h(\xi)\,d\xi$ and $g(\zeta)\,d\zeta$, which are normed for $x = (1,0)$ (or $x = (0,1)$ in the case $x \in \{0\}\times(0,\infty)$), depend on x. Moreover, the jump measure is not continuous as a function of x at the origin. Finally, note the linearity of the jump terms in $\rho\theta_2$ (the amount of mass of *the other type* that immigrates to site 0). The second case of Proposition 2.1, where $x \in \{0\}\times(0,\infty)$, can similarly be interpreted by interchanging the roles of x_1 and x_2 (and the roles of θ_1 and θ_2). The somewhat odd representation of $\mathcal{A}_\Theta f$ at $x = (0,0)$ can be accepted by the following argument: Corollary A.4 of the Appendix shows that the generator has an additive decomposition. The infinitesimal change which is triggered by the migration of both particle types is the same as the infinitesimal change caused by the migration of particle type 1 only, plus the infinitesimal change just caused by the migration of particle type 2. Thus, if $x_2 = 0$ migration of type 1 has to be represented by a first-order differential operator on the first coordinate; if $x_1 = 0$ migration of type 2 is described by $\partial_2 f$.

We define a stochastic process X as a solution of the martingale problem for this operator $\mathcal{A}_\Theta$. By $D_L[0,\infty)$ we denote the space of càdlàg-paths with values in L, and we rather look for a probability measure P on $D_L[0,\infty)$ such that the coordinate process defined on $D_L[0,\infty)$ is a solution of the martingale problem for $(\mathcal{A}_\Theta\,,\,\nu)$, where ν denotes a specified initial distribution on L; that means $PX_{\cdot,0,0} = \nu$. See [EK86] p.173,174 for more details.

2.2 Proposition *For each ν there exists a solution of the martingale problem for $(\mathcal{A}_\Theta, \nu)$ with sample paths in $D_L[0,\infty)$.*

PROOF. In order to use Theorem IV.5.4 in [EK86] on p.199 we check its assumptions. The space L is obviously locally compact and separable. We consider its one-point compactification $\hat{L} \,=\, L \cup \{\infty\}$. The operator $\mathcal{A}_\Theta$ is in particular defined for twicely continuously differentiable functions that vanish at infinity. A subset of these functions is surely dense in the space of continuous functions that vanish at infinity. Let $f \in \mathcal{C}^2(L)$ vanish at infinity. Assume $\sup_{x\in L} f(x) \,=\, f(y) \,\geq\, 0$, for some $y \,\in\, L$. Then

$$\begin{aligned}\mathcal{A}_\Theta f(y) \;&=\; \lim_{t\searrow 0} \frac{E_{y_1+(\rho\theta_1-y_1)t\,,\,y_2+(\rho\theta_2-y_2)t}\Big[f\big(D_1,D_2\big)\Big] \,-\, f(y)}{t} \\ &\leq\; \lim_{t\searrow 0} \frac{E_{y_1+(\rho\theta_1-y_1)t\,,\,y_2+(\rho\theta_2-y_2)t}\Big[f(y)\Big] \,-\, f(y)}{t} \;\leq\; 0\,.\end{aligned}$$

Hence, $\mathcal{A}_\Theta$ satisfies the positive maximum principle; see [EK86] p.165 for a definition. Then the above-mentioned theorem provides a solution of the martingale problem with sample paths having values in $\hat{L} \,=\, L \cup \{\infty\}$.

But moreover, $(\ _L, 0)$ is in the bp-closure of the graph of $\mathcal{A}_\Theta$; in fact, 1 is an element of its domain. (One might choose a sequence f_n which is constant and equal to 1 on $[0,n)\times\{0\} \cup \{0\}\times[0,n)$ with a smooth step to 0. Then for fix $x \in L$ we have $f_n(x) \to 1$ and $|\mathcal{A}_\Theta f_n(x)| \leq \varepsilon$, for n large enough, since the tails of the integrals in $\mathcal{A}_\Theta$ must vanish). Since $\nu(L) \,=\, 1$, Theorem IV.3.8 of [EK86] p.179, yields $P\big(X_{\cdot,0,\cdot} \in D_L[0,\infty)\big) \,=\, 1$. □

One might object that we only found a measure that describes the dynamics on site 0. But recall that the process on site 1 is constant and equals $\Theta = (\theta_1, \theta_2) \,\in\, [0,\infty)^2$. Therefore, we can simply choose dirac measure δ_Θ which charges the constant path $t \mapsto (\theta_1,\theta_2)$ in $D_{[0,\infty)^2}[0,\infty)$ and consider the product measure $P \times \delta_\Theta$ on $D_E[0,\infty)$, where $E := L \times [0,\infty)^2$. Instead, one might prefer to call $\big((X_{1,0,t}\,,\, X_{2,0,t})\big)_{t\geq 0}$ a model for one colony (with drift towards Θ).

Next, we want to show a duality for our process $X_t = \big((X_{1,0,t}\,,\, X_{2,0,t})\,,\,(\theta_1,\theta_2)\big)$. In the same spirit as for the finite variance model – cf. Section 1.1 – we choose a dual process Y, which obeys the same dynamics as X but with a different migration kernel. $Y_t \,=\, \big((Y_{1,0,t}, Y_{2,0,t})\,,\,(Y_{1,1,t}, Y_{2,1,t})\big)$ migrates according to

$$Q^* \;=\; \begin{pmatrix} -\kappa & \rho \\ 0 & 0 \end{pmatrix}, \tag{2.3}$$

the transpose of Q. Hence, for the generator acting on site 0 we have to choose $\acute{u} = -\kappa y_{1,0}$ and $\acute{v} = -\kappa y_{2,0}$. Note that we will choose $(y_{1,0}, y_{2,0}) \,\in\, L$. Therefore, if we compare with Proposition 2.1 – or more precisely with the results in the Appendix – we observe that the generator on site 0 only consists of drift terms. The same is true on site 1 since here the 'branching parameter' equals zero. Thus, for smooth functions $f : L \times [0,\infty)^2 \longrightarrow \mathbb{R}$, we can identify the pregenerator

$$\mathcal{A}^* f(y) \;=\; \sum_{\alpha=1}^{2} -\kappa y_{\alpha,0}\,\partial_{\alpha,0} f(y) \;+\; \sum_{\alpha=1}^{2} \rho\, y_{\alpha,0}\,\partial_{\alpha,1} f(y)\,, \tag{2.4}$$

where $\partial_{\alpha,i} f(y) \;=\; \frac{\partial}{\partial y_{\alpha,i}}\, f(y)$, $i \in \{0,1\}$.

Now, recall the definition of Mytnik's duality function F in Equation (1.8) on page 4. Define the map $H \,:\, (\,L\times[0,\infty)^2\,) \times (\,L\times[0,\infty)^2\,) \longrightarrow \mathbb{C}$ by

$$H(x,y) \;:=\; H\big((x_{\cdot,0}, x_{\cdot,1}), (y_{\cdot,0}, y_{\cdot,1})\big) \;:=\; F(x_{\cdot,0}, y_{\cdot,0})\, F(x_{\cdot,1}, y_{\cdot,1}). \tag{2.5}$$

for $x \,=\, \big((x_{1,0}, x_{2,0}), (x_{1,1}, x_{2,1})\big) \,\in\, L \times [0,\infty)^2$ and $y \,=\, \big((y_{1,0}, y_{2,0}), (y_{1,1}, y_{2,1})\big) \,\in\, L \times [0,\infty)^2$. Then, we can prove the following duality relation.

2.3 Proposition *Let X and Y as defined above, starting in x and y, respectively, where $x, y \in L \times [0,\infty)^2$ and with $x_{\cdot,1} = (\theta_1, \theta_2)$. Then*

$$E_{P_x}\Big[H(\,X_t\,,\,Y_0\,)\Big] \;=\; E_{P_y}\Big[H(\,X_0\,,\,Y_t\,)\Big]\,. \tag{2.6}$$

Proof. We compute $\mathcal{A}_\Theta H(\cdot, y)$. Recall that $x_{\cdot,1} = \Theta$ and $\mathcal{A}_\Theta$ only acts on site 0. Set $\mu(t) = (\mu_{1,t}\,,\,\mu_{2,t})$, where $\mu_{\alpha,t} \,=\, x_{\alpha,0} + t(\rho\theta_\alpha - \kappa x_{\alpha,0})$, for $\alpha \in \{1,2\}$. Let $D \sim DP_{\mu(t)}$ and $C \sim DP_{y_{\cdot,0}}$. We use Lemma 1.2(b) and then the fact that $DP_{y_{\cdot,0}} = \delta_{y_{\cdot,0}}$ for $y_{\cdot,0} \in L$.

$$\begin{aligned}
\mathcal{A}_\Theta H\big((\cdot,\Theta),(y_{\cdot,0},y_{\cdot,1})\big)(x_{\cdot,0}) \;&=\; \lim_{t\searrow 0} \frac{E_{\mu(t)}\big[F(D, y_{\cdot,0})\big] \;-\; F(x_{\cdot,0}, y_{\cdot,0})}{t}\, F(\Theta, y_{\cdot,1}) \\
&=\; \lim_{t\searrow 0} \frac{E_{y_{\cdot,0}}\big[F(\mu(t), C)\big] \;-\; F(x_{\cdot,0}, y_{\cdot,0})}{t}\, F(\Theta, y_{\cdot,1}) \\
&=\; \lim_{t\searrow 0} \frac{F(\mu(t), y_{\cdot,0}) \;-\; F(x_{\cdot,0}, y_{\cdot,0})}{t}\, F(\Theta, y_{\cdot,1})\,,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\searrow 0} \frac{F(\mu(t), y_{\cdot,0}) \;-\; F(x_{\cdot,0}, y_{\cdot,0})}{t} \\
&\qquad =\; F(x_{\cdot,0}, y_{\cdot,0})\Big[-(\rho\theta_1 - \kappa x_{1,0})(y_{1,0} + y_{2,0}) \;-(\rho\theta_2 - \kappa x_{2,0})(y_{1,0} + y_{2,0}) \\
&\qquad\qquad + i(\rho\theta_1 - \kappa x_{1,0})(y_{1,0} - y_{2,0}) \;- i(\rho\theta_2 - \kappa x_{2,0})(y_{1,0} - y_{2,0})\Big].
\end{aligned}$$

Note that the expression on the r.h.s. is bounded in $x_{\cdot,0} \in L$. Next, we write $x_{\alpha,1}$ instead of θ_α, for $\alpha \in \{1,2\}$, to simplify notation. We use $Q \,=\, (q_{jk})_{j,k\,\in\,\{0,1\}}$ given by (2.1), recall $q_{01} \,=\, 0 \,=\, q_{11}$. Then, we obtain

$$\begin{aligned}
&\mathcal{A}_\Theta H\big((\,\cdot\,, x_{\cdot,1})\,,\,(y_{\cdot,0}, y_{\cdot,1})\,\big)(x_{\cdot,0}) \\
&\quad =\; H(x,y)\Big\{-\sum_{\alpha,\beta=1}^{2}\sum_{j,k=0}^{1} q_{jk}\, x_{\alpha,j}\, y_{\beta,k} \;+\; i\sum_{\alpha,\beta=1}^{2}\sum_{j,k=0}^{1} (-1)^{\alpha+\beta} q_{jk}\, x_{\alpha,j}\, y_{\beta,k}\Big\}\,.
\end{aligned} \tag{2.7}$$

For the dual process Y with generator $\mathcal{A}^*$ given by (2.4) and migration matrix $Q^* \,=\, (q^*_{jk})_{j,k=0,1}$

as in (2.3) we obtain the following.

$$
\begin{aligned}
&\mathcal{A}^* H(x,\cdot)(y) \\
&\quad= \sum_{\alpha=1}^{2} q^*_{00}\, y_{\alpha,0} \Big[-(x_{1,0}+x_{2,0}) \;+\; i\,(-1)^{\alpha+1}(x_{1,0}-x_{2,0}) \Big] H(x,y) \\
&\qquad + \sum_{\alpha=1}^{2} q^*_{01}\, y_{\alpha,0} \Big[-(x_{1,1}+x_{2,1}) \;+\; i\,(-1)^{\alpha+1}(x_{1,1}-x_{2,1}) \Big] H(x,y) \\
&\quad= H(x,y)\Big\{ -\sum_{\alpha,\beta=1}^{2} \sum_{j=0}^{1} q^*_{0j}\, y_{\alpha,0}\, x_{\beta,j} \;+\; i \sum_{\alpha,\beta=1}^{2} \sum_{j=0}^{1} (-1)^{\alpha+\beta} q^*_{0j}\, y_{\alpha,0}\, x_{\beta,j} \Big\} \\
&\quad= H(x,y)\Big\{ -\sum_{\alpha,\beta=1}^{2} \sum_{j,k=0}^{1} q^*_{kj}\, y_{\alpha,k}\, x_{\beta,j} \;+\; i \sum_{\alpha,\beta=1}^{2} \sum_{j,k=0}^{1} (-1)^{\alpha+\beta} q^*_{kj}\, y_{\alpha,k}\, x_{\beta,j} \Big\}.
\end{aligned} \tag{2.8}
$$

For the last line recall $q^*_{10} = 0 = q^*_{11}$. Then observe $q_{jk} = q^*_{kj}$, for $j,k \in \{0,1\}$, and we infer

$$
\mathcal{A}_\Theta H(\cdot,y)(x) \;=\; \mathcal{A}^* H(x,\cdot)(y). \tag{2.9}
$$

Then, Corollary 4.4.13 on p.195 of [EK86] implies (2.6). (To see this choose the functions α and β of this reference equal to zero, note that $|H| \le 1$, and observe that $\mathcal{A}_\Theta H(\cdot,y)(x)$ and $\mathcal{A}^* H(x,\cdot)(y)$ are bounded). □

Observe the similarities of Equation (2.7) above and Equation (1.2) on page 2 with $\gamma = 0$.

We remark that Proposition 2.3 implies weak uniqueness for the process X with paths in $D_E[0,\infty)$. The Markov property for X is then immediate. For example see [EK86] Theorem 4.4.2. X is even Feller. However, we are not going into greater details here since that will be a more interesting matter in the next chapter. But note again that by Lemma 1.4 it is enough to consider a dual process Y such that $Y_{\cdot,0,t} \in L$ for $t \ge 0$.

Jump-measure for the box $[0,K]^2$

For completeness, we present in this subsection the jump-measure for a process, similar to X but which is restricted to the box $[0,K]^2$, with $K > 1$, or more precisely to the boundary of the box, $\partial[0,K]^2$. The line of reasoning is as before, but we have to substitute the DP-distribution by the DP^K-distribution of Section 1.3. Let $L^K := \{0\}\times[K-1] \cup [K-1]\times\{0\}$. We identify the pregenerator $\mathcal{A}^K_{\acute{u},\acute{v}}$. For $(x_1,0) \in L^K$, with $x_1 > 0$, we need to compute

$$
\mathcal{A}^K_{\acute{u},\acute{v}} f(x_1,0) \;=\; \lim_{t\searrow 0} \frac{E_{x_1+\acute{u}t,\acute{v}t}\Big[f\big(D_1,D_2\big)\Big] \;-\; f(x_1,0)}{t}\,,
$$

for some function f on $\partial[0,K]^2$, where $\big(D_1,D_2\big) \sim DP^K_{x_1+\acute{u}t,\acute{v}t}$.

Here, we concentrate on the jump measure n on $\partial[0,K]^2$

$$
n(\,dy\,|\,(x_1,0)\,) \;:=\; \lim_{t\searrow 0} \frac{1}{t}\, P_{x_1+\acute{u}t,\acute{v}t}[B_{\tau^K} \in dy]
$$

where B is planar Brownian motion and $\tau^K := \inf\{t > 0 : \; B_t \notin (0,K)^2\}$. We use the densities given in Remark 1.8. We denote by n_1, n_2, n_3 and n_4 the restriction of n on the sides $(0,K)\times\{0\}$, $\{K\}\times(0,K)$, $(0,K)\times\{K\}$ and $\{0\}\times(0,K)$, respectively. We do not indicate the dependency on $(x_1,0)$, $\acute{u}$ and $\acute{v}$. First, note that $\lim\limits_{t\searrow 0} \frac{1}{t}\sin(\frac{n\pi}{K}\,t\acute{v}) = \acute{v}\frac{n\pi}{K}$ and $\lim\limits_{t\searrow 0} \frac{1}{t}\sinh(\frac{n\pi}{K}\,t\acute{v}) = \acute{v}\frac{n\pi}{K}$. Then, on $\{K\}\times(0,K)$:

$$\begin{aligned} n_2(\{K\}\times dy_2) \;&:=\; \lim_{t\searrow 0}\frac{1}{t}\sum_{n=1}^{\infty}\sin\Big(\frac{n\pi}{K}\,t\acute{v}\Big)\sinh\Big(\frac{n\pi}{K}\,(x_1+t\acute{u})\Big)\frac{2}{K\sinh(n\pi)}\sin\Big(\frac{n\pi}{K}\,y_2\Big)\,dy_2 \\ &=\; \acute{v}\sum_{n=1}^{\infty}\sinh\Big(\frac{n\pi}{K}\,x_1\Big)\frac{2\,n\pi}{K^2\sinh(n\pi)}\sin\Big(\frac{n\pi}{K}\,y_2\Big)\,dy_2\,. \end{aligned}$$

Note that the series converges since $x_1 \le K-1$. This also justifies the interchange of limit and summation. Similarly, on $(0,K)\times\{K\}$:

$$\begin{aligned} n_3(dy_1\times\{K\}) \;&:=\; \lim_{t\searrow 0}\frac{1}{t}\sum_{n=1}^{\infty}\sin\Big(\frac{n\pi}{K}\,(x_1+t\acute{u})\Big)\sinh\Big(\frac{n\pi}{K}\,t\acute{v}\Big)\frac{2}{K\sinh(n\pi)}\sin\Big(\frac{n\pi}{K}\,y_1\Big)\,dy_1 \\ &=\; \acute{v}\sum_{n=1}^{\infty}\sin\Big(\frac{n\pi}{K}\,x_1\Big)\frac{2\,n\pi}{K^2\sinh(n\pi)}\sin\Big(\frac{n\pi}{K}\,y_1\Big)\,dy_1\,, \end{aligned}$$

and on $\{0\}\times(0,K)$:

$$\begin{aligned} n_4(\{0\}\times dy_2) \;&:=\; \lim_{t\searrow 0}\frac{1}{t}\sum_{n=1}^{\infty}\sin\Big(\frac{n\pi}{K}\,t\acute{v}\Big)\sinh\Big(n\pi\,\big(1-\frac{x_1+t\acute{u}}{K}\big)\Big)\frac{2}{K\sinh(n\pi)}\sin\Big(\frac{n\pi}{K}\,y_2\Big)\,dy_2 \\ &=\; \acute{v}\sum_{n=1}^{\infty}\sinh\Big(n\pi\,\big(1-\frac{x_1}{K}\big)\Big)\frac{2\,n\pi}{K^2\sinh(n\pi)}\sin\Big(\frac{n\pi}{K}\,y_2\Big)\,dy_2\,, \end{aligned}$$

for $x_1 > 0$. It remains to show existence of the measure on $(0,K)\times\{0\}$. As in the case of Proposition 2.1 we expect n_1 to be σ-finite only. To ensure existence of a limit we use the strong Markov property as in [KM07] Equation (3.16). Since $\tau^K \le T = \inf\{t > 0 : \; B_{1,t}B_{2,t} = 0\}$ we can write

$$\begin{aligned} DP_{x_1+\acute{u}t,\acute{v}t}(d\xi) \;&=\; P_{x_1+\acute{u}t,\acute{v}t}\big(B_T \in d\xi\big) \;=\; \int_{\partial[0,K]^2} DP^K_{x_1+\acute{u}t,\acute{v}t}(d\zeta)\,DP_\zeta(d\xi) \\ &=\; \int_{(0,K)\times\{0\}} DP^K_{x_1+\acute{u}t,\acute{v}t}(d\zeta)\,\delta_\zeta(d\xi) \;+\; \int_{\{K\}\times(0,K)} DP^K_{x_1+\acute{u}t,\acute{v}t}(d\zeta)\,DP_\zeta(d\xi) \\ &\quad+\; \int_{(0,K)\times\{K\}} DP^K_{x_1+\acute{u}t,\acute{v}t}(d\zeta)\,DP_\zeta(d\xi) \;+\; \int_{\{0\}\times(0,K)} DP^K_{x_1+\acute{u}t,\acute{v}t}(d\zeta)\,\delta_\zeta(d\xi)\,, \end{aligned}$$

hence,

$$\begin{aligned} n_1 \;=\; \lim_{t\searrow 0}\frac{1}{t}DP^K_{x_1+\acute{u}t,\acute{v}t}\Big|_{(0,K)\times\{0\}} \;&=\; \lim_{t\searrow 0}\frac{1}{t}\,DP_{x_1+\acute{u}t,\acute{v}t} \;-\; \int_0^K n_2(\{K\}\times d\zeta_2)\,DP_{(K,\zeta_2)} \\ &\quad-\; \int_0^K n_3(d\zeta_1\times\{K\})\,DP_{(\zeta_1,K)}\,, \end{aligned}$$

and the limits on the r.h.s. exist by Proposition 2.1 and the calculations above.

Thus, for $\mathcal{A}^K_{\acute{u},\acute{v}}$ we infer $(0 < x_1 \le K-1)$

$$\begin{aligned}
\mathcal{A}^K_{\acute{u},\acute{v}} f(x_1,0) \;&=\; \lim_{t\searrow 0} \frac{E_{x_1+\acute{u}t,\acute{v}t}\Big[f\big(B_{\tau^K}\big) \,-\, f(x_1,0) \,-\, (B_{1,\tau^K}-x_1)\partial_1 f(x_1,0)\Big]}{t} \\
&\quad + \;\; \lim_{t\searrow 0} \frac{E_{x_1+\acute{u}t,\acute{v}t}\Big[(B_{1,\tau^K}-x_1)\partial_1 f(x_1,0)\Big]}{t} \\
&= \int_0^K \big[f(y_1,0) \,-\, f(x_1,0) \,-\, (y_1-x_1)\partial_1 f(x_1,0)\big]\, n_1(dy_1\times\{0\}) \\
&\quad + \int_0^K \big[f(K,y_2) \,-\, f(x_1,0) \,-\, (K-x_1)\partial_1 f(x_1,0)\big]\, n_2(\{K\}\times dy_2) \\
&\quad + \int_0^K \big[f(y_1,K) \,-\, f(x_1,0) \,-\, (y_1-x_1)\partial_1 f(x_1,0)\big]\, n_3(dy_1\times\{K\}) \\
&\quad + \int_0^K \big[f(0,y_2) \,-\, f(x_1,0) \,+\, x_1\partial_1 f(x_1,0)\big]\, n_4(\{0\}\times dy_2) \\
&\quad + \; \acute{u}\,\partial_1 f(x_1,0)
\end{aligned}$$

since $E_{x_1+\acute{u}t,\acute{v}t}\big[B_{1,\tau^K}-x_1\big] \;=\; \acute{u}t.$

Similar expressions can be derived for $\mathcal{A}^K_{\acute{u},\acute{v}} f$ if we evaluate at $(0,x_2)$, where $0 < x_2 \le K-1$. For $x \in \partial[0,K]^2 \setminus L^K$ we set $\mathcal{A}^K_{\acute{u},\acute{v}} f(x) = 0$, that means the process is stopped if it leaves L^K. At the origin we certainly have $\mathcal{A}^K_{\acute{u},\acute{v}} f((0,0)) \;=\; \acute{u}\,\partial_1 f(0,0) \,+\, \acute{v}\,\partial_2 f(0,0)$.

2.2 Transition probabilities and Invariant distribution

For the process $X_t = \big((X_{1,0,t}\,,\, X_{2,0,t})\,,\,(\theta_1,\theta_2)\big)$ with values in $L\times\{(\theta_1,\theta_2)\}$ as introduced in the former section we can compute the transition probabilities and the invariant distribution.

2.4 Lemma *Fix $x \ge 0$ and let $X_{\cdot,0,0} \;=\; (x,0)$. Then for all $y \ge 0$ we have*

$$\begin{aligned}
&E_{(x,0)}\Big[e^{-\,y\,(X_{1,0,t}+X_{2,0,t}) \,+\, i\,y\,(X_{1,0,t}-X_{2,0,t})}\Big] \\
&\qquad\qquad = \;\exp\Big\{-\,y\,\big[\mu_x(t)+\nu_0(t)\big] \;+\; i\,y\,\big[\mu_x(t)-\nu_0(t)\big]\Big\}\,,
\end{aligned} \tag{2.10}$$

where $\mu_x(t) = x\,e^{-\kappa t} + (1-e^{-\kappa t})\frac{\rho}{\kappa}\theta_1$ and $\nu_0(t) = (1-e^{-\kappa t})\frac{\rho}{\kappa}\theta_2$. In particular

$$\lim_{t\to\infty} E_{(x,0)}\Big[e^{-\,y\,(X_{1,0,t}+X_{2,0,t}) \,+\, i\,y\,(X_{1,0,t}-X_{2,0,t})}\Big] \;=\; e^{-\,y\,\frac{\rho}{\kappa}\,(\theta_1+\theta_2) \,+\, i\,y\,\frac{\rho}{\kappa}\,(\theta_1-\theta_2)}. \tag{2.11}$$

Proof. We use the duality relation (2.6). For the dual process Y we choose the initial condition $Y_{\alpha,1,0} \;=\; 0$ on site 1, for both types $\alpha \in \{1,2\}$. On site 0 we start with $Y_{1,0,0} \;=\; y$ and $Y_{2,0,0} \;=\; 0$. Then Proposition 2.3 implies

$$\begin{aligned}
&E_{(x,0)}\Big[e^{-\,y\,(X_{1,0,t}+X_{2,0,t}) \,+\, i\,y\,(X_{1,0,t}-X_{2,0,t})}\Big] \;= \\
&E_{(y,0)}\Big[e^{-\,x\,(Y_{1,0,t}+Y_{2,0,t}) \,+\, i\,x\,(Y_{1,0,t}-Y_{2,0,t})}\; e^{-\,(\theta_1+\theta_2)(Y_{1,1,t}+Y_{2,1,t}) \,+\, i\,(\theta_1-\theta_2)\,(Y_{1,1,t}-Y_{2,1,t})}\Big].
\end{aligned} \tag{2.12}$$

It remains to identify Y_t for $t > 0$. Due to the lack of a catalyst the dual process Y and its limit for $t \to \infty$ can easily be computed. Namely, the mass of the type-1-particle just migrates according to Q^* and that means $Y_{1,\cdot,\cdot}$ obeys the differential equations

$$\begin{aligned} dY_{1,0}(t) &= -\kappa\, Y_{1,0}(t)\, dt\,, \\ dY_{1,1}(t) &= \rho\, Y_{1,0}(t)\, dt\,, \end{aligned} \tag{2.13}$$

hence,

$$\begin{aligned} Y_{1,0,t} &= y\, e^{-\kappa t}\,, \\ Y_{1,1,t} &= y\, \frac{\rho}{\kappa}\, (1 - e^{-\kappa t})\,, \end{aligned} \tag{2.14}$$

and $(\, Y_{1,0,t}\,,\, Y_{1,1,t}\,) \longrightarrow (\, 0\,,\, \frac{\rho}{\kappa}\, y\,)$ as $t \to \infty$. Putting this in Equation (2.12) gives the assertion. □

Combining Equation (1.12) of Lemma 1.2 with Lemma 1.4 we identify the DP-distributions as transition probabilities and invariant distribution of $(X_{\cdot,0,t})_{t\geq 0}$.

2.5 Corollary *Let* $X_{\cdot,0,0} := (x_1, x_2) \in L$. *Then*

$$\mathcal{L}[X_{\cdot,0,t}] \;=\; DP_{(\,\mu_{x_1}(t)\,,\,\nu_{x_2}(t)\,)}\,, \tag{2.15}$$

where $\mu_{x_1}(t) = x_1\, e^{-\kappa t} + (1 - e^{-\kappa t})\frac{\rho}{\kappa}\theta_1$ *and* $\nu_{x_2}(t) = x_2\, e^{-\kappa t} + (1 - e^{-\kappa t})\frac{\rho}{\kappa}\theta_2$. *In particular*

$$\lim_{t\to\infty} \mathcal{L}[X_{\cdot,0,t}] \;=\; DP_{(\,\frac{\rho}{\kappa}\theta_1\,,\,\frac{\rho}{\kappa}\theta_2\,)}\,. \tag{2.16}$$

Now, we adapt the finite variance mutually catalytic branching model of Dawson and Perkins to our case of two colonies. Denote by Z^γ the process with branching parameter $\gamma > 0$ on site 0, variance zero on site 1 and migration operator (2.1). Let $Z^\gamma_{\cdot,0,0} = (x_1, x_2) \in [0,\infty)^2$ and $\Theta = (\theta_1, \theta_2) \in [0,\infty)^2$. The process Z^γ then satisfies

$$\begin{aligned} dZ^\gamma_{\alpha,0,t} &= \big(\rho\,\theta_\alpha \;-\; \kappa Z^\gamma_{\alpha,0,t}\big)dt \;+\; \sqrt{\gamma\, Z^\gamma_{1,0,t}\, Z^\gamma_{2,0,t}}\; dB_{\alpha,0,t}\,, \\ Z^\gamma_{\alpha,1,t} &\equiv \theta_\alpha\,, \end{aligned} \tag{2.17}$$

for all $t \geq 0$ and types $\alpha \in \{1,2\}$, where $(B_{\alpha,0,t})_{t\geq 0}$ are two independent standard Brownian motions. For Z^γ we denote the dual process by Y^γ. It has migration operator Q^* and satisfies

$$\begin{aligned} dY^\gamma_{\alpha,0,t} &= -\kappa\, Y^\gamma_{\alpha,0,t}\, dt \;+\; \sqrt{\gamma\, Y^\gamma_{1,0,t}\, Y^\gamma_{2,0,t}}\; dW_{\alpha,0,t}\,, \\ dY^\gamma_{\alpha,1,t} &= \rho\, Y^\gamma_{\alpha,0,t}\, dt\,, \end{aligned} \tag{2.18}$$

for $t \geq 0$, $\alpha \in \{1,2\}$ and independent standard Brownian motions $(W_{\alpha,0,t})_{t\geq 0}$. Then, [DP98] Theorem 2.4 reads as

$$E_{((x_1,x_2),(\theta_1,\theta_2))}\big[\, H(\, Z^\gamma_t\,,\, Y^\gamma_0\,)\,\big] \;=\; E_{((y_{10},y_{20}),(y_{11},y_{21}))}\big[\, H(\, Z^\gamma_0\,,\, Y^\gamma_t\,)\,\big] \tag{2.19}$$

for $t > 0$, where $Z^\gamma_{\cdot,\cdot,0} = ((x_1,x_2),(\theta_1,\theta_2)) \in [0,\infty)^2 \times [0,\infty)^2$ and $Y^\gamma_{\cdot,\cdot,0} = ((y_{10},y_{20}),(y_{11},y_{21})) \in [0,\infty)^2 \times [0,\infty)^2$ and H as in (2.5). If we choose $Y^\gamma_{\cdot,1,0} = (0,0)$ and $Y^\gamma_{\cdot,0,0} = (y, 0)$, with $y \geq 0$, as we did for Y in the proof of Lemma 2.4, see Equations (2.13) and (2.14), both dual processes, Y and Y^γ, coincide. Therefore, by the duality relation (for finite gamma), we infer

$$E_{(x_1,x_2)}\Big[e^{-\,y\,(Z^\gamma_{1,0,t}+Z^\gamma_{2,0,t})\,+\,i\,y\,(Z^\gamma_{1,0,t}-Z^\gamma_{2,0,t})}\Big] \\ = \exp\Big\{-\,y\,\big[\mu_{x_1}(t)+\nu_{x_2}(t)\big] \;+\; i\,y\,\big[\mu_{x_1}(t)-\nu_{x_2}(t)\big]\Big\}, \tag{2.20}$$

where $\mu_{x_1}(t) = x_1\,e^{-\kappa t}+(1-e^{-\kappa t})\frac{\rho}{\kappa}\theta_1$ and $\nu_{x_2}(t) = x_2\,e^{-\kappa t}+(1-e^{-\kappa t})\frac{\rho}{\kappa}\theta_2$. In this case we do not have to choose $x_1x_2 = 0$. But note that $Z^\gamma_{\cdot,0,t}$ is not concentrated on L. Hence, the mixed Laplace-Fourier transforms of (2.20) do not determine the distribution of $Z^\gamma_{\cdot,0,t}$ if we choose $Y^\gamma_{\cdot,0,0} \in L$ only. But by Corollary 2.5 and Lemma 1.2 for any random variable $D = (D_1, D_2)$ with distribution $DP_{(y_1,y_2)}$, where $(y_1, y_2) \in [0,\infty)^2$, we have

$$\begin{aligned} E\Big[F\big(\,(\,\mu_{x_1,t}\,,\,\nu_{x_2,t}\,)\,,\,D\,\big)\Big] \;&=\; \int DP_{(y_1,y_2)}(d\xi_1,d\xi_2)F\big(\,(\,\mu_{x_1,t}\,,\,\nu_{x_2,t}\,)\,,\,(\xi_1,\xi_2)\,\big) \\ &=\; \int DP_{(\,\mu_{x_1,t}\,,\,\nu_{x_2,t}\,)}(d\zeta_1,d\zeta_2)F\big(\,(\,\zeta_1\,,\,\zeta_2\,)\,,\,(y_1,y_2)\,\big) \\ &=\; E\Big[F\big(\,(\,X_{1,0,t}\,,\,X_{2,0,t}\,)\,,\,(y_1,y_2)\,\big)\Big], \end{aligned} \tag{2.21}$$

$(x_1, x_2) \in L$. Finally, note that by Lemma 2.3 of [DP98] Mytnik's duality functions are not only separating but also convergence determining. With this in mind we can prove the following.

2.6 Lemma *Let $Z^\gamma_{\cdot,0,0} = X_{\cdot,0,0} := (x_1, x_2) \in L$. Then for each $t \geq 0$*

$$\mathcal{L}[Z^\gamma_{\cdot,0,t}] \underset{\gamma\to\infty}{\longrightarrow} \mathcal{L}[X_{\cdot,0,t}]\,.$$

PROOF. The strategy is to use the duality relation for finite variance and investigate the limit of the dual Y^γ of (2.18). To this end let $Y^\gamma_{\cdot,1,0} = (0,0)$ and $Y^\gamma_{1,0,0} = y_1$ and $Y^\gamma_{2,0,0} = y_2$, where $y_1, y_2 \geq 0$. Observe that

$$\tilde{Y}^\gamma_{\alpha,0,t} \;:=\; e^{\kappa t}\,Y_{\alpha,0,t}\,, \quad t \geq 0\,, \tag{2.22}$$

are local martingales, $\alpha \in \{1,2\}$, since by Itô's Lemma, see for instance [KS91] Theorem 3.3.6, p.153 and choose the function $f(t,x) := e^{\kappa t}\,x$, we have

$$\begin{aligned} \tilde{Y}^\gamma_{\alpha,0,t} \;&=\; y_\alpha \;+\; \int_0^t \kappa e^{\kappa s}\,Y^\gamma_{\alpha,0,s}\,ds \;+\; \int_0^t e^{\kappa s}\,dY_{\alpha,0,s} \\ &=\; y_\alpha \;+\; \int_0^t \kappa e^{\kappa s}\,Y^\gamma_{\alpha,0,s}\,ds \;+\; \int_0^t e^{\kappa s}\big(-\kappa\,Y^\gamma_{\alpha,0,s}\big)\,ds \;+\; \int_0^t e^{\kappa s}\,\sqrt{\gamma\,Y^\gamma_{1,0,s}\,Y^\gamma_{2,0,s}}\;dW_{\alpha,0,s} \\ &=\; y_\alpha \;+\; \int_0^t e^{\kappa s}\sqrt{\gamma\,Y^\gamma_{1,0,s}\,Y^\gamma_{2,0,s}}\;dW_{\alpha,0,s}\,. \end{aligned}$$

In particular $E[Y^\gamma_{\alpha,0,t}] = e^{-\kappa t}y_\alpha$. The application of Itô's Lemma also implies that $(\tilde{Y}^\gamma_{1,0,t}\,, \tilde{Y}^\gamma_{2,0,t})$ solves the SDE

$$d\tilde{Y}^\gamma_{\alpha,0,t} \;=\; \sqrt{\gamma\,\tilde{Y}^\gamma_{1,0,t}\,\tilde{Y}^\gamma_{2,0,t}}\;dW_{\alpha,0,t}\,, \quad \alpha \in \{1,2\}, \quad t \geq 0\,, \tag{2.23}$$

with initial condition $\tilde{Y}^\gamma_{\cdot,0,0} = (y_1, y_2)$. Next, compare $(\tilde{Y}^\gamma_{\alpha,0,t})_{t\geq 0}$ with $({}^\gamma\overrightarrow{Y}_{\alpha,0,t})_{t\geq 0}$ given by

$${}^\gamma\overrightarrow{Y}_{\alpha,0,t} \;:=\; \tilde{Y}^1_{\cdot,0,\gamma\,t} \;=\; y_\alpha \;+\; \int_0^{\gamma\,t} \sqrt{\tilde{Y}^1_{1,0,s}\,\tilde{Y}^1_{2,0,s}}\;dW_{\alpha,0,s}\,,$$

$\alpha \in \{1,2\}$. Using Brownian scaling, the accelerated process satisfies

$$\begin{aligned} {}^{\gamma}\overrightarrow{Y}_{\alpha,0,t} &= y_\alpha + \int_0^{\gamma t} \sqrt{\tilde{Y}^1_{1,0,s}\,\tilde{Y}^1_{2,0,s}}\, dW_{\alpha,0,s} \\ &= y_\alpha + \int_0^{t} \sqrt{\gamma\ {}^{\gamma}\overrightarrow{Y}_{1,0,s}\,{}^{\gamma}\overrightarrow{Y}_{2,0,s}}\, dB_{\alpha,0,s} \end{aligned}$$

for some Brownian motion $(B_{\alpha,0,s})_{s\geq 0}$.
Hence, both processes $(\tilde{Y}^{\gamma}_{1,0,t}, \tilde{Y}^{\gamma}_{1,0,t})_{t\geq 0}$ and $({}^{\gamma}\overrightarrow{Y}_{1,0,t}\,,\,{}^{\gamma}\overrightarrow{Y}_{2,0,t})_{t\geq 0}$ are weak solutions to SDE (2.23). [DP98] implies weak uniqueness for this SDE (take migration equal to nil) and therefore we obtain

$$\mathcal{L}\big[\,(\tilde{Y}^{\gamma}_{1,0,t}, \tilde{Y}^{\gamma}_{1,0,t})_{t\geq 0}\,\big] \;=\; \mathcal{L}\big[\,({}^{\gamma}\overrightarrow{Y}_{1,0,t}\,,\,{}^{\gamma}\overrightarrow{Y}_{2,0,t})_{t\geq 0}\,\big]\,. \tag{2.24}$$

Now, recall that $t \mapsto {}^{\gamma}\overrightarrow{Y}_{\alpha,0,t} \;=\; \tilde{Y}^1_{\alpha,0,\gamma t}$ is a nonnegative local martingale, hence, a supermartingale, cf. [RW2] IV.14.3 on p.22. Therefore, it provides an a.s. limit, see [RW1] Theorem II.69.1 on p.176. So, for any $t > 0$ we have

$$\lim_{\gamma\to\infty} \tilde{Y}^1_{\alpha,0,\gamma t} \;=\; \tilde{Y}^1_{\alpha,0,\infty} \quad P\text{-a.s.}$$

Moreover, $(\,\tilde{Y}^1_{1,0,\infty}\,,\,\tilde{Y}^1_{2,0,\infty}\,) \;\sim\; DP_{(y_1,y_2)}$. This is due to [DP98], compare with the proofs of their Theorems 1.4 and 1.5 (pp.1103/1104 and pp.1111/1112): To see this note that $(\tilde{Y}^1_{1,0,\gamma t}\,,\,\tilde{Y}^1_{2,0,\gamma t}\,)_{\gamma\geq 0}$ is a continuous local martingale, hence, it can be represented as a time-changed Brownian motion by the Dubins-Schwarz Theorem, cf. [RY91] pp.181-184 Theorem V.1.10. Of course, $(\tilde{Y}^1_{1,0,\gamma t}\,,\,\tilde{Y}^1_{2,0,\gamma t}\,)$ stays constant after one particle dies out. That means when planar Brownian motion $B_\gamma = (B_{1,\gamma}, B_{2,\gamma})$ first hits $\partial[0,\infty)^2$. Note that for both components of planar Brownian motion we have time-change $C_\gamma \;=\; \langle \tilde{Y}^1_{\alpha,0,t\,\cdot}\rangle_\gamma \;=\; \int_0^\gamma \tilde{Y}^1_{1,0,t\,s}\,\tilde{Y}^1_{2,0,t\,s}\,ds$, $\alpha \in \{1,2\}$. Then by the results in Section 1.2 $(B_{1,\gamma\wedge T}, B_{2,\gamma\wedge T})_{\gamma\geq 0}$ is uniformly integrable and $(B_{1,T}, B_{2,T}) \;\sim\; DP_{(y_1,y_2)}$, where $T \;:=\; \inf\{t > 0 : B_{1,t}B_{2,t} \;=\; 0\,\}$. Thus, for any $t > 0$ we infer that $(\,{}^{\gamma}\overrightarrow{Y}_{1,0,t}\,,\,{}^{\gamma}\overrightarrow{Y}_{1,0,t}\,)$ converges in distribution to $DP_{(y_1,y_2)}$ as $\gamma \to \infty$.

For site 1 recall that by (2.18) and transformation (2.22) we have

$$Y^{\gamma}_{\alpha,1,t} \;=\; \int_0^t \rho\, Y^{\gamma}_{\alpha,0,s}\, ds \;=\; \int_0^t \rho\, e^{-\kappa s}\, \tilde{Y}^{\gamma}_{\alpha,0,s}\, ds\,.$$

For $\tilde{Y}^1_{\alpha,0,\gamma s}$ instead of $\tilde{Y}^{\gamma}_{\alpha,0,s}$ observe that P-a.s. by dominated convergence

$$\lim_{\gamma\to\infty} \int_0^t \rho\, e^{-\kappa s}\, \tilde{Y}^1_{\alpha,0,\gamma s}\, ds \;=\; \int_0^t \rho\, e^{-\kappa s}\, \tilde{Y}^1_{\alpha,0,\infty}\, ds \;=\; \rho\,(1\,-\,e^{-\kappa t})\,\tilde{Y}^1_{\alpha,0,\infty}$$

since $t \mapsto \tilde{Y}^1_{\alpha,0,\gamma t}$ is continuous and converges as $t \to \infty$, hence $\sup\limits_{0\leq t\leq\infty} \tilde{Y}^1_{\alpha,0,\gamma t} < \infty$ P-a.s. Now, note that both processes, $(\tilde{Y}^1_{\alpha,0,\gamma s})_{s\geq 0}$ and $(Y^{\gamma}_{\alpha,0,s})_{s\geq 0}$, have continuous paths. Then by (2.24) it is allowed to substitute $\tilde{Y}^1_{\alpha,0,\gamma s}$ for $Y^{\gamma}_{\alpha,0,s}$ so that

$$\lim_{\gamma\to\infty} \mathcal{L}[Y^{\gamma}_{\alpha,1,t}] \;=\; \mathcal{L}\big[\rho\,(1\,-\,e^{-\kappa t})\,\tilde{Y}^1_{\alpha,0,\infty}\big].$$

Since weak convergence of distributions implies the convergence of the mixed Laplace-Fourier transforms of (2.20) we have

$$\begin{aligned}
&\lim_{\gamma\to\infty} E_{(x_1,x_2)}\Big[e^{-(y_1+y_2)(Z^{\gamma}_{1,0,t}+Z^{\gamma}_{2,0,t})+i(y_1-y_2)(Z^{\gamma}_{1,0,t}-Z^{\gamma}_{2,0,t})}\Big]\\
&= E_{(y_1,y_2)}\Big[e^{-(x_1+x_2)(e^{-\kappa t}Y^1_{1,0,\infty}+e^{-\kappa t}Y^1_{2,0,\infty})+i(x_1-x_2)(e^{-\kappa t}Y^1_{1,0,\infty}-e^{-\kappa t}Y^1_{2,0,\infty})}\\
&\qquad\times e^{-(\theta_1+\theta_2)\big(\rho(1-e^{-\kappa t})Y^1_{1,0,\infty}+\rho(1-e^{-\kappa t})Y^1_{2,0,\infty}\big)+i(\theta_1-\theta_2)\big(\rho(1-e^{-\kappa t})Y^1_{1,0,\infty}-\rho(1-e^{-\kappa t})Y^1_{2,0,\infty}\big)}\Big]\\
&= E_{(y_1,y_2)}\Big[e^{-[(x_1+x_2)e^{-\kappa t}+(\theta_1+\theta_2)\rho(1-e^{-\kappa t})](Y^1_{1,0,\infty}+Y^1_{2,0,\infty})}\\
&\qquad\qquad\times e^{i[(x_1-x_2)e^{-\kappa t}+(\theta_1-\theta_2)\rho(1-e^{-\kappa t})](Y^1_{1,0,\infty}-Y^1_{2,0,\infty})}\Big].
\end{aligned}$$

Comparing with equation (2.21) we have for all $(y_1,y_2)\in[0,\infty)^2$

$$\lim_{\gamma\to\infty} E\Big[F\big((Z^{\gamma}_{1,0,t},\,Z^{\gamma}_{2,0,t}),\,(y_1,y_2)\big)\Big] = E\Big[F\big((X_{1,0,t},\,X_{2,0,t}),\,(y_1,y_2)\big)\Big], \tag{2.25}$$

which forces the limit-distribution to be concentrated on L. What is more, it is the distribution of $(X_{1,0,t},\,X_{2,0,t})$. □

Chapter 3

Construction of the process for countably many colonies

It took numerous trials to find the appropriate method for establishing existence and uniqueness for a mutually catalytic branching model with infinite variance on some countable graph S. For sure, the Lévy-type generator of Section 2.1 suggests to work with a martingale problem approach since we expect the process to be of pure jump type rather than a diffusion or an Itô process (in the sense of [SV79] Section 4.3 p.92, say).

So our first try was to use the usual Hille-Yoshida theory as in [EK86] Theorem 4.5.4, which provides existence of solutions to martingale problems for a large class of operators. However, this result needs a state space which is locally compact. And this will not be the case in our setting. As state space we wish to choose the positive cone of a weighted ℓ^1-sequence space, the so-called Liggett-Spitzer space, in which the k-th coordinate gives the mass of particles at site $k \in S$, see page 29 below. We expect this choice to be more natural than the state space used in [DP98], since we only deal with the discrete model, with a so-called super-random walk. That means the set of sites S is understood to be countably infinite – for example, choose the lattice $\mathbb{Z}$ instead of the real line $\mathbb{R}$.

It seemed, another possible way was to mimic the existence theorem for the finite variance process on the lattice – see [DP98] Appendix p.1127 – which is due to [SS80]. That is, we should define a sequence $(S_n)_n$ of finite sets which exhausts S and a sequence of processes X^n that are restricted to S_n. Then it is expected to have convergence of X^n to a process on S. However, note that in [DP98] it was not possible to ensure existence on S_n simply by quoting standard theorems as given in [RW2]; see sections V.23 and V.24 (Theorems V.23.5 and V.24.1, namely), since the diffusions matrix is degenerated. In our case we cannot use standard existence theorems for Lévy-type generators (on $\mathbb{R}^d$) like in [EK86] Theorem 8.3.3 because the Lévy measure $\nu(d\xi, x)$ as given in Section 2.1 is not continuous (in zero) as a function of x. But there is much more to find in this area; see, for example, [Ap04] Chapter 6 for an introduction and [KX95] for a much more general setting. The processes discussed in [Bas88], [Bas04] and [Bas07] or [St75] seemed to be pretty close to our situation at first glance.

The right argument for constructing our process was to use a weak convergence technique, as

Leonid Mytnik suggested to me. Then existence will follow through proving tightness. The only problem was to find the proper approximation scheme. The trick here is not to manipulate the jump measure, as the Lévy approach suggests, but to define a simplified process for short time intervals of length $\varepsilon > 0$ and go forth in time step by step. Then as ε approaches zero the limit is supposed to be the process in question. There are plenty of approximation schemes of this Peano kind in the literature. For example, in [DP98], proof of Theorem 6.1, p.1134, both types of particles evolve during a small time like independent processes with branching rate given by the former fixed state of the other type. In [FX01] the processes of different types are catalytic super-Brownian motions with frozen, smoothed and truncated branching rate functions, on small time intervals. For the construction of a superprocess with killing, Fleischmann and Mytnik did the following: On small time intervals only one type is affected by the killing that is provided by the other type. The roles of the types are alternated on subsequent intervals. Then the interval length is shrinked to zero. See [FM03]. Note that all three examples consider continuous site spaces, which means superprocesses with $S = \mathbb{R}$. In retrospect, we wished we had paid more attention to these continuous site models. However, for our model we will use the most rigorous simplification. During a small time interval there will be no random fluctuation, i.e. there is no interplay between particle type 1 and type 2. At the small time interval's end the proper tool to implement interaction of both types was presented in Section 1.2, namely, the DP-distribution. This method is rather the same as the one for the construction of the generator for the process on one colony as in Chapter 2. But there is more to appreciate. First calculations suggested that the infinite rate branching model should satisfy a similar martingale problem as the model of Dawson and Perkins with branching parameter zero; see Section 1.1. Due to properties of the DP-distribution our approximation scheme maintains this property and bequeaths it to the limit process. Moreover, since the limit of the approximating processes is, in some way, degenerated, the martingale problem will give uniqueness via the same duality argument as in the model of Dawson and Perkins.

The first section of this chapter defines the approximation scheme. Tightness of this scheme is established in the second section, and the third one deals with the martingale problem for the limit process, duality, uniqueness and the Markov property.

3.1 Approximation Processes

We retain the notation introduced in Section 1.1. Let the matrix $Q = (q_{jk})_{j,k\in S}$ govern the migration of particles. We assume $q_{jk} \geq 0$ if $j \neq k$ and $\bar{\lambda} := \sup_{j\in S} |q_{jj}| < \infty$. Let $Q^* = (q^*_{jk})$ denote the transpose of Q, i.e. $q^*_{jk} = q_{kj}$ for all $j, k \in S$. Since we are dealing with infinitely many sites, which means S is countably infinite, it will be necessary to impose additional restrictions on Q and to restrict the class of configurations which are permitted. As a matter of fact, the state space for our processes will be a version of the so-called *Liggett-Spitzer space*. We will define this next. Recall $L := \{0\} \times [0, \infty) \cup [0, \infty) \times \{0\} = [0, \infty)^2 \setminus (0, \infty)^2$.

3.1 Definition *Let $\gamma = (\gamma_j)_{j\in S}$ be a strictly positive, finite measure on S such that*

$$\sum_{j\in S} \gamma_j\,|q^*_{jk}| \;\le\; M\,\gamma_k \quad \forall\, k\in S \tag{3.1}$$

for some constant M with $\bar\lambda \vee 1 < M < \infty$; we write $\gamma|Q^*| \;\le\; M\gamma$ for short. Then set

$$\mathbb{E}_\gamma \;:=\; \Big\{ x \,\in\, \big([0,\infty)\times[0,\infty)\big)^S \,:\, \sum_{\alpha=1}^{2}\sum_{k\in S} x_{\alpha,k}\,\gamma_k \;<\; \infty \Big\},$$

$$\mathbb{L}_\gamma \;:=\; \Big\{ x \,\in\, L^S \,:\, \sum_{\alpha=1}^{2}\sum_{k\in S} x_{\alpha,k}\,\gamma_k \;<\; \infty \Big\} \;=\; \mathbb{E}_\gamma \cap L^S\,.$$

$\mathbb{E}_\gamma$ is given the smallest σ-algebra such that the map $x \mapsto x_{\alpha,k}$ is measurable for each $(\alpha,k) \in \{1,2\}\times S$, which is the Borel σ-algebra in fact. The topology for $\mathbb{E}_\gamma$ (or $\mathbb{L}_\gamma$) is provided by the metric

$$\|x-y\|_\gamma \;=\; \sum_{\alpha=1}^{2}\sum_{k\in S} |x_{\alpha,k}-y_{\alpha,k}|\gamma_k\,,$$

where $x,y \,\in\, \mathbb{E}_\gamma$ (or $\mathbb{L}_\gamma$). Similarly, we will write $\|x_{\alpha,\cdot}-y_{\alpha,\cdot}\|_\gamma \;=\; \sum_{k\in S}|x_{\alpha,k}-y_{\alpha,k}|\gamma_k$ for the distance of type α.

The construction of these weighted ℓ^1-sequence spaces is due to Liggett and Spitzer [LS81] and is described in detail in [Li85] Chapter IX.1. Note that it is not always necessary to assume $\sum_{k\in S}\gamma_k \;<\; \infty$; cf. [GKW02] p.25. Compare with [Li85] Lemma IX.1.6 to see that the weight function $\gamma \,=\, (\gamma_k)_{k\in S}$ can be found as

$$\gamma_k \;=\; \sum_{n=0}^{\infty}\sum_{j\in S} q^{(n)}_{kj}\,\beta_j\,M^{-n}\,,$$

where $Q^{(n)} \;=\; \big(q^{(n)}_{jk}\big)_{j,k}$ denotes the n-fold composition of the transition matrix and $\beta = (\beta_k)_{k\in S}$ is a strictly positive summable function on S. It is even possible to choose two different weight functions for both types of particles $x_{1,\cdot}$ and $x_{2,\cdot}$; cf. [GKW99] p.7.

With an abuse of notation we write $\|x\|_\gamma := \langle x,\gamma\rangle$ for the 'norm' of $x\in\mathbb{E}_\gamma$. But note that $\mathbb{E}_\gamma$ does not possess a vector space structure. In the sense of this γ-norm Q^* can be considered as a bounded linear operator. Due to growth condition (3.1), for $x\in\mathbb{E}_\gamma$, we have

$$|\langle Q^*x\,,\,\gamma\rangle| \;\le\; M\,\langle x\,,\,\gamma\rangle \;=\; M\,\|x\|_\gamma.$$

Hence, we write $\|Q^*\| \le M$. But note that $Q^*x = (Q^*x_1, Q^*x_2)$ need not be an element of $\mathbb{E}_\gamma$ since it is possibly a signed vector.

Next, consider the deterministic differential equation

$$\begin{aligned} d\bar X_{\alpha,k,t} \;&=\; \big(\bar X_{\alpha,\cdot,t}Q\big)_k\,dt\,, \qquad \alpha=1,2, \quad k\in S,\\ \bar X_{\cdot,\cdot,0} \;&=\; x \,\in\, \mathbb{E}_\gamma \quad \big(\text{ or } \mathbb{L}_\gamma\big). \end{aligned} \tag{3.2}$$

Since Q satisfies (3.1) and $x \,\in\, \mathbb{E}_\gamma$ the differential equation (3.2) has a unique deterministic solution given by

$$\bar X_{\alpha,k,t} \;=\; \big(e^{tQ^*}x_\alpha\big)_k\,, \quad k\in S, \tag{3.3}$$

for each $\alpha \in \{1,2\}$. Moreover, (3.1) implies $\gamma(Q^*)^{(n)} \le M^n\gamma$ and hence, $\langle e^{tQ^*}x_\alpha\,,\gamma\rangle \le e^{tM}\langle x_\alpha\,,\gamma\rangle$. Let I denote the identity matrix. Since $q_{jk} \ge 0$ for all $j \ne k$ the matrix $Q + \bar{\lambda}I$ has only non-negative entries and so has $\exp\{Q^* + \bar{\lambda}I\}$. Then the same is true for $\exp(tQ^*) = \exp\{tQ^* + \bar{\lambda}I\}\,e^{-\bar{\lambda}}I$. Hence, Q generates the semigroup $P_t = \exp(tQ^*)$ where $P_t x \in \mathbb{E}_\gamma$ if $x \in \mathbb{E}_\gamma$ and $\|P_t\| \le e^{tM}$. Thus, it holds that $\bar{X}_{\cdot,\cdot,t} \in \mathbb{E}_\gamma$ for all $t \ge 0$. – Compare with [Li85] Theorems IX.1.27 and IX.2.2 pp. 431-433 (to compare choose the identity matrix for Liggett's random variables $A_k(\cdot,\cdot)$ for all $k \in S$ and rather read Equation (3.2) as $d\bar{X}_{\alpha,k,t} = \big(Q^*\bar{X}_{\alpha,\cdot,t}\big)_k\,dt$).

To create a link with Section 1.1 we note that Equation (3.2) is a very special case of the model of Dawson and Perkins, namely, choose variance parameter $\gamma = 0$ in Equation (1.1). In order to construct a process with infinite variance instead of variance zero, however, we need to put a certain random mechanism into play. The key ingredient will be the DP-distribution, which does not possess a finite second moment. We will do this next.

Let $\varepsilon > 0$. At (deterministic) time ε we introduce a (random) jump according to the DP-distribution at each site $k \in S$ independently. That means we define a process X' by setting $X'_{\cdot,k,t} = \bar{X}_{\cdot,k,t}$ for $0 \le t < \varepsilon$. And at time $t = \varepsilon$ we let $X'_{\cdot,k,\varepsilon} \sim DP_{\bar{X}_{\cdot,k,\varepsilon}}$ for all $k \in S$. Given $X'_{\cdot,\cdot,\varepsilon}$, for times $t \ge \varepsilon$ the process $X'_{\cdot,\cdot,t}$ is defined to be the solution of (3.2) starting at time $t = \varepsilon$ with initial condition $X'_{\cdot,\cdot,\varepsilon}$.

Continuing, we can similarly define the second jump at time $t = 2\varepsilon$. X'' equals X' for $0 \le t < 2\varepsilon$, and conditioned on $X'_{\cdot,k,2\varepsilon}$ we have that $X''_{\cdot,k,2\varepsilon} \sim DP_{X'_{\cdot,k,2\varepsilon}}$. Iterating this procedure, up to time $T > 0$, we obtain a process with $\lfloor \frac{T}{\varepsilon} \rfloor$ jumps at times $\varepsilon,\, 2\varepsilon, \ldots, \lfloor \frac{T}{\varepsilon} \rfloor\varepsilon$. We will denote this process by X^ε. We hope to find a limit of these processes X^ε afterwards as ε tends to 0. This limit is supposed to be the process in question. In order to do this, we shall take a closer look at the properties of the approximating processes X^ε.

All processes considered here, are supposed to have paths in $D_{\mathbb{E}_\gamma}[0,T]$, the space of càdlàg functions on $[0,T]$ with values in $\mathbb{E}_\gamma$. By $\mathscr{D}$ we denote the Borel σ-algebra of $D_{\mathbb{E}_\gamma}[0,T]$. Since $\mathbb{E}_\gamma$ is separable, $\mathscr{D}$ is generated by the coordinate mappings $\pi_t : D_{\mathbb{E}_\gamma}[0,T] \longrightarrow \mathbb{E}_\gamma$, $\pi_t(\chi) = \chi_t$, see [EK86] Proposition III.7.1. We define the filtration $(\mathscr{F}_t)_{0\le t\le T}$ by setting $\mathscr{F}_t := \sigma(\pi_s : 0 \le s \le t)$. Analogously, we can take $\mathbb{L}_\gamma$ instead of $\mathbb{E}_\gamma$ or $[0,\infty)$ instead of $[0,T]$.

3.2 Lemma *Let X^ε be as described above. Then $(X^\varepsilon_t)_{0\le t\le T}$ is a time-inhomogeneous Markov process with sample paths in $D_{\mathbb{E}_\gamma}[0,T]$ if $X^\varepsilon_{\cdot,\cdot,0} = x \in \mathbb{E}_\gamma$. Furthermore, we have*

(a) for $k \in S$, $\alpha \in \{1,2\}$

$$E[X^\varepsilon_{\alpha,k,t}] \;=\; \sum_j p_t(k,j)E[X^\varepsilon_{\alpha,j,0}]\,, \quad t \ge 0, \tag{3.4}$$

where $\exp(tQ^*) = P_t = \big(p_t(j,k)\big)_{j,k\in S}$ *;*

(b) for $\alpha \in \{1,2\}$ and for $k \in S$

$$M^\varepsilon_{\alpha,k,t} \;:=\; X^\varepsilon_{\alpha,k,t} \;-\; X^\varepsilon_{\alpha,k,0} \;-\; \int_0^t \big(X^\varepsilon_{\alpha,\cdot,s}Q\big)_k\,ds\,, \quad t \ge 0, \tag{3.5}$$

is an $\mathscr{F}_t$-martingale and

$$M^\varepsilon_{\cdot,k,t} \;=\; \big(M^\varepsilon_{1,k,t}\,,\, M^\varepsilon_{2,k,t}\big) \;=\; -\sum_{j=1}^{\lfloor \frac{T}{\varepsilon}\rfloor}\big(\Delta X^\varepsilon_{\cdot,k,j\varepsilon}\big)\;_{[j\varepsilon,\infty)}(t)\,, \quad t \ge 0, \tag{3.6}$$

with the usual notation $\Delta X^\varepsilon_{\alpha,k,j\varepsilon} := X^\varepsilon_{\alpha,k,j\varepsilon} - X^\varepsilon_{\alpha,k,j\varepsilon-}$ *for the jump of* $X^\varepsilon_{\alpha,k,j\varepsilon}$ *at time* $j\varepsilon$, $j = 1,\dots,\lfloor \frac{T}{\varepsilon} \rfloor$, *where* $X^\varepsilon_{\cdot,k,j\varepsilon-}$ *denotes the left hand limit;*

(c) *for* $k \in S$ *and* $\alpha \in \{1,2\}$

$$Z^\varepsilon_{\alpha,k,t} := e^{-q_{kk}t}\, X^\varepsilon_{\alpha,k,t}, \quad t \ge 0, \tag{3.7}$$

is a (non-negative) $\mathscr{F}_t$*-submartingale and so is* $Z^\varepsilon_{\alpha,t} := \langle Z^\varepsilon_{\alpha,\cdot,t}, \gamma \rangle$.

(d) *Let* $K > 0$. *Then for all* $\varepsilon > 0$ *we have*

$$P\Big[\sup_{0\le s\le t} \langle X^\varepsilon_{1,\cdot,s} + X^\varepsilon_{2,\cdot,s}, \gamma \rangle > K\Big] \le \frac{e^{\bar\lambda t}\, e^{tM}}{K}\, E\big[\langle X^\varepsilon_{1,\cdot,0} + X^\varepsilon_{2,\cdot,0}, \gamma \rangle\big]. \tag{3.8}$$

Proof. Since the jumps are independent and X^ε obeys a deterministic flow between them, the Markov property becomes clear. But note that the Markov process is non-homogeneous in time, because jumps do not occur at exponential times but at deterministic ones.

For $y \in [0,\infty)^2$ and $Y \sim DP_y$, we have $E\big[Y - y\big] = (0,0)$, i.e. the jumps of $X_{\cdot,k,j\varepsilon}$ are centered random variables, for each $k \in S$ and all jump times $j\varepsilon$, $j = 1,\dots,\lfloor \frac{T}{\varepsilon} \rfloor$. Note that $\Delta X^\varepsilon_{\alpha,k,j\varepsilon} + X^\varepsilon_{\alpha,k,j\varepsilon-} \ge 0$. Next observe that if $x \in \mathbb{E}_\gamma$ we have $\langle \bar X_{\alpha,\cdot,t}, \gamma \rangle \le e^{tM} \langle x_\alpha, \gamma \rangle < \infty$ by (3.1). So for any finite sequence $S_N \nearrow S$ we obtain

$$\begin{aligned} E\big[\langle X^\varepsilon_{\alpha,\cdot,j\varepsilon}, \gamma \rangle\big] &= \lim_{N\to\infty} E\big[\langle \Delta X^\varepsilon_{\alpha,\cdot,j\varepsilon} + X^\varepsilon_{\alpha,\cdot,j\varepsilon-}, \gamma \ \ {}_{S_N} \rangle\big] \\ &= E\big[\langle X^\varepsilon_{\alpha,\cdot,j\varepsilon-}, \gamma \rangle\big] = \langle \bar X_{\alpha,\cdot,j\varepsilon-}, \gamma \rangle \le e^{j\varepsilon M} \langle x_\alpha, \gamma \rangle \end{aligned}$$

for $j = 1$ by monotone convergence, and hence, for all $j = 1,\dots,\lfloor \frac{T}{\varepsilon} \rfloor$ by induction. This yields part (a). In particular we have $X^\varepsilon_{\cdot,\cdot,t} \in \mathbb{E}_\gamma$ for all $t \ge 0$.

Rewriting (3.2) we have

$$\begin{aligned} \bar X_{\alpha,k,t} - \bar X_{\alpha,k,0} &= \int_0^t \big(\bar X_{\alpha,\cdot,s} Q\big)_k\, ds, && \text{for } 0 \le t < \varepsilon, \text{ and} \\ X'_{\alpha,k,t} - X'_{\alpha,k,\varepsilon} &= \int_\varepsilon^t \big(X'_{\alpha,\cdot,s} Q\big)_k\, ds, && \text{for } \varepsilon \le t < 2\varepsilon. \end{aligned}$$

This gives

$$X'_{\alpha,k,t} - X'_{\alpha,k,0} - \int_0^t \big(X'_{\alpha,\cdot,s} Q\big)_k\, ds = \big(\bar X_{\alpha,k,\varepsilon} - X'_{\alpha,k,\varepsilon}\big) \ \ {}_{[\varepsilon,\infty)}(t),$$

for $0 \le t < 2\varepsilon$ and Equation (3.6) follows immediatly. Obviously, $M^\varepsilon_{\cdot,k,t}$ has only finitely many jumps on the interval $[0,T]$ and is constant between these jumps. As we have already mentioned, the jumps at times $t = \varepsilon, 2\varepsilon, \dots, \lfloor \frac{T}{\varepsilon} \rfloor \varepsilon$ are centered, and, by construction, independent given $X^\varepsilon_{\cdot,\cdot,j\varepsilon-}$, for $j = 1,\dots,\lfloor \frac{T}{\varepsilon} \rfloor$, respectively, and we have $\Delta X^\varepsilon_{\cdot,k,j\varepsilon} = \Delta M^\varepsilon_{\cdot,k,j\varepsilon}$. Hence, the process $(M^\varepsilon_{\cdot,k,t})_{t\ge 0}$ is an $\mathscr{F}_t$-martingale.

For $t \in [0,\varepsilon)$ holds (3.2) and we have

$$\begin{aligned} e^{-q_{kk}t}\, X^\varepsilon_{\alpha,k,t} &= X^\varepsilon_{\alpha,k,0} + \int_0^t \Big[e^{-q_{kk}s}\big(X^\varepsilon_{\alpha,\cdot,s} Q\big)_k - q_{kk}\, e^{-q_{kk}s}\, X^\varepsilon_{\alpha,k,s}\Big]\, ds \\ &= X^\varepsilon_{\alpha,k,0} + \int_0^t e^{-q_{kk}s} \sum_{j\ne k} q_{jk}\, X^\varepsilon_{\alpha,j,s}\, ds. \end{aligned} \tag{3.9}$$

This shows that $e^{-q_{kk}t}\, X^{\varepsilon}_{\alpha,k,t}$ is increasing. At all subsequent jump times the deterministic factors $e^{-q_{kk}\, j\, \varepsilon}$, $j = 1,\ldots,\lfloor \frac{T}{\varepsilon} \rfloor$, let the jumps remain centered and independent. Hence, $Z^{\varepsilon}_{\alpha,k,t} = e^{-q_{kk}t}\, X^{\varepsilon}_{\alpha,k,t}$ is a submartingale w.r.t. $(\mathscr{F}_t)_{t\geq 0}$. Equation (3.9) shows that

$$\frac{d}{dt}E\big[Z^{\varepsilon}_{\alpha,k,t}\big] \;=\; \big(E[Z^{\varepsilon}_{\alpha,\cdot,t}]\,\tilde{Q}\,\big)_k$$

where $\tilde{Q} = (\tilde{q}_{jk})_{j,k\in S}$ with $\tilde{q}_{jk} := q_{jk}$ for $j \neq k$ and $\tilde{q}_{jj} := 0$ for $j \in S$. $\tilde{Q}$ also satisfies (3.1). Moreover, $\langle\, Z^{\varepsilon}_{\alpha,\cdot,t}\,,\,\gamma\,\rangle \;\leq\; e^{\bar{\lambda}t}\langle\, X^{\varepsilon}_{\alpha,\cdot,t}\,,\,\gamma\,\rangle$, and for $0 \leq s \leq t$ we have

$$E\big[\,Z^{\varepsilon}_{\alpha,t}\,|\mathscr{F}_s\big] \;\geq\; \lim_{N\to\infty} E\Big[\sum_{k\in S_N}\gamma_k\, Z^{\varepsilon}_{\alpha,k,t}\,\Big|\,\mathscr{F}_s\Big] \;\geq\; \lim_{N\to\infty}\sum_{k\in S_N}\gamma_k\, Z^{\varepsilon}_{\alpha,k,s} \;=\; Z^{\varepsilon}_{\alpha,s}$$

by dominated convergence. Hence, $Z^{\varepsilon}_{\alpha,t}$ is a submartingale. Then we can use Doob's submartingale inequality.

$$\begin{aligned} P\Big[\sup_{0\leq s\leq t}\langle\, X^{\varepsilon}_{1,\cdot,s} + X^{\varepsilon}_{2,\cdot,s}\,,\,\gamma\rangle \;>\; N\Big] \;&\leq\; P\Big[\sup_{0\leq s\leq t} Z^{\varepsilon}_{1,s} + Z^{\varepsilon}_{2,s} \;>\; N\Big] \\ &\leq\; N^{-1}e^{\bar{\lambda}t}\, E\big[\langle\, X^{\varepsilon}_{1,\cdot,t} + X^{\varepsilon}_{2,\cdot,t}\,,\,\gamma\,\rangle\big] \\ &\leq\; \frac{e^{\bar{\lambda}t}}{N}\, e^{tM}\,\big\langle\, E\big[X^{\varepsilon}_{1,\cdot,0} + X^{\varepsilon}_{2,\cdot,0}\big]\,,\,\gamma\,\big\rangle. \end{aligned} \tag{3.10}$$

Letting $N \to \infty$ we obtain

$$P\Big[\,X^{\varepsilon}_s \,\in\, \mathbb{E}_{\gamma} \quad \forall\, 0 \leq s \leq t\,\Big] \;=\; 1$$

which shows that the whole sample paths of X^{ε} are in $\mathbb{E}_{\gamma}$. The computation in (3.10) implies (d). □

Approximation on a finite box

Above, we constructed the process X^{ε} by adding jumps to the deterministic migration of Equation (3.2). The particular choice of jump distribution will allow us later to establish a Markov process with infinite variance. However, the lack of a finite second moment causes mathematical difficulties. To overcome this problem we compare X^{ε} with a process $X^{K,\varepsilon}$, which posesses a finite second moment and coincides with X^{ε} with probability close to 1. Here, we want to give a definition for the process $X^{K,\varepsilon}$. To this end we substitute the DP-distribution by the DP^K-distribution as prepared in Section 1.3.

Assume X^{ε} starts in $x = (x_{1,\cdot}, x_{2,\cdot}) \in \mathbb{L}_{\gamma}$. By Lemma 3.2(d) we have

$$P\Big[\sup_{0\leq s\leq T}\langle\, X^{\varepsilon}_{1,\cdot,s} + X^{\varepsilon}_{2,\cdot,s}\,,\,\gamma\rangle \;>\; \frac{K}{2}\,\Big] \;\leq\; \frac{2\,e^{\bar{\lambda}T}}{K}\, e^{TM}\langle\, x_{1,\cdot,0} + x_{2,\cdot,0}\,,\,\gamma\,\rangle\,, \tag{3.11}$$

which is arbirary small for $K > 0$ large enough. In particular, we can choose K large enough such that $\langle\, x_{1,\cdot,0} + x_{2,\cdot,0}\,,\,\gamma\,\rangle \;<\; K/2$. Then, with probability close to 1, we have

$$\big(X^{\varepsilon}_{1,k,s} + X^{\varepsilon}_{2,k,s}\big)\gamma_k \;\leq\; \langle\, X^{\varepsilon}_{1,\cdot,s} + X^{\varepsilon}_{2,\cdot,s}\,,\,\gamma\rangle \;\leq\; \frac{K}{2}\,,$$

for all $k \in S$ and all $s \in [0,T]$. For this reason we choose at each site $j \in S$ the quantity $K_j := K/2\gamma_j$. We define $X^{K,\varepsilon}_s$ as the solution of Equation (3.2) for $0 \leq s < \varepsilon$ starting in $x \in \mathbb{L}_{\gamma}$.

And for $s = \varepsilon$ we set independently $X^{K,\varepsilon}_{\cdot,j,\varepsilon} \sim DP^{K_j}_{X_{\cdot,j,\varepsilon-}}$ for all $j \in S$. For subsequent time intervalls $[m\varepsilon, (m+1)\varepsilon)$, $m = 1, 2, \ldots, \lfloor \frac{T}{\varepsilon} \rfloor \varepsilon$, we repeat this procedure with initial value $X^{K,\varepsilon}_{\cdot,\cdot,m\varepsilon}$ in the same manner as at the beginning of this section for the process X^ε. Then, we obtain a process $X^{K,\varepsilon}$ for which holds $X^{K,\varepsilon}_{\cdot,j,m\varepsilon} \in \partial[0, K_j]^2$ for all $j \in S$ and all times $m\varepsilon$, $m = 0, 1, \ldots, \lfloor \frac{T}{\varepsilon} \rfloor \varepsilon$ (Note that even if we might leave the box $[0, K_j]^2$ for some $j \in S$ it is still possible to define the jumps by DP^{K_j} since planar Brownian motion will definitely hit $\partial[0, K_j]^2$). Define the stopping times $\bar{\sigma}^K := \inf\big\{ s > 0 : \langle X^{K,\varepsilon}_{1\,\cdot,s} + X^{K,\varepsilon}_{2,\cdot,s} \,,\, \gamma \rangle \ \geq\ K/2 \,\big\}$ and $\sigma^K_{\alpha,j} := \inf\big\{ s > 0 : X^{K,\varepsilon}_{\alpha,j,s} \ \geq\ K_j \big\}$ for $j \in S$ and $\alpha \in \{1,2\}$ and set $\sigma^K := \inf_{j\in S, \alpha\in\{1,2\}} \sigma^K_{\alpha,j}$. Then $\bar{\sigma}^K \leq \sigma^K$ and $X^{K,\varepsilon}_{\cdot,\cdot,s\wedge\bar{\sigma}^K} \in \bigtimes_{j\in S} [0, K_j]^2$ for all $0 \leq s \leq T$ and the process stops when any coordinate hits $\partial[0, K_j]^2 \setminus L$. In particular we have $\langle\, X^{K,\varepsilon}_{1,\cdot,s} \,+\, X^{K,\varepsilon}_{2,\cdot,s} \,,\, \gamma \,\rangle \ \leq\ K/2$ for all $0 \leq s < \bar{\sigma}^K$.

Similar results as in Lemma 3.2 hold for the process $X^{K,\varepsilon}$. For instance, note that $E[Y] = y$ if $Y \sim DP^K_y$, and hence,

$$X^{K,\varepsilon}_{\alpha,k,t} \;=\; X^{K,\varepsilon}_{\alpha,k,0} \;+\; \int_0^t \big(X^{K,\varepsilon}_{\alpha,\cdot,s} Q \big)_k \, ds \;+\; M^{K,\varepsilon}_{\alpha,k,t}\,,$$

for $t \geq 0$, $k \in S$ and $\alpha \in \{1,2\}$, where $M^{K,\varepsilon}_{\alpha,k,t}$ is given as in (3.6) but with $\Delta X^{K,\varepsilon}_{\alpha,k,j\varepsilon}$ instead of $\Delta X^{\varepsilon}_{\alpha,k,j\varepsilon}$. In particular, $Z^{K,\varepsilon}_{\alpha,k,t} := e^{-q_{kk}t}\, X^{K,\varepsilon}_{\alpha,k,t}$ is a submartingale and

$$P\Big[\sup_{0\leq s\leq t} \langle\, X^{K,\varepsilon}_{1,\cdot,s} \,+\, X^{K,\varepsilon}_{2,\cdot,s} \,,\, \gamma \rangle \;>\; \frac{K}{2} \,\Big] \;\leq\; \frac{2\, e^{\bar{\lambda} t}}{K}\, e^{tM} \langle\, x_{1,\cdot,0} \,+\, x_{2,\cdot,0} \,,\, \gamma \,\rangle\,. \tag{3.12}$$

A priori, the construction of the processes $\big(X^{K,\varepsilon}_{\cdot,\cdot,s\wedge\bar{\sigma}^K} \big)_s$ and $(X^{\varepsilon}_{\cdot,\cdot,s})_s$ are self-contained. But it is possible to extract X^ε from the process $X^{K,\varepsilon}$ as follows: Set $X^\varepsilon_s \equiv X^{K,\varepsilon}_s$ for $0 \leq s < \varepsilon$. In view of Lemma 1.11 we define the jumps of $X^{K,\varepsilon}_{\cdot,j,s}$ as above, by using DP^{K_j} at jump times $s = m\varepsilon$, where $m = 1, 2, \ldots, \lfloor \frac{T}{\varepsilon} \rfloor \varepsilon$. For X^ε we use the DP-distribution *in addition*. That means, we set $DP_{X^{K,\varepsilon}_{\cdot,j,\varepsilon+}} = DP_{DP^{K}_{X^{K_\varepsilon}_{\cdot,j,\varepsilon-}}}$ for the jump of $X^\varepsilon_{\cdot,j,\varepsilon}$ at site $j \in S$. Recall that $DP_x = \delta_x$ if $x \in L$. This works until $X^\varepsilon_{\alpha,j,s} > K_j$ for some $j \in S$ and $\alpha \in \{1,2\}$. After this event, X^ε has to be constructed independently of $X^{K,\varepsilon}$ in the same manner as before, see p. 30. So define $\bar{\tau}^K := \inf\big\{ s > 0 : \langle X^\varepsilon_{1\,\cdot,s} + X^\varepsilon_{2,\cdot,s} \,,\, \gamma \rangle \ \geq\ K/2 \,\big\}$. And observe that in this way there is a coupling $\big(\, X^{K,\varepsilon} \,,\, X^\varepsilon \,\big)$ such that

$$X^{K,\varepsilon}_{\cdot,\cdot,s} \;=\; X^{\varepsilon}_{\cdot,\cdot,s} \qquad \text{for all } s < \; \bar{\tau}^K \wedge \bar{\sigma}^K. \tag{3.13}$$

By (3.11) and (3.12) we have $P\big(\bar{\tau}^K \wedge \bar{\sigma}^K \ \geq T \big) \ \longrightarrow\ 1$ as $K \to \infty$.

3.2 Tightness

In this section we want to establish the existence of a weak limit of the family $\big\{\, X^\varepsilon \,:\, 0 < \varepsilon \leq 1 \big\}$ as $\varepsilon \searrow 0$. Therefore we have to show tighness. But note that the DP-distribution charges only L for each site $k \in S$. So we expect the limit process to live in $\mathbb{L}_\gamma \,=\, \mathbb{E}_\gamma \cap L^S$; and this fact will be crucial to obtain uniqueness. Consequently, we define a process $\tilde{X}^\varepsilon$ which is piecewise constant and equals X^ε at its jump times. More precisely, let $X^\varepsilon_{\cdot,\cdot,0} \,=\, x \,\in\, \mathbb{L}_\gamma$ and set

$$\tilde{X}^\varepsilon_{\alpha,k,t} \;:=\; X^\varepsilon_{\alpha,k,\,j\varepsilon}\,, \quad \text{for } \ t \,\in\, [\, j\,\varepsilon \,,\, (j+1)\varepsilon \,)\,, \quad j \;=\; 0, 1, \ldots, \lfloor \tfrac{T}{\varepsilon} \rfloor, \tag{3.14}$$

for each $k \in S$ and $\alpha \in \{1,2\}$. Then $\tilde{X}^\varepsilon$ has paths in $D_{\mathbb{L}_\gamma}[0,T]$.

The subsequent lemmas will prepare the application of a standard theorem on relative compactness in the Skorohod space $D_{\mathbb{L}_\gamma}[0,T]$, see Theorem III.9.1 in [EK86] on p.142. That is, to show tightness of $\tilde{X}^\varepsilon$ it is enough to check a compact containment condition for $\tilde{X}^\varepsilon$ and tightness of the real valued processes $G(\tilde{X}^\varepsilon)$ for a sufficient rich class of functions G.

3.3 Lemma *Let K_C be a subset of $\mathbb{E}_\gamma$ (or $\mathbb{L}_\gamma$) with the properties*

(i) there is a constant $C>0$ such that $\langle x_{1,\cdot}+x_{2,\cdot}\,,\,\gamma\rangle \le C$ for all $x \in K_C$,

(ii) for all $\eta>0$ exists a finite $S_\eta \subseteq S$ such that for all $x\in K_C$ holds $\langle x_{1,\cdot}+x_{2,\cdot}\,,\,\gamma\ {}_{S\setminus S_\eta}\rangle \le \eta$.

Then K_C is precompact.

Proof. Let $(x^n)_n$ be a sequence in K_C. To show compactness we have to construct a subsequence converging to a point in K_C. Since in particular $0 \le x^n_{\alpha,k} \le C/\gamma_k$ we can find for each $k\in S$ and $\alpha\in\{0,1\}$ a subsequence such that $x^{n_j}_{\alpha,k}$ converges to some $\bar{x}_{\alpha,k}$ as $j\to\infty$. Set $\bar{x} = (\bar{x}_{1,k},\bar{x}_{2,k})_{k\in S}$. Choosing a diagonal sequence we may assume convergence in each coordinate. For notational convenience we still denote this subsequence by $(x^n)_n$. Then by Fatou's lemma we have $C \ge \liminf_{n\to\infty}\langle x^n_{1,\cdot}+x^n_{2,\cdot}\,,\,\gamma\rangle \ge \langle \bar{x}_{1,\cdot}+\bar{x}_{2,\cdot}\,,\,\gamma\rangle$ and $\eta \ge \liminf_{n\to\infty}\langle x^n_{1,\cdot}+x^n_{2,\cdot}\,,\,\gamma\ {}_{S\setminus S_\eta}\rangle \ge \langle \bar{x}_{1,\cdot}+\bar{x}_{2,\cdot}\,,\,\gamma\ {}_{S\setminus S_\eta}\rangle$ for any pair (η, S_η) as in property (ii) above. This implies $\bar{x}\in\overline{K_C}$. Finally let $\eta>0$ and observe that

$$\begin{aligned}\sum_{\alpha=1}^{2}\langle |x^n_{\alpha,\cdot}-\bar{x}_{\alpha,\cdot}|\,,\,\gamma\rangle \;&\le\; \sum_{\alpha=1}^{2}\langle |x^n_{\alpha,\cdot}-\bar{x}_{\alpha,\cdot}|\,,\,\gamma\ {}_{S_{\eta/3}}\rangle \;+\; \sum_{\alpha=1}^{2}\langle x^n_{\alpha,\cdot}+\bar{x}_{\alpha,\cdot}\,,\,\gamma\ {}_{S\setminus S_{\eta/3}}\rangle \\ &\le\; \eta/3 \;+\; 2\eta/3 \;=\; \eta\,,\end{aligned}$$

for n large enough. This gives $x^n \to \bar{x}\in\overline{K_C}$ in $\mathbb{E}_\gamma$. □

Note that $\mathbb{L}_\gamma$ is a closed subset of $\mathbb{E}_\gamma$. That means, if $K_C \subseteq \mathbb{E}_\gamma$ is compact (in $\mathbb{E}_\gamma$) then $K_C\cap\mathbb{L}_\gamma$ is compact in $\mathbb{L}_\gamma$. So in the sequel we might consider K_C as a compact set of $\mathbb{E}_\gamma$ as well as a compact set of $\mathbb{L}_\gamma$.

3.4 Lemma *For every $\eta>0$ and $T>0$ there exists a compact set $\Gamma_{\eta,T}\subseteq\mathbb{E}_\gamma$ for which*

$$\inf_{\varepsilon>0} P\Big[\ \tilde{X}^\varepsilon_t \in \Gamma_{\eta,T} \quad \text{for } 0\le t\le T\Big] \;\ge\; 1-\eta\,. \tag{3.15}$$

Proof. Let $\eta>0$ and $T>0$. We will characterise the compact set $\Gamma_{\eta,T}$ via Lemma 3.3.

We first choose $C>0$: By Lemma 3.2(c) $t\mapsto Z^\varepsilon_{\alpha,k,t} := e^{-q_{kk}t}X^\varepsilon_{\alpha,k,t}$ is a submartingale. Set $A^\varepsilon_1 := \Big\{\sup_{0\le t\le T}\sum_{\alpha=1}^{2}\langle X^\varepsilon_{\alpha,k,t}\,,\,\gamma\rangle \;>\; C\Big\}$. Recall that $\sup_{\varepsilon>0}E\big[\langle X^\varepsilon_{\alpha,\cdot,T}\,,\,\gamma\rangle\big] < \infty$ by Lemma 3.2. Then by Doob's submartingale inequality we obtain

$$\begin{aligned}\sup_{\varepsilon>0}P\big[A^\varepsilon_1\big] \;\le\; \sup_{\varepsilon>0}P\Big[\sup_{0\le t\le T}\sum_{\alpha=1}^{2}\langle Z^\varepsilon_{\alpha,\cdot,t}\,,\,\gamma\rangle \;>\; C\Big] \;&\le\; \frac{1}{C}\sup_{\varepsilon>0}E\Big[\sum_{\alpha=1}^{2}\langle Z^\varepsilon_{\alpha,\cdot,T}\,,\,\gamma\rangle\Big] \\ &\le\; \frac{e^{T\bar{\lambda}}}{C}\sup_{\varepsilon>0}E\Big[\sum_{\alpha=1}^{2}\langle X^\varepsilon_{\alpha,\cdot,T}\,,\,\gamma\rangle\Big] \\ &\le\; \frac{\eta}{2}\,,\end{aligned}$$

for C large enough.

Next, we choose a sequence of finite subsets $S_{\eta_N} \subseteq S$ for $\eta_N := \frac{\eta^2}{2^{2N}}$, $N = 2, 3, \ldots$, which will be consequently suitable for property (ii) in Lemma 3.3: Since $\sup_{\varepsilon>0} E\big[\langle X^\varepsilon_{\alpha,\cdot,T}\,,\,\gamma\rangle\big] < \infty$ there exists a finite subset $S_{\eta_N} \subseteq S$ such that for the complement $(S_{\eta_N})^c := S \setminus S_{\eta_N}$ we have

$$\sum_{\alpha=1}^{2} \sum_{k\,\in\,(S_{\eta_N})^c} E\big[\,\gamma_k\, X^\varepsilon_{\alpha,k,T}\,\big] \;<\; \frac{1}{e^{T\bar\lambda}}\,\frac{\eta^2}{2^{2N}}\,.$$

For such S_{η_N} we can set $A^\varepsilon_N := \Big\{\sup_{0\le t\le T}\sum_{\alpha=1}^{2}\langle X^\varepsilon_{\alpha,k,t}\,,\,\gamma_{\;(S_{\eta_N})^c}\rangle \;>\; \frac{\eta}{2^N}\Big\}$, $N = 2, 3, \ldots$, and obtain similar as above

$$\sup_{\varepsilon>0} P\big[\,A^\varepsilon_N\,\big] \;\le\; \sup_{\varepsilon>0} P\Bigg[\sup_{0\le t\le T}\sum_{\alpha=1}^{2}\langle Z^\varepsilon_{\alpha,\cdot,t}\,,\,\gamma_{\;(S_{\eta_N})^c}\rangle \;>\; \frac{\eta}{2^N}\Bigg] \;\le\; \frac{\eta}{2^N}.$$

The choice of C and $\big(S_{\eta_N}\big)_N$ give a characterisation of a compact set $\Gamma_{\eta,T}$ as in Lemma 3.3 and for its complement $(\Gamma_{\eta,T})^c$ we have

$$\Big\{\sup_{0\le t\le T} X^\varepsilon_t \;\in\; (\Gamma_{\eta,T})^c\Big\} \;=\; \bigcup_{N=1}^{\infty} A^\varepsilon_N\,.$$

This implies

$$\sup_{\varepsilon>0} P\Bigg[\sup_{0\le t\le T} X^\varepsilon_t \;\in\; (\Gamma_{\eta,T})^c\Bigg] \;\le\; \sum_{N=1}^{\infty}\frac{\eta}{2^N} \;=\; \eta\,,$$

and Equation (3.15), which is stated for the process $\tilde X^\varepsilon$, follows immediately. □

We denote by $\mathrm{Lip}_{fin}(\mathbb{L}_\gamma)$ the real valued, bounded, (uniformly) Lipschitz continuous functions on $\mathbb{L}_\gamma$ which depend on only finitely many coordinates.

3.5 Lemma $\mathrm{Lip}_{fin}(\mathbb{L}_\gamma)$ *is a dense subset of* $\mathcal{C}_b(\mathbb{L}_\gamma,\mathbb{R})$ *w.r.t. uniform convergence on compact sets.*

Proof. We have to show that for arbitrary $f \in \mathcal{C}_b(\mathbb{L}_\gamma,\mathbb{R})$, compact $\mathcal{K} \subseteq \mathbb{L}_\gamma$ and $\eta > 0$ there exists a function $g \in \mathrm{Lip}_{fin}(\mathbb{L}_\gamma)$ such that $\sup_{x\in\mathcal{K}} |f(x) - g(x)| < \eta$.

Denote by $f_{|\mathcal{K}}$ the restriction of f on $\mathcal{K}$. Obviously, $\mathrm{Lip}_{fin}(\mathbb{L}_\gamma)$ separates points (of $\mathbb{L}_\gamma$) and so does the set $Lip_{\mathcal{K}} := \{g_{|\mathcal{K}} : g \in \mathrm{Lip}_{fin}(\mathbb{L}_\gamma)\}$ for $x, y \in \mathcal{K}$. Moreover, $Lip_{\mathcal{K}} \subseteq \mathcal{C}_b(\mathcal{K},\mathbb{R})$ is an algebra. Then the Stone-Weierstraß Theorem, see [Kl06] on p.282, yields $\|f_{|\mathcal{K}} - g_{|\mathcal{K}}\|_\infty < \eta$ for some $g \in Lip_{\mathcal{K}}$, and we are done. □

3.6 Lemma *Let* $G \in \mathrm{Lip}_{fin}(\mathbb{L}_\gamma)$ *and set*

$$Y^\varepsilon_t \;:=\; G(\tilde X^\varepsilon_t)\,, \quad t \ge 0\,.$$

Then the family of processes $\{Y^\varepsilon_\cdot\}_{\varepsilon>0}$ *(with sample paths in* $D_{\mathbb{R}}[0,T]$ *) is relatively compact.*

Proof. In order to show the assertion we use the Aldous criterion, see [Al78] (in particular Equation (13)), or [JS87] p.356 VI.§4a. That means, we have to check the following two conditions:

(i) For each fixed $t \geq 0$, $\{Y_t^\varepsilon\}_{\varepsilon>0}$ is tight.

(ii) Let $\eta > 0$. Then there exists $\delta > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{\varepsilon \leq \varepsilon_0} P\Big[\,\big|Y^\varepsilon_{\tau+\delta'} - Y^\varepsilon_\tau\big| > \eta\Big] \leq \eta\,. \tag{3.16}$$

for each $0 \leq \delta' \leq \delta$, and each stopping time τ bounded by T.

For condition (i) just note that $|Y_t^\varepsilon| = |G(\tilde{X}_t^\varepsilon)|$ is bounded uniformly in $\varepsilon > 0$ since G is bounded. Then, for instance, use [Kl06] p.248 Beispiel 13.28(ii).

The second condition is more subtle. First note that we should have constructed the processes $\tilde{X}^\varepsilon$ up to time $T' > T + 1$ and we might assume w.l.o.g. $0 < \delta < 1$, so that $\tau + \delta' < T'$. To keep the notation as simple as possible we omit details like $\tau + \delta' \wedge T$. So let $\eta > 0$. Since G is Lipschitz continuous and depends on finitely many coordinates only, we have $|Y^\varepsilon_{\tau+\delta'} - Y^\varepsilon_\tau| \leq C\,\|(\tilde{X}^\varepsilon_{\tau+\delta'} - \tilde{X}^\varepsilon_\tau)\ _{S_N}\|_\gamma$, for some constant $C > 0$ and some finite set $S_N \subseteq S$, with $|S_N| = N$, say. Hence, it is enough to show that with probability larger than $1 - \eta$,

$$\big|\tilde{X}^\varepsilon_{\alpha,k,\tau+\delta'} - \tilde{X}^\varepsilon_{\alpha,k,\tau}\big| \leq \frac{1}{2\,C\,N\,\hat{\gamma}}\,\eta =: \eta^*\,, \quad \text{for all } k \in S_N \text{ and } \alpha \in \{1,2\},$$

where $\hat{\gamma} := \max_{j\in S_N} \gamma_j$.

For $k \in S_N$ we have

$$\big|\tilde{X}^\varepsilon_{\alpha,k,\tau+\delta'} - \tilde{X}^\varepsilon_{\alpha,k,\tau}\big| \leq \int_\tau^{\tau+\delta'} \big|\big(X^\varepsilon_{\alpha,\cdot,s}Q\big)_k\big|\,ds + \big|M^\varepsilon_{\alpha,k,\tau+\delta'} - M^\varepsilon_{\alpha,k,\tau}\big|\,, \tag{3.17}$$

where $M^\varepsilon_{\cdot,k,t} = -\sum_{j=1}^{\lfloor \frac{T}{\varepsilon}\rfloor} \big(\Delta X^\varepsilon_{\cdot,k,j\varepsilon}\big)\ _{[j\varepsilon,\infty)}(t)$ as in Lemma 3.2. We show that both terms on the r.h.s. of (3.17) will be smaller than $\eta^*/2$ with probability larger than $1 - \eta/2$ for all $0 \leq \delta' \leq \delta$ if δ is small enough.

For $x \in \mathbb{L}_\gamma$ and for $k \in S_N$ we obtain by Lemma 3.2

$$E_x\Big[\big|\big(X^\varepsilon_{\alpha,\cdot,s}Q\big)_k\big|\Big] \leq \sum_{j\in S} E_x\big[X^\varepsilon_{\alpha,j,s}\big]|q_{jk}| = \sum_{j\in S}\sum_{l\in S} p_s(j,l)x_{\alpha,l}|q_{jk}|\,,$$

which is finite and can be bounded from above uniformly for all $k \in S_N$ and $s \in [0,T]$ since

$$\sum_{k\in S}\sum_{j\in S}\sum_{l\in S} p_s(j,l)x_{\alpha,l}|q_{jk}|\gamma_k \leq M \sum_{j\in S}\sum_{l\in S} x_{\alpha,l}\,p_s(j,l)\,\gamma_j \leq M\,e^{M\,s}\,\langle x_{\alpha,\cdot}\,,\gamma\rangle.$$

Then note that $E\big[\langle X^\varepsilon_{\alpha,\cdot,\tau}\,,\gamma\rangle\big] \leq e^{\bar{\lambda}T}E\big[\langle X^\varepsilon_{\alpha,\cdot,T}\,,\gamma\rangle\big]$, so we can get rid of the initial condition $X^\varepsilon_{\alpha,\cdot,\tau}$. Then we can use the elementary Markov inequality and obtain for some constant $\bar{C} > 0$, which does neither depend on $k \in S_N$ nor on ε,

$$P\Big(\int_\tau^{\tau+\delta'} \big|\big(X^\varepsilon_{\alpha,\cdot,s}Q\big)_k\big|ds \geq \eta^*/2\Big) \leq \delta'\,\frac{2\bar{C}}{\eta^*} \leq \delta\,\frac{2\bar{C}}{\eta^*} \leq \frac{\eta}{2}\,,$$

for δ small enough.

Next, we deal with the martingale term in (3.17). The strategy is as follows: We wish to use a Doob inequality with finite second moments. Therefore we choose $K > 0$ large enough such that the total weighted mass of X^ε and $X^{K,\varepsilon}$ is smaller than K with probability close to 1. Then we can use the finite second moment of the truncated process $X^{K,\varepsilon}$.

We first choose K: According to Lemma 3.2(d) and Equation (3.12) there is a $K > 0$ such that

$$P\Big[\sup_{0\le t\le T} \sum_{\alpha=1}^{2} \langle X^\varepsilon_{\alpha,\cdot,t}\,,\,\gamma\rangle \;\ge\; K/2\Big] \;\le\; \frac{\eta}{8} \tag{3.18}$$

and

$$P\Big[\sup_{0\le t\le T} \sum_{\alpha=1}^{2} \langle X^{K,\varepsilon}_{\alpha,\cdot,t}\,,\,\gamma\rangle \;\ge\; K/2\Big] \;\le\; \frac{\eta}{8} \tag{3.19}$$

for all $\varepsilon > 0$. That means, the jumps of X^ε and $X^{K,\varepsilon}$ coincide with probability larger than $1-\eta/4$. This implies

$$P\Big(\,M^\varepsilon_t = M^{K,\varepsilon}_t \text{ for all } t\in[0,T]\Big) \;\ge\; 1-\eta/4\,,$$

where $M^{K,\varepsilon}_{\cdot,k,t} \;=\; -\sum_{j=1}^{\lfloor \frac{T}{\varepsilon}\rfloor} \big(\Delta X^{K,\varepsilon}_{\cdot,k,j\varepsilon}\big) \;\; {}_{[j\varepsilon,\infty)}(t)$. For $M^{K,\varepsilon}$ we can use Doob's inequality.

$$\begin{aligned}
P\Big(\sup_{0\le\delta'\le\delta} \big|M^{K,\varepsilon}_{\alpha,k,\tau+\delta'} \;-\; M^{K,\varepsilon}_{\alpha,k,\tau}\big| \;\ge\; \eta^*/2\Big) \;&\le\; \frac{16}{(\eta^*)^2}\, E\Big[\big(M^{K,\varepsilon}_{\alpha,k,\tau+\delta} \;-\; M^{K,\varepsilon}_{\alpha,k,\tau}\big)^2\Big]\\
&=\; \frac{16}{(\eta^*)^2}\, E\Big[E_{X^{K,\varepsilon}_{\cdot,k,\tau}}\Big[\,\langle M^{K,\varepsilon}_{\alpha,k,\cdot}\rangle_\delta\,\Big]\Big]\\
&=\; \frac{16}{(\eta^*)^2}\, E\Big[E_{X^{K,\varepsilon}_{\cdot,k,\tau}}\Big[\sum_{j=1}^{\lfloor\frac{\delta}{\varepsilon}\rfloor} E\Big[(\Delta X^{K,\varepsilon}_{\alpha,k,j\varepsilon})^2\,\Big|\,X^{K,\varepsilon}_{\cdot,\cdot,j\varepsilon-}\Big]\Big]\Big].
\end{aligned}$$

Next, we will give an upper bound for a single jump. We use the results stated in Section 1.3 for the DP^K-distribution; see page 12. Recall that

$$E\Big[(\Delta X^{K,\varepsilon}_{\alpha,k,j\varepsilon})^2\,\Big|\,X^{K,\varepsilon}_{\cdot,\cdot,j\varepsilon-}\Big] \;=\; K_k^2\, V\big(\,X^{K,\varepsilon}_{1,k,j\varepsilon-}/K_k\;,\;X^{K,\varepsilon}_{2,k,j\varepsilon-}/K_k\,\big)\,,$$

where $K_k \;=\; \frac{K}{\gamma_k}$, and that for $(x_1,x_2)\;\in\;[0,K_k]^2$ we have

$$V(x_1,x_2) \;\le\; C'\,x_1\,x_2\,\Big[1+\log(K_k)+|\log(x_1)|\wedge|\log(x_2)|\Big], \tag{3.20}$$

for some constant $C' > 0$. Since during time $(j-1)\varepsilon$ and $j\varepsilon$ particles migrate according to Q we have $(X^{K,\varepsilon}_{\alpha,\cdot,(j-1)\varepsilon}Q)_k \;\le\; \gamma_k^{-1}\langle\,(X^{K,\varepsilon}_{\alpha,\cdot,(j-1)\varepsilon}Q)_\cdot\,,\,\gamma\,\rangle \;\le\; \frac{M}{2}\,\hat K$, where $\hat K = \max_{j\in S_N}\gamma_j^{-1}K$. Next, we can choose $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ we have $\varepsilon\frac{M}{2}\hat K \;\le\; \min_{j\in S_N}\frac{K}{2\gamma_j}$. Hence, $X^{K,\varepsilon}_{\alpha,k,j\varepsilon-} \;\le\; X^{K,\varepsilon}_{\alpha,k,(j-1)\varepsilon} + \varepsilon\,M\,\hat K/2$, for $\alpha\in\{0,1\}$. Note that $X^{K,\varepsilon}_{1,k,(j-1)\varepsilon}$ or $X^{K,\varepsilon}_{2,k,(j-1)\varepsilon}$ equals zero. If $X^{K,\varepsilon}_{2,k,(j-1)\varepsilon} = 0$ we can estimate x_2 by $\varepsilon\,M\,\hat K/2$ in (3.20) and proceed

$$\begin{aligned}
V\Big(\,X^{K,\varepsilon}_{1,k,j\varepsilon-}\;,\;X^{K,\varepsilon}_{2,k,j\varepsilon-}\Big) \;&\le\; C'\,\hat K\cdot\varepsilon\,\frac{M\hat K}{2}\,\big[1+\log(\hat K)\big] \;+\; C'\,X^{K,\varepsilon}_{1,k,j\varepsilon-}\cdot\varepsilon\,\frac{M\hat K}{2}\,\log(X^{K,\varepsilon}_{1,k,j\varepsilon-})\\
&\le\; \tilde C\,\varepsilon,
\end{aligned}$$

for some constant $\tilde C$. We also used $x_1|\log(x_1)| \;\le\; \hat K\log(\hat K)$ for $x_1\;\in\;[0,\hat K]$. Interchanging the roles of x_1 and x_2 we obtain in case of $X^{K,\varepsilon}_{1,k,(j-1)\varepsilon} = 0$ the same bound. Note that this bound holds for all initial conditions $\tilde{X}^{K,\varepsilon}_{\cdot,\cdot,\tau}$. This gives

$$\frac{16}{(\eta^*)^2}\, E\Big[E_{X^{K,\varepsilon}_{\cdot,k,\tau}}\Big[\sum_{j=1}^{\lfloor\frac{\delta}{\varepsilon}\rfloor} E\Big[(\Delta X^{K,\varepsilon}_{\alpha,k,j\varepsilon})^2 \,\Big|\, X^{K,\varepsilon}_{\cdot,\cdot,j\varepsilon-}\Big]\Big]\Big] \;\le\; \frac{16}{(\eta^*)^2}\,\tilde{C}\sum_{j=1}^{\lfloor\frac{\delta}{\varepsilon}\rfloor}\varepsilon$$
$$\le\; \frac{16}{(\eta^*)^2}\,\tilde{C}\,\delta \;\le\; \frac{\eta}{4}$$

for δ small enough. □

Now, we are ready to summarise the previous results.

3.7 Proposition

(a) The family of processes $\{\tilde{X}^\varepsilon : 0 < \varepsilon \le 1\}$ with sample paths in $D_{\mathbb{L}_\gamma}[0,T]$ is tight. Hence, it provides a subsequence which converges weakly to some process X.

(b) The same weak convergence is true for the family of processes $\{X^\varepsilon : 0 < \varepsilon \le 1\}$.

Proof. Note that $\mathbb{L}_\gamma$ is Polish. Then $D_{\mathbb{L}_\gamma}[0,T]$ equipped with the Skorohod topology is complete and separable; cf. [EK86] Chapter III, Section 5, in particular Theorem III.5.6, p. 121. Hence, relative compactness and tightness are equivalent. Then (a) follows by Theorem III.9.1 in [EK86] on p.142 in combination with Lemmas 3.4, 3.5 and 3.6.

Recall that the two processes X^ε and $\tilde{X}^\varepsilon$ coincide on the time grid $t = 0, \varepsilon, 2\varepsilon, \ldots, \varepsilon\lfloor\frac{T}{\varepsilon}\rfloor$. Assume that the subsequence $\tilde{X}^{\varepsilon_n}$ converges weakly to X on the Skorokhod space $\big(D([0,T], \mathbb{E}_\gamma)\,,\, d\big)$. We want to show weak convergence of X^{ε_n} to X (on the Skorokhod space). Since uniform convergence implies convergence w.r.t. d, see [JS87] Prop. VI.1.17, we show

$$\sup_{s\le T} \|\, X^{\varepsilon_n}_s \;-\; \tilde{X}^{\varepsilon_n}_s \,\|_\gamma \;\xrightarrow[n\to 0]{P}\; 0\,. \tag{3.21}$$

This implies weak convergence of X^{ε_n} to X by Slutzky's Theorem; see e.g. [Kl06] Satz 13.18, p.243; here, we only need a metric space, compare with [JS87] VI.1.22 and VI.1.23.

To show (3.21) let $\eta > 0$. As in the proof of Lemma 3.4 we can choose a finite subset S_N of S, with $|S_N| = N \in \mathbb{N}$, say, such that

$$\sup_{0<\varepsilon\le 1} P\Big[\sup_{0\le t\le T}\sum_{\alpha=1}^{2}\langle X^{\varepsilon_n}_{\alpha,\cdot,t}\,,\,\gamma \quad {}_{S\setminus S_N}\rangle \;\ge\; \eta/4\Big] \;\le\; \eta/4.$$

And then the same is true for $\tilde{X}^{\varepsilon_n}$. It remains to show that, for $k \in S_N$,

$$P\Big[\sup_{0\le t\le T}\big|X^{\varepsilon_n}_{\alpha,k,t} \;-\; \tilde{X}^{\varepsilon_n}_{\alpha,k,t}\big| \;\ge\; \eta/4N\Big] \;\le\; \frac{\eta}{4N}\,,$$

for all $\varepsilon_n < \varepsilon'$, where ε' is small enough. But as seen in Lemma 3.6 the processes X^{ε_n} or $\tilde{X}^{\varepsilon_n}$ live with probability larger than $1 - \eta/4$ inside a box of size K/γ_j for all $j \in S$ if K is large enough. Set $\hat{K} = \max_{j\in S_N}\gamma_j^{-1}K$. Then inside the box we have for all $k \in S_N$

$$\sup_{0\le t\le T}\big|X^{K,\varepsilon_n}_{\alpha,k,t} \;-\; \tilde{X}^{K,\varepsilon_n}_{\alpha,k,t}\big| \;\le\; \varepsilon\frac{M}{2}\hat{K} \;\le\; \frac{\eta}{4N}\,,$$

for all $0 \le \varepsilon_n < \varepsilon'$, with ε' small enough – compare with the arguments proceeding Equation (3.20) on page 37. □

3.8 Remark *We cautiously considered weak convergence of subsequences of the processes* X^ε *(or* $\tilde{X}^\varepsilon$*),* $0 < \varepsilon \leq 1$*, for a fixed time horizon* $T > 0$*. The generalisation to* $D_{\mathbb{L}_\gamma}[0,\infty)$ *is then immediate. In fact, since* $T > 0$ *was arbitrary we can choose a sequence* $(T_j)_j$ *with* $T_j \nearrow \infty$ *as* $j \to \infty$*. Then for each* T_j *we have a convergent subsequence, and hence, by using a diagonal subsubsequence we obtain weak convergence on any compact time interval.* ◊

3.3 Martingale Problem and Uniqueness

We wish to show that the weak limit X of the processes $\tilde{X}^{\varepsilon_n}$ is a solution to a certain martingale problem. Mytnik's self-duality will give uniqueness for this martingale problem. Hence, we will have constructed a unique (strong) Markov process.

To establish a martingale problem and a self-duality relation, the process Y dual to X has to obey the same dynamics as X but with Q^* as migration kernel instead of Q. Therefore, we have to assume from now on that the growth condition (3.1) holds for Q as well as for its transpose Q^*. That means we assume both,

$$\gamma|Q^*| \leq M\gamma \quad \text{and} \quad \gamma|Q| \leq M\gamma, \tag{3.22}$$

for some positive constant M. Then existence for Y follows by the results of Section 3.2.

For the dual process Y, we want to choose a slightly smaller initial configuration. We say that $y = (\, y_{1,k}\,,\, y_{2,k}\,)_{k\in S} \in \mathbb{E}_\gamma$ has finite support if $(\, y_{1,k}\,,\, y_{2,k}\,) \neq (0,0)$ only for finitely many $k \in S$; and we write $y \in \mathbb{E}_{\text{fin}}$. We say that $y = (\, y_{1,k}\,,\, y_{2,k}\,)_{k\in S} \in \mathbb{E}_\gamma$ is γ-bounded if there is a constant $C = C_y > 0$ such that

$$y_{1,k} + y_{2,k} \leq C\,\gamma_k \tag{3.23}$$

for all $k \in S$. We write $y \in \mathbb{E}_b$ for short. Obviously, $\mathbb{E}_{\text{fin}} \subseteq \mathbb{E}_b \subseteq \mathbb{E}_\gamma$ since we assume γ to be summable. Analogously, we use the notations $\mathbb{L}_{\text{fin}}$ and $\mathbb{L}_b$ for the corresponding subsets of $\mathbb{L}_\gamma$. The subsets $\mathbb{L}_b$ and $\mathbb{E}_b$ will serve as some sort of pseudo-dual space of $\mathbb{L}_\gamma$ and $\mathbb{E}_\gamma$, respectively. With the notation $\|y\|_b := \sup_{\alpha,k} \frac{y_{\alpha,k}}{\gamma_k}$ for $y \in \mathbb{L}_b$ (or $\in \mathbb{E}_b$) we can write

$$\langle\, y_\alpha\,,\, P_t x_\alpha\,\rangle \leq C\langle\, \gamma\,,\, P_t x_\alpha\,\rangle \leq C\,e^{tM}\langle\, \gamma\,,\, x_\alpha\,\rangle,$$

hence, $\langle\, y_\alpha\,,\, P_t x_\alpha\,\rangle \leq \|y\|_b\|P_t\|\|x\|_\gamma$. Conversely, property (3.23) is preserved for the process Y taking the expectation into account,

$$E[Y_{\alpha,k,t}] = \sum_{j\in S} p^*_t(j,k)y_{\alpha,j} \leq C\,e^{tM}\,\gamma_k\,, \tag{3.24}$$

if $Y_{\cdot,\cdot,0} = y \in \mathbb{L}_b$ and where $P^*_t = (p^*_t(j,k))_{j,k\in S}$ denotes the semigroup for Y; compare with the discussion after Equation (3.3). Hence, $\langle\, P^*_t y_\alpha\,,\, x_\alpha\,\rangle \leq \|y\|_b\|P^*_t\|\|x\|_\gamma$.

For $y \in \mathbb{E}_b$ we can define Mytnik's duality functions $F(\cdot,y) : \mathbb{E}_\gamma \longrightarrow \mathbb{C}$ by

$$F(x,y) = \exp\Big\{-\langle\, x_1 + x_2\,,\, y_1 + y_2\,\rangle + i\langle\, x_1 - x_2\,,\, y_1 - y_2\,\rangle\Big\}, \tag{3.25}$$

and property (3.23) ensures that this expression is always well defined. Note that these functions have the following 'site-wise' multiplicative decomposition

$$\begin{aligned} F(x,y) \;&=\; \prod_{k\in S} F_k(x_{\cdot,k}\,,\,y_{\cdot,k}) \\ :=\; &\prod_{k\in S} \exp\bigl\{-(x_{1,k}+x_{2,k})(y_{1,k}+y_{2,k}) \;+\; i(x_{1,k}-x_{2,k})(y_{1,k}-y_{2,k})\bigr\}. \end{aligned} \tag{3.26}$$

Now we can state the important martingale property for the approximation processes X^ε w.r.t. the duality functions $F(\cdot,y)$.

3.9 Proposition *Let $y\in\mathbb{L}_b$ and $F(\cdot,y)$ as in (3.25). Define the operator $F(\cdot,y)\mapsto \mathscr{A}F(\cdot,y)$ by*

$$\begin{aligned} &\mathscr{A}F(\cdot,y)(x) \;=\; \mathscr{A}F(x,y) \;=\; \\ &\quad\Bigl[-\langle x_{1,\cdot}+x_{2,\cdot}\,,\,Q^*(y_{1,\cdot}+y_{2,\cdot})\rangle \;+\; i\langle x_{1,\cdot}-x_{2,\cdot}\,,\,Q^*(y_{1,\cdot}-y_{2,\cdot})\rangle\Bigr]F(x,y), \end{aligned} \tag{3.27}$$

for $x\in\mathbb{E}_\gamma$. Then the process X^ε has the following property:

For any $\bigl(F(\cdot,y)\,,\,\mathscr{A}F(\cdot,y)\bigr)$

$$t \;\mapsto\; F(X^\varepsilon_t,y) \;-\; F(X^\varepsilon_0,y) \;-\; \int_0^t \mathscr{A}F(X^\varepsilon_s,y)\,ds \tag{3.28}$$

is a martingale w.r.t. $(\mathscr{F}_t)_{t\ge 0}$ starting from 0 at time $t=0$.

Remark: The stated martingale property is exactly the same as in the martingale problem for the process X, the weak limit of the processes X^{ε_n} (or $\tilde{X}^{\varepsilon_n}$), see Proposition 3.16 below. However, we assume neither the class of functions $F(\cdot,y)$ being rich enough to characterize the finite-dimensional distributions of processes with paths in $D_{\mathbb{E}_\gamma}[0,T]$ nor existence of a unique process with martingale property (3.28). In fact, each element of the family of processes $\bigl\{X^\varepsilon \,:\, 0\le\varepsilon\le\infty\bigr\}$ satisfies the martingale property (3.28), including the deterministic heat flow, where we have variance parameter zero in Section 1.1. Compare with [EK86] Sections IV.4.3 and IV.4.4 for the notion martingale problem.

Proof. Note that (3.2) can be rewritten as

$$(\mathscr{G}f)(x) \;=\; \sum_{\alpha=1}^{2}\sum_{k\in S}\sum_{j\in S} x_{\alpha,j}\,q_{jk}\,\frac{\partial}{\partial x_{\alpha,k}} f(x) \tag{3.29}$$

for appropriate functions f on $\mathbb{E}_\gamma$. $\mathscr{G}$ applied to $F(\cdot,y)$ gives (3.27). Since $y\in\mathbb{L}_b$ and $|F(x,y)|\le 1$ we have by (3.22)

$$\begin{aligned} \sup_{0\le t\le T} E\Bigl[\,\bigl|\mathscr{A}F(X^\varepsilon_t,y)\bigr|\,\Bigr] \;&\le\; 2C\,M \sup_{0\le t\le T} E\Bigl[\,\langle\, X^\varepsilon_{1,\cdot,t}+X^\varepsilon_{2,\cdot,t}\,,\,\gamma\,\rangle\,\Bigr] \\ &\le\; 2C\,M\,e^{MT}\,\langle\, x_{1,\cdot}+x_{2,\cdot}\,,\,\gamma\,\rangle, \end{aligned} \tag{3.30}$$

where $X^\varepsilon_{\alpha,k,0} \;=\; x_{\alpha,k}$, for $\alpha\in\{1,2\}$, $k\in S$, denotes the initial conditon of X^ε. This shows that the process given by (3.28) is integrable.

For $t\in[0,\varepsilon)$ we have

$$F(X_t,y) \;=\; F(X_0,y) \;+\; \int_0^t \mathscr{A}F(X^\varepsilon_s,y)\,ds$$

by (3.2) and (3.29). And a similar equation holds for all subsequent time intervals $[\,j\varepsilon\,,\,(j+1)\varepsilon)$, $j = 1,\dots,\lfloor\frac{T}{\varepsilon}\rfloor$ modulo the random initial condition.

Hence, to complete the proof of (3.28) we need to take a look at the jumps at times $j\varepsilon$, $j = 1,\dots,\lfloor\frac{T}{\varepsilon}\rfloor$ similarly as we did for (3.5).

Define

$$M_t^{\varepsilon,F} \;:=\; -\sum_{j=1}^{\lfloor\frac{T}{\varepsilon}\rfloor} \Delta F(X^{\varepsilon}_{\cdot,\cdot,j\varepsilon}\,,\,y)\;\;{}_{[j\varepsilon,\infty)}(t)\,. \tag{3.31}$$

Note that $(M_t^{\varepsilon,F})_t$ is piecewise constant and $|M_t^{\varepsilon,F}| \le 2\frac{T}{\varepsilon}$ for $0 \le t \le T$. Let S_N be finite subsets of S with $S_N \nearrow S$ and define $y^N \in \mathbb{L}_{\text{fin}}$ by setting $y^N_{\alpha,k} = y_{\alpha,k}$ for $k \in S_N$ and $y^N_{\alpha,k} = 0$ for $k \in S\setminus S_N$, $\alpha \in \{1,2\}$. Then inductively by the independence of jumps and Lemma 1.2 we have

$$E\big[\,F\big(\,X^{\varepsilon}_{\cdot,\cdot,j\varepsilon}\,,\,y^N\,\big)\,\big|\,X^{\varepsilon}_{\cdot,\cdot,j\varepsilon-}\,\big] \;=\; F\big(\,X^{\varepsilon}_{\cdot,\cdot,j\varepsilon-}\,,\,y^N\,\big)$$

hence, by dominated convergence

$$E\big[\,F\big(\,X^{\varepsilon}_{\cdot,\cdot,j\varepsilon}\,,\,y\,\big)\,\big|\,X^{\varepsilon}_{\cdot,\cdot,j\varepsilon-}\,\big] \;=\; F\big(\,X^{\varepsilon}_{\cdot,\cdot,j\varepsilon-}\,,\,y\,\big)$$

for all times $j = 1,\dots,\lfloor\frac{T}{\varepsilon}\rfloor$. This and integrability yields that $M^{\varepsilon,F}$ is a martingale and we are done. □

The functions $F(\cdot,y)$ given by (3.25), where $y \in \mathbb{L}_b$, might not be rich enough to characterise distributions on $\mathbb{E}_\gamma$. But recall Proposition 3.7. The limit of the processes X^ε has only values in the smaller space $\mathbb{L}_\gamma$. The next Lemma will show that it is worth considering the above martingale problem. In fact, for measures on $\mathbb{L}_\gamma$ the family of duality functions $F(\cdot,y)$ indexed by $y \in \mathbb{L}_{\text{fin}}$ is rich enough to separate. For this reason we can characterize the one-dimensional distributions of X later on. The proof of the following result uses arguments as in [My96] Lemma 3.5 p.49 and [DP98] Lemma 2.3 p.1098.

3.10 Lemma *If μ and ν are probability laws on $\mathbb{L}_\gamma$ such that*

$$\int F(x,y)\mu(dx) \;=\; \int F(x,y)\nu(dx), \tag{3.32}$$

for all $F(\cdot,y)$ with $y \in \mathbb{L}_{\text{fin}}$, then $\mu = \nu$.

PROOF. Let us first consider the functions $F(\cdot,y)$ indexed by $y \in \mathbb{E}_{\text{fin}}$ instead of $\mathbb{L}_{\text{fin}}$. Let $\phi = (\phi_1,\phi_2)$ be a r.v. with values in $\mathbb{E}_\gamma$. Then note that the mixed Laplace-Fourier transform $E[F(\phi,y)]$ actually is supposed to characterize the distribution of $(\phi_1+\phi_2\,,\,\phi_1-\phi_2)$. But this is sufficient since there is a one-to-one correspondence between the distributions of (ϕ_1,ϕ_2) and $(\phi_1+\phi_2\,,\,\phi_1-\phi_2)$. Now set

$$A \;:=\; \big\{\,F(\cdot,y)\;:\;y \in \mathbb{E}_{\text{fin}}\,\big\}. \tag{3.33}$$

We need to show that the linear span of A is an algebra which separates points. Then by [EK86] Theorem 3.4.5(a), p.113, and linearity A separates distributions on $\mathbb{E}_\gamma$. Obviously, the linear span of A is stable under addition and by the structure of F stable under multiplication. Thus, it remains to show that it separates points. To this end let $x, x' \in \mathbb{E}_\gamma$ with $x \neq x'$. Then there is $k \in S$ such that $(x_{1,k}\,,\, x_{2,k}) \neq (x'_{1,k}\,,\, x'_{2,k})$. This implies $x_{1,k} + x_{2,k} \neq x'_{1,k} + x'_{2,k}$ or $x_{1,k} - x_{2,k} \neq x'_{1,k} - x'_{2,k}$. In the first case choose $\bar{y} \in \mathbb{E}_\gamma$ such that $(\bar{y}_{1,k}\,,\, \bar{y}_{2,k}) = (1,1)$ and $(\bar{y}_{1,j}\,,\, \bar{y}_{2,j}) = (0,0)$ for $j \neq k$. Then $F(x,\bar{y}) \neq F(x',\bar{y})$. In the second case choose $\bar{y}_{1,k} = \frac{1}{2\pi}(|x_{1,k} - x_{2,k}| + |x'_{1,k} - x'_{2,k}|)^{-1}$ and $\bar{y}_{2,k} = 0$ and $(\bar{y}_{1,j}\,,\, \bar{y}_{2,j}) = (0,0)$ for $j \neq k$. Then $\exp\{i(x_{1,k} - x_{2,k})(\bar{y}_{1,k} - \bar{y}_{2,k})\} \neq \exp\{i(x'_{1,k} - x'_{2,k})(\bar{y}_{1,k} - \bar{y}_{2,k})\}$ and again we obtain $F(x,\bar{y}) \neq F(x',\bar{y})$.

Finally, let μ and ν be distributions on $\mathbb{L}_\gamma$ which satisfy (3.32). Fix $F(\cdot, y)$ with $y \in \mathbb{E}_{\text{fin}}$. Choose a finite subset $S_N \subseteq S$ such that $(y_{1,j}\,,\, y_{2,j}) = (0,0)$ for all $j \in S \setminus S_N$. For coordinates $k \in S_N$ we use the argument as in the proof of Lemma 1.4(b). In fact, by the product structure of F and since $|F| \leq 1$ we can write

$$\begin{aligned}
\int F(x,y)\,\mu(dx) &= \int \prod_{k\in S_N} F_k(x_{\cdot,k}\,,\, y_{\cdot,k})\,\mu(dx) \\
&= \int \prod_{k\in S_N} \left[\int_L DP_{y_{\cdot,k}}(d\xi^{(k)}) F_k(x_{\cdot,k}\,,\, \xi^{(k)})\right] \mu(dx) \\
&= \prod_{k\in S_N} \int_L DP_{y_{\cdot,k}}(d\xi^{(k)}) \left[\int F_k(x_{\cdot,k}\,,\, \xi^{(k)})\,\mu(dx)\right] \\
&= \prod_{k\in S_N} \int_L DP_{y_{\cdot,k}}(d\xi^{(k)}) \left[\int F_k(x_{\cdot,k}\,,\, \xi^{(k)})\,\nu(dx)\right] \\
&= \int \prod_{k\in S_N} \left[\int_L DP_{y_{\cdot,k}}(d\xi^{(k)}) F_k(x_{\cdot,k}\,,\, \xi^{(k)})\right] \nu(dx) \\
&= \int \prod_{k\in S_N} F_k(x_{\cdot,k}\,,\, y_{\cdot,k})\,\nu(dx) \\
&= \int F(x,y)\,\nu(dx).
\end{aligned}$$

Hence, we showed that (3.32) for measures which charge only $\mathbb{L}_\gamma$ implies $\int F(x,y)\,\mu(dx) = \int F(x,y)\,\nu(dx)$ for all $y \in \mathbb{E}_{\text{fin}}$. Thus, by the preceding argumentation we obtain $\mu = \nu$. □

Next, we state three technical Lemmas. The first one (including its proof) is taken from the preprint of Klenke and Mytnik, [KM07]. We will apply it later for the function $h(x) = x^{p-1}$, where $1 < p < 2$, and the two random variables $X^\varepsilon_{1,k,t}$ and $X^\varepsilon_{2,k,t}$.

3.11 Lemma *Let U and V be non-positively correlated, non-negative, integrable random variables and assume that $h : [0,\infty) \longrightarrow [0,\infty)$ is concave, differentiable and monotone increasing. Then*

$$E\big[Uh(V)\big] \;\leq\; E[U]h(E[V]).$$

PROOF. Since h is concave, we have $\frac{h(z)-h(x)}{z-x} \geq \frac{h(y)-h(z)}{y-z}$ for $x < z < y$, compare with [Kl06] Satz 7.7. p.142. Letting $y \to z$ we obtain $h(z) - h(x) \geq (z-x)h'(z)$ and, hence,

$$h(x) \;\leq\; h(z) + (x-z)h'(z).$$

In case $x > z$ we obtain the same inequality since for $z > y$ we have $h'(z) = \lim_{y\to z} \frac{h(z)-h(y)}{z-y} \geq \frac{h(x)-h(z)}{x-z}$. With $z := E[V]$ we get

$$h(x) \;\leq\; h(E[V]) + (x - E[V])\,h'(E[V])\,,$$

for all $x \in [0,\infty)$. Then $U\,h(V) \le U\Big\{ h(E[V]) + (V - E[V])\,h'(E[V]) \Big\}$, and so

$$\begin{aligned} E\big[\,U\,h(V)\,\big] &\le E\Big[U\,h(E[V])\Big] + E\Big[U\,(V-E[V])\,h'(E[V])\Big] \\ &= E[U]\,h(E[V]) + h'(E[V])\Big\{ E[U\,V\,] - E[U]\,E[V] \Big\} \\ &\le E[U]\,h(E[V]) \end{aligned}$$

since $h' \ge 0$ and U and V are non-positively correlated. □

3.12 Lemma *Let $t \ge 0$. For sites $j,k \in S$ we have*

$$E\Big[\,X^\varepsilon_{1,j,t}\,X^\varepsilon_{2,k,t}\,\Big] \le E\big[\,X^\varepsilon_{1,j,t}\,\big]\,E\big[\,X^\varepsilon_{2,k,t}\,\big], \tag{3.34}$$

i.e., $X^\varepsilon_{1,j,t}$ and $X^\varepsilon_{2,k,t}$ are non-positively correlated random variables.

Proof. We assume X^ε starts in $x \in \mathbb{L}_\gamma$. This immediately gives $E\big[\,X^\varepsilon_{1,j,t}\,X^\varepsilon_{2,k,t}\,\big] = E\big[\,X^\varepsilon_{1,j,t}\,\big]\,E\big[\,X^\varepsilon_{2,k,t}\,\big]$ for $t=0$. We proceed inductively. Assume (3.34) holds for all jump times $t \in \{0,\varepsilon,2\varepsilon,\dots,(N-1)\varepsilon\}$ for some $N \in \mathbb{N}$. Then for $t \in \big(K\varepsilon\,,\,(K+1)\varepsilon\big)$ with $K \in \{0,1,2,\dots,N\}$ we can write

$$\begin{aligned} E\Big[\,X^\varepsilon_{1,j,t}\,X^\varepsilon_{2,k,t}\,\Big] &= \sum_{l\in S}\sum_{l'\in S} p_{t-K\varepsilon}(j,l)\,p_{t-K\varepsilon}(k,l')\,E\big[\,X^\varepsilon_{1,l,K\varepsilon}\,X^\varepsilon_{2,l',K\varepsilon}\,\big] \\ &\le \sum_{l\in S}\sum_{l'\in S} p_{t-K\varepsilon}(j,l)\,p_{t-K\varepsilon}(k,l')\,E\big[\,X^\varepsilon_{1,l,K\varepsilon}\,\big]\,E\big[\,X^\varepsilon_{2,l',K\varepsilon}\,\big] \\ &= E\big[\,X^\varepsilon_{1,j,t}\,\big]\,E\big[\,X^\varepsilon_{2,k,t}\,\big]. \end{aligned}$$

At jump time $t = N\varepsilon$ we have $X^\varepsilon_{1,j,t}\,X^\varepsilon_{2,k,t} = 0$ if $j=k$, hence, obviously $E\big[\,X^\varepsilon_{1,k,t}\,X^\varepsilon_{2,k,t}\,\big] \le E\big[\,X^\varepsilon_{1,k,t}\,\big]\,E\big[\,X^\varepsilon_{2,k,t}\,\big]$. If $j \ne k$ recall that jumps have mean zero and are independent on different sites. Then we have

$$\begin{aligned} E\big[\,X^\varepsilon_{1,j,N\varepsilon}\,X^\varepsilon_{2,k,N\varepsilon}\,\big] &= E\Big[E\big[\,\big(X^\varepsilon_{1,j,N\varepsilon-} + \Delta X^\varepsilon_{1,j,N\varepsilon}\big)\big(X^\varepsilon_{2,k,N\varepsilon-} + \Delta X^\varepsilon_{2,k,N\varepsilon}\big)\,\big|\,X_{\cdot,\cdot,N\varepsilon-}\,\big]\Big] \\ &= E\big[\,X^\varepsilon_{1,j,N\varepsilon-}\,X^\varepsilon_{2,k,N\varepsilon-}\,\big] + E\Big[\,X^\varepsilon_{1,j,N\varepsilon-}\,E\big[\Delta X^\varepsilon_{2,k,N\varepsilon}\,\big|\,X_{\cdot,\cdot,N\varepsilon-}\,\big]\Big] \\ &\quad + E\Big[\,X^\varepsilon_{2,k,N\varepsilon-}\,E\big[\Delta X^\varepsilon_{1,j,N\varepsilon}\,\big|\,X_{\cdot,\cdot,N\varepsilon-}\,\big]\Big] + E\big[\,\Delta X^\varepsilon_{1,j,N\varepsilon}\,\Delta X^\varepsilon_{2,k,N\varepsilon}\,\big] \\ &\le E\big[\,X^\varepsilon_{1,j,N\varepsilon-}\,\big]\,E\big[\,X^\varepsilon_{2,k,N\varepsilon-}\,\big] \\ &= E\big[\,X^\varepsilon_{1,j,N\varepsilon}\,\big]\,E\big[\,X^\varepsilon_{2,k,N\varepsilon}\,\big] \end{aligned}$$

by induction hypothesis and since the last three (of four) terms in the antepenultimate line vanish. This completes the proof. □

The following results are supposed to be very well known. We state them for easy reference.

3.13 Lemma *Let I be a countable index set. Let $a = (a_k)_{k\in I} \in [0,\infty)^I$.*

(a) Then, for $1 \le p \le 2$,

$$\Big(\sum_k a_k^2\Big)^{p/2} \le \sum_k a_k^p\,.$$

(b) *For a finite subset $J \subseteq I$ let $p > 1$. Then*

$$\Big(\sum_{k\in J} a_k\Big)^p \;\le\; |J|^{p-1} \sum_{k\in J} a_k^p\,,$$

where $|J|$ denotes the cardinality of J.

Proof. (a) Denote by $\|a\|_p := \left(\sum_k a_k^p\right)^{1/p}$ the ℓ^p-norm and let $e^{(k)} := (\delta_{jk})_{j\in I}$ be the unit vector with 1 in the k-th coordinate and 0 otherwise. Define the vector $b = (b_k)_{k\in I}$ by setting $b_k := a_k^p$, for all $k \in I$, and determine the sequences $c^{(k)}$ by $c^{(k)} := a_k^p \cdot e^{(k)}$. Then $\sum_k c^{(k)} = b$. Moreover, we can write

$$\Big(\sum_k a_k^2\Big)^{p/2} \;=\; \Big(\sum_k (a_k^p)^{2/p}\Big)^{p/2} \;=\; \|b\|_{2/p} \;\le\; \sum_k \|a_k^p\, e^{(k)}\|_{2/p} \;=\; \sum_k (a_k^p)^{\frac{2}{p}\frac{p}{2}} \;=\; \sum_k a_k^p$$

by the Minkowski inequality.
(b) By Jensen's inequality we have

$$\sum_{k\in J} a_k^p \;=\; |J| \sum_{k\in J} \frac{1}{|J|} a_k^p \;\ge\; |J| \Big(\sum_{k\in J} \frac{1}{|J|} a_k\Big)^p \;=\; |J|^{1-p} \Big(\sum_{k\in J} a_k\Big)^p,$$

and we are done. □

The integral version of Lemma 3.13(b) reads as follows: Let μ be a finite measure on some measurable space $(E, \mathcal{E})$ and f a non-negative $\mathcal{E}$-measurable function. Then, for $p > 1$,

$$\Big(\int f(x)\,\mu(dx)\Big)^p \;\le\; \mu(E)^{p-1} \int f(x)^p \mu(dx)\,. \tag{3.35}$$

We wish to show a similar result as in Proposition 3.9 for the weak limit of the processes X^ε. The martingale property in (3.28) will be preserved for the limit as ε tends to zero if we can establish uniformly bounded moments. This is our next aim.

3.14 Lemma *Fix $y \in \mathbb{L}_b$, $F(\cdot, y)$, $T > 0$ and $1 < p < 2$. Let $M^{\varepsilon,F}$ as in (3.31) above. Then*

$$\sup_{0<\varepsilon\le 1} E\Big[\sup_{0\le t\le T} \Big| M_t^{\varepsilon,F}\Big|^p\Big] \;<\; \infty,$$

i.e., $M^{\varepsilon,F}$ is bounded in $L^p(P)$.

Proof. Recall that $M^{\varepsilon,F}$ consists of only finitely many jumps on the time grid $t = j\varepsilon$, with $j = 0, 1, \dots, \lfloor \frac{T}{\varepsilon} \rfloor$, and so we rather consider it as a discrete-time martingale. Moreover, at each time $j\varepsilon$ we decompose the jump into $|S|$-many jumps, one jump at each site k. To this end we fix an enumeration of the sites $k \in S$. We use the following decomposition of the duality function F,

$$F(x, y) \;=\; \prod_{k\in S} F_k(x_{\cdot,k}\,,\, y_{\cdot,k})\,,$$

where $F_k(x_{\cdot,k}\,,\, y_{\cdot,k}) := \exp\big\{-(x_{1,k} + x_{2,k})(y_{1,k} + y_{2,k}) + i(x_{1,k} - x_{2,k})(y_{1,k} - y_{2,k})\big\}$ only depends on the k-th coordinate of x (and y). Then

$$\mathcal{Z}_n^{j\varepsilon} \;:=\; F(X^\varepsilon_{\cdot,\cdot,j\varepsilon-}\,,\, y) \prod_{k=1}^{n} F_k(\Delta X^\varepsilon_{\cdot,k,j\varepsilon}\,,\, y)\,, \quad \text{for } n \in \mathbb{N}_0 \cup \{\infty\},$$

is a martingal since the jumps at time $j\varepsilon$ are centered and independent on each site $k \in S$. Set $\mathcal{X}_n^{j\varepsilon} := \mathcal{Z}_n^{j\varepsilon} - \mathcal{Z}_{n-1}^{j\varepsilon}$, for $n \in \mathbb{N} \cup \{\infty\}$, and note that

$$\sum_{n=1}^{\infty} \mathcal{X}_n^{j\varepsilon} = \mathcal{Z}_\infty^{j\varepsilon} - \mathcal{Z}_0^{j\varepsilon} = \Delta F(X^\varepsilon_{\cdot,\cdot,j\varepsilon}\,,\, y).$$

For the square bracket of the discrete-time, mean zero martingale $N \mapsto \sum_{n=1}^{N} \mathcal{X}_n^{j\varepsilon}$ we have

$$\begin{aligned}
\Big[\sum_{n=1}^{\cdot} \mathcal{X}_n^{j\varepsilon}\Big]_N &= \sum_{n=1}^{N} (\mathcal{X}_n^{j\varepsilon})^2 = \sum_{n=1}^{N} (\mathcal{Z}_n^{j\varepsilon} - \mathcal{Z}_{n-1}^{j\varepsilon})^2 \\
&\le \sum_{n=1}^{N} \big| F_n(\Delta X^\varepsilon_{\cdot,n,j\varepsilon}\,,\, y) - F_n(0\,,\, y)\big|^2 \\
&\le \sum_{n=1}^{N} 2^2 (y_{1,n} + y_{2,n})^2 \left(|\Delta X^\varepsilon_{1,n,j\varepsilon}| + |\Delta X^\varepsilon_{2,n,j\varepsilon}|\right)^2 \qquad (3.36)
\end{aligned}$$

by the Lipschitz continuity of $F_n(\cdot, y)$. Finally, instead of $M^{\varepsilon,F}$ we consider the martingale $\sum_{j=1}^{\cdot} \sum_{n=1}^{\cdot} \mathcal{X}_n^{j\varepsilon}$ with directed (time-)index set $I = \big\{ (j,n) : \; j = 1, \ldots, \lfloor \frac{T}{\varepsilon} \rfloor, \, n \in \mathbb{N} \cup \{\infty\} \big\}$ with the obvious ordering $(j_1, n_1) < (j_2, n_2)$ if and only if $j_2 > j_1$ or if $j_1 = j_2$ and $n_2 > n_1$. For this martingale we apply the Burkholder-Davis-Gundy inequality and (3.36).

$$\begin{aligned}
&E\left[\sup_{0 \le t \le T} \left| \sum_{j=1}^{\lfloor \frac{T}{\varepsilon} \rfloor} \Delta F\big(X^\varepsilon_{\cdot,\cdot,j\varepsilon}\,,\, y\big) \;_{[j\varepsilon,\infty)}(t) \right|^p \right] \\
&\qquad \le E\left[\sup_{J \le \lfloor \frac{T}{\varepsilon} \rfloor,\, N \in \mathbb{N} \cup \{\infty\}} \left| \sum_{j=1}^{J} \sum_{n=1}^{N} \mathcal{X}_n^{j\varepsilon} \right|^p \right] \\
&\qquad \le E\left[\left(\sum_{j=1}^{\lfloor \frac{T}{\varepsilon} \rfloor} \sum_{k \in S} 2^2 (y_{1,k} + y_{2,k})^2 \left(|\Delta X^\varepsilon_{1,k,j\varepsilon}| + |\Delta X^\varepsilon_{2,k,j\varepsilon}|\right)^2 \right)^{p/2} \right].
\end{aligned}$$

We continue by using Lemma 3.13 (a) and (b). Hence,

$$E\left[\sup_{0 \le t \le T} \left| M_t^{\varepsilon,F} \right|^p \right] \le \sum_{j=1}^{\lfloor \frac{T}{\varepsilon} \rfloor} \sum_{k \in S} 2^{2p-1} (y_{1,k} + y_{2,k})^p \, E\Big[|\Delta X^\varepsilon_{1,k,j\varepsilon}|^p + |\Delta X^\varepsilon_{2,k,j\varepsilon}|^p \Big]. \qquad (3.37)$$

Here is, where Lemma 1.7 comes into play. For the jump of type 1 at site k we have

$$\begin{aligned}
E\Big[|\Delta X^\varepsilon_{1,k,j\varepsilon}|^p\Big] &\le 2\, C_p \, E\Big[(X^\varepsilon_{1,k,j\varepsilon-})^{p-1} X^\varepsilon_{2,k,j\varepsilon-} \;_{\{X^\varepsilon_{2,k,(j-1)\varepsilon}=0\}} \Big] \\
&\quad + \; 2\, C_p \, E\Big[(X^\varepsilon_{2,k,j\varepsilon-})^{p-1} X^\varepsilon_{1,k,j\varepsilon-} \;_{\{X^\varepsilon_{1,k,(j-1)\varepsilon}=0\}} \Big].
\end{aligned} \qquad (3.38)$$

Then by Lemma 3.12 and Lemma 3.11, with $h(x) = x^{p-1}$ for $x \ge 0$, we obtain for the first summand of the r.h.s. of (3.38)

$$\begin{aligned}
&E\Big[(X^\varepsilon_{1,k,j\varepsilon-})^{p-1} X^\varepsilon_{2,k,j\varepsilon-} \;_{\{X^\varepsilon_{2,k,(j-1)\varepsilon}=0\}} \Big] \\
&\qquad \le E\big[X^\varepsilon_{1,k,j\varepsilon-}\big]^{p-1} E\Big[X^\varepsilon_{2,k,j\varepsilon-} \;_{\{X^\varepsilon_{2,k,(j-1)\varepsilon}=0\}} \Big] \\
&\qquad = E\big[X^\varepsilon_{1,k,j\varepsilon-}\big]^{p-1} \sum_{l \ne k} p_\varepsilon(k,l) \, E\big[X^\varepsilon_{2,l,(j-1)\varepsilon}\big], \qquad (3.39)
\end{aligned}$$

recall $\big(p_t(k,l)\big)_{k,l\in S} = P_t = \exp(tQ^*)$. Just as well, for the second summand of the r.h.s. of Equation (3.38) we obtain

$$E\Big[(X^\varepsilon_{2,k,j\varepsilon-})^{p-1}\, X^\varepsilon_{1,k,j\varepsilon-} \;\; {}_{\{X^\varepsilon_{1,k,(j-1)\varepsilon}=0\}}\Big] \le E\big[X^\varepsilon_{2,k,j\varepsilon-}\big]^{p-1} \sum_{l\neq k} p_\varepsilon(k,l)\, E\big[X^\varepsilon_{1,l,(j-1)\varepsilon}\big]. \quad (3.40)$$

If we consider the jump of type 2 at site k, that is to say the expression $E\Big[|\Delta X^\varepsilon_{2,k,j\varepsilon}|^p\Big]$ on the r.h.s. of (3.37), obviously, similar estimates can be obtained along the lines of (3.38), (3.39) and (3.40).

Going back to Equation (3.37) we summarise. Since $\langle\, E[X^\varepsilon_{\alpha,\cdot,t}]\,,\,\gamma\,\rangle \;\le\; e^{MT}\langle x_\alpha,\gamma\rangle$ for $0\le t\le T$ we have $E[X^\varepsilon_{\alpha,k,t}]^{p-1} \;\le\; \frac{C'}{\gamma_k^{p-1}}$ for $\alpha \in \{1,2\}$. Then we use (3.23) and can find a constant $c>0$, which subsumes previous constants, such that

$$\begin{aligned}\sum_{j=1}^{\lfloor\frac{T}{\varepsilon}\rfloor}\sum_{k\in S} & 2^{2p-1}(y_{1,k}+y_{2,k})^p\, E\big[|\Delta X^\varepsilon_{1,k,j\varepsilon}|^p\big] \\ &\le\; c\sum_{j=1}^{\lfloor\frac{T}{\varepsilon}\rfloor}\sum_{k\in S}\sum_{l\in S}\gamma_k\, p_\varepsilon(k,l)(1-\delta_{kl})\Big(\, E\big[X^\varepsilon_{1,l,(j-1)\varepsilon}\big] \;+\; E\big[X^\varepsilon_{2,l,(j-1)\varepsilon}\big]\,\Big)\end{aligned} \quad (3.41)$$

by (3.39) and (3.40). To give an estimate for the r.h.s. of (3.41) we consider $z=(z_1,z_2)\in\mathbb{E}_\gamma$ to simplify notation. Then

$$\Big|\sum_{k\in S}\sum_{l\in S}\gamma_k\, p_\varepsilon(k,l)(1-\delta_{kl})z_{\alpha,l}\;\Big| \;\le\; |\langle P_\varepsilon z_\alpha,\gamma\rangle - \langle P_0 z_\alpha,\gamma\rangle| \;+\; \sum_{k\in S}\big|1-p_\varepsilon(k,k)\big|z_{\alpha,k}\,\gamma_k$$

since P_0 equals the identity matrix I. For the first summand we have $|\langle P_\varepsilon z_\alpha,\gamma\rangle - \langle P_0 z_\alpha,\gamma\rangle| = \varepsilon\,|\langle Q^* P_\epsilon z_\alpha,\gamma\rangle| \;\le\; \varepsilon\, M\, e^{\varepsilon M}\langle z_\alpha,\gamma\rangle$, where $0\le\epsilon\le\varepsilon$ (≤ 1). For the second summand note that

$$\begin{aligned}\sup_{k\in S}\big|p_\varepsilon(k,k)-1\big| \;&=\; \sup_{k\in S}\big\langle\, \delta_{k\cdot}\,,\, |e^{\varepsilon Q^*}-I\,|\,\delta_{k\cdot}\big\rangle \\ &\le\; \sup_{k\in S}\|\delta_{k\cdot}\|_b\cdot\|e^{\varepsilon Q^*}-I\|\cdot\|\delta_{k\cdot}\|_\gamma \\ &\le\; \varepsilon\, M\, e^{\varepsilon M} \;\le\; \varepsilon\, M\, e^{M}\end{aligned}$$

since we interpret $\delta_{k\cdot}$ as an element of $\mathbb{E}_\gamma$, hence, $\|\delta_{k\cdot}\|_\gamma=\gamma_k$, as well as an element of the 'dual space' $(\mathbb{E}_\gamma)'=\mathbb{E}_b$ with elements $y=(y_1,y_2)$ such that $\sup_{j\in S} y_\alpha\gamma_j^{-1}$ is finite, hence, $\|\delta_{k\cdot}\|_b=\gamma_k^{-1}$. Clearly, for the operator norm $\|e^{\varepsilon Q^*}-I\| \;\le\; \varepsilon\, M\, e^{\varepsilon M}$. Therefore

$$\sum_{k\in S}\big|1-p_\varepsilon(k,k)\big|z_{\alpha,k}\,\gamma_k \;\le\; \varepsilon\, M\, e^{M}\langle z_\alpha,\gamma\rangle.$$

Thus,

$$\Big|\sum_{k\in S}\sum_{l\in S}\gamma_k\, p_\varepsilon(k,l)(1-\delta_{kl})z_{\alpha,l}\;\Big| \;\le\; \varepsilon\, 2M\, e^{M}\langle z_\alpha,\gamma\rangle.$$

We apply this to (3.41) and resubstitute $E\big[X^\varepsilon_{\alpha,\cdot,(j-1)\varepsilon}\big]$ for z_α. Also recall $\langle E[X^\varepsilon_{\alpha,\cdot,(j-1)\varepsilon}]\,,\gamma\rangle \;\le\; e^{(j-1)\varepsilon\, M}\langle x_\alpha,\gamma\rangle \;\le\; e^{T\,M}\langle x_\alpha,\gamma\rangle$, for all $j=1,\ldots,\lfloor\frac{T}{\varepsilon}\rfloor$, where $x=(x_1,x_2)$ is the inital configuration of X^ε. Then

$$\begin{aligned}\sum_{j=1}^{\lfloor\frac{T}{\varepsilon}\rfloor}\sum_{k\in S} 2^{2p-1}(y_{1,k}+y_{2,k})^p\, E\big[|\Delta X^\varepsilon_{1,k,j\varepsilon}|^p\big] \;&\le\; c\sum_{j=1}^{\lfloor\frac{T}{\varepsilon}\rfloor}\varepsilon\, 2M\, e^{M}\, e^{TM}\langle x_1+x_2\,,\,\gamma\rangle \\ &\le\; 2\,cTMe^{(T+1)M}\langle x_1+x_2\,,\,\gamma\rangle \;<\; \infty.\end{aligned} \quad (3.42)$$

Surely, the same estimate can be computed for the jump of type 2. This gives an upper bound for the r.h.s. of (3.37) which is finite and does not depend on $\varepsilon \in (0,1]$. This completes the proof. □

3.15 Lemma *For* $1 < p < 2$ *we have*

$$\sup_{0<\varepsilon\le 1} E\Big[\langle X^\varepsilon_{1,\cdot,T} + X^\varepsilon_{2,\cdot,T}\,,\,\gamma\rangle^p\Big] \;<\; \infty.$$

Proof. According to Lemma 3.2 we can write

$$\begin{aligned}\langle X^\varepsilon_{\alpha,\cdot,T}\,,\,\gamma\rangle \;&=\; \langle x_\alpha,\gamma\rangle \;+\; \int_0^T |\langle (X^\varepsilon_{\alpha,\cdot,s}Q)_\cdot\,,\,\gamma\rangle|\,ds \;+\; \langle M^\varepsilon_{\alpha,\cdot,T}\,,\,\gamma\rangle \\ &\le\; \langle x_\alpha,\gamma\rangle \;+\; M\int_0^T \langle \gamma\,,\,X^\varepsilon_{\alpha,\cdot,s}\rangle\,ds \;+\; \big|\langle M^\varepsilon_{\alpha,\cdot,T}\,,\,\gamma\rangle\big|\,.\end{aligned}$$

By Lemma 3.13(b) we then have

$$\begin{aligned}E\Big[\langle X^\varepsilon_{1,\cdot,T} + X^\varepsilon_{2,\cdot,T}\,,\,\gamma\rangle^p\Big] \;\le\; C\bigg\{ &\sum_{\alpha=1}^2 \langle x_\alpha,\gamma\rangle^p \;+\; \sum_{\alpha=1}^2 E\Big[\big|\langle M^\varepsilon_{\alpha,\cdot,T}\,,\,\gamma\rangle\big|^p\Big] \\ &+\; E\bigg[\Big(M\int_0^T \langle X^\varepsilon_{1,\cdot,s} + X^\varepsilon_{2,\cdot,s}\,,\,\gamma\rangle\,ds\Big)^p\bigg]\bigg\}\end{aligned} \tag{3.43}$$

for some constant $C > 0$ which only depends on p. To give an upper bound on the r.h.s. of (3.43) compare $E\Big[\big|\langle M^\varepsilon_{\alpha,\cdot,T}\,,\,\gamma\rangle\big|^p\Big]$ with the results derived in the proof of Lemma 3.14. In fact, here we can directly consider the jumps $\gamma_k|\Delta X^\varepsilon_{\alpha,k,j\varepsilon}|$, in contrast to (3.36), and we obtain

$$E\Big[\big|\langle M^\varepsilon_{\alpha,\cdot,T}\,,\,\gamma\rangle\big|^p\Big] \;\le\; C' \sum_{k\in S}\sum_{j=1}^{\lfloor \frac{T}{\varepsilon}\rfloor} \gamma^p E\big[|\Delta X^\varepsilon_{\alpha,k,j\varepsilon}|^p\big],$$

which is again finite, uniformly in $\varepsilon \in (0,1]$. Compare with (3.42).

By Jensen's inequality for Lebesgue measure on $[0,T]$ and $x \mapsto x^p$, see Equation (3.35) after Lemma 3.13, we have

$$E\bigg[\Big(M\int_0^T \langle X^\varepsilon_{1,\cdot,s} + X^\varepsilon_{2,\cdot,s}\,,\,\gamma\rangle\,ds\Big)^p\bigg] \;\le\; M^pT^{p-1}\int_0^T E\Big[\langle X^\varepsilon_{1,\cdot,s} + X^\varepsilon_{2,\cdot,s}\,,\,\gamma\rangle^p\Big]\,ds\,.$$

Altogether, there are constants a and b which depend on the initial condition x, p and T but not on ε such that

$$E\Big[\langle X^\varepsilon_{1,\cdot,T} + X^\varepsilon_{2,\cdot,T}\,,\,\gamma\rangle^p\Big] \;\le\; a \;+\; b\int_0^T E\Big[\langle X^\varepsilon_{1,\cdot,s} + X^\varepsilon_{2,\cdot,s}\,,\,\gamma\rangle^p\Big]\,ds.$$

An application of Gronwall's Lemma, see for instance [RY91] p. 543, completes the proof. □

Now we are ready to apply the results of the former lemmas. Due to Remark 3.8 we consider the time-index set $[0,\infty)$.

3.16 Proposition *Let* X *– with paths in* $D_{\mathbb{L}_\gamma}[0,\infty)$ *– be any weak limit point of the family* $\big\{\, X^\varepsilon \,:\, 0 < \varepsilon \le 1 \,\big\}$. *Then* X *is a solution to the following martingale problem:*

Let $\mathscr{A}$ be given by (3.27). *For any* $F(\cdot, y)$, *where* $y \in \mathbb{L}_b$,

$$t \;\mapsto\; F(X_t, y) \;-\; F(X_0, y) \;-\; \int_0^t \mathscr{A} F(X_s, y)\, ds \tag{3.44}$$

is a martingale starting from 0 *at time* $t = 0$.

Proof. By Proposition 3.7 we can choose from $\left\{ X^\varepsilon \,:\, \varepsilon > 0 \right\}$ a subsequence $(X^{\varepsilon_n})_n$ converging in law on $D_{\mathbb{E}_\gamma}[0,T]$. Going to the Skorohod space, cf. Skorohod representation theorem [EK86] Theorem III.1.8, we may assume

$$X^{\varepsilon_n} \longrightarrow X \quad a.s. \text{ in } \; D_{\mathbb{E}_\gamma}[0,T]\,, \quad \text{as } \; n \to \infty.$$

By Proposition 3.9 we have

$$F(X_t^{\varepsilon_n}, y) \;-\; F(X_0^{\varepsilon_n}, y) \;-\; \int_0^t \mathscr{A} F(X_s^{\varepsilon_n}, y)\, ds \;=\; M_t^{\varepsilon_n, F},$$

with $M^{\varepsilon_n,F}$ as in (3.31) and $\mathscr{A}F(x,y)$ as in (3.27). Since $x \mapsto F(x,y)$ and $x \mapsto \mathscr{A}F(x,y)$ are continuous, a.s.-convergence of $F(X_t^{\varepsilon_n}, y)$, $F(X_0^{\varepsilon_n}, y)$ and $\mathscr{A}F(X_s^{\varepsilon_n}, y)$ to $F(X_t, y)$, $F(X_0, y)$ and $\mathscr{A}F(X_s, y)$, respectively, is immediate. Next, recall that $s \mapsto e^{s\bar\lambda}\langle\, X^\varepsilon_{1,\cdot,s} + X^\varepsilon_{2,\cdot,s}\,,\, \gamma\,\rangle$ is a submartingale by Lemma 3.2 c). Then we can use Doob's submartingale inequality. We have

$$\begin{aligned} E\Big[\Big(\sup_{0\le s\le T}\big|\mathscr{A}F(X_s^{\varepsilon_n}, y)\big|\Big)^p\Big] \;&\le\; 2^p E\Big[\Big(\sup_{0\le s\le T} e^{s\bar\lambda}\langle\, X^\varepsilon_{1,\cdot,s} + X^\varepsilon_{2,\cdot,s}\,,\, \gamma\,\rangle\Big)^p\Big] \\ &\le\; C_{p,T,\bar\lambda}\, E\Big[\langle\, X^\varepsilon_{1,\cdot,T} + X^\varepsilon_{2,\cdot,T}\,,\, \gamma\,\rangle^p\Big], \end{aligned}$$

for some constant $C_{p,T,\bar\lambda}$ that depends on $p \in (1,2)$, $T > 0$ and $\bar\lambda > 0$. Hence,

$$\sup_{0<\varepsilon\le 1} E\Big[\Big(\sup_{0\le s\le T}\big|\mathscr{A}F(X_s^{\varepsilon_n}, y)\big|\Big)^p\Big] \;<\; \infty$$

by Lemma 3.15. This yields

$$\lim_{n\to\infty}\int_0^t \mathscr{A}F(X_s^{\varepsilon_n}, y)\, ds \;=\; \int_0^t \mathscr{A}F(X_s, y)\, ds$$

P-a.s. Finally, we obtain convergence of the martingales $M^{\varepsilon_n,F}$ as $n \to \infty$. The limit is again a martingale, which is provided by Lemma 3.14 and results in [JS87] Chapter IX, §1a and §1b, see Proposition IX.1.12, p. 525. □

After we have established the martingale problem for X a similar version should hold for the dual process. We now phrase this martingale problem for Y and give an abridged proof to avoid excessive repetition. Furthermore, property (3.24) simplifies the proof and, chronologically speaking, we were able to establish the martingale problem for the dual process Y at first. However, note that the duality functions $F(\cdot, y)$ and $F(x, \cdot)$ are indexed by different sets, namely $y \in \mathbb{L}_b$ and $x \in \mathbb{L}_\gamma$, respectively.

Let Y^ε be the approximate processes for Y as described in Section 3.1.

3.17 Proposition *Let Y be any weak limit point of the family $\{ Y^\varepsilon \,:\, 0<\varepsilon\le 1 \}$. Let $x \in \mathbb{L}_\gamma$ and $F(x,\cdot)$ as in (3.25). Define the operator $F(x,\cdot) \mapsto \mathscr{A}^* F(x,\cdot)$ by*

$$\begin{aligned} \mathscr{A}^* F(x,\cdot)(y) \;=\; \mathscr{A}^* F(x,y) \;=& \\ \Big[-\langle Q(x_{1,\cdot}+x_{2,\cdot})\,,\,y_{1,\cdot}+y_{2,\cdot}\rangle \;+\; i\langle Q(x_{1,\cdot}-x_{2,\cdot})\,,\,y_{1,\cdot}-y_{2,\cdot}\rangle\Big] F(x,y), \end{aligned} \tag{3.45}$$

for $y \in \mathbb{L}_b$. Then Y is a solution to the following martingale problem:

For any $\big(F(x,\cdot)\,,\, \mathscr{A}^* F(x,\cdot)\big)$

$$t \;\mapsto\; F(x,Y_t) \,-\, F(x,Y_0) \,-\, \int_0^t \mathscr{A}^* F(x,Y_s)\,ds \tag{3.46}$$

is a martingale starting from 0 at time $t=0$.

Proof. Observe that the approximate processes Y^ε satisfies the following:

$$F(x,Y_t^\varepsilon) \;=\; F(x,Y_0^\varepsilon) \;+\; \int_0^t \mathscr{A}^* F(x,Y_s^\varepsilon)\,ds \;+\; M_t^{Y^\varepsilon,F}\,, \tag{3.47}$$

where the martingal $M_t^{Y^\varepsilon,F}$ is given by

$$M_t^{Y^\varepsilon,F} \;:=\; -\sum_{j=1}^{\lfloor \frac{T}{\varepsilon}\rfloor} \Delta F(x\,,\,Y^\varepsilon_{\cdot,\cdot,j\varepsilon}) \;{}_{[j\varepsilon,\infty)}(t)\,. \tag{3.48}$$

Integrability of the random variable $F(x,Y_t^\varepsilon)$ can be obtained as in (3.30) by (3.24).

To show that a similar equation as in (3.47) is still valid for the limit Y as ε approaches zero we need to establish higher moments. First, we check that the family $\{ \big(M_t^{\varepsilon,F}\big)_{0\le t\le T} \,:\, 0<\varepsilon\le 1 \}$ is bounded in $L^p(P)$, for $1<p<2$, and hence, is uniformly integrable. As in the proof of Lemma 3.14 we decompose F and define

$$\mathcal{Y}_n^{j\varepsilon} \;:=\; F(x\,,\,Y^\varepsilon_{\cdot,\cdot,j\varepsilon-}) \prod_{k=1}^n F_k(x\,,\,\Delta Y^\varepsilon_{\cdot,k,j\varepsilon})\,, \quad \text{for } n\in\mathbb{N}_0\cup\{\infty\},$$

for times $j\varepsilon$, $j=1,\dots,\lfloor\frac{T}{\varepsilon}\rfloor$. Then, we set $\mathcal{X}_n^{j\varepsilon} \;:=\; \mathcal{Y}_n^{j\varepsilon} \,-\, \mathcal{Y}_{n-1}^{j\varepsilon}$ for $n\in\mathbb{N}\cup\{\infty\}$. For the square bracket of the martingale $N\mapsto \sum\limits_{n=1}^N \mathcal{X}_n^{j\varepsilon}$ we have

$$\begin{aligned} \Big[\sum_{n=1}^{\cdot} \mathcal{X}_n^{j\varepsilon} \Big]_N \;&=\; \sum_{n=1}^N (\mathcal{X}_n^{j\varepsilon})^2 \;=\; \sum_{n=1}^N (\mathcal{Y}_n^{j\varepsilon} - \mathcal{Y}_{n-1}^{j\varepsilon})^2 \\ &\le\; \sum_{n=1}^N \big| F_n(x\,,\,\Delta Y^\varepsilon_{\cdot,l,j\varepsilon}) \,-\, F_n(x\,,\,0)\big|^2 \\ &\le\; \sum_{l=1}^N 2^2 (x_{1,l}+x_{2,l})^2 \left(|\Delta Y^\varepsilon_{1,n,j\varepsilon}| + |\Delta Y^\varepsilon_{2,n,j\varepsilon}|\right)^2 \end{aligned} \tag{3.49}$$

by the Lipschitz continuity of $F_n(x,\cdot)$. Then, by the Burkholder-Davis-Gundy inequality, by (3.49) and by Lemma 3.13 we get

$$\begin{aligned} E\Bigg[\sup_{0\le t\le T}\Big|M^{Y^\varepsilon,F}\Big|^p\Bigg] &\le E\Bigg[\sup_{J\le\lfloor\frac{T}{\varepsilon}\rfloor\,,\,N\in\mathbb{N}\cup\{\infty\}}\Big|\sum_{j=1}^{J}\sum_{n=1}^{N}\mathcal{X}_n^{j\varepsilon}\Big|^p\Bigg] \\ &\le c_p E\Bigg[\Big(\sum_{j=1}^{\lfloor\frac{T}{\varepsilon}\rfloor}\sum_{k\in S}2^2(x_{1,l}+x_{2,l})^2\big(|\Delta Y^\varepsilon_{1,n,j\varepsilon}|+|\Delta Y^\varepsilon_{2,n,j\varepsilon}|\big)^2\Big)^{p/2}\Bigg] \\ &\le c_p'\sum_{j=1}^{\lfloor\frac{T}{\varepsilon}\rfloor}\sum_{k\in S}(x_{1,k}+x_{2,k})^p\,E\Big[|\Delta Y^\varepsilon_{1,k,j\varepsilon}|^p+|\Delta Y^\varepsilon_{2,k,j\varepsilon}|^p\Big]. \end{aligned} \tag{3.50}$$

Lemma 1.7 implies

$$\begin{aligned} E\Big[|\Delta Y^\varepsilon_{1,k,j\varepsilon}|^p\Big] \;&\le\; C_p\,E\Big[(Y^\varepsilon_{1,k,j\varepsilon-})^{p-1}\,Y^\varepsilon_{2,k,j\varepsilon-}\;{}_{\{Y^\varepsilon_{2,k,(j-1)\varepsilon-}=0\}}\Big] \\ &\quad+\;C_p\,E\Big[(Y^\varepsilon_{2,k,j\varepsilon-})^{p-1}\,Y^\varepsilon_{1,k,j\varepsilon-}\;{}_{\{Y^\varepsilon_{1,k,(j-1)\varepsilon-}=0\}}\Big]. \end{aligned} \tag{3.51}$$

For the first summand of the r.h.s. (3.51) we obtain by Lemma 3.12 and Lemma 3.11

$$\begin{aligned} E\Big[(Y^\varepsilon_{1,k,j\varepsilon-})^{p-1}\,Y^\varepsilon_{2,k,j\varepsilon-}\;{}_{\{Y^\varepsilon_{2,k,(j-1)\varepsilon}=0\}}\Big] \;&\le\; E\big[Y^\varepsilon_{1,k,j\varepsilon-}\big]^{p-1}E\Big[\,Y^\varepsilon_{2,k,j\varepsilon-}\;{}_{\{Y^\varepsilon_{2,k,(j-1)\varepsilon}=0\}}\Big] \\ &\le\; C^{p-1}\,e^{TM(p-1)}\,\gamma_k^{p-1}\,\cdot\,\varepsilon\,C\,M\,e^{TM}\,\gamma_k \\ &=\; C^p\,M\,e^{TMp}\,\gamma_k^p\,\varepsilon \end{aligned} \tag{3.52}$$

by (3.24) and since $E\Big[\big(Y^\varepsilon_{2,\cdot,s}Q\big)_k\Big] \le C\,e^{TM}\sum_j\gamma_j q_{jk} \le C\,M\,e^{TM}\,\gamma_k$ for all $s\in[(j-1)\varepsilon\,,\,\varepsilon j]$ by (3.22) and (3.24). The same arguments are valid for the second summand of the r.h.s. of (3.51).

Similar, we can give bounds as in (3.51) and (3.52) for the jump of type 2, that is $E\Big[|\Delta Y^\varepsilon_{2,k,j\varepsilon}|^p\Big]$.

Hence, there exists a constant $\check{C}$ which depends on p, T, M but not on ε such that

$$\begin{aligned} E\Bigg[\sup_{0\le t\le T}\Big|M^{Y^\varepsilon,F}\Big|^p\Bigg] \;&\le\; \sum_{k\in S}\sum_{j=1}^{\lfloor\frac{T}{\varepsilon}\rfloor}(x_{1,k}+x_{2,k})^p\,\check{C}\,\varepsilon\,\gamma_k^p \\ &\le\; T\,\check{C}\sum_{k\in S}(x_{1,k}+x_{2,k})^p\,\gamma_k^p\;<\;\infty. \end{aligned} \tag{3.53}$$

This shows that the martingale $M^{Y^\varepsilon,F}$ is bounded in $L^p(P)$.

Next, observe that (3.53) allows to imitate the proof of Lemma 3.15 for Y^ε using Lemma 3.2. Thus, for $1<p<2$ holds

$$\sup_{0<\varepsilon\le 1}E\Big[\langle\,Y^\varepsilon_{1,\cdot,T}+Y^\varepsilon_{2,\cdot,T}\,,\,\gamma\,\rangle^p\Big]\;<\;\infty.$$

Then the assertion follows as in Proposition 3.16. □

Now we are ready to show duality of the processes X and Y w.r.t. the functions F. The proof follows [DP98] Theorem 2.4.

3.18 Proposition *Let X and Y start in $x\in\mathbb{L}_\gamma$ and $y\in\mathbb{L}_b$, respectively, i.e. $X_{\cdot,\cdot,0}=x$ and $Y_{\cdot,\cdot,0}=y$. Then, for each time $t\ge 0$,*

$$E\Big[\,F\big(\,X_t\,,\,y\,\big)\,\Big]\;=\;E\Big[\,F\big(\,x\,,\,Y_t\,\big)\,\Big]. \tag{3.54}$$

Proof. According to Propositions 3.16 and 3.17 there exist mean zero martingales M^X and M^Y such that

$$M_t^X \;=\; F(X_t,y) \;-\; F(X_0,y) \;-\; \int_0^t \mathscr{A}F(X_s,y)\,ds \tag{3.55}$$

and

$$M_t^Y \;=\; F(x,Y_t) \;-\; F(x,Y_0) \;-\; \int_0^t \mathscr{A}^*F(x,Y_s)\,ds\,. \tag{3.56}$$

Define

$$f(s,t) \;:=\; E\Big[\,F\big(\,X_t\,,\,Y_s\,\big)\,\Big].$$

Recall property (3.24) for Y_s. Hence, for fixed time $s \geq 0$ we can assume that $Y_s(\omega) \in \mathbb{L}_b$. By (3.55) we then have

$$f(s,t) \;=\; E\Big[F(X_0,Y_s)\Big] \;+\; E\Big[\int_0^t \mathscr{A}F(X_r,Y_s)\,dr\Big]. \tag{3.57}$$

Next observe that by (3.24)

$$\begin{aligned} E\Big[\big|\mathscr{A}F(X_r,Y_s)\big|\Big] \;&\leq\; E\Big[\langle X_{1,\cdot,r}+X_{2,\cdot,r}\,,\,\big|Q^*(Y_{1,\cdot,s}+Y_{2,\cdot,s})\big|\,\rangle\Big] \\ &\leq\; C\,M\,e^{sM}\,E\big[\langle X_{1,\cdot,r}+X_{2,\cdot,r}\,,\,\gamma\rangle\big] \\ &\leq\; C\,M\,e^{sM}\,2\,e^{rM}\langle x_{1,\cdot}+x_{2,\cdot}\,,\,\gamma\rangle \;\leq\; 2\,C\,M\,e^{2TM}\langle x_{1,\cdot}+x_{2,\cdot}\,,\,\gamma\rangle. \end{aligned}$$

Hence, we can interchange expectation and integration in (3.57) and obtain

$$f(s,t) \;=\; E\Big[F(X_0,Y_s)\Big] \;+\; \int_0^t E\Big[\mathscr{A}F(X_r,Y_s)\Big]\,dr. \tag{3.58}$$

From (3.56) we can similarly derive

$$f(s,t) \;=\; E\Big[F(X_t,Y_0)\Big] \;+\; \int_0^s E\Big[\mathscr{A}^*F(X_t,Y_r)\Big]\,dr. \tag{3.59}$$

In particular, $f(t,0) \,=\, E\big[F(X_0,Y_t)\big]$ by (3.58) and $f(0,t) \,=\, E\big[F(X_t,Y_0)\big]$ by (3.59). Finally we apply [EK86] Lemma 4.4.10 on p.192. To this end let $\partial_1 f$ and $\partial_2 f$ denote the partial derivatives of the absolutely continuous functions $f(\cdot,s)$ and $f(t,\cdot)$ respectively. Then by (3.59) and (3.58)

$$\begin{aligned} &E\big[F(X_0,Y_t)\big] \;-\; E\big[F(X_t,Y_0)\big] \\ &\qquad=\; f(t,0) \;-\; f(0,t) \\ &\qquad=\; \int_0^t \Big\{\partial_1 f(s,t-s) \;-\; \partial_2 f(s,t-s)\Big\}ds \\ &\qquad=\; \int_0^t \Big\{E\Big[\mathscr{A}^*F(X_{t-s},Y_s)\Big] \;-\; E\Big[\mathscr{A}F(X_{t-s},Y_s)\Big]\Big\}ds \;=\; 0 \end{aligned}$$

since $\mathscr{A}F(x,y) \,=\, \mathscr{A}^*F(x,y)$. This shows (3.54) and we are done. □

3.19 Remark *So far, all processes X^ε as well as X started in a deterministic initial condition, i.e., we assumed $X^\varepsilon_0 = x$ for all $\varepsilon > 0$, and hence, $X_0 = x$, where $x \in \mathbb{L}_\gamma$. Obviously, to construct X^ε it is possible to choose any initial distribution ν on $\mathbb{L}_\gamma$, instead. Hence, we let $X^\varepsilon_0 \sim \nu$. For the existence of the limit processes X we need to ensure that both the family of processes X^ε is relatively compact and the martingale property* (3.28) *is preserved in the limit. Thus, we impose the following two conditions on the initial distribution. For each $p \in [1, 2)$*

$$\int \big\{ \langle x_1, \gamma \rangle^p \;+\; \langle x_2, \gamma \rangle^p \big\} \nu(dx) \;<\; \infty, \tag{3.60}$$

and, for all $j, k \in S$

$$\int_{\mathbb{L}_\gamma} x_{1,j}\, x_{2,k}\, \nu(dx) \;\le\; \int_{\mathbb{L}_\gamma} x_{1,j}\, \nu(dx) \int_{\mathbb{L}_\gamma} x_{2,k}\, \nu(dx). \tag{3.61}$$

It is easy to check the arguments of Sections 3.1, 3.2 and 3.3 with initial distribution ν satisfying (3.60) *and* (3.61)*. For example note that in Lemma 3.2 Equations* (3.4) *to* (3.8) *hold in a similar way. Substitute expressions like $\langle x_1 + x_2\,,\, \gamma\rangle$ by $E\big[\langle X^\varepsilon_{1,\cdot,0} + X^\varepsilon_{2,\cdot,0}\,,\, \gamma\rangle\big]$. Observe that in any case the martingales $M^\varepsilon_{\alpha,k,t}$, $M^{\varepsilon,F}_t$ and $M^{Y^\varepsilon,F}_t$ of* (3.5), (3.31) *and* (3.48)*, respectively, start in zero. Of course, condition* (3.61) *ensures that Lemma 3.12 remains valid. Finally, the Gronwall argument of Lemma 3.15 works similar since the first term on the r.h.s. of Equation* (3.43) *is bounded by* (3.60)*. So from now on we assume existence of a solution to martingale problem* (3.44) *with any initial distribution which satisfys* (3.60) *and* (3.61)*.* ♢

3.20 Remark *Now we reverse the point of view. Consider any solution to the martingale problem (not necessarily the one(s) we constructed), i.e. assume existence of a probability measure P on $\big(D_{\mathbb{L}_\gamma}[0,T]\,,\,\mathscr{D}\big)$ with canonical process $(\chi_t)_t$ such that for any $F(\cdot, y)$, with $y \in \mathbb{L}_b$,*

$$M^F_t \;:=\; F(\chi_t, y) \;-\; F(\chi_0, y) \;-\; \int_0^t \mathscr{A} F(\chi_s, y)\, ds$$

is a martingale, where $F(\cdot, y)$ and $\mathscr{A}F$ are given by (3.25) *and* (3.27)*, respectively. We can approximate χ in the same spirit as we established X via X^ε. To this end let $\bar{X}^x$ be the deterministic solution to Equation* (3.3) *starting in $x \in \mathbb{L}_\gamma$ and let $0 = t^n_0 < t^n_1 < \ldots < t^n_{N^n} = T$ be any partition of the set $[0, T]$ with $t^n_{k+1} - t^n_k \le \frac{1}{n}$ for all $k = 0, 1, \ldots, N^n$, e.g. use an equidistant partition as on page 30 for X^ε. Then we can define the process χ^n by setting for $t \in [t^n_k, t^n_{k+1})$*

$$\chi^n_t \;:=\; \chi_{t_k} \;+\; \bar{X}^{\chi_{t_k}}_{t - t_k}\,.$$

Note that $\chi^n_{t^n_k} \equiv \chi_{t^n_k}$ for all $k = 0, 1, \ldots, N^n$. Recall that for $\bar{X}$ we have $F(\bar{X}^x_t, y) - F(\bar{X}^x_0, y) - \int_0^t \mathscr{A} F(\bar{X}_s, y) ds = 0$, hence, the family of processes $\big\{\chi^n : n \ge 1\big\}$ also has the above martingale property. Recall the remark after Proposition 3.9 and note that χ^n_t has values in $\mathbb{E}_\gamma$ but $y \in \mathbb{L}_\gamma$. In particular,

$$M^{n,F}_t \;:=\; F(\chi^n_t, y) \;-\; F(\chi^n_0, y) \;-\; \int_0^t \mathscr{A} F(\chi^n_s, y)\, ds$$

is piecewise constant and has jumps at times t_k^n, $k = 1, 2, \ldots, N^n$, and is given by

$$M_t^{n,F} \;=\; \sum_{k=1}^{N^n} \big\{ F(\chi^n_{t_k-}, y) - F(\chi_{t_k}, y) \big\} \;\; {}_{[t_k^n,\infty)}(t).$$

Moreover, we have

$$E\big[\, F(\chi_{t_{k+1}^n}, y) \,\big|\, \chi_{t_k^n} \,\big] \;=\; F\big(\, \chi_{t_k^n} + \bar{X}^{\chi_{t_k^n}}_{t_{k+1}^n - t_k^n} \,,\, y \,\big)$$

for any $y \in \mathbb{L}_b$. Since $\chi_{t_{k+1}^n}$ has values in $\mathbb{L}_\gamma$ this idenitifies the distribution of $\chi_{t_{k+1}^n}$ given $\chi_{t_k^n}$. In fact, for each site $k \in S$ we then have $\chi_{\cdot,k,t_{k+1}^n} \sim DP_a$ with $a = \chi_{\cdot,k,t_k^n} + \bar{X}^{\chi_{t_k^n}}_{t_{k+1}^n - t_k^n}$. This implies that we can imitate all the former arguments for X^ε. χ inherits the moment properties in the same manner as X from the family X^ε. We want to emphasise that Lemma 3.15 and Lemma 3.14 similarly hold for χ. In particular, $\bar{X}$ gives the mean of χ, and if the initial distribution of χ satisfies (3.60) then so does the distribution of χ_t and the distribution of χ_τ for any stopping time $\tau \leq T$ by a submartingale argument. Lemma 3.12 remains valid if the initial distribution of χ_t satisfies (3.61). Even duality holds for χ. And χ is a limit point of $(\chi^n)_n$ since the finite dimensional distributions eventually coincide on an appropriate dense subset, cf. [JS87] VI.§3b, p.350 ♢

Consider two processes X and X' with common initial distribution ν, which satisfys (3.60) and (3.61). Assume both processes are solutions to the martingale problem (3.44). Then, in view of Remark 3.20, we can apply the duality result of Proposition 3.18. This gives

$$E\Big[\, F\big(X_t \,,\, y\big) \Big] \;=\; E\Big[\, F\big(X_0 \,,\, Y_t\big) \Big] \;=\; E\Big[\, F\big(X'_0 \,,\, Y_t\big) \Big] \;=\; E\Big[\, F\big(X'_t \,,\, y\big) \Big] \tag{3.62}$$

for $t \geq 0$ and some dual process Y starting in $y \in \mathbb{L}_b$. By Lemma 3.10 the processes X and X' have the same one-dimensional distributions, i.e. for any Borel measurable set $\Gamma \subseteq \mathbb{L}_\gamma$ and any $t \geq 0$ we obtain

$$P_\nu\Big[\, X_t \,\in\, \Gamma \Big] \;=\; P'_\nu\Big[\, X'_t \,\in\, \Gamma \Big]. \tag{3.63}$$

As it is well known, in the context of martingale problems this is enough to characterize the finite-dimensional distributions of theses solutions, see e.g. Theorem 4.4.2 of [EK86]. However, we can not directly apply this result since it requires all functions to be bounded. But here, $x \mapsto \mathscr{A}F(x, y)$ is an unbounded function on $\mathbb{L}_\gamma$. For all that it was observed by Mytnik, compare with [My98a] p.972 Sec.2 or [My98b] pp.249-251, that the boundedness assumption can be relaxed. Instead, we only need that X_t (and X'_t), $t \geq 0$, has a finite p-th moment, $1 \leq p < 2$. The proof of the next Proposition follows Mytnik, see [My96] pp.46-48.

3.21 Proposition *Any solution to the martingale problem* (3.44) *is a Markov process and any two solutions have the same finite-dimensional distributions.*

Proof. Let X be a solution to the martingale problem (3.44). X induces a probability measure P on $D_{\mathbb{L}_\gamma}[0, T]$ with σ-algebra $\mathscr{D}$, hence we will consider X as the canonical process on $D_{\mathbb{L}_\gamma}[0, T]$. The martingale property in (3.44) holds w.r.t. the filtration $(\mathscr{F}_t)_{0 \leq t \leq T}$.

Fix $r \in [0,T]$, and let $F \in \mathscr{F}_r$ satisfy $P(F) > 0$. Define the conditional probabilities $P_{\mathscr{F}_r}$ and P_{X_r} on $\mathscr{D}$ by

$$P_{\mathscr{F}_r}(B) \;:=\; \frac{E\big[\;_F\, E[\;_B|\mathscr{F}_r]\big]}{P(F)} \tag{3.64}$$

and

$$P_{X_r}(B) \;:=\; \frac{E\big[\;_F\, E[\;_B|X_r]\big]}{P(F)} \tag{3.65}$$

for any $B \in \mathscr{D}$. Set $Y_s := X_{s+r}$ for $0 \le s \le T-r$. Note that for any Borel measurable $\Gamma \subseteq \mathbb{L}_\gamma$

$$\begin{aligned} P_{\mathscr{F}_r}\big[Y_0 \in \Gamma\big] \;&=\; \frac{E\big[\;_F\, E[\;_{\{X_r \in \Gamma\}}|\mathscr{F}_r]\big]}{P(F)} \\ &=\; \frac{E\big[\;_F\, E[\;_{\{X_r \in \Gamma\}}|X_r]\big]}{P(F)} \;=\; P_{X_r}\big[Y_0 \in \Gamma\big] \\ &=\; \frac{E\big[\;_{F \cap \{X_r \in \Gamma\}}\big]}{P(F)} \\ &=\; P\big[X_r \in \Gamma \big| F\big]\,, \end{aligned}$$

i.e. Y has under $P_{\mathscr{F}_r}$ and P_{X_r} the same initial distribution and this initial distribution satisfies the moment property (3.60) and, by Lemma 3.12, condition (3.61) since X satisfies it under P. We set for $0 \le s \le t \le T-r$

$$\eta^{Y,y}(s,t) \;:=\; F(Y_t,y) \;-\; F(Y_s,y) \;-\; \int_s^t \mathscr{A}F(Y_u,y)\,du. \tag{3.66}$$

Since X solves the martingale problem w.r.t. P and $Y_\cdot = X_{r+\cdot}$ we have

$$E\big[\eta^{Y,y}(s,t)\cdot G \,\big|\, \mathscr{F}_{r+s}\big] \;=\; 0 \tag{3.67}$$

for any $\mathscr{F}_{r+s}$-measurable, bounded function G. Then, since $\sigma(X_r) \subseteq \mathscr{F}_{r+s}$,

$$\begin{aligned} E_{P_{X_r}}\big[\eta^{Y,y}(s,t)\cdot G\big] \;&=\; \frac{E\big[\;_F\, E[\eta^{Y,y}(s,t)\cdot G|\,X_r\,]\big]}{P(F)} \\ &=\; \frac{E\big[\;_F\, E[\,E[\eta^{Y,y}(s,t)\cdot G|\mathscr{F}_{r+s}]\,|\,X_r\,]\big]}{P(F)} \;=\; 0 \end{aligned}$$

and similarly,

$$E_{P_{\mathscr{F}_r}}\big[\eta^{Y,y}(s,t)\cdot G\big] \;=\; 0$$

for any $\mathscr{F}_{r+s}$-measurable function G. Hence, Y is a solution to the martingale problem (3.44) w.r.t. P_{X_r} and $P_{\mathscr{F}_r}$ (and filtration $(\mathscr{F}_{r+s})_s$).

Now note that Y under P_{X_r} and $P_{\mathscr{F}_r}$ has the 'same finite moments' as X under P (since P_{X_r} and $P_{\mathscr{F}_r}$ are only conditional probabilities). So duality holds for Y. And this implies $E_{P_{X_r}}[F(Y_s,y)] = E_{P_{\mathscr{F}_r}}[F(Y_s,y)]$ for all $y \in \mathbb{L}_{\text{fin}}$. This determines the one-dimensional distributions of Y under P_{X_r} and $P_{\mathscr{F}_r}$. Hence, for any bounded measurable function f on $\mathbb{L}_\gamma$ we have

$$E_{P_{X_r}}[\,f(Y_s)\,] \;=\; E_{P_{\mathscr{F}_r}}[\,f(Y_s)\,] \tag{3.68}$$

by the usual arguments of linearity and monotonicity. This gives

$$\begin{aligned} E\big[\ {}_{F}\, E[f(X_{r+s})\,|\,X_r\,]\big] \;&=\; E_{P_{X_r}}[f(Y_s)]\,P(F) \\ &=\; E_{P_{\mathscr{F}_r}}[f(Y_s)]\,P(F) \;=\; E\big[\ {}_{F}\, E[f(X_{r+s})\,|\,\mathscr{F}_r\,]\big]. \end{aligned} \tag{3.69}$$

Now observe that (3.69) holds for any $F \in \mathscr{F}_r$ (it is obvious for $F \in \mathscr{F}_r$ with $P(F) = 0$), and therefore we have proved

$$E\big[f(X_{r+s})\,\big|\,X_r\,\big] \;=\; E\big[\,f(X_{r+s})\,|\,\mathscr{F}_r\,\big], \tag{3.70}$$

which is the Markov property.

Now let X and X' be two solutions to the martingale problem (3.44). Both processes induce probability measures P and P', respectively, on the space $\big(D_{\mathbb{L}_\gamma}[0,T]\,,\,\mathscr{D}\big)$ with filtration $(\mathscr{F}_t)_{0\le t\le T}$. We want to show that their finite-dimensional distributions coincide, that is

$$E_P\Bigg[\prod_{j=1}^{m}\ {}_{\{X_{t_j}\in B_j\}}\Bigg] \;=\; E_{P'}\Bigg[\prod_{j=1}^{m}\ {}_{\{X'_{t_j}\in B_j\}}\Bigg] \tag{3.71}$$

for all choices $t_j \in [0,T]$ and Borel measurable subsets B_j of $\mathbb{L}_\gamma$. We show (3.71) by induction.

For $m = 1$ Equation (3.71) is valid by the discussion following (3.62) since X and X' are both solutions to the martingale problem (3.44).

Next, assume that (3.71) holds for all $m \le n$. If for some j we have $P(X_{t_j} \in B_j) = 0$ then (3.71) holds obviously, so w.l.o.g. we assume that

$$P(X_{t_j} \in B_j) \;=\; P'(X'_{t_j} \in B_j) > 0,$$

for all j. We choose $0 \le t_1 < t_2 < \ldots < t_n$ and $B_1, \ldots, B_n$. Set $A := \big\{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\big\}$ and note that $A \in \mathscr{F}_{t_n}$. Similar as above, define the probability measures $\tilde{P}$ and $\tilde{P}'$ on $D_{\mathbb{L}_\gamma}[0,T]$ by

$$\tilde{P}(B) \;=\; \frac{E_P\big[\ {}_{B}\ \ {}_{A}\big]}{P(A)} \quad\text{and}\quad \tilde{P}'(B) \;=\; \frac{E_{P'}\big[\ {}_{B}\ \ {}_{A}\big]}{P'(A)}$$

for $B \in \mathscr{D}$. Set $\tilde{X}_s := X_{t_n+s}$ and $\tilde{X}'_s = X'_{t_n+s}$ for $s \in [0, T - t_n]$. Then, by the same arguments used above, $\tilde{X}$ and $\tilde{X}'$ are solutions to the martingale problem (3.44) w.r.t. $\tilde{P}$ and $\tilde{P}'$, respectively. Note that instead of G as in (3.67) it is enough to consider functions of the form ${}_{A}$, see [EK86] p.174 Equation 3.4. Then by induction hypotheses (with $m = n$) we have for any Borel measurable $B_{n+1} \subseteq \mathbb{L}_\gamma$,

$$\begin{aligned} \tilde{P}(\tilde{X}_0 \in B_{n+1}) \;&=\; \frac{E_P\big[\ {}_{\{\tilde{X}_0\in B_{n+1}\}}\ \ {}_{A}\big]}{P(A)} \\ &=\; \frac{E_P\Big[\ {}_{\{X_{t_n}\in B_n\cap B_{n+1}\}}\prod_{j=1}^{n-1}\ {}_{\{X_{t_j}\in B_j\}}\Big]}{P(A)} \\ &=\; \frac{E_{P'}\big[\ {}_{\{\tilde{X}'_0\in B_{n+1}\}}\ \ {}_{A}\big]}{P'(A)} \;=\; \tilde{P}'(\tilde{X}'_0 \in B_{n+1}), \end{aligned}$$

so $\tilde{X}$ and $\tilde{X}'$ have the same initial distributions. Note that this initial distribution satisfies the moment condition (3.60) since X_{t_n} satisfies it w.r.t. P (and w.l.o.g. $P(A) > 0$) and the same is true for condition (3.61). Then duality applies and hence

$$\tilde{P}\big(\tilde{X}_s \in B_{n+1}\big) = \tilde{P}'\big(\tilde{X}'_s \in B_{n+1}\big)$$

for all $0 \le s \le T - t_n$ and all Borel measurable $B_{n+1} \subseteq \mathbb{L}_\gamma$. Thus, with $s = t_{n+1} - t_n$,

$$\begin{aligned} E_P\Big[\prod_{j=1}^{n+1} {}_{\{X_{t_j} \in B_j\}}\Big] &= P(A)\,\tilde{P}\big(\tilde{X}_s \in B_{n+1}\big) \\ &= P(A)\,\tilde{P}\big(\tilde{X}_s \in B_{n+1}\big) = E_{P'}\Big[\prod_{j=1}^{n+1} {}_{\{X'_{t_j} \in B_j\}}\Big], \end{aligned}$$

which is (3.71) for $m = n+1$. □

Now we can summarize previous results. Recall the definition of the processes $\tilde{X}^\varepsilon$ in (3.14) on page 33.

3.22 Theorem *As ε tends to 0, the processes $\tilde{X}^\varepsilon$ converge weakly on $D_{\mathbb{L}_\gamma}[0,T]$ to a unique Markov process X which solves the martingale problem* (3.44).

Proof. By Proposition 3.7, cf. Equation (3.21), $\tilde{X}^\varepsilon$ is tight, and hence, provides weakly convergent subsequences which converge to some process X with paths in $\mathbb{L}_\gamma$. By the approximation via X^ε the limit X is a solution to the martingale problem (3.44), which turns out to be a Markov solution with unique finite-dimensional distributions, see Proposition 3.21. Since $\mathbb{L}_\gamma$ is complete and separable the finite-dimensional distributions uniquely determine the hole probability measure P of X; see [EK86] Proposition 3.7.1 and [JS87] Lemma VI.3.19. □

Due to duality we can also prove the Feller property and the strong Markov property of X. To state these results we recall some notation. Let P_x denote the law of X with initial distribution δ_x. If $f : \mathbb{L}_\gamma \longrightarrow \mathbb{R}$ is a bounded and measurable function, we write

$$\bar{P}_t f(x) = E_{P_x}\big[f(X_t)\big],$$

for $x \in \mathbb{L}_\gamma$ and $0 \le t < \infty$.

3.23 Lemma

(a) $\bar{P}_t : \mathcal{C}_b(\mathbb{L}_\gamma) \longrightarrow \mathcal{C}_b(\mathbb{L}_\gamma)$.

(b) *For any finite $(\mathscr{F}_t)$-stopping time τ, bounded measurable function f on $\mathbb{L}_\gamma$ and any $t \ge 0$*

$$E\big[\,f(X_{t+\tau})\,\big|\,\mathscr{F}_\tau\,\big] = \bar{P}_t f(X_\tau). \tag{3.72}$$

Proof. Let $f : \mathbb{L}_\gamma \longrightarrow \mathbb{R}$ be bounded and continuous. Fix $t \in [0,T]$. Obviously, $\bar{P}_t f(x) = E_{P_x}\big[f(X_t)\big]$ is uniformly bounded in $x \in \mathbb{L}_\gamma$. Let $(x^{(n)})_n \subseteq \mathbb{L}_\gamma$ be a sequence which converges to $x \in \mathbb{L}_\gamma$ (w.r.t. $\|\cdot\|_\gamma$). To proof continuity we have to show $\lim_n \bar{P}_t f(x^{(n)}) = \bar{P}_t f(x)$.

First note that

$$E_{x^{(n)}}\big[\langle X_{1,\cdot,t} + X_{2,\cdot,t}\,,\,\gamma\rangle\big] \;\le\; e^{tM}\langle\, x_1^{(n)} + x_2^{(n)}\,,\,\gamma\rangle, \tag{3.73}$$

and the r.h.s. is bounded uniformly in n since $(x^{(n)})_n$ converges. Let $\eta > 0$. Then (3.73) gives rise to a compact set $K_\eta \subseteq \mathbb{L}_\gamma$, as in Lemma 3.3, such that

$$\inf_n P_{x^{(n)}}\big(X_t \in K_\eta\big) \;\ge\; 1-\eta\,,$$

compare with the arguments in the proof of Lemma 3.4. Hence, the laws $P^{X_t}_{x^{(n)}} := P_{x^{(n)}}(X_t \in \cdot\,)$ on $\mathbb{L}_\gamma$ are tight. Next, observe that by the duality result of Proposition 3.18 we have for all duality functions $F(\cdot, y)$, with $y \in \mathbb{L}_{\mathrm{fin}}$,

$$\lim_n E_{x^{(n)}}\Big[F(X_t\,,\,y)\Big] \;=\; \lim_n E_y\Big[F(x^{(n)}\,,\,Y_t)\Big] \;=\; E_y\Big[F(x\,,\,Y_t)\Big] \;=\; E_x\Big[F(X_t\,,\,y)\Big]$$

since F is bounded and Y_t is a.s. γ-bounded.

Hence, all subsequences of $(P^{X_t}_{x^{(n)}})$ contain a weak convergent subsequence such that

$$\int F(\varkappa, y)\Big(\lim_j P^{X_t}_{x^{(n_{k_j})}}\Big)(d\varkappa) \;=\; \int F(\varkappa, y)P^{X_t}_x(d\varkappa)\,,$$

i.e. all limit points of $(P^{X_t}_{x^{(n)}})$ coincide by Lemma 3.10. A fortiori, $(P^{X_t}_{x^{(n)}})$ weakly converges to $P^{X_t}_x$. Then we have $\lim_n \bar{P}_t f(x^{(n)}) \;=\; \bar{P}_t f(x)$ by definition of weak convergence.

Now we prove part (b). Before Equation (3.72) makes good sense, we need to know that the map $x \mapsto P_x(B)$ is measurable for fix $B \in \mathscr{D}$. The standard tool for proving this is a Theorem of Kuratowski (see [Pa67] Corollary 3.3 p.22) as used in [SV79] Exercise 6.7.4 or [EK86] Theorem 4.4.6. However, we prefer to follow the proof of Theorem 4.5.19 in [EK86]; see pp.215-216 around Equations (5.93) and (5.94). To this end we recall that in Proposition 3.21 the Markov property for X was already established. Due to the duality relation we can also infer measurability. In fact, by part (a) above, the Markov property and Lemma 3.2(a) the map $(x,t) \mapsto E_{P_x}\big[f(X_t)\big]$ is continuous for any $f \in \mathcal{C}_b(\mathbb{L}_\gamma)$. Therefore the transition function $(t, x, \Gamma) \mapsto P_x(X_t \in \Gamma)$, with $t \in [0,T]$, $x \in \mathbb{L}_\gamma$ and Borel measurable $\Gamma \subseteq \mathbb{L}_\gamma$, matches all assumptions listed at the begining of Sec.1 in Chapter 4 of [EK86]. We are done by quoting Proposition 4.1.2 of the same reference, which uses a Dynkin class argument.

Next, compare with [EK86] Theorem 4.4.2(b) and (c): As in the proof of Proposition 3.21 we can define the conditional probabilities

$$P_{\mathscr{F}_\tau}(B) \;:=\; \frac{E\big[\ {}_F E[\ {}_B|\mathscr{F}_\tau]\big]}{P(F)} \qquad \text{and} \qquad P_{X_\tau}(B) \;:=\; \frac{E\big[\ {}_F E[\ {}_B|X_\tau]\big]}{P(F)}$$

if $F \in \mathscr{F}_\tau$ satisfies $P(F) > 0$, where $B \in \mathscr{D}$. Then note that $t \mapsto F(X_t, y) - F(X_0, y) - \int_0^t \mathscr{A}F(X_u, y)\,du$ is a uniformly integrable càdlàg-martingale and τ is a finite stopping time. Hence, we can apply the optional stopping theorem. With $Y_s := X_{\tau+s}$, note that

$$\eta^{Y,y}(s,t) \;:=\; F(Y_t, y) \,-\, F(Y_s, y) \,-\, \int_s^t \mathscr{A}F(Y_u, y)\,du \tag{3.74}$$

has the same properties as η of (3.66). Hence, $P_{\mathscr{F}_\tau}$ and P_{X_τ} define solutions to the martingale problem with the same initial distribution. We obtain

$$E\big[f(X_{\tau+s})\,\big|\,\mathscr{F}_\tau\,\big] \;=\; E\big[f(X_{\tau+s})\,\big|\,X_\tau\,\big]$$

for any bounded measurable function f on $\mathbb{L}_\gamma$, as in Equation (3.70). This together with the measurability gives (3.72). $\square$

Chapter 4

Outlook

In this last part of the thesis we would like to hint at further developments and questions in this area. Existence and uniqueness for the mutually catalytic super-random walk is, of course, only the basic issue. And in fact there is a great deal of room to maneuver.

- Here is a question that my thesis advisor has raised: Consider the one-colony model $(X_{\cdot,0,t})_{t\geq 0}$ from Chapter 2 with infinite branching rate. Does

  $$P_x\big(\, X_{1,0,s} + X_{2,0,s} \;>\; 0 \text{ for all } s \in [0,T]\,\big) \;=\; 1$$

  hold if X is started in any point $x \in L$? If we assume $\rho = 1 = \kappa$ in the migration matrix Q of (2.1) the answer might depend on the choice of parameters $(\theta_1, \theta_2) \in [0,\infty)^2$ like in the case of ordinary Feller diffusions with immigration, see [IW89] Chapter IV Example 8.2 pp.235–237. Certainly, if $(\theta_1,\theta_2) = (0,0)$ and $x = (0,0)$ then $X_{\cdot,0,\cdot} \equiv (0,0)$. If $(\theta_1,\theta_2) = (0,0)$ and $x \in L\setminus\{(0,0)\}$ then $t \mapsto X_{\cdot,0,t}$ does not hit $(0,0)$ in finite time. For any $t > 0$ fixed, Corollary 2.5 implies $P_x(\, X_{1,0,t} + X_{2,0,t} \;>\; 0\,) \;=\; 1$ if $(\theta_1,\theta_2) \neq (0,0)$. However, note what is mentioned in [DP98] on page 1093 three lines before Theorem 1.4: Planar Brownian motion, when it exits the first quadrant, *is bounded away from* $(0,0)$.

- In [DP98], Dawson and Perkins also proved existence for a mutually catalytic branching model on $\mathbb{R}$. A non-trivial existence in $\mathbb{R}^2$ was less obvious; see the discussion in [DF00] Section 3.6, but it was established later in [DE02a] and [DE02b]. The model in the plane, denoted by $\mathbb{X}$, say, has similar features to the infinite rate model we constructed even though it is a coninuous process: Consider time $t > 0$. For Lebesgue-almost all sites b it has infinite variance, $\operatorname{Var}\mathbb{X}_t(b) \;\equiv\; \infty$, and its distribution satisfies $\mathbb{X}_t(b) \sim DP$. In particular for almost all $b \in \mathbb{R}^2$ we have $\mathbb{X}_{1,t}(b)\,\mathbb{X}_{2,t}(b) \;=\; 0$. Compare with Theorem 17 in [DE02a] or Theorem 3.7 in [DF00]. However, the branching rate $\gamma > 0$ has to be sufficiently small. Is it possible to compare $\mathbb{X}$ with a lattice approximation by using our model X (on $\varepsilon\mathbb{Z}^2$ as $\varepsilon \searrow 0$)? Perhaps for large γ?

- For some time there has been interest in *symbiotic* catalytic branching, which means both one-dimensional Brownian motions $B_{1,k,\cdot}$ and $B_{2,k,\cdot}$ (at a site $k \in S$), as in Equation (1.1), are correlated, i.e. for $\varrho \in [-1,1]$,

$$E\big[B_{\alpha,k,t}\, B_{\beta,j,s}\big] \;=\; \varrho \min\{\, t\,,\, s\,\}\, \delta_0(k-j),$$

where $\alpha, \beta \in \{1,2\}$, $j,k \in S$ and $s,t > 0$; cf. [EF03] or [DFX05]. See [EF03] for the correspondence with the stepping stone model (if $\varrho = -1$) and the Anderson model (if $\varrho = 1$). Mytnik's duality only needs a slight change; see Section 2.3 of [EF03].

There are two topics we would like to discuss in more detail.

4.1 The particle version – discrete state space

Assume that the process X is supposed to describe the evolution of true particles on a countable site space S. Then, $X_{\alpha,k,t}$ has to have values in $\mathbb{N}_0$, for all $\alpha \in \{1,2\}$, $k \in S$ and $t \geq 0$. We maintain the infinite branching rate (or competition rate), which means if two particles of different types collide at one site, one of both types will immediately become extinct (at that site).

For simplicity, assume X starts in x with finitely many particles only, i.e., $\sum_{\alpha=1}^{2} \sum_{k\in S} x_{\alpha,k} < \infty$, and each site is only occupied by one type. That is to say $x_{1,k}x_{2,k} = 0$ for all $k \in S$. Then, to each particle we can associate an exponential clock (with rate 1, say). Let τ_1 be the time when the first clock rings. This is a well-defined time since there are only finitely many particles; and there is exactly one clock which rings first. In particular, we have $\tau_1 > 0$ almost surely. If the clock of a particle at site $j \in S$ rings first, the particle migrates (or jumps) randomly to a new site k according to some given distribution (for example, if $S = \mathbb{Z}^d$ we can choose a nearest neighbour $k \in \mathbb{Z}^d$ with probability $(2d)^{-1}$). Then, on site k we introduce the following 'catalytic' interaction between the types.

Assume that the particle, which jumped, is of type 1. If site k is only populated by particles of the same type (type 1), then the number of particles of that type at that site is increased by 1. If k is occupied with the opposite type (type 2), we choose a new configuration at that site by using the law of an independent, (continuous time rate 1) simple symmetric random walk $\xi_t = (\xi_{1,t}, \xi_{2,t})$ on $\mathbb{Z}^2$ stopped when it first hits the axes, i.e., $(\xi_t)_{t\geq 0}$ is stopped at time $T := \inf\big\{\, t > 0 \,:\, \xi_{1,t}\, \xi_{2,t} = 0\,\big\}$. Set

$$\mathrm{dp}_{(a_1,a_2)} \;:=\; \mathcal{L}[\xi_T]$$

when ξ is started in $(a_1, a_2) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then we can put $X_t = x$ for $t \in [0, \tau_1)$, and

$$X_{\alpha,l,\tau_1} \;=\; X_{\alpha,l,\tau_1-} \qquad \text{for all } \alpha \in \{1,2\} \;\text{ and }\; l \in S \setminus \{j,k\}.$$

If $j = k$ then put $X_{\cdot,j,\tau_1} = X_{\cdot,j,\tau_1-}$. If $j \neq k$ we set $X_{1,j,\tau_1} = X_{1,j,\tau_1-} - 1$ and $X_{2,j,\tau_1} = X_{2,j,\tau_1-}$ at site j. For site k we let

$$(\, X_{1,k,\tau_1}\,,\, X_{2,k,\tau_1}\,) \;\sim\; \mathrm{dp}_{(X_{1,k,\tau_1-}\,+\,1\,,\, X_{2,k,\tau_1-})}\,.$$

Note that if $X_{2,k,\tau_1-} = 0$ then $\mathrm{dp}_{(X_{1,k,\tau_1-}+1\,,\, X_{2,k,\tau_1-})} \;=\; \delta_{(X_{1,k,\tau_1-}+1\,,\,0)}$.

Of course, we can give a similar definition if the particle that jumps is of type 2. Moreover, we can iterate this procedure. Proceeding with configuration X_{τ_1} as the new initial condition we can define τ_2 – the time when the second particle jumps – and so on. We obtain a sequence of random times $0 =: \tau_0 < \tau_1 < \tau_2 < \tau_3 < \dots$ and a process X on $[0, \sup_n \tau_n)$ such that X is constant on each interval $[\tau_{n-1}, \tau_n)$.

Certainly, the dp-distribution plays a similar role as the DP-distribution before. So, as a first step, the properties of the dp-distribution have to be investigated. Let d be a random variable with $d = (d_1, d_2) \sim \mathrm{dp}_{(a_1,a_2)}$. If $E[d_\alpha] = a_\alpha$, for each $\alpha \in \{1,2\}$, then we can infer that $n \mapsto M_{\alpha,n} := \sum_{k\in S} X_{\alpha,k,\tau_n}$ is a (non-negative, discrete-time) martingale for each $\alpha \in \{1,2\}$, hence, $\sup_n M_{\alpha,n} < \infty$ almost surely. This implies that the process does not explode; in other words, we have $\sup_n \tau_n = \infty$. Is it possible to compute $c(a,b) := \mathrm{dp}_{(a_1,a_2)}\big((b_1,b_2)\big)$ explicitly, for $a = (a_1,a_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ and $b = (b_1,b_2) \in \mathbb{N}_0 \times \{0\} \cup \{0\} \times \mathbb{N}_0$? One might wish to use these constants to define a pregenerator for the process in the present situation. Alternatively, set up a stochastic differential equation of the pure jump type driven by a Poisson point process; compare with [Bir03] Equation (2.9) on page 12.

For an infinite initial condition compare with Chapter 3. If the migration of particles, formerly described by the matrix Q, satisfies a condition as in Equation 3.1 then we may introduce the spaces $\mathbb{E}_\gamma := \mathbb{E}_\gamma \cap (\mathbb{N}_0 \times \mathbb{N}_0)^S$ and $\mathbb{L}_\gamma := \mathbb{L}_\gamma \cap (\mathbb{N}_0 \times \mathbb{N}_0)^S$ w.r.t. some weight function γ as in Definition 3.1 on page 28. But note that an extension of the above particle process with values in $\mathbb{L}_{fin} := \mathbb{L}_{\text{fin}} \cap (\mathbb{N}_0 \times \mathbb{N}_0)^S$ to a process with values in $\mathbb{L}_\gamma$ by the well-known approximation procedure as in [Li85] Chapter IX. Theorem 1.14 (see also [Bir03] sections 2.1 and 2.2) should not work, due to the lack of monotonicity (cf. [Bir03] Assumption A and Lemma 1b).

Instead one might want to proceed as in Section 3.1 above. Let (infinitely many) particles migrate for a short time $\varepsilon > 0$. Use the space $\mathbb{E}_\gamma$. Then at time $t = \varepsilon$ apply the dp-distribution, independently at each site $k \in S$ with parameters given by the configuration at time $t = \varepsilon-$. To find an appropriate martingale problem for these processes, one has to investigate the properties of the dp-distributions w.r.t. a rich family of functions. Do Mytnik's duality functions $F(\cdot, y)$, where $y \in \mathbb{L}_{fin}$, separate measures on $\mathbb{L}_\gamma$? How does a duality for this process have to look like?

4.2 The mean field limit

We return to the continuous state space setting. In Chapter 2 we showed that, for fixed $t \geq 0$, the one-dimensional distributions of $Z^\gamma_{\cdot,0,t}$ (the process with finite branching rate $\gamma > 0$ with drift towards $\Theta \in (0,\infty)^2$) converge to the distribution of $X_{\cdot,0,t}$ (the process with infinite branching rate with drift towards the same Θ) as $\gamma \to \infty$, provided both processes start in the same initial condition $(x, \Theta) \in L \times (0,\infty)^2$; see Lemma 2.6. Then [EK86] Theorem IV.2.5 on p.167 suggests that $Z^\gamma \underset{\gamma\to\infty}{\Longrightarrow} X$. But note that the processes Z^γ are continuous while X is not. Hence, weak convergence cannot hold on the path space $D_E[0,\infty)$, where $E := L \times \{\Theta\}$, since in this case the limit process has to be continuous; compare with [JS87] Proposition VI.3.26(i) and (iii) on p.351. So, at the utmost it is possible to show convergence of the finite-dimensional distributions, for the one-colony model as well as on countably many sites.

However, in this section we would like to hint at a possible converse. Is it possible to find a procedure such that the process with an infinite branching rate converges to the process with a finite branching rate? My thesis advisor suspects this to be true if the so-called mean field limit is considered: Fix an enumeration of S. Let $S_N := \{1, 2, \ldots, N\}$, and let $Q^N = (q_{jk})_{j,k \in S_N}$ be given by $q_{jk} = \frac{1}{N}$ for $j \neq k$ and $q_{jj} = -1$, $j, k \in S_N$. Denote by X^N the process on S^N with infinite branching rate and migration Q^N. Then, investigate the dynamics of the limit (in the sense of finite-dimensional distributions, say) of $\frac{1}{N} \sum_{k=1}^{N} X^N_{\cdot,k,\,h(t)}$, for an appropriate scaling $h : [0, \infty) \to [0, \infty)$ of time.

This line of argument is inspired by the connection of voter model and Wright-Fisher diffusion. In [CK03] on page 504 it is mentioned that the voter model is the limit of the Wright-Fisher diffusions $(W_k)_{k \in S}$,

$$dW_{k,t} \;=\; (\mathcal{Q}W_{\cdot,t})_k \, dt \;+\; \sqrt{\kappa \, W_{k,t}(1 - W_{k,t})} \, dB_{k,t} \,, \quad k \in \mathbb{Z}^d,$$

as $\kappa \to \infty$. Note that W only consists of one type of particles; the matrix $\mathcal{Q}$ governs the migration. As above, this holds at most in the sense of finite-dimensional distributions. Conversely, the scaled voter model V^N on S_N converges to a Wright-Fisher diffusion on one colony with migration $\mathcal{Q} \equiv 0$. Compare with [Kl99] Satz 1.26, for the following result.

$$\lim_{N \to \infty} P\Big[\Big(\frac{\#\big|\{j \in S_N : V^N_{j,\,Nt} = 1\}\big|}{N} \Big)_{t \geq 0} \in A \Big] \;=\; P\big[(W_{1,t})_{t \geq 0} \in A \big] \,,$$

for any Borel subset A of $D_{[0,1]}[0, \infty)$, provided the initial conditions satisfy $\lim\limits_{N \to \infty} \frac{\#|\{j \in S_N : V^N_{j,0} = 1\}|}{N} \in [0, 1]$. There is an even finer result, which describes the clustering of the voter model in $\mathbb{Z}^2$; see [Kl99] Section 1.9 or [CG86] for more.

Appendix

A.1 Generator for one colony

In this section we list the computations for the generator presented at the beginning of Section 2.1. Recall the definition of the DP-distribution in Equation (1.5) at the end of Section 1.1. We set

$$\varphi_1(\, x_1 \,|\, u, v\,) \;:=\; \frac{4}{\pi}\, \frac{u\, v\, x_1}{4\, u^2\, v^2 \;+\; \big(x_1^2 + v^2 - u^2\big)^2}\,,$$

$$\varphi_2(\, x_2 \,|\, u, v\,) \;:=\; \frac{4}{\pi}\, \frac{u\, v\, x_2}{4\, u^2\, v^2 \;+\; \big(x_2^2 + u^2 - v^2\big)^2}\,,$$

for $u, v \;>\; 0$. The sum

$$\varphi(x_1, x_2 \,|\, u, v\,) \;:=\; \varphi_1(\, x_1 \,|\, u, v\,)\;\; {}_{[0,\infty)\times\{0\}}(x_1, x_2) \;+\; \varphi_2(\, x_2 \,|\, u, v\,)\;\; {}_{\{0\}\times[0,\infty)}(x_1, x_2)$$

gives the density of the DP-distribution with parameter (u, v) w.r.t. Lebesgue measure, see Lemma 1.1 Now, let $\acute{u}, \acute{v} \;\in\; \mathbb{R},\; \acute{v} \geq 0$. Let $f : L \longrightarrow \mathbb{R}$ be a bounded, real valued, appropriate smooth function and let (u_∞, v_∞) be a random variable with distribution $DP_{(u+\acute{u}t, \acute{v}t)}$. Consider

$$\frac{1}{t}\left\{ E_{u+\acute{u}t,\acute{v}t}\Big[f\big(u_\infty, v_\infty\big)\Big] \;-\; f(u,0)\right\}$$

$$= \;\frac{1}{t}\bigg\{\int f(x_1,x_2)\,\varphi(x_1,x_2\,|\,u+\acute{u}t,\acute{v}t)\,dx_1dx_2 \;-\; f(u,0)\bigg\}$$

$$= \;\frac{1}{t}\int\Big[f(x_1,x_2)\;-\;f(u,0)\Big]\varphi(x_1,x_2\,|\,u+\acute{u}t,\acute{v}t)\,dx_1dx_2$$

$$= \;\frac{1}{t}\int_0^\infty\Big[f(x_1,0)\;-\;f(u,0)\Big]\varphi_1(x_1\,|\,u+\acute{u}t,\acute{v}t)\,dx_1$$

$$+ \quad \frac{1}{t}\int_0^\infty\Big[f(0,x_2)\;-\;f(u,0)\Big]\varphi_2(x_2\,|\,u+\acute{u}t,\acute{v}t)\,dx_2$$

$$= \;\frac{1}{t}\int_0^\infty\Big[f(x_1,0)\;-\;f(u,0)\;-\;(x_1-u)\,\partial_1 f(u,0)\Big]\varphi_1(x_1\,|\,u+\acute{u}t,\acute{v}t)\,dx_1 \qquad (*1)$$

$$+ \quad \frac{1}{t}\int_0^\infty (x_1-u)\,\partial_1 f(u,0)\,\varphi_1(x_1\,|\,u+\acute{u}t,\acute{v}t)\,dx_1 \qquad (*2)$$

$$+ \quad \frac{1}{t}\int_0^\infty\Big[f(0,x_2)\;-\;f(u,0)\;+\;u\,\partial_1 f(u,0)\Big]\varphi_2(x_2\,|\,u+\acute{u}t,\acute{v}t)\,dx_2 \qquad (*3)$$

$$- \quad \frac{1}{t}\int_0^\infty u\,\partial_1 f(u,0)\,\varphi_2(x_2\,|\,u+\acute{u}t,\acute{v}t)\,dx_2\,. \qquad (*4)$$

Since we added the term $(x_1 - u)\,\partial_1 f(u,0)$ in the last line above, i.e., we substracted the mean, we can compute the limits of any of the four expressions (*1) to (*4) as t tends to zero, for an appropriate class of functions, for example, we choose $f \in \mathcal{C}_b^2(L)$.

A.1 Lemma *Let $f \in \mathcal{C}_b^2(L)$ and assume $\acute{v} > 0$.*

(a)
$$\frac{1}{t}\int_0^\infty \Big[f(x_1,0) - f(u,0) - (x_1-u)\,\partial_1 f(u,0)\Big]\varphi_1(x_1\,|\,u+\acute{u}t, \acute{v}t)\,dx_1$$
$$\underset{t\searrow 0}{\longrightarrow} \int_0^\infty \Big[f(x_1,0) - f(u,0) - (x_1-u)\,\partial_1 f(u,0)\Big]\,\overline{\varphi}_1(x_1\,|\,u,0\,;\,\acute{u},\acute{v}\,)\,dx_1\,,$$
where $\overline{\varphi}_1(x_1\,|\,u,0\,;\,\acute{u},\acute{v}\,) = \frac{4}{\pi}\,\frac{u\,\acute{v}\,x_1}{(x_1-u)^2\,(x_1+u)^2}$.

(b)
$$\frac{1}{t}\int_0^\infty (x_1-u)\,\partial_1 f(u,0)\,\varphi_1(x_1\,|\,u+\acute{u}t,\acute{v}t)\,dx_1 \quad \underset{t\searrow 0}{\longrightarrow} \quad \Big(\acute{u} + \tfrac{2}{\pi}\,\acute{v}\Big)\,\partial_1 f(u,0)\,.$$

(c)
$$\frac{1}{t}\int_0^\infty \Big[f(0,x_2) - f(u,0) + u\,\partial_1 f(u,0)\Big]\varphi_2(x_2\,|\,u+\acute{u}t,\acute{v}t)\,dx_2$$
$$\underset{t\searrow 0}{\longrightarrow} \int_0^\infty \Big[f(0,x_2) - f(u,0) + u\,\partial_1 f(u,0)\Big]\,\overline{\varphi}_2(x_2\,|\,u,0\,;\,\acute{u},\acute{v}\,)\,dx_2\,,$$
where $\overline{\varphi}_2(x_2\,|\,u,0\,;\,\acute{u},\acute{v}\,) = \frac{4}{\pi}\,\frac{u\,\acute{v}\,x_2}{(x_2^2+u^2)^2}$.

(d)
$$\frac{1}{t}\int_0^\infty u\,\partial_1 f(u,0)\,\varphi_1(x_2\,|\,u+\acute{u}t,\acute{v}t)\,dx_2 \quad \underset{t\searrow 0}{\longrightarrow} \quad \tfrac{2}{\pi}\,\acute{v}\,\partial_1 f(u,0)\,.$$

Proof. First, we will have a look at the densities φ_1 and φ_2. For any non-negative fix x_1 and x_2, respectively, we obviously have

$$\begin{aligned}
\lim_{t\searrow 0}\frac{1}{t}\,\varphi_1(x_1\,|\,u+\acute{u}t,\acute{v}t) &= \lim_{t\searrow 0}\frac{1}{t}\,\frac{4}{\pi}\,\frac{(u+\acute{u}t)\,\acute{v}t\,x_1}{4\,(u+\acute{u}t)^2\,(\acute{v}t)^2 + \big(x_1^2+(\acute{v}t)^2-(u+\acute{u}t)^2\big)^2}\\
&= \frac{4}{\pi}\,\frac{u\,\acute{v}\,x_1}{\big(x_1^2 - u^2\big)^2} = \overline{\varphi}_1(x_1\,|\,u,0\,;\,\acute{u},\acute{v}\,)\,,\\
\lim_{t\searrow 0}\frac{1}{t}\,\varphi_2(x_2\,|\,u+\acute{u}t,\acute{v}t) &= \lim_{t\searrow 0}\frac{1}{t}\,\frac{4}{\pi}\,\frac{(u+\acute{u}t)\,\acute{v}t\,x_2}{4\,(u+\acute{u}t)^2\,(\acute{v}t)^2 + \big(x_2^2+(u+\acute{u}t)^2-(\acute{v}t)^2\big)^2}\\
&= \frac{4}{\pi}\,\frac{u\,\acute{v}\,x_2}{\big(x_2^2 + u^2\big)^2} = \overline{\varphi}_2(x_2\,|\,u,0\,;\,\acute{u},\acute{v}\,)\,.
\end{aligned}$$

We first prove (d):

$$\begin{aligned}
\lim_{t\searrow 0}\frac{1}{t}\int_0^\infty u\,\partial_1 f(u,0)\,\varphi_2(x_2\,|\,u+\acute{u}t,\acute{v}t)\,dx_2 &= \partial_1 f(u,0)\,\frac{4\,u^2\,\acute{v}}{\pi}\int_0^\infty \frac{x_2}{\big(x_2^2+u^2\big)^2}\,dx_2\\
&= \partial_1 f(u,0)\,\frac{4\,u^2\,\acute{v}}{\pi}\,\frac{1}{2\,u^2}\\
&= \tfrac{2}{\pi}\,\acute{v}\,\partial_1 f(u,0)\,.
\end{aligned}$$

Note that it is allowed to interchange limit and integration. The functions $\varphi_2(\,x_2\,|\,u+\acute{u}t,\acute{v}t\,)$ and $\overline{\varphi}_2(x_2\,|\,u,0\,;\,\acute{u},\acute{v}\,)$ do not have any singularities for $x_2 \in [0,\infty)$ and $0 \le t < \varepsilon = \varepsilon(u)$, they are uniformly bounded, and decay like x_2^{-4} for $x_2 \to \infty$; compare with [BaF91] p.392 et seq. The same arguments are valid for (c) since the bracket $\Big[f(0,x_2) \,-\, f(u,0) \,+\, u\,\partial_1 f(u,0)\Big]$ under the integral is bounded.

Assertions (a) and (b) are more involved. At first, we modify the integral in (b).

$$\begin{aligned}
&\frac{1}{t}\int (x_1-u)\,\frac{4}{\pi}\,\frac{(u+\acute{u}t)\,\acute{v}t\,x_1}{4\,(u+\acute{u}t)^2\,(\acute{v}t)^2 \,+\, \big(x_1^2+(\acute{v}t)^2-(u+\acute{u}t)^2\big)^2}\;{}_{[0,\infty)\times\{0\}}(x_1,x_2)\,dx_1dx_2 \\
&\quad= \frac{4}{\pi}\int_0^\infty \frac{(u+\acute{u}t)\,\acute{v}\,x_1(x_1-u)}{4\,(u+\acute{u}t)^2\,(\acute{v}t)^2 \,+\, \big(x_1^2+(\acute{v}t)^2-(u+\acute{u}t)^2\big)^2}\,dx_1 \\
&\quad= \frac{4\,(u+\acute{u}t)\,\acute{v}}{\pi}\int_0^\infty \frac{x^2}{x^4+bx^2+c}\,dx \;-\; \frac{4\,u\,(u+\acute{u}t)\,\acute{v}}{\pi}\int_0^\infty \frac{x}{x^4+bx^2+c}\,dx\,,
\end{aligned}$$

where

$$\begin{aligned}
b &:= 2(\acute{v}t)^2 \,-\, 2(u+\acute{u}t)^2 \\
c &:= (\acute{v}t)^4+(u+\acute{u}t)^4+2(\acute{v}t)^2(u+\acute{u}t)^2 \;=\; \Big((\acute{v}t)^2 \,+\, (u+\acute{u}t)^2\Big)^2.
\end{aligned}$$

Set $d := 4c-b^2 \;=\; 16\,(u+\acute{u}t)^2\,(\acute{v}t)^2$. Clearly, for all sufficiently small $t>0$ we have $b<0$ and $c,d>0$. We treat both integrals above separately.

$$\begin{aligned}
\int_0^\infty \frac{x^2}{x^4+bx^2+c}\,dx &= \frac{1}{c}\int_0^\infty \frac{x^2}{\big(\frac{x}{c^{1/4}}\big)^4 \,+\, \frac{b}{\sqrt{c}}\,\big(\frac{x}{c^{1/4}}\big)^2 \,+\, 1}\,dx \\
&= c^{-1/4}\int_0^\infty \frac{y^2}{y^4\,+\,\alpha\,y^2\,+\,1}\,dy \\
&= c^{-1/4}\frac{\pi\sqrt{2}}{2\sqrt{2\alpha\,+\,4}}\,,
\end{aligned}$$

where $\frac{b}{\sqrt{c}} =: \alpha \in (-2,0)$, for $t>0$ small enough; and

$$\begin{aligned}
\int_0^\infty \frac{x}{x^4+bx^2+c}\,dx &= \frac{1}{2}\int_0^\infty \frac{1}{y^2+by+c}\,dy \\
&= \frac{1}{2}\,\frac{2}{\sqrt{d}}\,\arctan\left[\frac{2y+b}{\sqrt{d}}\right]\Bigg|_{y=0}^{\infty} \\
&= \frac{1}{\sqrt{d}}\left(\frac{\pi}{2}\,-\,\arctan\left[\frac{b}{\sqrt{d}}\right]\right).
\end{aligned}$$

For putting both results together note that $\arctan\Big[\frac{b}{\sqrt{d}}\Big] \;=\; \operatorname{arccot}\Big[\frac{\sqrt{d}}{b}\Big] \,-\, \pi$, since we assume $b<0$. Then

$$\begin{aligned}
&\frac{\sqrt{d}}{\pi\,t}\,c^{-1/4}\frac{\pi\sqrt{2}}{2\sqrt{2\alpha\,+\,4}} \;-\; \frac{u\sqrt{d}}{\pi\,t}\,\frac{1}{\sqrt{d}}\left(\frac{\pi}{2}\,-\,\arctan\left[\frac{b}{\sqrt{d}}\right]\right) \\
&\quad= \frac{\sqrt{d}}{2t\,c^{1/4}\sqrt{\frac{b}{\sqrt{c}}+2}} \;+\; \frac{u}{\pi\,t}\left(\operatorname{arccot}\left[\frac{\sqrt{d}}{b}\right] \,-\, \frac{\pi}{2}\right) \;-\; \frac{u}{t} \\
&\quad= \frac{\sqrt{d}\;-\;2\,u\sqrt{b+2\sqrt{c}}}{2\,t\sqrt{b+2\sqrt{c}}} \;+\; \frac{u}{\pi}\,\frac{\operatorname{arccot}\Big[\frac{\sqrt{d}}{b}\Big] \,-\, \frac{\pi}{2}}{t}\,.
\end{aligned}$$

For the first term of the r.h.s. above note that

$$\begin{aligned}\sqrt{b+2\sqrt{c}} \;&=\; \sqrt{2(\acute{v}t)^2 \;-\; 2(u+\acute{u}t)^2 \;+\; 2\big((\acute{v}t)^2 \;+\; (u+\acute{u}t)^2\big)}\\ &=\; \sqrt{4(\acute{v}t)^2} \;=\; 2\acute{v}t\,,\end{aligned}$$

hence, $\sqrt{d} \;-\; 2\,u\,\sqrt{b+2\sqrt{c}} \;=\; 4(u+\acute{u}t)(\acute{v}t) \;-\; 2\,u\,2\acute{v}t \;=\; 4\acute{u}\acute{v}\,t^2$. That yields for all $t > 0$

$$\frac{\sqrt{d} \;-\; 2\,u\,\sqrt{b+2\sqrt{c}}}{2\,t\,\sqrt{b+2\sqrt{c}}} \;=\; \frac{4\acute{u}\acute{v}\,t^2}{2\,t\;2\acute{v}t} \;=\; \acute{u}.$$

To simplify the second term set

$$G(t) \;:=\; \operatorname{arccot}\left[\frac{\sqrt{d}}{b}\right].$$

Then we have

$$\begin{aligned}
b = b(t) \;&=\; 2(\acute{v}t)^2 \;-\; 2(u+\acute{u}t)^2\,, & b(0) \;&=\; -2u^2\,,\\
\frac{d}{dt}\,b = b'(t) \;&=\; 4\acute{v}^2\,t \;-\; 4\acute{u}(u+\acute{u}t)\,, & b'(0) \;&=\; -4\acute{u}u\,,\\
d = d(t) \;&=\; 16\,(u+\acute{u}t)^2\,(\acute{v}t)^2\,, & d(0) \;&=\; 0\,,\\
\frac{d}{dt}\,d = d'(t) \;&=\; 32\Big[\acute{u}(u+\acute{u}t)(\acute{v}t)^2 \;+\; (u+\acute{u}t)^2(\acute{v}t)\,\acute{v}\Big]\,,
\end{aligned}$$

and $G(0) = \frac{\pi}{2}$. We will need

$$\frac{d'\,b}{2\sqrt{d}} \;=\; \frac{32\Big[\acute{u}(u+\acute{u}t)(\acute{v}t)^2 \;+\; (u+\acute{u}t)^2(\acute{v}t)\,\acute{v}\Big]\,b(t)}{8\,(u+\acute{u}t)(\acute{v}t)} \;=\; \Big(4\,\acute{u}\acute{v}\,t \;+\; 4\,\acute{v}\,(u+\acute{u}t)\Big)\,b(t).$$

Then we have

$$\frac{d}{dt}G(t) \;=\; -\,\frac{1}{1 \;+\; \left[\frac{\sqrt{d}}{b}\right]^2} \cdot \frac{\frac{1}{2\sqrt{d}}\,d'\,b \;-\; b'\sqrt{d}}{b^2} \;=\; \frac{1}{b^2 \;+\; d}\left(b'\,\sqrt{d} \;-\; \frac{d'\,b}{2\sqrt{d}}\right),$$

and, hence,

$$G'(0) \;=\; \frac{1}{4\,u^4}\Big(0 \;-\; 4\,u\,\acute{v}\,b(0)\Big) \;=\; \frac{2\,\acute{v}}{u}\,.$$

Finally we can conclude

$$\begin{aligned}&\lim_{t\searrow 0}\frac{1}{t}\int (x_1-u)\,\frac{4}{\pi}\,\frac{(u+\acute{u}t)\,\acute{v}t\,x_1}{4\,(u+\acute{u}t)^2\,(\acute{v}t)^2 \;+\; \big(x_1^2+(\acute{v}t)^2-(u+\acute{u}t)^2\big)^2}\;\;{}_{[0,\infty)\times\{0\}}(x_1,x_2)\,dx_1dx_2\\ &\qquad=\quad \acute{u} \quad+\quad \frac{2}{\pi}\,\acute{v}\end{aligned}$$

which yields (b).

To justify (a) recall that

$$\overline{\varphi}_1(x_1\,|\,u,0\,;\,\acute{u},\acute{v}\,) \;=\; \frac{4}{\pi}\,\frac{u\,\acute{v}\,x_1}{\big(x_1^2 \;-\; u^2\big)^2} \;=\; \frac{4}{\pi}\,\frac{u\,\acute{v}\,x_1}{\big(x_1 \;-\; u\big)^2\big(x_1 \;+\; u\big)^2}\,.$$

Note that $\overline{\varphi}_1$ has a singularity of order 2 at $x_1 = u$. But this is healed by the integrand in brackets. For x_1 in a neighbourhood of u we have

$$f(x_1,0) \;-\; f(u,0) \;-\; (x_1-u)\,\partial_1 f(u,0) \;=\; (x_1-u)^2\,\partial_{11} f(\xi)$$

for some $\xi = \xi(x_1)$. This yields a finite integral over the interval $[0, 2u+1]$, say. For $x_1 \in [2u+1\,,\,\infty)$ the function $\overline{\varphi}_1$ decreases like x_1^{-4} while $f(x_1,0) - f(u,0) - (x_1-u)\,\partial_1 f(u,0)$ only grows linearly. Then, just like in (c) we obtain (a), compare with [BaF91] p.392 et seq. □

The careful reader might have noticed that it is enough to assume $f \in \mathcal{C}_b^{1+\epsilon}$, where $\epsilon > 0$, for proving the lemma above. We additionally assumed $\acute{v} > 0$. The case $\acute{v} = 0$ is much simpler. For $\acute{v} = 0$ the DP-distribution degenerates to $\delta_{(u+\acute{u}t,0)}$. Hence, $\mathcal{A}_{\acute{u},0} f(u,0) = \acute{u}\,\partial_1 f(u,0)$ and the expressions derived are still valid.

By putting all four limits together, we can summarise the disscusion as follows. We obtain a pregenerator $\mathcal{A}_{\acute{u},\acute{v}}$ depending on the input data $\acute{u}$ and $\acute{v}$.

A.2 Corollary *Set*

$$h(y) \;:=\; \frac{4}{\pi}\frac{y}{(y-1)^2(y+1)^2} \qquad \text{and} \qquad g(y) \;:=\; \frac{4}{\pi}\frac{y}{(y^2+1)^2}\,.$$

Let $u > 0$. Then, for $\acute{u} \in \mathbb{R}$ and $\acute{v} \geq 0$, we have

$$\begin{aligned}
\mathcal{A}_{\acute{u},\acute{v}} f(u,0) \;&=\; \lim_{t\searrow 0} \frac{E_{u+\acute{u}t,\acute{v}t}\Big[f\big(u_\infty, v_\infty\big)\Big] \;-\; f(u,0)}{t} \\
&=\; \int_0^\infty \Big[f(x_1,0) \;-\; f(u,0) \;-\; (x_1-u)\,\partial_1 f(u,0)\Big]\,\overline{\varphi}_1(x_1\,|\,u,0\,;\,\acute{u},\acute{v}\,)\,dx_1 \\
&\qquad +\; \int_0^\infty \Big[f(0,x_2) \;-\; f(u,0) \;+\; u\,\partial_1 f(u,0)\Big]\,\overline{\varphi}_2(x_2\,|\,u,0\,;\,\acute{u},\acute{v}\,)\,dx_2 \\
&\qquad +\; \acute{u}\cdot\partial_1 f(u,0) \\
&=\; \frac{\acute{v}}{u^2}\int_0^\infty \Big[f(x_1,0) \;-\; f(u,0) \;-\; (x_1-u)\,\partial_1 f(u,0)\Big]\,h\Big(\frac{x_1}{u}\Big)\,dx_1 \\
&\qquad +\; \frac{\acute{v}}{u^2}\int_0^\infty \Big[f(0,x_2) \;-\; f(u,0) \;+\; u\,\partial_1 f(u,0)\Big]\,g\Big(\frac{x_2}{u}\Big)\,dx_2 \\
&\qquad +\; \acute{u}\cdot\partial_1 f(u,0)\,.
\end{aligned}$$

By symmetry, on $\{0\}\times(0,\infty)$ we have ($v > 0$, $\acute{u} \geq 0$, $\acute{v} \in \mathbb{R}$)

$$\begin{aligned}
\mathcal{A}_{\acute{u},\acute{v}} f(0,v) \;&=\; \frac{\acute{u}}{v^2}\int_0^\infty \Big[f(x_1,0) \;-\; f(0,v) \;+\; v\,\partial_2 f(0,v)\Big]\,g\Big(\frac{x_1}{v}\Big)\,dx_1 \\
&\qquad +\; \frac{\acute{u}}{v^2}\int_0^\infty \Big[f(0,x_2) \;-\; f(0,v) \;-\; (x_2-v)\,\partial_2 f(0,v)\Big]\,h\Big(\frac{x_2}{v}\Big)\,dx_2 \\
&\qquad +\; \acute{v}\cdot\partial_2 f(0,v)\,.
\end{aligned}$$

At the origin we have to adjust the above argument.

A.3 Lemma *Let $f \in \mathcal{C}_b^2(L)$ and $(\acute{u}, \acute{v}) \in [0,\infty)^2$. Then*

$$\mathcal{A}_{\acute{u},\acute{v}} f(0,0) \;=\; \acute{u} \cdot \partial_1 f(0,0) \;+\; \acute{v} \cdot \partial_2 f(0,0)\,.$$

Proof. Note that we here have to consider

$$\begin{aligned}
\mathcal{A}_{\acute{u},\acute{v}} f(0,0) \;&=\; \lim_{t\searrow 0} \frac{1}{t}\left\{ E_{\acute{u}t,\acute{v}t}\Big[f\big(u_\infty, v_\infty\big)\Big] \;-\; f(u,0)\right\}\\
&= \lim_{t\searrow 0} \frac{1}{t}\int_0^\infty \Big[f(x_1,0) \;-\; f(0,0) \;-\; x_1\,\partial_1 f(0,0)\Big]\varphi_1(x_1 \,|\, \acute{u}t, \acute{v}t)\,dx_1\\
&\qquad + \qquad \lim_{t\searrow 0} \frac{1}{t}\int_0^\infty x_1\,\partial_1 f(0,0)\,\varphi_1(x_1 \,|\, \acute{u}t, \acute{v}t)\,dx_1\\
&\quad + \quad \lim_{t\searrow 0} \frac{1}{t}\int_0^\infty \Big[f(0,x_2) \;-\; f(0,0) \;-\; x_2\,\partial_2 f(0,0)\Big]\varphi_2(x_2 \,|\, \acute{u}t, \acute{v}t)\,dx_2\\
&\qquad + \qquad \lim_{t\searrow 0} \frac{1}{t}\int_0^\infty x_2\,\partial_2 f(0,0)\,\varphi_2(x_2 \,|\, \acute{u}t, \acute{v}t)\,dx_2\,.
\end{aligned}$$

Observe that $\varphi_1(y \,|\, \acute{u}t, \acute{v}t) \;=\; \varphi_2(y \,|\, \acute{v}t, \acute{u}t)$. Hence it is enough to consider the integrals with respect to the first component x_1. The two other integrals are treated similarly. We follow the arguments in the proof of Lemma A.1. Let $\acute{u}, \acute{v} > 0$.

$$\begin{aligned}
\frac{1}{t}\int_0^\infty x_1\,\partial_1 f(0,0)\,\varphi_1(x_1 \,|\, \acute{u}t, \acute{v}t)\,dx_1 \;&=\; \partial_1 f(0,0)\,\frac{4\acute{u}\acute{v}\,t}{\pi}\int_0^\infty \frac{x^2}{x^4 \,+\, b\,x^2 \,+\, c}\,dx\\
&=\; \partial_1 f(0,0)\,\frac{4\acute{u}\acute{v}\,t}{\pi}c^{-1/4}\int_0^\infty \frac{y^2}{y^4 \,+\, \alpha y^2 \,+\, 1}dy\\
&=\; \partial_1 f(0,0)\,\frac{4\acute{u}\acute{v}\,t}{\pi}c^{-1/4}\frac{\pi\,\sqrt{2}}{2\sqrt{2\alpha+4}}\\
&=\; \partial_1 f(0,0)\,\frac{2\acute{u}\acute{v}\,t}{\sqrt{b+2\sqrt{c}}} \;=\; \acute{u}\,\cdot\partial_1 f(0,0),
\end{aligned}$$

where $b \,=\, 2(\acute{v}\,t)^2 \,-\, 2(\acute{u}\,t)^2\,,\; c \,=\, \big((\acute{v}\,t)^2 + (\acute{u}\,t)^2\big)^2$. In particular, note that $b + 2\sqrt{c} \;>\; 0$ and $\alpha = \frac{b}{\sqrt{c}} \in (-2,2)$. For the other integral we write

$$\begin{aligned}
&\frac{1}{t}\int_0^\infty \Big[f(x_1,0) \;-\; f(0,0) \;-\; x_1\,\partial_1 f(0,0)\Big]\varphi_1(x_1 \,|\, \acute{u}t, \acute{v}t)\,dx_1\\
&\quad= \frac{1}{t}\int_0^1 \Big[f(x_1,0) \;-\; f(0,0) \;-\; x_1\,\partial_1 f(0,0)\Big]\varphi_1(x_1 \,|\, \acute{u}t, \acute{v}t)\,dx_1\\
&\qquad + \frac{1}{t}\int_1^\infty \Big[f(x_1,0) \;-\; f(0,0) \;-\; x_1\,\partial_1 f(0,0)\Big]\varphi_1(x_1 \,|\, \acute{u}t, \acute{v}t)\,dx_1\,.
\end{aligned}$$

For fixed $x_1 > 0$ we have $\frac{1}{t}\varphi_1(x_1 \,|\, \acute{u}t, \acute{v}t) \;=\; t\,\frac{4}{\pi}\frac{\acute{u}\acute{v}\,x_1}{4\acute{u}^2\acute{v}^2\,t^4 + \big(x_1^2 + (\acute{v}^2 - \acute{u}^2)t^2\big)^2}$. This forces the integral over $[1,\infty)$ to vanish as $t \searrow 0$. Calamities arise only for $x_1 = 0$. Note that for x_1 in a neighbourhood of 0, we have $f(x_1,0) \,-\, f(0,0) \,-\, x_1\,\partial_1 f(0,0) \;=\; x_1^2\,\partial_{11} f(\xi)$, where $\xi = \xi(x_1)$. For simplicity we assume this neighbourhood contains the intervall $[0,1]$ (otherwise we might integrate over intervalls $[0,\varepsilon]$ and $[\varepsilon,\infty)$ with $\varepsilon = \varepsilon(f)$ for the chosen function f). Since $\partial_{11} f$ is bounded on some compact set containing $[0,1]$ we can estimate

$$\begin{aligned}
&\left| \frac{1}{t} \int_0^1 \Big[f(x_1,0) \,-\, f(0,0) \,-\, x_1\, \partial_1 f(0,0) \Big] \varphi_1(x_1 \,|\, \acute{u}t, \acute{v}t)\, dx_1 \right| \\
&\qquad \leq\, K\, t \int_0^1 \frac{x^3}{4\acute{u}^2 \acute{v}^2\, t^4 \,+\, \big(x_1^2 \,+\, (\acute{v}^2 - \acute{u}^2) t^2 \big)^2}\, dx \\
&\qquad \leq\, K\, t \int_0^1 \frac{x^3}{x^4 + b x^2 + c}\, dx\,,
\end{aligned}$$

for a constant $K > 0$; recall $b = 2(\acute{v}\, t)^2 \,-\, 2(\acute{u}\, t)^2$ and $c = \big((\acute{v}\, t)^2 + (\acute{u}\, t)^2 \big)^2$. We continue

$$\begin{aligned}
&= \frac{K}{4}\, t \int_0^1 \frac{4x^3 \,+\, 2bx}{x^4 + bx^2 + c}\, dx \quad - \quad \frac{K}{2}\, t\, b \int_0^1 \frac{x}{x^4 + bx^2 + c}\, dx \\
&= \frac{K}{4}\, t\, \ln\big(x^4 + bx^2 + c \big) \Big|_{x=0}^1 \quad - \quad \frac{K}{4}\, t\, b \int_0^1 \frac{1}{y^2 + by + c}\, dy \\
&= \frac{K}{4}\, t\, \ln\big(1 + b + c \big) \,-\, \frac{K}{4}\, t\, \ln(\, c\,) \quad - \quad \frac{K}{4}\, t\, b\, \frac{2}{\sqrt{d}} \arctan\left[\frac{2y + b}{\sqrt{d}} \right] \Bigg|_{y=0}^1 \\
&= \frac{K}{4}\, t\, \ln\big(1 + b + c \big) \,-\, \frac{K}{4}\, t\, \ln(\, c\,) \quad - \quad \frac{K\, t\, b}{2\sqrt{d}} \arctan\left[\frac{2 + b}{\sqrt{d}} \right] \,+\, \frac{K\, t\, b}{2\sqrt{d}} \arctan\left[\frac{b}{\sqrt{d}} \right],
\end{aligned}$$

where $d := 4c - b^2 \,=\, 16\, (\acute{u}t)^2\, (\acute{v}t)^2 \,>0$. Now check that

$$\begin{aligned}
&\lim_{t \searrow 0} t\, \ln\big(1 + b + c \big) \,=\, 0\,, \\
&\lim_{t \searrow 0} t\, \ln(\, c\,) \,=\, \lim_{t \searrow 0} 2\, t\, \ln\big((\acute{v}\, t)^2 + (\acute{u}\, t)^2 \big) \,=\, 0\,, \\
&\lim_{t \searrow 0} \frac{K\, t\, b}{2\sqrt{d}} \arctan\left[\frac{2 + b}{\sqrt{d}} \right] \,=\, 0\,, \\
&\lim_{t \searrow 0} \frac{K\, t\, b}{2\sqrt{d}} \arctan\left[\frac{b}{\sqrt{d}} \right] \,=\, 0\,.
\end{aligned}$$

Altogether we have shown

$$\lim_{t \searrow 0} \frac{1}{t} \int_0^\infty \Big[f(x_1,0) \,-\, f(0,0) \,-\, x_1\, \partial_1 f(0,0) \Big] \varphi_1(x_1 \,|\, \acute{u}t, \acute{v}t)\, dx_1 \,=\, 0\,.$$

Thus, for $\mathcal{A}_{\acute{u},\acute{v}} f(0,0)$ only the drift terms remain.

At the end, note that we assumed $\acute{u}, \acute{v} > 0$. In the case $\acute{u} = 0$ or $\acute{v} = 0$ the assertion is obvious. □

At last, reflecting on the results above, we observe some sort of linearity in the input data $\acute{u}$ and $\acute{v}$. Hence, the following decompostion.

A.4 Corollary *In any of the cases considered above, we have*

$$\mathcal{A}_{\acute{u},\acute{v}} f \,=\, \mathcal{A}_{\acute{u},0} f \,+\, \mathcal{A}_{0,\acute{v}} f \,=\, \acute{u}\, \mathcal{A}_{1,0} f \,+\, \acute{v}\, \mathcal{A}_{0,1} f\,.$$

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