Symplectic Lagrangian Fibrations

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Abstract

In this work we investigate the deformation theory of pairs of an irreducible symplectic manifold X together with a Lagrangian subvariety $Y \subseteq X$, where the focus is on singular Lagrangian subvarieties. Among other things, Voisin's results [**Voi92**] are generalized to the case of simple normal crossing subvarieties; partial results are also obtained for more complicated singularities. As done in Voisin's article we link the codimension of the subspace of the universal deformation space of X parametrizing those deformations where Y persists, to the rank of a certain map in cohomology. This enables us in some concrete cases to actually calculate or at least estimate the codimension of this particular subspace. In these cases the Lagrangian subvarieties in question occur as fibers or fiber components of a given Lagrangian fibration $f : X \longrightarrow B$. We discuss examples and the question of how our results might help to understand some aspects of Lagrangian fibrations.

Zusammenfassung

In der vorliegenden Arbeit wird die Deformationstheorie von Paaren von einer irreduzibel symplektischen Mannigfaltigkeit X und einer Lagrangeschen Untervarietät $Y \subseteq X$ untersucht, wobei das Hauptaugenmerk auf singulären Lagrangeschen Untervarietäten liegt. Die Resultate von Voisin [**Voi92**] werden unter Anderem auf den Fall von Untervarietäten mit einfachen normalen Überkreuzungen verallgemeinert; ebenfalls werden Teilergebnisse für kompliziertere Singularitäten erzielt. Wie bereits bei Voisin geschehen, können wir die Kodimension desjenigen Unterraumes des universellen Deformationsraumes von X, der Deformationen parametrisiert, bei denen Y mit deformiert, mit dem Rang einer gewissen Abbildung in Kohomologie in Verbindung bringen. Dies erlaubt es uns, in konkreten Fällen die Kodimension des besagten Unterraumes zu bestimmen oder wenigstens abzuschätzen. Dabei handelt es sich bei den Lagrangeschen Untervarietäten Y in der Regel um Fasern oder Faserkomponenten einer gegebenen Lagrangeschen Faserung $f : X \longrightarrow B$. Wir diskutieren Beispiele und gehen darauf ein, wie unsere Resultate dem Verständnis gewisser Aspekte Lagrangescher Faserungen förderlich sein könnten.

Contents

Introduction

Results and methods

The purpose of this work is to study Lagrangian fibrations on irreducible symplectic manifolds. Matsushita [**Mat99**] discovered that any non-trivial fibration from an irreducible symplectic manifold to a projective variety of smaller dimension is a Lagrangian fibration. Since then these fibrations have become an important tool in understanding symplectic manifolds. Matsushita's theorem is recalled as part of Theorem VII.1.1. It is also an instance of the fact that there are strong restrictions for the base of such a fibration. Most notably, if the base of a Lagrangian fibration on a projective irreducible symplectic manifold is smooth and projective, then it is isomorphic to the complex projective space $\mathbb{P}^n$ by a theorem of Hwang [**Hwa08**, Thm 1.2].
For a proper Lagrangian fibration of class C^∞ between differentiable manifolds, the Liouville-Arnol'd theorem [**Arn89**, Ch 10, § 49] says that the smooth fibers are compact tori, that is, diffeomorphic to the n-fold cartesian product

$$\left(\mathbb{S}^1\right)^{\times n} = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$$

of the circle $\mathbb{S}^1$. The holomorphic analogue of this theorem tells us that the smooth fibers of a proper holomorphic Lagrangian fibration are complex tori. As tori are studied quite well, singular fibers enter the focus. Every Lagrangian fibration on an irreducible symplectic manifold has singular fibers, see [**Hwa08**, Prop 4]. They are an important invariant of the fibration, hence also of the symplectic manifold itself and can for example be used in explicit calculations, see [**Moz06**] or [**Bea99**]. General singular fibers have been classified by Hwang-Oguiso [**HO09a**] and Matsushita [**Mat07**].
It turns out to be rewarding to forget that a given subvariety is a fiber of a Lagrangian fibration and simply consider it as an abstract Lagrangian subvariety. Then one can ask whether the results of Voisin's article [**Voi92**] still hold. In [**Voi92**] Voisin studied deformations of pairs $Y \subseteq X$ where X is an irreducible symplectic manifold and Y a complex Lagrangian submanifold. She found out that, roughly speaking, deformations of X where Y stays a complex submanifold are exactly those deformations, where Y stays

Lagrangian. If M_Y denotes the subspace of the universal deformation space M of X parametrizing deformations, where Y stays a complex submanifold, she expressed the codimension of M_Y in M as the rank of the restriction map $H^2(X,\mathbb{C}) \longrightarrow H^2(Y,\mathbb{C})$.

We generalize Voisin's results to Lagrangian subvarieties with simple normal crossings. Here and in the following a *variety* does not need to be irreducible. To give a precise formulation of our main results, we have to introduce some notations. Let $i : Y \hookrightarrow X$ be a Lagrangian subvariety with simple normal crossings, let $\nu : \widetilde{Y} \longrightarrow Y$ be the normalization and put $j := i \circ \nu$. Let $(M, 0)$ be the germ of the universal deformation space of X where 0 is the point corresponding to X. It is known to be smooth by the Bogomolov-Tian-Todorov theorem, see [**Bog78, Tia87, Tod89**]. If the representative M is chosen small enough and simply connected, the universal family over M is a $\mathcal{C}^\infty$-trivial fiber bundle. For small deformations X_t of X this gives a diffeomorphism $\alpha_t : X \longrightarrow X_t$ and a class $\omega_t \in H^2(X,\mathbb{C})$ corresponding to the symplectic form on X_t. We put $j_t := \alpha_t \circ j$ and denote by $(M_i, 0)$ the germ of the universal deformation space for locally trivial deformations of the inclusion $i : Y \hookrightarrow X$. It comes with a forgetful map $p : M_i \longrightarrow M$. Then we have

Theorem VI.5.3 — *Let $i : Y \hookrightarrow X$ be a simple normal crossing Lagrangian subvariety in a compact irreducible symplectic manifold X, $\nu : \widetilde{Y} \longrightarrow Y$ the normalization and $j = i \circ \nu$. Consider the germs at* 0 *of the complex subspaces*

$$M_Y := \operatorname{im}(p : M_i \longrightarrow M) \text{ and } M'_Y := \{t \in M : j_t^* \omega_t = 0\}$$

of M. Then $M'_Y = M_Y$ and this space is smooth at 0 *of codimension*

$$\operatorname{codim}_M M_Y = \operatorname{codim}_M M'_Y = \operatorname{rk}\left(j^* : H^2(X,\mathbb{C}) \longrightarrow H^2(\widetilde{Y},\mathbb{C})\right).$$

in M.

The definition of M_Y as the image of $M_i \longrightarrow M$ is a way of formalizing the phrase "Y stays complex". Similarly the defining equation for M'_Y formalizes the statement "Y stays Lagrangian". The definition of M_Y is quite subtle. We do not know a good definition for M_Y for an arbitrary Lagrangian subvariety Y. The problem is that we do not in general know whether the set $p(M_i)$ is an analytic subset of M, see Chapter VI.

Many of the intermediate steps in the proof of Theorem VI.5.3 are essentially as in [**Voi92**], but for the smoothness of M_Y we have to argue differently. For this we develop ideas of Ran [**Ran92b**], [**Ran92a**] by exploiting the interplay between deformation theory and Hodge theory. On the way we obtain the following result, which is unrelated to symplectic geometry and maybe of independent interest.

THEOREM III.4.3 — *Let* $S = \operatorname{Spec} R$ *where* R *is a local Artin* $\mathbb{C}$*-algebra with residue field* $\mathbb{C}$*, let* Y *be a proper simple normal crossing* $\mathbb{C}$*-variety and let* $g : \mathcal{X} \longrightarrow S$ *and* $f : \mathcal{Y} \longrightarrow S$ *be proper, algebraic* S*-schemes. Assume that* $\mathcal{Y} \longrightarrow S$ *is a locally trivial deformation of* Y *and* $\mathcal{X} \longrightarrow S$ *is smooth. Let* $i : \mathcal{Y} \longrightarrow \mathcal{X}$ *be an* S*-morphism. Then for all* p, q *the morphism* $i^* : R^q g_* \Omega^p_{\mathcal{X}/S} \longrightarrow R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}$ *has a free cokernel.*

The complex $\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ is $\Omega^\bullet_{\mathcal{Y}/S}$ modulo torsion, see Definition III.1.1. Like the de Rham complex in the smooth case, $\widetilde{\Omega}^\bullet_{\mathcal{Y}^{\mathrm{an}}/S}$ calculates cohomology with coefficients in the constant sheaf $\underline{R}_{Y^{\mathrm{an}}}$ for normal crossing varieties, where Y^{an} is the complex space associated to the variety Y. In particular, Theorem III.4.3 holds for smooth morphisms $\mathcal{Y} \longrightarrow S$.

Let us spend some words about the content and the structure of this treatise. For more detailed explanations we refer to the introductions of the individual chapters.

In Chapter I we explain the necessary ingredients from deformation theory. In particular, we define *locally trivial* deformations in the Zariski and analytic context and show that they give rise to isomorphic deformation functors, see Corollary I.5.1. The material is quite standard. We included it as we did not find in the literature the particular formulations of these results we need in later chapters.

As our central technical tool we introduce the notions of a *mixed Hodge structure* and a *mixed Hodge-Weil structure*, both *over a local Artin* $\mathbb{C}$*-algebra*, in Chapter II. These notions appear to be new. They axiomatize the Hodge theory of locally trivial deformations of simple normal crossing varieties. In combination with commutative algebra over Artin rings they are essentially used in the proof of Theorem III.4.3. Mixed Hodge structures over a local Artin $\mathbb{C}$-algebra R are intermediate objects between ordinary mixed Hodge structures and variations of mixed Hodge structures. Mixed Hodge structures and mixed Hodge-Weil structures over R are related to one another by Grothendieck's theory of Weil restriction [**Gro59, Gro60**]. If $R = \mathbb{C}$, then both notions specialize to the ordinary notion of a mixed Hodge structure.

Chapter III provides a construction of a mixed Hodge structure over a local Artin $\mathbb{C}$-algebra R on the cohomology of locally trivial deformations over $S = \operatorname{Spec} R$ of simple normal crossing varieties. In the absolute case $R = \mathbb{C}$ a very explicit construction of a mixed Hodge structure on the cohomology of a proper simple normal crossing variety is described in [**GS75**]. It uses the canonical semi-simplicial resolution of a simple normal crossing variety, see section III.2. We lift this resolution on the central fiber to a resolution by schemes which are smooth over S. In this framework we can prove Theorem III.4.3. Similar results hold for deformations of compact Kähler manifolds over an Artinian base.

In Chapter IV we recall the basic definitions and results regarding symplectic manifolds. An overview of the geometry of the universal deformation space M of an irreducible symplectic manifold X is given in Chapter V. There we discuss several important subspaces of M, most of which are defined by the persistence of certain (properties of) geometric objects on X under deformation. In this chapter we also explain and adapt Voisin's results from [**Voi92**], see section V.3. Only the definition and discussion of the space M_Y from Theorem VI.5.3 is postponed until Chapter VI, because it is quite involved and needs some preparation.

Chapter VI is devoted to the proof our main result, Theorem VI.5.3. We develop Ran's ideas and explain the T^1-lifting principle. We show that as a consequence of work of Flenner and Kosarew [**FK87**] there exists a universal locally trivial deformation for the inclusion of a Lagrangian subvariety $i : Y \hookrightarrow X$ in an irreducible symplectic manifold. The base of this deformation is the space M_i, which is shown to be smooth in Theorem VI.3.12. By construction, there is a canonical map $p : M_i \longrightarrow M$. We show that in case Y has simple normal crossings, this map factors as the composition of a smooth map $p : M_i \longrightarrow M_Y$ and a closed immersion of a submanifold $M_Y \hookrightarrow M$, see Theorem VI.4.3. For this one maybe needs to shrink M_i and M. Then, we prove Theorem VI.5.3 by assembling all theory developed and collected in the previous chapters. Furthermore, the projectivity of simple normal crossing Lagrangian subvarieties in an irreducible symplectic manifold is shown. This is used to apply algebraic arguments from the previous chapters to those subvarieties.

We give examples and applications to Lagrangian fibrations in Chapter VII. Our results can be applied to most types of the general singular fibers of a Lagrangian fibration in the sense of Hwang-Oguiso [**HO09a**]. By work of Matsushita [**Mat05, Mat09**] there is a certain subspace M_L of M, where a given fibration $f : X \longrightarrow B$ on the irreducible symplectic manifold X is preserved and this subspace is exactly the subspace M_Y for a smooth fiber Y of f. We show that if the reduction Y of a general singular fiber is preserved under deformation of X as a subvariety, then the fibration is preserved as well and deformations of Y are still contained in a fiber. We investigate

which of the general singular fibers show up generically in M_L. We produce several interesting questions in this direction and we believe that our results may be helpful to attack them.
In Chapter VIII we explain some related open problems which we encountered during this work and which we think should be guiding problems in the struggle to understand Lagrangian fibrations. We also pose some problems and questions regarding Hodge theory, locally trivial deformations and generalizations of our results.

Notations and conventions

We try to stick to the following notations and conventions throughout this work. We denote by k a field of characteristic zero. As usual, for a ring R we write $R[\varepsilon] := R[x]/x^2$ where $\varepsilon := x \mod (x^2)$. Set is the category of sets, Sch the category of schemes. For a scheme Z the category of schemes over Z is denoted by Sch/Z. The opposite category of a category $\mathscr{C}$, that is, the category whose objects are the objects of $\mathscr{C}$ and whose morphisms are obtained by reversing the morphisms of $\mathscr{C}$, is denoted by $\mathscr{C}^{\mathrm{op}}$.
An *algebraic scheme* is a separated scheme of finite type over a noetherian ring. The term *algebraic variety* will stand for a separated reduced k-scheme of finite type. In particular, a variety may have several irreducible components. Similarly, a *complex variety* will be a separated reduced complex space. If there is no danger of confusion, we will drop the adjectives *algebraic* respectively *complex*. A *fibration* is a proper morphism with connected fibers from a variety to a normal variety. For a scheme Z and a ring R the R-valued points $\mathrm{Mor}_{\mathrm{Sch}}(\operatorname{Spec} R, Z)$ of Z are denoted by $Z(R)$. The subscheme defined by a sheaf of ideals $\mathcal{I}$ will be denoted by $V(\mathcal{I})$. If $\mathcal{I}$ is generated by sections $f_1, \ldots, f_n \in \Gamma(Z, \mathcal{O}_Z)$ we will also write $V(f_1, \ldots, f_n)$ for $V(\mathcal{I})$. For an Artin ring R we do not distinguish between a quasi-coherent sheaf on $S = \operatorname{Spec} R$ and its R-module of global sections. A complex space or algebraic scheme Y of equidimension n is called a *normal crossing variety* if for every closed point $y \in Y$ there is an $r \in \mathbb{N}_0$ such that $\widehat{\mathcal{O}}_{Y,y} \cong k[[y_1, \ldots, y_{n+1}]]/(y_1 \cdot \ldots \cdot y_r)$. It is called a *simple normal crossing variety* if in addition every irreducible component is nonsingular.
Let X be a scheme of finite type over $\mathbb{C}$. We write X^{an} for the complex space associated to X. For us a complex space is always separated and is allowed to have nilpotent elements in the structure sheaf. For a quasi-coherent $\mathcal{O}_X$-module F we denote by F^{an} the associated $\mathcal{O}_{X^{\mathrm{an}}}$-module $\varphi^* F$ where $\varphi : X^{\mathrm{an}} \longrightarrow X$ is the canonical morphism of ringed spaces.

If A is an abelian group and X a topological space, $\underline{A}_X$ will denote the constant sheaf on X with values in A.

The symbol $\square$ marks the end of a proof. If for some reason a proof is omitted, this will be indicated by the appearance of $\square$ at the end of the respective statement.

CHAPTER I

Deformation theory

We summarize elementary results from deformation theory. Although the material is pretty standard, we include it as notations and terminology are not uniform. The fundamental reference is [**Sch68**]. A detailed exposition is given in [**Ser06**], where most of the proofs are found or obtained by easy variations. In the presentation we restrict ourselves to algebraic schemes, but analogous results hold also true in the category of complex spaces. Not completely standard might be Lemma I.5.1, which shows that over the complex numbers locally trivial deformations defined in Zariski- and Euclidean topology respectively give rise to isomorphic deformation functors. We are aware that the results we give are not the most general; they are taylored for the applications we have in mind.

I.1. Generalities

Let k be a fixed algebraically closed field. By Art_k we denote the category of local Artinian k-algebras with residue field k. The maximal ideal of an element $R \in \mathrm{Art}_k$ will be denoted by $\mathfrak{m}$, sometimes we will also phrase this as $(R, \mathfrak{m}) \in \mathrm{Art}_k$. We write $\widehat{\mathrm{Art}}_k$ for the category of local noetherian k-algebras with residue field k, which are complete with respect to the $\mathfrak{m}$-adic topology. There is a natural inclusion $\mathrm{Art}_k \hookrightarrow \widehat{\mathrm{Art}}_k$ and as in the proof of [**Mat80**, (28.J) Cor 1] every $R \in \widehat{\mathrm{Art}}_k$ is a homomorphic image of a power series ring $k[[x_1, \ldots, x_m]]$. In particular, every $R \in \mathrm{Art}_k$ is a finitely generated k-algebra as $\mathfrak{m}^k = 0$ for $k \gg 0$. Moreover, every element $R \in \widehat{\mathrm{Art}}_k$ can be written as a limit of objects in Art_k.

$$R = \varprojlim_{n \in \mathbb{N}} R/\mathfrak{m}^n$$

This explains why a category of *noetherian* algebras is denoted by $\widehat{\mathrm{Art}}_k$. A *small extension* in Art_k is an exact sequence

$$0 \longrightarrow J \longrightarrow R' \longrightarrow R \longrightarrow 0,$$

where $R' \longrightarrow R$ is a surjection in Art_k and the maximal ideal $\mathfrak{m}'$ of R' annihilates J, that is, $\mathfrak{m}'.J = 0$. Because of this last condition, the R'-module structure on J factors through $R'/\mathfrak{m}' = R/\mathfrak{m} = k$.

Definition I.1.1. A *deformation functor* or *functor of Artin rings* is a functor $D : \mathrm{Art}_k \longrightarrow \mathrm{Set}$ with $D(k) = \{\star\}$, where Set is the category of sets. The set $t_D = D(k[\varepsilon])$ is called the *tangent space* of D. This terminology is justified below.

For a deformation functor D and $R \in \mathrm{Art}_k$ we consider the canonical map

$$D(R \times_k k[\varepsilon]) \longrightarrow D(R) \times_{D(k)} D(k[\varepsilon]) = D(R) \times t_D. \tag{I.1.1}$$

If this is a bijection for every R, then t_D can be endowed with a canonical k-vector space structure by [**Sch68**], see also [**Ser06**, Ch 2.2]. Moreover, if $0 \longrightarrow J \longrightarrow R' \longrightarrow R \longrightarrow 0$ is a small extension in Art_k we can form the algebra $k[J] = k \oplus J$ where $J^2 = 0$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be a k-basis of J. The isomorphism

$$\begin{gathered} k[\varepsilon_1] \times_k k[\varepsilon_2] \times_k \ldots \times_k k[\varepsilon_n] \longrightarrow k[J], \\ (\lambda + \lambda_1\varepsilon_1, \ldots, \lambda + \lambda_n\varepsilon_n) \mapsto \lambda + \sum_i \lambda_i \varepsilon_i, \end{gathered} \tag{I.1.2}$$

of k-algebras induces an isomorphism $t_D \otimes_k J \cong t_D \times \ldots \times t_D \longrightarrow D(k[J])$. The algebra homomorphism

$$k[J] \times_k R' \longrightarrow R', \qquad (x, a) \mapsto x - x_0 + a, \tag{I.1.3}$$

where x_0 is the first component of x with respect to $k[J] = k \oplus J$, induces an action of $t_D \otimes J = D(k[J])$ on $D(R')$. This action preserves the fibers of $D(R') \longrightarrow D(R)$. Morphisms of deformation functors for which (I.1.1) is bijective induce k-linear maps between their tangent spaces and are equivariant with respect to their respective actions. Every deformation functor $D : \mathrm{Art}_k \longrightarrow \mathrm{Set}$ has a unique extension $\widehat{D} : \widehat{\mathrm{Art}}_k \longrightarrow \mathrm{Set}$ given by

$$\widehat{D}(R) := \varprojlim_{n \in \mathbb{N}} D(R/\mathfrak{m}^n).$$

We call $\widehat{D}$ the *completion* of D. For an element $u_n \in D(R/\mathfrak{m}^n)$ and $R' \in \mathrm{Art}_k$ the map

$$\mathrm{Hom}_k(R/\mathfrak{m}^n, R') \longrightarrow D(R'), \quad \varphi \mapsto D(\varphi)(u_n)$$

defines a morphism $U_n : \mathrm{Hom}_k(R/\mathfrak{m}^n, \cdot) \longrightarrow D$ of functors. We will abbreviate a functor of the form $\mathrm{Hom}_k(R, \cdot)$ by h_R. By [**Ser06**, Lem 2.2.2] this process is compatible with taking limits and thus gives a bijection between elements u of $\widehat{D}(R)$ and morphisms of functors $U : h_R \longrightarrow D$. An element $u \in \widehat{D}(R)$ for some $R \in \widehat{\mathrm{Art}}_k$ is called a *formal element* of D.

Definition I.1.2. A deformation functor D is said to be *prorepresentable*, if there is a complete local noetherian k-algebra R, such that $D \cong h_R$. Let

$R \in \widehat{\mathrm{Art}}_k$ and $u \in \widehat{D}(R)$ be a formal element of D. If the corresponding $U : h_R \longrightarrow D$ is an isomorphism, we call u a *universal* formal element. In this case, D is prorepresented by R. A weaker notion is that of versality. We call u a *versal* formal element, if the morphism U is smooth. This means that the map

$$D(B) \longrightarrow D(A) \times_{h_R(A)} h_R(B) \tag{I.1.4}$$

is surjective for every surjection $B \longrightarrow A$ in Art_k. We say that u is a *semi-universal* formal element, if u is a versal formal element and the corresponding map $h_R(k[\varepsilon]) \longrightarrow D(k[\varepsilon])$ is bijective. A prorepresentable deformation functor $D \cong h_R$ is called *unobstructed*, if R is a smooth k-algebra.

Remark I.1.3. —

(1) Let $D_1, D_2 : \mathrm{Art}_k \longrightarrow \mathrm{Set}$ be functors of Artin rings and let a morphism $\eta : D_1 \longrightarrow D_2$ of functors be given. If the D_j, $j = 1, 2$ are prorepresentable, say $D_j = \mathrm{Hom}_{\mathrm{Art}}(R_j, \cdot)$, then η induces a ring homomorphism $\eta^{\#} : R_1 \longrightarrow R_2$.
(2) If D is a prorepresentable deformation functor, then the action (I.1.3) is free and transitive on the non-empty fibers of a small extension by [**Ser06**, Prop 2.3.4 (b)].

Definition I.1.4. If $D : \mathrm{Art}_k \longrightarrow \mathrm{Set}$ is a deformation functor, $R' \longrightarrow R$ is a morphism in Art_k and $\eta \in D(R)$ then we will write

$$D(R')_\eta := \varphi^{-1}(\eta) \subseteq D(R')$$

where $\varphi : D(R') \longrightarrow D(R)$ is the map induced by $R' \longrightarrow R$.

I.1.5. Curvilinear deformations. Here we work out a criterion for a prorepresentable deformation functor to be unobstructed. The criterion seems to be "well-known": it is widely used but we were unable to find an explicit proof in the literature.

Let R be a complete local noetherian k-algebra with maximal ideal $\mathfrak{m}$ and $A_n := k[t]/t^{n+1}$. Suppose R has the following lifting property for all $n \in \mathbb{N}$:

$$\begin{tikzcd} & A_{n+1} \arrow[d] \\ R \arrow[r] \arrow[ur, dashed, "\exists"] & A_n \end{tikzcd} \tag{I.1.5}$$

That is, for every k-algebra homomorphism $R \longrightarrow A_n$ there is a k-algebra homomorphism $R \longrightarrow A_{n+1}$ making (I.1.5) commutative. In this case we say that R is *curvilinearly smooth* over k. We have

Lemma I.1.6. If R is curvilinearly smooth over k, then R is a smooth k-algebra.

Proof. As R is a complete, noetherian k-algebra we may write $R = S/I$ where $S = k[[x_1, \ldots, x_m]]$, $m = \dim_k \mathfrak{m}/\mathfrak{m}^2$ and $I \subseteq S$ is an ideal. In this case we have that $I \subseteq \mathfrak{n}^2$, where $\mathfrak{n} \subseteq S$ is the maximal ideal. Assume that $I \neq 0$. Then there is a maximal $\ell \in \mathbb{N}$ with $I \subseteq \mathfrak{n}^\ell$. This inclusion implies that for every choice of $\lambda_1, \ldots, \lambda_m \in k$ the algebra homomorphism

$$S \longrightarrow k[t], \quad x_i \mapsto \lambda_i\, t$$

descends to an algebra homomorphism $R \longrightarrow A_\ell$. Let $\varphi : S \longrightarrow A_{\ell+1}$ be the composition with the canonical morphism $k[t] \longrightarrow A_{\ell+1}$. Then the curvilinear lifting property requires that there is a lift ψ in the diagram

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d] & A_{\ell+1} \arrow[d] \\ R \arrow[r] \arrow[ur, dashed, "\psi"] & A_\ell \end{tikzcd}$$

Let us take $f \in I \setminus \mathfrak{n}^{\ell+1}$ and let f_ℓ be its degree ℓ term. In other words, $f_\ell \in k[x_1, \ldots, x_m]$ is the unique homogeneous polynomial of degree ℓ with the property that

$$f = f_\ell + f_{\ell+1} \quad \text{for some } f_{\ell+1} \in \mathfrak{n}^{\ell+1}.$$

By assumption $f_\ell \neq 0$. The existence of $\psi : R \longrightarrow A_{\ell+1}$ requires that

$$0 = \varphi(f) = f_\ell(\lambda_1, \ldots, \lambda_m)$$

for every choice of λ_i. But as k is an infinite field, this is only possible if the polynomial $f_\ell = 0$ contradicting the choice of f. Thus $I = 0$ and R is smooth over k. □

The following corollary is just another way to phrase the previous lemma and the above lifting property.

Corollary I.1.7. Let D be a prorepresentable deformation functor. Then D is unobstructed, if for all n the map $D(A_{n+1}) \longrightarrow D(A_n)$ is surjective. □

I.2. Deformations of schemes

Let X be an algebraic k-scheme, $R \in \mathrm{Art}_k$ and $S = \operatorname{Spec} R$.

Definition I.2.1. A *deformation of* X over S is a flat S-scheme $\mathcal{X} \longrightarrow S$ together with an isomorphism $X \longrightarrow \mathcal{X} \times_S k$. An isomorphism between deformations $\mathcal{X} \longrightarrow S$, $\mathcal{X}' \longrightarrow S$ is an isomorphism $\varphi : \mathcal{X} \longrightarrow \mathcal{X}'$ of S-schemes with the property that the induced diagram

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ \mathcal{X} \times_S k \arrow[rr, "\varphi \times_S k"] & & \mathcal{X}' \times_S k \end{tikzcd}$$

is commutative. The functor

$$D_X : \mathrm{Art}_k \longrightarrow \mathrm{Set}, \quad R \mapsto \{\text{deformations of } X \text{ over } S\} / \sim$$

where $\sim$ is the relation of isomorphism, is called *functor of deformations of* X. A deformation $\mathcal{X} \longrightarrow S$ over $S = \operatorname{Spec} R$, $R \in \mathrm{Art}_k$ is called *(Zariski) locally trivial*, if for every $x \in X$ there is an open subset $U \subseteq X$ with $x \in U$ and an isomorphism

$$\begin{tikzcd} \mathcal{X}_{|U} \arrow[rr, "\cong"] \arrow[dr] & & X_{|U} \times_k S \arrow[dl] \\ & S & \end{tikzcd}$$

In other words, $\mathcal{X}$ induces the trivial deformation on U. Note that the topological spaces underlying X and $\mathcal{X}$ are the same as R is Artinian. The functor

$$D_X^{\mathrm{lt}} : \mathrm{Art}_k \longrightarrow \mathrm{Set}, \quad R \mapsto \{\text{locally trivial deformations of } X \text{ over } S\} / \sim$$

where $\sim$ is the relation of isomorphism, is called *functor of locally trivial deformations of* X.

We will often use the isomorphism $X \longrightarrow \mathcal{X} \times_S k$ to identify X with $\mathcal{X} \times_S k$. If X is smooth over k, every deformation of X is locally trivial, see [**Har77**, II.2, Ex 8.6]. If X is proper over k, then every deformation $\mathcal{X} \longrightarrow S$ over $S = \operatorname{Spec} R$, $R \in \mathrm{Art}_k$ is proper by the valuative criterion for properness. The following result is proven as Corollary 2.6.4 in [**Ser06**] for projective X. The proof there works for proper X as well.

Proposition I.2.2. If X is smooth and proper over k and $H^0(X, T_X) = 0$, the functor $D_X = D_X^{\mathrm{lt}}$ is prorepresentable. $\square$

Let X be an algebraic k-scheme and let $g : \mathcal{X} \longrightarrow S$ be a deformation of X over $S = \operatorname{Spec} R$. We put

(I.2.1)
$$T^1_{\mathcal{X}/R} := R^1 g_* T_{\mathcal{X}/S}, \quad T^1 := T^1_{X/k} = H^1(X, T_X), \quad T^2 := H^2(X, T_X).$$

Next we construct a map $T^1_{\mathcal{X}/R} \longrightarrow D_X(R[\varepsilon])_{\mathcal{X}}$. We write $S[\varepsilon] := \operatorname{Spec} R[\varepsilon]$. As S is affine, $R^1 g_* T_{\mathcal{X}/S} \cong \check{H}^1(\mathcal{X}, T_{\mathcal{X}/S})$. So let $\mathcal{X} = \bigcup_i U_i$ be a covering by open affines $U_i = \operatorname{Spec} A_i$ and let an element $\delta \in R^1 g_* T_{\mathcal{X}/S}$ be given. We represent δ by a Čech-1-cocyle $\{(\delta_{ij}, U_{ij})\}$ where $U_{ij} = U_i \cap U_j$ and $\delta_{ij} \in \Gamma(U_{ij}, T_{\mathcal{X}/S})$. We regard the δ_{ij} as R-derivations of $A_{ij} := \Gamma(U_{ij}, \mathcal{O}_{\mathcal{X}})$ and define an automorphism φ_{ij} of the scheme $U_{ij}[\varepsilon] := U_{ij} \times_S S[\varepsilon] = \operatorname{Spec}(A_{ij} \otimes_R R[\varepsilon])$ via $R[\varepsilon]$-linear extension of the map

$$A_{ij} \longrightarrow A_{ij} \otimes_R R[\varepsilon], \quad a \mapsto a + \varepsilon \delta_{ij}(a).$$

As δ is a cocycle, the isomorphisms φ_{ij} can be used to glue the schemes $U_i[\varepsilon] := \operatorname{Spec}(A_i \otimes_R R[\varepsilon])$ along the open subschemes $U_{ij}[\varepsilon]$. In this way we obtain a flat scheme $S[\varepsilon]$-scheme $\mathcal{X}_\delta$, which is an extension of $\mathcal{X}$. One can show that the map

$$T^1_{\mathcal{X}/R} \longrightarrow D_X(R[\varepsilon])_{\mathcal{X}}, \quad \delta \mapsto \mathcal{X}_\delta \tag{I.2.2}$$

is well-defined, where $D_X(R[\varepsilon])_{\mathcal{X}}$ is the fiber over $\mathcal{X}$ in the sense of Definition I.1.4. Furthermore we have

Lemma I.2.3. *Let $0 \longrightarrow J \longrightarrow R' \longrightarrow R \longrightarrow 0$ be a small extension in Art_k. Assume that X is smooth over k. Then there is a natural isomorphism $T^1 \xrightarrow{\cong} t_{D_X}$. Moreover, the following holds. Let $\mathcal{X}' \longrightarrow S$ be a deformation of X over $S' = \operatorname{Spec} R'$ such that $\mathcal{X}' \times_{S'} S = \mathcal{X}$. Then:*

(1) *The map $T^1_{\mathcal{X}/R} \longrightarrow D_X(R[\varepsilon])_{\mathcal{X}}$ from (I.2.2) is a bijection and the diagram*

$$\begin{tikzcd} T^1_{\mathcal{X}'/R'} \arrow[r] \arrow[d] & T^1_{\mathcal{X}/R} \arrow[d] \\ D_X(R'[\varepsilon])_{\mathcal{X}'} \arrow[r] & D_X(R[\varepsilon])_{\mathcal{X}} \end{tikzcd}$$

is commutative, where we obtain $T^1_{\mathcal{X}'/R'} \longrightarrow T^1_{\mathcal{X}/R}$ by applying $R^1 g_*$ to the natural map $T_{\mathcal{X}'/S'} \longrightarrow T_{\mathcal{X}'/S'}$.

(2) *There are natural maps $D_X(R) \longrightarrow T^2 \otimes J$ such that*

$$D_X(R') \longrightarrow D_X(R) \longrightarrow T^2 \otimes J$$

is an exact sequence of pointed sets. Furthermore, for every diagram

$$\begin{tikzcd} 0 \arrow[r] & J_1 \arrow[r] \arrow[d, "\phi"'] & R'_1 \arrow[r] \arrow[d] & R_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & J_2 \arrow[r] & R'_2 \arrow[r] & R_2 \arrow[r] & 0 \end{tikzcd}$$

of small extensions the diagram

$$\begin{tikzcd} D_X(R_1') \arrow[r] \arrow[d] & D_X(R_1) \arrow[r] \arrow[d] & T^2 \otimes J_1 \arrow[d, "\mathrm{id}\otimes\phi"] \\ D_X(R_2') \arrow[r] & D_X(R_2) \arrow[r] & T^2 \otimes J_2 \end{tikzcd}$$

commutes.

Proof. ad (1). This is a straight forward generalization of [**Ser06**, Thm 2.4.1]. The bijectivity of $T^1_{\mathcal{X}/R} \longrightarrow D_X(R[\varepsilon])_{\mathcal{X}}$ is obtained by reversing the construction of this map. It works as follows. Take a covering $\mathcal{X} = \bigcup_i U_i$ with open affines $U_i = \operatorname{Spec} A_i$. As X is smooth over k, also $\mathcal{X} \longrightarrow S$ is smooth and so the A_i will be smooth R-algebras. Let $\widetilde{\mathcal{X}}$ be a deformation of $\mathcal{X}$ over $R[\varepsilon]$. We will show that $\mathcal{X}$ is locally trivial. We have induced deformations $\widetilde{\mathcal{X}}|_{U_i} = \operatorname{Spec} A_i'$ of U_i, hence a diagram

$$\begin{tikzcd} & A_i' \arrow[d, "\varphi"] \\ A_i \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & A_i \end{tikzcd}$$

for which $\ker\varphi$ is nilpotent. By smoothness we obtain a lifting $A_i \longrightarrow A_i'$ and the deformation $\widetilde{\mathcal{X}}|_{U_i}$ is trivial by flatness. Thus, $\widetilde{\mathcal{X}}$ is given by gluing the schemes $U_i \times_S S[\varepsilon]$ where $S[\varepsilon] = \operatorname{Spec} R[\varepsilon]$. The corresponding $R[\varepsilon]$-algebra isomorphisms

$$\theta_i : A_i \otimes_R R[\varepsilon] \longrightarrow \Gamma(U_i, \mathcal{O}_{\widetilde{\mathcal{X}}})$$

restrict to the identity modulo ε. We put $U_{ij} = U_i \cap U_j = \operatorname{Spec} A_{ij}$ and obtain isomorphisms

$$\theta_{ij} : A_{ij} \otimes_R R[\varepsilon] \longrightarrow A_{ij} \otimes_R R[\varepsilon] = A_{ij} \oplus \varepsilon A_{ij}$$

defined by $\theta_{ij} := \theta_j|_{U_{ij}} \circ \theta_i^{-1}|_{U_{ij}}$, which satisfy the cocycle conditions. By $R[\varepsilon]$-linearity such a morphism is uniquely determined by the restriction to $A_{ij} \longrightarrow A_{ij} \oplus \varepsilon A_{ij}$ and has the form $\mathrm{id} + \varepsilon\delta_{ij}$. A direct calculation shows that $\delta_{ij} \in \operatorname{Der}_R(A_{ij}) = \Gamma(U_{ij}, T_{\mathcal{X}/S})$. The derivations δ_{ij} form a Čech-1-cocyle and define a class in $H^1(\mathcal{X}, T_{\mathcal{X}/S})$ from which the deformation $\widetilde{\mathcal{X}}$ can be reconstructed up to isomorphism. The rest of the argument is as in the absolute case $R = k$, see [**Ser06**, Thm 2.4.1].
ad (2). This is [**Ser06**, Prop 2.4.6]. □

We call T^1 the *tangent space*, T^2 the *obstruction space* and $T^1_{\mathcal{X}/R}$ a *relative tangent space* of D_X. By the lemma above there is no ambiguity in these names as $t_D \cong T^1$.

I.3. Deformations of morphisms

Let $i : Y \longrightarrow X$ be a k-morphism of algebraic k-schemes Y, X. Let $(R, \mathfrak{m}) \in \mathrm{Art}_k$ and $S = \operatorname{Spec} R$.

DEFINITION I.3.1. A *deformation of* i over S is a diagram

(I.3.1)
$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr] & \mathcal{X} \arrow[d] \\ & S \end{tikzcd}$$

where $\mathcal{X} \longrightarrow S$ and $\mathcal{Y} \longrightarrow S$ are flat S-schemes together with isomorphisms $\mathcal{X} \times_S \operatorname{Spec} R/\mathfrak{m} \longrightarrow X$, $\mathcal{Y} \times_S \operatorname{Spec} R/\mathfrak{m} \longrightarrow Y$ and $I \times_S \operatorname{Spec} R/\mathfrak{m} \cong i$ under these isomorphisms. An isomorphism between deformations of i is defined in the obvious way. The functor

$$D_i : \mathrm{Art}_k \longrightarrow \mathrm{Set}, \quad R \mapsto \{\text{deformations of } i \text{ over } S\} \,/\sim$$

where $\sim$ is the relation of isomorphism, is called *functor of deformations of* i. A deformation $I : \mathcal{Y} \longrightarrow \mathcal{X}$ as in (I.3.1) is called *(Zariski) locally trivial*, if for every $x \in X$, $y \in Y$ with $i(y) = x$ there are open subsets $U \subseteq X$, $V \subseteq Y$ with $y \in V$, $i(V) \subseteq U$ and an isomorphism

$$\begin{tikzcd} \mathcal{X}_{|U} \arrow[rr, "\cong"] \arrow[dr] & & X_{|U} \times_k S \arrow[dl] \\ & S & \\ \mathcal{Y}_{|V} \arrow[uu, "I_{|V}"] \arrow[ur] \arrow[rr, "\cong"] & & Y_{|V} \times_k S \arrow[ul] \arrow[uu, "i_{|V} \times_k \mathrm{id}"'] \end{tikzcd}$$

In other words, $I : \mathcal{Y} \longrightarrow \mathcal{X}$ induces the trivial deformation on V and U. The functor

$$D_i^{\mathrm{lt}} : \mathrm{Art}_k \longrightarrow \mathrm{Set}, \quad R \mapsto \{\text{locally trivial deformations of } i \text{ over } S\} \,/\sim$$

where $\sim$ is the relation of isomorphism, is called *functor of locally trivial deformations of* i.

I.3.2. Sheaves controlling the deformations of a closed immersion. Let $i : Y \hookrightarrow X$ be a closed immersion of algebraic k-schemes. Recall that i is said to be a *regular embedding*, if for every $y \in Y$ there is an affine open neighbourhood $U = \operatorname{Spec} A$ of $i(y)$ in X such that the ideal of $i(Y) \cap U$ is generated by a regular sequence in A, see for example Appendix D of [**Ser06**]. The scheme Y is called a *locally complete intersection*, if for every point $y \in Y$ there is a regular scheme X' and a regular embedding

$\operatorname{Spec}\mathcal{O}_{Y,y} \hookrightarrow X'$. This is an absolute notion in contrast to the relative notion of a regular embedding.

Suppose now that X is smooth and proper and Y is reduced and proper. We will express the tangent space of D_i^{lt} in terms of cohomology of certain sheaves. Let $(R, \mathfrak{m}) \in \mathrm{Art}_k$, $S = \operatorname{Spec} R$ and let

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "f"'] & \mathcal{X} \arrow[d, "g"] \\ & S \end{tikzcd} \tag{I.3.2}$$

be a deformation of i. Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{Y}$ in $\mathcal{X}$.

Assume that Y is a locally complete intersection. Then so is $\mathcal{Y}$ and $\mathcal{I}/\mathcal{I}^2$ is locally free. We have an exact sequence of sheaves on $\mathcal{Y}$

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}} \longrightarrow \Omega_{\mathcal{Y}/S} \longrightarrow 0. \tag{I.3.3}$$

Note that as Y is reduced, the map $d \otimes_R k$ is injective, hence also d is injective. We obtain the exact sequence

$$0 \longrightarrow T_{\mathcal{Y}/S} \longrightarrow T_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}} \xrightarrow{d^\vee} N_{\mathcal{Y}/\mathcal{X}} \longrightarrow T^1_{\mathcal{Y}/S} \longrightarrow 0, \tag{I.3.4}$$

where $N_{\mathcal{Y}/\mathcal{X}} := \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathcal{Y}})$. The sheaf $T^1_{\mathcal{Y}/S} := \operatorname{coker} d^\vee$ is supported on the singular locus of Y. We define the *equisingular normal sheaf*

$$N'_{\mathcal{Y}/\mathcal{X}} := \ker(N_{\mathcal{Y}/\mathcal{X}} \longrightarrow T^1_{\mathcal{Y}/S}). \tag{I.3.5}$$

Taking the preimage T_I of $T_{\mathcal{Y}/S}$ under the natural map $T_{\mathcal{X}/S} \longrightarrow T_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}}$ we obtain the exact sequence of sheaves on $\mathcal{X}$

$$0 \longrightarrow T_I \longrightarrow T_{\mathcal{X}/S} \longrightarrow N'_{\mathcal{Y}/\mathcal{X}} \longrightarrow 0. \tag{I.3.6}$$

The sheaf T_I is the relative version of a sheaf which is called the *sheaf of germs of tangent vectors to $\mathcal{X}$ which are tangent to $\mathcal{Y}$ along $\mathcal{Y}$* in [**Ser06**, 3.4.4]. It controls locally trivial deformations of a closed immersion in the sense of Lemma I.3.4.

The following satement is given in [**Ser06**, Rem 3.4.18] for projective schemes X and Y. As in the case of deformations of schemes, the proof carries over to proper schemes.

PROPOSITION I.3.3. Assume that $i : Y \hookrightarrow X$ is a closed immersion of proper k-schemes, that X is smooth and that $H^0(X, T_i) = 0$. Then the functor D_i^{lt} is prorepresentable. $\square$

Let $0 \longrightarrow J \longrightarrow R' \longrightarrow R \longrightarrow 0$ be a small extension in Art_k, $i : Y \hookrightarrow X$ be a closed immersion of proper algebraic k-schemes X and Y where Y is a reduced locally complete intersection and X is smooth over k. Let

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "f"'] & \mathcal{X} \arrow[d, "g"] \\ & S \end{tikzcd}$$

be a deformation of i over $S = \operatorname{Spec} R$. As for deformations of schemes we introduce tangent and obstruction spaces

$$T^1_{I/R} := R^1 g_* T_I, \qquad T^1 := T^1_{i/k} = H^1(X, T_i), \qquad T^2 := H^2(X, T_i). \tag{I.3.7}$$

One constructs a natural map

$$T^1_{I/R} \longrightarrow D_i(R[\varepsilon])_I, \tag{I.3.8}$$

where $D_i(R[\varepsilon])_I$ is the fiber over I in the sense of Definition I.1.4, similar to the one in (I.2.2).

Lemma I.3.4. Let $0 \longrightarrow J \longrightarrow R' \longrightarrow R \longrightarrow 0$ be a small extension in Art_k and let $i : Y \hookrightarrow X$ be a closed immersion of proper algebraic k-schemes X and Y where Y is a reduced locally complete intersection and X is smooth over k. Then there is a natural isomorphism $T^1 \xrightarrow{\cong} t_{D_i}$. More precisely the following holds. Let $I' : \mathcal{Y}' \hookrightarrow \mathcal{X}'$ a deformation of i over R' such that $I' \times_{S'} S = I$ where $S' = \operatorname{Spec} R'$. Then:

(1) The map $T^1_{I/R} \longrightarrow D_i(R[\varepsilon])_I$ from (I.3.8) is a bijection and the diagram

$$\begin{tikzcd} T^1_{I'/R'} \arrow[r] \arrow[d] & T^1_{I/R} \arrow[d] \\ D_i(R'[\varepsilon])_{I'} \arrow[r] & D_i(R[\varepsilon])_I \end{tikzcd}$$

is commutative, where we obtain $T^1_{I'/R'} \longrightarrow T^1_{I/R}$ by applying $R^1 g_*$ to the natural map $T_{I'} \longrightarrow T_I$.

(2) There are natural maps $D_i(R) \longrightarrow T^2 \otimes J$ such that

$$D_i(R') \longrightarrow D_i(R) \longrightarrow T^2 \otimes J$$

is an exact sequence of pointed sets. Furthermore, for every diagram

$$\begin{tikzcd} 0 \arrow[r] & J_1 \arrow[r] \arrow[d, "\phi"] & R'_1 \arrow[r] \arrow[d] & R_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & J_2 \arrow[r] & R'_2 \arrow[r] & R_2 \arrow[r] & 0 \end{tikzcd}$$

of small extensions the diagram

$$\begin{tikzcd} D_i(R_1') \arrow[r] \arrow[d] & D_i(R_1) \arrow[r] \arrow[d] & T^2 \otimes J_1 \arrow[d, "\mathrm{id}\otimes\phi"] \\ D_i(R_2') \arrow[r] & D_i(R_2) \arrow[r] & T^2 \otimes J_2 \end{tikzcd}$$

commutes.

Proof. ad (1): This is a straight forward generalization of [**Ser06**, Prop 3.4.17]. The sections of T_I are exactly those sections of $T_{\mathcal{X}/S}$ restricting to sections of $T_{\mathcal{Y}/S}$ on $\mathcal{Y}$. Sections of $T_{\mathcal{Y}/S}$ correspond to locally trivial deformations of Y over $S[\varepsilon] := \operatorname{Spec} R[\varepsilon]$ extending $\mathcal{Y}$. Now the proof works as the one of Lemma I.2.3.
ad (2): This is [**Ser06**, Prop 3.4.17]. □

I.4. Deformations of morphisms with fixed target

Let $i : Y \longrightarrow X$ be a k-morphism of algebraic k-schemes Y, X. Let $(R, \mathfrak{m}) \in \mathrm{Art}_k$ and $S = \operatorname{Spec} R$.

Definition I.4.1. A *deformation of i with target X* over S is a diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "I"] \arrow[dr] & X \times_k S \arrow[d, "p_2"] \\ & S \end{tikzcd} \tag{I.4.1}$$

where $\mathcal{Y} \longrightarrow S$ is flat, together with an isomorphism $\mathcal{Y} \times_S \operatorname{Spec} R/\mathfrak{m} \longrightarrow Y$ such that $I \times_S \operatorname{Spec} R/\mathfrak{m} \cong i$ under this isomorphism. If i is a closed immersion, the diagram (I.4.1) is also called a *deformation of Y as a subscheme of X* over R. An isomorphism between deformations of i with target X is defined in the obvious way. The functor

$$D_{i/X} : \mathrm{Art}_k \longrightarrow \mathrm{Set}, \quad R \mapsto \{\text{deformations of } i \text{ over } S \text{ with target } X\} / \sim$$

where $\sim$ is the relation of isomorphism, is called *functor of deformations of i with target X*. A deformation $I : \mathcal{Y} \longrightarrow X \times_k S$ as in (I.4.1) is called *(Zariski) locally trivial*, if for every $y \in Y$ there is an open subset $V \subseteq Y$

with $y \in V$ and an isomorphism

$$\begin{tikzcd} & X \times_k S \arrow[d] & \\ & S & \\ \mathcal{Y}_{|V} \arrow[uur, "I_{|V}"] \arrow[ur] \arrow[rr, "\cong"] & & Y_{|V} \times_k S \arrow[uul, "i_{|V} \times_k \mathrm{id}"'] \arrow[ul] \end{tikzcd}$$

In other words, $I : \mathcal{Y} \longrightarrow X \times_k S$ induces the trivial deformation on V. The functor $D^{\mathrm{lt}}_{i/X} : \mathrm{Art}_k \longrightarrow \mathrm{Set}$,

$$R \mapsto \{\text{locally trivial deformations of } i \text{ over } S \text{ with target } X\} \,/ \sim$$

where $\sim$ is the relation of isomorphism, is called *functor of locally trivial deformations of i with target X*.

Let $i : Y \hookrightarrow X$ be a closed immersion of proper algebraic k-schemes X and Y. The following satement is proven in [**Ser06**, Cor 3.2.2] for deformations of i with target X. The case of locally trivial deformations is proven in the same way.

Proposition I.4.2. If $i : Y \hookrightarrow X$ is a closed immersion of proper k-schemes. Then the functors $D_{i/X}$ and $D^{\mathrm{lt}}_{i/X}$ are prorepresentable. $\square$

Let $0 \longrightarrow J \longrightarrow R' \longrightarrow R \longrightarrow 0$ be a small extension in Art_k. Let $i : Y \hookrightarrow X$ be a closed immersion of proper algebraic k-schemes X and Y and

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "f"'] & X \times S \arrow[d, "g"] \\ & S \end{tikzcd}$$

be a deformation i over $S = \operatorname{Spec} R$. As tangent and obstruction spaces will serve

$$T^1_{I/X/R} := R^0 f_* N_{\mathcal{Y}/X\times S}, \quad T^1 := T^1_{i/X/k} = H^0(Y, N_{Y/X}), \qquad \text{(I.4.2)}$$
$$T^2 := H^1(Y, N_{Y/X}).$$

As in [**Ser06**, Prop 3.2.1] one constructs a natural map

$$T^1_{I/X/R} \longrightarrow D_{i/X}(R[\varepsilon])_I, \qquad \text{(I.4.3)}$$

where $D_{i/X}(R[\varepsilon])_I$ is the fiber over I in the sense of Definition I.1.4.

Lemma I.4.3. There is a natural isomorphism $T^1 \xrightarrow{\cong} t_{D_{i/X}}$. More precisely the following holds. Let $I' : \mathcal{Y}' \hookrightarrow X \times S'$ be a deformation of i with fixed target X over $S' = \operatorname{Spec} R'$ such that $I' \times_{S'} S = I$. Then:

(1) The map $T^1_{I/X/R} \longrightarrow D_{i/X}(R[\varepsilon])_I$ from (I.4.3) is a bijection and the diagram

$$\begin{tikzcd} T^1_{I'/X/R'} \arrow[r] \arrow[d] & T^1_{I/X/R} \arrow[d] \\ D_{i/X}(R'[\varepsilon])_{I'} \arrow[r] & D_{i/X}(R[\varepsilon])_I \end{tikzcd}$$

is commutative, where we obtain $T^1_{I'/X/R'} \longrightarrow T^1_{I/X/R}$ by applying $R^0 f_*$ to the natural map $N_{\mathcal{Y}'/X\times S'} \longrightarrow N_{\mathcal{Y}/X\times S}$.

(2) Assume that i is a regular embedding. Then there are natural maps $D_{i/X}(R) \longrightarrow T^2 \otimes J$ such that

$$D_{i/X}(R') \longrightarrow D_{i/X}(R) \longrightarrow T^2 \otimes J$$

is an exact sequence of pointed sets. Furthermore, for every diagram

$$\begin{tikzcd} 0 \arrow[r] & J_1 \arrow[r] \arrow[d, "\phi"'] & R'_1 \arrow[r] \arrow[d] & R_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & J_2 \arrow[r] & R'_2 \arrow[r] & R_2 \arrow[r] & 0 \end{tikzcd}$$

of small extensions the diagram

$$\begin{tikzcd} D_{i/X}(R'_1) \arrow[r] \arrow[d] & D_{i/X}(R_1) \arrow[r] \arrow[d] & T^2 \otimes J_1 \arrow[d, "\mathrm{id}\otimes\phi"] \\ D_{i/X}(R'_2) \arrow[r] & D_{i/X}(R_2) \arrow[r] & T^2 \otimes J_2 \end{tikzcd}$$

commutes.

PROOF. ad (1): We will show more generally that given an arbitrary deformation $I : \mathcal{Y} \hookrightarrow \mathcal{X}$ of i over $S = \operatorname{Spec} R$ there is a natural bijection between $R^0 f_* N_{\mathcal{Y}/\mathcal{X}}$ and the set of isomorphism classes of deformations of I over $R[\varepsilon]$. The proof is a straight forward generalization of [**Ser06**, Prop 3.2.1(ii)]. As for $R = k$ one reduces to the case where $\mathcal{Y}$ and $\mathcal{X}$ are affine. Assume $\mathcal{X} = \operatorname{Spec} A$, $\mathcal{Y} = \operatorname{Spec} B$ and $B = A/I$ where $I = (f_1, \dots, f_N)$. Consider the exact sequence

$$0 \longrightarrow M \longrightarrow A^N \longrightarrow I \longrightarrow 0$$

where the standard basis of A^N maps to the generators of I and M is the module of relations. We write $(r_1, \dots, r_N)$ for an element of $M \subseteq A^N$. Taking $\operatorname{Hom}_A(\cdot, B)$ we obtain an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(I, B) \longrightarrow \operatorname{Hom}_A(A^N, B) \xrightarrow{\varphi} \operatorname{Hom}_A(M, B).$$

Here we observe that

$$\ker\varphi \cong \operatorname{Hom}_A(I,B) \cong \operatorname{Hom}_A(I/I^2,B) \cong R^0 f_* N_{\mathcal{Y}/\mathcal{X}}$$

and $\operatorname{Hom}_A(A^N,B)\cong B^N$. An element of $\ker\varphi \subseteq B^N$ can be represented by an N-tuple $(h_1,\dots,h_N)\in A^N$ with

$$\sum_i h_i r_i \in I \text{ for every } (r_1,\dots,r_N)\in M.$$

This means that we find $s_1,\dots,s_N\in A$ with

$$\sum_i h_i r_i = -\sum_i f_i s_i$$

or equivalently,

$$\sum_i (f_i+\varepsilon h_i)\cdot(r_i+\varepsilon s_i) = 0$$

in $A\otimes_R R[\varepsilon]$. Then it follows from Corollary A.1.3 that the ideal

$$(f_1+\varepsilon h_1,\dots,f_N+\varepsilon h_N)\subseteq A\otimes_R R[\varepsilon]$$

defines a first-order deformation of $\mathcal{Y}$ as a subscheme of $\mathcal{X}$, because every relation among the f_i extends to a relation among the $f_i+\varepsilon h_i$. As all arguments may be reversed, we obtain the other direction.
ad (2): This is [**Ser06**, Prop 3.2.6]. □

In the same way one shows the following

Lemma I.4.4. If (I.4.2) is replaced with

$$T^1_{I/X/R} = R^0 f_* N'_{\mathcal{Y}/X\times S},\quad T^1 := T^1_{i/X/k} = H^0(Y,N'_{Y/X}),\qquad T^2 = H^1(Y,N'_{Y/X}), \tag{I.4.4}$$

where $N'_{Y/X}$ is the equisingular normal sheaf defined in section I.3.2, the analogous statements to Lemma I.4.3 hold for locally trivial deformations of i with fixed target X, i.e. with $D_{i/X}$ replaced by $D^{\mathrm{lt}}_{i/X}$ everywhere. □

Remark I.4.5. There are natural morphisms $D_{i/X}\longrightarrow D_i$ and $D_i\longrightarrow D_X$ of functors. They preserve local triviality and therefore induce morphisms $D^{\mathrm{lt}}_{i/X}\longrightarrow D^{\mathrm{lt}}_i$ and $D^{\mathrm{lt}}_i\longrightarrow D^{\mathrm{lt}}_X$ of functors.

I.5. Deformation theory in the analytic category

Some general remarks about the comparison between the algebraic and the analytic category are in order. As already mentioned, the deformation functors discussed may also be defined in the category of complex spaces and analogues of the results of the previous sections hold true in this context as well.

If $g : \mathcal{X} \longrightarrow S$ is an algebraic S-scheme where $S = \operatorname{Spec} R$ with $R \in \mathrm{Art}_{\mathbb{C}}$, one can associate a complex S-space $g^{\mathrm{an}} : \mathcal{X}^{\mathrm{an}} \longrightarrow S$ to it. Note that $S^{\mathrm{an}} = S$ as a locally ringed space for every Artinian S-scheme. As $T^{\mathrm{an}}_{\mathcal{X}/S} = T_{\mathcal{X}^{\mathrm{an}}/S}$ the natural map $R^i g_* T_{\mathcal{X}/S} \longrightarrow R^i g^{\mathrm{an}}_* T_{\mathcal{X}^{\mathrm{an}}/S}$ is an isomorphism for proper g by [**SGA1**, Exp XII, Thm 4.2]. The functor $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{an}}$ induces natural transformations

$$D_X \longrightarrow D_{X^{\mathrm{an}}}, \quad D_i \longrightarrow D_{i^{\mathrm{an}}}, \quad D_{i/X} \longrightarrow D_{i^{\mathrm{an}}/X^{\mathrm{an}}}. \tag{I.5.1}$$

When $k = \mathbb{C}$ one may also define functors $D^{\mathrm{lt}}_X, D^{\mathrm{lt}}_i$ and $D^{\mathrm{lt}}_{i/X}$ where Y and X are complex spaces and $i : Y \longrightarrow X$ is a morphism of complex spaces. In the definition of local triviality, the sets U and V from Definitions I.2.1, I.3.1 and I.4.1 are requested to be open sets in the Euclidean topology instead.

LEMMA I.5.1. Let $i : Y \hookrightarrow X$ be a closed immersion of proper algebraic $\mathbb{C}$-schemes, let X be smooth and let Y be a reduced locally complete intersection. Then the morphism

$$\mathrm{an} : D^{\mathrm{lt}}_i \longrightarrow D^{\mathrm{lt}}_{i^{\mathrm{an}}}$$

from (I.5.1) is an isomorphism of functors.

PROOF. We put $D := D^{\mathrm{lt}}_i, D^{\mathrm{an}} := D^{\mathrm{lt}}_{i^{\mathrm{an}}}$ and write T^1 and T^2 for tangent and obstruction spaces of D^{lt}_i, see Lemma I.3.4. Let $R \in \mathrm{Art}_k$. We will show by induction on the length $\lg(R)$ that $\mathrm{an}_R : D(R) \longrightarrow D^{\mathrm{an}}(R)$ is an isomorphism. Let $0 \longrightarrow J \longrightarrow R \xrightarrow{p} R'' \longrightarrow 0$ be a small extension in Art_k. Consider the diagram

$$\begin{tikzcd} D(R) \arrow[r, "D(p)"] \arrow[d, "\mathrm{an}_R"] & D(R'') \arrow[r] \arrow[d, "\mathrm{an}_{R''}", "\cong"'] & T^2 \otimes J \arrow[d, hook] \\ D^{\mathrm{an}}(R) \arrow[r, "D^{\mathrm{an}}(p)"] & D^{\mathrm{an}}(R'') \arrow[r] & \left(T^2\right)^{\mathrm{an}} \otimes J \end{tikzcd}$$

We have a simply transitive action of $T^1 \otimes J$ on the fibers of $D(R) \longrightarrow D(R'')$, a simply transitive action of $\left(T^1\right)^{\mathrm{an}} \otimes J$ on the fibers of $D^{\mathrm{an}}(R) \longrightarrow D^{\mathrm{an}}(R'')$ and an isomorphism $T^1 \otimes J \longrightarrow \left(T^1\right)^{\mathrm{an}} \otimes J$ such that an_R is equivariant. Now the claim follows from a version of the five-lemma for pointed sets, where we use the group actions and the fact that $\mathrm{an}_{R''}$ is an isomorphism by induction. □

REMARK I.5.2. The proof above is quite general. It shows, roughly speaking, that deformation functors with the same tangent and obstruction spaces are isomorphic. Therefore it also holds, when D^{lt}_i is replaced by one of the following functors and the below-mentioned assumptions hold.

- D_X^{lt}, where X is a proper k-scheme, which is a reduced, locally complete intersection.
- $D_{i/X}$, where $i : Y \hookrightarrow X$ is a closed immersion of proper k-schemes, Y is a reduced locally complete intersection and X is smooth.
- $D_{i/X}^{\mathrm{lt}}$, where $i : Y \hookrightarrow X$ is a closed immersion of proper k-schemes, Y is a reduced locally complete intersection and X is smooth.

Furthermore, we see that the notion of local triviality does not depend on whether one uses Zariski or Euclidean open sets to define it, as long as one sticks to deformations over local Artin algebras.

I.5.3. Universality of deformations. We will conclude this chapter with some more terminology. Let $\pi : \mathfrak{X} \longrightarrow M$ be a flat morphism of complex spaces or algebraic schemes and fix a closed point $0 \in M$. We call π or $\mathfrak{X}$ a *deformation* of $X = \pi^{-1}(0)$. If M is the spectrum of an object in $\mathrm{Art}_{\mathbb{C}}$, this definition clearly coincides with Definition I.2.1 or rather with its analogue in the analytic category. Let now

$$R = \widehat{\mathcal{O}_{M,0}} \in \widehat{\mathrm{Art}}_{\mathbb{C}}$$

be the completion of $\mathcal{O}_{M,0}$ at 0 and $\mathfrak{m} \subseteq R$ be the maximal ideal. Taking the limit

$$\varprojlim_{n \in \mathbb{N}} \mathfrak{X} \times_R R/\mathfrak{m}^n$$

defines a formal element $u \in \widehat{D}_X(R)$.

Definition I.5.4. We say that the deformation $\pi : \mathfrak{X} \longrightarrow M$ is *universal at* 0 respectively *versal at* 0 respectively *semi-universal at* 0, if u is a universal respectively versal respectively semi-universal formal element in the sense of Definition I.1.2. Furthermore, we call π *universal* respectively *versal* respectively *semi-universal*, if it has this property for every point $t \in M$. Sometimes, we call $\mathfrak{X}$ instead of π universal or versal or semi-universal. The base space M of a universal deformation is sometimes called the *universal deformation space* and similarly for versal or semi-universal deformations.

If $\pi : \mathfrak{X} \longrightarrow M$ is universal at 0, then the ring $R_X = \widehat{\mathcal{O}_{M,0}}$ prorepresents the functor D_X. We spell this out in a special case of deformations of a closed immersion in Lemma VI.3.4. Similar statements also hold true for deformations of other types of objects and similar terminology is applied.

CHAPTER II

Weil restriction and Hodge theory over an Artin ring

In this chapter we axiomatize the Hodge theory of locally trivial deformations $f : \mathcal{Y} \longrightarrow S$ of an algebraic variety Y over an Artinian scheme $S = \operatorname{Spec} R$, where $R \in \mathrm{Art}_{\mathbb{C}}$. We introduce the notion of a *mixed Hodge structure over* R, see Definition II.2.1. Similar to an ordinary mixed Hodge structure, it consists of a free R-module H, which has a real structure, together with two filtrations on H and conditions on their graded objects. This concept plays a major role in the proofs of our main results. As far as we know, it has not been studied before. Mixed Hodge structures over R are intermediate objects between ordinary mixed Hodge structures and variations of mixed Hodge structures. For $R = \mathbb{C}$ we recover the notion of a mixed Hodge structure in the usual sense. In Chapter III we construct such a structure on the de Rham cohomology of a simple normal crossing variety, but the construction should be possible in greater generality.

The purpose of this concept is to carry out Hodge theoretic arguments infinitesimally. Concrete instances of this idea are treated in Chapter III, here we work out the abstract framework. The problem for $R \neq \mathbb{C}$ is that there is no analogue of the complex conjugation on the R-module H. We will cure this by introducing the notion of a *mixed Hodge-Weil structure over* R', where R' is now a local Artin $\mathbb{R}$-algebra with residue field $\mathbb{R}$. This notion is a formalization of the Weil restriction of a mixed Hodge structure over R and there is a canonical complex conjugation.

The foundations of the modern theory of Weil restriction were laid by Grothendieck in [**Gro59, Gro60**]. In our special case Weil restriction is a process of associating an $\mathbb{R}$-scheme $S_{\mathbb{C}/\mathbb{R}}$ to a $\mathbb{C}$-scheme S such that the $\mathbb{R}$-valued points of $S_{\mathbb{C}/\mathbb{R}}$ are exactly the $\mathbb{C}$-valued points of S. Of course, the correct way to phrase this is the language of functors and representability. However, we think of Weil restriction simply as the algebro-geometric analogue of the process of regarding a complex manifold as a differentiable manifold.

We extend the concept of Weil restriction to modules. We are not aware that this has been done systematically before. Nevertheless, it is an elementary byproduct of the functorial treatment of Weil restriction. The purpose of this is to be able to study the properties of a mixed Hodge structure over R on an R-module H through the mixed Hodge-Weil structure on $H_{\mathbb{C}/\mathbb{R}}$ over $R_{\mathbb{C}/\mathbb{R}}$ obtained by Weil restriction. Therefore, we establish some comparison results between R-modules and their Weil restrictions.
Let us begin with the discussion of the theory of Weil restriction.

II.1. Weil restriction

Let $S \xrightarrow{f} Z \xrightarrow{p} W$ be morphisms of schemes and consider the functor

$$\text{(II.1.1)} \qquad \underline{S_{Z/W}} : (\mathrm{Sch/W})^{\mathrm{op}} \longrightarrow \mathrm{Set}, \quad S' \mapsto \mathrm{Mor}_{\mathrm{Sch/Z}}(S' \times_W Z, S).$$

In fact, we have $\mathrm{Mor}_{\mathrm{Sch/S'\times_W Z}}(S' \times_W Z, S \times_W Z) = \mathrm{Mor}_{\mathrm{Sch/Z}}(S' \times_W Z, S)$, which follows from the universal property of the fiber product. Therefore, the functor $\underline{S_{Z/W}}$ coincides with the one defined by Grothendieck in [**Gro59**, C.2, pp.12]. The functor $\underline{S_{Z/W}}$ is representable in the following cases.

(1) If $S \longrightarrow Z$ is proper and flat and $S \longrightarrow W$ is quasiprojective, this functor is representable by [**Gro60**, 4.c., p.20] by a W-scheme $S_{Z/W}$.
(2) Suppose that $Z \longrightarrow W$ is finite and locally free, i.e. finite, flat and of finite presentation, and that moreover for each $x \in W$ and each finite set of points $P \subseteq S \times_W k(x)$ there is an affine open $U \subseteq S$ containing P. Then $\underline{S_{Z/W}}$ is representable by an W-scheme $S_{Z/W}$ by [**BLR90**, 7.6, Thm 4].

The W-scheme $S_{Z/W}$ is called the *Weil restriction* of S.

II.1.1. Properties of Weil restriction. We will collect some properties of the process of Weil restriction. If not otherwise stated, proofs are found in [**BLR90**, Ch 7.6]. Recall that a presheaf of sets on Sch/Z is a functor $(\mathrm{Sch/Z})^{\mathrm{op}} \longrightarrow \mathrm{Set}$. The category of presheaves of sets on Sch/Z is denoted by $\mathrm{Psh}(Z)$. By the Yoneda embedding a Z-scheme S may be interpreted as a prescheaf of sets on Sch/Z via

$$\underline{S} : (\mathrm{Sch/Z})^{\mathrm{op}} \longrightarrow \mathrm{Set}, \quad T \mapsto \mathrm{Mor}_{\mathrm{Sch/Z}}(T, S).$$

We will not distinguish between S and $\underline{S}$. There is a notion of pushforward of presheaves along the morphism $p : Z \longrightarrow W$. Pushforward is the functor

$$p_* : \mathrm{Psh}(Z) \longrightarrow \mathrm{Psh}(W), \quad F \mapsto \big(S' \mapsto F(S' \times_W Z)\big)$$

and it coincides with Weil restriction on the full subcategory Sch/Z, i.e. $S_{Z/W} = p_*S$. Representability means that

$$\text{(II.1.2)} \qquad \mathrm{Mor}_{\mathrm{Sch/Z}}(S' \times_W Z, S) = \mathrm{Mor}_{\mathrm{Sch/W}}(S', S_{Z/W}).$$

In other words, $S \mapsto S_{Z/W}$ is right adjoint to the pullback $S' \mapsto p^*S' = S' \times_W Z$. In particular for a Z-scheme S there is a canonical morphism $\eta : S_{Z/W} \times_W Z \longrightarrow S$. If $p : Z \longrightarrow W$ is proper, flat and of finite presentation, then p_* preserves open and closed immersions.
We will now specialize to $Z = \operatorname{Spec} \mathbb{C}$ and $W = \operatorname{Spec} \mathbb{R}$. In this case every quasi-projective $\mathbb{C}$-scheme S has a Weil restriction. We write $S_{\mathbb{C}/\mathbb{R}}$ instead of $S_{Z/W}$. The functor p_* sends affine schemes to affine schemes, in other words, p_*S is representable by an affine scheme $S_{\mathbb{C}/\mathbb{R}}$. If $S = \operatorname{Spec} R$ we will write $R_{Z/W}$ for the cooordinate ring of $S_{Z/W}$. Equation (II.1.2) in particular gives $S(\mathbb{C}) = S_{Z/W}(\mathbb{R})$. If $S = \operatorname{Spec} R$, the morphism η from the adjointness property gives a canonical ring homomorphism $\eta : R \longrightarrow R_{\mathbb{C}/\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.
Let $S = \cup_i U_i$ be a covering by open affine subschemes, such that for given $t_1, t_2 \in S$ there is an index i_0 with $t_1, t_2 \in U_{i_0}$. The proof of representability in [**BLR90**, 7.6, Thm 4] shows that under this assumption the $(U_i)_{\mathbb{C}/\mathbb{R}}$ will cover $S_{\mathbb{C}/\mathbb{R}}$. For $R = \mathbb{C}[z_1, \ldots, z_n]/(f_1, \ldots, f_k)$ we have

$$\text{(II.1.3)} \qquad R_{\mathbb{C}/\mathbb{R}} = \mathbb{R}[x_1, y_1, \ldots, x_n, y_n]/(g_1, h_1, \ldots, g_k, h_k)$$

where $f_j = g_j + ih_j$ when we evaluate at $z_k = x_k + iy_k$.
If we define $\overline{S} := S \times_\sigma \mathbb{C}$ where $\sigma : \mathbb{C} \longrightarrow \mathbb{C}$ is the complex conjugation then (II.1.2) tells us that there is a canonical isomorphism $\overline{S}_{\mathbb{C}/\mathbb{R}} \cong S_{\mathbb{C}/\mathbb{R}}$ and by [**Sch94**, Ch 1, 4.11.3] there is a canonical isomorphism $S_{\mathbb{C}/\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} \longrightarrow S \times_{\mathbb{C}} \overline{S}$ such that η is identified with projection on the first factor. In particular, η is faithfully flat as the projection $S \times_{\mathbb{C}} \overline{S} \longrightarrow S$ is faithfully flat.

Lemma II.1.2. If R is a local Artin $\mathbb{C}$-algebra with residue field isomorphic to $\mathbb{C}$, then $R_{\mathbb{C}/\mathbb{R}}$ is a local Artin $\mathbb{R}$-algebra with residue field isomorphic to $\mathbb{R}$.

Proof. By (II.1.3) we see that $R_{\mathbb{C}/\mathbb{R}}$ is an $\mathbb{R}$-algebra of finite type. A maximal ideal $\mathfrak{m} \subset R_{\mathbb{C}/\mathbb{R}}$ will define a homomorphism $R_{\mathbb{C}/\mathbb{R}} \longrightarrow R_{\mathbb{C}/\mathbb{R}}/\mathfrak{m} = k$, where k is a finite field extension of $\mathbb{R}$ by Hilbert's Nullstellensatz. So $k = \mathbb{R}$ or $\mathbb{C}$. By the defining property of Weil restriction we have $\mathrm{Hom}_{\mathbb{R}}(R_{\mathbb{C}/\mathbb{R}}, \mathbb{R}) = \mathrm{Hom}_{\mathbb{C}}(R, \mathbb{C})$ and $\mathrm{Hom}_{\mathbb{R}}(R_{\mathbb{C}/\mathbb{R}}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(R, \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(R, \mathbb{C} \times \mathbb{C})$ both of which consist of one element. But the composition of the morphism $R \longrightarrow \mathbb{R}$ with the inclusion $\mathbb{R} \subseteq \mathbb{C}$ is the unique morphism $R \longrightarrow \mathbb{C}$. Thus, $R_{\mathbb{R}}$ is a local ring with unique maximal ideal $\mathfrak{m}$ and residue field $\mathbb{R}$. As $R_{\mathbb{C}/\mathbb{R}}$ is of finite type, $R_{\mathbb{C}/\mathbb{R}} = P/I$ where P is a polynomial ring and $I \subseteq P$

an ideal. The preimage $\mathfrak{n}$ of $\mathfrak{m}$ under the natural map $P \longrightarrow R_{\mathbb{C}/\mathbb{R}}$ is the unique maximal ideal of P containing I. Let $I \subseteq \mathfrak{p} \subseteq \mathfrak{n}$ be a minimal prime ideal containing I. As P is a Jacobson ring by the general form of the Nullstellensatz, see [**Eis95**, Thm 4.19], the ideal $\mathfrak{p}$ is the intersection of maximal ideals, so that $\mathfrak{p} = \mathfrak{n}$. Taking a primary decomposition of I we see that $\mathfrak{n}^k \subseteq I$ for some k, so $R_{\mathbb{C}/\mathbb{R}} = P/I$ is Artinian. □

Definition II.1.3. Let S be a $\mathbb{C}$-scheme, F be a quasi-coherent sheaf of $\mathcal{O}_S$-modules, denote by $q : S_{\mathbb{C}/\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} \longrightarrow S_{\mathbb{C}/\mathbb{R}}$ the canonical projection and let $\eta : S_{\mathbb{C}/\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} \longrightarrow S$ be as in II.1.1. We define the $S_{\mathbb{C}/\mathbb{R}}$-module

$$F_{\mathbb{C}/\mathbb{R}} := q_* \eta^* F$$

and call it the *Weil restriction* of F.

If $S = \operatorname{Spec} R$ and M is an R-module, then $M_{\mathbb{C}/\mathbb{R}} = M \otimes_R (R_{\mathbb{C}/\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})$ considered as an $R_{\mathbb{C}/\mathbb{R}}$ module. In the special case $M = H \otimes_{\mathbb{C}} R$ for some $\mathbb{C}$-vector space H, we find $M_{\mathbb{C}/\mathbb{R}} = H \otimes_{\mathbb{R}} R_{\mathbb{C}/\mathbb{R}}$. Weil restriction for modules has the following useful property.

Lemma II.1.4. The functor $F \mapsto F_{\mathbb{C}/\mathbb{R}}$ is faithfully exact, i.e. the sequence $K' \longrightarrow K \longrightarrow K''$ is exact if and only if $K'_{\mathbb{C}/\mathbb{R}} \longrightarrow K_{\mathbb{C}/\mathbb{R}} \longrightarrow K''_{\mathbb{C}/\mathbb{R}}$ is exact.

Proof. The morphism η is faithfully flat as noted at the end of section II.1.1. Therefore, η^* is faithfully exact. Also q_* is faithfully exact, as q is affine. □

Lemma II.1.5. Let $(R, \mathfrak{m})$ be a local Artin $\mathbb{C}$-algebra and F be a finitely generated R-module. Then F is a free R-module if and only if $F_{\mathbb{C}/\mathbb{R}}$ is a free $R_{\mathbb{C}/\mathbb{R}}$-module.

Proof. We will argue separately for η^* and q_*. For brevity we write $(R', \mathfrak{m}')$ instead of $(R_{\mathbb{C}/\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}, \mathfrak{m}_{\mathbb{C}/\mathbb{R}} \otimes_R \mathbb{C})$. Clearly $\eta^* F = F \otimes_R R'$ is free, if F is. Suppose $\eta^* F$ is free. We take a minimal set of generators for F and obtain a surjection $\varphi : R^n \longrightarrow F$ for some n. By Nakayama's Lemma $n = \dim_{\mathbb{C}} F \otimes_R R/\mathfrak{m}$ and as $F \otimes_R R' \otimes_{R'} R'/\mathfrak{m}' = F \otimes_R R/\mathfrak{m} \otimes_{R/\mathfrak{m}} R'/\mathfrak{m}'$ this is the rank of $\eta^* F$. But as η^* is faithfully exact, $\eta^* \ker \varphi = \ker \eta^* \varphi = 0$. So $\ker \varphi = 0$ and F is free.
Let F' be an R'-module. If F' is free as an R'-module, then it is free as an $R_{\mathbb{C}/\mathbb{R}}$-module, for R' is free over $R_{\mathbb{C}/\mathbb{R}}$. Suppose F' is free as an $R_{\mathbb{C}/\mathbb{R}}$-module. Since F' is an $R' = R_{\mathbb{C}/\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$-module, the submodule $\mathfrak{m}_{\mathbb{C}/\mathbb{R}} F'$ is a $\mathbb{C}$-vector space. Thus $\mathfrak{m}_{\mathbb{C}/\mathbb{R}} F' = \mathfrak{m}' F'$. If we take $x_1, \dots, x_k \in F'$ whose residue classes modulo $\mathfrak{m}_{\mathbb{C}/\mathbb{R}}$ form a $\mathbb{C}$-basis of $F'/\mathfrak{m}_{\mathbb{C}/\mathbb{R}} F'$, then F

is freely generated over $R_{\mathbb{C}/\mathbb{R}}$ by $x_1, ix_1, \ldots, x_k, ix_k$. In other words, F is freely generated over R' by $x_1, \ldots, x_k$. So F' is a free R'-module. $\square$

EXAMPLE II.1.6. For the projective space $S = \mathbb{P}^1_{\mathbb{C}}$ of lines in $\mathbb{C}^2$ one finds that $S_{\mathbb{C}/\mathbb{R}}$ is isomorphic over $\mathbb{R}$ to the quadric Q in $\mathbb{P}^3_{\mathbb{R}}$ given by

$$x_1 x_2 - x_0^2 - x_3^2 = 0.$$

To verify this, take the universal line bundle $\mathcal{L}$ on $\mathbb{P}^1_{\mathbb{C}}$. It fits into an exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathbb{C}^2 \otimes_{\mathbb{C}} \mathcal{O}_S \longrightarrow \mathcal{Q} \longrightarrow 0$$

of locally free sheaves on S. Taking the Weil restriction of this sequence, we obtain the sequence

$$0 \longrightarrow \mathcal{L}_{\mathbb{C}/\mathbb{R}} \longrightarrow q_*\left(\mathbb{C}^2 \otimes_{\mathbb{C}} \mathcal{O}_{S'}\right) \longrightarrow \mathcal{Q}_{\mathbb{C}/\mathbb{R}} \longrightarrow 0,$$

locally free sheaves on $S_{\mathbb{C}/\mathbb{R}}$, where $S' = S_{\mathbb{C}/\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ and $q : S' \longrightarrow S_{\mathbb{C}/\mathbb{R}}$ is the canonical morphism. Moreover,

$$q_*\left(\mathbb{C}^2 \otimes_{\mathbb{C}} \mathcal{O}_{S'}\right) = \mathbb{C}^2 \otimes_{\mathbb{R}} \mathcal{O}_{S_{\mathbb{C}/\mathbb{R}}}$$

and $\mathrm{rk}\mathcal{L}_{\mathbb{C}/\mathbb{R}} = 2$. So $S_{\mathbb{C}/\mathbb{R}}$ parametrizes 2-dimensional real subspaces in $\mathbb{C}^2$ and we obtain a classifying morphism $\varphi : S_{\mathbb{C}/\mathbb{R}} \longrightarrow \mathrm{Gr}(2,4)_{\mathbb{R}}$ to the corresponding Grassmanian. By construction, $S_{\mathbb{C}/\mathbb{R}}$ parametrizes exactly those 2-dimensional subspaces, which are complex lines, hence φ is injective. To see that φ is a closed embedding, it suffices to show that its set-theoretical image is a smooth closed subvariety. We fix the $\mathbb{R}$-basis e_1, ie_1, e_2, ie_2 where e_1, e_2 are the standard basis vectors of $\mathbb{C}^2$. Consider a real plane $W \subseteq \mathbb{C}^2$ spanned by two vectors

$$w = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix}, \; w' = \begin{pmatrix} \alpha_3 \\ \beta_3 \\ \alpha_4 \\ \beta_4 \end{pmatrix} \in W$$

and let $\mathrm{Gr}(2,4)_{\mathbb{R}} \hookrightarrow \mathbb{P}^5_{\mathbb{R}}$ be the Plücker embedding. The Plücker coordinates of the point W are exactly the 2×2-minors of the matrix

$$\begin{pmatrix} \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 & \alpha_4 & \beta_4 \end{pmatrix}$$

If W is a complex line in $\mathbb{C}^2$, then we can choose an $\mathbb{R}$-basis of W of the form w, w' with $w' = iw$. Thus, the Plücker coordinates of W are the 2×2 minors of the matrix

$$\begin{pmatrix} \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\ -\beta_1 & \alpha_1 & -\beta_2 & \alpha_2 \end{pmatrix}$$

These are

$$m_{12} = \alpha_1^2 + \beta_1^2 \qquad m_{13} = -\alpha_1\beta_2 + \alpha_2\beta_1 \qquad m_{14} = \alpha_1\alpha_2 + \beta_1\beta_2$$
$$m_{23} = -\alpha_1\alpha_2 - \beta_1\beta_2 \qquad m_{24} = -\alpha_1\beta_2 + \alpha_2\beta_1 \qquad m_{34} = \alpha_2^2 + \beta_2^2.$$

We observe that $m_{14} = -m_{23}$ and $m_{13} = m_{24}$. These two equations cut out a $\mathbb{P}^3_{\mathbb{R}} \subseteq \mathbb{P}^5_{\mathbb{R}}$, which contains the image of $S_{\mathbb{C}/\mathbb{R}}$. If we eliminate m_{14} and m_{24}, then the usual Plücker quartic

$$m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23} = 0$$

takes the form $m_{12}m_{34} - m_{13}^2 - m_{23}^2 = 0$, which is what we claimed up to renaming the variables. A calculation shows that there are no additional relations. Note, that indeed $Q \times_{\mathbb{R}} \mathbb{C} \cong \mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_C$ so $S_{\mathbb{C}/\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} \cong S \times \overline{S}$ as claimed in section II.1.1.

The usage of Weil restriction here may be seen as an analogue of the process of regarding a complex manifold as a differentiable manifold. The points in complex space of the former correspond to points in real space of the latter.

II.2. Hodge-Weil theory

We introduce the notion of mixed Hodge structure and mixed Hodge-Weil structure *over an Artin ring*. The term *mixed Hodge structure* with no further decoration will always stand for the classical notion, which we recorded in Definition B.1.2.

DEFINITION II.2.1. Let R be a local Artin $\mathbb{C}$-algebra with residue field $\mathbb{C}$. A *mixed Hodge structure over* R is a triple $\mathcal{H} = (H_{\mathbb{R}}, F^\bullet, W_\bullet)$, which consists of a finite dimensional $\mathbb{R}$-vectorspace $H_{\mathbb{R}}$ and two filtrations $F^\bullet$ and $W_\bullet$ on $H := (H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} R$. These are a finite decreasing filtration

$$H \supseteq \ldots \supseteq F^p \supseteq F^{p+1} \supseteq \ldots \supseteq 0$$

and a finite increasing filtration

$$0 \subseteq \ldots \subseteq W_m \subseteq W_{m+1} \subseteq \ldots \subseteq H$$

satisfying the following properties.

(1) All graded objects $\mathrm{Gr}_F^p \mathrm{Gr}_m^W H$ are free R-modules.
(2) The fiber $\mathcal{H} \otimes_R \mathbb{C} = (H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}, F^\bullet \otimes_R \mathbb{C}, W_\bullet \otimes_R \mathbb{C})$ over the unique point of $S = \operatorname{Spec} R$ is a mixed Hodge structure.

Note, that condition (1) implies that the W_m and the F^p are free R-modules. We will also call $\mathcal{H} \otimes_R \mathbb{C}$ the *central fiber* of $\mathcal{H}$. In case $\mathcal{H} \otimes_R \mathbb{C}$ is a pure Hodge structure of weight k, we call $\mathcal{H}$ a *pure Hodge structure* over R of weight k.

By Lemma A.2.3 the freeness of $\mathrm{Gr}_F^p H$ and $\mathrm{Gr}_m^W H$ is automatic as soon as $F^p H$ and $W_m H$ are free. However the following example shows that freeness of $\mathrm{Gr}_F^p \mathrm{Gr}_m^W H$ is not automatic.

EXAMPLE II.2.2. We take $R = \mathbb{C}[t]/t^2$ and $H_{\mathbb{R}} = \mathbb{R}^3$. We define a Hodge filtration and weight filtration on $H = H_{\mathbb{R}} \otimes_{\mathbb{R}} R = R^3$ as follows.

$$H = F^0 \supseteq F^1 = \left\langle \begin{pmatrix} 1 \\ 1+t \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} \right\rangle \supseteq F^2 = \left\langle \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} \right\rangle$$

and

$$H = W_1 \supseteq W_0 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

where $\langle \cdot \rangle$ denotes the R-span and all other F^p, W_m are zero. A calculation shows that indeed $(H_{\mathbb{R}}, F^p/t, W_m/t)$ is a mixed Hodge structure with $h^{1,0} = h^{0,1} = h^{0,0} = 1$ as only non-zero Hodge numbers. But

$$W_0 \cap F^1 = \left\langle \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} \right\rangle,$$

which is not a free module. As $W_0 \cap F^2 = 0 = W_{-1}$ we have $W_0 \cap F^1 = F^1 \mathrm{Gr}_0^W = \mathrm{Gr}_F^1 \mathrm{Gr}_0^W$ and $(H_{\mathbb{R}}, F^\bullet, W_\bullet)$ does not define a mixed Hodge structure over R.

DEFINITION II.2.3. Let R be a local Artin $\mathbb{C}$-Algebra and $\mathcal{H} = (H_{\mathbb{R}}, F, W)$, $\mathcal{H}' = (H'_{\mathbb{R}}, F', W')$ be mixed Hodge structures over R. A *morphism of mixed Hodge structures over* R is a linear map $f_{\mathbb{R}} : H_{\mathbb{R}} \longrightarrow H'_{\mathbb{R}}$ such that the induced morphism $f = f_{\mathbb{R}} \otimes \mathrm{id}_R : H \longrightarrow H'$ preserves both filtrations, i.e. $f(F^p) \subseteq F^{p\prime}$ and $f(W_m) \subseteq {W_m}'$. Here again $H = (H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} R$ and H' is defined analogously. We will often call f instead of $f_{\mathbb{R}}$ a morphism of mixed Hodge structures over R when there is no danger of confusion.

REMARK II.2.4. —

(1) If $\mathcal{H} = (H_{\mathbb{R}}, F, W)$ is a pure Hodge structure of weight k over an Artin ring R, then Nakayama's Lemma implies that W is a trivial filtration, i.e. $H = W_k \supseteq W_{k-1} = 0$. We will therefore suppress W in the notation and speak of a pure Hodge structure $\mathcal{H} = (H_{\mathbb{R}}, F)$ over R.

(2) There is a complex conjugation $H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ defined by $\overline{h \otimes \lambda} := h \otimes \overline{\lambda}$. However this does not canonically extend to an $\mathbb{R}$-linear map $H \longrightarrow H$, as H is a tensor product over $\mathbb{C}$ and complex conjugation is only $\mathbb{R}$-linear.

The notion of a mixed Hodge structure over R is an infinitesimal version of a variation of mixed Hodge structures. The problem in replacing the base manifold S of the variation with an Artin ring R (or rather its spectrum $S = \operatorname{Spec} R$) is that there is just one point in S and simply asking the fiber over that point to be a mixed Hodge structure is not enough for our purposes. It is known that the (pointwise) complex conjugates $\overline{F^p}$ of the Hodge filtration of a variation of Hodge structures do not in general form holomorphic vector bundles in case S is a complex manifold. This is because there is no analogue of the complex conjugate of an holomorphic vector bundle in the algebraic category. To have a substitute we introduce the following notion.

Definition II.2.5. Let R be a local Artin $\mathbb{R}$-algebra with residue field $\mathbb{R}$. A *mixed Hodge-Weil structure* over R is a triple $\mathcal{H} = (H_{\mathbb{R}}, F^\bullet, W_\bullet)$, which consists of a finite dimensional $\mathbb{R}$-vectorspace $H_{\mathbb{R}}$ and two filtrations $F^\bullet$ and $W_\bullet$ on $H := (H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{R}} R$. These are a finite decreasing filtration

$$H \supseteq \ldots \supseteq F^p \supseteq F^{p+1} \supseteq \ldots \supseteq 0$$

and a finite increasing filtration

$$0 \subseteq \ldots \subseteq W_m \subseteq W_{m+1} \subseteq \ldots \subseteq H$$

satisfying the following properties.

(1) All graded objects $\mathrm{Gr}_F^p \mathrm{Gr}_m^W H$ are free R-modules.
(2) The fiber $\mathcal{H} \otimes_R \mathbb{R} = (H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}, F^\bullet \otimes_R \mathbb{R}, W_\bullet \otimes_R \mathbb{R})$ over the unique point of $S = \operatorname{Spec} R$ is a mixed Hodge structure.

Note, that as in Definition II.2.1 condition (1) implies that the W_m and the F^p are free R-modules. We will also call $\mathcal{H} \otimes_R \mathbb{R}$ the *central fiber* of $\mathcal{H}$. In case $\mathcal{H} \otimes_R \mathbb{C}$ is a pure Hodge structure of weight k, we call $\mathcal{H}$ a *pure Hodge-Weil structure* over R of weight k.

Definition II.2.6. Let R be a local Artin $\mathbb{R}$-Algebra with residue field $\mathbb{R}$ and $\mathcal{H} = (H_{\mathbb{R}}, F, W)$, $\mathcal{H}' = (H'_{\mathbb{R}}, F', W')$ be mixed Hodge-Weil structures over R. A morphism of mixed Hodge-Weil structures over R is a linear map $f : H_{\mathbb{R}} \longrightarrow H'_{\mathbb{R}}$ such that the induced morphism $f_R = f \otimes \mathrm{id}_R : H \longrightarrow H'$ preserves both filtrations, i.e. $f_R(F^p) \subseteq F^{p\prime}$ and $f_R(W_m) \subseteq {W_m}'$. Here again $H = (H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}) \otimes_\mathbb{R} R$ and H' is defined analogously. We will write f instead of f_R when there is no danger of confusion.

Remark II.2.7. —

(1) As in the Hodge-case, we write $\mathcal{H} = (H_\mathbb{R}, F)$ for a pure Hodge-Weil structure.

(2) The complex conjugation $H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \longrightarrow H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ extends canonically to an $\mathbb{R}$-linear map $H \longrightarrow H$. Since morphisms of mixed Hodge-Weil structures are defined over $\mathbb{R}$, they are compatible with complex conjugation.

Recall that for a local Artin $\mathbb{C}$-Algebra R the ring $R_{\mathbb{C}/\mathbb{R}}$ is a local Artin $\mathbb{R}$-Algebra with residue with residue field $\mathbb{R}$ by Lemma II.1.2. Therefore the statement of the following Lemma makes sense.

Lemma II.2.8. Let $\mathcal{H} = (H_\mathbb{R}, F^\bullet, W_\bullet)$ be a mixed Hodge structure over a local Artin $\mathbb{C}$-Algebra R. Then $\mathcal{H}_{\mathbb{C}/\mathbb{R}} = \left(H_\mathbb{R}, F^\bullet_{\mathbb{C}/\mathbb{R}}, \left(W_{\mathbb{C}/\mathbb{R}}\right)_\bullet\right)$ is a mixed Hodge-Weil structure over $R_{\mathbb{C}/\mathbb{R}}$ and the central fibers of $\mathcal{H}$ and $\mathcal{H}_{\mathbb{C}/\mathbb{R}}$ are isomorphic as mixed Hodge structures. Moreover the Weil restriction of a morphism of mixed Hodge structures is a morphism of mixed Hodge-Weil structures.

Proof. The remark after Definition II.1.3 tells us that

$$H_{\mathbb{C}/\mathbb{R}} = (H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \otimes_\mathbb{C} R)_{\mathbb{C}/\mathbb{R}} = (H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}) \otimes_\mathbb{R} R_{\mathbb{C}/\mathbb{R}}. \tag{II.2.1}$$

By Lemma II.1.4 we see that the $F^p_{\mathbb{C}/\mathbb{R}}$ and $(W_m)_{\mathbb{C}/\mathbb{R}}$ are submodules of $H_{\mathbb{C}/\mathbb{R}} = (H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}) \otimes_\mathbb{R} R_{\mathbb{C}/\mathbb{R}}$. By Lemma II.1.5 the modules $\left(\mathrm{Gr}^p_F \mathrm{Gr}^W_m H\right)_{\mathbb{C}/\mathbb{R}}$ are free and by Lemma II.1.4 they are the graded objects of the filtrations $F^p_{\mathbb{C}/\mathbb{R}}$ and $(W_m)_{\mathbb{C}/\mathbb{R}}$. Let $\mathfrak{m}'$ be the maximal ideal of $R_{\mathbb{C}/\mathbb{R}}$. As $R_{\mathbb{C}/\mathbb{R}}/\mathfrak{m}' = \mathbb{R}$ we see from II.2.1 that $H_{\mathbb{C}/\mathbb{R}} \otimes_\mathbb{R} R_{\mathbb{C}/\mathbb{R}}/\mathfrak{m}' = H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$. For the same reason $F^p_{\mathbb{C}/\mathbb{R}} \otimes \mathbb{R} = F^p \otimes \mathbb{C}$ and $(W_m)_{\mathbb{C}/\mathbb{R}} \otimes \mathbb{R} = W_m \otimes \mathbb{C}$ so that $\mathcal{H}_{\mathbb{C}/\mathbb{R}} \otimes \mathbb{R}$ is a mixed Hodge structure. The proof also shows the statement about the central fibers and the statement about morphisms is immediate from the functoriality of the Weil restriction. □

Lemma II.2.9. Let R be a local Artin $\mathbb{R}$-Algebra with residue field $\mathbb{R}$ and $\mathcal{H} = (H_\mathbb{R}, F^\bullet)$ a pure Hodge-Weil structure of weight k. Then

$$H = F^p \oplus \overline{F^{q+1}}, \qquad \forall p, q, p+q = k, \tag{II.2.2}$$

$$H = \bigoplus_{p+q=k} H^{p,q}, \qquad H^{p,q} = F^p \cap \overline{F^q} \qquad \text{and} \tag{II.2.3}$$

$$F^p = \bigoplus_{r \geq p} H^{r,k-r}. \tag{II.2.4}$$

In particular the last statement implies that the $H^{p,q}$ are free and lift the subquotients $\mathrm{Gr}_F^p H$ to subobjects of H.

PROOF. As $\mathcal{H} \otimes_R \mathbb{R}$ is a pure Hodge structure, we have

$$H \otimes_R \mathbb{R} = F^p \otimes_R \mathbb{R} \oplus \overline{F^{q+1} \otimes_R \mathbb{R}} \quad \forall p, q, p+q=k.$$

Hence (II.2.2) follows from Nakayama's Lemma. Now (II.2.2) implies (II.2.3) just as in the case of ordinary Hodge structures. We will recall the proof. Let $\alpha \in F^p \subseteq H$ and write $\alpha = \beta + \gamma$ where $\beta \in F^{p+1}$, $\gamma \in \overline{F^{k-p}}$ according to $H = F^{p+1} \oplus \overline{F^{k-p}}$. Then $\gamma = \alpha - \beta \in F^p \cap \overline{F^{k-p}} = H^{p,k-p}$. This shows that $F^p = F^{p+1} \oplus H^{p,q}$, and (II.2.3) and (II.2.4) follow by induction on p. $\square$

LEMMA II.2.10. Let R be a local Artin $\mathbb{C}$-Algebra, let $\mathcal{H} = (H_{\mathbb{R}}, F, W)$ and $\mathcal{H}' = (H'_{\mathbb{R}}, F', W')$ be mixed Hodge structures over R and let $f : H \longrightarrow H'$ be a morphism of mixed Hodge structures over R. Then $f^{p,q} := f|_{H^{p,q}}$ satisfies $f^{p,q}(H^{p,q}) \subseteq (H')^{p,q}$ and $f = \sum_{p,q} f^{p,q}$. Moreover, all $f^{p,q}$ have constant rank in the sense of Definition A.2.1.

PROOF. By (II.2.3) the image of $f^{p,q}$ in contained in $(H')^{p,q}$, because f is defined over $\mathbb{R}$ and preserves the Hodge filtration. Again, as f is defined over $\mathbb{R}$ its cokernel is

$$\operatorname{coker} f = \operatorname{coker}\left(f_{\mathbb{R}} : H_{\mathbb{R}} \longrightarrow H'_{\mathbb{R}}\right) \otimes_{\mathbb{R}} R,$$

so it is free. Then

$$\operatorname{coker} f = \bigoplus_{p,q} \operatorname{coker} f^{p,q}$$

implies that $\operatorname{coker} f^{p,q}$ is free. So the claim follows from Lemma A.2.2. $\square$

CHAPTER III

Mixed Hodge structures for normal crossing varieties

Let $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_{\mathbb{C}}$ and let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of a proper simple normal crossing $\mathbb{C}$-variety Y. We will construct a complex $\widetilde{\Omega}^{\bullet}_{\mathcal{Y}/S}$, which calculates the cohomology with coefficients in the constant sheaf $\underline{R}_{Y^{\mathrm{an}}}$ on Y^{an}. It may be seen as a replacement for the de Rham complex $\Omega^{\bullet}_{\mathcal{Y}/S}$ for smooth f. In the absolute case $S = \operatorname{Spec}\mathbb{C}$ the complex $\widetilde{\Omega}^{\bullet}_{Y/\mathbb{C}}$ has been studied in great detail. As in [**Fri83**, Lem 1.5], we will construct a resolution for $\widetilde{\Omega}^{k}_{\mathcal{Y}/S}$ from a semi-simplicial resolution of $\mathcal{Y} \longrightarrow S$ all of whose terms are smooth over S. For this in turn will show that the canonical resolution of the central fiber extends.
Using the resolution of the complex $\widetilde{\Omega}^{\bullet}_{\mathcal{Y}/S}$ thus obtained, we contruct a mixed Hodge structure over R on $H^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}})$ where Y^{an} denotes the complex space associated to Y. The construction is very explicit and is a direct generalization of the treatment in the absolute case $R = \mathbb{C}$ described in [**GS75**, § 4].
We know that Hodge numbers of compact Kähler manifolds are constant in families. In [**Del68**] Deligne established among other things an algebraic analogue of this fact over arbitrary, in particular, over non-reduced base schemes. Deligne showed that for a smooth and proper morphism $f : \mathcal{Y} \longrightarrow S$ of algebraic schemes the $\mathcal{O}_S$-modules $R^q f_* \Omega^p_{\mathcal{Y}/S}$ are locally free and compatible with arbitrary base change. This serves as a basis to carry out the following Hodge theoretic arguments over Artinian schemes.
Let $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_{\mathbb{C}}$. We show in Theorem III.4.3 that for a smooth and proper morphism $g : \mathcal{X} \longrightarrow S$, a locally trivial deformation $f : \mathcal{Y} \longrightarrow S$ of a proper simple normal crossing variety and an S-morphism $i : \mathcal{Y} \longrightarrow \mathcal{X}$ the induced morphism

$$i^* : R^q g_* \Omega^p_{\mathcal{X}/S} \longrightarrow R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}$$

has a free cokernel, or equivalently, has constant rank, see Definition A.2.1 and Lemma A.2.2. The essence of the argument can be captured by looking at the case where $\mathcal{Y} \longrightarrow S$ is smooth. In fact, also the proof is by reduction to the smooth case. There the morphism i^* from above is identified as a graded component of a certain morphism $H_X \longrightarrow H_Y$ of pure Hodge structures over R. Such morphisms have constant rank by Lemma II.2.10. After Weil restriction, the graded pieces can be lifted to direct summands and therefore have constant rank as well. The comparison results between Hodge and Hodge-Weil structures from Chapter II allow to conclude that i^* itself has constant rank.

III.1. The complex $\widetilde{\Omega}^{\bullet}_{\mathcal{Y}/S}$

Definition III.1.1. Let $\mathcal{Y}$ be an algebraic scheme, let $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_k$ and let $f : \mathcal{Y} \longrightarrow S$ be a morphism of schemes. Assume that f is smooth on a dense open subset of $\mathcal{Y}$. We define the subsheaf $\tau^k_{\mathcal{Y}/S} \subseteq \Omega^k_{\mathcal{Y}/S}$ to be the subsheaf of sections supported on the singular locus of $Y = \mathcal{Y} \times_S k$. By abuse of language we speak of it as the *torsion subsheaf* of $\Omega^k_{\mathcal{Y}/S}$. We put $\widetilde{\Omega}^k_{\mathcal{Y}/S} := \Omega^k_{\mathcal{Y}/S}/\tau^k_{\mathcal{Y}/S}$. In the same way we define $\widetilde{\Omega}^k_{\mathcal{Y}/S}$ when $\mathcal{Y}$ is a complex space.

If $\alpha \in \Omega^k_{\mathcal{Y}/S}$ vanishes on Y^{reg}, then $d\alpha$ vanishes there as well. Therefore, $\tau^{\bullet}_{\mathcal{Y}/S} \subseteq \Omega^{\bullet}_{\mathcal{Y}/S}$ is a subcomplex and $\widetilde{\Omega}^{\bullet}_{\mathcal{Y}/S}$ is a complex. Next we will show that irreducible components of a variety extend to flat subschemes on locally trivial deformations. This will take some commutative algebra.

Lemma III.1.2. Let A be a reduced noetherian ring and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the pairwise distinct minimal prime ideals of A. Then $\operatorname{Ann} \mathfrak{p}_j = \cap_{i \neq j} \mathfrak{p}_i$ for each j.

Proof. Let $A_i = A/\mathfrak{p}_i$ and $\phi : A \longrightarrow A_1 \times \ldots \times A_n$ be the canonical map. It is injective, because $\cap_i \mathfrak{p}_i = \mathrm{nil}(A) = 0$. Suppose $a \in \cap_{i \neq j} \mathfrak{p}_i$, $b \in \mathfrak{p}_j$ and write $\phi(a) = (a_1, \ldots, a_n)$ and $\phi(b) = (b_1, \ldots, b_n)$. Then $\phi(ab) = (a_1 b_1, \ldots, a_n b_n) = 0$ because $a_i = 0$ for $i \neq j$ and $b_j = 0$. But ϕ is injective, hence $ab = 0$, in other words $a \in \operatorname{Ann} \mathfrak{p}_j$, so $\operatorname{Ann} \mathfrak{p}_j \supseteq \cap_{i \neq j} \mathfrak{p}_i$.
Let $a \in \operatorname{Ann} \mathfrak{p}_j$. Then for every $b \in \mathfrak{p}_j$ we have $0 = \phi(ab) = (a_1 b_1, \ldots, a_n b_n)$ in the above notation, where $b_j = 0$. As the $\mathfrak{p}_i$ are minimal and pairwise distinct, $\mathfrak{p}_j \backslash \mathfrak{p}_k \neq \emptyset$ for every $k \neq j$. If we fix k and choose $b \in \mathfrak{p}_j \backslash \mathfrak{p}_k$, then $b_k \neq 0$. So $a_k b_k = 0$ implies that $a_k = 0$ as A_k is an integral domain, so $a \in \mathfrak{p}_k$. Choosing different b we see that $a \in \cap_{i \neq j} \mathfrak{p}_i$ completing the proof. $\square$

Lemma III.1.3. Let A be a reduced noetherian ring, $\mathfrak{p} \subseteq A$ be a minimal prime ideal and $\psi : \mathfrak{p} \longrightarrow A/\mathfrak{p}$ be an A-module homomorphism. Then $\psi = 0$.

Proof. Let $\mathfrak{p}, \mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the pairwise distinct minimal prime ideals of A and $N := \operatorname{im} \psi \subseteq A/\mathfrak{p}$. We will show that $N = 0$. By Lemma III.1.2 we have $\operatorname{Ann} \mathfrak{p} = \cap_i \mathfrak{p}_i$. So $\mathfrak{p} \notin \operatorname{supp}(\mathfrak{p}) = V(\operatorname{Ann} \mathfrak{p})$, for otherwise $\cap_i \mathfrak{p}_i \subseteq \mathfrak{p}$

and thus $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some i as $\mathfrak{p}$ is prime, contradicting the fact that $\mathfrak{p} \neq \mathfrak{p}_i$ and $\mathfrak{p}$ is minimal. Thus, $\mathfrak{p} \otimes_A A_{\mathfrak{p}} = 0$ and the surjection

$$0 = \mathfrak{p} \otimes_A A_{\mathfrak{p}} \twoheadrightarrow N \otimes_A A_{\mathfrak{p}}$$

yields that $N_{\mathfrak{p}} = N \otimes_A A_{\mathfrak{p}} = 0$. Therefore, N is torsion. This implies $N = 0$, as it is an $A/\mathfrak{p}$-submodule of the torsion-free module $A/\mathfrak{p}$. $\square$

Lemma III.1.4. Let A be a reduced noetherian ring, $\mathfrak{p} \subseteq A$ a minimal prime ideal, $R \in \mathrm{Art}_k$ and $\mathfrak{P} \subseteq A \otimes_k R$ an ideal such that $A \otimes_k R/\mathfrak{P}$ is a flat deformation of $A/\mathfrak{p}$ over R. Then $\mathfrak{P} = \mathfrak{p} \otimes R$.

Proof. Let $\mathfrak{m} \subseteq R$ be the maximal ideal. As R is Artinian, there is $n \in \mathbb{N}$ such that $\mathfrak{m}^n = 0$. So we may argue inductively and assume that $\mathfrak{P}/\mathfrak{m}^k = \mathfrak{p} \otimes R/\mathfrak{m}^k \subseteq A \otimes R/\mathfrak{m}^k$. By flatness, we obtain the commutative diagram

(III.1.1)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathfrak{p} \otimes \mathfrak{m}^k/\mathfrak{m}^{k+1} \arrow[r] \arrow[d] & A \otimes \mathfrak{m}^k/\mathfrak{m}^{k+1} \arrow[r] \arrow[d] & A/\mathfrak{p} \otimes \mathfrak{m}^k/\mathfrak{m}^{k+1} \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & \mathfrak{P}/\mathfrak{m}^{k+1} \arrow[r] \arrow[d] & A \otimes R/\mathfrak{m}^{k+1} \arrow[r, "\varphi"] \arrow[d] & A \otimes R/(\mathfrak{P} + \mathfrak{m}^{k+1}) \arrow[r] \arrow[d, "\chi"] & 0 \\
0 \arrow[r] & \mathfrak{p} \otimes R/\mathfrak{m}^k \arrow[r] \arrow[d] & A \otimes R/\mathfrak{m}^k \arrow[r] \arrow[d] & A/\mathfrak{p} \otimes R/\mathfrak{m}^k \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. If we denote the inclusion

$$\mathfrak{p} \otimes R/\mathfrak{m}^{k+1} \hookrightarrow A \otimes R/\mathfrak{m}^{k+1}$$

by ψ, then $\varphi \circ \psi$ factors as

$$\begin{tikzcd}
 & A/\mathfrak{p} \otimes \mathfrak{m}^k/\mathfrak{m}^{k+1} \arrow[d, "\pi"] \\
\mathfrak{p} \otimes R/\mathfrak{m}^{k+1} \arrow[ur, dashed, "\exists"] \arrow[r, "\varphi \circ \psi"'] & A \otimes R/(\mathfrak{P} + \mathfrak{m}^{k+1})
\end{tikzcd}$$

Indeed, this can be seen as follows. Consider the commutative diagram

$$\begin{tikzcd}
 & \mathfrak{p} \otimes R/\mathfrak{m}^{k+1} \arrow[r, "\psi"] \arrow[d] & A \otimes R/\mathfrak{m}^{k+1} \arrow[r, "\varphi"] \arrow[d] & A \otimes R/(\mathfrak{P} + \mathfrak{m}^{k+1}) \arrow[r] \arrow[d, "\chi"] & 0 \\
0 \arrow[r] & \mathfrak{p} \otimes R/\mathfrak{m}^k \arrow[r] & A \otimes R/\mathfrak{m}^k \arrow[r] & A/\mathfrak{p} \otimes R/\mathfrak{m}^k \arrow[r] & 0
\end{tikzcd}$$

Then $\chi \circ \varphi \circ \psi = 0$ as the bottom row is exact. Therefore $\varphi \circ \psi$ factors through $\ker \chi$ as claimed.
Now observe that $\mathfrak{p} \otimes R/\mathfrak{m}^{k+1} \longrightarrow A/\mathfrak{p} \otimes \mathfrak{m}^k/\mathfrak{m}^{k+1}$ is zero by Lemma III.1.3, hence so is $\varphi \circ \psi$. Therefore ψ factors through $\ker \varphi = \mathfrak{P}/\mathfrak{m}^{k+1}$ as

$$\begin{tikzcd}
\mathfrak{p} \otimes R/\mathfrak{m}^{k+1} \arrow[r, hook, "\psi"] \arrow[d, dashed] & A \otimes R/\mathfrak{m}^{k+1} \\
\mathfrak{P}/\mathfrak{m}^{k+1} \arrow[ur]
\end{tikzcd}$$

But $\mathfrak{p} \otimes R/\mathfrak{m}^{k+1} \longrightarrow \mathfrak{P}/\mathfrak{m}^{k+1}$ becomes an isomorphism after tensoring with $R/\mathfrak{m}^k$, thus it is itself an isomorphism by flatness of $\mathfrak{P}/\mathfrak{m}^{k+1}$, see [**Ser06**, Lem A.4]. $\square$

Lemma III.1.5. Let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of a reduced noetherian scheme Y over an Artinian base $S = \operatorname{Spec} R$, $R \in \mathrm{Art}_k$. Then the irreducible components Y_α of Y lift uniquely to subschemes $\mathcal{Y}_\alpha \hookrightarrow \mathcal{Y}$ flat over S. Moreover, each $\mathcal{Y}_\alpha$ is a locally trivial deformation of Y_α.

Proof. Let $Y = \cup_i U_i$ be an open affine covering of Y such that there are R-algebra isomorphisms $\theta_i : A_i \otimes_k R \longrightarrow \Gamma(U_i, \mathcal{O}_{\mathcal{Y}})$ where $A_i := \Gamma(U_i, \mathcal{O}_Y)$. An irreducible component Y_α of Y gives a minimal prime ideal $\mathfrak{p}^i_\alpha$ in each A_i. We define $\mathcal{Y}^i_\alpha$ to be the closed subscheme of $\mathcal{Y}|_U$ whose ideal is $\theta_i(\mathfrak{p}_\alpha)$. Then $\mathcal{Y}^i_\alpha$ is a flat lifting of $Y_\alpha|_{U_i}$ for all i. Therefore, on $U_{ij} := U_i \cap U_j$ also $\mathcal{Y}^j_\alpha|_{U_{ij}}$ is a flat lifting of $Y_\alpha|_{U_{ij}}$ for all j. Then by Lemma III.1.4 we conclude that $\mathcal{Y}^i_\alpha|_{U_{ij}} = \mathcal{Y}^j_\alpha|_{U_{ij}}$ and so the $\mathcal{Y}^i_\alpha$ are the restrictions of a closed subscheme $\mathcal{Y}_\alpha$ of $\mathcal{Y}$. The argument also shows that $\mathcal{Y}_\alpha$ is unique. $\square$

III.2. Semi-simplicial resolutions

We define the notion of a semi-simplicial resolution for locally trivial deformations of simple normal crossing varieties. For this we recall some standard notions. Fix an arbitrary category $\mathscr{C}$.

Definition III.2.1. The *semi-simplicial category* Δ is the category whose objects are the ordered sets $[n] = \{0, \ldots, n\} \subseteq (\mathbb{N}_0, \leq)$ and whose morphisms are strictly increasing maps. A *semi-simplicial object* in $\mathscr{C}$ is a functor $X : \Delta^{\mathrm{op}} \longrightarrow \mathscr{C}$. A *semi-cosimplicial object* in $\mathscr{C}$ is a functor $X : \Delta \longrightarrow \mathscr{C}$.

A semi-simplicial object in $\mathscr{C}$ is given by objects $Y^n := Y([n])$ in $\operatorname{Ob}\mathscr{C}$ and morphisms $Y(\iota)$ for every $\iota \in \operatorname{Mor}\Delta$. In Δ there are exactly $n+1$ morphisms $[n-1] \to [n]$, all of them injective. So dually, there are morphisms $d^j : Y^n \longrightarrow Y^{n-1}$ for $j = 0, \ldots, n$. As every morphism $[n] \to [m]$ in Δ

can be factored as $[n] \to [n+1] \to \ldots \to [m]$, a semi-simplicial object is determined by the objects Y^n and the morphisms $d^j : Y^n \longrightarrow Y^{n-1}$ between them. We will write $Y^\bullet$ for a semi-simplicial object. An object Y in $\mathscr{C}$ may be considered as a trivial semi-simplicial object, that is, as the simplicial object having $Y^n = Y$ for all n and all $d^j = \mathrm{id}_Y$. Dual statements hold, if Y is a semi-cosimplicial object in $\mathscr{C}$.

A *morphism of semi-simplicial objects* is a natural transformation of functors. Such a morphism $a : Y \longrightarrow X$ is determined by a collection of morphisms $a_k : Y^k \longrightarrow X^k$ in $\mathscr{C}$ compatible with the d^j. For a morphism $a : Y^\bullet \longrightarrow Y$ from a semi-simplicial object to a trivial semi-simplicial object the morphism a_m coincides with the composition

$$Y^m \xrightarrow{d^{i_1}} Y^{m-1} \xrightarrow{d^{i_2}} \ldots \xrightarrow{d^{i_r}} Y^{m-r} \xrightarrow{a_{m-r}} Y, \quad 0 \le i_k \le m-k+1$$

for every choice of the i_k. We also say that $a : Y^\bullet \longrightarrow Y$ is an *augmentation* of $Y^\bullet$ to Y or that $Y^\bullet$ is *augmented* towards Y. We will also write an augmented semi-simplicial object $Y^\bullet \longrightarrow Y$ in the form

$$\ldots \substack{\rightarrow\\\rightarrow\\\rightarrow} Y^{[1]} \rightrightarrows Y^{[0]} \rightarrow Y \ .$$

Finally, if $\mathscr{C}$ is an additive category, we put

$$\delta := \delta^n := \sum_{j=0}^{n} (-1)^j d^j : Y^n \longrightarrow Y^{n-1} \qquad \text{(III.2.1)}$$

for a semi-simplicial object $Y_\bullet$ and

$$\delta := \delta_{n-1} := \sum_{j=0}^{n} (-1)^j d_j : Y_{n-1} \longrightarrow Y_n \qquad \text{(III.2.2)}$$

for a semi-cosimplicial object $Y_\bullet$. A calculation shows that these make $Y^\bullet$ and $Y_\bullet$ into complexes.

Definition III.2.2. Let S be a $\mathbb{C}$-scheme and $\mathcal{Y} \longrightarrow S$ be a proper scheme over S. A *semi-simplicial resolution* of $\mathcal{Y}$ over S is a semi-simplicial S-scheme $\mathcal{Y}^\bullet$ together with a morphism $a : \mathcal{Y}^\bullet \longrightarrow \mathcal{Y}$ of semi-simplicial S-schemes such that all $a_k : \mathcal{Y}^k \longrightarrow \mathcal{Y}$ are proper and $\mathcal{Y}^k \longrightarrow S$ is smooth for all k.

Note that for $S = \operatorname{Spec} \mathbb{C}$ this definition does *not* coincide with Deligne's definition [**Del71, Del74**]. Deligne defines semi-simplicial resolutions for varieties over $\mathbb{C}$. In his definition such a resolution is *of cohomological descent*. He uses it to construct a functorial mixed Hodge structure on the cohomology an algebraic $\mathbb{C}$-variety. In this treatise, we do not need to include any properties in the definition of such a resolution as all our resolutions are explicitly given. Moreover, we prove all our Hodge theoretical statements "by hand". However, we took Deligne's definition as a guideline, to see in practice which resolutions are the right ones to look at. Certainly, it requires further investigation to figure out a good class of S-schemes, where one can built a relative Hodge theory in the spirit of [**Del71, Del74**], see the discussion in section VIII.3.

III.2.3. Canonical resolution for locally trivial deformations of simple normal crossing varieties. Let Y be a proper simple normal crossing k-variety and let $Y = \cup_i Y_i$ be a decomposition into irreducible components. By definition this means that the Y_i are smooth over k. Let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y over $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_k$. Lemma III.1.5 allows us to write

$$\mathcal{Y} = \bigcup_{i=1}^{n} \mathcal{Y}_i$$

with flat S-schemes $\mathcal{Y}_i$. This union is a decomposition into irreducible components and $\mathcal{Y}_i$ is a locally trivial deformation of Y_i. By flatness

$$\mathcal{Y}^{[0]} := \coprod_i \mathcal{Y}_i \longrightarrow S$$

is smooth as well. For a subset $I \subseteq [n]$ we put

$$\mathcal{Y}^I := \bigcap_{i\in I} \mathcal{Y}_i, \quad \mathcal{Y}^{[k]} := \coprod_{|I|=k+1} \mathcal{Y}^I. \tag{III.2.3}$$

Here by $\mathcal{Y}_i \cap \mathcal{Y}_j$ we denote the scheme $\mathcal{Y}_i \times_{\mathcal{Y}} \mathcal{Y}_j$. There exists one map $a_k : \mathcal{Y}^{[k]} \longrightarrow \mathcal{Y}$ over S and $k+1$ canonical maps $d_j : \mathcal{Y}^{[k]} \longrightarrow \mathcal{Y}^{[k-1]}$ for $j = 0, \ldots, k$ over S coming from the $k+1$ inclusions $[k] \hookrightarrow [k+1]$. In other words, the collection of the $\mathcal{Y}^{[k]}$ together with the d_j is a semi-simplicial S-scheme and the a_k form an augmentation of $\mathcal{Y}^{[\bullet]}$ to $\mathcal{Y}$.

Lemma III.2.4. The semi-simplicial S-scheme $\mathcal{Y}^{[\bullet]}$ together with the augmentation $a : \mathcal{Y}^{[\bullet]} \longrightarrow \mathcal{Y}$ is a semi-simplicial resolution of $\mathcal{Y}$. We call it the *canonical resolution* of $\mathcal{Y}$ over S.

Proof. We have to show that all $\mathcal{Y}^{[m]} \longrightarrow S$ are smooth morphisms. Lemma III.1.5 tells us (or rather the choice of $\mathcal{Y}_i$, which was made using Lemma III.1.5) that $\mathcal{Y}_i$ is a flat deformation of the smooth variety Y_i and therefore smooth as well. For $m \geq 1$ we use that $\mathcal{Y}^{[m]}$ is a disjoint union of schemes of the form $\mathcal{Y}^I = \mathcal{Y}_{i_0} \times_{\mathcal{Y}} \ldots \times_{\mathcal{Y}} \mathcal{Y}_{i_m}$, where $I = \{i_0, \ldots, i_m\}$ and $|I| = m+1$. Moreover, smoothness is a local property, so let us assume that all schemes are affine, say

$$\mathcal{Y} = \operatorname{Spec} \mathcal{A}, \quad \mathcal{Y}_i = \operatorname{Spec} \mathcal{A}_i, \quad Y = \operatorname{Spec} A, \quad Y_i = \operatorname{Spec} A_i,$$

where

$$A = \mathcal{A} \otimes_R k, \quad A_i = \mathcal{A}_i \otimes_R k$$

for $S = \operatorname{Spec} R$. But all morphisms $\mathcal{Y}_i \longrightarrow \mathcal{Y}$ are S-morphisms and $\mathcal{Y}_i$ respectively $\mathcal{Y}$ are locally trivial deformations of Y_i respectively Y. Thus, we may assume that $\mathcal{A}_i \cong A_i \otimes_k R$ and $\mathcal{A} \cong A \otimes_k R$. Note that by Lemma III.1.4 the trivialization $\mathcal{A} \cong A \otimes_k R$ already induces an isomorphism $\mathcal{A}_i \cong A_i \otimes_k R$ so that we obtain an R-algebra isomorphism

$$\Gamma(\mathcal{Y}^I, \mathcal{O}_{\mathcal{Y}^I}) = \mathcal{A}_{i_0} \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} \mathcal{A}_{i_m} \cong (A_{i_0} \otimes_A \ldots \otimes_A A_{i_m}) \otimes_k R.$$

The ring $A_{i_0} \otimes_A \ldots \otimes_A A_{i_m}$ is the coordinate ring of the smooth k-variety $Y^I := \mathcal{Y}^I \times_S k = Y_{i_0} \times_Y \ldots \times_Y Y_{i_m}$. Smoothness of Y^I is immediate from the normal crossing condition. This shows that also $\mathcal{Y}^I$ is smooth over $S = \operatorname{Spec} R$ completing the proof. $\square$

III.2.5. Semi-cosimplicial resolution for $\widetilde{\Omega}^p_{\mathcal{Y}/S}$. For $\mathcal{Y}$ as in section III.2.3 the semi-simplicial S-scheme $\mathcal{Y}^{[\bullet]}$ induces semi-cosimplicial $\mathcal{O}_{\mathcal{Y}}$-modules $a_*\Omega^p_{\mathcal{Y}^{[\bullet]}/S}$. The formula (III.2.2) makes

$$a_*\Omega^p_{\mathcal{Y}^{[\bullet]}/S} : \qquad a_{0*}\Omega^p_{\mathcal{Y}^{[0]}/S} \xrightarrow{\delta_0} a_{1*}\Omega^p_{\mathcal{Y}^{[1]}/S} \xrightarrow{\delta_1} \ldots$$

a complex. The augmentation $a : \mathcal{Y}^{[\bullet]} \longrightarrow \mathcal{Y}$ induces a coagumentation

$$\Omega_{\mathcal{Y}/S} \xrightarrow{a_0^*} a_{0*}\Omega^p_{\mathcal{Y}^{[0]}/S} \xrightarrow{\delta_0} a_{1*}\Omega^p_{\mathcal{Y}^{[1]}/S} \xrightarrow{\delta_1} \ldots.$$

As $\mathcal{Y}^{[0]} \longrightarrow S$ is smooth, the morphism a_0^* factors through $\widetilde{\Omega}^p_{\mathcal{Y}/S}$ from Definition III.1.1. Clearly, the composition $\delta_0 \circ a_0^*$ is zero and we obtain a complex

$$0 \longrightarrow \tau^k_{\mathcal{Y}/S} \longrightarrow \Omega^k_{\mathcal{Y}/S} \longrightarrow a_{0*}\Omega^k_{\mathcal{Y}^{[0]}/S} \longrightarrow a_{1*}\Omega^k_{\mathcal{Y}^{[1]}/S} \longrightarrow \ldots \tag{III.2.4}$$

All following theory is based on the important

Lemma III.2.6. Let Y be a proper simple normal crossing $\mathbb{C}$-variety and $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation over an Artinian base $S = \operatorname{Spec} R$ with $R \in \mathrm{Art}_{\mathbb{C}}$.

(1) The sequence (III.2.4) is exact and so is the sequence with $\mathcal{Y}$ replaced by $\mathcal{Y}^{\mathrm{an}}$.
(2) $\widetilde{\Omega}^\bullet_{\mathcal{Y}^{\mathrm{an}}/S}$ is a resolution of the constant sheaf $\underline{R}_{Y^{\mathrm{an}}}$.

Proof. The question is local in $\mathcal{Y}$, so we may assume that $\mathcal{Y} = Y \times S$ is the trivial deformation. Then the resolution (III.2.4) is simply the pullback of the analogous resolution for Y along the flat morphism $Y \times S \longrightarrow Y$. This implies the claim, as the statement of the lemma is true for Y by [**Fri83**, Prop 1.5]. $\square$

LEMMA III.2.7. Let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of a simple normal crossing variety Y over an Artinian base $S = \operatorname{Spec} R$ with $R \in \operatorname{Art}_{\mathbb{C}}$.

(1) The canonical map $\left(\widetilde{\Omega}^k_{\mathcal{Y}/S}\right)^{\mathrm{an}} \longrightarrow \widetilde{\Omega}^k_{\mathcal{Y}^{\mathrm{an}}/S}$ is an isomorphism.
(2) The canonical map $R^i f_* \widetilde{\Omega}^k_{\mathcal{Y}/S} \longrightarrow R^i f^{\mathrm{an}}_* \widetilde{\Omega}^k_{\mathcal{Y}^{\mathrm{an}}/S}$ is an isomorphism.

PROOF. We clearly have $\left(\Omega_{\mathcal{Y}/S}\right)^{\mathrm{an}} \cong \Omega_{\mathcal{Y}^{\mathrm{an}}/S}$. Now (1) follows from (1) of Lemma III.2.6 because analytification is an exact functor by [**SGA1**, Exp XII, Prop 1.3.1] and compatible with taking the wedge product. Moreover (1) implies (2) by [**SGA1**, Exp XII, Thm 4.2]. □

REMARK III.2.8. In [**Ser56**] several comparison theorems are proven for projective varieties over $\mathbb{C}$. A generalization of Serre's work to proper schemes of finite type over $\mathbb{C}$ is given in Raynaud's exposé [**SGA1**, Exp XII]. The references in the proof are to generalizations of Serre's results [**Ser56**, Prop 10] and [**Ser56**, Thm 1].

LEMMA III.2.9. Let Y be an algebraic scheme and $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y over $S = \operatorname{Spec} R$ where $R \in \operatorname{Art}_k$. Let $S' \longrightarrow S$ be a morphism where $S' = \operatorname{Spec} R'$ with $R' \in \operatorname{Art}_k$. Consider the fiber product $\mathcal{Y}' = \mathcal{Y} \times_S S'$ and denote by $\sigma : \mathcal{Y}' \longrightarrow \mathcal{Y}$ the induced morphism, then the canonical morphism $\sigma^* \widetilde{\Omega}_{\mathcal{Y}/S} \longrightarrow \widetilde{\Omega}_{\mathcal{Y}'/S'}$ is an isomorphism.

PROOF. As the canonical morphism $\sigma^* \Omega_{\mathcal{Y}/S} \longrightarrow \Omega_{\mathcal{Y}'/S'}$ is always an isomorphism, we only have to check that the torsion is preserved. The question is local in $\mathcal{Y}$, so the claim follows immediately from local triviality. □

The following result is due to Deligne, see [**Del68**, Thm 5.5], for smooth morphisms $f : \mathcal{Y} \longrightarrow S$. His proof also works in our situation. As his arguments are part of the proof of a more general statement, we reproduce them here.

THEOREM III.2.10 (**Deligne**). *Let Y be a simple normal crossing variety, which is proper over $\mathbb{C}$, let $S = \operatorname{Spec} R$ for $R \in \operatorname{Art}_{\mathbb{C}}$ and let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y over S. Let $S' = \operatorname{Spec} R'$ for $R' \in \operatorname{Art}_{\mathbb{C}}$ and let $S' \longrightarrow S$ be a morphism. Then the following holds.*

(1) *The associated spectral sequence*

$$E_1^{p,q} = R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S} \Rightarrow R^{p+q} f_* \widetilde{\Omega}^\bullet_{\mathcal{Y}/S} = H^{p+q}(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}}) \tag{III.2.5}$$

degenerates at E_1.

(2) *The R-modules $R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}$ are free and compatible with arbitrary base change in the sense that for $\mathcal{Y}' = \mathcal{Y} \times_S S'$ the morphism*

$$R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S} \otimes_R R' \longrightarrow R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}'/S'}$$

is an isomorphism.

Proof. We argue as in [**Del68**], Théorème 5.5. By [**Del68**, (3.5.1)] a complex K of R-modules satisfies

$$\lg_R(H^n(K)) \leq \lg(R) \dim_{\mathbb{C}}(H^n(K \otimes^{\mathbb{L}}_R \mathbb{C}))$$

and $H^n(K)$ is a free R-module, if equality holds. Here lg denotes the length of a module. To apply this to the E_1-term of the spectral sequence (III.2.5) we need [**EGAIII2**], Théorème (6.10.5) saying that there is a bounded below complex L of free R-modules and an isomorphism of ∂-functors $R^q f_* \left(\widetilde{\Omega}^p_{\mathcal{Y}/S} \otimes f^* Q \right) \longrightarrow H^q(L \otimes Q)$ in the bounded complex Q of quasi-coherent R-modules. Here we use that $\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ is flat over R. Let $\bar{f} : Y \longrightarrow \operatorname{Spec} \mathbb{C}$ be the restriction of f to the central fiber. We will compare the spectral sequence (III.2.5) with the spectral sequence of $\bar{f}$. Again by [**Del68**, (3.5.1)] we have

$$\begin{aligned} \lg_R(R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}) &= \lg_R(H^q(L)) \\ &\leq \lg(R) \dim_{\mathbb{C}}(H^q(L \otimes_R \mathbb{C})) \\ &= \lg(R) \dim_{\mathbb{C}}(R^q \bar{f}_* \widetilde{\Omega}^p_{Y/\mathbb{C}}) \end{aligned} \tag{III.2.6}$$

and $R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}$ is a free R-module, if equality holds. The last equality is the base change property for $\widetilde{\Omega}_{\mathcal{Y}/S}$ from Lemma III.2.9. We have

$$\begin{aligned} \lg(R^n f_* \widetilde{\Omega}^\bullet_{\mathcal{Y}/S}) &\leq \sum_{p+q=n} \lg_R(R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}) \\ &\leq \lg(R) \sum_{p+q=n} \dim_{\mathbb{C}}(R^q \bar{f}_* \widetilde{\Omega}^p_{Y/\mathbb{C}}) \\ &= \lg(R) \dim_{\mathbb{C}}(R^n \bar{f}_* \widetilde{\Omega}^\bullet_{Y/\mathbb{C}}), \end{aligned}$$

where the first inequality comes from the existence of the spectral sequence, the second inequality is (III.2.6) and the last equality comes from the degeneration of the spectral sequence for Y, which is [**Fri83**, Prop 1.5]. But Lemma III.2.6 (2) implies that $\lg(R^n f_* \widetilde{\Omega}^\bullet_{\mathcal{Y}/S}) = \lg(R) \dim_{\mathbb{C}}(R^n \bar{f}_* \widetilde{\Omega}^\bullet_{Y/\mathbb{C}})$, so we have equality everywhere. Hence (1) and the first assertion of (2) follows. The second assertion of (2) follows from the first by [**EGAIII2**, (7.8.5)]. $\square$

Remark III.2.11. The proof only uses the following abstract properties of the complex $\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$.

- $\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ is flat over S.
- For $\sigma : S' \longrightarrow S$ and $\mathcal{Y}' = \mathcal{Y} \times_S S' \longrightarrow S'$ we have $\sigma^*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} \cong \widetilde{\Omega}^\bullet_{\mathcal{Y}'/S'}$.
- There is a functorial quasi-isomorphism $\underline{R}_{Y^{\mathrm{an}}} \longrightarrow \widetilde{\Omega}^\bullet_{\mathcal{Y}^{\mathrm{an}}/S}$.
- For $f : Y \longrightarrow \mathrm{Spec}(\mathbb{C})$ the spectral sequence

$$E_1^{p,q} = R^q f_* \widetilde{\Omega}^p_Y \quad \Rightarrow \quad R^{p+q} f_* \widetilde{\Omega}^\bullet_Y$$

  degenerates at E_1.

In particular, the proof works for smooth $f : \mathcal{Y} \longrightarrow S$ where $\Omega_{\mathcal{Y}/S} = \widetilde{\Omega}_{\mathcal{Y}/S}$.

III.3. Pure Hodge structures on smooth families

Let $f : \mathcal{Y} \longrightarrow S$ be a smooth and proper morphism where $S = \mathrm{Spec}\, R$ for $R \in \mathrm{Art}_{\mathbb{C}}$. We are going to put a pure Hodge structure over R on $H^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}})$. We define a decreasing filtration

$$F^p\Omega^\bullet_{\mathcal{Y}/S} := \Omega^{\geq p}_{\mathcal{Y}/S}, \tag{III.3.1}$$

which is called the *Hodge filtration.* It gives rise to a filtration $F^pH^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}})$ on $H^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}})$ by putting

$$F^pR^kf_*\Omega^\bullet_{\mathcal{Y}/S} = \mathrm{im}\left(R^kf_*F^p\Omega^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf_*\Omega^\bullet_{\mathcal{Y}/S}\right) \tag{III.3.2}$$

and using the isomorphisms $H^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}}) \longrightarrow R^kf^{\mathrm{an}}_*\Omega^\bullet_{\mathcal{Y}^{\mathrm{an}}/S}$ from [**Del68**, Lem 5.5.3] and $R^kf_*\Omega^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf^{\mathrm{an}}_*{\Omega^\bullet_{\mathcal{Y}/S}}^{\mathrm{an}} \cong R^kf^{\mathrm{an}}_*\Omega^\bullet_{\mathcal{Y}^{\mathrm{an}}/S}$ from [**SGA1**, Exp XII, Thm 4.2].

Lemma III.3.1. Let $f : \mathcal{Y} \longrightarrow S$ be a smooth and proper morphism and $S = \mathrm{Spec}\, R$, $R \in \mathrm{Art}_{\mathbb{C}}$. Then

$$\mathcal{H}^k(\mathcal{Y}) = \left(H^k(Y^{\mathrm{an}}, \mathbb{R}), F^pH^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}})\right)$$

is a pure Hodge structure of weight k over R, whose central fiber is the usual Hodge structure on $H^k(Y^{\mathrm{an}}, \mathbb{R})$. Moreover, the canonical morphism $R^kf_*F^p\Omega^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf_*\Omega^\bullet_{\mathcal{Y}/S}$ is injective, so that

$$R^kf_*F^p\Omega^\bullet_{\mathcal{Y}/S} \cong F^pH^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}}).$$

If $g : \mathcal{X} \longrightarrow S$ is smooth and proper, every S-morphism $i : \mathcal{Y} \longrightarrow \mathcal{X}$ induces a morphism $i^* : \mathcal{H}^k(\mathcal{X}) \longrightarrow \mathcal{H}^k(\mathcal{Y})$ of pure Hodge structures.

Proof. The filtration defined in (III.3.2) is the one, whose graded objects are found on E_∞ of the spectral sequence (III.2.5). By [**Del68**, Thm

5.5] we have $E_\infty = E_1$, so $\mathrm{Gr}_F^p R^k f_*\Omega^\bullet_{\mathcal{Y}/S} = R^{k-p} f_*\Omega^p_{\mathcal{Y}/S} = R^{k-p} f_*\mathrm{Gr}_F^p\Omega^\bullet_{\mathcal{Y}/S}$. The same theorem tells us that $R^{k-p}f_*\Omega^p_{\mathcal{Y}/S}$ is free. Therefore using

$$\begin{tikzcd}
0 \arrow[r] & R^k f_* F^{p+1}\Omega^\bullet_{\mathcal{Y}/S} \arrow[r] \arrow[d] & R^k f_* F^p\Omega^\bullet_{\mathcal{Y}/S} \arrow[r] \arrow[d] & R^k f_*\mathrm{Gr}_F^p\Omega^\bullet_{\mathcal{Y}/S} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^{p+1}R^k f_*\Omega^\bullet_{\mathcal{Y}/S} \arrow[r] & F^p R^k f_*\Omega^\bullet_{\mathcal{Y}/S} \arrow[r] & \mathrm{Gr}_F^p R^k f_*\Omega^\bullet_{\mathcal{Y}/S} \arrow[r] & 0
\end{tikzcd}$$

we find inductively that $R^k f_* F^p\Omega^\bullet_{\mathcal{Y}/S} \cong F^p R^k f_*\Omega^\bullet_{\mathcal{Y}/S}$ and that these are free submodules. Again by [**Del68**, Thm 5.5], all graded objects are compatible with base change and therefore restrict to a pure Hodge structure on the central fiber. The statement about morphisms is clear. □

Corollary III.3.2. There is a natural isomorphism

$$R^{k-p}f_*\Omega^p_{\mathcal{Y}/S} \longrightarrow \mathrm{Gr}_F^p H^k(Y),$$

where $H^k(Y) = H^k(Y^{\mathrm{an}}, \mathbb{R}) \otimes_{\mathbb{R}} R$.

Proof. Consider the sequences

(III.3.3)
$$\begin{tikzcd}
0 \arrow[r] & R^k f_*\Omega^{\geq p+1}_{\mathcal{Y}/S} \arrow[r] \arrow[d, "\cong"] & R^k f_*\Omega^{\geq p}_{\mathcal{Y}/S} \arrow[r] \arrow[d, "\cong"] & R^{k-p} f_*\Omega^p_{\mathcal{Y}/S} \arrow[r] \arrow[d, dashed, "\exists ?"] & 0 \\
0 \arrow[r] & F^{p+1}H^k(Y) \arrow[r] & F^p H^k(Y) \arrow[r] & \mathrm{Gr}_F^p H^k(Y) \arrow[r] & 0
\end{tikzcd}$$

where the first two vertical maps are isomorphisms by Lemma III.3.1. These isomorphisms imply that the upper sequence is exact on the left. As it is part of the long exact sequence associated to the sequence

$$0 \longrightarrow \Omega^{\geq p+1}_{\mathcal{Y}/S} \longrightarrow \Omega^{\geq p}_{\mathcal{Y}/S} \longrightarrow \Omega^p_{\mathcal{Y}/S}[-p] \longrightarrow 0$$

of complexes, injectivity at $(k+1)$-st direct image yields surjectivity at the k-th, hence exactness of the upper sequence. Therefore, the morphism $R^{k-p}f_*\Omega^p_{\mathcal{Y}/S} \longrightarrow \mathrm{Gr}_F^p H^k(Y)$ exists and by the five-lemma it is an isomorphism. □

According to Lemma II.2.8 the Weil restriction

$$\mathcal{H}^k(\mathcal{Y})_{\mathbb{C}/\mathbb{R}} = \Big(H^k(Y^{\mathrm{an}}, \mathbb{R}), F^p_{\mathbb{C}/\mathbb{R}}\Big)$$

of $\mathcal{H}^k(\mathcal{Y})$ is a pure Hodge-Weil structure. So by Lemma II.2.9 the submodules $H^{p,q}(Y) := F^p_{\mathbb{C}/\mathbb{R}} \cap \overline{F^q_{\mathbb{C}/\mathbb{R}}}$ of $H^k(Y)_{\mathbb{C}/\mathbb{R}} = \big(H^k(Y^{\mathrm{an}}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}\big) \otimes_{\mathbb{R}} R_{\mathbb{C}/\mathbb{R}}$ lift the subquotients $\mathrm{Gr}^p_{F_{\mathbb{C}/\mathbb{R}}} H^k(Y)_{\mathbb{C}/\mathbb{R}} = \big(\mathrm{Gr}_F^p H^k(Y)\big)_{\mathbb{C}/\mathbb{R}}$.

Corollary III.3.3. There is a natural isomorphism

$$\left(R^{k-p}f_*\Omega^p_{\mathcal{Y}/S}\right)_{\mathbb{C}/\mathbb{R}} \longrightarrow H^{p,q}(Y)$$

compatible with morphisms in the sense that for every smooth and proper $g : \mathcal{X} \longrightarrow S$ and every $i : \mathcal{Y} \longrightarrow \mathcal{X}$ over S the diagram

$$\begin{tikzcd}
\left(R^{k-p}f_*\Omega^p_{\mathcal{Y}/S}\right)_{\mathbb{C}/\mathbb{R}} \arrow[r, "i^*_{\mathbb{C}/\mathbb{R}}"] \arrow[d] & \left(R^{k-p}g_*\Omega^p_{\mathcal{X}/S}\right)_{\mathbb{C}/\mathbb{R}} \arrow[d] \\
H^{p,q}(Y) \arrow[r, "i^{p,q}"] & H^{p,q}(X)
\end{tikzcd}$$

commutes.

Proof. This follows directly by applying Weil restriction to the diagram (III.3.3) and using Lemma II.2.9. □

Recall that a module homomorphism has constant rank if and only if its cokernel is free by Lemma A.2.2.

Proposition III.3.4. Let $f : \mathcal{Y} \longrightarrow S$, $g : \mathcal{X} \longrightarrow S$ be proper and smooth over an Artinian base $S = \operatorname{Spec} R$, $R \in \mathrm{Art}_{\mathbb{C}}$ and let $i : \mathcal{Y} \longrightarrow \mathcal{X}$ be an S-morphism. Then the induced morphisms $i^* : R^q g_*\Omega^p_{\mathcal{X}/S} \longrightarrow R^q f_*\Omega^p_{\mathcal{Y}/S}$ have constant rank.

Proof. By Lemma III.3.1 we know that the morphism i induces a morphism $\mathcal{H}^k(\mathcal{X}) \longrightarrow \mathcal{H}^k(\mathcal{Y})$ between the pure Hodge structures over R associated to $\mathcal{X}$ and $\mathcal{Y}$. Taking Weil restrictions this gives a morphism $\mathcal{H}^k(\mathcal{X})_{\mathbb{C}/\mathbb{R}} \longrightarrow \mathcal{H}^k(\mathcal{Y})_{\mathbb{C}/\mathbb{R}}$ of Hodge-Weil structures by Lemma II.2.8. Let $i^{p,q} : H^{p,q}(X) \longrightarrow H^{p,q}(Y)$ be the induced map.
By Corollary III.3.3 the diagram

$$\begin{tikzcd}
\left(R^{k-p}f_*\Omega^p_{\mathcal{Y}/S}\right)_{\mathbb{C}/\mathbb{R}} \arrow[r, "i^*_{\mathbb{C}/\mathbb{R}}"] \arrow[d, "\cong"] & \left(R^{k-p}g_*\Omega^p_{\mathcal{X}/S}\right)_{\mathbb{C}/\mathbb{R}} \arrow[r] \arrow[d, "\cong"] & \operatorname{coker} i^*_{\mathbb{C}/\mathbb{R}} \arrow[r] \arrow[d] & 0 \\
H^{p,q}(Y) \arrow[r, "i^{p,q}"] & H^{p,q}(X) \arrow[r] & \operatorname{coker} i^{p,q} \arrow[r] & 0
\end{tikzcd}$$

commutes and the first two vertical maps are isomorphisms. Therefore also the third vertical map is an isomorphism. We know that $\operatorname{coker} i^{p,q}$ is free by Lemma II.2.10, hence so is $\operatorname{coker} i^*_{\mathbb{C}/\mathbb{R}}$. Now the claim follows from Lemma II.1.5, as $\operatorname{coker} i^*_{\mathbb{C}/\mathbb{R}} = (\operatorname{coker} i^*)_{\mathbb{C}/\mathbb{R}}$ by Lemma II.1.4. □

Proposition III.3.4 together with Lemma II.2.10 can be seen as a formalization of the idea that if S is the base manifold of a small deformation and

$t \in S$, the maps $H^q(X_t, \Omega^p_{X_t}) \longrightarrow H^q(Y_t, \Omega^p_{Y_t})$, the rank of which is semicontinuous in t, add up to the topological map $H^i(X, \underline{R}_X) \longrightarrow H^i(Y, \underline{R}_Y)$ by the Hodge decomposition. The rank of the latter is certainly constant and by semi-continuity the summands also have constant rank.

III.4. Mixed Hodge structures on normal crossing families

By Lemma III.2.6 there is a quasi-isomorphism $\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} \simeq \mathfrak{s}((a_\bullet)_*\Omega^\bullet_{\mathcal{Y}^{[\bullet]}/S})$, where $\mathfrak{s}(\cdot)$ denotes the single complex associated to a double complex. Therefore we may define filtrations

$$W_{-m}\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} := \mathfrak{s}((a_{\geq m})_*\Omega^\bullet_{\mathcal{Y}^{[\geq m]}/S}) \quad \text{and} \quad F^p\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} := \widetilde{\Omega}^{\geq p}_{\mathcal{Y}/S}.$$

These give rise to filtrations $F^pH^k(Y,R)$ and $W_mH^k(Y,R)$ on $H^k(Y,R)$ by putting

$$W_mR^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} = \operatorname{im}\left(R^kf_*W_{m-k}\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}\right) \tag{III.4.1}$$

and

$$F^pR^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} = \operatorname{im}\left(R^kf_*F^p\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}\right). \tag{III.4.2}$$

and using the isomorphisms $H^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}}) \longrightarrow R^kf^{\mathrm{an}}_*\widetilde{\Omega}^\bullet_{\mathcal{Y}^{\mathrm{an}}/S}$ from (2) of Lemma III.2.6 and $R^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf^{\mathrm{an}}_*\widetilde{\Omega}^\bullet_{\mathcal{Y}^{\mathrm{an}}/S}$ from (2) of Lemma III.2.7.

Lemma III.4.1. *Let Y be a proper simple normal crossing variety over $\mathbb{C}$ and let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y over $S = \operatorname{Spec} R$ for $R \in \mathrm{Art}_\mathbb{C}$. Then*

$$\mathcal{H}^k(\mathcal{Y}) = (H^k(Y^{\mathrm{an}}, \mathbb{R}), W_mH^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}}), F^pH^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}})) \tag{III.4.3}$$

is a mixed Hodge structure over R. Moreover, $R^kf_*F^p\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ is injective.

Proof. Literally as in the pure case, see Lemma III.3.1 and Corollary III.3.2, one shows that the R-modules $\mathrm{Gr}^p_FR^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ are free and isomorphic to $R^kf_*\mathrm{Gr}^p_F\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} = R^kf_*\widetilde{\Omega}^p_{\mathcal{Y}/S}$ and that $R^kf_*F^p\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ is injective. As only difference one has to use Theorem III.2.10 instead of [**Del68**, Thm 5.5]. To verify that (III.4.3) is a mixed Hodge structure over R, we have to show that the graded objects $\mathrm{Gr}^W_m\mathrm{Gr}^p_FH^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}})$ are free R-modules, or equivalently that the $\mathrm{Gr}^W_m\mathrm{Gr}^p_FR^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ are free R-modules, and that the central fiber is a mixed Hodge structure in the ordinary sense. The free R-module $R^kf_*\widetilde{\Omega}^p_{\mathcal{Y}/S}$ is the abutment of the spectral sequence

$$E_1^{k,m} = R^mf_*\left(a_{k*}\Omega^p_{\mathcal{Y}^{[k]}/S}\right) \Rightarrow R^{k+m}f_*\widetilde{\Omega}^p_{\mathcal{Y}/S} \tag{III.4.4}$$

induced by the resolution (III.2.4) for fixed p. The filtration defined in (III.4.1) induces a weight filtration $\mathrm{Gr}_F^p R^k f_* \widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ in the obvious way and the graded objects with respect to this filtration are the E_∞ terms of the spectral sequence (III.4.4). By [**Del68**, Thm 5.5] the R-modules $E_1^{k,m}$ are free and compatible with base change. Moreover, the differential d_1 on $E_1^{k,m}$ is given by the semi-simplicial differential

$$\delta : R^m f_* \left(a_{k*} \Omega^p_{\mathcal{Y}^{[k]}/S} \right) \longrightarrow R^m f_* \left(a_{k*} \Omega^p_{\mathcal{Y}^{[k+1]}/S} \right).$$

This morphism has constant rank by Proposition III.3.4. Hence $E_2^{k,m}$ is free, too, and compatible with base change by Lemma A.2.6. In the case $R = \mathbb{C}$ the spectral sequence is known to degenerate at E_2, see [**PS08**, Thms 3.12, 3.18]. As all E_2-terms of (III.4.4) are compatible with base change we have for all n that

$$\begin{aligned}
\sum_{k+m=n} \lg_R \left(E_2^{k,m} \right) &= \lg_R(R) \sum_{k+m=n} \dim_{\mathbb{C}} \left(E_2^{k,m} \otimes \mathbb{C} \right) \\
&= \lg_R(R) \dim_{\mathbb{C}} \left(R^n f_* \widetilde{\Omega}^p_{Y/\mathbb{C}} \right) \\
&= \lg_R \left(R^n f_* \widetilde{\Omega}^p_{\mathcal{Y}/S} \right).
\end{aligned}$$

Thus, the spectral sequence III.4.4 also degenerates at E_2 and the R-modules $E_\infty^{k,m} = \mathrm{Gr}_m^W R^{k+m} f_* \widetilde{\Omega}^p_{\mathcal{Y}/S} = \mathrm{Gr}_m^W \mathrm{Gr}_F^p R^{k+m} f_* \widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ coincide with the free R-modules $E_2^{k,m}$. Again, as all graded objects are compatible with base change, $\mathcal{H}$ restricts to a mixed Hodge structure on the central fiber, which is the usual mixed Hodge structure on Y. □

For later use we isolate an observation from the proof of the previous lemma.

Corollary III.4.2. Let Y be a proper simple normal crossing variety over $\mathbb{C}$ and let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y over $S = \operatorname{Spec} R$ for $R \in \mathrm{Art}_{\mathbb{C}}$. Then the spectral sequence (III.4.4) degenerates at E_2. □

Now we are able to deduce the main result of this chapter.

Theorem III.4.3. *Let $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_{\mathbb{C}}$, let Y be a proper simple normal crossing $\mathbb{C}$-variety and let $g : \mathcal{X} \longrightarrow S$ and $f : \mathcal{Y} \longrightarrow S$ be proper, algebraic S-schemes. Assume that $\mathcal{Y} \longrightarrow S$ is a locally trivial deformation of Y and $\mathcal{X} \longrightarrow S$ is smooth. Let $i : \mathcal{Y} \longrightarrow \mathcal{X}$ be an S-morphism. Then for all p, q the morphism $i^* : R^q g_* \Omega^p_{\mathcal{X}/S} \longrightarrow R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}$ has constant rank.*

Proof. Let $\ldots \Rrightarrow \mathcal{Y}^{[1]} \rightrightarrows \mathcal{Y}^{[0]} \to \mathcal{Y}$ be the semi-simplicial resolution of $\mathcal{Y}$ over S from Lemma III.2.4. This means in particular that $\mathcal{Y}^{[0]}$ is a locally trivial deformation of the normalization. By Theorem III.2.10 the

R-modules $R^q g_* \Omega^p_{\mathcal{X}/S}$, $R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}$ and $R^q f_* \Omega^p_{\mathcal{Y}^{[k]}/S}$ are free and compatible with base change. By Corollary III.4.2 we know that the spectral sequences (III.4.4) degenerate at E_2 for each p. As

$$E_2^{0,q} = \ker\left(R^q f_* \Omega^p_{\mathcal{Y}^{[0]}/S} \longrightarrow R^q f_* \Omega^p_{\mathcal{Y}^{[1]}/S}\right)$$

this implies that the first row in
(III.4.5)

$$\begin{tikzcd} 0 \arrow[r] & W_{p+q-1} R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S} \arrow[r] & R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S} \arrow[r, "\eta"] & R^q f_* \Omega^p_{\mathcal{Y}^{[0]}/S} \arrow[r, "\delta"] & R^q f_* \Omega^p_{\mathcal{Y}^{[1]}/S} \\ & & R^q g_* \Omega^p_{\mathcal{X}/S} \arrow[u, "i^*"] \arrow[ur, "\varphi"'] & & \end{tikzcd}$$

is exact.
Here $\operatorname{im} i^*$ does not intersect $W_{p+q-1} R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}$, as it does not on the central fiber. This last claim can be shown using Deligne's weak splitting as follows. We denote $X := \mathcal{X} \times_S \mathbb{C}$ and put $H_Y^{p+q} := H^{p+q}(Y, \mathbb{C})$ and $H_X^{p,q} := H^{p+q}(X, \mathbb{C})$. We identify H_Y^{p+q} and H_X^{p+q} with the hypercohomology of $\widetilde{\Omega}^\bullet_Y$ respectively $\Omega^\bullet_X$ and obtain

$$\begin{tikzcd} \left(W_{p+q-1} R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}\right) \otimes \mathbb{C} \arrow[r, hook] \arrow[d, equal] & \left(R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}\right) \otimes \mathbb{C} \arrow[d, equal] & \left(R^q g_* \Omega^p_{\mathcal{X}/S}\right) \otimes \mathbb{C} \arrow[l, "i^*"'] \arrow[d, equal] \\ W_{p+q-1} R^q f_* \widetilde{\Omega}^p_Y \arrow[r, hook] \arrow[d, equal] & R^q f_* \widetilde{\Omega}^p_Y \arrow[d, equal] & R^q g_* \Omega^p_X \arrow[l] \arrow[d, equal] \\ W_{p+q-1} \mathrm{Gr}_F^p H_Y^{p+q} \arrow[r, hook] & \mathrm{Gr}_F^p H_Y^{p+q} & \mathrm{Gr}_F^p H_X^{p+q} \arrow[l] \end{tikzcd}$$

Deligne's weak splitting [**PS08**, Ex 3.3 and Lem-Def 3.4] is a decomposition

$$H_Y^k = \bigoplus_{r,s} I_Y^{r,s}$$

such that

$$F^p H_Y^k = \bigoplus_{r \geq p} I_Y^{r,s} \quad \text{and} \quad W_m H_Y^k = \bigoplus_{r+s \leq m} I_Y^{r,s}.$$

The subspaces $I_Y^{r,s} \subseteq H_Y^{r+s}$ project isomorphically onto the subquotients $\mathrm{Gr}^W_{r+s} \mathrm{Gr}^r_F H_Y$. The Deligne weak splitting is preserved under morphisms of mixed Hodge structures. As the Hodge structure on H_X^{p+q} is pure of weight $p+q$, this yields $\operatorname{im} i^* \subseteq I_Y^{p,q}$ and therefore

$$\operatorname{im} i^* \cap W_{p+q-1} \mathrm{Gr}_F^p H_Y^{p+q} \ \subseteq \ I_Y^{p,q} \cap \bigoplus_{r+s \leq p+q-1} I_Y^{r,s} = 0.$$

as claimed. We come back to diagram (III.4.5) and observe that φ has constant rank by Proposition III.3.4. Also η has constant rank as δ has constant rank by Proposition III.3.4 and hence $\operatorname{coker} \eta = \ker \delta$ is free. As

$\operatorname{im} i^* \cap \ker \eta = 0$, Lemma A.2.5 implies that i^* has constant rank completing the proof. □

III.5. The case of Kähler manifolds

When X is a compact Kähler manifold, Theorem III.2.10 is true as well. But the references to [**EGAIII2**] have to be replaced.

THEOREM III.5.1 (**Deligne**). *Let X be a compact Kähler manifold, let $S = \operatorname{Spec} R$ for $R \in \mathrm{Art}_{\mathbb{C}}$ and let $g : \mathcal{X} \longrightarrow S$ be a deformation of X over S. Let $S' = \operatorname{Spec} R'$ for $R' \in \mathrm{Art}_{\mathbb{C}}$ and let $S' \longrightarrow S$ be a morphism. Then the following holds.*

(1) *The associated spectral sequence*

$$E_1^{p,q} = R^q g_* \Omega^p_{\mathcal{X}/S} \Rightarrow R^{p+q} g_* \Omega^\bullet_{\mathcal{X}/S} = H^{p+q}(X, \underline{R}_X) \tag{III.5.1}$$

degenerates at E_1.

(2) *The R-modules $R^q g_* \Omega^p_{\mathcal{X}/S}$ are free and compatible with arbitrary base change in the sense that for $\mathcal{X}' = \mathcal{X} \times_S S'$ the morphism*

$$R^q g_* \Omega^p_{\mathcal{X}/S} \otimes_R R' \longrightarrow R^q g_* \Omega^p_{\mathcal{X}'/S'}$$

is an isomorphism.

PROOF. This works literally as in the proof of Theorem III.2.10 We only have to replace the reference to [**EGAIII2**, Thm 6.10.5] by [**BS77**, Ch 3, Thm 4.1] and the reference to [**EGAIII2**, 7.8.5] by [**BS77**, Ch 3, Cor 3.10]. The rest of the proof of Theorem III.2.10 goes through, if we note that the spectral sequence associated to $\Omega^\bullet_X$ degenerates as X is a compact Kähler manifold. □

With this theorem at hand, the analogues of the other results of this chapter hold true as well. We record them for the sake of referenceability.
With the definitions (III.3.1) and (III.3.2) the analogue of Lemma III.3.1 holds.

LEMMA III.5.2. Let $g : \mathcal{X} \longrightarrow S$ be a deformation of a compact Kähler manifold X, where $S = \operatorname{Spec} R$, $R \in \mathrm{Art}_{\mathbb{C}}$. Then

$$\mathcal{H}^k(\mathcal{X}) := \Big(H^k(X, \mathbb{R}), F^p H^k(X, \underline{R}_X)\Big)$$

is a pure Hodge structure of weight k over R. Moreover the morphism $R^k g_* F^p \Omega^\bullet_{\mathcal{X}/S} \longrightarrow R^k g_* \Omega^\bullet_{\mathcal{X}/S}$ is injective, so that $R^k g_* F^p \Omega^\bullet_{\mathcal{X}/S} \cong F^p H^k(X, \underline{R}_X)$. If $h : \mathcal{Z} \longrightarrow S$ is a deformation of a compact Kähler manifold, every S-morphism $i : \mathcal{X} \longrightarrow \mathcal{Z}$ induces a morphism $i^* : \mathcal{H}^k(\mathcal{Z}) \longrightarrow \mathcal{H}^k(\mathcal{X})$ of pure Hodge structures.

PROOF. The reference to [**Del68**, Thm 5.5] has to be replaced by Theorem III.5.1. Then the proof of Lemma III.3.1 works literally. □

The proof of the analogues to Corollary III.3.2 and Corollary III.3.3 is straight forward. As this and abstract Hodge-Weil theory is all we needed for the proof of Proposition III.3.4, we have

PROPOSITION III.5.3. Let X and Y be compact Kähler manifolds and let $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_{\mathbb{C}}$. Let $f : \mathcal{Y} \longrightarrow S$ and $g : \mathcal{X} \longrightarrow S$ be deformations of Y and X over S and let $i : \mathcal{Y} \longrightarrow \mathcal{X}$ be an S-morphism. Then the induced morphisms $i^* : R^q g_* \Omega^p_{\mathcal{X}/S} \longrightarrow R^q f_* \Omega^p_{\mathcal{Y}/S}$ have constant rank. □

The analogue of Theorem III.4.3 will be formulated in a way that is appropriate for applications in Chapter VI.

THEOREM III.5.4. *Let X be a compact Kähler manifold, let Y be a simple normal crossing proper algebraic $\mathbb{C}$-variety and let $S = \operatorname{Spec} R$ for some $R \in \mathrm{Art}_{\mathbb{C}}$. Let $g : \mathcal{X} \longrightarrow S$ be a deformation of X over S, let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y over S and let $i : \mathcal{Y}^{\mathrm{an}} \longrightarrow \mathcal{X}$ be an S-morphism. Then for all p, q the morphism $i^* : R^q g_* \Omega^p_{\mathcal{X}/S} \longrightarrow R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}^{\mathrm{an}}/S}$ has constant rank.* □

III.6. Lifting the normalization

Let Y be a k-variety, let $S = \operatorname{Spec} R$ for $R \in \mathrm{Art}_k$ and let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y. We will show that the normalization $\widetilde{Y}$ of Y has a lifting to a flat S-scheme.
Let $0 \longrightarrow J \longrightarrow R \longrightarrow R' \longrightarrow 0$ be a small extension in Art_k with $J = t \cdot R$ and let A be a flat R-algebra. Tensoring with A we obtain an exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A' \longrightarrow 0,$$

where flatness implies that $I = J \otimes A$. As $t \cdot \mathfrak{m} = 0$ the A-module structure on I factors through the projection $p : A \longrightarrow A \otimes_R R/\mathfrak{m} =: A_0$, where $\mathfrak{m} \subseteq R$ is the maximal ideal, and write $p(a) =: \bar{a}$. Let $\sigma : A \longrightarrow A$ be an R-algebra automorphism that restricts to the identity on A_0 and let $\delta \in \mathrm{Der}_k(A_0)$. Then $\sigma' = \sigma + t\delta \circ p$ is again an R-algebra automorphism as the following calculation shows.

$$\begin{aligned}
\sigma'(a)\,\sigma'(b) &= \sigma(ab) + t\sigma(a)\delta(\bar{b}) + t\sigma(b)\delta(\bar{a}) \\
&= \sigma(ab) + t\left(a\delta(\bar{b}) + b\delta(\bar{a})\right) \\
&= (\sigma + t\delta \circ p)(ab) \\
&= \sigma'(ab)
\end{aligned}$$

Here we used that $\sigma(a)t = at = \bar{a}t$. We will frequently write $\sigma + t\delta$ for simplicity. The following lemma is a straight forward generalization of [**Ser06**, Lem 1.2.6].

Lemma III.6.1. Let A be a flat R-algebra. Let $0 \longrightarrow J \longrightarrow R \longrightarrow R' \longrightarrow 0$ be a small extension in Art_k with $J = t \cdot R$ and $A' = A \otimes_R R'$. Let σ_1, σ_2 be R-algebra automorphisms of A which restrict to the identity on $A_0 = A \otimes_R k$ and coincide on A'. Then there is a derivation $\delta \in \mathrm{Der}_k(A_0)$ such that $\sigma_1 = \sigma_2 + t\delta$.

Proof. It is enough to show that $\sigma := \sigma_2^{-1} \circ \sigma_1 = \mathrm{id} + t\delta$ beacuse σ_2 restricts to id on A_0 and hence $\sigma_2 \circ t\delta = t\delta$. We define $t\delta := \sigma - \mathrm{id}$. This is clearly R-linear and easily seen to be a derivation. $\square$

Let Y be a k-scheme, let $S = \operatorname{Spec} R$ for $(R, \mathfrak{m}) \in \mathrm{Art}_k$ and let $f : \mathcal{Y} \longrightarrow S$ be a deformation of Y. We denote by $T^0_{\mathcal{Y}/S} := \mathfrak{m}T_{\mathcal{Y}/S} \subseteq T_{\mathcal{Y}/S}$ the subsheaf of sections vanishing on Y and by $\mathrm{Aut}(\mathcal{Y}/S)^0 \subseteq \mathrm{Aut}(\mathcal{Y}/S)$ be the subgroup sheaf of automorphisms of $\mathcal{Y}$ over S that restrict to the identity on Y. Note that the continuous map underlying an automorphism $\varphi \in \mathrm{Aut}(\mathcal{Y}/S)$ is the identity on Y. Therefore we will identify such a φ with the corresponding automorphism of $\mathcal{O}_{\mathcal{Y}}$. The following lemma seems to be well-known, but we were unable to find a reference.

Lemma III.6.2. Let Y be a k-scheme, let $S = \operatorname{Spec} R$ for $(R, \mathfrak{m}) \in \mathrm{Art}_k$ and let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y. Then

$$T^0_{\mathcal{Y}/S} \longrightarrow \mathrm{Aut}(\mathcal{Y}/S)^0, \quad \vartheta \mapsto \exp(\vartheta) = \mathrm{id} + \vartheta + \frac{1}{2}\vartheta \circ \vartheta + \dots$$

is an isomorphism of sheaves of sets. In particular the logarithm

$$\log \varphi = \sum_{n=1}^{\infty} \frac{(\mathrm{id} - \varphi)^n}{n}$$

of an automorphism $\varphi \in \mathrm{Aut}(\mathcal{Y}/S)^0$ is a derivation.

Proof. Note that $\exp : T^0_{\mathcal{Y}/S} \longrightarrow \mathrm{Aut}(\mathcal{Y}/S)^0$ is well-defined as $\mathfrak{m}^n$ and hence also $\vartheta^{\circ n}$ vanish for $n \gg 0$ The statement is local in $\mathcal{Y}$ hence we may assume that $\mathcal{Y} = \operatorname{Spec} A$ is affine. The proof is by induction on the length $\lg(R)$. Let $0 \longrightarrow J \longrightarrow R \longrightarrow R' \longrightarrow 0$ be a small extension in Art_k with $J = t \cdot R$. We write $S' = \operatorname{Spec} R'$ and $\mathcal{Y}' = \mathcal{Y} \times_S S' = \operatorname{Spec} A'$ where $A' = A \otimes_R R'$. Let $\sigma \in \mathrm{Aut}(\mathcal{Y}/S)^0$ be given and $\sigma' \in \mathrm{Aut}(\mathcal{Y}'/S')^0$ be the restriction of σ to $\mathcal{Y}'$. By the inductive hypothesis there is $\vartheta' \in T^0_{\mathcal{Y}'/S'}$ with

$\exp(\vartheta') = \sigma'$. Now we have

$$\begin{aligned} T_{\mathcal{Y}'/S'} &= \mathrm{Hom}_{A'}(\Omega_{A'/R'}, A') \\ &= \mathrm{Hom}_{A'}(\Omega_{A/R} \otimes_A A', A') \\ &= \mathrm{Hom}_A(\Omega_{A/R}, A') \end{aligned}$$

and in order to lift ϑ' to some $\vartheta \in T_{\mathcal{Y}/S}$ we consider the exact sequence

$$\begin{array}{ccccc} \mathrm{Hom}_A(\Omega_{A/R}, A) & \longrightarrow & \mathrm{Hom}_A(\Omega_{A/R}, A') & \longrightarrow & \mathrm{Ext}^1_A(\Omega_{A/R}, JA). \\ \| & & \| & & \\ T_{\mathcal{Y}/S} & & T_{\mathcal{Y}'/S'} & & \end{array}$$

In general $T_{\mathcal{Y}/S} \longrightarrow T_{\mathcal{Y}'/S'}$ does not need to be surjective. But here $\mathcal{Y} \longrightarrow S$ is a locally trivial deformation of $Y = \mathcal{Y} \times_S k \longrightarrow \mathrm{Spec}(k)$, so we may assume that $A = A_0 \otimes_k R$ for some k-algebra A_0 and hence $A' = A_0 \otimes_k R'$. This implies that

$$\begin{aligned} T_{\mathcal{Y}'/S'} &= \mathrm{Hom}_A(\Omega_{A/R}, A') = \mathrm{Hom}_A(\Omega_{A_0/k} \otimes_{A_0} A, A') \\ &= \mathrm{Hom}_{A_0}(\Omega_{A_0/k}, A_0 \otimes_k R') = \mathrm{Hom}_{A_0}(\Omega_{A_0/k}, A_0) \otimes_k R' \\ &= T_Y \otimes_k R' \end{aligned}$$

and in the same way that $T_{\mathcal{Y}/S} = T_Y \otimes_k R$. So $T_{\mathcal{Y}/S} \longrightarrow T_{\mathcal{Y}'/S'}$ is surjective for a locally trivial deformation. Let $\vartheta \in T_{\mathcal{Y}/S}$ be a lift of ϑ'. Clearly, $\vartheta \in T^0_{\mathcal{Y}/S} = \mathfrak{m}T_{\mathcal{Y}/S}$ as $\vartheta' \in \mathfrak{m}'T_{\mathcal{Y}'/S'}$ and the preimage of $\mathfrak{m}'$ is $\mathfrak{m}$. The automorphisms $\exp(\vartheta)$ and σ both restrict to σ' on $\mathcal{Y}'$. Thus by Lemma III.6.1 there is $\delta \in \mathrm{Der}_k(A_0) = T_Y$ such that $\sigma = \exp(\vartheta) + t\delta = \exp(\vartheta + t\delta)$. As $\vartheta \in T^0_{\mathcal{Y}/S}$ we have $\vartheta + t\delta \in T^0_{\mathcal{Y}/S}$. So $\exp : T^0_{\mathcal{Y}/S} \longrightarrow \mathrm{Aut}(\mathcal{Y}/S)^0$ is surjective.
For injectivity assume $\exp(\vartheta) = \mathrm{id}$. Then $0 = \exp(\vartheta) - \mathrm{id} = \vartheta + \frac{1}{2}\vartheta \circ \vartheta + \ldots$. If $\vartheta \neq 0$ there is $k \in \mathbb{N}$ such that $\vartheta \in \mathfrak{m}^k T_{\mathcal{Y}/S} \backslash \mathfrak{m}^{k+1} T_{\mathcal{Y}/S}$. Then this equation means $\vartheta = 0 \mod \mathfrak{m}^{k+1}$, a contradiction. $\square$

The isomorphism $\exp : T^0_{\mathcal{Y}/S} \longrightarrow \mathrm{Aut}(\mathcal{Y}/S)^0$ of sheaves of sets is in general not an isomorphism of groups as $\mathrm{Aut}(\mathcal{Y}/S)^0$ is non-commutative in general.

Lemma III.6.3. Let Y be a reduced algebraic k-scheme and $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y over an Artinian base $S = \mathrm{Spec}\, R$. Then there exists a scheme $\widetilde{\mathcal{Y}}$ and a morphism $\nu : \widetilde{\mathcal{Y}} \longrightarrow \mathcal{Y}$ such that the composition $\mu = f \circ \nu : \widetilde{\mathcal{Y}} \longrightarrow S$ is a locally trivial deformation of the normalization $\widetilde{Y}$ of Y. Moreover, the pair $(\widetilde{Y}, \nu)$ is unique. In particular, μ is flat.

Proof. By Lemma III.1.5, we may assume that Y is integral. Let $Y = \cup_i U_i$ be a covering by open affine subsches $U_i = \mathrm{Spec}\, A_i$ and put

$A_{ij} = \Gamma(U_{ij}, \mathcal{O}_Y)$. The scheme $\mathcal{Y}$ is determined by the $\mathcal{Y}_i := \mathcal{Y}|_{U_i}$ and by the collection of R-algebra isomorphisms $\theta_{ij} := \theta_j^{-1} \circ \theta_i : A_{ij} \otimes_k R \longrightarrow A_{ij} \otimes_k R$, which satisfy the cocyle conditions on the triple intersections U_{ijk} and which reduce to the identity modulo $\mathfrak{m}$. Let $\widetilde{A}_i$ be the normalization of A_i, then we will obtain $\nu : \widetilde{\mathcal{Y}} \longrightarrow \mathcal{Y}$ by gluing $\nu_i : \mathrm{Spec}(\widetilde{A}_i) \longrightarrow \mathrm{Spec}(A_i \otimes R)$ along $\nu_i^{-1}(U_{ij}) \longrightarrow \nu_j^{-1}(U_{ij})$. Therefore we have to find a morphism $\widetilde{\theta}_{ij}$ to make the diagramm

$$\begin{tikzcd} A_{ij} \otimes R \arrow[r, "\theta_{ij}"] \arrow[d] & A_{ij} \otimes R \arrow[d] \\ \widetilde{A}_{ij} \otimes R \arrow[r, dashed, "\exists\, \widetilde{\theta}_{ij}?"] & \widetilde{A}_{ij} \otimes R \end{tikzcd}$$

commutative, where $\widetilde{A}_{ij}$ is the normalization and the cocycle conditions remain valid. Observe that such $\widetilde{\theta}_{ij}$ and also $\widetilde{\mathcal{Y}}$ will be unique. The θ_{ij} are determined by a section $s_{ij} \in \Gamma(U_{ij}, T^0_{\mathcal{Y}/S})$, hence the existence of $\widetilde{\theta}_{ij}$ is implied by Lemma III.6.2 as follows. The restriction of $\mathcal{Y} \longrightarrow S$ to U_{ij} is a trivial deformation, so $\Gamma(U_{ij}, T^0_{\mathcal{Y}/S}) = \Gamma(U_{ij}, T_Y \otimes_k \mathfrak{m})$ and the s_{ij} are determined by derivations on Y. These derivations extend to the normalization $\widetilde{Y}$ by a theorem of Seidenberg, see [**Sei66**, p. 168]. Thus, s_{ij} extends to $\widetilde{A}_{ij} \otimes R$ determining a morphism $\widetilde{\theta}_{ij}$. $\square$

CHAPTER IV

Symplectic geometry

We recall basic definitions an results regarding irreducible symplectic manifolds.

IV.1. Basic definitions and results

Definition IV.1.1. An *irreducible symplectic manifold* is a compact, connected, simply connected Kähler manifold X with $H^0(X, \Omega_X^2) = \mathbb{C}\cdot\omega$ where ω is symplectic, i.e. everywhere non-degenerate.

We fix once and for all a holomorphic symplectic form ω on X. Irreducible symplectic manifolds are quite special. As ω is symplectic, they are even-dimensional. By contraction, the symplectic form gives an isomorphism

$$\omega' : T_X \xrightarrow{\cong} \Omega_X.$$

and its top exterior power is by definition nowhere vanishing, hence a trivialization of the canonical bundle

$$\omega^{\wedge n} : \mathcal{O}_X \xrightarrow{\cong} \omega_X,$$

where $\dim X = 2n$. As a consequence, $\mathrm{c}_1(X) = \mathrm{c}_1(T_X) = \mathrm{c}_1(\omega_X) = 0$. As $\pi_1(X) = 0$, also

$$H_1(X, \mathbb{Z}) = \pi_1^{\mathrm{ab}}(X) = 0$$

and thus by the universal coefficient theorem also $H^1(X, \mathbb{C}) = 0$. Hodge decomposition and the Dolbeault isomorphism imply

$$H^1(X, \mathcal{O}_X) = 0 = H^0(X, \Omega_X).$$

Again using the symplectic form we find that $H^0(X, T_X) = 0$; a fact that will guarantee the existence of a universal formal deformation for such manifolds, see Proposition I.2.2.

Gradually one is led to believe that such manifolds have a very special geometry. Another manifestation of the fact that symplectic manifolds are central in the study of compact complex manifolds with vanishing first chern class is Bogomolov's famous decomposition theorem [**Bog74**]. The version stated below is found in [**Bea83**, Thm 2].

THEOREM IV.1.2 (**Bogomolov**). *Let X be a compact Kähler manifold with $c_1(X) = 0$. Then there is a finite etale covering $X' \longrightarrow X$ and an isomorphism*

$$X' \xrightarrow{\cong} T \times \prod_i V_i \times \prod_j X_j,$$

where T is a complex torus, V_i are Calabi-Yau manifolds and X_j are irreducible symplectic manifolds.

Here Calabi-Yau is meant in the strong sense. A Calabi-Yau manifold is a simply connected projective manifold of dimension ≥ 3 such that $H^{p,0}(V_i) = 0$ holds for all $0 < p < \dim V_i$.
Irreducible symplectic manifolds have a differential geometric counterpart: compact hyperkähler manifolds. These are compact Riemannian manifolds X of dimension $\dim_{\mathbb{R}} X = 4n$ with holonomy group exactly $\mathrm{Sp}(n)$, the unitary symplectic group. The group $\mathrm{Sp}(n)$ is the compact real form of $\mathrm{Sp}(2n, \mathbb{C})$ and is obtained as $\mathrm{Sp}(n) = \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n)$. The following theorem connects these two classes of manifolds.

THEOREM IV.1.3. *Let X be a compact Kähler manifold of dimension $2n$. The following conditions on X are equivalent.*

- *X admits a Kähler metric with holonomy exactly $\mathrm{Sp}(n)$.*
- *X is simply connected and admits a symplectic form which is unique up to scalars.*

This theorem is proven in [**Bea83**, Prop 4] as a consequence of Yau's theorem [**Yau78**] and Bogomolov's Theorem IV.1.2, see also [**GHJ03**, Thm 23.5]. We will not dwell on this side of the theory; the article [**Huy99**] however can also serve as an introduction.

DEFINITION IV.1.4. A closed subvariety Y of a symplectic manifold (X, ω) is called *Lagrangian*, if $\dim Y = \frac{1}{2} \dim X$ and $i_{\mathrm{reg}}^* \omega = 0$ where $i_{\mathrm{reg}} : Y_{\mathrm{reg}} \longrightarrow X$ is the inclusion of the regular part of Y.

If Y is smooth, the above definition coincides with the classical definition of a Lagrangian submanifold, namely that for every $y \in Y$ the tangent space $T_{Y,y} \subseteq T_{X,y}$ is Lagrangian, i.e. maximal isotropic with respect to ω_y.

DEFINITION IV.1.5. Let X be a symplectic manifold and B a normal complex space. A proper morphism $f : X \longrightarrow B$ is a *Lagrangian fibration*, if $f_* \mathcal{O}_X = \mathcal{O}_B$ and if the reduction of every fiber of f is a Lagrangian subvariety of X.

We postpone the discussion of the theory of Lagrangian fibrations until Chapter VII. We only want to mention here Voisin's argument that a Lagrangian submanifold of X is always projective, even if X is only Kähler, see [**Cam06**, Prop 2.1].

PROPOSITION IV.1.6 (**Voisin**). Let $Y \hookrightarrow X$ be the inclusion of a Lagrangian submanifold into an irreducible symplectic manifold. Then Y is projective.

This result will be generalized to some special types of singular Lagrangian subvarieties in Chapter VI.

IV.2. Examples of symplectic manifolds

There are not many examples known of irreducible symplectic manifolds and to find new examples is one of the most difficult problems in this area. Up to deformation all known examples are K3 surfaces, Hilbert schemes of points on K3 surfaces, generalized Kummer varieties and two sporadic examples constructed by O'Grady from moduli spaces of sheaves on K3 and abelian surfaces [**O'G99, O'G03**].

IV.2.1. K3 surfaces. A K3 surface is a compact complex surface S with $\omega_X \cong \mathcal{O}_X$ and $H^1(S, \mathcal{O}_S) = 0$. In dimension 2 irreducible symplectic manifolds are exactly K3 surfaces. The non-trivial part is to see that K3 surfaces are irreducible symplectic, see [**K3a85**].
K3 surfaces which admit a fibration where the generic fiber is an elliptic curve are called *elliptic*. Such a fibration is automatically Lagrangian. An example of an elliptic K3 surface is obtained as follows. Take generic homogenous polynomials $f_0, f_1, f_2 \in \mathbb{C}[x, y, z]$. Then

$$\left\{s^2 f_0 + st f_1 + t^2 f_2 = 0\right\} \subseteq \mathbb{P}^1 \times \mathbb{P}^2$$

can be seen to define an elliptic K3 surface via the projection to $\mathbb{P}^1$. Another example is the Fermat surface

$$S = \left\{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\right\} \subseteq \mathbb{P}^3.$$

The map $x_i \mapsto x_i^2$ gives a morphism from S to the ruled surface

$$\mathbb{P}^3 \supseteq \left\{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\right\} \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Now a projection to one of the factors gives a fibration on S.

IV.2.2. Hilbert schemes. Let S be a K3 surface. Then the Hilbert scheme $\mathrm{Hilb}^n(S)$ of n points on S is an irreducible symplectic manifold, see [**Bea83**, Thm 3]. If S is elliptic, then also $\mathrm{Hilb}^n(S)$ will admit a Lagrangian fibration. It is obtained as the composition of the Hilbert-Chow morphism $\varrho : \mathrm{Hilb}^n(S) \longrightarrow \mathrm{Sym}^n S$ to the symmetric product and the morphism $\mathrm{Sym}^n S \longrightarrow \mathrm{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$ obtained by applying the functor Sym to the elliptic fibration of S.
Other examples of at least rational Lagrangian fibrations on Hilbert schemes are obtained as follows, see [**Bea91**]. Take a quartic K3 surface $S \subseteq \mathbb{P}^3$. Then every subscheme $\xi \hookrightarrow S \subseteq \mathbb{P}^3$ of length 3 spans a plane H_ξ unless it is contained in a line $\ell \subseteq S \subseteq \mathbb{P}^3$. Therefore we obtain a rational fibration

$$\mathrm{Hilb}^3(S) \dashrightarrow (\mathbb{P}^3)^\vee$$

to the dual projective space by mapping $\xi \mapsto H_\xi$. If S is a double covering $S \longrightarrow \mathbb{P}^2$ of $\mathbb{P}^2$, then we have a rational fibration

$$\mathrm{Hilb}^2(S) \dashrightarrow (\mathbb{P}^2)^\vee$$

defined by sending a subscheme $\xi \hookrightarrow S$ of length 2 to the line ℓ_ξ determined by the image of ξ in $\mathbb{P}^2$. This is well-defined only if ξ is not contained in a fiber of $S \longrightarrow \mathbb{P}^2$.

IV.2.3. Generalized Kummer varieties. Let A be an abelian surface. Then the Hilbert scheme $\mathrm{Hilb}^n(A)$ of n points on A has a symplectic structure just as in the K3 case, but is not irreducible symplectic as $\pi_1(\mathrm{Hilb}^n(A))$ is non-trivial. Consider the morphisms

$$\mathrm{Hilb}^n(A) \longrightarrow \mathrm{Sym}^n A \xrightarrow{+} A,$$

where the first is the Hilbert-Chow morphism and the second is the addition in the group law on A. The composition $\mathrm{Hilb}^n(A) \longrightarrow A$ is a holomorphic fiber bundle and the fiber $K_{n-1}(A)$ is an irreducible symplectic manifold, the *generalized Kummer variety.* This is shown in [**Bea83**, Thm 4]. If $A \longrightarrow E$ is a fibration to an elliptic curve, then similar to the case of the Hilbert scheme of a K3 surface also $K_{n-1}(A)$ will admit a Lagrangian fibration.

IV.3. The Beauville-Bogomolov quadratic form

There is a quadratic form $q_X : H^2(X, \mathbb{R}) \longrightarrow \mathbb{C}$ on the second cohomology of an irreducible symplectic manifold defined by
(IV.3.1)

$$q_X(\alpha) = \frac{n}{2} \int_X \alpha^2.\omega^{n-1}.\bar{\omega}^{n-1} + (1-n) \left(\int_X \alpha.\omega^n.\bar{\omega}^{n-1} \right) \left(\int_X \alpha.\omega^{n-1}.\bar{\omega}^n \right).$$

It was discovered by Beauville [**Bea83**] and is called the *Beauville-Bogomolov-*form, see also [**GHJ03**, Def 22.8 ff]. It resembles the intersection pairing on surfaces, in particular it coincides with the intersection pairing up to a scalar, if X is a K3 surface. This form has a rich theory. It is to be held responsible for many things we know about irreducible symplectic manifolds. On the other side its existence is mysterious and a natural explanation for it would be a wonderful thing to have.
We will also write q_X for the extension of this form to $H^2(X,\mathbb{C})$. Using the Hodge decomposition $H^2(X,\mathbb{C}) = \mathbb{C}\omega\oplus H^{1,1}(X)\oplus\mathbb{C}\bar{\omega}$, writing $\alpha \in H^2(X,\mathbb{C})$ accordingly as $\alpha = \lambda\omega + \beta + \mu\bar{\omega}$ with $\lambda, \mu \in \mathbb{C}$ and normalizing ω such that $\int_X \omega^n.\bar{\omega}^n = 1$ the formula (IV.3.1) takes the form

$$q_X(\alpha) = \lambda\mu + \int_X \beta^2.\omega^{n-1}.\bar{\omega}^{n-1}. \tag{IV.3.2}$$

We list two properties of q_X; for proofs see § 23 of Huybrecht's lectures in [**GHJ03**] and references therein. We will not distinguish between the quadratic form q_X and the bilinear form obtained from q_X by polarization.

Proposition IV.3.1. The Beauville-Bogomolov form q_X is non-degenerate and has signature $(3, b_2(X)-3)$ on $H^2(X,\mathbb{R})$. If κ is a Kähler class, then q_X is positive on the subspace generated by κ and the real and imaginary parts $\mathrm{Re}\,\omega$ and $\mathrm{Im}\,\omega$ of the symplectic form. Moreover, q_X can be renormalized to be a primitive integral quadratic form on $H^2(X,\mathbb{Z})$.

Proposition IV.3.2. There are constants a_p only depending on the deformation type of X such that

$$a_p\; q_X(\alpha)^{n-p} = \int_X \mathrm{c}_p(X).\alpha^{2(n-p)}$$

for all $\alpha \in H^2(X,\mathbb{R})$ where $\mathrm{c}_p(X)$ are the Chern classes of X.

CHAPTER V

The universal deformation space of an irreducible symplectic manifold

In this chapter we explain existence and basic properties of a universal deformation space M for an irreducible symplectic manifold X. We define and discuss certain subspaces. Those results of Voisin's article [**Voi92**], which we use in Chapter VI, are explained and for convenience proofs are reproduced.

V.1. Existence and properties of the universal deformation space

By Kuranishi's theorem [**Kur62**] for every compact complex space X there exists a versal deformation space M for deformations of X. This means that there is a flat morphism $\pi : \mathfrak{X} \longrightarrow M$ of complex spaces and a point $0 \in M$ with $\pi^{-1}(0) = X$ and π is versal at 0 in the sense of Definition I.5.4. Moreover, π is also versal in an open neighbourhood of $0 \in M$. As $H^0(X, T_X) = 0$ for irreducible symplectic manifolds, M is a universal deformation space, that is, π is universal at 0. We call $\pi : \mathfrak{X} \longrightarrow M$ the *universal family*. Close to $0 \in M$ the fibers of π are again irreducible symplectic manifolds, see [**Bea83**, § 8]. Universality at 0 implies that

$$\dim_0 M = \dim t_{D_X} = \dim H^1(X, T_X) = \dim H^1(X, \Omega_X) = \dim H^{1,1}(X),$$

where the third equality comes from the isomorphism $T_X \xrightarrow{\cong} \Omega_X$ induced by the symplectic form. The universal deformation space M of X is known to be smooth by the Bogomolov-Tian-Todorov theorem [**Bog78, Tia87, Tod89**], see also [**GHJ03**, Thm 14.10] for an introduction. It is convenient to consider the pointed complex space $(M, 0)$ as a germ of complex spaces. This means that instead of $(M, 0)$ we look at the equivalence class of $(M, 0)$ in the category of pointed complex spaces, where two pointed complex spaces are considered equivalent if they are biholomorphic locally around their respective distinguished points. As M is smooth at 0 we may take as a representative of M a polydisc at the origin in $\mathbb{C}^n$.

V.1.1. Local Torelli theorem. Let M be a simply connected representative of the universal deformation space of X and consider the universal family $\pi : \mathfrak{X} \longrightarrow M$. Then π is a C^∞-trivial fiber bundle by Ehresmann's theorem. We choose once and for all a trivialization

$$\begin{tikzcd} X \times M \arrow[rr, "\alpha", "C^\infty \cong"'] \arrow[dr] & & \mathfrak{X} \arrow[dl, "\pi"] \\ & M & \end{tikzcd} \tag{V.1.1}$$

and a relative symplectic form $\omega \in R^0\pi_*\Omega^2_{\mathfrak{X}/M}$. We put $X_t := \pi^{-1}(t)$ and write $\omega_t := \omega|_{X_t}$ for the symplectic form on X_t. The restriction of α to the fiber over $t \in M$ is a diffeomorphism $\alpha_t : X \longrightarrow X_t$. Consider the *period map*

$$\mathcal{P} : M \longrightarrow \mathbb{P}\left(H^2(X, \mathbb{C})\right), \quad t \mapsto [\alpha_t^*\omega_t] \tag{V.1.2}$$

Beauville showed in [**Bea83**, Thm 5] that a local Torelli theorem holds for irreducible symplectic manifolds. This means that locally at 0 the period map identifies M with its period domain

$$Q_X := \left\{[v] \in \mathbb{P}(H^2(X, \mathbb{C})) : q_X(v, v) = 0, q_X(v, \bar{v}) > 0\right\} \tag{V.1.3}$$

where q_X is the Beauville-Bogomolov form on $H^2(X, \mathbb{C})$, see section IV.3.

V.2. Subspaces of M

V.2.1. Hodge bundles and the Gauß-Manin connection. The material presented here is taken from [**VoiI**, Ch 5.1.2], where we also refer to for proofs. We use the notation of section V.1 and define the vector bundle $\mathscr{H}^k$ on M as

$$\mathscr{H}^k := R^k\pi_*\underline{\mathbb{C}}_{\mathfrak{X}} \otimes \mathcal{O}_M.$$

We obtain a filtration by subbundles $\mathscr{F}^p\mathscr{H}^k$ of $\mathscr{H}^k$ by applying the isomorphism

$$R^k\pi_*\Omega^\bullet_{\mathfrak{X}/M} \cong R^k\pi_*\underline{\mathbb{C}}_{\mathfrak{X}} \otimes \mathcal{O}_M$$

to the subbundles

$$\operatorname{im}\left(R^k\pi_*\Omega^{\geq p}_{\mathfrak{X}/M} \longrightarrow R^k\pi_*\Omega^\bullet_{\mathfrak{X}/M}\right)$$

of $R^k\pi_*\Omega^\bullet_{\mathfrak{X}/M}$. The fiber of $\mathscr{F}^p\mathscr{H}^k$ at $t \in M$ is canonically identified with the p-term $F^pH^k(X_t)$ of the Hodge filtration on $H^k(X_t)$. We define the bundles

$$\mathscr{H}^{p,q} := \mathscr{F}^p\mathscr{H}^{p+q}/\mathscr{F}^{p+1}\mathscr{H}^{p+q}$$

The fiber of $\mathscr{H}^{p,q}$ at $t \in M$ is canonically identified with $H^q(X_t, \Omega^p_{X_t})$. The bundle $\mathscr{H}^k$ as well as the bundles $\mathscr{H}^{p,q}$ are sometimes called *Hodge bundles.* There is a local system $\mathscr{H}^k_{\mathbb{C}} := R^k\pi_*\underline{\mathbb{C}}_{\mathfrak{X}} \hookrightarrow \mathscr{H}^k$ and the associated flat connection

$$\nabla : \mathscr{H}^k \longrightarrow \mathscr{H}^k \otimes \Omega_M$$

is called the *Gauß-Manin* connection. It can be calculated as follows. If we take local sections $s_1, \dots, s_m \in \mathscr{H}_{\mathbb{C}}$ which trivialize $\mathscr{H}^k$ over some open subset, any section $s \in \mathscr{H}$ can be written as a sum $\sum_i f_i s_i$ with local sections $f_i \in \mathcal{O}_M$. Then by definition we have $\nabla(s) = \sum_i s_i \otimes df_i$ where d is just the ordinary differential $d : \mathcal{O}_M \longrightarrow \Omega_M$. In particular, the sheaf $\mathscr{H}_{\mathbb{C}}$ is exactly the sheaf of flat sections of $\mathscr{H}$. The Gauß-Manin connection fulfills the so-called *Griffiths transversality*

$$\nabla\left(\mathscr{F}^p \mathscr{H}^k\right) \subseteq \mathscr{F}^{p-1}\mathscr{H}^k \otimes \Omega_M.$$

Therefore, it induces morphisms $\bar{\nabla}_p : \mathrm{Gr}^p_{\mathscr{F}} \mathscr{H}^k \longrightarrow \mathrm{Gr}^{p-1}_{\mathscr{F}} \mathscr{H}^k \otimes \Omega_M$ between the graded objects of the filtration. This map is $\mathcal{O}_M$-linear and therefore corresponds to a map $\bar{\nabla}_p : \mathrm{Gr}^p_{\mathscr{F}} \mathscr{H}^k \longrightarrow \mathrm{Hom}(T_M, \mathrm{Gr}^{p-1}_{\mathscr{F}} \mathscr{H}^k)$. By a theorem of Griffiths its fiber at the point $t \in M$ can be identified with the map

$$H^{k-p}(X_t, \Omega^p_{X_t}) \longrightarrow \mathrm{Hom}\left(H^1(X_t, T_{X_t}), H^{k-p-1}(X_t, \Omega^p_{X_t})\right) \tag{V.2.1}$$

given by cup-product and contraction.

V.2.2. Hodge loci. Let $\beta \in H^k(X, \mathbb{C})$ be a cohomology class of type (p, q) with respect to the Hodge decomposition $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$. Suppose that M is simply connected. Then the local system $\mathscr{H}^k_{\mathbb{C}}$ from section V.2.1 is trivial and β extends to a global section of $\mathscr{H}^k_{\mathbb{C}}$ also denoted by β. We interpret β as a flat section of $\mathscr{H}^k$ and write β_t for its fiber at t. The following definition and some basic properties can be found in [**VoiII**, Ch 5.3].

Definition V.2.3. The *Hodge locus* associated to β is the complex subspace $M_\beta \hookrightarrow M$ defined by the vanishing of the induced section

$$\bar{\beta} : \mathcal{O}_M \longrightarrow \mathscr{H}^k \longrightarrow \mathscr{H}^k / \mathscr{F}^p \mathscr{H}^k.$$

So the Hodge locus M_β is the locus of all $t \in M$, where $\beta_t \in F^p H^k(X_t)$. If β is an integral or at least real cohomology class of Hodge type (p, p), then

$$M_\beta = \{t \in M \mid \beta_t \in H^{p,p}(X_t)\} \tag{V.2.2}$$

as β is fixed under complex conjugation and $F^p H^{2p}(X_t) \cap \overline{F^p H^{2p}(X_t)} = H^{p,p}(X_t)$.

V.2.4. Subspaces of M associated to Lagrangian subvarieties. Let $i : Y \hookrightarrow X$ be the inclusion of a Lagrangian subvariety in an irreducible symplectic manifold X of dimension $2n$. Let M be a simply connected representative of the universal deformation space of X, let $0 \in M$ be the point corresponding to X and let $\pi : \mathfrak{X} \longrightarrow M$ be the universal family. Following Voisin [**Voi92**], we define three subspaces of M associated to Y.

We denote by $\nu : \widetilde{Y} \longrightarrow Y$ a resolution of singularities and by $j = i \circ \nu$ the composition. We take a relative symplectic form $\omega \in R^0\pi_*\Omega^2_{\mathfrak{X}/M}$ and write $\omega_t := \omega|_{X_t}$ for the symplectic form on the fiber $X_t = \pi^{-1}(t)$. For the C^∞-trivialization α of the universal family from (V.1.1) we put $j_t = \alpha_t \circ j$.

Definition V.2.5. We define

$$M'_Y := \left\{ t \in M \mid j_t^* \omega_t = 0 \text{ in } H^2(\widetilde{Y}, \mathbb{C}) \right\}. \tag{V.2.3}$$

The Lagrangrian property of Y means $j_0^*\omega_0 = 0$.

If $[Y] \in H^{2n}(X, \mathbb{Z})$ denotes the Poincaré dual of the fundamental cycle of Y, we write μ_0 for the map $H^2(X, \mathbb{C}) \longrightarrow H^{2+2n}(X, \mathbb{C})$ given by cup product with $[Y]$. This map is a morphism of Hodge structures and can be factored as

$$\mu_0 : H^2(X, \mathbb{C}) \xrightarrow{j^*} H^2(\widetilde{Y}, \mathbb{C}) \xrightarrow{j_*} H^{2+2n}(X, \mathbb{C}).$$

Let us lift $[Y]$ to a flat section of $\mathscr{H}^2$. Then μ_0 can be extended to a map $\mu : \mathscr{H}^2 \longrightarrow \mathscr{H}^{2+2n}$. Interpreting the relative symplectic form ω as a section $\omega : \mathcal{O}_M \longrightarrow \mathscr{H}^2$ we give the following definition.

Definition V.2.6. We put $M'_{[Y]} := V(\mu \circ \omega)$. In other words,

$$M'_{[Y]} = \{ t \in M \mid \mu(\omega)_t = 0 \} = \{ t \in M \mid [Y]_t \cup \omega_t = 0 \}. \tag{V.2.4}$$

By the Lagrangian property we have $\mu_0(\omega_0) = 0$, so $0 \in M'_{[Y]}$.
Finally, we denote by $M_{[Y]}$ the Hodge locus associated to the class $[Y]$ of Y in $H^{2n}(X, \mathbb{C})$, see section V.2.2. As $[Y]$ is integral and of type (n, n), its Hodge locus is set-theoretically given by

$$M_{[Y]} = \{ t \in M \mid [Y]_t \in H^{n,n}(X_t) \}, \tag{V.2.5}$$

where as above $[Y]_t$ is the restriction to the fiber over t of the unique flat section of $\mathscr{H}^{2n}$ extending $[Y]$. In particular, we have $0 \in M_{[Y]}$.

Remark V.2.7. Observe that the spaces M'_Y, $M'_{[Y]}$ and $M_{[Y]}$ may be defined for arbitrary subvarieties $Y \hookrightarrow X$. Singularities do not cause any harm, as $M'_{[Y]}$ and $M_{[Y]}$ only depend on the class $[Y]$ and M'_Y is defined via a resolution of singularities. As we are only interested in the germs at 0 of these subspaces, we may and will assume that M'_Y, $M'_{[Y]}$ and $M_{[Y]}$ are connected.
Let us collect some simple observations on the relation among the spaces M'_Y, $M'_{[Y]}$ and $M_{[Y]}$. As $\mu = j^* j_*$ we have $M'_Y \subseteq M'_{[Y]}$. If $Y = \cup_i Y_i$ is a decomposition into irreducible components, then $M'_Y = \cap_i M'_{Y_i}$ as a direct consequence of the definitions. Moreover, the inclusions $M'_{[Y]} \supseteq \cap_i M'_{[Y_i]}$ and $M_{[Y]} \supseteq \cap_i M_{[Y_i]}$ are immediate.

V.2.8. Line bundles and Hodge loci of divisors. The subspaces of M defined here will be used in applications of Chapter VII. Let L be a line bundle on X. By [**Huy99**, 1.14], there is a universal deformation space M_L for deformations of the pair (X, L). Thus there is a family $\pi_L : \mathcal{X} \longrightarrow M_L$ and a line bundle $\mathcal{L}$ on the total space $\mathcal{X}$ which have the universal property for deformations of (X, L). If L is non-trivial, M_L is a smooth hypersurface in M. This is a consequence of $H^1(X, \mathcal{O}_X) = 0$, see [**Huy99**].
When we identify M with the quadric Q_X from (V.1.3), then M_L is given by the equation $q_X(\mathrm{c}_1(L), \cdot) = 0$ in $Q_X \subseteq \mathbb{P}\left(H^2(X, \mathbb{C})\right)$ as explained in [**GHJ03**, Lem 26.3]. More generally, if $\beta \in H^2(X, \mathbb{R}) \subseteq H^2(X, \mathbb{C})$ is a class of type $(1,1)$ with respect to the Hodge decomposition $H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$, then under this identification the Hodge locus $M_\beta \hookrightarrow M$ parametrizing deformations of X, where β remains of type $(1,1)$ is given by the equation $q_X(\beta, \cdot) = 0$ in $Q_X \subseteq \mathbb{P}\left(H^2(X, \mathbb{C})\right)$.

V.3. Voisin's results adapted

Essentially everything in this section is taken from [**Voi92**], but with some slight modifications to our situation. So unless the contrary is explicitly stated, all results presented are Voisin's. We will freely use the notations of section V.2.

PROPOSITION V.3.1. $M_{[Y]} = {M'}_{[Y]}$ as sets.

PROOF. We first show ${M'}_{[Y]} \subseteq M_{[Y]}$. For $t \in M'_{[Y]}$ we write $[Y]_t = \sum_{p+q=2n} [Y]_t^{p,q}$ with respect to the Hodge decomposition at t. We want to show that $[Y]_t = [Y]_t^{n,n}$. As $[Y]$ is integral, we have $\overline{[Y]_t^{p,q}} = [Y]_t^{q,p}$ and so it suffices to show that $[Y]_t^{p,q} = 0$ for $p < n$. As ω_t is of type $(2,0)$ on X_t the assumption $\mu(\omega_t) = 0$ gives $\omega_t \cup [Y]_t^{p,q} = 0$ for all p, q. But $\omega_t^k \cup : \Omega_{X_t}^{n-k} \longrightarrow \Omega_{X_t}^{n+k}$ is an isomorphism for $k \geq 0$, which can be seen pointwise by linear algebra. Hence the map $\omega_t \cup$ is injective for $p < n$, which yields that $[Y]_t^{p,q} = 0$ for $p < n$, as needed.
For the inclusion $M_{[Y]} \subseteq {M'}_{[Y]}$ it suffices to show that $M_{[Y]} \cap {M'}_{[Y]}$ is nonempty and open in $M_{[Y]}$ as it is automatically closed and we may assume that $M_{[Y]}$ is connected, see Remark V.2.7. This is the only point where we use that Y is Lagrangian, namely for the nonemptiness. For $t \in M_{[Y]}$ the morphism $\mu : H^2(X_t, \mathbb{C}) \longrightarrow H^{2n+2}(X_t, \mathbb{C})$ is a morphism of Hodge structures of degree (n, n) and hence gives morphisms $\mu^{p,q} : \mathscr{H}^{p,q} \longrightarrow \mathscr{H}^{p+n,q+n}$ for $p + q = 2$. By semi-continuity they satisfy $\mathrm{rk}\mu^{p,q}(t') \geq \mathrm{rk}\mu^{p,q}(t)$ for all t' in a small neighborhood U of t. As $\mu = \mu^{2,0} \oplus \mu^{1,1} \oplus \mu^{0,2}$ as a C^∞-morphism on U, the rank of the summands remains constant in t. So as

for $t = 0 \in M_{[Y]} \cap M'_{[Y]}$ we have $\mu^{2,0} = 0 = \mu^{0,2}$ this remains true in a neigbourhood and so the claim follows. $\square$

Proposition V.3.2. The varieties $M_{[Y]}$ and $M'_{[Y]}$ are smooth near $t = 0$ and their codimension in M is $r_{[Y]} = \operatorname{rk}\left(\mu : H^2(X,\mathbb{C}) \longrightarrow H^{2n+2}(X,\mathbb{C})\right)$. In particular, $M_{[Y]} = M'_{[Y]}$ as varieties by the preceding proposition.

Proof. We argue only for $M'_{[Y]}$, the case of $M_{[Y]}$ is similar. Consider the sheaf $\mathscr{H}_\mu := \mu(\mathscr{H}^2) \subseteq \mathscr{H}^{2n+2}$. As μ is defined on the level of local systems its rank is locally constant, so this is a vector bundle of rank $r_{[Y]}$. The variety $M'_{[Y]}$ is defined by the vanishing of the section $\mu(\omega) \in \mathscr{H}_\mu$, hence $\operatorname{codim} M'_{[Y]} \leq r_{[Y]}$. So it suffices to show that the rank of the system of equations $\mu(\omega) = 0$ is equal to $r_{[Y]}$. Recall that the Gauß-Manin connection is given by the differential d if we trivialize with flat sections. This implies that for μ to have rank $r_{[Y]}$ at 0 the $\bar{\nabla}_{\chi,0}(\mu_0(\omega_0))$ for $\chi \in T_{M,0} = H^1(X, T_X)$ have to span a vector space of dimension $r_{[Y]}$.
We have $\nabla_\chi(\mu(\omega_t)) = \mu(\nabla_\chi \omega_t)$ and by (V.2.1) the Gauß-Manin connection $\bar{\nabla} : \mathscr{F}^2\mathscr{H}^2 \longrightarrow \operatorname{Hom}(T_M, \mathscr{F}^1\mathscr{H}^2/\mathscr{F}^2\mathscr{H}^2)$ at t is identified with the morphism

$$H^0(\Omega^2_{X_t}) \longrightarrow \operatorname{Hom}\left(H^1(T_{X_t}), H^1(\Omega_{X_t})\right)$$

given by the cup product and contraction. As ω_0 is non-degenerate and of type $(2,0)$ the $\nabla_\chi \omega_t$ span the whole of $H^{1,1}(X)$ at $t = 0$. $\square$

Lemma V.3.3. The tangent space of M'_Y at 0 is given by
(V.3.1)

$$T_{M'_Y,0} = \ker\left(j^* \circ \omega' : H^1(X, T_X) \xrightarrow{\omega'} H^1(X, \Omega_X) \xrightarrow{j^*} H^1(\widetilde{Y}, \Omega_{\widetilde{Y}})\right)$$

where ω' is the isomorphism induced by the symplectic form on X.

Proof. Locally at $0 \in M$ the space M'_Y is cut out by the equation $j_t^*\omega_t = 0$. Therefore the tangent space at 0 is given by the equation

$$0 = \left(\nabla j_t^* \omega_t\right)|_{t=0} = j^*\left(\nabla \omega_t\right)|_{t=0}.$$

The Gauß-Manin conection at 0 can be identified with the map

$$H^0(X, \Omega^2_X) \longrightarrow \operatorname{Hom}(H^1(X, T_X), H^1(X, \Omega_X)), \quad \psi \mapsto (u \mapsto \psi(u))$$

given by cup product and contraction, which concludes the proof. $\square$

Lemma V.3.4. Let X be an irreducible symplectic manifold of dimension $\dim X = 2n$. Let $Y \subseteq X$ be an irreducible Lagrangian subvariety, let $\nu : \widetilde{Y} \longrightarrow Y$ a resolution of singularities and put $j = i \circ \nu$. If $n \geq 2$, assume that there is a Kähler class $\kappa \in H^2(X,\mathbb{R})$ such that $j^*\kappa$ is a Kähler form on $\widetilde{Y}$. Then

$$\ker\left(\mu : H^2(X,\mathbb{C}) \longrightarrow H^{2n+2}(X,\mathbb{C})\right) = \ker\left(j^* : H^2(X,\mathbb{C}) \longrightarrow H^2(\widetilde{Y},\mathbb{C})\right).$$

PROOF. We show equality of the respective kernels with real coefficients. From $\mu = j_*j^*$ we immediately have $\ker j^* \subseteq \ker \mu$. For the other inclusion we choose a Kähler class $\kappa \in H^2(X,\mathbb{R})$. We have to show that j_* is injective on $\operatorname{im} j^*$.

Assume $n = 1$. As $\widetilde{Y}$ is connected, $H^2(\widetilde{Y},\mathbb{C}) \cong \mathbb{C}$ and the map $j_* : H^2(\widetilde{Y},\mathbb{C}) \longrightarrow H^2(X,\mathbb{C})$ is given by $1 \mapsto [Y]$. As X is Kähler, $[Y] \neq 0$. So j_* is injective and the claim follows.

If $n \geq 2$ and define a bilinear form

$$q_\kappa(\alpha,\beta) := \int_{\widetilde{Y}} (j^*\kappa^{n-2}).\alpha.\beta \qquad \alpha,\beta \in H^2(\widetilde{Y},\mathbb{C})$$

on $H^2(\widetilde{Y},\mathbb{C})$. For $\alpha,\beta \in H^2(X,\mathbb{R})$ this gives

$$q_\kappa(j^*\alpha, j^*\beta) = \int_{\widetilde{Y}} j^*(\kappa^{n-2}.\alpha.\beta) = \int_X j_*j^*(\kappa^{n-2}.\alpha.\beta) = \int_X \kappa^{n-2}.\mu(\alpha).\beta.$$

So if $\mu(\alpha) = 0$, then $q_\kappa(j^*\alpha, j^*\beta) = 0$ for all $\beta \in H^2(X,\mathbb{R})$. To conclude that $j^*\alpha = 0$ it would be sufficient to see that q_κ is non-degenerate on $\operatorname{im} j^* \subseteq H^2(\widetilde{Y},\mathbb{R})$. On the whole of $H^2(\widetilde{Y},\mathbb{R})$ the form q_κ is non-degenerate by the Hodge index theorem, see [**VoiI**, Thm 6.33]. Here we need that $j^*\kappa$ is a Kähler class. That q_κ remains non-degenerate on the subspace $\operatorname{im} j^*$ can also be deduced as follows. As we have seen $\operatorname{im} j^* \subseteq H^{1,1}(\widetilde{Y},\mathbb{R}) := H^{1,1}(\widetilde{Y}) \cap H^2(\widetilde{Y},\mathbb{R})$ and on $H^{1,1}(\widetilde{Y},\mathbb{R})$ the form q_κ is non degenerate and has signature $(1, h^{1,1}-1)$. We know that $q_\kappa(j^*\kappa, j^*\kappa) > 0$ and so q_κ is negative definite on $j^*\kappa^\perp$. Write $j^*\alpha = c \cdot j^*\kappa + \alpha'$ where $\alpha' \in j^*\kappa^\perp$. The decomposition shows that $\alpha' \in \operatorname{im} j^*$ as well. Then if $j^*\alpha \neq 0$ at least one of the numbers $q_\kappa(j^*\alpha, j^*\kappa)$, $q_\kappa(j^*\alpha,\alpha')$ is nonzero and so $\mu(\alpha) \neq 0$ completing the proof. □

In [**Voi92**] the condition that $j^*\kappa$ be a Kähler class is automatic, as Y is a smooth submanifold there. We show that the condition is fulfilled in the following cases.

LEMMA V.3.5. Let X be an irreducible symplectic manifold, let $Y \subseteq X$ be an irreducible Lagrangian subvariety, let $\nu : \widetilde{Y} \longrightarrow Y$ a the normalization and put $j = i \circ \nu$. Assume that one of the following holds.

(1) Y has normal crossing singularities.
(2) X is projective and $\widetilde{Y}$ is smooth.

Then there is a Kähler class κ on X such that $j^*\kappa$ is a Kähler class on $\widetilde{Y}$.

Proof. Assume that Y has normal crossings. If κ is a Kähler class represented by a positive $(1,1)$-form ψ, then $j^*\psi$ is also positive. Indeed, analytically locally in $\widetilde{Y}$ the map j is a closed embedding and positivity of forms is a local property. The class of $j^*\psi$ is $j^*\kappa$ and thus positive.
If X is projective and the normalization $\widetilde{Y}$ of Y is smooth, then we can take the first Chern class of any ample line bundle for κ, as the pullback of an ample line bundle along a finite morphism is ample. □

Corollary V.3.6. Let X be an irreducible symplectic manifold, let $Y \subseteq X$ be an irreducible Lagrangian subvariety, let $\nu : \widetilde{Y} \longrightarrow Y$ the normalization and put $j = i \circ \nu$. We have

$$\ker\left(\mu : H^2(X,\mathbb{C}) \longrightarrow H^{2n+2}(X,\mathbb{C})\right) = \ker\left(j^* : H^2(X,\mathbb{C}) \longrightarrow H^2(\widetilde{Y},\mathbb{C})\right)$$

in each of the following cases:

(1) $\dim X = 2$.
(2) Y has normal crossing singularities.
(3) X is projective and $\widetilde{Y}$ is smooth.

Proof. The case $\dim X = 2$ is contained in Lemma V.3.4 and in the other cases Lemma V.3.5 guarantees that Lemma V.3.4 can be applied. □

So in these cases we can imitate [**Voi92**, Prop 1.7].

Proposition V.3.7. Let $Y \subseteq X$ be an irreducible Lagrangian subvariety, let $\widetilde{Y} \longrightarrow Y$ be the normalization and assume that one of the following cases holds:

(1) $\dim X = 2$.
(2) Y has normal crossing singularities.
(3) X is projective and $\widetilde{Y}$ is smooth.

Then we have $M'_{[Y]} = M'_Y$. In particular, M'_Y is smooth at 0.

Proof. We observed that $M'_Y \subseteq M'_{[Y]}$ in Remark V.2.7. As $M'_{[Y]}$ is smooth by Proposition V.3.2 it suffices to show that $M'_Y \supseteq M'_{[Y]}$ holds set-theoretically. By definition $t \in M'_{[Y]}$ if $\omega_t \cup [Y]_t = 0$ and $t \in M'_Y$ if $j_t^*\omega_t = 0$. But $\omega_t \cup [Y]_t = 0$ if and only if $j_t^*\omega_t = 0$ by Lemma V.3.4. □

CHAPTER VI

Deformations of Lagrangian subvarieties

Let X be an irreducible symplectic manifold and let $i : Y \hookrightarrow X$ be the inclusion of a Lagrangian subvariety. In this chapter we construct a universal deformation space M_i for locally trivial deformations of i as an easy application of results of Flenner and Kosarew [**FK87**]. It comes with a canonical map $p : M_i \longrightarrow M$, where M is the universal deformation space of X. If Y has simple normal crossings, we prove smoothness of M_i in Theorem VI.3.12. Moreover, we show that the image M_Y of p is well-defined and smooth in Theorem VI.4.3. The difficulty is that these objects are germs of complex spaces and taking the image does in general depend on the chosen representative.

The proofs of our smoothness results are elaborations of Ran's ideas [**Ran92b**], [**Ran92a**] and the method is related to the T^1-lifting principle. Here, all theory of the preceding chapters is put together. The main point is to show that the cohomology sheaves $T^1_{I/R}$ for a deformation $I : \mathcal{Y} \hookrightarrow \mathcal{X}$ of i over $R \in \mathrm{Art}_{\mathbb{C}}$ are free R-modules and compatible with base change, see Lemma I.3.4. The strategy is to link them to other modules via the exact sequence (VI.3.5). These modules can be related to Hodge theory using the symplectic form.

These smoothness results play an important role in the proof of our main result, Theorem VI.5.3. It generalizes Voisin's theorem [**Voi92**] to simple normal crossing Lagrangian subvarieties. The other ingredients in the proof are essentially the same as Voisin's.

One can find similar or related smoothness statements in the literature, for which full proofs never seemed to be written down, or this has happened only more or less explicitly in proves of other statements. As those facts are freely used in the literature, it might be helpful to have an explicit reference. So we use the opportunity to collect other smoothness results in section VI.6 with proofs derived from our main results.

VI.1. Projectivity of Lagrangian subvarieties

If $Y \subseteq X$ is a smooth Lagrangian subvariety, then Y is projective by Proposition IV.1.6. If $Y \subseteq X$ is a singular Lagrangian subvariety, it is natural to ask whether Y is still projective. Later on we want to apply the results of the preceding sections, which involve mixed Hodge structures on the cohomology of Y, to Lagrangian subvarieties in symplectic manifolds. Therefore we at least need to know that Y is an algebraic variety or more precisely, that $Y = \mathfrak{Y}^{\mathrm{an}}$ for an algebraic variety $\mathfrak{Y}$. We have

Lemma VI.1.1. Let $i : Y \hookrightarrow X$ be a complex Lagrangian subvariety in an irreducible symplectic manifold and let $\nu : \widetilde{Y} \longrightarrow Y$ be the normalization. There is a line bundle L on Y such that $\mathrm{c}_1(\nu^* L) = \nu^* i^* \lambda$ for some Kähler class λ on X. If moreover Y has normal crossings, then $\nu^* L$ is ample and $\widetilde{Y}$ is projective. In particular, $\widetilde{Y}$ is a projective algebraic variety.

Proof. Isomorphism classes of line bundles on Y are classified by the group $H^1(Y, \mathcal{O}_Y^\times)$, see [**GR77**, Kap V, § 3.2]. This cohomology group appears in the commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & H^1(Y,\mathcal{O}_Y^\times) \arrow[r] & H^2(Y,\mathbb{Z}) \arrow[r] \arrow[d] & H^2(Y,\mathcal{O}_Y) \arrow[r] & \dots \\
& & H^2(Y,\mathbb{C}) \arrow[r] & \mathbb{H}^2(Y,\widetilde{\Omega}_Y^\bullet) \arrow[u] & \\
& & & \mathbb{H}^2(Y,\widetilde{\Omega}_Y^{\geq 1}) \arrow[u] &
\end{tikzcd}$$

where the first line is the long exact sequence associated to the exponential sequence, see [**GR77**, Kap V, § 2.4], and the right vertical column comes from the short exact sequence

$$0 \longrightarrow \widetilde{\Omega}_Y^{\geq 1} \longrightarrow \widetilde{\Omega}_Y^\bullet \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Here we need that $\widetilde{\Omega}_Y^0 = \mathcal{O}_Y$. This is true, as Y is reduced, because then Y does not have embedded points. To obtain a holomorphic line bundle L on Y it is sufficient to find a class $\alpha \in H^2(Y, \mathbb{Z})$, such that the image in $\mathbb{H}^2(Y, \widetilde{\Omega}_Y^\bullet)$ comes from $\mathbb{H}^2(Y, \widetilde{\Omega}_Y^{\geq 1})$. Such L will have $\mathrm{c}_1(L) = \alpha$.
Let $H_X := \mathrm{im}(i^* : \mathbb{H}^2(X, \Omega_X^\bullet) \longrightarrow \mathbb{H}^2(Y, \widetilde{\Omega}_Y^\bullet))$ where $i : Y \hookrightarrow X$ is the inclusion. From the spectral sequence for $\Omega^\bullet$ we obtain maps

$$\begin{tikzcd}
H^0(X,\Omega_X^2) \arrow[r] \arrow[d] & \mathbb{H}^2(X,\Omega_X^\bullet) \arrow[d, "i^*"] & H^2(X,\mathbb{C}) \arrow[l, "\cong"'] \arrow[d] \\
H^0(Y,\widetilde{\Omega}_Y^2) \arrow[r] & \mathbb{H}^2(Y,\widetilde{\Omega}_Y^\bullet) & H^2(Y,\mathbb{C}) \arrow[l]
\end{tikzcd}$$

As Y is Lagrangian and by definition $\widetilde{\Omega}^2_Y$ is torsion free we have $i^*\omega = 0$ in $H^0(Y, \widetilde{\Omega}^2_Y)$ where $\omega \in H^0(X, \Omega^2_X)$ is the symplectic form on X. By Hodge-decomposition $\mathbb{H}^2(X, \Omega^\bullet_X) \cong H^2(X, \mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$ and by Dolbeault's theorem $H^0(X, \Omega^2_X) \cong H^{2,0}(X)$ we see that $H^{2,0}(X) \cong \mathbb{C}\omega$ maps to zero under i^*. From the left square of the above diagram, we see that also the complex conjugate $H^{0,2}(X) \cong \mathbb{C}\overline{\omega}$ maps to zero, as the map $H^2(X, \mathbb{C}) \longrightarrow H^2(Y, \mathbb{C})$ is defined over $\mathbb{R}$. Thus

$$\begin{aligned} H_X &= \operatorname{im}(i^* : H^2(X, \mathbb{C}) \longrightarrow \mathbb{H}^2(Y, \widetilde{\Omega}^\bullet_Y)) \\ &= \operatorname{im}(i^* : H^{1,1}(X) \longrightarrow \mathbb{H}^2(Y, \widetilde{\Omega}^\bullet_Y)). \end{aligned} \tag{VI.1.1}$$

Let $H_{X,\mathbb{R}} = \operatorname{im}(i^* : H^2(X, \mathbb{R}) \longrightarrow \mathbb{H}^2(Y, \widetilde{\Omega}^\bullet_Y))$. The last description in (VI.1.1) implies that $i^*(\mathcal{K}_X)$ is open in $H_{X,\mathbb{R}}$ where $\mathcal{K}_X$ is the Kähler cone of X. Indeed, $\mathcal{K}_X$ is open in $H^{1,1}(X)_\mathbb{R} = H^{1,1}(X) \cap H^2(X, \mathbb{R})$ and the map $H^{1,1}(X) \longrightarrow H_X$ is surjective. Therefore also $H^{1,1}(X)_\mathbb{R} \longrightarrow H_{X,\mathbb{R}}$ is surjective and so that $i^*(\mathcal{K}_X)$ is open in $H_{X,\mathbb{R}}$. We show next that $i^*(\mathcal{K}_X)$ meets the image of $H^2(Y, \mathbb{Z})$. Let us consider

$$H_{X,\mathbb{Q}} = \operatorname{im}(i^* : H^2(X, \mathbb{Q}) \longrightarrow \mathbb{H}^2(Y, \widetilde{\Omega}^\bullet_Y)) \subseteq H_X.$$

This is dense in $H_{X,\mathbb{R}}$ as $H^2(X, \mathbb{Q})$ is dense in $H^2(X, \mathbb{R})$ and so it meets $i^*(\mathcal{K}_X)$, say in $\alpha' \in H_{X,\mathbb{Q}} \cap i^*(\mathcal{K}_X)$. Then a multiple $\alpha = m \cdot \alpha'$ is contained in $\operatorname{im}(H^2(X, \mathbb{Z}) \longrightarrow \mathbb{H}^2(Y, \widetilde{\Omega}^\bullet_Y)) \cap i^*\mathcal{K}_X$ and we obtain a line bundle L on Y with the desired property by using the exponential sequence as explained above.

Now, suppose that Y has normal crossings. Then we have $\mathrm{c}_1(\nu^*L) = \nu^*\alpha$, where we define $\nu^* : \mathbb{H}^2(Y, \widetilde{\Omega}^\bullet_Y) \longrightarrow H^2(\widetilde{Y}, \mathbb{C})$ as the composition of the natural map $\mathbb{H}^2(Y, \widetilde{\Omega}^\bullet_Y) \longrightarrow \mathbb{H}^2(\widetilde{Y}, \Omega^\bullet_{\widetilde{Y}})$ with the inverse of the isomorphism $H^2(\widetilde{Y}, \mathbb{C}) \longrightarrow \mathbb{H}^2(\widetilde{Y}, \Omega^\bullet_{\widetilde{Y}})$. To show that ν^*L is ample, we will show that the class $\nu^*\alpha \in H^2(\widetilde{Y}, \mathbb{C})$ is represented by a positive $(1,1)$-form. Indeed, classes in $\mathcal{K}_X$ may be represented by $(1,1)$-forms, whose restrictions to every submanifold are positive. Since $\alpha \in i^*\mathcal{K}_X$, there is such a positive $(1,1)$-form ψ on X, whose cohomology class $[\psi] \in H^2(X, \mathbb{C})$ restricts to α on Y. Analytically locally in $\widetilde{Y}$, the composition $i \circ \nu : \widetilde{Y} \longrightarrow X$ is a closed immersion as Y has normal crossings. Thus, $\nu^*\psi$ is positive, as positivity is a local property. Pullback of differential forms is compatible with taking cohomology classes, so $\nu^*\alpha$ is represented by the positive form $\nu^*\psi$. This concludes the proof via Kodaira's embedding theorem and Chow's theorem. $\square$

PROPOSITION VI.1.2. If $Y \subseteq X$ is a complex Lagrangian simple normal crossing subvariety in a symplectic manifold, then Y is a projective algebraic variety.

PROOF. As the normalization $\widetilde{Y}$ is a projective algebraic variety by Lemma VI.1.1, so is every component Y_i of Y. If we knew that Y itself were an algebraic variety, we would be done, because an algebraic variety is projective, if all its irreducible components are. By induction on the number of irreducible components, we may write $Y = Y_1 \cup Y_2$ where Y_1 is irreducible and both Y_i are projective algebraic varieties. In particular $\Sigma = Y_1 \cap Y_2$ is a projective algebraic variety. Then the diagram

$$\begin{tikzcd}
\Sigma \arrow[r] \arrow[d] & Y_1 \arrow[d] \\
Y_2 \arrow[r] & Y
\end{tikzcd}$$

is cocartesian in the category of complex spaces. This is clear for the underlying topological space and is shown by an explicit computation using local descriptions $Y_2 \cong \{x_1 \cdot \ldots \cdot x_k = 0\}$, $Y_1 \cong \{x_{k+1} = 0\}$.
In detail it works like this. Suppose $n = \dim Y$ and Y is locally isomorphic to $x_1 \ldots x_{k+1} = 0$ in a small polydisc in $\Delta \subseteq \mathbb{C}^{n+1}$ with coordinates $x_1, \ldots, x_{n+1}$. Let $B = \mathcal{O}_{\mathbb{C}^{n+1}}(\Delta)$. We have to show that the diagram

$$\begin{tikzcd}
B/(x_1 \ldots x_k, x_{k+1}) & B/(x_{k+1}) \arrow[l] \\
B/(x_1 \ldots x_k) \arrow[u] & B/(x_1 \ldots x_{k+1}) \arrow[l] \arrow[u]
\end{tikzcd}$$

is cartesian. Therefore it suffices to show that given $f, g \in B$ representing $[f] \in B/(x_1 \ldots x_k)$ and $[g] \in B/(x_{k+1})$ with $[f] = [g]$ modulo $(x_1 \ldots x_k, x_{k+1})$ there is $h \in B$ unique up to $(x_1 \ldots x_{k+1})$ mapping to $[f]$ respectively $[g]$. By the condition on f and g there are $\alpha, \beta \in B$ with $f - g = -\alpha x_1 \ldots x_k + \beta x_{k+1}$. Then $h = f + \alpha x_1 \ldots x_k = g + \beta x_{k+1}$ maps to $[f]$ respectively $[g]$. For uniqueness assume we find $h' \in B$ mapping to $[f], [g]$, then $h' - h = x_1 \ldots x_k \gamma = x_{k+1} \delta$ for some $\gamma, \delta \in B$. Thus $x_1 \ldots x_{k+1}$ divides $h' - h$ so that $[h] = [h']$ in $B/(x_1 \ldots x_{k+1})$.
Hence by [**Fer03**, Scolie 4.3] the above diagram is also cocartesian in the category of ringed spaces. Thus Y is an algebraic variety by [**Fer03**, Thm 5.4] and projective by [**Har70**, Ch 1, Prop 4.3, 4.4]. $\square$

VI.2. Deformation theory on symplectic manifolds

Suppose $g : \mathcal{X} \longrightarrow S$ is a deformation of an irreducible symplectic manifold X over $S = \operatorname{Spec} R$ for $R \in \mathrm{Art}_k$. The symplectic form ω_0 on X extends to a section $\omega \in R^0 g_* \Omega^2_{\mathcal{X}/S}$, as this module is free. For the same reason all such extensions differ by a factor in $R^\times$. As ω is nondegenerate on the central fiber and R is local, it is non-degenerate everywhere.

Lemma VI.2.1. Let $i : Y \hookrightarrow X$ be a simple normal crossing Lagrangian subvariety. If $I : \mathcal{Y} \hookrightarrow \mathcal{X}$ is a locally trivial deformation of i over S, then $\mathcal{Y}$ is Lagrangian with respect to the symplectic form ω on $\mathcal{X}$.

Proof. Let $\tilde{f} : \widetilde{\mathcal{Y}} \longrightarrow S$ be the locally trivial deformation of the normalization of Y obtained from Lemma III.1.5. Note, that Y is projective by Proposition VI.1.2, so Lemma III.1.5 can be applied. As Y has simple normal crossings, $f \circ \nu : \widetilde{\mathcal{Y}} \longrightarrow S$ is smooth and for $j = i \circ \nu$ the restriction $j^* : R^0 g_* \Omega^2_{\mathcal{X}/S} \longrightarrow R^0 f_* \Omega^2_{\widetilde{\mathcal{Y}}/S}$ has constant rank by Proposition III.3.4. As $\mathrm{rk}(j^* \otimes \mathbb{C}) = 0$ on the central fiber, j^* is identically zero and thus $\mathcal{Y}$ is Lagrangian. $\square$

Lemma VI.2.2. Let $i : Y \hookrightarrow X$ be a locally complete intersection Lagrangian subvariety in an irreducible symplectic manifold X, let $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_{\mathbb{C}}$ and let

(VI.2.1)
$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "f"'] & \mathcal{X} \arrow[d, "g"] \\ & S \end{tikzcd}$$

be a locally trivial deformation of i over S. Then the symplectic form $\omega \in R^0 g_* \Omega^2_{\mathcal{X}/S}$ induces a morphism between the exact sequences from (I.3.3) to (I.3.4).

(VI.2.2)
$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}/\mathcal{I}^2 \arrow[r] \arrow[d, dashed, "\omega^{-1}"] & \Omega_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}} \arrow[r] \arrow[d, "\omega^{-1}"] & \Omega_{\mathcal{Y}/S} \arrow[r] \arrow[d, dashed, "\omega'"] & 0 & \\ 0 \arrow[r] & T_{\mathcal{Y}/S} \arrow[r] & T_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}} \arrow[r] & N_{\mathcal{Y}/\mathcal{X}} \arrow[r] & T^1_{\mathcal{Y}/S} \arrow[r] & 0. \end{tikzcd}$$

Proof. Since ω is non-degenerate, the map $\omega^{-1} : \Omega_{\mathcal{X}/S} \longrightarrow T_{\mathcal{X}/S}$ is an isomorphism. This will induce the other morphisms in the diagram as explained below. The composition $\varphi : \mathcal{I}/\mathcal{I}^2 \longrightarrow N_{\mathcal{Y}/\mathcal{X}} = \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathcal{Y}})$ is zero at smooth points. This follows from linear algebra and the remark after Definition IV.1.4. So $M := \operatorname{im} \varphi$ is torsion. But $\mathcal{Y}$ is a locally complete intersection, so $\mathcal{I}/\mathcal{I}^2$ is locally free and by [**Mat80**, 16, Thm 30] the submodule M is zero. So the restriction of ω^{-1} to $\mathcal{I}/\mathcal{I}^2$ factors through $T_{\mathcal{Y}/S}$. Once we have this, we obtain a morphism $\omega' : \Omega_{\mathcal{Y}/S} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}$, as the first line of (VI.2.2) is exact, by lifting sections to $\Omega_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}}$. □

Corollary VI.2.3. In the situation of the preceding lemma assume in addition that $f : \mathcal{Y} \longrightarrow S$ is smooth. Then ω gives an isomorphism $\omega' : \Omega_{\mathcal{Y}/S} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}$.

Proof. As f is smooth, $T^1_{\mathcal{Y}/S} = 0$. So (VI.2.2) gives a surjection $\omega : \Omega_{\mathcal{Y}/S} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}$. As both $\Omega_{\mathcal{Y}/S}$ and $N_{\mathcal{Y}/\mathcal{X}}$ are locally free, the claim follows. □

Note that $\mathcal{I}/\mathcal{I}^2 \longrightarrow T_{\mathcal{Y}/S}$ is not in general an isomorphism as $\Omega_{\mathcal{Y}/S} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}$ might have a kernel. The following Proposition determines this kernel.

Proposition VI.2.4. Let $i : Y \hookrightarrow X$ be a locally complete intersection Lagrangian subvariety in an irreducible symplectic manifold X, let $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_{\mathbb{C}}$ and let $I : \mathcal{Y} \hookrightarrow \mathcal{X}$ be a locally trivial deformation of i over S as in (VI.2.1). Let $\omega' : \Omega_{\mathcal{Y}/S} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}$ be as in (VI.2.2) and let $N'_{\mathcal{Y}/\mathcal{X}}$ be the equisingular normal sheaf defined in (I.3.5). Then the diagram

(VI.2.3)
$$\begin{tikzcd} \Omega_{\mathcal{Y}/S} \arrow[r, "\omega"] \arrow[d] & N_{\mathcal{Y}/\mathcal{X}} \\ \widetilde{\Omega}_{\mathcal{Y}/S} \arrow[r, dashed, "\exists\widetilde{\omega}"'] & N'_{\mathcal{Y}/\mathcal{X}} \arrow[u, hook] \end{tikzcd}$$

can be completed and $\widetilde{\omega} : \widetilde{\Omega}_{\mathcal{Y}/S} \longrightarrow N'_{\mathcal{Y}/\mathcal{X}}$ is an isomorphism. The analogue is true in the analytic setting.

Proof. As $\mathcal{Y}$ is a locally complete intersection, $N_{\mathcal{Y}/\mathcal{X}}$ is locally free, hence Cohen-Macaulay. Therefore it has no embedded primes by [**Mat80**, 16, Thm 30], hence $\tau^1_{\mathcal{Y}/S}$ maps to zero and $\widetilde{\omega}$ exists. But as ω is an isomorphism at smooth points of f, the support of $\ker\omega$ is contained in the singular locus of f, hence $\ker\omega \subseteq \tau^k_{\mathcal{Y}/S}$ and $\widetilde{\omega}$ is injective. Moreover $\widetilde{\Omega}_{\mathcal{Y}/S}$ maps onto $\ker(N_{\mathcal{Y}/\mathcal{X}} \longrightarrow T^1_{\mathcal{Y}/S})$ by (VI.2.2), hence is identified with $N'_{\mathcal{Y}/\mathcal{X}}$. All arguments are equally valid in the analytic category. □

This proposition determines the sheaf $\widetilde{\Omega}_{\mathcal{Y}/S}$ as one of the main objects in our studies. The complex $\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ calculates the cohomology with coefficients in the constant sheaf $\underline{R}_Y$ by Lemma III.2.6 and is used to put a mixed Hodge structure on the cohomology groups $H^k(Y, \underline{R}_Y)$, see section III.4.

VI.3. The space M_i and the T^1-lifting principle

VI.3.1. The space M_i. Let $i : Y \hookrightarrow X$ be the inclusion of a closed subvariety in an irreducible symplectic manifold. We explain here, why a universal deformation space M_i for locally trivial deformations of i in the sense of Definition I.5.4 exists as a germ of complex spaces. Given a deformation of a holomorphic map over a complex space S, the existence of a complex subspace of S parametrizing locally trivial deformations is due to [**FK87**], as we explain in Remark VI.3.3. So we have to pick a suitable candidate for S, obtain a subspace M_i and argue about universality.
For an irreducible symplectic manifold X there exists a universal deformation space M as explained in Chapter V. Let $\pi : \mathfrak{X} \longrightarrow M$ be the universal family and $q : \mathscr{D}(\mathfrak{X}/M) \longrightarrow M$ the relative Douady space of π constructed by Pourcin, see [**Pou69**, Thm2]. The Douady space is the complex analogue to the Hilbert scheme and parametrizes complex subspaces of $\mathfrak{X}$ relative over M. The inclusion $Y \hookrightarrow X \hookrightarrow \mathfrak{X}$ gives a point $0 \in \mathscr{D}(\mathfrak{X}/M)$.

Proposition VI.3.2. There is a complex subspace $M_i \hookrightarrow \mathscr{D}(\mathfrak{X}/M)$ containing 0, whose germ at 0 is a universal deformation space for locally trivial deformations of i. Furthermore, there is a canonical map $p : M_i \longrightarrow M$.

Proof. Let $I : \mathcal{Y} \longrightarrow \mathcal{X}$ be a locally trivial deformation of $Y \hookrightarrow X$ over a complex space S. After shrinking S we may assume that I is fiberwise a closed embedding and that there is a holomorphic classifying map $S \longrightarrow M$ so that $\mathcal{X} = \mathfrak{X} \times_M S$. Then we obtain a classifying map $S \longrightarrow \mathscr{D}(\mathfrak{X}/M)$ such that I is the pullback of $\mathscr{U} \hookrightarrow \mathfrak{X} \times_M \mathscr{D}(\mathfrak{X}/M)$, where $\mathscr{U}$ is the universal family of subspaces of $\mathfrak{X}$ over $\mathscr{D}(\mathfrak{X}/M)$. Now it follows from [**FK87**, Thm 5.3] that there is a maximal complex subspace M_i of $\mathscr{D}(\mathfrak{X}/M)$ parametrizing locally trivial deformations. The map p is simply the composition of the inclusion $M_i \hookrightarrow \mathscr{D}(\mathfrak{X}/M)$ and the projection $\mathscr{D}(\mathfrak{X}/M) \longrightarrow M$. □

Remark VI.3.3. Flenner and Kosarew show in [**FK87**] that every compact complex space X has a semi-universal locally trivial deformation $\mathcal{X}^{\mathrm{lt}} \longrightarrow M^{\mathrm{lt}}$ in the sense of space germs [**FK87**, (0.3) Cor]. The space M^{lt} is constructed as a subspace of a semi-universal deformation space M of X. If $\mathcal{X} \longrightarrow M$ is the semi-universal deformation over M, Flenner and Kosarew show that for every point $x \in X \subseteq \mathcal{X}$ there is a maximal subgerm $M_x \hookrightarrow M$, the so-called *trivial locus* for x, such that the map of germs $(\mathcal{X}, x) \longrightarrow (M_x, 0)$ is the trivial deformation [**FK87**, (0.2) Cor]. Then the subspace

$$M^{\mathrm{lt}} := \bigcap_{x \in X} M_x \tag{VI.3.1}$$

has the semi-universality property for locally trivial deformations. They remark [**FK87**, p. 630] that in the same article they also obtain similar results for deformations of other types of analytic objects and introduce the concept of *data of structure preserving maps* to have a unified treatment. In particular, this notion can be used to handle deformations of holomorphic maps as explained in [**FK87**, (1.5) Example]. If

$$\begin{tikzcd} \mathcal{Y} \arrow[rr, "F"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & M_f & \end{tikzcd}$$

is a universal or semi-universal deformation of a map $f : Y \longrightarrow X$, then [**FK87**, Thm 5.3] shows the existence of a trivial locus $M_y \hookrightarrow M_f$, that is, a subspace where $(\mathcal{Y}, y) \xrightarrow{F} (\mathcal{X}, f(y))$ is the trivial deformation of the map germ $(Y, y) \xrightarrow{f} (X, f(y))$. This theorem holds under the condition that the relative T^1 has finite rank, which is fulfilled for proper maps. So one can define the subspace $\cap_{y \in Y} M_y$ of M_f, which in complete analogy to (VI.3.1) is a universal or semi-universal deformation for locally trivial deformations of f.

By construction there is a forgetful morphism $p : M_i \longrightarrow M$ of complex spaces together with $p^\# : \mathcal{O}_{M,0} \longrightarrow \mathcal{O}_{M_i,0}$. Let $R_X = \widehat{\mathcal{O}_{M,0}}$ and $R_i = \widehat{\mathcal{O}_{M_i,0}}$ be the completions at 0 and denote again $p^\# : R_X \longrightarrow R_i$.

Lemma VI.3.4. The algebras R_i and R_X prorepresent D_i^{lt}, D_X so that

$$D_i^{\mathrm{lt}} = \mathrm{Hom}(R_i, \cdot) \quad \text{and} \quad D_X^{\mathrm{lt}} = \mathrm{Hom}(R_X, \cdot)$$

and the ring homomorphism $R_X \longrightarrow R_i$ induced by the map of functors is just $p^\#$.

Proof. We argue for R_i only, the case of R_X is similar. Every locally trivial deformation $\mathcal{Y} \hookrightarrow \mathcal{X}$ over $S = \operatorname{Spec} R$ with $R \in \mathrm{Art}_{\mathbb{C}}$ is itself a morphism of complex spaces, so there is a classifying map $\varphi : S \longrightarrow M_i$. As the underlying map of topological spaces is just the inclusion $\{0\} \hookrightarrow M_i$, the morphism φ is determined by the ring homomorphism $\mathcal{O}_{M_i,0} \longrightarrow R$. This morphism decends to $\varphi^\# : \mathcal{O}_{M_i,0}/\mathfrak{m}^k \longrightarrow R$ for some $k \in \mathbb{N}$, where $\mathfrak{m}$ is the maximal ideal. But $\mathcal{O}_{M_i,0}/\mathfrak{m}^k \overset{\cong}{\to} R_i/\mathfrak{m}_i^k$ where $\mathfrak{m}_i = \mathfrak{m}R_i$ and so φ is determined by a morphism $R_i \longrightarrow R$. The last statement is clear, as a morphism of complete local rings is determined by its truncations modulo powers of the maximal ideal. $\square$

REMARK VI.3.5. The results of Proposition VI.3.2 and Lemma VI.3.4 depend on X being symplectic only in so far that there exists a universal deformation space M for X, which is true for every compact complex manifold with $H^0(X, T_X) = 0$. The results do not at all depend on Y being Lagrangian.

VI.3.6. The T^1-lifting Principle. In this section we will prove smoothness of M_i at 0. Therefore we use Ran's T^1-lifting principle [**Ran92a**], a technique to prove unobstructedness of a given prorepresentable deformation functor $D : \mathrm{Art}_k \longrightarrow \mathrm{Set}$ having relative tangent spaces T^1_R and an obstruction space T^2 as in Lemma I.3.4. Ran's ideas were developed further by Kawamata [**Kaw92, Kaw97**]. The method works in two steps.
The *first step* works for every prorepresentable deformation functor D, which has an obstruction space T^2. Put $A_n := k[t]/t^{n+1}$ and let $A_{n+1} \longrightarrow A_n$ be the canonical projection. To prove unobstructedness of D it suffices to show that the induced map $D(A_{n+1}) \longrightarrow D(A_n)$ is always surjective by Corollary I.1.7. However we want to replace this by a different criterion. Therefore we introduce the k-algebras $B_n := A_n[\varepsilon]$ and $C_n := A_n[\varepsilon]/\varepsilon t^n$. There are canonical projections $C_n \longrightarrow B_{n-1}$ and $B_n \longrightarrow C_n \longrightarrow A_n$. The last one is split by the inclusion $A_n \longrightarrow B_n$.

LEMMA VI.3.7. Let $B_n \longrightarrow C_n$ be the canonical surjection. If the induced map $D(B_n) \longrightarrow D(C_n)$ is surjective, then $D(A_{n+1}) \longrightarrow D(A_n)$ is surjective.

PROOF. We have a morphism of small extensions in Art_k:

(VI.3.2)
$$\begin{tikzcd}
0 \arrow[r] & (t^{n+1}) \arrow[r] \arrow[d] & A_{n+1} \arrow[r] \arrow[d, "\delta"] & A_n \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & (\varepsilon t^n) \arrow[r] & B_n \arrow[r] & C_n \arrow[r] & 0
\end{tikzcd}$$

where $\delta(t) = t + \varepsilon$. The morphism $(t^{n+1}) \longrightarrow (\varepsilon t^n)$ is multiplication by $n+1$ and hence an isomorphism as $\operatorname{char} k = 0$. If we apply D to diagram (VI.3.2), we obtain

(VI.3.3)
$$\begin{tikzcd}
D(A_{n+1}) \arrow[r] \arrow[d, "\delta"] & D(A_n) \arrow[r] \arrow[d, "\delta"] & T^2 \otimes (t^{n+1}) \arrow[d, "\cong"] \\
D(B_n) \arrow[r] & D(C_n) \arrow[r] & T^2 \otimes (\varepsilon t^n)
\end{tikzcd}$$

Since $D(B_n) \longrightarrow D(C_n)$ is surjective, $D(C_n) \longrightarrow T^2 \otimes (\varepsilon t^n)$ is the zero map. The claim now follows by diagram chase. □

For an element $\xi_n \in D(A_n)$ we denote by $\xi_n|_{A_{n-1}}$ the image of ξ_n under the canonical map $D(A_n) \longrightarrow D(A_{n-1})$. Recall that $D(B_n)_{\xi_n} = \varphi_B^{-1}(\xi_n)$ where $\varphi_B : D(B_n) \longrightarrow D(A_n)$ is the canonical map.

LEMMA VI.3.8. The morphism $D(B_n) \longrightarrow D(C_n)$ is surjective if for all $\xi_n \in D(A_n)$ and $\xi_{n-1} := \xi_n|_{A_{n-1}}$ the map

$$D(B_n)_{\xi_n} \longrightarrow D(B_{n-1})_{\xi_{n-1}}$$

between the fibers over ξ_n and ξ_{n-1} is surjective.

PROOF. To see this, we consider the diagram

$$\begin{tikzcd} D(B_n) \arrow[r] \arrow[d] \arrow[dd, bend right, "\chi"'] & D(A_n) \arrow[d, equal] \\ D(C_n) \arrow[r, "\varphi_C"] \arrow[d, "\psi"] & D(A_n) \arrow[d] \\ D(B_{n-1}) \arrow[r, "\varphi_B"] & D(A_{n-1}) \end{tikzcd} \tag{VI.3.4}$$

where all morphisms are induced by the canonical projections, see section VI.3.6. Let $\eta \in D(C_n)$ be given and put $\xi_n := \varphi_C(\eta) \in D(A_n)$. The lower square is cocartesian, as D is prorepresentable and already the square of rings is cocartesian. Therefore the restriction of ψ to the fiber $D(C_n)_{\xi_n} = \varphi_C^{-1}(\xi_n)$ gives a bijection

$$D(C_n)_{\xi_n} \xrightarrow{\psi} D(B_{n-1})_{\xi_{n-1}}$$

onto the fiber over ξ_{n-1}. By assumption, $D(B_n)_{\xi_n} \longrightarrow D(B_{n-1})_{\xi_{n-1}}$ is surjective. Hence, there is $\eta' \in D(B_n)_{\xi_n}$ with $\chi(\eta') = \psi(\eta)$, so η' is a preimage of η and the claim follows. □

We summarize Lemma I.1.6, Lemma VI.3.7 and Lemma VI.3.8 in

LEMMA VI.3.9. Let D be a prorepresentable deformation functor, which has an obstruction space T^2. Then D is unobstructed, if for all $\xi_n \in D(A_n)$ and $\xi_{n-1} := \xi_n|_{A_{n-1}}$ the map

$$D(B_n)_{\xi_n} \longrightarrow D(B_{n-1})_{\xi_{n-1}}$$

is surjective. □

REMARK VI.3.10. We have seen in Chapter I that the following functors are prorepresentable and have an obstruction space if the below-mentioned assumptions hold.

- D_i^{lt} by Lemma I.3.3 and Lemma I.3.4, if $i : Y \hookrightarrow X$ is a closed immersion of proper schemes, Y is a reduced locally complete intersection, X is smooth and we have $H^0(X, T_i) = 0$.
- $D_X = D_X^{\mathrm{lt}}$ by Lemma I.2.2 and Lemma I.2.3, if X is a smooth and proper k-scheme with $H^0(X, T_X) = 0$.
- $D_{i/X}$ by Lemma I.4.2 and Lemma I.4.3, if $i : Y \hookrightarrow X$ is a closed immersion of proper schemes, Y is a reduced locally complete intersection and X is smooth.
- $D_{i/X}^{\mathrm{lt}}$ by Lemma I.4.2 and Lemma I.4.4, if $i : Y \hookrightarrow X$ is a closed immersion of proper schemes, Y is a reduced locally complete intersection and X is smooth.

Therefore, the previous lemma applies in these cases.

The *second step* of the T^1-lifting principle is to actually prove surjectivity of the map $D(B_n)_{\xi_n} \longrightarrow D(B_{n-1})_{\xi_{n-1}}$ for all ξ_n and ξ_{n-1} as in Lemma VI.3.9. This is not in general fulfilled and needs more input from the concrete geometric situation. We deduce this for $D = D_i^{\mathrm{lt}}$ from the fact that the cohomology sheaves controlling the deformation problem are locally free and compatible with base change, see Lemma VI.3.11. To achieve this, we link some related deformation problems by an exact sequence.
Consider a locally complete intersection Lagrangian subvariety $i : Y \hookrightarrow X$ in an irreducible symplectic manifold X. Let $S = \operatorname{Spec} R$ for $R \in \mathrm{Art}_{\mathbb{C}}$ and let

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "f"'] & \mathcal{X} \arrow[d, "g"] \\ & S \end{tikzcd}$$

be a locally trivial deformation of i over S. Consider the long exact sequence

$$0 \longrightarrow R^0 g_* T_I \longrightarrow R^0 g_* T_{\mathcal{X}/S} \longrightarrow R^0 f_* N'_{\mathcal{Y}/\mathcal{X}} \longrightarrow R^1 g_* T_I \longrightarrow \dots \tag{VI.3.5}$$

obtained from the sequence (I.3.6). If Y is a simple normal crossing subvariety, then $\mathcal{Y} \hookrightarrow \mathcal{X}$ is Lagrangian by Lemma VI.2.1. The symplectic form gives an isomorphism $T_{\mathcal{X}/S} \cong \Omega_{\mathcal{X}/S}$ and by Proposition VI.2.4 we have $N'_{\mathcal{Y}/\mathcal{X}} \cong \widetilde{\Omega}_{\mathcal{Y}/S}$. Moreover, the module $R^0 g_* \Omega_{\mathcal{X}/S}$ is free and compatible with base change by Theorem III.5.1. This gives $R^0 g_* \Omega_{\mathcal{X}/S} \otimes_R k = H^0(X, \Omega_X) = 0$, where the last equality holds as X is irreducible symplectic. By Nakayama's Lemma this implies $R^0 g_* \Omega_{\mathcal{X}/S} = 0$. Put together this gives the following long exact sequence

$$0 \longrightarrow R^0 f_* \widetilde{\Omega}_{\mathcal{Y}/S} \longrightarrow R^1 g_* T_I \longrightarrow R^1 g_* \Omega_{\mathcal{X}/S} \longrightarrow R^1 f_* \widetilde{\Omega}_{\mathcal{Y}/S} \longrightarrow \dots \tag{VI.3.6}$$

LEMMA VI.3.11. Let $i : Y \hookrightarrow X$ be a simple normal crossing Lagrangian subvariety in an irreducible symplectic manifold and let $I : \mathcal{Y} \hookrightarrow \mathcal{X}$ be a locally trivial deformation of i over $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_{\mathbb{C}}$. Then the modules $R^k g_* T_I$ are free for all k and all morphisms in (VI.3.6) have constant rank. In particular, all morphisms in (VI.3.5) have constant rank.

PROOF. By Theorem III.5.1 we know that $R^k g_* \Omega_{\mathcal{X}/S}$ is free. By Proposition VI.1.2 we know that Y is a projective variety, so Theorem III.2.10 applies and $R^k f_* \widetilde{\Omega}_{\mathcal{Y}/S}$ is free. Then by Theorem III.5.4 also the cokernel (and hence the kernel) of $R^k g_* \Omega_{\mathcal{X}/S} \longrightarrow R^k f_* \widetilde{\Omega}_{\mathcal{Y}/S}$ is free. So if we break up the sequence (VI.3.6) into pieces and use that if $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ is exact and F', F'' are free, then so is F we obtain freenes of $R^k g_* T_I$ for all k. $\square$

Thus, the T^1-lifting principle may be applied.

THEOREM VI.3.12. *Let Y be a Lagrangian simple normal crossing subvariety. Then the complex space M_i is smooth at* 0.

PROOF. We put $D := D_i^{\mathrm{lt}}$ and denote by A_n, B_n and C_n the algebras introduced in section VI.3.6. For $\xi_n \in D(A_n)$ we put $\xi_{n-1} := \xi_n|_{A_{n-1}}$. By Lemma VI.3.9 the functor D is unobstructed, if for all $\xi_n \in D(A_n)$ the map

$$D(B_n)_{\xi_n} \longrightarrow D(B_{n-1})_{\xi_{n-1}}$$

is surjective. For a given class $\xi_n \in D(A_n)$ take a deformation locally trivial

$$\begin{tikzcd} \mathcal{Y}_n \arrow[r, hook, "i_n"] \arrow[dr, "f"'] & \mathcal{X}_n \arrow[d, "g"] \\ & S_n \end{tikzcd}$$

of i over $S_n = \operatorname{Spec} A_n$ representing ξ_n. Let $i_{n-1} : \mathcal{Y}_{n-1} \hookrightarrow \mathcal{X}_{n-1}$ be the restriction of i_n to S_{n-1}. Then by Lemma I.3.4 the diagram

$$\begin{tikzcd} R^1 g_* T_{i_n} \arrow[r] \arrow[d] & R^1 g_* T_{i_{n-1}} \arrow[d] \\ D_i(B_n)_{i_n} \arrow[r] & D_i(B_{n-1})_{i_{n-1}} \end{tikzcd}$$

is commutative and the vertical maps are bijections. By Lemma VI.3.11 the module $R^1 g_* T_{i_n}$ is free and hence by [**EGAIII2**, Prop 7.8.5] it is compatible with base change. This means that $R^1 g_* T_{i_{n-1}} = R^1 g_* T_{i_n} \otimes_{A_n} A_{n-1}$. Clearly, $R^1 g_* T_{i_n} \longrightarrow R^1 g_* T_{i_n} \otimes_{A_n} A_{n-1}$ is surjective, which completes the proof. $\square$

VI.4. Definition and Smoothness of M_Y

Let $p : M_i \longrightarrow M$ be the canonical morphism from section VI.3.1. We learned in Chapter V that M is smooth. Also M_i is smooth by Theorem VI.3.12, so p is just a holomorphic map between complex manifolds. We prove that its differential Dp has constant rank in a neighbourhood 0. This makes the image M_Y of p well-defined and smooth. Our proof is an elaboration of an idea of Ran from [**Ran92a**] related to the T^1-lifting principle.
First let us spend a word about why the definition of the image of a morphism of space germs is subtle. The naive definition of the image could be to choose representatives $p : M_i \longrightarrow M$, take the image and then its germ. But this is not well-defined because shrinking M_i does not commute with taking the image. The following example will illustrate this.

Example VI.4.1. Let $f : \mathbb{C} \longrightarrow \mathbb{C}^2$ be given by $t \mapsto (t^2-1, t^3-t)$. The image of f is the nodal cubic $x^2+x^3 = y^2$ and the preimages of the node are $t = \pm 1$. So the image is not smooth, although the differential $D_t f : \mathbb{C} \longrightarrow \mathbb{C}^2$ has constant rank 1 everywhere. But if we take $U \subseteq \mathbb{C}$ around 1 small enough, then $f(U)$ will be smooth, namely one of the two branches of the nodal singularity. Thus, shrinking does not commute with taking the image.

Lemma VI.4.2. Let $U' \subseteq \mathbb{C}^m$ be an open neighbourhood of a point $x_0 \in \mathbb{C}^m$ and let $p : U' \longrightarrow \mathbb{C}^n$ be a holomorphic map such that the differential Dp has constant rank k on U'. Then there are open neighbourhoods $U \subseteq U'$ of x_0 and $V \subseteq \mathbb{C}^n$ of $p(x_0)$ such that $p(U) \subseteq V$ is a closed k-dimensional submanifold and $p : U \longrightarrow p(U)$ is a smooth morphism.

Proof. We may assume that $x_0 = 0$ and $p(x_0) = 0$. By a suitable choice of coordinates in $\mathbb{C}^m$ and $\mathbb{C}^n$ we achieve that the differential of p at 0 is given by

$$Dp(0) = \begin{pmatrix} \mathbf{1}_{k\times k} & 0 \\ 0 & 0 \end{pmatrix}$$

We consider the decompositions of

$$\mathbb{C}^m = \mathbb{C}^k \times \mathbb{C}^{m-k} \quad \text{and} \quad \mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k} \tag{VI.4.1}$$

corresponding to these coordinates. We write $p = (p_1, p_2)$ according to the decomposition $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k}$. As Dp has constant rank k on U', the differential Dp_1 still has constant rank k in a neighbourhood of 0. Therefore the differential of the map

$$\varphi := (p_1, \mathrm{id}) : U' \longrightarrow \mathbb{C}^k \times \mathbb{C}^{m-k}$$

has rank m in a neighbourhood of 0. Thus, we find open neighbourhoods $U \subseteq U'$ of 0 and $W \subseteq \mathbb{C}^k \times \mathbb{C}^{m-k}$ of 0 such that $\varphi|_U : U \longrightarrow W$ is biholomorphic. Put $\psi := \varphi^{-1}$ and $g := p \circ \psi$. Then the diagram

$$\begin{tikzcd} U \arrow[rr, "p=(p_1,p_2)"] \arrow[d, "\varphi"] & & \mathbb{C}^k \times \mathbb{C}^{n-k} \arrow[d, "\mathrm{pr}_1^n"] \\ W \arrow[u, bend left, "\psi"] \arrow[urr, "g"] \arrow[rr, "\mathrm{pr}_1^m"'] & & \mathbb{C}^k \end{tikzcd}$$

is commutative by construction, where pr_1^m and pr_1^n are the projections on the first factor in the decompositions (VI.4.1) of $\mathbb{C}^m$ and $\mathbb{C}^n$. In particular, $g_1 := g \circ \mathrm{pr}_1^n = \mathrm{pr}_1^m$. Let u, v be coordinates on $W \subseteq \mathbb{C}^k \times \mathbb{C}^{m-k}$ and write $g = (g_1, g_2) = (\mathrm{pr}_1^m, g_2)$ with respect to the decomposition $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k}$. Then

$$Dg(u,v) = \begin{pmatrix} \mathbf{1}_{k\times k} & 0 \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{pmatrix}$$

But $\mathrm{rk}Dg(x) = \mathrm{rk}Dp\left(\psi(x)\right) = k$ for all $x \in W$, so $\frac{\partial g_2}{\partial u} = 0$ identically on W and $g_2(u,v) = g_2(u)$. This means that if we put $V := \mathrm{pr}_1^m(W) \times \mathbb{C}^{n-k}$, then

$$g(W) = \left\{(z,w) \in V \mid w = g_2(z)\right\}.$$

Observe that $V \subseteq \mathbb{C}^n$ is open and $g(W)$ is a closed submanifold of V. This completes the proof. □

For a subvariety $i : Y \hookrightarrow X$ of an irreducible symplectic manifold X we denote by $0 \in M$ and $0 \in M_i$ the points corresponding to X and i.

Theorem VI.4.3. *Let $i : Y \hookrightarrow X$ be a Lagrangian simple normal crossing subvariety in an irreducible symplectic manifold X. Then there are open neighbourhoods $U \subseteq M_i$ of $0 \in M_i$ and $V \subseteq M$ of $0 \in M$ such that $M_Y := p(U) \subseteq V$ is a closed submanifold and $p : U \longrightarrow M_Y$ is a smooth morphism.*

Proof. By Theorem VI.3.12 and the Bogomolov-Tian-Todorov theorem we know that M_i and M are smooth at 0. By Lemma VI.4.2 we have to show that the differential Dp of $p : M_i \longrightarrow M$ has constant rank in a neighborhood of 0. The rank of Dp is constant near 0, if the stalk of

$$\mathrm{coker}(p_* : T_{M_i} \longrightarrow p^*T_M)$$

at 0 is free. Freeness may be tested after completion, so we have to verify that $p_* : T_{R_i} \longrightarrow T_{R_X}$ has constant rank, where $R_X = \widehat{\mathcal{O}_{M,0}}$ and $R_i = \widehat{\mathcal{O}_{M_i,0}}$, compare to Lemma VI.3.4. By the local criterion for flatness [**Ser06**, Thm A.5] this follows, if

$$T_{R_i} \otimes_{R_i} R_i/\mathfrak{m}_i^n \longrightarrow T_{R_X} \otimes_{R_X} R_i/\mathfrak{m}_i^n \tag{VI.4.2}$$

has constant rank for all n. Let $\eta : R_i \longrightarrow A$ be a $\mathbb{C}$-algebra homomorphism corresponding to a locally trivial deformation

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "f"'] & \mathcal{X} \arrow[d, "g"] \\ & S \end{tikzcd}$$

of i over $S = \operatorname{Spec} A$ and let $q : A[\varepsilon] \longrightarrow A$ be given by $\varepsilon \mapsto 0$. Then

$$\begin{aligned} D_i^{\mathrm{lt}}(A[\varepsilon])_\eta &= \operatorname{Hom}(R_i, A[\varepsilon])_\eta = \operatorname{Der}_{\mathbb{C}}(R_i, A) = \operatorname{Hom}_{R_i}(\Omega_{R_i/k}, A) \\ &= T_{R_i} \otimes_{R_i} A \end{aligned}$$

where $\operatorname{Hom}(R_i, A[\varepsilon])_\eta$ are those morphisms, which composed with q give η. Similarly, we find that

$$D_X(A[\varepsilon])_\xi = T_{R_X} \otimes_{R_X} A$$

for $\xi : R_X \longrightarrow A$. Now let $A = R_i/\mathfrak{m}_i^n$, let $\eta : R_i \longrightarrow R_i/\mathfrak{m}_i^n$ be the canonical projection and let $\xi = \eta \circ p^{\#}$ where $p^{\#} : R_X \longrightarrow R_i$ is the canonical map. Then we have a another description of the fibers over η and ξ

$$D_i^{\mathrm{lt}}(A[\varepsilon])_\eta = R^1 g_* T_I \quad \text{and} \quad D_X(A[\varepsilon])_\xi = R^1 g_* T_{\mathcal{X}/S}$$

by Lemma I.3.4 and Lemma I.2.3. Moreover, the map (VI.4.2) is identified with the map

$$R^1 g_* T_I \longrightarrow R^1 g_* T_{\mathcal{X}/S}.$$

from (VI.3.5), which is of constant rank by Lemma VI.3.11. This completes the proof. □

VI.5. Main results

Let $i : Y \hookrightarrow X$ be the inclusion of a simple normal crossing Lagrangian subvariety. We denote by $\nu : \widetilde{Y} \longrightarrow Y$ the normalization and by $j = i \circ \nu$ the composition. Let M be a simply connected representative of the universal deformation space of the irreducible symplectic manifold X. Let M_i be the universal deformation space for locally trivial deformations of i as in section VI.3.1 and let $p : M_i \longrightarrow M$ be the canonical map. Assuming we shrinked M_i and M sufficiently we define

$$M_Y := \operatorname{im}(M_i \longrightarrow M) \tag{VI.5.1}$$

as in Theorem VI.4.3. In [**Voi92**] the space M_Y was defined pointwise and later identified as the image of a suitable component of the relative Douady space of the universal family over M. Then, Voisin concluded that M_Y is analytic using properness of the relative Douady space over M. The difference is that we consider locally trivial deformations here, where no Douady space is known to exist. Therefore, at the moment we can only show analyticity of M_Y for simple normal crossing subvarieties using Theorem VI.4.3.

LEMMA VI.5.1. Suppose Y has simple normal crossings. Then

$$\ker\left(j^*: H^1(\Omega_X) \longrightarrow H^1(\Omega_{\widetilde{Y}})\right) = \ker\left(i^*: H^1(\Omega_X) \longrightarrow H^1(\widetilde{\Omega}_Y)\right),$$

where $\nu: \widetilde{Y} \longrightarrow Y$ is the normalization.

PROOF. As $j^* = \nu^* \circ i^*$ the inclusion $\supseteq$ is obvious. For the other direction it suffices to show that ν^* is injective on $\operatorname{im} i^*$. By Proposition VI.1.2 the subvariety Y is projective, hence by [**Del71, Del74**] there is a functorial mixed Hodge structure on $H^k_Y := H^k(Y, \mathbb{C})$ for every k. We denote by $F^\bullet$ the Hodge filtration on H^2_Y and by $W_\bullet$ the weight filtration on H^2_Y. As a special case of Corollary III.3.2 we deduce that

$$H^1(\widetilde{\Omega}_Y) = \mathrm{Gr}^1_F H^2_Y = F^1 H^2_Y / F^2 H^2_Y.$$

Let $\ldots \Rrightarrow Y^{[1]} \rightrightarrows Y^{[0]} \to Y$ be the canonical semi-simplicial resolution from Lemma III.2.4. Note that $\widetilde{Y} = Y^{[0]}$. Consider the weight spectral sequence associated to the first graded objects of the Hodge filtration given by

$$E_1^{r,s} = H^s(Y^{[r]}, \Omega^1_{Y^{[r]}}) \Rightarrow H^{r+s}(Y, \widetilde{\Omega}^1_Y) \tag{VI.5.2}$$

By [**PS08**, Thm 3.12 (3)] it degenerates on E_r if the weight spectral sequence degenerates at E_r. In their notation the spectral sequence is denoted by $E(\mathrm{Gr}^1_F, W)$. So because of Corollary III.4.2 both spectral sequences degenerate at E_2. The differential $d_1 : E_1^{0,1} \longrightarrow E_1^{0,1}$ is given by $\delta: H^1(\Omega_{Y^{[0]}}) \longrightarrow H^1(\Omega_{Y^{[1]}})$ and degeneration at E_2 tells us that

$$\begin{aligned}\mathrm{Gr}^W_2 \mathrm{Gr}^1_F H^2_Y = F^1H^2_Y/(W_1F^1H^2_Y + F^2H^2_Y) = E_\infty^{0,1} = E_2^{0,1} \\ = \ker\left(H^1(\Omega_{Y^{[0]}}) \longrightarrow H^1(\Omega_{Y^{[1]}})\right).\end{aligned}$$

In other words, as $W_2\mathrm{Gr}^1_F H^2_Y = \mathrm{Gr}^1_F H^2_Y = H^1(\widetilde{\Omega}_Y)$ there is an exact sequence

$$0 \longrightarrow W_1 \mathrm{Gr}^1_F H^2_Y \longrightarrow H^1(\widetilde{\Omega}_Y) \xrightarrow{\nu^*} H^1(\Omega_{Y^{[0]}}) \longrightarrow H^1(\Omega_{Y^{[1]}}),$$

so that $\ker \nu^* = W_1 \mathrm{Gr}_F^1 H_Y^2$. But $H_X^2 := H^2(X, \mathbb{C})$ has pure weight two because X is smooth. In particular, $W_1 \mathrm{Gr}_F^1 H_X^2 = 0$. Morphisms of mixed Hodge structures are strict with respect to both filtrations, so we have

$$0 = i^*(W_1 \mathrm{Gr}_F^1 H_X^2) = \mathrm{im}\, i^* \cap W_1 \mathrm{Gr}_F^1 H_Y^2 = \mathrm{im}\, i^* \cap \ker \nu^*$$

hence ν^* is injective on $\mathrm{im}\, i^*$ and we deduce $\ker i^* = \ker j^*$ completing the proof. □

The following lemma generalizes [**Voi92**, Lem 2.3] to the normal crossing case.

LEMMA VI.5.2. Suppose Y has simple normal crossings. Then we have $T_{M'_Y,0} = T_{M_Y,0}$ for the Zariski tangent spaces at $0 \in M_Y \cap M'_Y$.

PROOF. By Lemma V.3.3 the tangent space of M'_Y at 0 is

$$T_{M'_Y,0} = \ker \left(j^* \circ \omega' : H^1(X, T_X) \longrightarrow H^1(\widetilde{\Omega}_Y) \right).$$

By Lemma VI.5.1 we have

$$T_{M'_Y,0} = \ker \left(i^* \circ \omega' : H^1(X, T_X) \longrightarrow H^1(\Omega_{\widetilde{Y}}) \right),$$

where $\widetilde{Y} \longrightarrow Y$ is the normalization. On the other hand, M_Y is the smooth image of $p : M_i \longrightarrow M$ so that

$$\begin{aligned} T_{M_Y,0} &= \mathrm{im}\,(p_* : T_{M_i,0} \longrightarrow T_{M,0}) \\ &= \mathrm{im}\,\left(H^1(X, T_i) \longrightarrow H^1(X, T_X)\right) \\ &= \ker \left(H^1(X, T_X) \xrightarrow{\alpha} H^1(Y, N'_{Y/X}) \right) \end{aligned}$$

where the third equality holds because the sequence (VI.3.5) is exact.
By (VI.2.2) and Proposition VI.2.4 we have a commutative diagram

$$\begin{tikzcd} H^1(X, \Omega_X) \arrow[r, "j^*"] & H^1(Y, \widetilde{\Omega}_Y) \arrow[d, "\tilde{\omega}"] \\ H^1(X, T_X) \arrow[u, "\omega'"] \arrow[r, "\alpha"] & H^1(Y, N'_{Y/X}) \end{tikzcd}$$

where the vertical maps are isomorphisms. This implies that

$$T_{M_Y,0} = \ker(\alpha) = \ker(\tilde{\omega} \circ j^* \circ \omega') = \ker(j^* \circ \omega') = T_{M'_Y,0}$$

and completes the proof. □

THEOREM VI.5.3. *Let $i : Y \hookrightarrow X$ be a simple normal crossing Lagrangian subvariety in a compact irreducible symplectic manifold X, $\nu : \widetilde{Y} \longrightarrow Y$ the* *normalization and $j = i \circ \nu$. Then $M'_Y = M_Y$ and this space is smooth at* 0 *of codimension*

$$\text{(VI.5.3)} \quad \operatorname{codim}_M M_Y = \operatorname{codim}_M M'_Y = \operatorname{rk}\left(j^* : H^2(X, \mathbb{C}) \longrightarrow H^2(\widetilde{Y}, \mathbb{C})\right).$$

in M.

Proof. Assume that $Y = \cup_i Y_i$ is a decomposition into irreducible components. As a direct consequence of Lemma III.1.5 we have $M_Y \subseteq \cap_i M_{Y_i}$. In section V.2.4 we defined the subspaces M'_Y, $M'_{[Y]}$ and $M_{[Y]}$ of M associated to a Lagrangian subvariety Y of X. We have

$$\text{(VI.5.4)} \qquad \begin{tikzcd} M'_Y \arrow[r, hook, dashed] \arrow[d, equal] & M'_{[Y]} \arrow[r, equal] & M_{[Y]} \\ \bigcap_i M'_{Y_i} \arrow[r, equal] & \bigcap_i M'_{[Y_i]} \arrow[r, equal] \arrow[u, hook] & \bigcap_i M_{[Y_i]} \arrow[u, hook] \end{tikzcd}$$

where the vertical relations were observed in Remark V.2.7, the horizontal equalities on the right were shown in Proposition V.3.2 and the left lower equality holds as Y has simple normal crossings by Proposition V.3.7. As a consequence, we obtain the upper left inclusion.

As M_{Y_i} is smooth, in particular reduced, for each i we have that $M_{Y_i} \subseteq M_{[Y_i]}$ so that

$$M_Y \subseteq \bigcap_i M_{Y_i} \subseteq \bigcap_i M_{[Y_i]} = M'_Y.$$

Therefore, we find

$$\dim M_Y \leq \dim M'_Y \leq \dim T_{M'_Y,0} = \dim T_{M_Y,0}$$

where the last equality comes from Lemma VI.5.2. As M_Y is smooth by Theorem VI.4.3 we have equality everywhere. In particular, as M'_Y is smooth by Proposition V.3.7 we have $M_Y = M'_Y$.

The statement about the codimension follows from the description (V.3.1) of the tangent space of M'_Y. □

Remark VI.5.4. Let us analyse the proof of Theorem VI.5.3. The definition of M_i in section VI.3.1 works for an arbitrary Lagrangian subvariety. The spaces M'_Y, $M'_{[Y]}$ and $M_{[Y]}$ can be defined for arbitrary Lagrangian subvarieties, too, and the inclusion relations in (VI.5.4) hold as soon as Proposition V.3.7 can be applied. So let us assume that we are in one of the following cases:

(1) $\dim X = 2$.
(2) Y has normal crossing singularities.
(3) X is projective and $\widetilde{Y}$ is smooth.

Furthermore, we fix smooth, connected and simply connected representatives M'_Y, $M'_{[Y]}$, $M_{[Y]}$ and M such that M'_Y is a closed subvariety of $M'_{[Y]} = M_{[Y]}$ and $M_{[Y]}$ is a closed subvariety of M. Moreover we fix a connected representative M_i with a map $p : M_i \longrightarrow M$.
Depending on this choice, we define a subset M_Y^{set} by $M_Y^{\mathrm{set}} := p(M_i)$. Then set-theoretically, we have $M_Y^{\mathrm{set}} \subseteq M_{[Y]}$ as in the proof of the previous theorem. As $M_{[Y]}$ is a closed subvariety of M, the Zariski-closure $M_Y := \overline{M_Y^{\mathrm{set}}}$ is contained in $M_{[Y]}$. In particular, from the discussion above we deduce

$$M_Y \subseteq \cap_i M_{Y_i} \subseteq \cap_i M_{[Y_i]} = \cap_i M'_{[Y_i]} = \cap_i M'_{Y_i} = M'_Y.$$

Note that if Y has simple normal crossings this last chain of inclusions together with Theorem VI.5.3 implies that this definition coincides with the one from Theorem VI.4.3.
Although this definition of the subspace M_Y depends on choices, the inclusion $M_Y \subseteq M'_Y$ holds for every such choice. As the codimension of M'_Y is still described by (VI.5.3) we can in any case estimate the codimension of the locus M_Y in M where a certain subvariety is preserved. This fact together with the applications in Chapter VII justify such a despicable definition.

VI.6. More smoothness results

The following results are not new. They appeared in the literature somewhere somehow, but sometimes in a different disguise, sometimes with proofs only sketched. As they are easily obtained via the T^1-lifting technique, we include them to have a reference and a full proof. Let X be an irreducible symplectic manifold and let $i : Y \hookrightarrow X$ be a connected Lagrangian subvariety.

Theorem VI.6.1. *Assume that Y is smooth. Then the Douady space $\mathscr{D}(X)$ of X is smooth at $[Y]$.*

Proof. Let $S_n = \operatorname{Spec} A_n$ where $A_n = \mathbb{C}[t]/t^{n+1}$ and let

$$\begin{tikzcd} \mathcal{Y}_n \arrow[r, hook, "i_n"] \arrow[dr, "f"'] & X \times S_n \arrow[d, "g"] \\ & S_n \end{tikzcd}$$

be a deformation of Y inside X over S_n. In order to apply the T^1-lifting technique, we will show that the A_n-module $T^1_{i_n/X/A_n} = R^0 f_* N_{\mathcal{Y}_n/X\times S_n}$ is compatible with base change. By Lemma VI.2.1 we know that $\mathcal{Y}_n$ is Lagrangian in $X \times S_n$. As f is smooth, Corollary VI.2.3 tells us that $\Omega_{\mathcal{Y}_n/S_n} \longrightarrow N_{\mathcal{Y}_n/X\times S_n}$ is an isomorphism. Thus, $T^1_{i_n/X/A_n} \cong R^0 f_* \Omega_{\mathcal{Y}_n/S_n}$

is a free module and compatible with arbitrary base change by [**Del68**, Thm 5.5]. We denote by $i_{n-1} : \mathcal{Y}_{n-1} \longrightarrow X \times S_{n-1}$ the restriction of i_n to S_{n-1}. Then the map

$$R^0 f_* \Omega_{\mathcal{Y}_n/S_n} \longrightarrow R^0 f_* \Omega_{\mathcal{Y}_{n-1}/S_{n-1}} = R^0 f_* \Omega_{\mathcal{Y}_n/S_n} \otimes_{A_n} A_{n-1}$$

is surjective. By Lemma I.4.3 there is a commutative diagram

$$\begin{tikzcd} R^0 f_* \Omega_{\mathcal{Y}_n/S_n} \arrow[r] \arrow[d] & R^0 f_* \Omega_{\mathcal{Y}_n/S_n} \otimes_{A_n} A_{n-1} \arrow[d] \\ D_{i/X}(A_n[\varepsilon])_{i_n} \arrow[r] & D_{i/X}(A_{n-1}[\varepsilon])_{i_{n-1}} \end{tikzcd}$$

where the vertical maps are bijections. So the map

$$D_{i/X}(A_n[\varepsilon])_{i_n} \longrightarrow D_{i/X}(A_{n-1}[\varepsilon])_{i_{n-1}}$$

is surjective as well and the claim follows from Lemma VI.3.9 and Remark VI.3.10. □

THEOREM VI.6.2. *Let X be an irreducible symplectic manifold and let $g : \mathcal{X} \longrightarrow S$ be a deformation of X over a connected complex space S. Assume that $X = g^{-1}(s_0)$ for some point $s_0 \in S$ and that every fiber of g is an irreducible symplectic manifold. Let $i : Y \hookrightarrow X$ be a smooth Lagrangian submanifold, let $p : \mathscr{D}(\mathcal{X}/S) \longrightarrow S$ be the relative Douady space of g and let $0 \in \mathscr{D}(\mathcal{X}/S)$ be the point corresponding to Y. Then there are open neighbourhoods $U \subseteq \mathscr{D}(\mathcal{X}/S)$ of 0 and $V \subseteq S$ of s_0 such that $S_Y := p(U) \subseteq V$ is a closed subvariety and the restriction $p : U \longrightarrow S_Y$ is a smooth morphism.*

PROOF. The problem is local in S, so we may shrink S and assume that there is a pullback diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "g"'] & \mathfrak{X} \arrow[d, "\pi"'] \\ S \arrow[r, "\varphi"] & M \end{tikzcd}$$

where $\pi : \mathfrak{X} \longrightarrow M$ is the universal family over the universal deformation space of X. Let $q : \mathscr{D}_\pi \longrightarrow M$ be the relative Douady space of π. As Y is smooth, every deformation over an Artinian base is locally trivial. Thus the the space M_i from section VI.3.1 is an open neighbourhood of the point $y \in \mathscr{D}_\pi$ corresponding to Y. Let $o_i \in M_i$ be the point corresponding to i and let $o_X \in M$ be the point corresponding to X. By Theorem VI.4.3 there are open neighbourhoods $U' \subseteq M_i$ of o_i and $V' \subseteq M$ of o_X such that $M_Y = q(U') \subseteq V'$ is a closed submanifold and $q : U' \longrightarrow M_Y$ is a smooth morphism. We put $V := \varphi^{-1}(V')$ and $S_Y := \varphi^{-1}(M_Y)$. Then $S_Y \hookrightarrow V$ is a closed immersion as $M_Y \hookrightarrow V'$ is. Consider the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "q_S"'] & U' \arrow[d, "q"] \\ S_Y \arrow[r] \arrow[d, hook] & M_Y \arrow[d, hook] \\ V \arrow[r, "\varphi"] & V' \end{tikzcd}$$

where the outer square is a pullback. Note that as the formation of the relative Douady space is compatible with fiber products, there is an open immersion $U \hookrightarrow \mathscr{D}(\mathcal{X}/S)$. As the lower square and the outer square are pullbacks, so is the upper square. In particular, q_S is smooth as smoothness is preserved under basechange. This completes the proof. $\square$

Corollary VI.6.3. Let $i : Y \hookrightarrow X$ be a smooth Lagrangian submanifold in an irreducible symplectic manifold and let

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "f"'] & \mathcal{X} \arrow[d, "g"] \\ & S \end{tikzcd}$$

be a deformation of i over a smooth connected complex space S. Then the relative Douady space $\mathscr{D}(\mathcal{X}/S)$ is smooth at Y.

Proof. The existence of $\mathcal{Y} \longrightarrow S$ implies that in the notation of Theorem VI.6.2 the variety S_Y is an open subset of S. Then the claim follows from Theorem VI.6.2 as a composition of smooth maps is smooth. $\square$

CHAPTER VII

Applications to Lagrangian fibrations

In this section we give some examples and applications of Theorem VI.5.3 to Lagrangian fibrations. We also pose some questions regarding singular fibers, which hopefully support the quest for setting up a program to understand Lagrangian fibrations. A large part of these results only uses Voisin's original theorem or could possibly be deduced from it with a little more effort. However, with Theorem VI.5.3 at hand the arguments become more conceptual and we have another hint that similar statements might be true in greater generality.

VII.1. Properties of Lagrangian fibrations

Due to Matsushita and Hwang much is known about the structure of morphisms with domain an irreducible symplectic manifold.

Theorem VII.1.1 (**Hwang, Matsushita**). *Let X be an irreducible symplectic manifold of dimension $2n$. If B is a normal projective variety with $0 < \dim B < 2n$ and $f : X \longrightarrow B$ is a surjective morphism with connected fibers, then*

(1) $\dim B = n$, *B has only $\mathbb{Q}$-factorial log-terminal singularities, $-K_B$ is ample, the Picard number $\varrho(B)$ is one, the map f is equidimensional and every irreducible component of the reduction of a fiber is a Lagrangian subvariety. In particular, the fact that f is equidimensional implies that it is flat if B is smooth.*
(2) *If X is projective and B is smooth, then $B = \mathbb{P}^n$.*

Moreover, if B is only assumed to be a Kähler manifold, it is automatically projective.

Thus, such f is a Lagrangian fibration in the sense of Definition IV.1.5. Matsushitas contribution (1) is earlier, see [**Mat99, Mat00, Mat01, Mat03**], and (2) is due to Hwang [**Hwa08**, Thm 1.2]. As we mentioned in the introduction, the holomorphic Liouville-Arnol'd theorem shows that every smooth fiber is a complex torus. Moreover by Proposition IV.1.6 every

smooth fiber is projective, hence an abelian variety. It is worthwhile to remark that there is no example known of a Lagrangian fibration $f : X \longrightarrow B$ on an irreducible symplectic manifold, where B is not isomorphic to $\mathbb{P}^n$.
We want to study the geometry of the singular fibers. We know by [**Hwa08**, Prop 4.1] and [**HO09a**, Prop 3.1] that the analytic subset

$$D = \{t \in \mathbb{P}^n : X_t \text{ is singular}\}$$

is nonempty and of pure codimension one. We call D the *discriminant locus* of f. It seems difficult to describe its geometry in general, special cases are treated in [**Thi08**] and [**Saw08b**]. On the other hand due to [**HO09a**] the structure of a general singular fiber is known. The proof uses analytic methods and the statement is local in the base.

Theorem VII.1.2 (**Hwang-Oguiso**). *Let Y be a general singular fiber of a Lagrangian fibration $f : X \longrightarrow B$ over a polydisc B with $\dim X = 2n$. The normalization $\widetilde{Y}$ of Y_{red} is smooth, the Albanese variety $\mathrm{Alb}(\widetilde{Y_i})$ is of dimension $n-1$ for every irreducible component $\widetilde{Y_i}$ of $\widetilde{Y}$ and the Albanese map $\mathrm{alb} : \widetilde{Y} \longrightarrow \mathrm{Alb}(\widetilde{Y})$ is either a $\mathbb{P}^1$-bundle or an elliptic fiber bundle. In the latter case Y_{red} is smooth.*

Following [**HO09a**] we call the image of a fiber of $\widetilde{Y} \longrightarrow \mathrm{Alb}(\widetilde{Y})$ under the map $\widetilde{Y} \longrightarrow Y$ a *characteristic curve.* A *characteristic cycle* is a maximal (maybe infinite) connected union of characteristic curves. Note that by definition Y is a disjoint union of its characteristic cycles. For each characteristic curve C there is a unique irreducible component of $f^{-1}(D)$ containing C. We denote by $r = r(C)$ the multiplicity of divisor $f^{-1}(D) \subseteq X$. Fix a characteristic cycle Z. Hwang and Oguiso define the cycle

$$\Theta := \sum_{C \subseteq Z} r(C)C$$

where the sum runs over all characteristic curves $C \subseteq Z$. Hwang and Oguiso show in [**HO09a**, Thm 1.4] and [**HO09b**, Thm 1.1]

Theorem VII.1.3 (**Hwang-Oguiso**). *There is a $\mathbb{C}^{n-1}$-action on each general singular fiber Y which acts transitively on the set of characteristic cycles. The singularities of Y are locally trivial deformations of the singularities of Θ. If $n = \gcd(r_i)$ then $n \leq 6$ and $\frac{1}{n}\Theta$ is either a Kodaira-singular fiber of an elliptic surface or it is of type I_∞. The latter means that $\Theta = \sum_{i=1}^{\infty} C_i$ with $C_i = \mathbb{P}^1$ for all i and $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| = 1$ in which case C_i and C_j intersect transversally in one point.*

Remark VII.1.4. In fact Hwang and Oguiso give more detailed classification results, see [**HO09a**, Thm 1.4] and [**HO09b**, Thm 1.1]. Similar results on the structure of a general singular fiber were obtained independently by Matsushita [**Mat07**] in the projective case. His results are a little stronger, but more technical and his classification might not yet be complete.

VII.2. Deformations of fibers of Lagrangian fibrations

Let X be an irreducible symplectic manifold and let $f : X \longrightarrow B$ be a Lagrangian fibration. We will keep this notation throughout this section. We will need the following classical result.

Lemma VII.2.1. Let $i : Y \hookrightarrow X$ be a Lagrangian subvariety in an irreducible symplectic manifold X and assume that Y is contracted to a point by f. Let S be a connected complex space and suppose we are given a diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "p"'] & \mathcal{X} \arrow[r, "F"] \arrow[d, "\pi"] & P \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

where $I : \mathcal{Y} \hookrightarrow \mathcal{X}$ is a deformation of i over S, q is a proper morphism of complex spaces and F is a proper morphism extending f. Assume that for every $s \in S$ the morphism $F_s : \mathcal{X}_s \longrightarrow P_s$ obtained by base change is a Lagrangian fibration. If $\mathcal{Y} \longrightarrow S$ has connected fibers, then also $F(\mathcal{Y}_s)$ is set-theoretically a point for all $s \in S$.

Proof. By Theorem VII.1.1 a Lagrangian fibration is equidimensional. Then the Lemma is just a special case of the Rigidity Lemma [**KM98**, Lem 1.6]. $\square$

As explained in section V.2.8, the universal deformation space M_L of pairs (X, L) where L is a non-trivial line bundle on X, is a smooth hypersurface in M. The following result was probably first proven by Matsushita [**Mat05**] in the projective case and uses Voisin's result [**Voi92**]. See [**Mat09**, Prop 2.1] for the most general statement. We sketch the proof leaving the difficult part to Matsushita [**Mat09**].

Lemma VII.2.2. Let $f : X \longrightarrow B$ Lagrangian fibration and assume that B is projective. Let T be a smooth fiber of f and let $L = f^*A$ be the pullback of a very ample line bundle on B. Then $M_T = M_L$. In particular, M_T is a smooth hypersurface in M. Moreover, there is a complex variety P, a projective morphism $P \longrightarrow M$ and a diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "F"] \arrow[dr, "\pi"'] & & P \arrow[dl] \\ & M_L & \end{tikzcd}$$

where $\pi : \mathfrak{X} \longrightarrow M_L$ is the restriction of the universal family to M_L and F is a family of Lagrangian fibrations extending f.

Sketch of proof. Let $\mathcal{L}$ denote the universal line bundle on the total space of the universal family $\pi : \mathfrak{X} \longrightarrow M_L$. For the above choice of L we know by [**Mat09**, Cor 1.2] that $\pi_*\mathcal{L}$ is locally free and that there exists a family of Lagrangian fibrations $F : \mathfrak{X} \longrightarrow \mathbb{P}(\pi_*\mathcal{L})$ extending f.

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "F"] \arrow[dr, "\pi"'] & & \mathbb{P}(\pi_*\mathcal{L}) \arrow[dl, "q"] \\ & M_L & \end{tikzcd}$$

So if T is a smooth fiber of f, then $M_L \subseteq M_T$. For example any local section of q gives a deformation of T over M_L. By Voisin's theorem M_T is smooth of codimension equal to rk $\big(i^* : H^2(X, \mathbb{C}) \longrightarrow H^2(T, \mathbb{C})\big)$, where $i : T \hookrightarrow X$ is the inclusion. This rank is certainly ≥ 1, as the Kähler class restricts to a non-trivial element. As M_T contains the smooth hypersurface M_L, we have $M_L = M_T$. □

Suppose now that X is projective and $f : X \longrightarrow \mathbb{P}^n$ is a Lagrangian fibration. Let $D \subseteq \mathbb{P}^n$ be the discriminant divisor of f. We will assume that the reduced fiber $Y := (X_t)_{\mathrm{red}}$ has a smooth normalization $\widetilde{Y}$. In this situation we can define the space M_Y as in Remark VI.5.4, after necessary choices were made. By the classification result of Hwang and Oguiso the normalization $\widetilde{Y}$ is always smooth for general singular fibers. Let $Y = \cup_{i\in I} Y_i$ be a decomposition into irreducible components Y_i. In the situation of Lemma VII.2.2 we deduce

Lemma VII.2.3. Under the assumptions above we have $M_Y \subseteq M_L$, so if the reduced fiber is preserved as a subvariety, then the fibration is preserved. Moreover, locally trivial deformations of Y remain fiber components.

Proof. By Remark VI.5.4 we have $M_Y \subseteq M'_Y \subseteq \cap_i M'_{[Y_i]}$. But for a smooth fiber T of f we have $\sum_i n_i[Y_i] = [T]$ and so

$$\cap_i M'_{[Y_i]} \subseteq M'_{[\sum_i n_i Y_i]} = M'_{[T]} = M_T = M_L,$$

where the first two relations follow directly from Definition V.2.6, the third equality is Voisin's theorem and the last equality is Lemma VII.2.2. Put together this gives $M_Y \subseteq M_L$. The last claim follows from Lemma VII.2.1. $\square$

Example VII.2.4. A model of a general singular fiber with characteristic cycle of type I_k, $k \in \mathbb{N} \cup \{\infty\}$ and one irreducible component is obtained as follows. Take a $\mathbb{P}^1$-bundle $p : Z \longrightarrow A$ over an abelian variety A of dimension $k-1$ and suppose p has two disjoint sections $\iota_1, \iota_2 : A \longrightarrow Z$. We obtain a variety Y by glueing ι_1 with $\iota_2 \circ t_k$ where $t_k : A \longrightarrow A$ is the translation with an k-torsion point, $k \in \mathbb{N} \cup \{\infty\}$. Observe that Y has normal crossing singularities. Moreover, the sequence

$$A \overset{\iota_1}{\underset{\iota_1}{\rightrightarrows}} Z \longrightarrow Y$$

is a semi-simplicial resolution for Y. Using this resolution one can establish a theory analogous to Chapter III. Thus, if Y is a Lagrangian subvariety of an irreducible symplectic manifold X one can prove the analogue of Theorem VI.3.12 and Theorem VI.4.3. So in the notation of these theorems there is a smooth submanifold $M_Y \subseteq M$ and a smooth morphism $p : M_i \longrightarrow M_Y$. One shows the analogue of Theorem VI.5.3 in the same way, namely that $M_Y = M'_Y$ and $\operatorname{codim}_M M_Y = \operatorname{rk}\left(H^2(X, \mathbb{C}) \longrightarrow H^2(Z, \mathbb{C})\right)$.
For some k such singular fibers show up in Jacobian fibrations, see for example [**Bea99**, 1.2] or [**Saw08b**, sec 2], as the compactified Jacobian of a singular curve with a single node. Let $f : X \longrightarrow B$ be a Lagrangian fibration with a singular fiber Y of this kind. Let T be a smooth fiber of f an denote by $[T] \in H^{2n}(X, \mathbb{C})$ its cohomology class. As T and Y are both fibers of f they are algebraically equivalent as cycles, hence $[Y] = [T]$. Then Lemma VII.2.2 tells us that $\operatorname{codim}_M M_Y = 1$. Hence this type of singular fiber will be preserved under deformation whenever the fibration is preserved.

One important question regarding singular fibers seems the following

Question VII.2.5. Given a singular fiber X_t of a Lagrangian fibration $f : X \longrightarrow B$. What is the codimension of M_{X_t} in M? In other words what is the codimension of the locus in M, where this type of singular fiber is preserved?

This might be interesting for several reasons. One reason is that there are several results assuming the general singular fibers to be of a special kind, see [**HO10**], [**Saw08b**], [**Saw08a**], [**Thi08**]. If we knew that complicated general singular fibers only show up in higher codimension in M, we could always deform to such special situations.

VII.3. Codimension estimates

Let $f : X \longrightarrow \mathbb{P}^n$ be a Lagrangian fibration on an irreducible symplectic manifold, let and $Y = (X_t)_{\mathrm{red}}$ for $t \in D$ as before. In order to calculate or at least estimate the codimension of M_Y in M, we associate a divisor to each irreducible component of Y. Let D_0 be an irreducible component of the discriminant locus containing t and let $X_0 := X \times_B D_0$. Let $Y = \cup_{i \in I} Y_i$ and $X_0 = \cup_{j \in J} X_j$ be decompositions into irreducible components and consider the surjective map of sets $j : I \longrightarrow J$ mapping $i \in I$ to the unique $j = j(i) \in J$ with $Y_i \subseteq X_j$. In this way we associate a divisor in X to every irreducible component Y. We show that if Y_i is preserved under a deformation of X, then so is $X_{j(i)}$.
I am very grateful to Keiji Oguiso for explaining the following lemma.

Lemma VII.3.1. Let $f : X \longrightarrow \mathbb{P}^n$ be a Lagrangian fibration of a projective irreducible symplectic manifold X. Let $X_0 = \bigcup_{j \in J} X_j$ where $J = \{1, \ldots, r\}$ and let $i : Y = (X_t)_{\mathrm{red}} \hookrightarrow X$ for $t \in D_0 \subseteq \mathbb{P}^n$ be the reduction of a general singular fiber contained in X_0. Then

$$\operatorname{rk}\left(j^* : H^2(X, \mathbb{C}) \longrightarrow H^2(\widetilde{Y}, \mathbb{C})\right) \geq r,$$

where $\nu : \widetilde{Y} \longrightarrow Y$ is the normalization and $j = \nu \circ i$. More precisely, the subspace of $H^2(X, \mathbb{C})$ generated by the classes of the divisors X_j maps onto a subspace of of dimension $\geq r - 1$ not containing the class of the ample divisor.

Proof. If we take a general line $\ell \subseteq \mathbb{P}^n$, then the fiber product $X_\ell = X \times_{\mathbb{P}^n} \ell$ is smooth by Kleiman's theorem [**Kle74**, 2. Thm]. As $t \in D_0$ is general, there is such a line with $t \in \ell$. Let H be a very ample divisor on X and let $H_1, \ldots, H_{n-1} \in |H|$ be general. Then the intersection $S = X_\ell \cap H_1 \cap \ldots \cap H_{n-1}$ is a smooth surface by Bertini's theorem. By construction it comes with a morphism $g : S \longrightarrow \mathbb{P}^1 \cong \ell$.
Consider the diagram

$$\text{(VII.3.1)} \qquad \begin{array}{ccc} H^2(X, \mathbb{C}) & \xrightarrow{j^*} & H^2(\widetilde{Y}, \mathbb{C}) \\ \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \varrho_Y} \\ H^2(S, \mathbb{C}) & \xrightarrow{j_S^*} & H^2(\widetilde{F}, \mathbb{C}) \end{array}$$

where $F = Y \cap H_1 \cap \ldots \cap H_{n-1} \subseteq S$ and $\widetilde{F} \longrightarrow F$ is the normalization. Note that $\widetilde{Y}$ is smooth by Theorem VII.1.2 and $\widetilde{F}$ is smooth, as F is a curve. Let $Y = \bigcup_{i=1}^{s} Y_i$ and $F = \bigcup_{\lambda=1}^{q} F_\lambda$ be decompositions into irreducible components where $s = \#I$. We put

$$F(i) := Y_i \cap H_1 \cap \ldots \cap H_{n-1} = \bigcup_{\lambda \in \Lambda_i} F_\lambda,$$

where $\Lambda_i \subseteq \Lambda := \{1, \ldots, q\}$ is the subset of all λ such that $F_\lambda \subseteq Y_i$. If the H_k are general enough, the irreducible components F_λ of $F(i)$ are mutually distinct for all i. In other words, Λ is the disjoint union of the Λ_i. Indeed, one only has to verify that no irreducible component of $Y_i \cap Y_j \cap H_1 \ldots \cap H_{k-1}$ is contained in H_k for all i, j, and k.
We will show that the subspace $V \subseteq H^2(X, \mathbb{C})$ spanned by the X_j and H maps surjectively onto an r-dimensional subspace in $H^2(\widetilde{F}, \mathbb{C})$. This would imply the claim by diagram (VII.3.1).
Let $n_j \in \mathbb{N}$ be the multiplicity of $X_0 = f^{-1}(D_0)$ along X_j. Then

$$X_0 = \sum_j n_j X_j \quad \text{and} \quad X_t = \sum_i n_{j(i)} Y_i$$

as cycles, where as above $j(i)$ is the unique $j \in J$ with $Y_i \subseteq X_j$. Recall that $\Lambda = \coprod_i \Lambda_i$ is a dijoint union. So

$$n_\lambda := n_{j(i)} \quad \text{for} \quad \lambda \in \Lambda_i$$

is well-defined and we have

$$F = \sum_\lambda n_\lambda F_\lambda.$$

As $F = \bigcup_{\lambda=1}^{q} F_\lambda$ we obtain $\widetilde{F} = \bigcup_{\lambda=1}^{q} \widetilde{F}_\lambda$ where $\widetilde{F}_\lambda$ is the normalization of F_λ. Thus,

$$H^2(\widetilde{F}, \mathbb{C}) \cong \bigoplus_{\lambda=1}^{q} H^2(\widetilde{F}_\lambda, \mathbb{C}) \cong \mathbb{C}^q.$$

If we denote the intersection pairing on S by $(\cdot, \cdot)_S$, then under this isomorphism $j_S^* : H^2(S, \mathbb{C}) \longrightarrow H^2(\widetilde{F}, \mathbb{C})$ is given by

$$\alpha \mapsto ((\alpha, F_1)_S, \ldots, (\alpha, F_q)_S).$$

Let $\{x_\lambda \mid \lambda \in \Lambda\} \subseteq H^2(\widetilde{F}, \mathbb{C})^\vee$ be the dual basis of the basis of $H^2(\widetilde{F}, \mathbb{C})$ obtained corresponding to the standard basis of $\mathbb{C}^q \cong H^2(\widetilde{F}, \mathbb{C})$. By Zariski's Lemma [**BHPV04**, Ch III, Lem 8.2] the subspace $W \subseteq H^2(S, \mathbb{C})$ spanned by the classes of the F_λ maps surjectively to the hyperplane of $\mathbb{C}^q$ given by

$\sum_\lambda n_\lambda x_\lambda = 0$, So the subspace of $H^2(S,\mathbb{C})$ spanned by the classes of the F_λ and $H|_S$ maps surjectively onto $\mathbb{C}^q$. We have

$$\varrho_Y(j^*X_j) = j_S^*\varrho(X_j) = \big((\varrho(X_j),F_\lambda)_S\big)_\lambda$$

As the Λ_i are mutually disjoint, so are the $\Lambda_j := \bigcup_{i:j(i)=j} \Lambda_i$. We see from

$$(\varrho(X_j),F_\lambda)_S = \sum_{\mu\in\Lambda_j} (F_\mu,F_\lambda)_S$$

that the subspace of $H^2(X,\mathbb{C})$ generated by the X_j surjects onto a subspace of $\mathbb{C}^q$ of dimension $\geq r-1$. The claim follows as the image of V does not contain $j_S^*(H|_S)$. □

Denote by $M_{X_j} \subseteq M$ the Hodge locus associated to the class $[X_j] \in H^2(X,\mathbb{C})$, see Definition V.2.3. This is the locus where $[X_j]$ remains of type $(1,1)$. We identify M locally with its period domain

$$Q_X := \big\{[v]\in\mathbb{P}(H^2(X,\mathbb{C})) : q_X(v,v)=0, q_X(v,\bar v)>0\big\}$$

by the local Torelli theorem as in section V.1.1. Then M_{X_j} is given by the equation $q(X_j,\cdot)=0$ as explained in section V.2.8 For $K\subseteq I$ suppose $Y_K = \bigcup_{i\in K} Y_i$ and let

$$r_K = |\{j(i) \mid i\in K\}|\,.$$

We obviously have $r_K \leq r_I = r$.

Corollary VII.3.2. With the notation above

$$\begin{aligned}
\operatorname{codim} M_Y &\geq \operatorname{codim} M'_Y \geq r\\
\operatorname{codim} M_{X_t} &\geq \operatorname{codim} M'_{X_t} \geq r\\
\operatorname{codim} M_{Y_K} &\geq \operatorname{codim} M'_{Y_K} \geq r_K\\
\operatorname{codim} M_{Y_K} &\geq \operatorname{codim} M'_{Y_K} \geq r_K+1 \quad \text{if} \quad Y_K\neq Y.
\end{aligned}$$

Proof. This follows from Theorem VI.5.3, Lemma VII.3.1 and the fact that the subvariety of M given by $q_X(V,\cdot)=0$ has codimension $\dim V$ for a sub vector space $V\subseteq H^2(X,\mathbb{C})$. For the last statement one uses that by Zariski's Lemma the map j_S^* from the proof of Lemma VII.3.1 is surjective, if $Y_K\neq Y$. □

Assume as above that X is projective, that $f: X\longrightarrow \mathbb{P}^n$ is a Lagrangian fibration and that the reduced fiber $Y := (X_t)_{\mathrm{red}}$ over general $t\in D$ has a smooth normalization $\widetilde{Y}$. Let $L = f^*A$ be the pullback of an ample divisor

A on $\mathbb{P}^n$ and let $Y = \cup_{i \in I} Y_i$ be a decomposition into irreducible components. As clearly $M_{X_t} \subseteq M_Y$ we have using Lemma VII.2.3 and Lemma VII.3.1

$$M_{X_t} \subseteq M_Y \subseteq \bigcap_{i \in I} M_{Y_i} \subseteq \bigcap_{j \in J} M_{X_j} \subseteq M_L \subseteq M. \tag{VII.3.2}$$

This chain of inequalities suggests the following

QUESTION VII.3.3. Which of the first three inclusions in (VII.3.2) are strict? Can we formulate conditions, under which this can be determined?

The following example illustrates that in general $M_{Y_i} \subsetneq M_{X_{j(i)}}$. This is already a consequence of Corollary VII.3.2.

EXAMPLE VII.3.4. Let $S \longrightarrow \mathbb{P}^1$ be an elliptic K3 surface having a singular fiber S_t of Kodaira type I_3 over some $t \in \mathbb{P}^1$. Then there is an induced fibration $f : X = \mathrm{Hilb}^2(S) \longrightarrow \mathrm{Sym}^2\mathbb{P}^1 = \mathbb{P}^2$. For $D_0 = t + \mathbb{P}^1 \subseteq \mathrm{Sym}^2\mathbb{P}^1$ the pullback X_0 consists of three irreducible components X_1, X_2, X_3 having the intersection graph of I_3. The singular fiber over $t + p \in D_0$ for general $p \in \mathbb{P}^1$ is

$$Y = X_{t+p} = S_t \times S_p = Y_1 \cup Y_2 \cup Y_3,$$

where $Y_i \cong \mathbb{P}^1 \times S_p$. Let λ denote the restriction of the Kähler class to Y_1 and α_2, α_3 be the restrictions of the classes of X_2, X_3 to Y_1. Then $\alpha_2.\alpha_3 = 0$ as can be read off the intersection graph but $\alpha_2.\lambda > 0$. So λ and α_3 are linearly independent, hence

$$\mathrm{codim}_M M_{Y_1} = \mathrm{rk}\left(j^* : H^2(X, \mathbb{C}) \longrightarrow H^2(Y_1, \mathbb{C})\right) \geq 2 > 1 = \mathrm{codim}_M M_{X_1}$$

already by Voisin's theorem. As $\mathrm{im}\, j^* \subseteq H^{1,1}(Y_1)$ and the latter is 2-dimensional, we have $\mathrm{codim}_M M_{Y_1} = 2$. This however was to be expected, as $Y_1 \neq Y$ and hence $M_{Y_1} \geq r_1 + 1 = 2$ by Corollary VII.3.2. It also tells us, that $\mathrm{codim}_M M_Y \geq 3$. We will see in Theorem VII.3.8 that S_t is preserved as a singular fiber in codimension three in the deformation space of S, hence on a 17-dimensional submanifold. As we can always take the Hilbert scheme of these K3 surfaces, we have $\dim M_Y \geq 17$, hence $3 \leq \mathrm{codim}_M M_Y \leq 4$.

REMARK VII.3.5. Corollary VII.3.2 shows that besides the elliptically fibered case generically only singular fibers with a characteristic cycle Θ of Kodaira type I to IV or elliptic fiber bundles show up. Indeed, the other Kodaira fibers have at least two irreducible components with distinct multiplicities. So X_0 has to have at least two irreducible components as well.

In view of Lemma VII.3.1 it seems that the codimension of M_Y is rather influenced by the number of irreducible components of X_0 than by the number of irreducible components of Y. Thus, a very interesting and important question the following

Question VII.3.6. Let $Y = \cup_{i\in I} Y_i$ and $X_0 = \cup_{j\in J} X_j$ as in the beginning of section VII.3. Is then $\#I = \#J$?

There is no obvious reason, why these numbers should be equal, but in all examples we know they are equal. If always $\#I = \#J$ were the case, all singular fibers of type $I_m, m \geq 2$ could be excluded to show up generically. So we pose

Question VII.3.7. Which of the general singular fibers of Hwang-Oguiso show up in codimension one in M? Note that as $\operatorname{codim} M_L = 1$ and there are always singular fibers, there have to be fibers which show up in codimension one in M. We already met one of them in Example VII.2.4.

In the case of K3 surfaces, the situation becomes easier. As the fibers are divisors we have $I = J$ in the notation of section VII.3. So we know that $H^2(\widetilde{Y}, \mathbb{C}) \cong \mathbb{C}^r$ where $r = \#J$. Singular fibers of elliptic surfaces were classified by Kodaira [**Kod63**]. The canonical bundle formula for elliptic fibrations [**BHPV04**, Thm 12.1] rules out multiple fibers for elliptic K3 surfaces, as they would contribute to the canonical divisor.

Theorem VII.3.8. *Let X be a K3 surface and let $f : X \longrightarrow \mathbb{P}^1$ be a Lagrangian fibration. Let X_t be a singular fiber of f over $t \in \mathbb{P}^1$ and put $Y = (X_t)_{\mathrm{red}}$. Then the codimension of M'_Y in M is equal to the number of irreducible components of Y. Moreover, we always have $\operatorname{codim}_M M_Y \geq \operatorname{codim}_M M'_Y$ and we have $\operatorname{codim}_M M_Y = \operatorname{codim}_M M'_Y$ for fibers of type I_n, I_n^*, II^*, III^* and IV^*. Also we have $M_Y = M_{X_t}$ in all cases.*

Proof. Let $\widetilde{Y} \longrightarrow Y$ be the normalization and $r := \#J$ be the number of irreducible components of Y. We have $\dim H^2(\widetilde{Y}, \mathbb{C}) = r$. Lemma VII.3.1 and Remark VI.5.4 show that $\operatorname{codim}_M M'_Y = r$. For fibers of type I_n, I_n^*, II^*, III^* and IV^* the equality $\operatorname{codim}_M M_Y = \operatorname{codim}_M M'_Y$ follows from Theorem VI.5.3, as Y has normal crossings for all those fibers.
To see that $M_Y = M_{X_t}$ observe that if X_t is preserved under deformation, then so is Y. Moreover, under such a deformation the fibration is preserved by Lemma VII.2.2 and Y and X_t remain fiber components. In particular, the intersection graph of Y has to be a subgraph of X_t, but there are no non-trivial inclusions among the intersection graphs of the non-reduced Kodaira singular fibers. This completes the proof. □

Example VII.3.9. For singular fibers Y of type II, III and IV a direct calculation shows that the spectral sequence (III.2.5) from Theorem III.2.10 associated to $\widetilde{\Omega}^{\bullet}_Y$ does not degenerate at E_1. In fact,

$$\dim H^1(\widetilde{\Omega}_Y) > \dim H^1(\widetilde{Y}, \Omega_{\widetilde{Y}})$$

and Lemma VI.5.1 and Lemma VI.5.2 leave the possibly that $T_{M_Y,0} \subsetneq T_{M'_Y,0}$. In addition, one can for example explicitly write down a polynomial $f_{3,2} \in \mathbb{C}[x,y,z,u,v]$ homogeneous of bidegree $(3,2)$ defining an elliptic K3 surface in $\mathbb{P}^2 \times \mathbb{P}^1$ with type II singular fiber, that deforms to an elliptic K3 having only type I_1 singular fibers. This means that in this example the singular fiber Y does not show up in codimension one in M and so $M_Y \subsetneq M'_Y$. This means that either $T_{M_Y,0} \subsetneq T_{M'_Y,0}$ or M_Y is singular or both.

VII.4. $\mathrm{Hilb}^2(K3)$

We will now have a look at a particular example of a Lagrangian fibration of the irreducible symplectic manifold $\mathrm{Hilb}^2(S)$ where S is a K3 surface. We start with an elliptic K3 surface

$$g : S \longrightarrow \mathbb{P}^1.$$

For $t \in \mathbb{P}^1$ the fiber will be denoted E_t. It is generically an elliptic curve and degenerates over the discriminant $D(g)$ to one of the singular fibers classified by Kodaira [**Kod63**]. Now consider the 2nd Hilbert scheme $X = \mathrm{Hilb}^2(S) = S^{[2]}$. The fibration g induces a fibration

$$f : S^{[2]} \longrightarrow \mathbb{P}^2.$$

Note that $\mathrm{Hilb}^n(S)$ is not functorial in S. But the symmetric product is, and so the Lagrangian fibration f is obtained as the composition $g^{(2)} \circ \varrho$ of the Hilbert-Chow morphism $\varrho : S^{[2]} \longrightarrow \mathrm{Sym}^2 S$, see [**Leh04**], and the morphism $g^{(2)} : \mathrm{Sym}^2 S \longrightarrow \mathrm{Sym}^2 \mathbb{P}^1 = \mathbb{P}^2$, where we write

$$\mathrm{Sym}^n S = S^{\times n} / \mathfrak{S}_n = S \times \ldots \times S / \mathfrak{S}_n$$

for the n-th symmetric product and $\mathfrak{S}_n$ for the n-th symmetric group.

VII.4.1. Singular fibers. Let us decribe the fibers of f set theoretically. We denote the fiber over $(t_1, t_2) \in \mathrm{Sym}^2 \mathbb{P}^1$ by $X_{(t_1,t_2)}$. If $t_1 \neq t_2 \in \mathbb{P}^1$, then

$$X_{(t_1,t_2)} = E_{t_1} \times E_{t_2}. \tag{VII.4.1}$$

In this case the fiber $X_{(t_1,t_2)}$ becomes singular if one of the t_i happens to lie in $D(g)$. Thus for every point $t \in D(g)$ the line $\ell_t = t + \mathbb{P}^1$ is contained in

$D(f)$, hence ℓ_t is a component of $D(f)$. If $t_1 = t_2 = t$ and $t \notin \Delta(g)$ then

$$X_{(t_1,t_2)} = \mathrm{Sym}^2 E_t \cup_{E_t} \mathbb{P}(\Omega_S|_{E_t}), \tag{VII.4.2}$$

where $\mathbb{P}(\Omega_S|_{E_t})$ is the projective space of lines in $T_S|_{E_t}$. This contributes a conic to $Q \subseteq D(f)$, given by the diagonal imbedding $\mathbb{P}^1 \longrightarrow \mathrm{Sym}^2\mathbb{P}^1 = \mathbb{P}^2$, $p \mapsto 2p$. Summing up we have

$$D(f) = Q \cup \bigcup_{t \in D(g)} \ell_t.$$

Let us now fix one fiber $E = E_t$ and describe the two components of the above type of singular fiber. The curve $E = \mathbb{P}(\Omega_S|_E) \cap \mathrm{Sym}^2 E$ is a section of $\mathbb{P}(\Omega_S|_E)$-component and diagonally embedded in $\mathrm{Sym}^2 E$. Also $\mathrm{Sym}^2 E$ is a $\mathbb{P}^1$-bundle as we will explain next, but here the curve E is a four-section.

VII.4.2. $\mathrm{Sym}^2 E$ **component.** Let us fix a point $p_0 \in E$ defining a group structure on E with p_0 as unit element. We will write the group additively. The bundle structure on $\mathrm{Sym}^2 E$ is given by

$$\mathrm{Sym}^2 E \longrightarrow E, \quad (e_1, e_2) \mapsto e_1 + e_2,$$

hence the fiber over $p \in E$ as a set is

$$F_p = \{p_1 + p_2 : p_1 + p_2 = p\}.$$

Moreover we see that the curve

$$E \hookrightarrow \mathrm{Sym}^2 E, \quad e \mapsto (e, e),$$

is a four-section inside $\mathrm{Sym}^2 E$ as the composition $E \longrightarrow \mathrm{Sym}^2 E \longrightarrow E$, $e \mapsto 2e$ is four–to–one. It is classical that the assignment

$$E \longrightarrow \mathrm{Pic}^0(E), \quad p \mapsto \mathcal{O}_E(D_p - D_{p_0})$$

is an isomorphism of group schemes, where we write D_p for the divisor on E associated to the point p. We interpret $\mathrm{Sym}^n S$ as a parameter scheme for effective 0-cycles of degree n on S. Thus, $\mathrm{Sym}^n E$ can be interpreted as a parameter space for effective divisors of degree n on the curve E. In this description the addition map $\mathrm{Sym}^n E \longrightarrow E$ is given by

$$D_{p_1} + .. + D_{p_n} \mapsto \bigotimes_{i=1}^{n} \mathcal{O}_E(D_{p_i} - D_{p_0}) = \mathcal{O}_E(D_{\sum p_i} - D_{p_0}). \tag{VII.4.3}$$

If we fix $p \in E$, then the right hand side of (VII.4.3) is $\cong \mathcal{O}_E(D_p - D_{p_0})$ if and only if

$$D_{p_1} + .. + D_{p_n} - n \cdot D_{p_0} \sim D_p - D_{p_0},$$

where $\sim$ means linear equivalence, in other words

$$\sum_i D_{p_i} \sim (n-1)\cdot D_{p_0} + D_p.$$

The left hand side are exactly the divisors in the linear system

$$\left|H^0((n-1)\cdot D_{p_0} + D_p)\right|$$

and we summarize the discussion in

Lemma VII.4.3. Let $p : E\times E \longrightarrow E$ be the projection on the first factor and set $F := p_*\mathcal{O}_{E\times E}((n-1)\cdot E_0 + \Delta)$, where $E_0 = E\times\{p_0\}$ and Δ is the diagonal. Then

$$\begin{tikzcd} \mathbb{P}(F^\vee) \arrow[rr] \arrow[dr] & & \mathrm{Sym}^n E \arrow[dl, "+"] \\ & E & \end{tikzcd}$$

is an isomorphism of projective budles over E. $\square$

VII.4.4. Divisors on Hilb**.** If D is an effective divisor on S, we obtain a divisor D_{Hilb} on $S^{[n]}$ which may be described set theoretically as the locus of all subschemes whose support has nontrivial intersection with D. We will give a more functorial description of D_{Hilb}.
A morphism $T \longrightarrow S^{[n]}$ corresponds to a family Z of subschemes of S parametrized by T, that is, a diagram

(VII.4.4)
$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr] & T\times S \arrow[r, "q"] \arrow[d, "p"] & S \\ & T & \end{tikzcd}$$

such that $Z \longrightarrow T$ is flat and finite of degree n. If we push forward the natural morphism

(VII.4.5)
$$\mathcal{O}_Z \longrightarrow \mathcal{O}_Z \otimes q^*\mathcal{O}_S(D)$$

to T, take determinants we obtain the line bundle to the effective divisor we were looking for:

(VII.4.6)
$$\mathcal{O}_T(D_{\mathrm{Hilb}}) := \det(p_*(\mathcal{O}_Z \otimes q^*\mathcal{O}_S(D))) \otimes \det(p_*\mathcal{O}_Z)^{-1}$$

By construction it comes with a section and it measures where (VII.4.5) fails to be an isomorphism. The family Z can be reconstructed from the sheaf of algebras $p_*\mathcal{O}_Z$ as its relative Spec. This description holds for any T and we are particularly interested in the case $T \subseteq S^{[n]}$. So the task will be to describe the (universal) family $p_*\mathcal{O}_Z$ on T.

VII.4.5. The universal family on $\mathrm{Sym}^2 E$. Let us describe the universal family for the component $\mathrm{Sym}^2 E$ of a singular fiber of the map $f : S^{[2]} \longrightarrow \mathbb{P}^2$ of the type described in (VII.4.2). We know by Lemma VII.4.3 that $\mathbb{P} := \mathrm{Sym}^2 E = \mathbb{P}(F^\vee)$, where F is given by the pushforward of the sheaf $F_0 := \mathcal{O}_{E\times E}(E_0 + \Delta)$ by the projection on the first factor. So we have the diagram

$$\text{(VII.4.7)} \qquad \begin{array}{ccccc} \mathbb{P}\times E & \xrightarrow{\pi\times\mathrm{id}} & E\times E & \xrightarrow{q} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} & & \\ \mathbb{P} & \xrightarrow{\pi} & E & & \end{array}$$

On $\mathbb{P}$ consider the canonical morphism $\mathcal{O}_{\mathbb{P}}(-1) \xrightarrow{\alpha} \pi^* F$, pull it back with p and use the evaluation of relative global sections to obtain

$$p^*\mathcal{O}_{\mathbb{P}}(-1) \longrightarrow p^*\pi^* F = (\pi\times\mathrm{id})^* p^* p_* F_0 \longrightarrow (\pi\times\mathrm{id})^* F_0.$$

The map α is the inclusion of the tautological line bundle over $\mathbb{P}$ and as explained in VII.4.2 sections of $F = p_* F_0$ parametrize length 2 subschemes in E. So the subsheaf $\mathcal{O}_{\mathbb{P}\times E}(-Z) := p^*\mathcal{O}_{\mathbb{P}}(-1)\otimes(\pi\times\mathrm{id})^* F_0^\vee$ of $\mathcal{O}_{\mathbb{P}\times E}$ is the ideal sheaf of the universal family Z. Let us consider the exact sequence

$$\text{(VII.4.8)} \qquad 0 \longrightarrow \mathcal{O}_{\mathbb{P}\times E}(-Z) \longrightarrow \mathcal{O}_{\mathbb{P}\times E} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

Let D be a divisor on S and let D' be a divisor on $\mathbb{P}\times E$ with $\mathcal{O}_{\mathbb{P}\times E}(D') = (\pi\times\mathrm{id})^*(\mathcal{O}_S(D)|_E)$. In the same vein we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}\times E}(-Z) & \longrightarrow & \mathcal{O}_{\mathbb{P}\times E} & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\phi} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}\times E}(D'-Z) & \longrightarrow & \mathcal{O}_{\mathbb{P}\times E}(D') & \longrightarrow & \mathcal{O}_Z(D') & \longrightarrow & 0 \end{array}$$

and by (VII.4.6) we are interested in the determinant of the pushforward of $N = \operatorname{coker}\phi$. The diagram above gives a resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}\times E}(-Z) \longrightarrow \mathcal{O}_{\mathbb{P}\times E}(D'-Z)\oplus\mathcal{O}_{\mathbb{P}\times E} \longrightarrow \mathcal{O}_{\mathbb{P}\times E}(D') \longrightarrow N$$

of N. As $[D_{\mathrm{Hilb}}] = \mathrm{c}_1(\det p_* N) \in H^2(\mathbb{P},\mathbb{Z})$ is simply the degree 1 part of $\mathrm{ch}\,(p_*\mathcal{O}_Z(D') - p_*\mathcal{O}_Z)$, it is given by

(VII.4.9)

$$\begin{aligned} & p_*\mathrm{ch}\left(\mathcal{O}_{\mathbb{P}\times E}(-Z) - \mathcal{O}_{\mathbb{P}\times E}(D'-Z) - \mathcal{O}_{\mathbb{P}\times E} + \mathcal{O}_{\mathbb{P}\times E}(D')\right) \\ &= p_*\mathrm{ch}\left(\mathcal{O}_{\mathbb{P}\times E}(D') - \mathcal{O}_{\mathbb{P}\times E}\right) + p_*\mathrm{ch}\left(\mathcal{O}_{\mathbb{P}\times E}(-Z) - \mathcal{O}_{\mathbb{P}\times E}(D'-Z)\right) \\ &= p_*\left(e^{D'} - 1\right) + p_*\left(\mathrm{ch}\,(\mathcal{O}_{\mathbb{P}\times E}(-Z))\,\mathrm{ch}\left(\mathcal{O}_{\mathbb{P}\times E} - \mathcal{O}_{\mathbb{P}\times E}(D')\right)\right) \\ &= p_* D' + p_*\left(\mathrm{ch}\,(p^*\mathcal{O}_{\mathbb{P}}(-1))\,\mathrm{ch}\left((\pi\times\mathrm{id})^* F_0^\vee\right)\cdot\left(-D'\right)\right) \end{aligned}$$

where we simply write D' for the class of D' in cohomology. Note that we used twice that $(D')^2 = 0$, which holds as already $(D|_E)^2 = 0$. Using the projection formula and diagram (VII.4.7) we obtain

$$\begin{aligned} &\mathrm{ch}(p_*\mathcal{O}_Z(D') - p_*\mathcal{O}_Z) \\ &= \pi^* p_* q^*(D|_E) - \left(1 - \xi + \frac{\xi^2}{2}\right) p_* \left(\mathrm{ch}\left((\pi \times \mathrm{id})^* F_0^\vee\right) \cdot D'\right), \end{aligned} \tag{VII.4.10}$$

where $\xi = \mathrm{c}_1(\mathcal{O}_{\mathbb{P}}(1))$. If $d = E.D = \deg(D|_E)$ and if we denote by f the class of a fiber of π and by δ the class of $q^*(D|_E)$, then in degree 1 we have

(VII.4.11)

$$\begin{aligned} &\mathrm{ch}(p_*\mathcal{O}_Z(D') - p_*\mathcal{O}_Z)_1 \\ &= -\left((1-\xi)\, p_*(\pi \times \mathrm{id})^* \left(\left(1 - (E_0 + \Delta) + \frac{1}{2}(E_0 + \Delta)^2\right) \cdot \delta\right)\right)_1 \\ &= -\left((1-\xi)\, p_*(\pi \times \mathrm{id})^* \left(\delta - \Delta.\delta + \frac{1}{2}\Delta^2.\delta\right)\right)_1 \\ &= \xi \pi^* p_* \delta + \pi^* p_* (\Delta.\delta) \\ &= d \cdot (\xi + f). \end{aligned}$$

Here we used that $p_*\delta = d$ and $p_*(\Delta.\delta) = \delta$ as Δ is the diagonal.

Let $j : \mathbb{P} \longrightarrow X = \mathrm{Hilb}^2(S)$ correspond to the universal family $Z \subseteq \mathbb{P} \times S$. We want to calculate $\mathrm{rk}(j^* : H^2(X, \mathbb{Q}) \longrightarrow H^2(\mathbb{P}, \mathbb{Q}))$. As is well known $H^2(\mathbb{P}, \mathbb{Q}) = \mathbb{Q}^2 = \mathrm{Num}_{\mathbb{Q}}\, \mathbb{P}$, so all cohomology is algebraic. For ruled surfaces the numerical Picard group is generated by the class ξ of a section and the class f of a fiber. From (VII.4.11) we see that $\xi + f \in \mathrm{im}\, j^*$. Also $\mathrm{c}_1(p_*\mathcal{O}_Z)$ is in $\mathrm{im}\, j^*$ as already the universal family over Z is the pullback of the universal family over X. In order to show that it is linearly independent of $\xi + f$ we calculate this class using the exact sequence (VII.4.8).The class $\mathrm{c}_1(p_*\mathcal{O}_Z)$ is the degree 1 part of

$$\begin{aligned} \mathrm{ch}(p_*\mathcal{O}_Z) &= p_*\mathrm{ch}\left(\mathcal{O}_{\mathbb{P}\times E} - \mathcal{O}_{\mathbb{P}\times E}(-Z)\right) \\ &\quad - p_* \left(\mathrm{ch}(p^*\mathcal{O}_{\mathbb{P}}(-1)) \cdot \mathrm{ch}\left((\pi \times \mathrm{id})^* F_0^\vee\right)\right) \\ &\quad - \left(\mathrm{ch}(\mathcal{O}_{\mathbb{P}}(-1))\right) \cdot p_*(\pi \times \mathrm{id})^* \mathrm{ch}\left(F_0^\vee\right) \\ &\quad - \left(1 - \xi + \frac{1}{2}\xi^2\right) \cdot \pi^* p_* \mathrm{ch}\left(F_0^\vee\right) \end{aligned}$$

Therefore we find that

$$\mathrm{c}_1(p_*\mathcal{O}_Z) = -\pi^* p_* \left(\mathrm{ch}\left(F_0^\vee\right)_2\right) + \xi \cdot \pi^* p_* \left(\mathrm{ch}\left(F_0^\vee\right)_1\right).$$

The first term is equal to $\deg\left(p_*\left(\mathrm{ch}\left(F_0^\vee\right)_2\right)\right)\cdot f$ and

$$\deg\left(p_*\left(\mathrm{ch}\left(F_0^\vee\right)_2\right)\right)=\frac{1}{2}\int_{E\times E}(-E_0-D)^2=1.$$

For the second term note that $p_*\left(\mathrm{ch}\left(F_0^\vee\right)_1\right)=\left(p_*\mathrm{ch}\left(F_0^\vee\right)\right)_0=\mathrm{rk}Rp_*\left(F_0^\vee\right)$. By Grothendieck-Riemann-Roch we have

$$p_*\mathrm{ch}\left(F_0^\vee\right)=\mathrm{ch}\left(Rp_*F_0^\vee\right).$$

From $F_0=\mathcal{O}_{E\times E}(E_0+\Delta)$ we deduce that

$$\begin{aligned}
Rp_*F_0^\vee &= Rp_*\mathcal{O}_{E\times E}(-E_0)-Rp_*\mathcal{O}_\Delta(-E_0)\\
&= Rp_*\left(\mathcal{O}_{E\times E}-\mathcal{O}_{E_0}\right)-p_*\mathcal{O}_\Delta(-E_0)\\
&= \mathcal{O}_E-R^1p_*\mathcal{O}_{E\times E}-\mathcal{O}_E-\mathcal{O}_E(-p_0)\\
&= -R^1p_*\mathcal{O}_{E\times E}-\mathcal{O}_E(-p_0),
\end{aligned}$$

where $p_0=\Delta\cap E_0$. So we see that $\mathrm{rk}Rp_*\left(F_0^\vee\right)=-2$, which implies that

$$\mathrm{c}_1\left(p_*\mathcal{O}_Z\right)=-f-2\xi.$$

So $\mathrm{c}_1(p_*\mathcal{O}_Z)$ is linearly independent from $[D_{\mathrm{Hilb}}]$ for a divisor D on S intersecting E nontrivially and hence j^* is surjective. By Voisin's theorem, this means that Sym^2E is preserved in a codimension 2 subset of the deformation space M of X.

CHAPTER VIII

Open problems

Let $f : X \longrightarrow B$ be a Lagrangian fibration of an irreducible symplectic manifold. Besides the problems left open in chapter VII, one of the most important open problems in the study of singular fibers of Lagrangian fibrations seems to be to understand the geometry of the *discriminant locus*. This is defined as the analytic subset

$$D = \{t \in B : X_t \text{ is singular}\}$$

of B. Although we do not at all contribute to solving those problems in this work, we would like to put them into context. The most important problems regarding the discriminant from our point of view is to understand its deformation behaviour and to control its degree. Finally, it would be valuable to have more explicit descriptions of families of Lagrangian fibrations.

VIII.1. Discriminant locus - deformation behaviour

For any proper morphism $f : X \longrightarrow B$ of complex spaces there is the notion of a discriminant *subspace* – as opposed to a discriminant *locus*. This means that D is going to have a structure of a complex space instead of an analytic set only. It is explained by Tessier for finite morphisms in [**Tei77**], but there he also claims that this works for proper morphisms in the same way. I am very grateful to Duco van Straten for pointing out this reference.
First, one defines the *image* of f, see [**Tei77**, § 1, p 572], then the *critical subspace* of $C = C(f) \hookrightarrow X$, see [**Tei77**, § 2, p 587], and then the discriminant subspace $D = D(f) \hookrightarrow B$ as the image of the critical subspace, see [**Tei77**, § 2, p 588].

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & X \arrow[d] \\ D \arrow[r, hook] & B \end{tikzcd}$$

The definition uses fitting ideals and it is compatible with arbitrary base change. Thus, if

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "F"] \arrow[dr, "\pi"'] & & \mathbb{P}(\pi_*\mathcal{L}) \arrow[dl, "q"] \\ & M_L & \end{tikzcd}$$

is the family of Lagrangian fibrations of the restriction of the universal family of deformations of X to M_L as in Lemma VII.2.2, then $D(F) \times_{M_L} \{0\} = D(f)$ where $0 \in M_L$ is the point corresponding to X.

Question VIII.1.1. Is $D(F) \longrightarrow M_L$ a flat deformation of $D(f)$?

This question seems to be important for the following reason: if the map $D(F) \longrightarrow M_L$ were a flat deformation of $D(f)$ and we could somehow control this deformation, we could replace the problem of deforming a Lagrangian subvariety $i : Y \hookrightarrow X$ in a locally trivial way by the problem of deforming a Lagrangian fibration $f : X \longrightarrow B$ together with a point $t \in D(f)$, possibly avoiding problems with bad singularities in the fibers.
If $B = \mathbb{P}^n$ we know by [**HO09a**, Prop 3.1] and [**Hwa08**, Prop 4.1] that $D(f)$ is non-empty of pure codimension one. Then a positive answer to Question VIII.1.1 would in particular imply that the degree of $D(f)$ remains constant.

Question VIII.1.2. Does $D(f) \hookrightarrow \mathbb{P}^n$ have embedded points? Can it be defined as the vanishing of a section in some line bundle? Can this line bundle be described canonically in terms of the geometry of f?

Formulated in a different way, we could ask if it is possible to show that Teissier's description of the ideal sheaf of D can be shown to be a line bundle. The following question is – similar to those in sections VII.2 and VII.3 – trying to deform to an easier situation.

Question VIII.1.3. Is there a small deformation of $f' : X' \longrightarrow B'$ of f such that $D(f')$ is irreducible?

It is not clear to us, whether this is always possible. The reason is that we cannot avoid the following situation: suppose $D(f) = D_1 \cup D_2$ and the divisors $X_1 = f^{-1}(D_1)$ and $X_2 = f^{-1}(D_2)$ are irreducible. Then their classes in $H^2(X, \mathbb{Q})$ are multiples of one another as $\varrho(B) = 1$ and so one is preserved under deformation if and only if the other one is.

VIII.2. Discriminant locus - degree

Let $f : X \longrightarrow \mathbb{P}^n$ be a Lagrangian fibration on a projective irreducible symplectic manifold. Another important problem is to calculate or at least estimate the degree of the discriminant divisor. We will derive an expression relating the degree of the irreducible components of the discriminant divisor with certain intersection numbers on X. A definition of the multiplicity of the singularity is proposed. The ideas are basically contained in [**Saw08b**], where only special singular fibers are considered. But in the non-principally polarized case the arguments presented there seem to be incomplete. We do not give any satisfying answers, but we try to put Sawon's approach on a more general footing. For a good reason this chapter is called *Open problems*.

Recall from Proposition IV.3.2, that for the chern classes $\mathrm{c}_p(X)$ of X there are constants a_p only depending on the deformation type of X such that for any class $\alpha \in H^2(X, \mathbb{R})$

$$q_X(\alpha)^{n-p} = a_p \cdot \int_X \alpha^{2(n-p)}.\mathrm{c}_p(X)$$

where q_X is the Beauville-Bogomolov form. So given classes $\alpha, \beta \in H^2(X, \mathbb{R})$ we have

$$(q_X(\alpha)^n)^{n-1} \left(q_X(\beta)^{n-1}\right)^n = a_0^{n-1} a_1^n \left(\int_X \alpha^{2n}\right)^{n-1} \left(\int_X c_2(X).\beta^{2n-2}\right)^n$$

On the other side

$$\left(q_X(\alpha)^{n-1}\right)^n (q_X(\beta)^n)^{n-1} = a_0^{n-1} a_1^n \left(\int_X c_2(X).\alpha^{2n-2}\right)^n \left(\int_X \beta^{2n}\right)^{n-1},$$

so that

$$\left(\int_X \alpha^{2n}\right)^{n-1} \left(\int_X c_2(X).\beta^{2n-2}\right)^n = \left(\int_X c_2(X).\alpha^{2n-2}\right)^n \left(\int_X \beta^{2n}\right)^{n-1}.$$

If we take $\alpha = \sigma + t_1\bar{\sigma}$ and $\beta = A + t_2 H$ where A is an f-ample divisor, H is the pullback of an ample divisor and the t_i are indeterminates, we obtain

(VIII.2.1)

$$\left(\int_X (\sigma\bar{\sigma})^n\right)^{n-1} \left(\int_X c_2.H^{n-1}.A^{n-1}\right)^n = \left(\int_X c_2.(\sigma\bar{\sigma})^{n-1}\right)^n \left(\int_X A^n.H^n\right)^{n-1}$$

by comparing the coefficients of $(t_1t_2)^{n(n-1)}$. In [**Saw08b**, Lem 2] it is shown using Roszansky-Witten techniques that

$$\text{(VIII.2.2)} \qquad \frac{\left(\int_X c_2(X).(\sigma\bar{\sigma})^{n-1}\right)^n}{\left(\int_X (\sigma\bar{\sigma})^n\right)^{n-1}} = \frac{24^n (n!)^2}{n^n} \int_X \sqrt{\hat{A}}$$

where $\sqrt{\hat{A}} = 1 + \frac{c_2}{24} + \frac{7c_2^2 - c_4}{5760} + \ldots$ is the square root of the $\hat{A}$-genus. Combining (VIII.2.1) and (VIII.2.2) and taking the n-th root we end up with

$$\text{(VIII.2.3)} \qquad \frac{\int_X c_2(X).H^{n-1}.A^{n-1}}{\left(\int_X A^n.H^n\right)^{\frac{n-1}{n}}} = \sqrt[n]{\frac{24^n (n!)^2}{n^n} \int_X \sqrt{\hat{A}}} =: \gamma(X)$$

where the right hand side does not depend on the ample divisor A.
The term $\int_X A^n.H^n$ is equal to the product $\deg_B(H) \cdot \deg_{X_\eta}(A)$ of the degrees of H on B and of A on a general fiber X_η of f. The integral

$$\int_X c_2(X).H^{n-1}.A^{n-1}$$

is more interesting and urges us to understand $\mathrm{c}_2(X)$. Similar to Lemma VI.2.2 one can show that there is a morphism of short exact sequences

(VIII.2.4)
$$\begin{tikzcd}
0 \arrow[r] & f^*\Omega_B \arrow[r] \arrow[d, hook] & \Omega_X \arrow[r] \arrow[d, "\cong"'] & \Omega_{X/B} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{X/B} \arrow[r] & T_X \arrow[r] & f^*T_B &
\end{tikzcd}$$

where by definition, the lower sequence is the dual of the upper sequence. In general the lower sequence fails to be exact on the right. In any case, this gives a complex

$$f^*\Omega_B \longrightarrow \Omega_X \cong T_X \longrightarrow f^*T_B$$

which is now a complex of locally free sheaves. We will refer to it as the ΩT-complex and declare it to live in degrees 0 to 2. It is exact precisely at the points where f is smooth.
In this way, we can calculate the Chern character of X using the exact sequences

$$0 \to T_{X/B} \to T_X \xrightarrow{f_*} f^*T_B \to H^2(\Omega T) \to 0$$
$$0 \to f^*\Omega_B \to T_{X/B} \to H^1(\Omega T) \to 0$$

So we have

(VIII.2.5) $\mathrm{ch}(T_X) = f^*\mathrm{ch}(T_B) + f^*\mathrm{ch}(\Omega_B) + \mathrm{ch}(H^1(\Omega T)) - \mathrm{ch}(H^2(\Omega T)).$

We consider degree one terms only and we find

$$0 = \mathrm{c}_1(X) = \underbrace{f^*\mathrm{c}_1(T_B) + f^*\mathrm{c}_1(\Omega_B)}_{=0} + \mathrm{c}_1(H^1(\Omega T)) - \mathrm{c}_1(H^2(\Omega T)),$$

and we obtain $\mathrm{c}_1(H^1(\Omega T)) = \mathrm{c}_1(H^2(\Omega T))$. Since $\mathrm{ch}_2 = \frac{\mathrm{c}_1^2 - 2\mathrm{c}_2}{2}$ this and (VIII.2.5) yields
(VIII.2.6)
$$\begin{aligned}
\mathrm{c}_2(X) &= \frac{\mathrm{c}_1^2(X)}{2} - \mathrm{ch}_2(T_X) \\
&= -\mathrm{ch}_2(H^1(\Omega T)) + \mathrm{ch}_2(H^2(\Omega T)) - f^*\mathrm{ch}_2(\Omega_B) - f^*\mathrm{ch}_2(T_B) \\
&= \mathrm{c}_2(H^1(\Omega T)) - \mathrm{c}_2(H^2(\Omega T)) + 2\mathrm{c}_2(B) - \mathrm{c}_1^2(B).
\end{aligned}$$

As $B = \mathbb{P}^n$ we can express this in terms of the pullback of a hyperplane H.

$$f^*\mathrm{c}(\mathbb{P}^n) = (1+H)^{n+1} = 1 + (n+1)H + \binom{n+1}{2}H^2 + \dots,$$

so $2\mathrm{c}_2(B) - \mathrm{c}_1^2(B) = (n+1)nH^2 - (n+1)^2H^2$ leading to

$$\mathrm{c}_2(X) = \mathrm{c}_2(H^1(\Omega T)) - \mathrm{c}_2(H^2(\Omega T)) - (n+1)H^2.$$

With this the numerator of the left hand side of (VIII.2.3) reads

$$\begin{aligned} &\int_X \left(\mathrm{c}_2(H^1(\Omega T)) - \mathrm{c}_2(H^2(\Omega T)) - (n+1)H^2\right).H^{n-1}.A^{n-1} \\ &= \int_X \mathrm{c}_2(H^1(\Omega T)).H^{n-1}.A^{n-1} - \int_X \mathrm{c}_2(H^2(\Omega T)).H^{n-1}.A^{n-1} \end{aligned}$$

because $H^{n+1} = 0$ so that we have to calculate $\delta(X) := \mathrm{c}_2(H^1(\Omega T)) - \mathrm{c}_2(H^2(\Omega T))$ and (VIII.2.3) becomes

$$\text{(VIII.2.7)} \qquad \frac{\int_X \delta(X).H^{n-1}.A^{n-1}}{\left(\int_X A^n.H^n\right)^{\frac{n-1}{n}}} = \gamma(X).$$

As remarked above, the complex ΩT is exact at smooth points of f, hence the cycle $\delta(X)$ is supported on the singular locus of the singular fibers. Assume the discriminant divisor decomposes as

$$D(f) = \sum_i D_i,$$

where D_i are irreducible components. Accordingly, we will have a decomposition

$$\delta(X) = \sum_i \delta_i(X) + \delta'$$

where $\delta_i(X)$ is supported over D_i and δ' is supported over a subset of B codimension ≥ 2. The cycle H^{n-1} is the pullback of a line $\ell \subseteq \mathbb{P}^n$. So δ' does not play a role for the calculation of (VIII.2.7) as $H^{n-1}.\delta' = 0$.
Each cycle $\delta_i(X)$ gives an algebraic family of cycles $\{S_i(t)\}_{t\in D_i}$ in X parametrized by D_i. We write $S_i \equiv S_i(t)$, where $\equiv$ denotes algebraic equivalence. So we find

$$\delta_i(X).H^{n-1} = \sum_{t\in \ell\cap D_i} S_i(t) \equiv \#\{t \in \ell \cap D_i\} \cdot S_i = \deg(D_i)\cdot S_i.$$

Going back to (VIII.2.7) the discussion gives

$$\frac{1}{\left(\int_X A^n.H^n\right)^{\frac{n-1}{n}}} \sum_i \int_X \deg(D_i) S_i.A^{n-1} = \gamma(X)$$

which suggests to define weights

$$w_i := \frac{1}{\left(\int_X A^n.H^n\right)^{\frac{n-1}{n}}} \int_X S_i.A^{n-1},$$

so that we have the nice formula

$$\sum_{i=1}^{r} w_i \deg(D_i) = \gamma(X).$$

As said, the whole sum does not depend on A but a priori it might happen that the single w_i depend on A. This sounds unlikely, but how to show that this does not occur? Calculations in the case of $\mathrm{Hilb}^2(S)$ for an elliptic K3 surface S indicate that the decomposition does indeed not depend on the choice of A, but some work remains to be done to settle this question.

VIII.3. Generalizations

As our main results are built from many small pieces from different areas, it should be obvious that there is ample space for generalizations. Several calculations support this opinion. We want to explain, in which directions we plan to generalize.

Let us first look at the Hodge theoretic part. Our goals are to generalize Theorem III.4.3 and to construct mixed Hodge structures over Artin rings similar to Lemma III.4.1 for more general classes of singularities and deformations.

If we continue to work with the comples $\widetilde{\Omega}^{\bullet}_{\mathcal{Y}/S}$, then first of all we want to leave simple normal crossing singularities behind. The "simple" is annoying, i.e. the condition that irreducible components be smooth. The validity of our results should – at least in the algebraic category – only depend on the local structure of the singularities, not on the global geometry. The following class of singularities seems to be a promising candidate.

Definition VIII.3.1. A variety Y is said to have *transversal singularities*, if the following conditions are satisfied. For every point $y \in Y$ there is an isomorphism

$$\widehat{\mathcal{O}}_{Y,y} \cong k[[x_1, \ldots, x_n]]/I,$$

where $I = (f_1, \ldots, f_k)$ and for each m with $1 \le m \le k$ there is a subset $I_m \subseteq \{1, \ldots, n\}$ such that

$$f_m = \prod_{i \in I_m} x_i.$$

It can be used in inductive proofs just like normal crossing singularities in [**Fri83**, Lem 1.5]. The reason is this. If locally $Y = \cup_i Y_i$ is a decomposition in irreducible components, then the restriction $Y' \cap Y_j$ of $Y' = \cup_{i \neq j} Y_i$ to the irreducible component Y_j has again transversal singularities. With this definition at hand we want to solve

Problem VIII.3.2. Let $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_{\mathbb{C}}$, let Y be a proper $\mathbb{C}$-variety with transversal singularities and let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y. Then show that $\widetilde{\Omega}^{\bullet}_{\mathcal{Y}^{\mathrm{an}}/S}$ is quasi-isomorphic to the constant sheaf $\underline{R}_{Y^{\mathrm{an}}}$ and that the analogue of spectral sequence (III.2.5) associated to $\widetilde{\Omega}^{\bullet}_{\mathcal{Y}^{\mathrm{an}}/S}$ degenerates at E_1. Use this complex to put a mixed Hodge structure over R on the $H^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}})$.

A solution tho this problem can be used for

Problem VIII.3.3. Let $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_{\mathbb{C}}$, let X and Y be proper $\mathbb{C}$-variety with transversal singularities, let $g : \mathcal{X} \longrightarrow S$ and $f : \mathcal{Y} \longrightarrow S$ be locally trivial deformations of X and Y and let $i : \mathcal{Y} \longrightarrow \mathcal{X}$ be an S-morphism. Then show that the morphism $i^* : R^q g_* \widetilde{\Omega}^p_{\mathcal{X}/S} \longrightarrow R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}$ has constant rank for all p, q.

It does not sound absurd that these two results might generalize to locally trivial deformations of arbitrary varieties. Therefore one has to take a completely new approach and understand resolution of singularities. Then it might be possible to develop a theory in the spirit of [**Del71, Del74**] in the relative locally trivial situation. The first step would be

Problem VIII.3.4. Let Y be a proper $\mathbb{C}$-variety and let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y. Find a replacement for $\widetilde{\Omega}^{\bullet}_{\mathcal{Y}/S}$ to generalize and solve the preceding problems.

The question is how to find a replacement for the semi-simplicial resolution from Lemma III.2.4 in the normal crossing case. This seems to be important, because the resolution was explicitly used in our proofs. Here is a proposal how to obtain such a resolution for transversal singularities. Take the normalization $Y^{[0]} \longrightarrow Y$ and replace $Y^{[k]}$ with the k-fold fiber product

$$Y^{[0]} \times_Y Y^{[0]} \times_Y \ldots \times_Y Y^{[0]}.$$

This is a semi-simplicial object with infinitly many terms.
Working with the complex $\widetilde{\Omega}^{\bullet}_{\mathcal{Y}/S}$ would keep the relation to symplectic geometry via Proposition VI.2.4. Once we established all the properties of $\widetilde{\Omega}^{\bullet}_{\mathcal{Y}/S}$ for the class of locally trivial deformations of transversal singularities, Voisin's theorem is should be valid in this situation as well. The restriction posed by the unnatural assumptions of Lemma V.3.4 should only be a technical problem. More difficult seems to be the correct definition of M_Y.

Problem VIII.3.5. Show that the image of the natural map $p : M_i \longrightarrow M$ from Proposition VI.3.2 is a closed subvariety of M. Decide whether it is smooth or not and how its codimension can be calculated.

Then of course one should settle the applications to singular fibers of a Lagrangian fibration $f : X \longrightarrow B$. It should be possible to decide whether for the reduction Y of a general singular fiber over $t \in B$ the codimension $\operatorname{codim}_M M_Y$ is equal to the number $\#J$ of irreducible components of the divisor X_0 as defined in section VII.3. To decide whether $\#J$ coincides with the number $\#I$ of irreducible components of Y seems to be more difficult.
In this treatise, locally trivial deformations play a major role. So we want to pose what we consider important problems regarding locally trivial deformations. One is to study local triviality in the large, that is, not only over Artin rings. In the analytic category the notion of local triviality is reasonable in the sense of germs, see [**FK87**]. In the algebraic category it is not reasonable to define local triviality by replacing the Artinian scheme in Definition I.2.1 by an affine scheme, as this is rather restrictive. Instead we propose

Definition VIII.3.6. Let S be a k-scheme and $\mathcal{Y} \longrightarrow S$ be a scheme over S. We say that $\mathcal{Y} \longrightarrow S$ is a *locally trivial family* parametrized by S, if for every k-morphism $S' \longrightarrow S$ of a local Artinian k-scheme S' the morphism f in the diagram

$$\begin{tikzcd} \mathcal{Y} \times_S S' \arrow[r] \arrow[d, "f"'] & \mathcal{Y} \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

is a locally trivial deformation in the sense of Definition I.2.1.

With this definition we pose the problem of constructing a Hilbert scheme for locally trivial deformations.

Problem VIII.3.7. Let $Y \hookrightarrow X$ be a closed immersion. Construct a *Hilbert scheme for locally trivial deformations* of Y in X. This should be the unique maximal connected subscheme $\mathrm{Hilb}^{\mathrm{lt}}(Y \hookrightarrow X)$ of $\mathrm{Hilb}(X)$ containing the point corresponding to Y with the property that the restriction $\mathscr{U} \longrightarrow \mathrm{Hilb}^{\mathrm{lt}}(Y \hookrightarrow X)$ of the universal family is a locally trivial family in the sense of Definition VIII.3.6. Moreover, $\mathrm{Hilb}^{\mathrm{lt}}(Y \hookrightarrow X)$ should have the universal property with respect to connected locally trivial families.

If $\mathrm{Hilb}^{\mathrm{lt}}(Y \hookrightarrow X)$ existed, it would certainly not be proper in general as the following example shows.

Example VIII.3.8. Let $f : X \longrightarrow B$ be a morphism of smooth and projective algebraic $\mathbb{C}$-varieties with $\dim B = 1$ and let Y be a smooth fiber of f. As f is flat, there is a classifying map $\varphi : B \longrightarrow \mathrm{Hilb}(X)$ to the Hilbert scheme of X. This map is clearly injective. Let $V \subseteq B$ be the maximal open subset such that $f_V : X_V \longrightarrow V$ is smooth where $X_V = f^{-1}(V)$ and $f_V = f|_{X_V}$. Then f_V is a locally trivial deformation of Y. On the other hand for every $b \in D := B \setminus V$ and every neighbourhood $V' \subseteq B$ of b the map $f_{V'}$ is not a locally trivial deformation of $X_b := f^{-1}(b)$. It is a smoothing of X_b as $V' \cap V \neq \emptyset$. So if $D \neq \emptyset$ and if there were a Hilbert scheme $\mathrm{Hilb}^{\mathrm{lt}}(Y \hookrightarrow X)$ for locally trivial deformations the classifying map $V \longrightarrow \mathrm{Hilb}^{\mathrm{lt}}(Y \hookrightarrow X)$ would not extend to B. Hence $\mathrm{Hilb}^{\mathrm{lt}}(Y \hookrightarrow X)$ cannot be proper by the valuative criterion for properness. Clearly, such morphisms f exist in abundance.

It is obvious that the definition of a locally trivial family can be extended to other types of algebraic objects. Problem VIII.3.7 is related to a question of Flenner and Kosarew, see [**FK87**, p. 630]
If $k = \mathbb{C}$ we would like to know the answer to the following question.

Question VIII.3.9. Let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial family over a $\mathbb{C}$-scheme S in the sense of Definition VIII.3.6. Suppose that S^{an} is simply connected and f is proper. Is then $f^{\mathrm{an}} : Y^{\mathrm{an}} \longrightarrow S^{\mathrm{an}}$ a C^{∞}-trivial fiber bundle for S small enough and a suitable notion of a diffeomorphism between complex spaces?

APPENDIX A

Commutative algebra

A.1. Flatness

The statements and proofs below are a straight forward generalization of the corresponding material from [**Ser06**], Appendix A. We will include them for convenience.

Proposition A.1.1. Let $R \in \mathrm{Art}$, R' be an Artinian local R-algebra and $R' \twoheadrightarrow R$ a surjective R-algebra morphism. The following are equivalent for an R'-module M.

(1) M is flat.
(2) $\mathrm{Tor}_1^{R'}(M, R) = 0$ and $M \otimes_{R'} R$ is flat over R.

Proof. Flatness is stable under base change, so the implication (1) $\Rightarrow$ (2) is clear. The local criterion for flatness as formulated in [**Mat80**, (20.C) Thm 49] yields (2) $\Rightarrow$ (1). $\square$

Let A be a flat noetherian R-algebra, $R \in \mathrm{Art}_k$, and $I \subseteq A$ an ideal. Let R' be an Artinian local R-algebra and $R' \twoheadrightarrow R$ a surjective R-algebra morphism. Let $\mathfrak{A} = A \otimes_R R'$ and $\mathfrak{I} \subseteq \mathfrak{A}$ be an ideal with $\mathfrak{A}/\mathfrak{I} \otimes_{R'} R \cong A/I$. We want to have conditions on $\mathfrak{I}$ that guarantee flatness of $\mathfrak{A}/\mathfrak{I}$ over R'.

Theorem A.1.2. *Assume that A/I is flat over R. Let Π be a presentation*

$$A^n \longrightarrow A^N \longrightarrow A \longrightarrow A/I \longrightarrow 0 \tag{A.1.1}$$

of A/I as an A module. Then the following conditions are equivalent for an ideal $\mathfrak{I} \subseteq \mathfrak{A}$:

(1) *$\mathfrak{A}/\mathfrak{I}$ is R'-flat and $\mathfrak{A}/\mathfrak{I} \otimes_{R'} R \cong A/I$.*
(2) *There is a presentation $\Pi' : \mathfrak{A}^n \xrightarrow{\varphi} \mathfrak{A}^N \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{A}/\mathfrak{I} \longrightarrow 0$ such that $\Pi = \Pi' \otimes_{R'} R$.*
(3) *There is a complex $\Pi' : \mathfrak{A}^n \xrightarrow{\varphi} \mathfrak{A}^N \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{A}/\mathfrak{I} \longrightarrow 0$ which is exact except possibly at $\mathfrak{A}^N$ such that $\Pi = \Pi' \otimes_{R'} R$.*

Proof. We obtain $\mathrm{Tor}_1^{R'}(\mathfrak{A}/\mathfrak{I}, R)$ as $H_1(\Pi' \otimes_{R'} R) = H_1(\Pi)$ which is zero as Π is exact. Here we think of A/I as sitting in degree 1 using homological convention, i.e. degrees ascending from left to right. As A/I is flat over R, the implication (2) $\Rightarrow$ (1) follows from Proposition A.1.1.

Let us verify (1) $\Rightarrow$ (2). We will first construct a surjective morphism $\mathfrak{A}^N \longrightarrow \mathfrak{I}$. As $\mathfrak{A}/\mathfrak{I}$ is flat over R' so is $\mathfrak{I}$ by the long exact Tor-sequence and by tensoring $0 \longrightarrow \mathfrak{I} \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{A}/\mathfrak{I} \longrightarrow 0$ with R we obtain an isomorphism $\mathfrak{I} \otimes_{R'} R \xrightarrow{\cong} I$, in particular a surjective morphism $\mathfrak{I} \longrightarrow I$. So by lifting generators of I we obtain a morphism $\mathfrak{A}^N \longrightarrow \mathfrak{I}$ which by tensoring with R gives the surjective morphism $A^N \longrightarrow I$ from Π. So $\mathfrak{A}^N \longrightarrow \mathfrak{I}$ is surjective by Nakayama's Lemma.
Taking the kernel we obtain an exact sequence $0 \longrightarrow K \longrightarrow \mathfrak{A}^N \longrightarrow \mathfrak{I} \longrightarrow 0$. As $\mathfrak{I}$ is a flat lifting of I, applying the functor $\otimes_{R'} R$ gives $K \otimes_{R'} R \cong \ker(A^N \longrightarrow I)$. Lifting generators of $\ker(A^N \longrightarrow I)$ to K yields a surjective morphism $\mathfrak{A}^n \longrightarrow K$ and as above when tensored with R this morphism is just $A^n \longrightarrow \ker(A^N \longrightarrow I)$.
The implication (2) $\Rightarrow$ (3) is obvious. We conclude with (3) $\Rightarrow$ (1). If Π' is not exact at $\mathfrak{A}^N$ we add finitely many generators to have a presentation

$$\bar{\Pi}: \qquad \mathfrak{A}^m \xrightarrow{\phi} \mathfrak{A}^N \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{A}/\mathfrak{I} \longrightarrow 0.$$

So by construction $\mathfrak{A}^n \subseteq \mathfrak{A}^m$ and we have a factorization

$$\begin{tikzcd} \mathfrak{A}^m \arrow[r, "\phi"] & \mathfrak{A}^N \\ \mathfrak{A}^n \arrow[u, hook] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Hence

$$\operatorname{im}(\varphi \otimes_{R'} R) \subseteq \operatorname{im}(\phi \otimes_{R'} R) \subseteq \ker(A^N \longrightarrow A) = \operatorname{im}(\varphi \otimes_{R'} R)$$

where the last equality comes from $\Pi' \otimes_{R'} R = \Pi$. So $\operatorname{im}(\phi \otimes_{R'} R) = \ker(A^N \longrightarrow A)$ implying that $\bar{\Pi} \otimes_{R'} R = \Pi$, hence $\bar{\Pi} \otimes_{R'} R$ is exact. So as above we conclude that $\operatorname{Tor}_1^{R'}(\mathfrak{A}/\mathfrak{I}, R) = H_1(\bar{\Pi} \otimes_{R'} R) = H_1(\Pi) = 0$ and $\mathfrak{A}/\mathfrak{I}$ is R'-flat. $\square$

Corollary A.1.3. In the situation of the preceding Theorem assume that $I = (f_1, \ldots, f_N) \subseteq A$ and $\mathfrak{I} = (F_1, \ldots, F_N) \subseteq \mathfrak{A}$ with $F_i \mapsto f_i$ under $R' \longrightarrow R$. Then every relation $r_1 f_1 + \ldots + r_N f_N = 0$ in A lifts to a relation

$$R_1 F_1 + \ldots + R_N F_N = 0$$

in $\mathfrak{A}$ if and only if $\mathfrak{A}/\mathfrak{I}$ is flat over R' and $\mathfrak{A}/\mathfrak{I} \otimes_{R'} R \cong A/I$.

Proof. As $F_i \mapsto f_i$ the sequence

$$\mathfrak{A}^N \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{A}/\mathfrak{I} \longrightarrow 0$$

gives

$$\pi: \qquad A^N \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

when tensored with R. When we complete π to a presentation Π as in (A.1.1), the lifting of relations is an other way of saying that Π lifts to a presentation Π' as in (3) of Theorem A.1.2 which implies the claim. $\square$

A.2. Modules over Artin rings

Let R be a noetherian ring and $\varphi : F \longrightarrow G$ be a morphism between finitely generated free R-modules. We define $I_j(\varphi) = \operatorname{im}(\varphi' : \Lambda^j F \otimes (\Lambda^j G)^\vee \longrightarrow R)$, where φ' is induced by $\Lambda^j\varphi : \Lambda^j F \longrightarrow \Lambda^j G$. If we interpret φ as a matrix, then $I_j(\varphi)$ is the ideal generated by all $j \times j$-minors of φ. If F and G are finitely generated but not necessarily free, the definition still makes sense if G is projective. One defines the *rank of* φ as $\mathrm{rk}\varphi := \max\{i : I_i(\varphi) \neq 0\}$.

Definition A.2.1. Let R be a noetherian ring and $\varphi : F \longrightarrow G$ be a morphism between finitely generated R-modules. Suppose G is projective. We say that φ has *constant rank* k, if $I_k(\varphi) = R$ and $I_{k-1}(\varphi) = 0$. We say that φ has *constant rank*, if there is some k such that φ has constant rank k.

A characterization of this property is given by the following Lemma, the proof of which is found at [**Eis95**, Prop 20.8].

Lemma A.2.2. Let R be a noetherian ring and $\varphi : F \longrightarrow G$ be a morphism between finitely generated R-modules. Suppose G is projective. Then φ has constant rank, if and only if $\operatorname{coker}\varphi$ is a projective R-module. $\square$

If G is projective and $\varphi : F \longrightarrow G$ is of constant rank, then $\operatorname{im}\varphi$ is projective. If moreover F is projective, then also $\ker\varphi$ is projective. We will mostly be concerned with local noetherian rings, where projectivity is equivalent to freeness. We show next, that over local Artin rings the inclusion of a free submodule is of constant rank.

Lemma A.2.3. Let $(R, \mathfrak{m})$ be a local Artin ring with residue field k and $F_1 \subseteq F$ two finitely generated free R-modules. Then F/F_1 is free and $\varphi : F_1 \otimes k \longrightarrow F \otimes k$ is injective.

Proof. If $F_1 \longrightarrow F$ is injective, then F/F_1 is free if and only if $\varphi : F_1 \otimes k \longrightarrow F \otimes k$ is injective. This holds over any local noetherian ring by [**Ser06**, Cor A.6]. As both F_1 and F are free, the diagramm

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}F_1 \arrow[r] \arrow[d, hook] & F_1 \arrow[r] \arrow[d, hook] & F_1 \otimes k \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & \mathfrak{m}F \arrow[r] & F \arrow[r] & F \otimes k \arrow[r] & 0
\end{tikzcd}$$

has exact rows. If φ is not injective, then there is $x_1 \in F_1 \cap \mathfrak{m}F$ with $x_1 \notin \mathfrak{m}F_1$. Because of this last property we find $x_2, \ldots, x_k \in F$ such that $x_1, x_2, \ldots, x_k$ is a basis of F by Nakayama's Lemma. In particular, if $\alpha x_1 = 0$ for some non-zero $\alpha \in R$ implies that $\alpha = 0$. The case $R = k$ is trivial, so we may assume that the maximal ideal $\mathfrak{m}$ is non-zero. So there is $0 \neq \alpha \in \operatorname{Ann}\mathfrak{m}$. Therefore we have $\alpha x_1 \in \alpha\mathfrak{m}F = 0$, a contradiction. □

COROLLARY A.2.4. Let $(R, \mathfrak{m})$ be a local Artin ring with residue field k and $F_1, F_2 \subseteq F$ be two free submodules in a finitely generated free R-module. Then $F_1 \cap F_2 = 0$ if and only if $F_1 \otimes k \cap F_2 \otimes k = 0$.

PROOF. The condition $F_1 \cap F_2 = 0$ means that $F_1 \oplus F_2 \longrightarrow F$ is injective. This implies injectivity of $F_1 \otimes k \oplus F_2 \otimes k \longrightarrow F \otimes k$ by Lemma A.2.3, hence $F_1 \otimes k \cap F_2 \otimes k = 0$. The converse again follows from [**Ser06**, Cor A.6] over any local noetherian ring. □

LEMMA A.2.5. Let R be a local Artin ring and

$$\begin{tikzcd} F \arrow[d, "\psi"'] \arrow[dr, "\varphi"] & \\ G \arrow[r, "\eta"] & H \end{tikzcd}$$

a diagram of R-modules where G, H are free, η has constant rank and $\operatorname{im}\psi \cap \ker\eta = 0$. Then φ has constant rank if and only if ψ has.

PROOF. We may assume that ψ is injective since replacing F by $\operatorname{im}\psi$ does not change any cokernel. As η has constant rank, $\ker\eta$ and $\operatorname{coker}\eta = H/\eta(G)$ are free. When we consider the two exact sequences

$$0 \longrightarrow G/(F \oplus \ker\eta) \longrightarrow H/\varphi(F) \longrightarrow H/\eta(G) \longrightarrow 0$$

and

$$0 \longrightarrow \ker\eta \longrightarrow G/F \longrightarrow G/(F \oplus \ker\eta) \longrightarrow 0$$

we see that $H/\varphi(F)$ is free if and only if $G/(F \oplus \ker\eta)$ is free. By Lemma A.2.3 this is the case if and only if G/F is free. This proves the Lemma. □

LEMMA A.2.6. Let R be a local Artin ring and

$$H' \overset{d_1}{\to} H \overset{d_2}{\to} H''$$

a complex of free R-modules, i.e. $d_2 \circ d_1 = 0$. If the d_i have constant rank, then the cohomology $\ker d_2 / \operatorname{im} d_1$ is free.

Proof. Consider the diagram

$$\begin{tikzcd}
H' \arrow[r, "d_1"] \arrow[d] & H \arrow[r, "d_2"] & H'' \\
F \arrow[r, "\psi"] \arrow[ur, hook, "\varphi"] & G \arrow[u, hook, "\eta"'] &
\end{tikzcd}$$

where $F = \operatorname{im} d_1$ and $G = \ker d_2$. Here F and G are free by the remarks following Definition A.2.1, hence the claim follows from Lemma A.2.3. $\square$

APPENDIX B

Hodge theory

For completeness, we record the definition of a mixed Hodge structure and comment on a difference to our definition of a mixed Hodge structure over an Artin ring.

B.1. Mixed Hodge structures on singular varieties

Definition B.1.1. A pure Hodge structure $(H_{\mathbb{Z}}, F^\bullet)$ of weight $m \in \mathbb{Z}$ consists of the following data: a finitely generated $\mathbb{Z}$-module $H_{\mathbb{Z}}$ and a finite decreasing filtration $H = F^0 \supseteq F^1 \supseteq \ldots$ on $H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ such that

$$F^p \oplus \overline{F^{q+1}} = H$$

for all p, q with $p + q = m$. Note that we have a complex conjugation on $H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ defined by $\overline{h \otimes \lambda} = h \otimes \overline{\lambda}$.

In this definition we could as well start with a finitely generated Λ-module H_Λ for some subring $\Lambda \subseteq \mathbb{R}$ instead of $H_{\mathbb{Z}}$. Such a datum will be called a pure Λ-Hodge structure.

Definition B.1.2. A mixed Hodge structure $(H_{\mathbb{Z}}, F^\bullet, W_\bullet)$ consists of the following data: a finitely generated $\mathbb{Z}$-module $H_{\mathbb{Z}}$, a finite decreasing filtration $H = F^0 \supseteq F^1 \supseteq \ldots$ on $H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ and a finite increasing filtration $\ldots \subseteq W_m \subseteq \ldots \subseteq H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ such that for each $m \in \mathbb{Z}$ the filtration $F^p \mathrm{Gr}^W_m$ defined by

$$F^p \mathrm{Gr}^W_m = \mathrm{im}\,(F^p \cap W_m \longrightarrow W_m / W_{m-1})$$

defines a pure $\mathbb{Q}$-Hodge structure of weight m on Gr^W_m.

If in this definition $\mathbb{Q}$ is replaced by a ring Λ with $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{R}$ we speak of a mixed Λ-Hodge structure. In this sense, the mixed Hodge structures in the central fiber of our mixed Hodge structures over Artin rings from Definition II.2.1 is a mixed $\mathbb{R}$-Hodge structure. The condition that $\Lambda \subseteq \mathbb{R}$ is necessary for Definition B.1.2 to be reasonable: in order that Gr^W_m can be a pure Hodge structure, we need to have a complex conjugation. On H we have one coming from the real structure and this descends to Gr^W_m only if W_m and W_{m-1} are fixed, in other words, if they are themselves defined over $\mathbb{R}$.

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