Spectral and variational characterizations of solutions to semilinear eigenvalue problems

Dissertation
zur Erlangung des Grades
“Doktor der Naturwissenschaften”

am Fachbereich Mathematik
der Johannes Gutenberg-Universität
in Mainz

Tobias Weth
geboren in Rheinberg

Mainz, 2001

Tag der mündlichen Prüfung:
30. November 2001

Contents

Chapter 1

Introduction

It is hard to overestimate the importance of extremum principles in mathematics and physics. The idea that 'every effect in nature follows a maximum or minimum rule', as formulated by Leonhard Euler in 1744 [29], was a guiding light in the development of various theories in geometry and physics, starting from the famous least action principle of Fermat and Maupertuis. Geodesics, minimal surfaces, integral curves in Hamiltonian mechanics and stationary states in quantum mechanics, all these share the property of being a critical point of some functional ψ defined on a suitable manifold M. However, while minimization problems have a history prior to the 18th century, the search for unstable extrema is a more recent topic. Presumably, Birkhoff [14] was the first to introduce a *minimax principle* for a critical level *larger* than the global minimum, proving the existence of one closed geodesic on a surface of genus 0. The main idea behind minimax principles for nonlinear problems is the observation that critical levels often reveal a change in the topology of the corresponding sublevel set. This idea was set on a strong foundation by Ljusternik and Schnirelman [54], who where also the first to observe that a symmetry of ψ under the action of a compact topological group is reflected by a richer topology of sublevel sets. In the particular case where ψ is an even functional defined on the unit sphere S_1 of some Hilbert space $\mathcal{H}$, the *Ljusternik-Schnirelman levels*

$$c_n := \inf_{\substack{A\in\Sigma(S_1)\\ \gamma(A)\geq n}} \sup_{u\in A} \psi(u) \qquad (n \in \mathbb{N}) \tag{1.1}$$

arise as natural candidates for critical values of ψ (cf. [77]). Here $\Sigma(S_1)$ denotes the system of closed and symmetric subsets of S_1, and γ denotes the Krasnosel'skii genus. The relation (1.1) bears a resemblance to the Courant-Fischer principle, which states that under local compactness conditions a selfadjoint semibounded operator A on $\mathcal{H}$ has a sequence of eigenvalues given by

$$\lambda_n(A) := \inf_{\substack{V\leq\mathcal{D}(A)\\ \dim V\geq n}} \sup_{v\in V\cap S_1} (Av|v), \tag{1.2}$$

with $(\cdot|\cdot)$ denoting the scalar product in $\mathcal{H}$. Indeed, if we consider the energy functional $u \mapsto \psi(u) := \frac{1}{2}(Au|u)$ associated with the eigenvalue problem $Au = \lambda u$, then a comparison of (1.1) and (1.2) yields $c_n = \frac{1}{2}\mu_n$. Therefore the values c_n generalize the notion of a minimax eigenvalue for nonlinear problems, and this property has been stressed especially by Zeidler [77], [78]. Inspired by this observation, one might ask if even a form of 'nonlinear spectral theory' exists. For instance, the following questions may arise in this context:

(i) Can one characterize solutions to some *nonlinear* problems using the spectral theory of related *linear* problems ?

(ii) Is it possible to extend the notion of a spectral projection or a generalized eigenspace in a meaningful way ?

Considerations like this have motivated the present thesis, which pursues a 'spectral theoretic' approach to semilinear problems with variational structure. Such an approach has been proposed recently by Heid and Heinz [33], and the thesis is strongly influenced by their work. Let us explain the main idea in light of the following Dirichlet problem:

$$\Big(-\Delta + f(x,|u|)\Big)u = \lambda u, \qquad u \in W_0^{1,2}(\Omega). \tag{1.3}$$

Here $W^{1,2}(\Omega)$ denotes the usual Sobolev space on a domain $\Omega \subset \mathbb{R}^N$, and f is a real-valued continuous function. Moreover we assume that the nonlinearity in (1.3) is definite, precisely:

(M) $f(x,\cdot) : [0,\infty[\to \mathbb{R}$ is either nondecreasing for all $x \in \Omega$, or nonincreasing for all $x \in \Omega$.

In the first case, equation (1.3) is called *sublinear*, and in the second case it is called *superlinear*. To describe the spectral aspects of this equation, we cast it in an abstract functional analytic framework. For this replace $L^2(\Omega)$ by an arbitrary real Hilbert space $\mathcal{H}$, $W^{1,2}(\Omega)$ by a dense subspace $X \subset H$ and $-\Delta$ by a semibounded selfadjoint operator A_0 having X as its form domain. Suppose that for each $u \in X$ we are given a symmetric A_0-form compact perturbation $B(u)$ which depends continuously on u in the sense of quadratic forms and such that $B(u) = B(-u)$. Then we may build the form sum $A(u)$ of A_0 and $B(u)$ and consider the equation

$$A(u)u = \lambda u \qquad u \in \mathcal{D}(A(u)) \subset X. \tag{1.4}$$

Actually one has to impose certain growth conditions on f to treat (1.3) as a special case of (1.4), but we omit the details at this point. Moreover we just remark that, by generalizing condition (M) in a suitable way, we adapt the notion of sub- and superlinearity to this abstract context (cf. Sec. 6). Finally we assume that (1.4) is of variational type. By this we mean that the nonlinear operator $u \mapsto A(u)u$, extended as a map from X to its dual X^*, arises as derivative of a functional $\psi \in C^1(X)$. As a consequence, (1.4) is precisely the Euler-Lagrange equation of the functional $\psi_\lambda \in C^1(X)$ defined by

$$\psi_\lambda(u) = \psi(u) - \frac{\lambda}{2}\|u\|^2 \qquad (u \in X)$$

(here and in the following, $\|\cdot\|$ denotes the norm in $\mathcal{H}$). We now introduce, for given $n \in \mathbb{N}$, the following *spectral characterization* problem:

$(SC)_n$ Find a solution u of (1.4) such that $\lambda = \lambda_n(A(u))$, i.e. λ equals the n-th minimax eigenvalue of the operator $A(u)$.

In the sequel, we abbreviate $\lambda_n(A(u))$ to $\lambda_n(u)$. Of course, some justifications are needed to pose this problem in a precise way, and the appropriate framework is developed in Chapter 5. The fundamental interest of this characterization lies in the fact that solutions u of $(SC)_n$ share any property enjoyed by an n-th eigenfunction of a related *linear* problem. Depending on the precise framework, this gives rise to certain geometric properties of the function u. For instance, for elliptic PDEs of the type (1.3), every solution u of $(SC)_n$, $n \geq 2$ *changes sign*. Further *nodal properties* of u may be deduced from famous *Courant's nodal domain theorem* (cf. [22] and section 14.3), which states that the number of nodal domains (i.e. of connected components of the set $\{x \in \Omega | u(x) \neq 0\}$) of the n-th eigenfunction is bounded above by n. Moreover, refined information on the number of nodal domains is available in the one-dimensional or radially symmetric case.

Nodally characterized solutions to equations like (1.3) received much attention in recent years. This interest is mainly due to the fact that in case $\Omega = \mathbb{R}^N$ these solutions describe 'standing' or 'traveling' waves arising in nonlinear equations of the Schrödinger or Klein-Gordon type, see for instance [9],[11],[23],[41] and the references therein. In particular it is worth reviewing briefly the relationships between nodal properties and minimax energy levels which are already known. For *superlinear* Sturm-Liouville problems, Nehari [57] introduced a minimax principle involving functions with a fixed number of zeroes whose position is varied. His method was extended to problems on unbounded intervals by Ryder [62], and versions suitable for higher dimensional radially symmetric problems were developed in [69] and [9]. As a matter of fact, the characteristic energies defined by Nehari coincide with Ljusternik-Schnirelman levels on an appropriate manifold, see Section 8.3. Coffman [18] was the first to observe a relationship of this kind, but he identified Nehari's numbers with the Ljusternik-Schnirelman levels of a *different* auxiliary functional.
For *sublinear* problems on a compact interval, Hempel [37] proved the existence of nodal solutions also using a variational principle which involves the position of zeroes, and his critical levels were subsequently identified with Ljusternik-Schnirelman levels (cf. [19]). Sublinear problems on unbounded intervals and radially symmetric problems were treated by Heinz [35],[36] with the help of refined versions of Ljusternik-Schnirelman theory.
For equations like (1.3) in higher dimensions without any symmetry assumptions, sign properties of solutions are far from being well understood. In fact, even the nodal structure of the Dirichlet eigenfunctions of the Laplacian on an arbitrary domain Ω has not been clarified satisfactorily so far. Moreover, all nodal constructions available on intervals run up against serious handicaps in higher dimensions. Results establishing the existence of *sign changing* solutions have been obtained just recently (cf. [7],[10],[11],[15],[23]). In particular we mention the *critical point theory on partially ordered Hilbert spaces* as developed by Bartsch [7], which yields promising general results on sign changing solutions. This method strongly relies on the fact that for a large class of superlinear equations the negative gradient flow of the associated energy functional leaves the cone of positive functions in $C^1(\Omega)$ invariant.

Equation (1.4) may also be considered together with the *normalization* condition

$$\|u\| = R, \tag{1.5}$$

for given $R > 0$, which amounts to an isoperimetric side condition for equation (1.3). The case $R = 1$ is especially interesting for problems arising in quantum mechanics, where such solutions describe the density of stationary states. The standard variational procedure to obtain normalized solutions is the investigation of the functional ψ *restricted* to the sphere

$$S_R := \{u \in H \mid \|u\| = R\}$$

Indeed, if u is a critical point of $\psi|_{S_R}$, then u solves (1.4), but in this case the corresponding 'eigenvalue' λ enters as an unknown Lagrangian multiplier. As a consequence, we may *not* prescribe λ, which is the prize we pay by imposing condition (1.5). Nevertheless, we might still be able to establish $\lambda = \lambda_n(u)$ for given $n \in \mathbb{N}$, so that u is a solution of our spectral characterization problem.
Heid and Heinz [33] already examined such a variant of problem $(SC)_n$, however they could only treat the sublinear case under considerably strong restrictions. In [34] we removed some of these restrictions and presented new applications.
In this thesis, the arguments are worked out in an again more comprehensive and unified way. Moreover, we consider the sublinear type *as well as* the superlinear type of (1.4), and we construct solutions (u, λ) of $(SC)_n$ *either* with prescribed eigenvalue λ *or* with prescribed norm $\|u\|$. Besides, we construct *minimizing sets* corresponding to minimax characterizations of the form (1.1) with the help of spectral projections, and we derive useful inequalities relating the levels c_n to 'frozen Rayleigh quotients' of the form $\frac{(A(u)v|v)}{\|v\|^2}$ for $u, v \in X$.
In the second part of the thesis, we apply our results to elliptic PDEs of second order, moreover we consider integro-differential equations with a *nonlocal nonlinearity* of convolution type. Beyond mere existence results for these problems, we derive a deeper understanding of the solution set in view of nodal properties. Moreover, new connections between the existence of nodally characterized solutions and the nondegeneracy of Ljusternik-Schnirelman levels are derived, generalizing results from the linear theory.
Finally, the thesis provides affirmative and clarifying answers to the questions raised in [33, p.49].

We now briefly review our abstract results, considering first the sublinear problem together with the normalization condition (1.5). The basic idea is to detect solutions of $(SC)_n$ as elements of a *spectral fixed point set*. More precisely, define

$$\mathcal{P} := \{u \in X \mid P_n(u)u = u\},$$

where $P_n(u)$ denotes the generalized eigenprojection associated with the first n eigenvalues of the operator $A(u)$. Hence $\mathcal{P}$ may be regarded as some kind of 'generalized eigenspace' for the nonlinear problem. Indeed, if $A(u) = A$ does not depend on u, then $\mathcal{P}$ is the usual generalized eigenspace associated with the first n eigenvalues of A. However, in general $\mathcal{P}$ is not a vector space, and there even is no immediate evidence that $\mathcal{P}$ contains some nonzero element at all. However, assuming that

$$\lambda_n(u) < \lambda_{n+1}(u) \qquad (u \in X) \tag{1.6}$$

and considering $R = 1$ for simplicity, we show the following property:

$$(CP) \qquad \begin{cases} \mathcal{P} \cap S_1 \text{ is compact, } \gamma(\mathcal{P} \cap S_1) = n \quad \text{(In particular, } \mathcal{P} \cap S_1 \text{ is nonempty).} \\ c_n = \max \psi(\mathcal{P} \cap S_1) \\ \text{Every } u \in \mathcal{P} \cap S_1 \text{ satisfying } \Psi(u) = c_n \text{ solves problem } (SC)_n. \end{cases}$$

(cf. Sec. 6.1). In particular, this property provides solutions to $(SC)_n$, but the assertion is much stronger. In fact, (CP) naturally extends fundamental features of a generalized eigenspace V_n associated with the first n eigenvalues of some semibounded selfadjoint operator A. Moreover, (CP)

asserts that $\mathcal{P} \cap S_1$ is a *minimizing set* in view of the minimax characterization (1.1).
In the *superlinear case*, we prove properties dual to those listed in (CP). Recall that, equivalently to (1.2), the values $\lambda_n(A)$ are also given by

$$\lambda_n(A) = \sup_{\substack{V \leq \mathcal{D}(A) \\ codim\, V \leq n-1}} \inf_{v \in V \cap S_1} (Av|v) \tag{1.7}$$

A similar complementary description exists for the values c_n, involving the dual genus γ^*(cf. Sec. 3.2). Setting

$$\mathcal{Q} := \{u \in X \mid (I - P_{n-1})(u)u = u\},$$

and assuming

$$\lambda_{n-1}(u) < \lambda_n(u) \qquad (u \in X) \tag{1.8}$$

as well as a certain boundedness condition (cf. p. 50), we establish the following:

$$(CP)^- \qquad \begin{cases} \mathcal{Q} \cap S_1 \text{ is closed, } \gamma^*(\mathcal{Q} \cap S_1) \leq n-1 \\ c_n = \inf \psi(\mathcal{Q} \cap S_1), \text{ and this infimum in attained} \\ \text{Every } u \in \mathcal{Q} \cap S_1 \text{ satisfying } \Psi(u) = c_n \text{ solves problem } (SC)_n. \end{cases}$$

Again solutions to $(SC)_n$ are provided, and features of spectral subspaces are regained. We remark that, in contrast to the sublinear case, $\mathcal{Q} \cap S_1$ is *not* compact, hence local compactness of minimizing sequences has to be ensured. We will prove this compactness with the help of elementary spectral estimates. We also note that the assumption (1.8) may be weakened in applications, cf. Chapter 9.
The strategy to solve problem $(SC)_n$ for *fixed* λ is similar, However, to explore minimax principles, we now replace the functional ψ by ψ_λ, and we relate the corresponding values c_n to a different 'manifold' in place of S_1 in (1.1). The appropriate choice of this manifold depends on the position of λ with respect to the spectrum σ of $A(0)$. If $\lambda < \inf \sigma$, then we consider the *Nehari manifold*

$$\mathcal{N} := \{u \in X \setminus \{0\} \mid (A(u)u|u) = \lambda \|u\|^2\},$$

which is a closed subset of X containing all solutions of (1.4). If on the other hand $\lambda > \inf \sigma$, then we use the set

$$S := \{u \in X \mid \lambda_n(u) = \lambda\},$$

which to our knowledge has not been considered before for constrained minimax principles. Note that in general S is not a differentiable manifold. Nevertheless, either referring to $\mathcal{N}$ or to S, we show properties similar to (CP) and $(CP)^-$.

We now give an outline of how the thesis is organized. In Chapter 2 we commence by proving an analog of Courant's nodal theorem for unconstrained *superlinear* equations. By this we extend and complement known results based on Morse-theoretic arguments. Moreover, the proof gives a first view on how linear and nonlinear minimax principles may be compared with the help of condition (M).
In the subsequent chapter we turn to the abstract part of our thesis, starting with the investigation of even and continuous maps from a Banach space into the associated Grassmannian manifold. To

these maps we assign special fixed point sets whose topological properties are explored. As a prerequisite, we recall basic properties of the Grassmannian manifold and relatively compact subsets.
In Chapter 4 we establish the continuous dependence of eigenvalues and spectral projections for families of form compact perturbations. This is necessary to apply the results of Chapter 3 to our spectral fixed point sets $\mathcal{P}$ and $\mathcal{Q}$ as given above.
The examination of semilinear eigenvalue problems starts in Chapter 5, where we give a precise definition of problem $(SC)_n$. The Chapters 6 and 7 contain our most important results in the abstract 'functional analytic' framework. In particular we provide criteria for the solvability of problem $(SC)_n$, and we establish the above-mentioned properties $(CP), (CP)^-$.
The second part of the thesis, starting with Chapter 8, is devoted to applications to elliptic differential (and integro-differential) equations of second order. First we deal with a periodic boundary value problem involving nonlinear Hill's equation, which features the interesting phenomena of persistent eigenvalue gaps. More precisely, if $n \in \mathbb{N}$ is odd, then

$$\lambda_n(u) < \lambda_{n+1}(u)$$

for every $u \in X$, and therefore all of our abstract results apply in full strength. On the other hand, in view of the occurrence of double eigenvalues topological degree methods and, in particular, the global bifurcation results of Rabinowitz [58] do *not* apply here. The chapter closes with a brief note on nonlinear Sturm-Liouville problems, in particular including the announced identification of Nehari's characteristic numbers.
In Chapters 9-11 we consider three different types of superlinear problems defined on $\mathbb{R}^N$. First we are concerned with *normalized* solutions to superlinear Schrödinger equations, and we derive new results on nodal solutions for the radial *and* nonradial case. In particular we complement results derived for *unconstrained* equations (cf. [9] and [21] for the radial case, [11] for the nonradial case).
In Chapter 10 we then turn to *nonlocal* superlinear equations of Choquard type, where the nonlinearity is given by a convolution integral. For this type of equation Lions [50] established the existence of infinitely many radial solutions, but every nodal information on these solutions is new. Note in particular that all *local* reasoning is doomed to fail, and therefore nodal properties can *not* be shown neither by ODE dynamics nor by local variational techniques. Our method does not rely on locality, and therefore we derive existence and characterizations of nodal solutions.
Subsequently we treat a generalized Emden-Fowler equation, which might be seen as the limit case of a superlinear Schroedinger equation approaching the infimum of the essential spectrum. A variational framework for this equation is naturally given on the space $D^{1,2}(\mathbb{R}^N)$. This space does *not* arise in classical selfadjoint eigenvalue problems, and at first glance this seems to be an obstacle for our approach. We circumvent this problem by considering a family of related *bounded* operators $D^{1,2}(\mathbb{R}^N)$. Actually this family shows even nicer uniform properties than semibounded operators arising in an L^2-theory. In the radial case, our work improves results of Chabrowski [17] and Naito [56], whereas in the nonradial case our results seem to be basically new.
The final two chapters are concerned with sublinear equations on $\mathbb{R}^N$, which are mainly considered in approximations for quantum mechanical systems of many electrons (see [52] and the references therein). Nodal solutions for these kind of equations have been found so far either by bifurcation arguments (cf. [70] and [71]) or by a fixed point approach (see [76] and [52, Sec. III.3] for an improved version). Compared to these techniques, our approach gives additional information on minimax values and permits a relaxation of the growth conditions imposed on the nonlinearity.

Finally the thesis contains a rather extensive appendix. In the first three parts we furnish prerequisites from the theory of *linear* elliptic PDEs. Most of these are known or at least not surprising, but in standard references the assumptions on the coefficients are too restrictive for our purposes. In the last part we collect recent results on compact embeddings of Sobolev spaces in weighted L^p-spaces, and we deduce compactness properties of nonlinear operator valued maps.

We remark that we mainly restricted our attention to problems posed on the whole space, but in general our results carry over to *bounded* domains $\Omega \subset \mathbb{R}^N$. However, in view of nodal properties the case $\Omega = \mathbb{R}^n$ is more interesting, since many of the available methods encounter (at least technical) obstacles in this case.

Closing this introduction with a short outlook, we like to suggest a further investigation of spectral characterizations for nonlinear elliptic equations which goes beyond the scope of the present thesis. In view of our method, it remains to clarify to which extend our evenness and nondegeneracy assumptions (cf. (1.6) and (1.8)) can be relaxed. In addition, it is also interesting to examine eigenvalues positioned in *gaps* of the essential spectrum. Indeed, minimax principles for this type of eigenvalues have been developed very recently (see [32] and [26]), and maybe there exists an analogous connection with certain minimax values of the associated *strongly indefinite* nonlinear functional. For instance, one might think of the variational values defined by Benci [12].
Beside from such abstract extensions, many different applications of the present method are conceivable. As an example we mention fourth order elliptic equations which on intervals also exhibit nice nondegeneracy properties. Moreover one may consider second order systems of ordinary differential equations, as done in [33] for the sublinear case.
We finally mention the open question whether an odd superlinear problem posed on $\mathbb{R}^n$, either of local or of nonlocal nature, has an infinite number of *sign changing* solutions. We guess that this question can be answered by some form of spectral investigation.

1.1 Notation and conventions

General conventions

Supposing that the underlying space and the notion of distance is understood, $B_R(x)$ represents an open ball of radius R centered at x. The notions 'measurable' and 'measure' stand for 'Lebesgue-measurable' and 'Lebesgue-measure', respectively. All considered functions are *real-valued*. Moreover we call a function $f : \mathbb{R}^N \to \mathbb{R}$ *analytic* in case that it is *real analytic*. Unless otherwise stated, all occurring vector spaces are understood as *real* vector spaces.

Abstract notions

If $(X, \|\cdot\|)$ is a Banach space, we denote by X^* the *topological* dual of X and by $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$ the dual pairing.

To a functional $\Psi : X \to \mathbb{R}$ we assign the sublevel sets

$$\Psi^c = \{u \in X \mid \Psi(u) \leq c\} \subset X$$

for each $c \in \mathbb{R}$.
We write $\Psi \in C^1(X)$ in case that Ψ is Fréchet differentiable with continuous derivative $d\Psi : X \to X^*$. Supposing that this is true, we call the equation

$$d\Psi(u) = 0 \qquad u \in X \tag{1.9}$$

the *Euler-Lagrange equation* of Ψ. Conversely, being given an equation of the form (1.9), we say that Ψ is the corresponding *energy functional*.
Finally, Ψ satisfies the *Palais-Smale condition* (*PS condition* in short) *at a level* $c \in \mathbb{R}$ if any sequence $(u_n)_{n\in\mathbb{N}}$ in X with $\Psi(u_n) \to c$ and $d\Psi(u_n) \to 0$ in X^* admits a convergent subsequence.
If A is a densely defined (unbounded) linear Operator in X, we denote by $\mathcal{D}(A)$ the domain of A. Moreover we assign to A the set $\sigma(A)$ (resp. $\sigma_p(A)$, $\sigma_{ess}(A)$ and $\sigma_c(A)$) defined as the *spectrum* (resp. the *point spectrum*, *essential spectrum* and *continuous spectrum*) of the *complexification* of A.
For $\delta > 0$, the δ-neighborhood of a set $A \subseteq X$ is written $U_\delta(A)$, i. e. we have

$$U_\delta(A) := \{x \in X | \operatorname{dist}(x, A) < \delta\},$$

and similarly for subsets of any other metric space. Moreover, for a subspace V of X and $R > 0$ we write $B_R V$ in place of $B_R(0) \cap V$.
For a subset $D \subset X$ we denote by $\overline{D}$ resp. ∂D the closure resp. the boundary of D.
We briefly write $D \leq X$ to express that D is a *subspace* of X.
We say that D is *weakly compact* if every sequence $(u_n)_n \subset D$ contains a subsequence which converges weakly to some $u \in D$ (Strictly speaking, D is weakly *sequentially* compact in this case, but for simplicity we allow this slight abuse of notation).
As usual, let $\mathcal{L}(X)$ be the space of bounded linear operators in X. More generally, for two normed spaces E, F the normed space of bounded linear operators $E \to F$ will be denoted by $\mathcal{L}(E, F)$. For $T \in \mathcal{L}(E, F)$, we shall write $\mathcal{N}(T)$ resp. $\mathcal{R}(T)$ for the kernel resp. the range of the linear map T. Moreover, the letter $T^* \in \mathcal{L}(F^*, E^*)$ stands for the dual operator of T.
Finally, a (nonlinear) map $N : E \to F$ is called *completely continuous* if N is compact and continuous.
Moreover, N is called *strongly continuous* if $N(u_n) \to N(u)$ in F whenever $u_n \rightharpoonup u$ in E.
Although inconsistent, the latter notation is not common practice in the special case $F = \mathbb{R}$. Instead, it is customary to call N weakly (sequentially) continuous in this case, hence we will do so as well. Furthermore we call N weakly lower semicontinuous provided that $F = \mathbb{R}$ and that $u_n \rightharpoonup u$ implies $u \leq \liminf N(u_n)$.
We remark that if E is *reflexive*, then the strong continuity of N implies that it is completely continuous. If in addition N is *linear*, then both properties are equivalent to the mere compactness of N.

Elementary notions

The letter $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) denotes the set of positive (resp. negative) real numbers. Moreover, we denote by $O(\cdot)$ and $o(\cdot)$ the usual Landau symbols.

For $x \in \mathbb{R}^N$ we write $|x|$ for the Euklidian norm of x. An open and connected subset Ω of $\mathbb{R}^N$ is called a *domain*. For domains Ω_1, Ω_2 we write $\Omega_1 \subset\subset \Omega_2$ in case that $\overline{\Omega_1}$ is compact and $\overline{\Omega_1} \subset \Omega_2$. If $f : \Omega \to \mathbb{R}$ is continuous, a *nodal domain* N_f of f is defined as a connected component of the set

$$\{x \in \Omega \mid f(x) \neq 0\}.$$

Hence N_f is open and connected, and there holds $f(x) = 0$ for every $x \in \partial N_f \cap \Omega$.

Functions and spaces of functions

Let $\Omega \subset \mathbb{R}^N$ denote a measurable subset.
For an arbitrary subset $\Omega' \subset \Omega$ we denote by $1_{\Omega'} : \Omega \to \mathbb{R}$ the *characteristic function* of Ω', that is

$$1_{\Omega'}(x) = \begin{cases} 1 & \text{for} \quad x \in \Omega' \\ 0 & \text{for} \quad x \in \Omega \setminus \Omega'. \end{cases}$$

A function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is called a *Caratheodory function* if $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $x \in \Omega$ and $f(\cdot, t) : \Omega \to \mathbb{R}$ is measurable for all $t \in \mathbb{R}$.
For $1 \leq p \leq \infty$ we denote by $p' \in [1, \infty]$ the conjugate exponent, i.e. $p' = \frac{p}{p-1}$. Moreover, let $L^p(\Omega)$ be the usual Lebesgue space which norm is denoted by $\|\cdot\|_p$ independently of Ω. Moreover we use the symbol $(\cdot|\cdot)_2$ for the scalar product in $L^2(\Omega)$.
In the following let $\Omega \subset \mathbb{R}^N$ be a domain, and let $C_0^\infty(\Omega)$ stand for the vector space of C^∞-functions with compact support in Ω. We denote for $k \in \mathbb{N}$ by $W^{k,p}(\Omega)$ the usual Sobolev space of function $u \in L^p(\Omega)$ possessing distributional derivatives $D^\alpha u \in L^p(\Omega)$ for $\alpha \in \mathbb{N}_0^N$, $|\alpha| \leq k$. If $1 \leq p < \infty$, then $W^{k,p}(\Omega)$ is a reflexive, separable Banach space with norm $\|\cdot\|_{W^{k,p}}$ given by

$$\|\cdot\|_{W^{k,p}}^p = \sum_{|\alpha| \leq k} \|D^\alpha u\|_p.$$

Moreover, if $f : \Omega \to \mathbb{R}$ is a measurable function, we say $f \in L^p_{loc}(\Omega)$ resp. $f \in W^{k,p}_{loc}(\Omega)$ if for every $\eta \in C_0^\infty(\Omega)$ the function ηf defines an element of $L^p(\Omega)$, $W^{k,p}(\Omega)$, respectively.
Next we define for $1 \leq p < \infty$ and $k \in \mathbb{N}$ the space $W_0^{k,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$, hence $W_0^{k,p}(\Omega)$ is a reflexive separable Banach space as well.
Finally we focus on the case $\Omega = \mathbb{R}^N$.
For $N \geq 3$ we define $D^{1,2}(\mathbb{R}^N)$ as the space of functions $u \in L^{\frac{2N}{N-2}}$ with first distributional derivatives belonging to $L^2(\mathbb{R}^N)$. By Sobolev's inequality, $D^{1,2}(\mathbb{R}^N)$ becomes a Hilbert space with the scalar product

$$(u|v) = \int_{\mathbb{R}^N} \nabla u \nabla v.$$

For a given measurable a.e. positive function $w : \mathbb{R}^N \to \mathbb{R}$ and $1 \leq p < \infty$ we will denote by $L^p_w(\mathbb{R}^N)$ the space of measurable functions u satisfying

$$\|u\|_{L^p_w}^p := \int_{\mathbb{R}^N} w(x)|u|^p = \|w^{\frac{1}{p}}\, u\,\|_p^p < \infty,$$

where a.e. coinciding functions are identified. In L^p_w Hölder's inequality can be written as usual, that is

$$\int_{\mathbb{R}^N} w(x)|u_1(x)\cdot \ldots \cdot u_n(x)| \leq \|u_1\|_{L^{p_1}_w} \cdot \ldots \cdot \|u_1\|_{L^{p_k}_w},$$

whenever $\sum_{i=1}^k p_i = 1$ and $u_i \in L^{p_i}_w$, $i = 1, \ldots, k$.
If $a :]0,\infty[\to \mathbb{R}$ is an a.e. positive measurable function, we will also write L^p_a in place of $L^p_{a(|\cdot|)}$ for simplicity.
Finally, calling a function $f : \mathbb{R}^N \to \mathbb{R}$ *rapidly decreasing*, we mean that it is a Schwartz function.

Conventions on Sobolev embeddings, weak solutions and elliptic regularity

In the sequel let Ω be a domain.
Let $f \in W^{k,p}_{loc}(\Omega)$ for some $k \in \mathbb{N}$, $1 \leq p < \infty$. Saying that some property of f is a consequence of *Sobolev embeddings*, we refer to the following well known facts (see e.g. [1, Theorem 5.4]):

(i) Suppose that $kp < n$ and $1 \leq q \leq \frac{np}{n-kp}$. Then $f \in L^q_{loc}(\Omega)$. Moreover, the space $W^{k,p}_0(\Omega)$ is continuously embedded in $L^q(\Omega)$ ($W^{k,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ in short).

(ii) Suppose that $0 \leq m < k - \frac{n}{p} < m+1$. Then f is represented by an element of $C^m(\Omega)$ (which we denote by the same letter f). Moreover $W^{k,p}_0(\Omega) \hookrightarrow C^m(\overline{\Omega})$.

Being given a Caratheodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, we consider the equation

$$-\Delta u = f(x,u) \qquad x \in \Omega. \tag{1.10}$$

A function $u \in W^{1,2}_{loc}(\Omega)$ is called a weak solution of (1.10) if $f(\cdot, u(\cdot)) \in L^1_{loc}(\Omega)$ and for all $\varphi \in C^\infty_0(\Omega)$ the following identity holds:

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega f(x,u)\varphi.$$

Unless otherwise stated, the term 'solution' always means a 'weak solution'.
Finally, suppose that $f \in L^p_{loc}(\Omega)$ for some $1 < p < \infty$, and that $u \in W^{1,2}_{loc}(\Omega)$ is a (weak) solution of the equation

$$-\Delta u = f.$$

Then $u \in L^{2,p}_{loc}(\Omega)$ (cf. [40, p. 214]), and we will refer to this property of u as a consequence of *elliptic regularity*.

Chapter 2

A Courant type nodal theorem for unconstrained superlinear problems

We consider the Dirichlet problem

$$-\Delta u = f(x, |u|)u \qquad u \in X := W_0^{1,2}(\Omega), \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ is a (not necessarily bounded) domain, $N \geq 2$ and $f : \Omega \times [0, \infty[\to \mathbb{R}$ is a real valued Caratheodory function satisfying

(M) $f(x, \cdot)$ is nondecreasing on $[0, \infty[$ for a.e. $x \in \Omega$.

(G) There are constants $q \in]0, \frac{4}{N-2}]$, $C > 0$ such that $|f(x,t)| \leq C(1 + |t|^q)$ (resp. $q \in]0, \infty[$ in case $N = 2$).

With $F : \Omega \times [0, \infty[$ defined by

$$F(x,t) = \int_0^t f(x,s)s \, ds,$$

the energy functional $\psi : X \to \mathbb{R}$ corresponding to equation (2.1) is given by

$$\psi(u) = \int_\Omega |\nabla u(x)|^2 \, dx - \int_\Omega F(x, |u(x)|) \, dx.$$

Indeed, standard arguments (cf. [68, Theorem C.1]) show that $\psi \in C^1(X)$ and that critical points of ψ solve (2.1) weakly. Moreover, each one of these solutions is continuous and bounded in Ω by Lemma 2.3 below. As a consequence, the nodal domains of u are defined in a meaningful way, see Sec. 1.1. Consider the increasing sequence $(\beta_n)_n$ of minimax values given by

$$\beta_n = \inf_{\substack{V \leq X \\ \dim V \geq n}} \sup \psi(V) \qquad \in [0, \infty].$$

The following theorem relates these values to the number of nodal domains of solutions to (2.1).

Theorem 2.1. *Suppose that u is a weak solution of (2.1) such that $0 < \psi(u) \le \beta_n$ for some $n \in \mathbb{N}$. Then u has at most n nodal domains.*

Remark 2.2. (a) Equation (2.1) has been considered by many authors in view of existence and multiplicity of solutions. Early results are due to Ambrosetti and Rabinowitz [3], who have shown that ψ possesses infinitely many critical points provided that Ω is bounded and in addition to (G) there holds

(i) $q < \frac{4}{N-2}$

(ii) There exists $\eta > 2$ and $R > 0$ such that $0 < \eta F(x,t) \le f(x,t)t^2$ for $t \ge R$ and a.e. $x \in \Omega$.

More precisely, they introduced an increasing sequence $(b_n)_n$ of critical values given by an appropriate minimax principle (see [3, p.357]), and they show that for every $n \in \mathbb{N}$ the sublevel set ψ^{b_n} contains at least n critical points of ψ. As a matter of fact, it is easy to check that $b_n \le \beta_n$ for every n. Hence a combination of the Ambrosetti-Rabinowitz result and Theorem 2.1 yields a sequence $(u_n)_n$ of solutions to (2.1) such that u_n has at most n nodal domains.
(b) To the authors knowledge, nodal estimates for solutions to superlinear PDEs without symmetry were first considered by Benci and Fortunato, cf. [13]. They used information on the Morse index of certain critical points which can be obtained by involved deformation type arguments. It is worth discussing the necessary requirements for such an approach. Very often (cf. [13], [8], [7]) an equation of the form

$$-\Delta u = g(x,u) \tag{2.2}$$

is considered, whereas the following assumptions are made

(i) $g \in C^1(\Omega, \mathbb{R}^N)$

(ii) $g(x,0) = 0$, and $\partial_2 g(x,t) > \frac{g(x,t)}{t}$ for every $x \in \Omega$, $t \neq 0$.

While (i) ensures that the corresponding energy functional is of class C^2, (ii) implies that the Morse index of a solution is an upper bound for the number of nodal domains (cf. [13]).
However, if g is *odd* and satisfies (i) and (ii), then one can easily write $g(x,t) = f(x,|t|)t$ with a function f satisfying (M), hence Theorem 2.1 is applicable as well. Indeed, we claim that, for odd superlinear equations, Theorem 2.1 is a simpler and more general tool to derive upper estimates on nodal domains than Morse theory. Note in particular the following:

(i) While Morse type arguments show that, on a suitable minimax level for ψ, there is at least *one* solution with the desired nodal information, Theorem 2.1 provides this information for *all* solutions on this level and below.

(ii) In some cases (cf. [8], [7]), Morse theory requires that solutions are *isolated* (resp. it requires that minimax values are *nondegenerate*), which is very difficult to check.

(iii) To apply Morse theory, one needs that ψ is of class C^2, whereas Theorem 2.1 applies for C^1-functionals.

We finally mention that Theorem 2.1 immediately furnishes the nodal properties proven in [13]. Moreover, the assumptions concerning isolation in Theorem 1.1. and Theorem 7.3 of [7] seem to be superfluous.

The proof of Theorem 2.1 requires the following two lemmas.

Lemma 2.3. *Every weak solution $u \in X$ of (2.1) is (globally) bounded and continuous in Ω.*

Proof. Combining (G) with Sobolev embeddings, we infer that $|f(\cdot,|u(\cdot)|)| \le \tilde{C} + a(\cdot)$ with a suitably chosen positive function $a \in L^{\frac{N}{2}}(\Omega)$. Hence Lemma 14.2 yields $u \in L^q(\Omega)$ for every $2 \le q < \infty$. In particular there is a number $s > \frac{N}{2}$ such that $f(\cdot,|u(\cdot)|) \in L^s(\Omega \cap B_2(x))$ for every $x \in \Omega$, and the $L^s(\Omega \cap B_2(x))$-norm of $f(\cdot,|u(\cdot)|)$ does not depend on x. An application of [49, Theorem 13.1] now yields that $u \in L^\infty(\Omega \cap B_1(x))$, and that $\sup\limits_{x \in \Omega \cap B_1(x)} |u(x)|$ does not depend on x (here the boundary condition '$u \equiv 0$ on $\partial\Omega$' enters in an essential way). Hence u is globally bounded in Ω, and the continuity of u now easily follows by virtue of Lemma 14.1. □

Lemma 2.4. *For $x \in \Omega$ and $s, t \in [0,\infty[$ there holds*

$$(F(x,s) - F(x,t)) \ge f(x,t)(s^2 - t^2). \tag{2.3}$$

Proof. Put $\tilde{f}(x,t) := f(x,\sqrt{t})$ for $x \in \Omega, t \in [0,\infty[$ as well as

$$\tilde{F}(x,t) := \int_0^t \tilde{f}(x,\tau)\, d\tau \qquad (x \in \Omega, t \in [0,\infty[)$$

Then differentiation shows

$$\tilde{F}(x,t^2) = 2F(x,t) \qquad (x \in \Omega,\ t \in [0,\infty[).$$

Moreover, since $\tilde{f}(x,\cdot)$ is increasing on $[0,\infty[$, the function $\tilde{F}(x,\cdot)$ is convex on $[0,\infty[$. In particular there holds

$$\begin{aligned} 2(F(x,s) - F(x,t)) = \tilde{F}(x,s^2) - \tilde{F}(x,t^2) &\ge \partial_2 F(x,t^2)(s^2 - t^2) \\ &= \tilde{f}(x,t^2)(s^2 - t^2) \\ &= f(x,t)(s^2 - t^2) \end{aligned} \tag{2.4}$$

for $s, t \in [0,\infty[$ and a.e. $x \in \Omega$. □

In fact an inequality of the form (2.3) is crucial for all applications to be considered in this thesis. We now may complete the

Proof of Theorem 2.1. Put $V(x) := f(x,|u(x)|)$ for $x \in \Omega$. Then Lemma 2.3 yields $V \in L^\infty(\Omega)$, in particular V is a W-admissible potential, see Section 14.3.1. Hence $-\Delta - V$ is uniquely given as a selfadjoint semibounded operator on $L^2(\Omega)$ with form domain X (cf. Remark 14.6). Moreover, every eigenfunction of $-\Delta - V$ is continuous by standard elliptic regularity (cf. Lemma 14.1).
Now consider

$$\lambda_n := \inf_{\substack{V \le X \\ \dim V = n}} \sup_{v \in V} \frac{\|\nabla v\|_2^2 + \int_\Omega V v^2}{\|v\|_2^2},$$

and assume that, contrary to our claim, u has more than n nodal domains. Then Theorem 14.7 forces

$$\lambda_n < 0. \tag{2.5}$$

Choose an n-dimensional subspace $W \subset X$ such that

$$\sup_{v \in V} \frac{\|\nabla v\|_2^2 + \int_\Omega V v^2}{\|v\|_2^2} \leq \frac{\lambda_n}{2}.$$

In particular, $\|\nabla w\|_2^2 \leq \|V\|_\infty \|w\|_2^2$ for every w in W, and therefore Lemma 2.4 yields

$$\begin{aligned}
2(\psi(u) - \psi(w)) &\geq \int_\Omega |\nabla u(x)|^2 \, dx - \int_\Omega V(x) u^2(x) \, dx - \\
&\quad \left(\int_\Omega |\nabla w(x)|^2 \, dx - \int_\Omega V(x) w^2(x) \, dx \right) \\
&= -\left(\int_\Omega |\nabla w(x)|^2 \, dx - \int_\Omega V(x) w^2(x) \, dx \right) \\
&\geq -\frac{\lambda_n}{2} \|w\|_2^2 \\
&\geq c\|w\|_X^2,
\end{aligned}$$

where $\|\cdot\|_X$ denotes the usual $W^{1,2}(\Omega)$-norm and $c := \frac{|\lambda_n|}{4} \min\{1, \frac{1}{\|V\|_\infty}\} > 0$. As a consequence,

$$\psi(u) \geq \psi(w) + \frac{c}{2} \|w\|_X^2$$

for every $w \in W$. From this we conclude $\psi(u) > \sup \psi(W)$, since ψ is continuous in X and $\sup \psi(W) \geq \beta_n > 0 = \psi(0)$. In particular this forces $\psi(u) > \beta_n$, contrary to our assumption. Hence u has at most n nodal domains, as claimed. $\square$

Chapter 3

Topological properties of families of subspaces

In this chapter, let $(X, \|\cdot\|)$ denote a real Banach space, and let $\|\cdot\|$ also stand for the induced norm in $\mathcal{L}(X)$. We are concerned with continuous and even maps on X which take either projections or subspaces of X as values. To each of these maps we assign a special fixed point set in X, and we aim to show that this set has a 'sufficiently rich' topology. A precise formulation will be given with the help of the Krasnosel'kii genus and a related dual genus. Dealing with systems of subsets of X, we in particular consider the Hausdorff distance d_h. Recall that for two subsets $A, B \subset X$ we may write $d_h(A, B)$ in the form

$$d_h(A,B) = \inf\{\delta > 0 \mid A \subset U_\delta(B) \text{ and } B \subset U_\delta(A)\}.$$

3.1 The Grassmannian manifold and relatively compact subsets

We first recall basic facts on projections. As usual, we denote a linear operator $P \in \mathcal{L}(X)$ a (continuous) projection if $P^2 = P$, which implies that X splits in a topological direct sum $X = \mathcal{N}(P) \oplus \mathcal{R}(P)$. The following observation is standard.

Lemma 3.1. *If $P, Q \in \mathcal{L}(X)$ are projections with $\|P - Q\| < 1$, then there exists a topological isomorphism (i.e. a linear homeomorphism) $T \in \mathcal{L}(X)$ such that $T^{-1}PT = Q$.*

Proof. Setting $T := PQ - (I-P)(I-Q) \in \mathcal{L}(X)$, we immediately verify $PT = PQ = TQ$. Moreover there holds $T = P + Q - I$ and hence $T^2 = I - (P-Q)^2$. Since $\|P - Q\| < 1$, the operator $I - (P-Q)^2$ is a topological isomorphism, and therefore T is one as well. $\square$

We restrict our attention to projections of finite rank. In particular we denote for arbitrary $n \in \mathbb{N}$ by $\Pi_n(X)$ the set of all projections $P \in \mathcal{L}(X)$ of rank n, and we infer:

Lemma 3.2. $\Pi_n(X)$ *is a closed subset of $\mathcal{L}(X)$, hence it becomes a complete metric space with the metric induced by the norm of $\mathcal{L}(X)$.*

Proof. Consider a sequence $(P_k) \subset \Pi_n(X)$ and $P \in \mathcal{L}(X)$ such that $P_k \to P$ in $\mathcal{L}(X)$. Since $(A,B) \longmapsto AB$ defines a continuous bilinear map $\mathcal{L}(X) \times \mathcal{L}(X) \longrightarrow \mathcal{L}(X)$, there holds $P^2 = P$, hence P is a projection. For sufficiently large k we have $\|P_k - P\| < 1$ which by Lemma 3.1 implies that P has the same rank as P_k, that is $P \in \Pi_n(X)$. □

Next we introduce the *Grassmannian manifold* $G_n(X)$, which is defined as the set of all n-dimensional subspaces of X. We endow $G_n(X)$ with the *gap* metric

$$\Theta(V,W) := d_h(B_1V, B_1W) = \max\left\{ \max_{v\in V, \|v\|\le 1} \operatorname{dist}(v, B_1W)\,,\ \max_{w \in W, \|w\|\le 1} \operatorname{dist}(v, B_1V) \right\}.$$

It is well known that $(G_n(X), \Theta)$ is a complete metric space (see [42, pp. 197] as well as [27, p. 17]). Moreover we have

Lemma 3.3. *(a) If $V, W \in G_n(X)$ and $P, Q \in \mathcal{L}(X)$ are projections with $\mathcal{R}(P) = V$ and $\mathcal{R}(Q) = W$, then $\Theta(V,W) \le 2\|P - Q\|$.*

(b) If X is a Hilbert space and $V, W \in G_n(X)$, then $\Theta(V,W) = \|P_V - P_W\|$, where P_V resp. P_W denote the orthogonal projections on V resp. W.

(c) If X is finite dimensional, then $G_n(X)$ is compact.

Proof. (a): For $v \in V$, $\|v\| \le 1$ there holds

$$\operatorname{dist}(v, B_1W) = \inf_{w\in B_1W} \|v - w\| \le 2 \inf_{w \in W} \|v - w\| \le 2\|v - Qv\| \le 2\|Pv - Qv\| \le 2\|P - Q\|,$$

and in the same way $\operatorname{dist}(w, B_1V) \le 2\|Q - P\|$ for $w \in W$, $\|w\| \le 1$. Thus $\Theta(V,W) \le 2\|P - Q\|$.
(b): If X is a Hilbert space, then Θ can obviously be written as

$$\Theta(V,W) = \max\left\{ \max_{v\in V, \|v\|=1} \|v - P_W v\|\,,\ \max_{w\in W, \|w\|=1} \|w - P_V w\| \right\}.$$

Combining this identity with the argument carried out in [2, pp.96], we deduce the assertion.
(c): By passing to an equivalent norm, X becomes a Hilbert space, and by (b) we may view $G_n(X)$ as a closed bounded subset of $\mathcal{L}(X)$. Hence it is compact. □

We also remark that obviously

$$d_h(B_RV, B_RW) = R\Theta(V,W)$$

for $R > 0$, $V, W \in G_n(X)$. As a consequence of Lemma 3.3(a) we deduce

Lemma 3.4. *The map $\mathcal{R} : \Pi_n(X) \to G_n(X)$ defined by $P \mapsto \mathcal{R}(P)$ is continuous.*

Dealing with the Grassmannian manifold, we also introduce the *universal n-plane bundle* γ_n over $G_n(X)$ which total space is

$$\{(V, v) \in G_n(X) \times X \mid v \in V\}.$$

Hence, γ_n is a continuous n-dimensional subbundle of the trivial vector bundle $G_n(X) \times X$ (see [46, p.97]), and its projection is induced by the canonical projection $G_n(X) \times X \to G_n(X)$. In the next section we will use some basic properties of vector bundles for which we refer the reader to the books by Atiyah [5] and Husemoller [38]. We close this preliminary section with a note on relative compactness, but for this we have to impose an additional assumption on the Banach space X.

Definition 3.5. *The Banach space X is said to have the $\mathcal{P}$-property if for every compact set $C \subset X$ and arbitrary $\varepsilon > 0$ there is a projection $Q \in \mathcal{L}(X)$ of finite rank such that $\|v - Qv\| \leq \varepsilon$ for all $v \in C$.*

It should be noted that every Hilbert space obviously has the $\mathcal{P}$-property, moreover every separable Banach space with a Schauder basis. Banach spaces with the $\mathcal{P}$-property allow the following useful characterization of relatively compact subsets of $G_n(X)$.

Lemma 3.6. *Suppose that X has the $\mathcal{P}$-property and let $R > 0$. Consider a subset $\mathcal{M} \subset G_n(X)$ and put $\mathcal{M}_R := \bigcup_{V \in \mathcal{M}} B_R V \subset X$. Then the following statements are equivalent:*

(i) $\mathcal{M}$ is relatively compact in $G_n(X)$.

(ii) $\mathcal{M}_R$ is relatively compact in X.

(iii) For every $\varepsilon > 0$ there is a projection $Q \in \mathcal{L}(X)$ of finite rank such that $Q\mathcal{M} := \{Q(V) \mid V \in \mathcal{M}\} \subset G_n(\mathcal{R}(Q))$ and $\Theta(V, Q(V)) < \varepsilon$ for every $V \in \mathcal{M}$.

Proof. (i) $\Longrightarrow$ (ii): Let $\varepsilon > 0$. By assumption, there are $V_1, ..., V_m \in \mathcal{M}$ such that $\mathcal{M} \subset \bigcup_{i=1}^{n} B_{\frac{\varepsilon}{R}}(V_i)$ (here $B_{\frac{\varepsilon}{R}}(V_i)$ denotes a ball in the opening metric). Define W as the span of $\bigcup_{i=1}^{n} V_i$, then W is finite dimensional. Consider arbitrary $v \in \mathcal{M}_R$. Then $v \in B_R V$ for some $V \in \mathcal{M}$, and there is V_i such that

$$\mathrm{dist}(v, B_R V_i) \leq d_h(B_R V, B_R V_i) = R\ \Theta(V, V_i) < \varepsilon.$$

Therefore $\mathcal{M}_R \subset U_\varepsilon(B_R W)$. Since $\varepsilon > 0$ was arbitrary, we infer that $\mathcal{M}_R$ is relatively compact.
(ii) $\Longrightarrow$ (iii): Without loss, let $0 < \varepsilon < 2$. Since X has the $\mathcal{P}$-property, there is a projection $Q \in \mathcal{L}(X)$ of finite rank such that $\|Qv - v\| < \frac{\varepsilon R}{4}$ for all $v \in \mathcal{M}_R$. In particular, if $v \in \mathcal{M}_R$ and $\|v\| = R$, then $\|Qv\| \geq R(1 - \frac{\varepsilon}{4}) > \frac{R}{2} > 0$. Hence $Q|_V$ is injective for every $V \in \mathcal{M}$, and therefore $Q\mathcal{M} \subset G_n(\mathcal{R}(Q))$. Moreover, for every $V \in \mathcal{M}$ there holds

$$\begin{aligned}
\Theta(V, Q(V)) &= \frac{2}{R} d_h(B_{\frac{R}{2}} V, B_{\frac{R}{2}} W) \\
&\leq \frac{4}{R} \max\{\max_{v \in B_{\frac{R}{2}} V} \mathrm{dist}(v, B_R(W)),\ \max_{w \in B_{\frac{R}{2}} W} \mathrm{dist}(w, B_R(V))\} \\
&\leq \frac{4}{R} \max\{\max_{v \in B_{\frac{R}{2}} V} \|v - Qv\|,\ \max_{v \in B_R V} \|Qv - v\|\} \\
&< \frac{4}{R} \frac{\varepsilon R}{4} \\
&= \varepsilon
\end{aligned}$$

(iii) $\Longrightarrow$ (i): Let $\varepsilon > 0$ and choose Q as provided by (iii). Since $\mathcal{R}(Q)$ is finite dimensional, $G_n(\mathcal{R}(Q))$ is compact by Lemma 3.3(c). Moreover $\mathcal{M} \subset U_\varepsilon(G_n(\mathcal{R}(Q)))$ by (iii), with both sets being viewed as subsets of $G_n(X)$. Hence $\mathcal{M}$ is relatively compact. $\square$

3.2 The Krasnosel'skii genus of fixed point sets

If a variational problem shows invariance under the free action of a compact Lie group G, then a detailed analysis of the arising level sets requires tools furnished by the associated *equivariant* topology (see [6] for a general framework). We will make use of this in the simplest case $G = \mathbb{Z}_2$ acting on X by reflection at the origin, and we recall some corresponding notations. A subset A of X is called *symmetric* if it is invariant under this action, i. e. if

$$\forall\, x : \quad x \in A \Longrightarrow -x \in A.$$

A map $h : A \to B$, where $A, B \subseteq X$ are symmetric subsets, is called *odd* if it is equivariant with respect to this action, i. e. if

$$h(-x) = -h(x) \qquad \forall\, x \in A.$$

We denote by Σ the family of all closed and symmetric subsets of $X \setminus \{0\}$, and for every $A \in \Sigma$ we define the *Krasnosel'skii genus* $\gamma(A)$ in the following way (cf. [68, p. 94]): If $A \neq \emptyset$, then

$$\gamma(A) = \begin{cases} \inf\{n \in \mathbb{N} \mid \text{There is a continuous and odd map } h : A \to \mathbb{R}^n \setminus \{0\}\} \\ \infty, \text{ if } \{...\} = \emptyset \end{cases}$$

moreover $\gamma(\emptyset) = 0$. Equivalently, some authors define γ by

$$\gamma(A) = \begin{cases} \inf\{n \in \mathbb{N} \mid \text{There is a continuous and odd map } h : A \to S^{n-1}\} \\ \infty, \text{ if } \{...\} = \emptyset \end{cases}$$

for $A \neq \emptyset$, where $S^{n-1} := \{x \in \mathbb{R}^n \mid |x| = 1\}$. We remark that for $A, B \in \Sigma$, $h : X \to X$ odd and continuous there holds (cf. [68, p.95]):

(1_γ) $\gamma(A) \geq 0$. Moreover $\gamma(A) = 0 \iff A = \emptyset$.

(2_γ) $A \subset B \implies \gamma(A) \leq \gamma(B)$

(3_γ) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$

(4_γ) $\gamma(A) \leq \gamma(\overline{h(A)})$

(5_γ) If A is compact, then $\gamma(A) < \infty$ and there is a neighborhood N of A in X such that $\overline{N} \in \Sigma$ and $\gamma(A) = \gamma(\overline{N})$.

That is, $\gamma : \Sigma \to \mathbb{N}_0 \cup \infty$ is a *definite, monotone, sub-additive, supervariant and "semicontinuous"* map. To put the definition of γ into perspective, we recall the following classical theorem (see e.g. [66, p. 266]):

Lemma 3.7. *There is no odd and continuous map $S^m \to S^n$ for $m > n$.*

As a consequence, we infer $\gamma(S^{n-1}) = n$ for every $n \in \mathbb{N}$. In the following Proposition we calculate the Krasnosel'skii genus of a fixed point set involving the Grassmannian manifold.

Proposition 3.8. *Let X have the $\mathcal{P}$-property and consider $S \in \Sigma$ bounded and such that for every finite-dimensional subspace U of X there is an odd homeomorphism h of $S \cap U$ onto the unit sphere in U. Fix a number $n \in \mathbb{N}$, and consider a continuous map $H : [0,1] \times S \to G_n(X)$ having the following properties:*

(i) $H(t, -y) = H(t, y)$ *for* $0 \leq t \leq 1$ *and every* $y \in S$*,*

(ii) $H(0, \cdot)$ *is constant on* S*,*

(iii) *the range* $H([0,1] \times S)$ *is a relatively compact subset of* $G_n(X)$*.*

Put $V(y) := H(1, y)$ *and*

$$K := \{y \in S | y \in V(y)\}.$$

Then $K \in \Sigma, K$ is compact, and $\gamma(K) = n$. In particular, $K \neq \emptyset$.

Remark 3.9. Proposition 3.8 is an extension of [33, Proposition 2.1] which was restricted to *Hilbert spaces* X. We also remark that in [33] the homotopy H was supposed to factorize continuously via *orthogonal* projections. Even though we will still apply Proposition 3.8 only to Hilbert spaces X in this thesis, we nevertheless consider families of *nonorthogonal* projections which arise naturally in Chapter 5. In combination with Lemma 3.4, Proposition 3.8 will prove useful for the treatment of these families.

Proof of Prop. 3.8. Fix $R > 0$ such that $S \subset B_R X$. Because of (i) the set K is symmetric. Moreover, K is closed. Indeed, if $(y_j) \subset K$ converges to $y \in X$, then $y \in S$ and

$$\begin{aligned} \operatorname{dist}(y, B_R V(y)) &= \lim_{j\to\infty} \operatorname{dist}(y_j, B_R V(y)) \\ &\leq \lim_{j\to\infty} d_h(B_R V(y_j), B_R V(y)) \\ &= R \lim_{j\to\infty} \Theta(V(y_j), V(y)) \\ &= 0, \end{aligned}$$

since the map $V : S \to G_n(X)$ is continuous. Hence $y \in V(y)$, and K is closed. Moreover, K is compact. To see this, note that $V(S) \subset H([0,1] \times S)$ is relatively compact in $G_n(X)$ by assumption (iii), hence Lemma 3.6 implies that $\bigcup_{y \in S} B_R V(y)$ is relatively compact in X. This set contains K, and therefore K is compact.
Next we show

$$\gamma(K) \leq n. \tag{3.1}$$

The compactness of K implies that, identifying antipodal points in K, we get a compact (and Hausdorff) topological space K'. Because of (i), the restriction $\mathcal{H}$ of H to $[0,1] \times K$ factors in the form

$$\mathcal{H} : [0,1] \times K \xrightarrow{\alpha} [0,1] \times K' \xrightarrow{\hat{H}} G_n(X),$$

where $\alpha(t,y) := (t,[y])$, and where $\hat{H}$ is a continuous function (here and in the following we put $[y] := \{y,-y\}$). Hence

$$\hat{V} := \hat{H}(1,\cdot) : K' \to G_n(X)$$

is a *nullhomotopic* map by (ii), which implies that the pull-back $\xi := \hat{V}^*\gamma_n$ is a trivializable vector bundle (see [38, p. 29], actually only the paracompactness of K' is required !). Observe that the total space E of ξ can be written as

$$\{([y],v) \in K' \times X \mid y \in K,\, v \in V(y)\}.$$

Let $\tau : E \to K' \times \mathbb{R}^n$ be a trivialization of ξ, and define a map $\varphi : K \to \mathbb{R}^n$ as the composition

$$\varphi : K \xrightarrow{\sigma} E \xrightarrow{\tau} K' \times \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

where σ is given by

$$\sigma(y) := ([y],y),$$

and where the last arrow is canonical projection. Since τ is linear on fibers, it clearly follows that φ is *odd* and continuous. Moreover, $\varphi(y) \neq 0$ for all $y \in K$. Thus $\gamma(K) \le n$, i.e. (3.1) holds.
It remains to show

$$\gamma(K) \ge n, \tag{3.2}$$

which however is the most difficult part. We will prove this in two steps: First we assume that X is finite-dimensional, and afterwards we treat the general case by an approximation argument based on assumption (iii).

First step: The case $\dim X < \infty$.

Assume that $\dim X = N+1 \ge n$. Passing to an equivalent norm if necessary, we may suppose that X is a Hilbert space. By assumption there is an odd homeomorphism h of S onto the unit sphere S^N in X. For $x \in X \setminus \{0\}$, we denote by $\langle x \rangle$ the span of x, considered as a point of real projective N-space $\mathbb{RP}^N$. Because of assumption (i) the map H now factors in the form

$$H : [0,1] \times S \xrightarrow{\alpha} [0,1] \times \mathbb{RP}^N \xrightarrow{\tilde{H}} G_n(X),$$

where $\alpha(t,y) := (t, \langle h(y) \rangle)$, and where $\tilde{H}$ is continuous. Now, let $\gamma_n^\perp$ be the *orthogonal* complement of γ_n in the trivial bundle $G_n(X) \times X$. Thus, the total space of $\gamma_n^\perp$ consists of the pairs $(V,v) \in G_n(X) \times X$ such that $v \in V^\perp$, and the projection is induced by the canonical projection $G_n(X) \times X \to G_n(X)$. By assumption (ii), the map

$$\tilde{V}(\cdot) := \tilde{H}(1,\cdot) : \mathbb{RP}^N \to G_n(X)$$

is nullhomotopic, hence the pull-back $\tilde{\xi} := \tilde{V}^*\gamma_n^\perp$ is trivializable. Note that the total space of $\tilde{\xi}$ can be written as

$$\begin{aligned} \tilde{E} &= \{(p,v) \in \mathbb{RP}^N \times X \mid v \in \tilde{V}(p)^\perp\} \\ &= \{(\langle h(y) \rangle, v) \mid y \in S, v \in V(y)^\perp\}. \end{aligned}$$

Denote $P(y) \in \mathcal{L}(X)$ the orthogonal projection on $V(y)$ for $y \in S$, and observe that by Lemma 3.3(b) the map $P : S \to \Pi_n(X)$ is continuous. Let $\tilde{\tau} : \tilde{E} \to \mathbb{RP}^N \times \mathbb{R}^{N+1-n}$ be a trivialization of ξ, and define a map $\tilde{\varphi} : S \to \mathbb{R}^{N+1-n}$ as the composition

$$\tilde{\varphi} : S \xrightarrow{\tilde{\sigma}} \tilde{E} \xrightarrow{\tilde{\tau}} \mathbb{RP}^N \times \mathbb{R}^{N+1-n} \longrightarrow \mathbb{R}^{N+1-n},$$

where $\tilde{\sigma}$ is given by

$$\tilde{\sigma}(y) := (\langle h(y) \rangle, y - P(y)y)$$

and where the last arrow is canonical projection. Since h is odd and $P(-y) = P(y)$ by assumption, it clearly follows that $\tilde{\varphi}$ is *odd* and continuous. Now consider $B \in \Sigma$ such that $B \subseteq S$ and $B \cap K = \emptyset$. Then, for every $y \in B$, we have $y - P(y)y \neq 0$ and hence $\tilde{\varphi}(y) \neq 0$. Thus the restriction of $\tilde{\varphi}$ to B is an odd map $B \to \mathbb{R}^{N+1-n} \setminus \{0\}$ and hence we find

$$\gamma(B) \leq N + 1 - n. \tag{3.3}$$

By property (5_γ) it follows that there exists $\delta > 0$ such that

$$\gamma(\overline{U_\delta(K)}) = \gamma(K).$$

We choose such a $\delta > 0$ and take $B := S \setminus U_\delta(K)$ (in case $K = \emptyset$ we take $B = S$). Then $B \in \Sigma$ and $B \cap K = \emptyset$, so we have (3.3). Moreover, $S \subseteq B \cup \overline{U_\delta(K)}$ and $\gamma(S) = \gamma(S^N) = N + 1$ because of the odd homeomorphism h. Thus by property (3_γ) we obtain

$$N + 1 = \gamma(S) \leq \gamma(B) + \gamma(\overline{U_\delta(K)}) = \gamma(B) + \gamma(K) \leq N + 1 - n + \gamma(K),$$

and hence $\gamma(K) \geq n$. Thus we have proved (3.2) in the finite-dimensional case.

Second step: The general case.

Choose a strictly decreasing null sequence $(\varepsilon_j)_j \subseteq]0, \infty[$. Since X has the $\mathcal{P}$-property and $H([0,1] \times S) \subset G_n(X)$ is relative compact, we may choose projections $Q_j \in \mathcal{L}(X)$ of finite rank such that the condition (iii) of Lemma 3.6 is satisfied corresponding to $\varepsilon_j > 0$. Moreover define $H_j : [0,1] \times S \to G_n(\mathcal{R}(Q_j))$ by $H_j(t,y) := Q_j(H(t,y))$. Then H_j is continuous for every $j \in \mathbb{N}$, as follows directly from the definition of the opening metric. Set finally $V_j(\cdot) := H_j(1, \cdot) : S \to G_n(\mathcal{R}(Q_j))$ and $K_j := \{y \in S \cap \mathcal{R}(Q_j) \mid y \in V_j(y)\}$. According to the finite-dimensional version of Prop. 3.8 which has already been established, we know that $K_j \in \Sigma$ is compact and that

$$\gamma(K_j) = n \tag{3.4}$$

for every j. To complete the proof, we claim:

$(*)$ $\quad \mathcal{M} := \bigcup_{j \in \mathbb{N}} V_j(S)$ is relatively compact in $G_n(X)$.

Indeed, $V(S)$ is relatively compact, and $V_j(S) \subset G_n(\mathcal{R}(Q_j))$ is relatively compact for every j by Lemma 3.3(c). Moreover,

$$\mathcal{M} \subset U_{\varepsilon_j}\left(V(S) \cup \bigcup_{i=1}^{j} V_i(S)\right),$$

for every j, which yields the relative compactness of $\mathcal{M}$.
By $(*)$ and Lemma 3.6 we conclude that $\mathcal{M}_R := \bigcup_{j\in\mathbb{N}, y\in S} B_R V_j(y)$ is relatively compact in X, hence $Z := \overline{\mathcal{M}_R} \cap S \subset X$ is compact. Moreover, $K_j \in Z$ for every $j \in \mathbb{N}$. Now let $\mathcal{Z}$ be the metric space of all non-empty closed subsets of the compact space Z, equipped with the Hausdorff distance d_h. As is well known (see e.g. [53]), this space is again compact. Thus, after passing to a suitable subsequence, we may assume that we have a limit

$$K_\infty = \lim_{j\to\infty} K_j$$

with respect to the Hausdorff distance. Since reflection at the origin induces a homeomorphism $\mathcal{Z} \to \mathcal{Z}$ and $\Sigma \cap \mathcal{Z}$ is the fixed point set of that homeomorphism, $\Sigma \cap \mathcal{Z}$ is closed in $\mathcal{Z}$, and, in particular, $K_\infty \in \Sigma$. Moreover it follows from (5_γ) and the definition of d_h that

$$\gamma(K_\infty) \geq n.$$

Thus the desired result follows from

$$K \supseteq K_\infty. \tag{3.5}$$

To see this, consider an arbitrary $y \in K_\infty$ and note that by definition of the Hausdorff distance there exist $y_j \in K_j$, $j \in \mathcal{N}$ such that $y = \lim_{j\to\infty} y_j$. Hence also $V(y) = \lim_{j\to\infty} V(y_j)$ in $G_n(X)$ and therefore

$$\begin{aligned}
\text{dist}(y, B_R V(y)) &= \lim_{j\to\infty} \text{dist}(y_j, B_R V(y)) \\
&\leq \lim_{j\to\infty} d_h(B_R V_j(y_j), B_R V(y)) \\
&= R \lim_{j\to\infty} \Theta(V_j(y_j), V(y)) \\
&\leq R \lim_{j\to\infty} \Big[\Theta(V_j(y_j), V(y_j)) + \Theta(V(y_j), V(y))\Big] \\
&\leq R \lim_{j\to\infty} \Big[\varepsilon_j + \Theta(V(y_j), V(y))\Big] \\
&= 0.
\end{aligned}$$

Thus $y \in V(y)$, i.e. $y \in K$. Since $y \in K_\infty$ was arbitrary, we established (3.5) and therefore (3.2) as well. This finally completes the proof. $\square$

Corollary 3.10. *Let* $S_1 := \{v \in X \mid \|v\| = 1\}$ *be the unit sphere, and let*

$$\rho : X \setminus \{0\} \to S_1 \qquad v \mapsto v/\|v\|$$

be the radial projection. The assertions of Prop. 3.8 remain true when S *is replaced by a closed subset* $\tilde{S}$ *of* $X \setminus \{0\}$ *such that* ρ *restricts to a homeomorphism* $\tilde{S} \to S_1$.

Proof. Let $h : \tilde{S} \to S_1$ be the odd homeomorphism obtained by restricting ρ to $\tilde{S}$. Given a continuous $H : [0,1] \times \tilde{S} \to G_n(X)$ satisfying conditions (i) – (iii) from Prop. 3.8, we define $\tilde{H} : [0,1] \times S_1 \to G_n(X)$ by

$$\tilde{H}(t,z) := H(t, h^{-1}(z)).$$

Then obviously $S = S_1$ and $\tilde{H}$ satisfy all the assumptions of Prop. 3.8, and hence the set

$$\tilde{K} := \{z \in S_1 | z \in \tilde{H}(1,z)\}$$

has the desired properties. But the relations

$$y \in H(1,y) \cap \tilde{S}$$

and

$$z \in \tilde{H}(1,z) \cap S_1$$

are evidently equivalent via the substitution $z = h(y)$. This means that $K = h^{-1}(\tilde{K})$, whence the result. $\square$

The final part of this section is concerned with properties of a *noncompact* fixed point set involving the kernels of finite-range projections. As an appropriate topological measure we consider a *dual* genus. For this, fix $S \in \Sigma$ and put $\Sigma(S) := \{A \in \Sigma \mid A \subset S\}$. For $A \in \Sigma(S)$ we define

$$\gamma^*(A) := \sup\{\gamma(B) \mid B \in \Sigma(S), B \cap A = \emptyset\} \quad \in \mathbb{N} \cup \infty.$$

Clearly the values of γ^* depend crucially on the special choice of S. However, in our applications the role of S will be clear, hence we do not express this dependency in our notation. A basic observation is the following:

Lemma 3.11. *Let $S \in \Sigma$.*
If $W \subset X$ is a closed subspace of codimension n, then $W \cap S \in \Sigma(S)$ and $\gamma^*(W \cap S) \le n$.

Proof. By assumption, there exists a continuous projection $P \in \Pi_n(X)$ such that $\mathcal{N}(P) = W$. For arbitrary $B \in \Sigma(S)$, $B \cap W = \emptyset$ the restriction $P|_B : B \to P(X)$ is odd and symmetric, and $P(y) \neq 0$ for every $y \in B$. Since $\dim P(X) = n$, this forces $\gamma(B) \le n$, and we conclude $\gamma^*(W \cap S) \le n$. $\square$

Proposition 3.12. *Let $S \in \Sigma$. Fix a number $n \in \mathbb{N}$ and consider a continuous map $H : [0,1] \times S \to \Pi_n(X)$ having the following properties:*

(i) $H(t,-y) = H(t,y)$ *for* $0 \le t \le 1$ *and every* $y \in S$,

(ii) $H(0,\cdot)$ *is constant on* S.

Put $P := H(1,\cdot)$ and

$$K := \{y \in S \mid y \in \mathcal{N}(P(y))\}.$$

Then $K \in \Sigma(S)$ and $\gamma^*(K) \le n$.

Proof. Clearly $K \in \Sigma(S)$. To prove $\gamma^*(K) \leq n$, we proceed similar as in the proof of Proposition 3.8. Since $\operatorname{dist}(0, S) > 0$, a topological Hausdorff space S' is built by identifying antipodal points in S. Moreover, since S is paracompact (as a metric subspace of X), S' is paracompact as well. Now H factors in the form

$$H : [0,1] \times S \xrightarrow{\alpha} [0,1] \times S' \xrightarrow{\hat{H}} \Pi_n(X),$$

with $\alpha(t, y) := (t, [y])$ and a continuous map $\hat{H}$. Define $\hat{\eta} : [0,1] \times S' \to G_n(X)$ by

$$\hat{\eta}(t,p) := \mathcal{R}(\hat{H}(t,p)).$$

Then $\hat{\eta}$ is continuous by virtue of Lemma 3.4. Hence (ii) implies that

$$\hat{V} := \hat{\eta}(1, \cdot) : S' \to G_n(X)$$

is *nullhomotopic*. Hence the paracompactness of S' implies that the pull-back $\xi := \hat{V}^*\gamma_n$ is trivializable (see [38, p.29] again). The total space of ξ can be written as

$$E \quad = \quad \{([y], v) | y \in S, v \in \mathcal{R}(P(y))\}.$$

Using a trivialization $\tau : E \to S' \times \mathbb{R}^n$ of ξ, we define an odd and continuous map $\varphi : S \to \mathbb{R}^n$ as the composition

$$\varphi : S \xrightarrow{\sigma} E \xrightarrow{\tau} S' \times \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

where $\sigma(y) := ([y], P(y)y)$, and where the last arrow is canonical projection.
By construction, $P(y)y \neq 0$ whenever $y \notin K$. Hence $\gamma(K) \leq n - 1$, as claimed.

□

Chapter 4

Uniform perturbation theory for selfadjoint operators

Let $\mathcal{H}$ denote a real infinite-dimensional Hilbert space with scalar product $(\cdot|\cdot)$ and norm $\|\cdot\|$, and let $A_0 : \mathcal{D}(A_0) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator in $\mathcal{H}$ which is selfadjoint and bounded from below. Let X be the form domain of A_0, and let X^* be its topological dual. Since the range of the inclusion $i : X \to \mathcal{H}$ is dense in $\mathcal{H}$, the canonical identification of $\mathcal{H}$ with its dual leads to the following embeddings:

$$X \overset{i}{\hookrightarrow} \mathcal{H} \overset{i^*}{\hookrightarrow} X^*$$

We therefore regard all the vector spaces defined above as subspaces of X^*. In particular that means that if $v \in \mathcal{H}$, we refer to v also as an element of X^* instead of writing i^*v.
Put $m := -\inf \sigma(A_0) + 1$, and denote by W the square root of the selfadjoint positive operator $A_0 + mI : \mathcal{D}(A_0) \subset H \to H$. Then W is selfadjoint on $\mathcal{H}$ with domain $\mathcal{D}(W) = X$, and X becomes a Hilbert space with the scalar product

$$(u|v)_X := (Wu|Wv) \qquad (u, v \in X).$$

Speaking of the Hilbert space X, we always refer to this inner product. With the notation $W^* : H \to X^*$ for the dual of W, the canonical isometric isomorphism $J : X \to X^*$ can be written as $J = W^*W$. Indeed, for $u, v \in X$ we have

$$\langle W^*Wu, v\rangle = (Wu, Wv) = (u, v)_X,$$

where $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$ stands for the dual pairing. Moreover, note that

$$\|u\|_X \geq \|u\| \qquad \forall\, u \in X$$

and

$$\|u\| \geq \|u\|_{X^*} \qquad \forall\, u \in H.$$

Finally we remark that the operator $\hat{A} := J - mI : X \to X^*$ is precisely the unique continuous extension of the continuous densely defined operator $i^*A_0 : \mathcal{D}(A_0) \subset X \to X^*$.

4.1 Families of form compact perturbations

In the sequel let $\mathcal{K}_S(X, X^*)$ denote the closed subspace of *compact* operators $B \in \mathcal{L}(X, X^*)$ which in addition satisfy

$$\langle Bv, w\rangle = \langle Bw, v\rangle \qquad \text{for all } v, w \in X. \tag{4.1}$$

We endow $\mathcal{K}_S(X, X^*)$ with the norm of $\mathcal{L}(X, X^*)$, in this way it becomes a real Banach space. To each $B \in \mathcal{K}_S(X, X^*)$ we assign the quadratic form $[\cdot,\cdot]_B : X \times X \to \mathbb{R}$ defined by

$$[v, w]_B := \langle(\hat{A} + B)v, w\rangle = (v|w)_X - m(v|w) + \langle Bv, w\rangle.$$

The following Lemma provides uniform bounds for the forms $[\cdot,\cdot]_B$ related to a compact subset of $\mathcal{K}_S(X, X^*)$.

Lemma 4.1. *Consider a compact subset $M \subset \mathcal{K}_S(X, X^*)$. Then there are positive constants $a, b, c \in \mathbb{R}$ such that*

$$a\|v\|_X^2 \le [v, v]_B + c\|v\|^2 \le b\|v\|_X^2 \tag{4.2}$$

for all $B \in M,\ v \in X$.

Proof. First we claim that for every $\varepsilon > 0$ there exists a number $K := K(M, \varepsilon)$ such that

$$|\langle Bv, v\rangle| \le \varepsilon\|v\|_X^2 + K\|v\|^2 \qquad \forall\, B \in M,\ v \in X. \tag{4.3}$$

Assuming in contrary that this is false, we would find $\varepsilon_0 > 0$ and sequences $(B_n)_n \subset M$ as well as $(v_n)_n \subset X$ such that $\|v_n\|_X = 1$ and

$$|\langle B_n v_n, v_n\rangle| > \varepsilon_0 + n\|v_n\|^2 \qquad \forall n. \tag{4.4}$$

for all n. Passing to suitable subsequences, we may assume that $B_n \to T \in \mathcal{K}_S(X, X^*)$ and that $Tv_n \to w \in X^*$. Hence

$$\limsup_{n\in\mathbb{N}} |\langle B_n v_n, v_n\rangle| \le \limsup_{n\in\mathbb{N}} |\langle Tv_n, v_n\rangle| \le \|w\|_{X^*}, \tag{4.5}$$

and therefore $\|v_n\| \to 0$ by (4.4). Since the range of $i^* : H \to X^*$ is dense in X^*, we infer $v_n \rightharpoonup 0$ in X, and this forces $w = \lim\limits_{n\to\infty} Tv_n = 0$ by the compactness of T. This however contradicts (4.4) and (4.5) for sufficiently large n, and therefore (4.3) holds true.
Now, applying (4.3) with $0 < \varepsilon < 1$, we infer

$$\begin{aligned} \|v\|_X^2 &= [v, v]_B - \langle Bv, v\rangle + m\|v\|^2 \\ &\le [v, v]_B + |\langle Bv, v\rangle| + m\|v\|^2 \\ &\le [v, v]_B + \varepsilon\|v\|_X^2 + (K + |m|)\|v\|^2. \end{aligned}$$

and hence

$$a\|v\|_X^2 \le [v, v]_B + c\|v\|^2$$

with $a := 1 - \varepsilon$ and $c := K + |m|$. Moreover

$$\begin{aligned} [v,v]_B &\le \|v\|_X^2 + |\langle Bv, v\rangle| + |m|\|v\|^2 \\ &\le \|v\|_X^2 + \varepsilon\|v\|_X + (K + |m|)\|v\|^2 \\ &\le (1 + \varepsilon + K + |m|)\|v\|_X^2. \end{aligned}$$

Therefore

$$[v,v]_B + c\|v\|^2 \le b\|v\|_X,$$

with $b := 1 + \varepsilon + K + c + m$, and this completes the proof. □

In particular Lemma 4.1 asserts that for every $B \in \mathcal{K}_S(X, X^*)$ the quadratic form $[\cdot,\cdot]_B$ is closed, symmetric and bounded from below in $\mathcal{H}$ with domain X. Hence there exists a unique selfadjoint operator A_B in $\mathcal{H}$ with form domain X and such that

$$(A_B v|w) = [v,w]_B \qquad \text{for all } v, w \in \mathcal{D}(A_B).$$

By slight abuse of notations, we will sometimes call this operator the form sum of A and B (even though B is not given as an operator in $\mathcal{H}$!).
Next we define the nondecreasing sequence of values

$$\mu_k(B) := \inf_{\substack{V \le X \\ \dim V = k}} \sup_{v \in V} \frac{[v,v]_B}{(v|v)},$$

for each $B \in \mathcal{K}_S(X, X^*)$ and set

$$\mu_\infty := \inf \sigma_{ess}(A_B)$$

with the additional convention $\mu_\infty = \infty$ if $\sigma_{ess}(A_B)$ is void. Indeed, $\sigma_{ess}(A_B)$ only depends on A_0, as asserted by the following Lemma.

Lemma 4.2. *For every $B \in \mathcal{K}_S(X, X^*)$ there holds*

$$\sigma_{ess}(A_B) = \sigma_{ess}(A_0) = \{\lambda \in \mathbb{R} \mid \hat{A} - \lambda I : X \to X^* \textit{ is not a Fredholm operator } \} \tag{4.6}$$

Proof. For every $B \in \mathcal{K}_S(X, X^*)$ the operator $\hat{A} - \lambda I : X \to X^*$ is Fredholm if and only if $\hat{A} + B - \lambda I : X \to X^*$ is. Hence it suffices to show

$$\sigma_{ess}(A_B) = \{\lambda \in \mathbb{R} \mid \hat{A} + B - \lambda I \text{ is not Fredholm } \} \tag{4.7}$$

for every $B \in \mathcal{K}_S(X, X^*)$. Applying Lemma 4.1 to the singleton $M := \{B\}$, we may pick $c = c(B) > 0$ such that $A_B + cI$ is a strictly positive operator in $\mathcal{H}$. Now it is easy to verify that

$$\begin{aligned} \sigma_{ess}(A_B) &= \{\lambda \in \mathbb{R} \mid \lambda > -c \text{ and } \frac{1}{\lambda + c} \in \sigma_{ess}((A_B + cI)^{-1})\} \\ &= \{\lambda \in \mathbb{R} \mid \lambda > -c \text{ and } \frac{1}{\lambda + c} I - (A_B + cI)^{-1} \text{ is not Fredholm } \}, \end{aligned} \tag{4.8}$$

since $(A_B + cI)^{-1} : H \to H$ is *bounded* and symmetric. Moreover the following identity holds:

$$\frac{1}{\lambda + c} I - (A_B + cI)^{-1} = \frac{1}{\lambda + c} [(A_B + cI)^{-\frac{1}{2}}]^* [\hat{A} + B - \lambda I](A_B + cI)^{-\frac{1}{2}}.$$

Here $(A_B + cI)^{-\frac{1}{2}} : H \to X$ and its dual $[(A_B + cI)^{-\frac{1}{2}}]^* : X^* \to H$ are topological isomorphisms because of (4.2), and hence (4.7) follows from (4.8). □

Next we observe:

Lemma 4.3. *For every $n \in \mathbb{N}$ the function $\mu_n(\cdot) : \mathcal{K}_S(X, X^*) \to \mathbb{R}$ is continuous.*

Proof. Consider $B_j, B \in \mathcal{K}_S(X, X^*)$, $j \in \mathbb{N}$ such that $B_j \to B$. Since $M := \{B_j, B \mid j \in \mathbb{N}\}$ is compact, we may choose positive constants $a,\ b,\ c$ such that (4.2) holds with respect to M. Now if $0 < \delta < a$ and $j \in \mathbb{N}$ are such that

$$\|B - B_j\| < \delta, \tag{4.9}$$

we have

$$\begin{aligned} [v, v]_{B_j} &= [v, v]_B + \langle (B_j - B)v, v \rangle \\ &\leq [v, v]_B + \delta \|v\|_X^2 \\ &\leq [v, v]_B + \frac{\delta}{a}\left([v, v]_{B_j} + c\|v\|^2\right) \end{aligned}$$

and hence

$$\frac{a - \delta}{a}[v, v]_{B_j} \leq [v, v]_B + \frac{c\delta}{a}\|v\|^2$$

for all $v \in X$. It now follows from the definition of μ_n that

$$\mu_n(B_j) \leq \frac{a}{a - \delta}\mu_n(B) + \frac{c\delta}{a - \delta}. \tag{4.10}$$

Interchanging the roles of B and B_j we see that (4.9) implies

$$\mu_n(B) \leq \frac{a}{a - \delta}\mu_n(B_j) + \frac{c\delta}{a - \delta},$$

hence

$$\mu_n(B_j) \geq \frac{a - \delta}{a}\mu_n(B) - \frac{c\delta}{a}. \tag{4.11}$$

Combining (4.10) and (4.11) we conclude

$$\lim_{j \to \infty} \mu_n(B_j) = \mu_n(B).$$

□

Now fix $B \in \mathcal{K}_S(X, X^*)$ and recall that if $\mu_n(B) < \mu_\infty$, then $\mu_n(B)$ is an eigenvalue of A_B. More precisely, two different cases occur (see e.g. [24, p.90]):

(I) $\mu_k(B) < \mu_\infty$ for all $k \in \mathbb{N}$ and $\lim_{k \to \infty} \mu_k(B) = \mu_\infty$. Moreover, all the $\mu_k(B)$ are eigenvalues of A_B, each repeated a number of times equal to its multiplicity.

(II) Either $\mu_k(B) = \mu_\infty$ for all k, or there is a number $k_0 \in \mathbb{N}$ such that $\mu_{k_0}(B) < \mu_\infty$ and $\mu_k(B) = \mu_\infty$ for $k > k_0$. Then $\mu_1(B), ..., \mu_{k_0}(B)$ are eigenvalues of A_B, each repeated a number of times equal to its multiplicity.

Now, for arbitrary $B \in \mathcal{K}_S(X, X^*)$ and $n \in \mathbb{N}$ we denote by $P_n(B) \in \mathcal{L}(X)$ resp. $Q_n(B) \in \mathcal{L}(X)$ the spectral projections associated with the operator A_B and the interval $]-\infty, \mu_n(B)]$ resp. the interval $[\mu_n(B), \infty[$ (More precisely, $P_n(B)$ and $Q_n(B)$ are defined as the *restrictions* of these spectral projections to the form domain $X \subset H$ of A_B). Moreover we denote

$$V_n(B) := \mathcal{R}(P_n(B)).$$

Note that, if B and n are such that

$$\mu_n(B) < \mu_{n+1}(B), \tag{4.12}$$

then the numbers $\mu_1(B), ..., \mu_n(B)$ are eigenvalues of A_B, and $V_n(B)$ is the span of the corresponding eigenvectors. In particular there holds $P_n(B) \in \Pi_n(X)$ and $V_n(B) \in G_n(X)$. Moreover we have $\mathcal{R}(Q_{n+1}(B)) = \mathcal{N}(P_n(B)) = V_n(B)^\perp \cap X$ in this case, where $\perp$ denotes the orthogonal complement in $\mathcal{H}$ (not in X).

Proposition 4.4. *Consider $n \in \mathbb{N}$ and $D \subset \mathcal{K}_S(X, X^*)$ such that (4.12) holds for every $B \in D$. Then we have:*

(a) The map $P_n : D \to \Pi_n(X)$ is continuous.

(b) If D is relatively compact in $\mathcal{K}_S(X, X^*)$ and

$$\sup_{B \in D} \mu_n(B) < \mu_\infty, \tag{4.13}$$

then the set $V_n(D) \subset G_n(X)$ is relatively compact.

We remark that, in case that (4.12) even holds for every B in the closure $\overline{D}$ of D, then (b) is an immediate consequence of (a) and Lemma 3.4. The general case is slightly more involved.

Proof of Proposition 4.4. (a) Consider $B_j, B \in D$, $j \in \mathbb{N}$ such that $B_j \to B$. Hence $M := \{B_j, B \mid j \in \mathbb{N}\}$ is compact in $\mathcal{K}_S(X, X^*)$, and we may choose positive constants a, b, c such that (4.2) holds with respect to M. Now fix a closed Jordan curve Γ in the complex plane surrounding $\mu_1(B), ..., \mu_n(B)$ but no other eigenvalue of A_B. Then (4.12) and Lemma 4.3 imply that for j large enough we have

$$P_n(B_j) = \frac{1}{2\pi i} \int_\Gamma (\lambda I - A_{B_j})^{-1} \, d\lambda \tag{4.14}$$

(Actually this is true for the complexification of $P_n(B_j)$, but for the sake of brevity we do not express this in the notation). The convergence $P_n(B_j) \to P_n(B)$ now follows provided that

$$(\lambda I - A_{B_j})^{-1} \to (\lambda I - A_B)^{-1} \qquad (j \to \infty)$$

uniformly in $\lambda \in \Gamma$ with respect to the norm of $\mathcal{L}(H, X)$. By [59, Theorem VIII.25(c)], this uniform convergence holds with respect to the norm of $\mathcal{L}(H)$. To establish the stronger convergence, note that

$$(\lambda I - A_{B_j})^{-1} = -(A_{B_j} + cI)^{-1}(I - (c + \lambda)(\lambda I - A_{B_j})^{-1}),$$

and the same holds true for B in place of B_j. Thus it suffices to show

$$(A_{B_j} + cI)^{-1} \to (A_B + cI)^{-1} \qquad (j \to \infty) \tag{4.15}$$

in $\mathcal{L}(H, X)$. To prove this, recall that (4.2) implies that $(A_B + cI)^{1/2} : X \to H$ is an isomorphism. Hence, for j large enough we have

$$1 > \left\| [(A_B + cI)^{-1/2}]^*(B_j - B)(A_B + cI)^{-1/2} \right\|_{\mathcal{L}(H)} \longrightarrow 0.$$

As a consequence,

$$\left(I + [(A_B + cI)^{-1/2}]^*(B_j - B)(A_B + cI)^{-1/2} \right)^{-1} \longrightarrow I$$

in $\mathcal{L}(H)$. Using now

$$(A_{B_j} + cI)^{-1} = (A_B + cI)^{-1/2} \left(I + [(A_B + cI)^{-1/2}]^*(B_j - B)(A_B + cI)^{-1/2} \right)^{-1} (A_B + cI)^{-1/2},$$

we conclude that (4.15) holds. Thus (a) is proven.
(b) Consider an arbitrary sequence $(B_j)_j \subset D$. We have to show that $(V_n(B_j))_j$ contains a subsequence which is converging in $G_n(X)$.
First we may assume that, by passing to a subsequence, there holds $B_j \to B \in \mathcal{K}_S(X, X^*)$. By Lemma 4.3 and (4.13) we have $\lim\limits_{j\to\infty} \mu_n(B_j) \to \mu_n(B) < \mu_\infty$, in particular $\mu_m(B) < \mu_{m+1}(B)$ for some number $m \geq n$. This forces $\mu_m(B_j) < \mu_{m+1}(B_j)$ for $j > j_0$, provided that $j_0 \in \mathbb{N}$ is chosen large enough. Applying (a) we conclude

$$P_m(B_j) \to P_m(B) \qquad (j \to \infty, j > j_0),$$

and hence

$$V_m(B_j) \to V_m(B) \qquad (j \to \infty, j > j_0).$$

by Lemma 3.4. Now fix $R > 0$. Then the set

$$M_R := \bigcup_{j > j_0} B_R V_m(B_j)$$

is relatively compact in X by Lemma 3.6, hence also the set

$$\bigcup_{j > j_0} B_R V_n(B_j) \subset M_R.$$

Again by Lemma 3.6 we conclude that $\{V_n(B_j) \mid j > j_0\} \subset G_n(X)$ is relatively compact. Thus $(V_n(B_j))_j$ contains a convergent subsequence, as required. $\square$

Corollary 4.5. *Consider $m, n \in \mathbb{N}$, $m \geq n \geq 2$ and $D \subset K_S(X, X^*)$ such that for all $B \in D$ there holds*

$$\mu_{n-1}(B) < \mu_n(B) \leq \mu_m(B) < \mu_{m+1}(B).$$

Denote by $P(B)$ the spectral projection associated with A_B and the interval $[\mu_n(B), \mu_m(B)]$. Then the map $P : D \to \Pi_{m-n+1}(X)$ is continuous.

Proof. This holds since $P = P_m - P_{n-1}$, whereas P_m and P_{n-1} are continuous by Proposition 4.4. $\square$

We close this section with a note on weak lower semicontinuity.

Lemma 4.6. *Let $\lambda < \mu_\infty$. Then the functional*

$$w \mapsto \langle \hat{A}w, w \rangle - \lambda \|w\|^2$$

is weakly lower semicontinuous on X.

Proof. Put

$$n := \max\{j \in \mathbb{N} \mid \mu_j(0) \leq \lambda\} < \infty.$$

Then $\mu_n(0) < \mu_{n+1}(0)$, hence $P_n(0)$ and $Q_{n+1}(0)$ are complementary projections in X. Moreover, for $w \in \mathcal{D}(A_0)$ there holds

$$\langle (\hat{A} - \lambda I)Q_{n+1}(0)w, w \rangle = ((A_0 - \lambda I)Q_{n+1}(0)w|w) \geq 0,$$

and by continuity we infer that $w \mapsto \langle (\hat{A} - \lambda I)Q_{n+1}(0)w, w \rangle$ defines the square of a seminorm in X, in particular this functional is weakly lower semicontinuous. Moreover, if $w_j \rightharpoonup w$ in X, then $(\hat{A} - \lambda I)P_n(0)w_j \to (\hat{A} - \lambda I)P_n(0)w$ in X^*, hence also

$$\langle (\hat{A} - \lambda I)P_n(0)w_j, w_j \rangle \to \langle (\hat{A} - \lambda I)P_n(0)w, w \rangle.$$

The assertion now follows from the decomposition

$$\langle \hat{A}w, w \rangle - \lambda \|w\|^2 = \langle (\hat{A} - \lambda I)P_n(0)w, w \rangle + \langle (\hat{A} - \lambda I)Q_{n+1}(0)w, w \rangle$$

which holds for every $w \in X$. $\square$

4.2 Remarks on the bounded case

Having examined the continuous dependence of eigenvalues and spectral projections associated with semibounded operators in $\mathcal{H}$, we will now state related results for *bounded* symmetric operators in X. Indeed, in some of the following chapters we are naturally led to study operators $G \in \mathcal{L}(X)$ which are symmetric with respect to the scalar product *in* X. However, the following results *do not* rely on the fact that X is the form domain of some semibounded operator A_0, i.e., they hold for X being an arbitrary (infinite-dimensional) real Hilbert space.

Denote $\mathcal{L}_S(X) \subset \mathcal{L}(X)$ the real Banach space of bounded symmetric operators in X. For each $G \in \mathcal{L}_S(X)$ there is a *decreasing* sequence of values

$$\sigma_k(G) := \sup_{\substack{V < X \\ \dim V = k}} \inf_{v \in V} \frac{(Gv|v)_X}{(v|v)_X} \qquad (k \in \mathbb{N}), \tag{4.16}$$

as well as $\sigma_\infty(G) = \sup \sigma_{ess}(G)$. Moreover, for fix $G \in \mathcal{L}_S(X)$, we have the following two alternatives:

(I) $\sigma_k(G) > \sigma_\infty(G)$ for all $k \in \mathbb{N}$ and $\lim\limits_{k\to\infty} \sigma_k(G) = \sigma_\infty(G)$. Moreover, all the $\sigma_k(G)$ are eigenvalues of G, each repeated a number of times equal to its multiplicity.

(II) Either $\sigma_k(G) = \sigma_\infty$ for every $k \in \mathbb{N}$, or there is a number $k_0 \in \mathbb{N}$ such that $\sigma_{k_0}(G) > \sigma_\infty(G)$ and $\sigma_k(G) = \sigma_\infty(G)$ for $k > k_0$. Then $\sigma_1(G), ..., \sigma_{k_0}(G)$ are eigenvalues of G, each repeated a number of times equal to its multiplicity.

We denote by $\tilde{P}_n(G) \in \mathcal{L}_S(X)$ resp. $\tilde{Q}_n(G) \in \mathcal{L}_S(X)$ the spectral projections associated with the operator G and the interval $[\sigma_n(G), \sigma_1(G)]$ resp. the interval $]-\infty, \sigma_n(G)]$. Note that, if G and n are such that

$$\sigma_n(G) > \sigma_{n+1}(G), \tag{4.17}$$

then the numbers $\sigma_1(G), ..., \sigma_n(G)$ are eigenvalues of G. Moreover $\tilde{P}_n(G) \in \Pi_n(X)$ in that case, and $\mathcal{R}(\tilde{Q}_{n+1}(G)) = \mathcal{N}(\tilde{P}_n(G)) = \mathcal{R}(\tilde{P}_n(G))^\perp$, where $\perp$ now denotes the orthogonal complement *in* X. In view of the considerations of the previous section is is no surprise that the following three statements hold.

Lemma 4.7. *For every $n \in \mathbb{N}$ the function $\sigma_n(\cdot) : \mathcal{L}_S(X) \to \mathbb{R}$ is continuous.*

Proposition 4.8. *Consider $n \in \mathbb{N}$ and $D \subset \mathcal{L}_S(X)$ such that*

$$\sigma_n(G) > \sigma_{n+1}(G)$$

holds for every $G \in D$. Then the map $\tilde{P}_n : D \to \Pi_n(X)$ is continuous.

Corollary 4.9. *Consider $m, n \in \mathbb{N}$, $m \geq n \geq 2$ and $D \subset \mathcal{L}_S(X)$ such that for all $u \in D$ there holds*

$$\sigma_{n-1}(G) > \sigma_n(G) \geq \sigma_m(G) > \sigma_{m+1}(G).$$

Denote $\tilde{P}(G)$ the spectral projection associated with G and the interval $[\sigma_m(G), \sigma_n(G)]$. Then the map $\tilde{P} : D \to \Pi_{m-n+1}(X)$ is continuous.

We omit the proofs.

Chapter 5

Introducing a spectral characterization problem

We keep using the notations introduced in Chapter 4. In addition, we now consider a (nonlinear) map $B : X \to \mathcal{L}(X, X^*)$ which satisfies the fundamental hypothesis:

(H1) B is continuous, and $B(0) = 0$.

(H2) $B(u) \in \mathcal{L}(X, X^*)$ is a compact operator for every $u \in X$.

(H3) $\langle B(u)v, w\rangle = \langle B(u)w, v\rangle$ for all $u, v, w \in X$.

We may summarize these hypothesis by assuming that $B : X \to \mathcal{K}_S(X, X^*)$ is a continuous map satisfying $B(0) = 0$. Hence, for every $u \in X$, we may build $A_{B(u)}$ as in the previous chapter, but from now on we will simply write $A(u)$ for this operator. Moreover we write $[\cdot,\cdot]_u$, $\mu_n(u)$, $P_n(u)$, $V_n(u)$ in place of $[\cdot,\cdot]_{B(u)}$, $\mu_n(B(u))$, $P_n(B(u))$, $V_n(B(u))$, respectively. We now define a nonlinear eigenvalue problem featuring a spectral characterization:

Definition 5.1. *Let $n \in \mathbb{N}$. A vector $u \in X$ is called a solution of problem $(SC)_n$ if $u \in \mathcal{D}(A(u))$ and*

$$A(u)u = \mu_n(u)u$$

We remark that $u \in X$ is a solution of Problem $(SC)_n$ if and only if

$$(\hat{A} + B(u))u = \mu_n(u)u,$$

both sides being viewed as elements of X^*. Before we turn to this problem directly, we state some direct implications of Lemma 4.3, Proposition 4.4 and Corollary 4.5.

Lemma 5.2. *For every $n \in \mathbb{N}$ the function $\mu_n(\cdot) : X \to \mathbb{R}$ is continuous. If $B : X \to \mathcal{L}(X, X^*)$ is strongly continuous, then $\mu_n(\cdot) : X \to \mathbb{R}$ is weakly sequentially continuous.*

Proposition 5.3. *Consider $n \in \mathbb{N}$ and $D \subset X$ such that*

$$\mu_n(u) < \mu_{n+1}(u)$$

holds for every $u \in D$. Then:

(a) *The map $P_n : D \to \Pi_n(X)$ is continuous.*

(b) *If D is bounded in X, $B : X \to \mathcal{L}(X, X^*)$ is compact and*

$$\sup_{u \in D} \mu_n(u) < \infty,$$

then the set $V_n(D) \subset G_n(X)$ is relatively compact.

(c) *If D is a weakly compact in X and $B : X \to \mathcal{L}(X, X^*)$ is strongly continuous, then the set $V_n(D) \subset G_n(X)$ is compact.*

Corollary 5.4. *Consider $m, n \in \mathbb{N}$, $m \geq n \geq 2$ and $D \subset X$ such that for all $u \in D$ there holds*

$$\mu_{n-1}(u) < \mu_n(u) \leq \mu_m(u) < \mu_{m+1}(u).$$

Denote by $P(u)$ the spectral projection associated with $A(u)$ and the interval $[\mu_n(u), \mu_m(u)]$. Then the map $P : D \to \Pi_{m-n+1}(X)$ is continuous.

Note that, if $u \in X$ is a solution of $(SC)_n$ for given $n \in \mathbb{N}$, then in particular

$$u \in V_n(u). \tag{5.1}$$

We suspect that in general the set of all $u \in X$ satisfying (5.1) has a very complicated topological structure. However, in case that

(H4) $B(u) = B(-u)$ for every $u \in X$,

we get some view on this structure by applying the results from Section 3.2. Our first result in this spirit is the following.

Theorem 5.5. *Assume that in addition to (H1)- (H4) the map $B : X \to \mathcal{L}(X, X^*)$ is compact, and let $n \in \mathbb{N}$. Moreover consider an open, bounded and symmetric subset $D \subset X$ containing $0 \in X$ and such that*

$$\mu_n(u) < \mu_{n+1}(u)$$

for all $u \in \overline{D}$ and

$$\sup_{u \in \overline{D}} \mu_n(u) < \mu_\infty.$$

Finally suppose that for every finite dimensional subspace U of X the set $\partial D \cap U$ is homeomorphic to the unit sphere in U by radial projection.
Then the set $K := \{u \in \partial D \,|\, u \in V_n(u)\}$ has the following properties:
$K \in \Sigma$, K is compact, and $\gamma(K) = n$. In particular K is nonempty.

Note that if $n = 1$ in Theorem 5.5, then K consists of solutions to $(SC)_n$.

Proof of Theorem 5.5. Denoting $S := \partial D$ we infer that $S \in \Sigma$, i.e. S is a closed and symmetric subset of $X \setminus \{0\}$. Now define a map $H : [0, 1] \times S \to G_n(X)$ by

$$H(t, u) := V_n(tu).$$

In view of Proposition 5.3, Lemma 3.4 and our assumptions we infer that $\mathcal{H}$ is continuous, and that $H([0,1] \times S)$ is relatively compact in $G_n(X)$. Furthermore, $H(t,u) = H(t,-u)$ by (H4). Finally, $H(0,\cdot) : S \to G_n(X)$ is constant. Therefore the assumptions of Proposition 3.8 are satisfied, and an application of this Proposition yields precisely the assertion. □

Next we note that the conclusions of Theorem 5.5 also hold under slightly different assumptions. More precisely:

Theorem 5.6. *Assume that in addition to (H1)- (H4) the map $B : X \to \mathcal{L}(X, X^*)$ is strongly continuous, and let $n \in \mathbb{N}$. Moreover consider an open, bounded and symmetric subset $D \subset X$ such that $0 \in D$, $\overline{D}$ is weakly compact and*

$$\mu_n(u) < \mu_{n+1}(u) \qquad \text{for all } u \in \overline{D}.$$

Finally suppose that for every finite dimensional subspace U of X the set $\partial D \cap U$ is homeomorphic to the unit sphere in U by radial projection.
Then the set $K := \{u \in \partial D \,|\, u \in V_n(u)\}$ has the following properties:
$K \in \Sigma$, K is compact, and $\gamma(K) = n$. In particular K is nonempty.

This is proven by the same arguments as above, with Proposition 5.3(c) now yielding the desired relative compactness property.
Closing this section, we state a basic observation which we will use frequently in the following chapters.

Lemma 5.7.

(a) *If $B : X \to \mathcal{L}(X, X^*)$ is a* compact *map (i.e., it is completely continuous in view of (H1)), then the nonlinear operator $\tilde{B} : X \to X^*$ defined by $\tilde{B}(u) := B(u)u$ is completely continuous as well.*

(b) *If $B : X \to \mathcal{L}(X, X^*)$ is strongly continuous, then $\tilde{B}$ is strongly continuous as well.*

Proof. (a) Obviously $\tilde{B}$ is continuous. To see that $\tilde{B}$ is compact, consider an arbitrary bounded sequence $(u_n)_n \subset X$. Passing to a subsequence, we may assume that $u_n \rightharpoonup u \in X$ and that $B(u_n)$ converges to an operator $B_0 \in \mathcal{K}_S(X, X^*)$. In particular $B_0 u_n \to B_0 u$ in X^* and therefore

$$\|\tilde{B}(u_n) - B_0 u\|_{X^*} \leq \|B(u_n) - B_0\| \|u_n\|_X + \|B_0(u_n - u)\|_{X^*} \to 0.$$

Therefore $\tilde{B}$ is compact, as claimed.
(b) Similar. □

Chapter 6

Abstract sublinear equations

So far we did not impose any sign condition on the nonlinearity. In other words, a map $B : X \to \mathcal{L}(X, X^*)$ satisfies (H1)-(H4) if and only if $-B$ does. However, in the sequel we suppose that B is *nonnegative* in a certain sense. Precisely we impose the following hypothesis:

(CC) ('Comparison Condition') There is a map $\varphi : X \to \mathbb{R}$ such that for arbitrary vectors $u, v \in X$ there holds

$$2(\varphi(v) - \varphi(u)) \geq \langle B(u)v, v\rangle - \langle B(u)u, u\rangle.$$

We assume that (H1)-(H4) and (CC) are in force throughout this chapter.
To put (CC) into perspective, we first derive basic consequences. Observe that, by adding a suitable constant, we can normalize φ so as to have

$$\varphi(0) = 0, \tag{6.1}$$

Moreover, using (H1) and taking $u = 0$ in (CC), we obtain

$$\varphi(v) \geq 0 \qquad (v \in X), \tag{6.2}$$

whereas the choice $v = 0$ in (CC) leads to

$$0 \leq \varphi(u) \leq \frac{1}{2}\langle B(u)u, u\rangle \qquad (u \in X). \tag{6.3}$$

As indicated by the following Lemma, condition (CC) also forces a *variational* framework.

Lemma 6.1. *There holds $\varphi \in C^1(X)$, and*

$$d\varphi(u) = B(u)u : X \to X^* \qquad (u \in X). \tag{6.4}$$

However, in the following we will not use the differentiability of φ (at least not explicitly), since our approach does not rely on arguments based on deformations or general gradient flow investigations.

Proof of Lemma 6.1. Since the map $u \mapsto B(u)u$ is continuous by (H1), it suffices to show that φ is Gateaux differentiable with Gateaux derivative given by (6.4). Therefore consider $u, v \in X$ and $t > 0$. Then, using (CC) and (H3), we infer

$$\begin{aligned} 2\frac{\varphi(u+tv)-\varphi(u)}{t} &\geq \frac{1}{t}\big[\langle B(u)(u+tv),(u+tv)\rangle - \langle B(u)u,u\rangle\big] \\ &= 2\langle B(u)u,v\rangle + t\langle B(u)v,v\rangle \end{aligned}$$

as well as

$$\begin{aligned} 2\frac{\varphi(u+tv)-\varphi(u)}{t} &\leq \frac{1}{t}\big[\langle B(u+tv)(u+tv),(u+tv)\rangle - \langle B(u+tv)u,u\rangle\big] \\ &= 2\langle B(u+tv)u,v\rangle + t\langle B(u+tv)v,v\rangle. \end{aligned}$$

Passing to the limit $t \to 0$ we derive (6.4). □

We remark that (6.1) and Lemma 6.1 imply

$$\varphi(u) = \int_0^1 \langle B(tu)tu, u\rangle \, dt$$

for every $u \in X$, and therefore

$$\langle B(u)u,u\rangle > 0 \iff \varphi(u) > 0 \tag{6.5}$$

by virtue of (6.3). In applications to differential equations we will see that (CC) is closely related to *convexity*. We also state an abstract criterion in this spirit, which however is too restrictive for most of our applications.

Lemma 6.2. *[33, p. 32] Consider a* convex *functional of the form*

$$\varphi = \Phi \circ q,$$

where $\Phi \in C^1(X)$ and where q is a continuous X-valued quadratic form on X, i. e.

$$q(y) = b(y,y) \qquad\qquad (y \in X)$$

for a unique symmetric continuous bilinear map $b : X \times X \to X$.
Then, if $B : X \to \mathcal{L}(X, X^*)$ is given by

$$\langle B(y)v, w\rangle := 2\langle d\Phi(q(y)), b(v,w)\rangle$$

for $v, w, y \in X$, there holds (CC) for Φ and B.

Now observe that condition (H4) and (CC) in particular imply that φ is *even*, that is, $\varphi(u) = \varphi(-u)$ for every $u \in X$. The same is true for the functional $\psi : X \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{2}\langle \hat{A}u, u\rangle + \varphi(u) = \frac{1}{2}(\|u\|_X^2 - m\|u\|^2) + \varphi(u).$$

By Lemma 6.1 we infer that $\psi \in C^1(X)$ with derivative given by

$$d\psi(u) = (\hat{A} + B(u))u \tag{6.6}$$

for $u \in X$. An important tool for the upcoming investigations is the inequality

$$2(\psi(v) - \psi(u)) \geq [v,v]_u - [u,u]_u \qquad (u, v \in X), \tag{6.7}$$

which is an immediate consequence of (CC). Defining the *Rayleigh quotient at* $u \in X$ by

$$\rho_u(v) := \frac{[v,v]_u}{\|v\|^2} \qquad (v \in X \setminus \{0\}),$$

we can reformulate inequality (6.7) as

$$2(\psi(v) - \psi(u)) \geq \|v\|^2 \rho_u(v) - \|u\|^2 \rho_u(u) \qquad (u, v \in X \setminus \{0\}), \tag{6.8}$$

and this will be the more suitable form for comparing linear and nonlinear minimax principles. Moreover we remark that

$$\rho_u(u) \geq \frac{2\psi(u)}{\|u\|^2} \geq \rho_0(u) \geq \inf \sigma(A_0) \tag{6.9}$$

for every $u \in X \setminus \{0\}$ by (6.3).
In the following sections we investigate the level sets of ψ as well as of

$$\psi_\lambda(u) := \psi(u) - \frac{\lambda}{2}\|u\|^2 \qquad (u \in X)$$

for $\lambda \in \mathbb{R}$. To treat these cases simultaneously, we define for an arbitrary even and continuous functional $\Psi : X \to \mathbb{R}$ together with an arbitrary closed and symmetric subset $S \subset X \setminus \{0\}$ the *Ljusternik-Schnirelman* levels $c_n(\Psi, S)$ by

$$c_n(\Psi, S) := \inf_{\substack{A \in \Sigma(S) \\ \gamma(A) \geq n}} \sup_{u \in A} \Psi(u) \qquad \in \mathbb{R} \cup \{\pm\infty\}.$$

Note that $c_n(\Psi, S)$ also has a *dual* characterization given by

$$c_n(\Psi, S) = \sup_{\substack{A \in \Sigma(S) \\ \gamma^*(A) \leq n-1}} \inf_{u \in A} \Psi(u). \tag{6.10}$$

This is due to the easily-verified identity

$$c_n(\Psi, S) = \inf\{c \in \mathbb{R} |\ \gamma(S \cap \Psi^c) \geq n\} = \sup\{c \in \mathbb{R} |\ \gamma(S \cap \Psi^c) < n\},$$

where ψ^c is defined as the sublevel set of ψ, cf. Section 1.1. In the following two sections we will pursue the search for solutions of $(SC)_n$ which satisfy additional side conditions.

6.1 Spectrally characterized solutions with prescribed norm

We fix $R > 0, n \in \mathbb{N}$ for this section. In the following we are concerned with solutions u of problem $(SC)_n$ satisfying the additional side condition

$$\|u\| = R.$$

For this we define

$$S_R := \{u \in X | \; \|u\| = R\}$$

and

$$K := \{u \in S_R \mid u \in V_n(u)\}.$$

In order to simplify the notation, we just write c_n in place of $c_n(\psi, S_R)$ in this section. Our aim is to establish the following property:

(CP) K is compact, $\gamma(K) = n$, $c_n = \max\limits_{u \in K} \psi(u)$ and every $u \in \psi^{-1}(c_n) \cap K$ is a solution of $(SC)_n$.

In particular this property furnishes solutions of $(SC)_n$. First we observe that by (6.2) there holds

$$\psi(u) + \frac{m}{2}\|u\|^2 \geq \frac{1}{2}\|u\|_X^2 \geq \frac{1}{2}\|u\|^2, \tag{6.11}$$

in particular $c_n \geq \frac{(1-m)R^2}{2}$ for every n. Moreover $c_n < \infty$, since S_R contains compact subsets of genus n.

Proposition 6.3. *Let $u \in X$.*

(a) If $u \in S_R$, then

$$\psi(u) - c_n \leq \frac{R^2}{2}\Big(\rho_u(u) - \mu_n(u)\Big). \tag{6.12}$$

(b) If $u \in K$, then $\psi(u) \leq c_n$.

(c) If $\|u\| \leq R$ and $u \in V_n(u)$, then $\psi(u) \leq \max\{c_n, 0\}$.

(d) If $u \in \psi^{-1}(c_n) \cap K$, then u is a solution of problem $(SC)_n$.

Proof. We prove (a),(b) and (c) simultaneously. For this fix $u \in X$ with $\|u\| \leq R$, and suppose first that $\mu_n(u)$ is an eigenvalue of $A(u)$ (which implies that $\mu_1(u), ..., \mu_{n-1}(u)$ are eigenvalues as well). Choose pairwise orthogonal eigenvectors $u_1, \dots, u_{n-1}$ corresponding to $\mu_1(u), \dots, \mu_{n-1}(u)$. Let $W \subseteq V_n(u)$ be the span of $u_1, \dots, u_{n-1}$, and put $W^\perp := \{v \in X \mid (v|w) = 0 \; \forall w \in W\}$. Then clearly

$$\mu_n(u) = \inf_{v \in S_R \cap W^\perp} \rho_u(v),$$

whereas $\gamma^*(S_R \cap W^\perp) = n - 1$. By (6.10) and (6.8) we therefore obtain

$$\begin{aligned} 2(c_n - \psi(u)) &\geq \inf_{v \in S_R \cap W^\perp} 2(\psi(v) - \psi(u)) \\ &\geq \inf_{v \in S_R \cap W^\perp} R^2 \rho_u(v) - \|u\|^2 \rho_u(u) \\ &\geq R^2 \mu_n(u) - \|u\|^2 \rho_u(u) \end{aligned}$$

Now if either $u \in S_R$ or $\rho_u(u) \geq 0$, then (6.12) follows. If moreover $u \in V_n(u)$, then (6.12) evidently yields $\psi(u) \leq c_n$, since $\rho_u(u) \leq \mu_n(u)$ in this case. On the other hand, if $\rho_u(u) < 0$, then $\psi(u) < 0$ by virtue of (6.9). This establishes (a), (b) and (c).
Now consider the case that $\mu_n(u)$ is not an eigenvalue of $A(u)$, hence $\mu_n(u) = \mu_\infty \in \sigma_c(A(u))$. We then define W just as the span of all eigenvectors of $A(u)$ corresponding to eigenvalues below μ_∞, so that there holds $m := \dim W + 1 \leq n$. Observe that again we have

$$\mu_n(u) = \inf_{v \in S_R \cap W^\perp} \rho_u(v),$$

whereas now

$$\inf_{v \in S_R \cap W^\perp} \psi(v) \leq c_m \leq c_n,$$

hence (a), (b) and (c) are derived as in the first case.
Finally, to prove (d), suppose that $u \in K \cap \psi^{-1}(c_n)$. Then (a) yields $\rho_u(u) = \mu_n(u)$, which is possible only if u is an eigenvector of $A(u)$ with eigenvalue $\mu_n(u)$. □

Corollary 6.4. *Consider $\tilde{R} > 2[\max\{c_n, 0\} + mR^2]$ and*

$$D(R, \tilde{R}) := \{u \in X \mid \|u\| < R \textit{ and } \|u\|_X < \tilde{R}\}.$$

Then $K = \{u \in \partial D(R, \tilde{R}) \mid u \in V_n(u)\}$.

Proof. Fix $u \in X$ satisfying $u \in V_n(u)$ and $\|u\| \leq R$. Then $\psi(u) \leq \max\{c_n, 0\}$ by Prop. 6.3(c), and therefore $\|u\|_X < \tilde{R}$ in view of (6.11). Hence

$$u \in \partial D(R, \tilde{R}) \qquad \Longleftrightarrow \qquad u \in S_R.$$

and from this the assertion follows. □

Combining these observations with Proposition 5.6, we now may formulate the main result of this section.

Theorem 6.5. *Suppose that $B : X \to \mathcal{L}(X, X^*)$ is strongly continuous. Moreover assume that there is $\tilde{R} > 2[\max\{c_n, 0\} + mR^2]$ such that*

$$\mu_n(u) < \mu_{n+1}(u)$$

for all $u \in \overline{D(R, \tilde{R})}$. Then condition (CP) holds true.

Proof. Set $D := D(R, \tilde{R})$. Clearly $0 \in D$, and D is open, bounded and symmetric. Moreover, $\overline{D}$ is closed, bounded and convex, hence it is weakly compact. Finally, ∂D is homeomorphic to the unit sphere in X by radial projection, since ∂D is the unit sphere with respect to some equivalent norm on X. Hence an application of Theorem 5.6 yields the following properties for the set $\tilde{K} = \{u \in \partial D \mid u \in V_n(u)\}$:

$$\tilde{K} \in \Sigma,\ \tilde{K} \text{ is compact and } \gamma(\tilde{K}) = n.$$

But actually K coincides with $\tilde{K}$ by Corollary 6.4, so the same holds for K. In particular, ψ attains its maximum on K, and by definition of c_n there holds $\max\limits_{u \in K} \psi(u) \geq c_n$. But actually equality holds by virtue of Prop. 6.3(b). Finally, Prop. 6.3(d) ensures that every $u \in \psi^{-1}(c_n)$ is a solution of $(SC)_n$. Hence (CP) holds true. □

6.2 Spectrally characterized solutions with prescribed eigenvalue

Fix $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ with

$$\mu_n(0) < \lambda < \mu_\infty. \tag{6.13}$$

We are now interested in solutions u of problem $(SC)_n$ which in addition satisfy

$$\mu_n(u) = \lambda.$$

In particular, such a solution u is a critical point of ψ_λ. We need the following further assumptions:

$(CC)_1$ For arbitrary $u, v \in X$ the function $t \mapsto \langle B(tu)v, v\rangle$ is nondecreasing on $[0, \infty[$ and strictly increasing once it takes positive values.

(FG) There holds
$$(\lambda - \mu_1(0))\|v\|^2 < \lim_{t\to\infty} \langle B(tv)v, v\rangle \leq \infty$$
for all $v \in X \setminus \{0\}$.

(UC) If $u, v \in X$ are such that $B(u) \neq 0$ and $v \neq 0$ is an eigenfunction of $A(u)$, then $\langle B(u)v, v\rangle > 0$.

We remark that, in applications to differential equations, (UC) can easily be derived from unique continuation properties. Condition $(CC)_1$ implies that for every $u \in X$ the function

$$t \mapsto \mu_n(tu)$$

is nondecreasing on $[0, \infty[$. We set

$$S_\lambda := \{u \in X \mid \mu_n(u) = \lambda\}$$

and we claim that the values $c_n := c_n(\psi_\lambda, S_\lambda)$ contain solutions of $(SC)_n$ in their ψ-level set. More precisely, setting

$$K_\lambda := \{u \in S_\lambda \mid u \in V_n(u)\},$$

we will prove, under appropriate assumptions, the following property:

$(CP)_\lambda$ K_λ is compact, $\gamma(K_\lambda) = n$, $c_n = \max_{u \in K_\lambda} \psi_\lambda(u)$ and every $u \in \psi_\lambda^{-1}(c_n) \cap K_\lambda$ is a solution of $(SC)_n$.

Note that (6.13) and Lemma 5.2 imply that S_λ is a closed and symmetric subset of $X \setminus \{0\}$. Moreover:

Lemma 6.6. *The radial projection $S_\lambda \to S := \{w \in X \mid \|w\|_X = 1\}$ is injective.*

Proof. (a) Suppose in contradiction that there is $u \in X \setminus \{0\}$ and $0 < t < 1$ such that $u, tu \in S_\lambda$, i.e. $\mu_n(u) = \mu_n(tu) = \lambda$. In particular $B(u) \neq 0$, since otherwise $\mu_n(u) = \mu_n(0) < \lambda$. Denote by $v_1, ..., v_n$ a choice of orthonormalized eigenvectors corresponding to $\mu_1(u), ..., \mu_n(u)$, and define $V \subset X$ as the span of $v_1, ..., v_n$. Now consider arbitrary $v \in V$, $\|v\| = 1$. If

$$[v, v]_u = \mu_n(u), \tag{6.14}$$

then $v \in \mathcal{D}(A(u))$ and $A(u)v = \mu_n(u)v$. Hence $\langle B(u)v, v\rangle > 0$ by (UC), and therefore

$$\langle B(tu)v, v\rangle < \langle B(u)v, v\rangle$$

by $(CC)_1$. As a consequence,

$$[v, v]_{tu} < \mu_n(u). \tag{6.15}$$

On the other hand, if $[v, v]_u < \mu_n(u)$, then (6.15) holds as well. By a simple compactness argument we conclude

$$\mu_n(tu) \leq \sup_{v \in V,\, \|v\|_X = 1} [v, v]_{tu} < \mu_n(u),$$

which is a contradiction. This proves the claim. $\square$

Next we prove an inequality similar to Proposition 6.3(a):

Proposition 6.7. *Let $u \in S_\lambda$. Then*

$$\psi_\lambda(u) - c_n \leq \frac{\|u\|^2}{2}(\rho_u(u) - \lambda).$$

Proof. By (6.13) we infer that $\lambda = \mu_n(u)$ is an eigenvalue of $A(u)$, hence $\mu_1(u), ..., \mu_{n-1}(u)$ are eigenvalues as well. Let, as in the proof of Prop. 6.3, $W \subseteq V_n(u)$ be the span of of pairwise orthogonal eigenvectors corresponding to $\mu_1(u), \ldots, \mu_{n-1}(u)$, and denote $W^\perp := \{v \in X \mid (v|y) = 0\ \forall y \in W\}$. Then

$$\rho_u(v) \geq \lambda \qquad \forall\, v \in W^\perp, \tag{6.16}$$

whereas $\gamma^*(W^\perp \cap S_\lambda) \leq n - 1$ by virtue of Lemma 3.11. Therefore (6.10) and (6.8) yield

$$\begin{aligned}
2(c_n - \psi_\lambda(u)) &\geq \inf_{v \in S \cap W^\perp} 2(\psi_\lambda(v) - \psi_\lambda(u)) \\
&\geq \inf_{v \in S_R \cap W^\perp} \|v\|^2(\rho_u(v) - \lambda) - \|u\|^2(\rho_u(u) - \lambda) \\
&\geq -\|u\|^2(\rho_u(u) - \lambda),
\end{aligned}$$

which shows the assertion. $\square$

Corollary 6.8. *Let $u \in K_\lambda$. Then:*

(a) $\psi_\lambda(u) \le c_n$

(b) If $\psi_\lambda(u) = c_n$, then u is a solution of problem $(SC)_n$.

Proof. (a) Since $u \in V_n(u)$, there holds $\rho_u(u) \le \mu_n(u) = \lambda$. Hence Proposition 6.7 yields $\psi_\lambda(y) \le c_n$.
(b) If $\psi_\lambda(u) = c_n$, then $\rho_u(u) = \lambda = \mu_n(u)$ by Proposition 6.7, which is possible only if u is an eigenvector of $A(u)$ with eigenvalue $\mu_n(u)$.

□

Using assumption (FG), we now ensure that, at least for certain vectors $u \in X$, the value $\mu_n(u)$ is pushed up to the level λ.

Lemma 6.9. *If $B : X \to \mathcal{L}(X, X^*)$ is strongly continuous, then the set*

$$C := \{u \in X \mid u \in V_n(u),\ \mu_n(u) \le \lambda\}$$

is bounded in X.

Proof. Assume in contradiction that $\|u_j\|_X \to \infty$ for a sequence $(u_j)_j \subset C$. Since $\rho_0(u_j) \le \mu_n(u_j)$ remains bounded, $v_j := \frac{u_j}{\|u_j\|}$ defines a sequence $(v_j)_j$ which is bounded in X and normalized in $\mathcal{H}$, i.e. $\|v_j\| = 1$ for all j. Moreover, since $v_j \in V_n(u_j)$, there are numbers $\lambda_j \in [\mu_1(0), \lambda]$ such that

$$\langle \hat{A} v_j, v_j\rangle + \langle B(u_j)v_j, v_j\rangle = \lambda_j.$$

Passing to a subsequence, we may assume that $v_j \rightharpoonup v$ in X and $\lambda_j \to \hat{\lambda}$. Using (FG), $(CC)_1$ and Lemma 5.7(b), we also find $c > \hat{\lambda} - \mu_1(0)$ and $t > 0$ such that

$$\begin{aligned} \limsup_j \langle B(u_j)v_j, v_j\rangle &\ge \limsup_j \langle B(tv_j)v_j, v_j\rangle \\ &= \langle B(tv)v, v\rangle \\ &\ge c\|v\|^2. \end{aligned}$$

Now pick $\varepsilon > 0$ such that $\hat{\lambda} + \varepsilon < \mu_\infty$. Then the functional $w \mapsto \langle \hat{A} w, w\rangle - (\hat{\lambda} + \varepsilon)\|w\|^2$ is weakly lower semicontinuous on X by Lemma 4.6, and therefore

$$\begin{aligned} 0 &\ge \varepsilon\|v\| - \varepsilon \\ &\ge \langle \hat{A} v, v\rangle - \hat{\lambda}\|v\|^2 - \liminf_j[\langle \hat{A} v_j, v_j\rangle - (\hat{\lambda} + \varepsilon)\|v_j\|^2] - \varepsilon \\ &= \langle \hat{A} v, v\rangle - \hat{\lambda}\|v\|^2 - \liminf_j[-\langle B(u_j)v_j, v_j\rangle] \\ &= \langle \hat{A} v, v\rangle - \hat{\lambda}\|v\|^2 + \limsup_j[\langle B(u_j)v_j, v_j\rangle] \\ &\ge \langle \hat{A} v, v\rangle + (c - \hat{\lambda})\|v\|^2 \end{aligned}$$

However, since $(c - \hat{\lambda}) > -\mu_1(0)$, the last expression is nonnegative, and it vanishes if and only if $v = 0$. This forces a contradiction, and thus the lemma is proved. □

Theorem 6.10. *Suppose that $B : X \to \mathcal{L}(X, X^*)$ is strongly continuous, and that for all $u \in X$ with $\mu_n(u) \le \lambda$ there holds*

$$\mu_n(u) < \mu_{n+1}(u).$$

Then condition $(CP)_\lambda$ holds true.

Proof. In view of Lemma 6.9 we may choose $\tilde{R} > 0$ such that $C \subset B_{\tilde{R}}(0) \subset X$. Put $D := \{u \in X \mid \mu_n(u) < \lambda,\ \|u\|_X < \tilde{R}\}$. Then D is an open, bounded and symmetric subset of X containing $x = 0$. Moreover, by virtue of $(CC)_1$ there holds $\overline{D} = \{u \in X \mid \mu_n(u) \le \lambda,\ \|u\|_X \le \tilde{R}\}$, hence Lemma 5.2 implies that $\overline{D}$ is weakly compact. Hence, in order to apply Theorem 5.6, we just need to ensure the following

$(*)$ If $U \subset X$ is a finite dimensional subspace, then $\partial D \cap U$ is homeomorphic to the unit sphere S_U in U by radial projection.

To prove $(*)$, note that for every $u \in S_U$ there is a number $t \le \tilde{R}$ such that $tu \in \partial D$, hence the radial projection $U \cap \partial D \to S_U$ is surjective. To show injectivity, suppose in contradiction that there would exist $u, tu \in \partial D$, $t < 1$. In particular $\|u\|_X \le \tilde{R}$ and therefore $\|tu\| < \tilde{R}$. Hence $\mu_n(tu) = \lambda > \mu_n(0)$, and Lemma 6.6 now implies that $\mu_n(u) > \mu_n(tu) = \lambda$. This however contradicts $u \in \overline{D}$. We conclude noting that $U \cap \partial D$ is compact, hence the fact that the radial projection $U \cap \partial D \to S$ is continuous and bijective implies that it has a continuous inverse.
In view of $(*)$ Theorem 5.6 now yields that the set $\tilde{K} := \{u \in \partial D \mid u \in V_n(u)\}$ has the following properties:

$$\tilde{K} \in \Sigma,\ \tilde{K} \text{ is compact and } \gamma(\tilde{K}) = n. \tag{6.17}$$

Moreover, since $\tilde{K} \subset C \subset B_{\tilde{R}}X$, there holds $\tilde{K} \subset K_\lambda$. On the other hand,

$$K_\lambda \subset S_\lambda \cap C \subset S_\lambda \cap B_{\tilde{R}}X \subset \partial D,$$

the last inclusion being a consequence of Lemma 6.6. Hence also $K_\lambda \subset \tilde{K}$, and therefore both sets coincide. As a consequence, (6.17) is also true K_λ in place of $\tilde{K}$, and in particular ψ_λ attains its maximum on K_λ. By the very definition of c_n there holds $\max\limits_{u \in K} \psi_\lambda(u) \ge c_n$, hence equality holds by virtue of Corollary 6.8(a). Finally, Corollary 6.8(b) ensures that every $u \in \psi^{-1}(c_n)$ is a solution of $(SC)_n$.
Hence, $(CP)_\lambda$ holds true and the Theorem is proved. $\square$

Chapter 7

Abstract superlinear equations

We now deal with equations where, compared to the previous chapter, the nonlinear part carries the opposite sign. More precisely, assuming that $B : X \to \mathcal{L}(X, X^*)$ satisfies (H1)-(H4),(CC) and being given some $n \in \mathbb{N}$, we now intend to find vectors $u \in X \cap \mathcal{D}(A_{-B(u)})$ with the property

$$u \in \mathcal{D}(A_{-B(u)}) \quad \text{and} \quad A_{-B(u)}u = \mu_n(-B(u))u, \tag{7.1}$$

or, equivalently,

$$(\hat{A} - B(u))u = \mu_n(-B(u))u$$

in X^* (Here we used the notations of Chapter 4). The investigation of such a *superlinear* problem has to be done in an essentially different way, nevertheless we will recognize some kind of *duality* to the sublinear case. In order to keep the notation simple, we now *redefine* some of the symbols we used in Chapter 6 such that they fit in the present context. To be precise, we put

$$\psi(u) = \frac{1}{2}\|u\|_X^2 - \varphi(u) \qquad (u \in X),$$

and

$$\psi_\lambda(u) = \psi(u) - \frac{\lambda}{2}\|u\|^2 \qquad (u \in X), \tag{7.2}$$

for $\lambda \in \mathbb{R}$. Moreover we will write $A(u)$, $[\cdot,\cdot]_u$, $\mu_n(u)$, $P_n(u)$, $Q_n(u)$, $V_n(u)$ in place of $A_{-B(u)}$, $[\cdot,\cdot]_{-B(u)}$, $\mu_n(-B(u))$, $P_n(-B(u))$, $V_n(-B(u))$, respectively. Finally we put

$$\rho_u(v) := \frac{[v,v]_u}{\|v\|^2}$$

for $u \in X$, $v \in X \setminus \{0\}$, and we say that $u \in X \cap D(A(u))$ is a solution of problem $(SC)_n$ if u satisfies (7.1), that is, if

$$u \in \mathcal{D}(A(u)) \quad \text{and} \quad A(u)u = \mu_n(u)u.$$

As a consequence of Lemma 6.1, there holds

$$d\psi(u) = (\hat{A} - B(u))u \qquad (u \in X) \tag{7.3}$$

and

$$d\psi_\lambda(u) = d\psi(u) - \lambda u \qquad (u \in X). \tag{7.4}$$

Clearly the inequalities (6.7) resp. (6.8) now have to be replaced by

$$2(\psi(v) - \psi(u)) \le [v,v]_u - [u,u]_u \qquad (u,v \in X) \tag{7.5}$$

resp.

$$2(\psi(v) - \psi(u)) \le \|v\|^2 \rho_u(v) - \|u\|^2 \rho_u(u) \qquad (u,v \in X \setminus \{0\}). \tag{7.6}$$

Considering the superlinear case, we require the additional assumption

(H5) $\langle B(u)v, v\rangle \ge 0$ for all $u, v \in X$.

However, in case that (CC) is already satisfied, (H5) does not seem to be a strong further restriction. Indeed, in our applications (CC) and (H5) are always satisfied simultaneously. Note that (H5) in particular implies

$$\mu_n(u) \le \mu_n(0) \qquad \forall\, u \in X,\ n \in \mathbb{N}. \tag{7.7}$$

7.1 Spectrally characterized solutions with prescribed norm

Let $R > 0$ be given. Consider $S_R = \{u \in X \mid \|u\| = R\}$ and $c_n := c_n(\psi, S_R)$ similar as in Section 6.1. We define

$$K^- := \{u \in S_R \mid Q_n(u)u = u\},$$

suppressing the dependency on $n \in \mathbb{N}$ in our notation. Our aim is to establish the following property:

$(CP)^-$ $K^- \in \Sigma(S_R)$, $\gamma^*(K^-) \le n-1$, ψ takes its minimum on K^-, $c_n = \min\limits_{u \in K^-} \psi(u)$ and every $u \in \psi^{-1}(c_n) \cap K^-$ is a solution of $(SC)_n$.

Note that this property reveals some kind of duality to condition (CP) from Section 6.1. In particular it provides solutions of $(SC)_n$ again. In addition to (H1)-(H5) and (CC) we require the following condition:

(BB) There are constants $0 < a < 1,\ b > 0$ such that

$$\langle B(u)u, u\rangle \le a\|u\|_X^2 + b$$

for $u \in S_R$.

Remark 7.1. A sufficient condition for (BB) is the existence of numbers $a, b > 0, q \in [0,2[$ such that

$$\langle B(u)u, u\rangle \le a\|u\|_X^q + b \tag{7.8}$$

for $u \in S_R$.

As a matter of fact, a growth condition like (BB) is usually imposed for the variational treatment of superlinear problems on spheres in the $\mathcal{H}$-norm (cf. [74, p.413] and the references quoted there). In particular it guarantees that ψ is bounded from below on S_R, hence $c_n > -\infty$ for every $n \in \mathbb{N}$. More precisely there holds

Lemma 7.2.

$$\psi(u) \;\geq\; \frac{R^2}{2}\rho_u(u) \;\geq\; -\frac{b+mR^2}{2}$$

for all $u \in S_R$. Moreover, $\psi^c \cap S_R$ is bounded in X for every $c \in \mathbb{R}$.

Proof. Combining (6.3) and (BB), we deduce

$$\begin{aligned}
\psi(u) &= \frac{1}{2}(\|u\|_X^2 - mR^2) - \varphi(u) \\
&\geq \frac{1}{2}\Big(\|u\|_X^2 - mR^2 - \langle B(u)u, u\rangle\Big) = \frac{R^2}{2}\rho_u(u) \\
&\geq \frac{1}{2}[(1-a)\|u\|_X^2 - mR^2 - b] \qquad (7.9) \\
&\geq -\frac{b+mR^2}{2}
\end{aligned}$$

for $u \in S_R$. Moreover, since $0 < a < 1$, (7.9) implies that $\psi^c \cap S_R$ is bounded in X for every $c \in \mathbb{R}$. □

Now, as a first step to establish property $(CP)^-$, we observe:

Lemma 7.3. $K^- \in \Sigma(S_R)$.

Proof. Evidently K^- is symmetric, hence it remains to show that K^- is closed. For this consider a sequence $(u_k)_k \subset K^-$ such that $u_k \to u$ in X. In particular $u \in S_R$, since S_R is closed. Pick $j \in \mathbb{N}$ minimal such that $\mu_j(u) = \mu_n(u)$. If $j = 1$, then clearly $u \in K^-$. Hence consider the case $1 < j \leq n$. From Proposition 5.3(a) we infer that $P_{j-1}(u_k) \to P_{j-1}(u)$ in $\mathcal{L}(X)$, and consequently $P_{j-1}(u)u = \lim_{k\to\infty} P_{j-1}(u_k)u_k = 0$, since $u_k \in K^-$. We conclude that $Q_n(u)u = (I - P_{j-1}(u))u = u$, hence $u \in K^-$. □

Next we show an inequality analogous to Proposition 6.3(a).

Proposition 7.4. *Let $u \in S_R$, $n \in \mathbb{N}$. Then*

$$\psi(u) - c_n \geq \frac{R^2}{2}\Big(\rho_u(u) - \mu_n(u)\Big).$$

Proof. We first consider the case that $\mu_n(u)$ is an eigenvalue of $A(u)$. Choose pairwise orthogonal eigenvectors $u_1, \ldots, u_n$ corresponding to $\mu_1(u), \ldots, \mu_n(u)$, and let W be the span of $u_1, \ldots, u_n$. Then clearly

$$\mu_n(u) = \sup_{v \in S_R \cap W} \rho_u(v),$$

whereas $\gamma(S_R \cap W) = n$. By (7.6) we therefore obtain

$$\begin{aligned} 2(c_n - \psi(u)) &\leq \sup_{v \in S_R \cap W} 2(\psi(v) - \psi(u)) \\ &\leq \sup_{v \in S_R \cap W} R^2(\rho_u(v) - \rho_u(u)) \\ &\leq R^2(\mu_n(u) - \rho_u(u)). \end{aligned}$$

To complete the proof, consider the case that $\mu_n(u)$ is not an eigenvalue of $A(u)$, hence $\mu_n(u) = \mu_\infty \in \sigma_c(A(u))$. Then, for arbitrary $\varepsilon > 0$, we may still pick an n-dimensional subspace W such that $\rho_u(v) \leq \mu_\infty + \varepsilon$ for all $v \in W$. In the same way as above we now have

$$\begin{aligned} 2(c_n - \psi(u)) &\leq \sup_{v \in S_R \cap W} 2(\psi(v) - \psi(u)) \\ &\leq \sup_{v \in S_R \cap W} R^2(\rho_u(v) - \rho_u(u)) \\ &\leq R^2(\mu_n(u) + \varepsilon - \rho_u(u)) \end{aligned}$$

Letting $\varepsilon \to 0$, we again obtain the assertion. □

Corollary 7.5. *Let $u \in K^-$. Then:*

(a) $\psi(u) \geq c_n$

(b) If $\psi(u) = c_n$ then u is a solution of $(SC)_n$.

Proof. (a) Since $u \in \mathcal{R}(Q_n(u))$, there holds $\rho_u(u) \geq \mu_n(u)$. Hence Proposition 7.4 yields $\psi(u) \geq c_n$.
(b) If $\psi(u) = c_n$, then $\rho_u(u) = \mu_n(u)$ by Proposition 7.4, which is possible only if u is an eigenvector of $A(u)$ with eigenvalue $\mu_n(u)$.

□

Lemma 7.2 and Prop. 7.4 also yield

$$\mu_n(u) \geq \frac{2}{R^2}(c_n - \psi(u)) - \frac{b + mR^2}{R^2}$$

for $u \in S_R$ and every $n \in \mathbb{N}$. In particular we infer that the functions μ_n are bounded from below on a constraint sublevel set $\psi^c \cap S_R$, $c \in \mathbb{R}$ arbitrary. Hence they remain bounded by virtue of (7.7), and combined with the following estimate this fact proves useful for investigating minimizing sequences for ψ in K^-.

Lemma 7.6. *There holds*

$$\|d\psi(u) - \mu_n(u)u\|_{X^*} \leq R(1 + |m| + |\mu_1(u)|)^{1/2}\Big(\rho_u(u) - \mu_n(u)\Big)^{\frac{1}{2}} \tag{7.10}$$

for $u \in K^-$.

We postpone the proof until the end of the section, exploiting first the benefits of this inequality.

Proposition 7.7. *Suppose that*

$$\gamma^*(K^-) \leq n-1. \tag{7.11}$$

Then $K^- \neq \emptyset$ and

$$\inf_{u \in K^-} \psi(u) = c_n. \tag{7.12}$$

Moreover, if $(u_j)_j \subset K^-$ is a minimizing sequence for ψ in K^-, then $(u_j)_j \subset X$ and $(\mu_n(u_j))_j \subset \mathbb{R}$ are bounded sequences, and $\lim_{j\to\infty} \|d\psi(u_j) - \mu_n(u_j)u_j\|_{X^*} = 0$.

Proof. First, (7.12) follows directly from a combination of Corollary 7.5(a) with the relations (7.11) and (6.10). Moreover, if $(u_j)_j$ is a minimizing sequence for ψ in K^-, then Lemma 7.2 implies that $(u_j)_j$ is bounded, and so are the sequences $(\mu_n(u_j))_j$ and $(\mu_1(u_j))_j$. Moreover, Proposition 7.4 yields

$$o(1) = \psi(u_j) - c_n \quad \geq \quad \frac{R^2}{2}(\rho_{u_j}(u_j) - \mu_n(u_j)) \geq 0, \tag{7.13}$$

hence $\lim_{j\to\infty} [\rho_{u_j}(u_j) - \mu_n(u_j)] = 0$. By virtue of (7.10) we conclude that $\lim_{j\to\infty} \|d\psi(u_j) - \mu_n(u_j)u_j\|_{X^*} = 0$, as claimed. □

Theorem 7.8. *Suppose that (7.11) is valid, and that $B : X \to \mathcal{L}(X, X^*)$ is a compact map. Finally assume that $c_n < \frac{R^2}{2}\mu_\infty$.*
Then Condition $(CP)^-$ holds true.

Proof. In view of Corollary 7.5(b) and (7.12) we only need to insure that ψ attains its minimum on K^-. To this end, consider a minimizing sequence $(u_j)_j$ for ψ in K^-. Then the sequences $(u_j)_j \subset X$ and $(\mu_n(u_j))_j \subset \mathbb{R}$ are bounded by Prop. 7.7. In view of Lemma 5.7(a) we therefore may assume that, passing to a subsequence, $\mu_n(u) \to \lambda$ and $B(u_j)u_j \to w \in X^*$. Hence also

$$(\hat{A} - \lambda I)u_j = d\psi(u_j) - \mu_n(u_j) + B(u_j)u_j + (\mu_n(u_j) - \lambda)u_j \to w$$

again by Prop. 7.7. Moreover

$$\begin{aligned} \lambda &= \lim_{j\to\infty} \mu_n(u_j) \\ &= \lim_{j\to\infty} \rho_{u_j}(u_j) \\ &\leq \lim_{j\to\infty} \frac{2}{R^2}\psi(u_j) \\ &= \frac{2}{R^2}c_n \\ &< \mu_\infty. \end{aligned}$$

By virtue of Lemma 4.2 we conclude that the operator $\hat{A} - \lambda I : X \to X^*$ is Fredholm, in particular it is *proper* when restricted to a bounded subset. Hence, passing again to a subsequence, we may assume that $u_j \to \tilde{w} \in K^-$, and $\psi|_{K^-}$ attains its minimum at $\tilde{w}$. □

Finally we give a criterion for (7.11) to hold:

Proposition 7.9. *Suppose that* $\mu_{n-1}(u) < \mu_n(u)$ *for all* $u \in D := \{y \in X \mid \|y\| \le R\}$. *Then* $\gamma^*(K^-) \le n-1$.

Proof. By assumption and Proposition 5.3, the function $P_{n-1} : D \to \Pi_{n-1}(X)$ is well defined and continuous. Hence $H : [0,1] \times S_R \to \Pi_{n-1}(X)$, defined by $H(t,u) := P_{n-1}(tu)$ is well defined and continuous as well. Since $H(t,u) = H(t,-u)$, the assertion follows from Proposition 3.12 and the fact that $\mathcal{N}(P_{n-1}(u)) = \mathcal{R}(Q_n(u))$ for $u \in D$. □

We close the section with the

Proof of Lemma 7.6. Fix $u \in K^-$. In order to keep the notation simple, put $\mu_1 := \mu_1(u)$, $\mu_n := \mu_n(u)$ and $Q_n := Q_n(u)$. We show

$$\|(\hat{A} - B(u))w - \mu_n w\|_{X^*} \le \|w\|(1 + |m| + |\mu_1|)^{1/2}\Big(\rho_u(w) - \mu_n\Big)^{\frac{1}{2}} \tag{7.14}$$

for every $w \in \mathcal{R}(Q_n)$, which in particular implies (7.10). However it suffices to ensure (7.14) for $w \in \mathcal{R}(Q_n) \cap \mathcal{D}(A(u))$, since this space is dense in $\mathcal{R}(Q_n)$ and both sides of (7.14) are continuous real-valued functions in $w \in X$. Therefore we have

$$\|(\hat{A} - B(u) - \mu_n I)w\|_{X^*} = \sup_{v \in X, \|v\|_X = 1} \langle(\hat{A} - B(u) - \mu_n I)w, v\rangle = \sup_{v \in X, \|v\|_X = 1} ((A(u) - \mu_n I)w|v)$$

Again by continuity we may take the supremum over vectors $v \in \mathcal{D}(A(u))$, and from $((A(u) - \mu_n I)w, v) = ((A(u) - \mu_n I)w, Q_n v)$ we infer

$$\|(\hat{A} - B(u) - \mu_n I)w\|_{X^*} \le \sup_{\substack{v \in \mathcal{D}(A(u)) \\ \|v\|_X = 1}} ((A(u) - \mu_n I)w|Q_n v).$$

Note that $((A(u) - \mu_n I) \cdot |\cdot)$ defines a semidefinite scalar product on the subspace $\mathcal{R}(Q_n)$, and the corresponding Cauchy-Schwarz inequality implies

$$((A(u) - \mu_n I)w|Q_n v) \le ((A(u) - \mu_n I)w|w)^{1/2}((A(u) - \mu_n I)Q_n v|Q_n v))^{1/2}.$$

However, for every $v \in X$ we have

$$\begin{aligned} ((A(u) - \mu_n I)Q_n v|Q_n v) &= ((A(u) - \mu_n I)v|v) - ((A(u) - \mu_n I)(I - Q_n)v|(I - Q_n)v) \\ &\le ((A(u) - \mu_n I)v|v) + (\mu_n - \mu_1)((I - Q_n)v|(I - Q_n)v) \\ &\le \|v\|_X^2 - \langle B(u)v, v\rangle - (m + \mu_n)\|v\|^2 + (\mu_n - \mu_1)\|v\|^2 \\ &\le (1 + |m| + |\mu_1|)\|v\|_X^2 \end{aligned}$$

by virtue of (H5). We conclude that

$$\begin{aligned} \|(\hat{A} - B(u) - \mu_n I)w\|_{X^*} &\le ((A(u) - \mu_n I)w|w)^{1/2}(1 + |m| + |\mu_1|)^{1/2} \\ &= \|w\|(1 + |m| + |\mu_1|)^{1/2}\Big(\rho_u(w) - \mu_n\Big)^{\frac{1}{2}}. \end{aligned}$$

□

7.2 Spectrally characterized solutions with prescribed eigenvalue

Consider $n \in \mathbb{N}, \lambda \in \mathbb{R}$ *fixed*. We now intend to find solutions u of problem $(SC)_n$ which in addition satisfy

$$\mu_n(u) = \lambda. \tag{7.15}$$

We will be able to give reasonable criteria for the existence of such solutions. However, proving those criteria seem to require a different approach involving *related* eigenvalue problems for *bounded* linear operators *in* X. Let us motivate this: Note that an estimate in the spirit of Proposition 7.4 always involves the norm $\|\cdot\| = \|\cdot\|_H$. This did not cause any problem since we imposed the constraint $\|u\| = R$. However, to detect solutions of $(SC)_n$ satisfying (7.15), we have to replace this constraint. Indeed, if

$$\lambda < \mu_1(0) \tag{7.16}$$

for instance, we rather explore minimax principles on the *Nehari manifold*

$$\begin{aligned} \mathcal{N} &:= \{u \in X \setminus \{0\} \mid \rho_u(u) = \lambda\} \\ &= \{u \in X \setminus \{0\} \mid [u,u]_u = \lambda \|u\|^2\}, \end{aligned} \tag{7.17}$$

which then is a *closed* and symmetric subset of $X \setminus \{0\}$ containing all such solutions. However we are not able to control the norm $\|\cdot\|$ on $\mathcal{N}$ without further unpleasant restrictions. To circumvent this problem, we will replace the operators $A(u), u \in X$ by a family of bounded symmetric operators on X. We first illustrate the general procedure for the special case $\lambda = 0$ and $m = 0$ (i.e., $\inf \sigma(A_0) = 1$): In this case, for every $u \in X$ there holds

$$u \in \mathcal{D}(A(u)), \qquad A(u)u = 0. \tag{7.18}$$

if and only if

$$Ju = B(u)u \qquad \text{in } X^*, \tag{7.19}$$

that is, if and only if u is an eigenfunction of the eigenvalue problem

$$J^{-1}B(u)v = \sigma v \qquad \text{in } X \tag{7.20}$$

associated with the eigenvalue $\sigma = 1$. We remark that by (H2), (H3) and (H5), the operator $J^{-1}B(u) \in \mathcal{L}(X)$ is compact, symmetric and nonnegative for every $u \in X$, therefore its spectrum consists of a *decreasing* sequence of eigenvalues given by

$$\sigma_k(u) := \sup_{\substack{V \leq X \\ \dim V = k}} \inf_{v \in V} \frac{(J^{-1}B(u)v|v)_X}{(v|v)_X} = \sup_{\substack{V \leq X \\ \dim V = k}} \inf_{v \in V} \frac{\langle B(u)v, v\rangle}{(v|v)_X}$$

Moreover, it is easy to see that

$$\mu_n(u) = 0 \qquad \Longleftrightarrow \qquad \sigma_n(u) = 1 \qquad (u \in X). \tag{7.21}$$

Hence u *is a solution of* $(SC)_n$ *with* $\mu_n(u) = 0$ *if and only if* $u = v$ *solves (7.20) with* $\sigma = \sigma_n(u) = 1$. This justifies the study of the operators $J^{-1}B(u) \in \mathcal{L}(X)$ instead of our original operator family $A(u)$, $u \in X$. Furthermore we may treat eigenvalue problems of the form (7.20) in a more general framework, *not* assuming that X arises as the form domain of some semibounded selfadjoint operator A_0. Indeed, in order to study equations of the *Emden-Fowler* type (see Chapter 11), we need to consider the Hilbert space $X := D^{1,2}(\mathbb{R}^N)$ which is not of this form.
The plan to proceed is as follows: First we reformulate the results of Section 4.2 for operator-valued maps in order to treat a related eigenvalue problem in a more general setting. Equipped with the appropriate tools, we then return to the problem of finding solutions to $(SC)_n$ which satisfy (7.15).

7.2.1 On a related eigenvalue problem

Only for this subsection let us assume that X is an *arbitrary* real Hilbert space with scalar product $(\cdot|\cdot)_X$, and let $J : X \to X^*$ denote the canonical isometric isomorphism. We consider a nonlinear map $B : X \to \mathcal{L}(X, X^*)$ satisfying (H1)-(H4), (CC) and the following assumptions:

$(CC)_1$ For arbitrary $u, v \in X$, the function $t \mapsto \langle B(tu)v, v\rangle$ is nondecreasing on $[0, \infty[$ and increases strictly once it takes positive values.

$(CC)_2$ There is an $\eta > 2$ such that $0 \leq \eta\varphi(u) \leq \langle B(u)u, u\rangle$ for all $u \in X$.

Note that $(CC)_1$ and (H1) also imply (H5). Referring to the notations of Section 4.2, we define a continuous map $G : X \to \mathcal{L}_S(X)$ by $G(u) := J^{-1}B(u)$ for every $u \in X$. Indeed, (H3) implies that $G(u)$ is a *symmetric* operator for every u, moreover it is compact and nonnegative in view of (H2) and (H5). Hence the nonzero eigenvalues $\sigma_k(u) := \sigma_k(G(u))$, $k \in \mathbb{N}$ are given by (4.16), which now may be written as

$$\sigma_k(u) = \sup_{\substack{V \leq X \\ \dim V = k}} \inf_{v \in V} \frac{\langle B(u)v, v\rangle}{(v|v)_X}.$$

For the sake of brevity, we also write $\tilde{P}_n(u)$, $\tilde{Q}_n(u)$ in place of $\tilde{P}_n(G(u))$, $\tilde{Q}_n(G(u))$, respectively (cf. Section 4.2).
In the rest of the section we are interested in elements $u \in X$ satisfying

$$G(u)u = u \tag{7.22}$$

as well as

$$\sigma_n(u) = 1. \tag{7.23}$$

Since this problem corresponds, in a vague sense, to problem $(SC)_n$ for the special case $m = 0$, it is consistent to define

$$[v, w]_u := (v|w)_X - \langle B(u)v, w\rangle \qquad (u, v, w \in X)$$

and $\psi : X \to \mathbb{R}$ by

$$\psi(u) = \frac{1}{2}\|u\|_X^2 - \varphi(u) \qquad (u \in X).$$

From Lemma 6.1 we infer

$$d\psi(u) = (J - B(u))u \qquad (u \in X),$$

hence (7.22) holds if and only if $d\psi(u) = 0$, and every nontrivial solution of (7.22) is contained in the set

$$\mathcal{N} := \{u \in X \setminus \{0\} \mid [u,u]_u = 0\}.$$

Clearly $\mathcal{N}$ is symmetric, moreover it is *closed* in X, since B is continuous and $B(0) = 0$. In the sequel we explore the minimax values $c_n := c_n(\psi, \mathcal{N})$ on $\mathcal{N}$. For this we remark that ψ is positive on $\mathcal{N}$, more precisely

$$\psi(u) \geq \left(\frac{1}{2} - \frac{1}{\eta}\right)\|u\|_X^2. \tag{7.24}$$

for $u \in \mathcal{N}$ by virtue of $(CC)_2$. In particular we infer that $c_n \geq 0$ for all $n \in \mathbb{N}$. Moreover we recall the standard estimate

$$\varphi(tu) \geq t^\eta \varphi(u) \qquad \text{for } u \in X,\ t \geq 1. \tag{7.25}$$

The following Lemma gives some view on the geometry of $\mathcal{N}$.

Lemma 7.10. *If $V \subset X$ is a finite dimensional subspace such that for every $v \in V \setminus \{0\}$ there is a number $t > 0$ such that $\langle B(tv)v, v\rangle > 0$, then $V \cap \mathcal{N}$ is homeomorphic to the unit sphere $S_V \subset V$ by radial projection. In particular $\gamma(V \cap \mathcal{N}) = \dim V$.*

Proof. First observe that the radial projection $V \cap \mathcal{N} \to S_V$ is injective by $(CC)_1$. Moreover, by assumption and (6.5) we find for every $v \in S_V$ a positive number t such that $\varphi(w) > 0$ for $w = tv$, hence $(CC)_1$ and (7.25) yield

$$\langle B(sw)sw, sw\rangle \geq 2s^\eta \varphi(w) \geq \|sw\|_X^2 \tag{7.26}$$

for $s = s(w) > 0$ large enough. In particular there is a unique $t_v > 0$ such that $t_v v \in \mathcal{N}$. Since S_V is compact and B is continuous, the set $\{t_v \,|v \in S_V\}$ is bounded. Hence $V \cap \mathcal{N}$ is compact and the radial projection is continuous and bijective considered as a map $V \cap \mathcal{N} \longrightarrow S_V$. Thus it is a homeomorphism. □

Now, to derive a spectral estimate analogous to Prop. 7.4, we need the following further assumption:

$(CC)_3$ If $u, v \in X$ are such that v is a finite sum of eigenvectors of $G(u)$ corresponding to *positive* eigenvalues, then $\langle B(tv)v, v\rangle > 0$ for some $t > 0$.

Using this and Lemma 7.10, we derive

Proposition 7.11. *Let $u \in \mathcal{N}$ with $\sigma_n(u) \geq 1$. Then*

$$\psi(u) - c_n \geq \frac{d^2}{2}(\sigma_n(u) - 1),$$

where $d := \inf_{u \in \mathcal{N}} \|u\|_X > 0$.

Proof. Since $u \in \mathcal{N}$, (7.5) implies

$$2(\psi(v) - \psi(u)) \le [v, v]_u \tag{7.27}$$

for every $v \in X$. Now choose pairwise orthogonal eigenvectors $v_1, \dots, v_n \in X$ corresponding to $G(u)$ and the eigenvalues $\sigma_1(u), \dots, \sigma_n(u)$, and let V be the span of $v_1, ..., v_n$. Since $\sigma_i(u) \ge 1$ for $i = 1, ..., n$, a simultaneous view on $(CC)_3$ and Lemma 7.10 ensures that $\gamma(V \cap \mathcal{N}) = n$. From this we infer

$$\begin{aligned} 2(c_n - \psi(u)) &\le 2 \sup_{v \in \mathcal{N} \cap V} (\psi(v) - \psi(u)) \\ &\le \sup_{v \in \mathcal{N} \cap V} [v, v]_u \\ &\le (1 - \sigma_n(u)) \inf_{v \in \mathcal{N} \cap V} \|v\|_X^2 \\ &= d^2(1 - \sigma_n(u)), \end{aligned}$$

using again that $\sigma_n(u) \ge 1$. □

We now fix $n \in \mathbb{N}$, and we introduce the spectral fixed point set

$$K_{\mathcal{N}} := \{u \in \mathcal{N} \mid \tilde{Q}_n(u)u = u\}.$$

Not surprisingly, $K_{\mathcal{N}}$ has similar properties as K^- in Section 7.1. In particular, the following three assertions are to be compared with Corollary 7.5, Lemma 7.3 and Lemma 7.6.

Lemma 7.12. *For $u \in K_{\mathcal{N}}$ there holds:*

(a) $\psi(u) \ge c_n$

(b) If $\psi(u) = c_n$, then u satisfies (7.22) and (7.23).

Proof. Note that $u \in K_{\mathcal{N}}$ implies $\sigma_n(u) \ge 1$, therefore (a) follows directly from Proposition 7.11. Moreover, if in addition $\psi(u) = c_n$, then Proposition 7.11 implies $\sigma_n(u) = 1$, and by combining the relations $(G(u)u|u)_X = \sigma_n(u|u)_X$ and $u \in \mathcal{R}(\tilde{Q}_n(u))$ we conclude $G(u)u = u$. □

Lemma 7.13. $K_{\mathcal{N}} \in \Sigma(\mathcal{N})$.

Proof. Evidently $K_{\mathcal{N}}$ is symmetric. The closedness of $K_{\mathcal{N}}$ is seen similarly as in Lemma 7.3: Consider a sequence $(u_k)_k \subset K_{\mathcal{N}}$ such that $u_k \to u$ in X. In particular $u \in \mathcal{N}$, since $\mathcal{N}$ is closed. Pick $j \in \mathbb{N}$ minimal such that $\sigma_j(u) = \sigma_n(u)$. If $j = 1$, then clearly $u \in K$. Hence consider the case $1 < j \le n$. By Proposition 4.8 we infer that $\tilde{P}_{j-1}(u_k) \to \tilde{P}_{j-1}(u)$ in $\mathcal{L}(X)$, and therefore $\tilde{P}_{j-1}(u)u = \lim_{k\to\infty} \tilde{P}_{j-1}(u_k)u_k = 0$, since $u_k \in K_{\mathcal{N}}$. We conclude that $\tilde{Q}_n(u)u = (I - \tilde{P}_{j-1}(u))u = u$, hence $u \in K_{\mathcal{N}}$. □

Lemma 7.14. *There holds*

$$\|d\psi(u)u\|_{X^*} \le \|u\|_X \left[(\sigma_n(u) - 1) + [(\sigma_n(u) - 1)\sigma_n(u)]^{\frac{1}{2}} \right]$$

for every $u \in K_{\mathcal{N}}$.

Proof. Fix $u \in K_{\mathcal{N}}$. Then clearly $\sigma_n(u) \geq 1$, and $\sigma_n(u)(\cdot|\cdot)_X - (G(u)\cdot|\cdot)_X$ defines a (semidefinite) scalar product on the subspace $\mathcal{R}(\tilde{Q}_n(u)) \subset X$. Set $w := u - G(u)u \in \mathcal{R}(\tilde{Q}_n(u))$, then the corresponding Cauchy-Schwarz inequality yields

$$\begin{aligned}\sigma_n(u)(u|w)_X - (G(u)u|w)_X &\leq [\sigma_n(u)\|u\|_X^2 - (G(u)u|u)_X]^{\frac{1}{2}} \times \\ &\quad [\sigma_n(u)\|w\|_X^2 - (G(u)w|w)_X]^{\frac{1}{2}} \\ &\leq [(\sigma_n(u)-1)\|u\|_X^2]^{\frac{1}{2}}[\sigma_n(u)\|w\|_X^2]^{\frac{1}{2}} \\ &\leq \|u\|_X[(\sigma_n(u)-1)\sigma_n(u)]^{\frac{1}{2}}\|w\|_X.\end{aligned}$$

Therefore

$$\begin{aligned}\|w\|_X^2 &= (u|w)_X - (G(u)u|w)_X \\ &\leq \sigma_n(u)(u|w)_X - (G(u)u|w)_X + |1-\sigma_n(u)|(u|w)_X \\ &\leq \|u\|_X[(\sigma_n(u)-1)\sigma_n(u)]^{\frac{1}{2}}\|w\|_X + (\sigma_n(u)-1)\|u\|_X\|w\|_X,\end{aligned}$$

hence

$$\|d\psi(u)u\|_{X^*} = \|(J-B(u))u\|_{X^*} = \|w\|_X \leq \|u\|_X\left[(\sigma_n(u)-1) + [(\sigma_n(u)-1)\sigma_n(u)]^{\frac{1}{2}}\right].$$

□

In view of the preceding considerations we are in a position to prove

Proposition 7.15. *Assume that $\gamma(\mathcal{N}) \geq n$ and that*

$$\gamma^*(K_{\mathcal{N}}) \leq n-1. \tag{7.28}$$

Then $K_{\mathcal{N}} \neq \emptyset$ and

$$\inf_{u \in K_{\mathcal{N}}} \psi(u) = c_n. \tag{7.29}$$

Moreover, if $(u_j) \subset K_{\mathcal{N}}$ is a minimizing sequence for ψ in $K_{\mathcal{N}}$, then $\|d\psi(u_j)\|_{X^*} \to 0$.

Proof. First, (7.29) follows directly from (7.28), (6.10) and Corollary 7.12(a). Let $(u_j)_j$ be a minimizing sequence for ψ in $K_{\mathcal{N}}$. By (7.24) we observe that $(u_j)_j$ is bounded. Moreover Proposition 7.11 yields

$$o(1) = \psi(u_j) - c_n \geq \frac{d^2}{2}(\sigma_n(u_j)-1) \geq 0,$$

hence $\lim_{j\to\infty} \sigma_n(u_j) = 1$. The assertion now follows from Lemma 7.14. □

Now we easily deduce our main theorem.

Theorem 7.16. *Assume that $\gamma(\mathcal{N}) \geq n$ and that (7.28) is valid. Moreover suppose that ψ satisfies the PS condition at the level c_n. Then ψ attains its minimum c_n on $K_{\mathcal{N}}$, and every minimizer u satisfies (7.22) and (7.23).*

Proof. In view of Proposition 7.15 every minimizing sequence for ψ in $K_{\mathcal{N}}$ is a PS sequence at the level c_n, hence it contains a convergent subsequence by assumption. Since $K_{\mathcal{N}}$ is closed, the corresponding limit is a minimizer for ψ in $K_{\mathcal{N}}$. Now the assertion follows from Corollary 7.12(b). □

Corollary 7.17. *Assume that $\gamma(\mathcal{N}) \geq n$ and that (7.28) is valid. Moreover assume that $B : X \to \mathcal{L}(X, X^*)$ is compact. Then ψ attains its minimum c_n on $K_{\mathcal{N}}$, and every minimizer u satisfies (7.22) and (7.23).*

Proof. Consider a minimizing sequence $(u_j)_j$ for ψ in $K_{\mathcal{N}}$. Then (u_j) is bounded by (7.24), hence $B(u_j)u_j \to w \in X^*$ after passing to a subsequence. Since $d\psi(u_j) = (J - B(u_j))u_j$ for all j, Prop. 7.15 yields $\lim\limits_{j\to\infty} u_j = J^{-1}w$. In view of Lemma 7.13 we conclude that $\psi|_{K_{\mathcal{N}}}$ attains its minimum at $J^{-1}(w)$. Again the assertion now follows from Corollary 7.12. □

7.2.2 The case $\lambda < \mu_1(0)$

We now return to problem $(SC)_n$, as introduced on page 49. For this we fix λ with (7.16), and we consider $\mathcal{N}$ as defined in (7.17). Moreover we put $c_n := c_n(\psi_\lambda, \mathcal{N})$, and we aim to establish the existence of solutions u of $(SC)_n$ such that $\mu_n(u) = \lambda$ and $\psi_\lambda(u) = c_n$.
First we remark that (7.16) implies that

$$(u|v)_\lambda := \langle \hat{A}u, v\rangle - \lambda(u|v) \qquad (u, v \in X)$$

defines a scalar product on X which is equivalent to the original scalar product $(\cdot|\cdot)_X$, i.e. the induced norms $\|\cdot\|_\lambda$ and $\|\cdot\|_X$ are equivalent. The idea is now to apply the results of Section 7.2.1 *to the Hilbert space* $[X, (\cdot|\cdot)_\lambda]$. To this end, we assume that (H1)-(H4), (CC), $(CC)_1$ and $(CC)_2$ hold. Note that these assumptions stay invariant under a change to an equivalent scalar product. However, the canonical isometric isomorphism $X \to X^*$ is now given by $J_\lambda := \hat{A} - \lambda I$ in place of J. As a consequence, we have to consider the operators $G(u) := J_\lambda^{-1}B(u) \in \mathcal{L}(X)$ which are compact and symmetric *with respect to* $(\cdot|\cdot)_\lambda$. Denoting by

$$\sigma_k(u) = \sup_{\substack{V \leq X \\ \dim V = k}} \inf_{v \in V} \frac{(G(u)v|v)_\lambda}{(v|v)_\lambda} = \sup_{\substack{V \leq X \\ \dim V = k}} \inf_{v \in V} \frac{\langle B(u)v, v\rangle}{(v|v)_\lambda}$$

the nonzero eigenvalues of $G(u)$, the fundamental relationship to problem $(SC)_n$ is given as follows.

Lemma 7.18. *Let $u, v \in X$. Then:*

(i) $v \in \mathcal{D}(A(u))$, $A(u)v = \lambda v$ *if and only if* $G(u)v = v$.

(ii) $\mu_n(u) = \lambda$ *if and only if* $\sigma_n(u) = 1$.

Proof. This is a simple consequence of the definition of the values $\mu_n(\cdot)$ and $\sigma_n(\cdot)$. □

As a consequence, u is a solution of $(SC)_n$ satisfying $\mu_n(u) = \lambda$ *if and only if*

$$G(u)u = u \quad \text{and} \quad \sigma_n(u) = 1.$$

Moreover, in view of (7.2) there holds

$$\psi_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 - \varphi(u),$$

whereas $\mathcal{N}$ can be written as

$$\mathcal{N} = \{u \in X \setminus \{0\} \mid [u, u]_u^\lambda = 0\}$$

with

$$[v, w]_u^\lambda := (v|w)_\lambda - \langle B(u)v, w\rangle \qquad (u, v, w \in X).$$

Consequently, we put $c_n := c_n(\psi_\lambda, \mathcal{N})$ and denote by $\tilde{Q}_n(u)$ the spectral projection associated with the operator $G(u)$ and the interval $[0, \sigma_n(u)]$. Moreover, considering $n \in \mathbb{N}$ fixed, we define $K_{\mathcal{N}} := \{u \in \mathcal{N} \mid \tilde{Q}_n(u)u = u\}$.

Now we apply the results of Section 7.2.1, replacing $[X, (\cdot|\cdot)_X]$ by $[X, (\cdot|\cdot)_\lambda]$ and ψ by ψ_λ. Recall that for this we finally require the following assumption:

$(CC)_3$ If $u \in X$ and $0 \neq v \in X$ are such that v is a finite sum of eigenvectors of $G(u)$ corresponding to *positive* eigenvalues, then $\langle B(tv)v, v\rangle > 0$ for some $t > 0$.

We obtain the following results analogous to Proposition 7.11, Theorem 7.16 and Corollary 7.17.

Proposition 7.19. *For $u \in \mathcal{N}$ with $\sigma_n(u) \geq 1$ there holds*

$$\psi_\lambda(u) - c_n \geq \frac{d_\lambda^2}{2}(\sigma_n(u) - 1),$$

where $d_\lambda := \inf\limits_{v \in \mathcal{N}} \|v\|_\lambda > 0$.

Theorem 7.20. *Assume that $\gamma(\mathcal{N}) \geq n$ and that $\gamma^*(K_{\mathcal{N}}) \leq n - 1$. Moreover suppose that ψ_λ satisfies the PS-condition at the level c_n. Then $K_{\mathcal{N}}$ is nonempty, and*

$$\inf_{u \in K_{\mathcal{N}}} \psi_\lambda(u) = c_n.$$

Moreover, ψ_λ attains its minimum c_n on $K_{\mathcal{N}}$, and every minimizer u is a solution of $(SC)_n$ with $\mu_n(u) = \lambda$.

Corollary 7.21. *Assume that $\gamma(\mathcal{N}) \geq n$ and that $\gamma^*(K_{\mathcal{N}}) \leq n - 1$. Moreover assume that $B : X \to \mathcal{L}(X, X^*)$ is compact. Then the assertions of Theorem 7.20 hold true again.*

7.2.3 The case $\lambda \geq \mu_1(0)$

Let $n \in \mathbb{N}, \lambda \in \mathbb{R}$ be fixed. Again we intend to find solutions u of problem $(SC)_n$ with $\mu_n(u) = \lambda$, but in place of (7.16) we now suppose

$$\mu_1(0) \leq \lambda < \mu_n(0). \tag{7.30}$$

Throughout this subsection we assume that (H1)-(H4), (CC) and the following (slightly stronger) variant of $(CC)_1$ is in force:

$(CC)_1'$ For arbitrary $u, v \in X$, the function $t \mapsto \langle B(tu)v, v\rangle$ is nondecreasing on $[0, \infty[$. Moreover, if $\langle B(tu)v, v\rangle > 0$ for some $t > 0$, then this function increases strictly on $[t, \infty[$ and $\lim\limits_{s\to\infty} \langle B(su)v, v\rangle = \infty$.

We remark that $(CC)_1'$ and (H1) imply (H5) in particular. Setting now

$$S_\lambda := \{u \in X \mid \mu_n(u) = \lambda\}$$

we infer that $S_\lambda \subset X \setminus \{0\}$ is closed and symmetric by Lemma 5.2. In the following we analyze the minimax values $c_n := c_n(\psi_\lambda, S_\lambda)$ for ψ_λ on the set S_λ. To this end, consider for each $u \in X$ the operator

$$G(u) := J^{-1}(B(u) + (m + \lambda)I) : X \to X.$$

By (H2), (H3), (H5) and (7.30) we infer that $G(u)$ is a bounded selfadjoint and *positive definite* linear operator in the Hilbert space X. Moreover, the map $G : X \to \mathcal{L}(X)$ is continuous. Now define the decreasing sequence of *positive* values

$$\sigma_k(u) := \sup_{\substack{V \leq X \\ \dim V = k}} \inf_{v\in V} \frac{(G(u)v|v)_X}{(v|v)_X} = \sup_{\substack{V \leq X \\ \dim V = k}} \inf_{v\in V} \tilde{\rho}_u(v), \tag{7.31}$$

where

$$\tilde{\rho}_u(v) := \frac{\langle [B(u) + (m + \lambda)I]v, v\rangle}{(v|v)_X}.$$

Similar as in Section 7.2.2, a simple comparison of minimax values shows that u is a solution of $(SC)_n$ satisfying $\mu_n(u) = \lambda$ if and only if

$$G(u)u = u \quad \text{and} \quad \sigma_n(u) = 1 \tag{7.32}$$

Moreover we have $S_\lambda = \{u \in X \mid \sigma_n(u) = 1\}$. To proceed, we require the following additional assumptions:

(UC) If $u, v \in X$ are such that $B(u) \neq 0$ and $v \neq 0$ is an eigenfunction of $G(u)$, then $\langle B(u)v, v\rangle > 0$.

$(UC)_1$ If $W_u \subset X$ is an subspace spanned by finitely many eigenfunctions of $G(u)$ for some $u \in X$, then for all $v, w \in W_u \setminus \{0\}$ there is $t > 0$ with $\langle B(tv)w, w\rangle > 0$.

As already mentioned in Section 6.2, (UC) and $(UC)_1$ are closely related to unique continuation properties in applications to differential equations. In the following Lemma we use (UC) and $(UC)_1$ to establish crucially important topological properties of S_λ.

Lemma 7.22.

(a) The radial projection $S_\lambda \to S := \{w \in X \mid \|w\|_X = 1\}$ is injective.

(b) Consider an n-dimensional subspace $W \subset X$ such that for all $u, v \in W \setminus \{0\}$ there exists $t > 0$ with $\langle B(tu)v, v\rangle > 0$. Then $W \cap S_\lambda$ is homeomorphic to the unit sphere S_W in W by radial projection. In particular $\gamma(W \cap S_\lambda) = n$.

(c) If $W_u \subset X$ is an subspace spanned by n linearly independent eigenfunctions of $G(u)$ for some $u \in X$, then $\gamma(W_u \cap S_\lambda) = n$.

Proof. (a) Suppose in contradiction that there is $u \in X \setminus \{0\}$ and $t > 1$ such that $u, tu \in S_\lambda$, i.e. $\sigma_n(u) = \sigma_n(tu) = 1$. In particular $B(u) \neq 0$, since otherwise $\sigma_n(u) = \frac{\lambda}{\mu_n(0)} < 1$ by (7.30). Denote by $v_1, ..., v_n$ a choice of X-orthonormalized eigenfunctions corresponding to $\sigma_1(u), ..., \sigma_n(u)$, and let $V \subset X$ be the span of $v_1, ..., v_n$. Consider arbitrary $v \in V$, $\|v\|_X = 1$. If

$$(G(u)v|v)_X = \sigma_n(u),$$

then $G(u)v = \sigma_n(u)v$. Hence $\langle B(u)v, v\rangle > 0$ by (UC), and $(CC)_1'$ yields

$$\langle B(tu)v, v\rangle > \langle B(u)v, v\rangle$$

and therefore

$$(G(tu)v|v)_X > \sigma_n(u). \tag{7.33}$$

On the other hand, if $(G(u)v|v)_X > \sigma_n(u)$, then (7.33) holds as well. By a simple compactness argument we conclude

$$\sigma_n(tu) \geq \inf_{v \in V,\, \|v\|_X = 1} (G(tu)v|v)_X > \sigma_n(u),$$

in contradiction. This proves (a).
(b) Consider arbitrary $v \in S_W$. Then we may pick $t > 0$ such that

$$\inf_{w \in S_W} \langle B(tv)w, w\rangle > 0.$$

From $(CC)_1'$ we infer that

$$\sigma_n(sv) \geq \inf_{w \in S_W} \langle B(sv)w + (m + \lambda)w, w\rangle \longrightarrow \infty \qquad (s \to \infty),$$

whereas $\sigma_n(0) = \frac{\lambda}{\mu_n(0)} < 1$. Hence there is $s = s_v > 0$ such that $\sigma_n(s_v v) = 1$, i.e., $s_v v \in S_\lambda$. Moreover, the set $\{s_v \,|\, v \in S_W\}$ is bounded, since S_W is compact and $\sigma_n(\cdot) : X \to \mathbb{R}$ is continuous (cf. Lemma 4.7). Hence $W \cap S_\lambda$ is compact, and the radial projection $W \cap S_\lambda \to S_W$ is continuous and bijective. Thus it is a homeomorphism.
(c) This follows immediately from (b) and $(UC)_1$. $\square$

Proposition 7.23. *Let $u \in S_\lambda$. Then*

$$\psi_\lambda(u) - c_n \geq \frac{\|u\|_X^2}{2}(1 - \tilde{\rho}_u(u)).$$

Proof. By (7.5) we have

$$\begin{aligned} 2(\psi_\lambda(v) - \psi_\lambda(u)) &\leq [v,v]_u - \lambda\|v\|^2 - ([u,u]_u - \lambda\|u\|^2) \\ &= \|v\|_X^2(1 - \tilde{\rho}_u(v)) - \|u\|_X^2(1 - \tilde{\rho}_u(u)) \end{aligned} \tag{7.34}$$

for $u, v \in X$. Now since $\mu_n(u) = \lambda < \mu_n(0) \leq \mu_\infty$, we infer that $\mu_n(u) = \lambda$ is an eigenvalue of $A(u)$. Hence $\sigma_n(u) = 1$ is an eigenvalue of $G(u)$, and $\sigma_1(u), ..., \sigma_{n-1}(u)$ are eigenvalues of $G(u)$ as well (cf. Section 4.2). Choose pairwise X-orthogonal eigenvectors $v_1, \dots, v_n$ corresponding to $\sigma_1(u), \dots, \sigma_n(u)$, and let W be the span of $u_1, \dots, u_n$. Then

$$\tilde{\rho}_u(v) \geq \sigma_n(u) = 1 \qquad \forall\, v \in W \setminus \{0\},$$

whereas Lemma 7.22(c) yields $\gamma(W \cap S_\lambda) = n$. Using (7.34) we conclude

$$\begin{aligned} 2(c_n - \psi_\lambda(u)) &\leq \sup_{v \in S_\lambda \cap W} 2(\psi_\lambda(v) - \psi_\lambda(u)) \\ &\leq \sup_{v \in S_\lambda \cap W} \|v\|_X^2(1 - \tilde{\rho}_u(v)) - \|u\|_X^2(1 - \tilde{\rho}_u(u)) \\ &\leq -\|u\|_X^2(1 - \tilde{\rho}_u(u)). \end{aligned}$$

□

Let, as usual, $\tilde{Q}_n(u)$ stand for the spectral projection associated with the operator $G(u) \in \mathcal{L}_S(X)$ and the interval $[0, \sigma_n(u) = 1]$ (cf. Section 4.2 again). Moreover set

$$K_\lambda^- := \{u \in S_\lambda \mid \tilde{Q}_n(u)u = u\}.$$

Then $K_\lambda^- \in \Sigma(S_\lambda)$, which follows similarly as Lemma 7.13. Moreover we infer from Proposition 7.23:

Corollary 7.24. *If $u \in K_\lambda^-$, then*

(a) $\psi_\lambda(u) \geq \max\{0, c_n\}$

(b) If $\psi_\lambda(u) = c_n$ then u is a solution of $(SC)_n$ satisfying $\mu_n(u) = \lambda$.

Proof. (a) Since $u \in \mathcal{R}(\tilde{Q}_n(u)) \cap S_\lambda$, there holds $\tilde{\rho}_u(u) \leq 1$. Hence Proposition 7.23 yields $\psi_\lambda(u) \geq c_n$. Moreover, applying (7.5) to $v = 0$ yields

$$2\psi_\lambda(u) \geq \|u\|_X^2(1 - \tilde{\rho}_u(u)) \geq 0.$$

(b) If $\psi_\lambda(u) = c_n$, then $\tilde{\rho}_u(u) = 1$ by Proposition 7.23, which is possible only if u is an eigenvector of $G(u)$ with eigenvalue $\sigma_n(u) = 1$. Thus the assertion follows. □

Similar as in the preceding sections, we also need an estimate for $d\psi$ in K_λ^-.

Lemma 7.25. *For $u \in K_\lambda^-$ there holds*

$$\|d\psi_\lambda(u)\|_{X^*} \leq \|u\|_X (1 - \tilde{\rho}_u(u))^{\frac{1}{2}}.$$

Proof. Fix $u \in K_\lambda^-$ and note that $J^{-1} d\psi_\lambda(u) = u - G(u)u \in X$, hence

$$\|d\psi_\lambda(u)\|_{X^*} = \|u - G(u)u\|_X.$$

Since $\sigma_n(u) = 1$, we can define a semidefinite scalar product on the subspace $\mathcal{R}(\tilde{Q}_n(u)) \subset X$ by $(\cdot|\cdot)_X - (G(u) \cdot |\cdot)_X$. Set $w := u - G(u)u \in \mathcal{R}(\tilde{Q}_n(u))$, then the corresponding Cauchy-Schwarz inequality implies

$$\begin{aligned}
\|w\|_X^2 &= \|u - G(u)u\|_X^2 \\
&= (u|w)_X - (G(u)u|w)_X \\
&\leq [\|u\|_X^2 - (G(u)u|u)_X]^{\frac{1}{2}} [\|w\|_X^2 - (G(u)w|w)_X]^{\frac{1}{2}} \\
&\leq \|u\|_X (1 - \tilde{\rho}_u(u))^{\frac{1}{2}} \|w\|_X.
\end{aligned}$$

Therefore

$$\|d\psi_\lambda(u)\|_{X^*} = \|w\|_X \leq \|u\|_X (1 - \tilde{\rho}_u(u))^{\frac{1}{2}}.$$

□

The preceding observations furnish the tools for proving the following.

Proposition 7.26. *Suppose that $\gamma(S_\lambda) \geq n$ and that*

$$\gamma^*(K_\lambda^-) \leq n - 1 \tag{7.35}$$

Then $K_\lambda^- \neq \emptyset$ and

$$\inf_{u \in K_\lambda^-} \psi_\lambda(u) = c_n \geq 0. \tag{7.36}$$

Moreover, if $(u_j) \subset K_\lambda^-$ is a minimizing sequence for ψ_λ in K_λ^-, then $\|d\psi_\lambda(u_j)\|_{X^*} \to 0$.

Proof. First, (7.36) follows directly from (7.35), (6.10) and Corollary 7.24. Let $(u_j)_j$ be a minimizing sequence for ψ_λ in K_λ^-. Proposition 7.23 and Lemma 7.25 yield

$$\begin{aligned}
o(1) = \psi_\lambda(u_j) - c_n &\geq \frac{\|u\|_X^2}{2}(1 - \tilde{\rho}_u(u)) \\
&\geq \frac{\|d\psi_\lambda(u)\|_{X^*}^2}{2},
\end{aligned}$$

hence $\|d\psi_\lambda(u_j)\|_{X^*} \to 0$. □

Theorem 7.27. *Suppose that $\gamma(S_\lambda) \geq n$ and $\gamma^*(K_\lambda^-) \leq n-1$. Moreover assume that ψ_λ satisfies the Palais-Smale condition at the level c_n. Then*

$$\inf_{u \in K_\lambda^-} \psi_\lambda(u) = c_n \geq 0,$$

ψ_λ takes its minimum on K_λ^-, and every minimizer is a solution to $(SC)_n$ with $\mu_n(u) = \lambda$.

Proof. In view of Corollary 7.24 and (7.36) we only need to insure that ψ_λ attains its minimum on K_λ^-. Indeed, if $(u_j)_j$ is a minimizing sequence for ψ_λ in K_λ^-, then $\|d\psi_\lambda(u_j)\|_{X^*} \to 0$ by Proposition 7.26, whereas $(\psi_\lambda(u_j))_j$ remains bounded by (7.36). By assumption, $u_j \to u \in K_\lambda^-$ after passing to a subsequence. Hence $\psi_\lambda|_{K_\lambda^-}$ attains its minimum at u. □

Finally we give a criterion for (7.35) to hold:

Proposition 7.28. *Suppose that $\sigma_{n-1}(u) > \sigma_n(u)$ for all $u \in D := \{u \in X \mid \sigma_n(u) \leq 1\}$. Then $\gamma^*(K_\lambda^-) \leq n-1$.*

Proof. Denote $\tilde{P}_{n-1}(u)$ the spectral projection associated with the X-selfadjoint operator $G(u)$ and the eigenvalues $\sigma_1(u), ..., \sigma_{n-1}(u)$. Then $\tilde{P}_{n-1} : D \to \Pi_{n-1}(X)$ is continuous by Lemma 4.8. Hence $H : [0,1] \times S_\lambda \to \Pi_{n-1}(X)$ defined by $H(t,u) := \tilde{P}_{n-1}(tu)$ is continuous as well. Since $H(t,u) = H(t,-u)$, the assertion follows from Proposition 3.12 and the fact that $\mathcal{N}(\tilde{P}_{n-1}(u)) = \mathcal{R}(\tilde{Q}_n(u))$ for $u \in D$. □

Chapter 8

Periodic solutions of a nonlinear Hill's equation

In this chapter we consider a one-dimensional *periodic* equation, i.e. we are interested in 1-periodic solutions of the equation

$$(NH\pm) \qquad -(p(x)u')' + q(x)u \pm f(x,|u|)u = \lambda u, \qquad x \in \mathbb{R},$$

where $p,\ q : \mathbb{R} \to \mathbb{R}$ are given 1-periodic continuous functions, $p \in C^1(\mathbb{R})$ being positive everywhere. Moreover, $f : \mathbb{R} \times [0, \infty[\to \mathbb{R}$ is continuous and 1-periodic in the x-variable. As a matter of convenience, we assume

$$f(x, 0) \equiv 0 \text{ on } \mathbb{R}, \tag{8.1}$$

which can be arranged by taking q appropriately. We also need the following crucial condition:

(M) For every $x \in \mathbb{R}$, $f(x, \cdot)$ is nondecreasing on $[0, \infty[$.

To cast this problem in the framework of our abstract considerations, put $H := L^2([0, 1])$ and

$$X := \{u \in W^{1,2}([0, 1]) \mid u(0) = u(1)\}$$

(by Sobolev embeddings, $W^{1,2}([0, 1])$ consists of continuous functions). In view of our assumptions, the operator $A_0 := -\frac{d}{dx}(p\frac{d}{dx}) + q$ with domain

$$\mathcal{D}(A_0) = \{u \in W^{2,2}([0, 1]) \mid u(0) = u(1),\ u'(0) = u'(1)\} \subset H$$

is selfadjoint an bounded from below. Moreover, X is precisely the form domain of A_0. In accordance to Chapter 4 we put $m := -\inf \sigma(A_0) + 1$, and we endow X with the scalar product

$$(u|v)_X := \int_0^1 [pu'v' + (q + m)uv] \qquad (u, v \in X). \tag{8.2}$$

Note that the induced norm $\|\cdot\|_X$ is equivalent to the standard $W^{1,2}([0, 1])$-norm. Define

$$F(x, t) := \int_0^t f(x, s)s\, ds \qquad (x \in \mathbb{R}, t > 0).$$

Then we have:

Lemma 8.1. *(a) The map $B : X \to \mathcal{L}(X, X^*)$ given by*

$$\langle B(u)v, w\rangle = \int_0^1 f(x, |u(x)|)v(x)w(x)dx \tag{8.3}$$

is strongly *continuous, and $B(u) \in \mathcal{L}(X, X^*)$ is a compact linear operator for each $u \in X$.*

(b) The functional $\varphi : X \to \mathbb{R}$ defined by

$$\varphi(u) = \int_0^1 F(x, |u(x)|)\, dx \qquad (u \in X)$$

is continuous. Moreover, φ and B satisfy (CC).

Proof. (a) Denote by C the space of continuous 1-periodic functions equipped with the $\|\cdot\|_\infty$-norm, and denote by $i : X \to C$ the Sobolev embedding which is strongly continuous. Clearly B factorizes in the form

$$X \overset{i}{\hookrightarrow} C \overset{b}{\to} \mathcal{L}(C, C^*) \overset{j}{\to} \mathcal{L}(X, X^*),$$

where C^* denotes the dual of C, j maps an operator $h \in \mathcal{L}(C, C^*)$ to i^*hi and $\langle b(u)v, w\rangle$ is given by the right hand side of (8.3) for $u, v, w \in C$. It therefore suffices to prove that b is continuous. This however follows from the estimate

$$|\langle (b(u_1) - b(u_2))v, w\rangle| \le \|f(\cdot, |u_1(\cdot)|) - f(\cdot, |u_2(\cdot)|)\|_\infty \|v\|_\infty \|w\|_\infty$$

and the fact that f is *uniformly* continuous on subsets of the form $[0,1] \times [0, K]$ with $K > 0$ arbitrary.
(b) As in the proof of Lemma 2.4 one uses (M) to deduce the inequality

$$2[F(x, |v(x)|) - F(x, |u(x)|)] \ge f(x, |u(x)|)(v^2(x) - u^2(x)) \tag{8.4}$$

for $u, v \in X$ and $x \in [0,1]$. Integrating (8.4) yields precisely (CC). □

8.1 The sublinear case

We consider $(NH+)$. Thus, we are dealing with a sublinear equation, and we introduce the functionals $\psi, \psi_\lambda : X \to X^*$ defined by

$$\psi(u) = \frac{1}{2}\int_0^1 \Big(p(x)u'(x)^2 + q(x)u^2(x)\Big)dx + \varphi(u).$$

and

$$\psi_\lambda(u) = \psi(u) - \frac{\lambda}{2}\|u\|^2.$$

Here $\|\cdot\|$ denotes the norm in $\mathcal{H}$, i.e. $\|u\|^2 = \int\limits_0^1 u^2$ for $u \in H$. As a consequence of Lemma 8.1, the abstract conditions (H1)-(H4) and (CC) are satisfied. Hence we may define $A(u)$, $[\cdot,\cdot]_u$, $\mu_n(u)$, $P_n(u)$, $V_n(u)$ as well as problem $(SC)_n$ as in Chapter 6.

Remark 8.2. (a) Note that $\mathcal{D}(A(u)) = \mathcal{D}(A_0)$ for every $u \in X$. Moreover, if $u \in \mathcal{D}(A_0)$ satisfies $A(u)u = \lambda u$ for some $\lambda \in \mathbb{R}$, then u can be extended to a *classical* solution of $(NH\pm)$, as follows from the assumptions imposed on p, q and f. In this case we identify u with its extension and call u a solution of $(NH\pm)$.
(b) A well known fact on this periodic problem is that there holds

$$\mu_1(u) < \mu_2(u) \leq \mu_3(u) < \mu_4(u) \leq \dots , \qquad (u \in X) \tag{8.5}$$

i.e., $\mu_n(u) < \mu_{n+1}(u)$ *if* $n \in \mathbb{N}$ *is odd*. Moreover, every eigenfunction corresponding to $\mu_{2m}(u)$ or $\mu_{2m+1}(u)$ has exactly $2m$ simple zeroes in $[0, 1[$, $m = 1, 2, ...$ (for a proof of these assertions, see e.g. [28]). Finally we have $\mu_\infty = \infty$, since the operator A_0 has compact resolvent.

8.1.1 Solutions with prescribed norm

Let $R > 0$ be given. We are now concerned with solutions (u, λ) to $(NH+)$ which satisfy

$$\int_0^1 u^2(x)\, dx = R^2, \tag{8.6}$$

i.e. $\|u\| = R$. Putting $S_R := \{u \in X \mid \|u\| = R\}$ and $c_n := c_n(\psi, S_R)$ for $n \in \mathbb{N}$, we have:

Theorem 8.3. *Let* $n \in \mathbb{N}$ *be* odd*. Then*

(a) $c_n < c_{n+1}$

(b) There is a solution $(u, \lambda) \in S_R \times \mathbb{R}$ *of* $(NH+)$ *such that* $\psi(u) = c_n$*, and* u *has exactly* $n - 1$ *simple zeroes in* $[0, 1[$.

(c) If $(u, \lambda) \in S_R \times \mathbb{R}$ *is a solution of* $(NH+)$ *with* $\psi(u) > c_n$*, then* u *has at least* $n + 1$ *simple zeroes in* $[0, 1[$.

Proof. Since n is odd, there holds $\mu_n(u) < \mu_{n+1}(u)$ for every $u \in X$. Moreover, B is strongly continuous by Lemma 8.1(i). Hence we may apply Theorem 6.5 which implies that property (CP) holds for ψ and the set $K := \{u \in S_R \mid u \in V_n(u)\}$. In particular K contains a solution u of $(SC)_n$ satisfying $\psi(u) = c_n$. By Remark 8.2 we conclude that u has precisely $n - 1$ simple zeroes, as claimed in (b). Moreover, since $\rho_u(u) = \mu_n(u) < \mu_{n+1}(u)$, we infer $c_n = \psi(u) < c_{n+1}$ from Prop. 6.3(a). Hence (a) holds true as well.
Finally, suppose that $(u, \lambda) \in S_R \times \mathbb{R}$ is a solution of $(NH+)$ with *at most* $n - 1$ simple zeroes. Then $\rho_u(u) = \lambda \leq \mu_n(u)$, and therefore $\psi(u) \leq c_n$ again by Prop. 6.3(a). This yields (c), and the proof is complete. □

8.1.2 Solutions with prescribed eigenvalue

Next we are concerned with solutions u to $(NH+)$ for *given* parameter λ. Therefore fix $n \in \mathbb{N}$ and $\lambda > \mu_n(0)$. Put $S_\lambda := \{u \in X \mid \mu_n(u) = \lambda\}$ and $c_n := c_n(\psi_\lambda, S_\lambda)$. We are in a position to prove:

Theorem 8.4. *Suppose that* $n \in \mathbb{N}$ *is* odd*, and that in addition to (M) the nonlinearity* f *satisfies* *(i) For all* $x \in [0,1]$ *there holds*

$$\lambda - \mu_1(0) < \lim_{t \to \infty} f(x,t) \leq \infty$$

(ii) If $x \in [0,1]$, $t \in [0,\infty[$ *are such that* $f(x,t) > 0$*, then* $f(x,\cdot)$ *is strictly increasing on* $[t,\infty[$.

Then there is a solution u *of* $(NH+)$ *such that* $\psi(u) = c_n$ *and* u *has exactly* $n-1$ *simple zeroes in* $[0,1[$.

Proof. We apply the results from Section 6.2. For this we remark that (i) yields (FG) by virtue of Lebesgue's monotone convergence theorem. To show (UC), note that if $v \neq 0$ is an eigenfunction of some $u \in X$ with $B(u) \neq 0$, then v solves the equation

$$(p(x)v')' + q(x)v + f(x,|u|)v = \mu v,$$

for some $\mu \in \mathbb{R}$, which implies that $v \in C^2(\mathbb{R})$, and v can only vanish on a discrete subset of $\mathbb{R}$. Hence $\langle B(u)v, v\rangle > 0$, and (UC) holds. Finally, $(CC)_1$ is a direct implication of (ii).
In view of (8.5) we may now apply Theorem 6.10, which yields property $(CP)_\lambda$ for ψ_λ and the set $K_\lambda := \{u \in S_\lambda \mid u \in V_n(u)\}$. In particular K_λ contains a solution u of $(SC)_n$ with $\psi_\lambda(u) = c_n$, and u has the desired nodal property. $\square$

8.2 The superlinear case

We now consider (NH-), and we introduce the functionals $\psi, \psi_\lambda : X \to X^*$ defined by

$$\psi(u) = \frac{1}{2}\int_0^1 \Big(p(x)u'^2 + q(x)u^2\Big)dx - \varphi(u).$$

and

$$\psi_\lambda(u) = \psi(u) - \frac{\lambda}{2}\|u\|^2.$$

As a consequence of Lemma 8.1, the abstract conditions (H1)-(H4) and (CC) are satisfied. Hence we may define $A(u)$, $[\cdot,\cdot]_u$, $\mu_n(u)$, $P_n(u)$, $V_n(u)$ as well as problem $(SC)_n$ as in Section 7, that is with respect to the superlinear case. Note that Remark 8.2 is still valid with respect to this notations.

8.2.1 Solutions with prescribed norm

Let $R > 0$ be given. We are concerned with solutions (u,λ) to $(NH-)$ which satisfy the side condition (8.6). Therefore put $S_R := \{u \in X \mid \|u\| = R\}$ and $c_n := c_n(\psi, S_R)$. The following result reflects some kind of duality to the sublinear case, see Theorem 8.3:

Theorem 8.5. *Let* $n \in \mathbb{N}$ *be* even. *Moreover assume that there are numbers* $a, b > 0$ *and* $0 \leq q < 4$ *such that*

$$|f(x,t)| \leq at^q + b. \tag{8.7}$$

Then

(a) $c_n > c_{n-1}$.

(b) There is a solution $(u,\lambda) \in S_R \times \mathbb{R}$ *of* $(NH-)$ *such that* $\psi(u) = c_n$ *and* u *has exactly* n *simple zeroes in* $[0,1[$.

(c) If $(u,\lambda) \in S_R \times \mathbb{R}$ *is a solution of* $(NH-)$ *with* $\psi(u) < c_n$*, then* u *has at most* $n-2$ *simple zeroes in* $[0,1[$.

Proof. We apply the results from Section 7.1, showing first that condition (BB) is satisfied. Fix $u \in S_R$. From 8.7 we deduce

$$\langle B(u)u,u\rangle = \int_0^1 f(x,|u|)u^2\,dx \le a\int_0^1 |u|^{q+2} + bR^2 \le (a\|u\|_\infty^q + b)R^2. \tag{8.8}$$

Now consider arbitrary $t \in \mathbb{R}$ and $t-1 < s \le t$ such that $u^2(s) = \min u^2$. Then

$$\begin{aligned}
2u^2(t) &= u^2(t) + u^2(t-1)\\
&= 2u^2(s) + 2\int_s^{t-1} u'(\xi)u(\xi)\,d\xi + 2\int_s^t u'(\xi)u(\xi)\,d\xi\\
&\le 2u^2(s) + 2\int_{t-1}^t |u'(\xi)||u(\xi)|\,d\xi\\
&\le 2R^2 + 2R\|u'\|_2.
\end{aligned}$$

Hence $\|u\|_\infty^2 \le R(R+\|u\|_X)$, and combining this with (8.8) yields

$$\langle B(u)u,u\rangle \le \tilde{a}\|u\|_X^{\frac{q}{2}} + \tilde{b}$$

with $\tilde{a},\tilde{b}$ only depending on a,b and R. Since $\frac{q}{2} < 2$ by assumption, we derive (BB) in view of Remark 7.1.
Now, since n is even, there holds $\mu_{n-1}(u) < \mu_n(u)$ for every $u \in X$. Applying Proposition 7.9, we thus infer that $\gamma^*(K^-) \le n-1$ for the set $K^- := \{u \in S_R \mid Q_n(u)u = u\}$. Noting that B is compact by Lemma 8.1(i) and that $\mu_\infty = \infty$, we may apply Theorem 7.8 which yields property $(CP)^-$ for ψ and K^-. In particular K contains a solution u of $(SC)_n$ satisfying $\psi(u) = c_n$. By Remark 8.2 we conclude that $(u,\mu_n(u))$ has the properties claimed in (b). Moreover, since $\rho_u(u) = \mu_n(u) > \mu_{n-1}(u)$, we infer $c_n = \psi(u) > c_{n-1}$ from Prop. 7.4. Hence (a) holds true as well.
Finally, suppose that (u,λ) is an arbitrary solution of $(NH-)$, (8.6) with *at least* n simple zeroes. Then $\rho_u(u) = \lambda \ge \mu_n(u)$, and therefore $\psi(u) \ge c_n$ again by Prop. 7.4. This yields (c), and the proof is complete. ☐

8.2.2 Solutions with prescribed eigenvalue: The case $\lambda \ge \mu_1(0)$.

Next we are concerned with solutions u to $(NH-)$ for *given* parameter $\lambda \ge \mu_1(0)$, and we fix $n \in \mathbb{N}$ such that $\mu_n(0) > \lambda$. Moreover we define

$$S_\lambda := \{u \in X \mid \mu_n(u) = \lambda\}$$

and consider $c_n := c_n(\psi_\lambda, S_\lambda)$.

Theorem 8.6. *Suppose that $n \in \mathbb{N}$ is* even*, and that in addition to (M) the nonlinearity f satisfies*

(i) There is an $\eta > 2$ such that $0 \le \eta \int\limits_0^t f(x,s)s\,ds \le f(x,t)t^2$ for all $x,\ t$.

(ii) If $x \in [0,1],\ t \in [0,\infty[$ is such that $f(x,t) > 0$, then $f(x,\cdot)$ is strictly increasing on $[t,\infty[$.

(iii) $f \not\equiv 0$, i.e. there is $x \in \mathbb{R}$, $t \in (0,\infty)$ with $f(x,t) > 0$.

Then there is a solution u of $(NH-)$ such that $\psi_\lambda(u) = c_n$, and u has exactly n simple zeroes in $[0,1[$.

We will prove this result with the tools from Section 7.2.3. In view of (8.2) the canonical isometric isomorphism $J : X \to X^*$ equals, in distributional sense, the map

$$u \mapsto -(pu')' + (q+m)u.$$

Moreover, the operator-valued map $G : X \to \mathcal{L}(X)$ considered in Section 7.2.3 is given by

$$G(u) := J^{-1}[B(u) + (m+\lambda)I] \qquad (u \in X).$$

We recall that for all $u \in X$ the operator $G(u)$ is bounded, symmetric and *positive definite*. Moreover, since the embeddings $X \hookrightarrow H \hookrightarrow X^*$ are *compact*, we deduce that $G(u)$ is compact as well. As a consequence, the numbers $\sigma_k(u)$ defined by (7.31) are all eigenvalues of $G(u)$.

Lemma 8.7. *If $n \in \mathbb{N}$ is even, then $\sigma_n(u) < \sigma_{n-1}(u)$ for every $u \in X$.*

Proof. Note that σ is an eigenvalue of $G(u)$ *if and only if* $\tau = \frac{1}{\sigma}$ is an eigenvalue of weighted problem

$$-(pv')' + qv = \tau r(x) v, \tag{8.9}$$

with a uniformly positive weight $r(x) = f(x,|u(x)|) + m + \lambda$. Hence the assertion follows from [28, Theorem 2.3.1]. $\square$

Combining Lemma 8.7 and Proposition 7.28, we deduce

$$\gamma^*(K_\lambda^-) \le n-1, \tag{8.10}$$

for K_λ^- being defined as in Section 7.2.3. Next we establish a unique continuation property for sums of eigenfunctions of $G(u)$.

Lemma 8.8. *Let $u \in X$, and consider a finite sum $v = \sum\limits_{i=1}^n v_i$ of (nonzero) eigenfunctions v_i of $G(u)$ corresponding to pairwise different eigenvalues ξ_i.*
Then v does not vanish on any open subset of $]0,1[$.

Proof. We recall that $\xi_i > 0$ for $i = 1, ..., n$, since $G(u)$ is positive definite. Hence, every v_i is a classical solution of

$$-(pv_i')' + qv_i = \frac{1}{\xi_i}[f(x, |u(x)|) + m + \lambda]v_i \qquad (i = 1, ..., n).$$

In particular $v \in C^2(]0,1[)$. Moreover, assuming in contradiction that $v = 0$ on an open subset $M \subset]0,1[$, we would have

$$0 = -(pv')' + qv = \sum_{i=1}^{n}[-(pv_i')' + qv_i] = [f(\cdot, |u(\cdot)|) + m + \lambda] \sum_{i=1}^{n} \frac{1}{\xi_i} v_i \qquad \text{on } M$$

This however implies $\sum_{i=1}^{n} \frac{1}{\xi_i} v_i = 0$ on M. Iterating this argument, we derive

$$\sum_{i=1}^{n} \frac{1}{\xi_i^j} v_i = 0 \qquad \text{on } M \text{ for all } j \in \mathbb{N},$$

which implies $v_i = 0$ on M for $i = 1, ..., n$. However, since v_i solves a linear ODE with regular coefficients, we infer $v_i = 0$ for all i. Hence $v = 0$, as claimed. □

Now we are prepared for the

Proof of Theorem 8.6. As a consequence of Lemma 8.8 and (iii), the conditions (UC) and $(UC)_1$ from Section 7.2.3 are satisfied. In order to establish $(CC)_1'$ we have to use (i) and (ii): For arbitrary $u, v \in X$ the function $\tau : [0, \infty[\to [0, \infty[$ given by $\tau(t) := \langle B(tu)v, v\rangle$ is nondecreasing in view of (M). Now suppose that

$$\tau(t_0) > 0 \tag{8.11}$$

for some $t_0 > 0$, i.e. the function $f(\cdot, t_0 u(\cdot))v(\cdot)$ does not vanish identically. In view of (ii) we then infer that τ increases strictly on $[t_0, \infty[$, hence it remains to show that $\lim_{s\to\infty} \tau(s) = \infty$. For this we pick $s_1, s_2 > 0$ sufficiently large such that

$$\Omega_u := \{x \in [0,1] \mid f(x, s_1|) > 0,\ v(x) \neq 0,\ s_2|u(x)| \geq s_1\}$$

is a set of positive measure. Now a standard upshot of (ii) is that $F(x, st) \geq s^\eta F(x,t)$ for $t \geq 0$, $s \geq 1$, cf. (7.25). Combining this with (ii), we get for $x \in \Omega_u$ and $s \geq s_2$

$$\begin{aligned} f(x, |su(x)|)v^2(x) &\geq \eta \frac{F(x, s|u(x)|)}{|su(x)|^2} v^2(x) \\ &\geq \eta (\frac{s|u(x)|}{s_1})^{\eta-2} F(x, s_1) v^2(x), \end{aligned}$$

hence

$$\langle B(su)v, v\rangle \geq \eta(\frac{s}{s_1})^{\eta-2} \int_{\Omega_u} |u(x)|^{\eta-2} F(x, s_1) v^2(x)\, dx.$$

Since $\eta > 2$ and the integral on the right hand side is positive, we conclude $\lim_{s\to\infty} \tau(s) = \infty$. Hence $(CC)_1'$ is valid.
Next we show

$$\gamma(S_\lambda) \geq n.$$

For this pick an n-dimensional subspace $V \subset X$ of analytic functions (e.g. trigonometric polynomials). Using (iii), we find for every nonzero $u \in V$ a number $t > 0$ such that $\langle B(tu)v, v\rangle > 0$ for all $v \in V$, $v \neq 0$. In view of Lemma 7.22(b) we infer $\gamma(V \cap S_\lambda) = n$, hence $\gamma(S_\lambda) \geq n$.
Since ψ_λ satisfies the PS condition by virtue of the subsequent Lemma, we now may apply Theorem 7.27. This in particular yields a solution u of $(SC)_n$ with $\mu_n(u) = \lambda$ and $\psi_\lambda(u) = c_n$, and u has precisely n zeroes by Remark 8.2. □

Lemma 8.9. *ψ_λ satisfies the PS condition.*

Proof. Let $(u_n) \subset X$ be a sequence such that

$$\psi_\lambda(u_n) \to c \in \mathbb{R}, \qquad d\psi_\lambda(u_n) \to 0 \in X^*.$$

We first show that $(u_n)_n$ possesses a subsequence which is bounded in X. This is somewhat involved due to our (weak) growth assumptions and the fact that λ might be an eigenvalue of A_0. Note first that for every $v \in X$, $n \in \mathbb{N}$ there holds

$$\begin{aligned}
\langle B(u_n)u_n, v\rangle &\leq \|v\|_\infty \int_0^1 f(x, |u_n(x)|)|u_n(x)|\, dx \\
&\leq C_1\|v\|_X (\int_0^1 f(x, |u_n(x)|)u_n^2(x)\, dx + C_2) \\
&= C_1\|v\|_X (\langle B(u_n)u_n, u_n\rangle + C_2),
\end{aligned}$$

with constants $C_1, C_2 > 0$, hence

$$\|B(u_n)u_n\|_{X^*} \leq C_1(\langle B(u_n)u_n, u_n\rangle + C_2). \tag{8.12}$$

On the other hand, we have

$$\begin{aligned}
\langle B(u_n)u_n, u_n\rangle &= -\langle d\psi_\lambda(u_n), u_n\rangle + 2\psi_\lambda(u_n) + 2\varphi(u_n) \\
&\leq o(1)\|u_n\|_X + O(1) + \frac{2}{\eta}\langle B(u_n)u_n, u_n\rangle,
\end{aligned}$$

and therefore

$$\langle B(u_n)u_n, u_n\rangle \leq o(1)\|u_n\|_X + O(1), \tag{8.13}$$

since $\eta > 2$. Combining (8.12) and (8.13) we get

$$\|B(u_n)u_n\|_{X^*} \leq o(1)\|u_n\|_X + O(1). \tag{8.14}$$

Now pick $\varepsilon > 0$ such that $]\lambda - \varepsilon, \lambda + \varepsilon[$ contains no eigenvalue of A_0 *different* from λ. Denote by Q^+, P, Q^- the spectral projections associated with A_0 and the sets $]-\infty, \lambda - \varepsilon]$, $\{\lambda\}$, $[\lambda + \varepsilon, \infty[$, respectively. Moreover put $Q := Q^- + Q^+$ and $\hat{A}_\lambda := \hat{A} - \lambda I : X \to X^*$ for the sake of brevity. Then spectral theory yields

$$\begin{aligned}
\|Qu_n\|_X^2 &= \|Q^-u_n\|_X^2 + \|Q^+u_n\|_X^2 \\
&= \langle \hat{A}_\lambda Q^-u_n, Q^-u_n\rangle + (m+\lambda)\|Q^-u_n\|^2 + \langle \hat{A}_\lambda Q^+u_n, Q^+u_n\rangle + (m+\lambda)\|Q^+u_n\|^2 \\
&\le \left(1 - \frac{m+\lambda}{\varepsilon}\right)\langle \hat{A}_\lambda Q^-u_n, Q^-u_n\rangle + \left(1 + \frac{m+\lambda}{\varepsilon}\right)\langle \hat{A}_\lambda Q^+u_n, Q^+u_n\rangle \\
&= \left(1 - \frac{m+\lambda}{\varepsilon}\right)\langle \hat{A}_\lambda u_n, Q^-u_n\rangle + \left(1 + \frac{m+\lambda}{\varepsilon}\right)\langle \hat{A}_\lambda u_n, Q^+u_n\rangle \\
&\le C_3\langle \hat{A}_\lambda u_n, -Q^-u_n + Q^+u_n\rangle \\
&\le C_3\|\hat{A}_\lambda u_n\|_{X^*}\|Qu_n\|_X
\end{aligned}$$

with a constant $C_3 > 0$. This implies

$$\begin{aligned}
\|Qu_n\|_X &\le C_3\|\hat{A}_\lambda u_n\| \\
&\le C_3\|d\psi_\lambda u_n\|_{X^*} + \|B(u_n)u_n\|_{X^*} \\
&\le O(1) + o(1)\|u_n\|_X
\end{aligned}$$

by (8.14). Now assume that $\|u_n\|_X \to \infty$. Then, putting $v_n := \frac{u_n}{\|u_n\|_X}$, we have $\|v_n\|_X = 1$ for all n and $\|Qv_n\|_X \to 0$. Since $\mathcal{R}(P)$ is finite dimensional, we infer that $v_n \to v \in \mathcal{R}(P)$, $\|v\|_X = 1$ after passing to a subsequence. By (iii) and Lemma 8.8 there exists $t > 0$ such that $C_4 := \langle B(tv)v, v\rangle > 0$. Hence we conclude that for sufficiently large n we have

$$\langle B(u_n)u_n, u_n\rangle \ge \|u_n\|_X^2\langle B(tv_n)v_n, v_n\rangle \ge \frac{C_4}{2}\|u_n\|_X^2.$$

Combining this again with (8.13), we infer

$$\|u_n\|_X^2 \le o(1)\|u_n\|_X + O(1),$$

hence this subsequence is bounded in X, as desired.
Extracting an appropriate subsequence, we now have $B(u_n)u_n \to w \in X^*$ (cf. Lemma 5.7), and therefore

$$\hat{A}_\lambda u_n = d\psi_\lambda(u_n) + B(u_n)u_n \to w.$$

Since $\hat{A}_\lambda : X \to X^*$ is a Fredholm operator by Lemma 4.2, it is a proper map when restricted to a bounded subset. Hence, passing again to a subsequence, we may assume that $u_n \to u \in X$, as required. □

8.2.3 Solutions with prescribed eigenvalue: The case $\lambda < \mu_1(0)$.

We now assume that $\lambda < \mu_1(0)$, and we consider $\mathcal{N} := \{u \in X \setminus \{0\} \mid \rho_u(u) = \lambda\}$ and $c_n := c_n(\psi_\lambda, \mathcal{N})$ as in Section 7.2.2. With this notations there holds:

Theorem 8.10. *Consider $n \in \mathbb{N}$ even, and suppose that in addition to (M) the nonlinearity f satisfies the assumptions (i)-(iii) of Theorem 8.6. Then:*

(a) $c_n > c_{n-1}$.

(b) *There is a solution* (u, λ) *of* $(NH-)$ *such that* $\psi_\lambda(u) = c_n$ *and* u *has exactly* n *simple zeroes in* $[0, 1[$.

(c) *If* u *is a solution of* $(NH-)$ *with* $\psi_\lambda(u) < c_n$*, then* u *has at most* $n-1$ *simple zeroes in* $[0, 1[$.

We prove this theorem by applying Section 7.2.2. For this recall that conditions $(CC)_1$ and $(CC)_2$ are immediate consequences of (i) and (ii). Moreover, note that the scalar product $(\cdot|\cdot)_\lambda$ introduced in Section 7.2.2 can be written as

$$(u|v)_\lambda := \int_0^1 [p(u'v') + (q - \lambda)uv]\, dx.$$

Denoting by J_λ the canonical isometric isomorphism $X \to X^*$ associated with this scalar product, the operator family under consideration is given by $G(u) := J_\lambda^{-1} B(u)$ $(u \in X)$. Now observe that condition $(CC)_3$ is ensured by the following Lemma.

Lemma 8.11. *Let* $u \in X$*, and consider a finite sum* $v = \sum\limits_{i=1}^{n}$ *of (nonzero) eigenfunctions* v_i *of* $G(u)$ *corresponding to pairwise different positive eigenvalues. Then* v *does not vanish on the set* $I(f) := \{x \in [0, 1] \mid \exists\, t > 0$ *with* $f(x, t) > 0\}$.

Proof. Suppose in contradiction that $v = 0$ on $I(f)$. By a similar argument as in the proof of Lemma 8.8 this forces $v_i = 0$ on the set $\{x \in [0, 1] \mid f(x, |u(x)|) \neq 0\}$ for $i = 1, ..., n$. Hence $G(u)v_i = 0$ for $i = 1, ..., n$ in contradiction to the assumptions. □

Now fix $n \in \mathbb{N}$ even, and define $\tilde{Q}_n(u)$ as in Section 7.2.2. We recall that our abstract results involved the set $K_\mathcal{N} := \{u \in \mathcal{N} \mid \tilde{Q}_n(u)u = u\}$. In the present situation we observe:

Lemma 8.12. $\gamma^*(K_\mathcal{N}) \leq n - 1$.

Proof. First note that, if $u \in \mathcal{N}$, then $f(\cdot, |u|) \not\equiv 0$ on $[0, 1]$. Thus, testing with an n-dimensional subspace of X consisting of analytic functions (e.g. trigonometric polynomials), we infer $\sigma_n(u) > 0$. We claim

$$\sigma_n(u) < \sigma_{n-1}(u) \tag{8.15}$$

for every $u \in \mathcal{N}$. If in contradiction $\sigma_n(u) = \sigma_{n-1}(u)$, then also $\mu_n(\tilde{A}(u)) = \mu_{n+1}(\tilde{A}(u)) = \lambda$ for the operator

$$\tilde{A}(u) := \frac{\partial}{\partial x}(p\frac{\partial}{\partial x}) + q - \frac{1}{\sigma_n(u)} f(x, |u|) : H \subset D(A_0) \to H.$$

This however contradicts [28, Theorem 2.3.1]. As a consequence of (8.15), there holds $K_\mathcal{N} = \{u \in X \mid u \in \mathcal{N}(\tilde{P}_{n-1}(u))\}$. Hence we may deduce the assertion from Proposition 3.12, once we have shown that $\tilde{P}_{n-1} : \mathcal{N} \to \Pi_{n-1}(X)$ is homotopic to a constant map via an even homotopy. To this end, denote for $t \in [0, 1]$, $u \in X$ by $H_1(t, u) \in \mathcal{L}(X)$ the spectral projection associated with the operator $G_t(u) := J_\lambda^{-1}(B(u) + tI)$ and the interval $[\sigma_{n-1}(G_t(u)), \infty[$ (now using the notation of Section 4.2). For $t > 0$ and $u \in X$ the operator $G_t(u)$ is strictly positive, hence we conclude

$$\sigma_n(G_t(u)) < \sigma_{n-1}(G_t(u)) \tag{8.16}$$

as in the proof of Lemma 8.7. However, in view of (8.15), we infer that (8.16) holds for also for $t = 0$ *and* $u \in \mathcal{N}$. Hence Lemma 4.8 implies that $H_1 : [0,1] \times \mathcal{N} \to \Pi_n(X)$ is continuous. Next we define $H_2 : [0,1] \times \mathcal{N} \to \Pi_n(X)$ by

$$H_2(x, s) = \tilde{P}_{n-1}(G_1(su)).$$

Then H_2 is also continuous by virtue of (8.16) and Lemma 4.8. Piecing together H_2 and $\tilde{H}_1(\cdot,\cdot) := H_1(1 - \cdot, \cdot)$, we get a homotopy $H : [0,2] \times \mathcal{N} \to \Pi_n(X)$ with $H(0,\cdot) \equiv const \in \Pi_n(X)$ and $H(2,\cdot) = \tilde{P}_{n-1}(\cdot) : \mathcal{N} \to \Pi_n(X)$. Since H is even in the second variable, the assertion follows from Proposition 3.12. $\square$

Next we assert that

$$\gamma(\mathcal{N}) = \infty. \tag{8.17}$$

This is easily seen by Lemma 7.10. Indeed, let $k \in \mathbb{N}$ be given, and consider again a k-dimensional subspace consisting of analytic functions. Then, for each $v \in V \setminus \{0\}$, assumption (iii) furnishes a number $t > 0$ such that $\langle B(tv)v, v\rangle > 0$. Hence $\gamma(V \cap \mathcal{N}) = k$, and therefore $\gamma(\mathcal{N}) \geq k$.
We now may easily complete the

Proof of Theorem 8.10. In view of (8.17) and Lemma 8.12 we may apply Corollary 7.21, which in asserts that $K_{\mathcal{N}}$ contains a solution u of $(SC)_n$ satisfying $\psi_\lambda(u) = c_n$. By Remark 8.2 we conclude that u has precisely n simple zeroes, as claimed in (b). Moreover, since $\sigma_{n-1}(u) > \sigma_n(u) = 1$ by (8.15), we infer $c_n = \psi_\lambda(u) > c_{n-1}$ from Prop. 7.19. Hence (a) holds true as well.
Finally, suppose that u is an arbitrary solution of $(NH-)$ with *at least* n simple zeroes. Then $u \in \mathcal{N}$ and $\sigma_n(u) \geq 1$, hence $\psi_\lambda(u) \geq c_n$ again by Proposition 7.19. This yields (c), and the proof is complete. $\square$

8.3 Remarks on nonlinear Sturm-Liouville problems and characteristic numbers

In this section we make some comments on the Dirichlet problem

$$(SL\pm) \qquad -(p(x)u')' + q(x)u \pm f(x, |u|)u = \lambda u, \qquad x \in [0,1]$$

$$u(0) = u(1) = 0, \tag{8.18}$$

where we are given continuous functions $q : [0,1] \to \mathbb{R}$ and $f : [0,1] \times [0, \infty[\to \mathbb{R}$ as well as an everywhere positive function $p \in C^1([0,1])$. Moreover we assume

(M) $f(0) = 0$, and $f(x, \cdot)$ is nondecreasing on $[0, \infty[$ for a. e. $x \in [0,1]$.

Hence we consider the same class of second order ODEs as in the preceding sections, but now the periodicity assumptions are replaced by Dirichlet boundary conditions. Consequently we now consider $H := L^2[0,1]$ and $X := W_0^{1,2}([0,1])$, then the operator $A_0 := -\frac{d}{dx}(p\frac{d}{dx}) + q$ with domain

$$\mathcal{D}(A_0) = W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$$

is selfadjoint and bounded from below in $\mathcal{H}$, and X is precisely the form domain of A_0. Note that, by virtue of the assumptions imposed on the data, weak solutions $u \in X$ of $(SL\pm)$ are in fact *classical* C^2 solutions. The main difference to the periodic problem lies in the fact that now all eigenvalues are nondegenerate, i. e. that

$$\mu_n(u) < \mu_{n+1}(u) \tag{8.19}$$

for *every* $n \in \mathbb{N}$ and every $u \in X$. Moreover, the corresponding n-th eigenfunctions possesses precisely $n-1$ simple zeroes in $]0,1[$ (cf. Remark 8.2(b)). This is true irrespectively of whether the μ_n are defined according to the sub- or to the superlinear case. Therefore we may evidently repeat all the preceding considerations for *arbitrary* $n \in \mathbb{N}$. We omit the details and just give a view on some of the arising results. Therefore put

$$\psi^{\pm}(u) := \frac{1}{2}\int_0^1 \Big(p(x)u'^2 + q(x)u^2\Big)dx \pm \varphi(u).$$

and

$$\psi_\lambda^{\pm}(u) := \psi^{\pm}(u) - \frac{\lambda}{2}\|u\|^2$$

for $u \in X$, where $\varphi : X \to \mathbb{R}$ is given by

$$\varphi(u) := \int_0^1 \int_0^{|u(x)|} f(x,s)s\,ds\,dx.$$

Then there holds:

Theorem 8.13. *Let $R > 0$, $n \in \mathbb{N}$ and put $S_R := \{u \in X \,|\, \|u\| = R\}$ ($\|\cdot\|$ denoting the L^2-norm). Moreover set $c_n^{\pm} := c_n(\psi^{\pm}, S_R)$. Then*

(a) $c_n^+ < c_{n+1}^+$

(b) There is a solution $(u,\lambda) \in S_R \times \mathbb{R}$ of $(SL+)$ such that $\psi^+(u) = c_n^+$, and u has exactly $n-1$ simple zeroes in $]0,1[$.

(c) If $(u,\lambda) \in S_R \times \mathbb{R}$ is a solution of $(SL+)$, with $\psi^+(u) > c_n^+$, then u has at least n simple zeroes in $]0,1[$.

Moreover, if there are numbers $a, b > 0$ and $0 \le q < 4$ such that

$$|f(x,t)| \le at^q + b$$

for $x \in [0,1]$, $t \in [0,\infty[$, then also *(d)* $c_n^- < c_{n+1}^-$.

(e) There is a solution $(u,\lambda) \in S_R \times \mathbb{R}$ *of* $(SL-)$ *such that* $\psi^-(u) = c_n^-$ *and* u *has exactly* $n-1$ *simple zeroes in* $]0,1[$.

(f) If $(u,\lambda) \in S_R \times \mathbb{R}$ *is a solution of* $(SL-)$ *with* $\psi^-(u) < c_n^-$*, then* u *has at most* $n-2$ *simple zeroes in* $]0,1[$.

Theorem 8.14. *Suppose that* f *satisfies the additional conditions*

(i) There is an $\eta > 2$ *such that* $0 \le \eta \int\limits_0^t f(x,s)s\,ds \le f(x,t)t^2$ *for all* $x,\ t$.

(ii) If $x \in [0,1],\ t \in [0,\infty[$ *is such that* $f(x,t) > 0$*, then* $f(x,\cdot)$ *is strictly increasing on* $[t,\infty[$.

(iii) $f \not\equiv 0$*, i.e. there is* $x \in [0,1]$*,* $t \in (0,\infty)$ *with* $f(x,t) > 0$.

Consider $\lambda < \inf \sigma(A_0)$ *and*

$$\mathcal{N} := \{u \in X \setminus \{0\} \mid \int_0^1 [pu'^2 + (q-\lambda)qu^2] = \int_0^1 f(x,|u|)u^2\}$$

as well as $c_n := c_n(\psi_\lambda^-, \mathcal{N})$*. Then for every* $n \in \mathbb{N}$ *there holds*

(a) $c_n < c_{n+1}$.

(b) There is a solution (u,λ) *of* $(SL-)$ *such that* $\psi_\lambda^-(u) = c_n$ *and* u *has exactly* $n-1$ *simple zeroes in* $]0,1[$.

(c) If u *is a solution of* $(NH-)$ *with* $\psi_\lambda^-(u) < c_n$*, then* u *has at most* $n-1$ *simple zeroes in* $]0,1[$.

Remark 8.15. (a) In view of global bifurcation results due to Rabinowitz [58], the above-stated Theorems presumably do not contain any new *existence* result for nodal solutions. However, the nodal characterization by Ljusternik-Schnirelman levels complements results of Coffman (cf. [20] and [18]) in a clarifying way. In particular we infer that Nehari's characteristic numbers defined by

$$\tau_n := \min_{u \in \Gamma_n} \psi_\lambda^-(u)$$

coincide with the Ljusternik-Schnirelman levels $c_n := c_n(\psi_\lambda^-, \mathcal{N})$ of ψ_λ^- on the Nehari manifold $\mathcal{N}$ (at least under the stronger assumptions which Nehari originally imposed on the problem, cf. [57]). We recall that, as defined by Nehari, Γ_n denotes the class of all continuous and piecewise differentiable functions $u \in X$ such that

(i) u has at least $n-1$ zeroes $a_1 < a_2 < \dots < a_{n-1}$ in $]0,1[$ and

(ii) $u \cdot 1_{[a_j, a_{j+1}]} \in \mathcal{N}$ for $j = 0, \dots, n$, with $a_0 := 0$ and $a_n := 1$

Nehari [57] in particular proves the existence of an ψ_λ^--minimizer $u_1 \in \Gamma_n$ which solves $(SL-)$, and from Theorem 8.14(c) we deduce that $\tau_n = \psi_\lambda^-(u_1) \geq c_n$. On the other hand, Theorem 8.14(b) yields a solution u_2 of $(SL-)$ such that $u_2 \in \Gamma_n$ and $\psi_\lambda^-(u_2) = c_n$, and therefore $c_n \geq \tau_n$ by the very definition of τ_n.
Note that Coffman [18] first observed a relationship between Ljusternik-Schnirelman theory and Nehari's method, but he identified the characteristic numbers with Ljusternik-Schnirelman levels of a *different* auxiliary functional.
(b) Clearly one obtains analogous results replacing the Dirichlet conditions (8.18) by a general set of separated boundary conditions

$$\alpha_1 u(0) + \alpha_2 u'(0) = 0, \qquad \alpha_1^2 + \alpha_2^2 > 0,$$
$$\beta_1 u(1) + \beta_2 u'(1) = 0, \qquad \beta_1^2 + \beta_2^2 > 0.$$

Indeed, in this case problem $(SL\pm)$ can again be formulated via a family of selfadjoint operators with nondegenerate eigenvalues and such that the n-th eigenfunctions possesses exactly $n-1$ simple zeroes. Hence we may repeat the preceding reasonings once more.

Chapter 9

Normalized solutions to superlinear Schrödinger equations

We are interested in solutions (u, λ) of the equation

$$(NS) \qquad -\Delta u - f(x, |u|)u = \lambda u, \quad u \in W^{1,2}(\mathbb{R}^N)$$

satisfying the additional side condition

$$\|u\|_2 = R,$$

$R > 0$ being given.
We assume that $f : \Omega \times [0, \infty[\to \mathbb{R}$ is a Caratheodory function satisfying

(M) $f(0) = 0$, and $f(x, \cdot)$ is nondecreasing on $[0, \infty[$ for a. e. $x \in \mathbb{R}^N$.

Concerning the regularity of weak solutions to (NS) there holds:

Lemma 9.1. *Suppose that there is $\beta \in]0, \frac{4}{N-2}[$ and $C > 0$ such that*

$$f(x, t) \leq C(1 + t^\beta) \tag{9.1}$$

for a.e. $x \in \mathbb{R}^N$, $t \geq 0$, and suppose that $u \in W^{1,2}(\mathbb{R}^N)$ is a weak solution of (NS) for some $\lambda \in \mathbb{R}$. Then $u \in W^{2,s}_{loc}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ for every $s < \infty$.

Proof. Using (9.1) and Sobolev embeddings, we easily derive that $f(\cdot, |u(\cdot)) \in L^q_{loc}(\mathbb{R}^N)$ for some $q > \frac{N}{2}$, hence u is continuous in view of Lemma 14.1. As a consequence, $g = \lambda u + f(\cdot, |u|)u \in L^\infty_{loc}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, whereas u weakly solves

$$-\Delta u = g.$$

Hence standard elliptic regularity yields $u \in W^{2,s}_{loc}(\mathbb{R}^N)$ for all $s \geq 1$. However, $W^{2,s}_{loc}(\mathbb{R}^N)$ is embedded in $C^1(\mathbb{R}^N)$ for sufficiently large s. $\square$

9.1 The radial case

In this section we assume in addition to (M) that $N \geq 2$ and that f is radially symmetric, i.e. it can be written in the form $f(x,t) = \mathfrak{f}(|x|,t)$ with a Caratheodory function $\mathfrak{f} : [0,\infty[\times[0,\infty[\to \mathbb{R}$. Moreover we assume

(G) There are numbers $\beta_1, \beta_2 \in]0, \frac{4}{N}[$ and $C > 0$ such that $f(x,t) \leq C(t^{\beta_1} + t^{\beta_2})$ for a.e. $x \in \mathbb{R}^N$, $t \geq 0$.

(D) There exist positive constants A, t_0, r_0 and numbers $0 \leq \tau < 2$, $0 < \sigma < \frac{2(2-\tau)}{N}$ such that

$$\mathfrak{f}(r,t) \geq A r^{-\tau} t^{\sigma} \qquad \text{for} \quad 0 \leq t \leq t_0,\ r > r_0.$$

Now define $\mathcal{H}$, X as the closed subspaces consisting of radially symetric functions in $L^2(\mathbb{R}^N)$, $W^{1,2}(\mathbb{R}^N)$, respectively. Then X is precisely the form domain of the selfadjoint operator $A_0 : \mathcal{D}(A_0) \subset \mathcal{H} \to \mathcal{H}$ defined by

$$\begin{aligned} \mathcal{D}(A_0) &= \{u \in W^{2,2}(\mathbb{R}^N) \mid u \text{ radially symmetric}\} \\ A_0 u &= -\Delta u. \end{aligned} \tag{9.2}$$

We furthermore consider

$$F(r,t) := \int_0^t \mathfrak{f}(r,s) s\, ds$$

for $r, t \in [0,\infty[$.

Lemma 9.2. *Set*

$$\langle B(u)v, w\rangle := \int_{\mathbb{R}^N} \mathfrak{f}(|x|, |u(x)|) v(x) w(x)\, dx$$

for $u, v, w \in X$. Then:

(a) B is a well defined strongly continuous map $X \to \mathcal{L}(X, X^*)$. Moreover, $B(u) \in \mathcal{L}(X, X^*)$ is compact for every $u \in X$.

(b) For every $u \in X$ the integral

$$\varphi(u) := \int_{\mathbb{R}^N} F(|x|, |u(x)|)\, dx$$

exists. Moreover, $\varphi : X \to \mathbb{R}$ and B satisfy (CC).

Proof. (a) Let us first assume that

$$f(x,s) \leq C s^{\beta} \qquad \forall\ s \geq 0,\ x \in \mathbb{R}^N \tag{9.3}$$

with a constant $\beta \in]0, \frac{4}{N}[$. Since $2 < \beta < 2^*$, the Sobolev embedding $i : X \to L^{\beta+2}(\mathbb{R}^N)$ is compact (see [51, Proposition 1.1]). Moreover, B factorizes in the form

$$X \overset{i}{\hookrightarrow} L^{\beta+2}(\mathbb{R}^N) \xrightarrow{f_*} L^{\frac{\beta+2}{\beta}}(\mathbb{R}^N) \xrightarrow{b} \mathcal{L}(L^{\beta+2}(\mathbb{R}^N), L^{(\beta+2)'}(\mathbb{R}^N)) \xrightarrow{j} \mathcal{L}(X, X^*). \tag{9.4}$$

Here f_* and b are given by $f_*(u)(x) := f(x, u(x))$ and $b(u)v := uv$ respectively, and j maps a linear operator $h \in \mathcal{L}(L^{\beta+2}(\mathbb{R}^N), L^{(\beta+2)'}(\mathbb{R}^N))$ to i^*hi. By (G) the substitution operator f_* is bounded, hence it is continuous (see [47, Theorem 2.1]). Moreover, b is a continuous linear operator by Hölder's inequality, and evidently j is continuous as well. Hence (9.4) shows that B is strongly continuous and that $B(u) \in \mathcal{L}(X, X^*)$ is a *compact* linear operator for every $u \in X$.
In the general case, note that (G) permits to write $f = f_1 + f_2$, where f_1 and f_2 satisfy (9.3) with $\beta = \beta_1$ and $\beta = \beta_2$ respectively. According to this decomposition, B splits in a sum $B_1 + B_2$, where the operators B_i have the desired properties by the argument from above. Hence the assertion is true for B as well.
(b) As in the proof of Lemma 2.4 one uses (M) to deduce the inequality

$$2\Big(F(|x|, |v(x)|) - F(|x|, |u(x)|)\Big) \geq \mathfrak{f}(|x|, |u(x)|)(v^2(x) - u^2(x)) \tag{9.5}$$

for $u, v \in X$ and $x \in \mathbb{R}^N$. Integrating (9.5) over $\mathbb{R}^N$ yields precisely (CC). $\square$

As a consequence of Lemma 9.2, the map B satisfies (H1)-(H5) and (CC), hence we may consider $A(u)$, $\mu_n(u)$, $Q_n(u)$, ρ_u and problem $(SC)_n$ as defined in Chapter 7. We observe that solutions of $(SC)_n$ carry nodal information, more precisely:

Lemma 9.3. *(a) For every $u \in X$ the eigenvalues of $A(u)$ are nondegenerate. Moreover, if $\mu_n(u)$ is an eigenvalue of $A(u)$ and v a corresponding eigenfunction, then v has precisely n nodal domains.*

(b) If $u \in X$ is a solution of $(SC)_n$ for some $n \in \mathbb{N}$, then u solves (NS) weakly with $\lambda = \mu_n(u)$, and u has precisely n nodal domains.

Proof. (a) The assertion follows from Theorem 14.8 as soon as we have established that $V := f(\cdot, |u(\cdot)|)$ is a *radial W-admissible potential* (cf. Definition 14.5). For this note that, by virtue of (G) and Lemma 14.16(b), the relation (14.9) holds for any $\alpha < 0$. Therefore the admissibility follows in view of Lemma 9.2(a).
(b) This follows from (a) and the very definition of problem $(SC)_n$. $\square$

Now define the functional $\psi : X \to \mathbb{R}$ by

$$\psi(u) = \frac{1}{2}\|\nabla u\|_2^2 - \varphi(u)$$

and consider the minimax values $c_n := c_n(\psi, S_R)$ for ψ on the sphere

$$S_R := \{u \in X \mid \|u\|_2 = R\}.$$

The main result of this section reads as follows.

Theorem 9.4. *Let $R > 0$. Then*

(a) $c_n < c_{n+1}$ *for every* $n \in \mathbb{N}$.

(b) There exist (radial) solutions $(u_n, \lambda_n) \in S_R \times \mathbb{R}^-$ *of (NS) such that* $\psi(u_n) = c_n$*, and* u_n *has precisely* n *nodal domains.*

(c) If $(u, \lambda) \in S_R \times \mathbb{R}^-$ *is a radial solution of (NS) with* $\psi(u) < c_n$ *for some* $n \in \mathbb{N}$*, then* u *has* at most $n-1$ *nodal domains.*

Remark 9.5. (a) While standard Ljusternik-Schnirelman theory on the sphere S_R yields the mere existence of infinitely many normalized radial solutions to (NS) under the above conditions, the nodal information provided by Theorem 9.4 is new. In fact, we are only aware of a related result of Heinz [35] referring to a *sublinear* equation.
(b) Condition (D) is closely related to the existence of normalized solutions to (NS) with *small* prescribed L^2-norm. Indeed, [72, Theorem 4.8] asserts that a condition like D is necessary in case that R is small. To illustrate this, let us consider the nonlinearity

$$|f(x,t)| = \frac{t^\sigma}{1+|x|^\tau} \qquad (t > 0,\ x \in \mathbb{R}^N).$$

Then, if $N \geq 3$ and $\sigma > \frac{2(2-\tau)}{N} \geq (2-\tau)\max\{\frac{1}{N}, \frac{\sigma}{2}\}$, there is a constant $C > 0$ such that $\|u\|_2 \geq C$ whenever u solves (NS) weakly with some $\lambda < 0$ (C does not depend on λ, cf. [72, Theorem 4.8]). Moreover, a similar result holds for $N = 2$.

In the rest of the section we prove Theorem 9.4 with the tools supplied by Section 7.1. For this we first need to ensure:

Lemma 9.6. *For arbitrary* $R > 0$ *there holds condition (BB), i.e., there are constants* $a \in [0,1[$ *and* $b > 0$ *such that*

$$\langle B(u)u, u\rangle \leq a\|u\|_X^2 + b.$$

for every $u \in S_R$.

Proof. We use the well known multiplicative Sobolev inequality due to Gagliardo, Nirenberg and Golovkin, which asserts that for $p \in]2, \frac{2N}{N-2}[$ there is a constant $K = K(p,N)$ such that

$$\|u\|_p \leq K\|\nabla u\|_2^\theta \|u\|_2^{1-\theta} \tag{9.6}$$

with $\theta = N(\frac{1}{2} - \frac{1}{p})$ (see [73] and the references therein). Indeed, in view of (G) we deduce with $\beta = \max\limits_{i=1,2} \beta_i$

$$\begin{aligned}\langle B(u)u, u\rangle &\leq C(\|u\|_{\beta_1+2}^{\beta_1+2} + \|u\|_{\beta_2+2}^{\beta_2+2}) \\ &\leq 2C(\|u\|_{\beta+2}^{\beta+2} + \|u\|_2^2) \\ &\leq 2C(K^{\beta+2}R^{(\beta+2)(1-\theta)}\|u\|_X^{(\beta+2)\theta} + R^2)\end{aligned}$$

whereas $(\beta+2)\theta < 2$. Hence (BB) follows from Remark 7.1. $\square$

By Lemma 7.2 we now infer that ψ is bounded from below on S_R, hence $c_n > -\infty$ for every $n \in \mathbb{N}$. Next we show that these values are strictly smaller than zero, which is the infimum of the essential spectrum of A_0. This is of crucial importance to ensure local compactness (cf. the proof of Theorem 7.8).

Lemma 9.7. *For arbitrary $R > 0$, $n \in \mathbb{N}$ there holds $c_n < 0$.*

Proof. Pick an n-dimensional subspace $Z \subset X$ consisting of bounded continuous functions with support in $\mathbb{R}^N \setminus B_{r_0}(0)$. For $v \in Z$ define $j_k v \in X$ by $j_k v(x) := k^{-\frac{N}{2}} v(\frac{x}{k})$, and note that $\|j_k v\|_2 = \|v\|_2$ for every $v \in Z$. Put $Z_R := \{v \in Z \mid \|v\| = R\}$, and observe that there is a number $k_0 \in \mathbb{N}$ such that $\|j_k v\|_\infty \le t_0$ for all $v \in Z_R$, $k \ge k_0$. For these values of k and v assumption (D) implies

$$
\begin{aligned}
\varphi(j_k v) &\ge A \int_{\mathbb{R}^N \setminus B_{r_0}(0)} |x|^{-\tau} \int_0^{|j_k v(x)|} t^{\sigma+1}\, dt\, dx \\
&= \frac{A}{\sigma+2} \int_{\mathbb{R}^N \setminus B_{r_0}(0)} |x|^{-\tau} |j_k v(x)|^{\sigma+2}\, dx \\
&= \frac{A}{\sigma+2} k^{-\frac{N}{2}(\sigma+2)} \int_{\mathbb{R}^N \setminus B_{r_0}(0)} |x|^{-\tau} |v(\frac{x}{k})|^{\sigma+2}\, dx \\
&= \frac{A}{\sigma+2} k^{-\frac{N}{2}(\sigma)} \int_{\mathbb{R}^N \setminus B_{\frac{r_0}{k}}(0)} |kx|^{-\tau} |v(x)|^{\sigma+2}\, dx \\
&\ge \frac{A}{\sigma+2} k^{-\frac{N}{2}\sigma-\tau} \int_{\mathbb{R}^N \setminus B_{r_0}(0)} |x|^{-\tau} |v(x)|^{\sigma+2}\, dx \qquad (9.7)
\end{aligned}
$$

Note that the integral in (9.7) is positive for every $v \in Z_R$. Using this and the fact that Z_R is compact, we find constants $c_1, c_2 > 0$ such that for every $v \in Z_R$ there holds

$$\varphi(j_k v) \ge c_1 k^{-\frac{N}{2}(\sigma)-\tau}$$

as well as

$$\|\nabla j_k v\|_2^2 = k^{-2} \|\nabla v\|^2 \le c_2 k^{-2}.$$

Now $\frac{N}{2}\sigma + \tau < 2$ by assumption, hence

$$
\begin{aligned}
\sup_{v \in Z_R} \psi(j_k v) &= \sup_{v \in Z_R} \left[\frac{1}{2} \|\nabla j_k v\|_2^2 - \varphi(j_k v) \right] \\
&\le \frac{c_2}{2} k^{-2} - c_1 k^{-\frac{N}{2}(\sigma)-\tau} \\
&< 0.
\end{aligned}
$$

for k large enough. Since $\gamma(\{j_k v \mid v \in Z_R\}) = n$, we conclude $c_n < 0$, as claimed. □

Now fix $n \in \mathbb{N}$ and put $K^- := \{u \in S_R \mid \tilde{Q}_n(u) u = u\}$ as in section 7.1. Recall that, in order to apply Theorem 7.8, we have to show that $\gamma^*(K) \le n - 1$. However, since the 'unperturbed operator' $A_0 = -\Delta$ has no eigenvalues, Prop. 7.9 does not apply in the present situation. Instead, we proceed by a direct construction which involves the evaluation of (continuous) eigenfunctions at $x = 0$.

Lemma 9.8. *Consider $i \in \mathbb{N}$ and $D \subset X$ such that for $u \in D$ there holds $\mu_i(u) < 0$, which by Lemma 9.3(a) implies that $\mu_i(u)$ is a nondegenerate eigenvalue of $A(u)$. Denote $\hat{P}_i(u)$ the spectral projection associated with this eigenvalue. Then the function*

$$\begin{array}{ccc} D & \to & \mathbb{R} \\ u & \mapsto & \hat{P}_i(u)u|_{x=0} \end{array}$$

is odd and continuous.

Proof. Oddness is clear. To show continuity, consider a sequence $(u_j) \subset D$ such that $u_j \to u \in D$. To abbreviate the notation, we write $h_j := \hat{P}_i(u_j)u_j$ and $h := \hat{P}_i(u)u$. As an easy consequence of Corollary 4.5 we infer that $h_j \to h$ in X. Denote $\mathcal{U} := \{h, h_j \mid j \in \mathbb{N}\}$, in particular $\mathcal{U}$ is bounded in X. Moreover, using (G) and Sobolev embeddings, we infer that the sequence of functions $V_j := f(\cdot, |u_j|(\cdot)) - \mu_i(u_j)$, $j \in \mathbb{N}$ is bounded in $L^q(B_2(0))$ for some $q > \frac{N}{2}$. Applying Lemma 14.1(b) to $\Omega = B_2(0)$, $\Omega' = B_1(0)$ in particular yields that $h_j(0) \to h(0)$, as claimed. □

Lemma 9.9. $\gamma^*(K^-) \leq n - 1$.

Proof. Setting $\mathcal{O}_i := \{u \in S_R \,|\mu_i(u) < 0, \hat{P}_i(u)u|_{x=0} \neq 0\}$ for $i = 1, ..., n-1$, we infer by Lemma 9.8 that $\mathcal{O}_i$ is *open* and symmetric and that $\gamma(C) \leq 1$ for every closed and symmetric subset $C \subset \mathcal{O}_i$. Moreover,

$$\hat{P}_i(u)u|_{x=0} = 0 \quad \Longleftrightarrow \quad \hat{P}_i(u)u = 0$$

by virtue of Lemma 14.3, which implies that $S_R \setminus K^- \subset \bigcup_{i=1}^{n-1} \mathcal{O}_i$.

Now consider an arbitrary closed and symmetric subset $A \subset S_R \setminus K^-$. Then $A \subset \bigcup_{i=1}^{n-1} \mathcal{O}_i$. Moreover, since A is paracompact, this covering can be shrunk, i.e. there are open and symmetric subsets $\tilde{\mathcal{O}}_i$ such that $\overline{\tilde{\mathcal{O}}_i} \subset \mathcal{O}_i$ and $A \subset \bigcup_{i=1}^{n-1} \tilde{\mathcal{O}}_i$. Recalling that $\gamma(\overline{\tilde{\mathcal{O}}_i}) \leq 1$ for each i, we conclude $\gamma(A) \leq n-1$. Thus $\gamma^*(K^-) \leq n-1$, as claimed. □

We now have all necessary tools for the

Proof of Theorem 9.4. We start with the proof of (b), which we deduce from Theorem 7.8. For this let $n \in \mathbb{N}$ be given, and recall that the relation (7.11) holds by Lemma 9.9. Moreover, $\mu_\infty = 0$, hence Lemma 9.7 ensures that $c_n < \frac{R^2}{2}\mu_\infty$, as required. Recalling finally that B is compact by Lemma 9.2, we may apply Theorem 7.8 which yields condition $(CP)^-$. In particular there is a solution $u_n \in K^- \subset S_R$ of $(SC)_n$, and by virtue of Lemma 9.3(b) we conclude that u_n has precisely n nodal domains, as claimed in (b).
Moreover, suppose that $(u, \lambda) \in S_R \times \mathbb{R}$ is a solution of (NS) having more than n nodal domains. Then $\rho_u(u) = \lambda = \mu_{n+j}(u)$ for some $j \in \mathbb{N}$, whereas $\mu_{n+j}(u) > \mu_n(u)$ by Lemma 9.3(a). By Proposition 7.4 we infer that $\psi(u) \geq c_{n+j} \geq c_{n+1}$ *as well as* $\psi(u) > c_n$. Hence, every solution (u, λ) of (NS) satisfying *either* $\psi(u) \leq c_n$ *or* $\psi(u) < c_{n+1}$ has at most n nodal domains. Hence (c) holds in particular, but combined with (b) this also forces (a). □

9.2 The nonradial case

In this section we assume that in addition to (M) there holds the following assumption:

(G') There is a number $0 < \beta < \frac{4}{N}$ and a function $w \in L^\infty(\mathbb{R}^N)$ such that $\lim\limits_{|x|\to\infty} w(x) = 0$ and $f(x,t) \leq w(x)(1+t^\beta)$ for a.e. $x \in \mathbb{R}^N,\ t \geq 0$.

The behavior at infinity is required to overcome the lack of compactness which is an inconvenient feature of the *non-symmetric* setting. As in the radial case, we also require lower bounds on the growth of f, precisely we assume:

(D') There exists positive constants A, t_0, d as well as numbers $\tau < 2$, $0 < \sigma < \frac{2(2-\tau)}{N}$ and a point $x_0 \in \mathbb{R}^N$ such that $|x_0| > d$ and

$$f(x,t) \geq A|x|^{-\tau} t^\sigma \qquad \text{for} \quad 0 \leq t \leq t_0,\ x \in \mathcal{C} := \{sy \mid s \geq 1,\ |y - x_0| \leq d\}.$$

We emphasize that, as in the radial case, a growth condition in the form of (D') is necessary for the existence of normalized solutions to (NS) with *small* prescribed L^2-norm (cf. Remark 9.13 and [72, Theorem 4.8]). For this reason, condition (D') is familiar in the context of bifurcation from the essential spectrum, see [74, p. 431].
We now put $X := W^{1,2}(\mathbb{R}^N)$, and as in the radial case we have

Lemma 9.10. *Define*

$$\langle B(u)v, w\rangle := \int_{\mathbb{R}^N} f(x, |u(x)|)v(x)w(x)\, dx$$

for $u, v, w \in X$. Then:

(i) B is a well defined strongly continuous map $X \to \mathcal{L}(X, X^*)$. Moreover, $B(u) \in \mathcal{L}(X, X^*)$ is compact for every $u \in X$.

(ii) For every $u \in X$ the integral

$$\varphi(u) := \int_{\mathbb{R}^N} \int_0^{|u(x)|} f(x,t)t\, dt\, dx$$

exists. Moreover, $\varphi : X \to \mathbb{R}$ and B satisfy (CC).

Proof. Note that by (G') we may write $f = f_1 + f_2$ such that

$$f_i(x,s) \leq w(x)s^{\beta_i} \quad (x \in \mathbb{R}^N,\ s \in [0,\infty[,\ i = 1,2)$$

with $\beta_1 = 0$ and $\beta_2 = \beta$. Hence the assertion follows from Lemma 14.22 and the remark following it. ∎

Again we infer that the conditions (H1)-(H5) and (CC) are valid. Setting

$$A_0 := -\Delta : W^{2,2}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$

we therefore may consider $A(u)$, $\mu_n(u)$, $Q_n(u)$, ρ_u etc. as defined in Chapter 7.
Now fix $u \in X$ arbitrary. Note that (G') and Sobolev embeddings yield that

$$V := f(\cdot, |u(\cdot)|) \in L^q_{loc}(\mathbb{R}^N) \qquad \text{for some } q > \frac{N}{2}$$

and in view of Lemma 9.10(a) we deduce that V is a W-admissible potential (cf. Sec. 14.3.1). Hence Theorem 14.7 yields that every eigenfunction of $A(u)$ associated with $\mu_n(u)$ has at most n nodal domains (note that every such eigenfunction is continuous by virtue of Lemma 14.1). In particular, every eigenfunction associated with $\mu_1(u)$ does not change sign, hence it is positive by the strong Harnack inequality, see [67, Theorem C.1.3]. This immediately implies that

$$\mu_1(u) < \mu_2(u) \qquad \text{whenever} \quad \mu_1 < \mu_\infty, \tag{9.8}$$

and that every eigenfunction associated to $\mu_n(u)$, $n \geq 2$ changes sign. In particular we have proven:

Lemma 9.11. *If $u \in X$ is a solution of $(SC)_n$ for some n, then u has at most n nodal domains. If $n \geq 2$, then u changes sign.*

In our main theorem we state relationships between nodal properties of solutions

$$u \in S_R = \{u \in X \mid \|u\|_2 = R\}$$

of (NS) and the minimax values $c_n := c_n(\psi, S_R)$ for the functional $\psi : X \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{2}\|\nabla u\|_2^2 - \varphi(u)$$

(cf. Sec. 7.1).

Theorem 9.12. *Let $R > 0$. Then:*

(a) *If $(u, \lambda) \in S_R \times \mathbb{R}$ is a weak radial solution of (NS) satisfying either $\psi(u) \leq c_n$ or $\psi(u) < c_{n+1}$ for some $n \in \mathbb{N}$, then u has* at most n *nodal domains.*

(b) *For $n = 1, 2$ there are solutions $(u_n, \lambda_n) \in S_R \times \mathbb{R}$ of (NS) such that $\psi(u_n) = c_n$ and u_n has precisely n nodal domains.*

(c) $c_1 < c_2$

Remark 9.13. (a) As a matter of fact, (NS) has infinitely many normalized solutions under the above conditions, as can be deduced by standard Ljusternik-Schnirelman theory on the sphere S_R. In contrary, nodal information on normalized solutions is new. However, in addition to our upper bounds on nodal domains, we can only prove the existence of one *sign changing* solution. We nevertheless suspect that (NS) possesses an infinite number of normalized sign changing solutions. This guess is encouraged by results of Bartsch [7], who established the existence of infinitely many sign changing solutions for *unconstrained* solutions of a similar equation on a *bounded* domain. Note also that, in the *unconstrained* case, Bartsch and Wang [11] proved the existence of one sign changing solution to (NS), whereas they do not need f to be even.

In the rest of this section we prove Theorem 9.12, noting that some arguments work similar as in the radial case. Indeed, the proof of Lemma 9.6 just carries over, hence we infer

Lemma 9.14. *Condition (BB) is satisfied for arbitrary $R > 0$.*

Again we conclude $c_n > -\infty$ for every $n \in \mathbb{N}$. Moreover:

Lemma 9.15. *For arbitrary $R > 0$, $n \in \mathbb{N}$ there holds $c_n < 0$.*

Proof. We may proceed similar as in the radial case: Fix $n \in \mathbb{N}$, $R > 0$, and pick an n-dimensional subspace $Z \subset X$ consisting of rapidly decreasing *analytic* functions (e.g. a span of Hermite functions). For $v \in Z$ define $j_k v \in X$ by $j_k v(x) := k^{-\frac{N}{2}} v(\frac{x}{k})$, and note that $\|j_k v\|_2 = \|v\|_2$ for every $v \in Z$. Put $Z_R := \{v \in Z \mid \|v\| = R\}$, and observe that there is a number $k_0 \in \mathbb{N}$ such that $\|j_k v\|_\infty \leq t_0$ for all $v \in Z_R,\ k \geq k_0$. For these values of k and v assumption (D') implies

$$\begin{aligned}
\varphi(j_k v) &\geq A \int_{\mathcal{C}} |x|^{-\tau} \int_0^{|j_k v(x)|} t^{\sigma+1}\, dt\, dx \\
&= \frac{A}{\sigma+2} k^{-\frac{N}{2}(\sigma+2)} \int_{\mathcal{C}} |x|^{-\tau} |v(\frac{x}{k})|^{\sigma+2}\, dx \\
&= \frac{A}{\sigma+2} k^{-\frac{N}{2}\sigma-\tau} \int_{k^{-1}\mathcal{C}} |x|^{-\tau} |v(x)|^{\sigma+2}\, dx \\
&\geq \frac{A}{\sigma+2} k^{-\frac{N}{2}\sigma-\tau} \int_{\mathcal{C}} |x|^{-\tau} |v(x)|^{\sigma+2}\, dx
\end{aligned} \tag{9.9}$$

The latter inequality follows since $k^{-1}\mathcal{C} \supset \mathcal{C}$. Note that the integral in (9.9) is positive for every $v \in Z_R$. Using this and the fact that Z_R is compact, we find constants $c_1, c_2 > 0$ such that for every $v \in Z_R$ there holds

$$\varphi(j_k v) \geq c_1 k^{-\frac{N}{2}(\sigma)-\tau}$$

as well as

$$\|\nabla j_k v\|_2^2 = k^{-2} \|\nabla v\|^2 \leq c_2 k^{-2}.$$

As in the radial case we conclude $c_n < 0$. □

Lemma 9.16. *Let $n \in \{1, 2\}$. Then $\gamma^*(K^-) \leq n - 1$ for the set*

$$K^- := \{u \in S_R \,|Q_n(u)u = u\}.$$

Proof. The assertion is trivial for $n = 1$, since $K^- = S_R$ in this case.
In case $n = 2$ define $\mathcal{O} := \{u \in S_R \,|\mu_1(u) < 0\}$. Since $\mu_\infty = 0$, there holds $S_R \setminus K^- \subset \mathcal{O}$. By (9.8), $\mu_1(u)$ is a *nondegenerate* eigenvalue for $u \in \mathcal{O}$. Hence 4.8 yields that the spectral projection $\hat{P}_1(u) \in \mathcal{L}(X)$ onto the eigenspace of $\sigma_1(u)$ depends continuously on $u \in \mathcal{O}$. (here $\mathcal{O}$ is endowed with the topology of X). Now pick an everywhere positive function $v \in X$ and define h: $\mathcal{O} \to \mathbb{R}$ by

$$u \overset{h}{\mapsto} \int_{\mathbb{R}^N} v P_1(u) u.$$

Clearly h is odd and continuous, and for $u \in \mathcal{O}$ there holds

$$h(u) = 0 \quad \Longleftrightarrow \quad u \in K^-.$$

Hence $\gamma^*(K^-) \leq 1$, as claimed. □

In view of the above considerations we may easily complete the

Proof of Theorem 9.4. Starting again with the proof of (b), we apply Theorem 7.8 for $n = 1, 2$. To this end, note that (7.11) holds by Lemma 9.16, whereas $c_n < \frac{R^2}{2}\mu_\infty$ by Lemma 9.15. We conclude that property $(CP)^-$ is valid for $n = 1, 2$. Hence there are solutions $u_n \in K^- \subset S_R$ of $(SC)_n$, which in particular are weak solutions to (NS) corresponding to $\lambda = \mu_n(u)$. Moreover, u_n has precisely n nodal domains by Lemma 9.15, hence (b) holds.
Next, let $n \in \mathbb{N}$ and suppose that (u, λ) is a solution of (NS) with more than n nodal domains. Then $\rho_u(u) = \lambda = \mu_{n+j}(u)$ for some $j \in \mathbb{N}$, whereas $\mu_{n+j}(u) > \mu_n(u)$. By Proposition 7.4 we infer that $\psi(u) \geq c_{n+j} \geq c_{n+1}$ *as well as* $\psi(u) > c_n$. This shows (a), and (c) is an immediate consequence of (a) and (b). $\square$

Chapter 10

Equations of Choquard type

In this chapter we are concerned with a general form of *so-called* Choquard's equation

$(CH) \quad -\Delta u - (u^2 * V)u = \lambda u, \quad u \in W^{1,2}(\mathbb{R}^3),$

where we assume that V is a radially symmetric measurable function satisfying

(V_1) $V \in L^1(\mathbb{R}^3) + L^p(\mathbb{R}^3)$ for some $1 < p < \infty$.

(V_2) $\int\limits_{\mathbb{R}^3} (\xi * V)(x)\xi(x)\,dx \geq 0$ for every $\xi \in C_0^\infty(\mathbb{R}^3)$.

$(V3)$ $V \geq 0, V \not\equiv 0$.

Note that if V (viewed as a tempered distribution) has a positive Fourier transform, then (V2) is satisfied. This follows from the identity

$$\int_{\mathbb{R}^3} (V * \xi)\xi = \int_{\mathbb{R}^3} (V \hat{*} \xi)\overline{\hat{\xi}} = (2\pi)^{N/2} \int_{\mathbb{R}^3} \hat{V}|\hat{\xi}|^2$$

Important examples for potentials V satisfying $(V_1) - (V_3)$ are $V(x) := \frac{C}{|x|^\alpha}$ with constants $C > 0$, $\alpha \in]0, 3[$, as well as $V(x) := \frac{C}{|x|} e^{-\mu|x|}$ with $C, \mu > 0$ (see [50] and the references quoted there). Note that for every such V equation (CH) remains invariant under the action of the noncompact group of translations $u \mapsto u(\cdot + \tau)$, $\tau \in \mathbb{R}^3$. To avoid problems arising from this noncompactness, we focus on radially symmetric solutions of (CH). Denoting by $\mathcal{H}$ resp. X the Hilbert spaces consisting of the radially symmetric functions in $L^2(\mathbb{R}^3)$ resp. $W^{1,2}(\mathbb{R}^3)$, we observe:

Lemma 10.1.

(i) There is a strongly *continuous (nonlinear) operator* $X \to \mathcal{L}(X, X^*)$ *defined by*

$$\langle B(u)v, w\rangle := \int_{\mathbb{R}^3} (u^2 * V)(x)v(x)w(x)\,dx \tag{10.1}$$

(ii) For $u, v \in X$ *there holds*

$$\langle B(u)v, v\rangle \leq \langle B(u)u, u\rangle^{\frac{1}{2}} \langle B(v)v, v\rangle^{\frac{1}{2}}.$$

(iii) Define $\varphi : X \to \mathbb{R}$ by

$$\varphi(u) := \frac{1}{4}\langle B(u)u, u\rangle.$$

Then φ and B satisfy (CC).

Proof. (i) Without loss, we may assume that $V \in L^p(\mathbb{R}^3)$ for some $p \in [1, \infty[$. Setting $q = \frac{2p}{2p-1}$, we infer that the linear operator $c_* : L^q \to L^{2p}$ defined by

$$w \mapsto w * V$$

is continuous by convolution inequalities. Let $i : X \to L^{2q}$ denote the Sobolev embedding, which is strongly continuous since $2 < 2q < 6$ (see [51, Proposition 1.1]). Now B factorizes in the form

$$X \overset{i}{\hookrightarrow} L^{2q} \longrightarrow L^q \xrightarrow{c_*} L^{2p}(\mathbb{R}^3) \xrightarrow{b} \mathcal{L}(L^{2q}, L^{(2q)'}) \xrightarrow{j} \mathcal{L}(X, X^*), \tag{10.2}$$

Here the second arrow is given by the continuous map $u \mapsto u^2$, and the *linear* maps b and j are defined by

$$b(u)v := uv$$

and

$$j(u) = i_2^* u i_2$$

($i^* : L^{(2q)'} \to X^*$ denoting the dual of i). By Hölder's inequality, b is well defined and continuous. Thus the factorization shows that B is strongly continuous and $B(u) \in \mathcal{L}(X, X^*)$ is a *compact* linear operator for every $u \in X$.
(ii) Since B is continuous, it suffices to prove the assertion for

$$u, v \in \mathcal{C} := \{w \in C_0^\infty(\mathbb{R}^3) \mid w \text{ radially symmetric}\}.$$

Note that (V_2) implies that

$$(\xi_1|\xi_2)_* := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \xi_1(x)\xi_2(y)V(x-y)\,dx\,dy$$

defines a semidefinite scalar product on $\mathcal{C}$. The associated Cauchy-Schwarz inequality yields

$$\langle B(u)v, v\rangle = (u^2, v^2)_* \le \sqrt{(u^2, u^2)_*}\sqrt{(v^2, v^2)_*} = \langle B(u)u, u\rangle^{\frac{1}{2}}\langle B(v)v, v\rangle^{\frac{1}{2}},$$

as claimed.
(iii) By (ii) there holds

$$\begin{aligned} 2\varphi(v) - 2\varphi(u) &= \frac{1}{2}(\langle B(v)v, v\rangle - \langle B(u)u, u\rangle) \\ &\ge \langle B(u)u, u\rangle^{\frac{1}{2}}\langle B(v)v, v\rangle^{\frac{1}{2}} - \langle B(u)u, u\rangle \\ &\ge \langle B(u)v, v\rangle - \langle B(u)u, u\rangle. \end{aligned}$$

Hence (CC) is satisfied by φ and B. $\square$

The following Lemma provides more detailed information on the 'convolution operator' $u \mapsto u^2 * V$.

Lemma 10.2. *(a) There is a constant $c_1 > 0$ such that*

$$|(u^2 * V)(x)| \leq c_1 \|u\|_X^2 \left(1 + \frac{1}{|x|}\right) \tag{10.3}$$

for every $x \in \mathbb{R}^3 \setminus \{0\}$, $u \in X$.

(b) If $u \in X \cap L^\infty(\mathbb{R}^3)$, then $u^2 * V \in L^\infty(\mathbb{R}^3)$.

(c) If $u \in X \cap C^1(\mathbb{R}^3)$ and $\frac{\partial}{\partial r} u \in X \cap L^\infty(\mathbb{R}^3)$, then $u^2 * V \in C^1(\mathbb{R}^3)$.

Proof. We may write $V = V_1 + V_2$ with radially symmetric functions $V_1 \in L^\infty(\mathbb{R}^3)$, $V_2 \in L^1(\mathbb{R}^3)$. Then

$$|(u^2 * V_1)(x)| \leq \|V_1\|_\infty \|u\|_2^2 \leq \|V_1\|_\infty \|u\|_X^2,$$

for all x, and by (14.18) there holds for $x \neq 0$

$$\begin{aligned} |(u^2 * V_2)(x)| &\leq \int_{\mathbb{R}^3} u^2(x-y)|V_2(y)|\,dy \\ &\leq C^2 \|u\|_X^2 \int_{\mathbb{R}^3} \frac{1}{|x-y|}|V_2(y)|\,dy \\ &= C^2 \|u\|_X^2 \int_{\mathbb{R}^3} \frac{1}{\max\{|x|,|y|\}}|V_2(y)|\,dy \qquad (10.4) \\ &\leq \frac{C^2\|u\|_X^2}{|x|}\|V_2\|_1. \end{aligned}$$

Here (10.4) is established by carrying out the spherical integration and using the radial symmetry of $|V_2(\cdot)|$. We conclude that (a) holds with $c_1 := \|V_1\|_\infty + C^2\|V_2\|_1$. Moreover, if in addition $u \in L^\infty(\mathbb{R}^3)$, then $|(u^2 * V_2)(x)| \leq \|u\|_\infty^2 \|V_2\|_1$ for all x, hence $u^2 * V \in L^\infty$, as claimed in (b). Assertion (c) follows by similar arguments, since the assumptions allow to 'differentiate under the integral'. □

We point out that Lemma 10.1(i) and Lemma 10.2(a) in particular ensure that for every $u \in X$ the function $u^2 * V$ is a radial W-admissible potential, which is required to apply the nodal criteria of Sec. 14.3.1.
Moreover, Lemma 10.1 implies the validity of (H1)-(H5) and (CC), as usual. Hence, defining A_0 as in (9.2), we may treat equation (CH) in the framework of Section 7.2.2. In particular we refer freely to the notations $A(u)$, $\mu_n(u)$, etc.. We are interested in solutions of $(SC)_n$, since we expect them to carry nodal information. Indeed:

Lemma 10.3. *If $u \in X$ is a solution of problem $(SC)_n$ for some $n \in \mathbb{N}$, then $u \in W^{2,2}(\mathbb{R}^3) \cap C^2(\mathbb{R}^3)$, and u is a classical solution of (CH). Moreover, u has precisely n nodal domains.*

Proof. Clearly u is a weak solution of (CH). Since $u^2 * V \in L^q_{loc}(\mathbb{R}^3)$ for some $q > \frac{3}{2}$ by Lemma 10.2(a), we infer $u \in C(\mathbb{R}^3)$ from Lemma 14.1. In view of 14.18 we deduce $u \in L^\infty(\mathbb{R}^3)$, hence $\lambda u + (u^2 * V)u \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ by Lemma 10.2(b). Now standard elliptic regularity yields $u \in W^{2,2}(\mathbb{R}^3) \cap W^{2,p}_{loc}(\mathbb{R}^3)$ for every $1 \leq p < \infty$, and in particular $u \in C^1(\mathbb{R}^3)$ by Sobolev embeddings. Moreover, $u \in W^{2,2}(\mathbb{R}^3)$ implies that the radial derivative $\frac{\partial}{\partial r}u$ is an element of X. In view of (14.18) we infer that $\frac{\partial}{\partial r}u \in L^\infty(\mathbb{R}^3)$, and therefore Lemma 10.2(c) yields $\lambda u + (u^2 * V)u \in C^1(\mathbb{R}^3)$. Applying elliptic regularity once more, we finally establish $u \in C^2(\mathbb{R}^3)$, moreover u solves (CH) classically.
The nodal property now follows from Theorem 14.8. □

Remark 10.4. In [50, p. 1064] it is asserted that $u \in C^\infty(\mathbb{R}^3)$ for every weak solution $u \in X$ of (CH).

In our main theorem below we relate the existence of solutions with prescribed nodal properties to minimax values of the functional

$$\psi_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^3} [|\nabla u|^2 - \lambda u^2] - \varphi(u) \qquad (u \in X)$$

on the Nehari set

$$\mathcal{N} := \left\{ u \in X \mid \int_{\mathbb{R}^3} [|\nabla u|^2 - \lambda u^2] = \int_{\mathbb{R}^3} (u^2 * V)u^2 \right\} - \{0\}.$$

As usual we focus on the Ljusternik-Schnirelman levels $c_n := c_n(\psi_\lambda, \mathcal{N})$.

Theorem 10.5. *Fix $\lambda < 0$. Then*

(a) $c_n < c_{n+1}$ *for all* $n \in \mathbb{N}$.

(b) There exist classical radial solutions $u_n \in X$, $n \in \mathbb{N}$ *of (CH) such that* $\psi_\lambda(u_n) = c_n$ *and* u_n *has precisely* n *nodal domains.*

(c) If $u \in X$ *is a weak radial solution of (CH) satisfying* $\psi_\lambda(u) < c_{n+1}$ *for some* $n \in \mathbb{N}$*, then* u *has* at most n *nodal domains.*

Remark 10.6. The mere existence of infinitely many radial solutions to (CH) has been established by Lions [50], whereas nodal information on solutions is a basically new feature. In view of the nonlocal nature of (CH) all *local* reasoning fails, and therefore any techniques relying on ODE dynamics (cf. for instance [41]) and also local variational methods (see [9] and [21]) do not apply here.

The proof of Theorem 10.5 is based on abstract results stated in Section 7.2.2. We start with the following observation:

Lemma 10.7. *There holds* $\gamma(\mathcal{N}) = \infty$.

Proof. We prove this by constructing for given $m \in \mathbb{N}$ an m-dimensional subspace $V_m \subset X$ such that for every $v \in V_m \setminus \{0\}$ there holds

$$\langle B(v)v, v\rangle > 0, \tag{10.5}$$

which implies $\gamma(\mathcal{N} \cap V_m) = m$ by Lemma 7.10. To this end, choose m linearly independent functions $w_i : [0, \infty[\to \mathbb{R}$ which are real analytic on $]0, \infty[$, rapidly decreasing at infinity and satisfying $w_i'(0) = 0$ (e.g. one can take linear combinations of Hermite functions). Define $v_i \in X$ by $v_i(x) := w_i(|x|)$ and define $V_m \subset X$ as the span of the v_i. Since any $v \in V_m \setminus \{0\}$ can only vanish on a set of measure zero, there holds (10.5) by virtue of (V3). This proves the Lemma. $\square$

Next we remark that conditions $(CC)_1$ and $(CC)_2$ are obvious features of this special nonlinearity (with $\eta = 4$). Moreover, since $\lambda < 0$, we may pass to an equivalent scalar product $(\cdot|\cdot)_\lambda$ on X given by

$$(u|v)_\lambda := \int_{\mathbb{R}^3} \nabla u \nabla v - \lambda \int_{\mathbb{R}^3} uv.$$

Denoting (in accordance to Section 7.2.2) by $J_\lambda : X \to X^*$ the canonical isometric isomorphism with respect to this scalar product, we consider the symmetric operators $G(u) := J_\lambda^{-1} B(u) \in \mathcal{L}(X)$, and we use the corresponding notations $\sigma_n(u)$, $\tilde{Q}_n(u)$ and $\tilde{\rho}(u)$ for $u \in X$ as defined in this section. Next observe that condition $(CC)_3$ follows directly from Lemma 10.1(ii) and the following Lemma.

Lemma 10.8. *Let $u \in X$, and consider a finite sum $v = \sum_{i=1}^{n} v_i$ of (nonzero) eigenfunctions v_i of $G(u)$ corresponding to pairwise different* positive *eigenvalues ξ_i.*
Then $\langle B(u)v, v\rangle \neq 0$.

Proof. Every v_i is a weak solution to

$$-\Delta v_i - \lambda v_i = \frac{1}{\xi_i}(u^2 * V)(x) v_i \qquad (i = 1, ..., n). \tag{10.6}$$

Recalling that $u^2 * V \in L^q_{loc}(\mathbb{R}^3)$ for some $q > \frac{3}{2}$, we infer that v_i is continuous by Lemma 14.1. Hence $v_i \in W^{2,q}_{loc}(\mathbb{R}^3)$ for every i by elliptic regularity, and (10.6) holds pointwise a.e.. Now suppose in contradiction that $\langle B(u)v, v\rangle = 0$. Then $v(x) = 0$ a.e. on

$$\mathcal{M} := \{x \in \mathbb{R}^3 \mid (u^2 * V)(x) > 0\}$$

and by virtue of [31, Lemma 7.7] we deduce that

$$0 = -\Delta v(x) - \lambda v(x) = (u^2 * V)(x) \sum_{i=1}^{n} \frac{1}{\xi_i} v_i(x)$$

for a.e. $x \in \mathcal{M}$, hence $\sum_{i=1}^{n} \frac{1}{\xi_i} v_i(x) = 0$ for a.e. $x \in \mathcal{M}$. Iterating this argument, we derive for all $j \in \mathbb{N}$ the relation

$$\sum_{i=1}^{n} \frac{1}{\xi_i^j} v_i(x) = 0 \qquad \text{for a.e. } x \in \mathcal{M}.$$

This clearly yields $v_i(x) = 0$ for a.e. $x \in \mathcal{M}$, $i = 1, ..., n$. In view of (10.6) we conclude

$$-\Delta v_i(x) - \lambda v_i(x) = 0 \qquad \text{for a.e. } x \in \mathbb{R}^3,\ i = 1, ..., n$$

Hence $v_i = 0$ for all i, which contradicts the assumptions. $\square$

We proceed by exploring the spectral fixed point set $K_{\mathcal{N}} := \{u \in \mathcal{N} \mid \tilde{Q}_n(u)u = u\}$. To this end, we denote by $\hat{P}_i(u)$ the eigenprojection associated the operator $G(u)$ and the eigenvalue $\sigma_n(u)$ (cf. Lemma 4.9). The following assertions should be compared with Lemma 9.8.

Lemma 10.9.

(i) For every $u \in \mathcal{N}$, $i \in \mathbb{N}$ *there holds* $\sigma_i(u) > 0$, *and* $\sigma_i(u)$ *is a nondegenerate eigenvalue of* $G(u)$.

(ii) For every $i \in \mathbb{N}$ *the map*

$$\begin{array}{rcl} \mathcal{N} & \to & \mathbb{R} \\ u & \mapsto & \hat{P}_i(u)u|_{x=0} \end{array} \tag{10.7}$$

is odd and continuous.

Proof. (i) Since for every $u \in \mathcal{N}$ the function $u^2 * V$ does not vanish identically, we infer $\sigma_i(u) > 0$ by testing with an i-dimensional subspace $V_i \subset X$ as constructed in the proof of Lemma 10.7. Moreover, $\sigma_i(u)$ is nondegenerate in view of Theorem 14.9.
(ii) Oddness is clear. To prove continuity, consider a sequence $(u_j) \subset D$ such that $u_j \to u \in D$. Put $h_j := \hat{P}_n(u_j)u_j$ and $h := \hat{P}_n(u)u$. Since $\sigma_{i-1}(u) > \sigma_i(u) > \sigma_{i+1}(u)$ by (i), Corollary 4.9 implies that $h_j \to h$ in X. By Lemma 10.2(a) and the continuity of the function σ_n we infer that the set

$$\mathcal{V} := \{\frac{1}{\sigma_n} u_j^2 * V - \lambda \mid j \in \mathbb{N}\} \cup \{\frac{1}{\sigma_n} u^2 * V - \lambda \mid j \in \mathbb{N}\}$$

is a bounded subset of $L^q(B_2(0))$ for some $q > \frac{3}{2}$, whereas $\mathcal{U} := \{h, h_j \mid j \in \mathbb{N}\}$ is a compact subset of $W^{1,2}(\mathbb{R}^3)$. Applying Lemma 14.1(b) to $\Omega = B_2(0)$ and $\Omega' = B_1(0)$, we in particular infer $h_j(0) \to h(0)$, as desired. $\square$

Corollary 10.10. $\gamma^*(K_{\mathcal{N}}) \le n - 1$.

Proof. Define a map $g : \mathcal{N} \to \mathbb{R}^{n-1}$ by

$$g(u) = (P_1(u)u|_{x=0}, ..., P_{n-1}(u)u|_{x=0}).$$

Then g is odd and continuous by Lemma 10.9(b), moreover

$$g(u) = 0 \qquad \Longleftrightarrow \qquad P_i(u)u \quad \text{for } i = 1, ..., n-1$$

by Lemma 14.3. Hence $g(u) \ne 0$ whenever $u \in \mathcal{N} \setminus K_{\mathcal{N}}$, and from this we conclude $\gamma^*(K_{\mathcal{N}}) \le n - 1$. $\square$

We now easily complete the proof of

Proof of Theorem 10.5. We start with (b), letting $n \in \mathbb{N}$ be given. Note that the assumptions of Corollary 7.21 are satisfied in view of Lemma 10.7 and Corollary 10.10. Hence we infer the existence of a solution $u_n \in K_{\mathcal{N}} \subset \mathcal{N}$ of $(SC)_n$, which by virtue of Lemma 10.3 has the properties asserted in (b).
Moreover, suppose that u is a solution of (CH) having more than n nodal domains. Then Theorem 14.9 yields $\tilde{\rho}_u(u) = 1 = \sigma_{n+j}(u)$ for some $j \in \mathbb{N}$, whereas $\sigma_{n+j}(u) < \sigma_n(u)$. By Proposition 7.19 we infer that $\psi_\lambda(u) \geq c_{n+j}(\mathcal{N}) \geq c_{n+1}(\mathcal{N})$ *as well as* $\psi_\lambda(u) > c_n(\mathcal{N})$. Hence, every solution u of (EF) satisfying *either* $\psi_\lambda(u) \leq c_n(\mathcal{N})$ *or* $\psi_\lambda(u) < c_{n+1}(\mathcal{N})$ has at most n nodal domains. Hence (c) holds in particular, and combined with (b) this also forces (a). □

10.1 Remarks on the normalized case

We now turn to the question if, for prescribed $R > 0$ and $n \in \mathbb{N}$, there exists a radial solution (u, λ) of (CH) such that

$$\|u\|_2 = R \tag{10.8}$$

and u has precisely n nodal domains. Note that the case $R = 1$ is of special interest for applications in quantum mechanical models involving many bosons (cf. [30]).
We first remark that if V has a special *homogeneity*, then equation (CH) has nice scaling properties which we summarize in the following lemma.

Lemma 10.11. *(cf. [50])*
Suppose that $V(x) = C|x|^{-\alpha}$ with $C > 0$ and $\alpha \in]0, 3[$, and consider a solution $(u, \lambda) \in X \times \mathbb{R}^-$ of (CH). Then for every $\beta > 0$ the pair $(u_\beta, \beta^2\lambda)$ is a solution of (CH) as well, where $u_\beta \in X$ is defined by $u_\beta(x) = \beta^{\frac{5-\alpha}{2}} u(\beta x)$.

Combining this observation with Theorem 10.5(b), we establish L^2-bifurcation of infinitely many continuous branches (classified by the number of nodal domains) from the point $(u_0 = 0, \lambda_0 = 0)$ *in case* $\alpha \in]0, 2[$, whereas in case $\alpha \in]2, 3[$ those branches emanate (in a vague sense) from $(u_0 = 0, -\infty)$. In either case all branches cross the sphere $S_R := \{u \in X \mid \|u\|_2 = R\}$ for every $R > 0$, hence the following is an immediate consequence:

Corollary 10.12. *Let $R > 0$ and suppose that $V(x) = C|x|^{-\alpha}$ with $C > 0$ and $\alpha \in]0, 3[$, $\alpha \neq 2$. Then there exists a solution (u, λ) of (CH) such that $\|u\|_2 = R$ and u has precisely n nodal domains.*

To deal with inhomogeneous potentials V, we have to strengthen our assumptions. In the sequel we will impose (V_2), (V_3) and the following stronger version of (V_1):

(V_1') $V \in L^{p_1}(\mathbb{R}^3) + L^{p_2}(\mathbb{R}^3)$ for some $p_1,\ p_2 \in]\frac{3}{2}, \infty[$.

Without loss we assume that $p_1 < p_2$. Moreover we require the condition

$$\lim_{r \to \infty} r^2 V(r) = \infty \tag{10.9}$$

Note that these assumptions are in particular satisfied for V behaving essentially like a Coulomb potential. Now consider the functional

$$\psi(u) := \int_{\mathbb{R}^3} |\nabla u|^2 - \varphi(u) \qquad (u \in X)$$

and the Ljusternik-Schnirelman levels $c_n := c_n(\psi, S_R)$ on the sphere

$$S_R := \{u \in X \mid \|u\|_2 = R\}$$

for $R > 0$. Then we have the following analog of Theorem 9.4.

Theorem 10.13. *Fix $R > 0$. Then*

(a) $c_n < c_{n+1}$ *for all* $n \in \mathbb{N}$.

(b) There exist classical radial solutions $(u_n, \lambda_n) \in S_R \times \mathbb{R}^-$, $n \in \mathbb{N}$ *of (CH) such that* $\psi(u_n) = c_n$ *and* u_n *has precisely* n *nodal domains.*

(c) If $(u, \lambda) \in S_R \times \mathbb{R}^-$ *is a weak radial solution of (CH) with* $\psi(u) < c_{n+1}$ *for some* $n \in \mathbb{N}$, *then* u *has* at most n *nodal domains.*

Remark 10.14. As in the unconstrained case, Lions [50] established the mere existence of infinitely many *normalized* solutions, whereas the nodal information supplied by Theorem 10.13 is a basically new feature.

Proof of Theorem 10.13. First observe that condition (BB) from Section 7.1 is satisfied. Indeed, writing $V = V_1 + V_2$ with $V_1 \in L^{p_1}(\mathbb{R}^3)$ and $V_2 \in L^{p_2}(\mathbb{R}^3)$, we deduce

$$\begin{aligned} \langle B(u)u, u\rangle &\leq \|V_1 * u^2\|_{2p_1} \|u^2\|_{\frac{2p_1}{2p_1-1}} + \|V_2 * u^2\|_{2p_2} \|u^2\|_{\frac{2p_2}{2p_2-1}} \\ &\leq \|V_1\|_{p_1} \|u^2\|^2_{\frac{2p_1}{2p_1-1}} + \|V_2\|_{p_2} \|u^2\|^2_{\frac{2p_2}{2p_2-1}} \\ &= \|V_1\|_{p_1} \|u\|^4_{\frac{4p_1}{2p_1-1}} + \|V_2\|_{p_2} \|u\|^4_{\frac{4p_2}{2p_2-1}} \end{aligned}$$

from convolution inequalities, noting that $2 < \frac{4p_i}{2p_i-1} < 3$ for $i = 1, 2$ by (V_1'). Therefore 9.6 yields constants $C_1, C_2 > 0$ such that for every $u \in S_R$ there holds

$$\langle B(u)u, u\rangle \leq C_1(\|V_1\|_{p_1} + \|V_2\|_{p_2}) \|\nabla u\|_2^{4\theta} + C_2$$

with $0 < \theta := N(\frac{1}{2} - \frac{2p_1-1}{4p_1}) < \frac{1}{2}$. Hence (BB) follows in view of Remark 7.1.
As a consequence (cf. Lemma 7.2), ψ is bounded from below on S_R, and hence $c_n > \infty$ for every $n \in \mathbb{N}$. Moreover, in [50, Corollary 3] it is shown that (10.9) implies

$$c_n < 0 \qquad \text{for every } n \in \mathbb{N}.$$

Hence we may deduce Theorem 10.13 from Theorem 7.8 and Proposition 7.4 in precisely the same way as done in Section 9.1. Indeed, note that proving $\gamma^*(K^-) \leq n - 1$ only requires that the regularity and uniqueness criteria from Sections 14.1 and 14.2 are applicable. In other words, we require that the functions $u^2 * V$ are radial W-admissible potentials belonging to $L^q_{loc}(\mathbb{R}^3)$ for some $q > \frac{3}{2}$. This however has already been ensured in the previous subsection. $\square$

Chapter 11

Generalized Emden-Fowler equations

We consider the semilinear elliptic equation

$$(EF) \qquad -\Delta u = f(x,|u|)u \qquad u \in D^{1,2}(\mathbb{R}^N)$$

for $N \geq 3$. We assume that $f : \mathbb{R}^N \times [0,\infty[\to \mathbb{R}$ is a Caratheodory function which satisfies

($\mathcal{F}_1$) For a.e. $x \in \mathbb{R}^N$ there holds $f(x,0) = 0$, moreover $f(x,\cdot)$ is nondecreasing on $[0,\infty[$ and strictly increasing once it takes positive values.

($\mathcal{F}_2$) There is $\eta > 2$ such that $0 \leq \eta \int_0^t f(x,s)s\,ds \leq f(x,t)t^2$ for $t \geq 0$.

($\mathcal{F}_3$) $f \not\equiv 0$, i.e., the set

$$I(f) = \{x \in \mathbb{R}^N \mid \exists\, t > 0 \text{ s.t. } f(x,t) > 0\}$$

has positive measure.

11.1 The radial case

In this section we assume that f is radially symmetric, i.e. it can be written in the form $f(x,t) = \mathfrak{f}(|x|,t)$. Moreover we assume in addition to ($\mathcal{F}_1$)-($\mathcal{F}_3$) that

(A) $\mathfrak{f}(\cdot,t) \in L^\infty_{loc}(]0,\infty[)$ for every $t > 0$, and there is $\beta > 0$ such that

$$\frac{\mathfrak{f}(r,t)}{t^\beta} = o(r^\alpha) \begin{cases} r \to 0 \\ r \to \infty \end{cases}$$

uniformly in $t > 0$ with $\alpha = \frac{\beta}{2}(N-2) - 2$.

Note that condition (A) admits both sub- *and* supercritical nonlinearities, depending on the behavior in the radial space variable. We restrict our attention to radially symmetric solutions u of (EF), and we denote by X the Hilbert space consisting of radially symmetric functions in $D^{1,2}(\mathbb{R}^N)$. The following is a mere reformulation of Lemma 14.18.

Lemma 11.1. *Define*

$$\langle B(u)v, w\rangle := \int_{\mathbb{R}^N} \mathfrak{f}(|x|, |u(x)|)v(x)w(x)\ dx \tag{11.1}$$

for $u, v, w \in X$. Then:

(a) B is a well defined strongly continuous map $X \to \mathcal{L}(X, X^*)$. Moreover, $B(u) \in \mathcal{L}(X, X^*)$ is compact for every $u \in X$.

(b) B satisfies (CC) with respect to the functional $\varphi : X \to \mathbb{R}$ given by

$$\varphi(u) = \int_{\mathbb{R}^N} \int_0^{|u(x)|} \mathfrak{f}(|x|, t)t\, dt\, dx.$$

As a consequence, B and φ satisfy (H1)-(H5) and (CC), and we may consider $G(u)$, $\sigma_n(u)$, $\tilde{Q}_n(u)$, $\tilde{\rho}_u$ as defined in Section 7.2.1 for $u \in X$. The next lemma asserts that nodal solutions of (EF) arise as solutions of the spectral characterization problem posed by the relations (7.22) and (7.23).

Lemma 11.2. *If $u \in X$ satisfies (7.22) and (7.23) for some $n \in \mathbb{N}$, then u solves (EF) weakly. Moreover, $u \in \cap C(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$, and u has precisely n nodal domains.*

Proof. The relations (7.22) and (7.23) in particular imply that

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi = \int_{\mathbb{R}^N} f(\cdot, |u|)u\varphi$$

for all $\varphi \in X$. Using the radial symmetry of f, we infer that this also holds for every $\varphi \in C_0^\infty(\mathbb{R}^N)$, hence u is a weak solution of (EF). Since $f(\cdot, |u(\cdot)|) \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$ by (A) and (14.18), elliptic regularity yields $u \in W^{2,q}_{loc}(\mathbb{R}^N \setminus \{0\})$ for every $1 < q < \infty$, and therefore $u \in C^1(\mathbb{R}^N \setminus \{0\})$ by virtue of Sobolev embeddings. Now pick numbers $C, r > 0$ such that

$$\mathfrak{f}(|x|, |u(x)|) \le C|x|^\alpha |u(x)|^\beta \qquad \text{for } x \in B_r(0). \tag{11.2}$$

Using again (14.18), we obtain for $1 < p < \frac{N}{2}$ the relation

$$\begin{aligned}
\int_{B_r(0)} |f(\cdot, |u(\cdot)|)|^p &\le C^p \int_{B_1(0)} |x|^{p\alpha}|u(x)|^{p\beta} \\
&\le \tilde{C}\|u\|_X^{p\beta_1} \int_{B_r(0)} |x|^{p[\alpha - \frac{N-2}{2}\beta]}\, dx \\
&= \tilde{C}\|u\|_X^{p\beta_1} \int_{B_r(0)} |x|^{-2p}\, dx \\
&< \infty,
\end{aligned}$$

hence $f(\cdot, |u(\cdot)|) \in L^p_{loc}(\mathbb{R}^N)$, and therefore $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ by elliptic regularity. This yields $u \in L^s_{loc}(\mathbb{R}^N)$ for every $1 \le s < \infty$ by Sobolev embeddings. Combining this with (11.2) and recalling that $\alpha < 2$, we infer that $f(\cdot, |u(\cdot)|) \in L^q_{loc}(\mathbb{R}^N)$ for some $q > \frac{N}{2}$. Hence $u \in W^{2,q}_{loc}(\mathbb{R}^N)$ again by elliptic regularity, and u is continuous by Sobolev embeddings.

The nodal characterization now follows from Theorem 14.15, applied to $V := f(\cdot, |u(\cdot)|)$. Indeed, note that V satisfies (14.15) by (A) and the continuity of u, and combined with Lemma 11.1(a) this ensures that V is a radial D-admissible potential, as required in this theorem. $\square$

As in Chapter 10, the nodal structure of solutions to (EF) is related to the Ljusternik-Schnirelman levels $c_n := c_n(\psi, \mathcal{N})$ of the energy functional ψ on the Nehari set $\mathcal{N}$. We recall that in the present situation ψ and $\mathcal{N}$ are given as follows (cf. Section 7.2.1):

$$\begin{aligned} \psi(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \varphi(u) \qquad (u \in X), \\ \mathcal{N} &= \left\{ u \in X \setminus \{0\} \mid \int_{\mathbb{R}^N} |\nabla u|^2 = \int_{\mathbb{R}^N} \mathfrak{f}(|x|, |u|)u^2 \right\}. \end{aligned}$$

Now our main theorem has the same form as in Chapter 10.

Theorem 11.3.

(a) $c_n < c_{n+1}$ *for all* $n \in \mathbb{N}$.

(b) There exist (radial) solutions $u_n \in X$, $n \in \mathbb{N}$ *of (EF) s. t.* $\psi(u_n) = c_n$ *and* u_n *has precisely* n *nodal domains.*

(c) If $u \in X$ *is a solution of (EF) with* $\psi(u) < c_{n+1}$ *for some* $n \in \mathbb{N}$*, then* u *has* at most n *nodal domains.*

Remark 11.4. In view of the mere existence of nodal solutions, Theorem 11.3(b) improves results of Naito [56] and Chabrowski [17], who have considered the following (special) form of (EF):

$$-\Delta u - q(|x|)h(u) = 0 \qquad (u \in D^{1,2}(\mathbb{R}^N)). \tag{11.3}$$

In [56] the attention is restricted to homogeneous nonlinearities $h(u) = |u|^{\beta-1}u$, where $\beta > 1$ and

$$q \in C[0,\infty[\cap C^1(]0,\infty[) \quad q(r) > 0 \text{ for } r > 0 \tag{11.4}$$

as well as

$$\liminf_{r\to 0} \frac{rq'(r)}{q(r)} > -\frac{N+2-\beta(N-2)}{2} \tag{11.5}$$

$$\limsup_{r\to\infty} \frac{rq'(r)}{q(r)} < -\frac{N+2-\beta(N-2)}{2}. \tag{11.6}$$

Using ODE-techniques, Naito showed that if (11.4)-(11.6) hold, then for given $n \in \mathbb{N}$ there exists a radial solution u with precisely n nodal domains (cf. [56, Theorem 7]). Note that (11.4)-(11.6) imply ($\mathcal{F}_1$), ($\mathcal{F}_2$), ($\mathcal{F}_3$) and (A). On the other hand, if a nonlinearity of the form $q(|x|)|u|^{\beta-1}u, \beta > 1$ satisfies (A), we only require

$$q \in L^\infty_{loc}(]0,\infty[) \ , \ q \geq 0, \ q \not\equiv 0 \tag{11.7}$$

instead of (11.4) to ensure ($\mathcal{F}_1$), ($\mathcal{F}_2$), ($\mathcal{F}_3$). In particular we cover cases where q is singular at the origin, or where, for instance, supp q is contained in a very small interval $I \subset]0,\infty[$. In the latter case we conclude that the nodal solutions $u_n = u_n(r)$ found by Theorem 11.3 only change sign in I. This follows since u_n satisfies

$$(r^{N-1}u_n')' = 0, \quad u_n'(0) = 0 \quad \lim_{r\to\infty} u_n(r) = 0,$$

on $[0, \infty[\setminus I$, hence it can not change sign in this region.
Furthermore, we admit non-homogeneous nonlinearities h in (11.3). These are also considered in [17], but there it is assumed that h at least is not supercritical, and q has to be continuous on $[0, \infty[$ with $q(r) > 0$ for all positive r as well as $q(0) = 0$ and $\lim_{r \to \infty} q(r) = 0$. Note moreover that the methods of [17] do not carry over to growth conditions of the form (11.7). Nevertheless we also remark that in [17] no oddness of h is assumed, and in [56] also the case $0 < \beta < 1$ is considered, which can not be handled by our method.
(b) The growth rate imposed by (A) or equivalently by (11.5), (11.6) is optimal in a certain sense. Indeed, for the case $h(u) = |u|^{\beta-1}u,\ \beta > 1$, Kusano and Naito [43],[44] proved that if

$$\frac{rq'(r)}{q(r)} \geq -\frac{N+2-\beta(N-2)}{2} \quad r > 0,$$

then any classical radial solution of (11.3) does not change sign. Moreover they show that if

$$\frac{rq'(r)}{q(r)} < -\frac{N+2-\beta(N-2)}{2} \quad r > 0,$$

any classical radial solution of (11.3) has an infinite number of nodal domains.

To prove Theorem 11.3 we use the tools provided in Section 7.2.1, checking first that conditions $(CC)_1 - (CC)_3$ are fulfilled. Indeed, $(CC)_1$ and $(CC)_2$ are immediate consequences of ($\mathcal{F}_1$) and ($\mathcal{F}_2$). To ensure $(CC)_3$, suppose that $u, v \in X$ are such that v is a finite sum of eigenfunctions of $G(u)$ corresponding to positive eigenvalues. Then there is some $c > 0$ such that

$$\langle B(u)v, v\rangle = (G(u)v|v)_X = c(v|v)_X > 0,$$

which implies that $I(f) \cap \mathrm{supp}(v)$ is a set of positive measure. Hence $\langle B(tv)v, v\rangle > 0$ for $t > 0$ large enough, as required for $(CC)_3$.
Although the nonlinearity does not vanish identically due to $(\mathcal{F}_3)$, it may vanish on large subsets of $\mathbb{R}^N \times \mathbb{R}$. Hence the Nehari manifold $\mathcal{N}$ is not spherelike in general. Nevertheless there holds:

Lemma 11.5. $\gamma(\mathcal{N}) = \infty$

Proof. For given $m \in \mathbb{N}$ define an m-dimensional define the subspace $V_m \subset X$ just as in the proof of Lemma 10.7. Since $I(f)$ is a set of positive measure, any $v \in V_m \setminus \{0\}$ cannot vanish on $I(f)$, and we conclude that there is a number $t > 0$ such that

$$\langle B(tv)v, v\rangle > 0. \tag{11.8}$$

This implies $\gamma(\mathcal{N} \cap V_m) = m$ by virtue of Lemma 7.10, and since m was arbitrary, we conclude $\gamma(\mathcal{N}) = \infty$, as claimed. □

Now fix $n \in \mathbb{N}$ and put $K_{\mathcal{N}} := \{u \in \mathcal{N} \,|\tilde{Q}_n(u)u = u\}$. In order to apply Corollary 7.17, we need to ensure

$$\gamma^*(K_{\mathcal{N}}) \leq n - 1. \tag{11.9}$$

For this we define $\hat{P}_i(u)$ as the spectral projection associated the operator $G(u)$ and the eigenvalue $\sigma_i(u)$. One might try to establish (11.9) as in Chapter 10, using the maps $u \mapsto \hat{P}_i(u)u|_{x=0}$. However, assumption (A) does *not* guarantee that eigenfunctions of $G(u)$ are continuous in $x = 0$, which is an essential requirement to define these maps properly.
We will circumvent this problem in the following way: If $A \subset \mathcal{N} \setminus K_{\mathcal{N}}$ is an arbitrary odd and symmetric subset, we first deform A carefully such that $A \subset L^\infty(\mathbb{R}^N)$. Then we show $\gamma(A) \leq n - 1$ using the maps proposed above.
As a helpful tool we introduce for arbitrary $c > 0$ the 'cutoff map' $j_c : X \to X$, which is defined by

$$j_c(u)(x) := \min\{c, \max\{-c, u(x)\}\}.$$

Indeed $j_c(u) \in X$ for every $u \in X$, as can be deduced from [48, p. 54] for instance. Moreover, (14.18) implies that

$$\lim_{c \to \infty} j_c(u) \to u \quad \text{in } X \tag{11.10}$$

for every $u \in X$. Now we are ready to prove

Lemma 11.6. $\gamma^*(K_{\mathcal{N}}) \leq n - 1$.

Proof. Consider an arbitrary closed and symmetric subset $A \subset \mathcal{N} \setminus K_{\mathcal{N}}$. We have to show $\gamma(A) \leq n - 1$, i.e. there is a map $g : A \to \mathbb{R}^{n-1}$ such that

$$g \quad \text{is odd and continuous, and} \quad g(u) \neq 0 \quad \text{for every } u \in A. \tag{11.11}$$

First note that $u \in A \subset \mathcal{N}$ implies that $f(\cdot, |u(\cdot)|) \not\equiv 0$ does not vanish identically on $\mathbb{R}^N$. Moreover, the set $\mathcal{Q} := \{u \in X \mid \tilde{Q}_n(u)u = u\}$ is a closed subset of X, which can be seen by the same reasoning as in the proof of Lemma 7.13. Hence, using (11.10) and (14.18) we find for every $u \in A$ a symmetric neighborhood $U_u \subset A$ and a constant $c > 0$ such that

$$j_{c'}(v) \notin \mathcal{Q} \qquad \text{and} \qquad f(\cdot, |j_{c'}(v)(\cdot)|) \not\equiv 0$$

for every $v \in U_u$ and $c' \geq c$. Using a partition of unity consisting of even functions and subordinated to the thus-defined covering of A, we easily construct a continuous and even function $c : A \to]0, \infty[$ such that for every $u \in A$ there holds

(i) $j_{c(u)}(u) \notin \mathcal{Q}$

(ii) $f(\cdot, |j_{c(u)}(u)(\cdot)|) \not\equiv 0$.

This gives rise to an odd and continuous function $j : A \to X$ defined by

$$u \overset{j}{\longmapsto} j_{c(u)}(u) \qquad (u \in A)$$

Indeed, in view of [48, p.54, Cor. A.6], the continuity of j is evident. Note that $\sigma_i(v) > 0$ for every $v \in j(A)$, $i \in \mathbb{N}$, which in view of (ii) can be derived by testing with an i-dimensional subspace $V_i \subset X$ defined as in the proof of Lemma 10.7. Moreover, every eigenfunction Ψ of $G(v)$ associated with $\sigma_i(v)$ weakly solves

$$-\Delta\Psi = \frac{1}{\sigma_i(v)}\mathfrak{f}(|x|,|v(x)|)\Psi,$$

whereas $\lim\limits_{x\to 0}\frac{\mathfrak{f}(|x|,|v(x)|)}{|x|^\alpha} = 0$ in view of (A). Hence Lemma 14.3 yields that Ψ is uniquely determined up to a constant, which implies that $\sigma_i(v)$ is nondegenerate. Next we show:

(*) For every $i \in \mathbb{N}$ the map

$$\begin{aligned} \hat{g}_i : j(A) &\to \mathbb{R} \\ v &\mapsto \hat{P}_i(v)v|_{x=0} \end{aligned}$$

is continuous.

To this end, consider a sequence $(v_j) \subset j(A)$ such that $v_j \to v \in j(A)$. To abbreviate the notation, we write $h_j := \hat{P}_n(v_j)v_j$ and $h := \hat{P}_n(v)v$. Since $\sigma_{i-1}(v) > \sigma_i(v) > \sigma_{i+1}(v)$, Corollary 4.9 implies that $h_j \to h$ in X. Denote $\mathcal{U} := \{h, h_j \mid j \in \mathbb{N}\}$. Moreover, since the sequence (v_j) is bounded in $L^\infty(\mathbb{R}^N)$, condition (A) yields a constant $C > 0$ such that

$$\mathfrak{f}(|x|,|v_j(x)|) \le C|x|^\alpha$$

for $x \in B_2(0)$ and every j. Choosing $q > \frac{N}{2}$ such that $q\alpha > -N$, we infer that

$$\int_{B_2(0)} |f(x,|v_j(x)|)|^q\,dx \le C^q \int_{B_2(0)} |x|^{q\alpha}\,dx < \infty,$$

and the same can be shown for v in place of v_j. Furthermore, since $\lim\limits_{j\to\infty}\sigma_i(v_j) = \sigma_i(v) > 0$, we infer that the set

$$\mathcal{V} := \{\frac{1}{\sigma_i(v_j)} f(\cdot,|v_j|(\cdot)) \mid j \in \mathbb{N}\} \cup \{\frac{1}{\sigma_i(v)} f(\cdot,|v|(\cdot))\}$$

is a bounded subset of $L^q(B_2(0))$, whereas the set

$$\mathcal{U} := \{h_j|_{B_2(0)},\ h|_{B_2(0)} \mid j \in \mathbb{N}\}$$

is compact in $W^{1,2}(B_2(0))$. Hence, applying Lemma 14.1(b) to $\Omega = B_2(0)$ and $\Omega' = B_1(0)$, we in particular infer $h_j(0) \to h(0)$, and this establishes (*).
We complete the proof noting that

$$v \in j(A) \qquad \Longrightarrow \qquad \hat{g}_i(v) \neq 0 \quad \text{for some } i \in \{1, ..., n-1\},$$

which follows immediately from Lemma 14.3 and the fact that $j(A) \cap \mathcal{Q} = \emptyset$. Hence $g : A \to \mathbb{R}^{n-1}$, defined by its components

$$g_i := \hat{g}_i \circ j \qquad (i = 1, ..., n-1),$$

has the property (11.11), as required. Thus the assertion follows. □

It remains to complete the

Proof of Theorem 11.3. We check the hypothesis of Corollary 7.17 for given $n \in \mathbb{N}$: First, B is compact by Lemma 11.1(a). Moreover, Lemma 11.5 yields $\gamma(\mathcal{N}) \geq n$, whereas (7.28) holds by Lemma 9.9. Hence Corollary 7.17 applies, and in particular it yields $u_n \in K_{\mathcal{N}} \subset \mathcal{N}$ satisfying 7.22 and 7.23. By Lemma 11.2, the functions u_n have the properties claimed in (b).
Moreover, suppose that u is a solution of (EF) having more than n nodal domains. Then Theorem 14.15 implies that $\tilde{\rho}_u(u) = 1 = \sigma_{n+j}(u)$ for some $j \in \mathbb{N}$, whereas $\sigma_{n+j}(u) < \sigma_n(u)$. By Proposition 7.11 we infer that $\psi(u) \geq c_{n+j} \geq c_{n+1}$ *as well as* $\psi(u) > c_n$. Hence, every solution u of (EF) satisfying *either* $\psi(u) \leq c_n$ *or* $\psi(u) < c_{n+1}$ has at most n nodal domains. Thus (c) holds in particular, but combined with (b) this also forces (a). □

11.2 The nonradial case

In this section we assume in addition to $(\mathcal{F}_1)$-$(\mathcal{F}_3)$ that

(B) There are constants $a \in]0,2[$ and $C,\ \beta > 0$ such that for $t \geq 0$

$$f(x,t) \leq C\frac{1}{(1+|x|)^a}t^\beta, \qquad x \in \mathbb{R}^N, \quad t \geq 0\mathbb{R}$$

and

$$\frac{4-2a}{N-2} < \beta < \frac{4}{N-2}.$$

Condition (B) implies that the nonlinearity is *subcritical*. Moreover, for every $u \in D^{1,2}(\mathbb{R}^N)$ and $q \in [\frac{N}{2}, \frac{2N}{\beta(N-2)}[$ there holds

$$f(\cdot,|u(\cdot)|) \in L^q(\mathbb{R}^N). \tag{11.12}$$

More precisely, setting $p := \frac{2N}{N-2}$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |f(x,|u(x)|)|^q\,dx &\leq C\int_{\mathbb{R}^N} \frac{1}{(1+|x|)^{aq}}|u(x)|^{\beta q}\,dx \\ &\leq C\|u\|_p^{\beta q}\left(\int_{\mathbb{R}^N}\left(\frac{1}{1+|x|}\right)^{\tau' aq}\right)^{\frac{1}{\tau'}}dx \end{aligned} \tag{11.13}$$

where τ' is the conjugate exponent of $\tau := \frac{p}{\beta q} > 1$. Hence

$$\tau' aq = aq\frac{p}{p-\beta q} = 2a\frac{N}{\frac{2N}{q}-\beta(N-2)} \geq \frac{2a}{4-\beta(N-2)}N > N.$$

which implies the existence of the integral on the right hand side of (11.13).
Setting now $X := D^{1,2}(\mathbb{R}^N)$, we may prove that the nonlinearity has almost the same general properties as in the radial case (cf. Lemma 11.1).

Lemma 11.7. *Consider*

$$\langle B(u)v, w\rangle := \int_{\mathbb{R}^N} f(x, |u(x)|)v(x)w(x)\,dx$$

for $u, v, w \in X$. *Then:*

(i) $B : X \to \mathcal{L}(X, X^*)$ *is a well defined continuous map. Moreover,* $B(u) \in \mathcal{L}(X, X^*)$ *is compact for every* $u \in X$.

(ii) For every $u \in X$ *the integral*

$$\varphi(u) := \int_{\mathbb{R}^N} \int_0^{|u(x)|} f(x,t)t\,dt\,dx$$

exists. Moreover, $\varphi : X \to \mathbb{R}$ *and* B *satisfy* (CC).

Since the nonlinearity is *subcritical* and *vanishing at infinity*, one might expect that B is *strongly continuous* as in the radial case. Indeed, this is suggested by results of Schneider [64], but we did not examine this in detail. Instead, a local compactness property derived by Tshinanga [75] will be sufficient for our purposes.

Proof of Lemma 11.7. a) Set again $p := \frac{2N}{N-2}$ and consider the following factorization for B:

$$X \overset{i}{\hookrightarrow} L^p(\mathbb{R}^N) \xrightarrow{f_*} L^{\frac{N}{2}}(\mathbb{R}^N) \xrightarrow{b} \mathcal{L}(L^p(\mathbb{R}^N), L^{p'}(\mathbb{R}^N)) \xrightarrow{h} \mathcal{L}(X, X^*)$$

Here i denotes the Sobolev embedding, and $(f_*(u))(x) := f(x, |u(x)|)$. Moreover b is defined by $b(v)w := vw$ for $v \in L^{\frac{N}{2}}(\mathbb{R}^N)$, $w \in L^p(\mathbb{R}^N)$, while h maps a linear operator $S \in L^{\frac{N}{2}}(\mathbb{R}^N)$ to i^*Si (i^* denoting the dual of i). Obviously i, g and h are continuous linear operators. Moreover, (11.13) implies that f_* is a bounded substitution operator, hence it is continuous as well (cf. [47, p.22]). It remains to show that $B(u) \in \mathcal{L}(X, X^*)$ is compact for every $u \in X$. Since the compact linear operators form a closed subspace of $\mathcal{L}(X, X^*)$, it suffices to consider $u \in C_0^\infty(\mathbb{R}^N)$. Therefore suppose that u vanishes on $\mathbb{R}^N \setminus \Omega$, $\Omega \subset \mathbb{R}^N$ a smooth bounded domain. In this case $B(u)$ factorizes in the form

$$X \overset{r}{\hookrightarrow} W^{1,2}(\Omega) \overset{j}{\hookrightarrow} L^2(\Omega) \xrightarrow{k_u} L^2(\Omega) \xrightarrow{(r\circ j)^*} X^*$$

where r denotes the canonical restriction, j the *compact* Sobolev embedding, and k_u is given by $k_u(v) := f(\cdot, |u(\cdot)|)v$. Hence $B(u)$ is compact.
b) There holds

$$\int_0^{|u(x)|} f(x,t)t\,dt \le \frac{C}{(\beta+2)(1+|x|)^a}|u(x)|^{\beta+2} \tag{11.14}$$

where the right hand side of (11.14) is an L^1-function. Hence φ is well defined. Condition (CC) follows as usual from $(\mathcal{F}_1)$ (cf. Lemma 2.4). $\square$

By virtue of Lemma 11.7, conditions (H1)-(H5) and (CC) are satisfied for B and φ, and we may consider the compact operator $G(u)$ as well as $\sigma_n(u)$, $\tilde{Q}_n(u)$, $\tilde{\rho}_u$ as introduced in Section 7.2.1.

As a further consequence of Lemma 11.7(a) and (11.12) we infer that for every $u \in X$ the 'frozen potential' $V := f(\cdot, |u(\cdot)|)$ is *D-admissible* (cf. Sec. 14.3.2).
Now fix $u \in X$ arbitrary, and suppose that $\sigma_n(u) > 0$ for some $n \in \mathbb{N}$. If $v \in X$ is an eigenfunction of $G(u)$ corresponding to $\sigma_n(u)$, then v solves the equation

$$-\Delta v = \frac{1}{\sigma_n(u)} f(x, |u(x)|)v$$

in distributional sense. Hence, by virtue of (11.12) and Lemma 14.1 (applied to an arbitrary *bounded* domain $\Omega \subset \mathbb{R}^N$), we infer that v is continuous. Hence Theorem 14.7 yields that v has at most n nodal domains. In particular, every eigenfunction of $G(u)$ associated with $\sigma_1(u) > 0$ does not change sign, and this implies

$$\sigma_1(u) > \sigma_2(u) \qquad \text{whenever} \qquad \sigma_1(u) > 0. \tag{11.15}$$

Indeed, suppose in contrary that there are two X-orthogonal eigenfunctions v_1, v_2 associated with $\sigma_1(u)$, then

$$\begin{aligned} 0 &= \sigma_1(v_1|v_2)_X = (G(u)v_1|v_2)_X = \langle B(u)v_1, v_2\rangle \\ &= \int_{\mathbb{R}^N} f(x, |u(x)|)v_1(x)v_2(x)\, dx. \end{aligned}$$

However, by virtue of unique continuation properties, v_1 and v_2 do not vanish on a set of positive measure (More precisely, this follows by a combination of [25, Prop. 3] and [39, Theorem 6.3]). This forces $f(\cdot, |u|(\cdot)) \equiv 0$, contrary to $\sigma_1(u) > 0$.
A similar argument also shows that, for $n \geq 2$, every eigenfunction of $G(u)$ associated with $\sigma_n(u) > 0$ changes sign.
The above considerations in particular give rise to the following Lemma.

Lemma 11.8. *If $u \in \mathcal{N}$ satisfies (7.22) and (7.23) for some $n \in \mathbb{N}$, then $u \in W^{2,p}_{loc}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ for $1 \leq p < \infty$. Moreover, u has at most n nodal domains. If $n \geq 2$, then u changes sign.*

Proof. It only remains to prove that $u \in W^{2,p}_{loc}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ for $1 \leq p < \infty$. However, we have already seen that u is continuous, hence we infer $f(\cdot, |u(\cdot)|) \in L^\infty_{loc}(\mathbb{R}^N)$ from (B). Now elliptic regularity yields $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ for $1 \leq p < \infty$, and therefore $u \in C^1(\mathbb{R}^N)$ by virtue of Sobolev embeddings. □

As usual, we now relate nodal properties of solutions to the Ljusternik-Schnirelman values $c_n := c_n(\mathcal{N}, \psi)$ for the energy functional ψ on the Nehari manifold $\mathcal{N}$.

Theorem 11.9.

(a) *If $u \in X$ is a weak radial solution of (EF) such that either $\psi(u) \leq c_n$ or $\psi(u) < c_{n+1}$, then u has* at most *n nodal domains.*

(b) *For $n = 1, 2$ there exist solutions $u_n \in X$ of (EF) such that $\psi(u_n) = c_n$ and u_n has precisely n nodal domains.*

(c) $c_1 < c_2$

Remark 11.10. Under the assumptions stated above, it is already known (cf. [75]) that (EF) has an infinite number of solutions. However, the nodal information provided by Theorem 11.9 is new. As in Section 9.2, we can only prove the existence of one *sign changing* solution. We nevertheless suspect that (EF) possesses an infinite number of sign changing solutions.

In the remainder of this section we establish Theorem 11.9, following a similar strategy as in the radial case. However, the proof is simpler now, since condition (B) prevents from the regularity problems we had to encounter in the radial case. The following assertion is again easily derived from Lemma 7.10 by testing with (rapidly decreasing) analytic functions.

Lemma 11.11. *There holds* $\gamma(\mathcal{N}) = \infty$.

Lemma 11.12. *Consider* $K_{\mathcal{N}} := \{u \in \mathcal{N} \mid \tilde{Q}_2(u)u = u\}$. *Then* $\gamma^*(K_{\mathcal{N}}) \leq 1$.

Proof. For every $u \in \mathcal{N}$ the function $f(\cdot, |u(\cdot)|)$ does not vanish identically on $\mathbb{R}^N$, hence we deduce $\sigma_i(u) > 0$ for every $i \in \mathbb{N}$ by testing with analytic functions. In view of this we infer from (11.15) and 4.8 that the spectral projection $\hat{P}_1(u) \in \mathcal{L}(X)$ onto the eigenspace of $\sigma_1(u)$ depends continuously on $u \in \mathcal{N}$. Now pick a positive rapidly decreasing function $v \in X$ and observe that the map $h : \mathcal{N} \to \mathbb{R}$ defined by

$$u \overset{h}{\longmapsto} \int_{\mathbb{R}^N} v\hat{P}_1(u)u$$

is odd and continuous. Moreover, since $\hat{P}_1(u)u$ does not change sign, there holds

$$h(u) = 0 \quad \Longleftrightarrow \quad u \in K.$$

This implies $\gamma^*(K) \leq 1$, as claimed. □

We close the section with the

Proof of Theorem 11.3. To prove (b), we apply Theorem 7.17 in the cases $n \in 1, 2$. To this end, note that $\gamma(\mathcal{N}) \geq 2$ by Lemma 11.11. Moreover, Lemma 11.12 yields (7.28) in case $n = 2$, whereas this relation holds trivially in case $n = 1$. Finally, as proved in [75, Lemma 2.2.], the functional ψ satisfies the PS condition (Note that even though in [75] it is supposed that $f \in C(\mathbb{R}^N \times \mathbb{R})$, the proof also works if f is just a Caratheodory function). Therefore, for $n = 1, 2$ Theorem 7.17 in particular provides solutions u_n to (7.22) and (7.23). By Lemma 11.8 these solutions have the properties asserted in (b).
Now suppose that u is a solution of (EF) having more than n nodal domains. By Theorem 14.13 we infer that $\tilde{\rho}_u(u) = 1 = \sigma_{n+j}(u)$ for some $j \in \mathbb{N}$, whereas $\sigma_{n+j}(u) < \sigma_n(u)$. Therefore Proposition 7.11 yields $\psi(u) \geq c_{n+j} \geq c_{n+1}$ *as well as* $\psi(u) > c_n$. This shows (a), and (c) is an immediate consequence of (a) and (b). □

Chapter 12

Sublinear Schrödinger equations

We consider the radially symmetric semilinear elliptic equation

$$-\Delta u + q(|x|)u + \mathfrak{f}(|x|, |u|)u = \lambda u, \qquad u \in W^{1,2}(\mathbb{R}^N), \qquad N \geq 2 \tag{12.1}$$

together with the constraint

$$\|u\|_2 = R \tag{12.2}$$

for given $R > 0$. For the *linear* potential $q :]0, \infty[\to \mathbb{R}$ we assume

(I_ν) q is a continuous function on $]0, \infty[$ satisfying

$$\lim_{r \to \infty} q(r) = 0, \tag{12.3}$$

but also

$$0 > \limsup_{r \to \infty} q(r) r^\nu \geq -\infty \tag{12.4}$$

for some $\nu \in]0, 2[$, as well as

$$\int_0^\varepsilon r|q(r)|\, dr < \infty \qquad \text{in case } N \geq 3 \text{ resp.}$$
$$\int_0^\varepsilon r|ln(r)q(r)|\, dr < \infty \qquad \text{in case } N = 2.$$

A typical example is a potential of Coulomb type, i.e. $N = 3$ and $q(r) = -\frac{Z}{r}$, cf. Chapter 13.
We furthermore assume that $\mathfrak{f} :]0, \infty[\times [0, \infty[\to \mathbb{R}$ is a Caratheodory function satisfying

(M) For a.e. $r \in]0, \infty[$ there holds $\mathfrak{f}(r, 0) = 0$, and $\mathfrak{f}(r, \cdot)$ is nondecreasing on $[0, \infty[$.

Concerning the existence of nodal solutions to (12.1), (12.2) we have:

Theorem 12.1. *Suppose that* $\mathfrak{f} :]0,\infty[\times[0,\infty[\to \mathbb{R}$ *is continuous and that (M),* (I_ν) *and the following condition hold:*

(J_ν) *There are numbers* $r_0, \beta, c > 0$ *and* $\alpha \in \mathbb{R}$ *such that*

$$|\mathfrak{f}(r,t)| \le c r^\alpha t^\beta \qquad \text{for } t \ge 0, r \ge r_0$$

as well as one of following relations is valid:

(i)	$\alpha < 0$	*and*	$\alpha - N\frac{\beta}{2} < -\nu$
(ii)	$\alpha \ge 0$	*and*	$\alpha - (N-1)\frac{\beta}{2} \le -\frac{N-1}{N}\nu$

Then, for every $R > 0$ *and every* $n \in \mathbb{N}$ *there is a radially symmetric weak solution* (u,λ) *of (12.1),(12.2) such that* $\lambda < 0$, $u \in C(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$, *and* u *has precisely* n *nodal domains.*

Remark 12.2. (a) Whereas on bounded subsets $\mathfrak{f}$ may grow arbitrarily fast, condition (I_ν) controls the growth of $\mathfrak{f}$ *at infinity*. This is required to ensure that the corresponding 'frozen' linear eigenvalue problems have infinitely many eigenvalues below the essential spectrum (cf. Lemma 12.5).
(b) It is elucidating to consider the autonomous case (α =0) in particular. Then there must hold $\beta \ge \dfrac{2\nu}{N}$, i.e. the order of the nonlinearity near zero must be *at least* $1 + \dfrac{2\nu}{N}$. In the special case $N = 3$, $q(r) = -\frac{Z}{R}$ (i.e. $\nu = 1$), we recover the bound $\beta + 1 \ge \frac{5}{3}$ imposed by Lions [52, p. 36]. However, for $\alpha > 0$ the fixed point approach of [52, Sec. III.3] does not work any more, since in this case L^2-estimates are not enough to keep the corresponding eigenvalues away from zero.

We are not able to prove Theorem 12.1 directly, hence we will first introduce more restrictive growth conditions which allow us to cast the problem in the abstract setting of Section 6.1. In this setting we state a theorem which gives more detailed information on the solution set of (12.1), (12.2), and afterwards we will deduce Theorem 12.1 very easily by a priori estimates.
Note that condition (I_ν) guarantees that $q \in K_N$, the Kato class, hence q is $-\Delta$-form bounded with relative bound zero (cf. [4, Theorem 4.7]). Therefore, by the KLMN-Theorem (cf. [60]), the form sum $-\Delta + q$ is a well defined selfadjoint and semi-bounded operator in $L^2(\mathbb{R}^N)$ with form domain $W^{1,2}(\mathbb{R}^N)$. Moreover, the essential spectrum of this operator is $[0,\infty[$ as a consequence of (12.3), see [63, p. 218] for instance.
In the following let $\mathcal{H}$, X, $\mathcal{D}(A_0)$ denote the closed subspaces consisting of the radially symmetric functions in $L^2(\mathbb{R}^N)$,$W^{1,2}(\mathbb{R}^N)$, $\mathcal{D}(-\Delta + q)$, respectively. We define the operator $A_0 : \mathcal{D}(A_0) \subset H \to H$ as the restriction of $-\Delta + q$ to $\mathcal{D}(A_0)$, hence A_0 is selfadjoint and bounded from below. In accordance with Section 6.1 we put $m = -\inf \sigma(A_0) + 1$, and we endow X with the scalar product

$$(u|v)_X := \int_{\mathbb{R}^N} \nabla u \nabla v + \int_{\mathbb{R}^N} (q(x) + m) uv,$$

We introduce the following condition:

(J) For every $t > 0$ there holds $\mathfrak{f}(\cdot,t) \in L^\infty_{loc}(]0,\infty[)$, and $\mathfrak{f}$ can be written as a sum of Caratheodory functions $\mathfrak{f}_i$ satisfying (M) and

$$\frac{\mathfrak{f}_i(r,t)}{t^\beta} = O(r^\alpha) \quad \begin{cases} r \to 0 \\ r \to \infty \end{cases}$$

uniformly in $t > 0$ for numbers $\beta > 0$ and $\alpha \in]\frac{\beta}{2}(N-2) - 2, \frac{\beta}{2}(N-2)[$.

Note that, by virtue of Lemma 14.19 and the remark following it, assumption (J) ensures that $B : X \to \mathcal{L}(X, X^*)$ defined by

$$\langle B(u)v, w\rangle = \int_{\mathbb{R}^N} \mathfrak{f}(|x|, |u(x)|)v(x)w(x)dx \qquad (u, v, w \in X)$$

is a strongly continuous map satisfying (H1)-(H4). Moreover, (CC) holds for B and $\varphi : X \to \mathbb{R}$ given by

$$\varphi(u) = \int_{\mathbb{R}^N} \int_0^{|u(x)|} \mathfrak{f}(|x|, t)t\, dt\, dx.$$

Therefore we may refer to the notations $A(u)$, $\mu_n(u)$, ρ_u and $V_n(u)$ as well as to ψ and $c_n := c_n(\psi, S_R)$ as introduced in Section 6.1. We in particular recall that

$$S_R = \{u \in X \mid \|u\|_2 = R\}$$

in the present context. Now there holds:

Theorem 12.3. *Suppose that assumptions (I_ν), (M), (J) and (J_ν) are satisfied, and let $n \in \mathbb{N}$, $R > 0$. Then:*

(a) $c_n < c_{n+1}$

(b) There is a solution $(u, \lambda) \in S_R \times \mathbb{R}^-$ of (12.1) such that $\psi(u) = c_n$, $u \in C(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$, and u has precisely n nodal domains.

(c) Every solution $(u, \lambda) \in S_R \times \mathbb{R}$ of (12.1) with $\psi(u) > c_n$ has at least $n + 1$ nodal domains.

Assuming that we had already proved this, we easily complete the

Proof of Theorem 12.1. For arbitrary $c > 0$ define the function $\mathfrak{f}_c :]0, \infty[\times[0, \infty[\to \mathbb{R}$ by

$$\mathfrak{f}_c(r, s) = \begin{cases} \min\{c, \mathfrak{f}(r, s)\} & 0 < r \le r_0 \\ \mathfrak{f}(r, s) & r_0 < r < \infty \end{cases}$$

Clearly $\mathfrak{f}_c$ satisfies (M), (J) and J_ν. Therefore, in view of Theorem 12.3, to every $c > 0$ there corresponds a weak radial solution $(u_c, \lambda_c) \in S_R \times \mathbb{R}^-$ of (12.1), (12.2) with $\mathfrak{f}_c$ in place of $\mathfrak{f}$, and such that u_c has precisely n nodal domains. In particular u_c weakly solves

$$(-\triangle + V_c)u_c = 0,$$

with $V_c(x) := -\lambda_c + q(|x|) + \mathfrak{f}_c(|x|, |u_c(x)|)$ for $x \in \mathbb{R}^N$. However, by virtue of (M) the negative part V_c^- of V_c only depends on q, hence it is *uniformly* bounded in the norm of the Kato class K_N. By [67, Theorem C.1.2] we conclude that $|u_c(\cdot)|$ is uniformly bounded in $L^\infty(B_{r_0}(0))$ independent of c. Hence, for $c > 0$ large enough, u_c is the desired solution of (12.1), (12.2). It remains to show that $u_c \in C^2(\mathbb{R}^N \setminus \{0\})$. However this clearly follows since u_c, viewed as a function of $r = |x|$ solves on $]0, \infty[$ an ordinary differential equation of second order with continuous coefficients. $\square$

The proof of Theorem 12.3 will occupy the rest of the section, and from now on we always assume that (I_ν), (M), (J) and (J_ν) are satisfied.

Lemma 12.4. *Put $\mathcal{M} := \{x \in \mathbb{R}^n \mid |x| \geq r_0\}$. Then $u \in X$ implies that $\mathfrak{f}_{\mathcal{M}}(u) \in L^{\frac{N}{\nu}}(\mathcal{M})$, where $\mathfrak{f}_{\mathcal{M}}(u)(x) := \mathfrak{f}(x, |u(x)|)$ for $x \in \mathcal{M}$.*

Proof. First we recall that $u \in X$ implies that $u|_{\mathcal{M}} \in L^q(\mathcal{M})$ for every $2 \leq q \leq \infty$.
If $\alpha \geq 0$, then Lemma 14.16(b) yields

$$\begin{aligned} |\mathfrak{f}(x, |u(x)|)|^{\frac{N}{\nu}} &\leq c^{\frac{N}{\nu}} (r^{\alpha} |u(x)|^{\beta})^{\frac{N}{\nu}} \\ &\leq c^{\frac{N}{\nu}} \tilde{K}^{\frac{2\alpha}{N-1}\frac{N}{\nu}} \|u\|_X^{\frac{2\alpha}{N-1}\frac{N}{\nu}} |u(x)|^{(\beta - \frac{2\alpha}{N-1})\frac{N}{\nu}} \end{aligned}$$

for a.e. $x \in \mathcal{M}$ with $r = |x|$. However, $(\beta - \frac{2\alpha}{N-1})\frac{N}{\nu} \geq 2$ by (I_ν), hence the assertion follows in this case.
Next we assume that $0 > \alpha \geq -\nu$. By (I_ν) we then may pick $s < \frac{\nu}{\nu+\alpha}$ (resp. $s < \infty$ in case $\alpha = -\nu$) such that $\frac{\beta N s}{\nu} > 2$. Hence $s' > -\frac{\nu}{\alpha}$, and therefore

$$\begin{aligned} \int_{\mathcal{M}} |\mathfrak{f}(x, |u|)|^{\frac{N}{\nu}} &\leq c^{\frac{N}{\nu}} \int_{\mathcal{M}} r^{\frac{\alpha N}{\nu}} |u|^{\frac{\beta N}{\nu}} \\ &\leq c^{\frac{N}{\nu}} \left(\int_{\mathcal{M}} r^{\frac{s'\alpha N}{\nu}} \right)^{\frac{1}{s'}} \left(\int_{\mathcal{M}} |u|^{\frac{\beta N s}{\nu}} \right)^{\frac{1}{s}} < \infty \end{aligned}$$

by Hölder's inequality. Finally, if $\alpha < -\nu$, then

$$\begin{aligned} \int_{\mathcal{M}} |\mathfrak{f}(x, |u|)|^{\frac{N}{\nu}} &\leq c^{\frac{N}{\nu}} \int_{\mathcal{M}} r^{\frac{\alpha N}{\nu}} |u|^{\frac{\beta N}{\nu}} \\ &\leq c^{\frac{N}{\nu}} \left(\int_{\mathcal{M}} r^{\frac{\alpha N}{\nu}} \right) \|u|_{\mathcal{M}}\|_{\infty}^{\frac{\beta N}{\nu}} < \infty. \end{aligned}$$

Thus the assertion follows. $\square$

Lemma 12.5. *There holds*

$$\mu_n(u) < \mu_\infty \tag{12.5}$$

for every $u \in X$ and $n \in \mathbb{N}$.

Proof. Put $g := \mathfrak{f}_{\mathcal{M}}(u) \in L^{\frac{N}{\nu}}(\mathcal{M})$, and pick $K > 0, R_0 > \max\{r_0, 1\}$ such that $q(r) \leq -\dfrac{K}{r^\nu}$ for $r \geq R_0$. Consider an n-dimensional space $V \subset C_0^\infty$ of functions with support in

$$\{x \in \mathbb{R}^N \mid 1 < |x| < 2\}.$$

For $R > R_0$ and $\psi \in V$ define $\psi_R(x) := R^{-N/2}\psi(x/R)$, which implies that $\|\psi_R\|_2 = \|\psi\|_2$ and

$$\operatorname{supp} \psi_R \subset \{x \in \mathbb{R}^N \mid R < |x| < 2R\}.$$

We claim

$$\sup_{\substack{\psi \in V \\ \|\psi\|_2 = 1}} (A(u)\psi_R | \psi_R) < 0 \tag{12.6}$$

for R large enough, which clearly yields the assertion. To establish (12.6), note that $R > R_0$ implies

$$
\begin{aligned}
(A(u)\psi_R|\psi_R) &= ((-\Delta + q)\psi_R, \psi_R) + \int_{\mathbb{R}^N} g\psi_R^2 \\
&\leq (-\Delta\psi_R, \psi_R) + \int_{R\leq|x|\leq 2R} (-K|x|^{-\nu} + g(x))\psi_R^2(x)\,dx \\
&\leq R^{-2}(-\Delta\psi, \psi) + R^{-\nu}\Big(R^\nu \int_{1\leq|x|\leq 2} g(\frac{x}{R})\psi^2(x)\,dx - K\int_{1\leq|x|\leq 2} |x|^{-\nu}\psi^2\,dx\Big)
\end{aligned}
$$

hence (12.6) follows once we have shown that

$$\lim_{R\to\infty} R^\nu \int_{1\leq|x|\leq 2} g(\frac{x}{R})\psi^2(x)\,dx = 0$$

uniformly in $\psi \in V, \|\psi\|_2 = 1$. However, all norms on V being equivalent, it suffices to ensure

$$\lim_{R\to\infty} R^\nu \int_{1\leq|x|\leq 2} g(\frac{x}{R})\,dx = 0,$$

and this is true since

$$
\begin{aligned}
R^\nu \int_{1\leq|x|\leq 2} g(\frac{x}{R})\,dx &= R^{\nu-N} \int_{R\leq|x|\leq 2R} g(x)\,dx \\
&\leq R^{\nu-N}\Big(\int_{R\leq|x|\leq 2R} dx\Big)^{\frac{N-\nu}{N}} \|g\|_{L^{\frac{N}{\nu}}(|x|\geq R)} \\
&\leq C_0\|g\|_{L^{\frac{N}{\nu}}(|x|\geq R)} \quad \longrightarrow \quad 0
\end{aligned}
$$

for $R \to \infty$. Thus (12.6) holds, and the proof is complete. □

Summarizing the preceding lemmas, we now complete the

Proof of Theorem 12.3. Let $n \in \mathbb{N}, R > 0$ be given. We commence with the proof of (b): Recalling that B is strongly continuous by Lemma 14.19, we may apply Theorem 6.5 once we have shown

$$\mu_n(u) < \mu_{n+1}(u) \tag{12.7}$$

for every $u \in X$. For this note that $\mu_n(u)$ is an eigenvalue of $A(u)$ by Lemma 12.5, and every corresponding eigenfunction Ψ weakly solves

$$-\Delta\Psi = V\Psi$$

with $V(x) := -g(|x|) - \mathfrak{f}(|x|, |u(x)|) + \mu_n(u)$. By Lemma 14.3 we now infer that Ψ is continuous and uniquely determined up to a constant . Indeed, (J) and (14.19) ensure that $V \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$ and that (14.2) holds for some $\alpha > -2$, as required. Thus $\mu_n(u)$ is nondegenerate, and (12.7) holds.

Now Theorem 6.5 yields the validity of property (CP), in particular there exists a weak solution u of $(SC)_n$ with $\psi(u) = c_n$. In particular, u solves

$$-\Delta u = Vu$$

with V defined as above. Recalling that $B(u) \in \mathcal{L}(X, X^*)$ is compact, the above stated properties ensure that V is a radial W-admissible potential. Hence Theorem 14.8 establishes that u is a continuous function having precisely n nodal domains. Since $u \in C^1(\mathbb{R}^N \setminus \{0\})$ by elliptic regularity, the proof of b is complete. Moreover, since $\rho_u(u) = \mu_n(u) < \mu_{n+1}(u)$, Prop. 6.3(a) yields $c_n = \psi(u) < c_{n+1}$, as claimed in (a).
To prove (c), suppose that $(u, \lambda) \in S_R \times \mathbb{R}$ is a solution of (12.1) with at most n nodal domains. Then $\rho_u(u) \leq \mu_n(u)$ by Theorem 14.8, and hence $\psi(u) \leq c_n$ by Prop. 6.3(a). Thus the proof is complete. $\square$

Chapter 13

The Hartree equation and the TFW equation

As in the previous chapter we still consider a radially symmetric *sublinear* equation. However, we now focus on the case $N = 3$, and we are concerned with the eigenvalue problem

$$(TFW) \qquad \Big(-\Delta - \tfrac{Z}{|x|} + \mathfrak{f}(|x|,|u|) + u^2 * \tfrac{1}{|x|}\Big)u = \lambda u \qquad u \in W^{1,2}(\mathbb{R}^3) \qquad \|u\|_2 = R$$

Here we assume that Z is a positive constant. We emphasize the particular cases for which $R = 1$ and

(i) $\mathfrak{f} = 0$. Then (TFW) is known as the *restricted Hartree equation*.

(ii) $\mathfrak{f}(r,t) = \lambda t^p, \quad \lambda, p > 0$. Then we are dealing with the so called *Thomas-Fermi-Von Weizäcker equation* (TFW equation in short).

Both equations occur in approximative models for quantum mechanical systems involving many electrons.
As in the previous chapter, we assume that $\mathfrak{f} :]0,\infty[\times[0,\infty[\to \mathbb{R}$ is a Caratheodory function satisfying

(M) For every $r \in]0,\infty[$ there holds $\mathfrak{f}(r,0) = 0$, and $\mathfrak{f}(r,\cdot)$ is monotonically nondecreasing on $[0,\infty[$.

We will be concerned with radial solutions only, and we have the following result:

Theorem 13.1. *Suppose that (M) holds, that* $\mathfrak{f}$ *is continuous and that*

(J_1) *There are numbers* $r_0, \beta, c > 0$ *and* $\alpha \in \mathbb{R}$ *such that*

$$|\mathfrak{f}(r,t)| \le c r^{\alpha} t^{\beta} \qquad \textit{for } t \ge 0, r \ge r_0$$

as well as one of following conditions hold:

(i) $\qquad \alpha < 0 \qquad$ *and* $\qquad \alpha - \frac{3}{2}\beta < -1$

(ii) $\alpha \geq 0$ *and* $\alpha - \beta \leq -\frac{2}{3}$

Then, for every $0 < R \leq \sqrt{Z}$ *and every* $n \in \mathbb{N}$ *there is a radially symmetric solution* (u, λ) *of (TWF) such that* $\lambda < 0$, $u \in C(\mathbb{R}^3) \cap C^2(\mathbb{R}^3 \setminus \{0\})$, *and* u *has precisely* n *nodal domains.*

Remark 13.2. (a) As in the previous chapter, (J_1) controls the growth of $\mathfrak{f}$ only *at infinity*. However, only in case that $R < \sqrt{Z}$ we can ensure that the 'frozen' linear eigenvalue problems have infinitely many eigenvalues below zero (cf. Lemma 13.5). For the case $R = \sqrt{Z}$, a separate limiting argument is needed.
(b) The comments made in Remark 12.2(b) are also valid in the present case.

We now proceed as in the previous chapter: First we introduce more restrictive growth conditions which allow to cast the problem in the abstract setting of Section 6.1. Then we state an analog of Theorem 12.3, and afterwards we will deduce Theorem 13.1 very easily by a priori estimates.
Note that, concerning the linear part of problem (TWF), we are just dealing with the special case $N = 3$ and $q(r) = -\frac{Z}{r}$ in the notation of Chapter 12. Hence we keep using the symbols A_0, $\mathcal{H}$, X and m without further comment, and we recall how condition (J) can be written in the special case $N = 3$:

(J) For every $t > 0$ there holds $\mathfrak{f}(\cdot, t) \in L^\infty_{loc}(]0, \infty[)$, and $\mathfrak{f}$ can be written as a sum of Caratheodory functions $\mathfrak{f}_i$ satisfying (M) and

$$\frac{\mathfrak{f}_i(r,t)}{t^\beta} = o(r^\alpha) \quad \begin{cases} r \to 0 \\ r \to \infty \end{cases}$$

uniformly in $t > 0$ for numbers $\beta > 0$ and $\alpha \in]\frac{\beta}{2} - 2, \frac{\beta}{2}[$.

Then we have

Lemma 13.3. *Suppose that (J) is satisfied.*

(a) There is a strongly continuous map $B : X \to \mathcal{L}(X, X^*)$ *given by*

$$\langle B(u)v, w\rangle := \int_{\mathbb{R}^N} \Big(\mathfrak{f}(|x|, |u(x)|) + (u^2 * \frac{1}{|\cdot|})(x)\Big) v(x)w(x)\, dx$$

such that $B(u) \in \mathcal{L}(X, X^*)$ *is a compact linear operator for each* $u \in X$.

(b) Condition (CC) is satisfied for B *and the functional* $\varphi : X \to \mathbb{R}$ *given by*

$$\varphi(u) := \int_{\mathbb{R}^N} \int_0^{|u(x)|} \mathfrak{f}(|x|, t)t\, dt\, dx + \frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|}\, dx\, dy \qquad (u \in X).$$

Proof. Since $V := \frac{1}{|\cdot|}$ satisfies the conditions $(V_1) - (V_3)$ from Chapter 10, the assertion follows by combining Lemma 10.1 and Lemma 14.19. ☐

As a consequence of Lemma 13.3, (H1)-(H4) and (CC) are satisfied, and we may refer to the notations $A(u)$, $\mu_n(u)$, $V_n(u)$, ψ and $c_n := c_n(\psi, S_R)$ as defined in Section 6.1. We now have an analog of Theorem 12.3:

Theorem 13.4. *Suppose that assumptions (M), (J) and* (J_1) *are satisfied, and let* $0 < R < \sqrt{Z}$*,* $n \in \mathbb{N}$*. Then:*

(a) $c_n < c_{n+1}$

(b) There is a solution $(u, \lambda) \in S_R \times \mathbb{R}^-$ *of (TFW) such that* $\psi(u) = c_n$*,* $u \in C(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$*, and* u *has precisely* n *nodal domains.*

(c) Every solution $(u, \lambda) \in S_R \times \mathbb{R}$ *of (TFW) with* $\psi(u) > c_n$ *has at least* $n + 1$ *nodal domains.*

The remainder of the section is devoted to the proofs of Theorem 13.4 and Theorem 13.1. For this we assume that (M), (J) and (J_1) are in force from now on.

Lemma 13.5. *Suppose that* $0 < R < \sqrt{Z}$*, and let* $u \in X$ *with* $\|u\|_2 \leq R$*. Then*

$$\mu_n(u) < 0. \tag{13.1}$$

for every $n \in \mathbb{N}$*.*

Proof. Since u is radially symmetric, spherical integration yields

$$\begin{aligned} \int_{\mathbb{R}^N} u^2(y) \frac{1}{|x-y|}\, dy &= \int_{\mathbb{R}^N} u^2(y) \frac{1}{\max\{|x|, |y|\}}\, dy \\ &\leq \frac{1}{|x|} \|u\|_2^2 \\ &\leq \frac{R^2}{|x|} \end{aligned}$$

for every $x \in \mathbb{R}^3$, which implies that

$$(A(u)\psi|\psi) \leq (\tilde{A}(u)\psi|\psi) \tag{13.2}$$

for all $\psi \in C_0^\infty(\mathbb{R}^3)$, where

$$\tilde{A}(u) := -\Delta - \frac{Z - R^2}{|x|} + \mathfrak{f}(|x|, |u|).$$

However, since $Z > R^2$, the results of Chapter 12 apply to the operator valued map $\tilde{A}$ (with $\nu = 1$). Hence the assertion is a consequence of (13.2) and Lemma 12.5.

□

We now may complete the

Proof of Theorem 13.4. We proceed along the lines of the proof of Theorem 12.3:
Fix $\tilde{R} > 2[\max\{c_n, 0\} + mR^2]$ and consider $D := D(R, \tilde{R}) \subset X$. Then $\mu_n(u) < 0$ for every $u \in D$ by Lemma 13.5, and we again deduce that $\mu_n(u)$ is a *nondegenerate eigenvalue*. More precisely, now the corresponding 'frozen potential' can be written as

$$V_u(x) = -\frac{Z}{|x|} - \mathfrak{f}(|x|, |u(x)|) - (u^2 * \frac{1}{|\cdot|})(x) + \mu_n(u),$$

and convolution inequalities yield $u^2 * \frac{1}{|\cdot|} \in L^\infty(\mathbb{R}^3)$ for the additionally occurring term. Indeed, this follows from the fact that $\frac{1}{|\cdot|} \in L^2(\mathbb{R}^3) + L^4(\mathbb{R}^3)$, whereas $u^2 \in L^2(\mathbb{R}^3) \cap L^{\frac{4}{3}}(\mathbb{R}^3)$ for every $u \in X$. We again conclude that V is a radial W-admissible potential, and in particular Lemma 14.3 implies that

$$\mu_n(u) < \mu_{n+1}(u)$$

for every $u \in D$, as claimed. Recalling that B is strongly continuous, we may apply Theorem 6.5 to deduce the validity of property (CP). In particular this furnishes a solution u of $(SC)_n$ with $\psi(u) = c_n$. As in the proof of Theorem 12.3 we now derive the properties claimed for u in (b), and also (a) and (c) follow precisely by the same reasoning. $\square$

We close the chapter with the

Proof of Theorem 13.1. If $Z > R^2$, then we easily deduce the assertion along the lines of the proof of Theorem 12.1. Using this, we now treat the case $Z = R^2$ by a limiting argument which essentially is due to Lions (cf. [52]):
Consider a positive sequence $(\varepsilon_j)_j$ such that $\varepsilon_j \to 0$. Then, for every $j \in \mathbb{N}$, we already have a solution $(u_j, \mu_n(u_j))$ of (TFW) with Z replaced by $Z + \varepsilon_j$ and such that $\|u_j\|_2 = R$. From the equation we deduce that $(\|\nabla u_j\|_2)_j$ is a bounded sequence, i.e., $(u_j)_j$ is bounded in X. Passing to a subsequence, we may assume that $u_j \rightharpoonup u$ in X. Then Lemma 5.2 and the strong continuity of B yields $\lim\limits_{j\to\infty} \mu_n(u_j) = \mu_n(u)$, moreover $B(u_j)u_j \to B(u)u$ in X^* by Lemma 5.7(b). As a consequence, u weakly solves

$$\Big(-\Delta - \frac{Z}{|x|} + \mathfrak{f}(|x|, |u|) + u^2 * \frac{1}{|x|}\Big)u = \mu_n(u)u.$$

It remains to prove that $\|u\|_2 = R$. For this note that

$$\begin{aligned}
\mu_n(u)R^2 &= \lim_j \mu_n(u_j)R^2 \\
&= \lim_j \mu_n(u_j)\|u_j\|_2^2 \\
&= \lim_j \Big(\|\nabla u_j\|_2^2 - \int_{\mathbb{R}^3} \frac{Z+\varepsilon_j}{|x|}u_j^2 + \langle B(u_j)u_j, u_j\rangle\Big) \\
&\geq \|\nabla u\|_2^2 - \int_{\mathbb{R}^3} \frac{Z}{|x|}u^2 + \langle B(u)u, u\rangle \\
&= \mu_n(u)\|u\|_2^2,
\end{aligned}$$

hence either $\|u\|_2 = R$ or $\mu_n(u) \geq 0$. However, $\|u\|_2 < R$ would force $\mu_n(u) < 0$ by Lemma 13.5 (applied to $R' := \|u\|_2 < \sqrt{Z}$), and thus we conclude $\|u\|_2 = R$ in either case. $\square$

Chapter 14

Appendix

14.1 Notes on regularity

Lemma 14.1. *Suppose that $N \geq 2$ and $q > \frac{N}{2}$.*
Consider a domain $\Omega \subset \mathbb{R}^N$, a bounded *subset $\mathcal{V} \subset L^q(\Omega)$ and a* bounded *subset $\mathcal{U} \subset W^{1,2}(\Omega)$ with the following property:*

- *For each $u \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that u is a distributional solution of*

$$-\Delta u = Vu \qquad \text{in } \Omega.$$

Then $\mathcal{U} \subset W^{2,q}_{loc}(\Omega)$, and every $u \in \mathcal{U}$ is continuous in Ω. Moreover, if $\Omega' \subset\subset \Omega$, then

(a) $\mathcal{U}|_{\Omega'} := \{u|_{\Omega'} \mid u \in \mathcal{U}\}$ is bounded in $W^{2,q}(\Omega')$, and $\mathcal{U}|_{\overline{\Omega'}} := \{u|_{\overline{\Omega'}} \mid u \in \mathcal{U}\}$ is relatively compact in $C(\overline{\Omega'})$.

(b) If $(u_n)_n \subset \mathcal{U}$ is a sequence such that $u_n \to u \in \mathcal{U}$ in the $W^{1,2}(\Omega)$-norm, then $u_n \to u$ in $C(\overline{\Omega'})$.

Proof. As noted in [67, p.457], $L^q(\mathbb{R}^N)$ is continuously embedded in K_N, the Kato class (for the definition of K_N and its norm see [67, p. 453]). Hence [67, Theorem C.1.1] implies that every $u \in \mathcal{U}$ is continuous in Ω.
To prove (a), consider the special case $\Omega = B_R(0)$, $\Omega' = B_r(0)$ first, where $0 < r < R$. Pick $r_0 \in]r, R[$ and denote $\Omega_0 := B_{r_0}(0)$. Put $C_1 := \sup\limits_{V \in \mathcal{V}} \|V\|_{L^q(\Omega)}$. By [67, Theorem C.1.2] we infer that $\mathcal{U}|_{\Omega_0}$ is *bounded* in $L^\infty(\Omega_0)$, that is, $C_1 := \sup\limits_{u \in \mathcal{U}|_{\Omega_0}} \|V\|_{L^\infty(\Omega_0)} < \infty$. However, for every $u \in \mathcal{U}|_{\Omega_0}$ there is $V \in \mathcal{V}$ such that $-\Delta u = Vu$ in distributional sense on Ω_0, whereas $\|Vu\|_{L^q(\Omega_0)} \leq C_1 C_2$. Hence, $\mathcal{U}|_{\Omega}$ is a bounded subset of $W^{2,q}(\Omega')$ by the *Calderon-Zygmund inequality* [40, p.214]. Since $W^{2,q}(\Omega')$ is compactly embedded in $C(\overline{\Omega'})$, we conclude that $\mathcal{U}|_{\overline{\Omega'}}$ is relatively compact in $C(\overline{\Omega'})$.
Now consider general choices of Ω and Ω'. Note that for each $x \in \overline{\Omega'}$ there is a number $R = R(x) > 0$ such that $B_R(x) \subset \Omega$. Since $\overline{\Omega'}$ is compact, there are finitely many points $x_j \in \overline{\Omega'}$,

$j = 1, ..., n$ and corresponding positive numbers $0 < r_j < R_j$ such that

$$\overline{\Omega'} \subset \bigcup_{i=1}^{n} B_{r_j}(x_j) \subset \bigcup_{i=1}^{n} B_{R_j}(x_j) \subset \Omega.$$

In view of the first case the sets $\mathcal{U}|_{\overline{B_{r_j}(x_j)}}$ are relatively compact in $C(\overline{B_{r_j}(x_j)})$ for each $i \in \{1, ..., n\}$. From this one easily concludes that $\mathcal{U}_{\overline{\Omega'}}$ is relatively compact in $C(\overline{\Omega'})$. In the same way the boundedness of $\mathcal{U}|_{\Omega'}$ in $W^{2,q}(\Omega')$ is derived from the boundedness on the balls $B_{r_j}(x_j)$ which we have established already.
(b) Since $\{u_n|_{\overline{\Omega'}} \mid n \in \mathbb{N}\}$ is relatively compact in $C(\overline{\Omega'})$, it suffices to show $u_n(x) \to u(x)$ for every $x \in \overline{\Omega'}$. Assume in contradiction that there is $x_0 \in \overline{\Omega'}$, $\varepsilon > 0$ and a subsequence u_{n_k} such that $|u_{n_k}(x_0) - u(x_0)| > \varepsilon$. Without loss, we may assume $u_{n_k} \to u$ pointwise almost everywhere on Ω'. However, since u_{n_k} is *equicontinuous*, this implies $u_{n_k}(x_0) \to u(x_0)$ in contradiction. Hence (b) is proved. $\square$

Lemma 14.2. *Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$, $a \in L^{\frac{N}{2}}(\Omega)$ and $C > 0$. If $u \in W_0^{1,2}(\Omega)$ satisfies*

$$\int_\Omega \nabla u \nabla \varphi \leq C \int_\Omega |u\varphi| + \int_\Omega a(x)|u\varphi| \tag{14.1}$$

for every $\varphi \in W_0^{1,2}(\Omega)$, then $u \in L^q(\Omega)$ for every $2 \leq q < \infty$.

Proof. If $N = 2$, the assertion is just a consequence of Sobolev embeddings. Therefore suppose $N \geq 3$, and let $L, s \geq 0$. Applying (14.1) to $\varphi = u \min\{|u|^{2s}, L^2\} \in W_0^{1,2}(\Omega)$ yields

$$\begin{aligned}\int_\Omega |\nabla u|^2 \min\{|u|^{2s}, L^2\} + 2s \int_{|(x)|^s \leq L} |\nabla u|^2 |u|^{2s} &= \int_\Omega \nabla u \nabla \varphi \\ \leq C \int_\Omega |u|^2 \min\{|u|^{2s}, L^2\} + \int_\Omega a(x)|u|^2 \min\{|u|^{2s}, L^2\}&.\end{aligned}$$

Now suppose that $u \in L^{2s+2}(\Omega)$. Then we infer, with constants c_i depending on the L^{2s+2}-norm of u, that

$$\begin{aligned}\int_\Omega |\nabla(u \min\{|u|^s, L\})|^2 &= \int_\Omega |\nabla u|^2 \min\{|u|^{2s}, L^2\} + s^2 \int_{|(x)|^s \leq L} |\nabla u|^2 |u|^{2s} \\ &\leq c_1 + c_2 \int_\Omega a(x)|u|^2 \min\{|u|^{2s}, L^2\} \\ &\leq c_1 + c_2 K \int_\Omega |u|^2 \min\{|u|^{2s}, L^2\} + c_2 \int_{a(x) \geq K} a(x)|u|^2 \min\{|u|^{2s}, L^2\} \\ &\leq c_3(1+K) + c_2 \left(\int_{a(x) \geq K} |a(x)|^{\frac{N}{2}} \right)^{\frac{2}{N}} \left(\int_\Omega |u \min\{|u|^s, L\}|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}} \\ &\leq c_3(1+K) + \varepsilon(K) \int_\Omega |\nabla(u \min\{|u|^s, L\})|^2\end{aligned}$$

for every $K > 0$, where

$$\varepsilon(K) = c_2 \left(\int_{a(x) \geq K} |a(x)|^{\frac{N}{2}} \right)^{\frac{2}{N}} \to 0$$

for $K \to \infty$. Fix K such that $\varepsilon(K) = \frac{1}{2}$, then

$$\int_\Omega |\nabla(u \min\{|u|^s, L\})|^2 \leq 2c_3(1 + K)$$

for every L, and hence

$$\int_\Omega |u \min\{|u|^s, L\}|^{\frac{2N}{N-2}} \leq c_4$$

by Sobolev's inequality. Letting $L \to \infty$, we deduce that $u|u|^s \in L^{\frac{2N}{N-2}}$, and hence $u \in L^{\frac{(2s+2)N}{N-2}}(\Omega)$. The conclusion now follows by an iteration, starting with $s_0 = 0$. □

14.2 A uniqueness lemma for a linear radial problem with singularity at the origin

Dealing with radially symmetric measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ in the sequel, we will freely write $u(r) = u(x)$ for $r \in [0, \infty[$, $|x| = r$, i.e., we identify u with the associated function on the half-line. The next lemma is our main tool for showing that a radially symmetric setting provides *nondegeneracy* of eigenvalues.

Lemma 14.3. *Consider $N \geq 2$ and and a radially symmetric function $V \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$ such that*

$$\limsup_{x \to 0} \frac{|V(x)|}{|x|^\alpha} < \infty \tag{14.2}$$

for some $\alpha > -2$. Then every distributional solution $u \in W^{1,2}_{loc}(\mathbb{R}^N)$ of the equation

$$-\Delta u = Vu \tag{14.3}$$

is continuous. Moreover, for given $\lambda \in \mathbb{R}$, equation (14.3) has at most one *radially symmetric distributional solution $u \in W^{1,2}_{loc}(\mathbb{R}^N)$ with $u(0) = \lambda$.*
In particular, the trivial solution $u = 0$ is the only continuous radially symmetric weak solution with $u(0) = 0$.

Proof. From (14.2) we easily infer that $V \in L^p_{loc}(\mathbb{R}^N)$ for some $p > \frac{N}{2}$. Therefore every distributional solution $u \in W^{1,2}_{loc}(\mathbb{R}^N)$ of (14.3) is continuous on $\mathbb{R}^N$ by Lemma 14.1, moreover $u \in W^{2,p}_{loc}(\mathbb{R}^N) \cap W^{2,q}_{loc}(\mathbb{R}^N \setminus \{0\})$ for every $q < \infty$. Hence, if u is radially symmetric, then $u \in C([0, \infty[) \cap C^1(]0, \infty[)$ with *absolutely continuous* derivative u', and

$$(r^{N-1}u')' = -r^{N-1}V(r)u, \tag{14.4}$$

holds as an equation in $L^\infty_{loc}(]0,\infty[)$. We now claim

$$q := \lim_{r\to 0} r^{N-1}u'(r) = 0. \tag{14.5}$$

Indeed, note that $\limsup\limits_{r\to 0} |r^{N-1}V(r)u(r)| < \infty$ by virtue of (14.2) and Lemma 14.16(b). Hence (14.4) implies that $r \mapsto r^{N-1}u'(r)$ is uniformly continuous near $r=0$, in particular the limit (14.5) exists. Using this, we deduce

$$\begin{aligned} q^2 &= \lim_{r\to 0}\frac{1}{r}\int_0^r (s^{N-1}u'(s))^2\,ds \\ &\leq \limsup_{r\to 0} r^{N-2}\int_0^r s^{N-1}(u'(s))^2\,ds \\ &= \limsup_{r\to 0} r^{N-2}\int_{B_r(0)} |\nabla u|^2 \\ &= \limsup_{r\to 0} r^{N-2}\|u\|^2 \\ &= 0, \end{aligned}$$

hence $q=0$.
As a consequence, we may write

$$u'(s) = s^{-(N-1)}\int_0^s t^{N-1}V(t)u(t)\,dt$$

for $s>0$. Now for $s>0$ put $\rho(u,s) := \sup\limits_{0\leq t\leq s}|u(t)|$ and $C(s) := \sup\limits_{0\leq t\leq s}\frac{|V(t)|}{t^\alpha}$. Using the assumptions $\alpha > -2$ and $N \geq 2$, we infer

$$\int_0^s t^{N-1}|V(t)||u(t)|\,dt \leq \frac{1}{N+\alpha}s^{N+\alpha}C(s)\rho(u,s)$$

for every $s>0$, hence

$$\begin{aligned} \int_0^r |u'(s)|ds &\leq \int_0^r s^{-(N-1)}\int_0^s t^{N-1}|V(t)||u(t)|\,dt\,ds \\ &\leq \frac{1}{N+\alpha}\int_0^r s^{1+\alpha}C(s)\rho(u,s)\,ds \\ &\leq \frac{1}{(N+\alpha)(2+\alpha)}r^{2+\alpha}C(r)\rho(u,r) \end{aligned} \tag{14.6}$$

for every $r>0$. In particular u is absolutely continuous on $[0,\infty[$, and the relation

$$u(r) = u(0) + \int_0^r u'(s)ds \tag{14.7}$$

holds for every positive r. The proof is complete once we have shown that $u(0) = 0$ implies $u \equiv 0$. Indeed, if $u(0) = 0$, then (14.6) and (14.7) yield the inequality

$$\begin{aligned} |u(s)| &\leq \frac{1}{(N+\alpha)(2+\alpha)} s^{2+\alpha} C(s)\rho(u,s) \\ &\leq \frac{1}{(N+\alpha)(2+\alpha)} r^{2+\alpha} C(r)\rho(u,r) \end{aligned}$$

whenever $0 < s \leq r$. For sufficiently small r this implies $\rho(u,r) \leq d\rho(u,r)$ with some number $d = d(r) < 1$, hence $\rho(u,r) = 0$. In other words, u vanishes in a neighborhood of the origin. We conclude $u \equiv 0$, since on every compact subintervall of $]0,\infty[$ the function u solves a linear ordinary differential equation with bounded coefficients.

□

14.3 Nodal properties of eigenfunctions

As in the previous section, we frequently write $u(r)$ in place of $u(x)$ in case that $r = |x|$ and u is a radially symmetric function.

14.3.1 Operators with form domain $W_0^{1,2}(\Omega)$

We start by stating a fundamental prerequisite to derive nodal estimates:

Lemma 14.4. *Consider a domain $\Omega \subset \mathbb{R}^N$ and a continuous function $u \in W_0^{1,2}(\Omega)$. If Ω' is nodal domain of u, then $v \in W_0^{1,2}(\Omega)$ for the function $v : \Omega \to \mathbb{R}$ given by*

$$v(x) := \begin{cases} u(x) & \textit{for} \quad x \in \Omega' \\ 0 & \textit{for} \quad x \in \mathbb{R}^N \setminus \Omega' \end{cases}$$

and the weak derivative of v is given by

$$\nabla v = 1_{\Omega'} \nabla u,$$

Proof. This is a special case of [55, Lemma 1]. □

In the following we consider $N \geq 2$ and an arbitrary (not necessarily bounded) domain $\Omega \subset \mathbb{R}^N$. To shorten the notation, we put $W_0^{1,2} := W_0^{1,2}(\Omega)$. Moreover, in case that $\Omega = \mathbb{R}^N$, we set

$$W_r^{1,2} := \{u \in W^{1,2}(\mathbb{R}^N) \mid u \text{ radially symmetric}\}.$$

We introduce the notion of W*-admissible potentials*.

Definition 14.5.

(a) *A measurable function $V : \Omega \to \mathbb{R}$ is called a W*-admissible potential *if V can be written as a sum $V = V_1 + V_2$ such that $V_1 \in L^\infty(\Omega)$ and V_2 satisfies the following two conditions:*

(i) For all $v, w \in W_0^{1,2}$ there exists

$$\langle \tilde{V}_2 v, w \rangle := \int_\Omega V_2 v(x) w(x) \tag{14.8}$$

and the thus defined operator $\tilde{V}_2 : W_0^{1,2} \to (W_0^{1,2})^*$ is compact.

(ii) Either $V_2 \in L^{\frac{N}{2}}_{loc}(\Omega)$ or V_2 *is bounded on compact subsets of $\Omega \setminus \Gamma$, where Γ is a closed subset of measure zero such that $\Omega \setminus \Gamma$ is connected.*

(b) In case $\Omega = \mathbb{R}^N$ we call a radially symmetric function $V \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$ a radial W-admissible potential *if V can be written as a sum $V = V_1 + V_2$ of radially symmetric functions such that $V_1 \in L^\infty(\mathbb{R}^N)$ and V_2 satisfies the following two conditions:*

(i) Restricted to radial functions, (14.8) defines a compact operator $\tilde{V}_2 : W_r^{1,2} \to (W_r^{1,2})^*$.

(ii) There is $\alpha > -2$ such that

$$\limsup_{x \to 0} \frac{|V_2(x)|}{|x|^\alpha} < \infty. \tag{14.9}$$

Remark 14.6. (a) W-admissible potentials V are interesting for the following reasons. As a consequence of (i), the quadratic form q, defined on its domain $\mathcal{D}(q) = W_0^{1,2}$ by

$$q(u,v) := \int_\Omega \nabla u(x) \nabla v(x)\, dx + \int_\Omega V(x) u(x) v(x)\, dx,$$

is closed, symmetric and bounded from below in $L^2(\Omega)$ (cf. Lemma 4.1 and the remarks following it). Hence, to q corresponds a (unique) selfadjoint operator $H := -\Delta + V$ with $\mathcal{D}(H) \subset W_0^{1,2}$. Furthermore, condition (ii) implies that eigenfunctions u of H have (at least) the *weak unique continuation property*, i.e. if u vanishes on an open subset of Ω, then $u \equiv 0$. For a proof of the latter assertion, see [39, Theorem 6.3] and [67, p. 519] (cf. also [61, p. 240]).
(b) Analogous implications hold if $\Omega = \mathbb{R}^N$ and V is a *radial W-admissible potential*. Then $H := -\Delta + V$ is given in a natural way as a selfadjoint operator in the Hilbert space

$$L_r^2 := \{u \in L^2(\mathbb{R}^N) \mid u \text{ radially symmetric } \}$$

with form domain $W_r^{1,2}$. In addition, every eigenvalue of H is *nondegenerate* by virtue of Theorem 14.3, and eigenfunctions of H still have the weak unique continuation property.

Theorem 14.7. *Suppose that V is a W-admissible potential. Then, if u is a continuous eigenfunction of $H = -\Delta + V$ associated with an eigenvalue μ_n of the form*

$$\mu_n = \inf_{\substack{V \le W_0^{1,2} \\ \dim V = n}} \sup_{v \in V} \frac{\|\nabla v\|_2^2 + \int_\Omega V v^2}{\|v\|_2^2}, \tag{14.10}$$

the function u has at most n nodal domains.

Proof. Assume in contradiction that u has at least $n+1$ nodal domains $\Omega_1, ..., \Omega_{n+1}$. Recall that the Ω_i are *open* by the continuity of u, and in view of Lemma 14.4 we may define functions $v_i \in W_0^{1,2},\ i = 1, ..., n$ by

$$v_i(x) := \begin{cases} u(x) & \text{for} \quad x \in \Omega_i \\ 0 & \text{for} \quad x \in \mathbb{R}^N \setminus \Omega_i \end{cases}$$

Let Y denote the span of $v_1, ..., v_n$. Then $\dim Y = n$, and a direct calculation shows

$$(Hv|v)_2 = \mu_n \|v\|_2^2 \qquad \text{for all } v \in Y \tag{14.11}$$

Now choose orthonormalized eigenfunctions $u_1, ..., u_{n-1}$ of H corresponding to the eigenvalues $\mu_1, ..., \mu_{n-1}$, and let Z denote the span of $u_1, ..., u_{n-1}$. Since $\dim Z = n-1$, there exists $v \in Y \cap Z^\perp, v \neq 0$, which by (14.11) has to be an eigenfunction of H corresponding to μ_n. However, there holds $v(x) = 0$ for $x \in \Omega_{n+1}$, hence $v \equiv 0$ on Ω by the weak unique continuation property of v. Since this is a contradiction, u has at most n nodal domains. □

In the radial case, Theorem 14.7 can be refined by using the separation properties furnished by Lemma 14.10 below.

Theorem 14.8. *Suppose that $\Omega = \mathbb{R}^N$ and that V is a radial W-admissible potential. Then every eigenvalue of the selfadjoint operator $-\Delta + V$, defined on radial functions as in Remark 14.6(b), is nondegenerate. Moreover, if u is an eigenfunction corresponding to μ_n given by (14.10) (with $W_r^{1,2}$ in place of $W_0^{1,2}$), then u has precisely n nodal domains.*

Proof. Every eigenfunction of H corresponding to μ_n is a weak radial solution of

$$-\Delta u = (\mu_n - V)u,$$

hence u is continuous and unique up to a constant in view of Lemma 14.3. By an analogous reasoning as for proof of Theorem 14.7 (now respecting the rotational invariance of the problem), we infer that u has at most n nodal domains. Moreover, if u_{n-1} is an eigenfunction associated with $\sigma_{n-1} > \sigma_n$, then u_{n-1} weakly solves

$$-\Delta u_{n-1} = (\mu_{n-1} - V)u_{n-1}.$$

Since $V \in L^1_{loc}(\mathbb{R}^N \setminus \{0\})$ by (14.9), Lemma 14.10 ensures that u has at least one more zero than u_{n-1}. The assertion now follows by an inductive argument. □

For the next result fix $\lambda < 0$ and $\Omega = \mathbb{R}^N$, and endow $W_r^{1,2}$ with the scalar product

$$(u, v)_\lambda := \int_{\mathbb{R}^N} \nabla u \nabla v - \lambda \int_{\mathbb{R}^N} uv.$$

Moreover denote by $J_\lambda : W_r^{1,2} \to (W_r^{1,2})^*$ the canonical isometric isomorphism with respect to this scalar product, and for a radial W-admissible potential V consider the operator $\tilde{V} \in \mathcal{L}(W_r^{1,2}, (W_r^{1,2})^*)$ given by

$$\langle \tilde{V} v, w \rangle := \int_{\mathbb{R}^N} V v(x) w(x).$$

Then we have:

Theorem 14.9. *Suppose that $\Omega = \mathbb{R}^N$ and that V is a* nonnegative *radial W-admissible potential. Then for the n-th eigenvalue*

$$\sigma_n := \sup_{\substack{V \le W_r^{1,2} \\ \dim V = n}} \inf_{v \in V} \frac{(Gv|v)_\lambda}{(v|v)_\lambda}$$

of the operator $G := J_\lambda^{-1}\tilde{V} \in \mathcal{L}(W_r^{1,2})$ there holds:
If $\sigma_n > 0$, then σ_n is nondegenerate, and the associated (up to a constant) unique eigenfunction has precisely n nodal domains.

Proof. Note that u is an eigenfunction of G corresponding to $\sigma_n > 0$ if and only if λ is the n-th eigenvalue of the operator $\tilde{H} := -\Delta - \frac{1}{\sigma_n}V$ (restricted to radial functions), and u is a corresponding eigenfunction. Hence the assertion follows directly from Theorem 14.8. $\square$

We close this subsection by proving separation properties of Sturm type.

Lemma 14.10. *Consider $\Omega = \mathbb{R}^N$ and radial W-admissible potentials V_1, V_2 such that $V_1 \le V_2$ on $]0,\infty[$ and*

$(*)$ $\qquad V_1(r) < V_2(r)$ *whenever* $V_1(r) \neq 0 \qquad$ *($r \in]0,\infty[$).*

Moreover suppose that $u_1, u_2 \in W^{1,2}(\mathbb{R}^N)$ are radially symmetric weak solutions of the equations

$$-\Delta u_i = V_i u_i \qquad (i = 1, 2).$$

Then the following implications hold:

(i) If $0 < r_1 < r_2 < \infty$ satisfy $u_1(r_1) = u_1(r_2) = 0$ as well as $u_1(r) \neq 0$ for $r_1 < r < r_2$, then there is $\bar{r} \in (r_1, r_2)$ such that $u_2(\bar{r}) = 0$.

(ii) If $0 < \tilde{r}$ is such that $u_1(\tilde{r}) = 0$ as well as $u_1(r) \neq 0$ for $\tilde{r} < r < \infty$, then there is $\bar{r} \in (\tilde{r}, \infty)$ such that $u_2(\bar{r}) = 0$.

(iii) If $0 < \tilde{r}$ is such that $u_1(\tilde{r}) = 0$ as well as $u_1(r) \neq 0$ for $0 < r < \tilde{r}$, then there is $\bar{r} \in (0, \tilde{r})$ such that $u_2(\bar{r}) = 0$.

(iv) If $u_1(r) \neq 0$ for $0 < r < \infty$, then there is $\bar{r} > 0$ such that $u_2(\bar{r}) = 0$.

Proof. As in the proof of Lemma 14.3 we deduce that $u_i \in C([0,\infty[) \cap C^1(]0,\infty[)$ with absolutely continuous derivatives u_i', and that

$$(r^{N-1}u_i')' = -r^{N-1}g(r)u_i, \qquad (i = 1, 2)$$

considered as equations in $L^\infty_{loc}(0,\infty)$.
Suppose in contradiction that (i) is false. Then we may assume $u_1(r) > 0$, $r \in]r_1, r_2[$ as well as

$u_2(r) > 0$, $r \in]r_1, r_2[$. Moreover, the functions $r \to r^{N-1}u_1'(r)u_2(r)$ and $r \to r^{N-1}u_2'(r)u_1(r)$ are absolutely continuous on $[0, \infty[$ and there holds:

$$\begin{aligned} 0 &\leq \Big[-r^{N-1}u_1'(r)u_2(r)\Big]_{r_1}^{r_2} \\ &= \Big[r^{N-1}u_2'(r)u_1(r) - r^{N-1}u_1'(r)u_2(r)\Big]_{r_1}^{r_2} \\ &= \int_{r_1}^{r_2} \Big[(r^{N-1}u_2'(r))'u_1(r) - (r^{N-1}u_1'(r))'u_2(r)\Big]\,dr \\ &= \int_{r_1}^{r_2} r^{N-1}[V_1(r) - V_2(r)]u_1(r)u_2(r)\,dr. \end{aligned} \tag{14.12}$$

Since $V_1 \leq V_2$, we conclude that $V_1 = V_2$ on $]r_1, r_2[$. Hence $V_1 = 0$ on $]r_1, r_2[$ by assumption $(*)$, and therefore $(r^{N-1}u_1'(r))' = 0$ on $]r_1, r_2[$. This however contradicts $u(r_1) = u(r_2) = 0$, and thus (i) holds true.

(ii) Note first that, by Lemma 14.4, $\varphi := u_1 \cdot 1_{\{x \mid |x| \geq x_1)\}}$ defines an element of $W_0^{1,2}(\mathbb{R}^N)$. Moreover, without loss we have $u_1(r) > 0$, $r \in]\tilde{r}, \infty[$, hence there exists a sequence $(t_k) \subset]\tilde{r}, \infty[$, $t_k \to \infty$ such that $u_1'(t_k) \leq 0$ for all k. Now suppose in contradiction that $u_2(r) > 0$, $r \in [\tilde{r}, \infty[$. Then for each k there holds

$$\begin{aligned} \int_{|x|\geq\tilde{r}} V_1(|x|)u_1(x)u_2(x) &= \lim_{k\to\infty}\int_{\tilde{r}}^{t_k} r^{N-1}V_1(r)u_1(r)u_2(r)\,dr \\ &= \lim_{k\to\infty}\int_{\tilde{r}}^{t_k} (r^{N-1}u_1'(r))'u_2(r)\,dr \\ &= \lim_{k\to\infty}\left[\int_{\tilde{r}}^{t_k} r^{N-1}u_1'(r)u_2'(r)\,dr - r^{N-1}u_1(r)'u_2(r)\Big|_{\tilde{r}}^{t_k}\right] \\ &\geq \lim_{k\to\infty}\int_{\tilde{r}}^{t_k} r^{N-1}u_1'(r)u_2'(r)\,dr \\ &= \int_{|x|\geq\tilde{r}} \nabla u_2(x)\nabla u_1(x)\,dx \\ &= \int_{\mathbb{R}^N} \nabla u_2(x)\nabla\varphi(x)\,dx \\ &= \int_{\mathbb{R}^N} V_2(|x|)u_2(x)\varphi(x)\,dx\,dx \\ &= \int_{|x|\geq\tilde{r}} V_2(|x|)u_2(x)u_1(x)\,dx. \end{aligned}$$

Again we conclude that $V_1 = V_2$ on $]\tilde{r}, \infty[$, which, similar as in the proof of (i) yields a contradiction to $(*)$ and the fact that $u(\tilde{r}) = 0$. Hence (ii) is true as well.

(iii) and (iv) can be proved by similar arguments, using in addition that

$$\lim_{r\to 0} r^{N-1}u_1'(r)u_2(r) = 0 = \lim_{r\to 0} r^{N-1}u_2'(r)u_1(r),$$

cf. (14.5). □

14.3.2 Eigenvalue problems on $D^{1,2}(\mathbb{R}^N)$

Throughout this subsection, we assume $N \geq 3$. We consider eigenvalue problems defined on the Hilbert spaces $D^{1,2} := D^{1,2}(\mathbb{R}^N)$ and $D_r^{1,2} := \{u \in D^{1,2} \mid u \text{ radially symmetric}\}$ respectively. We recall that these spaces carry a canonical scalar product given by

$$(u|v) = \int_{\mathbb{R}^N} \nabla u \nabla v$$

Referring to this scalar product, we denote by $J : D^{1,2} \to (D^{1,2})^*$ the canonical isometric isomorphism.
We have the following analog of Lemma 14.4.

Lemma 14.11. *Consider a continuous function $u \in D^{1,2}$. If Ω' is nodal domain of u, then $v \in D^{1,2}$ for the function $v : \mathbb{R}^N \to \mathbb{R}$ given by*

$$v(x) := \begin{cases} u(x) & \textit{for} \quad x \in \Omega' \\ 0 & \textit{for} \quad x \in \mathbb{R}^N \setminus \Omega', \end{cases}$$

and the weak derivative of v is given by

$$\nabla v = 1_{\Omega'} \nabla u, \tag{14.13}$$

Proof. We apply Lemma 14.4 to the functions $u_n \in W^{1,2}(\mathbb{R}^N)$ defined by

$$u_n(x) := e^{-\frac{|x|^2}{n}} u(x) \qquad (x \in \mathbb{R}^N,\ n \in \mathbb{N}).$$

This yields $v_n \in W^{1,2}(\mathbb{R}^N) \subset D^{1,2}$ for the functions v_n given by

$$v_n(x) := \begin{cases} e^{-\frac{|x|^2}{n}} u(x) & \text{for} \quad x \in \Omega', \\ 0 & \text{for} \quad x \in \mathbb{R}^N \setminus \Omega \end{cases}$$

whereas their weak derivatives write as

$$\nabla v_n(x) = 1_{\Omega'}(x) \left(u(x) \frac{2e^{-\frac{|x|^2}{n}}}{n} x + e^{-\frac{|x|^2}{n}} \nabla u(x) \right).$$

Since the sequence $(\nabla v_n)_n$ converges to the right hand side of (14.13) in the L^2-norm, we deduce the assertion. $\square$

Next we redefine the notion of an admissible potential in a way that it suits to the present context.

Definition 14.12.

(a) A measurable function $V : \mathbb{R}^N \to \mathbb{R}$ is called a D-admissible potential *if the following two conditions hold*

(i) For all $v, w \in D^{1,2}$ *there exists*

$$\langle \tilde{V}v, w\rangle := \int_{\mathbb{R}^N} Vv(x)w(x) \tag{14.14}$$

and the thus defined operator $\tilde{V} : D^{1,2} \to (D^{1,2})^*$ *is compact.*

(ii) Either $V \in L^{\frac{N}{2}}_{loc}(\mathbb{R}^N)$ *or* V *is bounded on compact subsets of* $\mathbb{R}^N \setminus \Gamma$*, where* Γ *is a closed subset of measure zero such that* $\mathbb{R}^N \setminus \Gamma$ *is connected.*

(b) A radially symmetric function $V \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$ *is called a* radial D-admissible potential *if the following two conditions hold*

(i) Restricted to radial functions, (14.14) defines a compact operator $\tilde{V} : D^{1,2}_r \to (D^{1,2}_r)^*$.

(ii) There is $\alpha > -2$ *such that*

$$\limsup_{x \to 0} \frac{|V(x)|}{|x|^\alpha} < \infty. \tag{14.15}$$

We remark that, in contrast to Section 14.3.1, an L^∞-potential is not D-admissible in general. We have a nodal estimate in the spirit of Theorem 14.7.

Theorem 14.13. *Suppose that* V *is a* nonnegative D*-admissible potential. Then, if* $u \in D^{1,2}$ *is a continuous eigenfunction of the compact, symmetric and nonnegative operator* $G := J^{-1}\tilde{V} \in \mathcal{L}(D^{1,2})$ *corresponding to the eigenvalue*

$$\sigma_n := \sup_{\substack{V \le D^{1,2} \\ \dim V = n}} \inf_{v \in V} \frac{(Gv|v)}{(v|v)} \tag{14.16}$$

such that $\sigma_n > 0$*, then* u *has at most* n *nodal domains.*

Proof. We just have to adjust the proof of Theorem 14.7 to the present situation. Indeed, assume in contradiction that u has at least $n+1$ nodal domains $\Omega_1, ..., \Omega_{n+1}$. Using now Lemma 14.11, we define an n-dimensional subspace $Y \subset D^{1,2}$ spanned by the functions $v_i \in D^{1,2},\ i = 1, ..., n$ given by

$$v_i(x) := \begin{cases} u(x) & \text{for} \quad x \in \Omega_i \\ 0 & \text{for} \quad x \in \mathbb{R}^N \setminus \Omega_i. \end{cases}$$

Since again

$$(Gv|v) = \sigma_n \|v\|^2 \qquad \text{for all } v \in Y,$$

there exists an eigenfunction of $v \in Y$ of G associated to σ_n. Moreover $\sigma_n > 0$ by assumption, hence v is a weak solution of

$$-\Delta v = \frac{1}{\sigma_n} Vv. \tag{14.17}$$

Since V is D-admissible, v has the unique continuation property. Hence the fact that $v(x) = 0$ for $x \in \Omega_{n+1}$ forces $v \equiv 0$, and this is a contradiction. Thus u has at most n nodal domains. □

Again Theorem 14.13 can be refined in the *radial* case, now using the following analog of Lemma 14.10.

Lemma 14.14. *Consider $\Omega = \mathbb{R}^N$ and radial D-admissible potentials V_1, V_2 such that $V_1 \le V_2$ on $]0, \infty[$ and*

$$(*) \qquad V_1(r) < V_2(r) \text{ whenever } V_1(r) \neq 0 \qquad (r \in]0, \infty[).$$

Moreover suppose that $u_1, u_2 \in D^{1,2}$ are radially symmetric weak solutions of the equations

$$-\Delta u_i = V_i u_i \qquad (i = 1, 2).$$

Then the assertions (i)-(iv) of Lemma 14.10 hold true again.

Theorem 14.15. *Suppose that V is a nonnegative radial D-admissible potential.*
Then for the eigenvalue σ_n (defined as in (14.16) with $D^{1,2}$ replaced by $D_r^{1,2}$) of the operator $G := J^{-1}\tilde{V} \in \mathcal{L}(D_r^{1,2})$ there holds:
If $\sigma_n > 0$, then σ_n is nondegenerate, and the associated (up to a constant) unique eigenfunction has precisely n nodal domains.

Proof. Every eigenfunction of G corresponding to $\sigma_n > 0$ is a weak solution of

$$-\Delta u = \frac{1}{\sigma_n} V u,$$

hence u is continuous and unique up to a constant in view of Lemma 14.3. By analogous arguments as in the proof of Theorem 14.13 (now respecting the rotational invariance of the problem), we infer that u has at most n nodal domains. Finally, let u_{n-1} denote an eigenfunction associated with $\sigma_{n-1} > \sigma_n$, hence u_{n-1} weakly solves

$$-\Delta u_{n-1} = \frac{1}{\sigma_{n-1}} V u_{n-1}.$$

Since $V \in L^1_{loc}(\mathbb{R}^N \setminus \{0\})$ by (14.15), Lemma 14.14 shows that u has at least one more zero than u_{n-1}. The assertion now follows by an inductive argument. $\square$

14.4 Compact maps involving Sobolev spaces and weighted L^p-spaces

14.4.1 Radial functions

In the sequel we denote $W_r^{1,2} := \{u \in W^{1,2}(\mathbb{R}^N) \mid u \text{ radially symmetric}\}$, and for $N \ge 3$ we put $D_r^{1,2} := \{u \in D^{1,2}(\mathbb{R}^N) \mid u \text{ radially symmetric}\}$. We recall the following pointwise estimates:

Lemma 14.16. *(a) If $N \ge 3$, then every $u \in D_r^{1,2}$ is continuous on $\mathbb{R}^N \setminus \{0\}$, and there holds*

$$|u(r)| \le C_1 \|\nabla u\|_2 r^{\frac{2-N}{2}} \tag{14.18}$$

with a constant $C_1 = C_1(N) > 0$. Moreover, if $0 < r_0 < r_1 < \infty$, then $D_r^{1,2}$ is compactly embedded in $C([r_0, r_1])$ via the identification $u(r) = u(x)$ for $r = |x|$.

(b) *Every* $u \in W_r^{1,2}$ *is continuous on* $\mathbb{R}^N \setminus \{0\}$*, and if* $N \geq 2$*, then for every* $\tau \in [1,2[$ *there is a constant* $C_2 = C_2(\tau, N) > 0$ *such that*

$$|u(r)| \leq C_2(\|\nabla u\|_2 + \|u\|_2) r^{\frac{\tau - N}{2}}. \tag{14.19}$$

Moreover, if $0 < r_0 < r_1 < \infty$*, then* $W_r^{1,2}$ *is compactly embedded in* $C([r_0, r_1])$ *via the identification* $u(r) = u(x)$ *for* $r = |x|$*.*

For the proof of this Lemma we refer to [45, pp. 55] and [51]. We now formulate two results on compact embeddings. More general assumptions providing compact embeddings of $D_r^{1,2}$ resp. $W_r^{1,2}$ are given in [65] and [16].

Lemma 14.17. (a) *Consider* $N \geq 3$*,* $\beta > 0$*, and put* $\alpha = \frac{N-2}{2}\beta - 2$*. If* $a \in L^\infty_{loc}(]0,\infty[)$ *is a positive function satisfying*

$$\lim_{\substack{r \to \infty \\ r \to 0}} r^{-\alpha} a(r) = 0, \tag{14.20}$$

then $D_r^{1,2}(\mathbb{R}^N)$ *is compactly embedded in* $L_a^{\beta+2}$*.*

(b) *Consider* $N \geq 2$*,* $\beta > 0$*, and* $\alpha \in]\frac{N-2}{2}\beta - 2, \frac{N-2}{2}\beta[$*. If* $a \in L^\infty_{loc}(]0,\infty[)$ *is a positive function satisfying*

$$\lim_{\substack{r \to \infty \\ r \to 0}} r^{-\alpha} a(r) = 0,$$

then $W_r^{1,2}(\mathbb{R}^N)$ *is compactly embedded in* $L_a^{\beta+2}$*.*

Proof. (a) This has been proven in [65, Corollary 2.7].
(b) For $N \geq 3$ this follows from [65, Corollary 2.8], hence we restrict our attention to the case $N = 2$ here. Without loss, we may assume that $a(r) = r^\alpha$, $\alpha \in]-2, 0[$. Hence $a \in L^s(B_1(0))$ for some $s > 1$. Now consider the Banach space $C := L^{\beta+2}(\mathbb{R}^2) \cap L^{s'(\beta+2)}(\mathbb{R}^2)$, naturally endowed with the norm $\|u\|_C := \|u\|_{\beta+2} + \|u\|_{s'(\beta+2)}$. By [51, Proposition 1.1], $W_r^{1,2}$ is compactly embedded in C. Moreover, for $u \in C$ there holds

$$\begin{aligned} \int_{\mathbb{R}}^N a(x)|u(x)|^{\beta+2}\,dx &\leq \int_{B_1(0)} a(x)|u(x)|^{\beta+2}\,dx + \int_{\mathbb{R}^N \setminus B_1(0)} a(x)|u(x)|^{\beta+2}\,dx \\ &\leq \|a\|_{L^s(B_1(0))} \|u\|^{\beta+2}_{s'(\beta+2)} + \|u\|^{\beta+2}_{\beta+2}, \end{aligned}$$

hence C is continuously embedded in $L_a^{\beta+2}(\mathbb{R}^2)$. Hence $W_r^{1,2}$ is compactly embedded in $L_a^{\beta+2}(\mathbb{R}^2)$, as claimed. $\square$

Lemma 14.18. *Consider* N, β *and* α *as in Lemma 14.17(a) and suppose that*
$\mathfrak{f} :=]0,\infty[\times [0,\infty[\to \mathbb{R}$ *is a Caratheodory function satisfying*

(i) $\mathfrak{f}(r, \cdot)$ *is nondecreasing on* $[0,\infty[$ *for a.e.* $r \in]0,\infty[$*.*

(ii) $\mathfrak{f}(\cdot, t) \in L^\infty_{loc}(]0,\infty[)$ *for every* $t > 0$*.*

(iii) $\lim\limits_{\substack{r\to\infty\\ r\to 0}} \frac{|\mathfrak{f}(r,t)|}{r^\alpha t^\beta} = 0$ *uniformly in* $t > 0$.

Then:

(a) The relation

$$\langle B(u)v, w\rangle := \int_{\mathbb{R}^N} \mathfrak{f}(|x|, |u(x)|)v(x)w(x)\, dx \qquad (u, v, w \in D_r^{1,2}) \tag{14.21}$$

defines a strongly continuous map $B : D_r^{1,2} \to \mathcal{L}(D_r^{1,2}, (D_r^{1,2})^*)$. *Moreover,* $B(u) \in \mathcal{L}(D_r^{1,2}, (D_r^{1,2})^*)$ *is compact for every* $u \in D_r^{1,2}$.

(b) The integral

$$\varphi(u) := \int_{\mathbb{R}^N} \int_0^{|u(x)|} \mathfrak{f}(|x|, t)t\, dt\, dx \tag{14.22}$$

exists for every $u \in D_r^{1,2}$. *Moreover there holds*

$$2(\varphi(v) - \varphi(v)) \geq \langle B(u)v, v\rangle - \langle B(u)u, u\rangle \tag{14.23}$$

for $u, v \in D_r^{1,2}$.

Proof. (a) Consider an arbitrary bounded subset $M \subset D_r^{1,2}$. Using (i)-(iii) and (14.18), we find a positive function $a \in L^\infty(]0,\infty[)$ with the property (14.20) and such that $|\mathfrak{f}(r, |u(r)|)| \leq a(r)|u(r)|^\beta$ for every $u \in M, r > 0$. By virtue of Lemma 14.17 we have a compact embedding

$$D_r^{1,2} \hookrightarrow L_a^{\beta+2}. \tag{14.24}$$

In particular the range of the map $\mathfrak{f}_* : M \to L_a^{\frac{\beta+2}{\beta}}$ defined by $\mathfrak{f}_*(u)(r) = \frac{\mathfrak{f}(r,|u(r)|)}{a(r)}$ is *bounded* in $L_a^{\frac{\beta+2}{\beta}}$. Now consider a sequence $(u_n) \subset M$ and $u \in M$ such that $u_n \rightharpoonup u$. By Lemma 14.16 we infer $u_n(r) \to u(r)$ a. e. on $]0,\infty[$, hence $\mathfrak{f}_*(u_n)(r) \to \mathfrak{f}_*(u)(r)$ a. e. on $]0,\infty[$ as well. By Lebesgue's Theorem, we conclude

$$\int_{\mathbb{R}^N} a(|x|)|\mathfrak{f}_*(u_n)(x) - \mathfrak{f}_*(u)(x)|^{\frac{\beta+2}{\beta}}\, dx \quad \longrightarrow \quad 0,$$

that is, $\mathfrak{f}_*(u_n) \to \mathfrak{f}_*(u)$ strongly in $L_a^{\frac{\beta+2}{\beta}}$. Now since for $y \in M$ and $v, w \in D_r^{1,2}$ there holds

$$\langle B(y)v, w\rangle = \int_{\mathbb{R}^N} a(|x|)\mathfrak{f}_*(y)vw\, dx,$$

the compact embedding (14.24) and Hölder's inequality show that $B(u_n) \to B(u)$ in $\mathcal{L}(D_r^{1,2}, (D_r^{1,2})^*)$, and that $B(y)$ is a compact linear operator for every $y \in M$. However, $M \subset D_r^{1,2}$ was chosen as an arbitrary bounded subset, and therefore (a) holds true.

(b) Fix $u \in D^{1,2}_r$. Using (i)-(iii) and (14.18) again, we find a positive function $a \in L^\infty(]0,\infty[)$ with the property (14.20) and such that $|\mathfrak{f}(|x|,t)| \le a(|x|)t^\beta$ whenever $0 \le t \le |u(x)|$ and $x \in \mathbb{R}^N \setminus \{0\}$. Hence

$$\int_0^{|u(x)|} \mathfrak{f}(|x|,t)t\,dt \le a(|x|)\frac{|u(x)|^{\beta+2}}{\beta+2} \qquad \text{for } x \in \mathbb{R}^N \setminus \{0\},$$

and now the existence of the integral (14.22) follows by Lemma 14.17(a). Moreover, using (i), we derive for $u, v \in D^{1,2}_r$ and $x \in \mathbb{R}^N \setminus \{0\}$ the inequality

$$2\left(\int_0^{|v(x)|} \mathfrak{f}(|x|,t)t\,dt - \int_0^{|u(x)|} \mathfrak{f}(|x|,t)t\,dt\right) \ge \mathfrak{f}(|x|,|u(x)|)(v^2(x) - u^2(x)) \tag{14.25}$$

by the same argument as in the proof of Lemma 2.4. Integrating (14.25) over $\mathbb{R}^N$ directly leads to (14.23). □

Lemma 14.19. *Consider N, β and α as in Lemma 14.17(b), and suppose that $\mathfrak{f} :=]0,\infty[\times[0,\infty[\to \mathbb{R}$ is a Caratheodory function satisfying conditions (i)-(iii) from Lemma 14.18. Then:*

(a) The relation

$$\langle B(u)v, w\rangle := \int_{\mathbb{R}^N} \mathfrak{f}(|x|,|u(x)|)v(x)w(x)\,dx \qquad (u, v, w \in W^{1,2}_r) \tag{14.26}$$

defines a strongly continuous map $B : W^{1,2}_r \to \mathcal{L}(W^{1,2}_r, (W^{1,2}_r)^*)$. Moreover, $B(u) \in \mathcal{L}(W^{1,2}_r, (W^{1,2}_r)^*)$ is compact for every $u \in W^{1,2}_r$.

(b) For every $u \in W^{1,2}_r$ there exists the integral $\varphi(u)$ as given in (14.22). Moreover, (14.23) holds for every $u, v \in W^{1,2}_r$.

Proof. This can be derived from Lemma 14.17(b) in precisely the same way as in the proof of Lemma 14.18. □

Remark 14.20. Evidently the assertions of Lemma 14.18 and Lemma 14.19 remain valid under the weaker assumption that $\mathfrak{f}$ can be written as a sum of Caratheodory functions $\mathfrak{f}_i$ satisfying conditions (i)-(iii) with *different* constants α_i, β_i.

14.4.2 Nonradial functions

Lemma 14.21. *Consider $p \in [2, \frac{2N}{N-2}[$ and a positive function $w \in L^\infty(\mathbb{R}^N)$ such that $\lim\limits_{|x|\to\infty} w(x) = 0$. Then $W^{1,2}(\mathbb{R}^N)$ is compactly embedded in $L^p_w(\mathbb{R}^N)$.*

Proof. By assumption, $L^p(\mathbb{R}^N) \subset L^p_w(\mathbb{R}^N)$. Hence the ordinary Sobolev embedding extends to an embedding $i : W^{1,2}(\mathbb{R}^N) \to L^p_w(\mathbb{R}^N)$. We show that i is strongly continuous. For this suppose that $u_n \rightharpoonup u$ in $W^{1,2}(\mathbb{R}^N)$ and let $\varepsilon > 0$. Put $M := \sup\limits_{n\in\mathbb{N}} \|u_n\|_p$, then by assumption there is $R > 0$

such that $w(x) \leq \frac{\varepsilon}{M}$ for $|x| \geq R$. Moreover there holds $u_n|_{B_R(0)} \to u|_{B_R(0)}$ in $L^p(B_R(0))$, hence there is $n_0 \in \mathbb{N}$ such that

$$\int_{B_R(0)} w(x)|u_n(x) - u(x)|^p \, dx \leq \varepsilon. \tag{14.27}$$

For these n we infer

$$\begin{aligned}
\int_{\mathbb{R}^N} w(x)|u_n(x) - u(x)|^p \, dx &\leq \int_{B_R(0)} w(x)|u_n(x) - u(x)|^p \, dx \\
&\quad + \int_{\mathbb{R}^N \setminus B_R(0)} w(x)|u_n(x) - u(x)|^p \, dx \\
&\leq \varepsilon + \frac{\varepsilon}{M} \int_{\mathbb{R}^N} (|u_n(x) - u(x)|^p \, dx \\
&\leq 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude $u_n \to u$ in $L^p_w(\mathbb{R}^N)$. Hence i is strongly continuous, as claimed. □

In the following we abbreviate $W^{1,2}(\mathbb{R}^N)$ to $W^{1,2}$, and from Lemma 14.21 we derive an analog of Lemma 14.19 for the nonradial situation.

Lemma 14.22. *Consider w as in Lemma 14.21, $\beta \in [0, \frac{4}{N-2}[$ and suppose that $f : \mathbb{R}^N \times [0, \infty[\to \mathbb{R}$ is a Caratheodory function satisfying*

$$|f(x,t)| \leq w(x)t^\beta \qquad (x \in \mathbb{R}^N, \ t \geq 0) \tag{14.28}$$

. Then:

(a) The relation

$$\langle B(u)v, w\rangle := \int_{\mathbb{R}^N} f(x, |u(x)|)v(x)w(x) \, dx \qquad (u, v, w \in W^{1,2}) \tag{14.29}$$

defines a strongly continuous map $B : W^{1,2} \to \mathcal{L}(W^{1,2}, (W^{1,2})^*)$. Moreover, $B(u) \in \mathcal{L}(W^{1,2}, (W^{1,2})^*)$ is compact for every $u \in W^{1,2}$.

(b) For every $u \in W^{1,2}$ there exists the integral

$$\varphi(u) = \int_{\mathbb{R}^N} \int_0^{|u(x)|} f(x,t)t \, dt \, dx.$$

Moreover, if $f(x, \cdot)$ is nondecreasing on $[0, \infty[$ for a.e. $x \in \mathbb{R}^N$, then

$$2(\varphi(v) - \varphi(u)) \geq \langle B(u)v, v\rangle - \langle B(u)u, u\rangle$$

for every $u, v \in W^{1,2}$.

Proof. Again, this can be deduced from Lemma 14.21 by a similar reasoning as in the proof of Lemma 14.18. □

Remark 14.23. Evidently the assertion of Lemma 14.22 remain valid under the weaker assumption that f can be written as a sum of Caratheodory functions f_i satisfying condition (14.28) with *different* constants $\beta_i \in [0, \frac{4}{N-2}[$.

List of symbols

Abstract conditions and properties

Bibliography

[1] Adams, R. A., *Sobolev spaces*, Academic Press, New York-San Francisco-London, 1975

[2] Achieser, N. I. and Glasmann, I. M., *Theorie der linearen Operatoren im Hilbert-Raum*, Akademie-Verlag Berlin, 1968

[3] Ambrosetti, A. and Rabinowitz, P. H., Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **14** (1973), 349-381

[4] Aizenman, M. and Simon, B: Brownian Motion and Harnack Inequality for Schrödinger Operators. *Comm. Pure Appl. Math.* **35** (1982), 209–271.

[5] Atiyah, M. F., *K-Theory*, W. A. Benjamin, New York-Amsterdam, 1967

[6] Bartsch, T., *Topological methods for Variational Problems with Symmetries*, Springer-Verlag Berlin Heidelberg, 1993

[7] Bartsch, T., Critical point theory on partially ordered Hilbert spaces. *to appear in J. Funct. Anal.*

[8] Bartsch, T., Chang, K.-C. and Wang, Z.-Q., On the Morse indices of sign changing solutions of nonlinear elliptic problems. *Math. Z.* **233** (2000), 655–677

[9] Bartsch, T. and Willem, M., Infinitely many radial solutions of a semilinear elliptic problem on $\mathbb{R}^N$. *Arch. Rat. Mech. Anal.* **124** (1993), 261–276

[10] Bartsch, T. and Wang, Z.-Q., On the existence of sign changing solutions for semilinear Dirichlet problems. *Topol. Methods Nonlinear Anal.* **7** (1996), 115-131

[11] Bartsch, T. and Wang, Z.-Q., Sign changing solutions of nonlinear Schrödinger equations. *Topological Methods in Nonlinear Anal.* **13** (1999), 191-198

[12] Benci, V., On critical point theory for indefinite functionals in the presence of symmetries. *Trans. Amer. Math. Soc.* **274** (1982), 533-572

[13] Benci, V. and and Fortunato, D., A remark on the nodal regions of the solutions of some superlinear equations. *Proc. Roy. Soc. Edinburgh* **111A** (1989), 123-128

[14] Birkhoff, G. D., Dynamical systems with two degrees of freedom. *Trans. Amer. Math. Soc.* **18** (1917), 199-300

[15] Castro, A., Cossio, J. and Neuberger, J., Sign changing solutions for a superlinear Dirichlet problem. *Rocky Mountain J. Math.* **27** (1997), 1041-1053

[16] Chabrowski, J., On compact embeddings of radial Sobolev spaces and their applications. *Commun. in Partial Diff. Equat.* **17** (1992), 941-952

[17] Chabrowski, J., On nodal radial solutions of an elliptic problem involving critical Sobolev exponent. *Comment. Math. Univ. Carolinae* **37** (1996), 1-16

[18] Coffman, C. V., A minimum-maximum principle for a class of nonlinear integral equations. *J. Anal. Math.* **22** (1969), 391-419

[19] Coffman, C. V., On variational principles for sublinear boundary value problems. *J. Differential Equations* **17** (1975), 46-60

[20] Coffman, C. V., Ljusternik-Schnirelman theory, complementary principles and the Morse index. *Nonlinear Anal.* **12** (1988), 507-529.

[21] Conti, M., Merizzi, L., Terracini, S., Radial solutions of superlinear equations on $\mathbb{R}^N$. I. A global variational approach. *Arch. Rat. Mech. Anal.* **153** (2000), 291-316

[22] Courant, R. and Hilbert, D., *Methoden der mathematischen Physik I*, Springer-Verlag Berlin 1968

[23] Dancer, E. and Du, Y., Multiple solutions of some semilinear elliptic equations via the generalized Conley index. *J. Math. Anal. Appl.* **189** (1995), 848-871

[24] Davies, E. B., *Spectral Theory and Differential Operators*, Cambridge University Press, 1995

[25] de Figueiredo, D. G. and Gossez, J. P., Strict monotonicity of eigenvalues and unique continuation. *Comm. Partial Diff. Equat.* **17** (1992), 339-346

[26] Dolbeault, J., Esteban, M. J., and Séré, E., On the eigenvalues of operators with gaps. Application to Dirac operators. *J. Funct. Anal.* **174**, (2000), 208-226

[27] Duady, A., Le probléme des modules pour les sous-espaces analytiques compacts d'un espace analytique donné. *Ann. Inst. Fourier* **16** (1966), 1-95

[28] Eastham, M. P. S., *The Spectral Theory of Periodic Differential equations*, Scottish Academic Press, Edinburgh, 1973

[29] Euler, L., Methodus inveniendi lineas curvas mximi minimive proprietate gaudentes sive solutio problematis isoperimetrici latissimo sensu accepti. Lausanne-Genève (1944), *Opera, Ser. I, Vol. 24 (ed. C. Carathéodory)*, Bern 1952

[30] Fröhlich, J., Tsai, T.-P., Yau, H.-T., On a classical limit of quantum theory and the non-linear Hartree equation. *Conférence Mosh Flato* 1999, **Vol. I** (Dijon), 189–207, *Math. Phys. Stud.*, **21**, Kluwer Acad. Publ., Dordrecht, 2000

[31] Gilbarg , D. and Trudinger, N.S., *Elliptic Partial Differential Equations of Second Order*, 2nd edition, Springer, Berlin-Heidelberg-New York-Tokyo 1983

[32] Griesemer, M. and Siedentop, H., A minimax principle for the eigenvalues in spectral gaps. *J. London Math. Soc. (2)* **60** (1999), 490-500

[33] Heid, M. and Heinz, H.-P., Nonlinear Eigenvalue Problems Admitting Eigenfunctions With Known Geometric Properties. *Topological Methods in Nonlinear Anal.* **13** (1999), 17-51

[34] Heid, M., Heinz, H.-P. and Weth, T., Nonlinear Eigenvalue Problems Of Schrödinger Type Admitting Eigenfunctions With Given Spectral Characteristics. *to appear in Math. Nach.*

[35] Heinz, H. P., Nodal properties and variational characterizations of solutions to nonlinear Sturm-Liouville problems. *J. Differential Equations* **62** (1986), 299-333

[36] Heinz, H. P., Free Ljusternik-Schnirelman theory and the bifurcation diagrams of certain singular nonlinear problems. *J. Differential Equations* **66** (1986), 263-300

[37] Hempel, J., Multiple solutions for a class of nonlinear boundary value problems. *Indiana Univ. Math. J.* **20** (1971), 983-996

[38] Husemoller, D., *Fibre Bundles*, McGraw-Hill, New York 1966

[39] Jerison, D. and Kenig, C. E., Unique continuation and absence of positive eigenvalues for Schrödinger operators. *Annals of Mathematics* **121** (1985), 463-494

[40] Jost, J., *Partielle Differentialgleichungen*, Springer-Verlag Berlin Heidelberg 1998

[41] Jones, C.R.K.T. and Küpper, T., On infinitely many solutions of a semilinear elliptic equation. *SIAM J. Math. Anal.* **17** (1986), 803-835

[42] Kato, T., *Perturbation theory for linear operators*, Springer-Verlag Berlin Heidelberg 1966

[43] Kusano, T. and Naito, M., Positive entire solutions of second order superlinear elliptic equations. *Hiroshima Math J.* **16** (1986), 361-366

[44] Kusano, T. and Naito, M., Oscillatory theory of entire solutions of second order superlinear elliptic equations. *Funkcial. Ekvac.* **30** (1987), 269-282.

[45] Kuzin, I. and Pohozaev, S., *Entire Solutions of Semilinear Elliptic Equations*, Birkhäuser-Verlag Basel 1991

[46] Koschorke, U., Infinite dimensional K-Theory and characteristic classes of Fredholm bundle maps. *Proc. Sympos. Pure Math.*, vol. XV, Amer. Math. Soc., Providence, R.I. (1970), 95-133

[47] Krasnosel'skii, M.A., *Topological Methods in the Theory of Integral Equations*, Pergamon Press, Oxford-London-New York-Paris 1964

[48] Kinderlehrer, D. and Stampaccia, G., *An introduction to variational inequalities and their applications*, Academic Press, New York-London-Toronto-Sydney-San Francisco 1980

[49] Ladyzhenskaya, O. A., and Ural'tseva, N. N., *Linear and Quasilinear Elliptic Equations*, Academic Press, New York and London 1968

[50] Lions, P.-L., The Choquard equation and related questions. *Nonlinear Anal., Theory, Meth. Appl.* **Vol. 5, No 3** (1980), 1063-1073

[51] Lions, P.-L., Symétrie et compacité dans les espaces de Sobolev. *J. Funct. Anal.*, **49** (1982), 315-334

[52] Lions, P.-L., Solutions of Hartree-Fock Equations for Coulomb Systems. *Commun. Math. Phys.* **109** (1987), 33-97

[53] Michael, E., Topologies on spaces of subsets. *Trans. Amer. Math. Soc.* **71** (1951), 152-182

[54] Ljusternik, L. and Schnirelman, L., Méthodes topologiques dans les problèmes variationelles. *Actualites Sci. Industr.* **188**, Paris (1934)

[55] Müller-Pfeiffer, E., On the number of nodal domains for elliptic differential operators. *J. London Math. Soc. (2)*, **31**, (1985), 91-100

[56] Naito, Y., Bounded solutions with prescribed number of zeros for the Emden-Fowler differential equation. *Hiroshima Math. J.* **24** (1994), 177-220

[57] Nehari, Z., Characteristic values associated with a class of nonlinear second-order differential equations. *Acta. Math.* **105** (1961), 141-175

[58] Rabinowitz, P. H., Some global results for nonlinear eigenvalue problems. *J. Funct. Anal.* **7** (1971), 487-513

[59] Reed, M. and Simon, B., *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York, 1972

[60] Reed, M. and Simon, B., *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press, New York, 1975

[61] Reed, M. and Simon, B., *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press, New York, 1978

[62] Ryder, G. H., Boundary value problems for a class of nonlinear differential equations. *Pacific J. Math. 22* (1967), 477-503

[63] Schechter, M., *Spectra of partial differential operators*, North-Holland, Amsterdam, 1971

[64] Schneider, M., Compact embeddings and indefinite semilinear elliptic problems. *to appear in Nonlinear Anal.*

[65] Schneider, M., *Entire solutions of semilinear elliptic problems with indefinite nonlinearities*, Ph.D. Thesis, Johannes Gutenberg-Universität Mainz, 2001

[66] Spanier, E. H., *Algebraic Topology*, McGraw-Hill, New York 1966

[67] Simon, B., Schrödinger Semigroups. *Bulletin of the A.M.S.* **vol. 5, No. 3** (1982), 447–526

[68] Struwe, M., *Variational Methods*, 2nd edition, Springer-Verlag Berlin Heidelberg New York 1996

[69] Struwe, M., Superlinear elliptic boundary value problems with rotational symmetry. *Arch. Math.* **39** (1982), 233-240

[70] Stuart, C., Existence theory for the Hartree equation. *Arch. Rat. Mech. Anal.* **51** (1973), 60-69

[71] Stuart, C., An example in nonlinear functional analysis: the Hartree equation. *J. Math. Anal. Appl.* **49** (1975), 725-733

[72] Stuart, C., Bifurcation in $L^p(\mathbb{R}^N)$ for a semilinear elliptic equation. *Proc. Lond. Math. Soc. III* **57, No. 3** (1988), 511-541

[73] Stuart, C., Bifurcation from the essential spectrum for some noncompact nonlinearities. *Math. Meth. Appl. Sci.* **11** (1989), 525-542

[74] Stuart, C., *Bifurcation from the essential spectrum*, in: Matzeu, M. and Vignoli, A., editors, Topological Methods in Nonlinear Analysis II, Progress in Nonlinear Differential Equations and their Applications, 27, Birkhäuser Boston, M. A. 1997

[75] Tshinanga, S.B., Positive and Multiple Solutions of Subcritical Emden-Fowler Equations. *Differential and Integral Equations* **9** (1996), 363-370

[76] Wolkowisky, J. H., Existence of solutions of Hatree equations for N electrons. An application of the Schauder-Tychonoff theorem. *Indiana Univ. Math. J.* **22** (1972), 551-568

[77] Zeidler, E., *Nonlinear Functional Analysis And Its Applications, Vol. III: Variational Methods And Optimization*, Springer-Verlag New York 1986

[78] Zeidler, E., Ljusternik-Schnirelman theory on general level sets. *Math. Nachr.* **129** (1986), 235-259.