Deriving Deligne-Mumford Stacks with Perfect Obstruction Theories

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Zusammenfassung

Schnitttheorie auf Modulräumen hat in vielen Bereichen der enumerativen Geometrie zu immensen Fortschritten geführt. Für einige wichtige Probleme, allen voran das Zählen von stabilen Abbildungen und das Zählen von stabilen Garben, ist es notwendig, statt mit der Fundamentalklasse des Modulraums mit einer virtuellen Fundamentalklasse zu arbeiten. Die entscheidende Bedingung für die Existenz einer solchen virtuellen Fundamentalklasse ist, dass ein 2-Term Komplex die Deformationstheorie des Modulraums kontrolliert.

Für Modulräume mit dieser Eigenschaft hat Kontsevich 1994 vermutet, dass es derivierte Versionen dieser Modulräume gibt. Ein weiteres Indiz für die Existenz dieser Räume stammt aus der Theorie der derivierten algebraischen Geometrie. Dort wird vermutet, dass für jedes Paar bestehend aus einem Modulraum und einem Komplex, der die Deformationen des Modulraums kontrolliert, unter gewissen Zusatzbedingungen ein derivierter Modulraum existiert, der den gewählten Komplex als Kotangentialkomplex besitzt. In dieser Arbeit wird eine Form dieser nötigen zusätzlichen Bedingungen formuliert. Darüber hinaus wird gezeigt, dass diese Bedingungen für jeden Modulraum, dessen Deformationstheorie durch einen 2-Term Komplex kontrolliert wird, erfüllt sind. Schließlich werden die derivierten Modulräume mit den vorgegebenen Eigenschaften konstruiert.

Summary

Intersection theory on moduli spaces has lead to immense progress in certain areas of enumerative geometry. For some important areas, most notably counting stable maps and counting stable sheaves, it is important to work with a virtual fundamental class instead of the usual fundamental class of the moduli space. The crucial prerequisite for the existence of such a class is a two-term complex controlling deformations of the moduli space.

Kontsevich conjectured in 1994 that there should exist derived version of spaces with this specific property. Another hint at the existence of these spaces comes from derived algebraic geometry. It is expected that for every pair of a space and a complex controlling deformations of the space their exists, under some additional hypothesis, a derived version of the space having the chosen complex as cotangent complex. In this thesis one version of these additional hypothesis is identified. We then show that every space admitting a two-term complex controlling deformations satisfies these hypothesis, and we finally construct the derived spaces.

Contents

Chapter 1

Introduction

In the modern take on enumerative problems, intersection theory on moduli spaces has been the most successful approach. It has put a firm ground to many classical arguments and has led to new results. One of the greatest success stories has been the development of Gromov-Witten invariants and quantum cohomology, showing that the modular operad acts on the cohomology of any smooth projective variety. The moduli space used to produce Gromov-Witten invariants is the moduli space of stable maps. Performing intersection theory on this moduli space turns out to be a very difficult task though. In general, this space is very singular. But it gets even worse. The definition of Gromov-Witten invariants originally comes from symplectic geometry. For nice classes of symplectic manifolds, the moduli space used in the symplectic world that corresponds to the moduli space of stable maps is smooth of a certain dimension. Also, deformation theory gives an expected dimension for the moduli space of stable maps, which coincides with the dimension of the moduli space used in symplectic geometry. So to produce meaningful invariants we need the moduli spaces of stable maps to have this dimension. But it turns out that the moduli space of stable maps can have larger dimension than we expect both from symplectic geometry and deformation theory. This ruins all hope of using intersection theory on this moduli space to produce meaningful invariants, since the fundamental class will have the wrong degree. So what to do?

As long as a singular space is still a local complete intersection, we have nice formulas for the fundamental class. The cotangent complex of a local complete intersection is a two-term complex of vector bundles, and a beautiful theorem of Verdier says that

$$\tau(\mathcal{O}_X) = \mathrm{td}(\mathbb{L}_X^\vee) \cap [X],$$

which is very close to what we know for smooth varieties. Kontsevich observed that although the moduli space of stable maps can be worse than a local complete intersection, and thus the cotangent complex will have cohomology in infinite degrees, there is a very nice two-term complex of vector bundles that serves as a replacement for the cotangent complex. The key property of this two-term complex is that it controls deformations and obstruction of the moduli space just as the cotangent complex does. To quote from the original source:

> "The general scheme described in 1.4 can be applied in other situations: moduli of vector bundles on algebraic curves and surfaces, moduli of complex structures on surfaces, moduli of vector bundles on Calabi-Yau 3-folds. The common property of all such examples is that the natural complex whose 1-st cohomology group is equivalent to the tangent space to the appropriate moduli space, has trivial cohomology in degrees greater than or equal to 3."
> [Kon95, p.10]

Kontsevich further suggested that using this two-term complex, it should be possible to find local presentations of the moduli space as intersection of submanifolds of an ambient manifold in a coherent way. Gluing these local presentations should lead to a derived version of the moduli space and a virtual structure sheaf.

> "Globally, we can cover $Z = \overline{M}_{g,k}(V, \beta)$ by finitely many open sets: and on each of them we have an equivalence class of representations as intersections of manifolds. It is almost clear that different representations on intersections of open sets are equivalent modulo homotopy and higher homotopies between homotopies on multiple intersections. Unfortunately, we do not know how to formulate all this precisely." [Kon95, p.7]

Locally, this virtual structure sheaf should be the Tor-sheaf of the local presentation as intersection of submanifolds in the ambient manifold. Using this extra data, and writing E for the complex serving as replacement for the ill-behaved cotangent complex, Kontsevich suggested to use the formula

$$\tau(\mathcal{O}_X^{\mathrm{vir}}) = \mathrm{td}(E^\vee) \cap [X]^{\mathrm{vir}}$$

to define a virtual fundamental class having the dimension expected both from symplectic geometry and deformation theory.

This proposal of Kontsevich dates from 1994. In the meantime, a virtual fundamental class solving all problems has been constructed by Li-Tian and Behrend-Fantechi, although with different methods than described by Kontsevich. To solve similar problems on the symplectic side, Fukaya and Oh have introduced the notion of a Kuranishi structure to define virtual chains. Such a structure consists of covering the moduli space with local descriptions as zero-sets of sections of a vector bundle on a smooth space, which is a special case of the description proposed by Kontsevich. Using these descriptions, they develop a theory of virtual chains by locally perturbing the moduli space. Recently, Joyce has suggested that Kuranishi structures could be conveniently encoded using derived geometry. His slogan is that a space with a Kuranishi structure is a "derived orbifold with corners". The long-term goal is to develop a theory in which virtual cycles are formed without locally perturbing the moduli space. The virtual cycle then is just the fundamental cycle of the derived space itself.

A further hint at the existence of derived versions of certain moduli spaces comes from the recent development of the foundations of derived algebraic geometry by Lurie, Toën and Vezzosi. For many examples of moduli spaces it was possible to find derived moduli spaces with the feature that the cotangent complex of the derived moduli space is precisely the complex normally used to study the deformation theory of the classical moduli space. The construction of these examples always consists of finding the right continuation of the moduli functor to simplicial rings. Kontsevich's proposal for the existence of derived moduli spaces does not depend on some modular interpretation of the space, but instead only on the existence of a chosen 2-term complex controlling deformations and obstructions. Although the existence of these spaces seems natural from the viewpoint of Joyce and Kontsevich it comes as a surprise from the abstract viewpoint of derived algebraic geometry. For a general Artin n-stack it is expected that the data consisting of a (underived) stack, a choice of obstruction theory of arbitrary length having a structure of co-differential graded Lie algebroid, and a map of co-differential graded Lie algebroids to the cotangent complex is equivalent to the category of derived Artin n-stacks. In the situation of a Deligne-Mumford stack with a chosen 2-term complex chosen as obstruction theory there is no such apparent structure of co-differential graded Lie algebroid on the obstruction theory. Thus it is a priori by no means clear that derived versions of the moduli space exist. Nevertheless, from the special geometry of Deligne-Mumford stacks with a 2-term obstruction theory as opposed to the general case of Artin stacks with an obstruction theory of arbitrary length, we do have an additional geometric structure which serves as a replacement of the co-differential graded Lie algebroid structure. This special geometric feature is the covering of the moduli space with a system of presentation as zero-sets of sections of vector bundles mentioned above, or in short a Kuranishi strucure. Such a structure is not known to exist in the general case of Artin stacks or for obstruction theories that are longer than 2-term. In fact, using the expectation mentioned above, locally the Kuranishi structure immediately gives rise to a structure of co-differential graded Lie algebroid on the obstruction theory together with a morphism of co-differential graded Lie algebroids to the cotangent complex, since taking the homtopy fiber product of the presentation as zero-set gives a local derived extension of the moduli space.

The main result of this thesis is that for the pair of a Deligne-Mumford stack along with a two-term complex controlling deformations and obstruction derived versions of these moduli spaces indeed always exist. Our method of proof closely follows the suggestions of Kontsevich and Joyce. We show that every such space carries a system of local presentations of zero-sets of sections of vector bundles. These give rise to local derived versions of the moduli space. Pointwise this implies that every two-term complex controlling deformations and obstructions actually admits the structure of a differential graded Lie algebra whose Maurer-Cartan functor recovers the local moduli space. Using the conjectural correspondence above, this actually locally comes from a morphism of co-dg Lie algebroids from the obstruction theory to the cotangent complex. We then go on to glue these local derived extensions to one global derived moduli space.

Hopefully, this is only the start of the story. The precise relationship between the derived moduli space and the virtual fundamental class is not yet absolutely clear. The existence of a derived moduli space that is the derived version of a local complete intersection provides all the necessary data on the classical moduli space to define virtual fundamental classes, either by Kontsevich's formula or by the approach of Li and Tian or Behrend and Fantechi. Whether these coincide is still not settled. The most satisfactory solution would be to define the fundamental class of the derived moduli space appropriately. This would take the virtual out of the virtual fundamental class.

Another line of development is clarifying what happens if the chosen complex is longer than two-term. This situation comes up naturally if one wants to study Gromov-Witten invariants of singular projective varieties or Donaldson-Thomas invariants of four folds. The derived versions of the moduli spaces needed to define these invariants exist, but their geometry is not yet understood. In the case of a two-term complex, one had classical local complete intersections as analogies to find appropriate formulas. Once the complex becomes longer, there no longer exist classical analogues. This is because the cotangent complex of a variety can either be up to two-term or have infinite terms. Thus derived moduli spaces lead to fascinating new classes of spaces and fills the gap between local complete intersections and more singular spaces.

Summary of the chapters

Chapters 2 and 3 give a quick tour through the theory of virtual fundamental classes and derived algebraic geometry. I have not supplied any proofs, partly because I do not know how to prove the results and because they can be found in the original articles. Instead I have tried to give intermediate results that show the logical structure between the results.

Starting from chapter 4 my own work starts. Chapter 4 uses homotopy fiber products of schemes to provide easy examples where intersection theory, virtual fundamental classes and derived algebraic geometry meet. It includes a proof of Kontsevich's formula for the virtual fundamental class for homotopy fiber products. In chapter 5 an algebraic version of Fukaya's and Oh's theory of Kuranishi structures is introduced. We show that every space admitting a perfect obstruction theory also carries such an algebraic Kuranishi structure. Using the Kuranishi algebraic structure, it is immediate to define local derived versions of the moduli space. In chapter 6 we glue these local derived versions to one global derived moduli space. Chapter 7 contains some observations concerning virtual pullbacks and differential graded lie algebras.

Finally, in an appendix we have collected some facts about model categories which we have made use of, including a summary of left Bousfield localization and homotopy pushouts and homotopy fiber products.

Notation and conventions

- A derived extension of a scheme or stack X is denoted by X'. This hopefully makes it clear that there is no functorial correspondence between schemes and derived extensions, as there can be many such extensions.

- Let X be a Deligne-Mumford stack and E a sheaf on X. For an étale morphism $f : U \to X$ we will denote f^*E by $E|_U$.
- The symbol $\mathbb{L}_X$ will always denote the cotangent complex of X.
- All schemes and stacks are assumed to be of finite type and locally Noetherian over a base k.
- We briefly recall some finiteness conditions on the derived category of $\mathcal{O}_X$-modules. A complex is *pseudo-coherent* if it has coherent cohomology. We say that a complex $E \in D(X)$ is of *finite Tor-amplitude in* $[a, b]$ if for any $\mathcal{O}_X$-module M we have that $H^i(M \otimes^{\mathbb{L}} E) = 0$ for $i \notin [a, b]$. The complex $E \in D(X)$ is of *perfect amplitude in* $[a, b]$ if it is pseudo-coherent and of Tor-amplitude in $[a, b]$. A *strictly perfect* complex is a bounded complex of locally free sheaves of finite rank. A complex is perfect of amplitude in $[a, b]$ if and only if it is locally isomorphic in $D(X)$ to a strictly perfect complex concentrated in degrees $[a, b]$.
- We will often use the convention $E_i = (E^{-i})^\vee$ where dual make sense.
- I know this is not standard, but I find it very helpful to distinguish between a locally free sheaf and a vector bundle. In [Ful98], the vector bundle is denoted by a typed letter, and sheaf of sections by a calligraphic letter. Since the use of calligraphic letters for sheaves is forbidden by Mainz tradition, I adopted Grothendieck's notation of writing $\mathbb{V}(E)$ for $\operatorname{Spec} \operatorname{Sym} E$. Thus $\mathbb{V}(E)$ has sheaf of sections $E^\vee$. Unfortunately this collides with the convention used in [LT98], which is exactly opposite.
- Since we will extensively use fiber products we introduce a special notation for the index category corresponding to fiber products. Throughout this text, we let $\mathbf{D} = \{a \xrightarrow{\delta} b \xleftarrow{\epsilon} c\}$, where only non-identity morphisms are shown.

Chapter 2

The Virtual Fundamental Class

To define Gromov-Witten invariants, one has to perform intersection theory on the moduli space of stable maps. Unfortunately, the dimension of this moduli space can be larger than expected from symplectic geometry or deformation theory. To remedy this situation, the theory of virtual fundamental classes was developed by Li and Tian in [LT98] and by Behrend and Fantechi in [BF97]. In this chapter we give a quick tour through their results.

For this chapter we fix an algebraically closed field of characteristic zero. All schemes and stacks are assumed to be of finite type over k, and point will always mean k-point.

2.1 Kuranishi Families

In the theory of Donaldson invariants, the technique of the Kuranishi map is used to obtain smooth moduli spaces of the expected dimension. Typically, there is some complex along with a holomorphic map from one cohomology group to another. The zero-set of this holomorphic map then is the moduli space. For instance, if we take E to be a holomorphic bundle on some complex compact surface Z, we have the complex

$$\Omega^0_Z(\operatorname{End}_0 E) \to \Omega^{0,1}_Z(\operatorname{End}_0 E) \to \Omega^{0,2}_Z(\operatorname{End}_0 E),$$

there is a map $\psi\colon H^1 \to H^2$ defined on a neighborhood of 0 in H^1, and the moduli space of connections is isomorphic to the zero-set of ψ. Precise statements can be found in [DK90, Prop. 6.4.3].

On a formal level, this picture immediately carries over to Algebraic Geometry. We want to study the local geometry at a point x of a moduli space X, which we assume to be a Deligne-Mumford stack.

Definition 2.1. Let $\mathbf{Art}$ denote the category of local Artinian algebras over k with residue field k. The maximal ideal of an Artinian algebra A will be denoted by m_A.

Definition 2.2. Let X be a Deligne-Mumford stack, and x a point of X. The groupoid-valued deformation functor X_x associated to x in X is

$$\begin{aligned} \mathbf{Art} &\longrightarrow \mathbf{Grpds} \\ A &\longmapsto \alpha \in X(A) \text{ such that } \alpha|_{\mathrm{Spec}\,(A/m_A)} = x. \end{aligned}$$

Remark 2.3. The tangent space to X at x is simply $X_x(k[\epsilon]/\epsilon^2)$. This vector space will be denoted by $T^1_{X,x}$.

There exists another vector space that tells us something about the local geometry of X at x. This vector space measures whether we can lift maps from fat points to x to even fatter points. If X is smooth at x, this is of course always possible.

Definition 2.4. ([FGI$^+$05]) Let X_x be the deformation functor introduced above. An *obstruction space* for X_x is a vector space $T^2_{X,x}$ such that for any small extension

$$0 \to I \to B \to A \to 0$$

in $\mathbf{Art}$ there is a functorial exact sequence of groups and sets

$$T^1_{X,x} \otimes_k I \to X_x(B) \to X_x(A) \overset{\mathrm{ob}}{\to} T^2_{X,x} \otimes_k I.$$

Remark 2.5. From the exact sequence it immediately follows that a lifting of an element of $X_x(A)$ to an element of $X_x(B)$ exists if and only if the obstruction map vanishes.

We can now define an algebraic version of the Kuranishi map.

Definition 2.6. Let X be a Deligne-Mumford stack and x a point of X. Fix a tangent-obstruction theory $T^1_{X,x}$ and $T^2_{X,x}$. Let $\widehat{X}_x$ be the formal completion of X at x, and $\widehat{T}^1_0$ the formal completion of the vector space $T^1_{X,x}$ at zero. A *Kuranishi model* for $\widehat{X}_x$ is a morphism

$$\phi\colon \widehat{T}^1_0 \to T^2_{X,x}$$

such that the zero-locus of ϕ is isomorphic to $\widehat{X}_x$.

For the intrinsic choice of an obstruction theory Kuranishi models always exist.

Proposition 2.7. *Let X be a Deligne-Mumford stack and x a point of X. Then the formal completion $\widehat{X}_x$ has a Kuranishi model.*

Proof. By the definition of a Deligne-Mumford stack there exists an étale morphism from an affine scheme $U = \mathrm{Spec}\ A \to X$ which induces an isomorphism of complete local rings $\widehat{\mathcal{O}}_{X,x} \simeq \widehat{\mathcal{O}}_{U,p}$ for some point p of U. It therefore suffices to study the formal completion of U at p.
Let m be the maximal ideal of p in A and $d = \dim(m/m^2)$ the dimension of the tangent space. Then there exists an isomorphism

$$k[[x_1, \ldots, x_d]]/J \to \widehat{\mathcal{O}}_{U,p}.$$

Let n be the maximal ideal in $k[[x_1, \ldots, x_d]]$. Recall that the local ring $\mathcal{O}_{U,p}$ defines a deformation functor. Then $T^1_{X,x} = (m/m^2)^\vee$ is the tangent space of the deformation functor, and $T^2_{X,x} = (J/nJ)^\vee$ is an obstruction space ([FGI$^+$05, Theorem 6.1.19]). Let $r = \dim(J/nJ)^\vee$. By Nakayama's Lemma, J has at most r generators. Let $f_1, \ldots, f_r$ be the generators. Let T^1_0 be the formal completion of $T^1_{X,x}$ at zero. We can now define the Kuranishi map

$$\phi : \widehat{T}^1_0 \to T^2_{X,x},$$

which on the algebraic side maps a coordinate function y_i to the relation f_i. □

The number $d - r$ is the difference of the number of equations and the number of relations needed to define the Kuranishi model. In general, this difference can of course vary as we let a point move over X. This situation changes if the space we are studying admits a *perfect tangent-obstruction complex* in the terminology of Li and Tian. For the precise definition we refer to [LT98, Definition 1.2, 1.3]. Roughly speaking it is a two-term complex such that at every point x of X one cohomology group is the tangent space of x and the other cohomology group is an obstruction space. An immediate consequence of the existence of such a complex is that difference of equations and relations needed to define a Kuranishi model is constant.
Given such a complex, we can improve the description of the Kuranishi model.

Lemma 2.8. *Let X be a Deligne-Mumford stack with a perfect tangent-obstruction complex, and x a point of X. Denote by T^1 the tangent space of x and by T^2 the obstruction space given by the tangent-obstruction complex. Then there exists a Kuranishi model*

$$\phi\colon \widehat{T}^1_0 \to T^2.$$

Remark 2.9. Expressed slightly differently, we have written the local completion of X at x as the fiber product

$$\begin{tikzcd}
\widehat{X}_x \arrow[r] \arrow[d] & \widehat{T}^1_0 \arrow[d, "\phi"] \\
0 \arrow[r] & T^2.
\end{tikzcd}$$

Applying intersection theory bluntly to this situation ignoring all issues of formality we obtain a cycle of dimension $d - r$ in $\widehat{X}_x$. This cycle is defined by pulling back the normal cone of $\widehat{X}_x$ in $\widehat{T}^1_0$ from the pullback of the tangent space of 0 in T^2.

Definition 2.10. Let X be a Deligne-Mumford stack and x a point of X. Assume given a Kuranishi model $\phi : \widehat{T}^1_0 \to T^2$ for the formal completion of X at x. Then let C^ϕ denote the normal cone of $\widehat{X}_x$ in $\widehat{T}^1_0$.

So far the discussion has been purely point-wise. The next step is to move on from points to affine schemes. We first define what we would like to have in the end.

Definition 2.11. Let S be an affine scheme, p a point of S and $\widehat{S}_p$ the formal completion of S at p. Assume given a perfect tangent obstruction complex $E_1 \to E_2$. Define T^i to be the cohomology of $E_1 \to E_2$ at p. Consider the following diagram.

$$\begin{tikzcd}
\mathbb{V}(T^2) \times \widehat{S} \arrow[d] & \mathbb{V}(E_2^\vee) \times_S \widehat{S} \arrow[l, "j"'] \arrow[r, "r"] \arrow[d] & \mathbb{V}(E_2^\vee) \arrow[d] \\
\widehat{S}_p \arrow[r, "="] & \widehat{S}_p \arrow[r] & S.
\end{tikzcd}$$

The *virtual normal cone* is the unique (if it exists) cycle C^E in $\mathbb{V}(E_2^\vee)$ such that

$$r^*[C^E] = j^*[C^\phi]$$

holds.

The key step in producing the cycle C^E is to generalize the Kuranishi description of the formal completion of S at p to a Kuranishi description of the formal completion of S in $S \times S$. The main difference is that the Kuranishi map is no longer defined as a map from the tangent to the obstruction space, but instead is a map from the components the tangent-obstruction complex.

Definition 2.12. Let S be an affine scheme with a perfect tangent-obstruction complex $E_1 \to E_2$. A *relative Kuranishi model* for S in $S \times S$ is a morphism

$$\Phi \colon \widehat{\mathbb{V}(E_1^\vee)}_0 \to \mathbb{V}(E_2^\vee)$$

such that zero-set of Φ is isomorphic to the completion of S in $S \times S$.

Once we have such a relative Kuranishi model, we can define the virtual normal cone. Since the completion of S in $S \times S$ is described as the zero-set of a function with domain $\widehat{\mathbb{V}(E_1^\vee)}_0$, it naturally is a subspace of $\widehat{\mathbb{V}(E_1^\vee)}_0$.

Theorem 2.13. *Let S be an affine scheme with a perfect tangent-obstruction complex $E_1 \to E_2$.*

(i) Relative Kuranishi models exist.

(ii) Let Z be the formal completion of S in $S \times S$, and let $\Phi \colon \widehat{\mathbb{V}(E_1^\vee)}_0 \to \mathbb{V}(E_2^\vee)$ be a relative Kuranishi model. Let $C_{Z/\widehat{\mathbb{V}(E_1^\vee)}_0}$ be the normal cone of Z in $\widehat{\mathbb{V}(E_1^\vee)}_0$. The pullback of $C_{Z/\widehat{\mathbb{V}(E_1^\vee)}_0}$ to S defined by

$$\begin{tikzcd}
C^E \arrow[r] \arrow[d] & C_{Z/\widehat{\mathbb{V}(E_1^\vee)}_0} \arrow[d] \\
S \arrow[r] & Z
\end{tikzcd}$$

is the virtual normal cone.

This result is just the starting point. It remains to be checked that the virtual normal cone is independent of the choice of the Kuranishi model, independent of the quasi-isomorphism class of tangent-obstruction complex, and finally that it glues. Assuming all these results and assuming the existence of a global resolution of the tangent-obstruction complex, we can define the virtual fundamental class.

Definition 2.14. Let X be a Deligne-Mumford stack with a perfect tangent-obstruction complex that is globally given by a two-term complex $E_1 \to E_2$ of locally free sheaves. Then the *virtual fundamental class* is

$$s_0^*([C^E]) = [X]^{\mathrm{vir}}$$

where s_0 is the zero-section of $\mathbb{V}(E_2^\vee)$.

2.2 The Intrinsic Normal Cone

In the previous section we saw that the key ingredient in constructing a virtual fundamental class is producing a cone inside a vector bundle. This cone was described locally and then patched to a global object. We now want to give a quick summary on the take of Behrend and Fantechi on this problem. They produce, for any Deligne-Mumford stack X, independent of the existence of a given obstruction theory, a cone stack over X. This cone stack is the intrinsic normal cone of X. Instead of constructing the intrinsic normal cone on affine charts and checking that it glues, this cone stack is a substack of another stack, the intrinsic normal sheaf, that immediately exists globally.

At first glance, the intrinsic normal cone is different from the virtual normal cone of Li and Tian. The virtual normal cone is in general positive dimensional, whereas the intrinsic normal cone is by definition always of pure dimension zero. The relationship between them emerges as soon as one chooses an obstruction theory. Choosing an obstruction theory immediately gives an immersion of the intrinsic normal cone into the vector bundle stack associated to the obstruction theory. Pulling back this inclusion along the canonical chart of the vector bundle stack gives the virtual normal cone of Li and Tian.

We now plunge into the details. We begin by describing the abstract machinery that associates a stack to a two term complex of locally free sheaves.

Definition 2.15. Let X be a Deligne-Mumford stack and F a coherent sheaf on X. The *abelian cone* associated to F is

$$\mathrm{Spec}\ \mathrm{Sym}\, F \to X.$$

Example 2.16. Let $X \to M$, where M is smooth, be an immersion that is not regular. Then the normal cone of X in M is in general not abelian.

Example 2.17. To any cone $C \to X$ we can associate an abelian cone $A(C)$ and a closed immersion $C \to A(C)$ over X. To see this, let $C = \operatorname{Spec} \bigoplus_{i \leq 0} S^i$, and define $A(C) = \operatorname{Spec} \operatorname{Sym} S^1$.

Definition 2.18. Let C be a cone over X, $\mathbb{V}(E)$ a vector bundle over X, and $\mathbb{V}(E) \to C$ a morphism of cones over X. This induces a morphism $\mathbb{V}(E) \to A(C)$ of abelian cones over X. If C is invariant in $A(C)$ under the action of E, then we say that C is an *E-cone*.

We generalize these definitions to stacks.

Definition 2.19. Let $\mathfrak{C} \to X$ be an algebraic stack over X with vertex $0\colon X \to \mathfrak{C}$ and $\mathbb{A}^1$-action $\gamma\colon \mathbb{A}^1 \times \mathfrak{C} \to \mathfrak{C}$.[1]

(i) Then $\mathfrak{C}$ is a *cone stack* if étale locally on X there exists a vector bundle $\mathbb{V}(E)$ and an E-cone C over X such that $\mathfrak{C} \cong [C/\mathbb{V}(E)]$, where the isomorphism respects the vertex and $\mathbb{A}^1$-action.

(ii) A cone stack is called *abelian* if C can be chosen to be an abelian cone.

(iii) A cone stack is called a *vector bundle stack* if C can be chosen to be a vector bundle.

We now state the theorem that explains where all abelian cones come from.

Theorem 2.20. *Let X be a Deligne-Mumford stack, and let $D^{-1,0}_{\mathrm{coh}}(\mathcal{O}_X)$ be the derived category of $\mathcal{O}_X$-modules of Tor-amplitude in* $[-1, 0]$ *and coherent cohomology. Denote by $\mathbf{AbConeSt}_X$ the category of abelian cone stacks over X. Then there is an equivalence of categories*

$$D^{-1,0}_{\mathrm{coh}}(\mathcal{O}_X) \longrightarrow \mathrm{Ho}(\mathbf{AbConeSt}_X).$$

The map is given by

$$[E^{-1} \to E^0] \mapsto [\operatorname{Spec} \operatorname{Sym} E^{-1} / \operatorname{Spec} \operatorname{Sym} E^0].$$

Remark 2.21. Since the category of abelian cone stacks is naturally a 2-category, it makes sense to pass to the homotopy category in the right hand side of the above equivalence.

Remark 2.22. Recall that in the definition of the virtual normal cone of Li and Tian it was necessary to check independence of the quasi-isomorphism class of the tangent-obstruction complex. The above theorem is the corresponding statement in the approach of Behrend and Fantechi.

We can give a quick and brief definition of the intrinsic normal sheaf.

[1] For the exact definitions of *vertex* and the $\mathbb{A}^1$-action in this context see [BF97, Definition 1.5].

Definition 2.23. Let X be a Deligne-Mumford stack. Then the cutoff of the cotangent complex at -1 $\tau_{\geq -1}\mathbb{L}_X$ is an element of $D^{-1,0}_{\mathrm{coh}}(\mathcal{O}_X)$. Define the *intrinsic normal sheaf* $\mathfrak{N}_X$ of X to be the abelian cone stack associated to the truncated cotangent complex under the above equivalence.

Example 2.24. Let X be a scheme with an immersion i into a smooth scheme M with ideal sheaf I. Then the intrinsic normal sheaf of X is $[\mathrm{Spec}\,(\mathrm{Sym}\, I/I^2)/i^*T_M]$. Since the quasi-isomorphism class of the cotangent complex is independent of the choice of immersion, we get an equivalent stack for any other choice of immersion.

We now define the intrinsic normal cone as a closed substack of the intrinsic normal sheaf.

Definition 2.25.

1. A *local embedding* of X is a diagram

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, "f"'] & M \\ X & \end{tikzcd}$$

   where

   (i) U is an affine scheme;

   (ii) $f\colon U \to X$ is an étale morphism;

   (iii) M is a smooth affine scheme;

   (iv) $i\colon U \to M$ is a closed immersion.

2. A *morphism of local embeddings* is a commutative diagram

$$\begin{tikzcd} & V \arrow[dl, "g"'] \arrow[r, "j"] \arrow[d, "p"] & N \arrow[d, "q"] \\ X & U \arrow[l, "f"] \arrow[r, "i"'] & M \end{tikzcd}$$

   where the rows are local embeddings. Denote the category of local embeddings of X by $\mathbf{LocEmb}_X$.

Remark 2.26. In a morphism of local embeddings as above the morphism p is automatically étale since f and g are étale and we have $g = f \circ p$.

Remark 2.27. Our definition of the category of local embeddings differs slightly from the definition in [BF97] as we do not assume the morphism q to be smooth. The purpose of this modification is to allow for simplicial objects in the category of local embeddings, for which we also need degeneracy morphisms to exist. In these degeracy morphisms the morphism q will be a closed immersion.

Remark 2.28. Given two local embeddings $X \xrightarrow{f} U \xhookrightarrow{i} M$ and $X \xrightarrow{g} V \xhookrightarrow{j} N$, we define the *product of local embeddings* to be

$$\begin{tikzcd} & V \arrow[r, "j"] \arrow[dl, "g"'] & N \\ X & V \times_X U \arrow[l] \arrow[u] \arrow[d] \arrow[r] & M \times N \arrow[u] \arrow[d] \\ & U \arrow[r, "i"'] \arrow[ul, "f"] & M \end{tikzcd}$$

This is not necessarily the categorical product of the local embedding.

Remark 2.29. Note that $V \times_X U \hookrightarrow M \times N$ is in fact again a local embedding since X is a Deligne-Mumford stack and as such has an unramified diagnoal.

Definition 2.30. Define the *intrinsic normal cone* $\mathfrak{C}_X$ to be the unique closed substack of $\mathfrak{N}_X$ such that for every local embedding

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d] & M \\ X & \end{tikzcd}$$

we have

$$\mathfrak{C}_X|_U = [C_{U/M}/i^*T_M].$$

Remark 2.31. The intrinsic normal cone is always of pure dimension zero. This follows from the local description, since the normal cone of U in M has the dimension of M.

Remark 2.32. Of course, one has to check that this unique closed substack exists. This is considerably easier than in the construction of the virtual normal cone. This is because it is easier to prove that something is a subobject of a given global object than gluing something from scratch.

We now briefly mention how to produce the virtual fundamental class from the intrinsic normal cone if a perfect obstruction theory exists.

Definition 2.33. Let X be a Deligne-Mumford stack. A *perfect obstruction theory* is a morphism in the derived category of $\mathcal{O}_X$-modules

$$\phi\colon E \longrightarrow \mathbb{L}_X$$

where

(i) The complex E is of perfect amplitude in $[-1, 0]$.

(ii) $h^0(\phi)$ is an isomorphism.

(iii) $h^1(\phi)$ is surjective.

A *virtually smooth stack* is a Deligne-Mumford stack together with a choice of a perfect obstruction theory.

Remark 2.34. The equivalence to the definition of Li and Tian is the equivalence of points 1 and 3 of [BF97, Theorem 4.5].

As already mentioned previously, it sometimes is advantageous to choose a specific resolution of the perfect obstruction theory as morphism of complexes.

Definition 2.35. 1. A *local resolution* of the complex E consists of an étale morphism $U \to X$ where U is an affine scheme, a two-term complex F of locally free sheaves on U and an isomorphism $F \to E_U \in D(U)$.

2. A *local resolution of* $\phi\colon E \to \mathbb{L}_X$ consists of an étale morphism $U \to X$ where U is an affine scheme, a local resolution F of E on U, an isomorphism $L \to \mathbb{L}_U$ in $D(U)$, and a morphism of complexes $F \to L$ such that

$$\begin{array}{ccc} F & \longrightarrow & L \\ \downarrow & & \downarrow \\ E_U & \xrightarrow{\phi} & \mathbb{L}_U \end{array}$$

commutes in $D(U)$.

3. A *global resolution* for a perfect obstruction theory E is a complex of locally free sheaves $F = [F^{-1} \to F^0]$ concentrated in degrees -1 and 0 with an isomorphism $F \to E \in D(X)$.

Remark 2.36. In most cases one chooses local resolutions of ϕ only for the morphism to the -1-truncation of the cotangent complex $\mathbb{L}_X$. This suffices for applications to virtual classes.

Remark 2.37. The morphism $\phi\colon E \to \mathbb{L}_X$ lives on the left hand side of the equivalence of Theorem 2.20. Denote by $\mathfrak{E}$ the vector bundle stack corresponding to E. Translating the conditions on the morphism on cohomology to the right hand side becomes the statement that $\mathfrak{N}_X \to \mathfrak{E}$ is a closed immersion of abelian cone stacks over X. Composing with the inclusion of $\mathfrak{C}_X$ in $\mathfrak{N}_X$ we obtain a closed immersion

$$\begin{array}{ccc} \mathfrak{C}_X & \longrightarrow & \mathfrak{E} \\ & \searrow \quad \swarrow & \\ & X & \end{array} .$$

We can now define the virtual fundamental class.

Definition 2.38. Let X be a Deligne-Mumford stack with a perfect obstruction theory $\phi\colon E \to \mathbb{L}_X$. Let s_0 be the zero section of $\mathfrak{E}$. Then the *virtual fundamental class* is defined by

$$[X]^{\mathrm{vir}} = s_0^*([\mathfrak{C}_X]).$$

Remark 2.39. This definition requires intersection theory for Artin stacks, which is available due to [Kre99].

Remark 2.40. Denote the ranks of E^i by e_i. Then the vector bundle stack $\mathfrak{E}$ has rank $e_{-1} - e_0$. Since the intrinsic normal cone has pure dimension zero, intersecting this cone with the zero section gives a class of dimension $0 - (e_{-1} - e_0) = e_0 - e_{-1}$ which is precisely the expected dimension.

In the case when global resolutions exist, we can reproduce the virtual normal cone of Li and Tian.

Remark 2.41. Let X be a Deligne-Mumford stack with a perfect obstruction theory $\phi\colon E \to \mathbb{L}_X$. Assume that the perfect obstruction theory is globally defined by a two-term complex of vector bundles $[E_{-1} \to E_0]$. Then we have a canonical atlas for $\mathfrak{E}$ given by the quotient map from $\mathbb{V}(E^{-1}) \to \mathfrak{E}$. We can then form the following fiber product:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathbb{V}(E^{-1}) \arrow[d] \\ \mathfrak{C}_X \arrow[r] & \mathfrak{E}. \end{tikzcd}$$

The pullback C is then the virtual normal cone defined by Li and Tian, [KKP03, Proposition 2].

Chapter 3

Derived Algebraic Geometry

Derived algebraic geometry is the study of spaces obtained by gluing simplicial commutative rings along weak equivalences.

Just as in classical algebraic geometry, there are several approaches how to tackle these spaces. One can either use the language of the functor of points or the language of locally ringed spaces. We here follow closely the exposition of [HAG-II], which uses the functor of points. For a treatment closer to locally ringed spaces, see [Lur11].

3.1 Two Motivating Theorems

3.1.1 Characterizing Sheaves Among All Functors

One of Grothendieck's key ideas, maybe inspired from functional analysis, was not to study the geometry of a space directly, but instead to study all possible kinds of morphisms into this space. Expressed in more technical terms, this means to study the functor represented by a space. Of course there are plenty of functors that definitely do not arise from any geometric origin. This makes it necessary to characterize those functors which really are of geometric nature. Grothendieck was interested in easily verifiable conditions that ensure that a functor is representable. In the long run, this was accomplished by Artin for algebraic spaces instead of schemes. We present here a theorem that accomplishes the task for schemes, but in general is hard to check.

In short, a functor is representable by a scheme if and only if it is a sheaf in the Zariski topology and if it has a Zariski atlas. We first introduce the notion of a Zariski atlas.

Definition 3.1. Let $\mathbf{Aff}_A$ be the category of affine schemes over a ring A, and let $f\colon F \to G$ be a natural transformation of contravariant set-valued functors on $\mathbf{Aff}_A$. Then f is a *Zariski open immersion* if f is representable and for all test morphisms $\mathrm{Spec}\ S \to G$ the induced map f' in the fiber square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f'"'] & F \arrow[d, "f"] \\ \mathrm{Spec}\ S \arrow[r] & G \end{tikzcd}$$

is a Zariski open immersion.

We can now define a Zariski atlas.

Definition 3.2. Let $\mathbf{Aff}_A$ be the category of affine schemes over a ring A, and let F be a contravariant set-valued functor on $\mathbf{Aff}_A$. A *Zariski atlas* for F is a natural transformation of functors

$$\coprod_i \mathrm{Spec}\ R_i \to F$$

which is an epimorphism of sheaves and such that each component $\mathrm{Spec}\ R_i \to F$ is a Zariski open immersion.

We now can state the theorem that characterizes the functors representable by schemes among all functors.

Theorem 3.3. *[EH00, Thm. VI-14] Let $F : \mathbf{Aff}_A^{\mathrm{op}} \to \mathbf{Sets}$ be a contravariant functor. Then F is representable by a scheme if and only if*

(i) F is a sheaf in the Zariski topology and

(ii) F has a Zariski atlas.

Thus we have two conditions: The topological condition of satisfying descent and the geometric condition of having an atlas. In the sequel we will try to carry both notions over to simplicial set valued functors on the category of simplicial commutative rings. It will be important to keep both steps apart, first taking care of the descent condition and later imposing the right atlas conditions.

3.1.2 Sheaves Without The Sheaf Condition

Writing down explicitly the sheaf conditions for an up-to-homotopy sheaf is an impossible task, since there will be much more to check than just the cocycle condition up to triple intersections. This problem is already apparent in the definition of the descent conditions for a groupoid-valued functor, where one independently has to check that morphisms and objects glue. One therefore seeks a construction of the category of sheaves without explicit descent conditions. We briefly sketch how to accomplish this for a sheaf of sets on a Grothendieck site. Concretely, we will define a class of weak equivalences in the presheaf category. Localizing these weak equivalences gives the category of sheaves.

Definition 3.4. Let $\mathbf{C}$ be a category with a Grothendieck topology τ, and denote the category of presheaves on $\mathbf{C}$ by $\mathbf{Pr}(\mathbf{C})$. A morphism $f\colon F \to G$ in $\mathbf{Pr}(\mathbf{C})$ is *locally surjective* if for any $X \in \mathbf{C}$ and any $\xi \in G(X)$ there exists a covering $\coprod_i U_i \to X$ such that for any i the restriction ξ_{U_i} is in the image of $f(U_i)\colon F(U_i) \to G(U_i)$.

Definition 3.5. Let $\mathbf{C}$ be a category with a Grothendieck topology τ. Let W be the set of morphisms in $\mathbf{Pr}(\mathbf{C})$ satisfying the following conditions:

(i) f is injective on all sections and

(ii) f is locally surjective.

A morphism $f \in W$ is called a *τ-local isomorphism.*

Applying the general theory of localizing a category along a set of weak equivalences we obtain a category $W^{-1}\mathbf{Pr}(\mathbf{C})$. Using this category we can describe the category of sheaves.

Theorem 3.6. *The localization functor $\mathbf{Pr}(\mathbf{C}) \to W^{-1}\mathbf{Pr}(\mathbf{C})$ has a right adjoint which is fully faithful and whose essential image consists of the sheaves on $\mathbf{C}$ with respect to the topology τ.*

We record as a guiding principle that the sheaf conditions can be formulated by localizing at appropriate morphisms.

3.2 Simplicial Commutative Rings

In this section we gather important facts about the category of simplicial commutative A-algebras over a fixed commutative ring A. The first important fact is the model category structure on this category, for which we refer to the appendix.

3.2.1 An Adjunction

The category of simplicial commutative A-algebras is closely tied to discrete A-algebras via an adjoint pair.

Proposition 3.7. *Let $\mathbf{sAlg}_A$ be the category of simplicial commutative A-algebras, and let $\mathbf{Alg}_A$ be the category of commutative A-algebras. Define*

$$i\colon \mathbf{Alg}_A \to \mathbf{sAlg}_A$$

by $i(R)_n = R$ and taking all simplicial morphisms to be the identity. Then

$$\mathbf{sAlg}_A \underset{i}{\overset{\pi_0}{\rightleftarrows}} \mathbf{Alg}_A.$$

is an adjoint pair.

Remark 3.8. Let R be in $\mathbf{sAlg}_A$. Applying the adjunction to the identity $\pi_0(R) \to \pi_0(R)$ we obtain a canonical morphism

$$R \to i(\pi_0(R)).$$

The geometric version of this morphism will play an important role later.

The homotopy groups of a simplicial commutative A-algebra are always abelian, and can be assembled to a graded abelian group.

Definition 3.9. Let R be a simplicial commutative A-algebra. Define the *homotopy algebra* by

$$\pi_*(R) := \bigoplus_i \pi_i(R)$$

The name is justified by the following proposition.

Proposition 3.10. *Let R be a simplicial commutative A-algebra. Then its homotopy algebra is a graded commutative A-algebra.*

Proof. (Sketch) Elements $a \in \pi_n(R)$ and $b \in \pi_m(R)$ are represented by the homotopy classes of a maps $S^n \to R$ and $S^m \to R$. Composing with the multiplication map on R, we obtain

$$S^n \times S^m \to R \times R \to R.$$

Since $* \times S^m$ and $S^n \times *$ both get mapped to the basepoint $0 \in R$, this morphism factorizes over the smash product $S^n \wedge S^m \simeq S^{n+m}$. Define $a \cdot b \in \pi_{n+m}(R)$ to be homotopy class of the resulting map. □

3.2.2 Modules

We will now study the category of modules over a simplicial commutative A-algebra.

Definition 3.11. Let $\mathbf{M}_{R,\geq 0}$ be the category of simplicial modules over the simplicial commutative A-algebra R.

Remark 3.12. Since R is a simplicial A-algebra, the category $\mathbf{M}_{R,\geq 0}$ has a forgetful functor to the category $\mathbf{sMod}_A$ of simplicial objects in the category of A-modules. This category is equivalent, via the Dold-Kan correspondence, to chain complexes of modules over A concentrated in non-negative degrees. This allows us to transfer many constructions from chain complexes to simplicial modules over R. In particular, we can compute for $M \in \mathbf{M}_{R,\geq 0}$

$$\pi_i(M) = H_i(N(M)),$$

where N is the normalization functor of the Dold-Kan correspondence and we have omitted the forgetful functor from the notation.

We next want to introduce the suspension and loop space functors. These behave better for the stabilization of $\mathbf{M}_{R,\geq 0}$. This stabilization can be constructed by homotopy theoretic methods, but we will give a more accessible ad hoc definition instead.

Definition 3.13. Let R be a simplicial commutative A-algebra, and let $N(R)$ be its normalization, which is a commutative differential graded algebra over A. Define $\mathbf{M}_R$ to be the category of unbounded differential graded modules over $N(R)$ with its natural model category structure.

We can now define the suspension and loop functors by transporting them from the familiar categories of chain complexes.

Definition 3.14. Let $M \in \mathbf{M}_{R,\geq 0}$, and view $N(M)$ as unbounded chain complex of modules over A. Define the *suspension*

$$M[1] := K(N(M)[1])$$

where K is an inverse to the normalization functor. The *loop* functor is defined by

$$M[-1] := K(N(M)[-1]).$$

Remark 3.15. We have $\pi_i(M[1]) = \pi_{i+1}(M)$ and $\pi_i(M[-1]) = \pi_{i-1}(M)$.

Definition 3.16. Let $M \in \mathbf{M}_R$.

(i) The module M is *discrete* if $\pi_i(M) = 0$ for $i \neq 0$.

(ii) The module M is *n-connected* if $\pi_i(M) = 0$ for $i \leq n$. We will say that M is *connective* if it is -1-connected.

(iii) The module M is n-truncated if $\pi_i(M) = 0$ for $i > n$.

We now turn to the tensor product of R-modules.

Definition 3.17. Let $M, N \in \mathbf{M}_{R,\geq 0}$. Define their tensor product $M \otimes_R N$ levelwise, i.e. let $M \otimes_R N$ be the simplicial R-module which in degree n is given by

$$(M \otimes_R N)_n = M_n \otimes_{R_n} N_n.$$

The left derived functor of the tensor product is given by

$$M \otimes_R^{\mathbb{L}} N = M \otimes_R Q(N)$$

where Q denotes cofibrant replacement.

As usual, we have the following adjunction.

Proposition 3.18. *Let $f\colon R \to S$ be a morphism of simplicial commutative A-algebras. Then their exists an adjunction*

$$\mathbf{M}_{R,\geq 0} \underset{f^*}{\overset{-\otimes_R S}{\rightleftarrows}} \mathbf{M}_{S,\geq 0}.$$

Remark 3.19. Using the smash product of spectra or by transport of structure we can define the tensor product and its derived version also for $\mathbf{M}_R$.

To compute the homotopy groups of a tensor product we have the following result by Quillen.

Theorem 3.20. *[Qui67, II.6 Theorem 6(b)] Let $M, N \in \mathbf{M}_R$. There exists a spectral sequence with $E_2^{p,q} = \left(\mathrm{Tor}^p_{\pi_*(R)}(\pi_*(M), \pi_*(N))\right)_q$, where the right hand side is the q-th graded piece of the Tor^p group calculated as the tensor product of graded modules over a graded ring. If M, N are connective, we have*

$$E_2^{p,q} \Rightarrow \pi_{p+q}(M \otimes^{\mathbb{L}}_R N).$$

We now come to the central notion of flatness.

Definition 3.21. Let M be a connective R-module. The module M is *flat* if

$$- \otimes^{\mathbb{L}}_R M$$

preserves homotopy pullbacks.

An equivalent formulation of flatness can be achieved with the following definition.

Definition 3.22. Let M be a connective R-module. The module M is *strong* if the morphism

$$\pi_*(R) \otimes_{\pi_0(R)} \pi_0(M) \to \pi_*(M)$$

is an isomorphism.

We can now give alternative characterizations of flatness.

Proposition 3.23. *Let M be a connective R-module. Then the following conditions are equivalent:*

(i) The module M is flat.

(ii) The module M is strong and $\pi_0(M)$ is flat as $\pi_0(R)$ module.

(iii) If N is a discrete A-module, then $N \otimes^{\mathbb{L}}_R M$ is discrete.

(iv) The $\pi_0(R)$-module $\pi_0(R) \otimes^{\mathbb{L}}_R M$ is discrete and flat.

Remark 3.24. This shows that the "global" properties of a flat R-module M are determined by the "local" properties of $\pi_0(M)$ over $\pi_0(R)$, a principle formulated in Lurie's thesis [Lur04, Remark before Prop. 2.5.3].

We move on to projective modules.

Definition 3.25. Let M be a connective R-module. The module M is *projective* if it is the retract of a free module $\bigoplus_{i \in I} R$.

Remark 3.26. This is a direct reformulation of the classical notion of being the direct summand of a free module.

3.2.3 Cotangent Complex

The key tool for generalizing important geometric notions like étale and smooth from the discrete to the simplicial case is the cotangent complex.

Definition 3.27. Let $f\colon R \to S$ be a morphism of simplicial commutative A-algebras. Let $Q_R(S)$ be a cofibrant replacement of S in the category of simplicial commutative A-algebras over R. Let $\Omega^1_{Q_R(S)/R}$ denote the level-wise application of Kähler differentials. Then

$$\mathbb{L}_{S/R} := \Omega^1_{Q_R(S)/R} \otimes_{Q_R(S)} S$$

is the *cotangent complex* of S over R.

The Kähler differentials are characterized by a universal property: they corepresent the functor that associates to a module derivations with values in the module. The same statement for the cotangent complex does not hold in classical commutative algebra, but becomes true if we pass to simplicial algebras.

Definition 3.28. Let R be a simplicial commutative A-algebra and M a connective R-module. Define the *trivial square zero extension of R by M* by applying the usual square extension levelwise.

Remark 3.29. The trivial square zero extension comes with a natural augmentation $R \oplus M \to R$.

Definition 3.30. Let $f\colon R \to S$ be a morphism of simplicial commutative A-algebras, and let M be a connective S-module. Using the augmentation map of the trivial square zero extension $S \oplus M$ we obtain a map

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathbf{sAlg}_{A,/R}}(S, S \oplus M) \to \mathbb{R}\underline{\mathrm{Hom}}_{\mathbf{sAlg}_{A,/R}}(S, S).$$

Now define the *derived derivations with values in M* as the homotopy fiber product of the diagram of simplicial sets

$$\begin{tikzcd}
\mathbb{R}\,\mathrm{Der}_R(S, M) \arrow[r] \arrow[d] & \mathbb{R}\underline{\mathrm{Hom}}_{\mathbf{sAlg}_{A,/R}}(S, S \oplus M) \arrow[d] \\
\mathrm{id} \arrow[r] & \mathbb{R}\underline{\mathrm{Hom}}_{\mathbf{sAlg}_{A,/R}}(S, S).
\end{tikzcd}$$

Remark 3.31. In discrete commutative algebra, their is an analogous definition of the module of Kähler differentials as sections of the augmentation map of the trivial square extension.

We can now characterize the cotangent complex by a universal property.

Proposition 3.32. *Let $f\colon R \to S$ be a morphism of simplicial commutative A-algebras. Then the functor*

$$\mathbb{R}\operatorname{Der}_R(S,-)\colon \mathbf{M}_{S,\geq 0} \to \mathbf{sSet}$$

is corepresented by $\mathbb{L}_{S/R}$.

Only having the cotangent complex suffices to define formally smooth and formally étale morphisms. We need an additional finiteness hypothesis.

Definition 3.33. Let $f\colon R \to S$ be a morphism of simplicial commutative A-algebras, and let $C\colon I \to \mathbf{sAlg}_{A,/R}$ be a filtered diagram. The morphism f is homotopically finitely presented if

$$\operatorname{hocolim}_{i\in I} \mathbb{R}\underline{\operatorname{Hom}}_{\mathbf{sAlg}_{A,/R}}(S, C_i) \to \mathbb{R}\underline{\operatorname{Hom}}_{\mathbf{sAlg}_{A,/R}}(S, \operatorname{hocolim}_{i\in I} C_i)$$

is a weak equivalence of simplicial sets.

We now arrive at the following definition.

Definition 3.34. Let $f\colon R \to S$ be a morphism of simplicial commutative A-algebras.

(i) The morphism f is smooth if $\mathbb{L}_{R/S}$ is projective and f is homotopically finitely presented.

(ii) The morphism f is étale if $\mathbb{L}_{R/S} \simeq 0$ and f is homotopically finitely presented.

Again using strongness we have an alternative characterization.

Proposition 3.35. *Let $f\colon R \to S$ be a morphism of simplicial commutative A-algebras. The morphism f is smooth (étale) if and only if S is strong as R-module and the induced morphism $\pi_0(R) \to \pi_0(S)$ is smooth (étale).*

We also can define the derived version of being a local complete intersection morphism.

Definition 3.36. Let $f\colon R \to S$ be a morphism of simplicial commutative A-algebras. The morphism f is *quasi-smooth* if for any discrete S-module M the groups

$$\pi_i(\mathbb{L}_{S/R} \otimes_S M)$$

vanish for $i > 1$ and f is homotopically finitely presented.

We wrap up by listing a number of cofiber sequences which allow for computations of the cotangent complex. These are a considerable improvement over the non-derived setting, since the formulas 2 and 3 of the following proposition are only valid with an additional flatness hypothesis for the underived setting.

Proposition 3.37. *(i) Let $R \to S \to T$ be morphisms of simplicial commutative A-algebras. Then we have the following cofiber sequence of T-modules:*

$$\mathbb{L}_{S/R} \otimes^{\mathbb{L}}_S T \to \mathbb{L}_{T/R} \to \mathbb{L}_{T/S}.$$

(ii) Let

$$\begin{tikzcd} S' & S \arrow[l] \\ R' \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

be a homotopy cofiber square of simplicial commutative A-algebras. Then

$$\mathbb{L}_{S/R} \otimes^{\mathbb{L}}_S S' \to \mathbb{L}_{S'/R'}$$

is a weak equivalence of S'-modules. Furthermore the following square is a homotopy pushout of S'-modules:

$$\begin{tikzcd} \mathbb{L}_{S'} & \mathbb{L}_S \otimes^{\mathbb{L}}_S S' \arrow[l] \\ \mathbb{L}_{R'} \otimes^{\mathbb{L}}_{R'} S' \arrow[u] & \mathbb{L}_R \otimes^{\mathbb{L}}_R S' \arrow[l] \arrow[u] \end{tikzcd}$$

3.3 Descent

We now want to develop the technique introduced in Section 3.1.2 to solve the problem of defining the appropriate descent conditions. We first take care of the easier case of simplicial set valued functors on affine schemes. The simplification comes from the fact that the domain category itself is not a model category. We thus have to formulate descent conditions only for ordinary coverings and not for up-to-homotopy coverings. The key difficulty will be to identify the right set of natural transformations such that the localized category is equivalent to the category of sheaves.

3.3.1 Higher Stacks

We begin by equipping the category of simplicial presheaves on affine schemes with a model category structure. This model category structure will *not* take into account any topology and will have to be modified later on.

Definition 3.38. Denote by $\mathbf{sPr}(\mathbf{Aff}_A)$ the category of simplicial presheaves on A-algebras.

Definition 3.39. Let $f\colon F \to G$ be a morphism in $\mathbf{sPr}(\mathbf{Aff}_A)$.

(i) The morphism f is a global projective fibration if it is a levelwise fibration.

(ii) The morphism f is a global projective weak equivalence if it is a levelwise weak equivalence.

(iii) Cofibrations are defined by the left lifting property with respect to acyclic global projective fibrations.

This defines a model category structure on $\mathbf{sPr}(\mathbf{Aff}_A)$ which is commonly called the *global projective model structure*. We will now modify this model category structure to incorporate a topology. Thus fix a topology, for example the étale topology, on $\mathbf{Aff}_A$. Denote this topology by τ. We first define the homotopy sheaves of a simplicial presheaf.

Definition 3.40. Let $F \in \mathbf{sPr}(\mathbf{Aff}_A)$.

(i) Define the presheaf $\pi_0^{\mathrm{pr}}(F)\colon \mathbf{Aff}_A^{\mathrm{op}} \to \mathbf{Set}$ by sending X to $\pi_0(F(X))$.

(ii) Let $X \in \mathbf{Aff}_A$ and fix an $s \in F(X)_0$. Define the presheaf
$$\pi_i^{\mathrm{pr}}(F, s)\colon (\mathbf{Aff}_A/X)^{\mathrm{op}} \to \mathbf{Set}$$
by mapping
$$f\colon Y \to X \text{ to } \pi_i(F(Y), f^*s).$$

These presheaves will be called the *homotopy presheaves of* F. The associated sheaves will be denoted by $\pi_0(F)$ and $\pi_i(F, s)$ and are called the *homotopy sheaves of* F.

Remark 3.41. Since we take the associated sheaves, the topology really comes into play now.

We now define the local weak equivalences we wish to localize at.

Definition 3.42. Let $f\colon F \to G$ be a morphism in $\mathbf{sPr}(\mathbf{Aff}_A)$. The morphism f is a local weak equivalence if

(i) the morphism $\pi_0(F) \to \pi_0(G)$ is an isomorphism of sheaves and

(ii) for any $X \in \mathbf{Aff}_A$, any $s \in F(X)_0$ and any $i > 0$ the morphism $\pi_i(F, s) \to \pi_i(G, f(s))$ is an isomorphism of sheaves on $\mathbf{Aff}_A/X$.

It is a theorem due to Jardine [Jar87] that the left Bousfield localization at the class of local weak equivalences exists. The resulting model category structure is called the *local projective model structure*. In [DHI04], a model category structure which is Quillen equivalent to the local projective model structure is constructed, in which the fibrant objects admit the following nice description. This model structure is called the *universal homotopy theory*.

Theorem 3.43. *Equip $\mathbf{sPr}(\mathbf{Aff}_A)$ with the universal homotopy model structure. Then an object F is fibrant if and only if*

(i) For any $X \in \mathbf{Aff}_A$ the simplicial set $F(X)$ is fibrant.

(ii) For any $X \in \mathbf{Aff}_A$ and any hypercovering $H \to X$ the morphism

$$F(X) \to \operatorname{holim}_{[n]\in\Delta} F(H_n)$$

is an equivalence of simplicial sets.

The second condition is of course the important one, since it is the up-to-homotopy version of the usual descent condition. Recalling that the homotopy category of a model category is equivalent to the category of fibrant objects with homotopy classes of maps, and in view of the second condition of the theorem, the following definition is reasonable.

Definition 3.44. Define the category of topological stacks on $\mathbf{Aff}_A$ to be the homotopy category of $\mathbf{sPr}(\mathbf{Aff}_A)$ with respect to the local model structure. This category will be denoted by $\mathbf{St}_A$.

3.3.2 Derived Stacks

We now turn to the case of simplicial presheaves on the category of simplicial commutative A-algebras. We will proceed as above, with the difference of one additional localization. This additional localization makes sure that descent for up-to-homotopy coverings holds.

Definition 3.45. Let $\mathbf{dAff}_A$ be the opposite category of the category of simplicial commutative A-algebras, and $\mathbf{sPr}(\mathbf{dAff}_A)$ the category of simplicial presheaves.

Using the same definition as above, endow the category $\mathbf{sPr}(\mathbf{dAff}_A)$ with the global projective model structure. By the Yoneda embedding, every morphism $X \to Y$ in $\mathbf{dAff}_A$ gives a morphism of representable presheaves. Recall that the category of simplicial commutative A-algebras is a model category, and it thus makes sense to speak about weak equivalences.

Definition 3.46. Let W be the set of all morphisms in $\mathbf{sPr}(\mathbf{dAff}_A)$ obtained from weak equivalences in $\mathbf{dAff}_A$ by the Yoneda embedding. Define the *model category of prestacks* to be the left Bousfield localization of $\mathbf{sPr}(\mathbf{dAff}_A)$ equipped with the global model structure at the set W.

More or less by definition, we have the following result.

Proposition 3.47. *An object F in the model category of prestacks is fibrant if and only if*

(i) For any X in $\mathbf{dAff}$, the simplicial set $F(X)$ is fibrant.

(ii) F preserves weak equivalences.

The second condition is of course the important one. It ensures that a one-element covering by a weak equivalence will also lead to equivalent simplicial sets. But up to now, we have not made any use of the topology on $\mathbf{dAff}_A$. We incorporate the topology in a similar way as in the case of higher stacks.

Definition 3.48. Let $F \in \mathbf{sPr}(\mathbf{dAff}_A)$, and equip $\mathrm{Ho}(\mathbf{dAff}_A)$ with a model topology τ. Define the homotopy presheaves as in Definition 3.40. Assume F to be fibrant in the model category of prestacks. Since F preserves weak equivalences, the homotopy presheaves descend to functors

$$\pi_0^{\mathrm{pr}}(F)\colon \mathrm{Ho}(\mathbf{dAff}_A) \to \mathbf{Sets}$$

and

$$\pi_i^{\mathrm{pr}}(F,s)\colon \mathrm{Ho}(\mathbf{dAff}_A) \to \mathbf{Sets}.$$

Since $\mathrm{Ho}(\mathbf{dAff}_A)$ is a Grothendieck site, we can form the associated sheaves $\pi_0(F)$ and $\pi_i(F,s)$, the homotopy sheaves of F. If F is not fibrant, define $\pi_0(F) := \pi_0(RF)$, where R is fibrant replacement, and the same for $\pi_i(F,s)$.

As above, we now localize the model category of prestacks at the local equivalences, which we define now.

Definition 3.49. Let $f\colon F \to G$ be a morphism in the model category of prestacks. The morphism f is a *projective local weak equivalence* if

(i) the morphism $\pi_0(F) \to \pi_0(G)$ is an isomorphism of sheaves and

(ii) for any $X \in \mathbf{dAff}_A$, any $s \in F(X)_0$and any $i > 0$ the morphism $\pi_i(F,s) \to \pi_i(G, f(s))$ is an isomorphism of sheaves on $\mathbf{Aff}_A/X$.

It is proven in [HAG-I] that the left Bousfield localization at the class of projective local weak equivalences exists. The resulting model category structure is called the *projective local model structure.* Again it is possible to characterize the fibrant objects.

Theorem 3.50. *[HAG-I, Corollary 4.6.3] Equip $\mathbf{sPr}(\mathbf{dAff}_A)$ with the local model structure. Then an object F is fibrant if and only if*

(i) For any $X \in \mathbf{Aff}_A$ the simplicial set $F(X)$ is fibrant.

(ii) F preserves weak equivalences.

(iii) For any $X \in \mathbf{Aff}_A$ and any hypercovering $H \to X$ the morphism

$$F(X) \to \mathrm{holim}_{[n]\in\Delta} F(H_n)$$

is an equivalence of simplicial sets.

Thus the goal of finding a model category in which the fibrant objects satisfy descent for up-to-homotopy coverings is accomplished, and we can make the following definition.

Definition 3.51. The category of derived stacks $\mathbf{dSt}_A$ is the homotopy category of $\mathbf{sPr}(\mathbf{dAff}_A)$ with respect to the local model structure.

Keeping in mind that the category of simplicial commutative A-algebras is enriched over simplicial sets, we can define the spectrum of a simplicial A-algebra.

Definition 3.52. Let $R \in \mathbf{sAlg}_A$. Then the functor $\underline{\mathrm{Hom}}(A, -)$ is an element in $\mathbf{sPr}(\mathbf{dAff}_A)$. Define its image in $\mathbf{dSt}_A$ to be $\mathbb{R}\mathrm{Spec}\ R$.

Remark 3.53. It is a consequence of the Yoneda lemma for model categories, proved in [HAG-I], that this functor is fully faithful.

Remark 3.54. The explicit formula for $\mathbb{R}\mathrm{Spec}\ R(X)$ where $X = \mathrm{Spec}\ S$ is

$$\mathbb{R}\mathrm{Spec}\ R(X) = \underline{\mathrm{Hom}}(QR, S)$$

where Q denotes cofibrant replacement.

3.4 Atlas conditions

Looking back to the first of our motivating theorems, we have to find the appropriate descent conditions and define atlases to find geometric objects. Having taken care of descent, we now define atlases. All results apply word for word also to the category $\mathbf{St}_A$.

The definition of an atlas is an inductive procedure, starting with the trivial case of being representable by affines. From here it is possible to move on to morphisms, defining a morphism to be representable if it is representable on any affine test object.

Definition 3.55. (i) Let $F \in \mathbf{dSt}_A$. We say that F is *0-representable* if $F \simeq \mathbb{R}\mathrm{Spec}\ R$, for $R \in \mathbf{sAlg}_A$.

(ii) A morphism $f\colon F \to G$ in $\mathbf{dSt}_A$ is *0-representable* if for any morphism $g\colon \mathbb{R}\mathrm{Spec}\ S \to G$ the fiber product $\mathbb{R}\mathrm{Spec}\ S \times_G F$ is 0-representable.

(iii) A 0-representable morphism $f\colon F \to G$ in $\mathbf{dSt}_A$ is *smooth (étale)* if for any morphism $g\colon \mathbb{R}\mathrm{Spec}\ S \to G$ the induced morphism from the fiber product $\mathbb{R}\mathrm{Spec}\ R \to \mathbb{R}\mathrm{Spec}\ S$ is smooth (étale).

Having completed the first step of the induction, we can go on to define n-representability.

Definition 3.56. (i) Let $F \in \mathbf{dSt}_A$. We say that F is n*-representable* if it has an $n-1$-representable smooth atlas, i.e. there exists a family $\mathbb{R}\mathrm{Spec}\ R_i$ and a morphism $p\colon \coprod \mathbb{R}\mathrm{Spec}\ R_i \to F$ such that

- $\coprod \pi_0(\mathbb{R}\mathrm{Spec}\ R_i) \to \pi_0(F)$ is an epimorphism of sheaves and
- each component $\mathbb{R}\mathrm{Spec}\ R_i \to F$ is $n-1$-representable and smooth (étale).

(ii) A morphism $f\colon F \to G$ in $\mathbf{dSt}_A$ is *n-representable* if for any morphism $g\colon \mathbb{R}\mathrm{Spec}\ S \to G$ the fiber product $\mathbb{R}\mathrm{Spec}\ S \times_G F$ is n-representable.

(iii) A 0-representable morphism $f\colon F \to G$ in $\mathbf{dSt}_A$ is *smooth (étale)* if for any morphism $g\colon \mathbb{R}\mathrm{Spec}\ S \to G$ there exists an $n-1$-atlas $\coprod \mathbb{R}\mathrm{Spec}\ R_i$ for the fiber product such that each composite morphism $\mathbb{R}\mathrm{Spec}\ R_i \to \mathbb{R}\mathrm{Spec}\ S$ is smooth (étale).

We now arrive at the key definition of geometric derived stacks.

Definition 3.57. Let $F \in \mathbf{dSt}_A$. Define F to be an *Artin n-Stack* if

(i) F is m-representable for some $m \in \mathbb{N}$ and

(ii) for all discrete A-algebras R the simplicial set $F(\mathrm{Spec}\ R)$ is n-truncated

If F is m-representable by an $m-1$-representable étale atlas, we will say that F is a Deligne-Mumford n-stack.

3.5 Properties of Morphisms

So far we have only defined étale and smooth morphisms. Of course, there are many other interesting properties of morphisms. In this section we briefly outline some facts about further interesting properties following [Lur04, Section 3.5]. The same material can also be found in [HAG-II, Section 1.3.6].

Definition 3.58. Let P be a property of modules of simplicial commutative rings. The property P is *stable under arbitrary (étale, smooth) base change* if whenever an R-module M has the property P and S is an arbitrary (étale, smooth) R-algebra, then the module $B \otimes_R^{\mathbb{L}} M$ has the property P.

Definition 3.59. Let P be a property of morphisms of simplicial commutative rings. The property P is *local on the source* for the étale (smooth) topology if the following conditions are satisfied:

(i) Given any collection of morphisms $f_i\colon R \to S_i$ having the property P, the product morphism $R \to \prod\limits_i S_i$ has the property P.

(ii) If $R \xrightarrow{f} S \xrightarrow{g} T$ is a pair of morphisms such that g is étale (smooth), and $g \circ f$ has the property P, then f has the property P.

Properties that are local on the source and stable under base change are readily generalized to derived geometric stacks.

Definition 3.60. Let $f\colon F \to G$ be a morphism of derived Artin stacks. Let P be any property that is local on the source for the smooth topology and stable under arbitrary base change. The morphism f has the property P if for any map $\mathbb{R}\mathrm{Spec}\ R \to G$ and any étale morphism $\mathbb{R}\mathrm{Spec}\ S \to \mathbb{R}\mathrm{Spec}\ R \times^h_G F$ the R-algebra S has the property P.

Most important for the applications we have in mind is the following result.

Proposition 3.61. *[Lur04, Prop. 3.5.8] Quasi-smoothness is local on the source and stable under arbitrary base change.*

3.6 Features

3.6.1 The Canonical Inclusion

Every discrete commutative A-algebra R gives rise to a simplicial commutative A-algebra $i(R)$ by setting $i(R)_n = R$ and taking all simplicial morphisms to be the identity. We also have a map in the opposite direction, taking a simplicial commutative A-algebra R to $\pi_0(R)$, which is a discrete commutative A-algebra. These functors are actually adjoint to each other, giving rise to the following adjunction:

$$\mathbf{Aff}_A \underset{\pi_0}{\overset{i}{\rightleftarrows}} \mathbf{dAff}_A.$$

By precomposition, we get an adjoint pair

$$\mathbf{sPr}(\mathbf{Aff}_A) \underset{i^*}{\overset{\pi_0^*}{\rightleftarrows}} \mathbf{sPr}(\mathbf{dAff}_A).$$

This adjunction gives rise to Quillen functors of both sides equipped with the appropriate local projective model structures, and we finally obtain the following adjunction:

$$\mathbf{St}_A \underset{t_0}{\overset{i}{\rightleftarrows}} \mathbf{dSt}_A.$$

Definition 3.62. The functor t_0 is called the *truncation functor*.

Remark 3.63. It immediately follows that $t_0(\mathbb{R}\mathrm{Spec}\ R) = \mathrm{Spec}\,(\pi_0(R))$.

Remark 3.64. In [HAG-II, Section 2.2.4] many properties of the truncation functor are proved, for instance that t_0 maps derived Artin n-stacks to Artin n-stacks. The key feature of the adjunction that we will constantly expose is that t_0 commutes with homotopy limits and colimits, whereas i only commutes with homotopy colimits. Thus taking homotopy limits *after* viewing a scheme as a derived scheme gives rise to genuinely new objects.

Definition 3.65. Let $F \in \mathbf{St}_A$. A *derived extension* of F is a derived stack $F' \in \mathbf{dSt}_A$ with an isomorphism

$$F \simeq t_0(F').$$

Given a derived extension of a stack F, there always is a canonical inclusion of F into its derived extension.

Definition 3.66. Let $F \in \mathbf{St}_A$, and let F' be a derived extension. The *canonical inclusion*

$$j\colon i(F) \to F'$$

is the morphism in $\mathbf{dSt}_A$ obtained by adjunction from the isomorphism $F \simeq t_0(F')$.

To study the canonical inclusion we make the following definition.

Definition 3.67. A morphism $f\colon F \to G$ in $\mathbf{dSt}_A$ is a *closed immersion* if it is representable and for any morphism $\mathbb{R}\mathrm{Spec}\ S \to G$ the fiber product $\mathbb{R}\mathrm{Spec}\ S \times^h_G F \simeq \mathbb{R}\mathrm{Spec}\ R$ induces an epimorphism of rings $\pi_0(S) \to \pi_0(R)$.

We can now formulate the key property of the canonical inclusion.

Proposition 3.68. *[HAG-II, Prop. 2.2.4.7] Let F be an Artin n-stack, and let F' be a derived extension. Then the canonical inclusion*

$$j\colon i(F) \to F'$$

is a closed immersion.

Remark 3.69. The results of this section suggest that the relationship between a derived scheme to a classical scheme is very much like the relationship between non-reduced to reduced schemes. The geometric intuition behind derived schemes is that the underlying topological space is glued from the π_0 components of simplicial rings, with the other π_i groups serve as generalized nilpotents.

The canonical inclusion makes the connection between derived stacks and perfect obstruction theories.

Proposition 3.70. *Let F' be a derived quasi-smooth Deligne-Mumford 1-stack. Let F be the truncation of F', and denote by $j\colon F \to F'$ the canonical inclusion. Then*

$$j^*\mathbb{L}_{F'} \to \mathbb{L}_F$$

is a perfect obstruction theory for F.

Proof. Since F' is a quasi-smooth Deligne-Mumford 1-stack, the cotangent complex $\mathbb{L}_{F'}$ is of Tor-amplitude in $[-1,0]$. A standard property of the cotangent complex is that it has coherent cohomology. Since pullback preserves Tor-amplitude, we conclude

$$j^*\mathbb{L}_{F'} \in D^{[-1,0]}_{\mathrm{coh}}(F).$$

The morphism j induces a morphism

$$j^*\mathbb{L}_{F'} \to \mathbb{L}_F.$$

It remains to verify that this morphism is surjective on H^1 and an isomorphism of H^0. This follows from [BF97, Theorem 4.5], especially from the equivalence of 1 and 3. The statement of point 3 follows from the fact that $\mathbb{L}_{F'}$ is an obstruction theory for derived small extension, thus in particular for classical small extensions which only see the truncation F. $\square$

3.6.2 The Virtual Structure Sheaf

Given a classical Artin stack along with a derived extension, it is possible to produce a left-over of the derived extension which lives on the classical part. This is the virtual structure sheaf of the truncation.

Definition 3.71. [Toë09, Section 4.2, point 6.] Let F' be a derived Artin n- stack with truncation $t_0(F') = F$. Fix an atlas $\coprod_i \mathbb{R}\mathrm{Spec}\ R_i \to F'$. Then $\coprod_i \mathrm{Spec}\ \pi_0(R_i)$ is an atlas for F. The graded $\pi_0(R_i)$ modules $\pi_*(R_i)$ glue to quasi-coherent sheaf on $h^0(F)$, called the virtual structure sheaf. This sheaf will be denoted by $\mathcal{O}_F^{\mathrm{vir}}$.

Remark 3.72. The term virtual structure sheaf is also used in a slightly different context. If X is a Deligne-Mumford stack with a perfect obstruction theory the K-theoretic Gysin map yields by the same construction as for the virtual fundamental class a sheaf on X which is also called the virtual structure sheaf. These do not have to coincide.

Remark 3.73. [Toë09, 4.4.3] It is not clear that the cohomology groups of the virtual structure sheaf are coherent and vanish starting from some value. This makes it impossible to define the class of the virtual structure sheaf in the G-theory of the truncation. One known case where the cohomology groups of the virtual structure sheaf do vanish starting from some value and are coherent is the case of quasi-smooth derived Deligne-Mumford 1-stacks over a field of characteristic zero.

3.6.3 Groupoid Presentation

A nice feature of derived schemes and stacks is that they can be described in terms of up-to-homotopy groupoids, which is very close to defining manifolds in terms of their charts. We will briefly review the groupoid presentation of classical schemes and stacks and then pass on to the derived version.

Groupoid schemes

We begin by defining a groupoid object in the category of sets.

Definition 3.74. A groupoid object in the category $\mathbf{Set}$ consists of two sets U and R, and five morphisms:

(i) the source $s\colon R \to U$,

(ii) the target $t\colon R \to U$,

(iii) the multiplication $m\colon R_t \times_s R \to R$

(iv) the unit $e\colon U \to R$,

(v) and the inverse $R \to R$

satisfying the obvious axioms guessed from their names.[1] The notation for such a groupoid is

$$[R \rightrightarrows U].$$

Here are two basic examples.

Example 3.75. Assume that $(s,t)\colon R \to U \times U$ is injective. Then R defines an equivalence relation on U.

Example 3.76. Define a category $\mathbf{C}$ with objects R and morphisms U. This category is a groupoid in the sense that every morphism is invertible.

Thus the new feature of groupoid objects in comparison to equivalence relations is that elements of U can have automorphisms, namely those elements of R with the same source and target. We now carry this description over to schemes.

Definition 3.77. A groupoid object in the category $\mathbf{Schemes}_A$ consists of two schemes U and R and the above mentioned five morphisms with the appropriate axioms.

If we take U to be a disjoint union of affine schemes and view R as defining an equivalence relation on this disjoint union, it is of course tempting to glue the affine schemes along the equivalence relation to a scheme. With some additional hypothesis this is possible.

Lemma 3.78. *Let $[R \rightrightarrows U]$ be a groupoid scheme where U is an affine scheme and assume that*

(i) s and t are Zariski open immersions and

(ii) R is a closed subscheme of $U \times U$ via the tuple (s,t).

Then the colimit of the diagram

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"'] & U \\ U & \end{tikzcd}$$

is a scheme.

Proof. This just amounts to checking the cocycle condition necessary for gluing. □

[1] For the precise statement of all axioms see [BCE+20, Chapter 3]

Remark 3.79. In the opposite vein, a scheme X defines a groupoid scheme. Take a cover $\coprod U_i \to X$ and define $U := \coprod U_i$ and $R := U \times_X U$. Source and target are the projections.

Remark 3.80. If we relax the first condition in the previous lemma and allow s and t to be étale, the groupoid scheme $[R \rightrightarrows U]$ is by definition [Art71, Definition 2.3] an algebraic space.

Dropping the condition that R is a closed subscheme of $U \times U$ allows points to have automorphisms and leads to the world of stacks. In general, we have the following result.

Proposition 3.81. *Let $[R \rightrightarrows U]$ be a groupoid scheme with s and t smooth (étale). Then the colimit*

$$\operatorname{colim}(R \rightrightarrows U)$$

in the category of groupoid valued functors is an algebraic (Deligne-Mumford) stack.

Segal Groupoids

We now move on to the up-to-homotopy version of a groupoid object. It will be a special kind of simplicial object in a model category. To state the definition, we need a special morphism in the category $\mathbf{\Delta}$.

Definition 3.82. Let $\sigma^{i,n}$ be the morphism in $\mathbf{\Delta}$ defined by

$$\begin{aligned}[1] &\to [n] \\ 0 &\mapsto i \\ 1 &\mapsto i+1\end{aligned}$$

Definition 3.83. Let $\mathbf{M}$ be a Model category. A *Segal groupoid object* in $\mathbf{M}$ is a simplicial object X_* such that

(i) $\forall n > 1$, the morphism

$$\sigma_i := \prod_{0 \le i < n} \sigma_{i,n} \colon X_n \to \underbrace{X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1}_{n \text{ times}}$$

is a weak equivalence, and

(ii) the morphism

$$d_0 \times d_1 \colon X_2 \to X_1{}_{d_0} \times^h_{d_0} X_1$$

is a weak equivalence.

Remark 3.84. On first sight this definition looks quite far from the definition of a groupoid object. The relationship becomes clearer if one views X_1 to play the role of R and X_0 to play the role of U. The source, target and identity is given by the simplicial structure morphisms between X_1 and X_0. It remains to define the multiplication and the inverse.

Definition 3.85. Let X_* be a Segal groupoid object.

(i) The *multiplication* is the following morphism in $\mathrm{Ho}(\mathbf{M})$:

$$m := d_1 \circ \sigma_2^{-1} \colon X_1 \times^h_{X_0} X_1 \to X_2 \to X_1.$$

(ii) The *inverse* in $\mathrm{Ho}(\mathbf{M})$ is given by

$$i := d_2 \circ (d_0 \times d_1)^{-1} \circ (\mathrm{id} \times^h s_0 d_0) \colon X_1 \to X_{1\, d_0} \times^h_{d_0} X_1 \to X_2 \to X_1.$$

The key theorem we will later use is the derived analog of the above results on groupoid schemes.

Theorem 3.86. *Let X_* be a Segal groupoid object in $\mathbf{dSt}_A$. Assume that*

(i) X_0 and X_1 are disjoint unions of n-stacks and

(ii) the morphism $d_0 \colon X_1 \to X_0$ is étale.

Then the homotopy colimit

$$\mathrm{hocolim}_{[n] \in \mathbf{\Delta}} X_n$$

is a Deligne-Mumford $n+1$ - stack.

Proof. This is a special case of [HAG-II, Prop. 1.3.4.2]. □

Remark 3.87. An apparent discrepancy to the above results is that we only required d_0 to be étale. But for Segal groupoids one can show that this implies that d_1 is étale.

Chapter 4

Homotopy Fiber Products of Schemes

Homotopy fiber products provide basic first examples of derived schemes. The key observation in this context is that the inclusion functor from schemes to derived schemes in general does not commute with limits. It thus makes a difference if we take the fiber product after viewing a diagram of schemes as a diagram of derived schemes.

Curiously, fiber products also provide the basic example of virtually smooth schemes. After reviewing some of the basic features of homotopy fiber products, we make the connection between these two areas. Taking the homotopy fiber product instead of the fiber product in the basic example for virtually smooth schemes gives a derived scheme, and we verify that the induced perfect obstruction theory on the ordinary fiber product coincides with the perfect obstruction theory of the virtually smooth scheme. Finally, for this example we show that the class given by Kontsevich's formula for the virtual fundamental class is indeed the virtual fundamental class of the virtually smooth scheme.

4.1 Generalities on Homotopy Fiber Products

The most basic question is whether homotopy fiber products exist. We restrict ourselves here to the case of derived schemes, although the result holds in greater generality.

Proposition 4.1. *[Lur04, Prop. 4.6.3] The category of derived schemes has all finite homotopy limits.*

In a simple example the homotopy fiber product is readily computed.

Lemma 4.2. *Let $X = \mathrm{Spec}\ A$, $Y = \mathrm{Spec}\ B$ and $Z = \mathrm{Spec}\ C$ be affine schemes over k. Then the homotopy limit is given by*

$$X \times^h_Z Y = \mathbb{R}\mathrm{Spec}\,(A \otimes^{\mathbb{L}}_C B).$$

Here $\otimes^{\mathbb{L}}$ denotes the derived tensor product in the category of simplicial k-algebras.

Proof. The category of affine derived schemes over k is dual to the category of simplicial k-algebras. Thus homotopy limits go to homotopy colimits. □

In the case treated in the previous lemma, the homotopy groups of the simplicial k-algebra $A \otimes^{\mathbb{L}}_C B$ boil down to a well-known object.

Lemma 4.3. *Let A, B, C be commutative k-algebras. Then*

$$\pi_i(A \otimes^{\mathbb{L}}_C B) = \mathrm{Tor}_i^C(A, B).$$

Proof. View A and B as simplicial modules over C. Under the Dold-Kan correspondence cofibrant replacements get mapped to projective resolutions and the homotopy groups become homology groups of the complexes. □

A huge advantage in derived algebraic geometry is that the base change formulas for the cotangent complex hold without any flatness assumption. We will use this to give an explicit formula for the cotangent complex of a homotopy fiber product.

Lemma 4.4. *Let X, Y and Z be schemes over k. Let*

$$\begin{tikzcd} X \times^h_Z Y \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X \arrow[r, "g"'] & Z \end{tikzcd}$$

be a homotopy cartesian diagram. Then the cotangent complex of $X \times^h_Z Y$ is given by

$$\mathbb{L}_{X \times^h_Z Y} \simeq \mathrm{cone}((f')^*\mathbb{L}_{X/Z}[1] \to (g')^*\mathbb{L}_Y).$$

Proof. To simplify notation, let $W := X \times^h_Z Y$.

The morphism $g\colon X \to Z$ gives the canonical cotangent sequence

$$\mathbb{L}_{X/Z}[1] \to g^*\mathbb{L}_Z \to \mathbb{L}_X.$$

This can be rewritten as a homotopy pushout square

$$\begin{tikzcd} \mathbb{L}_{X/Z}[1] \arrow[r] \arrow[d] & g^*\mathbb{L}_Z \arrow[d] \\ \star \arrow[r] & \mathbb{L}_X. \end{tikzcd} \tag{4.1}$$

Since our diagram is homotopy cartesian, we have the following homotopy pushout square on W ([HAG-II, Lemma 1.4.1.12])

$$\begin{tikzcd}
(f')^*g^*\mathbb{L}_Z \arrow[r] \arrow[d] & (g')^*\mathbb{L}_Y \arrow[d] \\
(f')^*\mathbb{L}_X \arrow[r] & \mathbb{L}_W.
\end{tikzcd}$$

Pulling back the diagram (4.1) to W by f' and piecing it together with the above diagram we arrive at

$$\begin{tikzcd}
(f')^*\mathbb{L}_{X/Z}[1] \arrow[r] \arrow[d] & (f')^*g^*\mathbb{L}_Z \arrow[r] \arrow[d] & (g')^*\mathbb{L}_Y \arrow[d] \\
\star \arrow[r] & (f')^*\mathbb{L}_X \arrow[r] & \mathbb{L}_W.
\end{tikzcd}$$

Since the two inner squares are homotopy pushouts, so is the outer and the claim follows. □

We finally come to the lemma that describes the value of homotopy fiber products to the problem of finding derived extensions.

Lemma 4.5. *Let X, Y and Z be schemes over k. Then the truncation of the homotopy fiber product $X \times^h_Z Y$ is the ordinary fiber product $X \times_Z Y$.*

Proof. Since the truncation functor commutes with homotopy limits we conclude

$$t_0(X \times^h_Z Y) = t_0(X) \times_{t_0(Z)} t_0(Y) = X \times_Z Y.$$

□

4.2 The Basic Example of Behrend and Fantechi

The previous section motivated why homotopy fiber products are good candidates for finding derived extensions. In the case of virtually smooth schemes, one asks for more though. It does not suffice to only find a derived extension, but one also wants this derived extension to have the fixed perfect obstruction theory built in as cotangent complex. In one of the most basic examples of virtually smooth schemes, this question is readily settled. To study this example, we have to impose some more conditions on the spaces used to define the fiber product.

In the following, let M and V be smooth schemes over k with a morphism $f\colon V \to M$. Finally fix a regular immersion $i\colon W \to M$. We now let X be the fiber product of the diagram

$$\begin{tikzcd}
X \arrow[r, "i'"] \arrow[d, "g"'] & V \arrow[d, "f"] \\
W \arrow[r, "i"'] & M.
\end{tikzcd} \tag{4.2}$$

This is the classical situation studied in intersection theory as in [Ful98]. Replacing the top morphism by its normal cone and the bottom cone by its normal bundle we can define the Gysin pullback of the fundamental class of V in the Chow group of X. This is defined by

$$f^!([V]) = s_0^*([C_{X/V}]),$$

where s_0 denotes the zero-section of the pullback of the normal bundle of W in M to X. This class is of dimension $\dim V - \operatorname{codim} W$. Note that the dimension of X can be much larger than this.

We now show how in this situation X carries a perfect obstruction theory and unravel the definition of the virtual fundamental class.

Lemma 4.6. *([BF97, Section 6, Basic Example]) Let X be as in the basic setup* (4.2). *Then*

$$[g^* N_{W/M}^\vee \to (i')^* \Omega_V^1] \longrightarrow \mathbb{L}_X$$

is a perfect obstruction theory for X.

We now relate the virtual fundamental class associated to the above perfect obstruction theory to the class produced by intersection theory.

Proposition 4.7. *Let X be as in the basic setup* (4.2). *Equip X with the perfect obstruction theory of the above lemma. Then*

$$f^!([V]) = [X]^{\mathrm{vir}}.$$

Proof. Using the immersion $i' \colon X \to V$ we calculate

$$\tau_{\geq -1} \mathbb{L}_X = [I/I^2 \to (i')^* \Omega_V^1].$$

Thus the intrinsic normal cone $\mathfrak{C}_X$ is

$$[C_{X/V}/(i')^* T_V].$$

Since the perfect obstruction theory has a global resolution, we can compute the virtual fundamental class by first forming the fiber product

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & g^* N_{W/M} \arrow[d] \\
{[C_{X/V}/(i')^* T_V]} \arrow[r] & {[g^* N_{W/M}/(i')^* T_V]}
\end{tikzcd}$$

and then pulling back the class of C from $g^* N_{W/M}$ by the zero-section. Obviously,

$$C = C_{X/V},$$

proving the claim. $\square$

By taking the homotopy fiber product in (4.2) and using Lemma 4.5 we have a derived extension of X. To verify that this derived extension induces a perfect obstruction theory on X we have to check that the derived extension is indeed quasi-smooth. We state this in larger generality than we actually need.

Proposition 4.8. *Let X, Y and Z be derived Deligne-Mumford stacks with X and Y quasi-smooth over k and Z smooth over k. Then the homotopy fiber product*

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g'"'] & X \arrow[d, "g"] \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

is quasi-smooth.

Proof. Since W is a homotopy fiber product we have a pushout of simplicial modules on W

$$\begin{tikzcd} (g')^*f^*\mathbb{L}_Z \arrow[r] \arrow[d] & (f')^*\mathbb{L}_X \arrow[d] \\ (g')^*\mathbb{L}_Y \arrow[r] & \mathbb{L}_W. \end{tikzcd}$$

This gives us a cofiber sequence

$$(g')^*f^*\mathbb{L}_Z \to (f')^*\mathbb{L}_X \oplus (g')^*\mathbb{L}_Y \to \mathbb{L}_W.$$

Shifting this by one we arrive at

$$(f')^*\mathbb{L}_X \oplus (g')^*\mathbb{L}_Y \to \mathbb{L}_W \to (g')^*f^*\mathbb{L}_Z[-1].$$

Since we assumed Z to be smooth, we have $\mathbb{L}_Z \simeq \Omega_Z$ which is of Tor-amplitude 0. After shifting, it thus is of Tor-amplitude ≥ -1. On the other hand, $(f')^*\mathbb{L}_X \oplus (g')^*\mathbb{L}_Y$ is also of Tor-amplitude ≥ -1 since we assumed X and Y to be quasi-smooth. Since we squeezed $\mathbb{L}_W$ in between these two, it must also be of Tor-amplitude ≥ -1. $\square$

Corollary 4.9. *Let X be as in the basic setup* (4.2)*, and let X' be the homotopy limit. Then X' is quasi-smooth.*

We now want to check that the induced perfect obstruction theory coincides with the one constructed in the basic example 4.6.

Corollary 4.10. *Let X be as in the basic setup* (4.2) *and define X' to be the homotopy limit of the diagram. Denote by $j\colon X \to X'$ the canonical inclusion. Then*

$$j^*\mathbb{L}_{X'} \simeq [g^*N^\vee_{W/M} \to (i')^*\Omega^1_V]$$

is a quasi-isomorphism.

Proof. Since $W \to M$ is by assumption a regular immersion, we have

$$\mathbb{L}_{W/M}[1] \simeq N^{\vee}_{W/M}$$

and since V is smooth we have

$$\mathbb{L}_V \simeq \Omega^1_V.$$

The claim then follows from Lemma 4.4. □

As in the case of the pair of a scheme and a perfect obstruction theory, we want to reproduce the class constructed using intersection theory from the derived extension. Of course it would be possible to equip the truncation with its induced perfect obstruction theory, but there is a more direct formula involving the virtual structure sheaf. A different and nicer formula for this class also involving the virtual structure sheaf will be proven in the next section.

Proposition 4.11. *Let X be as in the basic setup* (4.2). *Let $\mathcal{O}_X^{\mathrm{vir}}$ be the virtual structure sheaf on X induced by the derived extension X'. Denote by n the expected dimension of X and by $\tau(\,)_n$ the n-th graded part of the Riemann-Roch transformation. Then*

$$\tau\left(\sum(-1)^i[\pi_i(\mathcal{O}_X^{\mathrm{vir}})]\right)_n = f^!([V]) = [X]^{\mathrm{vir}} \in A_n(X)_{\mathbb{Q}}.$$

Proof. We have to show that

$$\tau(\mathrm{Tor}^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)) = f^!([V]) + \text{ lower terms}.$$

This statement is just the well-known fact that the intersection product defined by Serre's Tor Formula and Fulton's refined intersection product coincide and can be found in [Ful98, 20.4]. □

4.3 Virtual Classes for Homotopy Fiber Products

Besides conjecturing the existence of derived versions of certain moduli spaces, Kontsevich suggested an explicit formula for these classes once such derived extensions exist. It is a natural generalization for the fundamental class of a local complete intersection. For such a space we have

$$\tau(\mathcal{O}_X) = \mathrm{td}(\mathbb{L}_X^{\vee}) \cap [X].$$

In case the space is no longer a local complete intersection but admits a perfect obstruction theory, Kontsevich suggested that

$$\tau(\mathcal{O}_X^{\mathrm{vir}}) = \mathrm{td}(E^{\vee}) \cap [X]^{\mathrm{vir}}$$

should hold. We here verify this formula for perfect obstruction theories and derived extensions arising from the basic example of the previous sections. Here everything can be proven using methods from classical intersection theory, most notably the specialization morphism.

4.3.1 Specialization

Let $f\colon X \to Y$ be a closed immersion. Recall that there exists a basic deformation square

$$\begin{tikzcd} C_{X/Y} \arrow[r, "i"] \arrow[d, shift left, "p"] & M(X,Y) \arrow[d] \\ X \arrow[u, shift left, "s_0"] \arrow[r, "f"'] & Y, \end{tikzcd}$$

where i is an inclusion. The key property of $M(X,Y)$ is that

$$M(X,Y) \setminus C_{X/Y} = Y \times \mathbb{A}^1 \backslash \{0\}.$$

Applying the exact sequence for Chow groups, we get an exact sequence

$$A_{k+1}(C_{X/Y}) \to A_{k+1}(M(X/Y)) \to A_{k+1}(Y \times \mathbb{A}^1 \backslash \{0\}).$$

Since the normal bundle to i is trivial, we deduce that

$$i^* i_*\colon A_{k+1}(N_{X/Y}) \to A_k(N_{X/Y})$$

is zero and we thus get an induced map

$$\rho\colon A_{k+1}(Y \times \mathbb{A}^1 \backslash \{0\}) \to A_k(C_{X/Y}).$$

We can now define the specialization map.

Definition 4.12. Let $q\colon Y \times \mathbb{A}^1 \backslash \{0\} \to Y$ be the projection. The specialization morphism

$$\sigma_{X/Y}\colon A_k(Y) \to A_k(C_{X/Y})$$

is the composite $\rho \circ q^*$.

We record some facts about the specialization map in the following theorem.

Theorem 4.13. *([DV76])*

(i) Let $f\colon X \to Y$ and $g\colon Z \to Y$ be a closed immersions, and let

$$i\colon C_{X \times_Y Z/Z} \to C_{X/Y}$$

be the canonical morphism of normal cones. Then

$$\sigma_{X/Y}([Z]) = i_*([C_{X \times_Y Z/Z}]).$$

(ii) Let E be a vector bundle on Y, and $\alpha \in A_*(Y)$. Then

$$\sigma_{X/Y}(c_i(E) \cap \alpha) = c_i(p^* f^* E) \cap \sigma_{X/Y}(\alpha)$$

(iii) *There exists a specialization morphism $\sigma_{X/Y}$ in K-theory with*

$$\sigma_{X/Y}([\mathcal{O}_Z]) = i_*([\mathcal{O}_{C_{X\times_Y Z/Z}}]).$$

If f is a local complete intersection, then

$$\begin{tikzcd} K_0(Y) \arrow[r, "f^*"] \arrow[dr, "\sigma_{X/Y}"'] & K_0(X) \arrow[d, "p^*"] \\ & K_0(N_{X/Y}) \end{tikzcd}$$

commutes.

(iv) *Let τ denote the Riemann-Roch homomorphism. Then the following square commutes:*

$$\begin{tikzcd} K_0(Y) \arrow[r, "\tau_Y"] \arrow[d, "\sigma_{X/Y}"'] & A_*(Y)\otimes\mathbb{Q} \arrow[d, "\sigma_{X/Y}"] \\ K_0(C_{X/Y}) \arrow[r, "\tau_{C_{X/Y}}"] & A_*(C_{X/Y})\otimes\mathbb{Q}. \end{tikzcd}$$

4.3.2 Proof of the Conjecture

To prove the conjecture, we have to recall the following Riemann-Roch formula proved by Verdier.

Theorem 4.14. *([DV76]) Let $f\colon X\to Y$ be a local complete intersection. Then the following square commutes:*

$$\begin{tikzcd} K_0(Y) \arrow[r, "\tau_Y"] \arrow[d, "f^*"'] & A_*(Y)\otimes\mathbb{Q} \arrow[d, "\mathrm{td}\, T_f \cap \_"] \\ K_0(X) \arrow[r, "\tau_X"] & A_*(X)\otimes\mathbb{Q}. \end{tikzcd}$$

We can now prove Kontsevich's conjecture for homotopy fiber products.

Theorem 4.15. *Let X be the fiber product of*

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "g'"'] & V \arrow[d, "g"] \\ W \arrow[r, "f"'] & M, \end{tikzcd}$$

and assume that V and M are smooth and f is a local complete intersection. Let $X' = W\times^h_M V$, and denote by $\mathcal{O}_X^{\mathrm{vir}}$ the virtual structure sheaf. Then

$$\tau_X([\mathcal{O}_X^{\mathrm{vir}}]) = \mathrm{td}(j^*\mathbb{L}_{X'}^\vee)\cap[X]^{\mathrm{vir}}.$$

Proof. To simplify notation, let $C := C_{X/V}$ and $N := (g')^* N_{W/M}$. Let $i\colon C \to N$ be the inclusion, $q\colon C \to X$ and $\pi\colon N \to X$ the projections such that $q = \pi \circ i$. Finally let $s_0\colon X \to N$ be the zero-section. By definition, we have that

$$f^*([\mathcal{O}_V]) = \sum_i (-1)^i \operatorname{Tor}^i_{\mathcal{O}_M}(\mathcal{O}_V, \mathcal{O}_W).$$

By point 3 of Theorem 4.13 we have

$$\sigma_{W/M}(\mathcal{O}_V) = \pi^*(\mathcal{O}_X^{\mathrm{vir}}) = i_*([\mathcal{O}_C]).$$

Applying the Riemann-Roch formula of point 4 of Theorem 4.13 to f' we conclude

$$\begin{aligned}\tau_C(\mathcal{O}_C) &= \tau_C(\sigma_{X/V}([\mathcal{O}_V])) = \sigma_{X/V}(\mathrm{td}(T_V) \cap [V]) \\ &= q^* \mathrm{td}(T_V|_X) \cap \sigma_{X/V}([V]) = q^* \mathrm{td}(T_V|_X) \cap [C].\end{aligned}$$

Applying Grothendieck-Riemann-Roch and the projection formula we arrive at

$$\begin{aligned}\tau_N(i_*[\mathcal{O}_C]) &= i_* \tau_C([\mathcal{O}_C]) = i_*(q^* \mathrm{td}(T_V|_X) \cap [C]) \\ &= i_*(i^* \pi^* \mathrm{td}(T_V|_X) \cap [C]) = \pi^* \mathrm{td}(T_V|_X) \cap i_*([C]).\end{aligned}$$

Noticing that $s_0^*([C]) = X^{\mathrm{vir}}$ and using Theorem 4.14 we finally deduce

$$\begin{aligned}\tau_X(\mathcal{O}_X^{\mathrm{vir}}) &= \tau_X(i_*([\mathcal{O}_C])) = \mathrm{td}(N)^{-1} \cap s_0^*(\tau_N(i_*([\mathcal{O}_C]))) \\ &= \mathrm{td}(j^* \mathbb{L}_{X'}^\vee) \cap [X]^{\mathrm{vir}}.\end{aligned}$$

□

4.4 Zero-Sets of Sections

A special case of this basic example is the zero-set of a section of a vector bundle. In this case the virtual fundamental class is a localized version of the Euler class of the bundle.

We begin by defining the zero-set of a section. Throughout, M will be a smooth scheme over an algebraically closed field k, and $\mathbb{V}(E)$ a vector bundle over M.

Definition 4.16. Let $s:\ M \to \mathbb{V}(E)$ be a section. Dually, this gives a morphism of sheaves $E^\vee \to \mathcal{O}_M$. The image is an ideal sheaf I in $\mathcal{O}_M$. The closed subscheme of M defined by the ideal sheaf I is the zero-set of s and will be denoted by $Z(s)$.

Remark 4.17. Alternatively, a section s can be viewed as a morphism from the trivial line bundle to $\mathbb{V}(E)$. The scheme $Z(s)$ then is the locus where the rank of s drops from one to zero.

We recall the definition of the Euler class of a vector bundle.

Definition 4.18. Let $\mathbb{V}(E)$ be a rank e vector bundle on a smooth scheme M. Then the Euler class of $\mathbb{V}(E)$ is defined by

$$c_e(E) \cap [M].$$

By adding codimensions, we expect $Z(s)$ to have dimension $r := \dim(M) - \mathrm{rk}(E)$. An alternative way of characterizing the zero-set of a section is by defining it to be the fiber product

$$\begin{tikzcd} Z(s) \arrow[r] \arrow[d, "i"'] & M \arrow[d, "s"] \\ M \arrow[r, "s_0"'] & \mathbb{V}(E), \end{tikzcd}$$

where s_0 denotes the zero section. This puts in the situation of our basic setup (4.2). Applying the general intersection procedure of [Ful98] to this fiber square, we obtain a class $\mathbb{Z}(s) \in A_r(Z(s))$. We will now prove that this class is a localized version of the Euler class of $\mathbb{V}(E)$.

Remark 4.19. We briefly recall two facts from intersection theory [Ful98, Corollary 6.5]:

(i) Any section $s : M \to \mathbb{V}(E)$ is a regular closed immersion.

(ii) The Gysin morphism $s^!$ is independent of the choice of section.

We can now prove that the class $\mathbb{Z}(s)$ pushes forward to the Euler class.

Proposition 4.20. *Let $i : \mathbb{Z}(s) \to M$ be the inclusion. Then*

$$i_*(\mathbb{Z}(s)) = c_e(\mathbb{V}(E)) \cap [M].$$

Proof. By definition, we have that

$$\mathbb{Z}(s) = s_0^!([M]).$$

From the push-forward property of Gysin homomorphisms, the independence of the choice of section and the self-intersection formula, and the observation that the normal bundle to s_0 can be identified with $\mathbb{V}(E)$ we deduce that

$$i_* s_0^!([M]) = s_0^! s_*([M]) = s_0^* s_*([M]) = s^* s_*([M]) = c_e(\mathbb{V}(E)) \cap [M].$$

□

An incredibly geometric construction of the class $\mathbb{Z}(s)$ is provided by a special case of MacPherson's graph construction. To relate this construction to $\mathbb{Z}(s)$, we recall the following geometric description of the operation "intersection with the zero section".

Proposition 4.21. *([Ful98, Proposition 3.3]) Let X be a scheme and $\mathbb{V}(E)$ a rank e vector bundle on X with zero section s_0. Let $[C] \in A_k(\mathbb{V}(E))$, and let $[\overline{C}]$ be a class in $A_k(P(\mathbb{V}(E) \oplus 1))$ that restricts to $[C]$. Denote by ξ the universal quotient bundle on $P(\mathbb{V}(E) \oplus 1)$ and by $q : P(\mathbb{V}(E) \oplus 1) \to X$ the projection . Then*

$$s^*([C]) = q_*(c_e(\xi) \cap [\overline{C}]).$$

We now recall MacPherson's graph construction in this special case.

Remark 4.22. Multiplying a section by a factor, we can make the section more vertical. Letting this go to infinity, we expect to find a cycle sitting over the locus where s vanishes. In more mathematical terms, view the section s as a morphism from the trivial line bundle 1 to $\mathbb{V}(E)$. The graph of s is a 1-dimensional subspace of $\mathbb{V}(E) \oplus 1$. Define a morphism

$$\begin{aligned} \phi :\ M \times \mathbb{A}^1 &\to P(\mathbb{V}(E) \oplus 1) \times \mathbb{A}^1 \\ (m, \lambda) &\mapsto (\Gamma(\lambda s(m)), \lambda)\,, \end{aligned}$$

and compose with the immersion $P(\mathbb{V}(E) \oplus 1) \times \mathbb{A}^1 \to P(\mathbb{V}(E) \oplus 1) \times \mathbb{P}^1$. Define W to be the closure of the image of ϕ. Let i_∞ be the inclusion of $P(\mathbb{V}(E) \oplus 1)$ in $P(\mathbb{V}(E) \oplus 1) \times \mathbb{P}^1$ at infinity. Then

$$i_\infty^*[W] = [P(C \oplus 1)] + [\tilde{M}],$$

where $\tilde{M}$ projects birationally to M, and C is the normal cone to $Z(s)$ in M. The geometric intuition is that at infinity, the section broke into a vertical part sitting over the zero locus and a horizontal part extending over all of M. Throwing out the component $[\tilde{M}]$, we have indeed found a cycle sitting over the zero-locus of s. We can immediately reproduce $\mathbb{Z}(s)$ by setting

$$\mathbb{Z}(s) = q_*(c_e(\xi) \cap [P(C \oplus 1)]),$$

keeping in mind the above proposition.

Chapter 5

Algebraic Kuranishi Structures

In symplectic topology, Gromov-Witten invariants are defined using moduli spaces of J-holomorphic curves. These moduli spaces are equipped with an extra geometric structure, from which then virtual chains or virtual cycles are produced. In the approach of [FO99], this extra geometric structure is called a Kuranishi structure. Roughly speaking, providing such a Kuranishi structure means to cover the moduli space with a system of presentations as zero-sets of sections of the obstruction bundle.

In this chapter, we propose an algebraic version of these definitions. On the algebraic side, the geometric structure needed to define virtual cycles has long been identified as a perfect obstruction theory. We will show that any virtually smooth space, i.e. a space together with a choice of a perfect obstruction theory, immediately carries an algebraic Kuranishi structure.

5.1 Review of Kuranishi Structures in Symplectic Topology

Kuranishi structures were first introduced in [FO99] to define Gromov-Witten invariants on the symplectic side of the bridge. We here review the modifications introduced by Joyce in [Joy07]. They are there treated in much greater generality using manifolds with corners.

Definition 5.1. Let X be a paracompact topological space and p a point of X. A *Kuranishi neighbourhood of* p in X consists of (V_p, E_p, s_p, ψ_p), where

(i) V_p is a smooth orbifold, which may have boundary or corners;

(ii) $E_p \to V_p$ is an orbifold vector bundle;

(iii) $s_p\colon V_p \to E_p$ is a smooth section, the *Kuranishi map*;

(iv) ψ_p is a homeomorphism from $s_p^{-1}(0)$ to an open neighbourhood of p in X.

A Kuranishi neighbourhood of a point thus gives a homeomorphism from an open neighbourhood of p to the zero-set of a section of a vector bundle. To be able to define a Kuranishi atlas one has to introduce coordinate changes. We omit the exact definition since it is technically involved and differs from author to author. Once this step is accomplished, one can move on to define *Kuranishi structures*. A Kuranishi structure κ on a paracompact topological space X consists of a germ of Kuranishi neighbourhoods for all points of X along with germs of coordinate changes for all points close to a given point.

An important remark is that it follows from the definitions that the virtual dimension of X, defined as $\dim V_p - \dim E_p$ is independent of the chosen point p.

Definition 5.2. A *Kuranishi space* (X, κ) consists of a paracompact topological space X along with a Kuranishi structure κ on X.

5.2 Algebraic Kuranishi Structures

A perfect obstruction theory is an extra geometric structure on a space sufficient to produce virtual cycles. In this section we propose an algebraic version of the additional geometric structure described above, namely an algebraic Kuranishi structure. We will show that if a space carries a perfect obstruction theory, it automatically has an algebraic Kuranishi structure. All the results needed to prove these assertions are already contained in one form or the other in various papers of Behrend and Fantechi, [Beh09, BF97].

Instead of using germs to define Kuranishi neighbourhoods, we will instead define a category of Kuranishi neighbourhoods sitting above the étale site of a virtually smooth space.

Remark 5.3. Since there is no danger of confusing our notion of Kuranishi neighbourhoods with those used in symplectic topology, we will drop the adjective "algebraic" in the following.

Definition 5.4. A *Kuranishi neighbourhood* for a local embedding $X \xleftarrow{f} U \xhookrightarrow{i} M$ with ideal sheaf I consists of a diagram[1] $\kappa_U \colon \mathbf{D} \to \mathbf{Aff}_k$, where κ_U is

$$M \xrightarrow{s} \mathbb{V}(F) \xleftarrow{0} M$$

such that

(i) $\mathbb{V}(F)$ is a vector bundle on M;

(ii) 0 denotes the zero section of $\mathbb{V}(F)$;

(iii) s is a section $s \colon F \to \mathcal{O}_M$ of $\mathbb{V}(F)$;

[1] Recall that the notation $\mathbf{D}$ is reserved for the index category $a \xrightarrow{\delta} b \xleftarrow{\epsilon} c$.

(iv) the limit cone of κ_U is

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, "i"'] & M \arrow[d, "0"] \\ M \arrow[r, "s"'] & \mathbb{V}(F); \end{tikzcd}$$

(v) the induced perfect obstruction theory

$$\begin{tikzcd} F \arrow[r, "d\circ s"] \arrow[d, "s"'] & \Omega^1_M|_U \arrow[d, "="] \\ I/I^2 \arrow[r, "d"'] & \Omega^1_M|_U \end{tikzcd}$$

for U is a local resolution[2] of $\phi\colon E_U \to \tau_{\geq -1}\mathbb{L}_U$.

Remark 5.5. By the condition on the limit cone of κ_U it follows that the section $s\colon F \to \mathcal{O}_M$ is a surjection onto the ideal sheaf I of U in M.

Remark 5.6. If it is clear from the context we will often drop the morphism $U \to X$ from a local embedding $X \leftarrow U \to M$.

Definition 5.7. Let κ_U and κ_V be Kuranishi neighbourhoods with respect to the local embeddings $U \hookrightarrow M$ and $V \hookrightarrow N$ of X. Then a *morphism of Kuranishi neighbourhoods* is a natural transformation $\alpha\colon \kappa_V \to \kappa_U$ given by

$$\begin{tikzcd} N \arrow[r] \arrow[d, "\alpha(a)"'] & \mathbb{V}(G) \arrow[d, "\alpha(b)"'] & N \arrow[l] \arrow[d, "\alpha(c)"'] \\ M \arrow[r] & \mathbb{V}(F) & M \arrow[l] \end{tikzcd}$$

such that

(i) $\alpha(a) = \alpha(c)$;

(ii) $\alpha(b)$ is induced by a morphism of sheaves of $\mathcal{O}_N$-modules $\rho\colon (\alpha(a))^*F \to G$;

(iii) the morphism ρ induces a quasi-isomorphism of the induced perfect obstruction theories on V;

(iv) the induced morphism between the limits

$$V = \lim_{d\in\mathbf{D}} \kappa_V \longrightarrow \lim_{d\in\mathbf{D}} \kappa_U = U$$

is an étale morphism over X.

[2]see Definition 2.35 for the definition of local resolution.

Remark 5.8. Given a morphism of local embeddings

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "p"'] & N \arrow[d, "q"] \\ U \arrow[r, "i"'] & M \end{tikzcd}$$

along with Kuranishi neighbourhoods κ_V and κ_U, we will call a morphism of Kuranishi neighbourhoods $\alpha\colon \kappa_V \to \kappa_U$ a *lift* of the above morphisms of local embeddings if the induced morphism between the limits is p and $\alpha(a) = q$.

Remark 5.9. The definition of morphisms of Kuranishi neighbourhoods given here is inspired by [Joy11].

Remark 5.10. As the composition of two morphisms of Kuranishi neighbourhoods is again such a morphism, Kuranishi neighbourhoods form a subcategory of the diagram category $\mathbf{Aff}_k^{\mathbf{D}}$. We will denote this category by $\mathbf{Kur}_X$.

Definition 5.11. A *Kuranishi structure* κ for the virtually smooth Deligne-Mumford stack (X, E) is a local embedding $X \xleftarrow{f} U \xhookrightarrow{i} M$ such that f is surjective along with a Kuranishi neighbourhood for U. We will denote a virtually smooth Deligne-Mumford stack along with the choice of a Kuranishi structure by (X, E, κ).

With all definitions set we now want to prove that Kuranishi neighbourhoods indeed exist. This result is well-known and can be found in [Beh09, Prop. 3.13] and [MPT10, Appendix A].

Lemma 5.12. *Étale locally on* (X, E)*, Kuranishi neighbourhoods exist.*

Proof. Let $X \xleftarrow{f} U \xhookrightarrow{i} M$ be a local embedding of X with ideal sheaf I. Assume for the moment that there exists a vector bundle $\mathbb{V}(F^{-1})$ on M and a local resolution of $\phi\colon E \to \tau_{\geq -1}\mathbb{L}_X$ on U of the form

$$\begin{tikzcd} F_U^{-1} \arrow[r, "d_F"] \arrow[d, "\psi^{-1}"'] & F^0 \arrow[d, "\psi^0"] \\ I/I^2 \arrow[r, "d"'] & \Omega_M^1|_U. \end{tikzcd}$$

Assume further that ψ^0 is an isomorphism. Then in fact there is a Kuranishi neighbourhood κ_U for the local embedding $U \hookrightarrow M$: Since ψ^0 is an isomorphism, ψ^{-1} is surjective by the 5-lemma. As M is an affine scheme, we can lift ψ^{-1} to a section

$$s\colon F^{-1} \twoheadrightarrow I.$$

It thus suffices to find a local resolution of $\phi\colon E \to \tau_{\geq -1}\mathbb{L}_X$ of the form given above.

To prove that local resolutions of such a form exist, we begin with an arbitrary local embedding $X \xleftarrow{f} U \xhookrightarrow{i} M$ with ideal sheaf I. As U is affine, there exists a local resolution of ϕ of the form

$$\begin{tikzcd}
K' \arrow[r, "\gamma"] \arrow[d] & K \arrow[r] \arrow[d] & 0 \arrow[d] \\
G^{-1} \arrow[r, "d_G"] \arrow[d, "\psi^{-1}"'] & G^0 \arrow[r] \arrow[d, "\psi^0"] & \operatorname{coker} d_G \arrow[d, "\cong"] \\
I/I^2 \arrow[r, "d"] & \Omega^1_M|_U. \arrow[r] & \operatorname{coker} d
\end{tikzcd}$$

where G is a complex of projectives. Here K and K' denote the kernels of ψ^0 and ψ^{-1}. Now let $R \twoheadrightarrow I/I^2$ be a free resolution of I/I^2. As ψ^0 is an isomorphism on H^0, adding R to the complex G we can without loss of generality assume the ψ^0 is surjective. As K is the kernel of a surjection of projectives, it is again projective. We can thus choose a section of $\gamma\colon K' \to K$, making K into a submodule of G^{-1}. Now let F be the complex

$$F := [G^{-1}/K \to G^0/K],$$

which obviouly again consists of projectives and is quasi-isomorphic to G. Then $F^0 \to \Omega^1_M$ is an isomorphism and we have fulfilled our first requirement. It remains to show that F^{-1} is the restriction of a vector bundle on M. After possibly localizing U, we can assume that F^{-1} is free, where the claim obviously holds. $\square$

Corollary 5.13. *Every virtually smooth Deligne-Mumford stack (X, E) carries a Kuranishi structure.*

Proof. Since Kuranishi neighbourhoods exist locally, there exist Kuranishi neighbourhoods κ_{U_i} for local embeddings $U_i \hookrightarrow M_i$ such that $\coprod_{i \in I} U_i \to X$ is surjective. Define $U := \coprod_{i \in I} U_i$. Observe that since the fiber product commutes with direct product, we can compute

$$\coprod_{i \in I} M_i \times_{\coprod_{i \in I} \mathbb{V}(F_i)} \coprod_{i \in I} M_i = \coprod_{i \in I} \big(M_i \times_{\mathbb{V}(F_i)} M_i\big) = \coprod_{i \in I} U_i,$$

and it follows that the direct product of the Kuranishi neighbourhoods κ_{U_i} is a Kuranishi neighbourhood κ_U for U. $\square$

Our next aim concerns the existence of morphisms of Kuranishi neighbourhoods. We want to show that given a morphism of local embeddings, we can in a certain sense pull back a Kuranishi structure. Furthermore, given a smooth morphism of local embeddings along with two Kuranishi structures, we wish to prove that there always exists a morphism of Kuranishi neighbourhoods. Unfortunately these seem to only exist in a non-canonical manner, since many choices are involved.

Lemma 5.14. *Let*

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "p"'] & N \arrow[d, "q"] \\
U \arrow[r, "i"'] & M
\end{tikzcd}$$

be a smooth morphism of local embeddings. Assume we have a Kuranishi neighbourhood κ_U for the local embedding $i\colon U \hookrightarrow M$. Then there exists a Kuranishi neighbourhood κ_V for the local embedding $j\colon V \hookrightarrow N$.

Proof. We have an exact sequence

$$q^*I \longrightarrow J \longrightarrow \Omega^1_{N/M}.$$

As $\Omega^1_{N/M}$ is projective, we can choose a splitting $J \simeq \Omega^1_{N/M} \oplus q^*I$. We now define the vector bundle needed on N to be $G := q^*F \oplus \Omega^1_{N/M}$, and define the section as

$$s'\colon G \overset{(q^*s,\mathrm{id})}{\longrightarrow} q^*I \oplus \Omega^1_{N/M} \longrightarrow J.$$

Define κ_V to be $N \overset{s'}{\to} \mathbb{V}(G) \overset{0}{\leftarrow} N$. Using that the rows in the diagram

$$\begin{tikzcd}
& p^*F|_U \arrow[rr] \arrow[d] \arrow[ddl, bend right] & & G|_V \arrow[rr] \arrow[d] \arrow[ddl, bend right] & & \Omega^1_{N/M}|_V \arrow[d] \arrow[ddl, bend right] \\
& p^*I/I^2 \arrow[rr] \arrow[dl] & & J/J^2 \arrow[rr] \arrow[dl] & & \Omega^1_{N/M}|_V \arrow[dl] \\
p^*\Omega^1_M|_U \arrow[rr] & & \Omega^1_N|_V \arrow[rr] & & \Omega^1_{N/M}|_V &
\end{tikzcd}$$

are exact it is straightforward to check that κ_V is a Kuranishi neighbourhood. □

Remark 5.15. From the proof of the previous lemma it is immediate that not only is κ_V a Kuranishi neighbourhood, but there also exists a morphism of Kuranishi neighbourhoods $\alpha\colon \kappa_V \to \kappa_U$. A more general statement on the existence of Kuranishi neighbourhood appears in Proposition 5.20.

Remark 5.16. As a quick sanity check, we confirm that the expected dimension of the Kuranishi neighbourhood defined above is correct. Let $\dim(M) = m$, $\operatorname{rk} F = f$, $\dim N = n$ and $\operatorname{rk} G = g$. The expected dimension for U is $m - f$. As $n - g = n - (f + n - m) = m - f$, this is also the expected dimension of V.

Remark 5.17. The Kuranishi neighbourhood constructed on V in the above lemma is not canonical since it depends on the choice of the splitting.

We next want to prove the existence of morphisms of Kuranishi neighbourhoods. We first need two preparatory lemmas. The first will be used to lift morphisms from the derived category to honest morphisms of complexes.

Lemma 5.18. *Let* $\begin{tikzcd} A \arrow[r, "f"'] \arrow[rr, bend left, "h"] & B \arrow[r, "g"'] & C \end{tikzcd}$ *be morphisms of two-term complexes such that there exists a homotopy* $g \circ f \sim h$*. Assume further that* $g^{-1} \circ f^{-1} \colon A^{-1} \to C^{-1}$ *is surjective and that* A^0 *is projective. Then there exists a morphism* $\tilde{f} \colon A \to B$ *along with a homotopy* $\tilde{f} \sim f$ *such that*

$$g \circ \tilde{f} = h.$$

Proof. Let the dotted morphism k in the diagram

$$\begin{tikzcd} A^{-1} \arrow[r, "d_A"] \arrow[d, "f^{-1}"] \arrow[dd, bend right=60, "h^{-1}"'] & A^0 \arrow[d, "f^0"] \arrow[dd, bend left=60, "h^0"] \arrow[ddl, dotted, "k"'] \\ B^{-1} \arrow[r, "d_B"] \arrow[d, "g^{-1}"] & B^0 \arrow[d, "g^0"] \\ C^{-1} \arrow[r, "d_C"'] & C^0. \end{tikzcd}$$

denote the homotopy between $g \circ f$ and h. As $g^{-1} \circ f^{-1} \colon A^{-1} \to C^{-1}$ is surjective and A^0 is projective, there exists a lift of k to $\tilde{k} \colon A^0 \to B^{-1}$. Now define $\tilde{f}$ by the formulas

$$\tilde{f}^0 = f^0 + d_B \circ \tilde{k}$$
$$\tilde{f}^{-1} = f^{-1} + \tilde{k} \circ d_A.$$

□

The second lemma concerns lifting a morphism of vector bundles to a smooth ambient space.

Lemma 5.19. *Let* $U \hookrightarrow M$ *be a closed immersion of affine schemes with ideal sheaf* I*. Assume that* F *and* G *are locally free sheaves on* M *with morphisms*

$$F \xrightarrow{s} I \xleftarrow{s'} G$$

with $s \colon F \to I$ *surjective. Assume that we have a commutative diagram*

$$\begin{tikzcd} G|_U \arrow[rr, "\phi"] \arrow[dr, "s'|_U"'] & & F|_U \arrow[dl, "s|_U"] \\ & I/I^2. & \end{tikzcd}$$

Then there exists a morphism $\Phi\colon G \to F$ such that $\Phi|_U = \phi$ and such that

$$\begin{tikzcd}
G \arrow[rr, "\Phi"] \arrow[dr, "s'"'] & & F \arrow[dl, "s"] \\
& I &
\end{tikzcd}$$

commutes.

Proof. Denote the restriction morphisms $F \to F|_U$ by res. We then have a commutative diagram

$$\begin{tikzcd}
& G \arrow[d, "{(s',\phi\circ\mathrm{res})}"'] \arrow[dr, "0"] & \\
F \arrow[r, "{(s,\mathrm{res})}"'] & I \oplus F|_U \arrow[r, "{(\mathrm{res},-s|_U)}"'] & I/I^2.
\end{tikzcd}$$

We thus get a morphism $\Phi\colon G \to \ker(\mathrm{res}, -s|_U)$. A diagram chase in

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
I \otimes F \arrow[r, two heads] \arrow[d] & I^2 \arrow[d] \\
F \arrow[r, two heads] \arrow[d] & I \arrow[d] \\
F/IF \arrow[r, two heads] \arrow[d] & I/I^2 \arrow[d] \\
0 & 0
\end{tikzcd}$$

shows that the lower column is exact at $I \oplus F|_U$, and thus the claim follows. $\square$

Note that Φ is not unique.

We can now state the result on the existence of morphisms of Kuranishi neighbourhoods.

Proposition 5.20. *Let*

$$\begin{tikzcd}
& V \arrow[r, "j"] \arrow[dl, "f"'] \arrow[d, "p"] & N \arrow[d, "q"] \\
X & U \arrow[l, "g"] \arrow[r, "i"'] & M
\end{tikzcd}$$

be a morphism of local embeddings of (X, E). Let κ_U and κ_V be Kuranishi neighbourhoods for U and V. Then there exists a morphism of Kuranishi neighbourhoods $\alpha\colon \kappa_V \to \kappa_U$ lifting the morphism of local embeddings above.

Proof. Let $[G|_V \to \Omega^1_N|_V] \longrightarrow [J/J^2 \to \Omega^1_N|_V]$ be the perfect obstruction theory on V induced by κ_V, $[F|_U \to \Omega^1_M|_U] \longrightarrow [I/I^2 \to \Omega^1_M|_U]$ be the analogous object on U. Note that we have a canonical quasi-isomorphism of complexes $p^*[I/I^2 \to \Omega^1_M|_U] \longrightarrow [J/J^2 \to \Omega^1_N|_V]$ induced by p as p is étale. As the above morphisms of complexes are both local presentations of the perfect obstruction theory $\phi\colon E \to \tau_{\geq -1}\mathbb{L}_X$, we have the following diagram in $D(\mathcal{O}_V)$

$$\begin{tikzcd}
{[G|_V \to \Omega^1_N|_V]} \arrow[rrr] \arrow[dr, dotted, "a"] & & & {[J/J^2 \to \Omega^1_N|_V]} \arrow[dl, dotted] \\
& E_V \arrow[r, dotted, "\phi"] & \tau_{\geq -1}\mathbb{L}_V & \\
p^*[F|_U \to \Omega^1_M|_U] \arrow[rrr] \arrow[ur, dotted, "b"] & & & p^*[I/I^2 \to \Omega^1_M|_U] \arrow[ul, dotted] \arrow[uu]
\end{tikzcd}$$

The dotted morphisms are supposed to indicate morphisms that a priori only live in the derived category, whereas the solid morphisms are morphisms of complexes. In the derived category there clearly exists a morphism filling in the missing edge on the left by setting $\gamma := a^{-1} \circ b$. Denote by $K(\mathcal{O}_V)$ the category of complexes of $\mathcal{O}_V$-modules modulo homotopy equivalence. As V is an affine scheme and the complex $p^*[F|_U \to \Omega^1_M|_U]$ consists of projectives, γ is actually a morphism of complexes making the outer square commutative in $K(\mathcal{O}_V)$ by [Wei94, Corollary 10.4.7]. By Lemma 5.18 by modifying γ this square commutes as morphisms of complexes. Applying Lemma 5.19 we can lift the resulting morphism $p^*F|_U \to G_V$ to a morphism $p^*F \to G$ giving the desired morphism of Kuranishi neighbourhoods. $\square$

Chapter 6

Gluing the Local Models

In chapter 5, we showed that every virtual smooth Deligne-Mumford stack admits an extra geometric structure, which we called an algebraic Kuranishi structure. In the introduction to [Joy07], Joyce outlines the expectation that such structures should be conveniently arranged using some form of derived geometry. The extra information encoded by the Kuranishi structure is what the space would look like if it were perturbed locally, and such information can be conveniently handled in derived geometry. Using the material from chapter 4 it is clear how to do this locally. Providing an algebraic Kuranishi structure for a virtually smooth Deligne-Mumford stack consists of finding local presentations as zero-sections of vector bundles, thus writing locally the space as a fiber product. Taking the homotopy fiber product instead of the ordinary fiber product encodes locally precisely the information of what the space looks like after perturbation. It thus remains to take care of gluing these local derived extensions to one global derived space, which we will do in this chapter.

We begin by giving a precise definition of the homotopy limit. To ensure that this is indeed a functor we have to choose a fibrant replacement functor in the appropriate diagram category. This amounts to choosing functorial models for the homotopy limit. All the model categoric details involved can be found in Appendix A.3.

We will then fix one specific cover of our virtually smooth space that admits a Kuranishi structure. Using material from the previous chapter, we show that the double intersections, triple intersections and so on of this cover are infact Kuranishi neighbourhoods. Choosing these Kuranishi neighbourhoods is a non-canonical non-functorial procedure. We will then go on to show that these ambiguities disapper after taking the homotopy limit of the Kuranishi neighbourhoods. This will finally allow us to glue the homotopy limits of the Kuranishi neighbourhoods to one global derived space.

Recall that as in the previous chapters, $\mathbf{D}$ denotes the index category $\{a \xrightarrow{\epsilon} b \xleftarrow{\delta} c\}$.

Functorial models for homotopy limits are given by fixing a fibrant replacement functor in the following definition.

Definition 6.1. Let $i\colon \mathbf{Aff}_k \to \mathbf{dAff}_k$ be the inclusion functor. Equip the category $\mathbf{dAff}_k^{\mathbf{D}}$ with the injective model structure and let R be a fibrant replacement functor. Define

$$\begin{aligned} \operatorname{holim}_{d\in\mathbf{D}}\colon \mathbf{Kur}_X &\longrightarrow \mathbf{dAff}_k \\ \kappa_U &\longmapsto \operatorname{holim}_{d\in\mathbf{D}} \kappa_U \end{aligned}$$

to be the functor defined by the restriction to $\mathbf{Kur}_X$ of the composition

$$\mathbf{Aff}_k^{\mathbf{D}} \xrightarrow{i_*} \mathbf{dAff}_k^{\mathbf{D}} \xrightarrow{R} \mathbf{dAff}_k^{\mathbf{D}} \xrightarrow{\lim} \mathbf{dAff}_k.$$

From the properties of homotopy fiber products reviewed in chapter 4 we obtain the following properties of the homotopy limit of a Kuranishi neighbourhood.

Corollary 6.2. *Let $X \xleftarrow{f} U \xhookrightarrow{i} M$ be a local embedding and κ_U a Kuranishi neighbourhood. Let $U' := \operatorname{holim}_{d\in\mathbf{D}} \kappa_U$.*

(i) On the truncation we have $t_0(U') = U$.

(ii) Let $j_U : U \hookrightarrow U'$ be the canonical inclusion. Then

$$j_U^* \mathbb{L}_{U'} \simeq [F|_U \xrightarrow{ds} \Omega^1_M|_U].$$

(iii) The derived scheme U' is quasi-smooth.

We immediately verify that the morphism between the homotopy limits is independent of some of the choices we will later make.

Lemma 6.3. *Let $\alpha\colon \kappa_V \to \kappa_U$ be a natural transformation lifting the morphism of local embeddings*

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "p"'] & N \arrow[d, "q"] \\ U \arrow[r, "i"'] & M. \end{tikzcd}$$

Then the morphism $\operatorname{holim}_{d\in\mathbf{D}}(\alpha)\colon \operatorname{holim}_{d\in\mathbf{D}} \kappa_V \to \operatorname{holim}_{d\in\mathbf{D}} \kappa_U$ *is independent of* $\alpha(b)$.

Proof. Let $\kappa_U = (M \to \mathbb{V}(F) \leftarrow M)$, and let $\kappa_V = (N \to \mathbb{V}(G) \leftarrow N)$. Let R be the fibrant replacement functor in the diagram category $\mathbf{dAff}_k^{\mathbf{D}}$. Denote the fibrant replacement by $R(\kappa_U) = (M' \to \mathbb{V}(F)' \leftarrow M')$, and use the analogous notation for $R(\kappa_V)$. Let α and α' be two natural transformations lifting the morphism of local embeddings above which only differ at the level $\alpha(b)\colon \mathbb{V}(G) \to \mathbb{V}(F)$. We have to show that $R\alpha$ and $R\alpha'$ define the same object $\lim_{d\in\mathbf{D}} R\kappa_V \to R\kappa_U$ in the category of cones $\mathbf{dAff}_{\mathbf{k}/R\kappa_U}$. We have a diagram of the fibrant replacements

$$
\begin{tikzcd}
 & & & \lim_{d\in\mathbf{D}} R(\kappa_U) \arrow[dl, bend left] \arrow[dddl, bend left] \arrow[dddddl, bend left] \\
 & & M' \arrow[dd] & \\
 & N' \arrow[ur, "R\alpha(a)"] \arrow[d] & & \\
\lim_{d\in\mathbf{D}} R(\kappa_V) \arrow[uuurrr, dotted, bend left] \arrow[ur, "r"] \arrow[r, "s"] \arrow[dr, "t"'] & \mathbb{V}(G)' \arrow[r, shift left, "R\alpha(b)"] \arrow[r, shift right, "R\alpha'(b)"'] & \mathbb{V}(F)' & \\
 & N' \arrow[u] \arrow[dr, "R\alpha(c)"'] & & \\
 & & M' \arrow[uu, "R\kappa_U(\delta)"'] &
\end{tikzcd}
$$

The cone formed by the curved arrows on the right hand side is the terminal object of the category $\mathbf{dAff}_{\mathbf{k}/R\kappa_U}$. The morphisms r, s and t are induced by the universal property of the limit of $R\kappa_V$. Finally, the dotted arrow is induced by the universal property of the cone formed by the curved arrows as terminal object of $\mathbf{dAff}_{\mathbf{k}/R\kappa_U}$. As

$$R\alpha(b) \circ s = R\kappa_U(\delta) \circ R\alpha(c) \circ t = R\alpha'(b) \circ s$$

the cones $\lim_{d\in\mathbf{D}} R(\kappa_V) \to R\kappa_U$ defined by different choices of $\alpha(b)\colon \mathbb{V}(G) \to \mathbb{V}(F)$ define the same object in $\mathbf{dAff}_{\mathbf{k}/R\kappa_U}$, and thus the induced morphisms of the limits $\lim_{d\in\mathbf{D}} R(\kappa_V) \to \lim_{d\in\mathbf{D}} R(\kappa_U)$ coincide as both are induced by the unversal property of the cone defined by the curved arrows as the terminal object in $\mathbf{dAff}_{\mathbf{k}/R\kappa_U}$. □

A further important property of the induced morphism is the following. Assume we are given a morphism of local embeddings of X which both admit Kuranishi neighbourhoods, and that we have a natural transformation between these neighbourhoods lifting the given morphism of local embeddings. By the functoriality of the homotopy limit, we obtain a morphism between the homotopy limits of the Kuranishi neighbourhoods. We next want to settle that the truncation of the morphism between the homotopy limits is indeed the morphism over X that we started out with.

Lemma 6.4. *Let $\alpha\colon \kappa_V \to \kappa_U$ be a morphism of Kuranishi neighbourhoods lifting* *the local embedding*

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "p"'] & N \arrow[d, "q"] \\ U \arrow[r, "i"'] & M. \end{tikzcd}$$

Then

$$t_0(\operatorname{holim}_{d\in\mathbf{D}} \alpha) = p$$

Proof. Recall that by definition $p = \lim_{d\in\mathbf{D}} \alpha$. As the homotopy limit functor commutes with the truncation functor, we can conclude

$$t_0(\operatorname{holim}_{d\in\mathbf{D}} \alpha) = \operatorname{holim}_{d\in\mathbf{D}}(t_0(\alpha)) = \lim_{d\in\mathbf{D}} \alpha = p$$

□

A further key property we will make use of later is that the induced morphisms are étale. To prove this we again need a preparatory lemma.

Lemma 6.5. *Let A be a simplicial commutative k-algebra, and let M be a simplicial A-module. Assume that $M \otimes_A^L \pi_0(A) \simeq 0$ in $D(\pi_0(A))$. Then $M \simeq 0$.*

Proof. We have to show that $\pi_i(M) = 0$ for all i. We prove this by induction. By Quillen's tor spectral sequence, we have

$$\begin{aligned} 0 \simeq & \pi_0(M \otimes_A^{\mathbb{L}} \pi_0(A)) \\ \simeq & \pi_0(M) \otimes_{\pi_0(A)} \pi_0(A) \\ \simeq & \pi_0(M), \end{aligned}$$

proving the result for $n = 0$. Assume now that $\pi_i(M) = 0$ for $i < n$, that is that M is $n-1$-connected. Thus $M[n]$ is connective. We now conclude

$$\begin{aligned} 0 \simeq & \pi_n(M \otimes_A^{\mathbb{L}} \pi_0(A)) \\ \simeq & \pi_0(M[n] \otimes_A^{\mathbb{L}} \pi_0(A)) \\ \simeq & \pi_0(M[n]) \otimes_{\pi_0(A)} \pi_0(A) \\ \simeq & \pi_n(M). \end{aligned}$$

□

Remark 6.6. We can rephrase the previous lemma in geometric terms. Let $Y' = \mathbb{R}\mathrm{Spec}\ A$, let $Y = t_0(Y') = \mathrm{Spec}\,(\pi_0(A))$, and let $j_Y : Y \hookrightarrow Y'$ denote the canonical inclusion, which is dual to the morphism $A \to \pi_0(A)$. Let $M_{Y'}$ be the quasicoherent module obtained under the equivalence of categories $\phi\colon \mathbf{QCoh}_{Y'} \to \mathbf{M}_{A,\geq 0}$ of [HAG-II, Section 1.3.7]. As ϕ is compatible with pullback [HAG-II, p. 89], we can conclude

$$\phi(j^* M_{Y'}) = M \otimes_A^{\mathbb{L}} \pi_0(A).$$

Since ϕ maps equivalences to equivalences, the previous lemma then states that if $j_Y^* M_{Y'} \simeq 0$, then $M_{Y'} \simeq 0$.

We can now show that the induced morphism is indeed étale.

Proposition 6.7. *Let $\alpha\colon \kappa_V \to \kappa_U$ be a morphism of Kuranishi neighbourhoods lifting the local embedding*

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "p"'] & N \arrow[d, "q"] \\ U \arrow[r, "i"'] & M. \end{tikzcd}$$

Then $\operatorname{holim}_{d\in\mathbf{D}}(\alpha)$ *is an étale morphsim.*

Proof. Let $\operatorname{holim}_{d\in\mathbf{D}}\kappa_V = V'$, $\operatorname{holim}_{d\in\mathbf{D}}\kappa_U = U'$, and let $\operatorname{holim}_{d\in\mathbf{D}}\alpha = p'$. We have to show that $\mathbb{L}_{V'/U'} \simeq 0$. Denote the canonical inclusions by $j_U\colon U \hookrightarrow U'$ and the analogous notation for V. By Lemma 6.5, it suffices to show that $j^*\mathbb{L}_{V'/U'} \simeq 0$. The diagram

$$\begin{tikzcd} V \arrow[r, "j_V"] \arrow[d, "p"'] & V' \arrow[d, "p'"] \\ U \arrow[r, "j_U"'] & U' \end{tikzcd}$$

commutes since $t_0(p') = p$ by Lemma 6.4. Pulling back the cotangent sequence for p' back to V, we have the following exact triangle:

$$j_V^*(p')^*\mathbb{L}_{U'} \to j_V^*\mathbb{L}_{V'} \to j_V^*\mathbb{L}_{V'/U'}.$$

Using the commutativity of the above diagram and Lemma 4.10 it follows that the first two terms in the triangle are quasi-isomorphic to E_V and the map between them is a quasi-isomorphism. Thus $\mathbb{L}_{V'/U'} \simeq 0$. $\square$

This settles all properties of the homotopy limit of Kuranishi neighbourhoods we will make use of. We now show that once we have a cover of our virtually smooth space admitting a Kuranishi structure, also the whole nerve of the cover admits (in a non-canonical way) a Kuranishi structure.

For the remainder of this chapter, fix one algebraic Kuranishi structure κ for X. Let $U \to X$ be the corresponding étale cover of X and $U \hookrightarrow M$ the corresponding embedding. We now fix some notation. Let $N(U)\colon \Delta^{\mathrm{op}} \to \mathbf{Aff}_k$ be the nerve of this cover. By definition, $N(U)([0]) = U$ and we set

$$N(U)([i]) = R_i,\ i \geq 1.$$

Let $p_j\colon R_i \to R_{i-1}$ be the face maps, and $r_j\colon R_i \to R_{i+1}$ the degeneracies. On the other hand, we can also take the absolute nerve $N(M)$ of M over $\operatorname{Spec} k$. The i-th level $N(M)([i])$ is just the $i+1$-fold absolute product of M. We will denote

$$N(M)([i]) = M^{\times i+1},$$

by q_i the face maps, and by t_i the degeneracies. By taking repeated products[1] of local embeddings we obtain commutative diagrams

$$\begin{tikzcd} R_i \arrow[r] \arrow[d, "p_j"'] & M^{\times i+1} \arrow[d, "q_j"] \\ R_{i-1} \arrow[r] & M^{\times i} \end{tikzcd}$$

and

$$\begin{tikzcd} R_{i+1} \arrow[r] & M^{\times i+2} \\ R_i \arrow[r] \arrow[u, "r_j"] & M^{\times i+1}. \arrow[u, "t_j"'] \end{tikzcd}$$

We thus obtain a simplicial object $\widetilde{N(U)}$ in the category of local embeddings $\mathbf{LocEmb}_X$. Denote the face maps of this simplicial object by d_i and the degeneracies by s_i.

All of the face maps

$$\begin{tikzcd} R_i \arrow[r] \arrow[d, "p_j"'] & M^{\times i+1} \arrow[d, "q_j"] \\ R_{i-1} \arrow[r] & M^{\times i} \end{tikzcd}$$

satisfy the conditions of Lemma 5.14. Thus since U has a Kuranishi neighbourhood κ_U there exists a Kuranishi neighbourhood for R_1. Proceeding inductively, all R_i are in fact Kuranishi neighbourhoods. Fix one such Kuranishi neighbourhood κ_{R_i} for each R_i.

Each of the face maps s_i and degeneracy maps d_i occuring in $\widetilde{N(U)}$ satisfy the condition of Proposition 5.20. We can thus lift them to morphisms of Kuranishi neighbourhoods α_{s_i} and α_{d_i}. As the choice of Kuranishi neighbourhoods is non-canonical these will in general not fulfill the simplicial identities.

Remark 6.8. We pause briefly and recall how many choices we made and which ones are canonical. We have listed them in the order made. Objects which canonically depend on choices made previously are listed as canonical.

object	canonical	non-canonical
$U \to X$		✓
$U \hookrightarrow M$		✓
$N(U), p_i, r_i$	✓	
$N(M), q_i, t_i$	✓	
$\widetilde{N(U)}, d_i, s_i$	✓	
κ_U		✓
κ_{R_i}		✓
$\alpha_{d_i}, \alpha_{s_i}$		✓

[1]see Definition 2.25

One should note though that two of the three components of the natural transformations α_{d_i} and α_{s_i} are indeed canonical, since they are by definition the projections q_i and the inclusions t_i. Only the third morphism ocurring in the natural transformation is non-canonical, but in fact the morphisms between the homotopy limits induced by α_{d_i} and α_{s_i} are independent of this non-canonical morphism by Lemma 6.3.

Now define a mapping $X'_*\colon \Delta^{\mathrm{op}} \to \mathbf{dAff}_k$ by setting

$$\begin{aligned}
[0] &\longmapsto \operatorname{holim}_{d\in\mathbf{D}} \kappa_U \\
[i] &\longmapsto \operatorname{holim}_{d\in\mathbf{D}} \kappa_{R_i},\ i \geq 1
\end{aligned}$$

on objects and

$$\begin{aligned}
s^i &\longmapsto \operatorname{holim}_{d\in\mathbf{D}} \alpha_{s_i} \\
d^i &\longmapsto \operatorname{holim}_{d\in\mathbf{D}} \alpha_{d_i}
\end{aligned}$$

on morphisms. To cut down on notation, will decorate the objects obtained by a homotopy limit by a prime, e.g. $\operatorname{holim}_{d\in\mathbf{D}} \kappa_U = U'$ and $\operatorname{holim}_{d\in\mathbf{D}} \alpha_{s_i} = s'_i$.

In the remainder of this chapter will show that X'_* is an étale Segal groupoid in the category $\mathbf{dAff}_k$. We start off by verifying that X'_* indeed defines a functor.

Lemma 6.9. *The mapping* $X'_*\colon \Delta^{\mathrm{op}} \to \mathbf{dAff}_k$ *is a functor.*

Proof. Recall that every morphism in Δ has a unique factorization into face and degeneracy maps by [Wei94, Lemma 8.1.2]. Since we have already defined X'_* on these maps, to verify that X'_* is indeed a functor it suffices to check the simplicial identities. But these follow immediately from the observation that $\operatorname{holim}_{d\in\mathbf{D}} \alpha$ is independent of the choice made in the construction of the morphisms α_{s_i} and α_{d_i} by Lemma 6.3. We exemplarily verify the simplicial identity $d'_i \circ d'_j = d'_j \circ d'_i$.

Let $\kappa_{R_i} = \big(M^{\times i+1} \to \mathbb{V}(F_{i+1}) \leftarrow M^{\times i+1}\big)$. As usual, the objects of $R\kappa_{R_i}$ will be decorated by primes. The morphism $d'_i \circ d'_j$ is induced by the morphism of cones

$$\begin{tikzcd}
 & & & & R'_n \arrow[dl, bend left] \arrow[ddddl] \arrow[dddddddl] \\
 & & & (M^{\times n})' \arrow[ddd] & \\
 & & (M^{\times n+1})' \arrow[ur, "Rq_j"] \arrow[dd] & & \\
 & (M^{\times n+2})' \arrow[ur, "Rq_i"] \arrow[d] & & & \\
R'_{n+2} \arrow[uuuurrrr, dotted, bend left] \arrow[ur] \arrow[r] \arrow[dr] & \mathbb{V}(F_{n+2})' \arrow[r] & \mathbb{V}(F_{n+1})' \arrow[r] & \mathbb{V}(F_n)' & \\
 & (M^{\times n+2})' \arrow[u] \arrow[dr, "Rq_i"'] & & & \\
 & & (M^{\times n+1})' \arrow[uu] \arrow[dr, "Rq_j"'] & & \\
 & & & (M^{\times n})' \arrow[uuu] &
\end{tikzcd}$$

The morphism $d'_j \circ d'_i$ is induced by the corresponding morphism of cones with the roles of j and i reversed. By definition, $q_i \circ q_j = q_j \circ q_i$. Since R is a functor, we have $Rq_i \circ Rq_j = Rq_j \circ Rq_i$. Thus the two morphisms of cones coincide by Lemma 6.3, and for the induced morphisms on the limits we have $d'_i \circ d'_j = d'_j \circ d'_i$. $\square$

The next task is to verify that X'_* is a Segal groupoid.

Proposition 6.10. *The simplicial object X'_* is an étale Segal groupoid.*

Proof. Recall that a morphism of affine derived schemes is an equivalence if it is an isomorphism on the truncation and the relative cotangent complex vanishes.

We first verify the second of the Segal conditions. We have to show that the map γ in the diagram

$$\begin{tikzcd}
R'_2 \arrow[dr, "\gamma"] \arrow[drr, bend left, "d'_1"] \arrow[ddr, bend right, "d'_0"'] & & \\
& R'_1 \times^h_{U'} R'_1 \arrow[r] \arrow[d, "\beta"'] & R'_1 \arrow[d, "d'_0"] \\
& R'_1 \arrow[r, "d'_0"'] & U'
\end{tikzcd}$$

is an equivalence. Applying the truncation functor to this diagram we obtain the corresponding diagram for the ordinary groupoid $N(U)$. Thus γ is an isomorphism on the truncation.

The morphism d'_0 is étale by Proposition 6.7. Thus β is also étale since it is the pullback of an étale morphism. As $\beta \circ \gamma = d'_0$, it follows that γ is étale and hence the relative cotangent complex of γ vanishes. So γ is an equivalence.

The proof of the other Segal condition is exactly the same only involving more indices and is hence omitted. $\square$

Theorem 6.11. *Let (X, E) be a virtually smooth Deligne-Mumford stack. Then there exists a quasi-smooth derived Deligne-Mumford stack X' such that*

(i) the derived stack X' is quasi-smooth;

(ii) there is an equialence $t_0(X') \simeq X$ of stacks;

(iii) for all points $p\colon \operatorname{Spec} k \to X'$ we have

$$\mathbb{L}_{X',x} \simeq E_p$$

Proof. We continue to use the notation introduced above. In particular, we have fixed a Kuranishi structure κ with respect to the local embedding $U \hookrightarrow M$. We will continue to denote by X'_* the étale Segal groupoid produced above.

Define

$$X' = \operatorname{hocolim}_{i \in \Delta^{\mathrm{op}}} X'_*.$$

As X'_* is an étale Segal groupoid, the stack X' is indeed a derived Deligne-Mumford stack. Since quasi-smoothness is a local on the source, quasi-smoothness of X' follows from Proposisition 4.8 and Corollary 6.2. We move on to the statement on the truncation. Again by Corollary 6.2 along with Lemma 6.4, we have

$$t_0(X'_*) = N(U)$$

where $N(U)$ is the nerve of the étale covering $U \to X$. Since $U \to X$ is an étale cover, we have

$$\operatorname{hocolim}_{i \in \Delta^{\mathrm{op}}} N(U) \simeq X.$$

As the truncation functor commutes with homotopy colimits, we can compute

$$t_0(X') = t_0(\operatorname{hocolim} X'_*) \simeq \operatorname{hocolim}(t_0(X'_*)) = \operatorname{hocolim} N(U) \simeq X.$$

Finally, for the last statement we can assume that $\operatorname{Spec} k \to X'$ factors as

$$\begin{tikzcd} & U' \arrow[d, "\pi"] \\ \operatorname{Spec} k \arrow[ur, "q"] \arrow[r, "p"] & X' \end{tikzcd}$$

where π is the canonical projection and U' is the first level of the Segal groupoid X'_*, which is an atlas for X'_*. Since π is étale, we can conclude

$$p^*\mathbb{L}_{X'} = q^*\pi^*\mathbb{L}_{X'} \simeq q^*\mathbb{L}_{U'} \simeq E_p$$

where the last equivalence is by Corollary 6.2 and the definition of Kuranishi neighbourhood. $\square$

Example 6.12. Let C be a curve and V a smooth projective scheme. Then the scheme of morphisms $\operatorname{Mor}(C,V)$ is virtually smooth. The perfect obstruction theory is obtained from the diagram

$$\begin{tikzcd} \operatorname{Mor}(C,V) \times C \arrow[r, "f"] \arrow[d, "\pi"] & V \\ \operatorname{Mor}(C,V), & \end{tikzcd}$$

where f is the universal morphism by choosing $E = R\pi_* f^*\Omega^1_V$.[2] By the above result, there exists a derived scheme $X' = \operatorname{Mor}(C,V)'$ such that for any point $[g]$ we have

$$H^0(\mathbb{L}_{X',[g]})^\vee = H^0(C, f^*T_V)$$
$$H^1(\mathbb{L}_{X',[g]})^\vee = H^1(C, f^*T_V).$$

[2] For the definition of the morphism to the cotangent complex see [BF97].

Example 6.13. In the situation of the above example, let V be a $K3$-surface. Then there exists another choice of a perfect obstruction theory. This is the reduced theory of [MP07]. It gives an expected dimension which is 1 dimension larger. From our result, we conclude there are two *different* derived extensions of the scheme of morphisms in this case.

Chapter 7

Some Applications

7.1 Virtual Pullbacks

Given two virtually smooth schemes and a morphism between these schemes, it is a natural question to ask whether there exists some notion of pullback that maps the virtual class of the one scheme to the virtual class of the other. This question was addressed by Cristina Manolache in her thesis. The result she obtained is that such pullbacks only exist if a certain compatibility condition holds between the perfect obstruction theories.

We here want to verify that these compatibility conditions always hold for morphisms that arise as truncations of the derived versions of the virtually smooth schemes. The interesting part of this result is that it suggests that virtual classes should really live in the suitably defined homology of the derived extensions. From a practical point of view, it should be just as difficult to find a morphism between derived moduli spaces as finding a morphism on the truncations for which the compatibility condition of Manolache holds.

We first recall the construction of virtual pullbacks from [Man08]. The key idea is to use a specialization map similar to the one used in section 4.3, the major difference being that the target of the specialization map is the intrinsic normal cone. The deformation space necessary for defining the specialization morphism was introduced by Kresch in [Kre99]. For simplicity we work in the absolute case instead of working relative over an Artin stack as in [Man08].

Definition 7.1. Let $f\colon X \to Y$ be a morphism of Deligne-Mumford stacks, and let $\mathfrak{C}_{X/Y}$ be the intrinsic normal cone of f. Let M' be Kresch's deformation space with general fiber Y and special fiber $\mathfrak{C}_{X/Y}$. Define the specialization map

$$\sigma'_{X/Y}\colon A_*(Y) \to A_*(\mathfrak{C}_{X/Y})$$

using the same method as in section 4.3

If we now assume that the intrinsic normal cone $\mathfrak{C}_{X/Y}$ embeds into some vector bundle stack $\mathfrak{E}$, we can define a virtual pullback map depending on $\mathfrak{E}$.

Definition 7.2. Let $f\colon X \to Y$ be a morphism of Deligne-Mumford stacks, and assume that $i\colon \mathfrak{C}_{X/Y} \to \mathfrak{E}$ is a closed immersion into a rank n vector bundle stack. Let $s_0\colon X \to \mathfrak{E}$ be the zero section. Define the *virtual pull back* to be the composition of

$$f^!_{\mathfrak{E}}\colon A_*(G) \overset{\sigma'_{X/Y}}{\to} A_*(\mathfrak{C}_{X/Y}) \overset{i_*}{\to} A_*(\mathfrak{E}) \overset{s_0^*}{\to} A_{*-n}(X).$$

In case the morphism $f\colon X \to Y$ admits a relative perfect obstruction theory, there of course is a canonical choice of $\mathfrak{E}$ and we will drop it from the notation. For such a relative perfect obstruction theory, Manolache obtained the following result.

Theorem 7.3. *[Man08, Corollary 4] Let X, Y be Deligne-Mumford stacks admitting perfect obstruction theories E_X, E_Y, and let $f\colon X \to Y$ be a morphism. Assume that there exists a morphism $\phi\colon f^*E_Y \to E_X$ commuting with $f^*\mathbb{L}_Y \to \mathbb{L}_X$. Then f admits a relative perfect obstruction theory and*

$$f^!([Y]^{\mathrm{vir}}) = [X]^{\mathrm{vir}}.$$

We will now verify that for morphisms which are truncations of derived morphisms, the existence of ϕ with the above properties is automatic. This strongly supports that virtual classes really belong in the realm of derived algebraic geometry.

Theorem 7.4. *Let $f'\colon X' \to Y'$ be a morphism of quasi-smooth derived Deligne-Mumford stacks. Equip the truncations X and Y with the induced perfect obstruction theories. Then there exists a virtual pullback $f^!\colon A_*(Y) \to A_*(X)$ such that*

$$f^!([Y]^{\mathrm{vir}}) = [X]^{\mathrm{vir}}.$$

Proof. We have to produce the morphism ϕ of Theorem 7.3. Let

$$f^*\mathbb{L}_Y \to \mathbb{L}_X \to \mathbb{L}_{X/Y}$$

be the canonical sequence for f, and let

$$(f')^*\mathbb{L}_{Y'} \to \mathbb{L}_{X'} \to \mathbb{L}_{X'/Y'}$$

be the canonical sequence for f'. From the commutative square

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "j_Y"'] & Y' \end{tikzcd}$$

we deduce a morphism of exact triangles

$$\begin{tikzcd} j_X^*(f')^*\mathbb{L}_{Y'} \arrow[r] \arrow[d] & j_X^*\mathbb{L}_{X'} \arrow[r] \arrow[d] & \mathbb{L}_{X'/Y'} \arrow[d] \\ f^*\mathbb{L}_Y \arrow[r] & \mathbb{L}_X \arrow[r] & \mathbb{L}_{X/Y}. \end{tikzcd}$$

Now take ϕ to be the first horizontal morphism. □

Remark 7.5. Note that for morphisms of Deligne-Mumford stacks which are not truncations of derived morphisms, it is in general *not* possible to produce a relative perfect obstruction theory from the absolute ones. Thus a virtual pullback is not always defined.

Remark 7.6. It is surprising that the virtual pullback exists without a flatness assumption on f', since in classical intersection theory the pullback only exists for flat morphisms.

7.2 Extended Deformation Functors

A central thesis in the study of deformation functors over a field of characteristic zero is that every deformation functor should arise as the Maurer-Cartan elements of a differential graded Lie algebra. In general the process of finding an appropriate differential graded Lie algebra is a difficult task. In this brief application we will show that for a moduli problem admitting a perfect obstruction theory, the perfect obstruction theory itself admits the structure of a differential graded Lie-algebra after shifting and dualizing.

The result we will be a direct corollary of a theorem of Lurie [Lur10]. Before stating the theorem we have to recall some definitions. Lurie proposes the following replacement for the category of Artinian rings with a fixed residue field in the derived setting.

Definition 7.7. Let k be a field, and A a simplicial commutative algebra over k. The algebra A is *small* if

(i) for every $n \geq 0$, the homotopy group $\pi_n(A)$ is a finite dimensional k-vector space,

(ii) for some $n > 0$, the homotopy groups $\pi_n(A)$ vanish,

(iii) and the commutative ring $\pi_0(A)$ is a local ring with residue field k.

Remark 7.8. A discrete small k-algebra A is an Artinian ring with residue field k.

In classical deformation theory not all functors of Artin rings are considered, but only functors satisfying some conditions making them into a deformation functors. For instance, all authors require that a deformation functor should map the ground field k to a one-point set. The other conditions tend to vary from author to author. Lurie proposes the following conditions, and calls the resulting functors formal moduli problems.

Definition 7.9. Let k be a field, and $\mathbf{skAlg}_{\mathrm{sm}}$ the category of small simplicial commutative k-algebras. A functor $F\colon \mathbf{skAlg}_{\mathrm{sm}} \to \mathbf{sSet}$ is a *formal moduli problem* if

(i) the simplicial set $F(k)$ is contractible;

(ii) for all maps of small simplicial commutative k-algebras $\phi\colon A \to B$ and $\phi'\colon A' \to B$ which induce surjections $\pi_0(A) \to \pi_0(B)$ and $\pi_0(A') \to \pi_0(B)$, the canonical map

$$F(A \times_B^h A') \to F(A) \times_{F(B)}^h F(A')$$

is a weak equivalence of simplicial sets.

Lurie's theorem is stated as an equivalence of ∞-categories. For a brief introduction we refer to [Gro10].

Definition 7.10. Let $\mathbf{Lie}_{\mathrm{dg}}^k$ be the ∞-category which is the localization of the category of differential graded Lie algebras over k at the morphisms which are quasi-isomorphisms of chain complexes.

For the following definition we need the fact that the ∞-category of ∞-categories has an internal Hom object with the correct universal property.

Definition 7.11. View $\mathbf{skAlg}$ and $\mathbf{sSet}$ as ∞-categories. Let $\mathbf{Fun}_\infty(\mathbf{skAlg}, \mathbf{sSet})$ be the ∞-category of functors.

Lurie then defines the tangent complex of to a formal moduli problem.

Definition 7.12. Let F be a formal moduli problem, and let $k[n]$ be the n-fold shift of k in the category of simplicial commutative k-algebras. Define the n-th tangent space

$$T_F(n) := F(k \oplus k[n]).$$

It is a non-trivial fact that the collection of simplicial sets $T_X(n)$ form a spectrum and indeed an E_∞-module over k. Using that in characteristic 0 the category of E_∞-algebras over k is equivalent as ∞-category to the localization of commutative algebra objects in chain complexes at quasi-isomorphisms, we can view the spectrum T_X as a differential graded module over k. The main theorem then is:

Theorem 7.13. *Let k be a field of characteristic zero, and let $\mathbf{Moduli}$ denote the full subcategory of $\mathbf{Fun}_\infty(\mathbf{skAlg}_{\mathrm{sm}}, \mathbf{sSet})$ spanned by the formal moduli problems over k. Then there is an equivalence of ∞-categories*

$$\Phi\colon \mathbf{Moduli} \to \mathbf{Lie}_{\mathrm{dg}}^k$$

with $\Phi(F) = T_F[-1]$. The inverse to Φ is given by solutions to the Maurer-Cartan equation of a differential graded Lie algebra.

Remark 7.14. This statement can also be found in [Pri10].

Adding up this theorem with Theorem 6.11 we obtain the following Corollary.

Corollary 7.15. *Let k be a field of characteristic zero, and let X be a Deligne-Mumford stack over k with a perfect obstruction theory $E \to \mathbb{L}_X$. Let p be a k-point of X, and let E_p be the pullback of E to p. Then $E_p^\vee[-1]$ has the structure of a differential graded Lie algebra such that the induced deformation functor is X_p.*

Proof. By Theorem 6.11 there exists a derived stack X', along with the canonical inclusion $j\colon X \to X'$ such that $p^*\mathbb{L}_{X'} = E_p$ for every point p of X. Let

$$p' := j \circ p$$

be the corresponding point of X', and denote by $X'_{p'}$ the functor of small simplicial commutative k-algebras. Since it arises from a geometric object, it is a formal moduli problem. By Theorem 7.13, we obtain that

$$T_{X',p'}[-1] = (p')^*\mathbb{L}^\vee_{X'}[-1]$$

has the structure of a differential graded Lie algebra. The claim follows from

$$(p')^*\mathbb{L}^\vee_{X'}[-1] = p^*j^*\mathbb{L}^\vee_{X'}[-1] = E^\vee_p[-1].$$

Since the induced deformation functor of the differential graded Lie algebra is given by the Maurer-Cartan equation, we can calculate

$$t_0(\Phi^{-1}(E^\vee_p[-1])) = t_0(X'_{p'}) = X_p.$$

□

Remark 7.16. In fact it is possible to deduce the preceding corollary from a much weaker statement. By Lemma 5.12, we know that locally Kuranishi neighbourhoods exist. Taking the homotopy fiber product of this Kuranishi neighbourhood provides a local derived extension having E_p as cotangent complex. Thus E_p has the structure of a differential graded Lie algebra.

Appendix

A.1 Model Categories

In this appendix we collect some results on model categories that were used throughout the text. The exposition closely follows [GS07].

We begin with the definition of a retract.

Definition A.1. Let $f\colon A \to B$ and $g\colon X \to Y$ be morphisms in an arbitrary category $\mathbf{C}$. Then f is a *retract* of g if there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "="] & X \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "="'] & Y \arrow[r] & B. \end{tikzcd}$$

We now recall the definition of a model category.

Definition A.2. A *model category* is a category $\mathbf{C}$ with three types of specified morphisms: weak equivalences, fibrations and cofibrations, satisfying the following axioms.

M1 The category $\mathbf{C}$ is closed under limits and colimits.

M2 Each type of specified morphisms is closed under retracts.

M3 Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that any two of f, g or $g \circ f$ is a weak equivalence, then so is the third.

M4 Every lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where i is a cofibration and f is a fibration and either one of the two is a weak equivalence has a solution.

M5 Any morphism $f\colon X \to Y$ can be factored either as

(i) $X \xrightarrow{i} Z \xrightarrow{r} Y$, where i is a cofibration and r is a fibration and a weak equivalence,

(ii) or as $X \xrightarrow{j} W \xrightarrow{s} Y$ j is a cofibration and a weak equivalence and s is a fibration.

Remark A.3. The axioms of for a model category are self-dual. If we assume that $\mathbf{M}$ is a model category, then the opposite category $\mathbf{M}^{\mathrm{op}}$ is also a model category, where the cofibrations of $\mathbf{M}^{\mathrm{op}}$ are the fibrations of $\mathbf{M}$, the fibrations of $\mathbf{M}^{\mathrm{op}}$ are the cofibrations of $\mathbf{M}$, and the weak equivalences remain unchanged.

The primordial example of a model category is the category of chain complexes over a commutative ring A. We mention this result for more than just the sake of completeness, since it will play in important role in putting a model structure on the category of simplicial algebras.

Theorem A.4. *Let A be a commutative ring, and denote by $\mathbf{Ch}_*A$ the category of non-negatively graded chain complexes of A-modules*[1]*. Then $\mathbf{Ch}_*A$ is a model category where a morphism $f\colon M_\bullet \to N_\bullet$ is*

(i) a weak equivalence if it is a quasi-isomorphism;

(ii) a fibration if $M_n \to N_n$ is surjective for $n \geq 1$;

(iii) and a cofibration if $M_n \to N_n$ is an injection with a projective cokernel for $n \geq 0$.

Remark A.5. It immediately follows that the cofibrant objects of this model category are exactly the complexes of projective models. On the other hand, the fibrant objects are *not* the injective modules. To identify these a different model structure is needed.

The Dold-Kan correspondence [Wei94] gives an equivalence of categories between non-negatively graded chain complexes of A-modules and simplicial A-modules. Transporting the structure from the above theorem and giving an alternative description of the fibrations, we arrive at the following result. We omit the concrete description of the cofibrations.

Proposition A.6. *Let A be a commutative ring, and denote by $\mathbf{sMod}_A$ the category of simplicial A-modules. Then $\mathbf{sMod}_A$ is a model category where a morphism $f\colon M \to N$ is*

(i) a weak equivalence if $\pi_*M \to \pi_*N$ is an isomorphism;

(ii) a fibration if $M \to \pi_0(M) \times_{\pi_0 N} N$ is a surjection.

[1] The differential goes down, i.e. $C_0 \leftarrow C_1 \leftarrow \dots$

We now would like to lift this model structure to simplicial A-algebras. This is not so easy, and additional assumptions on the model structures are needed. The key property we need is being cofibrantly generated.

Definition A.7. Let $\mathbf{C}$ be a category and F a class of morphisms in $\mathbf{C}$. Then an object $A \in \mathbf{C}$ is *sequentially small with respect to* F if for any sequence $X_1 \to X_2 \to X_3 \to \dots$ of morphisms in F the natural morphism

$$\operatorname{colim} \operatorname{Hom}(A, X_n) \to \operatorname{Hom}(A, \operatorname{colim} X_n)$$

is an isomorphism.

Definition A.8. (i) A model category $\boldsymbol{C}$ is *cofibrantly generated* if there are sets of morphisms I and J such that

(a) the source of every morphism in I is small with respect to to the class of all cofibrations;

(b) a morphism $f\colon X \to Y$ is an acyclic fibration if and only if f has the right lifting property with respect to all morphisms in I;

(c) the source of every morphism in J is small with respect to the class of all acyclic cofibrations;

(d) a morphism $f\colon X \to Y$ is a fibration if and only if f has the right lifting property with respect to all morphisms in J.

(ii) A model category is *combinatorial* if it is cofibrantly generated and the underlying category is presentable.

Remark A.9. The smallest class of maps that is closed under coproducts, cobase change, sequential colimits and retracts and contains I are exactly the cofibrations. Thus the term cofibrantly generated.

Example A.10. In the category $\mathbf{Ch}_* A$, define $D(n)$, $n \geq 1$ to be the chain complex with $D(n)_k = 0$ for $k \neq n$, $n-1$ and differential

$$d = \mathrm{id}_A\colon D(n)_n = A \longrightarrow A = D(n)_{n-1}.$$

Now define the set J to consist of the morphisms $0 \to D(n)$, $n \geq 1$. Define $S(n)$ to be the complex having a copy of A in degree n and being 0 otherwise. The set I then is defined to consist of the inclusions

$$S(n-1) \to D(n),\ n \geq 1; \tag{A.1}$$

$$0 \to S(0). \tag{A.2}$$

The following theorem gives general conditions on when one can lift a model structure along a left adjoint.

Theorem A.11. *Let*

$$\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$$

be an adjoint pair, and assume that $\mathbf{C}$ *is a cofibrantly generated model category with chosen generating sets* I *and* J*. Suppose further that* G *commutes with sequential colimits, and that* $\mathbf{D}$ *has all limits and colimits. Define a morphism* $f\colon X \to Y$ *in* $\mathbf{D}$ *to be*

(i) a weak equivalence if Gf *is a weak equivalence;*

(ii) a fibration if Gf *is a fibration;*

(iii) a cofibration if it has the left lifting property with respect to the acyclic cofibrations.

If furthermore every thus defined cofibration that has the left lifting property with respect to all fibrations is a weak equivalence, then $\mathbf{D}$ *is a cofibrantly generated model category. The generating sets are* $\{Fi \mid i \in I\}$ *and* $\{Fj \mid j \in J\}$.

Let $\mathbf{sAlg}_A$ denote the category of simplicial commutative algebras over A. Applying the above theorem to the pair

$$\mathbf{sMod}_A \underset{\Phi}{\overset{\mathrm{Sym}}{\rightleftarrows}} \mathbf{sAlg}_A$$

where Sym is the level-wise symmetric algebra functor and Φ is the forgetful functor, and making use of the fact that the category $\mathbf{sAlg}_A$ is presentable we obtain the following corollary.

Corollary A.12. *Let* A *be a commutative ring and* $\mathbf{sAlg}_A$ *the category of simplicial commutative algebras over* A*. Then* $\mathbf{sAlg}_A$ *is a combinatorial model category where a morphism* $f\colon R \to S$ *is*

(i) a weak equivalence if $\pi_* R \to \pi_* S$ *is a weak equivalence;*

(ii) a fibration if $R \to \pi_0 R \times_{\pi_0 S} S$ *is a surjection.*

Remark A.13. The hypothesis of Theorem A.11 are not always satisfied. If for instance k is a field of characteristic $p > 0$, then the adjoint pair from differential graded algebras over k to chain complexes over k does not satisfy the hypothesis. What fails is that the symmetric algebra functor no longer preserves weak equivalences between cofibrant objects.

A.2 Left Bousfield Localization

A key technique in the construction of the categories of higher stacks and derived stacks is left Bousfield localization. We here collect some fundamental results. We have to add the assumption that our model category is simplicial. One of the consequences of being a simplicial model category is that we do not only have a set of morphisms, but instead have a Hom-object that is a simplicial set.
Assume $\mathbf{M}$ to be a simplicial model category. Suppose that we have certain set of morphisms S in $\mathbf{M}$ which for some reason we would like to promote to being weak equivalences. The left Bousfield localization of $\mathbf{M}$ at S, if it exists, will be a new model structure on $\mathbf{M}$ such that the morphisms in S have become weak equivalences and thus isomorphisms if we pass to the homotopy category. Note that we have never changed anything in the category: The left Bousfield localization of $\mathbf{M}$ at S still has exactly the same objects and morphisms as $\mathbf{M}$. We have only tweaked the sets of weak equivalences, fibrations and cofibrations.

Example A.14. ([DS95]) Let $\mathbf{sSet}$ be the category of simplicial sets with the standard model structure. Now fix a homology theory h_* on the category of spaces. Define the class S to consist of those morphisms of simplicial sets, whose geometric realizations have the property that $h_*(|f|)$ is an isomorphism. In the homotopy category of the left Bousfield localization not only the weak homotopy equivalences have become isomorphisms, but also the equivalences with respect to the homology theory h_*.

We now come to the exact definitions.

Definition A.15. Let $\mathbf{M}$ be a simplicial model category and S a set of morphisms in $\mathbf{M}$. A *left localization of* $\mathbf{M}$ *with respect to* S is a model category $L_S\mathbf{M}$ together with a left Quillen functor $a\colon \mathbf{M} \to L_S\mathbf{M}$ such that

(i) the total left derived functor $\mathbb{L}a\colon \operatorname{Ho}(\mathbf{M}) \to \operatorname{Ho}(L_S\mathbf{M})$ takes maps in S to isomorphisms in $\operatorname{Ho}(L_S\mathbf{M})$;

(ii) the functor a is initial among all left Quillen functors with this property.

The goal is to define an appropriate model structure on $\mathbf{M}$ so that we obtain a left localization.

Definition A.16. Let S be a set of morphisms in $\mathbf{M}$.

(i) An object $A \in \mathbf{M}$ is *S-local* if for all morphisms $f\colon X \to Y$ in S the induced map $f^*\colon \mathbb{R}\,\underline{\operatorname{Hom}}(Y, A) \to \mathbb{R}\,\underline{\operatorname{Hom}}(X, A)$ is a weak equivalence of simplicial sets.

(ii) A morphism $f\colon X \to Y$ is an *S-local equivalence* if for all S-local objects A the induced map $f^*\colon \mathbb{R}\,\underline{\operatorname{Hom}}(Y, A) \to \mathbb{R}\,\underline{\operatorname{Hom}}(X, A)$ is a weak equivalence of simplicial sets.

Remark A.17. Every morphism in S is an S-local equivalence.

Definition A.18. Let $\mathbf{M}$ be a simplicial model category and S a set of morphisms in $\mathbf{M}$. The left Bousfield localization of $\mathbf{M}$ at S, if it exists, is a structure of a model category $L_S\mathbf{M}$ on $\mathbf{M}$ such that

(i) the weak equivalences in $L_S\mathbf{M}$ are the S-local equivalences;

(ii) the cofibrations are the cofibrations of M;

(iii) the fibrations are the maps with the right lifting property with respect to the acyclic cofibrations.

For completeness we record the following results, wich are both attributed to Smith:

Lemma A.19. *[Bar10, Lemma 4.3] Let* $\mathbf{M}$ *be a simplicial model category and* S *a set of morphisms in* $\mathbf{M}$*. If the left Bousfield localization of* M *at* S *exists, it is a left localization of* M *at* S*.*

Theorem A.20. *[Bar10, Thm. 4.7] Let* $\mathbf{M}$ *be a simplicial model category and* S *a set of morphisms in* $\mathbf{M}$*. Then the left Bousfield localization of* $\mathbf{M}$ *at* S *exists.*

A.3 Some Facts on Homotopy Fiber Products

In this section we review some basic facts about homotopy fiber products and homotopy pushouts. The results are essential for chapter 6. The exposition follows [Lur09, Appendix A.2] and [DS95].

We begin by recalling some basic definitions. We will denote by $\mathbf{D}$ the category $a \to b \leftarrow c$. [2] For any category $\mathbf{C}$, the category of diagrams of shape $\mathbf{D}$ will be written as $\mathbf{C}^{\mathbf{D}}$. Recall that there always is the diagonal functor $\Delta\colon \mathbf{C} \to \mathbf{C}^{\mathbf{D}}$ given by mapping an object to the constant diagram of shape $\mathbf{D}$. The same definition of course also applies to the category of diagrams of shape $\mathbf{D}$.

Definition A.21. Let $\mathbf{C}$ be a category.

(i) A *pushout*, if it exists, is a left adjoint to the diagonal functor Δ,

$$\mathbf{C}^{\mathbf{D}^{\mathrm{op}}} \underset{\Delta}{\overset{\mathrm{colim}}{\rightleftarrows}} \mathbf{C}.$$

(ii) A *fiber product*, if it exists, is a right adjoint to the diagonal functor Δ,

$$\mathbf{C} \underset{\lim}{\overset{\Delta}{\rightleftarrows}} \mathbf{C}^{\mathbf{D}}.$$

[2] Only non-identity morphisms are shown.

Remark A.22. Let $p\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{C}$ be a functor. Denote by $\mathbf{C}_{p/}$ the category of objects under p. Objects in this category are also called cones under p. Then $\mathrm{colim}_{d\in\mathbf{D}}$ is the initial object of $\mathbf{C}_{p/}$.

Dually, given a functor $p\colon \mathbf{D} \to \mathbf{C}$, the limit $\lim_{d\in\mathbf{D}}$ is the terminal object of the category $\mathbf{C}_{/p}$ of objects over p, also called cones over p.

If we now assume that $\mathbf{C}$ is a model category it is possible to define a model structures on the categories of diagrams of shape $\mathbf{D}$ and $\mathbf{D}^{\mathrm{op}}$.

Definition A.23. Let $\mathbf{M}$ be a model category.

(i) Define a natural transformation $\alpha\colon F \to G$ in $\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}$ to be a

(a) *projective fibration* if the induced map $F(D) \to G(D)$ is a fibration in $\mathbf{M}$ for each $D \in \mathbf{D}^{\mathrm{op}}$;

(b) *weak equivalence* if the induced map $F(D) \to G(D)$ is a weak equivalence in $\mathbf{M}$ for each $D \in \mathbf{D}^{\mathrm{op}}$;

(c) *projective cofibration* if it has the left lifting property with respect to the acyclic fibrations.

(ii) Define a natural transformation $\alpha\colon F \to G$ in $\mathbf{M}^{\mathbf{D}}$ to be a

(a) *injective cofibration* if the induced map $F(D) \to G(D)$ is a cofibration in $\mathbf{M}$ for each $D \in \mathbf{D}$;

(b) *weak equivalence* if the induced map $F(D) \to G(D)$ is a weak equivalence in $\mathbf{M}$ for each $D \in \mathbf{D}$;

(c) *injective fibration* if it has the right lifting property with respect to the acyclic cofibrations.

Proposition A.24. *[DS95, Prop. 10.6, 10.7, 10.11, 10.12]*

(i) With the definitions above, the categories $\mathbf{M}^{\mathbf{D}^{op}}$ and $\mathbf{M}^{\mathbf{D}}$ are model categories. The model structures are called respectively the projective model structure *and the* injective model structure.

(ii) Equip $\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}$ with the projective model structure. Then (colim, Δ) is a Quillen adjunction.

(iii) Equip $\mathbf{M}^{\mathbf{D}}$ with the injective model structure. Then $(\Delta, \lim)$ is a Quillen adjunction.

Remark A.25. Using Remark A.3, we immediately obtain that the projective model structure on $\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}$ gives the injective model structure on $(\mathbf{M}^{\mathrm{op}})^{\mathbf{D}}$. Thus a homotopy pushout in $\mathbf{M}$ is the same as a homotopy fiber product in $\mathbf{M}^{\mathrm{op}}$.

Since the pairs (colim, Δ) and $(\Delta, \lim)$ are Quillen adjunctions, the functors colim and $\lim$ admit total left and total right derived functors respectively.

Definition A.26. Let $\mathbf{M}$ be a model category.

(i) The *homotopy pushout* is the total left derived functor of colim,

$$\operatorname{hocolim}\colon \operatorname{Ho}(\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}) \to \operatorname{Ho}(\mathbf{M}).$$

(ii) The *homotopy fiber product* is the total right derived functor of $\lim$,

$$\operatorname{holim}\colon \operatorname{Ho}(\mathbf{M}^{\mathbf{D}}) \to \operatorname{Ho}(\mathbf{M}).$$

Remark A.27. A crucial observation is that the domains of $\operatorname{hocolim}$ and holim are not $\operatorname{Ho}(\mathbf{M})^{\mathbf{D}^{\mathrm{op}}}$ and $\operatorname{Ho}(\mathbf{M})^{\mathbf{D}}$. So the homotopy pushout and homotopy fiber product are *not* the pushout and fiber product in the homotopy category. In general, homotopy categories of model categories tend to have only very few (co)limits.

Remark A.28. As in the case of the pushout and fiber product, there are adjunctions on the level of homotopy categories

$$\operatorname{Ho}(\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}) \underset{\mathbb{R}\Delta}{\overset{\operatorname{hocolim}}{\rightleftarrows}} \operatorname{Ho}(\mathbf{M})$$

and

$$\operatorname{Ho}(\mathbf{M}) \underset{\operatorname{holim}}{\overset{\mathbb{L}\Delta}{\rightleftarrows}} \operatorname{Ho}(\mathbf{M}^{\mathbf{D}}).$$

Since the homotopy pushout and homotopy fiber product are only defined as objects of the homotopy category it is difficult to get ones hands on them. If we add the assumption that our model category $\mathbf{M}$ is combinatorial we can improve on this situation. This ensures, in particular, that we have functorial cofibrant replacements in $\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}$.

Our goal for the application in chapter 6 is to take homotopy fiber products in the category of derived affine schemes in a functorial manner. As this category itself is not combinatorial, the following results are not directly applicable. But since this category is by definition the dual of the category of simplicial algebras, which is combinatorial, the duals of the following results apply. So for the following, we ask the reader to keep in mind the example of $\mathbf{M}$ being the category of simplicial algebras, $\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}$ the category of diagrams in simplicial algebras that describe pushouts, and $(\mathbf{M}^{\mathrm{op}})^{\mathbf{D}}$ the category describing fiber products of affine derived schemes.

The key result we will make use of is the following.

Proposition A.29. *[Lur09, Prop. A.2.8.2] Let* $\mathbf{M}$ *be a combinatorial model category, and equip* $\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}$ *with the projective model structure. Then* $\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}$ *is a combinatorial model category.*

The above result implies that we can choose a cofibrant replacement functor for the category $\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}$. Using Remark A.25, this gives us a fibrant replacement functor in the dual category $(\mathbf{M}^{\mathrm{op}})^{\mathbf{D}}$.

Let Q be a cofibrant replacement functor for the category $\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}$. We can then define a functorial model for the homotopy pushout at the level of model categories as opposed to the previous definition which was on the level of $\mathrm{Ho}(\mathbf{M}^{\mathbf{D}^{\mathrm{op}}})$ by the composition

$$\mathbf{M}^{\mathbf{D}^{\mathrm{op}}} \xrightarrow{Q} \mathbf{M}^{\mathbf{D}^{\mathrm{op}}} \xrightarrow{\mathrm{colim}} \mathbf{M}.$$

Dually, Let R be a fibrant replacement functor for the category $(\mathbf{M}^{\mathrm{op}})^{\mathbf{D}}$. We can then define a functorial model for the homotopy fiber product by the composition

$$(\mathbf{M}^{\mathrm{op}})^{\mathbf{D}} \xrightarrow{R} (\mathbf{M}^{\mathrm{op}})^{\mathbf{D}} \xrightarrow{\lim} \mathbf{M}.$$

Remark A.30. Let $p\colon \mathbf{D} \to \mathbf{M}^{\mathrm{op}}$ be a functor. Using Remark A.22, we could also define the homotopy limit to be the terminal object in $\mathbf{M}^{\mathrm{op}}_{/Rp}$.

We can state the following simple lemma.

Lemma A.31. *Let* $\mathbf{M}$ *be a combinatorial model category and let* $\alpha\colon F \to G$ *be a morphism in* $\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}$*. Then there is a morphism*

$$\mathrm{hocolim}_{D\in\mathbf{D}^{\mathrm{op}}}\,\alpha\colon\ \mathrm{hocolim}_{D\in\mathbf{D}^{\mathrm{op}}}\,F \to \mathrm{hocolim}_{D\in\mathbf{D}^{\mathrm{op}}}\,G$$

in $\mathbf{M}$ *lifting the canonical morphism in* $\mathrm{Ho}(\mathbf{M})$*. Moreover this is functorial in morphisms in* $\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}$*.*

Proof. Let Q be a cofibrant replacement functor for $\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}$. The claim immediately follows from

$$\mathrm{hocolim} = \lim \circ Q$$

and the functoriality of this construction. □

We immediately obtain the following dual version of the lemma.

Corollary A.32. *Let* $\mathbf{M}$ *be a combinatorial model category and let* $\alpha\colon F \to G$ *be a morphism in* $(\mathbf{M}^{\mathrm{op}})^{\mathbf{D}}$*. Then there is a morphism*

$$\mathrm{holim}_{D\in\mathbf{D}}\,\alpha\colon\ \mathrm{holim}_{D\in\mathbf{D}}\,F \to \mathrm{holim}_{D\in\mathbf{D}}\,G$$

in $\mathbf{M}^{\mathrm{op}}$ *lifting the canonical morphism in* $\mathrm{Ho}(\mathbf{M}^{\mathrm{op}})$*. Moreover this is functorial in morphisms in* $(\mathbf{M}^{\mathrm{op}})^{\mathbf{D}}$*.*

Remark A.33. The crucial point making the lemma work is that the morphism $\alpha\colon F \to G$ exists on the level of $\mathbf{M}^{\mathbf{D}^{\mathrm{op}}}$ as opposed to only on the level of $\mathrm{Ho}(\mathbf{M}^{\mathbf{D}^{\mathrm{op}}})$.

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